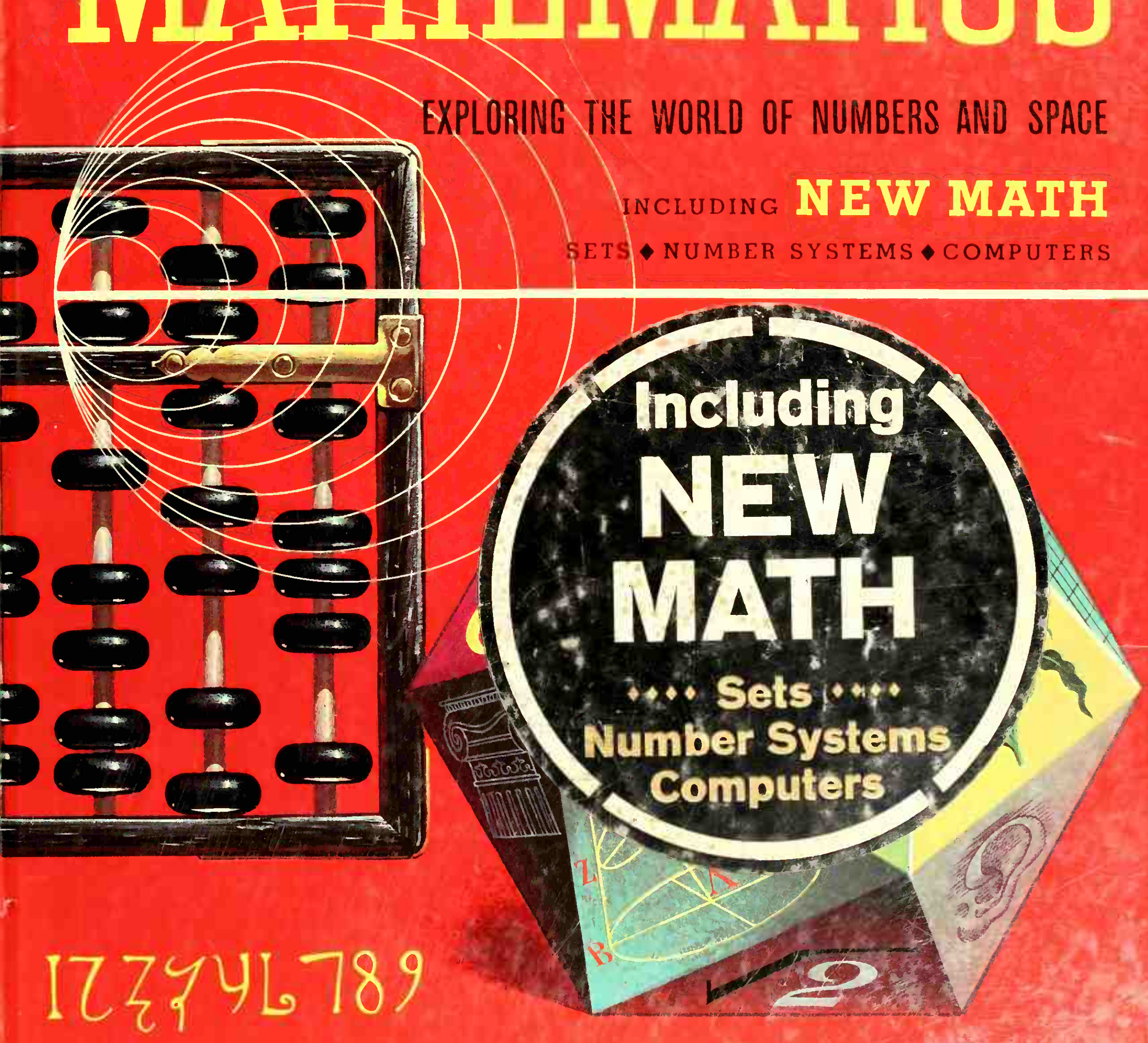
THE GIANT GOLDEN BOOK OF

MATHEMATICS

EXPLORING THE WORLD OF NUMBERS AND SPACE

INCLUDING **NEW MATH**

SETS ◆ NUMBER SYSTEMS ◆ COMPUTERS



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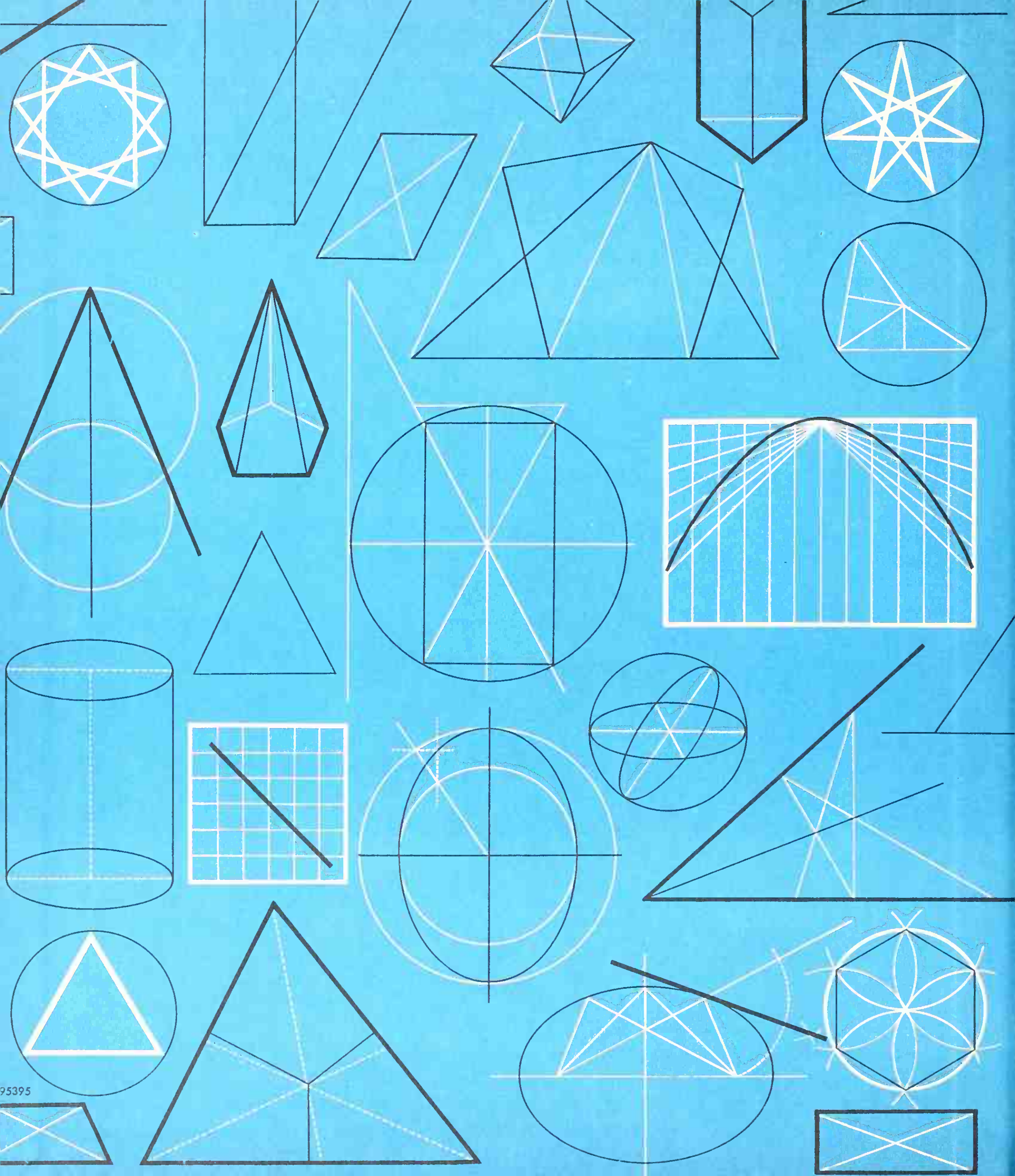

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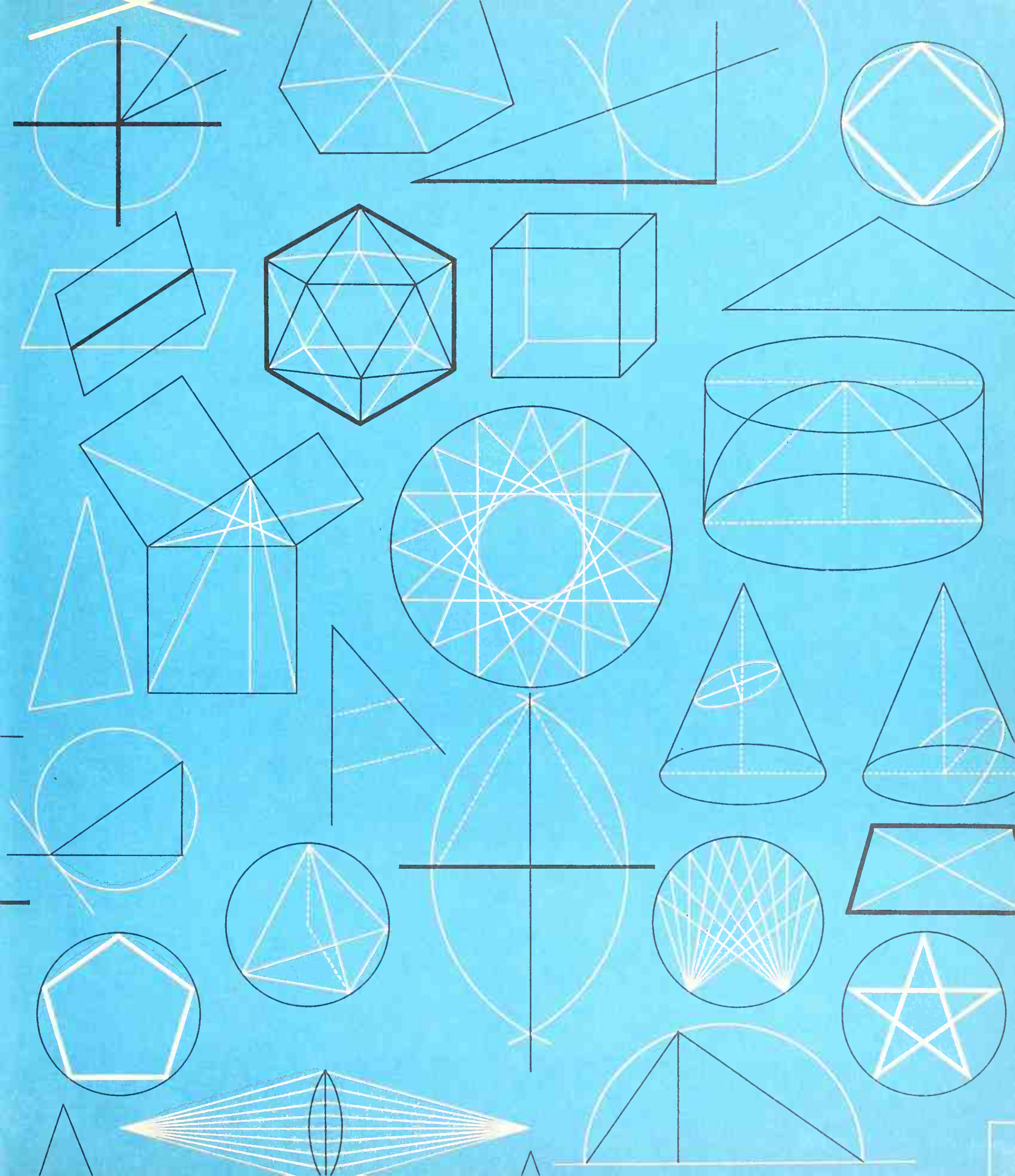

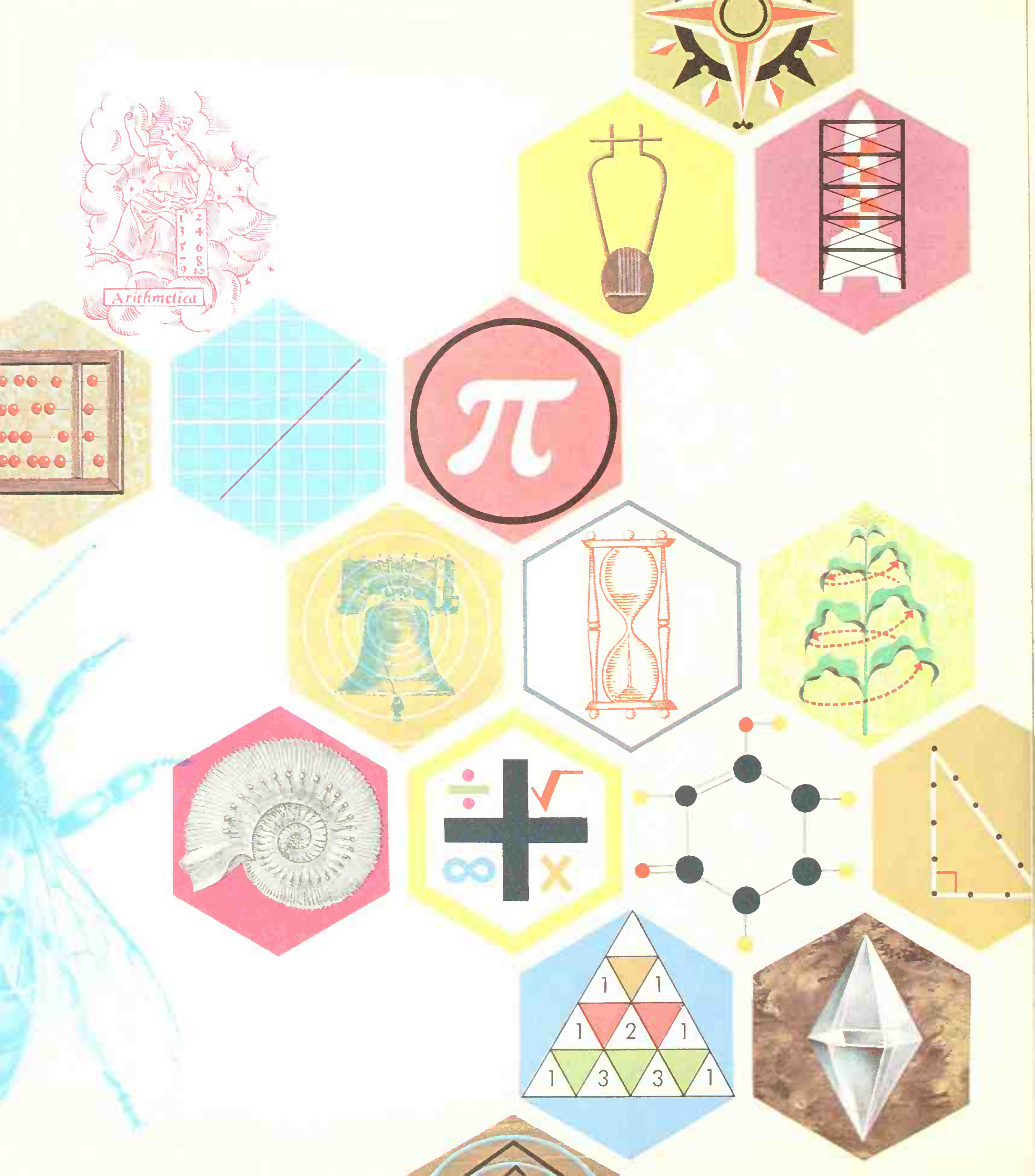
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THE
GIANT GOLDEN BOOK OF
MATHEMATICS

Exploring the World of Numbers and Space

by **IRVING ADLER, Ph.D.**

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with a foreword by **HOWARD F. FEHR**

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Mathematics is a world of wonder—a place where, with only a few numbers and points at our command, the most amazing formulas and geometric figures appear as out of a magician's hat. Mathematics is also a tool—a servant to our needs. When we wish to know how much? how many? how large? how fast? in what direction? with what chances?—the mathematician gives us a way to find the answer.

But above all mathematics is the Queen of Knowledge. It has its own logic—that is, a way of thinking. By applying this way of reasoning to numbers and to space, we can come up with ideas and conclusions that only the human mind can develop. These ideas often lead us to the hidden secrets of the ways in which nature works.

All this is revealed in the pages that follow, and I am happy to invite you to a glorious adventure in numbers and space as you read the words and study the beautiful pictures in this delightful book.

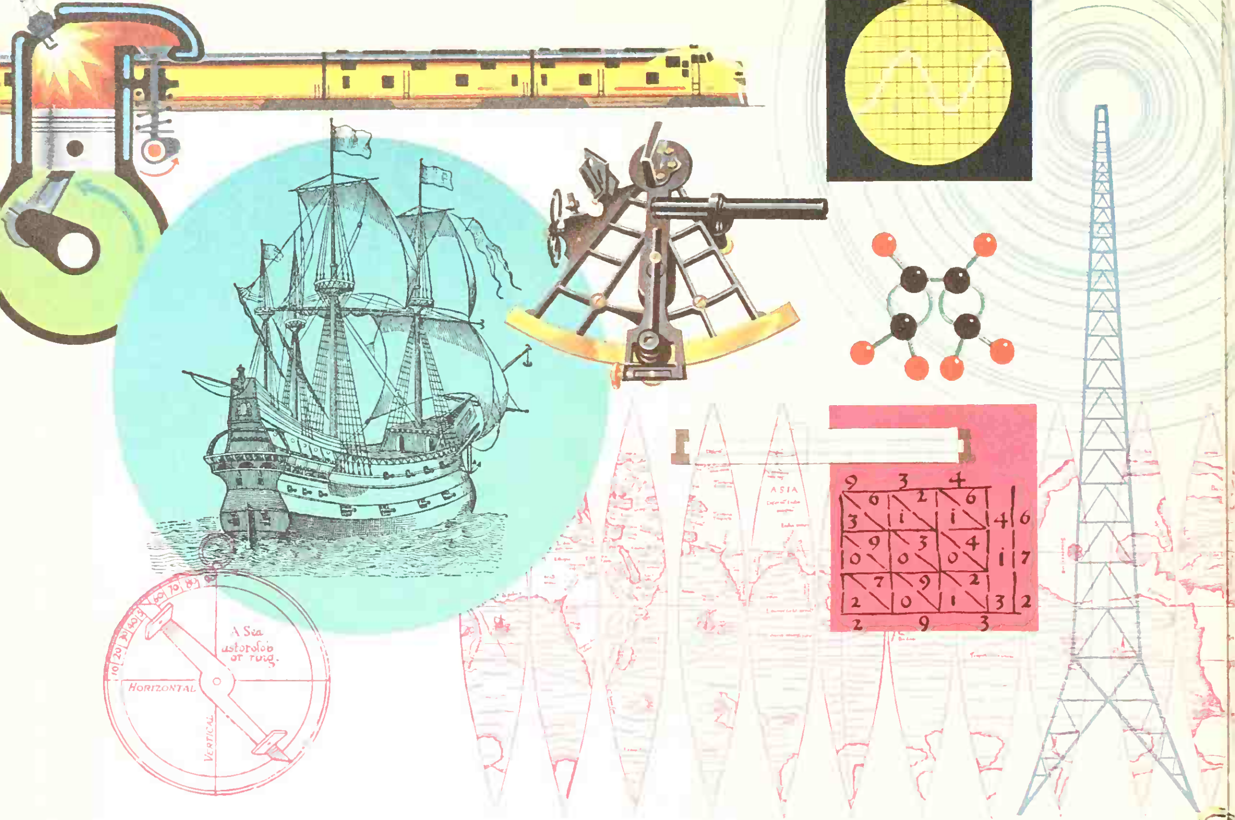
This book does not teach you ordinary arithmetic as you study it in school. It does tell you the extraordinary things that come from the use of what you study. It unfolds the story of man's struggle to explain the quantitative aspects of the world in which he lives. It tells the story of exceedingly small numbers, and numbers so large as to be beyond comprehension—from the infinitesimal—to the infinite. It takes you from a point, along a line, into a plane, out into space, and even beyond our space. And it shows how space itself was finally conquered by number.

This book also deals with practical things, such as how to make parallel lines and perpendicular lines, and where they are used. It describes the angles that a surveyor needs to know, and shows how to discover the speed at which a stone is falling or a rocket is traveling in space. How to find an area, or a volume, or how by the use of probability to predict one's chances of winning a game, are simply explained. But even more astounding is the unfolding of seemingly magical numbers for the interpretation of nature—a sea shell—a growing tree—a beautiful rectangle—the golden section. The arts of music and painting become the mathematics of harmonics and perspective, and the behavior of our entire universe is revealed as a mathematical system.

This is a book for inquisitive minds—those of young readers—and bright adults also—which if read and reread, each new time with more careful thought and study, will pay rich dividends in intellectual satisfaction. Each topic is only an initial episode that, if pursued by further study in school or other books, will reveal a knowledge on which the world of tomorrow is being built.

Because I teach this subject and train teachers to teach this subject; because I enjoy all mathematics to the utmost; and because I know the pleasure it gives—I welcome you to the pages that follow.

—HOWARD F. FEHR
Teachers College, Columbia University
Professor of Mathematics



The Science of Numbers and Space

At work or at play, we often have to answer questions like "How many?" "How big?" or "How far?" To answer such questions we have to use numbers. We have to know how numbers are related and how different parts of space fit together. To be sure our answers are correct, we try to think carefully. When we do these things, we are using mathematics.

Mathematics is the science in which we think carefully about numbers and space. It helps us keep score at a baseball game, measure the area of a floor, or decide which purchase is a better buy. It helps the engineer design a machine. It helps the scientist explore the secrets of nature. It supplies us with useful facts. It shows us short cuts for solving problems. It helps us understand the world we live in. It also gives us games and puzzles that we can do for fun.

Mathematics and Civilization

Mathematics grew up with civilization. It arose out of practical problems, and it helps people solve these problems.

In the days when men got their food by hunting, and gathering wild fruits, berries, and seeds, they had to count to keep track of their supplies. Counting, measuring, and calculating became more important when people became farmers and shepherds. Then people had to measure land and count their flocks.

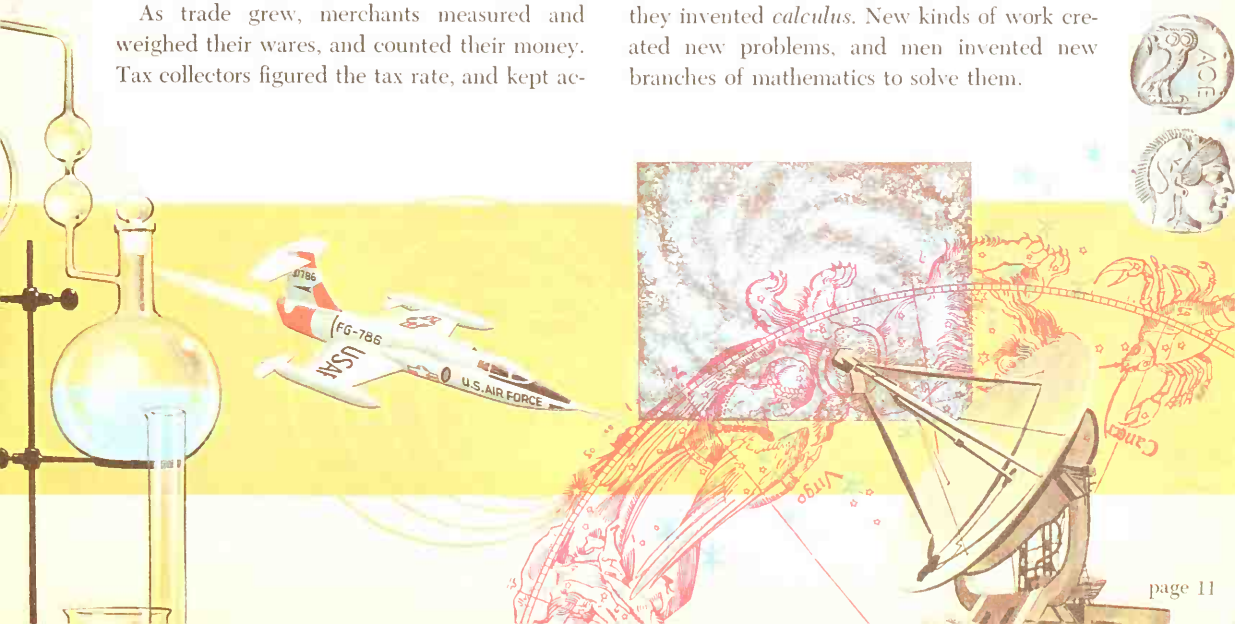
When they built irrigation dams and canals, they had to figure out how much earth to remove, and how many stones and bricks they would use. The overseers had to know in advance how much food to store up for the working force. Carpenters and masons had to measure and calculate as they built homes for the people, palaces for their rulers, and great tombs for their dead kings.

As trade grew, merchants measured and weighed their wares, and counted their money. Tax collectors figured the tax rate, and kept accounts. To deal with all these activities, men invented *arithmetic*, which studies numbers, and *geometry*, which studies space.

To predict the changes of the seasons, priests studied the motions of the sun, moon and stars. Navigators looked to the sky, too, for the stars that guided them from place to place. To help them in this work, men invented *trigonometry*, which relates distances to directions.

Commerce spread over the world. The same kinds of calculations often had to be repeated. To save time, some people worked out rules for doing them, and ways of doing many problems at once. This was the beginning of *algebra*.

As the centuries went by, men built machines and workshops. Scientists studied the earth, the sea, the air, and the sky. In these activities, people worked with things that move or change. To think accurately about motion and change, they invented *calculus*. New kinds of work created new problems, and men invented new branches of mathematics to solve them.



Numbers and How We Write Them

In the scene above, a team of primitive hunters has just killed some game with well-aimed arrows. The hunters can see at a glance that the set of animals killed doesn't match the set of men in the team. A man is left without an animal, so the hunters conclude that there are *more* men in the team than there are animals in the catch. Matching sets of objects in this way probably led to man's first mathematical ideas, the ideas of *more* and *less*.

We have many words in our language that grew out of our experience with trying to match sets. We distinguish a *single* person from a *couple*. A *lone* wolf is different from a *pack*. We also talk about a *pair* of socks or a *brace* of ducks. Words like single, couple, lone, pack, pair, and brace answer the question "How many?" At first this question used to get mixed up with the question "What kind?" So separate words like couple, pair, and brace were used to describe different kinds of objects.

But people soon learned that a couple of people matches a pair of socks or a brace of ducks, and that the matching has nothing to do with the kinds of things that they are. They realized that a couple, a pair, and a brace have something in common that makes it possible for them to match. This is how the idea of number arose. Today we use the number word *two* to answer the question "How many?" for any set that matches a couple, no matter what kind of objects are in the set.

Numbers were used long before there was any need to write them. The earliest written

An early form of the Arabic numerals

numbers we know about are found in the temple records of ancient Sumeria. Here priests kept track of the amount of taxes paid or owed, and of the supplies in the warehouses.

As time passed, men invented new and better ways of writing numbers. At first, men wrote them by making notches in a stick, or lines on the ground. We still use this system when we write the Roman numerals I, II, and III. We find it hidden, too, in our Arabic numerals 2 and 3. They began as sets of separated strokes. Then, when the strokes were written in a hurry, they were joined to each other.

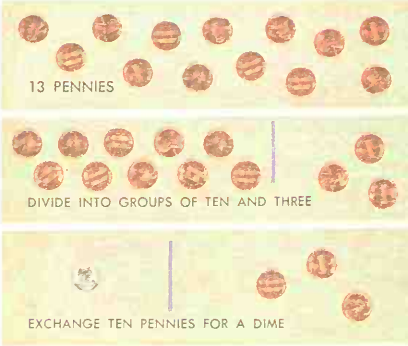
The Arabic numbers use only ten symbols, the digits 0, 1, 2, 3, 4, 5, 6, 7, 8, 9. But, with these ten digits, we can write down any number we like. We do this by breaking large numbers into groups, just as we do with money. We can separate thirteen pennies into groups of ten and three. We can exchange the ten pennies for a dime. Then we have *one* dime and *three* pennies. To write the number thirteen, we write 13. The 1 written in the second space from the right means one group of ten, just as one dime means one group of ten pennies.

Ancient records written in clay

If we have only ten pennies, they form a *group* of ten, with no additional pennies left over. To write the number ten, we put a 1 in the second space from the right to represent one group of ten. But to recognize this space as the *second* space, we must write something in the *first* space, even though there are no additional pennies beyond the group of ten. We write the digit 0 to represent "no pennies." If we didn't use the symbol 0 in this way, the whole system would not work.

The first people who recognized that they needed a symbol for the number zero were the people of ancient India. The Arabs learned it from the Indians, and then built it into the system of written numbers that we use today.

EGYPTIAN	𓏺	𓏻	𓏼	𓏽	𓏾	𓏿	𓐀	𓐁	𓐂	𓎆	𓍢
BABYLONIAN	𒁹	𒈫	𒐈	𒐉	𒐊	𒐋	𒐌	𒐍	𒐎	𒌋	𒁹𒐏
EARLY ROMAN	I	II	III	IIII	V	VI	VII	VIII	IX	X	C
CHINESE	一	二	三	四	五	六	七	八	九	十	百
HINDU	१	२	३	४	५	६	७	८	९	१०	१००
MAYAN	𝋡	𝋢	𝋣	𝋤	𝋥	𝋦	𝋧	𝋨	𝋩	𝋪	𝋥 𝋠
MODERN NUMERALS	1	2	3	4	5	6	7	8	9	10	100



Japanese merchants add and multiply on the Soroban, a kind of abacus

The Egyptians used the royal cubit as a unit of measurement

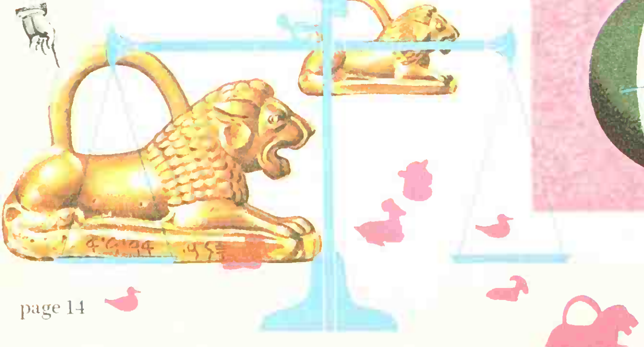
EGYPTIAN MEASURING JAR

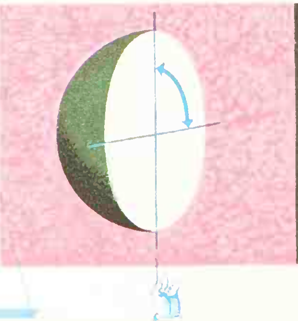
Standards and Measures

In ancient Egypt, a carpenter couldn't misplace his ruler, because it was attached to his body. The units of length that he used were on his arm. The *cubit* was the distance from the tip of his elbow to the tip of his middle finger. The *digit* was the width of one finger. These units fit together like this: Four digits equal one *palm*. Seven palms equal one cubit.

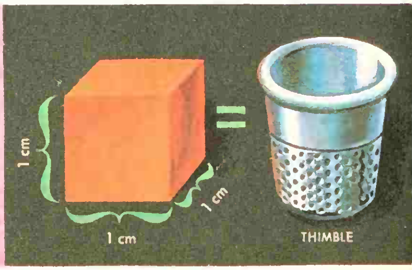
The units of length we use in the United States and England were once parts of the body, too. The *foot* was the length of a man's foot. The *inch* was the width of a thumb. The *yard* was the distance from a man's nose to the tip of his outstretched arm. If a customer with long arms bought cloth from a merchant with short arms, there was an argument about who should measure out the cloth. To avoid such arguments people began using *standard* units. The length of a standard unit is fixed by the government.

EARLY BRONZE WEIGHTS





Distance around meridian = four quadrants
One quadrant = ten million meters



The Metric System

In most countries of the world today the standard units of measurement are those of the *metric system*. The metric system was first adopted in France in 1795, and then spread to other countries. In this system, the standard units are based on measurements of the *earth* and *water*. The unit of length is called a *meter* and was derived from the distance around the earth in this way: A circle drawn on the surface of the earth through the north and south poles is called a *meridian*. One fourth of a meridian is called a *quadrant*. A quadrant was divided into ten million equal parts. The length of one of these parts was chosen to be a meter.

After scientists made careful measurements to find out how long the meter is, they measured its length between two scratches on a platinum bar. This bar, kept in a vault in Paris, is the official standard of length. A meter is about 39.37 inches long. For measuring small distances, it is subdivided into one hundred equal parts. Each part is called a *centimeter*. There are about 2½ centimeters in an inch.

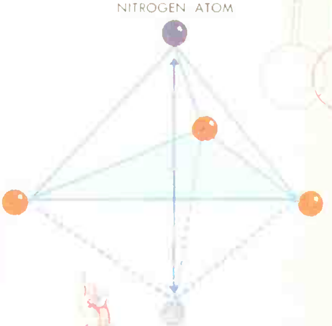
The unit of volume in the metric system is the *cubic centimeter* (abbreviated as cc.). It is the volume of a cube that is one centimeter high. A thimbleful is nearly equal to 1 cc. The unit of mass in the metric system is called a *gram*. It was chosen to be the mass of 1 cc. of water. One pound contains about 454 grams.

Units of Time

We use many different units of time. Each of them is based on some rhythm in nature, in which an interval of time is repeated over and over again. The *year* is based on the rhythm of the earth's motion around the sun. The *month* is based on the rhythm of the moon's motion around the earth. The *day* is based on the rhythm of the earth's rotation around its axis. The smaller units that we call an hour, a minute, and a second are obtained by subdividing the average length of a day.

We used to think of the day and its subdivisions as the best standard units, because the rotation of the earth was the most regular rhythm we knew about. We now have better units, based on very rapid rhythms inside molecules or atoms. These are used in molecular or atomic clocks. One, the ammonia clock, uses as its unit of time the period of a vibration inside an ammonia molecule. This period is so small that there are 23,870 million vibrations a second. With a molecular clock we can measure irregularities in the spinning of the earth.

Above are three rhythms in nature on which we base units of time.
One year = time of one round trip of the earth around the sun.
One month = time of one round trip of the moon around the earth.
One day = time of one complete turn of the earth around its axis



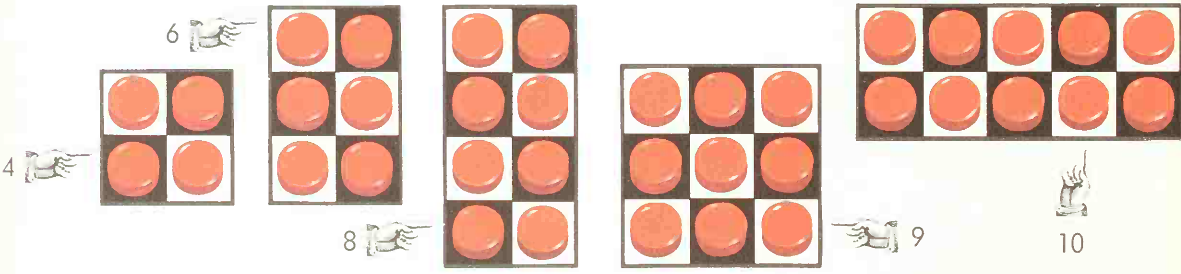
In the ammonia molecule, there are 23,870 million vibrations per second

Numbers We Cannot Split

You can make a "picture" of a whole number by using a line of checkers. To form the picture, use as many checkers as the number tells you to.

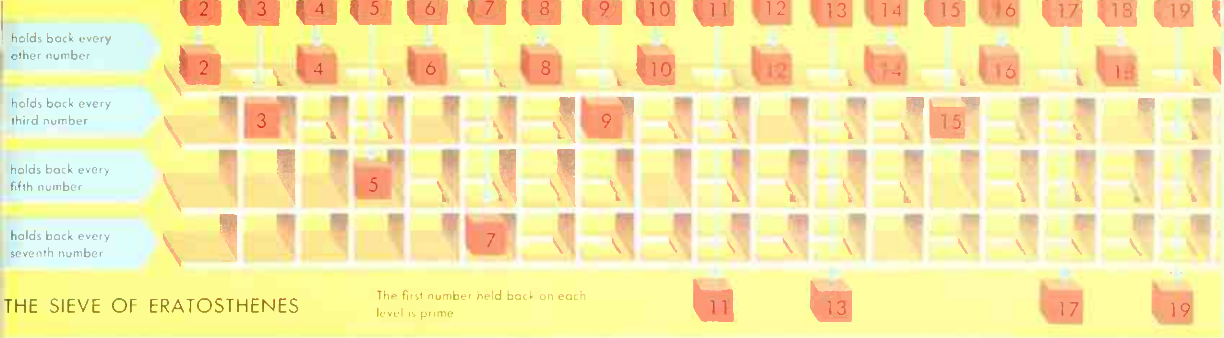
A line of four checkers can be split into two lines with two checkers each. If we put these lines under each other, the checkers form a rectangle. Rectangles can also be formed with 6, 8, 9 or 10 checkers. So we call these numbers *rectangle numbers*. The rectangle for the number 10 has 2 lines that have 5 checkers in each line. Notice that 2 × 5 = 10. *Every rectangle number is the product of smaller numbers.*

There are some numbers that cannot be split in this way. For example, we cannot arrange 7 checkers in a rectangle. We can arrange them in seven lines, with one checker in each line. But then they are still arranged in a single line, only now the line runs up and down instead of going from right to left. The number 7 is not a rectangle number. Numbers that cannot be pictured as rectangles are called *prime* numbers. This is because they cannot be written as the product of smaller numbers.



There is a simple way of finding out whether a number is a rectangle or prime number. This method is called the sieve of Eratosthenes, after the Greek scientist who devised the system, two centuries before the birth of Christ. Imagine all the whole numbers, starting with 2, arranged in order in a line. The number 2, which stands at the head of the line, is a prime number. Now count by 2's, and cross out every number you get. This removes the number 2, and all multiples of 2. They are numbers like 4, 6, 8, and so on, that form rectangles with two lines. Among the numbers that are left, the number 3 now stands at the head of the line. It is the next prime number. Now cross out the numbers you get when you count by 3's. They are numbers like 9 and 15, that form rectangles with three lines. Among the numbers that are left, the number 5 now stands at the head of the line. It is the third prime number.

Continue in this way, removing from the line the number at the head of the line, and all multiples of that number. After each family of



THE SIEVE OF ERATOSTHENES

The first number held back on each level is prime

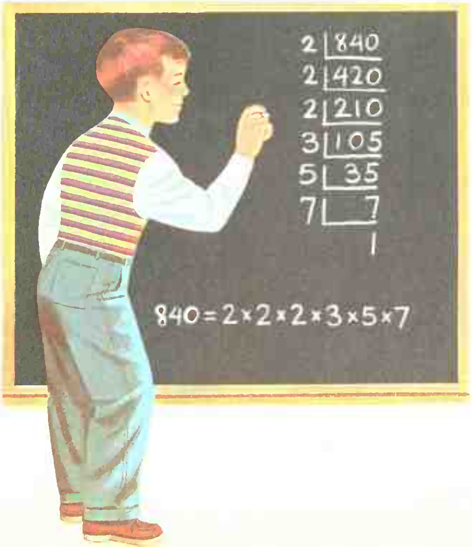
numbers is removed, the number that stands at the head of the line is the next prime number.

Every rectangle number can be written as the product of prime numbers. Thus, 6 = 2 × 3. In some cases a prime may have to be used as a multiplier or *factor* more than once. For example, 12 = 2 × 2 × 3.

To find the prime factors of a number, first try to divide it by the smallest prime number, 2. If the division comes out even, then try to divide 2 into the quotient. Keep on dividing by 2 until you get a quotient that 2 does not divide into. Then try dividing by 3. Continue in this way, using larger and larger prime numbers as divisors, until you get 1 as a quotient. The divisors you used in those cases where the divisions came out even will be the prime factors of the number. The example on the blackboard shows how to find the prime factors of 840.

How many prime numbers are there? This question was answered over two thousand years ago by the Greek mathematician Euclid. He proved that the number of primes is *unlimited*, by showing that no matter how many primes you find by the sieve of Eratosthenes, there are always some primes that are larger than those you have found. In fact, he said, multiply all the prime numbers up to and including the last one you found, and then add 1. If the number you get is prime, then it is a larger prime than those you had found before. If it is not prime, then it has prime factors. But none of the primes you had found before are factors, because when you divide any of them into the number, there is a remainder of 1 (the 1 that you added). So the prime factors of this number must be larger than the primes you had already found.

For example, suppose you have found the first four prime numbers, 2, 3, 5, 7. Take 2 × 3 × 5 × 7 + 1. This number is 211. It is not divisible by 2 or 3 or 5 or 7, because dividing by them gives a remainder of 1. So, if it is not itself a prime, its prime divisors must be larger than 7. It happens that 211 is a prime number.

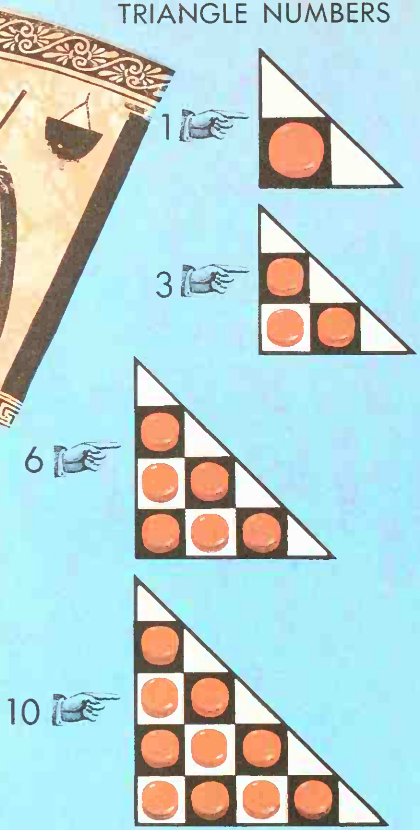


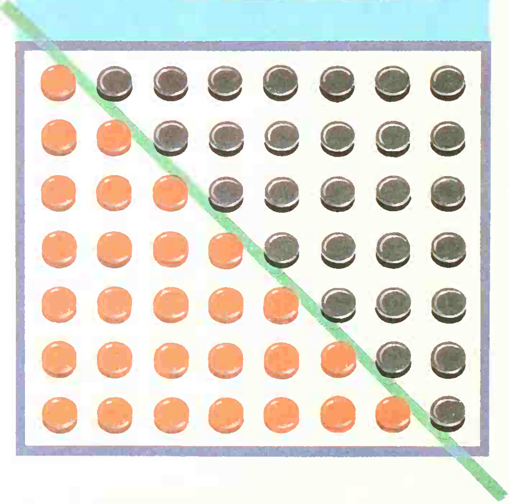
The Shapes of Numbers

Numbers, like people, come in many shapes. Some numbers form rectangles. There are others that form triangles, squares, or cubes.

Triangle Numbers

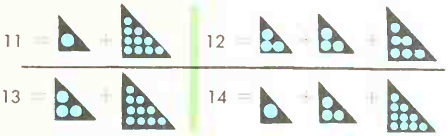
We find the numbers that form triangles by placing lines of checkers under each other. Put 1 checker in the first line, 2 checkers in the second line, 3 checkers in the third line, and so on. We get larger and larger triangles in this way. The number of checkers in a triangle is called a triangle number. The first four triangle numbers are 1, 3, 6, and 10. What is the seventh triangle number? One way to find out is to make the seventh triangle. Then count the number of checkers in it. But there is a short cut we can use. The drawing on the side shows the seventh triangle, with another one just like it placed next to it upside down. The two triangles together form a rectangle, so the triangle number is half of the rectangle number. The rectangle has seven





lines, and eight checkers in each line. So the rectangle number is 7 × 8, or 56. Half of that is 28. To find a triangle number, multiply the number of lines in the triangle by the next higher number, and then take half of the product. To find the eighth triangle number, take half of 8 × 9.

Most whole numbers are not triangle numbers. But even those that are not triangle numbers are related to them in a simple way. Each of them is the sum of two or three triangle numbers. For example, 11 = 1 + 10; 12 = 3 + 3 + 6; 13 = 3 + 10; 14 = 1 + 3 + 10. Find three triangle numbers that add up to 48.



Square Numbers

We form a square by making a rectangle in which the number of lines is the same as the number of checkers in each line. The smallest square has only one line, with one checker in the line. So the smallest square number is 1. The next square has two lines, with two checkers in each line. So the second square number is 2 × 2, or 4. The third square number is 3 × 3, or 9. To get a square number, multiply any number by itself. The seventh square number is 7 × 7, or 49. We call it "seven-squared" and sometimes write it as 7^2. The little two written in the upper right hand corner is a way of showing that the 7 is to be used as a multiplier twice. "Eight-squared" is written as 8^2, and means 8 × 8, or 64.

The square numbers are relatives of the odd numbers (numbers that cannot form rectangles with two lines). If you list the odd numbers in order, stop when you like, and add those you have listed, the sum is always a square number. The drawing above shows you why.

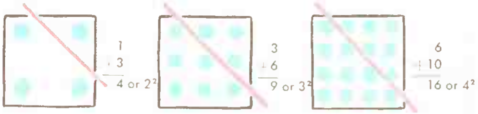
$1 = 1^2$

$1 + 3 = 2^2$

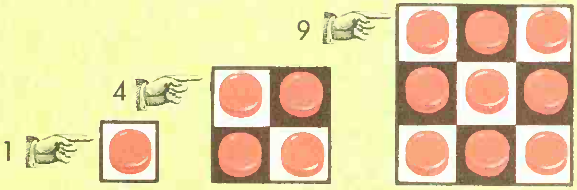
$1 + 3 + 5 = 3^2$

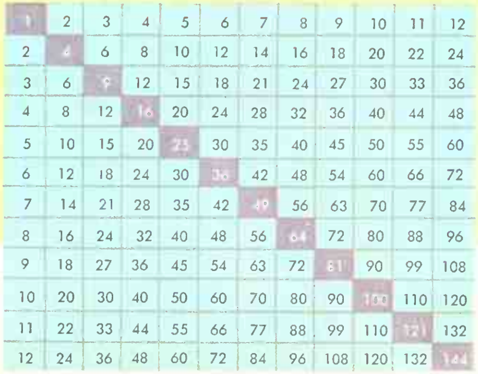
$1 + 3 + 5 + 7 = 4^2$

Square numbers are also relatives of the triangle numbers. Add any triangle number to the next higher triangle number. You always get a square number. The drawing below shows why.



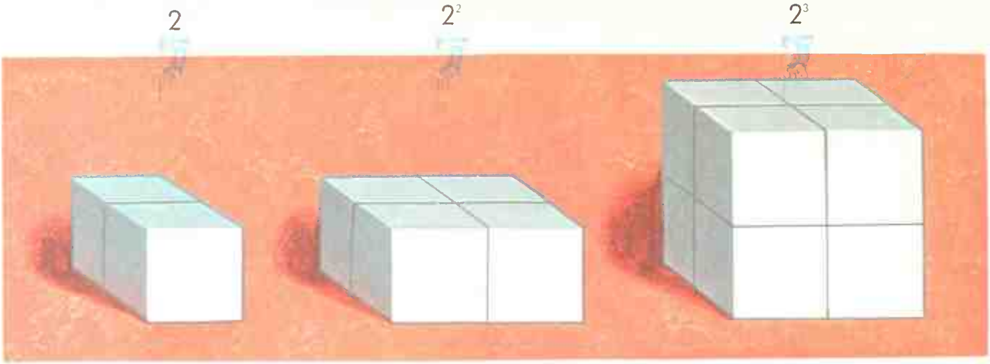
SQUARE NUMBERS





1	2	3	4	5	6	7	8	9	10	11	12
2	4	6	8	10	12	14	16	18	20	22	24
3	6	9	12	15	18	21	24	27	30	33	36
4	8	12	16	20	24	28	32	36	40	44	48
5	10	15	20	25	30	35	40	45	50	55	60
6	12	18	24	30	36	42	48	54	60	66	72
7	14	21	28	35	42	49	56	63	70	77	84
8	16	24	32	40	48	56	64	72	80	88	96
9	18	27	36	45	54	63	72	81	90	99	108
10	20	30	40	50	60	70	80	90	100	110	120
11	22	33	44	55	66	77	88	99	110	121	132
12	24	36	48	60	72	84	96	108	120	132	144

MULTIPLICATION TABLE
The square numbers are found on the diagonal

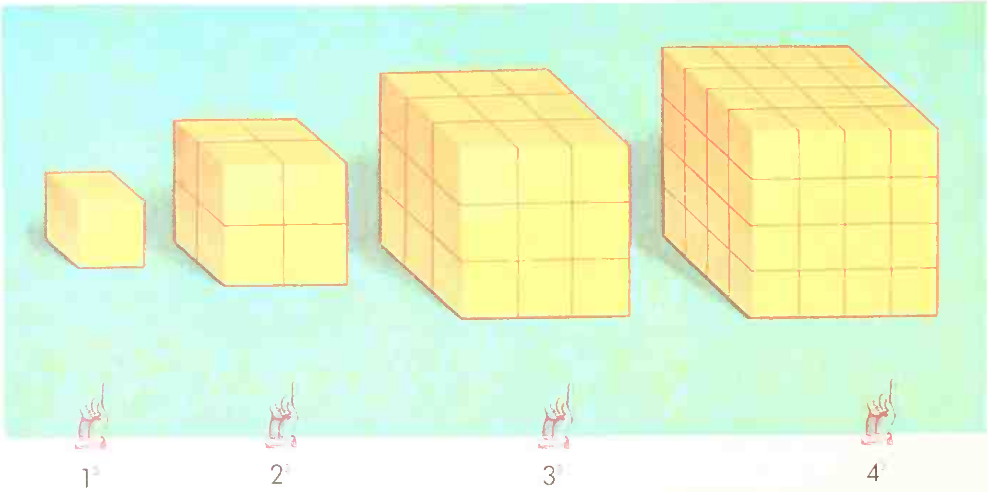


Cubic Numbers

If we use blocks instead of checkers, we can arrange them in lines to form a square, and pile the squares on top of each other in layers. When the number of layers equals the number of blocks in a line, we have a cube.

The number of blocks in a cube is called a cubic number. The smallest cubic number is 1. The second cubic number is 2 × 2 × 2, or 8. We call it "two-cubed," and sometimes write it as 2^3. The little three written in the upper right corner shows that the 2 is to be used as a multiplier three times. The fifth cubic number is "five-cubed." It is written as 5^3, and means 5 × 5 × 5, or 125. What does 6^3 mean? Compare the meanings of 2^3 and 3^2.

Two-squared, which is written as 2^2, is also called "two raised to the second power." Two-cubed, which is written as 2^3, is also called "two raised to the third power." In the same way, 2^4 is used as a short way of writing 2 × 2 × 2 × 2 (2 used as a multiplier four times), and is called "two raised to the fourth power." Multiplying out, we find that $2^4 = 16$. We read 2^5 as "two raised to the fifth power." What does it mean?



The Puzzle of the Reward

A wealthy king was once saved from drowning by a poor farm boy. To reward the boy, the king offered to pay him sums of money in thirty daily installments. But he offered the boy a choice of two plans of payment.

Under plan number 1, the king would pay $1 the first day, $2 the second day, $3 the third day, and so on, the payment increasing by $1 each day. Under plan number 2, the king would pay 1¢ the first day, 2¢ the second day, 4¢ the third day, and so on, the payment doubling each day. Which plan would give the boy the greatest reward?

We can answer the question by simply writing down the thirty installments under each plan, and then adding them up. But there is a shorter way of getting the answer, too. Under plan number 1, the total reward in dollars is the sum of all the whole numbers from 1 to 30. This is simply the thirtieth triangle number. According to the rule given on page 19, we can calculate it by multiplying 30 by 31, and then dividing by 2. The total reward under this plan would be $465.

Under plan number 2, the second installment in cents is 2; the third installment is 2×2 or 2^2; the fourth installment is $2 \times 2 \times 2$ or 2^3; each new installment is a higher power of 2, and the last installment is 2^{29}. A short cut for calculating the total sum is to write down what the reward would be if it were doubled, and then take away the single reward from the doubled reward:

Double reward:

$$\quad 2 + 2^2 + 2^3 + 2^4 \qquad\qquad + 2^{28} + 2^{29} + 2^{30}$$

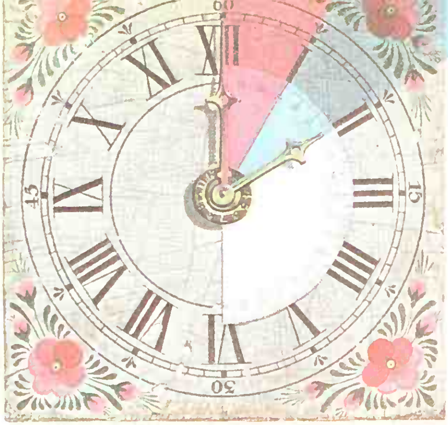
Single reward:

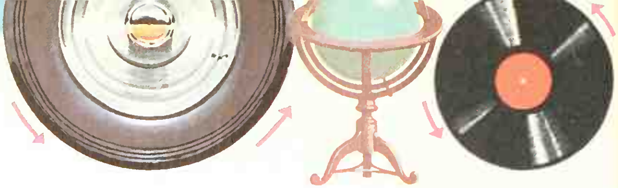
$$1 + 2 + 2^2 + 2^3 + 2^4 \qquad\qquad + 2^{28} + 2^{29}$$

When we subtract, those installments that are equal to each other cancel. Then we see that the difference is $2^{30} - 1$. We can calculate this number quickly by noticing that $2^5 = 32$; $2^{10} = 32 \times 32 = 1024$; $2^{30} = 1024 \times 1024 \times 1024 = 1{,}073{,}741{,}824$. Now we subtract 1 to find the total reward: 1,073,741,823 cents, or $10,737,418.23.

We see that an amount grows fast when it is doubled repeatedly. Keep this in mind when you try to answer the next puzzle: An amoeba is placed in an empty jar. After one second, the amoeba splits into two amoebas, each as big as the mother amoeba. After another second, the daughter amoebas split in the same way. As each new generation splits, the number of amoebas and their total bulk doubles each second. In one hour the jar is full. When is it half-full?

Splitting in two, or doubling, amoebas reproduce rapidly





Turns and Spins

There are many things that turn or spin. A wheel of a moving automobile turns. So does a phonograph turntable. The earth spins on its axis, and the minute hand of a clock rotates around the face. Since so many things turn, we often have to measure the amount of turning. An amount of rotation is called an *angle*. The unit we use for measuring an angle is called a *degree*. There are 360 degrees in one complete rotation.

To measure an angle we use a protractor.

In the drawing above, a protractor has been placed over the face of a clock. When the minute hand points to the 12 on the clock, it points to the zero on the protractor. As it moves away from the 12, it points out the number of degrees through which it has turned. It turns 30 degrees to reach the 1 on the clock. It turns 90 degrees to reach the 3. It turns 180 degrees, or half a complete rotation, to reach the 6.

There are two hands on the face of a clock. At each moment of the day there is an angle between them. The angle is the amount of rotation needed to turn one hand to the position of the other. At one o'clock, the angle between the hands of a clock is 30 degrees. At two o'clock, the angle is 60 degrees. What will the angle between the hands be at half-past two? The answer is printed upside down at the bottom of this page.

The face of a clock is like a circular race track around which the minute hand and the hour hand race against each other. They both start from the same position at 12 o'clock. But the minute hand moves faster than the hour hand, and gets ahead of it. The gap between them widens, until the hour hand is a full lap behind the minute hand. When this happens, the two hands are together again. What is the first time after twelve o'clock that this happens? It is not hard to figure out the answer.

The face of the clock is divided into 60 spaces. The hour hand moves around the face at a speed of 5 spaces an hour. The minute hand moves at a speed of 60 spaces an hour. The difference between 60 and 5 is 55. So, as the hour hand falls behind the minute hand, the gap between them widens at the rate of 55 spaces an hour. A full lap contains 60 spaces, so the gap becomes a full lap after $\frac{60}{55}$ of an hour, or $1\frac{1}{11}$ hours. One eleventh of an hour is $\frac{1}{11} \times 60$ minutes, or $5\frac{5}{11}$ minutes. So the first time the hands are together again is $5\frac{5}{11}$ minutes after one o'clock.

At half-past two, the minute hand points to the six, and the hour hand is halfway between the two and the three. The angle between them is 105°.

The Right Angle

The angle that we use most often is an angle of 90 degrees. We call it a right angle. We make bricks with right angles in each corner so they will stack easily in vertical piles. Then walls stand up straight instead of leaning over, and floors are level.

One way of making a right angle is to measure out 90 degrees with a protractor. There are other ways of making a right angle without using a protractor at all. A bricklayer makes a right angle with strings. He makes one string horizontal with the help of a level. He makes the other string vertical by hanging a weight from its end. A draftsman makes a right angle by drawing two circles that cross each other. He then draws a straight line between the points at which the circles cross, and another line between the centers of the circles.

In ancient Egypt, surveyors made a right angle by "rope-stretching." They used a long rope that was divided into twelve equal spaces by knots. One man held the two ends of the rope together. A second man held the knot that was three spaces from one end. A third man held the knot that was four spaces from the other end. When the rope was stretched tight, a right angle was formed.

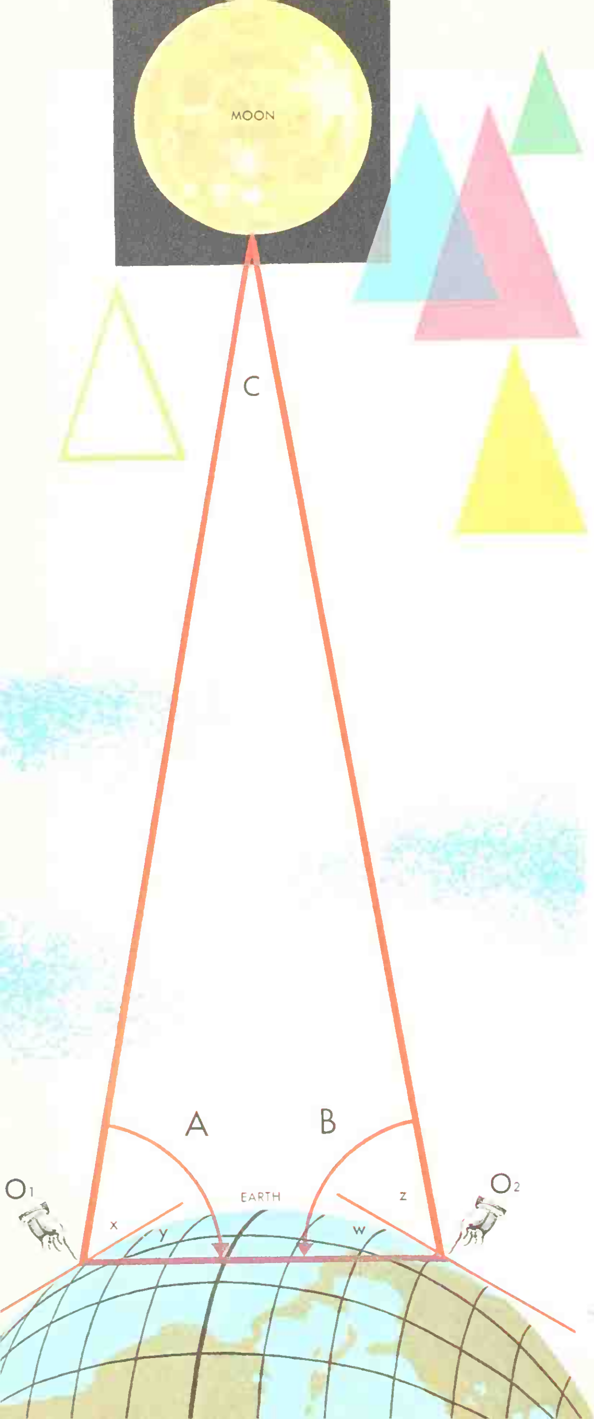
The simplest way to make a right angle is to fold a piece of paper. Fold it once. Then fold it again, so the crease falls on the crease.

Triangles and the Distance to the Moon

Triangles may have different sizes and shapes, but the three angles of any triangle always add up to the same amount. To see this for yourself, cut a triangle out of paper. Then tear off the three angles. Place them side by side, corner to corner, and edge to edge. You will see that they add up to *exactly 180 degrees.*

This is a useful fact to know, because it gives you a short cut for finding the angles of a triangle. You can find all three angles, even if you measure only two of them. For example, if one of the angles is 40 degrees, and the second one is 60 degrees, you can find the number of degrees in the third angle without measuring it. Simply add 40 to 60 and then subtract the result from 180.

This short cut is especially helpful if the third angle is out of reach. For example, suppose that two men, standing at separate places on the earth, look at the moon. The two men and the moon form a triangle. There is nobody on the moon to measure the angle up there. But we can calculate it from the angles we can measure on the earth. Knowing this angle is important to astronomers, because it helps them calculate the distance to the moon. If the moon were further away than it is, the angle would be smaller. If the moon were closer, the angle would be larger. The moon is approximately 240,000 miles away from the earth.

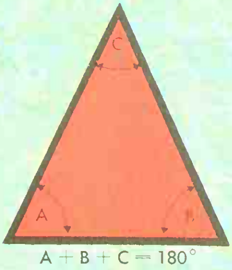


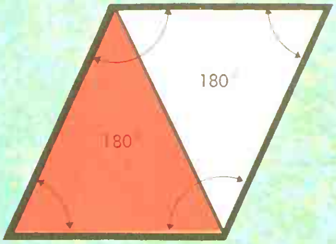
Once we know angles A and B, we can calculate angle C. Angle A = angle x + angle y. Angle B = angle z + angle w. Angle y and angle w can be calculated from the positions of the observers, O_1 and O_2, on earth.
Angle x = height of moon above horizon as seen by O_1
Angle z = height of moon above horizon as seen by O_2

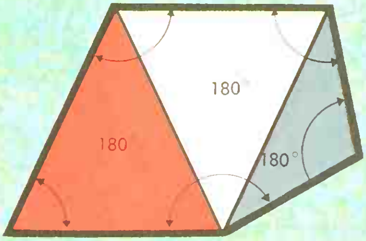
Figures with Many Sides

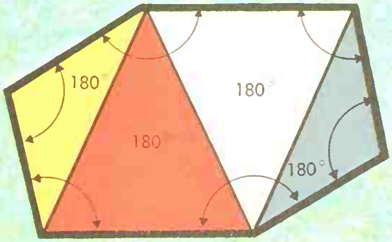
A closed figure with straight sides is called a *polygon.* The number of angles in a polygon is the same as the number of sides. A polygon with three sides is a *triangle.* One with four sides is called a *quadrilateral.* The names for some polygons with more than four sides are shown in the table below.

If we join opposite corners of a quadrilateral, two triangles are formed. If we add the angles of both triangles we have the sum of the angles of the quadrilateral. Since the angles of each triangle add up to 180 degrees, the angles of the quadrilateral add up to 2 × 180 degrees, or 360 degrees. A five-sided figure can be divided into three triangles, so its angles add up to 3 × 180 degrees. A six-sided figure can be divided into four triangles, so its angles add up to 4 × 180 degrees. To get the number of degrees in the sum of the angles of any polygon, take two less than the number of sides, and then multiply this number by 180.



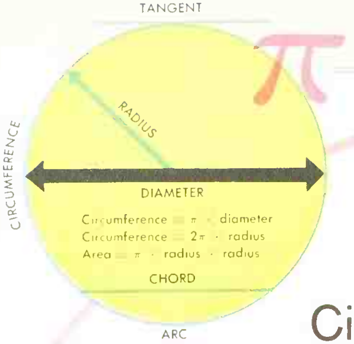






NAME	NUMBER OF SIDES	SUM OF ANGLES IN DEGREES
TRIANGLE	3	180°
QUADRILATERAL	4	2 × 180° = 360°
PENTAGON	5	3 × 180° = 540°
HEXAGON	6	4 × 180° = 720°
OCTAGON	8	6 × 180° = 1080°
DECAGON	10	8 × 180° = 1440°

A polygon may be divided into triangles, each of which contains 180°. To get the number of degrees in the sum of the angles of any polygon, take two less than the number of sides, and multiply by 180°



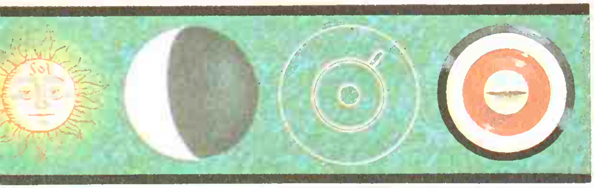
Circles and Toothpicks

We see circles everywhere. The wheels of automobiles, the rims of cups, and the faces of nickels and quarters are all circles. The sun and the full moon look like circles in the sky.

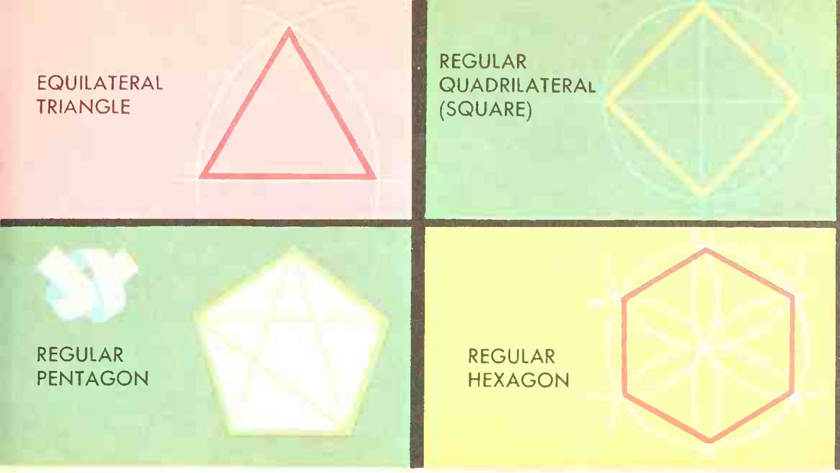
The distance across a circle, through its center, is called the *diameter* of the circle. The distance around the circle is called its *circumference*. Measure the diameter of a quarter, and you will find that it is about one inch long. You can measure the circumference of the quarter, too. First wind enough string around it to go around once. Then unwind the string, and measure it with a ruler. You will find that it is about three times as long as the diameter. Measure the circumference and diameter of the rim of a cup and you will get the same result. The circumference of any circle is a fixed number times the diameter. This fixed number cannot be written exactly as a fraction or decimal, so we use the Greek letter π (pi) to stand for it. It is almost equal to $3\frac{1}{7}$, or 3.14.

Strange as it may seem, there is a way of calculating the value of π by dropping a stick on the floor. The floor has to be made of planks of the same width. Use a thin stick, such as a toothpick, that is as long as the planks are wide. Simply drop the stick many times. Keep count of the number of times you drop it and the number of times it falls on a crack. Double the number of times you drop the stick and then divide by the number of times it fell on a crack. The result is your value of π.

For example, if you drop the stick 100 times, and it falls on a crack only 62 times, divide 200 by 62. The result is about 3.2. This is not a very accurate value of π. The more times you drop the stick, the more accurate a value you will get. When you drop the stick, whether or not it crosses a crack depends on where its center falls, and how it is turned around its center. When a stick turns around its center, it moves around a circle. That is why π, which is related to measuring a circle, is also related to the chance that the stick will cross a crack.



You can calculate π by dropping toothpicks on a wood floor ☞



REGULAR POLYGONS AND THEIR CONSTRUCTION

NAME	ANGLES	SUM OF ANGLES	SIZE OF EACH ANG
EQUILATERAL TRIANGLE	3	180°	60°
SQUARE	4	360°	90°
REGULAR PENTAGON	5	540°	108°
REGULAR HEXAGON	6	720°	120°

Equal Sides and Equal Angles

The symbol of the Office of Civil Defense is a triangle inside a circle. The sides of the triangle are all equal, and the angles are all equal. There are other polygons, too, that have equal sides and equal angles. We call them *regular polygons.* We come across them very often in everyday life. Some wall tiles are regular quadrilaterals, or squares. Some floor tiles are regular hexagons.

A regular polygon may have any number of sides, starting with three. One way of making a regular polygon is to calculate the number of degrees each of its angles should have, and then make these angles with a protractor, separating them with equal sides. The rule on page 50 about the angles of a polygon helps us make this calculation. If the figure has three sides, the angles must add up to 180 degrees. So each of the three angles must be 60 degrees. If the figure has four sides, the angles add up to 360 degrees. So each of the four angles must be 90 degrees. The angles for other regular polygons are shown in the table above.

However, there are short cuts for making the first few regular polygons. The regular polygon of three sides is called an *equilateral triangle.* You can make it with a ruler and compass by the method shown in the drawing. To make a square, first make a circle. Fold the paper so that the crease passes through the center of the circle. Now fold the paper again, in order to make a right angle at the center. Open up the paper, and join the points where the creases cross the circle.

To make a regular pentagon, cut a long strip of paper of uniform width. Then tie it into a knot as shown in the drawing, and press the knot flat. To make a regular hexagon, draw a circle, and then mark off pieces on the circle, with your compass open the same width you used to make the circle. There will be six equal pieces. Join their ends to make the hexagon.

Regular hexagons, equilateral triangles, and squares are often used as floor tiles because they fit together well

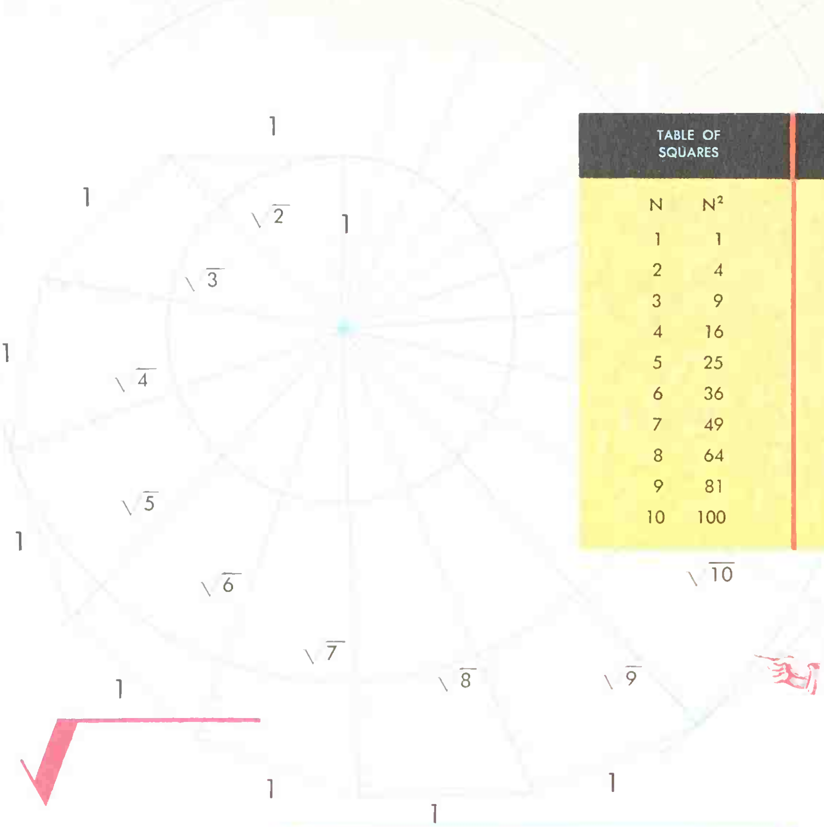


TABLE OF SQUARES

N	N^2
1	1
2	4
3	9
4	16
5	25
6	36
7	49
8	64
9	81
10	100

TABLE OF SQUARE ROOTS

N	$\sqrt{N}$
1	1
4	2
9	3
16	4
25	5
36	6
49	7
64	8
81	9
100	10

TABLE OF SQUARE ROOTS

N	$\sqrt{N}$
1	1.00
2	1.41
3	1.73
4	2.00
5	2.24
6	2.45
7	2.65
8	2.83
9	3.00
10	3.16

Right triangles are used here to construct in succession lines whose lengths are $\sqrt{2}$, $\sqrt{3}$, and so on. See rule of Pythagoras, page 34

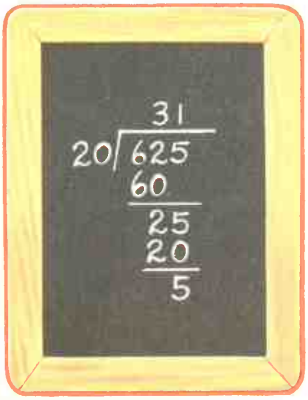
Square Root

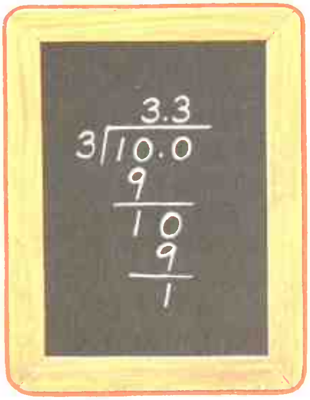
When two people are related to each other, we can describe the relationship in two opposite ways. If Mr. Smith is Peter's father, we can also say that Peter is Mr. Smith's son. In the same way, when numbers are related to each other, we can describe the relationship in two opposite ways. We show how 4 is related to 2 by saying that "four is two-squared." We can also say it in the opposite direction by saying that "two is the square root of four." Since nine is equal to "three-squared," we can also say that three is the square root of nine. We use the special symbol $\sqrt{\ }$ to mean "the square root of." So, $\sqrt{16}$ is read as "the square root of 16," and it stands for 4, because $4^2 = 4 \times 4 = 16$.

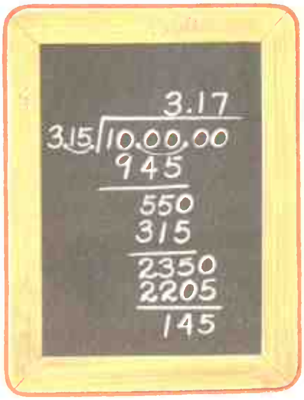
In the multiplication table on page 19, the square numbers are those that appear on the diagonal. We can list them separately in a table of squares like the one that is printed above. In this table, the whole numbers are listed in the first column, and the square of each number appears in the second column. If we interchange the columns, it becomes a table of square roots. Then, for each number that appears on the left, its square root appears to the right of it. But in this new table, we no longer find every whole number in the first column. The numbers 1, 4, and 9, for example, are listed, but the numbers 2, 3, 5, 6, 7, and 8 are not. They do not appear because they are not the squares of whole numbers, or, to say it in the opposite direction, their square roots are not whole numbers. These numbers have square roots that can be written approximately as decimal fractions. Since 2 is between 1 and 4, $\sqrt{2}$ lies between $\sqrt{1}$ and $\sqrt{4}$, that is, between 1 and 2. Since 7 lies between 4 and 9, $\sqrt{7}$ lies between $\sqrt{4}$ and $\sqrt{9}$, that is, between 2 and 3.

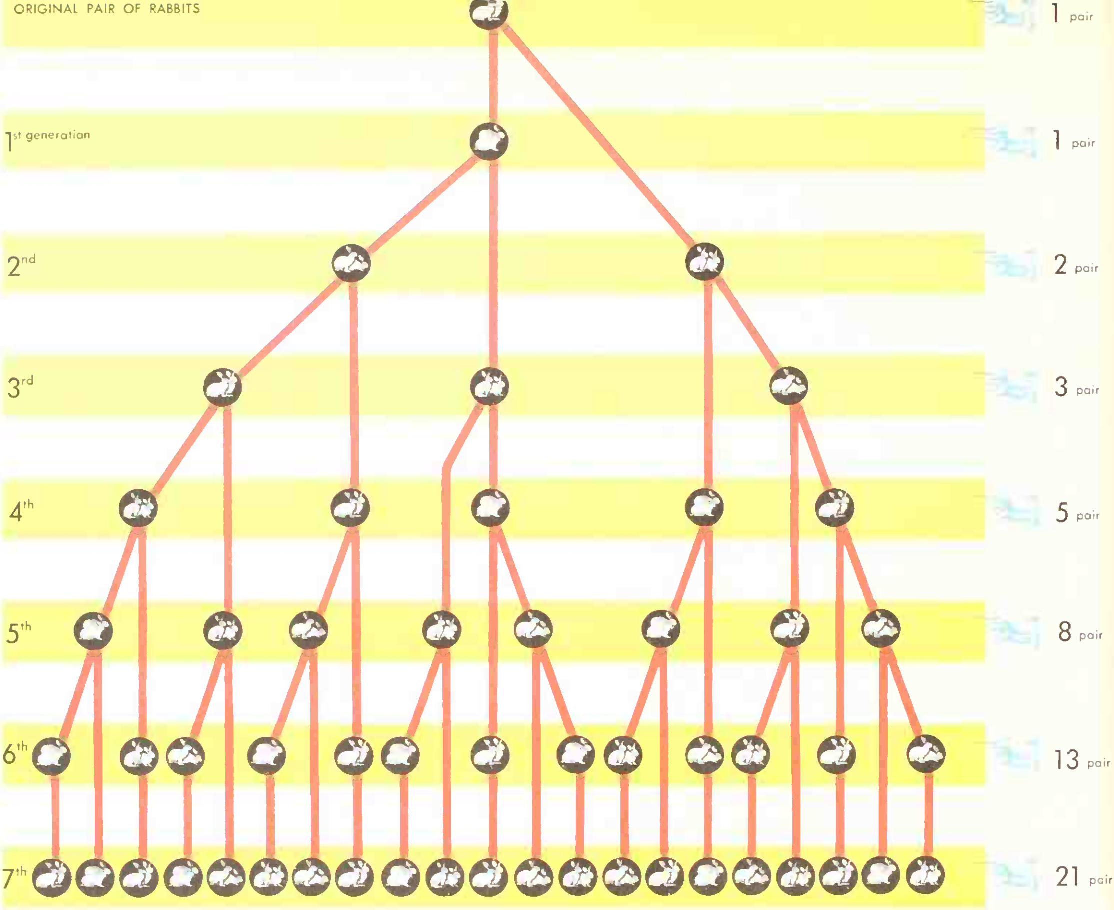
There are many methods for finding these in-between square roots. We shall use a method that may be described as "getting the right answer from a wrong guess." To show how it works, let us try it out first on a number that is the square of a whole number. Suppose we want to find the square root of 625. We take a guess, and say it is 20. Now we check our guess by dividing 20 into 625. If our guess is right, the answer we get by dividing should come out the same as the divisor. But it doesn't. It comes out about 31 instead. But this gives us a hint on how we can correct our bad guess. Now we know that the answer should be between 20 and 31. If we try the number 25, we find that it really is the square root of 625. By multiplying 25 times 25, we get 625.

Now let us use the same method to get an approximate value for the square root of 10. We take a guess and say it is 3. Dividing 3 into 10.0, we get 3.3. So a better guess is the average of 3 and 3.3. This number is 3.15. Now, to test how good a guess 3.15 is, we divide it into 10.0000. The quotient comes out 3.17, so a better guess would be the average of 3.15 and 3.17, which is 3.16. This is the best answer we can get with two decimal places. If we want a more accurate answer with more decimal places, we simply continue the process, checking each new guess by dividing it into 10. Approximate square roots of the numbers from 1 to 10 are shown in the third table on the preceding page.









Rabbits, Plants and the Golden Section

A man bought a pair of rabbits, and bred them. The pair produced one pair of offspring after one month, and a second pair of offspring after the second month. Then they stopped breeding. Each new pair also produced two more pairs in the same way, and then stopped breeding. How many new pairs of rabbits did he get each month?

To answer this question, let us write down in a line the number of pairs in each generation of rabbits. First write the number 1 for the single pair he started with. Next we write the number 1 for the pair they produced after a month. The

Pine cones have Fibonacci ratios of $\frac{5}{8}$, $\frac{8}{13}$

next month, both pairs produced, so the next number is 2. We now have three numbers in a line: 1, 1, 2. Each number represents a new generation. Now the first generation stopped producing. The second generation (1 pair) produced 1 pair. The third generation (2 pairs) produced 2 pairs. So the next number we write is 1 + 2, or 3. Now the second generation stopped producing. The third generation (2 pairs) produced 2 pairs. The fourth generation (3 pairs) produced 3 pairs. So the next number we write is 2 + 3, or 5. Each month, only the last two generations produced, so we can get the next number by adding the last two numbers in the line. The numbers we get in this way are called *Fibonacci numbers.* The first twelve of them are:

1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144

They have very interesting properties, and keep popping up in many places in nature and art.

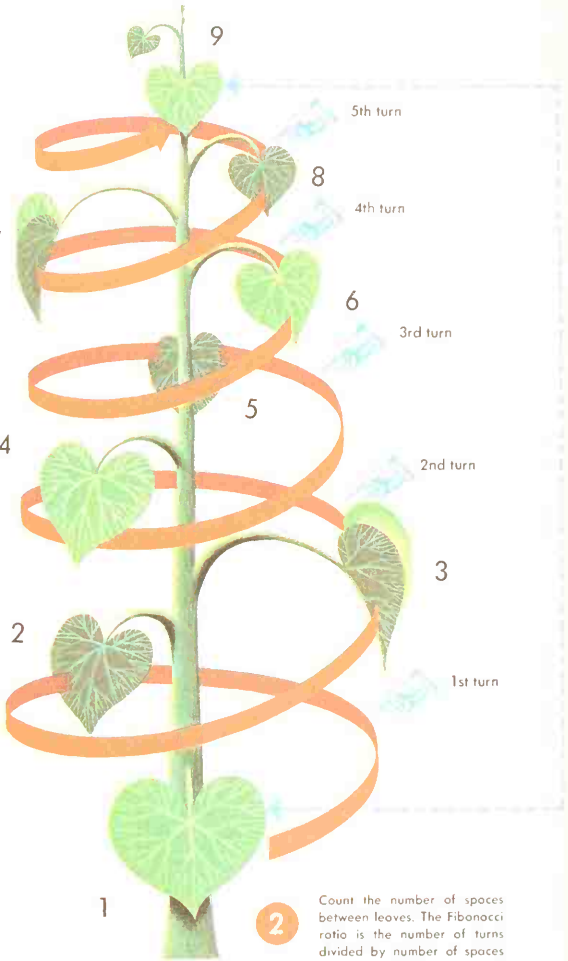
Here is one of the curious properties of these numbers. Pick any three numbers that follow each other in the line. Square the middle number and multiply the first number by the third number. The results will always differ by 1. For example, if we take the numbers 3, 5, 8, we get $5^2 = 5 \times 5 = 25$, while $3 \times 8 = 24$.

If we divide each number by its right hand neighbor, we get a series of fractions:

$$\frac{1}{1}\ \frac{1}{2}\ \frac{2}{3}\ \frac{3}{5}\ \frac{5}{8}\ \frac{8}{13}\ \frac{13}{21}\ \frac{21}{34}\ \frac{34}{55}\ \frac{55}{89}\ \frac{89}{144}$$

These fractions are related to the growth of plants. When new leaves grow from the stem of a plant, they spiral around the stem. The spiral turns as it climbs. The amount of turning from one leaf to the next is a fraction of a complete rotation around the stem. *This fraction is always one of the Fibonacci fractions.* Nature spaces

① Starting with a leaf such as number 1 in the diagram, count the number of turns around the stem taken by higher leaves in succession until you reach the leaf (here leaf 9) which is directly above the first

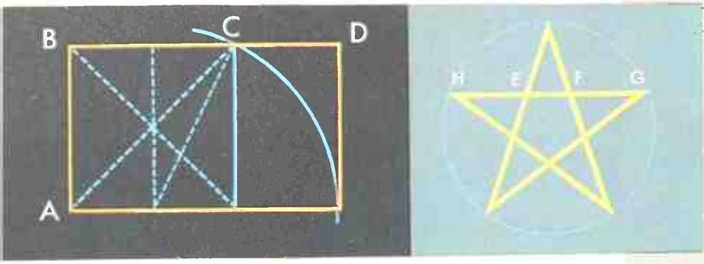


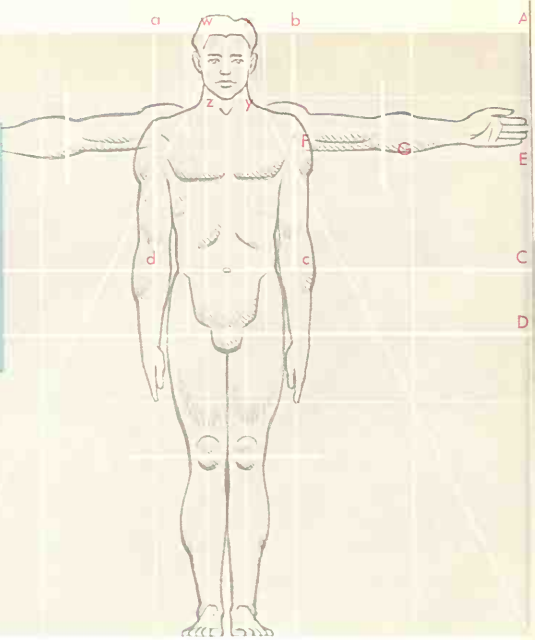
② Count the number of spaces between leaves. The Fibonacci ratio is the number of turns divided by number of spaces

In the example above there are five complete turns and eight spaces from leaf 1 to leaf 9. The Fibonacci ratio for this plant is $\frac{5}{8}$. Fibonacci was an Italian mathematician of the thirteenth century

Normal daisies usually have the Fibonacci ratio $\frac{21}{34}$

$$\frac{\sqrt{5-1}}{2} = \Phi$$





The construction of the "golden section," with the ratios $\frac{AB}{BD} = \frac{BC}{BD} = \frac{CD}{BC} = \Phi$. The lines of the five-pointed star are broken up in ratios: $\frac{EF}{EG} = \frac{FG}{FH} = \frac{FH}{HG} = \Phi$.

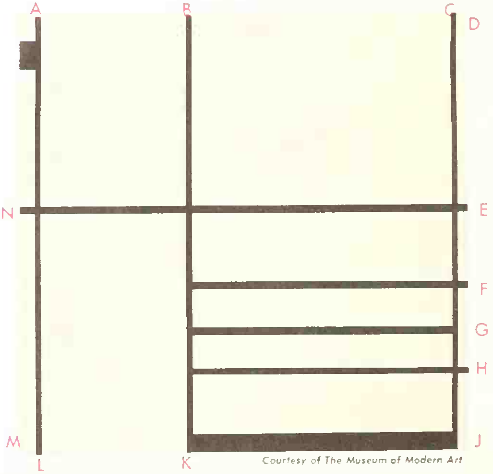
Living things often show surprising relationships to the golden section. The diagram of the athlete to the right shows ratios: $\frac{AC}{CB} = \frac{CB}{AB} = \frac{AE}{AD} = \frac{ab}{bA} = \frac{bA}{aA} = \frac{FG}{GE} = \Phi$. Rectangles abcd and wxyz are "golden rectangles." The same ratios are evident in the spacing of the knuckles and the wrist joint of the average hand

the leaves in this way so that the higher leaves do not shade the lower leaves too much.

The same fractions come up in art. For example, not all rectangles are equally pleasing to the eye. Some look too long and narrow. A square looks too stubby and fat. There is a shape between these extremes that looks the best. In this best-looking of rectangles the ratio of the width to the length is about the same as the ratio of the length to the sum of the width and length. It is called the *golden section.*

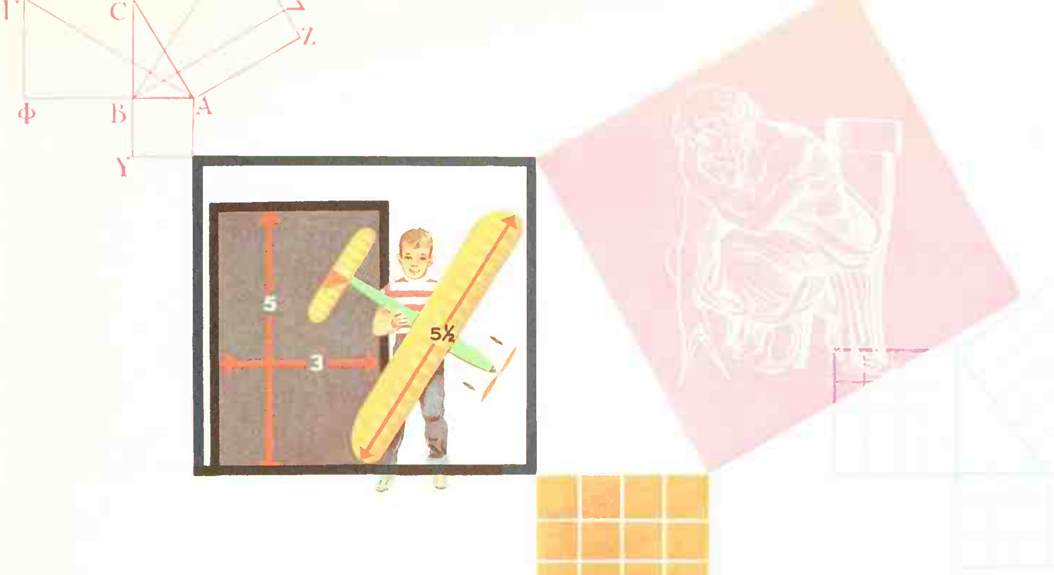
There is a formula that gives directions for calculating the golden section. The directions are: Subtract 1 from the square root of 5, and divide by 2. The square root table on page 28 shows that the square root of 5 is approximately 2.24. Subtracting 1, and dividing by 2, we get .62 as an approximate value of the golden section.

Fibonacci fractions are close to the golden section. In fact, the further out they are in the series, the closer they get to it. The fraction $\frac{5}{8}$ is closer to the golden section than $\frac{3}{5}$. The fraction $\frac{8}{13}$ is closer than $\frac{5}{8}$, and so on. In the design below, the golden section was used several times either to divide lines or to form rectangles.



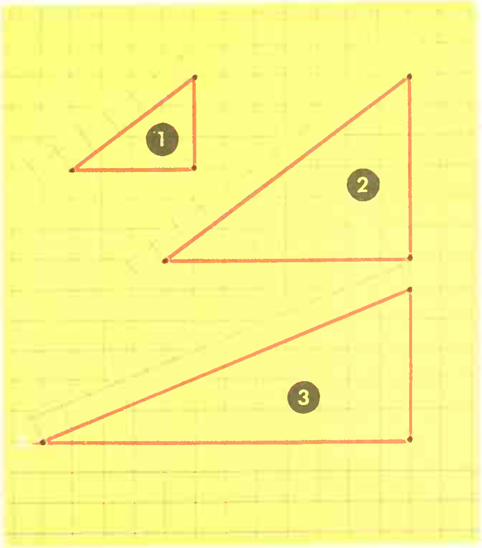
Courtesy of The Museum of Modern Art

Mondrian's *Black, White and Red* has these ratios equal to the golden section: $\frac{BC}{AC} = \frac{AB}{BC} = \frac{BC}{DJ} = \frac{EF}{EG} = \frac{FG}{EF} = \frac{HJ}{GJ} = \frac{GH}{HJ} = \frac{KL}{MN} = \frac{EH}{KJ}$



Getting through the Doorway

Sandy was building a large model airplane out in the little shed which was his workshop. As he was about to glue the wings to the body of the plane, Sandy thought, "I wonder if I'll be able to get the plane through the shed doorway after I put the wings on. The wingspread is $5\frac{1}{2}$ feet across, and the shed doorway is 3 feet wide and 5 feet high."



We can help Sandy solve his problem by finding out how the sides of a right triangle are related to each other. On a sheet of graph paper, make a right triangle four units wide (first leg) and three units high (second leg). Measure the hypotenuse (the longest side). It will be five units long. Now make two more right triangles as shown in the diagram. Measure the hypotenuse of each triangle:

	Leg	Leg	Hypotenuse
①	4	3	5
②	8	6	10
③	12	5	13

Look at the numbers for each triangle. There doesn't seem to be any obvious connection between them. But there is a hidden connection. It shows itself when we square each number:

$(\text{first leg})^2$	$(\text{second leg})^2$
$4 \times 4 = 16$	$3 \times 3 = 9$
$8 \times 8 = 64$	$6 \times 6 = 36$
$12 \times 12 = 144$	$5 \times 5 = 25$

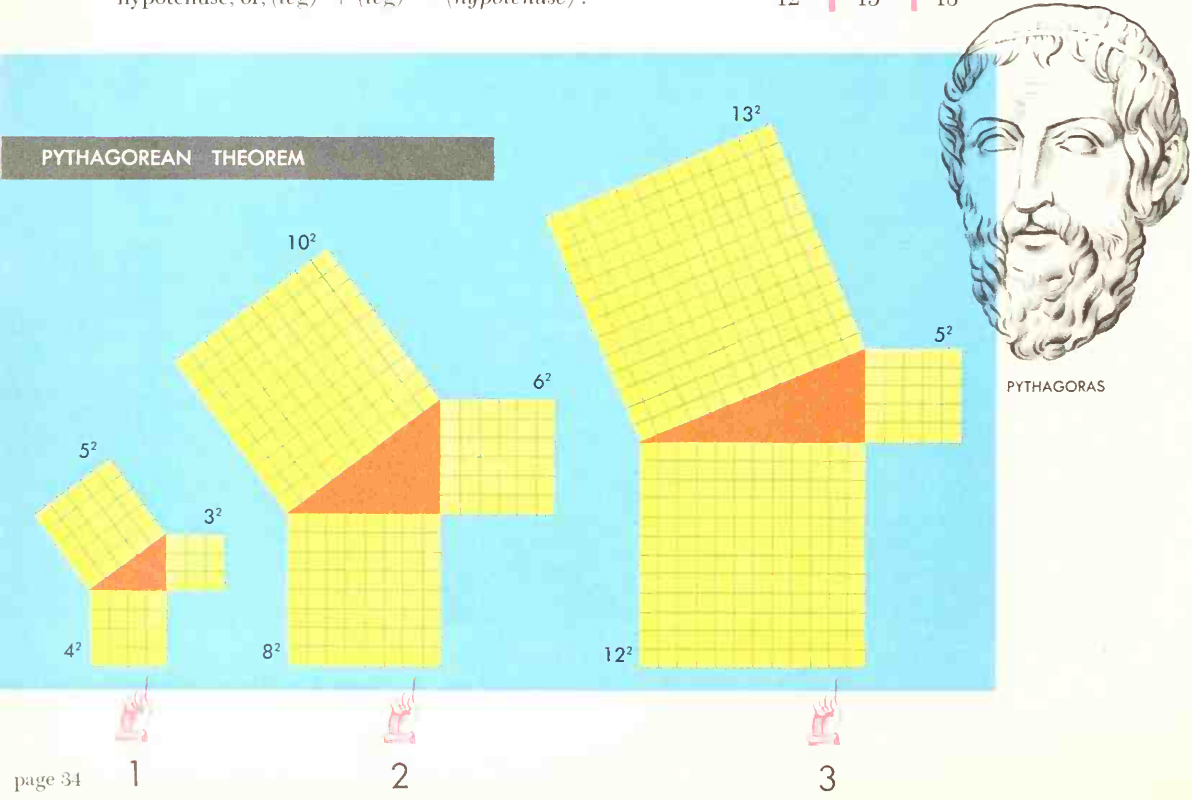
$(\text{hypotenuse})^2$	
$5 \times 5 = 25$	and $16 + 9 = 25$
$10 \times 10 = 100$	$64 + 36 = 100$
$13 \times 13 = 169$	$144 + 25 = 169$

These are examples of a rule discovered about 2500 years ago by a Greek mathematician named Pythagoras. The rule says that in every right triangle, the square of one leg plus the square of the other leg equals the square of the hypotenuse, or, $(leg)^2 + (leg)^2 = (hypotenuse)^2$.

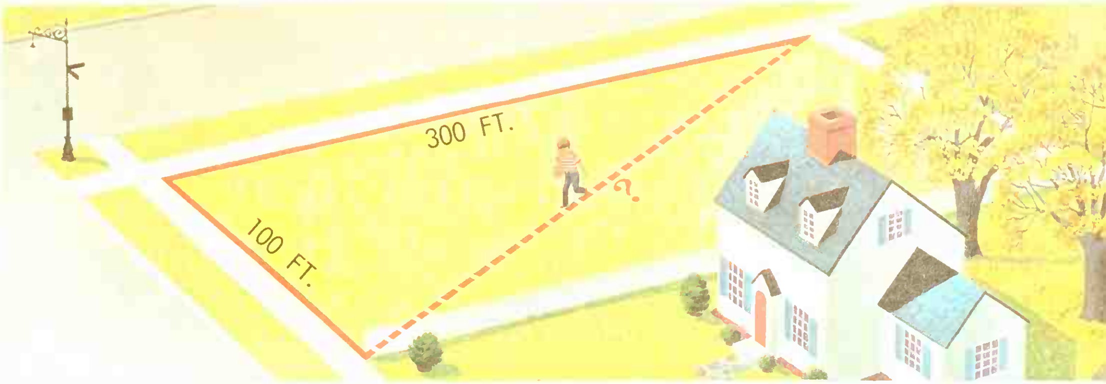
This rule helps us solve Sandy's problem. The width, the height, and the diagonal of the shed doorway form a right triangle. Its legs are 3 feet and 5 feet. $3^2 + 5^2 = 9 + 25 = 34$. Because 34 is the square of the diagonal through which the airplane must pass, we must square the wingspread of the plane in order to see whether it is smaller than the diagonal of the doorway. The wingspread is $5\frac{1}{2}$ feet. $(5\frac{1}{2})^2 = 5\frac{1}{2} \times 5\frac{1}{2} = \frac{11}{2} \times \frac{11}{2} = \frac{121}{4} = 30\frac{1}{4}$. This result is less than 34, so the airplane will go through.

Here are three sets of numbers. Only two obey the rule of Pythagoras. Which are they?

9	12	15
8	15	17
12	15	18



PYTHAGORAS



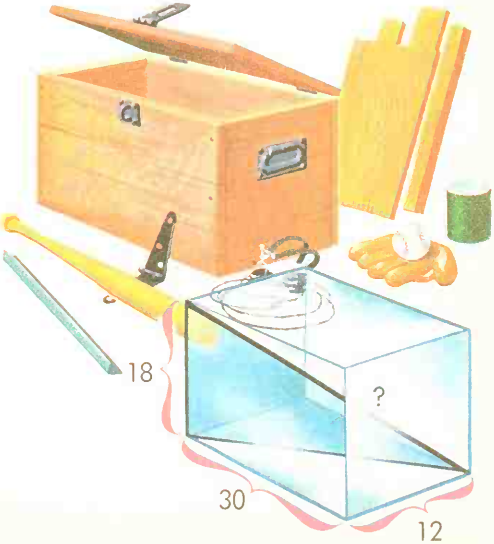
The Short Cut

There is an empty lot next to Sandy's house. The lot is 300 feet long and 100 feet wide. When Sandy comes home from school, he cuts across the lot diagonally. How much distance does he save this way?

The rule of Pythagoras helps us answer this question. The width, the length, and the diagonal of the lot form a right triangle. Expressed in hundreds of feet, the legs of the right triangle are 1 and 3. Using the rule of Pythagoras, we find that $(\text{hypotenuse})^2 = 1^2 + 3^2 = 1 + 9 = 10$. Then the number of hundred feet in the diagonal is $\sqrt{10}$, or 3.16. (See the table of square roots on page 28.) So the diagonal has a length of 316 feet. Along the sides of the lot, the distance is 400 feet. By taking the short cut, Sandy saves a distance of 84 feet.

The Trunk

Sandy is storing in an old trunk scraps of wood and metal that he thinks he may find a use for later. The trunk is 12 inches wide, 30 inches long, and 18 inches high. What is the longest piece of metal that can fit into the closed trunk? The answer to this question is the length of the diagonal of the trunk. We can find this diagonal by using an extension of the rule of Pythagoras: $(\mathit{length})^2 + (\mathit{width})^2 + (\mathit{height})^2 = (\mathit{diagonal})^2$. In this case we have $30^2 + 12^2 + 18^2 = (\text{diagonal})^2$. Then $(\text{diagonal})^2 = 900 + 144 + 324 = 1368$. The number of inches in the diagonal is $\sqrt{1368}$, or almost 37 inches. $(37 \times 37 = 1369.)$

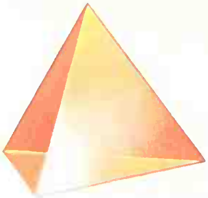


Salt and Diamonds

Many minerals form beautiful crystals with smooth flat faces and sharp edges. In some of these crystals, the faces are regular polygons that have the same size and shape, with the same number of polygons at each corner. A solid that is built in this way is called a *regular solid*. There are exactly five regular solids. Their names show the number of faces that they have.

The *tetrahedron* (four faces) is made of triangles, with three triangles at each corner. The *hexahedron* or cube (six faces) is made of squares, with three squares at each corner. The *octahedron* (eight faces) is made of triangles, with four triangles at each corner. The *dodecahedron* (twelve faces) is made of pentagons, with three pentagons at each corner. The *icosahedron* (twenty faces) is made of triangles, with five triangles at each corner.

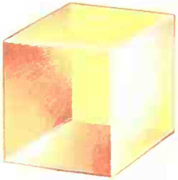
An interesting characteristic of all solids with flat faces is that if you add the number of corners to the number of faces of any one of these



TETRAHEDRON

OCTAHEDRON

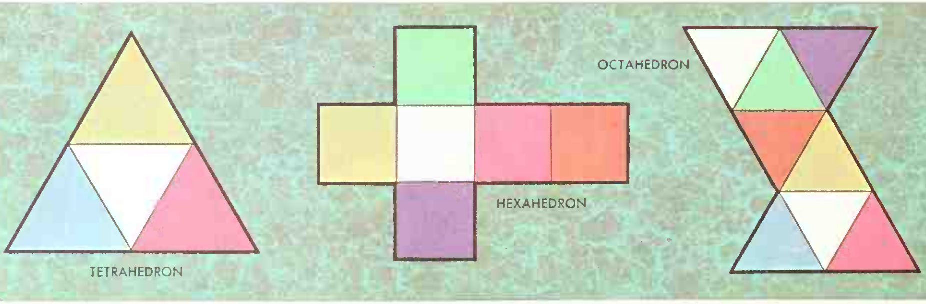
ICOSAHEDRON



HEXAHEDRON (CUBE)

DODECAHEDRON

Salt crystals are actually cubes



Patterns for making the five regular solids

solids, you will get the number of edges in the solid plus 2. Try it with the cube shown in the picture. There are eight corners, and six faces, so the sum of these numbers is 14. Now count the number of edges.

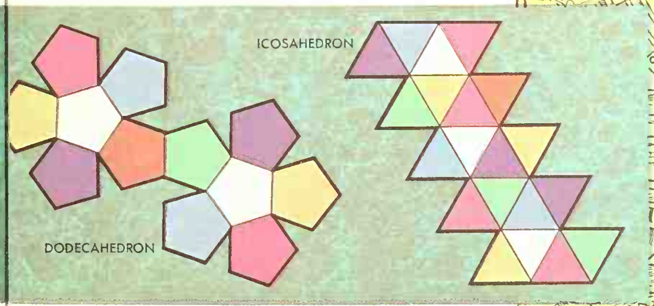
If you look at table salt under a magnifying glass, you will see that each crystal is a cube. A diamond crystal is an octahedron.

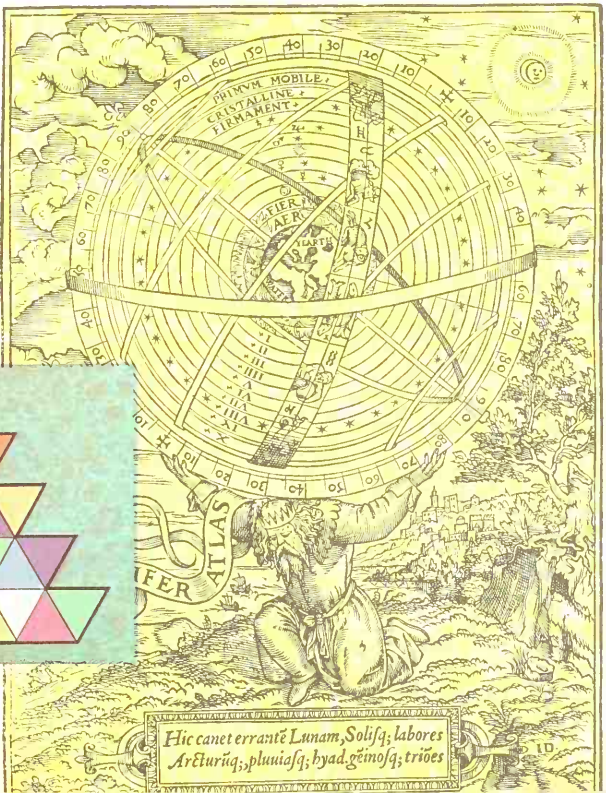
The regular solids make interesting decorations. Some are now made for sale as paperweights. There are calendars printed on a dodecahedron, with each month on a separate face. You can make a model for each of the regular solids by using the patterns shown here. First make an equilateral triangle, a square, and a regular pentagon on cardboard, and cut them out. Then you can make each figure as many times as you have to, and in the right position, by tracing around the cardboard form. When a pattern is complete, cut it out, and make creases on the lines. After you fold it up, seal it by binding the edges with adhesive tape.

Theories That Failed

There are exactly five regular solids, no more and no less. This fact has fascinated people ever since it became known. It led some to believe that the regular solids must have a special meaning in nature. In ancient Greece, the philosophers who were followers of Pythagoras connected it with the theory that the universe is built of four elements, earth, air, fire, and water.

Pythagoras is depicted on an ancient coin of Samos





An old print showing the ancient theory of the universe. From Cuningham's *The Cosmographical Glasse*

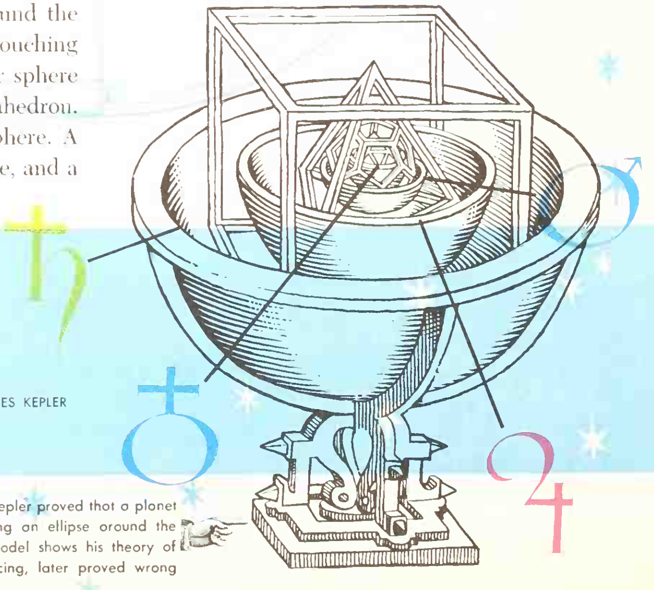
They said that earth is made of cubes, air is made of octahedrons, fire is made of tetrahedrons, and water is made of icosahedrons. The dodecahedron was the symbol of the universe as a whole. We know now that the structure of the universe is far more complicated. There are about one hundred chemical elements, not only four. It is interesting, though, that crystals formed by some combinations of the elements do have the shapes of regular solids.

The regular solids appear, too, in one of the theories of Johannes Kepler, the great astronomer of the sixteenth century. Kepler knew of the existence of six planets: *Mercury*, *Venus*, *Earth*, *Mars*, *Jupiter*, and *Saturn*. He thought that there weren't any others, and wondered why there should be exactly six of them. Since there are five spaces separating the six planets from each other, and there are five regular solids, he thought there must be a connection between these two facts.

He advanced the theory that the solids are related to the spacing of the planets in this way: He pictured the earth on a sphere around the sun. Around this sphere, with its faces touching the sphere, is a dodecahedron. A larger sphere passes through the corners of the dodecahedron. *Mars*, Kepler said, is on this second sphere. A tetrahedron surrounds the second sphere, and a third sphere surrounds the tetrahedron. On this third sphere lies *Jupiter*. A cube surrounds the third sphere, and a fourth sphere surrounds the cube. *Saturn* lies on the fourth sphere. Then he started again from the sphere that has the earth on it, and worked inwards toward the sun. There is an icosahedron inside the sphere, with a fifth sphere inside the icosahedron. The fifth sphere marks the position of *Venus*. In this sphere lies an octahedron, which in turn surrounds a sixth sphere, on which the innermost planet *Mercury* moves.

Kepler's neat little theory has been spoiled by the fact that his spheres don't quite match the actual distances of these six planets from the sun. Besides, we now know of three other planets: *Uranus*, *Neptune*, and *Pluto*. But, while this theory failed, his other theories about the motion of the planets were very successful. Kepler was the first to show that the orbit of each planet is an oval-shaped figure known as an *ellipse*. (See the drawing on page 54.)

JOHANNES KEPLER



Johannes Kepler proved that a planet travels along an ellipse around the sun. This model shows his theory of planet spacing, later proved wrong


The knotted cords, or *quipu*, have been used since ancient times by the Peruvian shepherd for recording the numbers of his flock

Five Number Systems

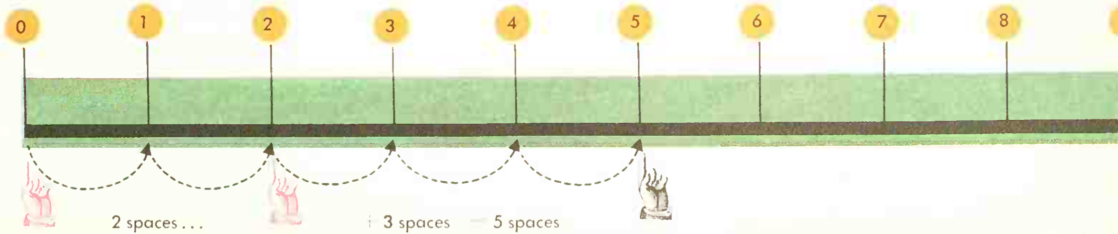
A popular toy made in many countries is a nest of dolls of different sizes. Each doll except the smallest one is like a hollow box. When you open it, you find another doll inside. Our number system is like this nest of dolls. It consists of five number systems, one within the other. The oldest of these systems is the smallest one. It grew up into the larger systems in several jumps, as new numbers were added to meet new needs.

Numbers for the Shepherd

A shepherd uses numbers to count his flock. The numbers used for counting, like 0, 1, 2, 3, 4, and so on, are called *natural numbers*. They make up the smallest and oldest of the number systems we use. An important fact about this system is that we can add or multiply any two natural numbers, and the answer will also be a natural number.

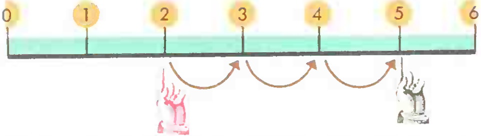
There is a way of picturing the natural numbers as points on a line. Start at any point on the line and then mark off more points at equal intervals to the right of it. Attach a number to each of these points by counting spaces from the starting point. If we imagine the line extending indefinitely to the right, then there is a point

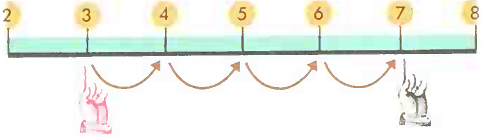
SYSTEM OF NATURAL NUMBERS



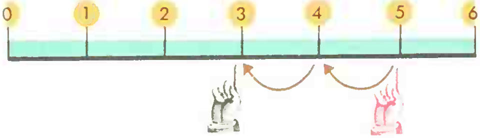
on it for every natural number. We can use this picture for doing problems in addition.

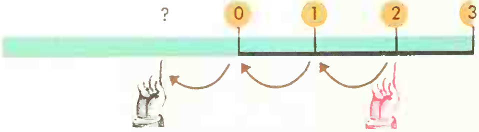
To add 2 + 3, place your finger at the 2, and move it 3 spaces to the right. Then your finger lands at 5, which is the answer. To add 3 + 4, start at the 3, and move your finger 4 spaces to the right. Under this scheme, *you add a number by moving that number of spaces to the right.*



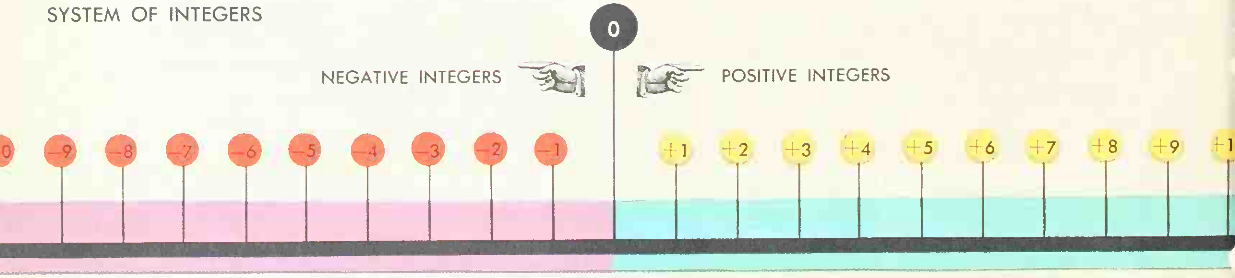


You can do subtraction on the line, too, by moving to the left instead. To do the problem 5 − 2, start at the 5 and move 2 spaces to the left. You land at the point marked 3, so 3 is the answer. But you have trouble if you try to do 2 − 5. Start at the 2, and move 5 spaces to the left. You land at a point that has no number attached to it. This shows that subtraction is not always possible in the natural number system.





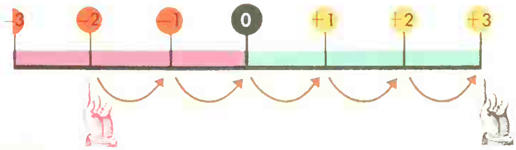
Trying to subtract 5 from 2 is not as silly as it may sound. If you have only two sheep in a pen, you cannot remove five sheep. But if you have only two dollars, you can lose five dollars, and end up with a debt of three dollars. So while the problem 2 − 5 may not arise for a shepherd, it may for his bookkeeper. To be able to do such an example, we need a larger number system.

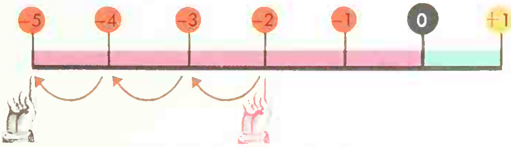


Numbers for the Bookkeeper

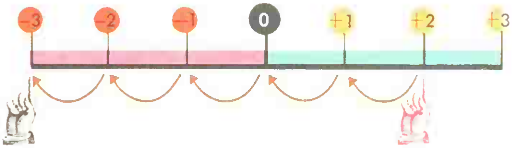
Our picture of the natural number system suggests an easy way to enlarge it. So far we have counted off equal spaces only to the right of 0. Now let's count off equal spaces to the left of 0. We have a new set of points marked off, with a number attached to each point. To distinguish them from the old numbers, let us put a plus sign before each number that lies to the right of 0, and a minus sign before each number that lies to the left of 0. The new enlarged system is called the system of *integers*. In this system, the old natural numbers acquire a new name. We call them *positive integers*. The other numbers, which lie to the left of 0, are called *negative integers*.

To do addition in the enlarged system, we extend the scheme in which we add a number by moving along the line. To add a positive integer, which is the same as a natural number, follow the old rule of moving to the right. To add a negative integer, move to the left instead. The example (−2) + (+5) means start at −2 and move 5 spaces to the right. The answer is +3. The example (−2) + (−3) means start at −2 and move your finger 3 spaces to the left. The answer is −5.





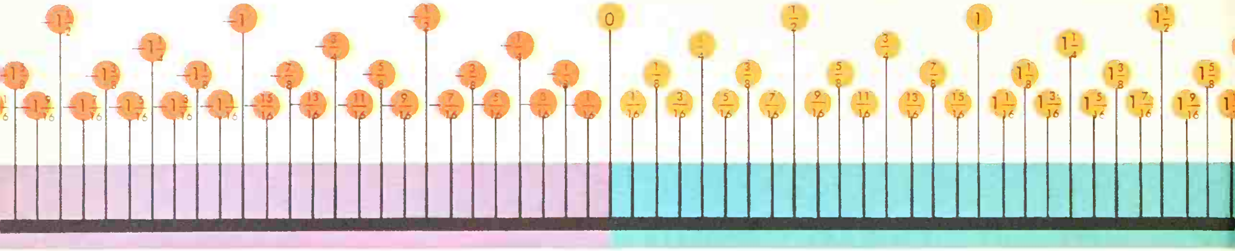
We do subtraction examples with integers by turning every subtraction example into an addition example, according to this rule: *To subtract a number, add the number that has the opposite sign*. Let us try this rule out on the example we could not do in the natural number system, 2 − 5. The natural number 2 is the same as the positive integer +2, and 5 is the same as +5. So we rewrite the problem in this way: (+2) − (+5). According to our rule, subtracting (+5) is the same as adding (−5). So, starting at +2, move 5 spaces to the left. You land at −3, and that is the answer. In the system of integers, every subtraction example has an answer.



To multiply two integers, we think of one of them as a multiplier, and look at it for directions telling us what to do with the other one. First we disregard the signs, and multiply as if we were working with natural numbers. Then we pay attention to the signs in this way. If the sign of the multiplier is +, it tells us to *keep* the sign of the other number for the answer. If the sign of the multiplier is −, it tells us to *change* the sign of the other number, and attach it to the answer.

For example, to multiply (−2) × (−3), think of the −2 as the multiplier. Since 2 × 3 = 6, the answer will be either +6 or −6. To find out which it is, we look at the signs. The sign of the multiplier is −, so it tells us to change the sign of the −3 and attach it to the answer. Then the answer is +6. With this rule, every multiplication example in the system of integers has an answer that is an integer.

Division is like multiplication done back-



wards. When we say (+6) ÷ (+2), it is like asking, "What number multiplied by +2 will give you +6?" We find an answer to this question, (+3), without any trouble. But when we say (+5) ÷ (+2), we run into difficulty. This problem asks, "What number multiplied by (+2) will give you (+5)?" In the system of integers, there is no such number. So we see that division is not always possible in this system.

Numbers for the Carpenter

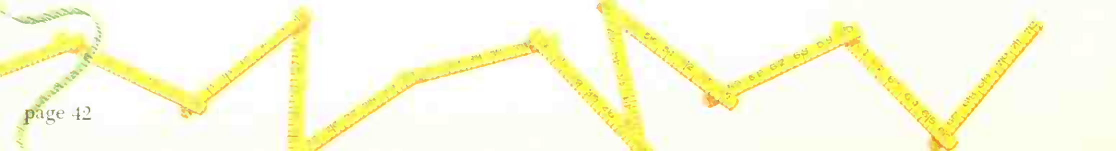
The edge of a carpenter's ruler is like the line on which we have pictured the natural numbers (or positive integers). The numbers 0, 1, 2, 3, and so on are placed at intervals of one inch, starting from the left end of the ruler. When the carpenter measures the length of a board, he puts the zero of his ruler at one end of the board, and looks at the position of the other end. If it lies next to the integer 8, he knows that the length of the board is 8 inches. But sometimes the end of the board lies between two integers. In that case, neither integer gives a very good answer.

To give a better answer for the length, we mark off points between the integers to divide the space between them into equal parts. To attach numbers to these points, we need a new kind of number. We enlarge our number system by putting in fractions. Halfway between 8 and 9 we put $8\frac{1}{2}$. Halfway between 8 and $8\frac{1}{2}$ we put $8\frac{1}{4}$, and so on. The enlarged system is called the system of *rational numbers*. Every rational number can be written as a fraction in more than one way. For example, $\frac{1}{2}$ can also be written as $\frac{4}{8}$ or $\frac{5}{10}$. The integers belong to the system of rational numbers, and they, too, can be written as fractions. The integer 2, for example, can be written as $\frac{2}{1}$ or $\frac{6}{3}$.

In the system of integers, we could not find an answer for the division problem 5 ÷ 2. But in the system of rational numbers we do find an answer to this problem. It is the fraction $\frac{5}{2}$. In the system of rational numbers, division by a number that is different from zero is always possible.

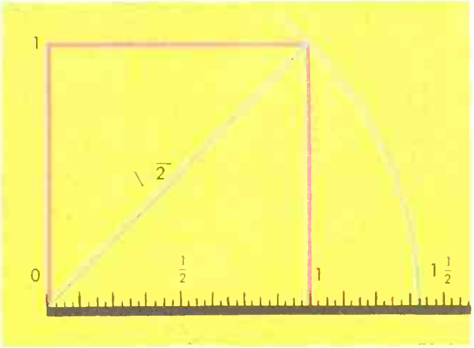
A Number for Every Point

To picture the natural number system, we chose a point on a line and called it 0. Then we attached the other natural numbers at equal intervals to the *right* of the 0. The system of integers gave us more numbers, so that we could attach some at equal intervals to the *left* of the 0. The system of rational numbers gave us still more, so that we could place numbers *between* the integers. Now we have numbers distrib-



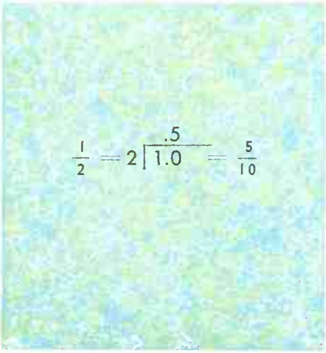
This symbol, meaning "one part of . . .," was used by Egyptians to express a fraction. It was used in combination with a number, as shown below:

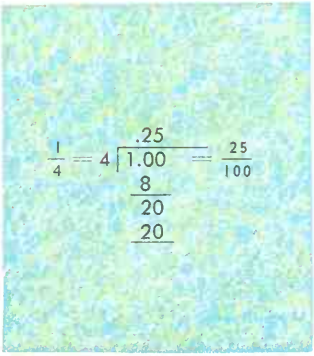
uted rather thickly along the number line, with each number attached to a point. Does the rational number system give us enough numbers to assign a number to every point on the line? Over two thousand years ago, the mathematicians of ancient Greece already understood that it does not. If we make a square, each of whose sides has length 1, we can figure out the length of its diagonal by the rule of Pythagoras (see page 34). We find that the length of the diagonal is $\sqrt{2}$. The diagram above shows how we can locate a point on the number line whose distance from 0 is equal to this length. But it can be shown that there is no rational number that is equal to $\sqrt{2}$. So, in the rational number system, there is no number we can attach to this point. So, we have to expand our number system again.

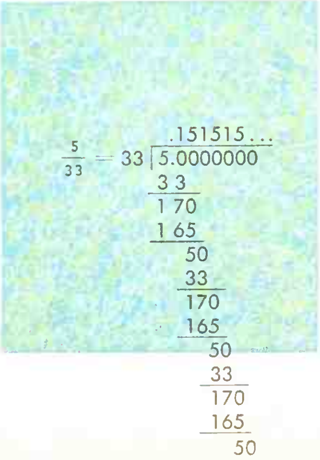


The clue to this next extension of the number system is found in another way in which rational numbers can be written. We can convert a common fraction into a decimal fraction by means of division, as shown in the drawing below. The fraction $\frac{1}{2}$ can be written as .5. The fraction $\frac{1}{4}$ can be written as .25. But to write the fraction $\frac{1}{3}$ we need the decimal .33333 . . ., that has an endless chain of 3's. We call a decimal like this one an *infinite decimal*. The fraction $\frac{1}{2}$ can be written as an infinite decimal, too, by writing it as .500000. . . . The fraction $\frac{5}{33}$ can be written as the infinite decimal .15151515. . . .

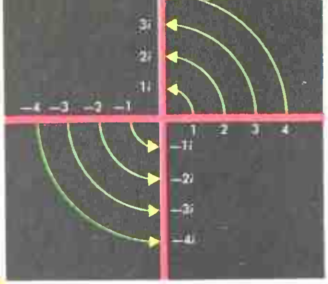
If we examine the infinite decimals we get from rational numbers by long division, we find an interesting feature in them all. Each ends up as a repeating decimal. For example, the decimal .49999 . . . repeats the 9 over and over again. But there are some infinite decimals that do not have a repeating pattern. We get a larger number system by using *all infinite decimals*, whether they repeat or not. This expanded system, made up of all infinite decimals, is called the *real number* system. In this system, we finally get a number for every point on the number line.

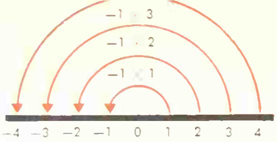




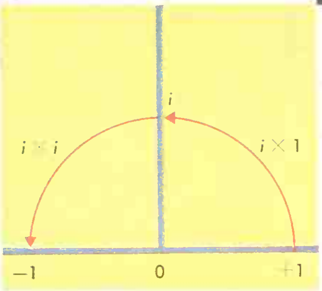


Rotation through 90° multiplies each number by *i*





Rotation through 180° multiplies each number by −1



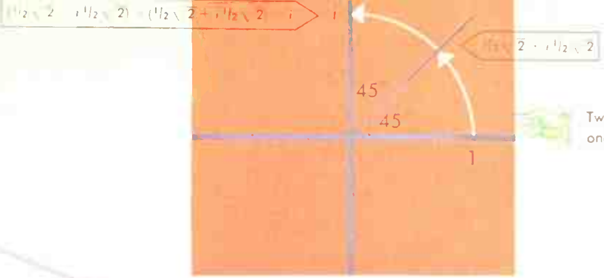
Two 90° rotations equal one 180° rotation

Numbers for the Electrician

The electric current brought to you in wires by the electric company is produced in coils of wire that are rotating in a magnetic field. To study the changes in the current, electricians find it convenient to use numbers to represent rotations. For example, a rotation of 360 degrees can be represented by the number 1. A rotation of 180 degrees can be represented by the number – 1. Performing two rotations, one after the other, is like multiplying their numbers. Thus, $(-1) \times (-1) = 1$, which checks with the fact that two rotations of 180 degrees are like a single rotation of 360 degrees.

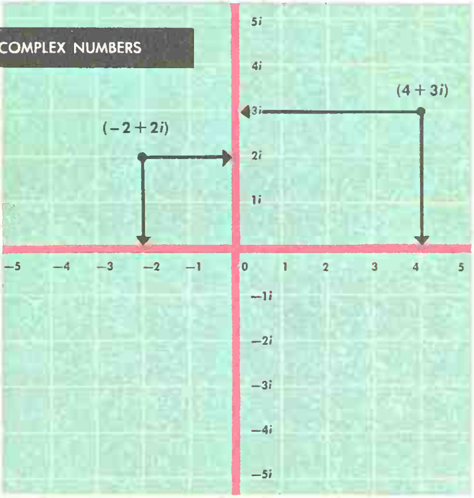
In this scheme, what number can stand for a rotation of 90 degrees? Whatever it is, when it is multiplied by itself, the product should come out −1, which is the number that represents a rotation of 180 degrees. But no real number, when multiplied by itself, can give −1 as a product. This is so because of the rules for multiplying real numbers. The number 0 times itself gives 0 as a product. A positive number times itself gives a positive number as a product. A negative number times itself also gives a positive number as the product. So no real number times itself can give the negative number −1 as a product. This means that there is no real number that can represent a rotation of 90 degrees. To be able to represent every rotation by a number we have to extend the number system once more. The enlarged number system is called the *complex number* system.

In this system, the number that stands for a 90 degree rotation is represented by the letter *i*, and has the property that $i \times i = -1$. Every complex number is written as a real number plus *i* times another real number. A rotation of 45 degrees is written as $\frac{1}{2}\sqrt{2} + i\frac{1}{2}\sqrt{2}$. When we multiply this number by itself, and make use of the fact that $\sqrt{2} \times \sqrt{2} = 2$, while $i \times i = -1$, we find that the product is *i*. This checks with the fact that two rotations of 45 degrees



Two 45° rotations equal one 90° rotation

SYSTEM OF COMPLEX NUMBERS



combined are equal to one rotation of 90 degrees.

To picture real numbers we used the points on a line. To picture complex numbers, we need all the points of a plane. The picture is formed in this way. First draw a horizontal line in the plane and put the real numbers on it. Call this line the *axis of reals*. Now rotate the line 90 degrees counterclockwise. The rotation multiplies every real number by the number *i*. In this way, we attach to each point on the vertical line through 0 a real number multiplied by *i*. These products are called *imaginary numbers*, and the vertical line is called the *axis of imaginaries*.

Now, in order to attach a number to any point in the plane, draw a horizontal line and a vertical line through that point. The vertical line points out a real number on the axis of reals. The horizontal line points out an imaginary number on the axis of imaginaries. The sum of these two numbers is the complex number attached to the point.

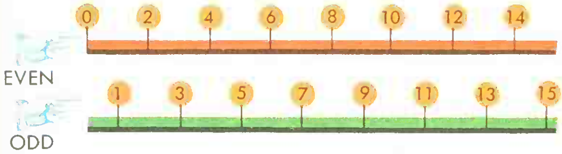
Now we have the complete nest of five number systems. Listing them from smallest to largest, with each number system lying within the next larger one, they are: natural numbers, integers, rational numbers, real numbers, and complex numbers.

Miniature Number Systems

Now that we have seen five distinct number systems, it will not be surprising that there are many more. Some of them are little toy number systems, containing only a few numbers. We shall find examples of them hidden within the natural number system.

A System with Two Numbers

Some natural numbers, like 0, 2, 4, 6, 8, and so on, are divisible by 2. They make up the family of *even* numbers. Those that are not divisible by 2, like 1, 3, 5, 7, and so on, make up the family of *odd* numbers. We can use these two families to build a number system that has only two numbers in it. In this system, each family is thought of as a single number, and we have a way of adding two families, or multiplying two families. To add two families, pick any member of each family, and add them. Then see what family the sum belongs to. For example, to add the odd family to the odd family, add an odd number to an odd number. The result is an even number. So the odd family plus the odd family equals the even family. In the same way, we find that odd plus even equals odd; even plus odd equals odd; and even plus even equals even. These results can be summarized in the following addition table.



+	even	odd
even	even	odd
odd	odd	even

×	even	odd
even	even	even
odd	even	odd

To multiply two families, pick a number from each family and multiply them. Then see which family the product belongs to. The results are shown in the multiplication table above.

In these tables we use the words *odd* and *even* as symbols for the odd family and the even family. We can also use the numerals 1 and 0 as symbols for the families. When you divide an odd number by 2, the remainder is always equal to 1. When you divide an even number by 2, the remainder is always equal to 0. So we can use the remainder that belongs to each family as the symbol for the family. In this scheme, 0 stands for *even*, and 1 stands for *odd*. Using these symbols, the addition and multiplication tables for our little system look like this:

Number System with Only Two Numbers

+	0	1
0	0	1
1	1	0

×	0	1
0	0	0
1	0	1

A System with Three Numbers

To get a number system with three numbers, we break the natural number system into three families, which we shall call 0, 1 and 2. The 0 family this time consists of all natural numbers that give a remainder of 0 when you divide by 3. The members of this family include 0, 3, 6, 9, 12, and so on. The 1 family consists of those numbers, like 1, 4, 7, etc., that give a remainder of 1 when you divide by 3. The 2 family consists of numbers like 2, 5, 8, etc., that give a remainder of 2 when you divide by 3. We add or multiply these families by the same rules used for the odd and even families: Add or multiply any representatives of the families, and then find the family the sum or product belongs to. Following these rules, we get the tables below.

Number System with Only Three Numbers

+	0	1	2
0	0	1	2
1	1	2	0
2	2	0	1

·	0	1	2
0	0	0	0
1	0	1	2
2	0	2	1

If we divide the natural numbers by 4, the possible remainders are 0, 1, 2, and 3. By putting into a single family all the numbers that have the same remainder when you divide by 4, we now get four separate families which we may call 0, 1, 2, and 3. If we divide by 5, the possible remainders are 0, 1, 2, 3, and 4, and we get five different remainder families. So division by 4 leads to a number system with four numbers in it; division by 5 to a number system with five numbers. Here are tables for these systems:

Number System with Only Four Numbers

+	0	1	2	3
0	0	1	2	3
1	1	2	3	0
2	2	3	0	1
3	3	0	1	2

×	0	1	2	3
0	0	0	0	0
1	0	1	2	3
2	0	2	0	2
3	0	3	2	1

Number System with Only Five Numbers

+	0	1	2	3	4
0	0	1	2	3	4
1	1	2	3	4	0
2	2	3	4	0	1
3	3	4	0	1	2
4	4	0	1	2	3

·	0	1	2	3	4
0	0	0	0	0	0
1	0	1	2	3	4
2	0	2	4	1	3
3	0	3	1	4	2
4	0	4	3	2	1

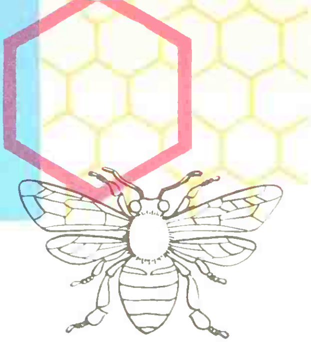
In each of these miniature number systems made up of remainder families, the tables show us how to do addition and multiplication. We find from the addition tables, too, that subtraction is always possible in them. For example, in the number system with only five numbers, let us try the subtraction problem $1 - 4$. We interpret this problem to mean "What number added to 4 gives 1 as the sum?" The table shows that $4 + 2 = 1$, so $1 - 4 = 2$, in this system. What is $1 - 3$ equal to in the system with five numbers? What is it equal to in the system with four numbers?

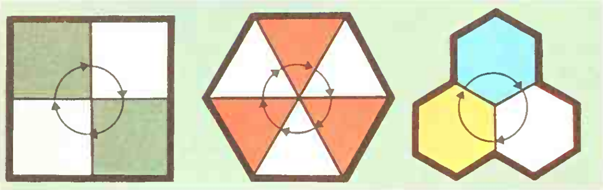
While subtraction is always possible in all the number systems made up of remainder families, division by a number different from zero is always possible only in some of them. In the number system with five numbers, for example, we can find an answer to the problem $1 \div 2$. Interpreting it to mean "What number times 2 equals 1?" we find that the answer is 3. But, in the number system with four numbers, there is no answer to this problem, because, in this system, no number times 2 gives 1 as a product. It turns out that division by a number different from zero is always possible in a number system built out of remainder families *only if the number of families* (which is the same as the divisor used to form the families) *is a prime number.*

There is a way of picturing these small number systems by means of points. Just as the natural number system can be represented by points on a straight line, each of these little systems can be represented by points on a circle. Addition can be carried out in the pictures by moving clockwise. Multiplication can be carried out by repeated addition. For example, to multiply 3×2, start at 0 and move clockwise in three steps, moving two spaces in each step.

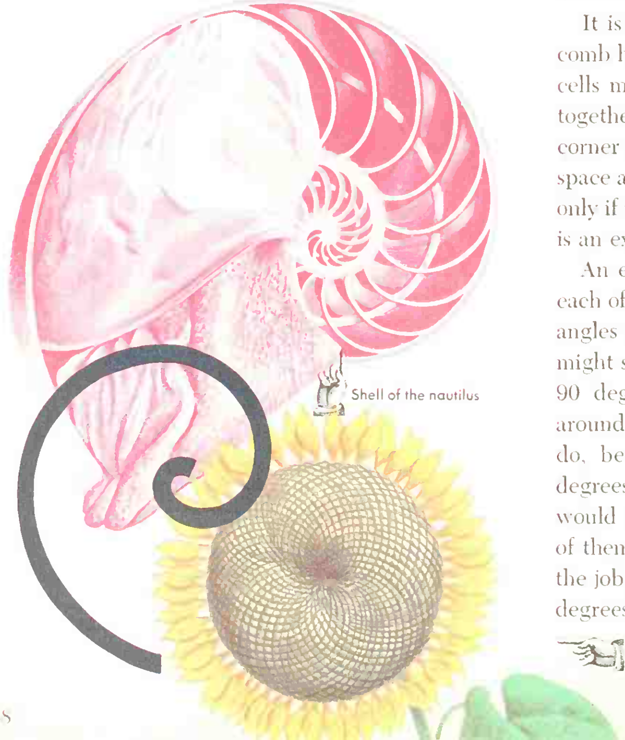
Mathematics in Nature

Snowflake crystal





Only squares, equilateral triangles, or regular hexagons can be fitted together as tiles. In the bee's cell, Nature has chosen the roomiest shape for storage



Shell of the nautilus

Nature has made some beautiful models of the curves and polygons and solids that are studied in mathematics. We find them in the sky, on the earth, and in the sea. Some are living things; others are dead matter.

At the top of this page we see a snowflake. All snowflakes, when crystallized, are built around the same form, the regular *hexagon*. Next to the snowflake we see the hexagon again, in a honeycomb built by bees. Each hexagon in the comb is a cell in which the bees store their honey. The cells fit together like the tiles on a bathroom floor.

It is no accident that the cells in the honeycomb have the shape they do. The shape of the cells must permit three or more of them to fit together at a corner. The angles that lie at one corner must add up to 360 degrees to fill the space around it. So a regular polygon may serve only if the number of degrees in one of its angles is an exact divisor of 360.

An equilateral triangle might serve, because each of its angles contains 60 degrees, so six triangles can fit together around a point. A square might serve, because each of its angles contains 90 degrees, so four squares can fit together around a point. A regular pentagon would not do, because each of its angles contains 108 degrees. Three of these angles lying side by side would not be enough to fill 360 degrees. Four of them would be too much. A hexagon can do the job, because each of its angles contains 120 degrees, so three hexagons fit together at one

The seeds of the sunflower are arranged in a pattern formed by spirals winding from its center corner. No regular polygon with more than six sides would do, because then each angle would contain more than 120 degrees, and three or more of them could not possibly fit together in the 360 degrees around a point. So we see that the only regular polygons that may serve as cells are equilateral triangles, squares, or regular hexagons. Of these three possibilities, the regular hexagon is best, because it stores the most honey between the walls of wax.

Beneath the honeycomb is a shell of the nautilus, an animal that lives in the sea. It has been cut open to show the chambers inside. The curve that winds out from the center is called a *spiral.* At the bottom of the page we see many spirals like it, winding out in two directions from the center of the giant sunflower.

In 1943 the ground opened up in a cornfield near Parícutin, Mexico. Hot lava rose up out of the ground and spread over the field. Layer fell on layer, and as the cinders rolled to the ground, they formed a perfect *cone.*

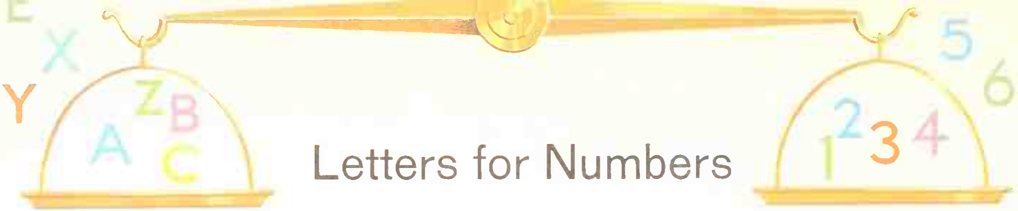
In the sky above, the moon, the sun, and the stars are all *spheres.* We can see the spherical shape clearly in the moon, which is nearest to the earth.

We have already seen some regular solids in dead matter, like crystals of diamond or salt. There are more of them among living things, too. The pictures on the side show the skeletons of some radiolarians. They are tiny animals that live in the sea. The floors of the Pacific and Indian Oceans are covered with such skeletons, left by animals that lived millions of years ago. The skeleton at the left is an almost perfect *octahedron,* or eight-faced solid. The one on the right is a *dodecahedron,* with twelve faces. The third is an *icosahedron,* with twenty faces.

Cinders from an erupting volcano roll to the ground to form a cone

The sun, the moon and the stars are spheres

These skeletons of radiolarians, which are tiny sea animals, are shaped like regular solids



Letters for Numbers

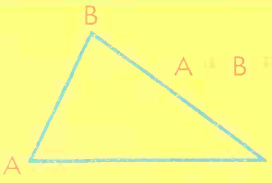
We know that $1 + 2 = 2 + 1$, $2 + 3 = 3 + 2$, and $4 + 7 = 7 + 4$. We can make any number of true statements like this. Simply write a first number plus a second number on one side of the equals sign. On the other side of the equals sign let the numbers change places.

Instead of writing each statement separately, we can write them all at once in this way. Let the letter a stand for any number. Let the letter b stand for any other number. Then we simply write: $a + b = b + a$. When we do this, we have taken the step from arithmetic to *algebra*.

n TH TRIANGLE NUMBER

$$\frac{n(n + 1)}{2}$$

ANGLES OF A TRIANGLE



$$A + B + C = 180°$$

ANGLES OF A POLYGON

n = NUMBER OF SIDES IN A POLYGON

s = SUM OF ITS ANGLES

$$s = (n-2) \times 180°$$

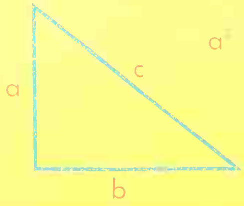
ANGLES OF A REGULAR POLYGON

n = NUMBER OF ANGLES

A = NUMBER OF DEGREES IN ONE ANGLE

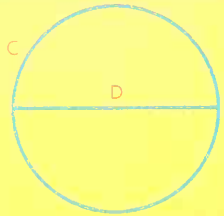
$$A = \frac{(n - 2)\,180}{n}$$

RULE OF PYTHAGORAS



$$a^2 + b^2 = c^2$$

CIRCUMFERENCE OF A CIRCLE



C = CIRCUMFERENCE

D = DIAMETER

$$C = \pi D$$

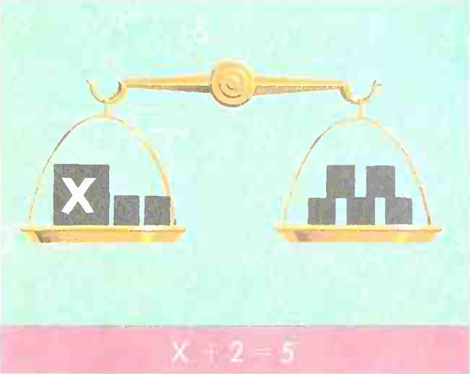
$$(\pi = 3.1416)$$

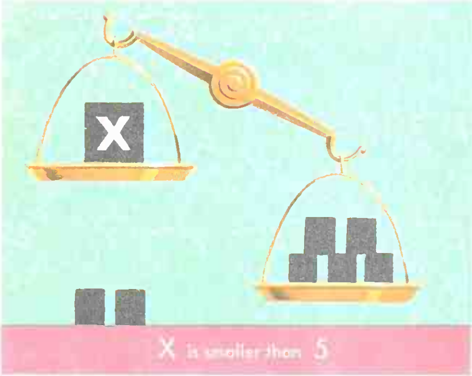
In algebra, we let letters stand for numbers. It is like using a code for saying many things in a small space. In this code, we do not use × to mean "times," because we may mix it up with the letter x. We show multiplication by using a dot instead, or by writing the multipliers side by side with no symbol between them. In this code, $a \cdot b$ means: "the number that a stands for, multiplied by the number that b stands for." When the same multiplier is used over and over again, we use the same short way of writing the product that was used for square numbers and cubic numbers on pages 19 to 20. When we write x^4, we mean $x \cdot x \cdot x \cdot x$. Some of the rules described in earlier sections of this book are shown in code on the preceding page.

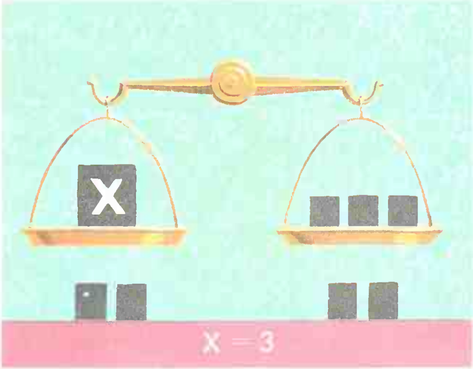
Here is a statement in code that is not always true: $x + 2 = 5$. It is not true if x stands for 7, because $7 + 2$ is not 5. It is true if x stands for 3. A statement like this is called an *equation*. To solve the equation means to find the number which makes it a true statement.

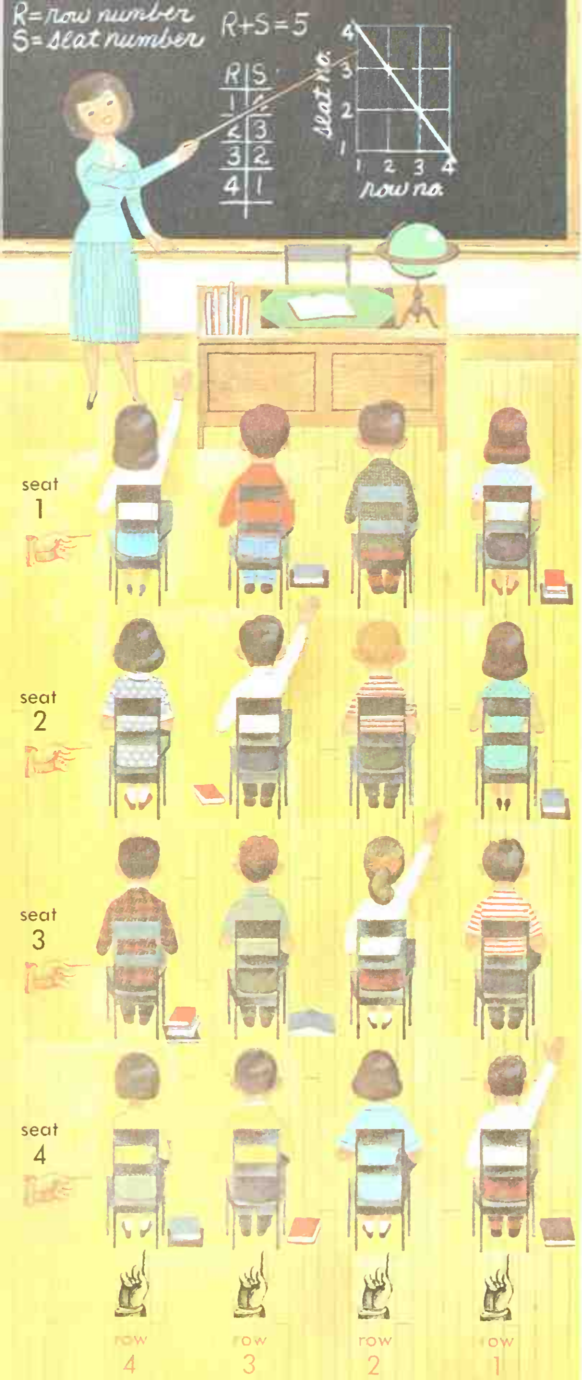
An equation resembles a balance scale. The $x + 2$ is supposed to balance the 5 the way equal weights balance on a scale. If we change one weight on a scale, we can make the weights balance again by changing the other weight in the same way. This is a hint on how we can solve an equation: Simply change both sides of the equation in the same way. Since 5 is the same as $3 + 2$, the equation $x + 2 = 5$ says: $x + 2 = 3 + 2$. If we take away 2 from both sides, they will still balance each other. In this way we find that $x = 3$ is the answer. To solve the equation $3x = 12$, we divide both sides by 3, and we get $x = 4$ as the answer.

Can you solve the equation $3x - 4 = 8$? To find the answer, add 4 to each side of the equation, and then divide each side by 3.









Your Number in Space

Many big cities are divided into blocks by streets running in one direction, and avenues crossing them at right angles. In these cities you can locate any street corner by mentioning two numbers: the number of the street and the number of the avenue that cross there. In the map on the right we can describe the corner marked by the cross by using the pair of numbers (3,4), if we agree that the first is the street number and the second is the avenue number.

In the same way, we can locate any seat in a classroom by mentioning two numbers: the row number and the seat number. In the classroom in the picture, the rows are numbered from right to left, and the seats are numbered from front to back. The teacher has just said to the class, "Raise your hand if your row number and seat number add up to 5." The locations of the pupils who raised their hands are given by the pairs of numbers, (1,4), (2,3), (3,2), and (4,1), where the first number in each pair is the row number. If we let r stand for row number, and s stand for seat number, we can list these locations in the table shown on the blackboard. We can also describe these locations by the equation: $r + s = 5$. Notice that the pupils whose hands are up are arranged in a straight line. The equation is a description of the locations on this line. Also, the line is a picture of the number pairs described by the equation.

This is an example of an important discovery made in the seventeenth century by the great French mathematician and philosopher René Descartes. An equation with two unknowns can be pictured by means of a line (straight or

curved), called a *graph*. Also, every line is described by means of an equation. The branch of mathematics that grew out of this discovery is called *analytic geometry*.

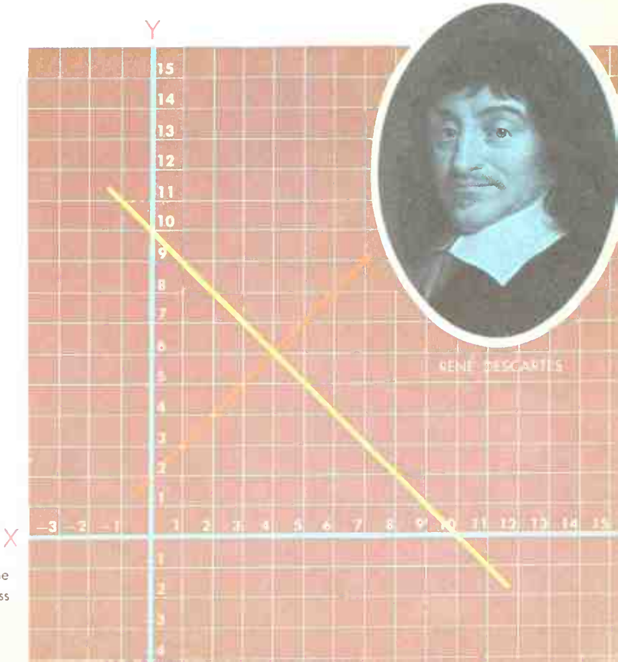
The connection between a line and its equation is usually shown in this way: The paper on which the line is drawn is divided into squares by two sets of lines that cross each other, like streets and avenues. To number these lines, we pick out one line in each set and call it the zero line, or axis. Then we number the lines by counting boxes away from each axis, in both directions. In one direction we attach a plus sign to each number. In the opposite direction we use a minus sign, so we can tell them apart. Then each intersection is described by a pair of numbers, telling you how far it is right or left and up or down from the axes. We call the right or left number the x number. We call the up or down number the y number. Numbers with fractions describe points that are between the lines.

The graphs of equations give us a convenient way of solving some problems by means of a picture. To show how this is done, let us solve this puzzle: The length of a rectangle is two inches more than the width, and the sum of the length and width is 10 inches. Find the length and width.

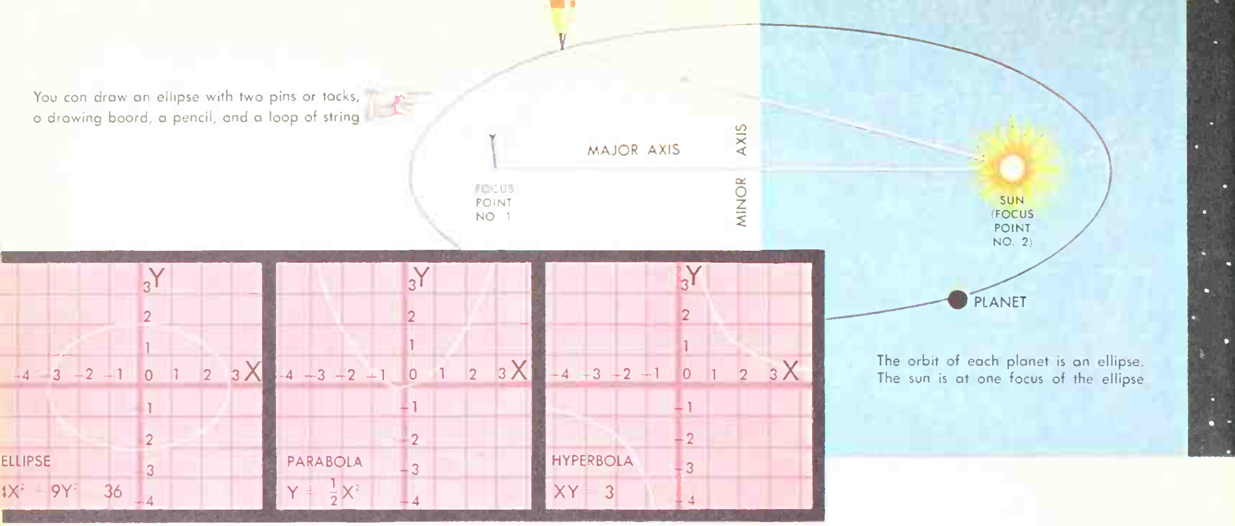
We begin by letting y stand for the number of inches in the length, while x stands for the number of inches in the width. The fact that the length is two inches more than the width leads to the equation $y = x + 2$. The graph of this equation is shown in red in the diagram below. The fact that the sum of the length and width is 10 inches leads to the equation $x + y = 10$. The graph of this equation is shown in yellow. The two graphs cross at a single point. At this point, $x = 4$ and $y = 6$. This is the only pair of numbers that satisfies both equations and gives an answer to the puzzle. The width is 4 inches, and the length is 6 inches.

X = NUMBER OF INCHES IN WIDTH
Y = NUMBER OF INCHES IN LENGTH

Y = X + 2		X + Y = 10	
if X = 0	Y = 2	if X = 0	Y = 10
if X = 1	Y = 3	if X = 3	Y = 7
if X = 2	Y = 4	if X = 7	Y = 3
if X = 3	Y = 5	if X = 10	Y = 0



The answer to the puzzle is found at the point where the red and yellow lines cross


You can draw an ellipse with two pins or tacks, a drawing board, a pencil, and a loop of string

The orbit of each planet is an ellipse. The sun is at one focus of the ellipse

Bridges, Planets and Whispering Galleries

There are three graphs printed above, together with their equations. The names of these curves are *ellipse*, *parabola*, and *hyperbola*. We meet them often in nature and in things we make.

An earth satellite, launched with a speed of 18,000 miles per hour, follows an ellipse around the earth. The paths of the planets around the sun are ellipses. The paths of some comets are parabolas.

To make an ellipse quickly and accurately, place two thumbtacks through your paper on a drawing board. Put a closed loop of string around the tacks and hold the point of your pencil against the string, inside the loop. Then move your pencil so that the string is always pulled tight. The points where the tacks are, are called the *foci* of the ellipse.

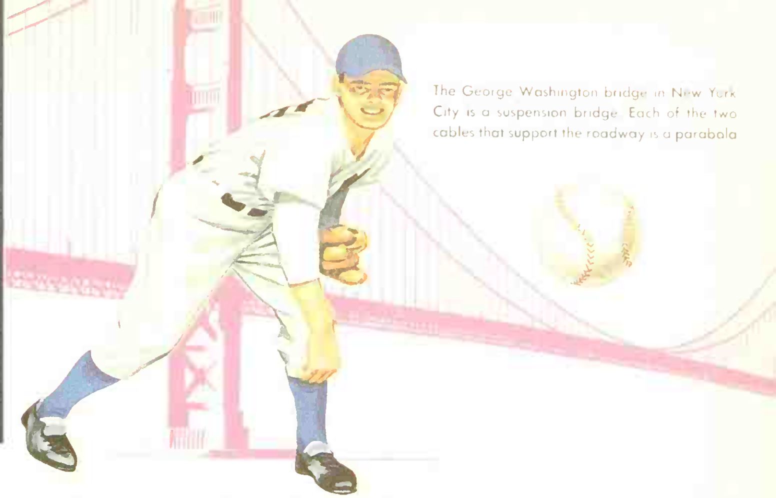
You can also make an ellipse by folding paper. First make a circle on transparent paper. Then choose a point (*P*) inside the circle, but not at the center. Fold the paper over so that *P* falls on the circle, and press the crease flat. Now move *P* around the circle, a short distance each time, and make a new crease for each position. After *P* has gone around the circle, you will find the creases surround an ellipse. You can also make a hyperbola in the same way if you choose the point *P* outside the circle, and follow the same procedure you did to make the ellipse.

An ellipse that spins around the line through its foci sweeps over a surface shaped like a football. Some rooms, like the Mormon Tabernacle in Salt Lake City, Utah, have domes with this shape. In these rooms, if a person whispers near one focus, he can be heard at the other focus, although he cannot be heard at many places in between.

This happens because the dome catches any

Ellipse formed by tilting a glass

The orbit of an earth satellite—the moon, or man's artificial satellites—is an ellipse



The George Washington bridge in New York City is a suspension bridge. Each of the two cables that support the roadway is a parabola

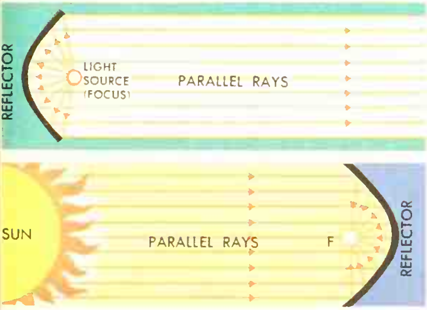
sound coming from one focus and reflects it back to the other. In this way, the sound that was scattered and weakened as it spread out from one focus through the chamber is concentrated at the other focus and restored to almost its original loudness.

Here are some more places where we find these curves:

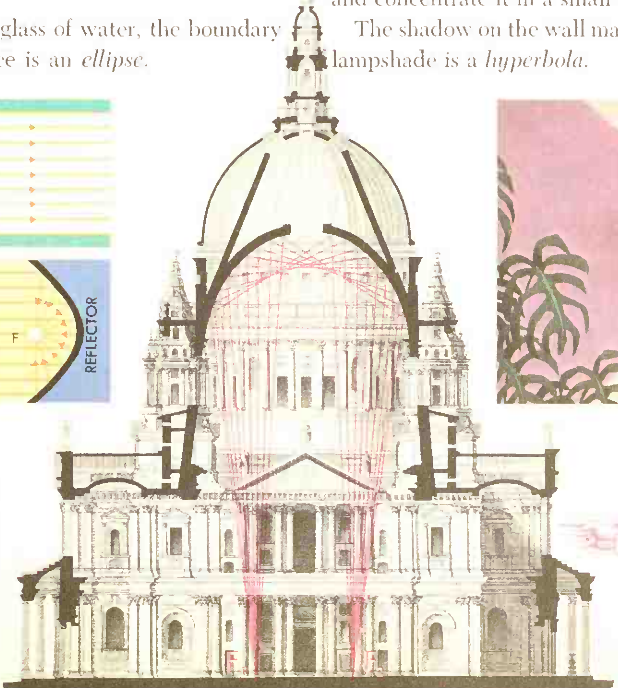
When you tilt a glass of water, the boundary of the water surface is an *ellipse*.

The path of a ball thrown through the air is a *parabola*. The cable supporting the roadway of a suspension bridge is a *parabola*. The roadway is a *parabola*, too, and a cross-section of a searchlight reflector is a *parabola*. The same type of parabolic reflector is used in solar furnaces to catch sunlight falling on a large area and concentrate it in a small spot.

The shadow on the wall made by a cylindrical lampshade is a *hyperbola*.



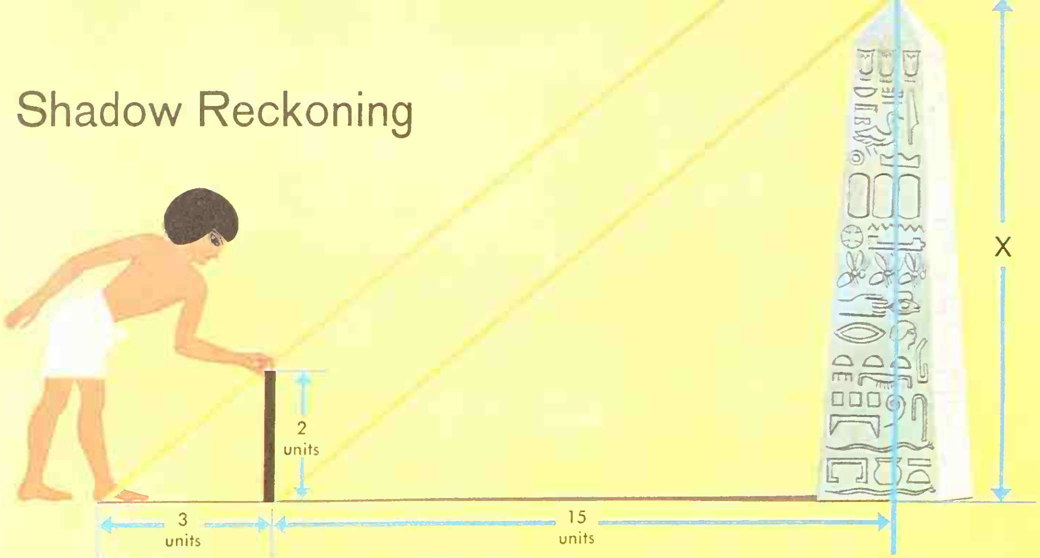
In a searchlight, a parabolic reflector is used to send out parallel rays. In a reflecting telescope, however, it is used to catch parallel rays and bring them to a focus



A hyperbola is formed where the cone of light from the lamp shade is cut by the flat surface of the wall

In many domes such as this one, a whisper at F_1, after being reflected twice by the dome, can be heard clearly at F_2

Shadow Reckoning

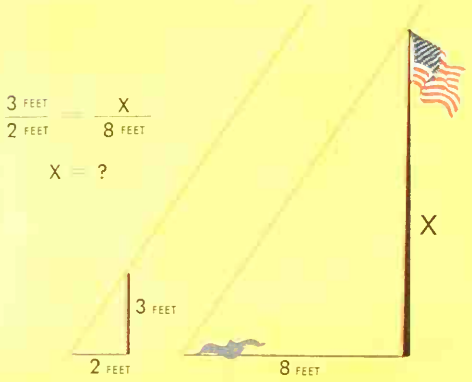


If you stand out in the sunshine before sunset, you cast a long shadow. Your shadow is like a picture of you, rolled flat, painted black, and stretched out on the ground. If you look at the shadows of other objects, you see that they are stretched in the same way. If you are 5 feet tall, and your shadow at some moment is 10 feet long, we compare these lengths by writing the fraction $\frac{5}{10}$. We call this fraction a *ratio*. Since the shadows of all objects are stretched in the same way, you get the same ratio when you compare the height of anything with the length of the shadow it casts at the same time. This fact leads to a way of measuring the height of a tall object by measuring its shadow instead.

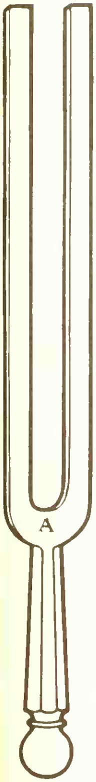
Let us note a simple fact about equal ratios. The ratios $\frac{5}{10}$ and $\frac{7}{14}$ are equal, because they both reduce to $\frac{1}{2}$. If you multiply the 5 by the 14, you get 70. If you multiply the 10 by the 7, you get 70, too. *In equal ratios, you get equal products when you multiply the numerator of one ratio by the denominator of the other.*

This rule helps us do shadow reckoning. If a tower casts a shadow 15 feet long at the same time that a 2-foot stick casts a 3-foot shadow, how high is the tower? Let us call the height of the tower x. When we compare it to the length of its shadow, we get the ratio $\frac{x}{15}$. For the stick, the ratio of height to shadow is $\frac{2}{3}$. To show that they are equal we write $\frac{x}{15} = \frac{2}{3}$. The rule tells us that $3 \cdot x = 2 \cdot 15$, or $3x = 30$. Dividing both sides by 3, we find that $x = 10$. So the height of the tower is 10 feet.

Problem: If a flagpole casts a shadow that is 8 feet long at the same time that a 3-foot stick casts a 2-foot shadow, how high is the flagpole?



Vibrations, Wheels and Waves

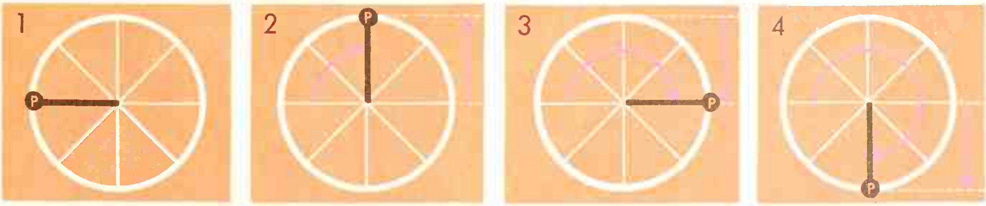


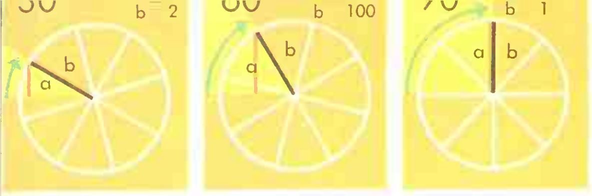
The world is full of vibrations which create disturbances in space. When these disturbances move, they move in the form of waves. When a ship glides across a lake, it sends water waves rolling away from its prow. When we speak, the vibrations of our vocal cords send sound waves out into the air. Light consists of waves sent out by vibrations within the atom.

To study these different kinds of waves, scientists compare them with a model wave that is easily produced with the help of a turning wheel. They find that the simplest of the waves are just like the model wave. The more complicated waves are often made up of several simple waves that are combined.

Let us see how a turning wheel can produce the model wave. First we pick out one spoke of the wheel, and watch it as the wheel turns with a steady speed. We shall pay special attention to the point on the spoke that is at the rim of the wheel. In the diagram this point is called *P*. As the wheel turns, this point is sometimes above the level of the center of the wheel, and sometimes below it. It keeps moving up and down with a steady rhythm. This up and down motion is the vibration we shall use to form a model wave.

Let us trace out the motion of the point *P* as the wheel makes one full turn. When the spoke is horizontal, *P* is on the same level as the center of the wheel. So its height above this level is 0. As the wheel turns, the height of *P* at first increases. The height is greatest after the wheel has turned 90 degrees, and the spoke is vertical. Then the height decreases as the wheel turns some more. When the amount of turning reaches 180 degrees, the spoke is horizontal again, and the height of *P* is 0 once more. Then *P* begins to fall below the level of the center of the wheel. It reaches its lowest position when





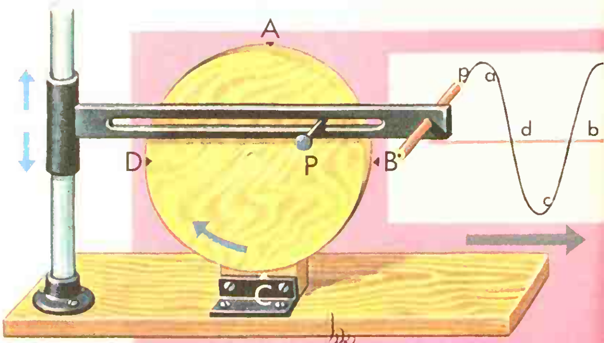
the wheel has turned 270 degrees. Then it rises again. After one complete turn of 360 degrees, the rising-falling cycle starts again.

The greatest height that *P* ever reaches is the length of the spoke of the wheel. In each position, the height it reaches may be described as a fraction of the length of the spoke. This fraction is obtained by dividing the height of *P* by the length of the spoke. When *P* is above the level of the center, we represent its height by a positive number. When *P* is below the level of the center, we represent its height by a negative number. So the fraction, too, may come out positive or negative. The size of the fraction depends on the angle through which the wheel has turned, starting from the position in which the spoke is horizontal.

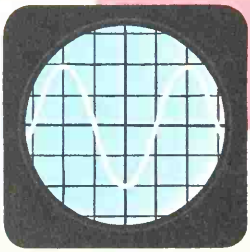
To show that the fraction is related to the angle, we mention the angle in the name that is given to this fraction. It is called the *sine of the angle*. When the angle is 0 degrees, the height of *P* is 0, so the sine of 0 degrees is 0. When the angle is 30 degrees, measurement shows that the height of *P* is half the length of the spoke. So the sine of 30 degrees is $\frac{1}{2}$ (or .50). When the angle is 60 degrees, the height of *P* is about .87 times the length of the spoke, so the sine of 60 degrees is about .87. When the angle is 90 degrees, the height of *P* is the same as the length of the spoke, so the sine of 90 degrees is 1. The values of the sine for other angles can be calculated the same way.

Now we use the motion of *P* to produce a wave. First we drive a nail into the spoke at *P*. We insert this nail in a long groove cut into a horizontal bar that is free to slide up and down, guided by a vertical post that supports it. As the wheel turns, the nail moves to the left or right in the groove, and pushes the bar up or down. The up and down motion of the bar simply copies the up and down motion of the point *P*. Now we attach a pencil to one end of the bar, and move a sheet of paper to the right as the pencil point presses against it. As the wheel turns, the pencil traces a wavy line on the paper. This line is called a *sine wave*.

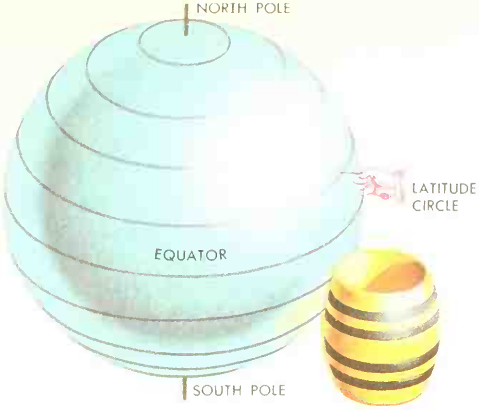
The drawing below shows some waves on the face of an *oscilloscope* tube. A stream of electrons is moving back and forth in the tube, producing a glowing line on the face. The wave is produced by making the stream move up and down as it moves back and forth. Scientists study different vibrations by converting them into electrical signals that they can feed into an oscilloscope tube. Then they examine the shape of the wave formed on the face of the tube.

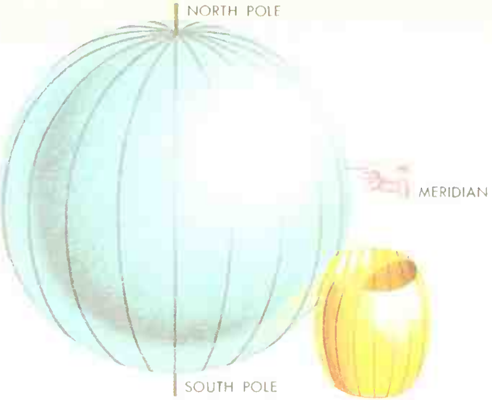


As P passes points A, B, C, D, it moves the bar so that the pencil (p) traces the sine wave a, b, c, d, on the moving sheet of paper



The sine wave as it appears on the oscilloscope screen





Our Home the Earth

The earth is a sphere. We describe the position of any point on its surface by means of two numbers, called the *latitude* and *longitude* of the point. They fix the position of a place on the earth in the same way that the numbers of a street and avenue that cross each other fix the position of a street corner in a big city. They identify the place as the intersection of a *latitude circle* and a *meridian.*

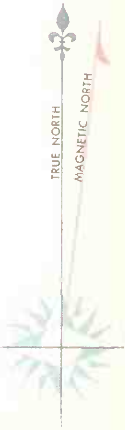
Latitude Circles: The earth is spinning like a top. The North Pole and South Pole lie on the axis around which it spins. A circle around the earth, halfway between the poles, is called the *equator.* The latitude circles girdle the earth like hoops around a barrel. Points that are on the same latitude circle are the same distance from the equator. The latitude of a point is the number of degrees through which a radius of the earth would have to turn, north or south, to move from the equator to the latitude circle.

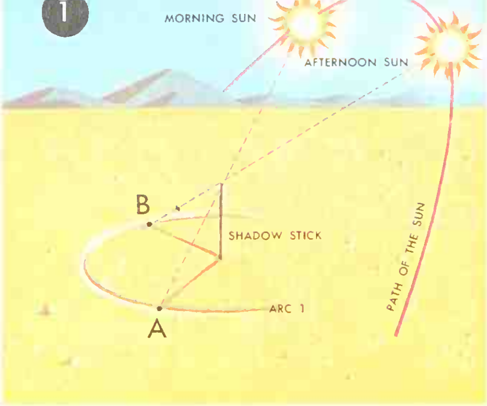
Meridians: The meridians are half-circles that join the North Pole to the South Pole. The zero meridian is the one that passes through Greenwich, England, where a naval observatory is located. We can imagine the Greenwich meridian turning east or west to reach the position of any other meridian. The number of degrees through which it would have to turn is called the longitude of that meridian.

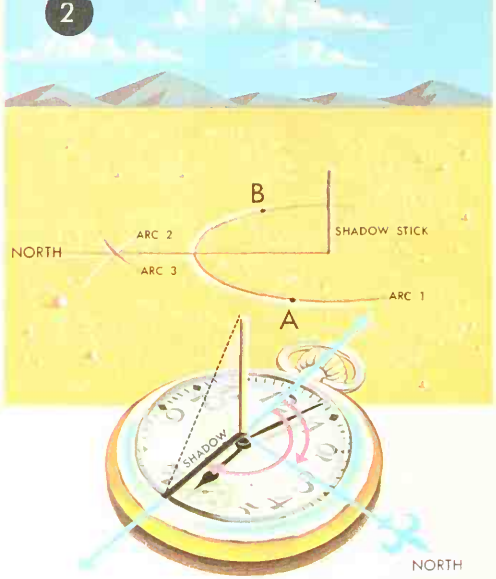
Through every point on the earth's surface, except the North and South Poles, there is only one meridian. In one direction, it leads to the North Pole. This is the direction we call North. When you stand outdoors where you live, how can you tell which way is north? You may think that a magnetic compass will answer this question for you, but it won't.

A magnetic compass, even if it works without interference, doesn't point to the North Pole. It points to the *magnetic* north pole, which is somewhere in Hudson Bay. Besides, nearby masses of iron, like the steel frame of a building, interfere with its operation. So the direction pointed out as north by a compass is usually wrong. The compass is useful for finding true north only if you know what the error of the compass is, so you can subtract the error.

To find the direction of true north, you have to get help from the rotation of the earth. One way of doing this is to use a shadow stick, driven vertically into the ground. The rotation of the earth makes the sun seem to rise in the east,





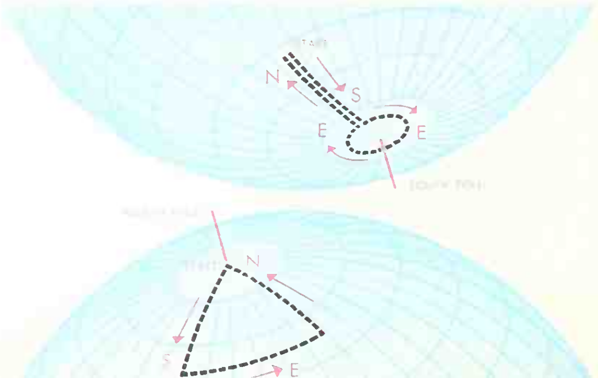


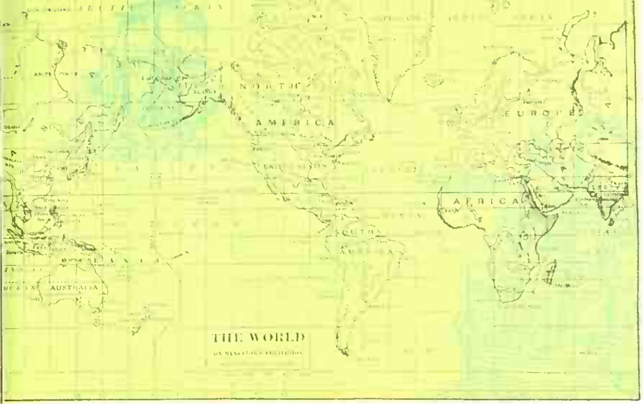
cross the sky, and set in the west. As the sun changes its position in the sky, the shadow of the stick turns and also changes in length. When the shadow is shortest, it points to true north. This happens at about noon.

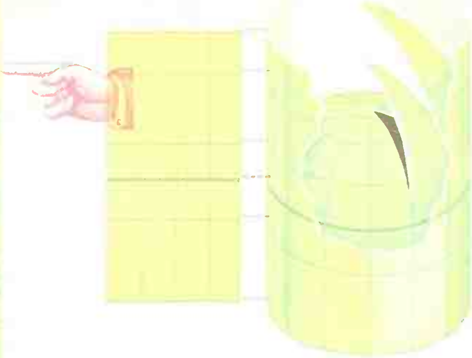
The noontime shadow changes in length very slowly. So it is difficult to locate north accurately by watching the noontime shadow. You can do it more accurately by watching the shadow in the morning and afternoon, when it changes length more rapidly. With the help of a rope tied to the shadow stick, make a circle on the ground around the stick. Locate the positions of the shadow, in the morning and in the afternoon, when the end of the shadow lies on the circle. True north is halfway between these positions, and can be located with a rope and stakes as shown in the drawings.

A fairly accurate and quick way of locating north uses a small shadow stick and a watch. Hold the watch level, with a thin stick standing vertically over the center of the watch. Turn the watch until the hour hand is under the shadow. Then the direction of north will be halfway between the shadow and the twelve.

Locating north is easiest at the South Pole. There every direction is north! At the North Pole, every direction is south. This peculiar fact about the poles inspired this well-known puzzle: A hunter leaves his tent and walks one mile south and then one mile east. He shoots a bear and then heads north with it. After traveling one mile, he arrives back at his tent. What is the color of the bear? The answer was supposed to be "White," because whoever invented the puzzle thought the only place where a path that goes one mile south, then one mile east, and then one mile north could form a closed loop is near the North Pole, where polar bears are white. (See the diagram.) It could also happen near the South Pole, as shown below, except that there are no land mammals in Antarctica.

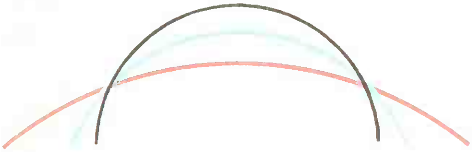




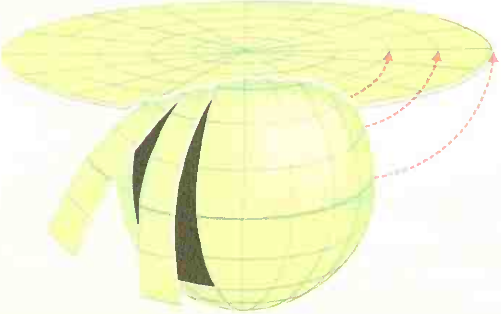


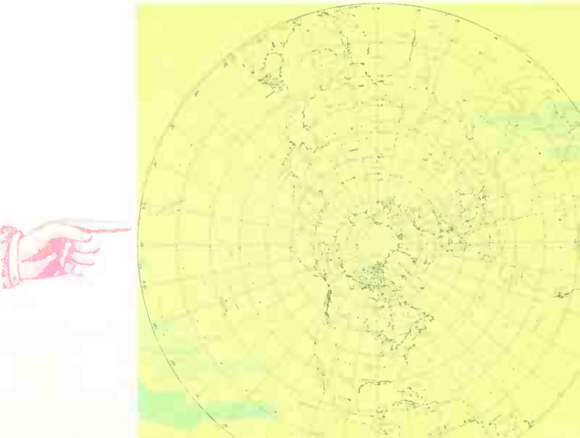
Distances on the Earth

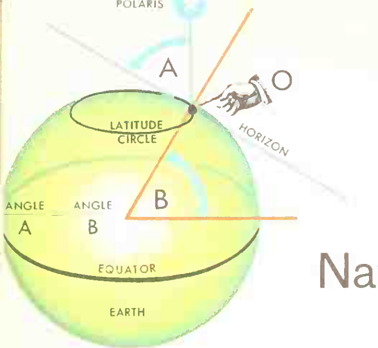
On a flat surface, the shortest path between two points is a straight line. The earth is not flat, so there are no straight paths on it. On the earth, the shortest path between two points is the one that curves the least. The larger a circle is, the less it curves. So the shortest path between two points on the earth is along the largest circle that joins them. Such a circle is called a *great circle*. Meridians and the equator are examples of such great circles on the earth.



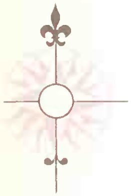
In the days before the airplane, the usual way of going to Europe from North America was by ship across the Atlantic Ocean. Ship lanes ran approximately east and west, and we used maps known as Mercator maps which were built around this east-west way of traveling. Now that we have airplanes, it is possible to travel shorter routes between Europe and North America by following great circle paths. Many of these great circle paths run more nearly north and south than east and west. They are demonstrated by means of a circumpolar map that shows what the earth looks like from above the North Pole. The circumpolar map shows that places that seemed to be far apart on the old maps are really quite close to each other.





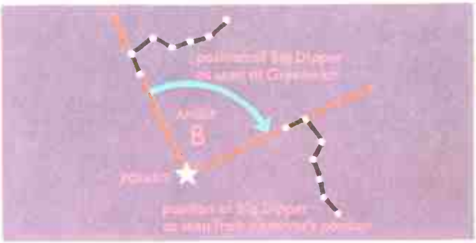


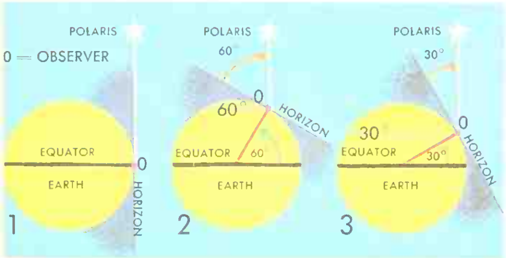
Navigation

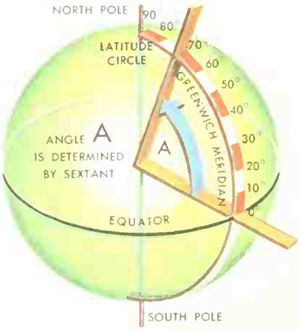


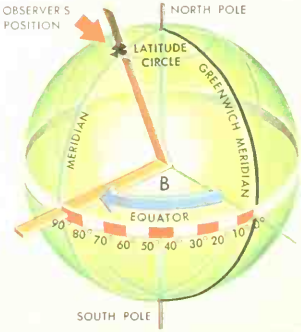
A navigator has two kinds of problems to solve. One is to find out where he is on the earth. The other is to figure out what course his ship should take to go from one place to another. His tools for solving these problems are a compass, a sextant, a clock, and an almanac. His compass, after he corrects for its error, tells him which way is north, so he can measure directions correctly. With his sextant he can measure the height of the sun, the moon, or a star above the horizon. His clock tells him the time in Greenwich, England. His almanac tells him how the sky looks at Greenwich any day of the year, any time of the day. With all this information, he can figure out the answer to his problems.

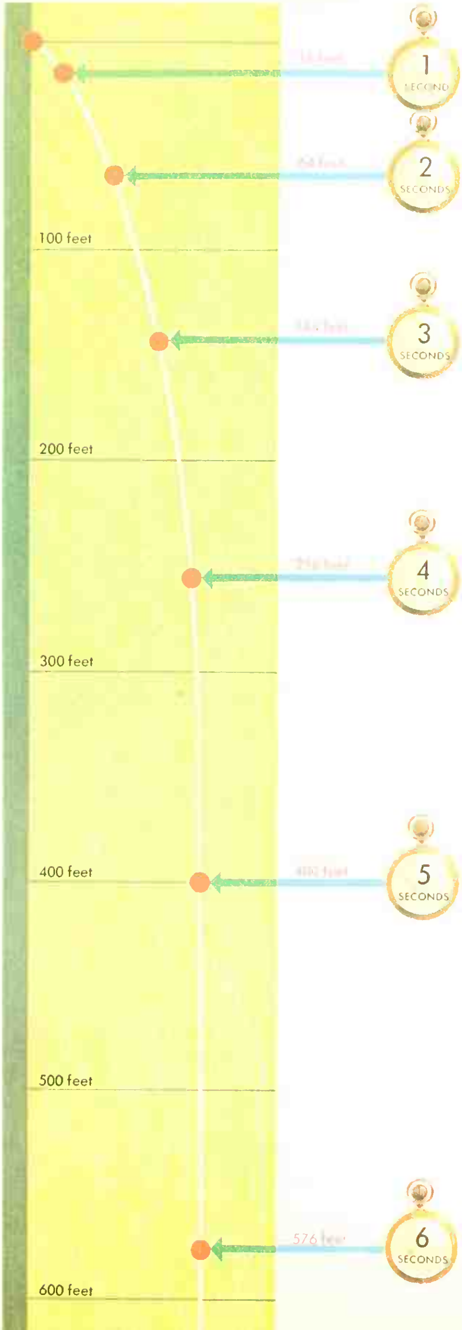
Let us see how he can locate his position on the earth. The earth is a sphere spinning on its axis. The axis points almost directly to Polaris, the North Star. The diagram below shows men in different positions on the earth looking at Polaris. The man on the equator sees Polaris on his horizon. For the others, the line to Polaris makes an angle with the horizon. The further north the man is, the larger the angle is. So, measuring this angle tells him how far north he is above the equator. If the angle is 60 degrees, then he knows he is somewhere on the latitude circle 60 degrees above the equator. Now he has to find out where he is on that circle. This circle is crossed by the meridian that passes through Greenwich, England. His clock tells him the time at Greenwich. His almanac tells him what the sky looks like there. The sky he sees above him looks different. Compared to the sky as seen from Greenwich, it looks as though it were turned through an angle. The amount of this turning tells him how far around the earth he is from the place where the meridian through Greenwich crosses his latitude circle. This information fixes his position.











Mathematics for a Changing World

We live in a world of change. People and things move about. Animals and plants grow in size and numbers. When things change, we often have to know the *speed* with which they change. It is easy to calculate it if the speed is steady. Suppose a car travels 120 miles in 3 hours, moving at a steady speed. Then its speed must be 40 miles an hour. We find it by using the rule: *Speed = Distance ÷ Time*. But speeds are not always steady. A speed can change, too. For example, when something falls from a great height, the longer it falls, the faster it falls. How do we find its speed in this case?

Average Speed = distance / time

Suppose we drop a stone from a cliff and take a movie of its fall. The movie shows us where the stone is at any time. After 1 second, the stone has fallen 16 feet. After 2 seconds, it has fallen 64 feet. How far it gets in 3, 4 and 5 seconds is shown in the table below. This table gives us the rule: *Distance* = *16* × *(time)*^{2}. From this rule we can figure out how far the stone falls in any number of seconds. In 6 seconds, for example, the distance would be $16 \times 6^2 = 16 \times 6 \times 6 = 576$ feet.

distance = $16 \times (\text{time})^2$

TIME OF FALL IN SECONDS	DISTANCE IN FEET
1	$16 = 16 \times 1^2$
2	$64 = 16 \times 2^2$
3	$144 = 16 \times 3^2$
4	$256 = 16 \times 4^2$
5	$400 = 16 \times 5^2$

Distance the stone fell

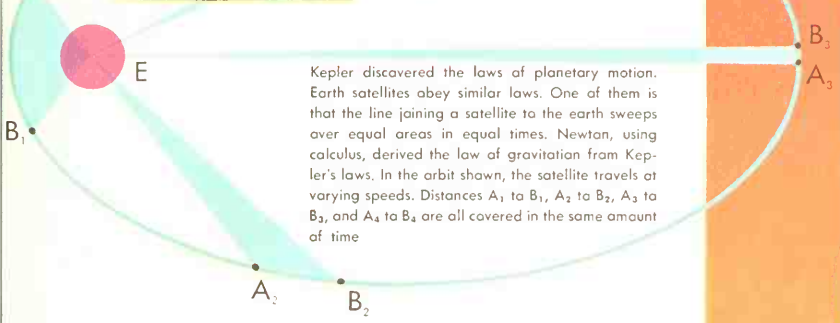
During the 1st second the stone fell 16 feet.
During the 2nd second, the stone fell $64 - 16 = 48$ feet.
During the 3rd second, the stone fell $144 - 64 = 80$ feet.
During the 4th second, the stone fell $256 - 144 = 112$ feet.
During the 5th second, the stone fell $400 - 256 = 144$ feet.

Now let us see how far the stone falls during one second alone. During the first second, it fell 16 feet. During the first two seconds it fell 64 feet. To find out how far it fell in the second second alone, we take away the 16 feet it fell in the first second: $64 - 16 = 48$. The table shows the calculation for each of the other seconds. So we see that the stone fell 16 feet during the first second, 48 feet during the second second, 80 feet during the third second, and so on. Its speed increases as it falls.

How fast is the stone falling after three seconds? We may use our table to get an estimate, or rough answer. During the third second, the stone fell 80 feet. So our estimate is that, after three seconds, its speed was 80 feet per second. But we know that our estimate is wrong, because the speed was changing all through that second. Eighty feet per second is only an average speed. The stone actually moved more slowly than that at the beginning of the second, and it moved faster than that at the end of the second. We can get a better estimate by getting its average speed during a shorter period of time, when the speed had less of a chance to change. For example, during the first $2\frac{1}{2}$ seconds, the stone fell $16 \times 2\frac{1}{2} \times 2\frac{1}{2}$ feet, or 100 feet. During the first three seconds it fell 144 feet. So during the last half-second of this three-second fall, the stone fell $144 - 100$ feet, or 44 feet. Now our estimate of its speed after 3 seconds is 44 feet per half-second, or 88 feet per second. By the same kind of calculation, we find that its average speed during the last quarter-second of its three-second fall is 92 feet per second. Its average speed during the last eighth of a second

Distance fallen in almost 3 seconds		Distance and average speed in last part of 3-second fall		
TIME OF FALL IN SECONDS	DISTANCE IN FEET	PART OF SECOND	DISTANCE IN FEET	AVERAGE SPEED IN FEET PER SECOND
$2\frac{1}{2}$	$16 \times 2\frac{1}{2} \times 2\frac{1}{2} = 100$	Last $\frac{1}{2}$ second	$144 - 100 = 44$	$44 \times 2 = 88$
$2\frac{3}{4}$	$16 \times 2\frac{3}{4} \times 2\frac{3}{4} = 121$	Last $\frac{1}{4}$ second	$144 - 121 = 23$	$23 \times 4 = 92$
$2\frac{7}{8}$	$16 \times 2\frac{7}{8} \times 2\frac{7}{8} = 132\frac{1}{4}$	Last $\frac{1}{8}$ second	$144 - 132\frac{1}{4} = 11\frac{3}{4}$	$11\frac{3}{4} \times 8 = 94$
$2\frac{15}{16}$	$16 \times 2\frac{15}{16} \times 2\frac{15}{16} = 138\frac{1}{16}$	Last $\frac{1}{16}$ second	$144 - 138\frac{1}{16} = 5\frac{15}{16}$	$5\frac{15}{16} \times 16 = 95$

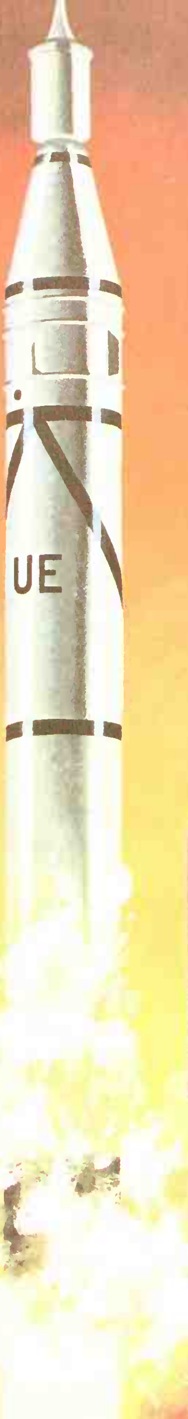
SIR ISAAC NEWTON



Kepler discovered the laws of planetary motion. Earth satellites obey similar laws. One of them is that the line joining a satellite to the earth sweeps over equal areas in equal times. Newton, using calculus, derived the law of gravitation from Kepler's laws. In the orbit shown, the satellite travels at varying speeds. Distances A_1 to B_1, A_2 to B_2, A_3 to B_3, and A_4 to B_4 are all covered in the same amount of time

is 94 feet per second. Its average speed during the last sixteenth of a second is 95 feet per second. Each new estimate is wrong, but it is less wrong than the one before it. We gradually sneak up on the correct answer. It turns out to be 96 feet per second. During the seventeenth century, the English scientist Newton and the German philosopher Leibnitz invented an efficient method of sneaking up on the right answer. It shows that *the speed of a falling body is 32 times the number of seconds it has fallen.* This checks with our result, because 32 × 3 = 96.

The branch of mathematics that uses the method of Newton and Leibnitz is called *differential calculus.* When Sputnik, the first earth satellite, began flying around the earth, scientists used calculus to figure out how fast it was moving. They needed calculus to do it because Sputnik's speed was changing all the time. Calculus is used every day by physicists, astronomers, and engineers whenever they study changes in which the change itself is changing.



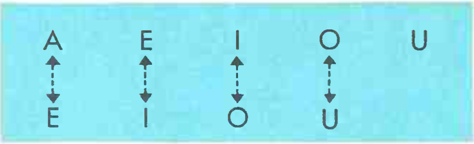
A rocket launching a satellite

WILHELM LEIBNITZ

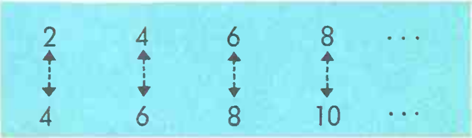
Finite and Infinite

When we count the objects in a collection or set, we say the natural numbers in order, assigning each number in turn to an object in the set. To count the set of vowels in the alphabet, we say, "*A*, 1; *E*, 2; *I*, 3; *O*, 4; *U*, 5." Because we run through the complete set of vowels in this way, we say that the set is *finite*. This means that counting it comes to an end. We can try to count the set of even natural numbers by assigning natural numbers to them, in this way: "2, 1; 4, 2; 6, 3; 8, 4;" and so on. In this case, the counting never comes to an end, because no matter how many even numbers we count, there are always more left over. So we say that the set of even numbers is *infinite*.

Infinite sets differ from finite sets in an important way. If you remove a member of a finite set, what is left will not match the original set. For example, the set of vowels *e*, *i*, *o*, *u* does not match the complete set of vowels, *a*, *e*, *i*, *o*, *u*.



If you remove a member of an infinite set, however, what is left will still match the original set.



The diagram below shows that the set of even numbers left over when 2 is removed matches the original set.

Infinite Series

If you have a finite collection of numbers, you can add them without any trouble. You arrange them in a line, and then, as you run down the line, you add each number into the sum, until you reach the end of the line. For example, to add $3 + 5 + 12 + 4$, you add 5 to 3, to get 8, then add 12 to the 8 to get 20, and then add 4 to the 20 to get 24. The last result is the sum. But you run into trouble with an infinite collection of numbers to add, because you never come to the end of the line. Try to add this infinite collection of ones: $1 + 1 + 1 + 1 \ldots$, where the dots show the series does not end.

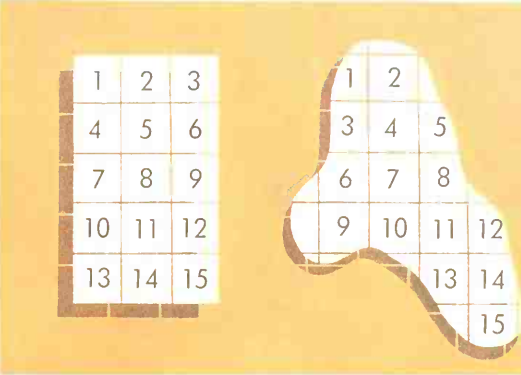
As you run down the line, you get as partial sums the numbers 1, 2, 3, 4, and so on. The partial sums get larger and larger without any limit. In this case, the series has no sum. But there are some infinite series which do have a sum. In fact, we met one before, on page 43. The infinite decimal .3333 . . . is really the infinite series $.3 + .03 + .003 + .0003 + \ldots$, and it has the sum $\frac{1}{3}$. We say $\frac{1}{3}$ is the sum because, as we add in more and more terms of the series, the partial sums come closer and closer to $\frac{1}{3}$. We say the partial sums approach $\frac{1}{3}$ as a limit.

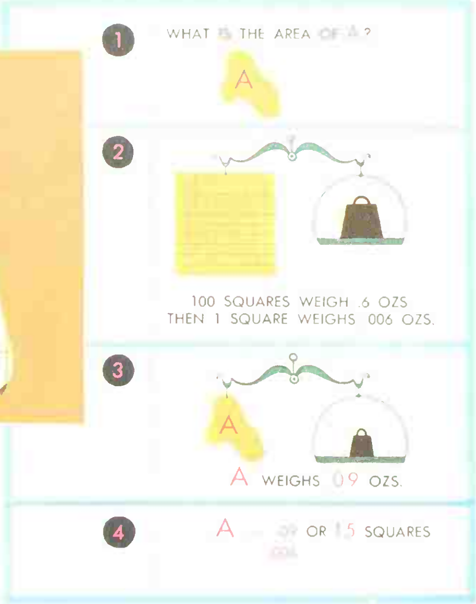
The number π, which is described on page 26, can be expressed in several different ways as the sum of an infinite series. One of the series relates π to the odd numbers in this way:

$$\pi/4 = \frac{1}{1} - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} + \ldots.$$

The same number, $\pi/4$, can also be written as the product of an infinite collection of fractions:

$$\pi/4 = \frac{2}{3} \times \frac{4}{3} \times \frac{4}{5} \times \frac{6}{5} \times \frac{6}{7} \times \frac{8}{7} \times \ldots$$



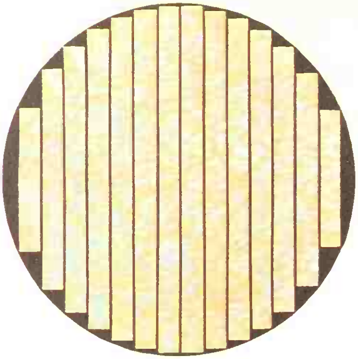


Measuring Areas and Volumes

To measure an area, we divide it into squares that are one unit wide, and then count the number of squares. For example, to measure the area of a rectangle that is three units wide and five units long, we divide it into squares, as shown in the diagram. Counting the squares, we find that the area of this rectangle is fifteen square units. In this case, we could have used a short cut for counting the squares. Since there are three rows of squares, with five squares in each row, the number of squares must be 3 × 5. The same short cut can be used for any rectangle. Simply multiply the number of units in the length by the number of units in the width. This rule is usually written as the formula *Area = length × width.*

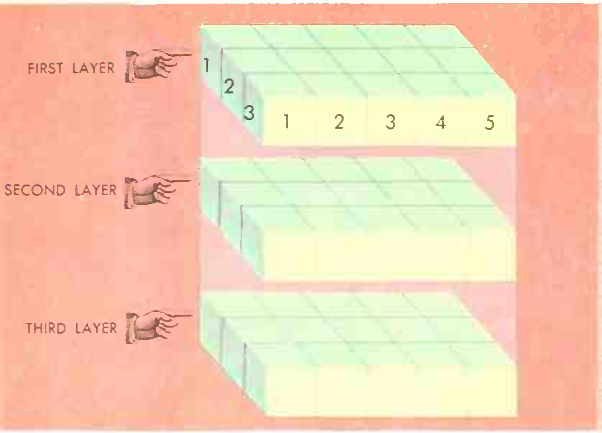
How would you find an area enclosed by an irregular curve? One way is to cover it with a network of lines that divide it into unit squares, and then count the squares. Where more than half a square is inside the area, count it as a full square. Where less than half a square is inside, do not count it at all. In this way we get an approximate value of the area. Here, too, we can use a short cut. Using cardboard of uniform thickness, cut out a square ten units long and ten units wide. Then weigh it on a sensitive scale. Suppose the weight turns out to be .6 of an ounce. Since the square contains 100 unit squares, each unit square weighs .006 of an ounce. Now draw the irregular area you want to measure on cardboard of the same thickness. Cut the area out, and weigh it. The weight will tell you the area, since each .006 of an ounce of weight represents one square unit of area. Another way of finding an irregular area is

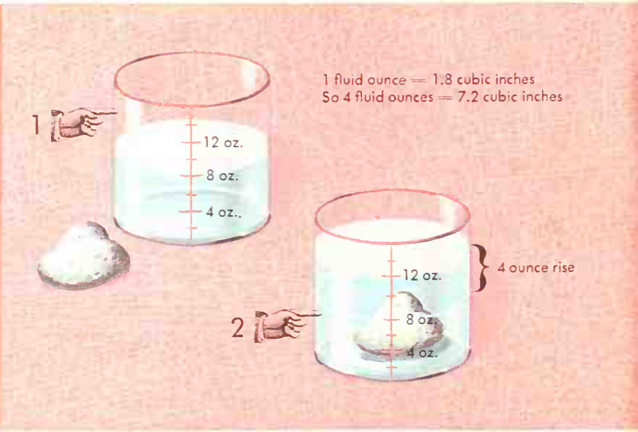
to cross it with evenly spaced parallel lines. Then, between each pair of parallel lines, draw the largest rectangle that will fit inside the area. The area of a rectangle can be calculated by means of the formula *Area = length × width*. After that, adding up the areas of the rectangles gives an approximate value for the irregular area. To get a better approximation, repeat the process, using thinner rectangles. By taking thinner and thinner rectangles, we can get an infinite sequence of approximations, approaching the actual area as a limit. A special branch of mathematics called *integral calculus* has the job of finding out what the limit is.



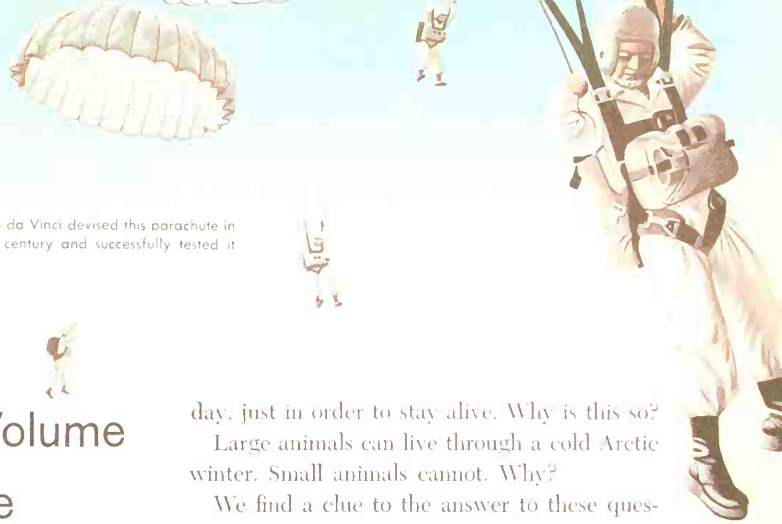
To measure a volume, we divide it into unit cubes, and then count them. For example, a rectangular box three units wide, five units long, and four units high, would yield sixty unit cubes. There is a short cut that we can use for counting them. The box contains four layers, and each layer contains three rows with five unit cubes in each row. So the total number of unit cubes is 4 × 3 × 5. This short cut is expressed in the formula *Volume = length × width × height*.

Measuring the volume of an irregular solid is a more difficult problem. But there are interesting short cuts that are sometimes useful. For example, suppose you wanted to measure the volume of a stone. First measure out a volume of water in a measuring cup. Then put the stone into the cup. The water level rises in the cup, and the increase in volume is the volume of the stone. To calculate the volume in cubic inches, make use of the fact that one fluid ounce equals 1.8 cubic inches. If the water level rises from the 8-ounce mark to the 12-ounce mark, then you know the volume of the stone is 7.2 cubic inches.





Leonardo da Vinci devised this parachute in the 16th century and successfully tested it



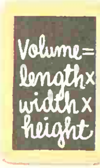
Surface and Volume in Nature

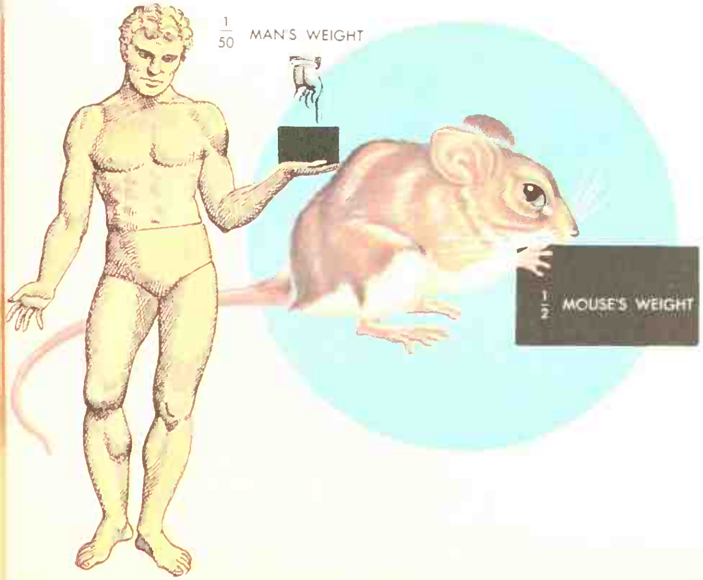
If a man falls from a height of three thousand feet, and has no parachute, he will surely be killed when he strikes the ground. But a mouse can fall from the same height, and simply get up and walk away, unhurt. Why?

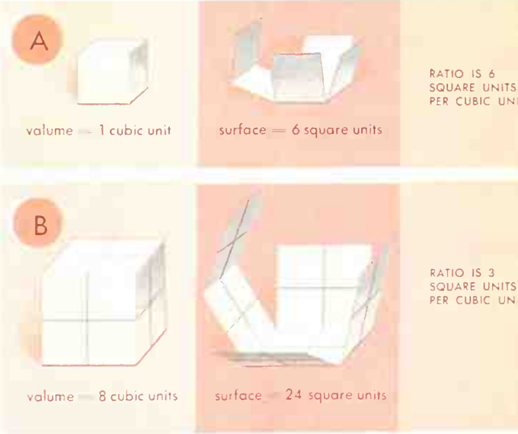
A man can be well fed by eating about one fiftieth of his weight in food every day. A mouse has to eat one half of his weight in food each day, just in order to stay alive. Why is this so?

Large animals can live through a cold Arctic winter. Small animals cannot. Why?

We find a clue to the answer to these questions by first comparing the surface and volume of large and small bodies. Examine a cube whose edge is two inches long. Using the formula *Volume = length × width × height*, we find that its volume is 2 × 2 × 2, or 8 cubic inches. Its surface is made up of six squares, each of which contains 2 × 2 square inches. So the total surface is 24 square inches. This cube has three square inches of surface for every cubic inch of volume. Now examine a cube whose







edge is only one inch long. Its volume is one cubic inch; its surface is six square inches. So this cube has six square inches of surface for every cubic inch of volume, or twice as much surface per cubic inch as the larger cube has. *Small bodies have more surface per unit volume than large bodies with the same shape.*

The Built-in Parachute

If a man falls from a great height, he is pulled down by a force equal to his weight. The size of his weight depends on his volume. As he falls, the air resists his fall. The size of this air resistance depends on his surface. It is not strong enough to balance his weight, so the man falls faster and faster until he strikes the ground. However, if he has a parachute, the open parachute exposes a large surface to the air. Then there is a large air resistance which soon balances the man's weight, and he floats down gently. A mouse, because it is so small, has much more surface compared to its weight than a man has. Its body surface acts as a built-in parachute, and it floats gently to the ground.

Heat Production and Loss

A man and a mouse are both warm-blooded animals whose bodies must remain warm to stay alive. Heat is produced in each part of the living body. The amount of heat produced depends on the volume of the body. At the same time, heat is constantly lost through the surface of the body. The amount of heat that is lost depends on the size of the surface. Both the man and the mouse eat food partly as fuel for the chemical fires inside them that replace the heat that is lost. Because the mouse has more surface for each cubic inch of volume than the man does, he loses his heat faster, and has to replace it faster. That is why he has to eat so much food every day.

Making up for lost heat is a greater problem for animals living in a cold climate than it is for animals that live in a warm climate. A bear living in the Arctic regions has a large volume producing heat, and a small surface, compared to his volume, losing it. So he can manage to keep himself both warm and alive. But a mouse has only a small volume producing heat, and a large surface, compared to this volume, losing it. In the Arctic regions, where the heat loss would be greater than it is in warmer parts of the earth, he wouldn't be able to keep up with the loss at all. That is why mice and other small mammals cannot stay alive through the cold Arctic winter.

Having a large body is a heat-saving advantage in a cold climate. For this reason, animals that live in the far north tend to be larger than their cousins that live near the equator. This interesting fact of biology has its roots in the mathematics of surfaces and volumes.

the four possible throws

Fires, Coins and Pinball Machines

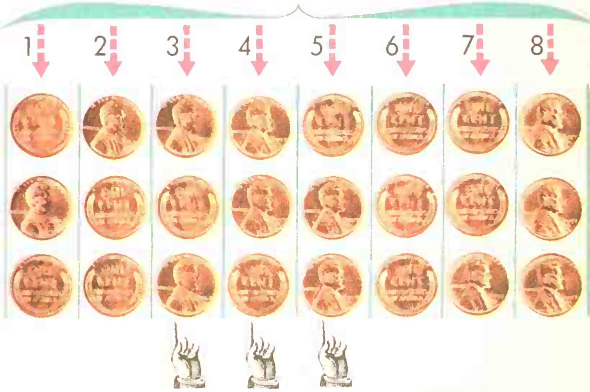
When a house burns down, thousands of dollars' worth of property is destroyed. The loss is too much for any one person to bear. So people join together to share the loss. They do this through insurance. Each person pays in to the insurance company a small sum each year. Then, if somebody's house burns down, the money is there, ready to be paid out to him to cover the loss. To know how much money each person should pay in, the insurance company must know first what the chance is that a fire might break out.

Figuring out this chance is done with the help of branches of mathematics called *probability* and *statistics*. A life insurance company uses these branches of mathematics to calculate the chance that a person will die in any given year. A pension fund uses them to figure out how long pensions will probably be paid to people who retire after a certain age.

Figuring out the chance that something will happen is like looking into the future. It is done by using common sense and a knowledge of what happened in the past. To see how it works in a simple case, let us try to foresee what happens when you toss a coin. The coin has two faces, head and tail. Common sense and experience join to tell us that, out of a large number of tosses, about half will come out heads, and the rest will be tails. Saying it another way: on the average, one out of two tosses will come out heads. So we say the chance of getting a head is $\frac{1}{2}$.

If we toss two coins, there are three possible results. We may get two heads, or two tails, or one head and one tail. What is the chance of getting each of these results? It is *not* one out of three. If we use two different coins (say a penny and a dime), we see that there are really four possible results. Throwing the penny first and the dime second, we might get head-head, or head-tail, or tail-head, or tail-tail. The chance of getting two heads is one out of four, or $\frac{1}{4}$. The chance of getting two tails is also $\frac{1}{4}$. The chance

the eight possible throws

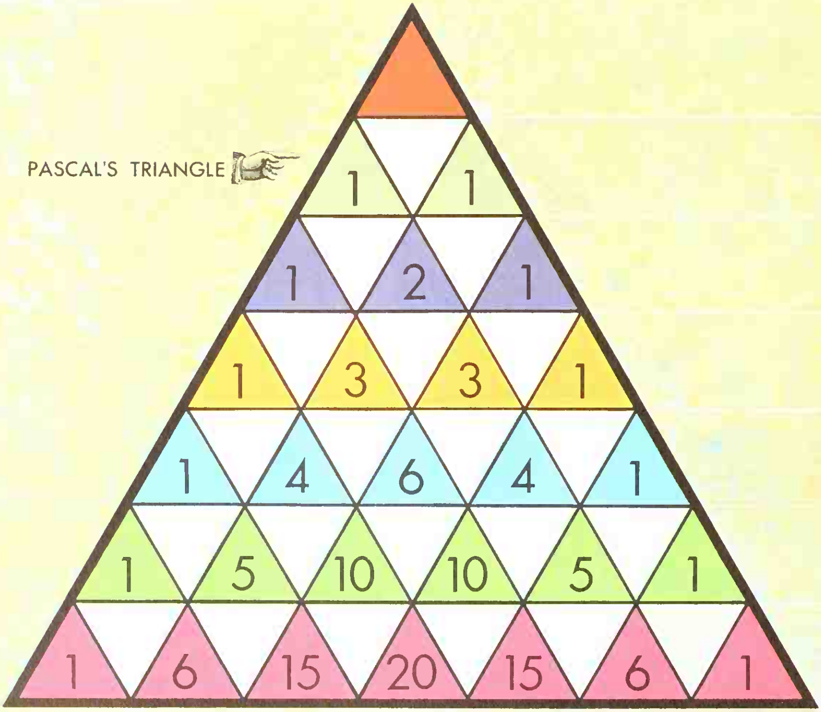


of getting one head and one tail is two out of four, or $\frac{1}{2}$.

What is the chance of getting two heads and a tail when you toss three coins? To answer this question, we must first notice that there are three ways of getting two heads and a tail. We may get head-head-tail, or head-tail-head, or tail-head-head. Compare this number with the total number of ways three coins can fall. This number is eight, since each coin can fall in two ways, and $2 \times 2 \times 2 = 8$. So the chance of getting two heads and a tail is $\frac{3}{8}$.

There is a short cut for finding the chance of getting any special combination. It is summed up in the arrangement of numbers known as *Pascal's triangle.* Pascal, a French philosopher and mathematician of the 17th century, was for a time interested in roulette and other games of chance. This interest led him to discover certain important rules about the probabilities of getting heads or tails on the toss of a coin. His findings are described in a triangular formation of numbers which shows easily the chance of getting heads or tails, or any combination of

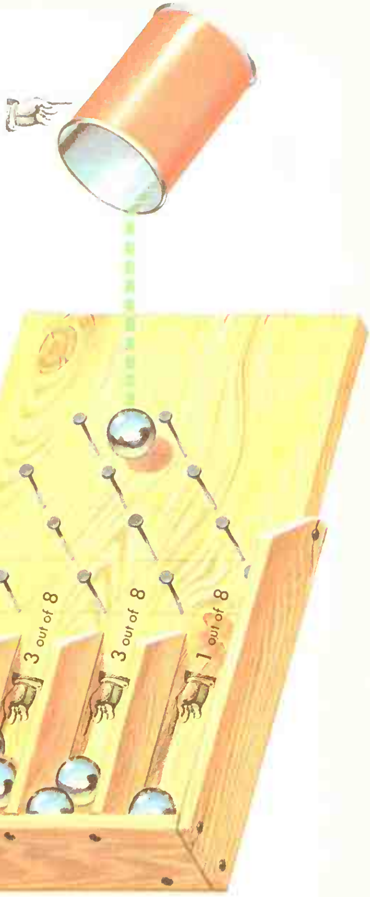
PASCAL'S TRIANGLE



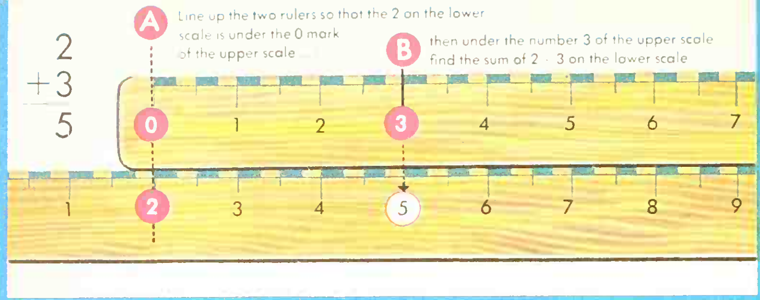
them, on a given number of tosses of a coin. Each line in the triangle is obtained from the line above it in this way: Write a 1 at each end; and under each pair of numbers that are side by side, write their sum. The first line is for tossing one coin; the second line for two coins; the third line for three coins; and so on. The first number in a line is for all heads. The next number is for one less head, and one more tail, and so on down the line. To figure the probabilities for tossing four coins, use the fourth line. For two heads and two tails, use the third number in that line. Compare this number to the sum of all the numbers in that line. The chance of getting two heads and two tails is six out of sixteen, or $\frac{3}{8}$.

1 COIN	If you toss one coin, the chance of getting heads is 1 out of 2, or ½.
2 COINS	If you toss two coins, your chance of getting 2 heads is 1 out of 4; of getting 1 head and 1 tail, 2 out of 4 or ½; of getting 2 tails, 1 out of 4.
3 COINS	If you toss three coins, your chances are: all heads, 1 out of 8; 2 heads and 1 tail, 3 out of 8; 2 tails and 1 head, 3 out of 8; all tails, 1 out of 8.
4 COINS	If four coins are tossed, there is 1 chance in 16 of getting all heads or all tails; 4 out of 16 of getting 3 heads and 1 tail, or 3 tails and 1 head; 6 out of 16 of getting 2 heads and 2 tails.
5 COINS	In five tosses, chances are: 1 out of 32 for all heads or all tails; 5 out of 32 for 4 heads & 1 tail, or 4 tails & 1 head; 10 out of 32 for 3 heads & 2 tails, or 3 tails & 2 heads.
6 COINS	In six tosses, the chances for all heads or all tails are 1 in 64; for 5 heads and 1 tail, or the reverse, 6 out of 64; 4 heads and 2 tails (or reverse), 15 out of 64; 3 and 3, 20 out of 64.

Eight balls poured out of this can are most likely to fall as indicated below



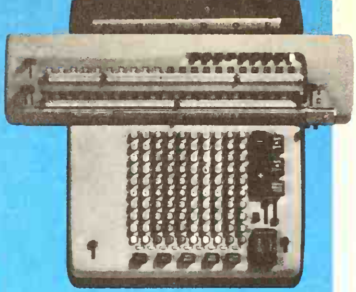
A pinball machine can be made of nails arranged as shown in the drawing above. If small metal balls are dropped into this machine at the top, and allowed to collect in vertical columns at the bottom, how many balls are likely to collect in each of the columns? Pascal's triangle gives the answer to this question.



Line up the two rulers so that the 2 on the lower scale is under the 0 mark of the upper scale ...

then under the number 3 of the upper scale find the sum of 2 + 3 on the lower scale

Two rulers can be used as a simple "adding machine



A modern calculating machine

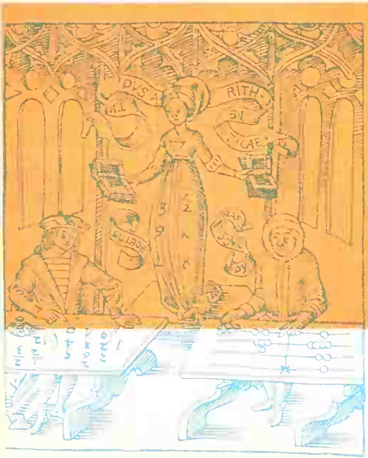
Calculating Machines

When we solve a problem, we try to figure it out in the shortest and easiest way. The easiest way to solve a problem is not to work on it at all. Let a machine do it for you, instead.

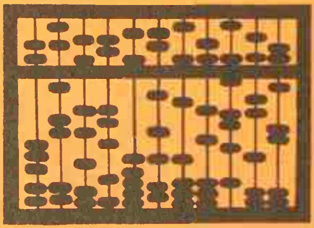
You can make your own adding machine out of two ordinary rulers. Simply place one ruler next to the other, edge to edge. Now your machine is ready for use. If you want to add 2 and 3, place the zero-edge of the upper ruler over the 2 on the lower ruler. Then locate the 3 on the upper ruler. Use the line that belongs to the 3 as a pointer. It points out the answer on the lower ruler.

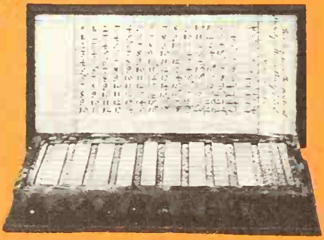
The Stick That Multiplies

By making a slight change in the rulers, we can turn them into a machine that multiplies. We get a hint on how to do it from one of the things we learned on page 20. A short way of writing $2 \cdot 2 \cdot 2 \cdot 2$ is 2^4. But $2 \cdot 2 \cdot 2 \cdot 2 = 16$. So 2^4 is another way of writing 16. The 4, which

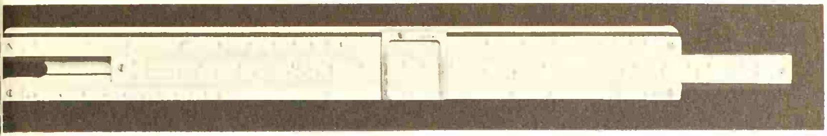


A Chinese abacus



Napier's rods, invented in 1617, were used to multiply numbers

Two methods of computing used during the Middle Ages: numerals and counters. Counter reckoning was really a form of the abacus. John Napier, a Scotsman, made an early mechanical device. (Drawing from print c. 1503)

A straight slide rule



A circular slide rule

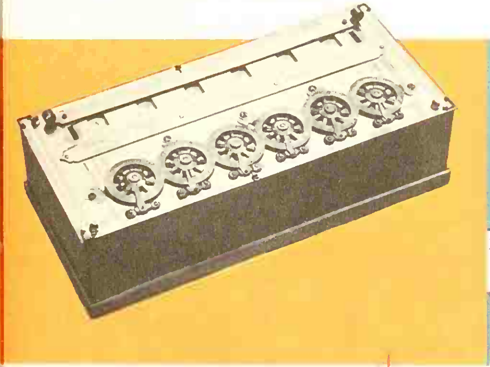
tells us how many two's to multiply to get 16, is called the *logarithm* of 16. In the same way, 2^3 is another way of writing 8, and the logarithm of 8 is 3. To multiply 16 by 8, we multiply 2^4 by 2^3. That means take 2·2·2·2 times 2·2·2. Replacing the word "times" by a multiplication sign, we get 2·2·2·2·2·2·2. The short way of writing this result is 2^7. This number, multiplied out, is 128, and its logarithm is 7.

Notice that while the numbers 16 and 8 were multiplied to get 128, their logarithms, 4 and 3, were *added* to get 7. This is our hint. We know already that two rulers can add the distances that are measured on them. So we make up a special pair of rulers in which the distance of each number from the end of the ruler is equal to the logarithm of the number. The rulers will add the logarithms. And adding the logarithms is like multiplying the numbers.

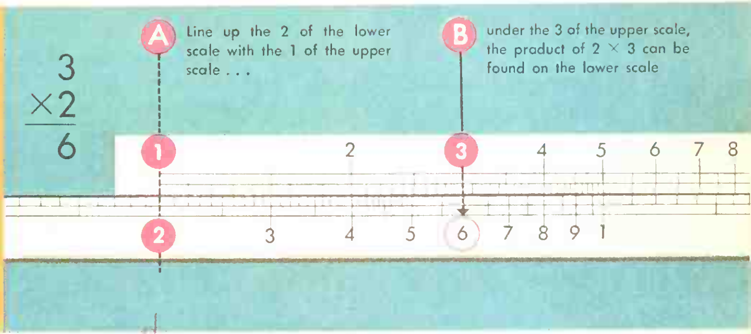
A pair of special rulers made up in this way is called a *slide rule*. Slide rules are used by people in many different kinds of work—engineers, architects, printers, and anyone else who has to make many rapid calculations.

Counting Wheel

Another simple calculating machine is the *odometer* in a car, which tells you how many miles the car has traveled. It is made up of a series of wheels placed side by side. The numbers from 0 to 9 are printed on the rim of each wheel. One of these numbers on each wheel shows through the little window on the dashboard. The wheel on the right counts tenths of a mile. When the car travels one tenth of a mile, the wheel turns around just enough to move the next higher number into place at the window. After 9 tenths of a mile, the number 9 shows through the window. After the next tenth, the wheel turns the 0 into place, and, at the same time, turns the wheel next to it one space.



The first calculating machine (1642) which automatically carried the tens was invented by Pascal, philosopher and mathematician. It could add figures up to six places



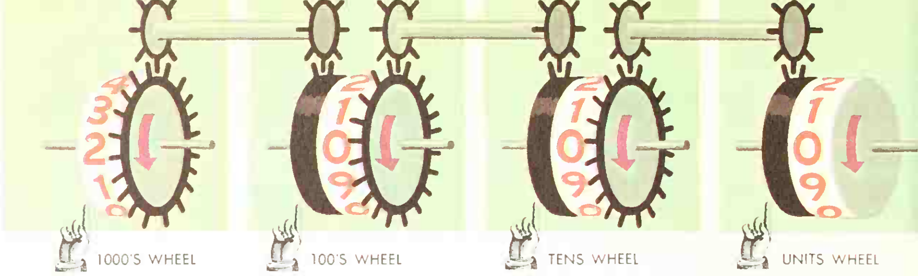
A Line up the 2 of the lower scale with the 1 of the upper scale . . .

B under the 3 of the upper scale, the product of 2 × 3 can be found on the lower scale

Notice that the numbers on the slide rule are not equally spaced. The slide rule multiplies numbers by adding their logarithms

ODOMETER

An odometer records the number of miles a car has traveled. For each mile, the units wheel advances 1 space. Ten spaces make a complete turn. When a turn is completed, the "fingers" on the edge turn the gear above. This makes the tens wheel advance 1 space. Thus, 10 spaces on the units wheel are exchanged for 1 space on the tens wheel



The effect is to exchange ten spaces on the first wheel for one space on the second wheel. In the same way the second wheel, after completing a full turn, exchanges ten spaces for one space on the third wheel. So, while the first wheel counts tenths of a mile, the second wheel counts whole miles, the third wheel counts tens of miles, the fourth wheel counts hundreds of miles, and so on.

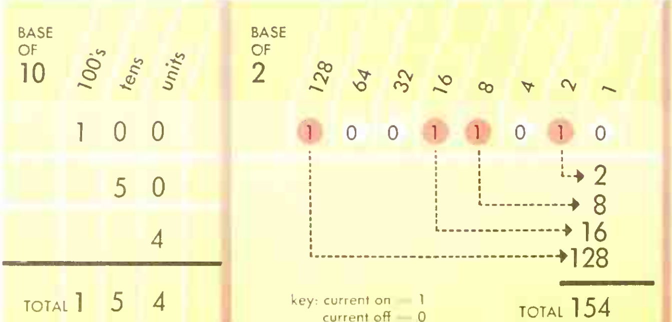
Most desk calculators work in the same way. They are simply counting machines. They add two numbers the way people add on their fingers. They count out the first number, and then, starting where the first number leaves off, they count out the second number. They multiply by adding the same number many times. To multiply 4 × 5, for example, a desk calculator adds 5, 5, 5, and 5.

The fastest calculators are the machines that work electronically. They are counting machines. Instead of having a series of turning wheels, they use a series of electrical circuits. They keep count by turning currents on and off in these circuits. Just as in the odometer one wheel passes the count on to the next by turning it, in electronic calculators one circuit passes the count on to the next by turning its current on or off. Each wheel in an odometer has ten positions, so the odometer builds up large numbers in groups of ten. Each circuit in an electronic calculator has just two positions, on and off. So an electronic calculator builds up large numbers in groups of two. Although this is a slower way of counting, electronic calculators work very rapidly, because electric currents travel almost as fast as light.

The odometer, desk calculator, and counting machine are all called *digital machines*, because they make all their calculations by simple steps of counting repeated over and over again, the way a person would who counts on his fingers. There is another type of calculator which measures instead of counts. These machines first convert numbers into such quantities as length, angle, and electric current. Then they combine the quantities, and convert the result back into a number. The slide rule is one example of this type of calculating machine.

ELECTRONIC CALCULATOR

Electronic calculators build up large numbers by counting electrical pulses in groups of two. The numbers can be shown by a panel of lights. A single pulse turns on the number 1 light on the right. A second pulse turns off this light and turns on the number 2 light. A third pulse turns on the 1 light again. The number of pulses is shown by adding the numbers of the lights that are on

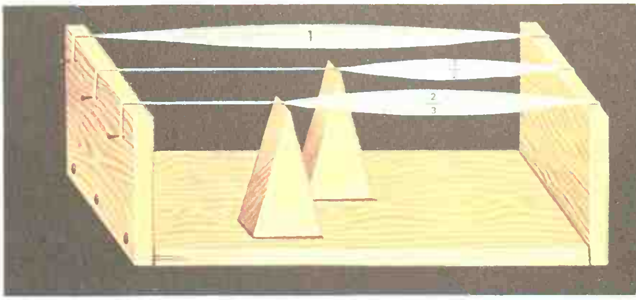


EVERY NUMBER CAN BE SHOWN IN THE BINARY SYSTEM, USING ONLY 1'S AND 0'S. SOME EXAMPLES ARE SHOWN BELOW:

1 = 1	7 = 111
2 = 10	8 = 1000
3 = 11	9 = 1001
4 = 100	10 = 1010
5 = 101	25 = 11001
6 = 110	50 = 110010

Pythagoras and his followers made many discoveries in music as well as in mathematics

It was Pythagoras who found that when the length of a vibrating string is halved, the tone an octave higher is sounded. A $\frac{2}{3}$ division produces the dominant tone



Mathematics and Music

A musical tone is made by a vibration. For example, if you stretch a string tight, and then pluck it, the string vibrates and produces a tone. What the tone sounds like depends on the number of vibrations that the string makes in a second. That is why tones are related to numbers, and music and mathematics are partners. The number of vibrations per second is called the frequency of the tone.

When a song is written, it is usually composed out of a family of tones called a *key*. To see how the tones in a key are related, let us actually build one.

The most important tone in a key is the one on which the song ends. It is called the *tonic*. Let us choose as tonic the tone made by a string that vibrates 256 times a second. We call this tone C. If we cut the string in half, it vibrates twice as fast. The tone made by this shorter string is also called C. Its frequency is 512 vibrations per second. If we double the frequency again, we get another tone called C, whose frequency is 1024. *We use the same name for two tones, when the frequency of one is double the frequency of the other.* The frequency 256 is double 128, and this in turn is double 64, and so on. So we get more tones that are called C by dividing by two. We give these tones the same name because we think of them as the same tone played at different levels.

Now let us vibrate a string whose length is two-thirds the length of the original one. The tone it produces is the tonic's closest relative. We call it the *dominant*. Its frequency is $1\frac{1}{2}$ times as great as the frequency of the tonic. This number $1\frac{1}{2}$ is the basis for building up a key. *A key is a family of tones in which each tone is followed by its dominant.* In order to find the dominant of any tone in the chain, we simply multiply its frequency by $1\frac{1}{2}$.

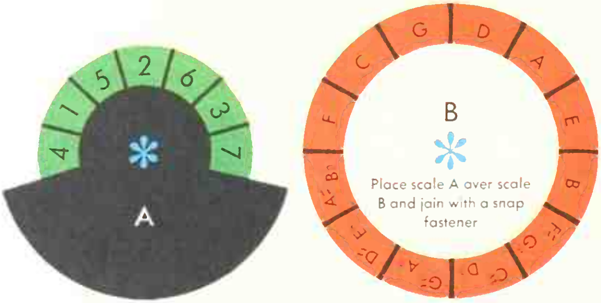
We began with the C whose frequency is 256. Its dominant has a frequency of 384, and is called G. G's dominant has a frequency of 576, and is called D. D's dominant has a frequency of 864, and is called A. A's dominant has a frequency of 1296, and is called E. E's dominant has a frequency of 1944, and is called B. Now we start at C again, and go the other way. The frequency 256 is $1\frac{1}{2}$ times as great as the frequency 171. The tone with this last frequency is called F, and C is the dominant of F. The seven tones we have named make up the key of C. They are listed in order in the first column of the table.

We started with the tone C that has a frequency of 256. The next higher C has a frequency of 512. We can get all of the tones of our key to lie between these limits. We can do this, because, when the frequency of a tone is too high or too low, we can replace it by a tone with the same name whose frequency is half as large or twice as large. Our G is not too high, so we keep it. All the others except F are too high, so we divide by 2 over and over again until the frequency lies between 256 and 512. The frequency of F is too low, so we double it. The results are shown in the third column of the table. Now, if we arrange the tones in order of frequency, starting with C (256) and ending with C (512), we have a ladder of tones climbing from C to the next higher C. This ladder is called a *scale*. We have them now in this order: C(256), D(288), E(324), F(342), G(384), A(432), B(486), C(512). This is the order of the white keys on a piano keyboard

To build up a scale that starts with another tonic or keynote instead of C, we have to bring in the black keys on the piano. They are named after the white keys that are near them. When a black key carries the name of a white key that it follows, it is called sharp (#). When it carries the name of a white key that comes after it, it is called flat (♭).

To find the scale that begins with any tone, we can use the wheels that are printed on this page. Copy them on cardboard. Punch a hole through the center of each copy, and fasten them together with a snap fastener. Turn the small wheel until the 1 lies next to the tone you want to use as tonic. Then the numbers from 2 to 7 point out the other tones in the scale, numbered in the correct order.

	Chain of Dominants	Key of C
NAME	FREQUENCY	FREQUENCY BETWEEN 256 AND 512
F	171	342
C	256	256
G	384	384
D	576	288
A	864	432
E	1296	324
B	1944	486

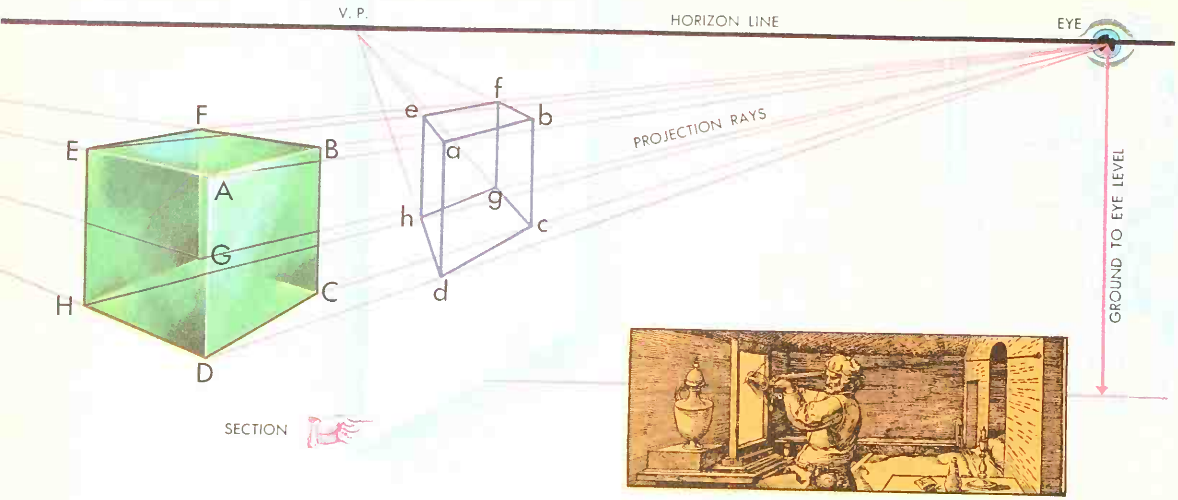


You can make your own device for finding the scale beginning with any tone. Trace these wheels on paper and cut out. Make hole in center for snap fastener

Mathematics and Art

Compare the two pictures that are on this page. The one at the left has been drawn to look like an old Egyptian painting. Everything in it looks flat, and each part looks as though it were right on top of the next part. It is hard to tell at a glance which elements are supposed to be nearer to you.

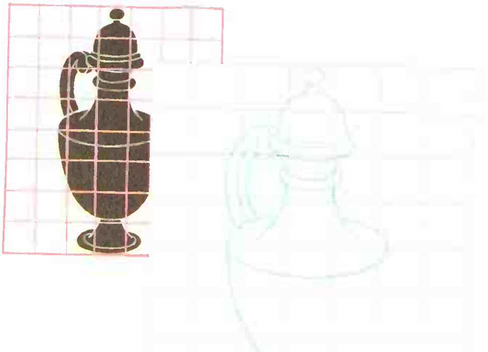
The other picture is a painting by the Italian artist Gentile Bellini. You can see that the people are closer than the building. You can also see that there is a feeling of distance between the different parts of the painting. The space in Bellini's painting looks much more real than that in the Egyptian painting because he used mathematics when he laid it out on his canvas.



The great German artist Albrecht Dürer said, "Geometry is the right foundation of painting." To make a painting look real, the painter thinks of his canvas as a "window" through which he is looking at a scene that is beyond it. He reasons in this way: Each point of the scene sends a ray of light to the eye of the person looking at it. These rays of light pass through the "window" between the eye and the scene. The place where a ray crosses the window is the place where the point it comes from will appear in the picture. The collection of rays going from the scene to the eye is called a *projection*. The picture formed where the window crosses the projection is called a *section*. To figure out what the section will look like is a problem of *perspective*. The rules of perspective were worked out with the help of geometry.

In Bellini's painting you see how he used two of the rules of perspective: The further away something is, the smaller it looks. Parallel lines that go off into the distance, like straight railroad tracks, look as though they come to a point.

Mathematics helped art through the science of perspective. But then art repaid its debt. This is because the study of perspective led to the development of a new branch of mathematics called *projective geometry*.



A picture can be enlarged with the help of two sheets of tracing paper marked off in squares. Make one set of squares larger than the other

Mathematics for Fun

There are mathematical cards that seem to perform amazing feats. You can make a set by following these directions. Use four square cards, each six inches wide. Make a margin of one inch at each edge, and divide the center space into one-inch squares. Then copy the pattern for each card. Cut out the boxes that are marked "Cut out." Notice that one card has numbers written on the back as well as the front. Be sure to place them as shown.

Now that the cards are ready, ask someone to think of a number from 1 to 15. Show him the front of each card, and ask him if his number is on the card. If he says "yes," put the card down on the table with the word "yes" on top. If he says "no," turn the card so that the "no" will be on top. Stack the cards one over the other, with the card that has no hole in it put down last, over the others. Pick up the cards and turn the stack over. The correct number will show through a window in the cards.

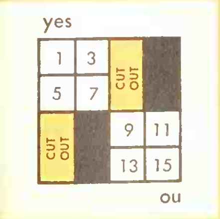
Magic Squares

17	24	1	8	15
23	5	7	14	16
4	6	13	20	22
10	12	19	21	3
11	18	25	2	9

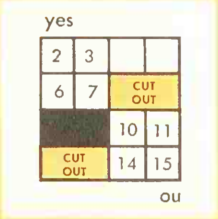
The arrangement of numbers shown above is called a five-by-five magic square. It uses all the whole numbers from 1 to 25. The "magic" property of the square is this: If you add the numbers in any row, or any column, or either diagonal (joining opposite corners), you always get the same sum. Try it.

Use the numbers from 1 to 9 to make a three-by-three magic square. Arrange them in three rows, with three numbers in each row. Placed properly, the rows, columns, and diagonals will add up to the same sum. We can figure out this sum in advance. Adding *all* the numbers from 1 to 9 is like finding the 9th triangle number. Using the rule on page 19, we multiply 9 by 10, and then divide by 2. The result is 45. Since the numbers are spread out in three rows, we can find the sum of one row by dividing by 3. So each row should add up to 15.

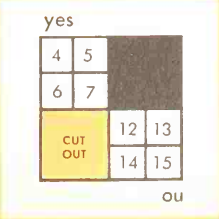
CARD NO. 1



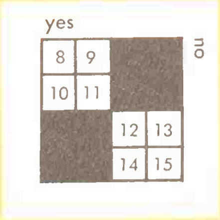
CARD NO. 2



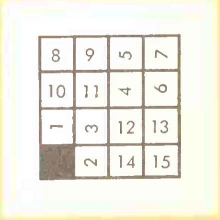
CARD NO. 3



CARD NO. 4 (FRONT)



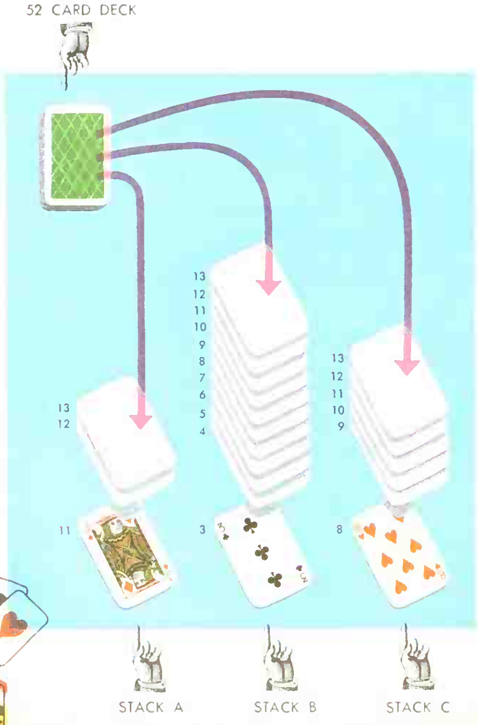
CARD NO. 4 (BACK)

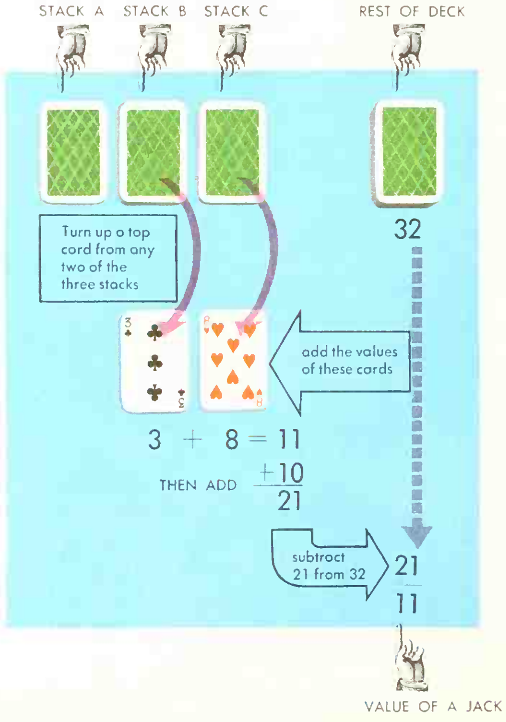


A Card Trick

There are many card tricks that are worked mathematically. This one is easy to do, and looks very mysterious. Use a deck of 52 cards, and shuffle it well. Ask someone in your audience to count out three stacks of cards from the deck, while you turn your back. He should follow these directions: "Put the first card face up, and start counting with the number on the card. Think of an Ace as 1, Jack as 11, Queen as 12, and King as 13. Count out more cards on top of it until you reach 13. If the first card is a 6, for example, you will reach 13 when you have put seven more cards on top of it. If the first card is a King, you won't have to put any more cards on top of it. Then turn the first stack over and start a new stack, until there are three stacks on the table, *face down*."

Now ask for the cards that are left over. Count them, and remember the number. Ask someone in the audience to turn up the top card in any two of the stacks. Then you tell them, without looking at it, what the top card on the other stack is. You figure it out mentally in this way: Add the numbers of the two cards turned up, and add ten to the result. Then subtract the sum from the number you got by counting the left-over cards. For example, if the cards turned up happen to be 3 and 8, and the number of left-over cards is 32, you would add 3 + 8 + 10, giving you 21. When you subtract 21 from 32, you get 11. So you now know that the top card of the third stack is a Jack.



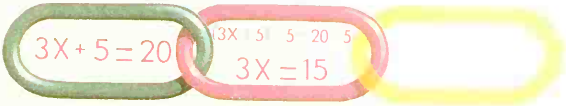


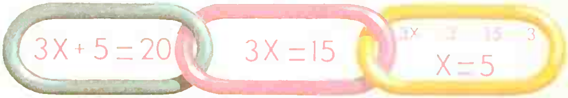
Proving It

Many statements may be made about numbers or space. Some of the statements are true, and some are false. We find out which ones are true by following the rules of *logic*, or careful thinking.

One type of proof often used in mathematics is a kind of *chain reasoning*. In this type of proof we move forward towards our result through a series of steps, each of which leads to the next, like links in a chain. We use this kind of reasoning, for example, when we solve an equation like: $3x + 5 = 20$. Our problem is to find out what number x stands for if the equation is a true statement. The equation says that the number which $3x + 5$ stands for, and the number 20, are equal. This is the first link in the chain.

We join it to the second link with the help of a rule that we know is true. This rule says that if we subtract the same number from equal numbers, we get equal results. So we subtract 5 from both numbers, and get the equation $3x = 15$. This is the second link in the chain. We join it to the third link by using another rule that we know is true. This rule says that if we divide equal numbers by the same number, we get equal results. So we divide by 3, and get the equation $x = 5$. This is the third link in the chain. The three-link chain tells us then, that if $3x + 5 = 20$, then x must be equal to 5.



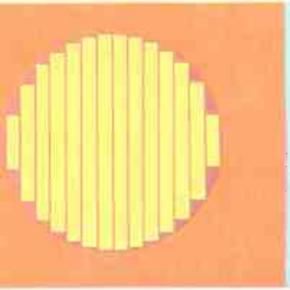


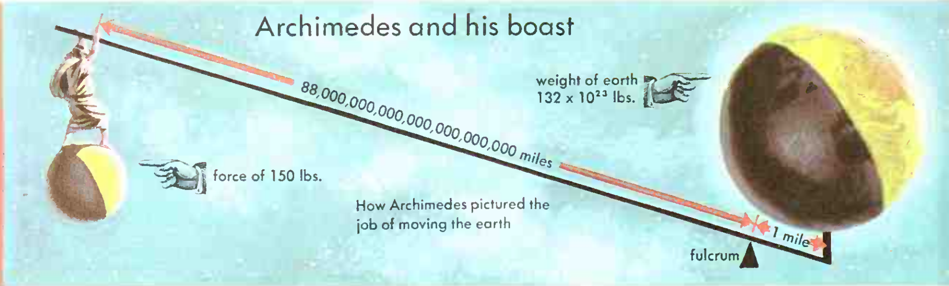
There is another kind of proof in which we back into our result instead of moving forward to it. We use a process of *elimination*. We first list a series of statements, chosen so that we are sure that one of them must be true. Next, we eliminate all the statements except one by proving they are false. Then the statement that is left must be true.

Here is an example of reasoning by elimination. Suppose there are more than twelve people in a room. Then we shall prove that at least two of them have their birthdays in the same month. First we list two statements: (1) At least two of the people in the room have birthdays in the same month; (2) No two of the people in the room have birthdays in the same month. We are sure one of these statements must be true. Now, if statement (2) is true, it means that the people in the room all have birthdays in *different* months. But, if there are more than twelve people in the room, and their birthdays are in different months, it means there are more than twelve different months. But this is impossible. So we eliminate statement (2). Once we have eliminated the second statement, we are sure that statement (1) must be true, because it is the only one that is left.

Three Great Mathematicians

Mathematics is a growing science. New questions are always coming up, rising partly from practical problems, and partly from problems of pure theory. In each generation, men have developed new ideas and methods to answer these questions. Thousands of men have shared in this work. Among the greatest were Archimedes, Isaac Newton, and Carl Friedrich Gauss.





Archimedes and his boast

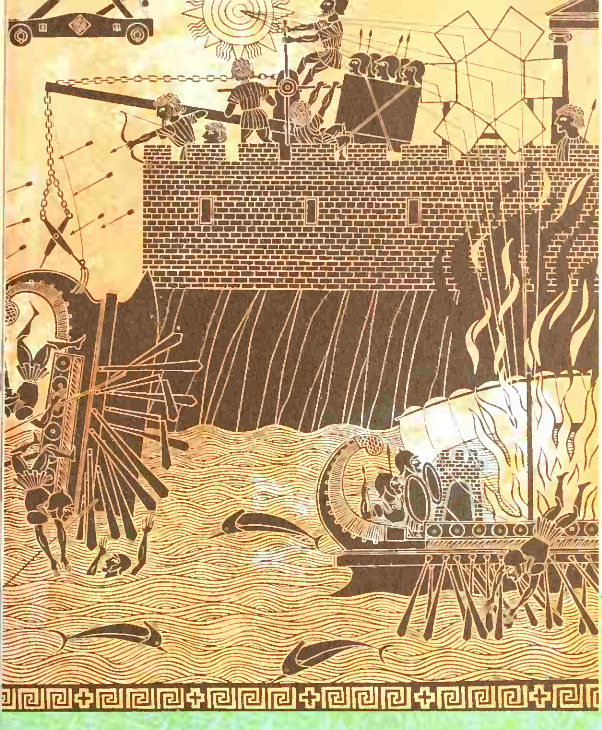
How Archimedes pictured the job of moving the earth

Moving the Earth

Archimedes was a citizen of Syracuse, a city on the island now known as Sicily. He was born in 287 B.C., and died at the age of 75. He made great discoveries in both mathematics and physics. He worked out a way of measuring the area within a closed curve by using a method almost like the method of modern calculus. He sliced the area into thin strips, and added the largest rectangles that could be drawn in these strips. By taking thinner and thinner strips, he got better and better approximations of the area within the curve. He made a careful study of levers, by means of which a small force can be used to balance a large weight. After his discovery of how this can be done, he is reported to have said, "Give me a place to stand on, and I can move the earth."

The King of Syracuse once ordered a crown made of pure gold. When the crown was finished, the King suspected that some silver had

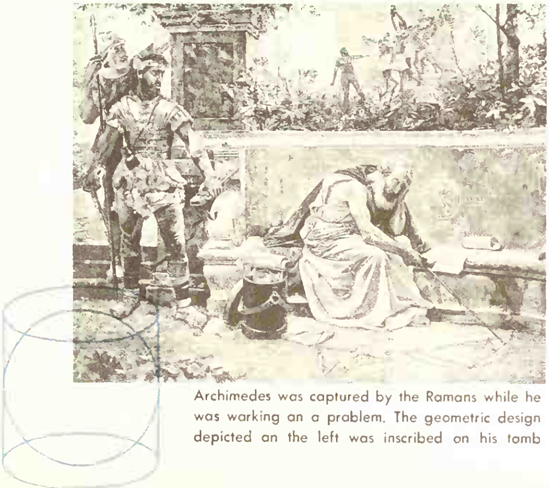
The screw of *Archimedes*, a kind of water pump used in ancient times



At the Battle of Syracuse, the machines of Archimedes created havoc among the Roman galleys. Above, one of Archimedes' machines lifts a ship out of the water.

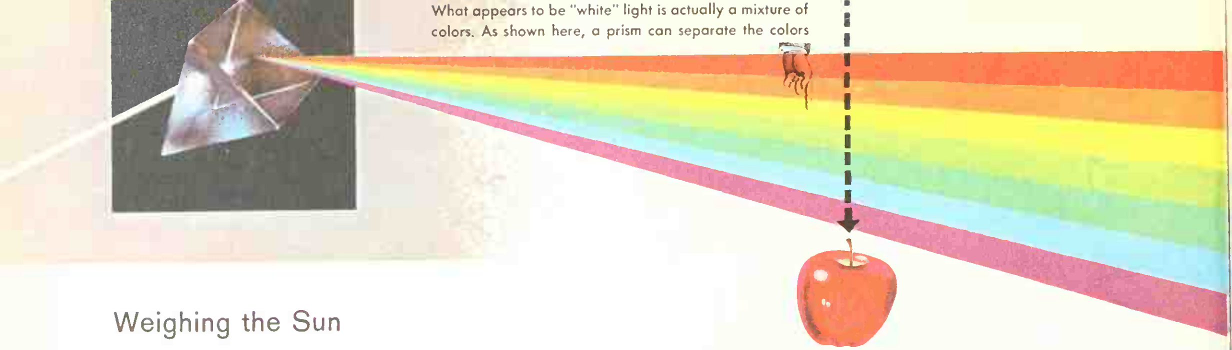
been mixed with the gold. He asked Archimedes to figure out a way of checking whether the crown was all gold, or not. One day, while Archimedes was at the public bath, he noticed how the level of the water rose when he stepped into it. He suddenly realized how he could solve the problem of the crown, and became so excited about his discovery that he ran home, naked, shouting, "Eureka!" (I have found it!). His idea was that he could measure the volume of the crown by putting it into a dish of water. (See page 68). If the crown contained any silver, the volume of the crown would be greater than the volume of an equal weight of pure gold.

How Archimedes proved that the king's crown was not pure gold ☞



Archimedes was captured by the Romans while he was working on a problem. The geometric design depicted on the left was inscribed on his tomb

When Syracuse was attacked by Rome, its soldiers defended the city with the help of great machines invented by Archimedes. According to the stories told in Roman history books, they used giant catapults to hurl great stones at the Roman ships. They had giant claws that lifted ships out of the water and smashed them against the rocks on the shore. However, in spite of these mechanical marvels, the Romans eventually won the war. When Syracuse was captured, Archimedes was killed by a Roman soldier.



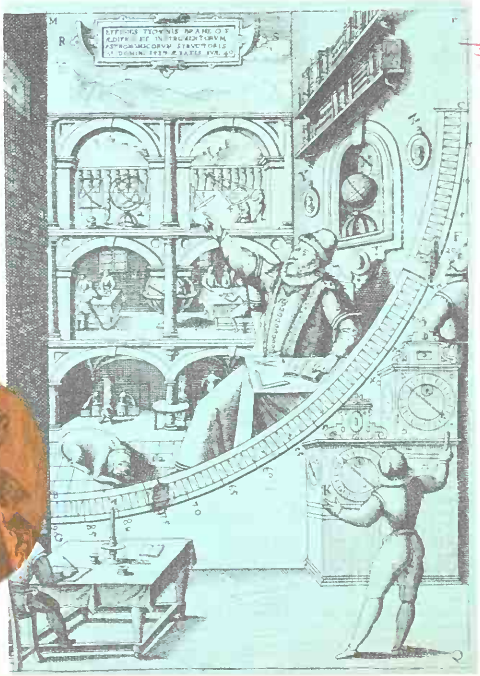
What appears to be "white" light is actually a mixture of colors. As shown here, a prism can separate the colors

Weighing the Sun

Isaac Newton was born in England in 1642, and died in 1727. We remember him chiefly for three great discoveries: He showed that white light is a mixture of colors; he discovered the law of gravitation and the laws of motion that now bear his name; and he invented calculus, the mathematical tool for studying motion. He made all three discoveries before he was 24 years old.

Newton's law of gravitation was the fourth link in a chain of great discoveries in astronomy.

NEWTON



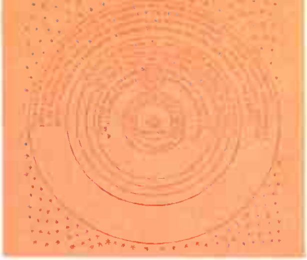
Tycho Brahe's accurate observations of the motions of the planets helped lead to the formulation of Sir Isaac Newton's law of gravitation. An old print depicts Tycho's Observatory

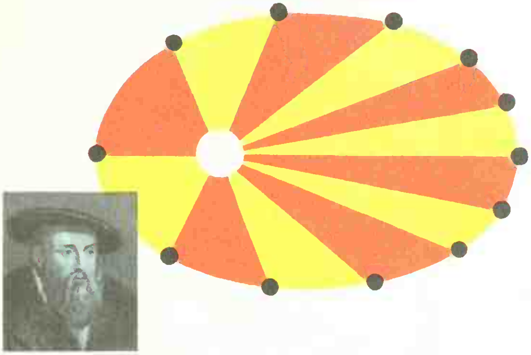
BRAHE

Newton was the first man to weigh the sun. He did it mathematically — using his law of gravitation

The ancient model of the solar system, and on the right, the modern one proposed by Copernicus

COPERNICUS





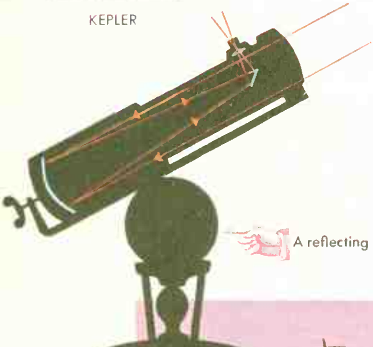
KEPLER

Copernicus supplied the first link by proposing the theory that the earth was a planet revolving around the sun. Tycho Brahe supplied the second link by making very accurate observations of the apparent motions of the sun and planets in the sky. Kepler supplied the third link by discovering, from Tycho's tables, the rules according to which the planets move. He found that the path of each planet is an ellipse, that the speed of a planet increases as it comes closer to the sun, and that the time it takes for a planet to make a round trip around the sun is related to its distance from the sun.

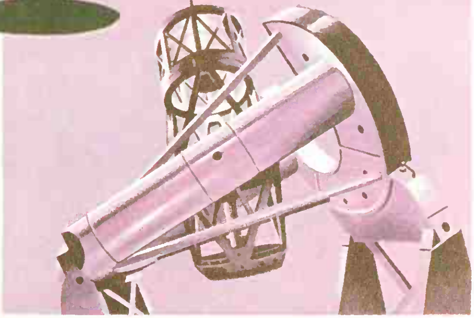
Newton supplied the fourth link when he showed by mathematical reasoning that the planets would move in this way only if the sun were pulling on each of them, and found the formula for calculating the strength of this pull. Using his formula, Newton was the first man to weigh the sun.

Newton also has an important invention to his credit. He invented the reflecting telescope, which uses a curved mirror instead of a lens. The great 200-inch telescope in the observatory on Mt. Palomar is a telescope of this type.

In October, 1957, our use of Newton's laws of gravitation and motion took a new turn. For almost three hundred years we had used them to explain the movements of the sun, the moon, the planets, and the stars. Now they help us launch man-made moons and planets that take their places alongside those found in nature.



A reflecting telescope uses a parabolic mirror



The two-hundred-inch telescope on Mt. Palomar is a reflecting telescope

A Mathematical Giant

The greatest mathematician of all time was Carl Friedrich Gauss. He was born into a poor family in Brunswick, Germany, in 1777. When he died, in 1855, he was world-famous as a mathematician, astronomer and physicist.

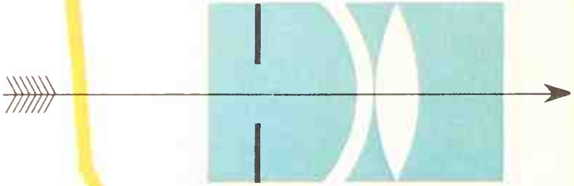
Gauss showed his talent for mathematics when he was very young. Once, at the age of three, young Carl listened attentively as his father, foreman of a group of bricklayers, calculated wages. Everyone was astounded when the boy called out that one of his father's figures was wrong—and then gave the correct figure. His father did the calculation over again, and found that Carl was right.

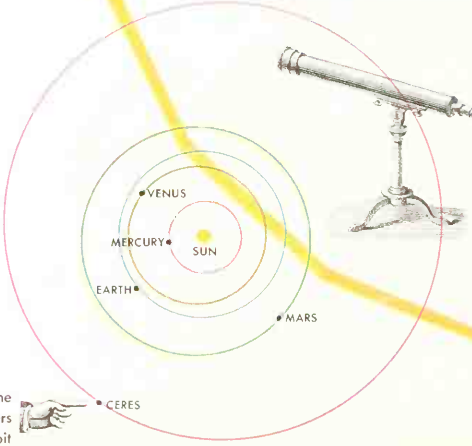
In elementary school, Gauss' teacher once asked the class to add all the numbers from 1 to 100. As soon as the teacher finished stating the problem, Gauss, who was then nine years old, wrote the answer on his slate and put it on the teacher's desk. He had figured it out mentally, using his own short cut. This short cut is the method for finding triangle numbers that is explained on page 19. Gauss' outstanding ability as a student led the Duke of Brunswick to sponsor his higher education.

Two of Gauss' earliest discoveries are among his best known. In 1796 he showed that a regular polygon of 17 sides can be constructed by means of a straight edge and compasses. In 1799, in his Ph.D. thesis, he gave the first flawless proof of what is known as the Fundamental Theorem of Algebra, that every algebraic equation has a solution.

Gauss made great contributions in many branches of mathematics. He did outstanding work in the theory of numbers, the theory of functions, probability and statistics, and the


Carl Friedrich Gauss and Wilhelm Eduard Weber, co-workers in the study of magnetism, were the first to construct an electromagnetic telegraph


In the field of optics, Gauss designed a lens for correction of astigmatism



When Ceres was lost behind the glare of the sun, Gauss' calculations enabled astronomers to rediscover the planet by tracing its orbit geometry of curved surfaces. He was one of the pioneers in constructing a new system of geometry, now known as non-Euclidean geometry, in which the sum of the angles of a triangle is less than 180 degrees.

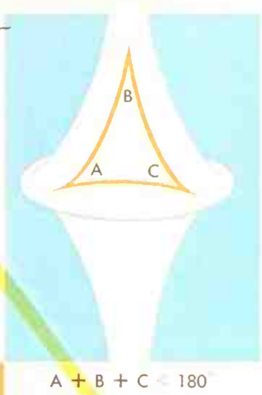
At the same time that he was making discoveries in pure mathematics, Gauss always had a strong interest in practical applications. He spent many years working as an astronomer. He was active as a surveyor and map maker. Working closely with Wilhelm Weber, the physicist, he studied electricity and magnetism. He also studied the structure of crystals.

In 1801, after the planetoid Ceres was discovered, astronomers lost track of it when it passed the sun. Gauss calculated the orbit of the planetoid, using a new method he had devised. Guided by Gauss' calculations, the astronomers found Ceres again in 1802.

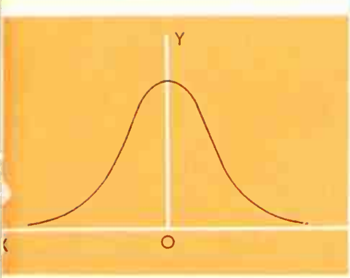
Gauss made a study of the magnetism of the earth. From his observations and calculations, he predicted where the south magnetic pole could be found. Navigators later found that this prediction was correct.

Gauss was an inventor as well as a theoretical scientist. He invented the *heliotrope*, a surveyor's instrument that uses a mirror for flashing sunlight across a great distance. Together with Weber he invented the *telegraph* at about the same time that Samuel Morse, working independently, developed the telegraph in America. Later, he invented a kind of *magnetometer*, an instrument that measures the strength of a magnetic field.

Gauss has been called "the prince of mathematicians." In honor of his great work, his name has been made part of the international language of science. The unit of magnetic field strength is now known as a *gauss*.

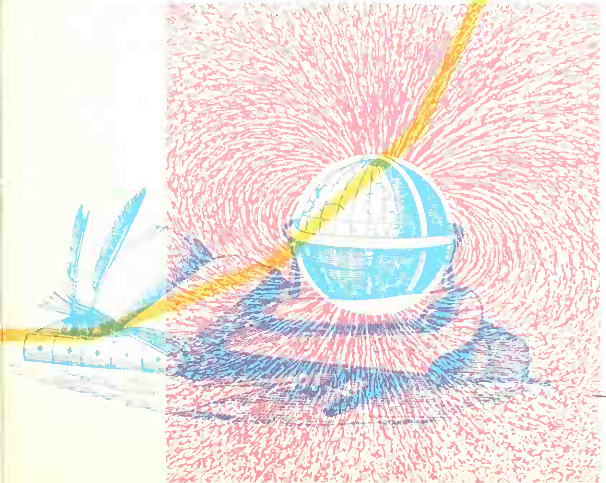


The pseudo-sphere is a curved surface whose geometry is non-Euclidean. On this surface, the sum of the angles of a triangle is less than 180°



The curve of probability, diagramed above, was of interest to Gauss in his studies of probability and statistics

An important advance in Euclidean constructions was Gauss' proof that a regular polygon of 17 sides could be constructed with a straight edge and compass



Gauss invented the bifilar magnetometer, an instrument used for measuring the strength of magnetic fields. The lines of force in the magnetic field surrounding the earth are illustrated in this picture

Mathematics in Use Today

Mathematics is part of our daily lives.

A housewife uses mathematics when she goes shopping: she compares prices, figures out her bills, and counts her change.

A bookkeeper uses mathematics to keep track of a company's income and expenses.

A machinist uses mathematics when he plans his work. He must measure and figure to know how to set his tool, so that it will cut out parts with the right size and shape.

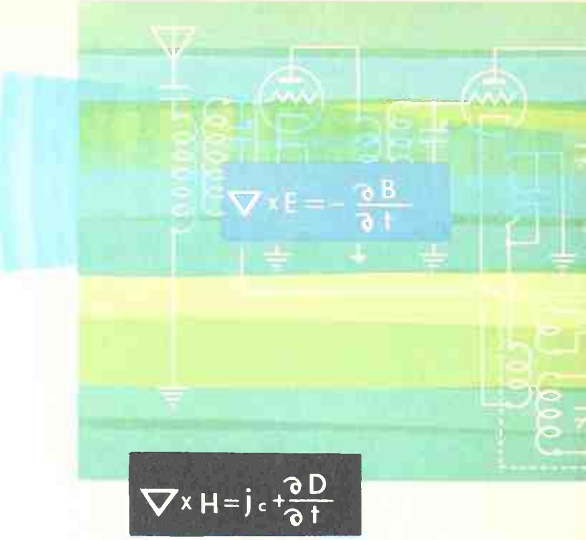
An engineer uses mathematics when he designs a new machine. He uses formulas to help him pick the right parts. Equations help him predict how they will work.

An airplane pilot uses mathematics to help him chart his course. He must figure distances and directions to know how to get from one place to another.

A farmer uses mathematics to figure out how much seed, feed, and fertilizer he needs. Like any businessman, he also has to reckon his accounts.

An astronomer uses mathematics to figure out how far away the stars are. He uses equations to help explain how stars are formed, what makes them shine, and how they change as they grow older.

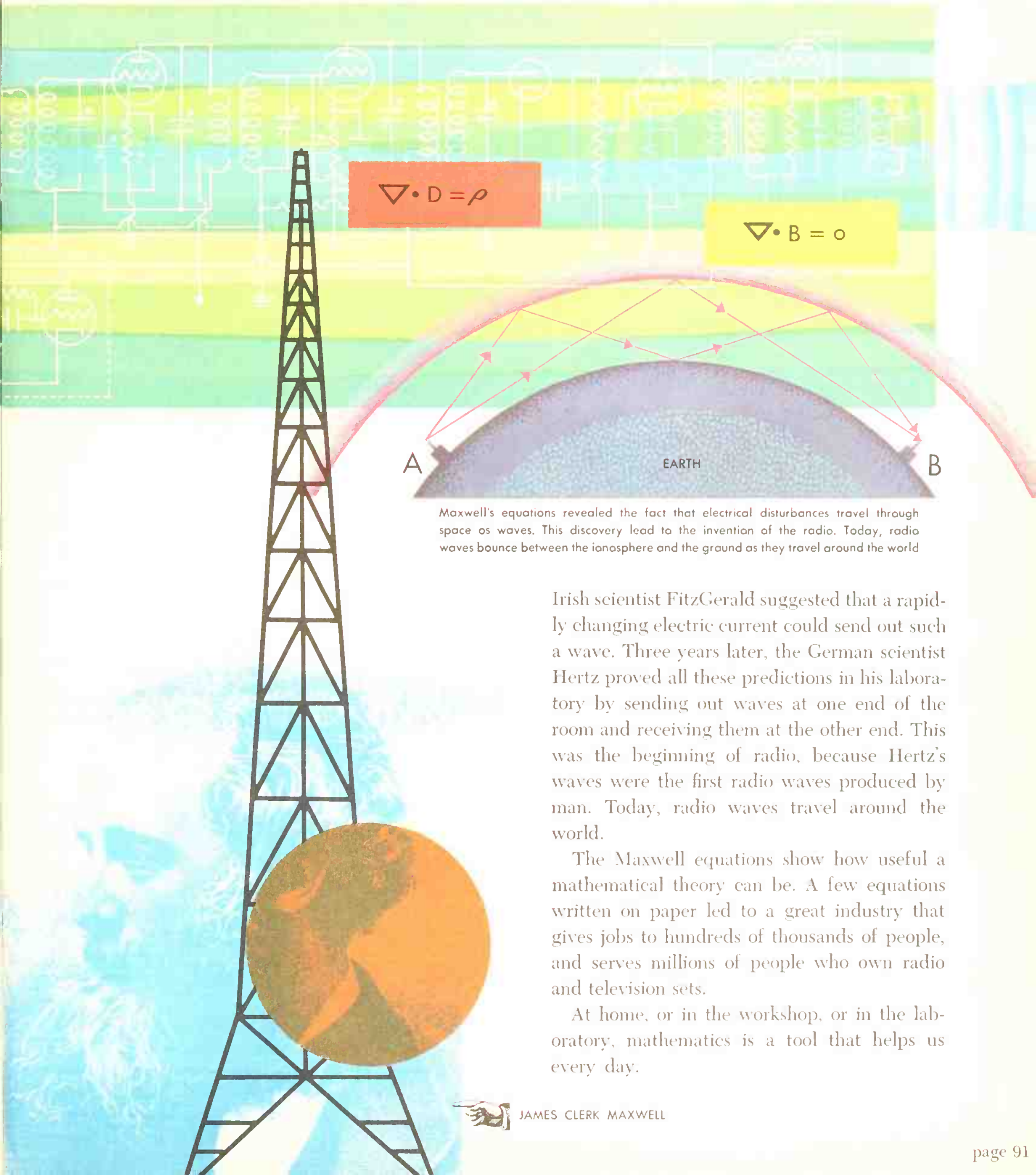
The physicist uses mathematics to explore the mysteries of the atom. His experiments give him facts. His equations show how these facts are related. Often his equations lead to new facts that were never known before. New facts often lead to new inventions. Then new inventions produce new products for our use.



Equations That Built an Industry

The strange-looking symbols at the top of this page are equations written in a mathematical shorthand. They are known as Maxwell's equations. Because of these equations, you can sit at home and hear a concert being played a thousand miles away, or watch a baseball game on your television screen.

During the 1870's, James Clerk Maxwell, a British scientist, was studying the behavior of electricity and magnetism. He found that he could sum up their properties in these four equations. After he discovered these equations, the equations told him something that nobody had suspected before. They told him that electrical disturbances travel through space as waves, moving with the speed of light. In 1883, the



Maxwell's equations revealed the fact that electrical disturbances travel through space as waves. This discovery lead to the invention of the radio. Today, radio waves bounce between the ionosphere and the ground as they travel around the world

Irish scientist FitzGerald suggested that a rapidly changing electric current could send out such a wave. Three years later, the German scientist Hertz proved all these predictions in his laboratory by sending out waves at one end of the room and receiving them at the other end. This was the beginning of radio, because Hertz's waves were the first radio waves produced by man. Today, radio waves travel around the world.

The Maxwell equations show how useful a mathematical theory can be. A few equations written on paper led to a great industry that gives jobs to hundreds of thousands of people, and serves millions of people who own radio and television sets.

At home, or in the workshop, or in the laboratory, mathematics is a tool that helps us every day.

☞ JAMES CLERK MAXWELL

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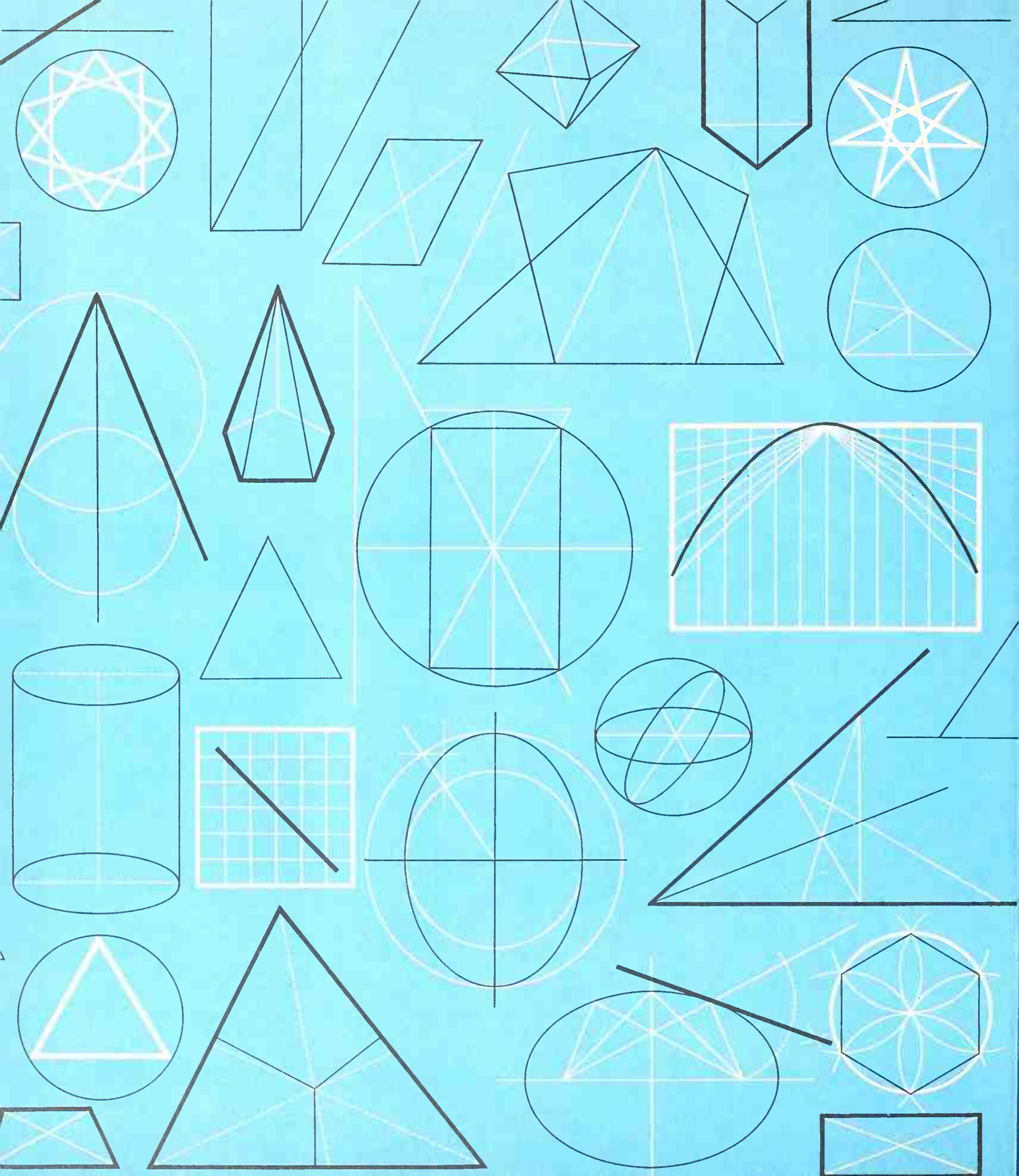

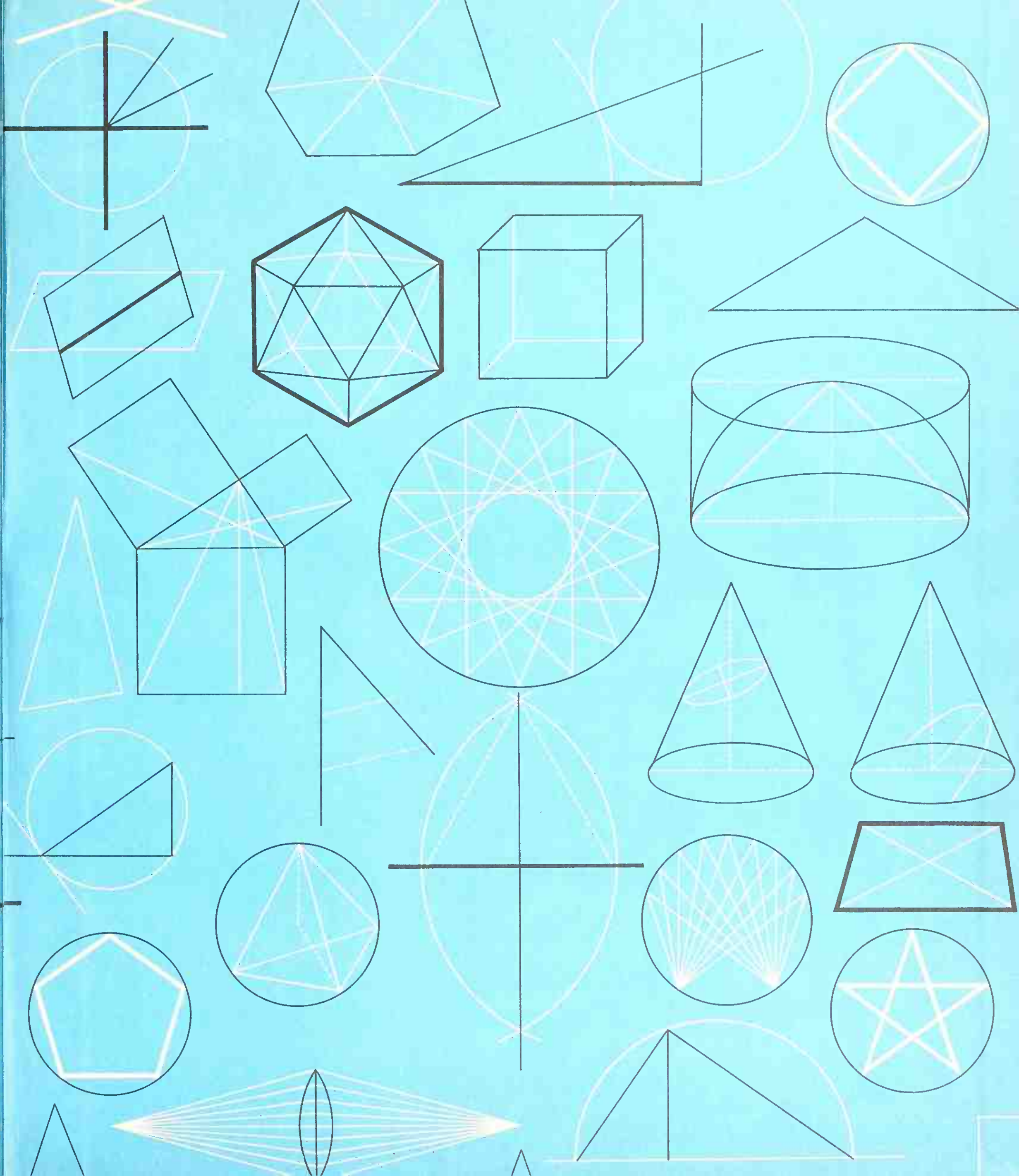

This lively and authoritative book, with more than 200 full-color pictures, introduces young readers to the fascinating science of numbers and space. Words and pictures explain the fundamental ideas of arithmetic, algebra, geometry, trigonometry, and calculus, as well as mathematics in nature and art. The reader will see why mathematics is important to our daily lives, and how men have used it across the centuries, from the Pyramid builders of ancient Egypt to astronomers of today. Plus mathematical tricks and games.