Chapter 9

9.1

1. $x(t) = 7e^{5t}$, by Fact 9.1.1.

2. $x(t) = -e \cdot e^{-0.71t} = -e^{1-0.71t}$, by Fact 9.1.1.

3. $P(t) = 7e^{0.03t}$, by Fact 9.1.1.

4. This is just an antiderivative problem: $y(t) = 0.8\frac{t^2}{2} + C = 0.4t^2 + C$, and $C = -0.8$, so that $y(t) = 0.4t^2 - 0.8$.

5. $y(t) = -0.8e^{0.8t}$, by Fact 9.1.1.

6. $x\,dx = dt$

   $\frac{x^2}{2} = t + C$, and $\frac{1}{2} = 0 + C$, so that $\frac{x^2}{2} = t + \frac{1}{2}$

   $x^2 = 2t + 1$

   $x(t) = \sqrt{2t+1}$

7. $x^{-2}\,dx = dt$

   $-x^{-1} = t + C$

   $-\frac{1}{x} = t + C$, and $-1 = 0 + C$, so that

   $-\frac{1}{x} = t - 1$

   $x(t) = \frac{1}{1-t}$; note that $\lim\limits_{x \to 1^-} x(t) = \infty$.

8. $x^{-1/2}\,dx = dt$

   $2x^{1/2} = t + C$, and $2\sqrt{4} = 0 + C$, so that $2x^{1/2} = t + 4$.

   $x(t) = \left(\frac{t}{2} + 2\right)^2$ for $t \geq -4$.

9. $x^{-k}\,dx = dt$

   $\frac{1}{1-k}x^{1-k} = t + C$, and $\frac{1}{1-k} = C$, so that

   $\frac{1}{1-k}x^{1-k} = t + \frac{1}{1-k}$

   $x^{1-k} = (1-k)t + 1$

   $x(t) = ((1-k)t+1)^{1/1-k}$.

10. $\cos x\, dx = dt$

    $\sin x = t + C$, and $C = 0$.

    $x(t) = \arcsin(t)$ for $|t| < 1$.

11. $\frac{dx}{1+x^2} = dt$

    $\arctan(x) = t + C$ and $C = 0$.

    $x(t) = \tan(t)$ for $|t| < \frac{\pi}{2}$.

12. We want $e^{kt} = 3^t$ or $e^k = 3$ or $k = \ln(3) : \frac{dx}{dt} = \ln(3)x$.

13. a. The debt in millions is $0.45(1.06)^{212} \approx 104,245$, or about 100 billion dollars.

    b. The debt in millions is $0.45e^{0.06 \cdot 212} \approx 150,466$, or about 150 billion dollars.

14. a. $x(t) = e^{-\frac{t}{8270}}$, by Fact 9.1.1

    If T is the half-life, then $e^{-\frac{T}{8270}} = \frac{1}{2}$ or $-\frac{T}{8270} = \ln\left(\frac{1}{2}\right)$ or $T = -8270\ln\left(\frac{1}{2}\right) \approx 5732$.

    The half-life is about 5732 years.

    b. We want to find t such that $e^{-\frac{t}{8270}} = 1 - 0.47 = 0.53$ or $-\frac{t}{8270} = \ln(0.53)$ or $t = -8270\ln(0.53) \approx 5250$. The Iceman died about 5000 years before A.D. 1991, or about 3000 B.C. The Austrian expert was wrong.

15. If $P(t) = P_0 e^{\frac{k}{100}t}$, then the doubling time T satisfies the equation $P(T) = P_0 e^{\frac{k}{100}T} = 2P_0$ or $e^{\frac{k}{100}T} = 2$ or $\frac{k}{100}T = \ln(2)$ or $T = \frac{100}{k}\ln(2) \approx \frac{69}{k}$ since $\ln(2) \approx 0.69$.

16. See Figure 9.1.

17. See Figure 9.2.

18. See Figure 9.3.

19. See Figure 9.4.

20. $A\vec{x} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -x_2 \\ x_1 \end{bmatrix}$. See Figure 9.5.

    It appears that the trajectories will be circles. If we start at $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ we will trace out the unit circle $\vec{x}(t) = \begin{bmatrix} \cos(t) \\ \sin(t) \end{bmatrix}$.

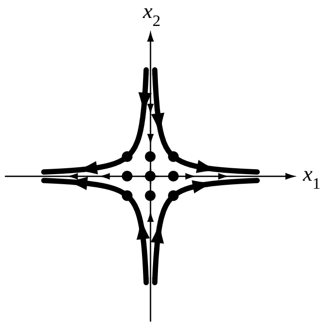

Figure 9.1: for Problem 9.1.16.

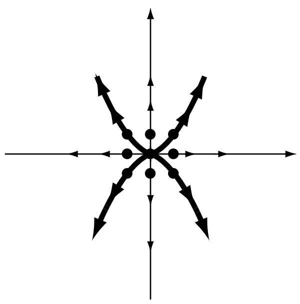

Figure 9.2: for Problem 9.1.17.

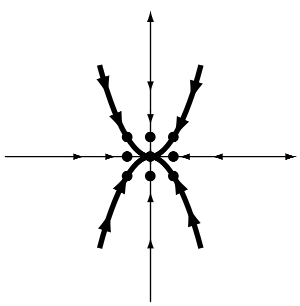

Figure 9.3: for Problem 9.1.18.

We can verify that $\frac{d\vec{x}}{dt} = \begin{bmatrix} -\sin(t) \\ \cos(t) \end{bmatrix}$ equals $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \vec{x}(t) = \begin{bmatrix} -\sin(t) \\ \cos(t) \end{bmatrix}$, as claimed.

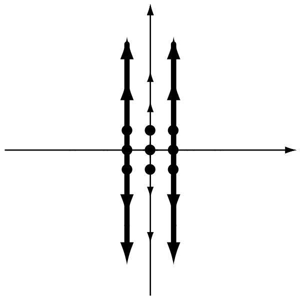

Figure 9.4: for Problem 9.1.19.

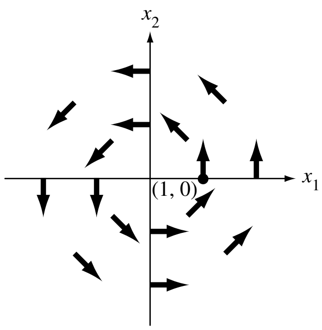

Figure 9.5: for Problem 9.1.20.

21. $A\vec{x} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_2 \\ 0 \end{bmatrix}$ (see Figure 9.6).

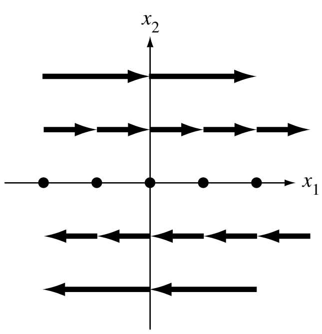

Figure 9.6: for Problem 9.1.21.

The trajectories will be horizontal lines. If we start at $\begin{bmatrix} p \\ q \end{bmatrix}$, then the horizontal velocity will be q, so that $\vec{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} p+qt \\ q \end{bmatrix}$. We can verify that $\frac{d\vec{x}}{dt} = \begin{bmatrix} q \\ 0 \end{bmatrix}$ equals $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \vec{x}(t) = \begin{bmatrix} q \\ 0 \end{bmatrix}$, as claimed.

22. We are told that $\frac{d\vec{x}_1}{dt} = A\vec{x}_1$ and $\frac{d\vec{x}_2}{dt} = A\vec{x}_2$. Let $\vec{x}(t) = \vec{x}_1(t) + \vec{x}_2(t)$. Then $\frac{d\vec{x}}{dt} = \frac{d\vec{x}_1}{dt} + \frac{d\vec{x}_2}{dt} = A\vec{x}_1 + A\vec{x}_2 = A(\vec{x}_1 + \vec{x}_2) = A\vec{x}$, as claimed.

23. We are told that $\frac{d\vec{x}_1}{dt} = A\vec{x}_1$. Let $\vec{x}(t) = k\vec{x}_1(t)$. Then $\frac{d\vec{x}}{dt} = \frac{d}{dt}(k\vec{x}_1) = k\frac{d\vec{x}_1}{dt} = kA\vec{x}_1 = A(k\vec{x}_1) = A\vec{x}$, as claimed.

24. We are told that $\frac{d\vec{x}}{dt} = A\vec{x}$. Let $\vec{c}(t) = e^{kt}\vec{x}(t)$. Then $\frac{d\vec{c}}{dt} = \frac{d}{dt}(e^{kt}\vec{x}) = \left(\frac{d}{dt}e^{kt}\right)\vec{x} + e^{kt}\frac{d\vec{x}}{dt} = ke^{kt}\vec{x} + e^{kt}A\vec{x} = (A + kI_n)(e^{kt}\vec{x}) = (A + kI_n)\vec{c}$, as claimed.

25. We are told that $\frac{d\vec{x}}{dt} = A\vec{x}$. Let $\vec{c}(t) = \vec{x}(kt)$. Using the chain rule we find that $\frac{d\vec{c}}{dt} = \frac{d}{dt}(\vec{x}(kt)) = k\frac{d\vec{x}}{dt}\big|_{kt} = kA(\vec{x}(kt)) = kA\vec{c}(t)$, as claimed.

   To get the vector field $kA\vec{c}$ we scale the vectors of the field $A\vec{x}$ by k.

26. $\lambda_1 = 3$, $\lambda_2 = -2$; $\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} -2 \\ 3 \end{bmatrix}, c_1 = 5, c_2 = -1$, so that $\vec{x}(t) = 5e^{3t}\begin{bmatrix} 1 \\ 1 \end{bmatrix} - e^{-2t}\begin{bmatrix} -2 \\ 3 \end{bmatrix}$.

27. Use Fact 9.1.3.

   The eigenvalues of $A = \begin{bmatrix} -4 & 3 \\ 2 & -3 \end{bmatrix}$ are $\lambda_1 = -6$ and $\lambda_2 = -1$, with associated eigenvectors $\vec{v}_1 = \begin{bmatrix} -3 \\ 2 \end{bmatrix}$ and $\vec{v}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. The coordinates of $\vec{x}(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ with respect to $\vec{v}_1$ and $\vec{v}_2$ are $c_1 = -\frac{1}{5}$ and $c_2 = \frac{2}{5}$.

   By Fact 9.1.3 the solution is $\vec{x}(t) = -\frac{1}{5}e^{-6t}\begin{bmatrix} -3 \\ 2 \end{bmatrix} + \frac{2}{5}e^{-t}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

28. $\lambda_1 = 2$, $\lambda_2 = 10$; $\vec{v}_1 = \begin{bmatrix} -3 \\ 2 \end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$; $c_1 = -\frac{1}{8}$, $c_2 = \frac{5}{8}$, so that $\vec{x}(t) = -\frac{1}{8}e^{2t}\begin{bmatrix} -3 \\ 2 \end{bmatrix} + \frac{5}{8}e^{10t}\begin{bmatrix} 1 \\ 2 \end{bmatrix}$.

29. $\lambda_1 = 0$, $\lambda_2 = 5$; $\vec{v}_1 = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$; $c_1 = -2$, $c_2 = 1$,

so that $\vec{x}(t) = -2\begin{bmatrix}-2\\1\end{bmatrix} + e^{5t}\begin{bmatrix}1\\2\end{bmatrix} = \begin{bmatrix}4\\-2\end{bmatrix} + e^{5t}\begin{bmatrix}1\\2\end{bmatrix}$.

30. $\lambda_1 = 0$, $\lambda_2 = 5$, $\vec{v}_1 = \begin{bmatrix}-2\\1\end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix}1\\2\end{bmatrix}$; $c_1 = -1$, $c_2 = 0$, so that $\vec{x}(t) = -\begin{bmatrix}-2\\1\end{bmatrix} = \begin{bmatrix}2\\-1\end{bmatrix}$.

31. $\lambda_1 = 1$, $\lambda_2 = 6$, $\lambda_3 = 0$; $\vec{v}_1 = \begin{bmatrix}1\\-2\\1\end{bmatrix}$. Since $\vec{x}(0) = \vec{v}_1$ we need not find $\vec{v}_2$ and $\vec{v}_3$.

$c_1 = 1, c_2 = c_3 = 0$, so that $\vec{x}(t) = e^t\begin{bmatrix}1\\-2\\1\end{bmatrix}$.

In Exercises 32 to 35, find the eigenvalues and eigenspaces. Then determine the direction of the flow along the eigenspaces (outward if $\lambda > 0$ and inward if $\lambda < 0$). Use Figure 11 of Section 9.1 as a guide to sketch the other trajectories.

32. See Exercise 26 and Figure 9.7.

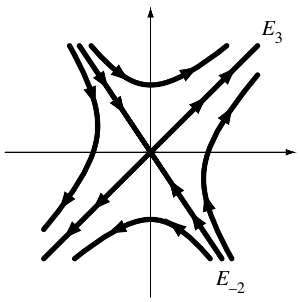

Figure 9.7: for Problem 9.1.32.

33. See Exercise 27 and Figure 9.8.

34. See Exercise 28 and Figure 9.9.

35. See Exercise 29 and Figure 9.10.

In Exercises 36 to 39, find the eigenvalues and eigenspaces (the eigenvalues will always be positive). Then determine the direction of the flow along the eigenspaces (outward if $\lambda > 1$ and inward if $1 > \lambda > 0$). Use Figure 11 of Section 7.1 as a guide to sketch the other trajectories.

36. See Figure 9.11.

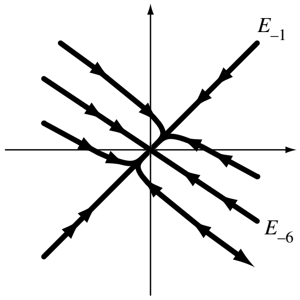

Figure 9.8: for Problem 9.1.33.

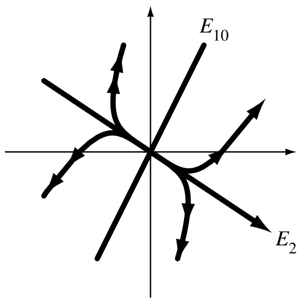

Figure 9.9: for Problem 9.1.34.

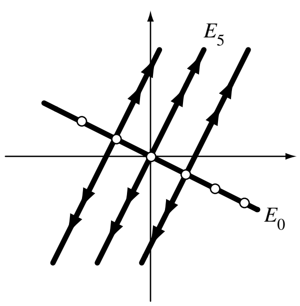

Figure 9.10: for Problem 9.1.35.

37. See Figure 9.12.

38. See Figure 9.13.

39. See Figure 9.14.

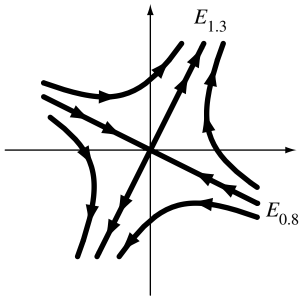

Figure 9.11: for Problem 9.1.36.

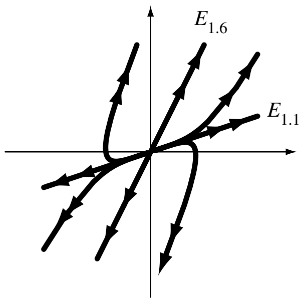

Figure 9.12: for Problem 9.1.37.

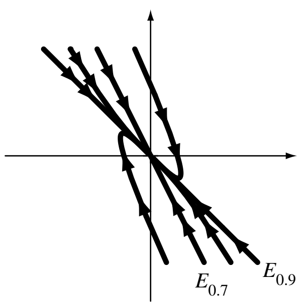

Figure 9.13: for Problem 9.1.38.

40. $\vec{x}(t) = e^{2t}\begin{bmatrix} 2 \\ 3 \end{bmatrix} + e^{3t}\begin{bmatrix} 3 \\ 4 \end{bmatrix}$

We want a 2×2 matrix A with eigenvalues $\lambda_1 = 2$ and $\lambda_2 = 3$ and associated eigenvectors

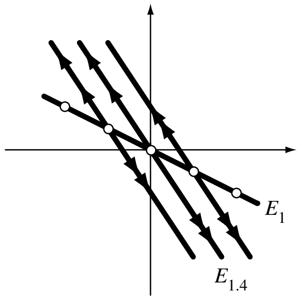

Figure 9.14: for Problem 9.1.39.

$\vec{v}_1 = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$ and $\vec{v}_2 = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$; that is $A \begin{bmatrix} 2 & 3 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 4 & 9 \\ 6 & 12 \end{bmatrix}$ or $A = \begin{bmatrix} 4 & 9 \\ 6 & 12 \end{bmatrix} \begin{bmatrix} 2 & 3 \\ 3 & 4 \end{bmatrix}^{-1} =$ $\begin{bmatrix} 4 & 9 \\ 6 & 12 \end{bmatrix} \begin{bmatrix} -4 & 3 \\ 3 & -2 \end{bmatrix} = \begin{bmatrix} 11 & -6 \\ 12 & -6 \end{bmatrix}$.

41. The trajectories are of the form $\vec{x}(t) = c_1 e^{\lambda_1 t} \vec{v}_1 + c_2 e^{\lambda_2 t} \vec{v}_2 = c_1 \vec{v}_1 + c_2 e^{\lambda_2 t} \vec{v}_2$. See Figure 9.15.

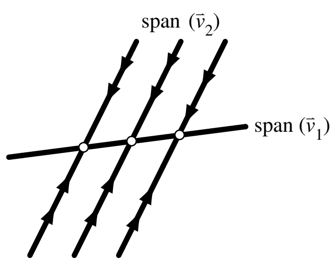

Figure 9.15: for Problem 9.1.41.

42. a. The term $0.8x$ in the second equation indicates that species y is helped by x, while species x is hindered by y (consider the term $-1.2y$ in the first equation). Thus y preys on x.

b. See Figure 9.16.

c. If $\frac{y(0)}{x(0)} < 2$ then both species will prosper, and $\lim_{t \to \infty} \frac{y(t)}{x(t)} = \frac{1}{3}$.

If $\frac{y(0)}{x(0)} \geq 2$ then both species will die out.

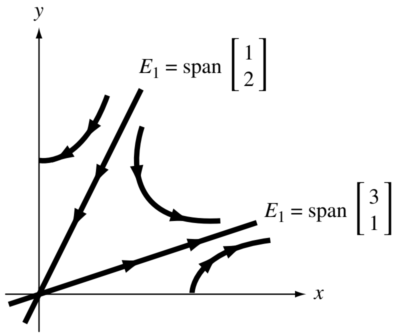

Figure 9.16: for Problem 9.1.42b.

43. a. These two species are *competing* as each is hindered by the other (consider the terms $-y$ and $-2x$).

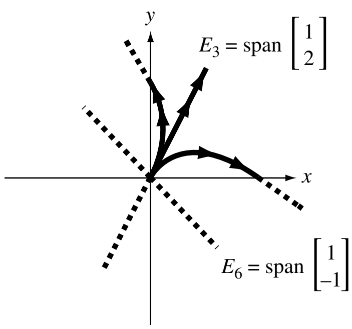

Figure 9.17: for Problem 9.1.43b.

b. Although only the first quadrant is relevant for our model, it is useful to consider the phase portrait in the other quadrants as well. See Figure 9.17.

c. If $\frac{y(0)}{x(0)} > 2$ then species y wins (x will die out); if $\frac{y(0)}{x(0)} < 2$ then x wins. If $\frac{y(0)}{x(0)} = 2$ then both will prosper and $\frac{y(t)}{x(t)} = 2$ for all t.

44. a. The two species are in symbiosis: Each is helped by the other (consider the terms $4y$ and $2x$).

b. See Figure 9.18.

c. Both populations will prosper and $\lim_{t\to\infty} \frac{y(t)}{x(t)} = \frac{1}{2}$, regardless of the initial populations.

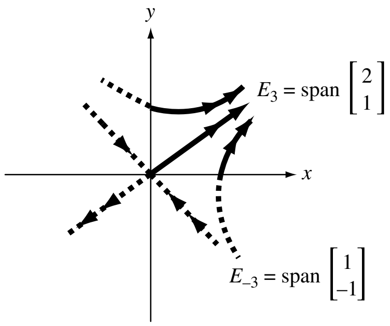

Figure 9.18: for Problem 9.1.44b.

45. a. Species y has the more vicious fighters, since they kill members of species x at a rate of 4 per time unit, while the fighters of species x only kill at a rate of 1.

    b. See Figure 9.19.

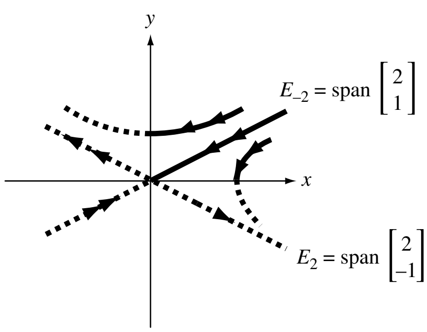

Figure 9.19: for Problem 9.1.45b.

    c. If $\frac{y(0)}{x(0)} < \frac{1}{2}$ then x wins; if $\frac{y(0)}{x(0)} > \frac{1}{2}$ then y wins; if $\frac{y(0)}{x(0)} = \frac{1}{2}$ nobody will survive the battle.

46. Look at the *phase portrait* in Figure 9.20.

47. a. The two species are in symbiosis: Each is helped by the other (consider the positive terms kx and ky).

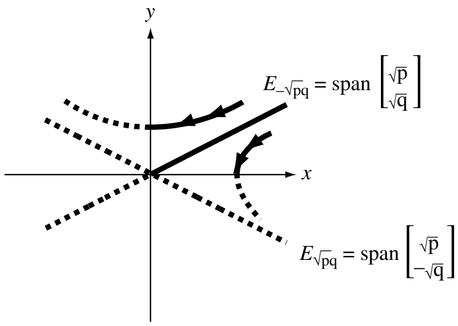

Figure 9.20: for Problem 9.1.46.

b. $\lambda_{1,2} = \frac{-5\pm\sqrt{9+4k^2}}{2}$

Both eigenvalues are negative if $\sqrt{9+4k^2} < 5$ or $9+4k^2 < 25$ or $4k^2 < 16$ or $k < 2$ (recall that k is positive).

If $k = 2$ then the eigenvalues are -5 and 0.

If $k > 2$ then there is a positive and a negative eigenvalue.

c. See Figure 9.21.

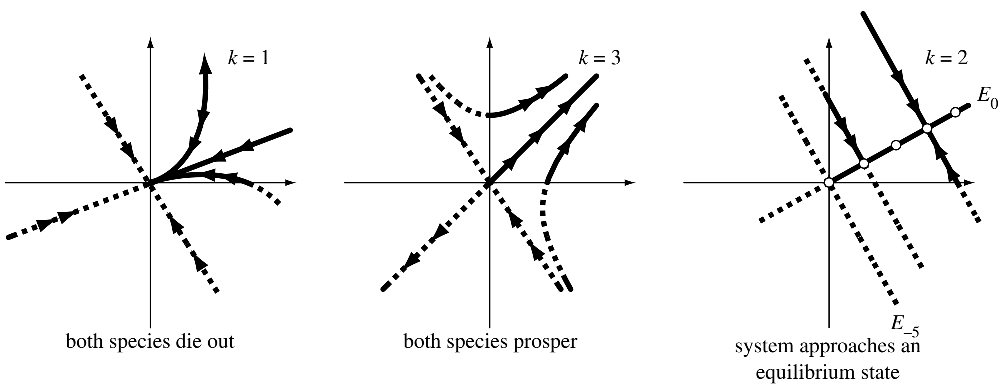

Figure 9.21: for Problem 9.1.47c.

48. a. Symbiosis

b. $\lambda_{1,2} = \frac{-5\pm\sqrt{9+4k}}{2}$

Both eigenvalues are negative if $\sqrt{9+4k} < 5$ or $9+4k < 25$ or $4k < 16$ or $k < 4$.

If $k = 4$ then the eigenvalues are -5 and 0.

If $k > 4$ then there is a positive and a negative eigenvalue.

c. $k = 1$: See corresponding figure in Exercise 47 and Figure 9.22.

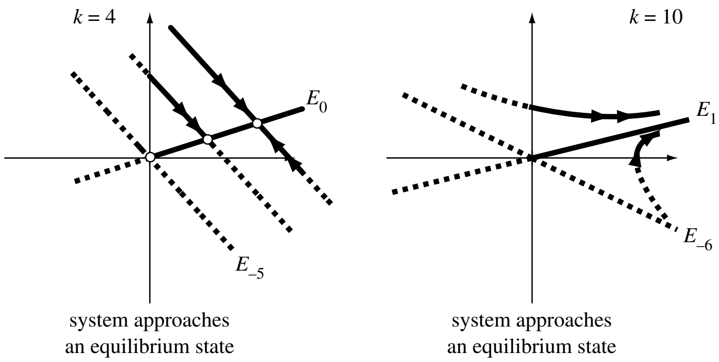

Figure 9.22: for Problem 9.1.48c.

49. $A = \begin{bmatrix} -1 & -0.2 \\ 0.6 & -0.2 \end{bmatrix}$, $\lambda_1 = -0.4$, $\lambda_2 = -0.8$

$E_{-0.4} = \text{span} \begin{bmatrix} -1 \\ 3 \end{bmatrix}$, $E_{-0.8} = \text{span} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

$\begin{bmatrix} g(0) \\ h(0) \end{bmatrix} = 15 \begin{bmatrix} -1 \\ 3 \end{bmatrix} + 45 \begin{bmatrix} 1 \\ -1 \end{bmatrix}$, so that $c_1 = 15, c_2 = 45$.

$\begin{bmatrix} g(t) \\ h(t) \end{bmatrix} = 15e^{-0.4t} \begin{bmatrix} -1 \\ 3 \end{bmatrix} + 45e^{-0.8t} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$, so that

$g(t) = -15e^{-0.4t} + 45e^{-0.8t}$

$h(t) = 45e^{-0.4t} - 45e^{-0.8t}$. See Figure 9.23.

50. We want both eigenvalues λ_1 and λ_2 to be negative, so that $\text{tr}(A) = \lambda_1 + \lambda_2 < 0$ and $\det(A) = \lambda_1\lambda_2 > 0$. Conversely, if $\text{tr}(A) < 0$ and $\det(A) > 0$, then the two eigenvalues $\lambda_{1,2} = \frac{\text{tr}(A) \pm \sqrt{(\text{tr}A)^2 - 4\det(A)}}{2}$ are both negative.

Answer: $\text{tr}(A) < 0$ and $\det(A) > 0$.

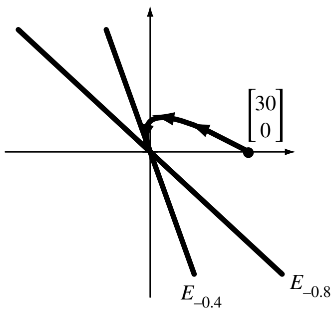

Figure 9.23: for Problem 9.1.49.

51. ith component of $\frac{d}{dt}(S\vec{x}) = \frac{d}{dt}(s_{i1}x_1(t) + s_{i2}x_2(t) + \cdots + s_{in}x_n(t))$

$= s_{i1}\frac{dx_1}{dt} + s_{i2}\frac{dx_2}{dt} + \cdots + s_{in}\frac{dx_n}{dt}$

$=$ ith component of $S\frac{d\vec{x}}{dt}$

52. The solutions of $\frac{d\vec{x}}{dt} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\vec{x}$ are of the form $\begin{bmatrix} p+qt \\ q \end{bmatrix}$, where $\vec{x}(0) = \begin{bmatrix} p \\ q \end{bmatrix}$, by Exercise 21. Since $\begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix} = \lambda I_2 + \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, the solutions of the given system are of the form $\vec{x}(t) = e^{\lambda t}\begin{bmatrix} p+qt \\ q \end{bmatrix}$, by Exercise 24. The zero state is a stable equilibrium solution if and only if $\lambda < 0$. The case $\lambda = 0$ is discussed in Exercise 21. See Figure 9.24.

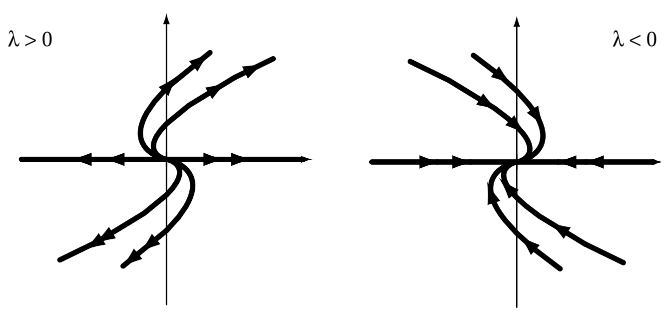

Figure 9.24: for Problem 9.1.52.

53. For the initial value $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$, the system $\frac{d\vec{x}}{dt} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \vec{x}$ has the solution $\vec{x}(t) = \begin{bmatrix} \cos(t) \\ \sin(t) \end{bmatrix}$, by Exercise 20; the system $\frac{d\vec{x}}{dt} = \begin{bmatrix} 0 & -q \\ q & 0 \end{bmatrix}$ has the solution $\vec{x}(t) = \begin{bmatrix} \cos(qt) \\ \sin(qt) \end{bmatrix}$, by Exercise 25; and the system $\frac{d\vec{x}}{dt} = \begin{bmatrix} p & -q \\ q & p \end{bmatrix} \vec{x}$ has the solution $\vec{x}(t) = e^{pt} \begin{bmatrix} \cos(qt) \\ \sin(qt) \end{bmatrix}$, by Exercise 24 $\left(\text{write } \begin{bmatrix} p & -q \\ q & p \end{bmatrix} = pI_2 + \begin{bmatrix} 0 & -q \\ q & 0 \end{bmatrix}\right)$. See Figure 9.25.

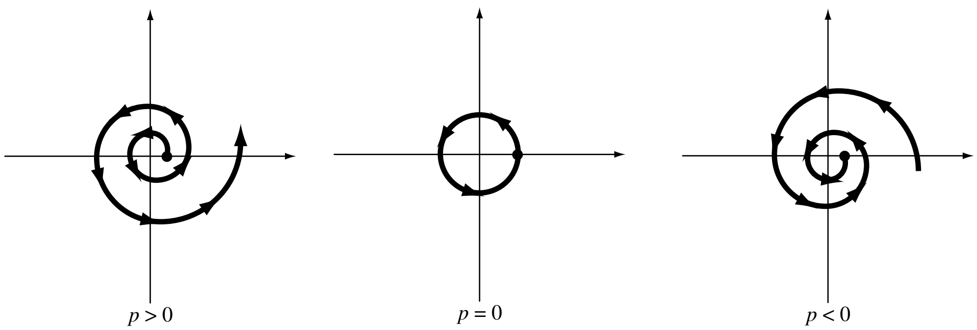

Figure 9.25: for Problem 9.1.53.

54. $A = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}$, $\lambda_1 = -1, \lambda_2 = -2; \vec{v}_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ and $\vec{v}_2 = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$. See Figure 9.26.

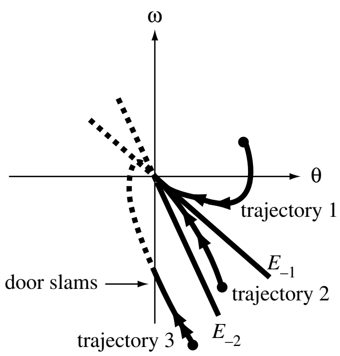

Figure 9.26: for Problem 9.1.54.

In the case of trajectory 3 the door will slam: Initially the door is opened just a little (θ is small) and given a strong push to close it (ω is large negative). More generally, the door will slam if the point $\begin{bmatrix} \theta(0) \\ \omega(0) \end{bmatrix}$ representing the initial state is located below the line $E_{-2} = \text{span}\begin{bmatrix} 1 \\ -2 \end{bmatrix}$, that is, if $\frac{\omega(0)}{\theta(0)} < -2$.

55. $A = \begin{bmatrix} 0 & 1 \\ -p & -q \end{bmatrix}, \lambda_{1,2} = \frac{1}{2}\left(-q \pm \sqrt{q^2 - 4p}\right)$; note that both eigenvalues are negative.

$E_{\lambda_1} = \text{span}\begin{bmatrix} 1 \\ \lambda_1 \end{bmatrix}$ and $E_{\lambda_2} = \begin{bmatrix} 1 \\ \lambda_2 \end{bmatrix}$.

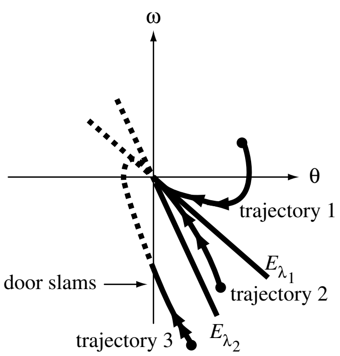

Figure 9.27: for Problem 9.1.55.

See Figure 9.27. In the case of trajectory 3 the door will slam: Initially the door is opened just a little (θ is small) and given a strong push to close it (ω is large negative). More generally, the door will slam if the point $\begin{bmatrix} \theta(0) \\ \omega(0) \end{bmatrix}$ representing the initial state is located below the line $E_{\lambda_2} = \text{span}\begin{bmatrix} 1 \\ \lambda_2 \end{bmatrix}$, that is, if $\frac{\omega(0)}{\theta(0)} < \lambda_2$.

9.2

1. By Euler's formula (Fact 9.2.2), $e^{2\pi i} = \cos(2\pi) + i\sin(2\pi) = 1$.

2. By Euler's formula (Fact 9.2.2), $e^{\frac{1}{2}\pi i} = \cos\left(\frac{\pi}{2}\right) + i\sin\left(\frac{\pi}{2}\right) = i$.

3. $r = \sqrt{(-1)^2 + 1^2} = \sqrt{2}$

   $\theta = \frac{3\pi}{4}$, so that $z = \sqrt{2}e^{\frac{3}{4}\pi i}$. See Figure 9.28.

4. $e^{3it} = \cos(3t) + i\sin(3t)$

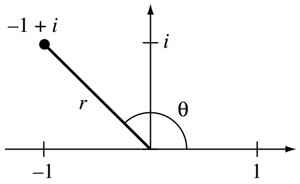

Figure 9.28: for Problem 9.2.3.

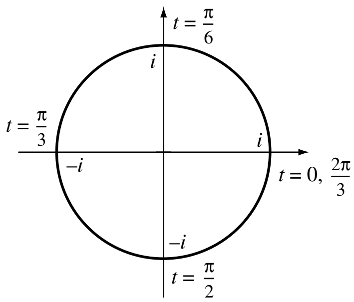

Figure 9.29: for Problem 9.2.4.

Period T given by $3T = 2\pi$ or $T = \frac{2\pi}{3}$. See Figure 9.29.

5. $e^{-0.1t-2it} = e^{-0.1t}e^{-2it} = e^{-0.1t}(\cos(2t) - i\sin(2t))$ spirals inward, in clockwise direction. See Figure 9.30.

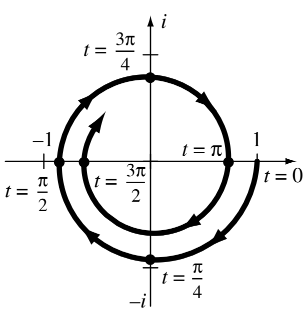

Figure 9.30: for Problem 9.2.5.

6. $\lambda_{1,2} = \pm i$; $E_i = \ker\begin{bmatrix} 3-i & -2 \\ 5 & -3-i \end{bmatrix} = \text{span}\begin{bmatrix} 3+i \\ 5 \end{bmatrix}$ and $E_{-i} = \text{span}\begin{bmatrix} 3-i \\ 5 \end{bmatrix}$.

General solution:

$\vec{x}(t) = c_1 e^{it}\begin{bmatrix} 3+i \\ 5 \end{bmatrix} + c_2 e^{-it}\begin{bmatrix} 3-i \\ 5 \end{bmatrix}$

If $c_1 = c_2 = 1$ then $\vec{x}(t) = (\cos(t) + i\sin(t))\begin{bmatrix} 3+i \\ 5 \end{bmatrix} + (\cos(t) - i\sin(t))\begin{bmatrix} 3-i \\ 5 \end{bmatrix} = \begin{bmatrix} 6\cos(t) - 2\sin(t) \\ 10\cos(t) \end{bmatrix}$.

7. $\det(A) = -2$, so that the zero state is not stable, by Fact 9.2.5.

8. Recall that all eigenvalues of a symmetric matrix are real. The zero state is stable if (and only if) all eigenvalues are negative (by Fact 9.2.4); this is the case if (and only if) the matrix is negative definite (by Fact 8.2.2).

9. The eigenvalues are conjugate complex, $\lambda_{1,2} = p \pm iq$, and $\mathrm{tr}(A) = 2p < 0$, so that p is negative. By Fact 9.2.4, the zero state is stable.

10. If $A = \begin{bmatrix} a & b \\ b & c \end{bmatrix}$ then $q(x_1, x_2) = ax_1^2 + 2bx_1x_2 + cx_2^2$, and the system takes the form $\left| \begin{array}{l} \frac{dx_1}{dt} = 2ax_1 + 2bx_2 \\ \frac{dx_2}{dt} = 2bx_1 + 2cx_2 \end{array} \right|$.

   a. The matrix of the system is $\begin{bmatrix} 2a & 2b \\ 2b & 2c \end{bmatrix}$, which is $2A$, so that $\frac{d\vec{x}}{dt} = \mathrm{grad}(q) = 2A\vec{x}$.

   b. By Fact 8.2.7, the level curves are ellipses. From multivariable calculus we know that grad (q) is perpendicular to the level curve, pointing inwards, so that all trajectories will approach the origin: The zero state is stable. See Figure 9.31.

   c. By Fact 8.2.7, the level curves are hyperbolas, as shown in Figure 9.32.

   $\lambda_1 > 0 > \lambda_2$

   d. The zero state is a stable equilibrium solution of the system $\frac{d\vec{x}}{dt} = \mathrm{grad}(q) = 2A\vec{x}$ if (and only if) the eigenvalues of $2A$ (and A) are negative. This means that the quadratic form $q(\vec{x}) = \vec{x} \cdot A\vec{x}$ is negative definite.

11. a. $q(\vec{x}) = 2a_{i1}x_ix_1 + 2a_{i2}x_ix_2 + \cdots + a_{ii}x_i^2 + \cdots + 2a_{in}x_ix_n +$ terms not involving x_i, so that $\frac{\partial q}{\partial x_i} = 2a_{i1}x_1 + 2a_{i2}x_2 + \cdots + 2a_{ii}x_i + \cdots + 2a_{in}x_n$ and $\frac{d\vec{x}}{dt} = \mathrm{grad}(q) = 2A\vec{x}$.

   The matrix of the system is $B = 2A$.

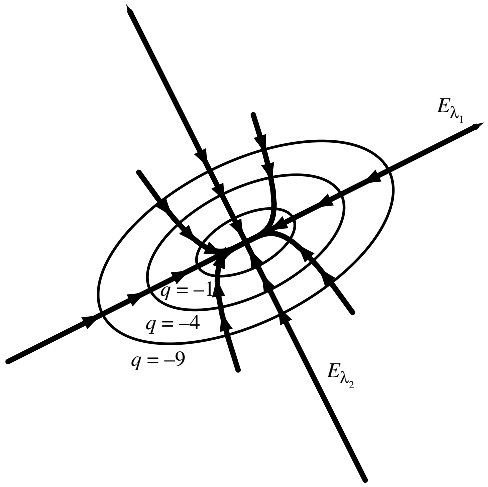

Figure 9.31: for Problem 9.2.10b.

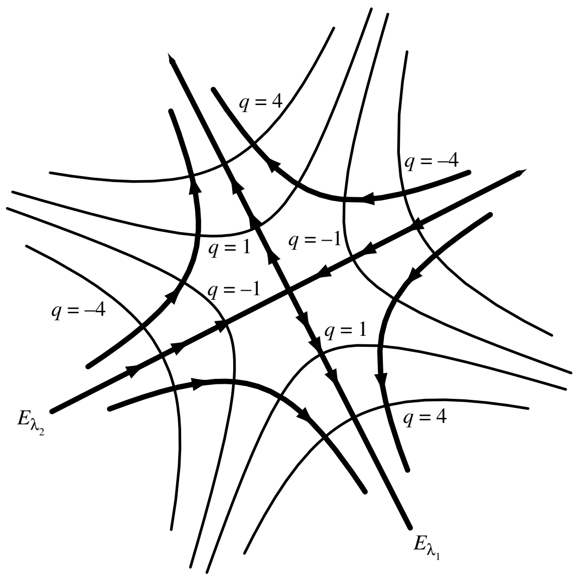

Figure 9.32: for Problem 9.2.10c.

d. The zero state is a stable equilibrium solution of the system $\frac{d\vec{x}}{dt} = \text{grad}(q) = 2A\vec{x}$ if (and only if) all the eigenvalues of $2A$ are negative. This means that the quadratic form $q(\vec{x}) = \vec{x} \cdot A\vec{x}$ is negative definite.

12. We will show that the real parts of all the eigenvalues are negative, so that the zero state is a stable equilibrium solution. Now the characteristic polynomial of A is $f_A(\lambda) = -\lambda^3 - 2\lambda^2 - \lambda - 1$. It is convenient to get rid of all these minus signs: The eigenvalues are the solutions of the equation $g(\lambda) = \lambda^3 + 2\lambda^2 + \lambda + 1 = 0$. Since $g(-1) = 1$ and $g(-2) = -1$, there will be an eigenvalue λ_1 between -2 and -1. Using calculus (or a graphing calculator), we see that the equation $g(\lambda) = 0$ has no other real solutions. Thus there must be two complex conjugate eigenvalues $p \pm iq$. Now the sum of the eigenvalues is $\lambda_1 + 2p = \text{tr}(A) = -2$, and $p = \frac{-2-\lambda_1}{2}$ will be negative , as claimed. The graph of $g(\lambda)$ is shown in Figure 9.33.

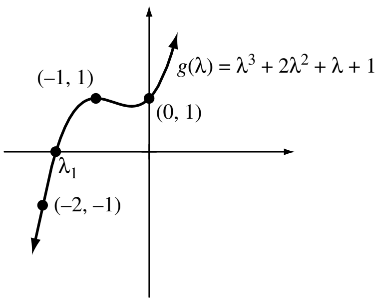

Figure 9.33: for Problem 9.2.12.

13. Recall that the zero state is stable if (and only if) the real parts of all eigenvalues are negative. Now the eigenvalues of A^{-1} are the reciprocals of those of A; the real parts have the same sign $\left(\text{if } \lambda = p + iq, \text{ then } \frac{1}{\lambda} = \frac{1}{p+iq} = \frac{p-iq}{p^2+q^2}\right)$.

14. a. For $i > 1$, $\frac{dx_i}{dt} = -k_i x_i + x_{i-1}$. This means that in the absence of quantity $x_{i-1}(t)$, the quantity $x_i(t)$ will decay exponentially, but the presence of x_{i-1} helps x_i to grow.

For $i = 1$, the beginning of the loop, $\frac{dx_1}{dt} = -k_1 x_1 - b x_n$, so that the presence of x_n contributes to the decrease of x_1.

b. If $n = 2$ then the matrix of the system is $A = \begin{bmatrix} -k_1 & -b \\ 1 & -k_2 \end{bmatrix}$ with $\text{tr}(A) = -k_1 - k_2 < 0$ and $\det(A) = k_1 k_2 + b > 0$, so that the zero state is stable, by Fact 9.2.5.

c. No; consider the case $k_1 = k_2 = k_3 = 1$ for simplicity; then the matrix of the system is

$A = \begin{bmatrix} -1 & 0 & b \\ 1 & -1 & 0 \\ 0 & 1 & -1 \end{bmatrix}$ and $f_A(\lambda) = (\lambda+1)^3 - b$. If b exceeds 1, then the matrix A will have a positive eigenvalue, so that the zero state is not stable.

15. The eigenvalues are $\lambda_1 = \text{tr}(A) > 0$ and $\lambda_2 = 0$. See Figure 9.34.

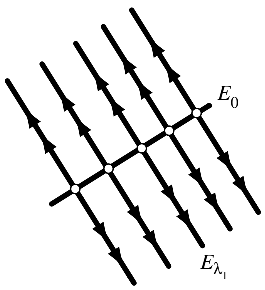

Figure 9.34: for Problem 9.2.15.

16. If $A = \begin{bmatrix} 0 & 1 \\ a & b \end{bmatrix}$ then $\text{tr}(A) = b$ and $\det(A) = -a$. By Fact 9.2.5, the zero state is stable if a and b are both negative.

17. If $A = \begin{bmatrix} -1 & k \\ k & -1 \end{bmatrix}$ then $\text{tr}(A) = -2$ and $\det(A) = 1 - k^2$. By Fact 9.2.5, the zero state is stable if $\det(A) = 1 - k^2 > 0$, that is, if $|k| < 1$.

18. If $\lambda_1, \lambda_2, \lambda_3$ are real and negative, then $\text{tr}(A) = \lambda_1+\lambda_2+\lambda_3 < 0$ and $\det(A) = \lambda_1\lambda_2\lambda_3 < 0$. If λ_1 is real and negative and $\lambda_{2,3} = p \pm iq$, where p is negative, then $\text{tr}(A) = \lambda_1 + 2p < 0$ and $\det(A) = \lambda_1(p^2+q^2) < 0$. Either way, both trace and determinant are negative.

19. False, consider $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -4 \end{bmatrix}$.

20. Use Fact 9.2.6, with $p = 0, q = \pi; a = 1, b = 0$.

$\vec{x}(t) = [\,\vec{w} \quad \vec{v}\,] \begin{bmatrix} \cos(\pi t) & -\sin(\pi t) \\ \sin(\pi t) & \cos(\pi t) \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = (\cos(\pi t))\vec{w} + (\sin(\pi t))\vec{v}$. See Figure 9.35.

21. a. $\left| \begin{matrix} \frac{db}{dt} = 0.05b + & s \\ \frac{ds}{dt} = & 0.07s \end{matrix} \right|$ and $\begin{bmatrix} b(0) \\ s(0) \end{bmatrix} = \begin{bmatrix} 1,000 \\ 1,000 \end{bmatrix}$

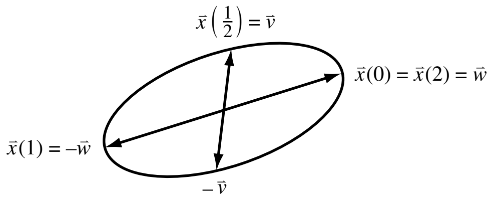

Figure 9.35: for Problem 9.2.20.

b. $\lambda_1 = 0.07, \lambda_2 = 0.05; \vec{v}_1 = \begin{bmatrix} 50 \\ 1 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}; \vec{x}(0) = 1,000\vec{v}_1 - 49,000\vec{v}_2$; so that

$b(t) = 50,000e^{0.07t} - 49,000e^{0.05t}$ and $s(t) = 1,000e^{0.07t}$.

22. $\lambda_1 = 3, \lambda_2 = 0.5; E_3 = \text{span}\begin{bmatrix} 1 \\ -1 \end{bmatrix}, E_{0.5} = \text{span}\begin{bmatrix} 0 \\ 1 \end{bmatrix}$

System is discrete so choose VII.

23. $\lambda_{1,2} = -\frac{1}{2} \pm i, r > 1$, so that trajectory spirals outwards. Choose II.

24. $\lambda_1 = 3, \lambda_2 = 0.5, E_3 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, E_{0.5} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

System is continuous, so choose I.

25. $\lambda_{1,2} = -\frac{1}{2} \pm i$; real part is negative so that trajectories spiral inwards in the counterclockwise direction $\left(\text{if } \vec{x} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \text{ then } \frac{d\vec{x}}{dt} = \begin{bmatrix} -1.5 \\ 2 \end{bmatrix}\right)$. Choose IV.

26. $\lambda_1 = 1, \lambda_2 = -2; E_1 = \text{span}\begin{bmatrix} 0 \\ 1 \end{bmatrix}, E_{-2} = \text{span}\begin{bmatrix} 1 \\ -1 \end{bmatrix}$.

System is continuous so choose V.

27. $\lambda_{1,2} = \pm 3i$, $E_{3i} = \text{span}\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix} + i\begin{bmatrix} 0 \\ -1 \end{bmatrix}\right)$, so that $p = 0$, $q = 3$, $\vec{w} = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$, $\vec{v} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

Now use Fact 9.2.6:

$$\vec{x}(t) = e^{0t}\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} \cos(3t) & -\sin(3t) \\ \sin(3t) & \cos(3t) \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} \sin(3t) & \cos(3t) \\ -\cos(3t) & \sin(3t) \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix}$$

28. $\lambda_{1,2} = \pm 6i, E_{6i} = \text{span}\left(\begin{bmatrix}2\\0\end{bmatrix} + i\begin{bmatrix}0\\3\end{bmatrix}\right)$, so that

$$\vec{x}(t) = \begin{bmatrix}0 & 2\\3 & 0\end{bmatrix}\begin{bmatrix}\cos(6t) & -\sin(6t)\\ \sin(6t) & \cos(6t)\end{bmatrix}\begin{bmatrix}a\\b\end{bmatrix} = \begin{bmatrix}2\sin(6t) & 2\cos(6t)\\3\cos(6t) & -3\sin(6t)\end{bmatrix}\begin{bmatrix}a\\b\end{bmatrix}.$$

29. $\lambda_{1,2} = 2 \pm 4i, E_{2+4i} = \text{span}\left(\begin{bmatrix}1\\0\end{bmatrix} + i\begin{bmatrix}0\\1\end{bmatrix}\right)$, so that

$$\vec{x}(t) = e^{2t}\begin{bmatrix}0 & 1\\1 & 0\end{bmatrix}\begin{bmatrix}\cos(4t) & -\sin(4t)\\ \sin(4t) & \cos(4t)\end{bmatrix}\begin{bmatrix}a\\b\end{bmatrix} = e^{2t}\begin{bmatrix}\sin(4t) & \cos(4t)\\ \cos(4t) & -\sin(4t)\end{bmatrix}\begin{bmatrix}a\\b\end{bmatrix}.$$

30. $\lambda_{1,2} = -2 \pm 3i, E_{-2+3i} = \text{span}\left(\begin{bmatrix}5\\3\end{bmatrix} + i\begin{bmatrix}0\\1\end{bmatrix}\right)$, so that

$$\vec{x}(t) = e^{-2t}\begin{bmatrix}0 & 5\\1 & 3\end{bmatrix}\begin{bmatrix}\cos(3t) & -\sin(3t)\\ \sin(3t) & \cos(3t)\end{bmatrix}\begin{bmatrix}a\\b\end{bmatrix} = e^{-2t}\begin{bmatrix}5\sin(3t) & 5\cos(3t)\\ \cos(3t)+3\sin(3t) & -\sin(3t)+3\cos(3t)\end{bmatrix}\begin{bmatrix}a\\b\end{bmatrix}.$$

31. $\lambda_{1,2} = -1 \pm 2i, E_{-1+2i} = \text{span}\left(\begin{bmatrix}1\\0\end{bmatrix} + i\begin{bmatrix}0\\-1\end{bmatrix}\right)$, so that $p = -1, q = 2, \vec{w} = \begin{bmatrix}0\\-1\end{bmatrix}$,

$\vec{v} = \begin{bmatrix}1\\0\end{bmatrix}$. Now $\begin{bmatrix}1\\-1\end{bmatrix} = \vec{x}(0) = \vec{w} + \vec{v}$, so that $a = 1$ and $b = 1$.

Then $\vec{x}(t) = e^{-t}\begin{bmatrix}0 & 1\\-1 & 0\end{bmatrix}\begin{bmatrix}\cos(2t) & -\sin(2t)\\ \sin(2t) & \cos(2t)\end{bmatrix}\begin{bmatrix}1\\1\end{bmatrix} = e^{-t}\begin{bmatrix}\sin(2t)+\cos(2t)\\ \sin(2t)-\cos(2t)\end{bmatrix}$.

See Figure 9.36.

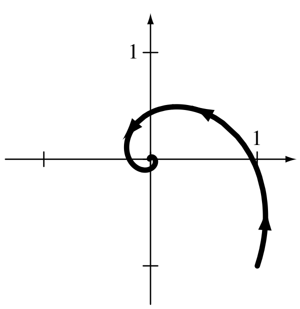

Figure 9.36: for Problem 9.2.31.

32. $\lambda_{1,2} = \pm 2i, E_{2i} = \text{span}\left(\begin{bmatrix}1\\0\end{bmatrix} + i\begin{bmatrix}0\\2\end{bmatrix}\right), \vec{x}(0) = 0\begin{bmatrix}0\\2\end{bmatrix} + 1\begin{bmatrix}1\\0\end{bmatrix}$, so that $a = 0$ and $b = 1$.

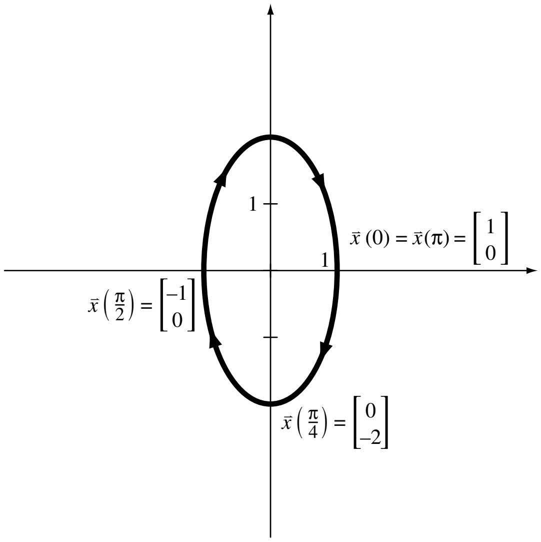

Figure 9.37: for Problem 9.2.32.

$\vec{x}(t) = \begin{bmatrix} 0 & 1 \\ 2 & 0 \end{bmatrix} \begin{bmatrix} \cos(2t) & -\sin(2t) \\ \sin(2t) & \cos(2t) \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} \cos(2t) \\ -2\sin(2t) \end{bmatrix}$. See Figure 9.37.

33. $\lambda_{1,2} = \pm i, E_i = \text{span}\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix} + i \begin{bmatrix} 0 \\ 1 \end{bmatrix}\right)$

$a = 1, b = 0$, so that $\vec{x}(t) = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \cos(t) & -\cos(t) \\ \sin(t) & \cos(t) \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \sin(t) \\ \sin(t) + \cos(t) \end{bmatrix}$

$= \cos(t) \begin{bmatrix} 0 \\ 1 \end{bmatrix} + \sin(t) \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. See Figure 9.38.

34. $\lambda_{1,2} = 1 \pm 2i$, $E_{1+2i} = \text{span}\left(\begin{bmatrix} 3 \\ -2 \end{bmatrix} + i \begin{bmatrix} 1 \\ 0 \end{bmatrix}\right)$

$a = 1, b = 0$, so that $\vec{x}(t) = e^t \begin{bmatrix} 1 & 3 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} \cos(2t) & -\sin(2t) \\ \sin(2t) & \cos(2t) \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = e^t \begin{bmatrix} \cos(2t) + 3\sin(2t) \\ -2\sin(2t) \end{bmatrix}$.
See Figure 9.39.

35. If $z = f + ig$ and $w = p + iq$ then $zw = (fp - gq) + i(fq + gp)$, so that $(zw)' = (f'p + fp' - g'q - gq') + i(f'q + fq' + g'p + gp')$.

Also $z'w = (f' + ig')(p + iq) = (f'p - g'q) + i(f'q + g'p)$ and $zw' = (f + ig)(p' + iq') = (fp' - gq') + i(gp' + fq')$. We can see that $(zw)' = z'w + zw'$, as claimed.

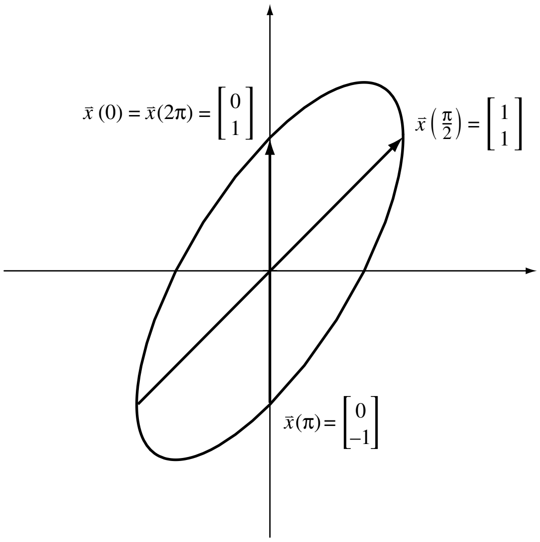

Figure 9.38: for Problem 9.2.33.

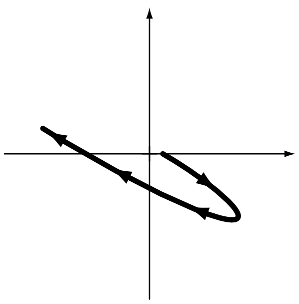

Figure 9.39: for Problem 9.2.34.

36. $A = \begin{bmatrix} 0 & 1 \\ -b & -c \end{bmatrix}$ and $f_A(\lambda) = \lambda^2 + c\lambda + b$, with eigenvalues $\lambda_{1,2} = \frac{-c \pm \sqrt{c^2 - 4b}}{2}$.

a. If $c = 0$ then $\lambda_{1,2} = \pm i\sqrt{b}$. The trajectories are ellipses. See Figure 9.40.

The block *oscillates harmonically*, with period $\frac{2\pi}{\sqrt{b}}$. The zero state fails to be asymptotically stable.

b. $\lambda_{1,2} = \frac{-c \pm i\sqrt{4b - c^2}}{2}$

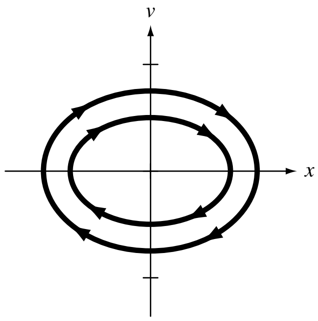

Figure 9.40: for Problem 9.2.36a.

The trajectories spiral inwards, since $\text{Re}(\lambda_1) = \text{Re}(\lambda_2) = -\frac{c}{2} < 0$. This is the case of a *damped oscillation*. The zero state is asymptotically stable. See Figure 9.41.

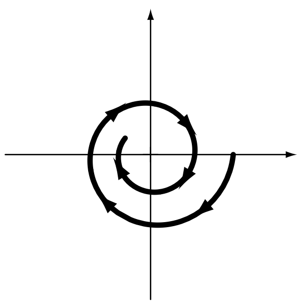

Figure 9.41: for Problem 9.2.36b.

c. This case is discussed in Exercise 9.1.55. The zero state is stable here.

37. a. $\frac{1}{z(t)}$ is differentiable when $z(t) \neq 0$, since both the real and the imaginary parts are differentiable $\left(\text{if } z = p + iq \text{ then } \frac{1}{z} = \frac{p-iq}{p^2+q^2}\right)$. To find $\left(\frac{1}{z}\right)'$, apply the product rule to the equation $z\left(\frac{1}{z}\right) = 1$: $z'\left(\frac{1}{z}\right) + z\left(\frac{1}{z}\right)' = 0$, so that $\left(\frac{1}{z}\right)' = -\frac{z'}{z^2}$.

b. $\left(\frac{z}{w}\right)' = \left(z\frac{1}{w}\right)' = z'\frac{1}{w} + z\left(\frac{1}{w}\right)' = \frac{z'}{w} - \frac{zw'}{w^2} = \frac{z'w - zw'}{w^2}$

38. a. $\left(\frac{z_1}{z_2}\right)' = \frac{{z'}_1 z_2 - z_1 {z'}_2}{z_2^2} = \frac{\lambda z_1 z_2 - \lambda z_1 z_2}{z_2^2} = 0$, so that $\frac{z_1(t)}{z_2(t)} = k$, a constant.

Now $z_1(t) = kz_2(t)$; substituting $t = 0$ gives $1 = z_1(0) = kz_2(0) = k$, so that $z_1(t) = z_2(t)$, as claimed.

b. Let $z_2(t) = e^{pt}(\cos(qt) + i\sin(qt))$ be the solution constructed in the text (se Page 409). Since $z_2(t) \neq 0$ for all t, this is the only solution, by part a.

39. Let $A = \begin{bmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{bmatrix}$. We first solve the system $\frac{d\vec{c}}{dt} = (A - \lambda I_3)\vec{c} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \vec{c}$, or $\frac{dc_1}{dt} = c_2(t)$, $\frac{dc_2}{dt} = c_3(t)$, $\frac{dc_3}{dt} = 0$.

$c_3(t) = k_3$, a constant, so that $\frac{dc_2}{dt} = k_3$ and $c_2(t) = k_3t + k_2$. Likewise $c_1(t) = \frac{k_3}{2}t^2 + k_2t + k_1$.

Applying Exercise 9.1.24, with $k = -\lambda$, we find that $\vec{c}(t) = e^{-\lambda t}\vec{x}(t)$ or $\vec{x}(t) = e^{\lambda t}\vec{c}(t)$

$= e^{\lambda t} \begin{bmatrix} k_1 + k_2t + \frac{k_3}{2}t^2 \\ k_2 + k_3t \\ k_3 \end{bmatrix}$ where k_1, k_2, k_3 are arbitrary constants. The zero state is stable if (and only if) the real part of λ is negative.

40. a. $B(t) = 1000(1 + 0.05i)^t = 1000(r(\cos\theta + i\sin\theta))^t = 1000r^t(\cos(\theta t) + i\sin(\theta t))$, where

$r = \sqrt{1 + 0.05^2} > 1$ and $\theta = \arctan(0.05) \approx 0.05$. See Figure 9.42.

b. $B(t) = 1000e^{0.05i} = 1000(\cos(0.05t) + i\sin(0.05t))$. See Figure 9.42.

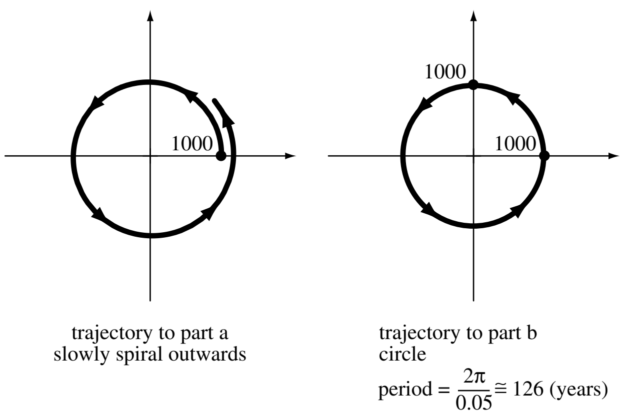

Figure 9.42: for Problem 9.2.40.

c. We would choose an account with annual compounding, since the modulus of the balance grows in this case. In the case of continuous compounding the modulus of the balance remains unchanged.

9.3

1. The characteristic polynomial of this differential equation is $\lambda - 5$, so that $\lambda_1 = 5$. By Fact 9.3.8 the general solution is $f(t) = Ce^{5t}$, where C is an arbitrary constant.

2. The solutions of $\frac{dx}{dt} + 3x = 0$ are of the form $x(t) = Ce^{-3t}$, where C is an arbitrary constant, and the differential equation $\frac{dx}{dt} + 3x = 7$ has the particular solution $x_p(t) = \frac{7}{3}$, so that the general solution is $x(t) = Ce^{-3t} + \frac{7}{3}$ (where C is a constant). Alternatively, we could use Fact 9.3.13.

3. Use Fact 9.3.13, where $a = -2$ and $g(t) = e^{3t}$:

   $f(t) = e^{-2t} \int e^{2t}e^{3t}\, dt = e^{-2t} \int e^{5t}\, dt = e^{-2t}\left(\frac{1}{5}e^{5t} + C\right) = \frac{1}{5}e^{3t} + Ce^{-2t}$, where C is a constant.

4. We can look for a sinusoidal solution $x_p(t) = P\cos(3t) + Q\sin(3t)$, as in Example 7. P and Q need to be chosen in such a way that $-3P\sin(3t) + 3Q\cos(3t) - 2P\cos(3t) - 2Q\sin(3t) = \cos(3t)$ or $\left| \begin{array}{l} -2P + 3Q = 1 \\ -3P - 2Q = 0 \end{array} \right|$ with solution $P = -\frac{2}{13}$ and $Q = \frac{3}{13}$. Since the general solution of $\frac{dx}{dt} - 2x = 0$ is $x(t) = Ce^{2t}$, the general solution of $\frac{dx}{dt} - 2x = \cos(3t)$ is $x(t) = Ce^{2t} - \frac{2}{13}\cos(3t) + \frac{3}{13}\sin(3t)$, where C is an arbitrary constant.

5. Using Fact 9.3.13, $f(t) = e^t \int e^{-t}t\, dt = e^t(-te^{-t} - e^{-t} + C) = Ce^t - t - 1$, where C is an arbitrary constant.

6. Using Fact 9.3.13, $f(t) = e^{2t} \int e^{-2t}e^{2t}\, dt = e^{2t} \int dt = e^{2t}(t + C)$, where C is an arbitrary constant.

7. By Definition 9.3.6, $p_T(\lambda) = \lambda^2 + \lambda - 12 = (\lambda + 4)(\lambda - 3)$.

   Since $p_T(\lambda)$ has distinct roots $\lambda_1 = -4$ and $\lambda_2 = 3$, the solutions of the differential equation are of the form $f(t) = c_1e^{-4t} + c_2e^{3t}$, where c_1 and c_2 are arbitrary constants (by Fact 9.3.8).

8. $p_T(\lambda) = \lambda^2 + 3\lambda - 10 = (\lambda + 5)(\lambda - 2) = 0$

   $x(t) = c_1e^{-5t} + c_2e^{2t}$, where c_1, c_2 are arbitrary constants.

9. $p_T(\lambda) = \lambda^2 - 9 = (\lambda - 3)(\lambda + 3) = 0$

   $f(t) = c_1 e^{3t} + c_2 e^{-3t}$, where c_1, c_2 are arbitrary constants.

10. $p_T(\lambda) = \lambda^2 + 1 = 0$ has roots $\lambda_{1,2} = \pm i$. By Fact 9.3.9, $f(t) = c_1 \cos(t) + c_2 \sin(t)$, where c_1, c_2 are arbitrary constants.

11. $p_T(\lambda) = \lambda^2 - 2\lambda + 2 = 0$ has roots $\lambda_{1,2} = 1 \pm i$. By Fact 9.3.9, $x(t) = e^t(c_1 \cos(t) + c_2 \sin(t))$, where c_1, c_2 are arbitrary constants.

12. $p_T(\lambda) = \lambda^2 - 4\lambda + 13 = 0$ has roots $\lambda_{1,2} = 2 \pm 3i$.

    By Fact 9.3.9, $f(t) = e^{2t}(c_1 \cos(3t) + c_2 \sin(3t))$, where c_1, c_2 are arbitrary constants.

13. $p_T(\lambda) = \lambda^2 + 2\lambda + 1 = (\lambda + 1)^2 = 0$ has the double root $\lambda = -1$. Following the strategy on page 429, we find $f(t) = e^{-t}(c_1 t + c_2)$, where c_1, c_2 are arbitrary constants.

14. $p_T(\lambda) = \lambda^2 + 3\lambda = \lambda(\lambda + 3) = 0$ has roots $\lambda_1 = 0, \lambda_2 = -3$.

15. By integrating twice we find $f(t) = c_1 + c_2 t$, where c_1, c_2 are arbitrary constants.

16. By Fact 9.3.10, the differential equation has a particular solution of the form $f_p(t) = P\cos(t) + Q\sin(t)$. Plugging f_p into the equation we find

    $(-P\cos(t) - Q\sin(t)) + 4(-P\sin(t) + Q\cos(t)) + 13(P\cos(t) + Q\sin(t)) = \cos(t)$ or

    $\left| \begin{array}{c} 12P + 4Q = 1 \\ -4P + 12Q = 0 \end{array} \right|$, so

    $P = \frac{3}{40}$

    $Q = \frac{1}{40}$.

    Therefore, $f_p(t) = \frac{3}{40}\cos(t) + \frac{1}{40}\sin(t)$.

    Next we find a basis of the solution space of $f''(t) + 4f'(t) + 13f(t) = 0$. $p_T(\lambda) = \lambda^2 + 4\lambda + 13 = 0$ has roots $-2 \pm 3i$. By Fact 9.3.9, $f_1(t) = e^{-2t}\cos(3t)$ and $f_2(t) = e^{-2t}\sin(3t)$ is a basis of the solution space.

    By Fact 9.3.4, the solutions of the original differential equation are of the form $f(t) = c_1 f_1(t) + c_2 f_2(t) + f_p(t) = c_1 e^{-2t}\cos(3t) + c_2 e^{-2t}\sin(3t) + \frac{3}{40}\cos(t) + \frac{1}{40}\sin(t)$, where c_1, c_2 are arbitrary constants.

17. By Fact 9.3.10, the differential equation has a particular solution of the form $f_p(t) = P\cos(t) + Q\sin(t)$. Plugging f_p into the equation we find $(-P\cos(t) - Q\sin(t)) + 2(-P\sin(t) + Q\cos(t)) + P\cos(t) + Q\sin(t) = \sin(t)$ or $\left| \begin{array}{c} 2Q = 0 \\ -2P = 1 \end{array} \right|$, so $\begin{array}{c} P = -\frac{1}{2} \\ Q = 0 \end{array}$.

Therefore, $f_p(t) = -\frac{1}{2}\cos(t)$.

Next we find a basis of the solution space of $f''(t) + 2f'(t) + f(t) = 0$. In Exercise 13 we see that $f_1(t) = e^{-t}$, $f_2(t) = te^{-t}$ is such a basis.

By Fact 9.3.4, the solutions of the original differential equation are of the form $f(t) = c_1 f_1(t) + c_2 f_2(t) + f_p(t) = c_1 e^{-t} + c_2 te^{-t} - \frac{1}{2}\cos(t)$, where c_1, c_2 are arbitrary constants.

18. We follow the approach outlined in Exercises 16 and 17.

- Particular solution $f_p = \frac{1}{10}\cos(t) + \frac{3}{10}\sin(t)$
- Solutions of $f''(t) + 3f'(t) + 2f(t) = 0$ are $f_1(t) = e^{-t}$ and $f_2(t) = e^{-2t}$.
- The solutions of the original differential equation are of the form $f(t) = c_1 e^{-t} + c_2 e^{-2t} + \frac{1}{10}\cos(t) + \frac{3}{10}\sin(t)$, where c_1 and c_2 are arbitrary constants.

19. We follow the approach outlined in Exercise 17.

- Particular solution $x_p(t) = \cos(t)$
- Solutions of $\frac{d^2x}{dt^2} + 2x = 0$ are $x_1(t) = \cos(\sqrt{2}t)$ and $x_2(t) = \sin(\sqrt{2}t)$.
- The solutions of the original differential equation are of the form $x(t) = c_1\cos(\sqrt{2}t) + c_2\sin(\sqrt{2}t) + \cos(t)$, where c_1 and c_2 are arbitrary constants.

20. $p_T(\lambda) = \lambda^3 - 3\lambda^2 + 2\lambda = \lambda(\lambda - 1)(\lambda - 2) = 0$ has roots $\lambda_1 = 0, \lambda_2 = 1, \lambda_3 = 2$.

By Fact 9.3.8, the general solution is $f(t) = c_1 + c_2 e^t + c_3 e^{2t}$, where c_1, c_2, c_3 are arbitrary constants.

21. $p_T(\lambda) = \lambda^3 + 2\lambda^2 - \lambda - 2 = (\lambda - 1)(\lambda + 1)(\lambda + 2) = 0$ has roots $\lambda_1 = 1, \lambda_2 = -1, \lambda_3 = -2$.

By Fact 9.3.8, the general solution is $f(t) = c_1 e^t + c_2 e^{-t} + c_3 e^{-2t}$, where c_1, c_2, c_3 are arbitrary constants.

22. $p_T(\lambda) = \lambda^3 - \lambda^2 - 4\lambda + 4 = (\lambda - 1)(\lambda - 2)(\lambda + 2) = 0$ has roots $\lambda_1 = 1, \lambda_2 = 2, \lambda_3 = -2$.

By Fact 9.3.8, the general solution is $f(t) = c_1 e^t + c_2 e^{2t} + c_3 e^{-2t}$, where c_1, c_2, c_3 are arbitrary constants.

23. General solution $f(t) = Ce^{5t}$

Plug in: $3 = f(0) = Ce^0 = C$, so that $f(t) = 3e^{5t}$.

24. General solution $x(t) = Ce^{-3t} + \frac{7}{3}$ (see Exercise 2).

    Plug in: $0 = x(0) = C + \frac{7}{3}$, so that $C = -\frac{7}{3}$ and $x(t) = -\frac{7}{3}e^{-3t} + \frac{7}{3}$.

25. General solution $f(t) = Ce^{-2t}$

    Plug in: $1 = f(1) = Ce^{-2}$, so that $C = e^2$ and $f(t) = e^2e^{-2t} = e^{2-2t}$.

26. General solution $f(t) = c_1e^{3t} + c_2e^{-3t}$ (see Exercise 9), with $f'(t) = 3c_1e^{3t} - 3c_2e^{-3t}$

    Plug in: $0 = f(0) = c_1 + c_2$ and $1 = f'(0) = 3c_1 - 3c_2$, so that $c_1 = \frac{1}{6}, c_2 = -\frac{1}{6}$, and $f(t) = \frac{1}{6}e^{3t} - \frac{1}{6}e^{-3t}$.

27. General solution $f(t) = c_1\cos(3t) + c_2\sin(3t)$ (Fact 9.3.9)

    Plug in: $0 = f(0) = c_1$ and $1 = f\left(\frac{\pi}{2}\right) = -c_2$, so that $c_1 = 0, c_2 = -1$, and $f(t) = -\sin(3t)$.

28. General solution $f(t) = c_1e^{-4t} + c_2e^{3t}$, with $f'(t) = -4c_1e^{-4t} + 3c_2e^{3t}$

    Plug in: $0 = f(0) = c_1 + c_2$ and $0 = f'(0) = -4c_1 + 3c_2$, so that $c_1 = c_2 = 0$ and $f(t) = 0$.

29. General solution $f(t) = c_1\cos(2t) + c_2\sin(2t) + \frac{1}{3}\sin(t)$, so that $f'(t) = -2c_1\sin(2t) + 2c_2\cos(2t) + \frac{1}{3}\cos(t)$ (use the approach outlined in Exercise 17)

    Plug in: $0 = f(0) = c_1$ and $0 = f'(0) = 2c_2 + \frac{1}{3}$, so that $c_1 = 0$, $c_2 = -\frac{1}{6}$, and $f(t) = -\frac{1}{6}\sin(2t) + \frac{1}{3}\sin(t)$.

30. a. k is a positive constant that depends on the rate of cooling of the coffee (it varies with the material of the cup, for example).

    A is the room temperature.

    b. $T'(t) + kT(t) = kA$

    Constant particular solution: $T_p(t) = A$

    General solution of $T'(t) + kT(t) = 0$ is $T(t) = Ce^{-kt}$.

    General solution of the original differential equation: $T(t) = Ce^{-kt} + A$

    Plug in: $T_0 = T(0) = C + A$, so that $C = T_0 - A$ and $T(t) = (T_0 - A)e^{-kt} + A$.

31. $\frac{dv}{dt} + \frac{k}{m}v = g$

    constant particular solution: $v_p = \frac{mg}{k}$

General solution of $\frac{dv}{dt} + \frac{k}{m}v = 0$ is $v(t) = Ce^{-\frac{k}{m}t}$.

General solution of the original differential equation: $v(t) = Ce^{-\frac{k}{m}t} + \frac{mg}{k}$

Plug in: $0 = v(0) = C + \frac{mg}{k}$, so that $C = -\frac{mg}{k}$ and $v(t) = \frac{mg}{k}\left(1 - e^{-\frac{k}{m}t}\right)$

$\lim_{t\to\infty} v(t) = \frac{mg}{k}$ (the "terminal velocity"). See Figure 9.43.

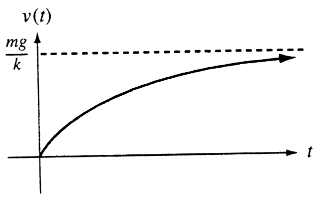

Figure 9.43: for Problem 9.3.31.

32. $\frac{dB}{dt} = kB - r$ or $\frac{dB}{dt} - kB = -r$

$\uparrow$ $\uparrow$
interest withdrawals

constant particular solution $B_p = \frac{r}{k}$

General solution of $\frac{dB}{dt} - kB = 0$ is $B(t) = Ce^{kt}$

General solution of the original differential equation: $B(t) = Ce^{kt} + \frac{r}{k}$

Plug in: $B_0 = B(0) = C + \frac{r}{k}$, so that $C = B_0 - \frac{r}{k}$ and $B(t) = \left(B_0 - \frac{r}{k}\right)e^{kt} + \frac{r}{k}$

if $B_0 > \frac{r}{k}$ then interest will exceed withdrawals and balance will grow.

if $B_0 < \frac{r}{k}$ then withdrawals will exceed interest and account will eventually be depleted.

if $B_0 = \frac{r}{k}$ then the balance will remain the same.

The graphs in Figure 9.44 show the three possible scenarios.

33. By Fact 9.3.9, $x(t) = c_1\cos\left(\sqrt{\frac{g}{L}}t\right) + c_2\sin\left(\sqrt{\frac{g}{L}}t\right)$, with period $P = \frac{2\pi}{\sqrt{\frac{g}{L}}} = 2\pi\frac{\sqrt{L}}{\sqrt{g}}$. It is required that $2 = P = 2\pi\frac{\sqrt{L}}{\sqrt{g}}$ or $L = \frac{g}{\pi^2} \approx 0.994$ (meters).

34. a. We will take downward forces as positive.

Let g = acceleration due to gravity,

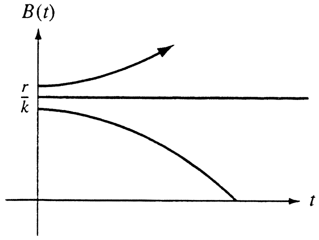

Figure 9.44: for Problem 9.3.32.

$\rho =$ density of block

$a =$ length of edge of block

Then (weight of block) = (mass of block) $\cdot\, g =$ (density of block)(volume of block) $g = \rho a^3 g$

buoyancy = (weight of displaced water) = (mass of displaced water) $\cdot\, g =$ (density of water) (volume of displaced water) $g = 1a^2x(t)g = a^2gx(t)$

b. Newton's Second Law of Motion tells us that

$m\frac{d^2x}{dt^2} = F =$ weight − buoyancy $= \rho a^3 g - a^2gx(t)$, where $m = \rho a^3$ is the mass of the block.

$\rho a^3\frac{d^2x}{dt^2} = \rho a^3 g - a^2gx(t)$

$\frac{d^2x}{dt^2} = g - \frac{g}{\rho a}x(t)$

$\frac{d^2x}{dt^2} + \frac{g}{\rho a}x = g$

constant solution $x_p = \rho a$

general solution (use Fact 9.3.9): $x(t) = c_1\cos\left(\sqrt{\frac{g}{\rho a}}t\right) + c_2\sin\left(\sqrt{\frac{g}{\rho a}}t\right) + \rho a$

Now $c_2 = 0$ since block is at rest at $t = 0$.

Plug in: $a = x(0) = c_1 + \rho a$, so that $c_1 = a - \rho a$ and

$x(t) = (a - \rho a)\cos\left(\sqrt{\frac{g}{\rho a}}t\right) + \rho a \approx 2\cos(11t) + 8$ (measured in centimeters)

c. The period is $P = \frac{2\pi}{\sqrt{\frac{g}{\rho a}}} = \frac{2\pi\sqrt{\rho a}}{\sqrt{g}}$. Thus the period increases as ρ or a increases (denser wood or larger block), or as g decreases (on the moon). The period is independent of the initial state.

35. a. $p_T(\lambda) = \lambda^2 + 3\lambda + 2 = (\lambda + 1)(\lambda + 2) = 0$ with roots $\lambda_1 = -1$ and $\lambda_2 = -2$, so $x(t) = c_1 e^{-t} + c_2 e^{-2t}$.

b. $x'(t) = -c_1 e^{-t} - 2c_2 e^{-2t}$

Plug in: $1 = x(0) = c_1 + c_2$ and $0 = x'(0) = -c_1 - 2c_2$, so that $c_1 = 2, c_2 = -1$ and $x(t) = 2e^{-t} - e^{-2t}$. See Figure 9.45.

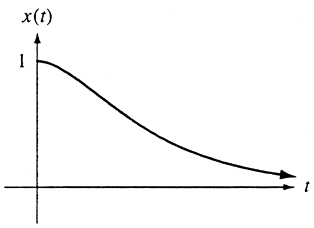

Figure 9.45: for Problem 9.3.35b.

c. Plug in: $1 = x(0) = c_1 + c_2$ and $-3 = x'(0) = -c_1 - 2c_2$, so that $c_1 = -1, c_2 = 2$, and $x(t) = -e^{-t} + 2e^{-2t}$. See Figure 9.46.

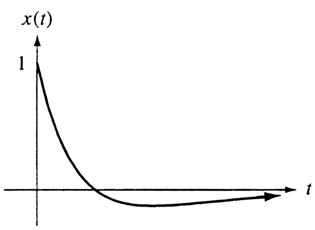

Figure 9.46: for Problem 9.3.35c.

d. The oscillator in part (b) never reaches the equilibrium, while the oscillator in part (c) goes through the equilibrium once, at $t = \ln(2)$. Take another look at Figures 9.45 and 9.46.

36. $f_T(\lambda) = \lambda^2 + 2\lambda + 101 = 0$ has roots $\lambda_{1,2} = -1 \pm 20i$.

By Fact 9.3.9, $x(t) = e^{-t}(c_1 \cos(20t) + c_2 \sin(20t))$.

Any nonzero solution goes through the equilibrium infinitely many times. See Figure 9.47.

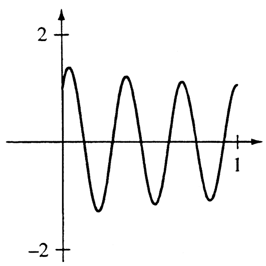

Figure 9.47: for Problem 9.3.36.

37. $f_T(\lambda) = \lambda^2 + 6\lambda + 9 = (\lambda + 3)^2$ has roots $\lambda_{1,2} = -3$.

Following the method of Example 10, we find the general solution $x(t) = e^{-3t}(c_1 + c_2 t)$ with $x'(t) = e^{-3t}(c_2 - 3c_1 - 3c_2 t)$.

Plug in: $0 = x(0) = c_1$, and $1 = x'(0) = c_2 - 3c_1$, so that $c_1 = 0, c_2 = 1$, and $x(t) = te^{-3t}$. See Figure 9.48.

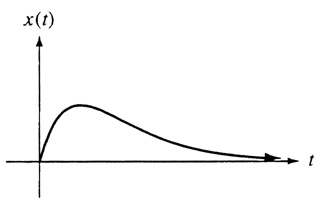

Figure 9.48: for Problem 9.3.37.

The oscillator does not go through the equilibrium at $t > 0$.

38. a. $(D - \lambda)(p(t)e^{\lambda t}) = [p(t)e^{\lambda t}]' - \lambda p(t)e^{\lambda t} = p'(t)e^{\lambda t} + \lambda p(t)e^{\lambda t} - \lambda p(t)e^{\lambda t} = p'(t)e^{\lambda t}$, as claimed.

b. Applying the result from part (a) m times we find $(D - \lambda)^m (p(t)e^{\lambda t}) = p^{(m)}(t)e^{\lambda t} = 0$, since $p^{(m)}(t) = 0$ for a polynomial of degree less than m.

c. By Fact 9.3.3, we are looking for m linearly independent functions. By part (b), the functions $e^{\lambda t}, te^{\lambda t}, t^2 e^{\lambda t}, \ldots, t^{m-1}e^{\lambda t}$ do the job.

d. Note that the kernel of $(D-\lambda_i)^{m_i}$ is contained in the kernel of $(D-\lambda_1)^{m_1}\cdots(D-\lambda_r)^{m_r}$, for any $1 \le i \le r$. Therefore, we have the following basis:

$$e^{\lambda_1 t}, te^{\lambda_1 t}, \ldots, t^{m_1-1}e^{\lambda_1 t},$$

$$e^{\lambda_2 t}, te^{\lambda_2 t}, \ldots, t^{m_2-1}e^{\lambda_2 t}, \ldots$$

$$e^{\lambda_r t}, te^{\lambda_r t}, \ldots, t^{m_r-1}e^{\lambda_r t}.$$

39. $f_T(\lambda) = \lambda^3 + 3\lambda^2 + 3\lambda + 1 = (\lambda+1)^3 = 0$ has roots $\lambda_{1,2,3} = -1$. In other words, we can write the differential equation as $(D+1)^3 f = 0$.

By Exercise 38, part (c), the general solution is $f(t) = e^{-t}(c_1 + c_2 t + c_3 t^2)$.

40. $f_T(\lambda) = \lambda^3 + \lambda^2 - \lambda - 1 = (\lambda+1)^2(\lambda-1) = 0$ has roots $\lambda_{1,2} = -1,\ \lambda_3 = 1$.

In other words, we can write the differential equation as $(D+1)^2(D-1) = 0$.

By Exercise 38, part (d), the general solution is $x(t) = e^{-t}(c_1 + c_2 t) + c_3 e^t$.

41. We are looking for functions x such that $T(x) = \lambda x$, or $T(x) - \lambda x = 0$. Now $T(x) - \lambda x$ is an nth-order linear differential operator, so that its kernel is n-dimensional, by Fact 9.3.3. Thus λ is indeed an eigenvalue of T, with an n-dimensional eigenspace.

42. a. We need to solve the second-order differential equation $Tx = D^2 x = \frac{d^2x}{dt^2} = \lambda x$. This differential equation has a two-dimensional solution space E_λ for any λ, so that all λ are eigenvalues of T.

if $\lambda > 0$ then $E_\lambda = \text{span}\left(e^{\sqrt{\lambda}t}, e^{-\sqrt{\lambda}t}\right)$

if $\lambda = 0$ then $E_\lambda = \text{span}(1, t)$

if $\lambda < 0$ then $E_\lambda = \text{span}\left(\sin\left(\sqrt{-\lambda}t\right), \cos\left(\sqrt{-\lambda}t\right)\right)$

b. Among the eigenfunctions of T we found in part (a), we seek those of period 1. In the case $\lambda < 0$ the shortest period is $P = \frac{2\pi}{\sqrt{-\lambda}}$. Now 1 is a period if $P = \frac{2\pi}{\sqrt{-\lambda}} = \frac{1}{k}$ for some positive integer k, or, $\lambda = -4\pi^2 k^2$. Then $E_\lambda = \text{span}(\cos(2\pi kt), (\sin(2\pi kt))$.

In the case $\lambda > 0$ there are no periodic solutions. In the case $\lambda = 0$ we have the constant solutions, so that $\lambda = 0$ is an eigenvalue with $E_0 = \text{span}(1)$.

Summary:

$\lambda = -4\pi^2 k^2$ is an eigenvalue, for $k = 1, 2, 3, \ldots$, with $E_\lambda = \text{span}(\cos(2\pi kt), (\sin(2\pi kt))$.
$\lambda = 0$ is an eigenvalue, with $E_0 = \text{span}(1)$.

43. a. Using the approach of Exercise 17, we find $x(t) = c_1 e^{-2t} + c_2 e^{-3t} + \frac{1}{10}\cos t + \frac{1}{10}\sin t$.

b. For large $t, x(t) \approx \frac{1}{10}\cos t + \frac{1}{10}\sin t$.

44. a. Using the approach of Exercises 16 and 17 we find $x(t) = e^{-2t}(c_1 \cos t + c_2 \sin t) - \frac{1}{40}\cos(3t) + \frac{3}{40}\sin(3t)$.

b. For large $t, x(t) \approx -\frac{1}{40}\cos(3t) + \frac{3}{40}\sin(3t)$.

45. We can write the system as $\left| \begin{array}{l} \frac{dx_1}{dt} = x_1 + 2x_2 \\ \frac{dx_2}{dt} = x_2 \end{array} \right| \begin{array}{l} x_1(0) = 1 \\ x_2(0) = -1 \end{array}$.

The solution of the second equation, with the given initial value, is $x_2(t) = -e^t$.

Now the first equation takes the form $\frac{dx_1}{dt} - x_1 = -2e^t$.

Using Example 9 (with $a = 1$ and $c = -2$) we find $x_1(t) = e^t(-2t + C)$.

plug in: $1 = x_1(0) = C$, so that $x_1(t) = e^t(1 - 2t)$ and $\vec{x}(t) = e^t \begin{bmatrix} 1 - 2t \\ -1 \end{bmatrix}$.

46. We can write the system as $\left| \begin{array}{rcr} \frac{dx_1}{dt} & = & 2x_1 + 3x_2 + x_3 \\ \frac{dx_2}{dt} & = & x_2 + 2x_3 \\ \frac{dx_3}{dt} & = & x_3 \end{array} \right| \begin{array}{rcr} x_1(0) & = & 2 \\ x_2(0) & = & 1 \\ x_3(0) & = & -1 \end{array}$.

We solve for x_2 and x_3 as in Exercise 45:

$x_2(t) = e^t(1 - 2t)$

$x_3(t) = -e^t$

Now the first equation takes the form $\frac{dx_1}{dt} - 2x_1 = 3e^t(1 - 2t) - e^t = e^t(2 - 6t), x_1(0) = 2$.

We use Fact 9.3.13 to solve this differential equation:

$$x_1(t) = e^{2t} \int e^{-2t} e^t (2 - 6t)\, dt = e^{2t} \int (2e^{-t} - 6te^{-t})\, dt = e^{2t}[-2e^{-t} + 6te^{-t} + 6e^{-t} + C]$$

plug in: $2 = x_1(0) = (-2 + 6 + c)$, so that $c = -2$ and $x_1(t) = e^{2t}(4e^{-t} + 6te^{-t} - 2) = 4e^t + 6te^t - 2e^{2t}$.

$$\vec{x}(t) = \begin{bmatrix} 4e^t + 6te^t - 2e^{2t} \\ e^t - 2te^t \\ -e^t \end{bmatrix}$$

47. a. We start with a *preliminary remark* that will be useful below: If $f(t) = p(t)e^{\lambda t}$, where $p(t)$ is a polynomial, then $f(t)$ has an antiderivative of the form $q(t)e^{\lambda t}$, where $q(t)$ is another polynomial. We leave this remark as a calculus exercise.

The function $x_n(t)$ satisfies the differential equation $\frac{dx_n}{dt} = a_{nn}x_n$, so that $x_n = Ce^{a_{nn}t}$, which is of the desired form.

Now we will show that x_k is of the desired form, assuming that $x_{k+1}, \ldots, x_n$ have this form. x_k satisfies the differential equation $\frac{dx_k}{dt} = a_{kk}x_k + a_{k,k+1}x_{k+1} + \cdots + a_{kn}x_n$ or $\frac{dx_k}{dt} - a_{kk}x_k = a_{k,k+1}x_{k+1} + \cdots + a_{kn}x_n$.

Note that, by assumption, the function on the right-hand side has the form $p_1(t)e^{\lambda_1 t} + \cdots + p_m(t)e^{\lambda_m t}$. If we set $a_{kk} = a$ for simplicity, we can write $\frac{dx_k}{dt} - ax_k = p_1(t)e^{\lambda_1 t} + \cdots + p_m(t)e^{\lambda_m t}$.

By Fact 9.3.13, the solution is

$$x_k(t) = e^{at} \int e^{-at}(p_1(t)e^{\lambda_1 t} + \cdots + p_m(t)e^{\lambda_m t})\, dt$$

$$= e^{at} \int (p_1(t)e^{(\lambda_1 - a)t} + \cdots + p_m(t)e^{(\lambda_m - a)t})\, dt$$

$= e^{at}(q_1(t)e^{(\lambda_1 - a)t} + \cdots + q_m(t)e^{(\lambda_m - a)t} + C) = q_1(t)e^{\lambda_1 t} + \cdots + q_m(t)e^{\lambda_m t} + Ce^{at}$ as claimed (note that a is one of the λ_i). The constant C is determined by $x_k(0)$. Note that we used the *preliminary remark* in the second to last step.

b. It is shown in introductory calculus classes that $\lim_{t\to\infty}(t^m e^{\lambda t}) = 0$ if and only if λ is negative (here m is a fixed positive integer). In light of part (a), this proves the claim.

48. There is an invertible S such that $S^{-1}AS = B$ is upper triangular. Then the system $\frac{d\vec{x}}{dt} = A\vec{x} = SBS^{-1}\vec{x}$ can be written as $\frac{d}{dt}(S^{-1}\vec{x}) = B(S^{-1}\vec{x})$ or $\frac{d\vec{u}}{dt} = B\vec{u}$, where $\vec{u} = S^{-1}\vec{x}$. Note that B has the m distinct diagonal entries $\lambda_1, \ldots, \lambda_m$.

a. By Exercise 47, the system $\frac{d\vec{u}}{dt} = B\vec{u}$ has a unique solution $\vec{u}(t)$. Then the system $\frac{d\vec{x}}{dt} = A\vec{x}$ has the unique solution $\vec{x}(t) = S\vec{u}(t)$.

b. It suffices to note that $\lim_{t\to\infty}\vec{x}(t) = \vec{0}$ if and only if $\lim_{t\to\infty}\vec{u}(t) = \vec{0}$, where $\vec{u} = S^{-1}\vec{x}$.