

## Chapter 3

### 3.1

1. Find all  $\vec{x}$  such that  $A\vec{x} = \vec{0}$ :

$$\left[ \begin{array}{ccc|c} 1 & 2 & 0 & 0 \\ 3 & 4 & 0 & 0 \end{array} \right] \longrightarrow \left[ \begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{array} \right], \text{ so that } x_1 = x_2 = 0.$$

$$\ker(A) = \{\vec{0}\}.$$

2. Find all  $\vec{x}$  such that  $A\vec{x} = \vec{0}$ :

$$\left[ \begin{array}{ccc|c} 2 & 3 & 0 & 0 \\ 6 & 9 & 0 & 0 \end{array} \right] \longrightarrow \left[ \begin{array}{ccc|c} 1 & \frac{3}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right], \text{ so that } \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -\frac{3t}{2} \\ t \end{bmatrix}$$

$$\text{Setting } t = 2 \text{ we find } \ker(A) = \text{span} \begin{bmatrix} -3 \\ 2 \end{bmatrix}.$$

3. Find all  $\vec{x}$  such that  $A\vec{x} = \vec{0}$ ; note that all  $\vec{x}$  in  $\mathbb{R}^2$  satisfy the equation, so that  $\ker(A) = \mathbb{R}^2 = \text{span}(\vec{e}_1, \vec{e}_2)$ .
4. Find all  $\vec{x}$  such that  $A\vec{x} = \vec{0}$ , or  $x_1 + 2x_2 + 3x_3 = 0$ .

$$\text{The solutions are of the form } \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -2t - 3r \\ t \\ r \end{bmatrix} = t \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} + r \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}, \text{ so that}$$

$$\ker(A) = \text{span} \left( \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix} \right).$$

5. Find all  $\vec{x}$  such that  $A\vec{x} = \vec{0}$ .

$$\left[ \begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 1 & 2 & 3 & 0 \\ 1 & 3 & 5 & 0 \end{array} \right] \longrightarrow \left[ \begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right]; \begin{array}{l} x_1 = x_3 \\ x_2 = -2x_3 \end{array}; \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} t \\ -2t \\ t \end{bmatrix}$$

$$\ker(A) = \text{span} \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}.$$

6. Find all  $\vec{x}$  such that  $A\vec{x} = \vec{0}$ .

$$\begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}; x_1 + x_2 + x_3 = 0$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -r-t \\ r \\ t \end{bmatrix} = r \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

$$\ker(A) = \text{span} \left( \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right).$$

7. Find all  $\vec{x}$  such that  $A\vec{x} = \vec{0}$ . Since  $\text{rref}(A) = I_3$  we have  $\ker(A) = \{\vec{0}\}$ .

8. Find all  $\vec{x}$  such that  $A\vec{x} = \vec{0}$ . Solving this system yields  $\ker(A) = \text{span} \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$ .

9. Find all  $\vec{x}$  such that  $A\vec{x} = \vec{0}$ . Solving this system yields  $\ker(A) = \{\vec{0}\}$ .

10. Solving the system  $A\vec{x} = \vec{0}$  we find that  $\ker(A) = \text{span} \begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \end{bmatrix}$ .

11. Solving the system  $A\vec{x} = \vec{0}$  we find that  $\ker(A) = \text{span} \begin{bmatrix} -2 \\ 3 \\ 1 \\ 0 \end{bmatrix}$ .

12. Solving the system  $A\vec{x} = \vec{0}$  we find that  $\ker(A) = \text{span} \left( \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ -1 \\ 1 \\ 0 \end{bmatrix} \right)$ .

13. Solving the system  $A\vec{x} = \vec{0}$  we find that  $\ker(A) = \text{span} \left( \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ -2 \\ -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right)$ .

14. By Fact 3.1.3, the image of  $A$  is the span of the column vectors of  $A$ :

$$\text{im}(A) = \text{span} \left( \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} \right).$$

15. By Fact 3.1.3, the image of  $A$  is the span of the columns of  $A$ :

$$\text{im}(A) = \text{span} \left( \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 4 \end{bmatrix} \right).$$

Since any two of these vectors span all of  $\mathbb{R}^2$  already, we can write

$$\text{im}(A) = \text{span} \left( \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right).$$

16. By Fact 3.1.3, the image of  $A$  is the span of the column vectors of  $A$ :

$$\text{im}(A) = \text{span} \left( \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ 3 \\ 3 \end{bmatrix} \right).$$

Since these three vectors are parallel, we need only one of them to span the image:

$$\text{im}(A) = \text{span} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

17. By Fact 3.1.3,  $\text{im}(A) = \text{span} \left( \begin{bmatrix} 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 4 \end{bmatrix} \right) = \mathbb{R}^2$  (the whole plane).

18. By Fact 3.1.3,  $\text{im}(A) = \text{span} \left( \begin{bmatrix} 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 4 \\ 12 \end{bmatrix} \right) = \text{span} \begin{bmatrix} 1 \\ 3 \end{bmatrix}$  (a line in  $\mathbb{R}^2$ ).

19. Since the four column vectors of  $A$  are parallel, we have  $\text{im}(A) = \text{span} \begin{bmatrix} 1 \\ -2 \end{bmatrix}$ , a line in  $\mathbb{R}^2$ .

20. Since the three column vectors of  $A$  are parallel, we have  $\text{im}(A) = \text{span} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ , a line in  $\mathbb{R}^3$ .

21. By Fact 3.1.3,  $\text{im}(A) = \text{span} \left( \begin{bmatrix} 4 \\ 1 \\ 5 \end{bmatrix}, \begin{bmatrix} 7 \\ 9 \\ 6 \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \\ 8 \end{bmatrix} \right)$ .

We must simply find out how many of the column vectors are not redundant to determine a basis of the image. We can determine this by taking the rref of the matrix:

$$\begin{bmatrix} 4 & 7 & 3 \\ 1 & 9 & 2 \\ 5 & 6 & 8 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \text{ which shows us that all three column vectors are independent: the span is all of } \mathbb{R}^3.$$

22. Compare with the solution to Exercise 21.

$$\begin{bmatrix} 2 & 1 & 3 \\ 3 & 4 & 2 \\ 6 & 5 & 7 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

This computation shows that the third column vector of  $A$ ,  $\vec{v}_3$ , is a linear combination of the first two. Thus, only the first two vectors are independent, and the image is a plane in  $\mathbb{R}^3$ .

23.  $\text{im}(T) = \mathbb{R}^2$  and  $\ker(T) = \{\vec{0}\}$ , since  $T$  is invertible (see Summary 3.1.8).

24.  $\text{im}(T)$  is the plane  $x + 2y + 3z = 0$ , and  $\ker(T)$  is the line perpendicular to this plane, spanned by the vector  $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$  (compare with Examples 5 and 9).

25.  $\text{im}(T) = \mathbb{R}^2$  and  $\ker(T) = \{\vec{0}\}$ , since  $T$  is invertible (see Summary 3.1.8).

26. Since  $\lim_{t \rightarrow \infty} f(t) = \infty$  and  $\lim_{t \rightarrow -\infty} f(t) = -\infty$ , we have  $\text{im}(f) = \mathbb{R}$ .

A careful proof involves the intermediate value theorem (see Exercise 2.2.47c),

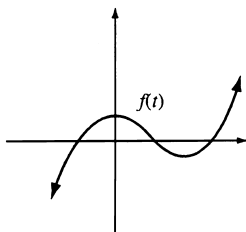


Figure 3.1: for Problem 3.1.26.

Any horizontal line intersects this graph at least once (compare with Example 3 and see Figure 3.1).

27. Let  $f(x) = x^3 - x = x(x^2 - 1) = x(x - 1)(x + 1)$ .

Then  $\text{im}(f) = \mathbb{R}$ , since

$$\lim_{x \rightarrow \infty} f(x) = \infty \text{ and } \lim_{x \rightarrow -\infty} f(x) = -\infty$$

but the function fails to be invertible since the equation  $f(x) = 0$  has three solutions,  $x = 0, 1$ , and  $-1$ .

28. This ellipse can be obtained from the unit circle by means of the linear transformation with matrix  $\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$ , as shown in Figure 3.2 (compare with Exercise 2.2.49).

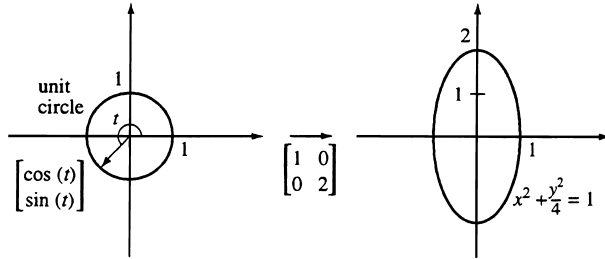


Figure 3.2: for Problem 3.1.28.

We obtain the parametrization  $\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} \cos(t) \\ \sin(t) \end{bmatrix} = \begin{bmatrix} \cos(t) \\ 2 \sin(t) \end{bmatrix}$  for the ellipse.

We can check that  $x^2 + \frac{y^2}{4} = \cos^2(t) + \frac{4 \sin^2(t)}{4} = 1$ .

29. Use spherical coordinates (see any good text on multivariable calculus):  $f \begin{bmatrix} \phi \\ \theta \end{bmatrix} = \begin{bmatrix} \sin(\phi) \cos(\theta) \\ \sin(\phi) \sin(\theta) \\ \cos(\phi) \end{bmatrix}$

30. By Fact 3.1.3,  $A = \begin{bmatrix} 1 \\ 5 \end{bmatrix}$  does the job. There are many other possible answers: any nonzero  $2 \times n$  matrix  $A$  whose column vectors are scalar multiples of vector  $\begin{bmatrix} 1 \\ 5 \end{bmatrix}$ .

31. The plane  $x + 3y + 2z = 0$  is spanned by the two vectors  $\begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}$  and  $\begin{bmatrix} -3 \\ 1 \\ 0 \end{bmatrix}$ , for example.

Therefore,  $A = \begin{bmatrix} -2 & -3 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}$  does the job. There are many other correct answers.

32. By Fact 3.1.3,  $A = \begin{bmatrix} 7 \\ 6 \\ 5 \end{bmatrix}$  does the job. There are many other correct answers: any nonzero

$3 \times n$  matrix  $A$  whose column vectors are scalar multiples of  $\begin{bmatrix} 7 \\ 6 \\ 5 \end{bmatrix}$ .

33. The plane is the kernel of the linear transformation  $T \begin{bmatrix} x \\ y \\ z \end{bmatrix} = x + 2y + 3z$  from  $\mathbb{R}^3$  to  $\mathbb{R}$ .

34. To describe a subset of  $\mathbb{R}^3$  as a kernel means to describe it as an intersection of planes (think about it). By inspection, the given line is the intersection of the planes

$$\begin{aligned} x + y &= 0 & \text{and} \\ 2x + z &= 0. \end{aligned}$$

This means that the line is the kernel of the linear transformation  $T \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x + y \\ 2x + z \end{bmatrix}$  from  $\mathbb{R}^3$  to  $\mathbb{R}^2$ .

35.  $\ker(T) = \{\vec{x} : T(\vec{x}) = \vec{v} \cdot \vec{x} = 0\} =$  the plane with normal vector  $\vec{v}$ .

$\text{im}(T) = \mathbb{R}$ , since for every real number  $k$  there is a vector  $\vec{x}$  such that  $T(\vec{x}) = k$ , for example,  $\vec{x} = \frac{k}{\vec{v} \cdot \vec{v}} \vec{v}$ .

36.  $\ker(T) = \{\vec{x} : T(\vec{x}) = \vec{v} \times \vec{x} = \vec{0}\} =$  the line spanned by  $\vec{v}$

(see Fact A.10b in the Appendix)

$\text{im}(T) =$  the plane with normal vector  $\vec{v}$

By Fact A.10a,  $T(\vec{x}) = \vec{v} \times \vec{x}$  is in this plane, for all  $\vec{x}$  in  $\mathbb{R}^3$ . Conversely, for every vector  $\vec{w}$  in this plane there is an  $\vec{x}$  in  $\mathbb{R}^3$  such that  $T(\vec{x}) = \vec{w}$ , namely  $\vec{x} = -\frac{1}{\vec{v} \cdot \vec{v}} T(\vec{w})$  (verify this!).

37.  $A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ ,  $A^2 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ ,  $A^3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ , so that

$\ker(A) = \text{span}(\vec{e}_1)$ ,  $\ker(A^2) = \text{span}(\vec{e}_1, \vec{e}_2)$ ,  $\ker(A^3) = \mathbb{R}^3$ , and

$\text{im}(A) = \text{span}(\vec{e}_1, \vec{e}_2)$ ,  $\text{im}(A^2) = \text{span}(\vec{e}_1)$ ,  $\text{im}(A^3) = \{\vec{0}\}$ .

38. a. If a vector  $\vec{x}$  is in  $\ker(A^k)$ , that is,  $A^k\vec{x} = \vec{0}$ , then  $\vec{x}$  is also in  $\ker(A^{k+1})$ , since  $A^{k+1}\vec{x} = AA^k\vec{x} = A\vec{0} = \vec{0}$ .

Therefore,  $\ker(A) \subseteq \ker(A^2) \subseteq \ker(A^3) \subseteq \dots$

Exercise 37 shows that these kernels need not be equal.

- b. If a vector  $\vec{y}$  is in  $\text{im}(A^{k+1})$ , that is,  $\vec{y} = A^{k+1}\vec{x}$  for some  $\vec{x}$ , then  $\vec{y}$  is also in  $\text{im}(A^k)$ , since we can write  $\vec{y} = A^k(A\vec{x})$ . Therefore,  $\text{im}(A) \supseteq \text{im}(A^2) \supseteq \text{im}(A^3) \supseteq \dots$

Exercise 37 shows that these images need not be equal.

39. a. If a vector  $\vec{x}$  is in  $\ker(B)$ , that is,  $B\vec{x} = \vec{0}$ , then  $\vec{x}$  is also in  $\ker(AB)$ , since  $AB(\vec{x}) = A(B\vec{x}) = A\vec{0} = \vec{0}$ :

$$\ker(B) \subseteq \ker(AB).$$

Exercise 37 (with  $A = B$ ) illustrates that these kernels need not be equal.

- b. If a vector  $\vec{y}$  is in  $\text{im}(AB)$ , that is,  $\vec{y} = AB\vec{x}$  for some  $\vec{x}$ , then  $\vec{y}$  is also in  $\text{im}(A)$ , since we can write

$$\vec{y} = A(B\vec{x}):$$

$$\text{im}(AB) \subseteq \text{im}(A).$$

Exercise 37 (with  $A = B$ ) illustrates that these images need not be equal.

40. For any  $\vec{x}$  in  $\mathbb{R}^m$ , the vector  $B\vec{x}$  is in  $\text{im}(B) = \ker(A)$ , so that  $AB\vec{x} = \vec{0}$ . If we apply this fact to  $\vec{x} = \vec{e}_1, \vec{e}_2, \dots, \vec{e}_m$ , we find that all the columns of the matrix  $AB$  are zero, so that  $AB = 0$ .

41. a.  $\text{rref}(A) = \begin{bmatrix} 1 & \frac{4}{3} \\ 0 & 0 \end{bmatrix}$ , so that  $\ker(A) = \text{span} \begin{bmatrix} -4 \\ 3 \end{bmatrix}$ .

$$\text{im}(A) = \text{span} \begin{bmatrix} 0.36 \\ 0.48 \end{bmatrix} = \text{span} \begin{bmatrix} 3 \\ 4 \end{bmatrix}.$$

Note that  $\text{im}(A)$  and  $\ker(A)$  are perpendicular lines.

- b.  $A^2 = A$

If  $\vec{v}$  is in  $\text{im}(A)$ , with  $\vec{v} = A\vec{x}$ , then  $A\vec{v} = A^2\vec{x} = A\vec{x} = \vec{v}$ .

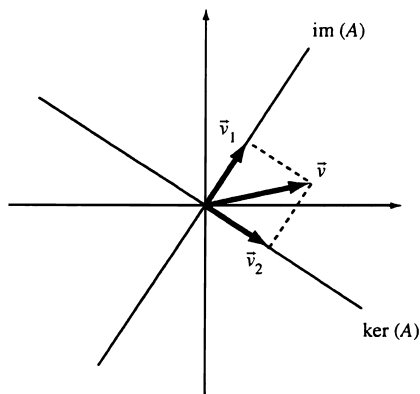


Figure 3.3: for Problem 3.1.41c.

- c. Any vector  $\vec{v}$  in  $\mathbb{R}^2$  can be written uniquely as  $\vec{v} = \vec{v}_1 + \vec{v}_2$ , where  $\vec{v}_1$  is in  $\text{im}(A)$  and  $\vec{v}_2$  is in  $\text{ker}(A)$ . (See Figure 3.3.) Then  $A\vec{v} = A\vec{v}_1 + A\vec{v}_2 = \vec{v}_1$  ( $A\vec{v}_1 = \vec{v}_1$  by part b,  $A\vec{v}_2 = \vec{0}$  since  $\vec{v}_2$  is in  $\text{ker}(A)$ ), so that  $A$  represents the *orthogonal projection* onto  $\text{im}(A) = \text{span} \begin{bmatrix} 3 \\ 4 \end{bmatrix}$ .

42. Using the hint, we see that the vector  $\vec{y} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix}$  is in the image of  $A$  if

$$\begin{array}{rclcl} y_1 & -3y_3 & +2y_4 & = & 0 & \text{and} \\ y_2 & -2y_3 & +y_4 & = & 0. \end{array}$$

This means that  $\text{im}(A)$  is the kernel of the matrix  $\begin{bmatrix} 1 & 0 & -3 & 2 \\ 0 & 1 & -2 & 1 \end{bmatrix}$ .

43. Using our work in Exercise 42 as a guide, we come up with the following procedure to express the image of an  $n \times m$  matrix  $A$  as the kernel of a matrix  $B$ :

If  $\text{rank}(A) = n$ , let  $B$  be the  $n \times n$  zero matrix.

If  $r = \text{rank}(A) < n$ , let  $B$  be the  $(n-r) \times n$  matrix obtained by omitting the first  $r$  rows and the first  $m$  columns of  $\text{rref}[A; I_n]$ .



44. a. Yes; by construction of the echelon form, the systems  $A\vec{x} = \vec{0}$  and  $B\vec{x} = \vec{0}$  have the same solutions (it is the whole point of Gaussian elimination not to change the solutions of a system).
- b. No; as a counterexample, consider  $A = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$ , with  $\text{im}(A) = \text{span}(\vec{e}_2)$ , but  $B = \text{rref}(A) = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ , with  $\text{im}(B) = \text{span}(\vec{e}_1)$ .
45. As we solve the system  $A\vec{x} = \vec{0}$ , we obtain  $r$  leading variables and  $m - r$  free variables. The “general vector” in  $\ker(A)$  can be written as a linear combination of  $m - r$  vectors, with the free variables as coefficients. (See Example 11, where  $m - r = 5 - 3 = 2$ .)
46. If  $\text{rank}(A) = r$ , then  $\text{im}(A) = \text{span}(\vec{e}_1, \dots, \vec{e}_r)$ . See Figure 3.4.

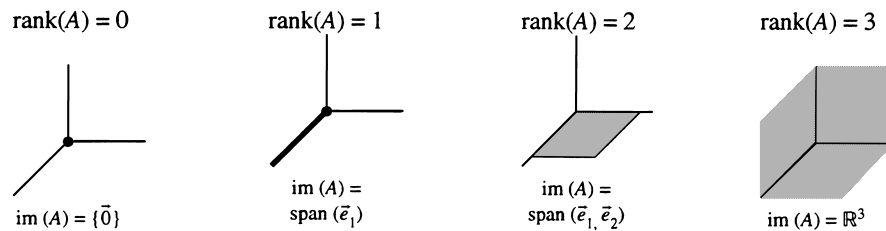


Figure 3.4: for Problem 3.1.46.

47.  $\text{im}(T) = L_2$  and  $\ker(T) = L_1$ .
48. a.  $\vec{w} = A\vec{x}$ , for some  $\vec{x}$ , so that  $A\vec{w} = A^2\vec{x} = A\vec{x} = \vec{w}$ .
- b. If  $\text{rank}(A) = 2$ , then  $A$  is invertible, and the equation  $A^2 = A$  implies that  $A = I_2$  (multiply by  $A^{-1}$ ).
- If  $\text{rank}(A) = 0$  then  $A = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ .
- c. First note that  $\text{im}(A)$  and  $\ker(A)$  are lines (there is one nonleading variable).
- By definition of a projection, we need to verify that  $\vec{x} - A\vec{x}$  is in  $\ker(A)$ . This is indeed the case, since
- $A(\vec{x} - A\vec{x}) = A\vec{x} - A^2\vec{x} = A\vec{x} - A\vec{x} = \vec{0}$  (we are told that  $A^2 = A$ ). See Figure 3.5.

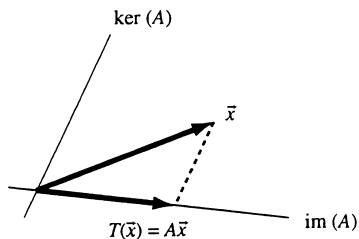


Figure 3.5: for Problem 3.1.48c.

49. If  $\vec{v}$  and  $\vec{w}$  are in  $\ker(T)$ , then  $T(\vec{v} + \vec{w}) = T(\vec{v}) + T(\vec{w}) = \vec{0} + \vec{0} = \vec{0}$ , so that  $\vec{v} + \vec{w}$  is in  $\ker(T)$  as well.

If  $\vec{v}$  is in  $\ker(T)$  and  $k$  is an arbitrary scalar, then  $T(k\vec{v}) = kT(\vec{v}) = k\vec{0} = \vec{0}$ , so that  $k\vec{v}$  is in  $\ker(T)$  as well.

50. From Exercise 38 we know that  $\ker(A^3) \subseteq \ker(A^4)$ . Conversely, if  $\vec{x}$  is in  $\ker(A^4)$ , then  $A^4\vec{x} = A^3(A\vec{x}) = \vec{0}$ , so that  $A\vec{x}$  is in  $\ker(A^3) = \ker(A^2)$ , which implies that  $A^2(A\vec{x}) = A^3\vec{x} = \vec{0}$ , that is,  $\vec{x}$  is in  $\ker(A^3)$ . We have shown that  $\ker(A^3) = \ker(A^4)$ .

51. We need to find all  $\vec{x}$  such that  $AB\vec{x} = \vec{0}$ . If  $AB\vec{x} = \vec{0}$ , then  $B\vec{x}$  is in  $\ker(A) = \{\vec{0}\}$ , so that  $B\vec{x} = \vec{0}$ .

Since  $\ker(B) = \{\vec{0}\}$ , we can conclude that  $\vec{x} = \vec{0}$ . It follows that  $\ker(AB) = \{\vec{0}\}$ .

52. Since  $C\vec{x} = \begin{bmatrix} A \\ B \end{bmatrix} \vec{x} = \begin{bmatrix} A\vec{x} \\ B\vec{x} \end{bmatrix}$ , we can conclude that  $C\vec{x} = \vec{0}$  if (and only if) both  $A\vec{x} = \vec{0}$  and  $B\vec{x} = \vec{0}$ . It follows that  $\ker(C)$  is the intersection of  $\ker(A)$  and  $\ker(B)$ :  $\ker(C) = \ker(A) \cap \ker(B)$ .

53. a. Using the equation  $1 + 1 = 0$  (or  $-1 = 1$ ), we can write the general vector  $\vec{x}$  in  $\ker(H)$  as

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \end{bmatrix} = \begin{bmatrix} p + r + s \\ p + q + s \\ p + q + r \\ p \\ q \\ r \\ s \end{bmatrix}$$

$$= p \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + q \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + r \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + s \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

$$\begin{array}{cccc} \uparrow & \uparrow & \uparrow & \uparrow \\ \vec{v}_1 & \vec{v}_2 & \vec{v}_3 & \vec{v}_4 \end{array}$$

b.  $\ker(H) = \text{span}(\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4)$  by part (a), and  $\text{im}(M) = \text{span}(\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4)$  by Fact 3.1.3, so that  $\text{im}(M) = \ker(H)$ .  $M\vec{x}$  is in  $\text{im}(M) = \ker(H)$ , so that  $H(M\vec{x}) = \vec{0}$ .

54. a. If no error occurred, then  $\vec{w} = \vec{v} = M\vec{u}$ , and  $H\vec{w} = H(M\vec{u}) = \vec{0}$ , by Exercise 53b.

If an error occurred in the  $i$ th component, then  $\vec{w} = \vec{v} + \vec{e}_i = M\vec{u} + \vec{e}_i$ , so that

$$H\vec{w} = H(M\vec{u}) + H\vec{e}_i = i\text{th column of } H.$$

Since the columns of  $H$  are all different, this method allows us to find out where an error occurred.

b.  $H\vec{w} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$  = seventh column of  $H$ : an error occurred in the seventh component of  $\vec{v}$ .

$$\text{Therefore } \vec{v} = \vec{w} + \vec{e}_7 = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} \text{ and } \vec{u} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}.$$

## 3.2

1. Not a subspace, since  $W$  does not contain the zero vector.

2. Not a subspace, since  $W$  contains the vector  $\vec{v} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$  but not the vector  $(-1)\vec{v} = \begin{bmatrix} -1 \\ -2 \\ -3 \end{bmatrix}$ .

3.  $W = \text{im} \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$  is a subspace of  $\mathbb{R}^3$ , by Fact 3.2.2.

4.  $\text{span}(\vec{v}_1, \dots, \vec{v}_m) = \text{im}[\vec{v}_1 \ \dots \ \vec{v}_m]$  is a subspace of  $\mathbb{R}^n$ , by Fact 3.2.2.
5. We have subspaces  $\{\vec{0}\}$ ,  $\mathbb{R}^3$ , and all lines and planes (through the origin). To prove this, mimic the reasoning in Example 2.

6. a. Yes!

- The zero vector is in  $V \cap W$ , since  $\vec{0}$  is in both  $V$  and  $W$ .
  - If  $\vec{x}$  and  $\vec{y}$  are in  $V \cap W$ , then both  $\vec{x}$  and  $\vec{y}$  are in  $V$ , so that  $\vec{x} + \vec{y}$  is in  $V$  as well, since  $V$  is a subspace of  $\mathbb{R}^n$ . Likewise,  $\vec{x} + \vec{y}$  is in  $W$ , so that  $\vec{x} + \vec{y}$  is in  $V \cap W$ .
  - If  $\vec{x}$  is in  $V \cap W$  and  $k$  is an arbitrary scalar, then  $k\vec{x}$  is in both  $V$  and  $W$ , since they are subspaces of  $\mathbb{R}^n$ . Therefore,  $k\vec{x}$  is in  $V \cap W$ .
- b. No; as a counterexample consider  $V = \text{span}(\vec{e}_1)$  and  $W = \text{span}(\vec{e}_2)$  in  $\mathbb{R}^2$ .

7. Yes; we need to show that  $W$  contains the zero vector. We are told that  $W$  is nonempty, so that it contains some vector  $\vec{v}$ . Since  $W$  is closed under scalar multiplication, it will contain the vector  $0\vec{v} = \vec{0}$ , as claimed.

8. We need to solve the system  $c_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + c_2 \begin{bmatrix} 2 \\ 3 \end{bmatrix} + c_3 \begin{bmatrix} 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ .

The general solution is  $\begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} t \\ -2t \\ t \end{bmatrix}$ .

Picking  $t = 1$  we find the nontrivial relation  $1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} - 2 \begin{bmatrix} 2 \\ 3 \end{bmatrix} + 1 \begin{bmatrix} 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ .

9. These vectors are linearly dependent, since  $\vec{v}_m = 0\vec{v}_1 + 0\vec{v}_2 + \dots + 0\vec{v}_{m-1}$ .
10. Linearly dependent, since  $\begin{bmatrix} 6 \\ 3 \end{bmatrix} = 3 \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ . Thus, the vector  $\begin{bmatrix} 6 \\ 3 \end{bmatrix}$  is redundant.
11. Linearly independent, since the two vectors are not parallel, and therefore not redundant.
12. Linearly dependent, since  $\begin{bmatrix} 0 \\ 0 \end{bmatrix} = 0 \begin{bmatrix} 7 \\ 11 \end{bmatrix}$ . Thus, the vector  $\vec{0}$  is redundant.

13. Linearly dependent, since the second vector is redundant  $\left(\begin{bmatrix} 1 \\ 2 \end{bmatrix} = 1 \begin{bmatrix} 1 \\ 2 \end{bmatrix}\right)$ .

14. Linearly independent, since  $\text{ref} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 2 \\ 0 & 0 & 3 \end{bmatrix} = I_3$  (use Fact 3.2.6).

15. Linearly dependent. By Fact 3.2.8, since we have three vectors in  $\mathbb{R}^2$ , at least one must be redundant. We can perform a straightforward computation to reveal that  $\vec{v}_3 = -\vec{v}_1 + 2\vec{v}_2$ .

16. Certainly  $\begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$  is not a multiple of  $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ , so it is not redundant. However,  $\begin{bmatrix} 6 \\ 5 \\ 4 \end{bmatrix} = 3 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + 1 \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$ , so  $\begin{bmatrix} 6 \\ 5 \\ 4 \end{bmatrix}$  is redundant. Thus, these vectors are linearly dependent.

17. Linearly independent. The first two vectors are clearly not redundant, and since  $\text{rref} \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 3 & 6 \end{bmatrix} = I_3$ , the last vector is also not redundant. Thus, the three vectors turn out to be linearly independent.

18. Linearly dependent, since  $\text{rref} \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & 7 \\ 1 & 4 & 10 \end{bmatrix} = \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ . So, we find that the vector  $\begin{bmatrix} 1 \\ 4 \\ 7 \\ 10 \end{bmatrix}$  turns out to be redundant.

19. Linearly dependent. First we see that  $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$  is not redundant, because it is first, and

non-zero. However,  $\begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix} = 2 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ , so it is redundant.

$\begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$  and  $\begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$  are clearly not redundant, but  $\begin{bmatrix} 3 \\ 4 \\ 5 \\ 0 \end{bmatrix} = 3 \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + 4 \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + 5 \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$ , so it is redundant.

20.  $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$  is redundant, simply because it is the zero vector.

$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$  is our first non-zero vector, and thus, is not redundant.

$\begin{bmatrix} 3 \\ 0 \\ 0 \end{bmatrix} = 3 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$  and is redundant.

$\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$  is not a multiple of  $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$  and is not redundant.

$\begin{bmatrix} 4 \\ 5 \\ 0 \end{bmatrix} = 4 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + 5 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$  and is redundant.

Similarly,  $\begin{bmatrix} 6 \\ 7 \\ 0 \end{bmatrix} = 6 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + 7 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$  and is also redundant.

However, by inspection,  $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$  is not a linear combination of  $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$  and  $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ , meaning that this last vector is not redundant. Thus, the seven vectors are linearly dependent.

21. Certainly, since the second vector equals the first, the second is redundant. So  $\vec{v}_1 = \vec{v}_2$ ,  $1\vec{v}_1 - 1\vec{v}_2 = \vec{0}$ , revealing that  $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$  is in  $\ker(A)$ .

22.  $\begin{bmatrix} 3 \\ 6 \end{bmatrix}$  is redundant, because  $\begin{bmatrix} 3 \\ 6 \end{bmatrix} = 3 \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ . So,  $3 \begin{bmatrix} 1 \\ 2 \end{bmatrix} - 1 \begin{bmatrix} 3 \\ 6 \end{bmatrix} = \vec{0}$ . Thus,  $\begin{bmatrix} 3 \\ -1 \end{bmatrix}$  is in the kernel of  $\begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix}$ .

23. The first column is  $\vec{0}$ , so it is redundant.  $1\vec{v}_1 = \vec{0}$ , so  $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$  is in  $\ker(A)$ .

24.  $\begin{bmatrix} 6 \\ 5 \\ 4 \end{bmatrix}$  is redundant, because  $\begin{bmatrix} 6 \\ 5 \\ 4 \end{bmatrix} = 3 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + 1 \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$ . Thus,  $3 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + 1 \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} - 1 \begin{bmatrix} 6 \\ 5 \\ 4 \end{bmatrix} = \vec{0}$

and  $\begin{bmatrix} 3 \\ 1 \\ -1 \end{bmatrix}$  is in the kernel of  $\begin{bmatrix} 1 & 3 & 6 \\ 1 & 2 & 5 \\ 1 & 1 & 4 \end{bmatrix}$ .

25. The third column equals the first, so it is redundant and  $\vec{v}_1 = \vec{v}_3$ , or  $1\vec{v}_1 + 0\vec{v}_2 - 1\vec{v}_3 = \vec{0}$ .

Thus,  $\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$  is in  $\ker(A)$ .

26.  $\begin{bmatrix} 2 \\ 3 \\ 0 \end{bmatrix} = 2 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + 3 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ , so  $\begin{bmatrix} 2 \\ 3 \\ 0 \end{bmatrix}$  is redundant. Now,  $2 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + 3 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} - 1 \begin{bmatrix} 2 \\ 3 \\ 0 \end{bmatrix} + 0 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \vec{0}$ ,

revealing that  $\begin{bmatrix} 2 \\ 3 \\ -1 \\ 0 \end{bmatrix}$  is in the kernel of  $\begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ .

27. A basis of  $\text{im}(A)$  is  $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ ,  $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ , by Fact 3.2.4.

28. The three column vectors are linearly independent, since  $\text{rref} \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 5 \\ 1 & 3 & 7 \end{bmatrix} = I_3$ .

Therefore, the three columns form a basis of  $\text{im}(A)(= \mathbb{R}^3)$ :

$$\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 5 \\ 7 \end{bmatrix}.$$

Another sensible choice for a basis of  $\text{im}(A)$  is  $\vec{e}_1, \vec{e}_2, \vec{e}_3$ .

29. The three column vectors of  $A$  span all of  $\mathbb{R}^2$ , so that  $\text{im}(A) = \mathbb{R}^2$ . We can choose any two of the columns of  $A$  to form a basis of  $\text{im}(A)$ ; another sensible choice is  $\vec{e}_1, \vec{e}_2$ .

30.  $\text{im}(A) = \text{span}(\vec{e}_1, \vec{e}_2)$

We can choose  $\vec{e}_1, \vec{e}_2$  as a basis of  $\text{im}(A)$ .

31. The two column vectors of the given matrix  $A$  are linearly independent (they are not parallel), so that they form a basis of  $\text{im}(A)$ .

32. By inspection, the first, third and sixth columns are redundant. Thus, a basis of the

image consists of the remaining column vectors:  $\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$ .

33.  $\text{im}(A) = \text{span}(\vec{e}_1, \vec{e}_2, \vec{e}_3)$ , so that  $\vec{e}_1, \vec{e}_2, \vec{e}_3$  is a basis of  $\text{im}(A)$ .

34. The fact that  $\begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$  is in  $\ker(A)$  means that

$$A \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} = [\vec{v}_1 \ \vec{v}_2 \ \vec{v}_3 \ \vec{v}_4] \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} = \vec{v}_1 + 2\vec{v}_2 + 3\vec{v}_3 + 4\vec{v}_4 = \vec{0}, \text{ so that } \vec{v}_4 = -\frac{1}{4}\vec{v}_1 - \frac{1}{2}\vec{v}_2 - \frac{3}{4}\vec{v}_3.$$

35. If  $\vec{v}_i$  is a linear combination of the other vectors in the list,  $\vec{v}_i = c_1\vec{v}_1 + \cdots + c_{i-1}\vec{v}_{i-1} + c_{i+1}\vec{v}_{i+1} + \cdots + c_n\vec{v}_n$ , then we can subtract  $\vec{v}_i$  from both sides to generate a nontrivial relation (the coefficient of  $\vec{v}_i$  will be -1).

Conversely, if there is a nontrivial relation  $c_1\vec{v}_1 + \cdots + c_i\vec{v}_i + \cdots + c_n\vec{v}_n = \vec{0}$ , with  $c_i \neq 0$ , then we can solve for vector  $\vec{v}_i$  and thus express  $\vec{v}_i$  as a linear combination of the other vectors in the list.

36. Yes; we know that there is a nontrivial relation  $c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_m\vec{v}_m = \vec{0}$ .

Now apply the transformation  $T$  to the vectors on both sides, and use linearity:

$$T(c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_m\vec{v}_m) = T(\vec{0}), \text{ so that } c_1T(\vec{v}_1) + c_2T(\vec{v}_2) + \cdots + c_mT(\vec{v}_m) = \vec{0}.$$

This is a nontrivial relation among the vectors  $T(\vec{v}_1), \dots, T(\vec{v}_m)$ , so that these vectors are linearly dependent, as claimed.

37. No; as a counterexample, consider the extreme case when  $T$  is the zero transformation, that is,  $T(\vec{x}) = \vec{0}$  for all  $\vec{x}$ . Then the vectors  $T(\vec{v}_1), \dots, T(\vec{v}_m)$  will all be zero, so that they are linearly dependent.

38. a. Using the terminology introduced in the exercise, we need to show that any vector  $\vec{v}$  in  $V$  is a linear combination of  $\vec{v}_1, \dots, \vec{v}_m$ . Choose a specific vector  $\vec{v}$  in  $V$ . Since we can find no more than  $m$  linearly independent vectors in  $V$ , the  $m+1$  vectors  $\vec{v}_1, \dots, \vec{v}_m, \vec{v}$  will be linearly dependent. Since the vectors  $\vec{v}_1, \dots, \vec{v}_m$  are independent,  $\vec{v}$  must be redundant, meaning that  $\vec{v}$  is a linear combination of  $\vec{v}_1, \dots, \vec{v}_m$ , as claimed.



- b. With the terminology introduced in part a, we can let  $V = \text{im} [\vec{v}_1 \ \cdots \ \vec{v}_m]$ .
39. Yes; the vectors are linearly independent. The vectors in the list  $\vec{v}_1, \dots, \vec{v}_m$  are linearly independent (and therefore non-redundant), and  $\vec{v}$  is non-redundant since it fails to be in the span of  $\vec{v}_1, \dots, \vec{v}_m$ .
40. Yes; by Fact 3.2.8,  $\ker(A) = \{\vec{0}\}$  and  $\ker(B) = \{\vec{0}\}$ . Then  $\ker(AB) = \{\vec{0}\}$  by Exercise 3.1.51, so that the columns of  $AB$  are linearly independent, by Fact 3.2.8.
41. To show that the columns of  $B$  are linearly independent, we show that  $\ker(B) = \{\vec{0}\}$ . Indeed, if  $B\vec{x} = \vec{0}$ , then  $AB\vec{x} = A\vec{0} = \vec{0}$ , so that  $\vec{x} = \vec{0}$  (since  $AB = I_m$ ).

By Fact 3.2.8,  $\text{rank}(B) = \# \text{ columns} = m$ , so that  $m \leq n$  and in fact  $m < n$  (we are told that  $m \neq n$ ). This implies that the rank of the  $m \times n$  matrix  $A$  is less than  $n$ , so that the columns of  $A$  are linearly dependent (by Fact 3.2.8).

42. We can use the hint and form the dot product of  $\vec{v}_i$  and both sides of the relation

$$c_1\vec{v}_1 + \cdots + c_i\vec{v}_i + \cdots + c_m\vec{v}_m = \vec{0}:$$

$$(c_1\vec{v}_1 + \cdots + c_i\vec{v}_i + \cdots + c_m\vec{v}_m) \cdot \vec{v}_i = \vec{0} \cdot \vec{v}_i, \text{ so that } c_1(\vec{v}_1 \cdot \vec{v}_i) + \cdots + c_i(\vec{v}_i \cdot \vec{v}_i) + \cdots + c_m(\vec{v}_m \cdot \vec{v}_i) = 0.$$

Since  $\vec{v}_i$  is perpendicular to all the other  $\vec{v}_j$ , we will have  $\vec{v}_i \cdot \vec{v}_j = 0$  whenever  $j \neq i$ ; since  $\vec{v}_i$  is a unit vector, we will have  $\vec{v}_i \cdot \vec{v}_i = 1$ . Therefore, the equation above simplifies to  $c_i = 0$ .

Since this reasoning applies to all  $i = 1, \dots, m$ , we have only the trivial relation among the vectors  $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_m$ , so that these vectors are linearly independent, as claimed.

43. Consider a linear relation  $c_1\vec{v}_1 + c_2(\vec{v}_1 + \vec{v}_2) + c_3(\vec{v}_1 + \vec{v}_2 + \vec{v}_3) = \vec{0}$ , or,  $(c_1 + c_2 + c_3)\vec{v}_1 + (c_2 + c_3)\vec{v}_2 + c_3\vec{v}_3 = \vec{0}$ . Since there is only the trivial relation among the vectors  $\vec{v}_1, \vec{v}_2, \vec{v}_3$ , we must have  $c_1 + c_2 + c_3 = c_2 + c_3 = c_3 = 0$ , so that  $c_3 = 0$  and then  $c_2 = 0$  and then  $c_1 = 0$ , as claimed.
44. Yes; this is a special case of Exercise 40 (recall that  $\ker(A) = \{\vec{0}\}$ , by Fact 3.1.7b).
45. Yes; if  $A$  is invertible, then  $\ker(A) = \{\vec{0}\}$ , so that the columns of  $A$  are linearly independent, by Fact 3.2.8.
46. Solve the system 
$$\left| \begin{array}{cccc|c} x_1 & + & 2x_2 & & 0 \\ & & & + & 3x_4 & + & 5x_5 & = & 0 \\ & & & x_3 & + & 4x_4 & + & 6x_5 & = & 0 \end{array} \right|.$$

The solutions are of the form

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} -2s - 3t - 5r \\ s \\ -4t - 6r \\ t \\ r \end{bmatrix} = s \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} -3 \\ 0 \\ -4 \\ 1 \\ 0 \end{bmatrix} + r \begin{bmatrix} -5 \\ 0 \\ -6 \\ 0 \\ 1 \end{bmatrix}.$$

The vectors  $\begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$ ,  $\begin{bmatrix} -3 \\ 0 \\ -4 \\ 1 \\ 0 \end{bmatrix}$ ,  $\begin{bmatrix} -5 \\ 0 \\ -6 \\ 0 \\ 1 \end{bmatrix}$  span the kernel, by construction, and they are linearly independent, by Fact 3.2.5. Therefore, the three vectors form a basis of the kernel.

47. By Fact 3.2.8, the rank of  $A$  is 3. Thus,  $\text{rref}(A) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ .

48. We can write  $3x_1 + 4x_2 + 5x_3 = [3 \ 4 \ 5] \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$ , so that  $V = \ker[3 \ 4 \ 5]$ .

To express  $V$  as an image, choose a basis of  $V$ , for example,  $\begin{bmatrix} 4 \\ -3 \\ 0 \end{bmatrix}$ ,  $\begin{bmatrix} 0 \\ 5 \\ -4 \end{bmatrix}$ .

Then,  $V = \text{im} \begin{bmatrix} 4 & 0 \\ -3 & 5 \\ 0 & -4 \end{bmatrix}$ .

There are other solutions.

49.  $L = \text{im} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

To write  $L$  as a kernel, think of  $L$  as the intersection of the planes  $x = y$  and  $y = z$ , that is, as the solution set of the system  $\begin{vmatrix} x - y & = 0 \\ y - z & = 0 \end{vmatrix}$ .

Therefore,  $L = \ker \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \end{bmatrix}$ .

There are other solutions.

50. The verification of the three properties listed in Definition 3.2.1 is straightforward. Alternatively, we can choose a basis  $\vec{v}_1, \dots, \vec{v}_p$  of  $V$  and a basis  $\vec{w}_1, \dots, \vec{w}_q$  of  $W$  (see Exercise 38a) and show that  $V + W = \text{span}(\vec{v}_1, \dots, \vec{v}_p, \vec{w}_1, \dots, \vec{w}_q)$  (compare with Exercise 4).

Indeed, if  $\vec{v} + \vec{w}$  is in  $V + W$ , then  $\vec{v}$  is a linear combination of  $\vec{v}_1, \dots, \vec{v}_p$  and  $\vec{w}$  is a linear combination of  $\vec{w}_1, \dots, \vec{w}_q$ , so that  $\vec{v} + \vec{w}$  is a linear combination of  $\vec{v}_1, \dots, \vec{v}_p, \vec{w}_1, \dots, \vec{w}_q$ . Conversely, if  $\vec{x}$  is in  $\text{span}(\vec{v}_1, \dots, \vec{v}_p, \vec{w}_1, \dots, \vec{w}_q)$ , then  $\vec{x} = (c_1\vec{v}_1 + \dots + c_p\vec{v}_p) + (d_1\vec{w}_1 + \dots + d_q\vec{w}_q)$ , so that  $\vec{x}$  is in  $V + W$ .

If  $V$  and  $W$  are distinct lines in  $\mathbb{R}^3$  (spanned by  $\vec{v}$  and  $\vec{w}$ , respectively), then  $V + W$  is the plane spanned by  $\vec{v}$  and  $\vec{w}$ .

51. a. Consider a relation  $c_1\vec{v}_1 + \dots + c_p\vec{v}_p + d_1\vec{w}_1 + \dots + d_q\vec{w}_q = \vec{0}$ .

Then the vector  $c_1\vec{v}_1 + \dots + c_p\vec{v}_p = -d_1\vec{w}_1 - \dots - d_q\vec{w}_q$  is both in  $V$  and in  $W$ , so that this vector is  $\vec{0}$ :  $c_1\vec{v}_1 + \dots + c_p\vec{v}_p = \vec{0}$  and  $d_1\vec{w}_1 + \dots + d_q\vec{w}_q = \vec{0}$ .

Now the  $c_i$  are all zero (since the  $\vec{v}_i$  are linearly independent) and the  $d_j$  are zero (since the  $\vec{w}_j$  are linearly independent).

Since there is only the trivial relation among the vectors  $\vec{v}_1, \dots, \vec{v}_p, \vec{w}_1, \dots, \vec{w}_q$ , they are linearly independent.

- b. In Exercise 50 we show that  $V + W = \text{span}(\vec{v}_1, \dots, \vec{v}_p, \vec{w}_1, \dots, \vec{w}_q)$ , and in part (a) we show that these vectors are linearly independent.

52. If  $a$ ,  $c$  and  $f$  are nonzero, then  $\text{rref} \begin{bmatrix} a & b & d \\ 0 & c & e \\ 0 & 0 & f \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ , and the three vectors are

linearly independent, by Fact 3.2.6. If at least one of the constants  $a$ ,  $c$  or  $f$  is zero, then at least one column of  $\text{rref}$  will not contain a leading one, so that the three vectors are linearly dependent.

53. The zero vector is in  $V^\perp$ , since  $\vec{0} \cdot \vec{v} = 0$  for all  $\vec{v}$  in  $V$ .

If  $\vec{w}_1$  and  $\vec{w}_2$  are both in  $V^\perp$ , then  $(\vec{w}_1 + \vec{w}_2) \cdot \vec{v} = \vec{w}_1 \cdot \vec{v} + \vec{w}_2 \cdot \vec{v} = 0 + 0 = 0$  for all  $\vec{v}$  in  $V$ , so that  $\vec{w}_1 + \vec{w}_2$  is in  $V^\perp$  as well.

If  $\vec{w}$  is in  $V^\perp$  and  $k$  is an arbitrary constant, then  $(k\vec{w}) \cdot \vec{v} = k(\vec{w} \cdot \vec{v}) = k \cdot 0 = 0$  for all  $\vec{v}$  in  $V$ , so that  $k\vec{w}$  is in  $V^\perp$  as well.

54. We need to find all vectors  $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$  in  $\mathbb{R}^3$  such that  $\begin{bmatrix} x \\ y \\ z \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = x + 2y + 3z = 0$ .

These vectors have the form  $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -2s - 3t \\ s \\ t \end{bmatrix} = s \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}$ .

Therefore,  $\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}$  is a basis of  $L^\perp$ .

55. We need to find all vectors  $\vec{x}$  in  $\mathbb{R}^5$  such that  $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{bmatrix} = x_1 + 2x_2 + 3x_3 + 4x_4 + 5x_5 = 0$ .

These vectors are of the form

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} -2a - 3b - 4c - 5d \\ a \\ b \\ c \\ d \end{bmatrix} = a \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + b \begin{bmatrix} -3 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + c \begin{bmatrix} -4 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} + d \begin{bmatrix} -5 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}.$$

The four vectors to the right form a basis of  $L^\perp$ ; they span  $L^\perp$ , by construction, and they are linearly independent, by Fact 3.2.5.

56. Consider a linear relation  $c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3 + c_4\vec{v}_4 = \vec{0}$  among the four given vectors. The last component of the vector on the left hand side is  $c_3$ , so that  $c_3 = 0$ . Now the fifth component on the left is  $c_1$ , so that  $c_1 = 0$ . The third component is now  $c_4$ , so  $c_4 = 0$ . It follows that  $c_2 = 0$  as well.

We have shown that there is only the trivial relation among the given vectors, so that they are linearly independent, regardless of the values of the constants  $a, b, \dots, m$ .

57. We will begin to go through the possibilities for  $j$  until we see a pattern:

$j = 1$ : Yes, because  $\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$  is in  $\ker(A)$  (the first column is  $\vec{0}$ ).

$j = 2$ : No, this would just be a multiple of the second column, and only  $\vec{0}$  if the  $j$ th component is zero.

$j = 3$ : Yes, since  $\begin{bmatrix} 0 \\ 2 \\ -1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$  is in  $\ker(A)$ .

At this point, we realize that we are choosing the redundant columns. Thus,  $j$  can also

be 6 and 7, because  $\begin{bmatrix} 0 \\ 3 \\ 0 \\ 4 \\ 5 \\ -1 \\ 0 \end{bmatrix}$ , and  $\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$  are in  $\ker(A)$ .

58. This occurs for each column,  $j$ , that is redundant. If  $\vec{x}$  is in the kernel, and the  $j$ th component of  $\vec{x}$  is the last non-zero component, then

$$x_1\vec{v}_1 + \cdots + x_j\vec{v}_j + x_{j+1}\vec{v}_{j+1} + \cdots + x_m\vec{v}_m = \vec{0}, \text{ but } x_{j+1} = \cdots = x_m = 0, \text{ so}$$

$$x_1\vec{v}_1 + \cdots + x_j\vec{v}_j = \vec{0}.$$

Thus, since  $x_j \neq 0$ ,  $\vec{v}_j = -\frac{x_1\vec{v}_1 + \cdots + x_{j-1}\vec{v}_{j-1}}{x_j}$  and  $\vec{v}_j$  is redundant. Conversely, if  $\vec{v}_j$  is

redundant, with  $\vec{v}_j = c_1\vec{v}_1 + \cdots + c_{j-1}\vec{v}_{j-1}$ , then the vector  $\vec{x} = \begin{bmatrix} c_1 \\ \vdots \\ c_{j-1} \\ -1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$  is in the kernel

of  $A$ . The last non-zero component of  $\vec{x}$  is the  $j$ th, as required.

### 3.3

1. Clearly the second column is just three times the first, and thus is redundant. Applying the notion of Kyle Numbers, we see:

$\begin{bmatrix} 3 & -1 \\ 1 & 3 \\ 2 & 6 \end{bmatrix}$ , so  $\begin{bmatrix} 3 \\ -1 \end{bmatrix}$  is in the  $\ker(A)$ . No other vectors belong in our list, so a basis of the kernel is  $\left(\begin{bmatrix} 3 \\ -1 \end{bmatrix}\right)$ , and a basis of the image is  $\left(\begin{bmatrix} 1 \\ 2 \end{bmatrix}\right)$ .

2. Using Kyle Numbers, we see that the second column is redundant:

$\begin{bmatrix} 4 & -1 \\ 1 & 4 \\ 2 & 8 \end{bmatrix}$ , so  $\begin{bmatrix} 4 \\ -1 \end{bmatrix}$  is in the  $\ker(A)$ . No other vectors belong in our list, so a basis of the kernel is  $\left(\begin{bmatrix} 4 \\ -1 \end{bmatrix}\right)$ , and a basis of the image is  $\left(\begin{bmatrix} 1 \\ 2 \end{bmatrix}\right)$ .

3. The two columns here are independent, so there are no redundant vectors. Thus,  $\emptyset$  is a basis of the kernel, and the two columns form a basis of the image:  $\left(\begin{bmatrix} 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 4 \end{bmatrix}\right)$ .

4. The first column is redundant. We use the following Kyle Numbers:

$\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 2 \end{bmatrix}$ , so  $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$  is in the  $\ker(A)$ . No other vectors belong in our list, so a basis of the kernel is  $\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right)$ , and a basis of the image is  $\left(\begin{bmatrix} 1 \\ 2 \end{bmatrix}\right)$ .

5. The first two vectors are non-redundant, but the third is a multiple of the first. We see:

$\begin{bmatrix} 3 & 0 & -1 \\ 1 & -2 & 3 \\ 2 & 4 & 6 \end{bmatrix}$ , so a basis of the kernel is  $\left(\begin{bmatrix} 3 \\ 0 \\ -1 \end{bmatrix}\right)$ , and a basis of the image consists of the non-redundant columns, or  $\left(\begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} -2 \\ 4 \end{bmatrix}\right)$ .

6. The first two vectors are non-redundant, but the third is a combination of the first two:

$\begin{bmatrix} 1 & 2 & -1 \\ 1 & 1 & 3 \\ 2 & 1 & 4 \end{bmatrix}$ , so a basis of the kernel is  $\left(\begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}\right)$ , and a basis of the image is  $\left(\begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix}\right)$ .

7. We immediately see fitting Kyle numbers for one relation:

$\begin{bmatrix} 2 & -1 & 0 \\ 1 & 2 & 3 \\ 1 & 2 & 4 \end{bmatrix}$ . Now, since the second column is redundant, we remove it from further inspection and keep a zero above it:

$$\begin{bmatrix} 1 & \begin{matrix} 0 \\ 2 \\ 2 \end{matrix} & 3 \\ 1 & \begin{matrix} 2 \\ 2 \\ 2 \end{matrix} & 4 \end{bmatrix}$$
. However, in this case, there are no more redundant vectors. Thus, a basis of the kernel is  $\left(\begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}\right)$ , and a basis of the image is  $\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 4 \end{bmatrix}\right)$ .

8. Here the second column is redundant, with Kyle Numbers as:

$$\begin{bmatrix} 3 & 1 \\ 1 & -3 \\ 2 & -6 \\ 3 & -9 \end{bmatrix}$$
. This reveals a basis of our kernel as  $\left(\begin{bmatrix} 3 \\ 1 \end{bmatrix}\right)$  and a basis of the image to be  $\left(\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}\right)$ .

9. The second column is redundant, and we can choose Kyle numbers as follows:

$$\begin{bmatrix} 2 & -1 & 0 \\ 1 & 2 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{bmatrix}$$
, but the third column is non-redundant. Thus, a basis of the kernel is  $\left(\begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}\right)$ , while a basis of the image is  $\left(\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}\right)$ .

10. The first column is redundant, and  $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$  is in the kernel:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 1 & 3 \end{bmatrix}$$
. No other columns are redundant, however, meaning that a basis of the kernel is  $\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right)$ , while a basis of the image is  $\left(\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}\right)$ .

11. Here it is clear that only the third column is redundant, since it is equal to the first.

Thus, a basis of the kernel is  $\left(\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}\right)$ , and  $\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}\right)$  is a basis of the image.

12. The third vector is the only redundant vector here, shown by:

$$\begin{bmatrix} 0 & 1 & -1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix}$$
. No other columns are redundant, however, meaning that a basis of the kernel is  $\left( \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} \right)$ , while a basis of the image is  $\left( \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right)$ .

13. Here we first see  $\begin{bmatrix} 2 & -1 & 0 \\ 1 & 2 & 3 \end{bmatrix}$ , then  $\begin{bmatrix} 3 & 0 & -1 \\ 1 & \cancel{2} & 3 \end{bmatrix}$ ,

so both the second and third columns are redundant, and a basis of the kernel is  $\left( \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \\ -1 \end{bmatrix} \right)$ .

This leaves  $([1])$  to be a basis of the image.

14. The first and the third columns are redundant, as the Kyle Numbers show us:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \end{bmatrix}$$
, then  $\begin{bmatrix} 0 & 2 & -1 \\ \cancel{0} & 1 & 2 \end{bmatrix}$ , so that a basis of the kernel is  $\left( \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ -1 \end{bmatrix} \right)$ . This

leaves  $([1])$  to be a basis of the image.

15. We quickly find that the third column is redundant, with the Kyle numbers

$$\begin{bmatrix} 2 & 2 & -1 & 0 \\ 1 & 0 & 2 & 0 \\ 0 & 1 & 2 & 0 \\ 1 & 0 & 2 & 0 \\ 0 & 1 & 2 & 0 \end{bmatrix}$$
, then see that the fourth column is also redundant,

$$\begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & \cancel{2} & 0 \\ 0 & 1 & \cancel{2} & 0 \\ 1 & 0 & \cancel{2} & 0 \\ 0 & 1 & \cancel{2} & 0 \end{bmatrix}$$
.

Thus, a basis of our kernel is  $\left( \begin{bmatrix} 2 \\ 2 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right)$ , while  $\left( \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix} \right)$  is a basis of our image.

16. The third column is redundant, as we find with



$$\begin{bmatrix} 3 & 2 & -1 & 0 \\ 1 & 1 & 5 & 1 \\ 0 & 1 & 2 & 2 \\ 0 & 1 & 2 & 3 \\ 0 & 1 & 2 & 4 \end{bmatrix}$$
. The fourth column, however, fails to be redundant.

Thus, a basis of our kernel is  $\left( \begin{bmatrix} 3 \\ 2 \\ -1 \\ 0 \end{bmatrix} \right)$ , while  $\left( \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} \right)$  is a basis of our image.

17. For this problem, we again successively use Kyle Numbers to find our kernel, investigating the columns from left to right. We initially see that the first column is redundant:

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & 0 & 3 \\ 0 & 0 & 0 & 1 & 4 \end{bmatrix}$$
, then the third column: 
$$\begin{bmatrix} 0 & 2 & -1 & 0 & 0 \\ 0 & 1 & 2 & 0 & 3 \\ 0 & 0 & 0 & 1 & 4 \end{bmatrix}$$
, followed by the fifth column: 
$$\begin{bmatrix} 0 & 3 & 0 & 4 & -1 \\ 0 & 1 & 0 & 0 & 3 \\ 0 & 0 & 0 & 1 & 4 \end{bmatrix}$$
.

Thus,  $\left( \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ -1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 3 \\ 0 \\ 4 \\ -1 \end{bmatrix} \right)$  is a basis of the kernel, and  $\left( \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right)$  is a basis of the image.

18. This matrix is already in rref, and we see that there are two columns without leading ones. These will be our redundant columns. Thus we see

$$\begin{bmatrix} 2 & 1 & 0 & 0 & 0 \\ 1 & -2 & 0 & -1 & 0 \\ 0 & 0 & 1 & 5 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$
, and 
$$\begin{bmatrix} -1 & 0 & 5 & -1 & 0 \\ 1 & -2 & 0 & -1 & 0 \\ 0 & 0 & 1 & 5 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$
.

Then  $\left( \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 5 \\ -1 \\ 0 \end{bmatrix} \right)$  is a basis of the kernel, and  $\left( \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right)$  is a basis of the image.

19. We see that the third column is redundant, and choose Kyle numbers as follows:

$$\begin{bmatrix} 5 & 4 & -1 & 0 & 0 \\ 1 & 0 & 5 & 3 & 0 \\ 0 & 1 & 4 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \text{ then we see that the fourth column is also redundant, } \begin{bmatrix} 3 & 2 & 0 & -1 & 0 \\ 1 & 0 & 5 & 3 & 0 \\ 0 & 1 & 4 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Thus,  $\left( \begin{bmatrix} 5 \\ 4 \\ -1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \\ 0 \\ -1 \\ 0 \end{bmatrix} \right)$  is a basis of the kernel, and  $\left( \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \right)$  is a basis of the image.

20. Although this matrix is not quite in rref, we can still quickly see that columns 2, 3, and 5 are the redundant columns:

$$\begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 5 & 3 & -3 \\ 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 5 & 0 & -1 & 0 & 0 \\ 1 & 0 & 5 & 3 & -3 \\ 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} -12 & 0 & 0 & 3 & -1 \\ 1 & 0 & 5 & 3 & -3 \\ 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

So,  $\left( \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 5 \\ 0 \\ -1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -12 \\ 0 \\ 0 \\ 3 \\ -1 \end{bmatrix} \right)$  is a basis of the kernel, and  $\left( \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \right)$  is a basis of the image.

21.  $\text{rref} \begin{bmatrix} 1 & 3 & 9 \\ 4 & 5 & 8 \\ 7 & 6 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & -3 \\ 0 & 1 & 4 \\ 0 & 0 & 0 \end{bmatrix}$ , which we can use to “spot” a vector in the kernel:

$\begin{bmatrix} -3 \\ 4 \\ -1 \end{bmatrix}$ . Since the third column is the only redundant one, this forms a basis of the kernel, and implies that the third column of  $A$  is also redundant. Thus, a basis of  $\text{im}(A)$  is  $\left( \begin{bmatrix} 1 \\ 4 \\ 7 \end{bmatrix}, \begin{bmatrix} 3 \\ 5 \\ 6 \end{bmatrix} \right)$ .

22.  $\text{rref} \begin{bmatrix} 2 & 4 & 8 \\ 4 & 5 & 1 \\ 7 & 9 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & -6 \\ 0 & 1 & 5 \\ 0 & 0 & 0 \end{bmatrix}$ . It is clear that the third vector is redundant, and we

quickly see that the vector  $\begin{bmatrix} -6 \\ 5 \\ -1 \end{bmatrix}$  is in the kernel. Since this is the only redundant

column,  $\left(\begin{bmatrix} -6 \\ 5 \\ -1 \end{bmatrix}\right)$  is a basis of the kernel. Thus, a basis of  $\text{im}(A)$  is  $\left(\begin{bmatrix} 2 \\ 4 \\ 7 \end{bmatrix}, \begin{bmatrix} 4 \\ 5 \\ 9 \end{bmatrix}\right)$ .

$$23. \text{ rref} \begin{bmatrix} 1 & 0 & 2 & 4 \\ 0 & 1 & -3 & -1 \\ 3 & 4 & -6 & 8 \\ 0 & -1 & 3 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 2 & 4 \\ 0 & 1 & -3 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Using the method of Exercises 17 and 19, we find the kernel:

$$\begin{bmatrix} 2 & -3 & -1 & 0 \\ 1 & 0 & 2 & 4 \\ 0 & 1 & -3 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \text{ then } \begin{bmatrix} 4 & -1 & 0 & -1 \\ 1 & 0 & 2 & 4 \\ 0 & 1 & -3 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

So a basis of  $\ker(A)$  is  $\left(\begin{bmatrix} 2 \\ -3 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 4 \\ -1 \\ 0 \\ -1 \end{bmatrix}\right)$ . The non-redundant column vectors of  $A$  form

a basis of  $\text{im}(A)$ :  $\left(\begin{bmatrix} 1 \\ 0 \\ 3 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 4 \\ -1 \end{bmatrix}\right)$ .

$$24. \text{ rref} \begin{bmatrix} 4 & 8 & 1 & 1 & 6 \\ 3 & 6 & 1 & 2 & 5 \\ 2 & 4 & 1 & 9 & 10 \\ 1 & 2 & 3 & 2 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}. \text{ Here our kernel is the span of only one}$$

vector:

$$\left(\begin{bmatrix} 2 \\ -1 \\ 0 \\ 0 \\ 0 \end{bmatrix}\right), \text{ while a basis of the image of } A \text{ is } \left(\begin{bmatrix} 4 \\ 3 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 9 \\ 2 \end{bmatrix}, \begin{bmatrix} 6 \\ 5 \\ 10 \\ 0 \end{bmatrix}\right).$$

$$25. \text{ rref} \begin{bmatrix} 1 & 2 & 3 & 2 & 1 \\ 3 & 6 & 9 & 6 & 3 \\ 1 & 2 & 4 & 1 & 2 \\ 2 & 4 & 9 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 0 & 5 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}. \text{ We will emulate Exercise 23 to find the}$$

$$\text{kernel: } \begin{bmatrix} 2 & -1 & 0 & 0 & 0 \\ 1 & 2 & 0 & 5 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \text{ then } \begin{bmatrix} 5 & 0 & -1 & -1 & 0 \\ 1 & 2 & 0 & 5 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

So a basis of  $\ker(A)$  is  $\left( \begin{bmatrix} 2 \\ -1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 5 \\ 0 \\ -1 \\ -1 \\ 0 \end{bmatrix} \right)$  and a basis of  $\text{im}(A)$  is  $\left( \begin{bmatrix} 1 \\ 3 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ 9 \\ 4 \\ 9 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \\ 2 \\ 2 \end{bmatrix} \right)$ .

26. a. We notice that each of the six matrices has two identical columns. In matrices  $C$  and  $L$ , the second column is identical to the third, so that  $\ker(C) = \ker(L) = \text{span} \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$ . In matrices  $H, T, X$  and  $Y$ , the first column is identical to the third, so that  $\ker(H) = \ker(T) = \ker(X) = \ker(Y) = \text{span} \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$ . Thus, only  $L$  has the same kernel as  $C$ .
- b. We observe that each of the six matrices in the list has two identical rows. For example, the first and the last row of matrix  $C$  are identical, so that any vector  $\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$  in  $\text{im}(C)$  will satisfy the equation  $y_1 = y_3$ . We can conclude that  $\text{im}(C) = \text{im}(H) = \text{im}(X) = \left\{ \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} : y_1 = y_3 \right\}$ ,  $\text{im}(L) = \left\{ \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} : y_1 = y_2 \right\}$ , and  $\text{im}(T) = \text{im}(Y) = \left\{ \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} : y_2 = y_3 \right\}$ .
- c. Our discussion in part *b* shows that the answer is matrix  $L$ .

27. Form a  $4 \times 4$  matrix  $A$  with the given vectors as its columns. We find that  $\text{rref}(A) = I_4$ , so that the vectors do indeed form a basis of  $\mathbb{R}^4$ , by Summary 3.3.9.
28. Form a  $4 \times 4$  matrix  $A$  with the given vectors as its columns. The matrix  $A$  reduces to

$$\begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & k - 29 \end{bmatrix}.$$

This matrix can be reduced further to  $I_4$  if (and only if)  $k - 29 \neq 0$ , that is, if  $k \neq 29$ .

By Summary 3.3.9, the four given vectors form a basis of  $\mathbb{R}^4$  unless  $k = 29$ .

29.  $x_1 = -\frac{3}{2}x_2 - \frac{1}{2}x_3$ ; let  $x_2 = s$  and  $x_3 = t$ . Then the solutions are of the form

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -\frac{3}{2}s - \frac{1}{2}t \\ s \\ t \end{bmatrix} = s \begin{bmatrix} -\frac{3}{2} \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} -\frac{1}{2} \\ 0 \\ 1 \end{bmatrix}.$$

Multiplying the two vectors by 2 to simplify, we obtain the basis  $\begin{bmatrix} -3 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix}$ .

30. Proceeding as in Exercise 29, we find the basis  $\begin{bmatrix} 1 \\ 2 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ 0 \\ 1 \end{bmatrix}$ .

31. Proceeding as in Exercise 29, we can find the following basis of  $V$ :  $\begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -4 \\ 0 \\ 0 \\ 1 \end{bmatrix}$ .

Now let  $A$  be the  $4 \times 3$  matrix with these three vectors as its columns. Then  $\text{im}(A) = V$  by Fact 3.1.3, and  $\ker(A) = \{\vec{0}\}$  by Fact 3.2.8, so that  $A$  does the job.

$$A = \begin{bmatrix} 1 & -2 & -4 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

32. We need to find all vectors  $\vec{x}$  in  $\mathbb{R}^4$  such that  $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \end{bmatrix} = 0$  and  $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 1 \\ 2 \\ 3 \end{bmatrix} = 0$ .

This amounts to solving the system  $\left| \begin{array}{cccc|c} x_1 & & -x_3 & +x_4 & = & 0 \\ & x_2 & +2x_3 & +3x_4 & = & 0 \end{array} \right|$ , which in turn amounts to finding the kernel of  $\begin{bmatrix} 1 & 0 & -1 & 1 \\ 0 & 1 & 2 & 3 \end{bmatrix}$ .

Using Kyle Numbers, we find the basis  $\begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ -3 \\ 0 \\ 1 \end{bmatrix}$ .

33. We can write  $V = \ker(A)$ , where  $A$  is the  $1 \times n$  matrix  $A = [c_1 \ c_2 \ \cdots \ c_n]$ .

Since at least one of the  $c_i$  is nonzero, the rank of  $A$  is 1, so that  $\dim(V) = \dim(\ker(A)) = n - \text{rank}(A) = n - 1$ , by Fact 3.3.7.

A “hyperplane” in  $\mathbb{R}^2$  is a line, and a “hyperplane” in  $\mathbb{R}^3$  is just a plane.

34. We can write  $V = \ker(A)$ , where  $A$  is the  $n \times m$  matrix with entries  $a_{ij}$ . Note that  $\text{rank}(A) \leq n$ . Therefore,  $\dim(V) = \dim(\ker(A)) = m - \text{rank}(A) \geq m - n$ , by Fact 3.3.7.
35. We need to find all vectors  $\vec{x}$  in  $\mathbb{R}^n$  such that  $\vec{v} \cdot \vec{x} = 0$ , or  $v_1x_1 + v_2x_2 + \cdots + v_nx_n = 0$ , where the  $v_i$  are the components of the vector  $\vec{v}$ . These vectors form a hyperplane in  $\mathbb{R}^n$  (see Exercise 33), so that the dimension of the space is  $n - 1$ .
36. No; if  $\text{im}(A) = \ker(A)$  for an  $n \times n$  matrix  $A$ , then  $n = \dim(\ker(A)) + \dim(\text{im}(A)) = 2 \dim(\text{im}(A))$ , so that  $n$  is an even number.
37. Since  $\dim(\ker(A)) = 5 - \text{rank}(A)$ , any  $4 \times 5$  matrix with rank 2 will do; for example,

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

38. a. The rank of a  $3 \times 5$  matrix  $A$  is 0,1,2, or 3, so that  $\dim(\ker(A)) = 5 - \text{rank}(A)$  is 2,3,4, or 5.
- b. The rank of a  $7 \times 4$  matrix  $A$  is at most 4, so that  $\dim(\text{im}(A)) = \text{rank}(A)$  is 0,1,2,3, or 4.
39. Note that  $\ker(C) \neq \{\vec{0}\}$ , by Fact 3.1.7a, and  $\ker(C) \subseteq \ker(A)$ . Therefore,  $\ker(A) \neq \{\vec{0}\}$ , so that  $A$  is not invertible.
40. We can choose a basis  $\vec{v}_1, \dots, \vec{v}_p$  in  $V$ , where  $p = \dim(V)$ . Then  $\vec{v}_1, \dots, \vec{v}_p$  are linearly independent vectors in  $W$ , so that  $\dim(V) = p \leq \dim(W)$ , by Fact 3.3.4a, as claimed.
41. We can choose a basis  $\vec{v}_1, \dots, \vec{v}_p$  of  $V$ , where  $p = \dim(V) = \dim(W)$ . Then  $\vec{v}_1, \dots, \vec{v}_p$  is a basis of  $W$  as well, by Fact 3.3.4c, so that  $V = W = \text{span}(\vec{v}_1, \dots, \vec{v}_p)$ , as claimed.
42. Consider a basis  $\vec{v}_1, \dots, \vec{v}_n$  of  $V$ . Since the  $\vec{v}_i$  are  $n$  linearly independent vectors in  $\mathbb{R}^n$ , they form a basis of  $\mathbb{R}^n$  (by parts (vii) and (ix) of Summary 3.3.9), so that  $V = \text{span}(\vec{v}_1, \dots, \vec{v}_n) = \mathbb{R}^n$ , as claimed. (Note that Exercise 42 is a special case of Exercise 41.)
43.  $\dim(V + W) = \dim(V) + \dim(W)$ , by Exercise 3.2.51b.
44. Suppose that  $V \cap W = \{\vec{0}\}$  and  $\dim(V) + \dim(W) = n$ .

Choose a basis  $\vec{v}_1, \dots, \vec{v}_p$  of  $V$  and a basis  $\vec{w}_1, \dots, \vec{w}_q$  in  $W$ ; note that  $p + q = n$ . By Exercise 3.2.51b, the  $n$  vectors  $\vec{v}_1, \dots, \vec{v}_p, \vec{w}_1, \dots, \vec{w}_q$  in  $\mathbb{R}^n$  are linearly independent, so that they form a basis of  $\mathbb{R}^n$  (by parts (vii) and (ix) of Summary 3.3.9). By Fact 3.2.10, any vector  $\vec{x}$  can be written uniquely as

$$\vec{x} = (c_1\vec{v}_1 + \dots + c_p\vec{v}_p) + (d_1\vec{w}_1 + \dots + d_q\vec{w}_q), \text{ with } \vec{v} = c_1\vec{v}_1 + \dots + c_p\vec{v}_p \text{ in } V \text{ and } \vec{w} = d_1\vec{w}_1 + \dots + d_q\vec{w}_q \text{ in } W, \text{ which gives the desired representation.}$$

Conversely, suppose  $V$  and  $W$  are complements. Let us first show that  $V \cap W = \{\vec{0}\}$  in this case.

Indeed, if  $\vec{x}$  is in  $V \cap W$ , then we can write  $\vec{x} = \vec{x} + \vec{0} = \vec{0} + \vec{x}$

$$\begin{array}{cccc} \uparrow & \uparrow & \uparrow & \uparrow \\ \text{in} & \text{in} & \text{in} & \text{in} \\ V & W & V & W \end{array}$$

Since this representation is unique (by definition of complements), we must have  $\vec{x} = \vec{0}$ , so that  $V \cap W = \{\vec{0}\}$ . By definition of complements, we have  $\mathbb{R}^n = V + W$ , so that  $n = \dim(V + W) = \dim(V) + \dim(W)$ , by Exercise 43.

45. Note that  $\text{im}(A) = \text{span}(\vec{v}_1, \dots, \vec{v}_p, \vec{w}_1, \dots, \vec{w}_q) = V$ , since the  $\vec{w}_j$  alone span  $V$ .

To find a basis of  $V = \text{im}(A)$ , we omit the redundant vectors from the list  $\vec{v}_1, \dots, \vec{v}_p, \vec{w}_1, \dots, \vec{w}_q$ , by Fact 3.2.4. Since the vectors  $\vec{v}_1, \dots, \vec{v}_p$  are linearly independent, none of them are redundant, so that our basis of  $V$  contains all vectors  $\vec{v}_1, \dots, \vec{v}_p$  and some of the vectors from the list  $\vec{w}_1, \dots, \vec{w}_q$ .

46. Use Exercise 45 with  $\vec{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$ ,  $\vec{v}_2 = \begin{bmatrix} 1 \\ 4 \\ 6 \\ 8 \end{bmatrix}$ , and  $\vec{w}_i = \vec{e}_i$  for  $i = 1, 2, 3, 4$ .

$$\text{Now rref} \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 2 & 4 & 0 & 1 & 0 & 0 \\ 3 & 6 & 0 & 0 & 1 & 0 \\ 4 & 8 & 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 2 & 0 & 0 & -\frac{1}{4} \\ 0 & 1 & -1 & 0 & 0 & \frac{1}{4} \\ 0 & 0 & 0 & 1 & 0 & -\frac{1}{2} \\ 0 & 0 & 0 & 0 & 1 & -\frac{3}{4} \end{bmatrix}.$$

Picking the non-redundant columns gives the basis  $\begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}, \begin{bmatrix} 1 \\ 4 \\ 6 \\ 8 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$ .

47. Using the terminology suggested in the hint, we need to show that  $\vec{u}_1, \dots, \vec{u}_m, \vec{v}_1, \dots, \vec{v}_p, \vec{w}_1, \dots, \vec{w}_q$  is a basis of  $V + W$ . Then  $\dim(V + W) + \dim(V \cap W) = (m + p + q) + m = (m + p) + (m + q) =$

$\dim(V) + \dim(W)$ , as claimed. Any vector  $\vec{x}$  in  $V + W$  can be written as  $\vec{x} = \vec{v} + \vec{w}$ , where  $\vec{v}$  is in  $V$  and  $\vec{w}$  is in  $W$ . Since  $\vec{v}$  is a linear combination of the  $\vec{u}_i$  and the  $\vec{v}_j$ , and  $\vec{w}$  is a linear combination of the  $\vec{u}_i$  and  $\vec{w}_j$ ,  $\vec{x}$  will be a linear combination of the  $\vec{u}_i, \vec{v}_j$ , and  $\vec{w}_k$ ; this shows that the vectors  $\vec{u}_1, \dots, \vec{u}_m, \vec{v}_1, \dots, \vec{v}_p, \vec{w}_1, \dots, \vec{w}_q$  span  $V + W$ .

To show linear independence, consider the relation  $a_1\vec{u}_1 + \dots + a_m\vec{u}_m + b_1\vec{v}_1 + \dots + b_p\vec{v}_p + c_1\vec{w}_1 + \dots + c_q\vec{w}_q = \vec{0}$ . Then the vector  $a_1\vec{u}_1 + \dots + a_m\vec{u}_m + b_1\vec{v}_1 + \dots + b_p\vec{v}_p = -c_1\vec{w}_1 - \dots - c_q\vec{w}_q$  is in  $V \cap W$ , so that it can be expressed uniquely as a linear combination of  $\vec{u}_1, \dots, \vec{u}_m$  alone; this implies that the  $b_i$  are all zero. Now our relation simplifies to  $a_1\vec{u}_1 + \dots + a_m\vec{u}_m + c_1\vec{w}_1 + \dots + c_q\vec{w}_q = \vec{0}$ , which implies that the  $a_i$  and the  $c_j$  are zero as well (since the vectors  $\vec{u}_1, \dots, \vec{u}_m, \vec{w}_1, \dots, \vec{w}_q$  are linearly independent).

48. By Exercise 47,  $\dim(V \cap W) = \dim(V) + \dim(W) - \dim(V + W) = 13 - \dim(V + W)$ .

The dimension of  $V + W$  is at least 7 (since  $W \subseteq V + W$ ) and at most 10 (since  $V + W \subseteq \mathbb{R}^{10}$ ); therefore the dimension of  $V \cap W$  is at least 3 and at most 6.

49. The nonzero rows of  $E$  span the row space, and they are linearly independent (consider the leading ones), so that they form a basis of the row space:  $[0 \ 1 \ 0 \ 2 \ 0]$ ,  $[0 \ 0 \ 1 \ 3 \ 0]$ ,  $[0 \ 0 \ 0 \ 1]$ .

50. As in Exercise 49, we observe that the nonzero rows of  $E$  form a basis of the row space, so that  $\dim(\text{row space of } E) = \text{rank}(E)$ .

51. a. All elementary row operations leave the row space unchanged, so that  $A$  and  $\text{rref}(A)$  have the same row space.

b. By part (a) and Exercise 50,  $\dim(\text{row space of } A) = \dim(\text{row space of } \text{rref}(A)) = \text{rank}(\text{rref}(A)) = \text{rank}(A)$ .

$$52. \text{rref}(A) = \begin{bmatrix} 1 & 0 & -1 & -2 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

By Exercises 50 and 51a,  $[1 \ 0 \ -1 \ -2]$ ,  $[0 \ 1 \ 2 \ 3]$  is a basis of the row space of  $A$ .

53. Using the terminology suggested in the hint, we observe that the vectors  $\vec{v}, A\vec{v}, \dots, A^n\vec{v}$  are linearly dependent (by Fact 3.2.8), so that there is a nontrivial relation  $c_0\vec{v} + c_1A\vec{v} + \dots + c_nA^n\vec{v} = \vec{0}$ .

We can rewrite this relation in the form  $(c_0I_n + c_1A + \dots + c_nA^n)\vec{v} = \vec{0}$ .



The nonzero vector  $\vec{v}$  is in the kernel of the matrix  $c_0I_n + c_1A + \cdots + c_nA^n$ , so that this matrix fails to be invertible.

54. We can use the approach outlined in Exercise 53, with  $\vec{v} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ , say.

$$\text{Then } \vec{v} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, A\vec{v} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \text{ and } A^2\vec{v} = \begin{bmatrix} -3 & -4 \\ 4 & -3 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -3 \\ 4 \end{bmatrix}.$$

We find the relation  $5\vec{v} - 2A\vec{v} + A^2\vec{v} = \vec{0}$ , so that the matrix  $5I_2 - 2A + A^2$  does the job.

55. If  $\text{rank}(A) = n$ , then the  $n$  non-redundant columns of  $A$  form a basis of  $\text{im}(A) = \mathbb{R}^n$ , so that the matrix formed by the non-redundant columns is invertible (by Fact 3.3.9).

Conversely, if  $A$  has an invertible  $n \times n$  submatrix  $B$ , then the columns of  $B$  form a basis of  $\mathbb{R}^n$  (again by Fact 3.3.9), so that  $\text{im}(A) = \mathbb{R}^n$  and therefore  $\text{rank}(A) = \dim(\text{im}(A)) = n$ .

56. Using the terminology suggested in the Exercise, we multiply the relation  $c_0\vec{v} + c_1A\vec{v} + \cdots + c_{m-1}A^{m-1}\vec{v} = \vec{0}$  with  $A^{m-1}$  and obtain  $c_0A^{m-1}\vec{v} = \vec{0}$  (all other terms vanish since  $A^m = 0$ ).

Since the vector  $A^{m-1}\vec{v}$  is nonzero (by construction), the scalar  $c_0$  must be zero, and our relation simplifies to  $c_1A\vec{v} + c_2A^2\vec{v} + \cdots + c_{m-1}A^{m-1}\vec{v} = \vec{0}$ .

Now we multiply both sides with  $A^{m-2}$  and obtain  $c_1A^{m-1}\vec{v} = \vec{0}$ , so that  $c_1 = 0$  as above. Continuing like this we conclude that all the  $c_i$  must be zero, as claimed.

57. As in Exercise 56, let  $m$  be the smallest positive integer such that  $A^m = 0$ . In Exercise 56 we construct  $m$  linearly independent vectors  $\vec{v}, A\vec{v}, \dots, A^{m-1}\vec{v}$  in  $\mathbb{R}^n$ ; now  $m \leq n$  by Fact 3.2.8.

Therefore  $A^n = A^m A^{n-m} = 0 A^{n-m} = 0$ , as claimed.

58. If the vectors  $\vec{w}_1, \dots, \vec{w}_q$  span an  $m$ -dimensional space  $V$  (with basis  $\vec{v}_1, \dots, \vec{v}_m$ ), then  $m \leq q$  by Fact 3.3.1 (since the vectors  $\vec{v}_i$  are linearly independent).

59. Prove Fact 3.3.4d: If  $m$  vectors  $\vec{v}_1, \dots, \vec{v}_m$  span an  $m$ -dimensional space  $V$ , then they form a basis of  $V$ . We need to show that the vectors  $\vec{v}_i$  are linearly independent. We will argue indirectly, assuming that the vectors are linearly dependent; this means that at least one of the vectors  $\vec{v}_i$  is redundant, say  $\vec{v}_p$ .

But then  $V = \text{span}(\vec{v}_1, \dots, \vec{v}_p, \dots, \vec{v}_m) = \text{span}(\vec{v}_1, \dots, \vec{v}_{p-1}, \vec{v}_{p+1}, \dots, \vec{v}_m)$ , contradicting Fact 3.3.4b.

60.  $\text{im}(A)$  is the plane onto which we project, so that  $\text{rank}(A) = \dim(\text{im}(A)) = 2$ .

61. a. Note that  $\text{rank}(B) \leq 2$ , so that  $\dim(\ker(B)) = 5 - \text{rank}(B) \geq 3$  and  $\dim(\ker(AB)) \geq 3$  since  $\ker(B) \subseteq \ker(AB)$ . Since  $\ker(AB)$  is a subspace of  $\mathbb{R}^5$ ,  $\dim(\ker(AB))$  could be

3, 4, or 5. It is easy to give an example for each case; for example, if  $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$  and

$$B = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \text{ then } AB = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \text{ and } \dim(\ker(AB)) = 3.$$

- b. Since  $\dim(\text{im}(AB)) = 5 - \dim(\ker(AB))$ , the possible values of  $\dim(\text{im}(AB))$  are 0, 1, and 2, by part a.

62. Write  $A = [\vec{v}_1 \ \dots \ \vec{v}_m]$  and  $B = [\vec{w}_1 \ \dots \ \vec{w}_m]$ , so that  $A + B = [\vec{v}_1 + \vec{w}_1 \ \dots \ \vec{v}_m + \vec{w}_m]$ . Any linear combination of the columns of  $A + B$ ,  $\vec{y} = c_1(\vec{v}_1 + \vec{w}_1) + \dots + c_m(\vec{v}_m + \vec{w}_m)$ , can be written as

$$\vec{y} = \underbrace{(c_1\vec{v}_1 + \dots + c_m\vec{v}_m)}_{\text{in im}(A)} + \underbrace{(c_1\vec{w}_1 + \dots + c_m\vec{w}_m)}_{\text{in im}(B)}$$

so that  $\text{im}(A + B) \subseteq \text{im}(A) + \text{im}(B)$  (see Exercise 3.2.50). Since  $\dim(V + W) \leq \dim(V) + \dim(W)$ , by Exercise 3.3.47, we can conclude that  $\text{rank}(A + B) = \dim(\text{im}(A + B)) \leq \dim(\text{im}(A)) + \dim(\text{im}(B)) = \text{rank}(A) + \text{rank}(B)$ .

Summary:  $\text{rank}(A + B) \leq \text{rank}(A) + \text{rank}(B)$ .

63. a. By Exercise 3.1.39b,  $\text{im}(AB) \subseteq \text{im}(A)$ , and therefore  $\text{rank}(AB) \leq \text{rank}(A)$ .

- b. Write  $B = [\vec{v}_1 \ \dots \ \vec{v}_m]$  and  $AB = [A\vec{v}_1 \ \dots \ A\vec{v}_m]$ . If  $r = \text{rank}(B)$ , then the  $r$  non-redundant columns of  $B$  will span  $\text{im}(B)$ , and the corresponding  $r$  columns of  $AB$  will span  $\text{im}(AB)$ , by linearity of  $A$ . By Fact 3.3.4b,  $\text{rank}(AB) = \dim(\text{im}(AB)) \leq r = \text{rank}(B)$ .

Summary:  $\text{rank}(AB) \leq \text{rank}(A)$ , and  $\text{rank}(AB) \leq \text{rank}(B)$ .

64. Same answer as Exercise 65.

65. Let  $\vec{v}_1, \dots, \vec{v}_6$  be the columns of matrix  $A$ . Following the hint, we observe that  $\vec{v}_5 = 4\vec{v}_1 + 5\vec{v}_2 + 6\vec{v}_4$ , which gives the relation  $4\vec{v}_1 + 5\vec{v}_2 + 6\vec{v}_4 - \vec{v}_5 = \vec{0}$ . Thus the vector

$$\vec{x} = \begin{bmatrix} 4 \\ 5 \\ 0 \\ 6 \\ -1 \\ 0 \end{bmatrix}$$

is in the kernel of matrix  $A$ . Since  $\vec{x}$  fails to be in the kernel of matrix  $B$ , the two kernels are different, as claimed.

66. We will freely use the terminology introduced in the hint. First we need to show that at least one of the column vectors  $\vec{a}_k$  and  $\vec{b}_k$  fails to contain a leading 1. If  $\text{rank}[\vec{a}_1 \ \cdots \ \vec{a}_{k-1}] = \text{rank}[\vec{b}_1 \ \cdots \ \vec{b}_{k-1}] = r$ , and if  $\vec{a}_k$  contains a leading 1, then  $\vec{a}_k$  is the standard vector  $\vec{e}_{r+1}$ ; likewise for  $\vec{b}_k$ . Since  $\vec{a}_k$  and  $\vec{b}_k$  are different vectors, they cannot both contain a leading 1. Without loss of generality, we can assume that  $\vec{a}_k$  fails to contain a leading 1, so that  $\vec{a}_k$  is redundant: We can write  $\vec{a}_k = c_1\vec{a}_1 + \cdots + c_{k-1}\vec{a}_{k-1}$ . Then the vector

$$\vec{x} = \begin{bmatrix} c_1 \\ \vdots \\ c_{k-1} \\ -1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \text{ is in the kernel of } A. \text{ We will show that } \vec{x} \text{ fails to be in the kernel of matrix}$$

$B$ , so that  $\ker(A) \neq \ker(B)$ , as claimed. Indeed,  $B\vec{x} = c_1\vec{b}_1 + \cdots + c_{k-1}\vec{b}_{k-1} - \vec{b}_k = c_1\vec{a}_1 + \cdots + c_{k-1}\vec{a}_{k-1} - \vec{b}_k = \vec{a}_k - \vec{b}_k \neq \vec{0}$ . We have used the fact that the first  $k-1$  columns of  $B$  are identical to those of  $A$ , while  $\vec{b}_k \neq \vec{a}_k$ .

67. Exercise 66 shows that if two matrices  $A$  and  $B$  of the same size are both in rref and have the same kernel, then  $A = B$ . Apply this fact to  $A$  and  $B = \text{rref}(M)$ .

### 3.4

- $\begin{bmatrix} 2 \\ 3 \end{bmatrix} = 2 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + 3 \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ , so  $[\vec{x}]_{\mathcal{B}} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$ .
- $\begin{bmatrix} 3 \\ -4 \end{bmatrix} = -4 \begin{bmatrix} 0 \\ 1 \end{bmatrix} + 3 \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ , so  $[\vec{x}]_{\mathcal{B}} = \begin{bmatrix} -4 \\ 3 \end{bmatrix}$ .

3.  $\begin{bmatrix} 31 \\ 37 \end{bmatrix} = 0 \begin{bmatrix} 23 \\ 29 \end{bmatrix} + 1 \begin{bmatrix} 31 \\ 37 \end{bmatrix}$ , so  $[\vec{x}]_{\mathcal{B}} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ .
4.  $\begin{bmatrix} 23 \\ 29 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 46 \\ 58 \end{bmatrix} + 0 \begin{bmatrix} 61 \\ 67 \end{bmatrix}$ , so  $[\vec{x}]_{\mathcal{B}} = \begin{bmatrix} \frac{1}{2} \\ 0 \end{bmatrix}$ .
5.  $\begin{bmatrix} 7 \\ 16 \end{bmatrix} = -4 \begin{bmatrix} 2 \\ 5 \end{bmatrix} + 3 \begin{bmatrix} 5 \\ 12 \end{bmatrix}$ , so  $[\vec{x}]_{\mathcal{B}} = \begin{bmatrix} -4 \\ 3 \end{bmatrix}$ .

This may not be as obvious as Exercises 1 and 3, but we can find our coefficients simply by reducing the matrix  $\begin{bmatrix} 2 & 5 & 7 \\ 5 & 12 & 16 \end{bmatrix}$ .

6.  $\begin{bmatrix} -4 \\ 4 \end{bmatrix} = 11 \begin{bmatrix} 1 \\ 2 \end{bmatrix} - 3 \begin{bmatrix} 5 \\ 6 \end{bmatrix}$ , so  $[\vec{x}]_{\mathcal{B}} = \begin{bmatrix} 11 \\ -3 \end{bmatrix}$ .

We arrive at this solution by reducing the matrix  $\begin{bmatrix} 1 & 5 & -4 \\ 2 & 6 & 4 \end{bmatrix}$ .

7. We need to find the scalars  $c_1$  and  $c_2$  such that  $\begin{bmatrix} 3 \\ 1 \\ -4 \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$ . Solving a linear system gives  $c_1 = 3$ ,  $c_2 = 4$ . Thus  $[\vec{x}]_{\mathcal{B}} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$ .

8. We need to find the scalars  $c_1$  and  $c_2$  such that  $\begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$ . Attempting to solve the linear system reveals an inconsistency;  $\vec{x}$  is not in the span of  $\vec{v}_1$  and  $\vec{v}_2$ .

9. We can solve this by inspection: Note that our first coefficient must be 3 because of the first terms of the vectors. Also, the second coefficient must be 2 due to the last terms.

However,  $3\vec{v}_1 + 2\vec{v}_2 = \begin{bmatrix} 3 \\ 3 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ -2 \\ 4 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ 4 \end{bmatrix}$ . Thus, we reason that  $\vec{x}$  is not in the span of  $\vec{v}_1$  and  $\vec{v}_2$ .

We can also see this by attempting to solve  $\begin{bmatrix} 1 & 0 & 3 \\ 1 & -1 & 3 \\ 0 & 2 & 4 \end{bmatrix}$ , which turns out to be inconsistent. Thus,  $\vec{x}$  is not in  $V$ .

10. Proceeding as in Example 1, we find  $[\vec{x}]_{\mathcal{B}} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$ .

11. Proceeding as in Example 1, we find  $[\vec{x}]_{\mathcal{B}} = \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \end{bmatrix}$ .
12. Proceeding as in Example 1, we find  $[\vec{x}]_{\mathcal{B}} = \begin{bmatrix} -3 \\ 5 \end{bmatrix}$ .
13. Here, we quickly see that since  $x_1 = 1 = 1c_1 + 0c_2 + 0c_3$ ,  $c_1$  must equal 1. We find  $c_2 = -1$  similarly, since  $x_2 = 1 = 2(1) + 1c_2 + 0c_3$ . Finally, now that  $x_3 = 3(1) + 2(-1) + 1c_3$ ,  $c_3$  must be zero.

$$\text{So } \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = 1 \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} - 1 \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} + 0 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \text{ and } [\vec{x}]_{\mathcal{B}} = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}.$$

14. We proceed by inspection here, noting that we need  $c_1 = 3$ , then see that  $c_2$  must be 4. Finally,  $c_3$  must be 6.

$$\text{Thus, } \begin{bmatrix} 3 \\ 7 \\ 13 \end{bmatrix} = 3 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + 4 \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} + 6 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \text{ and } [\vec{x}]_{\mathcal{B}} = \begin{bmatrix} 3 \\ 4 \\ 6 \end{bmatrix}.$$

15. This may be a bit too difficult to do by inspection. Instead we reduce  $\begin{bmatrix} 1 & 1 & 1 & 1 \\ 2 & 3 & 4 & 0 \\ 1 & 4 & 8 & 0 \end{bmatrix}$  to

$$\begin{bmatrix} 1 & 0 & 0 & 8 \\ 0 & 1 & 0 & -12 \\ 0 & 0 & 1 & 5 \end{bmatrix},$$

$$\text{revealing that } \vec{x} = 8\vec{v}_1 - 12\vec{v}_2 + 5\vec{v}_3, \text{ and } [\vec{x}]_{\mathcal{B}} = \begin{bmatrix} 8 \\ -12 \\ 5 \end{bmatrix}.$$

16. We reduce  $\begin{bmatrix} 1 & 1 & 1 & 7 \\ 1 & 2 & 3 & 1 \\ 1 & 3 & 6 & 3 \end{bmatrix}$  to  $\begin{bmatrix} 1 & 0 & 0 & 21 \\ 0 & 1 & 0 & -22 \\ 0 & 0 & 1 & 8 \end{bmatrix}$ ,

$$\text{revealing that } \vec{x} = 21\vec{v}_1 - 22\vec{v}_2 + 8\vec{v}_3, \text{ and } [\vec{x}]_{\mathcal{B}} = \begin{bmatrix} 21 \\ -22 \\ 8 \end{bmatrix}.$$

17. By inspection, we see that in order for  $\vec{x}$  to be in  $V$ ,  $\vec{x} = 1\vec{v}_1 + 1\vec{v}_2 - 1\vec{v}_3$  (by paying attention to the first, second, and fourth terms). Now we need to verify that the third terms “work out”. So,  $1(2) + 1(3) - 1(4) = 5 - 4 = 1 = x_3$ .

$$\text{Thus } \vec{x} \text{ is in } V, \text{ and } [\vec{x}]_{\mathcal{B}} = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}.$$

18. Here,  $\vec{x}$  is not in  $V$ , as we find an inconsistency while attempting to solve the system.

19 a.  $S = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$ , and we find the inverse  $S^{-1}$  to be equal to  $\frac{1}{2} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$ .

$$\text{Then } B = S^{-1}AS = \frac{1}{2} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 2 & 0 \\ 0 & -2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

b. Our commutative diagram:

$$\begin{array}{ccc} \vec{x} = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} & \xrightarrow{T} & T(\vec{x}) = A\vec{x} = c_1 A \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 A \begin{bmatrix} 1 \\ -1 \end{bmatrix} \\ & & = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} - c_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} \\ \downarrow & & \downarrow \\ [\vec{x}]_{\mathcal{B}} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} & \xrightarrow{T} & [T(\vec{x})]_{\mathcal{B}} = \begin{bmatrix} c_1 \\ -c_2 \end{bmatrix} \end{array}$$

So,  $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} c_1 \\ -c_2 \end{bmatrix}$ , and we quickly find  $B = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ .

c.  $B = [[T(\vec{v}_1)]_{\mathcal{B}} | [T(\vec{v}_2)]_{\mathcal{B}}] = \left[ \left[ \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right]_{\mathcal{B}} \quad \left[ \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right]_{\mathcal{B}} \right] = \left[ \begin{bmatrix} 1 \\ 1 \end{bmatrix}_{\mathcal{B}} \quad \begin{bmatrix} -1 \\ 1 \end{bmatrix}_{\mathcal{B}} \right] = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ .

20 a.  $S = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$ , and we find the inverse  $S^{-1}$  to be equal to  $\frac{1}{2} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$ .

$$\text{Then } B = S^{-1}AS = \frac{1}{2} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 2 & 2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 4 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}.$$

b. Our commutative diagram:

$$\vec{x} = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} \xrightarrow{T} T(\vec{x}) = A\vec{x} = c_1 A \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 A \begin{bmatrix} 1 \\ -1 \end{bmatrix} \\ = c_1 \begin{bmatrix} 2 \\ 2 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ 0 \end{bmatrix} = 2c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + 0 \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$\begin{array}{ccc} \downarrow & & \downarrow \\ [\vec{x}]_{\mathcal{B}} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} & \xrightarrow{T} & [T(\vec{x})]_{\mathcal{B}} = \begin{bmatrix} 2c_1 \\ 0 \end{bmatrix} \end{array}$$

So,  $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 2c_1 \\ 0 \end{bmatrix}$ , and we quickly find  $B = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}$ .

c.  $B = [[T(\vec{v}_1)]_{\mathcal{B}} [T(\vec{v}_2)]_{\mathcal{B}}] = \left[ \left[ \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right]_{\mathcal{B}} \quad \left[ \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right]_{\mathcal{B}} \right] = \left[ \left[ \begin{bmatrix} 2 \\ 2 \end{bmatrix} \right]_{\mathcal{B}} \quad \left[ \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right]_{\mathcal{B}} \right] = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}$ .

21 a.  $S = \begin{bmatrix} 1 & -2 \\ 3 & 1 \end{bmatrix}$ , and we find the inverse  $S^{-1}$  to be equal to  $\frac{1}{7} \begin{bmatrix} 1 & 2 \\ -3 & 1 \end{bmatrix}$ .

Then  $B = S^{-1}AS = \frac{1}{7} \begin{bmatrix} 1 & 2 \\ -3 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix} \begin{bmatrix} 1 & -2 \\ 3 & 1 \end{bmatrix} = \frac{1}{7} \begin{bmatrix} 7 & 14 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & -2 \\ 3 & 1 \end{bmatrix} = \frac{1}{7} \begin{bmatrix} 49 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 7 & 0 \\ 0 & 0 \end{bmatrix}$ .

b. Our commutative diagram:

$$\vec{x} = c_1 \begin{bmatrix} 1 \\ 3 \end{bmatrix} + c_2 \begin{bmatrix} -2 \\ 1 \end{bmatrix} \xrightarrow{T} T(\vec{x}) = A\vec{x} = c_1 A \begin{bmatrix} 1 \\ 3 \end{bmatrix} + c_2 A \begin{bmatrix} -2 \\ 1 \end{bmatrix} \\ = c_1 \begin{bmatrix} 7 \\ 21 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ 0 \end{bmatrix} = 7c_1 \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

$$\begin{array}{ccc} \downarrow & & \downarrow \\ [\vec{x}]_{\mathcal{B}} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} & \xrightarrow{T} & [T(\vec{x})]_{\mathcal{B}} = \begin{bmatrix} 7c_1 \\ 0 \end{bmatrix} \end{array}$$

So,  $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 7c_1 \\ 0 \end{bmatrix}$ , and we quickly find  $B = \begin{bmatrix} 7 & 0 \\ 0 & 0 \end{bmatrix}$ .

c.  $B = [[T(\vec{v}_1)]_{\mathcal{B}} [T(\vec{v}_2)]_{\mathcal{B}}] = \left[ \left[ \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix} \begin{bmatrix} 1 \\ 3 \end{bmatrix} \right]_{\mathcal{B}} \quad \left[ \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix} \begin{bmatrix} -2 \\ 1 \end{bmatrix} \right]_{\mathcal{B}} \right] \\ = \left[ \left[ \begin{bmatrix} 7 \\ 21 \end{bmatrix} \right]_{\mathcal{B}} \quad \left[ \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right]_{\mathcal{B}} \right] = \begin{bmatrix} 7 & 0 \\ 0 & 0 \end{bmatrix}$ .

22 a.  $S = \begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix}$ , and we find the inverse  $S^{-1}$  to be equal to  $\frac{1}{5} \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix}$ .

$$\begin{aligned} \text{Then } B = S^{-1}AS &= \frac{1}{5} \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} -3 & 4 \\ 4 & 3 \end{bmatrix} \begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix} = \frac{1}{5} \begin{bmatrix} 5 & 10 \\ 10 & -5 \end{bmatrix} \begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix} = \\ &= \frac{1}{5} \begin{bmatrix} 25 & 0 \\ 0 & -25 \end{bmatrix} = \begin{bmatrix} 5 & 0 \\ 0 & -5 \end{bmatrix}. \end{aligned}$$

b. Our commutative diagram:

$$\begin{array}{ccc} \vec{x} = c_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + c_2 \begin{bmatrix} -2 \\ 1 \end{bmatrix} & \xrightarrow{T} & T(\vec{x}) = A\vec{x} = c_1 A \begin{bmatrix} 1 \\ 2 \end{bmatrix} + c_2 A \begin{bmatrix} -2 \\ 1 \end{bmatrix} \\ & & = c_1 \begin{bmatrix} 5 \\ 10 \end{bmatrix} + c_2 \begin{bmatrix} 10 \\ -5 \end{bmatrix} = 5c_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} - 5c_2 \begin{bmatrix} -2 \\ 1 \end{bmatrix} \\ \downarrow & & \downarrow \\ [\vec{x}]_{\mathcal{B}} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} & \xrightarrow{T} & [T(\vec{x})]_{\mathcal{B}} = \begin{bmatrix} 5c_1 \\ -5c_2 \end{bmatrix} \end{array}$$

$$\text{So, } \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 5c_1 \\ -5c_2 \end{bmatrix}, \text{ and we quickly find } B = \begin{bmatrix} 5 & 0 \\ 0 & -5 \end{bmatrix}.$$

$$\begin{aligned} \text{c. } B &= [[T(\vec{v}_1)]_{\mathcal{B}} | [T(\vec{v}_2)]_{\mathcal{B}}] = \left[ \left[ \begin{bmatrix} -3 & 4 \\ 4 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right]_{\mathcal{B}} \quad \left[ \begin{bmatrix} -3 & 4 \\ 4 & 3 \end{bmatrix} \begin{bmatrix} -2 \\ 1 \end{bmatrix} \right]_{\mathcal{B}} \right] \\ &= \left[ \begin{bmatrix} 5 \\ 10 \end{bmatrix}_{\mathcal{B}} \quad \begin{bmatrix} 10 \\ -5 \end{bmatrix}_{\mathcal{B}} \right] = \begin{bmatrix} 5 & 0 \\ 0 & -5 \end{bmatrix}. \end{aligned}$$

23. a.  $S = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$ , and we find the inverse  $S^{-1}$  to be equal to  $\begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}$ .

$$\text{Then } B = S^{-1}AS = \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 5 & -3 \\ 6 & -4 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 4 & -2 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix}.$$

b. Our commutative diagram:

$$\begin{array}{ccc} \vec{x} = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 2 \end{bmatrix} & \xrightarrow{T} & T(\vec{x}) = A\vec{x} = c_1 A \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 A \begin{bmatrix} 1 \\ 2 \end{bmatrix} \\ & & = c_1 \begin{bmatrix} 2 \\ 2 \end{bmatrix} + c_2 \begin{bmatrix} -1 \\ -2 \end{bmatrix} = 2c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} - 1c_2 \begin{bmatrix} 1 \\ 2 \end{bmatrix} \\ \downarrow & & \downarrow \\ [\vec{x}]_{\mathcal{B}} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} & \xrightarrow{T} & [T(\vec{x})]_{\mathcal{B}} = \begin{bmatrix} 2c_1 \\ -c_2 \end{bmatrix} \end{array}$$



So,  $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 2c_1 \\ -c_2 \end{bmatrix}$ , and we quickly find  $B = \begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix}$ .

$$\begin{aligned} \text{c. } B &= [[T(\vec{v}_1)]_{\mathcal{B}} | [T(\vec{v}_2)]_{\mathcal{B}}] = \left[ \left[ \begin{bmatrix} 5 & -3 \\ 6 & -4 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right]_{\mathcal{B}} \quad \left[ \begin{bmatrix} 5 & -3 \\ 6 & -4 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right]_{\mathcal{B}} \right] \\ &= \left[ \left[ \begin{bmatrix} 2 \\ 2 \end{bmatrix} \right]_{\mathcal{B}} \quad \left[ \begin{bmatrix} -1 \\ -2 \end{bmatrix} \right]_{\mathcal{B}} \right] = \begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix}. \end{aligned}$$

24. a.  $S = \begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix}$ , and we find the inverse  $S^{-1}$  to be equal to  $\begin{bmatrix} 3 & -5 \\ -1 & 2 \end{bmatrix}$ .

$$\text{Then } B = S^{-1}AS = \begin{bmatrix} 3 & -5 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 13 & -20 \\ 6 & -9 \end{bmatrix} \begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix} = \begin{bmatrix} 9 & -15 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}.$$

b. Our commutative diagram:

$$\begin{array}{ccc} \vec{x} = c_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 5 \\ 3 \end{bmatrix} & \xrightarrow{T} & T(\vec{x}) = A\vec{x} = c_1 A \begin{bmatrix} 2 \\ 1 \end{bmatrix} + c_2 A \begin{bmatrix} 5 \\ 3 \end{bmatrix} \\ & & = c_1 \begin{bmatrix} 6 \\ 3 \end{bmatrix} + c_2 \begin{bmatrix} 5 \\ 3 \end{bmatrix} = 3c_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} + 1c_2 \begin{bmatrix} 5 \\ 3 \end{bmatrix} \\ \downarrow & & \downarrow \\ [\vec{x}]_{\mathcal{B}} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} & \xrightarrow{T} & [T(\vec{x})]_{\mathcal{B}} = \begin{bmatrix} 3c_1 \\ c_2 \end{bmatrix} \end{array}$$

So,  $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 3c_1 \\ c_2 \end{bmatrix}$ , and we quickly find  $B = \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}$ .

$$\begin{aligned} \text{c. } B &= [[T(\vec{v}_1)]_{\mathcal{B}} | [T(\vec{v}_2)]_{\mathcal{B}}] = \left[ \left[ \begin{bmatrix} 13 & -20 \\ 6 & -9 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} \right]_{\mathcal{B}} \quad \left[ \begin{bmatrix} 13 & -20 \\ 6 & -9 \end{bmatrix} \begin{bmatrix} 5 \\ 3 \end{bmatrix} \right]_{\mathcal{B}} \right] \\ &= \left[ \left[ \begin{bmatrix} 6 \\ 3 \end{bmatrix} \right]_{\mathcal{B}} \quad \left[ \begin{bmatrix} 5 \\ 3 \end{bmatrix} \right]_{\mathcal{B}} \right] = \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}. \end{aligned}$$

25. We will use the commutative diagram method here (though any method suffices).

$$\begin{aligned}
 \vec{x} = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 2 \end{bmatrix} &\xrightarrow{T} T(\vec{x}) = A\vec{x} = c_1 \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} \\
 &= c_1 \begin{bmatrix} 3 \\ 7 \end{bmatrix} + c_2 \begin{bmatrix} 5 \\ 11 \end{bmatrix} \\
 &= c_1 \left( -1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + 4 \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right) + c_2 \left( -1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + 6 \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right) \\
 &= (-c_1 - c_2) \begin{bmatrix} 1 \\ 1 \end{bmatrix} + (4c_1 + 6c_2) \begin{bmatrix} 1 \\ 2 \end{bmatrix} \\
 \downarrow & \qquad \qquad \qquad \downarrow \\
 [\vec{x}]_{\mathcal{B}} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} &\xrightarrow{T} [T(\vec{x})]_{\mathcal{B}} = \begin{bmatrix} -c_1 - c_2 \\ 4c_1 + 6c_2 \end{bmatrix}
 \end{aligned}$$

$$B \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} -c_1 - c_2 \\ 4c_1 + 6c_2 \end{bmatrix}, \text{ so } B = \begin{bmatrix} -1 & -1 \\ 4 & 6 \end{bmatrix}.$$

26. Let's build  $B$  "column-by-column":

$$\begin{aligned}
 B &= [[T(\vec{v}_1)]_{\mathcal{B}} | [T(\vec{v}_2)]_{\mathcal{B}}] \\
 &= \left[ \left[ \begin{bmatrix} 0 & 1 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right]_{\mathcal{B}} \quad \left[ \begin{bmatrix} 0 & 1 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right]_{\mathcal{B}} \right] \\
 &= \left[ \begin{bmatrix} 2 \\ 8 \end{bmatrix}_{\mathcal{B}} \quad \begin{bmatrix} 1 \\ 5 \end{bmatrix}_{\mathcal{B}} \right] = \begin{bmatrix} 6 & 4 \\ -4 & -3 \end{bmatrix}.
 \end{aligned}$$

27. We use a commutative diagram:

$$\begin{aligned}
 \vec{x} = c_1 \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix} + c_3 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} &\xrightarrow{T} T(\vec{x}) = A\vec{x} \\
 &= c_1 A \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix} + c_2 A \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix} + c_3 A \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \\
 &= c_1 \begin{bmatrix} 18 \\ 9 \\ -18 \end{bmatrix} + \vec{0} + \vec{0} = 9c_1 \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix} \\
 \downarrow & \qquad \qquad \qquad \downarrow \\
 [\vec{x}]_{\mathcal{B}} = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} &\xrightarrow{T} [T(\vec{x})]_{\mathcal{B}} = \begin{bmatrix} 9c_1 \\ 0 \\ 0 \end{bmatrix} \\
 B \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 9c_1 \\ 0 \\ 0 \end{bmatrix}, \text{ so } B = \begin{bmatrix} 9 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.
 \end{aligned}$$

28. Let's build  $B$  "column-by-column":

$$\begin{aligned}
 B &= [[T(\vec{v}_1)]_{\mathcal{B}}[T(\vec{v}_2)]_{\mathcal{B}}[T(\vec{v}_3)]_{\mathcal{B}}] \\
 &= \left[ \left[ \begin{bmatrix} 5 & -4 & -2 \\ -4 & 5 & -2 \\ -2 & -2 & 8 \end{bmatrix} \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix} \right]_{\mathcal{B}} \left[ \begin{bmatrix} 5 & -4 & -2 \\ -4 & 5 & -2 \\ -2 & -2 & 8 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \right]_{\mathcal{B}} \left[ \begin{bmatrix} 5 & -4 & -2 \\ -4 & 5 & -2 \\ -2 & -2 & 8 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ -2 \end{bmatrix} \right]_{\mathcal{B}} \right] \\
 &= \left[ \left[ \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \right]_{\mathcal{B}} \left[ \begin{bmatrix} 9 \\ -9 \\ 0 \end{bmatrix} \right]_{\mathcal{B}} \left[ \begin{bmatrix} 0 \\ 9 \\ -18 \end{bmatrix} \right]_{\mathcal{B}} \right] = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 9 \end{bmatrix}.
 \end{aligned}$$

29. Let's build  $B$  "column-by-column":

$$\begin{aligned}
 B &= [[T(\vec{v}_1)]_{\mathcal{B}}[T(\vec{v}_2)]_{\mathcal{B}}[T(\vec{v}_3)]_{\mathcal{B}}] \\
 &= \left[ \left[ \begin{bmatrix} -1 & 1 & 0 \\ 0 & -2 & 2 \\ 3 & -9 & 6 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right]_{\mathcal{B}} \left[ \begin{bmatrix} -1 & 1 & 0 \\ 0 & -2 & 2 \\ 3 & -9 & 6 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \right]_{\mathcal{B}} \left[ \begin{bmatrix} -1 & 1 & 0 \\ 0 & -2 & 2 \\ 3 & -9 & 6 \end{bmatrix} \begin{bmatrix} 1 \\ 3 \\ 6 \end{bmatrix} \right]_{\mathcal{B}} \right] \\
 &= \left[ \left[ \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \right]_{\mathcal{B}} \left[ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \right]_{\mathcal{B}} \left[ \begin{bmatrix} 2 \\ 6 \\ 12 \end{bmatrix} \right]_{\mathcal{B}} \right] = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}.
 \end{aligned}$$

30. Let's build  $B$  "column-by-column":

$$\begin{aligned}
 B &= [[T(\vec{v}_1)]_{\mathcal{B}}[T(\vec{v}_2)]_{\mathcal{B}}[T(\vec{v}_3)]_{\mathcal{B}}] \\
 &= \left[ \left[ \begin{bmatrix} 0 & 2 & -1 \\ 2 & -1 & 0 \\ 4 & -4 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right]_{\mathcal{B}} \left[ \begin{bmatrix} 0 & 2 & -1 \\ 2 & -1 & 0 \\ 4 & -4 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} \right]_{\mathcal{B}} \left[ \begin{bmatrix} 0 & 2 & -1 \\ 2 & -1 & 0 \\ 4 & -4 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix} \right]_{\mathcal{B}} \right] \\
 &= \left[ \left[ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right]_{\mathcal{B}} \left[ \begin{bmatrix} 0 \\ -1 \\ -2 \end{bmatrix} \right]_{\mathcal{B}} \left[ \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \right]_{\mathcal{B}} \right] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}.
 \end{aligned}$$

31. We can use a commutative diagram to see how this works:

$$\begin{array}{ccc}
 \vec{x} = c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3 & \xrightarrow{T} & T(\vec{x}) = \vec{v}_2 \times \vec{x} = c_1(\vec{v}_2 \times \vec{v}_1) + c_2(\vec{v}_2 \times \vec{v}_2) + c_3(\vec{v}_2 \times \vec{v}_3) \\
 & & = c_1(-\vec{v}_3) + c_2(\vec{0}) + c_3(\vec{v}_1) = c_3\vec{v}_1 - c_1\vec{v}_3 \\
 \downarrow & & \downarrow \\
 [\vec{x}]_{\mathcal{B}} = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} & \xrightarrow{T} & [T(\vec{x})]_{\mathcal{B}} = \begin{bmatrix} c_3 \\ 0 \\ -c_1 \end{bmatrix}
 \end{array}$$

$$B \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} c_3 \\ 0 \\ -c_1 \end{bmatrix}, \text{ so } B = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}.$$

32. Here we will build  $B$  column-by-column:

$$\begin{aligned} B &= [[T(\vec{v}_1)]_{\mathcal{B}} \quad [T(\vec{v}_2)]_{\mathcal{B}} \quad [T(\vec{v}_3)]_{\mathcal{B}}] \\ &= [[(\vec{v}_1 \times \vec{v}_3)_{\mathcal{B}} \quad (\vec{v}_2 \times \vec{v}_3)_{\mathcal{B}} \quad (\vec{v}_3 \times \vec{v}_3)_{\mathcal{B}}] = [[-\vec{v}_2]_{\mathcal{B}} \quad [\vec{v}_1]_{\mathcal{B}} \quad \vec{0}], \text{ since all three are perpendicular unit vectors.} \end{aligned}$$

$$\text{So, } B = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

33. Here we will build  $B$  column-by-column:

$$\begin{aligned} B &= [[T(\vec{v}_1)]_{\mathcal{B}} \quad [T(\vec{v}_2)]_{\mathcal{B}} \quad [T(\vec{v}_3)]_{\mathcal{B}}] \\ &= [[(\vec{v}_2 \cdot \vec{v}_1)\vec{v}_2]_{\mathcal{B}} \quad [(\vec{v}_2 \cdot \vec{v}_2)\vec{v}_2]_{\mathcal{B}} \quad [(\vec{v}_2 \cdot \vec{v}_3)\vec{v}_2]_{\mathcal{B}}] = [\vec{0} \quad [1\vec{v}_2]_{\mathcal{B}} \quad \vec{0}], \text{ since all three are perpendicular unit vectors.} \end{aligned}$$

$$\text{So, } B = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

34. Here we will build  $B$  column-by-column:

$$\begin{aligned} B &= [[T(\vec{v}_1)]_{\mathcal{B}} \quad [T(\vec{v}_2)]_{\mathcal{B}} \quad [T(\vec{v}_3)]_{\mathcal{B}}] \\ &= [[\vec{v}_1 - 2(\vec{v}_3 \cdot \vec{v}_1)\vec{v}_3]_{\mathcal{B}} \quad [\vec{v}_2 - 2(\vec{v}_3 \cdot \vec{v}_2)\vec{v}_3]_{\mathcal{B}} \quad [\vec{v}_3 - 2(\vec{v}_3 \cdot \vec{v}_3)\vec{v}_3]_{\mathcal{B}}] = [[\vec{v}_1]_{\mathcal{B}} \quad [\vec{v}_2]_{\mathcal{B}} \quad [-\vec{v}_3]_{\mathcal{B}}]. \end{aligned}$$

$$\text{So, } B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}. \text{ This is the reflection about the plane spanned by } \vec{v}_1 \text{ and } \vec{v}_2.$$

35. Using another commutative diagram:

$$\begin{aligned} \vec{x} = c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3 &\xrightarrow{T} T(\vec{x}) = c_1T(\vec{v}_1) + c_2T(\vec{v}_2) + c_3T(\vec{v}_3) \\ &= c_1(\vec{v}_1 - 2(\vec{v}_1 \cdot \vec{v}_1)\vec{v}_2) + c_2(\vec{v}_2 - 2(\vec{v}_1 \cdot \vec{v}_2)\vec{v}_2) + \\ &\quad c_3(\vec{v}_3 - 2(\vec{v}_1 \cdot \vec{v}_3)\vec{v}_2) \\ &= c_1(\vec{v}_1 - 2\vec{v}_2) + c_2(\vec{v}_2 - \vec{0}) + c_3(\vec{v}_3 - \vec{0}) \\ &= c_1\vec{v}_1 + (-2c_1 + c_2)\vec{v}_2 + c_3\vec{v}_3 \end{aligned}$$

$$\begin{array}{ccc} \downarrow & & \downarrow \\ [\vec{x}]_{\mathcal{B}} = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} & \xrightarrow{T} & [T(\vec{x})]_{\mathcal{B}} = \begin{bmatrix} c_1 \\ -2c_1 + c_2 \\ c_3 \end{bmatrix} \end{array}$$

So  $B = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ . This is a shear along the second term.

36. Here we will build  $B$  column-by-column:

$$B = [[T(\vec{v}_1)]_{\mathcal{B}} \quad [T(\vec{v}_2)]_{\mathcal{B}} \quad [T(\vec{v}_3)]_{\mathcal{B}}]$$

$$= [[\vec{v}_1 \times \vec{v}_1 + (\vec{v}_1 \cdot \vec{v}_1)\vec{v}_1]_{\mathcal{B}} \quad [\vec{v}_1 \times \vec{v}_2 + (\vec{v}_1 \cdot \vec{v}_2)\vec{v}_1]_{\mathcal{B}} \quad [\vec{v}_1 \times \vec{v}_3 + (\vec{v}_1 \cdot \vec{v}_3)\vec{v}_1]_{\mathcal{B}}] = [[\vec{v}_1]_{\mathcal{B}} \quad [\vec{v}_3]_{\mathcal{B}} \quad [-\vec{v}_2]_{\mathcal{B}}].$$

So,  $B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}$ . This is a 90-degree rotation around the line spanned by  $\vec{v}_1$ . The rotation is counterclockwise when looking from the positive  $\vec{v}_1$  direction.

37. We want a basis  $\mathcal{B} = (\vec{v}_1, \vec{v}_2)$  such that  $T(\vec{v}_1) = a\vec{v}_1$  and  $T(\vec{v}_2) = b\vec{v}_2$  for some scalars  $a$  and  $b$ . Then the  $\mathcal{B}$ -matrix of  $T$  will be  $B = [[T(\vec{v}_1)]_{\mathcal{B}} \quad [T(\vec{v}_2)]_{\mathcal{B}}] = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$ , which is a diagonal matrix as required. Note that  $T(\vec{v}) = \vec{v} = 1\vec{v}$  for vectors parallel to the line  $L$  onto which we project, and  $T(\vec{w}) = \vec{0} = 0\vec{w}$  for vectors perpendicular to  $L$ . Thus, we can pick a basis where  $\vec{v}_1$  is parallel to  $L$  and  $\vec{v}_2$  is perpendicular, for example,  $\mathcal{B} = \left( \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \end{bmatrix} \right)$ .

38. We want a basis  $\mathcal{B} = (\vec{v}_1, \vec{v}_2)$  such that  $T(\vec{v}_1) = a\vec{v}_1$  and  $T(\vec{v}_2) = b\vec{v}_2$  for some scalars  $a$  and  $b$ . Then the  $\mathcal{B}$ -matrix of  $T$  will be  $B = [[T(\vec{v}_1)]_{\mathcal{B}} \quad [T(\vec{v}_2)]_{\mathcal{B}}] = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$ , which is a diagonal matrix as required. Note that  $T(\vec{v}) = \vec{v} = 1\vec{v}$  for vectors parallel to the line  $L$  about which we reflect, and  $T(\vec{w}) = -\vec{w} = (-1)\vec{w}$  for vectors perpendicular to  $L$ . Thus, we can pick a basis where  $\vec{v}_1$  is parallel to  $L$  and  $\vec{v}_2$  is perpendicular, for example,  $\mathcal{B} = \left( \begin{bmatrix} 2 \\ 3 \end{bmatrix}, \begin{bmatrix} -3 \\ 2 \end{bmatrix} \right)$ .

39. Using the same approach as in Exercise 37, we want a basis,  $\vec{v}_1, \vec{v}_2, \vec{v}_3$  such that  $T(\vec{v}_1) = a\vec{v}_1, T(\vec{v}_2) = b\vec{v}_2$  and  $T(\vec{v}_3) = c\vec{v}_3$ . First we see that if  $\vec{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ , then  $T(\vec{v}_1) = \vec{v}_1$ . Next we notice that if  $\vec{v}_2$  and  $\vec{v}_3$  are perpendicular to  $\vec{v}_1$ , then  $T(\vec{v}_2) = -\vec{v}_2$  and  $T(\vec{v}_3) = -\vec{v}_3$ . So we can pick  $\vec{v}_2 = \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$  and  $\vec{v}_3 = \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}$ , for example.

40. From Exercise 37, we see that we want one of our basis vectors to be parallel to the line, while the others must be perpendicular the line. We can easily find such a basis:

$$\mathcal{B} = \left( \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \right).$$

41. We will use the same approach as in Exercises 37 and 39. Any basis with 2 vectors in the plane and one perpendicular to it will work nicely here! So, let  $\vec{v}_1, \vec{v}_2$  be in the plane.  $\vec{v}_1$  can be  $\begin{bmatrix} -1 \\ 3 \\ 0 \end{bmatrix}$ , and  $\vec{v}_2 = \begin{bmatrix} 0 \\ -2 \\ 1 \end{bmatrix}$  (note that these must be independent). Then  $\vec{v}_3$  should be perpendicular to the plane. We will use  $\vec{v}_3 = \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix}$ —the coefficient vector. This is perpendicular to the plane because all vectors perpendicular to  $\begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix}$  lie in the plane.

So, our basis is:  $\left( \begin{bmatrix} -1 \\ 3 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -2 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix} \right)$ .

42. From Exercise 38, we deduce that one of our vectors should be perpendicular to this plane, while two should fall inside it. Finding the perpendicular is not difficult: we simply take the coefficient vector:  $\begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix}$ . Then we add two linearly independent vectors on the plane,  $\begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$ , for instance. These three vectors form one possible basis.

43. By definition of coordinates (Definition 3.4.1),  $\vec{x} = 2 \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} + (-3) \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 4 \\ -3 \\ 2 \end{bmatrix}$ .

44. By definition of coordinates,  $\vec{x} = 2 \begin{bmatrix} 8 \\ 4 \\ -1 \end{bmatrix} + (-1) \begin{bmatrix} 5 \\ 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 11 \\ 6 \\ -1 \end{bmatrix}$ .

45. If  $\vec{v}_1, \vec{v}_2$  is a basis with the desired property, then  $\vec{x} = 2\vec{v}_1 + 3\vec{v}_2$ , or  $\vec{v}_2 = \frac{1}{3}\vec{x} - \frac{2}{3}\vec{v}_1$ . Thus we can make  $\vec{v}_1$  any vector in the plane that is not parallel to  $\vec{x}$ , and then let  $\vec{v}_2 = \frac{1}{3}\vec{x} - \frac{2}{3}\vec{v}_1$ .

For example, if we choose  $\vec{v}_1 = \begin{bmatrix} 3 \\ 2 \\ 0 \end{bmatrix}$ , then  $\vec{v}_2 = \frac{1}{3} \begin{bmatrix} -4 \\ -4 \\ -1 \end{bmatrix}$ .

46. As in Exercise 45, we can make  $\vec{v}_1$  any vector in the plane that is not parallel to  $\vec{x}$ , and

then let  $\vec{v}_2 = 2\vec{v}_1 - \vec{x}$ . For example, if we choose  $\vec{v}_1 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$ , then  $\vec{v}_2 = \begin{bmatrix} 1 \\ 1 \\ -3 \end{bmatrix}$ .

47. By Fact 3.4.4, we have  $A = SBS^{-1} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}^{-1} = \begin{bmatrix} d & c \\ b & a \end{bmatrix}$ .

48.  $[\vec{x}]_{\mathcal{B}} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$  means that  $\vec{x} = -\vec{v} + 2\vec{w}$ . See Figure 3.6

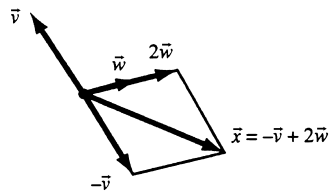


Figure 3.6: for Problem 3.4.48.

49.  $\vec{u} + \vec{v} = -\vec{w}$ , so that  $\vec{w} = -\vec{u} - \vec{v}$ , i.e.,  $[\vec{w}]_{\mathcal{B}} = \begin{bmatrix} -1 \\ -1 \end{bmatrix}$ .

50. a.  $\vec{OP} = \vec{w} + 2\vec{v}$ , so that  $[\vec{OP}]_{\mathcal{B}} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ ,  $\vec{OQ} = \vec{v} + 2\vec{w}$ , so that  $[\vec{OQ}]_{\mathcal{B}} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ .

b.  $\vec{OR} = 3\vec{v} + 2\vec{w}$ . See Figure 3.7.

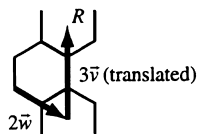


Figure 3.7: for Problem 3.4.50.

c. If the tip of  $\vec{u}$  is a vertex, then so is the tip of  $\vec{u} + 3\vec{v}$  and also the tip of  $\vec{u} + 3\vec{w}$  (draw a sketch!). We know that the tip  $P$  of  $2\vec{v} + \vec{w}$  is a vertex (see part a.). Therefore, the tip  $S$  of  $\vec{OS} = 17\vec{v} + 13\vec{w} = (2\vec{v} + \vec{w}) + 5(3\vec{v}) + 4(3\vec{w})$  is a vertex as well.

51. Let  $\mathcal{B} = (\vec{v}_1, \vec{v}_2, \dots, \vec{v}_m)$ . Then, let  $\vec{x} = a_1\vec{v}_1 + a_2\vec{v}_2 + \dots + a_m\vec{v}_m$  and  $\vec{y} = b_1\vec{v}_1 + b_2\vec{v}_2 + \dots + b_m\vec{v}_m$ . Then  $[\vec{x} + \vec{y}]_{\mathcal{B}} = [a_1\vec{v}_1 + a_2\vec{v}_2 + \dots + a_m\vec{v}_m + b_1\vec{v}_1 + b_2\vec{v}_2 + \dots + b_m\vec{v}_m]_{\mathcal{B}} = [(a_1 + b_1)\vec{v}_1 + (a_2 + b_2)\vec{v}_2 + \dots + (a_m + b_m)\vec{v}_m]_{\mathcal{B}}$

$$= \begin{bmatrix} a_1 + b_1 \\ a_2 + b_2 \\ \vdots \\ a_m + b_m \end{bmatrix} = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_m \end{bmatrix} + \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix} = [\vec{x}]_{\mathcal{B}} + [\vec{y}]_{\mathcal{B}}$$

52. Yes;  $T(\vec{x}) = [\vec{x}]_{\mathcal{B}} = S^{-1}\vec{x}$ , so that  $T$  is “given by a matrix.” (See Definition 2.1.1.)

53. By Definition 3.4.1, we have  $\vec{x} = S[\vec{x}]_{\mathcal{B}} = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} 7 \\ 11 \end{bmatrix} = \begin{bmatrix} 40 \\ 58 \end{bmatrix}$ .

54. Let  $Q$  be the matrix whose columns are the vectors of the basis  $\mathcal{T}$ . Then  $[[\vec{v}_1]_{\mathcal{T}} \dots [\vec{v}_n]_{\mathcal{T}}] = [Q^{-1}\vec{v}_1 \dots Q^{-1}\vec{v}_n] = Q^{-1}[\vec{v}_1 \dots \vec{v}_n]$  is an invertible matrix, so that the vectors  $[\vec{v}_1]_{\mathcal{T}} \dots [\vec{v}_n]_{\mathcal{T}}$  form a basis of  $\mathbb{R}^n$ .

55. By Definition 3.4.1, we have  $\vec{x} = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} [\vec{x}]_{\mathcal{B}}$  and  $\vec{x} = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix} [\vec{x}]_{\mathcal{R}}$ , so that  $\begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} [\vec{x}]_{\mathcal{B}} = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix} [\vec{x}]_{\mathcal{R}}$  and  $[\vec{x}]_{\mathcal{R}} = \underbrace{\begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}}_P [\vec{x}]_{\mathcal{B}}$ , i.e.,  $P = \begin{bmatrix} -\frac{1}{2} & 1 \\ \frac{1}{2} & 0 \end{bmatrix}$ .

56. Let  $S = [\vec{v}_1 \vec{v}_2]$  where  $\vec{v}_1, \vec{v}_2$  is the desired basis. Then by Fact 3.4.1,  $\begin{bmatrix} 1 \\ 2 \end{bmatrix} = S \begin{bmatrix} 3 \\ 5 \end{bmatrix}$  and  $\begin{bmatrix} 3 \\ 4 \end{bmatrix} = S \begin{bmatrix} 2 \\ 3 \end{bmatrix}$ , i.e.  $S \begin{bmatrix} 3 & 2 \\ 5 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}$ . Hence  $S = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} 3 & 2 \\ 5 & 3 \end{bmatrix}^{-1} = \begin{bmatrix} 12 & -7 \\ 14 & -8 \end{bmatrix}$ .  
The desired basis is  $\begin{bmatrix} 12 \\ 14 \end{bmatrix}, \begin{bmatrix} -7 \\ -8 \end{bmatrix}$ .

57. If we can find a basis  $\mathcal{B} = (\vec{v}_1, \vec{v}_2, \vec{v}_3)$  such that the  $\mathcal{B}$ -matrix of  $A$  is

$$B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \text{ then } A \text{ must be similar to } \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}. \text{ Because of the entries in the matrix } B, \text{ it is required that } A\vec{v}_1 = \vec{v}_1, A\vec{v}_2 = \vec{v}_2 \text{ and } A\vec{v}_3 = -\vec{v}_3. \text{ So, all we need for our basis is to pick independent } \vec{v}_1, \vec{v}_2 \text{ in the plane, and } \vec{v}_3 \text{ perpendicular to the plane.}$$

58. a. Consider a linear relation  $c_1A^2\vec{v} + c_2A\vec{v} + c_3\vec{v} = \vec{0}$ .

Multiplying  $A^2$  with the vectors on both sides and using that  $A^3\vec{v} = \vec{0}$  and  $A^4\vec{v} = \vec{0}$ , we find that  $c_3A^2\vec{v} = \vec{0}$  and therefore  $c_3 = 0$ , since  $A^2\vec{v} \neq \vec{0}$ .



Therefore, our relation simplifies to  $c_1 A^2 \vec{v} + c_2 A \vec{v} = \vec{0}$ .

Multiplying  $A$  with the vectors on both sides we find that  $c_2 A^2 \vec{v} = \vec{0}$  and therefore  $c_2 = 0$ . Then  $c_1 = 0$  as well. We have shown that there is only the trivial relation among the vectors  $A^2 \vec{v}$ ,  $A \vec{v}$ , and  $\vec{v}$ , so that these three vectors form a basis of  $\mathbb{R}^3$ , as claimed.

$$\text{b. } T(A^2 \vec{v}) = A^3 \vec{v} = \vec{0} \text{ so } [T(A^2 \vec{v})]_{\mathcal{B}} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

$$T(A \vec{v}) = A^2 \vec{v} \text{ so } [T(A \vec{v})]_{\mathcal{B}} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}.$$

$$T(\vec{v}) = A \vec{v} \text{ so } [T(\vec{v})]_{\mathcal{B}} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}.$$

Hence, by Fact 3.4.3, the desired matrix is  $\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ .

59. First we find the matrices  $S = \begin{bmatrix} x & y \\ z & t \end{bmatrix}$  such that  $\begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} x & y \\ z & t \end{bmatrix} = \begin{bmatrix} x & y \\ z & t \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix}$ , or

$\begin{bmatrix} 2x & 2y \\ 3z & 3t \end{bmatrix} = \begin{bmatrix} 2x & x+3y \\ 2z & z+3t \end{bmatrix}$ . The solutions are of the form  $S = \begin{bmatrix} -y & y \\ 0 & t \end{bmatrix}$ , where  $y$  and  $t$  are arbitrary constants. Since there are *invertible* solutions  $S$  (for example, let  $y = t = 1$ ), the matrices  $\begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$  and  $\begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix}$  are indeed similar.

60. First we find the matrices  $S = \begin{bmatrix} x & y \\ z & t \end{bmatrix}$  such that  $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x & y \\ z & t \end{bmatrix} = \begin{bmatrix} x & y \\ z & t \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ ,

or,  $\begin{bmatrix} x & y \\ -z & -t \end{bmatrix} = \begin{bmatrix} y & x \\ t & z \end{bmatrix}$ . The solutions are of the form  $S = \begin{bmatrix} y & y \\ -t & t \end{bmatrix}$ , where  $y$  and  $t$  are arbitrary constants. Since there are *invertible* solutions  $S$  (for example, let  $y = t = 1$ ), the matrices  $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$  and  $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$  are indeed similar.

61. We seek a basis  $\vec{v}_1 = \begin{bmatrix} x \\ z \end{bmatrix}$ ,  $\vec{v}_2 = \begin{bmatrix} y \\ t \end{bmatrix}$  such that the matrix  $S = [\vec{v}_1 \ \vec{v}_2] = \begin{bmatrix} x & y \\ z & t \end{bmatrix}$  satisfies

the equation  $\begin{bmatrix} -5 & -9 \\ 4 & 7 \end{bmatrix} \begin{bmatrix} x & y \\ z & t \end{bmatrix} = \begin{bmatrix} x & y \\ z & t \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ . Solving the ensuing linear system

$$\text{gives } S = \begin{bmatrix} -\frac{3z}{2} & \frac{z}{4} - \frac{3t}{2} \\ z & t \end{bmatrix}.$$

We need to choose  $z$  and  $t$  so that  $S$  will be invertible. For example, if we let  $z = 6$  and  $t = 1$ , then  $S = \begin{bmatrix} -9 & 0 \\ 6 & 1 \end{bmatrix}$ , so that  $\vec{v}_1 = \begin{bmatrix} -9 \\ 6 \end{bmatrix}$ ,  $\vec{v}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ .

62. We seek a basis  $\vec{v}_1 = \begin{bmatrix} x \\ z \end{bmatrix}$ ,  $\vec{v}_2 = \begin{bmatrix} y \\ t \end{bmatrix}$  such that the matrix  $S = [\vec{v}_1 \ \vec{v}_2] = \begin{bmatrix} x & y \\ z & t \end{bmatrix}$  satisfies the equation  $\begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix} \begin{bmatrix} x & y \\ z & t \end{bmatrix} = \begin{bmatrix} x & y \\ z & t \end{bmatrix} \begin{bmatrix} 5 & 0 \\ 0 & -1 \end{bmatrix}$ . Solving the ensuing linear system gives  $S = \begin{bmatrix} \frac{z}{2} & -t \\ z & t \end{bmatrix}$ . We need to choose both  $z$  and  $t$  nonzero to make  $S$  invertible. For example, if we let  $z = 2$  and  $t = 1$ , then  $S = \begin{bmatrix} 1 & -1 \\ 2 & 1 \end{bmatrix}$ , so that  $\vec{v}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ ,  $\vec{v}_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ .

63. First we find the matrices  $S = \begin{bmatrix} x & y \\ z & t \end{bmatrix}$  such that  $\begin{bmatrix} p & -q \\ q & p \end{bmatrix} \begin{bmatrix} x & y \\ z & t \end{bmatrix} = \begin{bmatrix} x & y \\ z & t \end{bmatrix} \begin{bmatrix} p & q \\ -q & p \end{bmatrix}$ , or,  $\begin{bmatrix} px - qz & py - qt \\ qx + pz & qy + pt \end{bmatrix} = \begin{bmatrix} px - qy & qx + py \\ pz - qt & qz + pt \end{bmatrix}$ . If  $q \neq 0$ , then the solutions are of the form  $S = \begin{bmatrix} -t & z \\ z & t \end{bmatrix}$ , where  $z$  and  $t$  are arbitrary constants. Since there are *invertible* solutions  $S$  (for example, let  $z = t = 1$ ), the matrices  $\begin{bmatrix} p & -q \\ q & p \end{bmatrix}$  and  $\begin{bmatrix} p & q \\ -q & p \end{bmatrix}$  are indeed similar. (If  $q = 0$ , then the two matrices are equal.)

64. If  $b$  and  $c$  are both zero, then the given matrices are equal, so that they are similar, by Fact 3.4.6.a. Let's now assume that at least one of the scalars  $b$  and  $c$  is nonzero; reversing the roles of  $b$  and  $c$  if necessary, we can assume that  $c \neq 0$ .

Let's find the matrices  $S = \begin{bmatrix} x & y \\ z & t \end{bmatrix}$  such that  $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x & y \\ z & t \end{bmatrix} = \begin{bmatrix} x & y \\ z & t \end{bmatrix} \begin{bmatrix} a & c \\ b & d \end{bmatrix}$ , or

$$\begin{bmatrix} ax + bz & ay + bt \\ cx + dz & cy + dt \end{bmatrix} = \begin{bmatrix} ax + by & cx + dy \\ az + bt & cz + dt \end{bmatrix}. \text{ The solutions are of the form}$$

$S = \begin{bmatrix} \frac{(a-d)z+b}{c} & z \\ z & t \end{bmatrix}$ , where  $z$  and  $t$  are arbitrary constants. Since there are *invertible*

solutions  $S$  (for example, let  $z = 1, t = 0$ ), the matrices  $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$  and  $\begin{bmatrix} a & c \\ b & d \end{bmatrix}$  are indeed similar.

65. a. If  $S = I_n$ , then  $S^{-1}AS = A$ .

b. If  $S^{-1}AS = B$ , then  $SBS^{-1} = A$ . If we let  $R = S^{-1}$ , then  $R^{-1}BR = A$ , showing that  $B$  is similar to  $A$ .

66. We build  $B$  “column-by-column”:

$$\begin{aligned} B &= \left[ \left[ T \begin{bmatrix} b \\ 1-a \end{bmatrix} \right]_{\mathcal{B}} \left[ T \begin{bmatrix} a-1 \\ b \end{bmatrix} \right]_{\mathcal{B}} \right] = \left[ \left[ \begin{bmatrix} ab+b-ba \\ b^2+a^2-a \end{bmatrix} \right]_{\mathcal{B}} \left[ \begin{bmatrix} a^2+b^2-a \\ ba-b-ab \end{bmatrix} \right]_{\mathcal{B}} \right] \\ &= \left[ \left[ \begin{bmatrix} b \\ 1-a \end{bmatrix} \right]_{\mathcal{B}} \left[ \begin{bmatrix} 1-a \\ -b \end{bmatrix} \right]_{\mathcal{B}} \right] = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}. \end{aligned}$$

Thus, this matrix represents the reflection about the line spanned by  $\begin{bmatrix} b \\ 1-a \end{bmatrix}$ . Note that the two vectors  $\begin{bmatrix} b \\ 1-a \end{bmatrix}$  and  $\begin{bmatrix} a-1 \\ b \end{bmatrix}$  are perpendicular.

67. The matrix we seek is  $\left[ \left[ T \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right]_{\mathcal{B}} \left[ T \begin{bmatrix} a \\ c \end{bmatrix} \right]_{\mathcal{B}} \right] = \left[ \left[ \begin{bmatrix} a \\ c \end{bmatrix} \right]_{\mathcal{B}} \left[ \begin{bmatrix} a^2+bc \\ ac+cd \end{bmatrix} \right]_{\mathcal{B}} \right] = \begin{bmatrix} 0 & bc-ad \\ 1 & a+d \end{bmatrix}$ .

68. Using Exercise 67 as a guide, consider the basis  $\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} a \\ c \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$ , and let  $S = \begin{bmatrix} 1 & 1 \\ 0 & 3 \end{bmatrix}$ .

69. The matrix of the transformation  $T(\vec{x}) = A\vec{x}$  with respect to the basis  $\begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \end{bmatrix}$  is

$$D = \left[ \left[ \begin{bmatrix} 3 \\ 6 \end{bmatrix} \right]_{\mathcal{B}} \left[ \begin{bmatrix} -2 \\ -1 \end{bmatrix} \right]_{\mathcal{B}} \right] = \begin{bmatrix} 3 & 0 \\ 0 & -1 \end{bmatrix}. \text{ Thus } S^{-1}AS = D \text{ for } S = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}.$$

70. Suppose such a basis  $\vec{v}_1, \vec{v}_2$  exists. If  $B = [[T(\vec{v}_1)]_{\mathcal{B}} [T(\vec{v}_2)]_{\mathcal{B}}]$  is upper triangular, of the form  $\begin{bmatrix} a & b \\ 0 & c \end{bmatrix}$ , then  $[T(\vec{v}_1)]_{\mathcal{B}} = \begin{bmatrix} a \\ 0 \end{bmatrix}$ , so that  $T(\vec{v}_1) = a\vec{v}_1$ , that is,  $T(\vec{v}_1)$  is parallel to  $\vec{v}_1$ . But this is impossible, since  $T$  is a rotation through  $\frac{\pi}{2}$ .

71. a. Note that  $AS = SB$ . If  $\vec{x}$  is in  $\ker(B)$ , then  $A(S\vec{x}) = SB\vec{x} = S\vec{0} = \vec{0}$ , so that  $S\vec{x}$  is in  $\ker(A)$ , as claimed.

b. We use the hint and observe that  $\text{nullity}(B) = \dim(\ker B) = p \leq \dim(\ker A) = \text{nullity}(A)$ , since  $S\vec{v}_1, \dots, S\vec{v}_p$  are  $p$  linearly independent vectors in  $\ker(A)$ . Reversing the roles of  $A$  and  $B$  shows that, conversely,  $\text{nullity}(A) \leq \text{nullity}(B)$ , so that the equation  $\text{nullity}(A) = \text{nullity}(B)$  holds, as claimed.

72. If  $A$  and  $B$  are similar  $n \times n$  matrices, then  $\text{rank}(A) = n - \text{nullity}(A) = n - \text{nullity}(B) = \text{rank}(B)$ , by Exercise 71 and the rank nullity theorem (Fact 3.3.7).

73. a. By inspection, we can find an orthonormal basis  $\vec{v}_1 = \vec{v}, \vec{v}_2, \vec{v}_3$  of  $\mathbb{R}^3$ :

$$\vec{v}_1 = \vec{v} = \begin{bmatrix} 0.6 \\ 0.8 \\ 0 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \vec{v}_3 = \begin{bmatrix} 0.8 \\ -0.6 \\ 0 \end{bmatrix}$$

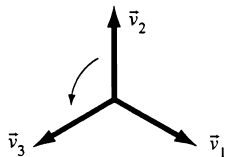


Figure 3.8: for Problem 3.4.73b.

b. Now  $T(\vec{v}_1) = \vec{v}_1, T(\vec{v}_2) = \vec{v}_3$  and  $T(\vec{v}_3) = -\vec{v}_2$  (see Figure 3.2), so that the matrix  $B$  of  $T$  with respect to the basis  $\vec{v}_1, \vec{v}_2, \vec{v}_3$  is

$$B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}. \text{ Then } A = SBS^{-1} = \begin{bmatrix} 0.36 & 0.48 & 0.8 \\ 0.48 & 0.64 & -0.6 \\ -0.8 & 0.6 & 0 \end{bmatrix}.$$

74. a.  $\vec{v}_0 + \vec{v}_1 + \vec{v}_2 + \vec{v}_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix} + \begin{bmatrix} -1 \\ 1 \\ -1 \end{bmatrix} + \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$

b. If  $\mathcal{B}$  is the basis  $\vec{v}_1, \vec{v}_2, \vec{v}_3$ , then  $\vec{v}_0 + \vec{v}_1 + \vec{v}_2 + \vec{v}_3 = \vec{0}$  (by part a) so  $\vec{v}_0 = -\vec{v}_1 - \vec{v}_2 - \vec{v}_3$ ,

i.e.  $[\vec{v}_0]_{\mathcal{B}} = \begin{bmatrix} -1 \\ -1 \\ -1 \end{bmatrix}.$

c.  $T(\vec{v}_2) = T(-\vec{v}_0 - \vec{v}_1 - \vec{v}_3) = -T(\vec{v}_0) - T(\vec{v}_1) - T(\vec{v}_3) = -\vec{v}_3 - \vec{v}_0 - \vec{v}_1 = \vec{v}_2$

Hence,  $T$  is a rotation through  $120^\circ$  about the line spanned by  $\vec{v}_2$ . Its matrix,  $B$ , is given by

$[[T(\vec{v}_1)]_{\mathcal{B}} [T(\vec{v}_2)]_{\mathcal{B}} [T(\vec{v}_3)]_{\mathcal{B}}]$  where

$$T(\vec{v}_1) = \vec{v}_0 = -\vec{v}_1 - \vec{v}_2 - \vec{v}_3 \text{ so } [T(\vec{v}_1)]_{\mathcal{B}} = \begin{bmatrix} -1 \\ -1 \\ -1 \end{bmatrix}$$

$$T(\vec{v}_2) = \vec{v}_2 \text{ so } [T(\vec{v}_2)]_{\mathcal{B}} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$$T(\vec{v}_3) = \vec{v}_1 \text{ so } [T(\vec{v}_3)]_{\mathcal{B}} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$\text{and } B = \begin{bmatrix} -1 & 0 & 1 \\ -1 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}.$$

$B^3 = I_3$  since if the tetrahedron rotates through  $120^\circ$  three times, it returns to the original position.

75.  $B = S^{-1}AS$ , where  $S = A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ . Thus  $B = A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$

76.  $B = S^{-1}AS$ , where  $S = A = \begin{bmatrix} \cos(t) & -\sin(t) \\ \sin(t) & \cos(t) \end{bmatrix}$ . Thus  $B = A = \begin{bmatrix} \cos(t) & -\sin(t) \\ \sin(t) & \cos(t) \end{bmatrix}$ .

77. Let  $S$  be the  $n \times n$  matrix whose columns are  $\vec{e}_n, \vec{e}_{n-1}, \dots, \vec{e}_1$ . Note that  $S$  has all 1's on "the other diagonal" and 0's elsewhere:

$$s_{ij} = \begin{cases} 1 & \text{if } i + j = n + 1 \\ 0 & \text{otherwise} \end{cases}$$

Also,  $S^{-1} = S$ .

Now,  $B = S^{-1}AS = SAS$ ; the entries of  $B$  are  $b_{ij} = s_{i,n+1-i}a_{n+1-i,n+1-j}s_{n+1-j,j} = a_{n+1-i,n+1-j}$ .

Answer:  $b_{ij} = a_{n+1-i,n+1-j}$

$B$  is obtained from  $A$  by reversing the order of the rows and of the columns.

78. Note first that the diagonal entry  $s_{ij}$  of  $S$  gives the unit price of good  $i$ .

If  $a_{ij}$  tells us how many dollars' worth of good  $i$  are required to produce one dollar's worth of good  $j$ , then  $a_{ij}s_{jj}$  tells us how many dollars' worth of good  $i$  are required to produce one unit of good  $j$ , and  $s_{ii}^{-1}a_{ij}s_{jj}$  is the number of units of good  $i$  required to produce one unit of good  $j$ . Thus  $b_{ij} = s_{ii}^{-1}a_{ij}s_{jj}$ , and  $B = S^{-1}AS$ .

## True or False

1. F; It's a subspace of  $\mathbb{R}^3$ .
2. T; by Definition 3.1.2.
3. T, by Summary 3.3.9.
4. F, by Fact 3.3.7.
5. T, by Summary 3.3.9.
6. F; The identity matrix is similar only to itself.
7. T; We have the nontrivial relation  $3\vec{u} + 3\vec{v} + 3\vec{w} = \vec{0}$ .
8. F; The columns could be  $\vec{e}_1, \vec{e}_2, \vec{e}_3, \vec{e}_4$  in  $\mathbb{R}^5$ , for example.
9. T, by Fact 3.3.2.
10. F; The nullity is  $6 - 4 = 2$ , by Fact 3.3.7.
11. T, by Fact 3.2.8.
12. T, by Summary 3.3.9.
13. F; The number  $n$  may exceed 4.
14. T, by Definition 3.2.1 ( $V$  is closed under linear combinations)
15. T, by Fact 3.4.6, parts b and c.
16. F; Let  $V = \text{span} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$  in  $\mathbb{R}^2$ , for example.
17. T, by Definition 3.2.3.
18. T, by Definition 3.2.1.
19. T; Check that  $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ .
20. T, by Fact 3.3.8.
21. F; We are unable to find an invertible matrix  $S$  as required in the definition of similarity.
22. F; Five vectors in  $\mathbb{R}^4$  must be dependent, by Fact 3.2.8.
23. T, by Definition 3.2.1 (all vectors in  $\mathbb{R}^3$  are linear combinations of  $\vec{e}_1, \vec{e}_2, \vec{e}_3$ ).
24. T; Use a basis with one vector on the line and the other perpendicular to it.

25. T, since  $AB\vec{v} = A\vec{0} = \vec{0}$ .
26. T, by Definition 3.2.3.
27. F; Suppose  $\vec{v}_2 = 2\vec{v}_1$ . Then  $T(\vec{v}_2) = 2T(\vec{v}_1) = 2\vec{e}_1$  cannot be  $\vec{e}_2$ .
28. F; Consider  $\vec{u} = \vec{e}_1$ ,  $\vec{v} = 2\vec{e}_1$ , and  $\vec{w} = \vec{e}_2$ .
29. T, since  $A^{-1}(AB)A = BA$ .
30. T, since both kernels consist of the zero vector alone.
31. T; Consider any basis  $\vec{v}_1, \vec{v}_2, \vec{v}_3$  of  $V$ . Then  $k\vec{v}_1, \vec{v}_2, \vec{v}_3$  is a basis as well, for any nonzero scalar  $k$ .
32. F; The identity matrix is similar only to itself.
33. F; Consider  $\begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ , but  $\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix}$ .
34. F; Let  $A = I_2$ ,  $B = -I_2$  and  $\vec{v} = \vec{e}_1$ , for example.
35. F; Let  $V = \text{span}(\vec{e}_1)$  and  $W = \text{span}(\vec{e}_2)$  in  $\mathbb{R}^2$ , for example.
36. T; If  $A\vec{v} = A\vec{w}$ , then  $A(\vec{v} - \vec{w}) = \vec{0}$ , so that  $\vec{v} - \vec{w} = \vec{0}$  and  $\vec{v} = \vec{w}$ .
37. T; Consider the linear transformation with matrix  $[\vec{w}_1 \ \dots \ \vec{w}_n][\vec{v}_1 \ \dots \ \vec{v}_n]^{-1}$ .
38. F; Suppose  $A$  were similar to  $B$ . Then  $A^4 = I_2$  were similar to  $B^4 = -I_2$ , by Example 7 of Section 3.4. But this isn't the case:  $I_2$  is similar only to itself.
39. F; Note that  $\mathbb{R}^2$  isn't even a subset of  $\mathbb{R}^3$ . A vector in  $\mathbb{R}^2$ , with two components, does not belong to  $\mathbb{R}^3$ .
40. T; If  $B = S^{-1}AS$ , then  $B + 7I_n = S^{-1}(A + 7I_n)S$ .
41. T; Let  $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ , for example, with  $\ker(A) = \text{im}(A) = \text{span}(\vec{e}_1)$ .
42. F; Consider  $I_n$  and  $2I_n$ , for example.
43. T; Matrix  $B = S^{-1}AS$  is invertible, being the product of invertible matrices.
44. T; Note that  $\text{im}(A)$  is a subspace of  $\ker(A)$ , so that  

$$\dim(\text{im } A) = \text{rank}(A) \leq \dim(\ker A) = 10 - \text{rank}(A).$$
45. T; Pick three vectors  $\vec{v}_1, \vec{v}_2, \vec{v}_3$  that span  $V$ . Then  $V = \text{im}[\vec{v}_1 \ \vec{v}_2 \ \vec{v}_3]$ .

46. T; Check that  $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$  is similar to  $\begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix}$ .
47. T; Pick a vector  $\vec{v}$  that is neither on the line nor perpendicular to it. Then the matrix of the linear transformation  $T(\vec{x}) = R\vec{x}$  with respect to the basis  $\vec{v}, R\vec{v}$  is  $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ , since  $R(R\vec{v}) = \vec{v}$ .
48. F; If  $B = S^{-1}AS$ , then  $B = (2S)^{-1}A(2S)$  as well.
49. T; Note that  $A(B - C) = 0$ , so that all the columns of matrix  $B - C$  are in the kernel of  $A$ . Thus  $B - C = 0$  and  $B = C$ , as claimed.
50. T; Suppose  $\vec{v}$  is in both  $\ker(A)$  and  $\text{im}(A)$ , so that  $\vec{v} = A\vec{w}$  for some vector  $\vec{w}$ . Then  $\vec{0} = A\vec{v} = A^2\vec{w} = A\vec{v} = \vec{v}$ , as claimed.
51. F; Suppose such a matrix  $A$  exists. Then there is a vector  $\vec{v}$  in  $\mathbb{R}^2$  such that  $A^2\vec{v} \neq \vec{0}$  but  $A^3\vec{v} = \vec{0}$ . As in Exercise 3.4.58a we can show that vectors  $\vec{v}, A\vec{v}, A^2\vec{v}$  are linearly independent, a contradiction (we are looking at three vectors in  $\mathbb{R}^2$ ).
52. T; The  $i$ th column  $\vec{a}_i$  of  $A$ , being in the image of  $A$ , is also in the image of  $B$ , so that  $\vec{a}_i = B\vec{c}_i$  for some  $\vec{c}_i$  in  $\mathbb{R}^m$ . If we let  $C = [\vec{c}_1 \ \cdots \ \vec{c}_m]$ , then  $BC = [B\vec{c}_1 \ \cdots \ B\vec{c}_m] = [\vec{a}_1 \ \cdots \ \vec{a}_m] = A$ , as required.
53. F; Think about this problem in terms of “building” such an invertible matrix column by column. If we wish the matrix to be invertible, then the first column can be any column other than  $\vec{0}$  (7 choices). Then the second column can be any column other than  $\vec{0}$  or the first column (6 choices). For the third column, we have *at most* 5 choices (not  $\vec{0}$  or the first or second columns, as well as possibly some other columns). For some choices of the first two columns there will be other columns we have to exclude (the sum or difference of the first two), but not for others. Thus, in total, fewer than  $7 \times 6 \times 5 = 210$  matrices are invertible, out of a total  $2^9 = 512$  matrices. Thus, most are not invertible.