Proceedings of Symposia in
PURE MATHEMATICS

Volume 33, Part 2

Automorphic Forms, Representations, and L-functions

A. Borel
W. Casselman
Editors

American Mathematical Society
Providence, Rhode Island

PROCEEDINGS OF THE SYMPOSIUM IN PURE MATHEMATICS
OF THE AMERICAN MATHEMATICAL SOCIETY

HELD AT OREGON STATE UNIVERSITY
CORVALLIS, OREGON
JULY 11–AUGUST 5, 1977

EDITED BY
A. BOREL
and
W. CASSELMAN

Prepared by the American Mathematical Society
with partial support from National Science Foundation Grant MCS 76-24539

Library of Congress Cataloging in Publication Data

Symposium in Pure Mathematics, Oregon State University, 1977.
Automorphic forms, representations, and *L*-functions.

(Proceedings of symposia in pure mathematics; v. 33)
Includes bibliographical references and index.
1. Automorphic forms–Congresses. 2. Lie groups–Congresses. 3. Representations of groups–Congresses. 4. *L*-functions–Congresses. I. Borel, Armand. II. Casselman, W., 1941– III. American Mathematical Society. IV. Title V. Series.
QA331.S937 1977 512′.7 78-21184
ISBN 0-8218-1435-4 (v.1)
ISBN 0-8218-1437-0 (v.2)

AMS (MOS) subject classifications (1970). Primary 10D20, 22E50, 12A70;
Secondary 20G25, 20G35, 14G10, 22E55, 22E45.

CONTENTS

Part 2

III. Automorphic representations and L-functions

IV. Arithmetical algebraic geometry and automorphic L-functions

Index

Proceedings of Symposia in Pure Mathematics
Vol. 33 (1979), part 2, pp. 3–26

NUMBER THEORETIC BACKGROUND

J. TATE

1. Weil Groups. If G is a topological group we shall let G^c denote the closure of the commutator subgroup of G, and $G^{ab} = G/G^c$ the maximal abelian Hausdorff quotient of G. Recall that if H is a closed subgroup of finite index in G there is a *transfer* homomorphism $t\colon G^{ab} \to H^{ab}$, defined as follows: if $s\colon H\backslash G \to G$ is any section, then for $g \in G$,

$$t(gG^c) = \prod_{x \in H\backslash G} h_{g,x} \quad (\text{mod } H^c),$$

where $h_{g,x} \in H$ is defined by $s(x)g = h_{g,x}s(xg)$.

(1.1) *Definition of Weil group.* Let F be a local or global field and $\bar{F}$ a separable algebraic closure of F. Let $E, E', \cdots$ denote finite extensions of F in $\bar{F}$. For each such E, let $G_E = \mathrm{Gal}(\bar{F}/E)$. A *Weil group for $\bar{F}/F$* is not really just a group but a triple $(W_F, \varphi, \{r_E\})$. The first two ingredients are a topological group W_F and a continuous homomorphism $\varphi\colon W_F \to G_F$ with dense image. Given W_F and φ, we put $W_E = \varphi^{-1}(G_E)$ for each finite extension E of F in $\bar{F}$. The continuity of φ just means that W_E is open in W_F for each E, and its having dense image means that φ induces a *bijection* of homogeneous spaces:

$$W_F/W_E \xrightarrow{\sim} G_F/G_E \approx \mathrm{Hom}_F(E, \bar{F})$$

for each E, and in particular, a group isomorphism $W_F/W_E \approx \mathrm{Gal}(E/F)$ when E/F is Galois. The last ingredient of a Weil group is, for each E, an isomorphism of topological groups $r_E\colon C_E \xrightarrow{\sim} W_E^{ab}$, where

$$C_E = \begin{cases} \text{The multiplicative group } E^* \text{ of } E \text{ in the local case,} \\ \text{the idele-class group } A_E^*/E^* \text{ in the global case.} \end{cases}$$

In order to constitute a Weil group these ingredients must satisfy four conditions:

(W_1) For each E, the composed map

$$C_E \overset{r_E}{\underset{\sim}{\longrightarrow}} W_E^{ab} \xrightarrow{\text{induced by } \varphi} G_E^{ab}$$

is the reciprocity law homomorphism of class field theory.

(W_2) Let $w \in W_F$ and $\sigma = \varphi(w) \in G_F$. For each E the following diagram is commutative:

AMS (MOS) subject classifications (1970). Primary 12A70.

$$\begin{array}{ccc} C_E & \xrightarrow{\;r_E\;} & W_E^{\mathrm{ab}} \\ {\scriptstyle \text{induced by } \sigma}\big\downarrow & & \big\downarrow{\scriptstyle \text{conjugation by } w} \\ C_{E^\sigma} & \xrightarrow{\;r_{E^\sigma}\;} & W_{E^\sigma}^{\mathrm{ab}} \end{array}$$

(W_3) For $E' \subset E$ the diagram

$$\begin{array}{ccc} C_{E'} & \xrightarrow{\;r_{E'}\;} & W_{E'}^{\mathrm{ab}} \\ {\scriptstyle \text{induced by inclusion } E'\subset E}\big\downarrow & & \big\downarrow{\scriptstyle \text{transfer}} \\ C_E & \xrightarrow{\;r_E\;} & W_E^{\mathrm{ab}} \end{array}$$

is commutative.

(W_4) The natural map

$$W_F \longrightarrow \operatorname{proj}\lim_E \{W_{E/F}\}$$

is an isomorphism of topological groups, where

(1.1.1) $$W_{E/F} \text{ denotes } W_F/W_E^c \quad (\textit{not } W_F/W_E),$$

and the projective limit is taken over all E, ordered by inclusion, as $E \to \bar{F}$.

This concludes our definition of Weil group. It is clear from the definition that if W_F is a Weil group for $\bar{F}/F$, then, for each finite extension E of F in $\bar{F}$, W_E (furnished with the restriction of φ and the isomorphisms $r_{E'}$ for $E' \supset E$) is a Weil group for $\bar{F}/E$.

If W_F is a Weil group, then for each $F \subset E' \subset E$ the diagram

(1.2.2) $$\begin{array}{ccc} C_E & \xrightarrow{\;r_E\;} & W_E^{\mathrm{ab}} \\ {\scriptstyle \text{norm, } N_{E/E'}}\big\downarrow & & \big\downarrow{\scriptstyle \text{induced by inclusion } W_E\subset W_{E'}} \\ C_{E'} & \xrightarrow{\;r_{E'}\;} & W_{E'}^{\mathrm{ab}} \end{array}$$

is commutative.

This follows from the fact that, when H is a normal subgroup of finite index in G, the composition

$$H^{\mathrm{ab}} \xrightarrow{\text{induced by inclusion}} G^{\mathrm{ab}} \xrightarrow{\text{transfer}} H^{\mathrm{ab}}$$

is the map which takes an element $h \in H$ into the product of its conjugates by representatives of elements of G/H. (In the notation of the first paragraph above, $h_{g,x} = s(x)\, gs(x)^{-1}$, if $g \in H \subset G$.)

(1.2) *Cohomology; construction of Weil groups.* Let W_F be a Weil group for $\bar{F}/F$. Then for each Galois E/F the group $W_{E/F} = W_F/W_E^c$ is an extension of $W_F/W_E = \mathrm{Gal}(E/F)$ by $W_E/W_E^c \approx C_E$. Let $\alpha_{E/F} \in H^2(\mathrm{Gal}(E/F), C_E)$ denote the class of this group extension. For each $n \in \boldsymbol{Z}$, let

(1.2.3) $$\alpha_n(E/F)\colon H^n(\mathrm{Gal}(E/F), \boldsymbol{Z}) \longrightarrow H^{n+2}(\mathrm{Gal}(E/F), C_E)$$

be the map given by cup product with $\alpha_{E/F}$. Since $C_{E'} \to C_E^{\mathrm{Gal}(E/E')}$ is a bijection for $F \subset E' \subset E$, the property (W_3) above implies, via an abstract cohomological theorem (combine the corollary of p. 184 of **[AT]** with Theorem 12, p. 154, of **[S1]**), that $\alpha_n(E/F)$ is an isomorphism for every n. Moreover, the canonical classes are interrelated by

$$\text{(1.2.4)} \qquad \mathrm{infl}\, \alpha_{E'/F} = [E{:}E']\alpha_{E/F} \quad \text{and} \quad \mathrm{res}\, \alpha_{E/F} = \alpha_{E/E'}$$

(for the first, use Theorem 6 on p. 188 of **[AT]**; the second is obvious). Thus, implicit in the existence of Weil groups is all the cohomology of class field theory.

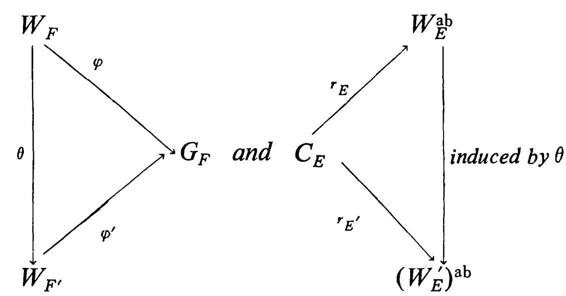
For example, taking $n = -1$ in (1.2.3) we find $H^1(\mathrm{Gal}(E/F), C_E) = 0$. Taking $n = 0$, we find $H^2(\mathrm{Gal}(E/F), C_E)$ is cyclic of order $[E{:}F]$, generated by $\alpha_{E/F}$. Taking $n = -2$ we find an isomorphism $G_{E/F}^{\mathrm{ab}} \approx C_F/N_{E/F}C_E$ which, by (W_1), is that given by the reciprocity law. For E/F cyclic, this isomorphism determines $\alpha_{E/F}$, and it follows that $\alpha_{E/F}$ is the "canonical" or "fundamental" class of class field theory. The same is true for arbitrary E/F as one sees by taking a cyclic E_1/F of the same degree as E/F, and inflating $\alpha_{E/F}$ and $\alpha_{E_1/F}$ to EE_1/F, where they are equal by (1.2.4).

Conversely, if we are given classes $\alpha_{E/E'}$ satisfying (1.2.4) and such that the maps (1.2.3) are isomorphisms, then we can construct a Weil group W_F as the projective limit of group extensions $W_{E/F}$ made with these classes. This construction is abstracted and carried out in great detail in Chapter XIV of **[AT]**. The existence of such classes $\alpha_{E/E'}$ is proved in **[AT]** and **[CF]**.

Thus, a Weil group exists for every F; to what extent is it unique?

(1.3) *Unicity.* A Weil group for $\bar{F}/F$ is unique up to isomorphism. More precisely:

(1.3.1) PROPOSITION. *Let W_F and W_F' be two Weil groups for $\bar{F}/F$. There exists an isomorphism $\theta\colon W_F \xrightarrow{\sim} W_F'$ such that the diagrams*



are commutative.

For each finite Galois E/F, let $I(E)$ denote the set of isomorphisms f such that the following diagram is commutative

$$\begin{array}{ccccccccc} 0 & \longrightarrow & C_E & \longrightarrow & W_{E/F} & \longrightarrow & \mathrm{Gal}(E/F) & \longrightarrow & 0 \\ & & \Big\downarrow{\scriptstyle \mathrm{id}} & & \Big\downarrow{\scriptstyle f} & & \Big\downarrow{\scriptstyle \mathrm{id}} & & \\ 0 & \longrightarrow & C_E & \longrightarrow & W'_{E/F} & \longrightarrow & \mathrm{Gal}(E/F) & \longrightarrow & 0 \end{array}$$

Since the two group extensions $W_{E/F}$ and $W'_{E/F}$ each have the same class, namely the canonical class $\alpha_{E/F}$, as their cohomology class, $I(E)$ is not empty. Since $H^1(\mathrm{Gal}(E/F), C_E) = 0$, an isomorphism $f \in I(E)$ is determined up to composing with an inner automorphism of $W_{E/F}$ by an element of $C_E \approx W_E^{ab}$. The center of $W_{E/F}$ is C_F, and C_E/C_F is compact. Hence $I(E)$, as principal homogeneous space for C_E/C_F, has a natural compact topology. For $E_1 \supset E$, the natural map $I(E_1) \to I(E)$ is continuous for this topology, since it is reflected in the norm map $N_{E_1/E}$, once we pick an element of $I(E_1)$. Hence the projective limit proj $\lim_E(I(E))$ is not empty. An element θ of this limit gives an isomorphism $W_F \xrightarrow{\sim} W_F'$ by (W_4), and has the required properties.

It turns out (cf. (1.5.2)) that θ is unique up to an inner automorphism of W_F by an element $w \in \mathrm{Ker}\ \varphi$, but we postpone the discussion of this question until after the next section.

(1.4) *Special cases.* We discuss now the special features of the four cases: F local nonarchimedean, F a global function field, F local archimedean, and F a global number field. In the first two of these, G_F is a completion of W_F; in the last two it is a quotient of W_F.

(1.4.1) *F local nonarchimedean.* For each E, let k_E be the residue field of E and $q_E = \mathrm{Card}(k_E)$. Let $\bar{k} = \bigcup_E k_E$. We can take W_F to be the dense subgroup of G_F consisting of the elements $\sigma \in G_F$ which induce on $\bar{k}$ the map $x \to x^{q_F^n}$ for some $n \in \boldsymbol{Z}$. Thus W_F contains the *inertia group* I_F (the subgroup of G_F fixing $\bar{k}$), and $W_F/I_F \approx \boldsymbol{Z}$. The topology in W_F is that for which I_F gets the profinite topology induced from G_F, and is open in W_F. The map φ: $W_F \to G_F$ is the inclusion, and the maps r_E: $E^* \to W_E^{ab}$ *are* the reciprocity law homomorphisms. Concerning the *sign* of the reciprocity law, our convention will be that $r_E(a)$ acts as $x \mapsto x^{\|a\|_E}$ on $\bar{k}$, where $\|a\|_E$ is the normed absolute value of an element $a \in E^*$. (If π_E is a uniformizer in E, then $\|\pi_E\|_E = q_E^{-1}$; thus our convention is that uniformizers correspond to the inverse of the Frobenius automorphism, as in Deligne **[D3]**, *opposite* to the convention used in **[D1]**, **[AT]**, **[CF]**, and **[S1]**.)

(1.4.2) *F a global function field.* Here the picture is as in (1.4.1). Just change "residue field" to "constant field", "inertia group I_F" to "geometric Galois group $\mathrm{Gal}(\bar{F}/F\bar{k})$", and define the norm $\|a\|_E$ of an idele class $a \in C_E$ to be the product of the normed absolute values of the components of an idele representing the class.

(1.4.3) *F local archimedean.* If $F \approx \boldsymbol{C}$ we can take $W_F = F^*$, φ the trivial map, r_F the identity.

If $F \approx \boldsymbol{R}$, we can take $W_F = \bar{F}^* \cup j\bar{F}^*$ with the rules $j^2 = -1$ and $jcj^{-1} = \bar{c}$, where $c \mapsto \bar{c}$ is the nontrivial element of $\mathrm{Gal}(\bar{F}/F)$. The map φ takes $\bar{F}^*$ to 1 and $j\bar{F}^*$ to that nontrivial element. The map $r_{\bar{F}}$ is the identity, and r_F is characterized by

$$r_F(-1) = jW_{\bar{F}}^c,$$
$$r_F(x) = \sqrt{x}\ W_{\bar{F}}^c, \quad \text{for } x \in F,\ x > 0.$$

($W_{\bar{F}}^c$ is the "unit circle" of elements $u \in \bar{F}$ with $\|u\| = N_{\bar{F}/F}\, u = 1$.)

(1.4.4) *F a global number field.* This is the only case in which there is, at present, no simple description of W_F, but merely the artificial construction by cocycles described in (1.2). This construction is due to Weil in **[W1]**, where he emphasizes the importance of the problem of finding a more natural construction, and proves the following facts. The map φ: $W_F \to G_F$ is surjective. Its kernel is the connected component of identity in W_F, isomorphic to the inverse limit, under the norm maps $N_{E/E'}$ of the connected components D_E of 1 in C_E. These norm maps $D_E \to D_{E'}$ are surjective, and $r_E(D_E) = (\operatorname{Ker}\varphi)W_E^c/W_E^c$ is the image of Ker φ in $W_{E/F}$. If E has r_1 real and r_2 complex places then D_E is isomorphic to the product of $\boldsymbol{R}$ with $r_1 + r_2 - 1$ solenoids and r_2 circles.

(1.4.5) Notice that in each of the four cases just discussed the subgroups of W_F which are of the form W_E for some finite extension E of F are just the open subgroups of finite index. Their intersection, Ker φ, is a divisible connected abelian group, trivial in the first two cases, isomorphic to $\boldsymbol{C}^*$ in the third, and enormous in the last case.

(1.4.6) In each case there is a homomorphism $w \mapsto \|w\|$ of W_F into the multiplicative group of strictly positive real numbers which reflects the norm or normed absolute value on C_F under the isomorphism $r_F : C_F \approx W_F^{ab}$. By (1.1.2) and the rule $\|N_{E/F}a\|_F = \|a\|_E$, the restriction of this "norm" function $\|w\|$ from W_F to a subgroup W_E is the norm function for W_E, so we can write simply $\|w\|$ instead of $\|w\|_F$ without creating confusion. In each case the kernel W_F^1 of $w \mapsto \|w\|$ is *compact*. In the first two cases, the image of $w \mapsto \|w\|$ consists of the powers of q_F, and W_F is a semidirect product $\boldsymbol{Z} \ltimes W_F^1$. In the last two cases $w \mapsto \|w\|$ is surjective, and in fact, W_F is a direct product $\boldsymbol{R} \times W_F^1$.

Let us refer to the first two cases as the "$\boldsymbol{Z}$-cases" and the last two as the "$\boldsymbol{R}$-cases". In the $\boldsymbol{Z}$-cases, φ is injective, but not surjective; in the $\boldsymbol{R}$-cases, φ is surjective, but not injective.

(1.5) *Automorphisms of Weil groups.* Let W_F be a Weil group for $\bar{F}/F$. Let $\operatorname{Aut}(\bar{F}, W_F)$ denote the set of pairs (σ, α), where $\sigma \in G_F$ is an automorphism of $\bar{F}/F$, and α is an automorphism of the group W_F such that the following diagrams are commutative, the second for all E:

$$\begin{array}{ccc} W_F & \xrightarrow{\ \varphi\ } & G_F \\ {\scriptstyle\alpha}\downarrow & & \downarrow{\scriptstyle\operatorname{Inn}(\alpha)} \\ W_F & \xrightarrow{\ \varphi\ } & G_F \end{array} \qquad \begin{array}{ccc} C_E & \xrightarrow{\ r_E\ } & W_E^{ab} \\ {\scriptstyle\text{induced by }\sigma}\downarrow & & \downarrow{\scriptstyle\text{induced by }\alpha} \\ C_{E^\sigma} & \xrightarrow{\ r_{E^\sigma}\ } & W_{E^\sigma}^{ab} \end{array}$$

Here $\operatorname{Inn}(\alpha)$ denotes the inner automorphism defined by α.

We shall call an automorphism of W_F *essentially inner* if it induces an inner automorphism on $W_{E/F}$ for each finite Galois E/F.

(1.5.1) PROPOSITION. *In the* $\boldsymbol{R}$*-cases* $\operatorname{Aut}(\bar{F}, W_F)$ *consists of the pairs* $(\varphi(w), \operatorname{Inn}(w))$, *for* $w \in W_F$.

In the $\boldsymbol{Z}$*-cases,* $\operatorname{Aut}(\bar{F}, W_F)$ *consists of the pairs* (σ, α_σ), *for* $\sigma \in G_F$, *where* α_σ *denotes the restriction of* $\operatorname{Inn}(\sigma)$ *to* W_F, *viewed as a subgroup of* G_F *via* φ. *This automorphism* α_σ *of* W_F *is not an inner automorphism if* $\sigma \notin W_F$, *but it is essentially inner in the sense of the definition above.*

(1.5.2) COROLLARY. *The isomorphism* θ *in* (1.3.1) *is unique in the* $\boldsymbol{Z}$*-cases, and is* *unique to an inner automorphism of* W_F *by an element of the connected component* $W_F^0 = \operatorname{Ker} \varphi$ *in the* $\boldsymbol{R}$*-cases.*

To prove (1.5.1) in the $\boldsymbol{R}$-cases, we note first that, since φ is surjective, we are reduced immediately to the case of the corollary: We must show that if $(1, \alpha) \in \operatorname{Aut}(\bar{F}, W_F)$, then $\alpha = \operatorname{Inn}(w)$ for some $w \in W_F^0$. Going back to the proof of (1.3.1) with $W_F' = W_F$ we find that the group of these α's is given by

$$\begin{aligned} \operatorname{proj}\lim_E (C_E/C_F) &= \operatorname{proj}\lim_E (C_E^1/C_F^1) && \text{(1 for norm 1)} \\ &= \operatorname{proj}\lim_E C_E^1/\operatorname{proj}\lim_E C_F^1 && \text{(by compacity)} \\ &= W^{0,1}/(Z \cap W^{\circ,1}) && \text{(existence theorem; 0 for connected component)} \\ &= W^0/Z \end{aligned}$$

as claimed, where Z is the center of W.

Suppose now we are in a $\boldsymbol{Z}$-case. Since φ is injective, i.e., $W_F \subset G_F$, it is clear that $\operatorname{Aut}(\bar{F}, W_F)$ consists only of the pairs (σ, α_σ). The center of G_F is 1, because $G_F/G_E \approx \operatorname{Gal}(E/F)$ acts faithfully on $C_E \subset G_E^{ab}$ for each finite Galois E/F. Hence, since W_F is dense in G_F, α_σ is not an inner automorphism of W_F unless $\sigma \in W_F$. However, α_σ does induce an inner automorphism of $W_{E/F}$ for finite E/F. Since W_F is dense in G_F it suffices to prove this last statement for σ close to 1, say $\sigma \in G_E$. Then α_σ induces an isomorphism of the group extension $0 \to C_E \to W_{E/F} \to \operatorname{Gal}(E/F) \to 0$ which is identity on the extremities, and hence is an inner automorphism by an element of C_E, since $H^1(\operatorname{Gal}(E/F), C_E) = 0$.

(1.6) *The local-global relationship.* Suppose now F is global. Let v be a place of F and F_v the completion of F at v. Let $\bar{F}$ (resp. $\bar{F}_v$) be a separable algebraic closure of F (resp. F_v) and let W_F (resp. W_{F_v}) be a Weil group for $\bar{F}/F$ (resp. for $\bar{F}_v/F_v$).

(1.6.1) Proposition. *Let* $i_v : \bar{F} \to \bar{F}_v$ *be an F-homomorphism. For each finite extension E of F in* $\bar{F}$, *let* $E_v = i(E)F_v$ *be the induced completion of E. There exists a continuous homomorphism* $\theta_v \colon W_{F_v} \to W_F$ *such that the following diagrams are commutative*

$$\begin{array}{ccc} W_{F_v} & \longrightarrow & G_{F_v} \\ \Big\downarrow \theta_v & & \Big\downarrow \text{induced by } i_v \\ W_F & \longrightarrow & G_F \end{array} \qquad \begin{array}{ccc} E_v^* & \xrightarrow{\sim} & W_{E_v}^{ab} \\ \Big\downarrow n_v & & \Big\downarrow \text{induced by } i_v \\ C_E & \xrightarrow{\sim} & W_E^{ab} \end{array}$$

where n_v *maps* $a \in E_v^*$ *to the class of the idele whose v-component is a and whose other components are* 1.

If F is a function field, then θ_v is unique. In the number field case, θ_v is unique up to composition with an inner automorphism of W_F defined by an element of the connected component $W_F^0 = \operatorname{Ker} \varphi$.

The proof of this is analogous to the proof of (1.3.1) and (1.5.1), using the standard relationship between global and local canonical classes, and the vanishing of $H^1(\mathrm{Gal}(E_v/F_v), C_E)$.

Combining (1.6.1) and (1.5.2) we obtain

COROLLARY. *The diagram*

$$\begin{array}{ccc} W_{F_v} & \longrightarrow & G_{F_v} \\ \theta_v \downarrow & & \downarrow i_v \\ W_F & \longrightarrow & G_F \end{array}$$

is unique up to isomorphism, and the automorphisms of it are the inner automorphisms of W, *defined by elements* $w \in W^0$, *which induce an automorphism of* W_{F_v} *(i.e., for* v *nonarchimedean,* w *fixed by* G_{F_v}*; for* v *archimedean,* w *a product of an element of* W^0 *by an element of* $(W^0)^{G_{F_v}}$*).*

2. Representations. Let G be a topological group. By a *representation* of G we shall mean, in this section, a continuous homomorphism $\rho: G \to \mathrm{GL}(V)$ where V is a finite-dimensional complex vector space. By a *quasi-character* of G we mean a continuous homomorphism $\chi: G \to C^*$. If (ρ, V) is any representation of G, then $\det \rho$ is a quasi-character which we may sometimes denote also by $\det V$. The map $V \mapsto \det V$ sets up a bijection between the isomorphism classes of representations V of dimension 1 and quasi-characters. Of course we can identify quasi-characters of G with quasi-characters of G^{ab}.

We let $M(G)$ denote the set of isomorphism classes of representations of G, and $R(G)$ the group of virtual representations. A function λ on $M(G)$ with values in an abelian group X can be "extended" to a homomorphism $R(G) \to X$ if and only if it is *additive*, i.e., satisfies $\lambda(V) = \lambda(V')\lambda(V'')$ whenever $0 \to V' \to V \to V'' \to 0$ is an exact sequence of representations of G.

(2.1) Let F be a local or global field, $\bar{F}$ an algebraic closure of F, and W_F a Weil group for $\bar{F}/F$. Let (ρ, V) be a representation of W_F. Since $W_F = \mathrm{proj\,lim}\,\{W_{E/F}\}$ and $\mathrm{GL}(V)$ has no nontrivial small subgroups, ρ must factor through $W_{E/F}$, for some finite Galois extension E of F in $\bar{F}$. It follows that if α is an essentially inner automorphism of W_F in the sense of (1.5), then $V^\alpha \approx V$. Thus essentially inner automorphisms act as identity on $M(W_F)$ and $R(W_F)$. By (1.5.1) we can therefore safely think of $M(W_F)$ as *a set depending only on* F, not on a particular choice of $\bar{F}$ or of Weil group W_F for $\bar{F}/F$, and the same for $R(W_F)$. In this sense, if v is a place of a global F, the "restriction" map $M(W_F) \to M(W_{F_v})$ induced by the map θ_v of Proposition (1.6.1) *depends only on* v, not on a particular choice of the maps i_v and θ_v in that proposition, and the same for $R(W_F) \to R(W_{F_v})$. We shall indicate this map by $\rho \mapsto \rho_v$ or $V \mapsto V_v$. (The independence from θ_v results from (1.6.1), and the independence from i_v, from (1.5.1).)

If E/F is any finite separable extension, we have *canonical* maps

$$R(W_F) \underset{\substack{\text{induction} \\ \text{ind}_{E/F}}}{\overset{\substack{\text{res}_{E/F} \\ \text{restriction}}}{\rightleftarrows}} R(W_E)$$

satisfying the usual Frobenius reciprocity, for we can identify W_E with a closed subgroup of finite index in W_F.

(2.2) *Quasi-characters and representations of Galois type.* Using the isomorphism $C_F \approx W_F^{\mathrm{ab}}$ we can identify quasi-characters of C_F with quasi-characters of W_F^{ab}. For example, we will denote by ω_s, for $s \in \boldsymbol{C}$, the quasi-character of W_F associated with the quasi-character $c \mapsto \|c\|_F^s$, where $\|c\|_F$ is the norm of $c \in C_F$. Thus $\omega_s(w) = \|w\|^s$ in the notation of (1.4).

On the other hand, since $\varphi\colon W_F \to G_F$ has dense image, we can identify the set $M(G_F)$ of isomorphism classes of representations of G_F with a subset of $M(W_F)$. We will call the representations in this subset "of Galois type". Thus, by (1.4.5), a representation ρ of W_F is of Galois type if and only if $\rho(W_F)$ is finite.

With these identifications, a character χ of G_F is identified with the character χ of C_F to which χ corresponds by the reciprocity law homomorphism.

(2.2.1) In the $\boldsymbol{Z}$-cases, i.e., if F is a global function field, or a nonarchimedean local field, then every *irreducible* representation ρ of W_F is of the form $\rho = \sigma \otimes \omega_s$, where σ is of Galois type. This is a general fact about irreducible representations of a group which is an extension of $\boldsymbol{Z}$ by a profinite group; some twist of ρ by a quasi-character trivial on the profinite subgroup has a finite image; see **[D3**, §4.10].

(2.2.2) If F is an archimedean local field, the quasi-characters of W_F, i.e., of $F^* \approx W_F^{\mathrm{ab}}$, are of the form $\chi = z^{-N}\omega_s$, where $z\colon F \to \boldsymbol{C}$ is an embedding and N an integer $\geqq 0$, restricted to be 0 or 1 if F is real. If F is complex, these are the only irreducible representations of $F^* = W_F$. If F is real, W_F has an abelian subgroup $W_{\bar{F}} = \bar{F}^*$ of index 2, and the irreducible representations of W_F which are not quasi-characters are of the form $\rho = \mathrm{Ind}_{\bar{F}/F}(z^{-N}\omega_s)$ with $N > 0$. (For $N = 0$ this induced representation is reducible:

$$\mathrm{Ind}_{\bar{F}/F}\omega_s = \omega_s \oplus x^{-1}\omega_{s+1} \tag{2.2.2.1}$$

where $x\colon F \to \boldsymbol{C}$ is the embedding of F in $\boldsymbol{C}$.)

(2.2.3) Suppose F is a global number field. A *primitive* (i.e., not induced from a proper subgroup) *irreducible* representation ρ of W_F is of the form $\rho = \sigma \otimes \chi$ where σ is of Galois type and χ a quasi-character.

Choose a finite Galois extension E of F big enough so that ρ factors through $W_{E/F} = W_F/W_E^c$. Since ρ is primitive and irreducible, $\rho(W_E^{\mathrm{ab}})$ must be in the center of $\mathrm{GL}(V)$, because W_E^{ab} is an abelian normal subgroup of $W_{E/F}$. In other words, the composed map $W_F \xrightarrow{\rho} \mathrm{GL}(V) \to \mathrm{PGL}(V)$ kills W_E and therefore gives a projective representation of $\mathrm{Gal}(E/F)$. This projective representation of $\mathrm{Gal}(E/F)$ can be lifted to a linear representation $\sigma_0\colon G_F \to \mathrm{GL}(V)$ (see **[S3**, Corollary of Theorem 4]). Let $\sigma = \sigma_0 \cdot \varphi$. The two compositions

$$W_F \underset{\sigma}{\overset{\rho}{\rightrightarrows}} \mathrm{GL}(V) \longrightarrow \mathrm{PGL}(V)$$

are equal; hence $\rho = \sigma \otimes \chi$ for some quasi-character χ.

(2.2.4) Note that, in all cases, global and local, the *primitive irreducible representations of W_F are twists of Galois representations by quasi-characters.*

(2.2.5) In the local nonarchimedean case one can say much more about the structure of primitive irreducible representations (see **[K]**). A first result of this sort is

(2.2.5.1) PROPOSITION. *Let F be local nonarchimedean and let V be a primitive irreducible representation of W_F. Then the restriction of V to the wild ramification group P is irreducible.*

This result is proved in **[K]** and **[B]**. The proof depends on the supersolvability of G_F/P.

(2.2.5.2) COROLLARY. *The dimension of V is a power of the residue characteristic p.*

Indeed, P is a pro-p-group.

(2.2.5.3) COROLLARY. *If U is an irreducible (not necessarily primitive) representation of W_F of degree prime to p, then U is monomial.*

Because U is induced from a primitive irreducible V whose dimension is prime to p and a p-power, hence 1.

(2.3) *Inductive functions of representations.* Let F be a local or global field. For a representation V of W_F, let $[V] \in R(W_F)$ denote the virtual representation determined by V. Let $R^0(W_F)$ denote the group of virtual representations of degree 0 of W_F, i.e., those of the form $[V]-[V']$, with $\dim V = \dim V'$.

(2.3.1) PROPOSITION. *The group $R(W_F)$ is generated by the elements of the form $\mathrm{Ind}_{E/F}[\chi]$ for E/F finite and χ a quasi-character of W_E. Similarly, $R^0(W_F)$ is generated by the elements of the form $\mathrm{Ind}_{E/F}([\chi] - [\chi'])$.*

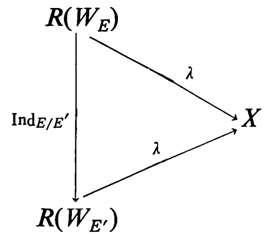
It suffices to prove the second statement, because $R(W_F) = R^0(W_F) + \boldsymbol{Z}\cdot[1]$. Let $R^0_*(W_F)$ denote the subgroup of $R^0(W_F)$ generated by the elements $\mathrm{Ind}_{E/F}([\chi] - [\chi'])$. By the degree 0 variant of Brauer's theorem **[D3**, Proposition 1.5] we have $R^0(G_F) \subset R^0_*(W_F)$. The formula $\mathrm{Ind}(\rho \otimes \mathrm{Res}\,\chi) = (\mathrm{Ind}\,\rho) \otimes \chi$ shows that $R^0_* \cdot \chi \subset R^0_*$ for each quasi-character χ of W_F.

To prove the proposition we must show for each irreducible representation ρ of W_F that $[\rho] - (\dim \rho)[1] \in R^0_*(W_F)$. For each ρ there is a finite extension E of F and a *primitive* irreducible representation ρ_E of W_E such that $\rho = \mathrm{Ind}_{E/F}\,\rho_E$. Then $[\rho] - (\dim\rho)\cdot[1]$ is the sum of $\mathrm{Ind}_{E/F}([\rho_E] - (\dim \rho_E)[1_E])$ and $(\dim \rho_E)\,(\mathrm{Ind}_{E/F}[1_E] - [E{:}F]\,[1_F])$. The latter is of Galois type, so by the transitivity of induction we are reduced to the case in which ρ is primitive and irreducible. But then $\rho = \sigma \otimes \chi$ with $\sigma \in R(G_F)$ and χ a quasi-character (2.2.4). If $n = \dim \rho = \dim \sigma$

$$[\rho] - n[1] = ([\sigma] - n[1])\,[\chi] + n([\chi] - [1])$$

and this is in $R^0_*(W_F)$ by the remarks above, since $[\sigma] - n\,[1] \in R^0(G_F)$.

(2.3.2) DEFINITION. Let F be a local or global field. Let λ be a function which assigns to each finite separable extension E/F and each $V \in M(W_E)$ an element $\lambda(V)$ in an abelian group X. We say λ is *additive over F* is for each E and each exact sequence $0 \to V' \to V \to V'' \to 0$ of representations of W_E we have $\lambda(V) = \lambda(V')\lambda(V'')$. When that is so we can define λ on virtual representations so that λ: $R(E) \to X$ is a homomorphism for each E. We say λ is *inductive over F* if it is additive over F and the diagram



is commutative for finite separable extensions $E/E'/F$. We say λ is *inductive in degree* 0 *over* F if the same is true with R replaced by R^0.

(2.3.3) REMARK. By (2.3.1) a λ which is inductive over F, or even only inductive in degree 0, is uniquely determined by its value on quasi-characters χ of W_E (i.e., of C_E), for all finite separable E/F. In [**D3**, §1.9] there is a discussion, for finite groups, of the relations a function λ of characters of subgroups must satisfy in order that it extend to an inductive function of representations.

(2.3.4) EXAMPLE. Let $a \in C_F$. Put

$$\lambda(V) = (\det V)(r_E(a)) \quad \text{for } V \in M(W_E).$$

Then λ is inductive in degree 0 over F. This follows from property (W_3) of Weil groups and the rule.

$$\det(\operatorname{Ind} V) = (\det V) \cdot \text{transfer}, \quad \text{for } V \text{ virtual of degree } 0$$

(cf. [**D3**, §1]).

(2.3.5) EXAMPLE. Suppose v is a place of a global field F, and λ is an inductive function over F_v. If we put for each finite separable E/F and each $V \in M(W_E)$

$$\lambda_v(V) = \prod_{w \text{ place of } E;\ w|v} \lambda(V_w)$$

we obtain an inductive function λ_v over F. If λ is only inductive in degree 0, then λ_v is inductive in degree 0.

Indeed, by a standard formula for the result of inducing from a subgroup and restricting to a different subgroup we have

$$(\operatorname{Ind}_{E/F} V)_v = \bigoplus_{w|v} \operatorname{Ind}_{E_w/F_v} V_w,$$

because if w_0 is one place of E over v, then the map $\sigma \mapsto \sigma w_0$ puts the set of double cosets $W_E \backslash W_F / W_{F_v}$ in bijection with the set of all such places, and for each σ we can identify $W_{E_{\sigma w_0}}$ with $(\sigma\ W_{F_v} \sigma^{-1}) \cap W_E$.

3. *L*-series, functional equations, local constants. The L-functions considered in this section are those associated by Weil [**W1**] to representations of Weil groups. They include as special cases the "abelian" L-series of Hecke, made with "Grössencharakteren" (= quasi-character of C_F), and the "nonabelian" L-functions of Artin, made with representations of Galois groups. Our discussion follows quite closely that of [**D3**, §§3, 4, 5] which we are just copying in many places.

(3.1) *Local abelian L-functions.* Let F be a local field.

For a quasi-character χ of F^* one defines $L(\chi) \in \boldsymbol{C}^* \cup \{\infty\}$ as follows.

(3.1.1) $F \approx \boldsymbol{R}$. For x the embedding of F in $\boldsymbol{C}$ and $N = 0$ or 1,

$$L(x^{-N}\omega_s) = \Gamma_{\mathbf{R}}(s) \overset{\text{defn}}{=} \pi^{-s/2}\,\Gamma(s/2).$$

(3.2.1) $F \approx \boldsymbol{C}$. For z an embedding of F in $\boldsymbol{C}$ and $N \geqq 0$,

$$L(z^{-N}\omega_s) = \Gamma_{\boldsymbol{C}}(s) \overset{\text{defn}}{=} 2(2\pi)^{-s}\Gamma(s).$$

(3.1.3) F nonarchimedean. For π a uniformizer in F,

$$L(\chi) = \begin{cases} (1 - \chi(\pi))^{-1}, & \text{if } \chi \text{ is unramified,} \\ 1, & \text{if } \chi \text{ is ramified.} \end{cases}$$

In every case, L is a meromorphic function of χ, i.e., $L(\chi\omega_s)$ is meromorphic in s, and L has no zeros.

(3.2) *Local abelian ε-functions.* We will denote by dx a Haar measure on F, by d^*x a Haar measure on F^* (e.g., $d^*x = \|x\|^{-1}\,dx$) and by ψ a nontrivial additive character of F.

Given ψ and dx, one has a "Fourier transform"

$$\hat{f}(y) = \int f(x)\,\psi\,(xy)\,dx.$$

The local functional equation

$$\text{(3.2.1)} \qquad \frac{\int \hat{f}(x)\,\omega_1\chi^{-1}(x)\,d^*x}{L(\omega_1\chi^{-1})} = \varepsilon(\chi, \psi, dx)\,\frac{\int f(x)\chi(x)\,d^*x}{L(\chi)}$$

defines a number $\varepsilon(\chi, \psi, dx) \in \boldsymbol{C}^*$ which is independent of f, for f's such that the two sides make sense. If f is continuous such that $f(x)$ and $\hat{f}(x)$ are $O(e^{-\|x\|})$ as $\|x\| \to \infty$, then the two sides make sense naively for χ such that $\chi(x) = \|x\|^{\sigma}$ with $0 < \sigma < 1$, and each side is a meromorphic function of χ. One takes the same multiplicative Haar measure d^*x on each side. The dependence of ε on ψ and dx comes from the dependence of the Fourier transform $\hat{f}(x)$ on ψ and dx. One finds

$$\text{(3.2.2)} \qquad \varepsilon(\chi, \psi, r\,dx) = r\varepsilon(\chi, \psi, dx), \qquad \text{for } r > 0,$$

$$\text{(3.2.3)} \qquad \varepsilon(\chi, \psi(ax), dx) = \chi(a)\|a\|^{-1}\,\varepsilon(\chi, \psi, dx) \quad \text{for } a \in F^*.$$

Easy computations carried out in **[T1]** and **[W2]** show that the function ε is given by (3.2.2), (3.2.3) and the following explicit formulas:

(3.2.4) $F \simeq \boldsymbol{R}$. Let x be the embedding of F in $\boldsymbol{C}$ and $N = 0$ or 1. For $\psi = \exp(2\pi i x)$ and dx the usual measure, $\varepsilon(x^{-N}\omega_s, \psi, dx) = i^N$.

(3.2.5) $F \approx \boldsymbol{C}$. Let z be an embedding of F in $\boldsymbol{C}$ and $N \geqq 0$. For $\psi = \exp(2\pi i\,\mathrm{Tr}_{\boldsymbol{C}/\boldsymbol{R}} z)$ and $dx = i dz \wedge d\bar{z}$ ($= 2\,da\,db$ for $z = a + bi$), $\varepsilon(z^{-N}\omega_s, \psi, dx) = i^N$.

(3.2.6) *F nonarchimedean.* Let $\mathcal{O}$ be the ring of integers in F. Put

$n(\psi) =$ the largest integer n such that $\psi(\pi^{-n}\mathcal{O}) = 1$,

$a(\chi) =$ the (exponent of the) conductor of χ ($= 0$ if χ is unramified, the smallest integer m such that χ is trivial on units $\equiv 1 \pmod{\pi^m}$ if π is ramified),

$c =$ an element of F^* of valuation $n(\psi) + a(\chi)$. If χ is unramified,

$$\text{(3.2.6.1)} \qquad \varepsilon(\chi, \psi, dx) = \frac{\chi(c)}{\|c\|}\int_{\mathcal{O}} dx.$$

(In particular, $\varepsilon(\chi, \psi, dx) = 1$, if $\int_{\mathcal{O}} dx = 1$, and $n(\psi) = 0$ when χ is unramified.)

For χ ramified,

$$(3.2.6.2)\qquad \begin{aligned}\varepsilon(\chi, \psi, dx) &= \int_{F^*} \chi^{-1}(x)\, \psi(x)\, dx \overset{\text{defn}}{=} \sum_{n\in\mathbf{Z}} \int_{\pi^n\mathcal{O}^*} \chi^{-1}(x)\, \psi(x)\, dx \\ &= \int_{c^{-1}\mathcal{O}^*} \chi^{-1}(x)\, \psi(x)\, dx.\end{aligned}$$

From these formulas one deduces, for χ arbitrary and ω unramified

$$(3.2.6.3)\qquad \varepsilon(\chi\omega, \psi, dx,) = \varepsilon(\chi, \psi, dx)\, \omega(\pi^{n(\psi)+a(\chi)}).$$

(3.3) *Local nonabelian L-functions.* We owe to Artin the discovery that there is an inductive (2.3.2) function L of representations of Weil groups of local fields such that $L(V) = L(\chi)$ when V is a representation of degree 1 corresponding to the quasi-character χ. The explicit description of L is as follows:

(3.3.1) *F archimedean.* Since L is additive, we can define it by giving its value on irreducible V. For F complex, $W_F = F^*$ is abelian, and the only irreducible V's are the quasi-characters χ, for which L has already been defined. For F real, the only irreducible V's which are not of dimension 1 are those of the form $V = \mathrm{Ind}_{\bar{F}/F}\, \chi$, where χ is a quasi-character of $\bar{F}^* = W_{\bar{F}}$ which is not invariant under "complex conjugation". For such V we put $L(V) = L(\chi)$, as we are forced to do in order that L be inductive.

(3.3.2) *F nonarchimedean.* Let I be the inertia subgroup of W_F. Let Φ be an "inverse Frobenius", i.e., an element of W_F such that $\|\Phi\| = \|\pi\|_F$. This condition determines Φ uniquely mod I and we put $L(V) = \det(1 - \Phi | V^I)^{-1}$, where V^I is the subspace of elements in V fixed by I.

A proof that the "nonabelian" function L defined as above is inductive can be found in [**D3**, Proposition 3.8] (as well as in [**A**]). In the archimedean case one uses the relation $\Gamma_{\mathbf{C}}(s) = \Gamma_{\mathbf{R}}(s)\Gamma_{\mathbf{R}}(s+1)$. Technically, in order that L have values in a *group*, we should view L as a function which associates with V the meromorphic function $s \mapsto L(V\omega_s)$, and take the X in Definition (2.3.2) to be the multiplicative group of nonzero meromorphic functions of s.

(3.4) *The local "nonabelian" ε-function,* $\varepsilon(V, \psi, dx)$. For this there is at present only an existence theorem (see below), no explicit formula.[1] This lack is not surprising if we recall that the formulas defining ε in (3.2) make essential use of the interpretation of χ as a quasi-character of F^*; if we think of χ as a quasi-character of W_F we have no way to define $\varepsilon(\chi, \psi, dx)$ without using the reciprocity law isomorphism $F^* \approx W_F^{ab}$. In fact it was his idea about "nonabelian reciprocity laws" relating representations of degree n of W_F to irreducible representations π of $\mathrm{GL}(n, F)$, and the possibility of defining $\varepsilon(\pi, \psi, dx)$ for the latter, which led Langlands to conjecture and prove a version of the following big

(3.4.1) THEOREM. *There is a unique function ε which associates with each choice of a local field F, a nontrivial additive character ψ of F, an additive Haar measure dx on F and a representation V of W_F a number $\varepsilon(V, \psi, dx) \in \mathbf{C}^*$ such that $\varepsilon(V, \psi, dx) = \varepsilon(\chi, \psi, dx)$ if V is a representation of degree 1 corresponding to a quasi-character χ, and such that if F is a local field and we choose for each finite separable extension E*

[1]Except for Deligne's expression in terms of Stiefel-Whitney classes for *orthogonal* representations [**D5**, Proposition 5.2].

of F an additive Haar measure μ_E on E, then the function which associates with each such E and each $V \in M(W_E)$ the number $\varepsilon(V, \psi \cdot \mathrm{Tr}_{E/F}, \mu_E)$ is inductive in degree 0 *over F in the sense of* (2.3.2).

The unicity of such an ε is clear, by (2.3.1); the problem is *existence*. The experience of Dwork and Langlands indicates that the local proof of existence, based on showing that the $\varepsilon(\chi, \psi, dx)$ satisfy the necessary relations, is too involved to publish completely. Deligne found a relatively short proof (see **[D3**, §4]; possibly also **[T2]**). It has two main ingredients, one global, one local: (1) the existence of a global $\varepsilon(V)$ coming from the global functional equation for $L(V)$ (cf. (3.5) below), and (2) the fact that if F is local nonarchimedean and α a wildly ramified quasi-character of F^*, there is an element $y = y(\alpha, \psi)$ in F^* such that for all quasi-characters χ of F^* with $a(\chi) \leqq \frac{1}{2}a(\alpha)$, we have $\varepsilon(\chi\alpha, \psi, dx) = \chi^{-1}(y)\varepsilon(\alpha, \psi, dx)$, a rather harmless function of χ.

Granting the existence of $\varepsilon(V, \psi, dx)$ the following properties of it are easy consequences of the corresponding properties of $\varepsilon(\chi, \psi, dx)$, via inductivity in degree 0 and (2.3.1).

(3.4.2) ε is additive in V, so makes sense for V virtual.

(3.4.3) $\varepsilon(V, \psi, r\,dx) = r^{\dim V}\varepsilon(V, \psi, dx)$, for $r > 0$. In particular, for V virtual of degree 0, $\varepsilon(V, \psi, dx) = \varepsilon(V, \psi)$ is independent of dx.

(3.4.4) $\varepsilon(V, \psi a, dx) = (\det V)(a)\,\|a\|^{-\dim V}\varepsilon(V, \psi, dx)$, for $a \in F^*$ (cf. (2.3.4)).

(3.4.5) $\varepsilon(V\omega_s, \psi, dx) = \varepsilon(V, \psi, dx) f(V)^{-s}\,\delta(\psi)^{-s\dim V}$, where:

$\delta(\psi) = q_F^{n(\psi)}$ in the nonarchimedean case and is characterized in the archimedean case by the fact that $\delta(\psi a) = \|a\|^{-1}\,\delta(\psi)$, and $\delta(\psi) = 1$ for ψ as in (3.2.4) and (3.2.5).

$f(V) = 1$ in the archimedean case, and $= q_F^{a(V)}$, the absolute norm of the Artin conductor of V in the nonarchimedean case. This f can be characterized as the unique function inductive in degree 0 such that $f(\chi) = q_F^{a(\chi)}$ for quasi-characters χ. For the well-known explicit formula for $a(V)$ in terms of higher ramification groups, see **[S1]** or **[D3**, (4.5)].

(3.4.6) Suppose F nonarchimedean, W unramified. Then

$$\varepsilon(V \otimes W, \psi, dx) = \varepsilon(V, \psi, dx)^{\dim W} \cdot \det W(\pi^{a(V) + \dim V n(\psi)}).$$

(3.4.7) Let V^* denote the dual of V and dx' the Haar measure dual to dx relative to ψ. Then

$$\varepsilon(V, \psi, dx)\,\varepsilon(V^*\omega_1, \psi(-x), dx') = 1.$$

In particular

$$|\varepsilon(V, \psi, dx)|^2 = f(V)(\delta(\psi)\,dx/dx')^{\dim V}, \quad \text{if } V^* = \bar{V},$$

i.e., if V is unitary.

(3.4.8) If E/F is a finite separable extension, V_E a virtual representation of degree 0 of W_E and V_F the induced representation of W_F, then $\varepsilon(V_F, \psi) = \varepsilon(V_E, \psi \circ \mathrm{Tr})$.

(3.5) *Global L-functions, functional equations.* Let F be a global field, ψ a nontrivial additive character of A_F/F, and dx the Haar measure on A_F such that $\int_{A_F/F} dx = 1$ (Tamagawa measure). Call ψ_v the local component of ψ at a place v, and let $dx = \prod_v dx_v$ be any factorization of dx into a product of local measures such that the ring of integers in F_v gets measure 1 for almost all v.

Let V be a representation of "the" global Weil group W_F, and put

(3.5.1) $$L(V, s) = \prod_v L(V_v\, \omega_s),$$

(3.5.2) $$\varepsilon(V, s) = \prod_v \varepsilon(V_v \omega_s, \psi_v, dx_v).$$

(3.5.3) THEOREM. *The product* (3.5.1) *converges for s in some right half-plane and defines a function* $L(V, s)$ *which is meromorphic in the whole s-plane and satisfies the functional equation*

(3.5.4) $$L(V, s) = \varepsilon(V, s)L(V^*, 1 - s)$$

where V^* *is the dual of* V.

For V a quasi-character χ this result was proved by Hecke. In the modern version of his proof (**[T1]**, **[W2]**) one shows by Poisson summation that for suitable functions f on A

$$\int_{A^*} \hat{f}(x)\omega_{1-s}\, \chi^{-1}(x)\, dx^* = \int_{A^*} f(x)\omega_s\, \chi(x)\, dx^*,$$

the integrals being defined for all s by analytic continuation. Taking $f = \prod f_v$ and using the local functional equation (3.2.1) (with χ replaced by $\chi\omega_s$) one finds that (3.5.4) holds in the "abelian" case, $V = \chi$.

At this point, even without having a theory of the local nonabelian $\varepsilon(V_v, \psi_v, dx_v)$'s, one gets, via (2.3.1), (2.3.5), and the inductivity of the local L's, that $L(V, s)$ is meromorphic in the whole plane for each V, being defined by the product (3.5.1) in a right half-plane, and that $L(V, s)$ is inductive as a function of V. It follows that

$$\varepsilon'(V, s) \overset{\text{defn}}{=} \frac{L(V, s)}{L(V^*, 1 - s)}$$

is inductive in V and satisfies $\varepsilon'(\chi, s) = \prod_v \varepsilon(\chi_v\omega_s, \psi_v, dx_v)$ for quasi-characters χ of A^*/F^*. It is this fact about the local $\varepsilon(\chi_v, \psi_v, dx_v)$'s—that their product over all v for a global χ has an inductive extension to all global V—that Deligne uses in his "global" proof of the existence of local nonabelian ε's. Once their existence is proved, we have $\varepsilon'(V, s) = \varepsilon(V, s)$ by the unicity of inductive functions since $\varepsilon(V, s)$, defined by the product (3.5.2), is inductive in degree 0 by (2.3.5).

(3.5.5) Hecke's global function $L(\chi, s)$ is entire if χ is not of the form ω_s. Artin conjectured (in the Galois case) that $L(V, s)$ is entire for any V which has no constituent of the form ω_s. Weil proved Artin's conjecture for function fields. Recently Langlands, using ideas of Saito and Shintani, made a first breakthrough in the number field case, treating certain V's of dimension 2 by base change, using the trace formula. (See *The solution of a base change problem for* GL(2) (*following Langlands, Saito, Shintani*), these PROCEEDINGS, part 2, pp. 115–133.) These methods work for all V's of dimension 2 for which the image of W_F in PGL(V) is the tetrahedral group. They also work for some octahedral cases, but a new idea will be needed to apply them in the nonsolvable icosahedral case. However, J. Buhler **[B]**, with the aid of the Harvard Science Center PDP11 and the main result of **[DS]**, has proved the Artin conjecture for one particular icosahedral V of conductor 800, by checking the existence of the corresponding modular form of weight 1 and level 800.

Although the Riemann hypothesis concerning the zeros of $L(\chi, s)$ has been proved by Weil in the function field case, there seems to be no breakthrough in sight in the number field case. The conjunction of the Artin conjecture for all V and the Riemann hypothesis for all χ is equivalent to the positivity of a certain distribution on W_F (cf. [W3]).

(3.6) *Comparison of different conventions for local constants.* The modern references for the material we have been discussing are Deligne [D3] and Langlands [L], and we have here followed the conventions of [D3]. Happily, the definition of L-functions, both local and global, in [D3] coincides with that in [L]. But Deligne's local constants $\varepsilon(V, \psi, dx)$, which we will designate in this section by ε_D instead of just ε, differ somewhat from Langlands' $\varepsilon(V, \psi)$ which we will denote by ε_L here. The relationship is

$$\varepsilon_L(V, \psi) = \varepsilon_D(V\omega_{1/2}, \psi, dx_\psi), \tag{3.6.1}$$

where dx_ψ is the additive measure which is self-dual with respect to ψ. The other way around we have

$$\varepsilon_D(V, \psi, dx) = (dx/dx_\psi)^{\dim V}\, \varepsilon_L(V\omega_{-1/2}, \psi). \tag{3.6.2}$$

In the nonarchimedean case the constant dx/dx_ψ is given explicitly by $q^{-n(\psi)/2}\int_{\mathcal{O}} dx$. Also, in that case if V corresponds to a quasi-character χ of F^* we have

$$\varepsilon_L(\chi, \psi) = \chi(c)\,\frac{\int_{\mathcal{O}^*}\chi^{-1}(u)\psi(u/c)\,du}{\left|\int_{\mathcal{O}^*}\chi^{-1}(u)\psi(u/c)\,du\right|}, \tag{3.6.3}$$

where c is an element of F^* of valuation $a(\chi) + n(\psi)$ as in (3.2.6).

Langlands puts

$$\varepsilon_L(s, V, \psi) \overset{\text{defn}}{=} \varepsilon_L(V\omega_{s-(1/2)}, \psi) = \varepsilon_D(V\omega_s, \psi, dx_\psi). \tag{3.6.4}$$

Then the "constant" $\varepsilon(V, s)$ in the global functional equation (3.5.4) is given by $\varepsilon(V, s) = \prod_v \varepsilon_L(s, V_v, \psi_v)$ for any nontrivial character ψ of A/F, because if dx_v is self-dual on F_v with respect to ψ_v for each place v, then $dx = \prod_v dx_v$ is self-dual on $\boldsymbol{A}$ with respect to ψ, and is therefore the Tamagawa measure on $\boldsymbol{A}$.

The behavior of ε_L under twisting by an unramified quasi-character is given by

$$\varepsilon_L(V\omega_s, \psi) = \varepsilon_L(V, \psi) f(V)^{-s}\,\delta(\psi)^{-s \dim V} \tag{3.6.5}$$

as in (3.4.5), but its dependence on ψ is according to

$$\varepsilon_L(V, \psi_a) = (\det V)(a)\varepsilon(V, \psi), \tag{3.6.6}$$

instead of as in (3.4.4). If V^* is the contragredient of V, then

$$\varepsilon_L(V, \psi)\varepsilon_L(V^*, \psi^{-1}) = 1. \tag{3.6.7}$$

Hence, by (3.6.6)

$$\varepsilon_L(V, \psi)\, \varepsilon_L(V^*, \psi) = (\det V)(-1) \tag{3.6.8}$$

and on the other hand,

$$|\varepsilon_L(V, \psi)| = 1, \quad \text{if } V \text{ is unitary.} \tag{3.6.9}$$

The ε_L-system has the advantage that it avoids carrying along the measure dx, but it has the following disadvantage: in the nonarchimedean case, if σ is a discontinuous automorphism of $\boldsymbol{C}$, then V^σ is again a representation of W_F, and ψ^σ an additive character, but $\varepsilon_L(V^\sigma, \psi^\sigma)$ is not in general equal to $\varepsilon_L(V, \psi)^\sigma$ (nor is $\varepsilon_L(0, V^\sigma, \psi^\sigma) = \varepsilon_L(0, V, \psi)^\sigma$). The trouble is that the absolute value (in (3.6.3)) may not be preserved by σ, and/or that the self-dual measure dx_ψ in (3.6.1) may involve $\sqrt{p}$, and hence may not be preserved by σ. If one does wish to eliminate the measure dx, it is probably preferable to define, say,

$$\varepsilon_1(V, \psi) = \varepsilon_D(V, \psi, dx_1), \tag{3.6.10}$$

where dx_1 is the measure for which $\mathcal{O}$ gets measure 1 in the nonarchimedean case, and is the measure described in (3.2.4) and (3.2.5) in the archimedean case. This convention has the minor disadvantage that the $\varepsilon(V)$ in the global functional equation is not equal to the product of the local $\varepsilon_1(V_v, \psi_v)$'s, but is, rather, a^{-1} times that product, where a is the square root of the discriminant for a number field, and is q^{g-1} for a function field of genus g with q elements in its constant field. But the $\varepsilon_1(V, \psi)$ has the advantage that in the nonarchimedean case we do have $\varepsilon_1(V^\sigma, \psi^\sigma) = \varepsilon_1(V, \psi)^\sigma$ for all automorphisms σ of $\boldsymbol{C}$. This is clear, by unicity (2.3.3) and the formula

$$\varepsilon_1(\chi, \psi) = \chi(c)q^{n(\psi)} \sum_{u \in \mathcal{O}^* \bmod \pi^{a(\chi)}} \chi(u)\left(\frac{u}{c}\right) \tag{3.6.11}$$

which follows from (3.2.6.1) and (3.2.6.2) where the notation is explained. Thus in the nonarchimedean case we can define, for V and ψ over any field $\boldsymbol{E}$ of characteristic 0 (an open subgroup of I acting trivially on V, and ψ trivial on some $\pi^n\mathcal{O}$), an $\varepsilon_1(V, \psi) \in \boldsymbol{E}^*$, in a unique way such that $\varepsilon_1(V^\alpha, \psi^\alpha) = \varepsilon_1(V, \psi)^\alpha$ for any homomorphism $\alpha\colon \boldsymbol{E} \to \boldsymbol{E}'$ and such that ε_1 is the old ε_1, given by (3.6.10), when $\boldsymbol{E} = \boldsymbol{C}$. So defined, $\varepsilon_1(V_E, \psi \cdot \mathrm{Tr}_{E/F})$ is inductive in degree 0 (2.3.2) for every field of scalars $\boldsymbol{E}$ of characteristic 0, and $\varepsilon_1(V, \psi)$ will be given by (3.6.11) if V corresponds to a quasi-character $\chi\colon F^* \to \boldsymbol{E}^*$.

In writing these notes I was tempted to shorten things a bit by using only $\varepsilon_1(V, \psi)$ instead of $\varepsilon_D(V, \psi, dx)$, but decided against it because (1) the ε_D-system avoids all choices and is the most general and flexible—any other system, like ε_L or ε_1 can be immediately described as a special case of ε_D; (2) the dependence of ε on dx shows "why" ε is inductive only in degree 0, and (3) in case our local field F is nonarchimedean, the ε_D-system, like the ε_1, works over any field $\boldsymbol{E}$ of characteristic 0, as soon as one defines the notion of Haar measure on F with values in $\boldsymbol{E}$ (cf. [**D3**, (6.1)]).

4. The Weil-Deligne group, λ-adic representations, L-functions of motives. The representations considered in §3 are just the beginning of the story. Those of Galois type are effective motives of degree 0—which Deligne calls *Artin motives* in his article [**D6**, §6] in these PROCEEDINGS—with coefficients in $\boldsymbol{C}$. We cannot discuss the notion of motive here (cf., e.g., [**D1**] and [**D6**] for this) but we do want to discuss the way in which L-functions and ε-functions are attached to motives of any degree. Only very special motives of degree $\neq 0$ correspond to the representations of W_F considered in §3, namely, those *of type* A_0, i.e., those which, after a finite extension E/F, correspond to direct sums of Hecke characters of type A_0 over E. (A candidate for a "motivic Galois group" for these is constructed by Langlands in these

PROCEEDINGS [L3].) The simplest motives not of this type are those given by elliptic curves with no complex multiplication; their L-functions are the "Hasse-Weil zeta-functions" which are not expressible in terms of Hecke's L-functions.

The procedure for attaching L-functions to motives in the form given it by Deligne [D3], [D6] can be outlined schematically as follows:

F is a global field.

v is a place of F.

$\boldsymbol{E}$ is a field of finite degree over $\boldsymbol{Q}$.

λ runs through the finite places of $\boldsymbol{E}$ whose residue characteristic is prime to char(F).

σ is an embedding of $\boldsymbol{E}$ in $\boldsymbol{C}$.

Motive M over F with cx. multn. by $\boldsymbol{E}$	$\Longrightarrow$	System $(H_\lambda(M))$ of λ-adic representations of G_F
$\Downarrow$ gd. fld. extension	l-adic coho.	$\Downarrow$ restriction
Motive M_v over F_v with cx. multn. by $\boldsymbol{E}$	$\Longrightarrow$	System $(H_\lambda(M_v))$ of λ-adic representations of G_{F_v}
$\Downarrow$ Hodge theory cf.(4.4), v archimedean		$\Downarrow$ v nonarchimedean; cf. (4.2)
Hodge structure Hdg(M_v) over F_v with cx. multn. by $\boldsymbol{E}$		
$\Downarrow$ (cf. (4.4))		Equiv. class $V(M_{v,\sigma})$ of repns. of the Weil-Deligne group W'_{F_v} over $\boldsymbol{C}$, invariant under Aut$(\boldsymbol{C}/\boldsymbol{E})$
Equiv. class $V(M_{v,\sigma})$ of repns. over $\boldsymbol{C}$ of the Weil group W_{F_v}		$\Downarrow$ cf. (4.1)
$\Downarrow$ cf. (3.3), (3.4)		
L- (i.e. Γ-) and ε-factors at the archimedean place v		L- and ε-factors at the nonarchimedean place v

In the next sections we discuss some of the steps and concepts indicated in the above chart. We begin with the Weil-Deligne group. This is a group scheme over $\boldsymbol{Q}$, but what counts, its points in and representations over fields of characteristic 0, can be described naively with no reference to schemes.

(4.1) *The Weil-Deligne group and its representations.* Let F be a nonarchimedean local field and let $\bar{F}$, $G_F = \mathrm{Gal}(\bar{F}/F)$, W_F (Weil group), and I (inertia group) have their usual meaning. For $w \in W_F$, let $\|w\|$ denote the power of q to which w raises elements of the residue field, as in (1.4.6). Thus we have $\|w\| = 1$ for $w \in I$, and $\|\Phi\| = q^{-1}$ for a geometric Frobenius element Φ. We view W_F as a group scheme over $\boldsymbol{Q}$ as follows: for each open normal subgroup J of I, we view W_F/J as a "discrete" scheme, and we put $W_F = \text{proj lim } (W_F/J)$, the limit taken over all J. In other words, we have

$$W_F = \coprod_{n \in \boldsymbol{Z}} \Phi^n I = \coprod_{n \in \boldsymbol{Z}} \operatorname{spec} A_n,$$

where A_n is the ring of locally constant $\boldsymbol{Q}$-valued functions on $\Phi^n I$.

(4.1.1) DEFINITION [**D3**, (8.3.6)]. The *Weil-Deligne group* W'_F is the group scheme over $\boldsymbol{Q}$ which is the semidirect product of W_F by $\boldsymbol{G}_a$, on which W_F acts by the rule $wxw^{-1} = \|w\|x$.

Let $\boldsymbol{E}$ be a field of characteristic 0. The group $W'_F(\boldsymbol{E})$ of points of W'_F with coordinates in $\boldsymbol{E}$ is just $\boldsymbol{E} \times W_F$ with the law of composition $(a_1, w_1)(a_2, w_2) = (a_1 + \|w_1\| a_2, w_1w_2)$ for $a_1, a_2 \in \boldsymbol{E}$ and $w_1, w_2 \in W_F$.

Let V be a finite-dimensional vector space over $\boldsymbol{E}$. A homomorphism of group schemes over $\boldsymbol{E}$

$$\rho' : W'_F \times_{\boldsymbol{Q}} \boldsymbol{E} \longrightarrow \mathrm{GL}(V)$$

determines, and is determined by, a pair (ρ, N) as in (4.1.2) below, such that, on points, $\rho'((a, w)) = \exp(aN)\cdot\rho(w)$. That is the explanation for the following definition:

(4.1.2) DEFINITION [**D3**, (8.4.1)]. Let $\boldsymbol{E}$ be a field of characteristic 0. A *representation of W'_F over $\boldsymbol{E}$* is a pair $\rho' = (\rho, N)$ consisting of:

(a) A finite-dimensional vector space V over $\boldsymbol{E}$ and a homomorphism $\rho : W_F \to \mathrm{GL}(V)$ whose kernel contains an open subgroup of I, i.e., which is continuous for the discrete topology in $\mathrm{GL}(V)$.

(b) A nilpotent endomorphism N of V, such that $\rho(w)N\rho(w)^{-1} = \|w\|N$, for $w \in W$.

(4.1.3) *Φ-semisimplicity*. Let $\rho' = (\rho, N)$ be a representation of W'_F over $\boldsymbol{E}$. Define $v : W_F \to \boldsymbol{Z}$ by $\|w\| = q^{-v(w)}$. There is a unique unipotent automorphism u of V such that u commutes with N and with $\rho(W_F)$ and such that $\exp(aN)\rho(w)u^{-v(w)}$ is a semisimple automorphism of V for all $a \in \boldsymbol{E}$ and all $w \in W_F—I$ [**D3**, (8.5)]. Then $\rho'_{ss} = (\rho u^{-v}, N)$ is called the *Φ-semisimplification* of ρ', and ρ' is called *Φ-semisimple* if and only if $\rho' = \rho'_{ss}$, i.e., $u = 1$, i.e., the Frobeniuses act semisimply. For this it is necessary and sufficient that the representation ρ of W_F be semisimple in the ordinary sense, because $\rho(\Phi)$ generates a subgroup of finite index in $\rho(W_F)$, and in characteristic 0 a representation of a group is semisimple if and only if its restriction to a subgroup of finite index is semisimple. In his article in these PROCEEDINGS, Borel discusses *admissible* morphisms $W'_F \to {}^LG$; when $G = \mathrm{GL}_n$, these are just our Φ-semisimple (ρ, N)'s.

(4.1.4) EXAMPLE. Sp(n) is the following representation (ρ, N) of W'_F over $\boldsymbol{Q}$.

$$V = \boldsymbol{Q}^n = \boldsymbol{Q}e_0 + \boldsymbol{Q}e_1 + \cdots + \boldsymbol{Q}e_{n-1},$$
$$\rho(w)e_i = \omega_i(w)e_i \qquad (= \|w\|^i e_i),$$
$$Ne_i = e_{i+1} \qquad (0 \leqq i < n - 1),\ Ne_{n-1} = 0.$$

(4.1.5) Given any (ρ, N), Ker N is stable under W_F. Hence (ρ, N) is irreducible $\Leftrightarrow$ $N = 0$ and ρ is irreducible. It is not hard to show that the Φ-semisimple indecomposable representations of W'_F are those of the form $\rho' \otimes \mathrm{Sp}(n)$ with ρ' irreducible. (The $\otimes$ is defined by $(\rho, N) \otimes (\rho_1, N_1) = (\rho \otimes \rho_1, N \otimes 1 + 1 \otimes N_1)$.)

(4.1.6) Let (ρ, N, V) be a representation of W'_F over $\boldsymbol{E}$. We put $V^I_N = (\mathrm{Ker}\, N)^I$ and define a local L-factor, a conductor, and a local constant by

$$Z(V, t) = \det(1 - \Phi t \mid V_N^I)^{-1}, \quad \text{and} \quad L(V, s) = Z(V, q^{-s}), \text{ when } \boldsymbol{E} \subset \boldsymbol{C};$$

$$a(V) = a(\rho) + \dim V^I - \dim V_N^I,$$

$$\varepsilon(V) = \varepsilon(\rho)\det(-\Phi \mid V^I/V_N^I),$$

and

$$\varepsilon(V, t) = \varepsilon(V)t^{a(V)}.$$

Here, for ε, the usual ψ and dx are understood, but omitted from the notation.

These quantities do not change if we replace V by its Φ-semisimplification; but note that they are *not* additive as functions of V, because V_N is not. If $N = 0$, they are the same as before.

One of the main reasons for introducing the Weil-Deligne group is the fantastic generalization of local class field theory embodied in:

(4.1.7) *Conjecture.* Let F be a nonarchimedean local field and n an integer $\geqq 1$. There is a (in fact more than one) natural bijection between isomorphism classes of Φ-semisimple representations of W_F' of degree n, and of irreducible admissible representations of $\mathrm{GL}(n, F)$.

For $n = 1$ this is local class field theory. For $n = 2$, it is discussed at length in [**D2**, (3.2)]. In this conjecture, for any n, the *irreducible* representations of W_F' (which are just irreducible representations of W_F) should correspond to the *cuspidal* representations of $\mathrm{GL}(n, F)$. I understand that Bernštein and Želevinsky have shown that the way in which arbitrary admissible representations of $\mathrm{GL}(n, F)$ are built out of cuspidal ones follows the same pattern as the way in which arbitrary Φ-semisimple representations of W_F' are built up out of irreducible ones. Thus the main problem is now the correspondence between irreducibles and cuspidals.

A more general conjecture, involving an arbitrary reductive group G rather than just $\mathrm{GL}(n)$, relates admissible representations of $G(F)$ to homomorphisms of W_F' into the "Langlands dual" of G (see Borel's article in these PROCEEDINGS). This more general conjecture is the nonarchimedean local case of "Langlands' philosophy".

(4.2) *λ-adic representations.* Now suppose l is a prime different from the residue characteristic p of F and let $t_l \colon I_F \to \boldsymbol{Q}_l$ be a nonzero homomorphism. (Such a t_l exists and is unique up to a constant multiple, because the wild ramification group P is a pro-p-group, and the quotient I/P is isomorphic to the product $\prod_{l' \neq p} \boldsymbol{Z}_{l'}$.) We have

$$t_l(w\sigma w^{-1}) = \|w\| \, t_l(\sigma), \quad \text{for } \sigma \in I, \, w \in W,$$

because conjugation by w induces raising to the $\|w\|$ power in I/P. Let Φ be an inverse Frobenius element (4.1.8). Suppose $\boldsymbol{E}_\lambda$ is a finite extension of $\boldsymbol{Q}_l$. A λ-adic representation of W_F is a finite-dimensional vector space V_λ over $\boldsymbol{E}_\lambda$ and a homomorphism of topological groups $\rho_\lambda \colon W_F \to \mathrm{GL}_{\boldsymbol{E}_\lambda}(V_\lambda)$ where $\mathrm{GL}_{\boldsymbol{E}_\lambda}(V_\lambda)$ has the λ-adic topology (i.e., the topology given by the valuation).

(4.2.1) THEOREM (DELIGNE [**D3**, §8]). *The relationship* $V_\lambda = V$ *and*

$$\rho_\lambda(\Phi^n \sigma) = \rho(\Phi^n \sigma)\exp(t_l(\sigma) \, N), \qquad \sigma \in I, \, n \in \boldsymbol{Z},$$

sets up a bijection between the set of λ-adic representations $(\rho_\lambda, V_\lambda)$ of W_F and the set of representations (ρ, N, V) of W_F' over E_λ. The corresponding bijection between isomorphism classes of each is independent of the choice of t_l and Φ.

To show that every ρ_λ is of this form one uses

(4.2.2) COROLLARY (GROTHENDIECK). *Let $(\rho_\lambda, V_\lambda)$ be a λ-adic representation of W_F. There exists a nilpotent endomorphism N of V_λ such that $\rho_\lambda(\sigma) = \exp(t_l(\sigma)N)$ for σ in an open subgroup of I.*

A proof of the corollary can be found in the appendix of **[ST]**. Here is a sketch. Since I is compact, $\rho_\lambda(I)$ stabilizes a "lattice" L in V_λ. Replacing F by a finite extension we can assume that $\rho_\lambda(I)$ fixes $L \pmod{l^2}$. Then $\rho_\lambda(I)$ is a pro-l-group, so is a homomorphic image of $t_l(I)$, since Ker t_l is prime to l. Choose $c \in \mathbf{Q}_l$ such that $ct_l(I) = \mathbf{Z}_l$. Then there is an $\alpha \in \mathrm{GL}(V_\lambda)$ fixing $L \pmod{l^2}$ such that

$$\rho_\lambda(\sigma) = \alpha^{ct_l(\sigma)} = \exp(t_l(\sigma)N)$$

for all $\sigma \in I$, where $N = c \log \alpha$. Conjugating by $\rho_\lambda(\Phi)$ we find $\rho_\lambda(\Phi)N\rho_\lambda(\Phi)^{-1} = q^{-1}N$. Thus the set of eigenvalues of N is stable under multiplication by q^{-1}. Since q is not a root of unity in characteristic 0, it follows that the only eigenvalue of N is zero, i.e., N is nilpotent.

(4.2.3) COROLLARY. *If V_λ is a semisimple λ-adic representation of W_F then some open subgroup of I acts trivially on V_λ, so V_λ can be viewed as an "ordinary" representation of W_F.*

For any V_λ the kernel of N is stable under W_F because $\rho_\lambda(w)N_{\rho_\lambda}(w)^{-1} = \|w\|N$. So if V_λ is irreducible, then $N=0$, and the statement follows from (4.2.2). A semisimple V_λ is a direct sum of irreducible subrepresentations.

(4.2.4) In view of (4.2.3), $\varepsilon(V_\lambda)$ and $a(V_\lambda)$ have meaning if V_λ is semisimple. For arbitrary V_λ, if $(\rho_\lambda, V_\lambda)$ and (ρ, N, V) correspond as in (4.2.1), we define the L- and ε-factors associated to V_λ to be those associated to V. These can be expressed directly in terms of V_λ as follows:

$$Z(V, t) = \det(1 - \Phi \mid V_\lambda^I) = Z(V_\lambda, t),$$

$$a(V) = a(V_\lambda^{ss}) + \dim(V_\lambda^{ss})^I - \dim V_\lambda = a(V_\lambda),$$

$$\varepsilon(V) = \varepsilon(V_\lambda^{ss}) \frac{\det(-\Phi \mid (V_\lambda^{ss})^I)}{\det(-\Phi \mid V_\lambda^I)} = \varepsilon(V_\lambda),$$

and

$$\varepsilon(V, t) = \varepsilon(V_\lambda)t^{a(V_\lambda)} = \varepsilon(V_\lambda, t),$$

where V_λ^{ss} is the semisimplification of V_λ in the ordinary sense. One can define a "Φ-semisimplification" of V_λ, analogous to that of V (4.1.3). The quantities on the right do not change if we replace V_λ by its Φ-semisimplification, but they are not additive in V_λ, because V_λ^I is not.

(4.2.4) *Motives.* Suppose now $\mathbf{E}$ is a finite extension of $\mathbf{Q}$. Let M be a motive with complex multiplication by $\mathbf{E}$, defined over our nonarchimedean local field F (**[D1]**, **[D6]**). Let n be the rank of M. Attached to M will be l-adic representations $H_l(M)$, vector spaces of dimension n over $\boldsymbol{Q}_l$ on which G_F acts continuously, one for each $l \neq \operatorname{char}(F)$. The field $\boldsymbol{E}$ will act on these, and for each l we get a decomposition $H_l(M) = \bigoplus_{\lambda | l} H_\lambda(M)$, where for each place λ of $\boldsymbol{E}$ above l, we put $H_\lambda(M) = E_\lambda \otimes_{E \otimes \boldsymbol{Q}_l} H_l(M)$, a vector space of dimension m over $\boldsymbol{E}$, where m is the rank of M over $\boldsymbol{E}$, given by $n = m[\boldsymbol{E}: \boldsymbol{Q}]$.

For each $l \neq p$ and each λ above l let $H'_\lambda(M)$ be the representation of W'_F over $\boldsymbol{E}_\lambda$ corresponding to $H_\lambda(M)$ by (4.2.1). If our motive M lives up to expectations, the system of λ-adic representations $H_\lambda(M)$ will be *compatible over* $\boldsymbol{E}$ in the sense that the system $H'_\lambda(M)$ is compatible over $\boldsymbol{E}$ in the following naive sense: for any two finite places λ, μ of $\boldsymbol{E}$ not over p and every commutative diagram

$$\begin{array}{ccccc} & & \boldsymbol{E}_\lambda & & \\ & \nearrow & & \searrow & \\ \boldsymbol{E} & & & & \boldsymbol{C} \\ & \searrow & & \nearrow & \\ & & \boldsymbol{E}_\mu & & \end{array}$$

the m-dimensional representations of W'_F over $\boldsymbol{C}$, $H'_\lambda(M) \otimes_{\boldsymbol{E}_\lambda} \boldsymbol{C}$ and $H'_\mu(M) \otimes_{\boldsymbol{E}_\mu} \boldsymbol{C}$, are isomorphic (or at the very least, have isomorphic Φ-semisimplifications). If so, then the isomorphism class of (the Φ-semisimplifications of) these representations depends only on the embedding $\boldsymbol{E} \subset \boldsymbol{C}$ in the diagram above.

We denote this isomorphism class by $V(M_\sigma)$, where σ denotes the embedding of $\boldsymbol{E}$ in $\boldsymbol{C}$ and $M_\sigma = M \otimes_{\boldsymbol{E},\sigma} \boldsymbol{C}$ is the motive of rank m with coefficients in $\boldsymbol{C}$ deduced from the original M, the action of $\boldsymbol{E}$ on it, and the embedding σ of $\boldsymbol{E}$ in $\boldsymbol{C}$, cf. [**D6**, 2.1]. Associated to $V(M_\sigma)$ as explained in (4.1.6) are the local quantities a, L, and ε which we shall denote by $L(M_\sigma, s)$, etc.

(4.3) *Reduction.* Let r be an integer $\geqq 0$ and X a projective nonsingular variety over F. In this paragraph we shall restrict our attention to the special motive $M = H^r(X)$ given by the r-dimensional cohomology of X, and we shall ignore any complex multiplication. For the moment F can be any field. Put $\bar{X} = X \times_F \bar{F}$, the scheme obtained by extending scalars from F to $\bar{F}$. For each prime $l \neq \operatorname{char}(F)$ the l-adic étale cohomology group $H^r(\bar{X}_{\text{et}}, \boldsymbol{Q}_l)$ is defined, and gives an l-adic representation of $G_F = \operatorname{Gal}(\bar{F}/F)$ (by functoriality, G_F acting on $\bar{X}$ through $\bar{F}$). In the notation of the previous paragraph we have now $\boldsymbol{E} = \boldsymbol{Q}$, $\lambda = l$, $H_l(M) = H^r(\bar{X}_{\text{et}}, \boldsymbol{Q}_l)$. I do not know to what extent the compatibility of the $H_l(M)$'s is known (assuming now again that F is local nonarchimedean), but the compatibility at least of their Φ-semisimplifications is known in one very important case—that of

(4.3.1) *Good reduction.* Let $\mathcal{O}$ be the ring of integers in F, and $k = \mathcal{O}/\pi\mathcal{O}$ the residue field. The scheme X is said to have *good reduction* if there exists a scheme $\mathfrak{X}$ projective and smooth over $\mathcal{O}$ such that $X = \mathfrak{X} \times_{\mathcal{O}} F$. Choosing such an $\mathfrak{X}$, one calls $\mathfrak{X} \times_{\mathcal{O}} k$ the *reduction* of X. Let us denote this reduction by X_0. Putting $\bar{X}_0 = X_0 \times_k \bar{k}$, where $\bar{k}$ is the residue field of $\bar{F}$, the base-change theorem gives a canonical isomorphism

$$(*) \qquad H_l(M) = H^r(\bar{X}, \boldsymbol{Q}_l) \approx H^r(\bar{X}_0, \boldsymbol{Q}_l)$$

compatible with the action of the Galois groups. Hence $H_l(M)$ is *unramified*, i.e., fixed by I, and the structure of $H_l(M)$ as representation of W_F is given by the action of Φ. Let $\varphi: X_0 \to X_0$ be the Frobenius *morphism*, and $\sigma: \bar{k} \to \bar{k}$ the Frobenius automorphism. The composition $\varphi \times \sigma$ acts on $\bar{X}_0 = X_0 \times \bar{k}$ by fixing points and by mapping $f \mapsto f^q$ in the structure sheaf. This map induces (a morphism canonically isomorphic to) the identity on the site $(\bar{X}_0)_{\text{et}}$, so the action of the Frobenius morphism φ on $H^r(\bar{X}_0, \boldsymbol{Q}_l)$ is the same as that of σ^{-1}, which is the one corresponding to our Φ under the isomorphism (*). That is why Deligne calls Φ the *geometric Frobenius.*

Deligne **[D4]** has proved Weil's conjecture, that the characteristic polynomial of φ acting on $H^r(\bar{X}_0, \boldsymbol{Q}_l)$ has coefficients in $\boldsymbol{Z}$, is independent of l, and that its complex roots have absolute value $q^{r/2}$. From the independence of l it follows in this case of good reduction that the Φ-semisimplifications of the $H_l(M)$'s form a compatible system; and the $H_l(M)$'s are known to be Φ-semisimple for $r = 1$.

It is natural to say that a motive M over F has *good reduction*, or is *unramified* if and only if $H_l(M) = H_l(M)^I$, i.e., if $V(M) = V(M)^I_N$. In case $M = H_1(A)$, A an abelian variety, this is equivalent to A having good reduction (criterion of Néron-Ogg-Shafaryevitch in **[ST]**).

Similarly we say M has *potential good reduction* $\Leftrightarrow N = 0$, and M has *semistable* reduction if $V(M) = V(M)^I$. Clearly this latter can always be achieved by a finite extension of the ground field.

(4.4) *F archimedean.* Let now M, $\boldsymbol{E}$, n, m be as in (4.2.4), but take F to be archimedean, instead of nonarchimedean. Let $z: F \to \boldsymbol{C}$ be the embedding of F in $\boldsymbol{C}$ if F is real, or one of the two isomorphisms of F on $\boldsymbol{C}$ if F is complex. Such a z gives us a motive M_z over $\boldsymbol{C}$ and M_z has a "Betti realization" $H_B(M_z)$ which is an n-dimensional vector space over $\boldsymbol{Q}$ whose complexification $H_B(M_z) \otimes \boldsymbol{C} = \bigoplus H^{pq}(M_z)$ is doubly graded in such a way that the map $1 \otimes c$ (c = complex conjugation) takes H^{pq} to H^{qp}. (For example, if $M = H^r(X)$ as in (4.3), then $H_B(M_z) = H^r(X_z^{\text{an}}, \boldsymbol{Q})$, where X_z^{an} is the complex analytic variety underlying the scheme $X \times_{F,z} \boldsymbol{C}$, and the complexification of this space, $H^r(X_z^{\text{an}}, \boldsymbol{C})$, is doubly graded by Hodge theory.)

Let $\bar{z} = c \circ z: F \to \boldsymbol{C}$ be the map conjugate to z. By transport of structure, there is an isomorphism $\tau: H_B(M_z) \to H_B(M_{\bar{z}})$ such that $\tau \otimes c$ preserves the bigrading on the complexifications; hence $\tau \otimes 1$ carries $H^{pq}(M_z)$ onto $H^{qp}(M_{\bar{z}})$. The field $\boldsymbol{E}$ of complex multiplications acts on $H_B(M_z)$ preserving the bigradation on the complexification, and τ is an $\boldsymbol{E}$-homomorphism. Let $\sigma: \boldsymbol{E} \to \boldsymbol{C}$. Putting $V_z(M_\sigma) = H_B(M_z) \otimes_{\boldsymbol{E},\sigma} \boldsymbol{C}$ we obtain a bigraded complex vector space of dimension m and a linear isomorphism $\tau \otimes 1: V_z(M_\sigma) \to V_{\bar{z}}(M_\sigma)$ taking V_z^{pq} to $V_{\bar{z}}^{qp}$.

There is a natural action of the Weil group W_F on these spaces as follows:

F complex. $z: F \approx \boldsymbol{C}$ an isomorphism, $W_F = F^*$, and W_F acts on V_z^{pq} by scalar multiplication via the character $z^{-p}(\bar{z})^{-q}$. Clearly, $\tau \otimes 1$ is W_F-equivariant, so the two representations $V_z(M_\sigma)$ and $V_{\bar{z}}(M_\sigma)$ are isomorphic. We let $V(M_\sigma)$ denote their isomorphism class.

F real. $z = \bar{z}: F \to \boldsymbol{C}$ is *the* embedding, and $W_F = \boldsymbol{C}^* \cup j\boldsymbol{C}^*$. This time $M_z = M_{\bar{z}}$, so we have only one space, $V_z(M_\sigma) = V_{\bar{z}}(M_\sigma)$, and $\tau \otimes 1$ is an automorphism of it. The action of W_F on it is as follows:

$u \in \boldsymbol{C}^*$ acts as multiplication by $u^{-p}(\bar{u})^{-q}$ on V_z^{pq}.

j acts as $i^{p+q}(\tau \otimes 1)$ on V_z^{pq}.

Again, let $V(M_\sigma)$ denote the equivalence class of this representation.

Notice that the representations obtained from motives via Hodge theory are very special, in that the p and q are integers.

Finally, define $L(M_\sigma, s)$ and $\varepsilon(M_\sigma, s)$ to be the L- and ε-factors associated to the representation $V(M_\sigma)$ as in §3. For a table making these explicit see **[D6**, 5.3].

(4.5) *F global.* Let F be a global field and M a motive with complex multiplication by $\boldsymbol{E}$, defined over F. For each place v of F, let M_v denote the restriction of M to F_v. Let $\sigma: \boldsymbol{E} \to \boldsymbol{C}$. The product $L(M_\sigma, s) = \prod_v L(M_{v,\sigma}, s)$ converges in a right half-plane. It is conjectured that it is meromorphic in the whole s-plane and satisfies the functional equation

$$L(M_\sigma, s) = \varepsilon(M_\sigma, s)\, L(M_\sigma^*, 1 - s)$$

with $M^* = \mathrm{Hom}(M, \boldsymbol{Q})$ and $\varepsilon(M_\sigma, s) = \prod_v \varepsilon(M_{v,\sigma}, s)$.

In the function field case this conjecture has been proved by Deligne. Let q be the number of elements in the constant field k of F. Grothendieck proved that for any given λ-adic representation V of G_F which is unramified at all but a finite number of places v, the corresponding L-function $L(V, s) = \prod_v L(V_v, s)$ is a rational function of q^{-s} (even a polynomial if $V^{\bar{G}} = 0$ and $V_{\bar{G}} = 0$, where $\bar{G}$ is the geometric Galois group, i.e., the kernel of the map of G_F to G_k), and satisfies a functional equation of the form $L(V, s) = \varepsilon(V, s)L(V^*, 1 - s)$ with an ε which is a monomial in q^{-s} of degree $\sum_v [k(v):k]\ a(V_v)$. Later, Deligne showed that Grothendieck's $\varepsilon(V, s)$ is equal to the product of the local $\varepsilon(V_v, s)$'s if $V = V_{\lambda_0}$ is a member of a family $(V_\lambda)_{\lambda \in \mathscr{L}}$ of λ-adic representations of G_F for some infinite set of places $\mathscr{L}$ of a number field $\boldsymbol{E}$, and the family is compatible in the following weak sense: for each $\lambda, \mu \in \mathscr{L}$ there is a finite set S of places of F such that for $v \notin S$, the representations V_λ and V_μ are unramified at v and the characteristic polynomals of Φ_v acting on V_λ and V_μ have coefficients in $\boldsymbol{E}$ and are equal. Deligne's method is to prove that Grothendieck's ε is congruent to the product of the local ε's modulo λ for all $\lambda \in \mathscr{L}$ and is therefore equal to that product. By (4.3) any λ-adic representation coming from l-adic cohomology, i.e., from a motive, is a member of a system which is weakly compatible in the above sense.

When $\dim(V) = 2$, then by Jacquet-Langlands (resp. Weil), Springer Lecture Notes 114 (resp. 189), these results show that $L(V, s)$ comes from an automorphic representation of (resp. modular form on) $\mathrm{GL}_2(\boldsymbol{A}_F)$. On the other hand, Drinfeld has recently shown that automorphic representations of GL_2 give rise to systems of l-adic representations occurring as constituents in tensor products of those coming from 1-dimensional l-adic cohomology, hence from motives. Thus for GL_2 over function fields, the equivalence between motives, compatible systems of l-adic representations, and automorphic representations is pretty well established.

In this connection it should be mentioned that Zarhin **[Z]** has proved the isogeny theorem over function fields: if two abelian varieties A and B over a global function field F give isomorphic l-adic representations, then they are isogenous; more precisely,

$$\boldsymbol{Q}_l \otimes \mathrm{Hom}_F(A, B) = \mathrm{Hom}_{G_F}(V_l(A), V_l(B)).$$

Over number fields our knowledge is not nearly so advanced. For Artin motives of rank 2, Langlands has made a beginning with the theory of base change (see the remarks (3.5.5)). For elliptic curves M over $\boldsymbol{Q}$, it is not even known whether

$L(M, s)$ has a meromorphic continuation throughout the s-plane, or whether the isogeny theorem is true. For a more detailed account of our ignorance, as well as of a few things which are known, see **[S4]**.

References

[A] E. Artin, *Zur Theorie der L-Reihen mit allgemeinen Gruppencharakteren*, Abh. Math. Sem. Univ. Hamburg **8** (1930), 292–306 (collected papers, pp. 165–179).

[AT] E. Artin and J. Tate, *Class field theory*, Benjamin, New York, 1967.

[B] J. Buhler, *Icosahedral Galois representations*, Ph.D. Thesis, Harvard Univ., (January 1977).

[CF] J. W. S. Cassels and A. Fröhlich, *Algebraic number theory*, Academic Press, London, 1967.

[D1] P. Deligne, *Les constantes des équations fonctionnelles*, Séminaire Delange-Pisot-Poitou 1969/70, 19 bis.

[D2] ———, *Formes modulaires et représentations de* GL(2), Antwerp II, Lecture Notes in Math., vol. 349, Springer-Verlag, 1973, pp. 55–106.

[D3] ———, *Les constantes des équations fonctionnelles des fonctions L*, Antwerp II, Lecture Notes in Math., vol. 349, Springer-Verlag, 1973, pp. 501–595.

[D4] ———, *La conjecture de Weil*. I, Inst. Hautes Etudes Sci. Publ. Math. **43** (1974), 273–307.

[D5] ———, *Les constantes locales de l'équation fonctionnelle de la fonction L d'Artin d'une représentation orthogonale*, Invent. Math. **35** (1976), 299–316.

[D6] ———, *Valeurs de fonctions L et périodes d'intégrales*, these Proceedings, part 2, pp. 313–346.

[DS] P. Deligne and J.-P. Serre, *Formes modulaires de poids* 1, Ann. Sci. Ecole Norm. Sup. (4) **7** (1974), 507–530.

[K] H. Koch, *Classification of the primitive representations of the Galois groups of local fields*, Invent. Math. **40** (1977), 195–216.

[L1] R. P. Langlands, *On Artin's L-functions*, Complex Analysis, Rice Univ. Studies No. 56, 1970, 23–28; also the mimeographed notes: *On the functional equation of Artin's L-functions.*

[L2] ———, *Modular forms and l-adic representations*, Lecture Notes in Math., vol. 349, Springer-Verlag, 1973, pp. 362–498.

[L3] ———, *Automorphic representations, Shimura varieties, and motives. Ein Märchen*, these Proceedings, part 2, pp. 205–246.

[S1] J.-P. Serre, *Corps locaux*, Hermann, Paris, 1962.

[S2] ———, *Facteurs locaux des fonctions zêta des variétés algébriques* (*Définitions et conjectures*), Séminaire Delange-Pisot-Poitou, 1969/70, no. 19.

[S3] ———, *Modular forms of weight one and Galois representations*, Proc. Durham Sympos. on Algebraic Number Fields, Academic Press, London, 1977, pp. 193–267.

[S4] ———, *Représentations l-adiques*, Algebraic Number Theory, Papers for the Internat. Sympos., Kyoto 1976 (S. Iyanaga, Ed.), Japan Soc. for the Promotion of Science, Tokyo, 1977, pp. 177–193.

[ST] J.-P. Serre and J. Tate, *Good reduction of abelian varieties*, Ann. of Math. (2) **88** (1968), 492–517.

[T1] J. Tate, *Fourier analysis in number fields and Hecke's zeta functions*, Thesis, Princeton Univ., (1950) (published in **[CF]**, 305–347).

[T2] ———, *Local constants*, Proc. Durham Sympos. on Algebraic Number Fields, Academic Press, London, 1977.

[W1] A. Weil, *Sur la théorie du corps de classes*, J. Math. Soc. Japan **3** (1951), 1–35.

[W2] ———, *Basic number theory*, Springer, New York, 1967.

[W3] ———, *Sur les formules explicites de la théorie des nombres*, Izv. Akad. Nauk SSSR Ser. Mat. **36** (1972), 3–18.

[Z] Ju. G. Zarhin, *Endomorphisms of abelian varieties over fields of finite characteristic*, Izv. Akad. Nauk SSSR Ser. Mat. **39** (2) (1975) = Math USSR-Izv. **9** (1975), 255–260.

Harvard University

Proceedings of Symposia in Pure Mathematics
Vol. 33 (1979), part 2, pp. 27–61

AUTOMORPHIC L-FUNCTIONS

A. BOREL

This paper is mainly devoted to the L-functions attached by Langlands [35] to an irreducible admissible automorphic representation π of a reductive group G over a global field k and to local and global problems pertaining to them. In the context of this Institute, it is meant to be complementary to various seminars, in particular to the $\mathbf{GL}_2$-seminars, and to stress the general case. We shall therefore start directly with the latter, and refer for background and motivation to other seminars, or to some expository articles on this topic in general [3] or on some aspects of it [7], [14], [15], [23].

The representation π is a tensor product $\pi = \bigotimes_v \pi_v$ over the places of k, where π_v is an irreducible admissible representation of $G(k_v)$ [11]. Accordingly the L-functions associated to π will be Euler products of local factors associated to the π_v's. The definition of those uses the notion of the L-group ${}^LG$ of, or associated to, G. This is the subject matter of Chapter I, whose presentation has been much influenced by a letter of Deligne to the author. The L-function will then be an Euler product $L(s, \pi, r)$ assigned to π and to a finite dimensional representation r of ${}^LG$. (If $G = \mathbf{GL}_n$, then the L-group is essentially $\mathbf{GL}_n(C)$, and we may tacitly take for r the standard representation r_n of $\mathbf{GL}_n(C)$, so that the discussion of $\mathbf{GL}_n$ can be carried out without any explicit mention of the L-group, as is done in the first six sections of [3].) The local L- and ε-factors are defined at all places where G and π are "unramified" in a suitable sense, a condition which excludes at most finitely many places. Chapter II is devoted to this case. The main point is to express the Satake isomorphism in terms of certain semisimple conjugacy classes in ${}^LG$ (7.1). At this time, the definition of the local factors at the ramified places is not known in general. For $\mathbf{GL}_n$ and r_n, however, there is a direct definition [19], [25]. In the general case, the most ambitious scheme is to associate canonically to an irreducible admissible representation of a reductive group H over a local field E a representation of the Weil-Deligne group W'_E of E into ${}^LH$, and then use L- and ε-factors associated to representations of W'_E [60]. This problem is the main topic of Chapter III.

The L-function $L(s, \pi, r)$ associated to π and r as above is introduced in §13. In fact, it is defined in general as a product of local factors indexed by almost all places of k. It converges absolutely in some right half-plane (13.3; 14.2). Some of the main

---

AMS (MOS) subject classifications (1970). Primary 10D20; Secondary 12A70, 12B30, 14G10, 22E50.

conjectural analytic properties (meromorphic continuation, functional equation), and the evidence known so far, are discussed in §14.

From the point of view of [**35**], a great many problems on automorphic representations and their L-functions are special cases of one, the so-called lifting problem or problem of functoriality with respect to L-groups. It is discussed in Chapter V. It is closely connected with Artin's conjecture (see §17 and the base-change seminar [**17**]). In §18 brief mention is made of some known or conjectured relations between automorphic L-functions and the Hasse-Weil zeta-function of certain varieties, to be discussed in more detail in the seminars on Shimura varieties [**8**], [**40**].

Thanks are due to H. Jacquet and R. P. Langlands for various very helpful comments on an earlier draft of this paper.

CONTENTS

CHAPTER I. L-GROUPS.

k is a field, $\bar{k}$ an algebraic closure of k, k_s the separable closure of k in $\bar{k}$, and Γ_k the Galois group of k_s over k. G is a connected reductive group, over $\bar{k}$ in 1.1, 1.2, 2.1, 2.2, over k otherwise.

§§1, 2 will be used throughout, §3 from Chapter III on. The reader willing to take on faith various statements about restriction of scalars need not read §§4, 5.

1. Classification. We recall first some facts discussed in [**58**].

1.1. There is a canonical bijection between isomorphism classes of connected reductive $\bar{k}$-groups and isomorphism classes of root systems. It is defined by associating to G the root datum $\psi(G) = (X^*(T), \varphi, X_*(T), \varphi^v)$ where T is a maximal torus of G, $X^*(T)$ ($X_*(T)$) the group of characters (1-parameter subgroups) of T and Φ (Φ^v) the set of roots (coroots) of G with respect to T.

1.2. The choice of a Borel subgroup $B \supset T$ is equivalent to that of a basis Δ of $\Phi(G, T)$. The previous bijection yields one between isomorphism classes of triples (G, B, T) and isomorphism classes of based root data $\psi_0(G) = (X^*(T), \Delta, X_*(T), \Delta^v)$. There is a split exact sequence

$$(1) \qquad (1) \longrightarrow \operatorname{Int} G \longrightarrow \operatorname{Aut} G \longrightarrow \operatorname{Aut} \psi_0(G) \longrightarrow (1).$$

To get a splitting, we may choose $x_\alpha \in G_\alpha$ ($\alpha \in \Delta$) and then have a canonical bijection

$$(2) \qquad \operatorname{Aut} \psi_0(G) \xrightarrow{\sim} \operatorname{Aut}(G, B, T, \{x_\alpha\}_{\alpha\in\Delta}).$$

Two such splittings differ by an inner automorphism Int t ($t \in T$).

1.3. Given $\gamma \in \Gamma_k$ there is $g \in G(k_s)$ such that $g \cdot {}^\gamma T \cdot g^{-1} = T$, $g \cdot {}^\gamma B \cdot g^{-1} = B$, whence an automorphism of $\psi_0(G)$, which depends only on γ. We let $\mu_G \colon \Gamma_k \to \operatorname{Aut} \psi_0(G)$ be the homomorphism so defined. If G' is a k-group which is isomorphic to G over $\bar{k}$ (hence over k_s), then $\mu_G = \mu_{G'} \Leftrightarrow G, G'$ are inner forms of each other.

1.4. Let $f\colon G \to G'$ be a k-morphism, whose image is a normal subgroup. Then f induces a map $\psi(f)\colon \psi(G) \to \psi(G')$ (contravariant (resp. covariant) in the first (last) two arguments). Given $B, T \subset G$ as above, there exists a Borel subgroup B' (resp. a maximal torus T') of G' such that $f(B) \subset B'$, $f(T) \subset T'$, whence also a map $\psi_0(f)$: $\psi_0(G) \to \psi_0(G')$.

2. Definition of the L-group.

2.1. The inverse system $\Psi_0^\vee$ to the based root datum $\Psi_0 = (M, \Delta, M^*, \Delta^\vee)$ is $\psi_0^\vee = (M^*, \Delta^\vee, M, \Delta)$. To the $\bar{k}$-group G we first associate the group ${}^LG^\circ$ over $\boldsymbol{C}$ such that $\psi_0({}^LG^\circ) = \psi_0(G)^\vee$. We let ${}^LT^\circ$, ${}^LB^\circ$ be the maximal torus and Borel subgroup defined by $\psi_0^\vee$, and say they define the canonical splitting of ${}^LG^\circ$.

Let f be as in 1.4. Then f also induces a map $\psi_0^\vee(f) : \psi_0(G')^\vee \to \psi_0(G)^\vee$. An algebraic group morphism of ${}^LG'^\circ$ into ${}^LG^\circ$ associated to it will be denoted ${}^Lf^\circ$. Given one, any other is of the form $\operatorname{Int} t \circ {}^Lf^\circ \circ \operatorname{Int} t'$ ($t \in {}^LT^\circ$, $t' \in {}^LT'^\circ$), and maps ${}^LT'^\circ$ (resp. ${}^LB'^\circ$) into ${}^LT^\circ$ (resp. ${}^LB^\circ$).

2.2. EXAMPLES. (1) Let $G = \mathbf{GL}_n$. Then ${}^LG^\circ = \mathbf{GL}_n$. In fact, let $M = \boldsymbol{Z}^n$ with $\{x_i\}$ its canonical basis. Let $\{e_i\}$ be the dual basis of $M^* = \boldsymbol{Z}^n$. Then $\Psi_0(\mathbf{GL}_n) = (M, \Delta, M^*, \Delta^\vee)$ with $\Delta = \{(x_i - x_{i+1}), 1 \leqq i < n\}$, $\Delta^\vee = \{(e_i - e_{i+1}), 1 \leqq i < n\}$, hence $\psi_0 = \psi_0^\vee$.

(2) Let G be semisimple and $\Psi_0(G) = (M, \Phi, M^*, \Phi^\vee)$. As usual, let $P(\Phi) \subset M \otimes \boldsymbol{Q}$ be the lattice of weights of Φ and $Q(\Phi)$ the group generated by Φ in M. Define $P(\Phi^\vee)$ and $Q(\Phi^\vee)$ similarly.

As is known G is simply connected (resp. of adjoint type) if and only if $P(\phi) = M$ (resp. $Q(\phi) = M$). Moreover

$$P(\Phi) = \{\lambda \in M \otimes \boldsymbol{Q} \,|\, \langle \lambda, \Phi^\vee \rangle \subset \boldsymbol{Z}\}, \qquad P(\Phi^\vee) = \{\lambda \in M^x \otimes \boldsymbol{Q} \,|\, \langle \lambda, \Phi \rangle \in \boldsymbol{Z}\}.$$

Therefore:

G simply connected $\Leftrightarrow {}^LG^\circ$ of adjoint type;

G of adjoint type $\Leftrightarrow {}^LG^\circ$ simply connected.

(3) Let G be simple. Up to central isogeny, it is characterized by one of the types $\boldsymbol{A}_n, \ldots, \boldsymbol{G}_2$ of the Killing-Cartan classification. It is well known that the map $\psi_0(G) \to \psi_0(G)^v$ permutes $\boldsymbol{B}_n$ and $\boldsymbol{C}_n$ and leaves all other types stable. Thus if $G = \mathbf{Sp}_{2n}$ (resp. $G = \mathbf{PSp}_{2n}$), then ${}^LG^\circ = \boldsymbol{SO}_{2n+1}$ (resp. ${}^LG^\circ = \mathbf{Spin}_{2n+1}$). In all other cases, $G \mapsto {}^LG^\circ$ preserves the type (but goes from simply connected group to adjoint group, and vice versa).

(4) Let again G be reductive and let $f: G \to G'$ be a central isogeny. Let

$$N = \operatorname{coker} f_*: X_*(T) \longrightarrow X_*(T') \qquad (T' = f(T)),$$
$$N' = \operatorname{coker} f^*: X^*(T') \longrightarrow X^*(T).$$

Then N and N' are isomorphic and $\ker {}^Lf^\circ = \operatorname{Hom}(N, \boldsymbol{C}^*) \xrightarrow{\sim} N$. In particular, ${}^Lf^\circ$ is an isomorphism if and only if f is one.

(5) Let $f: G \to G'$ be a central surjective morphism, $Q = \ker f$, and Q° the identity component of Q. Then $\ker {}^Lf^\circ \simeq Q/Q^\circ$.

If Q is connected, then $T'' = T \cap Q$ is a maximal torus of Q, and the injectivity of ${}^Lf^\circ$ follows from the fact that the exact sequence $1 \to T'' \to T \to T' \to 1$ necessarily splits. If Q is not connected, then $r: H = G/Q^\circ \to G'$ is a nontrivial separable isogeny, with kernel Q/Q°. ${}^Lf^\circ$ factors through ${}^Lr^\circ$ and, by the first part and (4), $\ker {}^Lf^\circ = \ker {}^Lr^\circ \cong Q/Q^\circ$. In particular, if we apply this to the case where $G' = G_{\mathrm{ad}}$ is the adjoint group of G, and use (2), we see that the derived group of ${}^LG^\circ$ is simply connected if and only if the center of G is connected. As an example, let $G = \mathbf{GSp}_{2n}$ be the group of symplectic similitudes on a $2n$-dimensional space. Then the derived group of ${}^LG^\circ$ is isomorphic to $\mathbf{Spin}_{2n+1}$. In fact, we have ${}^LG^\circ = (\mathbf{GL}_1 \times \mathbf{Spin}_{2n+1})/A$ where $A = \{1, a\}$ and $a = (a_1, a_2)$, with a_1 of order two in $\mathbf{GL}_1$ and a_2 the nontrivial central element of $\mathbf{Spin}_{2n+1}$. If $n = 2$, then $\mathbf{Spin}_{2n+1} = \mathbf{Sp}_{2n}$. It follows that if $G = \mathbf{GSp}_4$, then ${}^LG^\circ = \mathbf{GSp}_4(\boldsymbol{C})$.

2.3. We have canonically $\operatorname{Aut} \Psi_0 = \operatorname{Aut} \Psi_0^\vee$. Therefore we may view μ_G as a homomorphism of Γ_k into $\operatorname{Aut} \psi_0^\vee$. Choose a monomorphism

$$\operatorname{Aut} \psi_0^\vee \longrightarrow \operatorname{Aut}({}^LG^\circ, {}^LB^\circ, {}^LT^\circ) \tag{1}$$

as in 1.2(2). We have then a homomorphism

$$\mu_G': \Gamma_k \longrightarrow \operatorname{Aut}({}^LG^\circ, {}^LB^\circ, {}^LT^\circ).$$

The associated group to, or L-group of, G is then by definition the semidirect product

$${}^L(G/k) = {}^LG = {}^LG^\circ \rtimes \Gamma_k, \tag{2}$$

with respect to μ_G'. We note that μ_G' is well defined up to an inner automorphism by an element of ${}^LT^\circ$. The group ${}^LG$ is viewed as a topological group in the obvious way. The canonical splitting of ${}^LG^\circ$ (2.1) is stable under Γ_k.

We have a canonical projection ${}^LG \to \Gamma_k$ with kernel ${}^LG^\circ$. The splittings of the exact sequence

$$1 \longrightarrow {}^LG^\circ \longrightarrow {}^LG \xrightarrow{\nu_G} \Gamma_k \longrightarrow 1 \tag{3}$$

defined as in 1.2 via an isomorphism $\mathrm{Aut}\,\Psi_0^\vee \xrightarrow{\sim} \mathrm{Aut}({}^LG^\circ, {}^LB^\circ, {}^LT^\circ, \{x_\alpha\})$ are called admissible. They differ by inner automorphisms $\mathrm{Int}\, t$ ($t \in {}^LT^\circ$). Note that if G splits over k, then Γ_k acts trivially on ${}^LG^\circ$ and ${}^LG$ is simply the direct product of ${}^LG^\circ$ and Γ_k.

2.4. REMARKS. (1) So far, we can in this definition take ${}^LG^\circ$ over any field. We have chosen $\boldsymbol{C}$ since this is the most important case at present, but it is occasionally useful to use other local fields.

(2) There are various variants of this notion, which may be more convenient in certain contexts. For instance we can divide Γ_k by a closed normal subgroup which acts trivially on $\Psi^\vee$, hence on ${}^LG^\circ$, e.g., by $\Gamma_{k'}$ if k' is a Galois extension of k over which G splits. Then Γ_k is replaced by $\mathrm{Gal}(k'/k)$, and ${}^LG$ is a complex reductive Lie group.

We can also define a semidirect product ${}^LG^\circ \rtimes \Sigma$, for any group Σ endowed with a homomorphism into Γ_k, e.g., the Weil group of k, if k is a local or global field. In that case, we get the "Weil form" of ${}^LG$.

(3) Let G' be a k-group which is isomorphic to G over $\bar{k}$. Then G and G' are inner forms of each other if and only if ${}^LG$ is isomorphic to ${}^LG'$ over Γ_k. In fact, the first condition is equivalent to $\mu_G = \mu_{G'}$, and the latter is easily seen to be equivalent to the second condition. In particular, since two quasi-split groups over k which are inner forms of each other are isomorphic over k, it follows that if G, G' are quasi-split and ${}^LG \xrightarrow{\sim} {}^LG'$ over Γ_k then G and G' are k-isomorphic.

2.5. *Functoriality.* Let $f\colon G \to G'$ be a k-morphism whose image is a normal subgroup. Then $f_{\psi_0}\colon \psi_0(G) \to \psi_0(G')$ clearly commutes with Γ_k, hence so does $f_{\psi_0}^\vee : \psi_0(G')^v \to \psi_0(G)^v$ and ${}^Lf^\circ\colon {}^LG'^\circ \to {}^LG^\circ$. We get therefore a continuous homomorphism ${}^Lf : {}^LG' \to {}^LG$ such that

$$\begin{array}{ccc} {}^LG' & \xrightarrow{\ {}^Lf\ } & {}^LG \\ & {\scriptstyle \nu_{G'}} \searrow \quad \swarrow {\scriptstyle \nu_G} & \\ & \Gamma_k & \end{array}$$

is commutative, which extends ${}^Lf^\circ$.

2.6. *Representations.* For brevity, by *representation* of ${}^LG$ we shall mean a continuous homomorphism $r\colon {}^LG \to \mathbf{GL}_m(\boldsymbol{C})$ whose restriction to ${}^LG^\circ$ is a morphism of complex Lie groups.

Clearly, $\ker r$ always contains an open subgroup of Γ_k, hence r factors through ${}^LG^\circ \rtimes \Gamma_{k'/k}$, where k' is a finite Galois extension of k over which G splits. The group ${}^LG^\circ \rtimes \Gamma_{k'/k}$ is canonically a complex algebraic group and r is a morphism of complex algebraic groups.

3. Parabolic subgroups.

3.1. *Notation.* We let $\mathscr{P}(G/k)$ denote the set of parabolic k-subgroups of G, and write $\mathscr{P}(G)$ for $\mathscr{P}(G/\bar{k})$. Let $p(G/k)$ be the set of conjugacy classes (with respect to $G(\bar{k})$ or $G(k)$, it is the same) of parabolic k-subgroups, and $p(G) = p(G/\bar{k})$. Let $p(G)_k$ be the set of conjugacy classes of parabolic subgroups which are defined over k (i.e., if $P \in \sigma \in p(G)_k$, then ${}^\gamma P \in \sigma$ for all $\gamma \in \Gamma_k$). In particular $p(G/k) \hookrightarrow p(G)_k$. There is equality if G is quasi-split/k.

3.2. We recall there is a canonical bijection between $p(G)$ and the subsets of Δ.

Then $p(G)_k$ corresponds to the Γ_k-stable subsets of Δ and $p(G/k)$ to those Γ_k-stable subsets which contain the set Δ_0 of simple roots of a Levi subgroup of a minimal parabolic k-group. In particular we have $p(G/k) = p(G)_k$ if G is quasi-split over k. Given $P \in \mathscr{P}(G)$, we let $J(P)$ be the subset of Δ assigned to the class of P

Since two conjugate parabolic subgroups whose intersection is a parabolic subgroup are identical, we see in particular that if P is defined over k, $P' \supset P$, and the class of P' is defined over k, then P' is defined over k.

3.3. *Parabolic subgroups of* ${}^LG$. A closed subgroup P of ${}^LG$ is parabolic if $\gamma_G(P) = \Gamma_k$ and $P^\circ = {}^LG^\circ \cap P$ is a parabolic subgroup of ${}^LG^\circ$. Then $P = N_{{}^LG}(P^\circ)$. In other words, a parabolic subgroup is the normalizer of a parabolic subgroup P° of ${}^LG^\circ$, provided the normalizer meets every class modulo ${}^LG^\circ$. We say P is standard if it contains ${}^LB$. The standard parabolic subgroups are the subgroups

$$ {}^LP^\circ \rtimes \Gamma_k, \tag{1}$$

where ${}^LP^\circ$ runs through the standard parabolic subgroup of ${}^LG^\circ$ such that $J({}^LP^\circ) \subset \Delta^\vee$ is stable under Γ_k.

Every parabolic subgroup of ${}^LG$ is conjugate (under ${}^LG$ or, equivalently, ${}^LG^\circ$) to one and only one standard parabolic subgroup.

We let $\mathscr{P}({}^LG)$ be the set of parabolic subgroups of ${}^LG$ and $p({}^LG)$ the set of their conjugacy classes.

The given bijection $\Delta \leftrightarrow \Delta^\vee$ yields then, in view of 3.2, a bijection

$$ p(G)_k \leftrightarrow p({}^LG). \tag{2}$$

We shall say that a parabolic subgroup of ${}^LG$ is *relevant* if its class corresponds to one of $p(G/k)$ under this map. We let ${}^L\mathscr{P}({}^LG)$ be the set of relevant parabolic subgroups and ${}^Lp({}^LG)$ the set of their conjugacy classes, the *relevant conjugacy classes* of parabolic subgroups. Thus, by definition

$$ p(G/k) \leftrightarrow {}^Lp({}^LG). \tag{3}$$

Thus, if G and G' are inner forms of each other, $p({}^LG)$ and $p({}^LG')$ are the same, but ${}^Lp({}^LG)$ and ${}^Lp({}^LG')$ are not. If G' is quasi-split, then ${}^Lp({}^LG') \doteq p({}^LG')$; hence we have an injection

$$ {}^Lp({}^LG) \subset {}^Lp({}^LG') = p({}^LG'). \tag{4}$$

If $\mathscr{D}G$ is anisotropic over k, then ${}^Lp({}^LG)$ consists of G alone.

3.4. *Levi subgroups.* Let P be a parabolic subgroup of ${}^LG$. The unipotent radical N of P° is normal in P and will also be called the unipotent radical of P. Then $P/N \xrightarrow{\sim} P^\circ/N \rtimes \Gamma_k$. In fact, it follows from (1) that P is a split extension of N, and is the semidirect product of N by the normalizer in P of any Levi subgroup M° of P°. Those normalizers will be called the Levi subgroups of P.

Let $P \in \mathscr{P}(G/k)$, M a Levi k-subgroup of P. Let ${}^LP$ be the standard parabolic subgroup in the class associated to that of P (see (3)). Then ${}^LM$ may be identified to a Levi subgroup of ${}^LP$. In fact if M corresponds to $(X^*(T), J, X_*(T), J^v)$, then ${}^LM^\circ$ corresponds to $(X_*(T), J^v, X^*(T), J)$ and ${}^LM^\circ \ltimes \Gamma_k$ is equal to ${}^LM$ by definition and is a Levi subgroup of ${}^LP$, as defined above.

A Levi subgroup of a parabolic subgroup P of ${}^LG$ is *relevant* if P is.

For the sake of brevity, we shall sometimes say "Levi subgroup in G" for "Levi subgroup of a parabolic subgroup of G." Similarly for ${}^LG$.

3.5. LEMMA. *The proper Levi subgroups in ${}^LG$ are the centralizers in ${}^LG$ of tori in $\mathscr{D}({}^LG^\circ)$, which project onto Γ_k.*

Let M be a proper Levi subgroup in ${}^LG$. It is conjugate to a subgroup $\mathscr{Z}(S)^\circ \rtimes \Gamma_k$, where $S \subset {}^LT^\circ$ is the identity component of the kernel of a subset $J \subsetneqq \Delta^v$ stable under Γ_k. Let then S' be the one-dimensional subtorus of $S \cap \mathscr{D}({}^LG^\circ)$ on which the remaining simple roots are all equal. It is clear that $\mathscr{Z}(S')^\circ = \mathscr{Z}(S)$, and that S' is pointwise fixed under Γ_k. We have then $M = \mathscr{Z}(S')$.

Let now S be a nontrivial torus in $\mathscr{D}({}^LG^\circ)$ such that $\mathscr{Z}(S)$ meets every connected component of ${}^LG$. Fix an ordering on $X^*(S)$. There is a proper parabolic subgroup P° of ${}^LG^\circ$ of the form $\mathscr{Z}(S)^\circ \cdot U$, such that the weights of S in the unipotent radical U of P° are the roots of ${}^LG^\circ$ with respect to S which are positive for this ordering. U is normalized by $\mathscr{Z}(S)$; hence $\mathscr{Z}(S)\cdot U$ is a proper parabolic subgroup P of ${}^LG$, and then $\mathscr{Z}(S)$ is a Levi subgroup of P.

3.6. PROPOSITION. *Let H be a subgroup of ${}^LG$ whose projection on Γ_k is dense in Γ_k. Then the Levi subgroups in ${}^LG$ which contain H minimally form one conjugacy class with respect to the centralizer of H in ${}^LG^\circ$.*

Let C be the identity component of the centralizer of H in $\mathscr{D}({}^LG^\circ)$, and D a maximal torus of H. If $D = \{1\}$, then, by 3.5, H is not contained in any proper Levi subgroup in ${}^LG$, and there is nothing to prove. So assume $D \neq \{1\}$.

Let Γ' be a normal open subgroup of Γ_k which acts trivially on ${}^LG^\circ$. It is then normal in ${}^LG$, and $H\cdot\Gamma'$ projects onto Γ_k. Since $\mathscr{Z}(D)$ contains $H\cdot\Gamma'$, it projects onto Γ_k, hence is a proper Levi subgroup by 3.5. Let M be a Levi subgroup containing H. By 3.5, $M = Z(S)$, where S is a torus in $\mathscr{D}({}^LG^\circ)$. Then $S \subset C$, there exists $c \in C$ such that $c\cdot S\cdot c^{-1} \subset D$, hence $c\cdot M\cdot c^{-1} = \mathscr{Z}(S') \supset \mathscr{Z}(D)$.

4. Remarks on induced groups. (To be used mainly to discuss restriction of scalars in §5 and 6.4.)

4.1. Let A be a group, A' a subgroup of finite index of A and E a group on which A' operates by automorphisms. Then we let

$$\text{(1)} \qquad \operatorname{Ind}_{A'}^{A}(E) = I_{A'}^{A}(E) = \{f : A \longrightarrow E \mid f(a'a) = a'\cdot f(a)\ (a \in A;\ a' \in A')\}.$$

It is a group (composition being defined by taking products of values). It is viewed as an A-group by right translations:

$$\text{(2)} \qquad r_a f(x) = f(xa) \qquad (x, a \in A).$$

For $s \in A'\backslash A$, let

$$\text{(3)} \qquad E_s = \{f \in I_{A'}^{A}(E) \mid f(a) = 0 \text{ if } a \notin s\}.$$

Then E_s is a subgroup, $I_{A'}^{A}(E)$ is the direct product of the E_s's ($s \in A'\backslash A$), and these subgroups are permuted by A. The subgroup $E_{\bar{e}}$ is stable under A' and is isomorphic to E as an A' module under the map $f \mapsto f(e)$. The product of the E_s's ($s \in A'\backslash A$, $s \neq e$) is also stable under A'. We have therefore canonical homomorphisms

$$(4) \qquad E \rtimes A' \longrightarrow I(E) \rtimes A' \longrightarrow E \rtimes A'$$

whose composition is the identity.

4.2. Let B be a group, $\mu: B \to A$ a homomorphism. Let $B' = \mu^{-1}(A')$ and assume that μ induces a bijection: $B'\backslash B \xrightarrow{\sim} A'\backslash A$. Let E be a group on which A' operates by automorphisms, also viewed as a B'-group via μ. Then $f \mapsto \mu \circ f$ induces an isomorphism

$$(1) \qquad \mu' : I^A_{A'}(E) \xrightarrow{\sim} I^B_{B'}(E),$$

whose inverse is μ-equivariant.

This follows immediately from the definitions.

4.3. Let A, E be as before, C a group and $\nu: C \to A$ a homomorphism. Let $\varphi: C \to E \rtimes A$ be a homomorphism over A. The map $\phi: C \to E$ such that $\varphi(c) = (\phi(c), \nu(c))$ $(c \in C)$ is a 1-cocycle of C in E and $\varphi \mapsto \phi$ induces a bijection

$$H^1(C; E) \xrightarrow{\sim} \varphi_A(C, E),$$

where, by definition, $\varphi_A(C, E)$ denotes the set of homomorphisms $\varphi: C \to E \rtimes A$ over A, modulo inner automorphisms by elements of E.

4.4. Let A, A', B, B' and E be as in 4.2. We have a commutative diagram with exact rows

$$(1) \qquad \begin{array}{ccccccccc} 1 & \longrightarrow & I^A_{A'}(E) & \longrightarrow & I^A_{A'}(E) \rtimes A & \longrightarrow & A & \longrightarrow & 1 \\ & & \uparrow & & \uparrow & & \uparrow & & \\ 1 & \longrightarrow & E & \longrightarrow & E \ltimes A' & \longrightarrow & A' & \longrightarrow & 1 \end{array}$$

where the vertical maps are natural inclusions (4.1).

Let $\varphi: B \to I^A_{A'}(E) \rtimes A$ be a homomorphism over A. Using 4.1(4), we get by restriction a homomorphsim $\tilde{\varphi}: B' \to E \rtimes A'$ over A'.

4.5. Lemma. *The map* $\varphi \mapsto \tilde{\varphi}$ *of* 4.4 *induces a bijection* $\varphi_A(B, I^A_{A'}(E)) \xrightarrow{\sim} \varphi_{A'}(B', E)$.

We have, using 4.2, 4.3:

$$(1) \qquad \varphi_A(B, I^A_{A'}(E)) = H^1(B; I^A_{A'}(E)) = H^1(B; I^B_{B'}(E)),$$

$$(2) \qquad \Phi_{A'}(B', E) = H^1(B'; E).$$

By a variant of Shapiro's lemma, contained, e.g., in [**4**, 1.29]:

$$H^1(B; I^B_{B'}(E)) \xrightarrow{\sim} H^1(B'; E),$$

and it is clear that the isomorphisms (1), (2) carry this isomorphism over to $\varphi \mapsto \tilde{\varphi}$.

5. Restriction of scalars. *In this section,* k' *is a finite extension of* k *in* k_s, G' *is a connected* k'*-group, and* $G = \mathbf{R}_{k'/k}\, G'$.

5.1. The Galois group $\Gamma_{k'}$ of k_s over k' is an open subgroup (of finite index) of Γ_k and $\Sigma_{k',k} = \Gamma_{k'}\backslash\Gamma_k$ may be identified with the set of k-monomorphisms of k' into k_s. We have, in the notation of 4.1 (with $A = \Gamma_k$, $A' = \Gamma_{k'}$)

$$(1) \qquad G(\bar{k}) = I^{\Gamma_k}_{\Gamma_{k'}}(G'(\bar{k})) = \prod_{\sigma \in \Gamma_{k'}\backslash\Gamma_k} {}^\sigma G'(\bar{k}).$$

Assume G' to be reductive. Then we see easily that $\psi(G) = (M, \varphi, M^*, \varphi^\vee)$ is related to $\psi(G') = (M', \varphi', M'^*, \varphi'^\vee)$ by

$$M = I_{\Gamma_{k'}}^{\Gamma_k}(M'), \qquad \varphi = \bigcup_{a \in A' \setminus A} \varphi' \cdot a. \tag{2}$$

Similarly, if Δ' is a basis of φ', then

$$\Delta = \bigcup_a \Delta' \cdot a \tag{3}$$

is one for φ.

From this it follows that we have a natural isomorphism

$$^LG^\circ \xrightarrow{\sim} I_{\Gamma_{k'}}^{\Gamma_k}(^LG'^\circ). \tag{4}$$

We have then a commutative diagram

$$\begin{array}{ccccccccc} 1 & \longrightarrow & ^LG'^\circ & \longrightarrow & ^LG' = {}^LG'^\circ \rtimes \Gamma_{k'} & \longrightarrow & \Gamma_{k'} & \longrightarrow & 1 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 1 & \longrightarrow & ^LG^\circ = I_{\Gamma_{k'}}^{\Gamma_k}(^LG'^\circ) & \longrightarrow & ^LG = {}^LG^\circ \rtimes \Gamma_k & \longrightarrow & \Gamma_k & \longrightarrow & 1 \end{array} \tag{5}$$

5.2. The map $P' \mapsto R_{k'/k} P'$ induces a bijection between $\mathscr{P}(G'/k')$ and $\mathscr{P}(G/k)$. Moreover P' is a Borel subgroup of G' if and only if $R_{k'/k} P'$ is one of G. Hence G' is quasi-split/k' if and only if G is quasi-split over k (see [5, §6]).

Since $G(k) \xrightarrow{\sim} G'(k')$, we also get a bijection $p(G'/k') \xrightarrow{\sim} p(G/k)$.

If $J' \subset \Delta'$ is stable under $\Gamma_{k'}$ then $J = \bigcup_{a \in A' \setminus A} r_a(J')$ is stable under Γ_k. This map is easily seen to yield a bijection between $\Gamma_{k'}$-stable subsets of Δ' and Γ_k-stable subsets of Δ, whence also canonical bijections

$$p(^LG') \xrightarrow{\sim} p(^LG), \qquad {}^Lp(^LG') \xrightarrow{\sim} {}^Lp(^LG).$$

CHAPTER II. QUASI-SPLIT GROUPS. THE UNRAMIFIED CASE.

In this chapter, G is a connected reductive quasi-split k-group. From 6.2 *on, G is assumed to split over a cyclic extension k' of k, and σ denotes a generator of* $\mathrm{Gal}(k'/k)$.

6. Semisimple conjugacy classes in ^LG.

6.1. Assume B and T to be defined over k. Then the action of Γ_k on $X^*(T)$ or $X_*(T)$ given by μ_G coincides with the ordinary action. The greatest k-split subtorus T_d of T is maximal k-split in G, and its centralizer is T; in particular, T_d contains regular elements of G. Hence any element $w \in W$ which leaves T_d stable is completely determined by its restriction to T_d. It follows that $_kW$ may be identified with the subgroup of the elements of W which leave T_d stable or, equivalently, with the fixed-point set of Γ_k in W. If we go over to the L-group and identify canonically W with $W(^LG^\circ, {}^LT^\circ)$, then $_kW$ is also the fixed-point set of Γ_k in W, and it operates on the greatest subtorus S of $^LT^\circ$ which is pointwise fixed under Γ_k. The group S always contains regular elements; hence any element of $_kW$ is determined by its restriction to S. We let $_kN$ be the inverse image of $_kW$ in the normalizer N of $^LT^\circ$ in $^LG^\circ$.

6.2. LEMMA. *Every element $w \in {}_kW$ has a representative in $_kN$ which is fixed under σ.*

Write $\Delta^\vee = D_1 \cup \cdots \cup D_m$, where the D_i's are the distinct orbits of $\Gamma(k'/k)$ in $\Delta^\vee$. Let δ_i be the common restriction to S of the elements of D_i and S_i the identity component of the kernel of δ_i. Then ${}_kW$, viewed as a group of automorphisms of S, is generated by the reflections s_i to the S_i $(1 \leqq i \leqq m)$, and it suffices to prove the lemma for $w = s_i$ $(1 \leqq i \leqq m)$. [The "reflection" s_i is the unique element $\neq 1$ of W which leaves S stable, fixes S_i pointwise and is of order two.]

We let Lie(M) denote the Lie algebra of the complex Lie group M. For $\check{\alpha} \in \Delta^\vee$, let, as usual

$$(1) \qquad \mathfrak{g}_{\check{\alpha}} = \{X \in \mathrm{Lie}({}^LG^\circ) | \mathrm{Ad}\, t(X) = \check{\alpha}(t) \cdot X \ (t \in {}^LT^\circ)\}.$$

It is one-dimensional. Fix i between 1 and m. By construction of ${}^LG$, we can find nonzero elements $e_{\check{\alpha}} \in \mathfrak{g}_{\check{\alpha}}$ (resp. $e_{-\check{\alpha}} \in \mathfrak{g}_{-\check{\alpha}}$ $(\check{\alpha} \in D_i)$) which are permuted by σ. We have then

$$(2) \qquad [e_{\check{\alpha}}, e_{-\check{\alpha}}] = c \cdot \alpha,$$

where c is $\neq 0$, and independent of $\alpha \in D_i$ since $\Gamma(k'/k)$ is transitive on D_i. Here $X_*({}^LT^\circ) \otimes \boldsymbol{C}$ is identified with Lie(${}^LT^\circ$), and $\check{\alpha}$ with $\check{\alpha} \otimes 1$. The element

$$(3) \qquad f_{\pm i} = \sum_{\check{\alpha} \in D_i} e_{\pm \check{\alpha}},$$

is fixed under σ. Moreover, since the difference of two simple roots is not a root, we have

$$(4) \qquad h_i = [f_i, f_{-i}] = \sum_{\check{\alpha} \in D_i} [e_{\check{\alpha}}, e_{-\check{\alpha}}] = c \cdot \sum_{\check{\alpha} \in D_i} \alpha.$$

Using (3) and (4), we get

$$(5) \qquad [h_i, f_{\pm i}] = c \sum_{\check{\alpha}, \check{\beta} \in D_i} \langle \alpha, \beta \rangle \, e_{\pm \beta}.$$

By the transitivity of Gal(k'/k) on D_i, the number

$$(6) \qquad d = \sum_{\check{\alpha} \in D_i} \langle \alpha, \check{\beta} \rangle$$

is also independent of $\check{\beta} \in D_i$; therefore

$$(7) \qquad [h, f_{\pm i}] = c \cdot d \cdot f_{\pm i}.$$

We claim that $d \neq 0$, in fact that $d = 1, 2$. The irreducible components of D_i are permuted transitively by Gal(k'/k) and have a transitive group of automorphisms. Therefore they are of type A_1 or A_2. Then, accordingly, $d = 2$ or $d = 1$. It follows that h_i, f_i and f_{-i} span a three-dimensional simple algebra pointwise fixed under σ. Then so is the corresponding analytic subgroup G_i of ${}^LG^\circ$. The group G_i centralizes S_i and $S \cap G_i$ is a maximal torus of G_i, with Lie algebra spanned by h_i. Then the nontrivial element of $W(G_i, S \cap G_i)$ is the required element.

REMARK. An equivalent statement is proved, in a different manner, in [**35**, pp. 19–22].

6.3. We let $Y = {}^L(T_d)^\circ$. The group $X_*(T_d)$ may be identified to the fixed-point set of Γ_k in $X_*(T)$. The inclusion of $X_*(T_d) = X^*(Y)$ into $X_*(T) = X^*({}^LT^\circ)$ induces a surjective morphism ${}^LT^\circ \to Y$, to be denoted ν.

The map $A : t \mapsto t^{-1} \cdot {}^\sigma t$ is an endomorphism of ${}^LT^\circ$, whose differential dA at 1 is $(d\sigma - \mathrm{Id})$. Let

(1) $$U = (\ker A)^\circ, \qquad V = \operatorname{im} A.$$

Then U is pointwise fixed under σ, the Lie algebra of U (resp. V) is the kernel (resp. image) of dA. Since dA is semisimple, they are transversal to each other; hence

(2) $${}^LT^\circ = U \cdot V, \quad \text{and } U \cap V \text{ is finite.}$$

Moreover,

(3) $$V = \ker \nu, \qquad \nu(U) = Y.$$

In the rest of this chapter, we let ${}^LG$ stand for the "finite Galois form" ${}^LG^\circ \rtimes \operatorname{Gal}(k'/k)$ of the L-group. We now want to discuss the semisimple conjugacy classes in ${}^LG^\circ \rtimes \sigma$ with respect to ${}^LG^\circ$. We have

(4) $$g^{-1}\cdot(h \rtimes \sigma)\cdot g = g^{-1}\cdot h \cdot {}^\sigma g \rtimes \sigma \qquad (g, h \in {}^LG^\circ);$$

therefore ${}^LG^\circ$-conjugacy in ${}^LG^\circ \rtimes \sigma$ is equivalent to σ-conjugacy in ${}^LG^\circ$.

6.4. LEMMA. *Let $\nu' : {}^LT^\circ \rtimes \sigma \to Y$ be defined by $\nu'(t \times \sigma) = \nu(t)$ ($t \in {}^LT^\circ$). Then ν' induces a bijection*

(1) $$\bar{\nu} : ({}^LT^\circ \rtimes \sigma)/\operatorname{Int} {}_kN \xrightarrow{\sim} Y/{}_kW.$$

Let $n \in {}_kN$. By 6.2, we may write $n = w \cdot s$ with $w = {}^\sigma w$ and $s \in {}^LT^\circ$. Then the ${}^LT^\circ$-component of $n^{-1}(t \rtimes \sigma)\, n$ is

$$s^{-1}\cdot w^{-1}\cdot t \cdot w^\sigma s = s^{-1}\cdot {}^\sigma s \cdot (w^{-1}\cdot t \cdot w) \in V \cdot w^{-1}\cdot t \cdot w;$$

hence

$$\nu'(n^{-1}\cdot(t \rtimes \sigma)\cdot n) = \nu(w^{-1}\cdot t \cdot w) = w^{-1}\cdot \nu(t)\cdot w = w^{-1}\cdot \nu'(t \rtimes \sigma)\cdot w.$$

Thus ν' is equivariant with respect to the projection ${}_kN \to {}_kW$ and therefore induces a map of the left-hand side of (1) into the right-hand side of (1), which is obviously surjective. Let $t, t' \in {}^LT^\circ$ and assume that $\nu'(t \rtimes \sigma) = w^{-1}\cdot \nu'(t' \rtimes \sigma)\cdot w$ for some $w \in {}_kW$. Then we have $\nu(t) = \nu(w^{-1}\cdot t' \cdot w)$, where w is a representative of w fixed under σ, whence $t = v \cdot (w^{-1}\cdot t' \cdot w)$, with $v \in V$. We can write $v = s^{-1}\cdot {}^\sigma s$ for some $s \in {}^LT^\circ$, and get $t \rtimes \sigma = n^{-1}(t' \rtimes \sigma)n$, with $n = ws$.

6.5. LEMMA. *Let $({}^LG^\circ \rtimes \sigma)_{ss}$ be the set of semisimple elements in ${}^LG^\circ \rtimes \sigma$. Then the map*

$$\bar{\mu} : ({}^LT^\circ \rtimes \sigma)/\operatorname{Int}_k N \longrightarrow ({}^LG^\circ \times \sigma)_{ss}/\operatorname{Int} {}^LG^\circ,$$

induced by inclusion is a bijection.

By results of F. Gantmacher [**12**, Theorem 14], $\bar{\mu}$ is surjective. Let now $s, t \in {}^LT^\circ$ and $g \in {}^LG^\circ$ be such that $g^{-1}\cdot(s \rtimes \sigma)\cdot g = t \rtimes \sigma$, i.e., such that $g^{-1}\cdot s \cdot {}^\sigma g = t$. Using the Bruhat decomposition of ${}^LG^\circ$ with respect to ${}^LB^\circ$, we can write uniquely $g = u \cdot n \cdot v$, with u, v in the unipotent radical of ${}^LB^\circ$ and n in the normalizer N of ${}^LT^\circ$. These groups are stable under σ, and normalized by ${}^LT^\circ$. We have then

$$s \cdot {}^\sigma u \cdot {}^\sigma n \cdot {}^\sigma v = u \cdot n \cdot v \cdot t, \qquad (s \cdot {}^\sigma n \cdot s^{-1})\cdot s \cdot {}^\sigma n \cdot {}^\sigma v = u \cdot n \cdot t \cdot (t^{-1}\cdot v \cdot t);$$

hence ${}^\sigma n \cdot s^{-1} = n \cdot t$. Therefore the connected component of n in N is stable under

σ, i.e., n represents an element of ${}_kW$; hence $n \in {}_kN$, and $(t \rtimes \sigma)$ and $(s \rtimes \sigma)$ are conjugate under ${}_kN$.

REMARK. This proof was suggested to me by T. Springer.

6.6. If M is a complex affine variety, we let $\mathbf{C}[M]$ denote its coordinate algebra. The algebra $\mathbf{C}[Y]$ may be identified with the group algebra of $X^*(Y) = X_*(T_d)$. The quotient $Y/_kW$ is also an affine variety (in fact isomorphic to an affine space) and $\mathbf{C}[Y/_kW] = \mathbf{C}[Y]^{{}_kW}$.

Let $\text{Rep}({}^LG) \subset \mathbf{C}[{}^LG]$ be the subalgebra generated by the characters of finite dimensional holomorphic representations. Its elements are constant on conjugacy classes. In particular, they define by restriction functions on $({}^LG^\circ \rtimes \sigma)_{ss}/\text{Int}\,{}^LG^\circ$.

6.7. PROPOSITION. *The map*

$$\alpha = \bar{\mu} \circ \bar{\nu}^{-1} : Y/_kW \longrightarrow ({}^LG^\circ \rtimes \sigma)_{ss}/\text{Int}\,{}^LG^\circ$$

is a bijection, which induces an isomorphism of $\mathbf{C}[Y/_kW]$ *onto the algebra A of restrictions of elements of* $\text{Rep}({}^LG)$.

REMARK. We shall use 6.7 only when k is a nonarchimedean local field. In that case 6.7 is proved in [35, pp. 18–24].

PROOF. That α is bijective follows from 6.4, 6.5. We prove the second assertion as in [35]. Let ρ be a finite dimensional holomorphic representation of ${}^LG$ and f_ρ the function on ${}^LT^\circ$ defined by $f_\rho(t) = \text{tr}\,\rho(t \rtimes \sigma)$. It can be written as a finite linear combination $f = \sum c_\lambda \lambda$ of characters $\lambda \in X^*({}^LT^\circ)$. Since tr ρ is a class function on ${}^LG$, we have $f_\rho(s^{-1} \cdot t \cdot {}^\sigma s) = f_\rho(t)$ for all $s, t \in {}^LT^\circ$. By the linear independence of characters, it follows that if $c_\lambda \neq 0$, then λ is trivial on V (cf. 6.3(1)), hence is fixed under σ, i.e., may be identified to an element of $X^*(Y)$. Thus we may view f_ρ as an element of $\mathbf{C}[Y]$. But invariance by conjugation and 6.4 imply that $f \in \mathbf{C}[Y/_kW]$, whence a map $\beta : A \to \mathbf{C}[Y/_kW]$, which is obviously induced by α. There remains to see that β is surjective. Note that $\mathbf{C}[Y/_kW]$ is spanned, as a vector space, by the functions

$$(1) \qquad \varphi_\lambda = \sum_{w \in {}_kW} w \cdot \lambda,$$

where λ runs through a fundamental domain C of ${}_kW$ on $X_*(T_d)$. But it is standard that we may take for C the intersection of $X_*(T_d)$ with the Weyl chamber of W in $X_*(T)$ defined by B. Therefore every $\lambda \in C$ is a dominant weight for ${}^LG^\circ$ with respect to ${}^LT^\circ$. It is then the highest weight of an irreducible representation π_λ of ${}^LG^\circ$. Since it is fixed under σ, the representation ${}^\sigma\pi_\lambda : g \mapsto \pi_\lambda({}^\sigma g)$ is equivalent to π_λ. From this it is elementary that π_λ extends to an irreducible representation $\tilde{\pi}_\lambda$ of ${}^LG$ of the same degree as π_λ. The highest weight space is one-dimensional, stable under σ. Let c be the eigenvalue of σ on it. Then the trace gives rise to a function equal to $c \cdot \varphi_\lambda$ modulo a linear combination of functions φ_μ, with $\mu < \lambda$, in the usual ordering. That im β contains φ_λ ($\lambda \in C$) is then proved by induction on the ordering.

7. The Satake isomorphism and the L-group. Local factors.

7.1. We keep the previous notation and conventions. We assume moreover k to be a nonarchimedean local field, k' to be unramified over k, and σ to be the image of a Frobenius element Fr in Γ_k.

Let Q be a special maximal compact subgroup of $G(k)$ [61]. We assume $Q \cap T$ is the greatest compact subgroup of $T(k)$ and Q contains representatives of ${}_kW$. Let

$H(G(k), Q)$ be the Hecke algebra of locally constant, Q-bi-invariant, and compactly supported complex valued functions on $G(k)$. The Satake isomorphism provides a canonical identification $H \xrightarrow{\sim} \boldsymbol{C}[Y/_kW]$, hence also one of $Y/_kW$ with the characters of H **[6]**.

By 6.7, we have now a canonical isomorphism of H with the algebra A of restrictions of characters of finite dimensional representations of ${}^LG$ to semisimple ${}^LG^\circ$-conjugacy classes in $({}^LG^\circ \rtimes \sigma)$, hence also a canonical bijection between characters of $H(G(k), Q)$ and semisimple classes in ${}^LG^\circ \rtimes \sigma$. Furthermore, each such class can be represented by an element of the form (t, σ), with $t \in {}^LT^\circ$ fixed under σ (and is determined modulo the finite group $U \cap V$, in the notation of 6.3).

7.2. *Local factors.* Assume now that U is *hyperspecial* **[61]**. Let ψ be an additive character of k. Let (π, U_π) be an irreducible admissible representation of $G(k)$ of class 1 for Q and r a representation of ${}^LG$. Then the space of fixed vectors of Q in U_π is one-dimensional, acted upon by H via a character χ_π. To the latter is assigned by 7.1 a semisimple class S_χ in ${}^LG^\circ \rtimes \sigma$. We then put

$$(1) \qquad L(s, \pi, r) = \det(1 - r((g \rtimes \sigma)), q^{-s})^{-1}, \qquad \varepsilon(s, \pi, r, \psi) = 1,$$

where q is the order of the residue field, and (g, σ) any element of S_χ.

Chapter III. Weil Groups and Representations. Local Factors.

In this section, k is a local field, W_k (resp. W_k') the absolute Weil group (resp. Weil-Deligne group) of k. If H is a reductive k-group, then $\Pi(H(k))$ is the set of infinitesimal equivalence classes of irreducible admissible representations of $H(k)$.

G denotes a connected reductive k-group.

The main local problem is to define a *partition* of $\Pi(G(k))$ into finite sets $\Pi_{\varphi,G}$ or Π_φ indexed by the set $\Phi(G)$ of admissible homomorphisms of W_k' into ${}^LG$, modulo inner automorphisms (see §8 for $\Phi(G)$), and satisfying a certain number of conditions. So far, this has been carried out for any G if $k = \boldsymbol{R}, \boldsymbol{C}$ **[37]**, for tori over any k **[34]** and (essentially) for $G = \mathbf{GL}_2$ (cf. 12.2). §9 recalls the results for tori; §10 describes some of the conditions to be imposed on this parametrization; §11 summarizes the construction over $\boldsymbol{R}$ or $\boldsymbol{C}$. Such a parametrization would allow one to assign canonically local L- and ε-factors to any $\pi \in \Pi(G(k))$ and any complex representation of ${}^LG$. Two elements π, π' in the same set Π_φ would always have the same local factors, and are hence called *L-indistinguishable*. In the case of $\mathbf{GL}_n$ however, local factors have been defined in an a priori quite different way, so that the parametrization problem becomes subordinated to one concerning L- and ε-factors. This is discussed in §12.

8. Definition of $\Phi(G)$.

8.1. *Jordan decomposition in W_k'.* If $k = \boldsymbol{R}, \boldsymbol{C}$, then $W_k' = W_k$ and, by definition, every element of W_k is semisimple.

Let k be nonarchimedean. Then $x \in W_k'$ is said to be unipotent if and only if it belongs to G_a; the element x is semisimple if either $\varepsilon(x) \neq 0$ or x is in the inertia group. Here $\varepsilon: W_k' \to \boldsymbol{Z}$ is the canonical homomorphism $W_k' \to W_k \to k^* \to \boldsymbol{Z}$. Every element $x \in W_k'$ admits a unique Jordan decomposition $x = x_s \cdot x_u$ with x_s semisimple, x_u unipotent and $x_s x_u = x_u x_s$ **[60]**.

8.2. *The set* $\Phi(G)$. We consider homomorphisms $\alpha : W'_k \to {}^LG$ over Γ_k, i.e., such that the diagram

$$\begin{array}{ccc} W'_k & \xrightarrow{\alpha} & {}^LG \\ & \searrow \quad \swarrow & \\ & \Gamma_k & \end{array}$$

is commutative, and which satisfy moreover the following conditions:

(i) α is continuous, $\alpha(G_a)$ is unipotent, in ${}^LG^\circ$, and α maps semisimple elements into semisimple elements (in ${}^LG$: $x = (u, \gamma)$ is said to be semisimple if its image under any representation (2.6) is so).

(ii) If $\alpha(W'_k)$ is contained in a Levi subgroup of a parabolic subgroup P of ${}^LG$, then P is relevant (3.3).

Such α's are called admissible. We let $\Phi(G)$ be the set of their equivalence classes modulo inner automorphisms by elements of ${}^LG^\circ$.

If we write $\alpha(w) = (a(w), \nu(w))$ with $a(w) \in {}^LG^\circ$ then $w \mapsto a(w)$ is a 1-cocycle of W'_k (acting on ${}^LG^\circ$ via $W'_k \to \Gamma_k$) in ${}^LG^\circ$. It follows that

$$(1) \qquad \Phi(G) \subset H^1(W'_k;\ {}^LG^\circ).$$

Let H be a subgroup of W'_k. Then α: $W'_k \to {}^LG$ is said to be trivial on H if $\nu(H)$ acts trivially on ${}^LG^\circ$ and $\alpha(H) = \{1\}$. Note that if $\nu(H)$ acts trivially on ${}^LG^\circ$, then $a|_H$ is a homomorphism.

8.3. Assume G' is an inner quasi-split form of G. Then

$$(1) \qquad \Phi(G) \subset \Phi(G').$$

In fact ${}^LG \cong {}^LG'$ and ${}^Lp({}^LG') \supset {}^Lp({}^LG)$; therefore $\alpha \in \Phi(G) \Rightarrow \alpha \in \Phi(G')$.

8.4. Proposition. *Let k' be a finite separable extension of k; let G' be a connected reductive k-group and $G = R_{k'/k}G'$. Then there is a canonical bijection $\Phi(G) \xrightarrow{\sim} \Phi(G')$.*

We consider the situation of 5.2, 5.4 with $A = \Gamma_k$, $A' = \Gamma_{k'}$, $B = W'_k$, $B' = W'_{k'}$, $E = {}^LG'^\circ$. We have the injections (8.2):

$$\Phi(G) \subset H^1(W'_k;\ {}^LG^\circ), \qquad \Phi(G') \subset H^1(W'_{k'};\ {}^LG'^\circ).$$

Moreover ${}^LG^\circ = I_{\Gamma_{k'}}^{\Gamma_k}({}^LG'^\circ)$ (see 5.1); whence, by Shapiro's lemma and 5.2:

$$H^1(W'_k;\ {}^LG^\circ) \xrightarrow{\sim} H^1(W'_{k'};\ {}^LG'^\circ).$$

But it is clear that this isomorphism maps $\Phi(G)$ onto $\Phi(G')$.

8.5. Let $Z_L = C({}^LG^\circ)$. If a: $W_k \to Z_L$ and b: $W'_k \to {}^LG^\circ$ are 1-cocycles, then ab: $w \mapsto a(w)b(w)$ is again a 1-cocycle of W'_k in ${}^LG^\circ$. If a is continuous and b corresponds to $\varphi \in \Phi(G)$, then ab corresponds to an element of $\Phi(G)$. We get therefore maps

$$(1) \qquad H^1(W_k; Z_L) \times H^1(W'_k;\ {}^LG^\circ) \longrightarrow H^1(W'_k;\ {}^LG^\circ),$$

$$(2) \qquad H^1(W_k;\ Z_L) \times \Phi(G) \longrightarrow \Phi(G),$$

which define actions of the group $H^1(W_k; Z_L)$ on the sets $H^1(W'_k;\ {}^LG^\circ)$ and $\Phi(G)$.

8.6. PROPOSITION. *Let $\varphi: W_k' \to {}^LG$ be an admissible homomorphism. Then the Levi subgroups in ${}^LG$ which contain $\varphi(W_k')$ minimally form one conjugacy class with respect to the centralizer of $\varphi(W_k')$ in ${}^LG^\circ$.*

Since $\varphi(W_k')$ projects onto a dense subgroup of Γ_k by definition, this follows from 3.6.

REMARK. Formally, this also applies to the archimedean case, but the proof in that case is simpler [37, pp. 78–79]. In fact, the argument there applies in all cases to admissible homomorphisms of the Weil (rather than Weil-Deligne) group because $\varphi(W_k)$ is always fully reducible. In this case, the Levi subgroups which contain $\varphi(W_k)$ minimally are those of the parabolic subgroups which contain $\varphi(W_k)$ minimally. Those parabolic subgroups form therefore one class of associated groups.

9. The correspondence for tori.

9.1. Let T be a complex torus. A continuous homomorphism $\varphi: T \to C^*$ is described by a pair of elements $\lambda, \mu \in X^*(T) \otimes C$ such that $\lambda - \mu \in X^*(T)$, by the rule $\varphi(t) = t^\lambda \bar{t}^\mu$.

Similarly, a continuous homomorphism $\varphi: C^* \to T$ is given by $\mu, \nu \in X_*(T) \otimes C$ such that $\mu - \nu \in X_*(T)$; we have $\varphi(z) = z^\mu \bar{z}^\nu$, meaning that, for any $\lambda \in X^*(T)$, $\lambda \circ \varphi: C^* \to C^*$ is given by

$$\lambda(\varphi(z)) = z^{\langle\lambda, \mu\rangle} \bar{z}^{\langle\lambda, \nu\rangle}.$$

This can also be interpreted in the following way: identify $X_*(T) \otimes C$ with the Lie algebra $\mathrm{Lie}(T(C))$. Then the exponential map yields an isomorphism $(X_*(T) \otimes C)/2\pi i X_*(T) = T(C)$. Then $\mu, \nu \in \mathrm{Lie}(T(C))$ are such that $\varphi(e^h) = e^{h\cdot\mu + \bar{h}\cdot\mu}$ $(h \in C)$.

9.2. Let $G = T$ be a k-torus, and $l = \dim T$.

Any $\varphi \in \Phi(G)$ is trivial on G_a; hence

$$\Phi(G) = H^1_{\mathrm{ct}}(W_k;\, {}^LT^\circ) = H^1_{\mathrm{ct}}(W_k;\, X^*(T) \otimes C^*), \tag{1}$$

where H^1_{ct} refers to continuous cocycles.

On the other hand

$$\Pi(G) = \mathrm{Hom}((X_*(T) \otimes k_s^*)^{\Gamma_k}, C^*). \tag{2}$$

We have canonically [34, Theorem 1]

$$\Pi(G) = \Phi(G). \tag{3}$$

In fact, ${}^LT$ and W_k are replaced in [34] by a finite Galois form ${}^LT^\circ \rtimes \Gamma_{k'/k}$ and a relative Weil group $W_{k'/k}$, where k' is a finite Galois extension of k whose Galois group acts trivially on ${}^LT^\circ$; this is easily seen not to change $\Phi(G)$. The proof then consists in showing first that the transfer from $W_{k'/k}$ to k'^* yields an isomorphism

$$H_1(W_{k'/k}; X_*(T)) \xrightarrow{\sim} H_1(k'^*; X_*(T))^{\Gamma_{k'/k}} = (k'^* \otimes X_*(T))^{\Gamma_{k'/k}}, \tag{4}$$

and second that the pairing

$$H^1_{\mathrm{ct}}(W_{k'/k};\, {}^LT^\circ) \times H_1(W_{k'/k}; X_*(T)) \longrightarrow C^*, \tag{5}$$

associated to the evaluation map $(t, \lambda) \mapsto \lambda(t)$ $(t \in {}^LT^\circ;\ \lambda \in X_*(T))$ yields an isomorphism of the first group onto the group of characters of the second group, which is then (3) by definition.

For illustrations, we discuss some simple cases.

9.3. $k = \boldsymbol{C}$. Then $W_k = \boldsymbol{C}^*$ and $\Phi(G) = \mathrm{Hom}(\boldsymbol{C}^*, {}^LT^\circ)$. The correspondence follows from 9.1 since both $\mathrm{Hom}(\boldsymbol{C}^*, {}^LT^\circ)$ and $\mathrm{Hom}(T, \boldsymbol{C}^*)$ are canonically identified with $\{(\lambda, \mu) | \lambda, \mu \in X^*(T) \otimes \boldsymbol{C}, \lambda - \mu \in X^*(T)\}$.

9.4. $k = \boldsymbol{R}$. We have

$$(1) \qquad W_{\boldsymbol{R}} = \boldsymbol{C}^* \rtimes \{\tau\} \quad \text{with } \tau^2 = -1, \ \tau \cdot z \cdot \tau^{-1} = \bar{z} \ (z \in \boldsymbol{C}^*).$$

Put $\boldsymbol{C}^* = S \times \boldsymbol{R}^+$, with $S = \{z \in \boldsymbol{C}^*, z \cdot \bar{z} = 1\}$. Then $\mathrm{Int}\,\tau$ is the identity on $\boldsymbol{R}^+$, the inversion on S.

Write $\varphi(\tau) = (a, \sigma)$, where a is determined modulo σ-conjugacy, hence may be assumed to be fixed under σ (6.3(3)). We have then $\varphi(-1) = a^2$. Let μ, ν be the elements of $X^*(T) \otimes \boldsymbol{C}$ such that

$$(2) \qquad \varphi(z) = z^\mu \cdot \bar{z}^\nu \qquad (z \in \boldsymbol{C}^*), \ \mu - \nu \in X^*(T),$$

(see 9.1). We have $\varphi(\bar{z}) = \sigma(\varphi(z))$ $(z \in \boldsymbol{C}^*)$; hence $\nu = \sigma(\mu)$. Fix $h \in X^*(T) \otimes \boldsymbol{C}$ such that $a = \exp 2\pi i h$. Then the character π associated to φ is given by

$$(3) \qquad \pi(e^x) = \exp(\langle h, x - \sigma \cdot \bar{x} \rangle) \cdot \exp(\langle \mu, x + \sigma \cdot x \rangle)/2 \quad (x \in X_*(T) \otimes \boldsymbol{C})$$

[**37**, p. 27]. Here $^-$ denotes the complex conjugation of $\mathrm{Lie}(T(\boldsymbol{C})) = X_*(T) \otimes \boldsymbol{C}$ with respect to $X_*(T) \otimes \boldsymbol{R}$; hence $x \mapsto \sigma \cdot \bar{x}$ is the complex conjugation with respect to $\mathrm{Lie}(T(\boldsymbol{R}))$. It follows that $e^x \in T(\boldsymbol{R})$ if and only if $x - \sigma \cdot \bar{x} \in 2\pi i \cdot X_*(T)$.

EXAMPLES. (a) Let T be anisotropic over $\boldsymbol{R}$. Then $\sigma = -1$ and we may assume $a = 1, h = 0$. We have $e^x \in T(\boldsymbol{R})$ if x is purely imaginary and then (3) yields $\pi = \mu$. The fact that $\varphi(-1) = 1$ shows that $\mu \in X^*(T)$, confirming that $\Pi(T(\boldsymbol{R})) = X^*(T)$.

(b) Let T be split over $\boldsymbol{R}$. Then $\sigma = 1$, $\mu = \nu$, $\varphi(z) = (z \cdot \bar{z})^\mu$, $a^2 = 1$ and $h \in X^*(T)/2$. We have $e^x \in T(\boldsymbol{R})$ if and only if $x - \bar{x} \in 2\pi i \cdot X^*(T)$. It is then easily checked that π is given by μ on the connected identity component of $T(\boldsymbol{R})$, while its restriction to the torsion subgroup of $T(\boldsymbol{R})$ is the character naturally defined by h.

9.5. *The unramified case.* Let k be nonarchimedean, and assume T to split over an unramified extension k' of k. A character χ of $T(k)$ is said to be unramified if it is trivial on the greatest compact subgroup ${}^0T(k)$ of $T(k)$. On the other hand, $\varphi \in \Phi(T)$ is unramified if it is trivial (see 6.2) on the inertia group. The bijection $\Phi(T) \xrightarrow{\sim} \Pi(T)$ induces a bijection between the sets $\Phi_{\mathrm{unr}}(T)$ and $\Pi_{\mathrm{unr}}(T)$ of unramified elements [**34**]. In view of its importance, we describe it in more detail.

Given $t \in T(k')$, let $v(t) \in \mathrm{Hom}(X^*(T), \boldsymbol{Z})$ be defined by $v(t)(m) = \mathrm{ord}\, m(t)$ $(m \in X^*(T))$. It is well known, and easily deduced from Hilbert's Theorem 90, that $H^1(\Gamma_{k'/k}; o_{k'}^*) = 0$, where $o_{k'}^*$ is the group of units in the ring $o_{k'}$ of integers of k'. Since T splits over k', it follows that $H^1(\Gamma_{k'/k}; {}^0T(k')) = 0$. By Galois cohomology, this implies that $(T(k')/{}^0T(k'))^{\Gamma_k} = T(k)/{}^0T(k)$, therefore $t \mapsto v(t)$ yields a bijection

$$(1) \qquad T(k)/{}^0T(k) \xrightarrow{\sim} \mathrm{Hom}(X^*(T), \boldsymbol{Z})^{\Gamma_k} = X_*(T)^{\Gamma_k} = X_*(T_d),$$

where T_d is the greatest k-split torus of T (this can also be expressed by saying that the inclusion $T_d \subset T$ induces an isomorphism $T_d(k)/{}^0T_d(k) \xrightarrow{\sim} T(k)/{}^0T(k)$, cf. [6]). We have then

$$\Pi_{\text{unr}}(T(k)) = \operatorname{Hom}(X_*(T)^{\Gamma_k}, C^*) = Y. \tag{2}$$

The group Γ_k operates on ${}^LT^\circ$ via the cyclic group $\Gamma_{k'/k}$ which is generated by the image σ of a Frobenius element Fr. An unramified φ is completely determined by $\varphi(\text{Fr})$, which can be written $\varphi(\text{Fr}) = (t, \text{Fr})$, where $t \in {}^LT^\circ$ is determined up to conjugacy by ${}^LT^\circ$. Thus $\Phi_{\text{unr}}(T) = ({}^LT^\circ \rtimes \sigma)/\text{Int}\, {}^LT^\circ$; and elementary special case of 6.4 provides a canonical isomorphsim of the latter set onto Y, whence the desired isomorphism.

10. Desiderata. In order to formulate them, we need two preliminary constructions.

10.1. *The character χ_φ of $C(G)$ associated to $\varphi \in \Phi(G)$ (cf.* [37, *pp.* 20–34]). We want to associate canonically to $\varphi \in \Phi(G)$ a character of the center $C(G)$ of G. Let $\mathfrak{G}_{\text{rad}}$ be the greatest central torus of $\mathfrak{G}$. Then $G_{\text{rad}} \to G$ yields a surjective homomorphism ${}^LG \to {}^LG_{\text{rad}}$, whence a map $\Phi(G) \to \Phi(G_{\text{rad}})$. In view of 7.2, this allows us to associate to $\varphi \in \Phi(G)$ a character χ_φ of G_{rad}. Thus, if $C(G(k)) \subset G_{\text{rad}}(k)$, our problem is solved.

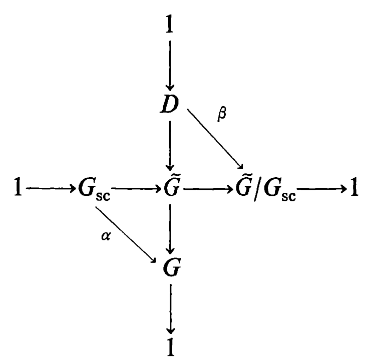
In the general case, G is enlarged to a bigger connected reductive G_1 generated by G and a central torus, whose center is a torus. One shows that $\Phi(G_1) \to \Phi(G)$ is surjective. Using the previous step, we get a character of $C(G_1)$, hence one of $C(G)$ by restriction. It is shown to be independent of the choice of G_1 (loc. cit.), and is χ_φ by definition.

The map $\varphi \mapsto \chi_\varphi$ is compatible with restriction of scalars [37, 2.11].

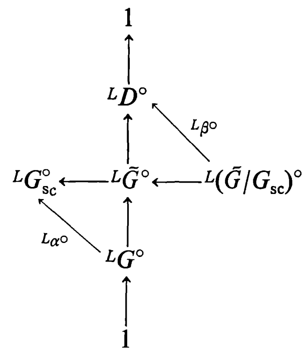
10.2. *The character π_α associated to $\alpha \in H^1(W_k'; Z_L)$* [37, *pp.* 34–36]. We recall that Z_L denotes the center of ${}^LG^\circ$ (8.5). We can always find a k-torus D such that $H^1(\Gamma_k; D) = 0$, and a k-group $\tilde{G}$ isogeneous to $G \times D$ such that there is an exact sequence

$$1 \longrightarrow D \longrightarrow \tilde{G} \longrightarrow G \longrightarrow 1. \tag{1}$$

Since $H^1(\Gamma_k; D) = 0$, the map $\mu: \tilde{G}(k) \to G(k)$ is surjective. Let G_{sc} be the universal covering of the derived group $\mathscr{D}G$ of G. We have a commutative diagram



Going over to L-groups, we get



Since $^LG_{sc}^\circ$ is of adjoint type (2.2), we see that $Z_L = \ker {}^L\alpha^\circ$. Moreover, it is easily seen that

(2) $$Z_L \cong \ker {}^L\beta^\circ.$$

This yields a map

(3) $$H^1(W_k'; Z_L) \longrightarrow \ker\{H^1(W_k'; {}^L(\tilde{G}/G_{sc})^\circ) \longrightarrow H^1(W_k'; {}^LD^\circ)\}.$$

This allows us to associate to $\alpha \in H^1(W_k'; Z_L)$ a character σ_α of $(\tilde{G}/G_{sc})(k)$ which is trivial on $D(k)$, hence a character π_α of $G(k) = \tilde{G}(k)/D(k)$. It can be shown to be independent of the choice of D. The map $\alpha \mapsto \pi_\alpha$ is compatible with restriction of scalars [**37**, 2.12] and satisfies:

(4) $$\chi_{\alpha\varphi} = \pi_\alpha \cdot \chi_\varphi \qquad (\alpha \in H^1(W_k'; Z_L),\ \varphi \in \Phi(G)).$$

10.3. *Conditions on the sets Π_φ*. (1) If $\pi \in \Pi_\varphi$, then $\pi(z) = \chi_\varphi(z)\cdot \mathrm{Id}$ $(z \in C(G))$.

(2) If $\varphi' = \alpha\cdot\varphi$ $(\varphi, \varphi' \in \Phi(G), \alpha \in H^1(W_k'; Z_L))$ (see 6.5), then $\Pi_{\varphi'} = \{\pi_\alpha \otimes \pi \mid \pi \in \Pi_\varphi\}$.

(3) The following conditions on a set Π_φ are equivalent:

(i) One element of Π_φ is square-integrable modulo $C(G)$.

(ii) All elements of Π_φ are square-integrable modulo $C(G)$.

(iii) $\varphi(W_k')$ is not contained in any proper Levi subgroup in ^LG.

(4) Assume $\varphi(G_a) = \{1\}$. The following conditions on a set Π_φ are equivalent:

(i) One element of Π_φ is tempered.

(ii) All elements of Π_φ are tempered.

(iii) $\varphi(W_k)$ is bounded.

(5) Let H be a connected reductive k-group and $\eta: H \to G$ a k-morphism with commutative kernel and cokernel. Let $\varphi \in \Phi(G)$ and $\varphi' = {}^L\eta \circ \varphi$. Then any $\pi \in \Pi_\varphi$, viewed as an $H(k)$-module, is the direct sum of finitely many irreducible admissible representations belonging to $\Pi_{\varphi'}$.

10.4. *The unramified case*. We say that $\varphi \in \Phi(G)$ is unramified if it is trivial, in the sense of 6.2, on G_a and on the inertia group I. If so, Im φ may be assumed to be in ^LT. Therefore, if $\Phi(G)$ contains an unramified element, then G is quasi-split (see 8.2 (ii)).

Assume now G to be quasi-split, to split over an unramified Galois extension

k' of k, and let $\varphi \in \Phi(G)$ be unramified. There exists $t \in ({}^LT^\circ)^{\Gamma_k}$ such that

(1) $$\varphi(\mathrm{Fr}) = (t, \mathrm{Fr}),$$

(9.5) and we have

(2) $$\varphi(w) = (t, \mathrm{Fr})^{\varepsilon(w)} \qquad (w \in W'_k),$$

where $\varepsilon : W'_k \to \boldsymbol{Z}$ is the canonical homomorphism. The element t defines an unramified character χ of a maximal k-torus T of a Borel k-subgroup B of G (9.5). It is then required that Π_φ consists of the constituents of the unramified normalized principal series $\mathrm{PS}(\chi)$ which have a nonzero vector fixed under some hyperspecial maximal compact subgroup. Conversely let (π, V) be an irreducible admissible representation with a nonzero vector fixed under some hyperspecial maximal compact subgroup. There exists then an unramified character χ of T such that (π, V) is a constituent of $\mathrm{PS}(\chi)$ (and χ is determined modulo the relative Weyl group). We have then $(\pi, V) \in \Pi_\varphi$, for the unramified φ which maps Fr to (t, Fr), where t represents χ (9.5). Note that if U is a special maximal compact subgroup of $G(k)$, then $G(k) = B(k) \cdot U$; hence the fixed-point set of U in $\mathrm{PS}(\chi)$ is at most one-dimensional. It follows that $\mathrm{PS}(\chi)$ has at most one irreducible constituent with nonzero fixed vectors under U.

This assignment is consistent with 7.2. Namely, if $\pi \in \Pi_\varphi$, then the semisimple class S_χ in ${}^LG^\circ \rtimes \sigma$ corresponding to the character of the Hecke algebra defined by π is indeed represented by $t \rtimes \sigma$. This follows from **[6]**.

REMARK. Originally, it was thought that Π_φ should consist of those constituents of $\mathrm{PS}(\chi)$ which had a nonzero fixed vector under some special maximal compact subgroup. However it was pointed out during the Institute by I. Macdonald that such representations may belong to the discrete series. If so, this condition would contradict 10.3(3). Upon a suggestion of J. Tits, this has led to the restriction to hyperspecial maximal compact subgroups made above. Those cannot belong to the discrete series, so that 10.3(3) and 10.4 are consistent.

10.5. EXAMPLE. Assume that $k = \boldsymbol{R}$ and that G is semisimple, possesses a Cartan subgroup T which is anisotropic over $\boldsymbol{R}$, and is an inner form of a split group. Then ${}^LG$ is the direct product of ${}^LG^\circ$ and Γ_k, the Weyl group W contains $-\mathrm{Id}$ and $G(\boldsymbol{R})$ has a discrete series. We want to describe the parametrization of the latter in terms of $\Phi(G)$. As the notation implies, we shall view ${}^LT$ as the L-group of T. Let $\varphi \in \Phi(G)$. It is given by a continuous homomorphism $\varphi' : W_{\boldsymbol{R}} \to {}^LG^\circ$. We may assume that $\mathrm{Im}\, \varphi'$ is contained in the normalizer of ${}^LT^\circ$. Let $n = \varphi(\tau)$ and let $w \in W$ be the element of W represented by n. Then $w^2 = 1$. Let $\mu, \nu \in X^*(T) \otimes \boldsymbol{C}$ be such that

(1) $$\varphi(z) = z^\mu \cdot \bar{z}^\nu \qquad (z \in \boldsymbol{C}^*),\ \mu - \nu \in X^*(T)$$

(see 9.1). We have

(2) $$\varphi(\bar{z}) = n \cdot \varphi(z) \cdot n^{-1} = z^{w \cdot \mu} \cdot \bar{z}^{w \cdot \nu};$$

hence $\mu = w \cdot \nu$, $\nu = w \cdot \mu$. Assume now that $\mathrm{Im}\, \varphi$ is not contained in any proper Levi subgroup in ${}^LG$ or, equivalently, that $\mathrm{Im}\, \varphi'$ is not contained in any proper Levi subgroup in ${}^LG^\circ$. Then $w = -\mathrm{Id}$ and μ is regular: in fact, the proper Levi subgroups in ${}^LG^\circ$ are the centralizers of nontrivial tori. This implies first that w does not fix pointwise any nontrivial torus in ${}^LT^\circ$, hence $w = -\operatorname{Id}$; if now μ were singular, then the centralizer of $\mu(\boldsymbol{C}^*)$ would contain a semisimple subgroup $H \neq \{1\}$ stable under Int n, the latter would leave pointwise fixed a torus $S \neq \{1\}$ of H, and Im φ' would be contained in the centralizer $Z(S)$ of S, a contradiction. Since $\nu = w \cdot \mu = -\mu$, we have

$$\lambda(\varphi(-1)) = (-1)^{\langle 2\mu, \lambda\rangle}, \quad \text{for all } \lambda \in X_*(T). \tag{3}$$

Let δ be half the sum of the roots α of G with respect to T such that $\langle \mu, \check{\alpha}\rangle > 0$. Then, Lemma 3.2 of **[37]** implies in particular

$$\lambda(\varphi'(-1)) = (-1)^{\langle 2\delta, \lambda\rangle}, \quad \text{for all } \lambda \in X_*(T). \tag{4}$$

It follows that

$$\mu \in \delta + X^*(T). \tag{5}$$

Therefore μ is among the elements of $X^*(T) \otimes \boldsymbol{Q}$ which parametrize the discrete series in Harish-Chandra's theorem. We then let Π_φ be the set of discrete series representations of $G(\boldsymbol{R})$ with infinitesimal character χ_μ. If $G(\boldsymbol{R})$ is compact, then Π_φ consists of the irreducible finite dimensional representation with dominant weight $\mu - \delta$. In that case, no proper parabolic subgroup of ${}^LG$ is relevant; hence $\Phi(G)$ consists of the φ considered here.

10.6. *Let* $G = \mathbf{GL}_n$, *k nonarchimedean.* Let ψ be an admissible representation of W_k'. If it is irreducible, then $\psi(G_a) = 1$. If it is indecomposable, then it is a tensor product $\rho \otimes \operatorname{sp}(m)$, where m divides n, ρ is irreducible of degree n/m, and $\operatorname{sp}(m)$ is m-dimensional, trivial on 1, maps a generator of the Lie algebra of G_a onto the nilpotent matrix with ones above the diagonal, zero elsewhere, and $w \in W_k'$ onto the diagonal matrix with entries $a(w)^i$ $(0 \leqq i < n)$ **[9**, 3.1.3]. If χ is a character of W_k (hence of k^*), and $\varphi = \chi \otimes \operatorname{sp}(n)$, then Π_φ consists of the special representation with central character determined by χ. In fact, the Weil-Deligne group came up for the first time precisely to fit the special representations of $\mathbf{GL}_2$ into the general scheme (see **[9]**).

11. Outline of the construction over *R*, *C*. We sketch here the various steps which yield the sets Π_φ when $k = \boldsymbol{R}$. For the proofs see **[37]**.

We note first that we may always assume $\varphi(W_k) \subset N({}^LT^\circ)$, and we can write (9.1)

$$\varphi(z) = z^\mu \bar{z}^\nu \qquad (z \in \boldsymbol{C}^*;\ \mu, \nu \in X^*(T) \otimes \boldsymbol{C},\ \mu - \nu \in X^*(T)).$$

11.1. Lemma. *Let* $\varphi \in \Phi(G)$. *Assume* $\varphi(W_{\boldsymbol{R}})$ *is not contained in any proper Levi subgroup in* ${}^LG$. *Then*

(i) *G has a Cartan k-subgroup C such that* $(\mathscr{D}G \cap S)(\boldsymbol{R})$ *is compact* **[28**, 3.1].

(ii) μ *is regular*; $\varphi(\boldsymbol{C}^*)$ *contains regular elements* **[37**, 3.3].

The group ${}^LC^\circ$ may be viewed as a maximal torus of ${}^LG^\circ$; hence there is an isomorphism ${}^LC \to^\sim {}^LT$ defined modulo an element of W. Therefore φ defines an orbit of W in $\Phi(C)$, hence, by 9.2, an orbit X_φ of W in $X(C(\boldsymbol{R}))$. [Note that W, which is defined in $G(\boldsymbol{C})$, operates on $C(\boldsymbol{R})$, since $C(\boldsymbol{R}) \cap \mathscr{D}G$ is compact, hence on $X(C(\boldsymbol{R}))$.]

11.2. Let $G_0 = C(\boldsymbol{R})((\mathscr{D}G)(\boldsymbol{R}))^\circ$. Let A_0 be the set of representations of G_0 which are square-integrable modulo the center, and have infinitesimal character $\chi_\lambda (\lambda \in X_\varphi)$. The induced representations $\pi = I_{G_0(\mathbf{R})}^{G(\mathbf{R})}(\pi_0)$ $(\pi_0 \in A_0)$ are irreducible [37, p. 50]. By definition, Π_φ is the set of equivalence classes of these representations [37, p. 54].

11.3. Let $\varphi \in \Phi(G)$. Let ${}^LM$ be a minimal relevant Levi subgroup containing Im φ. It is essentially unique (8.6). We assume ${}^LM \neq {}^LG$; we may view φ as an element of $\Phi(M)$. By 11.2, there is associated to it a finite set of $\Pi_{\varphi,M}$ of discrete series representations of M.

We may assume ${}^LM$ to be a Levi subgroup of a relevant parabolic subgroup ${}^LP$ corresponding to $P \in \mathscr{P}(G/k)$. Then $U = X^*(T) \otimes \mathbf{R} = X_*({}^LT^\circ) \otimes \mathbf{R}$. Let V be the subspace of elements of U which are orthogonal to roots of ${}^LM$, and fixed under Γ_k. It may be identified with the dual $\mathfrak{a}_P^*$ of the Lie algebra of a split component A of P.

Let ξ be the character of $C(M)$ defined by the elements of $\Pi_{\varphi,M}$. We may assume that $|\xi| \in \mathrm{Cl}(\mathfrak{a}_P^{*+})$. Let P_1 be the smallest parabolic k-subgroup containing P such that $|\xi|$, when restricted to $\mathfrak{a}_{P_1}$, is an element of the Weyl chamber $\mathfrak{a}_{P_1}^{*+}$. Let $M_1 = z(\mathfrak{a}_{P_1})$ and $P' = P \cap M_1$. Then P' is a parabolic subgroup of M_1. Moreover the restriction of $|\xi|$ to the split component $M_1 \cap A_P$ of P' is one; therefore, for each $\rho \in \Pi_{\varphi,M}$, the induced representation $\mathrm{Ind}_{P_1}^{M_1}(\rho)$ is tempered. Let Π'_φ be the set of all constituents of such representations. Then by definition, Π_φ is the set of Langlands quotients $J(P_1, \sigma)$ with $\sigma \in \Pi'_\varphi$ (cf. [37, p. 82]).

11.4. *Complex groups.* Assume now $k = \mathbf{C}$. Then $W_k = \mathbf{C}^*$, and $\Phi(G)$ may be identified to the set of homomorphisms of $\mathbf{C}^*$ into ${}^LT^\circ$, modulo the Weyl group W, i.e., to

$$(1) \qquad \{(\lambda, \mu), \text{ where } \lambda, \mu \in X^*(T) \otimes \mathbf{C}, \lambda - \mu \in X^*(T)\}$$

modulo the (diagonal) action of W. In this case Im φ is in the Levi subgroup ${}^LT$ of ${}^LB$, which is the ${}^LP$ of 11.3. The set $\Pi_{\varphi,M}$ consists of one character of T (cf. 9.1). Choose P_1, M, as in 11.3. Since the unitary principal series of a complex group are irreducible (N. Wallach), the set Π'_φ consists of one element. Hence so does Π_φ. Thus each Π_φ is a singleton. The classification thus obtained is equivalent to that of Zelovenko.

11.5. Let $G = \mathbf{GL}_n$, $k = \mathbf{R}$. In this case, it is also true that the tempered representations induced from discrete series are irreducible [22]; therefore each set Π'_φ (cf. 9.3) consists of only one element, hence so does Π_φ and we get a bijection between $\Phi(G)$ and $\Pi G(\mathbf{R})$.

Let $n = 2$. If φ is reducible, then Im φ is commutative; hence φ factors through $(W_{\mathbf{R}})^{\mathrm{ab}} = \mathbf{R}^*$ and is described by two characters μ, ν of $\mathbf{R}^*$. Then Π_φ consists of a principal series representation $\pi(\mu, \nu)$ (including finite dimensional representations, as usual). In particular there are three φ's with kernel $\mathbf{C}^*$, to which correspond respectively $\pi(1, 1)$, $\pi(\mathrm{sgn}, \mathrm{sgn})$ and $\pi(1, \mathrm{sgn})$, where sgn is the sign character. If φ is irreducible, then $\varphi(\tau)$ may be assumed to be equal to (s_0, τ), where s_0 is a fixed element of the normalizer of ${}^LT^\circ$ inducing the inversion on it. $\varphi(\mathbf{R}^+)$ belongs to the center of ${}^LG^\circ$, and $\varphi(S)$ is sum of two characters, described by two integers. Then Π_φ consists of a discrete series representation, twisted by a one-dimensional representation.

11.6. As is clear from these two examples, the main point to get explicit knowledge of the sets Π_φ is the decomposition of representations induced from tempered representations of parabolic subgroups. This last problem has been solved by A. Knapp and G. Zuckerman [29], [30].

11.7. *Remark on the nonarchimedean case*. Langlands' classification [37] is also valid over p-adic fields [57]. In view of 8.6, it is then clear that the last step (11.3) of the previous construction can also be carried out in the nonarchimedean case. Thus, besides the decomposition of tempered representations, the main unsolved problem in the p-adic case is the construction and parametrization of the discrete series.

12. Local factors.

12.1. Let $\pi \in \Pi(G(k))$ and r be a representation of ${}^LG$ (2.6). Assume that $\pi \in \Pi_\varphi$ for some $\varphi \in \Phi(G)$. For a nontrivial additive character ψ of k, we let

$$(1) \qquad L(s, \pi, r) = L(s, r \circ \varphi), \qquad \varepsilon(s, \pi, r) = \varepsilon(s, \pi, r, \psi) = \varepsilon(s, r \circ \varphi, \psi),$$

where on the right-hand sides we have the L- and ε-factors assigned to the representation $r \circ \varphi$ of W_k' [60]. In the unramified situation of 10.4, this coincides with the definition given in 7.2.

In view of what has been recalled so far, these local factors are defined if k is archimedean, or if k is nonarchimedean in the unramified case, or if G is a torus.

12.2. Let now $G = \mathbf{GL}_n$. In this case there are associated to $\pi \in \Pi(G(k))$ local factors $L(s, \pi)$ and $\varepsilon(s, \pi, \psi)$ defined by a generalization of Tate's method, in [25] for $n = 2$, in [19] for any n, which play a considerable role in the parametrization problem and in the local lifting. A natural question is then whether these factors can be viewed as special cases of 12.1, where $r = r_n$ is the standard representation of $\mathbf{GL}_n$, i.e., whether we have equalities

$$(1) \qquad L(s, \pi) = L(s, \pi, r_n), \qquad \varepsilon(s, \pi, \psi) = \varepsilon(s, \pi, r_n, \psi),$$

with the right-hand side defined by the rule of 12.1.

(a) Let $n = 2$. It has been shown in [25] that the equivalence class of π is characterized by the functions $L(s, \pi \otimes \chi)$, $\varepsilon(s, \pi \otimes \chi, \psi)$, where χ varies through the characters of k^*. In this case, the parametrization problem and the proof of (1) are part of the following problem:

(*) Given $\sigma \in \Phi(G)$, find $\pi = \pi(\sigma)$ such that

$$(2) \qquad L(s, \sigma \otimes \chi) = L(s, \pi \otimes \chi), \qquad \varepsilon(s, \sigma \otimes \chi, \psi) = \varepsilon(s, \pi \otimes \chi, \psi)$$

for all χ's, and prove that $\sigma \mapsto \pi(\sigma)$ establishes a bijection between $\Phi(G)$ and $\Pi(G(k))$.

This problem was stated and partially solved in [25]. The most recent and most complete results in preprint form are in [62]; they still leave out some cases of even residual characteristic, although some arguments sketched by Deligne might take care of them (see [63] for a survey).

As stated, the problem is local, but, except at infinity, progress was achieved first mostly by global methods: one uses a global field E whose completion at some place v is k, a reductive E-group H isomorphic to G over k, an element $\rho \in \Phi(H/k)$ whose restriction to ${}^L(H/k_v) = {}^LG$ is σ, chosen so that there exists an automorphic representation $\pi(\rho)$ with the L-series $L(s, \rho)$ (see §14 for the latter). This construction relies, among other things, on Artin's conjecture in some cases, and **[38]**. In fact, it was already shown in **[25]** that (*) for odd residual characteristics follows from Artin's conjecture, leading to a proof in the equal characteristic case. At present, there are in principle purely local proofs in the odd residue characteristic case **[63]**. Note also that the injectivity assertion is a statement on two-dimensional admissible representations of W_k', namely, whether such a representation σ is determined, up to equivalence, by the factors $L(s, \sigma \circ \chi)$ and $\varepsilon(s, \sigma \circ \chi, \psi)$. But, so far, the known proofs all use admissible representations of reductive groups **[63]**.

(b) For arbitrary n, (1) has been proved in the unramified case, for special representations, and by H. Jacquet for $k = \boldsymbol{R}, \boldsymbol{C}$ **[24]**.

(c) Local L- and ε-factors are also introduced for $G = \mathbf{GL}_2 \times \mathbf{GL}_2$ in **[21]**, at any rate for products $\pi \times \pi'$ of infinite dimensional irreducible representations. Partial extensions of this to $\mathbf{GL}_m \times \mathbf{GL}_n$ for other values of m, n are known to experts.

(d) For $n = 3$, $\pi \in \Pi(G(k))$ is again characterized uniquely by the factors $L(s, \pi \otimes \chi)$ and $\varepsilon(s, \pi \otimes \chi, \psi)$ **[27]**, **[46]**. For $n \geqq 4$ on, this is false **[46]**. However, it may be there are still such characterizations if χ is allowed to run through suitable elements of $\Pi(\mathbf{GL}_{n-1}(k))$ or maybe just $\Pi(\mathbf{GL}_{n-2}(k))$.

12.3. Local factors have also been defined directly for some other classical groups, in particular for $\mathbf{GSp}_4$ by F. Rodier **[48]**, extending earlier work of M. E. Novodvorsky and I. Piatetskii-Shapiro, for split orthogonal groups, in an odd number $2n + 1$ of variables by M. E. Novodvorsky **[41]**. In the latter case ${}^LG^\circ = \mathbf{Sp}_{2n}$, and in the unramified case, the local factors coincide (up to a translation in s) with those associated by 7.2 to the standard $2n$-dimensional representation of the L-group. See also **[42]**.

Chapter IV. The L-Function of an Automorphic Representation.

From now on, k is a global field, $\mathfrak{o} = \mathfrak{o}_k$ the ring of integers of k, A_k or A the ring of adeles of k, V (resp. V_∞, resp. V_f) the set of places (resp. infinite places, resp. finite places) of V. For $v \in V$, k_v, o_v and Nv have the usual meaning. Unless otherwise stated, G is a connected reductive k-group.

13. The L-function of an irreducible admissible representation of G_A.

13.1. Let π be an irreducible admissible representation of G_A and r a representation of ${}^LG$. There exists a finite Galois extension k' of k over which G splits and such that r factors through ${}^LG^\circ \rtimes \Gamma_{k'/k}$. We want to associate to π and r infinite Euler products $L(s, \pi, r)$ and $\varepsilon(s, \pi, r)$, whose factors are defined (at least) for almost all places of k.

Let $v \in V$. By restriction, r defines a representation r_v of $L(G/k_v) = {}^LG^\circ \rtimes \Gamma_k$. On the other hand, $\pi = \otimes_v \pi_v$, with $\pi_v \in \Pi(G(k_v))$ **[11]**. Assume the parametrization problem of Chapter III solved. Then there is a unique $\varphi_v \in \Phi(G/k_v)$ such that $\pi_v \in \Pi_{\varphi_v}$. Then we let

$$L(s, \pi, r) = \Pi_v L(s, \pi_v, r_v), \tag{1}$$

$$\varepsilon(s, \pi, r) = \Pi_v \varepsilon(s, \pi_v, r_v, \psi_v), \tag{2}$$

where ψ_v is an additive character of k_v associated to a given nontrivial additive character of k, and the factors on the right are given by 12.1(1).

The local problem is solved for archimedean v's, and for almost all finite v's (see below) so that the factors on the right are defined except for at most finitely many $v \in V_f$. For questions of convergence or meromorphic analytic continuation this does not matter, and we shall also denote such partial products by $L(s, \pi, r)$.

By 10.4, φ_v is well defined if the following conditions are fulfilled: G is quasi-split over k_v, $G(o_v)$ is a very special maximal compact subgroup of $G(k_v)$, k' is unramified over k, and π_v is of class one with respect to $G(o_v)$. All but finitely many $v \in V_f$ satisfy those conditions **[61]**.

13.2. THEOREM **[35]**. *Let π be an irreducible admissible unitarizable representation of G_A and r be a representation of ${}^LG$ (2.6). Then $L(s, \pi, r)$ converges absolutely for* Re s *sufficiently large.*

We may and do view r as a complex analytic representation of ${}^LG^\circ \rtimes \Gamma_{k'/k}$, where k' is a finite Galois extension of k over which G splits (2.7). We let V_1 be the set of $v \in V_f$ satisfying the conditions listed at the end of 13.1. We have to show that

(1) $$L' = \prod_{v \in V_1} L(s, \pi_v, r_v),$$

converges in some right half-plane.

Let Fr_v be the Frobenius element of $\Gamma_{k'_{v'}/k_v}$, where $v' \in V_{k'}$ lies over $v \in V_1$. We have

(2) $$\varphi_v(\mathrm{Fr}_v) = (t_v, \mathrm{Fr}_v), \quad \text{with } t_v \in {}^LT^\circ$$

and

(3) $$L(s, \pi_v, r_v) = (\det(1 - r((t_v, \mathrm{Fr}_v))N_v^{-s}))^{-1}.$$

To prove the theorem, it suffices therefore to show the existence of a constant $a > 0$ such that

(4) $$|\mu| \leqq (Nv)^a \quad \text{for every } v \in V_1 \text{ and eigenvalue } \mu \text{ of } r((t_v, \mathrm{Fr}_v)).$$

Let $n = [k' : k]$. Since we may assume t_v fixed under Γ_{k_v} (6.3), we have $t_v^n = (t_v, \mathrm{Fr}_v)^n$; hence it is equivalent to show (4) for all eigenvalues μ of $r(t_v)$. These are of the form t_v^λ, where λ runs through the set P_r of weights of r, restricted to ${}^LG^\circ$. Thus we have to show the existence of $a > 0$ such that

(5) $$|t_v|^{\mathrm{Re}\,\lambda} \leqq (Nv)^a \quad \text{for all } v \in V_1 \text{ and } \lambda \in P_r.$$

Let G' be a quasi-split inner k-form of G. Then ${}^LG = {}^LG'$, and G is isomorphic to G' over k_v for all $v \in V_1$. We may therefore replace G by G'; changing the notation slightly, we may (and do) assume G to be quasi-split over k. We then fix a Borel k-subgroup B of G and view ${}^LT$ as the L-group of a maximal k-torus T of G.

For a cyclic subgroup D of $\Gamma_{k'/k}$, let V_D be the set of $v \in V_1$ for which Γ_{k_v} is equal to the inverse image of D in Γ_k. The group $U = X_*(T)^D$ is then the group of one-parameter subgroups of a subtorus S of T such that S/k_v is a maximal k_v-split torus of G/k_v for all $v \in V_D$. The group

(6) $$Y = \mathrm{Hom}(U, \boldsymbol{C}^*) = \mathrm{Hom}(X_*(T)^D, \boldsymbol{C}^*) \qquad (v \in V_D),$$

is independent of v, and is the Y of §6 for G/k_v. The root datum $\psi(G/k_v)$, which is determined by the action of D, is also independent of $v \in V_D$.

Given $y \in Y$, let y_0 be a "logarithm" of y, i.e., an element of $\mathrm{Hom}(X_*(T)^D, \boldsymbol{C})$ such that

$$y(u) = Nv^{y_0(u)} = Nv^{\langle y_0, u\rangle}, \quad \text{for } u \in X_*(T)^D. \tag{7}$$

This element is determined modulo a lattice, but its real part $\mathrm{Re}\, y_0 \in \mathrm{Hom}(U, \boldsymbol{R})$, defined by

$$y(u) = Nv^{\langle \mathrm{Re}\, y_0, u\rangle} \tag{8}$$

is well defined. If y has values in $\boldsymbol{R}_+^*$, then we choose y_0 to be equal to its real part. The space $\mathfrak{a}^*$ is the dual of $\mathfrak{a} = U \otimes \boldsymbol{R}$ (the so-called real Lie algebra of S/k_v), and is acted upon canonically by ${}_kW$ as a reflection group. We let $\mathfrak{a}^{*+}$ be the positive Weyl chamber defined by B.

Let ρ_v be the unramified character of $T(k_v)$, given by $t \mapsto |\delta(t)|_v$, where $|\ |_v$ is the normalized valuation at v and δ half the sum of the positive roots. Then its real logarithm ρ_0 *is independent of* $v \in V_D$. In fact, it is a positive integral power of Nv whose exponent is determined by the k_v-roots, their multiplicities, and the indices q_α of the Bruhat-Tits theory [**61**]. But those are determined by the previous data and the action of Γ_{k_v} on the completed Dynkin diagram [**61**], which is also independent of $v \in V_D$. We write ρ_0 instead of $\rho_{v,0}$. We have $\rho_0 \in \mathfrak{a}^{*+}$.

The representation π_v is a constituent of an unramified principal series $\mathrm{PS}(\chi_v)$, where χ_v is an unramified character of $T(k_v)$, or, equivalently, of $S(k_v)$, determined up to a transformation by an element of ${}_kW$. Thus we may assume $\chi_{v,0}$ to be contained in the closure $\mathscr{C}\ell(\mathfrak{a}^{*+})$ of $\mathfrak{a}^{*+}$. Since π_v is unitary, the associated spherical function is bounded, and hence $\mathrm{Re}\,\chi_{v,0}$ is contained in the convex hull of ${}_kW(\rho_0)$, i.e., we have

$$\langle \rho_0 - \chi_{v,0}, \lambda\rangle \geqq 0, \quad \text{for all } \lambda \in \mathfrak{a}^{*+}. \tag{9}$$

(See remark following the proof.)

For $\lambda \in X^*({}^LT^\circ)$, let λ' be the restriction of λ to $X_*(T^D)$. In view of 10.4 and our conventions, we have then

$$|\lambda(t_v)| = Nv^{\langle \mathrm{Re}\,\chi_{v,0}, \lambda'\rangle}. \tag{10}$$

Let $\bar\lambda = {}_kW(\lambda') \cap \mathscr{C}\ell(\mathfrak{a}^{*+})$. Since $\mathrm{Re}\,\chi_{v,0} \in \mathscr{C}\ell(\mathfrak{a}^{*+})$, we have

$$Nv^{\langle \mathrm{Re}\,\chi_{v,0}, \lambda'\rangle} \leqq Nv^{\langle \mathrm{Re}\,\chi_{v,0}, \bar\lambda\rangle}. \tag{11}$$

Combined with (9), this implies

$$|\lambda(t_v)| \leqq Nv^{\langle \rho_0, \bar\lambda\rangle}. \tag{12}$$

If now λ runs through P_r, there are only finitely many possibilities for $\bar\lambda$, whence (4), with $a = \sup\langle \rho_0, \bar\lambda\rangle$ ($\lambda \in P_r$), for $v \in V_D$. Since V_1 is a finite union of such sets, this proves (4).

REMARK. The relation (9) is proved in [**35**, pp. 27–29] for the split case. For a general semisimple simply connected group, see I. Macdonald, *Spherical functions on a group of p-adic type*, Publ. Ramanujan Institute 2, Madras, Theorem 4.7.1, or H. Matsumoto, Lecture Notes in Math., vol. 590, Springer-Verlag, Berlin and New York, Proposition 4.4.11. In fact, we have used it for a general connected reductive group but the reduction to the case of simply connected semisimple groups is easily carried out by going over to the universal covering of the derived group.

13.3. COROLLARY. *Let P be a parabolic k-subgroup of G, $P = M \cdot N$ a Levi decomposition over k of P. Assume that π is a constituent of a representation* $\mathrm{Ind}\,_{P_A}^{G_A}(\sigma)$ *induced from a unitarizable irreducible admissible representation σ of M_A, viewed as a representation of P_A trivial on N_A. Then $L(s, \pi, r)$ is absolutely convergent in some right half-plane.*

We view ${}^LM$ as a subgroup of ${}^LG$ (3.3). Let r' be the restriction of r to ${}^LM$.

Let $v \in V_f$ be such that the conditions listed at the end of 13.1 are satisfied by M, G, σ_v and π_v. Then, by the transitivity of induction, it follows that there exists χ_v as in the above proof such that σ_v (resp. π_v) is the constituent of class 1 with respect to $M(o_v)$ (resp. $G(o_v)$) of the principal series $\mathrm{PS}(\chi_v)$ for $M(k_v)$ (resp. $G(k_v)$). Then $L(s, \pi_v, r) = L(s, \sigma_v, r')$ (7.2, 10.4). This being true for almost all v's, we are reduced to 13.2.

14. The L-function of an automorphic representation.

14.1. A smooth representation of G_A is *automorphic* if it is a subquotient of the regular representation of G_A in $G_k \backslash G_A$. It is cuspidal if it consists of cusp forms. If so, it is unitary modulo the center. We let $\mathfrak{A}(G/k)$ denote the set of equivalence classes of irreducible admissible automorphic representations of G_A. By Proposition 2 of **[39]**, every $\pi \in \mathfrak{A}(G/k)$ is a constituent of a representation induced from some cuspidal $\sigma \in \mathfrak{A}(M/k)$, where M is a Levi k-subgroup of a parabolic k-subgroup of G. Combined with 13.3 this yields the

14.2. THEOREM (LANGLANDS). *Let $\pi \in \mathfrak{A}(G/k)$ and r be a representation of ${}^LG$. Then $L(s, \pi, r)$ is absolutely convergent in some right half-plane.*

The L-function of an irreducible admissible automorphic representation will also be called an automorphic L-function.

14.3. There are several conjectures on the analytic character of $L(s, \pi, r)$ for automorphic π, all checked in some special cases, going back to the work of Hecke on L-series attached to Grössencharaktere and to modular forms.

(a) If $\pi \in \mathfrak{A}(G/k)$, then $L(s, \pi, r)$ admits a meromorphic continuation to the whole complex plane.

(b) Assume that π and G are such that the local solution to the local problem yields factors L and ε at all places. It is then conjectured that there is a functional equation $L(s, \pi, r) = \varepsilon(s, \pi, r) \cdot L(1 - s, \tilde{\pi}, r)$, where $\tilde{\pi}$ is the contragredient representation to π.

(c) In a number of cases, it has been shown that:

(*) If π is cuspidal, r irreducible nontrivial, then $L(s, \pi, r)$ is entire.

Here and there, conjectures to the effect that this should be a general phenomenon have been stated. However, there are counterexamples. Heuristically, one sees this is likely to happen if π is lifted from a cuspidal representation of a reductive group H (in the sense of V below) and the restriction of r to ${}^LH$ contains the trivial representation.

14.4. (a) Let $G = \mathbf{GL}_n$ and $r = r_n$ be the standard representation of $\mathbf{GL}_n(\boldsymbol{C})$. Then 14.3(b), (c) are proved in **[25]** for $n = 2$, in **[19]** for $n \geqq 2$, if L and ε are defined to be the products of the L- and ε-factors mentioned in 12.4. As recalled in 12.4, these are the same as those considered here at almost all places, and for $n = 2$, at all places.

(b) If $G = \mathbf{GL}_2 \times \mathbf{GL}_2$ and $r = r_2 \otimes r_2$, similar results are established by Jacquet in [21].

(c) Let $G = \mathbf{GL}_2$. If $r: \mathbf{GL}_2(C) \to \mathbf{GL}_3(C)$ is the adjoint representation, then 14.3(b), (c) are announced in [16]. This extends results of Shimura [54]. If $r = \mathrm{Sym}^3(r_2)$, $\mathrm{Sym}^4(r_2)$, then 14.3(b) is stated in [15], in the context of the global lifting (see V); for $\mathrm{Sym}^3(r_2)$, it is also proved in [51], in the framework of 14.5 below.

(d) Let k be a function field, $G = \mathbf{GL}_m \times \mathbf{GL}_n$ and $r = r_m \otimes r_n$. Let π (resp. π') be a cuspidal automorphic representation of the first (resp. second) factor. By the methods of [19], [26], [27], one can define L and ε, and (Jacquet dixit) show 14.3(b), and also the holomorphy, except when $m = n$ and π is contragredient to π'. These methods also yield further examples for other groups and for other representations. It is expected that similar results hold over number fields.

(e) 14.3(a) has also been checked when $G = \mathbf{PSp}(4)$ in some cases in [1], and, in general, in [42]. A functional equation is also established. 14.3(a), (b) are announced in [41] for orthogonal groups in an odd number of variables over functional fields, for the local factors mentioned in 12.3. For a survey and earlier references, see [43]. See also [44].

14.5. We describe some cases in which 14.3(a) has been verified in [33] (see also [18] for a survey). Let C be a split k-group, of adjoint type, endowed with its canonical $\mathfrak{o}$-structure. Fix a Borel subgroup B of C and a maximal torus T of B defined over $\mathfrak{o}$. Let P be a maximal proper standard parabolic subgroup and $P = M \cdot N$ its standard Levi decomposition. Since C is adjoint, it is easily seen that $C(M)$ is a torus. The group $M/C(M)$ is semisimple, split over k, of adjoint type, of rank equal to $\mathrm{rk}(C) - 1$. We let $G = M/C(M)$. The group ${}^LG^\circ$ is simply connected (2.2(2)). We have a natural inclusion ${}^LG \to {}^LM$, and ${}^LM$ is the Levi subgroup of a standard parabolic subgroup ${}^LP = {}^LM \cdot U$ with unipotent radical U (3.3). Let A be the split component of P in T, and ${}^LA^\circ$ the split component of ${}^LP^\circ$ in ${}^LT^\circ$. The group ${}^LA^\circ$ acts on the Lie algebra $\mathfrak{u}$ of U and its eigenspaces are irreducible ${}^LG^\circ$-modules. We let F_P denote the set of contragredient representations to these ${}^LG^\circ$-modules. The L-functions considered in [33] are of the form $L(s, \pi, r)$ with $r \in F_P$ and π an irreducible cuspidal automorphic representation of G. A number of examples are given in which $L(s, \pi, r)$ admits a meromorphic continuation. This is deduced from the results of [32]: let m be the length of a composition series of $\mathfrak{u}$ with respect to M. Then, for suitable numbering of the elements of F_P and strictly positive integers a_i, there is a relation

$$(1) \qquad M(s) = \prod_{1 \leq i \leq m} L(a_i s, \pi, r_i) \cdot L(s a_i + 1, \pi, r_i)^{-1},$$

where $M(s)$ is the intertwining operator occurring in the theory of Eisenstein series with respect to P, and is known to have a meromorphic continuation to the complex plane [32]. If $r = 1$, this and 13.2 yield the meromorphic continuation. In general, if we have the analytic continuation for all r_i's except one, (1) gives it for the remaining one.

14.6. The *converse problem* is to what extent automorphic representations can be characterized by analytic properties of their L-functions, or to give analytic conditions on a given L-function which will insure that it is automorphic. The first main result was Hecke's characterization of the Mellin transform of a parabolic modular form. Then came Weil's extension of this theorem to congruence subgroups [64], [65], its generalization in the context of representations in [25], and the extension to $\mathbf{GL}_3$ [46], [27]. In those results, conditions are imposed on the L-functions of π and of the twists $\pi \otimes \chi$ of π by characters. However, the analogous statement is false from $n = 4$ on [46]. It may remain true if one imposes conditions on the twist $\pi \otimes \rho$ of π by representations of $\mathbf{GL}_{n-1}$ or only of $\mathbf{GL}_{n-2}$. For results in that direction, over function fields, see [45].

Note however that in the general problem outlined here, one wishes rather to turn things around and deduce the analytical properties of some given L-series by showing directly that it is automorphic (see the seminars on base change and on zeta-functions of Shimura varieties [17], [8], [40]).

14.7. *Other problems.* (1) One "representation theoretic" form of "Ramanujan's conjecture" is the following: if $\pi = \bigotimes \pi_v$ is an irreducible nontrivial admissible cuspidal automorphic representation (and G is simple), then each π_v is tempered. It is now well known to be false for certain orthogonal or unitary groups, and even for one split group [20].

(2) Let π be a unitary irreducible representation of G_A. If $G = \mathbf{GL}_2$, then its multiplicity in the space of cusp forms ${}^0L_2(G(k)\backslash G(A))$ is at most one, "multiplicity one theorem" [25]. In fact there is even a "strong multiplicity one theorem" [38]: given π_v for almost all v's, there is at most one constituent π of the space of cusp-forms with those local factors.

The multiplicity one theorem has been proved for $\mathbf{GL}_n$ [52] and the strong form for $\mathbf{GL}_3$ [28]. It is unknown whether it is true for $\mathbf{SL}_2$. On the other hand, there are counterexamples for some inner forms of $\mathbf{SL}_2$ [31].

Chapter V. Lifting Problems.

Although the problems on automorphic L-functions discussed in §14 are only partially solved, the solutions provide practically all cases in which an L-series (automorphic or not) has been proved to have meromorphic or holomorphic analytic continuation with functional equation. This suggests trying, given an L-series and a reductive group G, to see whether G has an automorphic representation with the given L-series. Many instances of such questions can be viewed more precisely as special cases of the "lifting problem" or of the "problem of functoriality with respect to morphisms of L-groups." There is also a local version. For the sake of exposition, we shall start with the latter, but it should be borne in mind that the motivation and requirements stem from the global one, and that local and global are at present inextricably linked in many proofs. These questions were raised by Langlands in [35].

15. L-homomorphisms of L-groups.

15.1. Let E be a field and H, G connected reductive E-groups. A homomorphism $u\colon {}^LH \to {}^LG$ over Γ_k is said to be an L-homomorphism if it is continuous and if its restriction to ${}^LH^\circ$ is a complex analytic homomorphism of ${}^LH^\circ$ into ${}^LG^\circ$. Let E be local and G quasi-split. If $\varphi \in \Phi(H)$, then $u \circ \varphi \in \Phi(G)$. In fact, condition 8.2(i) is clearly satisfied, by $u \circ \varphi$, and so is 8.2(ii) because every parabolic subgroup of ${}^LG$

is relevant, G being assumed to be quasi-split. Therefore $\varphi \mapsto u \circ \varphi$ defines a map $\Phi(H) \to \Phi(G)$, to be denoted $\Phi(u)$.

15.2. Let $E = k$ be a global field. For $v \in V$, the Galois group Γ_{k_v} is a subgroup of Γ_k; hence the L-group of G viewed as a k_v-group, to be denoted ${}^L(G/k_v)$, is a subgroup of ${}^LG = {}^L(G/k)$. Thus, in particular, the L-homomorphism u of 15.1 defines by restriction an L-homomorphism u_v: ${}^L(H/k_v) \to {}^L(G/k_v)$, hence also a map $\Phi(u_v)$: $\Phi(H/k_v) \to \Phi(G/k_v)$ $(v \in V)$.

The "lifting problem" is, roughly speaking, whether such maps are mirrored by maps of representations in the local case, or of automorphic representations in the global case.

15.3. EXAMPLE: BASE CHANGE. Let H be a split over E, F a finite Galois extension of E, and $G = R_{F/E}H$. Then ${}^LG^\circ$ is a product of copies of ${}^LH^\circ$, indexed and permuted by $\Gamma_{F/E}$ (5.1). There is then a natural L-homomorphism u which is the identity on Γ_E and the diagonal map on ${}^LH^\circ$. If E is a local field, then W'_F is an open normal subgroup of W'_E, and the map $\Phi(u)$ may be viewed as given by the restriction to W'_F.

16. Local lifting.

16.1. Let $k = E$ be a local field, G quasi-split over E, H a connected reductive E-group and u: ${}^LH \to {}^LG$ an L-homomorphism. The problem of local lifting is, roughly, to establish a correspondence $\Pi(u)$: $\Pi(H(k)) \to \Pi(G(k))$ which preserves L- and ε-factors. If the local parametrization problem of III is solved, then $\Pi(u)$ is the map between indistinguishable classes which assigns $\Pi_{u\circ\varphi, G}$ to $\Pi_{\varphi,H}$ $(\varphi \in \Phi(H))$. The element $\Pi \in \Pi(G(k))$ is said to be *a lift of* $\pi \in \Pi(H(k))$ if $\Pi \in \Pi_{u\circ\varphi, G}$, where $\varphi \in \Phi(H)$ is such that $\pi \in \Pi_{\varphi,H}$. We have then

$$(1) \qquad L(s, \Pi, r) = L(s, \pi, r \circ \varphi), \qquad \varepsilon(s, \Pi, r, \psi) = \varepsilon(s, \pi, r \circ u, \psi).$$

for every representation r of ${}^LG$.

16.2. The local lifting is thus viewed as a map between classes of L-indistinguishable representations rather than one between representations. However it is possible to single out one lifting under assumptions which, in the global case, are satisfied almost everywhere: assume H, G to be quasi-split, split over an unramified extension F of E, endowed with an o_E-structure such that $H(o_E)$ and $G(o_E)$ are very special maximal compact subgroups, and π of class one with respect to $H(o_E)$. Then φ such that $\pi \in \Pi_{\varphi,H}$, and the set $\Pi_{u\circ\varphi, G}$ are well defined. Moreover, $\Pi_{u\circ\varphi, G}$ contains exactly one element of class one (with respect to $G(o_E)$), to be called the *natural lift* of π.

16.3. A full solution of the local parametrization problem does not seem to be in sight, and it is conceivable that it may require proving at the same time global results such as Artin's conjecture. Meanwhile, one wants to settle some approximations to it, notably to be able to prove some cases of Artin's conjecture. Note that if $G = \mathbf{GL}_n$, then the sets $\Pi_{\varphi,G}$ are either known or conjectured to consist of one element (12.2, 12.3). Such a lifting problem can then be stated as one of constructing a map u_*: $\Pi(H(k)) = \Pi(G(k))$ satisfying certain conditions. So far, there are two examples:

(a) Base change (cf. 15.3) when $H = \mathbf{GL}_2$ and F is cyclic of prime degree over E **[17]**, **[38]**, **[49]**, **[56]**. Besides some naturality conditions and 16.3, the main requirements relate the characters of π and of the hypothetical $u_*(\pi)$. The results also describe the fibres and the image of u_*. [Note that the results of **[38]** on this problem are used in **[62]**, so that we cannot invoke the solution of the local parametrization problem (12.4) for $\mathbf{GL}_2$ just to use the map $\Pi(u)$ of 16.1. If we could, then the local questions **[38]** would be mainly to relate the characters of π and $\Pi(u)(\pi)$.]

(b) $H = \mathbf{GL}_2$, $G = \mathbf{GL}_3$, and

$$(1) \qquad u\colon {}^LH^\circ = \mathbf{GL}_2(C) \longrightarrow {}^LG^\circ = \mathbf{GL}_3(C)$$

is given by the adjoint representation of ${}^LH^\circ$ (see **[16]**).

In this case, $\Pi = u_*(\pi)$ must be trivial on the center of ${}^LG^\circ$ and be such that the L- and ε-factors of $u_*(\pi) \otimes \chi$ (χ character of E^*) are certain given functions. There is at most one such Π(12.4(d)). In **[16]**, Π is stated to exist, except possibly if E has even residual characteristic and π is "extraordinary."

16.4. In 16.3(a), the lifting problem was connected with the existence of relations between characters. This is a direct connection between $\Pi(H)$ and $\Pi(G)$, which is of great importance for the use of the trace formula in proving or using the local or global lifting. We now mention two other examples of such relations. Assume that G is a quasi-split inner form of H. There is then an isomorphism $u\colon {}^LH \xrightarrow{\sim} {}^LG$ and an embedding $\Phi(u)\colon \Phi(H) \subset \Phi(G)$. If $f\colon H \to G$ is a k_s-isomorphism such that $f^{-1} \cdot {}^\gamma f$ is an inner automorphism of G for every $\gamma \in \Gamma_k$, then f establishes a bijection between conjugacy classes which are stable under Γ_k. Using results of Steinberg **[59]**, one then sees easily that maximal k-tori in H are isomorphic over k to maximal k-tori in G. This allows one in some cases to assign regular semisimple classes in $G(k)$ to such classes in $H(k)$, so that it makes sense to compare values of characters of $H(k)$ and of $G(k)$ on such classes.

(a) Let k be either $\boldsymbol{R}$ or nonarchimedean with odd residual characteristic. Let $G = \mathbf{GL}_2$ and H be the group of invertible elements in the quaternion algebra over k. The sets Π_φ are singletons, $\Phi(u)$ assigns to a (finite dimensional) irreducible representation π of $H(k)$ a discrete series representation π' of $G(k)$. In this case, the semisimple classes of $H(k)$ correspond to the elliptic classes in $G(k)$. It is proved in **[25]** that the characters of π and π' differ only by a sign on those classes.

(b) Let $k = \boldsymbol{R}$. For $\varphi \in \Phi(H)$, $\Phi(G)$, let χ_φ be the sum of the characters of the elements in Π_φ. Choose $\varphi \in \Phi(H)$ such that Π_φ consists of tempered representations. Then χ_φ and $\chi_{u\circ\varphi}$ are equal on the regular semisimple classes of $H(k)$, up to a sign depending only on H and G **[53**, 6.3].

16.5. We could also take the Weil forms of the L-groups. In that case an L-homomorphism, restricted to W_E, is assumed to satisfy the obvious analogue of 8.2(i). Take in particular the case where $H = \{1\}$. Then u is just an element of $\Phi(G)$. The lifting problem in this case is part of the local problem of III.

17. Global lifting.

17.1. Assume G to be quasi-split. Let H be a reductive k-group and $u\colon {}^LH \to {}^LG$ an L-homomorphism. Let $u_v\colon {}^L(H/k_v) \to {}^L(G/k_v)$ and $\Phi(u_v)\colon \Phi(H/k_v) \to \Phi(G/k_v)$ be the associated maps ($v \in V$) (see 15.1).

Let $\pi = \bigotimes_v \pi_v$ (resp. $\Pi = \bigotimes_v \Pi_v$) be an irreducible admissible representation of H_A (resp. G_A). Then Π is said to be a lift of π if Π_v is one of π_v for every $v \in V$ (16.1). If that is the case, then, for every representation r of ${}^LG$, we have

$$(1) \qquad L(s, \Pi, r) = L(s, \pi, r \circ u), \qquad \varepsilon(s, \Pi, r) = \varepsilon(s, \pi, r \circ u).$$

It is also usually requested that Π_v be the natural lift (16.2) of π_v for almost all v's. The question is then whether every automorphic π has a lift, which is automorphic, or, somewhat more ambitiously, whether there is a map u_*: $\mathfrak{A}(H/k) \to \mathfrak{A}(G/k)$ with reasonable properties, which sends $\pi \in \mathfrak{A}(H/k)$ onto a lift of π. One also wants to describe the fibres and the image of u_*.

In that degree of generality, the problem appears to be inaccessible at present. However, there are many results, old and recent, which are very striking illustrations of this principle, some of which will be extensively discussed in various seminars. Here, for orientation, and to give an idea of the scope of the problem, I shall list briefly some special cases, referring to the literature or to other seminars for more details.

REMARK. Let r be a representation of ${}^LH$ of degree n. Then it defines an L-homomorphism $u : {}^LH \to {}^L\mathbf{GL}_n = \mathbf{GL}_n(\boldsymbol{C}) \times \Gamma_k$ in the obvious way. A positive answer to the lifting problem would imply in particular that if π is an automorphic representation of H, then $L(s, \pi, r) = L(s, \Pi, r_n)$ where Π is an automorphic representation of $\mathbf{GL}_n$ and r_n the standard representation. This would therefore to a large extent reduce the study of automorphic L-functions to those of $\mathbf{GL}_n$, with respect to the standard representation.

17.2. Let $H = \{1\}$, $G = \mathbf{GL}_n$. Then an L-homomorphism u is just a continuous complex n-dimensional representation of Γ_k. The question is then whether the Artin L-series $L(s, u)$ is an automorphic L-series of $\mathbf{GL}_n$ (with respect to the standard representation of $\mathbf{GL}_n(\boldsymbol{C})$), which should be cuspidal if u is irreducible. In view of known results on $\mathbf{GL}_n$ (cf. 14.4) this would imply Artin's conjecture.

For $n = 1$, a positive answer is given by class-field theory. For $n = 2, 3$, a positive answer is equivalent to Artin's conjecture, since there are converses to Hecke theory **[25]**, **[65]**, **[27]**, **[46]**. For $n = 2$, it has been proved for dihedral or tetrahedral representations of Γ_k, and for some others over $\boldsymbol{Q}$ (see **[38]**, **[17]**, **[15]**).

17.3. Let k' be a Galois extension of k, n the degree of k' over k. Take $H = R_{k'/k}\mathbf{GL}_1$, $G = \mathbf{GL}_n$. There is a natural homomorphism f: ${}^LH^\circ \rtimes \Gamma_{k'/k}$ into the normalizer of a maximal torus ${}^LT^\circ$ of ${}^LG^\circ$. Since the former group is a quotient of ${}^LH$, and ${}^LG = {}^LG^\circ \times \Gamma_k$, we can define an L-homomorphism u: ${}^LH \to {}^LG$ by $u(h, \gamma) = (f(h), \gamma)$ $(h \in {}^LH^\circ, \gamma \in \Gamma_k)$. An automorphic representation of H is a Grössencharakter χ of k'. The problem is then whether the Artin L-series $L(s, \chi)$ is the L-series of an automorphic representation of G.

If $n = 2$, $k = \boldsymbol{Q}$, and k' is imaginary, this was proved by Hecke; π is associated to a cuspidal holomorphic automorphic form. If $n = 2$, $k = \boldsymbol{Q}$, and k' is real quadratic, this was established by H. Maass. π is then associated to a nonholomorphic automorphic form.

For $n = 3$, this is proved in **[26]**, **[27]**.

17.4. *Base change*. This is the global counterpart to 16.3(a). Let k' be a finite Galois extension of k. Assume H to be k-split and $G = R_{k'/k}H$. There is again an L-homomorphism u: ${}^LH \to {}^LG$ whose restriction to ${}^LH^\circ$ is a diagonal map. In this case $G(A)$ and $G(k)$ are canonically isomorphic to $H(A_{k'})$ and $H(k')$; therefore the problem is to associate an automorphic representation of $H(A_{k'})$ to an automorphic representation of $H(A_k)$. Again, it should be a counterpart to the restriction to $W_{k'}$ of homomorphisms $W_k \to {}^LH^\circ$.

If $H = \mathbf{GL}_2$ and k' is cyclic of prime degree, the lifting map u_* for representations is constructed in **[38]**, which also gives a description of its image and fibres.

This extends work of Doi-Naganuma, Jacquet **[21]** (on the quadratic case) and of Saito **[49]**, Shintani **[55]**, **[56]** (cf. **[17]**).

17.5. Let G be quasi-split, and H an inner form of G. Then ${}^LH = {}^LG$ and $\Phi(H/k_v) \subset \Phi(G/k_v)$ for all v's (8.3). Moreover, for almost all v's, H and G are isomorphic over k_v; hence $\Phi(H/k_v) = \Phi(G/k_v)$ and $\Pi(H(k_v)) = \Pi(G(k_v))$. The question is then, given $\pi = \otimes_v \pi_v$, is there an automorphic representation $\Pi = \otimes_v \Pi_v$ of G such that $\Pi_v = \pi_v$ for almost all v's?

If $G = \mathbf{GL}_2$ and H is the group of invertible elements of a quaternion algebra D over k, a positive answer is given by Jacquet-Langlands **[25]**. Note that, in that case, because of the "strong multiplicity one theorem," at most one Π may be associated to a given π in this way. The possible Π's are in fact the cuspidal automorphic representations for which Π_v belongs to the discrete series for all v's over which D does not split (loc. cit.).

17.6. If $G = \mathbf{GL}_2$, $G = \mathbf{GL}_3$ and u is given by the adjoint representation, as in 13.4, the global lifting problem has been solved by Gelbart-Jacquet **[16]**, the "local lifting" being the one of 16.2(b).

17.7. Let M be a Levi k-subgroup of a parabolic k-subgroup P of G. Then ${}^LM$ imbeds naturally into ${}^LG$ (3.3), whence an L-homomorphism $u\colon {}^LM \to {}^LG$. If π is cuspidal, then the analytic continuation and residues of Eisenstein series **[32]** are known to yield a unitary $u_*(\pi)$ in many cases, and, conjecturally, in general.

18. Relations with other types of L-functions.

18.1. In 17.2, the lifting problem amounts to identifying an Artin L-function with an automorphic L-function on $\mathbf{GL}_n$. One can also include in this problem more general representations of Weil groups if one passes to the Weil form of the L-groups. For simplicity, let us limit ourselves to relative Weil groups $W_{k'/k}$, where k' is a finite Galois extension of k over which H and G split. An L-homomorphism $u\colon {}^LH^\circ \rtimes W_{k'/k} \to {}^LG^\circ \rtimes W_{k'/k}$ is then a continuous homomorphism compatible with the projections on $W_{k'/k}$, whose restriction to ${}^LH^\circ$ is a complex analytic homomorphism into ${}^LG^\circ$, and such that, for $w \in W_{k'/k}$, $u(w) = (u'(w), w)$ with $u(w)$ semi-simple (cf. 8.2(i)).

If $H = \{1\}$, an L-homomorphism is said to be an admissible homomorphism of $W_{k'/k}$ into ${}^LG$. In analogy with the definition of $\Phi(G)$ in the local case, we can consider the set $\Phi_{k'/k}(G)$ of equivalence classes of such homomorphisms, modulo inner automorphisms of ${}^LG^\circ$, and then pass to a suitable limit $\Phi(G)$ over k'.

The lifting problem asks in this case to associate to any $\varphi \in \Phi(G)$ an automorphic representation π, such that, for any representation r of ${}^LG$, $L(s, \pi, r)$ is equal to the Artin-Hecke L-series of $r \circ u$. In particular, is every Artin-Hecke L-series that of an automorphic representation of $\mathbf{GL}_n$, with respect to the standard representation?

If G is a torus, then **[34]** provides a positive answer. In fact, in this case the irreducible admissible automorphic representations of G are the characters of $G(k)\backslash G(A)$, and **[34]** gives a homomorphism with finite kernel of $\Phi_{k'/k}(G)$ onto the set of such characters.

18.2. In the same vein, it is natural to ask whether Hasse-Weil zeta-functions (or even L-functions of compatible systems of l-adic representations of Galois groups) can be expressed in terms of automorphic L-functions. For elliptic curves over function fields, it is a theorem. That it should be the case for elliptic curves over

$\boldsymbol{Q}$ is the Taniyama-Weil conjecture; it has been checked in a number of special cases (see **[2]**, **[14]** for surveys from the classical and representation theoretic points of view respectively). Apart from that, this problem has been pursued mostly for Shimura curves and certain Shimura varieties; we refer to the corresponding seminars for a description of the present state of affairs.

Finally, one may ask whether it is possible to characterize a priori those automorphic representations whose L-series have an arithmetic or algebraico-geometric significance. A necessary condition if k is a number field is that for an infinite place v, π_v should be associated to a representation σ_v of W_{k_v} whose restriction to $\boldsymbol{C}^*$ is rational, $\boldsymbol{C}^*$ being viewed as real algebraic group, i.e., be of type A_0 in [3, 6.5]. If the L-series of π is to be an Artin L-series, then π should even be of type A_{00} (loc. cit.), i.e., σ_v should be trivial on $\boldsymbol{C}^*$. Let $k = \boldsymbol{Q}$. Then there are three possibilities for π_∞ (11.5). If $\pi_\infty = \pi(1, \text{sgn})$, then π corresponds to 2-dimensional representations of $\Gamma_{\boldsymbol{Q}}$ with odd determinant by the theorem of Deligne-Serre **[10]**, **[50]**. Modulo the Artin conjecture for such representations, the correspondence is bijective. However, I am not aware of any result for the other two possible values of π_∞. A positive answer would involve nonholomorphic automorphic forms. In **[36]**, it is shown in many cases for $\mathbf{GL}_2$ over $\boldsymbol{Q}$ that the L-series of a representation of type A_0 is that of a compatible system of l-adic representations of $\Gamma_{\boldsymbol{Q}}$. Over a function field, there is no condition such as A_0. In fact, for $\mathbf{GL}_2$, Drinfeld has shown that all irreducible admissible automorphic representations are associated to l-adic representations (see the lectures on his work by G. Harder and D. Kazhdan).

REFERENCES

1. A. N. Andrianov, *Dirichlet series with Euler product in the theory of Siegel modular forms of genus two*, Trudy Mat. Inst. Steklov **112** (1971), 73–94.

2. B. J. Birch and H. P. F. Swinnerton-Dyer, *Elliptic curves and modular functions of one variable*. IV, Lecture Notes in Math., vol. 476, Springer, New York, 1975, pp. 2–32.

3. A. Borel, *Formes automorphes et séries de Dirichlet* (*d'après R. P. Langlands*), Sém. Bourbaki, Exposé 466, (1974/75); Lecture Notes in Math., vol. 514, Springer, New York, pp. 189–222.

4. A. Borel and J.- P. Serre, *Théorèmes de finitude en cohomologie galoisienne*, Comment. Math. Helv. **39** (1964), 111–164.

5. A. Borel et J. Tits, *Groupes réductifs*, Inst. Hautes Etudes Sci. Publ. Math. **27** (1965), 55–151.

6. P. Cartier, *Representations of reductive $\mathfrak{p}$-adic groups*, these PROCEEDINGS, part 1, pp. 111–155.

7. W. Casselman, $\mathbf{GL}_n$, Proc. Durham Sympos. Algebraic Number Fields, London, Academic Press, New York, 1977.

8. ———, *The Hasse-Weil ζ-function of some moduli varieties of dimension greater than one*, these PROCEEDINGS, part 2, pp. 141–163.

9. P. Deligne, *Formes modulaires et représentations de* $\mathbf{GL}_2$, Modular Functions of One Variable, Lecture Notes in Math., vol. 349, Springer, New York, 1973, pp. 55–106.

10. P. Deligne and J.-P. Serre, *Formes modulaires de poids* 1, Ann. Sci. École Norm. Sup. (4) 7(1974), 507–530.

11. D. Flath, *Decomposition of representations into tensor products*, these PROCEEDINGS, part 1, pp. 179–183.

12. F. Gantmacher, *Canonical representations of automorphisms of a complex semi-simple group*, Mat. Sb. **5** (1939), 101–142. (in English)

13. S. Gelbart, *Automorphic functions on adele groups*, Ann. of Math. Studies, no. 83, Princeton Univ. Press, Princeton, N. J., 1975.

14. ———, *Elliptic curves and automorphic representations*, Advances in Math. **21** (1976), 235–292.

15. S. Gelbart, *Automorphic forms and Artin's conjecture*, Lecture Notes in Math., vol. 627, Springer-Verlag, Berlin and New York, 1977, pp. 241–276.

16. S. Gelbart and H. Jacquet, *A relation between automorphic forms on* $\mathbf{GL}_2$ *and* $\mathbf{GL}_3$, Proc. Nat. Acad. Sci. U.S.A. **73** (1976), 3348–3350.

17. P. Gérardin and J.-P. Labesse, *The solution of a base change problem for* GL_2 *(following Langlands, Saito, Shintani)*, these PROCEEDINGS, part 2, pp. 115–133.

18. R. Godement, *Fonctions automorphes et produits eulériens*, Sém. Bourbaki (1968/69), Exposé 349; Lecture Notes in Math., vol. 179, New York, 1970, pp. 37–53.

19. R. Godement and H. Jacquet, *Zeta-functions of simple algebras*, Lecture Notes in Math., vol. 260, Springer, New York, 1972.

20. R. Howe and I. I. Piatetski-Shapiro, *A counterexample to the 'generalized Ramanujan conjecture' for (quasi-) split groups*, these PROCEEDINGS, part 1, pp. 315–322.

21. H. Jacquet, *Automorphic Forms on* **GL(2)**. II, Lecture Notes in Math., vol. 278, Springer, New York, 1972.

22. ———, *Generic representations*, Non-Commutative Harmonic Analysis, Lecture Notes in Math., vol. 587, Springer, New York, 1977, pp. 91–101.

23. ———, *From* GL_2 *to* GL_n, U.S.-Japan Seminar on Number Theory (Ann Arbor, Mich., 1975).

24. ———, *Principal L-functions of the linear group*, these PROCEEDINGS, part 2, pp. 63–86.

25. H. Jacquet and R. P. Langlands, *Automorphic forms on* **GL(2)**, Lecture Notes in Math., vol. 114, Springer, New York, 1970.

26. H. Jacquet, I. Piatetskii-Shapiro and J. Shalika, *Construction of cusp forms on* GL_3, Univ. of Maryland Lecture Notes, no. 16, 1975.

27. ———, *Hecke theory for* GL_3, Ann. of Math. (to appear).

28. H. Jacquet and J. Shalika, *Comparaison des représentations automorphes du groupe linéaire*, C. R. Acad. Sci. Paris Ser. A **284** (1977), 741–744.

29. A. Knapp and G. Zuckerman, *Classification of irreducible tempered representations of semi-simple Lie groups*, Proc. Nat. Acad. Sci. U.S.A. **73** (1976), 2178–2180.

30. ———, *Normalizing factors, tempered representations and L-groups*, these PROCEEDINGS, part 1, pp. 93–105.

31. J. P. Labesse and R. P. Langlands, *L-indistinguishability for* SL_2, Institute for Advanced Study, Princeton, N. J. (preprint).

32. R. P. Langlands, *On the functional equations satisfied by Eisenstein series*, Lecture Notes in Math., vol. 544, Springer, New York.

33. ———, *Euler products*, Yale Univ. Press, 1967.

34. ———, *Representations of abelian algebraic groups*, Yale Univ., 1968 (preprint).

35. ———, *Problems in the theory of automorphic forms*, Lectures in Modern Analysis and Applications, Lecture Notes in Math., vol. 170, Springer, New York, 1970, pp. 18–86.

36. ———, *Modular forms and l-adic representations*, Modular Functions of One Variable. II, Lecture Notes in Math., vol. 349, Springer, New York, 1973, pp. 361–500.

37. ———, *On the classification of irreducible representations of real algebraic groups* (preprint).

38. ———, *Base change for* GL_2: *The theory of Saito-Shintani with applications*, Notes, Inst. Advanced Study, Princeton, N. J., 1975.

39. ———, *On the notion of an automorphic representation*, these PROCEEDINGS, part 1, pp. 203–207.

40. ———, *Automorphic representations, Shimura varieties, and motives. Ein Märchen*, these PROCEEDINGS, part 2, pp. 205–246.

41. M. E. Novodvorsky, *Théorie de Hecke pour les groupes orthogonaux*, C. R. Acad. Sci. Paris Sér. A, **285** (1975), 93–94.

42. ———, *Automorphic L-functions for the symplectic group* GSp(4), these PROCEEDINGS, part 2, pp. 87–95.

43. M. E. Novodvorsky and I. I. Piatetskii-Shapiro, *Rankin-Selberg method in the theory of automorphic forms*, Proc. Sympos. Pure Math., vol. 30, no. 2, Amer. Math. Soc., Providence, R. I., 1977, pp. 297–301.

44. I. Piatetski-Shapiro, *Euler subgroups*, Lie Groups and Their Representations, Proc. Summer School, Budapest, 1971.

45. ———, *Zeta-functions of* GL_n, Notes, Univ. of Maryland, College Park, Md.

46. ———, *Converse theorem for* GL_3, Notes, Univ. of Maryland, College Park, Md.

47. ———, *Multiplicity one theorems*, these PROCEEDINGS, part 1, pp. 209–212.

48. F. Rodier, *Les representations de* GSp(4, k), *où k est un corps local*, C. R. Acad. Sci. Paris **283** (1976), 429–431.

49. H. Saito, *Automorphic forms and algebraic extensions of number fields*, Lecture Notes in Math, vol. 8, Kinokuniya Book Store Co., Ltd., Tokyo, Japan, 1975.

50. J.-P. Serre, *Modular forms of weight one and Galois representations* (prepared in collaboration with C. J. Bushnell), Proc. Durham Sympos. Algebraic Number Fields (London), Academic Press, New York, 1977, pp. 193–268.

51. F. Shahidi, *Functional equations satisfied by certain L-functions*, Compositio Math. (to appear).

52. J. A. Shalika, *The multiplicity one theorem for* GL_n, Ann. of Math. (2) **100** (1974), 171–193.

53. D. Shelstad, *Characters and inner forms of a quasi-split group over F* (to appear).

54. G. Shimura, *On the holomorphy of certain Dirichlet series*, Proc. London Math. Soc. (3) **31** (1975), 79–98.

55. T. Shintani, *On liftings of holomorphic automorphic forms*, U.S.-Japan Seminar on Number Theory, (Ann Arbor, Mich., 1975).

56. ———, *On liftings of holomorphic cusp forms*, these PROCEEDINGS, part 2, pp. 97–110.

57. A Silberger, *The Langlands classification for reductive p-adic groups* (to appear).

58. T. A. Springer, *Reductive groups*, these PROCEEDINGS, part 1, pp. 3–27.

59. R. Steinberg, *Regular elements of semi-simple algebraic groups*, Inst. Hautes Etudes Sci. Publ. Math. **25** (1965), 49–80.

60. J. Tate, *Number theoretic background*, these PROCEEDINGS, part 2, pp. 3–26.

61. J. Tits, *Reductive groups over local fields*, these PROCEEDINGS, part 1, pp. 29–69.

62. J. B. Tunnell, *On the local Langlands conjecture for* GL(2), Thesis, Harvard Univ.

63. ———, *Report on the local Langlands conjecture for* GL_2, these PROCEEDINGS, part 2, pp. 135–138.

64. A. Weil, *Ueber die Bestimmung Dirichletscher Reihen durch Funktionalgleichungen*, Math. Ann. **168** (1967), 149–167.

65. ———, *Dirichlet series and automorphic forms*, Lecture Notes in Math., vol. 189, Springer, New York, 1971.

THE INSTITUTE FOR ADVANCED STUDY

Proceedings of Symposia in Pure Mathematics
Vol. 33 (1979), part 2, pp. 63–86

PRINCIPAL L-FUNCTIONS OF THE LINEAR GROUP

HERVÉ JACQUET

Regard the group $G_n = GL(n)$ as an algebraic group over some local or global field F. Then ${}^LG_n^0 = GL(n, C)$. Let r_n denote the natural representation of this last group on C^n; the L-functions attached to r_n play a central role in "Langlands philosophy". We review certain aspects of their theory, taking into account recent results on classification of representations.

1. Local nonarchimedean theory. We let F be a local nonarchimedean field. When this does not create confusion, we write G_n for $G_n(F)$ and use a similar notation for any F-group.

(1.1) Let π be an admissible representation of G_n on a complex vector space V; we denote by $\tilde{\pi}$ the representation contragredient to π, $\tilde{V}$ the space on which it operates, and $\langle \cdot, \cdot \rangle$ the canonical invariant bilinear form on $V \times \tilde{V}$. The representation $\tilde{\pi}$ is admissible, and irreducible if π is. Moreover $(\tilde{\pi})^{\sim} \cong \pi$. The functions

(1.1.1) $$g \longmapsto \langle \pi(g)v, \tilde{v} \rangle \qquad (v \in V, \tilde{v} \in \tilde{V})$$

and their linear combinations are the coefficients of π. Clearly if f is a coefficient of π, then the function $f^{\vee}$ defined by

(1.1.2) $$f^{\vee}(g) = f(g^{-1})$$

is a coefficient of $\tilde{\pi}$.

Let $M(p \times q, F)$ be the space of matrices with p rows, q columns and entries in F; denote by $\mathcal{S}(p \times q, F)$ the space of Schwartz-Bruhat functions on $M(p \times q, F)$.

If f is a coefficient of π, Φ is in $\mathcal{S}(p \times q, f)$, and s in C set

(1.1.3) $$Z(\Phi, s, f) = \int \Phi(g) \, |\det g|^s f(g) \, d^{\times}g.$$

The integral is extended to the group G_n and $d^{\times}g$ is a Haar measure on this group.

Below (Propositions (1.2), (1.4)) we state again the results of **[R.G.-H.J.]**.

(1.2) PROPOSITION. *Suppose π is irreducible.*

(1) *There is s_0 such that the integrals* (1.1.3) *converge absolutely in the half-plane* $\mathrm{Re}(s) > s_0$.

(2) *If the residual field of F has q elements, then the integrals represent rational*

AMS (MOS) subject classifications (1970). Primary 12A85, 22E50, 22E55, 12A80, 12B35.

functions of q^{-s}; as such, they admit a common denominator which does not depend on f or Φ.

(3) *Let $\psi \neq 1$ be an additive character of F. There is a rational function $\gamma(s, \pi, \psi)$ such that for all coefficients f of π and all Φ*

$$Z(\Phi^\wedge, 1 - s + \tfrac{1}{2}(n - 1), f^\vee) = \gamma(s, \pi, \psi)\, Z(\Phi, s, f),$$

where $\Phi^\wedge$ denotes the Fourier transform of Φ with respect to ψ.

In (3) $\Phi^\wedge$ is defined by

(1.2.4) $$\Phi^\wedge(x) = \int \Phi(y)\, \psi\,[\mathrm{tr}(yx)]\, dy,$$

where dy is the self-dual Haar measure on $M(n \times n, F)$. On the other hand the left-hand side in (3) has a meaning by (1) and (2) applied to $\tilde{\pi}$. Besides we could formulate Proposition (1.2) for the pair $(\pi, \tilde{\pi})$ rather than for π, the situation being symmetric in $\pi, \tilde{\pi}$; this symmetry will be apparent at each step of the proof anyway.

EXAMPLE (1.2.5). Suppose π is cuspidal. This condition is empty and therefore always satisfied if $n = 1$; in this case π is just a quasi-character of $F^\times$ and we know Proposition (1.2) from **[J.T. 1]** or **[A.W. 3]**. If $n > 1$, the coefficients of π are compactly supported modulo the center of G_n; we can then exploit this fact to obtain Proposition (1.2), the proof being essentially the same as in the case $n = 1$ (**[R.G.-H.J.]**, **[H.J. 1]**)

(1.3) The proof of Proposition (1.2) will be given in §2. For the time being, we derive some simple consequences of Proposition (1.2).

If f and Φ are as above and h is in G_n, then the functions f_1 and Φ_1 defined by

(1.3.1) $$f_1(g) = f(gh), \quad \Phi_1(x) = \Phi(xh)$$

are functions of the same type. Moreover if we assume (1.2.1), (1.2.2), then:

(1.3.2) $$Z(\Phi_1, s, f_1) = |\det h|^{-s}\, Z(\Phi, s, f).$$

It follows that the subvector space $I(\pi)$ of $C(q^{-s})$ spanned by integrals

(1.3.3) $$Z(\Phi, s + \tfrac{1}{2}(n - 1), f)$$

is in fact a fractional ideal of the ring $C[q^{-s}, q^s]$. Furthermore if we take f such that $f(e) \neq 0$ and Φ with support in a small enough neighborhood of e, we find that (1.3.3) is actually independent of s and $\neq 0$. Thus $I(\pi)$ contains the constants; in other words it admits a generator of the form $P(q^{-s})^{-1}$ with $P \in C[X]$. We will normalize P by demanding that $P(0) = 1$ and will set

(1.3.4) $$L(s, \pi) = P(q^{-s})^{-1}.$$

EXAMPLE (1.3.5). If π is supercuspidal then $L(s, \pi) = 1$ unless $n = 1$ and $\pi(x) = |x|^t$. Then (loc. cit.) $L(s, \pi) = (1 - q^{-s-t})^{-1}$.

Assume now Proposition (1.2) for $(\pi, \tilde{\pi})$. Because $I(\pi) \neq 0$, the factor γ is unique (which justifies the notation). Set

(1.3.6) $$\varepsilon(s, \pi, \psi) = \gamma(s, \pi, \psi) L(s, \pi)/L(1 - s, \tilde{\pi}).$$

Then (1.2.3) reads

$$(1.3.7)\quad \begin{aligned} &Z(\Phi^\wedge, 1 - s + \tfrac{1}{2}(n-1), f^\vee)/L(1-s, \tilde{\pi}) \\ &\qquad = \varepsilon(s, \pi, \psi) Z(\Phi, s + \tfrac{1}{2}(n-1), f)/L(s, \pi). \end{aligned}$$

In view of the definition of the L-factors, this implies that both $\varepsilon(s, \pi, \psi)$ and its inverse are in $C[q^{-s}, q^s]$. Thus $\varepsilon(s, \pi, \psi)$ is a monomial in q^{-s}.

Let also ω be the central quasi-character of π, that is to say the homomorphism $\omega: F^\times \to C^\times$ such that

$$(1.3.8)\quad \pi(a) = \pi(a \cdot 1_n) = \omega(a) 1_V \quad \text{for } a \in F^\times.$$

If the assertions of (1.2.3) are true for one choice of ψ, they are true for any other choice ψ'. In particular

$$(1.3.9)\quad \varepsilon(s, \pi, \psi') = \omega(b)|b|^{(s-1/2)n} \varepsilon(s, \pi, \psi) \quad \text{if } \psi'(x) = \psi(xb).$$

By exchanging π and $\tilde{\pi}$ we find also (if Proposition (1.2) is true for $(\pi, \tilde{\pi})$):

$$(1.3.10)\quad \varepsilon(1-s, \tilde{\pi}, \psi)\varepsilon(s, \pi, \psi) = \omega(-1).$$

Moreover let us denote, for any quasi-character χ, by $\pi \otimes \chi$ the representation $g \mapsto \pi(g)\chi(\det g)$ of G_n on V. Let also α_F or α be the module of F. Then:

$$(1.3.11)\quad L(s, \pi \otimes \alpha^t) = L(s + t, \pi),$$

$$(1.3.12)\quad \varepsilon(s, \pi \otimes \alpha^t, \psi) = \varepsilon(s + t, \pi, \psi).$$

The γ-factor satisfies similar identities.

2. Induced representations. Again F is a local nonarchimedean field.

(2.1) Let P be an F-parabolic subgroup of G_n. To be specific we take P to be standard of type $(n_1, n_2, \cdots, n_r)$, so that the matrices p in P have the form

$$(2.1.1)\quad p = \begin{pmatrix} m_1 & & & & u_{ij} \\ & m_2 & & & \\ & & \cdot & & \\ & & & \cdot & \\ & & & & \cdot \\ 0 & & & & m_r \end{pmatrix}, \quad m_i \in G_{n_i}.$$

We also set

$$(2.1.2)\quad \begin{aligned} U &= U_P \quad \text{(unipotent radical of } P\text{)}, \\ M &= M_P = P/U_P. \end{aligned}$$

Most of the time we identify M to the subgroup of $p \in P$ such that $u_{ij} = 0$. Then, in a natural way,

$$(2.1.3)\quad M \cong \prod G_{n_i} \qquad (1 \leqq i \leqq r).$$

If σ_i, $1 \leqq i \leqq r$, is an admissible representation of G_{n_i}, we can form the representation $\sigma = \times \sigma_i$ of $M = P/U$, regard it as a representation of P trivial on U, and induce it to G_n. The representation

$$(2.1.4)\quad \xi = I(G, P; \sigma) = I(G, P; \sigma_1, \sigma_2, \cdots, \sigma_r)$$

that we obtain is admissible. Its contragredient $\tilde{\xi}$ is equivalent to

$$\xi' = I(G, P; \tilde{\sigma}) = I(G, P;\ \tilde{\sigma}_1, \tilde{\sigma}_2, \cdots, \tilde{\sigma}_r). \tag{2.1.5}$$

More precisely let W be the space of σ. Then the space V of ξ consists of all functions F from G_n to W which are smooth and satisfy

$$F(pg) = \delta_P(p)^{1/2}F(g), \tag{2.1.6}$$

where δ_P is the module of P. The group G_n operates by right-shifts on V. The space V' of ξ' is defined similarly in terms of the space $\tilde{W}$ of $\tilde{\sigma}$. For $F \in V$, $\tilde{F} \in V'$ the function $\varphi(g) = \langle F(g), \tilde{F}(g)\rangle$ satisfies

$$\varphi(pg) = \delta_P(p)\varphi(g). \tag{2.1.7}$$

Denoting by

$$\varphi \longmapsto \int_{P\backslash G} \varphi(g)\, dg \tag{2.1.8}$$

a positive right-invariant form on the space of continuous functions satisfying (2.1.7), we may set

$$\langle F, \tilde{F}\rangle = \int_{P\backslash G} \langle F(g), \tilde{F}'(g)\rangle\, dg \tag{2.1.9}$$

and this is a pairing which allows us to identify V' to $\tilde{V}$ and ξ' to $\tilde{\xi}$.

Let R_F be the ring of integers of F and let

$$K = K_n = \mathrm{GL}(n, R_F). \tag{2.1.10}$$

Since $G_n = P \cdot K_n$, we may take (2.1.8) to be

$$\varphi \longmapsto \int_{K_n} \varphi(k)\, dk. \tag{2.1.11}$$

(2.2) It will be convenient to have a description of the coefficients of ξ in terms of those of σ. So let f be the coefficient of ξ determined by $F \in V$, $\tilde{F} \in \tilde{V} = V'$ ((1.1.1)); then the function $H\colon G_n \times G_n \to C$ defined by $H(g_1, g_2) = \langle F(g_1), \tilde{F}(g_2)\rangle$ satisfies the following conditions:

$$H(u_1mg_1, u_2mg_2) = \delta_P(m)H(g_1, g_2) \quad \text{for } g_i \in G_n,\ u_i \in U_P,\ m \in M_P; \tag{2.2.1}$$

(2.2.2) for any g_1, g_2 the function $m \longmapsto H(mg_1, g_2)$ is a coefficient of $\sigma \otimes \delta_P^{1/2}$;

(2.2.3) H is $K_n \times K_n$ finite on the right.

Moreover f is given by

$$f(g) = \int_{P\backslash G} H(hg, h)\, dh = \int_K H(kg, k)\, dk. \tag{2.2.4}$$

Conversely, if H is any function satisfying (2.2.1)–(2.2.3), then the function f defined by (2.2.4) is a coefficient of π. The coefficient $f^\vee$ of $\tilde{\xi}$ is then given by

(2.2.5) $$f^{\vee}(g) = \int_{P\backslash G} \tilde{H}(hg, h)\, dh \quad \text{where } \tilde{H}(g_1, g_2) = H(g_2, g_1)$$

and $\tilde{H}$ satisfies (2.2.1)–(2.2.3) with $\tilde{\sigma}$ instead of σ.

(2.3) Before formulating the main theorem of this section we remark that if π is an admissible representation which is perhaps not irreducible but admits a central quasi-character ((1.3.8)), then the assertions of (1.2) make sense for $(\pi, \tilde{\pi})$, although they may fail to be true. Below we assume that each σ_i admits a central quasi-character; thus ξ admits also a central quasi-character.

(2.3) PROPOSITION. *With the notations of* (2.1) *suppose the assertions of* (1.2) *are true for each pair* $(\sigma_i, \tilde{\sigma}_i)$. *Then they are true for* $(\xi, \tilde{\xi})$. *Moreover*:

$$I(\xi) = \prod_i I(\sigma_i), \quad \gamma(s, \xi, \psi) = \prod_i \gamma(s, \sigma_i, \psi).$$

The proof will occupy (2.4)–(2.6).

(2.4) Let f be given by (2.2.4). Then, exchanging the order of integrations, we get

$$\begin{aligned} Z(\Phi, s + \tfrac{1}{2}(n-1), f) &= \int_K dk \int_{G_n} \Phi(g)\,|\det g|^{s+(n-1)/2} H(kg, k)\, d^\times g \\ &= \int_K dk \int_{G_n} \Phi(k^{-1} g)\,|\det g|^{s+(n-1)/2} H(g, k)\, d^\times g \qquad (2.4.1) \\ &= \int dk\, dk' \int_P \Phi(k^{-1}pk')|\det p|^{s+(n-1)/2} H(pk', k)\, d_l p. \end{aligned}$$

If moreover we define (p being as in (2.1.1))

(2.4.2) $$\Psi(m_1, m_2, \cdots, m_r; k, k') = \int \Phi(k^{-1}pk') \otimes du_{ij},$$

(2.4.3) $$h(m_1, m_2, \cdots, m_r; k, k') = H(pk', k)\delta_P^{-1/2}(p),$$

then, after integrating in the variables u_{ij}, integral (2.4.1) can be written as

(2.4.4) $$\int dk\, dk' \int \Psi(m_1, m_2, \cdots, m_r; k, k') h(m_1, \cdots, m_r; k, k') \cdot \prod_i |\det m_i|^{s+(n_i-1)/2} \otimes d^\times m_i.$$

Because of the K-finiteness of the functions involved, for Φ and H given, (2.4.4) can be written as a sum over a finite set of $K \times K$ of the inner integrals. But for given k and k', $\Psi(m_1, m_2, \cdots, m_r; k, k')$ is a finite sum of products $\prod_i \Psi_i(m_i)$ with $\Psi_i \in \mathcal{S}(n_i \times n_i, F)$; similarly $h(m_1, m_2, \cdots, m_r; k, k')$ is a finite sum of products $\prod_i f_i(m_i)$ where f_i is a coefficient of σ_i ((2.2.2)). Thus (2.4.1) is a finite sum of products

(2.4.5) $$\prod_i Z(\Phi_i, s + \tfrac{1}{2}(n_i - 1), f_i),$$

where Φ_i is in $\mathcal{S}(n_i \times n_i, F)$ and f_i is a coefficient of σ_i. So we have proved (2.1.1), (2.1.2) for ξ, and even the inclusion $I(\xi) \subset \prod_i I(\sigma_i)$.

(2.5) Now we prove the reverse inclusion. Starting with an expression (2.4.5) we can certainly find $\Phi \in \mathcal{S}(n \times n, F)$ so that with the notations of (2.1.1):

$$\int \Phi(p) \otimes du_{ij} = \prod_i \Phi_i(m_i). \tag{2.5.1}$$

Moreover there are two K-finite functions η and η' on K such that

$$\iint \Phi(k^{-1}xk')\eta(k)\eta'(k')\, dk\, dk' = \Phi(x). \tag{2.5.2}$$

Then (2.4.5) is equal, for large Re s, to

$$\int \Phi(k^{-1}pk')|\det p|^{s+(n-1)/2} \prod_i f_i(m_i)\delta_P^{1/2}(p)\eta(k)\eta'(k')\, d_lp\, dk\, dk'. \tag{2.5.3}$$

Let dh be the normalized Haar measure on the compact group $K \cap P$. Changing k to hk, k' to $h'k'$ with h and h' in $K \cap P$ and then integrating over $(K \cap P) \times (K \cap P)$, we find that (2.5.3) is equal to

$$\begin{aligned} \int dh\, dh' \int \Phi(k^{-1}h^{-1}ph'k')|\det p|^{s+(n-1)/2} \\ \cdot \prod_i f_i(m_i)\delta_P^{1/2}(p)\eta(hk)\eta'(h'k')\, d_lp\, dk\, dk'. \end{aligned} \tag{2.5.4}$$

Now change p to hph'^{-1} and write h, h' in the form (2.1.1) with $h_i \in K_{n_i}$, $h_i' \in K_{n_i}$ instead of m_i. We see that (2.5.4) is equal to

$$\int \Phi[k^{-1}pk']H_1(p, k, k')|\det p|^{s+(n-1)/2}\, d_lp\, dk\, dk' \tag{2.5.5}$$

where

$$H_1(p, k, k') = \iint \eta(hk)\eta'(h'k') \prod_i f_i(h_i \cdot m_i \cdot h_i'^{-1})\, \delta_P^{1/2}(p)\, dh\, dh'.$$

It is easily verified that there is a function H satisfying (2.2.1)–(2.2.3) such that

$$H(pk, k') = H_1(p, k, k') \qquad (p \in P, k \in K, k' \in K').$$

If f is the corresponding coefficient of ξ ((2.2.4)), then, comparing (2.5.4) with (2.4.1), we find that

$$\prod_i Z(\Phi_i, s + \tfrac{1}{2}(n_i - 1), f_i) = Z(\Phi, s + \tfrac{1}{2}(n - 1), f).$$

So we have proved that $I(\xi) = \prod_i I(\sigma_i)$.

(2.6) Now we pass to the functional equation. In (2.4.1), let us replace f by $f^\vee$ and Φ by $\Phi^\wedge$. Then H is replaced by $\tilde{H}$ ((2.2.5)) and the identity (2.4.3) gives:

$$h(m_1^{-1}, m_2^{-1}, \cdots, m_r^{-1}; k', k) = \tilde{H}(pk, k')\delta_P^{-1/2}(p). \tag{2.6.1}$$

Similarly (2.4.2) gives

$$\Psi^\wedge(m_1, m_2, \cdots, m_r; k', k) = \int \Phi^\wedge(k^{-1}pk') \otimes du_{ij}, \tag{2.6.2}$$

where $\Psi^\wedge$ denotes the Fourier transform of Ψ with respect to each one of the variables m_i.

Instead of (2.4.1) and (2.4.4) we have now for Re s large enough

$$
(2.6.3)\quad
\begin{aligned}
&Z(\Phi^\wedge, s+\tfrac{1}{2}(n-1), f^\vee)\\
&= \int dk\, dk' \int \Phi^\wedge(k^{-1}pk')\, |\det p|^{s+(n-1)/2}\tilde{H}(pk', k)\, d_l p\\
&= \int dk\, dk' \int \Psi^\wedge(m_1, m_2, \cdots, m_r; k', k)\\
&\quad \cdot h(m_1^{-1}, m_2^{-1}, \cdots, m_r^{-1}; k', k) \prod_i |\det m_i|^{s+(n_i-1)/2} \otimes d^\times m_i.
\end{aligned}
$$

The functional equations for the representations σ_i and the remarks we made on h and Ψ imply now

$$Z(\Phi^\wedge, 1-s+\tfrac{1}{2}(n-1), f^\vee) = \prod_i \gamma(s, \sigma_i, \psi) Z(\Phi, s+\tfrac{1}{2}(n-1), f).$$

This concludes the proof of (2.3).

(2.7) Again let the notations be as in (2.3).

From (2.3) applied to ξ and $\tilde{\xi}$ we get

$$(2.7.1)\qquad L(s,\xi) = \prod_i L(s,\sigma_i), \qquad L(s,\tilde{\xi}) = \prod_i L(s,\tilde{\sigma}_i).$$

Then the last assertion of (2.3) reads

$$(2.7.2)\qquad \varepsilon(s,\xi,\psi) = \prod_i \varepsilon(s,\sigma_i,\psi).$$

Let π be an irreducible component of ξ; then any coefficient of π is a coefficient of ξ and $\tilde{\pi}$ is an irreducible component of $\tilde{\xi}$. It follows that Proposition (1.2) is true for $(\pi, \tilde{\pi})$. Moreover

$$(2.7.3)\qquad I(\pi) \subset I(\xi), \quad I(\tilde{\pi}) \subset I(\tilde{\xi}), \quad \gamma(s,\pi,\psi) = \gamma(s,\xi,\psi).$$

Thus there are two polynomials P, $\tilde{P}$ in $\boldsymbol{C}[X]$ such that

$$(2.7.4)\quad L(s,\pi) = P(q^{-s})L(s,\xi), \quad L(s,\tilde{\pi}) = \tilde{P}(q^{-s})L(s,\tilde{\xi}), \qquad P(0) = \tilde{P}(0) = 1.$$

Note that $P(q^{-s})$ divides $L(s,\xi)^{-1}$ in $\boldsymbol{C}[q^{-s}]$. Moreover

$$
(2.7.5)\quad
\begin{aligned}
\varepsilon(s,\pi,\psi) &= \gamma(s,\pi,\psi)\frac{L(s,\pi)}{L(1-s,\tilde{\pi})} = \gamma(s,\xi,\psi)\frac{L(s,\xi)P(q^{-s})}{L(1-s,\tilde{\xi})\tilde{P}(q^{-1+s})}\\
&= \varepsilon(s,\xi,\psi)\frac{P(q^{-s})}{\tilde{P}(q^{-1+s})}.
\end{aligned}
$$

Since the ε-factors are units of the ring $\boldsymbol{C}[q^{-s}, q^s]$, we find that

$$(2.7.6)\qquad P(X) = \prod(1-a_jX), \qquad \tilde{P}(X) = \prod(1-a_j^{-1}qX).$$

(2.8) In general, if π is an arbitrary irreducible admissible representation of G_n, it is a component of an induced representation of the form (2.1.4) where the σ_i are cuspidal. Since (1.2) is true for each σ_i ((1.2.5)), it is true for $(\xi, \tilde{\xi})$ ((2.3)) and thus for $(\pi, \tilde{\pi})$. So (1.2) is completely proved.

3. Computation of the L-factor. We have established (1.2) for any irreducible admissible representation π of G_n; we also know $\gamma(s,\pi,\psi)$ ((2.7.2)). It remains to compute the L-factor.

(3.1) *Square-integrable representations.* We have seen that if π is cuspidal then $L(s, \pi) = 1$ unless $n = 1$ and $\pi = \alpha^t$, in which case

(3.1.1) $$L(s, \pi) = (1-q^{-s-t})^{-1}.$$

We now compute $L(s, \pi)$ when π is essentially square-integrable. Recall that π is square-integrable if it admits a central *character* ω and its coefficients (which transform under ω) are square-integrable modulo the center. A representation π is essentially square-integrable if it has the form $\pi = \pi_0 \otimes \alpha^t$ where π_0 is square-integrable and t real. We now review the work of Bernstein and Zevelenski on the construction of such representations.

Let r be a divisor of n so that $n = jr$. Let P be the standard parabolic subgroup of G_n of type $(j, j, \cdots, j)$. Let also τ be a cuspidal representation of G_r. Set for $1 \leqq i \leqq r$, $\sigma_i = \tau \otimes \alpha^{i-1}$. Then the induced representation

(3.1.2) $$\xi = I(G, P; \sigma_1, \sigma_2, \cdots, \sigma_r)$$

admits a unique essentially square-integrable component π. All essentially square-integrable representations π are obtained in this way and r, τ are uniquely determined by π.

PROPOSITION (3.1.3). *With the above notations* $L(s, \pi) = L(s, \tau)$.

Indeed suppose $L(s, \tau) = 1$. Then $L(s, \sigma_i) = 1$ and by (2.7.1) $L(s, \xi) = 1$. By (2.7.4) we have then $L(s, \pi) = 1$. Suppose now $L(s, \tau) \neq 1$. Then $j = 1$, $r = n$, $\tau = \alpha^t$ and our assertion is nothing but Proposition (7.11) of **[R.G.-H.J.]**.

REMARK (3.1.4). If π is square-integrable then the poles of $L(s, \pi)$ are in the half-plane $\mathrm{Re}(s) \leqq 0$. This follows from (3.1.3) or from Proposition (1.3) of **[R.G.-H.J.]**.

(3.2) *Tempered representations.* Let again P be a parabolic subgroup of type $(n_1, n_2, \ldots, n_r)$. Let now σ_i be a square-integrable representation of G_{n_i}. Then it is well known that the induced representation

(3.2.1) $$\pi = I(G, P; \sigma_1, \sigma_2, \cdots, \sigma_r)$$

is irreducible (cf. for instance **[H.J. 2]**). The irreducible representations of this type are precisely the *tempered ones.* Note that if π is equivalent to another representation

(3.2.2) $$\pi' = I(G, P'; \sigma_1', \sigma_2', \cdots, \sigma_{r'}')$$

where the σ_i' are square-integrable, then P' and P are associate. More precisely $r = r'$, $P = MU_P$, $P = M'U_{P'}$, and there is an inner automorphism of G_n taking M to M' and $\sigma = \times\sigma_i$ to $\sigma' = \times\sigma_i'$; in other words one passes from $(M, (\sigma_i))$ to $(M', (\sigma_i'))$ by a permutation of the diagonal blocks of M and M'. Conversely if P' and the σ_i' are related to P and the σ_i in this way, then π' is equivalent to π.

By (2.3),

(3.2.3) $$L(s, \pi) = \prod_i L(s, \sigma_i), \quad L(s, \tilde{\pi}) = \prod_i L(s, \tilde{\sigma}_i),$$
$$\varepsilon(s, \pi, \psi) = \prod_i \varepsilon(s, \sigma_i, \psi).$$

REMARK (3.2.4). It follows from (3.2.3) and (3.1.4) that the poles of $L(s, \pi)$ for π tempered are contained in the half-plane $\mathrm{Re}(s) \leqq 0$.

A representation π is said to be essentially tempered if it has the form $\pi = \pi_0 \otimes \alpha^t$ where π_0 is tempered and t is real. Then

$$L(s, \pi) = L(s + t, \pi_0). \tag{3.2.5}$$

(3.3) *Langlands construction.* We now review the work of Silberger and Wallach which extends the results of Langlands to the p-adic case.

Let $Q = M_Q U_Q$ be a parabolic subgroup of type $(p_1, p_2, \cdots, p_r)$ and for each i, $1 \leqq i \leqq r$, τ_i an irreducible essentially tempered representation of G_{m_i}. Set $\tau = \times \tau_i$. Then $\tau_i = \tau_{i,0} \otimes \alpha^{t_i}$ where $\tau_{i,0}$ is tempered and t_i real. We assume that

$$t_1 > t_2 > \cdots > t_r. \tag{3.3.1}$$

Then the induced representation

$$\eta = I(G, Q; \tau_1, \tau_2, \cdots, \tau_r) \tag{3.3.2}$$

has a largest proper subrepresentation η' (possibly $\{0\}$); the irreducible representation $\pi = \eta/\eta'$ is noted

$$\pi = J(G, P; \tau_1, \tau_2, \cdots, \tau_r). \tag{3.3.3}$$

Every irreducible representation π has the form (3.3.3) where P (standard) and the τ_i are uniquely determined.

The space V' of η' may be described explicitly. Let W be the space of τ and $\tilde{W}$ the space of $\tilde{\tau}$. Then V' is the space of F in the space V of ξ such that

$$\int \langle F(\bar{u}g), \tilde{v}\rangle \, d\bar{u} = 0, \qquad \tilde{v} \in \tilde{W}, g \in G_n. \tag{3.3.4}$$

The integral is absolutely convergent and extended to $\bar{U} = \bar{U}_Q = U_{\bar{Q}}$, where $\bar{Q} = {}^t Q$ is the parabolic subgroup opposed to Q. It easily follows that the coefficients of π can be obtained by integrals similar to (2.2.4). Namely let H: $G_n \times G_n \to C$ be a function satisfying the following properties:

$$H(u_1 m g_1, \bar{u}_2 m g_2) = H(g_1, g_2), \qquad u_1 \in U_Q, \bar{u}_2 \in \bar{U}_Q, m \in M_Q; \tag{3.3.5}$$

(3.3.6) for any g_1, g_2 the function $m \mapsto H(mg_1, g_2)$ is a coefficient of $\tau \otimes \delta_Q^{1/2}$;

(3.3.7) H is $K_n \times K_n$ finite on the right.

Then the function f defined by the convergent integral

$$f(g) = \int_{M \backslash G} H(hg, h) \, dh = \int_{\bar{U} \times K} H(\bar{u}kg, k) \, dk \, d\bar{u} \tag{3.3.8}$$

is a coefficient of π and all coefficients of π can be obtained in this way for suitable H (see (3.6.6) and (3.6.7) for convergence questions).

Instead of Q we may consider $\bar{Q}$. Then if condition (3.3.1) is replaced by the similar condition with the inequalities reversed, the representation analogous to (3.3.3) is defined. For instance the quotient

(3.3.9) $$\pi' = J(G, \bar{Q}; \tilde{\tau}_1, \tilde{\tau}_2, \cdots, \tilde{\tau}_r)$$

of the induced representation

(3.3.10) $$\eta' = I(G, \bar{Q}; \tilde{\tau}_1, \tilde{\tau}_2, \cdots, \tilde{\tau}_r)$$

is defined. Its coefficients are given by integrals (3.3.8) where H satisfies (3.1.5)–(3.1.7) with $(\bar{Q}, \tilde{\tau})$ instead of (Q, τ).

In particular suppose f is the coefficient of π defined by (3.3.8) where H satisfies (3.3.5)–(3.3.7) for (Q, τ). Then

(3.3.11) $$f^{\vee}(g) = \int_{M\backslash G} \tilde{H}(hg, h)\, dh,$$

with $\tilde{H}(g_1, g_2) = H(g_2, g_1)$. But $\tilde{H}$ indeed satisfies (3.3.5)–(3.3.7) for $(\bar{Q}, \tilde{\tau})$. Thus $f^{\vee}$ is a coefficient of π' and

(3.3.12) $$\pi' = \tilde{\pi}.$$

Of course $\bar{Q}$ is also conjugate to the standard parabolic subgroup Q' of type $(n_r, n_{r-1}, \cdots, n_1)$ so that

(3.3.13) $$\tilde{\pi} = \pi' = J(G, Q'; \tilde{\tau}_r, \tilde{\tau}_{r-1}, \cdots, \tilde{\tau}_1).$$

(3.4) THEOREM. *Let the notations be as in* (3.3). *Then*

$$L(s, \pi) = \prod L(s, \tau_i), \quad L(s, \tilde{\pi}) = \prod L(s, \tilde{\tau}_i),$$

$$\varepsilon(s, \pi, \psi) = \prod \varepsilon(s, \tau_i, \psi).$$

It is enough to prove the assertion relative to $L(s, \pi)$; indeed the one for $L(s, \tilde{\pi})$ can be obtained by exchanging π and $\tilde{\pi}$ and the last assertion follows then from (2.7) and (cf. (2.3)) $\gamma(s, \eta, \psi) = \prod_i \gamma(s, \tau_i, \psi)$. The proof will occupy the rest of this section and (3.5). By (2.3) and (2.7) there are polynomials P and $\tilde{P}$ such that

$$L(s, \pi) = P(q^{-s}) \prod L(s, \tau_i), \quad L(s, \tilde{\pi}) = \tilde{P}(q^{-s}) \prod L(s, \tilde{\tau}_i).$$

It follows from (3.2.4) and (3.3.1) that $L(s, \tau_1)^{-1}$ and $\prod L(1-s, \tilde{\tau}_i)^{-1}$ are relatively prime. Thus $P(q^{-s})$ is prime to $L(s, \tau_1)^{-1}$ ((2.7.5)). So if $P \neq 1$ then

(3.4.1) $$\prod_{2 \leq i \leq r} L(s, \tau_i)$$

is not in $I(\pi)$. Therefore it will suffice to show that (3.4.1) is indeed in $I(\pi)$.

(3.5) The identity

(3.5.1) $$L(s, \pi) = \prod_{1 \leq i \leq r} L(s, \tau_i)$$

is trivial for $r = 1$. Assuming it is true for $r - 1$, let us prove it for r. As we have just seen, it suffices to show (3.4.1) is in $I(\pi)$.

Set $n_1 = p_1$, $n_2 = p_2 + \cdots + p_r$. Let Q' be the parabolic subgroup of type $(p_2, \cdots, p_r)$ in G_{n_2}. Then the representation

(3.5.2) $$\sigma_2 = J(G_{n_2}, Q'; \tau_2, \cdots, \tau_r)$$

is defined. Set also $\sigma_1 = \tau_1$, $\sigma = \sigma_1 \times \sigma_2$ and let P be the parabolic subgroup of type (n_1, n_2) in G_n. Then π is actually a quotient of

$$\xi = I(G, P; \sigma_1, \sigma_2). \tag{3.5.3}$$

More precisely it is easy to see that the coefficients of π are given by the integrals (3.3.8) where H is any function satisfying (3.3.5)–(3.3.7) with (P, σ) instead of (Q, τ).

If f is a coefficient of π defined in this way, then

$$\begin{aligned} &Z(\Phi, s + \tfrac{1}{2}(n-1), f) \\ &\quad = \iint dk\, d\bar{u} \int_{G_n} |\det g|^{s+(n-1)/2}\, \Phi(k^{-1}\bar{u}^{-1}g) H(g, k)\, d^\times g, \end{aligned} \tag{3.5.4}$$

where $\bar{u}$ is integrated over $\bar{U} = U_{\bar{P}}$. This is also

$$\begin{aligned} &\iint dk\, dk' \iint |\det m_1|^{s+(n_1-1)/2}\, |\det m_2|^{s+(n_2-1)/2} \\ &\quad \cdot \delta_P\begin{pmatrix} m_1 & 0 \\ 0 & m_2 \end{pmatrix}^{-1/2} H\left[\begin{pmatrix} m_1 & 0 \\ 0 & m_2 \end{pmatrix} k', k\right] d^\times m_1\, d^\times m_2 \\ &\quad \cdot \iint \Phi\left[k^{-1}\begin{pmatrix} m_1 & y \\ xm_1 & m_2 + xy \end{pmatrix} k'\right] dx\, dy. \end{aligned} \tag{3.5.5}$$

Here m_i ranges over G_{n_i}. We are going to see that given $\Phi_2 \in \mathscr{S}(n_2 \times n_2, F)$ and a coefficient f_2 of σ_2 there are H and Φ so that $Z(\Phi_2, s + \frac{1}{2}(n_2 - 1), f_2)$ is equal to (3.5.5). Since, by the induction hypothesis, $L(s, \sigma_2)$ is equal to (3.4.1), it will follow that (3.4.1) belongs to $I(\pi)$ which will conclude the proof.

There is a coefficient f_1 of π_1 and a Φ_1 in $\mathscr{S}(n_1 \times n_1, F)$ with support in a compact neighborhood of e such that

$$Z(\Phi_1, s + \tfrac{1}{2}(n_1 - 1), f_1) = 1. \tag{3.5.6}$$

There are also $\Phi_{12} \in \mathscr{S}(n_1 \times n_2, F)$, $\Phi_{21} \in \mathscr{S}(n_2 \times n_1, F)$, with supports in a neighborhood of 0 so that the function $\Phi \in \mathscr{S}(n \times n, F)$ defined by

$$\Phi\begin{pmatrix} m_1 & y \\ x & m_2 \end{pmatrix} = \Phi_1(m_1)\Phi_{12}(y)\Phi_{21}(x)\, \Phi_2(m_2) \tag{3.5.7}$$

satisfies

$$\int \Phi\begin{bmatrix} m_1 & y \\ xm_1 & m_2 + xy \end{bmatrix} dx\, dy = \Phi_1(m_1)\Phi_2(m_2). \tag{3.5.8}$$

There are also two K-finite functions ξ and ξ' on K such that

$$\iint \Phi(k^{-1}zk')\, \xi(k)\xi'(k')\, dk\, dk' = \Phi(z). \tag{3.5.9}$$

Then:

$$\begin{aligned} &Z(\Phi_2, s + \tfrac{1}{2}(n-1), f_2) \\ &\quad = \iint |\det m_1|^{s+(n_1-1)/2}\, |\det m_2|^{s+(n_2-1)/2} f_1(m_1) f_2(m_2) \\ &\qquad \cdot d^\times m_1 d^\times m_2 \iint \xi(k)\xi'(k')\, dk\, dk' \\ &\qquad \cdot \iint \Phi\left[k^{-1}\begin{pmatrix} m_1 & y \\ xm_1 & m_2 + xy \end{pmatrix} k'\right] dx\, dy. \end{aligned} \tag{3.5.10}$$

The proof is then finished as in (2.5). Namely let dh and dh' be the normalized Haar measures on $K \cap \bar{P}$ and $K \cap P$ respectively. For $h \in K \cap \bar{P}$, $h' \in K \cap P$ set

$$h = \begin{pmatrix} h_1 & 0 \\ * & h_2 \end{pmatrix}, \qquad h' = \begin{pmatrix} h_1' & * \\ 0 & h_2' \end{pmatrix};$$

then (3.5.10) is also equal to

$$\begin{aligned} &\iint dk\, dk' \iint |\det m_1|^{s+(n_1-1)/2}\, |\det m_2|^{s+(n_2-1)/2} \\ (3.5.11)\qquad &\cdot\, \delta_P\begin{pmatrix} m_1 & 0 \\ 0 & m_2 \end{pmatrix}^{-1/2} H_1\left[\begin{pmatrix} m_1 & 0 \\ 0 & m_2 \end{pmatrix}, k, k'\right] d^\times m_1\, d^\times m_2 \\ &\cdot \iint \Phi\left[k^{-1}\begin{pmatrix} m_1 & y \\ xm_1 & m_2 + xy \end{pmatrix} k'\right] dx\, dy \end{aligned}$$

where

$$\begin{aligned} &H_1\left[\begin{pmatrix} m_1 & 0 \\ 0 & m_2 \end{pmatrix}, k, k'\right] \\ (3.5.12)\qquad &= \delta_P\begin{pmatrix} m_1 & 0 \\ 0 & m_2 \end{pmatrix}^{1/2} \iint \xi(hk)\xi'(h'k') f_1(h_1 m_1 h_1'^{-1}) f_2(h_2 m_2 h_2'^{-1})\, dh\, dh'. \end{aligned}$$

There is H satisfying (3.3.5)–(3.3.7) for (P, σ) so that

$$(3.5.13)\qquad H\left[\begin{pmatrix} m_1 & 0 \\ 0 & m_2 \end{pmatrix} k', k\right] = H_1\left[\begin{pmatrix} m_1 & 0 \\ 0 & m_2 \end{pmatrix}, k, k'\right].$$

Comparing (3.5.11) to (3.5.5) we obtain our assertion.

(3.6) *Unramified representations.* Suppose π is unramified, that is contains the unit representation of K. Then B_n denoting the parabolic subgroup of type $(1,1,1, \cdots, 1)$, there are $u_i \in C$ so that π is the unique unramified component of

$$(3.6.1)\qquad \xi = I(G, B_n; \mu_1, \mu_2, \cdots, \mu_n), \qquad \mu_i = \alpha^{u_i}.$$

We may even assume $u_1 \geqq u_2 \geqq \cdots \geqq u_n$. Changing notations we may write the n-tuple of the μ_i in the form

$$(\nu_1^1, \nu_2^1, \cdots, \nu_{n_1}^1, \nu_1^2, \nu_2^2, \cdots, \nu_{n_2}^2, \cdots, \nu_1^r, \nu_2^r, \cdots, \nu_{n_r}^r)$$

where $n = n_1 + n_2 + \cdots + n_r$, $\nu_i^j = \alpha^{t_j}\lambda_i^j$, λ_i^j is a character, t_j is real and $t_1 > t_2 > \cdots > t_r$. Then $\tau_j = I(G_{n_j}, B_{n_j}; \nu_1^j, \nu_2^j, \cdots, \nu_{n_j}^j)$ is an essentially tempered unramified representation of G_{n_j} and in fact

$$(3.6.2)\qquad \pi = J(G, Q; \tau_1, \tau_2, \cdots, \tau_r),$$

if Q is the parabolic subgroup of type $(n_1, n_2, \cdots, n_r)$. This follows from an explicit computation: if H satisfies the conditions (3.3.5)–(3.3.7) for $(Q, \times\, \tau_j)$ and $H(k, k') = 1$ then

$$(3.6.3)\qquad \int H(\bar{u}, e)\, d\bar{u} \neq 0.$$

Thus the "J-component" of $\eta = I(G, Q; \tau_1, \tau_2, \cdots, \tau_r)$ is unramified; since $\xi \cong \eta$ we arrive at (3.6.2).

Thus

$$L(s, \pi) = \prod_i L(s, \mu_i), \quad L(s, \tilde{\pi}) = \prod L(s, \mu_i^{-1}),$$

(3.6.4)

$$\varepsilon(s, \pi, \psi) = \prod_i \varepsilon(s, \mu_i, \psi) \quad (= 1 \text{ if the exponent of } \psi \text{ is zero}).$$

For more precise results see **[R.G.-H.J.**, §6].

REMARK (3.6.5). Let the notations be as in (2.3). Suppose σ_i is irreducible unramified. Then ξ admits a unique unramified component π. It easily follows from (3.6.4) that $L(s, \pi) = \prod L(s, \sigma_i)$ with similar relations for $L(s, \tilde{\pi})$ and ε.

REMARK (3.6.6). Let π be a tempered representation of G_n. Let $\rho_n = I(G_n, B_n; 1, 1, \cdots, 1)$ and Ξ_n be the spherical function attached to ρ. Then any coefficient of π is majorized by a multiple of Ξ_n. Let the notations be now as in (3.3). Then the coefficients of τ_i are majorized by a multiple of $\Xi_{n_i} \otimes \alpha^{t_i}$. Moreover if H satisfies (3.3.5)–(3.3.7) for (Q, τ) then $|H| \leqq cH_0$ where H_0 satisfies (3.3.5) and

$$H_0(mk, k') = \delta_P^{1/2}(m) \prod_i \Xi_{n_i}(m_i).$$

Thus as far as convergence is concerned we may replace τ_i by $\rho_{n_i} \otimes \alpha^{t_i}$ and H by $H_0 > 0$. Together with the next lemma, this takes care of all problems of convergence.

LEMMA (3.6.7). *If Ω is a bounded set of $\mathscr{S}(n \times n, F)$, then there is $\Phi_0 \geqq 0$ in $\mathscr{S}(n \times n, F)$ so that $\Phi_0(k_1 x k_2) = \Phi_0(x)$ for $k_i \in K$ and $|\Phi| \leqq \Phi_0$.*

We may assume that Ω contains also all functions $x \mapsto \Phi(k_1 x k_2)$ with $\Phi \in \Omega$. Then there is $\Phi_1 \geqq 0$ so that $|\Phi| \leqq \Phi_1$ for all Φ in Ω **[A.W. 1**, Lemma 5]. It suffices to take $\Phi_0(x) = \iint \Phi_1(k_1 x k_2)\, dk_1\, dk_2$ where dk_i is the normalized Haar measure on K.

(3.7) According to "Langlands' philosophy" there should be a "natural bijection" $\sigma \mapsto \pi(\sigma)$ between the n-dimensional semisimple representations of the Weil-Deligne group W_F' and the irreducible admissible representations of $G_n(F)$. Moreover, one should have for $\pi = \pi(\sigma)$

$$L(s, \pi) = L(s, \sigma), \qquad \varepsilon(s, \pi, \psi) = \varepsilon(s, \sigma, \psi),$$

$$\pi(\sigma)^{\sim} = \pi(\tilde{\sigma}), \qquad \pi(\sigma \otimes \chi) = \pi(\sigma) \otimes \chi.$$

It is clear that if the map $\sigma \mapsto \pi(\sigma)$ could be defined for the irreducible representations of the Weil group, then the conjecture would be proved (cf. §5 below).

4. Local archimedean theory. In §§4, 5 the ground field F is local archimedean.

(4.1) We let K_n be $O(n, \boldsymbol{R})$ if $F = \boldsymbol{R}$ and $U(n)$ if $F = \boldsymbol{C}$. We donote by $\mathfrak{G}_n$ the Lie algebra of the real Lie group $G_n(F)$. We consider only admissible representations of the pair $(\mathfrak{G}_n, K_n)$ (as in **[N.W.]**) although we will often allow ourselves to speak of a representation of the group $G_n(F)$; in any case, we assume the reader to be thoroughly familiar with the relation between representations of the group $G_n(F)$ and of the pair $(\mathfrak{G}_n, K_n)$.

Let π be an admissible representation of $(\mathfrak{G}_n, K_n)$; then one can define the contragredient representation $\tilde{\pi}$, the coefficients of π, and its central quasi-character, even though π is not a representation of the group (cf. **[H.J.-R.L.**, §§5, 6]). Again if f is a coefficient of π then $f^{\vee}$ ((1.1.2)) is a coefficient of $\tilde{\pi}$.

As in the nonarchimedean case let $\mathscr{S}(p \times q, F)$ be the space of Schwartz functions on $M(p \times q, F)$. We also introduce the subspace $\mathscr{S}_0 = \mathscr{S}_0(p \times q, F)$ of functions Φ of the form (P being a polynomial),

$$\Phi(x) = P(x_{ij})\exp(-\pi \textstyle\sum x_{ij}^2) \qquad \text{if } F = \boldsymbol{R},$$

$$\Phi(z) = P(z_{ij}, \bar{z}_{ij})\exp(-2\pi \textstyle\sum z_{ij}\bar{z}_{ij}), \qquad \text{if } F = \boldsymbol{C}.$$

We can consider, for a given π, the integrals (1.1.3) where Φ is in $\mathscr{S}(p \times q, F)$ and f is a coefficient of π.

(4.2) PROPOSITION. *Suppose π is an irreducible admissible representation of* $(\mathfrak{G}_n, K_n)$.

(1) *There is s_0 so that the integrals* (1.1.3) *converge absolutely in the half-plane* $\mathrm{Re}(s) > s_0$.

(2) *For Φ in $\mathscr{S}_0(n \times n, F)$, the integrals are meromorphic functions of s in the whole complex plane. More precisely they can be written as polynomials in s times a fixed meromorphic function of s.*

(3) *Let $\psi \neq 1$ be an additive character of F; there is a meromorphic function $\gamma(s, \pi, \psi)$ so that, for all coefficients f of π and all $\Phi \in \mathscr{S}_0(n \times n, F)$,*

$$Z(\Phi^\wedge, 1 - s + \tfrac{1}{2}(n-1), f^\vee) = \gamma(s, \pi, \psi) Z(\Phi, s, f).$$

Again $\Phi^\wedge$ is defined by (1.2.4). If Φ is in $\mathscr{S}_0(n \times n, F)$ and ψ is defined by

$$\text{(4.2.4)} \qquad \begin{aligned} \psi(x) &= \exp(\pm 2i\pi x), && \text{if } F = \boldsymbol{R}, \\ \psi(z) &= \exp[\pm 2i\pi(z + \bar{z})], && \text{if } F = \boldsymbol{C}, \end{aligned}$$

then $\Phi^\wedge$ is still in $\mathscr{S}_0$; the left-hand side of (4.2.3) has then a meaning by (4.2.2) applied to $\tilde{\pi}$. If ψ is not given by (4.2.4) then some adjustment has to be made, that we leave to the reader (cf. (1.3.11), (1.3.12)). From now on, we assume ψ is given by (4.2.4).

EXAMPLE (4.2.5). If $n = 1$, then π is just a quasi-character of $F^\times$ and (4.2) is proved in **[J.T.]** or **[A.W. 2]**.

(4.3) The proof of Proposition (4.2) will be given below in (4.4). For the time being, we derive some simple consequences of (4.2).

By Lemma (3.6.7), the convergence of (1.1.3) for Re $s > s_0$ is actually uniform for Φ in a bounded set; thus, for Re $s > s_0$,

$$\text{(4.3.1)} \qquad \Phi \mapsto Z(\Phi, s, f)$$

is a distribution, depending holomorphically on s. Since $\mathscr{S}_0(n \times n, F)$ is dense in $\mathscr{S}(n \times n, F)$, it follows that if $f \neq 0$ then there is at least a $\Phi \in \mathscr{S}_0$ so that $Z(\Phi, s, f) \not\equiv 0$.

Suppose (4.2.2) has been proved. Let $L(s)$ be a meromorphic function of s so that, for all Φ in $\mathscr{S}_0$, $Z(\Phi, s + \frac{1}{2}(n-1), f)/L(s)$ is a polynomial; let also I be the subvector space of $\boldsymbol{C}[s]$ spanned by those ratios. By what we have just seen $I \neq \{0\}$. Moreover if Φ is in $\mathscr{S}_0$, so is the function Φ' defined by

$$\text{(4.3.2)} \qquad \Phi'(x) = \frac{d}{dt} \Phi(xe^{-t})\Big|_{t=0}.$$

Let ω be the central quasi-character of π; if we differentiate with respect to t the identity

(4.3.3)
$$Z(\Phi, s + \tfrac{1}{2}(n-1), f) = \int \Phi(xe^{-t}) f(x) |\det x|^{s+(n-1)/2} d^\times x \\ \omega(e^{-t}) \exp\left[-ts - \frac{t}{2}(n-1)\right]$$

and then set $t = 0$, we arrive at a relation of the form

(4.3.4) $(as + b)\, Z(\Phi, s + \tfrac{1}{2}(n-1), f) + Z(\Phi', s + \tfrac{1}{2}(n-1), f) = 0, \qquad a \neq 0.$

This shows that I is an ideal. Let P_0 be a generator of I and $L(s, \pi) = P_0(s) L(s)$. We see that the ratios $Z(\Phi, s + \tfrac{1}{2}(n-1), f)/L(s, \pi)$ are again polynomials, but this time they span $C[s]$ as a vector space. Up to multiplication by a constant, these properties characterize $L(s, \pi)$.

Assume now (4.2) proved for the pairs $(\pi, \tilde{\pi})$. Then γ is uniquely determined and one can define a factor ε as in (1.3.5); then the functional equation (4.2.3) reads as in (1.3.6). In view of the definition of the L-factors, this implies that both $\varepsilon(s, \pi, \psi)$ and its inverse are in $C[s]$. Thus $\varepsilon(s, \pi, \psi)$ is just a constant if ψ is given by (4.2.4); otherwise it is an exponential factor ((1.3.9)).

REMARK (4.3.5). For the time being the L- and ε-factors are defined up to multiplication by constants; of course these constants are related since γ is intrinsically defined. For $n = 1$, one may take the factors to be those given in **[J.T.1, 2]**.

REMARK (4.3.6). Relations (1.3.9)–(1.3.12) apply to the archimedean case.

(4.4). Let the notations be as in (2.1), the ground field F being now $\boldsymbol{R}$ or $\boldsymbol{C}$; it goes without saying that σ_i is now an admissible representation of $(\mathfrak{G}_{n_i}, K_{n_i})$. The induced representation ξ ((2.1.4)) can still be defined **[N.W.]** and its coefficients are given by the rule of (2.2), the only difference being that in (2.2.2) g_i must be in K_n and in (2.2.3) H must be C^∞ on $G_n \times G_n$. The remarks made before (2.3) apply and, as in (2.3), we have:

PROPOSITION (4.4.1). *With the notations of* (2.1) *suppose that each σ_i admits a central quasi-character and the assertions of* (4.2) *are true for each pair $(\sigma_i, \tilde{\sigma}_i)$. Then they are true for $(\xi, \tilde{\xi})$. Moreover, $\gamma(s, \xi, \psi) = \prod_i \gamma(s, \sigma_i, \psi)$ and one can take*

$$L(s, \xi) = \prod_i L(s, \sigma_i), \quad L(s, \tilde{\xi}) = \prod_i L(s, \tilde{\sigma}_i),$$
$$\varepsilon(s, \xi, \psi) = \prod_i \varepsilon(s, \sigma_i, \psi).$$

The proof is similar to the proof of (2.3). It suffices to observe that if Φ is in $\mathscr{S}_0(n \times n; F)$ the functions $x \mapsto \Phi(k_1 x k_2)$, $k_i \in K_n$, span a finite dimensional vector space of $\mathscr{S}_0$ and for each k, k' in K_n the functions (2.4.2) belong to the space $\bigotimes \mathscr{S}_0(n_i \times n_i, F)$.

Notations being as in (4.4.1), let π be an irreducible component of ξ. Then any coefficient of π is a coefficient of ξ and $\tilde{\pi}$ is a component of $\tilde{\xi}$. Thus (4.2) is true for $(\pi, \tilde{\pi})$. More precisely there are polynomials R and $\tilde{R}$ such that

(4.4.2) $$L(s, \pi) = R(s) \prod_i L(s, \sigma_i), \quad L(s, \tilde{\pi}) = \tilde{R}(s) \prod_i L(s, \tilde{\sigma}_i)$$

while

$$\gamma(s, \pi, \psi) = \prod_i \gamma(s, \sigma_i, \psi). \tag{4.4.3}$$

Since $\varepsilon(s, \pi, \psi)$ and $\varepsilon(s, \sigma, \psi)$ are constants R and $\tilde{R}$ have the form

$$R(s) = \prod_i (s - s_i), \quad \tilde{R}(s) = c\prod_i (1 - s - s_i). \tag{4.4.4}$$

Finally any given irreducible admissible representation π of G_n is a component of some induced representation ξ as in (4.4.1) with $n_i = 1$. Since (4.2) is then true for each $(\sigma_i, \tilde{\sigma}_i)$, it is true for $(\pi, \tilde{\pi})$ and (4.2) is proved.

(4.5) PROPOSITION. *For all s, $L(s, \pi) \neq 0$.*

PROOF. Let π be an irreducible admissible representation of G_n so that π is an irreducible representation of some induced representation ξ as in (4.4.1) with $n_i = 1$. We first show that for any Φ in $\mathscr{S}(n \times n, F)$ the ratio

$$Z(\Phi, s + \tfrac{1}{2}(n - 1), f)/L(s, \pi) \tag{4.5.1}$$

continues to an entire function of s.

Indeed let s_0 be such that for $\mathrm{Re}(s) > s_0$ (resp. $\mathrm{Re}(s) < 1 - s_0$) the integrals $Z(\Phi, s + \frac{1}{2}(n - 1), f)$ (resp. $Z(\Phi, 1 - s + \frac{1}{2}(n - 1), f^\vee)$) converge absolutely. Select s_1, s_2 such that $s_0 < s_1 < s_2$. Let also Q be a polynomial such that $Q(s)L(s, \xi)$ has no pole in the strip $1 - s_2 \leqq \mathrm{Re}(s) \leqq s_2$. Let Φ_i be a sequence of $\mathscr{S}_0$ converging to some element Φ of $\mathscr{S}$. Then, for a given f and a given polynomial R, the following limits are uniform in the domain $s_1 \leqq \mathrm{Re}(s) \leqq s_2$:

$$\lim_{i \to +\infty} R(s)Z(\Phi_i, s + \tfrac{1}{2}(n - 1), f) = R(s)Z(\Phi, s + \tfrac{1}{2}(n - 1), f), \tag{4.5.2}$$

$$\lim_{i, j \to +\infty} R(s)Z(\Phi_i - \Phi_j, s + \tfrac{1}{2}(n - 1), f) = 0. \tag{4.5.3}$$

Similarly $\Phi_i^\wedge - \Phi_j^\wedge \to 0$ so that, uniformly in the domain $1 - s_2 \leqq \mathrm{Re}(s) \leqq 1 - s_1$,

$$\lim_{i, j \to \infty} R(s)Z(\Phi_i^\wedge - \Phi_j^\wedge, 1 - s + \tfrac{1}{2}(n - 1), f^\vee) = 0. \tag{4.5.4}$$

Indeed by using repeatedly (4.3.4) one can reduce these assertions to the case $R = 1$; they are then obvious.

On the other hand,

$$Q(s)Z(\Phi_i - \Phi_j, s + \tfrac{1}{2}(n - 1), f)$$
$$= \varepsilon(s, \xi, \psi)^{-1} \frac{L(s, \xi)}{L(1 - s, \tilde{\xi})} Q(s)Z(\Phi_i^\wedge - \Phi_j^\wedge, 1 - s + \tfrac{1}{2}(n - 1), f^\vee).$$

The classical formula

$$\frac{\Gamma(x + iy)}{\Gamma(x' - iy)} \cong |y|^{x-x'} \qquad (|y| \to +\infty)$$

shows that $Q(s)L(s, \xi)/L(1 - s, \tilde{\xi})$ is bounded by a polynomial in the previous strip. Thus we also find, uniformly for $1 - s_2 \leqq \mathrm{Re}(s) \leqq 1 - s_1$,

$$\lim_{i, j \to +\infty} Q(s)Z(\Phi_i - \Phi_j, s + \tfrac{1}{2}(n - 1), f) = 0. \tag{4.5.5}$$

Now fix f and let $\varepsilon > 0$ be given; choose N so that for $i, j \geqq N$

$$|Q(s)Z(\Phi_i - \Phi_j, s + \tfrac{1}{2}(n - 1), f| \leqq \varepsilon,$$

for $s_1 \leqq \mathrm{Re}(s) \leqq s_2$ or $1 - s_2 \leqq \mathrm{Re}(s) \leqq 1 - s_1$. But now

$$Q(s)Z(\Phi_i - \Phi_j, s + \tfrac{1}{2}(n - 1), f) = Q(s)R(s)L(s, \xi)$$

where R is another polynomial. Since $L(s, \xi)$ decreases rapidly, on any vertical strips, we find that given (i, j) there is M so that, for $|\mathrm{Im}(s)| \geqq M$,

(4.5.6) $$|Q(s)Z(\Phi_i - \Phi_j, s + \tfrac{1}{2}(n - 1), f)| \leqq \varepsilon.$$

Thus by the maximum principle (4.5.6) is satisfied for $i, j \geqq N$ and $1 - s_2 \leqq \mathrm{Re}(s) \leqq s_2$. It follows that $Q(s)Z(\Phi_i, s + \frac{1}{2}(n - 1), f)$ is a Cauchy sequence for the topology of uniform convergence on the strip $1 - s_2 \leqq \mathrm{Re}(s) \leqq s_2$. Its limit $Z(s)$ is holomorphic on $1 - s_2 < \mathrm{Re}(s) < s_2$ and coincides with $Q(s)Z(\Phi, s + \frac{1}{2}(n - 1), f)$ on $s_1 \leqq \mathrm{Re}(s) < s_2$. Since s_2 is arbitrarily large the holomorphy of (4.5.1) follows.

Suppose now $L(s_0, \pi) = 0$ and $f \neq 0$. Then for each Φ, the meromorphic function $Z(\Phi, s + \frac{1}{2}(n - 1), f)$ vanishes at s_0; but if Φ has compact support contained in G_n then $Z(\Phi, s + \frac{1}{2}(n - 1), f)$ is defined for all s by the convergent integral (1.1.3). So we find that (1.1.3) vanishes for $s = s_0$ and $\Phi \in C_c^\infty(G_n)$; therefore $f \equiv 0$, a contradiction. Thus $L(s_0, \pi) \neq 0$. Q.E.D.

COROLLARY (4.5.8). *Suppose ξ is as in* (4.4.1) *and π is an irreducible component of ξ. Let R and $\tilde{R}$ be as in* (4.4.2). *Then if s_i is a zero of order m of R (resp. $\tilde{R}$) it is a pole of order $\geqq m$ of $L(s, \xi)$ (resp. $L(s, \tilde{\xi})$).*

5. Computation of the L-factor; archimedean case. Recall there is a "natural bijection" $\lambda \mapsto \pi(\lambda)$ between the set of classes of semisimple representations of the Weil group W_F of degree n and the admissible irreducible representations of $G_n(F)$ (**[R.L.]**, **[A.K.-G.Z]**, **[N.W.]**). In particular $\pi(\lambda)^\sim = \pi(\tilde{\lambda})$.

(5.1) THEOREM. *For any λ of degree n, let π be $\pi(\lambda)$. Then*

$$L(s, \pi) = L(s, \lambda), \qquad L(s, \tilde{\pi}) = L(s, \tilde{\lambda}),$$
$$\varepsilon(s, \pi, \psi) = \varepsilon(s, \lambda, \psi).$$

The proof of this theorem will occupy all of §5. Since the L-factor has not been normalized, it would be more correct to say that one can take $L(s, \pi)$ to be $L(s, \lambda)$ and that

$$\gamma(s, \pi, \psi) = \varepsilon(s, \lambda, \psi)L(1 - s, \tilde{\lambda})/L(s, \lambda).$$

(5.2) *Square-integrable representations.* Suppose λ is irreducible. Then $\pi(\lambda)$ is essentially square-integrable and conversely. More precisely, if $n = 1$ then λ is a quasi-character of $F^\times$, $\lambda = \pi$, and our assertion follows from the definition of the factors L and ε for λ. Suppose $n \neq 1$; then $n = 2$ and our assertion has been proved in [**H.J.-R.L.**, Theorem (13.1), Lemmas (13.24) and (5.17)]; actually in this case, the proof is a refinement of (4.4.1).

(5.3) *Tempered representations.* Suppose λ is unitary; then π is tempered and conversely. More precisely $\lambda = \bigoplus \lambda_i$ where λ_i is irreducible of degree n_i. Set $\sigma_i =$

$\pi(\lambda_i)$. Then $\pi = I(G, P; \sigma_1, \sigma_2, \cdots, \sigma_r)$ where P is the parabolic subgroup of type $(n_1, n_2, \cdots, n_r)$. By (4.4.1)

$$(5.3.1) \qquad L(s, \pi) = \prod L(s, \sigma_i) = \prod L(s, \lambda_i) = L(s, \lambda).$$

The factors $L(s, \tilde{\pi})$ and $\varepsilon(s, \pi, \psi)$ are computed similarly so our assertion follows again.

The case of the essentially tempered representations follows from (1.3.11), (1.3.12), and the relation

$$(5.3.2) \qquad \pi(\lambda \otimes \chi) = \pi(\lambda) \otimes \chi.$$

REMARK (5.3.3). It follows from (5.3.1) that, when π is tempered, the poles of $L(s, \pi)$ are in the half-plane $\mathrm{Re}(s) \leqq 0$.

(5.4) *General case.* In general, we may write $\lambda = \bigoplus \mu_i$, $\mu_i = \mu_{i,0} \otimes \alpha^{t_i}$ where $\mu_{i,0}$ is unitary of degree p_i, t_i real. We set then $\tau_{i,0} = \pi(\mu_{i,0})$, $\tau_i = \pi(\mu_i)$ so that by (5.3.2) $\tau_i = \tau_{i,0} \otimes \alpha^{t_i}$. We may assume (3.3.1) is satisfied. Then if Q is the parabolic subgroup of type $(p_1, p_2, \ldots, p_r)$ the induced representation η of (3.3.2) admits a unique irreducible quotient noted as in (3.3.3). That quotient is $\pi = \pi(\lambda)$.

By (4.4.1) and (5.3) we already know that

$$\gamma(s, \pi, \psi) = \prod_i \gamma(s, \tau_i, \psi) = \varepsilon(s, \lambda, \psi) L(1 - s, \tilde{\lambda}) / L(s, \lambda).$$

So it will suffice to compute $L(s, \pi)$. For exchanging then λ and $\tilde{\lambda}$ we will get $L(s, \tilde{\pi})$ (since $\tilde{\pi} = \pi(\tilde{\lambda})$) and the ε-factor from the γ-factor. So it will suffice to show that $L(s, \pi) = \prod_{1 \leqq i \leqq r} L(s, \tau_i)$. This is trivial if $r = 1$. So we may assume $r > 1$ and our assertion true for $r - 1$.

A priori, we have

$$L(s, \pi) = P(s) \prod L(s, \tau_i), \quad L(s, \tilde{\pi}) = \tilde{P}(s) \prod L(s, \tilde{\tau}_i)$$

where P and $\tilde{P}$ are polynomials related by (4.4.4). Let s_0 be a zero of order u of P. By (4.5.8) it is a pole of order $\geqq u$ of $\prod_i L(s, \tau_i)$ and by (4.4.4) a pole of order $\geqq u$ of $\prod_i L(1 - s, \tilde{\tau}_i)$. But $L(s, \tau_1)$ and $\prod L(1 - s, \tilde{\tau}_i)$ cannot have a common pole ((5.3.2), (5.3.3), (3.3.1)). Thus it will suffice to show that any pole of order u of $\prod_{i \geqq 2} L(s, \tau_i) = \prod_{i \geqq 2} L(s, \mu_i)$ which is not a pole of $L(s, \tau_1)$ is a pole of order $\geqq u$ of $L(s, \pi)$. This will be proved in (5.5) and (5.6).

(5.5) Let now the notations be as in (3.5). So set $n_1 = p_1$, $n_2 = n - n_1$, $\sigma_1 = \pi(\mu_1) = \tau_1$, $\sigma_2 = \pi(\mu_2 \oplus \ldots \oplus \mu_r)$, $\sigma = \sigma_1 \times \sigma_2$. By the induction hypothesis

$$(5.5.1) \qquad L(s, \sigma_2) = \prod_{2 \leqq i \leqq r} L(s, \mu_i).$$

Again π is a quotient of (3.5.3) where $P = MU$ is the parabolic subgroup of type (n_1, n_2). The coefficients of π are given by the absolutely convergent integrals

$$(5.5.2) \qquad f(g) = \int_{M \backslash G} H(hg, h)\, dh = \int_{K \times \bar{U}} H(\bar{u}kg, k)\, dk$$

where $\bar{U} = U_{\bar{P}}$, $\bar{P} = {}^tP$ and $H\colon G_n \times G_n \to C$ is any function satisfying the following properties:

$$(5.5.3) \qquad H(u_1 m g_1, \bar{u}_2 m g_2) = H(g_1, g_2), \qquad u_1 \in U, \bar{u}_2 \in \bar{U}, m \in M;$$

(5.5.4) for $k_i \in K_n$, the function $m \mapsto H(mk_1, k_2)$ is a coefficient of $\sigma \otimes \delta_P^{1/2}$,

(5.5.5) H is C^∞ and $K_n \times K_n$ finite on the right.

This being so if f is given by (5.5.2) then $Z(\Phi, s + \frac{1}{2}(n-1), f)$ is for Re(s) large enough, equal to (3.5.5) (convergence questions are left to the reader since they can be handled by (3.6.6) and (3.6.7)). Conversely let $\Phi \in \mathscr{S}(n \times n, F)$ and f_i a coefficient of σ_i be given. Set

$$A(\Phi, s, f_1, f_2) = \int |\det m_1|^{s+(n_1-1)/2} |\det m_2|^{s+(n_2-1)/2} f_1(m_1) f_2(m_2)\, \Phi\left[\begin{pmatrix} m_1 & y \\ xm_1 & m_2 + xy \end{pmatrix}\right] d^\times m_1\, d^\times m_2\, dx\, dy. \tag{5.5.6}$$

This multiple integral converges absolutely if Re(s) is large enough. We are going to show that

(5.5.7) *For any Φ, the quotient $A(\Phi, s, f_1, f_2)/L(s, \pi)$ continues to an entire function of s. If Φ is in $\mathscr{S}_0$, it is a polynomial in s.*

Assume first Φ is in $\mathscr{S}_0(n \times n, F)$. Then there are two K-finite functions ξ and ξ' on K_n such that (3.5.9) is satisfied. Let dh be the normalized Haar measure on the compact group $K \cap P = K \cap \bar{P} = K \cap M$; for h in this group write

$$h = \begin{pmatrix} h_1 & 0 \\ 0 & h_2 \end{pmatrix}, \qquad h_i \in K_{n_i}. \tag{5.5.8}$$

Then there is a function H satisfying (5.5.3)–(5.5.5) such that

$$H\left[\begin{pmatrix} m_1 & 0 \\ 0 & m_2 \end{pmatrix} k', k\right] = \delta_P^{1/2}\begin{pmatrix} m_1 & 0 \\ 0 & m_2 \end{pmatrix} \iint f_1(h_1 m_1 h_1'^{-1}) f_2(h_2 m_2 h_2'^{-1}) \xi'(h'k')\xi(hk)\, dh\, dh'. \tag{5.5.9}$$

If f is the coefficient of π corresponding to H by (5.5.2) we get, as in (3.5),

$$A(\Phi, s, f_1, f_2) = Z(\Phi, s + \tfrac{1}{2}(n-1), f). \tag{5.5.10}$$

This establishes (5.5.7) for Φ in $\mathscr{S}_0$.

Let us note also that there is for A a relation similar to (4.3.4) where Φ' is defined by:

$$\Phi'\begin{pmatrix} m_1 & x \\ y & m_2 \end{pmatrix} = \frac{d}{dt} \Phi\begin{pmatrix} m_1 e^{-t} & x \\ y e^{-t} & m_2 \end{pmatrix}\Bigg|_{t=0} \tag{5.5.11}$$

and is in $\mathscr{S}_0$ if Φ is. Thus the product of A by any polynomial is an integral of the same type with Φ replaced by Φ_1. If Φ is in $\mathscr{S}_0$ so is Φ_1. Moreover $\Phi \mapsto \Phi_1$ is continuous.

We are going to see now that there is a functional equation

$$B(\Phi^\wedge, 1 - s + \tfrac{1}{2}(n-1), f_1^\vee, f_2^\vee) = \gamma(s, \pi, \psi) A(\Phi, s, f_1, f_2) \tag{5.5.12}$$

where B is obtained by replacing in the definition of A the pair (P, σ) by the pair $(\bar{P}, \tilde{\sigma})$. It is then clear that (5.5.7) can be proved as the holomorphy of (4.5.1) (cf. proof of (4.5)).

To begin with it is clear that $\tilde{\pi}$ can be obtained from $(\bar{P}, \tilde{\sigma})$ as π is obtained from

(P, σ). (Cf. (3.3).) More precisely if f is a coefficient of π given by (5.5.2) then $f^\vee$ is given by (3.3.11) and $\tilde{H}$ (loc. cit.) satisfies (5.5.3)–(5.5.5) for $(\bar{P}, \tilde{\sigma})$.

This being so set

$$B(\Phi, s, f_1, f_2) = \int |\det m_1|^{s+(n_1-1)/2} |\det m_2|^{s+(n_2-1)/2} \tag{5.5.13}$$
$$f_1(m_1) f_2(m_2) \Phi\left[\begin{pmatrix} m_1 + yx & ym_2 \\ x & m_2 \end{pmatrix}\right] d^\times m_1\, d^\times m_2\, dx\, dy,$$

each time f_i is a coefficient of $\tilde{\sigma}_i$. Then for $\Phi \in \mathscr{S}_0$ (5.5.7) applies to B with $L(s, \tilde{\pi})$ instead of $L(s, \pi)$. Now let $\Phi \in \mathscr{S}_0$ and a coefficient f_i of σ_i be given; choose ξ, ξ' and H as above so that (5.5.9) is satisfied as well as (5.5.10) where f is given by (5.5.2). Then

$$\iint \Phi^\wedge[k^{-1}zk']\xi'(k)\xi(k')\, dk\, dk' = \Phi^\wedge(z).$$

Moreover $H^\sim$ being given as in (3.3.11) by $\tilde{H}(g_1, g_2) = H(g_2, g_1)$, we find

$$\tilde{H}\left[\begin{pmatrix} m_1 & 0 \\ 0 & m_2 \end{pmatrix} k', k\right]$$
$$= \delta_P^{1/2}\begin{pmatrix} m_1 & 0 \\ 0 & m_2 \end{pmatrix} \iint f_1^\vee[h_1 m_1 h_1'^{-1}] f_2^\vee[h_2 m_2 h_2'^{-1}] \xi(h'k')\xi'(hk)\, dh\, dh'.$$

Since $f^\vee$ is given by (3.3.11) we get

$$B(\Phi^\wedge, 1-s, f_1^\vee, f_2^\vee) = Z(\Phi^\wedge, 1 - s + \tfrac{1}{2}(n-1), f^\vee).$$

Comparing with (5.5.10) this concludes the proof of (5.5.12) and (5.5.7).

(5.6) Let $\mathscr{A}$ be the space of functions on $X = G_{n_1} \times \{M(n_2 \times n_2, F)\}$ which, in an obvious sense, are of compact support with respect to the first variable and of Schwartz type with respect to the second variable. We give to $\mathscr{A}$ the obvious topology. In particular $C_c^\infty(X)$ is a dense subset of $\mathscr{A}$. On the other hand we may regard $\mathscr{A}$ as a subspace of the space of Schwartz functions on $\{M(n_1 \times n_1, F)\} \times \{M(n_2 \times n_2, F)\}$. For Φ_1 in $C_c^\infty(G_{n_1})$, Φ_{12} in $C_c^\infty(M(n_1 \times n_2, F))$, Φ_{21} in $C_c^\infty(M(n_2 \times n_1, F))$, and Φ_2 in $\mathscr{S}(n_2 \times n_2, F)$ we may set

$$\Psi(m_1, x_2) = \Phi_1(m_1) \int \Phi_{21}(xm_1)\Phi_{12}(y)\Phi_2(x_2 + xy)\, dx\, dy. \tag{5.6.1}$$

We let $\mathscr{A}_0$ be the subspace of $\mathscr{A}$ spanned by these functions; it is easily checked that $\mathscr{A}_0$ contains a dense subspace of $C_c^\infty(X)$ so is dense in $\mathscr{A}$.

For Ψ in $\mathscr{A}$ and coefficients f_1, f_2 of σ_1, σ_2 we set

$$U(\Psi, s, f_1, f_2) = \iint \Psi(m_1, m_2) |\det m_1|^{s+(n_1-1)/2} |\det m_2|^{s+(n_2-1)/2} \tag{5.6.2}$$
$$\cdot f_1(m_1) f_2(m_2)\, d^\times m_1\, d^\times m_2.$$

The integral converges absolutely for $\mathrm{Re}(s)$ large enough. By (5.5.7) we know that

(5.6.3) *For Ψ in $\mathscr{A}_0$, the ratio $U(\Psi, s, f_1, f_2)/L(s, \pi)$ continues to an entire function of s.*

On the other hand, we are going to prove that

(5.6.4) *For Ψ in $\mathscr{A}$, the ratio $U(\Psi, s, f_1, f_2)/L(s, \sigma_2)$ continues to a holomorphic function of s. As such, it is a continuous function of (Ψ, s) on $\mathscr{A} \times \mathbf{C}$.*

Let us observe that the first assertion of (5.6.4) is trivial for Ψ in the (algebraic) tensor product

$$\mathscr{A}_1 = C_c^\infty(G_n) \otimes \mathscr{S}(n_2 \times n_2, F). \tag{5.6.5}$$

For if $\Psi = \Psi_1 \otimes \Psi_2$ the ratio is the product of an absolutely convergent integral by $Z(\Psi_2, s + \frac{1}{2}(n_2 - 1), f_2)/L(s, \sigma_2)$ which is a polynomial.

Moreover assume s_0 is a pole of order u of $L(s, \sigma_2)$; then there are f_1, f_2 and $\Psi \in \mathscr{A}_1$ so that the ratio of (5.6.4) has a pole of order u at s_0. Taking at the moment (5.6.4) for granted, assume s_0 is not a pole of $L(s, \sigma_1)$ and is a pole of order $< u$ of $L(s, \pi)$. Since $\mathscr{A}_0$ is dense in $\mathscr{A}$, it follows from (5.6.3) that the distribution $U(\cdot, s, f_1, f_2)$, which depends meromorphically on s, has a pole of order $< u$ at s_0. The same is true of the (scalar) meromorphic functions $U(\Psi, s, f_1, f_2)$. By taking Ψ in $\mathscr{A}_1$ we get a contradiction. This proves (5.1).

It remains therefore to prove (5.6.4). As noted, the first assertion is trivial for Ψ in $\mathscr{A}_1$, the ratio being then the product of a polynomial by a function bounded in vertical strips. Now there is a relation

$$(as + b)U(\Phi, s, f_1, f_2) + U(\Phi', s, f_1, f_2) = 0, \qquad a \neq 0, \tag{5.6.6}$$

where Φ' is in $\mathscr{A}_1$ if Φ is and depends continuously on Φ. Next introduce for Re s sufficiently small the integral

$$V(\Phi, s, f_1, f_2) = \int \Psi(m_1, m_2) \left|\det m_1\right|^{s+(n_1-1)/2} \left|\det m_2\right|^{1-s+(n_2-1)/2} f_1(m_1) f_2^\vee(m_2)\, d^\times m_1\, d^\times m_2.$$

Then, for $\Psi \in \mathscr{A}_1$,

$$V(\Psi^\wedge, s, f_1, f_2) = \gamma(s, \sigma_2, \psi) U(\Phi, s, f_1, f_2).$$

Again V satisfies a relation like (5.6.6). Finally, $\mathscr{A}_1$ is dense in $\mathscr{A}$. These remarks being made the proof of (5.6.4) is similar to the proof of the holomorphy of (4.5.1). This concludes the proof of (5.1).

6. Global theory. The field F is now an A-field.

(6.1) Let $\pi = \bigotimes_v \pi_v$ be an irreducible admissible representation of $G_n(A)$; again π may be a representation of some other object than the group **[I.P.S.]**. Then $\tilde{\pi} = \bigotimes \tilde{\pi}_v$. Set

$$L(s, \pi) = \prod_v L(s, \pi_v), \quad L(s, \tilde{\pi}) = \prod_v L(s, \tilde{\pi}_v). \tag{6.1.1}$$

Suppose that the central quasi-character ω of π is trivial on $F^\times$. Then set

$$\varepsilon(s, \pi) = \prod_v \varepsilon(s, \pi_v, \psi_v). \tag{6.1.2}$$

Here $\psi = \prod \psi_v$ is a nontrivial character of A/F; almost all factors in (6.1.2) are one and the product does not depend on ψ.

(6.2) THEOREM. *Suppose π is automorphic. Then the infinite products* (6.1.1) *converge absolutely in some right half-space. They continue to meromorphic functions of* *s in the whole complex plane. As such they satisfy* $L(s, \pi) = \varepsilon(s, \pi)L(1 - s, \tilde{\pi})$. *If F is a number field, they have finitely many poles and are bounded at infinity in vertical strips. If F is a function field whose field of constants has* Ω *elements, they are rational functions of* Ω^{-s}.

PROOF. If π is cuspidal this is Theorems 3, 4, 5, VII, §6 of **[A.W. 2]** for $n = 1$, and Theorem (13.8) of **[R.G.-H.J.]** for $n > 1$. If $n > 1$ and π is not cuspidal, there is a standard parabolic subgroup of type $(n_1, n_2, \cdots, n_r)$, and, for every i, an irreducible automorphic cuspidal representation σ_i of G_{n_i} such that the following conditions are satisfied: for any place v, π_v is an irreducible component of the induced representation $\xi_v = I(G_v, P_v; \sigma_{1v}, \sigma_{2v}, \cdots, \sigma_{rv})$. (Cf. **[R.L.]**.) Thus, for any place v,

$$L(s, \pi_v) = P_v(s) \prod_i L(s, \sigma_{i,v}),$$

$$L(s, \tilde{\pi}_v) = \tilde{P}_v(s) \prod_i L(s, \tilde{\sigma}_{i,v}),$$

$$\gamma(s, \pi_v, \psi_v) = \prod_i \gamma(s, \sigma_{i,v}, \psi_v),$$

where P_v is a polynomial in s if $F_v = \boldsymbol{R}$ or $\boldsymbol{C}$, and a polynomial in q_v^{-s} if F_v is non-archimedean with residual field of cardinality q_v. Moreover, for almost all v, π_v and the $\sigma_{i,v}$ are unramified so that ((3.6.5))

$$P_v = \tilde{P}_v = 1, \quad \varepsilon(s, \pi_v, \psi_v) = 1, \quad \text{almost all } v.$$

Thus

$$L(s, \pi) = \prod_v P_v(s) \prod_i L(s, \sigma_i),$$

$$L(s, \tilde{\pi}) = \prod_v \tilde{P}_v(s) \prod_i L(s, \tilde{\sigma}_i),$$

$$\varepsilon(s, \pi) = \prod_v \frac{\tilde{P}_v(1 - s)}{P_v(s)} \prod_i \varepsilon(s, \sigma_i).$$

Our assertions are then obvious, except the one on the boundedness of $L(s, \pi)$ in the number field case, which follows from the Phragmen-Lindelöf principle.

BIBLIOGRAPHY

[A.A.] A. Andrianov, *Zeta functions of simple algebras with non abelian characters*, Uspehi Mat. Nauk **23** (1968), 1–66 = Russian Math. Surveys **23** (4) (1968), 1–65.

[I.B.-A.Z. 1] I. N. Bernstein and A. V. Zelevinsky, *Representations of the group* GL(n, F) *where F is a non archimedean local field*, Uspehi Mat. Nauk **31** (3) (1976), 5–70 = Russian Math. Surveys **31** (3) (1976), 1–68.

[I.B.-A.Z. 2] ———, *Induced representations of the group* GL(n) *over a p-adic field*, Funkcional Anal. i Prilozen **10** (3) (1976), 74–75 = Functional Anal. Appl. **10** (3) (1976), 225–227.

[I.B.-A.Z. 3] ———, *Induced representations of reductive p-adic groups*. II, preprint.

[M.E.] M. Eichler, *Allgemeine Kongruenzklasseneinteilungen der Ideale einfacher Algebren über algebraischen Zahlkörpern und ihrer L-Reihen*, J. Reine Angew. Math. **179** (1938), 227–251.

[G.F. 1] G. Fujisaki, *On the zeta-functions of a simple algebra over the field of rational numbers*. Fac. Sci. Univ. Tokyo, Sect. 1 **7** (1958), 567–604.

[G.F. 2] ———, *On the L-functions of a simple algebra over the field of rational numbers*, Fac. Sci. Univ. Tokyo, Sect. 1 **9** (1962), 293–311.

[S.G.] S. Gelbart, *Fourier analysis on matrix spaces*, Mem. Amer. Math. Soc., vol. 108, Providence, R.I., 1971.

[R.G.] R. Godement, *Les fonction ζ des algèbres simples*. I, II, Sém. Bourbaki, 1957/1958, Exp. 171, 176.

[R.G.-H.J.] R. Godement and H. Jacquet, *Zeta functions of simple algebras*, Lecture Notes in Math., vol. 260, Springer-Verlag, 1972.

[E.H.] E. Hecke, *Über Modulfunktionen und die Dirichletschen Reihen mit Eulerscher Produktentwicklung*. I, II, Math. Ann. **114** (1937), 1–28; 316–351.

[K.H.] K. Hey, *Analytische Zahlentheorie in Systemen hyperkomplexer Zahlen*, Dissertation, Hamburg, 1929.

[K.I.] K. Iwasawa, *A note on L-functions*, Proc. Internat. Congr. Math., Cambridge, Mass., 1950, p. 322.

[H.J. 1] H. Jacquet, *Zeta functions of simple algebras (local theory)*, Harmonic Analysis on Homogeneous Spaces, Mem. Amer. Math. Soc., Providence, R.I., 1973, pp. 381–386.

[H.J. 2] ———, *Generic representations*, Non-Commutative Harmonic Analysis, Lecture Notes in Math., vol. 587, Springer-Verlag, Berlin, 1977, pp. 91–101.

[H.J.-R.L.] H. Jacquet and R. Langlands, *Automorphic forms on* GL(2), Lecture Notes in Math., vol. 114, Springer-Verlag, Berlin, 1970.

[M.K.] M. Kinoshita, *On the ξ-functions of a total matrix algebra over the field of rational numbers*, J. Math. Soc. Japan **17** (1965), 374–408.

[A.K.-G.Z.] A. W. Knapp and G. Zuckerman, *Classification theorems for representations of semi-simple Lie groups*, Non-Commutative Harmonic Analysis, Lecture Notes in Math., vol. 587, Springer-Verlag, Berlin, 1977, pp. 138–159.

[R.L.] R. P. Langlands, *On the classification of irreducible representations of real algebraic groups*, mimeographed notes, Institute for Advanced Study, Princeton (1973).

[H.M. 1] H. Maass, *Über eine neue Art von nicht-analytischen automorphen Funktionen und die Bestimmung Dirichletscher Reihen durch Functionalgleichungen*, Math. Ann. **121** (1949), 141–183.

[H.M. 2] ———, *Zetafunktionen mit Grössencharakteren und Kugelfunktionen*, Math. Ann. **134** (1957), 1–32.

[G.M. 1] G. N. Maloletkin, *Zeta functions of a semi-simple algebra over the field of rational numbers*, Dokl. Akad. Nauk SSSR **201** (3) (1971), 535–537 = Soviet Math. Dokl. **12** (6) (1971), 1678–1681.

[G.M. 2] ———, *Zeta functions of parabolic forms*, Mat. Sb. **86** (128) (1971), 622–643 = Math. USSR-Sb. **15** (4) (1971), 619–641.

[A.S.] A. Selberg, *Harmonic analysis and discontinuous groups in weakly symmetric riemannian spaces with applications to Dirichlet series*, J. Indian Math. Soc. **20** (1956), 47–87.

[H.S.] H. Shimizu, *On zeta functions of quaternion algebras*, Ann. of Math. (2) **81** (1965), 166–193.

[G.S.] G. Shimura, *On Dirichlet series and abelian varieties attached to automorphic forms*, Ann. of Math. (2) **76** (1962).

[E.S.] E. Stein, *Analysis in matrix spaces and some new representations of* SL(N, C), Ann. of Math. (2) **86** (1967), 461–490.

[T.T.] T. Tamagawa, *On the zeta functions of a division algebra*, Ann. of Math. (2) **77** (1963), 387–405.

[J.T. 1] J. Tate, *Fourier analysis in number fields and Hecke's zeta-functions*, Algebraic Number Theory, edited by J.W.S. Cassels and A. Fröhlich, Thompson, 1967, pp. 305–347.

[J.T. 2] ———, *Number theoretic background*, these PROCEEDINGS, part 2, pp. 3–26.

[N.W.] N. Wallach, *Representations of reductive Lie groups*, these PROCEEDINGS, part 1, pp. 71–86.

[G.W.] G. Warner, *Zeta functions on the real general linear group*, Pacific J. Math. **45** (1973), 681–691.

[A.W. 1] A. Weil, *Fonctions zeta et distributions*, Sém. Bourbaki, no. 312, (1965–66), 9 pp.

[A.W. 2] ———, *Sur certains groupes d'opérateurs unitaires*, Acta Math. **111** (1964), 144–211.

[A.W. 3] ———, *Basic number theory*, Springer-Verlag, Berlin, 1967.

[A.W. 4] A. Weil, *Dirichlet series and automorphic forms*, Lecture Notes in Math., vol. 189, Springer-Verlag, Berlin, 1971.

[E.W.] E. Witt, *Riemann-Rochscher Satz und ξ-Funktion im Hyperkomplexen*, Math. Ann. **110** (1934), 12–28.

[A.Z. 1] A. V. Zelevinskii, *Classification of irreducible non-cuspidal representations of the group* GL(n) *over p-adic fields*, Funkcional Anal. i Prilozen **11** (1) (1977), 67 = Functional Anal. Appl. **11** (1) (1977).

[A.Z. 2] ———, *Classification of representations of the group* GL(n) *over a p-adic field*, preprint. (Russian)

COLUMBIA UNIVERSITY

Proceedings of Symposia in Pure Mathematics
Vol. 33 (1979), part 2, pp. 87–95

AUTOMORPHIC L-FUNCTIONS FOR SYMPLECTIC GROUP $G\mathrm{Sp}(4)$

MARK E. NOVODVORSKY*

0. Introduction. The paper represents a talk delivered at the Summer Institute of the American Mathematical Society at Corvallis in 1977.

Let k be a global field, P the set of its normalized valuations, $A = \prod_{p\in P} k_p$ the adele ring of k, G a reductive algebraic group over k, $G_A = \prod_{p\in P} G_p$ the group of ideles of G, $\pi = \bigotimes_{p\in P} \pi_p$ a class of isomorphisms of irreducible admissible representations of G_A, ρ a finite dimensional representation of Langlands' dual group ${}^LG$. R. P. Langlands [1] defined L-function $L(\pi, \rho, s)$ as a certain Euler product,

$$L(\pi, \rho, s) = \prod_{p\in P} L(\pi_p, \rho, s), \tag{0.1}$$

convergent for all complex numbers s with sufficiently large real part if π is unitarizable. He conjectured that if π is the class of an automorphic representation, then $L(\pi, \rho)$ can be continued to a meromorphic function on the whole complex plane which satisfies the functional equation

$$L(\pi, \rho, s) = \varepsilon(\pi, \rho, s)L(\pi^*, \rho, 1 - s), \tag{0.2}$$

π^* being the contragradient representation to π and

$$\varepsilon(\pi, \rho, s) = \prod_{p\in P} \varepsilon(\pi_p, \rho, s)$$

another Euler product whose existence is a part of the conjecture. However, the definition of the local factors $L(\pi_p, \rho, s)$ depends on a parametrization problem (cf. Borel's paper, *Automorphic L-function*, these PROCEEDINGS, part 2, pp. 27–61) and, therefore, does not work at present until all the irreducible representations π_p of the groups G_p corresponding to nonarchimedean valuations p are of class 1. Therefore, in order to prove Langlands' conjecture one must first construct the Euler factors $L(\pi_p)$ and $\varepsilon(\pi_p)$.

A general approach to this problem was suggested by I. I. Piatetski-Shapiro [2]. Here we develop the ideas of [2] for the symplectic group $G = G\mathrm{Sp}(4)$ and prove Langlands' conjecture for the standard 4-dimensional representation ρ of its Langlands dual group ${}^LG = G\mathrm{Sp}(4, \boldsymbol{C})$. We consider also the group $G = G\mathrm{Sp}(4) \times \mathrm{GL}(2)$ and prove Langlands' conjecture for *generic* cuspidal automorphic representations π (cf. Piatetski-Shapiro's paper *Multiplicity one theorems*, these PRO-

AMS (MOS) subject classifications (1970). Primary 10D10, 22E55.

*Supported by NSF Grant MCS76–02160A1.

CEEDINGS, part 1, pp. 209–212) and the standard 8-dimensional representation ρ of its Langlands dual group $G\,\mathrm{Sp}(4, \boldsymbol{C}) \times \mathrm{GL}(2, \boldsymbol{C})$ (i.e. ρ is the tensor product of the standard 4- and 2-dimensional representations of the factors). Our results are complete only for functional global fields k. For number fields k we obtain meromorphic continuations of these automorphic L-functions but, because of certain difficulties with archimedean valuations, not the functional equations (0.2).

Since the methods of **[2]** are based on generalized Whittaker models, we consider generic and hypercuspidal automorphic representations separately (cf. the quoted paper of Piatetski-Shapiro).

The results for generic representations of $G\,\mathrm{Sp}(4)$ (§1) can be extended to all split orthogonal groups of Dynkin type B_n (Sp(4) covers $SO(5)$); in this form they were announced by the author (for char $k > 0$) in **[10]** and **[11]**. The results for hypercuspidal representations of $G\,\mathrm{Sp}(4)$ (§2) were obtained by I. I. Piatetski-Shapiro and the author. The results for $G\,\mathrm{Sp}(4) \times \mathrm{GL}(2)$ (§3) were announced by the author (for char $k > 0$) in **[12]**.

In this paper we assume that char $k \neq 2$.

1. G Sp(4), generic representations. Let $G \simeq G\,\mathrm{Sp}(4)$ be the group of similitudes of the bilinear form

$$\begin{pmatrix} 0 & & & 1 \\ & & 1 & \\ & -1 & & \\ -1 & & & 0 \end{pmatrix}$$

of four variables over k, Z the subgroup of all upper triangular unipotent matrices from G, ψ a nondegenerate[1] character of the group Z_A of ideles of Z which is trivial on principal ideles Z_k, p a nonarchimedean valuation of the field k. The space of all locally constant complex functions W on the group G_p which satisfy the equation

$$(1.1) \qquad W(zg) = \overline{\psi(z)}\,W(g) \quad \forall\, z \in Z_p,\ g \in G_p,$$

is denoted $\mathscr{W}_\psi(G_p)$; G_p acts in this space by right translations. A class π_p of isomorphisms of irreducible admissible representations of the group G_p is called *nondegenerate* if it contains the subrepresentation of G_p in an irreducible G_p-invariant subspace of $\mathscr{W}_\psi(G_p)$; this subspace then is called a *Whittaker model* of π_p and denoted by $\mathscr{W}_\psi(\pi_p)$.

THEOREM (I. M. GELFAND, D. A. KAZDAN **[3]**, F. RODIER **[4]**). *A Whittaker model of a nondegenerate class π_p is unique.*

Let π_p be a nondegenerate class of isomorphisms of irreducible admissible representations of the group G_p. Its restriction to the center C_p of G_p is proportional to a character, denote it ω_p. In view of the canonical isomorphism of the center C of the group $G\,\mathrm{Sp}(4)$ and the multiplicative group,

$$(1.2) \qquad C \ni \begin{pmatrix} \alpha & & & \\ & \alpha & & \\ & & \alpha & \\ & & & \alpha \end{pmatrix} \to \alpha \in k^*,$$

[1]That is, ψ is not trivial on the ideles of any horocycle subgroup of Z.

ω_p can be considered as a character of k_p^*. The contragradient class π_p^* coincides with $\pi_p \otimes \omega_p(\sigma_p)$ where σ is the unique homomorphism of $G\,\mathrm{Sp}(4)$ into k^* whose square is determinant (σ is the factor of similitude). Therefore, $\mathscr{W}_\psi(\pi_p^*)$ coincides with the set of functions

$$(1.3) \qquad W^*(g) = W(g)\cdot\omega_p^{-1}(\sigma_p(g)), \qquad g \in G_p,\ W \in \mathscr{W}_\psi(\pi_p).$$

We introduce the subgroups

$$H = \left\{\begin{pmatrix} \alpha & & & \\ & \alpha & & \\ & & 1 & \\ & & & 1 \end{pmatrix} \in G\right\} \simeq \mathrm{GL}(1),$$

$$U = \left\{\begin{pmatrix} 1 & & & \\ & 1 & & \\ & \beta & 1 & \\ & & & 1 \end{pmatrix} \in G\right\},$$

and consider the integral

$$(1.4) \quad \mathscr{I}_p(W, s) = \int_{(U\cdot H)_p} W(uh)\|\alpha\|^{s-1/2} d(uh), \qquad h = \begin{pmatrix} \alpha & & & \\ & \alpha & & \\ & & 1 & \\ & & & 1 \end{pmatrix},\ W \in \mathscr{W}_\psi(\pi_p).$$

THEOREM 1. *The integral* (1.4) *is absolutely convergent in a vertical half-plane* $\mathrm{Re}\, s \gg$ *and defines* $\mathscr{I}_p(W)$ *as a rational function in* q_p^{-s}, q_p *being the number of elements in the residue field of the valuation p. All the functions* $\mathscr{I}_p(W)$, $W \in \mathscr{W}_\psi(\pi_p)$, *admit a common denominator. There exists a rational in* q_p^{-s} *function* $\gamma(\pi_p)$ *such that*

$$\mathscr{I}_p(W, s) = \gamma(\pi_p, s)\mathscr{I}_p((\beta(W))^*, 1-s), \qquad W \in \mathscr{W}_\psi(G_p),$$

$$(1.5) \qquad \beta = \begin{pmatrix} 0 & \begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix} \\ \begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix} & 0 \end{pmatrix} \in G_A.$$

We denote by $Q_p(\pi_p)$ the polynomial in q_p^{-s} with constant term 1 which is the common denominator of smallest degree for functions $\mathscr{I}_p(W)$, $W \in \mathscr{W}_\psi(\pi_p)$, and define

$$(1.6) \qquad \begin{aligned} L(\pi_p, s) &= [Q(\pi_p, q^{-s})]^{-1}, \\ \varepsilon(\pi_p, s) &= \gamma(\pi_p, s)\cdot L(\pi_p, s)/L(\pi_p^*, 1-s). \end{aligned}$$

Let $\pi = \bigotimes_{p \in P} \pi_p$ be the class of isomorphisms of an automorphic *generic* cuspidal representation of the group G_A. Then all its nonarchimedean components π_p are nondegenerate and the Euler product

$$(1.7) \qquad L(\pi, s) = \prod_{p \in P\setminus S} L(\pi_p, s),$$

S being the subset of all archimedean valuations from P, is absolutely convergent in a vertical half-plane $\mathrm{Re}\, s \gg$; it coincides with the Euler product of Langlands corresponding to the standard 4-dimensional representation of Langlands' dual group ${}^LG \simeq G\,\mathrm{Sp}(4, \boldsymbol{C})$ if k is a functional global field and differs from it by a meromorphic (in the whole complex plane in s) factor if k is a number field. The factors $\varepsilon(\pi_p)$ are of the form aq_p^{-bs} and for almost all $p \in P\backslash S$ are identically equal to 1; therefore, the Euler product

$$\varepsilon(\pi, s) = \prod_{p\in P\backslash S} \varepsilon(\pi_p, s) \tag{1.8}$$

is, in fact, finite and defines $\varepsilon(\pi)$ as an entire function without zeros.

THEOREM 2. *The function* $L(\pi)$ *admits meromorphic continuation to the whole complex plane. If* char $k = p > 0$, *then this continuation is a polynomial in* p^s *and* p^{-s} *satisfying the functional equation*

$$L(\pi, s) = \varepsilon(\pi, s)L(\pi^*, 1 - s). \tag{1.8}$$

2. $G\,\mathrm{Sp}(4)$, **hypercuspidal representations.** This case has been investigated in cooperation with I. I. Piatetski-Shapiro.

For hypercuspidal representations, Whittaker models associated to nondegenerate characters of maximal unipotent subgroups do not work, and one has to consider other subgroups and generalized Whittaker models. It is convenient for our purposes to consider a different realization of the group $G = G\,\mathrm{Sp}(4)$. We choose a 2-dimensional semisimple k-algebra K and a 2-dimensional bilinear skew-symmetric form B in 2 variables over K; G can be realized now as the group of similitudes of the form $\mathrm{tr}_{K/k}B$. Now G contains the subgroup

$$G' \simeq \{g \in \mathrm{GL}(2, K) : \det g \in k\} \tag{2.1}$$

of all K-linear transformations from G; the subgroup of all upper triangular unipotent matrices from G' is denoted by U', the unipotent radical of the normalizer of U' in G is denoted by U; U is a 3-dimensional commutative horocycle subgroup in G. The algebra K has a unique nontrivial k-automorphism

$$\kappa \to \bar{\kappa}; \tag{2.2}$$

it induces a k-automorphism of the form $\mathrm{tr}_{K/k} B$ which is denoted τ, $\tau \in G$. We put

$$T' = \left\{\begin{pmatrix} \kappa & 0 \\ 0 & \bar{\kappa} \end{pmatrix}, \kappa \in K^*\right\} \subset G';$$

obviously, T' is isomorphic to the multiplicative group K^* of the algebra K. The groups G', U', and T' are normalized by τ.

Let p be a nonarchimedean valuation of the field k. We take a nontrivial character χ of the factor group U_p/U_p' and a quasi-character ξ' of the group T_p' whose restriction on the subgroup

$$\left\{\begin{pmatrix} \varepsilon & \\ & \bar{\varepsilon} \end{pmatrix} \in T_p', \varepsilon^{-1} = \bar{\varepsilon}\right\}$$

is unitary. If ξ' is τ-invariant, we continue it to a character ξ of the group T_p,

$$T = T' \cup \tau T' \qquad (\tau(\xi') = \xi'). \tag{2.3}$$

If ξ' is not τ-invariant, we put

$$T = T', \quad \xi = \xi' \qquad (\tau(\xi') \neq \xi'). \tag{2.4}$$

In both cases we define

$$Z = T \cdot U, \quad \psi(tu) = \xi(t) \cdot \chi(u) \quad \forall\, t \in T_p,\, u \in U_p; \tag{2.5}$$

Z is an algebraic k-subgroup of G, ψ is a quasi-character of the group Z_p. As in §1, we denote by $\mathscr{W}_\psi(G_p)$ the set of all complex locally constant functions W on the group G_p satisfying the equation (1.1), and an irreducible admissible subrepresentation of G_p in $\mathscr{W}_\psi$ by right translations is called the Whittaker model of its class of isomorphisms π_p and denoted $\mathscr{W}_\psi(\pi_p)$.

THEOREM 3. *Every class π_p of isomorphisms of irreducible admissible representations of the group G_p is either nondegenerate or has Whittaker model $\mathscr{W}_\psi(\pi_p)$ for some algebra K, subgroup T, and character ψ of the described type; for every fixed ψ this model is unique.*

REMARK 1. One class π_p can have Whittaker models corresponding to several of the characters ψ described in §§1 and 2.

REMARK 2. The group $\tilde{T}_p = T'_p \cup \tau T'_p$ is the stabilizer of χ in the normalizer of the group U_p in G_p; so, χ can be lifted to either a character or a 2-dimensional irreducible representation of the group $\tilde{T}_p \cdot U_p$. Theorem 3 can be reformulated as the theorem of the existence and uniqueness of Whittaker models associated to these 1- and 2-dimensional representations; such reformulation might be preferable logically but leads to more bulky constructions for Euler factors in case of 2-dimensional representations.

In the case when Z and ψ are defined by the formula (2.3), the uniqueness part of Theorem 3 was proven by I. I. Piatetski-Shapiro and the author **[8]** and by the author **[9]**. For supercuspidal representations π_p the uniqueness part of Theorem 3 follows from F. Rodier's Theorem 1 **[5]**.

Now we put

$$H = \left\{ \begin{pmatrix} \kappa & \\ & 1 \end{pmatrix},\ \kappa \in k^* \right\} \subset G' \subset \mathrm{GL}(2, k) \tag{2.6}$$

and consider the integral

$$\mathscr{I}_p(W, s) = \int_{H_p} W(h) \| h \|^s \, dh, \qquad \left\| \begin{pmatrix} \kappa & \\ & 1 \end{pmatrix} \right\| \stackrel{\text{def}}{=} \| \kappa \|. \tag{2.7}$$

THEOREM 4. *The integral* (2.7) *is absolutely convergent in a vertical half-plane* Re $s \gg$ *and defines $\mathscr{I}_p(W)$ as a rational function in q_p^{-s}. All the functions $\mathscr{I}_p(W)$, $W \in \mathscr{W}_\psi(\pi_p)$, admit common denominator.*

As in §1, we denote by $Q_p(\pi_p)$ the normalized common denominator of the smallest degree and put

$$L(\pi_p, s) = [Q(\pi_p, q^{-s})]^{-1}. \tag{2.8}$$

THEOREM 5. *Euler factor $L_p(\pi_p)$ does not depend on the choice of the subgroup Z and the character ψ of Z_p (particularly, if the class π_p is nondegenerate, the factors $L(\pi_p)$ defined in* §§1 *and* 2 *coincide).*

Let $\pi = \bigotimes_{p \in P} \pi_p$ be the class of isomorphisms of an automorphic cuspidal representation of the group G_A. Choosing for every nonarchimedean p a Whittaker model $\mathscr{W}_\psi(\pi_p)$, we obtain Euler factors $L(\pi_p)$ and define

$$L(\pi, s) = \prod_{p \in P \setminus S} L(\pi_p, s), \tag{2.9}$$

S being the set of archimedean valuations of k; this product is absolutely convergent in a vertical half-plane Re $s \gg$, it coincides with Langlands' Euler product corresponding to the standard 4-dimensional representation of Langlands' dual group ${}^L G \simeq G\,\mathrm{Sp}(4, \boldsymbol{C})$ if k is a functional field, and differs from it by a meromorphic (in the whole complex plane in s) factor if k is a number field.

THEOREM 6. *The function $L(\pi)$ admits meromorphic continuation to the whole complex plane. If* char $k = p > 0$, *then this continuation is a rational in p^{-s} function satisfying the functional equation*

$$L(\pi, s) = \varepsilon(\pi, s) L(\pi^*, 1 - s) \tag{2.10}$$

where $\varepsilon(\pi)$ is an entire function in s without zeros.

In fact, $\varepsilon(\pi)$ is an Euler product of some factors $\varepsilon(\pi_p)$ appearing in local functional equations; these equations are rather bulky and will appear in a joint work with I. I. Piatetski-Shapiro.

3. $G\,\mathrm{Sp}(4) \times \mathrm{GL}(2)$, generic representations. In this section we use a modification of H. Jacquet's treatment of induced representations applied in [7] to automorphic L-functions on GL(2) × GL(2).

The subgroups C and Z of $G\,\mathrm{Sp}(4)$, the character ψ of Z_A, the space $\mathscr{W}_\psi(G\,\mathrm{Sp}(4, k_p))$, p nonarchimedean, Whittaker model $\mathscr{W}_\psi(\pi_p)$ for a nondegenerate class π_p of irreducible admissible representations of $G\,\mathrm{Sp}(4, k_p)$, the homomorphism σ, and the character ω_p in this section are the same as in §1. We define the subgroup

$$H = \left\{ \begin{pmatrix} a & & b \\ & g & \\ c & & d \end{pmatrix} \in G\,\mathrm{Sp}(4), a, b, c, d \in k, g \in \mathrm{GL}(2) \right\} \tag{3.1}$$

and also its homomorphisms onto the group GL(2, k):

$$\begin{aligned} \phi_1 &: \begin{pmatrix} a & & b \\ & g & \\ c & & d \end{pmatrix} \to g, \\ \phi_2 &: \begin{pmatrix} a & & b \\ & g & \\ c & & d \end{pmatrix} \to \begin{pmatrix} a & b \\ c & d \end{pmatrix}. \end{aligned} \tag{3.2}$$

We denote by $\tilde{Z}$ the subgroup of all upper triangular unipotent matrices in GL(2,k_p). It is the image of $H \cap Z$ under the homomorphism ϕ_1, and the kernel

$$\left\{ \begin{pmatrix} 1 & & & b \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix} \right\}$$

of this homomorphism considered on the group $(H \cap Z)_A$ belongs to its commutator subgroup; therefore, every character of the group Z_A defines under ϕ_1 a character of $\tilde{Z}_A$ which we denote with the same symbol. As in §1, we denote by $\mathscr{W}_{\tilde{\psi}}(\mathrm{GL}(2, k_p))$ the space of all locally constant complex functions W satisfying the equation

$$W(zg) = \psi(z)W(g), \qquad z \in \tilde{Z}_p, g \in \mathrm{GL}(2, k_p) \tag{3.3}$$

and by $\mathscr{W}_{\tilde{\psi}}(\tilde{\pi}_p)$, $\tilde{\pi}_p$ a class of an infinite-dimensional irreducible admissible representations of $\mathrm{GL}(2, k_p)$, a subrepresentation of this group in $\mathscr{W}_{\psi}(\mathrm{GL}(2, k_p))$ which belongs to $\tilde{\pi}_p$; such a subrepresentation exists and is unique (cf. H. Jacquet, R. P. Langlands [6]). We denote by $\tilde{\omega}_p$ the restriction of $\tilde{\pi}_p$ on the center of the group $\mathrm{GL}(2, k_p)$; we consider $\tilde{\omega}_p$ as a character of the multiplicative group k_p^* in view of the canonical isomorphism

$$\mathrm{GL}(2) \ni \begin{pmatrix} \alpha & \\ & \alpha \end{pmatrix} \to \alpha \in k^*.$$

The space $\mathscr{W}_{\tilde{\psi}}(\tilde{\pi}_p^*)$, $\tilde{\pi}_p^* \simeq \tilde{\pi}_p \otimes \tilde{\omega}_p^{-1}(\det)$ being the contragradient representation to $\tilde{\pi}_p$, consists of the functions

$$W^*(g) = W(g) \cdot \omega_p^{-1}(\det(g)), \qquad g \in \mathrm{GL}(2, k_p). \tag{3.4}$$

For every locally constant complex function Φ on the plane $k_p \oplus k_p$ we define its Fourier transform

$$\hat{\Phi}(x, y) = \int_{k_p \oplus k_p} \Phi(u, v)\, \alpha(yu - xv)\, d(u, v) \tag{3.5}$$

where α is a fixed nontrivial character of the group A/k of classes of adeles; for every complex quasi-character μ of the multiplicative group k_p^* we put

$$f(h, \Phi, \mu) = \int_{k_p^*} \Phi((0, \kappa)\phi_2(h)) \cdot \mu(\kappa)\, d\kappa, \qquad h \in H_p. \tag{3.6}$$

If the integral (3.6) is absolutely convergent, then the function f satisfies, obviously, the equation

$$f\left(\begin{pmatrix} a & & b \\ & g & \\ 0 & & d \end{pmatrix} h, \Phi, \mu\right) = \mu(d^{-1}) f(h, \Phi, r), \qquad \begin{pmatrix} a & & b \\ & g & \\ d & & 0 \end{pmatrix}, h \in H_p. \tag{3.7}$$

Therefore, we can consider the integral

$$\begin{aligned} \mathscr{J}(W, \tilde{W}, \Phi, \mu_1(s), s) &= \int W(h)\, \tilde{W}(\phi_1(h)) f(h, \Phi, \mu_1) \| \phi_2(h) \|^s\, dh, \\ \mu_1(\kappa, s) &= \| \kappa \|^{2s} \omega(\kappa)\, \tilde{\omega}(\kappa), \qquad W \in \mathscr{W}_{\psi}(\pi_p),\ \tilde{W} \in \mathscr{W}_{\tilde{\psi}}(\tilde{\pi}_p), \end{aligned} \tag{3.8}$$

and a similar one for $\mathscr{J}(W^*, \tilde{W}^*, \hat{\Phi}, \mu_2(s), s)$,

$$\mu_2(\kappa, s) = \| \kappa \|^{2s} \omega^{-1}(\kappa)\, \tilde{\omega}^{-1}(\kappa).$$

THEOREM 7. *The integrals* (3.6) *and* (3.8) *are absolutely convergent in a vertical* *half-plane* Re $s \gg$ *and define* $\mathscr{J}(W, \tilde{W}, \Phi)$ *and* $\mathscr{J}(W^*, \tilde{W}^*, \hat{\Phi})$ *as rational functions in* q_p^{-s}. *There exists a rational in* q_p^{-s} *function* $\gamma(\pi_p, \tilde{\pi}_p)$ *such that*

$$(3.9)\qquad \mathscr{J}(W, \tilde{W}, \Phi, \mu_1(s), s) = \gamma(\pi_p, \tilde{\pi}_p, s)\, \mathscr{J}(W, \tilde{W}^*, \hat{\Phi}, \mu_2(1-s), 1-s) \quad \forall\, W \in \mathscr{W}_\psi(\pi_p),\ \tilde{W} \in \mathscr{W}_{\tilde{\psi}}(\tilde{\pi}_p),\ \Phi.$$

All the rational functions $\mathscr{J}(W, \tilde{W}, \Phi)$ *(for different* $W \in \mathscr{W}_\psi(\pi_p)$, $\tilde{W} \in \mathscr{W}_{\tilde{\psi}}(\tilde{\pi}_p)$, Φ*) admit a common denominator.*

As in §1, we denote by $Q(\pi_p, \tilde{\pi}_p)$ the normalized common denominator of the smallest degree in q^{-s} and define

$$(3.10)\qquad \begin{aligned} L(\pi_p \otimes \tilde{\pi}_p, s) &= [Q(\pi_p, \tilde{\pi}_p, q^{-s})]^{-1}, \\ \varepsilon(\pi_p \otimes \tilde{\pi}_p, s) &= \gamma(\pi_p, \tilde{\pi}_p, s) \cdot L(\pi_p \otimes \tilde{\pi}_p, s)/L(\pi_p^* \otimes \tilde{\pi}_p^*, 1-s). \end{aligned}$$

If $\pi \otimes \tilde{\pi}$ is the class of isomorphisms of a cuspidal *generic* irreducible representation of the group $G\,\mathrm{Sp}(4, k_p) \times \mathrm{GL}(2, k_p)$, $\pi = \bigotimes_{p \in P} \pi_p$, $\tilde{\pi} = \bigotimes_{p \in P} \tilde{\pi}_p$, then we put

$$(3.11)\qquad \begin{aligned} L(\pi \otimes \tilde{\pi}, s) &= \prod_{p \in P \setminus S} L(\pi_p \otimes \tilde{\pi}_p, s), \\ \varepsilon(\pi \otimes \tilde{\pi}, s) &= \prod_{p \in P \setminus S} \varepsilon(\pi_p \otimes \tilde{\pi}_p, s), \end{aligned}$$

S being the set of archimedean valuations of k. As before, the first product is absolutely convergent in a vertical half-plane Re $s \gg$. For a functional field k it coincides with Langlands' L-function $L(\pi, \rho, s)$ if ρ is the tensor product of the standard 4- and 2-dimensional representations of the complex groups $G\,\mathrm{Sp}(4, \boldsymbol{C})$ and $\mathrm{GL}(2, \boldsymbol{C})$; for a number field k these L-functions differ by a factor meromorphic in the whole complex plane. The second product (3.11) is, actually, finite and defines $\varepsilon(\pi \otimes \tilde{\pi})$ as an entire function without zeros.

THEOREM 8. *The function* $L(\pi \otimes \tilde{\pi})$ *admits meromorphic continuation on the whole complex plane. If* char $k = p > 0$, $L(\pi \otimes \tilde{\pi})$ *is a rational function in* p^{-s} *satisfying the equation*

$$(3.12)\qquad L(\pi \otimes \tilde{\pi}, s) = \varepsilon(\pi \otimes \tilde{\pi}, s)\, L(\pi^* \otimes \tilde{\pi}^*, 1-s).$$

REFERENCES

1. R. P. Langlands, *Problems in the theory of automorphic forms*, Lecture Notes in Math., vol. 170, Springer-Verlag, Berlin and New York, 1970.

2. I. I. Piatetski-Shapiro, *Euler subgroups*, Proc. Summer School Bolya-Janos Math. Soc. (Budapest, 1970), Halsted, New York, 1975.

3. I. M. Gelfand and D. A. Kazdan, *Representations of the group* GL(n, K) *where* K *is a local field*, Inst. Appl. Math., no. 242, Moscow, 1971.

4. F. Rodier, *Modèle de Whittaker des représentations admissibles des groupes réductifs p-adiques deployés*, C. R. Acad. Sci. Paris Sér. A **275** (1972), pp. 1045–1048.

5. ———, *Le représentations de* G Sp(4, k) *on* k *est un corps local*, C. R. Acad. Sci. Paris Sér. A **283** (1976), pp. 429–431.

6. H. Jacquet and R. P. Langlands, *Automorphic forms on* GL(**2**), Lecture Notes in Math., vol. 114, Springer-Verlag, Berlin and New York, 1970.

7. H. Jacquet, *Automorphic forms on* GL(2). Part 2, Lecture Notes in Math., vol. 278, Springer-Verlag, Berlin and New York, 1972.

8. M. E. Novodvorsky and I. I. Piatetskii-Shapiro, *Generalized Bessel models for the symplectic group of rank* 2, Mat. Sb. **90** (2) (1973), 246–256.

9. M. E. Novodvorsky, *On the theorems of uniqueness of generalized Bessel models*, Mat. Sb. **90** (2) (1973), 275–287.

10. ———, *Théorie de Hecke pour les groupes orthogonaux*, C. R. Acad. Sci. Paris Sér. A **285** (1977), 93–94.

11. ———, *Fonctions $\mathcal{J}$ pour des groupes orthogonaux*, C. R. Acad. Sci. Paris Sér. A **280** (1975), 1421–1422.

12. ———, *Fonction $\mathcal{J}$ pour G* Sp(4), C. R. Acad. Sci, Paris Sér. A **280** (1975), 191–192.

PURDUE UNIVERSITY

Proceedings of Symposia in Pure Mathematics
Vol. 33 (1979), part 2, pp. 97–110

ON LIFTINGS OF HOLOMORPHIC CUSP FORMS

TAKURO SHINTANI

Introduction. In **[10]**, H. Saito calculated the trace of "twisted" Hecke operators acting on the space of holomorphic cusp forms with respect to the Hilbert modular group over a cyclic totally real abelian field of prime degree. He discovered a striking identity between his "twisted trace formula" and the ordinary trace formula for the Hecke operators acting on the spaces of elliptic modular forms. Applying his identity he showed that elliptic modular forms are "lifted" to Hilbert modular forms (for the origin of "lifting" type results, see **[2]**). No less significantly, he characterized the space spanned by lifted forms. In the U.S.-Japan symposium "Applications of automorphic forms to number theory" which was held at Ann Arbor in June 1975, the author reported a representation theoretic interpretation and generalization of Saito's work (see **[13]**). Results presented in the author's talk were immediately generalized by Langlands in **[8]**. Moreover, Langlands discovered an unexpected application of "lifting theory" to the theory of Artin L-functions.

The present paper consists of two sections. In §1, we reproduce (with slight modifications) what the author presented at Ann Arbor. The second section is devoted to a few supplementary remarks.

The author wishes to express his hearty thanks to Professor H. Saito, who gave him detailed expositions of **[10]** before its publication. Two of the author's previous papers **[14]** and **[15]** are also motivated by **[10]**.

Notation. For a ring R, $R^\times$ is the group of units of R. We write $G(R) = G_R =$ GL(2, R). For a $\mathfrak{p}$-adic field k, $\mathfrak{o}(k) = \mathfrak{o}_k$ is the ring of integers of k. We denote by q_k the cardinality of the residue class field of k. For each $t \in k$ set $d(tx) = \alpha_k(t)\,dx = |t|_k\,dx$, where dx is an invariant measure of the additive group of k. For each locally compact totally disconnected group G, $C_0^\infty(G)$ is the space of compactly supported, locally constant functions on G. Let k be a field and F be a field extension of k. We denote by N_k^F the norm mapping from F to k.

1.

1.1. Let k be either a finite algebraic number field or a $\mathfrak{p}$-adic field. Let F be a commutative semisimple algebra over k of prime degree l. We assume that the group of automorphisms of F over k contains a cyclic subgroup $\mathfrak{g}$ of order l generated by σ. Then F is isomorphic either to a cyclic field extension of k or to the direct sum of l-copies of k. Set G_F = GL(2, F) and G_k = GL(2, k). We may regard

AMS (MOS) subject classifications (1970). Primary 10Dxx; Secondary 10D15, 10D20.

$\mathfrak{g}$ as a group of automorphisms of G_F (with the fixed point set G_k) in a natural manner. Two elements x and y of G_F are said to be *σ-twistedly conjugate* in G_F if there exists a $g \in G_F$ such that $y = g^\sigma x g^{-1}$. We denote by $x^{G_F,\sigma}$ the set consisting of all elements of G_F which are σ-twistedly conjugate to x in G_F. The set of σ-twisted conjugacy classes in G_F is denoted by $\mathscr{C}_\sigma(G_F)$. The usual conjugate class in G_F containing x is denoted by x^{G_F}. For each $x \in G_F$, set

$$N(x) = x^{\sigma^{l-1}} \cdot x^{\sigma^{l-2}} \cdots x^\sigma \cdot x.$$

It is easy to see that $N(g^\sigma \cdot x g^{-1}) = gN(x)g^{-1}$.

We denote by $\mathscr{C}(G_k)$ the set of conjugacy classes in G_k. For the proof of the next lemma see Lemma 3.4 and Lemma 3.6 of **[10]**.

LEMMA 1 (SAITO). *The notation and assumptions being as above, the mapping* $x^{G_F,\sigma} \mapsto N(x)^{G_F} \cap G_k$ *is an injection from* $\mathscr{C}_\sigma(G_F)$ *into* $\mathscr{C}(G_k)$. *If F is not a field, the mapping is a bijection.*

In the following, for each $\boldsymbol{c} \in \mathscr{C}_\sigma(G_F)$, we denote by $N(\boldsymbol{c}) \in \mathscr{C}(G_k)$ the image of $\boldsymbol{c}$ under the mapping given in Lemma 1. An $x \in G_k$ is said to be *regular* if the group of centralizers of x in G_k is a two-dimensional k-torus. We call a conjugate class in G_k *regular* if it consists of regular elements. A σ-twisted conjugacy class $\boldsymbol{c}$ in G_F is said to be *regular* if $N(\boldsymbol{c})$ is regular. We denote by $\mathscr{C}'_\sigma(G_F)$ (resp. $\mathscr{C}'(G_k)$) the set of regular σ-twisted conjugate classes (resp. regular conjugate classes) of G_F (resp. G_k).

1.2. We keep the notation in 1.1 and assume that k is a $\mathfrak{p}$-adic field. Let $G_F^\sim$ be the semidirect product of G_F with $\mathfrak{g}$. More precisely, $G_F^\sim$ is the group with the underlying set $\mathfrak{g} \times G_F$ with the composition rule given by $(\tau, g)(\tau', g') = (\tau\tau', g^{\tau'}g')$ $(\tau, \tau' \in \mathfrak{g}, g, g' \in G_F)$. It is immediate to see (via Lemma 1) that (σ, g_1) and (σ, g_2) are conjugate in $G_F^\sim$ if and only if g_1 and g_2 are σ-twistedly conjugate in G_F. Denote by $\hat{G}_k$ (resp. $\hat{G}_F$) the set of equivalence classes of irreducible admissible unitarizable representations of G_k (resp. G_F). For each $R \in \hat{G}_F$ and $\tau \in \mathfrak{g}$, let R^τ be the representation of G_F given by $R^\tau(x) = R(x^\tau)$. Then $R^\tau \in \hat{G}_F$. Thus, the group $\mathfrak{g}$ operates on $\hat{G}_F$. Denote by $\hat{G}_F^{\mathfrak{g}}$ the subset of $\mathfrak{g}$-fixed elements of $\hat{G}_F$. A representation of $G_F^\sim$ is said to be admissible if its restriction to G_F is admissible. Let $R^\sim$ be an irreducible admissible unitarizable representation of the group $G_F^\sim$. As is well known, the restriction R of $R^\sim$ to G_F is either irreducible or a direct sum of l-mutually inequivalent irreducible representations of G_F. In the former case $R^\sim$ is said to be of the first kind. Assume $R^\sim$ to be of the first kind and set $J_\sigma = R^\sim((\sigma, 1))$. Then

$$R(g^\sigma) = J_\sigma^{-1} R(g) J_\sigma \qquad (\forall\, g \in G_F). \tag{1.1}$$

Thus $R \in \hat{G}_F^{\mathfrak{g}}$. The relation (1.1) characterizes the operator J_σ up to a multiplication by an lth root of unity. Conversely let $R \in \hat{G}_F^{\mathfrak{g}}$. Then there exists a unitary operator of order l which satisfies (1.1). Hence, R is extended to an admissible irreducible unitarizable representation $R^\sim$ of $G_F^\sim$ of the first kind by setting $R^\sim((\sigma^m, g)) = J_\sigma^m R(g)$. For $\varphi \in C_0^\infty(G_F)$, denote by $R^\sim(\varphi)$ the operator given by

$$R^\sim(\varphi) = \int_{G_F} R^\sim((\sigma, g))\varphi(g)\, dg,$$

where dg is the invariant measure of G_F normalized so that the total volume of $G_{o(F)}$ is equal to 1.

It is shown that there exists a locally summable function $\chi(R^\sim)$ on G_F such that

$$\text{trace } R^\sim(\varphi) = \int_{G_F} \varphi(g)\chi(R^\sim)(g)\, dg.$$

It is proved that $\chi(R^\sim)(g)$ depends only upon the σ-twisted conjugate class of g in G_F and that $\chi(R^\sim)$ is defined only on $\mathscr{C}'_\sigma(G_F)$. For each $r \in \hat{G}_k$, denote by χ_r the character of r. Recall that $\chi_r(x)$ depends only upon the conjugate class of x and that χ_r is defined only on $\mathscr{C}'(G_k)$, assume further that the characteristic of the residue class field of k is not equal to 2.

THEOREM 1. *Let the notation and assumptions be as above.*

(1) *For each $R \in \hat{G}_F^{\mathfrak{g}}$, there exist an extension $R^\sim$ of R to a representation of $G_F^\sim$ and an $r \in \hat{G}_k$ such that*

$$\chi(R^\sim)(x) = \chi_r(N(x)), \tag{1.2}$$

where N is the norm mapping from $\mathscr{C}'_\sigma(G_F)$ to $\mathscr{C}'(G_k)$ given in Lemma 1.

(2) *For each $r \in \hat{G}_k$, there uniquely exists $R \in \hat{G}_F^{\mathfrak{g}}$ whose suitable extension $R^\sim$ to a representation of $G_F^\sim$ satisfies* (1.2).

For each $r \in \hat{G}_k$, we call $R \in \hat{G}_F^{\mathfrak{g}}$, which is related to r by (1.2), the *lifting* of r from $\hat{G}_k$ to $\hat{G}_F^{\mathfrak{g}}$. (Jacquet introduced in **[4]** the notion of lifting from a different viewpoint.)

REMARK 1. An analogue of Theorem 1 for finite general linear groups was given in **[14]**. Furthermore, an analogue of Theorem 1 for the case of $(F, k) = (\boldsymbol{C}, \boldsymbol{R})$ is given in **[15]** (resp. **[8]**) by a local (resp. global) method.

Let us describe the lifting from $\hat{G}_k$ to $\hat{G}_F^{\mathfrak{g}}$ in a concrete manner. If F is isomorphic (as a k-algebra) to the direct sum of l copies of k, G_F is isomorphic to the direct product of l-copies of G_k. It is immediate to see that the lifting of $r \in \hat{G}_k$ is given by $r\otimes \cdots \otimes r$ (the same is true even when the characteristic of the residue class field of k is equal to 2).

Next, we consider the case when F is a cyclic field extension of prime degree l. First, let us recall a description of $\hat{G}_k$. For details, see §§3 and 4 of Chapter 1 of **[5]**. For quasi-characters μ_1 and μ_2 of $k^\times$ such that $\mu_1\mu_2^{-1} \neq \alpha_k^{\pm 1}$, let $\rho(\mu_1, \mu_2)$ be the corresponding irreducible representation of G_k in the principal series. If $\mu_1\mu_2^{-1} = \alpha_k$, let $\sigma(\mu_1, \mu_2)$ be the corresponding special representation of G_k. For a quadratic extension K of k and a quasi-character ω of $K^\times$ such that $\omega' \neq \omega$ ($'$ denotes the conjugation of K with respect to k), let $\pi(\omega, K)$ be the corresponding absolutely cuspidal representation of G_k. If the characteristic of the residue class field of k is not equal to 2 (as we are now assuming), it is known that each infinite dimensional irreducible admissible representation of G_k is equivalent to some of $\rho(\mu_1, \mu_2)$, $\sigma(\mu_1, \mu_2)$ and $\pi(\omega, K)$. For each quasi-character μ of $k^\times$, we denote by $\boldsymbol{\mu}$ the quasi-character of $F^\times$ given by $\boldsymbol{\mu} = \mu \circ N_k^F$. For a quadratic extension $K \neq F$ of k and a quasi-character ω of $K^\times$, denote by $\boldsymbol{\omega}$ the quasi-character of $L^\times$ ($L = K\cdot F$) given by $\boldsymbol{\omega} = \omega\cdot N_k^L$.

PROPOSITION 1. *The notation and assumptions being as above, the lifting R of $r \in \hat{G}_k$ to $\hat{G}_F^{\mathfrak{g}}$ is given as follows*:

(1) *If* $r = \rho(\mu_1, \mu_2)$ (*resp.* $\sigma(\mu_1, \mu_2)$), $R = \rho(\boldsymbol{\mu}_1, \boldsymbol{\mu}_2)$ (*resp.* $R = \sigma(\boldsymbol{\mu}_1, \boldsymbol{\mu}_2)$).
(2) *If* $r = \pi(\omega, K)$ *and* $K \neq F$, $R = \pi(\boldsymbol{\omega}, L)$, *where* $L = F \cdot K$.
(3) *If* $r = \pi(\omega, K)$ *and* $K = F$, $R = \rho(\omega, \omega')$.

REMARK 2. The proof of the first part of Proposition 1 is straightforward (even when the characteristic of the residue class field of k is equal to 2).

1.3. We keep the notation in 1.2. In particular, k is a $\mathfrak{p}$-adic field (the residue class field of k may be of characteristic 2). For an $x \in G_F$, let $Z_\sigma(x)$ be the subgroup of G_F consisting of all elements of g of G_F satisfying $g^\sigma x g^{-1} = x$. Normalize invariant measures on G_F and $Z_\sigma(x)$ so that total volumes of their maximal compact open subgroups are all equal to 1. Denote by $d\dot{\boldsymbol{g}}$ the invariant measure on $G_F / Z_\sigma(x)$ given as the quotient of the normalized invariant measure of G_F by that of $Z_\sigma(x)$. For $\boldsymbol{f} \in C_0^\infty(G_F)$, set

$$\Lambda_\sigma(\boldsymbol{f}, x) = \int_{G_F/Z_\sigma(x)} \boldsymbol{f}(\dot{g}^\sigma x \dot{g}^{-1})\, d\dot{\boldsymbol{g}}. \tag{1.3}$$

It is shown that the integral is absolutely convergent. Moreover, for $\boldsymbol{c} \in \mathscr{C}'_\sigma(G_F)$, $\Lambda_\sigma(\boldsymbol{f}, x) = \Lambda_\sigma(\boldsymbol{f}, y)$ for any $x, y \in \boldsymbol{c}$. For each $\boldsymbol{c} \in \mathscr{C}'_\sigma(G)$, we put

$$\Lambda_\sigma(\boldsymbol{f}, \boldsymbol{c}) = \Lambda_\sigma(\boldsymbol{f}, x), \tag{1.4}$$

where x is an arbitrary element of $\boldsymbol{c}$. For each $x \in G_k$, let $Z(x)$ be the subgroup of centralizers of x in G_k. We normalize invariant measures on G_k and $Z(x)$ so that the total volumes of their maximal compact open subgroups are all equal to 1. Denote by $d\dot{g}$ the invariant measure on $G_k/Z(x)$ given as the quotient of the normalized invariant measure of G_k by that of $Z(x)$.

For $f \in C_0^\infty(G_k)$, set

$$\Lambda(f, x) = \int_{G_k/Z(x)} f(\dot{g} x \dot{g}^{-1})\, d\dot{g}. \tag{1.5}$$

It is known that the integral is absolutely convergent. For each $c \in \mathscr{C}'(G_k)$, we put

$$\Lambda(f, c) = \Lambda(f, x), \tag{1.6}$$

where x is an arbitrary element of c. It is easy to see that the right side of (1.6) is independent of the choice of $x \in c$.

PROPOSITION 2. *The notation being as above, for each* $\boldsymbol{f} \in C_0^\infty(G_F)$, *there exists an* $f \in C_0^\infty(G_k)$ *which satisfies the following conditions* (1) *and* (2):
(1) *For each regular* $\boldsymbol{c} \in C_\sigma(G_F)$, $\Lambda_\sigma(\boldsymbol{f}, \boldsymbol{c}) = \Lambda(f, N(\boldsymbol{c}))$.
(2) *For each* $c \in \mathscr{C}'(G_k) - N(\mathscr{C}'_\sigma(G_F))$, $\Lambda(f, c) = 0$.

REMARK 3. In Proposition 2, assume $\boldsymbol{f}$ is the characteristic function of $G_{\mathfrak{o}(F)}$. Assume further that F is either the unramified cyclic extension of degree l of k or is isomorphic to the direct sum of l copies of F. Then one may put f to be the characteristic function of $G_{\mathfrak{o}(k)}$.

REMARK 4. The correspondence $\boldsymbol{f} \mapsto f$ in Proposition 2 is, in a sense, dual to the lifting of irreducible characters of G_k given in Theorem 1.

1.4. Let k be a totally real algebraic number field of degree n and let k_A (resp. $k_A^\times$)

be the ring of adeles (resp. the group of ideles) of k. Denote by k_∞ (resp. $k_{A,0}$) the infinite (resp. finite) component of k_A. Then $k_A = k_\infty \oplus k_{A,0}$ and $k_A^\times = k_\infty^\times \times k_{A,0}^\times$. Moreover, both groups G_{k_∞} and $G_{k_A,0}$ are embedded into G_{k_A} in a natural manner and $G_{k_A} = G_{k_\infty} \times G_{k_A,0}$. Let $\{\infty_1, \cdots, \infty_n\}$ be the set of infinite places of k. For each $g \in G_{k_A}$, set $g = g_\infty g_0$ ($g_\infty \in G_\infty$, $g_0 \in G_{k_A,0}$) and $g_\infty = (g_{\infty,1}, \cdots, g_{\infty,n})$, where $g_{\infty,i} \in G_{\boldsymbol{R}}$ is the component of g_∞ corresponding to the infinite place ∞_i. For $g_{\infty,i}$ we introduce the following standard parametrization

$$g_{\infty,i} = \begin{pmatrix} t_i & \\ & \pm t_i \end{pmatrix}\begin{pmatrix} 1 & x_i \\ & 1 \end{pmatrix}\begin{pmatrix} \sqrt{y_i} & \\ & \sqrt{y_i}^{-1} \end{pmatrix}\begin{pmatrix} \cos\theta_i & \sin\theta_i \\ -\sin\theta_i & \cos\theta_i \end{pmatrix}$$

($t_i > 0$, $x_i \in \boldsymbol{R}$, $y_i > 0$, $\theta_i \in \boldsymbol{R}$).

Let Ω_i be the differential operator on G_{k_A} given by

$$\Omega_i = -y_i \frac{\partial^2}{\partial\theta_i \partial x_i} + y_i^2\left(\frac{\partial^2}{\partial x_i^2} + \frac{\partial^2}{\partial y_i^2}\right).$$

For $\theta = (\theta_1, \cdots, \theta_n) \in \boldsymbol{R}^n$, we denote by $\kappa(\theta)$ the element of G_{k_∞} whose ith component is

$$\begin{pmatrix} \cos\theta_i & \sin\theta_i \\ -\sin\theta_i & \cos\theta_i \end{pmatrix}.$$

Let χ be a unitary character of the group $k_A^\times/k^\times$ and let r be a function on the set of infinite places of k with values in the set of positive integers. Set $r_i = r(\infty_i)$. Denote by $S(\chi, r, k_A)$ the space of $\boldsymbol{C}$-valued functions f on G_{k_A} which satisfy the following conditions (1.7), (1.8) and (1.9).

(1.7) $f(g)$ is bounded, smooth with respect to g_∞ and locally constant with respect to g_0. Furthermore, $\Omega_i f = f r_i(r_i - 2)/4$ $(1 \leqq i \leqq n)$.

$$f\left(\gamma\begin{pmatrix} z & \\ & z \end{pmatrix}g\right) = \chi(z)f(g) \quad (\forall\, \gamma \in G_k,\ \forall\, z \in k_A^\times), \tag{1.8}$$

$$f(g\kappa(\theta)) = \exp\left(\sqrt{-1}\sum_{i=1}^n r_i\theta_i\right)f(g) \quad (\forall\, \theta \in \boldsymbol{R}^n). \tag{1.9}$$

Denote by $\mathfrak{p}(k)$ the set of finite primes of k. For each $v \in \mathfrak{p}(k)$, let k_v be the completion of k with respect to v and let $\mathfrak{o}_v$ be the ring of integers of k_v. Then the group $G_{k_A,0}$ is isomorphic to the restricted direct product $\prod_v G_{k_v}$. Normalize the Haar measure of $G_{k_A,0}$ so that the volume of $\prod_v G_{\mathfrak{o}_v}$ is equal to 1. For each $\varphi \in C_0^\infty(G_{k_A,0})$, we denote by $T(\varphi)$ the linear operator on $S(\chi, r, k_A)$ given by $(T(\varphi)f)(g) = \int_{G_{k_A,0}} f(gx)\varphi(x)\,dx$. It is shown that T_φ is of finite rank. Let F be a totally real cyclic extension of k of prime degree l. Set $\boldsymbol{\chi} = \chi \circ N_{k_A}^{F_A}$. Then $\boldsymbol{\chi}$ is a character of the group $F_A^\times/F^\times$. Extend r to a function $\boldsymbol{r}$ on the set of infinite places of F by setting $\boldsymbol{r}(\boldsymbol{w}) = r(w)$, where $\boldsymbol{w}$ is any infinite place of F and w is its restriction to k.

For each $\tau \in \mathfrak{g} = \mathrm{Gal}(F/k)$ (the Galois group of F with respect to k) and any function $\boldsymbol{f}$ on G_{F_A}, denote by $J_\tau \boldsymbol{f}$ the function on G_{F_A} given by $(J_\tau \boldsymbol{f})(g) = \boldsymbol{f}(g^\tau)$ $(g \in G_{F_A})$. It is easy to see that J_τ leaves the space $S(\boldsymbol{\chi}, \boldsymbol{r}, F_A)$ invariant. For each $v \in \mathfrak{p}(k)$, set $F_v = k_v \otimes_k F$. Let $\mathfrak{o}(F_v)$ be the maximal compact subring of F_v. The group $\mathfrak{g}$ operates on F_v in a natural manner. If v remains to be a prime in F, F_v is a cyclic field extension of k_v. If v splits in F, F_v is isomorphic to the direct sum of

l-copies of k_v. In both cases, F_v is embedded in $F_{A,0}$ in a natural manner. The group $G_{F_{A},0}$ is the restricted direct product $\prod_v G_{F_v}$. For each $g \in G_{F_A,0}$, we denote by $g_v \in G_{F_v}$ the v-component of g.

In the following we choose and fix a generator σ of $\mathfrak{g}$ once and for all. Let φ be an element of $C_0^\infty(G_{F_A,0})$ of the form $\varphi(g) = \prod_v \varphi_v(g_v)$, where $\varphi_v \in C_0^\infty(G_{F_v})$ and φ_v is the characteristic function of $G_{\mathfrak{o}(F_v)}$ except for a finite number of v. For each $v \in \mathfrak{p}(k)$, take $\varphi_v \in C_0^\infty(G_{k_v})$ which satisfies the equalities (1) and (2) of Proposition 2 for $f = \varphi_v$ and $f = \varphi_v$. We may assume that, except for a finite number of v, φ_v is the characteristic function of $G_{\mathfrak{o}_v}$ (see Remark 3 of 1.3). Denote by φ the function on $G_{k_A,0}$ given by $\varphi(g) = \prod_v \varphi_v(g)$. By class field theory, the group $k^\times N_k^F F_A^\times$ is a subgroup of index l of $k_A^\times$. Let $\chi_0\ (= 1), \chi_1, \cdots, \chi_{l-1}$ be l distinct characters of the group $k_A^\times/(k^\times N_k^F F_A^\times)$.

We are now ready to present an adelic version of Theorem 5.6 of **[10]**.

THEOREM 2. *The notation and assumptions being as above* (*in particular*, φ_v *and* φ_v *are related by* (1) *and* (2) *of Proposition* 1), *if* $r_1, r_2, \cdots, r_n > 2$,

$$(1.10) \qquad l \; Trace\; J_\sigma T(\varphi)|S(\chi, r, F_A) = \sum_{i=0}^{l-1} Trace\; T(\varphi)|S(\chi\chi_i, r, k_A).$$

REMARK 5. In **[10**, Theorem 5.6], k is the rational number field and F is a tamely ramified totally real abelian field of degree l. Furthermore, φ is assumed to be unramified. However, the correspondence $\varphi_v \to \varphi_v$ is *explicitly* described for *each* v. In that point, Theorem 5.6 of **[10]** is more precise than Theorem 2.

REMARK 6. If $l \neq 2$, $S(\chi\chi_i, r, k_A)$ is isomorphic to $S(\chi, r, k_A)$ as $G_{k_A,0}$-module.

1.5. Let us consider the representation theoretic meaning of both sides of the equality (1.10) of Theorem 2. Via the right regular representation: $g \mapsto T_g$, the group $G_{F_A,0}$ acts on $S(\chi, r, F_A)$. By a theorem of Jacquet-Langlands (see Proposition 10.9 and Proposition 11.1.1 of **[5]**), the space $S(\chi, r, F_A)$ decomposes into an algebraic direct sum of irreducible mutually inequivalent $G_{F_A,0}$-submodules. Denote by $\hat{G}_{F_A,0}(r, \chi)$ the set of equivalence classes of irreducible representations of $G_{F_A,0}$ realized on irreducible $G_{F_A,0}$-submodules of $S(\chi, r, F_A)$. For each $\pi \in \hat{G}_{F_A,0}(r, \chi)$, denote by $V(\pi, r, \chi)$ the irreducible $G_{F_A,0}$-submodule of $S(\chi, r, F_A)$ on which π is realized. We have $S(\chi, r, F_A) = \sum_\pi V(\pi, r, \chi)$ (an algebraic direct sum), where the summation with respect to π is over $\hat{G}_{F_A,0}(r, \chi)$. Denote by π^σ the representation of $G_{F_A,0}$ given by $\pi^\sigma(g) = \pi(g^\sigma)$. The obvious relation $J_\sigma T_{g^\sigma} = T_g J_\sigma$ implies that $\pi^\sigma \in \hat{G}_{F_A,0}(r, \chi)$ for each $\pi \in \hat{G}_{F_A,0}(r, \chi)$ and that $V(\pi^\sigma, r, \pi) = J_\sigma^{-1} V(\pi, r, \chi)$. Take a $\pi \in \hat{G}_{F_A,0}(r, \chi)$. It is known that, for each $v \in \mathfrak{p}(k)$, there exists $\pi_v \in \hat{G}_{F_v}$ such that π is equivalent to the restricted tensor product $\bigotimes_{v \in \mathfrak{p}(k)} \pi_v$ (except for a finite number of v, π_v is an unramified representation of G_{F_v}). Denote by $\hat{G}_{F_A,0}(r, \chi, \mathfrak{g})$ ($\mathfrak{g} = \mathrm{Gal}(F/k)$) the subset of $\hat{G}_{F_A,0}(r, \chi)$ consisting of all π such that $\pi^\sigma \cong \pi$. If $\pi^\sigma \neq \pi$, J_σ induces a cyclic permutation among $V(\pi^\tau, r, \chi)$ ($\tau \in \mathfrak{g}$). *Thus the trace of the restriction of* $J_\sigma T(\varphi)$ *to the subspace of* $S(\chi, r, F_A)$ *spanned by* $\{V(\pi, r, \chi);\ \pi \in \hat{G}_{F_A,0}(r, \chi) - \hat{G}_{F_A,0}(r, \chi, \mathfrak{g})\}$ *vanishes*. If $\pi^\sigma \cong \pi$, $\pi_v^\sigma \cong \pi_v$ for each $v \in \mathfrak{p}(k)$. There exists a linear operator $J_\sigma(\pi_v)$ of order l on the representation space of π_v such that

$$\pi_v(g^\sigma) = J_\sigma(\pi_v)^{-1} \pi_v(g) J_\sigma(\pi_v) \quad (\forall\, g \in G_{F_v}).$$

Such an operator $J_\sigma(\boldsymbol{\pi}_v)$ is unique up to a multiplication by an lth root of unity. For unramified $\boldsymbol{\pi}_v$, normalize $J_\sigma(\boldsymbol{\pi}_v)$ so that it fixes $\boldsymbol{\pi}_v(G_{\mathfrak{o}_v})$-invariant vectors in the representation space of $\boldsymbol{\pi}_v$. We may assume that the system of linear operators $\{J_\sigma(\boldsymbol{\pi}_v);\ v \in \mathfrak{p}(k)\}$ is normalized so that $J_\sigma | V(\boldsymbol{\pi}, \boldsymbol{r}, \boldsymbol{\chi}) \simeq \bigotimes_{v \in \mathfrak{p}(k)} J_\sigma(\boldsymbol{\pi}_v)$. For $\boldsymbol{\varphi} = \prod_v \boldsymbol{\varphi}_v \in C_0^\infty(G_{F_A,0})$, set $\boldsymbol{\pi}_v(\boldsymbol{\varphi}_v) = \int_{G_{F_v}} \boldsymbol{\varphi}_v(x)\boldsymbol{\pi}_v(x)\,dx$. Then $\boldsymbol{\pi}_v(\boldsymbol{\varphi}_v)$ is a linear operator of finite rank acting on the representation space of $\boldsymbol{\pi}_v$. Moreover

$$\text{Trace } J_\sigma T(\boldsymbol{\varphi}) | V(\boldsymbol{\pi}, \boldsymbol{r}, \boldsymbol{\chi}) = \prod_v \text{trace } J_\sigma(\boldsymbol{\pi}_v)\boldsymbol{\pi}_v(\boldsymbol{\varphi}_v).$$

Hence,

$$\text{(1.11)} \qquad \text{Trace } J_\sigma T(\boldsymbol{\varphi}) | S(\boldsymbol{\chi}, \boldsymbol{r}, F_A) = \sum_{\boldsymbol{\pi}} \prod_{v \in \mathfrak{p}(k)} \text{trace } J_\sigma(\boldsymbol{\pi}_v)\boldsymbol{\pi}_v(\boldsymbol{\varphi}_v),$$

where the summation with respect to $\boldsymbol{\pi}$ is over all $\hat{G}_{F_A,0}(\boldsymbol{r}, \boldsymbol{\chi}, \mathfrak{g})$. In a similar manner, we have

$$\text{(1.12)} \qquad \text{Trace } T(\varphi) | S(\chi, r, k_A) = \sum_{\pi} \prod_{v \in \mathfrak{p}(k)} \text{trace } \pi_v(\varphi_v)$$

for every $\varphi = \prod_v \varphi_v \in C_0^\infty(G_{k_A,0})$, where the summation with respect to $\pi = \bigotimes_v \pi_v$ is over all $\hat{G}_{k_A,0}(r, \chi)$ and

$$\pi_v(\varphi_v) = \int_{G_{k_v}} \varphi_v(x)\pi_v(x)\,dx.$$

Recall that in equalities (1.10), (1.11) and (1.12), $\boldsymbol{\varphi}_v$ and φ_v are related by the equalities (1) and (2) of Proposition 2 ($\forall\, v \in \mathfrak{p}(k)$). Thus, it is now natural to infer that a local implication of these equalities is Theorem 1. Furthermore, a global consequence is the following representation theoretic version of Theorem 3 of **[10]**.

THEOREM 3. *Assume $r_1, \ldots, r_n > 2$.*

(1) *For each $\pi = \bigotimes_v \pi_v \in G^\wedge_{k_A,0}(r, \chi)$ $(\pi_v \in \hat{G}_{k_v})$, there uniquely exists $\boldsymbol{\pi} = \bigotimes \boldsymbol{\pi}_v \in G^\wedge_{F_A,0}(\boldsymbol{r}, \boldsymbol{\chi}, \mathfrak{g})$ $(\boldsymbol{\pi}_v \in \hat{G}_F)$ such that for each odd place v of k, $\boldsymbol{\pi}_v$ is the lifting of π_v from $\hat{G}_{k_v}$ to $\hat{G}^{\mathfrak{g}}_{F_v}$.*

We call $\boldsymbol{\pi}$ the lifting of π from $G^\wedge_{k_A,0}(r, \chi)$ to $G^\wedge_{F_A,0}(\boldsymbol{r}, \boldsymbol{\chi}, \mathfrak{g})$.

(2) *If $l \neq 2$, for each $\boldsymbol{\pi} \in G^\wedge_{F_A,0}(\boldsymbol{r}, \boldsymbol{\chi}, \mathfrak{g})$, there uniquely exists $\pi \in G^\wedge_{k_A,0}(r, \chi)$ such that $\boldsymbol{\pi}$ is the lifting of π.*

(3) *If $l=2$, for each $\boldsymbol{\pi} \in G^\wedge_{F_A,0}(\boldsymbol{r}, \boldsymbol{\chi}, \mathfrak{g})$, there exists a $\pi \in G^\wedge_{k_A,0}(r, \chi)$ or $G_{k_A,0}(r, \chi\chi_1)$ (χ_1 is the character of order 2 of $k_A^\times/k^\times$ which corresponds to F in class field theory) such that $\boldsymbol{\pi}$ is the lifting of π.*

Moreover, π_1 and $\pi_2 \in \hat{G}_{k_A,0}(r, \chi)$ have the same lifting to $\hat{G}_{F_A,0}(\boldsymbol{r}, \boldsymbol{\chi}, \mathfrak{g})$ if and only if $\pi_1 = \pi_2$ or $\pi_1 = \pi_2 \otimes \chi_1$ (det).

2. In this section we expose the proof of Proposition 2. Then we indicate how Theorem 1 and Theorem 3 are made plausible by Theorem 2 (the proof of Theorem 2 is a (more or less obvious) modification of proofs of Theorem 1 and Theorem 5.6 of **[10]**).

2.1. In the following three subsections we use the notation in 1.1 and 1.3 without further comment. In particular, k is a $\mathfrak{p}$-adic field. Assume that F is isomorphic to the direct sum of l-copies of k. Then we may assume that G_F is the direct product of l copies of G_k and that, for each $x = (x_1, \cdots, x_l) \in G_F$, x^σ is given by $x^\sigma = (x_l, x_1, \cdots, x_{l-1})$. For any $f \in G_F$, set, for any $x \in G_k$,

$$f(x) = \int_{(G_k)^{l-1}} f((x_2, \cdots, x_l)^{-1} x, x_2, \cdots, x_l)\, dx_2 \cdots dx_l. \tag{2.1}$$

It is easy to see that $f \in C_0^\infty(G_k)$ and that $\Lambda_\sigma(f, c) = \Lambda(f, N(c))$ for *arbitrary* $c \in C'_\sigma(G_F)$. Thus the proof of Proposition 2 is quite straightforward for this case.

2.2. We summarize known results on orbital integrals on G_k. We denote by q_k the cardinality of the residue class field of k and by π a generator of the maximal ideal of $\mathfrak{o}(k)$. Denote by Q_k the set of isomorphism classes of two dimensional semisimple algebras over k. For each $K \in Q_k$, choose an embedding of K into $M(2, k)$ as a k-algebra. Via the embedding, we identify K with a subalgebra of $M(2, k)$. For each $x \in K$, denote by x' the image of x under the unique nontrivial k-algebra automorphism of K with respect to k. It is well known that

$$\mathscr{C}'(G_k) = \bigcup_{K \in Q_k} \bigcup_t (t)^{G_k} \quad \text{(disjoint union)},$$

where the union with respect to t is over a complete set of representatives of $(K^\times - k^\times)$ with respect to the action of automorphism groups of K with respect to k. Furthermore,

$$\mathscr{C}(G_k) - \mathscr{C}'(G_k) = \bigcup_{z \in k^\times} \left\{ \begin{pmatrix} z & \\ & z \end{pmatrix} \cup z\begin{pmatrix} 1 & 1 \\ & 1 \end{pmatrix}^{G_k} \right\} \quad \text{(disjoint union)}.$$

Let $\mathfrak{o}(K)$ be the maximal $\mathfrak{o}(k)$-order of K. For each nonnegative integer m, we denote by $\mathfrak{o}(K)_m$ the unique $\mathfrak{o}(k)$-order of K such that $[\mathfrak{o}(K), \mathfrak{o}(K)_m] = q_k^m$. Let $\mathfrak{o}(K)_m^\times$ be the group of units of $\mathfrak{o}(K)_m$. For each $t \in \mathfrak{o}(K)^\times - \mathfrak{o}(k)^\times$, there uniquely exists a nonnegative integer i such that $t \in \mathfrak{o}(K)_i^\times - \mathfrak{o}(K)_{i+1}^\times$. Set $i = i(t)$ for $t \in \mathfrak{o}(K)^\times - \mathfrak{o}(k)^\times$. For each $f \in C_0^\infty(G_k)$, set $\alpha(z) = f(z)$ and $\beta(z) = \Lambda(f, z(\begin{smallmatrix} 1 & 1 \\ & 1 \end{smallmatrix}))$ $(z \in k^\times)$ (cf. (1. 5)). Then both α and β are locally constant, compactly supported functions on $k^\times$. Furthermore, for each $K \in Q_k$, we denote by φ_K a function on $(K^\times - k^\times)$ given by $\varphi_K(t) = \Lambda(f, t)$ (note that φ_K does not depend upon a choice of an embedding of K into $M(2, k)$). Then φ_K is a locally constant function on $K^\times - k^\times$. The next lemma is a version of Lemma 6.2 of [7] (see also Theorem 2.1.1 and Theorem 2.2.2 of [12]). We include a proof.

LEMMA 2. *Let the notation and assumptions be as above. Then a triple* $\{\varphi_K; (K \in Q_k), \alpha, \beta\}$ *satisfies the following conditions*:

(1) *The support of* φ_K *is relatively compact in* $K^\times$.

(2) $\varphi_K(t) = \varphi_K(t')$ $(\forall\, t \in K^\times - k^\times)$.

(3) *For each* $z \in k^\times$ *there exists a neighborhood* $U(z)$ *of* z (*s.t.* $z^{-1}U(z) \subset \mathfrak{o}(K)^\times$) *such that*

$$\varphi_K(t) = \left\{1 - \frac{C(K)}{q_k - 1}\right\} \alpha(z) + C(K) q_k^{i(tz^{-1})-1} \beta(z) \tag{2.2}$$

for any $t \in U(z) - U(z) \cap k^\times$, *where we put* $C(K) = [\mathfrak{o}(K)^\times, \mathfrak{o}(K)_1^\times]$.

PROOF. Choose an $\omega \in \mathfrak{o}(K)$ so that $\{1, \omega\}$ is an $\mathfrak{o}(k)$-basis of $\mathfrak{o}(K)$. An $\mathfrak{o}(k)$-basis for $\mathfrak{o}(K)_m$ is given by $\{1, \pi^m\omega\}$ $(m = 0, 1, 2, \cdots)$. For each $t \in K$, there uniquely exists

$$\iota(t) = \begin{pmatrix} t_{11} & t_{12} \\ t_{21} & t_{22} \end{pmatrix} \quad \text{such that} \begin{pmatrix} t \\ t\omega \end{pmatrix} = \iota(t) \begin{pmatrix} 1 \\ \omega \end{pmatrix}.$$

Then the mapping $t \mapsto \iota(t)$ is an embedding of K into $M(2, k)$ as a k-algebra. It is known that every proper $\mathfrak{o}(K)_m$-ideal in K is principal (see Proposition 1 of [3]). Thus,

$$G_k = \bigcup_{m=0}^{\infty} G_{\mathfrak{o}(k)} \begin{pmatrix} 1 & \\ & \pi^m \end{pmatrix} \iota(K^\times) \quad \text{(disjoint union)}.$$

Hence, for any $f \in C_0^\infty(G_k)$ and any $t \in K^\times - k^\times$,

$$\Lambda(f, t) = \sum_{m=0}^{\infty} c_m f^\sim(\iota(t)_m) \tag{2.3}$$

where we set $f^\sim(x) = \int_{G(\mathfrak{o}_k)} f(uxu^{-1})\, du$, $c_m = [\mathfrak{o}(K)^\times, \mathfrak{o}(K)_m^\times]$,

$$\iota(t)_m = \begin{pmatrix} 1 & \\ & \pi^m \end{pmatrix} \iota(t) \begin{pmatrix} 1 & \\ & \pi^{-m} \end{pmatrix}$$

(cf. the proof of Lemma 7.3.2 of [5]). It is sufficient to prove the lemma for $z = 1$. Since f is locally constant and compactly supported, there exists a neighborhood U of 1 in $\mathfrak{o}(K)^\times$ such that

$$f^\sim(\iota(t)_m) = f^\sim\begin{pmatrix} 1 & \pi^{-m} t_{12} \\ 0 & 1 \end{pmatrix} = f^\sim\begin{pmatrix} 1 & \pi^{i(t)-m} \\ 0 & 1 \end{pmatrix} \qquad (m = 0, 1, 2, \cdots)$$

and that

$$f^\sim\begin{pmatrix} 1 & \pi^{i(t)+n} \\ 0 & 1 \end{pmatrix} = f(1_2) \qquad (n = 0, 1, \cdots)$$

for all $t \in U - U \cap k^\times$. We note that $\alpha(1) = f(1_2)$,

$$\beta(1) = \sum_{n \in \mathbf{Z}} q_k^{-n} f^\sim \begin{pmatrix} 1 & \pi^n \\ 0 & 1 \end{pmatrix}$$

and

$$[\mathfrak{o}(K^\times), \mathfrak{o}(K)_{m+1}^\times] = q_k [\mathfrak{o}(K)^\times, \mathfrak{o}(K)_m^\times] \qquad (m = 1, 2, \cdots).$$

Hence we have

$$\varphi_K(t) = \left\{1 - \frac{C(K)}{q_k - 1}\right\} \alpha(1) + C(K) q_k^{i(t)-1} \beta(1)$$

for any $t \in U - U \cap k^\times$.

A triple $\{\alpha, \beta, \varphi_K; K \in Q_k\}$ of locally constant compactly supported functions α, β on $k^\times$ and a system $\{\varphi_K; K \in Q_k\}$ of locally constant functions φ_K on $K^\times - k^\times$ is said to be an *admissible triple* if it satisfies the conditions (1), (2) and (3) of Lemma 2.

The next lemma, which is also a version of Lemma 6.2 of [7], follows from Corollary 1.1.4 of [12] and the previous lemma.

LEMMA 3. *Let $\{\alpha, \beta, \varphi_K; K \in Q_k\}$ be an admissible triple. Then there exists an $f \in C_0^\infty(G_k)$ such that $\varphi_K(t) = \Lambda(f, t)$ for any $t \in K^\times - k^\times$. Moreover, for such an f, $\alpha(z) = f(z \cdot 1_2)$ and $\beta(z) = \Lambda(f, z\begin{pmatrix} 1 & 1 \\ & 1 \end{pmatrix})$.*

2.3. Let F be a cyclic field extension of prime degree l of k. For each $K \in Q_k$, set $L = K \otimes_k F$. Then L is a two-dimensional semisimple algebra over F. A prescribed embedding of K into $M(2, k)$ is naturally extended to an embedding of L into $M(2, F)$. Furthermore, the norm mapping N_k^F from F to k is naturally extended to the norm mapping N_K^L from L to K. Set $L_1 = \{t \in L^\times, N_K^L t = 1\}$ and $F_1 = L_1 \cap F$ and $L' = \{t \in L^\times;\ N_K^L t \in K^\times - k^\times\}$. The following description of $\mathscr{C}_\sigma(G_F)$ is due to H. Saito (see Lemma 3.5 and Remark 3.8 of **[10]**).

LEMMA 4.

$$\mathscr{C}'_\sigma(G_F) = \bigcup_{K \in Q_k} \bigcup_t (t)^{G_F, \sigma}, \tag{1}$$

where the union with respect to t is taken over a complete set of representatives for L'/L_1 with respect to the action of the automorphism groups of L with respect to F.

(2) *If $l \neq 2$,*

$$\mathscr{C}'_\sigma(G_F) - \mathscr{C}_\sigma(G_F) = \bigcup_{z \in F^\times/F_1} (z \cdot 1_2)^{G_F, \sigma} \cup \bigcup_{z \in F^\times/F_1} z\begin{pmatrix} 1 & 1 \\ & 1 \end{pmatrix}^{G_F, \sigma}.$$

If $l = 2$,

$$\mathscr{C}_\sigma(G_F) - \mathscr{C}'_\sigma(G_F) = \bigcup_{z \in k^\times} \begin{pmatrix} & 1 \\ z & \end{pmatrix}^{G_F, \sigma} \cup \bigcup_{z \in F^\times/F_1} z\begin{pmatrix} 1 & 1 \\ & 1 \end{pmatrix}^{G_F, \sigma}.$$

For each $f \in C_0^\infty(G_F)$, we define functions α and β on $k^\times$ as follows: For $z \in N_k^F F^\times$, set $z = N_k^F \boldsymbol{z}$ ($\boldsymbol{z} \in F^\times$) and $\alpha(z) = \Lambda_\sigma(f, \boldsymbol{z} \cdot 1_2)$ and $\beta(z) = \Lambda_\sigma(f, \boldsymbol{z}(\begin{smallmatrix} 1 & 1 \\ & 1 \end{smallmatrix}))\ |l|_k$ (cf. (1.3)). For $z \in k^\times - N_k^F F^\times$, set $\beta(z) = 0$. If $l \neq 2$, set $\alpha(z) = 0$. If $l = 2$, set

$$\alpha(z) = -(q_k - 1)\, \Lambda_\sigma\left(f, \begin{pmatrix} & 1 \\ z & \end{pmatrix}\right).$$

It follows from Lemma 1 and Lemma 4 that α and β are well-defined functions on $k^\times$. Moreover, they are both locally constant and compactly supported.

For each $K \in Q_k$, we define a function φ_K on $K^\times - k^\times$ as follows: For $t \in (K^\times - k^\times) \cap N_K^L L$, set $t = N_K^L \boldsymbol{t}$ ($\boldsymbol{t} \in L^\times$) and $\varphi_K(t) = \Lambda_\sigma(f, \boldsymbol{t})$. For $t \notin (K^\times - k^\times) \cap N_K^L L^\times$, set $\varphi_K(t) = 0$. Lemma 1 and Lemma 4 again guarantee that φ_K is a well-defined function on $K^\times - k^\times$.

The proof of Proposition 2 is now reduced to the following Lemma 5.

LEMMA 5. *The notation being as above, $\{\alpha, \beta; \varphi_K; K \in Q_k\}$ is an admissible triple.*

For a subgroup V of $K^\times$ ($K \in Q_k$), set

$$W(V) = V \cup \left\{t\begin{pmatrix} 1 & 1 \\ & 1 \end{pmatrix};\ t \in V \cap k^\times\right\}. \tag{2.4}$$

The next lemma will be applied for the proof of Lemma 5.

LEMMA 6. *For a given $K \in Q_k$ and a given compact subset C_1 of G_F, there exist an open subgroup V of $K^\times$ and a compact subset C_2 of G_F such that $g^\sigma w g^{-1} \in C_1$ for some $w \in W(V)$ implies $g^\sigma g^{-1} \in C_2$.*

PROOF. There exists an open compact subgroup V_1 of $K^\times$ which satisfies the following conditions:

The mapping: $x \mapsto x^l$ is an *isomorphism* from V_1 to V_1^l whose inverse mapping is given by

$$x \mapsto x^{1/l} = \sum_{m=0}^{\infty} \binom{1/l}{m} (x-1)^m \tag{2.5}$$

(V_1 is so small that the series is absolutely and uniformly convergent on V_1).

Denote by C_3 the image of C_1 under the norm mapping: $x \mapsto N(x)$ (cf. 1.1). Since $N(g^\sigma w g^{-1}) = gN(w)g^{-1}$; $g^\sigma w g^{-1} \in C_1$, for some $w \in W(V_1)$ (cf. (2.4)) implies $gw^l g^{-1} \in C_3$. Put $C_4 = C_3 \cap \{gw^l g^{-1};\ w \in W(V_1),\ g \in G_F\}$. On C_4, the binomial series (2.5) is absolutely and uniformly convergent. Hence the closure C_5 of the image of C_4 under the mapping: $x \mapsto x^{1/l}$ is compact. It is easy to see that $gwg^{-1} = (gw^l g^{-1})^{1/l}$ for $w \in W(V_1)$. Hence $g^\sigma w g^{-1} \in C_1$ for some $w \in W(V_1)$ implies $g^\sigma g^{-1} \in C_1 C_5^{-1}$. Thus, we may put $C_2 = C_1 C_5^{-1}$ and $V = V_1$.

Proof of Lemma 5. It is easy to see that φ_K satisfies the conditions (1) and (2). For a $t_0 \in N_k^F \cdot F^\times$, take $z \in F^\times$ such that $t_0 = N_k^F z$. Let C_0 be the support of f and choose an open compact subgroup V of $K^\times$ which has the property described in Lemma 6 for $C_1 = z^{-1} C_0$. Then there exists a compact subset C_2 of G_F such that $zg^\sigma w g^{-1} \in C_0$ for $w \in W(V)$ implies $g^\sigma g^{-1} \in C_2$. ($W(V)$ is given by (2.4).)

Let $\{y_i;\ i \in I\}$ be a complete set of representatives for the double coset $G_{\mathfrak{o}(F)} \backslash G_F / G_k$. It follows from Lemma 1 that the mapping: $g \mapsto g^\sigma g^{-1}$ is a *homeomorphism* from G_F/G_k onto the closed subset $\{g \in G_F,\ N(g) = 1\}$ of G_F. Hence, there exists a *finite* subset I_0 of I such that

$$zg^\sigma w g^{-1} \in C_0 \text{ for a } w \in W(V) \text{ implies } g \in \bigcup_{i \in I_0} G(\mathfrak{o}_F) y_i G_k. \tag{2.6}$$

Then, for $t \in V - V \cap k^\times$, $Z_\sigma(t) = K^\times$ if V is small enough and

$$\varphi_K(t_0 t^l) = \Lambda_\sigma(f, zt) = \sum_{i \in I_0} \mu_i \Lambda(f_i, t),$$

where μ_i^{-1} is the invariant volume of $G_k \cap y_i^{-1} G(\mathfrak{o}_F) y_i$ in G_k, and $f_i \in C_0^\infty(G_k)$ is given by

$$f_i(g) = \int_{G(\mathfrak{o}_F)} f(zu^\sigma y_i^\sigma g y_i^{-1} u^{-1})\, du.$$

Since I_0 is *finite*, it follows from Lemma 2 that there exists a neighborhood $V_1 \subset V$ of 1 in $K^\times$ such that, for any $t \in V_1 - V_1 \cap k$ and any $i \in I_0$,

$$\Lambda(f_i, t) = \left\{1 - \frac{C(K)}{q_k - 1}\right\} f_i(1) + C(K) q_k^{i(t)-1} \Lambda\left(f_i, \begin{pmatrix} 1 & 1 \\ & 1 \end{pmatrix}\right).$$

However, (2.6) implies that

$$\sum_{i \in I_0} f_i(1) \mu_i = \Lambda_\sigma(f, z) = \alpha(t_0),$$

$$\sum_{i \in I_0} \mu_i \Lambda\left(f_i, \begin{pmatrix} 1 & 1 \\ & 1 \end{pmatrix}\right) = \Lambda_\sigma\left(f, z\begin{pmatrix} 1 & 1 \\ & 1 \end{pmatrix}\right) = |l|_k^{-1} \beta(t_0).$$

There exists a smaller neighborhood $V_2 \subset V_1$ such that, for $t \in V_2 - V_2 \cap k^\times$, $i(t^l) = \operatorname{ord} l + i(t)$.

Set $U = \{t_0 t^l;\ t \in V_2\}$. Then U is a neighborhood of t_0 in $K^\times$. We have proved that, for any $t \in U - U \cap k^\times$,

$$(2.7)\qquad \varphi_K(t) = \left\{1 - \frac{C(K)}{q_k - 1}\right\}\alpha(t_0) + C(K)q^{i(tt_0^{-1})-1}\beta(t_0).$$

Now take a $t_0 \in k^\times - N_k^F F^\times$. Set $L = F \otimes_k K$. If $l \neq 2$, $N_K^L L^\times \cap k^\times = N_k^F F^\times$. Hence there exists a neighborhood U of t_0 in K such that $U \cap N_K^L L^\times = \varnothing$. Hence, $\varphi_K(t) = 0$ for any $t \in U - U \cap k^\times$. Since $\alpha(t_0) = \beta(t_0) = 0$, the condition (3) is satisfied. Next assume $l = 2$. If K is not a field, there exists a neighborhood U of t_0 in $K^\times$ such that $U \cap N_L^K L^\times = \varnothing$. Hence $\varphi_K(t) = 0$ for any $t \in U - U \cap k^\times$. Since $\beta(t_0) = 0$ and $C(K) = q_k - 1$, the equality (2.7) is valid for any $t \in U - U \cap k^\times$.

Now assume that K is a quadratic extension of k. Then there exists $s_0 \in L^\times$ such that $t_0 = N_K^L s_0$. Furthermore, $Z_\sigma(s_0)$ is isomorphic to the multiplicative group of the division quaternion algebra over k.

The proof of the following Sublemma is quite similar to that of Lemma 6.

SUBLEMMA. *The notation and assumptions being as above, for a given compact subset C_1 of G_F, there exist an open subgroup V of $K^\times$ and a compact subset C_2 of G_F such that $g^\sigma s_0 v g^{-1} \in C_1$ for some $v \in V$ implies $g^\sigma s_0 g^{-1} \in C_2$.*

Normalize invariant measures of $Z_\sigma(s_0)$ and $K^\times$ so that volumes of their maximal compact subgroups are all equal to 1.

It follows easily from the Sublemma that there exists a neighborhood U_1 of s_0 in $L^\times$ such that the following integral is absolutely convergent for $s \in U_1$ and gives rise to a locally constant function on U_1: $\int_{G_F/K^\times} f(\dot{g}^\sigma s \dot{g}^{-1})\, d\dot{g}$. If $t = N_K^L s \notin k^\times$, the above integral is equal to $\varphi_K(t) = \Lambda_\sigma(f, s)$. If $s = s_0$, the above integral is equal to $\Lambda_\sigma(f, s_0) \times \int_{Z_\sigma(s_0)/K^\times} d\dot{x}$, where $d\dot{x}$ is the quotient measure of the invariant measure of $Z_\sigma(s_0)$ by that of $K^\times$. The volume of $Z_\sigma(s_0)/K^\times$ is equal to 1 or 2 according as K is ramified or not. On the other hand, $C(K)$ is equal to q_k or $q_k + 1$ according as K is ramified or not. Since $\alpha(t_0) = -(q_k - 1)\Lambda_\sigma(f, s_0)$ and $\beta(t_0) = 0$, we have shown that there exists a neighborhood U of t_0 such that (2.7) is valid for any $t \in U - U \cap k^\times$. The proof of Lemma 5 is now complete.

REMARK. The proof of Lemma 5 shows the following further relations between f and f of Proposition 2 when F is a field extension of k. If $z = N_k^F \boldsymbol{z}$ ($\boldsymbol{z} \in F^\times$), $f(z \cdot 1_2) = \Lambda_\sigma(f, \boldsymbol{z} \cdot 1_2)$ and

$$\Lambda\left(f, z\begin{pmatrix}1 & 1\\ & 1\end{pmatrix}\right) = |l|_k \Lambda_\sigma\left(f, \boldsymbol{z}\begin{pmatrix}1 & 1\\ & 1\end{pmatrix}\right).$$

If $k^\times \ni z \notin N_k^F F^\times$, $\Lambda(f, z(\begin{smallmatrix}1 & 1\\ & 1\end{smallmatrix})) = 0$ and

$$\begin{aligned} f(z \cdot 1_2) &= 0 && \text{if } l \neq 2,\\ &= -(q_k - 1)\Lambda_\sigma\left(f, \begin{pmatrix} & 1\\ z & \end{pmatrix}\right) && \text{if } l = 2.\end{aligned}$$

2.4. When F is isomorphic to the direct sum of l copies of k, the correspondence $f \to f$ of Proposition 2 is made explicit by (2.1). When F is the unramified cyclic extension of k of degree l, one can make the correspondence explicit if f is bi-$G_{o(F)}$-invariant.

In more detail, denote by $L_0(G_k, G_{o(k)})$ the set of all functions $f \in C_0^\infty(G_k)$ which are right and left $G_{o(k)}$-invariant. Then $L_0(G_k, G_{o(k)})$ becomes a commutative algebra with respect to the convolution product. For indeterminates X and Y, let $\boldsymbol{C}[X, Y, X^{-1}, Y^{-1}]_0$ be the subalgebra consisting of all elements of $\boldsymbol{C}[X, Y, X^{-1}, Y^{-1}]$ which are symmetric with respect to X and Y. For each $f \in L_0(G_k, G_{\mathfrak{o}(k)})$, put $\boldsymbol{F}(f, k)[X, Y] = \sum_{m,n\in \boldsymbol{Z}} C_{mn} X^m Y^n$, where

$$C_{mn} = q_k^{-(m-n)/2} \int_k f\left(\begin{pmatrix} \pi^m & \\ & \pi^n \end{pmatrix} \begin{pmatrix} 1 & x \\ & 1 \end{pmatrix}\right) dx.$$

Then it is known that the mapping: $f \mapsto \boldsymbol{F}(f, k)$ is an algebra isomorphism from $L_0(G_k, G_{\mathfrak{o}(k)})$ onto $\boldsymbol{C}[X, Y, X^{-1}, Y^{-1}]_0$ (cf. Theorem 3 of [**11**]).

Let F be the unramified cyclic extension of k of degree l. For each $f \in L_0(G_F, G_{\mathfrak{o}(F)})$, there uniquely exists a $\lambda(f) \in L_0(G_k, G_{\mathfrak{o}(k)})$ such that $\boldsymbol{F}(f, F)(X^l, Y^l) = \boldsymbol{F}(\lambda(f), k)(X, Y)$. The mapping: $f \mapsto \lambda(f)$ is a $\boldsymbol{C}$-algebra homomorphism from $L_0(G_F, G_{\mathfrak{o}(F)})$ into $L_0(G_k, G_{\mathfrak{o}(k)})$.

The homomorphism λ is discussed in [**6**] in a more general context. The following proposition is implicit in Saito [**10**]. In fact, considerable parts of 3.4—3.13 and 5.1—5.4 of [**10**] are devoted to the proof of Proposition 3.

PROPOSITION 3. *The notation and assumptions being as above, $f \in L_0(G_F, G_{\mathfrak{o}(F)})$ and $\lambda(f) \in L_0(G_k, G_{\mathfrak{o}})$ are related by* (1) *and* (2) *of Proposition* 2.

2.5. Let M be a finite subset of $\mathfrak{P}(k)$ containing all primes of k which ramify in F. Denote by $C_0^\infty(G_{F_A,0}, M)$ the subspace of $C_0^\infty(G_{F_A,0})$ spanned by $\varphi = \prod_v \varphi_v$ ($\varphi_v \in C_0^\infty(G_{F_v})$) such that, for each $v \notin M$, $\varphi_v \in L_0(G_{F_v}, G_{\mathfrak{o}(F_v)})$. For $\varphi = \prod_v \varphi_v \in C_0^\infty(G_{F_A,0}, M)$, we choose $\varphi = \prod_v \varphi_v \in C_0^\infty(G_{k_A,0})$ as follows:

For $v \in M$, we choose $\varphi_v \in C_0^\infty(G_{k_v})$ so that φ_v and φ_v are related by (1) and (2) of Proposition 2.

For $v \notin M$, if v remains to be prime in F, set $\varphi_v = \lambda(\varphi_v)$, where λ is the homomorphism from $L_0(G_{F_v}, G_{\mathfrak{o}(F_v)})$ to $L_0(G_{k_v}, G_{\mathfrak{o}(k_v)})$ introduced in 2.4. If v splits in F, denote by φ_v the function on G_{k_v} given by the right side of (2.1) for $f = \varphi_v$.

In both cases, $\varphi_v \in L_0(G_{k_v}, G_{\mathfrak{o}(k)})$ ($v \notin M$). For any unramified $\pi_v \in \hat{G}_{k_v}$, denote by $\pi_v^\sim$ the lifting of π_v from $\hat{G}_{k_v}$ to $\hat{G}_{F_v}$ (see Proposition 1).

It follows from the first part of Proposition 1 that

$$\text{trace } J_\sigma(\pi_v^\sim)\pi_v^\sim(\varphi_v) = \text{trace } \pi_v(\varphi_v) \qquad (\forall\, v \notin M). \tag{2.8}$$

Denote by $\hat{G}_{F_A,0}(r, \chi, M, \mathfrak{g})$ (resp. $\hat{G}_{k_A,0}(r, \chi, M)$) the subset of $\hat{G}_{F_A,0}(r, \chi, \mathfrak{g})$ (resp. $\hat{G}_{k_A,0}(r, \chi)$) consisting of all $\boldsymbol{\pi} = \bigotimes \boldsymbol{\pi}_v$ (resp. $\pi = \bigotimes \pi_v$) such that $\boldsymbol{\pi}_v$ (resp. π_v) is an unramified irreducible representation of G_{F_v} (resp. G_{k_v}) for any $v \notin M$. For $\varphi = \prod \varphi_v \in C_0^\infty(G_{F_A,0}, M)$, we have, by (1.10), (1.11) and (1.12) that

$$l \sum_{\boldsymbol{\pi}} \prod_v \text{trace } J_\sigma(\boldsymbol{\pi}_v)\, \boldsymbol{\pi}_v(\varphi_v) = \sum_{j=0}^{l-1} \sum_{\pi} \prod_v \text{trace } \pi_v(\varphi_v), \tag{2.9}$$

where the summation with respect to $\boldsymbol{\pi}$ is over all $\hat{G}_{F_A,0}(r, \chi, M, \mathfrak{g})$ and the summation with respect to π is over all $\hat{G}_{k_A,0}(r, \chi\chi_j, M)$.

For each $\boldsymbol{\pi} = \bigotimes \boldsymbol{\pi}_v \in \hat{G}_{F_A,0}(r, \chi, M, \mathfrak{g})$ denote by $X(\boldsymbol{\pi})$ the subset of $\bigcup_{j=0}^{l-1} \hat{G}_{k_A,0}(r, \chi\chi_j, M)$ consisting of all $\pi = \bigotimes \pi_v$ which satisfy the following condition:

For each $v \notin M$, the lifting of π_v from $\hat{G}_{k_v}$ to $\hat{G}_{F_v}^{\mathfrak{g}}$ is $\boldsymbol{\pi}_v$.

Then equalities (2.8) and (2.9) together with "the strong multiplicity one theorem" (see Theorem B of [9] and Theorem 2 of [1]) show the following:

$$l \prod_{v \in M} \operatorname{trace} J_\sigma(\pi_v)\pi_v(\varphi_v) = \sum_{\pi \in X(\pi)} \prod_{v \in M} \operatorname{trace} \pi_v(\varphi_v). \tag{2.10}$$

Here I must confess that I was too optimistic at Ann Arbor. I was erroneously convinced that both Theorem 1 and Theorem 3 are immediate consequences of the equality (2.10). Actually, highly nontrivial considerations are necessary to derive these theorems from (2.10). Anyway, far-reaching generalizations of Theorem 1 and Theorem 3 are established in [8].

References

1. W. Casselmann, *On some results of Atkin and Lehner*, Math. Ann. **201** (1973), 301–314.
2. K. Doi and H. Naganuma, *On the functional equation of certain Dirichlet series*, Invent. Math. **9** (1969), 1–14.
3. Y. Ihara, *Hecke polynomial as congruence ζ-functions in elliptic modular case*, Ann. of Math. (2) **85** (1967), 267–295.
4. H. Jacquet, *Automorphic forms on* GL(2). II, Lecture Notes in Math., vol. 278, Springer-Verlag, 1972.
5. H. Jacquet and R. Langlands, *Automorphic forms on* GL(2), Lecture Notes in Math., vol. 114, Springer-Verlag, 1970.
6. R. Langlands, *Problems in the theory of automorphic forms*, Lecture Notes in Math., vol. 170, Springer-Verlag, 1970, pp. 18–61.
7. ———, *Modular forms and l-adic representations*, Lecture Notes in Math., vol. 349, Springer-Verlag, 1973, pp. 361–500.
8. ———, *Base change for* GL(2), *the theory of Saito-Shintani with applications*, Institute for Advanced Study, Princeton, N.J., 1975.
9. T. Miyake, *On automorphic forms on* GL_2 *and Hecke operators*, Ann. of Math. (2) **94** (1971), 174–189.
10. H. Saito, *Automorphic forms and algebraic extensions of number fields*, Lectures in Mathematics, vol. 8, Kyoto Univ., Kyoto, Japan, 1975.
11. I. Satake, *Theory of spherical functions on reductive algebraic groups over* $\mathfrak{p}$*-adic fields*, Publ. Math. No. 18, Inst. Hautes Etudes Sci., 1963.
12. J. Shalika, *A theorem on semi-simple* $\mathfrak{p}$*-adic groups*, Ann. of Math. (2) **95** (1972), 226–242.
13. T. Shintani, *On liftings of holomorphic automorphic forms* (*a representation theoretic interpretation of the recent work of H. Saito*), Informal proceedings (available at Univ. Tokyo) for U.S.-Japan Seminar on Number Theory (Ann Arbor, 1975).
14. ———, *Two remarks on irreducible characters of finite general linear groups*, J. Math. Soc. Japan **28** (1976), 396–414.
15. ———, *On irreducible unitary characters of a certain group extension of* GL(2, $\boldsymbol{C}$), J. Math. Soc. Japan **29** (1977), 165–188.

University of Tokyo

Proceedings of Symposia in Pure Mathematics
Vol. 33 (1979), part 2, pp. 111–113

ORBITAL INTEGRALS AND BASE CHANGE

R. KOTTWITZ

Let F be a local field of characteristic 0 and let E be either $F \times \cdots \times F$ (l times) or a cyclic extension of F of degree l, where l is a prime. Embed F in $F \times \cdots \times F$ diagonally. Let $\mathcal{O}_F$ be the valuation ring of F, and let $\mathcal{O}_E$ be $\mathcal{O}_F \times \cdots \times \mathcal{O}_F$ if $E = F \times \cdots \times F$, and let it be the valuation ring of E if E is a field. If E is a field, let $\Gamma = \mathrm{Gal}(E/F)$, and let σ be a generator of Γ. If $E = F \times \cdots \times F$, let σ be the automorphism $(x_1, \cdots, x_l) \mapsto (x_2, \cdots, x_l, x_1)$ of E, and let Γ be the group of automorphisms of E generated by σ; again Γ is cyclic of order l.

Let $G = \mathrm{GL}_2$. The action of Γ on E induces an action of Γ on $G(E)$. We use the embedding of F in E to identify $G(F)$ with $G(E)^\Gamma$. Define a norm map $N: G(E) \to G(E)$ by putting $Ng = g^{\sigma^{l-1}} \cdots g^\sigma g$. This map was introduced by Saito [2]. It depends on the choice of σ.

LEMMA 1. *Let $g, x \in G(E)$. Then*
(i) $N(g^{-\sigma}xg) = g^{-1}(Nx)g$;
(ii) $(Nx)^\sigma = x(Nx)x^{-1}$;
(iii) $\det(Nx) = N_{E/F}(\det x)$;
(iv) *Nx is conjugate in $G(E)$ to an element of $G(F)$.*

The first three statements are easy calculations, and it is not hard to get (iv) from (ii). The equality (i) suggests the following definition: x, y in $G(E)$ are *σ-conjugate* if there exists $g \in G(E)$ such that $y = g^{-\sigma}xg$.

Statements (i) and (iv) together say that N induces a map from σ-conjugacy classes in $G(E)$ to conjugacy classes in $G(F)$. This map is always injective, and it is surjective if $E = F \times \cdots \times F$. It should also be noted that x is σ-conjugate to y if and only if (σ, x) is conjugate to (σ, y) in the semidirect product $\Gamma \ltimes G(E)$.

Choose Haar measures dg and dg_E on $G(F)$ and $G(E)$ respectively. If F is non-archimedean, normalize dg, dg_E so that $\mathrm{meas}(K_F) = \mathrm{meas}(K_E) = 1$, where $K_F = G(\mathcal{O}_F)$, $K_E = G(\mathcal{O}_E)$.

For any element γ of $G(F)$, let G_γ denote the centralizer of γ in G. We now give a definition which is due to Shintani [4].

DEFINITION. *Let $\varphi \in C_c^\infty(G(E))$ and $f \in C_c^\infty(G(F))$. We say that f is associated to φ if*

$$\text{(A)} \qquad \int_{G_\gamma(F)\backslash G(F)} f(g^{-1}\gamma g)\,\frac{dg}{dt} = \int_{G_\gamma(F)\backslash G(E)} \varphi(g^{-\sigma}\delta g)\,\frac{dg_E}{dt}$$

AMS (MOS) subject classifications (1970). Primary 22E50.

whenever $N\delta = \gamma$ *and* γ *is a regular element of* $G(F)$;

$$\text{(B)} \qquad \int_{G_\gamma(F)\backslash G(F)} f(g^{-1}\gamma g)\,\frac{dg}{dt} = 0$$

for every regular element γ *of* $G(F)$ *which is not a norm from* $G(E)$. *In* (A) *and* (B) dt *is a Haar measure on* $G_\gamma(F)$.

REMARKS. (1) For each regular $\gamma \in G(F)$ which is a norm, it is enough to check condition (A) of the definition for only one δ.

(2) For any element $\delta \in G(E)$, let $G^\sigma_\delta(E) = \{g \in G(E)\colon g^{-\sigma}\delta g = \delta\}$. It is not hard to see that if $\gamma = N\delta$ is a regular element of $G(F)$, then $G^\sigma_\delta(E) = G_\gamma(F)$. For $\gamma \in G(F)$, define $\varepsilon(\gamma)$ to be -1 if γ is central and is the norm of an element of $G(E)$ which is not σ-conjugate to a central element of $G(E)$, and define $\varepsilon(\gamma)$ to be 1 otherwise. It can be shown that if f is associated to φ, then

$$\text{(A}'\text{)} \qquad \int_{G_\gamma(F)\backslash G(F)} f(g^{-1}\gamma g)\,\frac{dg}{dt} = \varepsilon(\gamma) \int_{G^\sigma_\delta(E)\backslash G(E)} \varphi(g^{-\sigma}\delta g)\,\frac{dg_E}{dt'}$$

whenever $N\delta = \gamma$ belongs to $G(F)$, where dt' is a Haar measure on $G^\sigma_\delta(E)$ which depends only on the Haar measure dt on $G_\gamma(F)$;

$$\text{(B}'\text{)} \qquad \int_{G_\gamma(F)\backslash G(F)} f(g^{-1}\gamma g)\,\frac{dg}{dt} = 0$$

for every element γ of $G(F)$ which is not the norm of some element of $G(E)$.

LEMMA 2. (i) *Let* $\varphi \in C^\infty_c(G(E))$. *There is at least one* $f \in C^\infty_c(G(F))$ *which is associated to* φ.

(ii) *Let* $f \in C^\infty_c(G(F))$. *Then there exists some* $\varphi \in C^\infty_c(G(E))$ *to which* f *is associated if and only if* $\int_{G_\gamma(F)\backslash G(F)} f(g^{-1}\gamma g)\, dg/dt = 0$ *for every element* γ *of* $G(F)$ *which is not a norm from* $G(E)$ (*or equivalently, for every regular element* γ *of* $G(F)$ *which is not a norm from* $G(E)$).

Assume that F is nonarchimedean, and let $\mathscr{H}_F = \mathscr{H}(G(F), K_F)$ be the Hecke algebra of complex-valued compactly supported functions on $G(F)$ which are bi-invariant under K_F. Let $\mathscr{H}_E = \mathscr{H}(G(E), K_E)$. Saito [2] introduced a $\boldsymbol{C}$-algebra homomorphism $b\colon \mathscr{H}_E \to \mathscr{H}_F$ which we will now describe.

A function f in $\mathscr{H}_F$ gives rise to a function $f^\vee$ on the set D_F of isomorphism classes of unramified irreducible admissible representations of $G(F)$ by putting $f^\vee(\pi) = \operatorname{Tr}\pi(f)$ for $\pi \in D_F$. The set D_F is an algebraic variety over $\boldsymbol{C}$ (it is isomorphic to $\boldsymbol{C}^* \times \boldsymbol{C}^*$ divided by the action of the symmetric group S_2, which acts on the product by permuting the two factors). The map $f \mapsto f^\vee$ is a $\boldsymbol{C}$-algebra isomorphism, called the Satake isomorphism, of $\mathscr{H}_F$ with the algebra of regular functions on the variety D_F. This discussion applies to E as well; if E is a field, then D_E is again isomorphic to $\boldsymbol{C}^* \times \boldsymbol{C}^*$ divided by S_2, and if E is $F \times \cdots \times F$, then D_E is $D_F \times \cdots \times D_F$.

There is a map of algebraic varieties from D_F to D_E; if $E = F \times \cdots \times F$ the map is $\pi \mapsto \pi \otimes \cdots \otimes \pi$, and if E is a field it is $\pi(\tau) \mapsto \pi(\operatorname{Res}^{W_F}_{W_E}\tau)$ where $\pi(\tau)$ is the unramified representation of $G(F)$ corresponding to the unramified representation τ of the Weil group W_F of F, and $\pi(\operatorname{Res}^{W_F}_{W_E}\tau)$ is the unramified representation of

$G(E)$ corresponding to the restriction of τ to the Weil group W_E of E. So we get a C-algebra homomorphism from the algebra of regular functions on D_E to the algebra of regular functions on D_F, and hence also a C-algebra homomorphism b: $\mathscr{H}_E \to \mathscr{H}_F$.

REMARK. If $E = F \times \cdots \times F$, then $\mathscr{H}_E \simeq \mathscr{H}_F \otimes \cdots \otimes \mathscr{H}_F$, and $b(f_1 \otimes \cdots \otimes f_l) = f_1 * \cdots * f_l$.

LEMMA 3. *If F is nonarchimedean and E is unramified over F ($E = F \times \cdots \times F$ is allowed), then $b(\varphi)$ is associated to φ for all $\varphi \in \mathscr{H}_E$.*

This lemma is easy to prove for $E = F \times \cdots \times F$. It was first proved by Saito **[2]** when E is a field; it was subsequently proved by Langlands **[1]** using the buildings of $SL_2(F)$ and $SL_2(E)$.

Let A be the group of diagonal matrices contained in GL_2. To regular elements of A there are associated weighted orbital integrals which appear in the trace formula for GL_2 over a global field. We need to introduce a function λ_F on $G(F)$ whose logarithm is the weight factor in these integrals. Let $g \in G(F)$ and write $g = a\left(\begin{smallmatrix} 1 & x \\ 0 & 1 \end{smallmatrix}\right)k$ with $a \in A(F)$ and $k \in K_F$. Then $\lambda_F(g) = 1$ if $x \in \mathcal{O}_F$ and $\lambda_F(g) = |x|^{-2}$ otherwise. The function λ_E is defined in the same way on $G(E)$ in case E is a field.

LEMMA 4. *Suppose that F is nonarchimedean and that E is an unramified extension field of F. Then for any $\delta \in A(E)$ such that $N\delta$ is regular, and for any $\varphi \in \mathscr{H}_E$,*

$$l \int_{A(F)\backslash G(F)} b(\varphi)(g^{-1}(N\delta)g) \log \lambda_F(g) \frac{dg}{da} = \int_{A(F)\backslash G(E)} \varphi(g^{-\sigma}\delta g) \log \lambda_E(g) \frac{dg_E}{da}$$

where da is any Haar measure on $A(F)$.

This is proved in §3 of **[1]**.

REFERENCES

1. R. P. Langlands, *Base change for* GL(2), Notes, Institute for Advanced Study, 1975.

2. H. Saito, *Automorphic forms and algebraic extensions of number fields*, Lectures in Mathematics, Kyoto University, 1975.

3. ———, *Automorphic forms and algebraic extensions of number fields*, U.S.-Japan Seminar on Number Theory (Ann Arbor, 1975).

4. T. Shintani, *On liftings of holomorphic automorphic forms*, U.S.-Japan Seminar on Number Theory (Ann Arbor, 1975) (see these PROCEEDINGS, part 2, pp. 97–110).

RICHLAND, WASHINGTON

Proceedings of Symposia in Pure Mathematics
Vol. **33** (1979), part 2, pp. 115–133

THE SOLUTION OF A BASE CHANGE PROBLEM FOR GL(2) (FOLLOWING LANGLANDS, SAITO, SHINTANI)

P. GÉRARDIN AND J. P. LABESSE

These Notes present a survey of the results on the lifting of automorphic representations of GL(2) with respect to a cyclic extension of prime degree of the groundfield, and of some of its applications to the Artin conjecture, with some sketches of proofs. §§1–5 are devoted to the definitions and results on the lifting, §6 to the proof of the Artin conjecture in the tetrahedral case. The first part ends up with three appendices describing respectively two-dimensional representations of the Weil group (Appendix A), representations of GL(2) over a local field (Appendix B) and a global field (Appendix C). Part II gives some indications on the proof of the results on lifting. The main tools are the orbital and twisted orbital integrals, and a twisted trace formula **[Sa]**. The main references for the lifting are **[Sa]**, [S-1], [S-2], **[L]**. As a side remark, we would like to point out that the study of the example of the general linear group over a finite field [S-2] is illuminating.

I. Definitions, Theorems, Applications

1. Notation. Let F be a local or global field. Then W_F is the Weil group of F and, for F a p-field, W'_F is the Weil-Deligne group of F **[T]**. We recall that there exists a canonical surjective homomorphism $W_F \to C_F$ which identifies W_F^{ab} with C_F.

In all these notes, E is a Galois extension of F, cyclic of prime degree l, and Γ its Galois group (the "split case", where $E = F \times F \times \cdots \times F$ l-times and Γ is generated by $\sigma\colon (x_1, x_2, \cdots, x_l) \mapsto (x_2, \cdots, x_l, x_1)$ is handled easily and is left to the reader).

For F local, choose a nontrivial character ψ of the additive group of F, and define $\psi_{E/F} = \psi \circ \mathrm{Tr}_{E/F}$.

In the following we shall use systematically the notation given by Borel **[B]** about L-groups : $\Phi(G)$, $\cdots$, and by Tate **[T]** for Weil groups, the L- and ε-factors.

2. Base change for GL(1). From abelian class-field theory, there is a commutative diagram

$$\begin{array}{ccccccccc} & & 1 & \longrightarrow & W_E & \longrightarrow & W_F & \longrightarrow & \Gamma & \longrightarrow & 1 \\ & & & & \downarrow & & \downarrow & & \downarrow\wr & & \\ 1 & \longrightarrow & C_E^{1-\sigma} & \longrightarrow & C_E & \xrightarrow{N_{E/F}} & C_F & \longrightarrow & \Gamma & \longrightarrow & 1 \end{array}$$

AMS (*MOS*) *subject classifications* (1970). Primary 10D15, 12A70; Secondary 20E50, 20E55.

where σ is a generator of Γ.

The one-dimensional representations of W_F are given by the quasi-characters of $W_F^{ab} = C_F$; by composing a quasi-character of C_F with the norm, a map is defined called the lifting (or more precisely, the base change lift): $\chi \mapsto \chi_{E/F} = \chi \circ N_{E/F}$ which sends the set $\mathscr{A}(F)$ of quasi-characters of C_F in the set $\mathscr{A}(E)$ of quasi-characters of C_E.

The group Γ acts on the set of quasi-characters of C_E by:

$$({}^{\gamma}\theta)(z) = \theta(z^{\gamma}), \qquad \gamma \in \Gamma, z \in C_E;$$

the group $\hat{\Gamma}$ of characters of Γ can be identified with the set of characters of C_F which are trivial on $N_{E/F}C_E$; this group $\hat{\Gamma}$ acts on the set of quasi-characters of C_F by multiplication: $\chi \mapsto \chi\zeta$, $\zeta \in \hat{\Gamma}$. Then the exactness of the second line of the above diagram implies the following result:

PROPOSITION 1. *The lifting $\chi \mapsto \chi_{E/F}$ defines a bijection from the orbits of $\hat{\Gamma}$ in $\mathscr{A}(F)$ onto the invariant elements by Γ in $\mathscr{A}(E)$; moreover, the following relations hold*:

$$L(\chi_{E/F}) = \prod_{\zeta \in \hat{\Gamma}} L(\chi\zeta),$$

$$\varepsilon(\chi_{E/F}) = \prod_{\zeta \in \hat{\Gamma}} \varepsilon(\chi\zeta) \quad \textit{for } F \textit{ global},$$

$$\varepsilon(\chi_{E/F}, \psi_{E/F}) = \lambda_{E/F}(\psi)^{-1} \prod_{\zeta \in \hat{\Gamma}} \varepsilon(\chi\zeta, \psi) \quad \textit{for } F \textit{ local}.$$

3. Base change for GL(2) on the L-groups.

3.1. Let G be the group GL(2) over F, and $G_{E/F}$ be the group over F defined by restriction of scalars of the group GL(2) over E, so that $G_{E/F}(F) = \mathrm{GL}(2, E)$; the group $G_{E/F}$ is quasi-split, and there is a natural map:

$${}^{L}G = \mathrm{GL}(2, \boldsymbol{C}) \times \Gamma_F \longrightarrow {}^{L}G_{E/F} = \mathrm{GL}(2, \boldsymbol{C})^{\Gamma} \rtimes \Gamma_F;$$

here $\mathrm{GL}(2, \boldsymbol{C})^{\Gamma}$ is the set of applications from Γ in $\mathrm{GL}(2, \boldsymbol{C})$, and $\Gamma_F = \mathrm{Gal}(\bar{F}/F)$ acts by permutations of the coordinates [**B**, §5].

Given any $\rho \in \Phi(G)$, its restriction to W_E' defines an element $\rho_{E/F} \in \Phi(G_{E/F})$; the set $\Phi(G_{E/F})$ can be identified with the set $\Phi(G/E)$ [**B**], and the above map defines the application

$$\begin{aligned} \Phi(G) &\longrightarrow \Phi(G_{E/F}) = \Phi(G/E), \\ \rho &\longmapsto \rho_{E/F} \end{aligned}$$

called the base change.

For F local, let Fr_F be a Frobenius element in Γ_F; when E is unramified, then $\mathrm{Fr}_E = \mathrm{Fr}_F^l \in \Gamma_E$ is a Frobenius element for E. Moreover, if ρ is unramified and defined by $\mathrm{Fr}_F \mapsto s$, where s is a semisimple element in $\mathrm{GL}(2, \boldsymbol{C})$, then $\rho_{E/F}$ is unramified and is given by $\mathrm{Fr}_E \mapsto s^l$.

The properties of L- and ε-factors with respect to induction [**T**] show that:

$$L(\rho_{E/F}) = \prod_{\zeta \in \hat{\Gamma}} L(\rho \otimes \zeta),$$

$$\varepsilon(\rho_{E/F}) = \prod_{\zeta \in \Gamma} \varepsilon(\rho \otimes \zeta), \quad \text{for } F \text{ global},$$

$$\varepsilon(\rho_{E/F}, \psi_{E/F}) = \lambda_{E/F}(\psi)^{-2} \prod_{\zeta \in \Gamma} \varepsilon(\rho \otimes \zeta, \psi), \quad \text{for } F \text{ local}.$$

3.2. From the classification of the two-dimensional admissible representations of W'_F (see Appendix A), one has the following result:

PROPOSITION 2. (a) *The lifting* $\rho \in \Phi(G) \mapsto \rho_{E/F} \in \Phi(G_{E/F})$ *has for image the set of* Γ*-invariants in* $\Phi(G_{E/F})$.

(b) *In the following cases, the lifting is given by*

$$(\mu \oplus \nu)_{E/F} = \mu_{E/F} \oplus \nu_{E/F},$$
$$(\operatorname{Ind}_{W_K}^{W_F} \theta)_{E/F} = \operatorname{Ind}_{W_{KE}}^{W_E} \theta_{KE/K} \quad \text{for } E \neq K,$$
$$(\operatorname{Ind}_{W_K}^{W_F} \theta)_{E/F} = \theta \oplus {}^{\sigma}\theta \quad \text{for } E = K, 1 \neq \sigma \in \Gamma,$$

$(\chi \otimes \mathrm{sp}(2))_{E/F} = \chi_{E/F} \otimes \mathrm{sp}(2)$ (*for* F *nonarchimedean*).

(c) *Given a nondecomposable* $\rho \in \Phi(G)$, *the representations which have the same lifting are the* $\rho \otimes \zeta$ *for all* $\zeta \in \hat{\Gamma}$; *for* $\rho = \lambda \oplus \mu$, *the representations which have the same lifting are the* $\lambda\zeta \oplus \mu\zeta'$ *for all* $\zeta, \zeta' \in \hat{\Gamma}$.

4. Base change over a local field. In this section, we denote by σ a generator of $\Gamma = \mathrm{Gal}(E/F)$.

4.1. Let $\Pi(G)$ be the set of classes of admissible irreducible representations of $G(F)$. There is a conjectural bijection $\Phi(G) \rightleftarrows \Pi(G)$ [**B**]. The base change map $\Phi(G) \to \Phi(G_{E/F})$ must reflect a map $\Pi(G) \to \Pi(G_{E/F})$. The definition of base change for representations of $G(F)$ will be given in 4.3; since the image of $\Phi(G)$ is the set of Γ-invariant elements in $\Phi(G_{E/F})$, one studies first the admissible irreducible representations of $G(E)$ equivalent to their conjugates by Γ.

4.2. Let $\tilde{\pi}$ be an admissible irreducible representation of $G(E)$ such that ${}^{\sigma}\tilde{\pi} \simeq \tilde{\pi}$; then there exists an operator C on the space of $\tilde{\pi}$ such that $C^{-1}\tilde{\pi}(z)C = \tilde{\pi}(z^{\sigma})$, $z \in G(E)$, and $C^l = \mathrm{Id}$. This operator is determined up to an lth root of unity. The mapping $\tilde{\pi}' : (\sigma^m, z) \mapsto C^m\tilde{\pi}(z)$ defines an extension $\tilde{\pi}'$ of $\tilde{\pi}$ to the semidirect product $\Gamma \ltimes G(E)$.

PROPOSITION 3. *This representation has a character given by a locally integrable function* $\operatorname{Tr} \tilde{\pi}'$ *on* $\Gamma \ltimes G(E)$.

On $\sigma \times G(E)$, the character $\operatorname{Tr} \tilde{\pi}'$ defines a σ-invariant distribution on $G(E)$, i.e., invariant under σ-conjugation: $z \mapsto y^{-\sigma}zy$, $y, z \in G(E)$.

Let us state some properties of the σ-conjugation.

For $z \in G(E)$, put

$$N_{E/F,\sigma}z = z^{\sigma^{l-1}} \cdots z^{\sigma} \cdot z;$$

or simply $N(z)$ if no confusion can arise.

PROPOSITION 4. (a) $N_{E/F,\sigma}z$ *is conjugate in* $G(E)$ *to an element of* $G(F)$;

(b) $z \mapsto N_{E/F,\sigma}z$ *defines an injection of the set of* σ*-conjugacy classes of* $G(E)$ *into the set of conjugacy classes of* $G(F)$;

(c) *the elliptic classes of* $G(F)$ *obtained by* $N_{E/F,\sigma}$ *are those with determinant in* $N_{E/F}E^{\times}$; *the hyperbolic classes of* $G(F)$ *obtained are those whose eigenvalues are norms of* $E^{\times}$; *any unipotent class of* $G(F)$ *is in the image of* N.

4.3. *Definition of the base change for* GL(2) *over a local field.* Let $\pi \in \Pi(G)$ and $\tilde{\pi}$

be an irreducible admissible representation of $G(E)$ which is equivalent to its conjugate by σ. Then $\tilde{\pi}$ is called a base change lift of π, or a lifting of π, if either

(a) $\pi = \pi(\mu, \nu)$ and $\tilde{\pi} = \pi(\mu_{E/F}, \nu_{E/F})$, or

(b) there exists an extension $\tilde{\pi}'$ of $\tilde{\pi}$ to $\Gamma \ltimes G(E)$ such that $\operatorname{Tr} \tilde{\pi}'(\sigma \times z) = \operatorname{Tr} \pi(x)$ for $z \in G(E)$ whenever $N_{E/F,\sigma}\, z$ is conjugate in $G(E)$ to a regular semisimple element $x \in G(F)$.

Some of the notation in the following theorem is explained in Appendix B.

THEOREM 1 (BASE CHANGE FOR GL(2) OVER A LOCAL FIELD). (a) *Any* $\pi \in \Pi(G)$ *has a unique lifting* $\pi_{E/F} \in \Pi(G_{E/F})$, *and any* $\pi \in \Pi(G)$ *fixed by* Γ *is a lifting*;

(b) *the lifting is independent of the choice of the generator* σ *of* Γ;

(c) $\pi_{E/F} = \pi'_{E/F} \Leftrightarrow \pi' = \pi \otimes \zeta$ *for a* $\zeta \in \hat{\Gamma}$, *or* $\pi = \pi(\mu, \nu)$, $\pi' = \pi(\mu', \nu')$ *with* $\mu^{-1}\mu'$ *and* $\nu^{-1}\nu'$ *in* $\hat{\Gamma}$;

(d) $\omega_{\pi_{E/F}} = (\omega_\pi)_{E/F}$, $(\pi \otimes \chi)_{E/F} = \pi_{E/F} \otimes \chi_{E/F}$ *for any one-dimensional representation* χ *of* $F^\times$, $(\pi_{E/F})^\vee = (\pi^\vee)_{E/F}$ *(contragredient representations)*;

(e) *for* $E \supset F \supset k$ *with* E *and* F *Galois over* k, ${}^{\tilde{\gamma}}(\pi_{E/F}) = ({}^{\gamma}\pi)_{E/F}$ *for any* $\tilde{\gamma} \in \operatorname{Gal}(E/k)$, *with image* γ *in* $\operatorname{Gal}(F/k)$;

(f) *at least for* $\rho \in \Phi(G)$ *not exceptional*, $\pi(\rho)_{E/F} = \pi(\rho_{E/F})$.

5. Global base change.

5.1. Let $\Pi(G)$ be the set of classes of irreducible admissible automorphic representations of $G(A_F) = \mathrm{GL}(2, A_F)$, where A_F is the ring of adeles of the number field F. From the principle of functoriality **[B]**, the base change on L-groups should reflect a map from $\Pi(G)$ to $\Pi(G_{E/F})$, the set of irreducible admissible automorphic representations of $G(A_E) = \mathrm{GL}(2, A_E) = G_{E/F}(A_F)$; such a map must be compatible with the local data.

For any place v of F, put $E_v = E \otimes_F F_v$; it is a cyclic Galois extension of F or a product of l copies of F; in this latter case, define the lifting π_{E_v/F_v} of $\pi \in \Pi(G)$ by

$$\pi_{E_v/F_v} = \pi \otimes \cdots \otimes \pi \quad (l \text{ times}).$$

5.2. *Definition of the global base change for* GL(2). Let $\pi \in \Pi(G)$, $\tilde{\pi} \in \Pi(G_{E/F})$; then $\tilde{\pi}$ is called a lifting of π (or more precisely a base change lift of π) if, for every place v of F, $\tilde{\pi}_v$ is the lifting of π_v.

5.3. The notations used in the following theorem are those of Appendix C.

THEOREM 2 (GLOBAL BASE CHANGE FOR GL(2)). (a) *Every* $\pi \in \Pi(G)$ *has a unique lifting* $\pi_{E/F} \in \Pi(G_{E/F})$;

(b) *a cuspidal* $\tilde{\pi} \in \Pi(G_{E/F})$ *is a lifting if and only if it is fixed by* Γ, *and then, it is a lifting of cuspidal representations*; *a cuspidal* $\pi \in \Pi(G)$ *has a lifting which is cuspidal except for* $l = 2$ *and* $\pi = \pi(\operatorname{Ind}_{W_E}^{W_F} \theta)$ *and then* $\pi_{E/F} = \pi(\theta, {}^\sigma\theta)$;

(c) *for a cuspidal* $\pi \in \Pi(G)$, *the representations* π', *which have* $\pi_{E/F}$ *for lifting are the* $\pi' = \pi \otimes \zeta$ *with* $\zeta \in \Gamma$;

(d)
$$\begin{aligned} \omega_{\pi_{E/F}} &= (\omega_\pi)_{E/F} && \textit{(central quasi-characters)},\\ (\pi \otimes \chi)_{E/F} &= \pi_{E/F} \otimes \chi_{E/F} && \textit{(twisting by a quasi-character)},\\ (\pi_{E/F})^\vee &= (\check{\pi})_{E/F} && \textit{(contragredient representations)}; \end{aligned}$$

(e) *for* $E \supset F \supset k$ *with* E *and* F *Galois over* k, ${}^{\tilde{\gamma}}(\pi_{E/F}) = ({}^{\gamma}\pi)_{E/F}$ *for any* $\gamma \in \operatorname{Gal}(F/k)$ *image of* $\tilde{\gamma} \in \operatorname{Gal}(E/k)$;

(f) *if* $\pi = \pi(\rho)$ *for some* $\rho \in \phi(G)$, *then* $\pi_{E/F} = \pi(\rho_{E/F})$.

REMARK. There are examples of noncuspidal $\tilde{\pi} \in \Pi(G_{E/F})$, fixed by Γ which are not liftings (cf. [L, §10]).

6. Artin conjecture for tetrahedral type. Let ρ be a two-dimensional admissible representation of the Weil group of the number field F; we assume that its image modulo the center: $W_F \to \mathrm{GL}(2, C) \to \mathrm{PGL}(2, C)$ is the tetrahedral group $\mathfrak{A}_4$. This group is solvable: $1 \to D_4 \to \mathfrak{A}_4 \to C_3 \to 1$. The action of the cyclic group C_3 on the dihedral group D_4—the so-called mattress group—is given by the cyclic permutations of its nontrivial elements. The inverse image of D_4 in W_F is a normal subgroup of index 3, hence is the Weil group W_E of a cubic Galois extension E of F:

$$\begin{array}{ccccccccc} 1 & \longrightarrow & W_E & \longrightarrow & W_F & \longrightarrow & \mathrm{Gal}(E/F) & \longrightarrow & 1 \\ & & \downarrow & & \downarrow & & \downarrow \wr & & \\ 1 & \longrightarrow & D_4 & \longrightarrow & \mathfrak{A}_4 & \longrightarrow & C_3 & \longrightarrow & 1 \end{array}$$

The restriction $\rho_{E/F}$ of ρ to W_E has for image the dihedral group D_4; it is induced from a one-dimensional representation of a subgroup of index 2 in W_E, so that there is a corresponding cuspidal automorphic representation $\pi(\rho_{E/F})$ of $\mathrm{GL}(2, A_E)$.

The inner automorphisms of $\mathfrak{A}_4$ give an action of C_3 on D_4, and the action of $\Gamma = \mathrm{Gal}(E/F)$ fixes the class of $\rho_{E/F}$, hence also the class of $\pi(\rho_{E/F})$. From Theorem 2 (5.3), this representation is the base change lift of exactly three classes of irreducible cuspidal automorphic representations of $\mathrm{GL}(2, A_F)$, and their central character has for base change lift the central character of $\pi(\rho_{E/F})$, which is equal to $\det \rho_{E/F} = (\det \rho)_{E/F}$; there is only one of them, say π, with, the central character $\det \rho$.

The Artin conjecture for the representation ρ is the holomorphy of the corresponding L-function: $s \mapsto L(s, \rho)$. According to Jacquet-Langlands [**J-L**, Theorem 11.1, p. 350], the L-function $s \mapsto L(s, \pi)$ corresponding to cuspidal π is holomorphic ; hence, the Artin conjecture will be proved for ρ if we show the equality of these two L-functions. From [**J-L**, pp. 404–407], this will be done if the following assertion is shown to be true:

(c_0) $\pi_v = \pi(\rho_v)$ for each archimedean place v of F, and for almost all v.

As E is cubic over F, each infinite place v of F splits in E, so for $w|v$, we have the equations:

$$E_w = F_v, \quad (\rho_{E/F})_w = \rho_v, \quad (\pi(\rho_{E/F}))_w = \pi(\rho_v) = \pi_v,$$

which are more generally true for any place v of F which splits in E.

Now if v does not split and is unramified in E, and if moreover ρ_v is unramified, then so is the restriction ρ_{E_v/F_v} of ρ_v to W_{E_v}, and the representation $(\pi_{E/F})_v = \pi(\rho_{E_v/F_v})$ is unramified; since E_v/F_v is unramified this representation is the base change lift of unramified representations. This shows that π_v is unramified; call ρ_{π_v} a two-dimensional representation of W_{E_v}, such that $\pi_v = \pi(\rho_{\pi_v})$. We have shown that our assertion (c_0) is equivalent to:

(c_1) if v does not split and is unramified for ρ and E, ρ_v and ρ_{π_v} are equivalent.

The adjoint representation of PGL(2) defines an injection of PGL(2) in GL(3), hence a morphism A: GL(2) $\to$ GL(3). We observe now that the condition (c_1) is equivalent to the apparently weaker condition:

(c_2) if v is unramified for ρ and E, the three-dimensional representations $A\rho_v$ and $A\rho_{\pi_v}$ are equivalent.

In fact, call a (resp. b) the image of a Frobenius in W_{F_v} through ρ_v (resp. ρ_{π_v}); if (c_2) is satisfied, $a \in \boldsymbol{C}^\times b$; but det a = det b, hence $a = \pm b$. If $a = -b$ then, since ρ_v and ρ_{π_v} have the same restriction to W_{E_v}, a^3 is conjugate to $-a^3$, that is $\mathrm{Tr}(a^3) = 0$. Hence $A(a^3)$ is of order two, and this means that $A(a)$ is of order 6; but the image of $A\rho_v$ is in the tetrahedral group which has no element of order 6; so we have $a = b$, hence $\rho_v = \rho_{\pi_v}$. The introduction of $A\rho$ is motivated by the crucial observation, due to Serre, that this three-dimensional representation is induced by a one-dimensional representation of W_E; in fact, the tetrahedral group leaves invariant the set of the three lines joining the middles of the opposite edges of the tetrahedron. This means that $A\rho$ is induced by the one-dimensional representation θ of the stabilizer of one of these lines (obtained by restriction of $A\rho$); but this stabilizer is the pull-back of the dihedral group $D_4 \subset \mathfrak{A}_4$ in W_F, which is the subgroup W_E:

$$A\rho = \mathrm{Ind}\,_{W_E}^{W_F}\,\theta.$$

From **[J-PS-S]**, to such a three-dimensional irreducible monomial representation $A\rho$ of W_F is associated an irreducible cuspidal automorphic representation $\pi(A\rho)$ of GL(3, $\boldsymbol{A}_F$). On the other hand, the morphism A reflects a lifting from irreducible cuspidal automorphic representations of GL(2, $\boldsymbol{A}_F$) to automorphic representations of GL(3, $\boldsymbol{A}_F$); and, here, the representation $A\pi$ corresponding to π is cuspidal **[G-J-2]**. To prove (c_2) it suffices to show the condition:

(c_3) the lifting $A\pi$ is equivalent to $\pi(A\rho)$.

There is a practical criterion given by **[J-S]** to prove the equivalence of such representations: π_1 and π_2 are equivalent if and only if $L(s, \pi_1 \times \check{\pi}_2)$ has a pole at $s = 1$, where L is the L-function attached to the representation of $\mathrm{GL}_3(\boldsymbol{C}) \times \mathrm{GL}_3(\boldsymbol{C})$ in $\mathrm{GL}_9(\boldsymbol{C})$ given by the tensor product.

We shall prove that almost all local factors of $L(s, A\pi \times \pi(A\rho)^\vee)$ and of $L(s, \pi(A\rho) \times \pi(A\rho)^\vee)$ are equal; by nonvanishing properties of local factors **[J-S]**, this is enough to prove that $L(s, A\pi \times \pi(A\rho)^\vee)$ has a pole at $s = 1$, and hence (c_3) will be proved. If v is split, then $\pi_v = \pi(\rho_v)$ and, at least when π_v and ρ_v are unramified, $(A\pi)_v = \pi(A\rho_v)$: the local L-factors are then equal.

If v does not split in E and is unramified for E and ρ, the two local L-functions are those associated to the nine-dimensional representation of W_F given by $A\rho_{\pi_v} \otimes ((A\rho)_v^\vee)$ and $A\rho_v \otimes ((A\rho)_v^\vee)$; but we know that $A\rho_v = \mathrm{Ind}\,_{W_{E_v}}^{W_{F_v}}\,\theta_v$.

Now if U (resp. V) are representations of a group G (resp. a subgroup H) one has $U \otimes \mathrm{Ind}_H^G\, V \cong \mathrm{Ind}\,_H^G(V \otimes \mathrm{Res}\,_H^G U)$; also recall that the two representations ρ_{π_v} and ρ_v have equivalent restrictions to W_{E_v}; hence $A\rho_{\pi_v} \otimes (A\rho)_v^\vee$ is equivalent to $(A\rho_v) \otimes (A\rho)_v^\vee$ so that they have the same L-factor.

This concludes the proof of the Artin conjecture for the tetrahedral case.

The above proof is taken from a letter of Langlands to Serre (December 1975); it also contains some indications on a method to handle the octahedral case; however the latter requires some results on group representations which are not yet available. Still, a partial result is obtained **[L, §1]**:

Assume that ρ is of octahedral type; we use the fact that $\mathfrak{S}_4$ has a normal subgroup $\mathfrak{A}_4$ of index 2; hence there is a quadratic extension E of F for which the restriction $\rho_{E/F}$ is of type $\mathfrak{A}_4$. By the above theorem, $\pi(\rho_{E/F})$ exists and, by Theorem 2, it is the base change lift of two cuspidal admissible irreducible automorphic representations π' and π'' of $G(\boldsymbol{A}_F)$. Assume now that $F = \boldsymbol{Q}$, that E is totally real and that the complex conjugations in $\mathrm{Gal}(\bar{\boldsymbol{Q}}/\boldsymbol{Q})$ are sent by ρ into the class of $\left(\begin{smallmatrix}1 & 0\\ 0 & -1\end{smallmatrix}\right)$; in such a case the components of π' and π'' at the real place verify $\pi'_\infty = \pi''_\infty = \pi(\rho_\infty)$ with $\rho_\infty = 1 \oplus \mathrm{sign}$; then π' and π'' correspond to holomorphic automorphic forms of weight one.

This situation has been studied by Deligne-Serre **[D-S]**, and they show that $\pi' = \pi(\rho')$ and $\pi'' = \pi(\rho'')$ for some representations ρ', ρ'' of W_F in $\mathrm{GL}(2, \boldsymbol{C})$; now, by Theorem 2, $\pi(\rho')_{E/F} = \pi(\rho'_{E/F})$, and the same is true for π''; this shows that ρ' and ρ'' are the two representations which lift to $\rho_{E/F}$; hence either $\rho = \rho'$ or ρ''. Thus one concludes that either $\pi' = \pi(\rho)$ or $\pi'' = \pi(\rho)$, and this gives

THEOREM 4. *For a two-dimensional representation of $W_{\boldsymbol{Q}}$ which is of octahedral type and which sends the complex conjugation on $\left(\begin{smallmatrix}1 & 0\\ 0 & -1\end{smallmatrix}\right)$, and such that the above quadratic field E is real, the Artin conjecture is satisfied.*

Appendix A. List of the two-dimensional admissible representations of W'_F [D].

Notations. F is a local (resp. global) field, C_F is $F^\times$ (resp. the group of ideles classes of F); W'_F is the Weil-Deligne group of F **[T]**.

For F global, v a place of F, there is an injection $W_{F_v} \to W_F$ which defines an application $\rho \mapsto \rho_v$ from the two-dimensional admissible representations of W_F into those of W_{F_v}; if another representation ρ' of W_F satisfies $\rho'_v \sim \rho_v$ for all but a finite number of places, then ρ' is equivalent to ρ, and $\rho'_v \sim \rho_v$ for all v. The two-dimensional admissible representations of W_F are classified by the image of the inertia group in $\mathrm{PGL}(2, \boldsymbol{C})$, called the type of the representation.

(1) Cyclic type: $\mu \oplus \nu$ is the sum of the two one-dimensional representations of W_F defined by μ and ν; $\mu \oplus \nu \simeq \nu \oplus \mu$; $\det(\mu \oplus \nu) = \mu\nu$; $(\mu \oplus \nu) \otimes \chi = (\mu\chi) \oplus (\nu\chi)$; $(\mu \oplus \nu)^\vee = \mu^{-1} \oplus \nu^{-1}$. The L- and ε-functions verify:

$$L(\mu \oplus \nu) = L(\mu)L(\nu), \qquad \varepsilon(\mu \oplus \nu) = \varepsilon(\mu)\varepsilon(\nu) \quad (\text{if } F \text{ is global}),$$
$$\varepsilon(\mu \oplus \nu, \psi) = \varepsilon(\mu, \psi)\varepsilon(\nu, \psi) \quad (\text{if } F \text{ is local}),$$
$$\mu \oplus \nu = \bigotimes\nolimits_v (\mu_v \oplus \nu_v) \quad (\text{if } F \text{ is global}).$$

(2) Dihedral type: $\tau = \mathrm{Ind}_{W_K}^{W_F} \theta$, where θ is a quasi-character of C_K, and K a separable quadratic extension of F; τ is irreducible if and only if $\theta \neq {}^\sigma\theta$ $(1 \neq \sigma \in \mathrm{Gal}(K/F))$. For $\theta = {}^\sigma\theta$, let χ be a quasi-character of C_F such that $\theta = \chi \circ N_{K/F}$ and let $\hat{\sigma}$ be the character of C_F with Kernel $N_{K/F}C_K$. Then $\mathrm{Ind}_{W_K}^{W_F} \theta = \chi \oplus \chi\hat{\sigma}$; $\det \tau = \hat{\sigma} \cdot \theta|_{C_F}$; $\tau \otimes \chi = \mathrm{Ind}_{W_K}^{W_F} (\theta\chi \circ N_{K/F})$; $\check{\tau} = \mathrm{Ind}_{W_K}^{W_F}\theta^{-1}$; $L(\tau) = L(\theta)$, $\varepsilon(\tau) = \varepsilon(\theta)$ (F global), $\varepsilon(\tau, \psi) = \lambda_{K/F}(\psi)\varepsilon(\theta, \psi \circ \mathrm{Tr}_{K/F})$ (F local); for F global, $\tau = \bigotimes \tau_v$ with $\tau_v = \mathrm{Ind}_{W_{K_v}}^{W_{F_v}} \theta_v$ for K_v/F_v quadratic, and $\tau_v = \theta_v \oplus \theta_v$ for $K_v = F_v \times F_v$. The equivalences are: $\mathrm{Ind}_{W_1}^{W_1} \theta_1 \sim \mathrm{Ind}_{W_K}^{W_F} \theta_2 \Leftrightarrow$ either $K_1 = K_2$ and θ_1, θ_2 conjugate by $\mathrm{Gal}(K/F)$, or $K_1 \neq K_2$, $\theta_1{}^{\sigma_1}\theta_1^{-1}$ and $\theta_2{}^{\sigma_2}\theta_2^{-1}$ are of order 2 and $\theta_1 \circ N_{K_1K_2/K_1} = \theta_2 \circ N_{K_1K_2/K_2}$.

(3) Exceptional type: the image of the inertia group in $\mathrm{PGL}(2, \boldsymbol{C})$ is $\mathfrak{A}_4$ (tetrahedral type), $\mathfrak{S}_4$ (octahedral type), or $\mathfrak{A}_5$ (icosahedral type); they occur only for F

global or F nonarchimedean local of even residual characteristic; in this latter case, the icosahedral type does not occur.

(4) Special type (occurs only for F nonarchimedean local): $\chi \otimes \mathrm{sp}(2)$ for a quasi-character χ of $F^\times$ and sp(2) the representation of W'_F defined by

$$z \in \boldsymbol{C} \mapsto \begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad w \in W_F \mapsto \begin{pmatrix} |w| & 0 \\ 0 & 1 \end{pmatrix}.$$

Appendix B. List of admissible irreducible representations of GL(2, *F*), *F* local field [J-L].

Notations. $G = \mathrm{GL}(2, F)$, $|\ |$ is the absolute value defined by the dilatation of the Haar measure on F: $d(ax) = |a|\,dx$, and ψ is a nontrivial character of the additive group of F.

Representations. (1) Principal series $\rho(\mu, \nu)$, where μ, ν are quasi-characters of $F^\times$.

(a) *Definition.* Let $\rho(\mu, \nu)$ be the representation of G by right translations in the space of smooth functions for G such that

$$f(ang) = \mu(u)\nu(v)\,|uv^{-1}|^{1/2} f(g)$$

for any $g \in G$, $a = \left(\begin{smallmatrix} u & 0 \\ 0 & v \end{smallmatrix}\right) \in G$, $n \in \left(\begin{smallmatrix} 1 & * \\ 0 & 1 \end{smallmatrix}\right)$. When this representation is irreducible, then $\pi(\mu, \nu)$ is $\rho(\mu, \nu)$. When $\rho(\mu, \nu)$ is reducible, there are exactly two irreducible subquotients; one is finite dimensional and $\pi(\mu, \nu)$ is this one. The other one is denoted $\sigma(\mu, \nu)$.

(b) *Equivalences.* $\pi(\mu, \nu) \sim \pi(\nu, \mu)$. For $F = \boldsymbol{C}$, any irreducible admissible representation of G is equivalent to a $\pi(\mu, \nu)$.

(c) *Finite dimensional representations.* F nonarchimedean: they are one-dimensional, and are the $\pi(\mu, \nu)$ for $\mu\nu^{-1} = |\ |^{\pm 1}$: $\pi(\mu, \nu)(x) = |\det x|^{\pm 1/2} \nu(\det x)$, $x \in G$: the corresponding representations $\sigma(\mu, \nu)$ are called the special representations;

$F = \boldsymbol{R}$: They are the $\pi(\mu, \nu)$ with $\mu\nu^{-1}(a) = |a|^{\pm(n+1)} \cdot \mathrm{sign}(a)^n$ for integers $n \geqq 0$;

$F = \boldsymbol{C}$: they are the $\pi(\mu, \nu)$ with $\mu\nu^{-1}(a) = [a^{n+1}(\bar{a})^{m+1}]^{\pm 1}$ for integers $n \geqq 0$, $m \geqq 0$.

(d) *Other properties.*

Restriction to the center: $\omega_{\pi(\mu,\nu)} = \mu\nu$;

twisting by a quasi-character of $F^\times$: $\pi(\mu, \nu) \otimes \chi = \pi(\mu\chi, \nu\chi)$;

contragredient representation: $\pi(\mu, \nu)^\vee \sim \pi(\mu^{-1}, \nu^{-1})$;

local factors: $L(\pi(\mu, \nu)) = L(\mu)L(\nu)$, $\varepsilon(\pi(\mu, \nu), \psi) = \varepsilon(\mu, \psi)\varepsilon(\nu, \psi)$.

(2) *Weil representations.* $\pi(\tau)$, $\tau = \mathrm{Ind}_{W_K}^{W_F} \theta$, with θ a quasi-character of $K^\times$, and K a separable quadratic extension of F.

(a) *Definition.* Let G_K be the subgroup of index two in G defined by those elements which have a norm of $K^\times$ for determinant. Fix a nontrivial character ψ of the additive group of F. Then $\pi(\tau)$ is the class of the representation of G induced by the following representation $r(\theta, \psi)$ of G_K, in the space of smooth functions f on $K^\times$ such that $f(t^{-1}x) = \theta(t)f(x)$ for $t, x \in K^\times$, $N_{K/F}\ t = 1$:

$$\left(r(\theta, \psi)\begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} f\right)(x) = \psi(uN_{K/F}x) f(x), \quad u \in F,$$

$$\left(r(\theta, \psi)\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} f\right)(x) = \lambda_{K/F}(\psi) \int_{K^\times} f(y)\psi_{K/F}(xy^\sigma)d_\psi y,$$

$$\left(r(\theta, \psi)\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} f\right)(x) = \theta(b)f(bx) \quad \text{for } a = N_{K/F}b, b \in K^\times,$$

where $\psi_{K/F} = \psi \circ \mathrm{Tr}_{K/F}$, y^σ is the conjugate of y by the nontrivial element σ of $\mathrm{Gal}(K/F)$, $d_\psi y$ is the self-dual Haar measure on K with respect to the character $\psi_{K/F}$ and $\lambda_{K/F}(\psi)$ is the unitary part of the local factor $\varepsilon(\hat{\sigma}, \psi)$, where $\hat{\sigma}$ is the nontrivial character of $F^\times$ which is trivial on $N_{K/F}K^\times$.

(b) *Equivalences.*

(1) $\pi(\mathrm{Ind}_{W_K}^{W_F} \theta) = \pi(\mathrm{Ind}_{W_K}^{W_F} {}^\sigma\theta)$;

(2) $\pi(\mathrm{Ind}_{W_K}^{W_F} \theta) \sim \pi(\chi, \chi^{\hat{\sigma}})$ if $\theta = {}^\sigma\theta$ and χ is a quasi-character of $F^\times$ such that $\theta(a) = \chi(N_{K/F}\, a)$;

(3) Other equivalences: $\pi(\mathrm{Ind}_{W_{K_1}}^{W_F} \theta_1) = \pi(\mathrm{Ind}_{W_{K_2}}^{W_F} \theta_2)$ for $K_1 \neq K_2$, if and only if $\theta_1{}^{\sigma_1}\theta_1^{-1}$ and $\theta_2{}^{\sigma_2}\theta_2^{-1}$ are of order 2 and satisfy $\theta_1 \circ N_{K_1K_2/K_1} = \theta_2 \circ N_{K_1K_2/K_2}$.

(c) *Characterization.* If a nontrivial character χ of $F^\times$ fixes a class π of irreducible admissible representations of G: $\pi \otimes \chi = \pi$, then χ is of order 2, attached to a separable quadratic extension K of F and $\pi = \pi(\tau)$ where $\tau = \mathrm{Ind}_{W_K}^{W_F} \theta$ for a suitable θ, and conversely (cf. **[L**, §5]).

(d) *Other properties.* For $F = \boldsymbol{R}$, or nonarchimedean with odd residual characteristic, any irreducible admissible representation is a $\pi(\lambda, \mu)$ of a $\pi(\mathrm{Ind}_{W_K}^{W_F} \theta)$;

restriction to the center: $\omega_{\pi(\tau)} = \hat{\sigma} \cdot \theta|_{F^\times}$;

twisting by a quasi-character of $F^\times$: $\pi(\tau) \otimes \chi = \pi(\mathrm{Ind}_{W_K}^{W_F} \theta \,.\, \chi \circ N_{K/F})$,

contragredient representation: $\pi(\check{\tau}) = \pi(\mathrm{Ind}_{W_K}^{W_F} \theta^{-1})$,

local factors: $L(\pi(\tau)) = L(\theta)$, $\varepsilon(\pi(\tau), \psi) = \lambda_{K/F}(\psi)\varepsilon(\theta, \psi_{K/F})$.

(3) *Exceptional representations.* They occur only for F nonarchimedean of residual characteristic 2, and, up to twisting by quasi-characters of $F^\times$, their number is $4(|2|^{-2} - 1)/3$ for F of characteristic 0, infinite for F of characteristic 2. They are supercuspidal (see complements below) (cf. **[Tu]**).

(4) *Special representations.* They occur only for F nonarchimedean, and are the infinite dimensional subquotient $\sigma(\mu, \nu)$ of the reducible $\rho(\mu, \nu)$, that is for $\mu\nu^{-1} = |\ |^{\pm 1}$; one has $\sigma(\mu, \nu) \sim \sigma(\nu, \mu)$, $\sigma(\mu, \nu) = \chi\sigma(|\ |^{1/2}, |\ |^{-1/2})$ for $\chi = \mu|\ |^{-1/2}$.

Complements.

(1) Representations $\sigma(\mu, \nu)$. When the induced representation $\rho(\mu, \nu)$ is not irreducible, $\sigma(\mu, \nu)$ denotes any representation equivalent to the unique infinite dimensional subquotient of $\rho(\mu, \nu)$: for $F = \boldsymbol{R}$, the representations $\sigma(\mu, \nu)$ are the representations $\pi(\mathrm{Ind}\ _{C^\times}^{W_F} \theta)$, $\theta \neq {}^\sigma\theta$.

(2) For a two-dimensional admissible representation ρ of W_F, there is at most one irreducible admissible representation π of G often denoted $\pi(\rho)$ when it exists such that $\omega_\pi = \det \rho$, $L(\pi \otimes \chi) = L(\rho \otimes \chi)$ and $\varepsilon(\pi \otimes \chi, \psi) = \varepsilon(\rho \otimes \chi, \psi)$ for any quasi-character χ of $F^\times$; for ρ reducible, $\rho = \mu \oplus \nu$, then $\pi = \pi(\mu, \nu)$; for $\rho = \chi \otimes \mathrm{sp}(2)$ then $\pi = \sigma(\mu, \nu)$ with $\mu = \chi|\ |^{1/2}$, $\nu = \chi|\ |^{-1/2}$; for ρ dihedral $\rho = \mathrm{Ind}_{W_K}^{W_F} \theta$, then $\pi = \pi(\mathrm{Ind}_{W_K}^{W_F} \theta)$; in the remaining cases, that is when ρ is exceptional, the existence of π is still not completly settled, but base change techniques were used to prove it in many instances. Conversely, any π should be a $\pi(\rho)$.

Appendix C. Irreducible admissible automorphic representations of GL(2) [J-L].

Notations. F is a global field, A_F its ring of adeles, $C_F = A_F^\times/F^\times$ the group of ideles classes, $|\ |$ the absolute value on A_F.

(1) *Noncuspidal representations.*

(1.1) $\pi(\lambda, \mu)$ for λ, μ Grössencharakters of F: they are the following representations: $\pi(\lambda, \mu) = \otimes_v \pi(\lambda_v, \mu_v)$; the one-dimensional representations are the $\pi(\lambda, \mu)$ for $\lambda\mu^{-1} = |\ |^{\pm 1}$.

(1.2) Any noncuspidal irreducible admissible automorphic representation π of $\mathrm{GL}(2, A_F)$ has the following form: there are two Grössencharakters λ and μ of F, and a finite set S of places of F, such that the components π_v of π are given by

$$\pi_v = \pi(\lambda_v, \mu_v), \quad v \notin S, \qquad \pi_v = \sigma(\lambda_v, \mu_v), \quad v \in S,$$

where $\sigma(\lambda_v, \mu_v)$ denotes the infinite dimensional subquotient of the reducible $\rho(\lambda_v, \mu_v)$ (Appendix B).

(2) *Cuspidal representations* (*examples*).

(2.1) $\pi(\tau)$ with $\tau = \mathrm{Ind}_{W_K}^{W_F} \theta$ for a separable quadratic extension K of F and a Grössencharakter θ of K, not fixed under $\mathrm{Gal}(K/F)$, is the representation

$$\pi(\tau) = \bigotimes_v \pi(\mathrm{Ind}_{W_{K_v}}^{W_{F_v}} \theta_v)$$

with $\mathrm{Ind}_{W_{K_v}}^{W_{F_v}} \theta_v = \theta_v \oplus \theta_v$ for $K_v = F_v \times F_v$.

Properties. (1) Let $\pi \in \Pi(G)$; in order that there exist a nontrivial Grössencharakter χ of F such that $\pi \otimes \chi = \pi$, it is necessary and sufficient that there exist a separable quadratic extension E of F and a Grössencharakter θ of E such that

(a) χ is the character of C_F with Kernel $N_{E/F}C_E$;

(b) $\pi = \pi(\mathrm{Ind}_{W_E}^{W_F} \theta)$ (in particular $\pi = \pi(\tau, \tau\chi)$ if $\theta = {}^\sigma\theta$, where τ is a Grössencharakter of F which has θ for lifting to E);

(2) $\pi(\mathrm{Ind}_{W_{K_1}}^{W_F} \theta_1) = \pi(\mathrm{Ind}_{W_{K_2}}^{W_F} \theta_2) \Leftrightarrow$ either $K_1 = K_2$ and $\theta_2 = \theta_1$ or ${}^\sigma\theta_1$, or $K_1 \neq K_2$ then $\theta_1 \cdot {}^{\sigma_1}\theta_1^{-1}$ and $\theta_2 \cdot {}^\sigma\theta_2^{-1}$ are of order 2, and $(\theta_1) \circ N_{K_1K_2/K_1} = (\theta_2) \circ N_{K_1K_2/K_2}$.

(3.1) More generally let ρ be a two-dimensional admissible representation of W_F; we say that $\pi = \otimes\pi_v$ is $\pi(\rho)$ if $\pi_v \simeq \pi(\rho_v)$ for all v. The existence of such π when ρ is irreducible is related to the Artin conjecture for the $\rho \otimes \chi$, where χ is any quasi-character of C_F **[J-L**, §12**]**.

(3.2) Of course there are many other, more complicated, types of cuspidal representations: think of the classical Δ for example.

II. Base change for GL_2, a sketch of the proof.

1. The trace formula. In all the following we shall use notations close to those of **[G-J-1]** in these Proceedings.

Let F be a number field, E a cyclic extension of prime degree, put $Z_1 = N_{E/F}\, Z(A_E)$, where A_E denotes the ring of adeles of E, and $Z_1(F) = Z_1 \cap Z(F)$; as usual Z is the center of $\mathrm{GL}_2 = G$ and is identified with the multiplicative group. Since E/F is cyclic one has $Z_1(F) = N_{E/F}Z(E)$.

Choose a character ω of $Z_1/Z_1(F)$ and consider the space $L^2(Z_1 \cdot G(F)\backslash G(A), \omega) = L^2$ of functions on $G(F)\backslash G(A)$, which transform on Z_1 according to ω:

$$\varphi(z\gamma g) = \omega(z)\varphi(g), \qquad z \in Z_1,\ \gamma \in G(F),$$

and square-integrable on $Z_1 \cdot G(F)\backslash G(A)$.

In such a situation, which is slightly more general than the one studied in **[G-J-1]** (where $E = F$), one defines in an obvious way the spaces L^2_0 and L^2_{sp}. The restriction of the natural representation of $G(A)$ in L^2 to $L^2_0 \oplus L^2_{sp}$ will be denoted by r. If $f \in \mathscr{C}^\infty_c(Z_1\backslash G(A), \omega^{-1})$, the space of smooth functions on $G(A)$ compactly supported modulo Z_1 which transform according to ω^{-1} on Z_1, the operator $r(f)$ is of trace class. The Haar measures being chosen as in **[G-J-1**, §§6–7] we assume moreover that $\mathrm{vol}(Z_1 \cdot Z(F)\backslash Z(A)) = l$. Then $\mathrm{tr}\, r(f)$ is the sum of the expressions (i)—(vii) below (we assume that f is a tensor product of local functions f_v) (cf. **[L**, §8]).

(i)
$$\sum_{z \in Z_1(F)\backslash Z(F)} \mathrm{vol}(Z_1 \cdot G(F)\backslash G(A)) \cdot f(z),$$

(ii)
$$\sum_{\gamma \in \mathscr{E}} \varepsilon(\gamma)\, \mathrm{vol}(Z_1 \cdot G_\gamma(F)\backslash G_\gamma(A)) \int_{G_\gamma(A)\backslash G(A)} f(g^{-1}\gamma\, g)\, dg$$

where $\mathscr{E}$ is a set of representatives of the conjugacy classes of elliptic elements (i.e., whose eigenvalues are not in F) taken modulo $Z_1(F)$, and $\varepsilon(\gamma)$ is $\frac{1}{2}$ (resp. 1) if the equation $\delta^{-1}\gamma\delta = z\gamma$ has (resp. has not) a solution in $z \in Z_1(F) - \{1\}$.

(iii)
$$-\frac{1}{4} \sum_{\eta=(\mu,\mu);\eta \in D^\circ} \mathrm{tr}(M(\eta)\pi_\eta(f)),$$

where D° is the set of pairs $\eta = (\mu, \nu)$ of characters of $A^\times/F^\times$ such that $\mu\nu$ induces ω on Z_1, where $M(\eta)$ and π_η are defined in **[G-J-1**, §4], and with $\pi_\eta(f) = \int_{Z_1\backslash G(A)} f(g)\pi_\eta(g)\, dg$. A Haar measure $d\eta$ on D° is defined as in **[G-J-1**, §7-D] by considering D° as a union of homogeneous spaces under the group of characters of $A^\times/F^\times$ (with the dual Haar measure), acting by $\chi \cdot (\mu, \nu) = (\chi\mu, \chi^{-1}\nu)$.

This allows us to write the fourth term:

(iv)
$$\frac{1}{2} \int_{D^\circ} m^{-1}(\eta) \cdot m'(\eta)\mathrm{tr}(\pi_\eta(f))\, d\eta,$$

the derivative m' being computed as in **[G-J-1**, §7-D].

(v)
$$\sum_{z \in Z_1(F)\backslash Z(F)} l \cdot \lambda_0 \prod_v \frac{\int_{Z_v N_v\backslash G_v} f_v(g^{-1}zn_0 g)\, dg}{L(1, 1_v)}, \qquad \left(n_0 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\right),$$

where λ_0 and other notations are defined in **[G-J-I**, §7-B]. The remaining terms will not be written as in **[G-J-1]** since they are there expressed by noninvariant local distributions. For example the local distribution

$$\tfrac{1}{2} A_1(\gamma, f_v) = -\Delta(\gamma) \int_{|x_v|>1} f_v^K\begin{pmatrix} a & (a-b)x_v \\ 0 & b \end{pmatrix} \log|x_v|\, dx_v$$

where $\gamma = \left(\begin{smallmatrix} a & 0 \\ 0 & b \end{smallmatrix}\right)$, $\Delta_v(\gamma) = |(a-b)^2/ab|_v^{1/2}$ and $f_v^K(g) = \int_{K_v} f_v(k^{-1}gk)\, dk$, can be written $A_2(\gamma, f_v) + A_3(\gamma, f_v)$ where $A_2(\gamma, f_v)$ is invariant, and A_3 fits with other terms to provide an invariant expression.

More precisely one takes for a nonarchimedean place v:

$$A_2(\gamma, f_v) = \log|(a-b)/a|_v F(\gamma, f_v) + \Delta_v(\gamma) f_v^K(z) \int_{|x_v|>1} \log|x_v|\, dx_v - |a/b|^{1/2} \cdot |\tilde{\omega}_v|$$
$$\cdot \log|\tilde{\omega}_v| \cdot \int_{Z_v N_v\backslash G_v} f(gzn_0g^{-1})\, dg \qquad \left(z = \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}\right),$$

where $F(\gamma, f_v) = \Delta_v(\gamma) \int_{A_v\backslash G_v} f_v(g^{-1}\gamma g)\, dg$ and $\bar{\omega}_v$ is a uniformizing parameter for F_v.

If v is archimedean one takes

$$A_2(\gamma, f_v) = \log\left|1 - \frac{b}{a}\right|_v F(\gamma, f_v) - \frac{L'(1, 1_v)}{L(1, 1_v)^2} \int_{Z_vN_v\backslash G_v} f_v(g^{-1}zn_0g)\, dg.$$

This yields the term

$$\text{(vi)} \qquad - \sum_{\gamma\in Z_1\backslash A(F);\gamma\notin Z(F)} l\lambda_{-1} \sum_v A_2(\gamma, f_v) \prod_{w\neq v} F(\gamma, f_w).$$

It can be checked that $\gamma \mapsto A_3(\gamma, f_v)$ extends to a continuous map on $A(F_v)$ and one sees that the terms 6.34 in **[G-J-1]** minus our (v) plus 6.35 minus our (vi) yield the term

$$\sum_{\gamma\in Z_1\backslash A(F)} - \lambda_{-1} l \sum_v A_3(\gamma, f_v) \prod_{w\neq v} F(\gamma, f_w).$$

This can in turn be transformed by a kind of Poisson summation formula to

$$\int_{D^\circ} - \sum_v B_1(\eta_v, f_v) \prod_{w\neq v} \operatorname{tr} \pi_{\eta_w}(f_w)\, d\eta.$$

One has to add the term 6.36 of **[G-J-1]** minus our (iv) to get the final term:

$$\text{(vii)} \qquad 2 \int_{D^\circ} \sum_v B(\eta_v, f_v) \prod_{w\neq v} \operatorname{tr} \pi_{\eta_w}(f_w)\, d\eta.$$

2. The twisted trace formula. Here we shall use definitions and results of the Kottwitz lecture (see **[K]** in these PROCEEDINGS). As above we follow closely **[L, §8]**. Let $L^2(Z(A_E)\backslash G(A_E), \tilde{\omega}) = \tilde{L}^2$ where $\tilde{\omega} = \omega \circ N_{E/F}$. The Galois group $\Gamma = \mathrm{Gal}(E/F)$ acts on $\tilde{L}^2$ by ${}^\sigma\rho(x) = \rho(x^\sigma)$ for $\rho \in \tilde{L}^2$, $x \in G(A_E)$ and $\sigma \in \Gamma$.

Let R_d denote the restriction of the natural representation of $G(A_E)$ in $\tilde{L}^2$ to the discrete spectrum $\tilde{L}^2_0 \oplus \tilde{L}^2_{\mathrm{sp}}$. The projection commutes with the action of the Galois group; hence R_d can be extended to a representation R'_d of $\Gamma \ltimes G(A_E)$. Let $\phi \in \mathscr{C}_c^\infty(Z(A_E)\backslash G(A_E), \tilde{\omega}^{-1})$; then $R_d(\phi)$ is of trace class and $R'_d(\sigma)$ is unitary for $\sigma \in \Gamma$. The operator $R_d(\phi)$ can be represented by a kernel $K(\phi, x, y)$ and then the operator $R'_d(\sigma)R_d(\phi)$ is represented by the kernel $K(\phi, x^\sigma, y)$ and

$$\operatorname{tr}(R'_d(\sigma)R_d(\phi)) = \int_{Z(A_E)\backslash G(A_E)} K(\phi, x^\sigma, x)\, dx.$$

Assuming, as usual, that ϕ is a tensor product: $\phi = \otimes\phi_v$, one can proceed as in **[G-J-1]** to compute this integral; if $\sigma \neq 1$ it is the sum of the following terms (1)–(7):

$$\text{(1)} \qquad \sum_\delta \operatorname{vol}(Z(A)G^\sigma_\delta(E)\backslash G^\sigma_\delta(A_E)) \int_{Z(A_E)G^\sigma_\delta(A_E)\backslash G(A_E)} \phi(g^{-\sigma}\delta g)\, dg$$

where the sum runs over the σ-conjugacy classes of elements δ such that $N(\delta)$ is central, and G^σ_δ is the σ-centralizer of δ (cf. **[K]**).

$$\text{(2)} \qquad \sum_{\delta\in\mathscr{E}_\sigma} \varepsilon(\delta)\operatorname{vol}(Z(A)\cdot G^\sigma_\delta(E)\backslash G^\sigma_\delta(A_E)) \int_{Z(A_E)G^\sigma_\delta(A_E)\backslash G(A_E)} \phi(g^{-\sigma}\delta g)\, dg$$

where $\mathscr{E}_\sigma$ is a set of representatives of the σ-conjugacy classes that are not σ-conjugate to a triangular matrix, taken modulo $Z(E)$ in $G(E)$, and $\varepsilon(\delta)$ is $\frac{1}{2}$ or 1 according as the equation $\tau^{-\sigma}\delta\tau = z\delta$ has or not a solution in $Z(E)$ with $z \notin Z(E)^{1-\sigma}$.

(3) $$\frac{-1}{4} \sum_{^\sigma\eta=\eta^\vee} \operatorname{tr}(M_E(^\sigma\eta)\pi'_\eta(\sigma)\pi_\eta(\phi))$$

where $\eta^\vee = (\nu, \mu)$ if $\eta = (\mu, \nu)$ is a pair of characters of $A_E^\times/E^\times$ with $\mu\nu = \tilde{\omega}$. The representation π_η is realized in a space of functions on $G(A_E)$, the action of $\sigma \in \Gamma$ defines an operator $\pi'_\eta(\sigma)$ from the space of π_η to the space of π_{σ_η}; then $M_E(^\sigma\eta)$ intertwines π_{σ_η} and $\pi_{\sigma_\eta^\vee}$ but $^\sigma\eta^\vee = \eta$. Hence the product $M(^\sigma\eta)\ \pi'_\eta(\sigma)\pi_\eta(\phi)$ is a well-defined operator in the space of $\pi_{\sigma_\eta^\vee}$, and the above expression is meaningful.

(4) $$\frac{1}{2l^2} \int_{\eta\in D^\circ} m_E^{-1}(\tilde{\eta})m'_E(\tilde{\eta}) \operatorname{tr}(\pi'_{\tilde{\eta}}(\sigma)\pi_{\tilde{\eta}}(\phi))\, d\eta$$

where $\tilde{\eta} = (\mu \circ N_{E/F}, \nu \circ N_{E/F})$ if $\eta = (\mu, \nu)$. There are l^2 elements $\tilde{\eta}$ giving rise to the same $\tilde{\eta}$. The reader should be aware that our notation $\tilde{\eta}$ has not the same meaning as in [L].

(5) $$\lambda_0 \prod_v \theta^\sigma(0, \phi_v),$$

where $\theta^\sigma(0, \phi_v) = L(1, 1_v)^{-1} \iiint \phi_v(k^{-\sigma}t^{-\sigma}n^{-\sigma}\tilde{n}_0 ntk)t^{-2\rho}\, dn\, dt\, dk$, with $k \in \tilde{K}_v$, the standard maximal compact subgroup of $\mathrm{GL}_2(E \otimes F_v)$,

$$t = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}, \quad t^{-2\rho} = \left|\frac{a}{b}\right|^{-1} \quad \text{and} \quad \tilde{n}_0 = \begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix}$$

such that $\operatorname{tr}_{E/F} z = 1$. The integration is on $\tilde{K}_v \times Z(E_v)\backslash A(E_v) \times N(F_v)\backslash N(E_v)$, where $E_v = E \otimes F_v$.

(6) $$-\frac{\lambda_{-1}}{l} \sum_{\delta\in\mathscr{F}} \sum_v A_2^\sigma(\delta, \phi_v) \prod_{w\neq v} F^\sigma(\delta, \phi_w),$$

where $\mathscr{F} = \{\delta \in A^{1-\sigma}(E)Z(E)\backslash A(E) \mid N(\delta) \notin Z(F)\}$,

$$F^\sigma(\delta, \phi_w) = \Delta_w(\gamma) \int_{Z(E_v)A(F_v)\backslash G(E_v)} \phi_w(g^{-\sigma}\delta g)\, dg$$

and $\gamma = N(\delta)$. An explicit definition of $A_2^\sigma(\delta, \phi_v)$ will not be given here; we shall simply say that $A_2^\sigma(\delta, \phi_v) = lA_2(\gamma, f_v)$ if f_v is associated to ϕ_v under the base change correspondence (see [K]).

As above the remaining term can be written

(7) $$\frac{2}{l^2} \int_{\eta\in D^\circ} \sum_v B^\sigma(\tilde{\eta}_v, \phi_v) \prod_{w\neq v} \operatorname{tr} \pi_{\tilde{\eta}_w}(\phi_w).$$

For a definition and a detailed study of distributions A_2^σ and B^σ the reader is referred to [L, §7] (where the subscript σ is omitted) and to [K, Lemma 4].

3. The comparison. We assume now on that the function $f = \otimes f_v \in \mathscr{C}_c^\infty(Z_1\backslash G(A), \omega^{-1})$ and the function $\phi = \otimes\phi_v \in \mathscr{C}_c^\infty(Z(A_E)\backslash G(A_E), \tilde{\omega})$ are such that ϕ_v and f_v are "associated" in the sense defined in [K]; we consider $\mathscr{D}_1 = l \operatorname{tr}(R'_d(\sigma)R_d(\phi)) - \operatorname{tr} r(f)$.

PROPOSITION 1.

$$\mathscr{D}_1 = \frac{2}{l^2}\int_{D^\circ}\sum_v \Big(lB^\sigma(\tilde\eta_v,\phi_v) - \sum_{\eta\mapsto\tilde\eta} B(\eta_v,f_v)\Big)\prod_{w\neq v}\operatorname{tr}\pi_{\eta_w}(f_w)\,d\eta$$
$$-\frac{\delta_{l,2}}{2}\sum_{\eta=(\mu,{}^\sigma\mu);{}^\sigma\mu\neq\mu;{}^\sigma\mu\mu=\tilde\omega}\operatorname{tr}(M({}^\sigma\eta)\pi'_\eta(\sigma)\pi_\eta(\phi))$$

where

$$\delta_{l,2} = 1 \quad \text{if } l = 2,$$
$$= 0 \quad \text{if } l \neq 2.$$

The proof amounts to the comparison term by term of the expressions for $\operatorname{tr}(r(f))$ and $l\cdot\operatorname{tr}(R'_d(\sigma)R'_d(\phi))$.

For example to prove that $l\cdot(1) = $ (i) note that we work there with a sum over elements $\delta\in G(E)$ such that $N(\delta)$ is central in $G(F)$; hence $G^\sigma_\delta(E)$ is the set of F-points of a twisted inner form of G. Since we use Tamagawa measures,

$$l\operatorname{vol}(Z(A)G^\sigma_\delta(E)\backslash G^\sigma_\delta(A_E)) = l\operatorname{vol}(Z(A)G(F)\backslash G(A)) = \operatorname{vol}(Z_1G(F)\backslash G(A));$$

the expressions to be compared are products of the local analogues, and now, using properties A′ and B′ in [K] for associated functions and the fact that the number of places where a minus sign occurs is even, we obtain the desired results. To prove $l\cdot(2) = $ (ii) is even simpler since in that case $G^\sigma_\delta(E) = G_\gamma(F)$, where $\gamma = N(\delta)$: the twisting is trivial since G_γ is abelian.

To compare $l\cdot(3)$ and (iii) is slightly more complicated; we must distinguish two cases:

(a) $l\neq 2$; then ${}^\sigma\eta = \eta^\vee$ implies ${}^\sigma\eta = \eta = \eta^\vee$ and in such a case one has $M(\eta) = -1$. One the other hand one should note that $\operatorname{tr}\pi_{\eta_v}(f_v) = \int_{Z_v\backslash A_v} F(a, f_v)\eta_v(a)\,da$ and that for associated functions f_v and ϕ_v one has $F(a,f_v) = F^\sigma(b,\phi_v)$ if $a = N(b)$. Moreover

$$\operatorname{tr}\pi'_{\tilde\eta_v}(\sigma)\pi_{\tilde\eta_v}(\phi_v) = \int_{\tilde Z_v\backslash\tilde A_v} F^\sigma(b,\phi_v)\tilde\eta_v(b)\,db,$$

where $\tilde Z_v = Z(E\otimes F_v)$ and $\tilde A_v = A(E\otimes F_v)$. Then $\operatorname{tr}(\pi'_{\tilde\eta}(\sigma)\pi_{\tilde\eta}(\phi)) = \operatorname{tr}(\pi_\eta(f))$ if ϕ and f are associated and if $\tilde\eta = (\mu\circ N_{E/F}, \nu\circ N_{E/F})$ and $\eta = (\mu,\nu)$.

This yields $l\cdot(3) = $ (iii).

(b) If $l = 2$, the same arguments apply if $\eta = \eta^\vee = {}^\sigma\eta$, but there are other terms corresponding to $\eta = (\mu,\nu)$ with $\mu = {}^\sigma\nu \neq {}^\sigma\mu$ and then

$$l\cdot(3) - \text{(iii)} = -\frac{l}{4}\sum_{\mu\neq{}^\sigma\mu;\eta=(\mu,{}^\sigma\mu);{}^\sigma\mu\mu=\tilde\omega}\operatorname{tr}(M({}^\sigma\eta)\pi'_\eta(\sigma)\pi_\eta(\phi)).$$

To prove that $l\cdot(4) = $ (iv) we use the previous remarks and the fact that $m_E(\tilde\eta)^l = \prod_{\eta\mapsto\tilde\eta} m(\eta)$, where $\eta\mapsto\tilde\eta$ means that $\eta = (\mu,\nu)$ and $\tilde\eta = (\mu\circ N_{E/F}, \nu\circ N_{E/F}) = (\tilde\mu,\tilde\nu)$ and by definition:

$$m(\eta) = \frac{L(1,\mu^{-1}\nu)}{L(1,\nu^{-1}\mu)} \quad\text{and}\quad m_E(\tilde\eta) = \frac{L(1,\tilde\mu^{-1}\tilde\nu)}{L(1,\tilde\mu\tilde\nu^{-1})}.$$

Now $l\cdot(5) = $ (v) follows from the comparison of orbital and twisted orbital integrals on unipotent elements.

To conclude the proof of Proposition 1 it is enough to show that $l\cdot(6) = $ (vi), which in turn follows from the equality:

$$A_2(\gamma, f_v) = A_2^\sigma(\delta, \phi_v) \quad \text{if } \gamma = N(\delta).$$

4. The main theorem. The representation r is a discrete sum of unitary irreducible representations of $G(A)$ with multiplicity one:

$$r = \sum_{i\in I} \pi_i = \sum_{i\in I} \otimes\pi_{i,v}$$

and $\operatorname{tr} r(f) = \sum_{i\in I} \prod_v \operatorname{tr} \pi_{i,v}(f_v)$ for some set I.

Let v be a nonarchimedean place. Given f_v in the Hecke algebra of $G(F_v)$ then $\operatorname{tr} \pi_{i,v}(f_v)$ is zero unless $\pi_{i,v}$ is unramified and hence corresponds to the conjugacy class of some semisimple element $t_{i,v} \in \mathrm{GL}_2(C)$, the connected component of the L-group. Any function f_v in the Hecke algebra defines a rational class function $f_v^\vee$ on $\mathrm{GL}_2(C)$ such that $f_v^\vee(t_{i,v}) = \operatorname{tr} \pi_{i,v}(f_v)$. Choose a finite set V of places of F containing archimedean ones and assume f_v is in the Hecke algebra for $v \notin V$. Then one has

LEMMA 1. $\operatorname{tr} r(f) = \sum_i \prod_{v\in V} \operatorname{tr} \pi_{i,v}(f_v) \prod_{v\notin V} f_v^\vee(t_{i,v})$.

The representation R_d can also be written

$$R_d = \sum_{j\in J} \Pi_j = \sum_{j\in J} \bigotimes_v \Pi_{j,v}$$

where Π_j is an automorphic representation of $G(A_E)$ and $\Pi_{j,v}$ a representation of $G(F_v \otimes E)$. Since ${}^\sigma R_d \simeq R_d$ and is multiplicity free, σ permutes the Π_j. If ${}^\sigma\Pi_j \simeq \Pi_j$ one can restrict the operator $R_d'(\sigma)$ to the space of Π_j, and denote this restriction by $\Pi_j'(\sigma)$. The nonfixed Π_j do not contribute to the trace of $R_d'(\sigma)R_d(\phi)$ (a permutation matrix without fixed point has trace zero) and then

$$\operatorname{tr} R_d'(\sigma)R_d(\phi) = \sum_{{}^\sigma\Pi_j\simeq\Pi_j} \operatorname{tr} \Pi_j'(\sigma)\Pi_j(\phi).$$

Moreover $\Pi_j = \otimes\Pi_{j,v}$ and ${}^\sigma\Pi_{j,v} \simeq \Pi_{j,v}$; we can define $\Pi_{j,v}'(\sigma)$ up to lth roots of 1. If $\Pi_{j,v}$ is unramified, there is a canonical choice. If ϕ_v and f_v are associated in the Hecke algebras, we choose a semisimple element $t_{j,v} \in \mathrm{GL}_2(C)$ such that

$$\operatorname{tr} \Pi_{j,v}'(\sigma)\Pi_{j,v}(\phi_v) = \operatorname{tr} \Pi_{j,v}(\phi_v) = f_v^\vee(t_{j,v}),$$

and then for some big enough finite set V of places of F, we have:

LEMMA 2. $\operatorname{tr} R_d'(\sigma)R_d(\phi) = \sum_j \prod_{v\in V} \operatorname{tr} \Pi_{j,v}'(\sigma)\Pi_{j,v}(\phi_v) \prod_{v\notin V} f_v^\vee(t_{j,v})$.

REMARK. If v is nonarchimedean and split in E, then any f_v is associated to some ϕ_v; in fact in such a case ϕ_v may be taken to be $f_{w_1}\otimes f_{w_2}\otimes \cdots \otimes f_{w_l}$ where the w_i are the places of E above v, and $f_v = f_{w_1} * f_{w_2} * \cdots * f_{w_l}$ can be any smooth function on $G(F_v)$; the conjugacy class of $t_{j,v}$ is well defined by $\Pi_{j,v}$.

If v does not split in E, is unramified, and if ϕ_v and f_v are associated in the Hecke algebras, one can define a function $\phi_v^\vee$ on $\mathrm{GL}(2, C)$ as above; we have

$$f_v^\vee(t) = \phi_v^\vee(t^l) \quad \text{for } t \text{ semisimple in } \mathrm{GL}_2(C).$$

In such a case the conjugacy class of the $t_{j,v}$ above is not uniquely defined.

If $l \neq 2$ let $R = lR_d$.

Assume for a while $l = 2$; if μ is a character of $A_E^\times/E^\times$ such that ${}^\sigma\mu\mu = \bar{\omega}$ and ${}^\sigma\mu \neq \mu$, we consider $\tau_\mu = \pi_\eta$, where $\eta = (\mu, {}^\sigma\mu)$. As was said above, the operator $M({}^\sigma\eta)\pi'_\eta(\sigma) = \tau'_\mu(\sigma)$ maps the space of π_η into itself. One defines in such a way a representation τ'_μ of $\Gamma \ltimes G(A_E)$. One should remark that $\tau'_\mu \simeq \tau'_{{}^\sigma\mu}$. Let us denote by $\mathcal{T}$ the set of such representations (modulo equivalence) and let

$$R' = lR'_d \oplus \sum_{\tau'_\mu \in \mathcal{T}} \tau'_\mu.$$

An analogue of Lemma 2 can be stated.

We can now state the main theorem (cf. [L, Theorem 9.1]).

THEOREM 1. *Assume f and ϕ are associated; then* $\operatorname{tr} R'(\sigma)R(\phi) = \operatorname{tr} r(f)$.

(Recall that the definition of the correspondence "f and ϕ are associated" depends on the choice of a $\sigma \in \Gamma - \{1\}$.)

The proof of the theorem can be carried out as follows: consider the expression

(a) $$\operatorname{tr} R'(\sigma)R(\phi) - \operatorname{tr} r(f).$$

Thanks to Proposition 1 above, this is equal to

(b) $$\frac{2}{l^2} \int_{D^\circ} \sum_v \Big(lB(\tilde{\eta}_v, \phi_v) - \sum_{\eta_v \mapsto \tilde{\eta}_v} B(\eta_v, f_v) \Big) \prod_{w \neq v} \operatorname{tr} \pi_{\eta_w}(f_w)\, d\eta.$$

Using properties of some weighted orbital integrals [K, Lemma 4], one can prove that

$$lB(\tilde{\eta}_v, \phi_v) - \sum_{\eta_v \mapsto \tilde{\eta}_v} B(\eta_v, f_v) = 0,$$

at least when v is nonarchimedean, unramified in E, with ϕ_v and f_v associated and in the Hecke algebras. Hence there is a finite set of places V such that (b) can be written

(b′) $$\int_{D^\circ} \beta(\eta) \prod_{v \notin V} \operatorname{tr} \pi_{\eta_v}(f_v)\, d\eta$$

with some nice function β.

Now choose a place $v_0 \notin V$ split in E; then (b′) reduces to an absolutely convergent integral:

(b″) $$\int_{-\infty}^{+\infty} \delta(s) f_{v_0}^\vee \begin{pmatrix} aq^{is} & 0 \\ 0 & bq^{-is} \end{pmatrix} ds$$

where a, b depends on the central character ω_{v_0}. All we need to know is that $\delta(s)$ is some continuous, bounded and integrable function on the real line. On the other hand (a) can be written using Lemmas 1 and 2 above

(a″) $$\sum_{k=0}^{\infty} a_k f_{v_0}^\vee(t_k)$$

with $a_k \in C$ and $t_k \in \mathrm{GL}_2(C)$ semisimple elements corresponding to inequivalent unitary representations of $\mathrm{GL}_2(F_{v_0})$. The series, as the integral, is absolutely convergent. Now f_{v_0} is arbitrary in the Hecke algebra (since v_0 is split in E) and the

Hecke algebra separates inequivalent unramified representations. The Stone-Weierstrass theorem and easy majorations prove that all a_k are zero (which is a stronger statement than Theorem 1). All the desired results can now be xtracted from Theorem 1 and from some results on the characters of representations of $\Gamma \ltimes \mathrm{GL}_2(E_v)$ (cf. [L, §5]). We shall try to explain some of the steps.

5. Existence of weak liftings. Choose a finite set V of places of F containing all archimedean places and all places ramified in E. Assume that ϕ_v and f_v are associated and in the Hecke algebras for $v \notin V$. One can, using Lemmas 1 and 2, choose element $t_{n,v} \in \mathrm{GL}_2(\boldsymbol{C})$ for $v \in V$ and $n \in \boldsymbol{N}$ such that

$$\operatorname{tr} R'_d(\sigma)R_d(\phi) = \sum_n \alpha_n(\phi) \prod_{v\notin V} f_v^\vee(t_{n,v}),$$

$$\sum_{\tau'_\mu \in \mathscr{T}} \operatorname{tr} \tau'_\mu(\sigma)\tau_\mu(\phi) = \sum_n \beta_n(\phi) \prod_{v\notin V} f_v^\vee(t_{n,v}),$$

$$\operatorname{tr} r(f) = \sum_n \gamma_n(\phi) \prod_{v\notin V} f_v^\vee(t_{n,v});$$

we may assume moreover they are chosen such that the functions $T_n\colon (\phi_v)_{v\notin V} \mapsto \prod_{v\notin V} f_v^\vee(t_{n,v})$ on the product for $v \notin V$ of the Hecke algebras are distinct. (Recall the remark after Lemma 2.) Let $\delta_n = l\alpha_n + \beta_n - \gamma_n$; then the above theorem can be restated in the following form

$$\sum_{n\in \boldsymbol{N}} \delta_n(\phi)T_n(\phi) = 0.$$

Another use of density arguments, the T_n being distinct, yields

PROPOSITION 2. *For all n one has $\delta_n(\phi) = 0$.*

This can be read $l\alpha_n + \beta_n = \gamma_n$.

Assume that there exist a representation $\Pi = \bigotimes \Pi_v$, unramified outside V, occurring in $\tilde{L}_0$ such that $\operatorname{tr} \Pi_v(\phi_v) = f_v^\vee(t_{n,v})$ for some n with α_n not zero, and any $v \notin V$; then using the strong multiplicity one theorem [C] one concludes that such a Π is unique and satisfies $\Pi \simeq {}^\sigma\Pi$. The fact that $L(s, \Pi)$ is entire allows one to conclude that no other Π occurring in $\tilde{L}^2_{\mathrm{sp}}$ or in $\mathscr{T}$ has the property that $\operatorname{tr} \Pi_v(\phi_v) = f_v^\vee(t_{n,v})$, $v \notin V$. Then $\alpha_n(\phi) = \prod_{v\in V} \operatorname{tr} \Pi'_v(\sigma)\Pi_v(\phi_v)$ and $\beta_n(\phi) = 0$; since $l\alpha_n + \beta_n = \gamma_n$ we conclude that $\gamma_n(\phi)$ is not identically zero and hence there exists (at least) one π in $L^2_0 \otimes L^2_{\mathrm{sp}}$ such that $\pi = \bigotimes \pi_v$ and $\operatorname{tr} \pi_v(f_v) = f_v^\vee(t_{n,v})$ for $v \notin V$ and then Π_v is the lifting of π_v for $v \notin V$. We shall say that Π is a weak lifting of π if Π_v is a lifting of π_v for almost all v. We then have proved

THEOREM 2. *If Π is a cuspidal automorphic representation of $\mathrm{GL}_2(\boldsymbol{A}_E)$ such that $\Pi \simeq {}^\sigma\Pi$, then Π is the weak lifting of some automorphic representation π of $\mathrm{GL}_2(\boldsymbol{A})$.*

A direct study of L^2_{sp} allows one to prove:

PROPOSITION 3. *Any π occurring in L^2_{sp} lifts to a Π in $\tilde{L}^2_{\mathrm{sp}}$ and all σ-fixed representations in $\tilde{L}^2_{\mathrm{sp}}$ are obtained in this way.*

In the case $l = 2$, assume that $\Pi = \pi(\mu, {}^\sigma\mu)$ occurs in $\mathscr{T}$ and that μ_v is unramified for $v \notin V$; one can show using Proposition 2 and results on L-functions on $\mathrm{GL}_2 \times$

GL_2 that Π is the weak lifting of $\pi = \pi(\rho)$ where $\rho = \mathrm{Ind}_{W_E}^{W_F} \mu$ and that there exists $n \in N$ such that

$$\prod_{v \in V} \operatorname{tr} \Pi_v'(\sigma)\Pi_v(\phi_v) = \beta_n(\phi) = \gamma_n(\phi) = \prod_{v \in V} \operatorname{tr} \pi_v(f_v).$$

This can be used to show that at all places of F: $\operatorname{tr} \Pi_v'(\sigma)\Pi_v(\phi_v) = \operatorname{tr} \pi_v(f_v)$, and hence

THEOREM 3. *Let the fields E and F be either local or global. Then $\pi(\mu, {}^\sigma\mu)$ is the lifting of $\pi(\rho)$ where $\rho = \mathrm{Ind}_{W_E}^{W_F} \mu$.*

6. Any cuspidal π has a weak lifting. Assume for a while that some π occurring in L_0^2 has no weak lifting. Let V be a finite set of places of F including archimedean places and those where E or π are ramified. Let $f = \bigotimes f_v$ be associated to some ϕ and such that f_v is in the Hecke algebra for $v \notin V$. Consider the π_k in $L_0^2 \oplus L_{\mathrm{sp}}^2$ such that $\operatorname{tr} \pi_{k,v}(f_v) = \operatorname{tr} \pi_v(f_v)$ if $v \notin V$. Since the π_k have no weak lifting, Proposition 2 shows that the sum of the π_k gives a zero contribution to $\operatorname{tr} r(f)$:

$$\sum_k \prod_{v \in V} \operatorname{tr} \pi_{k,v}(f_v) \prod_{v \notin V} \operatorname{tr} \pi_v(f_v) = 0;$$

hence there is a set $V_1 \subset V$ such that $\sum_k \prod_{v \in V_1} \operatorname{tr} \pi_{k,v}(f_v) = 0$. One has to prove that this is impossible unless the sum is empty. The idea of the proof (by induction on the cardinality of V_1) is that characters of inequivalent representations are linearly independent, but the proof is complicated here by the fact that f_v cannot assume all values, since f_v must be associated to some ϕ_v (cf. [L, §9, pp. 25–30]). This yields

THEOREM 4. *Any cuspidal π has a weak lifting.*

7. Local liftings. To finish the proof of the global theorem on base change for cuspidal representations, one must show that the above weak liftings are liftings at all places. Let $\Pi \simeq {}^\sigma\Pi$, occurring in L_0^2, unramified outside a finite set V chosen as before; there is an n such that, according to Proposition 2, $l\alpha_n(\phi) = \gamma_n(\phi)$, which can be written, for some $V_1 \subset V$,

$$l \prod_{v \in V_1} \operatorname{tr} \Pi_v'(\sigma)\Pi_v(\phi_v) = \sum_k \prod_{v \in V_1} \operatorname{tr} \pi_{k,v}(f_v),$$

where the $\pi_k = \bigotimes \pi_{k,v}$ have Π as weak lifting. One then proves [L, §9, pp. 33–34]:

LEMMA 3. *If for some v the representation Π_v is the lifting of some π_v then it is the lifting of $\pi_{k,v}$ for all k, that is* $\operatorname{tr} \Pi_v'(\sigma)\Pi_v(\phi_v) = \operatorname{tr} \pi_{k,v}(f_v)$.

Then one may assume that V_1 does not contain such places.

On the other hand existence and properties of local liftings are easy to prove, or are deduced from Theorem 3 above, except for some supercuspidal representations (exceptional ones). Lemma 3 and this remark show that all desired local or global results (cf. part I of this paper) can be deduced from

THEOREM 5 [L, §9, PROPOSITION 9.6]). (a) *Every supercuspidal π_v has a lifting.*
(b) *If $\Pi_v \simeq {}^\sigma\Pi_v$ and is supercuspidal, then π_v is a lifting.*

Part (a) of this theorem is proved by embedding the local situation in an ad hoc global one, where the existence of liftings is known at all places except perhaps at one place, and to use the above equation with V_1, reduced to one element.

Part (b) then follows from the orthogonality relations of [**L**, §5].

The last paragraph of Langlands paper [**L**] is devoted to the proof of the existence of lifting for noncuspidal representations, using their explicit description (cf. Appendix C).

BIBLIOGRAPHY

[**B**] A. Borel, *Automorphic L-functions*, these PROCEEDINGS, part 2, pp. 27–61.

[**C**] W. Casselman. *On some results of Atkin and Lehner*, Math. Ann. **201** (1973), 301–334.

[**D**] P. Deligne, *Formes modulaires et représentations de* GL(2), Modular Functions of One Variable. II, Lecture Notes in Math., vol. 349, Springer-Verlag, Berlin, 1973.

[**D-S**] P. Deligne and J.P. Serre, *Formes modulaires de poids* 1, Ann. Sci. Ecole Norm. Sup. **4** (1974), 507–530.

[**G-J-1**] S. Gelbart and H. Jacquet, *Forms on* GL(2) *from the analytic point of view*, these PROCEEDINGS, part 1, pp. 213–251.

[**G-J-2**] ———, *A relation between automorphic forms on* GL(2) *and* GL(3), Proc. Nat. Acad. Sci. U. S. A. (1976).

[**J-L**] H. Jacquet and R. P. Langlands, *Automorphic forms on* GL(2), Lecture Notes in Math., vol. 114, Springer, Berlin, 1970.

[**J-PS-S**] H. Jacquet, I. I. Piatetskii-Shapiro and J. Shalika, *Construction of cusp forms on* GL(3), Lecture Notes, Maryland.

[**J-S**] H. Jacquet and J. Shalika, *Comparaison des formes automorphes du groupe linéaire*, C. R. Acad. Sci. Paris **284** (1977), 741–744.

[**K**] R. Kottwitz, *Orbital integrals and base change*, these PROCEEDINGS, part 2, pp.111–113.

[**L**] R. P. Langlands. *Base change for* GL(2), Institute for Advanced Study, Princeton, N. J., 1975 (preprint).

[**Sa**] H. Saito, *Automorphic forms and algebraic extensions of number fields*, Lectures in Math., no.8, Kinokuniya Book Store Co. Ltd., Tokyo, Japan, 1975.

[**S-1**] T. Shintani, *On liftings of holomorphic automorphic forms*, U.S.-Japan Seminar on Number Theory (Ann. Arbor, 1975).

[**S-2**] ———, *Two remarks on irreducible characters of finite general linear groups*. J. Math. Soc. Japan **28** (1976), 396–414.

[**S-3**] ———, *On liftings of holomorphic cusp forms*, these PROCEEDINGS, part 2, pp. 97–110.

[**T**] J. Tate, *Number theoretic background*, these PROCEEDINGS, part 2, pp. 3–26.

[**Tu**] J. Tunnell, *On the local Langlands conjecture for* GL(2), Invent. Math. **46** (1978), 179–198.

UNIVERSITÉ PARIS VII

Proceedings of Symposia in Pure Mathematics
Vol. 33 (1979), part 2, pp. 135–138

REPORT ON THE LOCAL LANGLANDS CONJECTURE FOR GL(2)

J. TUNNELL

Let $\Phi(\mathrm{GL}(2)/K)$ be the set of isomorphism classes of two-dimensional F-semisimple representations of the Weil-Deligne group W'_K of a nonarchimedean local field K. The purpose of this report is to discuss the following conjecture of Langlands relating $\Phi(\mathrm{GL}(2)/K)$ and the set $\Pi(\mathrm{GL}(2, K))$ of isomorphism classes of irreducible admissible representations of $\mathrm{GL}(2, K)$.

Conjecture. For each representation σ in $\Phi(\mathrm{GL}(2)/K)$ there exists a representation $\pi = \pi(\sigma)$ in $\Pi(\mathrm{GL}(2, K))$ such that the determinant of σ and the central quasi-character of π are equal and such that

$$L(\pi \otimes \chi) = L(\sigma \otimes \chi) \quad \text{and} \quad \varepsilon(\pi \otimes \chi) = \varepsilon(\sigma \otimes \chi)$$

for all quasi-characters χ of K^*. The map $\sigma \mapsto \pi(\sigma)$ is a bijection of $\Phi(\mathrm{GL}(2)/K)$ with $\Pi(\mathrm{GL}(2, K))$.

The existence statement in the conjecture was formulated in [**4**], where it was shown that for a given representation σ there is at most one representation π satisfying the desired conditions. The injectivity statement is equivalent to saying that a two-dimensional F-semisimple representation of W'_K is determined by its twisted L- and ε-factors and determinant. It is straightforward to show directly that reducible two-dimensional representations are in fact determined by the twisted L-factors alone. The known proofs of the injectivity statement for irreducible representations use admissible representation techniques.

As discussed in [**1**, 3.2.3] it follows from the work of Jacquet and Langlands that $\pi(\sigma)$ exists when σ is reducible, and that this establishes a bijection of the set of isomorphism classes of completely reducible (repectively reducible indecomposable) two-dimensional F-semisimple representations of W'_K with the set of isomorphism classes of principal series (respectively special) representations of $\mathrm{GL}(2, K)$.

Let $\Phi_{\mathrm{irr}}(\mathrm{GL}(2)/K)$ consist of isomorphism classes of irreducible two-dimensional representations of the Weil group W_K, and let $\Pi_{\mathrm{cusp}}(\mathrm{GL}(2, K))$ be the set of isomorphism classes of supercuspidal representations of $\mathrm{GL}(2, K)$. Members of these sets are characterized by the requirement that their twisted L-factors are all equal to 1. To prove the conjecture it is enough to verify that $\pi(\sigma)$ exists for all σ in $\Phi_{\mathrm{irr}}(\mathrm{GL}(2)/K)$ and to show that this establishes a bijection of $\Phi_{\mathrm{irr}}(\mathrm{GL}(2)/K)$ and

AMS (MOS) subject classifications (1970). Primary 12B30, 12B15; Secondary 10D99.

$\Pi_{\text{cusp}}(\text{GL}(2, K))$. This was first proved for local fields with odd residue characteristic (see §2), which suggested the bijectivity portion of the conjecture in general.

The conjecture has been proved for all nonarchimedean local fields except those extensions of $\boldsymbol{Q}_2$ of degree greater than 1 which do not contain the cube roots of unity. References for the proof are given in the following survey.

1. Existence of $\pi(\sigma)$. If σ is a representation induced from a one-dimensional representation of an index two subgroup of W_K there is an explicit construction (due to Weil) of an irreducible admissible representation $\pi(\sigma)$ of GL(2, K) with the desired properties [**4**, 4.7]. When K has odd residue characteristic all representations in $\Phi_{\text{irr}}(\text{GL}(2)/K)$ are induced from proper subgroups [**1**, 3.4.4], so the construction above applies.

For each local field of even residue characteristic there exist irreducible two-dimensional representations of the Weil group which are not induced from a proper subgroup [**11**, paragraph 29]. There is no explicit construction of $\pi(\sigma)$ known for such representations. The approach of Jacquet and Langlands to the existence of $\pi(\sigma)$ in this case is to imbed the local problem in a global one as follows. Let F be a global field and let ρ be an irreducible continuous two-dimensional complex representation of the Weil group W_F. For each place v of F let ρ_v be the restriction of ρ to the Weil group of the local field F_v.

THEOREM 1 [**4**, 12.2]. *If the global L-functions $L(s, \rho \otimes \chi)$ are holomorphic and bounded in vertical strips as functions of the complex variable s for all quasi-characters χ of $\boldsymbol{A}_F^*/F^*$ then $\pi(\rho_v)$ exists.*

Cases when the hypotheses of this theorem are met have been described in the discussion of Artin's Conjecture in the base change seminar at this conference. A homomorphism of a group to GL(2, $\boldsymbol{C}$) will be said to be of type H if the composition with the quotient map to PGL(2, $\boldsymbol{C}$) has image isomorphic to H. The results in brief are that Theorem 1 may be applied when ρ is induced from a proper subgroup (Artin), when F is a field of positive characteristic (Weil), when ρ is of A_4 type (Langlands-Jacquet-Gelbart), and when $F = \boldsymbol{Q}$, ρ is of S_4 type and the image of complex conjugation has determinant -1 (Langlands-Serre-Deligne).

THEOREM 2 [**10**, THEOREM A]. *Let K be a nonarchimedean local field which contains the cube roots of unity if it is a proper extension of $\boldsymbol{Q}_2$. Then $\pi(\sigma)$ exists for all σ in $\Phi(\text{GL}(2)/K)$.*

This theorem is proved by constructing a global field F, a place v of F such that $F_v \approx K$, and a representation ρ of W_F satisfying the hypotheses of Theorem 1 such that $\rho_v \approx \sigma$. The nonarchimedean local fields of even residue characteristic which do not contain the cube roots of unity are precisely those for which there exist representations of the Weil group of S_4 type [**11**, paragraph 25]. The fields excluded in Theorem 2 are those for which there are two-dimensional representations of the Weil group which are not restrictions of global representations known to satisfy the hypotheses of Theorem 1. Deligne has indicated that if l-adic representations can be associated to certain automorphic representations related to Shimura varieties arising from division algebras over a totally real field F, then the hypotheses of Theorem 1 will hold for representations of W_F of S_4 type with prescribed behavior at the infinite places. Theorem 2 will then be true for K a completion of F at a prime dividing two; since each finite extension of Q_2 is the completion of some totally real field, the proposition and Theorem 3 of §4 would then prove the conjecture in all cases.

2. Odd residue characteristic. The Plancherel formula for GL(2, K) shows that the supercuspidal representations of the form $\pi(\sigma)$ for σ in $\Phi_{\text{irr}}(\text{GL}(2)/K)$ exhaust the supercuspidal representations if K has odd residue characteristic. References [3], [8] and [9] contain treatments of the representation theory of GL(2, K) and related groups when K has odd residue characteristic.

The injectivity statement may be proved by examining the explicit formulas for characters of supercuspidal representations in the case of odd residue characteristic which are given in [7] and the references above (at least for SL(2) and PGL(2)). Suppose that σ is induced from a 1-dimensional representation λ of the Weil group W_E of a quadratic extension E of K. Denote the quasi-character of E^* corresponding to λ by the same symbol. The restriction of the character function of $\pi(\sigma)$ to a Cartan subgroup isomorphic to E^* is given by a formula involving λ and its Gal(E/K) conjugate. From the explicit form of the character it can be seen that $\pi(\sigma)$ determines λ up to Gal(E/K) conjugation, and hence determines the induced representation σ.

3. Positive characteristic. The discussion of §1 shows that $\pi(\sigma)$ exists for all representations of the Weil group of a local field of positive characteristic. The bijectivity assertion of the conjecture seems to have first been proved by Deligne [2] as a consequence of Drinfeld's results relating automorphic representations of GL(2) over global function fields to l-adic representations.

4. A method is presented in [10] that gives alternate proofs of Langlands' Conjecture for the cases discussed in §§2 and 3 and proves the conjectured bijection for all fields in Theorem 2.

PROPOSITION [**10**, 2.2]. *Let σ_1 and σ_2 be two-dimensional representations of the Weil group of a local field, each of which is the restriction of a global representation satisfying the hypotheses of Theorem* 1. *If $\pi(\sigma_1) \approx \pi(\sigma_2)$ then $\sigma_1 \approx \sigma_2$.*

The proposition above is proved by an inductive application of base change for GL(2). The assumptions of the hypothesis are necessary because the proofs of the base change results for local fields utilize global methods.

The following result holds for any nonarchimedean local field K.

THEOREM 3 [**10**, §§4 AND 5]. *There are partitions of $\Phi_{\text{irr}}(\text{GL}(2)/K)$ and $\Pi_{\text{cusp}}(\text{GL}(2, K))$ into finite sets Φ_λ and Π_λ respectively (indexed by a common set Λ) such that*

(1) *If $\sigma \in \Phi_\lambda$ and $\pi(\sigma)$ exists, then $\pi(\sigma) \in \Pi_\lambda$.*

(2) $\text{Card}(\Phi_\lambda) = \text{Card}(\Pi_\lambda)$ *for all $\lambda \in \Lambda$.*

The partition elements are determined by conditions on the Artin conductor and determinant (resp. conductor and central quasi-character) of elements in $\Phi_{\text{irr}}(\text{GL}(2)/K)$ (resp. Π_{cusp} (GL(2, K))). Since the conductor of a representation is determined by the twisted ε-factors, the first statement follows from the definitions.

The computation of the cardinality of the sets Φ_λ is done by constructing all irreducible two-dimensional representations of W_K as in **[11]** and counting those with a given Artin conductor and determinant. The sets Π_λ are studied by utilizing the correspondence between square-integrable representations of GL(2, K) and admissible representations of the group of invertible elements in the quaternion division algebra over K.

Theorem 2 together with the injectivity proposition and the counting results of Theorem 3 show that Φ_λ and Π_λ correspond bijectively by means of $\sigma \mapsto \pi(\sigma)$ for the local fields in the hypotheses to Theorem 2. This gives the known cases of the conjecture stated in the introduction.

5. Remarks. While the statement of Langlands' Conjecture is purely local, the proofs described above in the case of even residue characteristic utilize global methods (base change and cases of Artin's Conjecture). Only in the case of odd residue characteristic are the proofs described above purely local.

Cartier and Nobs have indicated a proof of the conjecture for the field Q_2 which is purely local. They calculate the necessary ε-factors for irreducible two-dimensional representations of W_{Q_2} and match them with the factors of supercuspidal representations of GL(2, Q_2) constructed in **[6]**. The supercuspidal representations are constructed by inducing finite dimensional representations of subgroups of GL(2, K) which are compact modulo the center. In **[5]** a similar construction of supercuspidal representations for GL(2, K) is given which should allow, in theory, the calculation of ε-factors and matching with the factors of representations of W_K to be done in general.

REFERENCES

1. P. Deligne, *Formes modulaires et representations de* GL(2), in *Modular functions of one variable*, Springer Lecture Notes, No. 349, 1973, pp. 55–106.

2. ———, *Review of "Elliptic A-modules" by V. Drinfeld*, Math. Reviews **52** (1976), #5580.

3. I. M. Gelfand, M. I. Graev and I. I. Piatetskii-Shapiro, *Representation theory and automorphic functions*, Saunders, 1969.

4. H. Jacquet and R. P. Langlands, *Automorphic forms on* GL(2), Springer Lecture Notes, No. 114, 1970.

5. P. Kutzko, *On the supercuspidal representations of* GL_2. I, II, Amer. J. Math. **100** (1978), 43–60.

6. A. Nobs, *Supercuspidal representations of* GL(2, Q_p), *including* $p = 2$ (to appear).

7. P. J. Sally, Jr. and J. A. Shalika, *Characters of the discrete series of representations of* SL(2) *over a local field*, Proc. Nat. Acad. Sci. U.S.A. **61** (1968), 1231–1237.

8. ———, *The Plancherel formula for* SL(2) *over a local field*, Proc. Nat. Acad. Sci. U.S.A. **63** (1969), 661-667.

9. A. J. Silberger, PGL(2) *over the p-adics*, Springer Lecture Notes, No. 166, 1970.

10. J. Tunnell, *On the local Langlands Conjecture for* GL(2), Thesis, Harvard University (1977).

11. A. Weil, *Exercises dyadiques*, Invent. Math. **27** (1974), 1–22.

PRINCETON UNIVERSITY

Proceedings of Symposia in Pure Mathematics
Vol. 33 (1979), part 2, pp. 141–163

THE HASSE-WEIL ζ-FUNCTION OF SOME MODULI VARIETIES OF DIMENSION GREATER THAN ONE

W. CASSELMAN

Introduction. Let k be a number field of finite degree, G a connected reductive group over k. By restriction of scalars, one may as well assume $k = \boldsymbol{Q}$. Let $K_{\boldsymbol{R}} \subseteq G(\boldsymbol{R})$ be the product of the center of $G(\boldsymbol{R})$ and a maximal compact subgroup. Then, in certain circumstances, one may assign to $X = G(\boldsymbol{R})/K_{\boldsymbol{R}}$ a $G(\boldsymbol{R})$-invariant complex structure; if Γ is an arithmetic subgroup of G small enough to contain no torsion then $\Gamma\backslash X$ will be a nonsingular algebraic variety. In many cases one can show that it has a model defined over a rather explicit number field, and under certain further assumptions—as Deligne and Langlands may explain—one can choose a *canonical* model defined over an abelian extension of a special number field E determined, roughly speaking, by G and the complex structure on X.

One might expect that the Hasse-Weil ζ-function of a canonical model is a product of L-functions of the sort Langlands associates to automorphic representations of $G(\boldsymbol{A})$. This turns out to be false (see the Introduction to **[22]**), but it is suggestive. The first result of this kind is due to Eichler, who showed that when $G = \mathrm{GL}_2(\boldsymbol{Q})$ and $\Gamma = \Gamma_0(N)$, then $\Gamma\backslash X$ has a model over $\boldsymbol{Q}$ and its Hasse-Weil ζ-function is, as far as all but a finite number of factors in its Euler product are concerned, a product of L-functions defined in this case by Hecke. This result was extended by others, notably Shimura, Kuga, Ihara, Deligne, and Langlands, to include: (a) other Γ in this G, (b) other G, (c) ζ-functions associated to nontrivial locally constant sheaves, and finally (d) factors of the ζ-function corresponding to primes where the variety behaves badly. With one exception—some unpublished work of Shimura—all this work is concerned with X of dimension one. (Refer—for a sampling—to **[11]**, **[28]**, **[14]**, **[6]**, and **[19]**.)

As a consequence of these results one had a generalization of Ramanujan's conjecture, applying the result of Deligne on the roots of the ζ-function of varieties over finite fields. A further consequence was a functional equation for and analytic continuation of the Hasse-Weil ζ-function concerned, which turned out—excepting again Shimura's example—to be a ζ-function associated in a particularly simple way to GL_2 or a quaternion algebra.

Over the past several years, Langlands has attacked the problem of varieties of dimension > 1, and what Milne and I are going to discuss in our lectures is the simplest case he deals with.

AMS (MOS) subject classifications (1970). Primary 10D20; Secondary 14G25, 22E55.

To be more precise, let:
F = a totally real field of degree, say, n;
$\mathfrak{o}_F$ = integers in F;
B = a quaternion algebra over F;
$\mathfrak{o}_B$ = a maximal order in B.
We recall that this means simply that B is an algebra of dimension four over F such that $B \otimes \bar{F} = M_2(\bar{F})$, where $\bar{F}$ is an algebraic closure of F. Since over any locally compact field there exists a unique quaternion division algebra, if v is any valuation of F then $B_v = B \otimes_F F_v$ is isomorphic either to $M_2(F_v)$ or to this unique division algebra, depending on whether or not B_v has 0-divisors. The algebra B is said to be split at v in the first case, ramified in the second. It is in fact split at all but a finite number of valuations; quadratic reciprocity says this number is even, and another classical result says that for each even set of valuations there is a unique quaternion algebra ramified at exactly those valuations. Thus B itself is a division algebra if and only if it is ramified somewhere. If v is nonarchimedean, the closure of $\mathfrak{o}_B$ in B_v is a maximal compact subring of B_v—unique if B is ramified at v, otherwise only unique up to conjugacy by an element of $B_v^\times$.

Let G be the algebraic group over Z defined by the multiplicative group of $\mathfrak{o}_B$. Thus for any ring R, $G(R) = (\mathfrak{o}_B \otimes R)^\times$. In particular, $G(\mathbf{Q}) \cong B^\times$ (canonically) and $G(\mathbf{R}) \cong \mathrm{GL}_2(\mathbf{R})^I \times (\mathbf{H}^\times)^J$ (noncanonically), where I is the set of real valuations of F where B is split, J those where it is ramified. For every rational finite prime p over which B does not ramify,

$$G(\mathbf{Z}_p) \cong \prod \mathrm{GL}_2(\mathfrak{o}_{F,\mathfrak{p}}), \qquad G(\mathbf{Q}_p) \cong \prod \mathrm{GL}_2(F_\mathfrak{p}),$$

where the product is over all primes $\mathfrak{p}$ of F dividing p.

Of course the simplest case is $B = M_2(F)$, but that is unfortunately the case we will not allow—i.e., *from now on we assume B to be a division algebra. Furthermore, we will assume B totally indefinite at the real primes of F*—i.e., that $J = \varnothing$. Langlands himself does not make these assumptions, but acknowledges gaps in the argument unless they hold.

Let Δ be a finite set of rational primes containing those over which either F or B ramifies, and let K_f be a compact open subgroup of $G(\mathbf{Z}_f)$ of the form $K_\Delta \cdot \prod_{p \notin \Delta} G(\mathbf{Z}_p)$, where K_Δ is a compact open subgroup of $\prod_{p \in \Delta} G(\mathbf{Z}_p)$.

Let Z be the center of G, and $Z_K = (Z(\mathbf{A}_f) \cap K_f) \cdot Z(\mathbf{R})$.

Consider $\mathbf{C}^\times$ as embedded in $\mathrm{GL}_2(\mathbf{R})$:

$$a + b\sqrt{-1} \mapsto \begin{pmatrix} a & -b \\ b & a \end{pmatrix}$$

and let $K_\mathbf{R}$ be the image of $(\mathbf{C}^\times)^I$ in $G(\mathbf{R})$. (This is the connected component of the $K_\mathbf{R}$ used before.) Then $G(\mathbf{R})/K_\mathbf{R}$ is a product of n copies of $\mathbf{C} - \mathbf{R}$.

Let K be $K_\mathbf{R} \cdot K_f$. The set ${}_KS(\mathbf{C}) = G(\mathbf{Q}) \backslash G(\mathbf{A})/K$ is (as will be explained later) the union of a finite number of compact complex analytic spaces of dimension n; it will be nonsingular if (as we assume from now on) K_f is small enough.

Milne (in his lectures at this Institute) will show that this space is the set of $\mathbf{C}$-valued points on a certain moduli scheme ${}_KS$ which is defined, smooth, and proper over Spec $Z[1/d]$, where d is the product of primes in Δ. He will also discuss the structure of its points over finite fields. What I will do is to identify its Hasse-Weil ζ-function as one of Langlands'; the basic idea in doing this will be to use the Selberg trace formula to calculate p-factors of Langlands' L-function, and Milne's results and an unorthodox application of the trace formula to calculate p-factors of Hasse-Weil—both for p not dividing Δ. (This will be a lot of trouble, and I should explain at the outset that, although when ${}_KS$ has dimension one it is possible to use a congruence relation to obtain the final result, this will not suffice in general.) One may then apply Deligne's results on the roots of the Frobenius to get a generalization of Ramanujan-Petersson; however, results about a functional equation for Hasse-Weil are incomplete.

In everything both Milne and I do we are essentially reporting on Langlands' work. The result we discuss is only a special case of a more complicated result of his involving subgroups of $B^\times$. We avoid, in treating this special case, problems of what he calls "L-indistinguishability," which perhaps he himself will talk about. His result is more general in other ways, too; I include remarks on this in the last section, where I have collected together a number of substantial parenthetical remarks.

My main reference is the summary **[21]**; complete statement and proofs are in **[22]**. Also relevant are Langlands' talk in the Hilbert problems Symposium **[20]** and his talks at this Institute.

1. Cohomology of ${}_KS(C)$ and representations.

1.1. Let ν be the reduced norm: $G \to \boldsymbol{G}_m$. Since B is a division algebra, the coset space $G(\boldsymbol{Q})\backslash G^1(A)$ is compact, where $G^1(A)$ is the kernel of the modulus homomorphism $|\nu| : x \to |\nu(x)|$. Since the image of $G^{\mathrm{conn}}(\boldsymbol{R})$, the connected compact of $G(\boldsymbol{R})$, under $|\nu|$ is all of $\boldsymbol{R}^{\mathrm{pos}}$, the set $G(\boldsymbol{Q})\backslash G(A)/G^{\mathrm{conn}}(\boldsymbol{R})$ is compact as well. Since K_f is open in $G(A_f)$, the set $G(\boldsymbol{Q})\backslash G(A)/G^{\mathrm{conn}}(\boldsymbol{R})K_f$ is finite. Therefore there exists a finite set $\mathscr{X}$ of elements of $G(A)$ such that $G(A) = \bigcup G(\boldsymbol{Q})xG^{\mathrm{conn}}(\boldsymbol{R})K_f$ $(x \in \mathscr{X})$. (Strong approximation gives one a better parametrization of $\mathscr{X}$, but we won't need that; see **[7]**.)

1.1.1. LEMMA. *For any $x \in G(A)$, the space $G(\boldsymbol{Q})\backslash G(\boldsymbol{Q})xG^{\mathrm{conn}}(\boldsymbol{R})K_f/K_f$ as a $G^{\mathrm{conn}}(\boldsymbol{R})$-space is isomorphic to $\Gamma_x\backslash G^{\mathrm{conn}}(\boldsymbol{R})$, where Γ_x is the image in $G^{\mathrm{conn}}(\boldsymbol{R})$ of $G(\boldsymbol{Q}) \cap G^{\mathrm{conn}}(\boldsymbol{R}) \cdot xK_fx^{-1}$.*

This is because $G^{\mathrm{conn}}(\boldsymbol{R})$ certainly acts transitively on this space and Γ_x is the isotropy subgroup of the coset $G(\boldsymbol{Q})xK_f$.

Let $\mathscr{H}$ be the upper half-plane in $\boldsymbol{C}$. As a consequence of the above:

1.1.2. PROPOSITION. *The $G^{\mathrm{conn}}(\boldsymbol{R})$-space $G(\boldsymbol{Q})\backslash G(A)/K_f$ is isomorphic to a disjoint union of spaces $\Gamma_x\backslash G^{\mathrm{conn}}(\boldsymbol{R})$ $(x \in \mathscr{X})$. The variety ${}_KS(\boldsymbol{C})$ is the disjoint union of the $\Gamma_x\backslash\mathscr{H}^n$.*

The same argument as that used to prove Lemma 2.1 of **[19]** may be applied to show that if K_f is only small enough, each Γ_x acts freely and ${}_KS(\boldsymbol{C})$ is nonsingular. *We assume this from now on.*

1.2. Let $\mathscr{A}_0$ be the space of automorphic forms on $G(\boldsymbol{Q})Z(\boldsymbol{R})\backslash G(A)$. It is a direct sum $\bigoplus\pi$ of irreducible, admissible, unitary representations of $G(A)$ (an abuse of language since not $G(\boldsymbol{R})$, but only its Lie algebra $\mathfrak{g}$, acts). If $\tilde{\mathfrak{g}}$ is the Lie algebra of

$\tilde{G}(\boldsymbol{R}) = G(\boldsymbol{R})/Z(\boldsymbol{R})$, then in fact the representation on $\mathscr{A}_0$ factors through $\tilde{\mathfrak{g}}$. Let $\tilde{\mathfrak{k}}$ be the Lie algebra of $\tilde{K}_{\boldsymbol{R}} = K_{\boldsymbol{R}}/K_{\boldsymbol{R}} \cap Z_{\boldsymbol{R}}$.

1.2.1. PROPOSITION. *The de Rham cohomology $H^*({}_KS(\boldsymbol{C}), \boldsymbol{C})$ is naturally isomorphic to $\bigoplus H^*(\tilde{\mathfrak{g}}, \tilde{\mathfrak{k}}, \pi_\infty) \otimes \pi_f^{K_f}$.*

The cohomology is the relative Lie algebra cohomology. The sum is over all constituents π of $\mathscr{A}_0$ (with multiplicity if necessary), which one factors as $\pi = \pi_\infty \otimes \pi_f$ where π_∞ is an irreducible admissible representation of $G(\boldsymbol{R})$ (more abuse of language as before) and π_f one of $G(\boldsymbol{A}_f)$.

PROOF SKETCH. Let $\mathscr{A}_0^{K_f}$ be the subspace of $\mathscr{A}_0$ of functions fixed by elements of K_f. Since $\tilde{\mathfrak{g}}$ commutes with K_f, this is a representation of $\tilde{\mathfrak{g}}$ which is clearly isomorphic to $\bigoplus \pi_\infty \otimes \pi_f^{K_f}$. Thus the proposition amounts to the claim that $H^*({}_KS(\boldsymbol{C}), \boldsymbol{C})$ is isomorphic to $H^*(\tilde{\mathfrak{g}}, \tilde{\mathfrak{k}}, \mathscr{A}_0^{K_f})$. Now from 1.1.2 it follows that $\mathscr{A}_0^{K_f}$ is the direct sum of subspaces $\mathscr{A}_{0,x}^{K_f}$ $(x \in \mathscr{X})$, where each $\mathscr{A}_{0,x}^{K_f}$ is the space $\mathscr{A}(\tilde{\Gamma}_x\backslash\tilde{G}^{\text{conn}}(\boldsymbol{R}))$ of $K_{\boldsymbol{R}}$-finite, $Z(\tilde{\mathfrak{g}})$-finite functions on $\tilde{\Gamma}_x\backslash\tilde{G}^{\text{conn}}(\boldsymbol{R})$ ($\tilde{\Gamma}_x$ is the image of Γ_x in $\tilde{G}$). Thus the proof of 1.2.1 reduces to:

1.2.2. LEMMA. *There exists a natural isomorphism*:

$$H^*(\tilde{\Gamma}_x\backslash\tilde{G}/\tilde{K}, \boldsymbol{C}) \cong H^*(\tilde{\mathfrak{g}}, \tilde{\mathfrak{k}}, \mathscr{A}(\tilde{\Gamma}_x\backslash\tilde{G})).$$

This sort of thing holds in fact for any semisimple Lie group $\tilde{G}$, cocompact $\tilde{\Gamma} \subset \tilde{G}$ and maximal compact $\tilde{K}$.

To see it: the cohomology of $X = \tilde{\Gamma}_x\backslash\tilde{G}/\tilde{K}$ is that of the de Rham complex, whose nth term is the space of C^∞ m-forms on X. Now the projection $\tilde{\Gamma}_x\backslash\tilde{G} \to X$ is a principal bundle, and the bundle of m-forms is associated to it and the K-space $\Lambda^m(\tilde{\mathfrak{g}}/\tilde{\mathfrak{k}})^\wedge$. Therefore C^∞ m-forms correspond to certain C^∞ functions from $\tilde{\Gamma}_x\backslash\tilde{G}$ to $\Lambda^m(\tilde{\mathfrak{g}}/\tilde{\mathfrak{k}})^\wedge$, which by means of an obvious duality may be thought of as $\tilde{K}$-linear maps from $\Lambda^m(\tilde{\mathfrak{g}}/\tilde{k})$ to $C^\infty(\tilde{\Gamma}_x\backslash\tilde{G})$. In short, the de Rham complex on X may be identified with a complex whose mth term is $\mathrm{Hom}_{\tilde{\mathfrak{k}}}(\Lambda^m(\tilde{\mathfrak{g}}/\tilde{\mathfrak{k}}), C^\infty(\tilde{\Gamma}_x\backslash\tilde{G}))$. But this is the mth term of the complex by which $H^*(\tilde{\mathfrak{g}}, \tilde{\mathfrak{k}}, C^\infty(\tilde{\Gamma}_x\backslash\tilde{G}))$ is calculated, and it turns out (by an explicit calculation) that the differentials are the same. Therefore 1.2.2 is true with $\mathscr{A}(\tilde{\Gamma}_x\backslash\tilde{G})$ replaced by $C^\infty(\tilde{\Gamma}\backslash\tilde{G})$. To get the final step, roughly, one recalls that according to Hodge theory cohomology classes may be represented uniquely by harmonic classes, and observes that these lift in the above process to elements of $\mathscr{A}$. (See Chapter IV of [3] for details on this and other points.)

1.3. Although the number of representations occurring in the sum in 1.2.1 is infinite, all but a finite number of terms vanish. To be more precise, we must say more about the relative Lie algebra cohomology of admissible representations. First of all, since $G(\boldsymbol{R}) \cong \mathrm{PGL}_2(\boldsymbol{R})^I$ (notation as in the introduction), each π_∞ factors as $\bigotimes \pi_{\infty,\iota}$, where each $\pi_{\infty,\iota}$ is an admissible representation of $\mathrm{PGL}_2(\boldsymbol{R})$. One has an easy Künneth formula:

$$H^*(\tilde{\mathfrak{g}}, \tilde{\mathfrak{k}}, \bigotimes \pi_{\infty,\iota}) \cong \bigotimes H^*(\mathfrak{sl}_2, \mathfrak{so}(2), \pi_{\infty,\iota})$$

so that the problem for $G(\boldsymbol{R})$ is reduced to one for $\mathrm{PGL}_2(\boldsymbol{R})$. (Note that the Lie algebra of PGL_2 is the same as that of SL_2.) For this: let C be the Casimir element in $U(\mathfrak{sl}_2)$, and recall that it lies in the center of $U(\mathfrak{sl}_2)$ and is centralized by the maximal compact $O(2)$, hence acts as a scalar on any irreducible admissible representation of $\mathrm{PGL}_2(\boldsymbol{R})$.

1.3.1. LEMMA. *Let (π, V) be an irreducible, admissible, unitary representation of* $\mathrm{PGL}_2(\boldsymbol{R})$. *Then*

$$\begin{aligned} H^*(\mathfrak{sl}_2, \mathfrak{so}_2, V) &\cong 0 && \text{if } \pi(C) \neq 0, \\ &\cong \mathrm{Hom}_{\mathfrak{so}_2}(\Lambda^*(\mathfrak{sl}_2/\mathfrak{so}_2), V) && \text{if } \pi(C) = 0. \end{aligned}$$

The idea of the proof here is that if π is unitary one can put a natural inner product on the complex $\mathrm{Hom}_{\mathfrak{so}_2}(\Lambda^*(\mathfrak{s}_{\mathfrak{l}_2}/\mathfrak{so}_2), V)$, such that $\pi(C) = dd^* + d^*d$ where d^* is the adjoint of d. (This is an observation good for all semisimple groups due to Kuga.) The lemma follows immediately.

For $\mathrm{PGL}_2(\boldsymbol{R})$, there are only three irreducible admissible representations π with $\pi(C) = 0$: (a) the trivial representation $\boldsymbol{C}$; (b) the character $\mathrm{sgn}(\det(g))$; (c) a single discrete series representation π_0, which may be identified with the quotient of the $O(2)$-finite functions on $\boldsymbol{P}^1(\boldsymbol{R})$ by the constant functions. Lemma 1.5 and an easy calculation concerning the restriction of these to $O(2)$ give:

1.3.2. COROLLARY. *If π is an irreducible admissible unitary representation of* $\mathrm{PGL}_2(\boldsymbol{R})$ *other than one of these then $H^*(\pi) = H^*(\mathfrak{sl}_2, \mathfrak{so}_2, \pi) = 0$. For these*:

(a) *when π is $\boldsymbol{C}$ or* sgn (det),

$$\begin{aligned} H^m(\pi) &\cong \boldsymbol{C}, && m = 0, \\ &\cong 0, && m = 1, \\ &\cong \boldsymbol{C}, && m = 2; \end{aligned}$$

(b) *when $\pi = \pi_0$*,

$$\begin{aligned} H^m(\pi) &\cong 0, && m = 0, \\ &\cong \boldsymbol{C} + \boldsymbol{C}, && m = 1, \\ &\cong 0, && m = 2. \end{aligned}$$

The assumption of unitarity is not in fact necessary (see [3]).

Hence if the irreducible admissible representation π_∞ of the original $G(\boldsymbol{R})$ is cohomologically nontrivial, it must be of the form $\otimes \pi_{\infty,\iota}$, where each $\pi_{\infty,\iota}$ is one of the above three representations, and its cohomology may be calculated accordingly.

If π is an irreducible admissible representation of $\mathrm{PGL}_2(\boldsymbol{R})$, set

$$\begin{aligned} m(\pi) &= 0 && \text{if } \pi \text{ is cohomologically trivial,} \\ &= 1 && \text{if } \pi \text{ is either } \boldsymbol{C} \text{ or sgn(det),} \\ &= -1 && \text{if } \pi \text{ is } \pi_0. \end{aligned}$$

If $\pi_\infty = \bigotimes \pi_{\infty,\iota}$ is an irreducible admissible representation of $G(\boldsymbol{R})$ define $m(\pi_\infty)$ to be $\prod m(\pi_{\infty,\iota})$. If $\pi = \pi_\infty \otimes \pi_f$ is an irreducible admissible representation of $G(\boldsymbol{A})$ where π_∞ is trivial on $Z(\boldsymbol{R})$, then define $m(\pi, K) = m(\pi_\infty) \cdot \dim \pi_f^{K_f}$. Thus the size of $m(\pi, K)$ is just $\dim \pi_f^{K_f}$ and its sign reflects the parity of its contribution to cohomology.

As a final remark let me point out that by applying the strong approximation theorem one can show that if $\pi = \otimes \pi_v$ (v over valuations of F) is an irreducible admissible automorphic representation of $G(\boldsymbol{A})$ and π_v is one-dimensional at a place where B is split, then π_v is one-dimensional everywhere. In particular, if one factor of π_∞ is one-dimensional so are all. As a consequence of this one recovers a result of Matsushima-Shimura [23] which says that the interesting cohomology of ${}_KS(C)$ occurs in the middle dimension.

2. The main theorem and some consequences.

2.1. I recall the L-group attached to G. First of all, if G is considered in the most straightforward way as a group over F —i.e., so that for any extension F' of F, $G(F') \cong (B \otimes_F F')^\times$ —then its L-group is just the direct product of ${}^LG^0 = \mathrm{GL}_2(\boldsymbol{C})$ and $\mathrm{Gal}(\bar{F}/F)$. This is because it is an inner twisting of $\mathrm{GL}_2(F)$. Since G as a group over $\boldsymbol{Q}$ is obtained from this one by restriction of scalars, its L-group ${}^LG^0$ is the one in some sense induced from this one: it is the semidirect product of $\mathrm{Gal}(\bar{\boldsymbol{Q}}/\boldsymbol{Q})$ by ${}^LG^\circ = \mathrm{GL}_2(\boldsymbol{C})^I$, where I is the set of embeddings of F into $\bar{\boldsymbol{Q}}$ and Gal acts on ${}^LG^0$ by permutation of factors. (Since $\bar{\boldsymbol{Q}}$ may be identified with a subfield of $\boldsymbol{C}$, this I is essentially the same as before.) Without any serious loss for our purposes we may (and will) replace Gal $(\bar{\boldsymbol{Q}}/\boldsymbol{Q})$ by Gal (F_{norm}/F), where F_{norm} is the smallest extension of F normal over $\boldsymbol{Q}$. It may be helpful if I point out that it is unramified over $\boldsymbol{Q}$ whenever F is. I recall also that if p is a prime of $\boldsymbol{Q}$ unramified in F and Φ is a Frobenius in $\mathrm{Gal}(F_{\mathrm{norm}}/F)$ over p, then the local L-group ${}^LG_{\boldsymbol{Q}_p} = {}^LG_p$ may be identified with the subgroup of ${}^LG_{\boldsymbol{Q}}$ whose image in $\mathrm{Gal}(F_{\mathrm{norm}}/F)$ is the cyclic subgroup generated by Φ. Thus one has an exact sequence $1 \to \mathrm{GL}_2(\boldsymbol{C})^I \to {}^LG_p \to \langle\Phi\rangle \to 1$. (Refer to [2] for everything about L-groups.)

In order to define L-functions associated to automorphic representations, one must also introduce finite-dimensional representations of ${}^LG$. There is only one (for each G) that we will be concerned with, and it is defined as follows: the space of this representation ρ is a tensor product of copies of $\boldsymbol{C}^2$, one for each element of I; the group ${}^LG^0 = \mathrm{GL}_2(\boldsymbol{C})^I$ acts through the standard representation on each factor, and $\mathrm{Gal}(\bar{\boldsymbol{Q}}/\boldsymbol{Q})$ (or $\mathrm{Gal}(F_{\mathrm{norm}}/\boldsymbol{Q})$, it makes no difference) acts by permuting the factors. This does indeed define a representation of ${}^LG$, since it is a semidirect product of these two groups. The dimension of this representation is 2^n. When $F = \boldsymbol{Q}$, for example, ${}^LG = \mathrm{GL}_2(\boldsymbol{C})$ and ρ is just the standard representation itself.

This particular choice of ρ may seem arbitrary, but in fact it was motivated originally (in Langlands' formulation) by general considerations about Shimura varieties (one should refer to the Introduction of [22] for a discussion of this point).

2.2. If X is any smooth, proper scheme over $\boldsymbol{Z}[1/d]$ for some $d \geqq 1$ and p is a prime not dividing d, then the zeta-function of X over $\boldsymbol{F}_p$ is defined to be the function $Z_p(s, X)$, rational in p^{-s}, such that (at least formally)

$$\log Z_p(s, X) = \sum_1^\infty \frac{1}{mp^{ms}} (\# X(\boldsymbol{F}_{p^m})).$$

As a consequence of the étale theory, this agrees with the definition in terms of l-adic cohomology:

$$Z_p(s, X) = \prod_{i=0}^{2\text{-}\dim X} \det(I - \Phi \cdot p^{-s})_{H^i(X, \boldsymbol{Q}_l)}$$

where Φ is the geometric Frobenius. At least as far as factors other than those dividing d are concerned, its Hasse-Weil zeta-function $Z(s, X)$ is the product of these $Z_p(s, X)$.

The main theorem of Langlands is:

2.2.1. THEOREM. *Up to prime factors in* Δ, *the Hasse-Weil zeta-function of* ${}_KS$ *agrees with* $\prod L(s - n/2, \pi, \rho)^{m(\pi, K)}$ *where the product is over all* π *occurring as constituents of* $\mathscr{A}_0$.

Of course one should consider the π with multiplicities if necessary. In fact, however, because of the theorem in §16 of [16] relating automorphic forms on G with those for GL_2, together with the result for GL_2 in §9 of [16], each π occurs exactly once.

In this product, $m(\pi, K) = 0$ for all but a finite number of π, since after all the cohomology of ${}_KS(C)$ is finite.

What is actually to be proven is a purely local result: for $p \notin \Delta$, the p-factors of $Z(s, {}_KS)$ and $\prod L(s - n/2, \pi, \rho)$ coincide. Now each $\pi = \otimes \pi_v$ with $m(\pi, K) \neq 0$ has the property that π_p for $p \notin \Delta$ is unramified, hence corresponds to an element $g(\pi_p)$ in the local L-group ${}^LG_p$. The p-factor of $L(s - n/2, \pi, \rho)$ is then $\det(I - \rho(g(\pi_p))/p^{s-n/2})^{-1}$. Upon expansion, the theorem reduces to a formal equation ($p \notin \Delta$):

$$\sum_{m=1}^{\infty} \frac{1}{mp^{ms}} (\#_K S(F_{p^m})) = \sum_{\pi \text{ in } \mathscr{A}_0} m(\pi, K) \sum_{m=1}^{\infty} p^{mn/2} \frac{1}{mp^{ms}} \operatorname{trace} \rho(g(\pi_p)^m).$$

This in turn amounts to an equation of coefficients (for $p \notin \Delta$, $m \geqq 1$):

$$\#_K S(F_{p^m}) = \sum_{\pi \text{ in } \mathscr{A}_0} m(\pi, K) p^{mn/2} \operatorname{trace} \rho(g(\pi_p)^m). \tag{2.1}$$

This is the form in which the theorem is actually proven. In these lectures I will give a complete proof only in the simplest case, when F is split over p.

2.3. Before beginning the proof, we give an example and some consequences.

The case $n = 2$ is the first interesting one, in the sense that, as already mentioned, the case $n = 1$ is an old result and can be (and has been) done more elementarily by means of a congruence relation. So, suppose for a while that F is quadratic over Q, and consider the possible p-factors occurring in $L(s, \pi, \rho)$. There are two cases: (1) when $p = \mathfrak{p}_1\mathfrak{p}_2$ splits in F and (2) when $p = \mathfrak{p}$ remains prime.

We look at (1) first. In this case $G(Q_p) \cong GL_2(F_{\mathfrak{p}_1}) \times GL_2(F_{\mathfrak{p}_2})$ and an unramified representation of this must be of the form $\pi_1 \otimes \pi_2$, where each π_i is an unramified representation of $GL_2(F_{\mathfrak{p}_i})$, hence corresponds to a pair of unramified characters (α_i, β_i) of $F_{\mathfrak{p}_i}^\times$. More precisely, π_i may be the whole principal series representation parametrized by the pair when it is irreducible, or the associated one-dimensional character of $GL_2(F_{\mathfrak{p}_i})$ when it is not. (By strong approximation, this last happens only when the global representation at hand is also one-dimensional.) In either case, observe that the local L-group ${}^LG_p$ is (because p splits) simply the direct product $GL_2(C) \times GL_2(C)$ and that the corresponding element $g(\pi_p)$ of ${}^LG_p$ is

$$\left(\begin{pmatrix} \alpha_1(p) & \\ & \beta_1(p) \end{pmatrix}, \begin{pmatrix} \alpha_2(p) & \\ & \beta_2(p) \end{pmatrix}\right).$$

The representation ρ is simply $C^2 \otimes C^2$, so the p-factor of $L(s - 1, \pi, \rho)$ is

$$\begin{aligned}(1 - \alpha_1(p)\alpha_2(p)p^{1-s})^{-1}(1 - \alpha_1(p)\beta_2(p)p^{1-s})^{-1}(1 - \beta_1(p)\alpha_2(p)p^{1-s})^{-1}\\ \cdot (1 - \beta_1(p)\beta_2(p)p^{-s})^{-1}.\end{aligned} \tag{2.2}$$

Next case (2). Here π_p is a single unramified representation of $\mathrm{GL}_2(F)$, say corresponding to the characters (α, β) of $F_{\mathfrak{p}}^\times/\mathfrak{o}_{\mathfrak{p}}^\times$. On the other hand, the local L-group is still a semidirect product of $\mathrm{GL}_2(\boldsymbol{C}) \times \mathrm{GL}_2(\boldsymbol{C})$ by the Galois group of order 2, since the Frobenius Φ generates Gal. I leave it as an exercise to verify that the element $g(\pi_p)$ of ${}^LG_p$ is

$$\left(\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}, \begin{pmatrix}\alpha(p) & 0\\ 0 & \beta(p)\end{pmatrix}\right)\Phi.$$

If e_1, e_2 are the standard basis elements of $\boldsymbol{C}^2$ then $g(\pi_p)$ acts as follows on the space of ρ (recall that Φ interchanges factors):

$$\begin{aligned} e_1 \otimes e_1 &\mapsto \alpha(p) e_1 \otimes e_1, & e_1 \otimes e_2 &\mapsto \alpha(p) e_2 \otimes e_1,\\ e_2 \otimes e_1 &\mapsto \beta(p) e_1 \otimes e_2, & e_2 \otimes e_2 &\mapsto \beta(p) e_2 \otimes e_2. \end{aligned}$$

Therefore the p-factor of $L(s-1, \pi, \rho)$ is

$$(1 - \alpha(p)p^{1-s})^{-1}(1 - \beta(p)p^{1-s})^{-1}(1 - \alpha(p)\beta(p)p^{2-s})^{-1}. \tag{2.3}$$

For $n > 2$, the analogous formulae can be rather complicated. Incidentally, the unpublished result of Shimura we have referred to before is precisely that the p-factors are of the form (2.2) and (2.3) for certain quadratic fields F.

For $F = \boldsymbol{Q}$ $(n = 1)$, one consequence of the expression of the Hasse-Weil zeta-function as an L-function associated to automorphic forms is that it has a functional equation and analytic continuation. For $n > 1$, this consequence is not automatic. For $n = 2$ it is already a difficult question only very recently settled by Asai **[1]** under some probably unnecessary restriction on F. Nothing seems to be known for $n > 2$. What is easier is to obtain a functional equation for the zeta-function of ${}_KS$ over F itself, at least for certain small degrees. This depends on a generalization of an idea of Rankin due to Jacquet and Shimura (see, for example, **[15]** and **[29]**).

3. The analytical trace formula. In this section I will develop a formula for the right side of equation (2.1), in the next one for the left, and in §5 the two will be compared. Beginning in §4 I will assume (although Langlands certainly does not) that p is split in F. This makes things much simpler. The argument will still involve several important ideas, but will avoid what is at once the most complicated and intriguing point of all, the use of paths in a certain Bruhat-Tits building. I hope that what remains is still of some interest. Langlands will presumably say something on this matter in his own talks (see **[17]**).

3.1. The right side of equation (2.1),

$$\sum_{\pi \text{ in } \mathscr{A}_0} m(\pi, K)\, p^{mn/2} \operatorname{trace} \rho(g(\pi_p)^m), \tag{3.1}$$

may be expressed as the trace of a certain operator R_f, for some $f \in C_c^\infty(G(A))$, acting on $\mathscr{A}_0$. Recall that $\mathscr{A}_0$ is a discrete direct sum of irreducible, admissible, unitary representations $\pi = \pi_{\boldsymbol{R}} \otimes \pi^p \otimes \pi_p$ of $G(A)$, and for any $f \in C_c^\infty(G(A)/Z_K)$ one has therefore trace $R_f = \sum_{\pi \text{ in } \mathscr{A}_0} \operatorname{trace} \pi(f)$, which is equal in turn to

$$\sum_{\pi \text{ in } \mathscr{A}_0} \operatorname{trace} \pi_{\boldsymbol{R}}(f_{\boldsymbol{R}}) \operatorname{trace} \pi^p(f^p) \operatorname{trace} \pi_p(f_p) \tag{3.2}$$

if f factors as $f_{\boldsymbol{R}} \cdot f^p \cdot f_p$. Recall that $m(\pi, K) = m(\pi_{\boldsymbol{R}}) \dim(\pi^p)^{K^p}$. A comparison of (3.1) with (3.2) suggests choosing the three factors of f so that

(3.3)(a) $$\text{trace } \pi(f_{\boldsymbol{R}}) = m(\pi)$$

for any irreducible admissible representation π of $G(\boldsymbol{R})/Z(\boldsymbol{R})$;

(3.3)(b) $$\text{trace } \pi(f^p) = \dim \pi^{K^p}$$

for any irreducible admissible representation π of $G(\boldsymbol{A}_f^p)$;

(3.3)(c) $$\begin{aligned} \text{trace } \pi(f_p^{(m)}) &= p^{mn/2} \text{ trace } \rho(g(\pi)^m) \quad && \pi \text{ unramified,} \\ &= 0 && \text{otherwise.} \end{aligned}$$

This is just what will be done.

The choice of f^p is simple:

$$f^p = (\text{meas } K^p)^{-1} \text{ char } K^p.$$

That of $f_p^{(m)}$ is not so explicit but just as simple: according to the Satake isomorphism [4] there exists a unique $f_p^{(m)} \in \mathscr{H}(G(\boldsymbol{Q}_p), G(\boldsymbol{Z}_p))$ satisfying (3.3)(c).

But the matter of $f_{\boldsymbol{R}}$ is more complicated.

3.2. If the representation π of $G(\boldsymbol{R})$ factors as $\bigotimes \pi_\iota$ according to $G(\boldsymbol{R}) \cong \mathrm{GL}_2(\boldsymbol{R})^I$ then $m(\pi) = \prod m(\pi_\iota)$. If $f = \prod f_\iota$ accordingly then trace $\pi(f) = \prod$ trace $\pi(f_\iota)$. Thus the problem of finding $f_{\boldsymbol{R}}$ reduces to the problem of finding $f \in C_c^\infty(\mathrm{PGL}_2(\boldsymbol{R}))$ such that

(3.4) $$\text{trace } \pi(f) = m(\pi)$$

for any irreducible admissible representation π of $\mathrm{PGL}_2(\boldsymbol{R})$. One can do this in several ways—for example directly from the Paley-Wiener theorem as in **[10]**, and also from considerations of orbital integrals (see **[26]**). But Serge Lang has pointed out an argument which is in some sense more elementary than either of these and which I present below.

From this point I follow Lang's book **[18**, Chapter VI, §7**]** rather closely.

The connected component of $\mathrm{PGL}_2(\boldsymbol{R})$ is $\mathrm{PSL}_2(\boldsymbol{R})$. Now the maximal compact of $\mathrm{SL}_2(\boldsymbol{R})$ corresponding to the one of $\mathrm{GL}_2(\boldsymbol{R})$ at hand is $SO(2)$; let $\varepsilon\colon SO(2) \to \boldsymbol{C}^\times$ be the fundamental character

$$\varepsilon\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} = \cos\theta + \sqrt{-1} \cdot \sin\theta.$$

The single discrete series representation π_0 of $\mathrm{PGL}_2(\boldsymbol{R})$ which is cohomologically nontrivial has the property that its restriction to $SO(2)$ is a direct sum $\bigoplus \varepsilon^n$ ($|n| \geqq 2$, n even). No other discrete series representation has ε^2 in its $SO(2)$-decomposition.

I recall that the Hecke algebra of all compactly supported bi-$SO(2)$-finite distributions acts on the space of any admissible representation of $\mathrm{SL}_2(\boldsymbol{R})$. If D satisfies $L_{k_1} R_{k_2} D = \varepsilon^m(k_1)\varepsilon^{-n}(k_2) D$ for all $k_1, k_2 \in K$, then for any such representation (π, V) the operator $\pi(D)$ takes all of V into the ε^m-eigenspace and annihilates all but the ε^n-eigenspace. In the special case $m = n$ and D amounts simply to integration against ε^{-n} on K, $\pi(D)$ is the identity on the ε^n-eigenspace. Therefore, by choosing $f \in C^\infty(\mathrm{SL}_2(\boldsymbol{R}))$ such that

$$f(k_1 g k_2) = \varepsilon^{-n}(k_1)\varepsilon^{-n}(k_2)f(g)$$

and approximating this D closely enough—i.e., choosing f positive, normalized, and with support close enough to K—one may assume that $\pi(f)$ is also the identity on this eigenspace.

Apply this to the case $m = n = 2$, $\pi = \pi_0$ to obtain $f_1 \in C_c^\infty(\mathrm{SL}_2(\boldsymbol{R}))$ such that (a) $f_1(k_1 g k_2) = \varepsilon^{-2}(k_1)\varepsilon^{-2}(k_2)f(g)$ for all $k_1, k_2 \in SO(2)$, $g \in \mathrm{SL}_2(\boldsymbol{R})$; (b) $\pi_0(f_1)$ is the identity on the ε^2-eigenspace and 0 on any other $SO(2)$-eigenspace. Because of the remark above about other discrete series representations of $\mathrm{PGL}_2(\boldsymbol{R})$ and ε^2, one even has

$$\text{(3.5)} \qquad \begin{aligned} \operatorname{trace} \pi(f_1) = 1, &\qquad \pi = \pi_0, \\ 0, &\qquad \pi \text{ is any discrete series representation of} \\ &\qquad \mathrm{PGL}_2(\boldsymbol{R}) \text{ other than the } \pi_0. \end{aligned}$$

At this point recall what Lang calls the Harish transform of an $f \in C_c^\infty(\mathrm{SL}_2(\boldsymbol{R}))$:

$$H_f(a) = |x - x^{-1}| \int_{A\backslash G} f(g^{-1}ag)\, dg \quad \text{for } a = \begin{pmatrix} x & \\ & x^{-1} \end{pmatrix},\ x \neq \pm 1.$$

Here A is the group of all diagonal matrices. This function H_f turns out to lie in $C_c^\infty(A)$ and even in $C_c^\infty(A)^W$, where W is the Weyl group. In fact, $f \to H_f$ induces an isomorphism of the space of functions $f \in C_c^\infty(\mathrm{SL}_2(\boldsymbol{R}))$ bi-invariant under $SO(2)$ with $C_c^\infty(A)^W$ [**18**, V, §2]. Thus, one may choose $f_2 \in C_c^\infty(\mathrm{SL}_2(\boldsymbol{R})SO(2))$ with $H_{f_1} = H_{f_2}$. Set $f = f_1 - f_2$, so $H_f = 0$.

Since a discrete series representation possesses no vectors $\neq 0$ fixed by $SO(2)$, equation (3.5) holds for f instead of f_1. Furthermore, if χ is any character of A and π_χ is the corresponding principal series representation of $\mathrm{SL}_2(\boldsymbol{R})$ then

$$\text{(3.7)} \qquad \operatorname{trace} \pi_\chi(f) = \int_A H_f(a)\chi(a)\, da = 0$$

[**18**, VII, §3]. Since any principal series representation of $\mathrm{PGL}_2(\boldsymbol{R})$ restricts to some π_χ on $\mathrm{PSL}_2(\boldsymbol{R})$, the same holds for all principal series representations of $\mathrm{PGL}_2(\boldsymbol{R})$.

Finally, if π is any irreducible finite-dimensional representation of $\mathrm{PGL}_2(\boldsymbol{R})$ then because its character and that of some unique discrete series representation add up to a principal series representation,

$$\text{(3.8)} \qquad \begin{aligned} \operatorname{trace} \pi(f) &= -1 \qquad \pi = \boldsymbol{C} \text{ or } \operatorname{sgn}(\det), \\ &= 0 \qquad \text{other finite-dimensional } \pi. \end{aligned}$$

Define now $f_{\boldsymbol{R}}$ on $\mathrm{PGL}_2(\boldsymbol{R})$ to be $-f$ on $\mathrm{PSL}_2(\boldsymbol{R})$ and 0 on the other connected component. Equations (3.5), (3.7), and (3.8) yield (3.4) for $f = f_{\boldsymbol{R}}$.

3.3. To recapitulate: the function $f = f_{\boldsymbol{R}} \cdot f^p \cdot f_p^{(m)} \in C^\infty(G(A)/Z_K)$ has the property that

$$\operatorname{trace} R_f|\mathscr{A}_0 = \sum_{\pi \text{ in } \mathscr{A}_0} m(\pi, K)p^{mn/2} \operatorname{trace} \rho(g(\pi_p)^m).$$

In §5 the trace will also be expressed by means of the Selberg trace formula. I will recall it here in a rather general form for purposes of contrast a little later: Let

$\mathscr{G}$ be any locally compact group, Γ a discrete subgroup with $\Gamma\backslash\mathscr{G}$ compact. The conjugacy class of any $\gamma \in \Gamma$ will be closed in $\mathscr{G}$ (a nice exercise) and so for any $F \in C_c^\infty(\mathscr{G})$ *the orbital integral* $\int_{\mathscr{G}_\gamma\backslash\mathscr{G}} F(g^{-1}\gamma g)\, dg$ is defined and finite, where $\mathscr{G}_\gamma$ is the centralizer of γ in $\mathscr{G}$. If $\Gamma_\gamma = \mathscr{G}_\gamma \cap \Gamma$, then clearly $\Gamma_\gamma\backslash\mathscr{G}_\gamma$ is compact also. *Under some mild assumption the operator* R_F *is of trace class on* $L^2(\Gamma\backslash\mathscr{G})$ and

$$\text{trace } R_F = \sum_{\{\gamma\}} \text{meas}(\Gamma_\gamma\backslash\mathscr{G}_\gamma) \int_{\mathscr{G}_\gamma\backslash\mathscr{G}} F(g^{-1}\gamma g)\, dg$$

where the sum is over all conjugacy classes in Γ. This will be applied in §5 to the case $\mathscr{G} = G(\boldsymbol{A})/Z_K$, $\Gamma = G(\boldsymbol{Q})/G(\boldsymbol{Q}) \cap Z_K$, F is the f just constructed.

4. The algebraic trace. In this and the next section I assume, as mentioned earlier, that p splits completely in F.

4.1. There exists a formula for $\#_K S(\boldsymbol{F}_{p^m})$ remarkably similar to that which the Selberg trace formula yields for the right-hand side of equation (2.1).

The set ${}_KS(\bar{\boldsymbol{F}}_p)$ is partitioned naturally by the equivalence relation of isogeny—i.e., two points lie in the same class when the abelian varieties-plus-structure that they parametrize are isogenous. The isogeny classes are of two types: (1) Those associated to systems $(F', \{\mathfrak{q}_i\})$, where F' is a totally imaginary quadratic extension of F which splits at some prime over p and the $\mathfrak{q}_i$ are certain ideals of F'—if $\mathfrak{p}_1, \cdots, \mathfrak{p}_r$ are the primes of F which split in F' then $\mathfrak{q}_i$ is one of the two prime factors of $\mathfrak{p}_i$. Two systems $(F_1', \{\mathfrak{q}_{1,i}\})$ and $(F_2', \{\mathfrak{q}_{2,i}\})$ determine the same isogeny class if and only if $F_1' = F_2'$ and $\{\mathfrak{q}_{1,i}\}$, $\{\mathfrak{q}_{2,i}\}$ are either equal or conjugate. (2) A single isogeny class associated to the quaternion division algebra D/F which is ramified precisely (a) where B/F is, (b) at all primes of F over p, and (c) at all real valuations of F. This latter case may be thought of as an amalgamation of all the quadratic imaginary F'/F which do not split over p.

If $Y = (F', \{\mathfrak{q}_i\})$ or D, the class of points in ${}_KS(\bar{\boldsymbol{F}}_p)$ associated to it will be denoted ${}_KS(Y)$. It is stable under the Frobenius; one may describe rather explicitly its structure together with this action. (For the parametrization of isogeny classes as well as the structure of each class, refer to Milne's lectures.) To each Y is associated: (1) A certain algebraic group $H = H_Y$ defined over $\boldsymbol{Q}$. If $Y = (F', \{\mathfrak{q}_i\})$ then H is (so to speak) the multiplicative group of F', while if $Y = D$ it is the multiplicative group of D. (2) A certain algebraic group $\bar{G}_Y = \bar{G}$ defined over $\boldsymbol{Q}_p$. First of all, for each prime $\mathfrak{p}$ of F over p one has

$\bar{G}_{\mathfrak{p}}$ = the multiplicative group of $F_{\mathfrak{p}}' \cong (\boldsymbol{F}_{\mathfrak{p}}^\times)^2$
if $Y = (F', \{\mathfrak{q}_i\})$ and $\mathfrak{p}$ splits in F',

= the multiplicative group of $D_{\mathfrak{p}}$ otherwise—
i.e., if $Y = (F', \{\mathfrak{q}_i\})$ and $\mathfrak{p}$ does not split in F' or if $Y = D$.

The group $\bar{G}$ is $\prod \bar{G}_{\mathfrak{p}}$. (3) A coset space $X = \prod X_{\mathfrak{p}}$, where $X_{\mathfrak{p}} = \bar{G}_{\mathfrak{p}}(F_{\mathfrak{p}})/\bar{G}_{\mathfrak{p}}(\mathfrak{o}_{\mathfrak{p}})$. On each $X_{\mathfrak{p}}$ define the transformation

$\Phi_{\mathfrak{p}}$ = multiplication by a uniformizing elements of $\mathfrak{q}$
if $Y = (F', \{\mathfrak{g}_i\})$, $\mathfrak{p}$ splits in F', and $\mathfrak{q}$ is the $\mathfrak{q}_i$ over $\mathfrak{p}$,

= multiplication by a uniformizing element of the prime ideal of $D_{\mathfrak{p}}$
otherwise.

Expressing $\bar{G}_{\mathfrak{p}}$ as $(F_{\mathfrak{p}}^{\times})^2$ in the first case, with the first factor corresponding to $\mathfrak{p}, \Phi_{\mathfrak{p}}$ is a little more explicitly multiplication by $(p, 1)$. Let $X = X_Y = \prod X_{\mathfrak{p}}$, $\Phi = \Phi_Y = \prod \Phi_{\mathfrak{p}}$.

Let $\bar{H}(\boldsymbol{Q}) = H(\boldsymbol{Q})/H(\boldsymbol{Q}) \cap Z_K$, $\bar{G}(\boldsymbol{A}_f)$ (an abuse of language) $= G(\boldsymbol{A}_f^p) \times \bar{G}(\boldsymbol{Q}_p)/Z_K \cap \boldsymbol{A}_f^{\times}$.

In every case, observe:

4.1.1. LEMMA. (a) *The algebraic group Z is canonically embedded in H;*

(b) *There exists a canonical class of equivalent embeddings of $H(\boldsymbol{Q})$ into $G(\boldsymbol{A}_f^p) \times \bar{G}(\boldsymbol{Q}_p)$, rendering $\bar{H}(\boldsymbol{Q})$ as a discrete subgroup of $\bar{G}(\boldsymbol{A}_f)$.*

Only the second needs explaining. There clearly exist natural embeddings of $H(\boldsymbol{Q})$ into $G(\boldsymbol{A}_f^p)$ and into $\bar{G}(\boldsymbol{Q}_p)$, all equivalent up to inner automorphism. The group $H(\boldsymbol{R})/Z(\boldsymbol{R})$ is compact in all cases (no accident) so that since $H(\boldsymbol{Q})$ is discrete in $H(\boldsymbol{A}) = H(\boldsymbol{A}_f) \times H(\boldsymbol{R})$ the group $H(\boldsymbol{Q})$ is discrete in $H(\boldsymbol{A}_f)/Z_K \cap \boldsymbol{A}_f^{\times}$, hence in $\bar{G}(\boldsymbol{A}_f)$.

Langlands' main result on the structure of ${}_K S(Y)$ is that *there is a canonical class of bijections between its points and the double coset space*

$$H(\boldsymbol{Q})\backslash G(\boldsymbol{A}_f^p) \times \bar{G}(\boldsymbol{Q}_p)/K^p \times \bar{G}(\boldsymbol{Z}_p) \cong \bar{H}(\boldsymbol{Q})\backslash \bar{G}(\boldsymbol{A}_f)/K^p \times \bar{G}(\boldsymbol{Z}_p)$$
$$\cong H(\boldsymbol{Q})\backslash G(\boldsymbol{A}_f^p) \times X/K^p.$$

The Frobenius on ${}_K S(Y)$ corresponds to Φ acting on this coset space through its action on X.

4.2. The number of points of ${}_K S(Y)$ rational over $\boldsymbol{F}_{p^m}$ is of course the number of fixed points of the mth power of the Frobenius. To express this conveniently requires a digression of some generality.

Suppose $\mathscr{G}$ to be any locally profinite group, Γ a discrete subgroup. Then $\mathscr{G}$ and hence also $C_c^\infty(\mathscr{G})$ acts on the space $C^\infty(\Gamma\backslash\mathscr{G})$ through the right regular representation. Explicitly, for every $F \in C_c^\infty(\mathscr{G})$, $f \in C^\infty(\Gamma\backslash\mathscr{G})$,

$$R_F f(x) = \int_{\mathscr{G}} F(g) f(xg)\, dg = \int_{\mathscr{G}} F(x^{-1}y) f(y)\, dy$$
$$= \int_{\Gamma\backslash\mathscr{G}} K_F(x, y) f(y)\, dy$$

where $K_F(x, y) = \sum_\Gamma F(x^{-1}\gamma y)$. The operator R_F (or, sometimes, F itself) will be said to possess a *formal trace* if $K_F(x, x)$ has compact support on $\Gamma\backslash\mathscr{G}$, and this trace is then defined to be $\int_{\Gamma\backslash\mathscr{G}} K_F(x, x)\, dx$. One can presumably relate this to other notions of trace, but all that is important here is that the number of fixed points of a power of the Frobenius can be expressed as such a trace and that one has a formula for it.

For each $\gamma \in \Gamma$, let $\mathscr{G}_\gamma$ be the centralizer of γ in $\mathscr{G}$ and $\Gamma_\gamma = \Gamma \cap \mathscr{G}_\gamma$. If F lies in $C_c^\infty(\mathscr{G})$, then its orbital integral $\int_{\mathscr{G}_\gamma\backslash\mathscr{G}} F(g^{-1}\gamma g)\, dg$ is defined in many different circumstances, but the most elementary hypothesis that guarantees this is that $F(g^{-1}\gamma g)$ has compact support on $\mathscr{G}_\gamma\backslash\mathscr{G}$; this in turn is assured by the assumption that the conjugacy class of γ is closed in $\mathscr{G}$.

4.2.1. PROPOSITION. *Let F be in $C_c^\infty(\mathscr{G})$. Suppose that the $\gamma \in \Gamma$ such that $F(g^{-1}\gamma g) \neq 0$ for some $g \in \mathscr{G}$ lie in only a finite number of conjugacy classes in Γ, and that for* *each such γ the space $\Gamma_\gamma\backslash\mathscr{G}_\gamma$ is compact and the conjugacy class of γ in $\mathscr{G}$ is closed. Then F has a formal trace which is given by the formula*

$$\sum_{\{\gamma\}} \operatorname{meas}(\Gamma_\gamma\backslash\mathscr{G}_\gamma) \int_{\mathscr{G}_\gamma\backslash\mathscr{G}} F(g^{-1}\gamma g)\, dg,$$

where the sum is over all conjugacy classes on Γ.

PROOF. Choose a compact open subgroup $\mathscr{G}_0 \subseteq \mathscr{G}$ small enough so that $\Gamma \cap \mathscr{G}_0 = \{1\}$ and F is bi-invariant under $\mathscr{G}_0$. Thus Γ acts freely on $\mathscr{G}/\mathscr{G}_0$. Let $\mathscr{X}$ be a set of representatives, so that $\mathscr{G}$ is the disjoint union of the $\Gamma x\, \mathscr{G}_0$ $(x \in \mathscr{X})$.

By one assumption on F, for each $x \in \mathscr{X}$ the sum $\sum_\Gamma F(x^{-1}\gamma x)$ may be restricted to a finite number of conjugacy classes in Γ, hence may be written as $\sum_{\{\gamma\}} \sum_{\Gamma_\gamma\backslash\Gamma} F(x^{-1}\delta^{-1}\gamma\delta x)$ where the first sum is over representatives of conjugacy classes and only a finite number need be taken into account. Because $\Gamma_\gamma\backslash\mathscr{G}_\gamma$ is compact and the conjugacy class of γ is closed in $\mathscr{G}$, the function $F(g^{-1}\gamma g)$ lies in $C_c^\infty(\Gamma_\gamma\backslash\mathscr{G})$; this implies that the above sum is nontrivial for only a finite subset of $\mathscr{X}$. Thus F does possess a formal trace, which is given by

$$(\operatorname{meas} \mathscr{G}_0) \cdot \sum_{x\in\mathscr{X}} \sum_{\{\gamma\}} \sum_{\Gamma_\gamma\backslash\Gamma} F(x^{-1}\delta^{-1}\gamma\delta x)$$

where in fact each sum may be restricted to a finite but arbitrarily large subset. Rearranging this it therefore becomes

$$\sum_{\{\gamma\}} \int_{\Gamma_\gamma\backslash\mathscr{G}} F(x^{-1}\gamma x)\, ds = \sum_{\{\gamma\}} \operatorname{meas}(\Gamma_\gamma\backslash\mathscr{G}_\gamma) \int_{\mathscr{G}_\gamma\backslash\mathscr{G}} F(x^{-1}\gamma x)\, dx.$$

4.3. Now take Γ to be $\bar{H}(\boldsymbol{Q})$, $\mathscr{G}$ to be $\bar{G}(\boldsymbol{A}_f)$—both corresponding to some isogeny class Y. Define

$$\begin{aligned} F^p &= (\operatorname{meas} K^p)^{-1} \operatorname{char} K^p, \\ F_{\mathfrak{p}}^{(m)} &= (\operatorname{meas} \bar{G}_{\mathfrak{p}}(\mathfrak{o}_{\mathfrak{p}}))^{-1} \operatorname{char}(\Phi_{\mathfrak{p}}{}^m \cdot \bar{G}(\mathfrak{o}_{\mathfrak{p}})), \\ F_p^{(m)} &= \prod F_{\mathfrak{p}}^{(m)}, \\ F^{(m)} &= F^p \cdot F_p^{(m)}. \end{aligned}$$

4.3.1. LEMMA. *The hypotheses of* 4.2.1 *are satisfied for this choice of Γ, $\mathscr{G}$, $F = F^{(m)}$ with $m \geqq 1$.*

When $Y = D$, this is clearly so since $\Gamma\backslash\mathscr{G}$ is actually compact. Thus let $Y = (F', \{\mathfrak{q}_i\})$. The only serious hypothesis to verify is the finiteness of the number of conjugacy classes with elements γ such that $F(g^{-1}\gamma g) \neq 0$ for some $g \in \mathscr{G}$. Now in this case $H(\boldsymbol{Q}) = (F')^\times$ and $\bar{G}(\boldsymbol{Q}_p) = \prod \bar{G}_{\mathfrak{p}}(F_{\mathfrak{p}})$ where $G_{\mathfrak{p}}$ is (a) $(F'_{\mathfrak{p}})^\times$ or (b) $D_{\mathfrak{p}}^\times$. In either case $F_p^{(m)}(g^{-1}\gamma g) = F(\gamma)$, and $F_p^{(m)}(\gamma) \neq 0$ if and only if (a) at each $\mathfrak{p}_i$ where F' splits γ has order m at $\mathfrak{q}_i$ and is a unit at $\bar{\mathfrak{q}}_i$; (b) where F' does not split γ is a generator of $\mathfrak{p}_D^m$. Furthermore, in order that $F^p(g^{-1}\gamma g) \neq 0$ for some $g \in G(\boldsymbol{A})$ it is necessary and sufficient that γ be a unit at all primes not dividing p. Thus if $F^{(m)}(g^{-1}\gamma g) \neq 0$ for some g, the order of γ is specified at all primes of F'. Any two such γ differ by a unit. But the units of F', modulo Z_K, are a finite set.

The function $F^{(m)}$ is important because of:

4.3.2. PROPOSITION. *The formal trace of $F^{(m)}$ is the number of fixed points of the mth power of the Frobenius on ${}_KS(Y)$.*

I leave this as an easy exercise.

5. The comparison.

5.1. From §3 one sees that the right-hand side of equation (2.1) is

$$\sum_{\{\gamma\}} \operatorname{meas}(\Gamma_\gamma \backslash \mathscr{G}_\gamma) \int_{\mathscr{G}_\gamma \backslash \mathscr{G}} F(g^{-1}\gamma g)\, dg \tag{5.1}$$

where $\Gamma = G(\boldsymbol{Q})/G(\boldsymbol{Q}) \cap Z_K$, $\mathscr{G} = G(\boldsymbol{A})/Z_K$, $F = f_{\boldsymbol{R}} \cdot f^p \cdot f_p^{(m)}$.

Let $\varphi\colon G(\boldsymbol{A}) \to \mathscr{G}$ be the canonical projection. It is easy to see that $\varphi(G_\gamma(\boldsymbol{A}))\backslash \mathscr{G}_\gamma$ is always compact, so that one may replace $\mathscr{G}_\gamma$ by $\varphi(G_\gamma(\boldsymbol{A}))$ in the above formula. But then one may note that $\varphi(G_\gamma(\boldsymbol{A}))\backslash \mathscr{G} \cong G_\gamma(\boldsymbol{A})\backslash G(\boldsymbol{A})$. Using the factorization of $G(\boldsymbol{A})$ each term may be written as the product of

(5.2) (a) $$\operatorname{meas}(G_\gamma(\boldsymbol{Q})Z_K \backslash G_\gamma(\boldsymbol{A})),$$

(5.2) (b) $$\int_{G_\gamma(\boldsymbol{A}_f^p)\backslash G(\boldsymbol{A}_f^p)} f^p(g^{-1}\gamma^p g)\, dg,$$

(5.2) (c) $$\int_{G_\gamma(\boldsymbol{Q}_p)\backslash G(\boldsymbol{Q}_p)} f_p^{(m)}(g^{-1}\gamma_p g)\, dg,$$

(5.2) (d) $$\int_{G_\gamma(\boldsymbol{R})\backslash G(\boldsymbol{R})} f_{\boldsymbol{R}}(g^{-1}\gamma_{\boldsymbol{R}} g)\, dg.$$

From §4 one sees that $\#_K S(\boldsymbol{F}_{p^m})$ may be expressed as

$$\sum_Y \sum_{\{\gamma\}} \operatorname{meas}\,(\Gamma_\gamma \backslash \mathscr{G}_\gamma) \int_{\mathscr{G}_\gamma \backslash \mathscr{G}} F^{(m)}(g^{-1}\gamma g)\, dg \tag{5.3}$$

where:

Y ranges over all systems $(F', \{\mathfrak{q}_i\})$, identifying a system and its conjugate, and the single D;

γ ranges over all conjugacy classes of $H_Y(\boldsymbol{Q})$;

$\mathscr{G} = \bar{G}_Y(\boldsymbol{A}_f) = G(\boldsymbol{A}_f^p) \times \bar{G}_Y(\boldsymbol{Q}_p)$;

$F^{(m)} = F^p \cdot F_p^{(m)}$

as at the end of §4. Just as above, each term here factors as the product of

(5.4)(a) $$\operatorname{meas}(H_\gamma(\boldsymbol{Q})(Z_K \cap \boldsymbol{A}_f^\times)\backslash G_\gamma(\boldsymbol{A}_f^p)\bar{G}_\gamma(\boldsymbol{Q}_p)),$$

(5.4)(b) $$\int_{G_\gamma(\boldsymbol{A}_f^p)\backslash G(\boldsymbol{A}_f^p)} F^p(g^{-1}\gamma^p g)\, dg,$$

(5.4)(c) $$\int_{\bar{G}_\gamma(\boldsymbol{Q}_p)\backslash \bar{G}(\boldsymbol{Q}_p)} F_p^{(m)}(g^{-1}\gamma_p g)\, dg.$$

The proof of the main theorem reduces to a comparison of measures and orbital integrals.

5.2. Even the comparison of measures is not trivial. For the moment let k be an arbitrary local field, ψ an additive character of k. If H is any finite-dimensional semisimple algebra over k, Tr its reduced trace, then $\psi \circ \mathrm{Tr}$ is an additive character of H. There exists on H a unique measure dx self-dual under the Fourier transform determined by this character, and this in turn gives rise to an invariant measure $d^\times x = dx/|x|_H$ on $H^\times$.

If now k is any global field, $\psi = \prod \psi_v$ a character of A_k/k, and H a semisimple algebra over k, one obtains thus measures on all the local groups $H_v^\times$ as well as on $H^\times(A_k)$, which is called the Tamagawa measure determined by ψ (§15 of **[16]**). Apply this construction in turn to the field F itself, quadratic extensions of F, and quaternion algebras over F (including $M_2(F)$). I will always assume such measures chosen from now on. One classical result is that the global measure of $H^\times(A)/A_F^\times \cdot H^\times(F)$ is independent of the quaternion algebra H (§16 of **[16]**, or the lectures **[12]** of Gelbart and Jacquet). I will also need a comparison of local measures:

5.2.1. LEMMA. *Let k be any nonarchimedean local field, H the unique quaternion division algebra over k. Then*

$$\text{meas } \mathfrak{o}_H^\times = (q - 1)^{-1} \text{ meas } \mathrm{GL}_2(\mathfrak{o}_k).$$

PROOF. Changing ψ only multiplies both measures by the same constant; choose ψ to be of conductor $\mathfrak{o}_k$. Let f be the characteristic function of $M_2(\mathfrak{o}_k)$. Then

$$\begin{aligned} \hat{f}(x) &= \int_{M_2(k)} f(y)\psi(\mathrm{Tr}(xy))\, dy \\ &= \text{meas} M_2(\mathfrak{o}_k) \begin{cases} 1, & x \in M_2(\mathfrak{o}_\mathfrak{k}), \\ 0, & x \notin M_2(\mathfrak{o}_\mathfrak{k}), \end{cases} \\ &= (\text{meas } M_2(\mathfrak{o}_\mathfrak{k})) \cdot f(x). \end{aligned}$$

Therefore meas $M_2(\mathfrak{o}_\mathfrak{k}) = 1$. The group $\mathrm{GL}_2(\mathfrak{o}_\mathfrak{k})$ is open in $M_2(\mathfrak{o}_\mathfrak{k})$ and since $x \in M_2(\mathfrak{o}_\mathfrak{k})$ lies in $\mathrm{GL}_2(\mathfrak{o}_\mathfrak{k})$ if and only if x mod $\mathfrak{p}_\mathfrak{k}$ is nonsingular,

$$\begin{aligned} \text{meas } \mathrm{GL}_2(\mathfrak{o}_\mathfrak{k}) &= \# \mathrm{GL}_2(\boldsymbol{F}_q)/\# M_2(\boldsymbol{F}_q) \\ &= (1 - 1/q)^2 (1 + 1/q). \end{aligned}$$

Now let f be the characteristic function of $\mathfrak{o}_H$. Then

$$\begin{aligned} \hat{f}(x) &= \int_{\mathfrak{o}_H} \psi(\mathrm{Tr}(xy))\, dy \\ &= \text{meas } \mathfrak{o}_H \begin{cases} 1, & x \in \mathfrak{p}_H^{-1}, \\ 0, & x \notin \mathfrak{p}_H^{-1}, \end{cases} \end{aligned}$$

or

$$\hat{f} = (\text{meas } \mathfrak{o}_H)(\text{characteristic function of } \mathfrak{p}_H^{-1}).$$

The Fourier transform of this in turn is $\hat{\hat{f}} = (\text{meas } \mathfrak{o}_H)(\text{meas}(\mathfrak{p}_H^{-1}))f$. Since $\text{meas}(\mathfrak{p}_H^{-1}) = q^2 \cdot \text{meas } \mathfrak{o}_H$, meas $\mathfrak{o}_H = q^{-1}$. Reasoning as above,

$$\text{meas } \mathfrak{o}_H^\times = (1 - 1/q)(1 + 1/q)(1/q).$$

The lemma follows.

One can similarly compare measures on $\mathrm{GL}_2(\boldsymbol{R})$ and $\boldsymbol{H}^\times$, but that will prove to be unnecessary.

5.2. Again for a while let k be any local field of characteristic 0, $\mathscr{G}$ either $\mathrm{GL}_2(k)$, the multiplicative group $H^\times(k)$ of the quaternion division algebra over k, $\mathrm{PGL}_2(k)$, or $H^\times/k^\times$. For any $x \in \mathcal{G}$ define $D(x) = |\alpha - \beta|^2/|\alpha\beta|$ of the eigenvalues of x (over an algebraic closure of k) are α, β. If T is any torus of $\mathcal{G}$, then for every regular $t \in T$ and $f \in C_c^\infty(\mathcal{G})$ define

$$O^T(f, t) = \int_{T\backslash\mathcal{G}} f(g^{-1}tg)\, dg.$$

The function $F(t) = O^T(f, t)$ satisfies

(5.5) (a) F is smooth on the regular elements of T,

(5.5) (b) $D^{1/2} F$ is locally bounded,

(5.5) (c) F has compact support on T.

If π is an irreducible admissible representation of $\mathcal{G}$ then the character ch_π of π also satisfies (5.5) (a)—(b). For any $f \in C_c^\infty(\mathcal{G})$,

$$\int f(g)\, dg = \frac{1}{2} \sum_{\{T\}} \int_T D(t) O^T(f, t)\, dt,$$

where the sum is over all conjugacy classes of tori in $\mathcal{G}$, and if π is an irreducible admissible representation then

(5.6) $$\operatorname{trace} \pi(f) = \frac{1}{2} \sum_{\{T\}} \int_T D(t)\, \mathrm{ch}_\pi(t) O^T(f, t)\, dt.$$

(For all this see §7 of **[16]**, and for more about orbital integrals see **[12]** and **[26]**.)

5.2.1. Lemma. *Suppose that $\mathcal{G} = H^\times$, F a conjugation-invariant function on $\mathcal{G}^{\mathrm{reg}}$ whose restriction to any torus T satisfies* (5.5) (a)—(c). *If*

$$\frac{1}{2} \sum_{\{T\}} \int_T D(t)\, \mathrm{ch}_\sigma(t) F(t)\, dt = 0$$

for all irreducible admissible representations of $\mathcal{G}$ then $F \equiv 0$.

Fix a torus T. The set of elements of $\mathcal{G}$ conjugate to regular elements of T is open. If $f \in C_c^\infty(T^{\mathrm{reg}})$, there exists $h \in C_c^\infty(\mathcal{G}^{\mathrm{reg}})$ with support in this set and such that $O^T(h) = f$. Then

$$\int_{\mathcal{G}} h(g) F(g)\, dg = \int_T D(t) O^T(h, t) F(t)\, dt$$
$$= \int_T D(t) f(t) F(t)\, dt.$$

But the characters of irreducible representations are complete in the space of central functions on $\mathcal{G}$, so that $f = \sum \hat{f}_\sigma \cdot \mathrm{ch}_\sigma$ (if k is nonarchimedean one can reduce this question to one about finite groups) and hence by hypothesis the above integral is 0. In other words, the integral of $D \cdot F$ against any $f \in C_c^\infty(T^{\mathrm{reg}})$ is 0, which implies that F itself is 0.

One may (and I will) identify conjugacy classes in $H^\times$ with elliptic and central conjugacy classes in $\mathrm{GL}_2(k)$. There exists (by §15 of **[16]**; see also **[12]**) a bijective correspondence $\sigma \leftrightarrow \pi$ between (a) irreducible, smooth (hence finite-dimensional) representations σ of $H^\times$ and (b) irreducible, admissible representations π of $\mathrm{GL}_2(k)$, space-integrable modulo Z, such that

(5.7) $$\mathrm{ch}_\sigma(t) = -\mathrm{ch}_\pi(t)$$

for all regular elliptic elements t. Representations of $H^\times/k^\times$ correspond to representations of $\mathrm{PGL}_2(k)$. When $k = \boldsymbol{R}$, the trivial representation of $H^\times$ corresponds to the single discrete series representation π_0 of PGL_2. For nonarchimedean k, the one-dimensional representations of $H^\times$ correspond to the Steinberg (or special) representations and all others correspond to absolutely cuspidal representations of GL_2.

5.2.2. PROPOSITION. *Let $f_{\boldsymbol{R}} \in C_c^\infty(\mathrm{PGL}_2(\boldsymbol{R}))$ be as in* §3. *Then for semisimple $x \in \mathrm{PGL}_2(\boldsymbol{R}) = \mathscr{G}$,*

$$\begin{aligned}\int_{\mathscr{G}_T\backslash\mathscr{G}} (g^{-1}xg)\, dg &= 0, && x \text{ hyperbolic},\\ &= (\mathrm{meas}\, \boldsymbol{C}^\times/\boldsymbol{R}^\times)^{-1}, && x \text{ elliptic},\\ &= -(\mathrm{meas}\, \boldsymbol{H}^\times/\boldsymbol{R}^\times)^{-1}, && x = 1.\end{aligned}$$

PROOF. The case of hyperbolic x fell out in the very construction of $f_{\boldsymbol{R}}$.
Recall that $f_{\boldsymbol{R}}$ has the property

$$\begin{aligned}\mathrm{trace}\, \pi(f_{\boldsymbol{R}}) &= 1, && \pi = 1 \text{ or } \mathrm{sgn}(\det),\\ &= -1, && \pi = \pi_0,\\ &= 0, && \text{otherwise}.\end{aligned}$$

Consider now the constant function $h_{\boldsymbol{R}}$ on $\boldsymbol{H}^\times/\boldsymbol{R}^\times$ with the value $(\mathrm{meas}\, \boldsymbol{H}^\times/\boldsymbol{R}^\times)^{-1}$. It satisfies

$$\begin{aligned}\mathrm{trace}\, \sigma(h_{\boldsymbol{R}}) &= 1, && \sigma = 1,\\ &= 0, && \text{otherwise}.\end{aligned}$$

Thus by (5.7) and remarks just afterwards trace $\pi(f_{\boldsymbol{R}}) = -\mathrm{trace}\, \sigma(h_{\boldsymbol{R}})$ whenever π and σ correspond to one another. This and equation (5.6) now yield

$$\int_{\boldsymbol{C}^\times/\boldsymbol{R}^\times} D(t)\, \mathrm{ch}_\pi(t)\, dt \int_{\boldsymbol{C}^\times\backslash\mathrm{PGL}_2(\boldsymbol{R})} f_{\boldsymbol{R}}(g^{-1}tg)\, dg$$
$$= -\int_{\boldsymbol{C}^\times/\boldsymbol{R}^\times} D(t)\mathrm{ch}_\sigma(t)\, dt \int_{\boldsymbol{C}^\times\backslash\boldsymbol{H}^\times} h_{\boldsymbol{R}}(g^{-1}tg)\, dt$$

whenever π and σ correspond. By (5.7) and 5.2.1 (or in this case just Fourier inversion on $\boldsymbol{C}^\times/\boldsymbol{R}^\times$),

$$\int_{\boldsymbol{C}^\times\backslash\mathrm{PGL}_2(\boldsymbol{R})} f_{\boldsymbol{R}}(g^{-1}tg)\, dg = \int_{\boldsymbol{C}^\times\backslash\boldsymbol{H}^\times} h_{\boldsymbol{R}}(g^{-1}tg)\, dg = (\mathrm{meas}\, \boldsymbol{C}^\times/\boldsymbol{R}^\times)^{-1}$$

for all $t \neq \pm 1 \in \boldsymbol{C}^\times/\boldsymbol{R}^\times$.

Since the orbital integrals of $f_{\boldsymbol{R}}$ vanish on hyperbolic elements, $f_{\boldsymbol{R}}$ and $-h_{\boldsymbol{R}}$ correspond to one another in the sense of **[12]** (the section on "matching orbital integrals"). Hence

$$f_{\boldsymbol{R}}(1) = -h_{\boldsymbol{R}}(1) = -(\mathrm{meas}\, \boldsymbol{H}^\times/\boldsymbol{R}^\times)^{-1}.$$

5.3. Now let k be nonarchimedean and local, and for the moment let $G = \mathrm{GL}_2(k)$,

$K = \mathrm{GL}_2(\mathfrak{o}_k)$. For each $m \geqq 0$ let $f^{(m)}$ be the unique function in the Hecke algebra $\mathscr{H}(G, K)$ defined according to the Satake isomorphism by trace $\pi(f^{(m)}) = q^{m/2}(\alpha^m + \beta^m)$ whenever $\pi = \pi(\alpha, \beta)$ is the unramified principal series representation associated to the characters $x \mapsto (\alpha^{\mathrm{ord}(x)}, \beta^{\mathrm{ord}(x)})$ of $k^\times$.

One can exhibit $f^{(m)}$ rather explicitly. For every $i, j \in \boldsymbol{Z}$ recall the Hecke operator

$$T(\mathfrak{p}^i, \mathfrak{p}^j) = (\text{meas}\, K)^{-1} \,\text{char}\, K\begin{pmatrix} \eta^i & \\ & \eta^j \end{pmatrix} K$$

where η is a generator of $\mathfrak{p}$. For $m \geqq 0$ recall also $T(\mathfrak{p}^m) = \sum T(\mathfrak{p}^i, \mathfrak{p}^j)$ $(i, j \geqq 0,$ $i + j = m)$. There are the more or less classical Hecke operators and satisfy the relation $T(\mathfrak{p})T(\mathfrak{p}^m) = T(\mathfrak{p}^{m+1}) + qT(\mathfrak{p}, \mathfrak{p})T(\mathfrak{p}^{n-1})$ which may be written formally as

$$(5.7) \qquad (I - T(\mathfrak{p})X + qT(\mathfrak{p}, \mathfrak{p})X^2)^{-1} = \sum T(\mathfrak{p}^m)X^m.$$

5.3.1. Lemma (Ihara). *One has*

$$f^{(0)} = 2 \cdot I, \quad f^{(1)} = T(\mathfrak{p}), \quad f^{(m)} = T(\mathfrak{p}^m) - qT(\mathfrak{p}, \mathfrak{p})T(\mathfrak{p}^{m-2}) \qquad (m \geqq 2).$$

Proof. The case $m = 0$ is clear. And it is well known that trace $\pi(T(\mathfrak{p})) = q^{1/2}(\alpha + \beta)$ if $\pi = \pi(\alpha, \beta)$, which gives the case $m = 1$.

Formally, then $T(\mathfrak{p}) \leftrightarrow q^{1/2}(\alpha + \beta)$ and, also elementary, $T(\mathfrak{p}, \mathfrak{p}) \leftrightarrow \alpha\beta$. Now

$$(1 - q^{1/2}\alpha X)^{-1} + (1 - q^{1/2}\beta X)^{-1} = \sum_0^\infty q^{m/2}(\alpha^m + \beta^m)X^m \leftrightarrow \sum f^{(m)} X^m$$

but this expression also equals

$$\frac{(1 - q^{1/2}\alpha X) + (1 - q^{1/2}\beta X)}{(1 - q^{1/2}\alpha X)(1 - q^{1/2}\beta X)} \leftrightarrow \frac{2 - T(\mathfrak{p})X}{1 - T(\mathfrak{p})X + q\,T(\mathfrak{p},\mathfrak{p})X^2}.$$

The proposition follows from this and (5.2).

5.3.2. Corollary. *If*

$$x = \begin{pmatrix} \eta^l & \\ & \eta^l \end{pmatrix}$$

then

$$\begin{aligned} f^{(m)}(x) &= -(\text{meas}\, K)^{-1}(q - 1), && m = 2l, \\ &= 0, && \textit{otherwise}. \end{aligned}$$

Continue to let H be the quaternion division algebra over k. From each $m \geqq 0$ let

$$h^{(m)} = (\text{meas}\, \mathfrak{o}_H)^{-1}\, \text{char}(\mathfrak{p}_H^m - \mathfrak{p}_H^{m+1}).$$

5.3.3. Proposition. *Let x be a semisimple element of G, $m \geqq 1$. Then*:
(a) *If x is hyperbolic*,

$$\begin{aligned} \int_{G_x \backslash G} f^{(m)}(g^{-1}xg)\, dg &= 1 \quad \textit{if } x = \begin{pmatrix} \eta^m & \\ & 1 \end{pmatrix} \textit{ or } \begin{pmatrix} 1 & \\ & \eta^m \end{pmatrix} \bmod (K \cap A), \\ &= 0 \quad \textit{otherwise}. \end{aligned}$$

(b) *If x is elliptic*

$$\int_{G_x \backslash G} f^{(m)}(g^{-1}xg)\, dg = \int_{H_x^\times \backslash H^\times} h^{(m)}(g^{-1}xg)\, dg.$$

(c) *If x is central,* $f^{(m)}(x) = -h^{(m)}(x)$.

PROOF. Recall (say from **[4]**) the relationship between the Satake isomorphism and orbital integrals: for $f \in \mathscr{H}(G, K)$

$$\tilde{f}(a) = D(a)^{1/2} \int_{A \backslash G} f(g^{-1}ag)\, dg, \qquad \text{trace}\, \pi_\chi(f) = \int_A \tilde{f}(a)\chi(a)\, da.$$

By definition of $f^{(m)}$, then,

$$\tilde{f}^{(m)}(a) = q^{m/2}, \qquad a = \begin{pmatrix} \eta^m & \\ & 1 \end{pmatrix} \text{ or } \begin{pmatrix} 1 & \\ & \eta^m \end{pmatrix},$$
$$= 0 \qquad \text{otherwise.}$$

Hence (a).

Claim (c) is immediate from 5.3.2 and 5.2.1. Only (b) is difficult. To begin, consider equation (5.6) as π ranges over all essentially square-integrable irreducible representations of GL_2:

$$\text{trace}\, \pi(f^{(m)}) = \frac{1}{2} \int_A D(a)\text{ch}_\pi(a)\, da \int_{A \backslash G} f^{(m)}(g^{-1}ag)\, dg$$
$$+ \frac{1}{2} \sum_{\{T\}} \int_T D(t)\text{ch}_\pi(t)\, dt \int_{T \backslash G} f^{(m)}(g^{-1}tg)\, dg$$

where now T ranges over all anisotropic tori. Similarly consider (5.6) as σ ranges over all irreducible admissible representations of $H^\times$:

$$\text{trace}\, \sigma(h^{(m)}) = \frac{1}{2} \sum_{\{T\}} \int_T D(t)\text{ch}_\sigma(t)\, dt \int_{T \backslash G} h^{(m)}(g^{-1}tg)\, dg.$$

Applying 5.2.1 and equation (5.7), it thus suffices to prove that

$$- \text{trace}\, \pi(f^{(m)}) + \frac{1}{2} \int_A D(a)\text{ch}_\pi(a)\, da \int_{A \backslash G} f^{(m)}(g^{-1}ag)\, dg = \text{trace}\, \sigma(h^{(m)})$$

whenever σ and π correspond. For this: (1) no π occurring here possesses K-fixed vectors $\neq 0$, so trace $\pi(f^{(m)}) = 0$ always. (2) If π is absolutely cuspidal then $\text{ch}_\pi(a) = 0$ unless $a \in A(\mathfrak{o}_k)$ (first observed in Lemma 6.4 of **[19]**; see also **[8]** and **[5]**). According to case (a) already done, $O^A(f) = 0$ on $A(\mathfrak{o})$, so the whole left-hand side vanishes. But in this case σ is not one-dimensional, hence not trivial on $\mathfrak{o}_H^\times$, so that the right-hand side vanishes also. (3) Suppose π is special, corresponding to the character $\sigma = \chi \circ N_M$ of $H^\times$. Thus $\sigma(h^{(m)}) = \chi(\eta^m)$ on the one hand and on the other

$$\frac{1}{2} \cdot \int_A D(a)\text{ch}_\pi(a)\, da \int_{A \backslash G} f^{(m)}(g^{-1}ag)\, dg = \chi(\eta^m)$$

also, by case (a) again.

5.4. *Return to earlier notation.* It will now turn out that there is a bijective correspondence between the nonzero terms of the two sums (5.1) and (5.3), and that corresponding terms agree. This will conclude our proof.

According to 5.2.2 the orbital integral of $f_{\boldsymbol{R}}$ in (5.2) is 0 unless $\gamma_{\boldsymbol{R}}$ is elliptic or central, so I will assume that to be the case from now on.

Suppose first that $\gamma \in G(\boldsymbol{Q})$ is central. The term in (5.1) corresponding to it is the product of

(5.8) (a) $\quad \operatorname{meas}(G(\boldsymbol{Q})Z_K\backslash G(\boldsymbol{A}))$,

(5.8) (b) $\quad f_p^{(m)}(\gamma_p) = (-1)^n (\operatorname{meas} D^\times(\boldsymbol{Z}_p))^{-1}$,

(5.8) (c) $\quad f^p(\gamma^p)$,

(5.8) (d) $\quad f_{\boldsymbol{R}}(\gamma_{\boldsymbol{R}}) = (-1)^n (\operatorname{meas} D^\times(\boldsymbol{R})/\boldsymbol{R}^\times)^{-1}$.

There is exactly one term in (5.7) corresponding to γ, among the terms indexed by $Y = D$. It is the product of

(5.9) (a) $\quad \operatorname{meas}(D^\times(\boldsymbol{Q})(Z_K \cap A_f^\times)\backslash D^\times(A_f))$,

(5.9) (b) $\quad F_p^{(m)}(\gamma_p) = (\operatorname{meas} D^\times(\boldsymbol{Z}_p))^{-1}$,

(5.9) (c) $\quad F^p(\gamma^p)$.

These products match since (5.8)(c) and (5.9)(c) are trivially equal and

$$\begin{aligned}\operatorname{meas}(G(\boldsymbol{Q})Z_K\backslash G(\boldsymbol{A})) &= \operatorname{meas}(D^\times(\boldsymbol{Q})Z_K\backslash D^\times(\boldsymbol{A}))\\ &= \operatorname{meas}(D^\times(\boldsymbol{Q})(Z_K \cap A_f^\times)\backslash D^\times(A_f)) \operatorname{meas}(Z(\boldsymbol{R})\backslash D^\times(\boldsymbol{R})).\end{aligned}$$

Now suppose $\gamma \in G(\boldsymbol{Q})$ such that $\gamma_{\boldsymbol{R}}$ is elliptic. Thus $F' = F(\gamma)$ is an imaginary quadratic extension. If this extension splits at no prime of F over p, let $Y = Y_\gamma = D$. Otherwise, $Y = Y_\gamma$ is one of the $(F', \{\mathfrak{q}_i\})$, and the $\mathfrak{q}_i$ must be specified. Now by the definition of $f_p^{(m)}$ as the product of the $f_{\mathfrak{p}}^{(m)}$ and 5.3.3, if the term for γ in (5.2) (c) does not vanish then $\gamma_{\mathfrak{p}}$ is conjugate in $\mathrm{GL}_2(F_{\mathfrak{p}})$ to

$$\begin{pmatrix}\eta^m & \\ & 1\end{pmatrix}$$

whenever $\mathfrak{p}$ splits in F'. In other words, $\gamma_{\mathfrak{p}}$ must have order m at some prime $\mathfrak{q}_\gamma$ of F' over $\mathfrak{p}$ and be a unit at the other; set $Y_\gamma = (F', \{\mathfrak{q}_\gamma\})$.

In either case the element γ clearly gives rise to an element of $H_{Y_\gamma}(\boldsymbol{Q})$, and it turns out to be an easy consequence of 5.2.2 and 5.3.3 that the corresponding terms in (5.2) and (5.4) agree.

6. Supplementary remarks.

6.1. When $F = \boldsymbol{Q}$, the main theorem follows from a *congruence relation* between Hecke operators and the Frobenius. Although such a relation is likely to hold very generally, it does not yield a formula for the ζ-function. I want to explain this a bit more carefully.

First of all, the integral Hecke operators always define algebraic correspondences on the scheme ${}_KS$. When $F = \boldsymbol{Q}$, the congruence relation says that modulo p $T(p) = \Phi + \Phi^* \cdot T(p, p)$ where Φ is the Frobenius and Φ^* its transpose (refer to §7.4 of **[28]**). An equivalent way of expressing this, as Langlands pointed out to me, is to say that in the ring of algebraic correspondences Φ is a root of the polynomial $X^2 - T(p)X + pT(p, p)$. In general one may consider the polynomial $\det(X - \rho(g))$ as a function of the semisimple element g of ${}^LG$. By the Satake isomorphism it corresponds to a polynomial whose coefficients are Hecke operators.

These will in fact be integral, and it looks not very difficult to show that Φ is a root. In more concrete terms, this will imply immediately that the roots of the Frobenius (acting on l-adic cohomology) lie among the roots of this polynomial, or that factors of $Z_p(X, {}_KS)$ lie among the factors of $\prod_{\pi \text{ in } \mathscr{A}_0} \det(I - \mathfrak{g}(g(\pi_p)))^{m(\pi, K^p)}$. Now when $F = \boldsymbol{Q}$, one obtains by a further relation which I have always found a little mysterious that in fact the roots coincide, but it is this second step that does not seem to generalize (I refer to 7.10(2) of **[28]**; see also Piatetskii-Shapiro's argument on pp. 333–336 of **[24]** for a representation-theoretic analogue). Langlands has remarked that there is in some sense a good reason for this difficulty, inasmuch as when problems of L-indistiguishability arise one may get in some sense only partial coincidence of the roots of the Frobenius with those of the Hecke polynomial.

Incidentally, in his proof of the main theorem in the cases already mentioned, Shimura also used the Selberg trace formula.

I might also mention that it is apparently Ihara who first applied the trace formula to matters of this kind—in **[14]** he showed, modulo some technical problems, that the Hasse-Weil ζ-function associated to certain sheaves on $\mathrm{SL}_2(\boldsymbol{Z})\backslash\mathscr{H}$ is a product of Hecke L-functions. Both he and Shimura used what one might call a global formulation. The first application involving representation theory and local orbital integrals is Langlands' Antwerp talk **[19]**; this earlier work seems to have played a role in Langlands' own further development as well as in Drinfeld's (note the remark in **[9]** on this point).

6.2. It is very little extra work to extend the proof of the main theorem to allow for nontrivial sheaves as well as a refinement which accounts for a factorization of the ζ-function according to the direct sum decomposition of $\mathscr{A}_0$.

Let me first sketch how to deal with sheaves. Suppose $\xi\colon G \to \mathrm{GL}_n$ to be a rational representation of G (i.e., rational over $\boldsymbol{Q}$) trivial on $N^1_{F/\boldsymbol{Q}}$, the kernel of the norm map from $F^\times$ to $\boldsymbol{Q}^\times$ (considered as algebraic groups) which is canonically embedded as scalars in the centre of G. Then one may associate to ξ a locally constant $\boldsymbol{Q}$-sheaf E_ξ on ${}_KS(\boldsymbol{C})$ and l-adic sheaves $E_\xi(\boldsymbol{Q}_l)$ on ${}_KS$ (as in a special case in **[19]**), and consider for each "good" p the p-factor of a ζ-function

$$\xi_p(s, {}_KS, \xi) = \prod_{i=0}^{2n} \det(I - (\Phi | H^i({}_KS, E_\xi(\boldsymbol{Q}_l)))p^{-s})^{(-1)^i}.$$

By the Lefschetz trace formula this satisfies

$$\log \zeta_p = \sum_{m=1}^{\infty} \frac{1}{mp^{ms}} \sum_{x \in {}_KS(\boldsymbol{F}_{p^m})} \operatorname{trace} \xi_x(\Phi^m).$$

In both these formulas Φ is the Frobenius, and for convenience I have written $\xi_x(\Phi^m)$ as the action of Φ^m on the stalk of E_ξ at the fixed point x of Φ^m.

On the other hand one can define numbers $m(\pi_{\boldsymbol{R}}, \xi)$ analogous to the numbers $m(\pi_{\boldsymbol{R}})$ in §1 satisfying, for example, $m(\pi_{\boldsymbol{R}}, \xi) = 0$ unless $\mathrm{Ext}^*_{(\mathfrak{g}, \mathfrak{k})}(\check{\xi}, \pi_{\boldsymbol{R}}) \neq 0$ and set $m(\pi) = m(\pi_{\boldsymbol{R}})m(\pi^p)$. The main theorem extends to:

$$L_p(s, {}_KS, \xi) = \prod_{\pi \text{ in } \mathscr{A}_0} L(s - n/2, \pi_p, \rho)^{m(\pi, \xi)}.$$

The proof is almost exactly the same as for ξ trivial; one uses an extended version of the trace formula for the trace of R_f, $f \in C_c^\infty(\mathscr{G})$, on the representation of $\mathscr{G}$ which is L^2-induced from ξ:

$$\operatorname{trace} R_f = \sum_{\{\gamma\}} \operatorname{meas}(\Gamma_\gamma \backslash \mathscr{G}_\gamma) \operatorname{Tr} \xi(\gamma) \int_{\mathscr{G}_\gamma \backslash \mathscr{G}} f(g^{-1} \gamma g)\, dg.$$

And one also uses a result of Langlands analogous to the one used above which gives not only the structure of ${}_KS(\bar{F}_p)$ but of E_ξ on ${}_KS(\bar{F}_p)$ (the natural guess is correct).

Now the second point. As Langlands shows, essentially, in **[19]**, corresponding to the decomposition $\mathscr{A}_0 = \bigoplus \pi$ one has a decomposition $H^*({}_KS(C), \bar{Q}) = \bigoplus H_\pi^*$. The only π which occur are those with $m(\pi) \neq 0$, and for a $\pi = \pi_R \otimes \pi_f$ which does occur the representation π_f may be defined over $\bar{Q}$. Corresponding in turn to this decomposition is one of $H^*({}_KS, \bar{Q}_l)$, and hence even of $H^*({}_KS \times \bar{F}_p, \bar{Q}_l)$. Another extension of the main theorem is that for good p

$$L_p(s, \pi, \rho) = \prod_{i=0}^{2n} \det(I - (\Phi | H_\pi^i(\bar{Q}))p^{-s})^{(-1)^i}.$$

To prove this: (1) one observes that the subspace $H_\pi^*(\bar{Q})$ is determined in $H^*({}_KS, \bar{Q})$ by the effect of operators in $\mathscr{H}(G(A_f^p), K^p)$ and (2) applies the reasoning given above, but allowing f^p to be an arbitrary element of this Hecke algebra.

It is this result which Piatetskii-Shapiro is concerned with proving (by means of a congruence relation) in **[24]**, with $F = Q$.

6.3. It has not escaped me that the result in §5 on the measures on GL_2 and $H^\times$ says, essentially, that the measures given by Jacquet-Langlands are multiples, by the same constant, of what Serre calls in **[25]** the Euler-Poincaré measures on each group. (This is not quite true since strictly speaking the E-P measures on these are zero: but there is a clear relationship between E-P measures on SL_2 and N_H^1.) Does this observation generalize? For real groups, I would expect a factor $\operatorname{card}(W_{G(C)}/W_K)$ to play a role, as it does in Harder's work **[13]**. Indeed, I suspect that Harder's paper does essentially relate inner twisting of measures to Serre's measures in the case when G is the inner twist of an anisotropic group. If G is not such an inner twist then Serre's measure is identically 0; how to describe the relationship of measures then?

As for comparison of orbital integrals, Drinfeld has a nice result for GL_n and division algebras of degree n (see Kazhdan's talk at this Institute). Diana Shelstad had proven in her thesis **[27]** many pretty results on real groups.

REFERENCES

1. T. Asai, *On certain Dirichlet series associated with Hilbert modular forms and Rankin's method*, Math. Ann. **226** (1977), 81–94.

2. A. Borel, *Automorphic L-functions*, these PROCEEDINGS, part 2, pp. 27–61.

3. A. Borel and N. Wallach, *Cohomology of discrete subgroups of semisimple groups*, Notes, Inst. Advanced Study, Princeton, N. J., 1977.

4. P. Cartier, *Lectures on representations of* $\mathfrak{p}$*-adic groups: A survey*, these PROCEEDINGS, part 1, pp. 111–155.

5. W. Casselman, *Characters and Jacquet modules*, Math. Ann. **230** (1977), 101–105.

6. P. Deligne, *Formes modulaires et représentations l-adiques*, Sém. Bourbaki 1968/69, no. 355, Lecture Notes in Math., vol. 179, Springer, New York.

7. P. Deligne, *Travaux de Shimura*, Sem. Bourbaki 1970/71, no. 389, Lecture Notes in Math., vol. 244, Springer, New York.

8. ———, *Le support du caractère d'une représentation supercuspidale*, C. R. Acad. Sci. Paris Sér. A-B **283** (1976), A155–A157.

9. V. G. Drinfeld, *Proof of the global Langlands conjecture for* GL(2) *over a function field*, Functional Anal. Appl. **61** (3) (1977), 74–75. (Russian)

10. M. Duflo and J-P. Labesse, *Sur la formule des traces de Selberg*, Ann. Sci. Ecole Norm. Sup. **4** (1971), 193–284.

11. M. Eichler, *Introduction to the theory of algebraic numbers and functions*, Academic Press, New York, 1966.

12. S. Gelbart and H. Jacquet, *Forms of* GL(2) *from the analytic point of view*, these PROCEEDINGS, part 1, pp. 213–251.

13. G. Harder, *A Gauss-Bonnet formula for discrete arithmetically defined groups*, Ann. Sci. Ecole Norm. Sup. **4** (1971), 409–455.

14. Y. Ihara, *Hecke polynomials as congruence ζ-functions in elliptic modular case*, Ann. of Math. (2) **85** (1967), 267–295.

15. H. Jacquet, *Automorphic forms for* GL(2) II, Lecture Notes in Math., vol. 278, Springer, New York.

16. H. Jacquet and R. P. Langlands, *Automorphic forms for* GL(2), Lecture Notes in Math., vol. 114, Springer, New York, 1973.

17. R. Kottwitz, *Combinatorics and Shimura varieties* mod p, these PROCEEDINGS, part 2, pp. 185–192.

18. S. Lang, $SL_2(\boldsymbol{R})$, Addison-Wesley, 1975.

19. R. P. Langlands, *Modular forms and l-adic representations*, Lecture Notes in Math., vol. 349, Springer, New York, 1973.

20. ———, *Some contemporary problems with origins in the Jugendtraum*, Proc. Sympos. Pure Math., vol. 28, Amer. Math. Soc., Providence, R.I., 1976, pp. 401–418.

21. ———, *Shimura varieties and the Selberg trace formula*, Canad. J. Math. (to appear)

22. ———, *On the zeta-functions of some simple Shimura varieties* (to appear).

23. Y. Matsushima and G. Shimura, *On the cohomology groups attached to certain vector-valued differential forms on the product of upper half-planes*, Ann. of Math. (2) **78** (1963), 417–449.

24. I. I. Piatetskii-Shapiro, *Zeta functions of modular curves*, Lecture Notes in Math., vol. 349, Springer, New York, 1973.

25. J-P. Serre, *Cohomologie des groupes discrets*, Prospects in Mathematics, Ann. of Math. Studies, no. 70, Princeton Univ. Press, Princeton, N. J., 1971.

26. D. Shelstad, *Orbital integrals for* $GL_2(\boldsymbol{R})$, these PROCEEDINGS, part 1, pp. 107–110.

27. ———, *Some character relations for real reductive algebraic groups*, Thesis, Yale Univ., 1974.

28. G. Shimura, *Introduction to the arithmetic theory of automorphic functions*, Princeton, 1971.

29. ———, *On the holomorphy of certain Dirichlet series*, Proc. London Math. Soc. **31** (1975), 79–98.

UNIVERSITY OF BRITISH COLUMBIA

Proceedings of Symposia in Pure Mathematics
Vol. 33 (1979), part 2, pp. 165–184

POINTS ON SHIMURA VARIETIES mod p

J. S. MILNE

There is associated to a reductive group G over $\boldsymbol{Q}$ with some additional structure a Shimura variety $S_{\boldsymbol{C}}$ defined over $\boldsymbol{C}$. In most cases it is known that $S_{\boldsymbol{C}}$ has a canonical model S_E defined over a specific number field E. For almost all finite primes v of E it is possible to reduce S_E modulo the prime and obtain a nonsingular variety S_v over a finite field $\boldsymbol{F}_{q_v}$. As is explained in [3], in order to identify the Hasse-Weil zeta-function of S_E or, more generally, of a locally constant sheaf on S_E it is necessary to have a description of $(S_v(\bar{\boldsymbol{F}}_{q_v}), \text{Frob})$ where $S_v(\bar{\boldsymbol{F}}_{q_v})$ is the set of points of S_v with coordinates in the algebraic closure $\bar{\boldsymbol{F}}_{q_v}$ of $\boldsymbol{F}_{q_v}$ and Frob is the Frobenius map $S_v(\bar{\boldsymbol{F}}_{q_v}) \to S_v(\bar{\boldsymbol{F}}_{q_v})$ which takes a point with coordinates $(a_1, \cdots, a_m)$ to $(a_1^{q_v}, \cdots, a_m^{q_v})$. To be useful, the description should be directly in terms of the group G.

Recently [13] Langlands has conjectured such a description of $(S_v(\bar{\boldsymbol{F}}_{q_v}), \text{Frob})$ for any Shimura variety S and any sufficiently good prime v. In [12] he has given a fairly detailed outline of a proof of the conjecture for those Shimura varieties which can be realized as coarse moduli schemes for problems involving only abelian varieties, (weak) polarizations, endomorphisms, and points of finite order. (So $G(\boldsymbol{Q})$ is of the form $\text{Aut}_B(H_1(A, \boldsymbol{Q}), \psi)$ where B is a semisimple $\boldsymbol{Q}$-algebra containing an order which acts on the abelian variety A, ψ is a Riemann form for A whose Rosati involution on $\text{End}(A) \otimes \boldsymbol{Q}$ stabilizes B, and Aut_B refers to B-linear automorphisms g of $H_1(A, \boldsymbol{Q})$ such that $\psi(gu, gv) = \psi(u, \mu(g)v)$ with $\mu(g)$ lying in some fixed algebra F contained in the centre of B and fixed by the Rosati involution; there is also a Hasse principle assumption.)

Earlier [**8**, Conjecture 1] Ihara had made a similar conjecture when S is a Shimura curve and had proved it when $G = \text{GL}_2$ [**9**, Chapter 5]. When $G = B^\times$, B a quaternion division algebra over $\boldsymbol{Q}$, Morita [**15**] proved Ihara's conjecture for all primes p of $E\,(= \boldsymbol{Q})$ not dividing the discriminant of B. Both he and Shimura have obtained partial results for more general quaternion algebras (unpublished). More recently Ihara has proved his conjecture for all Shimura curves and sufficiently good primes (announcement in [**11**]). While Ihara bases his proof on the Eichler-Shimura congruence relations, Morita's method, as described in [**10**], appears to be quite similar to that of Langlands.

In order to give some idea of the techniques Langlands uses in his proof I shall describe it in the case that G is the multiplicative group of a totally indefinite qua-

AMS (MOS) subject classifications (1970). Primary 14K10; Secondary 14G10, 14L05.

ternion algebra over a totally real number field. In §1 it is shown that S_C parametrizes, in a natural way, a family of abelian varieties with additional structure. The following section describes how Artin's representability criteria may be used to prove the existence of a variety S_Q over Q which is a canonical model for S_C and which, when reduced mod p, parametrizes a family of abelian varieties (with additional structure) in characteristic p. Thus the problem of describing $(S_p(\bar{F}_p)$, Frob) becomes one of describing this family. In §5 the Tate-Honda classification of isogeny classes of abelian varieties over finite fields is used to determine the isogeny classes in the family, and in §6 the individual isogeny classes are described. Since this requires the use of p-divisible groups and their Dieudonné modules, these are reviewed in §3.

Notation. F is a totally real number field of degree d over Q, B is a quaternion division algebra over F which is split everywhere at infinity, $b \mapsto b^*$ is a positive F-involution on B, and O_B is a maximal order in B.

G is the group scheme over Z such that $G(R) = (O_B^{\text{opp}} \otimes R)^\times$ for all rings R, where O_B^{opp} is the opposite algebra to O_B.

A is the ring of adèles for Q; $A = R \times A_f = R \times A_f^p \times Q_p^\times$; $A_f = Z_f \otimes Q$, $Z_f = \text{proj lim } Z/mZ$; $Z_f = Z_f^p \times Z_p$.

K is a (sufficiently small) open subgroup of $G(Z_f)$. Δ is a product of rational prime numbers such that if $p \nmid \Delta$ then p is unramified in F, B is split at all primes of F dividing p, and $K = K^pG(Z_p)$ where $K^p = K \cap G(A_f^p)$.

$S_C = {}_KS_C$ is the Shimura variety over C defined by G, K, and the map $h: C^\times \to G(R)$ defined in §1; thus its points in C are $S_C(C) = G(Q)\backslash G(A)/K_\infty K$ where K_∞ is the centralizer of h in $G(R)$.

If $V = V(Z)$ is a Z-module then $V(R) = V \otimes_Z R$ for any ring R.

1. S_C as a moduli scheme. Recall that an *abelian variety* over a field k is an algebraic group over k whose underlying variety is complete (and connected); its group structure is then commutative and the variety is projective. For example, an abelian variety of dimension one is an elliptic curve, and may be described by its equation, which is of the form

$$Y^2Z = X^3 + aXZ^2 + bZ^3, \qquad a, b \in k,\ 4a^3 + 27b^2 \neq 0.$$

It is impractical to describe abelian varieties of dimension greater than one by equations, but fortunately over C there is a classical description in terms of lattices in complex vector spaces. Let V be a lattice in C^g, i.e., V is the subgroup generated by an R-basis and so $V \otimes_Z R \approx C^g$. Then C^g/V is a compact complex-analytic manifold which becomes a commutative Lie group under addition. When $g = 1$ the Weierstrass $\wp$-function corresponding to V, and its derivative, define an embedding

$$z \mapsto (\wp(z),\ \wp'(z),\ 1): C/V \hookrightarrow P_C^2$$

of C/V as an algebraic subset of the projective plane. Thus C/V automatically has the structure of an algebraic variety and so is an abelian variety. This is no longer true if $g > 1$ for there may be too few functions on C^g/V to define an embedding of it into projective space. Since any meromorphic function on C^g/V is a quotient of theta functions on $\boldsymbol{C}^g$, $\boldsymbol{C}^g/V$ will be algebraic if and only if there exist enough theta functions. By definition, a theta function for V is a holomorphic function θ on $\boldsymbol{C}^g$ such that, for $v \in V$, $\theta(z + v) = \theta(z) \exp(2\pi i(L(z, v) + J(v)))$ where $L(z, v)$ is a $\boldsymbol{C}$-linear function of z and $J(v)$ depends only on v. One shows that $L(z, v)$ is additive in v, and so extends to a function $L: \boldsymbol{C}^g \times \boldsymbol{C}^g \to \boldsymbol{C}$ which is $\boldsymbol{C}$-linear in the first variable and $\boldsymbol{R}$-linear in the second. Set $E(z, w) = L(z, w) - L(w, z)$.

Then

(a) E is $\boldsymbol{R}$-valued, $\boldsymbol{R}$-bilinear, and alternating;

(b) E takes integer values on $V \times V$;

(c) the form $(z, w) \mapsto E(iz, w)$ is symmetric and positive.

(The symmetry is equivalent to having $E(iz, iw) = E(z, w)$ for all z, w; the positivity means $E(iz, z) \geqq 0$ for all z.)

A form satisfying these conditions is called a *Riemann form* for V and it is known that there exist enough functions to define a projective embedding of $\boldsymbol{C}^g/V$ if and only if there exists a Riemann form for V which is nondegenerate (and hence such that $E(iz, z)$ is positive definite). If $g = 1$ we may always define $E(z, w)$ to be the ratio of the oriented area of the parallelogram with sides Ow, Oz to that of a fundamental parallelogram for the lattice. Since this form always exists, and is unique up to multiplication by an integer, one rarely bothers to mention it. By contrast, if $g > 1$, a nondegenerate Riemann form will not usually exist and when it does, it will not be unique up to multiplication by an integer. However since $\boldsymbol{C}^g/V$ is compact the algebraic structure on $\boldsymbol{C}^g/V$ (but not the projective embedding) defined by a Riemann form is independent of the form.

Thus, given a lattice in $\boldsymbol{C}^g$ for which there exists a nondegenerate Riemann form, we obtain an abelian variety. Conversely, from an abelian variety A of dimension g we can recover a complex vector space W of dimension g and a lattice V in W for which there exists a nondegenerate Riemann form. W can be described (according to taste) as the Lie algebra $\mathrm{Lie}(A)$ of A, the tangent space $\boldsymbol{t}_A$ to A at its zero element, or as the universal covering space of the topological manifold $A(\boldsymbol{C})$. The lattice V can be described as the kernel of the exponential $\exp: \mathrm{Lie}(A) \to A(\boldsymbol{C})$, or as the fundamental group of $A(\boldsymbol{C})$ which, being commutative, is equal to $H_1(A, \boldsymbol{Z})$. We shall always regard the isomorphism $W/V \xrightarrow{\approx} A(\boldsymbol{C})$ as arising from the exact sequence,

$$0 \to H_1(A, \boldsymbol{Z}) \to \boldsymbol{t}_A \xrightarrow{\exp} A(\boldsymbol{C}) \to 0.$$

Since $H_1(A, \boldsymbol{Z})$ is a lattice in $\boldsymbol{t}_A$; we have $H_1(A, \boldsymbol{R}) = H_1(A, \boldsymbol{Z}) \otimes \boldsymbol{R} \xrightarrow{\approx} \boldsymbol{t}_A$. Thus A is determined by $H_1(A, \boldsymbol{Z})$ and the complex vector space structure on $H_1(A, \boldsymbol{R})$.

A complex structure on a real vector space $V(\boldsymbol{R})$ defines a homomorphism $h: \boldsymbol{C}^\times \to \mathrm{Aut}_{\boldsymbol{R}}(V(\boldsymbol{R}))$, $h(z) = (v \mapsto zv)$, and the complex structure is determined by h. Thus an abelian variety A is uniquely determined by the pair $(H_1(A, \boldsymbol{Z}), h)$ where $h: \boldsymbol{C}^\times \to \mathrm{Aut}(H_1(A, \boldsymbol{R}))$ is defined by the complex structure on $H_1(A, \boldsymbol{R}) = \boldsymbol{t}_A$. Moreover every pair $(V(\boldsymbol{Z}), h)$ for which there exists a Riemann form arises from an abelian variety.

Let $V(\boldsymbol{Z}) = H_1(A, \boldsymbol{Z})$. A point of finite order on A corresponds to an element of $V(\boldsymbol{R})$ some multiple of which is in $V(\boldsymbol{Z})$. More precisely, the group of points of finite order on A may be identified with $V(\boldsymbol{Q})/V(\boldsymbol{Z}) \subset V(\boldsymbol{R})/V(\boldsymbol{Z})$. For any integer $m > 0$, the group $A_m(\boldsymbol{C})$ of points of order m is equal to $m^{-1}V(\boldsymbol{Z})/V(\boldsymbol{Z}) \xrightarrow{\approx} V(\boldsymbol{Z}/m\boldsymbol{Z}) \approx (\boldsymbol{Z}/m\boldsymbol{Z})^{2\dim(A)}$. We define T_fA to be proj $\lim_m A_m(\boldsymbol{C}) = V(\boldsymbol{Z}_f)$ and, for any prime l, T_lA to be proj $\lim_m A_{l^m}(\boldsymbol{C}) = V(\boldsymbol{Z}_l)$; thus $T_fA = \prod_l T_lA$ and $T_lA \approx \boldsymbol{Z}_l^{2\dim(A)}$.

A homomorphism $A \to A'$ of abelian varieties induces a $\boldsymbol{C}$-linear map $\boldsymbol{t}_A \to \boldsymbol{t}_{A'}$ such that $H_1(A, \boldsymbol{Z})$ is mapped into $H_1(A', \boldsymbol{Z})$. Conversely, if A and A' correspond respectively to (V, h) and (V', h') then a map of $\boldsymbol{Z}$-modules $\alpha: V(\boldsymbol{Z}) \to V'(\boldsymbol{Z})$ extending to a $\boldsymbol{C}$-linear map $\alpha \otimes 1\colon V(\boldsymbol{R}) \to V'(\boldsymbol{R})$ (i.e., such that $\alpha \otimes 1 \circ h(z) = h'(z) \circ \alpha \otimes 1$ for all z) arises from a map of complex manifolds $A(\boldsymbol{C}) \to A'(\boldsymbol{C})$ and the compactness of $A(\boldsymbol{C})$ and $A'(\boldsymbol{C})$ implies that the map is algebraic. We write End(A) for the ring of endomorphisms of A and End°(A) for End$(A) \otimes_{\boldsymbol{Z}} \boldsymbol{Q}$. Since End°$(A)$ has a faithful representation on $H_1(A, \boldsymbol{Q})$, it is a finite-dimensional $\boldsymbol{Q}$-algebra; it is also semisimple, and its possible dimensions and structures are well understood.

To define a homomorphism $i\colon O_B \to \text{End}(A)$ when A corresponds to (V, h) is the same as to define an action of O_B on V such that h maps $\boldsymbol{C}^\times$ into $\text{Aut}_{O_B\otimes\boldsymbol{R}}(V(\boldsymbol{R}))$. When such an i is given we say that O_B *acts* on A provided $i(1) = 1$. Such an i induces an injection $i\colon B \hookrightarrow \text{End}°(A)$.

A nondegenerate Riemann form E for A defines an involution $\alpha \mapsto \alpha'$ of End°(A) by the rule $E(\alpha z, w) = E(z, \alpha' w)$; this is the *Rosati involution*, which is known to be positive, i.e., $\text{Tr}(\alpha\alpha') > 0$ for all $\alpha \neq 0$ where Tr denotes the reduced trace from End°(A) to $\boldsymbol{Q}$. Suppose O_B acts on A. We say that two Riemann forms E and E' on A are *F-equivalent* if there exist nonzero c, $c' \in O_F$ such that $E(u, cv) = E(u, c'v)$ for all u, $v \in V(\boldsymbol{Z})$, and we define a *weak polarization* of A to be an F-equivalence class Λ of nondegenerate Riemann forms. Since F is the centre of B, the Rosati involutions defined by any two elements of such a Λ induce the same map on $i(B)$. We shall be interested in triples (A, i, Λ) such that $E(i(b)u, v) = E(u, i(b^*)v)$ for $u, v \in V(\boldsymbol{R})$, $b \in B$, $E \in \Lambda$, i.e., we require that the Rosati involutions defined by Λ stabilize $i(B)$ and induce the given involution $b \mapsto b^*$ on B.

We next review some notations concerning B. The *main involution* $b \mapsto b^\iota$ of B is so defined that under any $\boldsymbol{R}$-isomorphism $B \otimes_F \boldsymbol{R} \xrightarrow{\approx} M_2(\boldsymbol{R})$, if b corresponds to $M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ then b^ι corresponds to $M^\iota = \left(\begin{smallmatrix} d & -b \\ -c & a \end{smallmatrix}\right)$; thus $b + b^\iota = \text{Tr}_{B/F}(b)$ and $bb^\iota = Nm_{B/F}(b)$. The Skolem-Noether theorem shows that there exists a $t \in B$ such that $b^* = t^{-1}b^\iota t = tb^\iota t^{-1}$ for all $b \in B$; automatically $t^2 \in F$ and the positivity of $b \mapsto b^*$ implies that $t^2 < 0$, i.e., t^2 has negative image under all embeddings $F \hookrightarrow \boldsymbol{R}$. We fix an isomorphism $B \otimes_{\boldsymbol{Q}} \boldsymbol{R} \xrightarrow{\approx} M_2(\boldsymbol{R}) \times \cdots \times M_2(\boldsymbol{R})$ such that if $b \leftrightarrow (M_1, \cdots, M_n)$ then $b^* \leftrightarrow (M_1^{\text{tr}}, \cdots, M_n^{\text{tr}})$ where M_i^{tr} is the transpose of M_i. Since

$$M^{\text{tr}} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}^{-1} M^\iota \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},$$

t maps to an element $(c_1\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right), \cdots, c_n\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right))$ with each $c_i \in \boldsymbol{R}$, and t may be chosen so that $c_i > 0$.

The next lemma implies that if O_B acts on a complex manifold $\boldsymbol{C}^{2d}/V$ then there is a Riemann form E for V whose corresponding Rosati involution induces $b \mapsto b^*$ on B and any two such forms are F-equivalent, i.e., that there is a unique weak polarization which is compatible with the O_B-action and the given involution.

LEMMA 1.1. *Let $V = V(\mathbf{Z})$ be a free $\mathbf{Z}$-module of rank $4d$ on which O_B acts. There is a nondegenerate alternating form ψ on $V(\mathbf{Q})$ such that:*

(a) $\psi(u, v) \in \mathbf{Z}$ *if* $u, v \in V(\mathbf{Z})$;

(*b*) $\psi(ut, u) < 0$ *for all* $u \neq 0$, $u \in V(\mathbf{R})$;

(*c*) $\psi(bu, v) = \psi(u, b^*v)$ *for all* $b \in B$ *and* $u, v \in V(\mathbf{Q})$;

(*d*) *for any B-automorphism α of $V(\mathbf{Q})$ there exists a $\mu(\alpha) \in F^\times$ such that $\psi(\alpha u, \alpha v) = \psi(u, \mu(\alpha)v)$ for all $u, v \in V(\mathbf{Q})$. Moreover' if ψ' is a second nondegenerate alternating form on $V(\mathbf{Q})$ satisfying* (c) *then there exists a $c \in F^\times$ such that $\psi(u, cv) = \psi'(u, v)$ for all $u, v \in V$.*

PROOF. $V(\mathbf{Q})$ has dimension one over B and so, after choosing an appropriate basis vector, we may identify $V(\mathbf{Q})$ with B and $V(\mathbf{Z})$ with a left ideal in O_B.

Define $\psi(u, v) = \mathrm{Tr}_{B/\mathbf{Q}}(uv^\iota t) = \mathrm{Tr}_{B/\mathbf{Q}}(utv^*)$. Then $\psi(u, v) = \mathrm{Tr}\,(utv^*) = \mathrm{Tr}(vt^*u^*) = \mathrm{Tr}(v(-t)u^*) = -\psi(v, u)$, and so ψ is alternating. (a) is obvious, and $\psi(ut, u) = \mathrm{Tr}_{B/\mathbf{Q}}(ut^2u^*) = \mathrm{Tr}_{F/\mathbf{Q}}(t^2\mathrm{Tr}_{B/F}(uu^*)) < 0$ for $u \neq 0$, which proves (b) and that ψ is nondegenerate. For (c) we note that $\psi(bu, v) = \mathrm{Tr}(utv^*b) = \mathrm{Tr}(ut(b^*v)^*) = \psi(u, b^*v)$. Finally, any B-automorphism α of $V(\mathbf{Q}) = B$ is multiplication on the right by an element $b \in B^\times$. Thus $\psi(\alpha u, \alpha v) = \mathrm{Tr}\,(ubb^\iota v^\iota t) = \psi(u, \mu(\alpha)v)$ with $\mu(\alpha) = Nm_{B/F}(b)$.

For the last part, consider the $\mathbf{Q}$-linear map $v \mapsto \psi'(1, v)\colon B \to \mathbf{Q}$. Since $\mathrm{Tr}_{B/\mathbf{Q}}\colon B \times B \to \mathbf{Q}$ is nondegenerate, there is a unique $b \in B$ such that $\psi'(1, v) = \mathrm{Tr}\,(btv^*)$ for all $v \in B$. Then $\psi'(u, v) = \psi'\,(1, u^*v) = \mathrm{Tr}\,(btv^*u) = \mathrm{Tr}(ubtv^*) = \mathrm{Tr}(ubv^\iota t)$. We also have $\psi'(1, v) = -\psi'(v, 1) = -\mathrm{Tr}\,(vbt) = -\mathrm{Tr}\,(t^\iota b^\iota v^\iota) = \mathrm{Tr}\,(b^\iota v^\iota t) = \mathrm{Tr}\,(b^\iota tv^*)$. Thus $b = b^\iota$, which implies that it is in F, and we may take $c = b$.

For the remainder of this section $V(\mathbf{Z})$ will be O_B regarded as an O_B-module and ψ will be as in the lemma. For any ring R we may identify $G(R)$ with $\mathrm{Aut}_{O_B\otimes R}(V(R))$ since any $O_B \otimes R$-endomorphism of $V(R) = O_B \otimes R$ is right multiplication by an element of $O_B \otimes R$. Define h to be the homomorphism $\mathbf{C}^\times \to G(\mathbf{R}) = \mathrm{Aut}_{B\otimes\mathbf{R}}(V(\mathbf{R}))$ such that $h(i)$ is right multiplication on $V(\mathbf{R}) = B \otimes \mathbf{R} \xrightarrow{\approx} M_2(\mathbf{R}) \times \dots \times M_2(\mathbf{R})$ by $(\left(\begin{smallmatrix}0 & 1\\ -1 & 0\end{smallmatrix}\right), \cdots, \left(\begin{smallmatrix}0 & 1\\ -1 & 0\end{smallmatrix}\right))$. Thus $K_\infty = \{(M_1, \cdots, M_d)\}$ with M_i of the form$\left(\begin{smallmatrix}a & b\\ -b & a\end{smallmatrix}\right)$, $a, b \in \mathbf{R}$. The form $E = \psi$ is a Riemann form for $(V(\mathbf{Z}), h)$, e.g., $\psi(iu, iv) = \psi(uh(i), vh(i)) = \psi(u, Nm_{B/F}(h(i))v) = \psi(u, v)$ and $\psi(iu, u) = \psi(ut/-(-t^2)^{1/2}, u) > 0$ for $u \neq 0$. Thus $(V(\mathbf{Z}), h)$ defines an abelian variety A. The action of O_B on $V(\mathbf{Z})$ induces a map $i\colon O_B \to \mathrm{End}(A)$ and the Rosati involution defined by the weak polarization Λ containing ψ induces $b \mapsto b^*$ on B.

Recall that K is an open subgroup of $G(\mathbf{Z}_f)$. Two isomorphisms $\phi_1, \phi_2\colon T_fA \xrightarrow{\approx} V(\mathbf{Z}_f)$ are *K-equivalent* if there is a $k \in K$ such that $\phi_1 = k\phi_2$. For example, if $K = K_m = \mathrm{Ker}(G(\mathbf{Z}_f) \to G(\mathbf{Z}/m\mathbf{Z}))$ then to give a K-equivalence class of isomorphisms $T_fA \to V(\mathbf{Z}_f)$ is the same as to give an isomorphism $A_m(\mathbf{C}) \xrightarrow{\approx} V(\mathbf{Z}/m\mathbf{Z})$, i.e., a level m structure.

THEOREM 1.2. *There is a one-one correspondence between the set of points $S_C(\mathbf{C}) = G(\mathbf{Q})\backslash G(\mathbf{A})/K_\infty K$ and the set of isomorphism classes of triples $(A, i, \bar{\phi})$ where A is an abelian variety of dimension $2d$, i defines an action of O_B on A, and $\bar{\phi}$ is a K-equivalence class of O_B-isomorphisms $T_fA \xrightarrow{\approx} V(\mathbf{Z}_f)$.*

REMARK 1.3. (a) We say that two triples $(A, i, \bar{\phi})$ and $(A', i', \bar{\phi}')$ are isomorphic if there exists an isomorphism $\alpha: A \to A'$ such that $\alpha \circ i(b) = i'(b) \circ \alpha$ for all $b \in O_B$ and $\phi' \circ (T_f\alpha) \in \bar{\phi}$ for all $\phi' \in \bar{\phi}'$.

(b) Normally when considering families of abelian varieties parametrized by Shimura varieties it is necessary to work with quadruples $(A, i, \Lambda, \bar{\phi})$ with Λ a (weak) polarization. This is not necessary in our case because, as we observed above, Λ always exists uniquely.

PROOF OF 1.2. We first show how to associate to any $g \in G(\boldsymbol{A})$ a triple $(A_g, i_g, \bar{\phi}_g)$. If $g = 1$ we take $(A, i, \bar{\phi})$ with (A, i) as defined before and $\bar{\phi}$ the class of the identity map $T_fA = V(\boldsymbol{Z}_f) \xrightarrow{\text{id}} V(\boldsymbol{Z}_f)$. We write a general g as $g = g_\infty g_f$, $g_\infty \in G(\boldsymbol{R})$, $g_f \in G(\boldsymbol{A}_f)$, and use g_∞ and g_f to modify respectively the complex structure on $V(\boldsymbol{R})$ and the lattice. Define $h_g: \boldsymbol{C}^\times \to G(\boldsymbol{R})$ by the formula $h_g(z) = g_\infty h(z) g_\infty^{-1}$ and define $gV(\boldsymbol{Z})$ to be the lattice $g_f V(\boldsymbol{Z}_f) \cap V(\boldsymbol{Q})$, the intersection taking place inside $V(\boldsymbol{A}_f)$. Then A_g is to be the abelian variety defined by the pair (gV, h_g). Since O_B still acts on $gV(\boldsymbol{Z})$ we have an obvious map $i_g: O_B \hookrightarrow \text{End}(A)$. We define $\phi_g: T_fA_g = g_fV(\boldsymbol{Z}_f) \xrightarrow{\approx} V(\boldsymbol{Z}_f)$ to be multiplication by g_f^{-1}.

If g is replaced by gk_∞ with $k_\infty \in K_\infty$ then h_g is unchanged since K_∞ is the centralizer of h in $G(\boldsymbol{R})$. If g is replaced by gk_f with $k_f \in K$ then h_g and $gV(\boldsymbol{Z})$ are unchanged while ϕ_g is replaced by $k_f^{-1}\phi_g$, which is K-equivalent to ϕ_g. If g is replaced by qg with $q \in G(\boldsymbol{Q})$ then $q^{-1}: V(\boldsymbol{R}) \to V(\boldsymbol{R})$ defines an isomorphism $(A_{qg}, \cdots) \xrightarrow{\approx} (A_g, \cdots)$. Thus $(A_g, \cdots)$ depends only on the double coset of g.

Conversely, an isomorphism $\alpha: (A_g, \cdots) \to (A_{g'}, \cdots)$ is induced by an isomorphism $V(\boldsymbol{R}) \to V(\boldsymbol{R})$ which sends $gV(\boldsymbol{Z})$ isomorphically onto $g'V(\boldsymbol{Z})$. In particular α defines a B-isomorphism $q: V(\boldsymbol{Q}) \to V(\boldsymbol{Q})$. Thus $q \in G(\boldsymbol{Q})$ and so, after replacing g' by $q^{-1}g'$ and α by $q^{-1}\alpha$, we may assume that the map $V(\boldsymbol{Q}) \to V(\boldsymbol{Q})$ corresponding to α is the identity. Thus $g_\infty h(z) g_\infty^{-1} = g'_\infty h(z) g'^{-1}_\infty$ for all z, and so $g_\infty^{-1} g'_\infty \in K_\infty$. Moreover, $gV(\boldsymbol{Z}) = g'V(\boldsymbol{Z})$ implies $g_f^{-1} g'_f \in G(\boldsymbol{Z}_f)$, and $g_f^{-1}: gV(\boldsymbol{Z}_f) \to V(\boldsymbol{Z}_f)$ being K-equivalent to $g'^{-1}_f: g'V(\boldsymbol{Z}_f) \to V(\boldsymbol{Z}_f)$ implies that $g_f^{-1} g'_f \in K$.

Finally we have to show that every $(A, i, \bar{\phi})$ arises from some g. Since B is a division algebra there is a B-isomorphism $H_1(A, \boldsymbol{Q}) \xrightarrow{\approx} V(\boldsymbol{Q})$ which we may use to identify $H_1(A, \boldsymbol{Q})$ with $V(\boldsymbol{Q})$. Then $H_1(A, \boldsymbol{Z})$ is a lattice in $V(\boldsymbol{Q})$ and so is of the form $g_fV(\boldsymbol{Z})$ for some $g_f \in G(\boldsymbol{A}_f)$. The isomorphism $V(\boldsymbol{R}) \approx \boldsymbol{t}_A$ induces a complex structure on $V(\boldsymbol{R})$, and we let $h': \boldsymbol{C}^\times \to \text{Aut}_{\boldsymbol{R}}(V(\boldsymbol{R}))$ be the corresponding map. Since B acts $\boldsymbol{C}$-linearly on $\boldsymbol{t}_A$, h maps into $\text{Aut}_{B \otimes \boldsymbol{R}}(V(\boldsymbol{R})) = G(\boldsymbol{R})$. Obviously there exists a $g_\infty \in G(\boldsymbol{R})$ such that $h'(z) = g_\infty h(z) g_\infty^{-1}$. Any $\phi \in \bar{\phi}$ is of the form $v \mapsto g_1^{-1} g_f^{-1} v: g_fV(\boldsymbol{Z}_f) \to V(\boldsymbol{Z}_f)$ for some $g_1 \in G(\boldsymbol{Z}_f)$. It is now clear that $(A, \cdots) \approx (A_g, \cdots)$ with $g = g_\infty g_f g_1$.

REMARK 1.4. (a) A map $\alpha: A \to A'$ of abelian varieties is an isogeny if it is surjective and has finite kernel; when O_B acts on A and A', α is called an O_B-isogeny if it commutes with the two actions. Clearly any isogeny (over $\boldsymbol{C}$) induces an isomorphism on the tangent spaces and so A_g is isogenous to $A_{g'}$ only if g_∞ and g'_∞ define the same double coset in $G(\boldsymbol{Q})\backslash G(\boldsymbol{R})/K_\infty$. On the other hand, the set $\text{End}_B^\circ(A)^\times \backslash G(\boldsymbol{A}_f)/K$ classifies the triples $(A, i, \bar{\phi})$ for which there is an O_B-isogeny $A \to A_1$. For example, if $g = g_f$ then, after replacing g_f by an integral multiple, we may assume that $gV(\boldsymbol{Z}) \subset V(\boldsymbol{Z})$. The identity map $V(\boldsymbol{R}) \to V(\boldsymbol{R})$ now defines an isogeny $A_g \to A_1$ with kernel $V(\boldsymbol{Z})/gV(\boldsymbol{Z})$ (cf. §6 below).

(b) In the case that $F = \boldsymbol{Q}$, the theorem may be strengthened. Consider the projection $V(\mathbf{R}) \times (G(\mathbf{A})/K_\infty K) \to G(\mathbf{A})/K_\infty K$. We give $G(\mathbf{A})/K_\infty K$ its usual complex structure and the copy of $V(\mathbf{R})$ over $gK_\infty K$ the complex structure defined by h_g. Inside each V_g we have a lattice $gV(\mathbf{Z})$, and these vary continuously with g. Thus we may divide out and obtain a map of complex manifolds $\mathscr{T} \to G(\mathbf{A})/K_\infty K$ such that the fibre over $gK_\infty K$ is the abelian variety A_g. We may now let $G(\mathbf{Q})$ act on both manifolds and divide out again to obtain an analytic family $\mathscr{A} \to S_{\mathbf{C}}$ of abelian varieties. Each fibre A_g has the structure defined by $(i_g, \bar{\phi}_g)$, and these vary continuously. In fact $\mathscr{A} \to S_{\mathbf{C}}$ is an algebraic family, i.e., $\mathscr{A}$ is an algebraic variety and the map is algebraic.

If $F \neq \mathbf{Q}$ the above construction fails because units of F may act on (A_g, i_g, ϕ_g) and so the action of $G(\mathbf{Q})$ on $\mathscr{T}$ is not free. However we may "rigidify" the situation as follows: consider quadruples $(A, i, \bar{\phi}, \varepsilon)$ where A, i, and $\bar{\phi}$ are as before and ε is an injection from the unique weak polarization Λ to $F^\times$ such that $\varepsilon(\psi') = c\varepsilon(\psi)$ if $\psi'(u, v) = \psi(u, cv)$. The isomorphism classes of quadruples are classified by $F^\times \times S_{\mathbf{C}}(\mathbf{C})$ which is a disjoint union of copies of $S_{\mathbf{C}}(\mathbf{C})$, one for each element of $F^\times$, on which $F^\times$ acts by permuting the copies. $F^\times \times S_{\mathbf{C}}$ may be regarded as a scheme over $\mathbf{C}$ which is an infinite disjoint union of varieties and the previous process gives an algebraic family of $\mathscr{A} \to F^\times \times S_{\mathbf{C}}$ of abelian varieties with structure.

References. The most elegant elementary and nonelementary treatments of abelian varieties over $\mathbf{C}$ are to be found respectively in **[20]** and **[17**, Chapter I]. Families of abelian varieties parametrized by Shimura varieties were extensively studied by Shimura in the 1960's (see his Annals papers of that period). They are also discussed briefly in **[4]**.

2. S **as a scheme over** $\mathbf{Z}[\Delta^{-1}]$. We shall see shortly that $S_{\mathbf{C}}$ has a model $S_{\mathbf{Q}}$ over $\mathbf{Q}$, i.e., that there is a scheme $S_{\mathbf{Q}}$ over $\mathbf{Q}$ whose defining equations, when considered over $\mathbf{C}$, give $S_{\mathbf{C}}$. There is no reason to believe that $S_{\mathbf{Q}}$ will be unique but Shimura has given conditions which will be satisfied by at most one model; such a model (when it exists) is said to be *canonical.* For example, let F' be a quadratic totally imaginary extension of F which splits B and let A_0 be the abelian variety of dimension d defined by the lattice $O_{F'} \subset F' \otimes \mathbf{R}$. Then $O_{F'}$ acts on A_0 and A_0 is said to have complex multiplication by F'. Let $A = A_0 \times A_0$. If we embed F' in B and choose a basis $\{e_1, e_2\}$ for B over F' with $e_1, e_2 \in O_B$, then we get a map $B \hookrightarrow M_2(F')$ $\subset M_2(\mathrm{End}^\circ(A_0)) = \mathrm{End}^\circ(A)$ sending O_B into $\mathrm{End}(A)$. Also we get a map $T_f A = (O_{F'} \oplus O_{F'}) \otimes \mathbf{Z}_f \xrightarrow{\phi} O_B \otimes \mathbf{Z}_f = V(\mathbf{Z}_f)$ (in the notation of §1). The triple $(A, i, \bar{\phi})$ defines a point of $S_{\mathbf{C}}$, and hence a point of $S_{\mathbf{Q}}$ with complex coordinates. For $S_{\mathbf{Q}}$ to be canonical these coördinates must be algebraic over $\mathbf{Q}$ and generate a certain explicitly described class field.

For the reasons explained in the introduction we would like to have a scheme S defined by equations in $\mathbf{Z}[\Delta^{-1}]$ which, when regarded over $\mathbf{Q}$, is the canonical model $S_{\mathbf{Q}}$ of $S_{\mathbf{C}}$, and which is such that it is possible to describe explicitly $(S(\bar{\mathbf{F}}_p)$, Frob) for any $p \nmid \Delta$. Such an S will define a functor $R \mapsto S(R)$ which associates to any ring R in which Δ is invertible the set of points of S with coordinates in R. (More generally, it associates to any scheme T over spec $\mathbf{Z}[\Delta^{-1}]$ the set $S(T)$ of maps $T \to S$.) Since the functor determines the scheme uniquely this suggests that in constructing S we should write down a functor $\mathscr{S}$ such that, in particular, $\mathscr{S}(\mathbf{C})$ $= S_{\mathbf{C}}(\mathbf{C}) = G(\mathbf{Q})\backslash G(\mathbf{A})/K_\infty K$ and try to prove that it is the points functor of a scheme. After §1 it is natural to define $\mathcal{S}(R)$ to consist of isomorphism classes of triples $(A, i, \bar{\phi})$ where each of the three objects is the analogue over R of the corresponding object over $\boldsymbol{C}$. Thus A is a projective abelian scheme of dimension $2d$ over R. Intuitively, A can be thought of as an algebraic family of abelian varieties, each of which is defined over a residue field of R. More precisely it is a projective smooth group scheme over spec R with geometrically connected fibres. As before i is to be a homomorphism $O_B \hookrightarrow \mathrm{End}(A)$ such that $i(1)$ is the identity map. We assume that A has a polarization whose Rosati involution induces $b \mapsto b^*$ on B. Two problems arise in defining $\bar{\phi}$ which may be best understood if we write $\phi\colon T_f A \to V(\boldsymbol{Z}_f)$ as a product $\prod \phi_l\colon \prod T_l A \to \prod V(\boldsymbol{Z}_l)$ of maps. Firstly, if p is not invertible in R there will never exist an isomorphism $\phi_p\colon T_p A \xrightarrow{\approx} V(\boldsymbol{Z}_p)$; thus we take ϕ_p to be a map defined only over $R[p^{-1}]$. Secondly, unless R is an algebraically closed field it is unrealistic to expect there to be an isomorphism $\phi_l\colon T_l A \to V(\boldsymbol{Z}_l)$ for any l, for this would imply that all coordinates of all l-power torsion points of A are in R. Instead we assume that $K \supset K_m = \mathrm{Ker}(G(\boldsymbol{Z}_f) \to G(\boldsymbol{Z}/m\boldsymbol{Z}))$ some m, and consider isomorphisms $\phi\colon A_m \xrightarrow{\approx} V(\boldsymbol{Z}/m\boldsymbol{Z})$ defined on some étale covering of R, two such isomorphisms ϕ_1 and ϕ_2 being K-equivalent if $\phi_1 = k\phi_2$, $k \in K$, locally on spec(R), and we take $\bar{\phi}$ to be a K-equivalence class in this new sense. It is necessary to put one extra condition on the triple $(A, i, \bar{\phi})$: if the R-algebra R' is such that $O_B \otimes R' \approx M_2(O_F \otimes R')$ then the two submodules of $\boldsymbol{t}_{A/R}$ corresponding to the idempotents $\left(\begin{smallmatrix}1&0\\0&0\end{smallmatrix}\right)$ and $\left(\begin{smallmatrix}0&0\\0&1\end{smallmatrix}\right)$ should be free $O_F \otimes R'$-modules of rank 1 locally on spec(R'). (This condition holds automatically if $R = \boldsymbol{C}$; for examples where it fails in an analogous situation in characteristic p, see [**18**, 1.29].)

Having defined our functor $\mathcal{S}$ we now have to see whether it is the points functor of a scheme. Generally speaking this is a very delicate question but M. Artin has given an often-manageable set of criteria for a functor to be the points functor of an algebraic space. An algebraic space is a slightly more general object than a scheme, but for our purposes it is just as good; it makes good sense to speak of its points with coordinates in a ring, and the proper and smooth base change theorems in étale cohomology, which are the theorems which allow us to compute Hasse-Weil zeta-functions by reducing modulo a prime, hold for algebraic spaces. (In fact, the algebraic spaces we get are almost certainly schemes, and this surely could be proved by using Mumford's methods [**16**] instead of Artin's.)

Consider first the case that $F = \boldsymbol{Q}$. Then Artin's criteria may be checked and show that there is an algebraic space S, proper and smooth over $\boldsymbol{Z}[\Delta^{-1}]$, such that $S(R) = \mathcal{S}(R)$ for any ring R in which Δ is invertible. In particular $S(\boldsymbol{C}) = \mathcal{S}(\boldsymbol{C}) = S_{\boldsymbol{C}}(\boldsymbol{C})$ and $S(\bar{\boldsymbol{F}}_p) = \mathcal{S}(\bar{\boldsymbol{F}}_p)$ for any p not dividing Δ. The algebraic family $\mathcal{A} \to S_{\boldsymbol{C}}$ constructed in 1.4(b) is an element of $\mathcal{S}(S_{\boldsymbol{C}}) = S(S_{\boldsymbol{C}})$, and so gives a map $S_{\boldsymbol{C}} \to S$. This induces a map $S_{\boldsymbol{C}} \to S \times \mathrm{spec}\, \boldsymbol{C}$ which is an isomorphism. Moreover it is known that $S_{\boldsymbol{Q}}$ is the canonical model.

When $F \neq \boldsymbol{Q}$, then a slightly weaker result holds, but one which is just as useful to us. Since there are nontrival automorphisms of $(A, i, \bar{\phi})$ there can be no algebraic space S with $S(R) = \mathcal{S}(R)$ for all R. However, there does exist an algebraic space S, proper and smooth over $\boldsymbol{Z}[\Delta^{-1}]$, and a functorial map $\mathcal{S}(R) \to S(R)$ which is an isomorphism whenever R is an algebraically closed field. Thus $S(\boldsymbol{C}) = \mathcal{S}(\boldsymbol{C}) = S_{\boldsymbol{C}}(\boldsymbol{C})$ as before, and $S_{\boldsymbol{Q}}$ is the canonical model of $S_{\boldsymbol{C}}$. To prove these facts one may "rigidify" the moduli problem as in the second paragraph of 1.4(b), make the constructions as in the case $F = \boldsymbol{Q}$, and then form quotients under the left action by $F^{\times}$, or else work directly with stacks.

Note that in either case, $S(\bar{\boldsymbol{F}}_p) = \mathscr{S}(\bar{\boldsymbol{F}}_p)$ has a description in terms of abelian varieties with additional structure.

References. **[1]** contains a short introduction to Artin's techniques for representing functors by algebraic spaces and **[2]** a more complete one. In **[5]** and **[18]** these techniques are applied to a situation which is very similar to ours. (In fact, it is almost identical; see §7 of the Introduction to **[5]**.) The basic definitions concerning abelian schemes can be found in **[16]**.

3. Finite group schemes, p-divisible groups, and Dieudonné modules. In the remaining sections we shall need to consider the finite subgroup *schemes* of an abelian variety and so, in this section, we review some of their properties. We fix a perfect field k of characteristic $p \neq 0$.

Let R be a finite k-algebra (so R is finite-dimensional as a vector space over k) and let $N = \operatorname{spec} R$. For any k-algebra R', a point of N in R' is simply a map of k-algebras $R \to R'$; thus $N(R') = \operatorname{Hom}_{k\text{-alg}}(R, R')$. If every $N(R')$ is given the structure of a commutative group in such a way that the maps $N(R') \to N(R'')$ induced by maps $R' \to R''$ are homomorphisms, then we call N, together with the family of group structures, a *finite group scheme* over k. As for affine algebraic groups, giving the family of group structures corresponds to giving a comultiplication map $R \xrightarrow{m} R \otimes_k R$.

EXAMPLE 3.1. (a) Any (commutative) finite group M can be regarded in an obvious way as an algebraic group over k and hence as a finite group scheme. Indeed, let R be a product of copies of k, one for each element of M, and let $N = \operatorname{spec} R$. Then N, as a set, is equal to M. The group law on M induces a comultiplication on R which, in turn, induces compatible group structures on $N(R')$ for all R'. If R' has no idempotents other than 0 and 1, then $N(R') = M$.

(b) $\boldsymbol{\mu}_{p^n} = \operatorname{spec} k[T]/(T^{p^n} - 1)$. Then $\boldsymbol{\mu}_{p^n}(R') = \{\zeta \in R' \mid \zeta^{p^n} = 1\}$ is a group under multiplication for any R', and these group structures make $\boldsymbol{\mu}_{p^n}$ into a finite group scheme. Note that $\boldsymbol{\mu}_{p^n}(R) = \{1\}$ if R has no nilpotents and, in particular, if R is an integral domain.

(c) $\boldsymbol{\alpha}_p = \operatorname{spec} k[T]/(T^p)$. Then $\boldsymbol{\alpha}_p(R') = \{a \in R' \mid a^p = 0\}$. As $(a + b)^p = a^p + b^p$ in any k-algebra, $\boldsymbol{\alpha}_p(R')$ is a group under addition, and these group laws make $\boldsymbol{\alpha}_p$ into a finite group scheme. Again $\boldsymbol{\alpha}_p(R')$ has only one element if R' has no nilpotents.

(d) $\boldsymbol{Z}/p\boldsymbol{Z} = \operatorname{spec} k[T]/(T^p - T)$. If R' has no idempotents other than 0 and 1 (e.g., R' an integral domain) then $(\boldsymbol{Z}/p\boldsymbol{Z})(R') = \boldsymbol{F}_p$, the prime subfield of R', which is a group under addition. This example is a special case of (a), because $k[T]/(T^p - T) = k[T]/T(T - 1) \cdots (T - (p - 1)) \approx k \times \cdots \times k$ (p copies).

The *rank* or *order* of a finite group scheme $N = \operatorname{spec} R$ is the dimension of R as a vector space over k. For example the order of the group scheme defined by M in 3.1(a) is the order of M, while the orders of $\boldsymbol{\mu}_{p^n}$, $\boldsymbol{\alpha}_p$, and $\boldsymbol{Z}/p\boldsymbol{Z}$ are p^n, p and p respectively.

A homomorphism from one finite group scheme $N_1 = \operatorname{spec} R_1$ to a second $N_2 = \operatorname{spec} R_2$ is a k-algebra homomorphism $R_2 \to R_1$ such that the induced maps $N_1(R') \to N_2(R')$ are all homomorphisms of commutative groups.

From now on we consider only finite group schemes of p-power order. The essential facts are the following.

Facts. 3.2.(a) They form an abelian category. Thus we may form kernels, quotients, etc. exactly as if we were working with a category of modules.

(b) When k is algebraically closed the only simple objects are $\boldsymbol{\mu}_p$, $\boldsymbol{\alpha}_p$, $\boldsymbol{Z}/p\boldsymbol{Z}$.

This means that any finite group scheme of p-power order has a composition series whose quotients are $\boldsymbol{\mu}_p$, $\boldsymbol{\alpha}_p$, or $\boldsymbol{Z}/p\boldsymbol{Z}$. There can be no homomorphism from one simple object to another of a different type.

(c) The category is self-dual, i.e., there is a contravariant functor $N \mapsto \hat{N}$ (= Cartier dual of N) which is an equivalence of the category with itself.

More precisely, for each N there is a pairing $N \times \hat{N} \to \boldsymbol{G}_m$ (= GL_1) such that, for any k-algebra R, the pairing induces isomorphisms $N(R) \xrightarrow{\approx} \mathrm{Hom}_R(\hat{N}, \boldsymbol{G}_m)$, $\hat{N}(R) \xrightarrow{\approx} \mathrm{Hom}_R(N, \boldsymbol{G}_m)$. For example, $(\boldsymbol{Z}/p\boldsymbol{Z})^\wedge = \boldsymbol{\mu}_p$ and the pairing $(\boldsymbol{Z}/p\boldsymbol{Z})(R) \times \boldsymbol{\mu}_p(R) \to \boldsymbol{G}_m(R)$ is $(n, \zeta) \mapsto \zeta^n$; $\hat{\boldsymbol{\alpha}}_p = \boldsymbol{\alpha}_p$ and the pairing $\boldsymbol{\alpha}_p(R) \times \boldsymbol{\alpha}_p(R) \to \boldsymbol{G}_m(R)$ is $(a, b) \to \exp(ab) = 1 + ab + \cdots + (ab)^{p-1}/(p-1)!$.

(d) $\mathrm{Hom}\,(\boldsymbol{Z}/p\boldsymbol{Z}, \boldsymbol{Z}/p\boldsymbol{Z}) = \boldsymbol{Z}/p\boldsymbol{Z}$, $\mathrm{Hom}\,(\boldsymbol{\mu}_p, \boldsymbol{\mu}_p) = \boldsymbol{Z}/p\boldsymbol{Z}$, $\mathrm{Hom}\,(\boldsymbol{\alpha}_p, \boldsymbol{\alpha}_p) = k$.

The statement for $\boldsymbol{Z}/p\boldsymbol{Z}$ is obvious, and that for $\boldsymbol{\mu}_p$ follows by Cartier duality. The map $\boldsymbol{\alpha}_p \to \boldsymbol{\alpha}_p$ corresponding to $c \in k$ is $(T \mapsto cT)$: $k[T]/(T^p) \to k[T]/(T^p)$ on the algebra of $\boldsymbol{\alpha}_p$ and $(a \mapsto ca)$: $\boldsymbol{\alpha}_p(R') \to \boldsymbol{\alpha}_p(R')$ on its points.

(e) If $0 \to N' \to N \to N'' \to 0$ is exact then $\mathrm{order}(N) = \mathrm{order}(N')\mathrm{order}(N'')$.

Let A be an abelian variety over k. For each n, $A_{p^n} \overset{\mathrm{df}}{=} \mathrm{Ker}(p^n\colon A \to A)$ is a finite group scheme of order $(p^n)^{2\dim(A)}$, i.e., the order is the same as when $p \neq$ characteristic(k). The system $A_p \hookrightarrow A_{p^2} \hookrightarrow \cdots$ is called the *p-divisible* (or *Barsotti-Tate*) *group* $A(p)$ of A. More generally, a p-divisible group of *height h* is a system of finite group schemes and maps $N = (N_1 \xrightarrow{i_1} N_2 \xrightarrow{i_2} N_3 \xrightarrow{i_3} \cdots)$ such that N_n has order p^{nh} and i_{n-1} identifies N_{n-1} with the kernel of $(N_n \xrightarrow{p^{n-1}} N_n)$. For example $\boldsymbol{Q}_p/\boldsymbol{Z}_p = (\boldsymbol{Z}/p\boldsymbol{Z} \to \boldsymbol{Z}/p^2\boldsymbol{Z} \to \cdots)$ and $\boldsymbol{\mu}_{p^\infty} = (\boldsymbol{\mu}_p \to \boldsymbol{\mu}_{p^2} \to \boldsymbol{\mu}_{p^3} \to \cdots)$ are p-divisible groups of height one. $A(p)$ is of height $2\dim(A)$. A homomorphism $\phi\colon N \to N'$ of p-divisible groups is a family of maps $\phi_n\colon N_n \to N'_n$ commuting with the maps i_n and i'_n.

Exercise 3.3. (k algebraically closed.) For any abelian variety A there are maps $\phi_n\colon A_{p^n} \to \hat{A}_{p^n}$ such that $\mathrm{Ker}(\phi_n) = \mathrm{Ker}(\phi_{n+1})$ for all sufficiently large n. Deduce that A has $\leqq p^{\dim A}$ points of order p, and that when equality holds $A(p) = (\boldsymbol{Q}_p/\boldsymbol{Z}_p)^{\dim(A)} \times (\boldsymbol{\mu}_{p^\infty})^{\dim(A)}$. (Such abelian variety is said to be *ordinary*.)

Let $W = W_k$ be the ring of Witt vectors over k; it is a complete discrete valuation ring of characteristic zero whose maximal ideal is generated by p and which has residue field k. There is a unique automorphism $a \mapsto a^{(p)}$ of W which induces the pth power map on k. If $k = \bar{\boldsymbol{F}}_p$ then W is the completion of the ring of integers in the maximal unramified extension $\boldsymbol{Q}_p^{un}$ of $\boldsymbol{Q}_p$ and $a \mapsto a^{(p)}$ is induced by the usual Frobenius automorphism of $\boldsymbol{Q}_p^{un}$ over $\boldsymbol{Q}_p$. Let $W[F, V]$ be the ring of noncommutative polynomials over W in which the relations $FV = p = VF$ and $Fa = a^{(p)}F$, $aV = Va^{(p)}$, hold for all $a \in W$. There is a contravariant functor, $N \mapsto DN =$ Dieudonné module of N, associating to each p-power order finite group scheme a $W[F, V]$-module which is of finite length as a W-module; D defines an antiequivalence of categories. The length of DN as a W-module is equal to the order of N. Thus manipulations with finite group schemes correspond exactly to manipulations with modules over the noncommutative ring $W[F, V]$. Examples:

$D(\boldsymbol{\mu}_p) = W/pW = k$; F acts as 0, V acts as 1;

$D(\boldsymbol{\alpha}_p) = k; F = 0, V = 0;$
$D(\boldsymbol{Z}/p\boldsymbol{Z}) = k; F = 1, V = 0.$
If N is unipotent and $pN = 0$, then $DN = \mathrm{Lie}(\hat{N})$; the bracket operation on $\mathrm{Lie}(\hat{N})$ is zero but it has the structure of a p-Lie-algebra and F acts as the "p-power" operation and V acts as zero. More generally, if N is unipotent and killed by p^n, then $DN = \mathrm{Hom}(N, W_n)$ where $W_1 = \boldsymbol{G}_a$ = the additive group and W_n = the Witt vectors of length n regarded as an algebraic group. There are canonical, nondegenerate, W-bilinear pairings $\langle\, , \rangle: DN \times D\hat{N} \to W \otimes \boldsymbol{Q}_p/\boldsymbol{Z}_p$ such that $\langle Fm, n\rangle = \langle m, Vn\rangle^{(p)}$, $\langle Vm, n\rangle^{(p)} = \langle m, Fn\rangle$.

The notion of Dieudonné module can be extended to p-divisible groups. On applying D to $N = (N_1 \to N_2 \to \cdots)$ we obtain a sequence of modules and maps $(DN_1 \leftarrow DN_2 \leftarrow \cdots)$, and we define $DN = \mathrm{proj\,lim}\, DN_n$. This is a $W[F, V]$-module which is free of finite rank equal to height(N) as a W-module.

In classifying p-divisible groups one begins by considering them up to isogeny: N and N' are *isogenous* if there is a surjective homomorphism $N \to N'$ with finite kernel or, equivalently, if there exists an injective homomorphism $DN' \to DN$ whose cokernel has finite length over W. If we write $W' = W[1/p] = W \otimes_{\boldsymbol{Z}_p} \boldsymbol{Q}_p$, $W'[F, F^{-1}] = W'[F, V] = W[F, V] \otimes_{\boldsymbol{Z}_p} \boldsymbol{Q}_p$, and $D'N$ for $DN \otimes_{\boldsymbol{Z}_p} \boldsymbol{Q}_p$ regarded as a $W'[F]$-module, then we see that N and N' are isogenous if and only if $D'N \approx D'N'$.

Let $\mathscr{M}$ be the category of $W'[F]$-modules whose objects occur as $D'N$ for some p-divisible group N. When k is algebraically closed one knows that $\mathscr{M}$ has exactly one simple object $D^\lambda = W'[F]/(F^r - p^s)$ for each rational number λ, $0 \leqq \lambda \leqq 1$, $\lambda = s/r$, $(r, s) = 1$. D^λ has dimension r over W', $\mathrm{End}(D^\lambda)$ is the unique division algebra over $\boldsymbol{Q}_p$ of degree r^2, and any $D \in \mathscr{M}$ can be written uniquely as a finite direct sum $D = (D^{\lambda_1})^{m_1} \oplus \cdots \oplus (D^{\lambda_t})^{m_t}$ with distinct λ_i. Then $\lambda_1, \cdots, \lambda_t$ are the *slopes* of D and $m_i r_i$, where $\lambda_i = s_i/r_i$, is the multiplicity of λ_i. We sometimes write $(D^{s/r})^m$ as $D^{sm/rm}$. Thus $D^{s/r}$ may now be a multiple of a simple module; it has slope s/r with multiplicity r and has dimension r over W'.

When k is algebraically closed and N is a p-divisible group over k, the slopes of $D'N$ are called the slopes of N. Clearly N is uniquely determined up to isogeny by its slopes and their multiplicities. For example, all p-divisible groups of height one are isogenous (in fact, isomorphic) to $\boldsymbol{\mu}_{p^\infty}$ or $\boldsymbol{Q}_p/\boldsymbol{Z}_p$ because $D'(\boldsymbol{\mu}_{p^\infty}) = D^1$ and $D'(\boldsymbol{Q}_p/\boldsymbol{Z}_p) = D^0$ are the only D^λ of dimension one over W'. There is only one simple D^λ of dimension two over W'; it is $D^{1/2} = D'(A(p))$ where A is a supersingular elliptic curve (cf. §5).

Let k have algebraic closure $\bar{k} \neq k$. Any p-divisible group N over k defines a p-divisible group $N_{\bar{k}}$ over $\bar{k}$ and it is known that $DN_{\bar{k}} \approx DN \otimes_{W_k} W_{\bar{k}}$. If $k = \boldsymbol{F}_q$ with $q = p^a$ then $F^a: DN \to DN$ is W-linear and so its characteristic polynomial $\det(T - F^a|DN) = \prod_1^{\mathrm{ht}(N)}(T - \alpha_i)$ is defined. The set of slopes of $D'N_{\bar{k}}$ is $\{\mathrm{ord}_q(\alpha_1), \mathrm{ord}_q(\alpha_2), \cdots\}$ where ord_q is the valuation of the algebraic closure of W'_k such that $\mathrm{ord}_q(q) = 1$.

If A is an abelian variety, we write DA for $DA(p)$. When A is defined over $\boldsymbol{F}_q$ the Frobenius endomorphism $\pi: (a_1, a_2, \cdots) \mapsto (a_1^q, a_2^q, \cdots)$ of A induces F^a on DA. The characteristic polynomial $P_A(T)$ of $\pi: A \to A$ in the sense of [**17**, §19] is $\det(T - F^a|DA)$. Thus the slopes of $D'A_{\bar{k}}$ can be read off from $P_A(T)$.

We can also define profinite group schemes $T_f A = \mathrm{proj\,lim} A_n$ and $T_l A = \mathrm{proj\,lim} A_{l^n}$. If $l \neq \mathrm{char}(k)$ and k is algebraically closed then $T_l A$ can be regarded (as before) as a free $\boldsymbol{Z}_l$-module of rank 2 dim(A). We write $T_fA = T_f^pA \times T_pA$.

Finally we note that to classify p-divisible groups up to isomorphism, it is necessary to classify the (F, V)-stable lattices in the objects of $\mathscr{M}$.

References. The best introduction to the subject matter of this section is [**6**].

4. $S(\bar{\boldsymbol{F}}_p)$ **as a family of abelian varieties.** Fix a prime p not dividing Δ. From §2 we know that points of $S(\bar{\boldsymbol{F}}_p)$ correspond to isomorphism classes of triples $(A, i, \bar{\phi})$ where A is an abelian variety of dimension $2d$ over $\bar{\boldsymbol{F}}_p$, i is an action of O_B on A, and $\bar{\phi}$ is a K^p-equivalence class of isomorphisms ϕ: $T_f^pA \to V(\boldsymbol{Z}_f^p)$ where $T_f^pA =$ proj $\lim_{p\nmid n} A_n(\bar{\boldsymbol{F}}_p)$. (Recall that ϕ_p: $T_pA \to V(\boldsymbol{Z}_p)$ is defined only over the ground ring with p inverted, and $\boldsymbol{F}_p[p^{-1}]$ is the zero ring.) The $O_B \otimes \bar{\boldsymbol{F}}_p$-module $\boldsymbol{t}_A$ satisfies the following condition:

(4.1) the subspaces corresponding to the idempotents $\left(\begin{smallmatrix}1&0\\0&0\end{smallmatrix}\right)$ and $\left(\begin{smallmatrix}0&0\\0&1\end{smallmatrix}\right)$ in $O_B \otimes \bar{\boldsymbol{F}}_p \approx M_2(\bar{\boldsymbol{F}}_p)$ are free $O_F \otimes \bar{\boldsymbol{F}}_p$-modules of rank 1.

If A is defined by equations $\Sigma a_{(i)}T^{(i)}$, let $A^{(p)}$ be the abelian variety over $\bar{\boldsymbol{F}}_p$ defined by the equations $\Sigma a_{(i)}^pT^i$. There is a Frobenius map $F = F_A$: $A \to A^{(p)}$ which takes a point with coordinates $(t_1, \cdots, t_n)$ to $(t_1^p, \cdots, t_n^p)$. The map F_A is a purely inseparable isogeny of degree p^{2d}, which means that A_F, the kernel of F, is a finite group scheme of order p^{2d} with only one point in any field (so that only $\boldsymbol{\alpha}_p$ and $\boldsymbol{\mu}_p$ occur in any composition series for it). As groups, $D(A) = D(A^{(p)})$, but the identity map $DA \to DA^{(p)}$ is (p)-linear, i.e., $am \mapsto a^{(p)}m$ for $a \in W$, $m \in DA$. The composite $DA \xrightarrow{\text{id}} DA^{(p)} \xrightarrow{DF_A} DA$ is a multiplication on the left by $F \in W[F, V]$ (see [**6**, p. 63]). Since F_A is zero on $\boldsymbol{t}_A$, $\boldsymbol{t}_A \approx \boldsymbol{t}_{A_F} \approx D\hat{A}_F \approx (DA_F)^* =$ dual(Coker($DA \xrightarrow{F} DA$)). Thus (4.1) may be checked on $DA/F(DA)$ instead of $\boldsymbol{t}_A$.

If $P \in S(\bar{\boldsymbol{F}}_p)$ corresponds to $(A, i, \bar{\phi})$ then, intuitively, we may think of the coordinates $(a_1, \cdots, a_r)$ of P as being the coefficients of the equations defining A. Thus Frob(P) corresponds to $(A^{(p)}, i^{(p)}, \phi^{(p)})$ where $i^{(p)}$ and $\bar{\phi}^{(p)}$ are such that F_A defines a map of triples $(A, i, \bar{\phi}) \to (A^{(p)}, i^{(p)}, \bar{\phi}^{(p)})$.

Finally we observe that there are "Hecke operators" acting. Let $g \in G(A_f)$ and suppose that K' is an open subgroup of $G(\boldsymbol{Z}_f)$ such that $g^{-1}K'g \subset K$; then $x \mapsto xg$: $G(A) \to G(A)$ induces a map $G(\boldsymbol{Q})\backslash G(A)/K_\infty K' \to G(\boldsymbol{Q})\backslash G(A)/K_\infty K$ which arises from a map of varieties $\mathscr{T}(g)$: ${}_{K'}S_C \to {}_KS_C$. If P corresponds to $(A', i', \bar{\phi}')$ then $\mathscr{T}(g)P$ corresponds to $(A, i, \bar{\phi})$ if there is an O_B-isogeny α: $A \to A'$ such that

$$\begin{array}{ccc} T_fA & \xrightarrow{T_f\alpha} & T_fA' \\ \downarrow{\scriptstyle\phi} & & \downarrow{\scriptstyle\phi'} \\ V(\boldsymbol{Z}_f) & \xrightarrow{mg} & V(\boldsymbol{Z}_f) \end{array}$$

commutes with m some positive integer. When we pass to S_p, only $G(A_f^p)$ continues to act: if $g \in G(A_f^p)$ and $P \in {}_{K'}S(\bar{\boldsymbol{F}}_p)$ and $\mathscr{T}(g)P \in {}_KS(\bar{\boldsymbol{F}}_p)$ correspond respectively to $(A', i', \bar{\phi}')$ and $(A, i, \bar{\phi})$ then there is an isogeny α: $A \to A'$ whose kernel has order prime to p and a commutative diagram

$$\begin{array}{ccc} T_f^pA & \xrightarrow{T_f^p\alpha} & T_fA' \\ \downarrow{\scriptstyle\phi} & & \downarrow{\scriptstyle\phi'} \\ V(\boldsymbol{Z}_f^p) & \xrightarrow{mg} & V(\boldsymbol{Z}_f^p) \end{array}$$

with m a positive integer prime to p. This definition is compatible with that over $\boldsymbol{C}$ in the sense that both mappings $\mathscr{T}(g)$ come by base change from a mapping $\mathscr{T}(g)\colon {}_{K'}S \times \operatorname{spec} \boldsymbol{Z}_{(p)} \to {}_{K}S \times \operatorname{spec} \boldsymbol{Z}_{(p)}$ where $\boldsymbol{Z}_{(p)} = \{m/n \in \boldsymbol{Q} | (n, p) = 1\}$. (More concretely, this means that if $(A, i, \bar{\phi})$ in characteristic zero specializes to $(\bar{A}, \bar{\imath}, \bar{\bar{\phi}})$ in characteristic p then $\mathscr{T}(g)(A, i, \bar{\phi})$ specializes to $\mathscr{T}(g)(\bar{A}, \bar{\imath}, \bar{\bar{\phi}})$.)

5. The isogeny classes. Fix a prime p not dividing Δ and consider pairs (A, i) where A is an abelian variety of dimension $2d$ over $\bar{\boldsymbol{F}}_p$ and i is a homomorphism $B \hookrightarrow \operatorname{End}^{\circ}(A)$ such that $i(1) = 1$. We write $A \sim A'$ if A and A' are isogenous, and $(A, i) \sim (A', i')$ if the pairs are B-isogenous in an obvious sense. $\mathscr{I}_p$ denotes the set of all B-isogeny classes and $(A, i) \otimes \boldsymbol{Q}$ the class containing (A, i). It will turn out (last paragraph below) that the map $(A, i, \bar{\phi}) \mapsto (A, i) \otimes \boldsymbol{Q}\colon S(\bar{\boldsymbol{F}}_p) \to \mathscr{I}_p$ is surjective and so, to describe $S(\bar{\boldsymbol{F}}_p)$, it suffices to describe $\mathscr{I}_p$ and the fibres of the map. The first is done in this section and the second in the next. Note that Frob (and $\mathscr{T}(g)$) preserves the fibres.

We first remark that, as in characteristic zero, there is a unique weak polarization on A inducing the given involution on B, and that it gives an F-equivalence class of pairings $A_n \times A_n \to \boldsymbol{G}_m$ for all n (cf. [**17**, §23]). In turn these pairings give an equivalence class of skew-symmetric pairings $\psi_l\colon T_l A \times T_l A \to T_l \boldsymbol{G}_m \approx \boldsymbol{Z}_l$ with nonzero discriminant for each $l \neq p$, and a similar pairing $\psi_p\colon DA \times DA \to W$; this last pairing satisfies the conditions $\psi_p(Fm, n) = \psi_p(m, Vn)^{(p)}$, $\psi_p(Vm, n)^{(p)} = \psi_p(m, Fn)$. All pairings satisfy $\psi(bm, n) = \psi(m, b^*n)$, $b \in B$.

The description of $\mathscr{I}_p$ will be based on the following classification of isogeny classes over a finite field. (Recall that an abelian variety over a field k is *simple* if it contains no nonzero, proper abelian subvariety defined over k and that any abelian variety is isogenous to a product of simple abelian varieties. If A is defined over $\boldsymbol{F}_q$ then $\pi = \pi_A$ is the Frobenius endomorphism $(a_1, a_2, \cdots) \mapsto (a_1^q, a_2^q \cdots)$.)

THEOREM 5.1. (a) *Let A be a simple abelian variety over $\boldsymbol{F}_q$ and let $E = \operatorname{End}^{\circ}(A)$. Then E is a division algebra with centre $\boldsymbol{Q}[\pi]$, π is an algebraic integer with absolute value $q^{1/2}$ under any embedding $\boldsymbol{Q}[\pi] \hookrightarrow \boldsymbol{C}$, and for any prime v of $\boldsymbol{Q}[\pi]$ the invariant of E at v is given by*

$$\begin{aligned} \operatorname{inv}_v(E) &= \tfrac{1}{2} && \text{if } v \text{ is real,} \\ &= 0 && \text{if } v | l,\ l \neq p, \\ &= \frac{\operatorname{ord}_v(\pi)}{\operatorname{ord}_v(q)} [\boldsymbol{Q}[\pi]_v : \boldsymbol{Q}_p] && \text{if } v | p. \end{aligned}$$

Moreover $2 \dim(A) = [\boldsymbol{Q}[\pi]: \boldsymbol{Q}][E: \boldsymbol{Q}[\pi]]^{1/2}$ *and* $e = [E: \boldsymbol{Q}[\pi]]^{1/2}$ *is the least common denominator of the* $\operatorname{inv}_v(E)$. *The characteristic polynomial $P_A(T)$ of π: $A \to A$ is $m(T)^e$ where $m(T)$ is the minimal polynomial of π over $\boldsymbol{Q}$.*

(b) *The simple abelian varieties A and A' over $\boldsymbol{F}_q$ are isogenous if and only if there is an isomorphism $\boldsymbol{Q}[\pi_A] \xrightarrow{\approx} \boldsymbol{Q}[\pi_{A'}]$ such that $\pi_A \mapsto \pi_{A'}$.*

(c) *Every algebraic integer π which has absolute value $q^{1/2}$ under any embedding $\boldsymbol{Q}[\pi] \hookrightarrow \boldsymbol{C}$ arises as the Frobenius endomorphism of a simple abelian variety A_π over $\boldsymbol{F}_q$.*

(d) *For any abelian varieties A and B over $\boldsymbol{F}_q$ and any prime l (including $l = p$) the canonical map*

$$\mathrm{Hom}(A, B) \otimes \boldsymbol{Z}_l \to \mathrm{Hom}(A(l), B(l)) = \mathrm{Hom}(T_l A, T_l B)$$

is an isomorphism. (If $l \neq p$ then $A(l)$ and $B(l)$ can be regarded as $\mathrm{Gal}(\bar{\boldsymbol{F}}_q/\boldsymbol{F}_q)$*-modules.)*

PROOF. The first part of (a) (the Riemann hypothesis) is due to Weil, part (c) to Honda, and the remainder to Tate; see **[17]**, **[21]**, [7], **[22]**, **[23]**.

For example, if in (c) we take $\pi = p^a$, $q = p^{2a}$ then we obtain an elliptic curve A_{p^a} such that $\mathrm{End}^\circ(A_{p^a})$ is a quaternion algebra over $\boldsymbol{Q}$ which is split everywhere except at p and the real prime. Any such elliptic curve is said to be *supersingular*.

It follows easily from (a) that if $\boldsymbol{Q}[\pi]$ has a real prime then either A is a supersingular elliptic curve or becomes isogenous to a product of two such curves over $\boldsymbol{F}_{q^2}$.

From now on we let p factor as $(p) = \mathfrak{p}_1 \cdots \mathfrak{p}_m$ in O_F, where the $\mathfrak{p}_i$ are distinct prime ideals, and we let d_i be the residue class degree of $\mathfrak{p}_i$ over p; thus $d = \sum d_i$.

PROPOSITION 5.2. *Let (A, i) be as above. The centralizer of B in* $\mathrm{End}^\circ(A)$ *is either:*

(a) *a quaternion algebra B' over F which splits except at the infinite primes, the primes where B is not split, and the $\mathfrak{p}_i$ for which d_i is odd, and there does not split; or*

(b) *a totally imaginary quadratic field extension F' of F which splits B.*

In the first case $A \sim A_0^{2d}$ where A_0 is a supersingular elliptic curve and in the second $A \sim A_0^2$ where A_0 is an abelian variety such that $F' \subset$ $\mathrm{End}^\circ(A_0)$.

PROOF. Suppose $A \sim A_0^r \times A_1$, $r \geqq 1$, where A_0 is a supersingular elliptic curve and $\mathrm{Hom}(A_0, A_1) = 0$. Then $\mathrm{End}^\circ(A) \approx M_r(E) \times \mathrm{End}^\circ(A_1)$, where $E = \mathrm{End}^\circ(A_0)$, and B embeds into $M_r(E)$. Consider $F \hookrightarrow M_r(E)$; we must have $d | 2r$, but $d = 2r$ is impossible because F does not split E [**19**, Theorem 10], and so $r = d$ or $2d$. The Skolem-Noether theorem shows that, when composed with an inner automorphism, the map $F \to M_r(E)$ factors through $M_r(\boldsymbol{Q})$. Thus the centralizer $C(F)$ of F in $M_r(E)$ is isomorphic to $M_{r/d}(F) \otimes E = M_{r/d}(E \otimes F)$. Let C be the centralizer of B in $M_r(E)$. Then $B \otimes_F C \approx C(F)$ because $C(F)$ and B are central simple algebras over F [**19**, §8]. It follows that either $r/d = 1$, $C = F$, and $B = E \otimes F$, or $r/d = 2$ and C is a quaternion algebra over F such that, in the Brauer group of F, $[B] + [C] = [E \otimes F]$. The first is impossible because B splits at infinite primes while E does not; thus the second holds, and this proves that case (a) of the proposition holds.

Next assume that $\mathrm{Hom}(A_0, A) = 0$ when A_0 is a supersingular elliptic curve, and fix a large subfield $\boldsymbol{F}_q$ of $\bar{\boldsymbol{F}}_p$ such that A and all its endomorphisms are defined over $\boldsymbol{F}_q$. From considering $A/\boldsymbol{F}_q$ we get a Frobenius endomorphism $\pi \in \mathrm{End}^\circ(A)$, and the assumption implies that there is no homomorphism $\boldsymbol{Q}[\pi] \to \boldsymbol{R}$. Consider

$$\begin{array}{ccccccc}
B & - & B[\pi] & - & B \otimes_F C & - & E \\
| & & | & & | & & \\
F & - & F[\pi] & - & C & & \\
| & & | & & & & \\
\boldsymbol{Q} & - & \boldsymbol{Q}[\pi] & & & &
\end{array}$$

where E is $\mathrm{End}^\circ(A)$ and C is the centralizer of B in E. Clearly, $F[\pi] = F$ would contradict our assumption. On the other hand we must have $[F[\pi]: F] \leqq 2$ and $F[\pi] = C$ for otherwise E would contain a commutative subring of dimension

$> 4d = 2\dim(A)$ over $\boldsymbol{Q}$, which is impossible by 5.1(a). Let $F' = C = F[\pi]$; it is a quadratic extension of F and can have no real prime because that would contradict our assumption. It splits B because, for any finite prime $l \neq p$, $(T_l A) \otimes_{\boldsymbol{Z}_l} \boldsymbol{Q}_l$ is free of rank 2 over $F'_l = F' \otimes \boldsymbol{Q}_l$, from which it follows that $B \otimes F'_l \approx M_2(F'_l)$, and we are assuming that B splits at any infinite prime or prime dividing p. Let e be an idempotent $\neq 0, 1$, in $(B \otimes F') \cap \mathrm{End}(A)$. Then $A_0 = eA$ is an abelian variety such that $A \sim A_0 \times A_0$. Since elements of F' commute with e, $F' \subset \mathrm{End}^\circ(A_0)$.

REMARK 5.3. In case (b) of 5.2, A_0 is isogenous to a power of a simple abelian variety, $A_0 \sim A_1^r$, because the centre of $E = \mathrm{End}^\circ(A)$ is a subfield of the field F'.

It follows that, for any pair (A, i) as above, A is isogenous to a power of a simple abelian variety and hence $\mathrm{End}^\circ(A)$ is a central simple algebra over the field $\boldsymbol{Q}[\pi]$. Let (A, i) and (A', i') be such that there exists an isogeny $\alpha: A \to A'$. The Skolem-Noether theorem shows that the map $B \xrightarrow{i} \mathrm{End}^\circ(A) \xrightarrow{\alpha_*} \mathrm{End}^\circ(A')$, where $\alpha_*(\gamma) = \alpha\gamma\alpha^{-1}$, differs from $i': B \to \mathrm{End}^\circ(A')$ by an inner automorphism $(\gamma \mapsto \beta\gamma\beta^{-1})$ of $\mathrm{End}^\circ(A')$. Thus $\beta\alpha$ is a B-isogeny $A \to A'$, and we have shown that $A \sim A'$ implies$(A, i) \sim (A', i')$.

We now consider in more detail the situation in 5.2(b). Let $\mathfrak{p}_1, \cdots, \mathfrak{p}_t$, $0 \leqq t \leqq m$, be the primes of F dividing p which split in F' and write $\mathfrak{p}_i = \mathfrak{q}_i\mathfrak{q}'_i$ for $i \leqq t$. Since $O_F \cap \mathrm{End}(A_0)$ and $\boldsymbol{Z}_p$ both act on $A_0(p)$, their tensor product does, and the splitting $F \otimes \boldsymbol{Q}_p \approx F_{\mathfrak{p}_1} \times \cdots \times F_{\mathfrak{p}_m}$ induces an isogeny $A_0(p) \sim A_0(\mathfrak{p}_1) \times \cdots \times A_0(\mathfrak{p}_m)$ and an isomorphism $D'A_0 \approx D'A_0(\mathfrak{p}_1) \times \cdots \times D'A_0(\mathfrak{p}_m)$. Clearly $A_0(\mathfrak{p}_i)$ has height $2d_i$ and so $D'(A_0(\mathfrak{p}_i))$ has dimension $2d_i$ over W'. Since $\psi_p(am, n) = \psi_p(m, an)$ for $a \in F$ the decomposition of $D'A_0$ is orthogonal for ψ_p, and ψ_p restricts to a non-degenerate form on each $D'(A_0(\mathfrak{p}_i))$. This implies that the set of slopes $\{\lambda_1, \lambda_2, \cdots\}$ of $D'(A_0(\mathfrak{p}_i))$ is invariant under $\lambda \mapsto 1 - \lambda$.

Fix an $i \leqq t$. As $F'_{\mathfrak{p}_i} \approx F'_{\mathfrak{q}_i} \times F'_{\mathfrak{q}'_i}$ acts on $D'A_0(\mathfrak{p}_i)$, $A_0(\mathfrak{p}_i)$ splits further: $A_0(\mathfrak{p}_i) \sim A_0(\mathfrak{q}_i) \times A_0(\mathfrak{q}'_i)$, $D'A_0(\mathfrak{p}_i) \approx D'(A_0(\mathfrak{q}_i)) \times D'(A_0(\mathfrak{q}'_i))$. Since $F'_{\mathfrak{q}_i} \subset \mathrm{End}^\circ(A(\mathfrak{q}_i))$ has degree $d_i = \mathrm{height}(A_0(\mathfrak{q}_i))$ over $\boldsymbol{Q}_p$, $A(\mathfrak{q}_i)$ is isogenous to a power of a simple p-divisible group: we may write $D'A(\mathfrak{q}_i) = D^{k_i/d_i}$, $0 \leqq k_i \leqq d_i$. Correspondingly, $D'A(\mathfrak{q}'_i) = D^{k'_i/d_i}$ with $k_i + k'_i = d_i$.

Fix an $i > t$. The $[F'_{\mathfrak{q}_i}: \boldsymbol{Q}_p] = 2d_i = \mathrm{height}\ A_0(\mathfrak{p}_i)$ and so, as above, $A_0(\mathfrak{p}_i)$ is isogenous to a power of a simple p-divisible group and we may write $DA_0(\mathfrak{p}_i) = D^{s/r}$. Since $s/r = 1 - s/r$ we must have $s/r = d_i/2d_i$. We write $k_i = d_i/2$.

Note that for some i, $1 \leqq i \leqq m$, we must have $k_i \neq d_i/2$ for otherwise all slopes of $A(p)$ would equal $\frac{1}{2}$. Then (see §3 and 5.1(a)) $|\pi/q^{1/2}| = 1$ for all primes v of $\boldsymbol{Q}[\pi]$ and so some power of it would equal one. On replacing $\boldsymbol{F}_q$ by a larger finite field we would have $\pi = q^{1/2}$, and this would imply that A is isogenous to a power of a supersingular elliptic curve, i.e., we would be in case (a). This means that $t \geqq 1$—at least one prime $\mathfrak{p}_i$ splits in F'.

THEOREM 5.4. *$\mathscr{I}_p$ contains one element for each pair $(F', (k_i)_{1 \leqq i \leqq m})$ where F' is a totally imaginary quadratic extension of F which splits B and is such that at least one $\mathfrak{p}_i$ splits in it; if $\mathfrak{p}_i$ splits in F' then k_i is an integer with $0 \leqq k_i \leqq d_i$ and otherwise $k_i = d_i/2$; for at least one i, $k_i \neq d_i - k_i$. When $\mathfrak{p}_i$ splits in F' we regard k_i and k'_i as being associated to $\mathfrak{q}_i$ and $\mathfrak{q}'_i$, and we do not distinguish between two pairs $(F', (k_i))$ and $(\bar{F}', (\bar{k}_i))$ which are conjugate over F. There is one additional "supersingular" element.*

For example, if $F = Q$ then there is the supersingular isogeny class and one class for each quadratic imaginary number field F' which splits B and in which p splits. If p splits completely in F then there is the supersingular class and one class for each totally imaginary quadratic extension F' of F of the right type and choice of one out of each pair of primes dividing a $\mathfrak{p}_i$ which splits in F'; one family of choices is not distinguished from the opposite family.

PROOF OF 5.4. We first construct an isogeny class $(A, i) \otimes Q$ corresponding to $(F', (k_1, \cdots, k_m))$. As before we let $\mathfrak{p}_i$, $1 \leqq i \leqq t$, be the primes dividing p which split in F'. Consider the ideal in $O_{F'}$,

$$\mathfrak{a} = \mathfrak{q}_1^{(f/d_1)k_1} \mathfrak{q}_1'^{(f/d_1)k_1'} \cdots \mathfrak{q}_{t+1}^{(f/d_{t+1})k_{t+1}} \cdots \mathfrak{q}_m^{(f/d_m)k_m}$$

where $f = 2d_1 \cdots d_m$. For some h, $\mathfrak{a}^h$ is principal, say $\mathfrak{a}^h = (\pi)$. If we write $a \mapsto \bar{a}$ for the nontrivial F-automorphism of F' then $\pi\bar{\pi} \in F$ and

$$(\pi\bar{\pi}) = (\mathfrak{q}_1^f \mathfrak{q}_1'^f \cdots \mathfrak{q}_m^f)^h \cap O_F = \mathfrak{p}_1^{fh} \cdots \mathfrak{p}_m^{fh} = (p^{fh}).$$

Thus $\pi\bar{\pi} = up^{fh}$ with u a unit in O_F. If u is a square in F then we may replace π by $\pi/u^{1/2}$ and obtain an equation $\pi\bar{\pi} = q$ with $q = p^{fh}$. If u is not a square then we replace π by π^2/u and obtain a similar equation with $q = p^{2fh}$. Note that the condition $k_i \neq k_i'$ for some i implies that $\pi \notin F$ and hence that $F' = F[\pi]$. Under any embedding $F' \hookrightarrow C$, F maps into R. Thus complex conjugation on C induces $a \mapsto \bar{a}$ on F'. In particular $\bar{\pi}$ is the complex conjugate of the complex number π and so $\pi\bar{\pi} = q$ implies that $|\pi| = q^{1/2}$.

Let A_π be the abelian variety corresponding, as in 5.1(c) to π, and let $E = \mathrm{End}^\circ(A_\pi)$. For any prime v of $Q[\pi]$

$$\begin{aligned} \mathrm{inv}_v(E) &= 0 && \text{if } v \nmid p, \\ &= (k_i/d_i)[Q[\pi]_v : Q_p] && \text{if } v \mid p \text{ and } \mathfrak{q}_i \mid v, \\ &= (k_i'/d_i)[Q[\pi]_v : Q_p] && \text{if } v \mid p \text{ and } \mathfrak{q}_i' \mid v. \end{aligned}$$

Let e_v be the denominator of $\mathrm{inv}_v(E)$ (when it is expressed in its lowest terms) and let e be the least common multiple of the e_v. Then $2\dim(A_\pi) = re$ where $r = [Q[\pi] : Q]$. Clearly $e_v | [F_{\mathfrak{q}}' : Q[\pi]_v]$ for any $\mathfrak{q}|v$, $v|p$, which implies (by class field theory) that F' splits E and (trivially) that e divides $[F' : Q[\pi]] = 2d/r$. As $[M_{2d/re}(E) : Q[\pi]] = (2d/re)^2 e^2 = [F' : Q[\pi]]^2$, F' embeds into $M_{2d/re}(E)$ [**19**, Theorem 10]. Let $A_0 = A_\pi^{2d/re}$. The characteristic polynomial $P_{A_\pi}(T)$ of π on A_π is $c_\pi(T)^e$ where $c_\pi(T)$ is the minimal polynomial of $\pi \in Q[\pi]$ over Q (5.1(a)). Thus $P_{A_0}(T)$ is $c_\pi(T)^{2d/r}$ which equals the characteristic polynomial of $\pi \in F'$ over Q. Corresponding to the splitting $F_p' = F_{\mathfrak{q}_1}' \times F_{\mathfrak{q}_1'}' \times \cdots$, we have $A_0(p) \sim A_0(\mathfrak{q}_1) \times A_0(\mathfrak{q}_1') \times \cdots$ and $P_{A_0}(T) = P_1(T)P_1'(T) \cdots$ where $P_i(T)$ (resp. $P_i'(T)$) is the characteristic polynomial of the image π_i of π in $F_{\mathfrak{q}_i}'$ (resp. π_i' of π in $F_{\mathfrak{q}_i'}'$) over Q_p. Thus (see §3) $A(\mathfrak{q}_i)$ has slopes equal to $\mathrm{ord}_q(\pi_i) = (fh/d_i)k_i/fh = k_i d_i$ and $A(\mathfrak{q}_i')$ has slopes equal to k_i'/d_i. Thus $A = A_0 \times A_0$, regarded as an abelian variety over $\bar{F}_p$, and the map i induced by $B \hookrightarrow M_2(F')$ represent an isogeny class corresponding to $(F', (k_1, \cdots, k_m))$.

(A_0^{2d}, i), where A_0 is a supersingular elliptic curve, represents the supersingular class.

Obviously if $(A, i) \sim (A', i')$ then both represent the supersingular class or correspond to the same pair $(F', (k_1, \cdots, k_m))$.

It remains to show that if (A, i) and (A', i') both correspond to $(F', (k_1, \ldots, k_m))$ then $(A, i) \sim (A', i')$. By considering A and A' to be defined over some finite subfield of $\bar{F}_p$ we get elements $\pi = \pi_A \in F'$ and $\pi' = \pi_{A'} \in F'$. The assumption implies that $\mathrm{ord}_{\mathfrak{q}}(\pi) = \mathrm{ord}_{\mathfrak{q}}(\pi')$ for all $\mathfrak{q}|p$, $\mathfrak{q}$ a prime of F', and 5.1(a) then shows that $|\pi/\pi'|_v = 1$ for all primes of F'. Thus π and π' differ by a root of 1 and so, after extending the finite field, we may take them to be equal. It follows that A and A', being isogenous to powers of the same abelian variety A_π, are themselves isogenous, and 5.3 completes the proof.

The proof that any class in $\mathscr{I}_p$ is represented by an element of $S(\bar{F}_p)$ requires the following lemma.

LEMMA 5.5. *Let $T \subset T_f A$ be such that $T_f A/T$ is finite; then there exists an isogeny $\alpha: A' \to A$ such that $T_f\alpha$ maps $T_f A'$ isomorphically onto T.*

PROOF. The finiteness of $T_f A/T$ means that, for all $n \gg 0$, the cokernel N of $T/nT \to T_f A/nT_f A$ is independent of n. Thus there is a map $A_n = T_f A/nT_f A \xrightarrow{\phi} N$. Define A' to be the cokernel of $a \mapsto (\phi(a), a): A_n \to N \times A$, and $\alpha: A' \to A$ to be $(b, a) \mapsto na$; then $(T_f\alpha)(T_f A') = T$.

Let (A, i) represent a class in $\mathscr{I}_p$ and let $O' = B \cap \mathrm{End}(A)$; it is an order in B. Regard (A, i) as being defined over a large finite field F_q; then $\mathrm{End}(A) \otimes Z_l \approx \mathrm{End}_{F_q}(T_l A)$ and $O' \otimes Z_l = (B \otimes Z_l) \cap \mathrm{End}_{F_q}(T_l A)$. For almost all l, $O' \otimes Z_l$ will equal $O_B \otimes Z_l$ and we take $T_l = T_l A$; for the remaining l we may choose a T_l of finite index in $T_l A$ which is stable under O_B, i.e., such that $\mathrm{End}_{F_q}(T_l) \cap (B \otimes Z_l) = O_B \otimes Z_l$. Note that $D(T_p/pT_p) = M$ is a $W[F, V]$-module of finite length over W. We may choose T_p such that M/FM satisfies (4.1). Let A' correspond to $T = \prod_l T_l$ as in the lemma. Then A' together with the obvious i and some $\bar{\phi}$ lies in $S(\bar{F}_p)$ and represents $(A, i) \otimes Q$.

6. An isogeny class. It remains to describe the set $Z = Z(A, i, \phi_A)$ of elements $(A', i', \bar{\phi}')$ of $S(\bar{F}_p)$ such that (A', i') is isogenous to a given pair (A, i). An O_B-isogeny $A' \xrightarrow{\alpha} A$ determines an injective map $T_f\alpha: T_f A' \to T_f A$ whose image Λ satisfies the following conditions:

(a) Λ is O_B-stable.

(b) $T_f A/\Lambda$ is a finite group scheme. (More precisely, $\mathrm{Coker}(\Lambda/n\Lambda \to T_f A/nT_f A) = \mathrm{Coker}(A'_n \to A_n) = \mathrm{Ker}(A' \xrightarrow{\alpha} A)$ for $n \gg 0$.)

(c) $D(\Lambda/p\Lambda)/FD(\Lambda/p\Lambda)$ satisfies (4.1). (For $\Lambda/p\Lambda = A'_p$ and so $D(\Lambda/p\Lambda)/FD(\Lambda/p\Lambda) = DA/F(DA)$.)

Consider all subobjects Λ of $T_f A$ satisfying (a), (b), (c). Any such Λ may be written $\Lambda = \Lambda^p \times \Lambda_p$ with Λ^p a Z_f^p-lattice in $T_f^p A$ (in the usual sense of modules over Z_f^p) and $\Lambda_p \subset T_p A$. We let Y be the set of pairs $(\Lambda, \bar{\phi})$ with Λ as above and $\bar{\phi}$ a K-equivalence class of isomorphisms $\Lambda^p \xrightarrow{\approx} V(Z_f^p)$. Since $(\Lambda, \bar{\phi})$ is determined by a pair $((\Lambda^p, \bar{\phi}), \Lambda_p)$, we may write $Y = Y^p \times Y_p$.

By 5.5, every $(\Lambda, \bar{\phi}) \in Y$ arises from a triple $(A', i', \bar{\phi}') \in S(\bar{F}_p)$ equipped with an isogeny $\alpha; A' \to A$. Thus we have a surjective map $Y \twoheadrightarrow Z \subset S(\bar{F}_p)$.

For n a positive integer we set $n(\Lambda, \bar{\phi}) = (n\Lambda, n\bar{\phi})$, and we define $Y \otimes Q$ to be the set of pairs (y, n) with $y \in Y$ and $n \in Z_{>0}$, where (y, n) and (y', n') are identified if $n'y = ny'$. Then we write $Y \otimes \mathbf{Q} = (Y^p \otimes \mathbf{Q}) \times (Y_p \otimes \mathbf{Q})$ where $Y^p \otimes \mathbf{Q}$ may be identified with the set of O_B-stable lattices Λ in $(T_f A) \otimes \mathbf{Q}$ equipped with a K-equivalence class of isomorphisms $\Lambda \xrightarrow{\approx} T_f A$.

There is an action of $H(\mathbf{Q}) = \mathrm{End}^\circ_{O_B}(A)^\times$ on $Y \otimes \mathbf{Q}$: for $\alpha \in H(\mathbf{Q})$ we choose a positive integer m such that $m\alpha$ is an isogeny of A and define $\alpha(\Lambda, \bar{\phi}, n) = (T_f(m\alpha), \overline{\phi T_f(m\alpha)^{-1}}, mn)$.

LEMMA 6.1. *The map $Y \to Z$ described above induces a bijection $H(\mathbf{Q})\backslash Y \otimes \mathbf{Q} \xrightarrow{\approx} Z$.*

PROOF. $(\Lambda, \bar{\phi}, n)$ and $(\Lambda', \bar{\phi}', n')$ map to the same element of $S(\bar{\mathbf{F}}_p)$ if and only if there exist O_B-isogenies

$$A' \underset{\alpha'}{\overset{\alpha}{\rightrightarrows}} A$$

and an O_B-isomorphism $\phi_0\colon T_f A' \to V(\mathbf{Z}_f)$ such that

$$n((T_f\alpha)T_f A', \overline{\phi_0(T_f\alpha)}) = (\Lambda, \bar{\phi}) \quad \text{and} \quad n'((T_f\alpha')T_f A', \overline{\phi_0(T_f\alpha')}) = (\Lambda', \bar{\phi}').$$

Then $\alpha'\alpha^{-1}$ makes sense as an element of $\mathrm{End}^\circ(A)$ and $\alpha'\alpha^{-1}(\Lambda, \bar{\phi}, n) = (\Lambda', \bar{\phi}', n')$.

LEMMA 6.2. *The map $G(\mathbf{A}_f^p) \to Y^p \otimes \mathbf{Q}$, $g \mapsto (g(T_f A), \phi_A g^{-1})$, induces a bijection $G(\mathbf{A}_f^p)/K \to Y^p \otimes \mathbf{Q}$.*

PROOF. Obvious.

LEMMA 6.3. *There is a one-one correspondence between $Y_p \otimes \mathbf{Q}$ and the set X of $W[F, V]$-submodules M of $D'A$ which are free of rank $4d$ over W, O_B-stable, and such that M/FM satisfies* (4.1).

PROOF. $p\colon \Lambda \to \Lambda$ induces maps $i_n\colon \Lambda/p^n\Lambda \hookrightarrow \Lambda/p^{n+1}\Lambda$ which define a p-divisible group $\Lambda(p) = (\Lambda/p^n\Lambda, i_n)$. The exact sequence $0 \to \Lambda \to T_p A \to N \to 0$ (N finite) gives rise to $0 \to N \to \Lambda(p) \to A(p) \to 0$. On applying D we get

$$0 \to DA \to D\Lambda(p) \to DN \to 0.$$

Since DN is torsion, we may identify $D'\Lambda(p)$ with $D'A$. To $(\Lambda, n) \in Y^p$ we associate $n^{-1}(D\Lambda(p)) \in X$.

THEOREM 6.4. *With the above notations,*

$$Z(A, i, \bar{\phi}) \approx H(\mathbf{Q})\backslash G(\mathbf{A}_f^p) \times X/K^p.$$

Frob *acts by sending $M \in X$ to FM; the Hecke operator $\mathcal{T}(g), g \in G(\mathbf{A}_f^p)$, "acts" by multiplication on the right on $G(\mathbf{A}_f^p)$.*

PROOF. This simply summarizes the above.

It remains to give a more explicit description of X. Note that, corresponding to the splitting $D'A \approx D'A(\mathfrak{p}_1) \times \cdots \times D'A(\mathfrak{p}_m)$, we have $X \approx X_1 \times \cdots \times X_m$. It is convenient to write $\bar{G}_i(\mathbf{Z}_p) = \mathrm{Aut}_{O_B}(A(\mathfrak{p}_i))$ and $\bar{G}_i(\mathbf{Q}_p) = \mathrm{End}^\circ_{O_B}(A(\mathfrak{p}_i))^\times = \mathrm{End}_{O_B}(D'A(\mathfrak{p}_i))^\times$. In the simplest cases $\bar{G}_i(\mathbf{Q}_p)$ acts transitively on the lattices $M \subset D'A(\mathfrak{p}_i)$ which belong to X_i, and in this case $X_i \approx \bar{G}_i(\mathbf{Q}_p)/\bar{G}_i(\mathbf{Z}_p)$. (To say that $\bar{G}_i(\mathbf{Q}_p)$ acts transitively means that each $A'(\mathfrak{p}_i)$ is isomorphic to $A(\mathfrak{p}_i)$ and not merely isogenous; cf. [**6**, p. 93].)

EXAMPLES 6.5. (a) $F = \boldsymbol{Q}$, F' is a quadratic extension of $\boldsymbol{Q}$, $(p) = \mathfrak{q}\mathfrak{q}'$ in F', and $(A, i, \bar{\phi})$ is in the isogeny class corresponding to $(F', (0))$.

Then $A(p) \approx (\boldsymbol{Q}_p/\boldsymbol{Z}_p)^2 \times (\boldsymbol{\mu}_{p^\infty})^2$ with $O_B \otimes \boldsymbol{Z}_p = M_2(\boldsymbol{Z}_p)$ acting in the obvious way on each factor. Thus $\bar{G}(\boldsymbol{Z}_p) = \{\left(\begin{smallmatrix} a & 0 \\ 0 & b \end{smallmatrix}\right) \mid a, b \in \boldsymbol{Z}_p^\times\}$ and $\bar{G}(\boldsymbol{Q}_p) = \{\left(\begin{smallmatrix} a & 0 \\ 0 & b \end{smallmatrix}\right) \mid a, b \in \boldsymbol{Q}_p^\times\}$. In this case $X = \bar{G}(\boldsymbol{Q}_p)/\bar{G}(\boldsymbol{Z}_p)$. Frob acts as $\left(\begin{smallmatrix} 1 & 0 \\ 0 & p \end{smallmatrix}\right)$.

(b) As above, except $(A, i, \bar{\phi})$ corresponds to $(F', (1))$.

Then $A(p) \approx (\boldsymbol{\mu}_{p^\infty})^2 \times (\boldsymbol{Q}_p/\boldsymbol{Z}_p)^2$ (i.e., in the splitting $F'_p = F'_{\mathfrak{q}} \times F'_{\mathfrak{q}'}$, $F'_{\mathfrak{q}}$ now corresponds to the $\boldsymbol{\mu}_{p^\infty}$ factor). $\bar{G}(\boldsymbol{Z}_p)$, $\bar{G}(\boldsymbol{Q}_p)$ and X are as before but Frob acts as $\left(\begin{smallmatrix} p & 0 \\ 0 & 1 \end{smallmatrix}\right)$.

(c) $F = \boldsymbol{Q}$, $(A, i, \bar{\phi})$ is in the supersingular class.

Then $D'A(p) \approx D^{1/2} \times D^{1/2}$ and $\mathrm{End}(D^{1/2}) = B'_p$, the unique division quaternion algebra over $\boldsymbol{Q}_p$. B acts through the embedding

$$B \otimes \boldsymbol{Q}_p \approx M_2(\boldsymbol{Q}_p) \xhookrightarrow{\text{canon}} M_2(B'_p).$$

Thus $\bar{G}(\boldsymbol{Q}_p)$, the centralizer of $B \otimes \boldsymbol{Q}_p$ in $M_2(B'_p)$, is $(B'_p)^\times$. Moreover $\bar{G}(\boldsymbol{Z}_p)$ may be taken to be $O^\times$ where O is the maximal order in B'_p. In this case $X \approx \bar{G}(\boldsymbol{Q}_p)/\bar{G}(\boldsymbol{Z}_p)$. Frob acts as multiplication by $\tilde{\omega}$, a generator of the maximal ideal of O.

(d) F arbitrary, p splits completely in F, $(p) = \mathfrak{p}_1 \cdots \mathfrak{p}_d$, $(A, i, \bar{\phi})$ corresponds to $(F', (k_1, \cdots, k_d))$.

Then $X \approx X_1 \times \cdots \times X_d$ where X_i is as in case (a) if $\mathfrak{p}_i$ splits in F' and $k_i = 0$, as in case (b) if $\mathfrak{p}_i$ splits and $k_i = 1$, and as in case (c) otherwise.

(e) The general case. For a statement of the result, see **[14]**. (This case is treated in detail in: J. Milne, *Etude d'une classe d'isogenie*, Séminaire sur les groupes réductifs et les formes automorphes, Université Paris VII (1977–1978).)

Added in proof (*November* 1978). The outline of a proof in [12] of the conjecture for those Shimura varieties which are moduli varieties is less complete than appeared at the time of the conference. The above proof (completed in the report referred to in 6.5(e)) for the case of the multiplicative group of a quaternion algebra differs a little from the outline in that it depends more heavily on the Honda-Tate classification of isogeny classes of abelian varieties over finite fields. The complete seminar referred to in 6.5(e), which redoes in greater detail much of the material in this article and [3], will be published in the series ***Publications Mathématiques de l'Université Paris* 7**.

BIBLIOGRAPY

1. M. Artin, *The implicit function theorem in algebraic geometry*, Colloq. in Algebraic Geometry, Bombay, 1969, pp. 13–34.

2. ———, *Théorèmes de représentabilité pour les espaces algébriques*, Presses de l'Université de Montréal, Montréal, 1973.

3. W. Casselman, *The Hasse-Weil ζ-function of some moduli varieties of dimension greater than one*, these PROCEEDINGS, part 2, pp. 141–163.

4. P. Deligne, *Travaux de Shimura*, Séminaire Bourbaki, 1970/71, no. 389.

5. P. Deligne and M. Rapoport, *Les schémas de modules de courbes elliptiques*, Antwerp II, Lecture Notes in Math., vol. 349, Springer, New York, pp. 143–316.

6. M. Demazure, *Lectures on p-divisible groups*, Lecture Notes in Math., vol. 302, Springer, New York, 1972.

7. T. Honda, *Isogeny classes of abelian varieties over finite fields*, J. Math. Soc. Japan **20** (1968), 83–95.

8. Y. Ihara, *The congruence monodromy problems*, J. Math. Soc. Japan **20** (1968), 107–121.

9. ———, *On congruence monodromy problems*, Lecture Notes, vols. 1, 2, Univ. Tokyo, 1968, 1969.

10. ———, *Non-abelian class fields over function fields in special cases*, Actes Congr. Internat. Math., vol. 1, Nice, 1970, pp. 381–389.

11. ———, *Some fundamental groups in the arithmetic of algebraic curves over finite fields*, Proc. Nat. Acad. Sci. U.S.A. **72** (1975), 3281–3284.

12. R. Langlands, *Letter to Rapoport* (Dated June 12, 1974—Sept. 2, 1974).

13. ———, *Some contemporary problems with origins in the Jugendtraum*, Proc. Sympos. Pure Math., vol. 28, Amer. Math. Soc., Providence, R. I., 1976, pp. 401–418.

14. ———, *Shimura varieties and the Selberg trace formula*, Canad. J. Math. **29** (1977), 1292–1299.

15. Y. Morita, *Ihara's conjectures and moduli space of abelian varieties*, Thesis, Univ. Tokyo (1970).

16. D. Mumford, *Geometric invariant theory*, Springer, 1965.

17. ———, *Abelian varieties*, Oxford, 1970.

18. M. Rapoport, *Compactifications de l'espace de modules de Hilbert-Blumenthal*, Compositio Math. (to appear).

19. J. P. Serre, Exposé 7, Séminaire Cartan 1950/51.

20. H. Swinnerton-Dyer, *Analytic theory of abelian varieties*, L.M.S. lecture notes **14** (1974).

21. J. Tate, *Endomorphisms of abelian varieties over finite fields*, Invent. Math. **2** (1966), 134–144.

22. ———, *Classes d'isogénie des variétés abéliennes sur un corps fini*, Séminaire Bourbaki, 1968–1969, no. 352.

23. W. Waterhouse and J. Milne, *Abelian varieties over finite fields*, Proc. Sympos. Pure Math., vol. 20, Amer. Math. Soc., Providence, R.I., 1971, pp. 53–64.

UNIVERSITY OF MICHIGAN

Proceedings of Symposia in Pure Mathematics
Vol. 33 (1979), part 2, pp. 185–192

COMBINATORICS AND SHIMURA VARIETIES mod p (BASED ON LECTURES BY LANGLANDS)

R. E. KOTTWITZ

One of the major problems in the study of Shimura varieties is that of expressing their Hasse-Weil zeta-functions as products of L-functions associated to automorphic forms. In **[1]** this problem has been solved for a certain class of Shimura varieties: those associated to an algebraic group G over $\boldsymbol{Q}$ which is the inverse image under $\mathrm{Res}_{F/\boldsymbol{Q}}(B^*) \xrightarrow{\text{norm}} \mathrm{Res}_{F/\boldsymbol{Q}}(F^*)$ of some connected $\boldsymbol{Q}$-subgroup of $\mathrm{Res}_{F/\boldsymbol{Q}}(F^*)$, where B is a totally indefinite quaternion algebra over a totally real number field F. In this paper we will discuss only the case $G = \mathrm{Res}_{F/\boldsymbol{Q}}(B^*)$. In this special case, the zeta-functions of the corresponding Shimura varieties can be expressed as products of automorphic L-functions associated to the group G itself (in general, groups besides G are needed as well). This case has already been discussed in **[2]** and the purpose of the present discussion is to provide further details on the combinatorial exercise referred to on the last page of that paper. This is worked out in detail in §4 of **[1]** for all of the subgroups of $\mathrm{Res}_{F/\boldsymbol{Q}}(B^*)$ mentioned previously, but even so it is interesting to carry out the exercise in our situation (with further simplifying assumptions added later) so that the main ideas can be understood more easily.

First we sketch the arguments of **[2]** which lead to the combinatorial exercise. Let K be a compact open subgroup of $G(\boldsymbol{A}_f)$ (where $\boldsymbol{A}_f$ is the ring of adeles of $\boldsymbol{Q}$ having component 0 at ∞). Let S_K be the Shimura variety obtained from G and K as in **[2]**. The variety S_K is defined over $\boldsymbol{Q}$ since B is totally indefinite, so its zeta-function $Z(s, S_K)$ is a product over the places v of $\boldsymbol{Q}$ of local factors $Z_v(s, S_K)$.

Let F' be a finite extension field of F which is Galois over $\boldsymbol{Q}$. For $\sigma \in \mathrm{Gal}(F'/\boldsymbol{Q})/\mathrm{Gal}(F'/F)$, let $H_\sigma = \mathrm{GL}_2(\boldsymbol{C})$ and let V_σ be the standard representation of H_σ on $\boldsymbol{C}^2$. Then ${}^LG^\circ = \prod_\sigma H_\sigma$ has a natural representation r on $V = \bigotimes_\sigma V_\sigma$, which may be extended to a representation of the L-group ${}^LG = \mathrm{Gal}(F'/\boldsymbol{Q}) \ltimes {}^LG^\circ$ by putting $r(\tau)(\bigotimes_\sigma v_\sigma) = \bigotimes_\sigma w_\sigma$ for $\tau \in \mathrm{Gal}(F'/\boldsymbol{Q})$ where $w_\sigma = v_{\tau^{-1}\sigma}$.

The result of **[2]** is that $Z(s, S_K) = \prod_\pi L(s - d/2, \pi, r)^{m(\pi,K)}$ up to a finite number of local factors, where $d = [F: \boldsymbol{Q}]$, π runs over the representations of $G(\boldsymbol{A})/Z(\boldsymbol{R})$ (where Z is the center of G) which occur in $L^2(G(\boldsymbol{Q})Z(\boldsymbol{R})\backslash G(\boldsymbol{A}))$, and $m(\pi, K)$ is an integer which is associated to π and K in a way that is described in **[2]**. The product over π is actually finite since $m(\pi, K)$ turns out to be 0 for all but a finite number of π. The precise result is

---

AMS (MOS) subject classifications (1970). Primary 05C05, 14G10, 22E50, 22E55.

(1) $$Z_p(s, S_K) = \prod_\pi L(s - d/2, \pi_p, r)^{m(\pi,K)}$$

where π_p is the local factor of π at a finite rational prime p which satisfies:

(A) F is unramified at p;

(B) B splits at every prime of F above p;

(C) $K = K_pK^p$ where K_p is a maximal compact subgroup of $G(\boldsymbol{Q}_p)$ and K^p is a compact open subgroup of $G(\boldsymbol{A}_f^p)$, $\boldsymbol{A}_f^p$ being the ring of adeles of $\boldsymbol{Q}$ with component 0 at p and ∞.

For the rest of this paper p will denote a fixed rational prime satisfying (A)–(C). Because of these assumptions about p, S_K has good reduction at p, and $m(\pi, K) = 0$ unless π_p is unramified. Take the logarithm of each side of (1), and use the power series expansion of $\log(1 - X)$. Comparing the coefficients of the powers of p^{-s} in the two sides, we see that (1) is equivalent to

(2) $$\text{Card } S_K(\boldsymbol{F}_{p^j}) = \sum_\pi m(\pi, K)p^{jd/2} \operatorname{tr}(r(g_{\pi_p})^j)$$

for all positive integers j, where π runs over all the representations of $G(\boldsymbol{A})/Z(\boldsymbol{R})$ which are unramified at p and which occur in $L^2(G(\boldsymbol{Q})Z(\boldsymbol{R})\backslash G(\boldsymbol{A}))$, and where g_{π_p} is an element of the semisimple conjugacy class of ${}^LG$ associated to π_p.

For every place v of $\boldsymbol{Q}$ choose a Haar measure dg_v on $G(\boldsymbol{Q}_v)$ such that the measure of $G(\boldsymbol{Z}_v)$ is 1 for almost all finite places v. Use dg_v to define the action of $L^1(G(\boldsymbol{Q}_v))$ on representations of $G(\boldsymbol{Q}_v)$. Choose a Haar measure dz_∞ on $Z(\boldsymbol{R})$, and use the quotient measure dg_∞/dz_∞ to define the action of $L^1(G(\boldsymbol{R})/Z(\boldsymbol{R}))$ on representations of $G(\boldsymbol{R})/Z(\boldsymbol{R})$. Let f^p be the characteristic function of the subset K^p of $G(\boldsymbol{A}_f^p)$ divided by the measure of K^p (with respect to $\prod_{v\neq p,\infty} dg_v$). It is pointed out in [2] that there exists $f_\infty \in C_c^\infty(G(\boldsymbol{R})/Z(\boldsymbol{R}))$ such that $m(\pi, K) = m(\pi)\operatorname{tr} \pi^p(f^pf_\infty)$ where $\pi^p = \bigotimes_{v\neq p}\pi_v$ and $m(\pi)$ is the multiplicity of π in $L^2(G(\boldsymbol{Q})Z(\boldsymbol{R})\backslash G(\boldsymbol{A}))$ (it is known that this multiplicity is 1, but we will not need to use this fact). By the theory of Hecke algebras, there exists a unique function $f_p^{(j)}$ in the Hecke algebra $\mathscr{H}(G(\boldsymbol{Q}_p), K_p)$ such that

$$\begin{aligned} \operatorname{tr} \pi_p(f_p^{(j)}) &= p^{jd/2} \operatorname{tr}(r(g_{\pi_p})^j) \quad \text{if } \pi_p \text{ is unramified,} \\ &= 0 \quad \text{if } \pi_p \text{ is ramified.} \end{aligned}$$

So the right side of (2) becomes $\sum_\pi m(\pi) \operatorname{tr} \pi(f_p^{(j)}f^pf_\infty)$, which can be evaluated in terms of orbital integrals by means of the trace formula. For $\gamma \in G(\boldsymbol{Q})$, let $d\bar{g}$ be the quotient of $\prod_v dg_v$ on $G(\boldsymbol{A})$ by some Haar measure dg_γ on $G_\gamma(\boldsymbol{A})$ (where as usual G_γ denotes the centralizer of γ in G), and let $m_\gamma = \text{meas}(G_\gamma(\boldsymbol{Q})Z(\boldsymbol{R})\backslash G_\gamma(\boldsymbol{A}))$ where the measure used is the quotient of dg_γ by the measure on $G_\gamma(\boldsymbol{Q})Z(\boldsymbol{R}) \approx G_\gamma(\boldsymbol{Q}) \times Z(\boldsymbol{R})$ which is the product of the Haar measure on $G_\gamma(\boldsymbol{Q})$ which gives points measure 1 with dz_∞. Applying the trace formula, we find that (2) is equivalent to

(3) $$\text{Card } S_K(\boldsymbol{F}_{p^j}) = \sum_\gamma m_\gamma \int_{G_\gamma(\boldsymbol{A})\backslash G(\boldsymbol{A})} f_p^{(j)}f^pf_\infty(g^{-1}\gamma g)\, d\bar{g}$$

where γ runs over a set of representatives of the conjugacy classes in $G(\boldsymbol{Q})$. It turns out that the integral $\int_{G_\gamma(\boldsymbol{R})\backslash G(\boldsymbol{R})} f_\infty(g^{-1}\gamma g)\, d\bar{g}_\infty$ is 0 unless γ is central or elliptic at ∞, and so the right side of (3) is

(4) $$\sum_{\gamma} a_{\gamma} \int_{G_{\gamma}(A_f)\backslash G(A_f)} f^p f_p^{(j)}(g^{-1}\gamma g)\, d\bar{g}_f$$

where $a_{\gamma} = m_{\gamma} \int_{G_{\gamma}(\boldsymbol{R})\backslash G(\boldsymbol{R})} f_{\infty}(g^{-1}\gamma g)\, d\bar{g}_{\infty}$ and γ runs over a set of representatives for the conjugacy classes in $G(\boldsymbol{Q})$ which are elliptic or central at ∞.

To go further we must know more about Card $S_K(\boldsymbol{F}_{p^j})$. To know this number for all j it is enough (in principle anyway) to know explicitly the set $S_K(\bar{\boldsymbol{F}}_p)$ and the action of the Frobenius Φ on this set.

Assumption. To make this description simpler, we assume from now on that p remains prime in F.

First of all, for every triple $\rho = (L, m', m'')$ consisting of a totally imaginary quadratic extension L of F which is split at p and can be embedded in B, and integers m', m'' such that $m'' > m' \geqq 0$ and $m' + m'' = d$, there is an associated Φ-stable subset Y_{ρ} of $S_K(\bar{\boldsymbol{F}}_p)$, and $S_K(\bar{\boldsymbol{F}}_p)$ is the disjoint union of the sets Y_{ρ} and another Φ-stable subset Y_0. Let I_{ρ} be L^* regarded as an algebraic group over $\boldsymbol{Q}$. It turns out that Y_{ρ} is isomorphic as a $\mathrm{Gal}(\bar{\boldsymbol{F}}_p/\boldsymbol{F}_p)$-set to $I_{\rho}(\boldsymbol{Q})\backslash(G(A_f^p) \times X_{\rho})/K^p$ for some set X_{ρ} on which $I_{\rho}(\boldsymbol{Q}_p)$ and Φ act, the two actions commuting with each other. An embedding $L \to B$ gives us a map $I_{\rho} \to G$, and $I_{\rho}(\boldsymbol{Q})$ acts on $G(A_f^p)$ via $I_{\rho}(\boldsymbol{Q}) \to I_{\rho}(A_f^p) \to G(A_f^p)$; also $I_{\rho}(\boldsymbol{Q})$ acts on X_{ρ} via $I_{\rho}(\boldsymbol{Q}) \to I_{\rho}(\boldsymbol{Q}_p)$, and K^p acts on the product $G(A_f^p) \times X_{\rho}$ by acting on $G(A_f^p)$ alone.

For any positive integer j and for $x \in X_{\rho}$, let $T_x^j = \{g \in I_{\rho}(\boldsymbol{Q}_p) \colon \Phi^j x = gx\}$, and let δ_x^j be the characteristic function of T_x^j. Choose a Haar measure μ on $I_{\rho}(\boldsymbol{Q}_p)$, and for $\gamma \in I_{\rho}(\boldsymbol{Q})$ let

(5) $$\varphi^{(j)}(\gamma) = \sum_{x \in X_{\rho} \bmod I_{\rho}(\boldsymbol{Q}_p)} \frac{1}{\mu(I_x)}\, \delta_x^j(\gamma)$$

where I_x is the stabilizer of x in $I_{\rho}(\boldsymbol{Q}_p)$. Choose a Haar measure ν on $G_{\gamma}(A_f^p)$. Let $\nu' = \prod_{v \neq p, \infty} dg_v$ where dg_v is the Haar measure on $G(\boldsymbol{Q}_v)$ chosen before, and recall that we have defined f^p to be the characteristic function of K^p divided by $\nu'(K^p)$. It is easy to show that the contribution of Y_{ρ} to $\mathrm{Card}(S_K(\boldsymbol{F}_{p^j}))$ is

(6) $$\sum_{\gamma \in I_{\rho}(\boldsymbol{Q})} b_{\gamma} \varphi^{(j)}(\gamma) \int_{G_{\gamma}(A_f^p)\backslash G(A_f^p)} f^p(g^{-1}\gamma g)\, \frac{d\nu'}{d\nu}$$

where $b_{\gamma} = \mathrm{meas}(I_{\rho}(\boldsymbol{Q})\backslash G_{\gamma}(A_f^p) \times I_{\rho}(\boldsymbol{Q}_p))$. The measure used is the quotient of $\nu \times \mu$ by the measure on $I_{\rho}(\boldsymbol{Q})$ which gives every point measure 1. We see that (4) and (6) have roughly the same form, but we must sum (6) over all triples $\rho = (L, m', m'')$, and it appears at first that there is nothing in (4) which corresponds to the sum over m', m''. But in fact we will now see that $f_p^{(j)}$ can be written in a natural way as a sum of functions, and it is this sum which corresponds to the sum over m', m''.

The function $f_p^{(j)}$ in the Hecke algebra $\mathscr{H} = \mathscr{H}(G(\boldsymbol{Q}_p), K_p)$ is characterized by the equation $\mathrm{tr}\, \pi_p(f_p^{(j)}) = p^{jd/2}\, \mathrm{tr}(r(g_{\pi_p})^j)$ for all unramified representations π_p of $G(\boldsymbol{Q}_p)$. Since p remains prime in F, F_p is a field (of degree d over $\boldsymbol{Q}_p$) and $G(\boldsymbol{Q}_p) \approx \mathrm{GL}_2(F_p)$ (since we assumed that B split at every prime of F above p). Regarding $\mathrm{GL}_2(F_p)$ as the F_p points of GL_2, we get an isomorphism $f \mapsto f^{\vee}$ from $\mathscr{H}$ to the algebra of polynomials in a, b, a^{-1}, b^{-1} which are symmetric in a, b (a and b are two indeterminates). Regarding $\mathrm{GL}_2(F_p)$ as the $\boldsymbol{Q}_p$ points of $\mathrm{Res}_{F/\boldsymbol{Q}}\, \mathrm{GL}_2$, we get an isomorphism $f \mapsto f^{\sim}$ from $\mathscr{H}$ to the algebra of functions on ${}^L G^{\circ} \times \{\Phi\}$ which are obtained by restricting linear combinations of characters of finite dimensional complex analytic representations of the complex Lie group ${}^LG$. The relation between $f^\vee$ and $f^\sim$ is

$$f^\sim\left(\begin{pmatrix} a_1 & 0 \\ 0 & b_1 \end{pmatrix} \times \cdots \times \begin{pmatrix} a_d & 0 \\ 0 & b_d \end{pmatrix} \times \Phi\right) = f^\vee\left(\begin{pmatrix} a_1 \cdots a_d & 0 \\ 0 & b_1 \cdots b_d \end{pmatrix}\right).$$

We want to find $f_p^{(j)} \in \mathscr{H}$ such that $(f_p^{(j)})^\sim(x) = p^{jd/2}\mathrm{tr}(r(x^j))$ for all $x \in {}^LG^\circ \times \{\Phi\}$. Let l be the greatest common divisor of j and d. Then for

$$x = \begin{pmatrix} a_1 & 0 \\ 0 & b_1 \end{pmatrix} \times \cdots \times \begin{pmatrix} a_d & 0 \\ 0 & b_d \end{pmatrix} \times \Phi,$$

we have

$$\begin{aligned} \mathrm{tr}\, r(x^j) &= \mathrm{tr}\, r\left(\begin{pmatrix} a_1 \cdots a_j & 0 \\ 0 & b_1 \cdots b_j \end{pmatrix} \times \begin{pmatrix} a_2 \cdots a_{j+1} & 0 \\ 0 & b_2 \cdots b_{j+1} \end{pmatrix} \times \cdots \times \Phi^j\right) \\ &= ((a_1 \cdots a_d)^{j/l} + (b_1 \cdots b_d)^{j/l})^l. \end{aligned}$$

It follows that $(f_p^{(j)})^\vee(\left(\begin{smallmatrix} a & 0 \\ 0 & b \end{smallmatrix}\right)) = p^{jd/2}(a^{j/l} + b^{j/l})^l$. For $k' \in \boldsymbol{Z}$ with $0 \leqq k' < l/2$, let $f_{j,k'}$ be the function in the Hecke algebra such that

$$f_{j,k'}^\vee = p^{jd/2}\binom{l}{h'}(a^{jk'/l}\, b^{jk''/l} + a^{jk''/l}\, b^{jk'/l})$$

where $k'' = l - k'$. If l is even, then for $k' = k'' = l/2$, let $f_{j,k'}$ be the function in the Hecke algebra such that $f_{j,k'}^\vee = p^{jd/2}\binom{l}{k'}a^{jk'/l}b^{jk'/l}$. By the binomial theorem we have $f_p^{(j)} = \sum_{k'=0}^{[l/2]} f_{j,k'}$.

Now consider a triple $\rho = (L, m', m'')$, and let $\gamma \in L - F$. Recall that m'' is determined by m' (since $m' + m'' = d$). Choose an embedding $L \to B$. Then γ gives us an element of $I_\rho(\boldsymbol{Q})$ and of $G(\boldsymbol{Q})$. We will show that the contribution of γ to (6) is 0 unless m' is divisible by d/l, in which case it is equal to the contribution of $k' = lm'/d$ and γ to the following rewritten version of (4):

$$\sum_\gamma \sum_{k'=0}^{[l/2]} a_\gamma \int_{G_\gamma(A_f^p)\backslash G(A_f^p)} f^p(g^{-1}\gamma g)\, \frac{d\nu'}{d\nu} \int_{G_\gamma(\boldsymbol{Q}_p)\backslash G(\boldsymbol{Q}_p)} f_{j,k'}(g^{-1}\gamma g)\, \frac{dg_p}{dg_\gamma}$$

where dg_γ is the Haar measure on $G_\gamma(\boldsymbol{Q}_p)$ corresponding to the Haar measure μ on $I_\rho(\boldsymbol{Q}_p)$ under the isomorphism $I_\rho(\boldsymbol{Q}_p) \xrightarrow{\sim} G_\gamma(\boldsymbol{Q}_p)$ induced by the embedding $L \to B$. It can easily be shown that $a_\gamma = b_\gamma$ with the choice of measures that we have made, so it is enough to show the following:

THEOREM. *With notations as above, $\varphi^{(j)}(\gamma) = 0$ unless d/l divides m', in which case it is $\int_{G_\gamma(\boldsymbol{Q}_p)\backslash G(\boldsymbol{Q}_p)} f_{j,k'}(g^{-1}\gamma g)(dg_p/dg_\gamma)$ where $k' = lm'/d$.*

We begin the proof of this theorem by evaluating $\int_{G_\gamma(\boldsymbol{Q}_p)\backslash G(\boldsymbol{Q}_p)} f_{j,k'}(g^{-1}\gamma g) \cdot (dg_p/dg_\gamma)$. This is easy since L splits at p and the hyperbolic orbital integrals of any $f \in \mathscr{H}$ can be read off from $f^\vee$. Let $\alpha, \beta \in F_p^*$ be the two eigenvalues of γ and assume without loss of generality that the valuation $v(\alpha)$ of α is less than or equal to the valuation $v(\beta)$ of β. Then the integral of $f_{j,k'}$ over the orbit of γ is equal to 0 unless $v(\alpha) = jk'/l$ and $v(\beta) = jk''/l$, in which case it is

$$|\alpha\beta/(\alpha - \beta)^2|_{F_p}^{1/2} p^{jd/2'}\binom{l}{k'}(\mathrm{meas}(I_0)^{-1})$$

where I_0 is the subgroup of $I_\rho(\boldsymbol{Q}_p)$ corresponding to $\mathcal{O}_{F_p}^* \times \mathcal{O}_{F_p}^*$ under $I_\rho(\boldsymbol{Q}_p) \xrightarrow{\sim} F_p^* \times F_p^*$. The final result is that the integral is equal to 0 unless $\nu(\alpha) = jk'/l$ and $\nu(\beta) = jk''/l$, in which case it is $\binom{l}{k'}p^{jk'd/l}\,\mu(I_0)^{-1}$.

We must now evaluate $\varphi^j(\gamma)$. This is the combinatorial exercise referred to in the beginning of this paper. First we will describe X_ρ and the actions of $I_\rho(\boldsymbol{Q}_p)$ and Φ on X_ρ. Let $\boldsymbol{Q}_p^{\mathrm{un}}$ be a maximal unramified extension of $\boldsymbol{Q}_p$ containing F_p, let $\mathcal{O}_p^{\mathrm{un}}$ be its valuation ring, let $\mathscr{B}'$ be the set of $\mathcal{O}_p^{\mathrm{un}}$-lattices in the two dimensional vector space $\boldsymbol{Q}_p^{\mathrm{un}} \oplus \boldsymbol{Q}_p^{\mathrm{un}}$ over $\boldsymbol{Q}_p^{\mathrm{un}}$, and let $\mathscr{B}$ be the set of classes of lattices in $\boldsymbol{Q}_p^{\mathrm{un}} \oplus \boldsymbol{Q}_p^{\mathrm{un}}$ (two lattices M_1, M_2 are said to be in the same class if there exists a nonzero element $c \in \boldsymbol{Q}_p^{\mathrm{un}}$ such that $M_1 = cM_2$). For any lattice M, let $\bar{M}$ denote the class of M. The set $\mathscr{B}$ is the set of vertices of the Bruhat-Tits building of $\mathrm{SL}_2(\boldsymbol{Q}_p^{\mathrm{un}})$. This building is a tree. The groups $\mathrm{GL}_2(\boldsymbol{Q}_p^{\mathrm{un}})$ and $\mathrm{Gal}\,(\boldsymbol{Q}_p^{\mathrm{un}}/\boldsymbol{Q}_p)$ act on $\mathscr{B}'$ and $\mathscr{B}$. Let

$$B = \begin{pmatrix} p^{m'} & 0 \\ 0 & p^{m''} \end{pmatrix},$$

and let σ be the Frobenius element of $\mathrm{Gal}\,(\boldsymbol{Q}_p^{\mathrm{un}}/\boldsymbol{Q}_p)$. Let X_ρ be the set of sequences $\{M_i\}_{i\in\boldsymbol{Z}}$ of elements M_i of $\mathscr{B}'$ satisfying:

(A) $M_i \supsetneqq M_{i-1} \supsetneqq pM_i$ for all $i \in \boldsymbol{Z}$;

(B) $B\sigma^d M_i = M_{i-d}$ for all $i \in \boldsymbol{Z}$.

The action of Φ on X_ρ is given by $\Phi\colon \{M_i\} \mapsto \{M_i'\}$ where $M_i' = M_{i-1}$. Identify $I_\rho(\boldsymbol{Q}_p)$ with $A(F_p)$, where $A = \{\begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix}\}$. The action of an element $\gamma \in I_\rho(\boldsymbol{Q}_p)$ on X_ρ is given by $\gamma\{M_i\} = \{\gamma M_i\}$.

Recall that we are trying to calculate $\varphi^j(\gamma) = \sum_x \mu(I_x)^{-1}\,\delta_x^j(\gamma)$, where the sum is taken over $x \in X_\rho \bmod I_\rho(\boldsymbol{Q}_p)$. Let I_1 be the subgroup of $I_\rho(\boldsymbol{Q}_p) = A(F_p)$ consisting of all matrices of the form

$$\begin{pmatrix} cp^n & 0 \\ 0 & c \end{pmatrix}$$

where $n \in \boldsymbol{Z}$ and $c \in F_p^*$. As before let $I_0 = A(\mathcal{O}_{F_p})$. Then $I_\rho(\boldsymbol{Q}_p) = I_0 \times I_1$. Furthermore $\mu(I_0) = \mu(I_x)\,[I_0\colon I_x] = \mu(I_x)\,[I_\rho(\boldsymbol{Q}_p)\colon I_1 I_x]$, so $\varphi^j(\gamma)$ is equal to $\mu(I_0)^{-1}\mathrm{Card}\, S(\gamma, j)$ where $S(\gamma, j) = \{x \in X_\rho \bmod I_1\colon \Phi^j x = \gamma x\}$.

Let $\mathfrak{A}$ be the apartment in $\mathscr{B}$ corresponding to the split torus A. Choose a point p_0 in $\mathfrak{A}$, and let $\mathscr{M} = \{x \in \mathscr{B}\colon p_0$ is the point of $\mathfrak{A}$ nearest $x\}$. Let $\mathscr{M}'$ be a set of representatives for the lattice classes in $\mathscr{M}$, or in other words, let $\mathscr{M}'$ be a subset of $\mathscr{B}'$ such that no two elements of $\mathscr{M}'$ are in the same class and such that $\{\bar{M}\colon M \in \mathscr{M}'\} = \mathscr{M}$. Then the set of $\{M_i\} \in X_\rho$ which satisfy

(C) $M_0 \in \mathscr{M}'$

is a set of representatives of $X_\rho \bmod I_1$. The condition $\Phi^j\{M_i\} = \gamma\{M_i\}$ just says

(D) $\gamma M_i = M_{i-j}$.

We can summarize this discussion by saying that $S(\gamma, j)$ is the set of all sequences of elements of $\mathscr{B}'$ satisfying (A)—(D). Any sequence $\{M_i\}$ in $S(\gamma, j)$ determines a sequence $\{x_i\} = \{\bar{M}_i\}$ of elements of $\mathscr{B}$ satisfying:

(A′) x_i is a neighbor of x_{i-1};

(B′) $B\sigma^d x_i = x_{i-d}$;

(C′) $x_0 \in \mathscr{M}$;

(D′) $\gamma x_i = x_{i-j}$.

Let $S'(\gamma, j)$ be the set of sequences $\{x_i\}$ satisfying (A′)—(D′). If the valuation of

$\det(\gamma)$ is not equal to j, then conditions (A) and (D) are incompatible, and $S(\gamma, j)$ is empty. So from now on we assume $v(\det \gamma) = j$. In this case, any element of $S'(\gamma, j)$ can be uniquely lifted to an element of $S(\gamma, j)$ (to check this one needs to use the fact that $v(\det B) = d$). So Card $S(\gamma, j) =$ Card $S'(\gamma, j)$ if $v(\det \gamma) = j$. Let $r, s \in \boldsymbol{Z}$ be such that $rj + sd = l$. Conditions (B′) and (D′) together are equivalent to the two conditions:

(E′) $\sigma^{dj/l} x_i = B^{-j/l} \gamma^{d/l} x_i$;

(F′) $\gamma^r B^s \sigma^{sd} x_i = x_{i-l}$.

We will now show that $S'(\gamma, j)$ is empty unless $B^{-j/l}\gamma^{d/l} \in I_0$. Suppose $S'(\gamma, j)$ is nonempty, and let $\{x_i\} \in S'(\gamma, j)$. Then p_0 is the point of $\mathfrak{A}$ closest to x_0, so τp_0 is the point of $\tau\mathfrak{A}$ closest to τx_0, where $\tau = \sigma^{-dj/l} B^{-j/l} \gamma^{d/l}$. But $\tau x_0 = x_0$ by (E′), and $\tau\mathfrak{A} = \mathfrak{A}$ since $\sigma\mathfrak{A} = \mathfrak{A}$, $B\mathfrak{A} = \mathfrak{A}$ and $\gamma\mathfrak{A} = \mathfrak{A}$. So τ fixes p_0. Since $\sigma^{-dj/l}$ fixes p_0, we conclude that $B^{-j/l}\gamma^{d/l}$ fixes p_0. But $B^{-j/l}\gamma^{d/l}$ is diagonal and its determinant is a unit (since we are assuming that $v(\det \gamma) = j$), so that the fact it fixes p_0 implies it belongs to I_0. So we have shown that $S'(\gamma, j)$ is empty unless $B^{-j/l}\gamma^{d/l} \in I_0$. Let $\alpha, \beta \in F_p^*$ be the diagonal entries of γ, so that $\gamma = \left(\begin{smallmatrix}\alpha & 0\\ 0 & \beta\end{smallmatrix}\right)$. Then $B^{-j/l}\gamma^{d/l} \in I_0$ if and only if $v(\alpha) = jm'/d$ and $v(\beta) = jm''/d$. In particular we see that $\varphi^j(\gamma) = 0$ for all γ unless jm'/d and jm''/d are integers. Recalling that $l = (j, d)$ and that $m' + m'' = d$, we see that $jm'/d, jm''/d \in \boldsymbol{Z}$ if and only if d/l divides m', and that if d/l does divide m', then $\varphi^j(\gamma)$ is still zero unless $v(\alpha) = jm'/d = jk'/l$ and $v(\beta) = jm''/d = jk''/l$. Comparing this with the orbital integrals we have computed, we find that all we have left to show is the following lemma.

LEMMA. *Suppose $B^{-j/l}\gamma^{d/l} \in I_0$. Then* Card $S'(\gamma, j) = \binom{l}{k'} p^{jk'd/l}$ *where $k' = lm'/d$.*

To prove this we first look more closely at conditions (E′) and (F′). Let $\mathscr{B}^\tau = \{x \in \mathscr{B} \colon \tau(x) = x\}$, where τ is the automorphism of $\mathscr{B}$ defined previously. Then (E′) simply says that $x_i \in \mathscr{B}^\tau$ for all $i \in \boldsymbol{Z}$. Let $\boldsymbol{Q}_p^c$ be the completion of $\boldsymbol{Q}_p^{\mathrm{un}}$. There exists a unique continuous action of $\mathrm{GL}_2(\boldsymbol{Q}_p^c)$ on $\mathscr{B}$ which extends the action of $\mathrm{GL}_2(\boldsymbol{Q}_p^{\mathrm{un}})$ on $\mathscr{B}$ ($\mathscr{B}$ is given the discrete topology). Each element of $\mathrm{GL}_2(\boldsymbol{Q}_p^c)$ acts by a simplicial automorphism of the tree $\mathscr{B}$. Using the fact that $B^{-j/l}\gamma^{d/l} \in I_0$, it is easy to show that there exists a diagonal matrix δ with diagonal entries in $\boldsymbol{Q}_p^c - \{0\}$ such that $\delta^{\sigma^{jd/l}} \delta^{-1} = B^{-j/l}\gamma^{d/l}$. A simple calculation shows that $\mathscr{B}^\tau = \delta\mathscr{B}^{\sigma^{dj/l}}$ where $\mathscr{B}^{\sigma^{dj/l}}$ denotes the set of fixed points of $\sigma^{dj/l}$ in $\mathscr{B}$. The subtree $\mathscr{B}^{\sigma^{dj/l}}$ of $\mathscr{B}$ can be canonically identified with the Bruhat-Tits building of SL_2 over the fixed field of $\boldsymbol{Q}_p^{\mathrm{un}}$ under $\sigma^{dj/l}$. So $\mathscr{B}^{\sigma^{dj/l}}$ is a tree in which every vertex has exactly $p^{dj/l} + 1$ neighbors. Since $\mathscr{B}^\tau$ is simply the translate of $\mathscr{B}^{\sigma^{dj/l}}$ by δ, it too is a tree in which every vertex has exactly $p^{dj/l} + 1$ neighbors.

A further consequence of $B^{-j/l}\gamma^{d/l} \in I_0$ is that $\mathscr{B}^\tau$ contains $\mathfrak{A}$. The reason for this is that $\sigma^{dj/l}$ fixes $\mathfrak{A}$ pointwise, as does every element of I_0.

In (F′) the automorphism $\gamma^r B^s \sigma^{sd}$ of $\mathscr{B}$ appears. We will need to know the effect of this automorphism on $\mathfrak{A}$. The automorphism σ^{sd} acts trivially on $\mathfrak{A}$. The automorphism B acts on $\mathfrak{A}$ by translation by $m'' - m'$. The automorphism γ^d acts on $\mathfrak{A}$ in the same way that B^j does, since $B^{-j/l}\gamma^{d/l} \in I_0$, and therefore γ acts on $\mathfrak{A}$ by translation by $(m'' - m')j/d$. So $\gamma^r B^s \sigma^{sd}$ acts on $\mathfrak{A}$ by translation by $l - 2k'$. This discussion has reduced our problem to that of proving the following lemma.

LEMMA. *Let T be a tree in which every vertex has $q + 1$ neighbors, and let $\mathfrak{A}$ be a* *subtree of T in which every vertex has 2 neighbors. Let $p_0 \in \mathfrak{A}$. Let l and k be nonnegative integers such that $l - 2k$ is positive. Let ξ be an automorphism of T which acts on $\mathfrak{A}$ by translation by $l - 2k$. Let $N(l, k)$ be the set of sequences of length $l + 1$ of points $x_0, \cdots, x_l$ in T satisfying:*

(i) *x_i is a neighbor of x_{i+1} for $i = 0, \cdots, l - 1$;*

(ii) *p_0 is the point of $\mathfrak{A}$ closest to x_0;*

(iii) *$x_l = \xi x_0$.*

Then $N(l, k) = \binom{l}{k} q^k$.

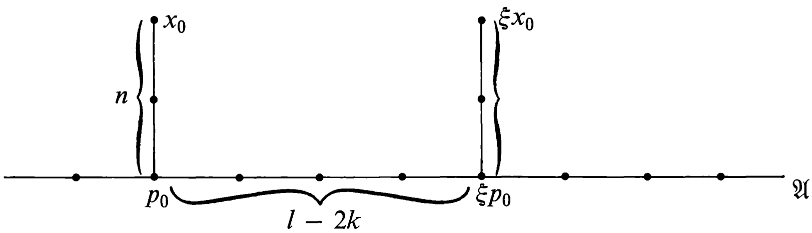
Let us agree to call a sequence of points $x_0, \cdots, x_l$ satisfying (i) a path of length l joining x_0 to x_l. Then

$$N(l, k) = \sum_{x_0 \in \mathscr{M}} \text{Card}\,\{\text{paths of length } l \text{ joining } x_0 \text{ to } \xi x_0\},$$

where $\mathscr{M}$ is the set of $x_0 \in T$ which satisfy (ii).

Let $a, b \in \boldsymbol{Z}$ with $a \geqq 2b \geqq 0$, and let x, y be two vertices of T at distance $a - 2b$ apart. Then the number of paths of length a joining x to y is independent of the choice of x and y, and will be denoted by $M(a, b)$.

The figure below shows that the distance between x_0 and ξx_0 is equal to $l - 2k + 2n$ where n is the distance from x_0 to p_0.



So $N(l, k) = \sum_{n=0}^{k} \text{Card}\,\{x_0 \in \mathscr{M}\colon \text{distance from } x_0 \text{ to } p_0 \text{ is } n\}\ M(l, k - n)$. The number of points x_0 at distance n from p_0 such that p_0 is the point of $\mathfrak{A}$ closest to x_0 is $(1 - q^{-1})q^n$ if $n > 0$ and is 1 if $n = 0$. Therefore

$$N(l, k) = M(l, k) + (1 - q^{-1}) \textstyle\sum_{n=1}^{k} q^n M(l, k - n).$$

The function $M(l, k)$ on the set of pairs of integers (l, k) such that $l \geqq 2k \geqq 0$ is characterized by the three properties:

(i) $M(l, 0) = 1$ for $l \geqq 0$;

(ii) $M(l, k) = (q + 1)M(l - 1, k - 1)$ for $l = 2k > 0$;

(iii) $M(l, k) = M(l - 1, k) + qM(l - 1, k - 1)$ for $l > 2k > 0$.

Condition (i) and the fact that (i), (ii), (iii) determine M uniquely are both obvious, and (ii), (iii) become obvious if it is observed that to give a path of length l from x to y is the same as to give a neighbor x' of x and a path of length $l - 1$ from x' to y.

For $l, k \in \boldsymbol{Z}$, let $m(l, k)$ be the coefficient of $X^l Y^k$ in the formal power series expansion of $f(X, Y) = (1 - qY)(1 - Y)^{-1}(1 - X - qXY)^{-1}$. We will show that $M(l, k) = m(l, k)$ for $l \geqq 2k \geqq 0$. It is enough to show that conditions (i)—(iii) hold with M replaced by m. From the definition of $f(X, Y)$ we get $(1 - X - qXY) \cdot f(X, Y) = (1 - qY)(1 - Y)^{-1}$. Comparing coefficients of $X^l Y^k$ on the two sides we see that $m(l, k) - m(l - 1, k) - qm(l - 1, k - 1) = 0$ if $l > 0$, which shows that m satisfies (iii). Setting $Y = 0$ in the definition of $f(X, Y)$ we get $\sum_{l=0}^{\infty} m(l, 0)X^l = (1 - X)^{-1} = 1 + X + X^2 + \cdots$, which shows that m satisfies (i).

We still have to show that m satisfies (ii). Let $g(X, Y) = (1 - X - qXY)^{-1}$ and let $n(l, k)$ be the coefficient of $X^l Y^k$ in the formal power series expansion of $g(X, Y)$. We have

$$(1 - X - qXY)^{-1} = \sum_{l=0}^{\infty} (X + qXY)^l = \sum_{l=0}^{\infty} \sum_{k=0}^{\infty} \binom{l}{k} X^{l-k}(qXY)^k = \sum_{l \geq 0; k \geq 0} \binom{l}{k} q^k X^l Y^k,$$

and therefore $n(l, k) = \binom{l}{k} q^k$. In particular for $k > 0$ we have

$$n(2k - 1, k) = \binom{2k - 1}{k} q^k = \binom{2k - 1}{k - 1} q^{k-1}\, q = qn(2k - 1, k - 1).$$

This shows that the coefficient of $X^{2k-1}Y^k$ in $(1 - qY)g(X, Y)$ is zero. But $(1 - qY) \cdot g(X, Y)$ is equal to $(1 - Y)f(X, Y)$, so the coefficient of $X^{2k-1}Y^k$ in $(1 - Y)f(X, Y)$ is zero, which means that

(7) $$m(2k - 1, k) = m(2k - 1, k - 1) \quad \text{for all } k > 0.$$

We have already seen that if $l > 0$, then $m(l, k) = m(l - 1, k) + qm(l - 1, k - 1)$. Take $l = 2k$ with $k > 0$. Then $m(2k, k) = m(2k - 1, k) + qm(2k - 1, k - 1)$. Combining this with equation (7) we see that m satisfies condition (ii).

Going back to our discusion of $N(l, k)$, we see that

$$N(l, k) = m(l, k) + (1 - q^{-1}) \sum_{n=1}^{\infty} q^n m(l, k - n)$$

since m agrees with M whenever M is defined and $m(l, k - n)$ is 0 for $n > k$. We may use the equation above to extend the definition of $N(l, k)$ to all integers l and k. With this convention we find that $\sum N(l, k)X^l Y^k$ is equal to

$$\begin{aligned}
&\sum_{l,k} m(l, k)X^l Y^k + (1 - q^{-1}) \sum_{n=1}^{\infty} q^n \sum_{l,k} m(l, k - n)X^l Y^k \\
&= f(X, Y) + (1 - q^{-1}) \sum_{n=1}^{\infty} q^n Y^n \sum_{l,k} m(l, k - n)X^l Y^{k-n} \\
&= f(X, Y)\left[1 + (1 - q^{-1}) \sum_{n=1}^{\infty} q^n Y^n\right] = f(X, Y)(1 - Y)(1 - qY)^{-1} = g(X, Y).
\end{aligned}$$

We have already computed the formal power series expansion of $g(X, Y)$. Looking back at the answer, we find that $N(l, k) = \binom{l}{k} q^k$. This proves the lemma.

REFERENCES

1. R. P. Langlands, *On the zeta-functions of some simple Shimura varieties*, Notes, Institute for Advanced Study, Princeton, N. J., 1977.

2. ———, *Shimura varieties and the Selberg trace formula*, Canad J. Math. (to appear).

UNIVERSITY OF WASHINGTON, SEATTLE

Proceedings of Symposia in Pure Mathematics
Vol. 33 (1979), part 2, pp. 193–203

NOTES ON L-INDISTINGUISHABILITY

(BASED ON A LECTURE OF R. P. LANGLANDS)*

D. SHELSTAD

These notes are intended as a brief discussion of the results of [4]. Although we consider essentially just groups G which are inner forms of SL_2, we emphasize formulations (cf. [7]) which suggest possible generalizations. We assume that F is a field of characteristic zero. In the case that F is local there are only finitely many irreducible admissible representations of $G(F)$ which are "L-indistinguishable" from a given representation. We structure this set, an L-packet, by considering not the characters of the members but rather sufficiently many linear combinations of these characters. In the case that F is global we consider certain L-packets of representations of $G(A)$ and describe the multiplicity in the space of cusp forms of a representation $\pi = \bigotimes_v \pi_v$ in terms of the position of the local representations π_v in their respective L-packets.

1. $\mathfrak{A}(T)$, $\mathfrak{D}(T)$ and $\mathscr{E}(T)$. Suppose that G is a connected reductive group defined over F, any field of characteristic zero, and that T is a maximal torus in G, also defined over F. Fix an algebraic closure $\bar{F}$ of F. Then we set $\mathfrak{A}(T) = \{g \in G(\bar{F})$: $\operatorname{ad} g^{-1}/T$ is defined over $F\}$ and $\mathfrak{D}(T) = T(\bar{F})\backslash\mathfrak{A}(T)/G(F)$. If $g \in \mathfrak{A}(T)$ then $\sigma \to g_\sigma = \sigma(g)g^{-1}$ is a continuous 1-cocycle of $\mathfrak{G} = \operatorname{Gal}(\bar{F}/F)$ in $T(\bar{F})$. The map $g \to (\sigma \to g_\sigma)$ induces an injection of $\mathfrak{D}(T)$ into a subgroup $\mathscr{E}(T)$ of $H^1(\mathfrak{G}, T(\bar{F}))$ defined as follows. Let T_{sc} be the preimage of T in the simply-connected covering group G_{sc} of the derived group of G. Then $\mathscr{E}(T)$ is the image of the natural homomorphism of $H^1(\mathfrak{G}, T_{sc}(\bar{F}))$ into $H^1(\mathfrak{G}, T(\bar{F}))$. If $H^1(\mathfrak{G}, G_{sc}(\bar{F})) = 1$ and so, in particular, if F is local and nonarchimedean then $\mathfrak{D}(T)$ coincides with $\mathscr{E}(T)$.

L-indistinguishability appears when G contains a torus T such that $\mathfrak{D}(T)$ is nontrivial.

2. Groups attached to G (local case). Assume now that F is local. Fix a finite Galois extension K of F over which T splits. We replace $\bar{F}$ by K and $\mathfrak{G}$ by $\mathfrak{G}_{K/F} = \operatorname{Gal}(K/F)$ in the definitions of the last section. An application of Tate-Nakayama duality then allows us to identify $\mathscr{E}(T)$ with the quotient of $\{\lambda \in X_*(T_{sc})$: $\sum_{\sigma \in \mathfrak{G}_{K/F}} \sigma\lambda = 0\}$ by

$$\Big\{\lambda \in X_*(T_{sc}) : \lambda = \sum_{\sigma \in \mathfrak{G}_{K/F}} \sigma\mu_\sigma - \mu_\sigma,\ \mu_\sigma \in X_*(T)\Big\},$$

AMS (MOS) subject classifications (1970). Primary 22E50, 22E55.

*This work has been partially supported by the National Science Foundation under Grant MCS76–08218.

$X_*(\ )$ denoting $\mathrm{Hom}(\mathrm{GL}_1,\ )$. A quasi-character κ on $X_*(T_{sc})$ trivial on this latter module defines, by restriction, a character on $\mathscr{E}(T)$. In [7] there is attached to each triple (G, T, κ) a quasi-split group over F (there denoted H). We will pursue this just in the case that G is an inner form of SL_2.

3. Groups attached to an inner form of SL_2 (local case). Suppose that G is an inner form of SL_2 and that F is local; we will continue with this assumption until §14. We have then two groups to consider: SL_2 and the group of elements of norm one in a quaternion algebra over F. We may take $\mathrm{PGL}_2(\boldsymbol{C})$ as ${}^LG^\circ$ (notation as in [1]) and the diagonal subgroup as distinguished maximal torus ${}^LT^\circ$.

Fix a maximal torus T in G, defined over F. A quasi-character κ from the last section is just a $\mathfrak{G}_{K/F}$-invariant quasi-character on $X_*(T)$. We fix an isomorphism between $X_*(T)$ and $X^*({}^LT^\circ) = \mathrm{Hom}({}^LT^\circ, \boldsymbol{C}^\times)$ as follows. If G is SL_2 and T the diagonal subgroup we use the map defined by the pairing between $X^*(T)$ and $X^*({}^LT^\circ)$; if T is arbitrary in SL_2 we choose a diagonalization and compose the induced map on $X^*(T)$ with that already prescribed. If G is the anisotropic form we may still regard T as a torus in SL_2 and proceed in the same way.

Using this isomorphism between $X_*(T)$ and $X^*({}^LT^\circ)$ we transfer κ to a quasi-character on $X^*({}^LT^\circ)$; using the canonical isomorphism between $\mathrm{Hom}(X^*({}^LT^\circ), \boldsymbol{C}^\times)$ and ${}^LT^\circ$ we then regard κ as an element of ${}^LT^\circ$. At the same time we transfer the action of $\mathfrak{G}_{K/F}$ on $X_*(T)$ to $X^*({}^LT^\circ)$ and ${}^LT^\circ$, writing σ_T for the new action of $\sigma \in \mathfrak{G}_{K/F}$.

Here are the possibilities. If T is split then $\mathfrak{G}_{K/F}$ acts trivially and κ is an arbitrary element of ${}^LT^\circ$. If T is anisotropic, suppose that T is defined by the quadratic extension E of F. We shall assume that K is some fixed large but finite Galois extension of F containing, in particular, E; $\mathfrak{G}_{K/F}$ acts on T through $\mathfrak{G}_{E/F}$. Let σ° be the nontrivial element of $\mathfrak{G}_{E/F}$ and $\alpha^\vee$ be a coroot for T in G. Then $\sigma^\circ\alpha^\vee = -\alpha^\vee$ so that $(\kappa(\alpha^\vee))^2 = 1$. Since $\alpha^\vee$ generates $X_*(T)$ there are then just two possibilities for κ. The nontrivial κ defines the element $\left(\begin{smallmatrix} -1 & 0 \\ 0 & 1 \end{smallmatrix}\right)_*$ of ${}^LT^\circ$; here, and throughout these notes, we use $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)_*$ to denote the image in $\mathrm{PGL}_2(\boldsymbol{C})$ of the matrix $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ in $\mathrm{GL}_2(\boldsymbol{C})$. The action of σ_T on ${}^LT^\circ$ is described as follows: if $\sigma \in \mathfrak{G}_{K/F}$ maps to the trivial element in $\mathfrak{G}_{E/F}$ under $\mathfrak{G}_{K/F} \to \mathfrak{G}_{E/F}$ then σ_T acts trivially and if σ maps to σ° then σ_T acts by

$$\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}_* \longrightarrow \begin{pmatrix} b & 0 \\ 0 & a \end{pmatrix}_* .$$

As an element of ${}^LT^\circ$, κ is σ_T-invariant, $\sigma \in \mathfrak{G}_{K/F}$. We define ${}^LH^\circ$ to be the connected component of the identity in the centralizer of κ in ${}^LG^\circ$. Whatever T, if κ is trivial then ${}^LH^\circ = {}^LG^\circ$ and if κ is nontrivial then ${}^LH^\circ = {}^LT^\circ$. Let $\sigma \in \mathfrak{G}_{K/F}$. Then since both ${}^LH^\circ$ and ${}^LT^\circ$ are invariant under σ_T we may multiply σ_T by an inner automorphism of ${}^LH^\circ$ to obtain an automorphism σ_H stabilizing ${}^LT^\circ$ and fixing each root, if any, of ${}^LT^\circ$ in ${}^LH^\circ$. The collection $\{\sigma_H, \sigma \in \mathfrak{G}_{K/F}\}$ defines a semidirect product ${}^LH = {}^LH^\circ \rtimes W_{K/F}$ where $W_{K/F}$, the Weil group of K/F, acts through $\mathfrak{G}_{K/F}$. In duality, we obtain a quasi-split group H over F. Specifically:

PROPOSITION. (a) *If κ is trivial (whatever T) then ${}^LH = {}^LG = {}^LG^\circ \times W_{K/F}$ and $H = \mathrm{SL}_2$.*

(b) *If T is split and κ nontrivial then ${}^LH = {}^LT^\circ \times W_{K/F}$ and $H = T$.*

(c) *If T is anisotropic and κ nontrivial then $^LH = {^LT^\circ} \rtimes W_{K/F}$ where $w \in W_{K/F}$ acts trivially on $^LT^\circ$ if w maps to 1 under $W_{K/F} \to \mathfrak{G}_{K/F} \to \mathfrak{G}_{E/F}$, and w acts by $\left(\begin{smallmatrix} a & 0 \\ 0 & b \end{smallmatrix}\right)_* \to \left(\begin{smallmatrix} b & 0 \\ 0 & a \end{smallmatrix}\right)_*$ if w maps to σ° (E, σ° as before); and $H = T$.*

To indicate that H is defined by (T, κ) we write $H = H(T, \kappa)$. Note that the choice (of diagonalization) made in defining the isomorphism between $X_*(T)$ *and* $X^*(^LT^\circ)$ does not affect $H(T, \kappa)$.

4. Embedding L-groups. For later use we specify an embedding ι_H of ^LH in ^LG. We refer to the proposition above. In (a) ι_H is to be the identity, in (b) ι_H is the inclusion and in (c) ι_H extends the inclusion of $^LT^\circ$ in $^LG^\circ$ by:

$$\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}_* \times w \longrightarrow \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}_* \times w$$

if $w \in W_{K/F}$ maps to 1 under $W_{K/F} \to \mathfrak{G}_{K/F} \to \mathfrak{G}_{E/F}$, and

$$\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}_* \times w \longrightarrow \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}_* \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}_* \times w$$

if w maps to σ°.

As remarked earlier there are analogues of these groups "H" for any connected reductive group G over F. In general, it need not be that ^LH embeds in ^LG; [7] indicates how to accommodate this (see Lemma 1 and the last several paragraphs).

5. Orbital integrals (normalization). We will transfer certain integrals from G to H. For this, normalizations are required. Fix a pair (T, κ).

We choose Haar measures dt and dg on $T(F)$ and $G(F)$ respectively. If $f \in C_c^\infty(G(F))$, $\delta \in \mathfrak{D}(T)$, and $\gamma \in T(F)$ is regular in G then we set

$$\Phi^\delta(\gamma, f) = \int_{h^{-1}T(F)h\backslash G(F)} f(g^{-1}h^{-1}\gamma hg) \frac{dg}{(dt)^h}$$

where $h \in \mathfrak{A}(T)$ represents δ and $(dt)^h$ is the measure on $h^{-1}T(F)h$ obtained from dt by means of ad h. For $\delta \in \mathscr{E}(T) - \mathfrak{D}(T)$ we set $\Phi^\delta(\ \ ,f) \equiv 0$. Recall that $\mathfrak{A}(T)$, $\mathfrak{D}(T)$, $\mathscr{E}(T)$ were defined in §1.

In the case that κ is trivial (whatever T) we define

$$\Phi^{T/\kappa}(\ \ ,f) = \Phi^{T/1}(\ \ ,f) = \varepsilon(G) \sum_{\delta \in \mathscr{E}(T)} \Phi^\delta(\ \ ,f)$$

where $\varepsilon(G) = 1$ if $G = \mathrm{SL}_2$ and $\varepsilon(G) = -1$ if G is anisotropic. We call $\Phi^{T/1}(\ ,\ )$ a "stable" orbital integral.

If T is split and κ nontrivial then we define

$$\Phi^{T/\kappa}(\gamma, f) = \frac{|\gamma_1 - \gamma_2|}{|\gamma_1\gamma_2|^{1/2}} \sum_{\delta \in \mathscr{E}(T)} \kappa(\delta)\Phi^\delta(\gamma, f)$$

where γ_1, γ_2 are the eigenvalues of γ.

Suppose that T is anisotropic and that κ is nontrivial. Then our normalization depends on two choices. First choose a nontrivial additive character ψ_F on F. If E is the quadratic extension of F attached to T then we define $\lambda(E/F, \psi_F)$ as in [5]. Also choose a regular element γ° of $T(F)$. Let κ' denote the quadratic character of $F^\times$ attached to E. Then we define

$$\Phi^{T/\kappa}(\gamma, f) = \lambda(E/F, \psi_F)\kappa'\left(\frac{\gamma_1 - \gamma_2}{\gamma_1^\circ - \gamma_2^\circ}\right)\frac{|\gamma_1 - \gamma_2|}{|\gamma_1\gamma_2|^{1/2}}\sum_{\delta\in\mathscr{E}(T)}\kappa(\delta)\Phi^\delta(\gamma, f)$$

where the order on the eigenvalues γ_1, γ_2 of γ and γ_1^0, γ_2^0 of γ° is prescribed by fixing a diagonalization of T. A different choice of ψ_F or γ° causes at most a sign change in the normalizing factor.

6. Transferring orbital integrals. The integrals $\Phi^{T/\kappa}(\ ,f)$ can be transferred to stable orbital integrals on $H = H(T, \kappa)$ in the following sense:

LEMMA. *If $f \in C_c^\infty(G(F))$ then there exists $f_H \in C_c^\infty(H(F))$ such that*

$$\Phi^{T/1}(\gamma, f_H) = \Phi^{T/\kappa}(\gamma, f)$$

and

(i) *if G is anisotropic and κ trivial (so that H is* SL_2*) then*

$$\Phi^{T'/1}(\ ,f_H) \equiv 0 \quad \textit{for } T' \textit{ split};$$

(ii) *if G is* SL_2 *and κ trivial (so that H is* SL_2 *again) then*

$$\Phi^{T'/1}(\ ,f_H) \equiv \Phi^{T'/1}(\ ,f)$$

for all T', provided that the Haar measures dt' are chosen consistently.

This is proved (in **[4]**) by a case-by-case argument. Note that if H is T then $\Phi^{T/1}(\ ,f_H)$ is just f_H itself.

For general (real) groups a formalism for transferring the $\Phi^{T/\kappa}(\ ,f)$ to H is developed in **[10]**. Some progress towards obtaining a result as above for any real group is made in **[9]** and **[10]**.

7. Stable distributions and a map. We define the space of stable distributions on $G(F)$ to be the closed subspace, with respect to simple convergence, of the space of all distributions on $G(F)$ generated by stable orbital integrals, that is, by the distributions $f \to \Phi^{T/1}(\gamma, f)$, where γ is a regular semisimple element of $G(F)$ and T denotes the torus containing γ.

The transfer of the $\Phi^{T/\kappa}(\ ,f)$ to H establishes a correspondence (f, f_H) between $C_c^\infty(G(F))$ and $C_c^\infty(H(F))$. Dual to this correspondence there is a well-defined map from the space of stable distributions on $H(F)$ to the space of invariant distributions on $G(F)$; that is, if Θ_H is a stable distribution on $H(F)$ then we may define a (conjugation-) invariant distribution Θ on $G(F)$ by the formula $\Theta(f) = \Theta_H(f_H)$, $f \in C_c^\infty(G(F))$. This map, $\Theta_H \to \Theta$, will be central to our study of L-indistinguishability.

We denote by χ_π the character of (the infinitesimal equivalence class of) an irreducible admissible representation of $G(F)$; we regard χ_π as a function on the regular semisimple elements of $G(F)$, using the same normalization of Haar measure as in §5. There is a simple way to determine whether χ_π, as distribution, is stable. Let $\tilde{G}$ be GL_2 in the case that G is SL_2, or the full multiplicative group of the underlying quaternion algebra in the case that G is anisotropic. Then a linear combination χ of characters is stable if and only if χ is invariant under $\tilde{G}(F)$ ... or (of relevance for generalizations (cf. **[9]**)) if and only if χ is invariant under $\mathfrak{A}(T)$, $T \subset G$.

8. Local L-packets and some stable characters. We assume that the Langlands correspondence has been proved for G; in fact enough has been proved in each of the cases being considered. In the following definitions we also allow G to be a torus. We denote by $\Phi(G)$ the set of equivalence classes of admissible homomorphisms of W_F' into ${}^LG^\circ \rtimes W_F'$ (notation and definitions as in [1]). To each $\{\varphi\} \in \Phi(G)$ there is attached a finite collection $\Pi_{\{\varphi\}}$ of irreducible admissible representations of $G(F)$, an L-packet. Two irreducible admissible representations of $G(F)$ are said to be L-indistinguishable if they belong to the same L-packet. In the case that G is an inner from of SL_2 there is a simpler definition: π_1 and π_2 are L-indistinguishable if and only if there exists $g \in \tilde{G}(F)$ such that π_2 is equivalent to $\pi_1 \circ \operatorname{ad} g$. We set $\chi_{\{\varphi\}} = \sum_{\pi \in \Pi\{\varphi\}} \chi_\pi$. Clearly:

PROPOSITION. *$\chi_{\{\varphi\}}$ is stable.*

Since the Langlands correspondence has been proved for any (connected reductive) real group [6] we can define characters $\chi_{\{\varphi\}}$ in that case. In general $\chi_{\{\varphi\}}$ need not be stable; however if the constituents of $\Pi_{\{\varphi\}}$ are tempered (cf. [3]) then $\chi_{\{\varphi\}}$ is stable [9].

From §12 on we will not need to distinguish in notation between an admissible homomorphism $\varphi: W_F' \to {}^LG^\circ \rtimes W_F'$ and its equivalence class; we will then denote both by φ and write Π_φ, χ_φ, etc.

9. Character identities (introduction). We return to the map of stable distributions on $H(F)$ to invariant distributions on $G(F)$. If $\{\varphi_H\} \in \Phi(H)$ then, as we have observed, $\chi_{\{\varphi_H\}}$ is stable. Its image in invariant distributions on $G(F)$ is represented by some function χ on the regular semisimple elements of $G(F)$. This function χ is computed by the Weyl integration formula. We write: $\chi_{\{\varphi_H\}} \approx \chi$.

Recall the embedding ι_H of ${}^LH$ in ${}^LG$ (§4). We could have used W_F' in place of $W_{K/F}$ in defining ${}^LG$, ${}^LH$, and ι_H. With these modifications, suppose that φ_H is an admissible homomorphism of W_F' into ${}^LH$. Then $\varphi = \iota_H \circ \varphi_H$ maps W_F' to ${}^LG$. Suppose that φ is admissible; recall that (local) admissibility imposes the condition that the image of φ lie only in parabolic subgroups of ${}^LG$ which are "relevant to G" (cf. [1]). Then we say that $\{\varphi\} \in \Phi(G)$ factors through $\{\varphi_H\} \in \Phi(H)$.

Suppose that $\{\varphi\}$ factors through $\{\varphi_H\}$. Then linear combinations of the characters of the representations in $\Pi_{\{\varphi\}}$ make natural candidates for χ ($\approx \chi_{\{\varphi_H\}}$).

10. "$S^\circ\backslash S$". We introduce a useful group. It is easier to work with a homomorphism $\varphi: W_F' \to {}^LG$ rather than an equivalence class $\{\varphi\}$. We exclude the case of sp(2) [1] and corresponding special representation of $G(F)$, and consider just an admissible homomorphism $\varphi: W_{K/F} \to {}^LG$ where K, ${}^LG$ (and ${}^LH$, ι_H) are as earlier.

We define S_φ to be the centralizer in ${}^LG^\circ$ of the image of φ; S_φ° will be the connected component of the identity in S_φ. If φ' is equivalent to φ then there exists $g \in {}^LG^\circ$ such that $S_{\varphi'} = gS_\varphi g^{-1}$ and $S_{\varphi'}^\circ = gS_\varphi^\circ g^{-1}$.

Suppose that $\varphi = \iota_H \circ \varphi_H$ where φ_H is an admissible homomorphism of $W_{K/F}$ into ${}^LH$ and $H = H(T, \kappa)$. We regard κ as an element of ${}^LT^\circ$. Recall that ${}^LH^\circ$ is the connected component of the identity in the centralizer of κ in ${}^LG^\circ$. By definition (§3), σ_H fixes κ, $\sigma \in \mathfrak{G}_{K/F}$. It follows then that κ lies in the center of the image of ${}^LH$ in ${}^LG$. Therefore κ centralizes the image of $\varphi_H(W_{K/F})$ in ${}^LG$; that is, κ centralizes

$\varphi(W_{K/F})$. We have then that $\kappa \in S_\varphi$. We define $s_\varphi(\kappa)$, or just $s(\kappa)$ when φ is understood, to be the coset of κ in $S_\varphi^\circ \backslash S_\varphi$. Recall that this quotient appeared in [3].

Suppose that φ' is equivalent to φ and that $\varphi' = \iota_H \circ \varphi'_H$ where $\varphi'_H : W_{K/F} \to {}^L H$ and $H = H(T', \kappa')$. Then $\kappa' \in S_{\varphi'}$. Write φ' as $\operatorname{ad} g \circ \varphi$, $g \in {}^L G^\circ$. Then $g^{-1}\kappa' g \in S_\varphi$. We define $s_\varphi(\kappa')$ to be the coset of $g^{-1}\kappa' g$ in $S_\varphi^\circ \backslash S_\varphi$. We will see that $S_\varphi^\circ \backslash S_\varphi$ is abelian. Therefore $s_\varphi(\kappa')$ is independent of the choice for g.

We continue with the same φ, φ'. The conjugation $\operatorname{ad} g$ induces an isomorphism between $S_\varphi^\circ \backslash S_\varphi$ and $S_{\varphi'}^\circ \backslash S_{\varphi'}$ which carries $s_\varphi(\kappa)$ to $s_{\varphi'}(\kappa)$. Because both groups are abelian this isomorphism is independent of the choice of g in the equivalence $\varphi' = \operatorname{ad} g \circ \varphi$. We may therefore regard $S_\varphi^\circ \backslash S_\varphi$ and the elements $s_\varphi(\kappa)$ as attached to $\{\varphi\}$.

11. Calculations. We compute explicitly $S_\varphi^\circ \backslash S_\varphi$ and the elements $s(\kappa) = s_\varphi(\kappa)$. We need take just one homomorphism $\varphi: W_{K/F} \to {}^L G$ from each equivalence class. If we write $\varphi(w)$ as $\varphi_1(w) \times w$, $w \in W_{K/F}$, then the homomorphism $\varphi_1: W_{K/F} \to {}^L G^\circ = \mathrm{PGL}_2(\boldsymbol{C})$ lifts to a two dimensional representation $\tilde{\varphi}_1$ of $W_{K/F}$ [7, Lemma 3].

(i) *Suppose that $\tilde{\varphi}_1$ is reducible.*

Then $\tilde{\varphi}_1$ factors through $\tau: W_{K/F} \to F^\times$ and is defined by a pair (μ, ν) of quasi-characters on $F^\times$. We take

$$\tilde{\varphi}_1(w) = \begin{pmatrix} \mu(\tau(w)) & 0 \\ 0 & \nu(\tau(w)) \end{pmatrix}, \qquad w \in W_{K/F}.$$

If $(\mu/\nu)^2 \neq 1$ then $S_\varphi = S_\varphi^\circ = {}^L T^\circ$ and if $\mu = \nu$ then $S_\varphi = S_\varphi^\circ = {}^L G^\circ$. However, if μ/ν has order two then $S_\varphi = {}^L T^\circ \rtimes \langle 1, (\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix})_* \rangle$; recall that $(\ )_*$ denotes the image of $(\ )$ in $\mathrm{PGL}_2(\boldsymbol{C})$. Hence $S_\varphi^\circ \backslash S_\varphi = \boldsymbol{Z}/(2)$.

Whatever μ/ν, φ factors through ${}^L T$, T a split torus. The corresponding κ ($(\begin{smallmatrix} z & 0 \\ 0 & 1 \end{smallmatrix})_*$, some $z \neq 1$) lies in S_φ° and $s(\kappa)$ is trivial. Note that (this or any) φ factors through H when $H = H(\ , 1)$; $s(1)$ is trivial also.

Suppose that μ/ν has order two. In (ii) we will show that a homomorphism equivalent to φ factors through ${}^L T$, where T is a torus defined by the quadratic extension E of F attached to μ/ν, and that the associated $s(\kappa) = s_\varphi(\kappa)$ is the coset of $(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix})_*$.

(ii) *Suppose that $\tilde{\varphi}_1$ factors through a representation $\tilde{\varphi}_2 = \mathrm{Ind}(W_{E/F}, E^\times, \theta)$ of $W_{E/F}$ where $E \subset K$ is a quadratic extension of F and θ is a quasi-character on $E^\times$.*

Then φ_1 factors through φ_2, the projective representation defined by $\tilde{\varphi}_2$. Let σ° be the nontrivial element of $\mathfrak{G}_{E/F}$ and define $\bar{\theta}(x) = \theta(x^{\sigma^\circ})$, $x \in E^\times$.

We realize $W_{E/F}$ explicitly as $\{x \times \rho; x \in E^\times, \rho \in \{1, \sigma^\circ\}\}$, with multiplication rule

$$(x \times \rho)(x' \times \rho') = x(x')^\rho a_{\rho,\rho'} \times \rho\rho'$$

where $a_{\rho,\rho'} = 1$ unless $\rho = \rho' = \sigma^\circ$ and $a_{\sigma^\circ,\sigma^\circ}$ is some (chosen) element α of $F^\times - Nm_{E/F}E^\times$. Then we may assume that

$$\tilde{\varphi}_2(x \times 1) = \begin{pmatrix} \theta(x) & 0 \\ 0 & \bar{\theta}(x) \end{pmatrix}, \qquad x \in E^\times,$$

and

$$\tilde{\varphi}_2(1 \times \sigma^\circ) = \begin{pmatrix} 0 & z \\ z & 0 \end{pmatrix}$$

where z is a chosen square root of $\theta(\alpha)$. Thus

$$\tilde{\varphi}_2(x \times 1) = \begin{pmatrix} \theta(x) & 0 \\ 0 & \bar{\theta}(x) \end{pmatrix}_*, \qquad x \in E^\times,$$

and

$$\varphi_2(1 \times \sigma^\circ) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}_*.$$

Let T be a torus attached to E. Then φ factors through ${}^LT$; indeed we chose the embedding ι_T of ${}^LT$ in ${}^LG$ (§4) so as to insure this. Recall that $T = H(T, \kappa)$ where κ, as element of ${}^LT^\circ$, is $\left(\begin{smallmatrix} -1 & 0 \\ 0 & 1 \end{smallmatrix}\right)_*$; $s(\kappa)$ is the coset of $\left(\begin{smallmatrix} -1 & 0 \\ 0 & 1 \end{smallmatrix}\right)_*$ in $S_\varphi^\circ/S_\varphi$.

To compute S_φ, suppose first that $(\theta/\bar{\theta})^2 \neq 1$. Then

$$S_\varphi = \left\{1, \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}_*\right\}$$

so that $S_\varphi^\circ\backslash S_\varphi = \boldsymbol{Z}/(2)$; we have already recovered the nontrivial element of $S_\varphi^\circ\backslash S_\varphi$ as an $s(\kappa)$.

Suppose now that $\theta/\bar{\theta}$ has order two. Then S_φ coincides with $\varphi_1(W_{K/F})$; that is,

$$S_\varphi = \left\{1, \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}_*, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}_*, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}_*\right\}.$$

Therefore $S_\varphi^\circ\backslash S_\varphi = \boldsymbol{Z}/(2) \oplus \boldsymbol{Z}/(2)$. We have recovered only $\left(\begin{smallmatrix} -1 & 0 \\ 0 & 1 \end{smallmatrix}\right)_*$ (and 1) as an $s(\kappa)$. To recover the remaining elements of $S_\varphi^\circ\backslash S_\varphi$ we recall homomorphisms φ', φ'': $W_{K/F} \to {}^LG$ which are equivalent to φ and factor through other ${}^LH$.

Let E_0 be the quadratic extension of E defined by $\theta/\bar{\theta}$. Then $F^\times/Nm_{E_0/F}E_0^\times = \boldsymbol{Z}/(2) \oplus \boldsymbol{Z}/(2)$ so that there are two distinct quadratic extensions E' and E'' of F which are distinct from E and contained in E_0. We pick a quasi-character θ' on $(E')^\times$ such that $\theta' \circ Nm_{E_0/E'} = \theta \circ Nm_{E_0/E}$ and $\theta'/\bar{\theta}'$ is the quadratic character attached to E_0/E' ($\bar{\theta}'(x) = \theta'(x^{\sigma'})$, $1 \neq \sigma' \in \mathfrak{G}_{E'/F}$); we pick θ'' similarly. Then $\tilde{\varphi}_2' = \operatorname{Ind}(W_{E'/F}, (E')^\times, \theta')$ and $\tilde{\varphi}_2'' = \operatorname{Ind}(W_{E''/F}, (E'')^\times, \theta'')$ define representations $\tilde{\varphi}_1'$, $\tilde{\varphi}_1''$ of $W_{K/F}$, each equivalent to $\tilde{\varphi}_1$, and hence homomorphisms φ', φ'' : $W_{K/F} \to {}^LG$, each equivalent to φ. We define φ_1', φ_2' and φ_1'', φ_2'' as we did φ_1, φ_2; we realize φ_2' and φ_2'' as we did φ_2. Then $\varphi_1(W_{K/F})$, $\varphi_1'(W_{K/F})$, $\varphi_1''(W_{K/F})$, S_φ, $S_{\varphi'}$ and $S_{\varphi''}$ all coincide and equal

$$\left\{1, \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}_*, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}_*, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}_*\right\}.$$

As with φ earlier, φ' (... φ'') factors through ${}^L(T')$ (... ${}^L(T'')$), where T' (...T'') is some torus attached to E' (... E''). Suppose $T' = H(T', \kappa')$ and $T'' = H(T'', \kappa'')$. Then, writing $\varphi = \operatorname{ad} g' \circ \varphi' = \operatorname{ad} g'' \circ \varphi''$, g', $g'' \in {}^LG^\circ$, we have

$$s(\kappa') = s_\varphi(\kappa') = g' \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}_* g'^{-1}$$

and

$$s(\kappa'') = s_\varphi(\kappa'') = g'' \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}_* g''^{-1}.$$

An elementary argument shows that none of the conjugations ad g', ad g'', ad$(g')^{-1}g''$ may fix $\left(\begin{smallmatrix}-1 & 0\\ 0 & 1\end{smallmatrix}\right)_*$ and so we conclude that

$$\{s(\kappa'), s(\kappa'')\} = \left\{\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}_*, \begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}_*\right\}.$$

Thus we have recovered each element of $S_\varphi^\circ\backslash S_\varphi$ as an "$s(\kappa)$".

We have one further case to consider: $\theta = \bar{\theta}$. We pick a quasi-character μ on $F^\times$ such that $\theta = \mu \circ Nm_{E/F}$. Then φ is equivalent to the homomorphism of type (i) above attached to the pair $(\mu, \zeta\mu)$ where ζ is the quadratic character of $F^\times$ defined by the extension E/F. We now call that homomorphsim φ'. We had that $S_{\varphi'}^\circ\backslash S_{\varphi'}$ has two elements and that $\left(\begin{smallmatrix}0 & 1\\ 1 & 0\end{smallmatrix}\right)_*$ is a representative for the nontrivial coset. We now recover this coset as an $s_{\varphi'}(\kappa)$. Let T be a torus attached to E. Then φ factors through ${}^LT$ and the associated κ is $\left(\begin{smallmatrix}-1 & 0\\ 0 & 1\end{smallmatrix}\right)_*$. If $\varphi' = \text{ad}\, g \circ \varphi$, $g \in {}^LG^\circ$, then

$$g\begin{pmatrix}-1 & 0\\ 0 & 1\end{pmatrix}_* g^{-1} \equiv \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}_* \qquad (\text{mod}\ {}^LT^\circ),$$

and we are done.

(iii) *Suppose that $\tilde{\varphi}_1$ is of tetrahedral or octahedral type.*

Then $S_\varphi = S_\varphi^\circ = 1$. This is a straightforward exercise.

In summary:

PROPOSITION. (1) *$S_\varphi^\circ\backslash S_\varphi$ is one of* 1, $\boldsymbol{Z}/(2)$ *or* $\boldsymbol{Z}/(2) \oplus \boldsymbol{Z}/(2)$.

(2) *For each element s of $S_\varphi^\circ\backslash S_\varphi$ there is a pair (T, κ) such that $s = s(\kappa)$.*

12. Character identities (continued). From now on we will not distinguish in notation between an admissible homomorphism φ: $W_{K/F} \to {}^LG$ and its equivalence class; that is, we denote both by φ ... as we have noted, $S_\varphi^\circ\backslash S_\varphi$ can be regarded as attached to the equivalence class.

Suppose that φ factors through φ_H, where $H = H(T, \kappa)$. Let $s = s(\kappa)$. Then, in the notation of §§8, 9:

LEMMA. *There exist integers $\langle s, \pi\rangle$, $\pi \in \Pi_\varphi$, such that*

$$\chi_{\varphi_H} \approx \sum_{\pi\in\Pi_\varphi} \langle s, \pi\rangle\, \chi_\pi.$$

The proof [4] is again a case-by-case argument. Here is a summary of the explicit results.

(A) $G = \mathrm{SL}_2$. We consider φ as in (i), (ii), (iii) of the last section.

(i) If $(\mu/\nu)^2 \neq 1$ or if $\mu = \nu$ then Π_φ contains one element which we denote by π; $S_\varphi^\circ\backslash S_\varphi = 1$ and $\langle 1, \pi\rangle = 1$. If μ/ν has order two then Π_φ has two elements which we denote by π_1 and π_2. Recall that $S_\varphi^\circ\backslash S_\varphi = \boldsymbol{Z}/(2)$; $\langle s, \pi_1\rangle$ and $\langle s, \pi_2\rangle$ are the two characters in s. These characters depend on the choices we made in normalizing orbital integrals (§5); that is, for some choices $\langle\ , \pi_1\rangle$ is the trivial character and for others $\langle\ , \pi_2\rangle$ is the trivial one.

(ii) We have already considered the case $\theta = \bar{\theta}$. If $\theta/\bar{\theta}$ is not of order two then Π_φ has two elements and $S_\varphi^\circ\backslash S_\varphi = \boldsymbol{Z}/(2)$; the result is as in (i).

Suppose that $\theta/\bar{\theta}$ has order two. Then we had that $S_\varphi^\circ\backslash S_\varphi$ is $\boldsymbol{Z}/(2) \oplus \boldsymbol{Z}(2)$; Π_φ has four elements, too. Suppose $\Pi_\varphi = \{\pi_i, i = 1, \cdots, 4\}$. Then we may (and do) normalize orbital integrals so that the $\langle\ , \pi_i\rangle$ are the four characters on $S_\varphi^\circ\backslash S_\varphi$.

(iii) Here Π_φ has one element, say π; $S_\varphi^\circ\backslash S_\varphi = 1$ and $\langle 1, \pi\rangle = 1$.

(B) G *anisotropic*. We have only to consider φ as in (ii) with $\theta \neq \bar{\theta}$, and (iii).

Suppose $F = \mathbf{R}$. Then $S_\varphi^\circ\backslash S_\varphi = \mathbf{Z}/(2)$ since we exclude (iii) also. However Π_φ has only one element, say π; $\langle s, \pi\rangle$ is a character in s, trivial or nontrivial according to our choice of normalizing factors for orbital integrals.

Suppose that F is nonarchimedean. Then Π_φ has the same number of elements as the corresponding L-packet in SL_2, and the result is as there, *except* when φ is of type (ii) with $\theta/\bar{\theta}$ of order two. Then the corresponding packet in SL_2 has four elements; Π_φ has only one element, say π. We must set $\langle 1, \pi\rangle = 2$ and $\langle s, \pi\rangle = 0$, $s \neq 1$; in particular, $\langle\ , \pi\rangle$ is not a character.

As for generalizing these identities we will report some progress for real groups in a forthcoming paper; in the case that κ is trivial, so that H is a quasi-split inner form of G, an appropriate identity is known provided that the constituents of Π_φ are tempered [9].

13. Structure of local L-packets. From the results of the last section we conclude:

PROPOSITION. *If G is the quasi-split form* (SL_2) *then the pairing $\langle\ ,\ \rangle$ identifies Π_φ* (*noncanonically*) *as the dual group of $S_\varphi^\circ\backslash S_\varphi$.*

If G is anisotropic we remark only the existence of the functions $\langle\ , \pi\rangle$, $\pi \in \Pi_\varphi$. With some qualifications, we can expect an analogue of the proposition for a general quasi-split real group (cf. [3]).

14. A global application. Suppose now that F is a global field and that G is SL_2. There are analogues for the groups H of the local case; again H may be a maximal torus in G, defined over F, or G itself.

We assume that K is some large but finite Galois extension of F and consider just homomorphisms φ: $W_{K/F} \to {}^LG$ defined by representations of the form $\mathrm{Ind}(W_{K/F}, W_{K/E}, \theta)$, where E is a quadratic extension of F contained in K and θ is a grossencharacter for E not factoring through $Nm_{E/F}$. If T is a torus attached to E then φ factors through ${}^LT$ (as earlier, provided that ${}^LT$ is correctly embedded in ${}^LG$). We define Π_φ to be the set of representations $\pi = \otimes_v \pi_v$ of $G(\mathbf{A})$ which are irreducible, admissible (as in [1]) and such that $\pi_v \in \Pi_{\varphi_v}$ for each place v. The set Π_φ is an L-packet *in the following sense.*

We define two irreducible admissible representations $\pi^1 = \otimes_v \pi_v^1$ and $\pi^2 = \otimes_v \pi_v^2$ of $G(\mathbf{A})$ to be L-indistinguishable if for all places v, π_v^1 and π_v^2 are L-indistinguishable and for almost all v, π_v^1 and π_v^2 are equivalent. An L-packet is an equivalence class for this relation.

We have singled out the packets Π_φ for the following reason. As is proved in [4], two representations from such a packet may appear with different multiplicities in the space of cusp forms for G. One may appear, with multiplicity one, and the other not appear. For the remaining L-packets the members do appear with equal multiplicity (conjectured to be either zero or one).

In [4] there is a formula for the multiplicity with which a representation from Π_φ appears in cusp forms; it is this we wish to discuss.

As in the local case, we define S_φ to be the centralizer of $\varphi(W_{K/F})$ in ${}^LG^\circ$ and S_φ° to be the connected component of the identity in S_φ. As in (ii) of §11 we have that either $S_\varphi^\circ\backslash S_\varphi = \mathbf{Z}/(2)$ or $S_\varphi^\circ\backslash S_\varphi = \mathbf{Z}/(2) \oplus \mathbf{Z}/(2)$. For each place v, S_φ embeds naturally in S_{φ_v} and S_φ° in $S_{\varphi_v}^\circ$. There is then a natural map of $S_\varphi^\circ \backslash S_\varphi$ into $S_{\varphi_v}^\circ \backslash S_{\varphi_v}$. In notation we will not distinguish between an element of $S_\varphi^\circ \backslash S_\varphi$ and its image in $S_{\varphi_v}^\circ \backslash S_{\varphi_v}$.

Recall that the local characters $\langle\ ,\pi_v\rangle$ depend on the choices we made when normalizing orbital integrals (§§5, 12). We chose a nontrivial additive character ψ_{F_v} on F_v and regular element $\gamma^\circ = \gamma_v^\circ$ of $T(F_v)$ for each torus T anisotropic over F_v. Instead, we now choose a nontrivial additive character ψ on $F\backslash A$ and for each torus T anisotropic over F a regular element γ° in $T(F)$. At each place v where T does not split we use ψ_v and γ° to specify $\Phi^{T/\kappa}(\ ,\ )$, κ nontrivial.

Fix $s \in S_\varphi^\circ \backslash S_\varphi$ and $\pi \in \Pi_\varphi$. Then we have $\langle s, \pi_v\rangle = 1$ for almost all v. Therefore we may define $\langle s, \pi\rangle = \Pi_v \langle s, \pi_v\rangle$. Then [**4**]:

PROPOSITION. (i) *$\langle s, \pi\rangle$ is independent of the choices made for ψ and γ°.*
(ii) *$\langle\ ,\ \rangle$ induces a (canonical) surjection of Π_φ onto the dual group of $S_\varphi^\circ \backslash S_\varphi$.*

Also:

THEOREM. *The multiplicity with which $\pi \in \Pi_\varphi$ appears in the space of cusp forms is*:

$$\frac{1}{[S_\varphi^\circ \backslash S_\varphi]} \sum_{s \in S_\varphi^\circ \backslash S_\varphi} \langle s, \pi\rangle .$$

That is, those π for which $\langle\ ,\pi\rangle$ is the trivial character appear with multiplicity one and those π for which $\langle\ ,\pi\rangle$ is nontrivial do not appear.

15. Afterword. More generally, and of relevance also for Shimura varieties (cf. [**8**]), we can consider inner forms of a group G such that $\mathrm{Res}_{E/F}\mathrm{SL}_2 \subset G \subset \mathrm{Res}_{E/F}\mathrm{GL}_2$ with E some finite Galois extension of F, and develop a local and a global theory in the same way. The analogous multiplicity formula is [**4**]:

$$\frac{d_\varphi}{[S_\varphi^\circ \backslash S_\varphi]} \sum_{s \in S_\varphi^\circ \backslash S_\varphi} \langle s, \pi\rangle$$

where d_φ is defined as follows. If $\varphi, \varphi' \colon W_F \to {}^LG$ are admissible homomorphisms (cf. [**1**, §16]) call φ and φ' locally equivalent everywhere if φ_v is equivalent to φ_v' for each place v. Call φ and φ' weakly globally equivalent if φ' is equivalent to $\omega\varphi$ where ω is a continuous 1-cocycle of W_F with values in the center of ${}^LG^\circ$ such that the restriction of ω to each local group is trivial. Then d_φ is the number of weak global equivalence classes in the everywhere local equivalence class of φ.

In [**8**] L-indistinguishability plays a role in relating the zeta-functions of certain Shimura varieties to automorphic L-functions. Briefly, in the case discussed in [**2**], where G is the full multiplicative group of some quaternion algebra over a totally real field and there is no L-indistinguishability, functions $L(s, \pi, \rho)$ appear (π an automorphic representation of $G(A)$, ρ a certain representation of ${}^LG$, fixed as in that lecture). In the general case of [**8**], where G is a subgroup of the full multiplicative group, we must consider at least some of the L-packets Π_φ where different multiplicities in cusp (...automorphic) forms occur. Then $\varphi\colon W_F \to {}^LG$ factors through $\iota_T\colon {}^LT \to {}^LG$, with T a nonsplit torus of G. If $\varphi = \iota_T \circ \varphi_T$, $\pi_T \in \Pi_{\varphi_T}$ and $\rho_T = \rho \cdot \iota_T$ then $L(s, \pi, \rho) = L(s, \pi_T, \rho_T)$. There is a natural decomposition $\rho_T = \rho_T^1 \oplus \rho_T^2$, where ρ_T^1, ρ_T^2 may be reducible, and it is the functions $L(s, \pi_T, \rho_T^i)$ which are relevant.

REFERENCES

1. A. Borel, *Automorphic L-functions*, these PROCEEDINGS, part 2, pp. 27–61.

2. W. Casselman, *The Hasse-Weil ζ-function of some moduli varieties of dimension greater than one*, these PROCEEDINGS, part 2, pp. 141–163.

3. A. W. Knapp and G. Zuckerman, *Normalizing factors, tempered representations and L-groups*, these PROCEEDINGS, part 1, pp. 93–105.

4. J-P. Labesse and R. P. Langlands, *L-indistinguishability for* SL(2) (to appear).

5. R. P. Langlands, *On Artin's L-functions*, Rice Univ. Studies, vol. 56, no. 2, pp. 23–28, 1970.

6. ———, *On the classification of irreducible representations of real algebraic groups* (to appear).

7. ———, *Stable conjugacy; definitions and lemmas* (to appear).

8. ———, *On the zeta functions of some simple Shimura varieties* (to appear).

9. D. Shelstad, *Characters and inner forms of a quasi-split group over* ***R***, Compositio. Math. (to appear).

10. ———, *Orbital integrals and a family of groups attached to a real reductive group*, Ann. Sci. Ecole Norm. Sup. (to appear).

COLUMBIA UNIVERSITY

Proceedings of Symposia in Pure Mathematics
Vol. 33 (1979), part 2, pp. 205–246

AUTOMORPHIC REPRESENTATIONS, SHIMURA VARIETIES, AND MOTIVES. EIN MÄRCHEN

R. P. LANGLANDS

1. Introduction. It had been my intention to survey the problems posed by the study of zeta-functions of Shimura varieties. But I was too sanguine. This would be a mammoth task, and limitations of time and energy have considerably reduced the compass of this report. I consider only two problems, one on the conjugation of Shimura varieties, and one in the domain of continuous cohomology. At first glance, it appears incongruous to couple them, for one is arithmetic, and the other representation-theoretic, but they both arise in the study of the zeta-function at the infinite places.

The problem of conjugation is formulated in the sixth section as a conjecture, which was arrived at only after a long sequence of revisions. My earlier attempts were all submitted to Rapoport for approval, and found lacking. They were too imprecise, and were not even in principle amenable to proof by Shimura's methods of descent. The conjecture as it stands is the only statement I could discover that meets his criticism and is compatible with Shimura's conjecture.

The statement of the conjecture must be preceded by some constructions, which have implications that had escaped me. When combined with Deligne's conception of Shimura varieties as parameter varieties for families of motives they suggest the introduction of a group, here called the Taniyama group, which may be of importance for the study of motives of CM-type. It is defined in the fifth section, where its hypothetical properties are rehearsed.

With the introduction of motives and the Taniyama group, the report takes on a tone it was not originally intended to have. No longer is it simply a matter of formulating one or two specific conjectures, but we begin to weave a tissue of surmise and hypothesis, and curiosity drives us on.

Deligne's ideas are reviewed in the fourth section, but to understand them one must be familiar at least with the elements of the formalism of tannakian categories underlying the conjectural theory of motives, say, with the main results of Chapter II of [**40**].

The present Summer Institute is predicated on the belief that there is a close relation between automorphic representations and motives. The relation is usually

AMS (MOS) subject classifications (1970). Primary 14G10, 14F99.

couched in terms of L-functions, and no one has suggested a direct connection. It may be provided by the principle of functoriality and the formalism of tannakian categories. This possibility is discussed in the highly speculative second section.

However there is a small class of automorphic representations which are certainly not amenable to this formalism. I have called them anomalous, and in order to make their significance clearer I have discussed an example of Kurokawa at length in the third section. Although the anomalous representations form only a small part of the collection of automorphic representations, they are frequently encountered, especially in the study of continuous cohomology, and so we have come full circle. Our long divagation has not been in vain, for we have acquired concepts that enable us to appreciate the global significance of the local examples described in the seventh section, which deals with the second of our original two problems.

At all events, I have exceeded my commission and been seduced into describing things as they may be and, as seems to me at present, are likely to be. They could be otherwise. Nonetheless it is useful to have a conception of the whole to which one can refer during the daily, close work with technical difficulties, provided one does not become too attached to it, but takes pains to ensure that it continues to conform to the facts, and is prepared to abandon it when that is called for. The views of this report are in any case not peculiarly mine. I have simply fused my own observations and reflections with ideas of others and with commonly accepted tenets.

I have also wanted to draw attention to the specific problems, on which expert advice would be of great help, and I hope that the report is sufficiently loosely written that someone familiar with continuous cohomology but not with arithmetic can turn to the seventh section, overlooking the first few pages, and see what the study of Shimura varieties needs from that theory, or that someone familiar with Shimura varieties but not automorphic representations or motives will be able to find the definition of the Serre group in the fourth section and the Taniyama group in the fifth, and then turn to read the sixth section.

Finally a word about what is not discussed in this report. The investigations of Kazhdan [**26**] and Shih [**46**] on conjugation of Shimura varieties are not reviewed; their bearing on the problem formulated here is not yet clear. L-indistinguishability is not discussed. Little is known, and that is described in another lecture [**44**]. Problems caused by noncompactness are ignored. There is a tremendous amount of material on compactification and on the cohomology of noncompact quotients, but no one has yet tried to bring it to bear on the study of the zeta-functions. The omission I regret most is that of a discussion of the reduction of the Shimura varieties modulo a prime [**36**]. Here there is a great deal to be said, especially about the structure of the set of geometric points in the algebraic closure of a finite field, starting with the work of Ihara on curves. I hope to report on this topic on another occasion.

2. Automorphic representations. Our present knowledge does not justify an attempt to fix a language in which the relations of automorphic representations with motives are to be expressed, Nonetheless that of tannakian categories [**40**] appears promising and it might be worthwhile to take a few pages to draw attention to the problems to be solved before it can be applied in the study of automorphic representations.

We first recall the rough classification of irreducible admissible representations of $\mathrm{GL}(n, F)$, F being a local field, and of automorphic representations of $\mathrm{GL}(n, A_F)$, F being a global field, reviewed in some of the other lectures ([3], [5], [35], [48], cf. also [7], [31]). In either case the representation π has a central character ω and $z \to |\omega(z)|$ may be extended uniquely from $Z(F)$, or $Z(A_F)$, to a positive character ν of $\mathrm{GL}(n, F)$, or of $\mathrm{GL}(n, A_F)$.

If F is a local field then π is said to be cuspidal if for any K-finite vectors u and v, in the space of π and its dual, $\nu^{-1}(g) \langle \pi(g)u, v\rangle$ is square-integrable on the quotient $Z(F)\backslash\mathrm{GL}(n, F)$. To construct an arbitrary irreducible admissible representation one starts from a partition $\{n_1, \cdots, n_r\}$ of n and cuspidal representations $\pi_1, \cdots, \pi_r$ of $\mathrm{GL}(n_i, F)$. If ω_i is the central character of π_i, there is a real number s_i such that $|\omega_i(z)| \equiv |z|^{s_i}$ if z lies in the centre of $\mathrm{GL}(n_i, F)$, a group isomorphic to $F^\times$. Changing the order of the partition, one supposes that $s_1 \geqq \cdots \geqq s_r$. The partition defines a standard parabolic subgroup P of $\mathrm{GL}(n)$ and $\sigma = \bigotimes \pi_i$ a representation of $M(F)$, because the Levi factor M of P is isomorphic to $\mathrm{GL}(n_1) \times \cdots \times \mathrm{GL}(n_r)$. The representation σ yields in the usual way an induced representation I_σ of $G(F)$. I_σ may not be irreducible, but it has a unique irreducible quotient, which we denote $\pi_1 \boxplus \cdots \boxplus \pi_r$. Every representation is of this form and $\pi_1 \boxplus \cdots \boxplus \pi_r \simeq \pi_1' \boxplus \cdots \boxplus \pi_s'$ if and only if $r = s$ and after renumbering $\pi_i \simeq \pi_i'$. Thus every representation can be represented uniquely as a formal sum, in the sense of this notation, of cuspidal representations. The representation $\pi_1 \boxplus \cdots \boxplus \pi_r$ is said to be tempered if all the s_i are 0. We can clearly define, in a formal manner, the sum of any finite number of representations.

We can in fact formally define an abelian catefory $\Pi(F)$ whose collection of objects is the union over n of the irreducible, admissible representations of $\mathrm{GL}(n, F)$. If $\pi = \pi_1 \boxplus \cdots \boxplus \pi_r$, $\pi' = \pi_1' \boxplus \cdots \boxplus \pi_s'$, with π_i and π_j' cuspidal, we set

$$\mathrm{Hom}(\pi, \pi') = \bigoplus_{\{i,\, j \mid \pi_i \sim \pi_j'\}} C.$$

The composition is obvious. The tempered representations form a subcategory $\Pi^\circ(F)$.

If F is a global field then π is said to be cuspidal if π is a constituent of the representation of $\mathrm{GL}(n, A_F)$ on the space of measurable cusp forms φ satisfying

(a) $\varphi(zg) = \omega(z)\varphi(g)$, $z \in Z(A_F)$,

(b) $\int_{Z(A_F)G(F)\backslash G(A_F)} \nu^{-2}(g)|\varphi(g)|^2\, dg < \infty$.

If $n_1, \cdots, n_r$ is a partition of n and $\pi_1, \cdots, \pi_r$ cuspidal representations of $\mathrm{GL}(n_i, A_F)$ we may again change the order so that $s_1 \geqq \cdots \geqq s_r$ and then construct the induced representation I_σ. Every automorphic representation is a constituent of some I_σ. For an adequate classification, one needs more. The following statement may eventually result from the investigations of Jacquet, Shalika, and Piatetskii-Shapiro, but has not yet been proved in general.

A. *If π is a constituent of I_σ and of $I_{\sigma'}$ then the partitions $\{n_1, \cdots, n_r\}$ and $\{n_1', \cdots, n_s'\}$ have the same number of elements, and, after a renumbering, $n_i = n_i'$ and $\pi_i \sim \pi_i'$.*

If $\pi_i = \otimes_v \pi_i(v)$ then one constituent of I_σ is the representation $\pi = \otimes_v \pi(v)$ with local components $\pi(v) = \pi_1(v) \boxplus \cdots \boxplus \pi_r(v)$. This representation will be denoted $\pi = \pi_1 \boxplus \cdots \boxplus \pi_r$, but the notation is not justified until statement A is proved. The representations of this form will be called isobaric and can again be used to define an abelian category $\boldsymbol{\Pi}(F)$.

We agree to call $\pi = \pi_1 \boxplus \cdots \boxplus \pi_r$ tempered if each of the cuspidal representations π_i has a unitary central character, that is, if each of the s_i is 0. However the language is only justified if we can prove the following statement, the strongest form of the conjecture of Ramanujan to which the examples of Howe-Piatetskii-Shapiro and Kurokawa allow us to cling.

B. *If $\pi = \otimes\pi(v)$ is a cuspidal representation with unitary central character then each of the factors $\pi(v)$ is tempered.*

The tempered representations form a subcategory $\boldsymbol{\Pi}^\circ(F)$ of $\boldsymbol{\Pi}(F)$.

It is clear that we have tried to define the categories $\boldsymbol{\Pi}(F)$ and $\boldsymbol{\Pi}^\circ(F)$ in such a way that if v is a place of F there are functors $\boldsymbol{\Pi}(F) \to \boldsymbol{\Pi}(F_v)$ and $\boldsymbol{\Pi}^\circ(F) \to \boldsymbol{\Pi}^\circ(F_v)$ taking π to its factor $\pi(v)$. However without a natural definition of the arrows, there is no unique way to define Hom $(\pi, \pi') \to$ Hom $(\pi(v), \pi'(v))$.

If I_σ is not irreducible it will have other constituents in addition to $\pi_1 \boxplus \cdots \boxplus \pi_r$. These automorphic representations will be called anomalous. Although the principle of functoriality may apply to them, there is considerable doubt that they can be fitted into a tannakian framework. Observe that statement A and the strong form of multiplicity one imply that if π is any automorphic representation there is a unique isobaric representation π' such that $\pi(v) \sim \pi'(v)$ for almost all v.

If F is a global field the purpose of the tannakian formalism would be to provide us with a reductive group over $\boldsymbol{C}$ whose n-dimensional representations, or rather their equivalence classes, are to correspond bijectively to the isobaric automorphic representations of GL(n, A_F). This hypothetical group will have to be very large, a projective limit of finite-dimensional groups. We denote it by $G_{\boldsymbol{\Pi}(F)}$.

The category Rep$(G_{\boldsymbol{\Pi}(F)})$ of the finite-dimensional representations over $\boldsymbol{C}$ of the algebraic group $G_{\boldsymbol{\Pi}(F)}$ would certainly be abelian, but in addition it is a category in which tensor products can be defined. Moreover there is a functor to the category of finite-dimensional vector spaces over C. If (φ, X), consisting of the space X and the representation φ of $G_{\boldsymbol{\Pi}(F)}$ on it, belongs to Rep $(G_{\boldsymbol{\Pi}(F)})$ one simply ignores φ. The tensor product satisfies certain conditions of associativity, commutativity, and so on, and the functor, called a fibre functor, is compatible with tensor products and other operations of the two categories.

A theorem of **[40]**, but not the principal one, asserts that, conversely, an abelian category with tensor products and a fibre functor is equivalent to the category of representations of a reductive group, provided certain natural axioms are satisfied. Thus it appears that if we are to be able to introduce $G_{\boldsymbol{\Pi}(F)}$ we will have to associate to each pair consisting of a cuspidal representation π of GL(n, A) and a cuspidal representation π' of GL(n', A) an isobaric representation $\pi \boxtimes \pi'$ of GL(nn', A). In general, if $\pi = \pi_1 \boxplus \cdots \boxplus \pi_r$ and $\pi' = \pi'_1 \boxplus \cdots \boxplus \pi'_s$ we would set

$$\pi \boxtimes \pi' = \boxplus_{i,j} (\pi_i \boxtimes \pi'_j).$$

In addition, we will have to associate to each isobaric representation π of $\mathrm{GL}(n, A)$, $n = 1, 2, \cdots$, a complex vector space $X(\pi)$ of dimension n, together with isomorphisms

$$X(\pi \boxplus \pi') \simeq X(\pi) \oplus X(\pi') \qquad X(\pi \boxtimes \pi') \simeq X(\pi) \otimes X(\pi').$$

There are a large number of conditions to be satisfied, among them one which is perhaps worth mentioning explicitly. Suppose π and π' are cuspidal and $\pi \boxtimes \pi' = \boxplus_{i=1}^{r} \pi_i$ with π_i cuspidal. Then the set $\{\pi_i, \cdots, \pi_r\}$ contains the trivial representation of $\mathrm{GL}(1, A)$ if and only if π' is the contragredient of π, when it contains this representation exactly once.

At the moment I have no idea how to define the spaces $X(\pi)$; indeed, no solid reason for believing that the functor $\pi \to X(\pi)$ exists. Even though the attempt to introduce the groups $G_{\Pi(F)}$ may turn out to be vain the prize to be won is so great that one cannot refuse to hazard it. One would like to show, in addition, that if π and π' are tempered then $\pi \boxtimes \pi'$ is also, and thus be able to introduce a group $G_{\Pi^\circ(F)}$ classifying the tempered automorphic representations. If Ω_F^+ is the group of multiplicative type whose module of rational characters is the module of positive characters of the topological group $F^\times \backslash I_F$ then $G_{\Pi(F)}$ will be a direct product $G_{\Pi^\circ(F)} \times \Omega_F^+$.

One will also wish to introduce, by a similar process, groups $G_{\Pi(F)}$ and $G_{\Pi^\circ(F)}$, attached to a local field F and classifying the irreducible, admissible representations of $\mathrm{GL}(n, F)$, $n = 1, 2, \cdots$, and the tempered representations of $\mathrm{GL}(n, F)$. The formalism is clearly intended to be such that if F_v is a completion of F there are homomorphisms $G_{\Pi(F_v)} \to G_{\Pi(F)}$ and $G_{\Pi^\circ(F_v)} \to G_{\Pi^\circ(F)}$ dual to $\Pi(F) \to \Pi(F_v)$ and $\Pi^\circ(F) \to \Pi^\circ(F_v)$.

If F is a local field the conjectured classification of the representations of $\mathrm{GL}(n, F)$ [3], verified when F is archimedean, provides a concrete description of the category $\Pi(F)$ with its product $\boxtimes$. If F is archimedean and W_F is the Weil group of F then $\Pi(F)$ is equivalent to the category of continuous semisimple representations σ of W_F on complex vector spaces X. The tensor product is the usual one $(\sigma, X) \otimes (\sigma', X') = (\sigma \otimes \sigma', X \otimes X')$ and the fibre functor is $(\sigma, X) \to X$.

Thus $G_{\Pi(F)}$ is a kind of algebraic hull of the topological group W_F. In particular there is a homomorphism $W_F \to G_{\Pi(F)}(\boldsymbol{C})$ whose image is Zariski-dense. The subcategory corresponding to $\Pi^\circ(F)$ is obtained by taking only those (σ, X) for which the image of $\sigma(W_F)$ is relatively compact.

If F is nonarchimedean one should take not the Weil group but a direct product $W_F' = \mathrm{SL}(2, \boldsymbol{C}) \times W_F$.

Conjecturally at least, σ is to be replaced by a continuous, semisimple representation of W_F' whose restriction to $\mathrm{SL}(2, \boldsymbol{C})$ is complex analytic. To obtain a category equivalent to $\Pi^\circ(F)$ one should take only those σ for which $\sigma(W_F)$ is relatively compact. Observe that in order to obtain a semisimple category we have replaced the group WD_F employed by Borel and Tate [47] by the group W_F'. If $w \to |w|$ is the usual positive character of the Weil group, there is an obvious homomorphism of WD_F into W_F' which takes $w \in W_F \subseteq WD_F$ to

$$\begin{pmatrix} |w|^{1/2} & 0 \\ 0 & |w|^{-1/2} \end{pmatrix} \times w.$$

Notice that according to these classifications there are homomorphisms of algebraic groups

$$G_{\Pi(F)} \to \mathrm{Gal}(\bar{F}/F) \quad \text{and} \quad G_{\Pi^\circ(F)} \to \mathrm{Gal}(\bar{F}/F),$$

the group on the right being a projective limit of finite groups. The principle of functoriality cannot be valid unless there are similar homomorphisms when F is a global field.

Let Ω_F, Ω_F^+, Ω_F^0, be the groups of multiplicative type whose modules of rational characters are, respectively, the module of all characters of $F^\times$, or $F^\times\backslash I_F$ if F is global, of all positive characters, or of all unitary characters. Then $\Omega_F = \Omega_F^+ \times \Omega_F^0$ and there will be homomorphisms

$$G_{\Pi(F)} \to \Omega_F, \quad G_{\Pi^\circ(F)} \to \Omega_F^0.$$

If F is nonarchimedean and local we may also define Ω_{un}, Ω_{un}^+, and Ω_{un}^0, by replacing the modules of characters of various types by the modules of unramified characters of the same type. The groups Ω_{un}, Ω_{un}^+, and Ω_{un}^0 contain a distinguished point over $\boldsymbol{C}$, the Frobenius Φ, which is simply the image of a uniformizing parameter in $F^\times$. In any case the formalism will certainly allow us to introduce for any representation σ of the algebraic group $G_{\Pi(F)}$ over $\boldsymbol{C}$ an L-function $L(s, \sigma)$ and if σ corresponds to the representation π of $\mathrm{GL}(n, A_F)$ then $L(s, \sigma) = L(s, \pi)$.

But the reasons for wishing to introduce the groups $G_{\Pi(F)}$ and $G_{\Pi^\circ(F)}$ and the associated formalism are not simply, or even primarily, aesthetic. There are problems which will be difficult to formulate exactly without them. Suppose, for example, that $\pi = \bigotimes_v \pi_v$ is an isobaric representation of $\mathrm{GL}(n, A)$ and each of the factors π_v is tempered. For almost all v, π_v is unramified and associated to a conjugacy class $\{g_v\} = \{g(\pi_v)\}$ in $\mathrm{GL}(n, \boldsymbol{C})$. Since π_v is supposed tempered this class meets the unitary group $U(n)$ and I may, as I prefer, regard it as a conjugacy class in $U(n)$. The general analytic analogue of the Tchebotarev theorem or the Sato-Tate conjecture would be a theorem or conjecture describing the asymptotic distribution of the classes $\{g_v\}$.

Suppose the formalism existed and π were associated to a representation σ of $G_{\Pi^\circ(F)}$. The image $\sigma(G_{\Pi^\circ(F)}(\boldsymbol{C}))$ would be a reductive subgroup $H(\boldsymbol{C})$ of $\mathrm{GL}(n, \boldsymbol{C})$ with a maximal compact subgroup K_H. For almost all v, σ_v would factor through $G_{\Pi^\circ(F_v)} \to \Omega_{\mathrm{un}}^0$ and $\sigma_v(\Phi_v)$ would be defined. Since its conjugacy class in $H(\boldsymbol{C})$ would meet K_H, it would define a conjugacy class in K_H, which we denote $\{\sigma_v(\Phi_v)\}$. Of course $\{\sigma_v(\Phi_v)\} \subseteq \{g_v\}$ and the asymptotic distribution of the classes $\{g_v\}$ can be inferred from that of the classes $\{\sigma_v(\Phi_v)\}$. There is a natural probability measure on the space of conjugacy classes in K_H. If X is a set of conjugacy classes and if the set $Y = \bigcup_{x\in X} x$ is measurable in K_H, one takes meas X = meas Y. It is natural to suppose that it is this measure which defines the asymptotic distribution of the classes $\{\sigma_v(\Phi_v)\}$. To verify the supposition, it will be necessary to establish that if ρ is any representation of $G_{\Pi^\circ(F)}$ over $\boldsymbol{C}$ then the order of the pole of $L(s, \rho)$ at $s = 1$ is equal to the multiplicity with which the trivial representation of $G_{\Pi^\circ(F)}$ occurs in ρ. If the existence of $G_{\Pi^\circ(F)}$ were established, it would be easy enough to deduce this from the recent results of Jacquet and Shalika **[25]**.

Within this formalism, the principle of functoriality asserts that if F is a local field and G a reductive group over F then any L-packet of representations of $G(F)$ is associated to a homomorphism φ: $G_{\Pi(F)} \to {}^LG$ of algebraic groups over $\boldsymbol{C}$ for which

$$\begin{array}{ccc} G_{\Pi(F)} & \xrightarrow{\varphi} & {}^LG \\ & \searrow \quad \swarrow & \\ & \mathrm{Gal}(\bar{F}/F) & \end{array}$$

is commutative. If $G_{\Pi(F)}$ is replaced by $G_{\Pi^\circ(F)}$, the L-packet should be tempered. If F is global, some caution will have to be exercised. If the L-packet Π consists of $\pi = \bigotimes \pi_v$ for which the π_v are always tempered, it should correspond to a φ: $G_{\Pi^\circ(F)} \to {}^LG$. Otherwise this may not be so, for a reason which will perhaps be clearer after an example of Kurokawa [**29**] is discussed in the next section. If there is a representation

$$\psi\colon {}^LG \to \mathrm{GL}(n) \times \mathrm{Gal}(\bar{F}/F)$$

and if the image $\psi_*(\Pi)$ of Π given by the principle of functoriality is not isobaric then Π can be associated to no φ.

One may nonetheless hope to prove, both locally and globally, that to each φ: $G_{\Pi(F)} \to {}^LG$ is associated an L-packet, provided φ commutes with the homomorphisms to the Galois group. If G is not quasi-split the local behaviour of φ with respect to parabolic subgroups will also have to be taken into account [**3**]. For archimedean fields one recovers the usual classification.

The few examples studied [**44**] suggest that questions about the multiplicity with which the elements of Π occur in the space of automorphic forms will have to be answered in terms of φ.

The principal reason for wishing to define the group $G_{\Pi(F)}$ is that it provides the only way visible at present to express completely the relation between automorphic forms and the conjectural theory of motives [**40**]. The category of motives over F, a local or a global field, is $\boldsymbol{Q}$-linear and tannakian, but it does not always possess a fibre functor over $\boldsymbol{Q}$ and seldom a single naturally defined one. Thus tannakian duality associates to it not a group over $\boldsymbol{Q}$ but a group-like object, a "gerbe" in the rustic terminology which has become so popular in recent years. Over $\boldsymbol{C}$ this object becomes a group $G_{\mathrm{Mot}(F)}$, and the relations between motives and automorphic representations will probably be adequately expressed by the existence of a homomorphism $\rho_F\colon G_{\Pi(F)} \to G_{\mathrm{Mot}(F)}$ defined over $\boldsymbol{C}$. The field F can be local or global. The local and global homomorphisms are to be compatible with each other and with the formation of L-functions. Both the image and the kernel of ρ_F will probably be rather large when F is a number field (cf. C.6.2 of [**43**]) but rather small when F is local.

3. Anomalous representations. Since the anomalous representations cause some difficulty in the study of the zeta-functions of Shimura varieties, it will be useful to acquire some feeling for them before going on. The brief remarks of the previous section suggest that an automorphic representation π of the reductive group G, or rather the L-packet Π containing it, should be called anomalous if for some homomorphism $\psi\colon {}^LG \to \mathrm{GL}(n, \boldsymbol{C}) \times \mathrm{Gal}(\bar{F}/F)$ the principle of functoriality takes

π or Π to an anomalous representation of $\mathrm{GL}(n, A)$. It may be that the counterexamples to the Ramanujan conjecture of Howe-Piatetskii-Shapiro **[19]** are anomalous in this sense.

Since their paper is not available to me as I write, I have to test this suggestion on another example, discovered by Kurokawa and quite explicit. The group G is to be the projective symplectic group in four variables over $\boldsymbol{Q}$. The L-group is then the direct product of ${}^LG^\circ$, the symplectic group in four variables and the galois group $\mathrm{Gal}(\bar{\boldsymbol{Q}}/\boldsymbol{Q})$. Since all the groups with which we shall deal in this section will be split, we may ignore the factor $\mathrm{Gal}\,(\bar{\boldsymbol{Q}}/\boldsymbol{Q})$.

Let G_1 be the product of PGL(2) over $\boldsymbol{Q}$ with itself, so that ${}^LG_1^\circ = \mathrm{SL}(2, \boldsymbol{C}) \times \mathrm{SL}(2, \boldsymbol{C})$ and let G_2 be GL(4) over $\boldsymbol{Q}$. Define $\varphi_1\colon {}^LG_1^\circ \to {}^LG$ to be the homomorphism

$$\begin{pmatrix}\alpha_1 & \beta_1\\ \gamma_1 & \delta_1\end{pmatrix} \times \begin{pmatrix}\alpha_2 & \beta_2\\ \gamma_2 & \delta_2\end{pmatrix} \to \begin{pmatrix}\alpha_1 & 0 & \beta_1 & 0\\ 0 & \alpha_2 & 0 & \beta_2\\ \gamma_1 & 0 & \delta_1 & 0\\ 0 & \gamma_1 & 0 & \delta_2\end{pmatrix}$$

and let φ_2 be the standard imbedding of ${}^LG^\circ$ in ${}^LG_2^\circ$ which is $\mathrm{GL}(4, \boldsymbol{C})$.

In order to analyze Kurokawa's example one must formulate his statements representation-theoretically. I state the facts necessary to this purpose, but have to ask that the reader understand the discrete series sufficiently well to verify them for himself. There is nothing to prove. It is simply a question of writing down explicitly for the special case of concern to us here some of the results of **[17]** and and **[18]**, and some definitions from **[31]**.

An automorphic representation $\pi = \bigotimes \pi_v$ of $G(\boldsymbol{A})$ is associated to a holomorphic form of weight k in the classical sense of **[39]** if and only if π_∞ is a member of the holomorphic discrete series and lies in an L-packet $\Pi_{\psi(k)}$, where the restriction of $\psi(k)$ to $\boldsymbol{C}^\times \subseteq W_{\boldsymbol{C}/\boldsymbol{R}}$ has the form

$$z \to \begin{pmatrix} z^\lambda \bar{z}^{-\lambda} & & & \\ & z^\mu \bar{z}^{-\mu} & & \\ & & z^{-\lambda}\bar{z}^{\lambda} & \\ & & & z^{-\mu}\bar{z}^{\mu}\end{pmatrix}$$

with $\lambda = (2k - 3)/2$, $\mu = \frac{1}{2}$. Since G is the projective group, k must be even. Notice that $z^\lambda \bar{z}^{-\lambda}$ is to be calculated as $z^{2\lambda}(z\bar{z})^{-\lambda} = \bar{z}^{-2\lambda}\,(z\bar{z})^\lambda$. We will eventually take $k = 10$.

On the other hand an automorphic representation $\pi = \bigotimes \pi_v$ of $\mathrm{PGL}(2, \boldsymbol{A})$ is associated to a holomorphic form of weight $2k - 2$ if and only if π_∞ belongs to the discrete series and to an L-packet $\Pi_{\varphi(k)}$, where

$$\varphi(k)\colon z \to \begin{pmatrix} z^\lambda \bar{z}^{-\lambda} & \\ & z^{-\lambda}\bar{z}^{\lambda}\end{pmatrix}$$

with λ as above. In particular φ_1 takes the L-packet $\Pi_{\varphi(k)} \otimes \Pi_{\varphi(2)}$, consisting in fact of a single representation, to $\Pi_{\psi(k)}$.

We want to apply the principle of functoriality to φ_1 and a very special automorphic representation π of $G_1(A)$. π must be a tensor product $\pi' \otimes \pi''$ of two representations of PGL(2, A). π'_∞ and π''_∞ will both be members of the discrete series, the first in $\Pi_{\varphi(10)}$ and the second in $\Pi_{\varphi(2)}$. In fact π' will be the automorphic representation associated to the cusp form of weight 18, but π'' will be anomalous or, to be more exact, its pull-back $\tilde{\pi}''$ to GL(2, A) will be anomalous. To construct it we begin with the partition $\{1, 1\}$ of 2 and the two characters

$$\eta: x \to |x|^{1/2}, \qquad \nu: x \to |x|^{-1/2}$$

of GL(1, A), and then construct the induced representation of GL(2, A) as in the preceding section. Any constituent $\tilde{\pi}''$ of the induced representation factors through a representation $\tilde{\pi}''$ of PGL(2, A). We so choose $\tilde{\pi}''$ that $\pi''_\infty \in \Pi_{\varphi(2)}$ while $\tilde{\pi}''_p = \eta_p \boxplus \nu_p$ for all p. The representation $\tilde{\pi}''$ is clearly anomalous.

The representation π is unramified. Thus the automorphic representation π° of $G(A)$ lies in the L-packet $\varphi_{1*}(\{\pi\})$ defined by π, φ_1 and the principle of functoriality if π°_p is unramified for all p and $\pi^\circ_p \in \varphi_{1*}(\{\pi_p\})$, and π°_∞ is in $\Pi_{\psi(10)}$. We take π° to be the representation defined by the cusp form χ_{10} of weight 10 [39]. Then π°_∞ lies in $\Pi_{\psi(10)}$. According to the definitions [3] the relation $\pi^\circ_p \in \varphi_{1*}(\{\pi_p\})$ is a statement about eigenvalues of Hecke operators. These statements have been verified for small primes by Kurokawa [29]. The necessary equalities are too complicated to be merely coincidences, and we may assume with some confidence that $\pi^\circ \in \varphi_{1*}(\{\pi\})$.

In any case the representation π° certainly is a counterexample to Ramanujan's conjecture. If $\varphi = \varphi_2 \circ \varphi_1$ then the principle of functoriality yields the same L-packet when applied to π and φ as it does when applied to π° and φ_2. Since G_2 is GL(4) the L-packet consists of a single representation. It is easily seen to be anomalous and to be equivalent almost everywhere to the isobaric representation $\pi' \boxplus \eta \boxplus \nu$. Thus π° itself is anomalous in the sense described at the beginning of the section.

4. Shimura varieties. In this section we review the definition of Shimura varieties, taken for now over $\boldsymbol{C}$, and their relations with motives. The point of view is Deligne's and most of what follows has been taken from his papers [9], [10], or learned in conversation with him. Of course the book of Saavedra Rivano [40] has again been a basic reference; many of the facts and definitions below will be found in it.

Recall that the data needed to define a Shimura variety are a connected reductive group G over $\boldsymbol{Q}$ and a homomorphism $h: \mathscr{R} \to G$ defined over $\boldsymbol{R}$. The symbol $\mathscr{R}$ is used to denote the group $\mathrm{Res}_{\boldsymbol{C}/\boldsymbol{R}}\mathrm{GL}(1)$. Thus we have a canonical isomorphism $\mathrm{GL}(1) \times \mathrm{GL}(1) \simeq \mathscr{R}$ over $\boldsymbol{C}$ and we may speak of the restriction of h to the first or the second factor. It is not h which matters but the set

$$\mathfrak{H} = \{\mathrm{Ad}\, g \circ h | g \in G(\boldsymbol{R})\}$$

and it will be best simply to let h denote an arbitrary element of $\mathfrak{H}$.

Recall that the pair (G, h) is subject to three conditions [10, §1.5]:

(a) *If w is the diagonal map* $\mathrm{GL}(1) \to \mathrm{GL}(1) \times \mathrm{GL}(1)$ *then the homomorphism* $h \circ w$ *is central.*

(b) *The Lie algebra $\mathfrak{G}$ of $G(\boldsymbol{C})$ is a direct sum $\mathfrak{G} = \mathfrak{p} + \mathfrak{k} + \bar{\mathfrak{p}}$ and if $(z_1, z_2) \in \mathscr{R}(\boldsymbol{C})$ then*

$$\begin{aligned} \text{ad } h(z)(X) &= z_1^{-1}z_2X, & X &\in \mathfrak{p}, \\ &= X, & X &\in \mathfrak{k}, \\ &= z_1z_2^{-1}X, & X &\in \bar{\mathfrak{p}}. \end{aligned}$$

In fact the summands $\mathfrak{p}$, $\mathfrak{k}$, *and* $\bar{\mathfrak{p}}$ *vary with* h, *and when it is useful to make the dependence on* h *explicit we write* $\mathfrak{p}_h$, $\mathfrak{k}_h$, *and* $\bar{\mathfrak{p}}_h$.

(c) *The adjoint action of* $h(i, -i)$ *on the adjoint group is a Cartan involution.*

Since $G(\boldsymbol{R})$ acts on the real manifold $\mathfrak{H}$ by conjugation every element of $\mathfrak{G}$ defines a complex vector field on $\mathfrak{H}$. Let X_h be the value of the vector field associated to X at $h \in \mathfrak{H}$. The complex structure on $\mathfrak{H}$ is so defined that the holomorphic tangent space at h is $\{X_h | X \in \bar{\mathfrak{p}}\}$ and the antiholomorphic tangent space is $\{X_h | X \in \mathfrak{p}\}$.

If $\boldsymbol{A}_f$ is the ring of adèles over $\boldsymbol{Q}$ whose component at infinity is 0 and K is an open compact subgroup of $G(\boldsymbol{A}_f)$ then $G(\boldsymbol{A}_f)/K$ is discrete and $X_K = \mathfrak{H} \times G(\boldsymbol{A}_f)/K$ is a complex manifold on which $G(\boldsymbol{Q})$ acts to the left. If K is sufficiently small then any $\gamma \in G(\boldsymbol{Q})$ with a fixed point lies in $Z(\boldsymbol{Q}) \cap K$ and thus fixes the whole manifold. We shall always assume that K is sufficiently small and then

$$\mathrm{Sh}_K(\boldsymbol{C}) = G(\boldsymbol{Q})\backslash\mathfrak{H} \times G(\boldsymbol{A}_f)/K$$

is a complex manifold, proved by Baily-Borel to be the set of complex points on an algebraic variety $\mathrm{Sh}_K = \mathrm{Sh}_K(G, h) = \mathrm{Sh}_K(G, \mathfrak{H})$ over $\boldsymbol{C}$.

Deligne anticipates that Sh_K will often be a moduli space for a family of motives over $\boldsymbol{C}$. This is sometimes so, the motives then being those attached to abelian varieties, but can certainly not yet be proved in general. Nonetheless there is a good deal to be learned from a rehearsal of the considerations that suggest such an interpretation of Sh_K. In essence one observes that $\mathrm{Sh}_K(\boldsymbol{C})$ is the parameter space for a family of polarized Hodge structures; the difficulty is to show that these Hodge structures all arise from motives.

A real Hodge structure V is a finite-dimensional vector space $V_{\boldsymbol{R}}$ over $\boldsymbol{R}$ together with a decomposition of its complexification $V_{\boldsymbol{C}} = \bigoplus_{p,q\in\boldsymbol{Z}} V^{p,q}$, satisfying $V^{q,p} = \bar{V}^{p,q}$. The collection of real Hodge structures forms a tannakian category over $\boldsymbol{R}$, whose associated group is $\mathscr{R}$. Indeed to a real Hodge structure V, one associates the representation σ of $\mathscr{R}$ defined by

$$\sigma(z_1, z_2)v = z_1^{-p}z_2^{-q}v, \qquad v \in V^{p,q}. \tag{4.1}$$

The relations $V^{q,p} = \bar{V}^{p,q}$ imply that σ is defined over $\boldsymbol{R}$. Conversely each representation of $\mathscr{R}$ that is defined over $\boldsymbol{R}$ yields a Hodge structure, the elements of $V^{p,q}$ being defined by (4.1). The real Hodge structure V is said to be of weight n if $V^{p,q} = 0$ whenever $p + q \neq n$. Certainly any Hodge structure is a direct sum $V = \bigoplus_n V^n$ with V^n of weight n.

We are however interested in the category of polarized rational Hodge structures. A rational Hodge structure V is formed by a finite-dimensional vector space $V_{\boldsymbol{Q}}$ over $\boldsymbol{Q}$, a direct sum decomposition $V_{\boldsymbol{Q}} = \bigoplus V_{\boldsymbol{Q}}^n$, and real Hodge structures of weight n on $V_{\boldsymbol{R}}^n = V_{\boldsymbol{Q}}^n \otimes_{\boldsymbol{Q}} \boldsymbol{R}$. There is a distinguished object of weight -2, the Tate object $\boldsymbol{Q}(1)$, in the category of rational Hodge structures. The underlying rational vector space is $\boldsymbol{Q}(1)_{\boldsymbol{Q}} = 2\pi i\boldsymbol{Q} \subseteq \boldsymbol{C}$ and, by definition,

$$\boldsymbol{Q}(1)^{-1,-1} = \boldsymbol{Q}(1)_{\boldsymbol{C}}.$$

It seems to be customary to identify the underlying vector space of $\mathbf{Q}(n) = \mathbf{Q}(1)^{\otimes n}$ with $(2\pi i)^n\mathbf{Q}$ and $\mathbf{Q}(n)_{\mathbf{R}}$ with $(2\pi i)^n\mathbf{R} \subseteq \mathbf{C}$. The factors $2\pi i$ have been chosen for reasons which need not concern us. It is no trouble to carry them along.

If V is a rational Hodge structure and σ the associated representation of $\mathscr{R}$ let C be $\sigma(-i, i)$ acting on $V_{\mathbf{R}}$. If V is of weight n, a polarization of V is a bilinear form $P : V \times V \to \mathbf{Q}(-n)$ satisfying:

(a) *For all u and v in* $V_{\mathbf{C}}$ *and all* $r \in R(\mathbf{C})$

$$P(\sigma(r)u, \sigma(r)v) = \sigma(r)P(u, v).$$

Thus the form is compatible with the Hodge structures.

(b) $P(v, u) = (-1)^n P(u, v)$.

(c) *The real-valued form* $(2\pi i)^n P(u, Cv)$ *on* $V_{\mathbf{R}}$ *is symmetric and positive-definite.*

A rational Hodge structure is said to be polarizable if each of its homogeneous components admits a polarization, a polarization of the full structure being defined by polarizations of the homogeneous components. The category $\mathscr{HOD}(\mathbf{Q})$ of polarizable Hodge structures is tannakian, with a natural fibre functor $\omega_{\mathrm{Hod}}\colon V \to V_{\mathbf{Q}}$ and an associated group G_{Hod}, reductive but overwhelmingly large. It does have factor groups of manageable size.

If V is a polarizable rational Hodge structure, one may take the tannakian category generated by V and $\mathbf{Q}(1)$ and the repeated formation of duals, sums, tensor products, and subobjects. The associated group is called the Mumford-Tate group of V and denoted by $\mathscr{MT}(V)$. It is finite-dimensional and reductive, and there is a surjection $G_{\mathrm{Hod}} \to \mathscr{MT}(V)$ defined over $\mathbf{Q}$. If σ is the representation of $\mathscr{R}$ attached to V then $\mathscr{MT}(V)$ is simply the smallest subgroup of the group of automorphisms of the rational vector space underlying V which contains $\sigma(\mathscr{R})$ and is defined over $\mathbf{Q}$ [37]. It is consequently connected.

The polarizable rational Hodge structures for which $\mathscr{MT}(V)$ is abelian play a particularly important role in the study of Shimura varieties. They are said to be of CM type. The second description of the groups $\mathscr{MT}(V)$ shows that the category of such Hodge structures is closed under sums and tensor products and thus is a tannakian category. The associated group has been studied at length in **[41]** and is often called the Serre group. At the risk of making a comparison with **[41]** difficult, for Serre himself employs a different notation, we shall denote the group by $\mathscr{S}$.

It is not difficult to describe $\mathscr{S}$. Let $\bar{\mathbf{Q}}$ be the algebraic closure of $\mathbf{Q}$ in $\mathbf{C}$ and let $\iota \in \mathrm{Gal}(\bar{\mathbf{Q}}/\mathbf{Q})$ be complex conjugation. To construct $X^*(\mathscr{S})$, the module of rational characters of $\mathscr{S}$, we start with the module M of locally constant integral-valued functions on $\mathrm{Gal}(\bar{\mathbf{Q}}/\mathbf{Q})$. The Galois group acts by right translation and

$$X^*(\mathscr{S}) = \{\lambda \in M \,|\, (\sigma - 1)(\iota + 1)\lambda = (\iota + 1)(\sigma - 1)\lambda = 0 \quad \forall \sigma \in \mathrm{Gal}(\bar{\mathbf{Q}}/\mathbf{Q})\}.$$

In particular if $\lambda \in X^*(\mathscr{S})$,

$$(\sigma - 1)\lambda(1) + (\sigma - 1)\lambda(\iota) = 0$$

because the left side is $(\iota + 1)(\sigma - 1)\lambda(1)$. The lattice of rational characters of $\mathscr{R}$ is canonically isomorphic to $\mathbf{Z} \oplus \mathbf{Z}$ and the homomorphism $h_0\colon \mathscr{R} \to \mathscr{S}$ dual to the homomorphism $X^*(\mathscr{S}) \to X^*(\mathscr{R})$ which sends λ to $(\lambda(1), \lambda(\iota))$ is defined over $\mathbf{R}$. The composite $h \circ w\colon \mathrm{GL}(1) \to \mathscr{S}$ is dual to the homomorphism $X^*(\mathscr{S}) \to \mathbf{Z}$ taking λ to $\lambda(1) + \lambda(\iota)$ and is defined over $\mathbf{Q}$.

To verify that the group $\mathscr{S}$ just defined in terms of its module of characters is the Serre group defined in terms of Hodge structures is easy enough. The existence of the two homomorphisms h_0 and $h_0 \circ w$ implies that every representation of $\mathscr{S}$ defined over $\boldsymbol{Q}$ defines a rational Hodge structure. It is enough to show that these are polarizable when the representation is irreducible. To obtain the irreducible representations, one takes a $\lambda \in X^*(\mathscr{S})$ and defines the field F by

$$\mathrm{Gal}(\bar{\boldsymbol{Q}}/F) = \{\sigma \in \mathrm{Gal}(\bar{\boldsymbol{Q}}/\boldsymbol{Q}) | \sigma\lambda = \lambda\}.$$

The underlying space of the representation is the vector space over $\boldsymbol{Q}$ defined by F, and the representation $r = r_\lambda$ is that defined symbolically by $r_\lambda(s): x \in F \to \lambda(s)x$. The weight of the associated Hodge structure is $-(\lambda(1) + \lambda(\iota)) = n$. There is an $\alpha \in F$ such that $\iota(\alpha) = (-1)^n\alpha$ and $(-1)^{\lambda(\iota)} i^n \alpha$ is totally positive. A possible polarization is

$$P(u, v) = (2\pi i)^{-n} \mathrm{Tr}_{F/\boldsymbol{Q}} u\alpha\iota(v).$$

Conversely suppose one has a rational Hodge structure V whose Mumford-Tate group $\mathscr{M}\mathscr{T}(V)$ is abelian. Since there is a homomorphism $\mathscr{R} \to \mathscr{M}\mathscr{T}(V)$, the coweight $\mathrm{GL}(1) \to \mathscr{R}$ of the group $\mathscr{R}$ defined by $z \to (z, 1)$ also defines a coweight $\nu^\vee$ of $\mathscr{M}\mathscr{T}(V)$. The lattice Y_* of coweights of $\mathscr{M}\mathscr{T}(V)$ is a $\mathrm{Gal}(\bar{\boldsymbol{Q}}/\boldsymbol{Q})$ module generated by $\nu^\vee$. We define an injective homomorphism of its lattice of rational characters V^* into $X^*(\mathscr{S})$ by sending $\nu \in Y^*$ to the element λ given by

$$\lambda(\sigma) = \langle \sigma\nu, \nu^\vee \rangle, \qquad \sigma \in \mathrm{Gal}(\bar{\boldsymbol{Q}}/\boldsymbol{Q}).$$

The dual homomorphism $\mathscr{S} \to \mathscr{M}\mathscr{T}(V)$ is surjective and V is defined by a rational representation of $\mathscr{S}$.

We return to the Shimura varieties $\mathrm{Sh}_K(\boldsymbol{C})$ and show how one attaches families of rational Hodge structures to $\mathrm{Sh}_K(\boldsymbol{C})$. Let G_0 be the largest quotient of G such that $(\sigma - 1)(\iota + 1)\nu^\vee = (\iota + 1)(\sigma - 1)\nu^\vee = 0$ for every coweight of the centre of G_0. Let ξ be a rational representation of G on V which factors through G_0.

To each $x = (h, g)$ in X_K we associate a triple $(V^x, \mathfrak{k}^x, \varphi^x)$. V^x is a rational Hodge structure whose underlying space is $V^x_{\boldsymbol{Q}} = V_{\boldsymbol{Q}}$, the Hodge structure being defined by the representation $\xi \circ h$ of $\mathscr{R}$. The third term φ^x is an isomorphism $V^x_{A_f} \to V_{A_f}$ and is given by $v \to \xi(g)^{-1}v$. It is only defined up to composition with an element of $\xi(K)$. The homogeneous components of $V^x_{\boldsymbol{Q}}$ are independent of x and may be written $V^n_{\boldsymbol{Q}}$. It follows from Lemma 2.8 of [**11**] that if $x \in X_K$ there is at least one collection P^x of bilinear forms $P^x_n : V^n_{\boldsymbol{Q}} \times V^n_{\boldsymbol{Q}} \to \boldsymbol{Q}(-n)_{\boldsymbol{Q}}$ which is invariant under the derived group of G and is a polarization of V^x. Let $\mathfrak{P}^x$ denote the collection of all such polarizations. If $g \in G(\boldsymbol{Q})$, $P^x \in \mathfrak{P}^x$, and $x' = \gamma x$ then the collection $P^{x'}$ given by $P^{x'}_n : (u, v) \to P^x_n(\xi(\gamma^{-1})u, \xi(\gamma^{-1})v)$ lies in $\mathfrak{P}^{x'}$.

We must also verify that the family $\{V^x\}$ over X_K is a family of rational Hodge structures in the sense of [**9**]. Otherwise it could not possibly be attached to a family of motives. There are two points to be verified. Let $V^x_{p,q}$ be the subspace of $V^x_{\boldsymbol{C}}$ of type p, q and set $V^x_p = \bigoplus_{p' \geq p} V^x_{p',q'}$. The space $V^x_p \subseteq V_{\boldsymbol{C}}$ must be shown to vary holomorphically with x. In other words if $v(x)$ is any local section of V^x_p and Y any antiholomorphic vector field then $Yv(x)$ also takes values in V^x_p. The condition of transversality must also be established, to the effect that for the same $v(x)$ and any holomorphic vector field Y the values of $Yv(x)$ lie in V^x_{p-1}.

There is certainly no harm in supposing that $v(x)$ takes values in $V^x_{p,q}$. Then one has to show that if $x^\circ = (h^\circ, g^\circ_f)$ is fixed and Y is the vector field defined by $X \in \mathfrak{G}$ then $Yv(x^\circ)$ lies in V^x_p when $X \in \mathfrak{p}_{h^\circ}$ and in V^x_{p-1} when $X \in \bar{\mathfrak{p}}_{h^\circ}$. Let K_{h° be the stabilizer of h° in $G(\boldsymbol{R})$. We represent X_K as the quotient $G(\boldsymbol{A})/K_{h^\circ} K$ and lift $v(x)$ to a function on $G(\boldsymbol{A})$ which we write as

$$(g, g_f) \to \xi(g)u(g, g_f), \qquad g \in G(\boldsymbol{R}),\ g_f \in G(\boldsymbol{A}_f).$$

The function u takes values in the constant space $V^{x^\circ}_{p,q}$. Moreover

$$Yv(x^\circ) = \xi(X)u(1, g^\circ_f) + Xu(1, g^\circ_f).$$

The second term lies in $V^{x^\circ}_{p,q}$ for all $X \in \mathfrak{G}$. If $X \in \mathfrak{p}$ then $\xi(X)V^{x^\circ}_{p,q} \subseteq V^{x^\circ}_{p+1,q-1}$ and if $X \in \bar{\mathfrak{p}}$ then $\xi(X)V^{x^\circ}_{p,q} \subseteq V^{x^\circ}_{p-1,q+1}$.

If $\gamma \in G(\boldsymbol{Q})$ and $x' = \gamma x$ then $v \to \xi(\gamma)v$ provides an isomorphism between $(V^x, \mathfrak{P}^x, \varphi^x)$ and $(V^{x'}, \mathfrak{P}^{x'}, \varphi^{x'})$. Thus, if $s \in S_K(\boldsymbol{C})$, any two elements of $\{(V^x, \mathfrak{P}^x, \varphi^x) \mid x \to s\}$ are canonically isomorphic, and we may take $(V^s, \mathfrak{P}^s, \varphi^s)$ to be any one of them, and redefine our family as a family of rational Hodge structures, with supplementary data, over the base $S_K(\boldsymbol{C})$. The locally constant sheaf $F_\xi(\boldsymbol{Q})$ of rational vector spaces underlying this family is the quotient of $V_{\boldsymbol{Q}} \times X_K$ by the action $\gamma\colon (v, x) \to (\xi(\gamma)v, \gamma x)$ of $G(\boldsymbol{Q})$. For this quotient to be well defined, the group K must be sufficiently small, for $G(\boldsymbol{Q}) \cap K_h K$ is then contained in the kernel of ξ for all h (cf. II.A.2 of **[41]**). It is here that the condition that ξ factors through G_0 intervenes.

When dealing with motives, one does not need to introduce the polarizations explicitly as part of the moduli problem, but in order to introduce a Hodge structure on the cohomology groups of the sheaves $F_\xi(\boldsymbol{Q})$ one must verify that on each connected component of $S_K(\boldsymbol{C})$ a locally constant section $s \to P^s$ can be defined. Let $G^\circ(\boldsymbol{R})$ be the connected component of $G(\boldsymbol{R})$ and choose $x^\circ = (h, g_f)$ in X_K. If

$$X^\circ_K = \{(\operatorname{Ad} g \circ h, g_f k) \mid g \in G^\circ(\boldsymbol{R}), k \in K\}$$

then the image of X°_K in $S_K(\boldsymbol{C})$ is open. If x and x' lie in X°_K and $x' = \gamma x$ then γ lies in

$$G(\boldsymbol{Q}) \cap G^\circ(\boldsymbol{R})K_h g_f K g_f^{-1}. \tag{4.2}$$

All we need do is find a collection $P = \{P_n\}$ such that $P \in \mathfrak{P}^x$ for all $x \in X^\circ_K$ and $P_n(\xi(\gamma)u, \xi(\gamma)v) = P_n(u, v)$ if γ lies in the group (4.2). Choose any P in $\mathfrak{P}^{x^\circ}$. Then $P \in \mathfrak{P}^x$ for all $x \in X^\circ_K$. There are certainly homomorphisms λ_n of G into the general linear group of V^n such that

$$P_n(\xi(\gamma)u, \xi(\gamma)v) = P_n(\lambda_n(\gamma)u, v).$$

The eigenvalues of $\lambda(\gamma)$ are positive if γ lies in $G^\circ(\boldsymbol{R})K_h$. Moreover λ is trivial on the derived group of G and factors through G_0. It therefore follows from the results of II.A.2 of **[41]** that each λ_n is the identity on all of (4.2).

Although we have defined the families $(V^s, \mathfrak{P}^s, \varphi^s)$ for any G, it is clear that $S_K(\boldsymbol{C})$ is not going to appear as a moduli space unless G is equal to G_0, and so for the rest of this section we assume this. The moduli problem is best formulated completely in the language of tannakian categories. We can drop the polarizations and retain only the pairs (V^s, φ^s), or (V^x, φ^x), but we now have to emphasize that (V^x, φ^x) is defined for every (finite-dimensional) representation ξ of G over $\boldsymbol{Q}$, and so we write $(\xi, V(\xi))$ for the representation and $(V^x(\xi), \varphi^x(\xi))$ for the pair (V^x, φ^x).

On the category $\mathscr{REP}(G)$ of finite-dimensional representations of G we have the natural fibre functor $\omega_{\mathrm{Rep}(G)}$: $(\xi, V(\xi)) \to V(\xi)_{\boldsymbol{Q}}$ and η^x: $(\xi, V(\xi)) \to V^x(\xi)$ is a $\otimes$-functor from $\mathscr{REP}(G)$ to $\mathscr{HOD}(\boldsymbol{Q})$ which satisfies $\omega_{\mathrm{Hod}} \circ \eta^x = \omega_{\mathrm{Rep}(G)}$. Since $\mathscr{HOD}(\boldsymbol{Q})$ and $\mathscr{REP}(G_{\mathrm{Hod}})$ are the same categories, η^x defines a homomorphism [**40**, II.3.3.1] φ^x: $G_{\mathrm{Hod}} \to G$ and η^x may be defined by $(\xi, V(\xi)) \to (\xi \circ \varphi^x, V(\xi))$. When we emphasize η^x, φ^x appears as an isomorphism of two fibre functors

$$\varphi^x\colon \omega^{\boldsymbol{A}_f}_{\mathrm{Hod}} \circ \eta^x \to \omega^{\boldsymbol{A}_f}_{\mathrm{Rep}(G)}.$$

However, when we emphasize φ^x, as we shall, then these two fibre functors are the same, for they are both obtained from $\omega_{\mathrm{Hod}} \circ \eta^x = \omega_{\mathrm{Rep}(G)}$ by tensoring with $\boldsymbol{A}_f$, and φ^x may be interpreted as an isomorphism of $\omega^{\boldsymbol{A}_f}_{\mathrm{Rep}(G)}$. Such an isomorphism is given by a $g^{-1} \in G(\boldsymbol{A}_f)$ [**40**, II.3]. This is the g appearing in $x = (h, g)$. Only the coset $gK \subseteq G(\boldsymbol{A}_f)$ is well defined.

We have arrived, by a rather circuitous route, at the conclusion that X_K parametrizes pairs (φ, g), φ being a homomorphism from G_{Hod} to G defined over $\boldsymbol{Q}$, and g in $G(\boldsymbol{A}_f)$ being specified only up to right multiplication by an element of K. In addition, φ is subject to the following constraint:

H. *The composition of* φ *with the canonical homomorphism* $\mathscr{R} \to G_{\mathrm{Hod}}$ *lies in* $\mathfrak{H}$.

If $\gamma \in G(\boldsymbol{Q})$ the pairs (φ, g) and $(\mathrm{ad}\,\gamma \circ \varphi, \gamma g)$ will be called equivalent. The variety $\mathrm{Sh}_K(\boldsymbol{C})$ parametrizes equivalence classes of these pairs.

One of the important tannakian categories is the category $\mathscr{MOT}(k)$ of motives over a field k. It cannot be constructed at present unless one assumes certain conjectures in algebraic geometry, referred to as the standard conjectures [**40**]. It is covariant in k, and rational cohomology together with its Hodge structure yields a $\otimes$-functor h_{BH}: $\mathscr{MOT}(\boldsymbol{C}) \to \mathscr{HOD}(\boldsymbol{Q})$. $\mathscr{MOT}(\boldsymbol{C})$ together with the fibre functor $\omega_{\mathrm{Mot}(\boldsymbol{C})}$ of rational cohomology also defines a group $G_{\mathrm{Mot}(\boldsymbol{C})}$ over $\boldsymbol{Q}$ and h_{BH} is dual to a homomorphism h^*_{BH}: $G_{\mathrm{Hod}} \to G_{\mathrm{Mot}(\boldsymbol{C})}$ defined over $\boldsymbol{Q}$.

Implicit in Deligne's construction is the hope that any homomorphism φ': $G_{\mathrm{Hod}} \to G$ satisfying H is a composite $\varphi' = \varphi \circ h^*_{BH}$. According to the Hodge conjecture, φ would be uniquely determined [**40**, VI.4.5] and $\mathrm{Sh}_K(\boldsymbol{C})$ would appear as the moduli space for pairs (φ, g), with g as before, but where φ is now a homomorphism from $G_{\mathrm{Mot}(\boldsymbol{C})}$ to G defined over $\boldsymbol{Q}$ and satisfying:

H$'$. *The composition of* φ *with the canonical homomorphism* $\mathscr{R} \to G_{\mathrm{Mot}(\boldsymbol{C})}$ *lies in* $\mathfrak{H}$.

This may be so but it will not be a panacea for all the problems with which the study of Shimura varieties is beset. So far as I can see, we do not yet have a moduli problem in the usual algebraic sense, and, in particular, no way of deciding over which field the moduli problem is defined. We can be more specific about this difficulty.

Suppose τ is an automorphism of $\boldsymbol{C}$. Then τ^{-1} defines a $\otimes$-functor $\eta(\tau)$: $\mathscr{MOT}(\boldsymbol{C}) \to \mathscr{MOT}(\boldsymbol{C})$. Let $G^{\tau}_{\mathrm{Mot}(\boldsymbol{C})}$ be the group defined by $\mathscr{MOT}(\boldsymbol{C})$ and the fibre functor $\omega_{\mathrm{Mot}(\boldsymbol{C})} \circ \eta(\tau)$. The dual of $\eta(\tau)$ is then an isomorphism over $\boldsymbol{Q}$:

$$\varphi(\tau)\colon G_{\mathrm{Mot}(\boldsymbol{C})} \to G^{\tau}_{\mathrm{Mot}(\boldsymbol{C})}.$$

The homomorphism φ has a dual, a $\otimes$-functor η: $\mathscr{REP}(G) \to \mathscr{MOT}(\boldsymbol{C})$ and the fibre functor $\omega_{\mathrm{Mot}(\boldsymbol{C})} \circ \eta(\tau) \circ \eta$ defines a group $G^{\tau,\varphi}$ over $\boldsymbol{Q}$. The $\otimes$-functor η then defines a dual

$$\varphi^\tau\colon G^\tau_{\mathrm{Mot}(C)} \to G^{\tau,\varphi}, \quad \text{and} \quad \varphi' = \varphi^\tau \circ \varphi(\tau)\colon G_{\mathrm{Mot}(C)} \to G^{\tau,\varphi}.$$

Moreover the two fibre functors $\omega^{A_f}_{\mathrm{Mot}(C)}$ and $\omega^{A_f}_{\mathrm{Mot}(C)} \circ \eta(\tau)$ are canonically isomorphic. As a consequence, there is a canonical isomorphism $G(A_f) \to G^{\tau,\varphi}(A_f)$. Let g' be the image of g.

The pair (φ', g') seems once again to define a solution to our moduli problem. The difficulty is that $G^{\tau,\varphi}$ may not be the group G or, even if it is, the composition of φ' with the canonical homomorphism may not lie in $\mathfrak{H}$. One of the purposes of the next two sections is to discover what $G^{\tau,\varphi}$ is likely to be.

5. The Taniyama group. There is one type of Shimura variety which is very easy to study, that obtained when G is a torus T. Then the set $\mathfrak{H}$ reduces to a single point $\{h\}$. For each open compact subgroup U of $T(A_f)$ the manifold $\mathrm{Sh}_U(C)$ consists of a finite set and $\mathrm{Sh}_U = \mathrm{Sh}_U(T, h)$ is zero-dimensional. In general a special point of $(G, \mathfrak{H})$ will be a pair (T, h) with $T \subseteq G$ and $h \in \mathfrak{H}$. If $U = K \cap T(A_f)$ then $\mathrm{Sh}_U(C)$ is a subset of $\mathrm{Sh}_K(C)$, the points of which have traditionally been referred to as special points, and I shall continue this usage. But it is best to give priority to the pair (T, h) rather than to the points of $\mathrm{Sh}_K(C)$ it defines. There are a number of unsolved problems about Shimura varieties and their special points that I want to describe in the next section. To formulate them some Galois cocycles have to be defined. Deligne has shown me that my original construction gave, in particular, a specific extension of $\mathrm{Gal}(\bar{Q}/Q)$ by the Serre group, $\mathcal{S}$, an extension I venture to call the Taniyama group and denote by $\mathcal{T}$. Since the cocycles needed are, as Rapoport observed, often easily defined in terms of $\mathcal{T}$, I begin by constructing it.

The group $\mathcal{S}$ is an algebraic group over Q, and $\mathcal{T}$ will also be defined over Q. Thus we will have an exact sequence

$$1 \to \mathcal{S} \to \mathcal{T} \to \mathrm{Gal}(\bar{Q}/Q) \to 1. \tag{5.1}$$

Recall that $X^*(\mathcal{S})$ is a module of functions on $\mathrm{Gal}(\bar{Q}/Q)$, and that the Galois action on $X^*(\mathcal{S})$ giving the structure of $\mathcal{S}$ as a group over Q is defined by right translation. We are still free to use left translation to define an algebraic action of $\mathrm{Gal}(\bar{Q}/Q)$ on $\mathcal{S}$, and it is this action which is implicit in (5.1). The extension will not split over Q but it will be provided with canonical splittings over each l-adic field Q_l, $\mathrm{Gal}(\bar{Q}/Q) \to \mathcal{T}(Q_l)$, which will fit together to give $\mathrm{Gal}(\bar{Q}/Q) \to \mathcal{T}(A_f)$.

Rather than attempting to work directly with $\mathcal{S}$, I choose a finite Galois extension L of Q, let $\mathcal{S}^L$ be the quotient group of $\mathcal{S}$ whose lattice of rational characters consists of all functions in $X^*(\mathcal{S})$ invariant under $G(\bar{Q}/L)$, and define extensions

$$1 \to \mathcal{S}^L \to \mathcal{T}^L \to \mathrm{Gal}(L^{ab}/Q) \to 1, \tag{5.2}$$

afterwards lifting to $\mathrm{Gal}(\bar{Q}/Q)$, and then passing to the limit.

To motivate the construction we suppose that the extension is defined and that there is a section $\tau \to a(\tau)$ of $\mathcal{T}^L \to \mathrm{Gal}(L^{ab}/Q)$ with $a(\tau) \in \mathcal{T}^L(L)$. Let $a(\tau_1)a(\tau_2) = d_{\tau_1,\tau_2}a(\tau_1\tau_2)$ with $d_{\tau_1,\tau_2} \in \mathcal{S}^L(L)$, and with

$$\tau_1(d_{\tau_2,\tau_3})d_{\tau_1,\tau_2\tau_3} = d_{\tau_1,\tau_2}d_{\tau_1\tau_2,\tau_3}. \tag{5.3}$$

Observe that the elements of the Galois group play two different roles. They are first of all elements of a quotient group of $\mathcal{T}^L$, and secondly they are automorphisms of L^{ab} and thus act on $\mathcal{T}^L(L^{ab})$, since $\mathcal{T}^L$ is defined over Q. In the first role they will be denoted by τ, perhaps with a subscript added, and in the second by ρ or σ.

We have $\rho(a(\tau)) = c_\rho(\tau)a(\tau)$ with $c_\rho(\tau) \in \mathcal{S}^L(L)$. Certainly

$$c_{\rho\sigma}(a(\tau)) = \rho(c_\sigma(\tau))c_\rho(\tau). \tag{5.4}$$

In addition

$$d_{\tau_1,\tau_2}c_\rho(\tau_1)\tau_1(c_\rho(\tau_2)) = \rho(d_{\tau_1,\tau_2})c_\rho(\tau_1\tau_2). \tag{5.5}$$

Conversely if we have collections $\{c_\rho(\tau)\}$ and $\{d_{\tau_1,\tau_2}\}$ satisfying (5.3), (5.4), and (5.5), we can construct $\mathcal{T}^L$ over $\boldsymbol{Q}$, together with the section a.

Any splitting $\mathrm{Gal}(L^{ab}/\boldsymbol{Q}) \to \mathcal{T}(\boldsymbol{A}_f)$ will be of the form $\tau \to b(\tau)a(\tau)$ with $b(\tau) \in \mathcal{S}^L(\boldsymbol{A}_f(L))$. In order that it be a splitting, we must have

$$b(\tau_1)\tau_1(b(\tau_2))d_{\tau_1,\tau_2} = b(\tau_1\tau_2). \tag{5.6}$$

If the $b(\tau)a(\tau)$ are to lie in $\mathcal{T}^L(\boldsymbol{A}_f)$ we must have

$$\rho(b(\tau))c_\rho(\tau) = b(\tau). \tag{5.7}$$

Again any collection $\{b(\tau)\}$ satisfying (5.6) and (5.7) defines a splitting, and it is our task to construct $\{b(\tau)\}$, $\{c_\rho(\tau)\}$, and $\{d_{\tau_1,\tau_2}\}$.

The group $\mathcal{S}$ is a quotient of G_{Hod} and thus is provided with a canonical homomorphism $h: \mathcal{R} \to \mathcal{S}$. Over $\boldsymbol{C}$ the group $\mathcal{R}$ is canonically isomorphic to $\mathrm{GL}(1) \times \mathrm{GL}(1)$. Restricting h to the first factor we obtain a coweight μ of $\mathcal{S}$, the canonical coweight. If $\lambda \in X^*(\mathcal{S})$ then $\langle \lambda, \mu \rangle = \lambda(1)$. Since $\mathcal{S}^L$ is a quotient of $\mathcal{S}$, μ also defines a coweight of $\mathcal{S}^L$, which for convenience will also be denoted by μ. If ν is any coweight of $\mathcal{S}^L$ and x any invertible element of L or of $\boldsymbol{A}_f(L)$ then x^ν will be the element of $\mathcal{S}^L(L)$, or $\mathcal{S}^L(\boldsymbol{A}_f(L))$, satisfying $\lambda(x^\nu) = x^{\langle \lambda, \nu \rangle}$ for all $\lambda \in X^*(\mathcal{S}^L)$. Recall that we have two actions of $\mathrm{Gal}(L/\boldsymbol{Q})$, or $\mathrm{Gal}(L^{ab}/\boldsymbol{Q})$, on $X^*(\mathcal{S}^L)$ or on

$$X_*(\mathcal{S}^L) = \mathrm{Hom}(X^*(\mathcal{S}^L), \boldsymbol{Z}),$$

the lattice of coweights. That defined by right translation we write $\nu \to \sigma\nu$, and that defined by left translation we write $\nu \to \nu\tau$. Thus $\langle \lambda, \sigma\mu \rangle = \lambda(\sigma^{-1})$ while $\langle \lambda, \mu\tau \rangle = \lambda(\tau)$, because an inverse intervenes in the action by left translation, and $\mu\tau^{-1} = \tau\mu$. Moreover $\sigma(x^\mu) = \sigma(x)^{\sigma\mu}$ while $\tau(x^\mu) = x^{\mu\tau}$. These aspects of the notation have to be emphasized because at some points our convention of distinguishing between ρ, σ on the one hand, and τ on the other, fails us.

For the study of Shimura varieties, it is best to take $\bar{\boldsymbol{Q}}$ to be the algebraic closure of $\boldsymbol{Q}$ in $\boldsymbol{C}$, and we shall do this. Thus we provide ourselves with an extension of the infinite valuation on $\boldsymbol{Q}$ to $\bar{\boldsymbol{Q}}$. There is a property of the Weil groups that will play a prominent role in our discussion. Let v be a valuation of $\bar{\boldsymbol{Q}}$ and hence of $\boldsymbol{Q}$, and let $\boldsymbol{Q}_v$ and $\bar{\boldsymbol{Q}}_v$ be the completions of $\boldsymbol{Q}$ and $\bar{\boldsymbol{Q}}$ with respect to v. Eventually v will be defined by the inclusion $\bar{\boldsymbol{Q}} \subseteq \boldsymbol{C}$, and $\boldsymbol{Q}_v$ will be $\boldsymbol{R}$ and $\bar{\boldsymbol{Q}}_v$ will be $\boldsymbol{C}$. In any case, the data provide us with imbeddings

$$F_v^\times \hookrightarrow C_F, \tag{5.8}$$

$$\mathrm{Gal}(F_v/\boldsymbol{Q}_v) \hookrightarrow \mathrm{Gal}(F/\boldsymbol{Q}), \tag{5.9}$$

if F is any finite Galois extension of $\boldsymbol{Q}$ in $\bar{\boldsymbol{Q}}$. The local and global Weil groups $W_{F_v/\boldsymbol{Q}_v}$ and $W_{F/\boldsymbol{Q}}$ are defined as extensions

$$1 \to F_v^\times \to W_{F_v/\mathbf{Q}_v} \to \mathrm{Gal}(F_v/\mathbf{Q}_v) \to 1$$

and

$$1 \to C_F \to W_{F/\mathbf{Q}} \to \mathrm{Gal}(F/\mathbf{Q}) \to 1.$$

We may imbed the arrows (5.8) and (5.9) in a commutative diagram

$$\begin{array}{ccccccccc} 1 & \to & F_v^\times & \to & W_{F_v/\mathbf{Q}_v} & \to & \mathrm{Gal}(F_v/\mathbf{Q}_v) & \to & 1 \\ & & \downarrow & & \downarrow{\scriptstyle I_F} & & \downarrow & & \\ 1 & \to & C_F & \to & W_{F/\mathbf{Q}} & \to & \mathrm{Gal}(F/\mathbf{Q}) & \to & 1 \end{array}$$

Moreover we may so choose the central arrows that they are compatible with field extensions and upon passage to the limit yield I: $W_{\mathbf{Q}_v} \to W_{\mathbf{Q}}$. It is the image of $W_{\mathbf{Q}_v}$ in $W_{\mathbf{Q}}$ that will be fixed, and I_F may be changed to $w \to xI_F(w)x^{-1}$ where $x \in C_F$ and $x\sigma(x)^{-1} \in F_v^\times$ for all $\sigma \in \mathrm{Gal}(F_v/\mathbf{Q}_v)$.

Now let v be the valuation given by $\bar{\mathbf{Q}} \subseteq \mathbf{C}$. Let $F_\infty^\times = \prod_{w|v} F_w^\times$. The natural map $F_\infty^\times \to C_F$ is an imbedding, and we sometimes regard $F_\infty^\times$ as a subgroup of C_F. If we take an element τ of $\mathrm{Gal}(\bar{\mathbf{Q}}/\mathbf{Q})$, lift to $W_{\mathbf{Q}}$, and then project to $W_{L/\mathbf{Q}}$ we obtain an element $w = w(\tau)$ of $W_{L/\mathbf{Q}}$ which is well defined modulo the connected component, and in particular modulo the closure of $L_\infty^\times$.

We choose a set of representatives w_σ, $\sigma \in \mathrm{Gal}(L/\mathbf{Q})$, for the cosets of C_L in $W_{L/\mathbf{Q}}$ in such a way that the following conditions are satisfied:

(a) $w_1 = 1$.

(b) *If* $\sigma \in \mathrm{Gal}(L_v/\mathbf{Q}_v)$ *then* $w_\sigma \in W_{L_v/\mathbf{Q}_v}$.

(c) *If* $\rho \in \mathrm{Gal}(L/\mathbf{Q})$ *and* $\sigma \in \mathrm{Gal}(L_v/\mathbf{Q}_v)$ *then* $w_\rho w_\sigma = a_{\rho,\sigma} w_{\rho\sigma}$ *with* $a_{\rho,\sigma} \in L_\infty^\times$.

To arrange the final condition we may choose a collection $\mathfrak{f}$ of representatives η for the cosets $\mathrm{Gal}(L/\mathbf{Q})/\mathrm{Gal}(L_v/\mathbf{Q}_v)$ and set $w_{\eta\sigma} = w_\eta w_\sigma$ if $\sigma \in \mathrm{Gal}(L_v/\mathbf{Q}_v)$. We suppose that $\mathfrak{f}$ contains 1. With this choice we also have:

(d) *If* $\{a_{\rho,\sigma}\}$ *is the cocycle defined by* $w_\rho w_\sigma = a_{\rho,\sigma} w_{\rho\sigma}$ *then* $a_{\eta,\rho} = 1$ *for* $\eta \in \mathfrak{f}$ *and* $\sigma \in \mathrm{Gal}(L_v/\mathbf{Q}_v)$.

If $w \in W_{L/\mathbf{Q}}$, let $w_\sigma w = c_\sigma(w) w_\sigma$, $c_\sigma(w) \in C_L$. If $w = w(\tau)$ we set

$$b_0(\tau) = \prod_{\sigma \in \mathrm{Gal}(L/\mathbf{Q})} c_\sigma(w)^{\sigma\mu}.$$

It lies in $C_L \otimes X_*(\mathscr{S}^L)$, but is not well defined, because w is not. However we can show that it is well defined if taken modulo $L_\infty^\times \otimes X_*(\mathscr{S}^L)$, and that, in addition, it behaves properly under extensions of the field L.

The ambiguity in w now has no effect, for we are only free to replace w by uw where $u = \lim_n u_n$ and u_n lies in the image of $L_\infty^\times U$, where U is a subgroup of the group of units of L defined by a strong congruence condition. But [**41**, II.A.2],

$$\prod_{\sigma \in \mathrm{Gal}(\bar{\mathbf{Q}}/\mathbf{Q})} \sigma(v)^{\sigma\mu} = 1$$

for all $v \in U$. Consequently

$$\prod_\sigma c_\sigma(uw)^{\sigma\mu} = \left(\prod_\sigma \sigma(u)^{\sigma\mu}\right)\left(\prod_\sigma c_\sigma(w)^{\sigma\mu}\right)$$

is congruent modulo $L_\infty^\times \otimes X_*(\mathscr{S}^L)$ to $\prod_\sigma c_\sigma(w)^{\sigma\mu}$.

Suppose the representatives w_σ are replaced by $e_\sigma w_\sigma$, and, hence, $a_{\rho,\sigma}$ by

$$a'_{\rho,\sigma} = e_\rho \rho(e_\sigma) e_{\rho\sigma}^{-1} a_{\rho,\sigma}.$$

If $\sigma \in \mathrm{Gal}(L_v/\boldsymbol{Q}_v)$ then $e_\sigma \in L_v^\times$ and $\rho(e_\sigma) \in L_\infty^\times$. Since $a_{\rho,\sigma}$ and $a'_{\rho,\sigma}$ must then both be in $L_\infty^\times$, we infer that $e_\rho \equiv e_{\rho\sigma} \pmod{L_\infty^\times}$ when $\sigma \in \mathrm{Gal}(L_v/\boldsymbol{Q}_v)$. Moreover $c_\sigma(w)$ is replaced by $c'_\sigma(w) = e_{\sigma\tau} e_\sigma^{-1} c(w)$ and $b_0(\tau)$ by

$$b'_0(\tau) = \left\{\prod_\sigma e_{\sigma\tau}^{\sigma\mu} e_\sigma^{-\sigma\mu}\right\} b_0(\tau).$$

The factor may be written

$$\prod_\sigma e_\sigma^{\sigma(\tau^{-1}-1)\mu} \equiv \prod_{\eta \in \mathfrak{f}} \prod_{\sigma \in \mathrm{Gal}(L_v/\boldsymbol{Q}_v)} e_\eta^{\eta\sigma(\tau^{-1}-1)\mu}.$$

Since

$$\sum_{\sigma \in \mathrm{Gal}(L_v/\boldsymbol{Q}_v)} \sigma(1-\tau^{-1})\mu = (1+\iota)(1-\tau^{-1})\mu = 0,$$

this change has no effect on $b_0(\tau)$. In this argument we have denoted the image in $\mathrm{Gal}(L/\boldsymbol{Q})$ of $\tau \in \mathrm{Gal}(\bar{\boldsymbol{Q}}/\boldsymbol{Q})$ by the same symbol, a practice we shall continue to indulge in.

If we modify I then w is replaced by xwx^{-1} with $x \in C_L$ and $x\sigma(x)^{-1} \in L_v^\times$ for all $\sigma \in \mathrm{Gal}(L_v/\boldsymbol{Q})$. Then $c_\sigma(xwx^{-1}) = \sigma(x)\sigma\tau(x^{-1})c_\sigma(w)$ and

$$\prod_\sigma \sigma(x)^{-\sigma\mu} \sigma\tau(x)^{\sigma\mu} = \prod_\sigma \sigma(x)^{\sigma(\tau^{-1}-1)\mu}.$$

Since $\sigma(x) \equiv x \pmod{L_\infty^\times}$ if $\sigma \in \mathrm{Gal}(L_v/\boldsymbol{Q}_v)$, the same argument as before shows that $b_0(\tau)$ is unchanged.

Finally suppose that $L \subseteq L'$. Then $b'_0(\tau) \in C_L \otimes X_*(\mathscr{S}^{L'})$ and $b_0(\tau) \in C_L \otimes X_*(\mathscr{S}^L)$ are both defined, and we must verify that $b'_0(\tau)$ is taken to $b_0(\tau)$ by the canonical mapping of the first group to the second. Either $L_v = \boldsymbol{R}$ or $L_v = \boldsymbol{C}$, and the two cases must be treated separately. Suppose first that $L_v = \boldsymbol{C}$ and hence that $\mathrm{Gal}(L'/L) \cap \mathrm{Gal}(L'_v/\boldsymbol{Q}_v) = \{1\}$. Since both $b_0(\tau)$ and $b'_0(\tau)$ are independent of the choices of coset representatives, we may choose those which make it easiest to verify that $b_0(\tau)$ is the image of $b'_0(\tau)$. Let $\mathfrak{e}$ be a set of representatives for the cosets $\mathrm{Gal}(L'/L)\backslash\mathrm{Gal}(L'/\boldsymbol{Q})/\mathrm{Gal}(L'_v/\boldsymbol{Q}_v)$ containing 1. Every element σ of $\mathrm{Gal}(L'/\boldsymbol{Q})$ may be written uniquely as a product $\sigma = \zeta\eta\rho$, $\zeta \in \mathrm{Gal}(L'/L)$, $\eta \in \mathfrak{e}$, $\rho \in \mathrm{Gal}(L'_v/\boldsymbol{Q}_v)$. We may suppose that $w'_\sigma = w'_\rho w'_\eta w'_\rho$ with $w'_1 = 1$ and $w'_\rho \in w_{L'_v/\boldsymbol{Q}_v}$. Let w' be $w(\tau)$ with respect to L', and w be $w(\tau)$ with respect to L. Then under the canonical map $\pi\colon W_{L'/\boldsymbol{Q}} \to W_{L/\boldsymbol{Q}}$ the element w' maps to w. If $\sigma \in \mathrm{Gal}(L/\boldsymbol{Q})$ it lifts to a unique element of $\mathrm{Gal}(L'/\boldsymbol{Q})$ of the form $\eta\rho$. We suppose that w_σ is the image of $w'_\eta w'_\rho$. Thus if

$$w'_\eta w'_\rho w' = d_{\eta,\rho}(w') w'_{\eta'} w'_{\rho'}$$

with $d_{\eta,\rho}(w') \in W_{L'/L}$ then $c_\sigma(w) = \pi(d_{\eta,\rho}(w'))$. On the other hand if $\sigma_1 = \zeta\eta\rho$ lies in $\mathrm{Gal}(L'/\boldsymbol{Q})$ then

$$w'_\zeta d_{\eta,\rho}(w') = c_{\sigma_1}(w') w'_{\zeta}.$$

Consequently, by the very definition of π,

$$c_\sigma(w) = \prod_{\sigma_1 \to \sigma} c_{\sigma_1}(w').$$

It follows immediately that $b_0(\tau)$ is the image of $b_0'(\tau)$.

If $L_v = \boldsymbol{R}$ then $X^*(\mathcal{S}^L) \simeq \boldsymbol{Z}$ and the Galois group acts trivially. Suppose we replace w_σ by $e_\sigma w_\sigma$ with $e_\sigma \in C_L$. Then $c_\sigma(w)$ is replaced by $e_\sigma e_{\sigma\tau}^{-1}$. Since

$$\prod_\sigma (e_\sigma e_{\sigma\tau}^{-1})^{\sigma\mu} = \prod_\sigma (e_\sigma e_{\sigma\tau}^{-1})^{\mu} = 1,$$

this has no effect on $b_0(\tau)$, and when defining $b_0(\tau)$ we need not suppose that the collection $\{w_\sigma\}$ is subject to the constraints (a), (b), and (c). If we want to define $b_0'(\tau)$, we may still need to be careful about the choice of the coset representatives w_σ', $\sigma \in \mathrm{Gal}(L'/\boldsymbol{Q})$. However, since we are only interested in the image of $b_0'(\tau)$ in $\mathcal{S}^L$, we may again ignore (a), (b), and (c). We choose a set $\mathfrak{e}$ of representatives for the cosets $\mathrm{Gal}(L'/L)\backslash\mathrm{Gal}(L'/\boldsymbol{Q})$, write $\sigma = \rho\eta$, $\rho \in \mathrm{Gal}(L'/L)$, $\eta \in \mathfrak{e}$, and take $w_\sigma' = w_\rho' w_\eta'$. If $\sigma \in \mathrm{Gal}(L/\boldsymbol{Q})$ is the image of η, we take $w_\sigma = \pi(w_\eta')$. The argument can now proceed as before.

Let $\tilde{b}(\tau)$ be a lift of $b_0(\tau)$ to $I_L \otimes X_*(\mathcal{S}^L) = \mathcal{S}^L(\boldsymbol{A}(L))$ and let $b(\tau)$ be the projection of $\tilde{b}(\tau)$ on $\mathcal{S}^L(\boldsymbol{A}_f(L))$. The element $b(\tau)$ is well defined modulo $\mathcal{S}^L(L)$ and, as we shall see, this bit of ambiguity will cause us no difficulty. But we have to fix one choice. The first point to verify is that

$$d_{\tau_1,\tau_2} = b(\tau_1)\tau_1(b(\tau_2))b(\tau_1\tau_2)^{-1}$$

lies in $\mathcal{S}^L(L)$. When verifying this, we may choose the liftings $\tilde{b}(\tau_1)$, $\tilde{b}(\tau_2)$, and $\tilde{b}(\tau_1\tau_2)$ in any way we like. We choose liftings $\tilde{c}_\sigma(w_1)$ and $\tilde{c}_\sigma(w_2)$ of $c_\sigma(w_1)$ and $c_\sigma(w_2)$ to I_L and take

$$\tilde{b}(\tau_1) = \prod_\sigma \tilde{c}_\sigma(w_1)^{\sigma\mu}, \qquad \tilde{b}(\tau_2) = \prod_\sigma \tilde{c}_\sigma(w_2)^{\sigma\mu}.$$

Since $c_\sigma(w_1w_2) = c_\sigma(w_1)c_{\sigma\tau_1}(w_2)$, we may take $\tilde{c}_\sigma(w_1w_2)$ to be $\tilde{c}_\sigma(w_1)\tilde{c}_{\sigma\tau_1}(w_2)$. Because

$$\tau_1^{-1}(\tilde{b}(\tau_2)) = \prod_\sigma \tilde{c}_\sigma(w_2)^{\sigma\tau_1^{-1}\mu} = \prod_\sigma \tilde{c}_{\sigma\tau_1}(w_2)^{\sigma\mu},$$

the element d_{τ_1,τ_2} will then be 1.

Finally we have to establish that the elements $c_\rho(\tau)$ defined by equation (5.7) lie in $\mathcal{S}^L(L)$, for equations (5.4) and (5.5) will then follow immediately. It will suffice to show that for any $w \in W_{L/\boldsymbol{Q}}$ and any $\rho \in \mathrm{Gal}(L/\boldsymbol{Q})$

$$\left\{\prod_\sigma c_\sigma(w)^{\sigma\mu}\right\}\left\{\prod_\sigma \rho(c_\sigma(w))^{-\rho\sigma\mu}\right\} \tag{5.10}$$

lies in $L_\infty^\times \otimes X_*(\mathcal{S}^L)$. Suppose $w = w_1w_2$ and w_1 projects to $\tau_1 \in \mathrm{Gal}(L/\boldsymbol{Q})$. Then $c_\sigma(w) = c_\sigma(w_1)c_{\sigma\tau_1}(w_2)$ and (5.10) is equal to

$$\left\{\prod_\sigma c_\sigma(w_1)^{\sigma\mu}\rho(c_\sigma(w_1))^{-\rho\sigma\mu}\right\}\tau_1\left\{\prod_\sigma c_\sigma(w_2)^{\sigma\mu}\rho(c_\sigma(w_2))^{-\rho\sigma\mu}\right\}.$$

Consequently we need only verify that (5.10) lies in $L_\infty^\times \otimes X_*(\mathcal{S}^L)$ for w in C_L and for $w = w_\tau$. If w lies in C_L then $c_\sigma(w) = \sigma(w)$ and $\prod_\sigma \sigma(w)^{\sigma\mu} = \prod_\sigma \rho\sigma(w)^{\rho\sigma\mu}$. The expression (5.10) is therefore equal to 1. If $w = w_\tau$ then $c_\sigma(w) = a_{\sigma,\tau}$ and

$$\prod_\sigma a_{\sigma,\tau}^{\sigma\mu}\rho(a_{\sigma,\tau})^{-\rho\sigma\mu} = \prod_\sigma a_{\rho,\sigma}^{\rho\sigma(\tau^{-1}-1)\mu}. \tag{5.11}$$

However it follows from condition (c) that $a_{\rho,\sigma} \equiv a_{\rho,\sigma\iota} \pmod{L_\infty^\times}$. Since $(1 + \iota) \cdot (1 - \tau^{-1})\mu = 0$, the right side of (5.11) lies in $L_\infty^\times \otimes X_*(\mathcal{S}^L)$.

Since $b(\tau)$, although defined for $\tau \in \mathrm{Gal}(\bar{Q}/Q)$, depends only on the image of τ in $\mathrm{Gal}(L^{ab}/Q)$, the groups $\mathcal{T}^L$ and $\mathcal{T}$ are now completely defined. The ambiguity in the $b(\tau)$ is easily seen to correspond to the ambiguity in the choice of the section $a(\tau)$.

If $E \subseteq \bar{Q}$ is any finite extension of Q, we let $\mathcal{T}_E$ be the inverse image of $\mathrm{Gal}(\bar{Q}/E)$ in $\mathcal{T}$. If $E \subseteq L^{ab}$ we may also introduce $\mathcal{T}^L_E$. The group $\mathcal{T}^L_L$ has been introduced by Serre **[41]**, who uses it to formulate some ideas of Taniyama. He makes its arithmetic significance quite clear, but his definition is sufficiently different from that given here that an explanation of the reasons for their equivalence is in order.

If x is an idèle of L then $\psi(x) = \prod_{\mathrm{Gal}(L/Q)} \sigma(x)^{\sigma\mu}$ is an element of $\mathcal{S}^L(A(L))$. By II.4 of **[41]** there is an open subgroup U of the group of idèles I_L such that $\psi(x) = 1$ if $x \in U \cap L^\times$. The standard map of I_L onto $\mathrm{Gal}(L^{ab}/L)$ restricts to U and if $\tau = \tau(x)$ is the image of x we may take $w = w(\tau)$ to be the image of x in C_L. Then $\psi(x)$ is a lifting of $b_0(\tau)$ to $\mathcal{S}^L(A(L))$. However $\psi(x)$ depends only on τ, and thus we may take $\tilde{b}(\tau) = \psi(x)$. Then $d_{\tau_1,\tau_2} = 1$ and $c_\rho(\tau) = 1$ if τ, τ_1, τ_2 lie in the image $\mathrm{Gal}(L^{ab}/F)$ of U. Here F is the finite extension of L defined by U. The elements $c_\rho(\tau)$ are in fact 1 for all $\tau \in \mathrm{Gal}(L^{ab}/L)$. If we choose a set of representatives $\mathfrak{e}$ for $\mathrm{Gal}(L^{ab}/F)\backslash\mathrm{Gal}(L^{ab}/L)$ and set $\tilde{b}(\tau\eta) = \tilde{b}(\tau)\tilde{b}(\eta)$, $\tau \in \mathrm{Gal}(L^{ab}/F)$, $\eta \in \mathfrak{e}$, then, in general, d_{τ_1,τ_2} will depend only on the images of τ_1, τ_2 in $\mathrm{Gal}(F/L)$, and $\mathcal{T}^L_L$ may be ob tained by pulling back an extension

$$1 \to \mathcal{S}^L \to \mathcal{T}_U \to \mathrm{Gal}(F/L) \to 1$$

to $\mathrm{Gal}(L^{ab}/L)$. Here $\mathcal{T}_U$ is the quotient of $\mathcal{T}^L_L$ by the normal subgroup $\{a(\tau) | \tau \in \mathrm{Gal}\,(L^{ab}/F)\}$. It is the extension $\mathcal{T}_U$ that Serre defines directly. He denotes it by the symbol $S_\mathfrak{m}$.

The map ψ defines a homomorphism of $L^\times/L^\times \cap U \simeq L^\times U/L^\times$ into $\mathcal{S}^L(L)$ and to verify that $\mathcal{T}_U$ is the group studied by Serre, we have only to verify that it can be imbedded in a commutative diagram

$$\begin{array}{ccccccccc} 1 & \to & L^\times U/L^\times & \to & I_L/L^\times & \to & I_L/L^\times U & \to & 1 \\ & & \downarrow & & \downarrow & & \downarrow\wr & & \\ 1 & \to & \mathcal{S}^L(L) & \to & \mathcal{T}_U(L) & \to & \mathrm{Gal}(F/L) & \to & 1 \end{array}$$

The right-hand arrow is $x \to \tau(x)^{-1}$. We do not have to pass to the quotient but may define the homomorphism

$$I_L/L^\times \to \mathcal{T}^L_L(L) \tag{5.12}$$

directly. The central arrow is then obtained by composing with the projection $\mathcal{T}^L_L(L) \to \mathcal{T}_U(L)$. The homomorphism (5.12) is

$$x \to \left\{\prod_\sigma \sigma(x)^{\sigma\mu}\right\} \tilde{b}(\tau)^{-1} a(\tau)^{-1}$$

if τ is the image of x in $\mathrm{Gal}(L^{ab}/L)$. The composition of our splitting $\mathrm{Gal}(L^{ab}/L) \to \mathcal{T}^L_L(Q_l)$ with $\mathcal{T}^L_L(Q_l) \to \mathcal{T}_U(Q_l)$ is either Serre's ε_l or its inverse, presumably its inverse, for we are so arranging matters that the eigenvalues of the Frobenius elements acting on the cohomology of algebraic varieties are greater than or equal to 1.

The cocycle $\rho \to c_\rho(\tau)$ certainly becomes trivial at every finite place, but is not necessarily trivial at the infinite place, v. Indeed under the isomorphism

$$H^1(\mathrm{Gal}(L_v/\mathbf{Q}_v), \mathscr{S}^L(L_v)) \simeq H^{-1}(\mathrm{Gal}(L_v/\mathbf{Q}_v), X_*(\mathscr{S}^L))$$

given by the Tate-Nakayama duality it corresponds to the element of the group on the right represented by $(1 - \tau^{-1})\mu$. If, as has been our custom, we denote the image of τ in $\mathrm{Gal}(L/\mathbf{Q})$ again by τ we may suppose that $w(\tau) = w_\tau$, for the class of $\{c_\rho(\tau)\}$ depends only on this image. According to the discussion of formula (5.11)

$$\prod_\sigma a_{\rho,\sigma}^{\rho\sigma(\tau^{-1}-1)\mu} = \prod_{\eta\in\mathfrak{f}} (a_{\rho,\eta\iota}a_{\rho,\eta}^{-1})^{\rho\eta(\tau^{-1}-1)\mu} = e_\rho(\tau)$$

lies in $L_\infty^\times \otimes X_*(\mathscr{S}^L) = \mathscr{S}^L(L_\infty)$. By definition, the classes of $\{c_\rho(\tau)\}$ and $\{e_\rho(\tau)\}$ are inverse to one another in $H^1(\mathrm{Gal}(L/\mathbf{Q}), \mathscr{S}^L(A(L)))$. Thus all we need do is calculate the projection of $e_\rho(\tau)$ on $\mathscr{S}^L(L_v)$ for $\rho \in \mathrm{Gal}(L_v/\mathbf{Q}_v)$.

Observe first of all that if we agree to choose coset representatives satisfying (d), then $a_{\rho,\eta\iota}\, a_{\rho,\eta}^{-1} = \rho(a_{\eta,\iota}^{-1})\, a_{\rho\eta,\iota} = a_{\rho\eta,\iota}$. Again if $\rho\eta = \eta_1\rho_1$ then

$$a_{\rho\eta,\iota} = a_{\eta_1\rho_1,\iota} = \eta_1(a_{\rho_1,\iota})a_{\eta_1,\rho_1\iota}\, a_{\eta_1,\rho_1}^{-1} = \eta_1(a_{\rho_1,\iota}).$$

Since $a_{\rho_1,\iota} \in L_v^\times$, the term on the right has a projection on $L_v^\times$ different from 1 only if $\eta_1 = 1$. Then η too equals 1, and so the projection of $e_\rho(\tau)$ on $\mathscr{S}^L(L_v)$ is

$$a_{\rho,\iota}^{\rho\iota(\tau^{-1}-1)\mu} = \prod_{\mathrm{Gal}(L_v/\mathbf{Q}_v)} a_{\rho,\sigma}^{\rho\sigma(\tau^{-1}-1)\mu},$$

in conformity with our assertion.

Suppose T is a torus over $\mathbf{Q}$, provided with a coweight μ such that

$$(5.13) \qquad (1 + \iota)(\tau - 1)\mu = (\tau - 1)(1 + \iota)\mu = 0$$

for all $\tau \in \mathrm{Gal}(\bar{\mathbf{Q}}/\mathbf{Q})$. Then there exists a unique homomorphism $\psi\colon \mathscr{S} \to T$ such that the composition of ψ with the canonical coweight of $\mathscr{S}$ is μ. We can transport the cocycles $\rho \to c_\rho(\tau)$ from $\mathscr{S}$ to T, obtaining cocycles $\{c_\rho(\tau, \mu)\}$ as well as $b(\tau, \mu) \in T(A_f(L))$. However if T and $h\colon \mathscr{R} \to T$ define a Shimura variety and μ is the restriction of h to the first factor of $\mathscr{R}$ the condition (5.13) will not necessarily be satisfied. Nonetheless, we may repeat the previous construction and define $b(\tau, \mu)$ and $\{c_\rho(\tau, \mu)\}$ for all τ such that $(1 + \iota)(\tau^{-1} - 1)\mu = 0$. This generalization is necessary for the treatment of those Shimura varieties $\mathrm{Sh}(G, h)$ for which G is not equal to G_0. It should perhaps be observed that $b(\tau, \mu)$ is not insensitive to the ambiguity in the choice of $w = w(\tau)$, although $\{c_\rho(\tau, \mu)\}$ is. However, if Z is the centre of G the ambiguity all lies in $Z(A_f) \cap K$, and may be ignored.

If $E \subseteq \mathbf{C}$ the motives over E whose associated Hodge structure is of CM type are themselves said to be of CM type. They form a tannakian category $\mathrm{CM}(E)$ with a natural fibre functor $\omega_{\mathrm{CM}(E)}$, given by rational cohomology. Since one expects that the natural functor $\mathrm{CM}(\bar{\mathbf{Q}}) \to \mathrm{CM}(\mathbf{C})$ is an equivalence, we may as well suppose that $E \subseteq \bar{\mathbf{Q}}$. According to the hopes expressed at the end of the previous section there should be an equivalence $\eta\colon \mathrm{CM}(\bar{\mathbf{Q}}) \to \mathscr{REP}(\mathscr{S})$ and an isomorphism $\omega_{\mathrm{Rep}(\mathscr{S})} \circ \eta \to \omega_{\mathrm{CM}(\bar{\mathbf{Q}})}$, which would enable us to identify $\mathscr{S}$ with the group $G_{\mathrm{CM}(\bar{\mathbf{Q}})}$, defined by the category $\mathrm{CM}(\bar{\mathbf{Q}})$ and the functor $\omega_{\mathrm{CM}(\bar{\mathbf{Q}})}$. If $E \subseteq \bar{\mathbf{Q}}$, one hopes that in the same way it will be possible to identify $\mathscr{T}_E$ with $G_{\mathrm{CM}(E)}$.

There are properties which this identification should have, and it is necessary to describe them explicitly. First of all, if $F \subseteq E$ the diagram

$$\begin{array}{ccccc} \mathcal{S} & \longrightarrow & \mathcal{T}_E & \longrightarrow & \mathcal{T}_F \\ \uparrow & & \updownarrow & & \updownarrow \\ G_{\mathrm{CM}(\bar{Q})} & \longrightarrow & G_{\mathrm{CM}(E)} & \longrightarrow & G_{\mathrm{CM}(F)} \end{array}$$

should be commutative.

The other properties are more complicated to describe. Suppose $\tau \in \mathrm{Gal}(\bar{Q}/Q)$ and τ takes E to E'. Its inverse then naturally defines a $\otimes$-functor $\eta(\tau)$ from $\mathrm{CM}(E')$ to $\mathrm{CM}(E)$. Let $G^{\tau}_{\mathrm{CM}(E')}$ be the group defined by the category $\mathrm{CM}(E')$ and the functor $\omega_{\mathrm{CM}(E)} \circ \eta(\tau)$. The dual of $\eta(\tau)$ is then an isomorphism $\varphi(\tau)\colon G_{\mathrm{CM}(E)} \to G^{\tau}_{\mathrm{CM}(E')}$. In terms of representations, $\eta(\tau)$ associates to every representation $(\xi', V(\xi'))$ of $G_{\mathrm{CM}(E')}$ a representation $(\xi, V(\xi))$ of $G_{\mathrm{CM}(E)}$. Since $\omega_{\mathrm{CM}(E')}$ and $\omega_{\mathrm{CM}(E)} \circ \eta(\tau)$ become isomorphic over $\bar{Q}$ there is a family of homomorphisms, one for each ξ', $\psi(\xi')\colon V(\xi')_{\bar{Q}} \to V(\xi)_{\bar{Q}}$, compatible with sums and tensor products. If $\psi'(\xi)$ is another possible family then there is a $t \in \mathcal{T}_{E'}(\bar{Q})$, such that $\psi'(\xi') = \psi(\xi')\xi'(t)$. Finally since the two functors $\omega^{A_f}_{\mathrm{CM}(E')}$ and $\omega^{A_f}_{\mathrm{CM}(E)} \circ \eta(\tau)$ are canonically isomorphic, arising as they do from the l-adic cohomology, there is a canonical family of isomorphisms $\psi_{A_f}(\xi')\colon V(\xi')_{A_f} \to V(\xi)_{A_f}$.

On the other hand, suppose $a(\tau) \in \mathcal{T}(\bar{Q})$ maps to τ. Let $\sigma(a(\tau)) = c_\sigma(\tau)a(\tau)$. We may use $a(\tau)$ to associate to every representation $(\xi', V(\xi'))$ of $\mathcal{T}_{E'}$ a representation $(\xi, V(\xi))$ of $\mathcal{T}_E$. The space $V(\xi)$ is obtained by twisting $V(\xi')$ by the cocycle $\{\xi'(c_\sigma(\tau)^{-1})\}$. The representation ξ is $t \to \xi'(a(\tau)ta(\tau)^{-1})$. This functor $(\xi', V(\xi')) \to (\xi, V(\xi))$ is to be (isomorphic to) that obtained from $\eta(\tau)$ by identifying $\mathcal{T}_E$ and $G_{\mathrm{CM}(E)}$ and $\mathcal{T}_{E'}$ and $G_{\mathrm{CM}(E')}$. Moreover one possible choice for $\psi(\xi')$ is to be the isomorphism $\psi_0(\xi')\colon V(\xi')_{\bar{Q}} \to V(\xi)_{\bar{Q}}$ implicit in the definition of $V(\xi)$. If $b(\tau)a(\tau)$ is the image of τ under the canonical splitting $\mathrm{Gal}(\bar{Q}/Q) \to \mathcal{T}(A_f)$, then $\psi_{A_f}(\xi')$ is to be $\psi_0(\xi') \circ \xi'(b(\tau))^{-1}$.

If $\tau \in \mathrm{Gal}(\bar{Q}/E)$ then $E' = E$ and $\eta(\tau)\colon \mathrm{CM}(E') \to \mathrm{CM}(E)$ is the identity functor. Thus the functor $(\xi', V(\xi')) \to (\xi, V(\xi))$ from $\mathrm{Rep}(T_{E'}) = \mathrm{Rep}(T_E)$ to $\mathrm{Rep}(T_E)$ must be canonically isomorphic to the identity. A final property, which seems to be independent of the preceding ones, is that this isomorphism should be given by $\xi' \to \xi$ and $\psi_0(\xi') \circ \xi'(a(\tau))\colon V(\xi') \to V(\xi)$. Observe that $a(\tau)$ is now in $\mathcal{T}_E$ and these transformations are defined over Q.

I assume that all this is so, just to see where it leads, and especially to see what it suggests about the groups $G^{\tau,\varphi}$ introduced in the previous section. But there are some lemmas to be verified first. I conclude the present section by describing a property of the Taniyama group whose significance was pointed out to me by Casselman. It will be needed to show that the zeta-functions of motives, and especially abelian varieties, of CM type can be expressed as products of the L-functions associated to representations of the Weil group.

The point is that there is a natural homomorphism φ of the Weil group W_Q of Q into $\mathcal{T}(C)$ and thus for any finite extension F of Q a homomorphism $\varphi_F\colon W_F \to \mathcal{T}_F(C)$. To define it we work at a finite level, defining $W_{L/Q} \to \mathcal{T}^L(C)$, and afterwards passing to the limit.

Fix for now a set of coset representatives w_σ which satisfies (a), (b), and (c). If $w \in W_{L/Q}$ we define $b_0(w) = \prod_{\sigma \in \mathrm{Gal}(L/Q)} c_\sigma(w)^{\sigma\mu}$. If τ is the image of w in $\mathrm{Gal}(L^{ab}/Q)$ then $b_0(w) \equiv b_0(\tau) \pmod{L^\times_\infty \otimes X^*(\mathcal{S}^L)}$, and we may lift $b_0(w)$ to $b(w)$ in $\mathcal{S}^L(A(L))$ in such a way that the projection of $b(w)$ in $\mathcal{S}^L(A_f(L))$ is $b(\tau)$. However a simple calculation shows that if τ_1, τ_2 are the images of w_1, w_2 then $b(w_1)\tau_1(b(w_2))b(w_1w_2)^{-1} \in \mathscr{S}^L(L)$. Since its projection on $S^L(\boldsymbol{A}_f(L))$ is equal to d_{τ_1, τ_2}, it is itself equal to d_{τ_1, τ_2}. If $b_v(w)$ is the projection of $b(w)$ in $\mathscr{S}^L(L_v) = \mathscr{S}^L(\boldsymbol{C})$, we may define φ by $\varphi\colon\ w \to b_v(w)a(\tau)$. If the coset representatives w_σ are changed then φ is replaced by $\varphi' =$ **ad** $a \circ \varphi$, $a \in \mathscr{S}^L(\boldsymbol{C})$, but this is of no importance.

If G_{W_F} is the group over $\boldsymbol{C}$ defined by the tannakian category of continuous, finite-dimensional, complex, semisimple representations of W_F then φ_F is the composite of the imbedding $W_F \to G_{W_F}(\boldsymbol{C})$ and an algebraic homomorphism ψ_F: $G_{W_F} \to \mathscr{T}_F$. Moreover if the principle of functoriality is valid, there is a surjection $G_{\Pi(F)} \to G_{W_F}$ and we can expect to have a diagram

$$\begin{array}{ccc} G_{\Pi(F)} & \xrightarrow{\rho_F} & G_{\mathrm{Mot}(F)} \\ \downarrow & & \downarrow \\ G_{W_F} & \xrightarrow[\psi_F]{} & \mathscr{T}_F \end{array}$$

whose two composite arrows differ by **ad** s, $s \in \mathscr{S}(\boldsymbol{C}) \subseteq \mathscr{T}_F(\boldsymbol{C})$.

6. Conjugation of Shimura varieties. The principal purpose of this section is to formulate a conjecture about the conjugation of Shimura varieties, a conjecture whose first justification is that it is a simple statement which implies what we need for the study of the zeta-functions at archimedean places and is compatible with all that we know. Some lemmas are necessary before it can be stated, and we shall see that these lemmas together with the hypothetical properties of the Taniyama group suggest an answer to the question that arose at the end of the fourth section. This answer in its turn throws new light on the conjecture, so that we can weave a consistent pattern of hypotheses, and our task will be ultimately to show that it has some real validity.

We need a construction, which we make in sufficient generality that it applies to all Shimura varieties and not just those associated to motives. Suppose the pair $(G, \mathfrak{H})$ defines a Shimura variety, T and $\bar{T}$ are two Cartan subgroups of G defined over $\boldsymbol{Q}$, and $h\colon \mathscr{R} \to T$, $\bar{h} \to \mathscr{R} \to \bar{T}$ both lie in $\mathfrak{H}$. Let μ and $\bar{\mu}$ be the coweights of T and $\bar{T}$ obtained by restricting h and $\bar{h}$ to the first factor of $\mathscr{R}$, and choose a finite Galois extension L which splits T and $\bar{T}$. Let $\tau \in \mathrm{Gal}(\bar{\boldsymbol{Q}}/\boldsymbol{Q})$; then the coweights $(1 + \iota)(\tau^{-1} - 1)\mu$ and $(1 + \iota)(\tau^{-1} - 1)\bar{\mu}$ are both central and they are equal. Choose $w = w(\tau)$ as before, and set

$$b_0(\tau, \mu) = \prod_\sigma c_\sigma(w)^{\sigma\mu}, \qquad b_0(\tau, \bar{\mu}) = \prod_\sigma c_\sigma(w)^{\sigma\bar{\mu}}.$$

Let $\tilde{b}(\tau, \mu)$ and $\tilde{b}(\tau, \bar{\mu})$ be liftings to $T(\boldsymbol{A}(L))$ and $\bar{T}(\boldsymbol{A}(L))$ and let $B(\tau) = B(\tau, \mu, \bar{\mu})$ be the projection of $\tilde{b}(\tau, \bar{\mu})^{-1}\, \tilde{b}(\tau, \mu)$ on $G(\boldsymbol{A}_f(L))$. Although we may not be able to define the cocycle $\{c_\rho(\tau, \mu)\}$, we can define $\{c_\rho(\tau, \mu_{\mathrm{ad}})\}$ if μ_{ad} is the composition of μ with the projection to the adjoint group.

It may be as well to check that $B(\tau)$ is indeed independent of the choice of w and of the coset representatives w_σ, provided the usual conditions (a), (b), and (c) are satisfied. If w_σ is replaced by $e_\sigma w_\sigma$ then, apart from a factor in $L_\infty^\times \otimes X_*(T)$, $b_0(\tau, \mu)$ is multiplied by $\prod_{\eta\in\mathfrak{f}} e_\eta^{\eta(1+\iota)(\tau^{-1}-1)\mu}$, and $b_0(\tau, \bar{\mu})$ by $\prod_{\eta\in\mathfrak{f}} e_\eta^{\eta(1+\iota)(\tau^{-1}-1)\bar{\mu}}$. Since these two terms are central and equal by assumption, the change has no effect on $B(\tau)$. If w is replaced by xwx^{-1} with $x\sigma(x^{-1}) \in L_v^\times$ for $\sigma \in \mathrm{Gal}(L_v/\boldsymbol{Q}_v)$, then $b_0(\tau, \mu)$ is modified by the product of an element in $L_\infty^\times \otimes X_*(T)$ and $\prod_\eta x^{\eta(1+\iota)(\tau^{-1}-1)\mu}\,\tau$. Since $b_0(\tau, \bar\mu)$ undergoes a similar modification, $B(\tau)$ is not affected. One can also show easily that $B(\tau)$ is not changed when L is enlarged; the argument is once again basically the same as that used to treat $b(\tau)$.

$B(\tau)$ does depend on the choice of $\tilde b(\tau, \mu)$ and $\tilde b(\tau, \bar\mu)$. These choices made, we will use them consistently to define $c_\rho(\tau, \mu_{\mathrm{ad}})$, $c_\rho(\tau, \bar\mu_{\mathrm{ad}})$, and, when

$$(1 + \iota)(\tau^{-1} - 1)\mu = (1 + \iota)(\tau^{-1} - 1)\bar\mu = 0,$$

$c_\rho(\tau, \mu)$, $c_\rho(\tau, \bar\mu)$. Thus $c_\rho(\tau, \mu_{\mathrm{ad}})$ is to be the projection of $\rho(\tilde b(\tau, \mu)^{-1})\tilde b(\tau, \mu)$ in $G_{\mathrm{ad}}(A_f(L))$. With these conventions, the ambiguity in $B(\tau)$ will cause no harm in the construction to be given next.

Let $G^{\tau,\mu}$ and $G^{\tau,\bar\mu}$ be the groups obtained from G by twisting with the cocycles $\{c_\rho(\tau, \mu_{\mathrm{ad}})^{-1}\}$ and $\{c_\rho(\tau, \bar\mu_{\mathrm{ad}})^{-1}\}$. We are going to verify the following:

FIRST LEMMA OF COMPARISON. (i) *If* $c_\rho = c_\rho(\tau, \mu_{\mathrm{ad}})$ *then*

$$\gamma_\rho = B(\tau)\mathbf{ad}\; c_\rho^{-1}(\rho(B(\tau)^{-1}))$$

lies in $G^{\tau,\mu}(L)$.

(ii) *The cocycle* $\{\gamma_\rho\}$ *in* $G^{\tau,\mu}(L)$ *bounds.*

(iii) *If* $(1 + \iota)(\tau^{-1} - 1)\mu = (1 + \iota)(\tau^{-1} - 1)\bar\mu = 0$ *then*

$$\gamma_\rho = c_\rho(\tau, \bar\mu)^{-1}\, c_\rho(\tau, \mu).$$

The third assertion is clear; it is the other two with which we must deal. The element γ_ρ can be obtained by projecting

$$\tilde b(\tau, \bar\mu)^{-1}\, \rho(\tilde b(\tau, \bar\mu))\rho(\tilde b(\tau, \mu)^{-1})\tilde b(\tau, \mu) \tag{6.1}$$

on $G(A_f(L))$. This makes it perfectly clear that γ_ρ is not affected if $w = w(\tau)$ is replaced by $xw(\tau)$ with $x \in C_L$. Thus we may assume that $w = w_\tau$, where τ is here also used to denote the image of τ in $\mathrm{Gal}(L/Q)$. Then we have to show that

$$\tilde b(\tau, \bar\mu)^{-1}\, \rho(\tilde b(\tau, \bar\mu)) \equiv \tilde b(\tau, \mu)^{-1}\, \rho(\tilde b(\tau, \mu)) \pmod{G(L_\infty)\, G(L)}.$$

It follows easily from (5.11) that if $\tilde a_{\rho,\eta}$ is a lift of $a_{\rho,\eta}$ to I_L, then both sides are congruent to

$$\prod_\eta \tilde a_{\rho,\eta}^{\rho\eta(1+\iota)(1-\tau^{-1})\mu} = \prod_\eta \tilde a_{\rho,\eta}^{\rho\eta(1+\iota)(1-\tau^{-1})\bar\mu}.$$

The desired equality follows.

To prove the second assertion we shall apply Hasse's principle, but for this we need a group G whose derived group G_{der} is simply connected. Let G_{sc} be the simply connected covering of G_{der}. The Cartan subgroup T defines T_{der} and T_{sc}. We have an imbedding $X_*(T_{\mathrm{sc}}) \to X_*(T)$ and $G_{\mathrm{der}} = G_{\mathrm{sc}}$ if and only if the quotient is torsion-free. If we can construct a diagram of $\mathrm{Gal}(L/Q)$-modules

$$\begin{array}{ccccccccc} 0 & \longrightarrow & X_*(T_{\mathrm{sc}}) & \longrightarrow & Q & \longrightarrow & P & \longrightarrow & 0 \\ & & \| & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & X_*(T_{\mathrm{sc}}) & \longrightarrow & X_*(T) & \longrightarrow & M & \longrightarrow & 0 \end{array}$$

in which P is torsion-free and $P \to M$ is a surjection whose kernel is a free

$\mathrm{Gal}(L/\boldsymbol{Q})$-module, then we can use it to define a central extension G' of G with $X_*(T') = \boldsymbol{Q}$. We will have $G'_{\mathrm{der}} = G'_{\mathrm{sc}} = G_{\mathrm{sc}}$, and $G'(\boldsymbol{R}) \to G(\boldsymbol{R})$ will be surjective. To construct the diagram, we choose an exact sequence of $\mathrm{Gal}(L/\boldsymbol{Q})$-modules

$$0 \longrightarrow N \longrightarrow P \longrightarrow M \longrightarrow 0,$$

with P torsion-free and N free. Then $\boldsymbol{Q}$ is the set of all (x, p) in $X_*(T) \oplus P$ for which x and p have the same image in M.

We lift μ to $\mu' = (\mu, \nu)$ with $\nu \in P$. If $\bar{\mu} = \mathbf{ad}\, g \circ \mu$ and g is the image of g', we set $\bar{\mu}' = \mathbf{ad}\, g' \circ \mu'$. If the assertion is valid for μ', $\bar{\mu}'$, and G', it is valid for μ, $\bar{\mu}$, and G. Consequently we may suppose that G_{der} is simply-connected.

There are two types of ambiguity in $B(\tau)$. It can be changed to $tB(\tau)$ with $t \in \bar{T}(L)$. Then $\{\gamma_\rho\}$ is replaced by $\{(t\gamma_\rho\ \mathbf{ad}\, c_\rho^{-1}(\rho(t)^{-1})\}$, and its cohomology class is not affected. We can also change $B(\tau)$ to $B(\tau)t$ with $t \in T(L)$. Then $\{c_\rho\}$ is replaced by $\{c'_\rho\}$ with $c'_\rho = \rho(t_{\mathrm{ad}}^{-1})c_\rho t_{\mathrm{ad}}$, t_{ad} being the image of t in G_{ad}, and $\{\gamma_\rho\}$ is replaced by $\{\gamma'_\rho\}$, with

$$\gamma'_\rho = \gamma_\rho\ \mathbf{ad}\ c_\rho^{-1}(\rho(t^{-1}))t.$$

If $\gamma_\rho = \delta\ \mathbf{ad}\ c_\rho^{-1}(\rho(\delta^{-1}))$ then

$$\gamma'_\rho = (\delta t)\mathbf{ad}^{-1}\ c'_\rho(\rho(\delta t)^{-1}).$$

Consequently the ambiguity in $B(\tau)$ has no effect on the assertion (ii).

Rather than prove (ii) directly for a given choice of the pair μ, $\bar{\mu}$, we want to prove it for a succession of pairs. For this one should first check that the validity of (ii) defines an equivalence relation. If μ and $\bar{\mu}$ are interchanged then $\{\gamma_\rho\}$ is replaced by $\{\gamma_\rho^{-1}\}$, and if $\gamma_\rho = \delta\ \mathbf{ad}\ c_\rho^{-1}(\rho(\delta^{-1}))$ then

$$\gamma_\rho^{-1} = \delta^{-1}\ \mathbf{ad}\ \bar{c}_\rho^{-1}(\rho(\delta)).$$

To show the transitivity, we introduce a new notation, denoting γ_ρ by $\gamma_\rho(\bar{\mu}, \mu)$, and c_ρ by $c_\rho(\mu)$. Suppose

$$\gamma_\rho(\mu_2, \mu_3) = t\ \mathbf{ad}\ c_\rho^{-1}(\mu_3)(\rho(t^{-1})),$$

$$\gamma_\rho(\mu_1, \mu_2) = s\ \mathbf{ad}\ c_\rho^{-1}(\mu_2)(\rho(s^{-1})).$$

Observing that

$$\gamma_\rho(\mu_2, \mu_3)\ \mathbf{ad}\ c_\rho^{-1}(\mu_3)(\rho(s^{-1}))\gamma_\rho(\mu_2, \mu_3)^{-1} = \mathbf{ad}\ c_\rho^{-1}(\mu_2)(\rho(s^{-1}))$$

and that $\gamma_\rho(\mu_1, \mu_3) = \gamma_\rho(\mu_1, \mu_2)\gamma_\rho(\mu_2, \mu_3)$, one deduces with little effort that

$$\gamma_\rho(\mu_1, \mu_3) = st\ \mathbf{ad}\ c_\rho(\mu_3)(\rho(st)^{-1}).$$

Transitivity established, we return to the original notation. If $\bar{\mu}$ is conjugate to μ under $G(\boldsymbol{Q})$, say $\bar{\mu} = \mathbf{ad}\, x \circ \mu$ then

$$\gamma_\rho = x\ \mathbf{ad}\ c_\rho^{-1}(x^{-1}) = x\ \mathbf{ad}\ c_\rho^{-1}(\rho(x^{-1})),$$

and certainly bounds. In general $\bar{\mu}$ and μ are not conjugate under $G(\boldsymbol{Q})$, but they are conjugate under $G(\boldsymbol{R})$. Since $G(\boldsymbol{Q})$ is dense in $G(\boldsymbol{R})$ we may take advantage of the transitivity and assume that they are conjugate under $G_{\mathrm{der}}(\boldsymbol{R})$.

It is now that the assumption that G_{der} is simply connected intervenes. If we are careful in our choice of $\tilde{b}(\tau, \mu)$ and $\tilde{b}(\tau, \bar{\mu})$, defining them by liftings of $c_\sigma(w)$ to I_L, then $B(\tau)$ and the γ_ρ will lie in $G_{\mathrm{der}}^{\tau,\mu}$. Moreover the cocycle $\{\gamma_\rho\}$ in $G_{\mathrm{der}}^{\tau,\mu}(L)$ certainly bounds at every finite place. Since we are applying Hasse's principle, we need only verify that it bounds at infinity as well.

One begins with a calculation similar to the one made while studying the cocycle $\{c_\rho(\tau)\}$. If $\rho \in \mathrm{Gal}(L/\boldsymbol{Q})$, set

$$e_\rho(\tau, \mu) = \prod_{\eta \in \mathfrak{f}} a_{\rho\eta, \iota}^{\rho\eta(\tau^{-1}-1)\mu}$$

and define $e_\rho(\tau, \bar{\mu})$ in a similar fashion. The projection of $\{\gamma_\rho\}$ on $G^{\tau,\mu}(L_\infty)$ is cohomologous to $e_\rho(\tau, \bar{\mu})e_\rho(\tau, \mu)^{-1}$. If $\rho \in \mathrm{Gal}(L_v/\boldsymbol{Q}_v)$ the projection of $e_\rho(\tau, \mu)$ on $G^{\tau,\mu}(L_v)$ is

$$f_\rho(\tau, \mu) = \prod_{\sigma \in \mathrm{Gal}(L_v/\boldsymbol{Q}_v)} a_{\rho,\sigma}^{\rho\sigma(\tau^{-1}-1)\mu}.$$

Thus all we need do is show that $f_\rho(\tau, \bar{\mu})f_\rho(\tau, \mu)^{-1}$ bounds in $G^{\tau,\mu}(L_v)$. Recall that the cocycle defining $G^{\tau,\mu}(L_v)$ is

$$h_\rho = \prod_{\sigma \in \mathrm{Gal}(L_v/\boldsymbol{Q}_v)} a_{\rho,\sigma}^{\rho\sigma(\tau^{-1}-1)\mu_{\mathrm{ad}}}.$$

For brevity, we write $\{\prod_\sigma a_{\rho,\sigma}^{\rho\sigma\nu}\} = \{\alpha_\rho(\nu)\}$. If $x \in G(\boldsymbol{R})$ we write $x(\mu) = \mathbf{ad}\, x \circ \mu$. If $x(\bar{\mu}) = \mu$ then

$$xf_\rho(\tau, \bar{\mu})f_\rho(\tau, \mu)^{-1} \operatorname{ad} h_\rho(\rho(x^{-1})) = \alpha_\rho((x\tau^{-1}x^{-1} - \tau^{-1})\mu),$$

because $\rho(x) = x$ and

$$f_\rho(\tau, \mu)^{-1} \operatorname{ad} h_\rho(x^{-1}) = x^{-1} f_\rho(\tau, \mu)^{-1}.$$

If w lies in the normalizer of T_{der} in $G_{\mathrm{der}}(L_v)$ then

$$w \operatorname{ad} h_\rho(\rho(w^{-1})) = w\rho(w^{-1})\alpha_\rho((w-1)(\tau^{-1}-1)\mu). \tag{6.2}$$

However $\{w\rho(w^{-1})\}$ is a cohomology class in $T_{\mathrm{der}}(L_v)$ or $T_{\mathrm{der}}^{\tau,\mu}(L_v)$, the two groups being equal, and it is shown in **[45]** that it is equivalent to $\{\alpha_\rho((w-1)\mu)\}$. Thus the class of (6.2) in $G_{\mathrm{der}}^{\tau,\mu}(L_v)$ is $\{\alpha_\rho((w-1)\tau^{-1}\mu)\}$. Since we may take w so that its action on the weights of T is the same as that of $x\tau^{-1}x^{-1}\tau$, the verification is complete.

There is another fact that we should verify while this proof is fresh in our minds. Suppose there is an ω in the Weyl group such that

$$\omega\mu = \tau^{-1}\mu. \tag{6.3}$$

Then $(1 + \iota)(\tau^{-1} - 1)\mu = 0$ and $\{c_\rho^{-1}(\tau)\} = \{c_\rho^{-1}(\tau, \mu)\}$ is defined. It bounds in $G(L)$. Once again it is enough to verify this when G_{der} is simply connected, although this time the modifications made to arrive at a simply-connected derived group would be different. It is no longer important that $P \to M$ be surjective, or that its kernel be free, but there must be a $\nu \in P$ which maps to μ and is fixed by $\mathrm{Gal}(\bar{\boldsymbol{Q}}/E)$.

The set of all τ which satisfy (6.3) for at least one ω form a group. Let $E = E(G, h) = E(G, \mathfrak{H})$ be its fixed field. Then $E \subseteq L$. In order to apply Hasse's principle we have to arrange that $\{c_\rho(\tau)\}$ lies in $G_{\mathrm{der}}(L)$ and that it obviously bounds in $G_{\mathrm{der}}(\boldsymbol{A}_f(L))$. Since $c_\rho(\tau)$ only depends on the image of τ in $\mathrm{Gal}(L/\boldsymbol{Q})$, we may calculate it when $w = w_\tau$. If $\mathfrak{S}$ is a set of coset representatives for $\mathrm{Gal}(L/\boldsymbol{Q})/\mathrm{Gal}(L/E)$, then

$$b_0(\tau) = \prod_{\nu \in \mathfrak{S}} \prod_{\sigma \in \mathrm{Gal}(L/E)} a_{\nu\sigma,\tau}^{\nu\sigma\mu}.$$

We write $a_{\nu\sigma,\tau} = \nu(a_{\sigma,\tau})a_{\nu,\sigma\tau}\, a_{\nu,\sigma}^{-1}$ and $\nu\sigma\mu = \nu\mu + \nu(\sigma - 1)\mu$. Now for $\sigma \in \mathrm{Gal}(L/E)$, $\nu(\sigma-1)\mu$ is a weight of the derived group and $\prod_\nu \prod_\sigma a_{\nu\sigma,\tau}^{\nu(\sigma-1)\mu}$ may be lifted to $T_{\mathrm{der}}(A(L))$. On the other hand $\prod_\nu \prod_\sigma a_{\nu,\sigma\tau}^{\nu\mu}\, a_{\nu,\sigma}^{-\nu\mu} = 1$. If $a = \prod_\sigma a_{\sigma,\tau}$ then a lies in C_E and lifts to a' in I_E. Moreover $\prod_\nu \nu(a)^{\nu\mu}$ lifts to $\prod_\nu \nu(a')^{\nu\mu}$ which lies in $T(A)$. If $\tilde{a}_{\nu\sigma,\tau}$ is a lifting of $a_{\nu\sigma,\tau}$ to I_L then we so choose $c_\rho(\tau)$ that

$$c_\rho(\tau) \equiv \left\{\prod_{\nu,\sigma} \rho(\tilde{a}_{\nu\sigma,\tau})^{-\rho\nu(\sigma-1)\mu}\right\}\left\{\prod_{\nu,\sigma} \tilde{a}_{\nu\sigma,\tau}^{\nu(\sigma-1)\mu}\right\} \tag{6.4}$$

modulo $T(L_\infty)$.

It remains to verify that the $\{c_\rho^{-1}(\tau)\}$ so defined bounds at the infinite place. The projection of the right side of (6.4) on $C_L \otimes X_*(T_{\mathrm{der}})$ is

$$\prod_{\nu,\sigma} a_{\rho,\nu\sigma}^{\rho\nu\sigma(\tau^{-1}-1)\mu} = \prod_{\sigma \in \mathrm{Gal}(L/\boldsymbol{Q})} a_{\rho,\sigma}^{\rho\sigma(\tau^{-1}-1)\mu}.$$

Thus, restricting $\{c_\rho^{-1}(\tau)\}$ to $\mathrm{Gal}(L/\boldsymbol{Q}_v)$ and projecting on $T_{\mathrm{der}}(L_v)$, we obtain a class cohomologous to $\{\alpha_\rho((\tau^{-1}-1)\mu)\} = \{\alpha_\rho((\omega-1)\mu)\}$. We have observed already that it is shown in **[45]** that the right side bounds in $G_{\mathrm{der}}(L_v)$.

Although there is one more consequence to be derived from (6.3), there are some things that must first be said about the general (T, h), or (T, μ).

We drop the assumption (6.3) for a while, and return to it later. We have associated to the pair (T, μ) and τ a twisted form $G^{\tau,\mu}$ of G. The twisting of T in G is trivial, and $T^\tau = T$ is a Cartan subgroup of $G^{\tau,\mu}$. Let μ^τ be $\tau^{-1}\mu$. There is a unique homomorphism $h^\tau: \mathscr{R} \to T^\tau$ whose restriction on the first factor is μ^τ and which is defined over $\boldsymbol{R}$.

The pair $(G^{\tau,\mu}, h^\tau)$ defines a Shimura variety.

The roots $\{\gamma\}$ of T in G are the same as the roots of T^τ in $G^{\tau,\mu}$. However the classification into compact and noncompact differs for the two pairs. The root γ of T in G is compact or noncompact according as $(-1)^{\langle\gamma,\mu\rangle}$ is 1 or -1. In order for $(G^{\tau,\mu}, h^\tau)$ to define a Shimura variety, γ must be compact or noncompact as a root of T^τ according as $(-1)^{\langle\gamma,\tau^{-1}\mu\rangle}$ is 1 or -1. However the ideas used in the proof of Lemma A.8 of **[34]** show that the type of γ is changed on passing from T, G to T^τ, $G^{\tau,\mu}$ if and only if $(-1)^{\langle\gamma,\tau^{-1}\mu-\mu\rangle} = -1$.

We are now almost ready to formulate a conjecture about the conjugation of Shimura varieties. After discussing the conjecture and its consequences, we shall show how it can be heuristically justified in terms of the Taniyama group and motives.

Recall that $\mathrm{Sh}_K(\boldsymbol{C}) = G(\boldsymbol{Q})\backslash\mathfrak{H} \times G(A_f)/K$. Thus if $g \in G(A_f)$ and $K_1 = g^{-1}Kg$, then right multiplication by g defines a morphism

$$\mathfrak{J}(g): \mathrm{Sh}_K(\boldsymbol{C}) \longrightarrow \mathrm{Sh}_{K_1}(\boldsymbol{C}).$$

It is algebraic and is called a Hecke correspondence. If τ is an automorphism of $\boldsymbol{C}$, we set $\mathrm{Sh}_K^\tau(G,h) = \mathrm{Sh}_K^\tau = \mathrm{Sh}_K \otimes_{\tau^{-1}} \boldsymbol{C}$, the Shimura varieties being at the moment only defined over $\boldsymbol{C}$. Then $\mathfrak{J}^\tau(g): \mathrm{Sh}_K^\tau \to \mathrm{Sh}_{K_1}^\tau$.

If $G = T$ is a torus and $K = U$ then Sh_U is 0-dimensional. Let τ also denote the element of $\mathrm{Gal}(\bar{\boldsymbol{Q}}/\boldsymbol{Q})$ defined by τ. If we set $U^\tau = U$ then

$$\mathrm{Sh}_U(\boldsymbol{C}) = T(\boldsymbol{A}_f)/U = T^\tau(\boldsymbol{A}_f)/U^\tau = \mathrm{Sh}_{U^\tau}.$$

This gives us an isomorphism $\mathrm{Sh}_U = \mathrm{Sh}_U(T, h) \to \mathrm{Sh}_{U^\tau} = \mathrm{Sh}_{U^\tau}(T^\tau, h^\tau)$. In addition there is the natural map τ from complex points of $\mathrm{Sh}_U^\tau(T, h)$ to complex points of $\mathrm{Sh}_U(T, h)$. Define $\varphi_\tau = \varphi_\tau(U, T, h)$ by the commutativity of the diagram

$$\begin{array}{ccc} & & \mathrm{Sh}_U(T, h) \\ & \nearrow^{\tau} & \big\downarrow \\ \mathrm{Sh}_U^\tau(T, h) & & \\ & \searrow_{\varphi_\tau} & \\ & & \mathrm{Sh}_{U^\tau}(T^\tau, h^\tau) \end{array}$$

In general $G^{\tau,\mu}$ and G are different. But $G^{\tau,\mu}$ is defined by the cocycle $\{c_\rho^{-1}(\tau, \mu_{\mathrm{ad}})\}$ and in $G_{\mathrm{ad}}(\boldsymbol{A}_f(L))$

$$c_\rho^{-1}(\tau, \mu_{\mathrm{ad}}) = b(\tau, \mu_{\mathrm{ad}})^{-1}\, \rho(b(\tau, \mu_{\mathrm{ad}})).$$

Thus

$$g \longrightarrow g^\tau = \mathbf{ad}\; b(\tau, \mu_{\mathrm{ad}})^{-1}(g)$$

defines an isomorphism of $G(\boldsymbol{A}_f)$ with $G^{\tau,\mu}(\boldsymbol{A}_f)$.

Conjecture. There is a family of biregular maps $\varphi_\tau = \varphi_\tau(K, G, h)$, $K \subseteq G(\boldsymbol{A}_f)$ *defined over* $\boldsymbol{C}$, *taking* $\mathrm{Sh}_K^\tau(G, h)$ *to* $\mathrm{Sh}_{K^\tau}(G^{\tau,\mu}, h^\tau)$, *and rendering the following diagrams commutative*:

(a)

$$\begin{array}{ccc} \mathrm{Sh}_U(T, h) & \hookrightarrow & \mathrm{Sh}_K(G, h) \\ \tau\big\uparrow & & \big\uparrow\tau \\ \mathrm{Sh}_U^\tau(T, h) & \hookrightarrow & \mathrm{Sh}_K^\tau(G, h) \\ \varphi_\tau\big\downarrow & & \big\downarrow\varphi_\tau \\ \mathrm{Sh}_{U^\tau}(T^\tau, h^\tau) & \hookrightarrow & \mathrm{Sh}_{K^\tau}(G^{\tau,\mu}, h^\tau) \end{array}$$

Here $U = T(\boldsymbol{A}_f) \cap K$ *and the* φ_τ *in the left column is* $\varphi_\tau(U, T, h)$.

(b)

$$\begin{array}{ccc} \mathrm{Sh}_K^\tau(G, H) & \xrightarrow{\ \mathfrak{F}^\tau(g)\ } & \mathrm{Sh}_{K_1}^\tau(G, h) \\ \varphi_\tau\big\downarrow & & \big\downarrow\varphi_\tau \\ \mathrm{Sh}_{K^\tau}(G^{\tau,\mu}, h^\tau) & \xrightarrow[\ \mathfrak{F}(g^\tau)\ ]{} & \mathrm{Sh}_{K_1^\tau}(G^{\tau,\mu}, h^\tau) \end{array}$$

The conjecture in this form refers to a specific T and a specific h factoring through T. If we choose another pair $(\bar{T}, \bar{h})$ we obtain another group $G^{\tau,\bar{\mu}}$ and another collection $\{\bar{\varphi}_\tau = \bar{\varphi}_\tau(K; G, \bar{h})\}$. The conjecture is inadequate as it stands, and must be supplemented by a statement relating φ_τ and $\bar{\varphi}_\tau$. Observe that there can be at most one family $\{\varphi_\tau\}$ satisfying the conditions (a) and (b).

We have already associated to the two pairs (T, h) and $(\bar{T}, \bar{h})$ a cocycle $\{\gamma_\rho\}$ which bounds in $G^{\tau,\mu}(L)$. Let $\gamma_\rho = u\ \mathrm{ad}\, c_\rho^{-1}(\rho(u^{-1}))$. Then $g \to ugu^{-1}$ defines an isomorphism of $G^{\tau,\mu}(\boldsymbol{Q})$ with $G^{\tau,\bar{\mu}}(\boldsymbol{Q})$ or of $G^{\tau,\mu}(\boldsymbol{A}_f)$ with $G^{\tau,\bar{\mu}}(\boldsymbol{A}_f)$. Set ${}^uK^\tau = uK^\tau u^{-1}$.

SECOND LEMMA OF COMPARISON. *The homomorphism* $\mathrm{ad}\ u \circ h^\tau$ *is conjugate under* $G^{\tau,\bar{\mu}}(\boldsymbol{R})$ *to* $\bar{h}^\tau$.

If this statement is true when u is replaced by a v in $G(L_v) = G(C)$ which also trivializes $\{\gamma_\rho\}$ restricted to $\mathrm{Gal}(L_v/Q_v)$ then it is true for u. An examination of the proof of the second property of $\{\gamma_\rho\}$ shows that we can take v to lie in $x^{-1}wT(L_v)$, x and w being as in that proof. Since $x^{-1}w(\tau^{-1}\mu) = \tau^{-1}\bar{\mu}$, we have $\mathrm{ad}\, v \circ h^\tau = \bar{h}^\tau$.

We infer that ad u carries $\mathfrak{H}^\tau$ to $\bar{\mathfrak{H}}^\tau$ and hence defines a bijection

$$G^{\tau,\mu}(\boldsymbol{Q})\backslash\mathfrak{H}^\tau \times G^{\tau,\mu}(\boldsymbol{A}_f)/K^\tau \longrightarrow G^{\tau,\bar{\mu}}(\boldsymbol{Q})\backslash\bar{\mathfrak{H}}^\tau \times G^{\tau,\bar{\mu}}(\boldsymbol{A}_f)/{}^uK^\tau$$

and an isomorphism

$$\psi\colon \mathrm{Sh}_{K^\tau}(G^{\tau,\mu}, h^\tau) \longrightarrow \mathrm{Sh}_{{}^{\mu}K^\tau}(G^{\tau,\bar{\mu}}, \bar{h}^\tau).$$

Since u and $B(\tau)$ both trivialize $\{\gamma_\rho\}$ in $G^{\tau,\mu}(\boldsymbol{A}_f(L))$ there is a y in $G^{\tau,\mu}(\boldsymbol{A}_f)$ with $u = B(\tau)y$. Let φ be the composite $\psi \circ \mathfrak{F}(y)$.

Supplement to the conjecture. The diagrams

$$\begin{array}{ccc} & \mathrm{Sh}_K^\tau(G, h) & \\ {\scriptstyle\varphi_\tau}\swarrow & & \searrow{\scriptstyle\bar{\varphi}_\tau} \\ \mathrm{Sh}_{K^\tau}(G^{\tau,\mu}, h^\tau) & \xrightarrow[\varphi]{\qquad} & \mathrm{Sh}_{K^\tau}(G^{\tau,\bar{\mu}}, \bar{h}^\tau) \end{array}$$

are commutative.

The conjecture as it stands certainly implies that the conjugate of a Shimura variety is again a Shimura variety. Together with its supplement, it implies the usual form of Shimura's conjecture **[10]**. To verify this one applies the Weil criterion **[49]** for descent of the field of definition. For this we need families of isomorphisms $f_\rho\colon \mathrm{Sh}_K^\rho(G, h) \longrightarrow \mathrm{Sh}_K(G, h)$ defined for automorphisms ρ of C over $E(G, h)$ and satisfying $f_{\sigma\rho} = f_\rho f_\sigma^\rho$.

Choose a Cartan subgroup T and an h which factors through it. We know that when τ fixes $E(G, h)$ the cocycle $\{c_\rho^{-1}(\tau, \mu)\}$ is defined and bounds in $G(L)$. Let $c_\rho^{-1}(\tau, \mu) = v\rho(v^{-1})$. Then $g \to vgv^{-1}$ is an isomorphism of $G(\boldsymbol{Q})$ with $G^{\tau,\mu}(\boldsymbol{Q})$. Methods which we have already used show easily that

The composite **ad** $v \circ h$ *is conjugate under* $G^{\tau,\mu}(\boldsymbol{R})$ *to* h^τ.

Consequently $\mathrm{ad}\, v$ defines an isomorphism $\mathfrak{H} \to \mathfrak{H}^\tau$ and then, as before, we obtain from

$$G(\boldsymbol{Q})\backslash\mathfrak{H} \times G(\boldsymbol{A}_f)/K \longrightarrow G^{\tau,\mu}(\boldsymbol{Q})\backslash\mathfrak{H}^\tau \times G^{\tau,\mu}(\boldsymbol{A}_f)/{}^vK$$

an isomorphism

$$S_K(G, h) \longrightarrow S_{vK}(G^{\tau,\mu}, h^\tau).$$

On the other hand $zv = b(\tau, \mu)^{-1}$ with $z \in G^{\tau,\mu}(\boldsymbol{A}_f)$. We define f_τ by the commutativity of

$$\begin{array}{ccc} \mathrm{Sh}_K^\tau(G, h) & \xrightarrow{\ f_\tau\ } & S_K(G, h) \\ {\scriptstyle\varphi_\tau}\downarrow & & \downarrow \\ \mathrm{Sh}_{K^\tau}(G^{\tau,\mu}, h^\tau) & \xrightarrow[\mathfrak{F}(z)]{\qquad} & \mathrm{Sh}_{vK}(G^{\tau,\mu}, h^\tau) \end{array}$$

I omit the calculations, lengthy but routine, by which it is deduced from the conjecture and its supplement that f_τ does not depend on the choice of T and h and that the cocycle condition $f_{\sigma\rho} = f_\rho f_\sigma^\rho$ is satisfied.

Up to now the S_K have been taken as varieties over $\boldsymbol{C}$, but by the criterion for descent we may now define them over $E(G, h)$ in such a way that the f_τ are simply the identity maps. It has to be verified that the models thus obtained are canonical, but the construction is clearly such that only the case that G is a torus T need be considered. Let a be the transfer of $w = w_\tau$ to C_E and a' a lifting of a to I_E. The proof that in this case $c_\rho(\tau, \mu)$ is trivial shows in fact that we may take it to be 1 and $b = b(\tau, \mu)$ to be $\prod_{\mathrm{Gal}(L/\boldsymbol{Q})/\mathrm{Gal}(L/E)} \nu(a')^{\nu\mu}$. We take v to be 1 and z to be $b^{-1} = b(\tau, \mu)^{-1}$. The composition

$$\mathrm{Sh}_U(T, h) \xrightarrow{\tau^{-1}} \mathrm{Sh}_U(T, h) \xrightarrow{\mathfrak{J}(b)} \mathrm{Sh}_U(T, h)$$

is then the identity for all τ fixing $E(T, h)$, and this is just the condition that $\mathrm{Sh}(T, h)$ be the canonical model.

Suppose $E(G, h) \subseteq \boldsymbol{R}$. If we take the canonical model for Sh_K then the complex conjugation defines an involution θ of the complex manifold $\mathrm{Sh}_K(\boldsymbol{C})$. It is necessary to have a concrete description of this involution in terms of the representation

$$\mathrm{Sh}_K(\boldsymbol{C}) = G(\boldsymbol{Q})\backslash\mathfrak{H} \times G(\boldsymbol{A}_f)/K,$$

and one purpose of the conjecture is to provide it.

Choose some special point (T, h). If $E(G, h) \subseteq \boldsymbol{R}$ then we may define $b(\iota, \mu)$ and $\{c_\rho(\iota, \mu)\}$. However the condition (c) on the coset representatives w_σ used to define $b(\iota, \mu)$ implies that $b(\iota, \mu) = 1$. We may also take $c_\rho(\iota, \mu)$ to be 1, and then v may be taken to be 1 as well. It follows that h and h^ι are conjugate in $G(\boldsymbol{R})$. Since $K_{h^\iota} = K_h$, η: ad $g \circ h^\iota \to$ ad $g \circ h$ is a well-defined map of $\mathfrak{H}$ to itself.

Consequence of the conjecture. The involution θ may be realized concretely as the mapping $(h, g) \to (\eta(h), g)$ of $G(\boldsymbol{Q})\backslash\mathfrak{H} \times G(\boldsymbol{A}_f)/K$ to itself.

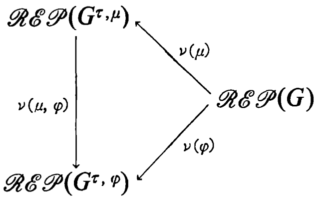
Since we are comparing two continuous mappings which commute with the $\mathfrak{J}(g)$, $g \in G(\boldsymbol{A}_f)$, it is enough to see that they coincide on the point in $S_U(T^\iota, h^\iota)$ represented by $(h^\iota, 1)$. Since f_ι is the identity and $v = z = 1$, φ_ι takes this point to the point in $\mathrm{Sh}_{K^\iota}(G^{\iota,\mu}, h^\iota) = \mathrm{Sh}_K(G, h)$ represented again by $(h^\iota, 1)$. It follows immediately from condition (a) of the conjecture that ι applied to the point represented by $(h^\iota, 1)$ is $(h, 1)$.

For each T and μ let $\nu(\mu)$ be the fibre functor from $\mathscr{REP}(G)$ to $\mathscr{REP}(G^{\tau,\mu})$ which takes $(\xi', V(\xi'))$ to $(\xi, V(\xi))$, with $\xi = \xi'$, and with $V(\xi)$ being the space obtained from $V(\xi')$ by changing the Galois action to σ: $x \to \xi'(c_\sigma(\tau, \mu)^{-1})\sigma(x)$. If $(\bar{T}, \bar{h})$ is another special point, and if $\nu(\mu, \bar{\mu})$ is defined by the diagram

$$\begin{array}{ccc}
\mathscr{REP}(G^{\tau,\mu}) & \xleftarrow{\ \nu(\mu)\ } & \\
\Big\downarrow{\scriptstyle \nu(\mu,\bar{\mu})} & & \mathscr{REP}(G) \\
\mathscr{REP}(G^{\tau,\bar{\mu}}) & \xleftarrow{\ \nu(\bar{\mu})\ } &
\end{array}$$

then, according to the first lemma of comparison, the two fibre functors $\omega_{\mathrm{Rep}(G^{\tau,\mu})}$ and $\omega_{\mathrm{Rep}(G^{\tau,\bar\mu})}\circ\nu(\mu,\bar\mu)$ are isomorphic.

On the other hand, we associated, at the end of the fourth section, to each pair (φ', g) a group $G^{\tau,\varphi}$ and a pair (φ', g'). The groups G and $G^{\tau,\varphi}$ are associated to the same tannakian category, and there is thus an equivalence of categories $\nu(\varphi)$: $\mathcal{REP}(G)\to\mathcal{REP}(G^{\tau,\varphi})$, determined up to isomorphism. If φ factors through the Serre group, then it can be factored through a Cartan subgroup $\bar T$ of G and defines a coweight $\bar\mu$ of T. The hypothetical properties of the Taniyama group imply that $G^{\tau,\varphi}$ may be taken to be $G^{\tau,\bar\mu}$ with $\nu(\varphi)$ being $\nu(\bar\mu)$. Thus we have a diagram



which is commutative up to isomorphism of functors. Moreover the two fibre functors $\omega_{\mathrm{Rep}(G^{\tau,\mu})}$ and $\omega_{\mathrm{Rep}(G^{\tau,\varphi})}\circ\nu(\mu,\varphi)$ are isomorphic. If we choose an isomorphism between them, we obtain [**40**, II.3.3] an isomorphism over $\boldsymbol{Q}$, $G^{\tau,\varphi}\to G^{\tau,\mu}$. Composing with φ' we obtain a homomorphism $\varphi''\colon G_{\mathrm{Mot}(C)}\longrightarrow G^{\tau,\mu}$. Since there are so many special points, it is not unreasonable to hope that $\nu(\mu,\varphi)$ exists for all φ, and that $\omega_{\mathrm{Rep}(G^{\tau,\mu})}$ and $\omega_{\mathrm{Rep}(G^{\tau,\varphi})}\circ\nu(\mu,\varphi)$ are always isomorphic.

The second lemma of comparison in conjunction with the hypothetical properties of the Taniyama group implies that the composition of φ'' with the canonical homomorphism $\mathcal{R}\to G_{\mathrm{Mot}(C)}$ lies in $\mathfrak{H}^\tau$ when φ factors through the Serre group, and once again we may surmise or hope that this will be so in general.

If φ_τ is the biregular map appearing in the conjecture then the composition $\varphi_\tau\circ\tau^{-1}$ defines a map from the set of complex points on $\mathrm{Sh}_K(G,h)$ to the set of complex points on $\mathrm{Sh}_{K^\tau}(G^{\tau,\mu},h^\tau)$. The idea is that $\varphi_\tau\circ\tau^{-1}$ will take (φ,g) to a pair (φ'',g''), by a process which can be defined within the moduli problem. We have just seen how to obtain φ'', at least at the hypothetical level at which we are working.

To obtain g'' we observe that we have two homomorphisms

$$\omega^{A_f}_{\mathrm{Rep}(G^{\tau,\varphi})}\circ\nu(\mu,\varphi)\longrightarrow\omega^{A_f}_{\mathrm{Rep}(G^{\tau,\mu})}. \tag{6.5}$$

One is obtained from the chosen isomorphism over $\boldsymbol{Q}$ by extending scalars to $\boldsymbol{A}_f$. The other is obtained by a lengthy composition. The g from which we start provides an isomorphism $\omega^{A_f}_{\mathrm{Mot}(C)}\circ\eta\to\omega^{A_f}_{\mathrm{Rep}(G)}$. The canonical isomorphism $\omega^{A_f}_{\mathrm{Mot}(C)}\circ\eta(\tau)\to\omega^{A_f}_{\mathrm{Mot}(C)}$ can be composed with η. Finally the definitions provide an isomorphism from $\omega_{\mathrm{Rep}(G^{\tau,\varphi})}\circ\nu(\varphi)$ to $\omega^{A_f}_{\mathrm{Mot}(C)}\circ\eta(\tau)\circ\eta$. Putting these all together, we obtain an isomorphism

$$\omega^{A_f}_{\mathrm{Rep}(G^{\tau,\varphi})}\circ\nu(\varphi)\longrightarrow\omega^{A_f}_{\mathrm{Rep}(G)}. \tag{6.6}$$

However we also have an isomorphism

$$\omega^{A_f}_{\mathrm{Rep}(G)}\longrightarrow\omega^{A_f}_{\mathrm{Rep}(G^{\tau,\mu})}\circ\nu(\mu). \tag{6.7}$$

It is given by the isomorphism $x\to\xi'(b(\tau,\mu)^{-1})x$ of $V(\xi')$ with $V(\xi)$. Composing

(6.6) and (6.7) we obtain a second isomorphism between the two fibre functors figuring in (6.5). According to general principles, it can be obtained by composing the first with an element $(g'')^{-1}$ in $G^{\tau,\mu}(A_f)$ **[40**, §II].

If one can establish, in some way or another, that the map $\psi_\tau\colon (\varphi, g) \to (\varphi'', g'')$ is really defined, then to prove the conjecture and its supplement one will only need to verify that the composite $\psi_\tau \circ \tau$ is complex analytic. However our purpose here has been to see how the wheels mesh, not to find the mainspring.

7. Continuous cohomology. If $G = G_0$ then, according to the principles of the fourth section, we should be able to attach to each point of $\mathrm{Sh}_K(\boldsymbol{C})$ an equivalence class of pairs (φ, g). Here φ is a homomorphism from $G_{\mathrm{Mot}(\boldsymbol{C})}$ to G defined over $\boldsymbol{Q}$ and if η is the associated θ-functor $\mathscr{REP}(G) \to \mathscr{MOT}(\boldsymbol{C})$, then g defines an isomorphism

$$\omega^{A_f}_{\mathrm{Mot}(\boldsymbol{C})} \circ \eta \to \omega^{A_f}_{\mathrm{Rep}(G)}.$$

In general we have mappings $\mathrm{Sh}_K(G, h) \to \mathrm{Sh}_{K_0}(G_0, h_0)$, and by pulling back we can associate to each point of $\mathrm{Sh}_K(\boldsymbol{C})$ a pair (φ, g) where g is again in $G(\boldsymbol{A}_f)$, but φ now takes $G_{\mathrm{Mot}(\boldsymbol{C})}$ to G_0.

If $(\xi, V(\xi))$ is a representation of G_0 over $\boldsymbol{Q}$ or, what is the same, a representation of G factoring through G_0, then to each point s of $\mathrm{Sh}_K(\boldsymbol{C})$ we may associate the motive $M(s, \xi)$ defined by $\xi \circ \phi$, together with the isomorphism

$$\omega^{A_f}_{\mathrm{Mot}(\boldsymbol{C})}(M(s, \xi)) \simeq V(\xi)_{\boldsymbol{A}_f}$$

defined by g. The variety $\mathrm{Sh}_K(G, h)$ should be defined over $E = E(G, h)$ and so should this family of motives. Suppose now that the variety $\mathrm{Sh}_K(G, h)$ is proper. In the best of all possible worlds, one might be able to form the cohomology groups $M^i, 0 \leqq i \leqq 2 \dim \mathrm{Sh}_K$, of the family $M(\cdot, \xi)$, which would again be motives, now over E, and thus correspond to representations σ^i of $G_{\mathrm{Mot}(E)}$. If the formalism of the second section were established, one could compose σ^i with ρ_F to obtain representations $\rho^i = \sigma^i \otimes \rho_F$ of $G_{\Pi(E)}$. Then the basic problem would simply be to describe the image $\rho^i(G_{\Pi(E)})$.

But we do not have all this formalism, and one of the principal reasons for studying Shimura varieties is the hope that by grappling with the specific arithmetic problems they pose we will obtain an insight that will help us with its construction. Informed by the general principles and hypotheses we are attempting to establish, we can try to formulate questions that are, at least in part, tractable and which if answered will confirm or, if the answer is other than expected, perhaps refute these principles.

In the present context we can first observe that even if the M^i remained undefined, the zeta-function $Z(z, M^i)$ can be defined directly in terms of the data at our disposal. It is a product over the places of E, $\prod_v Z_v(z, M^i)$. At a nonarchimedean place it can be defined by the l-adic representation of the Galois group on the ith cohomology group of the l-adic sheaf $F_\xi(\boldsymbol{Q}_l)$ associated to ξ, as in the papers **[30]** and **[34]**. Since our principal concern now is with the factors for the archimedean places, we need not enter into details.

The field E is contained in $\boldsymbol{C}$, and the archimedean places are obtained by applying automorphisms of $\boldsymbol{C}$, or of $\bar{\boldsymbol{Q}}$, to E. We first define the factor $Z_v(z, M^i)$ for the place v given by $E \subseteq \boldsymbol{C}$. We have seen that we can associate to ξ a locally constant sheaf $F_\xi(\boldsymbol{Q})$ over $\mathrm{Sh}_K(\boldsymbol{C})$. Moreover, we have an analytic family of polarized Hodge structures on $F_\xi(\boldsymbol{C}) = F_\xi(\boldsymbol{Q}) \otimes \boldsymbol{C}$. By a construction of Deligne **[12]**, **[50]** this defines a Hodge structure on the cohomology groups $H^i = H^i(\mathrm{Sh}_K(\boldsymbol{C}), F_\xi(\boldsymbol{C}))$.

In accordance with the ideas of Serre **[42]** the factor $Z_v(z, M^i)$ will be defined as $L(s, \rho^i)$ where ρ^i is a representation of the Weil group $W_{\boldsymbol{C}/E_v}$ on H^i. The Hodge structure on H^i defines a representation of $\mathscr{R}(\boldsymbol{R})$. Since $\boldsymbol{C}^\times = \mathscr{R}(\boldsymbol{R})$, this can be used to define ρ^i on $\boldsymbol{C}^\times \subseteq W_{\boldsymbol{C}/E_v}$. If E_v is equal to $\boldsymbol{C}$, this defines ρ^i completely. If $E_v = \boldsymbol{R}$ then to define ρ^i completely we also have to define $\rho^i(w)$, if w is the element of the Weil group which projects to ι and has square -1.

Since Sh_K is defined over E, ι also defines an involution on $\mathrm{Sh}_K(\boldsymbol{C})$ which we denoted by θ. What we need is a map of order two

$$\phi\colon \theta^* F \longrightarrow F, \qquad F = F_\xi(\boldsymbol{C}),$$

such that on each fibre

$$\theta^* F_s^{p,q} = F_{\theta(s)}^{p,q} \longrightarrow F_s^{q,p}.$$

The associated map ι^* on cohomology takes $H^{p,q}$ to $H^{q,p}$ and we set $\rho^i(w)$ equal to $(-1)^p \iota^*$.

To define ϕ we have to assume that the consequence of the conjecture which was described in the previous section is valid. It can be proved directly in several cases. Then θ can be obtained by taking the map $(h, g) \to (\eta(h), g)$ and passing to the quotient. Given (h, g), the fibres at the image of (h, g) and $(\eta(h), g)$ may both be identified with $V(\xi)_{\boldsymbol{C}}$ and ϕ is simply the identity map.

If we replace the imbedding $E \subseteq \boldsymbol{C}$ by $\tau^{-1}\colon E \to \boldsymbol{C}$, τ being an automorphism, then the complex manifold $\mathrm{Sh}_K(\boldsymbol{C})$ is replaced by $\mathrm{Sh}_K^\tau(\boldsymbol{C})$, which we may identitfy by means of φ_τ with $\mathrm{Sh}_{K^\tau}(\boldsymbol{C})$, the manifold associated to $\mathrm{Sh}_{K^\tau}(G^{\tau,\mu}, h^\tau)$. Thus the factor of the zeta-function defined by the place associated to $\tau^{-1}\colon E = E(G, h) \to \boldsymbol{C}$ can be calculated by replacing G by $G^{\tau,\mu}$, h by h^τ, and E by $\tau^{-1}(E) = E(G^{\tau,\mu}, h)$ with the place defined by its inclusion in $\boldsymbol{C}$. The space $V(\xi)$ has also to be twisted by the cocycle $\{\xi(c_\sigma(\tau, \mu)^{-1})\}$.

The function $Z(z, M^i)$ defined, the immediate problem is to show that it can be expressed as a product of L-functions associated to automorphic representations

$$Z(z, M^i) = \prod_j L(z - a_j, \pi_j, r_j). \tag{7.1}$$

Here $a_j \in \boldsymbol{C}$ is a translation, π_j is an automorphic representation of some group H_j, and r_j is a representation of the L-group ${}^L H_j$. The first step is to decide which H_j, which π_j, and which r_j intervene in the product.

The first step is to use the theory of continuous cohomology to compute the cohomology groups of the sheaves $F_\xi(\boldsymbol{C})$ together with their Hodge structure, and thus to compute $Z_v(z, M^i)$, v being again defined by $E \subseteq \boldsymbol{C}$. Using this together with an analysis of the L-packets of automorphic representations of G **[44]**, one searches for an identity (7.1) which is at least valid when both sides are replaced by their factors at v. An example is discussed in detail in **[34]**. The identity found, it must be verified for the local factors at the other places v'. If v' is an archimedean place, then the theory of continuous cohomology will allow us to compute $Z_{v'}(z, M^i)$ in terms of the automorphic representations of a $G^{\tau,\mu}$, a group which differs from G by an inner twisting. To make the comparison it will be necessary to have established the principle of functoriality for the pair G and $G^{\tau,\mu}$, and to have understood in detail how it manifests itself. This is bound up with the study of L-packets and is primarily an analytic problem, for we expect that the trace formula will give us a good purchase on it **[23]**.

At the finite places the identity (7.1) is difficult to treat as it stands, and for reasons familiar from topology one replaces the left side by

$$Z(z) = \prod_i Z(z, M^i)^{(-1)^i} \tag{7.2}$$

modifying the right side accordingly. The right side is then analyzed by the trace formula, at least if there is no ramification or, at worst, a mild sort **[8]**, **[14]**, **[30]**. I do not see at the moment any general way of dealing with a truly nonabelian situation, although a rather curious method has been discovered by Deligne for treating the group GL(2) **[13]**.

If there is no ramification, the factor of (7.2) at a finite place can be analyzed by the fixed point formulae of l-adic cohomology. Apart from combinatorial difficulties **[28]** the critical factor is to have a reasonably explicit description of the set of geometric points on $\mathrm{Sh}_K(G, h)$ in $\bar{\kappa}_\mathfrak{p}$, the algebraic closure of the residue field at a prime $\mathfrak{p}$ of E, together with the action of the Frobenius on it **[32]**. This is an idea first applied by Ihara **[20]**, who has since intensively studied the structure of this set for Shimura curves **[21]**, **[22]**.

Not much has been done when there is ramification. The first thing is to analyze in reasonably simple cases the manner in which the variety reduces badly. Some interesting discoveries have been made for curves **[8]**, **[14]**, **[15]**, but higher dimensional varieties behave in a more complicated manner. However tools are available for studying their reduction, and it is time to begin.

None of these steps will be easy to carry out. The study of L-packets is in an embryonic stage, and even the combinatorial problems will demand considerable ingenuity in their solution **[27]**. There is still a great deal to be learned from the study of specific examples.

The theory of continuous cohomology is itself in its infancy, and my purpose in this section is to draw attention to some problems which arise in the study of Shimura varieties and which the mature theory should resolve.

I begin by introducing a representation r of the L-group ${}^LG$ which will play a fundamental role in the discussion. The group ${}^LG$ is a semidirect product ${}^LG^\circ \times \mathrm{Gal}(\bar{Q}/Q)$. If T is a Cartan subgroup of G over Q then there is an isomorphism of $X_*(T)$, the lattice of coweights of T, with $X^*({}^LT^\circ)$, the lattice of weights of ${}^LT^\circ$, defined up to an element of the Weyl group. In particular if (T, h) is a special point then μ defines an orbit Θ in $X^*({}^LT^\circ)$, and Θ is independent of (T, h). Let r° be the representation of ${}^LG^\circ$ whose set of extreme weights is Θ. The group $\mathrm{Gal}(\bar{Q}/Q)$ acts on the weights of ${}^LT^\circ$ and preserves the set of dominant weights. The group $\mathrm{Gal}(\bar{Q}/E)$ fixes the set Θ. Theus $\mathrm{Gal}(\bar{Q}/E)$ fixes the dominant element $\mu^\vee$ in Θ, and we may extend r° to ${}^LG^\circ \times \mathrm{Gal}(\bar{Q}/E)$ in such a way that $\mathrm{Gal}(\bar{Q}/E)$ acts trivially on the weight space of $\mu^\vee$. The extended representation will also be called r°, and we define r to be the induced representation

$$r = \mathrm{Ind}({}^LG, {}^LG^\circ \times \mathrm{Gal}(\bar{Q}/E), r^\circ).$$

Led d be the dimension of $\mathrm{Sh}_K(G, h)$. One expects that the function (7.2) will be equal to a product of functions

$$L(z - d/2, \pi, \rho). \tag{7.3}$$

Here π is an automorphic representation of one of the groups H attached to G in [33]. There is, in general, an imbedding $\varphi\colon {}^LH \hookrightarrow {}^LG$ and ρ is a subrepresentation of $r \circ \varphi$.

If w is any place of $\boldsymbol{Q}$, let r_w be the restriction of r to the L-group ${}^LG_w$, which equals ${}^LG^\circ \times \mathrm{Gal}(\bar{Q}_w/Q_w)$. Implicit in this notation is an imbedding $\bar{Q} \subseteq \bar{Q}_w$. Then the double cosets in $\mathrm{Gal}(\bar{Q}/E)\backslash\mathrm{Gal}(\bar{Q}/Q)/\mathrm{Gal}(\bar{Q}_w/Q_w)$ parametrize the places of E dividing w, and $r_w = \bigoplus_{v|w} r_v$, with

$$r_v = \mathrm{Ind}({}^LG_w, {}^LG^\circ \times \mathrm{Gal}(\bar{Q}_w/E_v), r_v^\circ)$$

and $r_v^\circ(\tau) = r^\circ(\sigma_v\tau\sigma_v^{-1})$ if σ_v is some element in the coset defining v. The representation ρ will also be a direct sum $\rho = \bigoplus_{v|w}\rho_v$, and the function (7.3) will be a product $\prod_w\prod_{v|w} L(z - d/2, \pi_w, \rho_v)$. The factor corresponding to the place v is $L(z - d/2, \pi_w, \rho_v)$.

Since we shall only be interested in the place v given by $E \subseteq \boldsymbol{C}$, we shall write r and ρ instead of r_v and ρ_v. Moreover we shall write an automorphic representation of $G(\boldsymbol{A})$ (or of $H(\boldsymbol{A})$) as $\pi \otimes \pi_f$, π being a representation of $G(\boldsymbol{R})$ and π_f of $G(\boldsymbol{A}_f)$.

Any irreducible representation π of $G(\boldsymbol{R})$ lies in some L-packet Π_φ where φ is a homomorphism from $W_{\boldsymbol{C}/\boldsymbol{R}}$ to ${}^LG$. If α is the character of $W_{\boldsymbol{C}/\boldsymbol{R}}$ obtained by composing $W_{\boldsymbol{C}/\boldsymbol{R}} \to \boldsymbol{R}$ with the absolute value, let $\psi_1(\pi) = \alpha^{-d/2} \otimes (r \circ \varphi)$. Then $L(s - d/2, \pi, r) = L(s, \psi_1(\pi))$.

On the other hand, suppose $\pi \otimes \pi_f$ is an automorphic representation of $G(\boldsymbol{A})$. Let it act on the subspace $U \otimes U_f$ of the space of automorphic forms. If $h \in \mathfrak{H}$ then, according to the principles of continuous cohomology [4], its contribution to the cohomology of $\mathrm{Sh}_K(\boldsymbol{C})$ with values in $F_\xi(\boldsymbol{C})$ in dimension i is

$$\mathrm{Hom}_{K_h}(\Lambda^i\mathfrak{g}/\mathfrak{k} \otimes \tilde{V}, U) \otimes U_f^K. \tag{7.4}$$

Here $\mathfrak{k}$ is the Lie algebra of K_h and $\tilde{V}$ the dual of $V(\xi)$. The space U_f^K is the space of vectors in U_f fixed by K.

The action of $z \in \boldsymbol{C}^\times = \mathscr{R}(\boldsymbol{R})$ defining the Hodge structure sends $\varphi \otimes u$ to $\varphi' \otimes u$ with

$$\varphi'(X \otimes \tilde{v}) = \varphi(\eta(z)X \otimes \xi(h(z^{-1}))\tilde{v}).$$

Here $X \to \eta(z)X$ is the action which multiplies the exterior product of p holomorphic and q antiholomorphic vectors by $z^{-p}\bar{z}^{-q}$. Thus $\eta(z) = \mathrm{ad}\, h(z^{-1}) \circ \eta(\bar{z})$ and

$$\varphi'(X \otimes \tilde{v}) = \pi(h(z^{-1}))\varphi(\eta(\bar{z})X \otimes \tilde{v}).$$

If $E \subseteq \boldsymbol{R}$ we may extend this action of $\boldsymbol{C}^\times$ to an action of $W_{\boldsymbol{C}/\boldsymbol{R}} = W_{\boldsymbol{C}/E_v}$. Let $n \in G(\boldsymbol{R})$ be such that $\mathrm{ad}\, n \circ h = h^\iota$. Then the element of w which projects to ι and has square -1 sends $\varphi \otimes u$ to $\varphi' \otimes u$ with

$$\varphi'(X \otimes \tilde{v}) = \pi(n)\varphi(\mathrm{ad}\, n^{-1}(X) \otimes \xi(n^{-1})\tilde{v}).$$

In either case the representation of W_{C/E_v} on (7.4) factors as $\psi^i(\pi) \otimes 1$, where $\psi^i(\pi)$ acts on $\mathrm{Hom}_{K_h}(\Lambda^i \mathfrak{G}/\mathfrak{k} \otimes \tilde{V}, U)$. Let $\psi_2(\pi)$ be the element in the representation ring of $W_{C/R}$ defined by

$$\psi_2(\pi) = \bigoplus (-1)^i \operatorname{Ind}(W_{C/R}, W_{C/E_v}, \psi^i(\pi)).$$

Let $m(\pi_f, K)$ be the dimension of U_f^K. If for all π we had

$$\psi_2(\pi) = m(\pi)\psi_1(\pi) \tag{7.5}$$

we could expect a relation

$$Z(z) = \prod_{\pi} L(z - d/2, \pi, r)^{m(\pi_\infty)m(\pi_f, K)}. \tag{7.6}$$

Here, as a single exception, we have taken π to be a representation of $G(A)$, its component at infinity being denoted π_∞, and r to be the representation of ${}^L G$. However (7.5) is not always valid, and it is the true form of the relation between $\psi_1(\pi)$ and $\psi_2(\pi)$ that we must discover, for it is the clue to the correct expression of (7.2) as a product of L-functions associated to automorphic representations.

Let $\Pi(\xi) = \{\pi_1, \cdots, \pi_r\}$ be the set of discrete series representations with the same central and infinitesimal characters as ξ. Then $\Pi(\xi) = \Pi$ is an L-packet Π_φ and the representations $\psi_1(\pi_i)$ are all equal. We denote them by $\psi_1(\Pi)$. The continuous cohomology of the representations π_i is completely understood **[4]**, and it is a simple exercise to prove the following lemma.

LEMMA. $\bigoplus_{j=1}^r \psi_2(\pi_j) = (-1)^d \psi_1(\Pi)$.

Thus, in this case, the relation (7.5) fails when the L-packet has more than one element. In order to correct (7.6) one has to replace r by a subrepresentation. However r is in general irreducible as a representation of ${}^L G$, and so we have to introduce the groups ${}^L H$ of **[33]**, and begin the study of L-indistinguishability.

If we accept L-indistinguishability, but expect no other difficulties with the correction of (7.6), then we have to be prepared to prove that every irreducible component of $\psi_2(\pi)$ is a component of $\psi_1(\pi)$. But we will again be deceived. There is another difficulty.

It appears already in the simplest of the examples considered by Casselman **[6]** and Milne **[36]**, although they had no occasion to draw attention to it. Suppose G is the group associated to a quaternion algebra over Q which is split at infinity but not at p. Let $\pi_f = \pi_p \otimes \pi^p$, and suppose $\pi \otimes \pi_f$ is trivial on the centre, π_p is one-dimensional and trivial on the maximal compact subgroup K_p of $G(Q_p)$, and π is either one-dimensional or the first element of the discrete series. ξ is taken to be trivial. If $K = K^p K_p$ then $L(s - \frac{1}{2}, \pi \otimes \pi_f, r)$ should appear in the zeta-function $Z(s, \mathrm{Sh}_K)$ with the exponent $\pm m(\pi^p, K^p)$. Here $m(\pi^p, K^p)$ is the multiplicity with which the trivial representation of K^p occurs in π_f^p, and the sign is positive if π is one-dimensional and negative if it is the first element of the discrete series. As Casselman and Milne show in their lectures, this is so locally almost everywhere.

One can probably show without great difficulty that the local statement is correct at p as well when π belongs to the discrete series, for π then contributes to the cohomology in dimension one and $\pi_p = \pi(\sigma_p)$ where σ_p is a special representation of the thickened Weil group. In particular

$$L(z - \tfrac{1}{2}, \pi_p, r) = \frac{1}{1 - \varepsilon/p^z}, \qquad |\varepsilon| = 1.$$

However if π is one-dimensional then π contributes to the cohomology in dimensions zero and two and the corresponding local contribution to the zeta-function should be

$$\left\{\frac{1}{(1 - \varepsilon/p^z)(1 - \varepsilon p/p^z)}\right\}^{m(\pi^p, K^p)}.$$

The factor inside the brackets is not $L(z - \frac{1}{2}, \pi_p, r)$.

The difficulty is resolved if we realize that when π, and hence $\pi \otimes \pi_f$, is one-dimensional we should not be using $L(z - \frac{1}{2}, \pi \otimes \pi_f, r)$ at all but rather $L(z - \frac{1}{2}, \pi' \otimes \pi'_f, r)$ where $\pi' \otimes \pi'_f$ is the one-dimensional representation of $G'(\boldsymbol{A}) = \mathrm{GL}(2, \boldsymbol{A})$ defined by the same character of the idèle-class group as $\pi \otimes \pi_f$. Since $G'(\boldsymbol{Q}_v) \sim G(\boldsymbol{Q}_v)$ and $\pi'_v \sim \pi_v$ for almost all places v, the error of using $\pi \otimes \pi_f$ instead of $\pi' \otimes \pi'_f$ is not detected when one only considers the local zeta-function almost everywhere.

The significance of the considerations of the second and third sections begins to appear. The representation $\pi \otimes \pi_f$ and the representation $\pi'' \otimes \pi''_f$ of $G'(\boldsymbol{A})$ associated to it by the principle of functoriality are anomalous, because π''_v is one-dimensional for almost all places v while π''_p is infinite-dimensional. The isobaric representation equivalent to $\pi'' \otimes \pi''_f$ almost everywhere is $\pi' \otimes \pi'_f$. It was implicit in the discussion of the second section that anomalous representations would have nothing to do with motives, and so it should come as no surprise now that we must discard $\pi \otimes \pi_f$ and replace it by $\pi' \otimes \pi'_f$.

In this example π itself was not changed for $G(\boldsymbol{R}) \sim G'(\boldsymbol{R})$ and $\pi \sim \pi'$. However in general we must expect that π itself will have to be modified. Thus the proper factor will not be $L(z - d/2, \pi \otimes \pi_f, r)$ but $L(z - d/2, \pi' \otimes \pi'_f, r)$, where $\pi' \otimes \pi'_f$ is an automorphic representation of a group G' obtained from G by an inner twisting. Again π'_v will have to be equivalent to π_v almost everywhere.

Since at the moment we are primarily interested in the infinite place. we simply ask whether it is possible to find a candidate for π' or, rather, for an L-packet $\{\pi'\} = \Pi'$. There are apparently two conditions to be satisfied, the first arising from the compatibility of functional equations.

(a) *Let $\pi \in \Pi_\varphi$ and let $\{\pi'\} = \Pi_{\varphi'}$. For any additive character ψ of $\boldsymbol{R}$ and any representation σ of ${}^LG = {}^LG'$,*

$$\varepsilon'(z, \sigma \circ \varphi, \psi) = \varepsilon(z, \sigma \circ \varphi, \psi) \frac{L(1 - z, \sigma \circ \varphi)}{L(z, \sigma \circ \varphi)}$$

is equal to

$$\varepsilon'(z, \sigma \circ \varphi', \psi) = \varepsilon(z, \sigma \circ \varphi', \psi) \frac{L(1 - z, \sigma \circ \varphi')}{L(z, \sigma \circ \varphi')}.$$

(b) *It is possible to find a summand $\psi_0(\pi')$ of $\psi_1(\pi')$ which is such that $\psi_2(\pi) = \alpha\psi_0(\pi')$, $\alpha \in \boldsymbol{Z}$.*

These conditions are only tentative, and may have to be modified in the course of time, but they will serve for the explanation of our problem.

The first condition involves only φ and φ' and we begin by constructing some pairs that satisfy it. Fix an element w of $W_{C/R}$ that projects to ι and satisfies $w^2 = -1$. We may suppose that $\varphi(w) = a \times \iota$ with a in the normalizer of ${}^LT^\circ$ in ${}^LG^\circ$. Then $\varphi(w)$ also normalizes ${}^LT^\circ$. Let $\varphi(\iota)$ denote the transformation of $X^*({}^LT^\circ)$ or of $X_*({}^LT^\circ)$ defined by $\varphi(w)$. We may also suppose that φ takes $C^\times$ to ${}^LT^\circ$ and that $\varphi(z) = z^\Lambda \bar{z}^{\varphi(\iota)\Lambda}$ with $\Lambda \in X_*({}^LT^\circ) \otimes C$ and $\Lambda - \varphi(\iota)\Lambda \in X_*({}^LT^\circ)$.

The representation φ' will be defined in a similar way. Thus $\varphi'(w) = a' \times \iota$ with a' in the normalizer of ${}^LT^\circ$, and $\varphi'(z) = z^\Lambda \bar{z}^{\varphi'(\iota)\Lambda}$. Notice that Λ is to be the same for φ' as for φ. However a, which is given, is replaced by a', which we must now define.

We suppose that $\varphi(\iota)$ sends every root to its negative, and choose λ in $X_*({}^LT^\circ)$ such that $\lambda^\vee(a) = e^{2\pi i\langle\lambda,\lambda^\vee\rangle}$ for any weight $\lambda^\vee$ of ${}^LT^\circ$ which is orthogonal to all roots. We shall take a' to lie in ${}^LT^\circ$ and to be such that $\lambda^\vee(a') = e^{2\pi i\langle\lambda',\lambda^\vee\rangle}$ when $\lambda^\vee$ is orthogonal to all roots. Here λ' is still to be defined. If we also denote by a the operator on $X_*({}^LT^\circ) \otimes C$ defined by a and if we let q be one-half the sum of the positive roots then λ' is to be given by the equation

$$\lambda' = \frac{1+a}{2}\lambda - \frac{(1-a)(1+\varphi'(\iota))}{8}\Lambda + \frac{q}{2}.$$

Observe that the action of $\varphi'(\iota)$ is the same as that of ι.

We are assuming that φ is a given, well-defined homomorphism, and hence [31] that

$$\lambda + \varphi(\iota)\lambda \equiv \frac{\Lambda - \varphi(\iota)\Lambda}{2} - q \quad (\text{mod } X_*({}^LT^\circ)).$$

In order to show that φ' is also well defined we must verify that

$$\lambda' + \varphi'(\iota)\lambda' \equiv \frac{\Lambda - \varphi(\iota)\Lambda}{2} \quad (\text{mod } X_*({}^LT^\circ)). \tag{7.7}$$

We begin with the equations $a\varphi'(\iota) = \varphi'(\iota)a = \varphi(\iota)$ and $a(1 + \varphi(\iota)) = 1 + \varphi(\iota)$, remarking also that the square of both $\varphi(\iota)$ and $\varphi'(\iota)$ is the identity. We infer that the left side of (7.7) is equal to

$$\frac{(1+\varphi'(\iota))(1+\varphi(\iota))\lambda}{2} - \frac{(1-a)(1+\varphi'(\iota))}{4}\Lambda + q,$$

because $(1 + \varphi'(\iota))q/2 = q$. The sum is in turn congruent to

$$\frac{1-\varphi(\iota)}{2}\Lambda - \frac{(1-a)(1+\varphi'(\iota))}{4}\Lambda = \frac{1-\varphi'(\iota)}{2}\Lambda.$$

Consequently the homomorphsim φ' can be constructed whenever φ is defined and $\varphi(\iota)$ sends every root to its negative. The following lemma is valid in this generality.

LEMMA. *For any representation σ of the Weil form of ${}^LG$ and any nontrivial character ψ of $\boldsymbol{R}$,*

$$\varepsilon'(\tau, \sigma \circ \varphi, \psi) = \varepsilon'(\tau, \sigma \circ \varphi', \psi).$$

The proof is a computation based on the proof of Lemma 3.2 of [31] and on Chapters 5 and 6 of [24], but it is rather lengthy, and not worth including here.

Observe that we could have started with φ', defined by an a' in ${}^LT^\circ$, and, reversing the process, passed to λ and a. More generally if ${}^LM$ and ${}^LM'$ are two parabolic subgroups of ${}^LG$, and $\varphi: W_{C/R} \to {}^LM$ has an image which lies in no proper parabolic subgroup of ${}^LM$, then we can use the process to pass to a φ'' whose image lies in the minimal parabolic of ${}^LM$, and thus of ${}^LG$ or ${}^LM'$, and afterwards reverse it to pass from φ'' to a φ': $W_{C/R} \to {}^LM'$ whose image lies in no proper parabolic subgroup of ${}^LM'$.

My intention now is simply to show, by means of a few examples, how for a given π in some Π_φ one can choose one of the φ' just described so that the condition (b) is satisfied for the elements π' of $\Pi_{\varphi'}$. Of course the problem is to decide if such a choice is always possible. Without more examples or a general theorem, we cannot be at all confident that this is so.

If all the continuous cohomology of $\xi \otimes \pi$ is zero there is no difficulty satisfying (b). We take $\varphi' = \varphi$ and $\alpha = 0$. The simplest nontrivial example is obtained by taking ξ trivial and π trivial. Let $\pi \in \Pi_\varphi$. Then $\varphi(z) = z^q \bar{z}^{\varphi(\iota)q}$, with q equal again to one-half the sum of the positive roots. If ${}^LM$ is the parabolic subgroup of ${}^LG$ corresponding to the minimal parabolic of G over $\boldsymbol{R}$, then the image of φ lies in ${}^LM$ and $\varphi(\iota)$ takes every root of ${}^LT^\circ$ in ${}^LM$ to its negative. Define φ' as above, with $a' \in {}^LT^\circ$. G' can be taken to be the quasi-split form of G over $\boldsymbol{R}$. The continuous cohomology of π is all in even dimensions and all of type (p, p) for some p. To compute it one observes that it is the same as the cohomology of the compact dual, which can be computed by using Schubert cells. One verifies without difficulty that for $\pi' \in \Pi_{\varphi'}$ the representation $\psi_1(\pi')$, which depends in reality only on φ', is equivalent to $\psi_2(\pi)$.

If G is not quasi-split over $\boldsymbol{R}$ then φ' is different from φ. If G is not quasi-split over $\boldsymbol{R}$ then it is certainly not quasi-split over $\boldsymbol{Q}$, and the trivial representation of $G(\boldsymbol{A})$ is anomalous. Once again we see that the passage from π to π' is the local expression of the passage from an anomalous representation to one which is not anomalous.

Other interesting examples are the representations $\pi = J_{i,j}$ of $PSU(n, 1)$ discussed in Chapter XI of the notes of Borel-Wallach [4]. Take ξ trivial. In this case $\psi_2(\pi)$ is $(-1)^{i+j}$ times a representation induced from $\boldsymbol{C}^\times$, the representation of $\boldsymbol{C}^\times$ used having the weights

$$z^{-i}\,\bar{z}^{-j},\ z^{-i-1}\,\bar{z}^{-j-1},\ \cdots,\ z^{-(n-j)}\,\bar{z}^{-(n-i)}. \tag{7.8}$$

Here $0 \leqq i + j \leqq n - 1$ and $0 \leqq i, j$. Borel and Wallach lapse into vagueness at one point, and it may be that the roles of i and j should here be reversed, but that is of little consequence.

The group ${}^LG^\circ$ is $\mathrm{SL}(n + 1, \boldsymbol{C})$ and ${}^LT^\circ$ may be taken to be the group of diagonal matrices. The representation r° is the standard representation of $\mathrm{SL}(n + 1, \boldsymbol{C})$. It is easy enough to deduce from [4] that if $\pi \in \Pi_\varphi$ then

$$\varphi(z) = z^\Lambda \bar{z}^{\varphi(\iota)\Lambda}, \qquad z \in \boldsymbol{C}^\times,$$

with Λ being equal to $(n/2 - i,\ n/2,\ n/2 - 1,\ \cdots,\ n/2 - i + 1,\ n/2 - i - 1,\ \cdots,$

$-n/2+j+1,\ -n/2+j-1, \cdots,\ -n/2,\ -n/2+j)$. The numbers occurring here are $n/2, n/2-1, \cdots, -n/2$, but the order is somewhat unusual. The transformation $\varphi(\iota)$ is given by

$$(x_1, \cdots, x_{n+1}) \longrightarrow (-x_{n+1}, -x_2, \cdots, -x_n, -x_1).$$

We are of course using the obvious representation of the elements of $X_*({}^LT^\circ) \otimes \boldsymbol{C}$ as sequences of $n+1$ complex numbers whose sum is 0.

Suppose, to be definite, that $i \leqq j$. We will choose φ' to be such that the transformation $\varphi'(\iota)$ takes $(x_1, \cdots, x_{n+1})$ to $(-x_{n+1}, -x_n, \cdots, -x_{n-i+1}, -x_{n-j}, \cdots, -x_{i+2}, -x_{n-j+1}, \cdots, -x_{n-i}, -x_{i+1}, \cdots, -x_1)$. The indices within the gaps decrease or increase regularly by one. If $\pi' \in \Pi_{\varphi'}$ then the representation $\psi_1(\pi')$ is induced from a representation of $\boldsymbol{C}^\times$ with weights

$$\begin{aligned} &z^{-i}\bar{z}^{-j},\ 1,\ z^{-1}\bar{z}^{-1}, \cdots, z^{-i+1}\bar{z}^{-j+1};\\ &z^{-i-1}\bar{z}^{-j-1}, \cdots, z^{-(n-j-1)}\bar{z}^{-(n-i-1)};\\ &z^{-(n-j+1)}\bar{z}^{-(j-1)}, \cdots, z^{-(n-i)}\bar{z}^{-i};\\ &z^{-(n-i+1)}\bar{z}^{-(n-i+1)}, \cdots, z^{-n}\bar{z}^{-n}, z^{-(n-j)}\bar{z}^{-(n-i)}. \end{aligned} \tag{7.9}$$

Happily the set (7.8) is a subset of (7.9) and the condition (b) is satisfied.

It should be observed that the representation $\psi_0(\pi')$ that is chosen to satisfy (b) will have to be, except for some degenerate values of i and j, a proper subrepresentation of $\psi_1(\pi')$. This phenomenon will, I hope, be taken into account by L-indistinguishability. For example if ε is the element of ${}^LT^\circ$ with diagonal entries

$$1, \overbrace{-1, -1, \cdots, -1}^{i}, 1, \cdots, 1, \overbrace{-1, \cdots, -1}^{j}, 1$$

then ε commutes with $\varphi'(W_{\boldsymbol{C}/\boldsymbol{R}})$ and $\psi_0(\pi')$ may be taken to be the restriction of $\psi_1(\pi')$ to the $+1$ eigenspace of $r(\varepsilon)$.

REFERENCES

1. J. Arthur, *Eisenstein series and the trace formula*, these PROCEEDINGS, part 1, pp. 253–274.

2. W. Baily and A. Borel, *Compactification of arithmetic quotients of bounded symmetric domains*, Ann. of Math. (2) **84** (1966), 442–528.

3. A. Borel, *Automorphic L-functions*, these PROCEEDINGS, part 2, pp. 27–61.

4. A. Borel and N. Wallach, *Seminar on the cohomology of discrete subgroups of semi-simple groups*, Institute for Advanced Study (1976/77).

5. P. Cartier, *Representations of* $\mathfrak{p}$*-adic groups: A survey*, these PROCEEDINGS, part 1, pp. 111–155.

6. W. Casselman, *The Hasse-Weil* ζ*-function of some moduli varieties of dimension greater than one*, these PROCEEDINGS, part 2, pp. 141–163.

7. ———, GL(n), Algebraic Number Fields (L-Functions and Galois Properties), A. Fröhlich, ed., Academic Press, New York, 1977.

8. I. V. Cherednik, *Algebraic curves uniformized by discrete arithmetical subgroups*, Uspehi Mat. Nauk **30** (1975), 183–184. (Russian)

9. P. Deligne, *Travaux de Griffiths*, Sém. Bourbaki (1970).

10. ———, *Travaux de Shimura*, Sém. Bourbaki (1971).

11. ———, *La conjecture de Weil pour les surfaces* K3, Invent. Math. **15** (1972), 206–226.

12. ———, *Théorie de Hodge*. III, Inst. Hautes Études Sci. Publ. Math. **44** (1974), 5–78.

13. ———, Letter to Piatetski-Shapiro.

14. P. Deligne and M. Rapoport, *Les schémas de modules de courbes elliptiques*, Modular Functions of One Variable. II, Lecture Notes in Math., vol. 349, Springer, New York, 1972, pp. 143–316.

15. V. G. Drinfeld, *Coverings of p-adic symmetric spaces*, Functional Anal. Appl. **10** (1976), 29–40. (Russian)

16. J. Giraud, *Cohomologie non-abélienne*, Springer-Verlag, Berlin and New York, 1971.

17. Harish-Chandra, *Representations of semi-simple Lie groups*. VI, Amer. Math. J. **78** (1956), 564–628.

18. ———, *Discrete series for semi-simple Lie groups*. I, Acta Math. **113** (1965), 241–318; II, **116** (1966), 1–111.

19. R. Howe and I. Piatetski-Shapiro, *A counterexample to the "generalized Ramanujan conjecture" for (quasi-) split groups*, these Proceedings, part 1, pp. 315–322.

20. Y. Ihara, *Hecke polynomials as congruence ζ-functions in elliptic modular case*, Ann. of Math. (2) **85** (1967), 267–295.

21. ———, *On congruence monodromy problems*, vols. 1, 2, Lecture notes from Tokyo Univ., 1968, 1969.

22. ———, *Some fundamental groups in the arithmetic of algebraic curves over finite fields*, Proc. Nat. Acad. Sci. **72** (1975).

23. S. Gelbart and H. Jacquet, *Forms of* GL(2) *from the analytic point of view*, these Proceedings, part 1, pp. 213–251.

24. H. Jacquet and R. P. Langlands, *Automorphic forms on* GL(2), Lecture Notes in Math., vol. 114, Springer, New York, 1970.

25. H. Jacquet and J. Shalika, *A non-vanishing theorem for zeta-functions of* GL(n), Invent. Math. **38** (1976).

26. D. Kazhdan, *On arithmetic varieties*, Lie Groups and Their Representations, Halsted, New York, 1975.

27. R. Kottwitz, Thesis, Harvard Univ., 1977.

28. ———, *Combinatorics and Shimura varieties* mod p, these Proceedings, part 2, pp. 185–192.

29. N. Kurokawa, *Examples of eigenvalues of Hecke operators in Siegel modular forms*, 1977 (preprint).

30. R. P. Langlands, *Modular forms and l-adic representations*, Modular Forms of One Variable. II, Lecture Notes in Math., vol. 349, Springer, New York, 1973, pp. 361–500.

31. ———, *On the classification of irreducible representations of real algebraic groups*, Institute for Advanced Study, Princeton, N. J., 1973.

32. ———, *Some contemporary problems with origins in the Jugendtraum*, Proc. Sympos. Pure Math., vol. 28, Amer. Math. Soc., Providence, R. I., 1976, pp. 401–418.

33. ———, *Stable conjugacy: definitions and lemmas*, Canad. J. Math. (to appear).

34. ———, *On the zeta-functions of some simple Shimura varieties*, Canad. J. Math. (to appear).

35. ———, *On the notion of automorphic representation*, these Proceedings, part 1, pp. 203–207.

36. J. Milne, *Points on Shimura varieties* mod p, these Proceedings, part 2, pp. 165–184.

37. D. Mumford, *Families of abelian varieties*, Proc. Sympos. Pure Math., vol. 9., Amer. Math. Soc., Providence, R. I., 1966, pp. 347–351.

38. I. Piatetski-Shapiro, *Multiplicity one theorems*, these Proceedings, part 1, pp. 209–212.

39. H. Resnikoff and R. L. Saldaña, *Some properties of Fourier coefficients of Eisenstein series of degree two*, Crelle **265** (1974), 90–109.

40. N. Saavedra Rivano, *Catégories tannakiennes*, Lecture Notes in Math., vol. 265, Springer, New York, 1972.

41. J.-P. Serre, *Abelian l-adic representations and elliptic curves*, Benjamin, New York, 1968.

42. ———, *Facteurs locaux des fonctions zêta des variétés algébriques* (*définitions et conjectures*), Sém. Delange-Pisot-Poitou, 1970.

43. ———, *Répresentations l-adiques*, Algebraic Number Theory, Tokyo, 1977.

44. D. Shelstad, *Notes on L-indistinguishability*, these PROCEEDINGS, part 2, pp. 193–203.

45. ———, *Orbital integrals and a family of groups attached to a real reductive group*, 1977 (preprint).

46. K.-Y. Shih, *Conjugations of arithmetic automorphic function fields*, 1977 (preprint).

47. J. Tate, *Number theoretic background*, these PROCEEDINGS, part 2, pp. 3–26.

48. N. Wallach, *Representations of reductive Lie groups*, these PROCEEDINGS, part 1, pp. 71–86.

49. A. Weil, *The field of definition of a variety*, Amer. J. Math. **78** (1956), 509–524.

50. S. Zucker, *Hodge theory with degenerating coefficients*. I, 1977 (preprint).

THE INSTITUTE FOR ADVANCED STUDY

Proceedings of Symposia in Pure Mathematics
Vol. 33 (1979), part 2, pp. 247–290

VARIÉTÉS DE SHIMURA: INTERPRÉTATION MODULAIRE, ET TECHNIQUES DE CONSTRUCTION DE MODÈLES CANONIQUES

PIERRE DELIGNE

SOMMAIRE

Introduction. Cet article fait suite à [5], dont nous utiliserons les résultats essentiels (ceux des paragraphes 4 et 5). Dans une première partie, nous tentons de motiver les axiomes imposés aux systèmes (G, X) (2.1.1) à partir desquels sont définies les variétés de Shimura. On montre que, grosso modo, ils correspondent aux espaces de modules de structures de Hodge X^+ du type suivant.

(a) X^+ est une composante connexe de l'espace de toutes les structures de Hodge sur un espace vectoriel fixe V relativement auxquelles certains tenseurs $t_1 \cdots t_n$ sont de type (0, 0). Le groupe algébrique G est le sous-groupe de $\mathrm{GL}(V)$ qui fixe les t_i et X est l'orbite $G(\boldsymbol{R}) \cdot X^+$ de X^+ sous $G(\boldsymbol{R})$.

AMS (MOS) subject classifications (1970). Primary 10D20, 14G99; Secondary 20G15, 20G30.

(b) La famille de structure de Hodge sur V paramétrée par X^+ vérifie certaines conditions, vérifiées par les familles de structures de Hodge qui apparaissent naturellement en géométrie algébrique: pour une structure complexe convenable (et uniquement déterminée) sur X^+, c'est une variation de structure de Hodge polarisable.

L'espace X^+ est automatiquement un domaine hermitien *symétrique* ($=$ espace hermitien symétrique à courbure < 0). Les domaines hermitiens symétriques peuvent tous être ainsi décrits comme des espaces de modules de structures de Hodge (1.1.17), et je crois cette description très utile. Par exemple: le plongement d'un domaine hermitien symétrique D dans son dual $\check{D}$ (une variété de drapeaux) correspond à l'application (une structure de Hodge) $\mapsto$ (la filtration de Hodge correspondante). Les descriptions comme "domaine de Siegel de 3è espèce" s'interprètent en disant que, sous certaines hypothèses, si on superimpose à une structure de Hodge une filtration par le poids, on obtient une structure de Hodge mixte, d'où une application de D dans un espace de modules de structures de Hodge mixtes (cf. les constructions de [**1**, III, 4.1]). Ce dernier point ne sera ni mentionné, ni utilisé dans l'article.

Ce point de vue, et la description de certaines variétés de Shimura comme espaces de modules de variétés abéliennes, sont liés par le dictionnaire: il revient au même (équivalence de catégorie $A \mapsto H_1(A, \mathbf{Z})$) de se donner une variété abélienne ou une structure de Hodge polarisable de type $\{(-1, 0), (0, -1)\}$ (il s'agit ici de $\mathbf{Z}$-structures de Hodge sans torsion; par passage au dual ($A \mapsto H^1(A, \mathbf{Z})$), on peut remplacer $\{(-1, 0), (0, -1)\}$ par $\{(1, 0), (0, 1)\}$). Polariser la variété abélienne revient à polariser son H_1. Avec paramètres, de même, il revient au même de se donner un schéma abélien polarisé sur une variété complexe lisse S, ou une variation de structures de Hodge polarisée de type $\{(-1, 0), (0, -1)\}$ sur l'espace analytique S^{an}. Une famille analytique de variétés abéliennes, paramétrée par S^{an}, est automatiquement algébrique (ceci résulte de [**3**]). Pour interpréter des structures de Hodge de type plus compliqué, on aimerait remplacer les variétés abéliennes par des "motifs" convenables, mais il ne s'agit encore que d'un rêve.

Au numéro 1.2, nous donnons une description commode basée sur le formalisme précédent de la classification des domaines hermitiens symétriques, en terme de diagrammes de Dynkin et de leurs sommets spéciaux. Au numéro 1.3, nous classifions un certain type de plongements de domaines hermitiens symétriques dans le demi-espace de Siegel. Les résultats sont parallèles à ceux de Satake [**11**]. Une application de l'astuce unitaire de Weyl, pour laquelle nous renvoyons à [**7**], ramène la classification à la connaissance d'un fragment de la table, donnée par exemple dans Bourbaki [**4**], donnant l'expression des poids fondamentaux comme combinaison linéaire de racines simples.

Le lecteur désireux d'en savoir plus sur les variations de structure de Hodge, et la façon dont elles apparaissent en géométrie algébrique, pourra consulter [**6**] (dont nous ne suivons pas les conventions de signes); certains faits, énoncés dans [**6**], sont démontrés dans [**7**].

Aux numéros 2.1 et 2.2 nous définissons, dans un langage adélique, les *variétés de Shimura* ${}_KM_C(G, X)$ (notées ${}_KM_C(G, h)$ dans [**5**], pour h un quelconque élément de X), leur limite projective $M_C(G, X)$ et la notion de *modèle canonique*. Je renvoie au texte pour ces définitions, et dirai seulement qu'un modèle canonique de

$M_C(G, X)$ est un modèle de $M_C(G, X)$ sur le *corps dual* (2.2.1) $E(G, X)$, i.e. un schéma $M(G, X)$ sur $E(G, X)$ muni d'un isomorphisme $M(G, X) \otimes_{E(G,X)} \boldsymbol{C} \xrightarrow{\sim} M_C(G, X)$, cette façon de définir $M_C(G, X)$ sur $E(G, X)$ ayant des propriétés convenables ($G(\boldsymbol{A}^f)$-équivariance, et propriétés galoisiennes des *points spéciaux* (2.2.4)). On définit aussi la notion de modèle faiblement canonique (même définition que les modèles canoniques, avec $E(G, X)$ remplacé par une extension finie $E \subset \boldsymbol{C}$). Ils jouent un rôle technique dans la construction de modèles canoniques. Les différences apparentes entre les définitions de 2.1, 2.2 et celles de **[5]** proviennent d'un autre choix de conventions de signes (action à droite contre action à gauche, loi de réciprocité en théorie du corps de classes global...).

Pour une description heuristique, je renvoie à l'introduction de **[5]**. Pour une brève description, sur des exemples, de comment on passe du langage adélique à un langage plus classique, je renvoie à [**5**, 1.6—1.11, 3.14—3.16, 4.11—4.16].

Dans **[5]**, nous avions systématisé les méthodes introduites par Shimura pour construire des modèles canoniques. Dans la seconde partie du présent article, nous perfectionnons les résultats de **[5]**. Au numéro 2.6, nous déterminons l'action du groupe de Galois $\mathrm{Gal}(\bar{\boldsymbol{Q}}/E)$ sur l'ensemble des composantes connexes géométriques d'un modèle faiblement canonique (supposé exister) de $M_C(G, X)$ sur E, sans supposer, comme dans **[5]**, que le groupe dérivé de G est simplement connexe. Le point essentiel est la construction, donnée au numéro 2.4, d'un morphisme du type suivant. Soient G un groupe réductif (connexe) sur $\boldsymbol{Q}$, $\rho: \tilde{G} \to G$ le revêtement universel de son groupe dérivé G^{der} et M une classe de conjugaison, définie sur un corps de nombres E, de morphismes de $\boldsymbol{G}_m$ dans G. On construit un morphisme q_M du groupe des classes d'idèles de E dans le quotient abélien $G(\boldsymbol{A})/\rho\tilde{G}(\boldsymbol{A}) \cdot G(\boldsymbol{Q})$ de $G(\boldsymbol{A})$. Ce morphisme est fonctoriel en (G, M), et, si F est une extension de E, le diagramme

$$\begin{array}{ccc} C(F) & & \\ \downarrow{\scriptstyle N_{F/E}} & \searrow & \\ & & G(\boldsymbol{A})/\rho\tilde{G}(\boldsymbol{A}) \cdot G(\boldsymbol{Q}) \\ & \nearrow & \\ C(E) & & \end{array}$$

est commutatif. Si $\tilde{G}$ n'a pas de facteur G' sur $\boldsymbol{Q}$ tel que $G'(\boldsymbol{R})$ soit compact, on déduit du théorème d'approximation forte que

$$\pi_0(G(\boldsymbol{A})/\rho\tilde{G}(\boldsymbol{A})G(\boldsymbol{Q})) = \pi_0(G(\boldsymbol{A})/G(\boldsymbol{Q})),$$

et q_M fournit une action sur $\pi_0(G(\boldsymbol{A})/G(\boldsymbol{Q}))$ de $\pi_0 C(E)$, le groupe de Galois rendu abélien $\mathrm{Gal}(\bar{\boldsymbol{Q}}/E)^{\mathrm{ab}}$ d'après la théorie du corps de classes global.

La seconde idée nouvelle—en fait un retour au point de vue de Shimura—est l'observation suivante: les résultats de 2.6 permettent de reconstruire un modèle faiblement canonique $M_E(G, X)$ de $M_C(G, X)$ à partir de sa composante neutre $M^0_{\bar{\boldsymbol{Q}}}(G, X^+)$ (une composante connexe géométrique, dépendant du choix d'une composante connexe X^+ de X), munie de l'action (semi-linéaire) du sous-groupe H de $G(\boldsymbol{A}^f) \times \mathrm{Gal}(\bar{\boldsymbol{Q}}/E)$ qui la stabilise. Soient Z le centre de G, G^{ad} le groupe adjoint, $G^{\mathrm{ad}}(\boldsymbol{R})^+$ la composante neutre topologique de $G^{\mathrm{ad}}(\boldsymbol{R})$, et $G^{\mathrm{ad}}(\boldsymbol{Q})^+ = G^{\mathrm{ad}}(\boldsymbol{Q}) \cap G^{\mathrm{ad}}(\boldsymbol{R})^+$. L'adhérence $Z(\boldsymbol{Q})^-$ de $Z(\boldsymbol{Q})$ dans $G(\boldsymbol{A}^f)$ agissant trivialement sur $M_C(G, X)$, l'action de H sur $M_{\bar{Q}}^0(G, X^+)$ se factorise par $H/Z(Q)^-$. On peut en fait faire agir un groupe un peu plus gros, extension de $\mathrm{Gal}(\bar{Q}/E)$ par le complété de $G^{\mathrm{ad}}(Q)^+$ pour la topologie des images des sous-groupes de congruence de $G^{\mathrm{der}}(Q)$.

A isomorphisme unique près, cette extension ne dépend que de G^{ad}, G^{der} et de la projection $X^{+\mathrm{ad}}$ de X^+ dans G^{ad} (2.5). On la note $\mathscr{E}_E(G^{\mathrm{ad}}, G^{\mathrm{der}}, X^{+\mathrm{ad}})$. La composante neutre $M_C^0(G, X^+)$ est la limite projective des quotients de $X^{+\mathrm{ad}}$ par les sous-groupes arithmétiques de $G^{\mathrm{ad}}(Q)^+$, images de sous-groupes de congruence de $G^{\mathrm{der}}(Q)$. On vérifie enfin que les conditions que doivent vérifier son modèle $M_{\bar{Q}}^0(G, X^+)$ sur $\bar{Q}$, et l'action de $\mathscr{E}_E(G^{\mathrm{ad}}, G^{\mathrm{der}}, X^{+\mathrm{ad}})$, pour correspondre à un modèle faiblement canonique, ne dépendent que du groupe adjoint G^{ad}, de $X^{+\mathrm{ad}}$, du revêtement G^{der} de G^{ad} et de l'extension finie E (contenue dans C) de $E(G^{\mathrm{ad}}, X^{+\mathrm{ad}})$. Ces conditions définissent les modèles *faiblement canoniques* (resp. *canoniques*, pour $E = E(G^{\mathrm{ad}}, X^{+\mathrm{ad}})$) *connexes* (2.7.10).

Le problème de l'existence d'un modèle canonique ne dépend donc, en gros, que du groupe dérivé. Cette réduction au groupe dérivé est une version beaucoup plus commode de la maladroite méthode de modification centrale de h dans [**5**, 5.11].

En 2.3, nous construisons une provision de modèles canoniques à l'aide de plongements symplectiques, en invoquant [**5**, 4.21 et 5.7]. Les résultats de 1.3 nous permettent d'obtenir les plongements symplectiques désirés avec très peu de calculs. En 2.7, nous expliquons la réduction au groupe dérivé esquissée ci-dessus et nous déduisons de 2.3 un critère d'existence de modèles canoniques qui couvre tous les cas connus (Shimura, Miyake et Shih).

Dans l'article, nous utilisons l'équivalence entre modèles faiblement canoniques et modèles faiblement canoniques connexes pour transporter à ces derniers des résultats de [**5**] (unicité, construction d'un modèle canonique à partir d'une famille de modèles faiblement canoniques). Il eut été plus naturel de transposer les démonstrations, et de transposer de même la fonctorialité [**5**, 5.4], et le passage à un sous-groupe [**5**, 5.7] (évitant par là la sybilline proposition [**5**, 1.15]). Le manque de temps, et la lassitude, m'en ont empêché.

J'ai récemment démontré qu'on pouvait donner un sens purement algébrique à la notion de cycle rationnel de type (p, p), sur une variété abélienne A (sur un corps de caractéristique 0). On peut retrouver à partir de là le critère d'existence 2.3.1 de modèles canoniques, et donner une description modulaire des modèles obtenus avec son aide (cf. [**9**]). Cette description ne se prête malheureusement pas à la réduction modulo p. Cette méthode évite le recours à [**5**, 5.7], (et par là à [**5**, 1.15]) et fournit des renseignements partiels sur la conjugaison des variétés de Shimura.

0. Rappels, terminologie et notations.

0.1. Nous aurons à faire usage du théorème d'approximation forte, du théorème d'approximation réelle, du principe de Hasse et de la nullité de $H^1(K, G)$ pour G semi-simple simplement connexe sur un corps local non archimédien. Des indications bibliographiques sur ces théorèmes sont données dans [**5**, (0.1) à (0.4)]. Signalons en outre l'article de G. Prasad (*Strong approximation for semi-simple groups over function fields*, Ann. of Math. (2) **105** (1977), 553–572) prouvant le théorème d'approximation forte sur un corps global quelconque. Soit G un groupe semi-simple simplement connexe, de centre Z, sur un corps global K. Nous n'utiliserons le principe de Hasse pour $H^1(K, G)$ que pour les classes dans l'image de $H^1(K; Z)$. En particulier, les facteurs E_8 nous indiffèrent.

0.2. *Groupe réductif* signifiera toujours *groupe réductif connexe*. Un *revêtement* d'un groupe réductif est un revêtement *connexe*. Groupe *adjoint* signifiera groupe *réductif adjoint*. Si G est un groupe réductif, nous noterons G^{ad} son groupe adjoint, G^{der} son groupe dérivé, et $\rho: \tilde{G} \to G^{\mathrm{der}}$ le revêtement universel de G^{der}. Nous noterons souvent Z (ou $Z(G)$) le centre de G, et (conflit de notation) $\tilde{Z}$ celui de $\tilde{G}$.

0.3. Nous noterons par un exposant ^{0} une *composante connexe algébrique* (par exemple : Z^0 est la composante neutre du centre Z de G). L'exposant $^+$ désignera une *composante connexe topologique* (par exemple: $G(\boldsymbol{R})^+$ est la composante neutre topologique du groupe des points réels d'un groupe G). On notera aussi $G(\boldsymbol{Q})^+$ la trace de $G(\boldsymbol{R})^+$ sur $G(\boldsymbol{Q})$. Pour G réductif réel, nous noterons par un indice $_+$ l'image inverse de $G^{\mathrm{ad}}(\boldsymbol{R})^+$ dans $G(\boldsymbol{R})$. Même notation $_+$ pour la trace sur un groupe de points rationnels.

Pour X un espace topologique, nous notons $\pi_0(X)$ l'ensemble de ses composantes connexes, muni de la topologie quotient de celle de X. Dans l'article, l'espace $\pi_0(X)$ sera toujours discret, ou compact totalement discontinu.

0.4. Un *domaine hermitien symétrique* est un espace hermitien symétrique à courbure < 0 (i.e. sans facteur euclidien ou compact).

0.5. Sauf mention expresse du contraire, un *espace vectoriel* est supposé de dimension finie, et un *corps de nombres* est supposé de degré fini sur $\boldsymbol{Q}$. Les corps de nombres que nous aurons à considérer seront le plus souvent contenus dans $\boldsymbol{C}$; $\bar{\boldsymbol{Q}}$ désigne la clôture algébrique de $\boldsymbol{Q}$ dans $\boldsymbol{C}$.

0.6. On pose $\hat{\boldsymbol{Z}} = \operatorname{proj\,lim} \boldsymbol{Z}/n\boldsymbol{Z} = \prod_p \boldsymbol{Z}_p$, $\boldsymbol{A}^f = \boldsymbol{Q} \otimes \hat{\boldsymbol{Z}} = \prod_p \boldsymbol{Q}_p$ (produit restreint) et on note $\boldsymbol{A} = \boldsymbol{R} \times \boldsymbol{A}^f$ l'anneau des adèles de $\boldsymbol{Q}$. On désignera parfois encore par $\boldsymbol{A}$ l'anneau des adèles d'un corps global quelconque.

0.7. $G(K)$, $G \otimes_F K$, G_K : pour G un schéma sur F (par exemple un groupe algébrique sur F) et K une F-algèbre, on désigne par $G(K)$ l'ensemble des points de G à valeur dans K, et par G_K ou par $G \otimes_F K$ le schéma sur K déduit de G par extension des scalaires.

0.8. Nous normaliserons l'isomorphisme de réciprocité de la théorie du corps de classes global (= choisirons lui ou son inverse)

$$\pi_0 \boldsymbol{A}_E^*/E^* \xrightarrow{\sim} \operatorname{Gal}(\bar{\boldsymbol{Q}}/E)^{\mathrm{ab}}$$

de telle sorte que la classe de l'idèle égal à une uniformisante en v et à 1 aux autres places corresponde à un Frobenius géométrique (l'inverse d'une substitution de Frobenius) (cf. 1.1.6 et la justification loc. cit. 3.6, 8.12).

1. Domaines hermitiens symétriques.

1.1. *Espaces de modules de structures de Hodge.*

1.1.1. Rappelons qu'une structure de Hodge sur un espace vectoriel réel V est une bigraduation $V_{\boldsymbol{C}} = \bigoplus V^{pq}$ du complexifié de V, telle que V^{pq} soit le complexe conjugué de V^{qp}.

Définissons une action h de $\boldsymbol{C}^*$ sur $V_{\boldsymbol{C}}$ par la formule

$$(1.1.1.1) \qquad h(z)v = z^{-p}\bar{z}^{-q}v \quad \text{pour } v \in V^{pq}.$$

Les $h(z)$ commutent à la conjugaison complexe de $V_{\boldsymbol{C}}$, donc se déduisent par extension des scalaires d'une action, encore notée h, de $\boldsymbol{C}^*$ sur V. Regardons $\boldsymbol{C}$ comme une extension de $\boldsymbol{R}$, et soit $\boldsymbol{S}$ son groupe multiplicatif, considéré comme groupe algébrique réel (autrement dit, $\boldsymbol{S} = R_{\boldsymbol{C}/\boldsymbol{R}}\boldsymbol{G}_m$ (restriction des scalaires à la Weil)); on a $\boldsymbol{S}(\boldsymbol{R}) = \boldsymbol{C}^*$, et h est une action du groupe algebrique $\boldsymbol{S}$. On vérifie que cette construction définit une équivalence de catégories: (espaces vectoriels réels munis d'une structure de Hodge) $\to$ (espaces vectoriels réels munis d'une action du groupe algébrique réel $\boldsymbol{S}$).

A l'inclusion $\boldsymbol{R}^* \subset \boldsymbol{C}^*$ correspond une inclusion de groupes algébriques réels $\boldsymbol{G}_m \subset \boldsymbol{S}$. Nous noterons w_h (ou simplement w) la restriction de h^{-1} à $\boldsymbol{G}_m$, et l'appellerons le *poids* $w : \boldsymbol{G}_m \to \mathrm{GL}(V)$. On dit que V est homogène de poids n si $V^{pq} = 0$ pour $p + q \neq n$, i.e. $w(\lambda)$ est l'homothétie de rapport λ^n.

Nous noterons μ_h (ou simplement μ) l'action de $\boldsymbol{G}_m$ sur $V_{\boldsymbol{C}}$ définie par $\mu(z)v = z^{-p}v$ pour $v \in V^{pq}$. C'est un composé $\boldsymbol{G}_m \to \boldsymbol{S}_{\boldsymbol{C}} \to^h \mathrm{GL}(V_{\boldsymbol{C}})$.

La *filtration de Hodge* F_h (ou simplement F) est définie par $F^p = \bigoplus_{r \geq p} V^{rs}$. On dit que V est de *type* $\mathscr{E} \subset \boldsymbol{Z} \times \boldsymbol{Z}$ si $V^{pq} = 0$ pour $(p, q) \notin \mathscr{E}$.

Plus généralement, si A est un sous-anneau de $\boldsymbol{R}$ tel que $A \otimes \boldsymbol{Q}$ soit un corps (en pratique, $A = \boldsymbol{Z}$, $\boldsymbol{Q}$ ou $\boldsymbol{R}$), une *A-structure de Hodge* est A-module de type fini V, muni d'une structure de Hodge sur $V \otimes_A \boldsymbol{R}$.

EXEMPLE 1.1.2. L'exemple fondamental est celui où $V = H^n(X, \boldsymbol{R})$, pour X une variété kahlérienne compacte, et où $V^{pq} \subset H^n(X, \boldsymbol{C})$ est l'espace des classes de cohomologie représentées par une forme fermée de type (p, q). D'autres exemples utiles s'en déduisent par des opérations de produits tensoriels, passage à une structure de Hodge facteur direct, ou passage au dual. Ainsi, le dual $H_n(X, \boldsymbol{R})$ de $H^n(X, \boldsymbol{R})$ est muni d'une structure de Hodge de poids $-n$. L'homologie, ou la cohomologie, entières fournissent des $\boldsymbol{Z}$-structures de Hodge.

EXEMPLE 1.1.3. Les structures de Hodge de type $\{(-1, 0), (0, -1)\}$ sont celles pour lesquelles l'action h de $\boldsymbol{C}^* = \boldsymbol{S}(\boldsymbol{R})$ est induite par une structure complexe sur V; pour V de ce type, la projection pr de V sur $V^{-1,0} \subset V_{\boldsymbol{C}}$ est bijective, et vérifie $\mathrm{pr}(h(z)v) = z\ \mathrm{pr}(v)$.

EXEMPLE 1.1.4. Soit A un tore complexe; c'est le quotient L/Γ de son algèbre de Lie L par un réseau Γ. On a $\Gamma \otimes \boldsymbol{R} \xrightarrow{\sim} L$, d'où une structure complexe sur $\Gamma \otimes \boldsymbol{R}$. Considérons celle-ci comme une structure de Hodge (1.1.3). Via l'isomorphisme $\Gamma = H_1(A, \boldsymbol{Z})$, c'est celle de 1.1.2.

EXEMPLE 1.1.5. La structure de Hodge de Tate $\boldsymbol{Z}(1)$ est la $\boldsymbol{Z}$-structure de Hodge de type $(-1, -1)$ de réseau entier $2\pi i\boldsymbol{Z} \subset \boldsymbol{C}$. L'exponentielle identifie $\boldsymbol{C}^*$ à $\boldsymbol{C}/\boldsymbol{Z}(1)$, d'où un isomorphisme $\boldsymbol{Z}(1) = H_1(\boldsymbol{C}^*)$. La structure de Hodge $\boldsymbol{Z}(n) =_{\mathrm{dfn}} \boldsymbol{Z}(1)^{\otimes n}$ $(n \in \boldsymbol{Z})$ est la $\boldsymbol{Z}$-structure de Hodge de type$(-n, -n)$ de réseau entier $(2\pi i)^n\boldsymbol{Z}$. On note ...(n) le produit tensoriel de ... par $\boldsymbol{Z}(n)$ (*twist à la Tate*).

1.1.6. REMARQUE. La règle $h(z)v = z^{-p}\bar{z}^{-q}v$ pour $v \in V^{pq}$ est celle que j'utilise dans (*Les constantes des équations fonctionnelles des fonctions L*, Anvers II, Lecture Notes in Math., vol. 349, pp. 501–597) et l'inverse de celle de [6]. Elle est justifiée d'une part par l'exemple 1.1.4 ci-dessus, d'autre part par le désir que $\boldsymbol{C}^*$ agisse sur $\boldsymbol{R}(1)$ par multiplication par la norme (cf. loc. cit., fin de 8.12).

1.1.7. Une *variation de structure de Hodge* sur une variété analytique complexe S consiste en

(a) un système local V d'espaces vectoriels réels;

(b) en chaque point de S, une structure de Hodge sur la fibre de V en s, variant continûment avec s.

On exige que la filtration de Hodge varie holomorphiquement avec s, et vérifie l'axiome dit de *transversalité*: la dérivée d'une section de F^p est dans F^{p-1}.

Il sera souvent donné un système local $V_{\boldsymbol{Z}}$ de $\boldsymbol{Z}$-modules de type fini, tel que $V = V_{\boldsymbol{Z}} \otimes \boldsymbol{R}$. On parlera alors de *variation de $\boldsymbol{Z}$-structure de Hodge*. De même pour $\boldsymbol{Z}$ remplacé par un anneau A comme en 1.1.1.

Remarque 1.1.8. Regardons S comme étant une variété réelle, dont l'espace tangent en chaque point est muni d'une structure complexe, i.e. d'une structure de Hodge de type $\{(-1, 0), (0, -1)\}$. L'intégrabilité de la structure presque complexe de S s'exprime en disant que le crochet des champs de vecteurs est compatible à la filtration de Hodge du complexifié du fibré tangent: $[T^{0,-1}, T^{0,-1}] \subset T^{0,-1}$. De même, l'axiome des variations de structure de Hodge exprime que la dérivation (fibre tangent) $\otimes_{\boldsymbol{R}}$ (sections C^∞ de V) $\to$ (sections C^∞ de V) (ou plutôt le complexifié de cette application) est compatible aux filtrations de Hodge : $\partial_D F^p \subset F^p$ pour D dans $T^{0,-1}$ (holomorphie) et $\partial_D F^p \subset F^{p-1}$ pour D arbitraire (transversalité).

Principe 1.1.9. En géométrie algébrique, chaque fois qu'apparaît une structure de Hodge dépendant de paramètres complexes, c'est une variation de structure de Hodge sur l'espace des paramètres. L'exemple fondamental est 1.1.2, avec paramètres : si $f : X \to S$ est un morphisme propre et lisse, de fibres X_s kahlériennes, les $H^n(X_s, \boldsymbol{Z})$ forment un système local sur S, et la filtration de Hodge sur le complexifié $H^n(X_s, \boldsymbol{C})$ varie holomorphiquement avec s et vérifie l'axiome de transversalité.

1.1.10. Une *polarisation* d'une structure de Hodge réelle, de poids n, V est un morphisme $\Psi : V \otimes V \to \boldsymbol{R}(-n)$ tel que la forme $(2\pi i)^n \Psi(x, h(i)y)$ soit symétrique et définie positive. De même pour les $\boldsymbol{Z}$-structures de Hodge, en remplaçant $\boldsymbol{R}(-n)$ par $\boldsymbol{Z}(-n)$, ... Puisque $\Psi(h(i)x, y) = \Psi(x, h(-i)y)$ ($h(i)$ est trivial sur $\boldsymbol{R}(-n)$), et que $h(-i)y = (-1)^n h(i)y$, la condition de symétrie revient à : Ψ symétrique pour n pair, alterné pour n impair.

Les structures de Hodge qui apparaissent en géométrie algébrique sont des $\boldsymbol{Z}$-structures de Hodge homogènes et polarisables. Exemple fondamental : les théorèmes de positivité de Hodge assurent que $H^n(X, \boldsymbol{Z})$, pour X une variété projective et lisse, est polarisable (noter que $h(i)$ est l'opération notée C par Weil dans son livre sur les variétés kählériennes).

1.1.11. Soient des espaces vectoriels réels $(V_i)_{i \in I}$ et une famille de tenseurs $(s_j)_{j \in J}$ dans des produits tensoriels de puissances tensorielles des V_i et de leurs duaux. On s'intéresse aux familles de structures de Hodge sur les V_i, pour lesquelles les s_j sont de type (0, 0). Pour interpréter cette condition "type (0, 0)" dans des cas particuliers, noter que $f : V \to W$ est un morphisme si et seulement si, en tant qu'élément de $\mathrm{Hom}(V, W) = V^* \otimes W$, il est de type (0, 0).

Soit G le sous-groupe algébrique de $\prod \mathrm{GL}(V_i)$ qui fixe les s_j. D'après 1.1.1, une famille de structures de Hodge sur les V_i s'identifie à un morphisme $h : \boldsymbol{S} \to \prod \mathrm{GL}(V_i)$. Pour que les s_j soient de type (0, 0), il faut et il suffit que h se factorise par G : il s'agit de considérer les morphismes algébriques $h : \boldsymbol{S} \to G$.

On peut regarder G, plutôt que le système des V_i et s_j, comme l'objet primordial: si G est un groupe algébrique linéaire réel, il revient au même de se donner $h : \boldsymbol{S} \to$

G, ou de se donner sur chaque représentation V de G une structure de Hodge, fonctorielle pour les G-morphismes et compatible aux produits tensoriels (cf. Saavedra, [**10**, VI, §2]). Les morphismes w_h et μ_h de 1.1.1 deviennent des morphismes de $\boldsymbol{G}_m$ dans G, et de $\boldsymbol{G}_m$ dans $G_{\boldsymbol{C}}$ respectivement.

1.1.12. La construction 1.1.11 amène à considérer les espaces de modules de structures de Hodge du type suivant : on fixe un groupe algébrique linéaire réel G, et on considère une composante connexe (topologique) X de l'espace des morphismes (= homomorphismes de groupes algébriques sur $\boldsymbol{R}$) de $\boldsymbol{S}$ dans G.

Soit G_1 le plus petit sous-groupe algébrique de G par lequel se factorisent les $h \in X$: X est encore une composante connexe de l'espace des morphismes de $\boldsymbol{S}$ dans G_1. Puisque $\boldsymbol{S}$ est de type multiplicatif, deux éléments quelconques de X sont conjugués : l'espace X est une classe de $G_1(\boldsymbol{R})^+$-conjugaison de morphismes de $\boldsymbol{S}$ dans G. C'est aussi une classe de $G(\boldsymbol{R})^+$-conjugaison, et G_1 est un sous-groupe invariant de la composante neutre de G.

1.1.13. Vu 1.1.9 et 1.1.10, nous ne considérons que les X tels que, pour une famille fidèle V_i de représentations de G, on ait

(α) Pour tout i, la graduation par le poids de V_i (de complexifiée la graduation de $V_{i\boldsymbol{C}}$ par les $V_{i\boldsymbol{C}}^n = \bigoplus_{p+q=n} V_i^{pq}$) est indépendante de $h \in X$. Conditions équivalentes: $h(\boldsymbol{R}^*)$ est central dans $G(\boldsymbol{R})^0$; la représentation adjointe est de poids 0.

(β) Pour une structure complexe convenable sur X, et tout i, la famille de structures de Hodge définie par les $h \in X$ est une variation de structure de Hodge sur X.

(γ) Si V est la composante homogène d'un poids n d'un V_i, il existe $\Psi : V \otimes V \to \boldsymbol{R}(-n)$ qui, pour tout $h \in X$, soit une polarisation de V.

PROPOSITION 1.1.14. *Supposons vérifié* (α) *ci-dessus.*

(i) *Il existe une et une seule structure complexe sur X telle que les filtrations de Hodge des V_i varient holomorphiquement avec $h \in X$.*

(ii) *La condition* 1.1.13(β) *est vérifiée si et seulement si la représentation adjointe est de type* $\{(-1, 1), (0, 0), (1, -1)\}$.

(iii) *La condition* 1.1.13(γ) *est vérifiée si et seulement si* G_1 (*défini en* 1.1.12) *est réductif et que, pour $h \in X$, l'automorphisme intérieur* int $h(i)$ *induit une involution de Cartan de son groupe adjoint.*

(i) Soit V la somme des V_i. C'est une représentation fidèle de G. Une structure de Hodge est déterminée par la filtration de Hodge correspondante (plus la graduation par le poids si on n'est pas dans le cas homogène): en poids $n = p + q$, on a $V^{pq} = F^p \cap \bar{F}^q$. L'application φ de X dans la grassmannienne de $V_{\boldsymbol{C}}$: $h \mapsto$ la filtration de Hodge correspondante, est donc injective. Nous allons vérifier qu'elle identifie X à une sous-variété complexe de cette grassmannienne; ceci prouvera (i): la structure complexe sur X induite de celle de la grassmannienne est la seule pour laquelle φ soit holomorphe.

Soient L l'algèbre de Lie de G et $p : L \to \mathrm{End}(V)$ son action sur V. L'action p est un morphisme de G-modules, injectif par hypothèse. Pour tout $h \in X$, c'est aussi un morphisme de structures de Hodge. L'espace tangent à X en h est le quotient de L par l'algèbre de Lie du stabilisateur de h—à savoir le sous-espace L^{00} de L pour la structure de Hodge de L définie par h. L'espace tangent à la grassmannienne en $\varphi(h)$ est $\mathrm{End}(V_{\boldsymbol{C}})/F^0(\mathrm{End}(V_{\boldsymbol{C}}))$. Enfin, $d\varphi$ est le composé

$$\begin{array}{ccc} L/L^{00} & \xrightarrow{\ p\ } & \mathrm{End}(V)/\mathrm{End}(V)^{00} \\ \Big\downarrow \wr & & \Big\downarrow \wr \\ L_C/F^0L_C & \xrightarrow{\ p\ } & \mathrm{End}(V_C)/F^0\mathrm{End}(V_C) \end{array}$$

Puisque p est un morphisme injectif de structures de Hodge, $d\varphi$ est injectif; son image est celle de L_C/F^0L_C, un sous-espace complexe, d'où l'assertion.

(ii) L'axiome de transversalité signifie que l'image de $d\varphi$ est dans $F^{-1}\mathrm{End}(V_C)/F^0\mathrm{End}(V_C)$, i.e. que $L_C = F^{-1}L_C$.

Pour prouver (iii), nous ferons usage de [7, 2.8], rappelé ci-dessous. Rappelons qu'une *involution de Cartan* d'un groupe algébrique linéaire réel (non nécessairement connexe) G est une involution σ de G telle que la forme réelle G^σ de G (de conjugaison complexe $g \to \sigma(\bar{g})$) soit *compacte*: $G^\sigma(\boldsymbol{R})$ est compact et rencontre toutes les composantes connexes de $G^\sigma(\boldsymbol{C}) = G(\boldsymbol{C})$. Pour $C \in G(\boldsymbol{R})$ de carré central, une *C-polarisation* d'une représentation V de G est une forme bilinéaire Ψ G-invariante, telle que $\Psi(x, Cy)$ soit symétrique et défini > 0. Pour tout $g \in G(\boldsymbol{R})$, on a alors $\Psi(x, gCg^{-1}y) = \Psi(g^{-1}x, Cg^{-1}y)$: la notion de C-polarisation ne dépend que de la class de $G(\boldsymbol{R})$-conjugaison de C.

Rappel. 1.1.15 [7, 2.8]. *Soient G algébrique réel et $C \in G(\boldsymbol{R})$ de carré central. Les conditions suivantes sont équivalentes*:

(i) Int C *est une involution de Cartan de G*;

(ii) *toute représentation réelle de G est C-polarisable*;

(iii) *G admet une représentation réelle C-polarisable fidèle.*

On notera que la condition 1.1.15(i) entraîne que G^0 est réductif—pour avoir une forme compacte. Elle ne dépend que de la classe de conjugaison de C.

Prouvons 1.1.14(iii). Soit G_2 le plus petit sous-groupe algébrique de G par lequel se factorisent les restrictions des $h \in X$ à $U^1 \subset \boldsymbol{C}^*$. Pour qu'une forme bilinéaire $\Psi: V \otimes V \to \boldsymbol{R}(-n)$ vérifie 1.1.13(γ), il faut et il suffit que $(2\pi i)^n\Psi: V \otimes V \to \boldsymbol{R}$ soit invariant par les $h(U^1)$—donc par G_2—(ceci exprime que Ψ est un morphisme), et une $h(i)$-polarisation. D'après 1.1.15, 1.1.13(γ) équivaut à: int $h(i)$ est une involution de Cartan de G_2.

On en déduit d'abord que G_1 est réductif: G_2 l'est, pour avoir une forme compacte, et G_1 est un quotient du produit $\boldsymbol{G}_m \times G_2$. Puisque G_2 est engendré par des sous-groupes compacts, son centre connexe est compact: il est isogène au quotient de G_2 par son groupe dérivé. L'involution $\theta = \mathrm{int}\, h(i)$ est donc une involution de Cartan de G_2 si et seulement si c'en est une de son groupe adjoint, et on conclut en notant que G_1 et G_2 ont même groupe adjoint.

Les conditions en 1.1.14 ne dépendant que de (G, X), on a le

COROLLAIRE 1.1.16. *Les conditions* 1.1.13(α), (β), (γ) *ne dépendent pas de la famille fidèle choisie de représentations V_i.*

COROLLAIRE 1.1.17. *Les espaces X de* 1.1.13 *sont les domaines hermitiens symétriques.*

A. *Prouvons X de ce type.* On se ramène successivement à supposer:

(1) Que $G = G_1$: remplacer G par G_1 ne modifie ni X, ni les conditions 1.1.13.

(2) Que G est adjoint: par(1), G est réductif, et son quotient par un sous-groupe

central fini est le produit d'un tore T par son groupe adjoint G^{ad}. L'espace X s'identifie encore à une composante connexe de l'espace des morphismes de $\boldsymbol{S}/\boldsymbol{G}_m$ dans G^{ad} : si un tel morphisme se relève en un morphisme de $\boldsymbol{S}$ dans G, avec une projection donnée dans T, le relèvement est unique. Les conditions énoncées en 1.1.14 restent par ailleurs vérifiées.

(3) Que G est simple : décomposer G en produit de groupes simples G_i ; ceci décompose X en un produits d'espaces X_i relatifs aux G_i.

Soit donc G un groupe simple adjoint, et X une $G(\boldsymbol{R})^+$-classe de conjugaison de morphismes non triviaux $h : \boldsymbol{S}/\boldsymbol{G}_m \to G$, vérifiant les conditions de 1.1.14(ii), (iii). Le groupe G est non compact: sinon, int $h(i)$ serait trivial (par (iii)), Lie G serait de type (0, 0) (par (ii)) et h serait trivial. Soit $h \in X$. D'après (iii), son centralisateur est compact; il existe donc sur X une structure riemannienne $G(\boldsymbol{R})^+$-équivariante. D'après (ii), $h(i)$ agit sur l'espace tangent $\mathrm{Lie}(G)/\mathrm{Lie}(G)^{00}$ de X en h par -1: l'espace X est riemannien symétrique. On vérifie enfin qu'il est hermitien symétrique pour la structure complexe 1.1.14(i). Il est du type non compact (courbure < 0) car G est non compact.

B. Réciproquement, si X est un espace hermitien symétrique, et que $x \in X$, on sait que la multiplication par u ($|u| = 1$) sur l'espace tangent T_x à X en x se prolonge en un automorphisme $m_x(u)$ de X. Soient A le groupe des automorphismes de X, et $h(z) = m(z/\bar{z})$ pour $z \in \boldsymbol{C}^*$. Le centralisateur A_x de x commute à h, et la condition de 1.1.14(ii) est donc vérifiée: $\mathrm{Lie}(A_x)$ est de type (0, 0), et $T_x = \mathrm{Lie}(A)/\mathrm{Lie}(A_x)$ de type $\{(-1, 1), (1, -1)\}$. Enfin, on sait que A est la composante neutre de $G(\boldsymbol{R})$, pour G adjoint, et que l'espace riemannien symétrique X est à courbure < 0 si et seulement si la symétrie $h(i)$ fournit une involution de Cartan de G (voir Helgason [**8**]).

1.1.18. Indiquons deux variantes de 1.1.15 (cf. [**7**, 2.11]).

(a) On donne un groupe algébrique réel réductif (0.2) G, et une classe de $G(\boldsymbol{R})$-conjugaison de morphismes $h : \boldsymbol{S} \to G$. On suppose que w_h—noté w—est central, donc indépendant de h (condition 1.1.13(α)), et que int $h(i)$ est une involution de Cartan de $G/w(\boldsymbol{G}_m)$.

Puisque G est réductif, $w(\boldsymbol{G}_m)$ admet un supplément G_2 : un sous-groupe invariant connexe tel que G soit quotient de $w(\boldsymbol{G}_m) \times G_2$ par un sous-groupe central fini. Il est unique: engendré par le groupe dérivé et le plus grand sous-tore compact du centre. Il contient les $h(U_1)$ ($h \in X$), et int $h(i)$ en est une involution de Cartan. Si V est une représentation de G, sa restriction à G_2 admet donc une $h(i)$-polarisation Φ. Si V est de poids n, $w(\boldsymbol{G}_m)$ agit par similitudes, donc G de même : pour une représentation convenable de G sur $\boldsymbol{R}$, Φ est covariant. Pour cette représentation, $\boldsymbol{R}$ est de type (n, n); ceci permet de faire agir G sur $\boldsymbol{R}(n)$, de façon compatible à sa structure de Hodge, et de voir $\Psi = (2\pi i)^{-n}\Phi$ comme une *forme de polarisation G-invariante*: $V \otimes V \to \boldsymbol{R}(-n)$.

(b) Supposons que G se déduise par extension des scalaires à $\boldsymbol{R}$ de $G_{\boldsymbol{Q}}$ sur $\boldsymbol{Q}$, et que w soit défini sur $\boldsymbol{Q}$. Le groupe G_2 est alors défini sur $\boldsymbol{Q}$, car c'est l'unique supplément de $w(\boldsymbol{G}_m)$, et tout caractère de G/G_2 est défini sur $\boldsymbol{Q}$, car ce groupe est trivial ou isomorphe, sur $\boldsymbol{Q}$, à $\boldsymbol{G}_m$. Si une représentation (rationnelle) V de $G_{\boldsymbol{Q}}$ est de poids n, les formes bilinéaires G-invariantes $V \otimes V \to \boldsymbol{Q}(-n)$ forment un espace vectoriel F sur $\boldsymbol{Q}$. L'ensemble de celles qui sont de polarisation (rel. $h \in X$) est la trace sur F d'un ouvert de $F_{\boldsymbol{R}}$, et cet ouvert est non vide d'après (a). Il existe donc des *formes de polarisation* Ψ: $V \otimes V \to \boldsymbol{Q}(n)$ G-invariantes.

On prendra garde que les formes en (a) et (b) ne sont pas toujours de polarization pour tout $h' \in X$: si $h' = \mathrm{int}(g)(h)$, la formule $\Psi(x, h'(i)y) = g\Psi(g^{-1}x, h(i)g^{-1}y)$ montre que la forme $(2\pi i)^n\Psi(x, h'(i)y)$ est symétrique et définie—mais définie positive ou négative selon l'action de g sur $\boldsymbol{R}(-n)$.

1.2. *Classification.*

Dans la suite de ce paragraphe, nous utilisons la relation 1.1.17 entre domaines hermitiens symétriques et espaces de modules de structures de Hodge, pour reformuler certains résultats de **[1]** et **[8]**, et donner quelques compléments.

1.2.1. Considérons les systèmes (G, X) formés d'un groupe algébrique réel simple adjoint G, et d'une classe de $G(\boldsymbol{R})$-conjugaison X de morphismes de groupes algébriques réels $h : \boldsymbol{S} \to G$, vérifiant (les notations sont celles de 1.1.1, 1.1.11).

(i) La représentation adjointe Lie(G) est de type $\{(-1, 1), (0, 0), (1, -1)\}$ (en particulier, h est trivial sur $\boldsymbol{G}_m \subset \boldsymbol{S}$);

(ii) int $h(i)$ est une involution de Cartan;

(iii) h est non trivial ou—ce qui revient au même (cf. 1.1.17)—G est non compact.

D'après 1.1.17, les composantes connexes des espaces X ainsi obtenus sont les domaines hermitiens symétriques irréductibles.

L'hypothèse (ii) assure que les involutions de Cartan de G sont des automorphismes intérieurs, donc que G est une forme intérieure de sa forme compacte (cf. 1.2.3). En particulier, G, étant simple, est absolument simple.

La classe de $G(\boldsymbol{C})$-conjugaison de μ_h: $\boldsymbol{G}_m \to G_{\boldsymbol{C}}$ ne dépend pas du choix de $h \in X$. Nous la noterons M_X.

PROPOSITION 1.2.2. *Soit $G_{\boldsymbol{C}}$ un groupe algébrique complexe simple adjoint. A chaque système (G, X) formé d'une forme réelle G de $G_{\boldsymbol{C}}$, et de X vérifiant* 1.2.1(i), (ii), (iii) *associons M_X. On obtient ainsi une bijection entre classes de $G_{\boldsymbol{C}}(\boldsymbol{C})$-conjugaison de systèmes (G, X), et classes de $G_{\boldsymbol{C}}(\boldsymbol{C})$-conjugaison de morphismes nontriviaux μ: $\boldsymbol{G}_m \to G_{\boldsymbol{C}}$ vérifiant la condition suivante.*

(*) *Dans la représentation* ad μ *de $\boldsymbol{G}_m$ sur* Lie($G_{\boldsymbol{C}}$), *seuls apparaissent les caractères z, 1 et z^{-1}.*

Pour vérifier 1.2.2, nous utiliserons la dualité entre domaines hermitiens symétriques, et espaces hermitiens symétriques compacts:

1.2.3. Soient G une forme réelle de $G_{\boldsymbol{C}}$, X une classe de $G(\boldsymbol{R})$-conjugaison de morphismes de $\boldsymbol{S}/\boldsymbol{G}_m$ dans G, et $h \in X$. La forme réelle G correspond à une conjugaison complexe σ sur $G_{\boldsymbol{C}}$; definissons G^* comme la forme réelle de conjugaison complexe $\mathrm{int}(h(i))\sigma$:

$$G^*(\boldsymbol{R}) = \{g \in (\boldsymbol{C}) | g = \mathrm{int}(h(i))\sigma(g)\}.$$

Le morphisme h est encore défini sur $\boldsymbol{R}$, de $\boldsymbol{S}/\boldsymbol{G}_m$ dans G^* : on a $h(\boldsymbol{C}^*/\boldsymbol{R}^*) \subset G^*(\boldsymbol{R})$; définissons X^* comme la classe de $G^*(\boldsymbol{R})$-conjugaison de h. La construction $(G, X) \to (G^*, X^*)$ est une involution sur l'ensemble des classes de $G_{\boldsymbol{C}}(\boldsymbol{C})$-conjugaison des systèmes (G, X) formés d'une forme réelle G de $G_{\boldsymbol{C}}$, et d'une classe de $G(\boldsymbol{R})$-conjugaison de morphismes non triviaux de $\boldsymbol{S}/\boldsymbol{G}_m$ dans G. Elle échange les (G, X) comme en 1.2.2, et les (G, X) tels que G soit compact et que X vérifie 1.2.1(i).

On sait que les formes réelles compactes de $G_{\boldsymbol{C}}$ sont toutes conjuguées entre elles. Puisque si $g \in G_{\boldsymbol{C}}$ normalise une forme réelle G, on a $g \in G(\boldsymbol{R})$ (ceci parce que G est adjoint), la dualité ramène 1.2.2 à l'énoncé suivant:

LEMME 1.2.4. *Soit G une forme compacte de G_C. La construction $h \to \mu_h$ induit une bijection entre*

(a) *classes de $G(\boldsymbol{R})$-conjugaison de morphismes $h : \boldsymbol{S}/\boldsymbol{G}_m \to G$, vérifiant* 1.2.1(i), *et*

(b) *classes de $G_C(\boldsymbol{C})$-conjugaison de morphismes $\mu: \boldsymbol{G}_m \to G_C$, vérifiant* 1.2.2(*).

Soient T un tore maximal de G, et T_C son complexifié. On vérifie d'abord que l'application $h \to \mu_h$: $\mathrm{Hom}(\boldsymbol{S}/\boldsymbol{G}_m, T) \to \mathrm{Hom}(\boldsymbol{G}_m, T_C)$ est bijective. Si W est le groupe de Weil de T, on sait que

$$\mathrm{Hom}(U^1, T)/W \xrightarrow{\sim} \mathrm{Hom}(U^1, G)/G(\boldsymbol{R})$$

et

$$\mathrm{Hom}(\boldsymbol{G}_m, T_C)/W \xrightarrow{\sim} \mathrm{Hom}(\boldsymbol{G}_m, G_C)/G_C(\boldsymbol{C}).$$

L'application $h \to \mu_h$ induit donc une bijection

$$\mathrm{Hom}(\boldsymbol{S}/\boldsymbol{G}_m, G)/G(\boldsymbol{R}) \xrightarrow{\sim} \mathrm{Hom}(\boldsymbol{G}_m, G)/G_C(\boldsymbol{C}),$$

et, pour que h vérifie 1.2.1(i), il faut et il suffit que μ_h vérifie 1.2.2(*).

1.2.5. Soit G un groupe algébrique complexe simple adjoint. Nous allons énumérer les classes de conjugaison de morphismes non triviaux $\mu: \boldsymbol{G}_m \to G$ vérifiant 1.2.2(*), en terme du diagramme de Dynkin D de G. Rappelons que ce dernier est canoniquement attaché à G—en particulier, les automorphismes de G agissent sur D—On peut identifier ses sommets aux classes de conjugaison de sous-groupes paraboliques maximaux.

Soient T un tore maximal, $X(T) = \mathrm{Hom}(T, \boldsymbol{G}_m)$, $Y(T) = \mathrm{Hom}(\boldsymbol{G}_m, T)$ (le dual de $X(T)$ pour l'accouplement $X(T) \times Y(T) \to^0 \mathrm{Hom}(\boldsymbol{G}_m, \boldsymbol{G}_m) = \boldsymbol{Z}$), $R \subset X(T)$ l'ensemble des racines, B un système de racines simples, α_0 l'opposé de la plus grande racine et $B^+ = B \cup \{\alpha_0\}$. Les sommets de D sont paramétrés par B, et ceux du diagramme de Dynkin étendu D^+ par B^+.

Une classe de conjugaison de morphismes de $\boldsymbol{G}_m$ dans G a un unique représentant $\mu \in Y(T)$ dans la chambre fondamentale $\langle \alpha, \mu \rangle \geqq 0$ pour $\alpha \in B$. Il est uniquement déterminé par les entiers positifs $\langle \alpha, \mu \rangle$ ($\alpha \in B$) et, G étant adjoint, ceux-ci peuvent être prescrits arbitrairement. La condition 1.2.2(*), pour μ non trivial, se récrit

$$(*)' \qquad \langle \alpha_0, \mu \rangle = -1.$$

Ecrivons la plus grande racine comme combinaison linéaire de racines simples, $\sum_{\alpha \in B^+} n(\alpha)\alpha = 0$, avec $n(\alpha_0) = 1$, et appelons *spéciaux* les sommets de D^+ tels que, pour la racine correspondante $\alpha \in B^+$, on ait $n(\alpha) = 1$. On sait que le quotient du groupe des copoids par celui des coracines agit sur D^+, et de façon simplement transitive sur l'ensemble des sommets spéciaux. Les sommets spéciaux sont donc les conjugués sous $\mathrm{Aut}(D^+)$ du sommet correspondant à α_0, et leur nombre est l'indice de connexion $|\pi_1(G)|$ de G (cf. Bourbaki [**4**, VI, 2 ex 2 et 5a)]).

La condition (*)′ se récrit

(*)″ Pour une racine simple $\alpha \in B$ correspondant à un sommet spécial de D, on a $\langle \alpha, \mu \rangle = 1$. Pour les autres racines simples, $\langle \alpha, \mu \rangle = 0$.

1.2.6. Au total, les classes de $G_C(\boldsymbol{C})$-conjugaison de systèmes (G, X) comme en 1.2.2 sont paramétrées par les sommets spéciaux du diagramme de Dynkin D de G_C. En particulier, pour G une forme réelle donnée de G_C, X est déterminé par le sommet spécial $s(X)$ correspondant ($G(\boldsymbol{R}) \subset G_C(\boldsymbol{C})$ est en effet son propre normalisateur). Le sommet correspondant à $X^{-1} = \{h^{-1} | h \in X\}$ est le transformé de $s(X)$ par l'involution d'opposition.

Dans 1.2.3, G et G^* sont des formes intérieures l'un de l'autre. S'il existe X vérifiant 1.2.1(i), (ii), (iii), G est donc une forme intérieure de sa forme compacte. En d'autres termes, la conjugaison complexe agit sur le diagramme de Dynkin de G_C par l'involution d'opposition.

PROPOSITION 1.2.7. *Soit G un groupe algébrique réel simple adjoint, et supposons qu'il existe des morphismes $h : \boldsymbol{C}^*/\boldsymbol{R}^* \to G$ vérifiant* 1.2.1(i), (ii), (iii). *L'ensemble de ces morphismes a alors deux composantes connexes, échangées par $h \mapsto h^{-1}$. Chacune a pour stabilisateur la composante neutre $G(\boldsymbol{R})^+$ de G.*

L'hypothèse (ii) assure que le centralisateur K de $h(i)$ est un sous-groupe compact maximal de $G(\boldsymbol{R})$. En particulier, $\pi_0(K) \xrightarrow{\sim} \pi_0 G(\boldsymbol{R})$. Il a même algèbre de Lie que le centralisateur de h. Ce dernier est un groupe algébrique connexe—en tant que centralisateur d'un tore—et compact—en tant que sous-groupe du centralisateur de $h(i)$. Il est donc topologiquement connexe et $\mathrm{Centr}(h) = K^+ = K \cap G(\boldsymbol{R})^+$. Le centre de K^+ est de dimension 1 : le complexifié de K^+ est le centralisateur de μ_h donc, d'après $(*)''$, un sous-groupe de Levi d'un sous-groupe parabolique maximal. On peut aussi le déduire de ce que la représentation de K^+ sur $\mathrm{Lie}(G)/\mathrm{Lie}(K^+)$ est irréductible (cf. [**8**, preuve de V, 1.1]). Le morphisme h est donc un isomorphisme de $\boldsymbol{S}/\boldsymbol{G}_m$ avec le centre connexe de K^+, et, K^+ détermine h au signe près. A fortiori, $h(i)$ détermine h au signe près. Dès lors

(a) L'application $h \mapsto h(i)$ est 2 : 1.

(b) Elle envoie isomorphiquement l'orbite $G(\boldsymbol{R})^+/K^+$ de h sous $G(\boldsymbol{R})^+$ sur l'ensemble $G(\boldsymbol{R})/K$ de toutes les involutions de Cartan dans $G(\boldsymbol{R})$.

La proposition en résulte.

COROLLAIRE 1.2.8. *Soit (G, X) comme en* 1.2.1, *et s le sommet correspondant du diagramme de Dynkin de G_C.*

(i) *Si s n'est pas fixe par l'involution d'opposition, $G(\boldsymbol{R})$ et X sont connexes.*

(ii) *Si s est fixe par l'involution d'opposition, $G(\boldsymbol{R})$ et X ont deux composantes connexes; les composantes de X sont échangées par $h \mapsto h^{-1}$, et par les $g \in G(\boldsymbol{R}) - G(\boldsymbol{R})^+$.*

Signalons que le cas (i) est encore caractérisé par les conditions équivalentes

(i′) le système de racines relatif de G est de type C (plutôt que BC);

(i″) X est un domaine tube.

1.3. *Plongements symplectiques.*

1.3.1. Soit V un espace vectoriel réel, muni d'une forme alternée non dégénérée Ψ. Le *demi-espace de Siegel* S^+ correspondent admet la description suivante: c'est l'espace des structures complexes h sur V, telles que Ψ soit de type (1,1) (pour l'identification (1.1.3) entre structures complexes et structures de Hodge de type $\{(-1, 0), (0, -1)\}$) et que la forme $\Psi(x, h(i)x)$ soit symétrique et définie positive.

Si on remplace "défini positif" par "défini", le *double demi-espace de Siegel* $S^\pm$ obtenu est une classe de conjugaison de morphismes h: $\boldsymbol{S} \to C\mathrm{Sp}(V)$ ($C\mathrm{Sp}$ = similitudes symplectiques; dans [**5**], ce groupe est noté Gp).

1.3.2. Soient G un groupe algébrique réel adjoint (0.2) et X une classe de conjugaison de morphismes h: $\boldsymbol{S} \to G$. On suppose vérifiées les conditions (i), (ii) de 1.2.1, et on remplace (iii) par (iii') G est sans facteur compact.

Le système (G, X) est donc un produit de systèmes (G_ι, X_ι) comme en 1.2.1, et X_ι correspond à un sommet spécial du diagramme de Dynkin de $G_{\iota C}$ (1.2.6).

Considérons les diagrammes

$$(G, X) \longleftarrow (G_1, X_1) \longrightarrow (C\mathrm{Sp}(V), S^{\pm}),$$

où G est le groupe adjoint du groupe réductif G_1, et où X_1 est une classe de $G_1(\boldsymbol{R})$-conjugaison de morphismes de $\boldsymbol{S}$ dans G_1. On dispose d'une section $\tilde{G} \to G_1$, de sorte que V est une représentation de $\tilde{G}$. Notre but est la détermination 1.3.8 des représentations complexes irréductibles nontriviales W de $\tilde{G}$, qui est essentiellement équivalent à figurent dans la complexifiée d'une représentation ainsi obtenue. Ce problème celui résolu par Satake dans **[11]**.

LEMME 1.3.3. *Il suffit qu'existe* $(G_1, X_1) \to (G, X)$, *comme ci-dessus, et une représentation linéaire* (V, ρ) *de type* $\{(-1, 0), (0, -1)\}$ *de* G_1, *telle que* W *figure dans* V_C.

Remplaçant G_1 par le sous-groupe engendré par le groupe dérivé G_1' et par l'image de h, on se ramène à supposer que int $h(i)$ est une involution de Cartan de $G_1/w(\boldsymbol{G}_m)$. Il existe alors sur V une forme de polarisation Ψ (1.18(a)), telle que ρ soit un morphisme de (G_1, X_1) dans $(C\mathrm{Sp}(V), S^{\pm})$.

1.3.4. Considérons le système projectif $(H_n)_{n\in N}$ suivant: N est ordonné par divisibilité, $H_n = \boldsymbol{G}_m$, et le morphisme de transition de H_{nd} dans H_n est $x \mapsto x^d$ (lim proj H_n est le revêtement universel—au sens algébrique—de $\boldsymbol{G}_m$). Un *morphisme fractionnaire* de $\boldsymbol{G}_m$ dans un groupe H est un élément de lim inj $\mathrm{Hom}(H_n, H)$. De même pour le groupe $\boldsymbol{S}$. Pour $\mu\colon \boldsymbol{G}_m \to H$ fractionnaire, défini par $\mu^n\colon H_n = \boldsymbol{G}_m \to H$, et V une représentation linéaire de H, V est somme des sous-espaces V_a ($a \in (1/n)\boldsymbol{Z}$) tels que, via μ^n, $\boldsymbol{G}_m$ agisse sur V_a par multiplication par x^{na}. Les a tels que $V_a \neq 0$ sont les *poids* de μ dans V. De même, un morphisme fractionnaire h: $\boldsymbol{S} \to H$ détermine une décomposition de Hodge fractionnaire $V^{r,s}$ de V ($r, s \in \boldsymbol{Q}$).

LEMME 1.3.5. *Pour* $h \in X$, *soit* $\tilde{\mu}_h$ *le relèvement fractionnaire de* μ_h *à* $\tilde{G}_C$. *Les représentations* W *de* 1.3.2 *sont celles telles que* $\tilde{\mu}_h$ *n'ait que deux poids* a *et* $a + 1$.

La condition est nécessaire: Relevant h en $h_1 \in X_1$, on a $\mu_{h_1} = \tilde{\mu}_h \cdot \nu$, avec ν central. Sur V, μ_{h_1} a les poids 0 et 1. Si $-a$ est l'unique poids de ν sur W irréductible dans V_C, les seuls poids de $\tilde{\mu}_h$ sur W sont a et $a + 1$. Pour W non trivial, l'action de $\boldsymbol{G}_m$ via $\tilde{\mu}_h^n$ (n assez divisible) est non triviale (car G_C est simple), donc non centrale, et les deux poids a et $a + 1$ apparaissent.

La condition est suffisante: Prenons pour V l'espace vectoriel réel sous-jacent à W, et pour groupe G_1 le groupe engendré par l'image de $\tilde{G}$, et par le groupe des homothéties. Pour $h \in X$, de relèvement fractionnaire $\tilde{h}$ à $\tilde{G}$, soit $h_1(z) = h(z)z^{-a}\bar{z}^{1-a}$. Si W_a et W_{a+1} sont les sous-espaces de poids a et $a + 1$ de W, $\tilde{h}$ agit sur W_a (resp. W_{a+1}) par $(z/\bar{z})^a$ (resp. $(z/\bar{z})^{1+a}$), et h_1 par $\bar{z}$ (resp. z): h_1 est un vrai morphisme de $\boldsymbol{S}$ dans G_1, de projection h dans G, et V est de type $\{(-1, 0)\ (0, -1)\}$ rel. h_1. Il ne reste qu'à appliquer 1.3.3.

1.3.6. Traduisons la condition 1.3.5 en terme de racines. Soient T un tore maximal de G_C, $\tilde{T}$ son image inverse dans $\tilde{G}_C$, B un système de racines simples de T, et $\mu \in Y(T)$ le représentant dans la chambre fondamentale de la classe de conjugaison de μ_h ($h \in X$). Si α est le poids dominant de W, le plus petit poids est $-\tau(\alpha)$, pour τ

l'involution d'opposition. Il s'agit d'exprimer que $\langle \mu, \beta \rangle$ ne prend que deux valeurs a et $a + 1$, pour β un poids de W. Ces poids étant tous de la forme (α + une combinaison $\mathbf{Z}$-linéaire de racine), et les $\langle \mu, z \rangle$, pour r une racine, étant entiers, la condition s'exprime par $\langle \mu, -\tau(\alpha) \rangle = \langle \mu, \alpha \rangle - 1$, soit

$$\langle \mu, \alpha + \tau(\alpha) \rangle = 1. \tag{1.3.6.1}$$

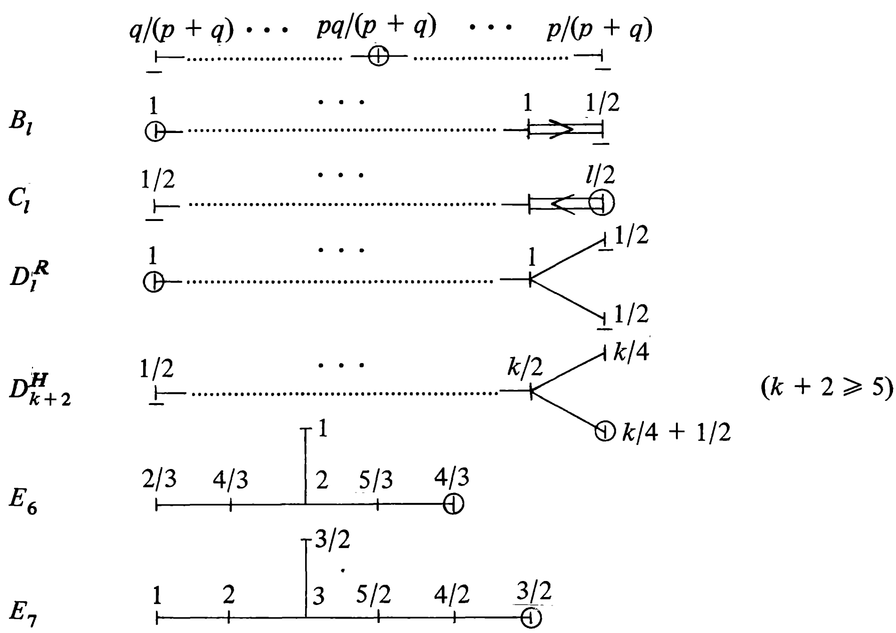
Déterminons les solutions de (1.3.6.1). Pour tout poids dominant α, $\langle \mu, \alpha + \tau(\alpha) \rangle$ est un entier, car $\alpha + \tau(\alpha)$ est combinaison $\mathbf{Z}$-linéaire de racines. Si $\alpha \neq 0$, il est > 0, sans quoi μ annulerait tous les poids de la représentation correspondante. Un poids dominant α vérifiant (1.3.6.1) ne peut donc être somme de deux poids:

LEMME 1.3.7. *Seuls les poids fondamentaux peuvent vérifier* 1.3.6.1.

1.3.8. D'après 1.3.7, les représentations W cherchées se factorisent par un facteur simple G_ι de G, et leur poids dominant est un poids fondamental; il correspond à un sommet du diagramme de Dynkin D_ι de $G_{\iota\mathbf{C}}$. La condition nécessaire et suffisante (1.3.6.1) ne dépend que de la projection de μ dans $G_{\iota\mathbf{C}}$; celle-ci correspond à un sommet spécial s de D_ι (1.2.6), et s à racine simple α_s. Le nombre $\langle \mu, \omega \rangle$, pour ω un poids, est le coefficient de α_s dans l'expression de ω comme combinaison $\mathbf{Q}$-linéaire de racines simples. Pour ω fondamental, ces coefficients sont donnés dans les tables de Bourbaki [**4**]. Ils sont donnés par la table suivante, où sont énumérés les diagrammes de Dynkin munis d'un sommet spécial (entouré). Chaque sommet correspond à un poids fondamental ω, et on l'a affecté du nombre $\langle \mu, \omega \rangle$. Les sommets correspondant aux poids qui vérifient (1.3.6.1) sont soulignés.

TABLE 1.3.9. Dans la table, ... indique une progression arithmétique.

A_{p+q-1} (sommet spécial en $p^{\text{ième}}$ position)



REMARQUES 1.3.10. (i) Pour G simple exceptionnel, aucune représentation W ne vérifie 1.3.2.

(ii) Pour G simple classique, sauf le cas D_l^H ($l \geqq 5$), les représentations W de 1.3.2 forment un système fidèle de représentations de $\tilde{G}$. Pour D^H, on obtient seulement une représentation fidèle d'un revêtement double de G (à savoir, la composante connexe algébrique du groupe des automorphismes d'un espace vectoriel sur $\boldsymbol{H}$ muni d'une forme antihermitienne non dégénérée—une forme intérieure de $SO(2n)$).

2. Variétés de Shimura.

2.0. *Préliminaires.*

2.0.1. Soient G un groupe, Γ un sous-groupe et $\varphi\colon \Gamma \to \Delta$ un morphisme. Supposons donnée une action r de Δ sur G, qui stabilise Γ, et telle que

(a) $r(\varphi(\gamma))$ est l'automorphisme intérieur int_γ de G;

(b) φ est compatible aux actions de Δ, sur Γ par r, et sur lui-même par automorphismes intérieurs: $\varphi(r(\delta)(\gamma)) = \mathrm{int}_\delta(\varphi(\gamma))$.

Formons le produit semi-direct $G \rtimes \Delta$. Les conditions (a), (b) reviennent à dire que l'ensemble des $\gamma \cdot \varphi(\gamma)^{-1}$ est un sous-groupe distingué, et on définit $G *_\Gamma \Delta$ comme le quotient de $G \rtimes \Delta$ par ce sous-groupe.

On notera que les hypothèses entraînent que $Z = \mathrm{Ker}(\varphi)$ est central dans G, et que $\mathrm{Im}(\varphi)$ est un sous-groupe distingué de Δ. Les lignes du diagramme

$$\begin{array}{ccccccccc} 0 & \longrightarrow & Z & \longrightarrow & \Gamma & \longrightarrow & \Delta & \longrightarrow & \Delta/\Gamma & \longrightarrow & 0 \\ & & \| & & \cap & & \cap & & \| & & \\ 0 & \longrightarrow & Z & \longrightarrow & G & \longrightarrow & G *_\Gamma \Delta & \longrightarrow & \Delta/\Gamma & \longrightarrow & 0 \end{array} \tag{2.0.1.1}$$

sont exactes, d'où un isomorphisme

$$\Gamma\backslash G \xrightarrow{\sim} \Delta\backslash G *_\Gamma \Delta \tag{2.0.1.2}$$

et, mise en évidence, une action à droite de $G *_\Gamma \Delta$ sur $\Gamma\backslash G$. Pour cette action, G agit par translations à droite, et Δ par l'action à droite r^{-1}.

Si G est un groupe topologique, que Δ est discret, et que l'action r est continue, la groupe $G *_\Gamma \Delta$, muni de la topologie quotient de celle de $G \rtimes \Delta$, est un groupe topologique, $G/\mathrm{Ker}(\varphi)$ en est un sous-groupe ouvert, et l'application (2.0.1.2) est un homéomorphisme.

La construction 2.0.1 garde un sens dans la catégorie des groupes algébriques sur un corps. Si G est un groupe réductif sur k, on a un isomorphisme canonique $G = \tilde{G} *_{Z(\tilde{G})} Z(G)$ (pour l'action triviale de $Z(G)$ sur $\tilde{G}$).

2.0.2. Soient G un groupe algébrique sur un corps k, et G^{ad} le quotient de G par son centre Z. L'action par automorphismes intérieurs de G sur lui-même $(x, y) \to xyx^{-1}\colon G \times G \to G$ est invariante par $Z \times \{e\}$ agissant par translation, donc se factorise par une action de G^{ad} sur G. Prendre garde que l'action de $\gamma \in G^{\mathrm{ad}}(k)$ sur $G(k)$ n'est pas nécessairement un automorphisme intérieur de $G(k)$ (la projection de $G(k)$ dans $G^{\mathrm{ad}}(k)$ n'est pas toujours surjective). Un exemple typique est l'action de $\mathrm{PGL}(n, k)$ sur $\mathrm{SL}(n, k)$.

De même, l'application "commutateur" $(x, y) = xyx^{-1}y^{-1}\colon G \times G \to G$ est invariante par $Z \times Z$ agissant par translation, et se factorise par une application "commutateur" $(\ ,\ )\colon G^{\mathrm{ad}} \times G^{\mathrm{ad}} \to G$.

Tout ceci, et le fait que ces "commutateurs" et "automorphismes intérieurs" vérifient les identités usuelles se voit au mieux par descente, i.e. en interprétant G comme un faisceau en groupes sur un site convenable, et G^{ad} comme le quotient de ce faisceau en groupes par son centre. En caractéristique 0, si on s'intéresse seulement aux points de G sur des extensions de k, il suffit d'utiliser la descente galoisienne—cf. 2.4.1, 2.4.2.

Variante. Pour G réductif sur k, les groupes G et $\tilde{G}$ ont même groupe adjoint, et les constructions précédentes pour G et $\tilde{G}$ sont compatibles. En particulier, l'application commutateur $(\ ,\ )\colon G \times G \to G$ a une factorisation canonique

$$(\ ,\ )\colon G \times G \longrightarrow G^{\text{ad}} \times G^{\text{ad}} \longrightarrow \tilde{G} \longrightarrow G.$$

On en déduit que le quotient de $G(k)$ par le sous-groupe distingué $\rho\tilde{G}(k)$ est abélien.

2.0.3. Soient k un corps global de caractéristique 0, $\boldsymbol{A}$ l'anneau de ses adèles, G un groupe semi-simple sur k et $N = \operatorname{Ker}(\rho\colon \tilde{G} \to G)$. Soient S un ensemble fini de places de k, $\boldsymbol{A}_S$ l'anneau des S-adèles (produit restreint étendu aux $v \notin S$) et posons $\Gamma_S = \rho\tilde{G}(\boldsymbol{A}_S) \cap G(k)$ (intersection dans $G(\boldsymbol{A}_S)$). C'est le groupe des éléments de $G(k)$ qui, en toute place $v \notin S$, peuvent se relever dans $\tilde{G}(k_v)$ (se rappeler que $\rho\colon \tilde{G}(\boldsymbol{A}) \to G(\boldsymbol{A})$ est propre).

La suite exacte longue de cohomologie identifie $G(k)/\rho\tilde{G}(k)$ à un sous-groupe de $H^1(\operatorname{Gal}(\bar{k}/k), N(\bar{k}))$, et $\Gamma_S/\rho\tilde{G}(k)$ aux éléments localement nul, en les places $v \notin S$, de ce sous-groupe. En particulier, $\Gamma_S/\rho\tilde{G}(k)$ est contenu dans le sous-groupe $H^1_\cdot(\operatorname{Gal}(\bar{k}/k), N(\bar{k}))$ des classes dont la restriction à tout sous-groupe monogène est triviale (argument et notations de **[12]**). Si $\operatorname{Im}\operatorname{Gal}(\bar{k}/k)$ est l'image de Galois dans $\operatorname{Aut} N(\bar{k})$, on a $H^1_\cdot(\operatorname{Gal}(\bar{k}/k), N(\bar{k})) = H^1_\cdot(\operatorname{Im}\operatorname{Gal}(\bar{k}/k), N(\bar{k}))$ (loc. cit.); en particulier, $\Gamma_S/\rho\tilde{G}(k)$ est fini.

PROPOSITION 2.0.4. (i) *Γ_S ne dépend que de l'ensemble des places $v \in S$ où le groupe de décomposition $D_v \subset \operatorname{Im}\operatorname{Gal}(\bar{k}/k)$ est non cyclique. En particulier, il ne change pas si on ajoute à S les places à l'infini.*

(ii) *$\Gamma_S/\rho\tilde{G}(k)$ s'identifie au sous-groupe du groupe fini $H^1_\cdot(\operatorname{Im}\operatorname{Gal}(\bar{k}/k), N(\bar{k}))$ formé des classes de restriction nulle à tout sous-groupe de décomposition D_v, $v \notin S$. En particulier, pour S grand, on a $\Gamma_S/\rho\tilde{G}(k) = H^1_\cdot(\operatorname{Im}\operatorname{Gal}(\bar{k}/k), N(\bar{k}))$.*

La restriction d'un élément de $H^1_\cdot(\operatorname{Im}\operatorname{Gal}(\bar{k}/k), N(\bar{k}))$ à un groupe de décomposition cyclique est automatiquement nulle, d'où (i). Pour (ii), on peut supposer que S contient les places à l'infini. Le principe de Hasse pour $\tilde{G}$ (pour des classes venant du centre) assure alors que tous les éléments du groupe (ii) sont effectivement réalisés comme classe d'obstruction.

COROLLAIRE 2.0.5. *Tout sous-groupe de S-congruence assez petit de $G(k)$ est dans Γ_S.*

Si U est un sous-groupe de S-congruence, $U/U \cap \rho\tilde{G}(k)$ est fini: l'obstruction à relever dans $\tilde{G}(k)$ meurt dans une extension galoisienne de degré et de ramification bornées, donc est dans $H^1(\operatorname{Gal}, N(\bar{k}))$ pour Gal un quotient fini de $\operatorname{Gal}(\bar{k}/k)$. Des conditions de S-congruence permettent alors de passer de ce H^1 à $\Gamma_S/\rho\tilde{G}(k)$, cf. **[12]**.

REMARQUE 2.0.6. On notera par ailleurs que si $\tilde{G}$ vérifie le théorème d'approximation forte rel. S, tout sous-groupe de S-congruence $U \subset \Gamma_S$ de $G(k)$ s'envoie sur $\Gamma_S/\rho\tilde{G}(k)$.

Corollaire 2.0.7. *Pour toute place archimédienne v, un sous-groupe de S-congruence assez petit U de $G(k)$ est dans la composante connexe topologique $G(k_v)^+$ de $G(k_v)$.*

Puisque $\tilde{G}(k_v)$ est connexe, on a $G(k_v)^+ = \rho\tilde{G}(k_v)$ et $U \subset \Gamma_S = \Gamma_{S\cup\{v\}} \subset G(k_v)^+$ (2.0.4 et 2.0.5).

Corollaire 2.0.8. *Le sous-groupe $G(k)\rho\tilde{G}(A_S)$ de $G(A_S)$ est fermé, topologiquement isomorphe à $\rho\tilde{G}(A_S)*_{\Gamma_S} G(k)$ (i.e. $\rho\tilde{G}(A_S)$ en est un sous-groupe ouvert).*

C'est un sous-groupe parce que, vu 2.0.2, $\rho\tilde{G}(A_S)$ est distingué dans $G(A_S)$, avec un quotient commutatif. Soit $T \supset S$ assez grand pour que $\tilde{G}(k)$ soit dense dans $\tilde{G}(A_T)$ (approximation forte). Notons k_{T-S} le produit des k_v pour $v \in T - S$. Pour K un sous-groupe compact ouvert de $G(A_T)$, on a

$$G(k)\rho\tilde{G}(A_S) = G(k)\rho(\tilde{G}(k)\cdot\tilde{G}(k_{T-S}) \times \rho^{-1}K) \subset G(k)(\rho\tilde{G}(k_{T-S}) \times K).$$

D'après 2.0.5, pour K assez petit, on a dans $G(A_T)$: $G(k) \cap K \subset \Gamma_T$, d'où dans $G(A_S)$: $G(k) \cap (\rho\tilde{G}(k_{T-S}) \times K) \subset \Gamma_S \subset \rho\tilde{G}(A_S)$. L'intersection de $G(k)\rho\tilde{G}(A_S)$ avec le sous-groupe ouvert $\rho\tilde{G}(k_{T-S}) \times K$ est donc contenue dans $\rho\tilde{G}(A_S)$, et le corollaire en résulte.

Corollaire 2.0.9. *Si $\tilde{G}$ vérifie le théorème d'approximation forte rel. S, l'adhérence de $G(k)$ dans $G(A_S)$ est $G(k)\cdot\rho\tilde{G}(A_S)$.*

2.0.10. Soit T un tore sur k et S un ensemble fini de places contenant les places archimédiennes. Soit $U \subset T(k)$ le groupe de S-unités. D'après un théorème de Chevaelly, tout sous-groupe d'indice fini de U est un sous-groupe de congruence (voir **[12]** pour une démonstration élégante). Il en résulte que si $T' \to T$ est une isogénie, l'image d'un sous-groupe de congruence pour T' est un sous-groupe de congruence pour T.

2.0.11. Soient G réductif sur k, $\rho: \tilde{G} \to G^{\mathrm{der}}$ le revêtement universel de son groupe dérivé, et Z^0 la composante neutre de son centre Z. Voici quelques corollaires de 2.0.10 (on suppose que l'ensemble fini S de places contient les places archimédiennes).

Corollaire 2.0.12. *Pour U d'indice fini dans le groupe des S-unités de $Z(k)$ il existe un sous-groupe compact ouvert K de $G(A_S)$ tel que*

$$G(k) \cap (K \cdot G^{\mathrm{der}}(A_S)) \subset G^{\mathrm{der}}(k) \cdot U.$$

Appliquons 2.0.10 à l'isogénie $Z^0 \to G/G^{\mathrm{der}}$: pour K petit, un élément γ de $G(k)$ dans $K\cdot G^{\mathrm{der}}(A_S)$ a dans $(G/G^{\mathrm{der}})(k)$ une image petite, pour la topologie des sous-groupes de S-congruence, donc peut se relever un petit élément z de $Z(k)$, et $\gamma = (\gamma z^{-1})\cdot z$.

Corollaire 2.0.13. *Le produit d'un sous-groupe de congruence de G^{der} et d'un sous-groupe d'indice fini du groupe des S-unités de $Z^0(k)$ est un sous-groupe de S-congruence de $G(k)$.*

COROLLAIRE 2.0.14. *Tout sous-groupe de S-congruence assez petit de $G(k)$ est contenu dans la composante neutre topologique $G(\boldsymbol{R})^+$ de $G(\boldsymbol{R})$.*

Appliquer 2.0.13, 2.0.7 à G^{der}, et 2.0.10 à Z^0.

2.0.15. On sait que $G^{\mathrm{der}}(k)\rho\tilde{G}(\boldsymbol{A})$ est ouvert dans $G(k)\rho\tilde{G}(\boldsymbol{A})$ (car image inverse de $\{e\} \subset$ le sous-groupe discret $G/G^{\mathrm{der}}(k)$ de $(G/G^{\mathrm{der}})(\boldsymbol{A})$). D'après 2.0.8, $G(k)\rho\tilde{G}(\boldsymbol{A})$ est donc un sous-groupe fermé de $G(\boldsymbol{A})$. On pose

$$\pi(G) = G(\boldsymbol{A})/G(k)\rho\tilde{G}(\boldsymbol{A}). \tag{2.0.15.1}$$

L'existence de commutateurs 2.0.2 montre que l'action de $G^{\mathrm{ad}}(k)$ sur $\pi(G)$, déduite de l'action 2.0.2 de G^{ad} sur G, est triviale.

2.1. *Variétés de Shimura.*

2.1.1. Soient G un groupe réductif, défini sur $\boldsymbol{Q}$, et X une classe de $G(\boldsymbol{R})$-conjugaison de morphismes de groupes algébriques réels de $\boldsymbol{S}$ dans $G_{\boldsymbol{R}}$. On suppose vérifiés les axiomes suivant (les notations sont celles de 1.1.1 et 1.1.11):

(2.1.1.1) Pour $h \in X$, $\mathrm{Lie}(G_{\boldsymbol{R}})$ est de type $\{(-1, 1), (0, 0), (1, -1)\}$.

(2.1.1.2) L'involution int $h(i)$ est une involution de Cartan du groupe adjoint $G^{\mathrm{ad}}_{\boldsymbol{R}}$.

(2.1.1.3) Le groupe adjoint n'admet pas de facteur G' défini sur $\boldsymbol{Q}$ sur lequel la projection de h soit triviale.

L'axiome 2.1.1 assure que le morphisme w_h ($h \in X$) est à valeurs dans le centre de G, donc est indépendant de h. On le note w_X, ou simplement w. Quelques simplifications apparaissent lorsqu'on suppose que:

(2.1.1.4) Le morphisme $w : \boldsymbol{G}_m \to G_{\boldsymbol{R}}$ est défini sur $\boldsymbol{Q}$.

(2.1.1.5) int $h(i)$ est une involution de Cartan du groupe $(G/w(\boldsymbol{G}_m))_{\boldsymbol{R}}$.

D'après 1.1.14(i), X admet une unique structure complexe telle que, pour toute représentation V de $G_{\boldsymbol{R}}$, la filtration de Hodge F_h de V varie holomorphiquement avec h. Pour cette structure complexe, les composantes connexes de X sont des domaines hermitiens symétriques. La preuve de 1.1.17 montre aussi que si l'on décompose $G^{\mathrm{ad}}_{\boldsymbol{R}}$ en facteurs simples, h se projette trivialement sur les facteurs compacts, et que chaque composante connexe de X est le produit d'espaces hermitiens symétriques correspondant aux facteurs non compacts. L'axiome 2.1.1.3 peut encore s'exprimer en disant que G^{ad} (resp. $\tilde{G}$, cela revient au même) n'a pas de facteur G' (défini sur $\boldsymbol{Q}$) tel que $G'(\boldsymbol{R})$ soit compact, et le théorème d'approximation forte assure que $\tilde{G}(\boldsymbol{Q})$ est dense dans $\tilde{G}(\boldsymbol{A}^f)$.

2.1.2. Les *variétés de Shimura* ${}_KM_C(G, X)$—ou simplement ${}_KM_C$—sont les quotients ${}_KM_C(G, X) = G(\boldsymbol{Q})\backslash X \times (G(\boldsymbol{A}^f)/K)$ pour K un sous-groupe compact ouvert de $G(\boldsymbol{A}^f)$. D'après 1.2.7, et avec les notations de 0.3, l'action de $G(\boldsymbol{R})$ sur X fait de $\pi_0(X)$ un espace principal homogène sous $G(\boldsymbol{R})/G(\boldsymbol{R})_+$. Puisque $G(\boldsymbol{Q})$ est dense dans $G(\boldsymbol{R})$ (théorème d'approximation réel), on a $G(\boldsymbol{Q})/G(\boldsymbol{Q})_+ \xrightarrow{\sim} G(\boldsymbol{R})/G(\boldsymbol{R})_+$, et, si X^+ est une composante connexe de X, on a

$${}_KM_C(G, X) = G(\boldsymbol{Q})_+\backslash X^+ \times (G(\boldsymbol{A}^f)/K).$$

Ce quotient est la somme disjointe, indexée par l'ensemble fini $G(\boldsymbol{Q})_+\backslash G(\boldsymbol{A}^f)/K$ de doubles classes, des quotients $\Gamma_g\backslash X^+$ du domaine hermitien symétrique X^+ par les images $\Gamma_g \subset G^{\mathrm{ad}}(\boldsymbol{R})^+$ des sous-groupes $\Gamma'_g = gKg^{-1} \cap G(\boldsymbol{Q})_+$ de $G(\boldsymbol{Q})_+$. Les Γ_g sont des groupes arithmétiques, d'où une structure d'espace analytique sur $\Gamma_g\backslash X^+$.

L'article [2] fournit une structure naturelle de variété algébrique quasi-projective sur ces quotients, donc sur ${}_K M_C(G, X)$. Si Γ_g est sans torsion (tel est le cas pour K assez petit); il résulte de [3] que cette structure est unique. Plus précisément, pour tout schéma réduit Z, un morphisme analytique de Z dans $\Gamma_g \backslash X^+$ est automatiquement algébrique.

2.1.3. On a

$$\pi_0\, {}_K M_C = G(\boldsymbol{Q}) \backslash \pi_0(X) \times (G(\boldsymbol{A}^f)/K) = G(\boldsymbol{Q}) \backslash G(\boldsymbol{A})/G(\boldsymbol{R})_+ \times K = G(\boldsymbol{Q})_+ \backslash G(\boldsymbol{A}^f)/K.$$

Puisque $G(\boldsymbol{A}^f)/K$ est discret, on peut remplacer $G(\boldsymbol{Q})_+$ par son adhérence dans $G(\boldsymbol{A}^f)$. La connexité de $\tilde{G}(\boldsymbol{R})$ assure que $\rho\tilde{G}(\boldsymbol{Q}) \subset G(\boldsymbol{Q})_+$. Par le théorème d'approximation forte pour $\tilde{G}$, $\rho\tilde{G}(\boldsymbol{Q})$ est dense dans $\rho\tilde{G}(\boldsymbol{A}^f)$, et $G(\boldsymbol{Q})_+^- \supset \rho\tilde{G}(\boldsymbol{A}^f)$. Dès lors,

$$(2.1.3.1) \qquad \begin{aligned} \pi_0\, {}_K M_C &= G(\boldsymbol{Q})_+ \rho\tilde{G}(\boldsymbol{A}^f) \backslash G(\boldsymbol{A}^f)/K \\ &= G(\boldsymbol{A}^f)/\rho\tilde{G}(\boldsymbol{A}^f) \cdot G(\boldsymbol{Q})_+ \cdot K = G(\boldsymbol{A}^f)/G(\boldsymbol{Q})_+ \cdot K, \end{aligned}$$

puisque $\rho\tilde{G}(\boldsymbol{A}^f)$ est un sous-groupe distingué, avec un quotient abélien. De même, posant $\bar{\pi}_0\pi(G) = \pi_0\pi(G)/\pi_0 G(\boldsymbol{R})_+$, on a

$$(2.1.3.2) \qquad \begin{aligned} \pi_0\, {}_K M_C &= G(\boldsymbol{A})/\rho\tilde{G}(\boldsymbol{A}) \cdot G(\boldsymbol{Q}) \cdot G(\boldsymbol{R})_+ \times K \\ &= \pi(G)/G(\boldsymbol{R})_+ \times K = \bar{\pi}_0\pi(G)/K. \end{aligned}$$

En particulier, $\pi_0\, {}_K M_C$ ne dépend que de l'image de K dans $G(\boldsymbol{A})/\rho\tilde{G}(\boldsymbol{A})$.

2.1.4. Pour K variable (de plus en plus petit), les ${}_K M_C$ forment un système projectif. Il est muni d'une action à droite de $G(\boldsymbol{A}^f)$: un système d'isomorphismes $g\colon {}_K M_C \xrightarrow{\sim} {}_{g^{-1}Kg} M_C$. Il est commode de considérer plutôt le schéma $M_C(G, X)$ —ou simplement M_C—limite projective des ${}_K M_C$. La limite projective existe parce que les morphismes de transition sont finis. Ce schéma est muni d'une action à droite de $G(\boldsymbol{A}^f)$, et il redonne les ${}_K M_C$: ${}_K M_C = M_C/K$.

Nous nous proposons de déterminer M_C et sa décomposition en composantes connexes.

DÉFINITION 2.1.5. *Fixons une composante connexe X^+ de X. La* composante neutre *M_C^0 de M_C est la composante connexe qui contient l'image de $X^+ \times \{e\} \subset X \times G(\boldsymbol{A}^f)$.*

DÉFINITION 2.1.6. *Soient G_0 un groupe adjoint sur $\boldsymbol{Q}$, sans facteur G_0' défini sur $\boldsymbol{Q}$ tel que $G_0'(\boldsymbol{R})$ soit compact et G_1 un revêtement de G_0. La* topologie *$\tau(G_1)$ sur $G_0(\boldsymbol{Q})$ est celle admettant pour système fondamental de voisinages de l'origine les images des sous-groupes de congruence de $G_1(\boldsymbol{Q})$.*

Nous noterons $^\wedge$ (rel. G_1), ou simplement $^\wedge$, la complétion pour cette topologie. Soit $\rho : \tilde{G}_0 \to G_1$ l'application naturelle, notons $^-$ l'adhérence dans $G_1(\boldsymbol{A}^f)$, et posons $\Gamma = \rho\tilde{G}_0(\boldsymbol{A}) \cap G_1(\boldsymbol{Q})$. Puisque $\tilde{G}_0(\boldsymbol{R})$ est connexe, $\Gamma \subset G_1(\boldsymbol{Q})^+$. On a (2.0.9, 2.0.14)

$$(2.1.6.1) \qquad G_0(\boldsymbol{Q})^\wedge(\text{rel. } G_1) = G_1(\boldsymbol{Q})^- *_{G_1(\boldsymbol{Q})} G_0(\boldsymbol{Q}) = \rho\tilde{G}_0(\boldsymbol{A}^f) *_\Gamma G_0(\boldsymbol{Q}),$$

$$(2.1.6.2) \qquad G_0(\boldsymbol{Q})^{+\wedge}(\text{rel. } G_1) = G_1(\boldsymbol{Q})_+^- *_{G_1(\boldsymbol{Q})_+} G_0(\boldsymbol{Q})^+ = \rho\tilde{G}_0(\boldsymbol{A}^f) *_\Gamma G_0(\boldsymbol{Q})^+.$$

PROPOSITION 2.1.7. *La composante neutre M_C^0 est la limite projective des quotients $\Gamma \backslash X^+$, pour Γ un sous-groupe arithmétique de $G^{\text{ad}}(\boldsymbol{Q})^+$, ouvert pour la topologie $\tau(G^{\text{der}})$.*

D'après 2.1.2, c'est la limite des $\Gamma\backslash X^0$, pour Γ l'image d'un sous-groupe de congruence de $G(\boldsymbol{Q})_+$. Le Corollaire 2.0.13 permet de remplacer G par G^{der}.

2.1.8. La projection de G dans G^{ad} induit un isomorphisme de X^+ avec une classe de $G(\boldsymbol{R})^+$-conjugaison de morphismes de $\boldsymbol{S}$ dans $G^{\mathrm{ad}}_{\boldsymbol{R}}$ et, d'après 2.1.7, $M^0_{\boldsymbol{C}}(G, X)$ ne dépend que de G^{ad}, G^{der} et de cette classe. Formalisons cette remarque. Soient G un groupe adjoint, X^+ une classe de $G(\boldsymbol{R})^+$-conjugaison de morphismes de $\boldsymbol{S}$ dans $G(\boldsymbol{R})$ qui vérifie (2.1.1), (2.1.2), (2.1.3) et G_1 un revêtement de G. Les *variétés connexes de Shimura* (rel. G, G_1, X^+) sont les quotients $\Gamma\backslash X^+$, pour Γ un sous-groupe arithmétique de $G(\boldsymbol{Q})^+$, ouvert pour la topologie $\tau(G_1)$. On note $M^0_{\boldsymbol{C}}(G, G_1, X^+)$ leur limite projective, pour Γ de plus en plus petit. On notera que l'action par transport de structure de $G(\boldsymbol{Q})^+$ sur $M^0_{\boldsymbol{C}}(G, G_1, X^+)$ se prolonge par continuité en une action du complété $G(\boldsymbol{Q})^{+\wedge}$ (rel. G_1).

Avec les notations de 2.1.7, et l'identification ci-dessus de X^+ avec une classe de $G(\boldsymbol{R})^+$-conjugaison de morphismes de $\boldsymbol{S}$ dans $G^{\mathrm{ad}}_{\boldsymbol{R}}$, on a

$$M^0_{\boldsymbol{C}}(G, X) = M^0_{\boldsymbol{C}}(G^{\mathrm{ad}}, G^{\mathrm{der}}, X^+).$$

2.1.9. Soient Z le centre de G, et $Z(\boldsymbol{Q})^-$ l'adhérence de $Z(\boldsymbol{Q})$ dans $Z(\boldsymbol{A}^f)$. D'après Chevalley (2.0.10), c'est le complété de $Z(\boldsymbol{Q})$ pour la topologie des sous-groupes d'indice fini du groupe des unités; il reçoit isomorphiquement l'adhérence de $Z(\boldsymbol{Q})$ dans $\pi_0 Z(\boldsymbol{R}) \times Z(\boldsymbol{A}^f)$.

Pour $K \subset G(\boldsymbol{A}^f)$ compact ouvert, on a $Z(\boldsymbol{Q}) \cdot K = Z(\boldsymbol{Q})^- \cdot K$ (dans $Z(\boldsymbol{A}^f)$), et

$$\begin{aligned} {}_K M_{\boldsymbol{C}} &= G(\boldsymbol{Q})\backslash X \times (G(\boldsymbol{A}^f)/K) = \frac{G(\boldsymbol{Q})}{Z(\boldsymbol{Q})}\Big\backslash X \times (G(\boldsymbol{A}^f)/Z(\boldsymbol{Q})\cdot K) \\ &= \frac{G(\boldsymbol{Q})}{Z(\boldsymbol{Q})}\Big\backslash X \times (G(\boldsymbol{A}^f)/Z(\boldsymbol{Q})^- \cdot K). \end{aligned}$$

L'action de $G(\boldsymbol{Q})/Z(\boldsymbol{Q})$ sur $X \times (G(\boldsymbol{A}^f)/Z(\boldsymbol{Q})^-)$ est propre. Ceci permet le passage à la limite sur K:

PROPOSITION 2.1.10. *On a*

$$M_{\boldsymbol{C}}(G, X) = \frac{G(\boldsymbol{Q})}{Z(\boldsymbol{Q})}\Big\backslash X \times (G(\boldsymbol{A}^f)/Z(\boldsymbol{Q})^-).$$

COROLLAIRE 2.1.11. *Si les conditions* 2.1.4 *et* 2.1.5 *sont vérifiées, on a* $M_{\boldsymbol{C}}(G, X) = G(\boldsymbol{Q})\backslash X \times G(\boldsymbol{A}^f)$.

Dans ce cas, $Z(\boldsymbol{Q})$ est discret dans $Z(\boldsymbol{A}^f)$ et $Z(\boldsymbol{Q})^- = Z(\boldsymbol{Q})$.

COROLLAIRE 2.1.12. *L'action à droite de $G(\boldsymbol{A}^f)$ se factorise par $G(\boldsymbol{A}^f)/Z(\boldsymbol{Q})^-$.*

2.1.13. Soient $G^{\mathrm{ad}}(\boldsymbol{R})_1$ l'image de $G(\boldsymbol{R})$ dans $G^{\mathrm{ad}}(\boldsymbol{R})$, et $G^{\mathrm{ad}}(\boldsymbol{Q})_1 = G^{\mathrm{ad}}(\boldsymbol{Q}) \cap G^{\mathrm{ad}}(\boldsymbol{R})_1$. L'action 2.0.2 de G^{ad} sur G induit une action (à gauche) de $G^{\mathrm{ad}}(\boldsymbol{Q})_1$ sur le système des ${}_K M_{\boldsymbol{C}}$

$$\mathrm{int}(\gamma)\colon {}_K M_{\boldsymbol{C}} \xrightarrow{\sim} {}_{\gamma K \gamma^{-1}} M_{\boldsymbol{C}},$$

et à la limite sur $M_{\boldsymbol{C}}$. Pour $\gamma \in G^{\mathrm{ad}}(\boldsymbol{Q})^+$, cette action stabilise la composante neutre (donc toutes les composantes, cf. ci-après) et y induit l'action 2.1.6.

Convertissons cette action en une action à droite, notée $\cdot\gamma$. Si γ est l'image de

$\delta \in G(\mathbf{Q})$, l'action $\cdot\gamma$ coïncide avec l'action de δ, vu comme élément de $G(\mathbf{A}^f)$: pour $u \in M_{\mathbf{C}}$ image de $(x, g) \in X \times G(\mathbf{A}^f)$, $u \cdot \gamma$ est image de

$$(\gamma^{-1}(x), \mathrm{int}_\gamma^{-1}(g)) = (\delta^{-1}(x), \delta^{-1}g\delta) \sim (x, g\delta) \bmod G(\mathbf{Q}) \quad \text{à gauche.}$$

Au total, nous obtenons ainsi une action à droite sur $M_{\mathbf{C}}$ du groupe

$$(2.1.13.1) \qquad \frac{G(\mathbf{A}^f)}{Z(\mathbf{Q})^-} *_{G(\mathbf{Q})/Z(\mathbf{Q})} G^{\mathrm{ad}}(\mathbf{Q})_1 = \frac{G(\mathbf{A}^f)}{Z(\mathbf{Q})^-} *_{G(\mathbf{Q})_+/Z(\mathbf{Q})} G^{\mathrm{ad}}(\mathbf{Q})^+.$$

PROPOSITION 2.1.14. *L'action à droite de $G(\mathbf{A}^f)$ sur $\pi_0 M_{\mathbf{C}}$ fait de $\pi_0 M_{\mathbf{C}}$ un espace principal homogène sous son quotient abélien $G(\mathbf{A}^f)/G(\mathbf{Q})_+^- = \bar{\pi}_0\pi(G)$.*

Cela résulte aussitôt par passage à la limite des formules 2.1.3.

2.1.15. Puisque $G^{\mathrm{ad}}(\mathbf{Q})$ agit trivialement sur $\pi(G)$ (2.0.15), et que $G^{\mathrm{ad}}(\mathbf{Q})^+$ stabilise au moins une composante connexe (2.1.13), le groupe $G^{\mathrm{ad}}(\mathbf{Q})^+$ les stabilise toutes. Pour l'action 2.1.13 du groupe (2.1.13.1) sur $M_{\mathbf{C}}$, le stabilisateur de chaque composante connexe est donc

$$(2.1.15.1) \qquad \frac{G(\mathbf{Q})_+^-}{Z(\mathbf{Q})^-} *_{G(\mathbf{Q})_+/Z(\mathbf{Q})} G^{\mathrm{ad}}(\mathbf{Q})^+ =_{2.0.13} G^{\mathrm{ad}}(\mathbf{Q})^{+\wedge}(\mathrm{rel.}\ G^{\mathrm{der}}).$$

Résumé 2.1.16. *Le groupe $G(\mathbf{A}^f)/Z(\mathbf{Q})^- *_{G(\mathbf{Q})/Z(\mathbf{Q})} G^{\mathrm{ad}}(\mathbf{Q})_1$ agit à droite sur $M_{\mathbf{C}}$. L'ensemble profini $\pi_0 M_{\mathbf{C}}$ est un espace principal homogène sous l'action du quotient abélien $G(\mathbf{A}^f/G(\mathbf{Q})_+^- = \bar{\pi}_0\pi(G)$ de ce groupe par l'adhérence de $G^{\mathrm{ad}}(\mathbf{Q})^+$. Cette adhérence est le complété de $G^{\mathrm{ad}}(\mathbf{Q})^+$ pour la topologie des images des sous-groupes de congruence de $G^{\mathrm{der}}(\mathbf{Q})$. L'action de ce complété sur la composante neutre, une fois convertie en une action à gauche, est l'action* 2.1.8.

2.2. *Modèles canoniques.*

2.2.1. Soient G et X comme en 2.1.1. Pour $h \in X$, le morphisme μ_h (1.1.1, complété par 1.1.11) est un morphisme sur $\mathbf{C}$ de groupes algébriques définis sur $\mathbf{Q}$: $\mu_h\colon \mathbf{G}_m \to G_{\mathbf{C}}$. Le *corps dual* (= reflex field) $E(G, X) \subset \mathbf{C}$ de (G, X) est le corps de définition de sa classe de conjugaison. Si X^+ est une composante connexe de X, on le notera parfois $E(G, X^+)$.

Soient (G', X') et (G'', X'') comme en 2.1.1. Si un morphisme $f\colon G' \to G''$ envoie X' dans X'', on a $E(G', X') \supset E(G'', X'')$.

2.2.2. Soient T un tore, E un corps de nombres, et μ un morphisme, défini sur E, de $\mathbf{G}_m$ dans T_E. Le groupe E^*, vu comme groupe algébrique sur $\mathbf{Q}$, est la restriction des scalaires à la Weil $R_{E/\mathbf{Q}}(\mathbf{G}_m)$. Appliquant $R_{E/\mathbf{Q}}$ à μ, on obtient $R_{E/\mathbf{Q}}(\mu)$: $E^* \to R_{E/\mathbf{Q}}T_E$.

On dispose aussi du morphisme norme $N_{E/\mathbf{Q}}: R_{E/\mathbf{Q}}T_E \to T$ (sur les points rationnels, c'est la norme, $T(E) \to T(\mathbf{Q})$). D'où par composition un morphisme $N_{E/\mathbf{Q}} \circ R_{E/\mathbf{Q}}(\mu) : E^* \to T$. Nous le noterons simplement $NR_E(\mu)$, ou même $NR(\mu)$. Si E' est une extension de E, μ est encore défini sur E', et

$$(2.2.2.1) \qquad NR_{E'}(\mu) = NR_E(\mu) \circ N_{E'/E}.$$

2.2.3. Soient en particulier T un tore, $h : \mathbf{S} \to T_{\mathbf{R}}$ et $X = \{h\}$. Si $E \subset \mathbf{C}$ contient $E(T, X)$, le morphisme μ_h est défini sur E, d'où un morphisme $NR(\mu_h) : E^* \to T$. Passant aux points adéliques modulo les points rationnels, on en déduit un homomorphisme du groupe des classes d'idèles $C(E)$ de E dans $T(\mathbf{Q})/T(\mathbf{A})$, et, par passage aux ensembles de composantes connexes, un morphisme

$$\pi_0 NR(\mu_h) : \pi_0 C(E) \to \pi_0(T(\mathbf{Q})/T(\mathbf{A})).$$

La théorie du corps de classe global identifie $\pi_0 C(E)$ au groupe de Galois rendu abélien de E.

Le groupe $\pi_0(T(\mathbf{Q})\backslash T(\mathbf{A}))$ est un groupe profini, limite projective des groupes finis $T(\mathbf{Q})\backslash T(\mathbf{A})/T(\mathbf{R})^+ \times K$ pour K compact ouvert dans $T(\mathbf{A}^f)$. C'est $\pi_0 T(\mathbf{R}) \times T(\mathbf{A}^f)/T(\mathbf{Q})^-$. Les variétés de Shimura ${}_K M_{\mathbf{C}}(T, X)$ sont les ensembles finis $T(\mathbf{Q})\backslash\{h\} \times T(\mathbf{A}^f)/K = T(\mathbf{Q})\backslash T(\mathbf{A}^f)/K$. Leur limite projective $T(\mathbf{A}^f)/T(\mathbf{Q})^-$, calculée en 2.1.10 est le quotient de $\pi_0(T(\mathbf{Q})\backslash T(\mathbf{A}))$ par $\pi_0 T(\mathbf{R})$.

Nous appellerons *morphisme de réciprocité* le morphisme $r_E(T, X)$: $\mathrm{Gal}(\bar{\mathbf{Q}}/E)^{\mathrm{ab}} \to T(\mathbf{A}^f)/T(\mathbf{Q})^-$ *inverse* du composé de l'isomorphisme de la théorie du corps de classe global (0.8), de $\pi_0 NR(\mu_h)$ et de la projection de $\pi_0 T(\mathbf{A})/T(\mathbf{Q})$ sur $T(\mathbf{A}^f)/T(\mathbf{Q})^-$. Il définit une action r_E de $\mathrm{Gal}(\bar{\mathbf{Q}}/E)^{\mathrm{ab}}$ sur les ${}_K M_{\mathbf{C}}(T, X)$: $\sigma \mapsto$ la translation à droite par $r_E(T, X)(\sigma)$.

Le cas universel (en E) est celui où $E = E(T, X)$: il résulte de (2.2.2.1) que l'action r_E de $\mathrm{Gal}(\bar{\mathbf{Q}}/E)$ est la restriction à $\mathrm{Gal}(\bar{\mathbf{Q}}/E) \subset \mathrm{Gal}(\bar{\mathbf{Q}}/E(T, X))$ de $r_{E(T,X)}$.

2.2.4. Soient G et X comme en 2.1.1. Un point $h \in X$ est dit *spécial*, ou *de type CM*, si $h: \mathbf{S} \to G(\mathbf{R})$ se factorise par un tore $T \subset G$ défini sur $\mathbf{Q}$, On notera que si T est un tel tore, l'involution de Cartan int $h(i)$ est triviale sur l'image de $T(\mathbf{R})$ dans le groupe adjoint, et que cette image est donc compacte. Le corps $E(T, \{h\})$ ne dépend que de h. C'est le *corps dual* $E(h)$ de h.

Nous transporterons cette terminologie aux points de ${}_K M_{\mathbf{C}}(G, X)$ et de $M_{\mathbf{C}}(G, X)$: pour $x \in {}_K M_{\mathbf{C}}(G, X)$ (resp. $M_{\mathbf{C}}(G, X)$), classe de $(h, g) \in X \times G(\mathbf{A}^f)$, la classe de $G(\mathbf{Q})$-conjugaison de h ne dépend que de x. Nous dirons que x est *spécial* si h l'est, que $E(h)$ est le *corps dual* $E(x)$ de x, et que la classe de $G(\mathbf{Q})$-conjugaison de h est le *type* de x.

Sur l'ensemble des points spéciaux de ${}_K M_{\mathbf{C}}(G, X)$ (resp. de $M_{\mathbf{C}}(G, X)$) de type donné, correspondant à un corps dual E, nous allons définir une action r de $\mathrm{Gal}(\bar{\mathbf{Q}}/E)$. Soient donc $x \in {}_K M_{\mathbf{C}}(G, X)$ (resp. $M_{\mathbf{C}}(G, X)$), classe de $(h, g) \in X \times G(\mathbf{A}^f)$, $T \subset G$ un tore défini sur $\mathbf{Q}$ par lequel se factorise h, $\sigma \in \mathrm{Gal}(\bar{\mathbf{Q}}/E)$, et $\tilde{r}(\sigma)$ un représentant dans $T(\mathbf{A}^f)$ de $r_E(T, \{h\})(\sigma) \in T(\mathbf{A}^f)/T(\mathbf{Q})^-$. On pose $r(\sigma)x =$ classe de $(h, \tilde{r}(\sigma)g)$. Le lecteur vérifiera que cette classe ne dépend que de x et de σ. L'action ainsi définie commute à l'action à droite de $G(\mathbf{A}^f)$ sur $M_{\mathbf{C}}(G, X)$.

2.2.5. Un *modèle canonique* $M(G, X)$ de $M_{\mathbf{C}}(G, X)$ est une forme sur $E(G, X)$ de $M_{\mathbf{C}}(G, X)$, muni de l'action à droite de $G(\mathbf{A}^f)$, telle que

(a) les points spéciaux sont algébriques;

(b) sur l'ensemble des points spéciaux de type τ donné, correspondant à un corps dual $E(\tau)$, le groupe de Galois $\mathrm{Gal}(\bar{\mathbf{Q}}/E(\tau)) \subset \mathrm{Gal}(\bar{\mathbf{Q}}/E(G, X))$ agit par l'action 2.2.4.

Par "forme" nous entendons: un schéma M sur $E(G, X)$, muni d'une action à droite de $G(\mathbf{A}^f)$, et d'un isomorphisme équivariant $M \otimes_{E(G,X)} \mathbf{C} \xrightarrow{\sim} M_{\mathbf{C}}(G, X)$.

Soit $E \subset \mathbf{C}$ un corps de nombres qui contient $E(G, X)$. Un modèle *faiblement canonique* de $M_{\mathbf{C}}(G, X)$ sur E est une forme sur E de $M_{\mathbf{C}}(G, X)$, muni de l'action à droite de $G(\mathbf{A}^f)$, qui vérifie (a) et

(b*) même condition que (b), avec $\mathrm{Gal}(\bar{\mathbf{Q}}/E(\tau))$ remplacé par $\mathrm{Gal}(\bar{\mathbf{Q}}/E(\tau)) \cap \mathrm{Gal}(\bar{\mathbf{Q}}/E)$.

2.2.6. Dans [**5**, 5.4, 5.5], nous inspirant de méthodes de Shimura, nous avons montré que $M_{\mathbf{C}}(G, X)$ admet au plus un modèle faiblement canonique sur E (pour $E(G, X) \subset E \subset \mathbf{C}$), et que, lorsqu'il existe, il est fonctoriel en (G, X).

2.3. *Construction de modèles canoniques.*

Dans ce numéro, nous déterminons des cas où s'applique le critère suivant, démontré dans [**5**, 4.21, 5.7], pour construire des modèles canoniques.

Critère 2.3.1. *Soient* (G, X) *comme en* 2.1.1, V *un espace vectoriel rationnel, muni d'une forme alternée non dégénérée* Ψ, *et* $S^{\pm}$ *le double demi-espace de Siegel correspondant* (*cf.* 1.3.1). *S'il existe un plongement* $G \hookrightarrow \mathrm{CSp}(V)$, *qui envoie* X *dans* $S^{\pm}$, *alors* $M_{\mathbf{C}}(G, X)$ *admet un modèle canonique* $M(G, X)$.

Proposition 2.3.2. *Soient* (G, X) *comme en* 2.1.1, $w = w_h$ $(h \in X)$ *et* (V, ρ) *une représentation fidèle de type* $\{(-1, 0), (0, -1)\}$ *de* G. *Si* int $h(i)$ *est une involution de Cartan de* $G_{\mathbf{R}}/w(\mathbf{G}_m)$, *il existe une forme alternée* Ψ *sur* V, *telle que* ρ *induise* $(G, X) \hookrightarrow (\mathrm{CSp}(V), S^{\pm})$.

Par hypothèse, la représentation fidèle V est homogène de poids -1. Le poids w est donc défini sur $\mathbf{Q}$, et on prend pour Ψ une forme de polarisation comme en 1.1.18(b).

Corollaire 2.3.3. *Soient* (G, X) *comme en* 2.1.1, $w = w_h$ $(h \in X)$, *et* (V, ρ) *une représentation fidèle de type* $\{(-1, 0), (0, -1)\}$ *de* G. *Si le centre* Z^0 *de* G *se déploye sur un corps de type CM, il existe un sous-groupe* G_2 *de* G, *de même groupe dérivé et par lequel se factorise* X, *et une forme alternée* Ψ *sur* V, *telle que* ρ *induise* $(G_2, X) \hookrightarrow (\mathrm{CSp}(V), S^{\pm}))$.

L'hypothèse sur Z^0 revient à dire que le plus grand sous-tore compact de $Z^0_{\mathbf{R}}$ est défini sur $\mathbf{Q}$. On prend G_2 engendré par le groupe dérivé, ce tore, et l'image de w, et on applique 2.3.2.

2.3.4. Soit (G, X) comme en 2.1.1, avec G $\mathbf{Q}$-simple adjoint. L'axiome (2.1.1.2) assure que $G_{\mathbf{R}}$ *est forme intérieure de sa forme compacte*. Exploitons ce fait.

(a) Les composantes simples de $G_{\mathbf{R}}$ sont absolument simples. Si on écrit G comme obtenu par restriction des scalaires à la Weil: $G = R_{F/\mathbf{Q}}G^s$ avec G^s absolument simple sur F, cela signifie que F est totalement réel. Posons les notations: $I =$ l'ensemble des plongements réels de F, et, pour $v \in I$, $G_v = G^s \otimes_{F,v} \mathbf{R}$, $D_v =$ diagramme de Dynkin de $G_{v\mathbf{C}}$. On a $G_{\mathbf{R}} = \prod G_v$, $G_{\mathbf{C}} = \prod G_{v\mathbf{C}}$, et le diagramme de Dynkin D de $G_{\mathbf{C}}$ est la somme disjointe des D_v. Le groupe de Galois $\mathrm{Gal}(\bar{\mathbf{Q}}/\mathbf{Q})$ agit sur D et I, de façon compatible à la projection de D sur I.

(b) La conjugaison complexe agit sur D par l'involution d'opposition. Celle-ci est centrale dans $\mathrm{Aut}(D)$. Dès lors, $\mathrm{Gal}(\bar{\mathbf{Q}}/\mathbf{Q})$ agit sur D via une action fidèle de $\mathrm{Gal}(K_D/\mathbf{Q})$, avec K_D totalement réel si l'involution d'opposition est triviale, quadratique totalement imaginaire sur un corps totalement réel sinon.

2.3.5. On a $X = \prod X_v$, pour X_v une classe de $G_v(\mathbf{R})$-conjugaison de morphismes de $\mathbf{S}$ dans G_v. Pour G_v compact, X_v est trivial. Pour G_v noncompact, X_v est décrit par un sommet s_v du diagramme de Dynkin D_v de $G_{v\mathbf{C}}$ (1.2.6).

Quelques notations: $I_c =$ l'ensemble des $v \in I$ tels que G_v soit compact, $I_{nc} = I - I_c$, D_c (resp. D_{nc}) $=$ la réunion des D_v pour $v \in I_c$ (resp. $v \in I_{nc}$), G_c (resp. G_{nc}) $=$ le produit des G_v pour $v \in I_c$ (resp. $v \in I_{nc}$); de même pour les revêtements universels; enfin, $\Sigma(X) =$ l'ensemble des s_v pour $v \in I_{nc}$. La définition 2.2.1 donne:

PROPOSITION 2.3.6. *Le corps dual de* (G, X) *est le sous-corps de* K_D *fixé par le sous-groupe de* $\mathrm{Gal}(K_D/\mathbf{Q})$ *qui stabilise* $\Sigma(X)$.

2.3.7. Supposons qu'il existe un diagramme

$$(G, X) \longleftarrow (G_1, X_1) \hookrightarrow (C\mathrm{Sp}(V), S^{\pm}). \tag{2.3.7.1}$$

Le revêtement universel $\tilde{G}$ de G se relève dans G_1, ce qui permet de restreindre la représentation V à $\tilde{G}$. Le quotient de $\tilde{G}$ qui agit fidèlement est par hypothèse le groupe dérivé de G_1. Appliquons 1.3.2, 1.3.8 au diagramme

$$(G_{nc}, X) \longleftarrow (\mathrm{Ker}(G_{1\mathbf{R}} \longrightarrow G_c)^0, X_1) \longrightarrow (C\mathrm{Sp}(V), S^{\pm}).$$

On trouve que les composantes irréductibles non triviales de la représentation $V_{\mathbf{C}}$ de $\tilde{G}_{nc\mathbf{C}}$ se factorisent par l'un des $\tilde{G}_{v\mathbf{C}}$ $(v \in I_{nc})$, et que leur poids dominant est fondamental, de l'un des types permis par la Table 1.3.9. L'ensemble des poids dominants des composantes irréductibles de la représentation $V_{\mathbf{C}}$ de $\tilde{G}_{\mathbf{C}}$ est stable par $\mathrm{Gal}(\bar{\mathbf{Q}}/\mathbf{Q})$. Puisque $\mathrm{Gal}(\bar{\mathbf{Q}}/\mathbf{Q})$ agit transitivement sur I, et que $I_{nc} \neq \varnothing$, on trouve que

(a) Toute composante irréductible W de $V_{\mathbf{C}}$ est de la forme $\bigotimes_{v \in T} W_v$, avec W_v une représentation fondamentale de $G_{v\mathbf{C}}$ $(v \in T \subset I)$, correspondant à un sommet $\tau(v)$ de D_v. Nous noterons $\mathscr{S}(V)$ l'ensemble des $\tau(T) \subset D$ pour $W \subset V_{\mathbf{C}}$ irréductible.

(b) Si $S \in \mathscr{S}(V)$, $S \cap D_{nc}$ est vide ou réduit à un seul point $s_S \in D_v$ $(v \in I_{nc})$, et, dans la Table 1.3.9 pour (D_v, s_v), s_S est l'un des sommets soulignés.

(c) $\mathscr{S}$ est stable par $\mathrm{Gal}(\bar{\mathbf{Q}}/\mathbf{Q})$. On a $\mathscr{S} \not\subset \{\varnothing\}$.

Si un ensemble de parties $\mathscr{S}$ de D vérifie (b) et (c), nous noterons $\tilde{G}(\mathscr{S})_{\mathbf{C}}$ le quotient de $\tilde{G}_{\mathbf{C}}$ qui agit fidèlement dans la représentation correspondante de $\tilde{G}_{\mathbf{C}}$. La condition (c) assure qu'il est défini sur $\mathbf{Q}$. Le cas le plus intéressant est celui où

(d) $\mathscr{S}$ est formé de parties à un élément.

Si $\mathscr{S}$ vérifie (b), (c), l'ensemble $\mathscr{S}'$ des $\{s\}$ pour $s \in S \in \mathscr{S}$ vérifie (b), (c), (d), et $\tilde{G}(\mathscr{S}')$ domine $\tilde{G}(\mathscr{S})$.

Dans la table ci-dessous—déduite de 1.3.9—nous donnons

La liste des cas où il existe $\mathscr{S}$ vérifiant (b), (c). D'après 1.3.10, ce ne peut être le cas que si G est de l'un des types A, B, C, D, et ces types seront successivement passés en revue.

L'ensemble $\mathscr{S}$ vérifiant (b), (c), (d) maximal, et le groupe $\tilde{G}(\mathscr{S})$ correspondant (il domine tous les $\tilde{G}(\mathscr{S})$, pour $\mathscr{S}$ vérifiant (b), (c)).

TABLE 2.3.8.

Types A, B, C. Le seul $\mathscr{S}$ vérifiant (b), (c), (d) est l'ensemble des $\{s\}$, pour s une extrémité—correspondant à une racine courte pour les types B, C—d'un diagramme D_v $(v \in I)$. Le revêtement $\tilde{G}(\mathscr{S})$ est le revêtement universel.

Type D_l $(l \geqq 5)$. Pour qu'il existe $\mathscr{S}$ vérifiant (b), (c), il faut et il suffit que les (G_v, X_v) $(v \in I_{nc})$ soient ou bien tous de type $\mathrm{D}_l^{\mathbf{R}}$, ou bien tous de type $\mathrm{D}_l^{\mathbf{H}}$. Distinguons ces cas :

Sous-cas $\mathrm{D}_l^{\mathbf{R}}$. Le $\mathscr{S}$ vérifiant (b), (c), (d) maximal est l'ensemble des $\{s\}$ pour s à l'extrémité "droite" d'un D_v. Le revêtement $\tilde{G}(\mathscr{S})$ est le revêtement universel.

Sous-cas $\mathrm{D}_l^{\mathbf{H}}$. L'unique $\mathscr{S}$ vérifiant (b), (c), (d) est l'ensemble des $\{s\}$ pour s

l'extrémité "gauche" d'un D_v. Le revêtement $\tilde{G}(\mathscr{S})$ de G est de la forme $R_{F/\mathbf{Q}}\tilde{G}^*$, pour $\tilde{G}^*$ le revêtement double de G forme de $SO(2l)$, cf. 1.3.10.

Type D_4. Remplaçons $\mathscr{S}$, vérifiant (d), par $S = \{s \mid \{s\} \in \mathscr{S}\}$. La condition (b) sur $\mathscr{S}$ devient: S est contenu dans l'ensemble E des extrémités de D, et $S \cap \Sigma(X) = \varnothing$. Pour la définition de $\Sigma(X)$ voir 2.3.5. La partie de E stable par $\mathrm{Gal}(\bar{\mathbf{Q}}/\mathbf{Q})$ maximale pour cette propriété est le complément de $\mathrm{Gal}(\bar{\mathbf{Q}}/\mathbf{Q}) \cdot \Sigma(X)$. Elle rencontre chaque D_v en 0, 1 ou 2 points. Dans le premier cas, il n'existe pas $\mathscr{S}$ vérifiant (b), (c). Dans le second (resp. 3è), elle (resp. son complément) est l'image d'une section τ de $X \to I$, invariante par Galois ; $\tau(I)$ est disjoint de (resp. contient) $\Sigma(X)$. Appelant $\tau(v)$ le sommet "gauche" de D_v, on retrouve la situation de D_l ($l \leqq 5$) :

Sous-cas $D_4^{\mathbf{R}}$. Il existe une section τ de $X \to I$ avec $\tau(I) \supset \Sigma(X)$. Cette section est alors unique, et la situation est la même qu'en $D_l^{\mathbf{R}}$, $l \geqq 5$.

Sous-cas $D_4^{\mathbf{H}}$. On donne une section τ de $X \to I$ avec $\tau(I) \cap \Sigma(X) = \varnothing$. Si on n'est pas dans le cas $D_4^{\mathbf{R}}$, cette section est unique, et l'unique $\mathscr{S}$ vérifiant (b), (c), (d) est l'ensemble des $\{s\}$ pour s l'extrémité "gauche" d'un D_v. Le revêtement $\tilde{G}(\mathscr{S})$ de G est de la forme $R_{F/\mathbf{Q}}\tilde{G}^s$, pour $\tilde{G}^s$ un revêtement double de G^s qui se décrit en terme de τ.

Pour la suite de ce travail, il nous sera commode de redéfinir le cas $D_4^{\mathbf{H}}$ comme excluant $D_4^{\mathbf{R}}$. Avec cette terminologie, il existe $\mathscr{S}$ vérifiant (b), (c) si et seulement si (G, X) est de l'un des types A, B, C, $D^{\mathbf{R}}$, $D^{\mathbf{H}}$ et, sauf pour le type $D^{\mathbf{H}}$, il existe $\mathscr{S}$ vérifiant (b), (c), (d) tel que $\tilde{G}(\mathscr{S})$ soit le revêtement universel de G.

2.3.9. Nous aurons à considérer des extensions quadratiques totalement imaginaires K de F, munies d'un ensemble T de plongements complexes: un au-dessus de chaque plongement réel $v \in I_c$. Un tel T définit une structure de Hodge h_T : $\mathbf{S} \to K_{\mathbf{R}}^*$ sur K (considéré comme un espace vectoriel rationnel, et sur lequel K^* agit par multiplication): si J est l'ensemble des plongements complexes de K, on a $K \otimes \mathbf{C} = \mathbf{C}^J$, et on définit h_T en exigeant que le facteur d'indice $\sigma \in J$ soit de type $(-1, 0)$ pour $\sigma \in T$, $(0, -1)$ pour $\bar{\sigma} \in T$, et $(0, 0)$ si σ est au-dessus de I_{nc}. Le résultat principal de ce numéro est la

PROPOSITION 2.3.10. *Soit (G, X) comme en* 2.1.1, *avec G $\mathbf{Q}$-simple adjoint, et de l'un des types* A, B, C, $D^{\mathbf{R}}$, $D^{\mathbf{H}}$. *Pour toute extension quadratique totalement imaginaire K de F, munie de T comme en* 2.3.9, *il existe un diagramme*

$$(G, X) \longleftarrow (G_1, X_1) \hookrightarrow (C\mathrm{Sp}(V), S^{\pm})$$

pour lequel

(i) *$E(G_1, X_1)$ est le composé de $E(G, X)$ et de $E(K^*, h_T)$.*

(ii) *Le groupe dérivé G_1' est simplement connexe pour G de type* A, B, C, $D^{\mathbf{R}}$, *et le revêtement de G décrit en* 2.3.8 *pour le type* $D^{\mathbf{H}}$.

Soit S le plus grand ensemble de sommets du diagramme de Dynkin D de $G_{\mathbf{C}}$ tel que $\{\{s\} \mid s \in S\}$ vérifie 2.3.7(b), (c). Nous l'avons déterminé en 2.3.8. Le groupe de Galois $\mathrm{Gal}(\bar{\mathbf{Q}}/\mathbf{Q})$ agit sur S, et on peut identifier S à $\mathrm{Hom}(K_S, \mathbf{C})$, pour K_S un produit convenable d'extension de $\mathbf{Q}$, isomorphes à des sous-corps de K_D puisque $\mathrm{Gal}(\bar{\mathbf{Q}}/K_D)$ agit trivialement sur D, donc sur S. En particulier, K_S est un produit de corps totalement réels ou CM. A la projection $S \to I$ correspond un morphisme $F \to K_S$.

Pour $s \in S$, soit $V(s)$ la représentation complexe de $\tilde{G}_{\mathbf{C}}$ de poids dominant le poids fondamental correspondant à s. La classe d'isomorphie de la représentation $\bigoplus V(s)$ est définie sur $\boldsymbol{Q}$. Ceci ne suffit pas à ce qu'on puisse la définir sur $\boldsymbol{Q}$; l'obstruction est dans un groupe de Brauer convenable. Toutefois, un multiple de cette représentation peut toujours être défini sur $\boldsymbol{Q}$. Soit donc V une représentation de $\tilde{G}$ sur $\boldsymbol{Q}$, avec $V_{\boldsymbol{C}} \sim \bigoplus V(s)^n$, pour n convenable. Nous noterons V_s l'unique facteur de $V_{\boldsymbol{C}}$ isomorphe à $V(s)^n$. Ces facteurs sont permutés par $\mathrm{Gal}(\bar{\boldsymbol{Q}}/\boldsymbol{Q})$ de façon compatible à l'action de $\mathrm{Gal}(\bar{\boldsymbol{Q}}/\boldsymbol{Q})$ sur S, et la décomposition $V_{\boldsymbol{C}} \sim \bigoplus V_s$ correspond donc à une structure de K_S-module sur V: sur V_s, K_S agit par multiplication par l'homomorphisme correspondant de K_S dans $\boldsymbol{C}$.

Notons $\tilde{G}'$ le quotient de $\tilde{G}$ qui agit fidèlement sur V. C'est le revêtement de G considéré en (ii).

Soit $h \in X$, et relevons h en un morphisme fractionnaire (1.3.4) de $\boldsymbol{S}$ dans $\tilde{G}'_{\boldsymbol{R}}$. On en déduit une structure de Hodge fractionnaire sur V, de poids 0. Soit $s \in S$, et v son image dans I. Le type de la décomposition de V_s est donné par la Table 1.3.9:

(a) si $v \in I_c$, V_s est de type $(0, 0)$;

(b) si $v \in I_{nc}$, V_s est de type $\{(r, -r), (r-1, 1-r)\}$ où r est donné par 1.3.9: c'est le nombre qui affecte le sommet s de D_v, muni du sommet spécial qui définit X_v.

On définit une structure de Hodge h_2 de V, en gardant V_s de type $(0, 0)$ pour $v \in I_c$, et, pour $v \in I_{nc}$, en renommant la partie de type $(r, -r)$ (resp. $(r-1, 1-r)$) de V_s comme étant de type $(0, -1)$ (resp. $(-1, 0)$). Si G_2 est le sous-groupe algébrique de $\mathrm{GL}(V)$ engendré par $\tilde{G}'$ et K_S^*, la structure de Hodge h_2 est un morphisme $\boldsymbol{S} \to G_{2\boldsymbol{R}}$. Notant X_2 sa classe de $G_2(\boldsymbol{R})$-conjugaison, on dispose de $(G_2, X_2) \to (G, X)$, et $E(G_2, X_2) = E(G, X)$.

Munissons $K \otimes_F V$ de la structure de Hodge h produit tensoriel de celle de V et de celle de K (2.3.9). On a $(K \otimes_F V) \otimes \boldsymbol{R} = \bigoplus_{v \in I}(K \otimes_{F,v} \boldsymbol{R}) \otimes_{\boldsymbol{R}} (V \otimes_{F,v} \boldsymbol{R})$. Cette décomposition est compatible à la structure de Hodge, et sur le facteur correspondant à $v \in I_c$ (resp, $v \in I_{nc}$), la structure de Hodge est le produit tensoriel d'une structure de type $\{(-1, 0), (0, -1)\}$ sur $K \otimes_{F,v} \boldsymbol{R}$ (resp. $V \otimes_{F,v} \boldsymbol{R}$) par une de type $\{(0, 0)\}$ sur $V \otimes_{F,v} \boldsymbol{R}$ (resp. $K \otimes_{F,v} \boldsymbol{R}$). Au total, h_3 est de type $\{(-1, 0), (0, -1)\}$. Si G_3 est le sous-groupe algébrique de $\mathrm{GL}(K \otimes_F V)$ engendré par K^* et G_2, la structure de Hodge h_3 est un morphisme $\boldsymbol{S} \to G_{3\boldsymbol{R}}$.

Si X_3 est la classe de conjugaison de h_3, on dispose de $(G_3, X_3) \to (G, X)$. Le groupe dérivé de G_3 est $\tilde{G}'$, et $E(G_3, X_3)$ est le composé de $E(G_2, X_2) = E(G, X)$ et de $E(K^*, h_T)$. Pour obtenir (G_1, X_1) cherché, il ne reste plus qu'à appliquer 2.3.3 à (G_3, X_3) et à sa représentation linéaire fidèle $K \otimes_F V$.

REMARQUE 2.3.11. La construction donnée se généralise pour fournir un diagramme (2.3.7.1) où $\mathscr{S}(V)$ est n'importe quel ensemble de parties de D vérifiant 2.3.7(b), (c). En gros :

(a) si $\mathscr{S}$ vérifie 2.3.7(b), (c), on définit $K_{\mathscr{S}}$ par $\mathrm{Hom}(K_{\mathscr{S}}, \boldsymbol{C}) = \mathscr{S}$, on construit une représentation V de $\tilde{G}$ telle que $\mathscr{S}(V) = \mathscr{S}$, et la décomposition $V_{\boldsymbol{C}} = \bigoplus V_S$ $(S \in \mathscr{S})$ fournit sur V une structure de $K_{\mathscr{S}}$-module;

(b) la structure de Hodge fractionnaire de V_S est de type $(0, 0)$ pour S au-dessus de I_c, de type $\{(r, -r), (r-1, 1-r)\}$ avec r décrit—comme ci-dessus—par le point de S au-dessus de I_{nc} sinon;

(c) on convertit $\{(r, -r), (r-1, 1-r)\}$ en $\{(0, -1), (-1, 0)\}$ comme plus haut;

(d) pour convertir le (0, 0) en $\{(-1, 0), (0, -1)\}$, on tensorise V, sur $K_{\mathscr{S}}$, avec $K'_{\mathscr{S}}$ de type CM muni d'une structure de Hodge convenable h.

Par cette méthode, on obtient pour (G_1, X_1) un groupe dérivé $\tilde{G}(\mathscr{S})$, et un corps dual composé de $E(G, X)$ et $E(K'^{*}_{\mathscr{S}}, h)$. Noter que, même pour $\mathscr{S}$ vérifiant (b), (c), (d), la conversion indiquée de (0, 0) est plus générale que celle de 2.3.10.

REMARQUE 2.3.12. Pour les types A, avec $\Sigma(X)$ fixe par l'involution d'opposition, B, C et D^{R}, le corps dual $E(G, X)$ est le sous-corps (totalement réel) de K_D fixe par le sous-groupe de $\mathrm{Gal}(K_D/\mathbf{Q})$ qui stabilise I_c. Si $I_c = \varnothing$, c'est $\mathbf{Q}$. Si I_c (resp. I_{nc}) est réduit à un élément v, c'est $v(F)$. Les $E(K, h_T)$ sont des extensions de $E(G, X)$.

REMARQUE 2.3.13. Pour ces types, et D^{H}, les V_s de 2.3.10, pour $v \in I_{nc}$, sont de type $\{(-\frac{1}{2}, \frac{1}{2}), (\frac{1}{2}, -\frac{1}{2})\}$. Ceci permet, dans 2.3.10, de remplacer G_2 par le sous-groupe de $\mathrm{GL}(V)$ engendré par F^* et $\tilde{G}'$. Si $I_c = \varnothing$, on peut même le remplacer par le sous-groupe de $\mathrm{GL}(V)$ engendré par $\mathbf{Q}^*$ et $\tilde{G}'$, et le critère 2.3.2 s'applique directement à ce groupe, d'où (G_1, X_1) avec $E(G_1, X_1) = E(G, X)$ ($= \mathbf{Q}$ hors le cas D^{H}).

2.4. *Lois de réciprocité*: *préliminaires*.

Les constructions de ce numéro nous permettrons, au numéro 2.6, de calculer la loi de réciprocité des modèles canoniques, i.e. l'action du groupe de Galois sur l'ensemble des composantes connexes géométriques.

Bien qu'elles s'expriment mieux dans le langage de la descente *fppf*, nous les avons exprimées dans celui de la descente galoisienne, le croyant plus familier aux non-géomètres. Ceci expose à quelques redites et inconséquences, et introduit des hypothèses parasites de séparabilité ou de caractéristique 0.

Soit G un groupe réductif sur un corps global k. Avec la notation de 2.0.15, notre but est de construire des morphismes canoniques des deux types suivant.

a. Pour k' une extension finie (qu'on supposera séparable) de k, et G' déduit de G par extension des scalaires à k', un morphisme norme

$$N_{k'/k}\colon \pi(G') \longrightarrow \pi(G). \tag{2.4.0.1}$$

b. Pour T un tore, et M une classe de conjugaison, définie sur k, de morphisms de T dans G, un morphisme

$$q_M : \pi(T) \longrightarrow \pi(G). \tag{2.4.0.2}$$

Si $m \in M(k)$, q_M sera le morphisme q_m induit par m; la difficulté est de montrer que ce morphisme ne dépend pas du choix de m, et de construire q_M même si M n'a pas de représentant défini sur k.

Les propriétés de fonctorialité de ces morphismes seront évidentes sur leur définition.

2.4.1. Nous utiliserons systématiquement le langage des torseurs (que je préfère à celui des cocycles), et celui de la descente galoisienne, sous la forme que lui a donnée Grothendieck (cf. **SGA 1**, ou **SGA 4$^{1/2}$**[Arcata]).

Descente galoisienne: Soit K une extension séparable finie d'un corps k. Pour construire un objet X sur k (par exemple un torseur), il suffit de construire (a) pour toute extension séparable k' de k, telle qu'il existe un morphisme de k-algèbre de K dans k', un objet $X_{k'}$ sur k'; (b) pour k'' une extension de k', un isomorphisme $\chi_{k'',k'}\colon X_{k'} \otimes k'' \xrightarrow{\sim} X_{k''}$; et de vérifier (c) une compatibilité $\chi_{k'''k''}\chi_{k''k'} = \chi_{k'''k'}$.

En pratique, cela signifie que pour construire X, on peut supposer l'existence d'objets auxiliaires qui n'existent que sur une extension séparable K de k—à charge de montrer le X construit ne dépend pas—à isomorphisme unique près—du choix d'un tel objet auxiliaire.

REMARQUE 1. La descente galoisienne est un cas particulier de la localisation en topologie étale; une construction comme en (a), (b), (c) ci-dessus sera souvent introduite par l'adverbe "localement".

EXEMPLE 2.4.2. Expliquons le relèvement canonique utilisé en 2.0.2 de l'application commutateur. L'usage de la descente galoisienne—plutôt que *fppf*—nous oblige à supposer que la projection de G sur G^{ad} est lisse, et à ne considérer que $(\, , \,) : G^{ad}(k) \times G^{ad}(k) \to G(k)$, plutôt que le morphisme $G^{ad} \times G^{ad} \to G$. Si x_1, $x_2 \in G^{ad}(k)$, on peut, localement, écrire $x_i = \rho(\tilde{x}_i)z_i$ avec z_i dans le centre de G. L'élément $\tilde{x}_i$ est unique, à multiplication par un élément du centre de $\tilde{G}$ près. Le commutateur de $\tilde{x}_1$ et $\tilde{x}_2$ ne dépend pas de l'arbitraire dans le choix des $\tilde{x}_i$, et on pose $(x_1, x_2) = \tilde{x}_1\tilde{x}_2\tilde{x}_1^{-1}\tilde{x}_2^{-1}$.

2.4.3. Pour G un groupe algébrique sur un corps k, un *G-torseur* est un schéma P sur k, muni d'une action à *droite* de G qui en fasse un espace principal homogène. Le G-torseur *trivial* G_d est G muni de l'action de G par translations à droite. On identifie les points $x \in P(k)$ aux *trivialisations* de P (isomorphismes $\varphi: G_d \xrightarrow{\sim} P$) par $\varphi(g) = xg$.

Si $f : G_1 \to G_2$ est un morphisme, et P un G_1-torseur, il existe un G_2-torseur $f(P)$ muni de $f : P \to f(P)$ vérifiant $f(pg) = f(p)f(g)$, et il est unique à isomorphisme unique près. Nous nous intéresserons à la catégorie $[G_1 \to G_2]$ des G_1-torseurs P munis d'une trivialisation de $f(P)$. Pour morphismes, on prend les isomorphismes de G_1-torseurs, compatibles à la G_2-trivialisation. On note $\boldsymbol{H}^0(G_1 \to G_2)$ le groupe des automorphismes de (G_{1d}, e) (c'est $\mathrm{Ker}(G_1(k) \to G_2(k))$) et $\boldsymbol{H}^1(G_1 \to G_2)$ l'ensemble (pointé par (G_{1d}, e)) des classes d'isomorphie d'objets.

Chaque $x \in G_2(k)$ définit un objet $[x]$ de $[G_1 \to G_2]$: le G_1-torseur trivial G_{1d}, muni de la trivialisation x de $f(G_{1d}) = G_{2d}$. Quand cela ne prête pas à confusion, nous le noterons simplement x. L'ensemble des morphismes de $[x]$ dans $[y]$ s'identifie à $\{g \in G_1(k) \mid f(g)x = y\}$: à g, associer $u \to gu$: $G_{1d} \to G_{1d}$. Un objet est de la forme $[x]$ si et seulement si, en tant que G_1-torseur, il est trivial—d'où une suite exacte

$$\begin{aligned} (2.4.3.1) \qquad & 1 \longrightarrow \boldsymbol{H}^0(G_1 \longrightarrow G_2) \longrightarrow G_1(k) \longrightarrow G_2(k) \\ & \longrightarrow \boldsymbol{H}^1(G_1 \longrightarrow G_2) \longrightarrow H^1(G_1) \longrightarrow H^1(G_2) \end{aligned}$$

(ceci ne décrit pas l'image inverse de $p \in H^1(G_1)$; pour la décrire, il faut procéder par torsion, comme dans [**13**]).

2.4.4. Si f est un épimorphisme, de noyau K, il revient au même de se donner le G_1-torseur P G_2-trivialisé par $x \in f(P)(k)$, ou le K-torseur $f^{-1}(x) \subset P$: le foncteur naturel $[K \to \{e\}] \to [G_1 \to G_2]$ est une équivalence.

Plus généralement, si g: $G_2 \to H$ induit un épimorphisme de G_1 sur H, et que $K_i = \mathrm{Ker}(G_i \to H)$, le foncteur naturel est une équivalence $[K_1 \to K_2] \to [G_1 \to G_2]$.

2.4.5. Si G est commutatif, la somme s: $G \times G \to G$ est un morphisme, et on définit la somme de deux G-torseurs par $P + Q = s(P \times Q)$. Si G_1 et G_2 sont commutatifs, on additionne de même les objets de $[G_1 \to G_2]$, qui devient une *catégorie de Picard* (strictement commutative) (**SGA 4**, XVIII, 1.4).

Tout ce qui précède vaut pour des faisceaux en groupes sur un topos quelconque.

2.4.6. Si k' est une extension finie de k (le cas où k'/k est séparable nous suffit) et G' un groupe algébrique sur k', le foncteur de restriction des scalaires à la Weil $R_{k'/k}$ est une équivalence de la catégorie des G'-torseurs avec celle des $R_{k'/k}G'$-torseurs. Ceci correspond au lemme de Shapiro $H^1(k', G') = H^1(k, R_{k'/k}G)$. Si G' se déduit par extension des scalaires de G-commutatif—sur k, on dispose d'un morphisme trace $R_{k'/k}G' \to G$—d'où un foncteur trace $\mathrm{Tr}_{k'/k}$ des G'-torseurs dans les G-torseurs. Plus généralement, pour $G_1 \to G_2$ un morphisme de groupes commutatifs on trouve un foncteur additif

$$\mathrm{Tr}_{k'/k}\colon [G'_1 \longrightarrow G'_2] \longrightarrow [G_1 \longrightarrow G_2]. \tag{2.4.6.1}$$

De tels foncteurs sont décrits avec une grande généralité dans **[SGA 4, XVII, 6.3]**. Pour k'/k séparable, on peut donner une définition simple par descente: localement, k' est somme $[k'\colon k]$ copies de k, $[G'_1 \to G'_2]$ s'identifie à la catégorie des $[k' : k]^{\text{uples}}$ d'objets de $[G_1 \to G_2]$, et $\mathrm{Tr}_{k'/k}$ à la somme.

Quand les groupes sont notés multiplicativement, on parlera plutôt de foncteur *norme* $N_{k'/k}$.

2.4.7. Soient G un groupe réductif (0.2) sur k, et $\rho : \tilde{G} \to G$ le revêtement universel de son groupe dérivé. Le cas particulier de 2.4.3 qui nous importe est $[\tilde{G} \to G]$. Soient G^{ad} le groupe adjoint de G, Z le centre de G, et $\tilde{Z}$ celui de $\tilde{G}$. Le morphisme $\tilde{G} \to G^{\mathrm{ad}}$ est un épimorphisme, d'où une équivalence (2.4.4)

$$[\tilde{Z} \longrightarrow Z] \longrightarrow [\tilde{G} \longrightarrow G]. \tag{2.4.7.1}$$

Puisque Z et $\tilde{Z}$ sont commutatifs, $[\tilde{Z} \to Z]$ est une catégorie de Picard strictement commutative (2.4.5). Utilisant l'équivalence (2.4.7.1), on fait de $[\tilde{G} \to G]$ aussi une telle catégorie. Nous allons calculer $[x_1] + [x_2]$ dans $[\tilde{G} \to G]$, et les données d'associativité et de commutativité. On suppose $\rho\colon \tilde{G} \to G^{\mathrm{der}}$ séparable pour pouvoir procéder par descente galoisienne. Ecrivant (localement) $x_i = \rho(g_i)z_i$, avec z_i central, on a des isomorphismes $g_i\colon [z_i] \to [x_i]$ et $g_1g_2\colon [z_1z_2] \to [x_1x_2]$, d'où un isomorphisme

$$[x_1] + [x_2] \xleftarrow{g_1+g_2} [z_1] + [z_2] = [z_1z_2] \xrightarrow{g_1g_2} [x_1x_2]. \tag{2.4.7.2}$$

Si on change de décomposition: $x_i = \rho(g'_i)z'_i$, avec $g_i = g'_iu_i$ $(u_i \in \tilde{Z})$, le diagramme

$$\begin{array}{ccccc} & & [z_1]+[z_2]=[z_1z_2] & & \\ & \swarrow^{g_1+g_2} & \downarrow{\scriptstyle u_1+u_2} \qquad \downarrow{\scriptstyle u_1u_2} & \searrow^{g_1g_2} & \\ [x_1]+[x_2] & & & & [x_1x_2] \\ & \nwarrow_{g'_1+g'_2} & & \nearrow_{g'_1g'_2} & \\ & & [z'_1]+[z'_2]=[z'_1z'_2] & & \end{array}$$

est commutatif. L'isomorphisme (2.4.7.2).

$$[x_1] + [x_2] = [x_1x_2] \tag{2.4.7.3}$$

est donc indépendant des choix faits. Le lecteur vérifiera facilement que, via cet isomorphisme, la donnée d'associativité est déduite de l'associativité du produit, et que la donnée de commutativité est $(x_1, x_2)\colon [x_2x_1] \to [x_1x_2]$. Il vérifiera aussi que si $y_i = \rho(g_i)x_i$ $(i = 1, 2)$, la somme des $g_i\colon [x_i] \to [y_i]$ est

$$g_1 + g_2 = g_1 \operatorname{int}_{x_1}(g_2)\colon [x_1x_2] \to [y_1y_2],$$

où int désigne l'action de G sur $\tilde{G}$ (définie par transport de structure, ou via l'action 2.0.2 de $G^{\mathrm{ad}} = \tilde{G}^{\mathrm{ad}}$).

La catégorie $[\tilde{G} \to G]$ étant de Picard, l'ensemble $\boldsymbol{H}^1(\tilde{G} \to G)$ des classes d'isomorphie d'objets est un groupe abélien. Les formules ci-dessus montrent que l'injection $G(k)/\rho(\tilde{G}(k)) \to \boldsymbol{H}^1(\tilde{G} \to G)$ est un homomorphisme. D'après (2.4.3.1), c'est un isomorphisme si $H^1(\tilde{G}) = 0$.

Soient k' une extension finie de k, et notons par un $'$ l'extension des scalaires à k'. L'équivalence (2.4.7.1) permet de déduire de 2.4.6.1 un foncteur trace (que nous baptiserons *norme*)

$$N_{k'/k}\colon [\tilde{G}' \longrightarrow G'] \longrightarrow [\tilde{G} \longrightarrow G]. \tag{2.4.7.4}$$

PROPOSITION 2.4.8. *Si k est un corps local ou global, le morphisme déduit de* (2.4.7.4) *par passage à l'ensemble des classes d'isomorphie d'objets induit un morphisme de $G(k')/\rho\tilde{G}''(k')$ dans $G(k)/\rho\tilde{G}(k)$*:

$$\begin{array}{ccc} G(k')/\rho\tilde{G}(k') & \longrightarrow & G(k)/\rho\tilde{G}(k) \\ \downarrow & & \downarrow \\ \boldsymbol{H}^1(\tilde{G}' \longrightarrow G') & \longrightarrow & \boldsymbol{H}^1(\tilde{G} \longrightarrow G). \end{array} \tag{2.4.8.1}$$

Si $H^1(\tilde{G}) = 0$, la flèche verticale droite est un isomorphisme, et l'assertion est évidente. Cette nullité vaut pour k local non archimédien. Pour k local archimédien, le seul cas intéressant est $k = \boldsymbol{R}$, $k' = \boldsymbol{C}$, et le diagramme commutatif

$$\begin{array}{ccc} Z(\boldsymbol{C}) & \longrightarrow\!\!\!\!\rightarrow & \boldsymbol{H}^1(\tilde{G}_{\boldsymbol{C}} \longrightarrow G_{\boldsymbol{C}}) \\ \downarrow{\scriptstyle N_{\boldsymbol{C}/\boldsymbol{R}}} & & \downarrow \\ Z(\boldsymbol{R}) & \longrightarrow & \boldsymbol{H}^1(\tilde{G} \longrightarrow G) \end{array}$$

montre que (2.4.8.1) est encore défini—à valeurs dans l'image de $Z(\boldsymbol{R})$ (et même de sa composante neutre).

Pour k global, et $x \in G(k')/\rho(\tilde{G}(k'))$, l'image de $N_{k'/k}(x) \in \boldsymbol{H}^1(\tilde{G} \to G)$ dans $H^1(\tilde{G})$ est donc localement nulle. D'après le principe de Hasse, elle est nulle. Nous n'utilisons ici le principe de Hasse que pour les classes de cohomologie dans l'image de $H^1(\tilde{Z})$, de sorte que les facteurs E_8 ne créent aucun trouble. L'image $N_{k'/k}(x)$ est donc dans $G(k)/\rho\tilde{G}(k)$, comme promis.

2.4.9. Pour k local non archimédien d'anneau des entiers V, k' non ramifié sur k, d'anneau des entiers V', et G réductif sur V, le morphisme 2.4.8 induit un morphisme de $G(V')/\rho\tilde{G}(V)$: on le voit en répétant les arguments qui précèdent sur V, la descente galoisienne étant remplacée par la localisation étale (ici, formellement identique à une descente galoisienne sur le corps résiduel).

On peut donc adéliser 2.4.8: pour k global, le produit restreint des morphismes 2.4.8, pour les complétés de k, est un morphisme

$$N_{k'/k}\colon G(\boldsymbol{A}')/\rho\tilde{G}(\boldsymbol{A}') \longrightarrow G(\boldsymbol{A})/\rho\tilde{G}(\boldsymbol{A}).$$

Divisant par le morphisme trace global, on obtient enfin le morphisme (2.4.0.1)

$$N_{k'/k}: \pi(G') \longrightarrow \pi(G).$$

De même que la construction du morphisme (2.4.0.1) repose sur celle du foncteur (2.4.7.1), celle de (2.4.0.2) reposera sur la

Construction 2.4.10. *Soient G réductif connexe sur k, ρ: $\tilde{G} \to G$ le revêtement universel du groupe dérivé, T un tore sur k, et M une classe de conjugaison, définie sur k, de morphisme de T dans G. On définira un foncteur additif*

$$q_M: [\{e\} \longrightarrow T] \longrightarrow [\tilde{G} \longrightarrow G].$$

Nous donnerons de la construction deux variantes.

1ère méthode. Localement, il existe m dans M. Posons $X(m) = Z \cdot m(T) \subset G$ et $Y(m) = \rho^{-1}X(m) = \tilde{Z} \cdot (\rho^{-1}\, m(T))^0$. Les groupes $X(m)$ et $Y(m)$, extensions de tores par un sous-groupe central de type multiplicatif, sont commutatifs. Ils donnent lieu à un diagramme

$$(1)_m \qquad \begin{array}{ccccc} \tilde{Z} & \longrightarrow & Y(m) & & \\ \downarrow & & \downarrow & & \\ Z & \longrightarrow & X(m) & \longleftarrow & T \end{array}$$

Si g dans G conjugue m en m', il conjugue $(1)_m$ en $(1)_{m'}$ (on le fait agir sur $\tilde{G}$ par l'action de $\tilde{G}^{\mathrm{ad}} = G^{\mathrm{ad}}$). De plus, l'isomorphisme $\operatorname{int}(g)$ de $(1)_m$ avec $(1)_{m'}$ ne dépend pas du choix de g: si g centralise m, il centralise Z, $m(T)$, $\tilde{Z}$, ainsi que $(\rho^{-1}(T))^0$ (un tore isogène à un sous-tore de T), donc $X(m)$ et $Y(m)$. Deux m dans M étant localement conjugués, ceci permet d'identifier entre eux les diagrammes $(1)_m$, et d'en déduire un diagramme unique

$$(1) \qquad \begin{array}{ccccc} \tilde{Z} & \longrightarrow & Y & & \\ \downarrow & & \downarrow & & \\ Z & \longrightarrow & X & \longleftarrow & T \end{array}$$

défini sur k.

On définit q_M comme étant le foncteur composé

$$q_M: [\{e\} \longrightarrow T] \longrightarrow [Y \longrightarrow X] \underset{2.4.4}{\overset{\sim}{\longleftarrow}} [\tilde{Z} \longrightarrow Z] \overset{\sim}{\longrightarrow} [\tilde{G} \longrightarrow G].$$

2ème méthode. On suppose $\rho: \tilde{G} \to G^{\mathrm{der}}$ séparable, pour procéder par descente galoisienne. Les objets de $[\{e\} \to T]$ n'ayant pas d'automorphismes, un foncteur de $[\{e\} \to T]$ dans $[\tilde{G} \to G]$ est simplement une loi qui à $t \in T(k)$ assigne un $\tilde{G}$-torseur G-trivialisé $q_M([t])$. Procédons par descente galoisienne. Localement il existe $m \in M(k)$. Soit q_m le foncteur $[t] \to [m(t)]$. Nous allons définir un système transitif d'isomorphismes entre les q_m. Ceci fait, nous pourrons définir q_M comme étant l'un quelconque des q_m.

Pour définir $\iota_{m',m}: q_m \xrightarrow{\sim} q_{m'}$, on choisit g tel que $m' = gmg^{-1}$ (nouvelle application de la méthode de descente) et on pose $\iota_{m',m}([t])$: $[m(t)] \to [m'(t)]$ est $(g, m(t)) \in \tilde{G}(k)$. On a bien $m'(t) = (\rho((g, m(t)))m(t)$, et il reste à vérifier que $(g, m(t))$ est indépendant de g.

L'espace C des g qui conjuguent m en m' est connexe et réduit, en tant que torseur sous le centralisateur du tore $m(T)$. La fonction $(g, m(t))$ de C dans $\tilde{G}$ a une projection $\rho((m, m(t))) = m'(t)m(t)^{-1}$ dans G constante. La fibre de ρ étant discrete, elle est constante.

La construction est récapitulée dans le diagramme commutatif

$$\text{(2.4.10.1)}\qquad \begin{array}{ccc} & q_M([t]) & \\ \swarrow & & \searrow \\ q_m([t]) & \xrightarrow{(g,\, m(t))} & q_{m'}([t]) \end{array} \qquad (q_m([t]) = [m(t)],\ m' = gmg^{-1}).$$

Construisons la donnée d'additivité de q_M. C'est la donnée, pour chaque $t_1, t_2 \in T(k)$, d'un isomorphisme $q_M([t_1t_2]) \to q_M([t_1]) + q_M([t_2])$, ces isomorphismes étant compatibles aux données d'associativité et de commutativité pour la somme dans $[\tilde{G} \to G]$. On les définit par descente: pour $m \in M$, et $m' = gmg^{-1}$, le diagramme suivant est commutatif

$$\begin{array}{ccccccc} & \nearrow & [m(t_1t_2)] & \xrightarrow{2.4.7} & [m(t_1)] + [m(t_2)] & \nwarrow & \\ q_M([t_1t_2]) & & \Big\downarrow{\scriptstyle (g,\, m(t_1t_2))} & & \Big\downarrow{\scriptstyle (g,\, m(t_1)) + (g,\, m(t_2))} & & q_M([t_1]) + q_M([t_2]) \\ & \searrow & [m'(t_1t_2)] & \xrightarrow{2.4.7} & [m'(t_1)] + [m'(t_2)] & \swarrow & \end{array}$$

(flèches obliques 2.4.10.1), et définit l'isomorphisme cherché indépendamment de m. Sa commutativité exprime une certaine identité, dans $\tilde{G}$, entre commutateurs (2.4.2), et la projection de cette identité dans G résulte de ce que les flèches écrites ont un sens. Pour la prouver, on note que localement (descente) c'est la projection d'une identité analogue pour $\tilde{G} \times Z^0$—de projection vraie dans $\tilde{G} \times Z^0$, donc vraie dans $\tilde{G}$.

La compatibilité à l'associativité et à la commutativité se voit par descente, en fixant m; on utilise que le commutateur 2.4.2 est trivial sur $m(T)$.

2.4.11. *Compléments.* (i) La construction 2.4.10 est compatible à l'extension des scalaires: notant par un $'$ l'extension des scalaires de k à une extension k' de k, on définit de façon évidente un isomorphisme de foncteurs additifs rendant commutatif le diagramme

$$\begin{array}{ccc} [\{e\} \longrightarrow T] & \xrightarrow{q_M} & [\tilde{G} \longrightarrow G] \\ \Big\downarrow & & \Big\downarrow \\ [\{e\} \longrightarrow T'] & \xrightarrow{q_{M'}} & [\tilde{G}' \longrightarrow G'] \end{array}$$

(ii) On peut répéter la construction 2.4.10 sur une base quelconque; la descente galoisienne est à remplacer par la localisation étale.

(iii) La construction 2.4.10 est compatible aux foncteurs normes. Pour définir l'isomorphisme de foncteurs additifs rendant commutatif le diagramme

$$\begin{array}{ccc} [\{e\} \longrightarrow T'] & \xrightarrow{q_{M'}} & [\tilde{G}' \longrightarrow G'] \\ \Big\downarrow & & \Big\downarrow{\scriptstyle 2.4.7.4} \\ [\{e\} \longrightarrow T] & \xrightarrow{q_M} & [\tilde{G} \longrightarrow G] \end{array}$$

(notations de (i), avec k'/k fini séparable), et vérifier ses propriétés, le plus simple est de procéder par descente: localement, k' devient une somme k^I de copies de k, $[\tilde{G}' \to G']$ devient $[\tilde{G}' \to G]^I$, la norme (trace) devient la somme, et tout est trivial.

2.4.12. Pour k local non archimédien, $H^1(\tilde{G}) = 0$ et, passant aux ensembles de classes d'isomorphie d'objets, on déduit de 2.4.10 un morphisme $q_M\colon T(k) \to G(k)/\rho\tilde{G}(k)$. Pour G réductif sur l'anneau des entiers V de k, T un tore sur V et M sur V, il induit un morphisme de $T(V)$ dans $G(V)/\rho\tilde{G}(V)$ (2.4.11(ii)).

Pour k archimédien, on peut rencontrer une obstruction dans $H^1(\tilde{G})$, mais elle disparaît pour x dans la composante neutre (topologique) $T(k)^+$ de $T(k)$: par 2.4.11(ii), elle dépend continument de x, et est nulle pour $x = e$. On a donc encore un morphisme $T(k)^+ \to G(k)/\rho\tilde{G}(k)$.

Pour k global, prenant le produit restreint de ces morphismes pour les complétés de k, on trouve

$$q_M\colon T(\boldsymbol{A})^+ \longrightarrow G(\boldsymbol{A})/\rho\tilde{G}(\boldsymbol{A}). \tag{2.4.12.1}$$

Si $T(k)^+ = \{x \in T(k) \mid \text{pour } v \text{ réel, } x \text{ est dans } T(k_v)^+\}$, le principe de Hasse, utilisé comme en 2.4.8, fournit

$$q_M\colon T(k)^+ \longrightarrow G(k)/\rho\tilde{G}(k). \tag{2.4.12.2}$$

Puisque $T(\boldsymbol{A})^+/T(k)^+ \xrightarrow{\sim} T(\boldsymbol{A})/T(k)$ (théorème d'approximation réel pour les tores), on obtient finalement par passage au quotient le morphisme (2.4.0.2) promis: $q_M\colon \pi(T) \to \pi(G)$.

2.5. *Application: une extension canonique.*

2.5.1. Soit G un groupe réductif sur $\boldsymbol{Q}$. On suppose que $\tilde{G}$ n'a pas de facteur G' (défini sur $\boldsymbol{Q}$) tel que $G'(\boldsymbol{R})$ soit compact. Le théorème d'approximation forte assure dès lors que $\tilde{G}(\boldsymbol{Q})$ est dense dans $\tilde{G}(\boldsymbol{A}^f)$. Le cas qui nous importe est celui d'un groupe comme en 2.1.1.

Reprenons le calcul de 2.1.3. Pour K compact ouvert dans $G(\boldsymbol{A}^f)$,

$$\begin{aligned}\pi_0(G(\boldsymbol{Q})\backslash G(\boldsymbol{A})/K) &= G(\boldsymbol{Q})\backslash\pi_0(G(\boldsymbol{R})) \times (G(\boldsymbol{A}^f)/K) = G(\boldsymbol{Q})\backslash G(\boldsymbol{A})/G(\boldsymbol{R})^+ \times K \\ &= G(\boldsymbol{Q})^+\backslash G(\boldsymbol{A}^f)/K\end{aligned} \tag{2.5.1.1}$$

et on peut remplacer $G(\boldsymbol{Q})$ (resp. $G(\boldsymbol{Q})^+$) par son adhérence dans $G(\boldsymbol{A}^f)$. Celle-ci contient $\rho\tilde{G}(\boldsymbol{A}^f)$, un sous-groupe distingué à quotient abélien de $G(\boldsymbol{A}^f)$, et, avec les notations de 2.0.15,

$$\pi_0(G(\boldsymbol{Q})\backslash G(\boldsymbol{A})/K) = \pi(G)/G(\boldsymbol{R})^+ \times K = \pi_0\pi(G)/K = G(\boldsymbol{A}^f)/G(\boldsymbol{Q})^+ \cdot K. \tag{2.5.1.2}$$

Passant à la limite sur K, on en déduit que

$$\pi_0(G(\boldsymbol{Q})\backslash G(\boldsymbol{A})) = \pi_0\pi(G) = G(\boldsymbol{A}^f)/G(\boldsymbol{Q})^{+-}.$$

Soit $\bar{\pi}_0\pi(G)$ le quotient $G(\boldsymbol{A}^f)/G(\boldsymbol{Q})_+^-$ de $\bar{\pi}_0\pi(G)$ par $\pi_0 G(\boldsymbol{R})_+$ (0.3). La suite

$$0 \longrightarrow G(\boldsymbol{Q})_+^-/Z(\boldsymbol{Q})^- \longrightarrow G(\boldsymbol{A}^f)/Z(\boldsymbol{Q})^- \longrightarrow \bar{\pi}_0\pi(G) \longrightarrow 0 \tag{2.5.1.3}$$

est exacte. L'action de G^{ad} sur G induit une action de $G^{\mathrm{ad}}(\boldsymbol{Q})$ sur cette suite exacte. L'existence du commutateur (2.0.2) montre que l'action de $G^{\mathrm{ad}}(\boldsymbol{A})$ sur $G(\boldsymbol{A})/\rho\tilde{G}(\boldsymbol{A})$ est triviale—à fortiori celle de $G^{\mathrm{ad}}(\boldsymbol{Q})$ sur $\bar{\pi}_0\pi(G)$. L'application du sous-groupe $G(\boldsymbol{Q})_+/Z(\boldsymbol{Q})$ de $G(\boldsymbol{Q})_+^-/Z(\boldsymbol{Q})^-$ dans $G^{\mathrm{ad}}(\boldsymbol{Q})^+$ vérifie les conditions de 2.0.1. Le groupe $G(\boldsymbol{Q})_+^-/Z(\boldsymbol{Q})^- *_{G(\boldsymbol{Q})_+/Z(\boldsymbol{Q})} G^{\mathrm{ad}}(\boldsymbol{Q})^+$ n'est autre que le complété de $G^{\mathrm{ad}}(\boldsymbol{Q})^+$ pour la topologie $\tau(G^{\mathrm{der}})$ (2.1.6). Appliquant de même la construction $*G^{\mathrm{ad}}(\boldsymbol{Q})^+$ au terme central de 2.5.1.3, on obtient finalement une extension

$$(2.5.1.4)\quad 0 \longrightarrow G^{\mathrm{ad}}(\boldsymbol{Q})^{+\wedge}(\mathrm{rel.}\ G^{\mathrm{der}}) \longrightarrow \frac{G(\boldsymbol{A}^f)}{Z(\boldsymbol{Q})^-} *_{G(\boldsymbol{Q})_+/Z(\boldsymbol{Q})} G^{\mathrm{ad}}(\boldsymbol{Q})^+ \longrightarrow \bar{\pi}_0\pi(G) \longrightarrow 0.$$

2.5.2. Du fait que G^{ad} n'est pas fonctoriel en G, la fonctorialité de cette suite est pénible à expliciter. Nous nous contenterons des deux cas suivant:

(a) Lorsqu'on ne considère que des groupes extension centrale d'un groupe adjoint donné, et des morphismes compatibles à la projection sur ce groupe adjoint, (2.5.1.4) est fonctoriel en G en un sens évident.

(b) Soit $H \subset G$ un tore et, contrairement aux conventions générales, *notons* H^{ad} *son image dans* G^{ad}. Tel étant le cas dans les applications, on suppose $H^{\mathrm{ad}}(\boldsymbol{R})$ compact—donc connexe puisque H^{ad} est connexe. Cette hypothèse assure que $H^{\mathrm{ad}}(\boldsymbol{Q})$ est discret dans $H^{\mathrm{ad}}(\boldsymbol{A}^f)$, et que $H(\boldsymbol{Q}) \subset G(\boldsymbol{Q})_+$. Posons $Z' = Z \cap H$. Le diagramme commutatif

$$\begin{array}{ccc} G(\boldsymbol{A}^f) & \longrightarrow & G(\boldsymbol{A}^f)/G(\boldsymbol{Q})_+^- = \bar{\pi}_0\pi(G) \\ \uparrow & & \uparrow \\ H(\boldsymbol{A}^f) & \longrightarrow & H(\boldsymbol{A}^f)/H(\boldsymbol{Q})^- = \bar{\pi}_0\pi(H) \end{array}$$

fournit un morphisme de suites exactes

$$(2.5.2.1)\quad \begin{array}{ccccccccc} 0 & \longrightarrow & G^{\mathrm{ad}}(\boldsymbol{Q})^{+\wedge}(\mathrm{rel.}G^{\mathrm{der}}) & \longrightarrow & \dfrac{G(\boldsymbol{A}^f)}{Z(\boldsymbol{Q})^-} *_{G(\boldsymbol{Q})_+/Z(\boldsymbol{Q})} G^{\mathrm{ad}}(\boldsymbol{Q})^+ & \longrightarrow & \bar{\pi}_0\pi(G) & \longrightarrow & 0 \\ & & \uparrow & & \uparrow & & \uparrow & & \\ 0 & \longrightarrow & H^{\mathrm{ad}}(\boldsymbol{Q}) & \longrightarrow & \dfrac{H(\boldsymbol{A}^f)}{Z'(\boldsymbol{Q})^-} *_{H(\boldsymbol{Q})/Z'(\boldsymbol{Q})} H^{\mathrm{ad}}(\boldsymbol{Q}) & \longrightarrow & \bar{\pi}_0\pi(H) & \longrightarrow & 0 \end{array}$$

2.5.3. Soient G comme en 2.5.1, E une extension finie de $\boldsymbol{Q}$, T un tore sur E et M une classe de conjugaison définie sur E de morphismes de T dans G: $M: T \to G_E$. Par passage aux π_0, le morphisme composé de (2.4.0.1) et (2.4.0.2)

$$N_{E/\boldsymbol{Q}}\, q_M : \pi\,(T) \longrightarrow \pi(G_E) \longrightarrow \pi(G)$$

fournit un morphisme $\pi_0\pi(T) \to \pi_0\pi(G) \to \bar{\pi}_0\pi(G)$. Nous noterons (G, M) son *inverse*, et $\mathscr{E}'_E(G, M)$ l'extension image inverse par $r(G, M)$ de l'extension (2.5.1.4):

$$\begin{array}{ccccccccc} 0 & \longrightarrow & G^{\mathrm{ad}}(\boldsymbol{Q})^{+\wedge}(\mathrm{rel.}\ G^{\mathrm{der}}) & \longrightarrow & \mathscr{E}'_E(G, M) & \longrightarrow & \pi_0\pi(T) & \longrightarrow & 0 \\ & & \| & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & G^{\mathrm{ad}}(\boldsymbol{Q})^{+\wedge}(\mathrm{rel.}\ G^{\mathrm{der}}) & \longrightarrow & \dfrac{G(\boldsymbol{A}^f)}{Z(\boldsymbol{Q})^-} *_{G(\boldsymbol{Q})_+/Z(\boldsymbol{Q})} G^{\mathrm{ad}}(\boldsymbol{Q})^+ & \longrightarrow & \bar{\pi}_0\pi(G) & \longrightarrow & 0 \end{array}$$

Soient $u: H \to G$ un morphisme comme en 2.5.2(a) ($H^{\mathrm{ad}} = G^{\mathrm{ad}}$), et N une classe de conjugaison, définie sur E, de morphismes de T dans H. On suppose que u envoie N dans M. Par fonctorialité, u définit alors un isomorphisme du quotient de $\mathscr{E}'_E(H, N)$ par $\mathrm{Ker}(G^{\mathrm{ad}}(\boldsymbol{Q})^{+\wedge}(\mathrm{rel.}\ G^{\mathrm{der}}) \to G^{\mathrm{ad}}(\boldsymbol{Q})^{+\wedge}(\mathrm{rel.}\ H^{\mathrm{der}}))$ avec $\mathscr{E}'_E(G, M)$.

Pour $H \to G$ un tore comme en 2.5.2(b), muni de $m: T \to H_E$ dans M, on trouve un morphisme d'extensions

$$\begin{array}{ccccccccc} 0 & \longrightarrow & G^{\mathrm{ad}}(\mathbf{Q})^{+\wedge}(\mathrm{rel.}\ G^{\mathrm{der}}) & \longrightarrow & \mathscr{E}'_E(G,M) & \longrightarrow & \pi_0\pi(T) & \longrightarrow & 0 \\ & & \uparrow & & \uparrow & & \| & & \\ 0 & \longrightarrow & H^{\mathrm{ad}}(\mathbf{Q}) & \longrightarrow & \cdots & \longrightarrow & \pi_0\pi(T) & \longrightarrow & 0 \end{array}$$

2.5.4. Soient E, G, T, M comme ci-dessus, avec G adjoint. Considérons les systèmes (G_1, M_1, u) formés d'une extension centrale $u: G_1 \to G$ de G ($G_1^{\mathrm{ad}} = G$), définie sur $\mathbf{Q}$, et d'une classe de conjugaison de morphismes M_1 de T dans G_1, définie sur E, qui relève M. Sur $\bar{\mathbf{Q}}$, pour m_1 dans M_1 d'image m dans M, le centralisateur de m_1 est l'image inverse du centralisateur de m: c'est vrai pour leurs algèbres de Lie, ils sont connexes en tant que centralisateurs de tores, et le centralisateur de m_1 contient le centre de G_1. On a donc $M_1 \xrightarrow{\sim} M$.

LEMME 2.5.5. *Il existe des systèmes (G_1, M_1, u) pour lesquels G_1^{der} est un revêtement arbitrairement prescrit de G.*

Il suffit de montrer qu'on peut obtenir le revêtement universel $\tilde{G}$. Sur $\bar{\mathbf{Q}}$, pour tout m dans M, la composante neutre de l'image inverse par m du revêtement $\tilde{G}$ de G est un revêtement $\pi: \tilde{T} \to T$ de T. Il ne dépend pas de m, donc est défini sur E, et M se relève en une classe de conjugaison $\tilde{M}$ de morphismes de $\tilde{T}$ dans $\tilde{G}_E$. Par passage au quotient par $\mathrm{Ker}(\pi)$, on déduit de $\tilde{M} \times \mathrm{Id}: \tilde{T} \to \tilde{G}_E \times \tilde{T}$ un relèvement $M_1': T \to (\tilde{G}_E \times \tilde{T})/\mathrm{Ker}(\pi)$. Ce relèvement est à valeurs dans un groupe G_E', défini sur E, de groupe adjoint G_E. Il reste à remplacer G_E' par un groupe défini sur $\mathbf{Q}$. Ecrivons $G_E' = \tilde{G}_E *_{\tilde{Z}_E} Z_E'$. L'idée est de remplacer Z_E' par le coproduit, sur $\tilde{Z}_E$, de ses conjugués: si on pose

$$Z = R_{E/\mathbf{Q}}(Z_E')/\mathrm{Ker}(\mathrm{Tr}_{E/\mathbf{Q}}: R_{E/\mathbf{Q}}(\tilde{Z}_E) \longrightarrow \tilde{Z})$$

et $G_1 = \tilde{G} *_{\tilde{Z}} Z$, on a $G_E' \subset G_{1E}$, et M_1' fournit le relèvement voulu.

Construction 2.5.6. *A isomorphisme unique près, l'extension $\mathscr{E}'(G_1, M_1)$ ne dépend que de M et G_1^{der}.*

Soient deux systèmes (G_1', M_1') et (G_1'', M_1''), de même groupe dérivé. Considérons la composante neutre G_1 du produit fibré de G_1' et G_1'' sur G, et al classe $M_1 = M_1' \times_M M_1''$. Le diagramme d'extensions

$$\mathscr{E}'(G_1', M_1') \xleftarrow{\sim} \mathscr{E}'(G_1, M_1) \xrightarrow{\sim} \mathscr{E}'(G_1'', M_1'')$$

fournit l'isomorphisme cherché.

DÉFINITION 2.5.7. *Soient G un groupe adjoint, G' un revêtement de G et M une classe de conjugaison, définie sur E, de morphismes de T dans G.* L'extension *$\mathscr{E}_E(G, G', M)$ de $\pi_0\pi(T)$ par le complété $G(\mathbf{Q})^{+\wedge}$ (rel. G') est l'extension $\mathscr{E}_E'(G_1, M_1)$, pour un quelconque système (G_1, M_1) comme en* 2.5.4, *tel que $G_1^{\mathrm{der}} = G'$.*

Pour F une extension de E, la classe M fournit, par extension des scalaires de E à F, une classe de conjugaison M_F de morphismes de T_F dans G_F; l'extension $\mathscr{E}_F(G, G', M)$ correspondante est image inverse de $\mathscr{E}_E(G, G', M)$ par la norme $N_{F/E}$: $\pi_0\pi(T_F) \to \pi_0\pi(T)$:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & G(\mathbf{Q})^{+\wedge}(\mathrm{rel.}\ G') & \longrightarrow & \mathscr{E}_F(G, G', M_F) & \longrightarrow & \pi_0\pi(T_F) & \longrightarrow & 0 \\ & & \| & & \downarrow & & \downarrow{\scriptstyle N_{F/E}} & & \\ 0 & \longrightarrow & G(\mathbf{Q})^{+\wedge}(\mathrm{rel.}\ G') & \longrightarrow & \mathscr{E}_E(G, G', M) & \longrightarrow & \pi_0\pi(T) & \longrightarrow & 0 \end{array}$$

Les extensions $\mathscr{E}_E(G, G', M)$ se déduisent toute de $\mathscr{E}_E(G, \tilde{G}, M)$ par passage au quotient: remplacer $G(\boldsymbol{Q})^{+\wedge}$ (rel. $\tilde{G}$) par son quotient $G(\boldsymbol{Q})^{+\wedge}$ (rel. G').

2.5.8. Soient $H \to G$ un tore, avec $H(\boldsymbol{R})$ compact, et $m \in M$, défini sur E, qui se factorise par H. Pour tout système $(G_1, M_1) \to (G, M)$ comme ci-dessus, soient H_1 la composante neutre de l'image inverse de H dans G_1, et m_1 l'élément de M_1 au-dessus de m (2.5.4.) Prenons l'image inverse par $r(H_1, \{m_1\})$ du morphisme d'extensions (2.5.2.1):

$$\begin{array}{ccccc} G(\boldsymbol{Q})^{+\wedge}\ (\text{rel. } G') & \longrightarrow & \mathscr{E}_E(G, G', M) & \longrightarrow & \pi_0\pi(T) \\ \uparrow & & \uparrow & & \| \\ H(\boldsymbol{Q}) & \longrightarrow & \cdots & & \pi_0\pi(T) \end{array}$$

On voit comme en 2.5.6 que, à isomorphisme unique près, ce diagramme ne dépend pas du choix de (G_1, M_1). Comme en 2.5.7, ce diagramme, rel. un revêtement G' de G, se déduit du même diagramme, rel. $\tilde{G}$, par passage au quotient. On a aussi la même fonctorialité en E qu'en 2.5.7. En particulier, pour m dans M, défini sur une extension F de E, qui se factorise par H on trouve un morphisme d'extensions

$$\begin{array}{ccccccccc} 0 & \longrightarrow & G(\boldsymbol{Q})^{+\wedge}(\text{rel. } G') & \longrightarrow & \mathscr{E}_E(G,G',M) & \longrightarrow & \pi_0\pi(T) & \longrightarrow & 0 \\ & & \uparrow & & \uparrow & & \uparrow{\scriptstyle N_{F/E}} & & \\ 0 & \longrightarrow & H(\boldsymbol{Q}) & \longrightarrow & \cdots & \longrightarrow & \pi_0\pi(T_F) & \longrightarrow & 0 \end{array} \tag{2.5.8.1}$$

Nous utiliserons ce diagramme de la façon suivante: si $\mathscr{E}_E(G, G', M)$ agit sur un ensemble V, et qu'un point $x \in V$ est fixe sous $H(\boldsymbol{Q}) \subset G(\boldsymbol{Q})$, il a un sens de demander qu'il soit fixe "par $\pi_0\pi(T_F)$" i.e. par le sous-groupe image de l'extension en 2ème ligne.

2.5.9. Spécialisons les hypothèses au cas qui nous intéresse. On part d'un système (G, X) comme en 2.1.1, avec G adjoint, et on fixe une composante connexe X^+ de X. On prend pour E une extension finie, contenue dans $\boldsymbol{C}$, de $E(G, X)$, et on fait $T = \boldsymbol{G}_m$, $M =$ la classe de conjugaison de μ_h, pour $h \in X$. Elle est définie sur E.

Le groupe $\pi(T)$ est le groupe des classes d'idèles de E, et la théorie du corps de classe global identifie $\pi_0\pi(T)$ à $\text{Gal}(\bar{\boldsymbol{Q}}/E)^{\text{ab}}$. Si G' est un revêtement de G, l'image inverse par le morphisme $\text{Gal}(\bar{\boldsymbol{Q}}/E) \to \text{Gal}(\bar{\boldsymbol{Q}}/E)^{\text{ab}}$ de l'extension $\mathscr{E}_E(G, G', M)$ est une extension

$$0 \longrightarrow G^{\text{ad}}(\boldsymbol{Q})^{+\wedge}(\text{rel. } G') \longrightarrow \mathscr{E}_E(G, G', X) \longrightarrow \text{Gal}(\bar{\boldsymbol{Q}}/E) \longrightarrow 0. \tag{2.5.9.1}$$

Le cas universel est celui où $E = E(G, X)$, et où $G' = \tilde{G}$: d'après 2.5.7, $\mathscr{E}_E(G, G', M)$ est l'image inverse de $\text{Gal}(\bar{\boldsymbol{Q}}/E) \subset \text{Gal}(\bar{\boldsymbol{Q}}/E(G, X))$ dans $\mathscr{E}_{E(G,X)}(G, G', X)$, et $\mathscr{E}_E(G, G', X)$ est un quotient de $\mathscr{E}_E(G, \tilde{G}, X)$.

2.5.10. Soit $h \in X^+$ un point spécial: h se factorise par $H \subset G$, un tore défini sur $\boldsymbol{Q}$. Puisque int $h(i)$ est une involution de Cartan, $H(\boldsymbol{R})$ est compact. On peut donc appliquer 2.5.8 à H et à μ_h (défini sur l'extension $E(H, h)$ de $E(G, X)$). Par image inverse, on déduit de (2.5.8.1) un morphisme d'extensions

$$\begin{array}{ccccc} G(\boldsymbol{Q})^{+\wedge}\ (\text{rel. } G') & \longrightarrow & \mathscr{E}_E(G, G', X) & \longrightarrow & \text{Gal}(\bar{\boldsymbol{Q}}/E) \\ \uparrow & & \uparrow & & \uparrow \\ H(\boldsymbol{Q}) & \longrightarrow & \cdots & \longrightarrow & \text{Gal}(\bar{\boldsymbol{Q}}/E \cdot E(H,h)) \end{array} \tag{2.5.10.1}$$

2.6. *La loi de réciprocité des modèles canoniques.*

2.6.1. Soient (G, X) comme en 2.1.1 et $E \subset \boldsymbol{C}$ un corps de nombres qui contient $E(G, X)$. Supposons que $M_{\boldsymbol{C}}(G, X)$ admette un modèle faiblement canonique $M_E(G, X)$ sur E. Le groupe de Galois $\mathrm{Gal}(\bar{\boldsymbol{Q}}/E)$ agit alors sur l'ensemble profini $\pi_0(M_{\boldsymbol{C}}(G, X))$ des composantes connexes géométriques de $M_E(G, X)$. Cette action commute à celle de $G(\boldsymbol{A}^f)$, par hypothèse définie que E. D'après 2.1.14, l'action (à droite) de $G(\boldsymbol{A}^f)$ fait de $\pi_0 M_{\boldsymbol{C}}(G, X)$ un espace principal homogène sous le quotient abélien $\bar{\pi}_0\pi G = G(\boldsymbol{A}^f)/G(\boldsymbol{Q})_+^-$. L'action de Galois est donc définie par un homomorphisme $r_{G,X}$ de $\mathrm{Gal}(\bar{\boldsymbol{Q}}/E)$ dans $\bar{\pi}_0\pi(G)$, dit de *réciprocité*. Convention de signe: l'action (à gauche) de σ coïncide avec l'action (à droite) de $r_{G,X}(\sigma)$. Ce morphisme se factorise par le groupe de Galois rendu abélien, identifié par la théorie du corps de classe global à $\pi_0\pi(\boldsymbol{G}_{mE})$, d'où

$$r_{G,X}\colon \pi_0\pi(\boldsymbol{G}_{mE}) \longrightarrow \bar{\pi}_0\pi(G). \tag{2.6.1.1}$$

2.6.2. Soit M la classe de conjugaison de μ_h, pour $h \in X$. Puisque $E \supset E(G, X)$, elle est définite sur E. Composant les morphismes 2.4.0, on obtient $N_{E/\boldsymbol{Q}}q_M$: $\pi(\boldsymbol{G}_{mE}) \to \pi(G_E) \to \pi(G)$.

Par passage au π_0, on en déduit

$$\pi_0 N_{E/\boldsymbol{Q}}q_M : \pi_0\pi(\boldsymbol{G}_{mE}) \longrightarrow \pi_0\pi(G) \longrightarrow \bar{\pi}_0\pi(G). \tag{2.6.2.1}$$

THÉORÈME 2.6.3. *Le morphisme* (2.6.1.1), *donnent l'action de* $\mathrm{Gal}(\bar{\boldsymbol{Q}}/E)$ *sur l'ensemble des composantes connexes géométriques d'un modèle faiblement canonique* $M_E(G, X)$ *de* $M_{\boldsymbol{C}}(G, X)$ *sur* E, *est l'inverse du morphisme* $\pi_0 N_{E/\boldsymbol{Q}}q_M$ *de* 2.6.2.

L'idée de la démonstration est que, pour chaque type τ de points spéciaux (2.2.4), on connaît l'action d'un sous-groupe d'indice fini Gal_τ de $\mathrm{Gal}(\bar{\boldsymbol{Q}}/E)$ sur les points spéciaux de ce type (par définition des modèles faiblement canoniques)—donc sur l'ensemble des composantes connexes puisque l'application qui à chaque point associe sa composante connexe est compatible à l'action de Galois. Que l'action de Gal_τ obtenue soit la restriction à Gal_τ de l'action définie par l'inverse de $\pi_0(N_{E/\boldsymbol{Q}}q_M)$ est vérifié en 2.6.4 ci-dessous, et il reste à vérifier que les Gal_τ engendrent $\mathrm{Gal}(\bar{\boldsymbol{Q}}/E)$.

Un type τ de points spéciaux est défini par $h \in X$ se factorisant par un tore $\iota\colon T \to G$ défini sur $\boldsymbol{Q}$. Le sous-groupe Gal_τ correspondant est $\mathrm{Gal}(\bar{\boldsymbol{Q}}/E) \cap \mathrm{Gal}(\bar{\boldsymbol{Q}}/E(T, h)) = \mathrm{Gal}(\bar{\boldsymbol{Q}}/E \cdot E(T, h))$. D'après [5, 5.1], pour toute extension finie F de $E(G, X)$, il existe (T, h) tel que l'extension $E(T, h)$ de $E(G, X)$ soit linéairement disjointe de F. Ceci est plus qu'assez pour assurer que les Gal_τ engendrent $\mathrm{Gal}(\bar{\boldsymbol{Q}}/E)$.

2.6.4. Soient T et h comme ci-dessus, et $\mu = \mu_h$. Le morphisme $\mu\colon \boldsymbol{G}_m \to T$ est défini sur $E(T, h)$ et le morphisme $\pi_0 NR(\mu_h)$ de 2.2.3 se déduit, par application du foncteur π_0, de $N_{E(T,h)/\boldsymbol{Q}} \circ q_\mu\colon \pi(\boldsymbol{G}_{mE(T,h)}) \to \pi(T)$. On en déduit que l'action de $\mathrm{Gal}(\bar{\boldsymbol{Q}}/E) \cap \mathrm{Gal}(\bar{\boldsymbol{Q}}/E(T, h))$ sur les points spéciaux de type τ est compatible à l'action de $\mathrm{Gal}(\bar{\boldsymbol{Q}}/E(T, h))^{\mathrm{ab}} = \pi_0\pi(\boldsymbol{G}_{mE(T,h)})$ sur $\pi_0(M_{\boldsymbol{C}}(G, X))$ déduite, par application du foncteur π_0, de l'inverse de

$$\iota \circ N_{E(T,h)/\boldsymbol{Q}} \circ q_\mu\colon \pi(\boldsymbol{G}_{mE(T,h)}) \longrightarrow \pi(T) \longrightarrow \pi(G).$$

De la fonctorialité de N et de q, il résulte que ce composé est $N_{E(T,h)/\boldsymbol{Q}} \circ q_M$:

$$\begin{array}{ccccc}
\pi(G_{mE(T,h)}) & \xrightarrow{q_\mu} & \pi(T_{E(T,h)}) & \longrightarrow & \pi(T) \\
 & \searrow^{q_M} & \downarrow & & \downarrow \\
 & & \pi(G_{E(T,h)}) & \longrightarrow & \pi(G)
\end{array}$$

égal à $N_{E(G,X)/\mathbf{Q}} \circ q_M \circ N_{E(T,h)/E(G,X)}$:

$$\begin{array}{ccccc}
\pi(G_{mE(T,h)}) & \xrightarrow{q_M} & \pi(G_{E(T,h)}) & \longrightarrow & \pi(G) \\
\downarrow & & \downarrow & & \| \\
\pi(G_{mE(G,X)}) & \xrightarrow{q_M} & \pi(G_{E(G,X)}) & \longrightarrow & \pi(G)
\end{array}$$

Puisque la norme $N_{E(T,h)/E(G,X)}$ correspond, via la théorie du corps de classe à l'inclusion de $\mathrm{Gal}(\bar{Q}/E(T, h))$ dans $\mathrm{Gal}(\bar{Q}/E(G, X))$, on a bien l'action promise.

2.7. *Réduction au groupe dérivé, et théorème d'existence.*

Dans ce numéro, schéma signifie "schéma admettant un faisceau inversible ample". Ceci nous permettra de passer sans scrupules au quotient par un groupe fini. La stabilité de cette condition sera évidente dans les applications, et je ne la vérifierai pas à chaque pas. Tout ceci n'est d'ailleurs qu'une question de commodité.

2.7.1. Soit Γ un groupe localement compact totalement discontinu. Nous nous intéresserons à des systèmes projectifs, munis d'une action à gauche de Γ, du type suivant.

(a) Un système projectif, indexé par les sous-groupes compacts ouverts K de Γ, de schémas S_K.

(b) Une action ρ de Γ sur ce système (définie par des isomorphismes $\rho_K(g) : S_K \xrightarrow{\sim} S_{gKg^{-1}}$).

(c) On suppose que $\rho_K(k)$ est l'identité pour $k \in K$. Pour L distingué dans K, les $\rho_L(k)$ définissent une action sur S_L du groupe fini quotient K/L, et on suppose que $(K/L)\backslash S_L \xrightarrow{\sim} S_K$.

Un tel système est déterminé par sa limite projective $S = \lim\operatorname{proj} S_K$, munie de l'action de Γ: on a $S_K = K\backslash S$. Nous appelerons S un *schéma muni d'une action à gauche continue de* Γ. On définit de même la continuité d'une action à droite par la condition $S = \lim\operatorname{proj} S/K$.

2.7.2. Soit π un ensemble profini, muni d'une action continue de Γ. On suppose que l'action est transitive, et que les orbites d'un sous-groupe compact ouvert sont ouvertes: pour $e \in \pi$, de stabilisateur Δ, la bijection $\Gamma/\Delta \to \pi$ est un homéomorphisme.

Si Γ agit continûment sur un schéma S, muni d'une application continue équivariante dans π, la fibre S_e est munie d'une action continue de Δ: pour K compact ouvert dans Γ, $K \cap \Delta\backslash S_e$ est la fibre en l'image de e de $K\backslash S \to K\backslash\pi$, et S_e est la limite de ces quotients.

LEMME 2.7.3. *Le foncteur $S \to S_e$ est une équivalence de la catégorie des schémas S, munis d'une action continue de Γ et d'une application continue équivariante dans π, avec la catégorie des schémas munis d'une action continue de Δ.*

Le foncteur inverse est le foncteur d'*induction* de Δ à Γ: formellement, $\mathrm{ind}_\Delta^\Gamma(T)$ est le quotient de $\Gamma \times T$ par Δ agissant par $\delta(\gamma, t) = (\gamma\delta^{-1}, \delta t)$; ceci a un sens parce que l'action de Δ sur Γ est propre; pour K compact ouvert dans Γ, on a

$$K\backslash \mathrm{Ind}_{\varDelta}^{\varGamma}(T) = K\backslash\varGamma \times T \quad \text{divisé par } \varDelta$$
$$= \coprod_{\gamma\in K\backslash \varDelta = K\backslash\pi} (\gamma K\gamma^{-1} \cap \varDelta)\backslash T.$$

La vérification détaillée est laissée au lecteur.

2.7.4. Soient E un corps, et F une extension galoisienne de E. Le groupe de Galois $\mathrm{Gal}(F/E)$ agit continûment sur $\mathrm{Spec}(F)$. Plus généralement, si X est un schéma sur E, il agit continûment (par transport de structure) sur $X_F = X \times_{\mathrm{Spec}(E)} \mathrm{Spec}(F)$. On a (descente galoisienne)

LEMME 2.7.5. *Le foncteur $X \to X_F$ est une équivalence de la catégorie des schémas sur E avec la catégorie des schémas sur F, munis d'une action continue de* $\mathrm{Gal}(F/E)$ *compatible à l'action de ce groupe de Galois sur F.*

2.7.6. Soient $E \hookrightarrow \bar{\boldsymbol{Q}}$ un corps de nombres, $\varGamma$ un groupe localement compact totalement discontinu, π un ensemble profini, muni d'une action de $\varGamma$ comme en 2.7.2, sauf qu'on prend ici une action à droite, et $e \in \pi$. On se donne aussi une action à gauche de $\mathrm{Gal}(\bar{\boldsymbol{Q}}/E)$, commutant à l'action de $\varGamma$. Soit $\varGamma_e$ le stabilisateur de e. Si on convertit l'action à droite de $\varGamma$ en une action à gauche, on obtient une action à gauche de $\varGamma \times \mathrm{Gal}(\bar{\boldsymbol{Q}}/E)$. Le stabilisateur de e, pour cette action, est une extension $\mathscr{E}$ de $\mathrm{Gal}(\bar{\boldsymbol{Q}}/E)$ par $\varGamma_e$.

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \varGamma_e & \longrightarrow & \mathscr{E} & \longrightarrow & \mathrm{Gal}(\bar{\boldsymbol{Q}}/E) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & \varGamma & \longrightarrow & \varGamma \times \mathrm{Gal}(\bar{\boldsymbol{Q}}/E) & \longrightarrow & \mathrm{Gal}(\bar{\boldsymbol{Q}}/E) & \longrightarrow & 0
\end{array}$$

2.7.7. Lorsque l'action de $\varGamma$ fait de π un espace principal homogène sous un quotient abélien $\pi(\varGamma)$ de $\varGamma$, l'action de Galois est définie par un morphisme r : $\mathrm{Gal}(\bar{\boldsymbol{Q}}/E) \to \pi(\varGamma)$, tel que $\sigma \cdot x = x \cdot r(\sigma)$, $\mathscr{E}$ ne dépend pas de e: $\varGamma_e$ est le noyau de la projection de $\varGamma$ sur $\pi(\varGamma)$, et l'extension $\mathscr{E}$ est l'image inverse, par r, de l'extension $\varGamma$ de $\pi(\varGamma)$ par $\varGamma_e$.

2.7.8. Considérons les schémas S sur E, munis d'une action à droite continue de $\varGamma$ et d'une application $\mathrm{Gal}(\bar{\boldsymbol{Q}}/E)$ et $\varGamma$-équivariante de $S_{\bar{\boldsymbol{Q}}}$ dans π. On note S_e la fibre en e. C'est un schéma sur $\bar{\boldsymbol{Q}}$, muni d'une action continue (à gauche) de l'extension $\mathscr{E}$, et l'action de $\mathscr{E}$ sur S_e est compatible à son action, via $\mathrm{Gal}(\bar{\boldsymbol{Q}}/E)$, sur $\bar{\boldsymbol{Q}}$. Combinant 2.7.3 et 2.7.5, on trouve

LEMME 2.7.9. *Le foncteur $S \to S_e$ est une équivalence de catégories.*

Le cas qui nous intéresse est celui où les S/K sont de type fini sur E, pour K compact ouvert dans $\varGamma$, et où l'application de $S_{\bar{\boldsymbol{Q}}}$ sur π identifie π à $\pi_0(S_{\bar{\boldsymbol{Q}}})$. Ces conditions correspondent à: les $K\backslash S_e$, pour K compact ouvert dans $\varGamma_e$, sont connexes et de type fini sur $\bar{\boldsymbol{Q}}$.

2.7.10. Soient G un groupe adjoint, G' un revêtement de G, X^+ une $G(\boldsymbol{R})^+$ — classe de conjugaison de morphismes de $\boldsymbol{S}$ dans $G_{\boldsymbol{R}}$, vérifiant les conditions de 2.1.1, et $E \subset \bar{\boldsymbol{Q}}$ une extension finie de $E(G, X^+)$. Un *modèle faiblement canonique* (*connexe*) de $M^0(G, G', X^+)$ sur E consiste en

(a) un modèle $M^0_{\bar{\boldsymbol{Q}}}$ de $M^0(G, G', X^+)$ sur $\bar{\boldsymbol{Q}}$, i.e. un schéma $M^0_{\bar{\boldsymbol{Q}}}$ sur $\bar{\boldsymbol{Q}}$, muni d'un isomorphisme du schéma sur $\boldsymbol{C}$ qui s'en déduit par extension des scalaires avec $M^0(G, G', X^+)$;

(b) une action continue de $\mathscr{E}_E(G, G', X^+)$ (2.5.9.1) sur le schéma $M^0_{\bar{Q}}$, compatible à l'action du quotient $\mathrm{Gal}(\bar{Q}/E)$ de $\mathscr{E}_E$ sur $\bar{Q}$, et telle que l'action du sous-groupe $G(Q)^{+\wedge}$ (rel. G') (une action $\bar{Q}$-linéaire cette fois) fournisse par extension des scalaires à C l'action 2.1.8;

(c) on exige que pour tout point spécial $h \in X^+$, se factorisant par un tore $H \to G$ défini sur Q, le point de $M^0(G, G', X^+)$ défini par h—fixe par $H(Q)$—soit défini sur $\bar{Q}$ et (en tant que point fermé de $M^0_{\bar{Q}}$) fixe par l'image de l'extension en deuxième ligne de (2.5.10.1) (rel. μ_h).

Lorsque $E = E(G, X)$, on parle de *modèle canonique* (*connexe*).

2.7.11. Les propriétés de fonctorialité suivantes sont immédiates.

(a) Soient des systèmes $(G_\iota, G'_\iota, X^+_\iota)$ comme en 2.7.10, en nombre fini, et $E \subset \bar{Q}$ un corps de nombres contenant les $E(G_\iota, X^+_\iota)$. Si les $M^0_{\bar{Q}\iota}$ sont des modèles faiblement canoniques, sur E, des $M^0_C(G_\iota, G'_\iota, X^+_\iota)$, leur produit est un modèle faiblement canonique de $M^0_C(\prod G_\iota, \prod G'_\iota, \prod X^+_\iota)$ sur E.

(b) Soient (G, G', X^+) comme en 2.7.10, et G'' un revêtement de G, quotient de G'. Si $M^0_{\bar{Q}}$ est un modèle faiblement canonique, sur E, de $M^0_C(G, G', X^+)$, son quotient par $\mathrm{Ker}(G(Q)^{+\wedge}$ (rel. G') $\to G(Q)^{+\wedge}$ (rel. G'')) est un modèle faiblement canonique, sur E, de $M^0_C(G, G'', X^+)$.

2.7.12. Soient G un groupe réductif sur Q, X comme en 2.1.1, X^+ une composante connexe de X et $E \subset \bar{Q}$ une extension finie de Q, contenant $E(G, X)$. Si $M_C(G, X)$ admet un modèle faiblement canonique $M_E(G, X)$ sur E, ce dernier est unique à isomorphisme unique près [5, 3.5]. L'action (2.0.2) de G^{ad} sur G induit donc une action, par transport de structure, de $G^{\mathrm{ad}}(Q)_+$ sur $M_E(G, X)$. Convertissons cette action en une action à droite. Combinée à l'action de $G(A^f)$, elle fournit une action à droite de

$$\frac{G(A^f)}{Z(Q)^-} *_{G(Q)/Z(Q)} G^{\mathrm{ad}}(Q)_+ = \frac{G(A^f)}{Z(Q)^-} *_{G(Q)_+/Z(Q)} G^{\mathrm{ad}}(Q)^+.$$

Après extension des scalaires à C, c'est l'action (2.1.13).

Soit π l'ensemble profini $\pi_0(M_{\bar{Q}}(G, X)) = \pi_0(M_C(G, X))$, et $e \in \pi$ la composante neutre (2.1.7) rel. X^+. Le foncteur 2.7.9 transforme $M_E(G, X)$, muni de la projection naturelle de $M_{\bar{Q}}(G, X)$ dans π, en un schéma $M^0(G, X)$ sur $\bar{Q}$, muni d'une action continue de l'extension (2.5.9.1).

PROPOSITION 2.7.13. *L'équivalence de catégories* 2.7.9 *fait se correspondre les modèles faiblement canoniques de $M(G, X)$ sur E et les modèles faiblement canoniques de $M^0(G^{\mathrm{ad}}, G^{\mathrm{der}}, X^+)$ sur E.*

Dans la définition 2.2.5 des modèles faiblement canoniques, nous avons imposé l'action d'un sous-groupe $\mathrm{Gal}(\bar{Q}/E(\tau)) \cap \mathrm{Gal}(\bar{Q}/E)$ de $\mathrm{Gal}(\bar{Q}/E)$ sur l'ensemble des points spéciaux de type τ. Ceux-ci forment une seule orbite sous $G(A^f)$, et l'action prescrite commute à l'action de $G(A^f)$. Dans la définition 2.2.5, on peut donc se contenter d'exiger que pour un point spécial de type τ ses conjugués par Galois soient comme prescrit. En particulier, il suffit de considérer les systèmes (H, h) formés d'un point spécial $h \in X^+$ se factorisant par un tore H défini sur Q, et, pour chaque système de ce type, de prescrire les conjugués sous $\mathrm{Gal}(\bar{Q}/E(H\{h\})) \cap \mathrm{Gal}(\bar{Q}/E)$ de l'image de $(h, e) \in X \times G(A^f)$ dans $M_C(G, X)$. On retrouve ainsi la variante [5, 3.13] de la définition: $M_C(H\{h\})$ a trivialement un modèle canonique (c'est un ensemble profini, et on prend le modèle sur $E(H,\{h\})$ pour lequel l'action de Galois sur ces points est l'action prescrite), et on impose au morphisme naturel $M_C(H, \{h\}) \to M_C(G, X)$ d'être défini sur $E \cdot E(H, \{h\})$.

Nous laissons au lecteur le soin de vérifier que l'équivalence de catégorie 2.7.9 transforme cette condition de fonctorialité en celle qui définit les modèles faiblement canoniques connexes.

2.7.14. Soient G un groupe algébrique réel adjoint, et X^+ une classe de $G(\boldsymbol{R})^+$-conjugaison de morphismes de $\boldsymbol{S}/\boldsymbol{G}_m$ dans $G_{\boldsymbol{R}}$. Notons M la classe de conjugaison de μ_h, pour $h \in X^+$: une classe de conjugaison de morphismes de $\boldsymbol{G}_m$ dans $G_{\boldsymbol{C}}$. Si G_1 est un groupe réductif de groupe adjoint G, un relèvement X_1^+ de X^+ en une classe de $G_1(\boldsymbol{R})^+$-conjugaison de morphismes de $\boldsymbol{S}$ dans $G_{\boldsymbol{R}}$ définit un relèvement $M(X_1^+)$ de M: la classe de conjugaison de μ_h, pour $h \in X_1^+$ (cf. 2.5.4).

LEMME 2.7.15. *La construction $X_1^+ \to M(X_1^+)$ met en bijection les relèvements de X^+ et ceux de M.*

La construction $h \to \mu_h$ est une bijection de l'ensemble des morphismes h de $\boldsymbol{S}$ dans un groupe réel G avec l'ensemble des morphismes μ de $\boldsymbol{G}_m$ dans $G_{\boldsymbol{C}}$ qui commutent à leur complexe conjugué: on a $h(z) = \mu(z)\,\bar{\mu}(\bar{z})$. Via ce dictionnaire, le problème devient de vérifier que si $\mu_1 : \boldsymbol{G}_m \to G_{\boldsymbol{C}}$ commute à $\bar{\mu}_1$. Cela résulte de la rigidité des tores: le morphisme int $\bar{\mu}_1(z)(\mu_1)$ coïncide avec μ_1 pour $z = 1$, et relève μ_1 pour toute valeur de z. Il est donc constamment égal à μ_1.

Ce dictionnaire permet de traduire 2.5.5 en le

LEMME 2.7.16. *Soient G, G' et X^+ comme en* 2.7.10. *Il existe un groupe réductif G_1, le groupe adjoint G et de groupe dérivé G', et une classe de $G_1(\boldsymbol{R})^0$ conjugaison X_1^+ de morphismes de $\boldsymbol{S}$ dans G qui relève X^+ et telle que $E(G, X^+) = E(G_1, X_1^+)$.*

2.7.17. Ce lemme, et l'équivalence 2.7.13, permettent de transporter aux modèles faiblement canoniques des variétés de Shimura connexes les résultats de [**5**] sur les modèles faiblement canoniques de variétés de Shimura, et établissent une équivalence entre les problèmes de construction correspondant.

COROLLAIRE 2.7.18. *Soient (G, X) comme en* 2.1.1, *X^+ une composante connexe de X et $E \subset \bar{\boldsymbol{Q}}$ une extension finie de $E(G, X)$. Pour que $M(G, X)$ admette un modèle faiblement canonique sur E, il faut et il suffit que $M^0(G^{\mathrm{ad}}, G^{\mathrm{der}}, X^+)$ en admette un. En particulier, l'existence d'un tel modèle ne dépend que de* $(G^{\mathrm{ad}}, G^{\mathrm{der}}, X^+, E)$.

COROLLAIRE 2.7.19. (*Cf.* [**5**, 5.5, 5.10, 5.10.2]). *Soient G, G', X^+ et E comme en* 2.7.10.

(i) *$M^0(G, G', X^+)$ admet au plus un modèle faiblement canonique sur E* (*unicité à isomorphisme unique près*).

(ii) *Supposons que, pour toute extension finie F de E, il existe une extension finie F' de E dans $\bar{\boldsymbol{Q}}$, linéairement disjointe de F, et un modèle faiblement canonique de $M^0(G, G', X^+)$ sur F'. Alors, il existe un modèle faiblement canonique de $M^0(G, G', X^+)$ sur E.*

Le corollaire 2.7.19 et 2.3.1, 2.3.10 fournissent de nombreux modèles canoniques.

THÉORÈME 2.7.20. *Soient G un groupe $\boldsymbol{Q}$-simple adjoint, G' un revêtement de G,* *et* X^+ *une* $G(\boldsymbol{R})^+$*-classe de conjugaison de morphismes de* $\boldsymbol{S}$ *dans* $G_{\boldsymbol{R}}$, *vérifiant* (2.1.1.1), (2.1.1.2), (2.1.1.3). *Dans les cas suivants,* $M^0(G, G', X^+)$ *admet un modèle canonique*

(a) G *est de type* A, B, C *et* G' *est le revêtement universel de* G.

(b) (G, X) *est de type* $\mathrm{D}^{\boldsymbol{R}}$ *et* G' *est le revêtement universel de* G.

(c) (G, X) *est de type* $\mathrm{D}^{\boldsymbol{H}}$, *et* G' *est le revêtement* 2.3.8 *de* G.

Appliquant 2.7.11, 2.7.18, on en déduit le

COROLLAIRE 2.7.21. *Soient* G *un groupe réductif,* X *une* $G(\boldsymbol{R})$*-classe de conjugaison de morphisme de* $\boldsymbol{S}$ *dans* $G_{\boldsymbol{R}}$, *vérifiant les conditions de* 2.1.1, *et* X^+ *une composante connexe de* X. *Pour que* $M(G, X)$ *admette un modèle canonique, il suffit que,* (G^{ad}, X^+) *soit un produit de système* (G_ι, X_ι^+) *du type considéré en* 2.7.20, *et que le revêtement* G^{der} *de* G^{ad} *soit un quotient du produit des revêtements des* G_ι *considérés en* 2.7.20.

BIBLIOGRAPHIE

Pour la bibliographie des articles de Shimura consacrés à la construction de modèles canoniques, je renvoie à [5].

1. A. Ash, D. Mumford, M. Rapoport and Y. Tai, *Smooth compactifications of locally symmetric varieties*, Math. Sci. Press, 1975.

2. W. Baily and A. Borel, *Compactification of arithmetic quotients of bounded symmetric domains*, Ann. of Math. (2) **84** (1966), 442–528.

3. A. Borel, *Some metric properties of arithmetic quotients of symmetric spaces and an extension theorem*, J. Differential Geometry **6** (1972), 543–560.

4. N. Bourbaki, *Groupes et algèbres de Lie*, Chapitres IV-VI, Hermann, 1968.

5. P. Deligne, *Travaux de Shimura*, Sém. Bourbaki Février 71, Exposé 389, Lecture Notes in Math., vol. 244, Springer-Verlag, Berlin, 1971.

6. ———, *Travaux de Griffiths*, Sém. Bourbaki Mai 70, Exposé 376, Lecture Notes in Math., vol. 180, Springer-Verlag, Berlin, 1971.

7. ———, *La conjecture de Weil pour les surfaces* K3, Invent. Math. **15** (1972), 206–226.

8. S. Helgason, *Differential geometry and symmetric spaces*, Academic Press, New York, 1962.

9. D. Mumford, *Families of abelian varieties*, Algebraic Groups and Discontinuous Subgroups, Proc. Sympos. Pure Math., vol. 9, Amer. Math. Soc., Providence, R.I., 1966, pp. 347–351.

10. N. Saavedra, *Catégories tannakiennes*, Lecture Notes in Math., vol. 265, Springer-Verlag, Berlin, 1972.

11. I. Satake, *Holomorphic imbedding of symmetric domains into a Siegel space*, Am. J. Math. **87** (1965), 425–461.

12. J. P. Serre, *Sur les groupes de congruence des variétés abéliennes*, Izv. Akad. Nauk SSSR Ser. Mat. **28** (1964), 3–20.

13. ———, *Cohomologie galoisienne*, Lecture Notes in Math., vol. 5, Springer-Verlag, Berlin, 1965.

SGA Sém. de Géométrie Algébrique du Bois-Marie. SGA1, SGA4 t.3 et SGA4$^{1/2}$ sont parus aux Lecture Notes in Math. nos. 224, 305, 569, Springer-Verlag, Berlin.

INSTITUT HAUTES ETUDES SCIENTIFIQUES, BURES-SUR-YVETTE

Proceedings of Symposia in Pure Mathematics
Vol. 33 (1979), part 2, pp. 291–311

CONGRUENCE RELATIONS AND SHIMURA CURVES

YASUTAKA IHARA

Introduction. Nonabelian reciprocity for the one-dimensional function field $\boldsymbol{K}$ over the finite field $\boldsymbol{F}_q$ has been investigated in two directions. One is the recent beautiful work of Drinfeld (realizing a part of Langlands' philosophy), which associates to each automorphic representation of $\mathrm{GL}_2(\boldsymbol{K}_A)$ a system of l-adic representations of the Galois group over $\boldsymbol{K}$. The other is older, and is essentially related to automorphic functions *in characteristic* 0. This is the direction suggested in my lecture notes **[5b]** as explicit conjectures (cf. also **[5a**, **c]**). Its aim is a sort of arithmetic uniformization theory for algebraic curves $\boldsymbol{X}$ over $\boldsymbol{F}_q$ by means of discrete subgroups Γ of $\mathrm{PSL}_2(\boldsymbol{R}) \times \mathrm{PGL}_2(k)$, where k is a $\mathfrak{p}$-adic field with $N(\mathfrak{p}) = q$. For example, if $\boldsymbol{X}_N$ is the canonical modular curve of level $N \not\equiv 0 \pmod{p}$ over $\boldsymbol{F}_{p^2}$, p a prime, then the space $\mathfrak{P}(\boldsymbol{X}_N)$ of all *ordinary* closed scheme-theoretic points of $\boldsymbol{X}_N$ can be expressed as the quotient $\Gamma_N \backslash \mathscr{H}$, where Γ_N is the modular group of level N over $\boldsymbol{Z}[1/p]$ and $\mathscr{H}$ is the space of all imaginary quadratic *subfields* M of $M_2(\boldsymbol{Q})$ with $(M/p) = 1$. This simultaneous uniformization $\mathfrak{P}(\boldsymbol{X}_N) = \Gamma_N \backslash \mathscr{H}$ is important, because this naturally describes all the Frobenius elements in the covering system $\{\boldsymbol{X}_N/\boldsymbol{X}_1\}$.[1] These were proved in **[5b**, Chapter 5 of Vols. 1, 2] by using several exquisite results of Deuring on the reduction of elliptic curves parametrized by points of $\mathscr{H}$. A simple characterization of the system $\{\boldsymbol{X}_N/\boldsymbol{X}_1\}$ was proved in **[6]**. The present work started with an observation that these can be proved with only the Kronecker congruence relation $T(p) \otimes \boldsymbol{F}_p = \Pi \cup {}^t\Pi$ as basis. This is not so surprising after all, but this observation should be *systematically* used due to the following reasons. First, Shimura curves also have congruence relations, due to Shimura, for almost all $\mathfrak{p}$. Secondly, there may exist curves other than Shimura curves having (unramified) congruence relations. Thirdly, it now seems important to regard congruence relations, not only as relations or theorems, but as *categorical objects*, because this category is very likely to be equivalent with the other two categories, that of Γ, and that of $\boldsymbol{X}$ with some additional structures, and the clarification of this equivalence seems significant.

The main purpose of this paper is to outline our theory which gives a systematic study of abstract congruence relations (or equivalently, CR-systems, §1). It mainly states that *whenever there is an unramified symmetric CR-system* $\mathscr{X}$ *over* $\mathfrak{o}$ (Definitions 1.1.1, 1.5.1), *there is a satisfactory arithmetic uniformization theory for an algebraic curve* $\boldsymbol{X}$ *over* $\boldsymbol{F}_q$ *by means of a discrete subgroup* Γ *of* $\mathrm{PSL}_2(\boldsymbol{R}) \times \mathrm{Aut}(\mathscr{T})$.

AMS (*MOS*) *subject classifications* (1970). Primary 14G15, 12A90; Secondary 14H30, 10D10.
[1]A brief review of this uniformization is given in §7.1.

Here, $\mathfrak{o}$ is the ring of integers of a $\mathfrak{p}$-adic field k ($N\mathfrak{p} = q$), and $\mathscr{T}$ is the *tree* associated with $\mathrm{PGL}_2(k)$ (Main Theorems I–III). This will then be applied to the case where X is a smooth reduction of a *Shimura curve* and Γ is a quaternionic discrete subgroup of $\mathrm{PSL}_2(\boldsymbol{R}) \times \mathrm{PGL}_2(k)$.

Some relations with other works are discussed in §7.2. Some of the proofs, including those which had been sketched in the previous announcement **[9a]**, are totally omitted. The details **[9b]** will appear shortly.

Notations and terminologies. $\mathfrak{o}$: a complete discrete valuation ring of characteristic 0 with finite residue field $\boldsymbol{F}_q$;

$\mathfrak{p}$: the maximal ideal of $\mathfrak{o}$;

k: the quotient field of $\mathfrak{o}$;

$\mathrm{Spec}\,\mathfrak{o} = \{\eta, s\}$ (η: the generic point, s: the closed point).

If Z is an $\mathfrak{o}$-scheme, $Z_\eta = Z \otimes_{\mathfrak{o}} k$ denotes its fiber over η (the general fiber) and $Z_s = Z \otimes_{\mathfrak{o}} \boldsymbol{F}_q$ denotes its fiber over s (the special fiber).

For any field F, $\bar{F}$ denotes its algebraic closure. If Z is a proper smooth irreducible algebraic curve over F, we write $\bar{Z} = Z \otimes_F \bar{F}$. If F' is the exact constant field of Z, $\bar{Z}$ consists of $[F' : F]$ connected components. The genus g of $\bar{Z}$ is defined by $g - 1 = [F' : F](\tilde{g} - 1)$, where $\tilde{g}$ is the genus of each component of $\bar{Z}$.

$k_d \subset \bar{k}$: the unique unramified extension of k with degree d;

$\mathfrak{o}_d$: the ring of integers of k_d;

$[q]$: the Frobenius automorphism of $\bigcup k_d$ over k.

1. Congruence relations.

1.1. Let X be a proper smooth irreducible algebraic curve over $\boldsymbol{F}_q$. We do not assume that X is absolutely irreducible. Let Π (resp. ${}^t\Pi$) be the graphs on $X \times_{\boldsymbol{F}_q} X$ of the qth power morphisms $X \to X$ (resp. $X \leftarrow X$). Consider Π, ${}^t\Pi$ and $\Pi \cup {}^t\Pi$ as closed reduced subschemes of $X \times_{\boldsymbol{F}_q} X$.

DEFINITION 1.1.1. A triple $(X_1, X_2; T)$ of two-dimensional integral $\mathfrak{o}$-schemes is called a congruence relation w.r.t. $(X, \mathfrak{o})$, if

(i) X_1, X_2 are proper smooth over $\mathfrak{o}$, and have X as the special fiber;

(ii) T is a closed subscheme of $S = X_1 \times_{\mathfrak{o}} X_2$, flat over $\mathfrak{o}$, such that $T \times_S (X \times_{\boldsymbol{F}_q} X) = \Pi \cup {}^t\Pi$.

Let $\mu: X_0 \to T$ be the normalization of the two-dimensional integral scheme T, and put $\varphi_i = \mathrm{pr}_i \circ \mu$ $(i = 1, 2)$, where pr_i are the projections of T to X_i. The system

$$\mathscr{X} = \{X_1 \xleftarrow{\varphi_1} X_0 \xrightarrow{\varphi_2} X_2\} \tag{1.1.2}$$

thus obtained will be called a *CR-system w.r.t.* $(X, \mathfrak{o})$. Since T is the image of X_0 in $X_1 \times_{\mathfrak{o}} X_2$, the association $(X_1, X_2; T) \to \mathscr{X}$ is invertible. Among the two equivalent notions, congruence relations and CR-systems, we shall use the latter more frequently.

1.2. Let $\boldsymbol{F}_{q^c}$ ($c \geqq 1$) be the exact constant field of X. Then it is easy to see that the exact constant rings of X_i ($i = 0, 1, 2$) are $\mathfrak{o}_c$. Moreover, since T is irreducible, and since the connected components of $X_1 \times_{\mathfrak{o}} X_2$ and of $X \times_{\boldsymbol{F}_q} X$ correspond bijectively, Π and ${}^t\Pi$ must lie on the same component of $X \times_{\boldsymbol{F}_q} X$. This shows that either $c = 1$ or $c = 2$.

DEFINITION 1.2.1. $\mathscr{X}$ belongs to Case 1 if $c = 1$, and Case 2 if $c = 2$.

We denote by g the genus of $\bar{X} = X \otimes_{\boldsymbol{F}_q} \bar{\boldsymbol{F}}_q$.

1.3. It follows easily from the definition that $X_{i\eta} = X_i \otimes_{\mathfrak{o}} k$ ($i = 0, 1, 2$) are proper smooth irreducible algebraic curves over k. Put $\bar{X}_{i\eta} = X_{i\eta} \otimes_k \bar{k}$, and let g_i denote the genus of $\bar{X}_{i\eta}$ ($i = 0, 1, 2$). Then $g_1 = g_2 = g$. Put $\varphi_{i\eta} = \varphi_i \otimes_{\mathfrak{o}} k$ ($i = 1, 2$). Then $\varphi_{i\eta}$ are finite k-morphisms of degree $q + 1$. Therefore, the Hurwitz formula gives

$$g_0 - 1 = (q + 1)(g - 1) + \tfrac{1}{2}\,\delta, \tag{1.3.1}$$

where δ is the degree (over k) of the differental divisor ("$\mathfrak{Differente}$") of $\varphi_{i\eta}$.

1.4. When $\boldsymbol{x}$ runs over all the $\boldsymbol{F}_{q^2}$-rational points of $\boldsymbol{X}$, $\boldsymbol{y} = (\boldsymbol{x}, \boldsymbol{x}^q)$ runs over all the geometric points belonging to the intersection $\Pi \cap {}^t\Pi$ on T_s. Consider $\boldsymbol{y}$ as a point of the two-dimensional scheme T. Then the local ring $\mathcal{O}_{T,\boldsymbol{y}}$ may or may not be normal, depending on $\boldsymbol{x}$.

DEFINITION 1.4.1. An $\boldsymbol{F}_{q^2}$-rational point $\boldsymbol{x}$ of $\boldsymbol{X}$ is called a *special point* if $\boldsymbol{y} = (\boldsymbol{x}, \boldsymbol{x}^q)$ is a normal point of T. The special points of $\boldsymbol{X}$ will be denoted by $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_H$. All other $\bar{\boldsymbol{F}}_q$-rational points of $\boldsymbol{X}$ are called the *ordinary points*.

The special fiber X_{0s} of the normalization X_0 of T consists of two irreducible components which can be identified with Π and ${}^t\Pi$ via μ. These two components meet at above those $\boldsymbol{y} = (\boldsymbol{x}, \boldsymbol{x}^q)$ for which $\boldsymbol{x}$ is special. Therefore, the coincidence of the Euler-Poincaré characteristics of $X_{0\eta}$ and of X_{0s} gives

$$g_0 - 1 = 2(g - 1) + H; \tag{1.4.2}$$

hence by (1.3.1), we obtain the following formula for the number of special points:

$$H = (q - 1)(g - 1) + \tfrac{1}{2}\,\delta. \tag{1.4.3}$$

By the Zariski connectedness theorem [**4**, III, 4.3.1], H is always positive.

1.5.

DEFINITION 1.5.1. (i) $\mathcal{X}$ is called *unramified*, if $\varphi_{1\eta}$ and $\varphi_{2\eta}$ are unramified.

(ii) $\mathcal{X}$ is called *symmetric*, if $X_2 = X_1$ and ${}^tT = T$, where ${}^tT$ is the transpose of T; or more precisely, if there exists a pair $(\mathscr{E}_1, \mathscr{E}_2)$ of mutually inverse $\mathfrak{o}$-isomorphisms $\mathscr{E}_1\colon X_1 \xrightarrow{\sim} X_2$, $\mathscr{E}_2\colon X_2 \xrightarrow{\sim} X_1$ which lift the identity map of $\boldsymbol{X}$ and for which $(\mathscr{E}_1 \times \mathscr{E}_2)(T) = {}^tT$. (It follows easily that such a symmetry $(\mathscr{E}_1, \mathscr{E}_2)$ is at most unique; [**9b**, §1].)

By (1.4.3), when $\mathcal{X}$ is unramified we have $H = (q - 1)(g - 1)$, and conversely. In particular, $g > 1$ when $\mathcal{X}$ is unramified.

1.6. When $\mathfrak{o} = \boldsymbol{Z}_p$, the ring of p-adic integers, we can prove, as an application of our previous work [**10**], the following *rigidity of unramified congruence relations.*

THEOREM 1.6.1. *Let* $\boldsymbol{X}$, *and a set* $\mathfrak{S}$ *of* $\boldsymbol{F}_{p^2}$*-rational points of* $\boldsymbol{X}$ *be given. Then there exists at most unique unramified CR-system* $\mathcal{X}$ *with respect to* $(\boldsymbol{X}, \boldsymbol{Z}_p)$ *which has* $\mathfrak{S}$ *as the set of special points. Moreover, when it exists, it is symmetric.*

Thus, when $\mathfrak{o} = \boldsymbol{Z}_p$, $\mathcal{X}$: *unramified* implies $\mathcal{X}$: *symmetric*. As for the existence, the question is more difficult. We already know a necessary condition $|\mathfrak{S}| = (q - 1)(g - 1)$ for the existence of $\mathcal{X}$. But this is not sufficient. The question is essentially connected with a certain differential on a formal lifting of $\boldsymbol{X}$ (see [**10**], [**7**]). I hope to discuss the further developments of [**10**] in the near future.

1.7. The most well-known congruence relation is the Kronecker congruence relation $(X_1, X_2; T)$, where $X_1 = X_2$ is the projective j-line over $\boldsymbol{Z}_p$ and T is the closed subscheme of $X_1 \times_{\boldsymbol{Z}_p} X_2$ defined by the modular equation $\Phi_p(j, j') = 0$ of order p. This is symmetric, but not unramified. As is well known, the ordinary (resp. special) points are the singular (resp. supersingular) j-invariants of elliptic curves over $\bar{\boldsymbol{F}}_p$, with the exception of the cusp which is also ordinary.

For other examples, see §6 and **[9a]**.

2. The first Galois theory. Let $\mathscr{X} = \{X_1 \overset{\varphi_1}{\leftarrow} X_0 \overset{\varphi_2}{\rightarrow} X_2\}$ be any CR-system w.r.t. $(\boldsymbol{X}, \mathfrak{o})$. The purpose of this section is to establish a fundamental Galois-theoretic property of the system

$$\mathscr{X}_\eta = \{X_{1\eta} \xleftarrow{\varphi_{1\eta}} X_{0\eta} \xrightarrow{\varphi_{2\eta}} X_{2\eta}\}$$

over k. This will be basic for our further studies.

2.1. Let K_i denote the function field of X_i $(i = 0, 1, 2)$. Then each K_i is an algebraic function field of one variable with exact constant field k_c (cf. §1.2). The morphisms φ_i $(i = 1, 2)$ induce the inclusions $K_i \hookrightarrow K_0$, and we have $K_0 = K_1K_2$, $[K_0: K_i] = q + 1$ $(i = 1, 2)$.

DEFINITION 2.1.1. L is the smallest Galois extension of K_0 such that L/K_1, L/K_2 are both Galois extensions.

Call V_i the Galois group of L/K_i $(i = 0, 1, 2)$. Note that $V_0 = V_1 \cap V_2$.

DEFINITION 2.1.2. $G_\mathfrak{p}^+$ is the subgroup of $\mathrm{Aut}(L/k)$ generated by V_1 and V_2.

The Krull topologies of V_i will be extended to the topology of $G_\mathfrak{p}^+$ defined by the characterization "V_i are open".

2.2. Suppose for a moment that $\mathscr{X}$ is the Kronecker CR-system (see §1.7). Then (in the usual sense of notations) $K_1 = \boldsymbol{Q}_p(j(\tau))$, $K_2 = \boldsymbol{Q}_p(j(p\tau))$, and L is the field generated over $\boldsymbol{Q}_p$ by all functions of the form $j(p^n\tau + b)$ $(n \in \boldsymbol{Z}, b \in \boldsymbol{Z}[1/p])$. This shows that

$$V_1 = \mathrm{PGL}_2(\boldsymbol{Z}_p), \qquad V_2 = \begin{pmatrix} 0 & -1 \\ p & 0 \end{pmatrix}^{-1} V_1 \begin{pmatrix} 0 & -1 \\ p & 0 \end{pmatrix},$$

and $G_\mathfrak{p}^+ = \mathrm{PGL}_2^+(\boldsymbol{Q}_p)$, where (in general)

$$\mathrm{PGL}_2^+(k) = \{g \in \mathrm{GL}_2(k);\ \mathrm{ord}_\mathfrak{p} \det(g) \equiv 0 \pmod 2\}/k^\times.$$

So, in this case, we already know the structure of $G_\mathfrak{p}^+$ relative to V_1, V_2. In fact, we know the structure of the related double coset ring and, as its consequence, we know such a property of $G_\mathfrak{p}^+$ that $G_\mathfrak{p}^+$ is the free product of V_1 and V_2 with amalgamated subgroup V_0 **[5b**, Vol. 1, Chapter 2, §28], **[16]**. We can show that these are *general phenomena* attached to the congruence relations.

2.3. To be precise, return to an arbitrary CR-system $\mathscr{X}$ and consider the disjoint union $(V_1\backslash G_\mathfrak{p}^+) \sqcup (V_2\backslash G_\mathfrak{p}^+)$ of two left coset spaces as a point-set. Call it $\mathscr{T}^0$. Two points V_1g, V_2g' (belonging to different coset spaces) are called "mates" if $V_1g \cap V_2g' \neq \emptyset$. Since $(V_i: V_0) = q + 1$ $(i = 1, 2)$, each point has exactly $q + 1$ mates. Consider the diagram $\mathscr{T} = \mathscr{T}(G_\mathfrak{p}^+; V_1, V_2)$ obtained from this point-set $\mathscr{T}^0$ by connecting each pair of mates by a segment. Then $G_\mathfrak{p}^+$ acts on $\mathscr{T}$ by the right multiplications, and the action is effective, due to the minimality of L.

THEOREM 2.3.1. *Let $\mathscr{X}$ be any CR-system and put $\mathscr{T} = \mathscr{T}(G_\mathfrak{p}^+; V_1, V_2)$. Then* (i) *$\mathscr{T}$ is connected and acyclic; in other words, for any points $A, B \in \mathscr{T}^0$, there exists a* by ι. Then $V_0 \iota$ is well defined, $\iota^{-1} V_0 \iota = V_0$, $\iota^2 \in V_0$, and $\iota^{-1} V_1 \iota = V_2$, $\iota^{-1} V_2 \iota = V_1$.

DEFINITION 2.7.1. When $\mathscr{X}$ is symmetric, $G_\mathfrak{p}$ is the group generated by $G_\mathfrak{p}^+$ and ι.

Note that $G_\mathfrak{p}$ is generated by V_1 and ι, and that $(G_\mathfrak{p}: G_\mathfrak{p}^+) = 2$. We shall identify $(V_1 \backslash G_\mathfrak{p}^+) \sqcup (V_2 \backslash G_\mathfrak{p}^+)$ with $V_1 \backslash G_\mathfrak{p}$, by $V_1 g \leftrightarrow V_1 g$, $V_2 g \leftrightarrow V_1 \iota g$ $(g \in G_\mathfrak{p}^+)$. Thus, $V_1 g$ and $V_1 g'$ are *mates* if and only if $g' g^{-1} \in V_1 \iota V_1$. This group $G_\mathfrak{p}$ acts on $\mathscr{T}$ effectively, and Theorem 2.3.1(ii) can be rewritten as:

(ii*) If $A, B, A', B' \in \mathscr{T}^0$ are such that $l(A, B) = l(A', B')$, then there exists $g \in G_\mathfrak{p}$ such that $A' = A^g$, $B' = B^g$.

As a corollary of Theorem 2.3.1(i) we obtain:

COROLLARY 2.7.2. *$G_\mathfrak{p}$ is the free product of V_1 and $V_0 \cup V_0 \iota$ with amalgamated subgroup V_0.*

When $\mathscr{X}$ is the Kronecker CR-system, we have (for a suitable extension ι):

$$G_\mathfrak{p} = \mathrm{PGL}_2(\boldsymbol{Q}_p), \qquad \iota = \begin{pmatrix} 0 & -1 \\ p & 0 \end{pmatrix}.$$

2.8. Now return to an arbitrary CR-system $\mathscr{X}$. When it is unramified, K_0/K_1, K_0/K_2 are unramified extensions of algebraic function fields, and this implies that L/K_0 is also unramified.

DEFINITION 2.8.1. $\mathscr{X}$ is called *almost unramified* if almost all prime divisors of K_0/k are unramified in L.

By the above remark, $\mathscr{X}$: *unramified* implies $\mathscr{X}$: *almost unramified.* I do not know whether the almost-unramifiedness also implies the symmetricity when $\mathfrak{o} = \boldsymbol{Z}_p$. This definition of almost-unramifiedness singles out the class of CR-systems that are related to automorphic functions. There are examples of $\mathscr{X}$ that are *not* almost unramified.

2.9. *The arithmetic fundamental group Γ.* Let $\mathscr{X}$ be almost unramified and symmetric. Then to each $\mathscr{X}$ and an embedding $\varepsilon: k \hookrightarrow \boldsymbol{C}$ ($\boldsymbol{C}$: the complex number field) we can associate a discrete subgroup Γ of

$$\mathrm{PSL}_2(\boldsymbol{R}) \times G_\mathfrak{p}, \tag{2.9.1}$$

determined up to conjugacy, in the following manner. Consider the set Σ of all those places ξ_C of L into $\boldsymbol{C} \cup (\infty)$ such that (i) ξ_C extends ε, and (ii) the valuation ring of ξ_C is either L itself or is a discrete valuation ring. Let $G_\mathfrak{p}$ act on Σ as $\xi_C \to g\xi_C$, where $(g\xi_C)(a) = \xi_C(a^g)$ $(\xi_C \in \Sigma, g \in G_\mathfrak{p}, a \in L)$. Then Σ carries a natural $G_\mathfrak{p}$-invariant complex structure (cf. [**5b**, Vol. 1, Chapter 2] or [**9b**]). Moreover, each connected component Σ_0 of Σ is isomorphic to the *complex upper half-plane* $\mathfrak{H}$, and $G_\mathfrak{p}$ acts transitively on the set of all connected components of Σ. Choose any connected component Σ_0 of Σ, and put

$$\Gamma = \{\gamma \in G_\mathfrak{p} \mid \gamma \cdot \Sigma_0 = \Sigma_0\}. \tag{2.9.2}$$

Then Γ acts effectively on $\Sigma_0 \cong \mathfrak{H}$, and hence Γ can be considered as a subgroup of $\mathrm{Aut}(\mathfrak{H}) \cong \mathrm{PSL}_2(\boldsymbol{R})$. On the other hand, Γ is naturally a subgroup of $G_\mathfrak{p}$. By these two embeddings, Γ will henceforth be considered as a subgroup of $\mathrm{PSL}_2(\boldsymbol{R}) \times G_\mathfrak{p}$. Up to conjugacy in $\mathrm{PSL}_2(\boldsymbol{R}) \times G_\mathfrak{p}$, Γ does not depend on the choices of Σ_0 and of the above two isomorphisms. (On the other hand, Γ depends essentially on ε.)

DEFINITION 2.9.3. Γ is called the arithmetic fundamental group belonging to $\mathscr{X}$ and ε.

Put

$$\begin{aligned} \Gamma^+ &= \Gamma \cap (\mathrm{PSL}_2(\boldsymbol{R}) \times G_\mathfrak{p}^+), \\ \Delta_i &= \Gamma \cap (\mathrm{PSL}_2(\boldsymbol{R}) \times V_i) \qquad (i = 1, 0, 2). \end{aligned} \tag{2.9.4}$$

Then the projections of Δ_i to $\mathrm{PSL}_2(\boldsymbol{R})$ are fuchsian groups of the first kind, and when $\mathscr{X}$ is unramified, they are nothing but the universal covering groups of $X_{i\eta} \otimes_{k_c} \boldsymbol{C}$.[2] Therefore, Γ^+ is discrete in the product group (2.9.1), and (in view of (2.9.6) below) moreover it is torsion-free when $\mathscr{X}$ is unramified. By the same reason, the quotient of the product group (2.9.1) modulo Γ^+ has finite invariant volume, and it is compact when $\mathscr{X}$ is unramified. The projection of Γ^+ in $\mathrm{PSL}_2(\boldsymbol{R})$ is dense, and that in $G_\mathfrak{p}^+$ is dense in $\overline{G_\mathfrak{p}^+}$.

Now for Γ itself: In Case 2, ι acts as an involution of k_2/k_1; hence $\Gamma = \Gamma^+$. In Case 1, the symmetry ι of K_0 can be extended to an element of Γ; call it ι again. Then $\Delta_0 \iota$ is well defined, and we have

$$\iota^{-1}\Delta_0\iota = \Delta_0, \quad \iota^2 \in \Delta_0; \qquad \iota^{-1}\Delta_1\iota = \Delta_2, \quad \iota^{-1}\Delta_2\iota = \Delta_1. \tag{2.9.5}$$

Therefore, Γ is generated by Γ^+ and ι. Since $G_\mathfrak{p}^+ = V_0\Gamma^+$, and $G_\mathfrak{p} = V_0\Gamma$ (Case 1), we obtain immediately from (2.3.2), (2.7.2) the following

COROLLARY 2.9.6. (i) *Γ^+ is the free product of Δ_1 and Δ_2 with amalgamated subgroup Δ_0.*

(ii) *In Case 2, we have $\Gamma = \Gamma^+$, and in Case 1, Γ is the free product of Δ_1 and $\Delta_0 \cup \Delta_0\iota$ with amalgamated subgroup Δ_0.*

When $\mathscr{X}$ is the Kronecker CR-system, we have

$$\Delta_1 = \mathrm{PSL}_2(\boldsymbol{Z}), \qquad \Delta_2 = \begin{pmatrix} 0 & -1 \\ p & 0 \end{pmatrix}^{-1} \Delta_1 \begin{pmatrix} 0 & -1 \\ p & 0 \end{pmatrix},$$

$$\Gamma^+ = \mathrm{PSL}_2(\boldsymbol{Z}[1/p]),$$

and

$$\Gamma = \{\gamma \in \mathrm{GL}_2(\boldsymbol{Z}[1/p]);\ \det\gamma \in p^{\boldsymbol{Z}}\}/\pm p^{\boldsymbol{Z}},$$

up to conjugacy in $\mathrm{PSL}_2(\boldsymbol{R}) \times \mathrm{PGL}_2(\boldsymbol{Q}_p)$.

3. The canonical liftings. Let $\mathscr{X} = \{X_1 \overset{\varphi_1}{\leftarrow} X_0 \overset{\varphi_2}{\rightarrow} X_2\}$ be any CR-system w.r.t. $(\boldsymbol{X}, \mathfrak{o})$. We shall study in detail the reduction mod $\mathfrak{p}$ of some special kind of places ξ of L/k. Roughly speaking, we consider those ξ whose stabilizer in $G_\mathfrak{p}^+$ is "big" (the condition [A] of §3.3). Our main goal is to outline the proof of Theorem 3.4.1, which states that the reduction mod $\mathfrak{p}$ induces a *bijection* between the set of all $G_\mathfrak{p}^+$-orbits $G_\mathfrak{p}^+ \cdot \xi$ of those ξ and that of all ordinary closed points (scheme-theoretic points) of $\boldsymbol{X} \otimes \boldsymbol{F}_{q^2}$. This is achieved by constructing a *canonical lifting* $\boldsymbol{x} \to \xi_i$ of each ordinary geometric point $\boldsymbol{x}$ of $\boldsymbol{X}$ to a geometric point ξ_i of $X_{i\eta}$ $(i = 1, 2)$. When $\mathscr{X}$ is unramified and symmetric, our main results are reformulated in terms of Γ (§3.14, Main Theorem I, and §3.15).

[2]When $c = 2$, $\otimes$ is w.r.t. the embedding $k \hookrightarrow \boldsymbol{C}$ determined by Σ_0.

3.1. *Reduction of $G_\mathfrak{p}^+$-orbits in* $\mathrm{Pl}(L/k)$. Denote by $\mathrm{Pl}(L/k)$ the set of all places of L into $\bar{k} \cup (\infty)$ over k. Let $\mathrm{Aut}(L/k)$ act on $\mathrm{Pl}(L/k)$ by $\xi \to g\xi$, where $(g\xi)(a) = \xi(a^g)$ $(a \in L,\ g \in \mathrm{Aut}(L/k))$. For each $\xi \in \mathrm{Pl}(L/k)$, denote by ξ_i its restriction to K_i $(i = 1, 2)$ considered as a point of $\bar{X}_{i\eta}$, and by ξ_{is} the unique specialization of ξ_i on $\bar{X}$. Since $T \otimes_\mathfrak{o} \boldsymbol{F}_q = \Pi \cup {}^t\Pi$, we have $\xi_{2s} = \xi_{1s}^{q^{\pm 1}}$. Since $G_\mathfrak{p}^+$ is generated by V_1 and V_2, the repeated use of this fact shows that for any $g \in G_\mathfrak{p}^+$ and $1 \leqq i, j \leqq 2$, $(g\xi)_{is}$ and ξ_{js} are conjugate over $\boldsymbol{F}_q$, and that when $i = j$ they are conjugate over $\boldsymbol{F}_{q^2}$. When $\mathscr{X}$ is symmetric, $(g\xi)_{is}$ and ξ_{js} are conjugate over $\boldsymbol{F}_q$ for any $g \in G_\mathfrak{p}$. We shall pay attention to the following mappings induced by $\xi \to \xi_{is}$ $(i = 1, 2)$:

$$(3.1.1)_i \qquad \mathrm{Pl}(L/k) \mapsto \{\text{Points of } \bar{X}\},$$

$$(3.1.2)_i \qquad G_\mathfrak{p}^+\backslash \mathrm{Pl}(L/k) \mapsto \{\boldsymbol{F}_{q^2}\text{-conjugacy classes of points of } \bar{X}\},$$

and when $\mathscr{X}$ is symmetric,

$$(3.1.3)_{\mathrm{symm}} \qquad G_\mathfrak{p}\backslash \mathrm{Pl}(L/k) \to \{\boldsymbol{F}_q\text{-conjugacy classes of points of } \bar{X}\}.$$

As for $(3.1.2)_i$ $(i = 1, 2)$ they are the transforms of each other by the involution of $\boldsymbol{F}_{q^2}/\boldsymbol{F}_q$.

3.2. For each $\xi \in \mathrm{Pl}(L/k)$, define its (transcendental) decomposition group D_ξ^+ and the inertia group I_ξ^+ by

$$(3.2.1) \qquad \begin{aligned} D_\xi^+ &= \{g \in G_\mathfrak{p}^+;\ g\xi \sim \xi\} = \{g \in G_\mathfrak{p}^+;\ \Theta_\xi^g = \Theta_\xi\}, \\ I_\xi^+ &= \{g \in G_\mathfrak{p}^+;\ g\xi = \xi\} = \{g \in D_\xi^+;\ g \text{ acts trivially on } \Theta_\xi/\mathfrak{m}_\xi\}, \end{aligned}$$

where $\sim$ is the equivalence of places, Θ_ξ is the valuation ring of ξ, and $\mathfrak{m}_\xi$ is its maximal ideal. When $\mathscr{X}$ is symmetric, we define D_ξ and I_ξ in the same manner, i.e., just by dropping the symbol + in (3.2.1). Obviously, I_ξ^+ (resp. I_ξ) is normal in D_ξ^+ (resp. D_ξ). For each $g \in G_\mathfrak{p}^+$, and $g \in G_\mathfrak{p}$ when $\mathscr{X}$ is symmetric, define its degree $\mathrm{Deg}(g)$ by

$$(3.2.2) \qquad \mathrm{Deg}(g) = \min_{A \in \mathscr{T}^0} l(A, A^g).$$

Then $\mathrm{Deg}(g)$ is a nonnegative integer, and it is even if and only if $g \in G_\mathfrak{p}^+$. Moreover, $\mathrm{Deg}(g) = 0$ if and only if g is $G_\mathfrak{p}^+$-conjugate to some element of V_1 or V_2. Put

$$(3.2.3) \qquad I_\xi^0 = \{g \in I_\xi^+;\ \mathrm{Deg}(g) = 0\}.$$

3.3. We consider the following condition [A] for $\xi \in \mathrm{Pl}(L/k)$:

[A] *I_ξ^0 forms a subgroup of I_ξ^+ with infinite index.*

If [A] is satisfied for ξ, then also for $g\xi$ ($g \in G_\mathfrak{p}^+$, resp. $G_\mathfrak{p}$). When $\mathscr{X}$ is unramified, then L/K_i^g $(g \in G_\mathfrak{p}^+)$ are unramified, so that the *usual* inertia groups $I_\xi^+ \cap g^{-1}V_i g$ are trivial. Therefore, $I_\xi^0 = \{1\}$. Therefore, the condition [A] is equivalent with

[A] ($\mathscr{X}$: unramified). *I_ξ^+ is an infinite group.*

Return to the general case of $\mathscr{X}$, and define $\mathrm{Pl}(L/k;\ [\mathrm{A}])$ as the set of all $\xi \in \mathrm{Pl}(L/k)$ satisfying [A]. It is stable under $G_\mathfrak{p}^+$ (resp. $G_\mathfrak{p}$). For each $\xi \in \mathrm{Pl}(L/k; [\mathrm{A}])$, define $\mathrm{Deg}^+(\xi)$ (resp. $\mathrm{Deg}(\xi)$ when $\mathscr{X}$: symmetric) as the minimum value of $\mathrm{Deg}(g)$ where g runs over all elements of $I_\xi^+ - I_\xi^0$ (resp. $I_\xi - I_\xi^0$). Then $\mathrm{Deg}^+(\xi)$ depends only on

$G_\mathfrak{p}^+ \cdot \xi$, and when $\mathscr{X}$ is symmetric, both $\mathrm{Deg}^+(\xi)$ and $\mathrm{Deg}(\xi)$ depend only on $G_\mathfrak{p} \cdot \xi$.

3.4. Now a main result of §3, in a form still implicit about the canonical liftings, is as follows.

THEOREM 3.4.1. (i) *For each* $i = 1, 2$, *the reduction map* $\xi \to \xi_{is}$ *induces a bijection between the set of all* $G_\mathfrak{p}^+$*-orbits in* $\mathrm{Pl}(L/k, [\mathrm{A}])$ *and that of all* $\boldsymbol{F}_{q^2}$*-conjugacy classes of ordinary points* (*see* §1.4) *of* $\bar{\boldsymbol{X}}$; *i. e.*,

$$\mathrm{red}_i\colon G_\mathfrak{p}^+\backslash \mathrm{Pl}(L/k, [\mathrm{A}]) \xrightarrow{\approx} \{\textit{ordinary closed points of } \boldsymbol{X} \underset{\boldsymbol{F}_q}{\otimes} \boldsymbol{F}_{q^2}\}.$$

Moreover, we have $\mathrm{Deg}^+(\xi) = 2 \cdot \mathrm{Deg}(\xi_{is}/\boldsymbol{F}_{q^2})$.

(ii) *When* $\mathscr{X}$ *is symmetric*, red_i ($i = 1, 2$) *induce one and the same bijection between the set of all* $G_\mathfrak{p}$*-orbits in* $\mathrm{Pl}(L/k; [\mathrm{A}])$ *and that of all* $\boldsymbol{F}_q$*-conjugacy classes of ordinary points of* $\bar{\boldsymbol{X}}$; *i.e.*,

$$\mathrm{red}_i\colon G_\mathfrak{p}\backslash \mathrm{Pl}(L/k; [\mathrm{A}]) \xrightarrow{\approx} \{\textit{ordinary closed points of } \boldsymbol{X}\}.$$

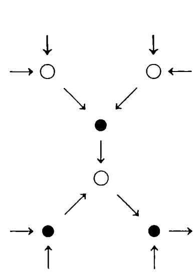
Moreover, we have $\mathrm{Deg}(\xi) = \mathrm{Deg}(\xi_{is}/\boldsymbol{F}_q)$.

(iii) *For any* $\xi \in \mathrm{Pl}(L/k; [\mathrm{A}])$, I_ξ^0 *is a normal subgroup of* I_ξ^+ *such that* $I_\xi^+/I_\xi^0 \simeq \boldsymbol{Z}$, *and* D_ξ^+/I_ξ^+ *is canonically isomorphic to the full Galois group of* $\Theta_\xi/\mathfrak{m}_\xi$ *over* $k_2 \cap (\Theta_\xi/\mathfrak{m}_\xi)$. *When* $\mathscr{X}$ *is symmetric*, I_ξ^0 *is also normal in* I_ξ, $I_\xi/I_\xi^0 \simeq \boldsymbol{Z}$, *and* D_ξ/I_ξ *is canonically isomorphic to the full Galois group of* $\Theta_\xi/\mathfrak{m}_\xi$ *over* k.

The proof will be outlined in the rest of §3. The main point is the construction of the inverse maps $\xi_{is} \to \xi$, the liftings of ξ_{is} *to such* ξ *that satisfy* [A] (see §§3.10–3.11). But before this, we shall give some preparatory materials.

3.5. *Rivers on the tree* $\mathscr{T}$. To look more closely at the $G_\mathfrak{p}^+$-orbits $G_\mathfrak{p}^+ \cdot \xi$ such that ξ_{is} are *ordinary*, the following notion of "river" on $\mathscr{T} = \mathscr{T}(G_\mathfrak{p}^+; V_1, V_2)$ is very helpful.

DEFINITION 3.5.1. A river on $\mathscr{T}$ is defined when each segment has a direction in such a manner that for each $A \in \mathscr{T}^0$ and its mates $B_0, B_1, \cdots, B_q$, precisely one of $\overline{AB_i}$ ($0 \leq i \leq q$), say $\overline{AB_0}$, is directed outward as $\overrightarrow{AB_0}$ and the rest are all directed inward; $\overleftarrow{AB_1}, \cdots, \overleftarrow{AB_q}$.



A river on $\mathscr{T}$ is determined uniquely by an arbitrary infinite flow going downstream

$$\circ\!\to\!\bullet\!\to\!\circ\!\to\!\bullet\!\to \cdots$$

on $\mathscr{T}$. It is clear that two such flows on $\mathscr{T}$ determine the same river if and only if they meet somewhere in their downstreams.

3.6. Let $\xi \in \mathrm{Pl}(L/k)$. Since ξ_{is} $(i = 1, 2)$ are mutually conjugate over $\boldsymbol{F}_q$, ξ_{1s} is ordinary if and only if ξ_{2s} is so. Call ξ *ordinary* when ξ_{is} are so. Each *ordinary* element $\xi \in \mathrm{Pl}(L/k)$ determines a river on $\mathscr{T}$, called $\mathrm{Riv}(\xi)$, in the following manner. Let $A = V_1 g_1$ and $B = V_2 g_2$ be mates, so that $V_1 g_1 \cap V_2 g_2 \neq \varnothing$. Take $g \in V_1 g_1 \cap V_2 g_2$. Let ζ be the restriction of $g\xi$ to K_0 considered as a point of $\bar{X}_{0\eta}$, and ζ_s be its unique specialization on $\bar{X}_{0s}$. Since ξ is ordinary, $g\xi$ is also ordinary. Therefore, ζ_s does not belong to $\Pi \cap {}^t\Pi$ (the intersection taken on X_{0s}). Therefore, ζ_s lies on just one of Π or ${}^t\Pi$. If it is Π, give the direction $\overrightarrow{AB}$, and if it is ${}^t\Pi$, give it the other way. It is easy to see that this defines a river $\rho = \mathrm{Riv}(\xi)$ on $\mathscr{T}$.

Now let $[A_0 \to A_1 \to \cdots]$ be any infinite flow in this river, and take $g \in D_\xi^+$ (resp. D_ξ when $\mathscr{X}$: symmetric). Then g leaves $\mathrm{Riv}(\xi)$ invariant, so that the g-transform of the above flow is another flow $[A_0^g \to A_1^g \to \cdots]$ of $\mathrm{Riv}(\xi)$. Therefore, they must meet somewhere in their downstream. Therefore, $A_i^g = A_{i+\delta}$ holds for some $\delta \in \boldsymbol{Z}$ if i is sufficiently large, and δ is independent of the choice of $[A_0 \to A_1 \to \cdots]$. The mapping $g \to \delta = \delta(g)$ defines a homomorphism $\delta\colon D_\xi^+ \to \boldsymbol{Z}$ (resp. $D_\xi \to \boldsymbol{Z}$), and it is easy to see that

$$\mathrm{Deg}(g) = |\delta(g)|, \qquad g \in D_\xi^+ \text{ (resp. } D_\xi). \tag{3.6.1}$$

In particular, I_ξ^0 is the kernel of $\delta|_{I_\xi^+}$ (resp. $\delta|_{I_\xi}$, since $I_\xi^0 \subset I_\xi^+$); hence I_ξ^0 is always a normal subgroup of I_ξ^+ (resp. I_ξ) and we have $I_\xi^+/I_\xi^0 \simeq (0)$ or $\boldsymbol{Z}$ (resp. $I_\xi/I_\xi^0 \simeq (0)$ or $\boldsymbol{Z}$) *whenever ξ is ordinary.*

3.7. *The mapping* χ. Let $i = 1, 2$. A point ξ_i of $\bar{X}_{i\eta}$ is called ordinary if its specialization $\xi_{is} \in \bar{X}$ is so. Let $(\bar{X}_{i\eta})^{\mathrm{ord}}$ denote the set of all ordinary points of $\bar{X}_{i\eta}$. The mapping χ of $\bar{X}_{1\eta}^{\mathrm{ord}} \sqcup \bar{X}_{2\eta}^{\mathrm{ord}}$ into itself is defined as follows. Let $\xi_1 \in \bar{X}_{1\eta}^{\mathrm{ord}}$ (resp. $\xi_2 \in \bar{X}_{2\eta}^{\mathrm{ord}}$), and let ζ (resp. ζ') be the unique point of $\bar{X}_{0\eta}$ such that $\bar{\varphi}_1(\zeta) = \xi_1$ (resp. $\bar{\varphi}_2(\zeta') = \xi_2$) and that its specialization ζ_s (resp. ζ'_s) on $\bar{X}_{0s}$ lies on Π (resp. ${}^t\Pi$). Here, $\bar{\varphi}_i = \varphi_i \otimes_k \bar{k}$. Then χ is defined by $\chi(\xi_1) = \bar{\varphi}_2(\zeta)$, $\chi(\xi_2) = \bar{\varphi}_1(\zeta')$. Thus, χ maps $\bar{X}_{1\eta}^{\mathrm{ord}}$ into $\bar{X}_{2\eta}^{\mathrm{ord}}$, and vice versa. Generically, χ is a q-to-one mapping. (An important p-adic analytic property, the rigidity of χ, has been investigated by Tate, Deligne and Dwork (cf. [3]) in the case where $\mathscr{X}$ is the Kronecker CR-system.)

3.8. *Illustration of* χ. Take any $\xi \in \mathrm{Pl}(L/k)$ which is ordinary, and put

$$(G_\mathfrak{p}^+ \cdot \xi)_i = \{(g\xi)_i;\, g \in G_\mathfrak{p}^+\} \qquad (i = 1, 2). \tag{3.8.1}$$

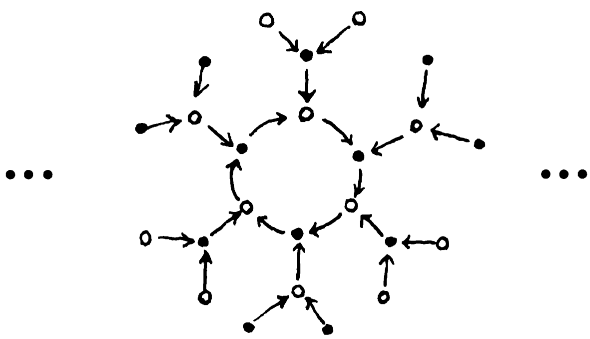
Then the disjoint union $(G_\mathfrak{p}^+ \xi)_1 \sqcup (G_\mathfrak{p}^+ \xi)_2$, which is a subset of points of $\bar{X}_{1\eta} \sqcup \bar{X}_{2\eta}$, can be naturally identified with

$$(V_1 \backslash G_\mathfrak{p}^+ / I_\xi^+) \sqcup (V_2 \backslash G_\mathfrak{p}^+ / I_\xi^+) = \mathscr{T}^0 / I_\xi^+. \tag{3.8.2}$$

Moreover, the action of χ on this set corresponds to the arrows defined from $\mathrm{Riv}(\xi)$ on $\mathscr{T}$;

$$\text{“}(G_\mathfrak{p}^+ \xi)_1 \sqcup (G_\mathfrak{p}^+ \xi)_2 \text{ with the action of } \chi\text{”} = \{\mathscr{T} \text{ with } \mathrm{Riv}(\xi)\}/I_\xi^+.$$

For example, if $I_\xi^0 = \{1\}$ and $I_\xi^+ \simeq \boldsymbol{Z}$, then this quotient looks like (for $q = 2$).

(3.8.3) 

The length of the central cycle is equal to $\mathrm{Deg}^{+}(\xi)$.

3.9. *The schemes* $T_{ij}(\mathfrak{p}^l)$. For each i, j $(1 \leqq i, j \leqq 2)$ and $l \geqq 0$ with

$$i - j \equiv l \pmod 2, \tag{3.9.1}$$

a closed subscheme $T_{ij}(\mathfrak{p}^l)$ of $X_i \otimes_{\mathfrak{o}} X_j$ is defined in the following way. Let A_1, A_2 be the points of $\mathscr{T}^0$ corresponding to V_1, V_2, respectively. Fix i, j and l satisfying (3.9.1), and let B be any point of $\mathscr{T}^0$ such that $l(A_i, B) = l$. Then $l(A_j, B)$ is even, so that $B = A_j^g$ with some $g \in G_{\mathfrak{p}}^{+}$. Note that the double coset $V_j g V_i$ is independent of the choice of B. Consider the ring homomorphism $K_i \otimes_k K_j \to K_i \cdot K_j^g \subset L$ defined by $\sum_\lambda u_\lambda \otimes v_\lambda \to \sum_\lambda u_\lambda v_\lambda{}^g$. Then the kernel defines a closed integral subscheme of $X_i \times_{\mathfrak{o}} X_j$ depending only on i, j, l. Call it $T_{ij}(\mathfrak{p}^l)$. It is easy to check that $T_{ji}(\mathfrak{p}^l) = {}^tT_{ij}(\mathfrak{p}^l)$. When $\mathscr{X}$ is symmetric, $T_{ij}(\mathfrak{p}^l)$ is symmetric and depends only on l.

PROPOSITION 3.9.2. *Let* $T_{ij}(\mathfrak{p}^l)_s$ *be the special fiber of* $T_{ij}(\mathfrak{p}^l)$. *Then* $T_{ij}(\mathfrak{p}^l)_s$ *is a closed subscheme of* $\boldsymbol{X} \times_{\boldsymbol{F}_q} \boldsymbol{X}$ *determined by the following two properties*: (i) *it is locally defined by a single equation*, (ii) *its irreducible components and their multiplicities are given by the following formula*:

$$T_{ij}(\mathfrak{p}^l)_s = (\Pi^l + {}^t\Pi^l) + \sum_{1 \leqq k < l/2} q^{k-1}(q-1)(\Pi^{l-2k} + {}^t\Pi^{l-2k}) + \varepsilon(l) q^{(l-2)/2}(q-1)\,\Delta,$$

where Π^r *is the graph of the* q^r*th power morphism of* $\boldsymbol{X}$, ${}^t\Pi^r$ *is its transposed graph*, Δ *is the diagonal of* $\boldsymbol{X} \times_{\boldsymbol{F}_q} \boldsymbol{X}$, *and* $\varepsilon(l) = 1$ (*resp.* 0) *according to* l: *even* (*resp.* l: *odd*).

3.10. *The canonical liftings*. Let $i = 1$ or 2, and $l \geqq 1$. For each *ordinary* point $\boldsymbol{x}$ of $\bar{X}$ with degree l over $\boldsymbol{F}_{q^2}$, we are going to define its *canonical lifting* $\xi_i \in \bar{X}_{i\eta}$. Define $j = 1$ or 2 by the congruence $i - j \equiv l \pmod 2$, and look at Proposition 3.9.2. Observe that the points of $\bar{X}$ with degree l over $\boldsymbol{F}_{q^2}$ are in a one-to-one correspondence $\boldsymbol{x} \leftrightarrow (\boldsymbol{x}, \boldsymbol{x}^{q^l})$ with those geometric points of $\Pi^l \cap {}^t\Pi^l$ not lying on any other irreducible components of $T_{ij}(\mathfrak{p}^l)_s$ than Π^l or ${}^t\Pi^l$.

PROPOSITION 3.10.1. *If* $\boldsymbol{x}$ *is ordinary*, $(\boldsymbol{x}, \boldsymbol{x}^{q^l})$ *is not normal on the two-dimensional scheme* $T_{ij}(\mathfrak{p}^l)$.

This can be proved easily by using a suitable morphism from the normalization of $T_{ij}(\mathfrak{p}^l)$ onto X_0. The following lemma, which is crucial to the canonical liftings, is an elementary exercise in two-dimensional local rings:

LEMMA 3.10.2. *Let R be any complete discrete valuation ring, and let k (resp. κ) denote its quotient field (resp. residue field). Put* Spec $R = \{\eta, s\}$ *(s: the closed point). Let Z be a two-dimensional integral scheme having a structure of a proper and flat R-scheme, and let z be a κ-rational ordinary double point of $Z_s = Z \otimes_R \kappa$ which is not normal on Z. Then there is a unique point ζ on Z_η which is not normal and which has z as its specialization. Moreover, ζ is a k-rational ordinary double point of Z_η.*

Since z is reduced on Z_s (being an ordinary double point) and since Z is flat over R, the two-dimensional local ring $\mathcal{O}_{Z,z}$ is a Cohen-Macaulay ring. This gives the existence of ζ, by the Serre's criterion for normality [4, IV, 5.8.6]. The uniqueness and the last assertion follow from the isomorphism $\hat{\mathcal{O}}_{Z,z} \simeq R[[X, Y]]/XY$ for the completion of $\mathcal{O}_{Z,z}$.

Now apply Lemma 3.10.2 for $R = \mathfrak{o}_{2l}$, $Z = T_{ij}(\mathfrak{p}^l) \otimes_{\mathfrak{o}} \mathfrak{o}_{2l}$ and $z = (x, x^{q^l})$ to obtain the following

THEOREM 3.10.3. *Let i, j, l and x be as at the beginning of* §3.10. *Then there exists a unique k_{2l}-rational point (ξ_i, ξ_j') of $T_{ij}(\mathfrak{p}^l)_\eta$ ($\xi_i \in \bar{X}_{i\eta}$, $\xi_j' \in \bar{X}_{j\eta}$) which is not normal and which lifts (x, x^{q^l}). Moreover, (ξ_i, ξ_j') is an ordinary double point of $T_{ij}(\mathfrak{p}^l)_\eta \otimes_k k_{2l}$.*

DEFINITION 3.10.4. ξ_i is the *canonical lifting* of x on $\bar{X}_{i\eta}$.

3.11. The following basic properties of the canonical liftings follow from the uniqueness of (ξ_i, ξ_j'). First, since $T_{ji}(\mathfrak{p}^l) = {}^tT_{ij}(\mathfrak{p}^l)$, the definition and the uniqueness of (ξ_i, ξ_j') tell us immediately that ξ_j' is the canonical lifting of x^{q^l} on $\bar{X}_{j\eta}$. More important properties will be given in the following

THEOREM 3.11.1. *Let x be an ordinary point of $\bar{X}$. For each $i = 1, 2$, let ξ_i be the canonical lifting of x on $\bar{X}_{i\eta}$. Put $d = \mathrm{Deg}(x/F_q)$. Then* (i) *ξ_i is k_d-rational, and is of degree d over k;* (ii) *$\xi_i^{[q]}$ is the canonical lifting of x^q on $\bar{X}_{i\eta}$;* (iii) *$\chi(\xi_1)$ (resp. $\chi(\xi_2)$) is the canonical lifting of x^q on $\bar{X}_{2\eta}$ (resp. $\bar{X}_{1\eta}$);* (iv) *ξ_i is the unique point of $\bar{X}_{i\eta}$ which specializes to x and satisfies $\chi^{2d}(\xi_i) = \xi_i$;* (v) *when $\mathscr{X}$ is symmetric, we have $\xi_1 = \xi_2$.*

In the case where $\mathscr{X}$ is the Kronecker CR-system, the canonical lifting is the same as the Deuring's lifting of (the j-invariant of) a nonsupersingular elliptic curve over $\bar{F}_p$ to (the j-invariant of) such an elliptic curve over $\bar{Q}_p$ that has the same endomorphism ring as the former. The above characterization (iv) follows the Dwork's characterization of the Deuring lifting given in [3].

3.12. Fix i ($1 \leqq i \leqq 2$), and consider the canonical lifting $x \to \xi_i$ of the ordinary points of $\bar{X}$ to the points of $\bar{X}_{i\eta}$. By Theorem 3.11.1 it induces the lifting of each ordinary closed point $(x, x^{q^2}, \cdots, x^{q^{2l-2}})$ ($l = \mathrm{Deg}(x/F_{q^2})$) of $X \otimes_{F_q} F_{q^2}$ to a closed point $(\xi_i, \xi_i^{[q]^2}, \cdots, \xi_i^{[q]^{2l-2}})$ of $X_{i\eta} \otimes_k k_2$. Moreover, since $\xi_i^{[q]^2} = \chi^2(\xi_i)$, the extensions of $\xi_i, \xi_i^{[q]^2}, \cdots$ to the elements of $\mathrm{Pl}(L/k)$ belong to one and the same $G_\mathfrak{p}^+$-orbit $G_\mathfrak{p}^+ \cdot \xi$. Since $\chi^{2l}(\xi_i) = \xi_i$, I_ξ^+ contains an element g with $\delta(g) = 2l$. From this we can show easily that $G_\mathfrak{p}^+ \cdot \xi \in \mathrm{Pl}(L/k; [A])$, $\mathrm{Deg}^+(\xi) = 2l$, and that D_ξ^+/I_ξ^+ induces the full automorphism group of $\mathcal{O}_\xi/\mathfrak{m}_\xi$ over $k_2 \cap (\mathcal{O}_\xi/\mathfrak{m}_\xi)$. So, as for Theorem 3.4.1, it remains to prove that "these $G_\mathfrak{p}^+ \cdot \xi$ *exhaust* $G_\mathfrak{p}^+\backslash \mathrm{Pl}(L/k; [A])$".

This can be checked in the following way. First, count the degree of the intersection product $T_{ii}(\mathfrak{p}^{2l})_\eta \cdot \Delta_\eta$ on $X_{i\eta} \times_k X_{i\eta}$ by using Proposition 3.9.2. Then count the number of those distinct points of $T_{ii}(\mathfrak{p}^{2l})_\eta \cdot \Delta_\eta$ that either belong to the $G_\mathfrak{q}^+$-transforms of canonical liftings, or do not correspond to an element of $\mathrm{Pl}(L/k;\, [\mathrm{A}])$, by using Riv($\xi$). We can check that the latter number reaches the former, and this gives rise to that "$G_\mathfrak{p}^+ \cdot \xi$ exhaust $G_\mathfrak{p}^+\backslash\mathrm{Pl}(L/k;\, [\mathrm{A}])$". The details **[9b]** will be published shortly.

3.13. One can deduce some more conclusions from Theorem 3.11.1. For example, for each $\xi \in \mathrm{Pl}(L/k;\, [\mathrm{A}])$, the residue field $\Theta_\xi/\mathfrak{m}_\xi$ is *abelian* over $(\Theta_\xi/\mathfrak{m}_\xi) \cap k_2$ (and also over k if $\mathscr{X}$: symmetric), and its norm group can be determined explicitly. Roughly speaking (i.e., disregarding ramifications), this is determined by the slope of $T_{ii}(\mathfrak{p}^{2l})_\eta$ at (ξ_i, ξ_i). At this stage, we use the work **[12]** of Lubin-Tate.

3.14. *The First Main Theorem.* Now let $\mathscr{X}$ be an *unramified symmetric* CR-system w.r.t. $(X, \mathfrak{o})$. Fix an embedding $\varepsilon\colon k \hookrightarrow C$, and let Γ be the arithmetic fundamental group belonging to $\mathscr{X}$ and ε (§2.9). In §§3.14–15, we shall give some reformulations of Theorem 3.4.1 and Theorem 3.11.1 in terms of Γ.

As in §2.9, take a connected component Σ_0 of Σ, identify Σ_0 with $\mathfrak{H}$, and consider $\Gamma = \{\gamma \in G_\mathfrak{p};\ \gamma\Sigma_0 = \Sigma_0\}$ as a group of transformations of $\mathfrak{H}$. For each $\tau \in \mathfrak{H}$, let Γ_τ denote its stabilizer in Γ, and put

$$\mathscr{H} = \{\tau \in \mathfrak{H};\ |\Gamma_\tau| = \infty\}. \tag{3.14.1}$$

Obviously, $\mathscr{H}$ is a Γ-stable subset of $\mathfrak{H}$. The points of $\mathscr{H}$ will be called *Γ-points* on $\mathfrak{H}$.

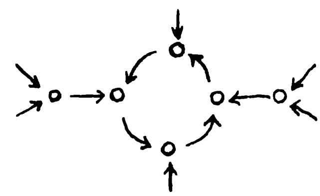
Main Theorem I. *Let $\mathfrak{P}(X)$ denote the set of all ordinary closed points of X. Then the reduction* mod $\mathfrak{p}$ *induces a bijection*

$$i_\Gamma\colon \Gamma\backslash\mathscr{H} \approx \mathfrak{P}(X). \tag{3.14.2}$$

This is a direct corollary of Theorem 3.4.1(ii), because there is a canonical bijection between $\Gamma\backslash\mathscr{H}$ and $G_\mathfrak{p}\backslash\mathrm{Pl}(L/k;\, [\mathrm{A}])$, defined as follows. First, an element of Σ is algebraic over k if it has a nontrivial stabilizer in $G_\mathfrak{p}$. Therefore, $\mathscr{H}$ is embedded in $\mathrm{Pl}(L/k)$. Moreover, for any $\tau \in \mathfrak{H} \cap \mathrm{Pl}(L/k)$, Γ_τ is nothing but the inertia group I_τ defined in §3.2. Since the condition [A] (for $\mathscr{X}$: unramified) is equivalent with $|I_\tau| = \infty$ (§3.3), this implies that $\mathfrak{H} \cap \mathrm{Pl}(L/k;\, [\mathrm{A}]) = \mathscr{H}$. Since $G_\mathfrak{p}$ acts transitively on the set of connected components of Σ, this induces the above bijection.

3.15. Although the Main Theorem I has a simple form, it does not contain our best knowledge on the reductions and the liftings of *geometric* points, which will now be described in terms of Γ as follows. Let Δ_i $(i = 1, 2)$ be the subgroups of Γ corresponding to V_i (see §2.9). Then $\Delta_i\backslash\mathfrak{H}$ can be identified with the set of points of $X_{iC} = X_{i\eta} \otimes_{k_c} C$, where $\otimes$ is w.r.t. the embedding $k_c \hookrightarrow C$ defined by the restriction of Σ_0 to k_c. Let $\boldsymbol{x}$ be a geometric point of X over F_q with degree d, and $\boldsymbol{P} = (\boldsymbol{x}, \boldsymbol{x}^q, \cdots, \boldsymbol{x}^{q^{d-1}})$ be the closed point of X containing $\boldsymbol{x}$. Let $\mathscr{H}^{\boldsymbol{P}}$ denote the set of all those points $\tau \in \mathscr{H}$ such that the Γ-orbit containing τ corresponds to $\boldsymbol{P}$ by i_Γ ((3.14.2)).

Case 1. In this case, Γ contains an involution ι satisfying (2.9.5), so that there is no distinction between the two choices of i. The quotient $\Delta_1\backslash\mathscr{H}^{\boldsymbol{P}} \subset X_{1C}$ is illustrated by the diagram:

(3.15.1)  $(q = 2, d = 4)$

where the arrow is the χ-mapping which is q-to-1, and the central cycle is of length d which consists of the canonical liftings of $\boldsymbol{x}^{q^\nu}$ $(0 \leqq \nu < d)$ on $\bar{X}_1$.

Case 2. In this case, the simultaneous $\mathfrak{o}_2$-structure for $\mathscr{X}$ gives mutually conjugate structures for X_1 and X_2. Therefore, X_{1C} and X_{2C} are generally nonisomorphic. The disjoint union $(\Delta_1 \backslash \mathscr{H}^P) \coprod (\Delta_2 \backslash \mathscr{H}^P)$ is illustrated by the diagram (3.8.3), where $\circ \in \Delta_1 \backslash \mathscr{H}^P$, $\bullet \in \Delta_2 \backslash \mathscr{H}^P$, the arrow is the χ-mapping, and the central cycle is of length d (which is even) which consists of the canonical liftings of $\boldsymbol{x}^{q^\nu}$ on X_{iC} (i is distinguished by the parity of ν).

Finally, in each case, Γ_τ is free cyclic and the homomorphism $\delta: D_\tau \to \boldsymbol{Z}$ of §3.6 induces an isomorphism $\Gamma_\tau \simeq d \cdot \boldsymbol{Z}$. Therefore, if γ_τ is a generator of Γ_τ, the degree d of $\boldsymbol{P}$ over $\boldsymbol{F}_q$ is equal to $\mathrm{Deg}(\gamma_\tau)$, where γ_τ is considered as an element of $G_\mathfrak{p}$. Moreover, we can distinguish the two generators of Γ_τ by the signs of $\delta(\gamma_\tau)$. This sign has also the following interpretation. Let $d/dz \to \lambda \cdot d/dz$ be the linear transform of the tangent space of $\mathfrak{H}$ at τ induced by γ_τ. Then λ belongs to $k^\times$ (via the embedding $\varepsilon: k \to \boldsymbol{C}$), and $\mathrm{ord}_\mathfrak{p}\lambda$ has the same sign as $\delta(\gamma_\tau)$.

4. The second Galois theory. The subject of second Galois theory is a system $\hat{f} = (f_1, f_0, f_2)$ of three finite étale morphisms f_i: $X_i^* \to X_i$ connecting two CR-systems $\mathscr{X} = \{X_1 \overset{\varphi_1}{\leftarrow} X_0 \overset{\varphi_2}{\to} X_2\}$ and $\mathscr{X}^* = \{X_1^* \overset{\varphi_1^*}{\leftarrow} X_0^* \overset{\varphi_2^*}{\to} X_2^*\}$ in a compatible way. The purpose of §4 is to present our main results on the two categorical equivalences induced by $\hat{f} \to \hat{f} \otimes \boldsymbol{C}$ and $\hat{f} \to \hat{f} \otimes \boldsymbol{F}_q$. The former is highly nontrivial.

4.1. In general, let $\mathscr{U} = \{U_1 \overset{\psi_1}{\hookleftarrow} U_0 \overset{\psi_2}{\hookrightarrow} U_2\}$ be a system formed of three schemes U_i $(i = 0, 1, 2)$ and two morphisms ψ_i: $U_0 \to U_i$ $(i = 1, 2)$. Let

$$\mathscr{U}^* = \{U_1^* \xleftarrow{\psi_1^*} U_0^* \xrightarrow{\psi_2^*} U_2^*\}$$

be another such system. We say that a pair $(\mathscr{U}^*, \hat{f})$ is a *finite étale covering of* $\mathscr{U}$, if $\hat{f}$ is a triple (f_1, f_0, f_2) of finite étale morphisms f_i: $U_i^* \to U_i$ $(i = 0, 1, 2)$ satisfying $f_i \circ \psi_i^* = \psi_i \circ f_0$ $(i = 1, 2)$ and $U_0^* \simeq U_i^* \times_{U_i} U_0$ (canonically; $i = 1, 2$).

Now let $\mathscr{X} = \{X_1 \overset{\varphi_1}{\leftarrow} X_0 \overset{\varphi_2}{\to} X_2\}$ be a CR-system w.r.t. $(\boldsymbol{X}, \mathfrak{o})$, and $\mathscr{X}^* = \{X_1^* \overset{\varphi_1^*}{\leftarrow} X_0^* \overset{\varphi_2^*}{\to} X_2^*\}$ be another CR-system w.r.t. $(\boldsymbol{X}^*, \mathfrak{o})$, *with the common base ring* $\mathfrak{o}$. Suppose that $(\mathscr{X}^*, \hat{f})$ is a finite étale covering of $\mathscr{X}$ and that the constituents f_i $(i = 0, 1, 2)$ of $\hat{f} = (f_1, f_0, f_2)$ are $\mathfrak{o}$-morphisms. Then we shall call $(\mathscr{X}^*, \hat{f})$ a *finite étale CR-covering of* $\mathscr{X}$. In this case, it follows from the definitions that the two finite étale morphisms f_{is}: $\boldsymbol{X}^* \to \boldsymbol{X}$ obtained from f_i $(i = 1, 2)$ by the base change $\otimes_\mathfrak{o} \boldsymbol{F}_q$ must coincide. This morphism f_{is} will be denoted by $\boldsymbol{f}$. It also follows easily that $\mathfrak{S}^* = \boldsymbol{f}^{-1}(\mathfrak{S})$, where $\mathfrak{S}$ (resp. $\mathfrak{S}^*$) is the set of special points w.r.t. $\mathscr{X}$ (resp. $\mathscr{X}^*$). When $\mathscr{X}$ is unramified, $\mathscr{X}^*$ is also unramified, because $\varphi_{i\eta}^*$ are the base changes of $\varphi_{i\eta}$. When $\mathscr{X}$ is symmetric, we can prove easily by using **[4,** IV 18.3.4**]** that $\mathscr{X}^*$ is also symmetric.

4.2. Now let $\mathscr{X}$ be any *unramified* symmetric CR-system w.r.t. $(\boldsymbol{X}, \mathfrak{o})$, and fix an embedding $\varepsilon: k \hookrightarrow \boldsymbol{C}$. Let Γ be the arithmetic fundamental group belonging to $\mathscr{X}$, ε. Then our main result on the second Galois theory reads as follows.

MAIN THEOREM II. *The following three categories* (i), (ii), (iii), *are canonically equivalent*:

(i) *Finite étale CR-coverings* $(\mathscr{X}^*, \hat{f})$ *of* $\mathscr{X}$.

(ii) *Subgroups* Γ^* *of* Γ *with finite indices.*

(iii) *Finite étale coverings* $\boldsymbol{f}: \boldsymbol{X}^* \to \boldsymbol{X}$, *with* $\boldsymbol{X}^*$: *connected, such that all points of* $\boldsymbol{X}^*$ *lying above the special points* $\boldsymbol{x}_1, \cdots, \boldsymbol{x}_H$ *of* $\boldsymbol{X}$ *are* $\boldsymbol{F}_{q^2}$*-rational points of* $\boldsymbol{X}^*$.

The equivalence functors are as follows; (i) $\to$ (ii) is the functor of taking the arithmetic fundamental group, and (i) $\to$ (iii) is the one described in §4.1. The proofs are omitted here. The equivalence (i) $\sim$ (iii) is proved in the same way as in **[6]**. The proof of (i) $\sim$ (ii) contains much more delicate points. The main point is our criterion (using some liftings of the Frobenius) for the good reduction of unramified coverings of curves **[8]**.

5. Simultaneous uniformizations and reciprocity. In §5, $\mathscr{X}$ is an *unramified symmetric* CR-system w.r.t. $(\boldsymbol{X}, \mathfrak{o})$, ε is a fixed embedding $k \hookrightarrow \boldsymbol{C}$, and Γ is the arithmetic fundamental group belonging to $\mathscr{X}$, ε.

5.1. Let Γ^* be a subgroup of Γ with finite index, $(\mathscr{X}^*, \hat{f})$ be the corresponding finite étale CR-covering of $\mathscr{X}$, and $(\boldsymbol{X}^*, \boldsymbol{f})$ be the corresponding finite étale covering of $\boldsymbol{X}$ (§4). Let $\mathscr{H}$ (resp. $\mathscr{H}^*$) be the set of all Γ-points (resp. Γ^*-points) on $\mathfrak{H}$ (§3.14). Then $\mathscr{H}^* = \mathscr{H}$, because $(\Gamma_\tau : \Gamma_\tau^*) < \infty$ for any $\tau \in \mathfrak{H}$. Let $\mathfrak{P}(\boldsymbol{X}^*)$ denote the set of all ordinary closed points of $\boldsymbol{X}^*$ w.r.t. $\mathscr{X}^*$. As we noted in §4, $\mathfrak{P}(\boldsymbol{X}^*)$ is the inverse image of $\mathfrak{P}(\boldsymbol{X})$ w.r.t. $\boldsymbol{f}$. By Main Theorem I for $\mathscr{X}^*$, we have a canonical bijection

(5.1.1) $$i_{\Gamma^*}: \Gamma^* \backslash \mathscr{H} \approx \mathfrak{P}(\boldsymbol{X}^*)$$

defined by the reduction mod $\mathfrak{p}$.

When $\boldsymbol{X}^*/\boldsymbol{X}$ is a Galois covering, and $\boldsymbol{P}^*$ is a closed point of $\boldsymbol{X}^*$, the Frobenius automorphism $((\boldsymbol{X}^*/\boldsymbol{X})/\boldsymbol{P}^*)$ is by definition the element σ of the Galois group $\mathrm{Gal}(\boldsymbol{X}^*/\boldsymbol{X})$ of $\boldsymbol{X}^*/\boldsymbol{X}$ that acts on the geometric points of $\boldsymbol{P}^*$ as the q^dth power map, where d is the degree over $\boldsymbol{F}_q$ of the closed point $\boldsymbol{P}$ of $\boldsymbol{X}$ lying below $\boldsymbol{P}^*$. (Since the action of elements σ of the Galois group on *points* are from the *left*, and that on *functions* are from the *right* connected by $f^\sigma(\xi) = f(\sigma\xi)$, the geometric and the arithmetic Frobeniuses do not have inverse expressions here.)

MAIN THEOREM III. (i) *The diagram*

(5.1.2) $$\begin{array}{ccc} \Gamma^*\backslash\mathscr{H} & \xrightarrow{\;i_{\Gamma^*}\;} & \mathfrak{P}(\boldsymbol{X}^*) \\ \Big\downarrow {\scriptstyle canon.} & & \Big\downarrow {\scriptstyle f} \\ \Gamma\backslash\mathscr{H} & \xrightarrow[\;i_\Gamma\;]{} & \mathfrak{P}(\boldsymbol{X}) \end{array}$$

is commutative; (ii) *when* Γ^* *is a normal subgroup of* Γ, *the natural action of* Γ/Γ^* *on* $\Gamma^*\backslash\mathscr{H}$ *and the action of* $\mathrm{Gal}(\boldsymbol{X}^*/\boldsymbol{X})$ *on* $\mathfrak{P}(\boldsymbol{X}^*)$ *correspond with each other through* i_{Γ^*} *and through the canonical isomorphism* $\Gamma/\Gamma^* \cong \mathrm{Gal}(\boldsymbol{X}^*/\boldsymbol{X})$ *of Main Theroem* II;

(iii) *moreover, when* Γ^* *is normal in* Γ, *the Frobenius automorphism of* $\boldsymbol{P}_\tau^* = i_{\Gamma^*}(\Gamma^* \cdot \tau)$ $(\tau \in \mathscr{H})$ *over* $\boldsymbol{X}$ *is given by* $\Gamma^* \cdot \gamma_\tau$, *where* γ_τ *is the generator of* Γ_τ *such that* $\delta(\gamma_\tau) < 0$ (§3.15);

$$\left(\frac{\boldsymbol{X}^*/\boldsymbol{X}}{\boldsymbol{P}_\tau^*}\right) = \Gamma^* \cdot \gamma_\tau. \tag{5.1.3}$$

Thus, as a universal expression of the Frobenius automorphism, we may write in a more suggestive way as

$$\left(\frac{/\boldsymbol{X}}{\tau}\right) = \gamma_\tau. \tag{5.1.4}$$

6. The case of Shimura curves. The above results apply to each Shimura curve for almost all $\mathfrak{p}$ by Theorem 6.3.3 which is the "$\mathfrak{p}$-canonical version" of the Shimura congruence relation.

6.1. Let F be a totally real algebraic number field of finite degree m, and B be a quaternion algebra over F which is unramified at *one* infinite place ∞_1 of F and ramified at all other infinite places $\infty_2, \cdots, \infty_m$ of F. As usual, the subscripts $\mathfrak{l}$, A and $\boldsymbol{R}$ indicate the localization, adelization, and the infinite part of the adele, respectively. The reduced norm for B/F (and its localization, etc.) will be denoted by $N(*)$. For any $b_{\boldsymbol{R}} \in B_{\boldsymbol{R}}^\times$, $N(b_{\boldsymbol{R}})$ is positive at $\infty_2, \cdots, \infty_m$. Let $(B_A^\times)^+$ denote the group of all $b_A \in B_A^\times$ with $N(b_{\boldsymbol{R}})$: *totally positive*, and put $(B_{\boldsymbol{R}}^\times)^+ = (B_A^\times)^+ \cap B_{\boldsymbol{R}}^\times$.

Now let $\mathfrak{L}$ be the function field over F constructed in Shimura **[15]** for B/F (the case "$n = r = 1$"). It is an infinitely generated 1-dimensional extension of F, and is obtained as the total field of arithmetic automorphic functions on B. In **[15]** Shimura established an isomorphism

$$\mathrm{Aut}(\mathfrak{L}/F) \cong (B_A^\times)^+/\overline{(B_{\boldsymbol{R}}^\times)^+ \cdot F^\times}, \tag{6.1.1}$$

where $^-$ denotes the topological closure. Recall that the algebraic closure of F in $\mathfrak{L}$ is the maximum abelian extension F^{ab} of F, and that the diagram

$$\begin{array}{ccc} (B_A^\times)^+ & \xrightarrow{\;t\;} & \mathrm{Aut}(\mathfrak{L}/F) \\ {\scriptstyle N^{-1}}\downarrow & & \downarrow{\scriptstyle \rho} \\ F_A^\times & \xrightarrow{\;c\;} & \mathrm{Aut}(F^{\mathrm{ab}}/F) \end{array} \tag{6.1.2}$$

is commutative, where t is the homomorphism defined by (6.1.1), c is the homomorphism defined by the class field theory, ρ is the restriction homomorphism, and $N^{-1}(b_A) = N(b_A)^{-1}$.

For each open compact subgroup $\mathfrak{U}$ of $\mathrm{Aut}(\mathfrak{L}/F)$, let $\mathfrak{L}^{\mathfrak{U}}$ (resp. $(F^{\mathrm{ab}})^{\mathfrak{U}}$) be its fixed field in $\mathfrak{L}$ (resp. F^{ab}). Then $\mathfrak{L}^{\mathfrak{U}}$ is an algebraic function field of one variable over $(F^{\mathrm{ab}})^{\mathfrak{U}}$. Let $S(\mathfrak{U})$ be the proper smooth curve with function field $\mathfrak{L}^{\mathfrak{U}}$. Then $S(\mathfrak{U})$ is called the *Shimura curve corresponding to* $\mathfrak{U}$.

6.2. Now let $\mathfrak{p}$ be a finite place of F at which B is *unramified*. Consider $F_{\mathfrak{p}}^\times$ as a (central) subgroup of $(B_A^\times)^+$ and let P be the closure of $t(F_{\mathfrak{p}}^\times)$. Then P is a central *compact* subgroup of $\mathrm{Aut}(\mathfrak{L}/F)$. Let $k^{(1)}$ (resp. $k^{(\infty)}$) be the decomposition field (resp. the inertia field) of $\mathfrak{p}$ in F^{ab}/F, and $k^{(d)}$ $(d \geqq 1)$ be the unique subextension of $k^{(\infty)}/k^{(1)}$ with degree d. Let $\mathfrak{L}^P$ be the fixed field of P in $\mathfrak{L}$. Then $\mathfrak{L}^P \cap k^{(\infty)} = k^{(2)}$.

Put

(6.2.1) $$B_A^\times = \prod_{\mathfrak{l}\neq\infty,\,\mathfrak{p}}' B_\mathfrak{l}^\times, \qquad F_A^\times = \prod_{\mathfrak{l}\neq\infty,\,\mathfrak{p}}' F_\mathfrak{l}^\times,$$

(where $\mathfrak{l} \neq \infty$ is the abbreviation for $\mathfrak{l} \neq \infty_1, \cdots, \infty_m$), and for each subgroup H of $B_A^\times$ (resp. $F_A^\times$) denote by $H^\sim$ the topological closure in $B_A^\times$ (resp. $F_A^\times$) of the projection of H in $B_A^\times$ (resp. in $F_A^\times$). Now look at the canonical isomorphism

(6.2.2) $$\mathrm{Aut}(\mathfrak{L}/F)/P \cong (B_A^\times)^+/\overline{(B_R^\times)^+ \cdot F_\mathfrak{p}^\times \cdot F^\times} \cong (B_\mathfrak{p}^\times/F_\mathfrak{p}^\times) \times (B_A^\times/(F^\times)^\sim).$$

Then the groups of (6.2.2) act on $\mathfrak{L}^P$ in the natural manner. Put

(6.2.3) $$B_A^{\times(1)} = \{b_A \in B_A^\times;\, N(b_A) \in ((F^\times)^+)^\sim\},$$

where $(F^\times)^+$ is the group of all totally positive elements of F. The isomorphism (6.2.2) induces

(6.2.4) $$\mathrm{Aut}(\mathfrak{L}/k^{(1)})/P \cong (B_\mathfrak{p}^\times/F_\mathfrak{p}^\times) \times (B_A^{\times(1)}/(F^\times)^\sim).$$

DEFINITION 6.2.5. For each open compact subgroup U_A of $B_A^{(1)}/(F^\times)^\sim$, the fixed field of U_A in $\mathfrak{L}^P$, with the natural action of $B_\mathfrak{p}^\times/F_\mathfrak{p}^\times$, will be called the *$B_\mathfrak{p}^\times/F_\mathfrak{p}^\times$-field associated with* U_A, and denoted by $L = L(U_A)$.

Thus, $L(U_A)$ is a one-dimensional infinitely generated extension over $k^{(1)}$ on which $B_\mathfrak{p}^\times/F_\mathfrak{p}^\times$ acts effectively.

DEFINITION 6.2.6. For each U_A, the group $\Gamma = \Gamma(U_A)$ is the subgroup of $B^\times/F^\times$ formed of all $\gamma \in B^\times/F^\times$ satisfying

$$\mathrm{pr}_{\infty_1}(\gamma) \in (B_{\infty_1}^\times)^+/\boldsymbol{R}^\times, \qquad \mathrm{pr}_A(\gamma) \in U_A,$$

where pr_{∞_1} (resp. pr_A) are the projections of $B^\times/F^\times$ into $B_{\infty_1}^\times/\boldsymbol{R}^\times$, $B_A^\times/(F^\times)^\sim$, respectively.

By the Eichler-Kneser approximation theorem, $\Gamma(U_A) = \Gamma(U_A')$ implies $U_A = U_A'$.

Now fix an isomorphism $B_\mathfrak{p} \simeq M_2(F_\mathfrak{p})$ (over $F_\mathfrak{p}$), which induces $B_\mathfrak{p}^\times/F_\mathfrak{p}^\times \simeq \mathrm{PGL}_2(F_\mathfrak{p})$. Let V_1, V_2 be the open compact subgroups of $B_\mathfrak{p}^\times/F_\mathfrak{p}^\times$ corresponding to $\mathrm{PGL}_2(\mathfrak{o}_\mathfrak{p})$, $w_\mathfrak{p}^{-1}\,\mathrm{PGL}_2(\mathfrak{o}_\mathfrak{p})w_\mathfrak{p}$ respectively, where $\mathfrak{o}_\mathfrak{p}$ is the ring of integers of $F_\mathfrak{p}$ and $w_\mathfrak{p} = \left(\begin{smallmatrix}0&1\\ \pi&0\end{smallmatrix}\right)$ (π: a prime element of $\mathfrak{o}_\mathfrak{p}$). Put $V_0 = V_1 \cap V_2$. Let U_A be an open compact subgroup of $B_A^{(1)}/(F^\times)^\sim$, put $L = L(U_A)$, and let K_i ($i = 0, 1, 2$) be the fixed fields of V_i in L (i.e., the fixed fields of $V_i \times U_A$ in $\mathfrak{L}^P$). Then each K_i is an algebraic function field of one variable over $k^{(1)}$, and $K_0 = K_1K_2$. Moreover, $w_\mathfrak{p}$ induces an involution ι of K_0 which is trivial on $k^{(1)}$ and inverts K_1 and K_2. As for L, it is the smallest extension of K_0 such that L/K_i ($i = 1, 2$) are both Galois extensions. Now let $X_{i\eta}$ ($i = 0, 1, 2$) be the proper smooth curves over $k^{(1)}$ with function fields K_i, and $\varphi_{i\eta}\colon X_{0\eta} \to X_{i\eta}$ ($i = 1, 2$) be the natural morphisms. Then $X_{1\eta} = X_{2\eta}$ (via ι) and ${}^tT_\eta = T_\eta$ for the image T_η of $X_{0\eta}$ in $X_{1\eta} \times_{k^{(1)}} X_{2\eta}$. The system

(6.2.7) $$\{X_{1\eta} \xleftarrow{\varphi_{1\eta}} X_{0\eta} \xrightarrow{\varphi_{2\eta}} X_{2\eta}\}$$

thus defined does not depend on the choice of the isomorphism $B_\mathfrak{p} \simeq M_2(F_\mathfrak{p})$. We shall call (6.2.7) the *$k^{(1)}$-system associated with* $(\mathfrak{p}, U_A)$.

Each extension $\tilde{\mathfrak{p}}$ of $\mathfrak{p}$ in $k^{(1)}$ defines a $\tilde{\mathfrak{p}}$-adic embedding of $k^{(1)}$ into $F_\mathfrak{p}$. Let $\mathscr{X}(\tilde{\mathfrak{p}}, U_A)_\eta$ denote the system of curves over $F_\mathfrak{p}$ obtained from (6.2.7) by the base change $\otimes F_\mathfrak{p}$ with respect to this embedding of $k^{(1)}$.

Question 6.2.8. *Does there exist a symmetric CR-system* $\mathscr{X}(\tilde{\mathfrak{p}}, U_A)$ *over* $\mathfrak{o}_\mathfrak{p}$ *which has* $\mathscr{X}(\tilde{\mathfrak{p}}, U_A)_\eta$ *as its general fiber*?

Shimura's work provides an affirmative answer to this question for "almost all $\mathfrak{p}$" in the sense described below, and it has been supplemented by Y. Morita to some results for "individual" $(\tilde{\mathfrak{p}}, U_A)$. But as far as the author knows, this question has not yet been solved for all cases. Note here that by our Main Theorem II (§4) and the last statement of §4.1, *when* (6.2.8) *is valid for* $(\tilde{\mathfrak{p}}, U_A)$ *and when* $\varphi_{i\eta}$ $(i = 1, 2)$ *are unramified, then this is also true for* $(\tilde{\mathfrak{p}}, U_A^*)$ *for all open subgroups* U_A^* *of* U_A.

6.3. Let $\mathfrak{p}$ be as in §6.2. An open compact subgroup $\mathfrak{U}$ of Aut $(\mathfrak{L}/F)$ is said to be *coprime with* $\mathfrak{p}$ if $\mathfrak{U}$ contains the image $t(\tilde{V}_\mathfrak{p})$ of a maximal compact subgroup $\tilde{V}_\mathfrak{p}$ of $B_\mathfrak{p}^\times$. In this case, $\tilde{V}_\mathfrak{p}$ is uniquely determined, and is called the $\mathfrak{p}$-*component* of $\mathfrak{U}$. Obviously, $\mathfrak{U}$ is coprime with almost all $\mathfrak{p}$.

Let $\mathfrak{U}$ be the coprime with $\mathfrak{p}$. Then $(F^{\mathrm{ab}})^{\mathfrak{U}} \subset k^{(\infty)}$. Put

$$\mathfrak{U}^{(P)} = (\mathfrak{U} \cap \mathrm{Aut}(\mathfrak{L}/k^{(\infty)}))\cdot P, \tag{6.3.1}$$

and $S(\mathfrak{U}^{(P)})$ be the proper smooth curve whose function field is the fixed field of $\mathfrak{U}^{(P)}$ in $\mathfrak{L}$. Then the exact constant field of $S(\mathfrak{U}^{(P)})$ is $k^{(2)}$, and

$$S(\mathfrak{U}^{(P)}) \otimes k^{(\infty)} \cong S(\mathfrak{U}) \otimes k^{(\infty)}, \tag{6.3.2}$$

where the tensor products are over the exact constant fields. This curve $S(\mathfrak{U}^{(P)})$, obtained from $S(\mathfrak{U})$ by dividing $S(\mathfrak{U}) \otimes k^{(\infty)}$ by the action of P, will be called the $\mathfrak{p}$-*canonical model* of $S(\mathfrak{U})$. If we consider $\mathfrak{U}^{(P)}$ as a subgroup of the group (6.2.4), then $\mathfrak{U}^{(P)}$ decomposes as $\mathfrak{U}^{(P)} = V_\mathfrak{p} \times U_A$, where $V_\mathfrak{p}$ is the image in $B_\mathfrak{p}^\times/F_\mathfrak{p}^\times$ of the $\mathfrak{p}$-component $\tilde{V}_\mathfrak{p}$ of $\mathfrak{U}$, and U_A is an open compact subgroup of $B_A^{\times(1)}$. We shall call U_A the A-*component* of $\mathfrak{U}$. By definitions, $S(\mathfrak{U}^{(P)})$ is $k^{(1)}$-isomorphic with $X_{i\eta}$ $(i = 1, 2)$ where $\{X_{1\eta} \xleftarrow{\varphi_{1\eta}} X_{0\eta} \xrightarrow{\varphi_{2\eta}} X_{2\eta}\}$ is the $k^{(1)}$-system associated with $(\mathfrak{p}, U_A)$.

Now, from the Shimura's congruence relation [**15**, (I) Theorem 2.23], we obtain just by passing to the $\mathfrak{p}$-canonical models, the following

THEOREM 6.3.3 (SHIMURA). *Let* $\mathfrak{U}$ *be any open compact subgroup of* Aut$(\mathfrak{L}/F)$ *which is an image of an open compact subgroup of* $(B_A^\times)^+$. *For each finite place* $\mathfrak{p}$ *of* F *at which* B *is unramified and such that* $\mathfrak{U}$ *is coprime with* $\mathfrak{p}$, *let* U_A *denote the* A-*component of* $\mathfrak{U}$. *Then Question* 6.2.8 *has an affirmative answer* (*for all extensions* $\tilde{\mathfrak{p}}$ *of* $\mathfrak{p}$) *for almost all* $\mathfrak{p}$.

In his Master's thesis [**13**], Y. Morita proved among others the following:

THEOREM 6.3.4 (MORITA). *When* $F = \boldsymbol{Q}$, *Question* 6.2.8 *has an affirmative answer for all* p *at which* B *is unramified and with which* $\mathfrak{U}$ *is coprime.*

6.4. Let $(\tilde{\mathfrak{p}}, U_A)$ be such that Question 6.2.8 has an affirmative answer. We shall see how the objects L, $G_\mathfrak{p}$, Γ, $\mathscr{H}$, etc. are given explicitly for $\mathscr{X} = \mathscr{X}(\tilde{\mathfrak{p}}, U_A)$. Note first that the base ring $\mathfrak{o} = \mathfrak{o}_\mathfrak{p}$ is the ring of integers of $F_\mathfrak{p}$.

(i) *The field* L *and the group* $G_\mathfrak{p}$ (§2). The field L for this case is given by $L(U_A) \otimes_{k^{(1)}} F_\mathfrak{p}$, and the group $G_\mathfrak{p}$ is nothing but $B_\mathfrak{p}^\times/F_\mathfrak{p}^\times \simeq \mathrm{PGL}_2(F_\mathfrak{p})$.

(ii) *The ramifications and* $(\mathscr{H}, \Gamma)$ (§§2, 3). It is clear that $\mathscr{X}$ is almost unramified. Moreover, if $B \not\simeq M_2(\boldsymbol{Q})$ and U_A is sufficiently small, then $\mathscr{X}$ is unramified. Let $\varepsilon: F_\mathfrak{p} \hookrightarrow \boldsymbol{C}$ be an extension of ∞_1. Then the group Γ and its two embeddings are given by $\Gamma = \Gamma(U_A)$, pr_{∞_1}, $\mathrm{pr}_{\mathfrak{p}}$, respectively. To give $\mathscr{H}$ explicitly, let $J_{\mathfrak{p}}(B)$ denote the set of all such quadratic extensions M of F *contained in* B that M is totally imaginary and that $((M/F)/\mathfrak{p}) = 1$. Put $(B^\times)^+ = B^\times \cap (B_A^\times)^+$. Then $(B^\times)^+$ acts on $\mathfrak{H}$ through the localization at ∞_1, and for each $M \in J_{\mathfrak{p}}(B)$ there is a unique common fixed point τ_M of $M^\times$ on $\mathfrak{H}$. Now $\mathscr{H}$ is given by

$$(6.4.1) \qquad \mathscr{H} = \{\tau_M;\ M \in J_{\mathfrak{p}}(B)\}.$$

Since $\tau_M \neq \tau_{M'}$ for $M \neq M'$, $\mathscr{H}$ can be identified with $J_{\mathfrak{p}}(B)$. The corresponding action of Γ on $J_{\mathfrak{p}}(B)$ is then given by $M \to \gamma M \gamma^{-1}$.

(iii) *The second Galois theory.* The open subgroups U_A^* of U_A correspond bijectively with the congruence subgroups Γ^* of $\Gamma(U_A)$. Our theory (§§2–5) is valid for any subgroups of $\Gamma(U_A)$ with finite indices. We do not know whether all such subgroups Γ^* are congruence subgroups, except for the case of $B = M_2(\boldsymbol{Q})$ where it was proved to be valid by Mennicke and Serre.

(iv) *The reciprocity.* For each $M \in J_{\mathfrak{p}}(B)$, the group Γ_τ for $\tau = \tau_M$ is given by $\Gamma \cap (M^\times/F^\times)$. Let $\mathfrak{p} = \mathfrak{p}_1\mathfrak{p}_2$ be the decomposition of $\mathfrak{p}$ in M. Then the degree $\mathrm{Deg}(\tau)$ is the smallest positive integer d such that $\mathfrak{p}_1^d$ is principal; $\mathfrak{p}_1^d = (\alpha)$. When $\mathscr{X}$ is unramified, Γ_τ is free cyclic and is generated by the class of $\alpha \pmod{F^\times}$. The two generators of Γ_τ are distinguished by the choice of $\mathfrak{p}_1$. *Choose* $\mathfrak{p}_1$ *to be the restriction of* $\tilde{\mathfrak{p}}$ *to* M and let γ_τ be the generator of Γ_τ represented by α (where $\mathfrak{p}_1^d = (\alpha)$). Then this is the generator describing the reciprocity of Main Theorem III.

Thus, we have described explicitly the objects of Main Theorems I, II, III for the case of Shimura curves.

A numerical example is given in **[9a]**.

7. Comments and remarks.

7.1. *Brief review of the elliptic modular case* (*cf.* **[5b**, *Chapter* 5 *in Vols.* 1, 2]; **[6]**). Let $X = \mathrm{Spec}\ \boldsymbol{Z}[j]$ be the affine j-line over $\boldsymbol{Z}$, and put $X_{\boldsymbol{C}} = X \otimes \boldsymbol{C}$. Let $\mathfrak{H}$ be the complex upper half-plane, $\Delta = \mathrm{PSL}_2(\boldsymbol{Z})$, and identify $\Delta\backslash\mathfrak{H}$ with $X_{\boldsymbol{C}}$ via the modular j-function.[3] Let $\mathfrak{M}$ be the set of all imaginary quadratic *subfields* of $M_2(\boldsymbol{Q})$. For each $M \in \mathfrak{M}$, denote by τ_M the common fixed point of elements of $M^\times$ in $\mathfrak{H}$, and put $j_M = j(\tau_M)$ which is a point of $X_{\boldsymbol{C}}$. Then j_M is determined by the Δ-conjugacy class of M in $\mathfrak{M}$, and is an algebraic integer.

Fix a prime number p, and its extension $\tilde{p}$ in the algebraic closure $\bar{\boldsymbol{Q}}$. Put $\boldsymbol{X} = X \otimes \boldsymbol{F}_p$. A geometric point of $\boldsymbol{X}$ will be called *special* (resp. *ordinary*), if it is the j-invariant of a supersingular (resp. singular) elliptic curve in the Deuring's sense **[1]**. As is well known, the special points are $\boldsymbol{F}_{p^2}$-rational. For each $M \in \mathfrak{M}$, denote by $\boldsymbol{j}_M$ the geometric point of $\boldsymbol{X}$ obtained by the reduction mod $\tilde{p}$ of j_M. Then:

(I) $\mathfrak{M} \ni M \to \boldsymbol{j}_M \in \boldsymbol{X} \otimes \bar{\boldsymbol{F}}_p$ is surjective.

(II) If $(M/p) \neq 1$, then $\boldsymbol{j}_M$ is special.

(III) If $(M/p) = 1$, then $\boldsymbol{j}_M$ is ordinary; moreover, when this is so, $\boldsymbol{j}_{M'}$ $(M' \in \mathfrak{M})$ is $\boldsymbol{F}_p$-conjugate with $\boldsymbol{j}_M$ if and only if M' and M are Γ-conjugate in $M_2(\boldsymbol{Q})$, where

$$(7.1.1) \qquad \Gamma = \{\gamma \in \mathrm{GL}_2(\boldsymbol{Z}[1/p]);\ \det(\gamma) \in p^{\boldsymbol{Z}}\}/\pm p^{\boldsymbol{Z}}.$$

[3]The constant multiple of j is normalized in such a way that $j(\sqrt{-1}) = 12^3$.

Therefore, if $\mathscr{H}$ denotes the set of all τ_M with $M \in \mathfrak{M}$ and $(M/p) = 1$, then the reduction mod $\bar{p}$ induces a bijection

(7.1.2) i_Γ: $\Gamma\backslash\mathscr{H} \approx \{F_p\text{-conjugacy classes of ordinary geometric points of } X\}$.

Moreover, for each F_p-conjugacy class $P = (j, j^p, \cdots, j^{p^{d-1}})$ of ordinary geometric points of X, the collection $\Delta\backslash\mathscr{H}^P$ of all points j_M such that $j_M \in P$ can be illustrated by the diagram (3.15.1), where the arrow $\xi \to \xi'$ is the unique $\bar{p}$-adic lifting of the pth power map satisfying $(\xi, \xi') \in T(p) \otimes \bar{Q}$, $T(p)$ being the Hecke correspondence $\subset X \times_Z X$ defined by the double coset $\Delta(^1{}_p)\Delta$. The mapping χ is generically p-to-1, the central cycle is of length d, and the distance k of j_M from the central cycle is the p-exponent of the conductor of $M_2(Z) \cap M$. Each ordinary geometric point $j \in X$ has a unique lifting in the central cycle, called the *canonical lijting* (or Deuring representative) of j. These are obtained by the modular reconsideration of the Deuring's results on the reduction of elliptic curves **[2]**.

(IV) Subgroups Γ^* of Γ with finite indices are categorically equivalent with those finite coverings $X^* \to X$ over F_{p^2} satisfying (a) the Igusa's ramification properties, and (b) all points of X^* lying above the special points of X are F_{p^2}-rational (cf. **[6]**).

(V) The reciprocity in this system of coverings of X is described as follows. Let P be as above, and $\tau = \tau_M \in \mathscr{H}^P$. Let Γ_τ be the stabilizer of τ in Γ. Then, modulo torsion (which occurs when $j = 0$ or 12^3), Γ_τ is isomorphic to Z and has a generator γ_τ represented (modulo p^Z) by an element of $M^\times$ which generates a *positive* power of $\bar{p} \in M$. The Γ-conjugacy class $\{\gamma_\tau\}_\Gamma$ is well-defined by P (modulo torsion), and $\{\gamma_\tau\}_{\bar{\Gamma}}$ is the Frobenius conjugacy class for P in this system of coverings. Here, $\bar{\Gamma}$ is the projective limit of all finite factor groups of Γ **[5b**, Vol. 2, Chapter 5].

By (IV), (V), the basic bijection (7.1.2) holds also between Γ^* and X^*.

7.2. *Remarks on its relations with other works.* The chief datum defining a Shimura variety (of GL(2)-type) is a quaternion algebra B over a totally real number field F. The dimension of the variety is the number of distinct infinite places of F at which B is split. In the Shimura theory, *the totally indefinite case and the one-dimensional case* are the two extremes. They coincide only when $F = Q$.

(A) *Morita's* contribution on the Shimura curves over F_p for the case $F = Q$ is briefly mentioned in **[5c]**.

(B) *Langlands*' conjecture on the explicit description of the points of Shimura varieties over finite fields, together with that of the Frobenius action, is (essentially) a higher dimensional generalization of our conjectures and results given in **[5b]**. Langlands discussed extensively, obtaining definitive results, the case where B is *totally indefinite* **[11]**. When $F = Q$, his results essentially coincide with the Morita's solution of our conjecture.[4] When $F \neq Q$, the object varieties are different.

(C) *With Drinfeld.* Let X be a Shimura curve over F_q, Γ be its arithmetic fundamental group, and consider the system of coverings $X^* \to X$ corresponding to *congruence* subgroups $\Gamma^* \subset \Gamma$. Then the Galois group of this system $\{X^*/X\}$ is an adelic compact group (without infinite and $\mathfrak{p}$ factors), and Γ is the "maximal global subgroup" of this adelic Galois group. Recall (Main Theorem III) that the Frobeniuses in $\{X^*/X\}$ are represented by the *global* element $\gamma_\tau \in \Gamma$.

[4]There is however a difference about the aspect of supersingular moduli; mainly because of the (current) difference in the standpoint.

Now let $F = \boldsymbol{Q}$. Then this adelic presentation of the Galois group gives a system of l-adic representations (not always in $GL_2(\boldsymbol{Q}_l)$ but in $GL_2(\bar{\boldsymbol{Q}}_l)$ when $l \mid D(B)$). It is very plausible that there exists an automorphic representation of $GL_2(\boldsymbol{K}_A)$ ($\boldsymbol{K} = \boldsymbol{F}_q(X)$) which corresponds with this system via the Drinfeld theorem. On the other hand, if $F \neq \boldsymbol{Q}$, then the reduced norm of γ_τ over F is not usually a power of p; hence it cannot correspond with an automorphic representation of GL_2 (perhaps possible for GL_{2m} ($m = [F:\boldsymbol{Q}]$)). At any rate, this relation would not reduce one theory to another.

References

1. M. Deuring, *Die Typen der Multiplikatorenringe elliptischer Funktionenkörper*, Abh. Math. Sem. Univ. Hamburg **14** (1941), 197–272.

2. ———, **(a)** *Teilbarkeitseigenschaften der singulären Moduln der elliptischen Funktionen und die Diskriminante der Klassengleichung*, Comment. Math. Helv. **19** (1947), 74–82.

(b) *Die Struktur der elliptischen Funktionenkörper und die Klassenkörper der imaginären quadratische Zahlkörper*, Math. Ann. **124** (1952), 393–426.

3. B. Dwork, *p-adic cycles*, Inst. Hautes Etudes Sci. Publ. Math. **37** (1969), 27–116.

4. A. Grothendieck, *Eléments de géométrie algébrique (EGA)*. I – IV, Inst. Hautes Etudes Sci. Publ. Math (1960–1967).

5. Y. Ihara, **(a)** *The congruence monodromy problems*, J. Math. Soc. Japan **20–1, 2** (1968), 107–121.

(b) *On congruence monodromy problems*, Lecture Notes, Univ. Tokyo, vols.1, 2, 1968, 1969.

(c) *Non-abelian classfields over function fields in special cases*, Actes du Congrès Internat. Math., Nice, 1970, Tome 1, pp. 381–389.

6. ———, *On modular curves over finite fields*, Discrete Subgroups of Lie Groups, Proc. Internat. Colloq., Bombay, Oxford Univ. Press, 1973, pp. 161–202.

7. ———, *On the differentials associated to congruence relations and the Schwarzian equations defining uniformizations*, J. Fac. Sci. Univ. Tokyo Sect. IA Math. **21** (1974), 309–332.

8. Y. Ihara and H. Miki, *Criteria related to potential unramifiedness and reduction of unramified coverings of curves*, J. Fac. Sci. Univ. Tokyo Sect. IA Math. **22** (1975), 237–254.

9. Y. Ihara, **(a)** *Some fundamental groups in the arithmetic of algebraic curves over finite fields*, Proc. Nat. Acad. Sci. U.S.A. **72** (1975), 3281–3284.

(b) *Congruence relations and Shimura curves*. II (to appear in J. Fac. Sci. Univ. Tokyo Sect. IA Math. 25–3).

10. ———, *On the Frobenius correspondences of algebraic curves*, Algebraic Number Theory, Papers contributed for the Kyoto Internat. Symposium, 1976; Japan Soc. Promotion of Sci., Tokyo, pp. 67–98.

11. R. P. Langlands, **(a)** *Shimura varieties and the Selberg trace formula* (to appear).

(b) *On the zeta functions of some simple Shimura varieties* (to appear).

12. J. Lubin and J. Tate, *Formal complex multiplication in local fields*, Ann. of Math. (2) **81** (1965), 380–387.

13. Y. Morita, *Ihara's conjectures and moduli space of abelian varieties*, Master's thesis, Univ. Tokyo (1970).

14. G. Shimura, *Construction of classfields and zeta functions of algebraic curves*, Ann. of Math. (2) **85** (1967), 58–159.

15. ———, *On canonical models of arithmetic quotients of bounded symmetric domains*. I, II, Ann. of Math. (2) **91** (1970), 144–222; **92** (1970), 528–549.

16. J.-P. Serre, *Arbres, amalgames*, SL_2, Astérisque **46**, Soc. Math. France, 1977.

University of Tokyo

Proceedings of Symposia in Pure Mathematics
Vol. 33 (1979), part 2, pp. 313–346

VALEURS DE FONCTIONS L ET PÉRIODES D'INTÉGRALES

P. DELIGNE

Dans cet article, j'énonce une conjecture (1.8, 2.8) reliant les valeurs de certaines fonctions L en certains points entiers à des périodes d'intégrales.

Les fonctions L considérées sont celles des motifs—un mot auquel on n'attachera pas un sens précis. Ceci inclut notamment les fonctions L d'Artin, les fonctions L attachées à des caractères de Hecke algébriques (= Grössencharakter de type A_0), et celles attachées aux formes modulaires holomorphes sur le demi-plan de Poincaré, supposées primitives (= new forms; on considère toutes les fonctions L, L_k, attachées aux puissances symétriques, Sym^k, de la représentation l-adique correspondante).

Cet article doit le jour à D. Zagier: pour son insistance à demander une conjecture, et pour la confirmation expérimentale qu'il en a donnée, sitôt émise, pour les fonctions L_3 et L_4 attachées à $\Delta = \sum \tau(n)q^n$ (voir **[18]**). C'est cette confirmation qui m'a donné la confiance nécessaire pour vérifier que la conjecture était compatible aux résultats de Shimura **[13]** sur les valeurs de fonctions L de caractères de Hecke algébriques.

0. Motifs. *Le lecteur est invité à ne consulter ce paragraphe qu'au fur et à mesure des besoins.* On y rappelle une partie du formalisme, dû à Grothendieck, des motifs. Pour les démonstrations, je renvoie à **[8]**.

0.1. La définition de Grothendieck des *motifs sur un corps k* a la forme suivante.

(a) Soit $\mathscr{V}(k)$ la catégorie des variétés projectives et lisses sur k. On construit une catégorie additive $\mathscr{M}'(k)$, pour laquelles les groupes $\mathrm{Hom}(M, N)$ sont des espaces vectoriels sur $\mathbf{Q}$, munie de

α. Un produit tensoriel $\otimes$, associatif, commutatif et distributif par rapport à l'addition des objects ("associatif" et "commutatif" n'est pas une propriété du foncteur $\otimes$, mais une donnée, soumise à certaines compatibilités—cf. Saavedra **[19]**);

β. Un foncteur contravariant H^*, de $\mathscr{V}(k)$ dans $\mathscr{M}'(k)$, bijectif sur les objets et transformant sommes disjointes en sommes, et produits en produits tensoriels (donnée d'un isomorphisme de foncteurs $H^*(X \times Y) = H^*(X) \otimes H^*(Y)$, compatible à l'associativité et à la commutativité).

Il s'agit, pour l'essentiel, de définir $\mathrm{Hom}(H^*(X), H^*(Y))$. Pour la définition, uti-

AMS (MOS) subject classifications (1970). Primary 10D25, 14K10.

lisée par Grothendieck, de ce groupe comme un groupe de classes de correspondances entre X et Y, voir 0.6.

(b) Rappelons (SGA 4 IV 7.5) qu'une catégorie additive est *karoubienne* si tout projecteur (= endomorphisme idempotent) est défini par une décomposition en somme directe, et que chaque catégorie additive a une *enveloppe karoubienne*, obtenue en lui adjoignant formellement les images des projecteurs. La catégorie $\mathscr{M}_{\text{eff}}(k)$ des *motifs effectifs sur k* est l'enveloppe karoubienne de $\mathscr{M}'(k)$.

(c) On définit le *motif de Tate* $\boldsymbol{Z}(-1)$ comme un facteur direct $H^2(\boldsymbol{P}^1)$ convenable de $H^*(\boldsymbol{P}^1)$. On vérifie que la symétrie (déduite de la commutativité de $\otimes$): $\boldsymbol{Z}(-1) \otimes \boldsymbol{Z}(-1) \to \boldsymbol{Z}(-1) \otimes \boldsymbol{Z}(-1)$ est l'identité, et que le foncteur $M \to M \otimes \boldsymbol{Z}(-1)$ est pleinement fidèle.

(d) La catégorie $\mathscr{M}(k)$ des motifs sur k se déduit de $\mathscr{M}_{\text{eff}}(k)$ en rendant inversible le foncteur $M \to M \otimes \boldsymbol{Z}(-1)$.

Notons (n) l'itéré $(-n)^{\text{ième}}$ de l'auto-équivalence $M \to M \otimes \boldsymbol{Z}(-1)$. La catégorie $\mathscr{M}(k)$ admet $\mathscr{M}_{\text{eff}}(k)$ comme sous-catégorie pleine, et tout object de $\mathscr{M}(k)$ est de la forme $M(n)$ pour M dans $\mathscr{M}_{\text{eff}}(k)$ et n un entier.

Par définition, si F est un foncteur additif de $\mathscr{M}'(k)$ dans une catégorie karoubienne $\mathscr{A}$, il se prolonge à $\mathscr{M}_{\text{eff}}(k)$. Si $\mathscr{A}$ est munie d'une auto-équivalence $A \to A(-1)$, et le prolongement de F d'un isomorphisme de foncteurs $F(M(-1)) = F(M)(-1)$, il se prolonge à $\mathscr{M}(k)$.

Exemple 0.1.1. Si k' est une extension de k, et F le foncteur $H^*(X) \to H^*(X \otimes_k k')$ de $\mathscr{M}'(k)$ dans $\mathscr{M}(k')$, on obtient le foncteur *extension des scalaires* de $\mathscr{M}(k)$ dans $\mathscr{M}(k')$. Si k' est une extension finie de k, on dispose du foncteur de restriction des scalaires à la Grothendieck

$$\amalg_{k'/k}: \mathscr{V}(k') \to \mathscr{V}(k): (X \to \operatorname{Spec}(k')) \mapsto (X \to \operatorname{Spec}(k') \to \operatorname{Spec}(k)).$$

On le prolonge en $F: H^*(X) \mapsto H^*(\amalg_{k'/k} X)$, d'où un foncteur de *restriction des scalaires* $R_{k'/k}: \mathscr{M}(k') \to \mathscr{M}(k)$.

Exemple 0.1.2. Soit $\mathscr{H}$ une "théorie de cohomologie", à valeurs dans une catégorie karoubienne $\mathscr{A}$, fonctorielle pour les morphismes dans $\mathscr{M}'(k)$. Le foncteur $\mathscr{H}$ se prolonge à $\mathscr{M}_{\text{eff}}(k)$. Si $\mathscr{A}$ est muni d'un produit tensoriel, que $\mathscr{H}$ vérifie une formule de Künneth et que la tensorisation avec $\mathscr{H}(\boldsymbol{Z}(-1))$ est une auto-équivalence de $\mathscr{A}$, il se prolonge à $\mathscr{M}(k)$. Ce prolongement est le foncteur "*réalisation d'un motif dans la théorie $\mathscr{H}$*".

Nous noterons (n) l'auto-équivalence de $\mathscr{A}$ itéré $(-n)^{\text{ième}}$ de la tensorisation avec $\mathscr{H}(\boldsymbol{Z}(-1))$. Pour la détermination de $\mathscr{H}(\boldsymbol{Z}(-1))$ dans diverses théories $\mathscr{H}$, voir 3.1.

0.2. Nous utiliserons les réalisations suivantes:

0.2.1. *Réalisation de Betti H_B.* Correspondant à $k = \boldsymbol{C}$, $\mathscr{A}$ = espaces vectoriels que $\boldsymbol{Q}$, $\mathscr{H}$ = cohomologie rationnelle: $X \mapsto H^*(X(\boldsymbol{C}), \boldsymbol{Q})$;

0.2.2. *Réalisation de de Rham H_{DR}.* Correspondant à k de caractéristique 0, $\mathscr{A}$ = espaces vectoriels sur k, $\mathscr{H}$ = cohomologie de de Rham : $X \mapsto \boldsymbol{H}^*(X, \Omega_X^*)$;

0.2.3. *Réalisation l-adique H_l.* Correspondant à k algébriquement clos, de caractéristique $\neq l$, $\mathscr{A}$ = espaces vectoriels sur $\boldsymbol{Q}_l$, $\mathscr{H}$ = cohomologie l-adique: $X \mapsto H^*(X, \boldsymbol{Q}_l)$.

Et leurs variantes:

0.2.4. *Réalisation de Hodge.* k est ici une clôture algébrique de $\boldsymbol{R}$, et $\mathscr{A}$ la catégorie des espaces vectoriels sur $\boldsymbol{Q}$, de complexifié $V \otimes k$ muni d'une bigraduation

$V \otimes k = \bigoplus V^{p,q}$ telle que V^{qp} soit le complexe conjugué de V^{pq}. Pour théorie de cohomologie, on prend le foncteur $X \mapsto H^*(X(k), \boldsymbol{Q})$ muni de la bigraduation de son complexifié $H^*(X(k), k)$ fournie par la théorie de Hodge. La réalisation de Betti est sous-jacente à celle de Hodge.

0.2.5. Pour k quelconque, et σ un plongement complexe de k, on note $H_\sigma(M)$ la réalisation de Betti du motif sur $\boldsymbol{C}$ déduit de M par l'extension des scalaires $\sigma: k \to \boldsymbol{C}$. Notons c la conjugaison complexe. On obtient par transport de structure un isomorphisme $F_\infty: H_\sigma(M) \xrightarrow{\sim} H_{c\sigma}(M)$, et $F_\infty \otimes c$ envoie H_σ^{pq} sur $H_{c\sigma}^{pq}$. Pour σ réel, F_∞ est une involution de $H_\sigma(M)$, dont le complexifié échange $H_\sigma^{pq}(M)$ et $H_\sigma^{qp}(M)$.

La *σ-réalisation de Hodge* est $H_\sigma(M)$, muni de sa bigraduation de Hodge et, si σ est réel, de l'involution F_∞. Pour $k = \boldsymbol{Q}$, ou $k = \boldsymbol{R}$ et σ le plongement identique, on remplace l'indice σ par B. Pour $k = \boldsymbol{R}$, et $M = H^*(X)$, l'involution F_∞ de $H_B^*(M) = H^*(X(\boldsymbol{C}), \boldsymbol{Q})$ est l'involution induite par la conjugaison complexe $F_\infty : X(\boldsymbol{C}) \to X(\boldsymbol{C})$.

0.2.6. *Réalisation de de Rham.* La cohomologie de de Rham est munie d'une filtration naturelle, la *filtration de Hodge*, aboutissement de la suite spectrale d'hypercohomologie

$$E_1^{pq} = H^q(X, \Omega^p) \Rightarrow H_{DR}^{p+q}(X).$$

De là, une filtration F sur $H_{DR}(M)$.

0.2.7. *Réalisation l-adique.* Si X st une variété sur un corps k, de clôture algébrique $\bar{k}$, le groupe de Galois $\mathrm{Gal}(\bar{k}/k)$ agit par transport de structure sur $H^*(X_{\bar{k}}, \boldsymbol{Q}_l)$. Si M est un motif sur k, et qu'on définit $H_l(M)$ comme la réalisation l-adique du motif sur $\bar{k}$ qui s'en déduit par extension des scalaires, ceci fournit une action de $\mathrm{Gal}(\bar{k}/k)$ sur $H_l(M)$.

0.2.8. *Réalisation adélique finie.* Pour k algébriquement clos, de caractéristique 0, on peut rassembler les cohomologies l-adiques en une cohomologie adélique

$$H^*(X, \boldsymbol{A}^f) = (\Pi H^*(X, \boldsymbol{Z}_l)) \otimes_{\boldsymbol{Z}} \boldsymbol{Q}.$$

On peut cumuler les variantes 0.2.7 et 0.2.8.

0.2.9. Dans les exemples 0.2.1, 0.2.2, 0.2.3, et leurs variantes, on peut remplacer la catégorie $\mathscr{A}$ par celle, $\mathscr{A}^*$, des objets gradués de $\mathscr{A}$: utiliser la graduation naturelle de H^*. Pour que la formule de Künneth fournisse un isomorphisme de foncteurs $\mathscr{H}(M \otimes N) \xrightarrow{\sim} \mathscr{H}(M) \otimes \mathscr{H}(N)$, compatible à l'associativité et à la commutativité, il faut prendre pour donnée de commutativité dans $\mathscr{A}^*$ celle donnée par la règle de Koszul.

0.3. Pour $k = \boldsymbol{C}$, le théorème de comparaison entre cohomologie classique et cohomologie étale fournit un isomorphisme $H^*(X(\boldsymbol{C}), \boldsymbol{Q}) \otimes \boldsymbol{Q}_l \xrightarrow{\sim} H^*(X, \boldsymbol{Q}_l)$. Si X est défini sur $\boldsymbol{R}$, cet isomorphisme transforme F_∞ (0.2.5) en l'action (0.2.7) de la conjugaison complexe. Pour un motif, on a de même $H_B(M) \otimes \boldsymbol{Q}_l \xrightarrow{\sim} H_l(M)$.

0.4. Pour $k = \boldsymbol{C}$, le complexe de de Rham holomorphe sur X^{an} est une résolution du faisceau constant $\boldsymbol{C}$. Par GAGA, on a donc un isomorphisme

$$H^*(X(\boldsymbol{C}), \boldsymbol{Q}) \otimes \boldsymbol{C} = H^*(X(\boldsymbol{C}), \boldsymbol{C}) \xrightarrow{\sim} \mathscr{H}^*(X^{\mathrm{an}}, \Omega^{*\mathrm{an}}) \xleftarrow{\sim} H^*(X, \Omega^*).$$

Pour un motif, on obtient un isomorphisme (compatible aux filtrations de Hodges 0.2.4 : $F^p = \bigoplus_{p' \geq p} V^{p'q'}$ et 0.2.6) $H_B(M) \otimes \boldsymbol{C} \to^{\sim} H_{DR}(M)$.

Soit plus généralement M un motif sur un corps k, et σ un plongement complexe de k. Appliquant ce qui précède au motif sur $\boldsymbol{C}$ déduit de M par extension des scalaires, on trouve un isomorphisme

$$I : H_\sigma(M) \otimes \boldsymbol{C} \xrightarrow{\sim} H_{DR}(M) \otimes_{k,\sigma} \boldsymbol{C}. \tag{0.4.1}$$

Notons le cas particulier $k = \boldsymbol{Q}$, où on trouve deux $\boldsymbol{Q}$-structures naturelles sur la réalisation de M en cohomologie complexe : l'une, $H_B(M)$, liée à la description de la cohomologie en terme de cycles, l'autre, $H_{DR}(M)$, liée à sa description en terme de formes différentielles algébriques.

0.5. Ce qu'il advient de ces réalisations et compatibilités par extension du corps des scalaires est clair. Pour la restriction des scalaires, les résultats sont les suivants. Soient k' une extension finie de k, M' un motif sur k', et $M = R_{k'/k}(M')$

$$H_\sigma(\mathrm{M}) = \bigoplus_\tau H_\tau(M'), \tag{0.5.1}$$

où la somme est étendue à l'ensemble $J(\sigma)$ des plongement complexes de k' qui prolongent σ. Cet isomorphisme est compatible aux bigraduations de Hodge, et à F_∞.

$$H_{DR}(M) = H_{DR}(M') \quad \text{(restriction des scalaires de } k' \text{ à } k). \tag{0.5.2}$$

Cet isomorphisme est compatible à la filtration de Hodge.

$$H_l(M) = \mathrm{Ind}\, H_l(M') \quad \text{(représentation induite de } \mathrm{Gal}(\bar{k}/k') \text{ à } \mathrm{Gal}(\bar{k}/k)). \tag{0.5.3}$$

(0.5.4) Via l'isomorphisme $k' \otimes_{k,\sigma} \boldsymbol{C} = \boldsymbol{C}^{J(\sigma)}$, et l'isomorphisme $H_{DR}(M') \otimes_{k,\sigma} \boldsymbol{C} = \bigoplus_{\tau \in J(\sigma)} H_{DR}(M') \otimes_{k',\tau} \boldsymbol{C}$ qui s'en déduit, (0.4.1) pour M et σ est la somme des isomorphismes (0.4.1) pour M' et τ $(\tau \in J(\sigma))$:

$$\begin{array}{ccc}
H_\sigma(M) \otimes \boldsymbol{C} & \xrightarrow[\sim]{(0.4.1)} & H_{DR}(M) \otimes_{k,\sigma} \boldsymbol{C} \\
\Big\| \scriptstyle (0.5.1) & & \Big\| \scriptstyle (0.5.2) \\
\bigoplus_\tau H_\tau(M') \otimes \boldsymbol{C} & \xrightarrow[\sim]{(0.4.1)} & \bigoplus_\tau H_{DR}(M') \otimes_{k',\tau} \boldsymbol{C}.
\end{array}$$

0.6. Pour X projectif et lisse sur k, notons $Z^d(X)$ l'espace vectoriel sur $\boldsymbol{Q}$ de base l'ensemble des sous-schémas fermés irréductibles de codimension d de X, et $Z^d_R(X)$ son quotient par une relation d'équivalence R. Pour k de caractéristique 0, une des définitions de Grothendieck des motifs s'obtient en faisant (pour X et Y connexes) $\mathrm{Hom}(H^*(Y), H^*(X)) = Z_R^{\dim(Y)}(X \times Y)$, R étant l'équivalence cohomologique.

En ce qui concerne la caractéristique p, signalons seulement deux difficultés: on ignore si l'équivalence cohomologique—en cohomologie l-adique ($l \neq p$)—est indépendante de l, et la définition de la classe d'un cycle pose des problèmes en cohomologie cristalline.

0.7. Soit X une variété projective et lisse complexe. Nous appellerons *cycle de Hodge* de codimension d sur X un élément de $H^{2d}(X(\boldsymbol{C}), \boldsymbol{Q})$ de type (d, d)—soit, ce qui revient au même, un élément de $H^{2d}(X(\boldsymbol{C}), \boldsymbol{Q})(d)$ de type $(0, 0)$.

Soient k un corps algébriquement clos et X une variété projective et lisse sur k. Posons

$$H^{2d}(X, k \times A^f)(d) = H^{2d}_{DR}(X)(d) \times H^{2d}(X, A^f)(d).$$

Pour $k = \boldsymbol{C}$, on appelle encore *cycle de Hodge* l'image dans

$$H^{2d}(X, k \times A^f)(d) = H^{2d}(X(\boldsymbol{C}), \boldsymbol{Q})(d) \otimes (k \times A^f)$$

d'un cycle de Hodge. Pour k algébriquement clos, admettant des plongements complexes, un *cycle de Hodge absolu* de codimension d sur X est un élément de $H^{2d}(X, k \times A^f)(d)$, tel que, pour tout plongement complexe σ de k, son image dans $H^{2d}(X \otimes_k \boldsymbol{C}, \boldsymbol{C} \times A^f)(d)$ soit de Hodge. On vérifie que

PROPOSITION 0.8. (i) *L'espace vectoriel sur $\boldsymbol{Q}$ $Z^d_{\text{ha}}(X)$ des cycles de Hodge absolu est invariant par extension des scalaires de k à un corps algébriquement clos k' (admettant encore un plongement complexe).*

(ii) *Pour k algébriquement clos de caractéristique* 0, *et X défini sur un sous-corps algébriquement clos k_0 de k, admettant un plongement complexe: $X = X_0 \otimes_{k_0} k$, on pose*

$$Z^d_{\text{ha}}(X) = Z^d_{\text{ha}}(X_0) \subset H^{2d}(X_0, k_0 \times A^f)(d) \subset H^{2d}(X, k \times A^f)(d).$$

D'après (i), *cette définition ne dépend pas des choix de X_0 et k_0.*

(iii) *Pour X défini sur un sous-corps k_0 de k: $X = X_0 \otimes_{k_0} k$, le groupe* $\text{Aut}(k/k_0)$ *agissant sur $H^{2d}(X, k \times A^f)(d)$ stabilise $Z^d_{\text{ha}}(X)$. Il agit sur $Z^d_{\text{ha}}(X)$ à travers un groupe fini, correspondant à une extension finie k_0' de k_0. On pose $Z^d_{\text{ha}}(X_0) = Z^d_{\text{ha}}(X)^{\text{Aut}(k/k_0)}$.*

0.9. Une notion utile de motif s'obtient en faisant (pour X et Y connexes)

$$\text{Hom}(H^*(Y), H^*(X)) = Z^{\dim(Y)}_{\text{ha}}(X \times Y).$$

Les composantes de Künneth de la diagonale de $X \times X$ sont absolument de Hodge. Ceci permet de décomposer $H^*(X)$ en une somme de motifs $H^i(X)$, de munir la catégorie des motifs d'une graduation (avec $H^i(X)$ de poids i) et de modifier la contrainte de commutativité pour $\otimes$ comme en [**19**, VI, 4.2.1.4]. On vérifie que, ceci fait, la $\otimes$-catégorie des motifs sur k est tannakienne, isomorphe à la catégorie des représentations d'un groupe proalgébrique réductif.

Pour $k = \boldsymbol{C}$, la conjecture suivante, plus faible que celle de Hodge, équivaut à dire que le foncteur "réalisation de Hodge" est une équivalence de la catégorie des motifs 0.9 avec la catégorie des structures de Hodge facteur direct de la cohomologie de variété algébriques (ou déduites par twist à la Tate de tels facteurs directs).

Espoir 0.10. Tout cycle de Hodge l'est absolument.

Si X est une variété abélienne à multiplication complexe par un corps quadratique imaginaire K, avec $\text{Lie}(X)$ libre sur $k \otimes K$, les méthodes de B. Gross [**7**] prouvent que certains cycles de Hodge non triviaux sont absolument de Hodge. Ajouté sur épreuves: partant de ce cas particulier, j'ai pu vérifier 0.10 pour les variétés abéliennes. J'ai vérifié aussi que la $\otimes$-catégorie de motifs (0.9) engendrée par les $H^\wedge(X)$, pour X une variété abélienne, contient les motifs $H^*(X)$, pour X une surface K3 ou une hypersurface de Fermat.

0.11. Le principal défaut de la définition 0.9 des motifs est qu'elle ne se prête pas à la réduction mod p. On ignore si un motif 0.9 sur un corps de nombres F fournit un système compatible de réprésentations l-adiques de $\text{Gal}(\bar{F}/F)$.

0.12. Nous utiliserons le mot "motif" de façon libre, sans nous préoccuper de faire rentrer les motifs considérés dans le cadre de Grothendieck. L'essentiel pour nous sera de disposer de réalisations $\mathscr{H}(M)$, pour les théories $\mathscr{H}$ considérées en 0.2, et d'avoir pour ces groupes le même formalisme que pour les $\mathscr{H}^*(X)$.

1. Enoncé de la conjecture (cas rationnel).

1.1. Soit M un motif sur $\mathbf{Q}$. Nous admettrons que les réalisations l-adiques $H_l(M)$ de M forment un système strictement compatible de représentations l-adiques, au sens de Serre [**11**, I.11]. A savoir : il existe un ensemble fini S de nombres premiers, tel que chaque $H_l(M)$ soit non ramifié en dehors de $S \cup \{l\}$, et que, notant par $F_p \in \mathrm{Gal}(\bar{\mathbf{Q}}/\mathbf{Q})$ un élément de Frobenius géométrique en p (l'inverse d'une substitution de Frobenius φ_p), le polynôme $\det(1 - F_p t, H_l(M)) \in \mathbf{Q}_l[t]$ $(p \notin S \cup \{l\})$ soit à coefficients rationnels, et indépendant de l. Notons $Z_p(M, t)$ son inverse, et posons $L_p(M, s) = Z_p(M, p^{-s})$.

La série de Dirichlet à coefficients rationnels donnée par le produit eulérien $L_S(M, s) = \prod_{p \notin S} L_p(M, s)$ converge pour $\mathscr{R}s$ assez grand. Pour s quelconque, on définit $L_S(M, s)$ par un prolongement analytique (qu'on espère exister).

Notre but est d'énoncer une conjecture donnant la valeur de $L_S(M, s)$ en certains points entiers, au produit près par un nombre rationnel. Puisque, pour s entier, p^{-s} est rationnel, le choix de S est sans importance—à ceci près qu'aggrandir S peut introduire des zéros mal venus.

1.2. Pour écrire proprement l'équation fonctionnelle conjecturale des fonctions L, il y a lieu de compléter le produit eulérien $L_S(M, s)$ par des facteurs locaux en $p \in S$, et à l'infini. La définition des facteurs locaux en $p \in S$ requiert une hypothèse additionnelle, que nous supposerons vérifiée:

(1.2.1) Soient p un nombre premier, $D_p \subset \mathrm{Gal}(\bar{\mathbf{Q}}/\mathbf{Q})$ un groupe de décomposition en p, $I_p \subset D_p$ son sous-groupe d'inertie et $F_p \in D_p$ un Frobenius géométrique. Le polynôme $\det(1 - F_p t, H_l(M)^{I_p}) \in \mathbf{Q}_l\,[t]$ $(l \neq p)$ est à coefficients rationnels, et est indépendant de l.

Posons $Z_p(M, t) = \det\,(1 - F_p t, H_l(M)^{I_p})^{-1} \in \mathbf{Q}(t)$, et $L_p(M, s) = Z_p(M, p^{-s})$. On définit

$$L(M, s) = \prod_p L_p(M, s). \tag{1.2.2}$$

Le facteur à l'infini $L_\infty(M, s)$ (essentiellement un produit de fonctions Γ) dépend de la réalisation de Hodge de M—en fait seulement de la classe d'isomorphie de l'espace vectoriel complexe $H_B(M) \otimes \mathbf{C}$, muni de sa décomposition de Hodge et de l'involution F_∞. Sa définition est rappelée en 5.2.

Posant $\Lambda(M, s) = L_\infty(M, s)L(M, s)$, l'équation fonctionnelle conjecturale des fonctions L s'écrit

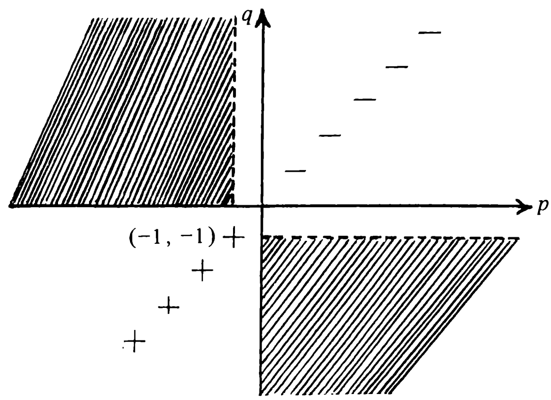
$$\Lambda(M, s) = \varepsilon(M, s)\Lambda(\check{M}, 1-s) \tag{1.2.3}$$

où $\check{M}$ est le dual de M (de réalisations les duales des réalisations de M) et où $\varepsilon(M, s)$, comme fonction de s, est le produit d'une constante par une exponentielle. Sa définition est rappelée en 5.2. Elle dépend d'une hypothèse additionnelle.

DÉFINITION 1.3. *Un entier n est* critique *pour M si ni $L_\infty(M, s)$, ni $L_\infty(\check{M}, 1-s)$ n'ont de pôle en $s = n$.*

Notre but est de conjecturer la valeur de $L(M, n)$ pour n critique, à multiplication par nombre rationnel près. On a $L(M(n), s) = L(M, n + s)$ (3.1.2), et de même pour L_∞. Ceci nous permet de ne considérer que les nombres $L(M) =_{\mathrm{dfn}} L(M, 0)$. Nous dirons que M est *critique* si 0 est critique pour M. On vérifie que pour que M soit critique, il faut et il suffit que les nombres de Hodge $h^{pq} =_{\mathrm{dfn}} \dim H^{pq}(M)$ de M, pour $p \neq q$, ne soient non nuls que pour (p, q) dans la partié hachurée du diagramme ci-dessous, et que F_∞ agisse sur $H^{p,p}$, par l'identité si $p < 0$, par -1 si $p \geqq 0$.

(1.3.1)



Supposons *M homogène de poids w*: $h^{pq} = 0$ pour $p + q \neq w$, et posons $\mathscr{R}(M) = -w/2$. Il résulte de la conjecture de Weil que, pour S assez grand, la série de Dirichlet $L_S(M, s)$ converge absolument pour $\mathscr{R}(M) + \mathscr{R}(s) > 1$. Pour $\mathscr{R}(M) + \mathscr{R}(s) = 1$, on est au bord du demi-plan de convergence, et on conjecture

(a) que $L_S(M, s)$ ne s'annule pas,

(b) que $L_S(M, s)$ est holomorphe, sauf si M est de poids pair $-2n$, et contient $\boldsymbol{Z}(n)$ en facteur: on s'attend alors à un pôle en $s = 1 - n$ (une valeur non critique).

L'analogie avec le cas des corps de fonctions, et les cas connus, mènent à croire que les facteurs locaux $L_p(M, s)$ (p quelconque, y compris ∞) n'ont de pôle que pour $\mathscr{R}(M) + \mathscr{R}(s) \leqq 0$. Si tel est le cas, (a), (b) et l'équation fonctionnelle conjecturale (1.2.3) impliquent que $L(M) \neq 0, \infty$ pour M critique et $\mathscr{R}(M) \neq \frac{1}{2}$. Pour $\mathscr{R}(M) = \frac{1}{2}$, $L(M)$ s'annulle parfois. Notre conjecture (1.8) est alors vide.

PROPOSITION 1.4. *Soit M un motif sur $\boldsymbol{R}$. Via l'isomorphisme* (0.4.1)

$$H_B(M) \otimes \boldsymbol{C} \xrightarrow{\sim} H_{DR}(M) \otimes_{\boldsymbol{R}} \boldsymbol{C},$$

$H_{DR}(M)$ s'identifie au sous-espace de $H_B(M) \otimes \boldsymbol{C}$ fixe par $c \to F_\infty \bar{c}$.

Prenons $M = H^*(X)$. La conjugaison complexe sur $H_{DR}(M)_{\boldsymbol{C}} = H^*(X_{\boldsymbol{C}}, \Omega^*)$ se déduit par fonctorialité de l'automorphisme antilinéaire F_∞ du schéma $X_{\boldsymbol{C}}$. Remontant la flèche composée (0.4) qui définit I, on l'identifie à l'involution déduite de l'automorphisme $(F_\infty, \bar{F}_\infty^*)$ de $(X(\boldsymbol{C}), \Omega^{*\mathrm{an}})$, puis au composé de F_∞^*: $H^*(X(\boldsymbol{C}), \boldsymbol{C}) \to H^*(X(\boldsymbol{C}), \boldsymbol{C})$ et de la conjugaison complexe sur les coefficients. Ceci vérifie 1.4.

1.5. Pour M un motif sur $\boldsymbol{R}$, nous noterons $H_B^+(M)$ (resp. $H_B^-(M)$) le sous-espace de $H_B(M)$ fixe par F_∞ (resp. où $F_\infty = -1$). Posons $d(M) = \dim H_B(M)$ et $d^\pm(M) = \dim H_B^\pm(M)$. Pour M un motif sur k, et σ un plongement complexe de k qui se factorise par $\boldsymbol{R}$, on note $H_\sigma^\pm(M)$, $d_\sigma^\pm(M)$ et $d(M)$ ces objets pour le motif sur $\boldsymbol{R}$ déduit de M par σ. Pour $k = \boldsymbol{Q}$, on omet la mention de σ.

Le corollaire suivant résulte aussitôt de 1.4, et de ce que tant F_∞ que la conjugaison complexe échangent H^{pq} et H^{qp}.

COROLLAIRE 1.6. *Soit M un motif sur $\boldsymbol{R}$. Pour la structure réelle $H_{DR}(M)$ de $H_B(M) \otimes \boldsymbol{C}$, les sous-espaces H^{pq} sont définis sur $\boldsymbol{R}$; le sous-espace $H_B^+(M) \otimes \boldsymbol{R}$ est réel, et $H_B^-(M) \otimes \boldsymbol{R}$ est purement imaginaire.*

1.7. Dans la fin de ce paragraphe, nous ne considérerons que des motifs sur $\boldsymbol{Q}$; sauf mention expresse du contraire, nous les supposons homogènes. Si leur poids w est pair, nous supposerons aussi que F_∞ agit sur $H^{pp}(M)$ ($w = 2p$) comme un scalaire: soit $+1$, soit -1. Cette hypothèse est vérifiée pour M critique. Puisque F_∞ échange H^{pq} et H^{qp}, elle assure que les dimenions $d^+(M)$ et $d^-(M)$ sont égales l'une à $\sum_{p>q} h^{pq}$, l'autre à $\sum_{p\geqq q} h^{pq}$. En particulier, ces dimensions sont égales à celles de sous-espaces F^+ et F^- figurant dans la filtration de Hodge. Posant $H_{DR}^\pm(M) = H_{DR}(M)/F^\mp$, on a encore $\dim H_{DR}^\pm(M) = d^\pm(M)$.

Il résulte de ce que F_∞ échange H^{pq} et H^{qp} que les applications composées

$$I^\pm\colon H_B^\pm(M)_C \longrightarrow H_B(M)_C \xrightarrow[\sim]{I} H_{DR}(M)_C \longrightarrow H_{DR}^\pm(M)_C \tag{1.7.1}$$

sont des isomorphismes. On pose

$$c^\pm(M) = \det(I^\pm), \tag{1.7.2}$$

$$\delta(M) = \det(I), \tag{1.7.3}$$

le déterminant étant calculé dans des bases rationnelles de $H_B^\pm$ et $H_{DR}^\pm$ (resp. H_B et H_{DR}). La définition de $\delta(M)$ ne requiert pas les hypothèses faites sur M.

D'après 1.6, I^+ est réel, i.e. induit $I^+\colon H_B^+(M)_{\boldsymbol{R}} \to H_{DR}^+(M)_{\boldsymbol{R}}$, tandis que I^- est purement imaginaire. Les nombres $c^+(M)$, $i^{d^-(M)}c^-(M)$ et $i^{d^-(M)}\delta(M)$ sont donc réels non nuls. A multiplication par un nombre rationnel près, il ne dépendent que de M.

Les *périodes* de M sont classiquement les $\langle\omega, c\rangle$ pour $\omega \in H_{DR}(M)$ et $c \in H_B(M)^\vee$. Par exemple, si X est une variété algébrique sur $\boldsymbol{Q}$, ω une n-forme sur X, définie sur $\boldsymbol{Q}$ et que c est un n-cycle sur $X(\boldsymbol{C})$, $\langle\omega, c\rangle = \int_c \omega$ est une période de $H^n(X)$. Exprimons $c^\pm(M)$ en terme de périodes. Le dual de $H_{DR}^\pm(M)$ est le sous-espace $F^\pm$ de $H_{DR}(M^\vee)$, où $M^\vee$ est le motif dual de M. Si la base choisie de $H_{DR}^\pm(M)$ a pour duale la base (ω_i) de $F^\pm(H_{DR}(M^\vee))$, et que (c_i) est la base choisie de $H_B^\pm(M)$, la matrice de $I^\pm$ est $\langle\omega_i, c_j\rangle$, et $c^\pm(M) = \det(\langle\omega_i, c_j\rangle)$.

Conjecture 1.8. *Si M est critique, $L(M)$ est un multiple rationnel de $c^+(M)$.*

2. Enoncé de la conjecture (cas général).

2.1. Jusqu'ici, nous n'avons considéré que des fonctions L données par des séries de Dirichlet à coefficients rationnels. Pour faire mieux, il nous faudra considérer des motifs à coefficient dans des corps de nombres.

Voici deux façons, équivalentes, pour construire la catégorie des motifs sur k, à coefficient dans un corps de nombres E, à partir de la catégorie des motifs sur k. Il s'agit d'une construction valable pour toute catégorie additive karoubienne (0.1(b)) dans laquelle les $\mathrm{Hom}(X, Y)$ sont des expaces vectoriels sur $\boldsymbol{Q}$.

A. Un motif sur k, à coefficient dans E, est un motif M sur k muni d'une structure de E-module: $E \to \mathrm{End}(M)$.

B. La catégorie $\mathscr{M}_{k,E}$, des motifs sur k, à coefficient dans E, est l'enveloppe karoubienne (cf. 0.1(b)) de la catégorie d'objets les motifs sur k–considéré comme objet de $\mathscr{M}_{k,E}$, un motif M se notera M_E—et de morphismes donnés par $\mathrm{Hom}(X_E, Y_E) = \mathrm{Hom}(X, Y) \otimes E$.

Passage de B *à* A. Pour X un motif, et V un espace vectoriel sur $\boldsymbol{Q}$ de dimension finie, on note $X \otimes V$ le motif, isomorphe à une somme de $\dim(V)$ copies de X, caractérisé par $\mathrm{Hom}(Y, X \otimes V) = \mathrm{Hom}(Y, X) \otimes V$ (ou par $\mathrm{Hom}(X \otimes V, Y) = \mathrm{Hom}(V, \mathrm{Hom}(X, Y))$). On passe de B à A en associant à M_E le motif $M \otimes E$, muni de sa structure de E-module naturelle.

Passage de A *à* B. Si M est muni d'une structure de E-module, on récupère sur M_E deux structures de E-module: celle déduite de celle de M, et celle qu'a tout objet de $\mathscr{M}_{k,E}$. L'objet de $\mathscr{M}_{k,E}$ correspondant à M est le plus grand facteur direct de M_E sur lequel ces structures coïncident. En détail: l'algèbre $E \otimes E$ est un produit de corps, parmi lesquels une copie de E dans laquelle $x \otimes 1$ et $1 \otimes x$ se projettent tous deux comme x. L'idempotent correspondant e agit sur M_E (qui est un $E \otimes E$-module) et son image est l'object de $\mathscr{M}_{k,E}$ qui correspond à M.

Les motifs à coefficient dans E sont le plus souvent donnés sous la forme A. La forme B à l'avantage de rester raisonnable pour E non de rang fini sur $\boldsymbol{Q}$. Elle est utile pour comprendre le formalisme tensoriel: on peut définir produit tensoriel et dual, pour les motifs à coefficient dans E, par leur fonctorialité et les formules $X_E \otimes_E Y_E = (X \otimes Y)_E$ et $(X_E)^\vee = (X^\vee)_E$. Dans le langage A, $X \otimes_E Y$ est le plus grand facteur direct de $X \otimes Y$ sur lequel coïncident les deux structures de E-module de $X \otimes Y$, et X est le dual usuel de $X^\vee$, muni de la structure de E-module transposée. Si on applique ces remarques à la catégorie des espaces vectoriels sur $\boldsymbol{Q}$, plutôt qu'à celle des motifs, on retrouve l'isomorphisme du F-dual d'un espace vectoriel sur E avec son $\boldsymbol{Q}$-dual, donné par

$$\omega \mapsto \text{la forme } \mathrm{Tr}_{E/\boldsymbol{Q}}(\langle \omega, v \rangle).$$

Nous avons défini en 0.1.1 des foncteurs de restriction et d'extension du corps k des scalaires. Ils transforment motifs à coefficient dans E en motifs à coefficient dans E. On dispose aussi de foncteurs de restriction et d'extension des coefficients: soit F une extension finie de E:

Extension des coefficients. Dans le langage A, c'est $X \mapsto X \otimes_E F$; dans le langage B, c'est $X_E \mapsto X_F$.

Restriction des coefficients. Dans le langage A, on restreint à E la structure de F-module.

Le lecteur prendra soin de ne pas confondre les rôles de k et E. Un exemple type qu'on peut retenir est celui du H_1 des variétés abéliennes sur k, à multiplication complexe par un ordre de E. En terme de la variété abélienne—prise à isogénie près—les foncteurs ci-dessus deviennent: extension du cops de base, restriction des scalaires à la Weil, construction $\otimes_E F$, qui multiplie la dimension par $[F\colon E]$, restriction à E de la structure de F-module.

2.2. Soit M un motif sur $\boldsymbol{Q}$, à coefficient dans un corps de nombres E. Pour chaque nombre premier l, la réalisation l-adique $H_l(M)$ de M est un module sur le complété l-adique E_l de E. Ce complété est le produit des complétés E_λ, pour λ idéal premier au-dessus de l, d'où une décomposition de $H_l(M)$ en un produit de E_λ-modules $H_\lambda(M)$.

On conjecture pour les $H_\lambda(M)$ une compatibilité analogue à 1.2.1; si elle est vérifiée, on peut pour chaque plongement complexe σ de E définir une série de Dirichlet à coefficients dans σE, convergente pour $\mathscr{R}s$ assez grand:

$$\begin{aligned} L(\sigma, M, s) &= \prod_p L_p(\sigma, M, s), \quad \text{où} \\ L_p(\sigma, M, s) &= \sigma Z_p(M, p^{-s}) \quad \text{avec } Z_p \in E(t) \subset E_\lambda(t) \text{ donné par} \\ Z_p(M, t) &= \det(1 - F_p t, H_\lambda(M)^{I_p})^{-1} \quad \text{pour } \lambda \nmid p. \end{aligned} \tag{2.2.1}$$

Pour σ variable, ces séries de Dirichlet se déduisent les unes des autres par conjugaison des coefficients. Nous regarderons le système des $L(\sigma, M, s)$ comme une fonction $L^*(M, s)$ à valeurs dans la $\boldsymbol{C}$-algèbre $E \otimes \boldsymbol{C}$, identifiée à $\boldsymbol{C}^{\mathrm{Hom}(E, \boldsymbol{C})}$ par

$$E \otimes \boldsymbol{C} \xrightarrow{\sim} \boldsymbol{C}^{\mathrm{Hom}(E, \boldsymbol{C})} : e \otimes z \mapsto (z, \sigma(e))_\sigma. \tag{2.2.2}$$

Cette fonction peut aussi être définie directement par un produit eulérien. On espère comme en 1.1 qu'elle admet un prolongement analytique en s.

Il y a lieu de compléter le produit eulérien $L(\sigma, M, s)$ par un facteur à l'infini $L_\infty(\sigma, M, s)$ dépendant de la réalisation de Hodge de M. Posant $\Lambda(\sigma, M, s) = L_\infty(\sigma, M, s)L(\sigma, M, s)$, l'équation fonctionnelle conjecturale des fonctions L s'écrit

$$\Lambda(\sigma, M, s) = \varepsilon(\sigma, M, s)\Lambda(\sigma, \check{M}, 1 - s), \tag{2.2.3}$$

où $\varepsilon(\sigma, M, s)$, comme fonction de s, est le produit d'une constante par une exponentielle. Les définitions de L_∞ et ε sont rappelées en 5.2. Comme ci-dessus, on regardera le système des Λ et celui des ε, pour σ variable, comme des fonctions Λ^* et ε^* à valeurs dans $E \otimes \boldsymbol{C}$.

Il résulte de 2.5 ci-dessous que $L_\infty(\sigma, M, s)$ est indépendant de σ, et que la fonction L_∞, pour le motif $R_{E/\boldsymbol{Q}}\, M$ déduit de M par restriction du corps des coefficients (2.1) est la puissance $[E\colon \boldsymbol{Q}]^{\text{ième}}$ de $L_\infty(\sigma, M, s)$. Ceci justifie la

PROPOSITION-DÉFINITION 2.3. *Soit M un motif sur $\boldsymbol{Q}$ à coefficient dans E. Un entier n est* critique *pour M si les conditions équivalentes suivantes sont vérifiées*

(i) *l'entier n est critique pour $R_{E/\boldsymbol{Q}}\, M$;*

(ii) *ni $L_\infty(\sigma, M, s)$, ni $L_\infty(\sigma, \check{M}(1), s)$ n'ont de pôle en $s = n$.*

On dit que M est critique *si* 0 *est critique pour M.*

Notre but est de conjecturer la valeur de $L^*(M) =_{\mathrm{dfn}} L^*(M, 0)$, pour M critique, à multiplication par un élément de E près. En d'autres termes, il s'agit de conjecturer simultanément les valeurs des $L(\sigma, M) =_{\mathrm{dfn}} L(\sigma, M, 0)$, à multiplication près par un système de nombres $\sigma(e)$, $e \in E$.

2.4. La réalisation $H_B(M)$ de M en cohomologie rationnelle est munie d'une structure de E-espace vectoriel. Sa dimension est le *rang* sur E de M. L'involution F_∞ est E-linéaire; les parties + et − sont donc des E-sous-espaces vectoriels. On note $d^+(M)$ et $d^-(M)$ leur dimension.

Le complexifié de $H_B(M)$ est un $E \otimes \boldsymbol{C}$-module libre. Identifiant $E \otimes \boldsymbol{C}$ à $\boldsymbol{C}^{\mathrm{Hom}(E,\boldsymbol{C})}$ (2.2.2), on en déduit une décomposition

$$H_B(M) \otimes \boldsymbol{C} = \bigoplus_\sigma H_B(\sigma, M),$$

avec

$$H_B(\sigma, M) = (H_B(M) \otimes \boldsymbol{C}) \otimes_{E \otimes \boldsymbol{C}, \sigma} \boldsymbol{C},$$

soit

$$H_B(\sigma, M) = H_B(M) \otimes_{E,\sigma} \boldsymbol{C}.$$

Les $H_B^{p,q}(M)$ de la décomposition de Hodge étant stables par E, chaque $H_B(\sigma, M)$ hérite d'une décomposition de Hodge $H_B(\sigma, M) = \bigoplus H_B^{p,q}(\sigma, M)$. L'involution F_∞ permute H^{pq} et H^{qp}, en particulier stabilise H^{pp}, qu'elle découpe en des parties $+$ et $-$. On note $h^{pq}(\sigma, M)$ la dimension de $H_B^{pq}(\sigma, M)$, et $h^{pp\pm}(\sigma, M)$ celle de $H_B^{pp\pm}(\sigma, M)$. La proposition suivante permet d'omettre σ de la notation.

PROPOSITION 2.5. *Les nombres $h^{pq}(\sigma, M)$ et $h^{pp\pm}(\sigma, M)$ sont indépendant de σ.*

On peut supposer, et on suppose, que M est homogène. Dans ce cas, $H^{pq}(M)$ s'identifie au complexifié du E-espace vectoriel $\mathrm{Gr}_F^p(H_{DR}(M))$: c'est un $E \otimes \boldsymbol{C}$-module libre, et la première assertion en résulte. Pour la seconde, on observe que $h^{pp\pm}(\sigma, M)$ est l'excès de $d^\pm(M)$ sur $\sum_{p>q} h^{pq}(\sigma, M)$.

2.6. Soit M un motif sur $\boldsymbol{Q}$ à coefficient dans E. Dans la fin de ce paragraphe, sauf mention expresse du contraire, nous supposons que $R_{E/\boldsymbol{Q}}\, M$ vérifie les hypothèses de 1.7. Les espaces $F^\pm$ et $H_{DR}^\pm$ sont cette fois des espaces vectoriels sur E. Les isomorphismes (0.4.1) et (1.7.1)

(2.6.1) $$I: H_B(M) \otimes \boldsymbol{C} \longrightarrow H_{DR}(M) \otimes \boldsymbol{C},$$
(2.6.2) $$I^\pm: H_B^\pm(M) \otimes \boldsymbol{C} \longrightarrow H_{DR}^\pm(M) \otimes \boldsymbol{C},$$

sont des isomorphismes de $E \otimes \boldsymbol{C}$-modules entre complexifiés d'espaces vectoriels sur E. On pose

$$c^\pm(M) = \det(I^\pm) \in (E \otimes \boldsymbol{C})^*, \qquad \delta(M) = \det(I) \in (E \otimes \boldsymbol{C})^*,$$

le determinant étant calculé dans des bases E-rationnelles de $H_B^\pm(M)$ et $H_{DR}^\pm(M)$ (resp. $H_B(M)$ et $H_{DR}(M)$). La définition de $\delta(M)$ ne requiert pas les hypothèses faites sur M. A multiplication par un élément de E^* près, ces nombres ne dépendent que de M. Il résulte à nouveau de 1.6 que $c^+(M)$, $i^{d^-(M)}\, c^-(M)$ et $i^{d^-(M)}\, \delta(M)$ sont dans $(E \otimes \boldsymbol{R})^*$.

Conjecture 2.7. *Posons $\mathscr{R}(M) = -\frac{1}{2}w$. Si M est critique,*

(i) *$L(\sigma, M, s)$ n'a jamais de pôle en $s = 0$, et ne peut s'annuler en $s = 0$ que pour $\mathscr{R}(M) = \frac{1}{2}$.*

(ii) *La multiplicité du zéro de $L(\sigma, M, s)$ en $s = 0$ est indépendante de σ.*

Pour (i), je renvoie à la discussion à la fin de 1.3. Que (ii) soit raisonnable m'a été suggéré par B. Gross.

Conjecture 2.8. *Pour M critique et $L(\sigma, M) \neq 0$, $L^*(M)$ est le produit de $c^+(M)$ par un élément de E^*.*

REMARQUE 2.9. Un motif M sur un corps de nombres k, à coefficient dans E, définit aussi une fonction $L^*(M, s)$. Ces fonctions sont couvertes par notre conjecture, vu l'identité $L^*(M, s) = L^*(R_{k/\mathbf{Q}} M, s)$, où $R_{k/\mathbf{Q}}$ est la restriction des scalaires de k à $\mathbf{Q}$ (appliquer (0.5.3)).

Les fonctions $L(\sigma, M, s)$ sont des produits eulériens, indexés par les places finies de k. Il y a lieu de les compléter par les facteurs à l'infini $L_v(\sigma, M, s)$, indexés par les places à l'infini, dont la définition est rappelée en 5.2. Ils dépendent en général de σ. Seul $L_\infty(\sigma, R_{k/\mathbf{Q}} M, s)$, produit sur v à l'infini des $L_v(\sigma, M, s)$, est indépendant de σ.

REMARQUE 2.10. Soient F une extension de E, ι le morphisme structural de E dans F et $\iota_{\mathbf{C}}$ son complexifié: $E \otimes \mathbf{C} \hookrightarrow F \otimes \mathbf{C}$. On a $L^*(M \otimes_E F, s) = \iota_{\mathbf{C}} L^*(M, s)$ et $c^{\pm}(M \otimes_E F) = \iota_{\mathbf{C}}\, c^{\pm}(M)$. La conjecture est donc compatible à l'extension du corps des coefficients. Pour F une extension galoisienne de E, de groupe de Galois G, le Théorème 90 de Hilbert $H^1(G, F^*) = 0$ assure que $((F \otimes \mathbf{C})^*/F^*)^G = (E \otimes \mathbf{C})^*/E^*$: la conjecture est invariante par extension du corps des coefficients.

REMARQUE 2.11. Si E est une extension de F, on a $L^*(R_{E/F}M, s) = N_{E/F}\, L^*(M, s)$. Les périodes $c^{\pm}$ vérifiant la même identité, la conjecture est compatible à la restriction des coefficients.

REMARQUE 2.12. Soit D une algèbre à division de rang d^2 sur E. Un motif M sur $\mathbf{Q}$, muni d'une structure de D-module, et de rang n sur D, définit une série de Dirichlet à coefficients dans E dont les facteurs eulériens sont presque tous de degré nd: pour λ une place finie de E, $H_\lambda(M)$ est un module libre sur le complété $D_\lambda = D \otimes_E E_\lambda$, et on reprend la définition (2.2.1) en posant

$$Z_p(M, t) = \det \mathrm{red}\,(1 - F_p t, H_\lambda(M)^{I_p})^{-1}$$

(si D_λ est une algèbre de matrices sur E_λ, et e un idémpotent indécomposable, le déterminant réduit d'un endomorphisme A d'un D_λ-module H est le déterminant, calculé sur E, de la restriction de A à eH; la définition dans le cas général procède par descente).

La liberté que nous donne 2.10 d'étendre le corps des coefficients met ces fonctions L elles aussi sous le chapeau 2.8: choisissant une extension $\iota\colon E \hookrightarrow F$ de E qui neutralise D, et un idempotent indécomposable e de $D \otimes_E F$, on a $\iota_{\mathbf{C}} L^*(M, s) = L^*(e(M \otimes_E F), s)$.

On peut aussi définir $c^+(M)$ directement dans ce cadre. C'est ce qui est expliqué en 2.13 ci-dessous.

La fin de ce paragraphe est inutile pour la suite.

2.13. Soit D une algèbre simple sur un corps E (voire une algèbre d'Azumaya sur un anneau...). Pour le formalisme tensoriel, il est commode de regarder les D-modules comme de "faux E-espaces vectoriels":

(a) Se donner un espace vectoriel V sur E revient à se donner, pour toute extension étale F de E, un F-espace vectoriel V_F, et des isomorphismes compatibles $V_G = V_F \otimes_F G$ pour G une extension de F. On prend $V_F = V \otimes_E F$. Par descente, il suffit de ne se donner les V_F que pour F assez grand.

(b) Soit W un D-module. Pour toute extension étale F de E, et tout F-isomorphisme $D \otimes F \sim \mathrm{End}_F(L)$, avec L libre, posons $W_{F,L} = \mathrm{Hom}_{D \otimes F}(L, W \otimes F)$ (les produits tensoriels sont sur E). On a un isomorphsime de $D \otimes F$-module $W \otimes_E F = L \otimes_F W_{F,L}$.

Si F est assez grand pour neutraliser D, L est unique à isomorphisme non unique près; la non-unicité est due aux homothéties, qui agissent trivialement sur End(L). C'est pourquoi la donnée des $W_{F,L}$ n'est pas du type (a); c'est un "faux espace vectoriel sur E".

Soient (W^α) une famille de D-modules, et T une opération tensorielle. Si les homothéties de L agissent trivialement sur $T(W^\alpha_{F,L})$, le F-espace vectoriel $T(W^\alpha_{F,L})$ est indépendant du choix de L et on obtient un système du type (a), d'où un espace vectoriel $T(W^\alpha)$ sur E.

EXAMPLE. Si W' et W'' sont deux D-modules de rang $n\cdot[D\colon E]^{1/2}$ sur E, on peut prendre $T = \mathbf{Hom}(\bigwedge^n W'_{F,L}, \bigwedge^n W''_{F,L})$; on obtient un espace vectoriel $\delta(W', W'')$ de rang 1 sur E, et tout homomorphisme $f\colon W' \to W''$ a un déterminant réduit $\det \mathrm{red}(f) \in \delta(W', W'')$.

Pour définir $c^+(M)$, on applique cette construction aux D-modules $H_B^+(M)$ et à $H_{DR}^+(M)$. Posons $\delta = \delta(H_B^+(M), H_{DR}^+(M))$. Le déterminant réduit de l'isomorphisme de $D \otimes \mathbf{C}$-modules $I^+\colon H_B^+(M)_{\mathbf{C}} \xrightarrow{\sim} H_{DR}^+(M)_{\mathbf{C}}$ est dans le $E \otimes \mathbf{C}$-module libre de rang 1 $\delta \otimes \mathbf{C}$. On pose $\det \mathrm{red}(I^+) = c^+(M)\cdot e$ pour e une base de δ.

3. Exemple: la fonction ζ.

3.1. Pour comprendre les diverses réalisations du motif de Tate $\mathbf{Z}(1)$, le plus simple est de l'écrire $\mathbf{Z}(1) = H_1(\mathbf{G}_m)$. Le groupe multiplicatif n'étant pas une variété projective, ceci ne rentre pas dans le cadre de Grothendieck, qui demande qu'on définisse plutôt $\mathbf{Z}(1)$ comme le dual du facteur direct $H^2(\mathbf{P}^1)$ de $H^*(\mathbf{P}^1)$.

La réalisation en $\mathbf{Z}_l$-cohomologie de $\mathbf{Z}(1)$ est le module de Tate $T_l(\mathbf{G}_m)$ de $\mathbf{G}_m$

$$\mathbf{Z}_l(1) = \mathrm{proj\,lim}\ \mu_{l^n}.$$

En réalisation de Hodge, on a $H_B(\mathbf{Z}(1)) = H_1(\mathbf{C}^*)$ isomorphe à $\mathbf{Z}$ (à $\mathbf{Q}$ plutôt, en homologie rationnelle). Ce groupe est purement de type $(-1, -1)$ et $F_\infty = -1$.

En réalisation de de Rham, $H_{DR}(\mathbf{Z}(1))$ est le dual de $H^1_{DR}(\mathbf{G}_m)$, isomorphe à $\mathbf{Q}$, de générateur la classe de dz/z. L'unique période de $H^1(\mathbf{G}_m)$ est

$$\oint \frac{dz}{z} = 2\pi i. \tag{3.1.1}$$

Sur $\mathbf{Z}_l(1)$, le Frobenius arithmétique φ_p ($p \neq l$) agit par multiplication par p. Le Frobenius géométrique agit donc par multiplication par p^{-1}; ceci justifie l'identité citée en 1.3

$$L(M(n), s) = L(M, n + s). \tag{3.1.2}$$

Puisque F_∞ agit sur $(-1)^n$ sur $H_B(\mathbf{Z}(n))$, $H_B^\varepsilon(\mathbf{Z}(n))$ est nul pour $\varepsilon = -(-1)^n$, et $c^\varepsilon(\mathbf{Z}(n)) = 1$. Pour $\varepsilon = (-1)^n$, d'après (3.1.1), $c^\varepsilon(\mathbf{Z}(n)) = \delta(\mathbf{Z}(n)) = (2\pi i)^n$:

$$\begin{aligned} &c^\varepsilon(\mathbf{Z}(n)) = (2\pi i)^n & &\text{pour } \varepsilon = (-1)^n, \\ &c^\varepsilon(\mathbf{Z}(n)) = 1, \quad \delta(\mathbf{Z}(n)) = (2\pi i)^n & &\text{pour } \varepsilon = -(-1)^n. \end{aligned} \tag{3.1.3}$$

3.2. La fonction $\zeta(s)$ est la fonction L attachée au motif unité $\mathbf{Z}(0) = H^*(\text{Point})$. Les entiers critiques pour $\mathbf{Z}(0)$ sont les entiers pair > 0, et les entiers impairs $\leqq 0$. A cause du pôle de $\zeta(s)$ en $s = 1$, 0 n'est pas un zéro trivial et il serait raisonnable de définir les entiers critiques comme incluant 0. La équation (3.1.3) et les valeurs connues de $\zeta(n) = L(\boldsymbol{Z}(n))$ pour n critique vérifient 1.8: $\zeta(n)$ est rationnel pour n impair $\leqq 0$, et un multiple rationnel de $(2\pi i)^n$ pour n pair $\geqq 0$.

4. Compatibilité à la conjecture de Birch et Swinnerton-Dyer.

4.1. Soient A une variété abélienne sur $\boldsymbol{Q}$, et d sa dimension. La conjecture de Birch et Swinnerton-Dyer [**15**] affirme notamment:

(a) $L(H^1(A), 1)$ est non nul si et seulement si $A(\boldsymbol{Q})$ est fini.

(b) Soit ω un générateur de $H^0(A, \Omega^d)$. Alors, $L(H^1(A), 1)$ est le produit de $\int_{A(\boldsymbol{R})}|\omega|$ par un nombre rationnel.

Le motif $H^1(A)(1)$ est isomorphe au dual $H_1(A)$ de $H^1(A)$: ceci traduit l'existence d'une polarization, autodualité de $H^1(A)$ à valueurs dans $\boldsymbol{Z}(-1)$. D'après 1.7, $c^+(H^1(A)(1))$ se calcule donc comme suit: si $\omega_1, \cdots, \omega_d$ est une base de $H^0(A, \Omega^1) = F^+H^1_{DR}(A)$, et $c_1, \cdots, c_d$ une base de $H_1(A(\boldsymbol{C}), \boldsymbol{Q})^+$, on a

$$c^+(H^1(A)(1)) = \det\langle \omega_i, e_j\rangle. \tag{4.1.1}$$

Désignant encore par e_i un cycle représentatif, on a $\langle \omega_i, e_j\rangle = \int_{e_j} \omega_i$.

Prenons pour base (e_i) une base sur $\boldsymbol{Z}$ de $H_1(A(\boldsymbol{R})^\circ, \boldsymbol{Z}) \subset H_1(A(\boldsymbol{C}), \boldsymbol{Z})$. Le produit de Pontryagin des e_i est representé par le d-cycle $A(\boldsymbol{R})^\circ$ dans $A(\boldsymbol{C})$, pour une orientation convenable, et, si ω est le produit extérieur des ω_i, le déterminant (4.1.1) est l'intégrale $\int_{A(\boldsymbol{R})^\circ} \omega$. On a

$$\int_{A(\boldsymbol{R})} |\omega| = [A(\boldsymbol{R}): A(\boldsymbol{R})^\circ]\left|\int_{A(\boldsymbol{R})^\circ} \omega\right|,$$

et 1.8 pour $H^1(A)(1)$ équivaut donc à 4.1(b) ci-dessus.

4.2. La conjecture de Birch et Swinnerton-Dyer donne la valeur exacte de $L(H^1(A), 1)$; la description du facteur rationnel en 4.1(b) à partir du motif $H^1(A)(1)$ a la forme suivante:

(a) La donnée de $M = H^1(A)(1)$ équivaut à celle de A à isogénie près. Il faut commencer par choisir A. Cela revient à choisir un réseau entier dans $H_B(M)$, dont les l-adifiés soient stables sous l'action de $\mathrm{Gal}(\bar{\boldsymbol{Q}}/\boldsymbol{Q})$.

(b) On choisit alors ω, par example comme produit extérieur des éléments d'une base d'un réseau entier dans $H_{DR}(M)$. Cet ω détermine la période, $c^+(M)$, et, pour chaque nombre premier p, un facteur rationnel $c_p(M)$. Les $c_p(M)$ sont presque tous égaux à 1, et la formule du produit assure que $c^+(M) \cdot \prod_p c_p(M)$ est indépendant de ω.

(c) Un autre nombre rationnel, $h(M)$, est défini en terme d'invariants cohomologiques de A; le nombre rationnel cherché est $h(M) \cdot \prod_p c_p(M)^{-1}$.

L'invariance de la conjecture par isogénie est par ailleurs un théorème non trivial.

4.3. Pour généraliser 4.1(a) à un motif de poids -1 M quelconque, il faudrait disposer d'un analogue de $A(\boldsymbol{Q})$. Le groupe $A(\boldsymbol{Q})$ peut s'interpréter comme le groupe des extensions de $\boldsymbol{Z}(0)$ par $H_1(A)$, dans la catégorie des 1-motifs [**6**, §10] sur $\boldsymbol{Q}$. Ceci suggère de considérer le groupe des extensions de $\boldsymbol{Z}(0)$ par M, dans une catégorie de motifs mixtes, parallèles aux structures de Hodge mixtes, mais on ne dispose même pas d'une définition conjecturale d'une telle catégorie!

J'observerai seulement que, dans toutes les théories cohomologiques usuelles, un cycle Y de dimension d, cohomologue à zéro, sur une variété algébrique propre et lisse X détermine un torseur sous $H^{2d-1}(X)(d)$: on dispose d'une suite exacte de cohomologie

$$0 \longrightarrow H^{2d-1}(X)(d) \longrightarrow H^{2d-1}(X-Y)(d) \xrightarrow{\partial} H^{2d}_Y(X)(d) \longrightarrow H^{2d}(X)(d),$$

Y définit une classe de cohomologie $\mathrm{cl}(Y) \in H^{2d}_Y(X)(d)$, d'image nulle dans $H^{2d}(X)(d)$, et on prend $\partial^{-1}\mathrm{cl}(Y)$. Cette construction correspond à celle qui à un diviseur de degré 0 sur une courbe associe un point de sa jacobienne.

5. Compatibilité à l'équation fonctionnelle. Soit E un corps de nombres. Dans $(E \otimes \boldsymbol{C})^*$, nous noterons $\sim$ la relation d'équivalence définie par le sous-groupe E^*.

PROPOSITION 5.1. *Soit M un motif sur $\boldsymbol{Q}$, à coefficient dans E. On suppose vérifiées les hypothèses de* 1.7. *On a alors*

$$c^+(M) \sim (2\pi i)^{-d^-(M)} \cdot \delta(M) \cdot c^+(\check{M}(1)).$$

Pour L un module libre de rang n sur un anneau commutatif A (A sera E, ou $E \otimes \boldsymbol{C}$), posons $\det L = \bigwedge^n L$. On étend par localisation cette définition au cas où L est seulement projectif de type fini (cette généralisation n'est pas indispensable à la preuve de 5.1). On a des isomorphismes canoniques

$$\det(L^\vee) = \det(L)^{-1} \tag{5.1.1}$$

et, pour tout facteur direct P et L,

$$\det(L) = \det(P) \cdot \det(L/P) \tag{5.1.2}$$

(on a désigné par $\cdot$ un produit tensoriel, et par $^{-1}$ un dual). Il y a ici des problèmes de signe, qu'on résoud au mieux en considérant $\det(L)$ comme un module gradué inversible, placé en degré le rang de L. Nos résultats finaux étant modulo $\sim$, nous ne nous en inquiéterons pas.

Pour X et Y de même rang, posons $\delta(X, Y) = \mathrm{Hom}(\det X, \det Y) = \det(X)^{-1} \cdot \det(Y)$. Le *déterminant* de $f: X \to Y$ est dans $\delta(X, Y)$. On déduit de (5.1.1) et (5.1.2) des isomorphismes

$$\delta(X, Y) = \delta(Y^\vee, X^\vee) \tag{5.1.3}$$

et, pour $F \subset X$ et $G \subset Y$,

$$\delta(X,Y) = \delta(F, Y/G) \cdot \delta(G, X/F)^{-1}. \tag{5.1.4}$$

LEMME 5.1.5. *Via l'isomorphisme* (5.1.3), *on a* $\det(u) = \det({}^t u)$.

LEMME 5.1.6. *Soient F et G des facteurs directs de X et Y. On suppose que l'isomorphisme $f: X \to Y$ induit des isomorphismes $f_F: F \to Y/G$ et $f_G^{-1}: G \to X/F$. Via l'isomorphisme* (5.1.4), *on a* $\det(f) = \det(f_F) \det(f_G^{-1})^{-1}$.

La vérification de ces lemmes est laissée au lecteur. Pour 5.1.6, notons seulement que, pour $G^1 = f^{-1}(G)$ et $F^1 = f(F)$, on a $X = F \oplus G^1$, $Y = G \oplus F^1$, et que f échange F et F^1, G et G^1.

Appliquons le Lemme 5.1.6 aux complexifiés de $H_B^+(M) \subset H_B(M)$ et de $F^- \subset H_{DR}(M)$. On trouve que, via l'isomorphisme (5.1.4), le déterminant de I : $H_B(M) \otimes \boldsymbol{C} \to H_{DR}(M) \otimes \boldsymbol{C}$ est le produit du déterminant de I^+: $H_B^+(M) \otimes \boldsymbol{C} \to$

$(H_{DR}(M)/F^-) \otimes \mathbf{C}$ et de l'inverse du déterminant du morphisme induit par l'inverse de I:

$$J^-: F^- \otimes \mathbf{C} \longrightarrow (H_B(M) \,/\, H_B^+(M)) \otimes \mathbf{C}.$$

Le morphisme J^- est le transposé du morphisme I^- pour le motif $\check{M}$ dual de M. Appliquant 5.1.5, et prenant des bases sur E des espaces $\delta(X, Y)$ en jeu, on obtient finalement que

$$\delta(M) \sim c^+(M) \cdot c^-(\check{M})^{-1}. \tag{5.1.7}$$

On en déduit 5.1 en appliquant au motif $\check{M}$ la formule suivante, conséquence de 3.1:

$$c^\pm(M) = (2\pi i)^{-d^\pm(M)\cdot n}\, c^{\pm(-1)^n}(M(n)). \tag{5.1.8}$$

Notons aussi, pour usage ultérieur, la formule analogue

$$\delta(M) = (2\pi i)^{-d(M)\cdot n}\, \delta(M(n)), \tag{5.1.9}$$

5.2. Rappelons la forme exacte de l'équation fonctionnelle conjecturale des fonctions L des motifs ([**12**], [**4**]). Nous nous placerons dans le cas général d'un motif sur un corps de nombres k, à coefficient dans un corps E muni d'un plongement complexe σ (cf. 2.9). La forme générale d'abord:

(a) Pour chaque place v de k, on définit un facteur local $L_v(\sigma, M, s)$. Pour v fini, la définition de $L_v(\sigma, M, s) = \sigma Z_v(M, Nv^{-s})$ $(Z_v(M, t) \in E(t))$ dépend d'une hypothèse de compatibilité analogue à (1.2.1). On a $Z_v(M, t) = \det(1 - F_v t, H_\lambda(M)^{I_v})^{-1}$. Pour v infini, induit par un plongement complexe τ, on obtient $L_v(\sigma, M, s)$ en décomposant $H_\tau(M) \otimes_{E,\sigma} \mathbf{C}$ en somme directe de sous-espaces minimaux stables par les projecteurs qui donnent la décomposition de Hodge, et par F_∞ pour v réel, en associant à chacun le "facteur Γ" de la Table (5.3), et en prenant leur produit. Pour v complexe, les sous-espaces minimaux sont de dimension un, d'un type (p, q). Pour v réel, il sont soit de dimension 2, de type $\{(p, q), (q, p)\}$, $p \neq q$, soit de dimension 1, de type (p, p), avec $F_\infty = \pm 1$. On note $\Lambda(\sigma, M, s)$ le produit des $L_v(\sigma, M, s)$.

(b) Soient Ψ un caractère non trivial du groupe $\mathbf{A} \otimes k/k$ des classes d'adèles de k, et Ψ_v ses composantes. Soient aussi, pour chaque place v de k, une mesure de Haar dx_v sur k_v. On suppose que, pour presque tout v, dx_v donne aux entiers de k_v la masse 1, et que le produit $\bigotimes dx_v$ des dx_v est la mesure de Tamagawa, donnant la masse 1 au groupe des classes d'adèles.

On définit des constantes locales $\varepsilon_v(\sigma, M, s, \Psi_v, dx_v)$, presque toutes égales à 1 et toutes, comme fonction de s, produit d'une constante par une exponentielle. Soit $\varepsilon(\sigma, M, s)$ leur produit (indépendant de Ψ et des dx_v).

(c) L'équation fonctionnelle conjecturale est

$$\Lambda(\sigma, M, s) = \varepsilon(\sigma, M, s)\Lambda(\sigma, \check{M}, 1 - s).$$

Pour définir les ε_v, une hypothèse additionnelle de compatibilité entre les $H_\lambda(M)$ est requise. Elle permet d'associer à v, σ, M une classe d'isomorphie de représentations complexes du groupe de Weil $W(\bar{k}_v/k_v)$ pour v infini, du groupe de Weil épaissi $'W(\bar{k}_v/k_v)$ pour v fini, et on prend le ε de [**4**, 8.12.4], avec $t = p^{-s}$.

Pour v infini, ceci revient à décomposer $H_\tau(M) \otimes_{E,\sigma} \mathbf{C}$ comme en (a), à associer à chaque sous-espace de la décomposition un facteur ε'_v, et à prendre leur produit. La table des ε'_v, pour un choix particulier de Ψ_v et de dx_v, est donnée en 5.3.

Pour v fini, on commence par restreindre les représentations $H_\lambda(M)$ à un groupe de décomposition $\mathrm{Gal}(\bar{k}_v/k_v) \subset \mathrm{Gal}(\bar{k}/k)$, puis au groupe de Weil $W(\bar{k}_v/k_v)$. Appliquant [**4**, 8.3, 8.4], on déduit de $H_\lambda(M)$ une classe d'isomorphie ρ_λ de représentations de $'W(\bar{k}_v/k_v)$ sur E_λ. Il est loisible ici, et utile, de la remplacer par sa F-semi-simplifiée [**4**, 8.6]. On demande que ces représentations, pour λ variable, soient compatibles, i.e., que si on étend les scalaires de E_λ à $\boldsymbol{C}$, par $\tilde{\sigma}: E_\lambda \to \boldsymbol{C}$ prolongeant σ, la classe d'isomorphie de la représentation obtenue soit indépendante de λ et de σ. C'est la classe d'isomorphie cherchée.

Remontons pour le lecteur les renvois internes de [**4**]. Une représentation de $'W$ est donnée par une représentation ρ du groupe de Weil dans $\mathrm{GL}(V)$, et par un endomorphisme nilpotent N de V. En terme de la constante locale [**4**, 4.1], de ρ, celle de (ρ, N) est donnée par

$$\varepsilon((\rho, N), s, \Psi, dx) = \varepsilon(\rho \otimes \omega_s, \Psi\, dx) \cdot \det(-FNv^{-s}, V^{\rho(I)}/\mathrm{Ker}(N)^{\rho(I)}).$$

Remarque 5.2.1. La fonction de s

$$\varepsilon((\rho, N), s, \Psi, dx)L((\rho, N)^\vee, 1 - s)L((\rho, N), s)^{-1}$$

est la même pour (ρ, N) et pour $(\rho, 0)$. Ceci permet d'énoncer l'équation fonctionnelle conjecturale des fonctions L en supposant seulement une compatibilité entre les semi-simplifiées des restrictions des $H_\lambda(M)$ à un groupe de décomposition.

5.3. Dans la table ci-dessous, on donne les facteurs locaux, et les constantes, associées aux divers types de sous-espaces minimaux de la réalisation de Hodge. Pour les constantes, on a supposé Ψ_v et la mesure dx_v choisis comme suit:

v réel: $\Psi_v(x) = \exp(2\pi ix)$, mesure dx,

v complexe: $\Psi_v(z) = \exp(2\pi i\ \mathrm{Tr}_{\boldsymbol{C}/\boldsymbol{R}}(z))$, mesure $|dz \wedge d\bar{z}|$,

soit pour $z = x + iy$, $\exp(4\pi ix)$ et $2dxdy$.

On utilise les notations $\Gamma_{\boldsymbol{R}}(s) = \pi^{-s/2}\,\Gamma(s/2)$, $\Gamma_{\boldsymbol{C}}(s) = 2 \cdot (2\pi)^{-s}\,\Gamma(s)$.

place	type	facteur Γ	constante (pour Ψ, dx ci-dessus)
complexe	(p, q) ou (q, p), $p \leqq q$	$\Gamma_{\boldsymbol{C}}(s-p)$	i^{q-p}
réelle	$\{(p, q), (q, p)\}$, $p < q$	$\Gamma_{\boldsymbol{C}}(s - p)$	i^{q-p+1}
	(p, q), $F_\infty = (-1)^{p+\varepsilon}$, $\varepsilon = 0$ ou 1	$\Gamma_{\boldsymbol{R}}(s+\varepsilon-p)$	i^ε

Le cas qui nous intéresse est celui où $k = \boldsymbol{Q}$. On peut dans ce cas prendre $\Psi_\infty(x) = \exp(2\pi ix)$, $\Psi_p(x) = \exp(-2\pi ix)$ (via l'isomorphisme $\boldsymbol{Q}_p/\boldsymbol{Z}_p$ = partie p-primaire de $\boldsymbol{Q}/\boldsymbol{Z}$), dx_∞ = mesure de Lebesgue dx, dx_p = la mesure de Haar sur $\boldsymbol{Q}_p$ donnant à $\boldsymbol{Z}_p$ la masse 1.

Proposition 5.4. *Si M est critique de poids w, on a, modulo un nombre rationnel indépendant de σ,*

$$L_\infty(\sigma, \check{M}(1))L_\infty(\sigma, M)^{-1} \sim (2\pi)^{-d^-(M)} \cdot (2\pi)^{-wd(M)/2}.$$

D'après 2.5, les $L_\infty(\sigma, \ )$ sont indépendant de σ. Ceci nous permet de ne vérifier 5.4 que pour σ fixé. La formule est compatible à la substitution $M \mapsto \check{M}(1)$: $\check{M}(1)$ est de poids $-2-w$, son d^- est $d^+(M)$ et abrégeant $d(M)$ et $d^\pm(M)$ en d et $d^\pm$, on a $d = d^+ + d^-$ et

$$\left(-d^- - \frac{wd}{2}\right) + \left(-d^+ - \frac{(-2-w)d}{2}\right) = 0.$$

Ceci permet de ne vérifier 5.4 que pour $w \geqq -1$.

Pour s entier, on a, modulo $\boldsymbol{Q}^*$,

$$\begin{aligned} &\Gamma_{\boldsymbol{R}}(s) \sim (2\pi)^{-s/2} && \text{pour } s \text{ pair} > 0, \\ &\Gamma_{\boldsymbol{R}}(s) \sim (2\pi)^{(1-s)/2} && \text{pour } s \text{ impair}, \\ &\Gamma_{\boldsymbol{C}}(s) \sim (2\pi)^{-s} && \text{pour } s > 0. \end{aligned} \tag{5.4.1}$$

Pour $w \geqq -1$, la puissance de 2π dans la contribution de chaque sous-espace de $H_B(M) \otimes \boldsymbol{C}$, comme en 5.2 (a), est donc donnée par

	$L_\infty(M)$	$L_\infty(\check{M}(1))$	$L_\infty(\check{M}(1))L_\infty(M)^{-1}$
$\{(pq), (qp)\}, p \leqq q$	p	$-1-p$	$-1-w$
(pp), p pair $\geqq 0$, $F_\infty = -1$	$\frac{p}{2}$	$-1-\frac{p}{2}$	$-1-\frac{w}{2}$
(pp), p impair $\geqq 0$, $F_\infty = -1$	$\frac{1+p}{2}$	$\frac{-1-p}{2}$	$-1-\frac{w}{2}$

La proposition en résulte aussitôt.

Posons $\det M = \bigwedge^{d(M)} M$ (puissance extérieure sur E).

PROPOSITION 5.5. $\varepsilon^*(M) \sim \varepsilon^*(\det M)$.

Pour des dx_v choisis comme suggéré en 5.4, nous prouverons plus précisément des équivalences

$$\varepsilon_v^*(M, \Psi_v, dx_v) \sim \varepsilon_v^*(\det M, \Psi_v, dx_v). \tag{5.5.1}$$

Posant $\eta_v(\sigma, M, \Psi_v) = \varepsilon_v(\sigma, M, \Psi_v, dx_v)\, \varepsilon_v^{-1}(\sigma, \det M, \Psi_v, dx_v)$, ceci équivaut à

$$\tau\eta_v(\sigma, M, \Psi_v) = \eta_v(\tau\sigma, M, \Psi_v) \tag{5.5.2}$$

pour tout automorphisme τ de $\boldsymbol{C}$.

Il résulte de [**4**, 5.4] que, si $a \in \boldsymbol{Q}_v^*$ est de valeur absolue 1, et qu'on pose $(\Psi_v \cdot a)(x) = \Psi_v(ax)$, on a

$$\eta_v(\sigma, M, \Psi_v) = \eta_v(\sigma, M, \Psi_v \cdot a) \quad (\text{pour } \|a\|_v = 1). \tag{5.5.3}$$

Pour τ un automorphisme de $\boldsymbol{C}$, et v fini, $\tau\Psi_v$ est de cette forme $\Psi_v \cdot a$ avec $\|a\|_v = 1$; de même, $\bar{\Psi}_\infty = \Psi_\infty \cdot (-1)$.

Pour v fini, la définition de ε_v est purement algébrique, d'où $\tau\eta_v(\sigma, M, \Psi_v) = \eta_v(\tau\sigma, M, \tau\Psi_v)$ et (5.5.2) résulte de (5.5.3).

Pour $v = \infty$, on a encore $\bar{\eta}_\infty(\sigma, M, \bar{\Psi}_\infty) = \eta_\infty(\bar{\sigma}, M, \bar{\Psi}_\infty) = \eta_\infty(\bar{\sigma}, M, \Psi_\infty)$; si on prend Ψ_∞ comme suggéré en 5.3, η_∞ est une puissance de i indépendante de σ, d'où $\eta_\infty(\sigma, M, \Psi_\infty) = \pm 1$, indépendant de σ, ce qui vérifie (5.5.2).

THÉORÈME 5.6. *Modulo la Conjecture* 6.6 *sur la nature des motifs de rang* 1, *la Conjecture* 2.8 *est compatible à l'équation fonctionnelle conjecturale des fonctions* L: *on a*

$$L_\infty^*(M)\, c^+(M) \sim \varepsilon^*(M) L_\infty^*(\check{M}(1))\, c^+(\check{M}(1)).$$

D'après 5.1, 5.3 et 5.5, cette formule équivaut à

$$(2\pi i)^{-d^-(M)} \cdot \delta(M) \sim (2\pi)^{-d^-(M)} \cdot (2\pi)^{-wd(M)/2} \cdot \varepsilon^*(\det M).$$

Posons $D = \det M$ et $\varepsilon = d^-(D)$. On a $\delta(M) = \delta(D)$, $d^-(M) \equiv \varepsilon \pmod 2$, et $wd(M)$ est le poids $w(D)$ de D, de sorte que la formule équivaut encore à

(5.6.1) $$\varepsilon^*(D) \sim (2\pi)^{w(D)/2} \cdot i^\varepsilon \cdot \delta(D).$$

Nous prouverons en 6.5 que (5.6.1) vaut pour une classe de motifs de rang 1 qui, conjecturalement (6.6), les englobe tous.

6. Exemple: les fonctions L d'Artin.

DÉFINITION 6.1. *La* catégorie des motifs d'Artin *est l'enveloppe karoubienne de la duale de la catégorie d'objets les variétés sur $\mathbf{Q}$ de dimension* 0, *de morphismes les correspondances définies sur $\mathbf{Q}$.*

Par définition, chaque variété de dimension 0, X, définit un motif d'Artin $H(X)$, le foncteur H est un foncteur contravariant pleinement fidèle,

$$H: (\text{variétés de dim } 0, \text{correspondances}) \longrightarrow (\text{motifs d'Artin})$$

et tout motif d'Artin est facteur direct d'un $H(X)$.

6.2. Explicitons cette définition. Soient $\bar{\mathbf{Q}}$ une clôture algébrique de $\mathbf{Q}$, et G le groupe de Galois $\mathrm{Gal}(\bar{\mathbf{Q}}/\mathbf{Q})$. Une variété de dimension 0 est le spectre d'un produit fini A de corps de nombres, et la théorie de Galois (sous la forme que lui a donnée Grothendieck) dit que le foncteur

$$X = \mathrm{spec}(A) \mapsto X(\bar{\mathbf{Q}}) = \mathrm{Hom}(A, \bar{\mathbf{Q}}):$$

(catégorie des variétés de dimension 0, sur $\mathbf{Q}$, et des morphismes de schémas) $\to$ (catégorie des ensembles finis munis d'une action continue de G) est une équivalence de catégorie. Le foncteur inverse est $I \mapsto$ spectre de l'anneau des fonctions G-invariantes de I dans $\bar{\mathbf{Q}}$.

Une *correspondance* d'une variété de dimension 0, X dans une autre, Y, est une combinaisons linéaire formelle à coefficients dans $\mathbf{Q}$ de composantes connexes de $X \times Y$. L'application

$$\sum a_i Z_i \mapsto \sum a_i(\text{fonction caractéristique de } Z_i(\bar{\mathbf{Q}}) \subset (X \times Y)(\bar{\mathbf{Q}}))$$

identifie correspondances et fonctions G-invariantes, à valeurs rationnelles, sur $(X \times Y)(\bar{\mathbf{Q}}) = X(\bar{\mathbf{Q}}) \times Y(\bar{\mathbf{Q}})$, et la composition des correspondances au produit matriciel.

Notons H le foncteur contravariant $X \mapsto$ espace vectoriel $\mathbf{Q}^{X(\bar{\mathbf{Q}})}$, muni de l'action naturelle de G; (correspondance $F: X \to Y) \mapsto$ le morphisme $F^*: \mathbf{Q}^{Y(\bar{\mathbf{Q}})} \to \mathbf{Q}^{X(\bar{\mathbf{Q}})}$ de matrice ${}^tF$. Il est pleinement fidèle, et identifie la catégorie des motifs d'Artin à celle des représentations rationnelles de G.

6.3. Dans ce modèle, si $\bar{\mathbf{Q}}$ est la clôture algébrique de $\mathbf{Q}$ dans $\mathbf{C}$, le foncteur "réalisation de Betti" H_B est le foncteur "espace vectoriel sous-jacent". La structure de Hodge est purement de type (0, 0), et l'involution F_∞ est l'action de la conjugaison complexe $F_\infty \in G$. On a en effet un isomorphisme, fonctoriel pour les correspondances,

$$H^*(X(C), Q) = Q^{X(C)} = Q^{X(\bar{Q})} = H(X).$$

Le foncteur "réalisation l-adique" H_l est le foncteur $H_l(V) = V \otimes Q_l$. On a en effet un isomorphisme, fonctoriel pour les correspondances,

$$H^*(X(\bar{Q}), Q_l) = Q_l^{X(\bar{Q})} = Q^{X(\bar{Q})} \otimes Q_l.$$

Calculons de même la réalisation de de Rham. Pour $X = \mathrm{Spec}(A)$, on a $H^*_{DR}(X) = A$. Ecrivant que $A = \mathrm{Hom}_G(X(\bar{Q}), \bar{Q}) = (Q^{X(\bar{Q})} \otimes \bar{Q})^G$, on obtient que

(6.3.1) $$H_{DR}(V) = (V \otimes \bar{Q})^G.$$

La formule (6.3.1) réalise $H_{DR}(V)$ comme un sous-espace de $V \otimes \bar{Q}$. Ce sous-espace est une Q-structure: on a $(V \otimes \bar{Q})^G \otimes \bar{Q} \xrightarrow{\sim} V \otimes \bar{Q}$. Après extension des scalaires à $\bar{Q}$, $H_B(V)$ et $H_{DR}(V)$ sont donc canoniquement isomorphes. Etendant les scalaires jusqu'à C, on trouve l'isomorphisme (0.4.1) (le vérifier pour $V = H(X)$).

6.4. Soit E une extension finie de Q. La *catégorie des motifs d'Artin à coefficient dans E* est la catégorie déduite de celle des motifs d'Artin comme en 2.1. Nous l'identifierons à la catégorie des E-espaces vectoriels de dimension finie, munis d'une action de G.

Les motifs d'Artin à coefficient dans E, de rang 1 sur E, correspondent aux caractères $\varepsilon: G \to E^*$. Nous allons calculer leurs périodes. Soient donc $\varepsilon: G \to E^*$ et f le conducteur de ε: ε se factorise par un caractère, encore noté ε, du quotient $(Z/fZ)^* = \mathrm{Gal}(Q(\exp(2\pi i/f))/Q)$ de G. Notons $[\varepsilon]$ l'espace vectoriel E_ε, de dimension 1 sur E, sur lequel G agit par ε. La somme de Gauss

$$g = \sum \varepsilon(u) \otimes \exp(2\pi iu/f) \in [\varepsilon] \otimes \bar{Q}$$

est non nulle, et invariante par G: c'est une base, sur E, de $H_{DR}([\varepsilon])$. Le déterminant de $I: H_B([\varepsilon]) \otimes C \xrightarrow{\sim} H_{DR}([\varepsilon]) \otimes C$, calculé dans les bases 1 et g, vaut g^{-1}. On sait que, pour tout plongement complexe σ de E, on a $\sigma g \cdot \bar{\sigma} g = f$, nombre rationnel indépendant de σ, d'où

(6.4.1) $$\delta([\varepsilon]) \sim \sum \varepsilon^{-1}(u) \otimes \exp(-2\pi iu/f) \in (E \otimes C)^*.$$

PROPOSITION 6.5. *Soit D le motif $[\varepsilon](n)$. C'est un motif sur Q, à coefficients dans E, de rang* 1 *et de poids* $-2n$. *Posant* $\varepsilon(-1) = (-1)^\eta$, *avec* $\eta = 0$ *ou* 1, *on a*

$$\varepsilon^*(D) \sim (2\pi)^{-n} i^{\eta-n} \delta(D).$$

On sait que la constante de l'équation fonctionnelle de la fonction L de Dirichlet $L(\sigma, [\varepsilon], s) = \sum \sigma\varepsilon^{-1}(n) \cdot n^{-s}$ (σ plongement complexe de E) est donnée par

$$\varepsilon(\sigma, [\varepsilon], s) = i^\eta \cdot f^s \cdot \sum \sigma\varepsilon(u)^{-1} \exp(-2\pi iu/f).$$

D'après (5.1.9), on a par ailleurs $\delta(D) = (2\pi i)^n \delta([\varepsilon])$. On conclut en appliquant (6.4.1) et en notant que pour s entier ($s = n$), f^n est rationnel indépendant de σ.

Cette proposition vérifie (5.6.1) pour les motifs $[\varepsilon](n)$. Pour achever la preuve de 5.6, il ne reste qu'à énoncer la

Conjecture 6.6. *Tout motif sur Q, à coefficient dans E et de rang* 1 *est de la forme* $[\varepsilon](n)$, *pour ε un caractère de G* = Gal $(\bar{Q}/Q)$ *à valeurs dans les racines de l'unité de E et n un entier.*

PROPOSITION 6.7. *La Conjecture* 2.8 *est vraie pour les fonctions L d'Artin.*

Pour les motifs d'Artin, on dispose de l'équation fonctionnelle des fonctions L. De plus, le déterminant d'un motif d'Artin, tordu à la Tate, est du type prédit en 6.6. Les arguments des paragraphes 5, 6 montrent donc la compatibilité de 2.8 à l'équation fonctionnelle, et il suffit de prouver 2.8 pour les motifs $V(n)$, pour V un motif d'Artin et n un entier $\leqq 0$. Si $V(n)$ est critique, F_∞ agit alors sur V par multiplication par $-(-1)^n$, et $c^+ = 1$ (cf. 3.1), et il s'agit de prouver que, pour tout plongement σ de E dans $\boldsymbol{C}$ et tout automorphisme de $\boldsymbol{C}$, on a $\tau L(\sigma, V(n)) = L(\tau\sigma, V(n))$.

On le déduit des résultats de Siegel **[14]**. Voir **[2**, 1.2].

7. Fonctions L attachées aux formes modulaires.

7.1. Posons $q = e^{2\pi i z}$, et soit $f = \sum a_n q^n$ une forme modulaire holomorphe cuspidale primitive (new form) de poids $k \geqq 2$, conducteur N et caractère ε. La série de Dirichlet $\sum a_n n^{-s}$ admet un développement en produit eulérien, de facteur local en $p \nmid N$ égal à $(1 - a_p p^{-s} + \varepsilon(p)p^{k-1} p^{-2s})^{-1}$.

Soit E le sous-corps de $\boldsymbol{C}$ engendré par les a_n. La forme f doit donner lieu à un motif $M(f)$ de rang 2 à coefficient dans E, de type de Hodge $\{(k-1, 0), (0, k-1)\}$, de déterminant $[\varepsilon^{-1}](1-k)$ (notations de 6.4) et de fonction L la série de Dirichlet $\sum a_n n^{-s}$.

Je n'ai pas essayé de définir $M(f)$ comme étant un motif au sens de Grothendieck. Une difficulté est que $M(f)$ apparaît de façon naturelle comme facteur direct dans la cohomologie d'une variété non compacte, ou encore comme facteur direct dans la cohomologie d'une courbe modulaire complète, à coefficient dans l'image directe d'un faisceau localement constant (plutôt, un système local de motifs!) sur la partie à distance finie de cette courbe modulaire. Ceci échappe au formalisme de Grothendieck, mais permet de définir les réalisations du motif $M(f)$.

7.2. Supposons tout d'abord que $k = 2$, et que ε est trivial. Soient X le demi-plan de Poincaré, N le conducteur de f, $\Gamma_0(N)$ le sous-groupe de $\mathrm{SL}(2, \boldsymbol{Z})$ formé des matrices dont la réduction mod N est de la forme $\left(\begin{smallmatrix} * & * \\ 0 & * \end{smallmatrix}\right)$, et posons $\omega_f = \sum a_n q^n \cdot dq/q = \sum a_n q^n \cdot 2\pi i dz$. La forme ω_f est une forme différentielle holomorphe sur la courbe complétée $\bar{M}(\Gamma_0(N))$ de $M(\Gamma_0(N)) = X/\Gamma_0(N)$. Elle est vecteur propre des correspondances de Hecke:

$$(7.2.1) \qquad T_n^* \omega_f = a_n \omega_f \quad (\text{pour } n \text{ premier à } N)$$

et est caractérisée à un facteur près par (7.2.1). Ce fait résulte du théorème fort de multiplicité 1 et de la théorie des formes primitives; le lecteur peut, s'il le préfère compléter (7.2.1) par une condition analogue pour n non premier à N, et faire de même ci-dessous; l'assertion devient alors élémentaire, car la condition (7.2.1) ainsi complétée détermine (à un facteur près) le développement de Taylor de ω_f en la pointe $i\infty$.

Abrégeons $M(\Gamma_0(N))$ en M et $\bar{M}(\Gamma_0(N))$ en $\bar{M}$. Ces courbes ont une $\boldsymbol{Q}$-structure naturelle, pour laquelle les correspondances de Hecke sont définies sur $\boldsymbol{Q}$. La forme ω_f est définie sur E, et sa conjuguée par un automorphisme σ de $\boldsymbol{C}$ est $\omega_{\sigma f}$ (appliquer σ aux coefficients).

Définissons le motif $M(f)$ come étant, dans le langage 2.1(B), le sous-motif de $H^1(\bar{M})_E$ noyau des endomorphismes $T_n^* - a_n$—si on veut ne considérer que des noyaux de projecteurs, remplacer $T_n^* - a_n$ par $P(T_n - a_n\Delta)^*$ pour P un polynôme convenable. On sait que $M(f)$ a les propriétés énoncées en 7.1.

Dans chacune des théories cohomologiques $\mathscr{H}$ qui nous intéressent, l'application

$$\mathscr{H}^1_c(M) \longrightarrow \mathscr{H}^1(\bar{M}) \tag{7.2.2}$$

est surjective, et les systèmes de valeurs propres des T_n^* sur le noyau n'apparaissent pas dans $\mathscr{H}^1(\bar{M})$. La $\mathscr{H}$-réalisation de $M(f)$ est donc encore le noyau commun, dans $\mathscr{H}^1_c(M) \otimes E$, des $T_n^* - a_n$.

Le groupe de cohomologie $H^1_c(M, \boldsymbol{Q})$ est muni d'une structure de Hodge mixte, de type $\{(0, 0), (0, 1), (1, 0)\}$, dont $H^1(\bar{M}, \boldsymbol{Q})$ est le quotient de type $\{(0, 1), (1, 0)\}$. En particulier, (7.2.2) induit un isomorphisme sur les sous-espaces F^1 de la filtration de Hodge; toute forme différentielle holomorphe ω sur $\bar{M}$ définit ainsi une classe de cohomologie à support propre sur M. Ceci peut se voir directement: si $\mathscr{T} \subset \mathscr{O}$ est l'idéal des pointes, $\boldsymbol{H}^*_c(M)$, en cohomologie de de Rham, est l'hypercohomologie sur $\bar{M}$ du complexe $\mathscr{T} \to \Omega^1$ (analytiquement, ce complexe est une résolution du faisceau constant $\boldsymbol{C}$ sur M, prolongé par 0 sur $\bar{M}$), et le $\boldsymbol{H}^1$ reçoit $H^\circ(\bar{M}, \Omega^1)$.

Supposons pour simplifier que $E = \boldsymbol{Q}$, et calculons $c^+(M(f)(1))$. Le motif $M(f)(1)$ est le dual de $M(f)$ et c^+ est donc une période de $M(f)$ (1.7): il faut intégrer ω_f contre une classe d'homologie rationnelle de $\bar{M}$, fixe par F_∞. Relevant $M(f)$ dans $\mathscr{H}^1_c(M)$, on voit qu'on peut plutôt intégrer ω_f contre une classe d'homologie sans support de M, fixe par F_∞. Toutes les intégrales non nulles de ce type seront commensurables.

Pour calculer F_∞, il est utile d'écrire M comme quotient de $X^\pm = \boldsymbol{C} - \boldsymbol{R}$ par le sous-groupe de $\mathrm{GL}(2, \boldsymbol{Z})$ formé des matrices ayant pour réduction mod N une matrice $\left(\begin{smallmatrix} * & * \\ 0 & * \end{smallmatrix}\right)$. La conjugaison complexe est alors induite par $z \to \bar{z}$, et l'image dans $M(\boldsymbol{C})$ de $i\boldsymbol{R}^+$ est un cycle sans support fixe par F_∞. La formule $L(M(f), 1) = -\int_0^{i\infty} \omega_f$ justifie 1.8 pour $M(f)(1)$: le second membre est, ou bien nul, ou bien $\sim c^+(M(f)(1))$.

REMARQUE 7.3. On a aussi

$$c^\pm(M(f)(1)) \sim \int_{a/b}^{i\infty} \omega_f \pm \int_{-a/b}^{i\infty} \omega_f .$$

7.4. Pour ε (et E) quelconque, il faut remplacer $\Gamma_0(N)$ par $\Gamma_1(N)$: le sous-groupe de $\mathrm{SL}(2, \boldsymbol{Z})$ formé des matrices de réduction mod N de la forme $\left(\begin{smallmatrix} 1 & * \\ 0 & 1 \end{smallmatrix}\right)$. Comme ci-dessus, pour calculer F_∞, il est plus commode de travailler avec $X^\pm$ et le sous-groupe de $\mathrm{GL}(2, \boldsymbol{Z})$ forme des matrices de réduction mod N de la forme $\left(\begin{smallmatrix} 1 & * \\ 0 & * \end{smallmatrix}\right)$. Par ailleurs, une dualisation apparait, cachée en 7.2 par la symétrie de la correspondance T_n. Pour une définition convenable de T_n, on a

(a) $M(f)$ est le noyau commun des $T_n^* - a_n$ dans $H^1(\bar{M})$, $\bar{M} = X/\Gamma_1(N)$.

(b) On a ${}^tT_n^*\omega_f = a_n\omega_f$, et $T_n^*\omega_f = \bar{a}_n\omega_f$ (noter la formule $\bar{a}_n = \varepsilon(n)^{-1}a_n$), de sorte que ω_f est dans la réalisation de de Rham de $M(\bar{f}) = M(f)^\vee(-1)$.

On trouve que $c^+(M(f)(1)) \in (E \otimes \boldsymbol{C})^*/E^* \in \boldsymbol{C}^{*\mathrm{Hom}(E,\boldsymbol{C})}/E^*$ est donné par le système de périodes

$$c^+(M(f)(1)) \sim \left(\int_0^{i\infty} \omega_{\sigma f}\right)_\sigma ,$$

si ce dernier est non nul. Noter que si l'une de ces intégrales s'annule, elles s'annulent toutes. Tel est le cas si et seulement si, dans la partie de l'homologie sans support tendue par le cycle $i\mathbf{R}^+$ et ses transformés par les T_n, le système de valeurs propres a_n pour les T_n n'apparaît pas. Ceci justifie 2.8, et, partiellement 2.7, pour $M(f)(1)$.

REMARQUE 7.5. Les systèmes de valeurs propres des T_n dans $\mathrm{Ker}(\mathscr{H}^1_c(M) \to \mathscr{H}^1(\bar{M}))$ sont liés aux séries d'Eisenstein, alors que ceux qui apparaissent dans $\mathscr{H}^1(\bar{M})$ sont liés aux formes paraboliques. C'est pourquoi ces ensembles sont disjoints. Il en résulte que la structure de Hodge mixte de $H^1_c(M, \mathbf{Q})$ est somme de structures de Hodge: l'extension de $H^1(\bar{M}, \mathbf{Q})$ par $\mathrm{Ker}(H^1_c(M, \mathbf{Q}) \to H^1(\bar{M}, \mathbf{Q}))$ splitte. Ceci, généralisé au cas de n'importe quel sous-groupe de congruence de $\mathrm{SL}(2, \mathbf{Z})$, équivaut au théorème de Manin [9] selon lequel la différence entre deux pointes est toujours d'ordre fini dans la jacobienne (cf. [**6**, 10.3.4, 10.3.8 et 10.1.3]). La démonstration donnée ne diffère d'ailleurs pas en substance de celle de Manin.

7.6. En poids k quelconque, il faut remplacer la cohomologie de $\bar{M}$ par la cohomologie de $\bar{M}$ à coefficient dans un faisceau convenable. Pour décrire ce qui se passe, je remplacerai M et $\bar{M}$ par M_n et $\bar{M}_n$, relatifs au groupe de congruence $\Gamma(n)$, avec n multiple de N et $\geqq 3$. On redescend ensuite à M et $\bar{M}$ en prenant les invariants par un groupe fini convenable. Soient $g\colon E \to M_n$ la courbe elliptique universelle et j l'inclusion de M_n dans $\bar{M}_n$. La cohomologie à considérer est $H^1(\bar{M}_n, j_*\mathrm{Sym}^{k-2}(R^1g_*\mathbf{Q}))$. Les réalisations de $M(f)$ sont facteur direct de ce groupe, calculé dans la théorie de cohomologie correspondante. On peut comme précédemment les relever dans $H^1_c(M_n, \mathrm{Sym}^{k-2}(R^1g_*\mathbf{Q}))$. Pour calculer $c^\pm$ du dual de $M(f)$. il faut alors intégrer la classe dans H^1_c définie par f contre des classes d'homologie sans support de M_n à coefficient dans le système local dual de $\mathrm{Sym}^{k-2}(R^1g_*\mathbf{Q})$. Si on prend le cycle image de $i\mathbf{R}^+$, muni de diverses sections du dual (une base), on trouve les intégrales d'Eichler

$$\int_0^{i\infty} f(q) \cdot \frac{dq}{q} \cdot (2\pi i z)^l \qquad (0 \leqq l \leqq k-2)$$

(périodes paires pour l pair, impaires pour l impair)—et on sait comment écrire $L(M(f), n)$ pour n critique en terme de ces périodes.

PROPOSITION 7.7. *Soit M un motif de rang* 2, *à coefficient dans E, de type de Hodge* $\{(a, b), (b, a)\}$ *avec $a \neq b$. Alors, $d^\pm\mathrm{Sym}^n(M)$ et $c^\pm\mathrm{Sym}^n M$ sont donnés par les formules suivantes*:

(1) *si* $n = 2l + 1 : d^\pm = l + 1$, *et*

$$c^\pm\mathrm{Sym}^n M = c^\pm(M)^{(l+1)(l+2)/2}\, c^\mp(M)^{l(l+1)/2}\, \delta(M)^{l(l+1)/2};$$

(2) *si* $n = 2l$: $d^+ = l + 1, d^- = l$, *et*

$$c^+\mathrm{Sym}^n M = (c^+(M)\, c^-(M))^{l(l+1)/2}\, \delta(M)^{l(l+1)/2},$$
$$c^-\mathrm{Sym}^n M = (c^+(M)\, c^-(M))^{l(l+1)/2}\, \delta(M)^{l(l-1)/2}.$$

C'est une question de simple algèbre linéaire, que nous traiterons par "analyse dimensionnelle".

(a) $H_B(M)$ est un espace vectoriel de rang 2 sur E, muni d'une involution F_∞, de la forme $\left(\begin{smallmatrix}1 & 0\\ 0 & -1\end{smallmatrix}\right)$ dans une base e^+, e^- convenable. Les $d^\pm\mathrm{Sym}^n M$ sont les dimensions des parties $+$ et $-$ de la puissance symétrique $n^{\text{ième}}$, de bases respectives $\{e^{+^n}, e^{+^{(n-2)}} e^{-^2}, \cdots\}$ et $\{e^{+^{(n-1)}} e^-, e^{+^{(n-3)}} e^{-^3}, \cdots\}$ —d'où les valeurs annoncées.

(b) Soit ω, η une base du dual de $H_{DR}(M)$, telle que ω annule $F^+H_{DR}(M)$. Le sous-espace $F^\pm$ de $H_{DR}(\mathrm{Sym}^n M)^\vee \sim \mathrm{Sym}^n H_{DR}(M)^\vee$ admet pour base les ω^n, $\omega^{n-1}\eta, \cdots, \omega^{n-d^\pm+1}\,\eta^{d^\pm-1}$, et

$$(7.7.1)^+ \qquad c^+\mathrm{Sym}^n M = \langle \omega^n \wedge (\omega^{n-1}\eta) \wedge \cdots, e^{+^n} \wedge (e^{+^{(n-2)}} e^{-^2}) \wedge \cdots \rangle,$$

$$(7.7.1)^- \qquad c^-\mathrm{Sym}^n M = \langle \omega^n \wedge (\omega^{n-1}\eta) \wedge \cdots, (e^{+^{(n-1)}} e^-) \wedge (e^{+^{(n-3)}} e^{-^3}) \wedge \cdots \rangle.$$

(c) Posons $V = H_B(M) \otimes \boldsymbol{C} \sim H_{DR}(M) \otimes \boldsymbol{C}$; c'est un $E \otimes \boldsymbol{C}$-module. Les formules ci-dessus montrent que $c^\pm\mathrm{Sym}^n M$ ne dépend que du $E \otimes \boldsymbol{C}$-module V, de sa base e^+, e^-, de $\omega \in V^*$, et de l'image $\bar{\eta}$ de η dans $V^*/\langle\omega\rangle$: le $d^\pm$-vecteur à gauche du produit scalaire (7.7.1) ne change pas si on remplace η par $\eta + \lambda\omega$. Par ailleurs, $\langle \omega, e^+\rangle$ et $\langle \omega, e^-\rangle$ sont inversibles, et $\bar{\eta}$ est une base de $V^*/\langle\omega\rangle$. Le système $(V, e^+, e^-, \omega, \bar{\eta})$ est donc décrit à isomorphisme près par les quantités $c^+ = \langle \omega, e^+\rangle$, $c^- = \langle \omega, e^-\rangle$ et $\delta = \langle \omega \wedge \eta, e^+ \wedge e^-\rangle$, dans $(E \otimes \boldsymbol{C})^*$. Remplacer e^+, e^-, ω, $\bar{\eta}$ par λe^+, μe^-, ω, $\nu\bar{\eta}/\lambda\mu$ remplace c^+, c^- et δ par λc^+, μc^- et $\nu\delta$. Pour $c^+ = c^- = \delta = 1$, le second membre de $(7.7.1)^\pm$ est dans $\boldsymbol{Q}^*$. Dans le cas général $c^\pm\mathrm{Sym}^n M$ est donc un multiple rationnel du produit de $c^+(M)$, $c^-(M)$ et $\delta(M)/c^+(M)c^-(M)$ aux puissances respectives les degrés auquels figurent e^+, e^- et η dans (7.7.1). On s'épargne la moitié du calcul en notant que remplacer F_∞ par $-F_\infty$ échange e^+ et e^-, donc c^+ et c^-, respecte δ, et échange les $(7.7.1)^\pm$ pour n impair, les conserve pour n pair.

$n = 2l + 1 : d^+ = d^- = l + 1,$

$$\begin{aligned}
\deg \eta \text{ dans } (7.7.1)^\pm &= 0 + 1 + \cdots + l = l(l+1)/2,\\
\deg e^+ \text{ dans } (7.7.1)^+ &= (2l+1) + (2l-1) + \cdots + 1 = (l+1)^2\\
&= \deg e^- \text{ dans } (7.7.1)^-,\\
\deg e^+ \text{ dans } (7.7.1)^- &= 2l + (2l-2) + \cdots + 0 = l(l+1)\\
&= \deg e^- \text{ dans } (7.7.1)^+;
\end{aligned}$$

$n = 2l: d^+ = l + 1, d^- = l,$

$$\begin{aligned}
\deg \eta \text{ dans } (7.7.1)^+ &= 0 + 1 + \cdots + l = l(l+1)/2,\\
\deg \eta \text{ dans } (7.7.1)^- &= 0 + l + \cdots + (l-1) = l(l-1)/2,\\
\deg e^\pm \text{ dans } (7.7.1)^+ &= 2l + (2l-2) + \cdots + 0 = l(l+1),\\
\deg e^\pm \text{ dans } (7.7.1)^- &= (2l-1) + (2l-3) + \cdots + 1 = l^2.
\end{aligned}$$

7.8. Cette proposition nous fournit une conjecture pour les valeurs aux entiers critiques de $L(\mathrm{Sym}^n M(f), s)$. Voici les formules:

$$(7.8.1) \qquad L(\sigma, M(f), s) = \sum \sigma a_n n^{-s} = \prod_p L_p(\sigma, M(f), s)$$

où pour presque tout p

$$L_p(M(f), s) = (1 - a_p p^{-s} + \varepsilon(p)p^{k-1-2s})^{-1} = ((1-\alpha'_p p^{-s})(1-\alpha''_p p^{-s}))^{-1},$$

avec ε un caractère de Dirichlet. On a $\bigwedge^2 M(f) = [\varepsilon^{-1}](1-k)$, d'où

$$(7.8.2) \qquad \delta(M(f)) \sim (2\pi i)^{1-k} \cdot \sum \varepsilon(u) \otimes \exp(-2\pi i u/F),$$

si ε est de conducteur F, et

$$(7.8.3) \qquad L^*(M(f), m) \sim (2\pi i)^m c^\pm(M(f)), \quad \pm = (-1)^m, \text{ pour } 1 \leq m \leq k-1.$$

(7.8.4) $$L(\sigma, \mathrm{Sym}^n M(f), s) = \prod_p L_p(\sigma, \mathrm{Sym}^n M(f), s)$$

où pour presque tout p

$$L_p(\mathrm{Sym}^n M(f), s)^{-1} = \prod_{i=0}^{n} (1 - \alpha'^i_p \alpha''^{n-i}_p p^{-s});$$

conjecturalement, pour m critique, on a

$$L^*(\mathrm{Sym}^n M(f), m) \sim (2\pi i)^{md^{\pm}\mathrm{Sym}^n M(f)} \cdot c^{\pm}\mathrm{Sym}^n M(f), \qquad \pm = (-1)^m,$$

où $c^{\pm}$ et $d^{\pm}$ sont donnés en terme de $c^{\pm}(M)$ et $\delta(M)$ (caractérisés par (7.8.2), (7.8.3)) par les formules 7.7.

Pour l'évidence numérique en faveur de cette conjecture, voir **[18]**.

8. Caractères de Hecke algébriques. Le lecteur trouvera dans [**3**, paragraphe 5], dont nous utiliserons les notations, les définitions essentielles relatives aux caractères de Hecke algébriques (= grössencharaktere de type A_0).

Conjecture 8.1. *Soient k et E deux extensions finies de $\boldsymbol{Q}$, et χ un caractère de Hecke algébrique de k à valeurs dans E.*

(i) *Il existe un motif $M(\chi)$ de rang un sur k, à coefficients dans E, tel que, pour toute place λ de E, la représentation λ-adique $H_\lambda M(\chi)$ soit celle définie par χ: le Frobenius géométrique en $\mathscr{P}$ premier au conducteur de χ et à la caractéristique résiduelle l de λ agit par multiplication par $\chi(\mathscr{P})$.*

(ii) *Ce motif est caractérisé à isomorphisme près par cette propriété.*

(iii) *Tout motif de rang* 1 *est de la forme $M(\chi)$.*

(iv) *Décomposons $k \otimes E$ en produit de corps: $k \otimes E = \prod K_i$, et écrivons la partie algébrique $\chi_{\mathrm{alg}}: k^* \to E^*$ de χ sous la forme $\chi_{\mathrm{alg}}(x) = \prod N_{K_i/E}(x)^{n_i}$. La décomposition de $k \otimes E$ induit une décomposition de $H_{DR}(M(\chi))$ en les $H_{DR}(M(\chi))_i = H_{DR}(M(\chi)) \otimes_{k\otimes E} K_i$; avec cette notation, la filtration de Hodge est la filtration par les $\bigoplus_{n_i \geqq p} H_{DR}(M(\chi))_i$.*

L'unicité 8.1(ii) impose aux $M(\chi)$ le formalisme suivant

(8.1.1) $$M(\chi'\chi'') \sim M(\chi') \otimes M(\chi').$$

(8.1.2) Si $\iota: E \longrightarrow E'$ est une extension finie de E, $M(\iota\chi)$ se déduit de $M(\chi)$ par extension des coefficients de E à E'.

(8.1.3) Si k' est une extension finie de k, $M(\chi \circ N_{k'/k})$ se déduit de $M(\chi)$ par extension des scalaires de k à k'.

REMARQUE 8.2. Posons les notations:

c = la conjugaison complexe, $\bar{\boldsymbol{Q}}$ = la clôture algébrique de $\boldsymbol{Q}$ dans $\boldsymbol{C}$, $S = \mathrm{Hom}(k, \bar{\boldsymbol{Q}}) = \mathrm{Hom}(k, \boldsymbol{C})$, $J = \mathrm{Hom}(E, \bar{\boldsymbol{Q}}) = \mathrm{Hom}(E, \boldsymbol{C})$. La décomposition de $k \otimes E$ en les K_i correspond à la partition de $S \times J$ en les orbites de $\mathrm{Gal}(\bar{\boldsymbol{Q}}/\boldsymbol{Q})$.

Tout homomorphisme algébrique $\eta: k^* \to E^*$ s'écrit sous la forme $\prod N_{K_i/E}(x)^{n_i}$. Si l'orbite sous $\mathrm{Gal}(\bar{\boldsymbol{Q}}/\boldsymbol{Q})$ de $(\sigma, \tau) \in S \times I$ correspond à K_i, i.e., si $\sigma \otimes \tau: k \otimes E \to \boldsymbol{C}$ se factorise par K_i, nous noterons $n(\eta; \sigma, \tau)$, ou simplement $n(\sigma, \tau)$, l'entier n_i. La fonction $n(\sigma, \tau)$ est constante sur les orbites de $\mathrm{Gal}(\bar{\boldsymbol{Q}}/\boldsymbol{Q})$. Si η est la partie algébrique d'un caractère de Hecke χ, on a de plus

(8.2.1) L'entier $w = n(\eta; \sigma, \tau) + n(\eta; c\sigma, \tau)$ est indépendant de σ et τ.

C'est le *poids* de χ (et de $M(\chi)$). On écrira parfois $n(\chi; \sigma, \tau)$ pour $n(\chi_{\text{alg}}; \sigma, \tau)$.

Réciproquement, un homorphisme η vérifiant (8.2.1) est presque de la forme χ_{alg}, pour χ un caractère de Hecke convenable:

(a) une de ses puissances l'est;

(b) il existe une extension finie $\iota: E \to E'$ de E telle que $\iota\eta$ le soit;

(c) il existe une extension finie k' de k telle que $\eta \circ N_{k'/k}$ le soit.

La règle 8.1(iv) permet de déduire la bigraduation de Hodge de $H_\sigma M(\chi)$ de sa structure de E-module : le facteur direct $H_\sigma M(\chi) \otimes_{E,\tau} \boldsymbol{C}$ de $H_\sigma M(\chi) \otimes \boldsymbol{C}$ est de type de Hodge (p, q), avec $p = n(\chi; \sigma, \tau)$ et $q = w - p = n(\chi; \sigma, c\tau)$.

EXEMPLE 8.3. Soit A une variété abélienne sur k, à multiplication complexe par E. On suppose $H_1(A)$ de rang 1 sur E (type CM). Il résulte de la théorie de Shimura et Tanayama que $H_1(A)$ vérifie la condition de 8.1(i), pour un caractère de Hecke algébrique χ de k à valeurs dans E, et que la partie algébrique χ_{alg} de χ se lit sur le $k \otimes E$-module Lie(A):

$$\chi_{\text{alg}}(x) = \det_E(x \otimes 1, \text{Lie}(A))^{-1}$$

EXEMPLE 8.4. Prenons pour k le corps des racines $n^{\text{ièmes}}$ de l'unité, soit V l'hypersurface de Fermat d'équation projective $\sum_{i=0}^{m} X_i^n = 0$ et soit M le motif "cohomologie primitive de dimension moitié de V". Soit G le quotient de μ_n^{m+1} par son sous-groupe diagonal. Ce groupe agit sur V par $(\alpha_i) * (X_i) = (\alpha_i X_i)$, et sur M par transport de structure. Décomposant M à l'aide de la décomposition de l'algèbre de groupe $\boldsymbol{Q}[G]$ en produit de corps, on obtient des motifs vérifiant 8.1(i) pour des caractères de Hecke algébriques convenables: ceux introduits par Weil dans son étude des sommes de Jacobi.

EXEMPLE 8.5. Le motif $\boldsymbol{Z}(-1)$ vérifie 8.1(i) pour χ = la norme. Pour χ d'ordre fini, un motif d'Artin convenable vérifie 8.1(i).

8.6. Pour chaque $\sigma \in S$, $H_\sigma(M(\chi))$ est de rang 1 sur E. Choisissons une base e_σ de chaque $H_\sigma(M(\chi))$. Outre sa structure de E-module, la somme des $H_\sigma(M(\chi)) \otimes \boldsymbol{C}$ a une structure naturelle de $k \otimes \boldsymbol{C} = \boldsymbol{C}^S$-module —au total, une structure de $k \otimes E \otimes \boldsymbol{C}$-module libre de rang 1, pour laquelle $e = \sum e_\sigma$ est une base.

8.7. La réalisation de de Rham $H_{DR}(M(\chi))$ est un $k \otimes E$-module libre de rang 1. Choisissons en une base ω. La somme des $I_\sigma: H_\sigma(M(\chi)) \otimes \boldsymbol{C} \xrightarrow{\sim} H_{DR}(M(\chi)) \otimes_{k,\sigma} \boldsymbol{C}$ est l'isomorphisme (0.4) de $k \otimes E \otimes \boldsymbol{C}$-module

$$I: \bigoplus_\sigma H_\sigma(M(\chi)) \otimes \boldsymbol{C} \xrightarrow{\sim} H_{DR}(M(\chi)) \otimes_{\boldsymbol{Q}} \boldsymbol{C}.$$

Soit ω une base du $k \otimes E$-module $H_{DR}(M(\chi))$, et posons $p'(\chi) = \omega/I(e) \in (k \otimes E \otimes \boldsymbol{C})^*$. Cette *période* dépend de χ, ω et e. Prise modulo $(E \otimes F)^*$ et E^{*S}, elle ne dépend que de χ. Nous noterons $p'(\chi; \sigma, \tau)$—ou simplement $p'(\sigma, \tau)$—la composante d'indice (σ, τ) de son image par l'isomorphisme $E \otimes F \otimes \boldsymbol{C} \xrightarrow{\sim} \boldsymbol{C}^{S \times J}$.

8.8. Soit A comme en 8.3 et calculons les périodes $p'(\sigma, \tau)$, modulo $\bar{\boldsymbol{Q}}^*$, pour le motif $H^1(A)$. On commence par étendre les scalaires de k à $\bar{\boldsymbol{Q}}^*$, à l'aide de σ. Si $n(\sigma, \tau) = 1$, il existe alors une 1-forme holomorphe ω définie sur $\bar{\boldsymbol{Q}}$ telle que $u^*\omega = \tau(u)\omega$ pour $u \in E$, et, pour $Z \in H_1(A(\boldsymbol{C}))$, on a $p'(\sigma, \tau) \sim \int_Z \omega$. On peut passer de là au cas général à l'aide de la formule $p'(\sigma, \tau) \cdot p'(\sigma, c\tau) \sim 2\pi i$.

8.9. La conjecture 8.1(ii) affirme en particulier que si deux motifs vérifient la condition de 8.1(i), ils ont même période p. Pour les motifs 8.4, les périodes s'expriment en terme de valeurs de la fonction Γ et, si on travaille mod $\bar{\boldsymbol{Q}}^*$, 8.1 suggère la conjecture de B. Gross [7] reliant certaine périodes à des produits de valeurs de la fonction Γ.

La comparaison de 8.4 et 8.5 mène à la Conjecture 8.11, 8.13 suivante. Le résultat annoncé après 0.10 permet de la démontrer.

8.10. Soient N un entier, $k = \boldsymbol{Q}(\exp(2\pi i/N))$, $\mathscr{P}$ un idéal premier de k, premier à N, $k_{\mathscr{P}}$ le corps résiduel et $q = N\mathscr{P} = |k_{\mathscr{P}}|$. On notera t l'inverse de la réduction mod $\mathscr{P}$, des racines $N^{\text{ième}}$ de 1 dans $k_{\mathscr{P}}$ à celles de k.

Pour $a \in N^{-1}\,\boldsymbol{Z}/\boldsymbol{Z}$, $a \neq 0$, considérons la somme de Gauss $g(\mathscr{P}, a, \Psi) = -\sum t(x^{-a(q-1)})\Psi(x)$. La somme est étendue à $k_{\mathscr{P}}^*$ et $\Psi\colon k_{\mathscr{P}} \to \boldsymbol{C}^*$ est un caractère additif non trivial.

Soit $\boldsymbol{a} = \sum n(a)\delta_a$ dans le groupe abélien libre de base $N^{-1}\boldsymbol{Z}/\boldsymbol{Z} - \{0\}$. Si $\sum n(a)a = 0$, le produit des $g(\mathscr{P}, a, \Psi)^{n(a)}$ est indépendant de Ψ et on pose $g(\mathscr{P}, \boldsymbol{a}) = g(\mathscr{P}, a, \Psi)^{n(a)}$. Weil **[16]** a montré que, comme fonction de $\mathscr{P}$, $g(\mathscr{P}, \boldsymbol{a})$ est un caractère de Hecke algébrique $\chi_{\boldsymbol{a}}$ de k à valeurs dans k. Notons $\langle a \rangle$ le représentant entre 0 et 1 de a dans $\boldsymbol{Z}/N$. Si $\boldsymbol{a} = \sum n(a)\delta_a$ verifie

$$(*) \qquad \text{Pout tout } u \in (\boldsymbol{Z}/N)^*, \text{ on a } \sum n(a)\,\langle ua \rangle = 0,$$

il résulte la détermination par Weil de la partie algébrique de $\chi_{\boldsymbol{a}}$ que $\chi_{\boldsymbol{a}}$ est d'ordre fini; on note encore $\chi_{\boldsymbol{a}}$ le caractère de $\mathrm{Gal}(\bar{\boldsymbol{Q}}/k)$ valant $\chi_{\boldsymbol{a}}(\mathscr{P})$ sur le Frobenius géométrique en $\mathscr{P}$.

Posons $\Gamma(\boldsymbol{a}) = \Gamma(\langle a \rangle)^{n(a)}$. Une première partie de la conjecture: algébraicité de $\Gamma(\boldsymbol{a})$ lorsque $\boldsymbol{a}$ vérifie $(*)$, a été prouvée par Koblitz et Ogus: voir l'appendice. On désire avoir, plus précisément:

Conjecture 8.11. *Si* $\boldsymbol{a}$ *vérifie* $(*)$, *on a* $\sigma\Gamma(\boldsymbol{a}) = \chi_{\boldsymbol{a}}(\sigma)\cdot\Gamma(\boldsymbol{a})$ *pour tout* $\sigma \in \mathrm{Gal}(\bar{\boldsymbol{Q}}/k)$.

Si $\boldsymbol{a}$ est invariant par un sous-groupe H de $(\boldsymbol{Z}/N)^*$, on peut préciser 8.11 en remplaçant $\mathrm{Gal}(\bar{\boldsymbol{Q}}/k)$ par $\mathrm{Gal}(\bar{\boldsymbol{Q}}/k^H)$, et en utilisant SGA $4^{1/2}$ 6.5 pour définir un caractère de ce groupe, à valeur dans les racines de l'unité de k^H. C'est ce que nous faisons ci-dessous.

8.12. Soient H un sous-groupe de $(\boldsymbol{Z}/N)^*$, $\mathscr{P}$ un idéal premier de k^H, premier à N, κ son corps résiduel et $\Psi\colon \kappa \to \boldsymbol{C}^*$ un caractère additif non trivial. Pour $a \in N^{-1}\,\boldsymbol{Z}/\boldsymbol{Z} - \{0\}$, soient $\mathscr{P}_{a,i}$ les idéaux premiers de $k^H(\exp(2\pi i a))$ au-dessus de $\mathscr{P}$. Si κ' est le corps résiduel en $\mathscr{P}_{a,i}$, et que $|\kappa'| = q'$, on note $g(\mathscr{P}_{a,i}, a, \Psi)$ la somme de Gauss $-\sum t(x^{-a(q'-1)})\Psi(\mathrm{Tr}_{\kappa'/\kappa}x)$ (somme étendue à κ'^*). Le produit $g(\mathscr{P}, a, \Psi) = \prod_i g(\mathscr{P}_{a,i}, a, \Psi)$ ne dépend que de l'orbite de a sous H.

Soit $\boldsymbol{a} = \sum n(a)\delta_a$, invariant par H, et vérifiant $\sum n(a)a = 0$. Pour chaque orbite 0 de H dans $N^{-1}\,\boldsymbol{Z}/\boldsymbol{Z} - \{0\}$, on note $n(0)$ la valeur constante de $n(a)$ sur 0. On pose

$$g(\mathscr{P}, \boldsymbol{a}) = \prod_{a \bmod H} g(\mathscr{P}, a, \Psi)^{n(a)}.$$

Comme fonction de $\mathscr{P}$, $g(\mathscr{P}, \boldsymbol{a})$ est un caractère de Hecke algébrique $\chi_{\boldsymbol{a}}$ de k^H à valeurs dans k^H. Pour le prouver, on se ramène par additivité à supposer les $n(a) \geqq 0$. On applique alors [3, 6.5] pour $F = k^H$, $\bar{F} = \bar{\boldsymbol{Q}}$, $k =$ notre k, et $I =$ une somme disjointes de copies d'orbites H: $n(0)$ copies de 0; pour $i \in I$, d'image a/N dans $N^{-1}\boldsymbol{Z}/\boldsymbol{Z}$, on prend pour λ_i le composé $\hat{\boldsymbol{Z}}(1)_{\bar{F}} \to (\boldsymbol{Z}/N)(1)_{\bar{F}} = \mu_N(k) \xrightarrow{x^a} k^*$; le caractère de Hecke obtenu est le produit de $\chi_{\boldsymbol{a}}$ par le caractère "signature de la représentation de permutation de H sur I". Un cas particulier de ce résultat figure déjà dans Weil **[17]**. Le caractère $\chi_{\mathbf{a}}$ de 8.10 est le composé de $\chi_{\mathbf{a}}$ ci-dessus avec la norme N_{k/k^H}.

Conjecture 8.13. *Soient $\mathscr{P}$ un idéal premier de k^H, premier à N et $F_{\mathscr{P}}$ un Frobenius géométrique en $\mathscr{P}$. Si $\mathbf{a}$ invariant sous H vérifie* (*), *on a*

$$F_{\mathscr{P}}\Gamma(\mathbf{a}) = g(\mathscr{P}, \mathbf{a}) \cdot \Gamma(\mathbf{a}).$$

En particulier, si le groupe des racines de l'unité de k^H est d'ordre N', on a $\Gamma(\mathbf{a})^{N'} \in k^H$.

On peut de 8.13 déduire la variante suivante, apparemment plus générale. On remplace la condition (*) par

$$(*') \qquad \sum n(a)\langle ua\rangle = k \quad \text{est un entier indépendant de } u \in (\mathbf{Z}/N)^*.$$

Le caractère de Hecke $N\mathscr{P}^{-k} \cdot g(\mathscr{P}, \mathbf{a})$ est alors d'ordre fini. On l'identifie à un caractère χ de $\mathrm{Gal}(\bar{\mathbf{Q}}/k^H)$, et on espère avoir

$$\sigma((2\pi i)^{-k}\, \Gamma(\mathbf{a})) = \chi(\sigma)\cdot((2\pi i)^{-k}\, \Gamma(\mathbf{a})).$$

8.14. Si k est un corps extension quadratique totalement imaginaire d'un corps totalement réel, soit, comme nous dirons, un corps de type CM, Shimura **[13]** a déterminé les valeurs critiques des fonctions L des caractères de Hecke algébriques de k, au produit près par un nombre algébrique. Il les exprime en terme de périodes de variétés abéliennes de type CM, à multiplication complexe *par k*. Dans la fin du paragraphe, nous montrons que son théorème est compatible à la Conjecture 2.8, qui les exprime en terme de périodes de motifs *sur k*, de rang 1.

8.15. Soient χ et $M(\chi)$ comme en 8.1. Excluons le cas 8.5, et supposons que $R_{k/\mathbf{Q}}\, M(\chi)$ vérifie 1.7. Le corps k est alors totalement imaginaire, et les nombres de Hodge h^{pp} sont nuls: avec les notations de 8.1 et 8.2, aucun n_i n'est égal à $w/2$. Notre première tâche est de calculer $c^+R_{k/\mathbf{Q}}M(\chi)$ en terme des périodes $p'(\chi;\, \sigma, \tau)$. Rappelons que c'est le déterminant de l'isomorphisme de $E \otimes \mathbf{C}$-module

$$I^+\colon H_B^+ R_{k/\mathbf{Q}}M(\chi) \otimes \mathbf{C} \xrightarrow{\ \sim\ } H_{DR}^+ R_{k/\mathbf{Q}}M(\chi) \otimes \mathbf{C},$$

calculé dans des bases définies sur E.

Choisissons les e_σ de 8.6 de sorte que $F_\infty e_\sigma = e_{c\sigma}$. Les $e_\sigma \pm e_{c\sigma}$ forment alors une base de $H_B^\pm R_{k/\mathbf{Q}}M(\chi) \subset H_B R_{k/\mathbf{Q}}\, M_\chi = \bigoplus_{\sigma\in S} H_\sigma M(\chi)$. Notons au passage que $d^+ = d^- = \frac{1}{2}[k\colon \mathbf{Q}]$. Soit $\bar{S}$ le quotient de S par $\mathrm{Gal}(\mathbf{C}/\mathbf{R})$. Pour calculer $c^+ = \det(I^+)$, nous utiliserons la base $(e_\sigma + e_{c\sigma})$ de H_B^+. Elle est indexée par $\bar{S}$.

D'après 8.1(iv), la filtration de Hodge de $H_{DR}R_{k/\mathbf{Q}}M(\chi) = H_{DR}M(\chi)$ se lit sur sa structure de $k \otimes E$-module: si on note $(k \otimes E)^+$ le facteur direct de $k \otimes E$ produit des K_i tels que $n_i < w/2$, le quotient $H_{DR}^+R_{k/\mathbf{Q}}M(\chi)$ de $H_{DR}R_{k/\mathbf{Q}}M(\chi)$ est le facteur direct correspondant:

$$H_{DR}^+R_{k/\mathbf{Q}}M(\chi) = H_{DR}M(\chi) \otimes_{k\otimes E} (k \otimes E)^+.$$

Soit ω comme en 8.7, et utilisons la structure de $k \otimes E \otimes \mathbf{C} = \mathbf{C}^{S\times J}$-module de $H_{DR}M(\chi)$ pour décomposer ω: $\omega = \sum \omega_{\sigma,\tau}$. On a par définition $I(e_\sigma) = \sum_\tau p'(\sigma,\tau)^{-1}\omega_{\sigma,\tau}$, et donc

$$I^+(e_\sigma + e_{c\sigma}) = \sum_{n(\sigma,\tau)<w/2} p'(\sigma, \tau)^{-1}\omega_{\sigma,\tau} + \sum_{n(\sigma,\tau)<w/2} p'(c\sigma, \tau)^{-1}\omega_{c\sigma,\tau};$$

c'est la somme, indexée par $\tau \in J$, de termes égaux à $p'(\sigma, \tau)^{-1}\omega_{\sigma,\tau}$ pour $n(\sigma, \tau) < w/2$, et à $p'(c\sigma, \tau)^{-1}\omega_{c\sigma,\tau}$ pour $n(\sigma, \tau) > w/2$. Pour $\bar{\sigma} \in \bar{S}$, de représentant σ, posons

$$\omega_{\bar{\sigma}} = \sum_{n(\sigma,\tau)<w/2} \omega_{\sigma,\tau} + \sum_{n(c\sigma,\tau)<w/2} \omega_{c\sigma,\tau}.$$

Les $I^+(e_\sigma + e_{c\sigma})$ sont des multiples des $\omega_{\bar{\sigma}}$ par des éléments de $E \otimes C$; les $\omega_{\bar{\sigma}}$ forment donc une base de H_{DR}^+. Dans les bases $e_\sigma + e_{c\sigma}$ et $\omega_{\bar{\sigma}}$, la matrice de I^+ est diagonale; son déterminant $\det'(I^+) \in (E \times C)^* = C^{*J}$ a pour coordonnées

$$\det'(I^+)_\tau = \prod_{n(\sigma,\tau)<w/2} p'(\sigma, \tau)^{-1}.$$

Soient les applications $E^{\bar{S}} \to E^S\colon 1_{\bar{\sigma}} \to 1_\sigma + 1_{c\sigma}$, pour $\bar{\sigma}$ image de σ, $E^S \otimes C \to^\sim k \otimes E \otimes C$, déduit de l'isomorphisme de $k \otimes C$ avec C^S, et la projection de $k \otimes E$ sur $(k \otimes E)^+$. Par composition, on obtient un isomorphisme de $E \otimes C$-modules:

$$E^{\bar{S}} \otimes C \xrightarrow{\sim} (k \otimes E)^+ \otimes C.$$

Nous noterons $D(\chi)$ son déterminant, calculé dans des bases définies sur E des deux membres. Identifiant H_{DR} à $k \otimes E$ à l'aide de la base ω, on voit que c'est le déterminant de l'application identique de $H_{DR}^+ \otimes C$, calculé dans la base $\omega_{\bar{\sigma}}$, à la source, et une base définie sur E, au but. Notant $D_\tau(\chi)$ ses composantes dans C^J, on trouve pour $c^+ = \det'(I^+) \cdot D(\chi)$ la formule suivante.

PROPOSITION 8.16. *On a*

$$c^+ R_{k/Q} M(\chi) = \Big(\prod_{n(\sigma,\tau)<w/2} p'(\sigma, \tau)^{-1} \cdot D_\tau(\chi) \Big)_{\tau \in J}.$$

REMARQUE 8.17. Supposons k de type CM, extension quadratique de k_0 totalement réel. Le quotient $\bar{S}$ de S s'identifie alors à l'ensemble des plongements complexes de k_0, et le diagramme

$$\begin{array}{ccccc} E^{\bar{S}} \otimes C & \xrightarrow{\sim} & k_0 \otimes E \otimes C & & \\ \downarrow & & \downarrow & & \\ E^S \otimes C & \longrightarrow & k \otimes E \otimes C & \longrightarrow & (k \otimes E)^+ \otimes C \end{array}$$

est commutatif. L'application composée $k_0 \otimes E \to (k \otimes E)^+$ est donc un isomorphisme, et $D(\chi)$ est encore le déterminant de $E^{\bar{S}} \otimes C \to^\sim k_0 \otimes E \otimes C$, déduit par extension des scalaires de Q à E de l'isomorphisme $C^{\bar{S}} \to^\sim k_0 \otimes C$. Ceci fournit pour $D(\chi)$, bien défini mod E^*, un représentant dans $(Q \otimes C)^* = C^* \subset (E \otimes C)^*$, à savoir le déterminant de l'inverse de la matrice (σa), pour $\sigma \in \bar{S}$ et a parcourant une base de k_0 sur Q. L'isomorphisme $C^{\bar{S}} \to^\sim k_0 \otimes C$ transforme la forme quadratique $\sum x_i^2$ en la forme $\operatorname{Tr}(xy)$. Ceci permet d'identifier $(\det(\sigma a))^2$ au discriminant de k_0 :

$$D(\chi) \sim \text{racine carrée du discriminant de } k_0.$$

8.18. Notons $p''(\chi; \sigma, \tau)$ l'image de $p'(\chi; \sigma, \tau)$ dans $C^*/\bar{Q}^*$. Elle ne dépend que de χ, σ et τ. Si un homomorphisme algébrique $\eta\colon k^* \to E^*$ vérifie (8.2.1), une de ses puissance est la partie algébrique d'un caractère de Hecke: $\eta^N = \chi_{\text{alg}}$. De plus, si $\chi'_{\text{alg}} = \chi''_{\text{alg}}$, χ' et χ'' ne diffèrent que par un caractère d'ordre fini et $\chi'^M = \chi''^M$ pour M convenable. On déduit de (8.1.1) que $p''(\chi^M; \sigma, \tau) = p''(\chi; \sigma, \tau)^M$, et ceci permet de poser sans ambiguité

$$p(\eta;\sigma,\tau) = p(\chi;\sigma,\tau)^{1/N}, \quad \text{pour } \eta^N = \chi_{\text{alg}}.$$

Ces périodes obéissent au formalisme suivant:

(8.18.1) $p(\eta'\eta'';\sigma,\tau) = p(\eta';\sigma,\tau)\,p(\eta'';\sigma,\tau)$.

(8.18.2) $p(\eta;\sigma,\tau)$ ne change pas quand on remplace E par une extension E' de E, et τ par un de ses prolongements à E'.

(8.18.3) $p(\eta;\sigma,\tau)$ ne change pas quand on remplace k par une extension k' de k, σ par un de ses prolongements à k', et χ par $\chi \circ N_{k'/k}$.

(8.18.4) Si α est un automorphisme de k, et β un automorphisme de E, on a $p(\eta;\sigma,\tau) = p(\beta\eta\alpha^{-1},\sigma\alpha^{-1},\tau\beta^{-1})$.

(8.18.5) Le complexe conjugué de $p(\eta;\sigma,\tau)$ est $p(\eta;\bar{\sigma},\bar{\tau})$.

(8.18.6) Pour $k = F = \mathbf{Q}$, $p(\text{Id};\text{Id},\text{Id}) = 2\pi i$.

Les formules (8.15.1) à (8.15.3) résultent de (8.1.1) à (8.1.3), (8.18.4) et (8.18.5) se voient par transport de structure, et (8.18.6) résulte de (8.5).

8.19. Soit $\eta : k^* \to E^*$ un homomorphisme vérifiant (8.2.1). On suppose aussi que $n(\eta;\sigma,\tau)$ ne vaut jamais $w/2$, ce qui permet de définir $(k \otimes E)^+$ comme en 8.15. Définissons $\eta^* : E^* \to k^*$ par

$$\eta^*(y) = \det_k(1 \otimes y, (k \otimes E)^+).$$

Cet homomorphisme vérifie encore (8.2.1), et

$$\begin{aligned} n(\eta^*;\sigma,\tau) &= 1 \quad \text{si } n(\eta;\sigma,\tau) < w/2, \\ &= 0 \quad \text{si } n(\eta;\sigma,\tau) > w/2. \end{aligned}$$

Si η^* est la partie algébrique d'un caractère de Hecke χ^*, $M(\chi^*)$ est le H^1 d'une variété abélienne sur E, à multiplication complexe par k, dont l'algèbre de Lie, comme $k \otimes E$-module, est isomorphe à $(k \otimes E)^+$.

Proposition 8.20. *Avec les hypothèses et notations de* 8.19, *prenons pour E un sous corps de* **C**, *et notons* 1 *l'inclusion identique de E dans* **C**. *On a*

$$\prod_{n(\sigma,1)<w/2} p(\eta;\sigma,1) = \prod_{\sigma} p(\eta^*;1,\sigma)^{n(\eta;\sigma,1)}.$$

Si $\iota : E \to E' \subset \mathbf{C}$ est une extension finie de E, le membre de gauche ne change pas quand on remplace η par $\iota\eta$ (8.18.2). On a $(\iota\eta)^* = \eta^* \circ N_{E'/E}$ et, par (8.18.3), le membre de droite ne change pas non plus.

Si $\iota : k \to k'$ est une extension finie de degré d de k, le membre de gauche est élevé à la puissance d quand on remplace k par k', et η par $\eta \circ N_{k'/k}$: un plongement complexe de k est induit par d plongements de k', et on applique (8.18.3). De même pour le membre de droite, par (8.18.2) et l'égalité $(\eta \circ N_{k'/k})^* = \iota\eta^*$.

Ces compatibilités nous ramènent à supposer que E est galoisien et que k est isomorphe à E. Pour chaque isomorphisme ω de k dans E, posons $n(\omega) = n(\eta; 1 \circ \omega, \omega)$. Notant additivement le groupe des homomorphismes de k^* dans E^*, on a $\eta = \sum n(\omega)\,\omega$. Puisque $p(\eta; 1 \circ \omega, 1) = p(\eta \circ \omega^{-1}; 1, 1)$ (8.15.4), on a

$$(8.20.1) \qquad \prod_{n(\sigma,1)<w/2} p(\eta;\sigma,1) = \prod_{n(\omega)<w/2} p(\eta \circ \omega^{-1};1,1) = p\Big(\sum_{n(\omega)<w/2} \eta \circ \omega^{-1};\, 1,\, 1\Big).$$

Par ailleurs,

$$\sum_{n(\omega)<w/2} \eta \circ \omega^{-1} = \sum_{n(\omega_1)<w/2;\, \omega_2} n(\omega_2)\omega_2 \circ \omega_1^{-1} = \sum_{\omega_2} n(\omega_2) \sum_{n(\omega_1)<w/2} \omega_2 \circ \omega_1^{-1}$$
$$= \sum_{\omega_2} n(\omega_2)\, \omega_2 \circ \eta^*.$$

Ceci permet de continuer (8.20.1) par

$$= \prod_{\omega} p(\omega \circ \eta^*; 1, 1)^{n(\omega)} = \prod_{\omega} p(\eta^*; 1, 1 \circ \omega)^{n(\omega)} = \prod_{\sigma} p(\eta^*; 1, \sigma)^{n(\eta;\sigma, 1)},$$

(nouvelle application de 8.15.4), et prouve 8.20.

8.21. Combinant 8.16 et 8.20, on trouve pour la composante d'indice 1 de $c^+ R_{k/\boldsymbol{Q}} M(\chi) \in (E \otimes \boldsymbol{C})^*/E^* = \boldsymbol{C}^{*J}/E^*$, l'expression suivante, mod $\bar{\boldsymbol{Q}}^*$

$$c_1^+ R_{k/\boldsymbol{Q}} M(\chi) \sim \prod_{\sigma} p(\chi^*_{\text{alg}}; 1, \sigma)^{-n(\chi;\sigma, 1)}.$$

Si χ (i.e., $M(\chi)$) est critique, la Conjecture 2.8 affirme donc que

$$L(1 \circ \chi, 0) \sim \prod_{\sigma} p(\chi^*_{\text{alg}}; 1, \sigma)^{-n(\chi;\sigma, 1)} \quad (\text{mod } \bar{\boldsymbol{Q}}^*).$$

Pour E assez grand, χ^*_{alg} est la partie algébrique d'un caractère de Hecke χ^*, et les périodes s'interprètent comme périodes d'intégrales abéliennes (8.19), (8.3). L'énoncé obtenu est celui que Shimura a prouvé pour k de type CM, ou abélien sur un corps de type CM (avec une restriction sur le poids).

REMARQUE 8.22. Si $\eta : k^* \to E^*$ vérifie (8.2.1), il existe des sous-corps k' de k et $\iota : E' \to E$ de E, soit de type CM, soit égaux à $\boldsymbol{Q}$, et une factorisation $\eta = \iota\eta' N_{k/k'}$. On a alors $p(\eta; \sigma, \tau) = p(\eta'; \sigma \mid k', \tau \mid k')$.

Si maintenant k et E sont de type CM (ou $\boldsymbol{Q}$), et qu'on note encore c leur conjugaison complexe, on a $c\sigma = \sigma c$, $c\tau = \tau c$, et $\eta c = c\eta$, d'où

$$p(\eta; \sigma, \tau)^- = p(\eta; c\sigma, c\tau) = p(\eta; \sigma c, \tau c) = p(c\eta c^{-1}; \sigma, \tau) = p(\eta; \sigma, \tau):$$

les périodes, a priori dans $\boldsymbol{C}^*/\bar{\boldsymbol{Q}}^*$, sont réelles, i.e., dans $\boldsymbol{R}^*/(\bar{\boldsymbol{Q}}^* \cap \boldsymbol{R}^*)$.

REMARQUE 8.23. Soient G le groupe de Galois de la réunion des extensions de type CM de $\boldsymbol{Q}$ dans $\bar{\boldsymbol{Q}} \subset \boldsymbol{C}$, et $c \in G$ la conjugaison complexe. C'est un élément central de G. Si φ est une fonction localement constante à valeurs entières sur G, nous poserons $\varphi^*(x) = \varphi(xc)$. Supposons que $\varphi + \varphi^*$ est constante. Soient G_1 un quotient fini de G tel que φ se factorise par une fonction φ_1 sur G_1, et k le corps correspondant. L'hypothèse faite signifie que l'endomorphisme $\Sigma\varphi_1(\sigma)\sigma$ de k^* vérifie (8.2.1). La période $p(\Sigma\varphi_1(\sigma)\sigma; 1,1)$ ne dépend pas du choix de G_1; on pose $P(\varphi) = p(\Sigma\varphi_1(\sigma)\, \sigma; 1,1)$. La fonctionnelle P est un homomorphisme dans $\boldsymbol{C}^*/\bar{\boldsymbol{Q}}^*$ du groupe des fonctions localement constante à valeurs entières sur G qui vérifient $\varphi + \varphi^* =$ constante.

Appendix by N. Koblitz and A. Ogus. Algebraicity of some products of values of the Γ function. Let $A_N = N^{-1} \boldsymbol{Z}/\boldsymbol{Z} - \{0\}$, and let $U_N = (\boldsymbol{Z}/N\boldsymbol{Z})^*$ operate on A_N in the obvious way. If $f: A_N \to \boldsymbol{C}$, define $\langle f \rangle : U_N \to \boldsymbol{C}$ by $\langle f \rangle(u) = \sum_{a \in A_N} \langle a \rangle f(ua)$, where $\langle a \rangle$ denotes the representative of a between 0 and 1.

We first compute $H_{\boldsymbol{Q}} = \{f: A_N \to \boldsymbol{Q}: \langle f \rangle \text{ is constant}\}$.

EXAMPLE. If n is a divisor of N and $a \in A_N$, consider the set $S_{n,a} = [\{a + k/n: k = 0, 1, \cdots, n-1\} \cup \{-na\}] \cap A_N$. It is easy to see that its characteristic function $\varepsilon_{n,a}$ is in $H_{\boldsymbol{Q}}$. Note that $S_{1,a} = \{a, -a\}$.

PROPOSITION. *H_Q is generated by $\{\varepsilon_{n,a}: n = 1$ or n is prime and $na \neq 0\}$.*

PROOF. The orbits of A_N under the action of U_N correspond to the divisors d of N with $1 < d \leqq N$: each orbit can be written uniquely in the form $U_N(1/d)$. The stabilizer subgroup I_d of $1/d$ is $\{u \in U_N: u \equiv 1 \bmod d\}$ and the orbit of $1/d$ is canonically isomorphic to U_d. Thus, a function f on A_N is determined by the collection of functions $f_d: U_d \to Q$ defined by $f_d(v) = f(v/d)$.

To prove the proposition, we first complexify, so as to be able to work with characters of U_N.

LEMMA. *If $f: A_N \to C$ and if χ is a character of U_N, then the inner product of $\langle f \rangle$ and χ is given by:*

$$(\langle f \rangle \mid \chi) = - \sum_d L(0, \chi_d) \mid I_d \mid \cdot (f_d \mid \chi_d),$$

where the sum is taken over those divisors d of N such that χ is pulled back from a character χ_d of U_d.

PROOF. We have

$$\begin{aligned}(\langle f \rangle \mid \chi) &= \sum_{u \in U_N} \langle f \rangle(u)\, \bar{\chi}(u) = \sum_{a \in A_N} \sum_{u \in U_N} \langle a \rangle f(ua)\, \bar{\chi}(u) \\ &= \sum_{d \mid N} \sum_{v \in U_d} \sum_{u \in U_N} \langle v/d \rangle f_d(u_d v)\, \bar{\chi}(u),\end{aligned}$$

where $u_d \in U_d$ is the image of u.

Now if we choose a set $U_d' \subseteq U_N$ of coset representatives for $U_N \to U_d$, we can write:

$$\sum_{u \in U_N} f_d(u_d v)\bar{\chi}(u) = \sum_{u' \in U_d'} \sum_{u \in I_d} f_d(u_d' v)\bar{\chi}(u')\bar{\chi}(u) = \sum_{u' \in U_d'} f_d(u_d' v)\bar{\chi}(u') \sum_{u \in I_d} \bar{\chi}(u).$$

Of course, this sum is zero unless χ is trivial on I_d, i.e., unless χ is pulled back from a character χ_d of U_d, in which case it is

$$\sum_{u' \in U_d} f_d(u'v)\bar{\chi}_d(u')|I_d| = \sum_{w \in U_d} f_d(w)\bar{\chi}_d(w)\chi_d(v)|I_d| = \chi_d(v)|I_d|(f_d|\chi_d).$$

If we substitute this into the above expression for $(\langle f \rangle | \chi)$ and note that $\langle v/d \rangle = |v|/d$, where $|v|$ is the smallest positive representative of $v \in U_d$, we find:

$$\begin{aligned}(\langle f \rangle|\chi) &= \sum_d \sum_{v \in U_d} |v|\chi_d(v)|I_d|d^{-1}(f_d|\chi_d) \\ &= - \sum_d L(0, \chi_d)|I_d|(f_d|\chi_d) \quad \text{as claimed.} \quad \square\end{aligned}$$

To prove the proposition, we let d_f be the largest divisor of N such that $f_d \neq 0$; it suffices to prove that any f in H can be written as $g + f'$, with g a linear combination of the ε's and $d_{f'} < d_f$. Let $d = d_f$; we shall show below that f_d is a linear combination of functions h_1 which factor through $U_d \to U_d/\{\pm 1\}$, and functions h_p which factor through $U_d \to U_{d/p}$ for some prime divisor p of d, $p \neq d$. Any function of the first type is invariant under ± 1, and hence the corresponding function on the orbit $U_N(1/d)$ is a linear combination of $\varepsilon_{1,a}$'s. A function of the second type is invariant under the kernel K of $U_d \to U_{d/p}$, which has order p if p divides d/p and order $p - 1$ otherwise. In the first case, $K = \{1 + kd/p: k = 0,\ldots,p - 1\}$, and in the second, it is this same set with one element deleted, namely, the value of $1 + kd/p$ which is divisible by p. Thus, the set $S_{p,w/d} = (Kw)/d$, with the addition of one or two elements in the orbit $U_N(p/d)$. Hence it is clear that the corresponding function on the orbit $U_N(1/d)$ can be written as a linear combination of $\varepsilon_{p,a}$'s and functions supported on the orbit of p/d, and we have obtained our desired decomposition $f = g + f'$.

It remains for us to prove the claim about f_d. If d is twice an odd number the map $U_d \to U_{d/2}$ is an isomorphism, and the claim is trivial. In the other cases, choose a primitive odd character χ_d of U_d, and let χ be the pull-back of χ_d to U_N. Since χ is nontrivial and $\langle f \rangle$ is constant, $(\langle f \rangle | \chi) = 0$. Let us look at the expression for this inner product provided by the lemma; the only nonzero terms come from those f_e such that $f_e \neq 0$ and $I_e \subseteq \mathrm{Ker}(\chi)$. But then $\mathrm{Ker}(\chi)$ contains I_e and I_d, hence also $I_d I_e = I_m$, where $m = \text{g.c.d.}(d, e)$. Since χ is primitive, $m = d$ and d divides e, and since $f_e = 0$ for $e > d$, $0 = (\langle f \rangle | \chi) = -L(0, \chi_d)|I_d|(f_d|\chi_d)$. Since χ_d is odd and primitive, we conclude that $(f_d|\chi_d) = 0$, i.e., f_d is orthogonal to every odd primitive character of U_d. It follows that f_d can be written as a linear combination of characters which are even (factor through $U_d/\{\pm 1\}$) or imprimitive (factor through some $U_{d/p}$). When $d = p$, we should also remark that the constant functions on U_p are also invariant under ± 1, and hence are already covered by the first case. □

THEOREM. *Suppose that $f: A_N \to \mathbf{Q}$ is such that $\langle f \rangle$ is constant : $\sum_a \langle a \rangle f(ua) = k$ for all $u \in U_N$. Then*

$$\Gamma(f) =_{\mathrm{def}} \pi^{-k} \prod_a \Gamma(\langle a \rangle)^{f(a)}$$

is an algebraic number.

PROOF. The map $\langle\ \rangle : H_{\mathbf{Q}} \to \mathbf{Q}$ sending any f to $\sum_a \langle a \rangle f(a)$ is evidently linear, and hence $\Gamma(f + g) = \Gamma(f)\Gamma(g)$, for f and g in $H_{\mathbf{Q}}$. By the proposition, any f in $H_{\mathbf{Q}}$ is a $\mathbf{Q}$-linear combination of the ε's; hence nf is a sum of ε's for some n. Since $\Gamma(f)^n = \Gamma(nf)$, we are reduced to checking the theorem for the ε's described in the proposition. Noting that $\langle \varepsilon_{1,a} \rangle = 1$ and that $\langle \varepsilon_{p,a} \rangle = (p + 1)/2$ if $pa \neq 0$, one has only to appeal to the identities:

$$\Gamma(x)\Gamma(1 - x) = \pi(\sin \pi x)^{-1}, \qquad \sum_{k=1}^{p-1} \Gamma\left(x + \frac{k}{p}\right) = p^{1/2-x}(2\pi)^{(p-1)/2}\,\Gamma(px). \quad \square$$

REMARK. Kubert has recently obtained a much more precise result expressed in a different terminology from which it follows that, if $H_{\mathbf{Q}}$ is $\mathbf{Z}$-valued, then $2f$ is a $\mathbf{Z}$-linear combination of the ε's (to appear).

BIBLIOGRAPHIE

1. M. V. Borovoi, *Sur l'action du groupe de Galois sur les classes de cohomologie rationnelles de type* (p, p) *des variétés abéliennes*, Mat. Sb. **94** (1974), 649–652. (Russian)

2. J. Coates and S. Lichtenbaum, *On l-adic zeta functions*, Ann. of Math (2) **98** (1973), 498–550.

3. P. Deligne, *Applications de la formule des traces aux sommes trigonométriques*, dans **SGA** $4^{1/2}$, 168–232.

4. ———, *Les constantes des équations fonctionnelles des fonctions L*, Modular Functions of One Variable. II, Lecture Notes in Math., vol. 349, Springer-Verlag, New York, 1973, pp. 501–595.

5. ———, *Formes modulaires et représentations l-adiques*, Séminaire Bourbaki 355 (Février 1969), Lecture Notes in Math., vol. 179, Springer-Verlag, New York, 139–172.

6. P. Deligne, *Théorie de Hodge*. III, Inst. Hautes Etudes Sci. Publ. Math. **44** (1974), 5–77.

7. B. Gross, *On the periods of abelian integrals and a formula of Chowla and Selberg*, Invent. Math. **45** (1978), 193–211.

8. Ju. I. Manin, *Correspondences, motifs and monoidal transformations*.

9. ———, *Points paraboliques et fonction zeta des courbes modulaires*, Izv. **36** (1972), 19–66. (Russian)

10. I. I. Piateckii-Shapiro, *Relations entre les conjectures de Tate et de Hodge pour les variétés abéliennes*, Mat. Sb. **85(4)** (1971), 610–620 = Math. USSR-Sb. **14** (1971), 615–625.

11. J. P. Serre, *Abelian l-adic representations and elliptic curves* (Mc Gill), Benjamin, New York, 1968.

12. ———, *Facteurs locaux des fonctions zêta des variétés algébriques* (*définitions et conjectures*), Sém. Delange-Pisot-Poitou 1969/70, exposé 19.

13. G. Shimura, *On some arithmetic properties of modular forms of one and several variables*, Ann. of Math. (2) **102** (1975), 491–515.

14. C. L. Siegel, *Berechnung von Zetafunktionen an ganzzahligen Stellen*, Nachr. Akad. Wiss. Göttingen Math. Phys. Kl. II **10** (1969), 87–102.

15. J. Tate, *On the conjecture of Birch and Swinnerton-Dyer and a geometric analogue*, Séminaire Bourbaki 306 (1965/66), Benjamin, New York; reproduit dans 10 *exposés sur la théorie des schémas*, North-Holland, Amsterdam, 1968.

16. A. Weil, *Jacobi sums as grössencharaktere*, Trans. Amer. Math. Soc. **73** (1952), 487–495.

17. ———, *Sommes de Jacobi et caractères de Hecke*, Nachr. Akad. Wiss. Göttingen Math. Phys. Kl. II **1** (1974), 1–14.

18. D. Zagier, *Modular forms whose Fourier coefficients involve zeta functions of quadratic fields*, Modular Forms of One Variable. VI, Lecture Notes in Math., vol. 627, Springer-Verlag, New York, 1977, pp. 105–169.

19. N. Saavedra, *Catégories tannakiennes*, Lecture Notes in Math., vol. 265, Springer-Verlag, New York, 1972.

SGA. *Séminaire de géométrie algébrique du Bois-Marie.*

SGA 4. par M. Artin, A. Grothendieck et J. L. Verdier, *Théorie des topos et cohomologie étale des schémas*, Lecture Notes in Math., vols. 259, 270, 305, Springer-Verlag, New York.

SGA $4^{1/2}$. par P. Deligne, *Cohomologie étale*, Lecture Notes in Math., vol. 569, Springer-Verlag, New York, 1977.

INSTITUT DES HAUTES ETUDES SCIENTIFIQUES, BURES SUR YVETTE, FRANCE

Proceedings of Symposia in Pure Mathematics
Vol. **33** (1979), part 2, pp. 347–356

AN INTRODUCTION TO DRINFELD'S "SHTUKA"

D. A. KAZHDAN

We begin with a formulation of the Langlands conjecture for GL(n) over a local field K of positive characteristic p. Fix a prime $l \neq p$ and denote by $\bar{Q}_l$ an algebraic closure of Q_l.

We introduce some notation. For any topological group G with a compact subgroup $U \subset G$ we will consider smooth $\bar{Q}_l$-representations of G, that is, homomorphisms $\rho: G \to \operatorname{Aut} V$, where V is a $\bar{Q}_l$-vector space such that the stabilizer of any vector $v \in V$ is open. We say that ρ is admissible if for every open compact subgroup $U \subset G$ we have $\dim V^U < \infty$, where V^U is the set of U-invariant vectors.

DEFINITION. We denote by $\hat{G}$ the set of equivalence classes of admissible irreducible representations of G and by $\hat{G}_n \subset G$ ($n = 1, 2, \cdots$) the subset consisting of classes of n-dimensional representations.

REMARK. It is easy to check that all standard results (see [7]) about admissible C-representations are true for our $\bar{Q}_l$-representations.

Now we can formulate the local unramified reciprocity law. Let $K_{\mathfrak{p}}$ be a nonarchimedean local field, $o = o_{\mathfrak{p}}$ its ring of integers, $\pi \in o$ a prime element, $k = o/(\pi)$ the residue field, $q = \operatorname{card} k$, and v the valuation on $K_{\mathfrak{p}}$ such that $v(\pi) = 1$.

Recall that an irreducible admissible representation of GL(n, $K_{\mathfrak{p}}$) is *unramified* if the subspace of GL(n, o)-invariant vectors has positive dimension.

REMARK. In this case this dimension is one.

The classification of the set $\hat{G}_{\mathfrak{p},\mathrm{un}}$ of equivalence classes of unramified representations of GL(n, $K_{\mathfrak{p}}$) is well known. Let $B \subset \mathrm{GL}(n, K_{\mathfrak{p}})$ be the subgroup of triangular matrices. To any element $x = (x_1, \cdots, x_n)$ ($x_i \in \bar{Q}_l^*$) we associate the character χ_x of B:

$$\chi_x(b) = (q^{1-n}x_1)^{\nu(b_{11})}\,(q^{2-n}x_2)^{\nu(b_{22})} \cdots x_n^{\nu(b_{nn})}$$

where b_{ii} are the diagonal elements of b. By $\tilde{\rho}_x$ we denote the induced representation $\operatorname{ind}_B^{\mathrm{GL}(n, K_{\mathfrak{p}})}\chi_x$ (which is realized in the space of locally constant functions on GL(n, $K_{\mathfrak{p}}$) such that $f(bg) = \chi_x(b)f(g)$ for all $b \in B$, $g \in \mathrm{GL}(n, K_{\mathfrak{p}})$). The representation $\tilde{\rho}_x$ is not necessarily irreducible, but it has a finite Jordan-Hölder series which contains exactly one unramified component. We denote it by ρ_x. The following statement is well known [1].

THEOREM 1. (a) *For every element s of the symmetric group S_n, $\rho_x \approx \rho_{s(x)}$.*
(b) *The map $\rho: (\bar{Q}_l^*)^n/S_n \to \hat{G}_{\mathfrak{p},\mathrm{un}}$ is an isomorphism.*

AMS (MOS) subject classifications (1970). Primary 10H10; Secondary 10D20, 22E55, 20G35.

REMARK. Our parametrization of $\hat{G}_{\mathfrak{p},\mathrm{un}}$ differs from the usual one by a shift and does not use fractional powers of q.

Denote by $\bar{K}_{\mathfrak{p}}$ a separable closure of $K_{\mathfrak{p}}$, and by $\bar{k}_{\mathfrak{p}}$ the corresponding algebraic closure of $k_{\mathfrak{p}}$.

We denote by $W_{K_{\mathfrak{p}}}$ the dense subgroup of $\mathrm{Gal}(\bar{K}_{\mathfrak{p}}/K_{\mathfrak{p}})$ consisting of elements which induce on $\bar{k}_{\mathfrak{p}}$ the map $x \to x^{q^n}$ for some $n \in \boldsymbol{Z}$, and we denote by $I_{K_{\mathfrak{p}}} \subset W_{K_{\mathfrak{p}}}$ the subgroup fixing $\bar{k}_{\mathfrak{p}}$. Thus $W_{K_{\mathfrak{p}}}$ is the Weil group of $K_{\mathfrak{p}}$ and $I_{\mathfrak{p}}$ the inertia group.

A representation of $W_{K_{\mathfrak{p}}}$ is said to be unramified if it is completely reducible and its restriction to $I_{K_{\mathfrak{p}}}$ is trivial.

We can now state and easily prove the local reciprocity law for unramified representations.

THEOREM-DEFINITION 2. *There is a natural* 1-1 *correspondence between* $\hat{G}_{\mathfrak{p},\mathrm{un}}$ *and the set of equivalence classes of n-dimensional unramified representations of* $W_{K_{\mathfrak{p}}}$.

REMARK. By an n-dimensional representation we mean a morphism into $\mathrm{GL}(n, \bar{\boldsymbol{Q}}_l)$.

PROOF-CONSTRUCTION. Let τ be an unramified semisimple representation of $W_{K_{\mathfrak{p}}}$. Since it is unramified, we may consider it as a representation of $W_{K_{\mathfrak{p}}}/I_{K_{\mathfrak{p}}} \simeq \boldsymbol{Z}$. It thus corresponds to an element α_τ in $\mathrm{GL}(n, \bar{\boldsymbol{Q}}_l)$ (the image of the Frobenius Fr). Since τ is semisimple, α_τ is semisimple, and so the conjugacy class of α_τ is completely determined by the set of eigenvalues $x = (x_1, \cdots, x_n) \in (\bar{\boldsymbol{Q}}_l^*)^n$. We associate to τ the representation $\rho = \rho_x$ of $\mathrm{GL}(n, K_{\mathfrak{p}})$, and we write $\rho \sim \tau$. □

This construction uses the explicit realization of unramified representations, but there exists a more canonical formulation. It is known (see **[1]**) that for every natural number m there exists a unique locally constant $\boldsymbol{Q}$-valued function χ_m on $\mathrm{GL}(n, K_{\mathfrak{p}})$ with compact support such that (a) χ_m is bi-$\mathrm{GL}(n, o)$-invariant, (b) for every $x = (x_1, \cdots, x_n) \in (\bar{\boldsymbol{Q}}_l^*)^n$:

$$\mathrm{Tr}\, \rho_x(\chi_m) = x_1^m + \cdots + x_n^m.$$

Now we can reformulate Theorem 2.

THEOREM 2′. *The irreducible unramified representation* ρ *of* $\mathrm{GL}(n, K_{\mathfrak{p}})$ *corresponds to the unramified n-dimensional representation* τ *of* $W_{K_{\mathfrak{p}}}$ *if and only if* $\mathrm{Tr}\,\rho(\chi_m) = \mathrm{Tr}\,\tau(\mathrm{Fr}^m)$ *for all natural numbers* $m > 0$.

Now consider the global case.

Let K be a global field, char $K = p$.

We may consider K as the field of rational functions on a smooth absolutely irreducible curve X over the finite field k. Let $q = \mathrm{card}\, k$. We denote by Π the set of all points of K. For every $\mathfrak{p} \in \Pi$ we denote by $K_{\mathfrak{p}}$ the completion of K at $\mathfrak{p}$. We denote by $\boldsymbol{A}$ the adele ring of K and consider the $\bar{\boldsymbol{Q}}_l$-space L of $\bar{\boldsymbol{Q}}_l$-valued locally constant functions f on $\mathrm{GL}(n, K)\backslash\mathrm{GL}(n, \boldsymbol{A})$ which are cuspidal. That is to say for every K-subspace $\Lambda \subset K^n$, $\Lambda \neq 0, K^n$, $\int_{U_\Lambda(k)\backslash U_\Lambda(\boldsymbol{A})} f(ux)\, du \equiv 0$ where U_Λ is the subgroup of $g \in \mathrm{GL}(n, K^n)$ such that $g\Lambda = \Lambda$ and g acts trivially on Λ and K^n/Λ. The group $\mathrm{GL}(n, \boldsymbol{A})$ acts by right shifts on L. The support of every element $f \in L$ is compact mod $\boldsymbol{A}^*$ (the center of $\mathrm{GL}(n, \boldsymbol{A})$) (see, e.g., Godement-Jacquet, Lecture Notes in Math., vol. 260, Springer, p. 142) and, consequently, the representation T of $\mathrm{GL}(n, \boldsymbol{A})$ on L is smooth.

We will call an irreducible admissible representation of $\mathrm{GL}(n, A)$ a *cuspidal* automorphic representation if it occurs as a subrepresentation of L. We denote by $\mathrm{GL}(n, A)_c^\wedge$ the set of these representations.

Recall from, say, [5] that any irreducible admissible representation (ρ, V) of $\mathrm{GL}(n, A)$ may be uniquely expressed as a restricted tensor product $\widehat{\otimes}\rho_\mathfrak{p}(\mathfrak{p} \in \Pi)$ of local irreducible admissible representations. If S is a finite subset of Π such that $\rho_\mathfrak{p}$ is unramified for $\mathfrak{p} \notin S$, we say ρ is unramified outside S.

The next statement (see [10]) is useful for the precise formulation of the Langlands Conjecture.

THEOREM 3. *Let ρ, ρ' be two cuspidal automorphic representations of* $\mathrm{GL}(n, A)$ *such that $\rho_\mathfrak{p} = \rho'_\mathfrak{p}$ for almost all $\mathfrak{p} \in \Pi$. Then $\rho = \rho'$.*

Now consider the Galois side. Let $\bar{K}$ be a separable algebraic closure of K, and let W_K be the dense subgroup of $\mathrm{Gal}(\bar{K}/K)$ consisting of elements which induce on $\bar{k}$ the map $x \to x^{q^n}$ for some $n \in \boldsymbol{Z}$. For every $\mathfrak{p} \in \Pi$ we have an imbedding $W_{K_\mathfrak{p}} \hookrightarrow W_K$ defined up to conjugation in W_K (see [12]).

DEFINITION. Let τ be a finite dimensional $\bar{\boldsymbol{Q}}_l$-representation of W_K. We say that τ is unramified at $\mathfrak{p}$ if the restriction $\tau_\mathfrak{p}$ of τ to $W_{K_\mathfrak{p}}$ is unramified.

REMARK 1. It is known that for every such τ there exists a finite set $S \subset \Pi$ such that τ is unramified outside S.

REMARK 2. If τ is unramified at $\mathfrak{p}$, then $\tau(\mathrm{Fr}\ \mathfrak{p})$ is well defined up to conjugation.

THEOREM 4 (SEE [12]). *Let τ and τ' be two irreducible representations of W_K, and suppose that, for almost all $\mathfrak{p} \in \Pi$, $\tau(\mathrm{Fr}\ \mathfrak{p})$ is conjugate to $\tau'(\mathrm{Fr}\ \mathfrak{p})$. Then τ is equivalent to τ'.*

We can now formulate the Langlands Conjecture. *There exists a one-one correspondence $\varphi : \rho \mathrel{\mathscr{L}} \tau$ between the set $\mathrm{GL}(n, A)_c^\wedge$ and the set $(W_K)_n^\wedge$ which has the following property*:

For every cuspidal automorphic ρ there exists a finite set $S \subset \Pi$ such that for all $\mathfrak{p} \in \Pi - S$ the representations ρ and τ are unramified outside S and $\rho_\mathfrak{p} \sim \tau_\mathfrak{p}$ (see Theorem 2).

REMARK 1. Theorems 3 and 4 imply the uniqueness of such a correspondence (if it exists).

REMARK 2. There is another conjecture, in which we ask for a one-one correspondence between cuspidal automorphic representations and systems of l-adic representations of W_K. It follows from results of Deligne [3] and Drinfeld that this is true for $n = 2$.

REMARK. We can deduce from the global conjecture the following local one.

Let M be a local field of positive characteristic.

There exists a one-one correspondence between super cuspidal representations ρ_M of $\mathrm{GL}(n, M)$ and irreducible n-dimensional representations τ_M of W_M. This correspondence is such that for every global field K, for every point $\mathfrak{p}_0 \in \Pi$ such that $M \simeq K_{\mathfrak{p}_0}$ and for every automorphic representation $\rho = \widehat{\otimes}\rho_\mathfrak{p}$ such that $\rho_{\mathfrak{p}_0}$ is super cuspidal, the restriction $\tau_{\mathfrak{p}_0}$ of the representation $\tau \sim \rho$ of W_K on $W_{K_{\mathfrak{p}_0}} \simeq W_M$ corresponds to $\rho_{\mathfrak{p}_0}$.

We can now ask ourselves a rather general question: how can we arrange a one-one correspondence between representations of two different groups, say G and W?

One way is for every representation ρ of G to define a function Γ_ρ on some set X and to do the same for every representation τ of W. Then we can say that $\rho \sim \tau$ iff $\Gamma_\rho \equiv \Gamma_\tau$. If $G = \mathrm{GL}(2, \boldsymbol{A})$ or $\mathrm{GL}(3, \boldsymbol{A})$, $W = W_K$, X is the set of characters of idele class group $K^*\backslash \boldsymbol{A}^*$ and Γ_ρ and Γ_τ are Γ-functions, this method works very well (see [7], [9]) and permits us for every system of l-adic irreducible 2 or 3-dimensional representations τ of W_K to find the corresponding representation ρ of $\mathrm{GL}(2, \boldsymbol{A})$ or $\mathrm{GL}(3, \boldsymbol{A})$. But so far this approach does not give us the possibility of going in the opposite direction and constructing a representation τ of W_K from a representation ρ of $\mathrm{GL}(n, \boldsymbol{A})$.

Another more explicit approach to producing a correspondence between representations of two groups G and W was first used by H. Weyl [13] who considered the case $G = \mathrm{GL}(n, \boldsymbol{R})$, $W = S_m$ = the symmetric group. He considered the representation of $G \times W$ on the tensor power $L = V^{\otimes m}$ of $V \simeq \boldsymbol{R}^m$. We can decompose L in a direct sum $L = \bigoplus L_i$ of $G \times W$-invariant irreducible subspaces. As is well known, L_i is a tensor product $L_i = \rho_i \otimes \tau_i$ where ρ_i are irreducible representations of $\mathrm{GL}(n, \boldsymbol{R})$ and τ_i are irreducible representations of S_m. He found that (for $n \geqq m$) in this way we get a one-one correspondence between representations of S_m and part of the representations of $\mathrm{GL}(n, \boldsymbol{R})$. This means that (a) $\{\tau_i\} = \hat{S}_m$, (b) if i, j are such that $\rho_i \simeq \rho_j$ then $i = j$. Drinfeld applies this method to the case when $G = \mathrm{GL}(2, \boldsymbol{A})$, $W = W_K$. The first idea would be to construct a $\bar{\boldsymbol{Q}}_l$-representation (T, L) [1] of $\mathrm{GL}(2, \boldsymbol{A}) \times W_K$ with the following properties. When we consider the decomposition of $L = \bigoplus L_i$ into a sum of $G \times W$-invariant irreducible subspaces and write each L_i as $\rho_i \otimes \tau_i$, $\rho_i \in \mathrm{GL}(\hat{2}, \boldsymbol{A})_a$, $\tau_i \in (\hat{W}_K)_2$ then (a) $\{\rho_i\} = \mathrm{GL}(2, \boldsymbol{A})_a^\wedge$; (b) for all i, $\tau_i \sim \rho_i$. But, unfortunately, such a representation (T, L) does not exist.

It might be useful to explain why not. To do this we must recall some definitions and results which are well known for representations of finite groups (see [11]) and can be easily restated and reproved for admissible representations.

So let G be a group and (ρ, V) be an irreducible admissible $\bar{\boldsymbol{Q}}_l$-representation of G. We denote by $M(\rho) \subset \bar{\boldsymbol{Q}}_l$ the field of definition of the equivalence class of ρ. That is, $M(\rho)$ is the field which corresponds to the subgroup $\mathfrak{g}_\rho \subset \mathrm{Gal}(\bar{\boldsymbol{Q}}_l/\boldsymbol{Q}_l)$ consisting of elements $G \in \mathrm{Aut}(\bar{\boldsymbol{Q}}_l : \boldsymbol{Q}_l)$ such that $\rho^\sigma \sim \rho$.

Definition. We say that ρ is *unobstructed* if it can be realized over $M(\rho)$. That means that there exist a vector space V_0 over $M(\rho)$ and a morphism $\rho_0 : G \to \mathrm{Aut}\, V_0$ such that $(\rho_0, V_0 \otimes_{M(\rho)} \bar{\boldsymbol{Q}}_l)$ is equivalent to (ρ, V).

The following two statements are well known (see [11]):

Lemma 1. *Suppose that (ρ, V) is a representation of G, $H \subset G$ is a subgroup and* $\dim V^H = 1$. *Then ρ is unobstructed.*

Lemma 2. *Suppose that L is the space of a smooth $\boldsymbol{Q}_l$-representation of $G \times W$, $(\rho, V) \in \hat{G}$, $(\tau, C) \in \hat{W}$ are such that $\mathrm{Hom}_W(C, L) \simeq V$ as a G-module. Suppose that ρ is unobstructed. Then τ is also unobstructed.*

Lemma 3. *Every irreducible representation of* $\mathrm{GL}(2, \boldsymbol{A})$ *is unobstructed.*

[1]Because (T, L) is uniquely determined up to isomorphism by its properties, it is natural to expect that it is defined over $\boldsymbol{Q}_l$ and not only over $\bar{\boldsymbol{Q}}_l$.

PROOF. For every positive ideal $\mathfrak{a}$ of K we denote by $\Gamma_{\mathfrak{a}} \subset \mathrm{GL}(2, A)$ the subgroup of matrices with integer elements which preserves the vector (0, 1) mod $\mathfrak{a}$. It is known (see [10]) that there exists an ideal $\mathfrak{a}$ such that $\dim V^{\Gamma_{\mathfrak{a}}} = 1$. So our lemma follows from Lemma 1. On the other hand, it is easy to construct a two-dimensional representation τ of W_K which is obstructed. For example, we can take τ to be induced from a one-dimensional representation of a subgroup of index two in W_K. In this case we can (see [1]) construct $\rho \in \mathrm{GL}(2, A)^{\wedge}_c$ such that $\rho \sim \tau$. So we see that a representation (T, L) with properties (a) and (b) does not exist.

What is to be done? We know that for every irreducible representation τ of any group H the representation $\tau \otimes \hat{\tau}$ (where $\hat{\tau}$ = contragredient of τ) of $H \times H$ is unobstructed. So we can try to realize $\bigoplus \rho \otimes \tau \otimes \hat{\tau}$ ($\rho \in \mathrm{GL}(2, A)^{\wedge}_c$, $\tau \sim \rho$). This is almost possible. Among other things, we must consider the direct integral instead of the direct sum.

To formulate precisely the result of Drinfeld we need some more definitions.

DEFINITION. Let (ρ, V) be a representation of $\mathrm{GL}(n, A)$. We say that ρ is a *graded* representation if we can write V as a direct sum $V = \bigoplus_{t \in \mathbf{Z}} V^t$ in such a way that for every $g \in \mathrm{GL}(n, A)$

$$\rho(g)V^t = V^{t+\nu(\det(g))}.$$

REMARK. It is clear that for any graded representation (ρ, V) the operators $\rho(f)$, $f \in C_c^\infty(\mathrm{GL}(n, A))$, are not of trace class and we cannot define the character Tr_ρ. But sometimes we can define the regularized character

$$\mathrm{\bar{T}r}_\rho(f) \stackrel{\text{def}}{=} \mathrm{Tr}\, P_0 \rho(f)\, P_0, \qquad f \in \varphi(\mathrm{GL}(n, A)),$$

where P_0 is the projection onto V^0.

It is not hard to check that the definition of $\mathrm{\bar{T}r}_\rho$ is independent of the gradation $V = \bigoplus V^t$.

Let (ρ, V) be a graded representation, and let $x \in \hat{A}^*$. We denote by V_x the quotient of V by the subspace generated by $\{x(\det g)v - \rho(g)v\}$, $v \in V, g \in \mathrm{GL}(n, A)$.

DEFINITION. We say that (ρ, V) is completely reducible if, for all $x \in \hat{A}^*$, V_x is a direct sum of irreducible representations.

The following result is easy.

LEMMA. *Let (ρ, V), (ρ', V') be two graded completely reducible representations such that $\mathrm{\bar{T}r}_\rho$ and $\mathrm{\bar{T}r}_{\rho'}$ exist and $\mathrm{\bar{T}r}_\rho = \mathrm{\bar{T}r}_{\rho'}$. Then $\rho \sim \rho'$.*

Now we can formulate the "constructive" variant of Drinfeld's result.

THEOREM 6. *There exists a representation (ρ, V) of $\mathrm{GL}(2, A) \times W_K \times W_K$ such that*

(a) *the restriction of ρ to $K^* \subset$ Center $\mathrm{GL}(2, A)$ is trivial;*

(b) *for every $x \in \hat{A}^*/K^*$, V_x is a direct sum of irreducible representations*

$$V_x = \bigoplus V_{x,i}, \qquad V_{x,i} = \rho_{x,i} \otimes \tau_{x,i} \otimes \hat{\tau}_{x,i};$$

(c) $\mathrm{GL}(2, A)^{\wedge}_c = \{\rho_{x,i}\}_{x,i}$;

(d) *for all (x, i), $\rho_{x,i} \sim \tau_{x,i}$.*

Of course, the proof of this theorem consists of two parts: (A) construction of (ρ, V) and (B) proof that (ρ, V) satisfies conditions (a)–(d).

We shall only briefly discuss both parts. First, we consider a geometric interpretation of $W_K \times W_K$. Let Λ be the field of functions on $X \times X$, $\Lambda_s \supset \Lambda$ its separable closure, $W_\Lambda \subset \mathrm{Gal}(\Lambda_s/\Lambda)$ a subgroup consisting of elements which induce the map $x \to x^{q^n}$, $n \in \boldsymbol{Z}$, on $\bar{k}$.

We have the natural map $\pi: W_\Lambda \to W_K \times W_K$, which is simply the restriction of any $\sigma \in W_\Lambda \subset \mathrm{Aut}(\Lambda_s/\Lambda)$ to $K_s \otimes K_s \subset \Lambda_s$. It is clear that $\mathrm{Im}\,\pi$ consists of pairs $(\sigma', \sigma'') \in W_K \times W_K$ such that $\sigma'|\bar{k} = \sigma''|\bar{k}$. Drinfeld defines an extension

$$(**) \qquad 0 \longrightarrow \dot{W}_\Lambda \longrightarrow \tilde{W}_\Lambda \longrightarrow \boldsymbol{Z} \longrightarrow 0$$

and an epimorphism $\tilde{\pi}: \tilde{W}_\Lambda \to W_K \times W_K$.

To do this consider the group $\check{W}_\Lambda$ of automorphisms γ of the algebraic closure $\bar{\Lambda}$ of Λ such that the restriction of γ to the perfect closure $\Lambda_{\mathrm{per}}^{K_{\mathrm{per}} \otimes K_{\mathrm{per}}} \subset \Lambda$ has the form $\mathrm{Fr}_1^m \mathrm{Fr}_2^n$ where Fr_1, Fr_2 are partial Frobeniuses on $K_{\mathrm{per}} \otimes K_{\mathrm{per}}$. We define $\tilde{W}$ to be $\check{W}_\Lambda/\{$the subgroup generated by the Frobenius$\}$.

We have the natural imbedding $W_\Lambda \to \check{W}_\Lambda$, and it gives the extension (**).

REMARK. Analogously, we may define groups $\check{W}_K$ and $\check{W}_K$, but in this case we will have $\tilde{W}_K = W_K$.

Our morphism π extends to $\tilde{\pi}: \tilde{W}_\Lambda \to \tilde{W}_K \times \tilde{W}_K = W_K \times W_K$.

THEOREM 7 (DRINFELD). (1) *$\tilde{\pi}$ is an epimorphism.*

(2) *For every finite group H and every homomorphism $\varphi: \tilde{W}_\Lambda \to H$, we can write φ as a composition $\varphi = \bar{\varphi} \circ \tilde{\pi}$, where $\bar{\varphi}$ is a homomorphism $\bar{\varphi}: W_K \times W_K \to H$.*

The proof of this theorem is not difficult, but in order to present it we would have to introduce some new definitions, and this would take too long. So we leave the proof to the reader.

Now we can try to imagine the possible construction of the representation (ρ, V).

As we know, we can consider K as a field of functions on a smooth, projective, absolutely irreducible curve X over a finite field k. Let $S = X \times X$. Suppose that we can define a projective system M_i, $i \in I$, of algebraic varieties over the generic point η of S such that: (a) we have the action of $\mathrm{GL}(2, \boldsymbol{A})$ on the projective limit $M = \mathrm{proj\,lim}\, M_i$ (we consider all M_i as Λ-varieties),

(b) we have liftings of partial Frobeniuses Fr_1, Fr_2 (as endomorphisms of $k(\eta)$) to the endomorphisms of M_i, for all $i \in I$, in such a way that $\mathrm{Fr}_1 \circ \mathrm{Fr}_2$ lifts to the Frobenius on M_i.

Then we can construct a representation (ρ_M, V_M) of $\mathrm{GL}(2, \boldsymbol{A}) \times W_K \times W_K$. To do this fix any integer r and consider $V_M = \mathrm{inj\,lim}\, H_c^r(M_i, \boldsymbol{Q}_l)$.[2] Then the group $\mathrm{GL}(2, \boldsymbol{A}) \times \tilde{W}_\Lambda$ acts naturally on V_M, and $\tilde{W}_\Lambda$ preserves the images of $H_c^r(M_i, \boldsymbol{Q}_l)$ in V_M for every $i \in I$. It follows now from Theorem 7 that the action of $\mathrm{GL}(2, \boldsymbol{A}) \times \tilde{W}_\Lambda$ can be factored through $\mathrm{GL}(2, \boldsymbol{A}) \times W_K \times \dot{W}$.

Drinfeld's construction actually follows this scheme. But the varieties M_i are not of finite type. To explain why this is so we have to remind ourselves what we want to realize. We want to obtain a representation (ρ, V) of $\mathrm{GL}(2, \boldsymbol{A}) \times W_K \times W_K$

[2] By H_c^r we mean l-adic cohomology with compact support.

such that its restriction to GL(2, A) will be four times the representation in the space L of cuspidal locally constant functions on GL(2, K)\GL(2, A). We see that in the very definition of L we describe it as a subspace in the bigger space $\check{L}$ of all locally constant functions with compact support on GL(2, K)\GL(2, A). So we can expect that the only way to realize the representation (ρ, V) is to realize it as a subspace of $(\check{\rho}, \check{V})$, and that the restriction of $(\check{\rho}, \check{V})$ to GL(2, A) will be a multiple of $\check{L}$; we can try to realize $\check{V}$ as proj lim $H_c^r(M_i, Q_l)$ and V as a subspace of $\check{V}$.

But the representation $\check{L}$ is "big". This means that for a compact open subgroup $U \subset$ GL(2, A) and a character $X \in (A^*/K^*)^\wedge$, the subspace $(\check{L}_x)^U$ of U-invariant vectors in $\check{L}_x$ is infinite dimensional. This means (see below) that dim $H^r(M_i^0, Q_l) = \infty$ [3] and M_i^0 cannot be a variety of a finite type.

After this long explanation of "why the construction cannot be very nice", we describe (but will not present) Drinfeld's construction.

For every positive divisor D on X, Drinfeld defines an algebraic space M_D over η, and for every $D' \supset D$ he defines a morphism $M_{D'} \to M_D$. This space is the union $M_D = \bigcup_{n=-\infty}^{\infty} M_D^n$ of connected irreducible components, all the spaces M_D^{2n} (resp. M_D^{2n+1}), $n \in Z$, are isomorphic. They have the following structure. (1) Each space M_D^n is a union $M_D^n = \bigcup M_D^{K, n-K}$ of surfaces of finite type $M_D^{K, n-K}$, (2) $M_D^{K+1, n-K-1} \supset M_D^{K, n-K}$, and (3) the difference $M_D^{K+1, n-K-1} - M_D^{K, n-K}$ is a union of affine lines. Drinfeld also defines compactifications $\bar{M}_D^{K, n-K}$ of each subspace $M_D^{K, n-K}$, and shows that the birational isomorphism $M_D^{K+1, n-K-1} \sim M_D^{K, n-K}$ can be extended to a morphism $\Pi_D^{K, n}: \bar{M}_D^{K+1, n-K-1} \to \bar{M}_D^{K, n-K}$ such that (a) the composition of π with the natural imbedding $M_D^{K, n-K} \hookrightarrow M_D^{K+1, n-K-1}$ is the identity and (b) the restriction of π to the boundary $\Gamma_D^{K, n} =^{\text{def}} \bar{M}_D^{K+1, n-K-1} - \bar{M}_D^{K+1, n-K-1}$ is a radical morphism. Define $\bar{M}_D =$ proj $\lim_K \bar{M}_D^{K, n-K}$ and $\bar{M} =$ proj $\lim_D \bar{M}_D$. Drinfeld shows that the partial Frobeniuses and GL(2, A) act naturally on $\bar{M}$. Now we can define

$$\check{V} = H^2(\bar{M}, Q_l) \overset{\text{def}}{=} \lim_{K, n, D} H^2(\bar{M}_D^{K, n-K}, Q_l).$$

To define V we consider the subspace $V_e \subset \check{V}$ which is generated by classes in $H^2(\bar{M}, Q_l)$ of rational curves on $\bar{M}$.

REMARK. It follows from (3) (and the existence of a GL(2, A)-action on $\bar{M}$) that we have an infinite number of rational curves on each $M_D^{K, n-K}$, so $M_D^{K, n-K}$ is an example of a surface which has an infinite number of rational curves on it without having a family. It is very possible that for large D, $M_D^{K, n-K}$ is a surface of generic type.

It can be proved that the restriction to V_e of the canonical cup-product on $H^2(\bar{M}, Q_l)$ is nondegenerate. Drinfeld defines V to be the orthogonal complement of $\check{V}_e$ in V (or you may consider V as the quotient space $\check{V}/V_e$).

Of course, the partial Frobeniuses preserve V_e, and as was explained before, we obtain a representation ρ of GL(2, A) $\times W_K \times W_K$ on V.

THEOREM 8. *The representation* (ρ, V) *satisfies the conditions of Theorem* 6.

We have not presented the actual construction of $\bar{M}$. But suppose it given. How

[3] M_0^i is a connected component of M_i.

can we prove that it gives us the corresponding representation (ρ, V) which realizes the Reciprocity Law?

The only known way to do this is based on the Trace Formula. We have the following general statement.

Let (ρ, V) be a graded representation of $\mathrm{GL}(n, A) \times W_K \times W_K$ such that

(a) For every Schwartz-Bruhat function on $\mathrm{GL}(n, A)$, $\mathrm{Tr}(f)$ exists.

(b) For any two places $\mathfrak{p}', \mathfrak{p}''$ of K, any number $m', m'' \in \boldsymbol{Z}^+$ and any S-B function f on $\mathrm{GL}(n, A^{\mathfrak{p}', \mathfrak{p}''})$, we have the equality

$$(!)\qquad \mathrm{Tr}(X^0_{\mathfrak{p}'} \otimes X^0_{\mathfrak{p}''} \otimes f) \times \mathrm{Fr}^{m'}_{\mathfrak{p}'} \times \mathrm{Fr}^{m''}_{\mathfrak{p}''})_V = \mathrm{Tr}(X^{m'}_{\mathfrak{p}'} \times \check{X}^{m''}_{\mathfrak{p}''} \times f)_L, \quad \text{where } \check{X}^m_{\mathfrak{p}}(g) \overset{\mathrm{def}}{=} X^m_{\mathfrak{p}}(g^{-1}).$$

Then the representation (ρ, V) realizes the Reciprocity Law.

REMARK. By $\mathrm{Fr}^{m'}_{\mathfrak{p}'}$ we denote the measure on W_K which can be defined in the following way. First of all, consider the local group $W_{K_{\mathfrak{p}'}}$. It contains the inertia group $I_{\mathfrak{p}'}$, and $W_{K_{\mathfrak{p}'}}/I_{\mathfrak{p}'} \approx \boldsymbol{Z}$. We denote by $\mu^{m'}_{\mathfrak{p}'}$ the $I_{\mathfrak{p}'}$-invariant measure on $W_{K_{\mathfrak{p}'}}$ which is concentrated on the preimage of m' and satisfies $\int_{W_{K_{\mathfrak{p}'}}} \mu^{m'}_{\mathfrak{p}'} = 1$. We have an imbedding $W_{K_{\mathfrak{p}'}} \hookrightarrow W_K$ (which is defined up to conjugation), and we denote by $\mathrm{Fr}^{m'}_{\mathfrak{p}'}$ the image of $\mu^{m'}_{\mathfrak{p}'}$ under this imbedding.

Given these remarks, the proof of this statement is rather standard, and we will not present it.

To apply it to our case we must first of all define a graded structure on V and secondly explain how to compute both sides.

The first part is easy. Our algebraic spaces M_D are disjoint unions of M^n_D. The same is true for $\bar{M}_D$ and $\bar{M} = \bigcup \bar{M}^n$. So we can write $V = \bigoplus V^t$, where $V^t = H^2(M^{2t} \cup M^{2t+1}, \boldsymbol{Q}_l)$. When you have the definition of $\bar{M}$, you will see that the condition $\rho(g)V^t = V^{t+\nu(\det g)}$, $g \in \mathrm{GL}(2, A)$, is satisfied.

To find the right side of $\mathrm{Tr}(X^{m'}_{\mathfrak{p}'} \times \check{X}^{m''}_{\mathfrak{p}''} \times f)$, we can apply the Selberg Trace Formula [7] which, fortunately, is known for $\mathrm{GL}(2, A)$.

To do this we need only (modulo rather complicated computations) to find "orbital integrals" of $X^m_{\mathfrak{p}}$, that is, for very conjugacy class $\Omega \subset \mathrm{GL}(2, K_{\mathfrak{p}})$ to find the integral $\int_\Omega X^m_{\mathfrak{p}}(w)\, dw$. The answer is well known (see [8]). In fact, Drinfeld has solved the corresponding problem for $\mathrm{GL}(n, K_{\mathfrak{p}})$. Let me describe his result, because it might be useful for people who try to prove the Reciprocity Law for $\mathrm{GL}(n)$.

THEOREM 9. (1) *If Ω is a nonsemisimple conjugacy class in* $\mathrm{GL}(n, K_{\mathfrak{p}})$, *then* $\int_\Omega X^m_{\mathfrak{p}}\, dw = 0$.

Let Ω be a semisimple class, $\gamma \in \Omega$. Then $Z_G(\gamma) = \prod \mathrm{GL}(n_i, L_i)$, where $L_i \supset K_{\mathfrak{p}}$ are finite extensions of $K_{\mathfrak{p}}$, $\sum n_i[L_i\colon K_{\mathfrak{p}}] = n$. So we can write $\gamma = \prod \gamma_i$, $\gamma_i \in L^*_i$.

(2) *If $\int_\Omega X^m_{\mathfrak{p}}(w)\, dw \neq 0$ then there exists j, such that $n_i\nu(N_{L_j/K_{\mathfrak{p}}}(\gamma_j)) = m$, and $\nu(N_{L_i:K_{\mathfrak{p}}}(\gamma_i)) = 0$ for all $i \neq j$. To define the $\int_\Omega$ we have to fix the Haar measure on Ω. Because $\Omega = Z_G(\gamma)\backslash \mathrm{GL}(n, K_{\mathfrak{p}})$, it is sufficient to fix Haar measure dr on $\mathrm{GL}(r, L)$ for all $r \in \boldsymbol{Z}^*$, $L \supset K_{\mathfrak{p}}$ a finite extension. We do this by the condition* meas $\mathrm{GL}(r, o_L) = 1$.

(3) *Let $\gamma \in \Omega_\gamma$ be a semisimple element satisfying the conditions in* (2). *Then*

$$\int_{\Omega_\gamma} X^m_{\mathfrak{p}}(w)\, dw = (1 - q_i)(1 - q_i^2) \cdots (1 - q_i^{n_i - 1})\, [l_i, k],$$

where l_i is the residue field of L_i and q_i = card l_i.

The proof of this theorem is based on the explicit formula for $X_{\mathfrak{p}}^m$ which Drinfeld found. He proved that

(1) $X_{\mathfrak{p}}^m(g) \neq 0 \Rightarrow g \in \mathrm{GL}(n, o_{\mathfrak{p}})$ and $\nu(\det g) = m$.

(2) If g satisfies (1), then $X_{\mathfrak{p}}^m(g) = (1-q)(1-q^2)\cdots(1-q^{r-1})$, where $n-r$ is the rank of the reduction of g mod π. □

To finish, we have to explain how to compute the left side of (!). Of course, it is sufficient to consider the case when f is the characteristic function of a set $U_D g U_D$ for $g \in G$, D a divisor of X, $U_D \subset \mathrm{GL}(n, A)$ the congruence compact open subgroup of integer adelic matrices equal to Id mod D. To do this, fix some $N \geqq 1$ and let ${}^N\bar{M}_D = \bigcup_n \bar{M}_D{}^{[(n+N)/2],[(n-N+1)/2]}$. The space ${}^N M_D$ is a graded union of algebraic spaces of finite type. By definition, we have the projection Π: $\bar{M} \to {}^N\bar{M}_D$. Denote by $\Gamma_{g,D} \subset {}^N\bar{M}_D \times {}^N\bar{M}_D$ the image of the map $\Pi \times \Pi \circ g$: $\bar{M} \to {}^N\bar{M}_D \times {}^N\bar{M}_D$. This $\Gamma_{g,D}$ defines a correspondence on ${}^N\bar{M}_D$. Drinfeld proved that the ${}^N\bar{M}_D$ are quasi-smooth surfaces (i.e., the étale cohomology of ${}^N\bar{M}_D$ with support in any geometric point is the same as in the smooth case). So $\Gamma_{g,D}$ defines an operator $A_{g,D}$ in $H^2({}^N\bar{M}_D, \bar{Q}_l)$.

The variety ${}^N\bar{M}_D$ is a surface over the generic point η of $X \times X$. For almost all closed points $(\mathfrak{p}', \mathfrak{p}'')$ of $X \times X$. it has good reduction R over $(\mathfrak{p}', \mathfrak{p}'')$. The cohomology of R is isomorphic to $H^*({}^N\bar{M}_D, \bar{Q}_l)$, and we can consider the operator $C_{g,D} = \mathrm{Fr}_{\mathfrak{p}'}^{m'} \otimes \mathrm{Fr}_{\mathfrak{p}''}^{m''} \otimes A_{g,D}$ on $H^2(R, \bar{Q}_l)$. The left side of (!) reduces to computing Tr C_{g,D_μ}. To do this, one applies the Lefschetz Trace Formula (see [**8**]).

It tells us that

$$\sum_{i=1} (-1)^i \operatorname{Tr}(C_{g,D} | H^i({}^N\bar{M}_D, \bar{Q}_l)) = \sum_{x \in \Delta \cap \Gamma_{g,D}} n(x),$$

where $\Delta \subset R \times R$ is the diagonal and $\bar{\Gamma}_{g,D}$ is the reduction of $\Gamma_{g,D}$.

First look at the left side. Using the existence of the two systems of rational curves on M, one proves that $H^1({}^N\bar{M}_D, \bar{Q}_l) = H^3({}^N\bar{M}_D, \bar{Q}_l) = 0$. It is easy to find H^0 and H^4, so it remains to compute the right side.

To do this, Drinfeld describes the geometric points of the reduction of $\bar{M}$ over $(\mathfrak{p}', \mathfrak{p}'')$ as a $\mathrm{Fr}_{\mathfrak{p}'} \times \mathrm{Fr}_{\mathfrak{p}''} \times \mathrm{GL}(2, A^{\mathfrak{p}',\mathfrak{p}''})$-module.[4] It gives us the description of $\Delta \cap \Gamma_{g,D}$. If R were smooth, we could conclude that for large m', m'', $n(x) = 1$. Using the fact that R is quasi-smooth, he proves that this is asymptotically correct (for large m', m''), and comparing with the Selberg Trace side, he deduces (!).

References

1. P. Cartier, *Representations of a $\mathfrak{p}$-adic group: A survey*, these PROCEEDINGS, part 1, pp. 111–155.

2. P. Deligne, *Formes modulaires et représentations de* GL(2), Lecture Notes in Math., vol. 349, Springer-Verlag, Berlin and New York.

3. ———, *Les constants des équations fonctionnelles des functions L*, Lecture Notes in Math., vol. 349, Springer-Verlag, Berlin and New York.

4. ———, *Cohomologie étale*, SGA $4\frac{1}{2}$, Lecture Notes in Math., vol. 569, Springer-Verlag, Berlin and New York.

[4] $A^{\mathfrak{p}',\mathfrak{p}''}$ are adèles without $\mathfrak{p}'$ and $\mathfrak{p}''$ components.

5. D. Flath, *Decomposition of representations into tensor products*, these PROCEEDINGS, part 1, pp. 179–183.

6. I. M. Gelfand, M. I. Graev and I. I. Piatetski-Shapiro, *Representation theory and automorphic functions*, Saunders, 1969.

7. H. Jacquet and R. P. Langlands, *Automorphic forms on* GL(2), Lecture Notes in Math., vol. 114, Springer-Verlag, Berlin and New York.

8. R. P. Langlands, *Modular forms and l-adic representations*, Lecture Notes in Math., vol. 349, Springer-Verlag, Berlin and New York.

9. I. I. Piatetski-Shapiro, *Converse theorem for* GL(3), Mimeo Notes No. 15, Univ. of Maryland, 1975.

10. ———, *Multiplicity one theorems*, these PROCEEDINGS, part 1, pp. 209–212.

11. J.-P. Serre, *Représentation linéaires des groupes finis*, Hermann, Paris, 1967.

12. J. Tate, *Number theoretic background*, these PROCEEDINGS, part 2, pp. 3–26.

13. H. Weyl, *The classical groups, their invariants and representations*, Princeton Univ. Press, Princeton, 1939.

HARVARD UNIVERSITY

Proceedings of Symposia in Pure Mathematics
Vol. 33 (1979), part 2, pp. 357–379

AUTOMORPHIC FORMS ON GL_2 OVER FUNCTION FIELDS (AFTER V. G. DRINFELD)

G. HARDER AND D. A. KAZHDAN

Introduction. This is a report on unpublished work of V. G. Drinfeld on automorphic forms over function fields. Our aim is to give a description of the so-called scheme of 'shtuka', to discuss the computation of the trace of Frobenius on the cohomology of this scheme in terms of the Selberg trace formula and the proof of the Ramanujan conjecture. We will not discuss the much more subtle question concerning the reciprocity law which involves the compactification of the above moduli scheme.

We shall not give detailed proofs, but we shall try to give references and hints so that the reader may fill the gaps.

1. The moduli space of *FH*-sheaves.

1.0. Let $X/\boldsymbol{F}_q$ be a smooth projective curve. Let $\bar{X} = X \times_{\boldsymbol{F}_q} \bar{\boldsymbol{F}}_q$; Drinfeld introduces some objects over $\bar{X}$ which he calls 'shtuka'. These objects are vector bundles over $\bar{X}$ together with some additional structure. These additional structure data involve the Frobenius and some kind of 'Hecke modifications'. Therefore we suggest to call these objects *FH*-sheaves.

1.1. *Vector bundles.* Let $X/\boldsymbol{F}_q$ be our given curve, let $K/\boldsymbol{F}_q$ be its field of meromorphic functions. We denote

$$\bar{X} = X \times_{\boldsymbol{F}_q} \bar{\boldsymbol{F}}_q, \quad K_\infty = K \cdot \bar{\boldsymbol{F}}_q.$$

The space of adeles of K (resp. K_∞) is denoted by A (resp. A_∞). The geometric points of $X/\boldsymbol{F}_q$ are denoted by $v, w \in X(\bar{\boldsymbol{F}}_q)$. For any $v \in X(\bar{\boldsymbol{F}}_q)$ we denote the completion of K_∞ with respect to v by K_v and its ring of integers by $\mathcal{O}_v$.

The valuations of the field K/F_q will be denoted by $\mathfrak{p}, \cdots$. Then $K_\mathfrak{p}$ will be the completion of K at $\mathfrak{p}$ and $\mathcal{O}_\mathfrak{p}$ its ring on integers. If $v \in X(\boldsymbol{F}_q)$ induces $\mathfrak{p}$ on K we write $v \to \mathfrak{p}$. The points v which lie over a fixed $\mathfrak{p}$ form an orbit under the action of the Frobenius on $X(\bar{\boldsymbol{F}}_q)$.

A vector bundle of rank d over $\bar{X}$ is a locally free sheaf of rank d over the structure sheaf $\mathcal{O}_{\bar{X}}$. We have an alternate way to describe vector bundles. Let us take a vector space V/K_∞ together with a basis $e_1, \cdots, e_d$. Let us assume that we have a family $\boldsymbol{M} = \{M_v\}_{v \in X(\bar{\boldsymbol{F}}_q)}$ of $\mathcal{O}_v$ lattices $M_v \subset V \otimes K_v$, s.t.

$$M_v = \bigoplus_{i=1}^{d} \mathcal{O}_v e_i \quad \text{for almost all } v.$$

AMS (MOS) subject classifications (1970). Primary 10D15, 14F05, 14H10.

We can associate a locally free sheaf $E/\bar{X}$ to this family of lattices by defining

$$\Gamma(\bar{U}, E) = \{x \in V \mid x \in M_v \text{ for all } v \in \bar{U}\}$$

where $\bar{U} \subset \bar{X}$ is an open subset.

On the other hand it is very easy to see that every vector bundle can be realized in this form.

Let us take for $\boldsymbol{M}_0$ the trivial family $M_{0,V} = \mathcal{O}_v e_1 \oplus \dots \oplus \mathcal{O}_v e_d$ for all $v \in X(\bar{\boldsymbol{F}}_q)$. Then we may use an adele $\boldsymbol{x} \in \mathrm{GL}_d(A_\infty)$ to produce a new family $\boldsymbol{x}\boldsymbol{M}_0 = \{x_v M_{0,v}\}_v$ and it is an easy exercise to check that we have a bijection $\mathrm{GL}_d(K_\infty)\backslash\mathrm{GL}_d(A_\infty)/\mathscr{K}_\infty \xrightarrow{\sim}$ {set of isomorphism classes of vector bundles of rank d on $\bar{X}$} where $\mathscr{K}_\infty = \prod_v \mathrm{GL}_d(\mathcal{O}_v)$.

1.2. *Level structures on vector bundles.* Let D be a positive divisor on $X/\boldsymbol{F}_q$, i.e., a divisor which is rational over $\boldsymbol{F}_q$. We define as usual $\mathcal{O}_{\bar{X}}(-D)$ = sheaf of germs of regular functions f with $\mathrm{div}(f) \geqq D$ = sheaf of ideals defining D, and for any vector bundle we define $E(-D) = E \otimes \mathcal{O}_{\bar{X}}(-D)$. A level structure (along D) on E is an $\mathcal{O}_{\bar{X}}$ isomorphism Ψ: $E/E(-D) \xrightarrow{\sim} (\mathcal{O}_{\bar{X}}/\mathcal{O}_{\bar{X}}(-D)^d)$. The isomorphism classes of vector bundles with a level structure along D are given by the elements in the coset space $\mathrm{GL}_d(K_\infty)\backslash\mathrm{GL}_d(A_\infty)/\mathscr{K}_\infty(D)$ where $\mathscr{K}_\infty(D)$ is the open congruence subgroup of $\mathscr{K}_\infty$ defined by $x \equiv 1 \bmod D$.

We are particularly interested in the two cases $d = 1$ and $d = 2$. A vector bundle $E/\bar{X}$ of rank 2 with level structure along D is called stable if for any line subbundle $L \subset E$ we have

$$\deg L < \tfrac{1}{2}(\deg E + \deg D).$$

It follows from Mumford's theory that we have a fine moduli scheme $\mathscr{M}_0 \to \mathrm{Spec}(\boldsymbol{F}_q)$ for the functor of stable bundles with level structure along D. (Compare **[6]**.)

If for any $\nu \in \boldsymbol{Z}$ we denote by $\mathscr{M}_D^{(\nu)}$ the stable vector bundles with deg det $E = \nu$ then the $\mathscr{M}_D^{(\nu)}$ are smooth quasi-projective schemes over $\boldsymbol{F}_q$.

1.3. *Modifications of vector bundles for* $d=2$. Let $E/\bar{X}$ be a vector bundle, let $v \in X(\bar{\boldsymbol{F}}_q)$. We want to define the notion of a modification of E at v. The fiber of our vector bundle $E/\bar{X}$ at v is a vector space $E_v = E \otimes (\mathcal{O}_v/\mathscr{M}_v) = E \otimes \bar{\boldsymbol{F}}_q(v)$ over $\bar{\boldsymbol{F}}_q$. A point $\alpha_v \in P^1(E_v)(\bar{\boldsymbol{F}}_q)$ defines a nonzero linear form $\alpha_v^*: E_v \to \bar{\boldsymbol{F}}_q$ which is unique up to a scalar. We define $E(\alpha_v)$ = {sheaf of germs of sections s of E for which $\alpha_v^*(s_v) = \alpha_v(s_v) = 0$}. This give us a new vector bundle on $\bar{X}$ and we have an exact sequence

$$0 \longrightarrow E(\alpha_v) \hookrightarrow E \longrightarrow \bar{\boldsymbol{F}}_q(v) \longrightarrow 0,$$

i.e., $E/E(\alpha_v)$ is a sheaf concentrated on v and the fiber over v is a one dimensional vector space over the residue field $\bar{\boldsymbol{F}}_q = \bar{\boldsymbol{F}}_q(v)$. We call $E(\alpha_v)$ a lower modification of E at v (in direction α_v).

We want to give an interpretation of this in terms of lattices. Let us assume that we have a family of lattices $\boldsymbol{M} = \boldsymbol{x}\boldsymbol{M}_0$ with $\boldsymbol{x} \in \mathrm{GL}_2(A_\infty)$. Let us pick an element

$$\alpha_v \in \mathrm{GL}_2(\mathcal{O}_v)\begin{pmatrix}\pi_v & 0\\ 0 & 1\end{pmatrix}\mathrm{GL}_2(\mathcal{O}_v)$$

where π_v is a uniformizing parameter. Then we put $\boldsymbol{\alpha}_v = (\cdots, 1, \cdots, \alpha_v, \cdots, 1 \cdots)$ and the bundle defined by $\boldsymbol{M}(\boldsymbol{\alpha}_v) = \boldsymbol{x}\boldsymbol{\alpha}_v \cdot \boldsymbol{M}_0$ defines obviously a modification of $\boldsymbol{M}$.

If we observe that the set

$$\mathrm{GL}_2(\mathcal{O}_v)\begin{pmatrix}\pi_v & 0\\ 0 & 1\end{pmatrix}\mathrm{GL}_2(\mathcal{O}_v)$$

divided by $\mathrm{GL}_2(\mathcal{O}_v)$ on the right is exactly our $\boldsymbol{P}^1(E_v)(\bar{\boldsymbol{F}}_q)$ we see that we have an interpretation of the notion of modification in terms of families of lattices.

If $w \in X(\bar{\boldsymbol{F}}_q)$ is another point we define upper modifications at w: For $\alpha_w \in \boldsymbol{P}^1(E_w)(\bar{\boldsymbol{F}}_q)$ we put $E(\tilde{\alpha}_w) = E(\alpha_w) \otimes \mathcal{O}_{\bar{X}}(w)$. In this case we have $E \hookrightarrow E(\tilde{\alpha}_w) \to \bar{\boldsymbol{F}}_q(w)$ and again we have an interpretation in terms of lattices $\boldsymbol{M} = \boldsymbol{x}\boldsymbol{M}_0 \to \boldsymbol{x}\boldsymbol{\alpha}_w M_0$ where $\tilde{\boldsymbol{\alpha}}_w = (\cdots 1 \cdots, \tilde{\alpha}_w, \cdots 1 \cdots)$ and

$$\tilde{\alpha}_w \in \mathrm{GL}_2(\mathcal{O}_w)\begin{pmatrix}\pi_w^{-1} & 0\\ 0 & 1\end{pmatrix}\mathrm{GL}_2(\mathcal{O}_w).$$

If $v \neq w$ are two geometric points and $\alpha_v \in \boldsymbol{P}^1(E_v)(\bar{\boldsymbol{F}}_q)$, $\alpha_w \in \boldsymbol{P}^1(E_w)(\bar{\boldsymbol{F}}_q)$ then we define $E(\alpha_v, \tilde{\alpha}_w) = (E(\alpha_v))(\tilde{\alpha}_w)$. We observe that $\deg E(\alpha_v, \tilde{\alpha}_w) = \deg E$.

Another important observation is that we have a canonical identification

$$E(\alpha_v, \tilde{\alpha}_w)\big|_{\bar{X}-\{v\}-\{w\}} = E\big|_{\bar{X}-\{v\}-\{w\}}.$$

This is clear from the definition of $E(\alpha_v, \tilde{\alpha}_w)$ in terms of locally free sheaves. Therefore we know: If D is a positive divisor on $X/\boldsymbol{F}_q$ and if $v, w \notin \mathrm{supp}(D)$ then a level structure ψ on E along D induces a level structure on $E(\alpha_v, \tilde{\alpha}_w)$ which will be denoted by the same letter ψ.

It is very easy to see that the data of such a double modification $E \to E(\alpha_v, \tilde{\alpha}_w) = E'$ is equivalent to give a diagram

$$\begin{matrix} E \xrightarrow{\phi_v} \mathcal{E} \\ E' \xrightarrow{\phi_w} \mathcal{E} \end{matrix} \qquad \mathcal{E} \text{ locally free of rank 2}$$

where $\deg \mathcal{E} = \deg E + 1 = \deg E' + 1$ and where $\mathcal{E}/\phi_v(E)$ is concentrated at w and $\mathcal{E}/\Phi_w(E')$ is concentrated at v. We just have to choose $\mathcal{E} = E(\tilde{\alpha}_w)$ and ϕ_v, ϕ_w are the inclusions.

1.4. *The FH-sheaves and their moduli space.* The Frobenius automorphism $x \to x^q$ of $\bar{\boldsymbol{F}}_q$ induces a map $\mathrm{Fr}: \mathrm{Spec}(\bar{\boldsymbol{F}}_q) \Rightarrow \mathrm{Spec}(\bar{\boldsymbol{F}}_q)$ and therefore a map $\mathrm{Id} \times \mathrm{Fr} : \bar{X} = X \times_{\boldsymbol{F}_q} \mathrm{Spec}(\bar{\boldsymbol{F}}_q) \to \bar{X}$.

If $E/\bar{X}$ is a vector bundle we denote by E^σ the pull-back under the above map, i.e., $E^\sigma = (\mathrm{Id} \times \mathrm{Fr})^* E$. If E has a level structure ψ along D then we get a level structure

$$\psi^\sigma: E^\sigma/E^\sigma(-D) \to (\mathcal{O}_{\bar{X}}/\mathcal{O}_{\bar{X}}(-D)^2)^\sigma = \mathcal{O}_{\bar{X}}/\mathcal{O}_{\bar{X}}(-D)^2$$

since $(\mathcal{O}_{\bar{X}}/\mathcal{O}_{\bar{X}}(-D))^2 = (\mathcal{O}_X/\mathcal{O}_X(-D))^2 \otimes_{\boldsymbol{F}_q} \bar{\boldsymbol{F}}_q$. A set of data $(E, \psi, \alpha_v, \alpha_w, \Phi)$ is called an *FH*-sheaf —or a 'shtuka'— if we have the following:

(1) D is a positive rational divisor and $v, w \in X(\bar{\boldsymbol{F}}_q)$, $v \neq w$ and $v, w \notin \mathrm{supp}(D)$.

(2) $\alpha_v \in \boldsymbol{P}^1(E_v)(\bar{\boldsymbol{F}}_q)$, $\alpha_w \in \boldsymbol{P}^1(E_w)(\bar{\boldsymbol{F}}_q)$.

(3) ψ is a level structure along D.

(4) $\Phi: (E^\sigma, \psi^\sigma) \xrightarrow{\sim} (E(\alpha_v, \tilde{\alpha}_w), \psi)$.

If we fix the two points v, w we call it an *FH*-sheaf over (v, w).

Two such *FH*-sheaves are isomorphic if the bundles are isomorphic and the isomorphism maps the corresponding data into each other.

REMARK. If $(E, \psi, \alpha_v, \alpha_w, \Phi)$ is an *FH*-sheaf and if (E, ψ) is sufficiently stable for instance, if for any line subbundle $L \subset E$ we have $\deg L \leqq \frac{1}{2}(\deg E + \deg D) - 4$ then the additional data α_v, $\tilde{\alpha}_w$, v, w and Φ are uniquely determined by the pair (E, ψ). This is again a very simple exercise.

Let us assume that we have given an *FH*-sheaf $(E, \alpha_v, \tilde{\alpha}_w, \Phi)$ and a rational positive divisor D on $X/\boldsymbol{F}_q$. We ask for the possible level structures on this *FH*-sheaf along D. The isomorphism Φ induces an isomorphism $\Phi_D\colon E^\sigma/E^\sigma(-D) \to E/E(-D)$ and we must have for our level structure map $\psi\colon E/E(-D) \to \mathcal{O}_{\bar{X}}/\mathcal{O}_{\bar{X}}(-D)^2$ that $\psi^{-1}\psi^\sigma = \Phi_D$. It follows from Lang's theorem that we can always solve this equation and the solutions form a principal homogeneous space and the action of $\mathrm{GL}_2(\mathcal{O}_X/\mathcal{O}_X(-D))$, i.e., all other solutions are of the form $g \circ \psi$ with $g \in \mathrm{GL}_2(\mathcal{O}_X/\mathcal{O}_X(-D)) = G(D)$.

We now want to show that we can construct a coarse moduli space for the functor of *FH*-sheaves. Let us fix two points $v, w \in X(\bar{\boldsymbol{F}}_q)$, different from each other and a D with $v, w \notin \mathrm{supp}(D)$. Let us denote for any $t \in \boldsymbol{N}$ by $\mathcal{M}_{D,t}$ the moduli space of vector bundles E with level structure along D which satisfy $\deg L \leqq \frac{1}{2}(\deg E + \deg D) - t$ for all line subbundles $L \subset E$. This is an open subscheme of our $\mathcal{M}_D$ above. If we look at a modification $E(\alpha_v, \tilde{\alpha}_w)$ of such a bundle it will correspond to a point in $\mathcal{M}_{d,t-1}$. One checks without too much difficulty that we find a closed subscheme over $\bar{\boldsymbol{F}}_q$, $\mathcal{H}_{v,w} \subset \mathcal{M}_{D,t} \times \mathcal{M}_{D,t-1}$ whose geometric points are the pairs of bundles $(E, E(\alpha_v, \tilde{\alpha}_w))$. If $\mathfrak{p}$, $\mathfrak{q}$ are the valuations of K induced by v and w and if $\boldsymbol{F}_{q^d}$ is the union of the residue fields of $\mathfrak{p}$ and $\mathfrak{q}$ then we have obviously that $\mathcal{H}_{v,w}$ is defined over $\boldsymbol{F}_{q^d}$.

According to a remark above we see that the directions α_v, α_w *of* the modifications are uniquely determined by (x, y) if $t \geqq 4$ (for instance). Moreover some simple infinitesimal arguments will prove that the tangent space at $\mathcal{H}_{v,w}$ in the point (x, y) projects surjectively to the tangent space of the first component and that the kernel of this projection is of dimension 2.

Now let Γ_{Fr} be the graph of the geometric Frobenius $(x, x^{(q)}) \subset \mathcal{M}_{D,t} \times \mathcal{M}_{D,t-1}$. Then it is clear that $M_{D,t}^{(v,w)} = \mathcal{H}_{v,w} \cap \Gamma_{\mathrm{Fr}}$ is a smooth two-dimensional scheme over $\boldsymbol{F}_{q^d}$ and the first components of its geometric points are exactly the bundles with a (uniquely determined) structure as *FH*-sheaves.

On $M_{D,t}^{(v,w)}$ we have an action of the finite group $G(D)$. If $D' \geqq D$ then we have an obvious mapping of some open part $\tilde{M}_{D',t}^{(v,w)} \subset M_{D',t}^{(v,w)}$ to $M_{D,t}^{(v,w)}$ (the stability condition changes as D becomes larger) and obviously we get

$$\begin{array}{ccc} \tilde{M}_{D',t}^{(v,w)} & \hookrightarrow & \tilde{M}_{D',t}^{(v,w)} \\ \downarrow & & \\ \tilde{M}_{D,t}^{(v,w)}/\ker(G(D') \longrightarrow G(D)) & \xrightarrow{\sim} & M_{D,t}^{(v,w)} \end{array}$$

and therefore we have $M_{D,t}^{(v,w)} \subsetneq M_{D',t}^{(v,w)}/\ker(G(D') \to G(D))$ and we may define

$$M_D^{(v,w)} = \operatorname*{inj\,lim}_{D' \geqq D;\, v,\, w \notin \mathrm{supp}(D')} M_{D',t}^{(v,w)}/\ker(G(D') \longrightarrow G(D))$$

for some $t \geqq 4$.

This altogether is a sketch of the proof of the following theorem.

THEOREM 1.4.1. (i) *The functor of FH-sheaves over* (v, w) *with level structure along* D *is representable as an inductive limit* $\{M_D^{(v,w)}\}/\boldsymbol{F}_{q^d}$ *of schemes. If we fix the degree of the determinant then it will be a limit of quasi-projective schemes which are quotients of smooth schemes by the action of finite groups.*

(ii) *If* $\Lambda_D = \bar{X} \times \bar{X} - \Delta - D \times \bar{X} - \bar{X} \times D$ *we can construct an inductive system* $M_D \to \Lambda_D$ *whose fiber over* (v, w) *is* $M_D^{(v,w)}$.

The assertion (ii) just follows from the fact that we can construct $\mathscr{H}_{v,w}$ with variable parameters v, w too.

REMARK. The formulation is a little bit sloppy since we are not working with the limit itself but rather with the inductive system. Later on we shall also discuss the projective system proj $\lim_D M_D^{(v,w)}$.

1.5. *Geometric properties of* $M_D^{(v,w)}$ *and* M_D.

1.5.1. Let first consider the case $d = 1$. A line bundle $L/\bar{X}$ has a structure as an FH-sheaf if $L^\sigma \to L\,(w - v)$ or equivalently $L^{-1} \otimes L^\sigma = \mathscr{O}_{\bar{X}}\,(w - v)$. This tells us that the 1-dimensional FH-sheaves form a principal homogenous space under the action of $\mathrm{Pic}_X(\boldsymbol{F}_q)$; this is the group of line bundles on X. Therefore the moduli space for 1-dimensional FH-sheaves over (v, w) is an infinite discrete set.

If we introduce a level structure it will become a principal space under the action of $\mathrm{Pic}_{X,D}(\boldsymbol{F}_q)$ of line bundles on X with a rational level structure along D.

If we fix D and vary v, w we find an infinite covering $P_D \to \Lambda_D$ whose Galois group of groups of "Decktransformationen" is exactly $\mathrm{Pic}_{X,D}(\boldsymbol{F}_q)$.

1.5.2. If $d = 2$ and (E, ψ) is an FH-sheaf with level structure then we see immediately that $(\det E, \det \psi)$ is a 1-dimensional FH-sheaf with level structure. This way we get a map $\det\colon M_D \to \tilde{P}_D$.

1.6. *Extension of FH-sheaves.* Now we shall assume that our two points v, w are not only different from each other but they are also not in the same orbit of the Frobenius map. This is the same as that v and w induce different valuations on K.

Let us assume that we have an FH-sheaf $(E, \psi, \alpha_v, \alpha_w, \Phi)$ and that we have a line subbundle $F \subset E$ such that $\Phi : F^\sigma \to F'$ where F' is the line subbundle of $E\,(\alpha_v, \tilde{\alpha}_w)$ corresponding to F (a line subbundle is given by its generic fiber). One checks very easily that there are two possibilities.

(i) α_v is nonzero on the fiber $F \otimes \bar{\boldsymbol{F}}_q(v) = F_v \subset E_v$. Then α_w is zero on F_w and $F' = F(w - v)$, i.e., F itself is a one-dimensional FH-sheaf. In this case we find that the quotient line bundle E/F is defined over $\boldsymbol{F}_q$. This follows from 1.5.2.

(ii) α_v is zero on F_v and then nonzero on F_w. In this case $F' = F = F^\sigma$ and the line bundle is defined over $\boldsymbol{F}_q$. In this case E/F is a one-dimensional FH-sheaf.

In the first case we call the FH-sheaf a left FH-extension and we shall write L instead of F and $0 \to L \to E \to H \to 0$. In the second case the FH-sheaf is a right extension and we shall write H_1 instead of F and $0 \to H_1 \to E \to L_1 \to 0$. We want to investigate a little bit more closely the conditions when an FH-sheaf is an extension and in how many ways it can be written as an extension.

Let us look at the generic fiber V/K_∞ of E. Then Φ induces a linear map Φ_k: $V^\sigma \to V$. Since $V^\sigma = V$ as an additive group this mapping Φ_K induces a map Φ_K:

$V \to V$ which is only σ^{-1}-linear, i.e.,

$$\Phi_K(\lambda^\sigma \cdot x) = \lambda \Phi_K(x).$$

Therefore—if we want our $(E, \psi, \alpha_v, \alpha_w, \Phi)$ to be an extension—this mapping Φ_K must have a one-dimensional invariant subspace $W \subset V$. Then we may choose a basis vector $e_1 \in W$ and $\bar{e}_2 \in V/W$ and we find $\Phi_K(e_1) = \lambda_1 e_1$, $\bar{\Phi}_K(\bar{e}_2) = \lambda_2 \bar{e}_2$.

PROPOSITION 1.6.1. *Let* $\mathfrak{p}$ *(resp.* $\mathfrak{q}$*) be the valuations induced by* v *(resp.* w*) on* K*. We assume* $\mathfrak{p} \neq \mathfrak{q}$*. Let* $\lambda_1, \lambda_1 \in KF_{q^n} = K_n$ *and*

$$\mu_i = \mathrm{Norm}_{K_n/K}(\lambda_i) \in K, \qquad i = 1, 2.$$

Then we have up to ordering μ_1, μ_2:

(1) μ_1 *has a zero at* $\mathfrak{p}$ *and is a unit at* $\mathfrak{q}$.

(2) μ_2 *has a pole at* $\mathfrak{q}$ *and is a unit at* $\mathfrak{p}$.

(3) *The vector space* V *decomposes in a unique way into a sum of two subspaces* $V = W \oplus W'$*, i.e., we can find a complement to* W.

(4) *If* L *(resp.* H_1*) is the subbundle of* E *induced by* W *(resp.* W'*) then we can write* $(F, \psi, \alpha_v, \alpha_w, \Phi)$ *in exactly one way as a left and in exactly one way as a right extension*:

$$0 \to L \to E \to H \to 0, \qquad 0 \to H_1 \to E \to L_1 \to 0.$$

The proof of this proposition follows from a local analysis of what happens at v and w.

We conclude this section with a discussion of the next obvious question. Given L, H or H_1, L_1 how can we describe the FH-extensions $0 \to L \to E \to H \to 0$, $0 \to H_1 \to E_1 \to L_1 \to 0$?

We discuss only the first case. Let us fix v, w and a divisor D, still with the standing assumption $\mathfrak{p} \neq \mathfrak{q}$, $v, w \notin \operatorname{supp} D$. Moreover we fix level structures

$$\psi_L\colon L/L(-D) \xrightarrow{\sim} \mathcal{O}_{\bar{X}}/\mathcal{O}_{\bar{X}}(-D), \qquad \psi_H\colon H/L(-D) \xrightarrow{\sim} \mathcal{O}_{\bar{X}}/\mathcal{O}_{\bar{X}}(-D).$$

The extensions of L by H are classified by the elements in $H^1(\bar{X}, \mathrm{Hom}(H, L))$ and the set

$$H^1(\bar{X}, \mathrm{Hom}(H, L)(-D)) = H^1(\bar{X}, H^{-1} \otimes L(-D))$$

classifies extensions with a splitting along the divisor D.

Therefore a vector bundle E which corresponds to an element $\xi \in H^1(\bar{X}, H^{-1} \otimes L(-D))$ is automatically equipped with a level structure along D.

We assume that D has large degree compared to $\deg(H^{-1} \otimes L)$. We ask for those ξ which give rise to a vector bundle with a (then unique) FH-structure. This means that we have to get a

$$\begin{array}{ccccccccc}
0 \to & L & \to & E & \to & H & \to 0 \\
 & \searrow^{i} & & \searrow & & & \\
0 \to & L(w) & \to & E & \to & H & \to 0 \\
 & \nearrow_{\Phi_L} & & \nearrow & & & \\
0 \to & L^\sigma & \to & E^\sigma & \to & H & \to 0
\end{array}$$

Here $\tilde{\Phi}_L$ is the composition $L^\sigma \xrightarrow{\Phi_L} L(w - v) \hookrightarrow L(w)$. We get a diagram

$$\begin{array}{lcl} H^1(\bar{X}, H^{-1} \otimes L(-D)) & \overset{i^*}{\searrow} & \\ & & H^1(\bar{X}, H^{-1} \otimes L(-D+W)) \\ H^1(\bar{X}, H^{-1} \otimes L^\sigma(-D)) & \underset{\Phi_L^*}{\nearrow} & \\ \| & & \\ H^1(\bar{X}, H^{-1} \otimes L(-D)) & & \end{array}$$

and to get an FH-sheaf we have to look for the classes ξ which satisfy

$$i^*(\xi) - \tilde{\Phi}_L^*(\xi^\sigma) = 0.$$

PROPOSITION 1.6.2. (1) *This equation defines a smooth one-dimensional group scheme over $\bar{F}_q$ which is isomorphic to G_a.*

(2) *It can be defined over the field of definition of (L, ψ_L, Φ_L) and (H, ψ_H).*

(3) *To each pair (L, ψ_L), (H, ψ_H) we find two affine lines $G_D^{L,H}$, $G_D^{H,L}$ of right and left FH-extensions with level structure along D.*

The proof of this proposition is again not very difficult: the assertion that the solutions of the above equation form a *connected* scheme is a little bit tricky.

REMARK 1. The FH-sheaves which we constructed as extensions have a particular kind of level structure in the sense that the level structure is adapted to the extension. The $\psi: E/E(-D) \to (\mathcal{O}_{\bar{X}}/\mathcal{O}_{\bar{X}}(-D))^2$ maps the $L/L(-D)$ to the first component in $(\mathcal{O}_{\bar{X}}/\mathcal{O}_{\bar{X}}(-D))^2$. One checks easily that any FH-extension with an adapted level structure can be obtained in the above way as a point on $G_D^{L,H}$ or $G_D^{H,L}$.

2. On the other hand we have an action of $\mathrm{GL}_2(\mathcal{O}_X/\mathcal{O}_X(-D))$ on the scheme of FH-sheaves with level structure along D (1.4). Then we may look at the translates of the affine lines $G_D^{L,H}$ and $G_D^{H,L}$ and we find some more affine lines on the moduli scheme. The points on the union of these lines correspond to the FH-extensions where the level structure is not necessarily adapted.

3. If we drop the assumption that the degree of D should be large, nothing is changed substantially. We can pass to a larger divisor and after that we divide by the corresponding group actions. We find quotients of affine lines by finite groups and this gives again affine lines.

4. If $E/\bar{X}$ is an FH-sheaf which contains a subbundle $F \subset C$ with $\deg F > \frac{1}{2} \deg E + 1$ then it is easy to see that Φ has to map F^σ to F', i.e., that E is necessarily an FH-extension. If $s \geqq 2$ is an integer and if $M_D^{(v,w)}(s)$ is the subscheme whose points are given by vector bundles E s.t. $\deg F \leqq \frac{1}{2} \deg E - s$ for all line subbundles $F \subset E$, then $M_D^{(v,w)}(s-1) - M_D^{(v,w)}(s)$ is a union of affine lines of left and right FH-extensions, where the subbundle is running over all FH-sheaves (resp. sheaves defined over F_q) of degree s and rank 1.

5. The affine lines $G_D^{L,H}$(resp. $G_D^{H_1,L_1}$) where $\deg L$ (resp. $\deg H_1$) is very small will contain a nonempty open piece where the bundles are stable but there will be more and more unstable points of higher and higher degree of instability if $\deg L$ (resp. $\deg H_1$) tends to $-\infty$.

2. The geometric points on the moduli space. In this section we want to discuss the main result which establishes a relation between the number of points on certain open pieces of our moduli spaces and the traces of Hecke operators on the space of cusp forms.

2.1. *The fundamental formula.* Our standing assumption will be that $v, w \in X(\bar{F}_q)$ and that v and w induce different valuations $\mathfrak{p}$ and $\mathfrak{q}$ on K. We consider the schemes $M_D^{(v,w)} \to \operatorname{Spec}(F_{q^d})$ where F_{q^d} is the union of the residue fields of $\mathfrak{p}$ and $\mathfrak{q}$. If $E(\psi, \alpha_v, \alpha_w, \Phi)$ is an FH-extension then we have two exact sequences

$$0 \to L \to E \to H \to 0, \qquad 0 \to H_1 \to E \to L_1 \to 0,$$

and we call the above extension strongly decomposed if $\deg L > \deg H$ or $\deg H_1 \geqq \deg L_1$. The nonstrongly decomposed FH-sheaves form an open subscheme $U_D \hookrightarrow M_D^{(v,w)}$. On U_D we have an action of the group P_D of line bundles on X with level structure along D. This group acts by tensorisation on U_D and $M_D^{(v,w)}$ and this action induces an action of the same group on the cohomology with compact supports $H_!^{\cdot}(\bar{U}_D, \bar{Q}_l)$ where $\bar{U}_D = U_D \times_{F_{q^d}} \bar{F}_q$ and $\bar{Q}_l$ is an algebraic closure of Q_l. Let $\mu \subset \bar{Q}_l$ be the group of roots of unity. For any finite character ω: $P_D \to \mu$ we denote by $H_!^{\cdot}(\bar{U}_D, \bar{Q}_l)_\omega$ the corresponding eigenspace in the cohomology. We are interested in the trace—i.e., the alternating sum of traces—of the powers Fr^m of the geometric Frobenius for $d|m$ on the cohomology

$$\operatorname{trace} \mathrm{Fr}^m | H_!^{\cdot}(\bar{U}_D, \bar{Q}_l)_\omega = \sum_{i=0}^{4} (-1)^i \operatorname{trace} \mathrm{Fr}^m | H_!^i(\bar{U}_D, \bar{Q}_l)_\omega.$$

REMARK. The scheme $U_D \to \operatorname{Spec}(F_{q^d})$ is not of finite type. We have the determinant map $\det U_D \to \tilde{P}_D$ introduced in 1.5.2. The fibers $U_D^{(\xi)}$ for $\xi \in \tilde{P}_D(\bar{F}_q)$ are quasi-projective schemes and if $a \in P_D$ then the tensorisation defines a map $U_D^{(\xi)} \to U_D^{(\xi a^2)}$.

If $P_{D,\omega}$ is the kernel of ω then

$$H_!^{\cdot}(\bar{U}_D, Q_l)_\omega \subset H_!^{\cdot}(\bar{U}_D, \bar{Q}_l)^{P_{D,\omega}} = \bigoplus_{\xi \in \tilde{P}_D / P_{D,\omega}^2} H_!^{\cdot}(\bar{U}_D^{(\xi)}, \bar{Q}_l).$$

The information contained in the above trace is therefore equivalent to some informations on the number of rational points of the $U_D^{(\xi)}$.

We now want to relate the above trace of Fr^m to the trace of some Hecke operators acting on a space of automorphic forms.

We choose an embedding $\mu \hookrightarrow C^*$. Then we can consider the given character ω also as a character with values in C^*. Let us put $\mathscr{K}_\mathfrak{p} = \mathrm{GL}_2(\mathcal{O}_\mathfrak{p})$, $\mathscr{K}_\mathfrak{q} = \mathrm{GL}_2(\mathcal{O}_\mathfrak{q})$ and $\mathscr{K}'_D \subset \prod_{\mathfrak{p}' \neq \mathfrak{p}, \mathfrak{q}} \mathrm{GL}_2(\mathcal{O}_{\mathfrak{p}'})$ to be the full congruence subgroup defined by D.

If $Z(A)$ is the centre of $\mathrm{GL}_2(A)$ we have an identification

$$Z(A)/Z(K) \cdot (Z(A) \cap \mathscr{K}_\mathfrak{p} \times \mathscr{K}_\mathfrak{q} \times \mathscr{K}'_D) \simeq P_D$$

and therefore we may view our character ω also as a complex valued character on $Z(A)$. We introduce the space

$$H_\omega = L^2_{\omega, \mathrm{disc}}(\mathrm{GL}_2(K) \backslash GL_2(A) / \mathscr{K}'_D \times \mathscr{K}_\mathfrak{p} \times \mathscr{K}_\mathfrak{q})$$

where ω is the above central character and the index disc means that we restrict our attention to the discrete spectrum [**1**, §4]. Our central character is unramified at $\mathfrak{p}$ and $\mathfrak{q}$. We introduce some algebras $\mathscr{H}_{\mathfrak{p},\omega}$, $\mathscr{H}_{\mathfrak{q},\omega}$ of Hecke operators.

The elements of $\mathscr{H}_{\mathfrak{p},\omega}$ are those functions f: $\mathrm{GL}_2(K_\mathfrak{p}) \to C$ which are compactly supported modulo the centre, and which satisfy

$$f(xz) = f(x)\, \omega(z), \qquad z \in Z(K_\mathfrak{p}),\ x \in \mathrm{GL}_2(K_\mathfrak{p}),$$

and which are $\mathscr{K}_{\mathfrak{p}}$ bi-invariant. Each such f defines an operator $T_f\colon H_\omega \to H_\omega$,

$$T_f(h)(y) = \int_{\mathrm{GL}_2(K_{\mathfrak{p}})/Z(K_{\mathfrak{p}})} h(yx) f(x^{-1})\, dx,$$

where $y \in \mathrm{GL}_2(A)$ and $\mathrm{GL}_2(K_{\mathfrak{p}}) \hookrightarrow \mathrm{GL}_2(A)$.

In these algebras we have for any $n \in \boldsymbol{Z}$ the elements $\Phi_n^{(\mathfrak{p})}$ which are defined as follows: If $\eta_{s_1, s_2} \neq 0$ is a function on $\mathrm{GL}_2(K_{\mathfrak{p}})$ which satisfies

$$\eta_{s_1, s_2}\left(\begin{pmatrix} t_1 & * \\ 0 & t_2 \end{pmatrix} k_{\mathfrak{p}}\right) = |t_1|_{\mathfrak{p}}^{s_1} |t_2|_{\mathfrak{p}}^{s_2} \left|\frac{t_1}{t_2}\right|_{\mathfrak{p}}^{1/2}$$

$k_{\mathfrak{p}} \in \mathrm{GL}_2(\mathcal{O}_{\mathfrak{p}})$ and has central character ω, i.e., $\omega(t) = |t|_{\mathfrak{p}}^{s_1+s_2}$ then

$$\int_{Z(K_{\mathfrak{p}})\backslash \mathrm{GL}_2(K_{\mathfrak{p}})} \eta_{s_1, s_2}(yx)\, \Phi_n^{(\mathfrak{p})}(x^{-1})\, dx = (N\mathfrak{p})^{|n|/2}(N\mathfrak{p}^{ns_1} + N\mathfrak{p}^{ns_2})\, \eta_{s_1, s_2}(y)$$

where $N\mathfrak{p}$ is cardinality of the residue field at $\mathfrak{p}$. This definition can be expressed in terms of representations. If we have an irreducible representation of class one $\rho_{\mathfrak{p}}\colon \mathrm{GL}_2(K_{\mathfrak{p}}) \to \mathrm{End}(H)$ which has central character ω then

$$\rho_{\mathfrak{p}}(T_{\Phi_n^{(\mathfrak{p})}})\, \eta = (N\mathfrak{p})^{|n|/2}(N\mathfrak{p}^{ns_1} + N\mathfrak{p}^{ns_2})\, \eta_{s_1, s_2}$$

where $\eta \neq 0$ is a $\mathscr{K}_{\mathfrak{p}}$-invariant vector and where s_1, s_2 are the parameters of this representation (compare [**1**, §3, B]).

We are now able to state the main equality.

Let us denote by $d_{\mathfrak{p}}$ (resp. $d_{\mathfrak{q}}$) the degree of the place $\mathfrak{p}$ (resp. $\mathfrak{q}$). The operator

$$\Phi_m = T_{\Phi_{m/d_{\mathfrak{p}}}^{(\mathfrak{p})}} \circ T_{\Phi_{-m/d_{\mathfrak{q}}}^{(\mathfrak{q})}} : H_\omega \to H_\omega$$

is of trace class (compare [**1**, §2]) and we have

$$\text{trace } \mathrm{Fr}^m | H_!(\bar{U}_D, \boldsymbol{Q}_1)_\omega = \text{trace } \Phi_m | H_\omega - \text{Eisenstein contribution in the trace formula for the operator } \Phi_m.$$

The Eisenstein contribution is the sum of the terms 6.36, 6.37 in [**1**]. We are here not quite consistent with the notations in [**1**]. We refer the reader to §2.3.

2.2. *Consequences of the main equality.* Before discussing the proof of the main equality we shall give some applications. We shall see that it has some geometric consequences for the schemes U_D and it implies the Ramanujan conjecture for cusp forms.

Let us pick a line bundle η over $X/\boldsymbol{F}_q$ of degree one. It generates an infinite cyclic group $\{\eta\} \subset P_D$ and $P_D/\{\eta\} = P_{0,D}$ is finite. Then it is rather easy to see that

$$\sum_{\omega \in \hat{P}_{0,D}} \text{trace } \mathrm{Fr}^m | H_!(\bar{U}_D, \bar{\boldsymbol{Q}}_1)_\omega = \text{trace } \mathrm{Fr}^m \Big(\bigoplus_{\xi \in \bar{P}_D; \deg \xi = 0, 1} H_!(\bar{U}_D^{(\xi)}, \bar{\boldsymbol{Q}}_1) \Big).$$

We express this trace in terms of the eigenvalues $\alpha_{i,\nu}$ of the Frobenius acting on the cohomology $H_!^i(\bar{U}_D^{(\xi)}, \bar{\boldsymbol{Q}}_1)$. Then we obtain $\sum_{i=1}^4 \sum_{\nu=1}^{b_i} (-1)^i \alpha_{i,\nu}^m$ for the above trace.

Now we look for a similar expression of the trace of Φ_m on the spaces H_ω. Here we know that H_ω decomposes further,

$$H_\omega = H_{\omega,\text{cusp}} \oplus H_{\omega,1} \bigoplus_\rho H_{\omega,\rho} \oplus \bigoplus_\chi \boldsymbol{C}\cdot\chi$$

where ρ runs over a finite set of irreducible representations of $\mathrm{GL}_2(A)$ in the space of cusp forms $L^2_{\omega,\text{cusp}}(\mathrm{GL}_2(K)\backslash\mathrm{GL}_2(A))$ and where χ: $\mathrm{GL}_2(A) \to \boldsymbol{C}$ is a function which factors over the determinant $\chi\ \mathrm{GL}_2(A) \to^{\det} I_K \to P_D \to^{\omega'} \mu \subset \boldsymbol{C}^*$ with $\omega'^2 = \omega$.

We compute the trace of Φ_m on each of the constituents.

The representation ρ has to be of class one at $\mathfrak{p}$ and $\mathfrak{q}$ and therefore $\sigma_\mathfrak{p}$ is obtained by induction from a representation

$$\rho_{s_1,s_2}: \begin{pmatrix} t_{1,\mathfrak{p}} & * \\ 0 & t_{2,\mathfrak{p}} \end{pmatrix} - |t_{1,\mathfrak{p}}|_\mathfrak{p}^{s_1} |t_{2,\mathfrak{p}}|_\mathfrak{p}^{s_2} |t_{1,\mathfrak{p}}/t_{2,\mathfrak{p}}|_\mathfrak{p}^{1/2}.$$

We define $\xi_\mathfrak{p}(\rho) = q^{s_1/d_\mathfrak{p}}$, $\xi'_\mathfrak{p}(\rho) = q^{s_2/d_\mathfrak{p}}$, etc. Then

$$\operatorname{trace} \Phi_m | H_{\omega,\rho} = (\dim H^{\mathscr{K}}_{\omega,\rho}) q^m (\xi_\mathfrak{p}(\rho)^m + \xi'_\mathfrak{p}(\rho)^m)\ (\xi_\mathfrak{q}(\rho)^m + \xi'_\mathfrak{q}(\rho)^m).$$

This follows from the definition of ϕ_m.

The contribution of the one-dimensional spaces C_χ is also easy to compute. This function χ is

$$\chi\begin{pmatrix} t_{1,\mathfrak{p}} & * \\ 0 & t_{2,\mathfrak{p}} \end{pmatrix} \to \chi(t_{1,\mathfrak{p}}\, t_{2,\mathfrak{p}}) = |t_{1,\mathfrak{p}}|_\mathfrak{p}^{s-1/2} |t_{2,\mathfrak{p}}|_\mathfrak{p}^{s+1/2} |t_{1,\mathfrak{p}}/t_{2,\mathfrak{p}}|^{1/2}.$$

We put $\eta_\mathfrak{p} = \chi(\pi_\mathfrak{p})^{1/d_\mathfrak{p}}$, etc., and get by definition

$$T_{\Phi_m}\chi = (\eta_\mathfrak{p}^m + \eta_\mathfrak{p}^m q^m)\,(\eta_\mathfrak{q}^m + \eta_\mathfrak{q}^m q^m).$$

We want to recall that $\eta_\mathfrak{p}$, $\eta_\mathfrak{q}$ are roots of unity.

It can be checked very directly from the trace formula that the Eisenstein contribution to the trace of Φ_m in the trace formula can be written as

$$\sum_{i=0}^{S} \varepsilon_i \eta_i^m = \text{Einsenstein contribution in the Selberg trace formula for the operator } \Phi_m,$$

with $\varepsilon_i = \pm 1$ and where the η_i are algebraic integers of weight 0, 1, 2, i.e., $|\eta_i| = q^{\nu_i}$, $\nu_i = 0,1,2$ (compare [**8**, 3.1]).

If we define $a(\rho) = \dim H^{\mathscr{K}}_{\omega,\rho}$ then our main equality reads as follows:

$$\sum_{i=0}^{4}\sum_{\nu=1}^{b_i}(-1)^i\alpha_{i,\nu}^m = \sum_\rho a(\rho) q^m((\xi_\mathfrak{p}(\rho)^m + \xi'_\mathfrak{p}(\rho)^m)(\xi_\mathfrak{q}(\rho)^m + \xi'_\mathfrak{q}(\rho)^m)$$
$$+ \sum_\chi(\eta_\mathfrak{p}^m + \eta_\mathfrak{p}^m q^m)(\eta_\mathfrak{q}^m + \eta_\mathfrak{q}^m q^m) + \sum_{i=0}^{S}\varepsilon_i\eta_i^m.$$

Now we are ready to derive some consequences from this equality. First we recall what is known about the $\alpha_{i,\nu}$.

The $\alpha_{i,\nu}$ are algebraic integers and we have $|\alpha_{i,\nu}| = q^{j/2}$ where j is an integer less than or equal to i. This number j is the weight of $\alpha_{i,\nu}$. (Compare [**9**, 14].)

On the other hand we have a simple

LEMMA 2.2.1. *If* $x_1, \cdots, x_n, x'_1, \cdots, x'_{n'}, y_1, \cdots, y_m, y'_1, \cdots, y'_{m'}$ *are complex numbers and if*

$$\sum_{i=1}^{n} x_i^k - \sum_{i=1}^{m} y_i^k = \sum_{i=1}^{n'} x_i'^k - \sum_{i=1}^{m'} y_i'^k, \qquad k = 1, 2, \cdots,$$

and if $x_i \neq y_j$ *and* $x_i' \neq y_j'$ *for all* i, j *then* $n = n'$, $m = m'$ *and up to order we have* $x_i = x_i', y_i = y_i'$.

Now we know that $\xi_{\mathfrak{p}}(\rho)\cdot\xi_{\mathfrak{p}}'(\rho)$ and $\xi_{\mathfrak{q}}(\rho)\cdot\xi_{\mathfrak{q}}'(\rho)$ are roots of unity because ω is finite. Moreover it follows from the theory of automorphic forms that $|\xi_{\mathfrak{p}}(\rho)| < q$ and $|\xi_{\mathfrak{q}}(\rho)| < q$. This means that the terms $(\eta_{\mathfrak{p}}\, \eta_{\mathfrak{q}})^m q^{2m}$ which occur in the contribution from $\boldsymbol{C}\cdot\chi$ are the only terms of weight 4 on the right-hand side. The number of these terms is equal to $2\,|P_{0,D}|$. This is clear since for each ω which is a square we have $2\,|P_{0,D}/P_{0,D}^2|$ solutions of $\omega'^2 = \omega$ in the group of characters of P_D.

On the other hand the terms of weight 4 on the left-hand side come from the top cohomology

$$\sum_{\xi,\, \deg(\xi)=0,1} \text{trace } \mathrm{Fr}^m \big| H_!^4(\bar{U}_D^{(\xi)}, \bar{\boldsymbol{Q}}_l) = \sum_{v=1}^{b_4} \alpha_{4,\nu}^m.$$

We shall see in the next section that the $\bar{U}_D^{(\xi)}/\bar{\boldsymbol{F}}_q$ are all nonempty. The number of ξ is again equal to $2\,|P_D|$. Therefore we find the $\bar{U}_D^{(\xi)}/\bar{\boldsymbol{F}}_q$ are irreducible.

On the other hand we see that the terms of weight 3 on the left-hand side occur with a minus sign since they have to come from $H_!^3(\bar{U}_D^{(\xi)}, \bar{\boldsymbol{Q}}_l)$. But our remark on the Einsenstein contribution shows that there are no terms with minus sign and weight 3 on the right-hand side. So there cannot be any such term on the left-hand side. This is easily seen to be equivalent to saying that for any smooth compactification $\bar{U}_D^{(\xi)} \hookrightarrow Y$ we have $H^3(\bar{Y}, \boldsymbol{Q}_l) = H^1(\bar{Y}, \boldsymbol{Q}_l) = 0$.

This absence of terms of weight 3 now implies directly that the terms $q^m\, \xi_{\mathfrak{p}}(\rho)$, etc., have to be of weight 2, i.e., $|\xi(\rho)| = 1$, etc.

We may summarize:

THEOREM 2.2.2. (1) *The schemes* $U_D^{(\xi)}/\boldsymbol{F}_{q^d}$ *are absolutely irreducible.*

(2) *The cohomology groups* $H_!^3(\bar{U}_D^{(\xi)}, \boldsymbol{Q}_l)$ *do not contain any classes of weight* 3.

Moreover we found:

Automorphic cusp forms on GL$_2(A)$ *over a function field satisfy the Ramanujan conjecture, i.e., for any* ρ *occurring in* $H_{\omega,\,\mathrm{cusp}}$ *the local components* $\rho_{\mathfrak{p}}$ *and* $\rho_{\mathfrak{q}}$ *are in the unitary principal series.*

REMARK. Drinfeld himself uses another argument to prove (2) in Theorem 2.2.2. It also follows from the existence of the two families of affine lines in $U_D/\boldsymbol{F}_q$.

2.3. *The proof of the main equality.* The proof of the main equality is obtained by a method which is similar to the one used in the theory of Shimura varieties. (Compare [3], [5].) We start from a description of the set of geometric points on the moduli space together with the action of the Frobenius and the Picard group on this set. The set of these geometric points will decompose into 'orbits'; each of these orbits will give a contribution to the trace of Frobenius. The contribution which comes from a fixed orbit may then be compared to a contribution to the trace of the Hecke operators in the trace formula.

2.3.1. *The set of geometric points.* For simplicity we want to assume that our two points v, $w \in X(\bar{\boldsymbol{F}}_q)$ are actually rational; this means that $\mathfrak{p}$ and $\mathfrak{q}$ have degree 1. This will simplify the exposition but not change the proof substantially.

In this case the moduli spaces $M_D^{(v,w)}$ are defined over $\boldsymbol{F}_q$ and since v, w are fixed we also want to drop the superscript (v, w). The vector bundles with level structure along D over $\bar{X}$ are in 1-1 correspondence to the double cosets (compare 1.2) $GL_2(K_\infty)\backslash GL_2(A_\infty)/\mathscr{K}_\infty(D)$ and this tells us that the set

$$M(\bar{\boldsymbol{F}}_q) = \operatorname*{proj\,lim}_{D,\, v,\, w \notin \operatorname{supp}(D)} M_D(\bar{\boldsymbol{F}}_q)$$

is a subset $M(\bar{\boldsymbol{F}}_q) \subset GL_2(K_\infty)\backslash GL_2(A_\infty)/GL_2(\mathcal{O}_v) \times GL_2(\mathcal{O}_w)$.

REMARK. For notations compare 1.1. We still consider v, w as valuation of K_∞, so K_v, K_w are completions of K_∞ and $\mathcal{O}_v$, $\mathcal{O}_w$ are the rings of integers in K_v, K_w.

It is clear that exactly those points $\boldsymbol{x} \in GL_2(A_\infty)$ give a point in $M(\bar{\boldsymbol{F}}_q)$ for which we can solve the equation

$$(*) \qquad \boldsymbol{x}^\sigma = \gamma^{-1}\boldsymbol{x}\boldsymbol{\alpha}_v\tilde{\boldsymbol{\alpha}}_w$$

where σ is the mapping induced by the action of the Frobenius on the constants in K_∞, where $\gamma \in GL_2(K_\infty)$ and where

$$\alpha_v \in GL_2(\mathcal{O}_v)\begin{pmatrix}\pi_v & 0\\ 0 & 1\end{pmatrix}GL_2(\mathcal{O}_v), \qquad \tilde{\alpha}_w \in GL(\mathcal{O}_w)\begin{pmatrix}\pi_w^{-1} & 0\\ 0 & 1\end{pmatrix}GL_2(\mathcal{O}_w),$$

and where $\boldsymbol{\alpha}_v$ (resp. $\tilde{\boldsymbol{\alpha}}_w$) is the adele which has v- (resp. w)-component α_v (resp. $\tilde{\alpha}_w$) and 1 elsewhere. We observe that γ, α_v, $\tilde{\alpha}_w$ are determined by $\boldsymbol{x}$.

If we have two solutions of $(*)$:

$$\boldsymbol{x}^\sigma = \gamma^{-1}\,\boldsymbol{x}\boldsymbol{\alpha}_v\tilde{\boldsymbol{\alpha}}_w,$$
$$\boldsymbol{x}'^\sigma = \gamma'^{-1}\,\boldsymbol{x}'\boldsymbol{\alpha}_v'\tilde{\boldsymbol{\alpha}}_w',$$

then $\boldsymbol{x}$ and $\boldsymbol{x}'$ correspond to the same point of $M(\bar{\boldsymbol{F}}_q)$ if and only if $\boldsymbol{x} = a^{-1}\boldsymbol{x}'\boldsymbol{k}_v\boldsymbol{k}_w$ with $a \in GL_2(K_\infty)$ and $k_v \in GL_2(\mathcal{O}_v)$, $k_w \in GL_2(\mathcal{O}_w)$. Then we find

$$\gamma' = a\gamma a^{-\sigma}, \quad \alpha_v' = k_v\alpha_v k_v^{-\sigma}, \quad \tilde{\alpha}_w' = k_w\tilde{\alpha}_w k_w^{-\sigma}.$$

(Here we take advantage of our assumption that v and w are rational.) The relation $\gamma' = a\gamma a^{-\sigma}$ means that γ and γ' are σ-conjugate and we shall say that α_v' is σ-$GL_2(\mathcal{O}_v)$-conjugate to α_v and $\tilde{\alpha}_w'$ is σ-$GL_2(\mathcal{O}_w)$-conjugate to $\tilde{\alpha}_w$. This gives us:

Each geometric point in $M(\bar{\boldsymbol{F}}_q)$ gives rise to the following data:

(i) a σ-conjugacy class $\{\gamma\}$ in $GL_2(K_\infty)$,

(ii) a σ-$GL_2(\mathcal{O}_v)$-conjugacy class $\{\alpha_v\}$ in

$$GL_2(\mathcal{O}_v)\begin{pmatrix}\pi_v & 0\\ 0 & 1\end{pmatrix}GL_2(\mathcal{O}_v),$$

(iii) a σ-$GL_2(\mathcal{O}_w)$-conjugacy class $\{\alpha_w\}$ in

$$GL_2(\mathcal{O}_w)\cdot\begin{pmatrix}\pi_w^{-1} & 0\\ 0 & 1\end{pmatrix}GL_2(\mathcal{O}_w).$$

If these sets of data are given, then we find a point if and only if γ is σ-conjugate to α_v (resp. $\tilde{\alpha}_w$) in $GL_2(K_v)$ (resp. $GL_2(K_w)$) and if γ is σ-conjugate to 1 outside of v and w.

Therefore we find the following strategy to describe the set of geometric points $M(\bar{F}_q)$.

(i) Describe the set of σ-conjugacy classes in $GL_2(K_\infty)$.

(ii) Given such a class describe the possible classes α_v, $\tilde{\alpha}_w$.

(iii) Given γ, α_v, $\tilde{\alpha}_w$ describe the set of geometric points which belong to this set of data, i.e., the solutions of $\boldsymbol{x}^\sigma = \gamma^{-1}\boldsymbol{x}\boldsymbol{\alpha}_v\tilde{\boldsymbol{\alpha}}_w$ with these specific γ, α_v, $\tilde{\alpha}_w$.

2.3.2. *σ-conjugacy.* We start to discuss (i). To do this we make use of the Saito-Shintani method of norms. (Compare [7].)

If $\gamma \in GL_2(K_\infty)$ then we can find an integer s.t. $\gamma \in GL_2(K\cdot \boldsymbol{F}_{q^n})$. If this is the case we define $N_n(\gamma) = \gamma\,\gamma^\sigma \cdots \gamma^{\sigma^{n-1}} = \gamma_0$. One can show without difficulties

(1) The conjugacy class of γ_0 is defined over K.

(2) If we σ-conjugate γ into $a\gamma a^{-\sigma} = \gamma'$ with $a \in GL_2(K\cdot \boldsymbol{F}_{q^n})$ then $aN_n(\gamma)a^{-1} = N_n(\gamma')$, and we may find an $a \in GL_2(K\boldsymbol{F}_{q^n})$, s.t. we have $N_n(\gamma') = N_n(a\gamma a^{-\sigma}) = \gamma_0 \in GL_2(K)$.

The following assertion is slightly less trivial.

(3) If $N_n(\gamma)$ and $N_n(\gamma')$ are conjugate in $GL_2(K_\infty)$ then γ and γ' are σ-conjugate.

To prove (3) one has to use Galois cohomology and class-field theory. We shall give enough hints for the proof of (3) in the following discussion.

We shall always assume that $N_n(\gamma) = \gamma_0 \in GL_2(K)$ and moreover we shall assume that γ_0 is semisimple. We are free to do so since we may pass from n to $n\cdot p$ which raises γ_0 into the pth power.

Therefore we end up with two possibilities:

γ_0 is central and then we say that the σ-conjugacy class is central.

γ_0 is semisimple and no power γ_0^k with $k \geqq 1$ is central. In this case we shall say that the σ-conjugacy class is not central. If γ_0 generates a field we shall call this class nondecomposed. If γ_0 sits in a split torus we call the class decomposed.

We consider the central conjugacy classes first. If $\gamma \in GL_2(K\boldsymbol{F}_{q^n})$ and $N_n(\gamma) = \gamma_0$ is a central element, then the image $\bar{\gamma}$ of γ in $PGL_2(K\boldsymbol{F}_{q^n})$ defines a 1-cocycle in $Z^1(K\boldsymbol{F}_{q^n}/K,\ PGL_2(K\boldsymbol{F}_{q^n}))$. Then it follows from Hilbert's Theorem 90 that γ and γ' are σ-conjugate if and only if $\bar{\gamma}$ and $\bar{\gamma}'$ define the same cohomology class in $H^1(K\cdot \boldsymbol{F}_{q^n}/K,\ PGL_2(K\boldsymbol{F}_{q^n}))$ and then we find that the central σ-conjugacy classes are in 1-1 correspondence with the elements of exponent 2 in the Brauer group of K. We shall not say very much about the case of noncentral σ-conjugacy classes. The only thing we want to mention is that if $\gamma_0 = N_n(\gamma)$ is noncentral, then we have either $\gamma_0 \in E^\times \hookrightarrow GL_2(K)$ where E/K is a quadratic extension or $\gamma_0 \in T(K)$ where T/K is a split torus. In both cases we must have that γ is in the centralizer of γ_0, i.e., $\gamma \in E_\infty^\times$ or $\gamma \in T(K_\infty)$.

2.3.3. *σ-$GL_2(\mathcal{O}_v)$-conjugacy.* We come to the discussion of (ii). Let us assume that we have $\alpha \in GL_2(K\boldsymbol{F}_{q^n})$; we ask:

Can we solve

$$x_v\,\alpha_v\,x_v^{-\sigma} = \gamma, \qquad x_v \in GL_2(K_v),\ \alpha_v \in GL_2(\mathcal{O}_v)\begin{pmatrix}\pi_v & 0\\ 0 & 1\end{pmatrix}GL_2(\mathcal{O}_v),$$

and how many σ-$GL_2(\mathcal{O}_v)$-conjugacy classes for α_v do we get?

Let us assume in the first case that

$$N_n(\gamma) = \gamma_0 = \varepsilon_\mathfrak{p}\cdot\ z \in K^\times$$

where z is a central element, i.e. $z \in K^\times$ and ε is a unit in $K_{\mathfrak{p}}^\times$.

Then we may modify γ into $u_v \gamma u_v^{-\sigma}$ with $u \in K_v^\times$ s.t.

$$N_n(u_v \gamma u_v^{-\sigma}) = u_v u_v^{-\sigma^n} N_n(\gamma) = u_v u_v^{-\sigma^n} \varepsilon_{\mathfrak{p}} z$$

and we can find a u such that $uu^{-\sigma^n} \varepsilon = 1$.

Then we get $x_v u \alpha_v u^{-\sigma} x_v^{-\sigma} = u \gamma u^{-\sigma}$ and we may change $\alpha_v \to u \alpha_v u^{-\sigma}$ since we do not change the σ-$\mathrm{GL}_2(\mathcal{O}_v)$-conjugacy class. This implies that we have

$$N_n(x_v \alpha_v x_v^{-\sigma}) = x_v N(\alpha_v) x_v^{-\sigma_n} = z \in K^\times.$$

If we pass to the projective linear group we find $\bar{x}_v N(\bar{\alpha}_v) \bar{x}_v^{-\sigma^n} = \mathbf{1}$, i.e.,

$$N(\bar{\alpha}_v) = \bar{x}_v^{\sigma^n} \bar{x}_v^{-1}.$$

But if we take n large with respect to the order given by divisibility we find $\bar{x}_v^{\sigma^n} \bar{x}_v^{-1}$ is close to one and therefore we find an element $\bar{k}_v \in \mathrm{PGL}_2(\mathcal{O}_v)$ s.t. $\bar{k}_v^{\sigma^n} \bar{k}_v^{-1} = \bar{x}_v^{\sigma^n} \bar{x}_v^{-1}$. Changing again $\alpha_v \to k_v \alpha_v k_v^{-\sigma}$ we find that within the σ-$\mathrm{GL}_2(\mathcal{O}_v)$-conjugacy class of our original α_v we may assume $N_n(\alpha_v) = z_v \in K_v^\times$. It is easy to see that then $2|n$ and

$$N_n(\alpha_v) = \begin{pmatrix} \pi_v^{n/2} & 0 \\ 0 & \pi_v^{n/2} \end{pmatrix}.$$

Since we know that $H^1(KF_{q^n}/K, \mathrm{PGL}_2(\mathcal{O}(F_{q^n}))) = 0$ it follows that in this case α_v is σ-conjugate to the element

$$\begin{pmatrix} 0 & \pi_v \\ 1 & 0 \end{pmatrix}.$$

We ask for which $\{\gamma\}$ the condition $N_n(\gamma) = \gamma_0 = \varepsilon_{\mathfrak{p}} \cdot z_{\mathfrak{p}}$ is fulfilled. First of all we observe that this is certainly true if $\{\gamma\}$ is a central class or if $\{\gamma\}$ is nondecomposed and the field generated by γ_0 is nonsplit at $\mathfrak{p}$. In the latter case we just take n to be even and consider the valuation. We say that $\{\gamma\}$ is nondecomposed at $\mathfrak{p}$.

Let us assume that γ_0 is noncentral and splits at $\mathfrak{p}$. In this case it is clear since γ has to be in the centralizer of γ_0 that γ_0 is conjugate in $\mathrm{GL}_2(K_{\mathfrak{p}})$ to

$$\begin{pmatrix} \pi_v^n & 0 \\ 0 & 1 \end{pmatrix}$$

and then α_v has to be σ-$\mathrm{GL}_2(\mathcal{O}_v)$-conjugate to

$$\begin{pmatrix} \pi_v^n & 0 \\ 0 & 1 \end{pmatrix}.$$

We shall say that $\{\gamma\}$ is decomposed at v. Therefore we find that we have only two possible σ-$\mathrm{GL}_2(\mathcal{O}_v)$-conjugacy classes for α_v namely

$$\left\{\begin{pmatrix} \pi_v & 0 \\ 0 & 1 \end{pmatrix}\right\}, \quad \left\{\begin{pmatrix} 0 & \pi_v \\ 1 & 0 \end{pmatrix}\right\}$$

and γ has to be σ-conjugate to one of them at v.

2.3.4. *The orbits.* Let us assume we have a global σ-conjugacy class $\{\gamma\}$ and α_v, $\tilde{\boldsymbol{\alpha}}_w$ and a solution $\boldsymbol{x}_0^\sigma = \gamma^{-1} \boldsymbol{x}_0 \boldsymbol{\alpha}_v \tilde{\boldsymbol{\alpha}}_w$. Using approximation we may very well assume that $x_{0v} \in GL_2(\mathcal{O}_v)$, $x_{0w} \in GL_2(\mathcal{O}_w)$. Then any other solution with the same γ, α_v, $\tilde{\alpha}_w$ is of the form $\boldsymbol{x} = x_0 \boldsymbol{y}$ where $\boldsymbol{y}$ has to satisfy

(i) If we write $\boldsymbol{y} = \boldsymbol{y}'\boldsymbol{y}_v\, \boldsymbol{y}_w$ where y' has component 1 at v and w then $\boldsymbol{y}' \in GL_2(A\mathfrak{p}, \mathfrak{q})$, i.e., y' is rational outside of $\mathfrak{p}$, $\mathfrak{q}$.

(ii) $y_v\alpha_v y_v^{-\sigma} = \alpha_v$, $y_w\alpha_w y_w^{-\sigma} = \tilde{\alpha}_w$.

We define the σ-centralizers

$$Z_\sigma(\alpha_v) = \{y_v \in GL_2(\mathcal{O}_v) | y_v\alpha_v y_v^{-\sigma} = \alpha_v\},$$
$$Z_\sigma(\tilde{\alpha}_w) = \{y_w \in GL_2(\mathcal{O}_w) | y_w\tilde{\alpha}_w y_w^{-\sigma} = \tilde{\alpha}_w\}.$$

We find that the solutions of $\boldsymbol{x}^\sigma = \gamma^{-1}\boldsymbol{x}\, \boldsymbol{\alpha}_v\, \tilde{\boldsymbol{\alpha}}_w$ form an orbit under the action of the group $GL_2(A_{\mathfrak{p},\mathfrak{q}}) \times Z_\sigma(\alpha_v) \times Z_\sigma(\tilde{\alpha}_w)$. We compute the stabilizer. To do this we observe first that in the nondecomposed case

$$Z_\sigma\left(\begin{pmatrix}0 & \pi_v\\ 1 & 0\end{pmatrix}\right) = D(K_v)^\times$$

where $D(K_v)^\times$ is the multiplicative group of the quaternion division algebra at v and $D(K_v)^\times \cap GL_2(\mathcal{O}_v) = D^{(1)}(K_v)$ is the maximal compact subgroup of element whose reduced norm has absolute value one. Then

$$\bar{D}_v = D(K_v)^\times/D^{(1)}(K_v) = \boldsymbol{Z}.$$

In the decomposed case we have

$$Z_\sigma\left(\begin{pmatrix}\pi_v & 0\\ 0 & 1\end{pmatrix}\right) = K_\mathfrak{p}^\times \times K_\mathfrak{p}^\times$$

and

$$Z_\sigma\left(\begin{pmatrix}\pi_v & 0\\ 0 & 1\end{pmatrix}\right) \cap GL_2(\mathcal{O}_v) = U_\mathfrak{p} \times U_\mathfrak{p}$$

where $U_\mathfrak{p}$ is the group of units. In this case we find

$$\bar{D}_v = Z_\sigma\left(\begin{pmatrix}\pi_v & 0\\ 0 & 1\end{pmatrix}\right) \Big/ Z_\sigma\left(\begin{pmatrix}\pi_v & 0\\ 0 & 1\end{pmatrix}\right) \cap GL_2(\mathcal{O}_v) = \boldsymbol{Z} \times \boldsymbol{Z}.$$

Since $Z_\sigma(\alpha_v) \cap GL_2(\mathcal{O}_v)$ acts obviously trivially on the points of the moduli space we find that the set of points in question is an orbit under $GL_2(A_{\mathfrak{p},\mathfrak{q}}) \times \bar{D}_v \times \bar{D}_w$. The σ-centralizer of γ

$$Z_\sigma(\gamma) = \{a \in GL_2(K_\infty) | a\gamma a^{-\sigma} = \gamma\}$$

is mapped by $a \to \boldsymbol{x}_0^{-1}a\boldsymbol{x}_0$ into $GL_2(A_{\mathfrak{p},\mathfrak{q}}) \times \bar{D}_v \times \bar{D}_w$ and one sees easily that this σ-centralizer is the stabilizer we are looking for. Therefore our orbit is $Z_\sigma(\gamma)\backslash GL_2(A_{\mathfrak{p},\mathfrak{q}}) \times \bar{D}_v \times \bar{D}_w$. Such an orbit $Z(\gamma)\backslash GL_2(A_{\mathfrak{p},\mathfrak{q}}) \times \bar{D}_v \times \bar{D}_w \subset M(\bar{\boldsymbol{F}}_q)$ is invariant under the action of the Frobenius. To see this we start from

$$\boldsymbol{x}^\sigma = \boldsymbol{x}_0^\sigma \boldsymbol{y}^\sigma = \gamma^{-1}\boldsymbol{x}_0\boldsymbol{y}\boldsymbol{\alpha}_v\tilde{\boldsymbol{\alpha}}_w.$$

We can choose $\alpha_v = \alpha_v^\sigma$, $\tilde{\alpha}_w = \tilde{\alpha}_w^\sigma$ and then it is obvious that for $\boldsymbol{x}' = \gamma\boldsymbol{x}^\sigma$ we have $\boldsymbol{x}' = \boldsymbol{x}_0\boldsymbol{y}\boldsymbol{\alpha}_v\tilde{\boldsymbol{\alpha}}_w$ and this implies that the Frobenius acts on $Z_\sigma(\gamma)\backslash GL_2(A_{\mathfrak{p},\mathfrak{q}}) \times \bar{D}_v \times \bar{D}_w$ by multiplication on the right by $\boldsymbol{\alpha}_v\tilde{\boldsymbol{\alpha}}_w \in \bar{D}_v \times \bar{D}_w$.

We recall that

$$\alpha_v = 1 \in \boldsymbol{Z} = \bar{D}_v, \qquad \text{nondecomposed case,}$$

$$\alpha_v = (1, 0) \in \boldsymbol{Z} \times \boldsymbol{Z} = D_v, \quad \text{decomposed case.}$$

The group of rational line bundles with level structure along D acts on M_d (compare 2.1) and this gives us an action of $P = \text{proj lim}_D P_D$, $\mathfrak{p}\cdot\mathfrak{q} \notin \text{supp}(D)$, on M. We notice that

$$P \xrightarrow{\sim} I_K/(U_\mathfrak{p} \times U_\mathfrak{q})\cdot K^\times$$

where I_K is the adele group, and $U_\mathfrak{p}$ (resp. $U_\mathfrak{q}$) are the units at $\mathfrak{p}$ (resp. $\mathfrak{q}$). Since I_K is isomorphic to the centre $Z(A) \subset \text{GL}_2(A)$ we see that the above action is on the orbits given by

$$\boldsymbol{x} \to \boldsymbol{x}z, \qquad z \in Z(A),$$

$$\boldsymbol{x} \in Z_\sigma(\gamma)\backslash \text{GL}_2(A_{\mathfrak{p},\mathfrak{q}}) \times \bar{D}_v \times \bar{D}_w.$$

The considerations have shown us so far that the σ-conjugacy class $\{\gamma\}$ determines the rest of our data and also determines the orbit. But we still have to discuss the question:

For which $\{\gamma\}$ do we get a nonempty orbit, i.e., for which $\{\gamma\}$ can we solve (*)?

Let us assume that our class $\{\gamma\}$ is central. Then it follows from $\boldsymbol{x}^\sigma = \gamma^{-1}\boldsymbol{x}\boldsymbol{\alpha}_v\tilde{\boldsymbol{\alpha}}_w$ that γ is a boundary at all places different from $\mathfrak{p}$ and $\mathfrak{q}$. Therefore it follows that the quaternion algebra (or better central algebra) defined by γ has to be the one which is nonsplit at $\mathfrak{p}$ and $\mathfrak{q}$ and split everywhere else. On the other hand we know that there is exactly one central σ-conjugacy class $\{\gamma\}$ which corresponds to the unique quaternion algebra which is nonsplit at $\mathfrak{p}$ and $\mathfrak{q}$. For this particular $\{\gamma\}$ we can solve (*) with

$$\alpha_v = \begin{pmatrix} 0 & \pi_v \\ 1 & 0 \end{pmatrix}, \qquad \tilde{\alpha}_w = \begin{pmatrix} 0 & \pi_w^{-1} \\ 1 & 0 \end{pmatrix}$$

and we get exactly one orbit this way. This orbit is called the supersingular orbit.

If γ is not central then γ determines either a quadratic field extension E/K or a split torus T/K. We shall always assume that E/K (resp. T/K) are embedded into GL_2 in such a way that

$$\text{units}(E_\mathfrak{p}) \hookrightarrow \text{GL}_2(\mathcal{O}_\mathfrak{p}), \qquad \text{units}(E_\mathfrak{q}) \hookrightarrow \text{GL}_2(\mathcal{O}_\mathfrak{q}),$$

resp.

$$\text{units}(T(K_\mathfrak{p})) = U_\mathfrak{p} \times U_\mathfrak{p} \hookrightarrow \text{GL}_2(\mathcal{O}_\mathfrak{p}),$$

$$\text{units}(T(K_\mathfrak{q})) = U_\mathfrak{q} \times U_\mathfrak{q} \hookrightarrow \text{GL}_2(\mathcal{O}_\mathfrak{q}).$$

Now we get from (*) that for n sufficiently large we have $\text{div}(N_n(\gamma)) = n\mathfrak{p} - n\mathfrak{q}$. This implies: If E/K does not split at $\mathfrak{p}$ and at $\mathfrak{q}$ then

$$N_n(\gamma) = a\cdot\varepsilon, \qquad a \in K^\times,\ \mathscr{E} \in \text{global units in } E^\times.$$

But then it is clear that $\{\gamma\}$ is a central class, this implies that those field extensions which are not split at $\mathfrak{p}$ or $\mathfrak{q}$ cannot contribute to the nonsupersingular orbits.

On the other hand we shall see that any extension which does split at one of the two places will contribute to the points by one or two orbits. To see this we assume for instance that $\mathfrak{p}$ decomposes in E, $\mathfrak{p} = \mathfrak{p}_1\mathfrak{p}_2$. Then we can find an element $\gamma \in E_\infty$, s.t. the divisor of γ is of the form

$$\operatorname{div}(\gamma) = -\mathfrak{q}' + \mathfrak{p}_1 - \mathfrak{a} + \mathfrak{a}^\sigma$$

where $\mathfrak{q}'$ is a prime in E_∞ which lies above $\mathfrak{q}$ and $\mathfrak{a}$ is a positive divisor which does not contain $\mathfrak{p}$ and $\mathfrak{q}$ in its support.

The existence of such a divisor $\mathfrak{a}$ follows again from Lang's theorem applied to the Jacobian of E.

The norm of γ will have the divisor $\operatorname{div}(N_n(\gamma)) = n\mathfrak{p}_1 - n\mathfrak{q}_1$ where $\mathfrak{q}_1 = \mathfrak{q}$ if $\mathfrak{q}$ is nonsplit and where $\mathfrak{q}_1$ is one of the divisors above $\mathfrak{q}$ in E if $\mathfrak{q}$ does split in E.

Since E/K has a nontrivial automorphism which can be realised as an inner automorphism of $GL_2(K)$ we see:

Each field extension E/K which is split at one of the places $\mathfrak{p}$ or $\mathfrak{q}$ contributes to the set of orbits. It gives one orbit, if it splits at exactly one of the places; it gives two orbits if it splits at both places.

2.3.5. *The computation of the trace of* Fr^m. We recall that we want to compute the trace $\mathrm{Fr}^m | H_!^\cdot(\bar{U}_D, \boldsymbol{Q}_l)_\omega$. To do this we proceed as follows: Our character ω contains an element $\lambda \in P_D$ of infinite order in its kernel. We divide the scheme $U_D/\boldsymbol{F}_q$ by the action of the infinite cyclic group $\{\lambda\}$. Then we get

$$U_D/\{\lambda\} = \bigcup_{\xi;\xi \in P_D/\{\lambda^2\}} U_D^{(\xi)} = Z.$$

The scheme $Z/\boldsymbol{F}_q$ is quasi-projective. (Compare 2.2.)

On this scheme $Z/\boldsymbol{F}_q$ we have an action of the finite group $P_D/\{\lambda\}$ which is induced by the tensorisation with line bundles and

$$H_!^\cdot(\bar{U}_D, \boldsymbol{Q}_l)_\omega = H_!^\cdot(\bar{Z}, \boldsymbol{Q}_l)_\omega \subset H_!^\cdot(\bar{Z}, \boldsymbol{Q}_l).$$

Therefore we get the information we want, if we are able to compute the traces of $\mathrm{Fr}^m \circ \delta$ on $H_!^\cdot(\bar{Z}, \boldsymbol{Q}_l)$ where δ runs over the elements of $P_D/\{\lambda\}$. To compute these traces we have to compute the number of fixed points of $\mathrm{Fr}^m \circ \delta$ on the set $Z(\bar{\boldsymbol{F}}_q)$. These fixed points can be counted on the individual orbits, which have been described above. To prove the main equality we shall compare these contributions to the corresponding ones in the Selberg trace formula.

We recall the description of an orbit from 2.3.4. It is a set $Z_\sigma(\gamma)\cdot\{\lambda\}\backslash GL_2(A_{\mathfrak{p},\mathfrak{q}}) \times \bar{D}_v \times \bar{D}_w$. The element δ acts on it by multiplication by an idele $\boldsymbol{\delta} \in Z(A)$ and Fr^m is multiplication by $(\alpha_v\, \tilde{\alpha}_w)^m$. Let $\mathscr{K}'_D \subset GL_2(A_{\mathfrak{p},\mathfrak{q}})$ be the open compact subgroup defined by D. We define the function $\Psi_{D,m,\omega}$ as the characteristic function of the set $\bigcup_{\nu\in\boldsymbol{Z}} \mathscr{K}'_D \cdot \boldsymbol{\lambda}^\nu(\alpha_v\, \tilde{\alpha}_w)^m \cdot \boldsymbol{\delta} \cdot Z(K)$ where $Z(K)$ is of course the centre of $GL_2(K)$. Then we get from very elementary considerations that the number of fixed points on our orbit is equal to

$$\sum_{a \text{ conj. classes in } Z_\sigma(\gamma)/Z(K)} \int_{Z_a(K)\backslash GL_2(A\mathfrak{p},\,\mathfrak{q})\times\bar{D}_v\times\bar{D}_w} \Psi_{D,m,\delta}(\boldsymbol{x}^{-1}\, a\boldsymbol{x})\, d\boldsymbol{x}$$

where $Z_a(K)$ is the centralizer of the element $a \in GL_2(K)$ in $GL_2(K)$. The Haar measure $d\boldsymbol{x}$ has to be chosen in such a way that the volume of the open compact subgroup $\mathscr{K}'_D \times \{0\} \times \{0\}$ becomes one.

Once we have the above formula for the number of fixed points we can easily derive a formula for the trace of Fr^m on $H_!(\bar{U}_D, \boldsymbol{Q}_l)_\omega$.

We define a new function $\Psi_{D,m,\omega}$: $\mathrm{GL}_2(A_{\mathfrak{p},\mathfrak{q}}) \times \bar{D}_v \times \bar{D}_w \to \boldsymbol{C}$ by

$$\begin{aligned} \Psi_{D,m,\omega}(\boldsymbol{x}) &= \omega(\boldsymbol{z}) \quad \text{if } \boldsymbol{x} = \boldsymbol{k}'_D \cdot \boldsymbol{z} \cdot (\alpha_v \, \tilde{\alpha}_w)^m, \ \boldsymbol{z} \in Z(A), \ \boldsymbol{k}'_D \in \mathscr{K}'_D, \\ &= 0 \qquad \text{otherwise.} \end{aligned}$$

Then it is quite easy to see that the contribution of the orbit to the trace of Fr^m is given by the expression

$$\sum_{a \text{ conj. classes in } Z_\sigma(\gamma)/Z(K)} \mathrm{vol}(Z_a(A)/Z_a(K) \cdot Z(A)) \cdot I_{D,m,\omega}(a)$$

where $I_{D,m,\omega}(a) = \int_{Z_a(A)\backslash \mathrm{GL}_2(A_{\mathfrak{p},\mathfrak{q}}) \times \bar{D}_v \times \bar{D}_w} \Psi_{D,m,\omega}(\boldsymbol{x}^{-1} a \boldsymbol{x}) \, d\tilde{\boldsymbol{x}}$ and $d\tilde{\boldsymbol{x}}$ is the quotient measure on $Z_a(A)\backslash \mathrm{GL}_2(A_{\mathfrak{p},\mathfrak{q}}) \times \bar{D}_v \times \bar{D}_w$, s.t. $dt d\tilde{\boldsymbol{x}} = d\boldsymbol{x}$ and $\mathrm{vol}_{dt}(Z_a(A) \cap \mathscr{K}_D \times \{0\} \times \{0\}) = 1$.

At least at this point the reader will realize that the above considerations do not hold for the decomposed orbit. The decomposed orbit will get a special treatment in 2.3.7. We mention that the points on the nondecomposed orbits are contained in $U(\bar{\boldsymbol{F}}_q)$.

Since the functions $\Psi_{D,m,\omega}$ are the products of local functions we find that the integral $I_{D,m,\omega}(a)$ will be a product of local integrals. We find the value

$$\begin{aligned} I_{D,m,\omega}(a) &= \left(\int_{Z_a(A_{\mathfrak{p},\mathfrak{q}})\backslash \mathrm{GL}_2(A_{\mathfrak{p},\mathfrak{q}})} \chi'_D(\boldsymbol{x}'^{-1} a \boldsymbol{x}') \, d\tilde{\boldsymbol{x}}' \right) \cdot \delta(a, m) \\ &= I'_{D,\omega}(a) \cdot \delta(m, a), \end{aligned}$$

where $\delta(m, a) = \delta_{\mathfrak{p}}(m, a) \cdot \delta_{\mathfrak{q}}(m, a)$ and $\delta_{\mathfrak{p}}(m, a) = \int_{Z_a(K_{\mathfrak{p}})\backslash Z_\sigma(\alpha_v)} \Psi_{D,m,\omega,\mathfrak{p}}(x_{\mathfrak{p}}^{-1} a x_{\mathfrak{p}}) \, d\tilde{x}_{\mathfrak{p}}$. Again we choose $d\tilde{x}_{\mathfrak{p}}$ to be the quotient measure $d\tilde{x}_{\mathfrak{p}} = dx_{\mathfrak{p}}/dt_{\mathfrak{p}}$ with

$$\mathrm{vol}_{dt_{\mathfrak{p}}}(Z_a(K_{\mathfrak{p}}) \cap \mathrm{GL}_2(\mathcal{O}_{\mathfrak{p}})) = \mathrm{vol}_{dx_{\mathfrak{p}}}(Z_\sigma(\alpha_v) \cap \mathrm{GL}_2(\mathcal{O}_v)) = 1.$$

Now we are ready for the comparison to the Selberg trace formula.

2.3.6. *Comparison to the Selberg trace formula.* We are now in a situation which has been studied by R. Langlands in several different papers. (Compare [**3**], [**5**].) The trace of the operator Φ_m on the Hilbert space $L^2_{\omega,\mathrm{disc}}(\mathrm{GL}_2(K)\backslash \mathrm{GL}_2(A)/\mathscr{K}_D)$ will be computed by means of the Selberg trace formula. For simplicity we shall assume that our character ω is the trivial character. This simplifies the notation considerably, we shall omit all the indices ω from now on.

The trace formula gives an expression for the trace which is a sum of contributions from the different conjugacy classes in $\mathrm{GL}_2(K)/K$ and some extra terms which are called the Eisenstein terms (compare [**1**, §6]). The Eisenstein terms can of course be ignored in the proof of the main equality.

First of all we consider those terms in the trace formula which come from the identity element and the anisotropic semisimple elements. Each such semisimple element defines a quadratic extension E/K. Then the comparison works out as follows.

(i) The supersingular contribution to the trace Fr^m = contribution of the identity element + contribution of all quadratic extensions which do not split at $\mathfrak{p}$ and $\mathfrak{q}$. (*Reminder*. These extensions do not give orbits!)

(ii) The sum of the contributions from the extensions which do split at $\mathfrak{p}$ or at $\mathfrak{q}$ to the trace of Fr^m = contribution of the same extensions to the Selberg trace.

Let us look at (i). The supersingular orbit is obtained from the quaternion algebra D/K which is not split exactly at the two places $\mathfrak{p}$ and $\mathfrak{q}$. From our discussion in 2.3.4 we get that $Z_\sigma(\gamma) = D(K)^*$ and $Z_\sigma(\alpha_v) = D(K_\mathfrak{p})^*$, $Z_\sigma(\tilde{\alpha}_w) = D(K_\mathfrak{q})^*$. We have

$$\bar{D}_v = \boldsymbol{Z} = D(K_\mathfrak{p})^*/D^{(1)}(K_\mathfrak{p}); \qquad \bar{D}_w = \boldsymbol{Z} = D(K_\mathfrak{q})^*/D^{(1)}(K_\mathfrak{q})$$

where $D^{(1)}$ means that the reduced norm should have absolute value one. The orbit is

$$D(K)\cdot D^{(1)}(K_\mathfrak{p})\cdot D^{(1)}(K_\mathfrak{q})\backslash D(A) = D(K)\backslash \mathrm{GL}_2(A_{\mathfrak{p},\mathfrak{q}}) \times \boldsymbol{Z} \times \boldsymbol{Z}.$$

The conjugacy classes in $D(K)^*/K^*$ are given by the identity element and the elements in E^*/K^* mod conjugation where E/K runs over those quadratic extensions which are nonsplit at $\mathfrak{p}$ and $\mathfrak{q}$. Therefore the contribution to the trace of Fr^m is $\mathrm{vol}(D(K)^*\cdot Z(A)\backslash D(A)^*)\cdot \Psi_{D,m}(e)$ + a sum over the quadratic extensions E/K which do not split at $\mathfrak{p}$ and $\mathfrak{q}$ and the contribution of E/K is equal to

$$\frac{1}{|\mathrm{AUT}(E)|}\sum_{a\in E^*/K^*} \mathrm{vol}(I_E/I_K\, E^*)\cdot I'_D(a)\cdot \delta(a, m).$$

The contribution in the Selberg trace formula looks very similar:

$$\mathrm{vol}(\mathrm{GL}_2(K)\cdot Z(A)\backslash \mathrm{GL}_2(A))\cdot \Phi_m(e);$$

for the central class and for any extension E/K we get

$$\frac{1}{|\mathrm{AUT}(E)|}\sum_{a\in E^*/K^*} \mathrm{vol}(I_E/I_K E^*)\cdot I'_D(a)\cdot \Theta(a, \Phi_m)$$

with $\Theta(a, \Phi_m) = \Theta_\mathfrak{p}(a, \Phi_m^{(\mathfrak{p})} \cdot \theta_\mathfrak{q}(a, \Phi_m^{(\mathfrak{q})})$ and

$$\Theta_\mathfrak{p}(a, \Phi_m^{(\mathfrak{p})}) = \int_{Z_a(K_\mathfrak{p})^*\backslash \mathrm{GL}_2(K_\mathfrak{p})} \Phi_m^{(\mathfrak{p})}\,(x_\mathfrak{p}^{-1}\, a x_\mathfrak{p})\, d\tilde{x}_\mathfrak{p}.$$

The decisive point is that we can compare the factors $\Psi_{D,m}(e)$ and $\Phi_m(e)$ and $\delta(a, m)$ and $\Theta(a, \Phi_m)$.

Again we look at the central classes first. We observe that

$$\mathrm{vol}(D(K)\cdot Z(A)\backslash D(A)^*) = (q - 1)^2\, \mathrm{vol}(\mathrm{GL}_2(K)\cdot Z(A)\backslash \mathrm{GL}_2(A)).$$

This equality is equivalent to the fact that the projective group of the division algebra and the projective linear group have the same Tamagawa numbers. The factor $(q - 1)^2$ is due to the fact that we normalised our measures in such a way that the volumes of $\mathscr{K}'_D \times D^{(1)}(K_\mathfrak{p}) \times D^{(1)}(K_\mathfrak{q})$ and $\mathscr{K}'_D \times \mathrm{GL}_2(\mathcal{O}_\mathfrak{p}) \times \mathrm{GL}_2(\mathcal{O}_\mathfrak{q})$ are both equal to one.

On the other hand it is clear that $\Psi_{D,m}(e) = 1$ if m is even and zero if m is odd. We claim

$$\begin{aligned} \Phi_m^{(\mathfrak{p})}(e) &= -(q-1) && \text{if } m \text{ is even,}\\ &= 0 && \text{if } m \text{ is odd.}\end{aligned}$$

To verify this assertion we introduce the functions $f_n \in \mathscr{H}_\mathfrak{p}$ for $n \geqq 0$.

$$f_n\left(\begin{pmatrix}\pi_{\mathfrak{p}}^{n+a} & 0\\ 0 & \pi_{\mathfrak{p}}^{a}\end{pmatrix}\right) = 1$$

and on the other $GL_2(\mathcal{O}_{\mathfrak{p}})$-double-cosets we put f_n equal to zero. Then one can check rather easily that

$$\Phi_n^{(\mathfrak{p})} = f_n - (q-1)\sum_{\nu=1}^{[n/2]} f_{n-2\cdot\nu}.$$

This formula is also obtained by inverting the formula in Lemma 3.1 in [**4**]. This implies of course our claim and we get for m even $\Phi_m(e) = (q-1)^2$. This proves the equality of the central contributions.

Let us now pick a noncentral element $a \in D_K$. We compute

$$\delta_{\mathfrak{p}}(a, m) = \int_{Z_a(K_{\mathfrak{p}})^*\backslash D(K_{\mathfrak{p}})^*} \Psi_{D,m,\mathfrak{p}}(x_{\mathfrak{p}}^{-1}\, a x_{\mathfrak{p}})\, d\tilde{x}_{\mathfrak{p}}.$$

The crucial observation is that the conjugacy class $\{x_{\mathfrak{p}}^{-1}\, a x_{\mathfrak{p}}\}_{x_{\mathfrak{p}}\in D(K_{\mathfrak{p}})^*}$ decomposes into 2 orbits under the action of the maximal compact subgroup $D^{(1)}(K_{\mathfrak{p}})$ if the element a generates an unramified extension and we get one such orbit if a generates a ramified extension. This implies

$$\begin{aligned}\delta_{\mathfrak{p}}(a, m) &= 2 \quad \text{if } m \text{ is even},\\ &= 0 \quad \text{if } m \text{ is odd};\end{aligned}$$

in case that a generates an extension which is unramified at $\mathfrak{p}$ and

$$\delta_{\mathfrak{p}}(a, m) = 1 \quad \text{for all } m > 0$$

if this extension is ramified at $\mathfrak{p}$.

On the other hand we can compute the orbital integrals $\theta_{\mathfrak{p}}(a, \Phi_m^{(\mathfrak{p})})$ and $\theta_{\mathfrak{q}}(a, \Phi_m^{(\mathfrak{q})})$ by using Langlands' formulas in [**4**, 3.x].

We observe that Lemma 3.5 implies that for $u = \begin{pmatrix} b & 1\\ 0 & b\end{pmatrix}$,

$$\int_{Z_u(k_{\mathfrak{p}})\backslash GL_2(K_{\mathfrak{p}})} \Phi_m^{(\mathfrak{p})}(x_{\mathfrak{p}}^{-1}\, u x_{\mathfrak{p}})\, d\tilde{x}_{\mathfrak{p}} = 0.$$

Now it follows from Lemma 3.6. that in case our element a generates an unramified extension at

$$\begin{aligned}\theta_{\mathfrak{p}}(a, \Phi_m^{(\mathfrak{p})}) &= 2 \quad \text{if } m \text{ is even},\\ &= 0 \quad \text{if } m \text{ is odd},\end{aligned}$$

and $\Theta_{\mathfrak{p}}(a, \Phi^{(\mathfrak{p})}) = 1$ if a generates a ramified extension [**4**, 3.7, 3.8].

We get that the corresponding terms on both sides are equal. At this point we want to say that we did not check the case of characteristic 2. To do this one has to compute the $\delta_{\mathfrak{p}}(a, m)$ and $\Theta_{\mathfrak{p}}(a, \Phi_m^{(\mathfrak{p})})$ in the case where a generates an inseparable extension.

Let us now look at the other nondecomposed orbits. They come from field extensions which are split at at least one of the places $\mathfrak{p}$ or $\mathfrak{q}$. Let $a \in GL_2(K)$ be an element which generates a field extension which splits at $\mathfrak{p}$. We have to compute $\delta_{\mathfrak{p}}(a, m)$ and $\theta_{\mathfrak{p}}(a, \Phi_m^{(\mathfrak{p})})$. Since the extension splits at $\mathfrak{p}$, we find $\bar{D}_v = \boldsymbol{Z} \times \boldsymbol{Z}$ and the image of a in D_v is given by $(\mathrm{ord}_{\mathfrak{p}'}(a), \mathrm{ord}_{\mathfrak{p}''}(a))$. We have to pick one of the two primes $\mathfrak{p}'$ out of the two primes $\mathfrak{p}'$, $\mathfrak{p}''$ which lie above $\mathfrak{p}$. Then the action of Fr^m is given by $(m, 0) \in D_v$. This means that

$$\begin{aligned} \delta_{\mathfrak{p}}(a, m) = \delta'_{\mathfrak{p}}(a, m) &= 1 \quad \text{if } \operatorname{ord}_{\mathfrak{p}'}(a) - \operatorname{ord}_{\mathfrak{p}''}(a) = m, \\ &= 0 \quad \text{otherwise.} \end{aligned}$$

On the other hand it follows from [**4**, Lemma 3.1] that

$$\begin{aligned} \Theta_{\mathfrak{p}}(a, \Phi_m^{(\mathfrak{p})}) &= 1 \quad \text{if } |\operatorname{ord}_{\mathfrak{p}'}(a) - \operatorname{ord}_{\mathfrak{p}''}(a)| = m, \\ &= 0 \quad \text{otherwise.} \end{aligned}$$

Let us assume that our extension does not split at $\mathfrak{q}$. Then we could also pick $\mathfrak{p}''$ instead of $\mathfrak{p}''$. We get the same orbit. Therefore we get twice the trace of Fr^m if we work with the function $\delta'_{\mathfrak{p}}(a, m) + \delta''_{\mathfrak{p}}(a, m)$ on the left-hand side of the trace formula. But obviously we have $\delta'_{\mathfrak{p}}(a, m) + \delta''_{\mathfrak{p}}(a, m) = \Theta_{\mathfrak{p}}(a, \Phi_m^{(\mathfrak{p})})$.

If we now recall that there is a factor $\frac{1}{2}$ on the side of the Selberg trace formula and that there is no such factor in the formula for Fr^m we see that the contributions of the given field on both sides are equal. The argument is very similar in the case that the field extension E/K generated by a is split at $\mathfrak{p}$ and at $\mathfrak{q}$. In that case we have seen in 2.3.4 that we get two orbits. These orbits correspond to the four possible ways of selecting primes above $\mathfrak{p}$ and $\mathfrak{q}$, where the selections conjugate under the action of the Galois group give the same orbit. From here on everything is the same as before and we get the desired equality of the contributions.

Now we have compared the contributions coming from the central and elliptic conjugacy classes. There are no contributions from the unipotent class on the side of the Lefschetz fixed point formula. But these contributions do also vanish in the Selberg trace formula. We have already mentioned above that the orbital integrals of $\Phi_m^{(\mathfrak{p})}$ over the unipotent class vanish. Since this happens at two places it follows from a standard argument (compare [**2**, p. 523]) that these terms are zero.

What remains to be done is the comparison of the terms coming from the decomposed orbit and the split semisimple elements. At this moment we have to recall the definition of $U/\boldsymbol{F}_q$ and these considerations deserve a new paragraph.

2.3.7. *The decomposed orbit.* Let T/K be the standard maximal torus of diagonal matrices in GL_2/K. We denote by N^+ (resp. N^-) the unipotent radicals of the Borel subgroups of upper (resp. lower) triangular matrices. Let us choose an element $\gamma \in K_\infty^\times$ s.t.

$$\operatorname{div}(\gamma) = \mathfrak{p} - \mathfrak{q} + \mathfrak{a} - \mathfrak{a}^\sigma.$$

If $\gamma \in K \cdot \boldsymbol{F}_{q^n}$ then $\gamma_0 = N_n(\gamma)$ has the divisor $\operatorname{div}(\gamma_0) = n\mathfrak{p} - n\mathfrak{q}$. We can find an idele $\boldsymbol{x}_0$ in I_{K_∞} such that

$$\begin{pmatrix} \boldsymbol{x}_0^\sigma & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \gamma^{-1} & 0 \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} \boldsymbol{x}_0 & 0 \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} \pi_v & 0 \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} \pi_w^{-1} & 0 \\ 0 & 1 \end{pmatrix}$$

and we put

$$\boldsymbol{x}_0 = \begin{pmatrix} \boldsymbol{x}_0 & 0 \\ 0 & 1 \end{pmatrix}, \qquad \gamma = \begin{pmatrix} \gamma & 0 \\ 0 & 1 \end{pmatrix}.$$

The points in $M(\bar{\boldsymbol{F}}_q)$ in the decomposed orbit are those which correspond to the elements of the set $\boldsymbol{x}_0 \boldsymbol{y}$ with $\boldsymbol{y} \in \mathrm{GL}_2(A_{\mathfrak{p},\mathfrak{q}}) \times T(K_{\mathfrak{p}}) \times T(K_{\mathfrak{q}})$ and the orbit is $\mathrm{T}(K)\backslash\mathrm{GL}_2(A_{\mathfrak{p},\mathfrak{q}}) \times (\boldsymbol{Z} \times \boldsymbol{Z}) \times (\boldsymbol{Z} \times \boldsymbol{Z})$. Here $(\boldsymbol{Z} \times \boldsymbol{Z}) = T(K_{\mathfrak{p}})/T(\mathcal{O}_{\mathfrak{p}})$ (resp. $T(K_{\mathfrak{q}})/T(\mathcal{O}_{\mathfrak{q}})$) and Fr^m acts by $(m, 0) \times (-m, 0) \in (\boldsymbol{Z} \times \boldsymbol{Z}) \times (\boldsymbol{Z} \times \boldsymbol{Z})$. We have to decide which points of this orbit belong to $U(\bar{\boldsymbol{F}}_q)$. To see this we write

$$y = t^+(y) \cdot n^+(y) \cdot k'^+(y),$$

resp.

$$y = t^-(y) \cdot n^-(y) \cdot k'^-(y)$$

with $t^+(y)$, $t^-(y) \in T(A_{p,q}) \times Z^4$, $n^+(y) \in N^+(A_{p,q})$ (resp. $n^-(y) \in N^-(A_{p,q})$) and $k'^+(y)$, $k'^-(y) \in \mathscr{K}'$. These two different ways of writing y correspond to the two different ways of getting the corresponding FH-sheaf as an FH-extension. To verify this we recall the situation in 1.1. We find that the FH-sheaf is given by the lattice

$$x_0 \cdot y \cdot M_0 = x_0 \cdot t^+(y) \cdot n^+(y) \cdot M_0 = x_0 \cdot t^-(y) \cdot n^-(y) \cdot M_0 = M.$$

In both cases we find a subsheaf of rank 1 in M, namely the one generated by the first, resp. second, basis vector. We call these subsheaves L (resp. H_1). Then it is clear that L is an FH-sheaf of rank 1 and H_1 is an invertible sheaf over X/F_q. If we write

$$t^+(y) = \begin{pmatrix} t_1^+ & 0 \\ 0 & t_2^+ \end{pmatrix}, \qquad t^-(y) = \begin{pmatrix} t_1^- & 0 \\ 0 & t_2^- \end{pmatrix}$$

we find (the degree of x_0 is zero!)

$$\deg(L) = -\deg(t_1^+) = -\log_q(|t_1^+|), \qquad \deg(H_1) = -\deg(t_2^-).$$

We get a point in $U(\bar{F}_q)$ if and only if (compare 2.1) $|t_1^+| < |t_2^+|$ and $|t_2^-| \leqq |t_1^-|$.

If π is a uniformizing element at a place of K then we have the following identity for all $f > 0$:

$$\begin{pmatrix} 1 & \pi^{-f} \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \pi^{-f} & 0 \\ 0 & \pi^{f} \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ \pi^{-f} & 1 \end{pmatrix} \cdot \begin{pmatrix} \pi^{f} & 1 \\ 1 & 0 \end{pmatrix}$$

and this identity allows us to compute the difference $\deg(t_1^-) - \deg(t_2^-) = \deg(t_1^+) - \deg(t_2^+) + 2 \cdot \lambda(n^+(y))$ where $\lambda(n^+(y)) = \sum_{q' \neq p,q} \lambda_{q'}(n_{q'}^+)$ and

$$\begin{aligned} \lambda_{q'}(n_{q'}^+) &= 0 && \text{if } n_{q'}^+ \in \mathcal{O}_{q'}, \\ &= \deg_{q'}(n_{q'}^+) && \text{if } n_{q'}^+ \notin \mathcal{O}_{q'}, \end{aligned}$$

i.e., if $n_{q'}^+ = \varepsilon_{q'} \pi_{q'}^{-f_{q'}}$ and $f_{q'} > 0$ then $\lambda_{q'}(n_{q'}^+) = -f_{q'}$. Then we find that $x_0 y = x_0 \cdot t^+(y) \cdot n^+(y) \cdot k'(y)$ gives a point in $U(\bar{F}_q)$ if and only if $\deg(t_1^+) > \deg(t_2^+)$ and if moreover $2 \cdot \lambda(n^+(y)) \leqq \deg(t_2^+) - \deg(t_1^+)$. We introduce the group

$$T^{(1)}(A) = \left\{ \begin{pmatrix} t_1 & 0 \\ 0 & t_2 \end{pmatrix} \middle| \, |t_1/t_2| = 1 \right\}$$

and we get from a simple counting argument that the contribution of the decomposed orbit to the trace of Fr^m is equal to

$$\sum_{a \in T(K)/K} 2 \operatorname{vol}(T^{(1)}(A)/T(K)\, Z(A)) \cdot \tilde{I}'_D(a) \delta(m, a)$$

where

$$\tilde{I}'_D(a) = -\int_{\mathscr{K}' \times N^+(A_{p,q})} \chi'_D(k' n^{-1} a n k') \cdot \lambda(n)\, dk'\, dn.$$

Again we compare this to the corresponding term in the Selberg trace formula where the corresponding integral is given by

$$\int_{\mathscr{K}\times N^+(A)} \chi_D \circ \Phi_m^{(\mathfrak{p})} \circ \Phi_m^{(\mathfrak{q})}(\boldsymbol{k}^{-1}\boldsymbol{n}^{-1}\boldsymbol{ank})\cdot\tilde{\lambda}(\boldsymbol{n})\, d\boldsymbol{k}\, d\boldsymbol{n}.$$

Here are some differences since in the second integral we are also integrating over the variables $n_\mathfrak{p}$ and $n_\mathfrak{q}$ and

$$\tilde{\lambda}(\boldsymbol{n}) = \sum_{\mathfrak{q}'\neq\mathfrak{p},\mathfrak{q}} \lambda_{\mathfrak{q}'}(n_{\mathfrak{q}'}) + \lambda_\mathfrak{p}(n_\mathfrak{p}) + \lambda_\mathfrak{q}(n_\mathfrak{q}) = \lambda(\boldsymbol{n}') + \lambda_\mathfrak{p}(n_\mathfrak{p}) + \lambda_\mathfrak{q}(n_\mathfrak{q}).$$

Now we come to the last point of the proof. We show that the integrals

$$\int \chi_D \circ \Phi_m^{(\mathfrak{p})} \circ \Phi_m^{(\mathfrak{q})}\, (\boldsymbol{k}^{-1}\boldsymbol{n}^{-1}\boldsymbol{ank})\lambda_\mathfrak{p}(\boldsymbol{n}_\mathfrak{p})\, d\boldsymbol{k}\, d\boldsymbol{n}$$

vanish and the same is true for $\mathfrak{q}$ instead of $\mathfrak{p}$. To see this we observe first of all that we can only get a nonzero contribution if $\mathrm{div}(a) = \pm m\mathfrak{p} - (\pm m\mathfrak{q})$. This is clear since a certainly has to be a unit outside of $\mathfrak{p}$ and $\mathfrak{q}$. Then the assertion follows from our knowledge about $\Theta_\mathfrak{p}(a, \Phi_m^{(\mathfrak{p})})$ and $\Theta_\mathfrak{q}(a, \Phi_m^{(\mathfrak{q})})$. But if this is so then we shall see that the integral $\int_{N^+(K_\mathfrak{p})} \Phi_m^{(\mathfrak{p})}\, (n_\mathfrak{p}^{-1}\, a n_\mathfrak{p})\, \lambda_\mathfrak{p}(n_\mathfrak{p})\, dn_\mathfrak{p}$ vanishes. This follows from the fact that the integrand itself is zero. To see this we start from the fact that $\lambda_\mathfrak{p}(n_\mathfrak{p}) = 0$ implies $n_\mathfrak{p} \notin \mathcal{O}_\mathfrak{p}$ but then one checks easily that $n_\mathfrak{p}^{-1}an_\mathfrak{p} = an_\mathfrak{p}^{-a}n_\mathfrak{p} \notin \mathrm{supp}\,(\Phi_m^{(\mathfrak{p})})$. This follows for instance from the formulas relating the $\Phi_m^{(\mathfrak{p})}$ and the f_n. Therefore the value of the integral in the Selberg trace formula is

$$I_D(a)\cdot\Theta_\mathfrak{p}(a, \Phi_m^{(\mathfrak{p})})\cdot\theta_\mathfrak{q}\,(a, \Phi_m^{(\mathfrak{q})}).$$

There is a factor $\frac{1}{2}$ in front of these terms in the Selberg trace formula, but the $\Theta_\mathfrak{p}(a, \Phi_m^{(\mathfrak{p})})\cdot\Theta_\mathfrak{q}(a, \Phi_m^{(\mathfrak{q})})$ is four times as often equal to one as $\delta(a, m)$ and this proves the equality of the decomposed contributions. This finishes the proof of the main equality.

REFERENCES

1. S. Gelbart and H. Jacquet, *Forms on* GL(2) *from the analytic point of view*, these PROCEEDINGS, part 1, pp. 213–251.

2. H. Jacquet and R. Langlands, *Automorphic forms on* GL(2), Lecture Notes in Math., vol. 114, Springer-Verlag, New York, 1970.

3. R. Langlands, *Modular forms and* l*-adic representations*. II, Proc. Summer School Antwerp, Lecture Notes 349, 1973, pp. 361–500.

4. ———, *Base change for* GL(2), Mimeographed Notes, Inst. for Advanced Study, Princeton, N.J., 1975.

5. ———, *On the zeta-functions of some Shimura varieties*, Mimeographed Notes, Inst. for Advanced Study, Princeton, N.J., 1977.

6. D. Mumford, Unpublished letter to Harder and Kazhdan.

7. H. Saito, *Automorphic forms and algebraic extensions of number* fields, Lecture Notes in Math., Kyoto University, 8, 1975.

8. P. Deligne, *La conjecture de Weil*, Inst. Hautes Etudes Sci. Publ. Math. **43** (1974), 273–307.

9. ———, *Poids dans la cohomologie des variétés algébriques*, Actes du Congrès Internat. des Math., Vancouver, 1974.

MATHEMATISCHES INSTITUT, BONN

HARVARD UNIVERSITY

AUTHOR INDEX

BCDEFGHIJ–CM–89876543210