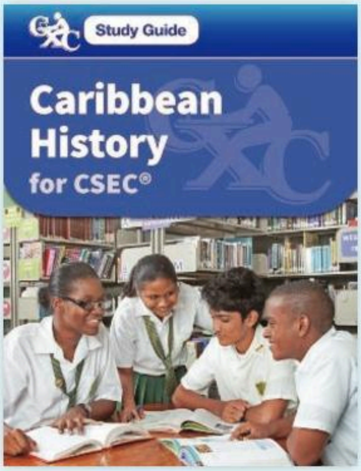
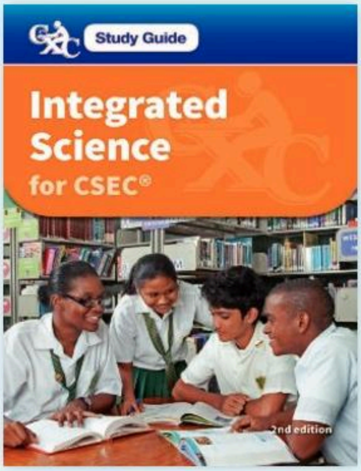
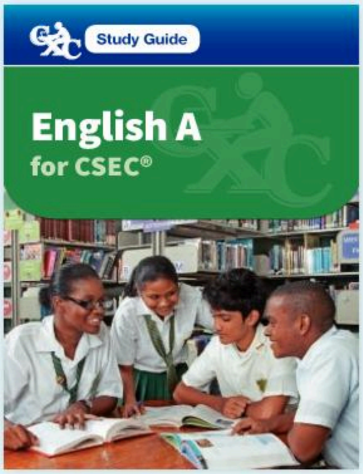
Study Guide

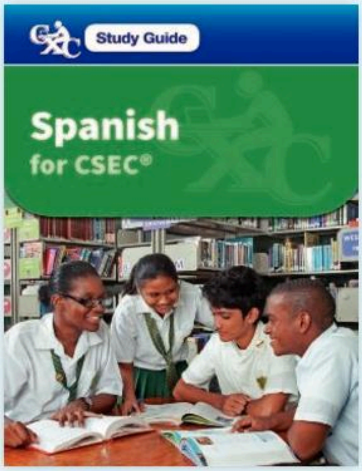
Mathematics

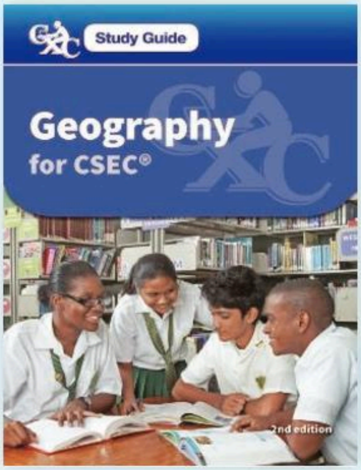
for CSEC® 2nd edition


Study Guide
Caribbean History for CSEC®


Study Guide
Integrated Science for CSEC®
2nd edition


Study Guide
English A for CSEC®

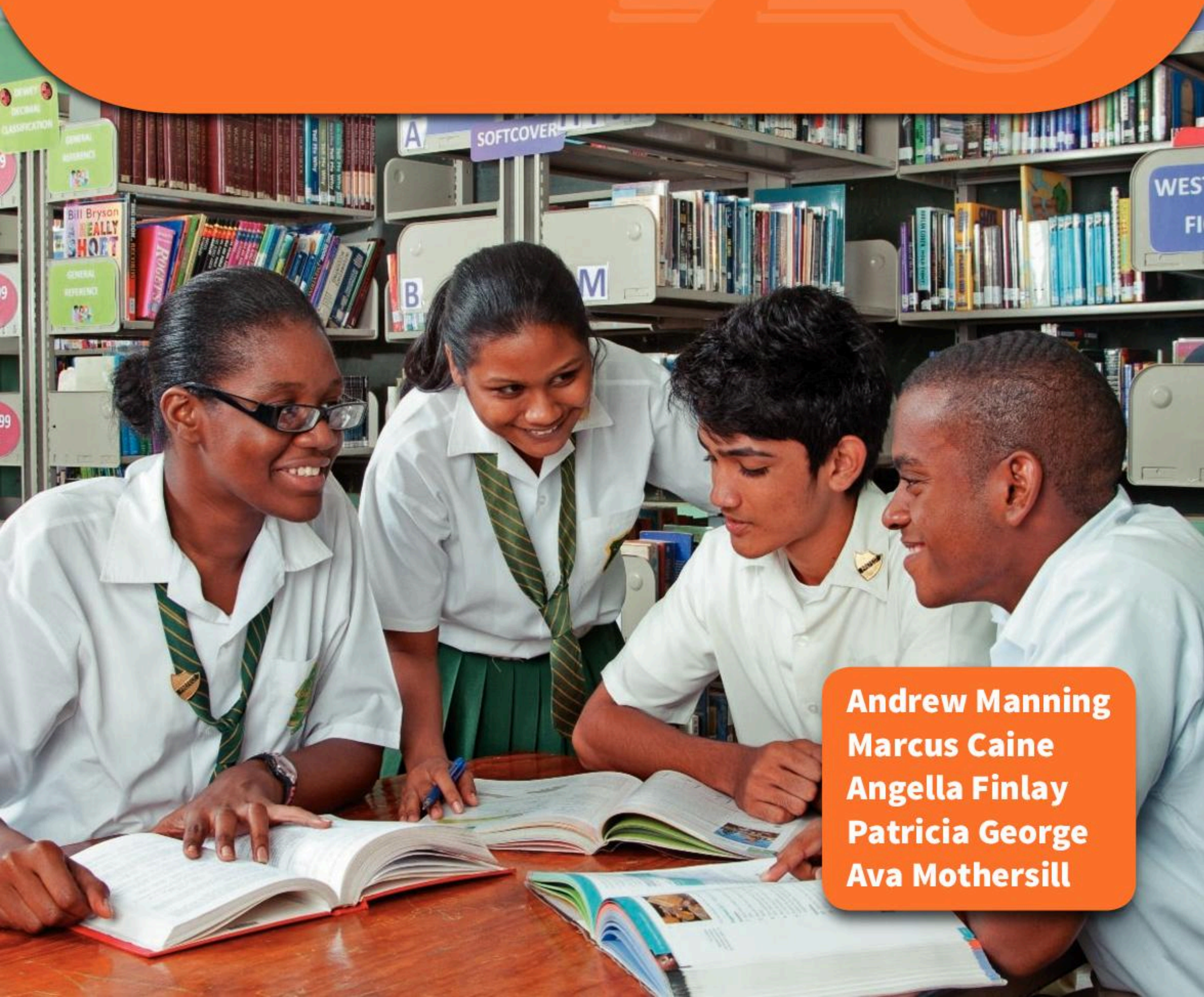

Study Guide
Spanish for CSEC®


Study Guide
Geography for CSEC®
2nd edition

Study Guide

Mathematics

for CSEC® 2nd edition



Andrew Manning
Marcus Caine
Angella Finlay
Patricia George
Ava Mothersill

OXFORD
UNIVERSITY PRESS

Great Clarendon Street, Oxford, OX2 6DP, United Kingdom

Oxford University Press is a department of the University of Oxford.
It furthers the University's objective of excellence in research, scholarship,
and education by publishing worldwide. Oxford is a registered trade mark of
Oxford University Press in the UK and in certain other countries

First published by Nelson Thornes Ltd in 2012
This edition published by Oxford University Press in 2017

British Library Cataloguing in Publication Data
Data available

978-0-1984-1452-0

1 3 5 7 9 10 8 6 4 2

Printed in China

Acknowledgements

Cover photograph: Mark Lyndersay, Lyndersay Digital, Trinidad
www.lyndersaydigital.com
Inside photograph: iStockphoto
Illustrations: Tech-Set Limited and Mike Bastin
Page make-up: Tech-Set Limited, Gateshead

Photographs
p47: iStockphoto
p49: iStockphoto/Jacqui Paterson

Contents

Contents

Additional support can be accessed online at
www.oxfordsecondary.com/9780198414520

Introduction

This Study Guide has been developed exclusively with the Caribbean Examinations Council (CXC®) to be used as an additional resource by candidates, both in and out of school, following the Caribbean Secondary Education Certificate (CSEC®) programme.

It has been prepared by a team with expertise in the CSEC® syllabus, teaching and examination. The contents are designed to support learning by providing tools to help you achieve your best in CSEC® Maths and the features included make it easier for you to master the key concepts and requirements of the syllabus. Do remember to refer to your syllabus for full guidance on the course requirements and examination format!

This Study Guide is accompanied by a support website which includes electronic activities to assist you in developing good examination techniques:

- **On your marks** activities provide sample examination-style short answer and essay type questions, with example candidate answers and feedback from an examiner to show where answers could be improved. These activities will build your understanding, skill level and confidence in answering examination questions.
- **Test yourself** activities are specifically designed to provide experience of multiple-choice examination questions and helpful feedback will refer you to sections inside the study guide so that you can revise problem areas.

This unique combination of focused syllabus content and interactive examination practice will provide you with invaluable support to help you reach your full potential in CSEC® Maths.

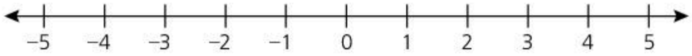
Access your support website at
www.oxfordsecondary.com/9780198414520

1 Number theory, computation, sets and consumer arithmetic

1.1 Types of number

LEARNING OUTCOMES

- Distinguish between different types of number
- Understand the relationship between different types of number



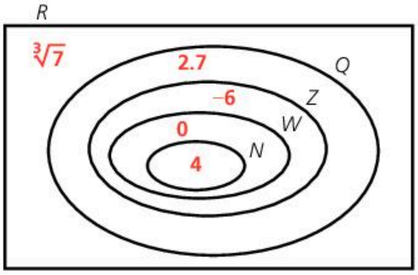
Structure

If we count backwards from 5, we don't have to stop at zero.

5, 4, 3, 2, 1, 0, −1, −2, −3, −4, −5, etc.

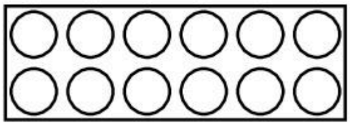
- The counting numbers (1, 2, 3, 4, 5, etc.) are called **natural numbers** (often given the symbol *N*).
- Zero is not a natural number.
- The natural numbers together with zero make up the **whole** numbers, identified by *W*.
- **Integers** (*Z*) are all the natural numbers, zero and the negative whole numbers (…, −7, −6, −5, −4, −3, −2, −1, 0, 1, 2, 3, …).
- **Rational numbers** (*Q*) are all the numbers that can be written as a fraction. They include the integers, terminating and recurring decimals (for example −3, −1, 0, 1, 4, $\frac{1}{2}$, $\frac{2}{3}$, $\frac{11}{8}$, 0.75, 5.333).
- **Irrational numbers** cannot be written exactly as a fraction or a decimal, as they never recur. Examples include π and $\sqrt{2}$.
- The **real numbers**, *R*, consist of the rational numbers and irrational numbers.

The diagram on the left shows a **Venn diagram** representation: examples are given in red.



Venn diagram

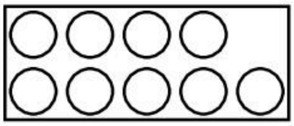
Factors and multiples, odd and even, prime and composite numbers



Twelve bottles could fit into a rectangular box that is 2 bottles wide and 6 bottles long.

This is because 12 is in the 2 times table. We say 12 is a **multiple** of 2. (12 = 2 × 6).

2 is a **factor** of 12, because 12 can be divided exactly by 2.



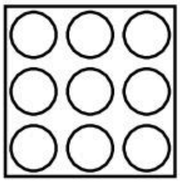
Nine bottles will not fit exactly into a rectangular box that is 2 bottles wide.

This is because 9 is not a multiple of 2.

Multiples of 2 (2, 4, 6, 8, 10, …) are called **even** numbers.

The last figure (or units digit) of an even number is always even (for example, 3**4** or 15**6**).

Numbers that are not multiples of 2 (1, 3, 5, 7, …) are called **odd** numbers. Odd numbers always have a units digit that is odd (for example, 6**7** or 9**9** or 245**1**).

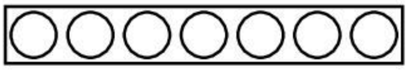


Nine bottles will fit in a 3 by 3 box.

This is because 9 is a multiple of 3. ($9 = 3 \times 3$).

Numbers that fit into a square box (e.g. $16 = 4 \times 4$, $100 = 10 \times 10$) are called **square numbers**.

A square number is the result of multiplying an integer by itself.



Seven bottles will **only** fit in a 1 by 7 crate.

This is because 7 is **only** a multiple of 1 and 7.

Numbers like 7, that are multiples of 1 and themselves and nothing else, are called **prime numbers**.

Composite numbers are natural numbers with more than two factors.

KEY POINTS

1. Natural numbers: 1, 2, 3, 4, …
2. Whole numbers: 0, 1, 2, 3, 4, …
3. Integers: … -3, -2, -1, 0, 1, 2, 3, …
4. Rational numbers: 2, -5, 0.7, $\frac{2}{7}$, $0.\dot{4}\dot{7}$, …
5. Irrational numbers: $\sqrt{7}$, π, …
6. Real numbers: The rational and irrational numbers together.
7. Multiples of n are the results of multiplying n by a natural number.
8. Even numbers are the multiples of 2.
9. Odd numbers are the natural numbers that are not multiples of 2.
10. Factors of n are the exact divisors of n.
11. A prime number has exactly two factors, 1 and itself.

ACTIVITY

In the 18th century, Goldbach (a famous mathematician) stated that every even number greater than 2 could be made by adding together two prime numbers.

Start to check this by expressing all the even numbers between 4 and 50 as the addition of two prime numbers (the same prime number can be used twice).

EXAM TIP

- 1 is not a prime number.
- 2 is the only even prime number.
- Do not confuse factors and multiples; factors are equal to or smaller while multiples are equal to or larger.

SUMMARY QUESTIONS

1 Which of the words natural; prime; integer; even; square; irrational apply to the number

a 1 **b** $\sqrt{8}$

c 16 **d** 5

e -7 **f** $\frac{5}{11}$?

2 Find two numbers that are factors of 20 and are prime numbers.

1.2

Highest common factor and lowest common multiple

LEARNING OUTCOMES

- List the factors and multiples of a positive integer
- Compute the HCF and LCM of two or more positive integers

Factors and multiples

The **factors** of a number are the integers by which it can be divided with no remainder.

The factors of 24 are 1, 2, 3, 4, 6, 8, 12 and 24.

Negative numbers are the product of a positive and negative number as you will see in Unit 3.2, so the factors of 24 are also −1, −2, −3, −4, −6, −8, −12 and −24.

In the same way, the factors of −6 are 1, −6. −1, 6, 2, −3, −2, 3. Note that pairs in the same colour multiply to −6.

EXAM TIP

When listing factors of a number, list pairs of factors that multiply to make that number, starting with 1.

The **prime factors** of a number are its factors that are prime numbers. The prime factors of 24 are 2 and 3. The **multiples** of a number are the results of multiplying it by an integer.

The multiples of 7 are 7, 14, 21, 28, 35, etc.

HCF and LCM

The **HCF (highest common factor)** of two or more numbers is the highest number that is a factor of both or all of them.

WORKED EXAMPLE 1

The factors of 56 are 1, 2, 4, 7, 8, 14, 28, 56

The factors of 84 are 1, 2, 3, 4, 6, 7, 12, 14, 21, 28, 42, 84

The common factors are 1, 2, 4, 7, 14, 28

28 is the HCF as it is the largest number in both lists.

EXAM TIP

Students often confuse HCF and LCM.

Remember an HCF is a FACTOR, so will be smaller than the original numbers.

The LCM is a MULTIPLE, so it cannot be smaller than the original numbers.

The **LCM (lowest common multiple)** of two or more numbers is the smallest number that is a multiple of both or all of them.

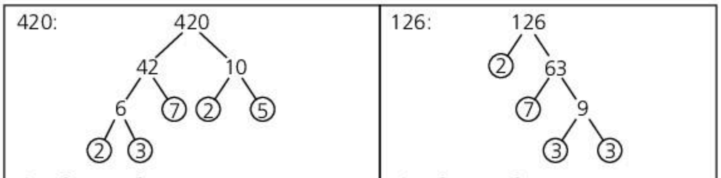
WORKED EXAMPLE 2

The multiples of 12 are 12, 24, 36, 48, 60, 72, 84, 96, 108, 120, ...

The multiples of 20 are 20, 40, 60, 80, 100, 120, ...

60 is the LCM as it is the smallest number in both lists.

Prime factors can be found from factor trees. A factor tree splits numbers into pairs of factors, with the branches ending with a prime number (circled).



Prime factors of 420 = 2 × 2 × 3 × 5 × 7

Prime factors of 126 = 2 × 3 × 3 × 7

Another method of finding prime factors is to keep dividing by prime numbers until the end result is 1:

$$\begin{array}{rl} & 1 \\ 7) & \overline{7} \\ 3) & \overline{21} \\ 5) & \overline{105} \\ 2) & \overline{210} \\ 2) & \overline{420} = 2 \times 2 \times 5 \times 3 \times 7 \end{array} \qquad \begin{array}{rl} & 1 \\ 3) & \overline{3} \\ 7) & \overline{21} \\ 3) & \overline{63} \\ 2) & \overline{126} = 2 \times 3 \times 7 \times 3 \end{array}$$

To find the HCF and LCM by using prime factors

Another way of finding the LCM and HCF is to split the numbers into prime factors.

HCF

List the prime factors, putting identical factors in line vertically.

Then select the factors that are common to BOTH lists.

WORKED EXAMPLE 3

$420 = 2 \times 2 \times 3 \quad \times 5 \times 7$

$126 = 2 \quad \times 3 \times 3 \quad \times 7$

$\text{HCF} = 2 \quad \times 3 \quad \times 7 = 42$

LCM

List the prime factors, putting identical factors in line vertically.

Then select the factors that appear in ONE LIST OR THE OTHER OR BOTH.

WORKED EXAMPLE 4

$420 = 2 \times 2 \times 3 \quad \times 5 \times 7$

$126 = 2 \quad \times 3 \times 3 \quad \times 7$

$\text{LCM} = 2 \times 2 \times 3 \times 3 \times 5 \times 7 = 1260$

SUMMARY QUESTIONS

1 Find the HCF of:

- **a** 60 and 84
- **b** 84 and 462
- **c** 60 and 462
- **d** 60, 84 and 462.

2 Find the LCM of:

- **a** 20 and 30
- **b** 20 and 18
- **c** 18 and 30
- **d** 18, 20 and 30.

3 Jane, Judy and John all love to play Carnival in Trinidad & Tobago, but they find it very expensive. Because of the cost, Jane plays every 2 years, Judy every 3 years and John every 5 years.

If they all played in 2009, in which year will they next play together?

ACTIVITY

Find **two** numbers with an HCF of 6 and an LCM of 180.

There are four different pairs to find!

KEY POINTS

1. **Prime factors** are found by repeatedly dividing a number by prime numbers until the end result is 1.
2. The HCF (Highest Common Factor) is the product of the factors that are common to BOTH lists.
3. The LCM (Lowest Common Multiple) is the product of the factors in one list or the other or both.

1.3

Operations with real numbers: natural numbers and decimals

LEARNING OUTCOMES

- Round off to a given number of decimal places
- Round off to a given number of significant figures
- Add and subtract natural numbers and decimals
- Multiply natural numbers and decimals
- Divide natural numbers and decimals

EXAM TIP

Estimating answers first helps you find mistakes in your calculations.

EXAM TIP

Significant figures are not the same as decimal places.

Rounding

The attendance at a cricket match was 3672.

Here is the newspaper headline.

The newspaper has rounded the attendance to the nearest thousand.

The attendance was between 3000 and 4000.

Halfway between is 3500, so the attendance was closer to 4000.

To round off, always look at the next digit to the right of the last digit of interest. If it is 5 or more, round up, otherwise round down.

Decimal places

Decimal places are the digits after the decimal point.

To round to a number of decimal places, the next decimal place tells us whether to round up or down.

Significant figures

All figures are significant except for leading zeros in a decimal.

WORKED EXAMPLE 1

4.3173 to 2 decimal places is 4.32. As the 3rd decimal place is 7, we round up.

0.004527 to 2 significant figures is 0.0045. The leading zeros (0.00) are not significant figures.

The 3rd significant figure is 2, so we round down.

Computation: Addition and subtraction

To add and subtract, it is important to align the digits according to their place value.

WORKED EXAMPLE 2

Two pipes have a length of 3.8 m and 1.46 m, respectively.

The total length is
3.8 m + 1.46 m

$$\begin{array}{rccc} & \text{U} & \text{t} & \text{h} \\ & 3\,. & 8 & \\ + & 1\,. & 4 & 6 \\ \hline & 5\,. & 2 & 6 \\ \hline & _1 & & \end{array}$$

Carry the ten tenths over as 1 unit

The difference in length is
3.8 m − 1.46 m

$$\begin{array}{rccc} & \text{U} & \text{t} & \text{h} \\ & 3\,. & {}^{7}\cancel{8} & {}^{1}0 \\ - & 1\,. & 4 & 6 \\ \hline & 2\,. & 3 & 4 \end{array}$$

Add a 0 to avoid mistakes

Create ten hundredths from one tenth

Computation: Multiplication

Natural numbers

WORKED EXAMPLE 3

472 × 23

$4\ 7\ 2$	
$\times\ 2\ 3$	
$1\ {}_{2}4\ 1\ 6$	Multiply 472 by 3
${}_{1}9\ 4\ 4\ 0$	Multiply 472 by 20, by inserting a zero and then multiplying 472 by 2
$1\ 0\ 8\ 5\ 6$	Add the two calculated lines together.

EXAM TIP

When multiplying, every digit of one number is multiplied by every digit of the other number.

Decimals

WORKED EXAMPLE 4

4.72 × 2.3

Multiply 472 × 23 as above. 472 × 23 = 10856

To position the decimal point:

The question 4.72 × 2.3 has a total of 3 places of decimals.

So you need 3 places of decimals in the answer: 10.856

Can you follow James' reasoning?

Computation: Division

Multiply both the divisor and dividend by 10 until the divisor is a natural number.

WORKED EXAMPLE 5

To divide 4.16 by 0.8: Multiply both by 10: 41.6 ÷ 8

$\begin{array}{r} 0 \\ 8\overline{)4^{4}1.6} \end{array}$	From the left: 4 ÷ 8 = 0 remainder 4
$\begin{array}{r} 05. \\ 8\overline{)4^{4}1.6} \end{array}$	41 ÷ 8 = 5 remainder 1
$\begin{array}{r} 05.2 \\ 8\overline{)4^{4}1.{}^{1}6} \end{array}$	16 ÷ 8 = 2

4.16 ÷ 0.8 = 5.2

ACTIVITY

Write down a 6-digit number by repeating a 3-digit number, e.g. 285285

Divide it by 7.

Divide the answer by 11.

Divide that answer by 13.

What do you notice?

KEY POINTS

1. You can round to decimal places or significant figures.
2. Use rounding to estimate.
3. To add or subtract, line up digits by place value.
4. To multiply decimals, position the decimal point in the answer by estimation or by adding the number of decimal places in the numbers being multiplied.
5. Never divide by a decimal. Multiply both numbers by 10 until the divisor is an integer.

SUMMARY QUESTIONS

1. Round 2.567 to:
   - **a** 2 decimal places
   - **b** 1 significant figure.
2. Multiply 2.6 by 1.8.
3. Divide 4.26 by 0.4.

1.4

Operations with real numbers: fractions

LEARNING OUTCOMES

- Understand how to find equivalent fractions
- Add and subtract fractions and mixed numbers
- Multiply a fraction by a whole number or a fraction
- Divide fractions and mixed numbers

Common fractions

Cancelling to lowest terms

A fraction is in its lowest terms when the numerator and denominator have no common factors.

Dividing both numerator and denominator by a common factor is called **cancelling**.

$$\frac{18}{24} = \frac{18 \div 6}{18 \div 6} = \frac{3}{4}$$

Equivalent fractions

Equivalent fractions are fractions which are the same part of a whole one.

Equivalent fractions can be found by multiplying or dividing the numerator and denominator by the same amount.

$$\frac{2}{5} = \frac{2 \times 3}{5 \times 3} = \frac{6}{15} = \frac{2 \times 5}{5 \times 5} = \frac{10}{25}$$

Adding and subtracting

Fractions can only be added or subtracted if they have a common denominator.

If the denominators are different, then use equivalent fractions to make the denominators equal.

Adding

$\frac{3}{5} + \frac{2}{3}$

$= \frac{9}{15} + \frac{10}{15}$

The denominator is 15 because it is the LCM of 3 and 5, the original denominators. $\frac{3}{5} = \frac{9}{15}$ (multiplying by 3), and $\frac{2}{3} = \frac{10}{15}$ (multiplying by 5).

$= \frac{19}{15} = 1\frac{4}{15}$

There are fifteen fifteenths in a whole one, so $\frac{19}{15}$ = a whole one and 4 more fifteenths.

Subtracting

$3\frac{1}{6} - 1\frac{3}{4}$

$= \frac{19}{6} - \frac{7}{4}$

When adding or subtracting, convert to mixed numbers. Just like adding, find a common denominator (12) and use equivalent fractions.

$= \frac{38}{12} - \frac{21}{12} = \frac{17}{12} = 1\frac{5}{12}$

Change back to a mixed number.

ACTIVITY

$\frac{1}{3}$ of the West Indian Cricket Squad for the 2010 tour of Sri Lanka were batsmen, $\frac{1}{5}$ were bowlers and the rest were all-rounders.

What fraction were all-rounders?

How many cricketers do you think there were altogether in the squad?

Multiplying and dividing

Any fractions can be multiplied or divided; there is no need to have a common denominator.

But mixed numbers must be changed into improper fractions.

To multiply a fraction by an integer, write the integer as a fraction with denominator of 1, e.g. write 3 as $\frac{3}{1}$.

$1\frac{2}{3} \times 2\frac{1}{4}$	When multiplying or dividing, any mixed numbers must be made into an improper fraction.
$= \frac{5}{3} \times \frac{9}{4}$	There are three thirds in a whole one, so $1\frac{2}{3}$ is $\frac{3}{3} + \frac{2}{3} = \frac{5}{3}$.
$= \frac{5}{1} \times \frac{3}{4}$	When **multiplying only**, you can cancel the numerator and denominator of different fractions. Divide the 9 and the 3 by 3.
$= \frac{15}{4} = 3\frac{3}{4}$	Multiply the numerators and multiply the denominators. Fifteen quarters make 3 whole ones and three more quarters.
$3\frac{1}{3} \div 1\frac{1}{4}$	Change to improper fractions.
$= \frac{10}{3} \div \frac{5}{4}$	When dividing, **invert the second fraction** (the divisor) and **multiply**.
$= \frac{10}{3} \times \frac{4}{5}$	Now you are **multiplying**, you can cancel the numerator and denominator of the different fractions.
$= \frac{2}{3} \times \frac{4}{1}$	In this case, cancel the 10 and 5 by dividing both by 5.
$= \frac{8}{3} = 2\frac{2}{3}$	

Always simplify your answers by writing them as mixed numbers where appropriate, and writing the fraction in its lowest terms.

EXAM TIP

- You do not need to find the LCM when multiplying or dividing, but you must when adding or subtracting.
- Learn how to work with fractions on your calculator.

ACTIVITY

Malcolm says that $8 \div \frac{1}{2} = 4$.
How could you convince him that he is wrong?

KEY POINTS

The chart below summarises how to calculate with fractions.

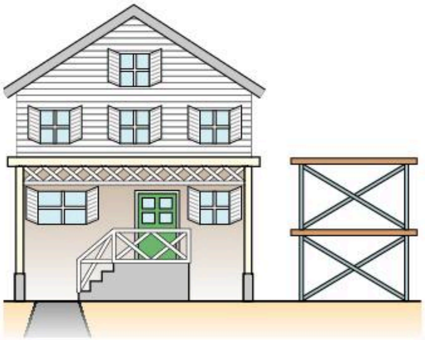
Addition	Subtraction	Multiplication	Division
Change any mixed numbers into improper fractions			
Find a common denominator (LCM)		Write any integers as a fraction with a denominator of 1	
Find equivalent fractions with LCM as denominator			Invert divisor
	If subtraction cannot be performed, break down a whole one and add to the first fraction	Cancel across fractions if possible	
Add or subtract the fractions		Multiply numerators, multiply denominators	
Cancel if possible			
Change to mixed number if possible			

SUMMARY QUESTIONS

1 Calculate:

a $1\frac{1}{2} - \frac{3}{4}$ **b** $2\frac{1}{4} \times 6$ **c** $3\frac{2}{3} + 1\frac{5}{6}$

2 A building has a height of $8\frac{1}{2}$ m.
How many scaffolding blocks of height $1\frac{1}{4}$ m will be needed to reach the top?



1.5 Real numbers

LEARNING OUTCOMES

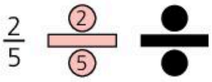
- Convert between fractions, decimals and percentages
- Express one quantity as a fraction or percentage of another
- Calculate a fraction or percentage of a quantity
- Calculate the whole from a fraction or percentage

Conversions between real numbers

Fractions and decimals

Fractions are a short way of writing a division.

This is why the division sign looks like a fraction.



$\frac{3}{8}$ means 3 ÷ 8.

To change a fraction to a decimal, divide the numerator by the denominator:

$$\begin{array}{r} 0.\,3\,7\,5 \\ 8\overline{)3.{}^{3}0{}^{6}0{}^{4}0} \end{array} \qquad \frac{3}{8} = 0.375$$

To change a decimal to a fraction, write the decimal part as a fraction with the place value of the last digit as denominator.

WORKED EXAMPLE 1

U t h
0. 8 5 has a last digit in the hundredths place,
so $0.85 = \frac{85}{100} = \frac{17}{20}$

Decimals and percentages

Decimals are part of a whole one, whereas percentages are part of 100.

Percentages are 100 times bigger than the equivalent decimal.

So, to change a decimal to a percentage, multiply by 100 (by moving the digits two places to the left).

$0.4 = 40\%$ $\qquad$ $0.575 = 57.5\%$

To change a percentage to a decimal, divide by 100 (by moving the digits two places to the right).

$80\% = 0.80 = 0.8$ $\qquad$ $24\% = 0.24$

EXAM TIP

- A fraction means division.
- A percentage can be written as a common fraction with a denominator of 100.
- Always give fractional answers in their lowest terms.

Fractions and percentages

To change a fraction to a percentage, first change it to a decimal (by dividing) and then to a percentage (by multiplying by 100).

WORKED EXAMPLE 2

$\frac{5}{9}$

$$\begin{array}{r} 0.\,5\,5\,5\,\ldots \\ 9\overline{)5.{}^{5}0{}^{5}0{}^{5}0} \end{array} = 55.555\ldots\% = 55.6\% \text{ (to 1 decimal place)}$$

To change a percentage to a fraction, write the percentage as a fraction of 100.

WORKED EXAMPLE 3

$48\% = \frac{48}{100} = \frac{12}{25}$ $\qquad$ $62.5\% = \frac{62.5}{100} = \frac{625}{1000} = \frac{125}{200} = \frac{25}{40} = \frac{5}{8}$

Expressing one quantity as a fraction or percentage of another

As well as meaning division, a fraction also means 'out of', so $\frac{2}{5}$ means 2 out of 5.

So $12 as a fraction of $30 is $\frac{12}{30} = \frac{2}{5}$

As a percentage, $\frac{12}{30} = \frac{2}{5} = 2 \div 5 = 0.4 = 40\%$

To calculate a fraction or a percentage of a quantity

Multiplication means 'lots of'. Three lots of $15 = 3 × $15 = $45.

When dealing with fractions, we just say 'two-thirds of $18' instead of 'two-thirds lots of $18'. In this sense, 'of' means multiply.

So to find $\frac{2}{3}$ of $18, calculate

$$\frac{2}{3} \times 18 = \frac{2}{3} \times \frac{18}{1} = \frac{2}{1} \times \frac{6}{1} = \frac{12}{1} = \$12$$

And 40% of 12 kg = 0.4 × 12 = 4.8 kg.

To calculate the whole from a fraction or percentage

Michael saves $\frac{2}{5}$ of his wages. He saves $64.

So $\frac{1}{5}$ of his wages is $64 ÷ 2 = $32. His wages are 5 × $32 = $160.

Marie pays a 5% deposit of $650 for a car.

So 1% = $650 ÷ 5 = $130. The car costs 100 × $130 = 13 000.

KEY POINTS

1 To change a fraction to a decimal, divide the numerator by the denominator.

2 To change a decimal to a fraction, divide the decimal part by the place value of the last figure, and simplify, e.g. $0.32 = \frac{32}{100} = \frac{8}{25}$

3 To change a decimal to a percentage, move the digits 2 places to the left to multiply by 100, e.g. 0.625 = 62.5%

4 To change a percentage to a decimal, move the digits two places to the right to divide by 100, e.g. 38% = 0.38.

5 To change a fraction to a percentage, change it to a decimal and then to a percentage.

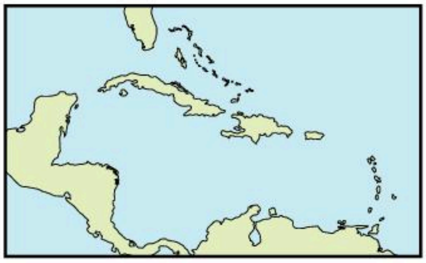
6 To change a percentage to a fraction, write the percentage as a fraction of 100 and simplify, e.g. $45\% = \frac{45}{100} = \frac{9}{25}$

ACTIVITY

The Caribbean has a population of about 40 million.

About 3 million of these live in Jamaica.

- What percentage of the Caribbean population lives in Jamaica?
- Research the population of some other Caribbean islands and find out what percentage of the total live on these islands.



Map of the Caribbean

SUMMARY QUESTIONS

1 Write

   a 30% as a fraction in its lowest terms

   b 0.45 as a percentage

   c 57.5% as a decimal and as a fraction in its lowest terms.

2 Which is greater, and by how much:
35% of $45 or $\frac{3}{8}$ of $40?

3 A school has 350 students.

56% of them are girls.
28 girls are left-handed.

What fraction of the girls are left-handed?

1.6

Money … more or less

LEARNING OUTCOMES

- Calculate discount, sales tax (VAT), profit and loss
- Express profit, loss, discount, sales tax as a percentage of some value
- Solve problems involving hire purchase, profit, loss and discount

Profit and loss

The cost price plus the profit gives the selling price.

The cost price minus the loss gives the selling price.

Profit and loss are expressed as **percentages of the cost price**.

If I buy a car for \$12 000 and sell it later for \$10 000, my loss is \$2000.

My percentage loss is

$$\frac{2000}{12\,000} = 0.16666 \ldots = 16.7\% \text{ (to 1 decimal place).}$$

WORKED EXAMPLE 1

Harry buys a chair for \$250 and wants to sell it to make a 15% profit.

His profit is 15% of \$250 = 0.15 × 250 = \$37.50, so he must sell it for \$250 + \$37.50 = \$287.50

Discount

Shops sometimes offer a discount; this is a percentage that they take off the price, for example in a sale.

WORKED EXAMPLE 2

Oliver sees some shoes in the sale. They normally cost \$70, but there is a 20% discount.

The discount is 20% of \$70, or 0.2 × \$70 = \$14, so the shoes cost \$70 − \$14 = \$56.

VAT

Many islands charge VAT, sales tax or consumption tax of around 15% on goods.

This is added on to the price that you pay for goods in the shop, so an item that might otherwise cost \$28 has 15% added on. 15% of \$28 = 0.15 × \$28 = \$4.20, so the item actually costs \$32.20.

Hire purchase

Sometimes you buy goods on hire purchase. This means paying some money to start with, called a deposit, followed by a number of monthly instalments. This usually costs more than buying it outright.

ACTIVITY

Find a hire purchase deal from a newspaper or catalogue.

Calculate the additional percentage you pay by using hire purchase.

WORKED EXAMPLE 3

Peter wanted to buy a motorbike for \$900. He paid a deposit of \$200 followed by 24 monthly instalments of \$35.

He paid \$200 + (24 × \$35) = \$1040

So he paid an extra \$140, or $\frac{140}{900} = 0.15555 \ldots$
$= 15.6\%$ (to 1 decimal place).

Reverse percentages

Percentage changes (profit, loss, discount, tax) are always based on the **original amount**.

WORKED EXAMPLE 4

Alison sees a handbag in a sale.

She wants to know how much it cost originally.

She cannot work it out by finding 25% of \$60, because \$60 is the sale price, and the 25% has already been taken off the original price.

She realises that, if 25% has been taken off, she has paid 75%, or 0.75.

So original price $\times$ 0.75 = \$60

Working back, \$60 $\div$ 0.75 = \$80

The original price was \$80.

Use this technique whenever you have to work back to the original price.

WORKED EXAMPLE 5

If a toy costs \$3.68 including VAT at 15%, you are paying 100% + 15% VAT = 115% or 1.15

Original price $\times$ 1.15 = price including VAT

So price including VAT $\div$ 1.15 = original price

\$3.68 $\div$ 1.15 = \$3.20

KEY POINTS

1. Profit, loss, discount and tax are all calculated as percentages.
2. Losses and discounts produce a final value lower than the original; profit and taxes result in a higher final value.
3. When working back to find the original amount, add or subtract the percentage to 100%, convert to a decimal and divide.

EXAM TIP

- All percentages are based on the original price.
- Hire purchase = deposit plus instalments

ACTIVITY

Melissa buys an article and aims to make a profit of 20%.

Unfortunately she cannot sell it, so she puts it in a sale with 20% off.

Explain to her why she makes a loss of 4%.

SUMMARY QUESTIONS

1. Jeffrey buys an article for \$60 and sells it for \$81. What is his percentage profit?
2. Chris buys an article for \$60. He then adds on 20% for his profit, and then has to add on 15% VAT. What is the selling price?
3. A freezer can be bought on hire purchase for a deposit of \$70 and 6 monthly payments of \$19.
   - **a** How much will it cost?
   - **b** This is 15% more than the original cash price. What is the original cash price?

1.7 Working with ratios

LEARNING OUTCOMES

- Compare two quantities using ratios
- Divide a quantity in a given ratio
- Convert from one set of units to another
- Convert using conversion tables, currency conversion

Ratio

A **ratio** is a way of comparing two or more quantities.

If a class has 12 boys and 18 girls, we say the ratio of boys to girls is 12 : 18 (we say this as '12 to 18').

Ratios can be simplified, like fractions. Dividing both quantities by 6

$$12 : 18 = 2 : 3.$$

This means that we can put the class into groups of 6; there would be 2 groups of boys and 3 groups of girls.

You can make a pineapple and orange smoothie for 4 people with these ingredients:

16 oz canned pineapple chunks
4 oz orange juice frozen concentrate
8 oz vanilla yogurt
4 oz water
4 ice cubes

The ratio of pineapple chunks : orange juice : vanilla yogurt is

$$16 : 4 : 8$$

or $4 : 1 : 2$

WORKED EXAMPLE 1

In a mixture for chocolate chip cookies, the ratio of chocolate chip to cookie batter is 2 : 5.

Angella has 400 g of chocolate chip. How much cookie batter will she need?

SOLUTION

400 g of chocolate chip is 2 parts of the mixture, so 1 part is 400 g ÷ 2 = 200 g.

So 5 parts of cookie batter = 5 × 200 g = 1000 g.
Both parts of the ratio are multiplied by 200.

Converting between units

To make conversions, we apply the principle of equivalent fractions.

WORKED EXAMPLE 2

8 km is approximately equal to 5 miles.

The ratio of km : miles is 8 : 5

So, to find the equivalent of 20 km in miles,

$$8 : 5 = 20 : x$$

As 20 ÷ 8 = 2.5, we multiply both numbers in the 8 : 5 ratio by 2.5.

$$8 : 5 = (8 \times 2.5) : (5 \times 2.5) = 20 : 12.5$$

So 20 km = 12.5 miles.

Alternatively, we could solve the equation

$\frac{8}{5} = \frac{20}{x}$ by using equivalent fractions.

$$\frac{8}{5} = \frac{8 \times 2.5}{5 \times 2.5} = \frac{20}{12.5}$$

WORKED EXAMPLE 3

One Barbadian dollar (BB \$1) is worth 1.34 Eastern Caribbean dollars (XC \$1.34).

The ratio of BB \$: XC \$ = 1 : 1.34.

So to find the value of XC \$200, we must solve

$$\frac{1}{1.34} = \frac{x}{200} \text{ using equivalent fractions.}$$

Multiplying both sides of the equation by 200:

$$\frac{200}{1.34} = \frac{200x}{200}$$

$$149.25373134 = x$$

So XC \$200 = BB \$149.25 (to 2 decimal places)

To divide a quantity in a ratio

WORKED EXAMPLE 4

In November 2010, Mavado had two songs in the Jamaican Top 10. Messiah was at number 2, while Stulla was at number 6. The sales of Messiah and Stulla were in the ratio 3 : 1.

The total sales of the two songs was 46 000.

The ratio 3 : 1 means that for every 3 sales of Messiah there was 1 sale of Stulla, that is, if put into equal piles there are 3 piles of Messiah and 1 pile of Stulla.

There are 4 equal piles altogether, so each pile contains 46 000 ÷ 4 = 11 500 CDs.

So the sales of Messiah were 3 × 11 500 = 34 500.

The sales of Stulla were 1 × 11 500 = 11 500.

SUMMARY QUESTIONS

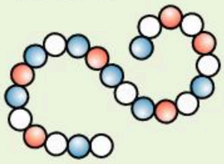
1 Divide \$45 in the ratio 2 : 3

2 Rob is mixing concrete using cement, sand and gravel in the ratio 1 : 2 : 3.
He uses 60 000 cm^3 of sand.
How much cement and gravel will he use?

3 One metre is approximately equal to 39 inches.
Andrew's height is 72 inches. What is his height in metres?

ACTIVITY

A necklace is made of red, blue and white beads.



The ratio of red beads to white beads is 2 : 3.

The ratio of red beads to blue beads is 3 : 4.

- What is the ratio of white beads to blue beads?
- Which is the most common colour of bead in the necklace?

EXAM TIP

- Give ratios in their simplest form.
- The order of quantities in a ratio is important.

KEY POINTS

1 Ratios do not tell you about the size of the quantities.

2 Ratio is about shares, or parts.

3 Ratio is useful when performing conversion calculations.

1.8

Standard form and indices

LEARNING OUTCOMES

- Find squares, cubes, square roots and cube roots
- Write a rational number in standard form
- Understand and use the laws of indices
- Calculate with numbers in standard form

Squares, cubes and roots

Multiplying a number by itself is called squaring the number.

7 squared is written as 7^2 and means $7 \times 7 = 49$.

The cube of a number is the result of multiplying three of the same number together.

4^3, or 4 cubed, means $4 \times 4 \times 4 = 64$.

The square root of a number is the inverse of the square.

The square root of 25 is written $\sqrt{25}$ and is equal to 5, as $5^2 = 25$.

$\sqrt{25}$ is also equal to -5, as $(-5)^2 = 25$.

Negative numbers do not have a real square root.

The cube root is the inverse of the cube.

$\sqrt[3]{512} = 8$, as $8^3 = 512$.

Your calculator has square, cube, square root and cube root keys.

Index notation

a^n means a is multiplied by itself n times.

$$4^5 = 4 \times 4 \times 4 \times 4 \times 4$$

The form a^n is called index notation. a is the base and n is the index, power or exponent.

The laws of indices

There are rules for combining powers of the same number.

$$4^5 \times 4^3 = (4 \times 4 \times 4 \times 4 \times 4) \times (4 \times 4 \times 4) = 4^8,$$

or generally $\boldsymbol{y^a \times y^b = y^{a+b}}$

$$7^6 \div 7^2 = \frac{7 \times 7 \times \cancel{7} \times \cancel{7} \times 7 \times 7}{\cancel{7} \times \cancel{7}} = 7^4,$$

or generally $\boldsymbol{y^a \div y^b = y^{a-b}}$

$(5^3)^4 = 5^3 \times 5^3 \times 5^3 \times 5^3 = 5^{12}$, or generally $\boldsymbol{(y^a)^b = y^{ab}}$

$y^4 \div y^4 = y^0$. But $y^4 \div y^4 = 1$, so $\boldsymbol{y^0 = 1}$

$y^3 \times y^{-3} = y^0 = 1$, so y^3 and y^{-3} are reciprocals.

This means that $y^{-3} = \frac{1}{y^3}$, or $\boldsymbol{y^{-a} = \frac{1}{y^a}}$

Standard form

When numbers get very large or very small, we often write them in standard form. Most calculators do this. Standard form is written as $A \times 10^n$, where $1 \leqslant A < 10$ and n is an integer.

WORKED EXAMPLE 1

The population of the Dominican Republic is 10 090 000.

Because $10\,090\,000 = 1.09 \times 10 \times 10 \times 10 \times 10 \times 10 \times 10 \times 10$, we write 10 090 000 as 1.09×10^7.

A grain of sand has a diameter of 0.007 cm.

$$\begin{aligned} 0.007 &= 7 \div (10 \times 10 \times 10) \\ &= 7 \div 10^3 = 7 \times 10^{-3}. \end{aligned}$$

The power of 10 tells how many places the digits have to move to return to their original positions:

3.2×10^5

$= 320\,000$

Digits move 5 places left

4.7×10^{-3}

$= 0.0047$

Digits move 3 places right

Calculating in standard form

To add or subtract numbers in standard form, the safest way is to change them out of standard form first.

WORKED EXAMPLE 2

$$\begin{aligned}(3.4 \times 10^5) + (6.7 \times 10^4) &= 340\,000 + 67\,000 \\ &= 407\,000 \\ &= 4.07 \times 10^5\end{aligned}$$

$$\begin{aligned}(4 \times 10^{-3}) - (9.2 \times 10^{-4}) &= 0.004 - 0.00092 \\ &= 0.00308 \\ &= 3.08 \times 10^{-3}\end{aligned}$$

To multiply or divide, the decimal parts and the powers of 10 can be dealt with separately.

ACTIVITY

Cuba has a population of 1.1×10^7, and an area of $1.1 \times 10^5\,\text{km}^2$. Barbados has a population of 2.8×10^5, and an area of $4.3 \times 10^2\,\text{km}^2$.

- In terms of area, how many times bigger than Barbados is Cuba?
- How many times the population of Barbados is the population of Cuba?
- Population density is the number of people for each km^2. Find the population density of Cuba and of Barbados.

SUMMARY QUESTIONS

1 Write:

a 0.00056 in standard form

b 1.4×10^5 as an ordinary number.

2 The Sun is approximately 1.5×10^8 kilometres from Earth. The speed of light is about 3×10^5 kilometres per second. How long does it take for light to get from the Sun to Earth?

3 Simplify:

a $4^6 \times 4^3$ **b** $3^{-2} \div 3^{-5}$ **c** $8^7 \times \frac{1}{8^5}$

EXAM TIP

When calculating in standard form, make sure the answer is still in standard form, i.e. the first part is between 1 and 9.

WORKED EXAMPLE 3

$$\begin{aligned}&(3.4 \times 10^6) \times (3 \times 10^3) \\ &= (3.4 \times 3) \times (10^6 \times 10^3) \\ &= 10.2 \times 10^9, \text{ or, in standard form,} \\ &= 1.02 \times 10^{10}\end{aligned}$$

WORKED EXAMPLE 4

$$\begin{aligned}&(5 \times 10^4) \div (2 \times 10^{-2}) \\ &= (5 \div 2) \times (10^4 \div 10^{-2}) \\ &= 2.5 \times 10^6\end{aligned}$$

KEY POINTS

1 Laws of indices:
- $y^a \times y^b = y^{a+b}$
- $y^a \div y^b = y^{a-b}$
- $(y^a)^b = y^{ab}$
- $y^0 = 1$
- $y^{-n} = \frac{1}{y^n}$

2 Standard form is written as $A \times 10^n$, where $1 \leqslant A < 10$ and n is an integer.

1.9 Ordering, patterns and sequences

LEARNING OUTCOMES

- Order a set of real numbers
- Generate a term of a sequence given a rule
- Derive a rule given the terms of a sequence

Ordering decimals

To order decimals, write them vertically, aligning place values.

WORKED EXAMPLE 1

Three parcels weigh 2.4 kg, 1.69 kg and 1.645 kg

To put them in order of weight:

U		**t**	**h**	**th**
2	.	4		
1	.	6	9	
1	.	6	4	5

2.4 is the largest, as it has the most units.

1.69 is larger than 1.645 as they have the same number of units and tenths, but 1.69 has more hundredths.
From largest to smallest: 2.4, 1.69, 1.645

Ordering fractions

Fractions can be put in order by changing them to equivalent fractions with a common denominator.

WORKED EXAMPLE 2

To order $\frac{3}{5}$, $\frac{3}{4}$, $\frac{5}{8}$, $\frac{7}{10}$:

A common denominator for 5, 4, 8 and 10 is 40.

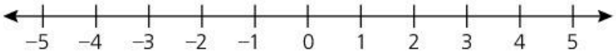
$\frac{3}{5} = \frac{24}{40}$, $\frac{3}{4} = \frac{30}{40}$, $\frac{5}{8} = \frac{25}{40}$, $\frac{7}{10} = \frac{28}{40}$,

so the correct order (smallest first) is $\frac{3}{5}$, $\frac{5}{8}$, $\frac{7}{10}$, $\frac{3}{4}$

Ordering integers

To order integers, draw or imagine a number line. The numbers increase in size from left to right.

To order 4, −3, 1 and −5:



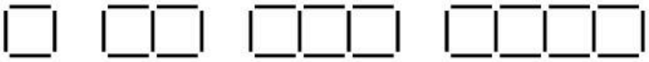
−5, −3, 1, 4 is the correct order.

EXAM TIP

To order integers, think about temperature: the colder it is, the lower the temperature.
−6 °C is colder than −5 °C, so −6 is lower than −5.

Generating a term of a sequence given a rule

Here is a sequence of squares made from line segments:



The number of line segments in each pattern is 4, 7, 10, 13.

We could describe this sequence as: start with 4 and then keep adding 3.

So the next term in the sequence would be 16.

This table shows the number of squares and the number of line segments.

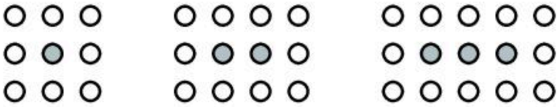
Number of squares, s	1	2	3	4
Number of line segments, l	4	7	10	13

The rule connecting s and l is $l = 3s + 1$. For example, when $s = 4$, $l = 3 \times 4 + 1 = 13$.

So for 10 squares ($s = 10$) we would need $3 \times 10 + 1 = 31$ line segments.

Deriving a rule given the terms in a sequence (the n^{th} term)

This pattern is made of grey and white circles.



The number of white circles is given in the table:

Pattern number, *n*	1	2	3
Number of white circles, *w*	8	10	12

To find the rule connecting n and w:

w is increasing by 2. Because of this, insert the 2 times table in between the rows of the table:

Pattern number, *n*	1	2	3	
2× table	2	4	6	×2
Number of white circles, *w*	8	10	12	+6

If we double n, we get the 2 times table. If we add 6 to the 2 times table, we get w.

So the rule is $w = 2n + 6$

We say the nth term of the sequence 8, 10, 12, … is $2n + 6$.

We can use the rule to work out that the 10th pattern would have $2 \times 10 + 6 = 26$ white circles.

ACTIVITY

Find the next number in this pattern:

1, 2, 5, 14, 41, 122, …

ACTIVITY

Michael sowed a bean in a pot.

One day he planted the bean outside. On the same day he sowed another bean in a pot.

Ten days later the bean plant in the pot was just 1 cm tall. The plant outside was already 38 cm tall.

Each evening Michael measured his two plants.

On the evening of the next day the little bean plant had grown another 2 cm so it was 3 cm tall. Each day it continued to grow double the amount it had grown the day before.

The outside plant grew at a steady 5 cm a day.

After how many days were the two plants the same height when Michael measured them in the evening? How tall were they?

KEY POINTS

1 Use a common denominator to order fractions.

2 Use a number line to order integers.

3 Many geometrical patterns can be described using numbers.

4 A rule for a sequence can be found when all members of a set fit into a pattern. The rule must work for all numbers.

5 The rule for a sequence can be written as $y = an + b$, where n is the position in the sequence, a is the increase between consecutive terms in the sequence. b can be found by solving an equation using a given term in the sequence.

SUMMARY QUESTIONS

1 Put these numbers in order of size:

- **a** 4.2, 4.08, 3.999, 4.19
- **b** $\frac{1}{3}, \frac{2}{9}, \frac{5}{12}, \frac{7}{18}$
- **c** −4, 7, −2, −5, 8

2 Write down the first five terms in the sequence where the nth term is $2n - 1$.

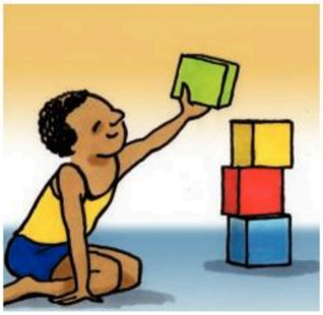
3 Find the rule for the nth term in these sequences:

- **a** 3, 7, 11, 15, 19, …
- **b** 7, 9, 11, 13, 15, …
- **c** 18, 15, 12, 9, 6, …

1.10 Properties of numbers and operations

LEARNING OUTCOMES

- Use properties of numbers and operations in computational tasks
- Understand identities and inverses
- Know and apply the order of operations to calculations
- Understand the commutative, distributive and associative rules



WORKED EXAMPLE 1

Brackets $6 \times (3 + 4) - 27 \div 3^2$

Indices

$= 6 \times 7 - 27 \div 3^2$

Dividing and **M**ultiplying

$= 6 \times 7 - 27 \div 9$

Adding and **S**ubtracting

$= 42 - 3$

$= 39$

Identities

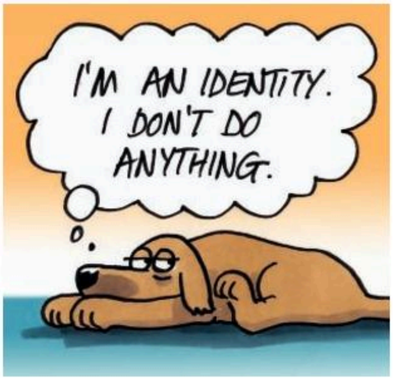
Identities are numbers which have no effect when a particular operation is used.

For multiplication and division, the identity is 1:

$7 \times 1 = 7; 6 \div 1 = 6$

For addition and subtraction, the identity is 0:

$5 + 0 = 5; 8 - 0 = 8$



Inverses

Inverses have an undoing effect.

- Addition and subtraction are **inverse operations**; $4 + 5 = 9$, so $9 - 5 = 4$.
- Multiplication and division are inverse operations; $4 \times 5 = 20$, so $20 \div 5 = 4$.

The **additive inverse** of a number is one which adds to it to equal the identity (0).

The additive inverse of 4 is -4, because $4 + (-4) = 0$.

The additive inverse is found by changing the number to its opposite sign.

The **multiplicative inverse** of a number is one which it is multiplied by to equal the identity (1).

The multiplicative inverse of 4 is $\frac{1}{4}$, because $4 \times \frac{1}{4} = 1$.

The multiplicative inverse is found by writing the number as an improper fraction and then inverting it.

The multiplicative inverse is also called the **reciprocal**.

The order of operations

The calculation of $2 + 3 \times 4$ gives an answer of 20 if you add $2 + 3$ and then multiply the answer by 4, or an answer of 14 if you multiply 3×4 and then add the answer to 2.

To avoid confusion, we **multiply** or **divide** before we **add** or **subtract**. To change this order, use **brackets**.

There are four stages to the order of operations, which are sometimes remembered as **BIDMAS** (see Worked Example 1).

The associative, commutative and distributive laws

- An operation, $*$, is **associative** if $(a * b) * c = a * (b * c)$.

WORKED EXAMPLE 2

Addition and multiplication are associative:

$$(4 + 3) + 2 = 4 + (3 + 2) = 9$$

$$(4 \times 3) \times 2 = 4 \times (3 \times 2) = 24$$

Subtraction and division are **not** associative:

$$(6 - 3) - 2 = 1;\ 6 - (3 - 2) = 5$$

$$(12 \div 4) \div 2 = 1.5;\ 12 \div (4 \div 2) = 6$$

- An operation, $*$, is **distributive** over another operation, $\triangle$, if $a * (b \triangle c) = (a * b) \triangle (a * c)$.

WORKED EXAMPLE 4

Multiplication is distributive over addition and subtraction:

$4 \times (5 + 2) = (4 \times 5) + (4 \times 2) = 28$

$6 \times (5 - 3) = (6 \times 5) - (6 \times 3) = 12$

Uses of these rules

Mental calculations can be made easier by using these rules.

Examples include:

Distributive rule:

$$\begin{aligned}(8 \times 13) + (8 \times 17) &= 8 \times (13 + 17)\\ &= 8 \times 30\\ &= 240\end{aligned}$$

Commutative and associative rules:

$$\begin{aligned}15 \times 9 \times 4 &= 15 \times 4 \times 9\\ &= 60 \times 9\\ &= 540\end{aligned}$$

- An operation, $*$, is **commutative** if $a * b = b * a$

WORKED EXAMPLE 3

Addition and multiplication are commutative:

$$4 + 3 = 3 + 4 = 7$$

$$4 \times 3 = 3 \times 4 = 12$$

Subtraction and division are **not** commutative:

$$6 - 3 = 3;\ 3 - 6 = -3$$

$$4 \div 2 = 2;\ 2 \div 4 = 0.5$$

ACTIVITY

Explain which laws are being used in each line here:

$$\begin{aligned}&34 \times 26\\ &= 34 \times (20 + 6)\\ &= (34 \times 20) + (34 \times 6)\\ &= (20 \times 34) + (6 \times 34)\\ &= 20 \times (30 + 4) + 6 \times (30 + 4)\\ &= 20 \times 30 + 20 \times 4 + 6 \times 30 + 6 \times 4\\ &= 600 + 80 + 180 + 24\\ &= 884\end{aligned}$$

KEY POINTS

1 The additive identity $= 0$, the multiplicative identity $= 1$.

2 The additive inverse of a is $-a$.

3 The multiplicative inverse (reciprocal) of $\frac{a}{b}$ is $\frac{b}{a}$.

4 The associative law: $(a * b) * c = a * (b * c)$

5 The commutative law: $a * b = b * a$

6 The distributive law: $a * (b \triangle c) = (a * b) \triangle (a * c)$

SUMMARY QUESTIONS

1 Calculate $12 - 2 \times (3 + 6 \div 2)$

2 Write down

a the multiplicative inverse and **b** the additive inverse of $\frac{3}{5}$.

3 Calculate mentally:

a $5 \times 19 \times 4$

b $(27 \times 18) - (27 \times 8)$

EXAM TIP

- Don't forget BIDMAS.
- When you cannot use a calculator, these properties and laws can provide a shortcut.

1.11 Bases

LEARNING OUTCOMES

- State the value of a digit in a numeral given its base
- Solve problems involving concepts in number theory

WORKED EXAMPLE 1

Creatures on the planet Smoosh have four fingers on each hand, so they count in eights.

Their place values are:

8^3	8^2	8^1	8^0
Five hundred and twelves	Sixty-fours	Eights	Units

A Smooshian wrote, 'I am 123 years old today'.

He is not one hundred and twenty-three. He is

(1 × 64) + (2 × 8) + (3 × 1) = eighty-three.

Our number system

Historically, we have always counted on our fingers. This is why our number system is based on 10.

Ten units make a Ten, ten Tens make a Hundred, ten Hundreds make a Thousand and so on.

So we could write our place values as powers of 10:

Thousands	Hundreds	Tens	Units
10^3	10^2	10^1	10^0

Other bases

Sometimes we need to work in other bases. Computers work in base 2, or binary. Feet and inches work in base 12.

To find the value of a digit in a numeral given its base

Write down the place value using index notation.

For example, the value of the 3 in 2312 (base 4):

4^3	4^2	4^1	4^0
2	3	1	2

The 3 is worth $3 \times 4^2 = 3 \times 16 = 48$.

To find the value of a numeral given its base

The value of 2312 (base 4) is:

$$\begin{aligned} &2 \times 4^3 + 3 \times 4^2 + 1 \times 4^1 + 2 \times 4^0 \\ &= 2 \times 64 + 3 \times 16 + 1 \times 4 + 2 \times 1 = 182 \end{aligned}$$

To change a base 10 number to a different base

WORKED EXAMPLE 2

Space-Captain Williams' space ship travels at 1010 km/h. He wants to write this down for the Smooshians, but he must write it in base 8.

He knows the place values in base 8 are:

8^3	8^2	8^1	8^0
Five hundred and twelves	Sixty-fours	Eights	Units

He starts with 1010

He subtracts $\underline{512}$ 1 × 512

498

He subtracts $\underline{448}$ because 448 is 7 × 64
50
He subtracts $\underline{48}$ because 48 is 6 × 8
Which leaves 2 or 2 × 1.

So 1010 = 1 × 512 + 7 × 64 + 6 × 8 + 2 × 1, or 1762 (base 8)

Space-Admiral Jones did it by dividing by 8 repeatedly:

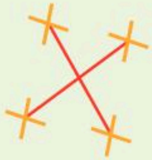
$$\begin{array}{r l} 1 & \text{remainder 7 sixty-fours} \\ 8\overline{)\ \ 15} & \text{remainder 6 eights} \\ 8\overline{)\ 126} & \text{remainder 2 units} \\ 8\overline{)1010} & \end{array}$$

He got the same result of 1762! Can you see it? Which method do you think is easier?

ACTIVITY

A New Year Cracker bursts into a star with 4 arms.

Each arm then bursts into 4 fresh arms.



Each of these then bursts into 4, and so on.

Write down the number of arms at each of the first **five** stages.

Adding and subtracting in other bases

Adding and subtracting work the same in any base. Just remember that the base indicates how many in one place equal one in the next place to the left.

WORKED EXAMPLE 3

$$\begin{array}{r l} 45 & \text{(base 8)} \\ +\ \underline{36} & \text{(base 8)} \\ \underline{3} & \text{(base 8)} \\ {\scriptstyle 1} & \end{array}$$

Step 1: 5 + 6 = eleven, which is 1 eight and 3 units, written 13. The 1 is carried.

$$\begin{array}{r l} 45 & \text{(base 8)} \\ +\ \underline{36} & \text{(base 8)} \\ \underline{103} & \text{(base 8)} \\ {\scriptstyle 1}\ \ & \end{array}$$

Step 2: 4 + 3 + (carried) 1 = eight, which is 1 eight and 0 units, written 10.

WORKED EXAMPLE 4

$$\begin{array}{r l} 1101 & \text{(base 2)} \\ -\ \underline{111} & \text{(base 2)} \\ \underline{0} & \end{array}$$

Step 1: 1 − 1 = 0. Then 0 − 1 cannot be done (without negative numbers) so we borrow from the 1 which is worth 2 in the next column.

$$\begin{array}{r l} \overset{0}{\cancel{1}}\overset{2}{1}01 & \text{(base 2)} \\ -\ \underline{111} & \text{(base 2)} \\ \underline{10} & \end{array}$$

Step 2: 2 − 1 = 1. Then 0 − 1 cannot be done (without negative numbers) so we borrow from the 1 which is worth 2 in the next column.

$$\begin{array}{r l} \overset{0}{\cancel{1}}\overset{22}{\cancel{1}}01 & \text{(base 2)} \\ -\ \underline{111} & \text{(base 2)} \\ \underline{110} & \end{array}$$

Step 3: 2 − 1 = 1. Then 0 − 0 = 0.

EXAM TIP

Express the place values in index notation to make the calculations easier.

KEY POINTS

1 The place value of the 1st digit (on the right) is always 1.
2 Place values on the left or right of a digit are found by multiplying (left) or dividing (right) the place value by the base.

SUMMARY QUESTIONS

1 What is the value of the digit 3 in 3210 (base 4)?
2 Write 321 (base 10) in base 8.
3 Find the (base 10) value of 1011 (base 2).
4 Calculate:
  a 101 + 11 (base 2)
  b 210 − 123 (base 4)

1.12

Interest, appreciation and depreciation

LEARNING OUTCOMES

- Understanding and calculating simple interest and compound interest
- Appreciation and depreciation

WORKED EXAMPLE 1

If you invest \$2000 for 3 years and receive \$180 interest, you can use the formula to calculate the rate of interest.

$P = 2000$, $I = 180$ and $T = 3$:

$$I = \frac{PRT}{100}$$

$$180 = \frac{2000 \times R \times 3}{100}$$

$$180 = 60R$$

$$R = 3\%$$

Simple interest

When you invest money in a bank, they usually pay you **interest**; that is, they give you some extra money. They pay you a percentage of what you invest. This percentage is called the **rate** of interest.

The amount you invest is called the **principal**.

The amount of interest that you receive, I, is given by the formula $I = \frac{PRT}{100}$, where P is the principal, R is the rate of interest per year, and T is the time in years that you invest the money.

Compound interest

Most banks pay **compound interest**. You also pay compound interest on loans, including credit cards. With compound interest, the interest is added to the principal at the end of the year.

WORKED EXAMPLE 2

If you borrow \$2000 at 3% per annum (per year), at the end of the first year you owe interest of 3% of $\$2000 = 0.03 \times 2000 = \$60$. You now owe a total of \$2060.

In the second year, you owe an extra 3% of $\$2060 = 0.03 \times 2060 = \$61.80$, so you then have a debt of \$2121.80.

In the third year, you are charged 3% of $\$2121.80 = \$63.65$, giving a total of \$2185.45, or interest of \$185.45.

A quick way to calculate compound interest is to use the method of adding a percentage that was introduced in 1.6.

WORKED EXAMPLE 3

If we add 3% to an amount, we have a total of 103% or, as a decimal, 1.03.

So $\$2000 \times 1.03$ would give the total after the first year's interest is added.

Three years' compound interest could be calculated as $\$2000 \times 1.03 \times 1.03 \times 1.03 = \$2000 \times 1.03^3$

A general formula is final amount, $F, = P \times \left(1 + \frac{R}{100}\right)^T$, where P = principal, R = rate of interest and T = time in years.

Appreciation and depreciation

When an item gains in value, we say it **appreciates**. The increase in value is called **appreciation**. Jewellery and antiques are examples of items that appreciate.

Many items become worth less every year – for example, a car. This is called **depreciation**.

WORKED EXAMPLE 4

A necklace cost \$200 in 2006. Every year it appreciates by 5%.

By 2010 it was worth $\$200 \times 1.05^4 = \$243.10$

1.05 represents a 5% increase (105%), and the power of 4 is for 4 years.

WORKED EXAMPLE 5

A car costs \$10 000 in the year 2010. It depreciates by 12% every year.

Its value in 2015 will be $\$10\,000 \times 0.88^5 = \$5277$.

0.88 represents a 12% decrease (100% − 12%), and the power of 5 is for 5 years.

ACTIVITY

Marsha buys two rings. They cost \$400 each.

The gold one appreciates by 4% a year, the silver one depreciates by 4% a year.

Which will happen first: the gold ring doubling its value or the silver ring halving its value?

EXAM TIP

- You must learn the formulae for simple and compound interest.
- Do not confuse the two.
- In the formulae, time is measured in years. Remember that 6 months is 0.5 years, not 0.6

KEY POINTS

1 Simple interest: $I = \frac{PRT}{100}$, and

2 Compound interest: $F = P \times \left(1 + \frac{R}{100}\right)^T$, where I = interest, P = principal, R = rate of interest, F = final amount and T = time in years.

- Appreciation and depreciation work in the same way as compound interest.

SUMMARY QUESTIONS

1 I can invest \$2000 for 5 years at 3.2% simple interest or 3% compound interest. Which is better and by how much?

2 Mary's beautiful antique vase cost her \$4000 in 2009. If it appreciates by 7% every year, how much is it worth in 2012?

3 I invest \$350 at 4% simple interest. How long will it take to earn \$100 interest?

1.13 Measures

LEARNING OUTCOMES

- Problems involving measures and money, including exchange rate
- The 12- and 24-hour clock

The metric system

All measures in the metric system use the same system.

Length is based on the **metre**, **capacity** on the **litre** and **mass** on the **gram**.

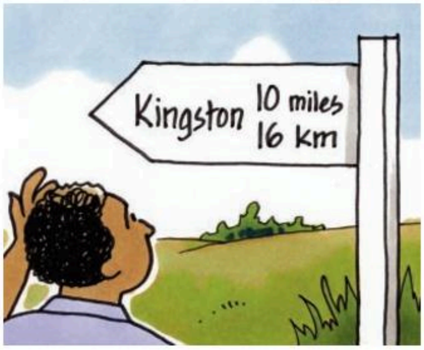
Common parts are thousandths (milli, m) and hundredths (centi, c), and the commonly used multiple is kilo, k (1000). So:

1000 mm = 1 m
100 cm = 1 m
10 mm = 1 cm
1000 m = 1 km

1000 mℓ = 1 ℓ
100 cℓ = 1 ℓ
10 mℓ = 1 cℓ
1000 ℓ = 1 kℓ
1000 kg = 1 tonne

1000 mg = 1 g
100 cg = 1 g
10 mg = 1 cg
1000 g = 1 kg

The imperial system



The imperial system of length

12 inches = 1 foot
3 feet = 1 yard
1760 yards = 1 mile

Metric and imperial comparisons

2.5 cm ≈ 1 inch
30 cm ≈ 1 foot
1 m ≈ 1 yard
8 km ≈ 5 miles

The imperial system of mass

16 ounces (oz) = 1 pound (lb)
14 lb = 1 stone
8 stone = 1 hundredweight (cwt)
20 cwt = 1 ton

Metric and imperial comparisons

30 g ≈ 1 oz
0.5 kg ≈ 1 lb
6 kg ≈ 1 stone
50 kg ≈ 1 cwt
1 tonne ≈ 1 ton

The imperial system of capacity

16 fluid ounces (fl. oz) = 1 pint
8 pints = 1 gallon

Metric and imperial comparisons

30 mℓ ≈ 1 fl. oz
0.5 ℓ ≈ 1 pint
4 litres ≈ 1 gallon

Problems involving conversion between units

Converting between different units involves ratios.

For example, to find the approximate metric equivalent of 12.5 stone:

WORKED EXAMPLE 1

The ratio of kg : stone is approximately 6 : 1

So

12.5 stone ≈ 6 × 12.5 = 75 kg

ACTIVITY

On a particular day, the exchange rate for Eastern Caribbean$1 was US$0.37, and €0.28.

Using this information, how many US$ was €1 worth?

Time

There are two ways of writing the time; the 12-hour system and the 24-hour clock.

- In the 12-hour clock, times between midnight and midday are followed by a.m. e.g. 7:45 a.m., and times between midday and midnight are followed by p.m.
- In the 24-hour clock, times are written as 4-digit numbers, with the first two digits indicating the hours and the last two indicating the minutes.
    - ◦ Midnight is written as 0000 and midday as 1200.
    - ◦ After midday, the hour does not reset to 0, but continues, so 1:00 p.m. is written as 1300.

So 2:30 in the afternoon is written as 2:30 p.m. (12-hour clock) or 1430 (24-hour clock).

A quarter to 8 in the morning is 7:45 a.m. (12-hour clock) or 0745 (24-hour clock).

When working with time, remember it is not metric. There are 60 minutes in an hour.

WORKED EXAMPLE 2

To find the length of a bus journey that starts at 0952 and ends at 1025:

From 0952 to 1000 is 8 minutes.

From 1000 to 1025 is 25 minutes.

Total time = 8 + 25 = 33 minutes.

EXAM TIP

- Treat currency and measure conversions as ratios.
- Remember time is not metric. 4 h 30 minutes = 4.5 hours, not 4.3 hours.
- Learn the key metric and imperial equivalents.

KEY POINTS

1 Milli = $\frac{1}{1000}$, centi = $\frac{1}{100}$, kilo = 1000.

2 a.m. = morning, p.m. = afternoon.

SUMMARY QUESTIONS

1 Write

a 4270 m in km **b** 33 cℓ in mℓ **c** 2.4 kg in g

2 Approximately how many miles is equivalent to 20 km?

3 **a** Write **i** 10:55 a.m. and **ii** 1:13 p.m. in the 24-hour clock.

b How many minutes is it from 10:55 a.m. to 1:13 p.m.?

1.14 Earning and spending money

LEARNING OUTCOMES

- Solve problems involving salaries and wages
- Solve problems involving taxes and utilities

Salaries and wages

Some people are paid a **weekly wage**; others are paid an **annual salary**.

Wages

People on a weekly wage usually have an **hourly rate of pay**, and a number of hours a week they have to work, the **basic week**.

Any extra hours worked are called **overtime**, and are usually paid at a higher rate.

For example:

WORKED EXAMPLE 1

Melissa earns \$9 per hour for a basic 35-hour week. Overtime is paid at time and a half.

Last week she worked a total of 41 hours.

Her weekly wage is calculated as follows:

Basic week: 35 hours @ \$9 per hour = \$315.

Overtime: $41 - 35 = 6$ hours. Rate = $\$9 \times 1.5 = \$13.50$ (time and a half means getting paid 1.5 times the usual rate).

Overtime = 6 hours @ \$13.50 = \$81.

Total weekly wage = $\$315 + \$81 = \$396$.

Salaries

People who earn a salary get a fixed amount of money for a year's work. The money is paid monthly, so the annual salary is divided by 12 and that amount is paid every month.

People on salaries cannot earn overtime, but they sometimes get a **bonus** at the end of the year.

WORKED EXAMPLE 2

Peter is the managing director of a company. He has an annual salary of \$75 000.

He receives an annual bonus of 1% of the company's profits.

Last year the company made a profit of \$1 240 000.

His monthly salary is $\$75\,000 \div 12 = \$6250$.

His bonus is 1% of \$1 240 000 = \$12 400, making his total earnings for the year \$87 400.

Taxes

People pay taxes so that the government can pay for services such as the police and hospitals.
The taxes vary in the Caribbean.

Income tax is tax paid on money you earn.

In Jamaica, people pay income tax at 25% on annual earnings over J $441 168 (US $5200), whereas there is no income tax in the Bahamas. In Barbados, income tax is charged at 20% on all income up to B $24 200 (US $12 100), and 35% on income above B $424 200.

Value Added Tax or **Consumption Tax** is tax you pay on most items you buy.

In Trinidad & Tobago, Value Added Tax (VAT) is 15%, in Jamaica it is called general consumption tax and is charged at 17.5%, but there is no VAT in the Virgin Islands.

There are many other taxes, which vary from island to island.

Utilities

Many utilities (for example, electricity, water and telephone) make a **fixed charge** plus a **cost per unit**.

For example, a telephone company might have a fixed charge of $40 per quarter (3 months) plus a cost of $0.08 per minute of call time.

A person making 342 minutes of calls in a quarter would pay

$$\$40 + (342 \times \$0.08) = \$67.36.$$

KEY POINTS

1. Salaries are increased by bonuses.
2. Salaries and wages are reduced by deductions for insurance and tax.
3. Wages are dependent on the hours worked, the hourly rate, and overtime.

SUMMARY QUESTIONS

Allison works a 35-hour week at $8.50 per hour, with overtime paid at time and a half.

She works 40 hours every week.

She pays income tax at the rate of 20%.

1. How much does she earn a year before tax?
2. How much does she earn a year after tax?
3. She makes radios that sell at $45 plus 15% VAT. How much do they cost including VAT?

ACTIVITY

- Draw a line graph to show the amount of income tax paid in Jamaica on earnings up to US $30 000.
- On the same axes, show the amount of income tax paid in Barbados on earnings up to US $30 000.
- On which salary (in US$) is the amount of tax paid the same in Jamaica and Barbados?

ACTIVITY

Paul and Christopher can choose between two different electricity tariffs.

They can either choose tariff A which has a fixed charge of $40 per quarter plus $0.36 per unit of electricity used, or tariff B which has a fixed charge of $20 per quarter plus $0.40 per unit of electricity used.

Paul uses 420 units of electricity per quarter, and Christopher uses 540 units.

- Which tariff should Paul choose? Which tariff should Christopher choose?
- Set up an equation and solve it to find the number of units that gives the same bill under each tariff.

EXAM TIP

- Make sure you make all necessary tax deductions from salaries.
- Do not mix units (e.g. dollars and cents) in calculations.

1.15 Sets

LEARNING OUTCOMES

- Use the concepts of sets to describe or list members
- Use and interpret set notation
- Represent a set in various forms
- Recognise and interpret Venn diagrams
- Apply the principles of membership of a set, cardinality, finite, infinite, universal, equal, equivalent sets

A **set** is a collection of objects, usually having something in common.

Examples might be the boys in a class, or the factors of 12, or the vowels in the alphabet.

The members of a set are called the **elements** of the set.

Set notation

A set is usually denoted by a capital letter, and the membership is either described or the elements are listed. The description or listing is written inside curly brackets.

So we might write $A = \{\text{even numbers less than } 12\}$, or $A = \{2, 4, 6, 8, 10\}$.

The symbol $\in$ means '**is an element of**'. We could write $4 \in A$.

$\notin$ means 'is not an element of'. We could say that $5 \notin A$.

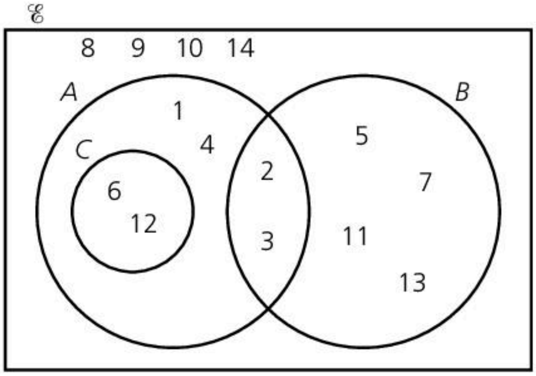
Venn diagrams

A **Venn diagram** is a diagrammatic representation of sets. A Venn diagram consists of a rectangle to represent the **universal set**, written as $\mathscr{E}$, the set containing all elements that are relevant to the topic.

Other sets are shown as circles.

The Venn diagram below is an example to demonstrate the principles of sets.

$\mathscr{E} = \{\text{natural numbers less than or equal to } 14\}$,
$A = \{\text{factors of } 12\}$, $B = \{\text{prime numbers}\}$ and $C = \{\text{multiples of } 6\}$.



Every element in set C is also in set A. We say C is a **subset** of A, or $C \subset A$. Both of the elements of C, multiples of 6, are also elements of A, factors of 12.

The vocabulary of sets

Set A has 6 elements. We say the **cardinality** of set A is 6 and write $n(A) = 6$.

The cardinality of B is also 6, and we say that A and B are **equivalent** sets as they both contain the same number of elements. We can write this as $A \equiv B$.

- **Equal** sets are sets that contain exactly the same elements. For example, {the letters in the word 'these'} = {the letters in the word 'sheet'} = {e, h, s, t}.
- An **empty** set, or **null** set, is one that contains no elements. For example {odd numbers that are multiples of 4} or {negative square numbers}. We use the symbol ∅ to represent a null set. The null set is a subset of every set.
- A **finite set** has a finite or countable number of elements.
- An **infinite set** has an infinite number of elements. It is impossible to count or list all the members of an infinite set since there is no end to the membership.
  For example {natural numbers} = {1, 2, 3, 4, 5, 6, …}

The set D = {letters in the word 'parallelogram'} is D = {p, a, r, l, e, o, g, m}, or D = {a, e, g, l, m, o, p, r}, as order is unimportant.

Sets do not contain the same element more than once.

EXAM TIP

- It is important to learn the vocabulary of sets.

ACTIVITY

A = {letters in 'BANANA'}
B = {letters in 'TANGERINE'}
C = {letters in 'CARIBBEAN'}
D = {letters in 'TURTLE'}
E = {letters in 'GRANITE'}

- Which two sets are equal?
- Which other set is equivalent to them?
- Which set is a subset of another set?
- Which two sets have no common elements?

KEY POINTS

1. The members of a set are called elements.
2. The number of elements in a set is the cardinality.
3. Sets can be finite or infinite.
4. The universal set, $\mathscr{E}$, contains all the elements being considered, the empty or null set, ∅ contains no elements.
5. Equal sets contain the same elements; equivalent sets have the same cardinality.

SUMMARY QUESTIONS

$\mathscr{E}$ = {natural numbers no greater than 25}
A = {1, 4, 9, 16, 25}
B = {3, 6, 9, 12, 15, 18, 21, 24}
C = {4, 8, 12, 16, 20, 24}
D = {5, 10, 15, 20, 25}

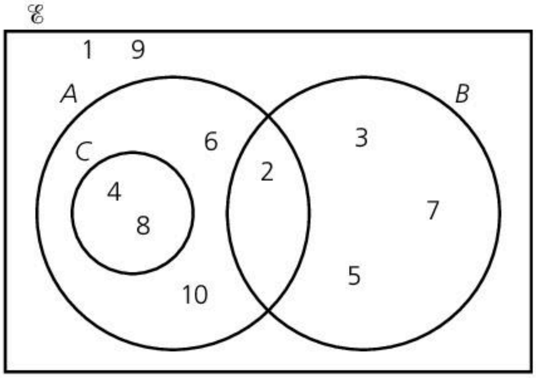
1. Describe sets A, B, C and D in words.
2. Which two sets are equivalent?
3. What can you say about E = {even numbers} and F = {multiples of 2}?

1.16 Combining sets

LEARNING OUTCOMES

- Represent a set in various forms
- Recognise and interpret Venn diagrams
- Apply the principles of complement of a set, subsets, intersection, disjoint sets, union of sets

Combinations of sets

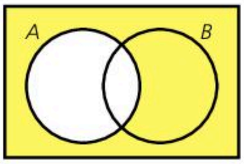


In the Venn diagram above:

ℰ = {natural numbers no greater than 10}
A = {even numbers}
B = {prime numbers}
C = {multiples of 4}

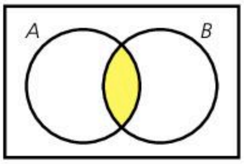
Sets B and C are **disjoint** sets. This means that they have no common elements. On the Venn diagram, the two circles do not cross.

So we draw circle C completely inside circle A.



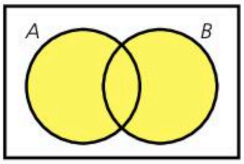
- The **complement** of A, written as A', is the set containing everything that is not in A. In this case, A' is {1, 3, 5, 7, 9}, or {odd numbers}.

  A' is shaded in the diagram.



- The **intersection** of A and B, written as $A \cap B$, is the set of elements in both A and B, which is where the circles intersect (or cross) on the Venn diagram.
  $A \cap B = \{2\}$, the only even prime number.

  $A \cap B$ is shaded in the diagram.



- The **union** of A and B, written as $A \cup B$, is the set of elements in either A or B (or both), which is everything in either circle on the Venn diagram.
  $A \cup B = \{2, 3, 4, 5, 6, 7, 8, 10\}$

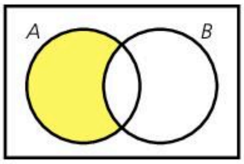
  $A \cup B$ is shaded in the diagram.

Describing regions

- The intersection of two sets A and B consists of the elements in BOTH A AND B.
- The union of two sets A and B consists of the elements in EITHER A OR B OR BOTH.
- The complement of a set consists of all the elements NOT in that set.

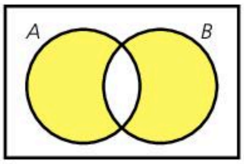
We can use this to help describe regions using set notation.

For example, to describe this region:



The shaded part is the section in A AND not in B.

In set notation, this is $A \cap B'$.



This one is more difficult to describe.

We try to describe it using words like OR, AND and NOT.

The shaded region here is everything in the union of A and B and not in the intersection.

That is $(A \cup B) \cap (A \cap B)'$

ACTIVITY

A = {letters in 'cucumber'}
B = {letters in 'melon'}
C = {letters in 'mango'}

- What is $A \cap B \cap C$?
- Melissa says that $\{A \cup B \cup C\}$ is equal to {letters in 'long car number'}. Is she correct?
- Make up your own puzzle like this.

KEY POINTS

1 A subset of a set is contained in the set.

2 The complement of a set A is written A' and contains everything except the elements of set A.

3 The union, $\cup$, of two sets contains anything in either set.

4 The intersection, $\cap$, contains those elements in both sets.

SUMMARY QUESTIONS

$\mathscr{E}$ = {natural numbers no greater than 25}
A = {1, 4, 9, 16, 25}
B = {3, 6, 9, 12, 15}
C = {4, 8, 12, 16, 20}
D = {5, 10, 15, 20, 25}

1 What is $A \cap C$?

2 What is $B \cup C$?

3 What is the cardinality of the set $\{A \cup B \cup C \cup D\}'$?

1.17 Venn diagrams

LEARNING OUTCOMES

- List subsets of a given set
- Find the number of subsets of a set with n elements
- Construct Venn diagrams
- Determine elements in intersections, unions and complements of sets
- Solve problems using Venn diagrams

EXAM TIP

The null set and A are subsets of A.

EXAM TIP

In any question, always make sure you understand the universal set.

Subsets

If all the elements of set A are also elements of set B, then we say A is a **subset** of B.

We write this as $A \subset B$.

The set A = {right-angled triangles} is a subset of B = {triangles}, which is itself a subset of C = {polygons}.

$$A \subset B \subset C$$

Number of subsets

- For a set A, the null set, $\varnothing$, and A itself are subsets of A.
- The set {a} has two subsets.
  They are: $\varnothing$, {a}
- The set {a, b} has four subsets.
  They are: $\varnothing$, {a}, {b}, {a, b}
- The set {a, b, c, d} has 16 subsets.
  They are: $\varnothing$, {a}, {b}, {c}, {d}, {a, b}, {a, c}, {a, d}, {b, c}, {b, d}, {c, d}, {a, b, c}, {a, b, d}, {a, c, d}, {b, c, d}, {a, b, c, d}

Generally, for a set with a cardinality of $\boldsymbol{n}$, there are **2^n subsets**.

Constructing Venn diagrams

Venn diagrams can be used to show the elements in a set or the number of elements in a set.

To construct a Venn diagram, we often need to determine the elements in an intersection.

WORKED EXAMPLE 1

If $\mathscr{E}$ = {natural numbers less than or equal to 20}, A = {even numbers}, B = {factors of 20} and C = {multiples of 5}, then:

$$A = \{2, 4, 6, 8, 10, 12, 14, 16, 18, 20\}$$

$$B = \{1, 2, 4, 5, 10, 20\}$$

$$C = \{5, 10, 15, 20\}$$

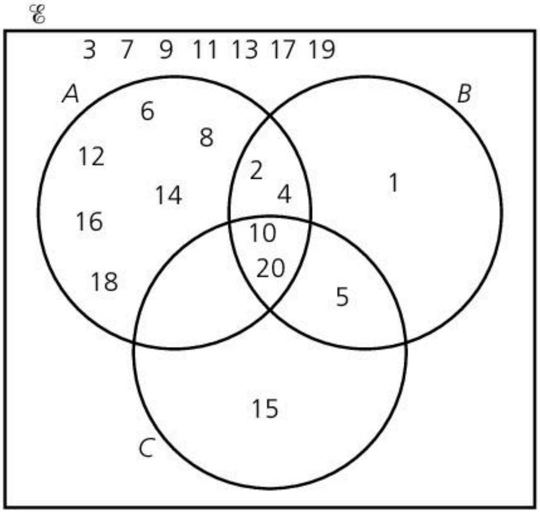
Then $A \cap B = \{2, 4, 10, 20\}$

$A \cap C = \{10, 20\}$

$B \cap C = \{5, 10, 20\}$

and $A \cap B \cap C = \{10, 20\}$

To construct the Venn diagram, it prevents mistakes to start with the intersections.



Solving problems using Venn diagrams

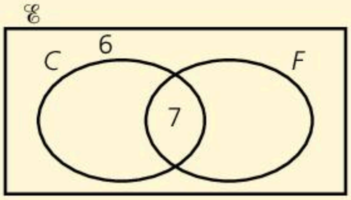
This is best explained using an example.

WORKED EXAMPLE 2

In a group of 40 boys, 16 play football and 25 play cricket.

6 boys play neither.

To find out how many play both, draw a Venn diagram.



We know the cardinality of ℰ is 40, and that there are 6 boys in $(C \cup F)'$ (outside the union of C and F).

So there must be $40 - 6 = 34$ in $(C \cup F)$.

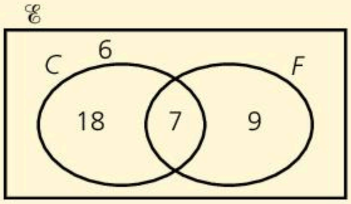
16 footballers + 25 cricketers = 41.
But this includes those who play both, twice.

So the extra $41 - 34 = 7$ make up $C \cap F$.

Now we can complete the Venn diagram:

Number playing football only $= 16 - 7 = 9$

Number playing cricket only $= 25 - 7 = 18$



ACTIVITY

A = {factors of 10} D = {prime numbers between 4 and 16}
B = {factors of 9} $E = A \cup C$
C = {factors of 4} $F = D \cup (A \cap B)$

- Place each set into the correct square on the grid below.

	Cardinality 3	Cardinality 4	Cardinality 5
Only contains odd elements			
Contains both odd and even elements			

KEY POINTS

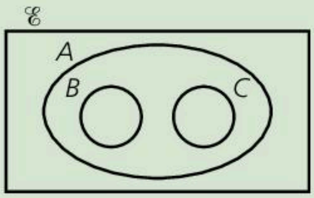
1. A Venn diagram can show the elements of sets or the cardinality of the sets.
2. If $A \subset B$, then every element of A is in B.
3. The number of subsets of a set with cardinality n is 2^n.
4. The null set and A are subsets of A.
5. The cardinality of $(A \cup B)$ = the cardinality of A + the cardinality of B − the cardinality of $(A \cap B)$.

SUMMARY QUESTIONS

1 A = {left-handed students}, B = {15-year-old students}
Joel is a left-handed 14-year-old.
Cynthia is a right-handed 15-year-old.
Philip is a right-handed 14-year-old.
Match the three students with the correct set.

$(A \cup B)'$ $A' \cap B$ $A \cap B'$

2



What does this Venn diagram tell you about

a A and B **b** $B \cap C$?

3 A class of 36 students contains 17 girls. Of the class, 11 have visited the USA. There are 12 boys in the class who have not visited the USA. Draw a Venn diagram to show this information and find out how many girls have visited the USA.

Module 1 Practice exam questions

SECTION 1: Non-calculator

1 From these numbers: 5, 6, 7, 8, 9, 10, choose:

- **a** a prime number
- **b** a square number
- **c** a multiple of 4
- **d** a factor of 27

2 Find the HCF and LCM of 42 and 56.

3 Multiply 4.2 by 2.9, giving your answer correct to 3 significant figures.

4 Calculate $\frac{2}{3} \times \left(1\frac{1}{4} - \frac{5}{8}\right)$

5 Write $\frac{5}{8}$ as a decimal.

6 Spencer and Mark share some money in the ratio 4 : 3.
Mark receives \$84.
How much are they sharing?

7 Simplify $\frac{3^5 \times 3^4}{3^2}$, writing your answer in index notation.

8 Write these numbers in order of size, starting with the smallest:

$3.7 \quad -3.4 \quad 3\frac{3}{4} \quad -3\frac{1}{2} \quad 3.687 \quad \sqrt{8}$

9 Write down the first five terms of the sequence that follows the rule:
Start with 3.
Then double it and add 2.

10 The area of Grenada is 470 000 m^2.
Write this in standard form.

11 An operation ✪ is defined as
a ✪ $b = 2a \times 3b$.
Show that the operation ✪ is

- **a** associative
- **b** commutative.

12 Calculate $4 + 6 \div 2 - 2 \times 3$

13 How many minutes is it from 0855 to 1025?

14 List all the subsets of $A = \{a, b, c\}$.

15 $\frac{2}{3}$ of the students in a class are boys. $\frac{3}{4}$ of the boys are right-handed.
There are 18 right-handed boys.
How many students are there in the class?

SECTION 2: Calculator

1 Calculate 45% of \$85.

2 Harry bought a vase for \$480.
Clara wants to buy the vase.
Harry says she can pay \$100 deposit followed by 12 monthly instalments of \$35.
What is Harry's percentage profit?

3 Melissa sold a car at a loss of 30%.
She sold it for \$1400.
What did she pay for it originally?

4 A New Zealand dollar is worth 1.54 Barbadian dollars.
How many New Zealand dollars would I get for 320 Barbadian dollars?

5 Find the *n*th term of the sequence

5, 8, 11, 14, 17, …

6 What is the multiplicative inverse (reciprocal) of $1\frac{1}{2}$?

SECTION 2: Calculator

7 The number 3215 is in base 6.
What is the value of the digit 2?

8 Charles invests \$600 at 4% per annum.
What is this worth after adding 3 years'

a simple interest

b compound interest?

9 A car, bought for \$5000, depreciates by 7% every year.
How much is the car worth after three years?

10 Complete this bill:

3 books	@ \$12.99 each
5 pencils	@ \$ 0.35 each
2 notepads	@ \$ 1.75 each
2.5 litres paint	@ \$ 4.50/litre
Subtotal:	
Sales tax @ 15%	
TOTAL	

11 Lionel earns \$14 per hour for a 35-hour week. Overtime is paid at time and a half.
Last week he worked for 42 hours.
How much did he earn?

12 This question is about the three sets, A, B and C.

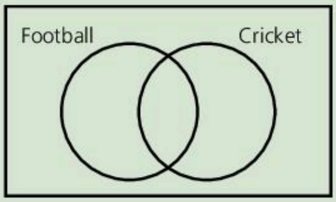
A = {factors of 24}
B = {multiples of 6}
C = {odd numbers}

a Which set is a finite set?

b What is the cardinality of $A \cap B$?

c What can you say about $B \cap C$?

13 In a group of 36 boys, 20 play cricket and 18 play football. 6 boys play neither.
Complete the Venn diagram to show this information.



14 Marie has an annual salary of \$45 000.
She also receives a bonus of 2% of the company profits.
Income tax is charged at 20% on all income up to \$24 200, and 35% on income beyond \$24 200.
Her company made a profit of \$890 000.
Calculate:

a her bonus

b the amount of income tax she pays.

15 For each statement below, say whether it is true or false.

a {prime numbers} $\subset$ {odd numbers}

b 16 $\in$ {square numbers}

c 15 $\in$ {multiples of 5} $\cap$ {factors of 100}

d {multiples of 7} $\cap$ {factors of 20} = $\varnothing$

16 Marjory earns an annual salary of \$55 000.
She pays income tax on her earnings as follows:

\$0–\$20 000 tax free.
\$20 000.01–\$42 000 taxed at 23%
\$42 000.01 and over taxed at 35%.

She also calculates that she spends \$23 000 a year on goods which have had 15% VAT included in the price.
What percentage of her salary is taken in tax (income tax and VAT)?

2 Measurement and statistics

2.1 Estimating area and scale drawing

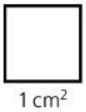
LEARNING OUTCOMES

- Estimate area of irregular plane figures
- Use maps and scale drawings to determine distances and areas

Estimating area

The **area** of a closed plane shape is the amount of surface inside it.

The area is measured in square units, such as cm^2.
A square with sides of 1 cm has area 1 cm^2.

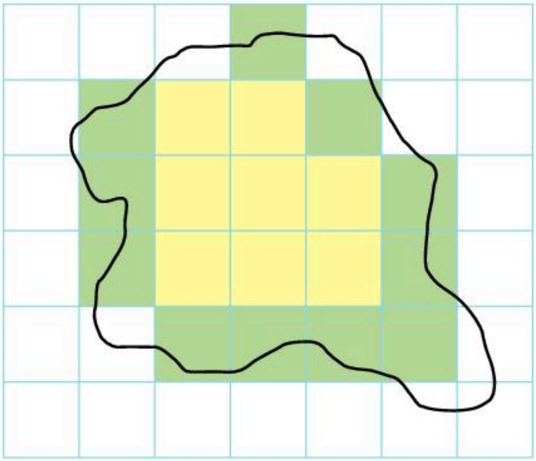


So, area can be thought of as the number of unit squares needed to cover a surface.

To estimate the area of a shape, trace it onto a centimetre-squared grid.

Count the number of squares inside the shape, and include any squares which are more than half inside the shape.

In the shape below, there are 8 complete squares (yellow) and 11 green squares which are more than half included in the shape.



We estimate the area as $8 + 11 = 19\,\text{cm}^2$.

This method is very useful for irregular shapes, and is often used in scale drawings.

Scale drawing

In a scale drawing, a scale is often written as a ratio, for example 1 : 10 000.

This means that the real world is 10 000 times greater than the drawing, so 1 cm on the drawing represents 10 000 cm or 100 m in the real world.

On other occasions, a scale is represented in words, for example '1 cm represents 10 m'.

WORKED EXAMPLE 1

Here is a map of Cuba, drawn on a centimetre-squared grid.

Scale: 1 cm represents 100 km

The distance from Santa Clara to Guatanamo on the map (marked in red) is 5.7 cm.

As each cm represents 100 km, the actual distance is 5.7 × 100 km = 570 km.

The area of Cuba, found by counting squares (shaded yellow), is 11 cm^2.

Each square represents an area of 100 km × 100 km = 10 000 km^2.

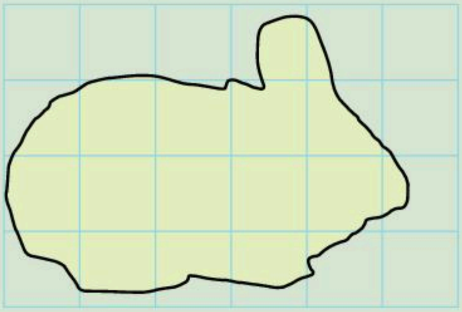
So the estimate of the area of Cuba is 11 × 10 000 km^2 or 110 000 km^2.

KEY POINTS

1. The area of a 2-dimensional shape is the amount of space inside it.
2. The area can be estimated by counting squares.
3. Use a map scale to calculate distances and areas on a map.

SUMMARY QUESTIONS

1. Estimate the area of this shape.
   The shape is the map of an island, drawn to a scale of 1 cm to 10 km.
2. Find the actual length of the island (the greatest distance across the island).
3. Estimate the area of the island.



ACTIVITY

- Find a map of your island.
- Trace it onto a centimetre-squared grid and work out the length, width and area of the island.
- Research the actual size to see how close your estimates are.

EXAM TIP

- Make sure you look closely at the scale on a map.
- Remember the area of a square is equal to the side length squared.

2.2

Perimeter and area

LEARNING OUTCOMES

- Calculate the perimeter of a polygon
- Use the area formulae for quadrilaterals and triangles to calculate areas or missing sides

The **perimeter** of a shape is the distance around the boundary of the shape. It is measured in units of length (mm, cm, etc.).

The **area** of a shape is the amount of flat (2-dimensional) space inside the shape.

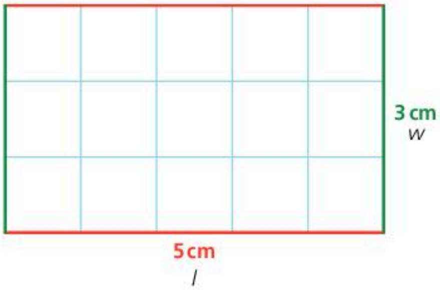
Area is measured in square units, such as square centimetres (cm^2), square millimetres (mm^2), or square metres (m^2). The area is the number of unit squares that would fill the shape.

To find the area of some special quadrilaterals, multiply the **base** and the **perpendicular height**.

Rectangles

The **area of a rectangle** or a square is found by multiplying the length by the width.

The perimeter is 5 cm + 3 cm + 5 cm + 3 cm = 16 cm.



$$A = l \times w$$

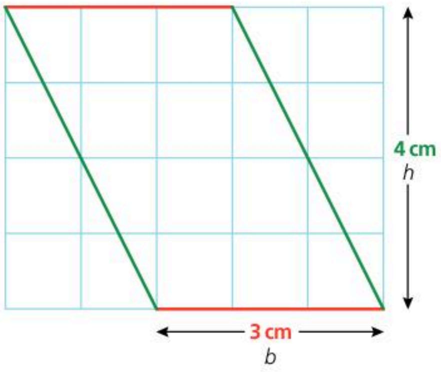
$$\text{Area} = 5\,\text{cm} \times 3\,\text{cm}$$
$$= 15\,\text{cm}^2$$

Parallelograms

The **area of a parallelogram** or a **rhombus** is found by multiplying the base by the perpendicular height.

$$A = b \times h$$

$$\text{Area} = 3\,\text{cm} \times 4\,\text{cm}$$
$$= 12\,\text{cm}^2$$

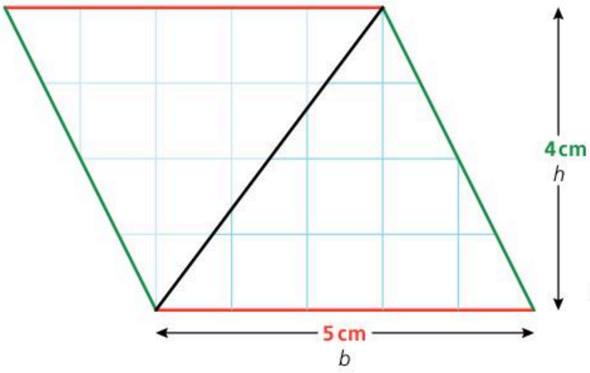


Measure the green sides; they are 4.5 cm long.

So the perimeter is 3 cm + 4.5 cm + 3 cm + 4.5 cm = 15 cm.

Area of a triangle

The area of a **triangle** is half the area of a parallelogram:



$$A = \frac{b \times h}{2}$$

$$\text{Area} = \frac{5\,\text{cm} \times 4\,\text{cm}}{2}$$
$$= 10\,\text{cm}^2$$

EXAM TIP

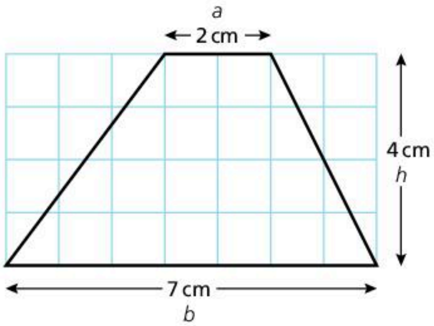
- Don't forget to include the correct units – area is always measured in square units.

Area of a trapezium

The **area of a trapezium** is found by multiplying the average width by the perpendicular height.

$$A = \frac{a + b}{2} \times h$$

$$\text{Area} = \frac{2 + 7}{2} \times 4 = 18\,\text{cm}^2$$

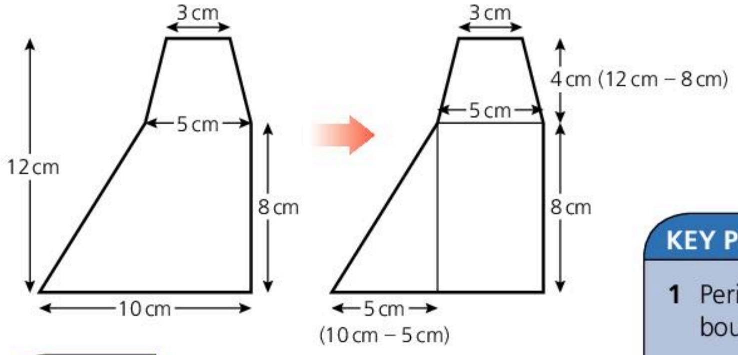


Compound shapes

To find the area of more complex shapes, cut them into shapes you can find the area of.

Add all the areas together to find the total area.

This shape can be split into a rectangle, a triangle and a trapezium:



Area of triangle

$$= \frac{b \times h}{2} = \frac{5 \times 8}{2} = 20\,\text{cm}^2$$

Area of rectangle

$$= l \times w = 5 \times 8 = 40\,\text{cm}^2$$

Area of trapezium

$$= \frac{a + b}{2} \times h = \frac{3 + 5}{2} \times 4 = 16\,\text{cm}^2$$

Total area $= 20 + 40 + 16 = 76\,\text{cm}^2$

ACTIVITY

Cut out three congruent (the same shape and size) paper parallelograms, with a base of 16 cm and a height of 12 cm.

1 Calculate the area of the parallelogram.

2 Make a single straight cut on one of the parallelograms so the two pieces can be rearranged to make a rectangle.

- Calculate the area of the rectangle.

3 Make a single straight cut on the next parallelogram so the two pieces can be rearranged to make a trapezium.

- Calculate the area of the trapezium.

4 Make a single straight cut on the third parallelogram so the two pieces can be rearranged to make a triangle.

- Calculate the area of the triangle.

KEY POINTS

1 Perimeter is the distance around the boundary of a shape.

2 Area is the amount of 2-dimensional space inside a shape.

3 Key formulae:

Area of a rectangle = length × width

Area of a parallelogram
= base × perpendicular height

Area of a triangle

$$= \frac{\text{base} \times \text{perpendicular height}}{2}$$

Area of a trapezium

$$= \frac{\text{sum of the parallel sides}}{2} \times \text{perpendicular height}$$

SUMMARY QUESTIONS

1 Find the area of a triangle with a base of 7 cm and a height of 5 cm.

2 A trapezium has parallel sides of length 5 cm and 9 cm. The area of the trapezium is 56 cm^2. Calculate the perpendicular height.

3 A rectangle has a perimeter of 30 cm and a length of 9 cm. Calculate the area of the rectangle.

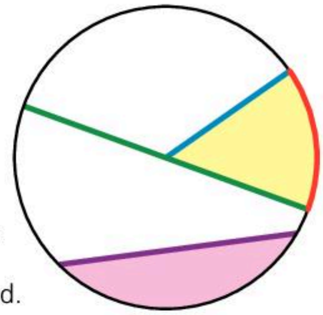
2.3 Circles

LEARNING OUTCOMES

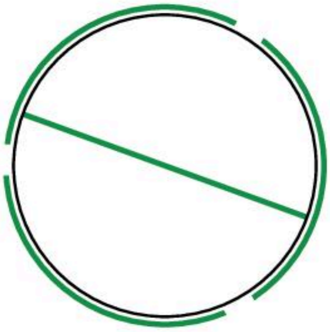
- Calculate the circumference of a circle
- Calculate the area of a circle
- Calculate the perimeter and area of a combination of polygons and circles
- Calculate the length of arc and area of sector

Parts of a circle

- The distance around the outside (perimeter) of a circle is called the **circumference**.
- A part of the circumference is an **arc**.
- A line crossing the circle through the centre is a **diameter**.
- A line from the centre to the circumference is a **radius** (plural: radii).
- A line from a point on the circumference to another point on the circumference is a **chord**. The diameter is a special chord.
- An area cut off by a chord is a segment.
- An area cut off by two radii is a sector.



Length of circumference and arc



Imagine wrapping the diameter around the circle.

It takes just over three diameters to reach around the circumference.

The exact number of diameters is called π (pi), and is approximately 3.14159

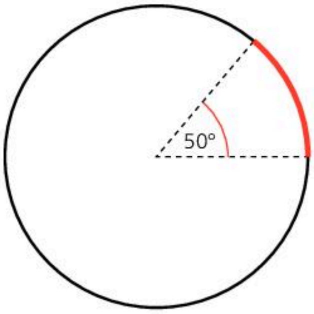
π is an **irrational** number (see 1.1), and so cannot be written down exactly.

Circumference = π × diameter, or $C = \pi D$.

The diameter is twice the radius, so alternatively $C = 2\pi r$.

If a circle has a radius of 2 cm, the circumference = $2\pi r$ = $2 \times \pi \times 2$ or approximately 12.6 cm (to 1 decimal place).

An **arc** is a fraction of the circumference of a circle.



The size of the fraction depends on the size of the angle at the centre.

If the angle is 50°, the arc is $\frac{50}{360}$ of the circumference, as the whole angle is 360°.

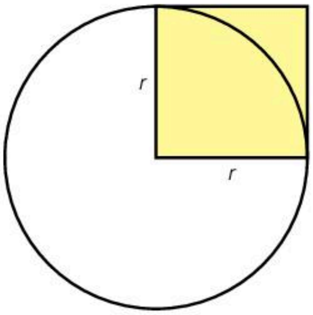
If the angle at the centre is θ, then

$$\text{Length of arc} = \frac{\theta}{360} \times \text{circumference} = \frac{\theta}{360}\, 2\pi r$$

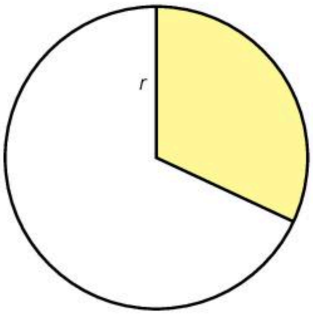
Area of circle and sector

Here is a circle, and a square with an area of radius × radius, or r^2.

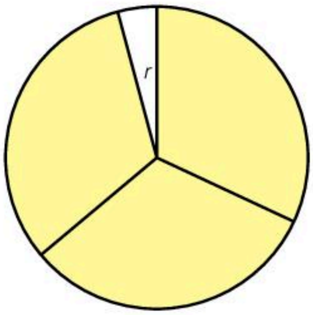
Imagine squeezing the square so it kept the same area, but fit inside the circle.



It would look like this.



We could fit just over three of these in the circle:



In fact, it would take exactly π of them.

So the area of the circle is π × the area of the yellow square, or $A = \pi r^2$.

Compound shapes

Some shapes need to be split into parts in order to find the area and perimeter.

WORKED EXAMPLE 1

The door shown is 76 cm wide and 2.2 m tall.

To find the area and perimeter, we split it into a rectangle and a semicircle.

The diameter of the semicircle is 76 cm, so the radius is 76 ÷ 2 = 38 cm.

The height of the rectangle is 220 − 38 = 182 cm.

Area of rectangle

$= l \times w = 182 \times 76$

$= 13\,832\ \text{cm}^2$

Area of semicircle

= area of circle ÷ 2

$= \pi r^2 \div 2$

$= \pi \times 38^2 \div 2$

$= 2268\ \text{cm}^2$ (to nearest cm^2)

Total area

$= 13\,832 + 2268 = 16\,100\ \text{cm}^2$

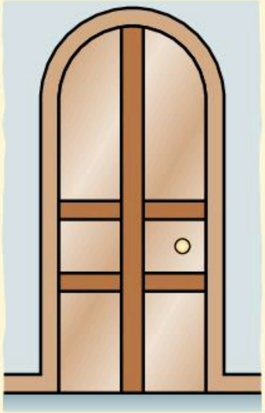
Perimeter

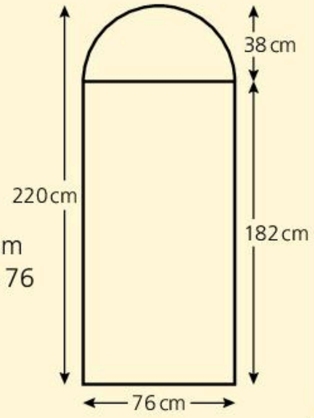
= semicircular top + 2 sides + bottom

= circumference ÷ 2 + (2 × 182) + 76

$= 2 \times \pi \times 38 \div 2 + 364 + 76$

= 559 cm (to nearest cm)





ACTIVITY

Queen's Park Savannah is Port of Spain's largest open space, and the world's largest traffic roundabout.

The perimeter is about 3.6 km.

- Calculate the area inside it.

EXAM TIP

- Do not confuse the formulae for circumference and area.
- Remember that the diameter is double the radius.
- When you are given a specific value for π (in a question), be sure to use it.
- Do not round off answers too soon.

KEY POINTS

1 Circumference of a circle = $2\pi r$

2 Area of a circle = πr^2

3 Length of arc with angle at centre $\theta = \frac{\theta}{360} 2\pi r$

4 Area of sector with angle at centre $\theta = \frac{\theta}{360} \pi r^2$

SUMMARY QUESTIONS

1 Find the area and circumference of a circle with radius 6 cm.

2 A circle has a circumference of 66 cm. Find the diameter, radius and area.

3 A semicircle has an area of $40\ \text{cm}^2$. Calculate the perimeter.

2.4 Surface area

LEARNING OUTCOMES

- Calculate surface area of solids (cube, cuboid, prism, cylinder, right pyramid and cone).

A 3-dimensional shape, or solid, has a number of faces or surfaces. For example:

- A cuboid has six rectangular faces.
- A cylinder has two circular faces and a curved surface.

When finding the surface area of a solid, it helps to draw a **net**.

Prisms

A **prism** is a solid with the same **cross-section** throughout its length.

A cylinder is a circular prism, as it has the same circular cross-section throughout its length.

A cube is a square prism and a cuboid is a rectangular prism.

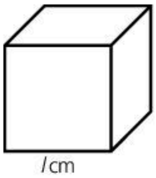
Cones and pyramids are not prisms as the cross-section changes throughout their lengths.

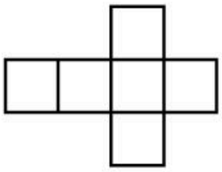
Surface area – prisms

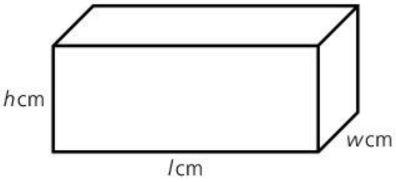
The **surface area** of a solid is the sum of the areas of each face.

A cube of side l cm has six faces. Each has an area of $l \times l = l^2$ cm^2.

Therefore, the surface area of that cube $= 6l^2$ cm^2.







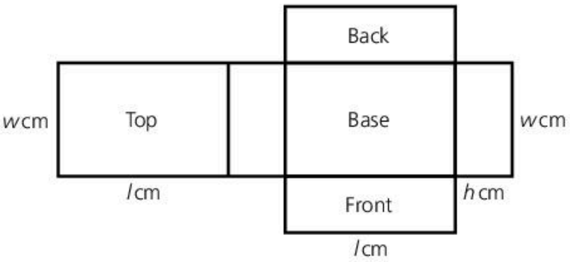
A cuboid of length l cm, width w cm and height h cm also has six faces.

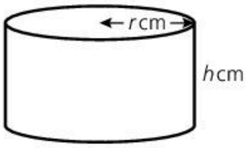
The top and bottom each have an area of $l \times w = lw$ cm^2

The front and back each of an area of $l \times h = lh$ cm^2

The two ends each have an area of $w \times h = wh$ cm^2

The surface area $= 2lw + 2lh + 2wh$ cm^2





For a cylinder of radius r cm and a height of h cm:

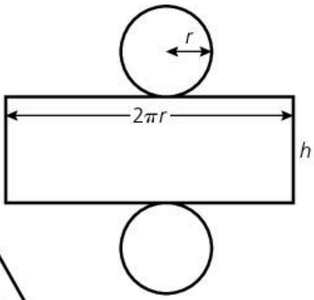
The top and bottom are each circles with an area of πr^2

The curved surface could be cut and opened into a rectangle.

The width is h cm and the length is the circumference of the circle, or $2\pi r$.

The area of the curved surface is $2\pi rh$.

The surface area of the cylinder $= 2\pi r^2 + 2\pi rh$.



Surface area – pyramid and cone

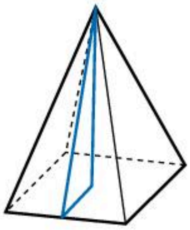
A right pyramid has its top vertex directly above the centre of the base.

The pyramid in the diagram has a square base with sides of x cm, and it has a perpendicular height of y cm.

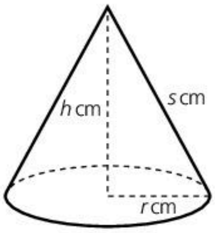
The height, h cm, of each triangular face can be found by applying Pythagoras (section 4.7) to the blue triangle: $h^2 = y^2 + (\frac{x}{2})^2$.

The surface area = area of all 4 triangles + area of square base

$$= 4 \times \frac{xh}{2} + x^2$$



The diagram shows a cone with radius r cm, height h cm and a slant height of s cm:



The base is a circle with area πr^2.

The curved surface is a sector of a circle:

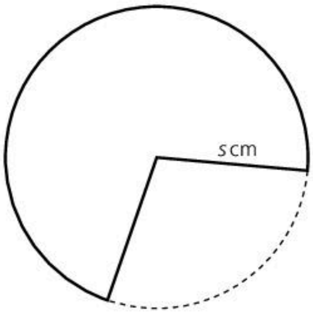
The circumference of the whole circle is $2\pi s$.

The circumference of the sector must equal the circumference of the base, or $2\pi r$.

So the fraction of the circle required is $\frac{2\pi r}{2\pi s} = \frac{r}{s}$

So the area of the sector is $\frac{r}{s} \times \pi s^2 = \pi rs$

Surface area of cone $= \pi r^2 + \pi rs$



Surface area – sphere

The surface area of a sphere with radius r cm is $4\pi r^2$.

KEY POINTS

1. Surface area is the sum of the areas of all surfaces.
2. Surface area of a cylinder $= 2\pi r^2 + 2\pi rh$
3. Surface area of a cone $= \pi r^2 + \pi rs$
4. Surface area of a sphere $= 4\pi r^2$

EXAM TIP

When calculating surface area, it helps to draw a net.

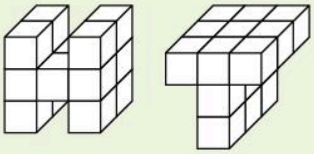
SUMMARY QUESTIONS

1. Find the surface area of a cuboid 4 cm long, 3 cm wide and 2 cm high.
2. A cone of radius 5 cm, height 12 cm and slant height 13 cm is stuck on top of a cylinder of radius 5 cm, height 12 cm. Find the surface area of the shape.
3. A cylinder has a height of 6 cm and a radius of 6 cm. A sphere has a radius of 6 cm.

   Show that the sphere and the cylinder have the same surface area.

ACTIVITY

Show that these two prisms have the same surface area:



2.5 Volume

LEARNING OUTCOMES

- Calculate volume of solids (prism, cylinder, cone, sphere, cube, cuboid)

Volume

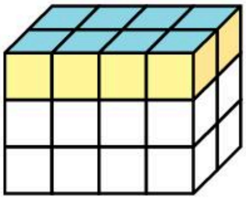
The **volume** of a solid is the amount of 3-dimensional space it occupies. It is measured in cubic units.

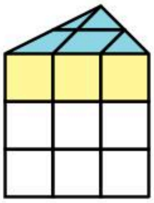
For example, a cubic centimetre, or 1 cm^3, is the space taken up by a cube of edge length 1 cm.

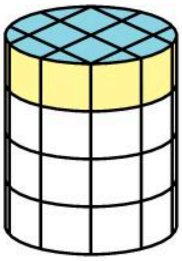
The volume of a solid can be thought of as the number of unit cubes that can be packed into the solid.

Volume of a prism

Below are three prisms, all split into centimetre cubes.



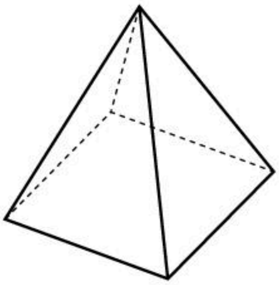




	Cuboid	Triangular prism	Cylinder
Area of blue top = total number of blue squares	$l \times w$ $= 4 \times 2$ $= 8\,cm^2$	$\frac{b \times h}{2}$ $= \frac{3 \times 2}{2}$ $= 3\,cm^2$	πr^2 $= \pi \times 2^2$ $\approx 12.57\,cm^2$ (to 2 d.p.)
The number of yellow and blue cubes is equal to the number of blue squares:			
Volume of the top layer (yellow)	$8\,cm^3$	$3\,cm^3$	$12.57\,cm^3$ (to 2 d.p.)
Total volume = volume of top layer × the number of layers	$= 8 \times 3$ $= 24\,cm^3$	$= 3 \times 3$ $= 9\,cm^3$	$\approx 12.57 \times 4$ $\approx 50.27\,cm^3$ (to 2 d.p)

For any prism, volume = area of top × height, or $V = Ah$

Volume of pyramids, cones and spheres



The volume of a pyramid or a cone is given by

Volume = $\frac{1}{3}$ base area × perpendicular height.

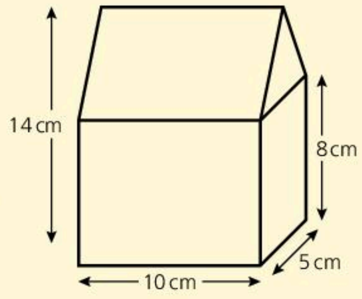
So:

- for a square-based pyramid with base of side l and height h
  $V = \frac{1}{3}l^2h$
- for a cone of radius r and height h
  $V = \frac{1}{3}\pi r^2h$
- for a sphere of radius r
  $V = \frac{4}{3}\pi r^3$

Volume of compound shapes

Compound shapes can be broken into simpler shapes.

WORKED EXAMPLE 1



The shape shown can be split into a cuboid and a triangular prism.

The cuboid has dimensions 10 cm × 5 cm × 8 cm
so, the area of the top is $10 \times 5 = 50\ \text{cm}^2$.

The volume = area × height = $50 \times 8 = 400\ \text{cm}^3$.

The prism has a triangular front of base 5 cm and height $(14 - 8) = 6$ cm.

The area of the triangle is $\frac{\text{base} \times \text{height}}{2} = \frac{5 \times 6}{2} = 15\ \text{cm}^2$.

The volume of the prism is area × length = $15 \times 10 = 150\ \text{cm}^3$.

Total volume = $400 + 150 = 550\ \text{cm}^3$.

Solving problems

Most problems can be solved by setting up an equation.

WORKED EXAMPLE 2

A sphere of radius 12 cm has the same volume as a cylinder of radius 16 cm.

To calculate the height of the cylinder, we set up an equation:

Volume of sphere $= \frac{4}{3}\pi r^3 = \frac{4}{3} \times \pi \times 12^3$

Volume of cylinder = Area of top × height
$= \pi r^2 \times h = 16^2\pi h$

So $\frac{4}{3} \times \pi \times 12^3 = 16^2\pi h$

Or $2304\pi = 256\pi h$

$h = \frac{2304\pi}{256\pi} = 9$ cm

ACTIVITY

A lady on holiday in Jamaica collects water from Dunns River Falls in a plastic container in the shape of a cuboid, measuring 30 cm long, 20 cm wide and 10 cm high.

She tips the water into a cylindrical drum with a radius of 50 cm and a height of 1 m.

- How many times will she have to fill the plastic container in order to fill the drum?

KEY POINTS

1 Volume is the amount of 3-D space inside a solid.

2 Volume of a prism = area of base × height

3 Volume of a cone $= \frac{1}{3}\pi r^2 h$

4 Volume of a sphere $= \frac{4}{3}\pi r^3$

SUMMARY QUESTIONS

1 Find the volume of a cuboid 6 cm long, 3.5 cm wide and 4 cm high.

2 A cylinder has a radius of 5 cm and a height of 7 cm. Calculate the volume of the cylinder.

3 A pyramid has a square base of side 8 cm. It has the same volume as a cube of side 6 cm. Calculate the height of the pyramid.

2.6

Units of measurement

LEARNING OUTCOMES

- Convert units of length, area, capacity, time, speed
- Use SI units for area, volume, mass, temperature, time

ACTIVITY

- Research the SI system.

SI units

The SI (Système International) units of measurement adopt a series of prefixes which can be used for length, mass and capacity.

The commonly used prefixes are kilo (k) meaning 1000, centi (c) meaning hundredth and milli (m) meaning thousandth.

So 1 mg (milligram) is $\frac{1}{1000}$ of a gram, 1 cℓ (centilitre) is $\frac{1}{100}$ of a litre and 1 km (kilometre) is 1000 metres.

Length	Mass	Capacity
10 mm = 1 cm	10 mg = 1 cg	10 mℓ = 1 cℓ
100 cm = 1 m	100 cg = 1 g	100 cℓ = 1 ℓ
1000 m = 1 km	1000 g = 1 kg	1000 ℓ = 1 kℓ

Additionally, 1000 kg = 1 tonne

There are other rarely used divisions and multiples; for example deci (d) is $\frac{1}{10}$.

Conversion between SI units

Converting within the SI system only requires multiplying or dividing by 10, 100 or 1000.

For example:

$$1.4\,\text{km} = 1.4 \times 1000\,\text{m} = 1400\,\text{m}$$

$$25\,\text{c}\ell = 25 \div 100\,\ell = 0.25\,\ell$$

Time

Time does not have units that are multiples of 10, 100 or 1000.

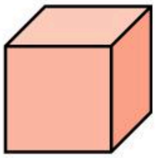
60 seconds = 1 minute
60 minutes = 1 hour
24 hours = 1 day
365.25 days ≈ 1 year

Area and volume

A square with sides of 1 cm has an area of 1 cm × 1 cm = 1 cm^2

It is also 10 mm × 10 mm = 100 mm^2

So 1 cm^2 = 100 mm^2



A cube with sides of 1 m has a volume of 1 m × 1 m × 1 m = 1 m^3

But it is also 100 cm × 100 cm × 100 cm = 1 000 000 cm^3

For area, the length conversions are squared.

For volume, the length conversions are cubed.

For example:

$$1\ \text{km} = 1000\ \text{m}$$

So $1\ \text{km}^2 = 1000^2\ \text{m}^2 = 1\,000\,000\ \text{m}^2$

And $1\ \text{km}^3 = 1000^3\ \text{m}^3 = 1\,000\,000\,000\ \text{m}^3$.

Most of the world uses the SI System, but most notably the United States of America do not.

The USA uses the imperial system:

Length	Mass	Capacity
12 inches = 1 foot 3 feet = 1 yard 1760 yards = 1 mile	16 ounces (oz) = 1 pound (lb) 100 lb = 1 hundredweight (cwt) 20 cwt = 1 ton	16 fluid ounces = 1 pint 8 pints = 1 gallon

EXAM TIP

- Remember to multiply when changing to a smaller unit, and divide when changing to a larger unit.
- Area and volume have different conversion rates to length.
- Be careful when working with time on a calculator. 2.25 hours is not 2 hours 25 minutes, but 2 hours 15 minutes.

Approximate equivalents of the two systems

Length	Mass	Capacity
1 inch ≈ 2.5 cm 1 yard ≈ 1 m 1 mile ≈ 1600 m 5 miles ≈ 8 km	1 oz ≈ 30 g 1 lb ≈ 450 g 2.2 lb ≈ 1 kg 1 ton ≈ 900 kg 1.1 ton ≈ 1 tonne	1 fluid ounce ≈ 30 mℓ 1 pint ≈ 0.5 ℓ 1 gallon ≈ 4 ℓ

ACTIVITY

Martinique has an area of 1128 km² or 436 sq. miles.

- Use this information to find out how many km² make up 1 square mile.
- How many km² are there in 25 square miles?
- Does this agree with the approximate equivalent 5 miles ≈ 8 km?

KEY POINTS

1. kilo (k) means 1000
2. centi (c) means hundredth
3. milli (m) means thousandth
4. The conversions for area are the squares of the length conversions.
5. The conversions for volume are the cubes of the length conversions.
6. It is important to know the approximate imperial equivalents.

SUMMARY QUESTIONS

1 A recipe for soup includes these ingredients:

8 litres of water
500 g salted beef
225 g breadfruit
50 g coco

What are these quantities in imperial units?

2 How many seconds are there in $2\frac{1}{4}$ hours?

3 A rectangle is 20 cm long and 10 cm wide. Write down the area in:

a cm² **b** mm² **c** square inches

2.7 Time, distance and speed

LEARNING OUTCOMES

- Solve problems involving time, distance and speed

EXAM TIP

- For speeds in km/h, the distance must be in kilometres and the time in hours.
- Remember 30 minutes is not 0.30 hours.

ACTIVITY

On 16 August 2009, Usain Bolt of Jamaica set a world record of 9.58 s for the 100 m sprint.

- Calculate his speed in m/s.
- At this speed, how far would he run in 1 minute?
- How far would he run in 1 hour?
- What was his speed in km/h?

Speed

Speed is a measure of how quickly something is moving.

A car is constantly changing speed, but we say we are travelling at '60 kilometres per hour'. This means that, if we continued travelling at that speed for an hour, we would travel 60 kilometres.

Speed is usually measured in km/h (kilometres per hour) or m/s (metres per second).

Average speed

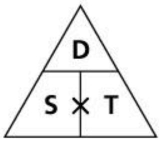
Because speed tends to change, we often talk of **average speed**. This is calculated by dividing the distance travelled by the time taken, so

$$\text{speed} = \frac{\text{distance}}{\text{time}}.$$

For example:

A bus takes 30 minutes to travel 18 km.

The average speed $= \frac{18\,\text{km}}{0.5\text{ hours}} = 36\,\text{km/h}$



The speed triangle is a useful way of remembering the connection between time, distance and speed: the letters for distance, speed and time go in alphabetical order:

To calculate the time travelled, cover up the T for time, and it shows that you must calculate $\frac{D}{S}$.

It also shows that distance = speed × time, and that speed $= \frac{\text{distance}}{\text{time}}$.

Distance–time graphs

Journeys are often shown on a distance–time graph.

WORKED EXAMPLE 1

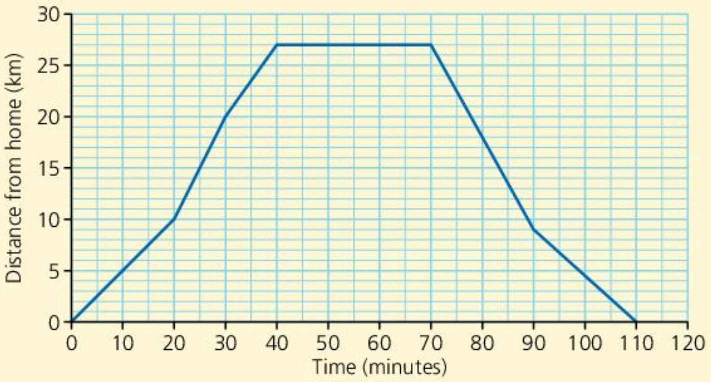
Here is a graph of Jason's journey to the supermarket and back.

Jason travels 10 km in the first 20 minutes.

To calculate a speed in km/h, first write the time in hours.

20 minutes is $\frac{1}{3}$ of an hour, so his speed for this part of the journey is

$$10 \div \frac{1}{3} = 30\,\text{km/h}.$$



It takes him 40 minutes to travel the 27 km to the supermarket, so his average speed for the journey is

$$27 \div \frac{2}{3} = 40.5 \text{ km/h}.$$

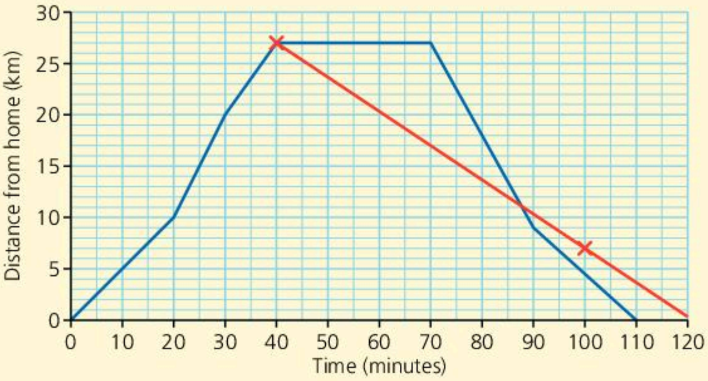
He is in the supermarket for 30 minutes, shown by the horizontal line. It is horizontal because he is not travelling towards or away from home.

The fastest he travelled was on the first part of the journey back home. We know this because it is the steepest part. He travelled $27 - 9 = 18$ km in 20 minutes, an average speed of $\frac{18}{\frac{1}{3}} = 54$ km/h

He met his sister Marsha in the supermarket. She left just as he arrived, and she travelled home at an average speed of 20 km/h.

We can plot her journey by joining her leaving point, (40, 27), with her position an hour later. She is 20 km closer to home, 60 minutes later, so she is at the point (100, 7).

Using the graph we can see that Jason passed Marsha after about 88 minutes, just over 11 km from home.

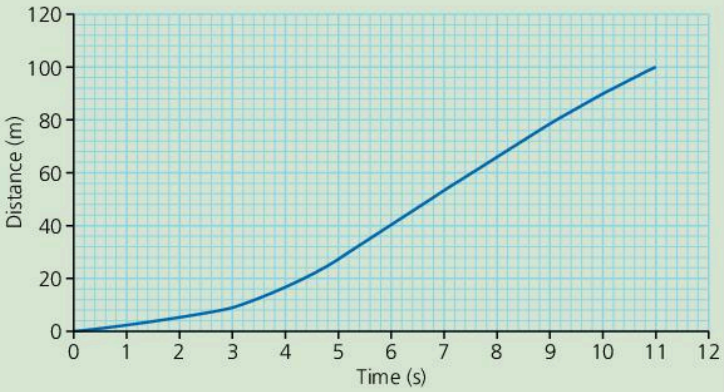


SUMMARY QUESTIONS

1 I travel at 52 km/h for 1 hour 15 minutes. How far do I travel?

2 Nadine cycles 10 km to work. She goes to work at an average speed of 25 km/h. She returns at 24 km/h. How much quicker is her journey to work than her journey home?

3 The graph shows the distance travelled by James when running the 100 m sprint.



a What was his average speed over the first 3 seconds?

b What was his average speed over the last 3 seconds?

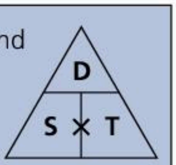
c What was his average speed over the whole race?

EXAM TIP

- The graph does not show that Jason was travelling uphill and downhill. It shows him travelling from home and towards home.
- The steeper the line, the greater the speed.

KEY POINTS

1 Time, distance and speed are connected:



2 Be consistent with units. Do not multiply a speed in km/h by a time in minutes.

3 A travel graph shows how distance from a point changes over time.

2.8 Measurement and accuracy

LEARNING OUTCOMES

- State the error associated with a given measurement
- Determine the range of possible values for a given measurement
- Solve measurement problems

ACTIVITY

An elevator has a sign saying 'Maximum weight: 320 kg'.

The four people in the elevator have weights of 83 kg, 82 kg, 78 kg and 76 kg (all to the nearest kg).

- Is it safe for them all to travel in the elevator at the same time?

Errors in measurement

No measurement, whether length, mass, time or other measure, can ever be wholly accurate.

They are affected by human errors, or the limitations of eyesight, and the inaccuracy of measuring equipment. Like all solids, a ruler will expand slightly when heated, so its length will vary according to the temperature. A ruler is unlikely to have smaller markings than millimetres, so it cannot be used to measure to a greater degree of accuracy.

Measurements given to a particular unit may actually be up to half a unit larger or smaller.

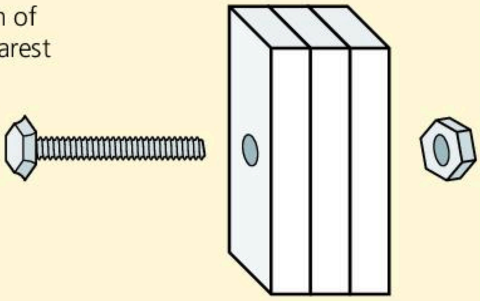
For example, a pencil measured as 12.6 cm long (to the nearest millimetre) might be anything between 12.55 and 12.65 cm long, half a millimetre more or less than the stated measurement.

Calculating with maximum and minimum values

WORKED EXAMPLE 1

Three metal plates, each of width 4.5 cm (to the nearest mm) are to be bolted together, with a bolt of length 14.0 cm (to the nearest mm).

Calculate how much thread will be showing to attach to the nut.



Solution

Each plate could be between 4.45 cm and 4.55 cm thick.

The total thickness is between 3 × 4.45 cm and 3 × 4.55 cm, or 13.35 cm and 13.65 cm.

The bolt is between 13.95 cm and 14.05 cm.

The greatest possible amount of thread will be showing if the plates are small (13.35 cm) and the bolt is large (14.05 cm).

There could be $14.05 - 13.35 = 0.7$ cm showing.

But if the plates are large and the bolt is small, the amount of thread might be only $13.95 - 13.65 = 0.3$ cm.

Here we are combining a maximum measurement with a minimum measurement.

WORKED EXAMPLE 2

In a lemonade factory, lemonade flows through a pipe at a rate of 200 mℓ per second, to the nearest 10 mℓ.

Each litre bottle is filled for 5 seconds, to the nearest second.

How much lemonade will the bottles contain?

Solution

The quantity of lemonade could be as much as
5.5 s × 205 mℓ/s = 1127.5 mℓ

It could be as little as 4.5 × 195 mℓ = 877.5 mℓ

WORKED EXAMPLE 3

Joe ran 100 m (to the nearest m) in 14 seconds (to the nearest second).

He ran somewhere between 99.5 m and 100.5 m, in a time between 13.5 seconds and 14.5 seconds.

He said, 'I might have run as far as 100.5 m as quickly as 13.5 seconds, so my average speed was 100.5 ÷ 13.5 = 7.44 m/s'.

His teacher said, 'But you may have only run 99.5 m, and it might have taken 14.5 seconds, so your speed might only be 99.5 ÷ 14.5 = 6.86 m/s'.

Again, we are combining a maximum measurement with a minimum measurement.

ACTIVITY

A circular cake has a radius of 15 cm to the nearest cm.

Jo is decorating it.

She has a green frill to go around the cake, and some pink icing to go on top.

- Find the greatest possible length for the frill and the greatest possible area for the icing.

SUMMARY QUESTIONS

1 A packet of butter weighs 250 g to the nearest 10 g.
24 packets are packed in a box.
Calculate the maximum and minimum weight of the 24 packets.

2 A bird flies at 2.5 m/s (to 1 decimal place) for 5 seconds (to the nearest second).
Calculate the maximum possible distance the bird has flown.

3 A rectangle has an area of 35 cm^2 to the nearest cm^2. The length is 7.5 cm to the nearest mm. Calculate the minimum width of the rectangle.

EXAM TIP

- Look out for words like 'maximum' and 'minimum' to recognise questions about accuracy in measurement.
- Do not always combine maximum values; sometimes you need to combine a maximum with a minimum.

KEY POINTS

1 A measurement given to an accuracy of one unit can always be half a unit bigger or smaller.

2 Maximum
$(a + b)$ = max a + max b

3 Minimum
$(a + b)$ = min a + min b

4 Maximum
$(a - b)$ = max a − min b

5 Minimum
$(a - b)$ = min a − max b

6 Maximum
$(a \times b)$ = max a × max b

7 Minimum
$(a \times b)$ = min a × min b

8 Maximum
$(a \div b)$ = max a ÷ min b

9 Minimum
$(a \div b)$ = min a ÷ max b

2.9 Data

LEARNING OUTCOMES

- Recognise types of data (discrete, continuous; ungrouped, grouped; qualitative, quantitative)
- Construct frequency tables (grouped/ungrouped, discrete/continuous)
- Determine class features for a data set (class interval, limits, boundaries, midpoint)

Types of data

Any information used to help decision-making is called **data**. Data is the plural of datum. We often collect data by survey, questionnaire or other means. There are different types of data:

- **Qualitative data** are not numerical, for example favourite music (steel band, reggae, calypso, pop, etc.)
- **Quantitative data** are numerical, such as heights, shoe sizes or ages.

  Quantitative data can be continuous or discrete:
- **Continuous data** can take on any value on a scale, for example, the mass of flour in a jar.
- **Discrete data** can only take on certain values, usually integers, such as a test score or a shoe size.

Data can be **grouped** or **ungrouped**.

Grouped data are put into categories. The method of grouping depends on whether the data are discrete or continuous.

WORKED EXAMPLE 1

Maisie was collecting the marks of students in a test. These are discrete data, so she sets up a frequency table like this:

The test scores are all integers, so the data are discrete.

Tallies are grouped in fives to make counting the frequencies easier.

Test score	Tally	Frequency
1–10	|	1
11–20	𝍸 |||	8
21–30	𝍸 𝍸 |	11
31–40	𝍸 ||	7
41–50	𝍸 |	6

WORKED EXAMPLE 2

Frankie was recording the heights of heartflowers in her garden.

These heights are continuous data. She must make sure there are no gaps in the table, so heights such as 10.7 cm or 19.95 cm are included.

A heartflower with a height of exactly 10 cm is recorded in the first row, as in the first row the height must be less than or equal to 10 cm. In the second row, the height must be greater than 10 cm so a heartflower of 10.1 cm is recorded in the second row.

Height, x (cm)	Tally	Frequency
$0<x\leq 10$	|	1
$10<x\leq 20$	||	2
$20<x\leq 30$	𝍸 |||	8
$30<x\leq 40$	𝍸 𝍸	10
$40<x\leq 50$	𝍸 𝍸 ||	12

Class limits and boundaries

When we round off continuous data, they appear to be discrete data.

For example, if the students in a class all measured their height to the nearest cm, then we could use this chart:

Height (cm)	Tally	Frequency
131–140		
141–150		
151–160		
161–170		
171–180		

In this table, the **class limits** of the first row are 131 cm and 140 cm.

But because the data have been rounded, the **class boundaries** are 130.5 cm and 140.5 cm, because any height from 130.5 cm up to (but less than) 140.5 cm will be included in this category.

Class interval

To calculate the class interval (or width of the class), subtract the lower class boundary from the upper class boundary.

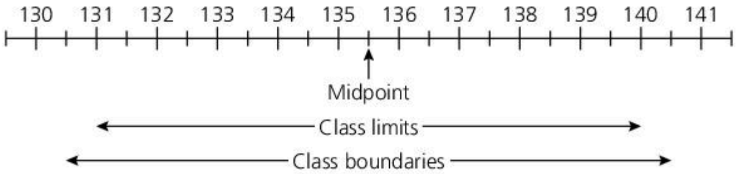
In the above example, the class interval is $140.5 - 130.5 = 10$ cm.

Midpoint of class

The midpoint is the mean of the class limits.

In the example, it is $\frac{131 + 140}{2} = 135.5$ cm

All the above information can be seen on this number line:



EXAM TIP

- Class boundaries and class limits are not necessarily the same.

ACTIVITY

In 1968, Tommie Smith (USA) set a World Record in the 200 m race of 19.8 seconds.

From 1977, all World Record timings for the 200 m were given to 2 decimal places instead of the 1 decimal place used previously.

Usain Bolt's (Jamaica) World Record set in 2009 was 19.19 seconds.

- What are the limits of Tommie Smith's time?
- What are the limits of Usain Bolt's time?

KEY POINTS

1 Data can be quantitative (numerical) or qualitative (non-numeric).

2 Quantitative data can be grouped or ungrouped.

3 Quantitative data can be discrete or continuous.

4 Class limits are the upper and lower measures stated in the frequency table.

5 Class boundaries go beyond the limits when continuous data have been rounded.

6 The midpoint of a class is the mean of the upper and lower limits.

SUMMARY QUESTIONS

All the questions are about these three incomplete tally charts.

Favourite fruit	
Apple	
Banana	
Cherry	

Mass of parcel (kg)	
6–10	
11–15	
16–20	

Length l (cm)	
$10 < l \leqslant 15$	
$15 < l \leqslant 20$	
$20 < l \leqslant 25$	

1 Which chart shows:
a continuous data **b** qualitative data?

2 What are the class width and the class boundaries for the first category in the Mass chart?

3 What is the midpoint of the first category in the Length chart?

2.10 Displaying information (1)

LEARNING OUTCOMES

- Construct and interpret bar charts
- Construct and interpret frequency polygons
- Construct and interpret line graphs

EXAM TIP

- When drawing bar charts, remember to choose a sensible scale.
- When interpreting bar charts, look carefully at the scale.

ACTIVITY

- Conduct a survey of 20 friends to find out their favourite sport.
- Put the results in a bar chart.

Bar charts

A bar chart has parallel rectangular bars or columns of the same width, usually with a space between them. Each bar shows the quantity of a different category of data.

Bar charts are used for qualitative data or discrete quantitative data, but not for continuous data.

WORKED EXAMPLE 1

A company uses many reams of paper in a year. The table gives the data for last year:

Quarter	Reams of paper used
Jan–Mar	270
April–Jun	330
Jul–Sep	190
Oct–Dec	450

First we need to choose the scale.

When choosing a scale, we choose a round number for each square on the axis.

The biggest number is 450, so we need a vertical scale that shows at least 450.

A scale of one square to 50 will take 9 squares.

Then we can draw the bar chart as shown.


Bar chart showing amount of paper used

Frequency polygons

A frequency polygon is used to represent quantitative data, usually continuous data.

WORKED EXAMPLE 2

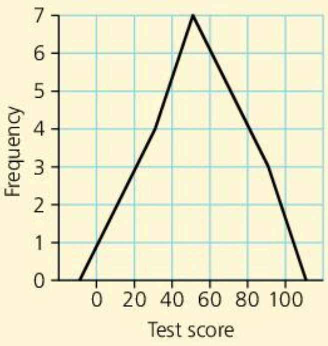
To draw a frequency polygon, plot the points (class midpoint, frequency) and join them. The end points are the midpoints of the intervals on either side.

It meets the x-axis at the centres of the adjacent empty bars.

For this data:

- the points are (10.5, 2), (30.5, 4), (50.5, 7), (70.5, 5) and (90.5, 3)
- the next interval on either side would be −19 to 0, and 101 to 120. So the end points are (−9.5, 0) and (110.5, 0).

Test Score	Frequency
1–20	2
21–40	4
41–60	7
61–80	5
81–100	3



Line graphs

A line graph is drawn by plotting points and connecting them with straight lines. Usually, if time is one of the quantities, it is placed on the horizontal axis.

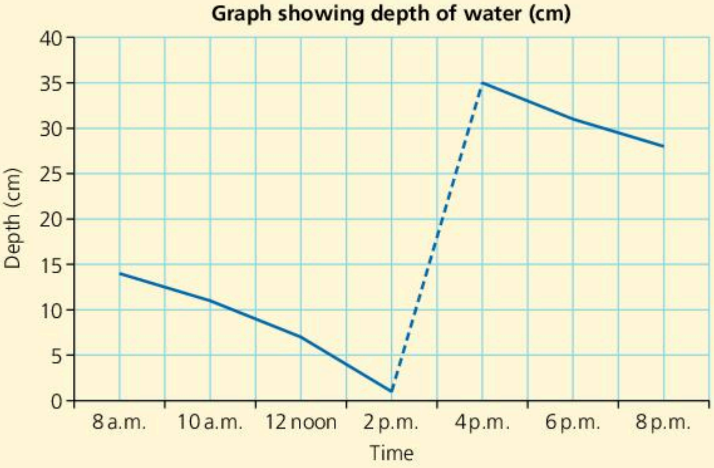
WORKED EXAMPLE 3

The depth of water in a water cooler is measured every 2 hours.

Here are the results:

Time	8 a.m.	10 a.m.	12 noon	2 p.m.	4 p.m.	6 p.m.	8 p.m.
Depth (cm)	14	11	7	1	35	31	28

As a line graph, the results look like this:



A line graph is useful as it sometimes allows us to estimate values between the readings. For example, we might estimate that the depth of water at 11 a.m. was 9 cm.

However, we need to take care. Look at the dashed section of the line. Joining these points makes it look as though it took 2 hours to refill the container, whereas it will have taken just a minute or two to replace the empty one with a full one. The graph should, in fact, be vertical at the moment the container is replaced but the graph does not tell us what time this happened, only that it was between 2 p.m. and 4 p.m.

SUMMARY QUESTIONS

1 Draw a bar chart to show this information from a survey:

Favourite sport	**Frequency**
Cricket	8
Tennis	3
Football	7
Athletics	6

2 Draw a frequency polygon to show these two sets of data showing information about passengers on a bus.

Age	1–10	11–20	21–30	31–40	41–50	51–60
Male frequency	1	3	5	7	5	3
Female frequency	0	4	4	8	7	5

3 Which type of chart would you draw to represent qualitative data?

KEY POINTS

1 Bar charts are for qualitative data or discrete data. There is a gap between the bars.

2 Frequency polygons allow two sets of data to be shown on the same axes.

3 Line graphs allow us to make estimates between the measured values.

2.11 Displaying information (2)

LEARNING OUTCOMES

- Construct and interpret histograms
- Construct and interpret pie charts

Histograms

A histogram looks similar to a bar chart.

A histogram is used when data are continuous.

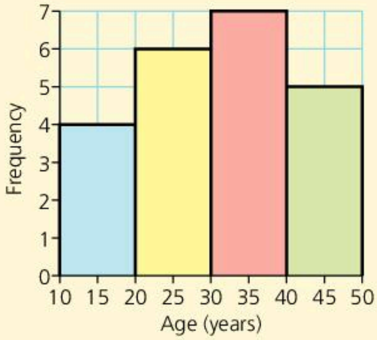
Because it is continuous, there are no gaps between the bars.

WORKED EXAMPLE 1

The table below shows the ages of some people on a bus.

Age (years)	Frequency
10–19	4
20–29	6
30–39	7
40–49	5

Here is the histogram.



The data are continuous, as you are classed as 19 right up to the day before your 20th birthday.

So the 10–19 bar occupies the space from 10 right up to 20 on the chart.

Pie charts

A pie chart is circular in shape. A pie chart is different from the other graphs because it does not show frequencies. Instead it shows proportions, or fractions, of the total.

Pie charts are particularly useful to show the results of a sample, as the actual frequencies are less important than the proportions.

WORKED EXAMPLE 2

A survey of people's favourite music gave these results:

Type of music	Frequency
Steel band	3
Reggae	7
Rap	9
Salsa	5

To put this information into a pie chart, we find the total size of the sample: 3 + 7 + 9 + 5 = 24

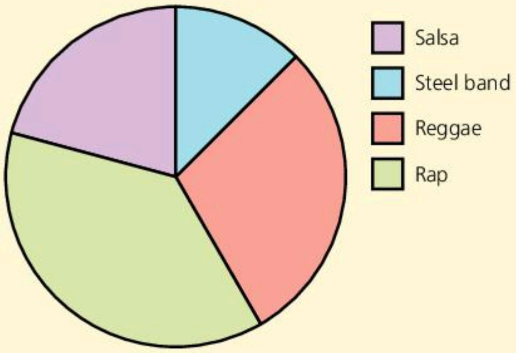
A full turn is 360°, so each person occupies 360 ÷ 24 = 15°.

To find the angle for each sector, we multiply this result by the frequency.

So the angles are:

Type of music	Frequency	Angle
Steel band	3	3 × 15 = 45°
Reggae	7	7 × 15 = 105°
Rap	9	9 × 15 = 135°
Salsa	5	5 × 15 = 75°

And the pie chart looks like this:



The chart does not show the actual quantities, but it shows quite clearly that rap was the most popular.

Because pie charts show fractions, we can use fractions to solve problems.

WORKED EXAMPLE 3

A pie chart shows that in a survey of 60 people, those who chose 'mango' as their favourite fruit were represented by an angle of 48°.

Calculate how many people chose mango.

Solution

The mango sector is $\frac{48}{360}$ of the circle.

So the number of people choosing mango is $\frac{48}{360}$ of 60.

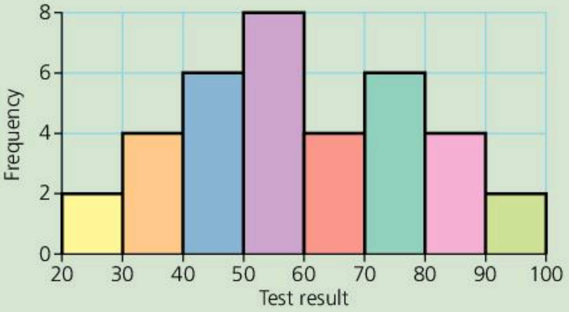
$$= \frac{48}{360} \times 60$$
$$= \frac{2}{15} \times 60$$
$$= 8 \text{ people.}$$

SUMMARY QUESTIONS

1 In a pie chart showing favourite ice cream, the angle for coconut is 54°. If 40 people took part in the survey, how many chose coconut?

2 In what way is a pie chart different from all the other charts?

3 In this histogram of test results, how many people took the test?



EXAM TIP

- Before drawing the pie chart, make sure the angles add up to 360°.
- Make sure you use your protractor correctly.
- If your pie chart is not filled, you have made a mistake. Check your calculations, and then check your measuring.

ACTIVITY

- Conduct a survey of 24 friends to find out their favourite type of music.
- Put the results in a pie chart.
- Put the results in a bar chart.
- What are the advantages of a bar chart?
- What are the advantages of a pie chart?

KEY POINTS

1 Histograms show continuous data, and there are no gaps between the bars.

2 Pie charts show proportions rather than frequencies.

2.12 Measures of central tendency

LEARNING OUTCOMES

- Determine the mode, median and mean from raw data and ungrouped frequency tables

An **average** is a single piece of information used to represent a set of data. There are three commonly used averages.

Mode

The **mode** is the most common piece of data.

A group of friends counted the number of letters in their first names.

Here are the results.

6 8 7 6 4 8 6 9 7 5 6 9

The mode is 6, as it occurs more than any other number.

Mean

The **mean** is the result of sharing the data equally.

WORKED EXAMPLE 1

In a library, the number of books on each of five shelves is:

45 54 37 63 36

The total number of books is $45 + 54 + 37 + 63 + 36 = 235$.

Dividing by 5 as there are five shelves, $235 \div 5 = 47$.

The mean number of books on a shelf is 47.

So the books could be arranged with 47 on each shelf.

The mean is calculated as $\text{mean} = \dfrac{\text{sum of data}}{\text{number of items of data}}$

Median

The **median** is the middle piece of data when put in order.

WORKED EXAMPLE 2

Jacob records the number of oranges on each branch on an orange tree.

The results are: 4 5 7 5 3 5 6 4 7 7 6

Put in order: 3 4 4 5 5 5 6 6 7 7 7

For a list of n numbers, the middle one is in position $\dfrac{n+1}{2}$.

There are 11 numbers in the list, so the middle one is $\dfrac{11+1}{2}$, or the 6th number. The median is 5.

If there is an even number of pieces of data, there is no middle item.

For the data 4, 6, 7, 8, 10, 14, there are 6 numbers, so the middle one is in position $\dfrac{6+1}{2} = 3.5$

This is halfway between the 3rd and 4th numbers, so we find the mean of those:

$$\frac{7+8}{2} = 7.5$$

Averages from frequency tables

WORKED EXAMPLE 3

One hundred children were asked how many days they had been absent from school in the last month. Here are the results.

Number of days absent	Frequency
0	35
1	33
2	24
3	5
4	3
TOTAL	100

- The **mode** (or the **modal number** of days) is 0, because this is the most common number, as it has the largest frequency of 35.
- The **median** is the middle of the 100 so we must find the person in position $\frac{100 + 1}{2} = 50.5$

  This shows that there is not a person in the middle, but a middle pair, in positions 50 and 51.

  The first 35 students have an absence rate of 0.
  The next 33 have an absence rate of 1. If we put these students in line, they will occupy positions 36 to 68, so both the 50th and 51st student has an absence rate of 1. So the median is 1.
- For the **mean**, we must add together all 100 values. There are 35 zeros, which make a total of 0.

  There are 33 ones, which total 33.

  The 24 twos have a sum of 48 and so on.

  It is quickest to add a column to the table:

Number of days absent	Frequency	Subtotal
0	35	$35 \times 0 = 0$
1	33	$33 \times 1 = 33$
2	24	$24 \times 2 = 48$
3	5	$5 \times 3 = 15$
4	3	$3 \times 4 = 12$
TOTAL	100	108

  So

  $$\text{Mean} = \frac{\text{sum of data}}{\text{number of items of data}} = \frac{108}{100} = 1.08$$

ACTIVITY

- Six natural numbers have a mean of 4, a median of 5 and a mode of 6. Find the six numbers.
- Six natural numbers have a median of 4, a mode of 5 and a mean of 6. Find the six numbers.

KEY POINTS

1 The MOde is the MOst common.

2 The MeDian is the MiDdle value.

3 The MEAN is $\frac{\text{sum of data}}{\text{number of items of data}}$

SUMMARY QUESTIONS

1 The mean height of four students is 182 cm. Calculate the total height of the four students.

2 Three numbers have a mean of 5 and a mode of 6. What are the three numbers?

3 Five people at a bus stop have a mean age of 44, a median of 43 and a mode of 47. A 47-year-old leaves the queue, and a 50-year-old joins the queue.
What happens to the mean, median and mode?

EXAM TIP

- The mode is not the highest frequency; it is the *value* that has the highest frequency.
- When using a frequency table, the median is not the middle row, but the value of the middle item.
- When using a frequency table, divide by the sum of the frequencies, not the number of rows.

2.13 Grouped data

LEARNING OUTCOMES

- Determine the modal class, median class and estimate of the mean from grouped frequency tables

Grouped frequency tables

Sometimes information is given to us as **grouped data**.

In these cases, we do not have the raw data, so we cannot calculate an average.

Instead, we can find the **modal class** (the most common group), the **median class** (the group containing the middle value), and an **estimated mean**.

We will discover how to estimate a median in unit 2.16.

The speeds of 50 motorists on a stretch of road are recorded in the table below. This information is used in the discussions below.

Speed (km/h)	**Frequency, *f***
51–55	2
56–60	3
61–65	6
66–70	18
71–75	12
76–80	9

Modal class

The modal class (or modal group) is the one which is most common.

So the modal class is the one with the largest frequency. In this case it is 66–70 km/h.

Median class

The median speed of 50 motorists is the speed of one in position $\frac{50+1}{2} = 25.5$, so the median is the speed of the 25th and 26th motorists.

There are 2 motorists with a speed of 51–55 km/h.

There are 5 motorists altogether with a speed between 51 and 60 km/h.

11 motorists had speeds between 51 and 65 km/h, and 29 had speeds up to 70 km/h.

So the 25th and 26th motorists are in the 66–70 km/h group.

The actual speeds are unknown as the data are grouped. 66–70 km/h is the median class as it contains the median drivers.

ACTIVITY

A computer helpline company states that 'On average we answer the telephone within 10 seconds'.

Here are the data for a particular day.

Time taken, *t* (sec)	**Frequency, *f***
1–5	76
6–10	87
11–20	21
21–30	11
31–60	4
61–120	1

- Is their claim justified?
- Explain your answer.

Estimate of the mean

We cannot find the exact mean as we do not have exact data. Instead we find an estimate of the mean.

To do this we assume that everyone in each group drives at the midway speed. Then the calculation is just the same as in the previous unit for finding the mean from a frequency table.

Speed (km/h)	**Frequency, *f***	**Midway value (*m*)**	$f \times m$
51–55	2	53	106
56–60	3	58	174
61–65	6	63	378
66–70	18	68	1224
71–75	12	73	876
76–80	9	78	702
TOTAL	50		3460

The **estimated mean** $= \frac{3460}{50} = 69.2$ km/h.

This is only an estimate as we have not used the actual speeds of motorists, instead using the middle value of each group in the table.

As there are likely to be some motorists travelling faster than the middle value and some travelling more slowly, the estimated mean is likely to be a good estimate.

EXAM TIP

- The modal class is the class with the highest frequency.
- The median class is the class that contains the middle value, not the middle class.
- An estimate of the mean is calculated, not guessed.

KEY POINTS

1 The modal group is the most common group.

2 To find the median class, find which class contains the median of all the data.

3 To estimate the mean, assume all items in the table have the middle value for the group, and then use the rule

$$\text{mean} = \frac{\text{sum of data}}{\text{number of items of data}}$$

SUMMARY QUESTIONS

Twenty friends compared the number of texts they sent in a day. Here are the results.

Number of texts sent	1–5	6–10	11–15	16–20	21–25
Frequency	2	4	5	8	1

1 Find the modal class.

2 Calculate an estimate of the mean.

3 Find the median class.

2.14 Which average?

LEARNING OUTCOMES

- Understand the 4 levels of measurement
- Determine which average is best to use in a given situation

Levels of measurement

There are 4 different levels of measurement.

The **nominal scale** is the most basic and means just putting data into categories.

Examples include flavours of ice cream or colours of doors.

There is no obvious order; red could come before or after blue.

You could find a mode of such data.

The **ordinal scale** is numerical but the numbers do not indicate equal intervals.

Imagine you and a friend are both asked to rank 5 different brands of cola from 1 (nicest) to 5 (least good).

You might both choose the same colas for positions 1 and 2. But you might think 1 is much better, whereas your friend might think it is only slightly better. The "distance" between 1 and 2 is different for each of you.

An ordinal scale is ordered but the distances between positions is unequal.

You could find the median and mode of such data.

If instead you are asked to score each cola from 1 (most dissatisfied) to 7 (very satisfied), you would try to split the 7 points into equal intervals. This is called an **interval scale**.

You could find the mean, median and mode of interval data.

A **ratio scale** is similar to an interval scale but has a true zero point, a point that means an absence of measure.

Length is a ratio scale, as 0 cm means no length.

Temperature is not a ratio scale, as 0^o does not mean no temperature. The temperature can go below 0^o.

You could find the mean, median and mode of such data.

Which average is best?

The purpose of an average is to represent a set of data with a single figure.

- The **mean** is the only average that uses every piece of data. If we know the mean and the number of pieces of data, we can calculate the sum. However, if there is an outlier (an extreme value), it will influence the mean whereas the median and mode will be unaffected.

It is best for the ordinal and ratio scales if there are no outliers.

- The **median** is easy to find from a list or frequency table, but not from grouped data. If a set of data consisted of 6 small numbers and 5 large numbers, for example 3, 4, 4, 5, 6, 55, 56, 58, 60, then the median will not be representative as it will reflect the larger of the groups.

It can be used for all scales except the nominal scale.

- The mode is the only average that can be used for qualitative data, as it does not depend on adding or ordering. But there might be more than one mode, or none at all, and if the sample is small it may not be representative of the data.

It can be used for all scales.

The table below summarises the strengths and weaknesses of each average.

	Strengths	**Weaknesses**	**Scales**
Mean	Uses all the data. Can be used to calculate the sum of the data.	Influenced by outliers or extreme values.	Interval Ratio
Median	Easy to calculate. Always central.	Does not take all values into account.	Ordinal Interval Ratio
Mode	Can be used for qualitative data.	Might not be representative of all the data, particularly for a small sample. Might be more than one mode, or none at all.	Nominal Ordinal Interval Ratio

ACTIVITY

Design a questionnaire about wages in a factory.
Try to include one question using each scale of measurement.

SUMMARY QUESTIONS

Three school friends carried out a survey about mathematics. Here are the first questions each one asked:

Sally's question: What was your percentage in the last maths examination?

Marcus' question: Number these subjects in order of preference, with 1 as your favourite and 4 as your least favourite:

English Maths History Art

James' question: How much do you enjoy maths on a scale of 1 to 10, where 1 means "not at all" and 10 means "I love it."

1 For each question, state whether the answers collected will be nominal, ordinal, interval or ratio.
2 Which average (mean, median, mode) would you use to summarise the answers given to each question?

KEY POINTS

1 The nominal scale is usually qualitative and can only be summarised by the mode.
2 The ordinal scale is an ordering but the intervals are not equal. It can be summarised with mode or median.
3 The interval scale has equal intervals.
4 The ratio scale has a true zero.

2.15 Measures of dispersion

LEARNING OUTCOMES

- Understand measures of dispersion (range, interquartile range, semi-interquartile range)
- Determine measures of dispersion for raw data

Measures of dispersion

Dispersion means how spread out is a set of data.

There are a number of ways of measuring dispersion.

Patrick and Michael both play cricket.

Patrick's scores over five innings are 35, 38, 38, 41 and 43.

Michael's scores over five innings are 0, 21, 38, 38 and 98.

They both have a mean score of 39, a median of 38 and a mode of 38.

But their scores are quite different; Michael's scores are more spread out.

Range

The **range** of a set of data is the difference between the largest value and the smallest.

Looking at the example above:

For Patrick, the range is $43 - 35 = 8$.

For Michael, the range is $98 - 0 = 98$.

This shows that Michael's scores are much more spread out than Patrick's.

The problem with the range is that it only takes into account two values, the largest and the smallest.

WORKED EXAMPLE 1

A 100-metre sprinter has times of 12.2 seconds, 12.0 seconds, 11.9 seconds, 12.3 seconds, 18.5 seconds and 12.1 seconds.

She tripped over in one race which accounts for the slow time of 18.5 seconds, but it affects the range hugely. Her range is $18.5 - 11.9 = 6.6$ seconds. Without the slow race, her range is $12.3 - 11.9 = 0.4$ seconds.

Interquartile range

- The median is the middle value of a set of data.
- The **lower quartile** cuts off the bottom quarter, and so for small data sets it is the value one quarter of the way from the smallest, in position $\frac{n+1}{4}$.
- The **upper quartile** cuts off the top 25% of data, and so for small data sets it is in position $\frac{3(n+1)}{4}$.

  For example, if there are 11 items, the median is the $\frac{11+1}{2} = 6$th item.

  The lower quartile will be in position $\frac{11+1}{4} = 3$.

  The upper quartile will be in position $\frac{3(11+1)}{4}$, or the 9th item.
- The **interquartile range** measures the difference between the upper quartile and the lower quartile. It shows the range of the middle 50% of the data, and so is not affected by outliers.

The semi-interquartile range

The **semi-interquartile range** is half of the interquartile range. Usually this gives a good estimate of the distance from the lower quartile to the median, and the median to the upper quartile.

ACTIVITY

Tulane took 27 maths tests. Here are her scores:

Score	Frequency
2	1
3	2
4	3
5	4
6	5
7	6
8	4
9	2
10	1

- Find her median score, the range and the interquartile range.

KEY POINTS

1 The range = largest data item − smallest data item.

2 The median is the middle data item.

3 The lower quartile is the data item one quarter of the way from the smallest.

4 The upper quartile is the data item one quarter of the way from the largest.

5 The interquartile range = upper quartile − lower quartile.

6 The semi-interquartile range = $\frac{\text{interquartile range}}{2}$

EXAM TIP

- To find the quartiles, data must be arranged in order.
- Always give the range and interquartile range as single numbers, for example, it might be 12, not 5 to 17.

SUMMARY QUESTIONS

Hilary makes badges to sell to tourists.

During a 15-day period, she records the number she makes and the number she sells:

Day	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**	**12**	**13**	**14**	**15**
Number made	23	20	19	25	20	18	19	17	25	27	10	18	21	15	13
Number sold	7	17	21	12	19	25	11	13	20	32	26	18	24	23	22

1 Find the range for each set of data.

2 Find the interquartile range for each set of data.

3 Which set of data is the more spread out?

2.16 Cumulative frequency

LEARNING OUTCOMES

- Construct a cumulative frequency table
- Construct a cumulative frequency graph
- Use a cumulative frequency graph to find the median, quartiles, range, interquartile range, semi-interquartile range
- Calculate the percentage above a given value from a cumulative frequency table or graph

Very often, we have to deal with a large amount of data. If the data are grouped, we can only estimate the mean. We can also estimate the median and quartiles by using cumulative frequency tables and graphs. In reality, quartiles and medians tend to be more useful for large data sets.

Cumulative frequency tables

A cumulative frequency table shows the sum of all frequencies to a given point.

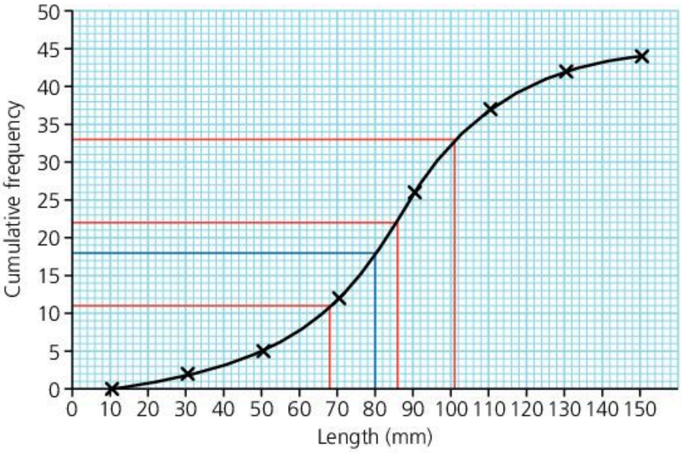
Here is a table showing the length of pencils in an office.

Length (mm)	Frequency
11–30	2
31–50	3
51–70	7
71–90	14
91–110	11
111–130	5
131–150	2

There are two pencils up to 30 mm long, five up to 50 mm long and 12 up to 70 mm long. These are **cumulative frequencies**, and can be added to the table:

Length (mm)	Frequency	Maximum length (upper boundary)	Cumulative frequency
11–30	2	30.5 cm	2
31–50	3	50.5 cm	5
51–70	7	70.5 cm	12
71–90	14	90.5 cm	26
91–110	11	110.5 cm	37
111–130	5	130.5 cm	42
131–150	2	150.5 cm	44

Cumulative frequency graphs



The last two columns of the table can be used to draw a cumulative frequency graph.

We know that there were no pencils less than 10.5 mm, so that is our starting point.

Because the sample is reasonably large, we find the quartiles and median by dividing the 44 pencils by 4:

> 44 ÷ 4 = 11, so the lower quartile is the length of the 11th pencil.

The median is the length of the 22nd pencil (11 × 2).

The upper quartile is the length of the 33rd pencil (11 × 3).

These values can be read from the red lines on the graph:

- The median is 86 mm
- The lower quartile is 68 mm and the upper quartile is 101 mm, so the interquartile range is 101 − 68 = 33 mm.
- The semi-interquartile range is $\frac{33}{2} = 16.5$ mm.

The graph can also be used to obtain other information.

For example, to find the percentage of pencils that are at least 80 mm long:

The blue line shows that there are 18 pencils shorter than 80 mm long.

So there are 44 − 18 = 26 pencils 80 mm or longer.

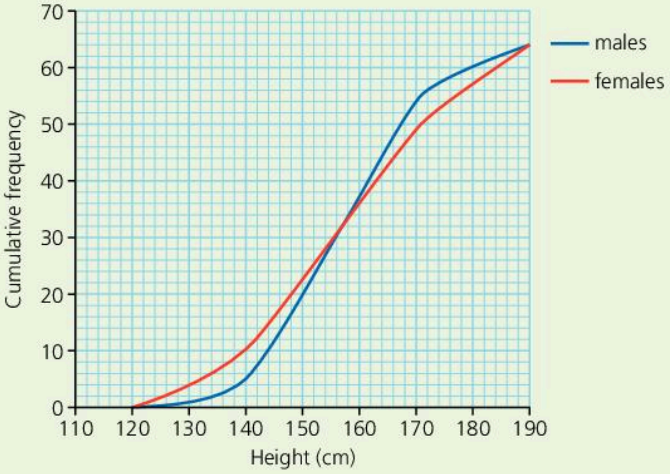
The percentage is $\frac{26}{44} \times 100 = 59.1\%$ (to 1 decimal place).

EXAM TIP

- The cumulative table has a column for maximum value, and the cumulative frequency includes all frequencies up to that value.
- Cumulative frequency graphs do not necessarily start from the origin (0, 0).

ACTIVITY

The cumulative frequency graph shows the heights of boys (in blue) and girls (in red) in a school.



- The graphs are different shapes. What does this difference tell you?

KEY POINTS

1. A cumulative frequency table shows the sum of the frequencies up to a given point.
2. A cumulative frequency graph always ascends from left to right.
3. The median and quartiles can be read from a cumulative frequency graph.

SUMMARY QUESTIONS

Andrew did a survey on the beach.

He measured the lengths of 60 pieces of seaweed.

The results of the survey are shown in the table.

Length of seaweed (cm)	Frequency
1–20	2
21–40	22
41–60	15
61–80	10
81–100	6
101–120	4
121–140	1

1. Draw a cumulative frequency table and graph to show this information.
2. Use your graph to find the lower and upper quartiles.
3. What percentage of the seaweed was over 75 cm long?

2.17 Summary statistics

LEARNING OUTCOMES

- use standard deviation to compare sets of data
- make inferences from summary statistics

Standard deviation

The range and interquartile range are both measures of dispersion, or how spread out the data are.

Each has a weakness.

The range only uses the largest and smallest values, and so is affected by outliers.

The interquartile range only uses the middle 50% of the data, and so gives no indication of how spread the top and bottom 25% are.

James and Henry both took 11 tests. Here are their scores, put in order:

											TOTAL
James	11	14	14	14	15	16	17	17	19	19	20	176
Henry	7	9	16	16	16	17	17	19	19	20	20	176

The range for James is 20 – 11 = 9. Henry's range is 20 – 7 = 13.

This suggests that Henry's scores are more varied.

But James' interquartile range is 19 – 14 = 5. Henry's is 19 – 16 = 3.

This suggests that James' scores are more varied.

Just as the mean is an average that uses all the data, the standard deviation is a measure of dispersion that uses every piece of data.

You do not need to be able to calculate the standard deviation for your examination. The example below is just to help you understand it.

WORKED EXAMPLE 1

Both sets of data have a total of 176, and so the mean is 176 ÷ 11 = 16.

The table below shows how much each score is above (+) or below (–) the mean.

											TOTAL
James	–5	–2	–2	–2	–1	0	+1	+1	+3	+3	+4	0
Henry	–9	–7	0	0	0	+1	+1	+3	+3	+4	+4	0

If we add these up, we get 0. This will be the case for any set of data.

So we square these scores to make them positive.

											TOTAL
James	25	4	4	4	1	0	1	1	9	9	16	74
Henry	81	49	0	0	0	1	1	9	9	16	16	182

The mean squared difference for each student is:
James = 74 ÷ 11 = 6.73 Henry = 182 ÷ 11 = 16.55

Finally, take the square root of the mean squared difference:
James = $\sqrt{6.73}$ = 2.59 Henry = $\sqrt{18}$ = 4.07

These figures are the **standard deviation**.

The steps were:

- Find the difference between each piece of the data and mean.
- Square those differences.
- Find the mean of those squares.
- Find the square root of the mean of the squares of the differences.
- This tells us that, **when all results are take into consideration**, Henry's scores are more spread out than James'.

Using summary statistics

When comparing sets of data, the central measures (mean, median and mode) give an overall impression of which set is higher.

The measures of dispersion (range, interquartile range and standard deviation) give an indication of which set is more varied.

WORKED EXAMPLE 2

The table shows data about the number of hours per week that a group of 20 males and 20 females spent watching television in a week.

	Male	**Female**
Mean	13.4	15.2
Median	13.5	15
Mode	15	14
Range	13	6
Interquartile range	4.5	3
Standard deviation	3.3	1.8

The mean and median both suggest that the females watch more on average than the males. The mode suggests otherwise, but as it is a small sample it is less reliable than the mean and median.

The males have a higher range, interquartile range and standard deviation. This means that the male results are more spread out whereas the females' figures will be closer together.

KEY POINTS

1. The standard deviation is measure of dispersion that sues every piece of data.
2. The averages (mean, median and mode) tell you which set of data is generally higher.
3. The measures of dispersion (range, interquartile range and standard deviation) tell you which set of data is more spread out.

SUMMARY QUESTIONS

Robert and Sarah took 6 maths tests each. Robert's mean score was 65 and his standard deviation was 12.

Sarah's mean score was 62 and her standard deviation was 16.

1. Who do you think did better overall? How can you tell?
2. Whose scores were more consistent (closer together)? How can you tell?
3. Who do you think had the highest single test score? Can you tell?

2.18 Probability

LEARNING OUTCOMES

- Understand the probability scale
- Determine experimental probabilities of events

ACTIVITY

- Put a counter in a START square.
- Roll a dice.

START 1	START 2	START 3	START 4	START 5	START 6

  - If you throw a 1, go DOWN a square.
  - If you throw a 2 or a 3, go LEFT a square.
  - If you throw a 4, 5 or 6, go RIGHT a square.
  - Carry on until your counter comes off the grid.
  - If it goes off to the LEFT or RIGHT you lose.
  - If it goes off the BOTTOM, you win.
- Keep playing the game, using different start squares.
- Which start square gives you the best chance of winning?
- Compare your results with others.

EXAM TIP

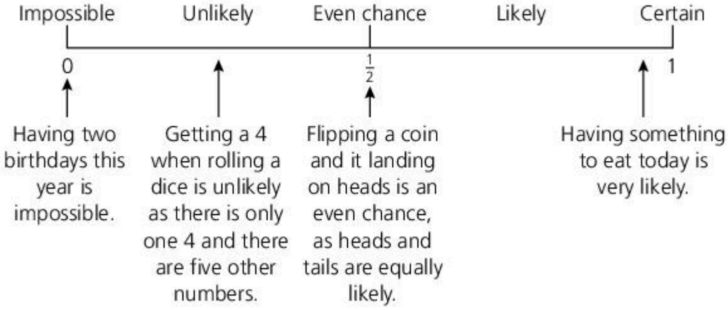
- The experimental probability

$$= \frac{\text{Number of times the event occurs}}{\text{Total number of trials}}$$

Probability is the measure of how likely an **outcome** is.

- An **impossible event** has a probability of 0.
- Something that is **certain** is given a probability of 1.
- All other probabilities lie between 0 and 1.

Probabilities can be represented by marking the position on a **probability scale**.



Experimental probability

When you flip a drawing pin, it might land point up or point down. It is not easy to say which is more likely.

WORKED EXAMPLE 1

Barbara performed an experiment.

She flipped a drawing pin 100 times.

It landed point up 38 times and point down 62 times.

So it landed point up 38 times out of 100, or $\frac{38}{100}$.

We say the experimental probability of the pin landing point up is $\frac{38}{100}$.

When finding an experimental probability, it is very important to perform the experiment a large number of times.

Probabilities can be written as fractions, decimals or percentages, so we could say the probability in the experiment above was $\frac{38}{100}$, 0.38 or 38%.

The meaning of probability

A probability of $\frac{38}{100}$ means that the outcome is unlikely, as it is less than $\frac{1}{2}$.

It means that, on average, we would expect the drawing pin to land point up 38 times out of every 100.

It is important to understand that this is merely an expectation; in the real world, the unexpected happens occasionally.

Using probability

Probability can be used to say how likely something is. It can also help us calculate the number of times an event will occur.

$$\text{Probability} = \frac{\text{Number of times the event occurs}}{\text{Total number of trials}}$$

So,

$$\text{Probability} \times \text{Total number of trials} = \text{Number of times the event occurs}$$

WORKED EXAMPLE 2

Polly wanted to win a cuddly toy at the fair.

To win, she had to hook a plastic duck with a winning number on the bottom.

She watched people playing. Out of 40 people, 6 won a cuddly toy.

There were 100 plastic ducks. Polly wanted to know how many had a winning number.

She decided that the experimental probability of winning was $\frac{6}{40} = \frac{3}{20}$

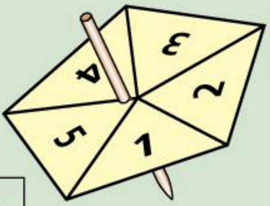
So $\frac{3}{20}$ of the ducks might be winners.

$\frac{3}{20}$ of 100 = 15.

So she thinks there are about 15 winning ducks.

SUMMARY QUESTIONS

1 Isobel made a spinner out of card and a cocktail stick.
She did not quite get the cocktail stick through the centre.
She recorded the results of 80 spins:



Score	1	2	3	4	5
Frequency	10	17	18	11	24

a Calculate the experimental probability that she scores a one.

b Calculate the experimental probability that she scores a five.

2 Marsha works in a factory making light bulbs.
She tests a sample of 50, and finds that 6 do not work.
How many are likely to work out of a box of 400 bulbs?

3 George has a bag containing some counters. He tries an experiment where he takes a counter, makes a note of its colour and then puts it back. He does this 100 times with these results:

Colour	Red	Blue	Black	White
Frequency	32	19	38	11

If there are 16 counters in the bag, how many of each colour are there likely to be?

KEY POINTS

1 Probability can be written as a fraction, decimal or percentage.

2 Probability is always between 0 (impossible) and 1 (certain).

3 Probability can be found by experiment if the number of trials is sufficiently large.

2.19 Probability theory

LEARNING OUTCOMES

- Determine the sample space for simple experiments (set of all possible outcomes)
- Determine theoretical probabilities of events
- Understand that the sum of the probabilities of all possibilities must equal 1

Equally likely outcomes

Sometimes we are faced with a number of events which are all equally likely.

For example, when we flip a coin, it is just as likely to land on heads as tails.

When we roll a dice, all six outcomes (1, 2, 3, 4, 5 and 6) are equally likely.

When all outcomes are equally likely, the probability of an event is:

$$\frac{\text{number of successful outcomes}}{\text{total number of possible outcomes}}.$$

The probability of a coin landing on heads is $\frac{1}{2}$, as there are two possible outcomes, heads and tails.

A dice has six possible outcomes, 1, 2, 3, 4, 5 and 6.

So the probability of it landing on 5 is $\frac{1}{6}$.

The probability of it landing on a number greater than 4 is $\frac{2}{6} = \frac{1}{3}$.

Sample space diagrams

When answering probability questions, it is often useful to list all the possible outcomes. This helps us to find the number of equally likely outcomes.

Imagine flipping a 10¢ coin and a 5¢ coin.

We might think there are three possible outcomes: both heads, both tails or one of each, so the probability of both coins landing on heads is $\frac{1}{3}$. However, this is incorrect.

The probability of both coins landing on heads is $\frac{1}{4}$.

The probability of getting a head and a tail is $\frac{2}{4} = \frac{1}{2}$.

A listing of all possible outcomes is called a **sample space diagram**.

A 2-way table is a quick and accurate way of constructing a sample space diagram.

The possible outcomes are:

10¢	5¢
H	h
H	t
T	h
T	t

This can be shown in a 2-way table:

		5¢	
		h	**t**
10¢	**H**	Hh	Ht
	T	Th	Tt

For example, here is a sample space diagram for rolling two dice:

		Blue dice					
		1	**2**	**3**	**4**	**5**	**6**
Red dice	**1**				🟩		
	2			🟩			
	3		🟩				
	4	🟩					
	5						
	6						

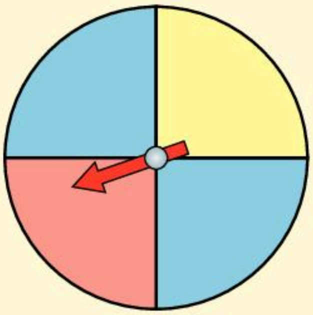
The green squares show the ways of scoring a total of 5.

The probability of scoring a total of 5 is $\frac{4}{36} = \frac{1}{9}$.

The sum of probabilities

The sum of the probabilities of all the possible outcomes of an event is 1.

WORKED EXAMPLE 1



With this spinner, the probability of it landing on red is $\frac{1}{4}$.

The probability of it landing on blue is $\frac{1}{2}$.

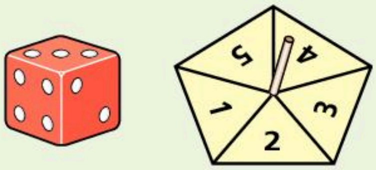
The probability of it landing on red or blue is $\frac{1}{4} + \frac{1}{2} = \frac{3}{4}$.

The probability of it landing on red, blue or yellow is $\frac{1}{4} + \frac{1}{2} + \frac{1}{4} = 1$.

They add up to 1 because it is certain it will land on red, blue or yellow.

ACTIVITY

A five-sided spinner and a dice are thrown. The scores are multiplied together.



- Which two scores are most likely? What is the probability of each of these scores?

WORKED EXAMPLE 2

Ben, Charles and David are having a race.

The probability that Ben wins is $\frac{1}{10}$.

The probability that Charles wins is $\frac{1}{2}$.

So the probability that Ben or Charles wins is $\frac{1}{10} + \frac{1}{2} = \frac{1}{10} + \frac{5}{10} = \frac{6}{10} = \frac{3}{5}$.

The sum of all the probabilities is 1.

So, the probability that David wins is $1 - \frac{3}{5} = \frac{2}{5}$.

EXAM TIP

- Remember how to add fractions by finding a common denominator.

KEY POINTS

1 When all the outcomes are equally likely, the probability of an event is $\frac{\text{number of successful outcomes}}{\text{total number of possible outcomes}}$.

2 A sample space diagram is a list of all possible outcomes.

3 The sum of the probabilities of all possible outcomes is 1.

SUMMARY QUESTIONS

Teri rolls a dice and flips a coin.

1 Draw a sample space diagram to show the possible outcomes.

2 What is the probability of getting an even number on the dice and a head on the coin?

3 Teri uses either a red dice, a blue dice or a green dice. The probability that she uses a red dice is 0.4, and the probability that she uses a blue dice is 0.55. What is the probability that she uses a green dice?

2.20

Combining probabilities

LEARNING OUTCOMES

- Use contingency tables
- Use the addition rule for exclusive events
- Use the multiplication rule for independent events

The multiplication rule

Independent events are events where the outcome of one event does not influence the other event.

Sophie rolls a fair 6-sided dice and flips a fair coin. These are independent events as the coin has an even chance of landing on heads or tails regardless of the score on the dice.

The sample space diagram shows all possible outcomes:

		Score on dice					
		1	**2**	**3**	**4**	**5**	**6**
Coin outcome	**Heads**	1 Heads	2 Heads	3 Heads	4 Heads	5 Heads	6 Heads
	Tails	1 Tails	2 Tails	3 Tails	4 Tails	5 Tails	6 Tails

There are 12 possible equally likely outcomes.

The probability of her rolling a 4 and flipping a tail $= \frac{1}{12}$.

The multiplication rule says:

To find the probability of 2 or more independent events occurring, multiply the probabilities of each event.

Probability of rolling a 4 and flipping a tail

= probability of rolling a 4 × probability of flipping a tail

$= \frac{1}{6} \times \frac{1}{2} = \frac{1}{12}$.

WORKED EXAMPLE 1

The probability that Brian is late for school tomorrow is $\frac{1}{10}$.

The probability that Sarah is late for school tomorrow is $\frac{1}{15}$.

Both events are independent.

The probability that Brian and Sarah are both on time

= probability that Brian is on time x probability that Sarah is on time

$= \left(1 - \frac{1}{10}\right) \times \left(1 - \frac{1}{15}\right) = \frac{9}{10} \times \frac{14}{15} = \frac{126}{150} = \frac{21}{25}$

The addition rule

Mutually exclusive events are events that cannot both happen at the same time.

When Sophie rolls her dice and flips her coin, she cannot roll a 3 and a 4 at the same time. They are mutually exclusive.

She can roll a 3 and flip a head at the same time. They are not mutually exclusive.

Brian arriving late for school and Sarah arriving late for school are not mutually exclusive as they both might be late!

The addition rule says:

To find the probability of either of 2 mutually exclusive events occurring, add the probabilities of each event.

WORKED EXAMPLE 2

Patricia is making a necklace using red beads, blue beads, white beads and black beads. She chooses each bead at random.

The table shows the probability of choosing a red, blue or white bead.

	Red	**Blue**	**White**	**Black**
Probability	0.2	0.35	0.3	

The probability of the next bead being white or black = probability of white + probability of black

$= 0.3 + (1 - 0.2 - 0.35 - 0.3) = 0.3 + 0.15 = 0.45$

KEY POINTS

1. Multiply probabilities of independent events to find the probability of both occurring.
2. Add probabilities of mutually exclusive events to find the probability of either occurring.

Contingency tables

A contingency table is a table for showing results in two different classes.

WORKED EXAMPLE 3

A company tests a new drug on 100 patients to see if it produces an improvement in a skin condition.

The results are shown below:

		Gender		
		Male	**Female**	**TOTAL**
Result	**Improvement**	32	16	48
	No improvement	24	28	52
	TOTAL	56	44	100

A patient selected at random has a probability of $\frac{48}{100} = \frac{12}{25}$ of showing improvement.

A male patient selected at random has a probability of $\frac{32}{56} = \frac{4}{7}$ of showing improvement.

A patient selected at random showed improvement. The probability that the patient was female is $\frac{16}{48} = \frac{1}{3}$.

SUMMARY QUESTIONS

The table shows the number of left-handed and right-handed students in a class.

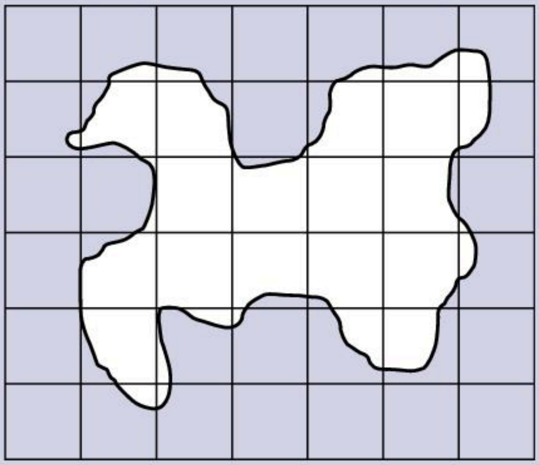
		Gender	
		Male	**Female**
Handedness	**Left-handed**	4	3
	Right-handed	18	15

Two students, one of each gender, are selected at random.

1. Calculate the probability that both students selected are right-handed.
2. What is the probability that at least one student selected is left-handed?
3. Calculate the probability that both students selected are left-handed, and use your answer to calculate the probability that exactly one student selected is left-handed.

Module 2 Practice exam questions

1 The island below is drawn to a scale of 1 cm to 5 km.
Estimate the area of the island in km^2.



2 A trapezium and a triangle have the same area.
The trapezium has parallel sides of 8 cm and 4 cm, and a height of 5 cm.
The triangle has a height of 12 cm.
Calculate the base of the triangle.

3 The diagram shows three-quarters of a circle of radius 10 cm, and two radii.
Calculate:

a the area

b the perimeter of the shape, giving your answers to 1 decimal place.

4 A cone has a radius of 5 cm and a height of 9 cm. Calculate its volume.

5 A cylinder has a radius of 9 cm and a height of 13 cm.
Calculate:

a the volume

b the surface area.

6 Daisy cycles 12 miles in 2 hours 15 minutes.

a Roughly how many kilometres is 12 miles?

b Give an approximate value for Daisy's speed in km/h.

7 Melissa travels 300 km at an average speed of 80 km/h.
Calculate the time it took in hours and minutes.

8 A water butt contains 250 gallons of water to the nearest 10 gallons.
The water runs out at a rate of 4 gallons per minute to the nearest integer.
Calculate:

a the maximum time

b the minimum time it will take for the butt to empty.

9 Match the data descriptions with the data types:

1 Shoe size
2 Shoe colour
3 Foot length

A Qualitative data
B Discrete data
C Continuous data

10 I fill in a questionnaire. One question says:

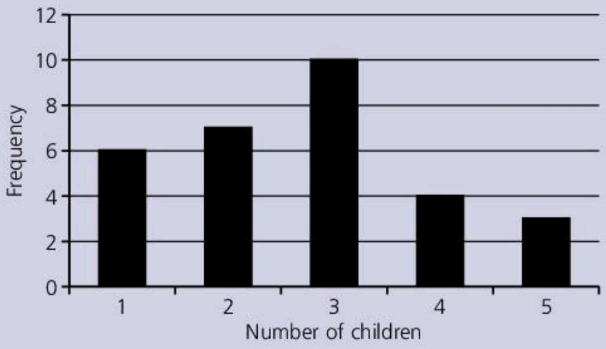
What is your height?
Up to 160 cm ☐
161 cm–170 cm ☐
171 cm–180 cm ☐
181 cm–190 cm ☐
Over 190 cm ☐

My height is 178.4 cm.

a Which category do I select?

b What are the class boundaries for this category?

11 The bar chart shows the number of children in the families in a school class.



Show this information in a pie chart.

12 Michael draws a pie chart to show the age of people in his street.
There are 8 people in the age group 40–60 out of a total of 40 residents.
What angle should he use for this category?

13 Five friends have a mean age of 17 years.
One friend moves away. The mean age of the remaining four is 19.
What is the age of the friend who moved away?

14 Bronwen owns a pet shop.
The table gives information about the weights of hamsters in Bronwen's shop.

Weight, *w*, of hamsters in g	**Number of hamsters**		
29–30	9		
31–32	5		
33–34	4		
35–36	2		

a Which is the modal group?

b Copy the table and use the two additional columns to calculate an estimate for the mean weight of the hamsters in Bronwen's shop.

c Draw a cumulative frequency graph to show the weights of the hamsters.

d Find the interquartile range of the weights.

15 I roll a 4-sided dice and a 6-sided dice.

a Draw a sample space diagram to show all possible outcomes.

b What is the probability that the two dice show the same score?

16 A bag contains red, blue and green counters.
The probability of choosing a red counter at random is $\frac{1}{4}$, and the probability of choosing a blue one is $\frac{2}{3}$.

a What is the probability of choosing a green one?

b If the bag contains 16 counters that are either red or green, how many counters are there of each colour?

17 Drissa cycles to work at an average speed of 12 km/h. The journey takes him 15 minutes. He does the return journey in 20 minutes. At what speed does he cycle home?

18 A cuboid has dimensions of 6 cm, 8 cm and 18 cm.
Find the dimensions of a cube which has the same surface area as the cuboid.

3 Algebra and relations, functions and graphs

3.1 Symbols for numbers

LEARNING OUTCOMES

- Use symbols and letters to represent numbers, operations and variables
- Translate statements expressed algebraically into words and vice versa

Why use symbols instead of numbers?

There are times when we use symbols or letters in place of numbers.

Sometimes it is used as a short hand when we do not know what a number is.

For example, we might say:

> I think of a number. I add three. I double the answer. I subtract 4. Then I take away the number I first thought of. The answer is 7.

If we use the letter n to stand for the number, we could write this in a much shorter way:

$$(n + 3) \times 2 - 4 - n = 7.$$

Remember we need to put brackets around the $(n + 3)$ to show we work it out before multiplying by 2.

Letters like 'n' in algebra represent specific numbers and are called **unknowns**.

On other occasions, we use a letter or symbol when it does not matter what the number is.

For example, the area of a triangle, A, is given by

$$A = \frac{b \times h}{2},$$

where b is the length of the base and h is the perpendicular height. The formula is true whatever the values of b and h happen to be.

Letters like 'b' and 'h' in algebra represent numbers, and are called **variables**, because their value is not fixed and can vary.

Algebraic notation

Using letters or symbols allows us to be concise. But we all need to write algebra in the same way or we will be misunderstood.

- To add a and b we write $a + b$ or $b + a$.
- To subtract h from 7, we write $7 - h$.

We avoid multiplication signs between letters and between a number and a letter.

- To multiply a by b we write ab.
- When multiplying a letter and a number, the number is always written first: m multiplied by 4 is written as $4m$, not $m4$.

Division is usually written as a fraction

- $a \div b$ is written as $\frac{a}{b}$.

Brackets are used to show an operation that must be carried out first.

- $3(a - b)$ means subtract b from a first as it is in brackets, and then multiply by 3.

Note that the 3 next to the bracket means multiply, just as in $3a$.

In an expression like ab^2, we carry out the square before multiplying. So

- ab^2 means $b \times b \times a$

but

- $(ab)^2$ means $(a \times b) \times (a \times b)$

Words to algebra

Robert's sister Marlene is two years older than Robert. Their mother Marcia is twice as old as Marlene. Marcia is three times as old as her younger daughter, Miriam.

We do not know their ages, but we can use algebra to show the connections.

If Robert is r years old,

Marlene is $r + 2$ years old as she is two years older.

Marcia is $2(r + 2)$ years old as her age is twice Marlene's.

Miriam is $\frac{2(r + 2)}{3}$ years old, one third of Marcia's age.

Algebra to words

The expression $\frac{2(x - 4)}{3} - 5$ looks a little daunting at first, but it is easier to understand if we use BIDMAS, which we met in 1.10.

Brackets first:	Start with x and subtract 4.
Indices:	there are none.
Division and Multiplication:	Multiply $(x - 4)$ by 2 and divide by 3.
Addition and Subtraction:	Subtract 5 from the answer.

We will meet this again in 3.2.

SUMMARY QUESTIONS

1 There are n coconuts on a tree. Another tree has two more coconuts.
How many coconuts are there on the second tree?

2 At 10 o'clock there were c cars in a car park.
Between 10 o'clock and 12 o'clock, 12 more cars arrived.
Between 12 o'clock and 3 o'clock, half the cars left.
How many cars were in the car park at 3 o'clock?

3 Find the value of $3 + 2 \times \frac{4^2}{2}$

ACTIVITY

The names and symbols we give to numbers are arbitrary, although for higher numbers there is a system.

The French call 98 'Quatre vingt dix huit', which translates to 'four twenties, ten and eight'.

Here are the numbers from 1 to 10 in Arawak.

1. Aba
2. Bian
3. Kabun
4. Biti
5. Abadakabo
6. Abatian
7. Bianitian
8. Kabunitian
9. Bistia
10. Bianidakabo

- What evidence is there that the Arawak people counted in fives? (Dakabo is an Arawak word for hand).

EXAM TIP

- Always use BIDMAS when performing calculations.

KEY POINTS

1 Letters and symbols can stand for unknown numbers, or for any number (variable).

2 ab means $a \times b$.

3 $\frac{a}{b}$ means $a \div b$.

3.2

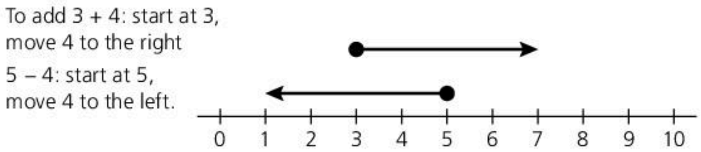
Directed numbers and substitution

LEARNING OUTCOMES

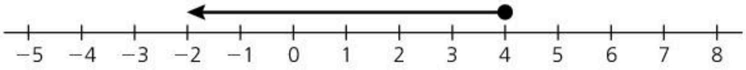
- Perform arithmetic operations involving directed numbers
- Substitute numbers for algebraic symbols in expressions

Negative numbers and the number line

Many children learn to add and subtract on a number line.



For questions like $4 - 6$, with negative answers, we need to extend the number line to the left, to introduce numbers below zero.



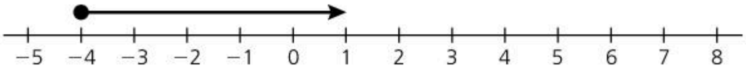
$4 - 6 = -2$

Addition and subtraction with negative numbers

All addition and subtraction questions can be solved on a number line, just like the ones above.

For example: $-4 + 5$

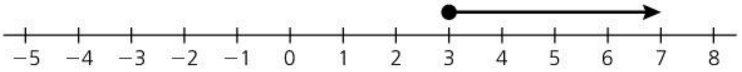
Start at the first part (-4) and move 5 to the right (+).



However, when the second number (to be added or subtracted) is negative, then you must move in the opposite direction.

For example: $3 - -4$

Start at 3, face left (−). But as the number to be subtracted is negative (-4), move 4 to the right instead.



Multiplication and division with negative numbers

We know that:

$2 \times 4 = 8$ $12 \div 3 = 4$ $2 \times 3 \times 5 = 30$

If one number is negative, the answer is negative:

$-2 \times 4 = -8$ $12 \div -3 = -4$ $2 \times -3 \times 5 = -30$

If two numbers are negative, the answer is positive:

$-2 \times -4 = 8$ $-12 \div -3 = 4$ $-2 \times 3 \times -5 = 30$

If three numbers are negative, the answer is negative:

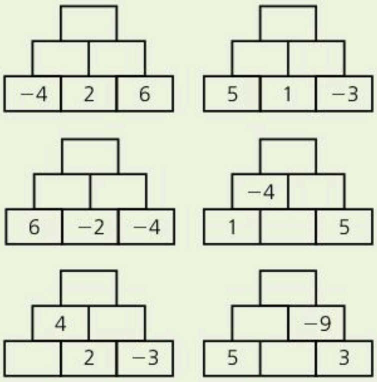
$-2 \times -3 \times -5 = -30$

In general:

if there is an **even** number of negative numbers the answer is **positive** and if there is an **odd** number of negative numbers the answer is **negative**.

ACTIVITY

- In the walls below, add two adjacent numbers to calculate the number in the brick above them.



- Now try again, but this time multiply two adjacent numbers to calculate the number in the brick above them.

Substitution

Substitution is where we replace letters in an expression with a number, and calculate the result.

For example, to find the value of $\frac{a - 2b}{c}$ if $a = 2$, $b = -3$ and $c = -4$, the value of

$$\begin{aligned}\frac{a - 2b}{c} &= \frac{2 - 2 \times -3}{-4}\\ &= \frac{2 - -6}{-4}\\ &= \frac{8}{-4}\\ &= -2\end{aligned}$$

EXAM TIP

- Remember to follow BIDMAS.

ACTIVITY

Work through this maze.

- You can only move horizontally or vertically.
- You can only move onto a square with a value of 5 when $a = 2$, $b = -1$ and $c = -3$.

START **5**	$a - c$	$5(a - b)$	$\frac{c^2 - b}{a}$	$b(2b + c)$	$a^3 + c$	$a - bc$
$ac + b$	$b - ac$	$a + c$	$b - 2c$	$c^2 - b$	$a^2 + 1$	$ab + c$
$a + bc$	$a^2 - b$	$2c + b$	$a + \frac{c}{b}$	$a + c$	$12 + ac$	$c - a$
$c^2 - 2a$	$ab - c$	$4a + c$	$5(a + b)$	$a^2 + b$	$(c - b)^2 - b$	$\frac{5c}{b - a}$
$b^2 + 2a$	$abc + b$	$ab^2 - c$	$\frac{2a^2 + b}{bc}$	$b + 2c$	$a^2 - b^2$	**FINISH** $c^2 - a^2$

KEY POINTS

1 To add or subtract directed numbers, use a number line, remembering to move backwards if the last number is negative.

2 To multiply or divide directed numbers, calculate the numerical answer first, and then if there is an even number of negative numbers the answer remains positive but if there is an odd number of negative numbers the answer will be negative.

3 Use BIDMAS when calculating and substituting.

SUMMARY QUESTIONS

1 Calculate:

a $4 + -5$ **b** $-2 - 4$ **c** $3 - -1$

d 6×-3 **e** $\frac{(-2)^2}{-4}$

2 If $a = -3$ and $b = -2$, evaluate:

a ab **b** ab^2 **c** a^2b **d** $(ab)^2$

3 Which of these can never be negative, whatever the value of n?

n^2 $-n^2$ n^3

3.3 Combining expressions

LEARNING OUTCOMES

- Perform four basic operations with algebraic expressions
- Use laws of indices to manipulate expressions
- Remove brackets (using the distributive law)
- Calculate with algebraic fractions

Collecting like terms

Like terms are terms containing identical variables or combinations of variables.

So $3x$, $6x$ and x are like terms. So are $4ab$, $-3ab$ and ba.

But a^2 and $2a$ are not, neither are ab^2 and a^2b because the powers of the corresponding letters are not the same.

Addition and subtraction

Only like terms can be added or subtracted, in much the same way only fractions with the same denominator can be added or subtracted.

So $3a + 2a = 5a$

$8b^2 - b^2 = 7b^2$ (because b^2 is the same as $1b^2$)

$6a - 2ab - 3a + 5ab = 6a - 3a - 2ab + 5ab = 3a + 3ab$.

Multiplication and division

Any terms can be multiplied or divided.

$$a \times b = ab$$

$$4a \times 2ab = 4 \times a \times 2 \times a \times b = 4 \times 2 \times a \times a \times b = 8a^2b$$

$$4a \div 2ab = \frac{4a}{2ab} = \frac{2}{b}$$

Laws of indices

There are five important rules when using indices:

1 Whenever we multiply powers of the same number, we add the indices.

$$a^4 \times a^3 = (a \times a \times a \times a) \times (a \times a \times a) = a^7$$

In general: $n^a \times n^b = n^{a+b}$

2 To raise a power to another power, multiply the powers.

$$(a^4)^3 = a^4 \times a^4 \times a^4 = a^{12}$$

In general: $(n^a)^b = n^{a \times b}$

3 Whenever we divide powers of the same number, we subtract the indices.

$$d^6 \div d^2 = \frac{d \times d \times d \times d \times d \times d}{d \times d} = d^4$$

In general: $n^a \div n^b = n^{a-b}$

4 Any number raised to the power 0 is 1.

$n^3 \div n^3 = 1$ as any number divided by itself is 1.

But, using the rule for subtraction of indices

$$n^3 \div n^3 = n^{3-3} = n^0 = 1$$

5 A negative power and the same positive power are reciprocals.

$a^5 \times a^{-5} = a^{5+-5} = a^0 = 1$, so a^5 and a^{-5} are reciprocals

$$a^{-5} = \frac{1}{a^5}$$

In general: $n^{-a} = \frac{1}{n^a}$

Using these rules allows us to simplify complex expressions:

$$(3a^2)^3 \times 4a^3b^2 \div 6a^4b^7$$
$$= 27a^6 \times 4a^3b^2 \div 6a^4b^7$$
$$= 108a^9b^2 \div 6a^4b^7$$
$$= 18a^5b^{-5} \text{ or } \frac{18a^5}{b^5}$$

Removing a pair of brackets

$$3(2a - b) = (2a - b) + (2a - b) + (2a - b) = 6a - 3b$$

Both $2a$ and $-b$ have been multiplied by 3.

We can always remove a pair of brackets by multiplying each term inside the brackets by the term outside.

For example

$$3a(2a - 5b + c) = 6a^2 - 15ab + 3ac$$

Algebraic fractions

We deal with algebraic fractions in exactly the same way as numeric fractions.

To add or subtract:

$$\frac{a}{2bc} - \frac{c}{ab}$$

First, find a common denominator.

We can use $2abc$, as $2bc \times a = ab \times 2c = 2abc$.

$$\frac{a}{2bc} - \frac{c}{ab} = \frac{a \times a}{2bc \times a} - \frac{c \times 2c}{ab \times 2c} = \frac{a^2}{2abc} - \frac{2c^2}{2abc}$$
$$= \frac{a^2 - 2c^2}{2abc}$$

To multiply or divide:

$$\frac{3a^3b}{2c^2} \div \frac{ab^3}{4cd}$$

When dividing, turn the second fraction upside down

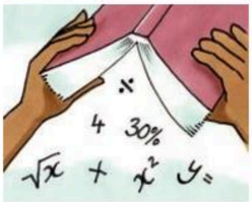
$$= \frac{3a^3b}{2c^2} \times \frac{4cd}{ab^3}$$

Multiply numerators, multiply denominators

$$= \frac{12a^3bcd}{2ab^3c^2}$$

And simplify

$$= \frac{6a^2d}{b^2c}$$



KEY POINTS

1. To multiply a term by a bracket, multiply each term in the brackets by the term outside.
2. The laws of indices:

$$n^a \times n^b = n^{a+b}$$
$$(n^a)^b = n^{ab}$$
$$n^a \div n^b = n^{a-b}$$
$$n^0 = 1$$
$$n^{-a} = \frac{1}{n^a}$$

EXAM TIP

- Remember to multiply the numerator and denominator by the same amount.

ACTIVITY

- Match the expressions on the left with the simplified versions on the right.

$2(4x - y)$	$12x^2y$
$\frac{2x}{y} \times \frac{y}{x}$	$\frac{2x^2 + y^2}{xy}$
$5x - 2y - 3x$	$2x - 2y$
$\frac{2x}{y} + \frac{y}{x}$	$12xy$
$4x \times 3y^2$	$8x - 2y$
$\frac{8xy^2}{3} \div \frac{2y}{9}$	$12x^2y^2$
$\frac{24x^3y}{2x}$	$12x^2y$
$\frac{4x^2y^3}{y} + \frac{8x^3y^2}{x}$	2

- Make up your own matching exercise like this one, and test it on a friend.

SUMMARY QUESTIONS

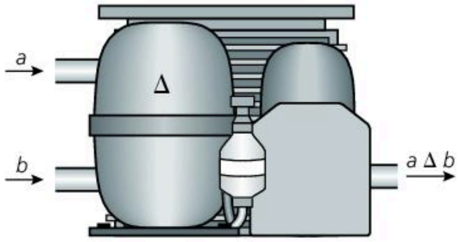
Simplify:

1. $5a - 2b + 3(b - 2a)$
2. $\frac{b}{2a} + \frac{d}{ac}$
3. $\frac{4b^2}{15a^4c} \times \frac{10a^2b^5}{3c^3}$

3.4 Binary operations

LEARNING OUTCOMES

- Perform binary operations by substitution



We have seen that letters and symbols can be used to represent numbers.

We can also use symbols to represent **operations**.

Addition, subtraction, multiplication and division are all **binary operations** because they operate on two numbers.

We can define our own binary operations.

The perimeter of a rectangle with sides of *a* cm and *b* cm could be written as

$$a \square b = 2a + 2b.$$

Remember that to find the area of a triangle we multiplied the base by the height and then divided by 2.

We could write this as a binary operation, Δ, so that

$$a \Delta b = \frac{a \times b}{2},$$

where *a* is the base length and *b* is the perpendicular height.

A triangle has base 5 cm and height 6 cm. For this triangle $a = 5$ and $b = 6$.

$$a \Delta b = 5 \Delta 6 = \frac{5 \times 6}{2} = 15\,\text{cm}^2.$$

The associative, commutative and distributive laws

You first met these in 1.10.

Here we are going to see if the rules apply to our own binary operations.

The commutative law

You might notice that $5 \Delta 6 = 6 \Delta 5$, which suggests it might be **commutative**. But to be sure, we need to know it is true for all values of *a* and *b*.

The operation is commutative if $a \Delta b = b \Delta a$.

To check this, we must use algebra rather than numerical examples. If it is true for *a* and *b*, then it will be true for whatever values of *a* and *b* we choose.

$$a \Delta b = \frac{ab}{2} = \frac{ba}{2} = b \Delta a$$

So Δ is commutative.

The associative law

The operation is associative if $(a \Delta b) \Delta c = a \Delta (b \Delta c)$.

$$(a \Delta b) \Delta c = \frac{ab}{2} \Delta c = \frac{\frac{ab}{2} \times c}{2} = \frac{abc}{4}$$

$$a \Delta (b \Delta c) = a \Delta \frac{bc}{2} = \frac{a \times \frac{bc}{2}}{2} = \frac{abc}{4}$$

So Δ is associative.

The identity

The identity is 2, as it does not change the value of a number:

$$2 \Delta a = \frac{2a}{2} = a$$

The distributive law

The **distributive** law involves two operations.

If we define the operation ☺ as $x ☺ y = 2x + y$, then

$$3 ☺ 4 = 2 \times 3 + 4 = 10.$$

To check whether Δ is distributive over ☺, we need to find out if $a \Delta (x ☺ y) = (a \Delta x) ☺ (a \Delta y)$.

$$a \Delta (x ☺ y) = a \Delta (2x + y) = \frac{a \times (2x + y)}{2} = \frac{2ax + ay}{2}$$

$$(a \Delta x) ☺ (a \Delta y) = \frac{ax}{2} ☺ \frac{ay}{2} = \frac{2ax}{2} + \frac{ay}{2} = \frac{2ax + ay}{2}$$

So $a \Delta (x ☺ y) = (a \Delta x) ☺ (a \Delta y)$, so Δ is distributive over ☺.

The distributive law is the rule we use to multiply a single term by a bracket:

$$a(b + c) = ab + ac.$$

To check if ☺ is distributive over Δ:

We need to check if $a ☺ (x \Delta y) = (a ☺ x) \Delta (a ☺ y)$.

$$a ☺ (x \Delta y) = a ☺ \frac{xy}{2} = 2a + \frac{xy}{2}$$

$$(a ☺ x) \Delta (a ☺ y) = (2a + x) \Delta (2a + y) = \frac{(2a + x)(2a + y)}{2}.$$

☺ is not distributive over Δ.

EXAM TIP

- Use algebra to determine whether operations are associative, commutative or distributive, as this will show whether the law is true for all numbers.

ACTIVITY

- Show that ☺ is not associative or commutative.
- Two operations are defined as a ✪ $b = 2ab$, and a ✿ $b = a + b - 1$.
- Show that both ✪ and ✿ are associative.
- Show that both ✪ and ✿ are commutative.
- Find the identity for each operation.
- Is one operation distributive over the other?

KEY POINTS

1. A binary operation is a method of combining two numbers.
2. The associative law: $(a*b)*c = a*(b*c)$
3. The commutative law: $a*b = b*a$
4. The distributive law: $a*(b \circ c) = (a*b) \circ (a*c)$
5. The Identity, I, is the number for which $I*a = a$, for all elements.
6. The inverse of a, a', is the element for which $a*a' = I$.

SUMMARY QUESTIONS

These questions are about the operations defined by

$a \clubsuit b = (a^2 + b^2)$, and

$x \heartsuit y = (x - y)$.

1 Find the value of

a $5 \clubsuit 4$

b $5 \heartsuit 4$.

2 Show that $\clubsuit$ is commutative.

3 Show that $\heartsuit$ is not associative.

3.5 Expanding and factorising

LEARNING OUTCOMES

- Apply the distributive law to factorise or expand algebraic expressions, e.g.
  $x(a + b) = xa + xb$,
  $(a + b)(x + y)$
  $= (a + b)x + (a + b)y$
  $= ax + bx + ay + by$
- Simplify algebraic expressions
- Factorise algebraic expressions by finding a common factor
- Rewrite a quadratic expression in the form $a(x + h)^2 + k$ (completing the square).

Removing a pair of brackets

A reminder:

We learnt in 3.3 that $3(2a - b)$ means $3 \times (2a - b) = 6a - 3b$.

Both $2a$ and $-b$ have been multiplied by 3.

Also $3a(2a - 5b + c) = 6a^2 - 15ab + 3ac$

We can expand and simplify more complex expressions by expanding carefully and then simplifying:

$$3a(2b - 4) - b(2a - 3)$$

$$= 6ab - 12a - 2ab + 3b$$

Note the last term is positive, as $-b \times -3 = +3b$

$$= 4ab - 12a + 3b$$

Factorising by finding a common factor

Factorising is the inverse of expanding; to factorise an expression, we split it into factors, which usually involves brackets.

Using the example above,

to factorise $6a - 3b$ we must turn it back into $3(2a - b)$.

$6a$ and $3b$ have a common factor of 3, as $6a = 3 \times 2a$ and $3b = 3 \times b$.

3 is the highest common factor of the two terms.

So

$$6a - 3b = 3 \times 2a - 3 \times b$$

$$= 3(2a - b)$$

Using the distributive law in reverse

An expression such as $8a - 4ab + 6a^2$ has three terms.

Each term has a factor of $2a$.

They also have common factors of 1, 2 and a, but $2a$ is the highest common factor.

$$8a - 4ab + 6a^2 = 2a \times 4 - 2a \times 2b + 2a \times 3a$$

$$= 2a(4 - 2b + 3a)$$

Using the distributive law

This is not as complicated as it seems. The numerical part of each term, 8, 4 and 6, have a common factor of 2, and the letter parts, a, ab and a^2, have an a in common, so $2a$ is the common factor which can be extracted from each term.

EXAM TIP

- When factorising, make sure you have extracted all common factors.
- You can always check your factorising by seeing if your answer expands to the original expression.

Expanding two expressions

Sometimes we have two pairs of brackets to multiply, for example:

$(3x + 2)(5x - 1)$

Multiplication is distributive over addition, so

$$\begin{aligned}(3x + 2) \times (5x - 1) &= 3x(5x - 1) + 2(5x - 1) && \text{Distributive law}\\ &= 15x^2 - 3x + 10x - 2 && \text{Expanding}\\ &= 15x^2 + 7x - 2 && \text{Simplifying}\end{aligned}$$

The four terms come from multiplying the

First term in each bracket: $3x \times 5x = 15x^2$

Outer terms: $3x \times -1 = -3x$

Inner terms: $+2 \times 5x = +10x$

Last terms: $+2 \times -1 = -2$

Completing the square

Not all expressions will factorise.

An expression in the form $ax^2 + bx + c$ can always be written in the form $a(x + h)^2 + k$.

This is useful when solving quadratic equations, as you will see in unit 3.9

Completing the square uses the identity $(x + n)^2 = (x + n)(x + n) = x^2 + 2nx + n^2$.

Replacing n with $\frac{b}{2}$ gives

$\left(x + \frac{b}{2}\right)^2 = x^2 + bx + \left(\frac{b}{2}\right)^2$, or $x^2 + bx = \left(x + \frac{b}{2}\right)^2 - \left(\frac{b}{2}\right)^2$

So $x^2 + bx + c = \left(x + \frac{b}{2}\right)^2 - \left(\frac{b}{2}\right)^2 + c$

WORKED EXAMPLE 1

To write $x^2 - 6x + 2$ in completed square form,

use the identity $x^2 + bx + c = \left(x + \frac{b}{2}\right)^2 + \left(\frac{b}{2}\right)^2 + c$

$b = -6$, $c = 2$.

So $x^2 - 6x + 2 = \left(\frac{x + (-6)}{2}\right)^2 - \left(\frac{(-6)}{2}\right)^2 + 2 = (x - 3)^2 - 9 + 2$

$= (x - 3)^2 - 7$

SUMMARY QUESTIONS

1 Expand $3x(2a - 4x - 7)$

2 Expand and simplify $2(3x - 4y) - 3(x - y)$

3 Expand $(3a - 4)(4a + 3)$

4 Factorise fully:

a $b^2 + 5b$ **b** $2xy - 3y$ **c** $36^2 - 9c$

EXAM TIP

- To expand two expressions, remember FOIL.
- Remember to simplify if possible.

ACTIVITY

Match the expressions on the left with the expansions on the right.

$(x + 2)(x - 2)$	$x^2 - 4x$
$(x - 1)(x + 4)$	$x^2 - 4$
$x(x - 4)$	$x^2 - 4x + 3$
$(x - 2)^2 + 3$	$x^2 + 3x - 4$
$(x - 3)(x - 1)$	$x^2 - 4x + 7$

KEY POINTS

1. Use FOIL to expand two expressions.
2. You can factorise by extracting a common factor.
3. Always factorise fully.
4. To complete the square, use $x^2 + bx + c = \left(x + \frac{b}{2}\right)^2 - \left(\frac{b}{2}\right)^2 + c$

3.6 Further factorising

LEARNING OUTCOMES

- Factorise algebraic expressions including difference between two squares
- Prove two algebraic expressions to be identical

EXAM TIP

- Take great care with the signs. You are multiplying to expand, but adding to simplify.

Factorising into two pairs of brackets

In 3.5 we found out how to factorise by finding a common factor.

We also saw that

$$(3x + 2)(5x - 1) = 15x^2 - 3x + 10x - 2 = 15x^2 + 7x - 2$$

To factorise, we need to turn $15x^2 + 7x - 2$ back into $(3x + 2)(5x - 1)$.

The expression $15x^2 + 7x - 2$ contains three terms:

- The x^2 term, $15x^2$, is the product of the First terms in each bracket, $3x \times 5x$
- The x term, $+7x$, is the sum of products of the Outer and Inner terms, $3x \times -1$ and $+2 \times 5x$
- The constant term, -2, is the product of the Last terms, $+2 \times -1$

WORKED EXAMPLE 1

To factorise $3x^2 - x - 4$:

$3x^2$ can only factorise into $3x \times x$.

4 can factorise into 4×1 or 2×2, but one factor must be negative as the constant is -4.

So the possibilities that yield $3x^2$ and -4 are listed below. On the right of each possibility is the sum of products of the Outer and Inner terms, which must make $-\boldsymbol{x}$

	OUTER	INNER	SUM
$(3x + 4)(x - 1)$	$3x \times -1 = -3x$	$+4 \times x = +4x$	$-3x + 4x = +x$
$(3x - 4)(x + 1)$	$3x \times +1 = 3x$	$-4 \times x = -4x$	$3x + -4x = -\boldsymbol{x}$
$(3x + 1)(x - 4)$	$3x \times -4 = -12x$	$+1 \times x = +1x$	$-12x + 1x = -11x$
$(3x - 1)(x + 4)$	$3x \times +4 = 12x$	$-1 \times x = -1x$	$12x + -1x = +11x$
$(3x + 2)(x - 2)$	$3x \times -2 = -6x$	$+2 \times x = +2x$	$-6x + 2x = -4x$
$(3x - 2)(x + 2)$	$3x \times +2 = 6x$	$-2 \times x = -2x$	$6x + -2x = +4x$

The correct factorisation is $(3x - 4)(x + 1)$ as the outer and inner terms add up to $-\boldsymbol{x}$.

The difference between two squares

The expression $4x^2 - 9$ looks different to the type above. This is because there is no term in x.

However, it factorises in the same way, and is called the difference between two squares, because $4x^2$ is $(2x)^2$ and 9 is 3^2.

It factorises to $(2x + 3)(2x - 3)$. The outer terms have a product of $-6x$ and the inner terms multiply to $+6x$. The sum of these is 0, which is why there was no term in x in the original expression.

Generally, any expression of the form $a^2x^2 - b^2$ factorises to $(ax + b)(ax - b)$.

EXAM TIP

- Do not confuse the factorising of $x^2 - 4 = (x + 2)(x - 2)$ with $x^2 - 4x = x(x - 4)$.
- Sometimes a difference between two squares is not immediately obvious, for example $5x^2 - 45 = 5(x^2 - 9)$ $= 5(x + 3)(x - 3)$.

ACTIVITY

The expressions $x^2 - 5x - 6$ and $x^2 - 5x + 6$ look very similar, but the factorisations are quite different. Find the factors of each expression.

Proving two algebraic expressions are equal

Algebraic expressions can often be written in more than one way. The easiest way to show that they are equal is to simplify the more complex expression until it resembles the simpler one.

WORKED EXAMPLE 2

Show that $(x + 3)^2 + (2x - 4)^2 = 5(x^2 - 2x + 5)$.

Solution

$$\begin{aligned}(x + 3)^2 + (2x - 4)^2 &= (x + 3)(x + 3) + (2x - 4)(2x - 4)\\ &= x^2 + 3x + 3x + 9 + 4x^2 - 8x - 8x + 16\\ &= 5x^2 - 10x + 25\\ &= 5(x^2 - 2x + 5)\end{aligned}$$

KEY POINTS

1 You can factorise by extracting a common factor, or as the product of two pairs of brackets, including the difference between two squares.

2 Always factorise fully.

SUMMARY QUESTIONS

1 Factorise fully:

a $b^2 + 5b - 24$ **b** $x^2 - 2x + 1$

2 Factorise fully:

a $3c^2 - 4c + 1$ **b** $25c^2 - 16$

3 Show that $x^2 + (2x - 3)(x + 3) = 3(x^2 + x - 3)$.

3.7

Changing the subject of a formula

LEARNING OUTCOMES

- Change the subject of a formula
- Change the subject of a formula including roots and powers
- Change the subject of a formula where the subject appears on both sides
- Solve word problems

The subject of a formula

Most formulae are written to enable us to calculate one specific variable. That variable is called the **subject of a formula**.

For example, the formula for the area of a triangle, $A = \frac{bh}{2}$, is written in such a way that if we know b and h, we can calculate A.

If, instead, we wanted to draw a triangle with a given area and base, we would need a formula to calculate the height. We would change the subject so that the formula showed how to calculate h.

Changing the subject of a formula

A formula tells us the calculations required to work out a given variable.

To change the subject, we undo the calculations by applying the inverse operation.

The formula $A = \frac{bh}{2}$ tells us that $b \boxed{\times h} \boxed{\div 2} = A$.

Using inverses, we deduce that $b = \boxed{\div h} \boxed{\times 2}\ A$ or $\frac{2A}{h} = b$.

Note that we 'undo' the last operation first. It's a bit like taking off your shoes and socks. Because you put your shoes on after your socks, it is easier to take your shoes off before removing your socks.

To perform the same changes using mathematical terminology, we rearrange a formula by performing the same operation to both sides of the formula.

WORKED EXAMPLE 1

The area of a trapezium, A, is given by the formula $A = \frac{(a + b)}{2}h$,

where a and b are the lengths of the parallel sides and h is the perpendicular height.

To make b the subject: in the formula, b is added to a, then divided by 2, then multiplied by h.

We apply inverses in the reverse order:

Divide both sides by h: $\frac{A}{h} = \frac{(a + b)}{2}h \div h$

Simplify: $\frac{A}{h} = \frac{(a + b)}{2}$

Multiply both sides by 2: $\frac{2A}{h} = \frac{2(a + b)}{2}$

Simplify: $\frac{2A}{h} = a + b$

Subtract a: $\frac{2A}{h} - a = a + b - a$

Simplify: $\frac{2A}{h} - a = b$, or $b = \frac{2A}{h} - a$

WORKED EXAMPLE 2

The time, T seconds, that it takes for a pendulum to swing is given by the formula

$$T = 2\pi\sqrt{\frac{l}{g}},$$

where l is the length in m and g is the acceleration due to gravity in m/s^2.

To make l the subject, first divide by 2π:

$$\frac{T}{2\pi} = \frac{2\pi}{2\pi}\sqrt{\frac{l}{g}} \qquad \text{Simplify:} \qquad \frac{T}{2\pi} = \sqrt{\frac{l}{g}}$$

Square both sides: $\left(\frac{T}{2\pi}\right)^2 = \left(\sqrt{\frac{l}{g}}\right)^2$ Simplify: $\left(\frac{T}{2\pi}\right)^2 = \frac{l}{g}$

Multiply by g: $g\left(\frac{T}{2\pi}\right)^2 = \frac{lg}{g}$ Simplify: $l = g\left(\frac{T}{2\pi}\right)^2$

EXAM TIP

- Remember to apply the inverse of the last operation first.

The subject appears on both sides of the formula

WORKED EXAMPLE 3

Suppose we wanted to draw a rectangle so that the numerical values of the area and perimeter were equal.

Area = perimeter

$lw = 2l + 2w$

To make l the subject:

First, we must remove the term containing l from the right-hand side by subtracting $2l$ from both sides:

$$lw - 2l = 2l + 2w - 2l$$

$$lw - 2l = 2w$$

$$l(w - 2) = 2w \qquad \text{Taking } l \text{ as a common factor}$$

$$\frac{l(w - 2)}{w - 2} = \frac{2w}{w - 2} \qquad \text{Dividing both sides by } (w - 2)$$

$$l = \frac{2w}{w - 2}$$

ACTIVITY

- Using the formula $l = \dfrac{2w}{w - 2}$, find the value of l if $w = 3$, $w = 4$, $w = 5$ and $w = 6$.
- Can you find a value of l and w so that the area of the rectangle is 25 cm^2 and the perimeter is 25 cm?

SUMMARY QUESTIONS

1 Make n the subject of this formula:

$$S = \frac{3n - 1}{2}$$

2 Make k the subject of this formula:

$$r = 3k^2 + p$$

3 A sphere of radius r has a surface area of $4\pi r^2$. A cube of side length l has the same surface area as the sphere. Write r in terms of l. (This means write a formula and make r the subject.)

KEY POINTS

1 To change the subject of a formula, perform the same operations on both sides.

2 Use inverse operations to undo the formula.

3 If the new subject is on both sides of the formula, remove it from one side and factorise.

3.8 Linear equations

LEARNING OUTCOMES

- Solve linear equations in one unknown
- Solve linear equations in one unknown containing brackets
- Solve linear equations in one unknown containing fractions
- Solve a simple linear inequality on one unknown

WORKED EXAMPLE 1

To solve the equation

$4x - 5 = 2x + 7$

The left side will contain the terms in x, so add 5 to both sides to undo the -5.

$4x - 5 + 5 = 2x + 7 + 5$

Simplify:

$4x = 2x + 12$

The right side will contain the terms without x, so subtract $2x$ from both sides:

$4x - 2x = 2x + 12 - 2x$

Simplify:

$2x = 12$

To find the value of x, divide both sides by 2:

$\frac{2x}{2} = \frac{12}{2}$

Or

$x = 6$

Linear equations

A **linear equation** is a mathematical sentence stating that two expressions are equal. They are recognisable because they contain an equal sign.

We are usually asked to solve an equation; this means we should calculate the value of the letter (or **unknown**) in the equation.

The method of solving a linear equation is similar to changing the subject of a formula, which you saw in 3.7. We use inverse operations, applied to both sides of the equation, to rearrange the terms. The aim is to remove all terms containing the unknown from one side of the equation, and remove all terms not containing the unknown from the other side.

Solving equations with brackets

Usually the simplest method to solve an equation containing brackets is to remove them:

WORKED EXAMPLE 2

$4(3 - 2a) = 2(a + 7)$

Remove brackets:

$12 - 8a = 2a + 14$

There are more unknowns on the right, so we will make this the side for the unknowns.

$12 - 8a + 8a = 2a + 14 + 8a$ Add 8 to both sides.

$12 = 10a + 14$ Simplified; now subtract 14

$12 - 14 = 10a + 14 - 14$

Simplify: $-2 = 10a$

To find a, divide both sides by 10:

$\frac{-2}{10} = \frac{10a}{10}$

or

$a = -\frac{1}{5}$

EXAM TIP

- Always aim to make the unknown positive by removing the smaller quantity. $-8a$ is smaller than $2a$, so we removed the $-8a$ by adding.

Equations containing fractions

The best technique for dealing with fractions is to multiply each term by a common denominator. The denominators will cancel, removing all the fractions.

WORKED EXAMPLE 3

$$\frac{5}{x+1} = \frac{1}{2-x}$$

The common denominator is $(x + 1)(2 - x)$

Multiplying each term by the common denominator:

$$\frac{5(x+1)(2-x)}{x+1} = \frac{1(x+1)(2-x)}{2-x}$$

Simplify the fractions by cancelling

$$5(2 - x) = 1(x + 1)$$

Expand the brackets

$$10 - 5x = x + 1$$

Add $5x$ to both sides to make the term in x positive

$$10 = 6x + 1$$

Subtract 1 from both sides

$$9 = 6x$$

Divide both sides by 6

$$x = 1.5$$

Solving problems using equations

The important steps are choosing what the unknown should represent, and formulating the equation. Consider this problem.

WORKED EXAMPLE 4

There are 376 stones in three piles. The second pile has 24 more stones than the first pile. The third pile has twice as many stones as the second. How many stones are there in each pile?

Solution

Let the first pile contain n stones.

Then the second pile contains $n + 24$ stones.

The third pile contains $2(n + 24)$ stones.

There are 376 stones altogether, so

$$n + n + 24 + 2(n + 24) = 376$$

$$n + n + 24 + 2n + 48 = 376$$

Removing brackets

$$4n + 72 = 376$$

Simplifying

$$4n + 72 - 72 = 376 - 72$$

Subtracting 72 from both sides

$$4n = 304$$

$$\frac{4n}{4} = \frac{304}{4}$$

Dividing by 4

$$n = 76$$

So the piles contain $n = 76$ stones, $n + 24 = 100$ stones and $2(n + 24) = 200$ stones.

ACTIVITY

A-to-B Taxis charge \$15 plus \$3.60 per mile.

Direct Taxis charge \$5 plus \$4 per mile.

- Solve an equation to find the distance for which both companies charge the same.
- What is the cost of this journey?

SUMMARY QUESTIONS

1 Solve the equation $3x - 2 = x + 8$

2 Solve the equation $\frac{3x-1}{4} = \frac{x}{2} - 1$

3 Solve the equation $\frac{4}{2x+3} = 4$

KEY POINTS

1 If there are fractions, simplify them.

2 Remove any brackets.

3 By using inverse operations, create a side for the unknowns and a side for the numbers.

4 Always keep the unknowns positive.

3.9 Quadratic equations

LEARNING OUTCOMES

- Solve quadratic equations by factorising
- Solve quadratic equations by completing the square
- Solve quadratic equations by formula

WORKED EXAMPLE 1

A rectangle has a length 4 cm greater than its width. The area is 45 cm^2.

If we call the width w cm, the length is $w + 4$ cm, and, as the area is found by multiplying the length and width, we arrive at the equation

$$w(w + 4) = 45$$

or, removing the brackets,

$$w^2 + 4w = 45$$

Quadratic equations

Equations with a squared term are called **quadratic equations**.

Solving quadratic equations by factorising

Quadratic equations can sometimes be solved by factorising. This method relies on a special property of the number 0:

If two numbers have a product of 0, then one of the numbers must be 0.

WORKED EXAMPLE 2

For the rectangle above, to solve $w^2 + 4w = 45$, make the equation equal zero by subtracting 45:

$$w^2 + 4w - 45 = 0$$

Then factorise the left-hand side

$$(w + 9)(w - 5) = 0$$

But if these two factors have a product of 0,

either $w + 9 = 0$ *or* $w - 5 = 0$

$w = -9$ or $w = 5$

Quadratic equations can have two solutions.

In this case, one solution does not make sense in relation to the original problem, as a rectangle cannot have a width of -9 cm.

So, the rectangle is 5 cm wide and 9 cm long.

Solving quadratic equations by completing the square

Not all expressions will factorise, so the above method will not always work.

You saw in unit 3.5 that $x^2 + bx + c = \left(x + \frac{b}{2}\right)^2 - \left(\frac{b}{2}\right)^2 + c$

$2x^2 - 4x + 1 = 0$ cannot be solved by factorising.

First, divide by 2 so the coefficient of x^2 (the number of the x^2 term) is 1:

$x^2 - 2x + \frac{1}{2} = 0$

Using the identity above, $b = -2$ and $c = \frac{1}{2}$

So $x^2 - 2x + \frac{1}{2} = 0$ becomes

$(x - 1)^2 - (-1)^2 + \frac{1}{2} = 0$

$(x - 1)^2 - \frac{1}{2} = 0$

$(x - 1)^2 = \frac{1}{2}$ adding $\frac{1}{2}$ to both sides

$x - 1 = +$ or $-\sqrt{\frac{1}{2}}$ Square roots can be positive or negative.

$x = +$ or $-\sqrt{\frac{1}{2}} + 1$

$x = \sqrt{\frac{1}{2}} + 1$ or $x = -\sqrt{\frac{1}{2}}$

$x = 1.71$ or $x = 0.29$ (to 2 decimal places)

WORKED EXAMPLE 3

$2x^2 - 4x + 1 = 0$ cannot be solved by factorising.

First, divide by 2 so the coefficient of x^2 (the number of the x^2 term) is 1:

$x^2 - 2x + \frac{1}{2} = 0$

Using the identity $x^2 + bx + c = \left(x + \frac{b}{2}\right)^2 - \left(\frac{b}{2}\right)^2 + c$, $b = -2$ and $c = \frac{1}{2}$

So $x^2 - 2x + \frac{1}{2} = 0$ becomes

$(x - 1)^2 - (-1)^2 + \frac{1}{2} = 0$

$(x - 1)^2 - \frac{1}{2} = 0$

$(x - 1)^2 = \frac{1}{2}$ adding $\frac{1}{2}$ to both sides

$x - 1 = \pm\sqrt{\frac{1}{2}}$ Square roots can be positive or negative.

$x = \pm\sqrt{\frac{1}{2}} + 1$

$x = \sqrt{\frac{1}{2}} + 1$ or $x = -\sqrt{\frac{1}{2}} + 1$

$x = 1.71$ or $x = 0.29$ (to 2 decimal places)

Solving quadratic equations by formula

Completing the square is really only useful if the coefficient of x^2 is 1.

For the general quadratic equation $ax^2 + bx + c = 0$,

we use the formula $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

ACTIVITY

- Solve the equation $x^2 - 4x + 3 = 0$ by
  - Factorising
  - Completing the square
  - Using the formula

WORKED EXAMPLE 4

To solve the equation $3x^2 - 5x + 1 = 0$:

$a = 3, b = -5, c = 1$

Substitute into $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$$x = \frac{-(-5) \pm \sqrt{(-5)^2 - 4 \times 3 \times 1}}{2 \times 3}$$

$$x = \frac{5 \pm \sqrt{25 - 12}}{6}$$

$$x = \frac{5 + \sqrt{13}}{6} \text{ or } x = \frac{5 - \sqrt{13}}{6}$$

$x = 1.43$ or $x = 0.23$ (to 2 decimal places)

EXAM TIP

- You must be able to use all three methods for solving quadratic equations, as examination questions might specify which method to use.
- If an examination question asks for answers to a number of decimal places or significant figures, this tells you that the quadratic will not factorise.

KEY POINTS

1. Quadratic equations have a term x^2.
2. Quadratic equations have two solutions.
3. Quadratic equations can be solved by factorising, completing the square or by using the formula

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

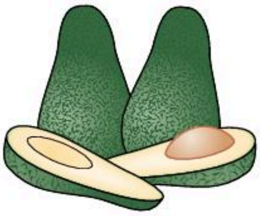
SUMMARY QUESTIONS

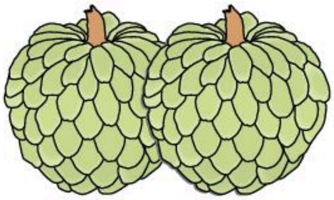
1. Solve the equation $x^2 + 2x - 15 = 0$ by factorising.
2. Solve the equation $x^2 + 4x - 7 = 0$ by completing the square.
3. A rectangular field has a width of x metres and a length of $2x + 3$ metres. Its area is 4000 m². Write an equation and solve it by formula to find the length and width of the field.

3.10 Simultaneous equations

LEARNING OUTCOMES

- Solve simultaneous linear equations in two unknowns algebraically
- Solve word problems





Simultaneous equations

I buy 3 avocadoes and 2 sugar-apples for $8.

If an avocado costs \$$x$ and a sugar-apple costs \$$y$, we can write

$$3x + 2y = 8$$

One solution is $x = 2$ and $y = 1$.

There are other solutions, for example $x = 1$ and $y = 2.5$.

In fact, on its own, the equation has an infinite number of solutions.

If we knew that 2 avocados and 5 sugar-apples cost $10.10, then

$$2x + 5y = 10.1$$

This, too, has other solutions, for example $x = 2.3$ and $y = 1.1$.

However, there is only one solution that satisfies both equations; $x = 1.8$ and $y = 1.3$.

A single equation with two unknowns has an infinite number of solutions. We can solve a pair of equations in two unknowns to give a unique solution. Such equations are called **simultaneous equations**.

Solving simultaneous equations by eliminating an unknown

WORKED EXAMPLE 1

$$3x + 4y = 22 \qquad (A)$$
$$2x - 3y = 9 \qquad (B)$$

The first step is to manipulate the equations so that one of the unknowns has the same coefficient. We multiply equation A by 2, and equation B by 3, so they both contain $6x$.

$2 \times (A)$ $\qquad 6x + 8y = 44$

$3 \times (B)$ $\qquad 6x - 9y = 27$

Now subtract the equations: $17y = 17$ $\qquad$ Note: $8y - -9y = 17y$

Divide both sides by 17: $y = 1$

Now we know the value of y, we can calculate x:

Equation (A): $3x + 4y = 22$

Substitute $y = 1$: $3x + 4 = 22$

Subtract 4: $3x = 18$

Divide by 3: $x = 6, y = 1$

We can check we are correct by substituting into equation (B):

$$2x - 3y = 9$$

Substitute $x = 6, y = 1$: $12 - 3 = 9$ $\qquad$ Correct.

EXAM TIP

- Remember to multiply all terms in the equation; don't forget the constant.
- Sometimes equations will already have the same coefficient of x or y, so there is no need to multiply the equations.

Now consider this example:

WORKED EXAMPLE 2

$5x + 4y = 18$ (A)

$3x - 2y = 13$ (B)

We can equate coefficients of y by multiplying (B) by 2:

(A) $5x + 4y = 18$

2 × (B) $\underline{6x - 4y = 26}$

$11x = 44$

Because the signs of the $4y$ terms are opposite, we add the equations to eliminate them.

Divide by 11 $\underline{x = 4}$

Equation (A) $5x + 4y = 18$

Substitute $x = 4$ $20 + 4y = 18$

Subtract 20 $4y = -2$

Divide by 4 $\underline{y = -\frac{1}{2}, x = 4}$

Check in (B) $3x - 2y = 13$

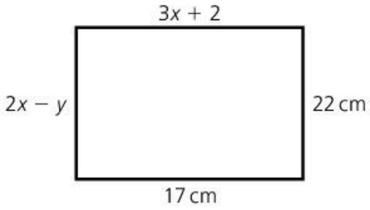
Substitute $y = -\frac{1}{2}, x = 4$ $12 - -1 = 13$ Correct.

EXAM TIP

- To eliminate an unknown: if the signs are the same, subtract the equations.
- If the signs are Different, then ADD the equations.

ACTIVITY

- Find the value of x and y in this rectangle.
- A square has a length of $a + 2b$ and width of $3a - 2b$. Its area is 196 cm^2. Calculate a and b.



KEY POINTS

1. Simultaneous equations can be solved by eliminating one unknown.
2. To eliminate the unknown, the coefficients must be equal in each equation.

SUMMARY QUESTIONS

1. Solve the simultaneous equations

   $a + 2b = 9$
   $3a - 2b = 19$

2. Solve the simultaneous equations

   $6x + 5y = 4$
   $4x + 3y = 3$

3. The sum of my age and my father's age is 56.
   In five years time, my father will be exactly twice my age.
   How old are we both now?

3.11 Further simultaneous equations

LEARNING OUTCOMES

- Solve simultaneous equations, one linear, one non-linear
- Solve word problems

In unit 3.10, we solved pairs of simultaneous equations.

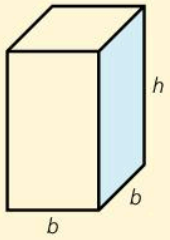
Sometimes a better method is to eliminate an unknown by substitution.

Solving simultaneous equations by substitution

This method is very useful when one of the equations is non-linear.

WORKED EXAMPLE 1

Suppose a cuboid has a square base of side b cm and a height of h cm.



The total length of its edges is 72 cm, and its surface area is 210 cm^2.

There are four vertical edges of length h and eight horizontal edges of length b, so:

$$4h + 8b = 72 \qquad \text{(A)}$$

The four vertical faces have an area of bh and the two horizontal faces have an area of b^2, so:

$$4bh + 2b^2 = 210 \qquad \text{(B)}$$

We cannot eliminate b or h in the same way as before, as (B) is non-linear.

However, we can rearrange (A) to make h the subject:

$$\begin{aligned} 4h + 8b &= 72 \\ 4h &= 72 - 8b && \text{Subtracting } 8b \\ h &= 18 - 2b && \text{Dividing by 4} \end{aligned}$$

We can substitute this into (B):

$$\begin{aligned} 4bh + 2b^2 &= 210 \\ 4b(18 - 2b) + 2b^2 &= 210 \end{aligned}$$

Expand to give: $72b - 8b^2 + 2b^2 = 210$

Simplify: $72b - 6b^2 = 210$

Rearrange: $0 = 6b^2 - 72b + 210$

Divide by 6: $b^2 - 12b + 35 = 0$

Factorise: $(b - 7)(b - 5) = 0$

So $b - 7 = 0$ or $b - 5 = 0$

$b = 7$ or $b = 5$

Substituting into (A):

If $b = 7$:
$$\begin{aligned} 4h + 8b &= 72 \\ 4h + 56 &= 72 \\ 4h &= 16 \\ h &= 4, b = 7 \end{aligned}$$

If $b = 5$:
$$\begin{aligned} 4h + 8b &= 72 \\ 4h + 40 &= 72 \\ 4h &= 32 \\ h &= 8, b = 5 \end{aligned}$$

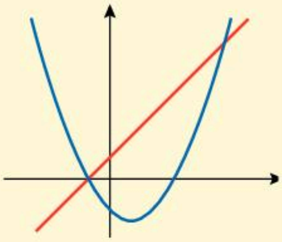
Because there was a quadratic equation involved, we get two pairs of solutions.

Finding intersections of graphs

Simultaneous equations can be used to find the points where two graphs intersect.

WORKED EXAMPLE 2

The diagram shows the graphs of $y = x^2 - 2x - 3$ and $y - 2x = 2$



To find the points of intersection, we solve the simultaneous equations

$y = x^2 - 2x - 3$ (1) and

$y - 2x = 2$ (2)

From equation (2):

$y = 2x + 2$

Substituting in to (1):

$$y = x^2 - 2x - 3$$

$$2x + 2 = x^2 - 2x - 3$$

$$2x + 2 - 2x - 2 = x^2 - 2x - 3 - 2x - 2$$ Subtract 2*x* and 2 from each side

$$0 = x^2 - 4x - 5$$

$$(x - 5)(x + 1) = 0$$ Factorising

$x - 5 = 0$ or $x + 1 = 0$

$x = 5$ or $x = -1$

Substituting into (2):

$y - 2x = 2$	$y - 2x = 2$
$x = 5$:	$x = -1$:
$y - 10 = 2$	$y + 2 = 2$
$y = 12$	$y = 0$

(5, 12) and (−1, 0) are the intersections.

We can check in (1):

$y = x^2 - 2x - 3$	$y = x^2 - 2x - 3$	
$x = 5, y = 12$:	$x = -1, y = 0$:	
$12 = 25 - 10 - 3$	$0 = 1 + 2 - 3$	Both are correct.

EXAM TIP

- A pair of linear equations will have a pair of solutions.
- A linear and non-linear equation will have two pairs of solutions.

SUMMARY QUESTIONS

1. Solve the equations $x + y = 5$ and $y = x^2 - 4x + 7$
2. A rectangle has a length of x cm and a width of y cm. The area is 40 cm^2 and the perimeter is 26 cm.
   So $xy = 40$ and $2x + 2y = 26$.
   Calculate the length and width of the rectangle.
3. The graphs of $y = x^2 - 3x + 4$ and $y = x^2 + 2x - 1$ cross at one point only.
   Solve the simultaneous equations to find the coordinates of the point where they cross.

KEY POINTS

1. Simultaneous equations can be solved by substitution.
2. If one equation is quadratic, there will be two pairs of solutions.
3. Calculate the second unknown from the linear equation.
4. Simultaneous equations can be used to find the points where two graphs intersect.

3.12 Variation

LEARNING OUTCOMES

- Represent direct and indirect variation symbolically
- Solve problems involving direct and inverse variation

Direct variation

A hose pipe dispenses 420 litres of water in 12 minutes.

So in 24 minutes it will dispense 840 litres of water.

There are two quantities here: the time in minutes and the volume of water in litres.

Doubling the time doubles the amount of water. If we multiply the time by 5, we multiply the amount of water by 5.

We say that the volume of water **varies directly** with the time.

Using v for the volume of water in litres and t for the time in minutes, we write it algebraically as $v \propto t$. We read this as "v is directly proportional to t".

We also write $v = kt$, where k is a constant.
When $v = 420$, $t = 12$, so $420 = 12k$, or $k = 35$.
So $v = 35t$.
The rate of flow is 35 litres/minute.

v varies directly with t, v is directly proportional to t, $v \propto t$ and $v = kt$ all mean the same thing.

Rates always involve variation.

Jonathan is paid \$15 per hour. His total pay, p, varies directly with the number of hours worked, h. In this case, $p = 15h$.

WORKED EXAMPLE 1

The distance I run in 10 minutes varies directly with my speed.

If I run at 5 mph, I cover a distance of 810 m.

As the distance, d m and speed, s mph vary directly, I can write

$d \propto s$ or $d = ks$.

When $s = 5$, $d = 810$.

Substituting into $d = ks$ gives $810 = 5k$, or $k = 162$

So $d = 162s$.

So running at 8 mph, I would run $d = 162 \times 8 = 1296$ m

More complex direct variation

WORKED EXAMPLE 2

A circle with a diameter of 5 cm has an area of 19.6 cm^2 (to 1 decimal place).

A circle with a diameter of 10 cm has an area of 78.4 cm^2 (to 1 decimal place).

Diameter (d)	Diameter^2 (d^2)	Area (A)
5	25	19.6
10	100	78.4

Doubling the diameter produces a 4-fold increase in area.

In fact the area varies directly with the square of the diameter.

$A \propto d^2$, or $A = kd^2$

Substituting $A = 19.6$ and $d = 5$,

$$19.6 = 25k$$
$$k = 0.784$$

So $A = 0.784d^2$

WORKED EXAMPLE 3

The volume of a sphere, V cm^3 varies directly as the cube of the radius, r cm.

When the radius is 5 cm, the volume is 523.6 cm^3. To find the radius of a sphere with a volume of 1000 cm^3:

$V \propto r^3$, or $V = kr^3$.

$$523.6 = 125k$$
$$k = 4.19 \text{ (to 2 decimal places)}$$
$$V = 4.19r^3$$

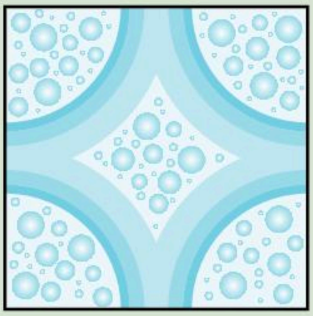
So for a sphere with a volume of 1000 cm^3,

$$1000 = 4.19r^3$$
$$r^3 = 238.66$$
$$r = \sqrt[3]{238.66}$$
$$r = 6.2 \text{ cm}$$

SUMMARY QUESTIONS

1 h varies directly with k.
Complete the table of values for h and k.

h	5	10		8
k	7		35	



2 The mass of a square tile varies directly with the square of the side length.
A tile with side length 10 cm has a mass of 340 g.
Find the mass of a tile with side length 15 cm.

3 CHALLENGE
x varies directly with the cube root of y.
When $x = 1$, $y = 27$.
Calculate y when $x = 3$.

EXAM TIP

- If a and b vary directly, then $\frac{a}{b}$ is constant.

ACTIVITY

A square has an area, A cm^2, and a diagonal of length d cm.

The area varies directly with the square of the length of the diagonal.

When the diagonal is 6 cm long, the area is 18 cm^2.

- Find the equation connecting A and d.
- Find the area of a square with a diagonal of 10 cm.
- Find the length of diagonal of a square with an area of 72 cm^2.

KEY POINTS

1 If x varies directly with y, then $x \propto y$, and $x = ky$.

2 k is a constant, and can be calculated given a pair of values for x and y.

3 Variation is sometimes referred to as proportion.

4 The variation might involve powers and roots, for example if x varies directly with the cube of y, then $x \propto y^3$ and $x = ky^3$.

3.13 Inverse variation

LEARNING OUTCOMES

- Represent indirect variation symbolically
- Solve problems involving inverse variation

Inverse variation

In 3.12, we saw how quantities can vary directly.

If one quantity doubles, so does the other.

Sometimes quantities vary **inversely**.

Imagine you have enough cat food to feed 4 cats for 6 days.

If you had 8 cats, the food would only last for 3 days.

Doubling the number of cats means the food lasts for half the time.

Inverse variation means that if one quantity is multiplied by n, the other is divided by n.

Assuming the cats eat once a day, you actually have enough food for 24 meals.

If c is the number of cats and d is the number of days the food will last, then

$$d = \frac{24}{c}.$$

When d and c vary inversely, we write

$$d \propto \frac{1}{c} \text{ or } d = \frac{k}{c}.$$

We can say d is inversely proportional to c, or d varies inversely with c.

EXAM TIP

- If a and b vary inversely, then ab is constant.

WORKED EXAMPLE 1

My journey to work is 20 km long.

If I travel at 30 km/h, it takes me 40 minutes.

If I travel at 60 km/h, it takes me 20 minutes.

Doubling the speed halves the time.

If I multiply the speed by 3, I divide the time by 3.

We say that the speed and time **vary inversely**.

Using s for speed in km/h and t for time in minutes, we write this as

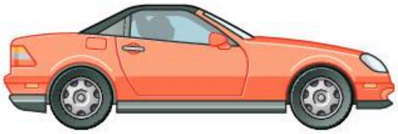
$$t \propto \frac{1}{s} \quad \text{or} \quad t = \frac{k}{s}.$$

Substituting $s = 30$, $t = 40$, we get

$$40 = \frac{k}{30}$$

so $\quad k = 1200$

The relationship is $t = \dfrac{1200}{s}$



ACTIVITY

The length, l cm, and width, w cm, of a rectangle vary so that the area is fixed at 40 cm^2.

- Find some values of l and w that fit.
- Write the equation that connects l and w.

More complex inverse variation

As with direct variation, inverse variation can involve powers or roots.

For example, the time it takes for a car to travel 200 m from rest varies inversely with the square root of the acceleration. If it accelerates at 4 m/s^2, it takes 10 seconds.

We can use this to find the time it takes if the acceleration is 3 m/s^2.

The time, t, varies inversely with the square root of the acceleration, a.

$$t \propto \frac{1}{\sqrt{a}}, \text{ or } t = \frac{k}{\sqrt{a}}.$$

$t = 10$ when $a = 4$, so

$$10 = \frac{k}{\sqrt{4}}$$

$$k = 20$$

So $\quad t = \dfrac{20}{\sqrt{a}}$

If $a = 3$, $t = \dfrac{20}{\sqrt{3}} = 11.5$ seconds (to 1 decimal place)

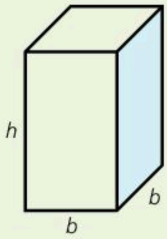
ACTIVITY

A cuboid has a square base and a volume of 36 cm^3.

- Write a rule or formula showing the connection between the base length and the height.
- Complete the table showing the possible base lengths and heights.

Base length, *b* (cm)	1	2	3	4
Height, *h* (cm)				2.25

- Describe the type of variation between the base length and the height.



SUMMARY QUESTIONS

1 w varies inversely with p.
Complete the table of values for w and p.

w	6	9		10
p	12		24	

2 The intensity of light, I, varies indirectly with the square of the distance from the light source, d. The intensity of light is 18 flux at a distance of 20 cm from the source. Calculate the distance from the light source where the intensity is 8 flux.

3 CHALLENGE
r varies inversely with the cube root of s.
When $r = 8$, $s = 27$.
Calculate s when $r = 12$.

KEY POINTS

1 If x varies inversely with y, then $x \propto \dfrac{1}{y}$, and $x = \dfrac{k}{y}$.

2 k is a constant, and can be calculated given a pair of values for x and y.

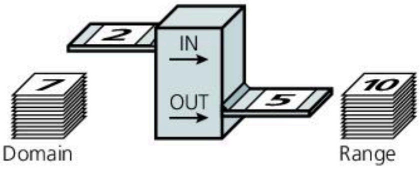
3 Variation is sometimes referred to as proportion.

4 The variation might involve powers and roots, for example if x varies indirectly with the cube of y, then $x \propto \dfrac{1}{y^3}$, and $x = \dfrac{k}{y^3}$.

3.14 Relations

LEARNING OUTCOMES

- Appreciate what makes a relationship
- Understand types of relationship
- Understand the terms domain, range, image, codomain
- Represent a relationship graphically, algebraically, set of ordered pairs, arrow diagrams



Mathematical relations

A relation is simply a relationship between two sets of elements. A list of all the students in a class matched to their heights is an example of a relation.

Imagine a machine; a set of numbers is fed in on cards, and for each number put in, there is an output.

This is a '+3' machine. 2 goes in, 5 comes out.

We could call this relation $y = x + 3$, using x to represent the input and y to represent the output.

Domain, codomain, range and image

All the numbers that can be fed in are called the **domain**. In this case, the domain might be all the real numbers.

The numbers that can be fed out are called the **range**.

If 2 maps onto 5, we say that 5 is the **image** of 2.

The relation $y = x^2$ might have a domain of the real numbers. We might define the **codomain**, or allowable outputs, as the real numbers. But in practice, the output for this relation can never be negative, so the range is the non-negative real numbers.

When defining a relation, we need to state the domain and codomain. The range will either be equal to or a subset of the codomain.

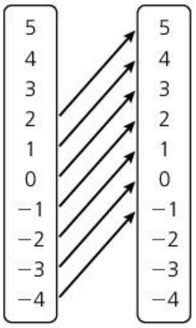
To be classed as a relation, every element of the domain must map to one or more elements in the codomain. In other words, every input must have at least one output.

As negative numbers do not have square roots, $y = \sqrt{x}$ is only a relation if we define the domain as the non-negative real numbers. Then, all the elements of the domain have a square root. If the domain was *all* the real numbers, then $y = \sqrt{x}$ would not be defined for negative values of x, and so would not be a relation.

Representing relations

The relation that adds 3 to the element of the domain can be written as an **equation**, $y = x + 3$.

This could be shown on an **arrow diagram**:



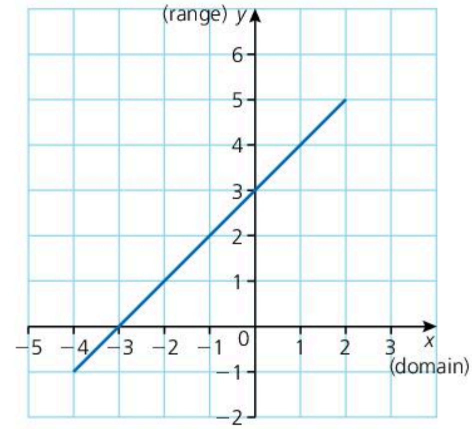
This is a **one–one** relation, as every member of the domain maps to a unique member of the range.

We can also represent the relation using a **table**.

x	−5	−4	−3	−2	−1	0	1	2	3	4	5
y	−2	−1	0	1	2	3	4	5	6	7	8

Ordered pairs such as (2, 5), (1, 4), (0, 3), (−1, 2) can also be used to represent a relation.

We can then represent the relation by **a graph**, by plotting the ordered pairs as shown on the right:



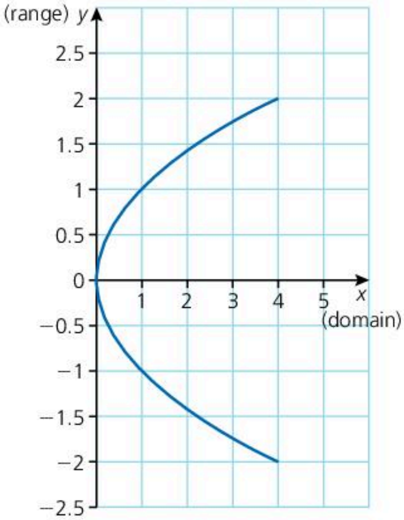
Some relations are **one–many**. This means that members of the domain can map to more than one member of the range.

An example is $y = \sqrt{x}$

Here, 4 maps to 2 and to −2.

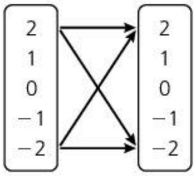
1 maps to 1 and −1.

On a graph it looks like this:

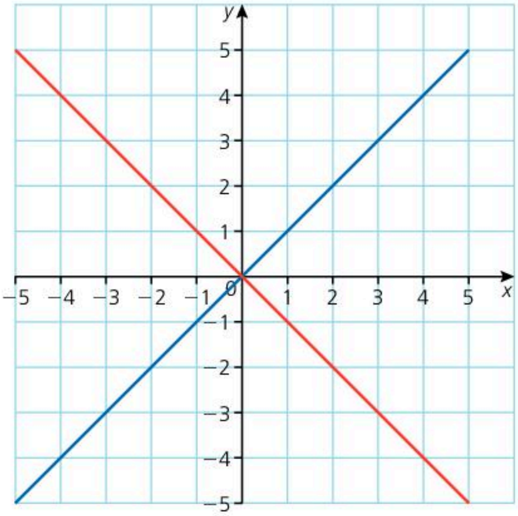


Some relations are **many–one**, such as $y = x^2$. Here, two members of the domain map to the same member of the range: (3, 9) and (−3, 9) are examples.

Some relations are **many–many**.
An example is $y = \sqrt{x^2}$.
This contains the example shown on the arrow diagram:



On a graph, it would look like this:



EXAM TIP

- The rule for a relation must be true for all elements of the domain.

SUMMARY QUESTIONS

1 In the relation $y = x^2 - 3$, what is the image of:

a 4 **b** −4?

2 What type of a relation is $y = x^2 - 3$?

3 What is the range of the relation $y = x^2 - 3$?

KEY POINTS

1 There are four types of relations; one–one, one–many, many–one and many–many.

2 The allowed inputs make up the domain, the allowed outputs the codomain.

3 The set of all actual outputs is called the range.

4 Relations can be shown as an equation, a set of ordered pairs or as a graph.

3.15 Functions

LEARNING OUTCOMES

- State the properties that define a function
- Distinguish between relationship and function by using ordered pairs, arrow diagrams and graphs
- Use function notation ($f{:}x \rightarrow$, $f(x) =$, $y = f(x)$)

Mathematical functions

A function is a relation where each element of the domain maps to exactly one element of the codomain.

So one–one and many–one relations are functions, but one–many and many–many relations are not.

Function notation

As with relations, we can use equations, arrow diagrams, tables, ordered pairs and graphs to show a function.

We can also use two other forms of algebraic notation.

The function $y = x^2 - 2x$ can also be written as

$$f(x) = x^2 - 2x, \text{ or } f : x \rightarrow x^2 - 2x$$

All three forms mean the same thing. The $f(x)$ notation reinforces that we are dealing with a function, and the $f : x \rightarrow$ notation suggests a mapping from one number to another.

The vertical line test

One way to make sure that a relation is a function is to use the vertical line test.

WORKED EXAMPLE 1

Consider the function $f(x) = x^2 - 3$.

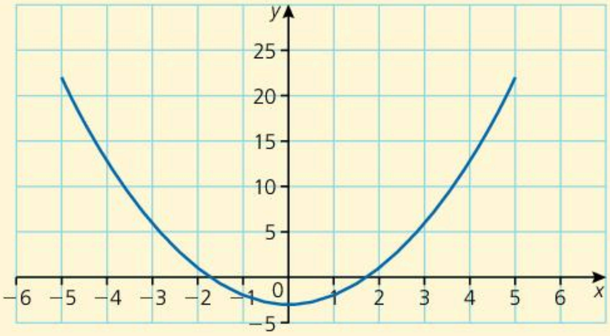
When $x = -5$, $f(x) = 22$.

We can write this as $f(-5) = 22$.

Here is a table of values of $f(x)$ for $x = -5$ to $x = 5$.

x	−5	−4	−3	−2	−1	0	1	2	3	4	5
f(x)	22	13	6	1	−2	−3	−2	1	6	13	22

Here is the graph of the function:



The **vertical line test** states that a relation is a function if there are no vertical lines that intersect the graph at more than one point.

In this case, any vertical line will cross the graph just once, so it is a function.

The inverse of a function

The inverse of a function $y = f(x)$ is $x = f(y)$.

It reverses the effect of the function f.

To find the inverse of the function $y = x^2 - 3$:

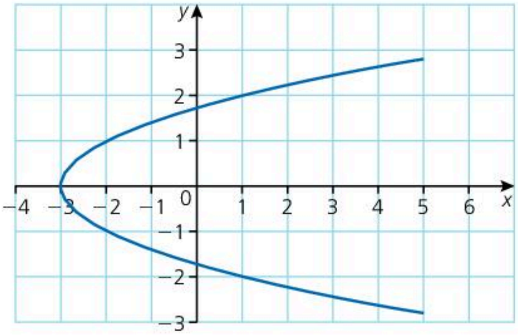
First, interchange x and y to give $x = y^2 - 3$.

Change the subject:

$$x + 3 = y^2, \text{ or } y = \sqrt{x + 3}$$

The function $y = x^2 - 3$ will have an inverse of $y = \sqrt{x + 3}$.

Because a square root can be positive or negative, the graph looks like this:

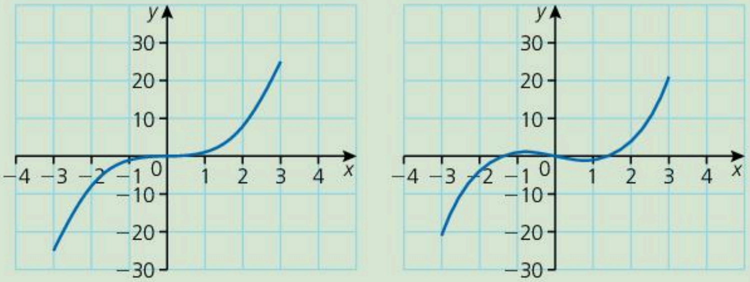


This fails the vertical line test, and so although it is a relation, it is not a function.

A many–one function will have an inverse that is a one–many relation.

SUMMARY QUESTIONS

Here are the graphs of $y = x^3$ and $y = x^3 - 2x$.



1 Which of the two functions is a one–one function?

2 Explain why the other function will not have an inverse function.

3 Choose a sensible domain so that $f: x \rightarrow \sqrt{x + 6}$ is a function.

ACTIVITY

$y = \sqrt{x + 3}$ is a function if we choose the domain and codomain carefully.

The codomain could be defined as the non-zero real numbers, so it is no longer a many–one relation.

- What limits must be placed on the domain?
- The domain of a function is the codomain of the inverse function and vice versa.
- What would the graph of $y = x^2 - 3$ look like now?

EXAM TIP

- Only one–one functions will have an inverse that is also a function.

KEY POINTS

1 A function is a one–one or many–one relation.

2 The inverse of a function $y = f(x)$ is found by writing $x = f(y)$, and then changing the subject to y.

3 The vertical line test can be used on a graph to check if the relation is a function.

3.16

Graphs of linear functions

LEARNING OUTCOMES

- Understand the concept of a linear function
- Draw and interpret graphs of linear functions
- Find the *x*- and *y*-intercepts of linear functions graphically and algebraically
- Find the gradient of a straight line (concept of slope)

ACTIVITY

- Make *y* the subject of these functions to decide whether they are linear or not.

$x + y = 7$ $\quad \frac{y}{x} = 3$

$xy = 5$ $\quad \frac{y}{x} = x$

$\frac{y + 7}{x} = 2$ $\quad y = \frac{2}{x}$

- If you can, use a graph plotter to check your answers.

EXAM TIP

- It helps to find three points on the line, in case you make a mistake. If the three points don't make a straight line, check your calculations!

What is a linear function?

A **linear function** is one that produces a straight-line graph.

We can recognise a linear function from its equation, as it will have one of three possible forms:

- $y = c$, where c is a constant (for example, $y = 4$, $y = -2$, $y = 0$)
- $x = c$, where c is a constant (for example, $x = 2$, $x = -1$, $x = \pi$)
- $y = mx + c$, where m and c are constants (for example, $y = 2x - 4$, $y = 3 - 5x$, $y = 2x$)

It is possible that the equation might be in a different form, but a linear function can always be arranged into one of the three forms.

For example, $x - 2y = 7$ can be rearranged as follows:

$$x = 7 + 2y$$
$$x - 7 = 2y$$
$$y = \tfrac{1}{2}x - 3\tfrac{1}{2}$$

And $\frac{x}{y} = 2$ can be rearranged to

$$x = 2y$$
$$y = \tfrac{1}{2}x$$

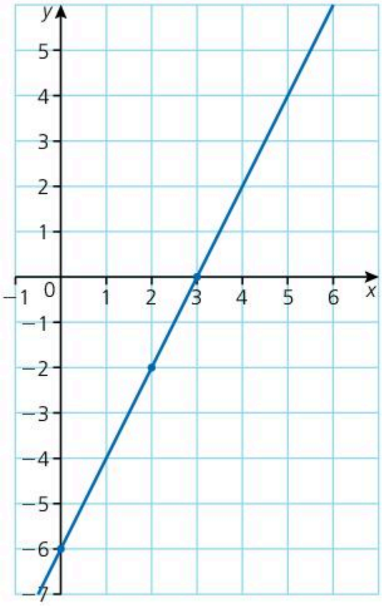
Drawing linear graphs

Once we know a function is linear, we only need to identify two points to be able to draw the graph, as there is only one straight line that passes through two given points.

Usually, an easy point to identify is the intercept on the *y*-axis. Substituting $x = 0$ into the function tells us the intercept on the *y*-axis as all points on the *y*-axis have an *x*-coordinate of 0.

We can choose any other value of *x* to find a second point on the line.

Or we can find the intercept on the *x*-axis by substituting $y = 0$.



For example, to draw the graph of $y = 2x - 6$.

When $x = 0$, $y = 2 \times 0 - 6 = -6$.

So the graph passes through $(0, -6)$.

When $x = 2$, $y = 2 \times 2 - 6 = -2$.

So the graph passes through $(2, -2)$.

When $y = 0$,

$$0 = 2x - 6$$
$$6 = 2x$$
$$x = 3.$$

The graph passes through $(3, 0)$.

EXAM TIP

- When drawing straight-line graphs, use a ruler.

The concept of gradient

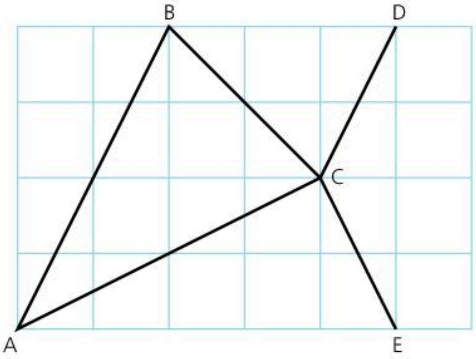
A linear (straight-line) graph has the same **gradient** along its entire length.

The gradient of a straight line is a measure of its steepness.

We measure the gradient of a line by finding the vertical distance for one unit of horizontal distance.

The gradient is calculated as:

$$\frac{\text{vertical distance between two points}}{\text{horizontal distance between the same two points}}$$



The gradient of AB $= \frac{4}{2} = 2$

This tells us that for every one unit horizontally, the line climbs 2 squares vertically.

The gradient of AC $= \frac{2}{4} = \frac{1}{2}$

A line which slopes downwards from left to right has a negative gradient.

The gradient of CE $= \frac{-2}{1} = -2$

The gradient of BC $= \frac{-2}{2} = -1$

The gradient of CD $= \frac{2}{1} = 2$

AB and CD are **parallel**, and so have the same gradient.

EXAM TIP

- When calculating gradient by counting squares pay special attention to the scale on the axes.

KEY POINTS

1. Linear (straight-line) graphs can be written as $y = c$, $x = c$ or $y = mx + c$, where m and c are constants.
2. To draw a linear graph, find three points and join them.
3. Substituting $y = 0$ gives the intercept on the x-axis.
4. Substituting $x = 0$ gives the intercept on the y-axis.
5. Parallel lines have the same gradient.

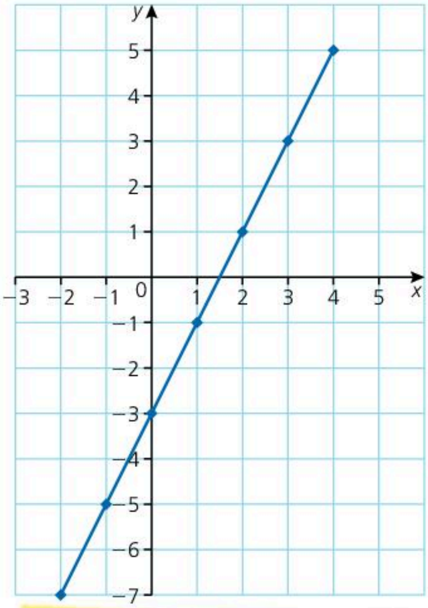
SUMMARY QUESTIONS

1. Find the intercepts on the axes of the graph of $y = 3x + 6$.
2. Draw the graphs of $y = 2x + 1$ and $y = 7 - x$.
3. Find the gradient of each graph in question **2**.

3.17 The equation of a straight line

LEARNING OUTCOMES

- Draw a graph given the equation of a straight line
- Find the equation of a straight line from a graph



EXAM TIP

- You can always check your answer by substituting. In the example, substitute $x = 3$ into $y = 7 - 2x$ to check that $y = 1$ when $x = 3$.

ACTIVITY

- Draw the graphs of:
  $y = 3x - 2$ $y = \frac{1}{2}x + 1$
  $y = 3 - x$
- For each graph, check that the coefficient of x is equal to the gradient, and that the constant is equal to the intercept on the y-axis.

The equation of a straight line

Consider the equation $y = 2x - 3$.

We can construct a table of values for x and y:

x	−2	−1	0	1	2	3	4
y	−7	−5	−3	−1	1	3	5

On the left is the graph of $y = 2x - 3$.

We saw in 3.16 how to find the gradient of a line. The gradient here is 2, as for every 1 unit the line moves to the right, it climbs 2 units upwards.

The graph has a gradient of 2 because of the $2x$ in the equation.

The table shows that, as x increases by 1, y increases by 2. This is the gradient.

The effect of the −3 in the equation is to dictate that the graph passes through (0, −3). When $x = 0$, $y = 2 \times 0 - 3 = -3$. This point is the intercept on the y-axis.

The equation of a straight line tells us two important facts about the graph:

A straight line with equation $\boldsymbol{y = mx + c}$ has gradient m and crosses the y-axis at $(0, c)$.

Drawing straight-line graphs

We saw in 3.16 that we need to find two points to draw a straight-line graph.

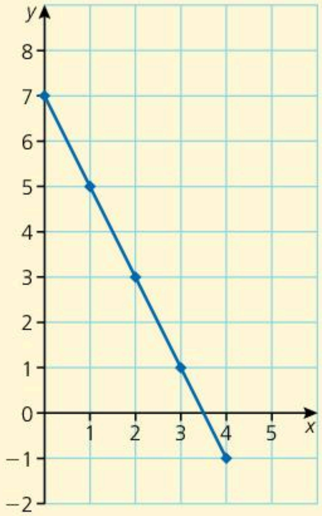
Now we can draw one straight from the equation, using the intercept and gradient.

WORKED EXAMPLE 1

To draw the graph of $y = 7 - 2x$:

We know it will cross the y-axis at (0, 7), with a gradient of −2, so we mark the point (0, 7), and then plot other points one unit to the right and two units down to represent a gradient of −2:

We join the points to complete the graph.

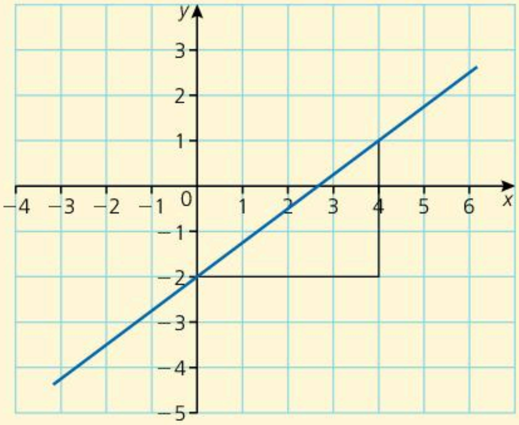


Finding the equation from a graph

We can easily find the equation of a graph using the concept of gradient and intercept.

WORKED EXAMPLE 2

Consider this graph:



The graph crosses the y-axis at -2.

To find the gradient, we take two points on the graph.

The gradient,

$$\frac{\text{vertical distance between two points}}{\text{horizontal distance between the same two points}}$$

can be written as $\frac{y_2 - y_1}{x_2 - x_1}$ given two points (x_1, y_1) and (x_2, y_2).

The graph passes through $(0, -2)$ and $(4, 1)$, so using these points, the gradient is $\frac{1 - -2}{4 - 0} = \frac{3}{4}$. So the gradient is $\frac{3}{4}$.

The equation of a straight line is given by $y = mx + c$, where m is the gradient and c is the intercept on the y-axis.

So the equation is $y = \frac{3}{4}x - 2$

SUMMARY QUESTIONS

1 What is the gradient of:

a $y = 3x - 1$ **b** $y = 9 - 3x$ **c** $x + 2y = 6$?

(Hint for **c**: make y the subject)

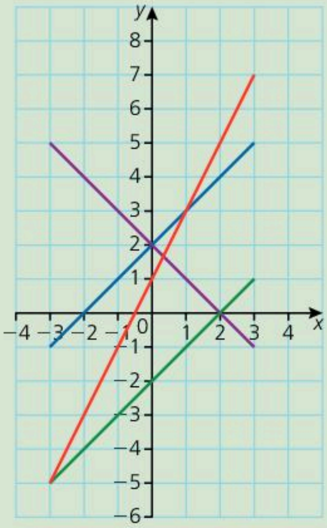
2 Match the graphs with the correct equation:

$y = x + 2$

$y = 2x + 1$

$x = y + 2$

$x + y = 2$



3 Write down the equation of a straight line which passes through $(0, 4)$ with a gradient of 2.

KEY POINTS

1 The gradient of a line is a measure of steepness, calculated as $\frac{y_2 - y_1}{x_2 - x_1}$.

2 The graph of $y = mx + c$ has a gradient of m and crosses the y-axis at $(0, c)$.

3.18 Solving problems using gradient and intercept

LEARNING OUTCOMES

- Find the equation of a straight line from the gradient and one point
- Find the equation of a straight line from the coordinates of two points
- Solve problems involving gradient of parallel and perpendicular lines

In unit 3.17 we discovered that the equation of a straight line is of the form $y = mx + c$, where m is the gradient and c is the intercept on the y-axis.

We can use this to find the equation of a straight line in a number of different situations.

To find the equation of a straight line from the gradient and a point

This is best shown through an example.

WORKED EXAMPLE 1

A straight line has gradient -2 and passes through the point (3, 1).

To find the equation:

In the equation $y = mx + c$, $m = -2$ (the gradient).

So the equation is $y = -2x + c$.

To find c, we know the equation passes through (3, 1), so when $x = 3$, $y = 1$.

Substituting $x = 3$, $y = 1$ into $y = -2x + c$ gives

$1 = -6 + c$

or $c = 7$.

The equation is $y = -2x + 7$.

To find the equation of a straight line from the coordinates of two points

The technique here is similar to the one above, except first we must find the gradient.

WORKED EXAMPLE 2

To find the equation of the line that passes through $(-4, 2)$ and $(6, 7)$:

The gradient is given by $\frac{y_2 - y_1}{x_2 - x_1} = \frac{7 - 2}{6 - -4} = \frac{5}{10} = \frac{1}{2}$.

The equation is $y = \frac{1}{2}x + c$.

We can use either point to find c; it might be easier to take the positive values (6, 7):

$7 = 3 + c$

$c = 4$

The equation is $y = \frac{1}{2}x + 4$

Parallel and perpendicular lines

Parallel lines must have the same gradient.

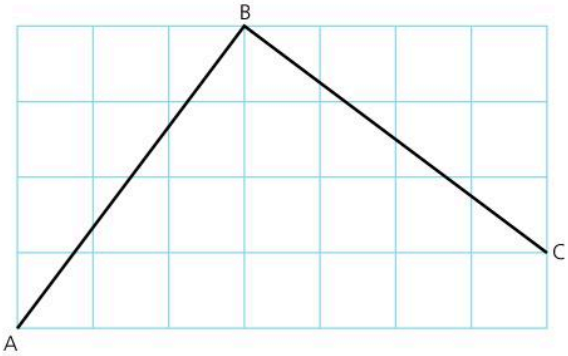
AB and BC are perpendicular.

The line AB has gradient of $\frac{4}{3}$.

The gradient of BC is $-\frac{3}{4}$. This is the negative reciprocal of the gradient of AB.

Generally,

> **if two lines are perpendicular, and the gradient of one line is m, the other has gradient $-\frac{1}{m}$.**



Problems involving gradient

WORKED EXAMPLE 1

A line has equation $y = 2x - 3$.

A line is parallel to $y = 2x - 3$ and passes through (3, 1).

Another line is perpendicular to $y = 2x - 3$ and passes through (3, 1).

Find the equations of these lines.

Solution

The parallel line will have the same gradient as $y = 2x - 3$, which is 2.

So the equation is $y = 2x + c$.

It passes through (3, 1), so when $x = 3$, $y = 1$.

Substituting, $1 = 6 + c$

$c = -5$

The equation is $y = 2x - 5$

The perpendicular line has a gradient of the negative reciprocal of 2, or $-\frac{1}{2}$.

So the equation is $y = -\frac{1}{2}x + c$

It passes through (3, 1), so when $x = 3$, $y = 1$.

Substituting,

$1 = -1\frac{1}{2} + c$ or $c = 2\frac{1}{2}$

The equation is $y = -\frac{1}{2}x + 2\frac{1}{2}$

ACTIVITY

Try to answer these questions without drawing an accurate diagram.

A quadrilateral has vertices A(6, 4), B (3, 6), C(−1, 0) and D(2, −2)

- Find the equations of AB, BC, CD and AD.
- Explain how the equations show that AB and CD are parallel.
- Explain how the equations show that AB and AD are perpendicular.
- What shape is ABCD?

SUMMARY QUESTIONS

A line joins (2, 5) and (6, −3).

1 Find the gradient of the line.

2 Find the equation of the line.

3 Find the equation of the line which passes through (2, 5), perpendicular to the original line.

KEY POINTS

1 Lines with the same gradient are parallel.

2 Perpendicular lines have gradients of m, $-\frac{1}{m}$.

3.19 Further properties of linear graphs

LEARNING OUTCOMES

- Determine the length and midpoint of a line segment from coordinates
- Solve graphically two linear equations in two variables

The length of a line segment

The mathematician thinks of a **line** as straight, continuing in both directions without end and having no thickness.

A **line segment** is part of a line. It has a beginning and an end, and a definite length.

The graph below shows a line segment that runs from (−3, −2) to (3, 4).

When we were finding gradients in 3.17, we used the horizontal and vertical lengths. The horizontal length is $(x_2 - x_1)$ and the vertical height is $(y_2 - y_1)$.

So, for this line segment:

the horizontal length is $3 - -3 = 6$
the vertical height is $4 - -2 = 6$.

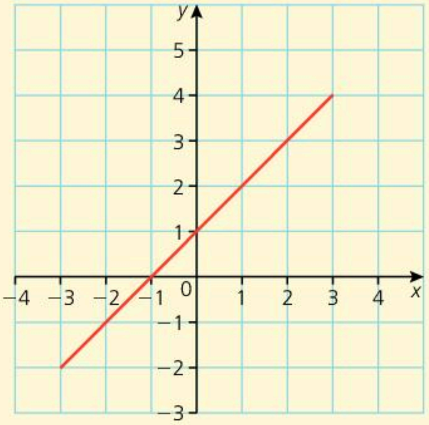
We can also use these lengths to find the length of the line segment.

WORKED EXAMPLE 1

The length of the line segment can be found by Pythagoras:

If the length is l, then:

$$6^2 + 6^2 = l^2$$
$$l^2 = 36 + 36$$
$$l = \sqrt{72}$$
$$l = 8.5 \text{ (to 1 d.p.)}$$



EXAM TIP

- Don't confuse the techniques for finding the length and the midpoint of a line segment.
- To find the length from a to b you subtract b − a before applying Pythagoras.
- To find the midpoint of a and b you add before dividing by 2.

The midpoint of a line segment

The midpoint of the line segment will have an *x*-coordinate exactly halfway between the *x*-coordinates of the ends of the line segment, and a *y*-coordinate exactly halfway between the *y*-coordinates of the ends of the line segment.

So we need to find the number halfway between −3 and 3 for the *x*-coordinate, and halfway between −2 and 4 for the *y*-coordinate.

The easiest way to find the number halfway between *a* and *b* is to find the mean of *a* and *b*.

So the midpoint of a line segment is found by finding the mean of the coordinates.

So the midpoint is given by $\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$.

In this case, the midpoint is $\left(\frac{-3 + 3}{2}, \frac{-2 + 4}{2}\right)$ or (0, 1).

Solving simultaneous equations graphically

In unit 3.10 we saw how to solve a pair of simultaneous equations algebraically.

We can also solve them by drawing a graph.

WORKED EXAMPLE 2

To solve the simultaneous equations

$y = 3x - 1$ and $y = 5 - x$

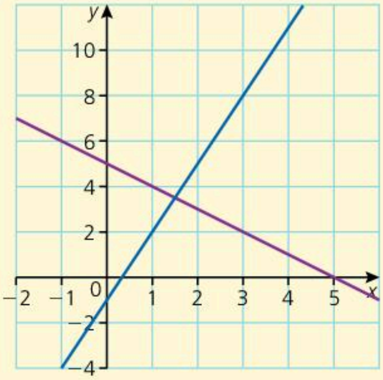
$y = 3x - 1$ has a gradient of 3 and passes through (0, −1).

$y = 5 - x$ has a gradient of −1 and passes through (0, 5).

Plotting the two graphs on the same axis gives this result:

The intersection of the graphs (where the lines cross) shows the point where the two equations share the same values.

So the solution to the equations is $x = 1.5$, $y = 3.5$



KEY POINTS

1 The length of a line segment joining (x_1, y_1) and (x_2, y_2) is $\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

2 The midpoint of the line segment is $\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$.

3 Simultaneous equations can be solved graphically by drawing graphs of the equations and finding the coordinates of the point of intersection.

SUMMARY QUESTIONS

1 Use a graph to solve the simultaneous equations $y = 2x + 1$ and $y = 7 - x$

Questions **2** and **3** are about the line segment which joins (3, −2) to (−1, 6).

2 Find the length and midpoint of the line segment.

3 Find the equation of the line which passes through the midpoint at right angles to the original line.

ACTIVITY

Daisy's electricity company lets her choose between two tariffs.

Tariff A: Fixed charge of $1 per week, plus $2 per kW of electricity used.

Tariff B: Fixed charge of $4 per week, plus $1 per kW of electricity used.

Tariff A has an equation $y = 2x + 1$, where y is the total cost and x is the number of kW used.

Tariff B has an equation $y = x + 4$.

- Plot the graphs on the same axes. For what number of kW is the cost the same for each tariff?

3.20 Inequalities

LEARNING OUTCOMES

- Represent the solution of linear inequalities using set notation, number line

WORKED EXAMPLE 1

To solve $3(5 - x) \leqslant x + 7$

$$\begin{aligned}
15 - 3x &\leqslant x + 7 \\
15 - 3x + 3x &\leqslant x + 7 + 3x \\
15 &\leqslant 4x + 7 \\
15 - 7 &\leqslant 4x + 7 - 7 \\
8 &\leqslant 4x \\
\frac{8}{4} &\leqslant \frac{4x}{4} \\
x &\geqslant 2
\end{aligned}$$

Solving linear inequalities

An inequality is similar to an equation, except that it tells us that two expressions are not equal.

There are four signs that we use:

- $<$ means 'is less than', so $-2 < 1$.
- $>$ means 'is greater than', so $\pi > 3$.
- $\leqslant$ means 'is less than or equal to', so if $x \leqslant 2$, x can be any value smaller than or equal to 2.
- $\geqslant$ means 'is greater than or equal to', so if $4 > x \geqslant 3$, x is greater than or equal to 3 and less than 4.

Inequalities are solved in exactly the same way as equations except when multiplying or dividing by a negative number. We must avoid multiplying or dividing an inequality by a negative number, as this changes the direction of an inequality.

For example, $3 > 2$, but multiplying both sides by -2 gives $-6 < -4$. Note that the inequality has been reversed.

So we must make sure the unknown term is always positive.

Representing an inequality

We saw above that inequalities can be solved in a very similar way to equations.

The main difference between an equation and an inequality is the format of the solutions. A linear equation with one unknown has one solution, a quadratic equation has up to two solutions, and a pair of linear simultaneous equations has a pair of solutions, one in each unknown.

An inequality does not have a single solution; instead, it has a range of solutions.

We can represent this in a number of ways.

WORKED EXAMPLE 2

Suppose we have the inequality $2x + 5 \geqslant 3x + 2 > x - 9$.

We can solve this by splitting it into two separate inequalities:

$$\begin{array}{rlcrl}
2x + 5 & \geqslant 3x + 2 & \text{and} & 3x + 2 & > x - 9 \\
2x + 3 & \geqslant 3x & & 3x & > x - 11 \\
3 & \geqslant x & & 2x & > -11 \\
x & \leqslant 3 & & x & > -5.5
\end{array}$$

The solutions tell us that x is any real number greater than -5.5 and less than or equal to 3.

Using **set notation**, the solution set for the example above is $\{x \in R: -5.5 < x \leqslant 3\}$.

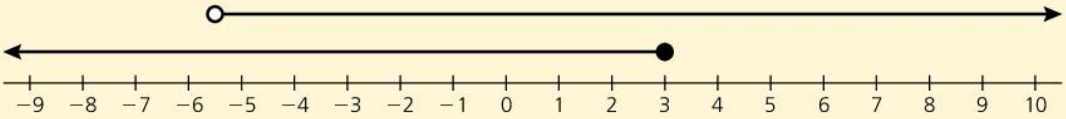
We read this as 'x is an element of the real numbers such that x is greater than -5.5 and less than or equal to 3'.

We could show the same information on a number line.

WORKED EXAMPLE 3

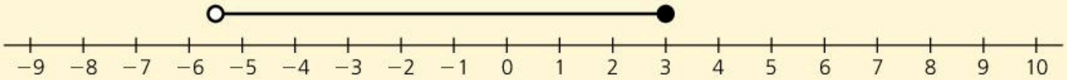
The top arrow represents $x > -5.5$; the circle is empty because x cannot equal 5.5

The lower arrow represents $x \leqslant 3$; the circle is filled, because x can equal 3.



But x must satisfy both conditions. This is true where the lines overlap.

So the number line representation looks like this:



ACTIVITY

Millie bought some bottles of lemonade. She also bought a box of chocolates that weighed 500 g.

Her bag is strong enough to carry up to 2000 g in weight.

Millie put two bottles of lemonade and the chocolates in the bag. The bag carried the weight.

Millie realised that, if a bottle of lemonade weighs b g, then $2b + 500 \leqslant 2000$.

- Solve this inequality.

She then took out the chocolates and put in a third bottle of lemonade. The bag broke.

- Write down an inequality to show this information, and solve it to find the range of possible weights for a bottle of lemonade.

EXAM TIP

- When the inequality is $<$ or $>$, use an unfilled circle on a number line.
- When the inequality is $\leqslant$ or $\geqslant$, use a filled circle.

SUMMARY QUESTIONS

1 Solve the inequality $2x + 4 < 4x - 3$.

2 Solve the inequality $x + 3 \leqslant 2x + 2 < x + 9$, writing your answer in set notation and as a number line diagram.

3 If $2y < 3y + 5$ and $4y + 2 \leqslant y - 7$, find all the possible integer values of y.

KEY POINTS

1 An inequality with one unknown can be written in set notation or shown on a number line.

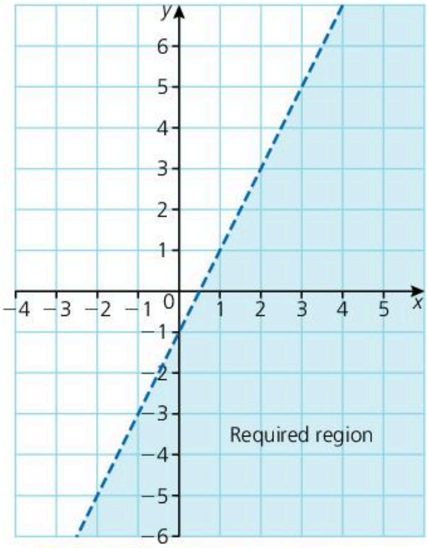
2 Filled circles are part of the solution; unfilled circles are not.

3.21

Inequalities with two unknowns

LEARNING OUTCOMES

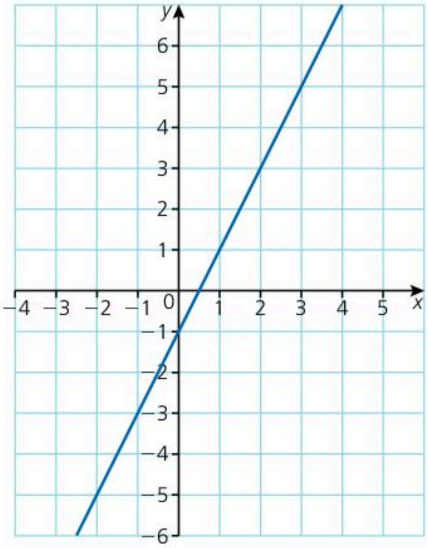
- Represent the solution of linear inequalities using a graph
- Draw a graph of linear inequality in two variables
- Use linear programming techniques



EXAM TIP

- When the inequality is < or >, use a dotted line on a graph.
- When the inequality is ⩽ or ⩾, use a solid line.

Linear inequalities in two variables



The inequality $y < 2x - 1$ contains two variables, and so is best represented on a graph.

First, we draw the graph of $y = 2x - 1$; it has an intercept of -1 on the y-axis and a gradient of 2.

The line shows where $y = 2x - 1$.

On one side of the line, $y < 2x - 1$ and on the other side $y > 2x - 1$.

To decide which side is where $y < 2x - 1$, we select a test point on one side. (0, 0) is a good point to use if it is not on the line. We substitute it in the given inequality. If the resulting statement is true then the side with the selected point is on the required side. If it is false then the other side is the required side.

Using (0, 0), which is above the line, we substitute $x = 0$ and $y = 0$ into $y < 2x - 1$, which gives

$0 < 2 \times 0 - 1$, or $0 < -1$.

This is false, so $y < 2x - 1$ is on the other side of the line.

The line is not part of the region, as y cannot equal $2x - 1$, so we use a dotted line.

We can indicate the region $y < 2x - 1$ by shading.

Linear programming

Linear programming is a method of solving problems by graphing inequalities.

WORKED EXAMPLE 1

Thomas and Sarah make two different types of toy.

A toy plane takes Thomas 20 minutes to assemble, and Sarah spends 30 minutes painting it.

Thomas takes 30 minutes to assemble a boat, and Sarah takes 15 minutes to paint it.

They can each work for up to 6 hours (or 360 minutes).

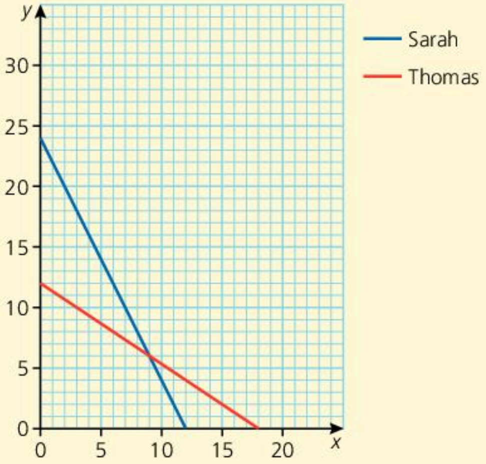
If they make x planes and y boats, the total time for Thomas is $20x + 30y \leqslant 360$

For Sarah, the total time is $30x + 15y \leqslant 360$

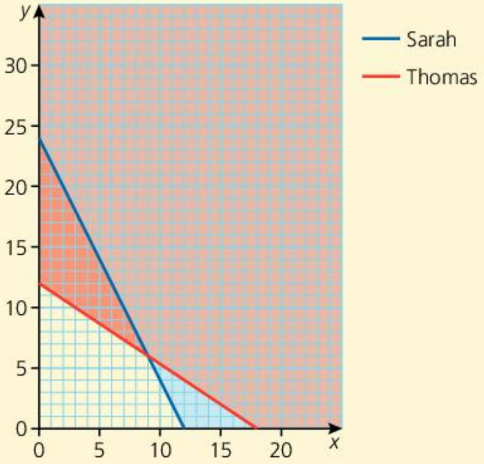
To draw the graphs:

	Thomas	Sarah
	$20x + 30y = 360$	$30x + 15y = 360$
If $x = 0$		
	$0 + 30y = 360$	$0 + 15y = 360$
	$y = 12$	$y = 24$
	(0, 12)	(0, 24)
If $y = 0$		
	$20x + 0 = 360$	$30x + 0 = 360$
	$x = 18$	$x = 12$
	(18, 0)	(12, 0)

Drawing the graphs shows the number of each toy they can make.



The lines show the maximum number they can make if they work the full number of hours available. If we shade the unwanted regions to leave the possible region, we get:



The possible numbers they can make are in a quadrilateral with vertices at (0, 0), (0, 12), (9, 6) and (12, 0). One of these vertices represents the optimum (best) result.

Every plane they sell makes them \$3 profit, and every boat makes them \$2 profit.

The profit for each vertex is:

(0, 0):	0 planes @ \$3 = \$0,	0 boats at \$2 = \$0.	Total profit = \$0
(0, 12):	0 planes @ \$3 = \$0,	12 boats at \$2 = \$24.	Total profit = \$24
(9, 6):	9 planes @ \$3 = \$27,	6 boats at \$2 = \$12.	Total profit = \$39
(12, 0):	12 planes @ \$3 = \$36,	0 boats at \$2 = \$0.	Total profit = \$36

So for maximum profit they make 9 planes and 6 boats.

SUMMARY QUESTIONS

1. Use a graph to show the region $y \geqslant 2x - 3$.
2. On a graph, show the region where $y > 1$, $x \geqslant -1$ and $x + 2y \leqslant 6$.
3. Bananas cost \$1, and pineapples cost \$3. I have \$30. Write down an inequality to show the number of bananas, b, and pineapples, p, I can buy.

KEY POINTS

1. Solid lines are part of the solution; dotted lines are not.
2. When linear programming, shade the unwanted regions, so the clear part is the solution set.

3.22 Composite and inverse functions

LEARNING OUTCOMES

- Derive composite functions e.g. fg, f^2
- State the relationship between function and its inverse
- Derive the inverse function f^{-1}, $(fg)^{-1}$
- Evaluate $f(a)$, $f^{-1}(a)$, $fg(a)$, $(fg)^{-1}(a)$
- Use the relationship $(fg)^{-1} = g^{-1}f^{-1}$

EXAM TIP

- In $gf(x)$, $f(x)$ is calculated first and the result put into $g(x)$.

Composite functions

Kevin and Maria are operating two function machines.

Kevin's machine carries out the function $f(x) = x^2 + 1$

Maria's machine carries out the function $g(x) = 2x - 3$

The output from Kevin's machine is fed into Maria's machine.

So if Kevin inputs 3, his output is

$$f(3) = 3^2 + 1 = 10.$$

This is fed into Maria's machine.

Her output is

$$g(10) = 2 \times 10 - 3 = 17.$$

We have used $f(x)$ as the input for Maria's machine.

So the overall effect of both machines is $g(f(x))$, which we write as $gf(x)$.

When two functions are combined in this way, they are called **composite functions**.

We can replace the composite function, $gf(x)$, with a single function:

$$\begin{aligned} g(x) &= 2x - 3, \text{ so} \\ gf(x) &= 2f(x) - 3 = 2(x^2 + 1) - 3 \\ &= 2x^2 + 2 - 3 \\ gf(x) &= 2x^2 - 1 \end{aligned}$$

We can check that $gf(3) = 17$:

$$gf(3) = 2(3)^2 - 1 = 17$$

If we connected the machines so that Maria's output was fed into Kevin's machine, we would get a different output:

$$\begin{aligned} fg(x) &= (g(x))^2 + 1 = (2x - 3)^2 + 1 \\ &= 4x^2 - 6x - 6x + 9 + 1 \\ fg(x) &= 4x^2 - 12x + 10 \end{aligned}$$

fg(x)* is not equal to *gf(x). In general, composite functions are not commutative.

Inverse functions

In the above example, $f(x)$ is a many–one function.

For example, $f(3) = f(-3) = 10$.

So $f(x)$ will not have an inverse function unless we define the domain of $f(x)$ as the non-negative real numbers. The codomain of the inverse function will then also be the non-negative real numbers.

To find the inverse of $f(x)$, written as $f^{-1}(x)$:

Write $f(x)$ as $y = x^2 + 1$

The inverse is $x = y^2 + 1$

Or $x - 1 = y^2$

$y = \sqrt{x - 1}$

So $f^{-1}(x) = \sqrt{x - 1}$

Similarly, to find $g^{-1}(x)$:

$g(x)$ is represented as $y = 2x - 3$, so

$g^{-1}(x)$ is $x = 2y - 3$

$x + 3 = 2y$

$y = \frac{x + 3}{2}$

$g^{-1}(x) = \frac{x + 3}{2}$

To find the inverse of $gf(x)$, we would have to find $g^{-1}(x)$ first, so $(gf)^{-1}(x) = f^{-1}(g^{-1}(x))$

The inverse of a function undoes the effect of the original function.

The function and its inverse are commutative.

So $f^{-1}(f(x)) = f(f^{-1}(x)) = x$

Squaring a function

If $g(x) = 2x - 3$, then $gg(x) = g(2x - 3) = 2(2x - 3) - 3 = 4x - 9$

We write $gg(x)$ as $g^2(x)$

Solving problems using inverse functions

We can solve problems by using the fact that **$f^{-1}f(x)$ is equal to the identity, or 1**.

WORKED EXAMPLE 1

Suppose $f(x) = 2x - 3$ and $fg(x) = x + 3$, and we want to find $g(x)$.

$fg(x) = x + 3$, so, operating on both sides by $f^{-1}(x)$

$f^{-1}fg(x) = f^{-1}(x + 3)$.

But $f^{-1}f(x) = 1$, the identity, so

$g(x) = f^{-1}(x + 3)$.

$f(x)$ is represented by $y = 2x - 3$, so

$f^{-1}(x)$ is represented by $x = 2y - 3$, or $y = \frac{x + 3}{2}$

$f^{-1}(x) = \frac{x + 3}{2}$, so

$g(x) = f^{-1}(x + 3) = \frac{(x + 3) + 3}{2} = \frac{x + 6}{2}$

ACTIVITY

- If $f(x) = 2x - 1$ and $g(x) = 2x + 1$, match the functions on the left with their equivalent function on the right:

$f^{-1}(x)$	$\frac{x + 1}{4}$
$g^{-1}(x)$	$4x + 1$
$fg(x)$	$4x + 3$
$gf(x)$	$\frac{x + 1}{2}$
$(fg)^{-1}(x)$	$4x - 3$
$(gf)^{-1}(x)$	$4x - 1$
$f^2(x)$	$\frac{x - 1}{2}$
$g^2(x)$	$\frac{x - 1}{4}$

SUMMARY QUESTIONS

1 If $f(x) = \frac{3x - 1}{2}$, find $f^{-1}(x)$.

2 If $f(x) = \frac{x}{2} - 3$ and $g(x) = 2x + 6$, find $fg(x)$.

What does your answer tell you about $f(x)$ and $g(x)$?

3 $f(x) = \frac{1}{x}$

a What value of x must be excluded from the domain?

b What is special about $f^{-1}(x)$?

KEY POINTS

1 $fg(x)$ means apply function g followed by function f.

2 $f^{-1}(x)$ is the inverse function of $f(x)$.

3 $(fg)^{-1}(x) = g^{-1}f^{-1}(x)$

4 $f^2(x) = ff(x)$

3.23

Quadratic functions

LEARNING OUTCOMES

- Draw graphs of quadratic functions
- Use graphs to find *y* given *x*
- Use graphs to find all possible values of *x* given *y*
- Use graphs to find the maximum or minimum value of a function, and the equation of axis of symmetry
- Draw and interpret graphs of quadratic functions to find the interval in the domain for which elements of the range may be > or < a given point

Drawing quadratic graphs

A quadratic function such as $y = x^2 - x + 1$ is not linear.

To plot such a graph, we need to calculate a number of values.

WORKED EXAMPLE 1

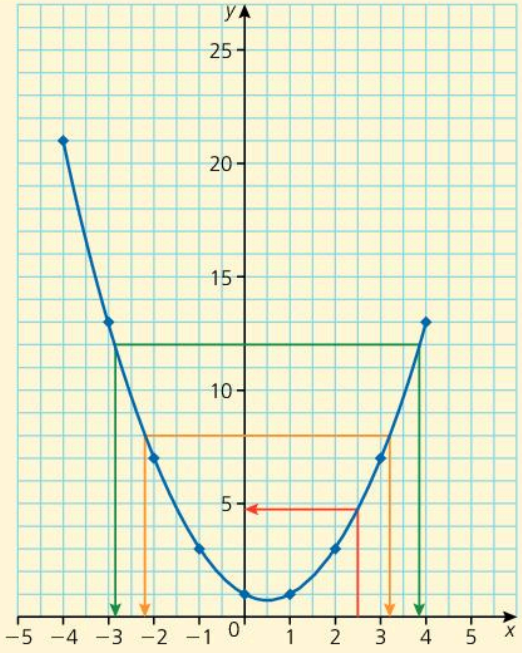
Suppose we wish to draw the graph of $y = x^2 - x + 1$ for values of x from -4 to 4.

We construct a table, showing each part of the equation on a separate line:

x	-4	-3	-2	-1	0	1	2	3	4	
x^2	16	9	4	1	0	1	4	9	16	Remember x^2 cannot be negative
$-x$	4	3	2	1	0	-1	-2	-3	-4	$-(-4) = 4$; $-(4) = -4$
$+1$	1	1	1	1	1	1	1	1	1	The constant term
$y = x^2 - x + 1$	21	13	7	3	1	1	3	7	13	The sum of the three parts

X has values from -4 to 4, and *y* has values from 1 to 21, although it will drop a little below 1 when we draw the graph.

We plot the points and join them with a smooth curve.



There is a symmetry to the graph, which can also be seen in the table.

The axis of symmetry is the line $x = 0.5$

The minimum value is found on this axis of symmetry, where $x = 0.5$

An accurate value of this minimum can be found by substituting $x = 0.5$ into the equation:

$$y = x^2 - x + 1 = (0.5)^2 - 0.5 + 1 = 0.75$$

EXAM TIP

- Quadratic graphs are curves. Do not join the points with a ruler.
- Quadratic graphs are always a U-shape. If the x^2 term is negative, the U is inverted.

Reading values from a graph

We can use the graph shown to match elements of the domain (x) to the range (y).

For example, when $x = 2.5$, we can follow the red line to find the corresponding value of y, which is 4.75.

To find elements in the domain which map to 12 in the range, follow the green arrows. The values of x are -2.9 and 3.9 to 1 decimal place.

This tells us that the solutions to the equation $x^2 - x + 1 = 12$ are $x = -2.9$ and $x = 3.9$ to 1 decimal place.

To solve the inequality $x^2 - x < 7$:

First, add 1 to both sides so that the left-hand side matches the equation of the graph:

$$x^2 - x + 1 < 8$$

From the graph, the solutions to $x^2 - x + 1 = 8$ are $x = -2.2$ and $x = 3.2$. This is shown by the orange line on the graph.

The graph shows that the region where $y < 8$ (the part of the curve below the line $y = 8$) is the part between these solutions, so the solution to the inequality is $-2.2 < x < 3.2$.

ACTIVITY

Draw the graph of $y = x^2 - 2x$, taking values of x from -2 to 4.

On the same set of axes, draw the graph of $y = x - 2$.

- What are the x-coordinates of the points of intersection?
- Rearrange the equation $x^2 - 2x = x - 2$ so that the right-hand side $= 0$.
- Solve the quadratic equation and compare your answers to the points of intersection of the graphs.

KEY POINTS

1 To draw a quadratic graph, start with a table. Calculate each term separately and then add them up.

2 You can use the graph to read intermediate values and solve inequalities.

SUMMARY QUESTIONS

1 Draw the graph of $y = x^2 + x - 3$, taking values of x from -3 to 3.

2 Use your graph to find the value of y when $x = 1.5$.

3 Use your graph to solve the inequality $x^2 + x - 3 < 2$.

3.24

More graphs of functions

LEARNING OUTCOMES

- Estimate the gradient at a given point
- Find the intercepts of a function
- Solve quadratic equations graphically
- Find the axis of symmetry, maximum or minimum value of a quadratic in the form $a(x + h)^2 + k$
- Sketch the graph of a quadratic in the form $a(x + h)^2 + k$ and determine the number of roots
- Draw and interpret the graphs of other non-linear functions $\left(y = \frac{a}{x}, y = \frac{a}{x^2}, y = ax^3\right)$

The gradient of a curve at a point

A quadratic graph is curved, and so the steepness, or gradient, is constantly changing as we move along the graph.

To estimate the gradient of a curve at a point, we draw a **tangent** at that point. A tangent is a line that just touches the curve without passing inside the curve.

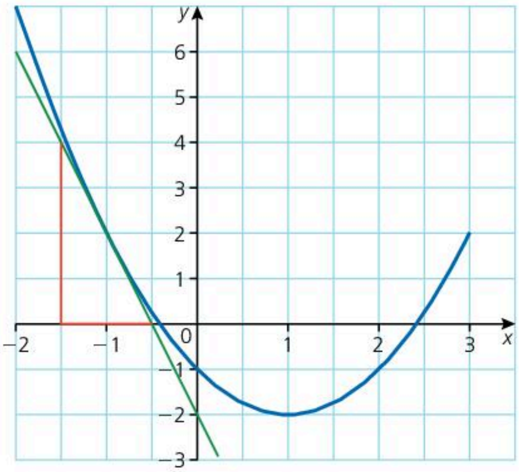
We then find the gradient of the tangent.

Here is the graph of $y = x^2 - 2x - 1$.

To find the gradient where $x = -1$, we draw the tangent at that point, shown in green.

The red lines show that the tangent has a vertical drop of -4 and a horizontal length of 1, so at $x = -1$ the gradient of the curve is

$$\frac{-4}{1} = -4.$$



Points where the tangent is horizontal have a gradient of 0. These are called **turning points**.

ACTIVITY

- Solve the equation $x^2 - 2x - 1 = 0$ by using the formula to check these solutions.
- What are the advantages and disadvantages of solving quadratic equations graphically?

The intercepts of a curve with the axes

The graph of $y = x^2 - 2x - 1$, crosses the y axis at $(0, -1)$. As with linear graphs, the constant term indicates the intercept on the y-axis.

The solutions to the quadratic equation $x^2 - 2x - 1 = 0$ can be found where $y = 0$.

These are the x coordinates where the graph crosses the x axis.

So, reading from the graph, the solutions to $x^2 - 2x - 1 = 0$ are $x = -0.4$ and $x = 2.4$.

Completed square form and sketching quadratic curves

Writing the equation $y = x^2 - 2x - 1$ in completed square form, we get $y = (x - 1)^2 - 2$.

Compare this to the minimum point of the graph, $(1, -2)$.

The minimum value of the equation $y = (x + a)^2 + b$ is at $(-a, b)$, and the axis of symmetry is the line $x = -a$.

This allows us to sketch curves without drawing up a table of values.

WORKED EXAMPLE 1

To sketch the graph of $y = x^2 + 4x + 5$:

The intercept on the y-axis is at (0, 5).

The coefficient of x^2 is positive so it has a U shape.

In completed square form, the equation is $y = (x + 2)^2 + 1$, so the minimum value is at $(-2, 1)$, and the curve is symmetrical about the line $x = -2$.

So the graph will look like the graph shown opposite:

The equation $x^2 + 4x + 5 = 0$ has no solutions, as the graph does not cross the x-axis.

Other curved graphs

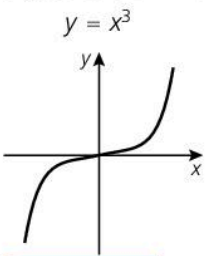
In 3.23, we drew graphs by constructing a table.

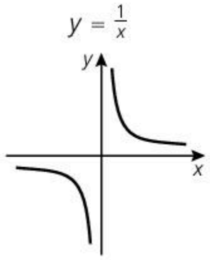
The same technique can be applied to draw the graphs of $y = x^3$, $y = \frac{1}{x}$, $y = \frac{1}{x^2}$.

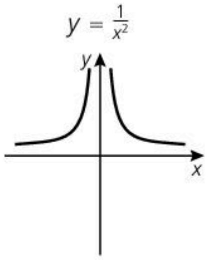
x	-4	-3	-2	-1	0	1	2	3	4
x^3	-64	-27	-8	-1	0	1	8	27	64
$\frac{1}{x}$	-0.25	-0.33	-0.5	-1		1	0.5	0.33	0.25
$\frac{1}{x^2}$	0.06	0.11	0.25	1		1	0.25	0.11	0.06

$y = \frac{1}{x}$ and $y = \frac{1}{x^2}$ are not defined when $x = 0$.

The graphs look like this:

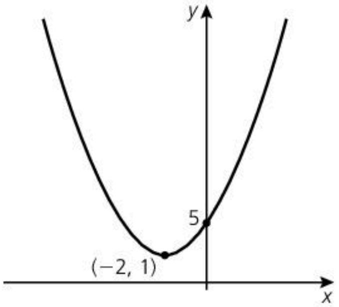






KEY POINTS

1. The gradient of a curve at a point is estimated from the gradient of the tangent at that point.
2. The graph of an equation in the form $y = ax^2 + bx + c$ will intersect the y-axis at $(0, c)$.
3. The graph of an equation in the form $y = a(x + h)^2 + k$ has a turning point at $(-h, k)$.
4. Quadratic graphs, cubic graphs, reciprocal graphs and reciprocal squared graphs each have their own identifiable shape.



ACTIVITY

The equation $x^2 + 4x + 5 = 0$ will not factorise, as there are no solutions.

- Use the completed square form, $(x + 2)^2 + 1 = 0$, to see what happens when you try to solve the equation by completing the square. How can you tell there are no solutions?

ACTIVITY

- Draw the graphs accurately to check the sketches above.

EXAM TIP

Learn the shapes of these graphs.

SUMMARY QUESTIONS

1. Write down the intercept on the y-axis of $y = x^2 - 4x + 3$.
2. Write $y = x^2 - 4x + 3$ in completed square form, and identify the coordinates of the minimum point.
3. Sketch the graph of $y = x^2 - 4x + 3$, identifying all the intercepts on the axes.

3.25 Travel graphs

LEARNING OUTCOMES

- Draw and interpret distance–time graphs
- Draw and interpret speed–time graphs to determine distance; time; speed; acceleration

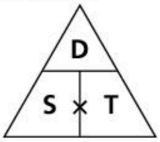
It is often useful to draw graphs of journeys, as we can read information from them.

Distance–time graphs

Marsha lives in Grenada. She travels from Sauteurs to Grenville to visit her aunt, a distance of 18 km.

She cycles for an hour and a half at 10 km/h, and then finishes the journey at 6 km/h. She stays with her aunt for an hour and 20 minutes. She returns home at a constant speed. The journey takes 2 hours.

Remember the triangle from 2.7:



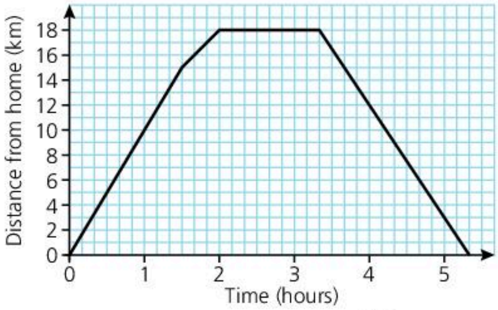
Here is the information about Marsha's journey in a table, with missing distances and times calculated.

Description	Time	Distance	Speed
A. Start journey	1.5 hours	Speed × time = 15 km	10 km/h
B. Finish journey	Distance ÷ speed = 0.5 hour	18 − 15 = 3 km	6 km/h
C. Stays with aunt	1 hour 20 min	0 km	0 km/h
D. Return	2 hours	18 km	

EXAM TIP

- Remember that time is not a metric unit; 1 hour 30 minutes is 1.5 hours, not 1.3 hours.
- A useful scale for time is to take 6 squares for 1 hour, and then each square is 10 minutes.

The information can be transferred to a graph:



The first part of the graph has a gradient of $\frac{15}{1.5}$ = 10 km/h.

The gradient of a distance-time graph represents the speed.

Speed–time graphs

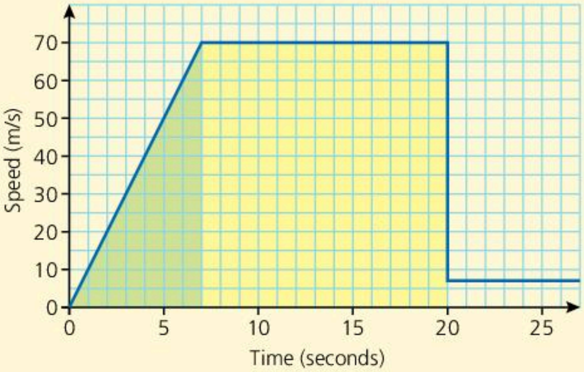
When speed is constantly changing, a speed–time graph is more useful.

WORKED EXAMPLE 1

Marc is a skydiver. He jumps from a plane at an altitude of 2000 m. His speed towards earth increases steadily from 0 m/s to 70 m/s after 7 seconds. He has then reached terminal velocity, and falls at a constant speed.

13 seconds later, he opens his parachute. His speed immediately drops to a constant 7 m/s until he lands.

Marc's speed–time graph looks like this:



The first section of the graph shows a steady increase in speed. This is called **acceleration**.

Acceleration is measured in m/s^2, read as 'metres per second per second'.

Marc's acceleration is $10\,m/s^2$, because every second his speed increases by 10 m/s.

This is the gradient of the graph.

The gradient of a speed–time graph represents the acceleration.

We know that distance = speed × time.

For the freefall section of the graph, that is $70\,m/s \times 13\,s = 910\,m$.

This is the area of the yellow section of the graph.

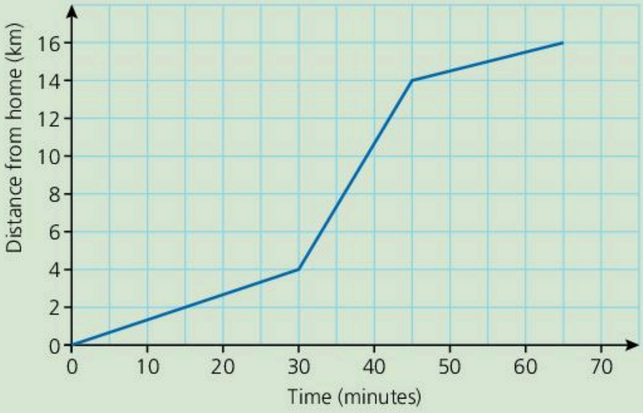
The area under a speed–time graph tells us the distance travelled.

So during the first stage, the distance travelled is the area of the green triangle,

$$\frac{\text{base} \times \text{height}}{2} = \frac{7 \times 70}{2} = 245\,m.$$

SUMMARY QUESTIONS

Here is a graph of Edward's journey to work. He walks to the bus stop, gets a bus and then walks the final part.



1 What is the average speed of the bus in km/h?

2 On which part of the journey did he walk faster; to the bus or from the bus?

3 Robin is running in the 100 m race. He accelerates from 0 m/s to 10 m/s in 2 seconds. He maintains a speed of 10 m/s for 8 seconds, and then starts to slow down, so that he stops in 5 seconds.
Show this information on a speed–time graph, and calculate how far he runs altogether.

ACTIVITY

- Use the graph to calculate how far Marc had fallen after
  - 1 second
  - 2 seconds
  - 3 seconds
  - 4 seconds
  - 5 seconds
  - 6 seconds
  - 7 seconds
- Use this information to draw a distance–time graph of the first 7 seconds of his descent. What shape is your graph?

KEY POINTS

1 The gradient of a distance–time graph shows the speed.

2 The gradient of a speed–time graph shows the acceleration.

3 The area under a speed–time graph shows the distance travelled.

Module 3 Practice exam questions

1 Rupert has \$$n$.
His sister Lucinda has \$2 more than Rupert.
Their brother Edward has twice as much as Lucinda.
How much money do they have altogether?

2 If $a = 2$, $b = -3$ and $c = 4$, find the value of $\frac{2a(1 - b)}{c}$

3 Simplify $\frac{2x}{3} + \frac{x + 1}{2}$

4 The binary operation ✲ is defined by
a ✲ $b = a(b - 2)$

a Find the value of 3 ✲ 5

b Find the value of 5 ✲ 3

c What do your answers tell you about ✲?

5 Expand $2x(x - 3y)$

6 Factorise fully $2x^2 + 6x - 20$

7 Make s the subject of the formula
$v^2 = u^2 + 2as$

8 Solve the equation $3x - 2 = x + 7$

9 Solve the equation $x^2 - 3x + 2 = 0$ by factorisation.

10 Solve the equation $x^2 - 4x - 6 = 0$ by completing the square.

11 Solve the simultaneous equations

$2a + b = 7$
$3a - 2b = 14$

12 A rectangular garden is l m long and w m wide.
The length and width are connected by the equations

$l + w = 20$
$l^2 + w^2 = 208$

Find the length and width of the garden.

13 x varies directly as the square of y.
When $x = 7$, $y = 2$.
Calculate x when $y = 5$.

14 Match the relations on the left with the types on the right.

Relation	**Type**
$y = x^2 - 2$	One–one
$y = 3x + 1$	Many–one
$y = \sqrt{x - 3}$	One–many

15 **a** Find the inverse of the function $f(x) = x^2 - 7$

b Explain why the inverse is not a function.

16 **a** Draw axes from -1 to 4 for x, and -5 to 10 for y.

b Draw the graphs of $y = 3x - 2$ and $y = x + 1$

c Use your graph to solve the simultaneous equations $y = 3x - 2$ and $y = x + 1$

17 Find the equation of the straight line that passes through $(-3, 1)$ and $(2, -9)$.

18 A line goes from $(-2, -1)$ to $(4, 7)$. Find:

a the length of the line

b the midpoint of the line.

19 Draw axes from -1 to 4 for x, and 0 to 10 for y.
Mark the region contained by $x > 1$, $y > 2$ and $y \leqslant 8 - 2x$

20 $f(x) = 2x + 1$ and $g(x) = x - 5$
Find:

a $f(3)$

b $g(-2)$

c $fg(x)$

d $f^{-1}(x)$

21 **a** Draw the graph of $y = x^2 - 2x$, taking values of x from -3 to 3.

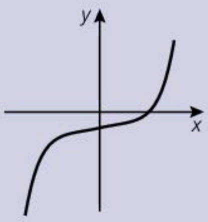
b Use your graph to find the value of y when $x = -1.5$

c Estimate the gradient at the point where $x = 2$.

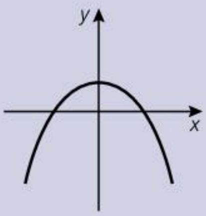
22 Match the graphs to their equations:

a $y = 2 - x^2$ **b** $y = \frac{2}{x^2}$ **c** $y = x^3 - 2$

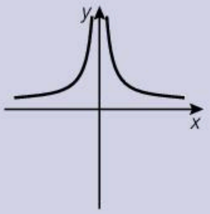
(1)



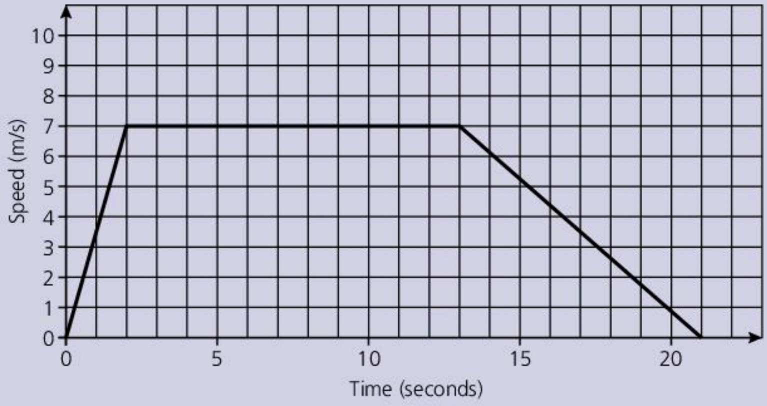
(2)



(3)



23 Here is a speed–time graph of Gina's race.



a Calculate Gina's acceleration over the first 2 seconds.

b Find the total distance Gina ran.

24 The mass of a coin varies directly with the square of the diameter.
A coin with diameter 2.5 cm has a mass of 25 g.
Calculate:

a the mass of a coin with a diameter of 3 cm

b the diameter of a coin with a mass of 9 g.

25 Calculate the points of intersection of the graphs of $y = x - 2$ and $x^2 + y^2 = 10$.

4 Geometry & trigonometry and vectors & matrices

4.1

Properties of lines and angles

LEARNING OUTCOMES

- Understand the terms point, line, ray, line segment, parallel lines, intersecting lines, perpendicular lines, curve, angle (acute, right, obtuse, reflex), plane, solid face, edge, vertex
- Draw and measure angles accurately
- Solve geometric problems using angle properties (vertically opposite, alternate, co-interior, at a point, complementary and supplementary)

Points and lines

- A **point** has a definite position but has no size. Of course this means that we would not be able to see it, so when we mark a point it does have a size, but mathematically we assume it has no length, width or height.
- A **line** is straight, continues in both directions without end, and has no thickness.
- A **ray** is straight, has a start point but continues to infinity.
- A **line segment** is part of a line; it has a beginning and end, and a definite length.
- **Parallel lines** are lines that are the same perpendicular distance apart along their entire length, and so never meet.
- **Intersecting lines** are lines that meet at a common point.
- **Perpendicular lines** meet at right angles.
- A **curve** is a smoothly flowing line with no sharp changes in direction.

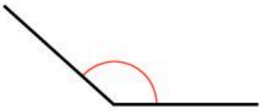
Angles

An **angle** is a measure of the turn from one direction to another.

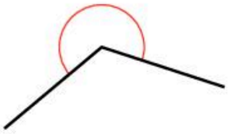
Angles are commonly measured in degrees. 360 degrees (360°) make a full turn, so 180° is a half turn and 90° is a quarter turn or **right angle**.

Acute
(less than 90°)

Right
(= 90°)



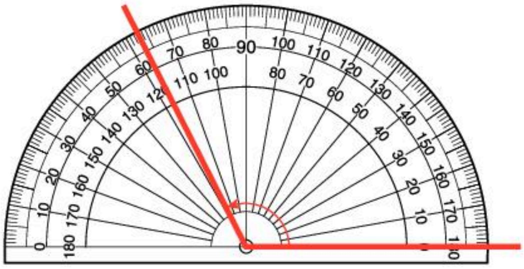
Obtuse
(greater than 90°, less than 180°)



Reflex
(greater than 180°)

We use a protractor to measure angles.

Protractors usually have two scales, one clockwise and the other anticlockwise. It is important to use the correct scale.



To measure an angle, place the protractor on the angle so that one of the zero lines exactly coincides with one arm of the angle, and the centre mark is exactly on the vertex.

The zero scale in the illustration is the inner scale, and to turn from this line is an anticlockwise turn. The angle is 117°.

To measure a reflex angle, instead measure the remaining part of the full turn, and subtract your answer from 360°.

In the illustration, the obtuse angle is 109°.

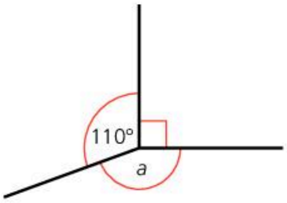
The reflex angle = 360 − 109 = 251°.

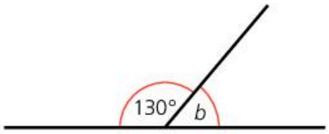
Angles at a point and angles on a straight line

A full turn is 360°, so angles that make up a full turn have a sum of 360°.

A half turn, or 180° angle, makes a straight line.
We can use these facts to calculate missing angles.

Look at these diagrams.





$$a + 90 + 110 = 360$$
$$a + 200 = 360$$
$$a = 160°$$

$$b + 130 = 180$$
$$b = 50°$$

Vertically opposite angles

When two lines cross, the angles opposite each other at the vertex are equal.

They are called **vertically opposite** angles.

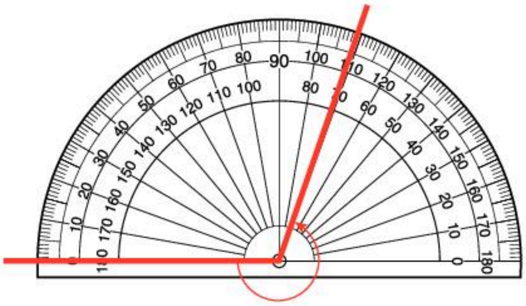
Angles *a* and *c* are equal, and angles *b* and *d* are equal.

Angles *a* and *b* add up to 180°. Two angles that add up to 180° are called **supplementary angles**.

Two angles that add up to 90° are called **complementary angles**.

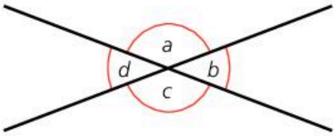
KEY POINTS

1. A point has a definite location but no size.
2. A line has no ends, a ray has one end and a line segment has two ends.
3. Parallel lines never meet, perpendicular lines meet at right angles.
4. Angles can be acute (< 90°), right (90°), obtuse (between 90° and 180°) or reflex (> 180°).
5. Angles at a point sum to 360°, angles on a straight line sum to 180°.
6. Vertically opposite angles are equal.
7. Complementary angles add up to 90°. Supplementary angles add up to 180°.

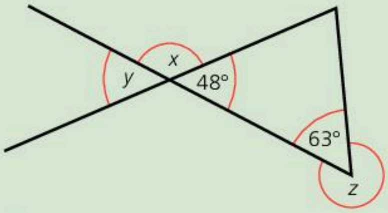


ACTIVITY

- A full turn is split into four angles, *a*, *b*, *c* and *d*, so that *b* is twice the size of *a*, *c* is three times the size of *a*, and *d* is four times the size of *a*. Find the size of each angle.
- Another full turn is split into five angles in the ratio 1 : 2 : 3 : 4 : 5. Find the size of each angle.
- Another full turn is split into nine angles in the ratio 1 : 2 : 3 : 4 : 5 : 6 : 7 : 8 : 9. Find the size of each angle.



SUMMARY QUESTIONS



1 Which of angles *x*, *y* and *z* is:

a reflex **b** acute

c obtuse?

2 Calculate the size of angle:

a *x* **b** *y* **C** *z*

3 Explain the difference between a line and a ray.

4.2 Parallel lines

LEARNING OUTCOMES

- Solve geometric problems using angles (alternate, adjacent, corresponding; co-interior; parallel, transversal)

Corresponding angles

Parallel lines are identified on a diagram with arrows.

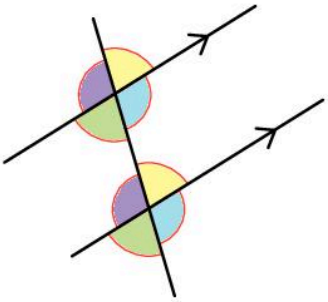
A **transversal** is a line that crosses a pair of parallel lines.

The transversal creates four pairs of **corresponding angles**.

Corresponding angles are pairs of angles in similar, or corresponding positions.

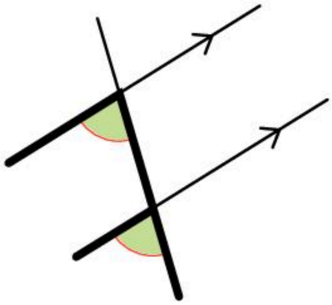
You can imagine sliding one of the pair on to the other.

Corresponding angles on parallel lines are equal.



In the diagram on the left, pairs of corresponding angles are marked in the same colour.

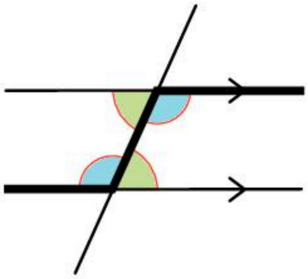
Corresponding angles always make an F-shape, although it might be upside down or back to front.



Alternate angles

A transversal on parallel lines creates two pairs of **alternate angles**. Alternate angles are between the parallel lines, on alternate sides of the transversal.

Alternate angles make a Z-shape, although the Z might be back to front or stretched.



Because of the rotational symmetry of the diagram, alternate angles on parallel lines are equal.

EXAM TIP

- Do not assume lines are parallel unless the question tells you.
- If you are asked to give reasons for your answer, use the correct mathematical vocabulary.
- If you are asked to calculate an angle, you must not measure it.

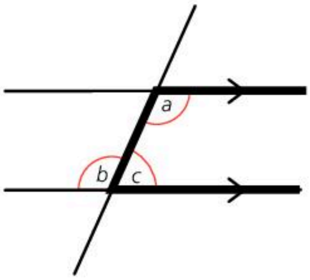
Co-interior angles

In the diagram:

$a = b$ (alternate angles on parallel lines)
$b + c = 180°$ (angles on a straight line)

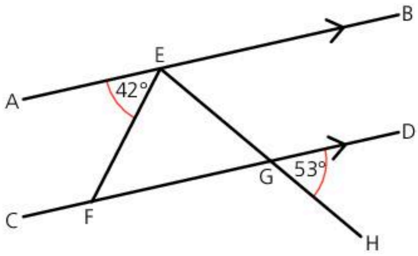
So

$a + c$ must equal 180°.



Angles like a and c inside a pair of parallel lines and on the same side of the transversal are supplementary, and are called co-interior angles. They make a C-like shape.

Solving problems



In complex diagrams, we use three letters to identify an angle. In the diagram, the angle of 42° is called angle AEF or FEA.

Imagine drawing the angle with a continuous movement; you would draw from A to E to F, so we name it AEF. The middle letter (E) indicates the vertex where the angle is. The first and last pairs of letters (AE and EF) show the lines which surround the angle. They are the arms of the angle.

Angle problems require the correct choice of angle fact. Sometimes you need to calculate other angles along the way. It helps to write any angles you calculate on the diagram.

WORKED EXAMPLE 1

To calculate angle FEG in the diagram above:

BEG = DGH = 53° Corresponding angles on parallel lines

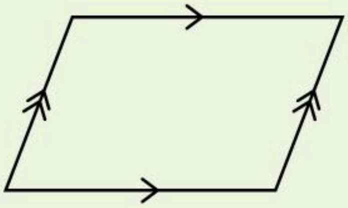
AEF + FEG + BEG = 180° Angles on a straight line

FEG = 180 − 42 − 53 = 85°.

ACTIVITY

A parallelogram consists of two pairs of parallel lines.

- Prove that the angles add up to 360°.
- Prove that the opposite angles are equal.



KEY POINTS

1 The angle ABC is the angle at B, between AB and BC.

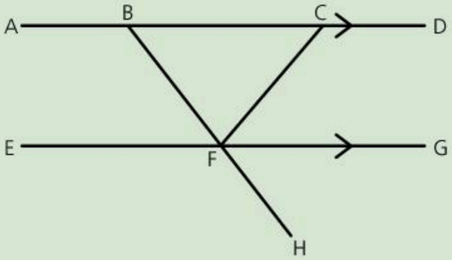
2 Corresponding angles on parallel lines are equal. They make an F shape.

3 Alternate angles on parallel lines are equal. They make a Z shape.

4 Co-interior angles are inside the parallel lines on the same side of the transversal. Co-interior angles are supplementary. They make a C shape.

SUMMARY QUESTIONS

Use the diagram to answer these questions.



1 Which angle is alternate to BCF?

2 Which angle is co-interior with BCF?

3 If angle FBC = 56° and CFG = 48°, calculate angle:

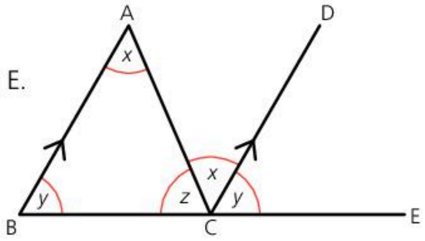
a DCF **b** CFH **c** EFH

4.3 Properties of triangles and quadrilaterals

LEARNING OUTCOMES

- Understand properties of triangles; equilateral, right, isosceles
- Understand properties of quadrilaterals; square, rectangle, rhombus, kite, parallelogram, trapezium
- Solve geometric problems using congruent triangles, similar figures

Triangles



ABC is a triangle.

BC is extended (or **produced**) to E.

CD is drawn parallel to BA.

We know angle BAC = ACD = x (alternate angles on parallel lines)

And ABC = DCE = y (corresponding angles on parallel lines)

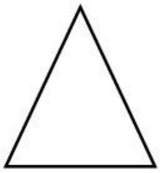
The three angles at C (ACB, ACD and DCE) show that:

$$x + y + z = 180° \text{ (angles on a straight line)}$$

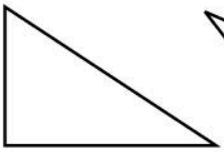
So the three angles in the triangle, $x + y + z = 180°$

The angle sum of any triangle is 180°.

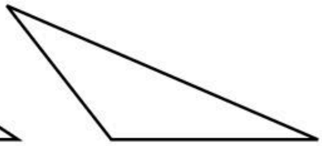
Triangles can have three acute angles (an **acute-angled triangle**), two acute angles and a right angle (a **right-angled triangle**) or two acute angles and an obtuse angle (an **obtuse-angled triangle**).



Acute-angled

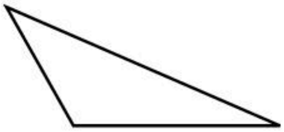


Right-angled

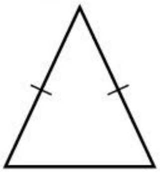


Obtuse-angled

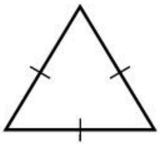
- A **scalene** triangle has all sides of different length and all angles different sizes.
- An **isosceles** triangle has two equal sides and two equal angles.
- An **equilateral** triangle has three equal sides and three angles of 60°.



Scalene



Isosceles



Equilateral

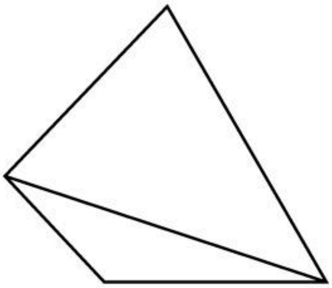
Quadrilaterals

Quadrilaterals are four-sided shapes.

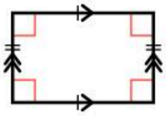
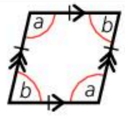
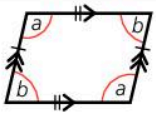
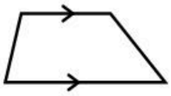
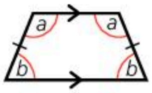
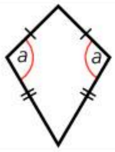
Quadrilaterals can be split into two triangles by a diagonal, and so the angle sum of all quadrilaterals is 2 × 180 = 360°.

Some quadrilaterals have special properties.

These properties include equal sides and equal angles. Their diagonals may be equal, they may bisect each other, and they may be perpendicular.



The table below lists those shapes and their properties.

Shape		Sides	Angles	Diagonals
Square		All sides equal. Opposite sides parallel.	All four angles = 90°	Diagonals equal, bisect, perpendicular.
Rectangle		Opposite sides equal. Opposite sides parallel.	All four angles = 90°	Diagonals equal, bisect.
Rhombus		All sides equal. Opposite sides parallel.	Opposite angles equal.	Diagonals bisect, perpendicular.
Parallelogram		Opposite sides equal. Opposite sides parallel.	Opposite angles equal.	Diagonals bisect.
Trapezium		One pair of opposite sides parallel.		
Isosceles trapezium		One pair of opposite sides parallel, the other pair equal.	Two pairs of equal angles. Opposite angles supplementary.	Diagonals equal.
Kite		Two pairs of equal adjacent sides.	One pair of angles equal.	One diagonal bisected, perpendicular.

ACTIVITY

- Draw each of the quadrilaterals shown in the table.
- Draw the diagonals to check the properties given in the table.

EXAM TIP

- The diagonals are perpendicular if there is a line of symmetry through opposite vertices.
- The diagonals are equal if there is a line of symmetry through opposite sides.
- The diagonals bisect if there is rotational symmetry (of order 2 or more).

SUMMARY QUESTIONS

1 An isosceles triangle has an angle of 70°. Find the size of the other two angles. (There are two possible answers; can you find them both?)

2 A quadrilateral has three angles of 70°.

a Calculate the fourth angle.

b What type of quadrilateral is it?

3 A quadrilateral has two pairs of equal angles and the diagonals are equal but do not bisect.What type of quadrilateral is it?

KEY POINTS

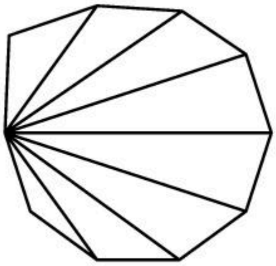
1 Triangles can be scalene, isosceles or equilateral; acute-angled, right-angled or obtuse-angled.

2 The special quadrilaterals are square, rectangle, rhombus, parallelogram, trapezium, isosceles trapezium and kite.

4.4 Properties of polygons

LEARNING OUTCOMES

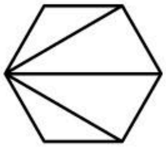
- Understand angle properties of polygons
- Solve problems in geometry



Polygons

A **polygon** is the general name for a closed shape made of three or more straight sides.

The angles in a polygon



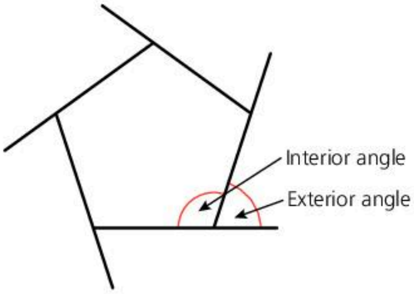
- A hexagon has 6 sides, and can be split into 4 triangles from a vertex. So the angles of a hexagon add up to 4 × 180 = 720°.
- A decagon has 10 sides, and can be split into 8 triangles from a vertex. So the angles of a decagon add up to 8 × 180 = 1440°.

Generally, an n-sided polygon can be split into $(n - 2)$ triangles.

So they have an angle sum of $(n - 2) \times 180°$.

These angles inside the polygon are called **interior angles**.

Exterior angles make a straight line with the interior angles.



So an exterior angle and the interior angle are supplementary (add up to 180°).

The exterior angles of any polygon add up to 360°.

A **regular polygon** has all sides equal and all angles equal.

So a regular hexagon has 6 equal angles of 720 ÷ 6 = 120°.

ACTIVITY

- Imagine – or draw – a triangle on the floor.
- Start halfway along one of the sides and walk along the sides of the triangle.
- By the time you return to your starting position, you will have made a complete turn, or 360°.
- Repeat with other polygons.

You always turn round once, because you turn through the exterior angles, which always add up to 360°.

Names and angle properties of polygons

Number of sides	Name	Angle sum
3	Triangle	180°
4	Quadrilateral	2 × 180 = 360°
5	Pentagon	3 × 180 = 540°
6	Hexagon	4 × 180 = 720°
7	Heptagon	5 × 180 = 900°
8	Octagon	6 × 180 = 1080°
9	Nonagon	7 × 180 = 1260°
10	Decagon	8 × 180 = 1440°

ACTIVITY

The maximum number of right angles in a triangle is 1, but in a quadrilateral is 4.

- Find the maximum number of (interior) right angles in a pentagon and in a hexagon.

Solving problems

The interior and exterior angle properties of polygons can be used to solve problems.

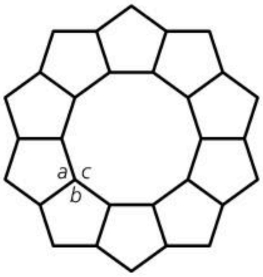
For example, a set of congruent regular pentagons are connected to make a loop.

To prove that they form a regular decagon at the centre:

The exterior angle of a regular pentagon = 360 ÷ 5 = 72°.

So the interior angles are all 180 − 72 = 108°.

Angles *a* and *b* are each 108°.
So the remaining angle *c* = 360 − (2 × 108) = 144°.



But the exterior angle of a regular decagon = 360 ÷ 10 = 36°.

And the interior angle = 180 − 36 = 144°.

But this is true for all the angles in the central shape.

So the central shape is a regular decagon.

EXAM TIP

- Regular means all sides equal and all angles equal.
- For regular polygons, the exterior angle = 360 ÷ number of sides, and the interior angle = 180 − exterior angle.
- For the sum of the interior angles, learn the formula or use the sequence.

KEY POINTS

1 A regular polygon has equal sides and equal angles

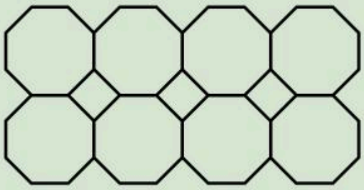
2 The interior angles of an *n*-sided polygon add up to $(n - 2) \times 180°$.

3 The exterior angles of a polygon add up to 360°.

SUMMARY QUESTIONS

1 A pentagon has three angles of 104° each, and the other two angles are equal. Calculate the size of one of these angles.

2 This tessellation pattern shows two regular octagons and a square meeting at each vertex. Use this fact to find the interior angle of a regular octagon.



3 A regular polygon has interior angles of 144°. How many sides does it have? (Hint: find the size of an exterior angle first.)

4.5

Constructing angles

LEARNING OUTCOMES

- Construct angles of 30°, 45°, 60°, 90°, 120°
- Construct parallel and perpendicular lines

There are a number of constructions we can perform without a ruler. These are sometimes called ruler-and-compasses constructions, but are really straight-edge and compasses constructions, as the ruler is used to draw straight lines rather than to measure. The compasses are used for marking equal distances.

Construct an angle of 60°

An equilateral triangle has angles of 60°. So, to construct an angle of 60°, we mark the vertices of an equilateral triangle.

First, draw a line and mark a point, A, near one end.

Open the compasses, put the point on A and make two arcs; one along the line, crossing it at B, and the other above the line.

Keeping the compasses open at the same distance, put the point on B and draw an arc to cut the other arc at C.

Join AC to complete the angle of 60°. (Diagram 1)

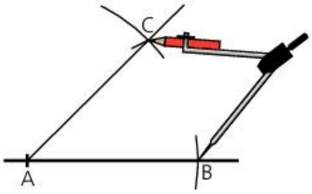


Diagram 1

Bisecting a line segment

To bisect a line segment *AB*, open the compasses to more than half the length *AB*.

With the compass point on *A*, draw arcs on either side of *AB*.

Repeat with the compass point on *B*, so the arcs intersect. (Diagram 2)

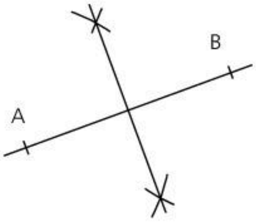


Diagram 2

The line joining the intersections bisects *AB*.

Bisecting an angle

To bisect (or cut exactly in half) an angle, put the point of the compasses on the vertex, A, and make equal marks at B and C along each arm of the angle. (Diagram 3)

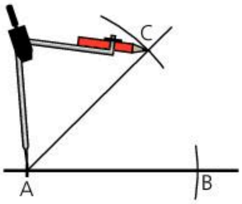


Diagram 3

Using B and C as centres, draw two arcs to cross inside the angle at D.

Draw the angle bisector from A through D.

AD bisects angle CAB because AD is a diagonal of the rhombus ABDC. (Diagram 4)

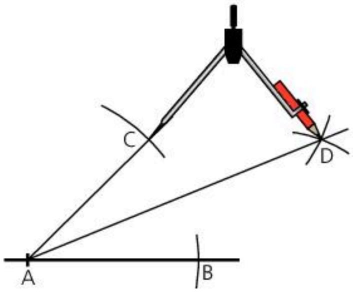


Diagram 4

Constructing a perpendicular to a line

To construct a perpendicular to a line, we simply bisect an angle of 180°.

To construct a right angle at *A*, place the compass point on *A* and make arcs *B* and *C*, at equal distances on either side of *A*.

Opening the compasses wider, make two equal arcs from *B* and *C* to cross on the same side of the line. Join *A* to the point of intersection to construct the perpendicular. (Diagram 5)

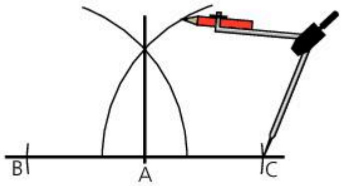


Diagram 5

Constructing other angles

- An angle of 45° can be constructed by bisecting an angle of 90°.
- An angle of 30° can be constructed by bisecting an angle of 60°.
- An angle of 120° can be constructed by constructing a 60° angle on a straight line; the obtuse angle is 120°.

Constructing a perpendicular from a point to a line

To construct a perpendicular from the point A to the line BC, put the point of the compasses on A and draw two arcs crossing BC at D and E. (Diagram 6)

With D and E as centres, draw two more equal arcs to intersect at F. AF is perpendicular to BC.

This is because AF is a diagonal of the rhombus ADFE. (Diagram 7)

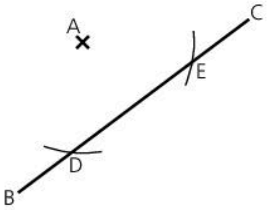


Diagram 6

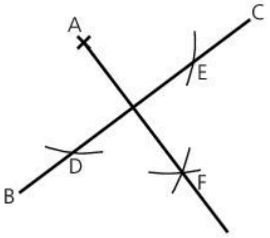


Diagram 7

To construct a pair of parallel lines

To construct a line through C parallel to AB, first draw a line from A through C.

Use compasses to draw three arcs of the same radius:

- Centre A, at D on AC
- Centre A, at E on AB
- Long arc, centre C at F on AC produced. (Diagram 8)

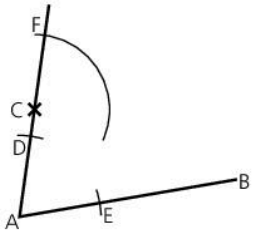


Diagram 8

Open the compasses to the distance DE. Put the point on F to draw an arc to cross the long arc at G. CG is parallel to AB. (Diagram 9)

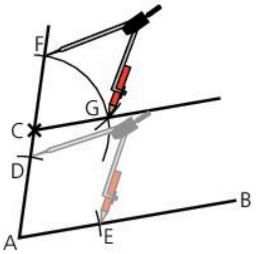


Diagram 9

ACTIVITY

Using straight-edge and compasses, construct angles of:

- 45°
- 30°
- 15°

What other angles could you construct?

EXAM TIP

- Always use a sharp pencil for constructions.
- Straight-edge and compasses constructions do not involve measuring.
- Make sure your compasses are tight enough so they do not slip.

KEY POINTS

1. All the angle constructions use symmetrical markings. Make sure you can see the symmetry to help you remember them.
2. The parallel lines construction relies on copying an angle, and uses the compasses to measure and copy a distance.

SUMMARY QUESTIONS

1. Construct an angle of 120°.
2. Bisect your angle of 120°. Check your accuracy by measuring with a protractor.
3. Explain how you could construct an angle of 105°.

4.6

Constructing triangles and polygons

LEARNING OUTCOMES

- Construct triangles
- Construct regular polygons
- Construct irregular polygons

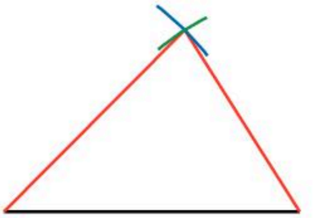
The constructions here involve a ruler, protractor and compasses.

Constructing triangles

We can construct a triangle given three pieces of information:

- The length of all three sides (SSS); or
- The length of two sides and the size of the angle between them (SAS); or
- The length of two sides and the size of an angle other than the angle in between them (SSA); or
- The length of one side and the size of two angles (AAS).

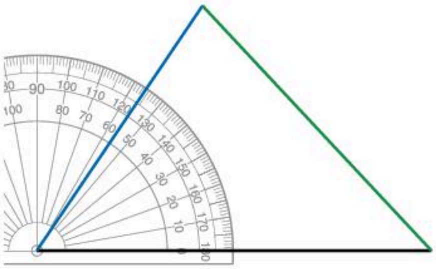
Constructing an SSS triangle



To draw a triangle with sides of 6 cm, 5 cm and 4 cm:

- Draw the longest side first (6 cm).
- With compasses open to 5 cm, put the point on one end of the line and draw an arc (shown in blue).
- With compasses open to 4 cm, put the point on the other end of the 6 cm line, and draw an arc to cross the first arc (shown in green).
- Finally, connect the two ends of the line to the intersection of the arcs (red).

Constructing an SAS triangle



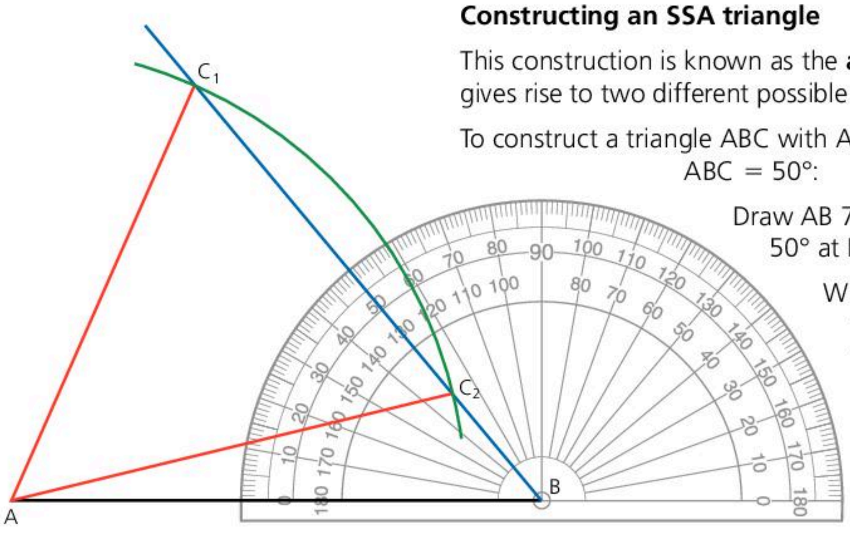
To draw a triangle with sides of 6 cm and 4 cm, with an angle between them of 56°:

- Draw the longest side (6 cm) first, and then use a protractor to measure an angle of 56° at one end.
- Extend this line to 4 cm (shown in blue) and join the remaining ends of the two lines (green) to complete the triangle.

Constructing an SSA triangle

This construction is known as the **ambiguous case**, as it usually gives rise to two different possible answers.

To construct a triangle ABC with AB = 7 cm, AC = 6 cm and ABC = 50°:



Draw AB 7 cm long, and mark an angle of 50° at B (shown in blue).

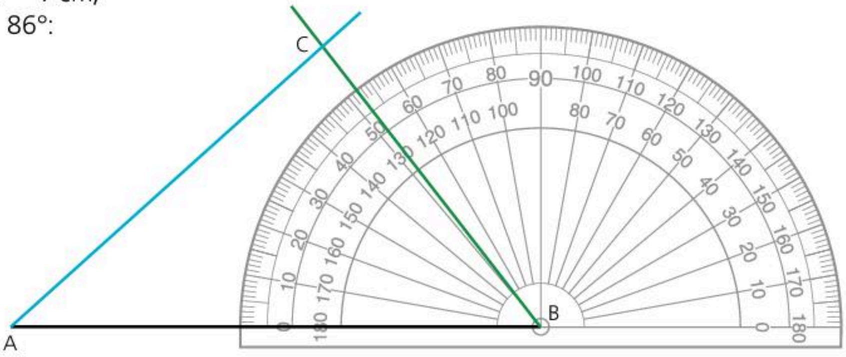
With compasses open to 6 cm and the point on A, draw an arc to cross the 50° line (shown in green).

The arc crosses at two points (labelled C_1 and C_2). Joining either of these to A gives a triangle matching the description.

Constructing an AAS triangle

To construct a triangle ABC with AB = 7 cm, angle BAC = 42° and angle ACB = 86°:

- Draw the line AB 7 cm long.
- Measure an angle of 42° at A (shown in blue).
- The angle at B = 180 − 42 − 86 = 52°, as the angles in a triangle add up to 180° (shown in green). Extend the lines if necessary so they intersect at C.



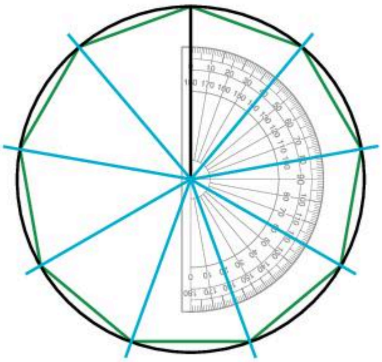
Constructing regular polygons

Regular polygons can be constructed inside a circle.

A full turn is 360°.

To construct a regular polygon with 9 sides (a nonagon), divide 360° by 9: 360 ÷ 9 = 40°.

- First, draw a circle and a radius. From the radius, measure angles of 40° (shown in blue).
- Join the ends of the radii to construct the regular nonagon (shown in green).



EXAM TIP

- Make sure you use the correct scale on the protractor by turning from the scale marked 0.
- Measure the length of a line from 0, not from 1.

Constructing irregular polygons

To construct irregular polygons including quadrilaterals, make a sketch first, and plan the order in which to carry out the construction.

KEY POINTS

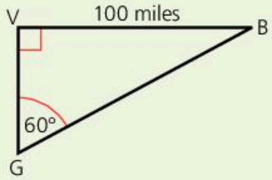
1. SSA triangles usually have two solutions, one acute-angled triangle and one obtuse-angled triangle.
2. Regular polygons can be drawn inside a circle by dividing 360° by the number of sides.

SUMMARY QUESTIONS

1. Construct a triangle ABC where AB = 9 cm, angle BAC = 72° and ABC= 64°. Measure the lengths of AC and BC.
2. Construct a regular octagon.
3. Construct a pentagon ABCDE with AB = BC = CD = DE = 7 cm, angle ABC = BCD = CDE = 112°, BAE = DEA = 102°. Measure the length of AE.

ACTIVITY

1. Construct a parallelogram with sides of 8 cm and 5 cm and an angle of 70°.
2. The diagram shows Barbados (B), Saint Vincent (V) and Grenada (G).



Using a scale of 1 cm to 10 miles, make an accurate drawing. Can you construct the angles without using a protractor?

4.7

Similarity and congruence

LEARNING OUTCOMES

- Use properties of similar triangles
- Identify and use congruent triangles

Similar triangles

If the three angles of one triangle are equal to the three angles of another triangle, then the two triangles are identical in shape even if they are different in size.

Such triangles are called **similar triangles**.

The lengths of each pair of corresponding sides are in the same ratio.

WORKED EXAMPLE 1

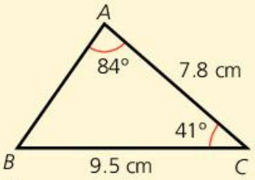
In triangle ABC, angle $B = 180 - 84 - 41 = 55°$

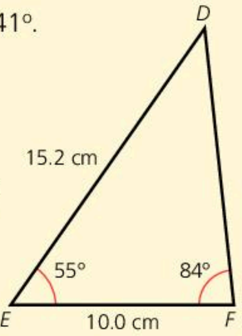
In triangle DEF, angle $D = 180 - 55 - 84 = 41°$.

So angle A = angle F

Angle B = angle E

Angle C = angle D





Corresponding sides are:

AB and FE, AC and FD, BC and ED.

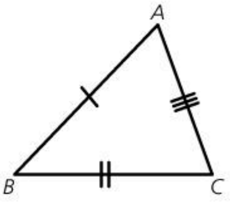
$\frac{ED}{BC} = \frac{15.2}{9.5} = 1.6$

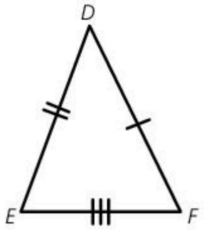
So the sides of DEF are $1.6 \times$ the sides of ABC.

$DF = 1.6 \times AC = 1.6 \times 7.8 = 12.48$ cm

$AB = EF \div 1.6 = 10.0 \div 1.6 = 6.25$ cm

Congruent triangles





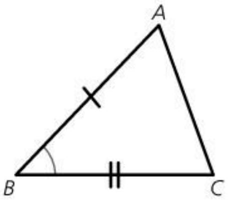
Triangles that are exactly the same shape and size are called **congruent triangles**.

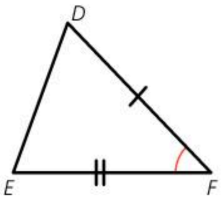
The three angles of one triangle are equal to the three angles of the other triangle.

The three sides of one triangle are equal to the three sides of the other triangle.

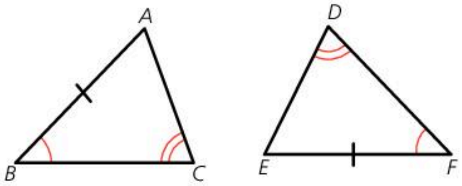
There are four sets of minimum requirements to be sure that two triangles are congruent:

1. **SSS**. The three **S**ides of one triangle are equal in length to the three sides of the other triangle.
2. **SAS**. The triangles have two pairs of equal **S**ides and the included **A**ngles (the angles between these two sides) are equal.

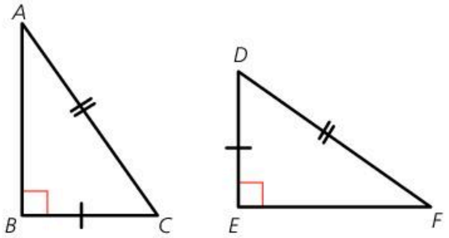




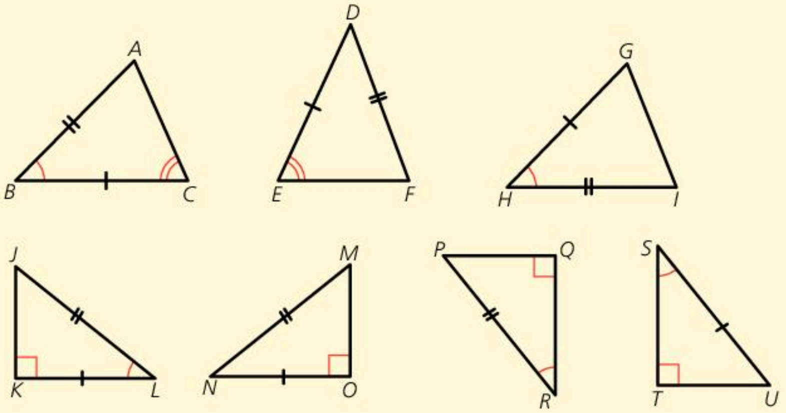
3. **AA Corr S**. The triangles have two pairs of equal **A**ngles and one pair of corresponding **S**ides equal.



4. **RHS**. The two triangles are **R**ight angled, have equal **H**ypotenuses and one other pair of equal **S**ides.



WORKED EXAMPLE 2



Triangles *ABC* and *FDE* are NOT congruent. The equal angles *C* and *E* are not between the equal sides and so this is not SAS.

Triangles *ABC* and *IHG* are congruent (SAS). The equal angles *B* and *H* are between the equal sides *BC*&*HG* and *BA*&*HI*.

Triangles *JKL* and *MON* are congruent (RHS). $K = O = 90°$, $JL = MN$ (hypotenuses) and $KL = ON$.

Triangles *JKL* and *PQR* are congruent (AA Corr S). $K = Q$, $L = R$ and $JL = PR$ which are corresponding sides.

Triangles *JKL* and *UTS* are NOT congruent. The hypotenuses *JL*&*US* are not equal so RHS does not apply. Angles *K*&*T* are equal, as are *L*&*S*, but the equal sides *KL*&*SU* are not in corresponding positions.

KEY POINTS

1 Similar triangles have equal angles so are the same shape. Corresponding sides are in the same proportion.

2 Congruent triangles are the same shape and size. The four proofs of congruence are SSS, SAS, AA Corr S and RHS.

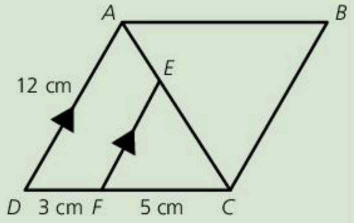
EXAM TIP

- For the SAS proof of congruence, the equal angles must be the angle between the equal sides.
- For the AA Corr S proof, the equal sides have to be in corresponding positions.

SUMMARY QUESTIONS

ABCD is a parallelogram.
EF is parallel to *AD*.

$AD = 12$ cm, $DF = 3$ cm and $FC = 5$ cm.



1 Prove that triangles *ACD* and *ABC* are congruent.

2 Prove that triangles *CEF* and *CAD* are similar.

3 Calculate the length of *EF*.

Pythagoras' theorem

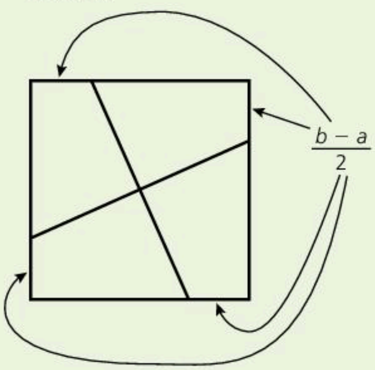
LEARNING OUTCOMES

- Use Pythagoras' theorem to find missing sides in right-angled triangles
- Use Pythagoras' theorem to solve problems

ACTIVITY

Draw a right-angled triangle.

- Draw a square on each side of the triangle.
- If the smallest square has side length *a* cm, the middle one *b* cm and the largest *c* cm, calculate $\frac{b-a}{2}$.
- Measure $\frac{b-a}{2}$ cm clockwise from each vertex of the middle (*b* cm) square. Join the marks as shown.
- Cut out the square and cut it into four pieces along the lines. Cut out the smallest square.
- Arrange the five pieces to exactly cover the largest square to show that Pythagoras was correct.



Pythagoras' theorem

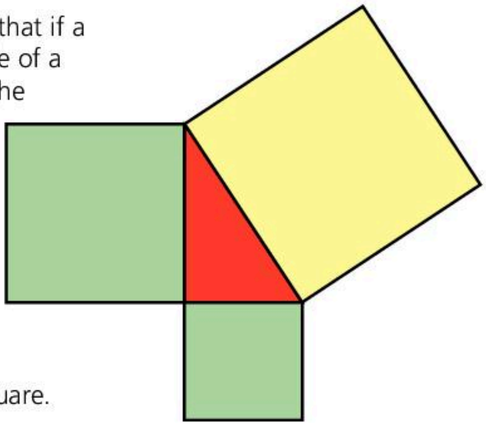
Pythagoras was a Greek who was alive around 500BC. He was a mathematician and philosopher, and is best remembered for his theorem about right-angled triangles.

Although the theorem had been used previously by Babylonians and Indians, it is believed that Pythagoras was the first person to prove the result.

Pythagoras' theorem states that if a square is drawn on each side of a right-angled triangle, then the sum of the areas of the two smaller squares is equal to the area of the larger square.

The red triangle is a right-angled triangle.

The sum of the areas of the green squares is equal to the area of the yellow square.



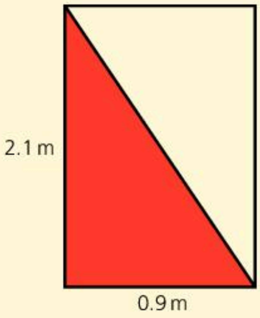
Using Pythagoras' theorem

WORKED EXAMPLE 1

Joe makes a rectangular wooden frame for a door.

To make it rigid, he wants to put in a diagonal piece.

He draws a sketch, and marks on it the length and width of the door.



Joe realises that he can use Pythagoras' theorem. To help, he colours the diagram to show the right-angled triangle.

He does not draw the squares.

He knows the area of the bottom square would be $0.9 \times 0.9 = 0.81\,\text{m}^2$.

The square on the left would be $2.1 \times 2.1 = 4.41\,\text{m}^2$.

So the large square would have an area of $0.81 + 4.41 = 5.22\,\text{m}^2$.

The diagonal$^2 = 5.22$, so the diagonal $= \sqrt{5.22}$, or 2.28 m (to the nearest cm).

The longest side of the triangle is always opposite the right angle. It is called the **hypotenuse**.

If the shorter sides of a right-angled triangle are *a* and *b*, and the **hypotenuse is *c***, then Pythagoras' theorem can be written as $\boldsymbol{a^2 + b^2 = c^2}$.

Solving problems using Pythagoras' theorem

Joe realised the diagonal of a rectangle created right-angled triangles. Isosceles triangles also can be split into two congruent right-angled triangles.

EXAM TIP

- Pythagoras' theorem only works for right-angled triangles.
- Always draw a diagram.
- Add the squares when finding the hypotenuse.
- Subtract the squares when finding one of the shorter sides.

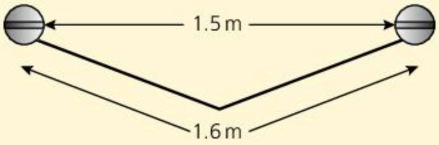
WORKED EXAMPLE 2

Charlotte is hanging a large painting on the wall.

She is going to hang it on a wire attached to two screws.

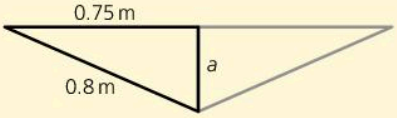
The screws are 1.5 m apart, and the wire is 1.6 m long.

The diagram shows the arrangement.



Charlotte wants to know how far down from the screws the wire will reach.

She draws a sketch. The triangle is isosceles, so she divides it into two congruent right-angled triangles.



Using Pythagoras' theorem,

$$a^2 + 0.75^2 = 0.8^2$$
$$a^2 + 0.5625 = 0.64$$
$$a^2 = 0.64 - 0.5625 = 0.0775$$
$$a = \sqrt{0.0775} = 0.278\text{ m}$$

The wire will reach 27.8 cm below the screws.

KEY POINTS

1. The longest side of a right-angled triangle is called the hypotenuse.
2. If a and b are perpendicular and c is the hypotenuse, then $a^2 + b^2 = c^2$.
3. Rectangles and isosceles triangles can be split into two congruent right-angled triangles.

SUMMARY QUESTIONS

1. A right-angled triangle has shorter sides of 6 cm and 8 cm. Find the length of the hypotenuse.
2. A square has a diagonal of 12 cm. Find the length of side of the square.
3. An equilateral triangle has sides of 8 cm. Calculate the area of the triangle.

4.9

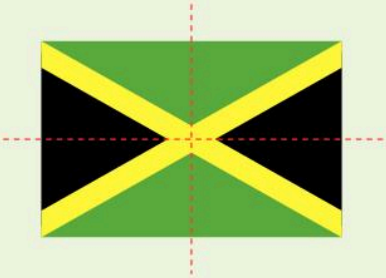
Symmetry, reflections and rotations

LEARNING OUTCOMES

- Identify reflection and rotation symmetry
- Find the image of an object or the object given the image

ACTIVITY

The Jamaican flag has two lines of symmetry and rotation symmetry of order 2.



- Make a display of flags that have symmetry. Mark any lines of symmetry and write the order of rotation symmetry.
- Design a school flag that possesses either or both forms of symmetry.

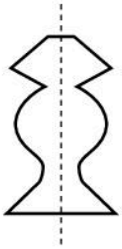
EXAM TIP

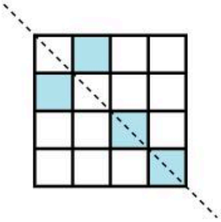
- Shapes without rotation symmetry are described as having rotation symmetry of order 1.

Reflection symmetry

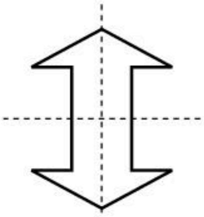
A shape has **reflection symmetry** if one half is a mirror image of the other half.

Here are two examples, with the **mirror line** shown as a dotted line.

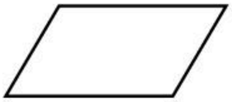




Some shapes have more than one line of reflection symmetry, while others have none.



Two lines of symmetry

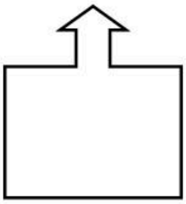


No lines of symmetry

Rotation symmetry

With rotation symmetry, the shape or image can be rotated and it still looks the same. How many matches there are as you go around once is called **the order of rotation symmetry**.

Here are some examples.



Order 1

Order 2

Order 3

Reflection

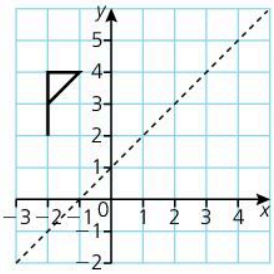
A **reflection** is a **congruent transformation**.

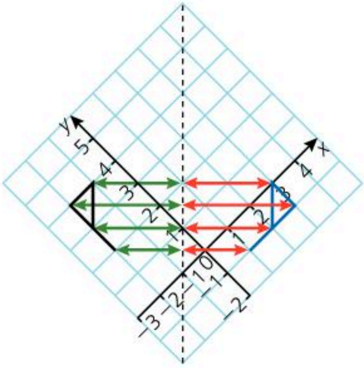
It changes the position of the shape without affecting its shape or size.

When we reflect a shape in a mirror line, the **image** (the result of the transformation) is the same perpendicular distance from the mirror as the **object** (the original shape), but on the opposite side of the mirror.

To reflect the flag in the mirror line $y = x + 1$:

- Turn the diagram so that the mirror line is vertical.
- Measure the horizontal distance of each vertex from the mirror (shown in green), and measure an equal distance the other side of the mirror (in red). Join the vertices to create the image (blue).





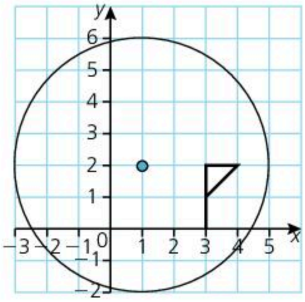
To define a reflection, we must name the mirror line.

EXAM TIP

- To reflect a shape, turn the paper so that the mirror line is vertical, as the reflection is then horizontal.

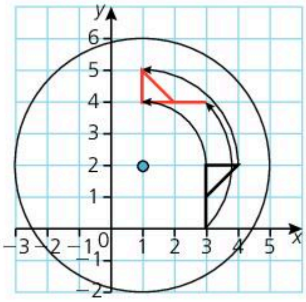
Rotation

Another congruent transformation is **rotation**. When we rotate a shape, it is as if the shape is on a large wheel. We need to know what angle to rotate it through, and the position of the centre of the 'wheel'.



To rotate the flag in the diagram 90° anticlockwise about the point (1, 2):

- Imagine a wheel, with the centre at (1, 2).
- Rotate the diagram 90° anticlockwise, and the flag is now horizontal, with the top of the flagpole 2 units directly above the centre of rotation. Use tracing paper if it helps.



EXAM TIP

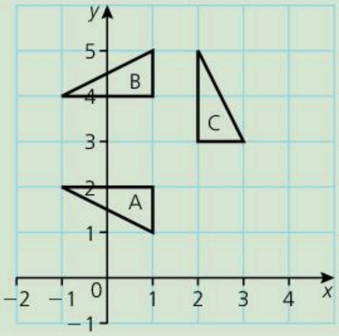
- To rotate a shape, use tracing paper or rotate the paper to see where the image should be.

SUMMARY QUESTIONS

1 Describe fully the transformation that maps A onto B.

2 Describe fully the transformation that maps B onto C.

3 Describe fully the transformation that maps A onto C.



KEY POINTS

1 A shape has reflection symmetry if there is a mirror line that splits the shape into an object and its reflection.

2 The order of rotation symmetry is the number of matches with the original shape as it rotates through 360°.

3 To define a reflection, give the equation of the mirror line.

4 To define a rotation, give the centre of rotation and the angle and direction of the rotation.

4.10 Further transformations

LEARNING OUTCOMES

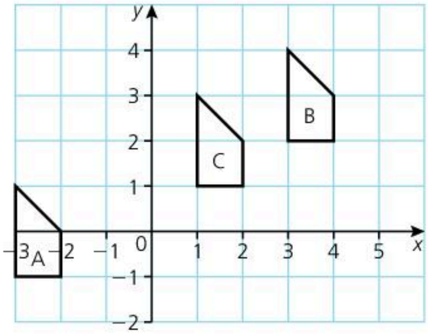
- Represent translations using vectors
- Recognise enlargements with positive, fractional and negative scale factors
- Locate the image under a combination of transformations

Translations

A **translation** is a congruent transformation. A translation changes the position of a shape by sliding it, but does not alter the **orientation** of the shape (it does not reflect or rotate it.) Translations are defined by a **column vector**. A column vector tells us how far to move the shape horizontally and vertically.

$$\begin{pmatrix} x \\ y \end{pmatrix} \text{ means } \begin{pmatrix} \text{move } x \text{ units to the right; a negative number means} \\ \text{move to the left} \\ \text{move } y \text{ units up; a negative number means move down} \end{pmatrix}$$

- The translation that moves A to B is $\begin{pmatrix} 6 \\ 3 \end{pmatrix}$; *B* to *C* is $\begin{pmatrix} -2 \\ -1 \end{pmatrix}$



EXAM TIP

- The centre of enlargement can be on, in or outside the shape.
- Draw the rays from the centre of enlargement to help with the drawing.
- To find the centre of enlargement given the object and image, draw the rays back to a point of intersection.

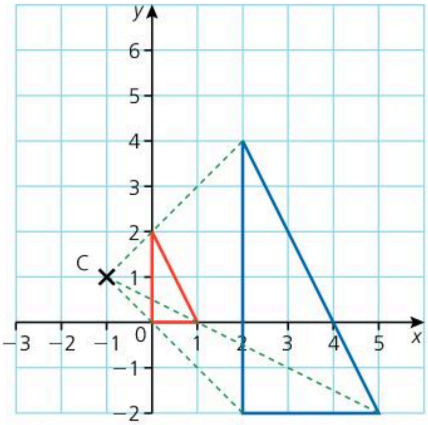
Enlargements

An enlargement is a **similar transformation** as the image is the same shape as the object with the same angles, but a different size. The original dimensions are increased by the same multiplying factor, or **scale factor**.

Additionally, the distance of each vertex from the **centre of enlargement** is increased by the same scale factor.

In the diagram, C is the centre of enlargement.

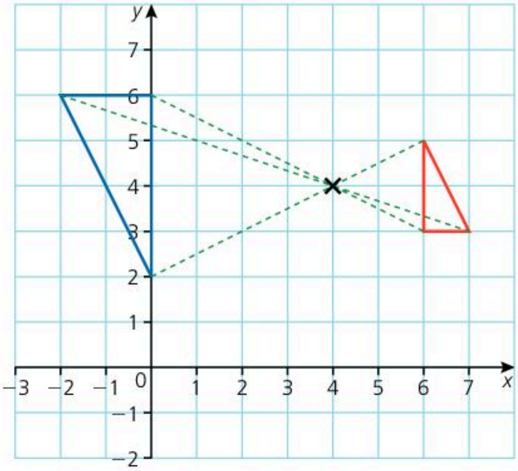
The blue triangle is an enlargement of the red triangle with a scale factor of 3.



Each side of the blue triangle is 3 times the length of the corresponding side of the red triangle, and each vertex of the blue triangle is 3 times the distance from C as the vertices of the red triangle are.

The scale factor can be a fraction; a scale factor of $\frac{1}{2}$ produces an image half the size of the object, half the distance from the centre of enlargement. Although the image is smaller than the object, we still call it an enlargement.

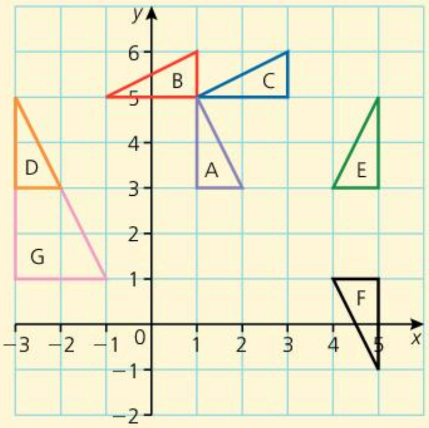
A negative scale factor produces an image on the other side of the centre of enlargement. The diagram shows an enlargement, scale factor −2, centre (4, 4).



Combining transformations

When we perform a transformation on an object, and then perform another transformation on the image, we can sometimes replace the two transformations with a single one.

WORKED EXAMPLE 1

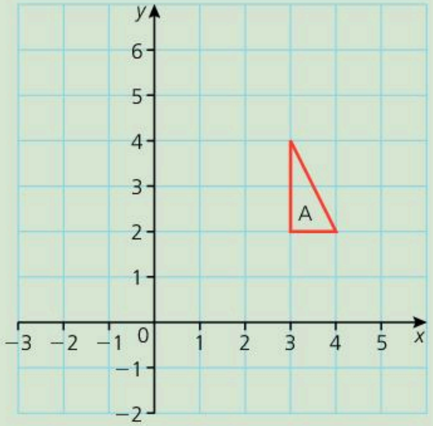


1 Rotate A 90° anticlockwise about (1, 5), image is C.
Translate C through $\begin{pmatrix}-2\\0\end{pmatrix}$, image is B.
Single transformation from A to B: Rotation 90° anticlockwise about (−1, 3).

2 Reflect A in $x = 3$, image is E.
Reflect E in $y = 2$, image is F.
Single transformation from A to F: 180° rotation about (3, 2).

3 Translate A through $\begin{pmatrix}-4\\0\end{pmatrix}$, image is D.
Enlarge D, scale factor 2, with centre (−3, 5), image is G.
Single transformation from A to G: enlargement, scale factor 2, centre (5, 5).

SUMMARY QUESTIONS



1 Translate triangle A through $\begin{pmatrix}-4\\1\end{pmatrix}$.
Label the image B.

2 Enlarge B with a scale factor of 2, centre (−3, 5). Label the image C.

3 Describe fully the single transformation that transforms A to C.

EXAM TIP

- Always draw a diagram.
- If a question asks for a single transformation, don't give a 2-stage answer.

ACTIVITY

Draw diagrams to show that these are true.

- Two translations can always be replaced by a single translation.
- Two perpendicular reflections can always be replaced by a rotation.
- Two parallel reflections can always be replaced by a translation.

KEY POINTS

1 Translations are defined by a column vector.

2 Enlargements are defined by a scale factor and centre of enlargement.

3 Some combinations of transformation can be replaced by a single transformation.

4.11

Three-dimensional shapes

LEARNING OUTCOMES

- Solve geometric problems using faces, edges, vertices
- Use classes of solids (prisms, pyramids, cylinders, cones, sphere)

Solids

Three-dimensional shapes are sometimes called **solids**.

A two-dimensional shape is a flat shape, such as a triangle or a circle.

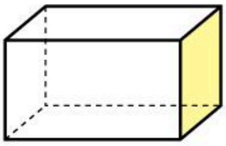
Three-dimensional shapes have volume as well as surface area.

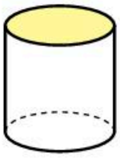
Classes of solid

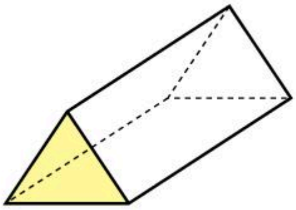
Some solids are the same shape throughout their length. If we take slices in one direction, we always get the same **cross-section**.

Solids of this type are called **prisms**.

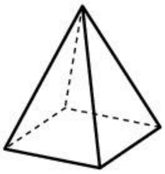
Here are some examples. The yellow cross-section is constant through the entire shape. Hidden edges are shown with dotted lines.

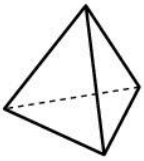

Cuboid

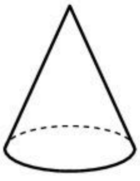

Cylinder


Triangular prism

Other shapes rise from a base to a point. These are the **pyramids**.


Square-based pyramid


Tetrahedron

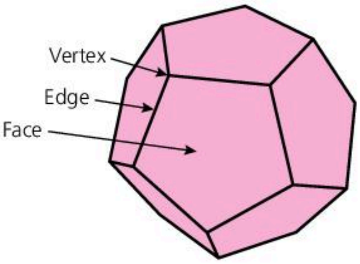

Cone

Other solids include the sphere.

Faces, edges and vertices

- A flat surface on a solid is called a **face**.
- Two faces meet at an **edge**.
- Edges meet at a **vertex**. The plural of vertex is **vertices**.

The dodecahedron shown here has many hidden faces, edges and vertices.



The whole shape is made of 12 pentagons, so it has 12 faces.

Each pentagon has five sides, so that makes a total of 12 × 5 = 60 sides.

Each edge is formed where two sides coincide, so there must be 60 ÷ 2 = 30 edges.

Each edge has two ends making a total of 60 ends. Each vertex is formed by three ends, so there must be 60 ÷ 3 = 20 vertices.

Leonhard Euler, an 18th-century Swiss mathematician, discovered the rule that for any **convex polyhedron**, the number of faces, *F*, edges, *E* and vertices, *V*, are connected by the formula

$$F + V = E + 2.$$

In this case, 12 + 20 = 30 + 2

A convex polyhedron is a solid with entirely flat faces with no reflex interior angles.

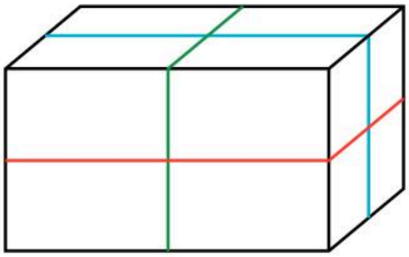
A cylinder is not a convex polyhedron because it has a curved surface.

Solids and symmetry

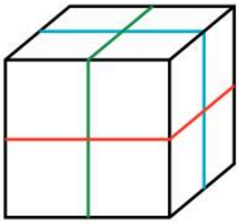
Two-dimensional shapes have mirror lines, or lines of symmetry, but 3-dimensional shapes have **planes of symmetry**. A plane is a flat surface.

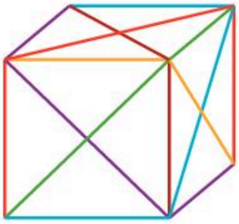
A cuboid has three planes of symmetry; that is, three planes where if the shape were cut in half, one half would be a mirror image of the other.

The cuboid has its planes of symmetry marked with coloured lines.



A cube has nine planes of symmetry: three through the midpoints of edges and six through vertices.





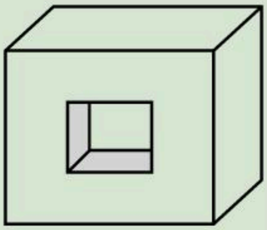
SUMMARY QUESTIONS

1 A cuboid has a square hole passing through the centre.

How many faces, edges and vertices does the shape have?

2 Does Euler's Law hold for this shape?

3 How many planes of symmetry does the shape have?



EXAM TIP

- Use the symmetry of shapes to help count hidden faces, edges and vertices.

ACTIVITY

- Count the number of faces, vertices and edges for
  - a cuboid
  - a square-based pyramid
  - a tetrahedron
  - a triangular prism.
- Check that your answers agree with Euler's Rule, $F + V = E + 2$.

ACTIVITY

Find the four planes of symmetry of:

- a triangular prism
- a square-based pyramid.

KEY POINTS

1 Solids have faces (flat surfaces), edges (where faces meet) and vertices (corners).

2 Euler's Rule:
faces + vertices = edges + 2

3 Solids have planes of symmetry – the surface where a knife would cut the shape into two mirror-image pieces.

4.12 Trigonometry

LEARNING OUTCOMES

- Determine the trig ratios in acute angles of right-angled triangles
- Use trig ratios in right-angled triangles in real-world (practical geometry and scale drawing, bearing, heights, distances, angle of elevation/depression)

ACTIVITY

A mnemonic such as 'Some Old Hairy Camels Are Hairier Than Others Are' can help you remember the rules.

- Can you make up a better one? It has to be something you will remember.

EXAM TIP

You must learn these three formulae.

ACTIVITY

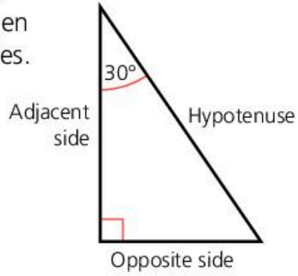
Draw a right-angled triangle with an angle of 30°.

- Measure the length of the side opposite the 30° angle.
- Measure the length of the hypotenuse.
- What do you notice?
- What happens if the right-angled triangle has an angle of 19.5°?

Similar right-angled triangles

All right-angled triangles with an angle of 30° are similar.

So if in one such triangle, the side **opposite** the 30° angle is half of the **hypotenuse**, then it will be true for all 30° right-angled triangles.



We say that, for an angle of 30°,

$$\frac{\text{opposite side}}{\text{hypotenuse}} = 0.5.$$

This is called the **sine** of 30°.

The third side is called the **adjacent** side, because it is adjacent, or next to, the angle of 30°.

The result of $\frac{\text{length of the adjacent side}}{\text{length of the hypotenuse}}$ is called the **cosine** of 30°,

and $\frac{\text{opposite side}}{\text{adjacent side}}$ is called the **tangent** of 30°.

These are usually abbreviated to the first three letters, so we write

$$\sin A = \frac{\text{opp}}{\text{hyp}}, \qquad \cos A = \frac{\text{adj}}{\text{hyp}}, \qquad \tan A = \frac{\text{opp}}{\text{adj}}$$

Using trigonometry

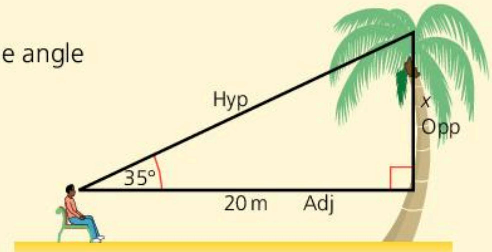
WORKED EXAMPLE 1

To find the length of a side (1)

Alan looks up at the top of a tree. The **angle of elevation** (the angle he looks up from the horizontal) is 35°.
He is 20 m from the foot of the tree.

He uses this information to find the height of the tree.
He draws a sketch, and labels the sketch.

He calls the side he wants to calculate x. He knows the adjacent side, and wants to calculate the opposite side. The formula with adjacent and opposite in is the tangent formula.



He writes: $\tan 35 = \frac{\text{opp}}{\text{adj}}$

His calculator tells him that tan 35 = 0.7002, so he writes:

$$0.7002 = \frac{x}{20}.$$

$$x = 0.7002 \times 20 = 14.0\text{ m}.$$

So the height of the tree is 14.0 m, plus the height of his eyes from the ground.

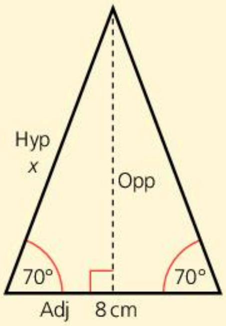
To find the length of a side (2)

An isosceles triangle has two angles of 70° and a base of 8 cm.

We are going to find the length of the other sides.

The dotted perpendicular line creates two congruent right-angled triangles with an adjacent side of 4 cm.

We know the adjacent side and want the hypotenuse, so we use cosine:



$$\cos 70 = \frac{\text{adj}}{\text{hyp}}$$

$$0.3420 = \frac{4}{x}$$

$$0.3420 \times x = 4 \quad \text{Multiplying both sides by } x$$

$$x = \frac{4}{0.3420} = 11.7\text{ cm (to 1 decimal place)}$$

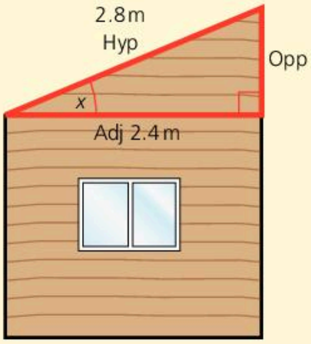
To calculate an angle

Melody measures the end of her shed. It is 2.4 m wide.

The roof is 2.8 m long.

Melody wants to calculate x, the angle of the roof with the horizontal.

She knows the adjacent and hypotenuse, so she chooses cosine.



$$\cos x = \frac{\text{adj}}{\text{hyp}}$$

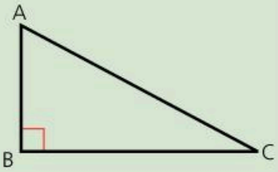
$$\cos x = \frac{2.4}{2.8} = 0.85714$$

As she knows the cosine and wants the angle, she must use the inverse function of cosine, $\cos^{-1} 0.85714$. Her calculator has an INVERSE key.

$$x = \cos^{-1} 0.85714 = 31.0° \text{ (to 1 decimal place)}.$$

SUMMARY QUESTIONS

These questions refer to triangle ABC.



1 AB = 12 cm, angle BAC = 67°. Calculate BC.

2 AB = 7 cm, angle ACB = 38°. Calculate AC.

3 AB = 4 cm, AC = 9 cm. Calculate angle BAC.

KEY POINTS

1 Trigonometry is used in right-angled triangles.

2 $\sin = \frac{\text{opp}}{\text{hyp}}$, $\cos = \frac{\text{adj}}{\text{hyp}}$, $\tan = \frac{\text{opp}}{\text{adj}}$

3 Identify the known or wanted sides and angles to choose the correct function.

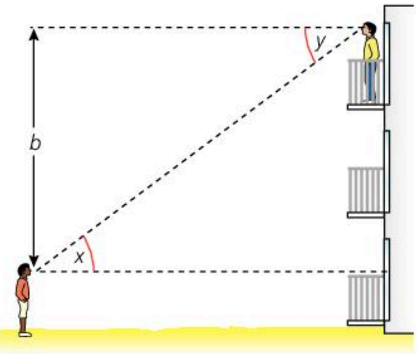
4.13 Further uses of trigonometry

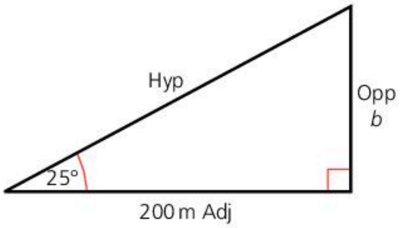
LEARNING OUTCOMES

- Use trigonometry in right-angled triangles with angles of elevation and depression
- Use trigonometry with bearings

Trigonometry has a number of practical uses. Here we will explore some of them.

Angles of elevation and depression





WORKED EXAMPLE 1

Peter is relaxing on his hotel balcony. John is standing on the beach.

When they look at each other, John looks up through angle x.

The angle upwards from the horizontal is called the **angle of elevation**.

Peter looks down through angle y.

The angle downwards from the horizontal is called the **angle of depression**.

The angles are equal, as they are alternate angles on parallel lines.

If John is 200 m from the hotel, and the angle of elevation is 25°, he can use trigonometry to calculate the height b.

He knows the adjacent side, and wants the opposite side, so he uses tangent.

$$\begin{aligned} \tan 25 &= \frac{\text{opp}}{\text{adj}} \\ 0.4663 &= \frac{b}{200} \\ b &= 93.3\text{ m (to 1 d.p.)} \end{aligned}$$

Bearings

We can use compass bearings to describe a direction, such as North, or South-East.

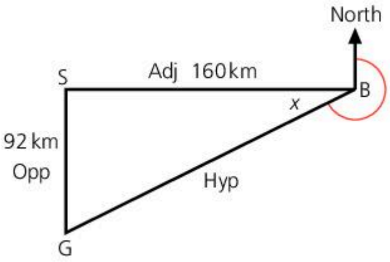
A more precise method is to use **angle bearings**.

Angle bearings are measured clockwise, from North. Bearings are always written with three digits, so a bearing of 72° is written as 072°.

Barbados is 160 km due East of Saint Vincent.

Saint Vincent is 92 km due North of Grenada.

To find the bearing of Grenada from Barbados, we first draw a sketch and label it.



We calculate angle x:

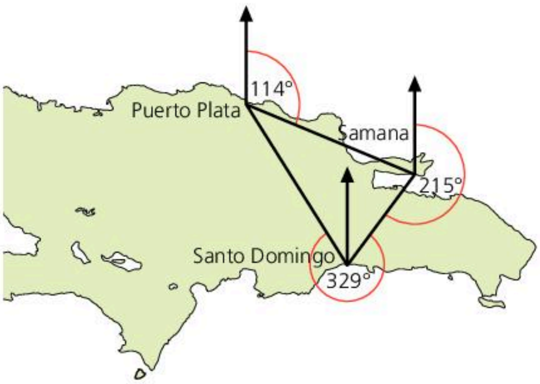
We know the opposite and adjacent, so we use tan:

$$\tan x = \frac{\text{opp}}{\text{adj}} = \frac{92}{160} = 0.575$$

$$x = \tan^{-1} 0.575 = 29.9°$$

The bearing (marked in red) is 270 − 29.9 = 240.1°

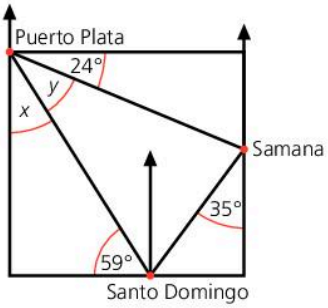
Here is a more complex example.



The bearing of Samana from Puerto Plata is 114°, Santo Domingo is on a bearing of 215° from Samana, and Puerto Plata from Santo Domingo is on a bearing of 329°.

We can draw a sketch map of this, and create right-angled triangles.

The angles of 24°, 35° and 59° have been calculated by subtracting 90°, 180° and 270° from the angles on the map. All the other angles in the diagram can now be calculated. This would allow us to use trigonometry to solve problems.

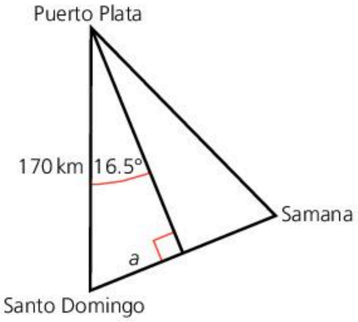


Samana and Santo Domingo are both 170 km from Puerto Plata, so the triangle is isosceles.

Angle x is $180 - 90 - 59 = 31°$

Angle $y = 90 - 24 - 31 = 35°$

So the triangle looks like this:



To find the distance a:

$$\sin 17.5 = \frac{\text{opp}}{\text{hyp}}$$

$$0.3007 = \frac{a}{170}$$

$$a = 51.12 \text{ km}$$

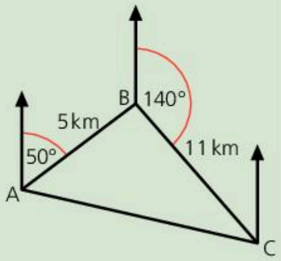
So the distance from Santo Domingo to Samana = 2 × 51.12 km = 102 km (to the nearest km).

KEY POINTS

1. Angles of elevation and depression are angles looking up or down from the horizontal.
2. Bearings are measured in a clockwise direction from North.

SUMMARY QUESTIONS

1. Howard looks down from a cliff to a boat through an angle of depression of 35°. The height of the cliff is 85 m. How far away is Howard from the boat?
2. The boat is 3 km from a port, on a bearing of 056° from the port. How far to the east of the port is the boat?
3. B is 5 km from A on a bearing of 050°. C is 11 km from B on a bearing of 140°.
   - **a** Explain how you know that angle ABC is a right angle.
   - **b** Find the bearing and distance of C from A.



ACTIVITY

A is on a bearing of 070° from B. The distance between A and B is 20 km.

- How far North of B is A?
- How far East of B is A?
- What is the bearing of B from A?
- What is the connection between the bearing of A from B and the bearing of B from A?

EXAM TIP

- Never round off until the end of a calculation. You might write down a rounded version, but keep the full display in your calculator.

4.14

The area of a triangle and trigonometry in three dimensions

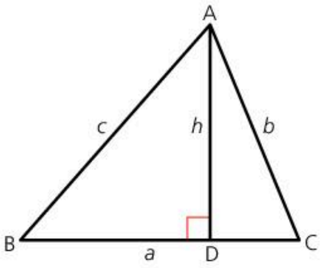
LEARNING OUTCOMES

- Find the area of a triangle using the formula $A = \frac{1}{2}ab \sin C$
- Solve practical problems involving heights and distances in 3-D situations

Trigonometry can be applied to other situations. In this unit we use trigonometry to find the area of a triangle, and also apply trigonometry to three dimensions.

The area of a triangle

We can use trigonometry to find the area of a triangle, given two sides and the angle between them.



In the diagram, Area $= \frac{1}{2}ah$.

$$\sin = \frac{\text{opp}}{\text{hyp}}$$

So $\sin C = \frac{h}{b}$, or $h = b \sin C$.

Substituting $h = b \sin C$ into the area formula, we get

$$\text{Area} = \tfrac{1}{2}ab \sin C$$

We could also write it as

$$\text{Area} = \tfrac{1}{2}bc \sin A, \text{ or}$$

$$\text{Area} = \tfrac{1}{2}ac \sin B$$

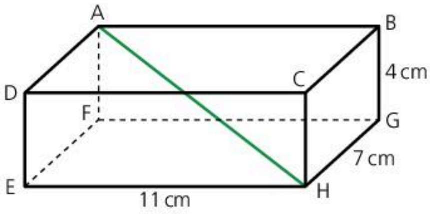
The formula uses two sides and the angle in between those two sides.

So, if $a = 9\,\text{cm}$, $b = 11$ cm and $C = 65°$.

$$\text{Area} = \tfrac{1}{2}ab \sin C$$

$$\text{Area} = \tfrac{1}{2} \times 9 \times 11 \times \sin 65 = 44.9\,\text{cm}^2$$

Trigonometry in three dimensions



We can apply trigonometry in three dimensions.

The major step is identifying right-angled triangles.

Look at the diagram of the cuboid ABCDEFGH.

We are going to find out which of these three angles are right angles:

ADC ADH ACE

A good way to find out is to imagine the cuboid with one of the sides of the angle horizontal. We then decide if the other side can be put in a vertical position.

- If ADEF is horizontal, then AD is horizontal and DC is vertical. ADC is a right angle.
- If DCHE is horizontal, then DH is horizontal and AD is vertical. ADH is a right angle.
- But if DCHE is horizontal, CE is horizontal but AC cannot be made vertical. So ACE is not a right angle.

WORKED EXAMPLE 1

In the cuboid ABCDEFGH, BG = 4 cm, GH = 7 cm and EH = 11 cm.

A rod rests inside the box from A to H.

We are going to calculate the angle between the rod and FH.

We know AF = 4 cm, and we can calculate FH by Pythagoras' theorem.

$$EH^2 + EF^2 = FH^2$$

$$FH^2 = 11^2 + 7^2 = 170$$

$$FH = \sqrt{170} = 13.04 \text{ cm (to 2 d.p.)}$$

$$\tan x = \frac{\text{opp}}{\text{adj}} = \frac{4}{13.04} = 0.307$$

$$x = \tan^{-1} 0.307 = 17.1° \text{ (to 1 d.p.)}$$

EXAM TIP

- Never round off until the end of a calculation. You might write down a rounded version, but keep the full display in your calculator.
- You often have to use Pythagoras' theorem and trigonometry in 3-D problems.

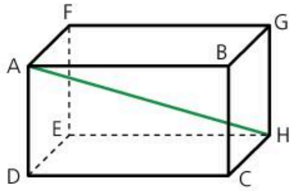
The angle between a line and a plane

Look at the cuboid ABCDEFGH at the start of this unit.

Sometimes you will be asked to find the angle between a line and a plane. For example, we might be asked to find the angle between AH and the plane DCHE.

To understand which angle this is, rotate the cuboid so DCHE is horizontal.

Imagine shining a light vertically at the shape.



AH would cast a shadow at DH.

The angle between AH and DCHE is the angle AHD.

KEY POINTS

1 The area of a triangle is given by $A = \frac{1}{2}ab \sin C$

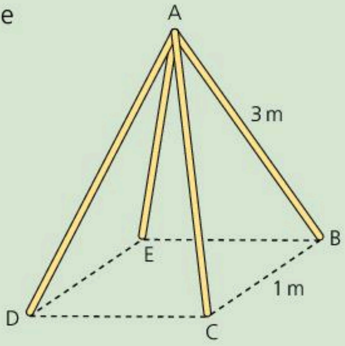
2 The angle between a line and a plane is the angle perpendicular to the plane.

SUMMARY QUESTIONS

1 Calculate the area of triangle ABC where AB = 9 cm, BC = 8.5 cm and angle ABC = 67°.

2 A pyramid has a square base ABCD, and a top vertex E. The base has edges of 10 cm, and the angle between AE and the base is 65°. Calculate the height of the pyramid.

3 Four 3-metre bamboo canes are used to make a support for some runner beans.
One end of each cane is stuck in the ground in the form of a square of side 1 m. The top ends are tied together.
Calculate the angle BAD.



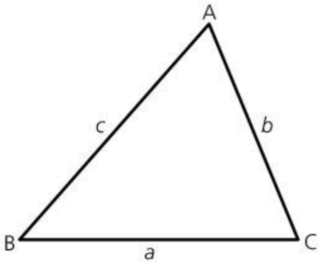
4.15 The sine rule

LEARNING OUTCOMES

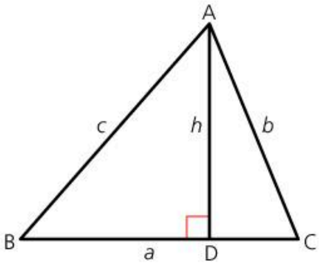
- Use the sine rule to calculate a side
- Use the sine rule to calculate an angle

Although sine, cosine and tangent are functions that can be used in right-angled triangles, we can develop formulae for use in any triangle.

We use a new notation, using lower case letters to label sides after the angle they are opposite.



The sine rule



If we draw a perpendicular line from A to meet BC at D, we create two right-angled triangles.

In triangle ABD:	**In triangle ADC:**
$c \sin B = h$	$b \sin C = h$

So $c \sin B = b \sin C$

$$\frac{c}{\sin C} = \frac{b}{\sin B}$$

Dividing both sides by sin B × sin C

By using a different perpendicular, we could show that

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

This is the sine rule.

The sine rule is used when the features we know and want to find include two sides and two angles.

Finding the length of a side with the sine rule

WORKED EXAMPLE 1

A child's slide has an angle of 35° with the horizontal, and the steps make an angle of 75° with the horizontal. The steps are 2.7 m long. We are going to find the length of the slide, x.

First label the vertices A, B and C, and the sides opposite them as a, b and c.

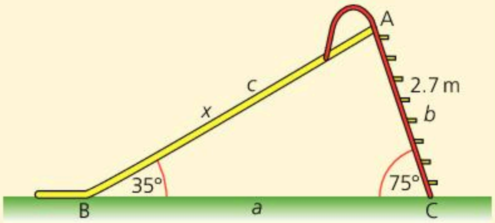
We know B, C and b, and need to find c.

So we choose the part of the sine rule containing B, b, C and c: $\frac{b}{\sin B} = \frac{c}{\sin C}$

Substituting $b = 2.7$, $B = 35°$ and $C = 75°$, $\frac{2.7}{\sin 35} = \frac{x}{\sin 75}$

$$\frac{2.7}{0.5736} = \frac{x}{0.9659}$$

$$4.707 = \frac{x}{0.9659}$$



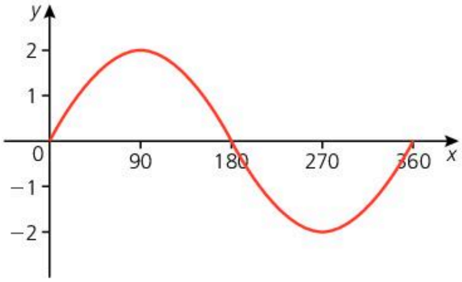
Multiply both sides by sin 75° (0.9659):

$x = 4.55$ m (to the nearest cm)

Finding the size of an angle using the sine rule

Because a triangle can have an obtuse angle, we need to be aware of the values of sin x when x is greater than 90°.

The graph of $y = \sin x$ looks like this:



Sin x can take values between −1 and 1.

The symmetry of the graph about $x = 90$ means that $\sin x = \sin(180 - x)$.

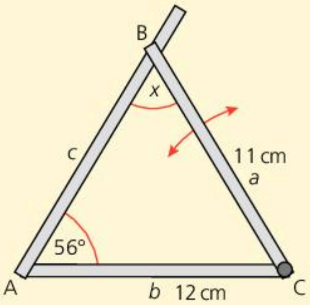
Because an acute angle and an obtuse angle have the same sine, this can lead to a question having more than one solution.

WORKED EXAMPLE 2

A triangular frame is to be made from three aluminium strips.

The base is 12 cm long, the left hand piece is welded at an angle of 56° and the third strip is 11 cm long and pivots from the right end.

We are going to find the angle x when the end of the right strip just meets the left strip.



We know A, a and b and want to find B.

So we use

$$\frac{\sin A}{a} = \frac{\sin B}{b} \quad \text{Substitute A, } a \text{ and } b\text{:}$$

$$\frac{\sin 56}{11} = \frac{\sin x}{12} \quad \text{Divide sin 56° by 11}$$

$$0.07537 = \frac{\sin x}{12}$$

$$\sin x = 0.9044$$

$$x = \sin^{-1} 0.9044 = 64.7°$$

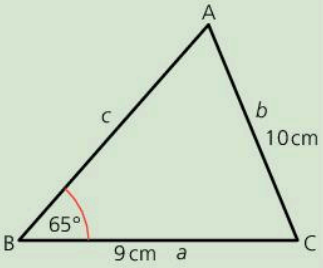
But, as $\sin x = \sin(180 - x)$, x could also be $180 - 64.7° = 115.3°$

SUMMARY QUESTIONS

1 Use the sine rule to calculate the size of angle A.

2 Calculate angle C.

3 Use the sine rule to calculate the length of c.



ACTIVITY

- Use your calculator to check that
  - sin 30° = sin 150°
  - sin 70° = sin 110°
  - sin 36° = sin 144°
- Try some other values to show that $\sin x = \sin(180 - x)$.

EXAM TIP

- When finding an angle, it is easier to turn the rule upside down.

KEY POINTS

1 The sine rule is $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$.

2 Invert the rule to find an angle.

3 The sine rule involves 2 sides and 2 angles.

4.16

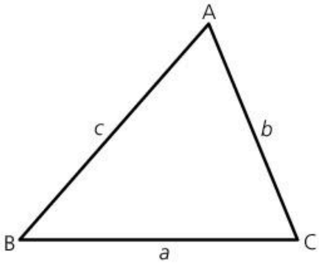
The cosine rule

LEARNING OUTCOMES

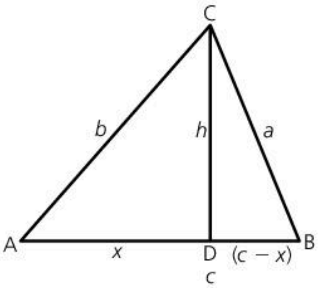
- Use the cosine rule to calculate a side
- Use the cosine rule to calculate an angle

The cosine rule can also be used in any triangle.

As with the sine rule, lower case letters label the sides after the angle they are opposite.



The cosine rule



Using a similar diagram to the one we used for the sine rule, we can apply Pythagoras' theorem to both right-angled triangles to obtain

$$h^2 = b^2 - x^2, \text{ and } h^2 = a^2 - (c - x)^2$$

So

$$a^2 - (c - x)^2 = b^2 - x^2$$

Or

$$a^2 = b^2 - x^2 + (c - x)^2$$
$$a^2 = b^2 - x^2 + c^2 - 2cx + x^2$$
$$a^2 = b^2 + c^2 - 2cx$$

But $x = b \cos A$

So $\mathbf{a^2 = b^2 + c^2 - 2bc \cos A}$

This is the **cosine rule**.

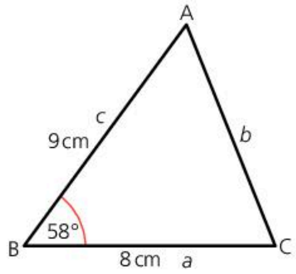
The cosine rule has three different forms, which can be derived by relabelling the diagram.

It can be written as

- $a^2 = b^2 + c^2 - 2bc \cos A$, or
- $b^2 = a^2 + c^2 - 2ac \cos B$ (by replacing A with B, a with b and b with a), or
- $c^2 = b^2 + a^2 - 2ba \cos C$ (by replacing A with C, a with c and c with a).

We use the cosine rule when the question gives us all three sides or two sides and the included angle.

Finding a side with the cosine rule



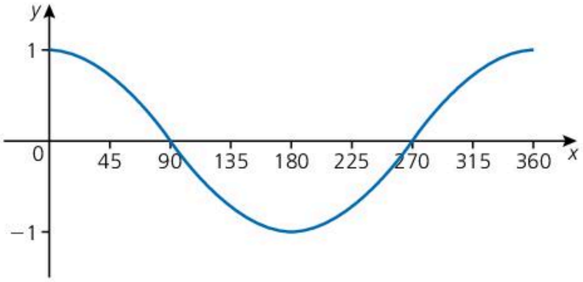
To find side b in the triangle ABC:

We know angle B, so we use the formula with B in it:

$$b^2 = a^2 + c^2 - 2ac \cos B$$
$$b^2 = 64 + 81 - 144 \cos 58$$
$$b^2 = 145 - 76.3083$$
$$b^2 = 68.6916$$
$$b = 8.3 \text{ cm (to 1 d.p.)}$$

The cosine graph

The graph of $y = \cos x$ looks like this:



It is a similar shape to the graph of $y = \sin x$, but it has been translated to the left.

The graph has reflection symmetry about $x = 180$, so $\cos x = \cos(360 - x)$.

There is rotational symmetry about (90, 0), so $\cos x = -\cos(180 - x)$.

Every angle between 0° and 180° has a discrete value of $\cos x$, so there is not the same ambiguity as there is with $\sin x$.

ACTIVITY

- Use your calculator to check that

$$\begin{aligned} \cos 30^\circ &= \cos 330^\circ \\ \cos 30^\circ &= -\cos 150^\circ \\ \cos 64^\circ &= \cos 296^\circ \\ &= -\cos 116^\circ \end{aligned}$$

- Try some other values to show that $\cos x = \cos(360 - x) = -\cos(180 - x)$.

EXAM TIP

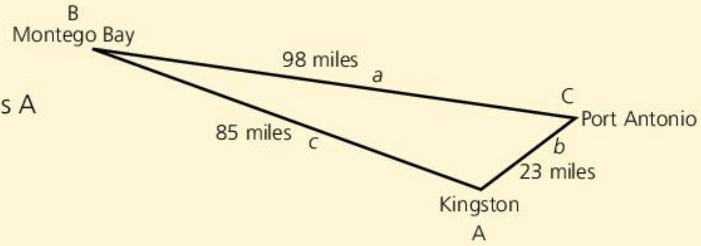
- Angles between 90° and 180° have a negative cosine.

Finding an angle with the cosine rule

WORKED EXAMPLE 1

Using the map we can find the angle A at Kingston.

$$a^2 = b^2 + c^2 - 2bc \cos \mathrm{A}$$



Substituting, we get

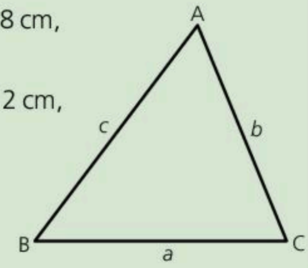
$$\begin{aligned} 9604 &= 529 + 7225 - 3910 \cos \mathrm{A} \\ 3910 \cos \mathrm{A} &= 529 + 7225 - 9604 \\ 3910 \cos \mathrm{A} &= -1850 \\ \cos \mathrm{A} &= \frac{-1850}{3910} = -0.4731 \\ \mathrm{A} &= \cos^{-1} -0.4731 = 118.2^\circ \end{aligned}$$

SUMMARY QUESTIONS

1 If angle B = 57°, $a = 9$ cm and $c = 8$ cm, calculate the length of b.

2 If $a = 8$ cm, $b = 7.5$ cm and $c = 12.2$ cm, calculate the size of angle C.

3 If $a = 9$ cm, $b = 8.5$ cm and $c = 10.4$ cm, calculate all three angles.



KEY POINTS

1 The cosine rule is $a^2 = b^2 + c^2 - 2bc \cos \mathrm{A}$.

2 The cosine rule is used when all three sides or two sides and the included angle are given.

4.17 Circle theorems

LEARNING OUTCOMES

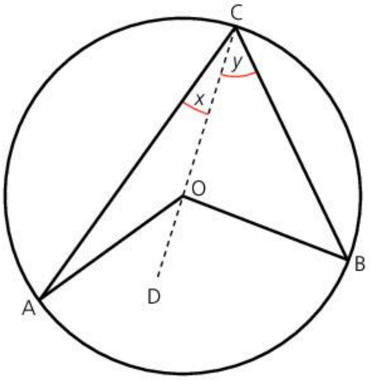
- Circle theorems:
  - Angle at centre = twice the angle at circumference
  - Angles in the same segment are equal
  - Angle in a semicircle = 90°
  - Opposite and exterior angles of cyclic quadrilaterals

Earlier on we considered the angle properties of polygons.

There are a number of facts about angles in a circle. These facts are known as the **circle theorems**.

Circle theorems

The angle at the centre is twice the angle at the circumference



In the diagram, A, B and C are points on the circumference of a circle, whose centre is O.

CO is extended to D, and angle ACO = x and BCO = y.

CO = AO as they are radii, so triangle ACO is isosceles.

So

Angle CAO = ACO = x

Angle AOC = $180 - 2x$ Angles in a triangle add up to 180°

Angle AOD = $2x$ as AOC + AOD = 180 Angles on a straight line

Similarly, triangle COB is isosceles, so CBO = y, COB = $180 - 2y$ and BOD = $2y$.

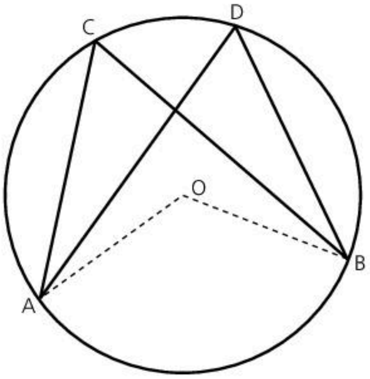
The angle at the centre from A and B, angle AOB, is $2x + 2y$.

The angle at the circumference from A and B, ACB, is $x + y$.

So the angle at the centre is twice the angle at the circumference.

This is true for all circles.

Angles in the same segment are equal



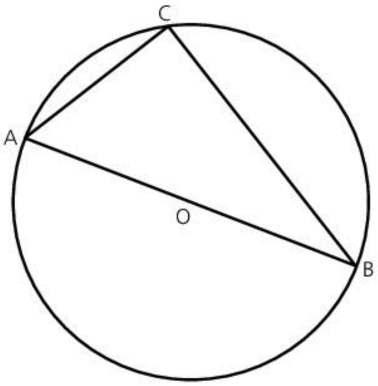
In this diagram, angles ACB and ADB share the vertices A and B, and also C and D lie on the circumference on the same side of AB.

Angle ACB = $\frac{1}{2}$AOB Angle at centre = 2 × angle at circumference

and angle ADB = $\frac{1}{2}$AOB Angle at centre = 2 × angle at circumference

so angle ACB = angle ADB.

The angle in a semicircle is a right angle



In this diagram, AOB is a diameter.

So the angle at the centre, AOB = 180°.

Angle ACB = $\frac{1}{2}$ of the angle at the centre, so

ACB = 90°

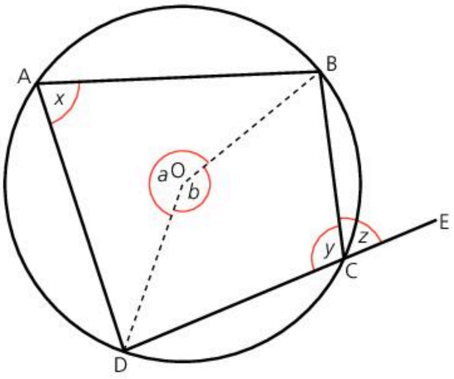
Cyclic quadrilaterals

If the four vertices of a quadrilateral lie on the circumference of a circle, it is called a **cyclic quadrilateral**.

ABCD is a cyclic quadrilateral.

BCE is an exterior angle.

Opposite angles of a cyclic quadrilateral are supplementary.



If BAD $= x$, and BCD $= y$,

$b = 2x$ Angle at centre = 2 × angle at circumference

and $a = 2y$ Angle at centre = 2 × angle at circumference

$a + b = 360°$ Angles round a point

so $2x + 2y = 360°$

or $x + y = 180°$

BAD + BCD = 180°

So the opposite angles of a cyclic quadrilateral are supplementary.

The exterior angle of a cyclic quadrilateral is equal to the interior opposite angle

$x + y = 180°$ Opposite angles of a cyclic quadrilateral

$y + z = 180°$ Angles on a straight line

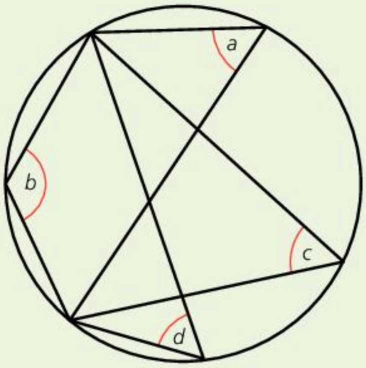
so $x = z$

So, the exterior angle of a cyclic quadrilateral is equal to the interior opposite angle.

ACTIVITY

Three of the four angles *a*, *b*, *c* and *d* are equal.

- Which three are equal, and what can you say about the fourth?



EXAM TIP

- When solving geometry problems, always give reasons for your answer.

KEY POINTS

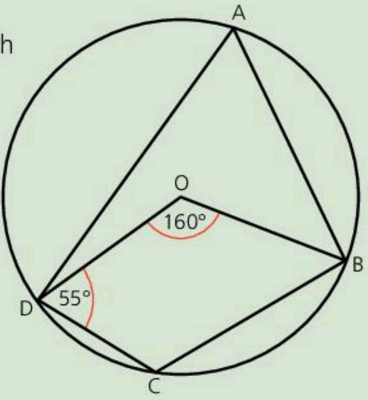
1. The angle at the centre = 2 × the angle at the circumference.
2. Angles in the same segment are equal.
3. The angle in a semicircle is a right angle.
4. Opposite angles of a cyclic quadrilateral are supplementary.
5. The exterior angle of a cyclic quadrilateral is equal to the interior opposite angle.

SUMMARY QUESTIONS

ABCD is a cyclic quadrilateral, with centre O.

Calculate angle:

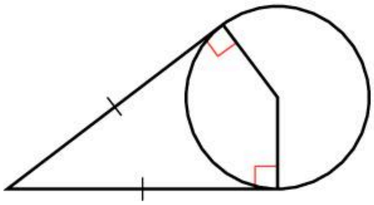
1. DAB
2. DCB
3. OBC



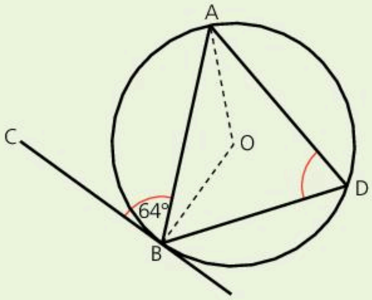
4.18 Tangents and chords

LEARNING OUTCOMES

- Circle theorems:
  - the tangent is perpendicular to the radius
  - tangents from a point are equal
  - alternate segment theorem
  - line from centre of circle to midpoint of chord is perpendicular to chord



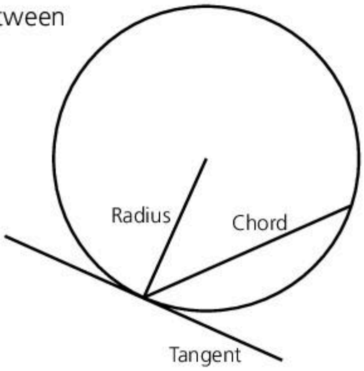
ACTIVITY



1. In the diagram, O is the centre of the circle. Use the diagram to demonstrate the alternate segment theorem by calculating angle
   ABO
   BAO
   AOB
   ADB
2. Repeat question **1**, but with angle ABC = 40°.
3. Repeat question **1**, but with angle ABC = x.

Here we will study the relationship between tangents, radii and chords.

- A **tangent** is a line that just touches a circle without entering it.
- A **chord** is a line joining two points on the circumference.
- A **radius** is a line from the circumference to the centre.

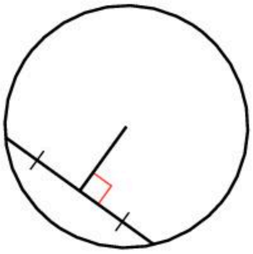


Tangent properties

- The angle between a tangent and the radius at the point of contact is 90°.
- From any point outside a circle, there are two tangents.
- The tangents from a point to a circle are equal in length.

Chord properties

- The line from the centre of the circle to the midpoint of a chord is perpendicular to the chord.
- The perpendicular from the centre of a circle to the chord bisects the chord.



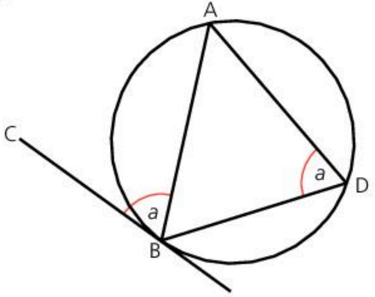
The alternate segment theorem

A chord splits a circle into two segments.

The diagram shows a tangent CB and a chord AB at the point of contact.

The angle ADB is in the **alternate segment** to the angle ABC, between tangent and chord.

- The angle between a tangent and chord is equal to the angle in the alternate segment.



Solving problems with circle theorems

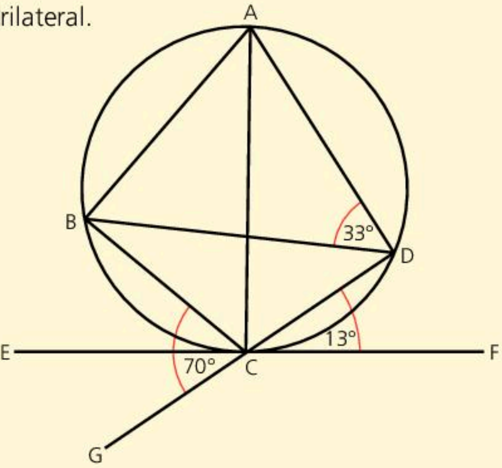
There are many problems that can be posed using the circle theorems. The theorems from 4.16 and those presented here need to be used together. Always annotate a diagram to mark any angles obtained.

WORKED EXAMPLE 1

ABCD is a cyclic quadrilateral.

EF is a tangent at C, and DCG is a straight line.

Angles
ADB = 33°,
DCF = 13° and
BCG = 70°.



1 Calculate:
   - **a** angle BAD
   - **b** angle BAC
   - **c** angle BDC

2 Prove that AC is a diameter.

Solution

1 **a** BAD = BCG = 70° (exterior angle of a cyclic quadrilateral = interior opposite angle)

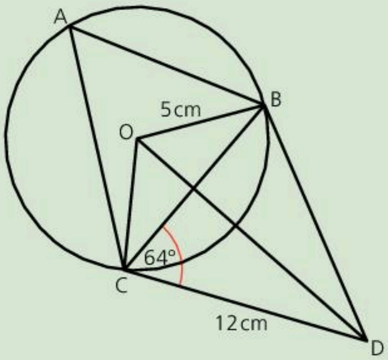
   b CAD = DCF = 13° (angle between tangent and chord = angle in the alternate segment)
   BAC = BAD − CAD = 70 − 13 = 57°

   c BDC = BAC = 57° (angles in the same segment are equal)

2 ADC = ADB + BDC = 33 + 57 = 90°.
So AC is a diameter (angle in a semicircle = 90°)

SUMMARY QUESTIONS

ABC are points on a circle, centre O and radius 5 cm. BD and CD are tangents to the circle such that DC = 12 cm and BCD = 64°.



Calculate:

1 the length of DO

2 the size of angle CAB

3 the size of angle OBC.

EXAM TIP

- Many of the circle theorems are derived from the angle at the centre = 2 × angle at circumference.
- The tangent, radius and chord properties highlight the symmetry of circles.
- The alternate segment theorem is most commonly forgotten. Make sure you understand it and remember it.
- When writing an angle use the three letter format, i.e. angle ABC not angle B.

KEY POINTS

1 The tangent is perpendicular to the radius at the point of contact.

2 Tangents from a point are equal.

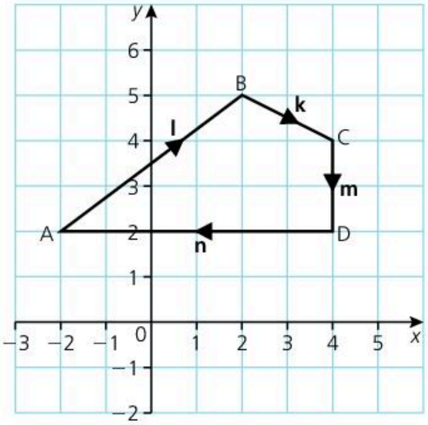
3 The angle between a tangent and a chord is equal to the angle in the alternate segment.

4 The line from the centre of a circle to the midpoint of a chord is perpendicular to the chord.

4.19 Vectors

LEARNING OUTCOMES

- Understand the concept of vectors
- Combine vectors (triangle or parallelogram law)
- Add and subtract column vectors
- Understand and use vector algebra
- Express a point P (a, b) as a position vector
- Determine the magnitude and direction of a vector



Vectors represent movement, and are used in transformation geometry to define a translation (see 4.9).

There are three common ways of representing a vector.

- The vector from B to C can be written as a column vector, $\begin{pmatrix}2\\-1\end{pmatrix}$.
- It can also be written as $\overrightarrow{BC}$, the arrow indicating the direction from B to C.
- It can also be written as **k**. Because we cannot easily write in bold type, when answering questions we indicate a vector by writing $\underset{\sim}{k}$.

Column vectors follow the conventions of coordinates. The first number represents a horizontal direction and the second number represents a vertical direction, so $\begin{pmatrix}2\\-1\end{pmatrix}$ means 2 to the right (a negative number would indicate left), and 1 down (because the number is negative it signifies downward movement).

However, coordinates and vectors have completely different properties:

Column vector	Coordinates
Represents a movement of a specific magnitude or size.	Has no movement – it signifies a point which remains static, and has no size.
Has a definite direction.	Has no movement and, therefore, no direction.
Has no position – there are an infinite number of $\begin{pmatrix}3\\-1\end{pmatrix}$ vectors.	Has a defined position. There is only one point (3, −1).

Position vectors

Position vectors give the vector from the origin to a point.

So the position vector of the point (*a*, *b*) is $\begin{pmatrix}a\\b\end{pmatrix}$.

Addition of vectors

The movement from A to B is represented by the vector $\begin{pmatrix}4\\3\end{pmatrix}$, and the movement from B to C by $\begin{pmatrix}2\\-1\end{pmatrix}$.

Adding the horizontal components and the vertical components we get

$$\overrightarrow{AB} + \overrightarrow{BC} = \begin{pmatrix}4\\3\end{pmatrix} + \begin{pmatrix}2\\-1\end{pmatrix} = \begin{pmatrix}6\\2\end{pmatrix}, \text{ or } \overrightarrow{AC}.$$

In vector terms, moving from A to B and then from B to C is identical to moving directly from A to C.

Similarly

$$\overrightarrow{AB} - \overrightarrow{BC} = \begin{pmatrix}4\\3\end{pmatrix} - \begin{pmatrix}2\\-1\end{pmatrix} = \begin{pmatrix}2\\4\end{pmatrix}.$$

Subtracting $\overrightarrow{BC}$ has the same effect as adding $\overrightarrow{CB}$.

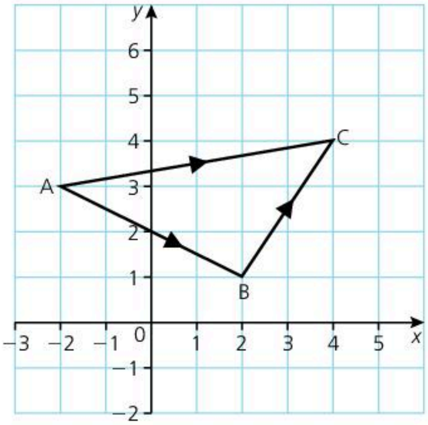
The triangle law and parallelogram law

To add two vectors, we can complete a triangle or a parallelogram.

To add $\overrightarrow{AB}$, $\begin{pmatrix}4\\-2\end{pmatrix}$ and $\overrightarrow{BC}$, $\begin{pmatrix}2\\3\end{pmatrix}$ draw the two vectors end-to-end, and complete the triangle:

$\overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{AC}$.

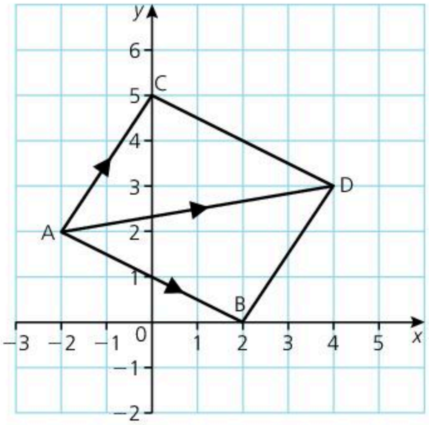
This is the triangle law for addition of vectors.



To add $\overrightarrow{AB}$ and $\overrightarrow{AC}$, complete the parallelogram ACDB.

$\overrightarrow{AB} + \overrightarrow{AC}$ = diagonal $\overrightarrow{AD}$.

This is the parallelogram law.

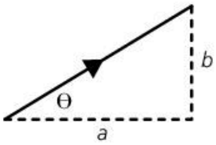


The magnitude and direction of a vector

The magnitude, or size, of a vector can be found using Pythagoras' theorem.

The vector $\begin{pmatrix}a\\b\end{pmatrix}$ has a horizontal component of a and a vertical component of b, so the magnitude is $\sqrt{(a^2 + b^2)}$.

The angle the vector makes with the horizontal, θ, is given by $\tan^{-1}\dfrac{b}{a}$.



KEY POINTS

1 Vectors have magnitude and direction, but no fixed position.

2 Vectors can be added or subtracted algebraically, by adding or subtracting the horizontal components, and the vertical components.

3 Vectors can be added graphically using the triangle law of the parallelogram law.

4 The position vector of (x, y) is $\begin{pmatrix}x\\y\end{pmatrix}$, and represents a movement from the origin to (x, y).

5 The magnitude of $\begin{pmatrix}x\\y\end{pmatrix}$ is $\sqrt{x^2 + y^2}$ and the angle with the horizontal is $\tan^{-1}\dfrac{y}{x}$.

ACTIVITY

- Draw your own grid, and mark four points A, B, C and D.
- Write down the column vectors $\overrightarrow{AB}$, $\overrightarrow{BC}$, $\overrightarrow{CD}$, $\overrightarrow{DA}$.
- Add the column vectors $\overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CD} + \overrightarrow{DA}$. Explain your answer.
- Write down the column vectors $\overrightarrow{BA}$, $\overrightarrow{CB}$, $\overrightarrow{DC}$, $\overrightarrow{AD}$.
- What is the connection between the column vector $\overrightarrow{XY}$ and $\overrightarrow{YX}$?

EXAM TIP

- When you square a negative number, the answer is positive.

SUMMARY QUESTIONS

If $\mathbf{a} = \begin{pmatrix}3\\-1\end{pmatrix}$, $\mathbf{b} = \begin{pmatrix}1\\5\end{pmatrix}$ and $\mathbf{c} = \begin{pmatrix}-4\\2\end{pmatrix}$, find

1 $\mathbf{a} + \mathbf{b}$

2 $\mathbf{c} - \mathbf{a}$

3 The magnitude of $\mathbf{d}$, where $\mathbf{a} + \mathbf{b} + \mathbf{d} = \mathbf{c}$

4.20 Vector geometry

LEARNING OUTCOMES

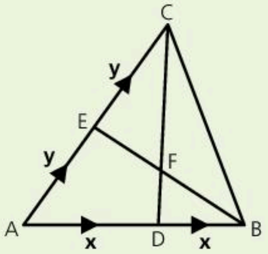
- Multiply a vector by a scalar
- Understand the meaning of collinearity
- Use vectors to solve problems in geometry

ACTIVITY

In triangle ABC, $\overrightarrow{AB} = 2\mathbf{x}$ and $\overrightarrow{AC} = 2\mathbf{y}$.

D and E are the midpoints of AB and AC.

$DF = \frac{1}{3}DC$



Find, in terms of **x** and **y**,

a $\overrightarrow{DC}$ **b** $\overrightarrow{DF}$

c $\overrightarrow{AF}$ **d** $\overrightarrow{BC}$

e $\overrightarrow{BG}$, where G is the midpoint of BC

f $\overrightarrow{AG}$

How can you tell that A, F and G are collinear?

Multiplying a vector by a scalar

In 4.18 we added and subtracted vectors.

Multiplying two vectors makes little sense, but we can multiply a vector by a **scalar** (or a single number).

For example, $3\begin{pmatrix}4\\1\end{pmatrix}$, or 3 times the vector $\begin{pmatrix}4\\1\end{pmatrix}$, $= \begin{pmatrix}12\\3\end{pmatrix}$.

This is the same as $\begin{pmatrix}4\\1\end{pmatrix} + \begin{pmatrix}4\\1\end{pmatrix} + \begin{pmatrix}4\\1\end{pmatrix}$.

So the resultant vector has three times the magnitude of the original vector, and goes in the same direction, so is parallel to the original vector.

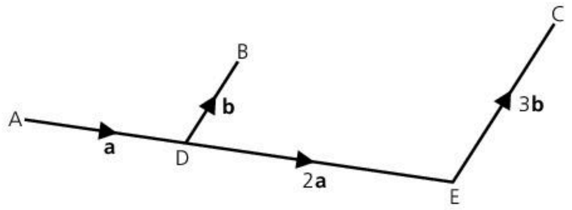
Generally

$$n\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}nx\\ny\end{pmatrix}$$

$\begin{pmatrix}x\\y\end{pmatrix}$ and $\begin{pmatrix}a\\b\end{pmatrix}$ are parallel if $\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}ka\\kb\end{pmatrix} = k\begin{pmatrix}a\\b\end{pmatrix}$, and the magnitude of $\begin{pmatrix}x\\y\end{pmatrix}$ is k times the magnitude of $\begin{pmatrix}a\\b\end{pmatrix}$.

Collinearity

Three points, A, B and C, are said to be **collinear** if they lie on a straight line.



In the diagram, $\overrightarrow{AD} = \mathbf{a}$, $\overrightarrow{DE} = 2\mathbf{a}$, $\overrightarrow{DB} = \mathbf{b}$ and $\overrightarrow{EC} = 3\mathbf{b}$.

To show that A, B and C are collinear:

$\overrightarrow{AB} = \overrightarrow{AD} + \overrightarrow{DB} = \mathbf{a} + \mathbf{b}$

$\overrightarrow{AC} = \overrightarrow{AD} + \overrightarrow{DE} + \overrightarrow{EC} = \mathbf{a} + 2\mathbf{a} + 3\mathbf{b} = 3\mathbf{a} + 3\mathbf{b} = 3(\mathbf{a} + \mathbf{b})$

So $\overrightarrow{AC} = \overrightarrow{3AB}$, so AC and AB are parallel and both pass through A.

So A, B and C are collinear.

Solving vector problems

The key to solving vector problems is to see them as a combination of geometry and algebra. A labelled diagram is essential, and algebraic notation explains what we are doing.

We use the **a**, **b**, **c** notation for vectors as we are often trying to prove general statements rather than calculate exact values.

WORKED EXAMPLE 1

ABC is a triangle.

The position vectors of A and B are $\mathbf{a}$ and $\mathbf{b}$, respectively, and $\overrightarrow{AC} = \mathbf{a} + 2\mathbf{b}$.

D is the midpoint of AB.

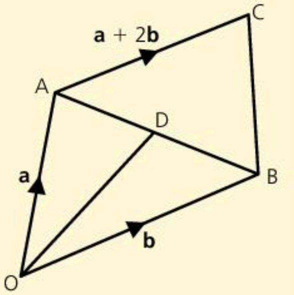
a Find the position vector of:
  i C
  ii D in terms of $\mathbf{a}$ and $\mathbf{b}$.

b Show that O, C and D are collinear.

c Find $\overrightarrow{BC}$ in terms of $\mathbf{a}$ and $\mathbf{b}$.

Solution

Start with a diagram:



a i $\overrightarrow{OC} = \overrightarrow{OA} + \overrightarrow{AC}$

$= \mathbf{a} + \mathbf{a} + 2\mathbf{b} = 2\mathbf{a} + 2\mathbf{b}$

ii $\overrightarrow{OD} = \overrightarrow{OA} + \overrightarrow{AD}$

$= \overrightarrow{OA} + \frac{1}{2}\overrightarrow{AB}$

$= \overrightarrow{OA} + \frac{1}{2}(\overrightarrow{AO} + \overrightarrow{OB})$

$= \mathbf{a} + \frac{1}{2}(-\mathbf{a} + \mathbf{b})$

$= \frac{1}{2}\mathbf{a} + \frac{1}{2}\mathbf{b} = \frac{1}{2}(\mathbf{a} + \mathbf{b})$

b $\overrightarrow{OC} = 2\mathbf{a} + 2\mathbf{b} = 4[\frac{1}{2}(\mathbf{a} + \mathbf{b})] = 4\overrightarrow{OD}$

OC and OD are parallel, both passing through O, so O, C and D are collinear.

c $\overrightarrow{BC} = \overrightarrow{BO} + \overrightarrow{OA} + \overrightarrow{AC}$

$= -\mathbf{b} + \mathbf{a} + \mathbf{a} + 2\mathbf{b}$

$= 2\mathbf{a} + \mathbf{b}$

EXAM TIP

- Always check that your answers seem realistic.
- Make sure you take direction into account.

SUMMARY QUESTIONS

ABCD is a trapezium, with AD parallel to and twice the length of BC. $\overrightarrow{AB} = \mathbf{b}$, and the diagonal $\overrightarrow{AC} = \mathbf{a}$.

1 Find, in terms of $\mathbf{a}$ and $\mathbf{b}$,
  a $\overrightarrow{BC}$ **b** $\overrightarrow{AD}$ **c** $\overrightarrow{CD}$

2 AC intersects BD at X, so that 2BX = XD
Find $\overrightarrow{BX}$ in terms of $\mathbf{a}$ and $\mathbf{b}$.

3 Find $\overrightarrow{AX}$ in terms of $\mathbf{a}$ and $\mathbf{b}$.

KEY POINTS

1 $n\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} nx \\ ny \end{pmatrix}$

2 Points are collinear if they lie on the same straight line.

3 To prove collinearity, show that vectors are multiples of each other (and so are parallel) and have a common point.

4.21 Matrices

LEARNING OUTCOMES

- Understand concepts of matrices (concept of matrix, row, column, order, types, practical use)
- Add and subtract matrices
- Multiply matrices; multiply matrix by scalar (non-commutativity of matrix multiplication)
- The determinant and inverse matrix

Matrices

A **matrix** (plural: **matrices**) is a rectangular array of numbers.

They have applications in the real world. For example, they are used in computer graphics to enable a 3-dimensional image to be drawn on a 2-dimensional screen.

Here we will focus on how to manipulate matrices, and 4.21 will show how we use them in transformations.

A matrix is described by the number of **rows** and **columns** it has.

A row is horizontal, a column is vertical.

So $\begin{pmatrix} 2 & -1 & 4 \\ 3 & 2 & -1 \end{pmatrix}$ is a 2 by 3 (or 2 × 3) matrix.

The column vectors we considered in 4.18 are 2 × 1 matrices.

Matrix addition and subtraction

Only matrices of the same size can be added or subtracted.

Components in the same position are added or subtracted.

So

$$\begin{pmatrix} 1 & 2 \\ -2 & 0 \end{pmatrix} + \begin{pmatrix} 2 & 0 \\ -1 & -3 \end{pmatrix} = \begin{pmatrix} 1+2 & 2+0 \\ -2+-1 & 0+-3 \end{pmatrix} = \begin{pmatrix} 3 & 2 \\ -3 & -3 \end{pmatrix}$$

and

$$\begin{pmatrix} 1 & 3 \\ -2 & 0 \end{pmatrix} - \begin{pmatrix} 2 & -1 \\ -3 & 1 \end{pmatrix} = \begin{pmatrix} 1-2 & 3--1 \\ -2--3 & 0-1 \end{pmatrix} = \begin{pmatrix} -1 & 4 \\ 1 & -1 \end{pmatrix}$$

Multiplying a matrix by a scalar

A matrix can be multiplied by a scalar by multiplying each component by the scalar.

$$3\begin{pmatrix} 2 & 3 \\ 0 & -2 \end{pmatrix} = \begin{pmatrix} 6 & 9 \\ 0 & -6 \end{pmatrix}$$

Multiplying two matrices

Matrix multiplication involves multiplying all the components in a row of the first matrix by all the components of a column of the second matrix. So the numbers of components in a row of the first matrix must match the number of components in a column of the second matrix.

Algebraically, we can only multiply an $a \times b$ matrix by a $b \times c$ matrix. The resultant will be an $a \times c$ matrix.

WORKED EXAMPLE 1

To multiply: Matrix **A** Matrix **B**

$$\begin{pmatrix} 1 & 0 & 3 \\ 2 & 4 & -1 \end{pmatrix} \qquad \begin{pmatrix} 1 & 2 \\ 2 & 3 \\ 1 & 0 \end{pmatrix}$$

Multiply the components of the first row of matrix **A** by the first column of matrix **B**, and add the results:

$1 \times 1 + 0 \times 2 + 3 \times 1 = 4$. This goes in the first row, first column of the answer.

First row, second column: $1 \times 2 + 0 \times 3 + 3 \times 0 = 2$

Second row, first column: $2 \times 1 + 4 \times 2 + -1 \times 1 = 9$

Second row, second column: $2 \times 2 + 4 \times 3 + -1 \times 0 = 16$

Answer: $\begin{pmatrix} 4 & 2 \\ 9 & 16 \end{pmatrix}$

The identity matrix

The identity matrix for an $m \times n$ matrix is an $n \times n$ matrix of zeros except for a diagonal of 1s from top left to bottom right.

For example, $\begin{pmatrix} 2 & 5 & -3 \\ 4 & 0 & 3 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 5 & -3 \\ 4 & 0 & 3 \end{pmatrix}$

The inverse and determinant of a 2 × 2 matrix

If the matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is multiplied by $\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$, the resultant is $\begin{pmatrix} ad - bc & 0 \\ 0 & ad - bc \end{pmatrix}$, or $(ad - bc)\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, where $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ is the identity.

So the inverse of $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is $\frac{1}{ad - bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$.

$ad - bc$ is called the **determinant** of the matrix.

KEY POINTS

1. An $n \times m$ matrix is a rectangular array of numbers.
2. Matrices of the same size can be added or subtracted by adding or subtracting corresponding components.
3. An $n \times m$ matrix can be multiplied by an $m \times p$ matrix to produce an $n \times p$ matrix.
4. To multiply two matrices, a row from the first matrix is multiplied by a column from the second, and the answers added.
5. The identity for an $m \times n$ matrix is an $n \times n$ matrix containing a diagonal of 1s, with the other components all 0.
6. The inverse of $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is $\frac{1}{ad - bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$, where $ad - bc$ is the determinant.

ACTIVITY

Matrix multiplication is not commutative.

- If $\mathbf{A} = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}$, and $\mathbf{B} = \begin{pmatrix} 1 & 0 \\ 2 & 3 \\ 1 & 2 \end{pmatrix}$, find

  A × **B**

  B × **A**

EXAM TIP

- Matrix multiplication is not commutative.

ACTIVITY

- Show that the determinant of $\begin{pmatrix} 3 & 1 \\ 4 & 2 \end{pmatrix}$ is 2.
- Write down the inverse matrix of $\begin{pmatrix} 3 & 1 \\ 4 & 2 \end{pmatrix}$.
- Show that $\begin{pmatrix} 3 & 1 \\ 4 & 2 \end{pmatrix}$ multiplied by its inverse is $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.
- What happens if you multiply the inverse by $\begin{pmatrix} 3 & 1 \\ 4 & 2 \end{pmatrix}$?

SUMMARY QUESTIONS

$\mathbf{A} = \begin{pmatrix} 4 & 0 & 1 \\ 2 & -3 & 1 \end{pmatrix}$, $\mathbf{B} = \begin{pmatrix} 2 & -3 \\ 1 & 3 \end{pmatrix}$,

$\mathbf{C} = \begin{pmatrix} 3 \\ 2 \\ -4 \end{pmatrix}$, $\mathbf{D} = \begin{pmatrix} 1 & -2 & 4 \\ 2 & 5 & -1 \end{pmatrix}$

1 Some of these can be calculated; others cannot. For those that can, find the answer.

a **A** − **D** **b** **B** + **C** **c** **AC**

d **CA** **e** **AB** **f** **BA**

2 Find the inverse matrix of **B**.

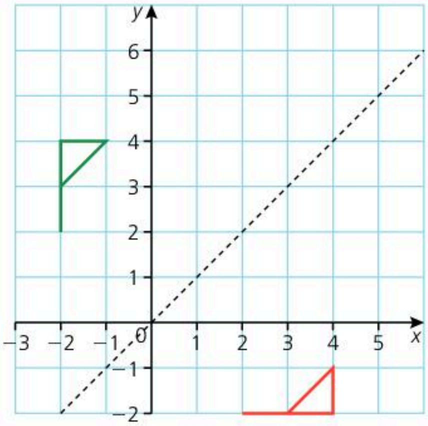
3 Does **B**(**A** + **D**) = **BA** + **BD**? What do we call this property?

4.22

Matrices and transformations

LEARNING OUTCOMES

- Determine the 2 × 2 matrix associated with specific transformations
- Determine the 2 × 2 matrix representation of the result of two transformations



EXAM TIP

- To produce these transformations, we always premultiply – that is, the transformation matrix is written first.

Matrices can be used to produce transformations.

Reflections

The green flag has vertices at (−2, 2), (−2, 4), (−1, 4) and (−2, 3).

As position vectors, these can be written as $\begin{pmatrix} -2 \\ 2 \end{pmatrix}$, $\begin{pmatrix} -2 \\ 4 \end{pmatrix}$, $\begin{pmatrix} -1 \\ 4 \end{pmatrix}$ and $\begin{pmatrix} -2 \\ 3 \end{pmatrix}$

or, as a 2 × 4 matrix, **P**: $\begin{pmatrix} -2 & -2 & -1 & -2 \\ 2 & 4 & 4 & 3 \end{pmatrix}$.

If we multiply this by matrix, **M**, $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, we arrive at, **MP** = **P**′

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} -2 & -2 & -1 & -2 \\ 2 & 4 & 4 & 3 \end{pmatrix} = \begin{pmatrix} 2 & 4 & 4 & 3 \\ -2 & -2 & -1 & -2 \end{pmatrix}.$$

The four columns are the position vectors of the red flag.

The matrix $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ produces a reflection in the line $y = x$,

and $\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$ produces the line $y = -x$.

The matrix $\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$ produces a reflection in the y axis, and $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ produces a reflection in the x axis.

Rotations

To rotate about the origin, $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ produces a 90° anticlockwise rotation and $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ produces a 90° clockwise rotation.

A 180° rotation is produced by the matrix $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$.

The matrix $\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$ produces an anticlockwise rotation through an angle of θ.

Inverses

If we reflect a shape in a mirror line, and then reflect the image in the same mirror line, we return to the starting point. This means that reflections are **self-inverse**; the inverse of a reflection is the same reflection.

Enlargements

We know that the identity matrix is $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. An enlargement, centre the origin, scale factor n is produced by the matrix $\begin{pmatrix} n & 0 \\ 0 & n \end{pmatrix}$.

Combinations of transformations

In the diagram on the next page, the green flag has been reflected in the y-axis to give the red image. The red flag has then been enlarged, centre the origin, by scale factor 2 to produce the blue image.

ACTIVITY

- Show that the inverse of $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, a 90° anticlockwise rotation about the origin, is $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ a 90° clockwise rotation about the origin.

The matrix for the position vectors of the green flag is

$\begin{pmatrix} -2 & -2 & -1 & -2 \\ 1 & 3 & 3 & 2 \end{pmatrix}$.

The reflection is achieved by the operation

$$\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} -2 & -2 & -1 & -2 \\ 1 & 3 & 3 & 2 \end{pmatrix} = \begin{pmatrix} 2 & 2 & 1 & 2 \\ 1 & 3 & 3 & 2 \end{pmatrix}.$$

The enlargement is represented by

$$\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} 2 & 2 & 1 & 2 \\ 1 & 3 & 3 & 2 \end{pmatrix} = \begin{pmatrix} 4 & 4 & 2 & 4 \\ 2 & 6 & 6 & 4 \end{pmatrix}$$

This could be written as $\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} -2 & -2 & -1 & -2 \\ 1 & 3 & 3 & 2 \end{pmatrix}$.

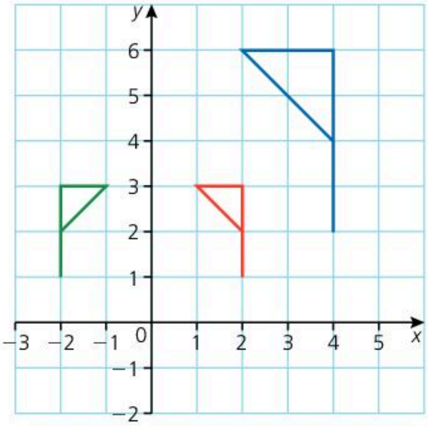
Multiplying the two transformation matrices,

$$\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} -2 & 0 \\ 0 & 2 \end{pmatrix}$$

This is the single transformation that maps the green flag to the blue:

$$\begin{pmatrix} -2 & 0 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} -2 & -2 & -1 & -2 \\ 1 & 3 & 3 & 2 \end{pmatrix} = \begin{pmatrix} 4 & 4 & 2 & 4 \\ 2 & 6 & 6 & 4 \end{pmatrix}$$

We saw in 4.21 that matrix multiplication is not commutative, so the order is important. We produce transformations by putting the transformation matrix in front. This is called premultiplying. When we put a matrix after another it is called postmultiplying.



EXAM TIP

- When combining matrices for two transformations, we premultiply, so the first transformation is on the right of the second.

ACTIVITY

- Draw your own shape on a grid, and write the position vectors of the vertices as a single matrix.
- Use your own 2 × 2 matrix to transform your shape.
- Find the inverse matrix and check it reverses the transformation.

KEY POINTS

1. Some reflections, rotations and enlargements can be defined by a 2 × 2 matrix.
2. If matrix **A** performs a transformation on an object **P** then the image **P**′ can be obtained using **AP** = **P**′.
3. If matrix **A** performs a transformation, and then matrix **B** transforms the image, **A** and **B** can be replaced by the single transformation matrix **BA**.

SUMMARY QUESTIONS

1. Draw the triangle with vertices (2, 1), (3, 3) and (0, 2). Label it A.
   Transform it with the matrix $\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$.
   Draw the image, and label it B.
2. Transform the image B with the matrix $\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$. Draw the new image, and label it C.
3. Find the single matrix that transforms A to C. Describe fully this transformation.

4.23 Matrix algebra

LEARNING OUTCOMES

- Use matrices to solve problems in arithmetic, algebra and geometry (linear equations up to 3 × 3)

ACTIVITY

- Solve the matrix equation $\begin{pmatrix} 3 & 1 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 4 \\ -2 \end{pmatrix}$ by
  - **a** manipulating the matrix to produce a 0 in the bottom row
  - **b** using the inverse matrix.

  Check your answers are the same for each method.

Solving simultaneous equations with matrices

Here is a pair of simultaneous equations:

$$3x - 2y = 7$$
$$4x - 3y = 10$$

We can write these equations as a matrix equation:

$$\begin{pmatrix} 3 & -2 \\ 4 & -3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 7 \\ 10 \end{pmatrix}$$

The determinant of $\begin{pmatrix} 3 & -2 \\ 4 & -3 \end{pmatrix} = (3 \times -3) - (-2 \times 4) = -1$.

So the inverse matrix is

$$-1\begin{pmatrix} -3 & 2 \\ -4 & 3 \end{pmatrix}$$

Premultiplying both sides of the equation by the inverse matrix:

$$-1\begin{pmatrix} -3 & 2 \\ -4 & 3 \end{pmatrix}\begin{pmatrix} 3 & -2 \\ 4 & -3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = -1\begin{pmatrix} -3 & 2 \\ -4 & 3 \end{pmatrix}\begin{pmatrix} 7 \\ 10 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = -1\begin{pmatrix} -1 \\ 2 \end{pmatrix}$$

The left-hand side can be simplified because a matrix multiplied by its inverse equals the identity matrix.

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$$

$$x = 1, y = -2$$

Finding the matrix that produces a transformation

Suppose the points A(1, 1) and B(1, 3) get transformed by a matrix to A′(3, 1) and B′(1, −1) respectively.

If the matrix that produced this transformation is $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$, then

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & 3 \end{pmatrix} = \begin{pmatrix} 3 & 1 \\ 1 & -1 \end{pmatrix}$$

We are going to use an inverse matrix.

The determinant of $\begin{pmatrix} 1 & 1 \\ 1 & 3 \end{pmatrix} = (1 \times 3) - (1 \times 1) = 2$, so the inverse matrix is

$$\frac{1}{2}\begin{pmatrix} 3 & -1 \\ -1 & 1 \end{pmatrix}$$

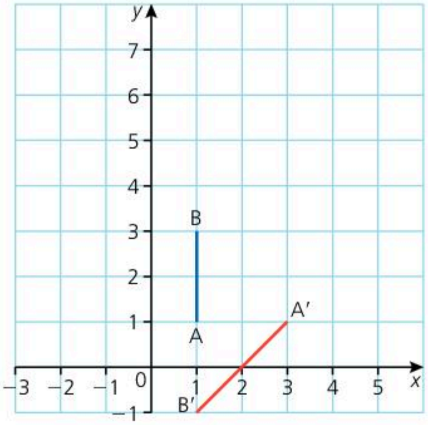
Postmultiplying both sides of the equation by the inverse gives

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & 3 \end{pmatrix} \times \frac{1}{2}\begin{pmatrix} 3 & -1 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} 3 & 1 \\ 1 & -1 \end{pmatrix} \times \frac{1}{2}\begin{pmatrix} 3 & -1 \\ -1 & 1 \end{pmatrix}$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & 3 \end{pmatrix}\begin{pmatrix} 1.5 & -0.5 \\ -0.5 & 0.5 \end{pmatrix} = \begin{pmatrix} 3 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 1.5 & -0.5 \\ -0.5 & 0.5 \end{pmatrix}$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 4 & -1 \\ 2 & -1 \end{pmatrix}$$

We have found the matrix that produces the transformation.



Solving matrix problems

Inverse matrices are very useful when solving problems.

They allow us to simplify matrix equations as shown in the above example.

WORKED EXAMPLE 1

Find the matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ such that

$$\begin{pmatrix} 1 & -3 \\ -2 & 5 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 4 & 1 \\ -2 & 4 \end{pmatrix}.$$

Solution

The inverse matrix of $\begin{pmatrix} 1 & -3 \\ -2 & 5 \end{pmatrix}$ is $-1\begin{pmatrix} 5 & 3 \\ 2 & 1 \end{pmatrix}$ or $\begin{pmatrix} -5 & -3 \\ -2 & -1 \end{pmatrix}$.

So we premultiply both sides of the equation by the inverse:

$$\begin{pmatrix} -5 & -3 \\ -2 & -1 \end{pmatrix}\begin{pmatrix} 1 & -3 \\ -2 & 5 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} -5 & -3 \\ -2 & -1 \end{pmatrix}\begin{pmatrix} 4 & 1 \\ -2 & 4 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} -14 & -17 \\ -6 & -6 \end{pmatrix}$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} -14 & -17 \\ -6 & -6 \end{pmatrix}$$

Remember that matrix multiplication is not commutative, so to find the matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ such that

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 & -3 \\ -2 & 5 \end{pmatrix} = \begin{pmatrix} 4 & 1 \\ -2 & 4 \end{pmatrix}$$

we would postmultiply by the inverse matrix

$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ such that

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 & -3 \\ -2 & 5 \end{pmatrix}\begin{pmatrix} -5 & -3 \\ -2 & -1 \end{pmatrix} = \begin{pmatrix} 4 & 1 \\ -2 & 4 \end{pmatrix}\begin{pmatrix} -5 & -3 \\ -2 & -1 \end{pmatrix}$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} -22 & -13 \\ 2 & 2 \end{pmatrix}$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} -22 & -13 \\ 2 & 2 \end{pmatrix}$$

EXAM TIP

- Remember that matrix multiplication is not commutative.
- A matrix multiplied by its inverse = the identity matrix.
- Premultiply or postmultiply depending on the position of the matrix you need to eliminate.

KEY POINTS

1. The equations

   $ax + by = e$, and
   $cx + dy = f$

   can be written as a matrix equation

   $\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} e \\ f \end{pmatrix}$
2. Rows can be multiplied by a constant, and rows can be added together or subtracted.
3. The inverse matrix is very useful when solving algebraic or geometric problems.

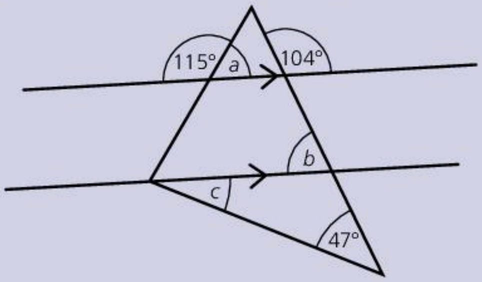
SUMMARY QUESTIONS

1. If $\begin{pmatrix} 3 & a \\ 1 & 2 \end{pmatrix}\begin{pmatrix} 3 \\ b \end{pmatrix} = \begin{pmatrix} 7 \\ 1 \end{pmatrix}$, find a and b.
2. Solve the simultaneous equations

   $3x - y = 0$
   $2x - 2y = -2$
3. Find the matrix that maps (1, 2) to (−3, 4) and (3, −1) to (5, −2).

Module 4 Practice exam questions

1 Calculate the angles marked *a*, *b* and *c* in the diagram below, stating the reasons.

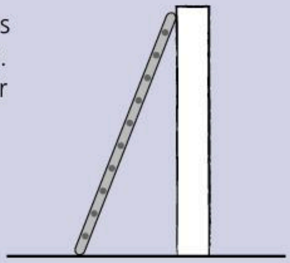


2 A regular polygon has interior angles of 150°. How many sides has the polygon?

3 An obtuse-angled isosceles triangle has an angle of 38°. Calculate the size of the other two angles.

4 Without using a protractor, construct an angle of 45°.

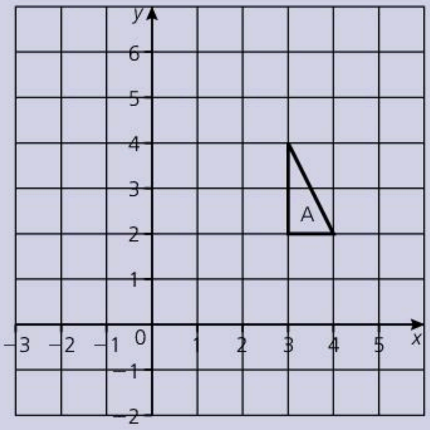
5 A 20-foot ladder leans against a vertical wall. The foot of the ladder is 7 feet from the bottom of the wall. How far up the wall does the ladder reach?



6 **a** Rotate triangle A 90° anticlockwise about (2, 1). Label the image B.

b Reflect B in the line $y = 1$. Label the image C.

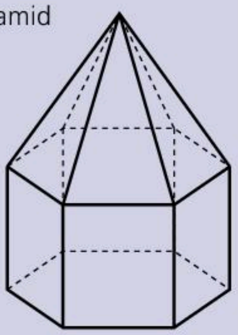
c What is the single transformation that maps A onto C?



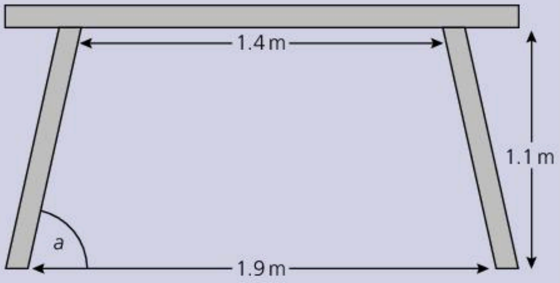
7 A regular hexagonal pyramid is attached to a regular hexagonal prism as shown.

a Find the number of faces, edges and vertices the solid possesses.

b Find the number of planes of symmetry for the shape.



8 A picnic table has a symmetrical side view as shown.
Calculate angle *a* between the leg and the horizontal.



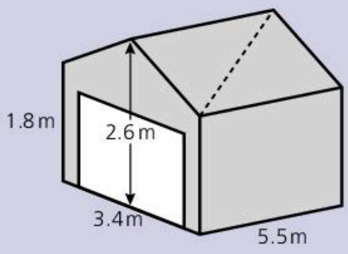
9 In triangle ABC, angle ABC = 51°, AB = 11 cm and BC = 14 cm.

a Calculate the length of AC.

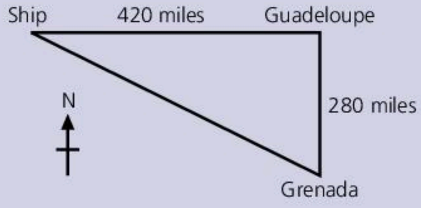
b Calculate the size of angle BAC.

c Calculate the area of the triangle ABC.

10 A garage has a base 5.5 m long and 3.4 m wide. It is 1.8 m tall at the sides, and 2.6 m tall at the highest point. The roof contains a diagonal support, as shown by a broken line. Calculate the angle between the wooden support and the horizontal.



11 A ship, sailing towards Grenada, is 420 miles due West of Guadeloupe. Guadeloupe is 280 miles due North of Grenada.
On what bearing is the ship sailing?

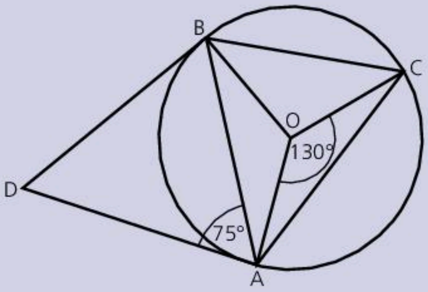


12 DA and DB are tangents to a circle, centre O. C is a point on the circumference such that angle AOC = 130°. Angle DAB = 75°.

a Calculate angle ACB, giving a reason for your answer.

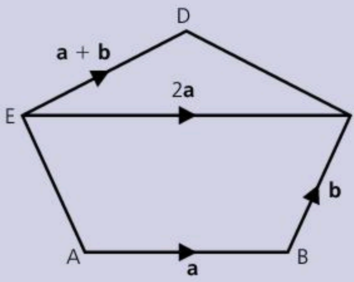
b Calculate angle ABC, giving a reason for your answer.

c Explain why AOBD is a cyclic quadrilateral.



13 ABCDE is a pentagon. $\overrightarrow{AB} = \mathbf{a}$, $\overrightarrow{BC} = \mathbf{b}$, $\overrightarrow{EC} = 2\mathbf{a}$ and $\overrightarrow{ED} = \mathbf{a} + \mathbf{b}$.

Prove that AE and CD are equal and parallel.



14 If $\overrightarrow{AB} = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$ and $\overrightarrow{AC} = \begin{pmatrix} -2 \\ 4 \end{pmatrix}$, write $\overrightarrow{BC}$ as a column vector.

15 If $\mathbf{A} = \begin{pmatrix} 2 & 1 \\ 0 & -1 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} -1 & 2 \\ 1 & 1 \end{pmatrix}$,

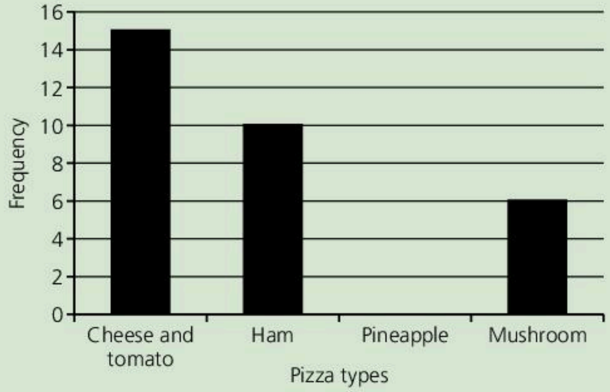
a Find

- **i** **AB**
- **ii** $\mathbf{A}^{-1}$, the inverse of **A**
- **iii** $\mathbf{B}^{-1}$, the inverse of **B**
- **iv** $(\mathbf{AB})^{-1}$, the inverse of **AB**

b Show that $\mathbf{B}^{-1}\mathbf{A}^{-1} = (\mathbf{AB})^{-1}$

Further exam practice 1

1 Lionel carried out a survey to find out what type of pizza people in his class liked best. He put the results in a bar chart.



a Five children chose pineapple. Draw the entry for pineapple that should go in the bar chart.

b How many more children chose cheese and tomato than chose mushroom?

c If the same information was shown in a pie chart, what angle would be used for ham?

2 Here are the marks of 12 children in a test.

7	7	8	8	8	8
9	10	10	12	12	12

a Calculate:
- **i** the range
- **ii** the median
- **iii** the mode
- **iv** the mean.

b To pass the test, you needed to score at least 9.
What percentage of the children passed the test?

3 **a** Simplify:
- **i** $6x - 4x$
- **ii** $5a - 4b + 2a + b$

b Work out the value of $3f + 2g$ when $f = 4$ and $g = -3$.

4 **a** Work out the volume of a cuboid 10 cm long, 4 cm wide and 6 cm high.

b Calculate the surface area of the cuboid.

c Write down the dimensions of a different cuboid with the same volume.

5 The table shows the waiting times (in minutes) of people catching a bus.

Waiting time (minutes)	Frequency
0 to 2	5
2 to 4	9
4 to 6	5
6 to 8	3
8 to 10	2

a Draw a histogram to show this information.

b Write down the modal class for these waiting times.

c One of these passengers is chosen at random.
What is the probability that this passenger waited 4 minutes or less?

d The probability that a passenger selected at random is male is $\frac{3}{8}$.
- **i** What is the probability that a passenger selected at random is female?
- **ii** How many males were in the survey?

6 **a** Complete this table for $y = x^2 + 2x$.

x	-3	-2	-1	0	1	2	3
x^2			1		1		
$+2x$			-2				
$y = x^2 + 2x$			-1				

b Draw the graph of $y = x^2 + 2x$ for values of x from -3 to 3 and y from -4 to 16.

c Write down:
- **i** the minimum value of y
- **ii** the equation of the axis of symmetry.

d On the same axes, draw the graph of $y = 2x + 3$.

e Use your graph to solve the simultaneous equations $y = x^2 + 2x$ and $y = 2x + 3$.

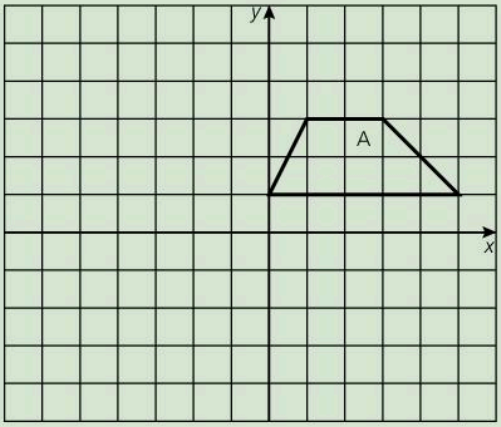
7 Solve the equations:

- **a** $5y - 3 = 27$
- **b** $11x - 3 = 4x + 18$
- **c** $\frac{x}{3} + 2 = \frac{x}{2}$

8 **a** To make pastry, flour and butter are mixed in the ratio 7 : 4.
To the nearest g, how much flour is needed to make 500 g of pastry?

b A packet of pastry contains 500 g correct to the nearest 5 g.
What is the least possible amount of pastry that could be contained in the carton?

9 **a** Trapezium A is drawn on a 1 cm square grid.
Work out the area of trapezium A.



b Rotate A 90° anticlockwise about the origin. Label the image B.

c Reflect B in the x axis. Label the image C.

d Describe fully the single transformation that maps A onto C.

10 **a** Write 56 as a product of prime factors.

b **i** Find the highest common factor (HCF) of 56 and 70.

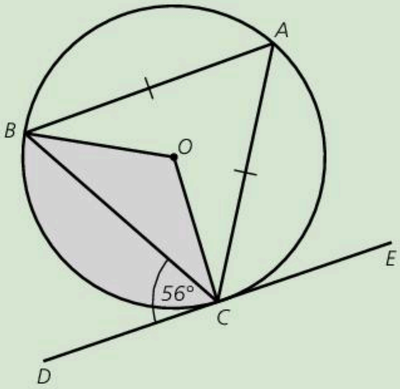
ii Find the lowest common multiple (LCM) of 56 and 70.

11 a varies inversely with the square of b.
When $a = 5$, $b = 6$.

- **a** Find an equation connecting a and b.
- **b** Calculate a when $b = 3$.
- **c** Calculate b when $a = 45$.

12 O is the centre of a circle.
ABC is an isosceles triangle such that A, B and C lie on the circle, and AB = AC.
DE is a tangent to the circle at C.
Angle BCD = 56°.

- **a** Calculate angle BAC.
- **b** Calculate angle BOC.
- **c** Calculate angle OCA.
- **d** If the radius of the circle is 8 cm, calculate the area of the shaded sector.



Further exam practice can be found online:
www.oxfordsecondary.com/9780198414520

Answers

Module 1: Summary questions

1.1

1 **a** natural, integer, square
b irrational
c natural, integer, even, square
d natural, prime, integer
e integer
f none
2 2 and 5

1.2

1 **a** 12 **b** 42 **c** 6 **d** 6
2 **a** 60 **b** 180 **c** 90 **d** 180
3 2039

1.3

1 **a** 2.57 **b** 3
2 4.68
3 10.65

1.4

1 **a** $\frac{3}{4}$
b $13\frac{1}{2}$
c $5\frac{1}{2}$
2 7 blocks will be needed $\left(8\frac{1}{2} \div 1\frac{1}{4} = 6\frac{4}{5}\right)$

1.5

1 **a** $\frac{3}{10}$ **b** 45% **c** 0.575, $\frac{23}{40}$
2 35% of \$45 is \$15.75, \$0.75 more than $\frac{3}{8}$ of \$40
3 $\frac{1}{7}$ (28 out of 196)

1.6

1 35%
2 \$82.80
3 **a** \$184 **b** \$160

1.7

1 \$18 and \$27
2 30 000 cm^3 of cement, 90 000 cm^3 of gravel
3 1.846 … m

1.8

1 **a** 5.6×10^{-4} **b** 140 000
2 500 seconds
3 **a** 4^9 **b** 3^3 **c** 8^2

1.9

1 **a** 3.999, 4.08, 4.19, 4.2
b $\frac{2}{9}, \frac{1}{3}, \frac{7}{18}, \frac{5}{12}$
c −5, −4, −2, 7, 8
2 1, 3, 5, 7, 9
3 **a** $4n - 1$ **b** $2n + 5$ **c** $-3n + 21$

1.10

1 0
2 **a** $\frac{5}{3}$ **b** $-\frac{3}{5}$
3 **a** $5 \times 19 \times 4 = 5 \times 4 \times 19 = 380$
b $27 \times 10 = 270$

1.11

1 192
2 501
3 11
4 **a** 1000 (base 2)
b 21 (base 4)

1.12

1 The simple interest is better by \$1.45 (\$320 − \$318.55)
2 \$4900(.17)
3 $7\frac{1}{7}$ years

1.13

1 **a** 4.27 km
b 330 ml
c 2400 g
2 12–13 miles
3 **a** **i** 1055 **ii** 1313
b 138 minutes

1.14

1 \$18 785
2 \$15 028
3 \$51.75

1.15

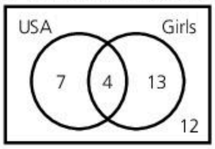
1 A = {square numbers}
B = {multiples of 3}
C = {multiples of 4}
D = {multiples of 5}
2 A and D
3 They are equal and infinite.

1.16

1 {4, 16}
2 {3, 4, 6, 8, 9, 12, 15, 16, 20}
3 12

1.17

1 Philip is $(A \cup B)'$, Cynthia is $A' \cap B$, Joel is $A \cap B'$
2 **a** B is a subset of A
b $B \cap C$ is the null set.
3



4 girls have visited the USA

Module 1: Practice exam questions

SECTION 1: Non-calculator

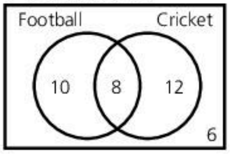
1 **a** 5 or 7 **b** 9
c 8 **d** 9
2 HCF = 14, LCM = 168
3 12.2
4 $\frac{5}{12}$
5 0.625
6 \$196
7 3^7
8 $-3\frac{1}{2}, -3.4, \sqrt{8}, 3.687, 3.7, 3\frac{3}{4}$
9 3, 8, 18, 38, 78
10 4.7×10^5
11 **a** $(a✪b)✪c = (2a \times 3b)✪c = 6ab✪c$
$= 12ab \times 3c = 36abc$
$a✪(b✪c) = a✪(2b \times 3c) = a✪(6bc)$
$= 2a \times 18bc = 36abc = (a✪b)✪c$
b $a✪b = 2a \times 3b = 6ab$
$b✪a = 2b \times 3a = 6ab = a✪b$
12 1
13 90 minutes
14 { } or ∅, {a}, {b}, {c}, {a, b}, {a, c}, {b, c}, {a, b, c}
15 36 students

SECTION 2: Calculator

1 \$38.25
2 $8\frac{1}{3}\%$
3 \$2000
4 \$207.79
5 $3n + 2$
6 $\frac{2}{3}$
7 72
8 **a** \$672 **b** \$674.92
9 \$4021.79
10

3 books	@ \$12.99 each	= \$38.97
5 pencils	@ \$0.35 each	= \$ 1.75
2 notepads	@ \$1.75 each	= \$ 3.50
2.5 litres paint	@ \$4.50/litre	= \$11.25
Subtotal:		\$55.47
Sales tax @ 15%		\$ 8.32
TOTAL		\$63.79

11 \$637
12 **a** A **b** 3
c It is empty (∅)
13



14 **a** \$17 800 **b** \$18 350
15 **a** False **b** True
c False **d** True
16 \$9610 income tax + \$3000 VAT = \$12 610 = 22.9%

Module 2: Summary questions

2.1

1 12 cm^2
2 53 km
3 1200 km^2

2.2

1 17.5 cm^2
2 8 cm
3 54 cm^2

2.3

1 Area = 113.1 cm^2, circumference = 37.7 cm
2 Diameter = 21.0 cm, radius = 10.5 cm, area = 346.6 cm^2 (all to 1 d.p.)
3 25.9 cm

2.4

1 Surface area = 52 cm^2
2 The surface area = 210π cm^2
3 The sphere and the cylinder both have a surface area of 144π cm^2.

2.5

1 84 cm^3
2 $175\pi \approx 549.8$ cm^3
3 $10\frac{1}{8}$ cm

2.6

1 2 gallons (16 pints) of water, 1.1 b (17 or 18 oz) salted beef, 8 oz breadfruit, 2 oz coco
2 8100 seconds
3 a 200 cm^2
b 20 000 mm^2
c 32 square inches

2.7

1 65 km
2 1 minute (24 minutes against 25 minutes)
3 a 3 m/s b $11\frac{1}{3}$ m/s
c $9\frac{1}{11}$ m/s

2.8

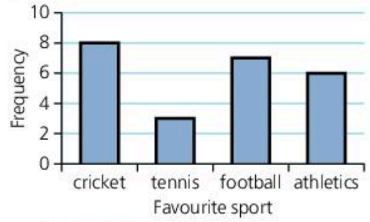
1 5880 g, 6120 g
2 14.025 m
3 4.57 cm (to 2 d.p.)

2.9

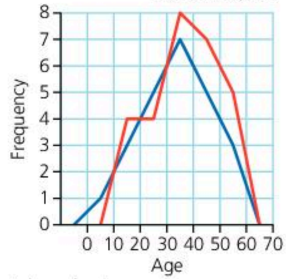
1 a Length, mass
b Favourite fruit
2 Width: 5 kg; lower boundary = 5.5 kg; upper boundary = 10.5 kg
3 12.5 cm

2.10

1



2



3 A bar chart

2.11

1 6
2 It does not show frequencies. Instead it shows proportions.
3 36

2.12

1 728 cm
2 3, 6, 6
3 The mean increases (by 0.6), the median is unchanged, and there will be no mode.

2.13

1 The modal class is 16–20
2 The estimate of the mean is 13.5
3 The median is in the class 11–15.

2.14

1 Sally mean or median; Marcus median or mode; James mean or median.

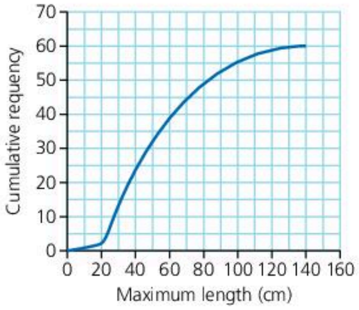
2.15

1 17 (made), 25 (sold)
2 6 (made), 11 (sold)
3 The sold data is more spread out, as shown by the larger range and interquartile range.

2.16

1

Maximum length (cm)	Cumulative frequency
20	2
40	24
60	39
80	49
100	55
120	59
140	60



2 Lower quartile = 31,
Upper quartile = 70
3 $\frac{13}{60} = 21.7\%$

2.17

1 Robert (higher mean)
2 Robert (smaller standard deviation)
3 Possibly Sarah, but you cannot be sure.

2.18

1 a $\frac{1}{8}$ b $\frac{3}{10}$
2 352
3 5 red, 3 blue, 6 black, 2 white

2.19

Teri rolls a dice and flips a coin.

1

	1	2	3	4	5	6
Head	Head, 1	Head, 2	Head, 3	Head, 4	Head, 5	Head, 6
Tail	Tail, 1	Tail, 2	Tail, 3	Tail, 4	Tail, 5	Tail, 6

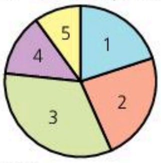
2 $\frac{3}{12} = \frac{1}{4}$
3 0.05

2.20

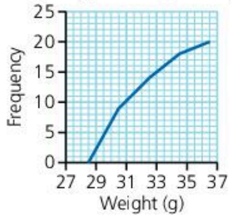
1 $\frac{15}{22}$ 2 $\frac{7}{22}$
3 Both left-handed $\frac{1}{33}$. So exactly one $= \frac{7}{22} - \frac{1}{33} = \frac{19}{66}$

Module 2: Practice exam questions

1 375 km^2
2 5 cm
3 a 235.6 cm^2 b 67.1 cm
4 235.6 cm^3
5 a 3308.1 cm^3 b 1244.1 cm^2
6 a 20 km b 9 km/h
7 3 hours 45 minutes
8 72.9 minutes, 54.4 minutes
9 1B, 2A, 3C.
10 a 171 cm–180 cm
b 170.5 cm, 180.5 cm
11



12 72°
13 9 years
14 a 29 g–30 g b 31.4 g
c



d 3.5 g
15 a

	1	2	3	4	5	6
1						
2						
3						
4						

b $\frac{4}{24} = \frac{1}{6}$
16 a $\frac{1}{12}$
b 12 red, 32 blue, 4 green
17 9 km/h
18 It is a 10 cm × 10 cm × 10 cm cube.

Module 3: Summary questions

3.1

1 $n + 2$
2 $\frac{c + 12}{2}$
3 19

3.2

1 a −1 b −6 c 4
d −18 e −1
2 a 6 b −12
c −18 d 36
3 n^2 is never negative

3.3

1 $b - a$
2 $\frac{bc + 2d}{2ac}$
3 $\frac{8b^7}{9a^2c^4}$

Answers

3.4

1 **a** $5 \clubsuit 4 = 5^2 + 4^2 = 41$
b $5 \heartsuit 4 = 5 - 4 = 1$
2 $a \clubsuit b = a^2 + b^2 = b^2 + a^2 = b \clubsuit a$
3 $(a \heartsuit b) \heartsuit c = (a - b) \heartsuit c = a - b - c$
$a \heartsuit (b \heartsuit c) = a \heartsuit (b - c) = a - (b - c)$
$= a - b + c$

3.5

1 $6ax - 12x^2 - 21x$
2 $3x - 5y$
3 $12a^2 - 7a - 12$
4 **a** $b(b + 5)$
b $y(2x - 3)$
c $9c(4c - 1)$

3.6

1 **a** $(b + 8)(b - 3)$
b $(x - 1)(x - 1) = (x - 1)^2$
2 **a** $(3c - 1)(c - 1)$
b $(5c + 4)(5c - 4)$
3 $x^2 + (2x - 3)(x + 3)$
$= x^2 + 2x^2 + 6x - 3x - 9$
$= 3x^2 + 3x - 9$
$= 3(x^2 + x - 3)$

3.7

1 $n = \dfrac{2S + 1}{3}$

2 $k = \sqrt{\dfrac{r - p}{3}}$

3 $4\pi r^2 = 6l^2$, so $r = \sqrt{\dfrac{3l^2}{2\pi}}$

3.8

1 $x = 5$
2 $x = -3$
3 $x = -1$

3.9

1 $x = -5$ or $x = 3$
2 $x = -2 \pm \sqrt{11}$, $x = -5.32$
or $x = 1.32$ (2 d.p.)
3 $x(2x + 3) = 4000$;
$2x^2 + 3x - 4000 = 0$; $x = 43.98$
or $x = -45.48$.
Field is 90.96 m × 43.98 m (2 d.p.)

3.10

1 $a = 7, b = 1$
2 $x = 1.5, y = -1$
3 I am 17 years old, my father is 39 years old.

3.11

1 $x = 1, y = 4$ or $x = 2, y = 3$
2 The rectangle is 8 cm × 5 cm
3 (1, 2)

3.12

1

h	5	10	**25**	8
k	7	**14**	35	**11.2**

2 765 g

3 729

3.13

1

w	6	9	**3**	10
p	12	**8**	24	**7.2**

2 30 cm **3** $s = 8$

3.14

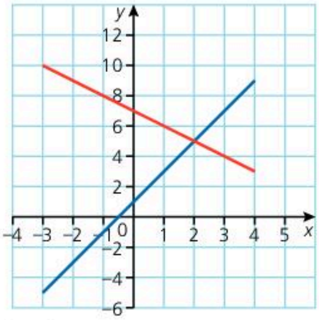
1 **a** 13 **b** 13
2 Many–one.
3 All the real numbers $\geqslant -3$

3.15

1 $y = x^3$ is a one–one function.
2 $y = x^3 - 2x$ is a many–one function, so its inverse will be one–many, and therefore not a function.
3 A possible answer is the real numbers $\geqslant -6$.

3.16

1 (0, 6) and (−2, 0)
2



3 2, −1

3.17

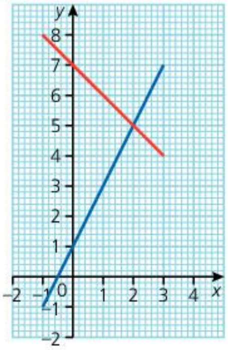
1 **a** 3
b −3
c −0.5
2 $y = x + 2$ blue; $y = 2x + 1$ red;
$x = y + 2$ green; $x + y = 2$ purple
3 $y = 2x + 4$

3.18

1 −2
2 $y = -2x + 9$
3 $y = \frac{1}{2}x + 4$

3.19

1

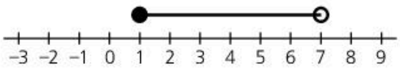


$x = 2, y = 5$

2 Length $= \sqrt{80} \approx 8.94$,
midpoint = (1, 2)
3 $y = \frac{1}{2}x + 1\frac{1}{2}$

3.20

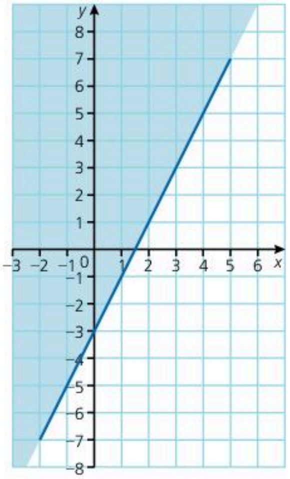
1 $x > 3.5$
2 $x \in \{R\text{: } 1 \leqslant x < 7\}$



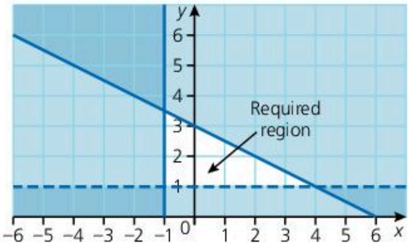
3 $y \in \{-4, -3\}$

3.21

1



2



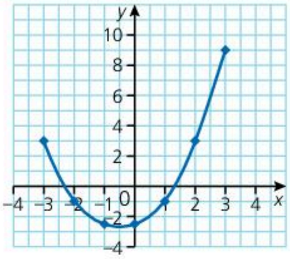
3 $b + 3p \leqslant 30$

3.22

1 $f^{-1}(x) = \dfrac{2x + 1}{3}$
2 $fg(x) = x$. $f(x)$ and $g(x)$ are inverses of each other.
3 **a** $x \neq 0$
b $f^{-1}(x) = f(x)$; $f(x)$ is self-inverse.

3.23

1



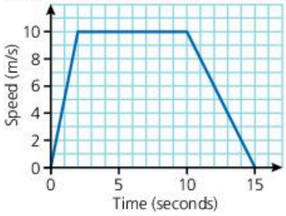
2 $y = 0.75$
3 $-2.8 < x < 1.8$

3.24

1 (0, 3)
2 $y = (x - 2)^2 - 1$; minimum point at $(2, -1)$
3 $x^2 - 4x + 3 = 0$ has solutions of $x = 1$ and $x = 3$, so the intercepts are (0, 3), (1, 0) and (3, 0)

3.25

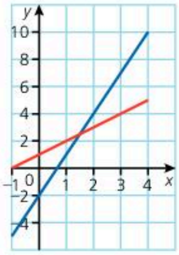
1 40 km/h
2 To the bus (8 km/h against 6 km/h)
3



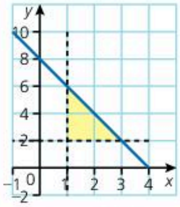
He runs 115 m altogether.

Module 3: Practice exam questions

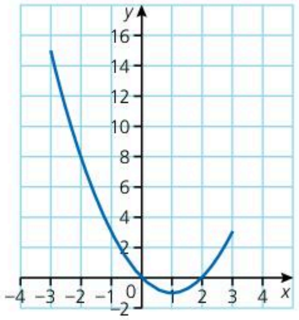
1 $\$4n + 6$ **2** 4
3 $\dfrac{7x + 3}{6}$
4 **a** 9 **b** 5
c ☼ is not commutative.
5 $2x^2 - 6xy$
6 $2(x + 5)(x - 2)$
7 $s = \dfrac{v^2 - u^2}{2a}$
8 $x = 4.5$ **9** $x = 1$ or $x = 2$
10 $x = 2 \pm \sqrt{10}$ ($x = 5.16$ or $x = -1.16$ to 2 d.p.)
11 $a = 4, b = -1$
12 12 m, 8 m **13** $x = 43.75$
14 $y = x^2 - 2$ Many–one;
$y = 3x + 1$ One–one;
$y = \sqrt{x - 3}$ One–many
15 **a** $y = \sqrt{x + 7}$
b It is a one–many relation.
16 **a, b**



c $x = 1.5$, $y = 2.5$
17 $y = -2x - 5$
18 **a** 10 **b** (1, 3)
19



20 **a** 7 **b** −7
c $fg(x) = 2x - 9$ **d** $f^{-1}(x) = \dfrac{x - 1}{2}$
21 **a**



b The exact value is 5.25
c 2
22 **a** 2 **b** 3 **c** 1
23 **a** 3.5 m/s² **b** 112 m
24 **a** 36 g
b 1.5 cm
25 $(3, 1)$ and $(-1, -3)$

Module 4: Summary questions

4.1

1 **a** z **b** y **c** x
2 **a** 132° **b** 48° **c** 297°
3 A line continues to infinity in both directions. A ray starts from a fixed point and continues to infinity in one direction.

4.2

1 CFG
2 CFE
3 **a** 132° **b** 104° **c** 124°

4.3

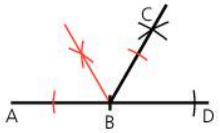
1 55° and 55° or 70° and 40°
2 **a** 150°
b It is a kite
3 An isosceles trapezium

4.4

1 114°
2 $\dfrac{360 - 90}{2} = 135°$.
3 $\dfrac{360}{180 - 144} = 10$ sides.

4.5

1 Construct CBD = 60°, so ABC = 120°.
2 Bisect ABC (shown in red).



3 Construct an angle of 90°. Adjacent to it construct an angle of 60°. Bisect the right angle, so that an angle of 45° and 60° together make 105°.

4.6

1 AC should be 11.6 cm, BC 12.3 cm
2 Use angles of 45° at the centre.

3 AE should be 8.7 cm

4.7

ABCD is a parallelogram. *EF* is parallel to *AD*.
1 $AD = BC$, $AB = CD$, (opposite sides of parallelogram), AC is common. Congruent SSS
2 $FEC = DAC$ (alternate angles),
$EFC = ADC$ (alternate angles),
$FCE = DCA$ (common)
3 7.5 cm

4.8

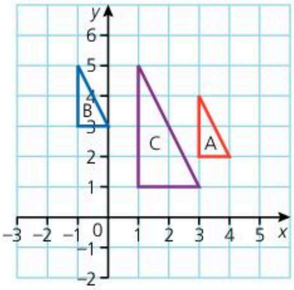
1 10 cm
2 $\sqrt{72} = 8.5$ cm (to 1 d.p.)
3 27.7 cm² (to 1 d.p.)

4.9

1 Reflection in $y = 3$
2 90° clockwise rotation about (1, 3)
3 Reflection in $y = 4 - x$.

4.10

1 Triangle B
2 Triangle C
3 Enlargement. Scale factor 2, centre (5, 3)



4.11

1 10 faces, 24 edges and 16 vertices
2 Yes; 10 + 16 = 24 + 2
3 3

4.12

1 28.3 cm **2** 11.4 cm **3** 63.6°

4.13

1 148.2 m (to 1 d.p.)
2 2.49 km (to 3 s.f.)
3 **a** The bearing of A from B = 50 + 180 = 230°
ABC = bearing of A from B − bearing of C from B
= 230 − 140 = 90°
b C is 12.1 km from A on a bearing of 115.6°

4.14

1 35.2 cm² **2** 15.2 cm **3** 27.3°

4.15

1 54.7° **2** 60.3° **3** 9.6 cm

4.16

1 8.2 cm **2** 103.8°
3 A = 55.8°, B = 51.4°, C = 72.9°

4.17

1 80° **2** 100° **3** 45°

4.18

1 13 cm **2** 64° **3** 26°

4.19

1 $\begin{pmatrix} 4 \\ 4 \end{pmatrix}$ **2** $\begin{pmatrix} -7 \\ 3 \end{pmatrix}$
3 $\sqrt{68} = 8.25$ (to 2 d.p.)

4.20

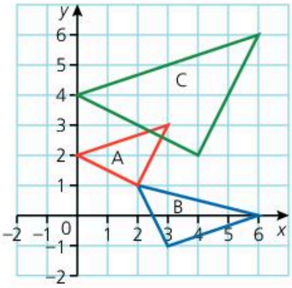
1 **a** $\mathbf{a} - \mathbf{b}$
b $2\mathbf{a} - 2\mathbf{b}$
c $\mathbf{a} - 2\mathbf{b}$
2 $\frac{2}{3}\mathbf{a} - \mathbf{b}$
3 $\overrightarrow{AX} = \overrightarrow{AB} + \overrightarrow{BX} = \mathbf{b} + \frac{2}{3}\mathbf{a} - \mathbf{b} = \frac{2}{3}\mathbf{a}$

4.21

1 **a** $\begin{pmatrix} 3 & 2 & -3 \\ 0 & -8 & 2 \end{pmatrix}$
c $\begin{pmatrix} 8 \\ -4 \end{pmatrix}$
f $\begin{pmatrix} 2 & 9 & -1 \\ 10 & -9 & 4 \end{pmatrix}$
b, **d** and **e** cannot be computed.
2 $\frac{1}{9}\begin{pmatrix} 3 & 3 \\ -1 & 2 \end{pmatrix}$
3 Yes, they both equal $\begin{pmatrix} -2 & -10 & 10 \\ 17 & 4 & 5 \end{pmatrix}$. Matrix multiplication is distributive over matrix addition.

4.22

1 and **2** on diagram
3 $\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$, an enlargement, scale factor 2, centre (0, 0)

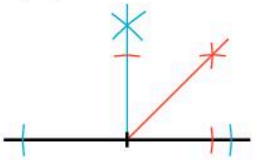


4.23

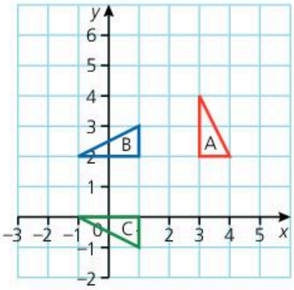
1 $a = 2, b = -1$
2 $x = 0.5, y = 1.5$
3 $\begin{pmatrix} 1 & -2 \\ 0 & 2 \end{pmatrix}$

Module 4: Practice exam questions

1 **a** 65°
b 76°
c 29°
2 12
3 38° and 104°
4 Construction should include the construction of a right angle (shown in blue), followed by bisection of right angle (red)



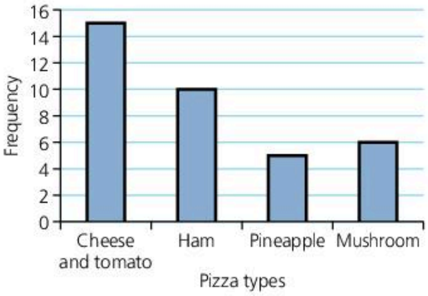
5 18.7 feet (to 1 d.p.)
6 **a**, **b** See diagram.
c Reflection in $y = 3 - x$



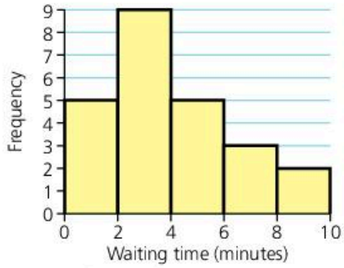
7 **a** 13 faces, 24 edges, 13 vertices.
b 6
8 77.2° (to 1 d.p.)
9 **a** 11.1 cm (to 1 d.p.)
b 78.6° (to 1 d.p.)
c 59.8 cm² (to 1 d.p.)
10 7.1° (to 1 d.p.)
11 123.7° (to 1 d.p.)
12 **a** 75° (alternate segment theorem)
b 65° (angle at centre = 2 × angle at circumference)
c DBO = DAO = 90° (angle between tangent and radius)
So opposite angles add up to 180°, so AODB is a cyclic quadrilateral.
13 $\overrightarrow{AE} = \overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CE} = \mathbf{a} + \mathbf{b} - 2\mathbf{a}$
$= \mathbf{b} - \mathbf{a}$
$\overrightarrow{CD} = \overrightarrow{CE} + \overrightarrow{ED} = -2\mathbf{a} + \mathbf{a} + \mathbf{b}$
$= \mathbf{b} - \mathbf{a}$
$\overrightarrow{AE} = \overrightarrow{CD}$, so AE and CD are equal and parallel.
14 $\begin{pmatrix} -5 \\ 3 \end{pmatrix}$
15 **a** **i** $\begin{pmatrix} -1 & 5 \\ -1 & -1 \end{pmatrix}$
ii $-\frac{1}{2}\begin{pmatrix} -1 & -1 \\ 0 & 2 \end{pmatrix}$
iii $-\frac{1}{3}\begin{pmatrix} 1 & -2 \\ -1 & -1 \end{pmatrix}$
iv $\frac{1}{6}\begin{pmatrix} -1 & -5 \\ 1 & -1 \end{pmatrix}$
b $\mathbf{B}^{-1}\mathbf{A}^{-1} = -\frac{1}{3}\begin{pmatrix} 1 & -2 \\ -1 & -1 \end{pmatrix}$
$\times -\frac{1}{2}\begin{pmatrix} -1 & -1 \\ 0 & 2 \end{pmatrix}$
$= \frac{1}{6}\begin{pmatrix} -1+0 & -1 & -4 \\ 1+0 & 1 & 2 \end{pmatrix}$
$= \frac{1}{6}\begin{pmatrix} -1 & -5 \\ 1 & -1 \end{pmatrix}$
$= (\mathbf{AB})^{-1}$

Further exam practice 1

1 **a**



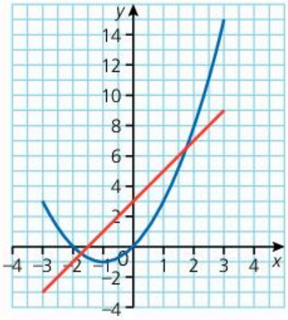
b 9
c 100°
2 **a** **i** 5
ii 8.5
iii 8
iv 9.25
b 50%
3 **a** **i** $2x$
ii $7a - 3b$
b 6
4 **a** 240 cm³
b 248 cm²
c Any three lengths that multiply to 240, for example 1 cm × 1 cm × 240 cm, or 5 cm × 6 cm × 8 cm
5 **a**



b 2 to 4 minutes
c $\frac{7}{12}$
d **i** $\frac{5}{8}$
ii 9
6 **a**

x	−3	−2	−1	0	1	2	3
x^2	9	4	1	0	1	4	9
$+ 2x$	−6	−4	−2	0	2	4	6
$y = x^2 + 2x$	3	0	−1	0	3	8	15

b, **d**



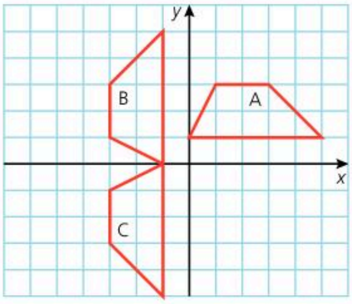
c **i** −1 **ii** $x = -1$

e $x = -1.7$, $y = -0.5$ or $x = 1.7$, $y = 6.5$

7 **a** $y = 6$
b $x = 3$
c $x = 12$

8 **a** 318 g **b** 497.5 g

9 **a** 7 cm^2
b, **c**



d Reflection in $y = -x$

10 **a** $2 \times 2 \times 2 \times 7$ or $2^3 \times 7$
b **i** 14
ii 280.

11 **a** $a = \dfrac{180}{b^2}$
b $a = 20$
c $b = 2$

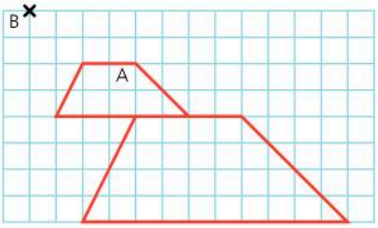
12 **a** 56°
b 112°
c 28°
d 62.6 cm^2 (to 3 s.f.)

Further exam practice 2

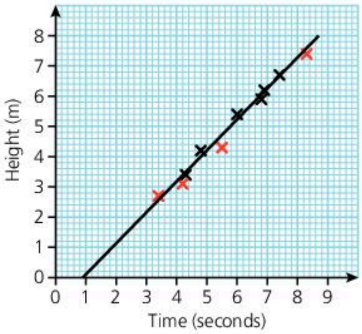
1 **a** 8
b 63
c $6\frac{2}{3}$

2 **a** 81 km/h
b 1325

3



4 **a**, **c**



b Strong, positive correlation
d 5.9 seconds

5 **a** 133.7 cm
b 1147.9 cm^2

6 \$49.82

7 $\frac{5}{12}$

8 **a** \$787.40
b 12.5%

9 **a**

	Male	**Female**
Youngest	20	**18**
Lower quartile	24	**23**
Median	27	25
Upper quartile	**33**	34
Oldest	56	51
Interquartile range	9	11
Range	**36**	33

b **i** True
ii False
iii False
iv You cannot tell.

10 **a** $x = 17$, $y = -3$
b $m = a^2 + 3$

11 **a** 11.6 cm
b 34.2°
c 15.1 cm^2
d 5.2 cm

12 $x = 3$

13 $\begin{pmatrix} 6 & -14 \\ -1 & 4 \end{pmatrix}$

14 **a** **i** $\{1, 3, 4, 5, 7, 9\}$
ii $\{3, 5, 7\}$
iii $\{4, 6, 8, 10\}$
b $A \cap (B \cup C) = \{1, 3, 5, 7, 9\} \cap \{1, 2, 3, 4, 5, 7, 9\} = \{1, 3, 5, 7, 9\}$
$(A \cap B) \cup (A \cap C) = \{1, 9\} \cup \{3, 5, 7\} = \{1, 3, 5, 7, 9\}$

15 **b** A = 88°, B = 52° and C = 40°.

16 **a** **i** 2**b**
ii 3**a**
iii **b**
iv 2**a**
b $\overrightarrow{CE} = 2\mathbf{a}$
$\overrightarrow{EA} = \overrightarrow{ED} + \overrightarrow{DA} = \mathbf{b} + \mathbf{a} - \mathbf{b} = \mathbf{a}$
$\overrightarrow{CE} = 2\overrightarrow{EA}$

17 **a** $2x^2 - 3$
b $(2x - 3)^2 = 4x^2 - 12x + 9$
c $\dfrac{x + 3}{2}$

Further exam practice 3

1 5×10^5

2 **a** **i** $3x(x - 7)$
ii $(x + 4)(x - 4)$
b $4x^2 - 4x - 3$

3 **a** \$16 000
b \$17 000

4 (−2, 0) and (1.5, 7)

5 20.25

6

	3	4.5	$\sqrt{5}$	$\sqrt{4}$	−3
Real number	✓	✓	✓	✓	✓
Integer	✓	✗	✗	✓	✓
Natural number	✓	✗	✗	✓	✗
Prime number	✓	✗	✗	✓	✗
Rational number	✓	✓	✗	✓	✓

7 **a** \$42
b 4.8% loss
c \$55

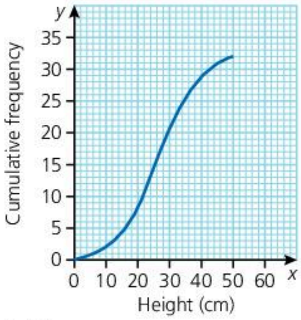
8 $2.5 < x \leqslant 4$

9 **a** **i** 7
ii 5
iii 13
iv 19
v 4
b $x > 1$

10 **a** **i** $\begin{pmatrix} 5 \\ 0 \end{pmatrix}$
ii $\begin{pmatrix} 5 \\ -3 \end{pmatrix}$
iii $\begin{pmatrix} 14 \\ -7 \end{pmatrix}$
b $5\mathbf{c} - \mathbf{b} = \begin{pmatrix} 14 \\ -7 \end{pmatrix} = 3.5\begin{pmatrix} 4 \\ -2 \end{pmatrix} = 3.5\mathbf{a}$

11 **a** $x = 3$
b (3, −2)
c $x = 3 \pm \sqrt{2}$ or $x = 1.59$ or $x = 4.41$ (to 3 s.f..)

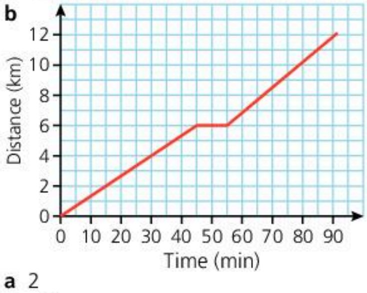
12 **a**



b **i** 26 cm
ii 34 − 20 = 14 cm

13 a 12.1 km

b



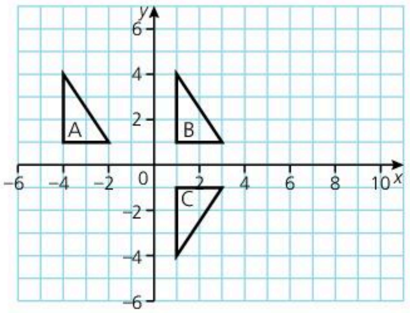
14 a 2

b -3

c $y = 2x - 2$

d $y = -\frac{1}{2}x - 2$

15 a, b

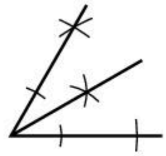


c $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

Further exam practice 4

1 $a = 114°$, $b = 43°$, $c = 55°$

2



3 a For every value of x, $y = x^2 - 4$ produces just one value of y. (It is a many–one mapping.)
But $y = \sqrt{x} - 4$ has values of x for which y is undefined (e.g. $x = -2$), and is also a one–many mapping (e.g. $x = 9$ gives $y = -1$ or -7), and so is not a function.

b A domain might be $\{R: x \geqslant 0\}$.
A codomain might be $\{R: y \geqslant -2\}$

4 a $\sqrt{89} \approx 9.4$ units

b $(4.5, -1)$

5 2 is even and prime and so the intersection is not empty.

6 a 30–40

b 37.8

7 a 2736.95 krone

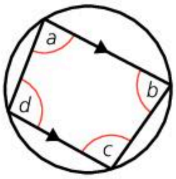
b \$40.19

8 a
$$\begin{aligned}(a\clubsuit b)\clubsuit c &= 2ab\clubsuit c\\ &= 2 \times 2abc\\ &= 4abc\end{aligned}$$
$$\begin{aligned}a\clubsuit(b\clubsuit c) &= a\clubsuit 2bc\\ &= 2 \times 2abc\\ &= 4abc\end{aligned}$$

b $a\heartsuit b = 2a - b$
$b\heartsuit a = 2b - a$, so it is not commutative.

c
$$\begin{aligned}a\clubsuit(b\heartsuit c) &= a\clubsuit(2b - c)\\ &= 2a(2b - c)\\ &= 4ab - 2ac\end{aligned}$$
$$\begin{aligned}(a\clubsuit b)\heartsuit(a\clubsuit c) &= 2ab\heartsuit 2ac\\ &= 2 \times 2ab - 2ac\\ &= 4ab - 2ac\end{aligned}$$
So ♣ is distributive over ♥.

9 $a + c = 180$ (opposite angles of cyclic quad)
$a + d = 180$ (co-interior angles on parallel lines)
So $d = c = 180 - a$
Similarly, $a = b = 180 - c$.
So the trapezium is isosceles.



10 223 (base 6)

11 a $x + 2y$

b $a + 12b$

c $12a^5$

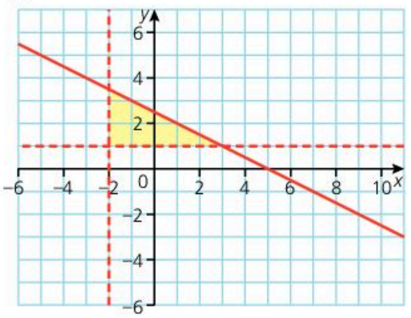
d $\frac{2b^2}{3a^2} = \frac{2}{3}b^2a^{-2}$

12 $\frac{x + 2}{2}$

13 a $\sqrt{40} \approx 6.3$ cm

b 19.1 cm^2 (to 1 d.p.)

14



15 23 minutes

16 a i 24

ii 44

iii 80

b $(a + 1)^2 - (a - 1)^2$
$= a^2 + 2a + 1 - (a^2 - 2a + 1)$
$= 4a$
$101^2 - 99^2 = 4 \times 100 = 400$

17 a i 64°

ii 64°

b 12 cm

18 a $\begin{pmatrix} 1 & 1 \\ 4 & -3 \end{pmatrix}$

b $\begin{pmatrix} 2 & 4 & 6 \\ 4 & 0 & -2 \end{pmatrix}$

19 a i 72°

ii 54°

b Each isosceles triangle can be split into two right-angled triangles.
Each right-angled triangle has base 4 cm and height of 4 tan 54°.
So each right-angled triangle has an area of $\frac{1}{2} \times 4 \times 4 \tan 54° = 8 \tan 54°$
There are 10 such triangles (two in each isosceles triangle), so total area $= 10 \times 8 \tan 54° = 80 \tan 54°$

20 a 14

b 36

c 24

Index

Study Guide

Mathematics for CSEC® 2nd edition

Less Stress, More Success

Developed exclusively with the Caribbean Examinations Council®, this Study Guide will provide you with additional support to maximise your performance in CSEC® Mathematics.

Written by a team of experts in the CSEC® Mathematics syllabus and examination, this Study Guide covers all the essential information in an easy-to-use double-page format. Each topic begins with the key learning outcomes from the syllabus and contains a range of features designed to enhance your study of the subject, such as:

- Examination-style practice questions to build confidence ahead of your examination
- Engaging activities that help you develop the analytical skills required for the examination
- Examination tips with essential advice on succeeding in your assessments
- Did You Know? boxes to expand your knowledge and encourage further study

This Study Guide is supported by additional online material, incorporating multiple-choice questions and examination-style questions with sample answers and examiner feedback, to aid revision and illustrate what is required to achieve top marks in CSEC® Mathematics examination.

The Caribbean Examinations Council (CXC®) has worked exclusively with Oxford University Press to produce a series of Study Guides across a wide range of subjects at CSEC® and CAPE®.

OXFORD
UNIVERSITY PRESS

How to get in touch:
web www.oup.com/caribbean
email schools.enquiries.uk@oup.com
tel +44 (0)1536 452620
fax +44 (0)1865 313472

ISBN 978-0-19-841452-0