CAMBRIDGE

ESSENTIAL Specialist Mathematics

Third edition

MICHAEL EVANS
JOSIAN ASTRUC
SUE AVERY
NEIL CRACKNELL
PETER JONES

Interactive student CD included

CHAPTER

1

A toolbox

Objectives

- To revise the properties of **sine**, **cosine** and **tangent**
- To revise methods for solving **right-angled triangles**
- To revise the **sine rule** and **cosine rule**
- To revise basic triangle, parallel lines and circle geometry
- To revise **arithmetic** and **geometric sequences**
- To revise **arithmetic** and **geometric series**
- To revise **infinite geometric series**
- To revise cartesian equations for **circles**
- To sketch graphs of **ellipses** from the general cartesian relation

$$\frac{(x-h)^2}{a^2}+\frac{(y-k)^2}{b^2}=1$$

- To sketch graphs of **hyperbolas** from the general cartesian relation

$$\frac{(x-h)^2}{a^2}-\frac{(y-k)^2}{b^2}=1$$

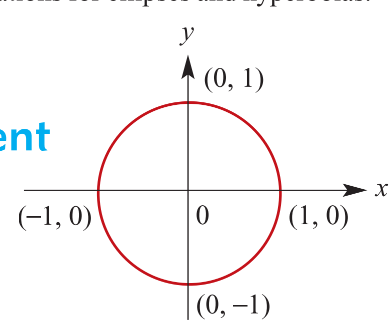
- To consider **asymptotic behaviour** of hyperbolas
- To work with **parametric equations** for circles, ellipses and hyperbolas

The first six sections of this chapter revise areas for which knowledge is required in this course, and which are referred to in the Specialist Mathematics Study Design.

The final section introduces cartesian and parametric equations for ellipses and hyperbolas.

1.1 Circular functions

Defining sine, cosine and tangent

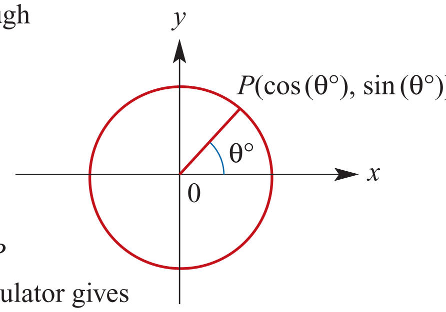


The unit circle is a circle of radius one with centre at the origin.

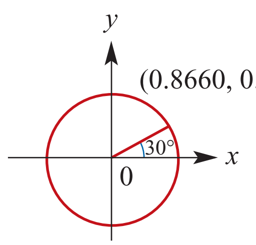
It is the graph of the relation $x^2 + y^2 = 1$.

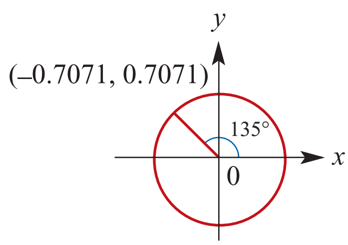
Sine and cosine may be defined for any angle through the unit circle.

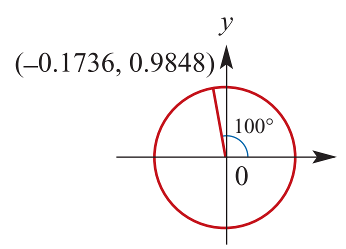
For the angle of $\theta°$, a point P on the unit circle is defined as illustrated opposite. The angle is measured in an anticlockwise direction from the positive direction of the x axis.



$\cos(\theta°)$ is defined as the x-coordinate of the point P and $\sin(\theta°)$ is defined as the y-coordinate of P. A calculator gives approximate values for these coordinates where the angle is given.







$\sin 30° = 0.5$ (exact value)

$\cos 30° = \frac{\sqrt{3}}{2} \approx 0.8660$

$\sin 135° = \frac{1}{\sqrt{2}} \approx 0.7071$

$\cos 135° = \frac{-1}{\sqrt{2}} \approx -0.7071$

$\cos 100° \approx -0.1736$

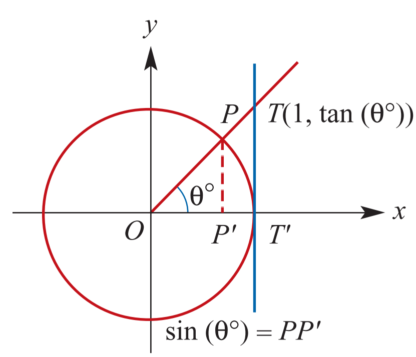
$\sin 100° \approx 0.9848$

$\tan(\theta°)$ is defined by $\tan(\theta°) = \frac{\sin(\theta°)}{\cos(\theta°)}$. The value of $\tan(\theta°)$ can be illustrated geometrically through the unit circle.

By considering similar triangles OPP' and OTT', it can be seen that

$$\frac{TT'}{OT'} = \frac{PP'}{OP'}$$

i.e.
$$TT' = \frac{\sin(\theta°)}{\cos(\theta°)} = \tan(\theta°)$$



For a right-angled triangle OBC, a similar triangle $OB'C'$ can be constructed that lies in the unit circle.

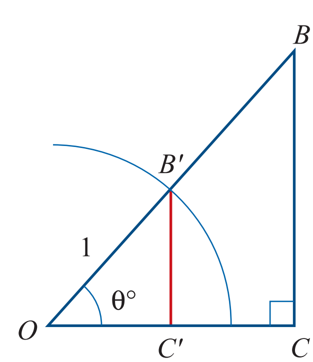
By the definition, $OC' = \cos(\theta°)$ and $C'B' = \sin(\theta°)$.

The scale factor is the length OB.

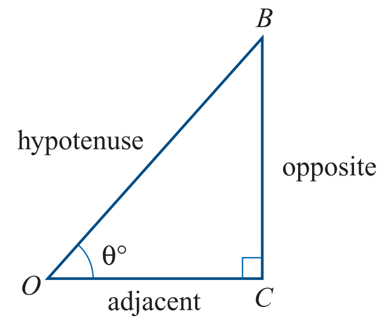
Hence $BC = OB \sin(\theta°)$ and $OC = OB \cos(\theta°)$.

This implies

$$\frac{BC}{OB} = \sin(\theta°) \quad \text{and} \quad \frac{OC}{OB} = \cos(\theta°)$$



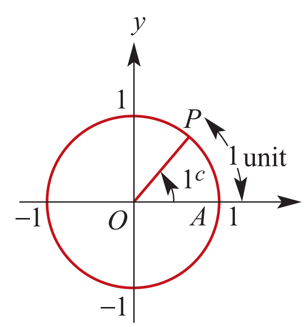
This gives the ratio definition of sine and cosine for a right-angled triangle. The naming of sides with respect to an angle θ° is as shown.



$$\sin\theta^\circ = \frac{\text{opp}}{\text{hyp}} \quad \left(\frac{\text{opposite}}{\text{hypotenuse}}\right)$$
$$\cos\theta^\circ = \frac{\text{adj}}{\text{hyp}} \quad \left(\frac{\text{adjacent}}{\text{hypotenuse}}\right)$$
$$\tan\theta^\circ = \frac{\text{opp}}{\text{adj}} \quad \left(\frac{\text{opposite}}{\text{adjacent}}\right)$$

Definition of a radian

In moving around the circle a distance of 1 unit from A to P the angle POA is defined. The measure of this angle is 1 radian.



One radian (written 1^c) is the angle **subtended** at the centre of the unit circle by an arc of length 1 unit.

Note: Angles formed by moving **anticlockwise** around the circumference of the unit circle are defined as **positive**. Those formed by moving in a **clockwise** direction are said to be **negative**.

Degrees and radians

The angle, in radians, swept out in one revolution of a circle is $2\pi^c$

$$\therefore \quad 2\pi^c = 360^\circ$$
$$\therefore \quad \pi^c = 180^\circ$$
$$\therefore \quad 1^c = \frac{180^\circ}{\pi} \quad \text{or} \quad 1^\circ = \frac{\pi^c}{180}$$

Henceforth the ^{c} may be omitted. Any angle is assumed to be measured in radians unless otherwise indicated.

The following table displays the conversions of some special angles from degrees to radians.

Angles in degrees	0	30	45	60	90	180	360
Angles in radians	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	π	2π

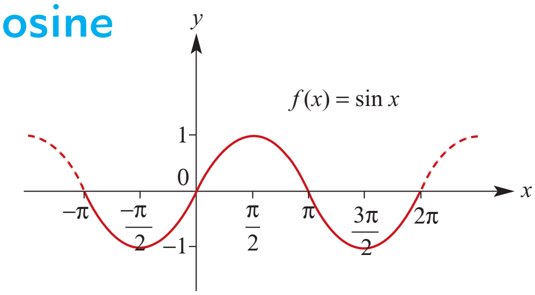
Some values for the trigonometric functions are given in the following table.

x in radians	$\sin x$	$\cos x$	$\tan x$
0	0	1	0
$\frac{\pi}{6}$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{3}}{3}$
$\frac{\pi}{4}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	1
$\frac{\pi}{3}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\sqrt{3}$
$\frac{\pi}{2}$	1	0	undefined

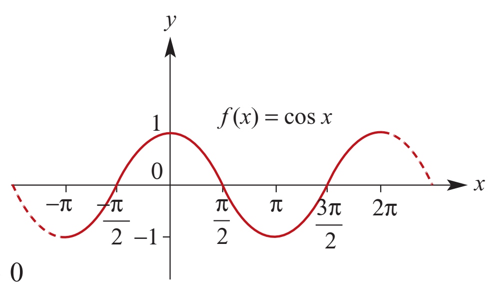
The graphs of sine and cosine

As $\sin x = \sin(x + 2\pi n)$, $n \in Z$, the function is **periodic** and the period is 2π.

A sketch of the graph of $f: R \to R$, $f(x) = \sin x$ is shown opposite. The amplitude is 1.



A sketch of the graph of $f: R \to R$, $f(x) = \cos x$ is shown opposite. The period of the function is 2π. The amplitude is 1.



For $y = a\cos(nx)$ and $y = a\sin(nx)$ $\quad a > 0, n > 0$

Period $= \frac{2\pi}{n}$, amplitude $= a$, range $= [-a, a]$

Symmetry properties for sine and cosine

From the graph of the functions or from the unit circle definitions, the following results may be obtained.

$\sin(\pi - \theta) = \sin\theta$	$\sin(\pi + \theta) = -\sin\theta$
$\cos(\pi - \theta) = -\cos\theta$	$\cos(\pi + \theta) = -\cos\theta$
$\sin(2\pi - \theta) = -\sin\theta$	$\sin(-\theta) = -\sin\theta$
$\cos(2\pi - \theta) = \cos\theta$	$\cos(-\theta) = \cos\theta$
$\sin(\theta + 2n\pi) = \sin\theta$	$\cos(\theta + 2n\pi) = \cos\theta$ for $n \in Z$
$\sin\left(\frac{\pi}{2} - \theta\right) = \cos\theta$	$\cos\left(\frac{\pi}{2} - \theta\right) = \sin\theta$

Example 1

a Change 135° into radians.

b Change 1.5^c into degrees, correct to two decimal places.

c Find the exact value of:

i $\sin \pi^c$ **ii** $\cos\left(\frac{7\pi}{4}\right)^c$

Solution

a $135° = \left(\frac{135 \times \pi}{180}\right)^c = \left(\frac{3\pi}{4}\right)^c$

Note that angles in radians which are expressed in terms of π are left in that form.

b $1.5^c = \left(\frac{1.5 \times 180}{\pi}\right)^\circ = 85.94^\circ$, correct to two decimal places.

c **i** $\sin \pi^c = 0$

ii $\cos\left(\frac{7\pi}{4}\right)^c = \cos\left(\frac{7\pi}{4} - 2\pi\right)^c = \cos\left(\frac{-\pi}{4}\right)^c = \cos\frac{\pi^c}{4} = \frac{\sqrt{2}}{2}$

Example 2

Find the exact value of:

a $\sin 150°$ **b** $\cos(-585°)$

Solution

a
$$\begin{aligned}\sin 150° &= \sin(180° - 150°)\\ &= \sin 30°\\ &= 0.5\end{aligned}$$

b
$$\begin{aligned}\cos(-585°) &= \cos 585°\\ &= \cos(585° - 360°)\\ &= \cos(225°)\\ &= -\cos 45°\\ &= -\frac{\sqrt{2}}{2}\end{aligned}$$

Example 3

Find the exact value of:

a $\sin\left(\frac{11\pi}{6}\right)$ **b** $\cos\left(\frac{-45\pi}{6}\right)$

Solution

a
$$\begin{aligned}\sin\left(\frac{11\pi}{6}\right) &= \sin\left(2\pi - \frac{11\pi}{6}\right)\\ &= -\sin\frac{\pi}{6}\\ &= -\frac{1}{2}\end{aligned}$$

b
$$\begin{aligned}\cos\left(\frac{-45\pi}{6}\right) &= \cos\left(-7\tfrac{1}{2} \times \pi\right)\\ &= \cos\left(-\frac{\pi}{2}\right)\\ &= 0\end{aligned}$$

The Pythagorean identity

For any value of θ

$$\cos^2\theta + \sin^2\theta = 1$$

Example 4

If $\sin x^\circ = 0.3$, $0 < x < 90$, find:

a $\cos x^\circ$ **b** $\tan x^\circ$

Solution

a
$$\sin^2 x^\circ + \cos^2 x^\circ = 1$$
$$0.09 + \cos^2 x^\circ = 1$$
$$\therefore \quad \cos^2 x^\circ = 0.91$$
$$\cos x^\circ = \pm\sqrt{0.91}$$

as $0 < x < 90$, $\cos x^\circ = \sqrt{0.91} = \sqrt{\dfrac{91}{100}} = \dfrac{\sqrt{91}}{10}$

b
$$\tan x^\circ = \frac{\sin x}{\cos x} = \frac{0.3}{\sqrt{0.91}} = \frac{3}{\sqrt{91}} = \frac{3\sqrt{91}}{91}$$

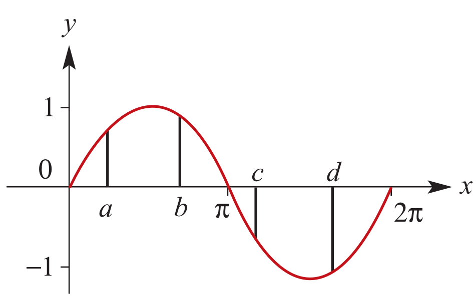
Solution of equations

If a trigonometric equation has a solution, then it will have a corresponding solution in each 'cycle' of its domain. Such equations are solved by using the symmetry of the graphs to obtain solutions within one 'cycle' of the function. Other solutions may be obtained by adding multiples of the period to these solutions.

Example 5

The graph of $y = f(x)$ where $f: R \to R$, $f(x) = \sin x$, $x \in [0, 2\pi]$ is shown.

Find the other x value which has the same y value as each of the pronumerals marked.



Solution

For $x = a$, the value is $\pi - a$.

For $x = b$, the other value is $\pi - b$.

For $x = c$, the other value is $2\pi - (c - \pi) = 3\pi - c$.

For $x = d$, the other value is $\pi + (2\pi - d) = 3\pi - d$.

Example 6

Solve the equation $\sin\left(2x + \dfrac{\pi}{3}\right) = \dfrac{1}{2}$ for $x \in [0, 2\pi]$.

Solution

Let $\theta = 2x + \frac{\pi}{3}$

Note $\quad 0 \le x \le 2\pi$

$$\Leftrightarrow \quad 0 \le 2x \le 4\pi$$

$$\Leftrightarrow \quad \frac{\pi}{3} \le 2x + \frac{\pi}{3} \le \frac{13\pi}{3}$$

$$\Leftrightarrow \quad \frac{\pi}{3} \le \theta \le \frac{13\pi}{3}$$

Therefore solving the equation $\sin\left(2x + \frac{\pi}{3}\right) = \frac{1}{2}$ for $x \in [0, 2\pi]$ is achieved by first solving the equation $\sin(\theta) = \frac{1}{2}$ for $\frac{\pi}{3} \le \theta \le \frac{13\pi}{3}$

Consider $\sin\theta = \frac{1}{2}$

$$\therefore \theta = \frac{\pi}{6} \text{ or } \frac{5\pi}{6} \text{ or } 2\pi + \frac{\pi}{6} \text{ or } 2\pi + \frac{5\pi}{6} \text{ or } 4\pi + \frac{\pi}{6} \text{ or } 4\pi + \frac{5\pi}{6} \text{ or } \ldots$$

The solutions $\frac{\pi}{6}$ and $\frac{29\pi}{6}$ are not required as they lie outside the restricted domain for θ.

$\therefore$ For $\frac{\pi}{3} \le \theta \le \frac{13\pi}{3}$

$$\theta = \frac{5\pi}{6} \text{ or } \frac{13\pi}{6} \text{ or } \frac{17\pi}{6} \text{ or } \frac{25\pi}{6}$$

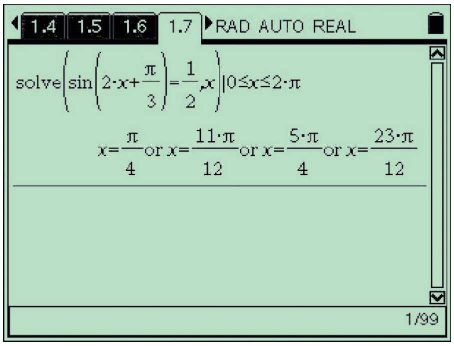
$$\therefore 2x + \frac{2\pi}{6} = \frac{5\pi}{6} \text{ or } \frac{13\pi}{6} \text{ or } \frac{17\pi}{6} \text{ or } \frac{25\pi}{6}$$

$$\therefore 2x = \frac{3\pi}{6} \text{ or } \frac{11\pi}{6} \text{ or } \frac{15\pi}{6} \text{ or } \frac{23\pi}{6}$$

$$\therefore x = \frac{\pi}{4} \text{ or } \frac{11\pi}{12} \text{ or } \frac{5\pi}{4} \text{ or } \frac{23\pi}{12}$$

Using a TI-Nspire calculator

Make sure the calculator is in Radian mode (**RAD** on top bar).
Complete as shown.

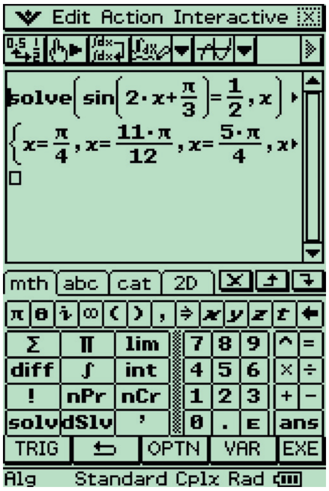


Using a Casio ClassPad calculator

Make sure the calculator is in Radian mode (**Rad** at bottom right of screen in Main window).

Enter $\sin\left(2x + \frac{\pi}{3}\right) = \frac{1}{2} \mid 0 \le x \le 2\pi$ and highlight the equation $\sin\left(2x + \frac{\pi}{3}\right) = \frac{1}{2}$.

From the **Interactive** menu choose **Equation/Inequality** and then **solve** and set the variable as x.



Sketch graphs

The graphs of functions defined by rules of the form $f(x) = a\sin(nx + \varepsilon) + b$ and $f(x) = a\cos(nx + \varepsilon) + b$ can be obtained from the graphs of $\sin x$ and $\cos x$ by transformations.

Example 7

Sketch the graph of h: $[0, 2\pi] \to R$, $h(x) = 3\cos\left(2x + \frac{\pi}{3}\right) + 1$.

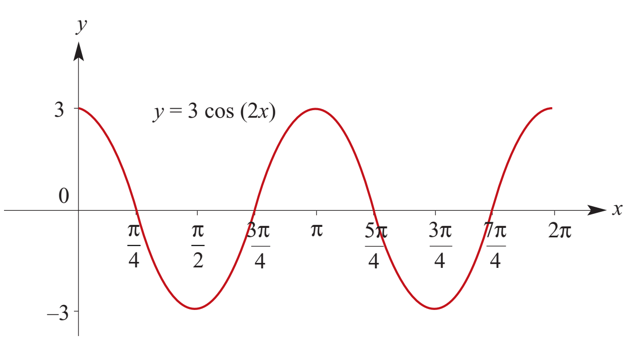
Solution

$$h(x) = 3\cos\left(2\left(x + \frac{\pi}{6}\right)\right) + 1$$

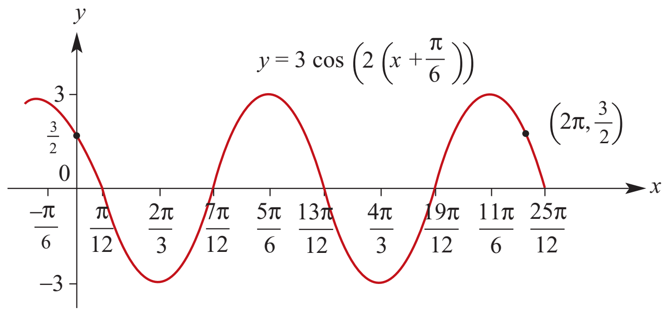
The transformations from the graph of $y = \cos x$ are

- a dilation from the y axis of factor $\frac{1}{2}$
- a dilation from the x-axis of factor 3
- a translation of $\frac{\pi}{6}$ in the negative direction of the x axis
- a translation of 1 in the positive direction of the y axis

The graph with the dilations applied is as shown below.



The translation $\frac{\pi}{6}$ in the negative direction of the x axis is then applied.

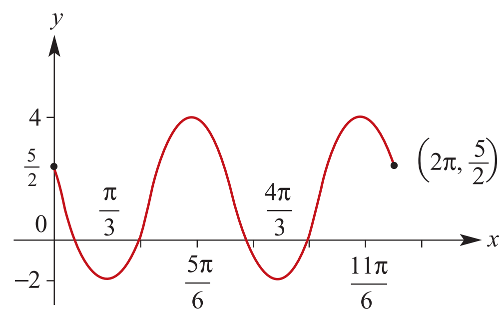


The final translation is applied and the graph is given for the required domain.

The x-axis intercepts are found by solving the equation.

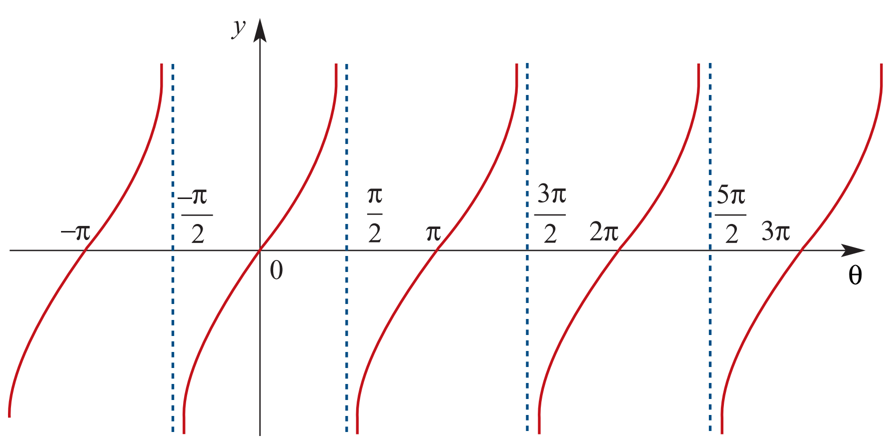
$$3\cos\left(2x + \frac{\pi}{3}\right) + 1 = 0$$

i.e. $$\cos\left(2x + \frac{\pi}{3}\right) = -\frac{1}{3}$$



The graph of tan

A sketch of the graph of $f: R\backslash\{(2n+1)\frac{\pi}{2}; n \in Z\} \to R,\ f(\theta) = \tan\theta$ is shown below.



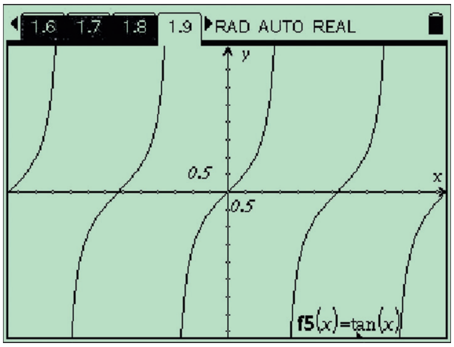
Note: $\theta = -\frac{\pi}{2}, \frac{\pi}{2}, \frac{3\pi}{2}$ and $\frac{5\pi}{2}$ are asymptotes.

Observations from the graph

- The graph repeats itself every π units, i.e. the period of tan is π.
- Range of tan is R.
- The vertical asymptotes have equation $\theta = (2k+1)\frac{\pi}{2}$ where $k \in Z$.

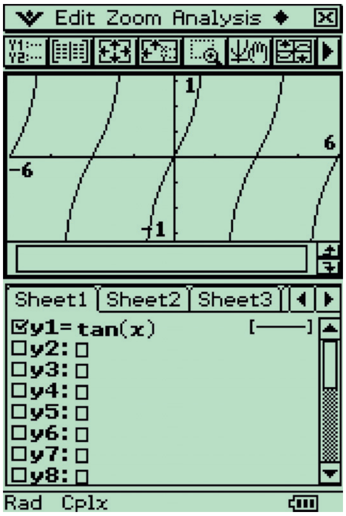
Using a TI-Nspire calculator

Open a **Graphs & Geometry** application and define $f1(x) = \tan(x)$.



Using a Casio ClassPad calculator

Choose Graph&Tab... from the Main menu.
Enter tan (x).
Check the tickbox and click on the button.
The window setting may be altered by using the button (if this button does not appear, click on the Graph window first to select it).



Symmetry properties for tan

From the definition of tan, the following results are obtained:

$\tan(\pi - \theta) = -\tan\theta \quad \tan(\pi + \theta) = \tan\theta \quad \tan(2\pi - \theta) = -\tan\theta \quad \tan(-\theta) = -\tan\theta$

Example 8

Find the exact values of:

a $\tan(330^\circ)$ **b** $\tan\left(\frac{4\pi}{3}\right)$

Solution

a $\tan(330^\circ) = \tan(360 - 30)^\circ$
$= -\tan(30)^\circ$
$= -\frac{\sqrt{3}}{3}$

b $\tan\left(\frac{4\pi}{3}\right) = \tan\left(\pi + \frac{\pi}{3}\right)$
$= \tan\left(\frac{\pi}{3}\right)$
$= \sqrt{3}$

Solutions of equations involving tan

The procedure here is similar to that used for solving equations involving sin and cos, except that only one solution needs to be selected then all other solutions are π or 180° apart.

Example 9

Solve the equations:

a $\tan x = -1$ for $x \in [0, 4\pi]$ **b** $\tan(2x - \pi) = \sqrt{3}$ for $x \in [-\pi, \pi]$

Solution

a $\tan x = -1$

Now $\tan\frac{3\pi}{4} = -1$.

Therefore $x = \frac{3\pi}{4}$ or $\frac{3\pi}{4} + \pi$ or $\frac{3\pi}{4} + 2\pi$ or $\frac{3\pi}{4} + 3\pi$.

Hence $x = \frac{3\pi}{4}$ or $\frac{7\pi}{4}$ or $\frac{11\pi}{4}$ or $\frac{15\pi}{4}$.

b $\tan(2x - \pi) = \sqrt{3}$.

Therefore $-2\pi \le 2x \le 2\pi$ and thus $-3\pi \le 2x - \pi \le \pi$ and $-3\pi \le \theta \le \pi$.

In order to solve $\tan(2x - \pi) = \sqrt{3}$ first solve $\tan\theta = \sqrt{3}$.

$\theta = \frac{\pi}{3}$ or $\frac{\pi}{3} - \pi$ or $\frac{\pi}{3} - 2\pi$ or $\frac{\pi}{3} - 3\pi$

$\therefore$ $\theta = \frac{\pi}{3}$ or $-\frac{2\pi}{3}$ or $-\frac{5\pi}{3}$ or $-\frac{8\pi}{3}$

and as $\theta = 2x - \pi$

$2x - \pi = \frac{\pi}{3}$ or $-\frac{2\pi}{3}$ or $-\frac{5\pi}{3}$ or $-\frac{8\pi}{3}$

Therefore $2x = \frac{4\pi}{3}$ or $\frac{\pi}{3}$ or $-\frac{2\pi}{3}$ or $-\frac{5\pi}{3}$

And $x = \frac{2\pi}{3}$ or $\frac{\pi}{6}$ or $-\frac{\pi}{3}$ or $-\frac{5\pi}{6}$.

Exercise 1A

1 **a** Change the following angles from degrees to exact values in radians:

i $720°$ **ii** $540°$ **iii** $-450°$ **iv** $15°$ **v** $-10°$ **vi** $-315°$

b Change the following angles from radians to degrees:

i $\left(\frac{5\pi}{4}\right)^c$ **ii** $\left(\frac{-2\pi}{3}\right)^c$ **iii** $\left(\frac{7\pi}{12}\right)^c$

iv $\left(\frac{-11\pi}{6}\right)^c$ **v** $\left(\frac{13\pi}{9}\right)^c$ **vi** $\left(\frac{-11\pi}{12}\right)^c$

2 Perform the correct conversion on each of the following, giving the answer correct to two decimal places.

a Convert from degrees to radians:

i $7°$ **ii** $-100°$ **iii** $-25°$

iv $51°$ **v** $206°$ **vi** $-410°$

b Convert from radians to degrees:

i 1.7^c **ii** -0.87^c **iii** 2.8^c

iv 0.1^c **v** -3^c **vi** -8.9^c

3 Find the exact value of each of the following:

a $\sin\left(\frac{2\pi}{3}\right)$ **b** $\cos\left(\frac{3\pi}{4}\right)$ **c** $\cos\left(-\frac{\pi}{3}\right)$

d $\cos\left(\frac{5\pi}{4}\right)$ **e** $\cos\left(\frac{9\pi}{4}\right)$ **f** $\sin\left(\frac{11\pi}{3}\right)$

g $\cos\left(\frac{31\pi}{6}\right)$ **h** $\cos\left(\frac{29\pi}{6}\right)$ **i** $\sin\left(-\frac{23\pi}{6}\right)$

4 Find the exact value of each of the following:

a $\sin(135°)$ **b** $\cos(-300°)$ **c** $\sin(480°)$

d $\cos(240°)$ **e** $\sin(-225°)$ **f** $\sin(420°)$

5 If $\sin(x°) = 0.5$ and $90 < x < 180$, find:

a $\cos(x°)$ **b** $\tan(x°)$

6 If $\cos(x°) = -0.7$ and $180 < x < 270$, find:

a $\sin(x°)$ **b** $\tan(x°)$

7 If $\sin(x) = -0.5$ and $\pi < x < \frac{3\pi}{2}$, find:

a $\cos(x)$ **b** $\tan(x)$

8 If $\sin(x) = -0.3$ and $\frac{3\pi}{2} < x < 2\pi$, find:

a $\cos(x)$ **b** $\tan(x)$

9 Solve each of the following for $x \in [0, 2\pi]$:

a $\sin x = -\frac{\sqrt{3}}{2}$ **b** $\sin(2x) = \frac{\sqrt{3}}{2}$

c $2\cos(2x) = -1$ **d** $\sin\left(x + \frac{\pi}{3}\right) = -\frac{1}{2}$

e $2\cos\left(2\left(x + \frac{\pi}{3}\right)\right) = -1$ **f** $2\sin\left(2x + \frac{\pi}{3}\right) = -\sqrt{3}$

10 Find the exact values of each of the following:

a $\tan\left(\frac{5\pi}{4}\right)$ **b** $\tan\left(-\frac{2\pi}{3}\right)$ **c** $\tan\left(-\frac{29\pi}{6}\right)$ **d** $\tan(240°)$

11 If $\tan x = \frac{1}{4}$ and $\pi \le x \le \frac{3\pi}{2}$, find the exact value of:

a $\sin x$ **b** $\cos x$ **c** $\tan(-x)$ **d** $\tan(\pi - x)$

12 If $\tan x = -\frac{\sqrt{3}}{2}$ and $\frac{\pi}{2} \le x \le \pi$, find the exact value of:

a $\sin x$ **b** $\cos x$ **c** $\tan(-x)$ **d** $\tan(x - \pi)$

13 Solve each of the following for $x \in [0, 2\pi]$:

a $\tan x = -\sqrt{3}$ **b** $\tan\left(3x - \frac{\pi}{6}\right) = \frac{\sqrt{3}}{3}$

c $2\tan\left(\frac{x}{2}\right) + 2 = 0$ **d** $3\tan\left(\frac{\pi}{2} + 2x\right) = -3$

14 Sketch the graphs of each of the following for the stated domain:

a $f(x) = \sin 2x, x \in [0, 2\pi]$ **b** $f(x) = \cos\left(x + \frac{\pi}{3}\right), x \in \left[\frac{-\pi}{3}, \pi\right]$

c $f(x) = \cos\left(2\left(x + \frac{\pi}{3}\right)\right), x \in [0, \pi]$ **d** $f(x) = 2\sin(3x) + 1, x \in [0, \pi]$

e $f(x) = 2\sin\left(x - \frac{\pi}{4}\right) + \sqrt{3}, x \in [0, 2\pi]$

15 Sketch the graphs of each of the following for $x \in [0, \pi]$, clearly labelling all intercepts with the axes and all asymptotes:

a $f(x) = \tan(2x)$ **b** $f(x) = \tan\left(x - \frac{\pi}{3}\right)$

c $f(x) = 2\tan\left(2x + \frac{\pi}{3}\right)$ **d** $f(x) = 2\tan\left(2x + \frac{\pi}{3}\right) - 2$

1.2 Solving right-angled triangles

Pythagoras' theorem

This well-known theorem is applicable to right-angled triangles and will be stated here without proof:

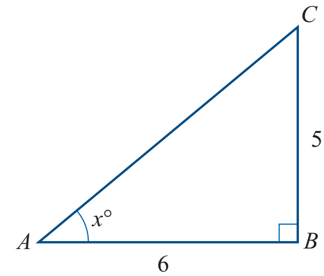
$$(\text{hyp})^2 = (\text{opp})^2 + (\text{adj})^2$$

Example 10

In triangle ABC, $\angle ABC = 90°$ and $\angle CAB = x°$, $AB = 6$ cm and $BC = 5$cm.

Find:

a AC **b** the trigonometric ratios related to $x°$ **c** x



Solution

a By Pythagoras' theorem, $AC^2 = 5^2 + 6^2 = 61$

$\therefore \quad AC = \sqrt{61}$ cm

b $\sin x° = \dfrac{5}{\sqrt{61}} \quad \cos x° = \dfrac{6}{\sqrt{61}} \quad \tan x° = \dfrac{5}{6}$

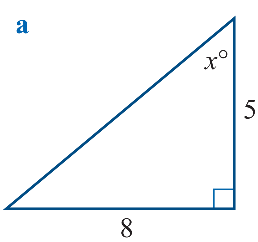
c $\tan x° = \dfrac{5}{6}$

$\therefore \quad x = 39.81$ (correct to two decimal places)

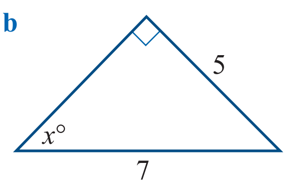
Exercise 1B

1 Find the trigonometric ratios $\tan x°$, $\cos x°$ and $\sin x°$ for each of the following triangles:

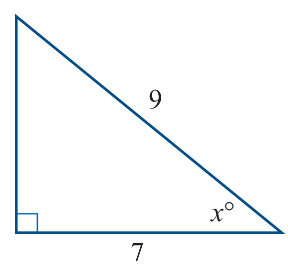
a



b

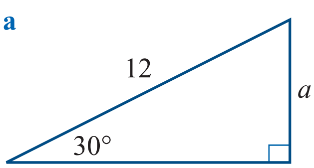


c

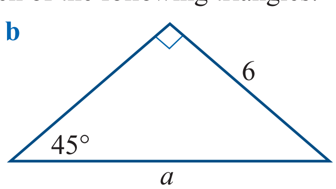


2 Find the exact value of a in each of the following triangles:

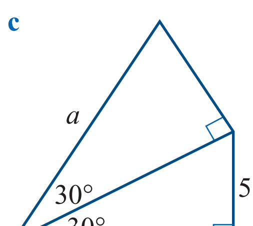
a



b

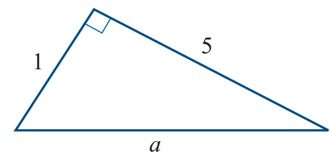


c

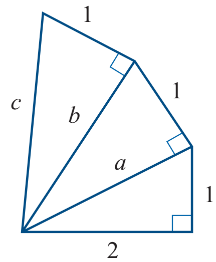


3 Find the exact value of the pronumerals for each of the following:

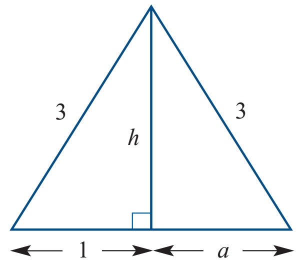
a



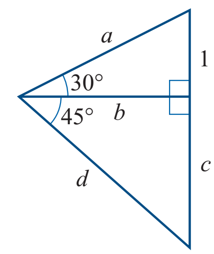
b



c



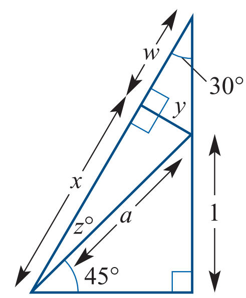
d



4 **a** Find the values of a, y, z, w and x.

b Hence deduce exact values for $\sin(15^\circ)$, $\cos(15^\circ)$ and $\tan(15^\circ)$.

c Find the exact values for $\sin(75^\circ)$, $\cos(75^\circ)$ and $\tan(75^\circ)$.



1.3 The sine and cosine rules

The sine rule

In section 1.2, methods for finding unknown lengths and angles for right-angled triangles were discussed. This section discusses methods for finding unknown quantities in non-right-angled triangles.

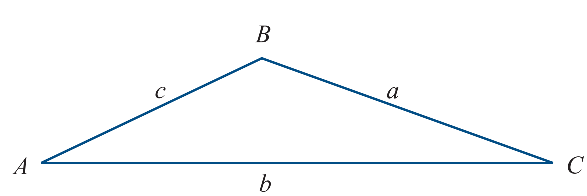
The sine rule is used to find unknown quantities in a triangle when one of the following situations arises:

- one side and two angles are given
- two sides and a non-included angle are given

In the first case a unique triangle is defined, but for the second it is possible for two triangles to exist.

Labelling convention

The following convention is followed in the remainder of this chapter. Interior angles are denoted by upper-case letters, and the length of the side opposite an angle is denoted by the corresponding lower-case letter.

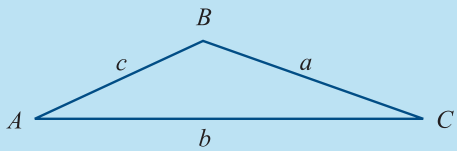


The magnitude of angle BAC is denoted by A.

The length of side BC is denoted by a.

The sine rule states that for triangle ABC

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$



A proof will only be given for the acute-angled triangle case. The proof for obtuse-angled triangles is similar.

Proof

In triangle ACD, $\sin A = \dfrac{h}{b}$

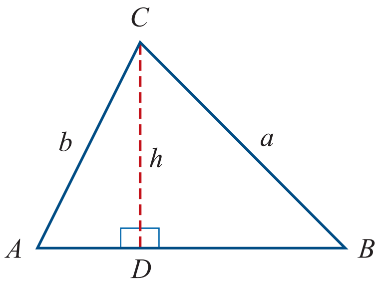
$\therefore \quad h = b \sin A$

In triangle BCD, $\sin B = \dfrac{h}{a}$

$\therefore \quad h = a \sin B$

$\therefore \quad a \sin B = b \sin A$

i.e. $\dfrac{a}{\sin A} = \dfrac{b}{\sin B}$

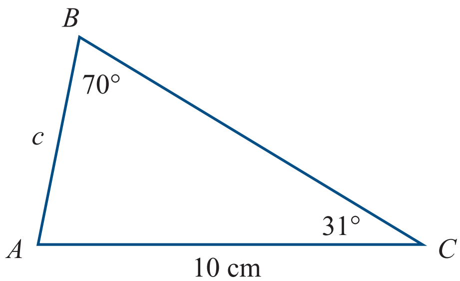


Similarly, starting with a perpendicular from A to BC would give

$$\frac{b}{\sin B} = \frac{c}{\sin C}$$

Example 11

Use the sine rule to find the length of AB.



Solution

$$\frac{c}{\sin 31^\circ} = \frac{10}{\sin 70^\circ}$$

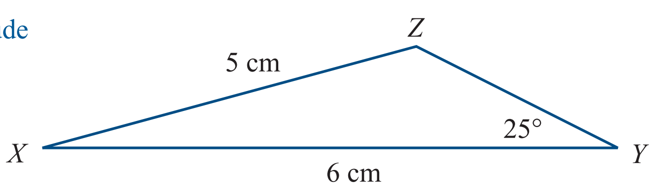
$$\therefore \quad c = \frac{10 \times \sin 31^\circ}{\sin 70^\circ}$$

$$\therefore \quad c = 5.4809\ldots$$

The length of AB is 5.48 cm correct to two decimal places.

Example 12

Use the sine rule to find the magnitude of angle XZY, given that $Y = 25^\circ$, $y = 5$ and $z = 6$.



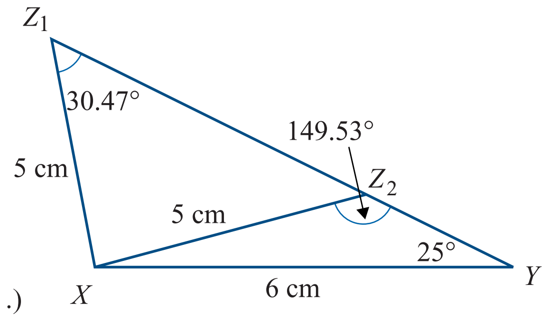
Solution

$$\frac{5}{\sin 25^\circ} = \frac{6}{\sin Z}$$

$$\therefore \quad \frac{\sin Z}{6} = \frac{\sin 25^\circ}{5}$$

$$\therefore \quad \sin Z = \frac{6 \times \sin 25^\circ}{5}$$

$$= 0.5071\ldots$$



$$\therefore \quad Z = \sin^{-1}(0.5071\ldots)$$

$$\therefore \quad Z = 30.4736\ldots \text{ or } 180 - 30.4736\ldots$$

$$\therefore \quad Z = 30.47^\circ \text{ or } Z = 149.53^\circ \text{ correct to two decimal places.}$$

Remember: $\sin(180 - \theta)^\circ = \sin \theta^\circ$

There are two solutions for the equation $\sin Z = 0.5071\ldots$

Note: When using the sine rule in the situation where two sides and a non-included angle are given, the possibility of two such triangles existing must be considered. Existence can be checked through the sum of the given angle and the calculated angle not exceeding 180°.

The cosine rule

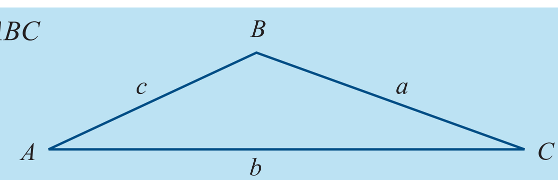
The cosine rule is used to find unknown quantities in a triangle when one of the following situations arises:

- two sides and an included angle are given
- three sides are given

The cosine rule states that for triangle ABC

$a^2 = b^2 + c^2 - 2bc \cos A$ or,

equivalently, $\cos A = \dfrac{b^2 + c^2 - a^2}{2bc}$



The symmetrical results also hold, i.e.

- $b^2 = a^2 + c^2 - 2ac \cos B$
- $c^2 = a^2 + b^2 - 2ab \cos C$

The result will be proved for acute-angled triangles. The proof for obtuse-angled triangles is similar.

Proof

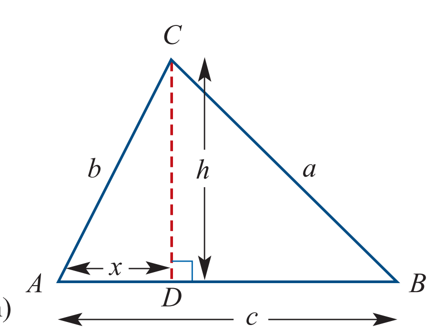
In triangle ACD

$$b^2 = x^2 + h^2 \text{ (Pythagoras' theorem)}$$

$$\cos A = \frac{x}{b} \text{ and therefore } x = b \cos A$$

In triangle BCD

$$a^2 = (c - x)^2 + h^2 \text{ (Pythagoras' theorem)}$$

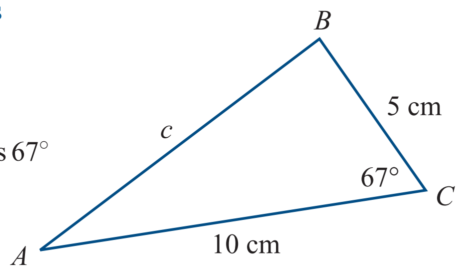


Expanding gives

$$a^2 = c^2 - 2cx + x^2 + h^2$$
$$= c^2 - 2cx + b^2 \qquad (\text{as } x^2 + h^2 = b^2)$$
$$\therefore \quad a^2 = b^2 + c^2 - 2bc\cos A \qquad (\text{as } x = b\cos A)$$

Example 13

For triangle ABC, find the length of AB in centimetres correct to two decimal places.



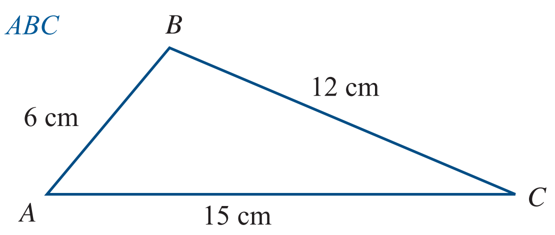
Solution

$$c^2 = 5^2 + 10^2 - 2 \times 5 \times 10 \cos 67^\circ$$
$$= 85.9268\ldots$$
$$\therefore c \approx 9.269$$

The length of AB is 9.27 cm correct to two decimal places.

Example 14

Find the magnitude of angle ABC for triangle ABC correct to two decimal places.



Solution

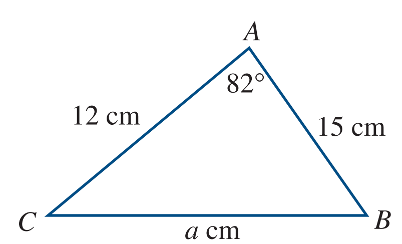
$$\cos B = \frac{a^2 + c^2 - b^2}{2ac}$$
$$= \frac{12^2 + 6^2 - 15^2}{2 \times 12 \times 6}$$
$$= -0.3125$$
$$\therefore \quad B = (108.2099\ldots)^\circ$$
i.e. $\quad B \approx 108.21^\circ$ correct to two decimal places.

The magnitude of angle ABC is $108^\circ\ 12'\ 36''$ (to the nearest second).

Example 15

In ΔABC, $\angle CAB = 82^\circ$, $AC = 12$ cm, $AB = 15$ cm.
Find, correct to two decimal places:

a BC **b** $\angle ACB$



Solution

a BC is found by applying the cosine rule:

$$\begin{aligned} a^2 &= b^2 + c^2 - 2bc\cos A \\ &= 12^2 + 15^2 - 2 \times 12 \times 15 \cos 82^\circ \\ &= 144 + 225 - 360 \times \cos 82^\circ \\ &= 318.897\ldots \end{aligned}$$

$BC = a = 17.86$ cm, correct to two decimal places.

b $\angle ACB$ is found by applying the sine rule pair:

$$\frac{a}{\sin A} = \frac{c}{\sin C}$$

$$\therefore \quad \sin C = \frac{c \sin A}{a} = \frac{15 \times \sin 82^\circ}{17.86}$$

$\therefore \quad \angle ACB = 56.28^\circ$, correct to two decimal places.

Note: 123.72° is also a solution to this equation but it is discarded as a possible answer as it is inconsistent with the information already given.

Exercise 1C

Skillsheet TEACHER

1 In triangle ABC, $\angle BAC = 73^\circ$, $\angle ACB = 55^\circ$ and $AB = 10$ cm. Find, correct to two decimal places:

a BC **b** AC

2 In triangle ABC, $\angle ABC = 58^\circ$, $AB = 6.5$ cm and $BC = 8$ cm. Find, correct to two decimal places:

a AC **b** $\angle BCA$

3 The adjacent sides of a parallelogram are 9 cm and 11 cm. One of its angles is 67°. Find the length of the longer diagonal, correct to two decimal places.

4 In ΔABC, $\angle ACB = 34^\circ$, $AC = 8.5$ cm and $AB = 5.6$ cm. Find, correct to two decimal places:

a the two possible values of $\angle ABC$ (one acute and one obtuse)

b BC in each case

5 In ΔABC, $\angle ABC = 35^\circ$, $AB = 10$ cm and $BC = 4.7$ cm. Find, correct to two decimal places:

a AC **b** $\angle ACB$

6 In ΔABC, $\angle ABC = 45^\circ$, $\angle ACB = 60^\circ$ and $AC = 12$ cm. Find AB.

7 In ΔPQR, $\angle QPR = 60^\circ$, $PQ = 2$ cm and $PR = 3$ cm. Find QR.

8 In ΔABC, $\angle ABC$ has magnitude 40°, $AC = 20$ cm and $AB = 18$ cm. Find the distance BC correct to 2 decimal places.

9 In ΔABC, $\angle ACB$ has magnitude 30°, $AC = 10$ cm and $AB = 8$ cm. Find the distance BC using the cosine rule.

10 In ΔABC, $AB = 5$ cm, $BC = 12$ cm and $AC = 10$ cm. Find:

a the magnitude of $\angle ABC$, correct to two decimal places

b the magnitude of $\angle BAC$, correct to two decimal places

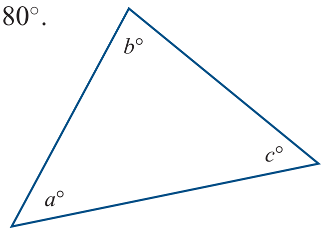
1.4 Geometry prerequisites

In the Specialist Mathematics study design it is stated that students should be familiar with several geometric results and be able to apply them in examples.

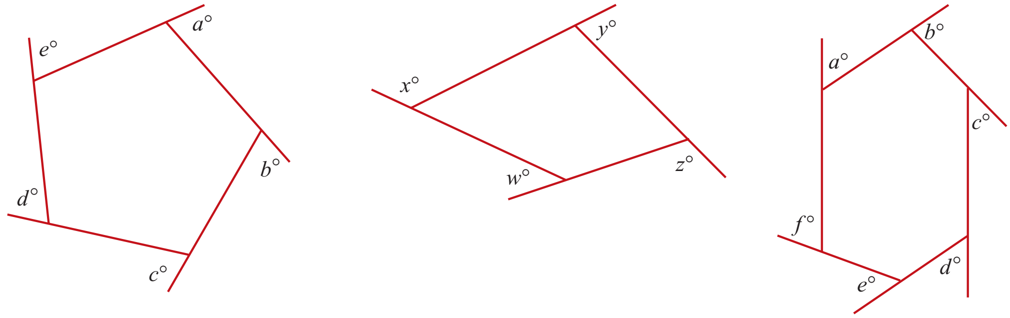
These results have been proved in earlier years' study. In this section they are listed.

- The sum of the interior angles of a triangle is 180°.

$$a + b + c = 180$$



- The sum of the exterior angles of a convex polygon is 360°.



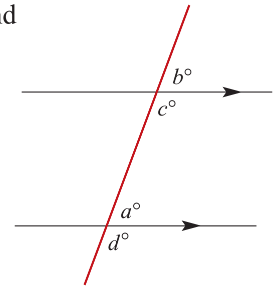
$a + b + c + d + e = 360$ $\quad$ $x + y + z + w = 360$ $\quad$ $a + b + c + d + e + f = 360$

- Corresponding angles of lines cut by a transversal are equal if, and only if, the lines are parallel.

$a = b$ and $c = d$

a and b are corresponding angles

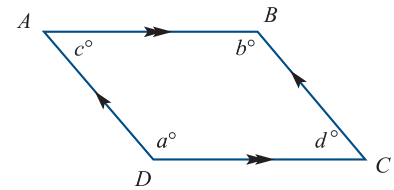
c and d are corresponding angles



- Opposite angles of a parallelogram are equal, and opposite sides are equal in length.

  $AB = DC \quad AD = BC$

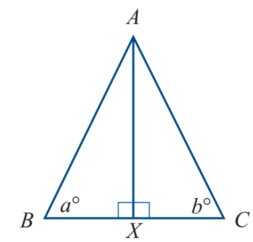
  $c = d$ and $a = b$



- The base angles of an isosceles triangle are equal.

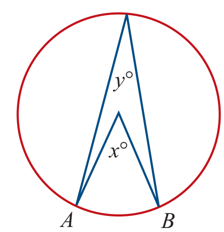
  $(AB = AC)$

  $a = b$

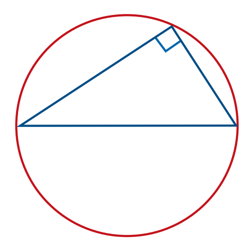


- The line joining the vertex to the midpoint of the base of an isosceles triangle is perpendicular to the base.
- The perpendicular bisector of the base of an isosceles triangle passes through the opposite vertex.
- The angle subtended by an arc at the centre of a circle is twice the angle subtended by the same arc at the circumference.

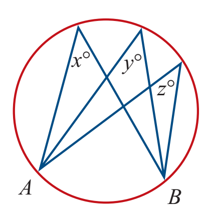
  $x = 2y$



- The angle in a semicircle is a right angle.

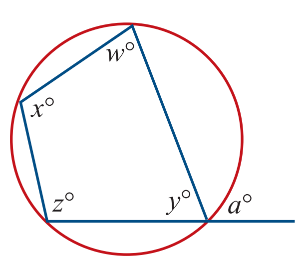


- Angles in the same segment of a circle are equal.



  $x = y = z$

- The sum of the opposite angles of a cyclic quadrilateral is 180°.



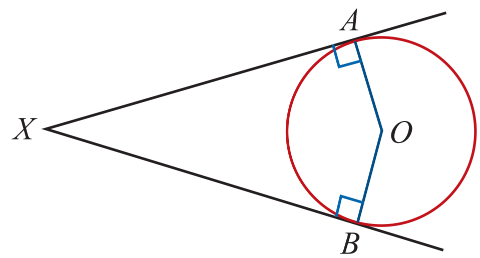
  $x + y = 180$

  $z + w = 180$

  $a = x$

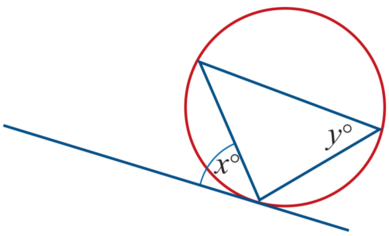
- An exterior angle of a cyclic quadrilateral and the interior opposite angle are equal.

- A tangent to a circle is perpendicular to the radius at the point of contact.
- The two tangents to a circle from an exterior point are equal in length.



$XA = XB$

- An angle between a tangent to a circle and a chord through the point of contact is equal to the angle in the alternate segment.

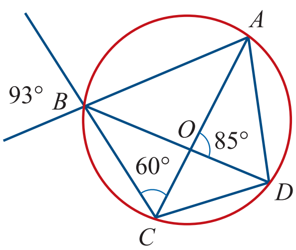


$x = y$

Example 16

Find the magnitude of each of the following angles:

a $\angle ABC$
b $\angle ADC$
c $\angle CBD$
d $\angle OCD$
e $\angle BAD$

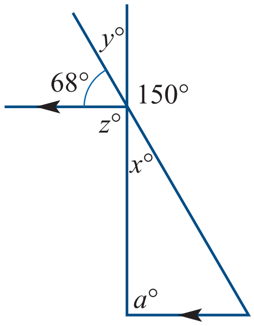


Solution

a $\angle ABC = 93°$ (vertically opposite)
b $\angle ADC = 87°$ (opposite angle of a cyclic quadrilateral)
c $\angle COB = 85°$ (vertically opposite)
$\angle CBD = [180 - (60 + 85)]° = 35°$ (angles of a triangle, ΔCBO)
d $\angle CAD = 35°$ (angle subtended by the arc CD)
$\angle ADC = 87°$ (From **b**)
$\angle OCD = [180 - (87 + 35)]° = 58°$ (angles of a triangle, $\Delta\ CAD$)
e $\angle BAD = [180 - (60 + 58)]° = 62°$ (opposite angles of a cyclic quadrilateral)

Exercise 1D

1 Find the value of a, y, z and x.



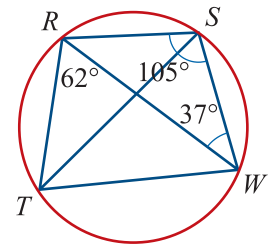
2 Find the magnitude of each of the following:

a $\angle RTW$

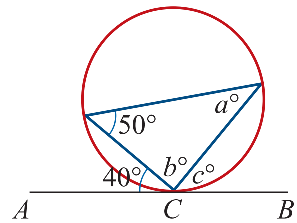
b $\angle TSW$

c $\angle TRS$

d $\angle RWT$



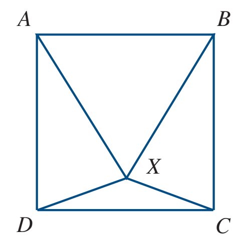
3 Find the value of a, b and c. AB is a tangent to the circle at C.



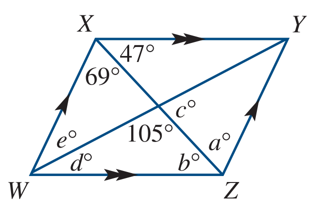
4 $ABCD$ is a square and ABX is an equilateral triangle.
Find the magnitude of:

a $\angle DXC$

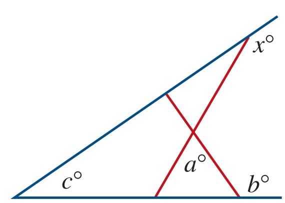
b $\angle XDC$



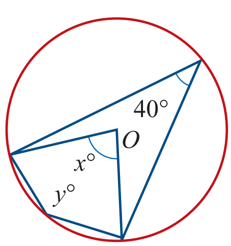
5 Find the values of a, b, c, d and e.



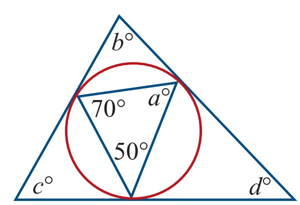
6 Find x in terms of a, b and c.



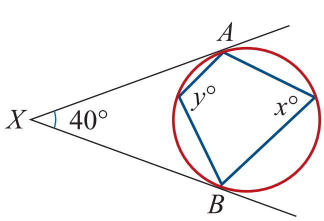
7 Find the values of x and y, given that O is the center of the circle.



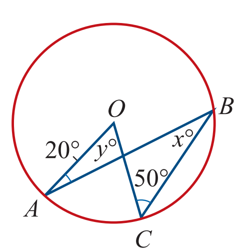
8 Find the values of a, b, c and d.



9 Find the values of x and y.



10 O is the centre of the circle. Find the values of x and y.



1.5 Sequences and series

The following are examples of sequences of numbers:

a 1, 3, 5, 7, 9 . . . **b** 0.1, 0.11, 0.111, 0.1111 . . . **c** $\frac{1}{3}, \frac{1}{9}, \frac{1}{27}, \frac{1}{81} \ldots$

d 10, 7, 4, 1, −2 . . . **e** 0.6, 1.7, 2.8, 3.9 . . .

Note that each sequence is a set of numbers, with order being important.

For some sequences of numbers a rule can be found connecting any number to the preceding number. For example:

for sequence A,	a rule is:	add 2
for sequence C,	a rule is:	multiply by $\frac{1}{3}$
for sequence D,	a rule is:	subtract 3
for sequence E,	a rule is:	add 1.1

The numbers of a sequence are called **terms**. The *n*th term of a sequence is denoted by the symbol t_n. So the first term is t_1, the 12th term is t_{12} and so on.

A sequence can be defined by specifying a rule which enables each subsequent term to be found using the previous term. In this case, the rule specified is called an **iterative rule** or a **difference equation**. For example:

sequence A can be defined by $t_1 = 1, t_n = t_{n-1} + 2$

sequence C can be defined by $t_1 = \frac{1}{3}, t_n = \frac{1}{3}t_{n-1}$

Example 17

Use the difference equation to find the first four terms of the sequence $t_1 = 3, t_n = t_{n-1} + 5$

Solution

$$t_1 = 3$$
$$t_2 = t_1 + 5 = 8$$
$$t_3 = t_2 + 5 = 13$$
$$t_4 = t_3 + 5 = 18$$

The first four terms are 3, 8, 13, 18.

Example 18

Find the difference equation for the following sequence.

$9, -3, 1, -\frac{1}{3} \ldots$

Solution

$$-3 = -\frac{1}{3} \times 9 \qquad \text{i.e.} \quad t_2 = -\frac{1}{3}t_1$$
$$1 = -\frac{1}{3} \times -3 \qquad \text{i.e.} \quad t_3 = -\frac{1}{3}t_2$$
$$\therefore \quad t_n = -\frac{1}{3}t_{n-1}, t_1 = 9$$

Alternatively, a sequence can be defined by a rule that is stated in terms of n. For example:

$t_n = 2n$ defines the sequence $t_1 = 2, t_2 = 4, t_3 = 6, t_4 = 8 \ldots$

$t_n = 2^{n-1}$ defines the sequence $t_1 = 1, t_2 = 2, t_3 = 4, t_4 = 8 \ldots$

Example 19

Find the first four terms of the sequence defined by the rule $t_n = 2n + 3$.

Solution

$$t_1 = 2 \times 1 + 3 = 5 \qquad t_2 = 2 \times 2 + 3 = 7$$
$$t_3 = 2 \times 3 + 3 = 9 \qquad t_4 = 2 \times 4 + 3 = 11$$

The first four terms are 5, 7, 9, 11.

Arithmetic sequences

A sequence in which each successive term is found by adding a constant value to the previous term is called an **arithmetic sequence**. For example, 2, 5, 8, 11 . . . is an arithmetic sequence.

An arithmetic sequence can be defined by a difference equation of the form

$$t_n = t_{n-1} + d \quad \text{where } d \text{ is a constant.}$$

If the first term of an artithmetic sequence $t_1 = a$ then the nth term of the sequence can also be described by the rule:

$$t_n = a + (n-1)d \quad \text{where } a = t_1 \text{ and } d = t_n - t_{n-1}$$

d is the **common difference**.

Example 20

Find the tenth term of the arithmetic sequence −4, −1, 2, 5 . . .

Solution

$$a = -4, d = 3, n = 10$$
$$t_n = a + (n-1)d$$
$$t_{10} = -4 + (10-1)3$$
$$= 23$$

Arithmetic series

The sum of the terms in a sequence is called a **series**. If the sequence in question is arithmetic, the series is called an **arithmetic series**. The symbol S_n is used to denote the sum of n terms of a sequence,

i.e. $$S_n = a + (a+d) + (a+2d) + \cdots + (a + (n-1)d)$$

If this sum is written in reverse order, then

$$S_n = (a + (n-1)d) + (a + (n-2)d) + \cdots + (a+d) + a$$

Adding these two expressions together gives $2S_n = n[2a + (n - 1)d]$

$$\therefore \quad S_n = \frac{n}{2}[2a + (n - 1)d]$$

and since the last term $l = t_n = a + (n - 1)d$

$$S_n = \frac{n}{2}(a + l)$$

Geometric sequences

A sequence in which each successive term is found by multiplying the previous term by a fixed value is called a **geometric sequence**.

For example, 2, 6, 18, 54 . . . is a geometric sequence.

A geometric sequence can be defined by an iterative equation of the form $t_n = rt_{n-1}$, where r is constant.

If the first term of a geometric sequence $t_1 = a$, then the nth term of the seqeunce can also be described by the rule

$$t_n = ar^{n-1} \quad \text{where } r = \frac{t_n}{t_{n-1}}$$

r is called the **common ratio**.

Example 21

Calculate the tenth term of the sequence 2, 6, 18, . . .

Solution

$$a = 2, r = 3, n = 10$$
$$t_n = ar^{n-1}$$
$$t_{10} = 2 \times 3^{(10-1)} = 39\,366$$

Geometric series

The sum of the terms in a geometric sequence is called a **geometric series**. An expression for S_n, the sum of n terms, of a geometric sequence can be found using a similar method to that used in the development of a formula for an arithmetic series.

Let $\quad S_n = a + ar + ar^2 + \cdots + ar^{n-1} \quad \ldots \boxed{1}$

Then $\quad rS_n = ar + ar^2 + ar^3 + \cdots + ar^n \quad \ldots \boxed{2}$

Subtract $\boxed{1}$ from $\boxed{2}$

$$rS_n - S_n = ar^n - a$$

$$\therefore \quad S_n(r - 1) = a(r^n - 1)$$

and $$S_n = \frac{a(r^n - 1)}{r - 1}$$

For values of r such that $-1 < r < 1$, it is often more convenient to use the alternative formula

$$S_n = \frac{a(1 - r^n)}{1 - r}$$

which is obtained by subtracting $\boxed{2}$ from $\boxed{1}$ above.

Example 22

Find the sum of the first nine terms of the geometric sequence $\frac{1}{3}, \frac{1}{9}, \frac{1}{27}, \frac{1}{81} \ldots$

Solution

$$a = \tfrac{1}{3}, r = \tfrac{1}{3}, n = 9$$

$$\therefore \quad S_9 = \frac{\frac{1}{3}\left(\left(\frac{1}{3}\right)^9 - 1\right)}{\frac{1}{3} - 1}$$

$$= \tfrac{-1}{2}\left(\left(\tfrac{1}{3}\right)^9 - 1\right)$$

$$\approx \tfrac{1}{2}(0.999\,949)$$

$$\approx 0.499\,975 \text{ (to 6 decimal places)}$$

Infinite geometric series

If the common ratio of a geometric sequence has a magnitude less than 1, i.e. $-1 < r < 1$, then each successive term of the sequence is closer to zero. e.g., 4, 2, 1, $\frac{1}{2}, \frac{1}{4} \ldots$

When the terms of the sequence are added, the corresponding series $a + ar + ar^2 + \cdots + ar^{n-1}$ will approach a limiting value,
i.e. as $n \to \infty$, $S_n \to$ a limiting value. Such a series is said to be **convergent**.

In example 22 above, it was found that for the sequence $\frac{1}{3}, \frac{1}{9}, \frac{1}{27}, \frac{1}{81} \ldots$ the sum of the first nine terms, S_9, was 0.499975.

For the same sequence, $S_{20} = 0.499\,999\,999\,9 \approx 0.5$

So even for a relatively small value of n (20), the sum approaches the limiting value of 0.5 very quickly.

Given that $\quad S_n = \dfrac{a(1 - r^n)}{1 - r}$

$$\Rightarrow S_n = \frac{a}{1 - r} - \frac{ar^n}{1 - r}$$

as $n \to \infty$, $r^n \to 0$ and hence $\dfrac{ar^n}{1 - r} \to 0$

It follows then that the limit as $n \to \infty$ of S_n is $\dfrac{a}{1 - r}$

So $$S_\infty = \frac{a}{1 - r}$$

This is also referred to as ‘the sum to infinity’ of the series.

Example 23

Find the sum to infinity of the series $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots$

Solution

$$r = \tfrac{1}{2},\ a = 1$$

$$\therefore \qquad S_\infty = \frac{1}{1 - \frac{1}{2}} = 2$$

Example 24

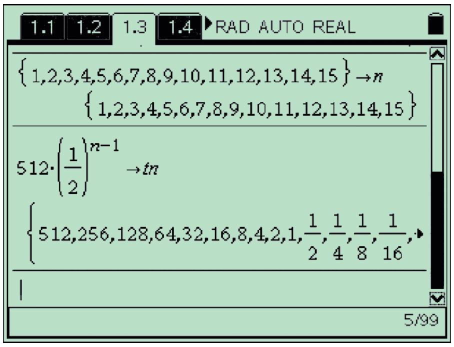
Graph the terms of the geometric sequence defined by:

$$a_n = 512(0.5)^{n-1} \text{ for } n = 1, 2 \ldots$$

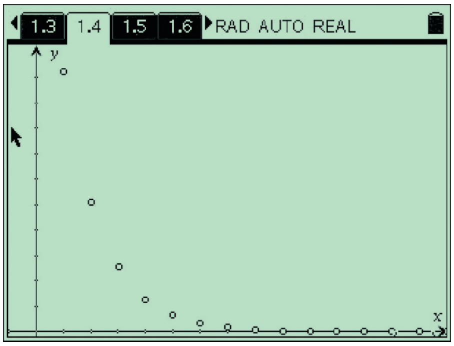
Solution

Using a TI-Nspire calculator

Complete as shown to generate the first 15 terms of the sequence of numbers defined by the rule $a_n = 512(0.5)^n$. Storing the resulting list will enable us to graph the sequence.

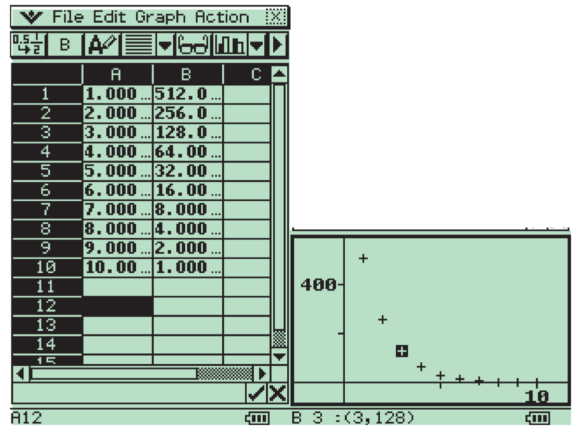


To graph the sequence, open a **Graphs & Geometry** application (home 2) and graph the sequence as a **Scatter Plot** (menu 3 4), using an appropriate **Window** (menu 4). n and tn are entered in the x and y boxes respectively. Note that it is possible to see the coordinates of the points using **Trace** (menu 5 1). To enter n, select the $x \in$ box and press **Enter**, and select n. This can also be achieved through a **Lists & Spreadsheet** application. This is completed by choosing **Sequence** from the **Graph Type** menu.



Using a Casio ClassPad calculator

The Casio Classpad Spreadsheet is an efficient way to produce and graph the sequence. It works in a similar way to a computer spreadsheet such as Microsoft Excel. Enter the values for n in column A, then in B1 enter the formula =512*(0.5)^A1. Highlight cell B2 and the cells below it and select **Edit** and then **Fill Range** to complete the sequence. Select the values in columns A and B. Click the arrow beside and select graph type to produce the graph.



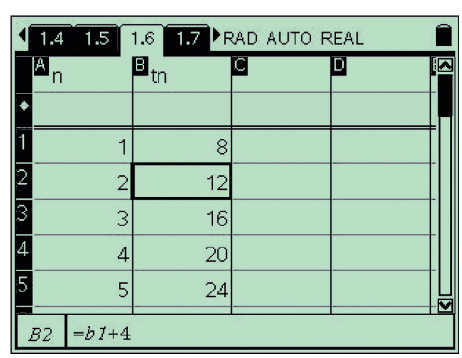
Example 25

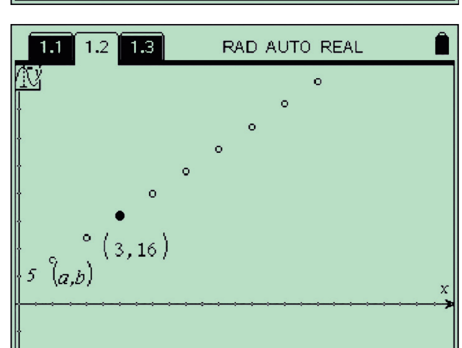
Use a calculator to generate terms of the arithmetic sequence with iterative formula $t_n = t_{n-1} + 4$, $t_1 = 8$.

Solution

Using a TI-Nspire calculator

This type of sequence is easiest handled in a **Lists & Spreadsheet** application (home 3).
Use the **arrows** (◀ ▶ ▲ ▼) to name the first two columns n and tn respectively.
Enter 1 in cell A1 and enter 8 in cell B1.
Enter =a1+1 in cell A2 and =b1+4 in cell B2.
Highlight the cells A2 and B2 using caps and the NavPad and use **Fill Down** (menu 3 3) to generate the sequence of numbers.
To graph the sequence, open a **Graphs & Geometry** application (home 2) and graph the sequence as a **Scatter Plot** (menu 3 4), using an appropriate **Window** (menu 4).
Note that it is possible to see the coordinates of the points using **Trace** (menu 5 1).

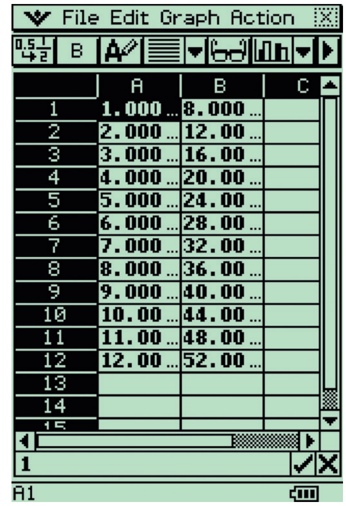




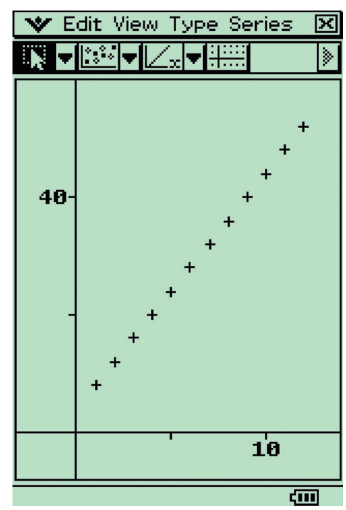
Using a Casio ClassPad calculator

Choose a ClassPad Spreadsheet. Enter 1 in cell A1 and $=$A1$+$1 in cell A2. Highlight cells A2 to A12 and then select **Fill Range** from the **Edit** menu. Enter 8 in cell B1 and $=$B1$+$4 in B2. Highlight cells B2 to B12 and select **Fill Range** from the **Edit** menu.

Shade the required cells to be graphed.



From the **Graph** menu select **Scatter**. Select **Resize** to see the graph.



Exercise 1E

1 A difference equation has rule $t_{n+1} = 3t_n - 1$, $t_1 = 6$. Find t_2 and t_3. Use a CAS calculator to find t_8.

2 A difference equation has rule $y_{n+1} = 2y_n + 6$, $y_1 = 5$. Find y_2 and y_3. Use a CAS calculator to find y_{10} and to plot a graph showing the first ten values.

3 The Fibonacci sequence is given by the difference equation $t_{n+2} = t_{n+1} + t_n$ where $t_1 = t_2 = 1$. Find the first ten terms of the Fibonacci sequence.

4 Find the sum of the first ten terms of an arithmetic sequence with first term 3 and common difference 4.

5 Find the sum to infinity of $1 - \frac{1}{3} + \frac{1}{9} - \frac{1}{27} + \ldots$

6 The first, second and third terms of a geometric sequence are $x + 5$, x and $x - 4$ respectively. Find:

a x **b** the common ratio

c the difference between the sum to infinity and the sum of the first ten terms

7 Find the sum of the first eight terms of a geometric sequence with first term 6 and common ratio -3.

8 Find the sum to infinity of the geometric sequence $a, \frac{a}{\sqrt{2}}, \frac{a}{2}, \frac{a}{2\sqrt{2}} \ldots$ in terms of a.

9 Consider the sum $S_n = 1 + \frac{x}{2} + \frac{x^2}{4} + \cdots + \frac{x^{n-1}}{2^{n-1}}$

a Calculate S_{10} when $x = 1.5$.

b **i** Find the possible values of x for which the infinite sum exists. Denote this sum by S.

ii Find the values of x for which $S = 2S_{10}$.

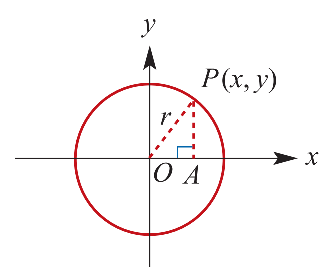
10 **a** Find an expression for the infinite geometric sum $1 + \sin\theta + \sin^2\theta + \ldots$

b Find the values of θ for which the infinite geometric sum is 2.

1.6 Circles

For a circle with centre the origin and radius r, if the point with coordinates (x, y) is on the circle then $x^2 + y^2 = r^2$.

The converse is also true, i.e. a point with coordinates (x, y) such that $x^2 + y^2 = r^2$ lies on the circle with centre the origin and radius r.



Applying Pythagoras' theorem to triangle OAP yields

$$r^2 = OP^2 = x^2 + y^2$$

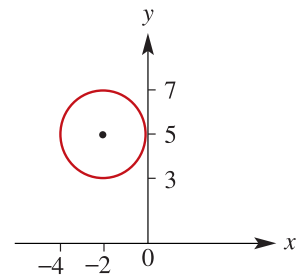
In general, the following result holds:

> The circle with centre (h, k) and radius r is the graph of the equation
>
> $$(x - h)^2 + (y - k)^2 = r^2$$
>
> This graph is obtained from the graph of $x^2 + y^2 = r^2$ by the translation defined by
>
> $$(x, y) \to (x + h, y + k)$$

Example 26

Sketch the graph of the circle with centre at (−2, 5) and radius 2, and state the cartesian equation for this circle.

Solution



The equation is

$$(x+2)^2 + (y-5)^2 = 4$$

which may also be written as

$$x^2 + y^2 + 4x - 10y + 25 = 0$$

Note: The equation $x^2 + y^2 + 4x - 10y + 25 = 0$ can be 'unsimplified' by completing the square.

$$x^2 + y^2 + 4x - 10y + 25 = 0$$

implies $\quad x^2 + 4x + 4 + y^2 - 10y + 25 + 25 = 29$

i.e. $\quad (x+2)^2 + (y-5)^2 = 4$

This suggests a general form of the equation of a circle.

$$x^2 + y^2 + Dx + Ey + F = 0$$

Completing the square gives

$$x^2 + Dx + \frac{D^2}{4} + y^2 + Ey + \frac{E^2}{4} + F = \frac{D^2 + E^2}{4}$$

i.e. $$\left(x + \frac{D}{2}\right)^2 + \left(y + \frac{E}{2}\right)^2 = \frac{D^2 + E^2 - 4F}{4}$$

If $D^2 + E^2 - 4F > 0$, then the equation represents a circle with centre $\left(\frac{-D}{2}, \frac{-E}{2}\right)$ and radius $\sqrt{\frac{D^2 + E^2 - 4F}{4}}$.

If $D^2 + E^2 - 4F = 0$, then the equation represents one point $\left(\frac{-D}{2}, \frac{-E}{2}\right)$.

If $D^2 + E^2 - 4F < 0$, then the equation has no graphical representation in the cartesian plane.

Example 27

Sketch the graph of $x^2 + y^2 + 4x + 6y - 12 = 0$. State the coordinates of the centre and the radius.

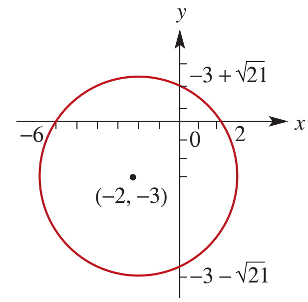
Solution

$$x^2 + y^2 + 4x + 6y - 12 = 0$$

$\therefore \quad x^2 + 4x + 4 + y^2 + 6y + 9 - 12 = 13$

i.e. $\quad (x + 2)^2 + (y + 3)^2 = 25$

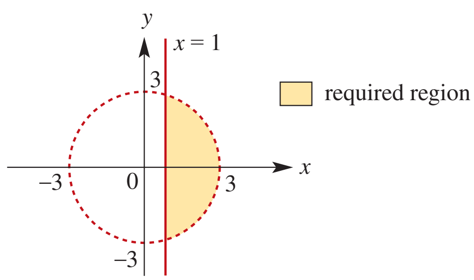
The circle has centre $(-2, -3)$ and radius 5.



Example 28

Sketch a graph of the region of the plane such that $x^2 + y^2 < 9$ and $x \geq 1$.

Solution



Exercise 1F

1 Find the equations of the circles with the following centres and radii:

a centre (2, 3); radius 1 **b** centre (−3, 4); radius 5

c centre (0, −5); radius 5 **d** centre (3, 0); radius $\sqrt{2}$

2 Find the radii and the coordinates of the centres of the circles with the following equations:

a $x^2 + y^2 + 4x - 6y + 12 = 0$ **b** $x^2 + y^2 - 2x - 4y + 1 = 0$

c $x^2 + y^2 - 3x = 0$ **d** $x^2 + y^2 + 4x - 10y + 25 = 0$

3 Sketch the graphs of each of the following:

a $2x^2 + 2y^2 + x + y = 0$ **b** $x^2 + y^2 + 3x - 4y = 6$

c $x^2 + y^2 + 8x - 10y + 16 = 0$ **d** $x^2 + y^2 - 8x - 10y + 16 = 0$

e $2x^2 + 2y^2 - 8x + 5y + 10 = 0$ **f** $3x^2 + 3y^2 + 6x - 9y = 100$

4 Sketch the graphs of the regions of the plane specified by the following:

a $x^2 + y^2 \leq 16$ **b** $x^2 + y^2 \geq 9$

c $(x - 2)^2 + (y - 2)^2 < 4$ **d** $(x - 3)^2 + (y + 2)^2 > 16$

e $x^2 + y^2 \leq 16$ and $x \leq 2$ **f** $x^2 + y^2 \leq 9$ and $y \geq -1$

5 The points (8, 4) and (2, 2) are the ends of a diameter of a circle. Find the coordinates of the centre and the radius of the circle.

6 Find the equation of the circle, centre (2, −3), which touches the x axis.

7 Find the equation of the circle which passes through (3, 1), (8, 2) and (2, 6).

8 Find the radii and coordinates of the centre of the circles with equations $4x^2 + 4y^2 - 60x - 76y + 536 = 0$ and $x^2 + y^2 - 10x - 14y + 49 = 0$, and find the coordinates of the points of intersection of the two curves.

9 Find the coordinates of the points of intersection of the circle with equation $x^2 + y^2 = 25$ and the line with equation:

a $y = x$ **b** $y = 2x$

1.7 Ellipses and hyperbolas

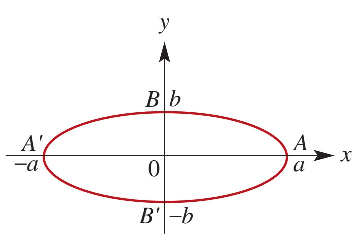
Ellipses

The curve with equation $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is an ellipse with centre the origin, x-axis intercepts $(-a, 0)$ and $(a, 0)$ and y-axis intercepts $(0, -b)$ and $(0, b)$.

If $a > b$ the ellipse will appear as shown in the diagram on the left.

If $b > a$ the ellipse is as shown in the diagram on the right.

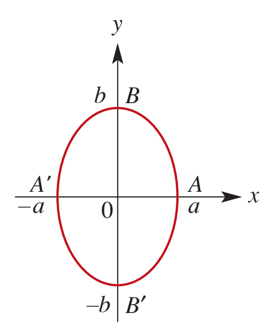
If $b = a$, the equation is that of a circle with centre the origin and radius a.



$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1; a > b$

AA' is the major axis

BB' is the minor axis



$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1; b > a$

AA' is the minor axis

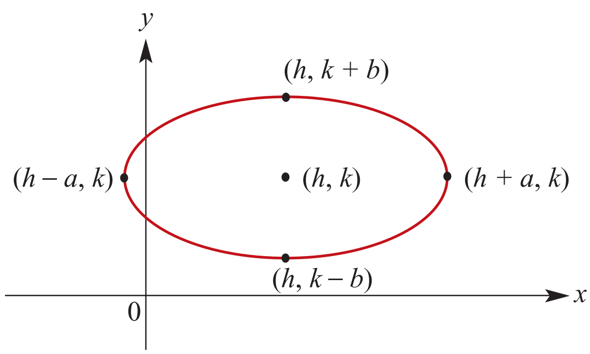
BB' is the major axis

The general cartesian form is as given below.

The curve with equation

$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$$

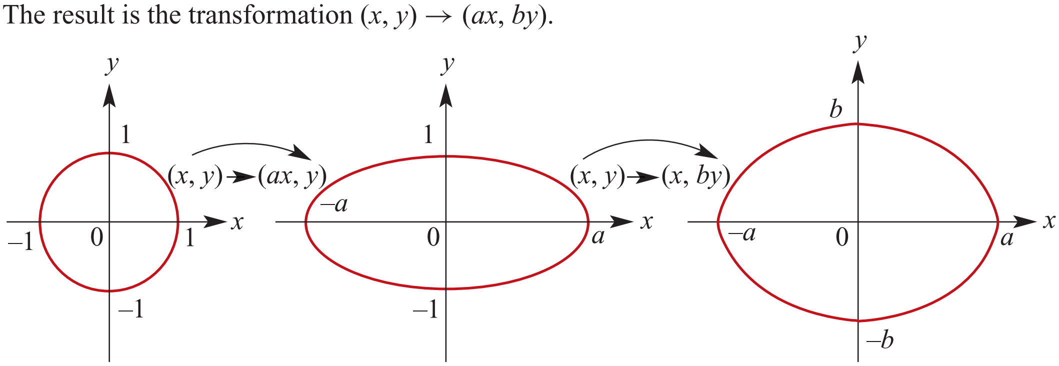
is an ellipse with centre (h, k). It is obtained by a translation of the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$. The translation is $(x, y) \to (x + h, y + k)$.



The ellipse with equation $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ can be obtained by applying the following dilations to the circle with equation $x^2 + y^2 = 1$:

- a dilation of factor a from the y axis, i.e. $(x, y) \to (ax, y)$
- a dilation of factor b from the x axis, i.e. $(x, y) \to (x, by)$

The result is the transformation $(x, y) \to (ax, by)$.



Example 29

Sketch the graph of each of the following. Give the axes intercepts and the coordinates of the centre.

a $\dfrac{x^2}{9} + \dfrac{y^2}{4} = 1$

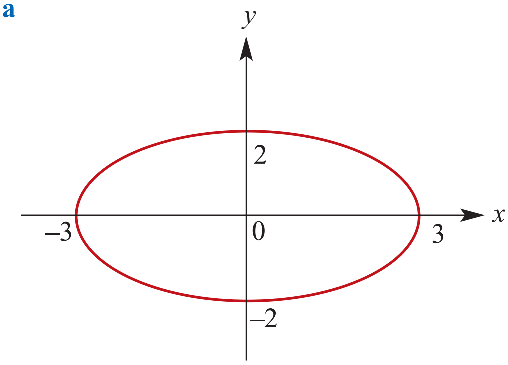
b $\dfrac{x^2}{4} + \dfrac{y^2}{9} = 1$

c $\dfrac{(x-2)^2}{9} + \dfrac{(y-3)^2}{16} = 1$

d $3x^2 + 24x + y^2 + 36 = 0$

Solution

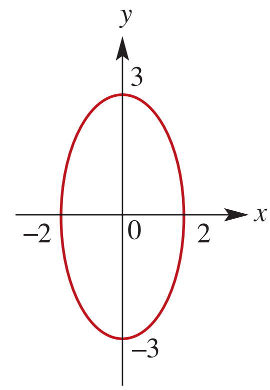
a



Centre (0, 0)

Axes intercepts $(\pm 3, 0)$ and $(0, \pm 2)$

b



Centre (0, 0)

Axes intercepts $(\pm 2, 0)$ and $(0, \pm 3)$

c Centre is at (2, 3)

When $x = 0$

$$\frac{4}{9} + \frac{(y-3)^2}{16} = 1$$

$$\therefore \quad \frac{(y-3)^2}{16} = \frac{5}{9}$$

$$\therefore \quad (y-3)^2 = \frac{16 \times 5}{9}$$

$$\therefore \quad y = 3 \pm \frac{4\sqrt{5}}{3}$$

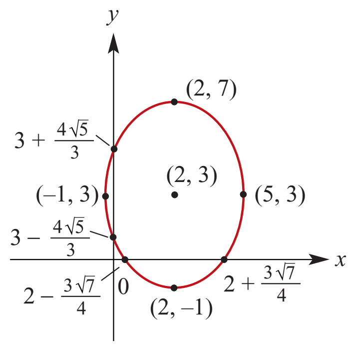
When $y = 0$

$$\frac{(x-2)^2}{9} + \frac{9}{16} = 1$$

$$\frac{(x-2)^2}{9} = \frac{7}{16}$$

$$(x-2)^2 = \frac{9 \times 7}{16}$$

$$x = 2 \pm \frac{3\sqrt{7}}{4}$$



d $3x^2 + 24x + y^2 + 36 = 0$

Completing the square yields

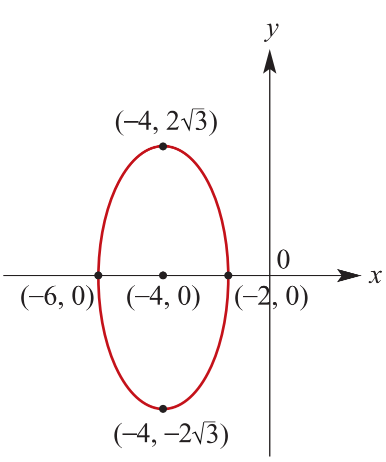
$$3[x^2 + 8x + 16] + y^2 + 36 - 48 = 0$$

i.e.
$$3(x+4)^2 + y^2 = 12$$

$$\frac{(x+4)^2}{4} + \frac{y^2}{12} = 1$$

$\therefore$ Centre $(-4, 0)$

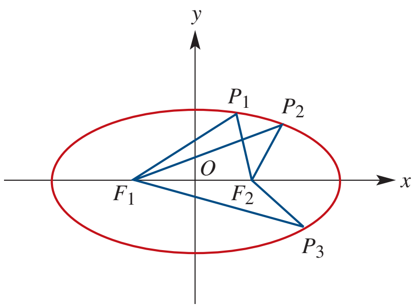
Axes intercepts $(-6, 0)$ and $(-2, 0)$



Defining an ellipse

In the previous section a circle was defined as a set of points which are all a constant distance from a given point (the centre). An ellipse can be defined in a similar way.

Consider the set of all points P such that $PF_1 + PF_2$ is equal to a constant k with $k > 2m$, and the coordinates of F_1 and F_2 are $(m, 0)$ and $(-m, 0)$ respectively. We can show that the equation describing this set of points is $\dfrac{x^2}{a^2} + \dfrac{y^2}{a^2 - m^2} = 1$ where $k = 2a$.



$$\begin{aligned} P_1F_1 + P_1F_2 &= P_2F_1 + P_2F_2 \\ &= P_3F_1 + P_3F_2 \end{aligned}$$

This can be pictured as a string of length $P_1F_1 + P_1F_2$ being attached by nails to a board at F_1 and F_2 and, considering the path mapped out by a pencil, extending the string so that it is taut, and moving 'around' the two points.

Let the coordinates of P be (x, y).

$$PF_1 = \sqrt{(x-m)^2 + y^2} \text{ and } PF_2 = \sqrt{(x+m)^2 + y^2}$$

and assume

$$PF_1 + PF_2 = k$$

Then

$$\sqrt{(x+m)^2 + y^2} + \sqrt{(x+m)^2 + y^2} = k$$

Rearranging and squaring gives

$$\begin{aligned} (x+m)^2 + y^2 &= k^2 - 2k\sqrt{(x-m)^2 + y^2} + (x+m)^2 + y^2 \\ \therefore \quad 4mx &= k^2 - 2k\sqrt{(x-m)^2 + y^2} \end{aligned}$$

Rearranging and squaring again gives

$$4k^2(x-m)^2 + 4k^2y^2 = k^4 - 8k^2mx + 16m^2x^2$$

Collecting like terms

$$\begin{aligned} 4(k^2 - 4m^2)x^2 + 4k^2y^2 &= k^2(k^2 - 4m^2) \\ \therefore \quad \frac{4x^2}{k^2} + \frac{4y^2}{k^2 - 4m^2} &= 1 \end{aligned}$$

Let $a = \dfrac{k}{2}$, then $\dfrac{x^2}{a^2} + \dfrac{y^2}{a^2 - m^2} = 1$

The points F_1 and F_2 are called the **foci** of the ellipse. The constant $k = 2a$ is called the **focal sum**.

If $a = 3$ and $m = 2$, the ellipse with equation $\frac{x^2}{9} + \frac{y^2}{5} = 1$ is obtained.

For an ellipse with equation $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ and $a > b$, the foci are at $(\pm\sqrt{(a^2 - b^2)}, 0)$.

Given an equation of the form $Ax^2 + By^2 + Cx + Ey + F = 0$, where A and B are both positive (or both negative), the corresponding graph is an ellipse or a point. If $A = B$ the graph is that of a circle. In some cases, as for the circle, no pairs (x, y) will satisfy the equation.

Hyperbolas

The curve with equation $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ is a hyperbola with centre at the origin. The axis intercepts are $(a, 0)$ and $(-a, 0)$.

The hyperbola has asymptotes $y = \frac{b}{a}x$ and $y = -\frac{b}{a}x$. An informal argument for this is as follows.

The equation $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ can be rearranged:

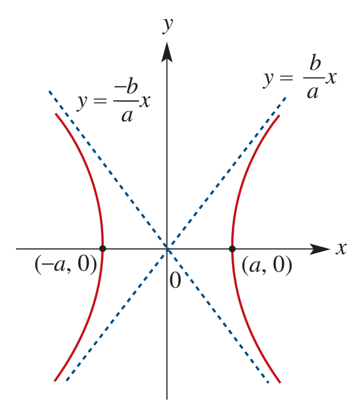
$$\frac{y^2}{b^2} = \frac{x^2}{a^2} - 1$$

$$\therefore \quad y^2 = \frac{b^2x^2}{a^2}\left(1 - \frac{a^2}{x^2}\right)$$

But as $x \to \pm\infty$, $\frac{a^2}{x^2} \to 0$

$$\therefore \quad y^2 \to \frac{b^2x^2}{a^2}$$

i.e. $$y \to \pm\frac{bx}{a}$$



$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

The general equation for a hyperbola is formed by suitable translations.

The curve with equation

$$\frac{(x-h)^2}{a^2} - \frac{(y-k^2)}{b^2} = 1$$

is a hyperbola with centre (h, k). The asymptotes are

$$y - k = \pm\frac{b}{a}(x - h)$$

This hyperbola is obtained from the hyperbola with equation $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ by the translation defined by $(x, y) \to (x + h, y + k)$.

Example 30

For each of the following equations, sketch the graph of the corresponding hyperbola, give the coordinates of the centre and the axes intercepts, and the equations of the asymptotes.

a $\frac{x^2}{9} - \frac{y^2}{4} = 1$

b $\frac{y^2}{9} - \frac{x^2}{4} = 1$

c $(x-1)^2 - (y+2)^2 = 1$

d $\frac{(y-1)^2}{4} - \frac{(x+2)^2}{9} = 1$

Solution

a $\frac{x^2}{9} - \frac{y^2}{4} = 1$

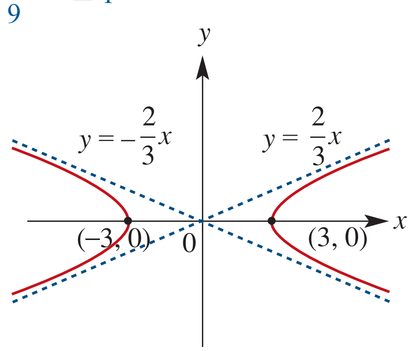
$\therefore \quad y^2 = \frac{4x^2}{9}\left(1 - \frac{9}{x^2}\right)$

Equation of asymptotes:

$$y = \pm\frac{2}{3}x$$

When $y = 0$, $x^2 = 9$ and therefore $x = \pm 3$.

Axes intercepts (3, 0) and (−3, 0), centre (0, 0).



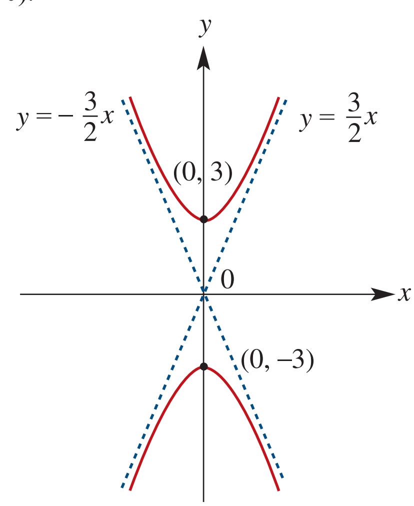
b $\frac{y^2}{9} - \frac{x^2}{4} = 1$ is the reflection of

$\frac{x^2}{9} - \frac{y^2}{4} = 1$ in the line $y = x$.

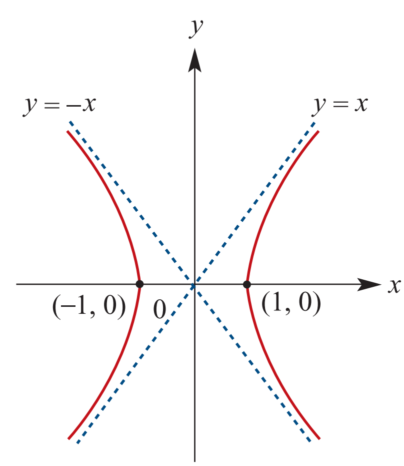
$\therefore$ asymptotes are $x = \pm\frac{2}{3}y$

i.e. $\quad y = \pm\frac{3}{2}x$

The y-axis intercepts are (0, 3) and (0, −3).



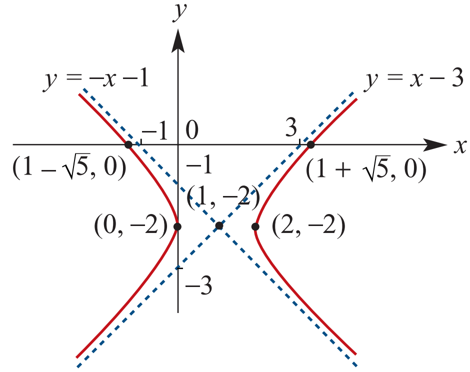
c $(x-1)^2 - (y+2)^2 = 1$. The graph of $x^2 - y^2 = 1$ is sketched first. The asymptotes are $y = x$ and $y = -x$. This hyperbola is called a **rectangular hyperbola** as its asymptotes are perpendicular. The centre is (0, 0) and the axes intercepts are at (1, 0) and (−1, 0)



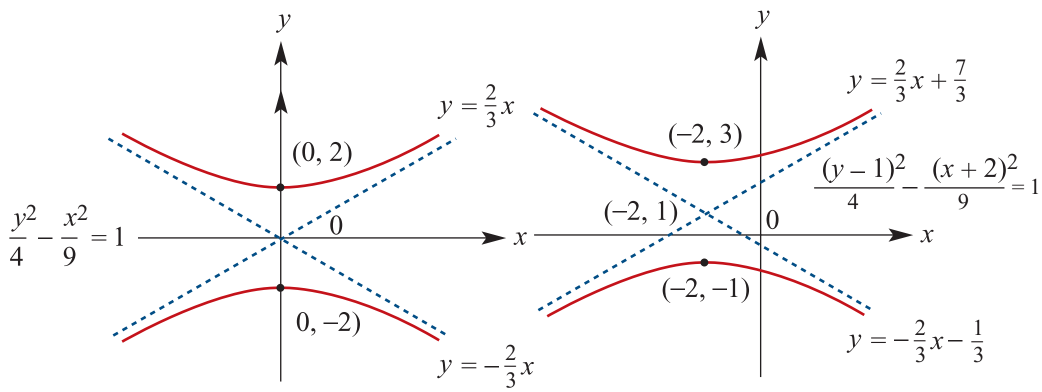
A translation of $(x, y) \to (x + 1, y - 2)$ is applied. The new centre is $(1, -2)$ and the asymptotes have equations $y + 2 = \pm (x - 1)$, i.e. $y = x - 3$ and $y = -x - 1$.

When $x = 0$, $y = -2$ and when $y = 0$

$$(x - 1)^2 = 5$$
$$x = 1 \pm \sqrt{5}$$



d $\dfrac{(y-1)^2}{4} - \dfrac{(x+2)^2}{9} = 1$ is obtained by translating the hyperbola $\dfrac{y^2}{4} - \dfrac{x^2}{9} = 1$ through the translation defined by $(x, y) \to (x - 2, y + 1)$.



Note that the asymptotes for $\dfrac{y^2}{4} - \dfrac{x^2}{9} = 1$ are the same as for those of the hyperbola $\dfrac{x^2}{9} - \dfrac{y^2}{4} = 1$. The two hyperbolas are called **conjugate hyperbolas**.

Defining a hyperbola

Hyperbolas can be defined in a manner similar to the methods discussed earlier in this section for circles and ellipses.

Consider the set of all points, P, such that $PF_1 - PF_2 = k$ where k is a constant and F_1 and F_2 are points with coordinates $(m, 0)$ and $(-m, 0)$ respectively.

Then the equation of the curve defined in this way is

$$\frac{x^2}{a^2} - \frac{y^2}{m^2 - a^2} = 1 \quad k = 2a$$

Exercise 1G

Skillsheet TEACHER

1 Sketch the graph of each of the following. Label the axes intercepts. State the coordinates of the centre.

a $\frac{x^2}{9} + \frac{y^2}{16} = 1$

b $25x^2 + 16y^2 = 400$

c $\frac{(x-4)^2}{9} + \frac{(y-1)^2}{16} = 1$

d $x^2 + \frac{(y-2)^2}{9} = 1$

e $9x^2 + 25y^2 - 54x - 100y = 44$

f $9x^2 + 25y^2 = 225$

g $5x^2 + 9y^2 + 20x - 18y - 16 = 0$

h $16x^2 + 25y^2 - 32x + 100y - 284 = 0$

i $\frac{(x-2)^2}{4} + \frac{(y-3)^2}{9} = 1$

j $2(x-2)^2 + 4(y-1)^2 = 16$

2 Sketch the graphs of each of the following. Label the axes intercepts and give the equations of the asymptotes.

a $\frac{x^2}{16} - \frac{y^2}{9} = 1$

b $\frac{y^2}{16} - \frac{x^2}{9} = 1$

c $x^2 - y^2 = 4$

d $2x^2 - y^2 = 4$

e $x^2 - 4y^2 - 4x - 8y - 16 = 0$

f $9x^2 - 25y^2 - 90x + 150y = 225$

g $\frac{(x-2)^2}{4} - \frac{(y-3)^2}{9} = 1$

h $4x^2 - 8x - y^2 + 2y = 0$

i $9x^2 - 16y^2 - 18x + 32y - 151 = 0$

j $25x^2 - 16y^2 = 400$

3 Find the coordinates of the points of intersection of $y = \frac{1}{2}x$ with:

a $x^2 - y^2 = 1$

b $\frac{x^2}{4} + y^2 = 1$

4 Show that there is no intersection point of the line $y = x + 5$ and the ellipse $x^2 + \frac{y^2}{4} = 1$.

5 Find the coordinates of the points of intersection of the curves $\frac{x^2}{4} + \frac{y^2}{9} = 1$ and $\frac{x^2}{9} + \frac{y^2}{4} = 1$. Show that the points of intersection are the vertices of a square.

6 Find the coordinates of the points of intersection of $\frac{x^2}{16} + \frac{y^2}{25} = 1$ and the line with equation $5x = 4y$.

7 On the one set of axes sketch the graphs of $x^2 + y^2 = 9$ and $x^2 - y^2 = 9$.

8 Sketch each of the following regions:

a $x^2 - y^2 \le 1$

b $x^2 - y^2 \ge 4$

c $y^2 \le \frac{x^2}{4} - 1$

d $\frac{x^2}{9} + \frac{y^2}{4} < 1$

e $x^2 - y^2 \leq 1$ and $x^2 + y^2 \leq 4$

f $\dfrac{(x-3)^2}{16} + \dfrac{y^2}{9} \leq 1$

g $x^2 - y^2 \leq 4$ and $\dfrac{x^2}{9} + y^2 \leq 1$

h $x^2 - y^2 > 1$ and $x^2 + y^2 \leq 4$

i $\dfrac{(x-2)^2}{9} + y^2 \leq 4$

j $\dfrac{x^2}{4} + y^2 \leq 1$ and $y \leq x$

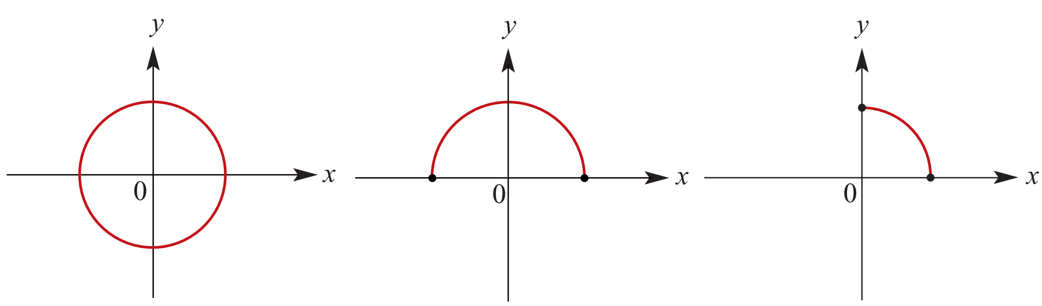
1.8 Parametric equations of circles, ellipses and hyperbolas

Circles

It is sometimes useful to express the rule of a relation in terms of a third variable, called a **parameter**. We have already seen in the work on circular functions that the unit circle can be expressed in cartesian form, i.e. $\{(x, y): x^2 + y^2 = 1\}$ or in the form $\{(x, y): x = \cos t, y = \sin t, \text{ with } t \in [0, 2\pi]\}$. The latter is called the set of parametric equations of the unit circle which give the coordinates (x, y) of all points on the unit circle.

The restriction for the values of t is unnecessary in the representation of the graph as $\{(x, y): x = \cos t, y = \sin t, \text{ with } t \in R\}$ gives the same points with repetitions since $\cos(2\pi + t) = \cos t$ and $\sin(2\pi + t) = \sin t$. If the set of values for t is the interval $[0, \pi]$, only the top half of the circle is obtained.

The set notation is often omitted, and in the following this will be done. The next three diagrams illustrate the graphs resulting from the parametric equations $x = \cos t$ and $y = \sin t$ for three different sets of values of t.



$t \in [0, 2\pi]$ $\quad$ $t \in [0, \pi]$ $\quad$ $t \in \left[0, \dfrac{\pi}{2}\right]$

In general, $x^2 + y^2 = a^2$, where $a > 0$, is the cartesian equation of a circle with centre at the origin and radius a. The parametric equations are $x = a \cos t$ and $y = a \sin t$. The minimal interval of t values to yield the entire circle is $[0, 2\pi]$.

The domain and range of the corresponding cartesian relation can be determined by the parametric equation determining the x value and the y value respectively. The range of the function with rule $x = a \cos t$, $t \in [0, 2\pi]$ is $[-a, a]$ and hence the domain of the relation $x^2 + y^2 = a^2$ is $[-a, a]$. The range of the function with rule $y = a \sin t$, $t \in [0, 2\pi]$ is $[-a, a]$ and hence the range of the relation $x^2 + y^2 = a^2$ is $[-a, a]$.

Example 31

A circle is defined by the parametric equations $x = 2 + 3\cos\theta$ and $y = 1 + 3\sin\theta$ for $\theta \in [0, 2\pi]$, Find the corresponding cartesian equation of the circle and state the domain and range of this relation.

Solution

The range of the function with rule $x = 2 + 3\cos\theta$ is $[-1, 5]$ and hence the domain of the corresponding cartesian relation is $[-1, 5]$. The range of the function with rule $y = 1 + 3\sin\theta$ is $[-2, 4]$ and hence the range of the corresponding cartesian relation is $[-2, 4]$.

Rewrite the equations as $\dfrac{x-2}{3} = \cos\theta$ and $\dfrac{y-1}{3} = \sin\theta$.

Square both sides of each of these equations and add:

$$\frac{(x-2)^2}{9} + \frac{(y-1)^2}{9} = \cos^2\theta + \sin^2\theta \text{ and therefore } \frac{(x-2)^2}{9} + \frac{(y-1)^2}{9} = 1$$

i.e. $(x-2)^2 + (y-1)^2 = 9$

Ellipses

It has been shown in the previous section that $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$, where a and b are positive real numbers, is the cartesian equation of an ellipse with centre at the origin, and x-axis intercepts $(\pm a, 0)$ and y-axis intercepts $(0, \pm b)$. The parametric equations for such an ellipse are $x = a\cos t$ and $y = b\sin t$. The minimal interval of t values to yield the entire ellipse is $[0, 2\pi]$.

The domain and range of the corresponding cartesian relation can be determined by the parametric equation determining the x value and the y value respectively. The range of the function with rule $x = a\cos t$ is $[-a, a]$ and hence the domain of the relation $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ is $[-a, a]$. The range of the function with rule $y = b\sin t$ is $[-b, b]$ and hence the range of the relation $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ is $[-b, b]$.

The proof that the two forms of equation yield the same graph uses the Pythagorean identity $\sin^2 t + \cos^2 t = 1$.

Let $x = a\cos t$ and $y = b\sin t$.

Therefore $\dfrac{x}{a} = \cos t$ and $\dfrac{y}{b} = \sin t$. Squaring both sides of each of these equations yields

$$\frac{x^2}{a^2} = \cos^2 t \quad \text{and} \quad \frac{y^2}{b^2} = \sin^2 t$$

Now, since $\sin^2 t + \cos^2 t = 1$ it follows that $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$.

Example 32

Find the cartesian equation of the curve with parametric equations $x = 3 + 3\sin t$, $y = 2 - 2\cos t$ with $t \in R$ and describe the graph.

Solution

$x = 3 + 3\sin t$ and $y = 2 - 2\cos t$, therefore $\dfrac{x-3}{3} = \sin t$ and $\dfrac{2-y}{2} = \cos t$.

Square both sides of each equation and add:

$$\frac{(x-3)^2}{9} + \frac{(2-y)^2}{4} = \sin^2 t + \cos^2 t$$

Hence $$\frac{(x-3)^2}{9} + \frac{(2-y)^2}{4} = 1$$

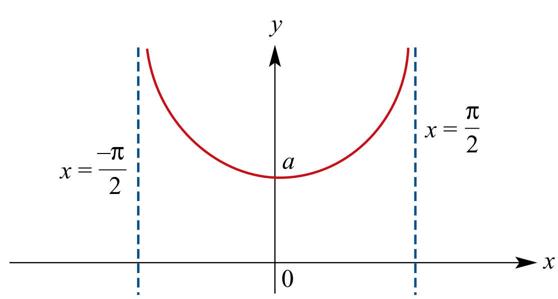
But $(2-y)^2 = (y-2)^2$ so this equation is more neatly written as

$$\frac{(x-3)^2}{9} + \frac{(y-2)^2}{4} = 1$$

Clearly this is an ellipse, with centre at (3, 2), and axes intercepts at (3, 0) and (0, 2).

Hyperbolas

The general cartesian equation for a hyperbola with 'centre' at the origin is $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$. The parametric equations are $x = a \sec t$ and $y = b \tan t$ where $\sec t = \dfrac{1}{\cos t}$ and $t \in \left(\dfrac{-\pi}{2}, \dfrac{\pi}{2}\right)$ gives the right-hand branch of the hyperbola. For the function with rule $x = a \sec t$ and domain $\left(\dfrac{-\pi}{2}, \dfrac{\pi}{2}\right)$ the range is $[a, \infty)$. (The sec function is discussed further in Chapter 3.)



The graph of $y = a \sec x$ is shown for the interval $\left(\dfrac{-\pi}{2}, \dfrac{\pi}{2}\right)$.

For the function with rule $y = b \tan t$ and domain $\left(\dfrac{-\pi}{2}, \dfrac{\pi}{2}\right)$ the range is R. The left branch of the hyperbola can be obtained for $t \in \left(\dfrac{\pi}{2}, \dfrac{3\pi}{2}\right)$.

The proof that the two forms of equation can yield the same graph uses a form of the Pythagorean identity $\sin^2 t + \cos^2 t = 1$. Divide both sides of this identity by $\cos t$. This yields $\tan^2 t + 1 = \sec^2 t$.

Consider $x = a \sec t$ and $y = b \tan t$. Therefore $\dfrac{x}{a} = \sec t$ and $\dfrac{y}{b} = \tan t$.

Square both sides of each equation to obtain $\dfrac{x^2}{a^2} = \sec^2 t$ and $\dfrac{y^2}{b^2} = \tan^2 t$.

Now, since $\tan^2 t + 1 = \sec^2 t$, it follows that $\dfrac{y^2}{b^2} + 1 = \dfrac{x^2}{a^2}$.

Therefore $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$.

Example 33

Find the cartesian equation of the curve with parametric equations $x = 3\sec t$, $y = 4\tan t$, where $t \in \left(\frac{\pi}{2}, \frac{3\pi}{2}\right)$ and describe the curve.

Solution

Now $x = 3\sec t$ and $y = 4\tan t$. Therefore $\frac{x}{3} = \sec t$ and $\frac{y}{4} = \tan t$. Square both sides of each equation to obtain $\frac{x^2}{9} = \sec^2 t$ and $\frac{y^2}{16} = \tan^2 t$. Add these two equations to obtain $\frac{y^2}{16} + 1 = \frac{x^2}{9}$.

So the cartesian form of the curve is $\frac{x^2}{9} - \frac{y^2}{16} = 1$.

The range of the function with rule $x = 3\sec t$ for $t \in \left(\frac{\pi}{2}, \frac{3\pi}{2}\right)$ is $(-\infty, -3]$.

Hence the domain for the graph is $(-\infty, -3]$.

This is the left branch of a hyperbola, with centre at the origin, and x intercept at $(-3, 0)$ and with asymptotes with equations $y = \frac{4x}{3}$ and $y = -\frac{4x}{3}$.

Example 34

Give parametric equations for each of the following:

a $x^2 + y^2 = 9$ **b** $\frac{x^2}{16} + \frac{y^2}{4} = 1$ **c** $\frac{(x-1)^2}{9} - \frac{(y+1)^2}{4} = 1$

Solution

a The parametric equations are $x = 3\cos t$ and $y = 3\sin t$ or $x = 3\sin t$ and $y = 3\cos t$.

There are infinitely many pairs of equations which determine the curve given by the cartesian equation $x^2 + y^2 = 9$. Others are $x = -3\cos(2t)$ and $y = 3\sin(2t)$.

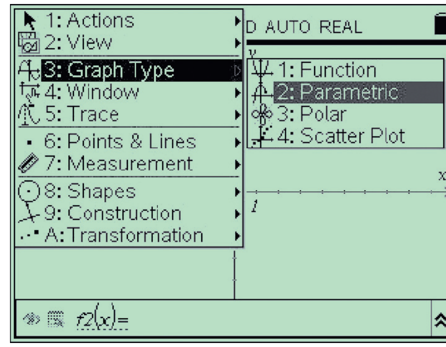
For $x = 3\cos t$ and $y = 3\sin t$ it is sufficient for t to be chosen for the interval $[0, 2\pi]$ to obtain the whole curve. For $x = -3\cos(2t)$ and $y = 3\sin(2t)$ it is sufficient for t to be chosen in the interval $[0, \pi]$.

b The obvious solution is $x = 4\cos t$ and $y = 2\sin t$

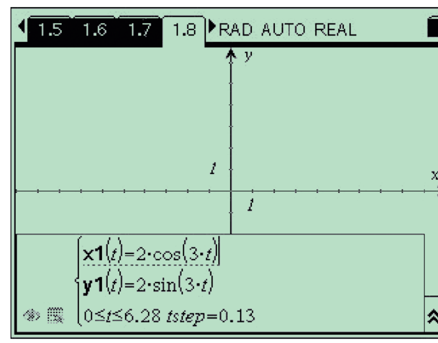
c The obvious solution is $x - 1 = 3\sec t$ and $y + 1 = 2\tan t$

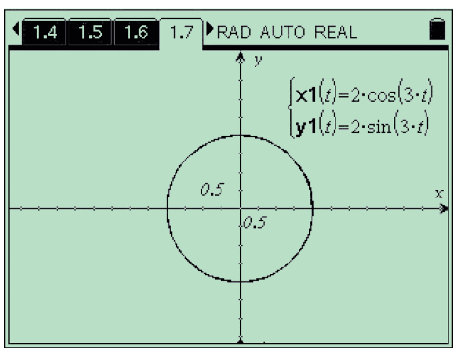
Using a TI-Nspire calculator

Open a **Graphs & Geometry** application (⌂ 2) and choose **Parametric** from the **Graph Type** menu (menu 3 2).



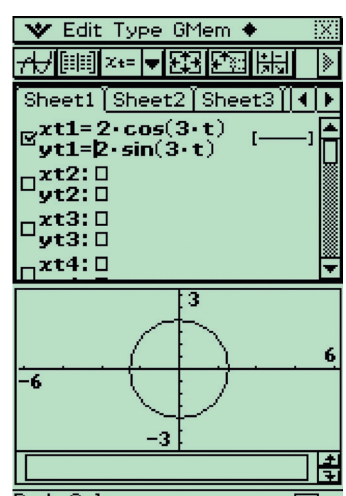
Enter $x1(t) = 2\cos(3t)$ and $y1(t) = 2\sin(3t)$ as shown right. The graph is shown below.





Using a Casio ClassPad calculator

Choose **Graph and Table** from the menu. From the **Type** menu select **ParamType**. Enter the equations as shown right and click the graph icon in the icon bar at the top of the screen.



Exercise 1H

1 Find the cartesian equation of the curve determined by the parametric equations $x = 2\cos 3t$ and $y = 2\sin 3t$, and determine the domain and range of the corresponding relation.

2 Determine the corresponding cartesian equation of the curve determined by each of the following parametric equations and sketch the graph of each of these.

a $x = \sec t,\ y = \tan t,\ t \in \left(\frac{\pi}{2}, \frac{3\pi}{2}\right)$

b $x = 3\cos 2t,\ y = -4\sin 2t$

c $x = 3 - 3\cos t,\ y = 2 + 2\sin t$

d $x = 3\sin t,\ y = 4\cos t,\ t \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

e $x = \sec t,\ y = \tan t,\ t \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

f $x = 1 - \sec(2t),\ y = 1 + \tan(2t),\ t \in \left(\frac{\pi}{4}, \frac{3\pi}{4}\right)$

3 Give parametric equations corresponding to each of the following:

a $x^2 + y^2 = 16$

b $\frac{x^2}{9} - \frac{y^2}{4} = 1$

c $(x-1)^2 + (y+2)^2 = 9$

d $\frac{(x-1)^2}{9} + \frac{(y+3)^2}{4} = 9$

4 A circle has centre (1, 3) and radius 2. If parametric equations for this circle are $x = a + b\cos(2\pi t)$ and $y = c + d\sin(2\pi t)$, where a, b, c and d are positive constants, state the values of a, b, c and d.

5 An ellipse has x-axis intercepts (−4, 0) and (4, 0) and y-axis intercepts (0, 3) and (0, −3). State a possible pair of parametric equations for this ellipse.

6 The graph of the circle with parametric equations $x = 2\cos 2t$ and $y = 2\sin 2t$ is dilated by a factor 3 from the x axis. For the image curve, state:

a a possible pair of parametric equations

b the cartesian equation

7 The graph of the ellipse with parametric equations $x = 3 - 2\cos\left(\frac{t}{2}\right)$ and $y = 4 + 3\sin\left(\frac{t}{2}\right)$ is translated 3 units in the negative direction of the x axis and 2 units in the negative direction of the y axis. For the image curve state:

a a possible pair of parametric equations

b the cartesian equation

8 Sketch the graph of the curve with parametric equations $x = 2 + 3\sin(2\pi t)$ and $y = 4 + 2\cos(2\pi t)$ for:

a $t \in \left[0, \frac{1}{4}\right]$

b $t \in \left[0, \frac{1}{2}\right]$

c $t \in \left[0, \frac{3}{2}\right]$

For each of these, state the domain and range.

Review

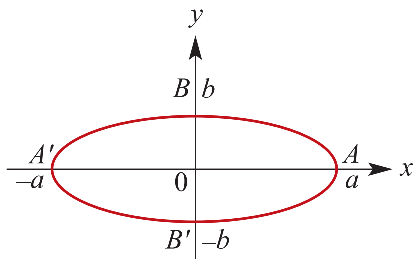
Summary of circles, ellipses and hyperbolas

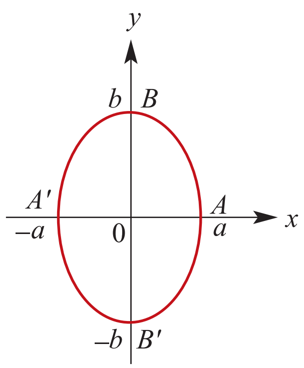
Circles

- The circle with centre at the origin and radius a is the graph of the equation $x^2 + y^2 = a^2$.
- The circle with centre (h, k) and radius a is the graph of the equation $(x - h)^2 + (y - k)^2 = a^2$.
- In general, $x^2 + y^2 = a^2$, where $a > 0$, is the cartesian equation of a circle with centre at the origin and radius a. The parametric equations are $x = a \cos t$ and $y = a \sin t$. The minimal interval of t values to yield the entire circle is $[0, 2\pi]$.
- The circle with centre (h, k) and radius a can be described through the parametric equations $x = h + a \cos t$ and $y = k + a \sin t$.

Ellipses

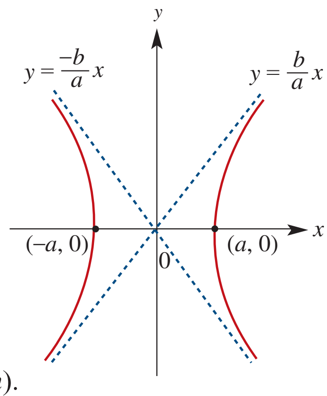
- The curve with equation $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is an ellipse with centre the origin, x-axis intercepts $(-a, 0)$ and $(a, 0)$ and y-axis intercepts $(0, -b)$ and $(0, b)$. For $a > b$ the ellipse will appear as shown in the diagram on the left. If $b > a$ the ellipse is as shown in the diagram on the right.





- The curve with equation $\frac{(x - h)^2}{a^2} + \frac{(y - k)^2}{b^2} = 1$ is an ellipse with centre (h, k).
- The ellipse with centre at the origin, and x-axis intercepts $(\pm a, 0)$ and y-axis intercepts $(0, \pm b)$ has parametric equations $x = a \cos t$ and $y = b \sin t$. The minimal interval of t values to yield the entire ellipse is $[0, 2\pi]$.
- The ellipse with centre (h, k) formed by translating the ellipse with equation $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ can be described through the parametric equations $x = h + a \cos t$ and $y = k + b \sin t$.

Hyperbolas



- The curve with equation $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ is a hyperbola with centre at the origin. The axis intercepts are $(a, 0)$ and $(-a, 0)$. The hyperbola has asymptotes $y = \frac{b}{a}x$ and $y = -\frac{b}{a}x$.
- The curve with equation $\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$ is a hyperbola with centre (h, k). The hyperbola has asymptotes $y - k = \frac{b}{a}(x - h)$ and $y - k = -\frac{b}{a}(x - h)$.
- The parametric equations for the hyperbola shown above are $x = a \sec t$ and $y = b \tan t$ where $\sec t = \frac{1}{\cos t}$.
- The hyperbola with centre (h, k) formed by translating the hyperbola with equation $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ can be described through the parametric equations $x = h + a \sec t$ and $y = k + b \tan t$.

Multiple-choice questions

1 The 3rd term of a geometric sequence is 4. If the 8th term is 128, then the 1st term is:

A 2 **B** 1 **C** 32 **D** 5 **E** none of these

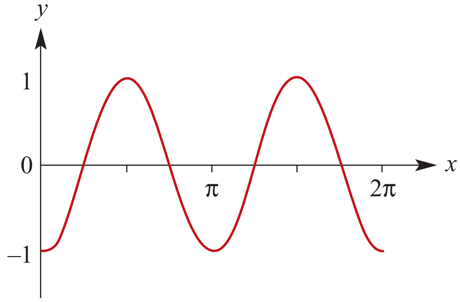
2 If the numbers 5, x and y are in arithmetic sequence then:

A $y = x + 5$ **B** $y = x - 5$ **C** $y = 2x + 5$
D $y = 2x - 5$ **E** none of these

3 If $2 \cos x^\circ - \sqrt{2} = 0$, the value of the acute angle x° is:

A 30° **B** 60° **C** 45° **D** 25° **E** 27.5°

4 The equation of the graph shown is:



A $y = \sin 2\left(x - \frac{\pi}{4}\right)$

B $y = \cos\left(x + \frac{\pi}{4}\right)$

C $y = \sin(2x)$

D $y = -2\sin(x)$

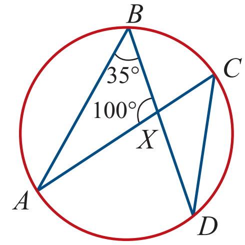
E $y = \sin\left(x + \frac{\pi}{4}\right)$

5 The exact value of the expression $\sin\left(\frac{2\pi}{3}\right) \times \cos\left(\frac{\pi}{4}\right) \times \tan\left(\frac{\pi}{6}\right)$ is:

A $\frac{1}{\sqrt{2}}$ **B** $\frac{1}{\sqrt{3}}$ **C** $\frac{\sqrt{2}}{4}$ **D** $\frac{\sqrt{3}}{2}$ **E** none of these

Review

6 In the diagram, A, B, C and D are points on the circle. $\angle ABD = 35^\circ$ and $\angle AXB = 100^\circ$. The magnitude of $\angle XDC$ is:



A 35° **B** 40° **C** 45°
D 50° **E** 55°

7 In a geometric sequence $t_2 = 24$ and $t_4 = 54$. The sum of the first 5 terms, if the common ratio is positive, is:

A 130 **B** 211 **C** 238 **D** 316.5 **E** 810

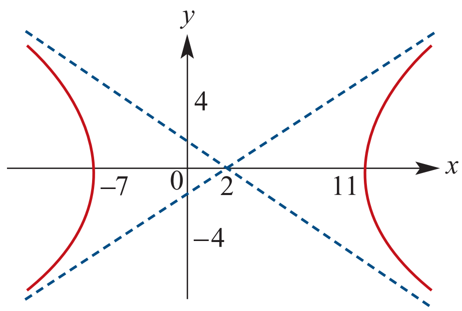
8 In a triangle ABC, $a = 30$, $b = 21$ and $\cos C = \frac{51}{53}$. The value of c to the nearest whole number is:

A 9 **B** 10 **C** 11 **D** 81 **E** 129

9 The coordinates of the centre of the circle with equation $x^2 - 8x + y^2 - 2y = 8$ are:

A $(-8, -2)$ **B** $(8, 2)$ **C** $(-4, -1)$ **D** $(4, 1)$ **E** $(1, 4)$

10 The equation of the graph shown is:



A $\dfrac{(x+2)^2}{27} - \dfrac{y^2}{108} = 1$

B $\dfrac{(x-2)^2}{9} - \dfrac{y^2}{34} = 1$

C $\dfrac{(x+2)^2}{81} - \dfrac{y^2}{324} = 1$

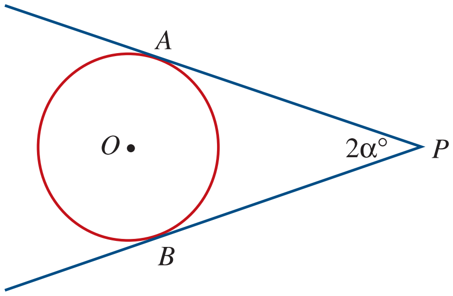
D $\dfrac{(x-2)^2}{81} - \dfrac{y^2}{324} = 1$

E $\dfrac{(x+2)^2}{9} - \dfrac{y^2}{36} = 1$

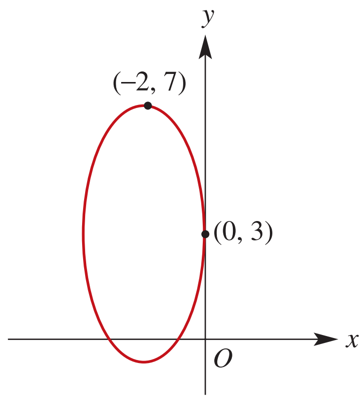
Short-answer questions (technology-free)

1 For the difference equation $f_n = 5f_{n-1}$, $f_0 = 1$, find f_n in terms of n.

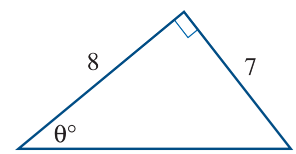
2 AP and BP are tangents to the circle with centre O. If $AP = 10$ cm, find OP.



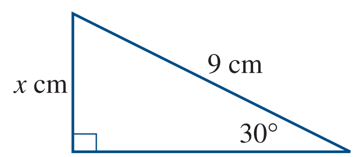
3 Write down the equation of the ellipse shown.



4 Find $\sin\theta°$.



5 Find x.



6 A circle has a chord of length 10 cm situated 3 cm from its centre. Find:

a the radius length

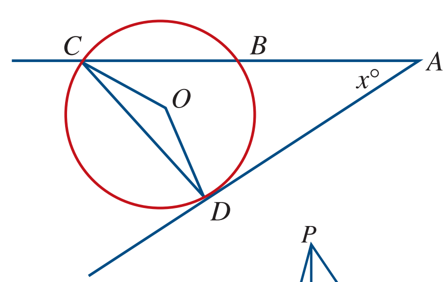
b the angle subtended by the chord at the centre

7 **a** Find the exact value of $\cos 315°$.

b Given that $\tan x° = \frac{3}{4}$ and $180 < x < 270$, find an exact value of $\cos x°$.

c Find an angle A ($A \neq 330$) such that $\sin A = \sin 330°$.

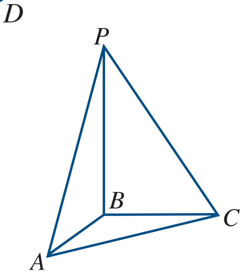
8 In the diagram, AD is a tangent to the circle with centre O, AC intersects the circle at B, and $BD = AB$.



a Find $\angle BCD$ in terms of x.

b If $AD = y$ cm, $AB = a$ cm and $BC = b$ cm, express y in terms of a and b.

9 ABC is a horizontal right-angled triangle with the right angle at B. P is a point 3 cm directly above B. The length of AB is 1 cm and the length of BC is 1 cm.
Find the angle which the triangle ACP makes with the horizontal.



10 **a** Solve $2\cos(2x + \pi) - 1 = 0$, $-\pi \leq x \leq \pi$.

b Sketch the graph of $y = 2\cos(2x + \pi) - 1$, $-\pi \leq x \leq \pi$, clearly labelling axes intercepts.

c Solve $2\cos(2x + \pi) < 1$, $-\pi \leq x \leq \pi$.

Review

11 The triangular base ABC of a tetrahedron has side lengths $AB = 15$ cm, $BC = 12$ cm and $AC = 9$ cm. If the apex D is 9 cm vertically above C, then find:

a the angle C of the triangular base

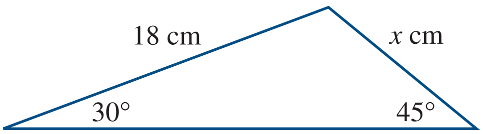
b the angles that the sloping edges make with the horizontal

12 Two ships sail from port at the same time. One sails 24 nautical miles due east in three hours, and the other sails 33 nautical miles on a bearing of 030° in the same time.

a How far apart are the ships three hours after leaving port?

b How far apart would they be in five hours if they maintained the same bearings and constant speed?

13 Find x.



14 An airport A is 480 km due east of airport B. A pilot flies on a bearing of 225° from A to C and then on a bearing of 315° from C to B.

a Make a sketch of the situation.

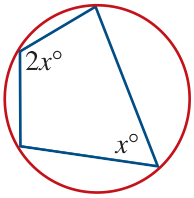
b Determine how far the pilot flies from A to C.

c Determine the total distance the pilot flies.

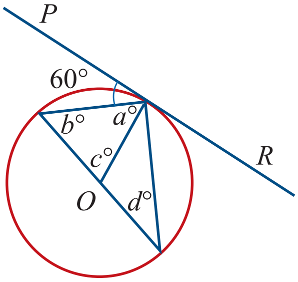
15 Find the equations of the asymptotes for the hyperbola with rule $x^2 - \frac{(y-2)^2}{9} = 15$.

16 A curve is defined by the parametric equations $x = 3\cos(2t) + 4$ and $y = \sin(2t) - 6$. Give the cartesian equation of the curve.

17 **a** Find the value of x.



b Find a, b, c and d, given that PR is a tangent to the circle with centre O.



18 A curve is defined by the parametric equations $x = 2\cos(\pi t)$ and $y = 2\sin(\pi t) + 2$. Give the cartesian equation of the curve.

19 **a** Sketch the graphs of $y = -2\cos x$ and $y = -2\cos\left(x - \frac{\pi}{4}\right)$ on the same set of axes, for $x \in [0, 2\pi]$.

b Solve $-2\cos\left(x - \frac{\pi}{4}\right) = 0$ for $x \in [0, 2\pi]$.

c Solve $-2\cos x \le 0$ for $x \in [0, 2\pi]$.

20 Find all angles θ, such that $0 \le \theta \le 2\pi$, where:

a $\sin\theta = \frac{1}{2}$ **b** $\cos\theta = \frac{\sqrt{3}}{2}$ **c** $\tan\theta = 1$

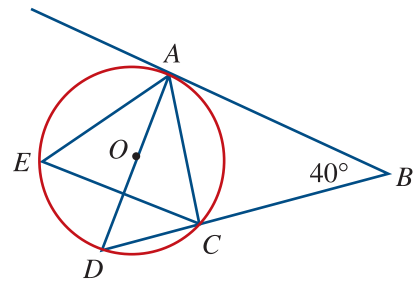
21 A circle has centre $(1, 2)$ and radius 3. If parametric equations for this circle are $x = a + b\cos(2\pi t)$ and $y = c + d\sin(2\pi t)$, where a, b, c and d are positive constants, state the values of a, b, c and d.

22 O is the centre of a circle with points A, C, D and E on the circle. Find:

a $\angle ADB$

b $\angle AEC$

c $\angle DAC$



23 Find the centre and radius of the circle with equation $x^2 + 8x + y^2 - 12y + 3 = 0$.

24 Find the x- and y-axes intercepts of the graph of the ellipse $\frac{x^2}{81} + \frac{y^2}{9} = 1$.

25 The first term of an arithmetic sequence is $(3p + 5)$ where p is a positive integer. The last term is $(17p + 17)$ and the common difference is 2.

a Find in terms of p:

i the number of terms **ii** the sum of the sequence

b Show that the sum of the sequence is divisible by 14 only when p is odd.

26 A sequence is formed by using rising powers of 3: $3^0, 3^1, 3^2, \ldots$

a Find the nth term. **b** Find the product of the first twenty terms.

Extended-response questions

1 A hiker walks from point A on a bearing of 010° for 5 km and then on a bearing of 075° for 7 km to reach point B.

a Find the length of AB.

b Find the bearing of B from the start point A.

A second hiker travels from point A on a bearing of 080° for 4 km to a point P, and then travels in a straight line to B.

c Find:

i the total distance travelled by the second hiker

ii the bearing on which the hiker must travel in order to reach B from P.

A third hiker also travels from point A on a bearing of 080° and continues on that bearing until he reaches point C. He then turns and walks towards B. In doing so, the two legs of the journey are of equal length.

d Find the distance travelled by the third hiker to reach B.

2 An ellipse is defined by the rule $\frac{x^2}{2} + \frac{(y+3)^2}{5} = 1$.

a Find:

i the domain of the relation **ii** the range of the relation

iii the centre of the ellipse.

E is an ellipse given by the rule $\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$. The domain of E is $[-1, 3]$ and its range is $[-1, 5]$.

b Find the values of a, b, h and k.

The line $y = x - 2$ intersects the ellipse E at A $(1, -1)$ and at P.

c Find the coordinates of the point P.

A line perpendicular to the line $y = x - 2$ is drawn at P. This line intersects the y axis at Q.

d Find the coordinates of Q.

e Find the equation of the circle through A, P and Q.

3 **a** Show that the circle with equation $x^2 + y^2 - 2ax - 2ay + a^2 = 0$ touches both the x axis and the y axis.

b Show that every circle that touches the x axis and y axis has an equation of a similar form.

c Hence show that there are exactly two circles passing through the point $(2, 4)$ and just touching the x axis and y axis and give their equations.

d State the coordinates of the centres of these two circles and give the radius of each of these circles.

e For each of the circles, find the gradient of the line which passes through the centre and the point $(2, 4)$.

f Find an equation to the tangent to each circle at the point $(2, 4)$.

4 A circle is defined by the parametric equation $x = a \cos\theta$ and $y = a \sin\theta$. Let P be the point with coordinates $(a \cos\theta, a \sin\theta)$.

a Find the equation of the straight line which passes through the origin and the point P.

b State the coordinates, in terms of θ, of the other point of intersection of the circle with the straight line through the origin and P.

c Find the equation of the tangent to the circle at the point P.

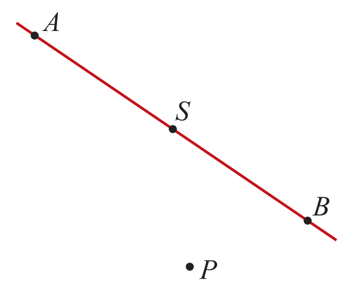
d Find the coordinates of the points of intersection A and B of the tangent with the x axis and y axis respectively.

e Find the area of triangle OAB in terms of θ if $0 < \theta < \frac{\pi}{2}$. Find the value of θ for which the area of this triangle is a minimum.

5 The line with equation $x = -a$ is the equation of the side BC of an equilateral triangle ABC circumscribing the circle with equation $x^2 + y^2 = a^2$.

- **a** Find the equations of AB and AC.
- **b** Find the equation of the circle circumscribing triangle ABC.

6 This diagram shows a straight track through points A, S and B, where A is 10 km northwest of B and S is exactly halfway between A and B. A surveyor is required to reroute the track through P from A to B to avoid a major subsidence at S.



The surveyor determines that A is on a bearing of 330° from P and B is on a bearing of 70° from P.
Assume that the region under consideration is flat. Find:

- **a** the magnitude of angles APB, PAB and PBA
- **b** the distance from P to B and from P to S
- **c** the bearing of S from P
- **d** the distance from A to B through P, if the surveyor chooses to reroute the track along a circular arc.

CHAPTER

2

Vectors

Objectives

- To understand the concept of a **vector**
- To apply basic operations to vectors
- To understand the zero vector
- To use the unit vectors $\boldsymbol{i}$ and $\boldsymbol{j}$ to represent vectors in two dimensions
- To use the fact that, if $\boldsymbol{a}$ and $\boldsymbol{b}$ are parallel, then $\boldsymbol{a} = k\boldsymbol{b}$ for a real value k, and to use the converse of this
- To use the unit vectors $\boldsymbol{i}$, $\boldsymbol{j}$ and $\boldsymbol{k}$ to represent vectors in three dimensions
- To understand the **triangle of vectors**, extending to the polygon of vectors
- To evaluate the **scalar product** of two vectors
- To understand the algebraic laws applicable to the scalar product
- To recognise the scalar product property of **two perpendicular vectors**
- To understand the concept of the angle between two vectors
- To understand **vector resolutes** and **scalar resolutes**
- To resolve a vector into **rectangular components**
- To apply vector techniques to proof in geometry

2.1 Introduction to vectors

In science or engineering, some of the things that are measured are completely determined by their magnitude. For example, mass, length and time are determined by a number and an appropriate unit of measurement.

e.g. length: 30 cm is the length of the page of a particular book
time: 10 s is the time for one athlete to run 100 metres

More is required to describe velocity, displacement or force. The **direction** must be recorded as well as the magnitude.

e.g. velocity: 60 km/h in a direction south-east

Quantities in two- or three-dimensional space that have direction as well as magnitude can be represented by arrows that point in the direction of the action and whose lengths give the magnitude of the quantity in terms of a suitably chosen unit.

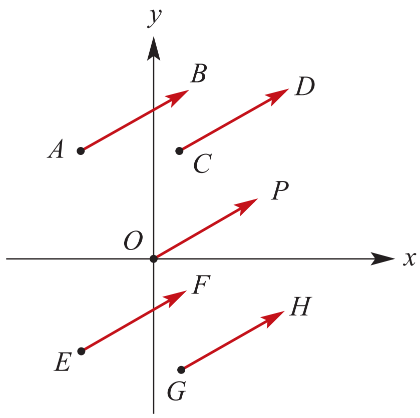
Arrows with the same length and direction are regarded as equivalent. These arrows are **directed line segments** and the sets of equivalent segments are called **vectors**.

The five directed line segments shown all have the same magnitude and direction.

A directed line segment from a point A to a point B is denoted by $\overrightarrow{AB}$.

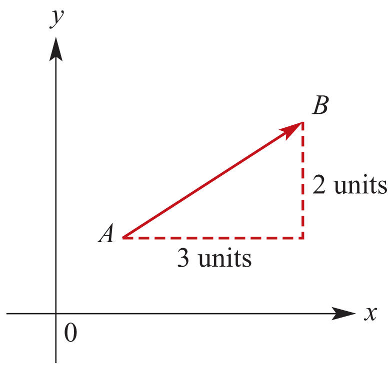
For simplicity of language this is also called vector $\overrightarrow{AB}$, i.e. the set of equivalent segments can be named through one member of that set.

Note: $\overrightarrow{AB} = \overrightarrow{CD} = \overrightarrow{OP} = \overrightarrow{EF} = \overrightarrow{GH}$



In Essential Advanced General Mathematics a column of numbers was introduced to represent the translation and it was called a vector. This is consistent with the approach here as the column of numbers corresponds to a set of equivalent directed line segments.

The column $\begin{bmatrix} 3 \\ 2 \end{bmatrix}$ corresponds to the directed line segment that goes 3 across and 2 up.



This notation will be used to represent a directed line segment in the first section of this chapter. Vectors are often denoted by a single bold-face roman letter. For example, the vector from A to B can be denoted by $\overrightarrow{AB}$ or by a single $\boldsymbol{v}$. That is, $\boldsymbol{v} = \overrightarrow{AB}$. When a vector is handwritten the notation is $\underset{\sim}{v}$.

Magnitude of vectors

The magnitude of vector $\overrightarrow{AB}$ is denoted by $|\overrightarrow{AB}|$, and for vector $\boldsymbol{v}$ the magnitude is denoted by $|\boldsymbol{v}|$. The magnitude of a vector is represented by the length of a directed line segment corresponding to the vector.

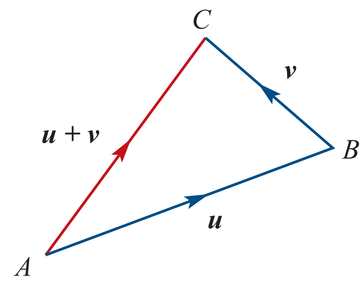
For $\overrightarrow{AB}$ in the diagram above, Pythagoras' theorem gives $|\overrightarrow{AB}| = \sqrt{3^2 + 2^2} = \sqrt{13}$.

In general, if $\overrightarrow{AB}$ is represented by the column vector $\begin{bmatrix} x \\ y \end{bmatrix}$ the magnitude, $|\overrightarrow{AB}|$, is equal to $\sqrt{x^2 + y^2}$.

Addition of vectors (the triangle of vectors)

Two vectors $\boldsymbol{u}$ and $\boldsymbol{v}$ can be added geometrically by drawing a line segment representing $\boldsymbol{u}$ from A to B and then a line segment from B to C representing $\boldsymbol{v}$.

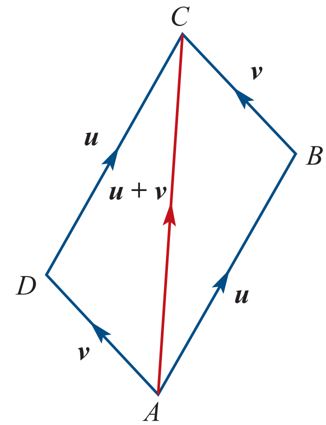
The sum $\boldsymbol{u} + \boldsymbol{v}$ is the vector from A to C. That is, $\boldsymbol{u} + \boldsymbol{v} = \overrightarrow{AC}$.



The same result is achieved if the order is reversed. This is represented in the diagram:

i.e. $\boldsymbol{u} + \boldsymbol{v} = \overrightarrow{AC}$

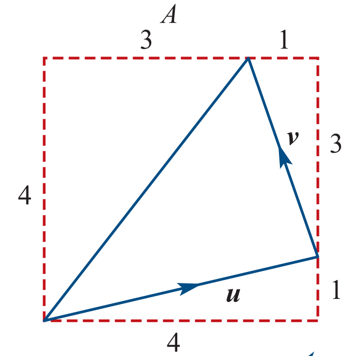
and $\boldsymbol{u} + \boldsymbol{v} = \boldsymbol{v} + \boldsymbol{u}$



The addition can also be achieved with the column vector notation. For example:

if $\boldsymbol{u} = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$ and $\boldsymbol{v} = \begin{bmatrix} -1 \\ 3 \end{bmatrix}$

then $\boldsymbol{u} + \boldsymbol{v} = \begin{bmatrix} 4 \\ 1 \end{bmatrix} + \begin{bmatrix} -1 \\ 3 \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$

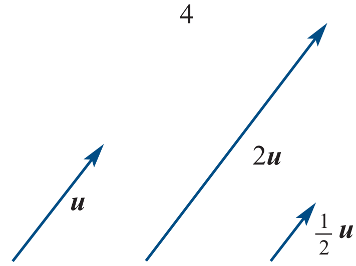


Scalar multiplication

Multiplication by a real number (scalar) changes the length of the vector. For example:

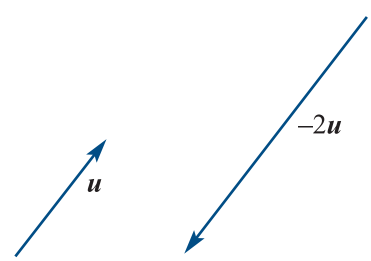
$2\boldsymbol{u} = \boldsymbol{u} + \boldsymbol{u}$ and $\frac{1}{2}\boldsymbol{u} + \frac{1}{2}\boldsymbol{u} = \boldsymbol{u}$

$2\boldsymbol{u}$ is twice the length of $\boldsymbol{u}$ and $\frac{1}{2}\boldsymbol{u}$ is half the length of $\boldsymbol{u}$.



The vector $k\boldsymbol{u}$, $k \in R^+$, has the same direction as $\boldsymbol{u}$, but its length is multiplied by a factor of k.

When a vector is multiplied by -2 the vector's direction is reversed and its length is doubled.



When a vector is multiplied by -1 the vector's direction is reversed and the length remains the same.

If $\quad \boldsymbol{u} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}, -\boldsymbol{u} = \begin{bmatrix} -3 \\ -2 \end{bmatrix}, 2\boldsymbol{u} = \begin{bmatrix} 6 \\ 4 \end{bmatrix}$ and $-2\boldsymbol{u} = \begin{bmatrix} -6 \\ -4 \end{bmatrix}$

If $\quad \boldsymbol{u} = \overrightarrow{AB}$ then $-\boldsymbol{u} = -\overrightarrow{AB} = \overrightarrow{BA}$.

The directed line segment $-\overrightarrow{AB}$ starts at B and finishes at A.

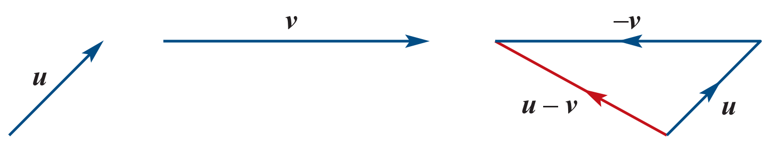
Zero vector

The **zero vector** is denoted by **0** and represents a line segment of zero length. The zero vector has no direction. The magnitude of the zero vector is 0. Note that $0 \times \boldsymbol{a} = \boldsymbol{0}$ and $\boldsymbol{a} + (-\boldsymbol{a}) = \boldsymbol{0}$.

In two dimensions, $\boldsymbol{0} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

Subtraction of vectors

In order to subtract $\boldsymbol{v}$ from $\boldsymbol{u}$, add $-\boldsymbol{v}$ to $\boldsymbol{u}$. For example:

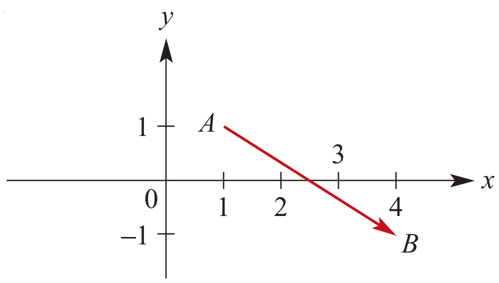


Example 1

Draw the directed line segment defined by $\begin{bmatrix} 3 \\ -2 \end{bmatrix}$ and state the magnitude of the corresponding vector.

Solution

$\begin{bmatrix} 3 \\ -2 \end{bmatrix}$ is the vector '3 across to the right and 2 down'.



Note: Here the vector starts at (1, 1) and finishes at (4, −1). It can start at any point. The magnitude of the vector $= \sqrt{3^2 + (-2)^2} = \sqrt{13}$

Example 2

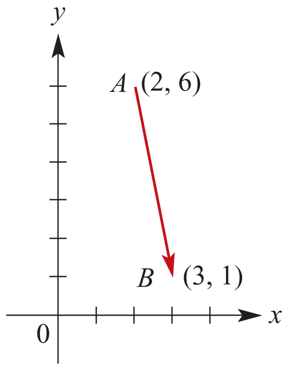
The vector $\boldsymbol{u}$ is defined by the directed line segment from (2, 6) to (3, 1). If $\boldsymbol{u} = \begin{bmatrix} a \\ b \end{bmatrix}$ find a and b.

Solution

From the diagram $\begin{bmatrix} 2 \\ 6 \end{bmatrix} + \boldsymbol{u} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$

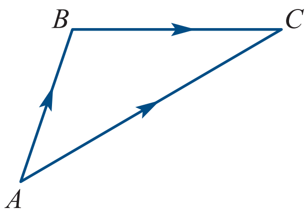
The vector $\boldsymbol{u} = \begin{bmatrix} 3-2 \\ 1-6 \end{bmatrix} = \begin{bmatrix} 1 \\ -5 \end{bmatrix}$

Hence $a = 1$ and $b = -5$.

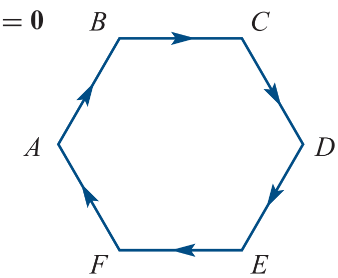


Polygons of vectors

For two vectors $\overrightarrow{AB}$ and $\overrightarrow{BC}$, $\overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{AC}$



For a polygon $ABCDEF$, $\overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CD} + \overrightarrow{DE} + \overrightarrow{EF} + \overrightarrow{FA} = \mathbf{0}$

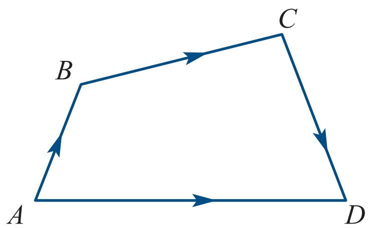


Example 3

Illustrate the vector sum $\overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CD}$ where A, B, C and D are points in the plane.

Solution

$$\overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CD} = \overrightarrow{AD}$$



Parallel vectors

The non-zero vectors $\boldsymbol{u}$ and $\boldsymbol{v}$ are said to be parallel if there exists $k \in R\backslash\{0\}$ such that $\boldsymbol{u} = k\boldsymbol{v}$.

If $\boldsymbol{u} = \begin{bmatrix} -2 \\ 3 \end{bmatrix}$ and $\boldsymbol{v} = \begin{bmatrix} -6 \\ 9 \end{bmatrix}$ then vector $\boldsymbol{u}$ is parallel to $\boldsymbol{v}$ as $\boldsymbol{v} = 3\boldsymbol{u}$.

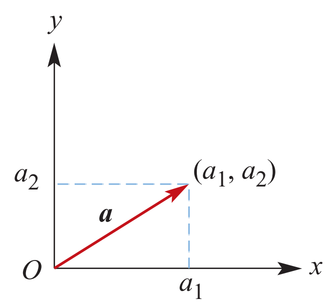
Position vectors

The point O, the origin, can be used as a starting point for a vector to indicate the position of a point in space relative to that point.

For a point A the position vector is $\overrightarrow{OA}$.

The two-dimensional vector $\boldsymbol{a} = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}$ is associated with the point (a_1, a_2).

The position vector representing $\boldsymbol{a}$ is the position vector which ends at point (a_1, a_2).



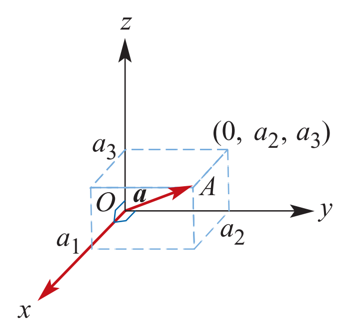
Vectors in three dimensions

The definition of vector given above is, of course, also valid in three dimensions. The properties which hold in two dimensions also hold in three dimensions.

For vectors in three dimensions, a third axis, denoted by z, is used. The x axis is drawn at an angle to indicate a direction out of the page and towards the reader.

The third axis is at right angles to the other two axes. $\boldsymbol{a}$ can be represented as a column vector.

$$\boldsymbol{a} = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \text{ and } \boldsymbol{a} = \overrightarrow{OA} \text{ the position vector of the point } A.$$

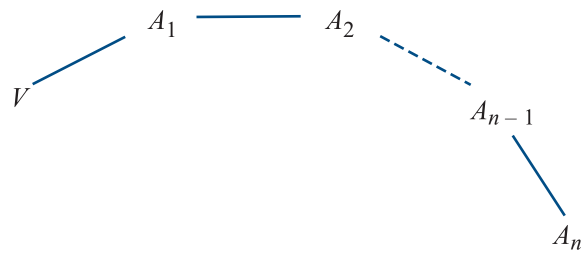


The position vector representing $\boldsymbol{a}$ is the position vector which ends at the point (a_1, a_2, a_3).

It is appropriate to summarise the following properties for vectors of the same dimension before proceeding.

- $\boldsymbol{a} + \boldsymbol{b} = \boldsymbol{b} + \boldsymbol{a}$ — commutative law for vector addition
- $(\boldsymbol{a} + \boldsymbol{b}) + \boldsymbol{c} = \boldsymbol{a} + (\boldsymbol{b} + \boldsymbol{c})$ — associative law for vector addition
- $\boldsymbol{a} + \boldsymbol{0} = \boldsymbol{a}$ — zero vector
- $\boldsymbol{a} + -\boldsymbol{a} = \boldsymbol{0}$ — $-\boldsymbol{a}$ is the opposite or inverse vector
- $m(\boldsymbol{a} + \boldsymbol{b}) = m\boldsymbol{a} + m\boldsymbol{b}$ — distributive law where $m \in R$
- $\boldsymbol{a}$ is parallel to $\boldsymbol{b}$ if there exists $k \in R \backslash \{0\}$ such that $\boldsymbol{a} = k\boldsymbol{b}$

Let $V, A_1, A_2 \ldots$ and A_n be points in space.



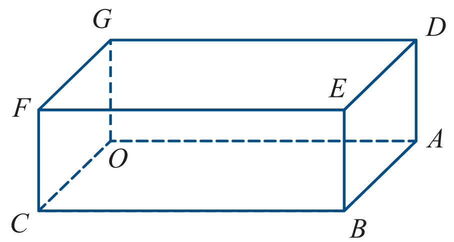
Then $\overrightarrow{VA_1} + \overrightarrow{A_1A_2} + \overrightarrow{A_2A_3} + \cdots \overrightarrow{A_{n-1}A_n} = \overrightarrow{VA_n}$

Example 4

OABCDEFG is a cuboid as shown. Let $\overrightarrow{OA} = \boldsymbol{a}$, $\overrightarrow{OG} = \boldsymbol{g}$ and $\overrightarrow{OC} = \boldsymbol{c}$.

Find the following vectors in terms of $\boldsymbol{a}$, $\boldsymbol{g}$, and $\boldsymbol{c}$:

a $\overrightarrow{OB}$ **b** $\overrightarrow{OF}$ **c** $\overrightarrow{GD}$
d $\overrightarrow{GB}$ **e** $\overrightarrow{FA}$



Solution

a $\overrightarrow{OB} = \overrightarrow{OA} + \overrightarrow{AB}$
$= \boldsymbol{a} + \boldsymbol{c}$ (as $\overrightarrow{AB} = \overrightarrow{OC}$)

b $\overrightarrow{OF} = \overrightarrow{OC} + \overrightarrow{CF}$
$= \boldsymbol{c} + \boldsymbol{g}$ (as $\overrightarrow{CF} = \overrightarrow{OG}$)

c $\overrightarrow{GD} = \overrightarrow{OA} = \boldsymbol{a}$

d $\overrightarrow{GB} = \overrightarrow{GO} + \overrightarrow{OA} + \overrightarrow{AB}$
$= -\boldsymbol{g} + \boldsymbol{a} + \boldsymbol{c}$

e $\overrightarrow{FA} = \overrightarrow{FG} + \overrightarrow{GO} + \overrightarrow{OA}$
$= -\boldsymbol{c} - \boldsymbol{g} + \boldsymbol{a}$

Example 5

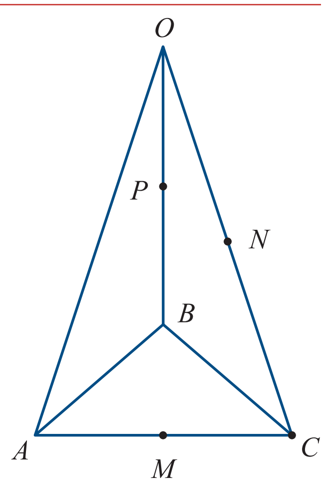
OABC is a tetrahedron.

$\overrightarrow{OA} = \boldsymbol{a}$, $\overrightarrow{OB} = \boldsymbol{b}$, $\overrightarrow{OC} = \boldsymbol{c}$

M is the midpoint of *AC*
N is the midpoint of *OC*
P is the midpoint of *OB*.

Find in terms of $\boldsymbol{a}$, $\boldsymbol{b}$ and $\boldsymbol{c}$:

a $\overrightarrow{AC}$ **b** $\overrightarrow{OM}$ **c** $\overrightarrow{CN}$ **d** $\overrightarrow{MN}$ **e** $\overrightarrow{MP}$



Solution

a $\overrightarrow{AC} = \overrightarrow{AO} + \overrightarrow{OC}$
$= -\boldsymbol{a} + \boldsymbol{c}$

b $\overrightarrow{OM} = \overrightarrow{OA} + \overrightarrow{AM}$
$= \overrightarrow{OA} + \frac{1}{2}\overrightarrow{AC}$
$= \boldsymbol{a} + \frac{1}{2}(-\boldsymbol{a} + \boldsymbol{c})$
$= \frac{1}{2}(\boldsymbol{a} + \boldsymbol{c})$

c $\overrightarrow{CN} = \frac{1}{2}\overrightarrow{CO}$
$= \frac{1}{2}(-\boldsymbol{c})$
$= -\frac{1}{2}\boldsymbol{c}$

d $\overrightarrow{MN} = \overrightarrow{MO} + \overrightarrow{ON}$
$= -\frac{1}{2}(\boldsymbol{a} + \boldsymbol{c}) + \frac{1}{2}\boldsymbol{c}$
$= \frac{1}{2}\boldsymbol{a} - \frac{1}{2}\boldsymbol{c} + \frac{1}{2}\boldsymbol{c}$
$= -\frac{1}{2}\boldsymbol{a}$
i.e. MN is parallel to AO

e $\overrightarrow{MP} = \overrightarrow{MO} + \overrightarrow{OP}$
$= -\frac{1}{2}(\boldsymbol{a} + \boldsymbol{c}) + \frac{1}{2}\boldsymbol{b}$
$= \frac{1}{2}(\boldsymbol{b} - (\boldsymbol{a} + \boldsymbol{c}))$

Linear dependence and independence

A set of vectors is said to be **linearly dependent** if one of its members can be expressed as a linear combination of the other vectors.

For example, the set of vectors $\boldsymbol{a}$, $\boldsymbol{b}$ and $\boldsymbol{c}$ is linearly dependent if there exist real numbers k, l and m, not all zero, such that $k\boldsymbol{a} + l\boldsymbol{b} + m\boldsymbol{c} = \boldsymbol{0}$.

A set of vectors is said to be **linearly independent** if it is not linearly dependent.

The vectors $\boldsymbol{a}$, $\boldsymbol{b}$ and $\boldsymbol{c}$ are linearly independent if the solution of the equation $k\boldsymbol{a} + l\boldsymbol{b} + m\boldsymbol{c} = \boldsymbol{0}$ is uniquely represented by $k = l = m = 0$.

Two simple facts about linear independence are:

- a set that contains the zero vector is linearly dependent
- a set with exactly two vectors is linearly independent if and only if one vector is not a scalar multiple of the other

An alternative practical definition of linear dependence of three vectors is given below.

Consider the set of vectors $\boldsymbol{a}$, $\boldsymbol{b}$ and $\boldsymbol{c}$.

If $\boldsymbol{a}$ and $\boldsymbol{b}$ can be observed to be independent, i.e. not parallel, then the set of vectors $\boldsymbol{a}$, $\boldsymbol{b}$ and $\boldsymbol{c}$ is linearly dependent if there exist real numbers m and n, not both zero, such that $\boldsymbol{c} = m\boldsymbol{a} + n\boldsymbol{b}$.

This representation of $\boldsymbol{c}$ in terms of two independent vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ is unique as demonstrated in this important result.

> Let $\boldsymbol{a}$ and $\boldsymbol{b}$ be two independent (not parallel) vectors.
> Then $m\boldsymbol{a} + n\boldsymbol{b} = p\boldsymbol{a} + q\boldsymbol{b}$ implies $m = p$ and $n = q$
> $m\boldsymbol{a} + n\boldsymbol{b}$ and $p\boldsymbol{a} + q\boldsymbol{b}$ may be considered as two possible representations of vector $\boldsymbol{c}$.

Proof

If $m\boldsymbol{a} + n\boldsymbol{b} = p\boldsymbol{a} + q\boldsymbol{b}$ then $(m - p)\boldsymbol{a} + (n - q)\boldsymbol{b} = \boldsymbol{0}$.

As $\boldsymbol{a}$ and $\boldsymbol{b}$ are independent vectors, then by the definition of linear independence, $(m - p) = 0$ and $(n - q) = 0$.

That is, $m = p$ and $n = q$.
$\boldsymbol{c}$ has a unique representation.

Example 6

Determine if the following sets of vectors are linearly dependent.

a $\boldsymbol{a} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$, $\boldsymbol{b} = \begin{bmatrix} 3 \\ -1 \end{bmatrix}$ and $\boldsymbol{c} = \begin{bmatrix} 5 \\ 6 \end{bmatrix}$

b $\boldsymbol{a} = \begin{bmatrix} 3 \\ 4 \\ -1 \end{bmatrix}$, $\boldsymbol{b} = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}$ and $\boldsymbol{c} = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$

Solution

a We note that $\boldsymbol{a}$ and $\boldsymbol{b}$ are not parallel.

Let $\boldsymbol{c} = m\boldsymbol{a} + n\boldsymbol{b}$

Then $5 = 2m + 3n$

$6 = m - n$

Solving the simultaneous equations we have $m = \frac{23}{5}$ and $n = \frac{-7}{5}$.

This set of vectors is linearly dependent.

Generally any set of three or more two-dimensional vectors will be linearly dependent.

b Again we note that $\boldsymbol{a}$ and $\boldsymbol{b}$ are not parallel.

Let $\boldsymbol{c} = m\boldsymbol{a} + n\boldsymbol{b}$

Then $-1 = 3m + 2n$

$0 = 4m + n$

$1 = -m + 3n$

Solving the first two equations we have $m = \frac{1}{5}$ and $n = \frac{-4}{5}$.

However, when these values are substituted in the third equation,

$-m + 3n = \frac{-13}{5} \neq 1$.

There are no solutions which satisfy the three equations

$\therefore$ the vectors are linearly independent.

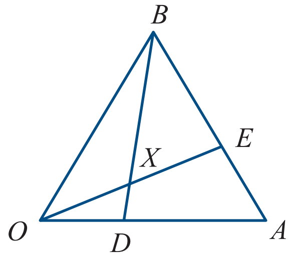
Example 7

Points A and B have position vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ respectively relative to an origin O. The point D is such that $\overrightarrow{OD} = k\overrightarrow{OA}$ and the point E is such that $\overrightarrow{AE} = l\overrightarrow{AB}$. The line segments BD and OE intersect at X. If $\overrightarrow{OX} = \frac{2}{5}\overrightarrow{OE}$ and $\overrightarrow{XB} = \frac{4}{5}\overrightarrow{DB}$

a Express $\overrightarrow{OX}$ in terms of $\boldsymbol{a}$, $\boldsymbol{b}$, k and l.

b Express $\overrightarrow{XB}$ in terms of $\boldsymbol{a}$, $\boldsymbol{b}$, k and l.

c Find k and l.



Solution

a
$$\begin{aligned}\overrightarrow{OX} &= \tfrac{2}{5}\overrightarrow{OE}\\ &= \tfrac{2}{5}(\overrightarrow{OA} + \overrightarrow{AE})\\ &= \tfrac{2}{5}(\overrightarrow{OA} + l\overrightarrow{AB})\\ &= \tfrac{2}{5}(\boldsymbol{a} + l(\overrightarrow{AO} + \overrightarrow{OB}))\\ &= \tfrac{2}{5}(\boldsymbol{a} + l(-\boldsymbol{a} + \boldsymbol{b}))\\ &= \tfrac{2}{5}((1-l)\boldsymbol{a} + l\boldsymbol{b})\end{aligned}$$

b
$$\begin{aligned}\overrightarrow{XB} &= \tfrac{4}{5}\overrightarrow{DB}\\ &= \tfrac{4}{5}(\overrightarrow{DO} + \overrightarrow{OB})\\ &= \tfrac{4}{5}(-\overrightarrow{OD} + \overrightarrow{OB})\\ &= \tfrac{4}{5}(-k\overrightarrow{OA} + \overrightarrow{OB})\\ &= \tfrac{4}{5}(-k\boldsymbol{a} + \boldsymbol{b})\\ &= -\tfrac{4}{5}k\boldsymbol{a} + \tfrac{4}{5}\boldsymbol{b}\end{aligned}$$

c Note $\overrightarrow{XB} = \overrightarrow{XO} + \overrightarrow{OB}$

$$\begin{aligned}\therefore \quad \overrightarrow{XB} &= \tfrac{-2}{5}[(1-l)\boldsymbol{a} + l\boldsymbol{b}] + \boldsymbol{b}\\ &= \tfrac{-2}{5}(1-l)\boldsymbol{a} + \left(1 - \tfrac{2l}{5}\right)\boldsymbol{b}\\ &= \tfrac{2}{5}(l-1)\boldsymbol{a} + \left(1 - \tfrac{2l}{5}\right)\boldsymbol{b}\end{aligned}$$

As $\boldsymbol{a}$ and $\boldsymbol{b}$ are independent vectors, $\overrightarrow{XB}$ has a unique representation in terms of $\boldsymbol{a}$ and $\boldsymbol{b}$.

$$\therefore \quad -\tfrac{4}{5}k\boldsymbol{a} + \tfrac{4}{5}\boldsymbol{b} = \tfrac{2}{5}(l-1)\boldsymbol{a} + \left(1 - \tfrac{2l}{5}\right)\boldsymbol{b}$$

Hence $\tfrac{-4}{5}k = \tfrac{2}{5}(l-1)$ $\quad \boxed{1}$ $\quad$ and $\tfrac{4}{5} = 1 - \tfrac{2l}{5}$ $\quad \boxed{2}$

From equation $\boxed{2}$

$$\frac{2l}{5} = \frac{1}{5}$$

$$\therefore \quad l = \tfrac{1}{2}$$

Substitute in $\boxed{1}$

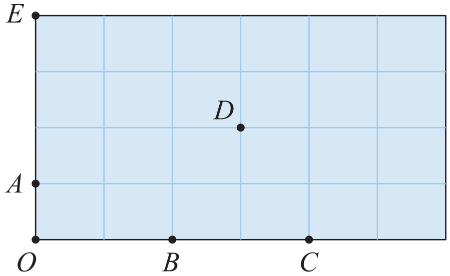
$$-\tfrac{4}{5}k = \tfrac{2}{5}(l-1)$$

$$\therefore \quad \tfrac{4}{5}k = \tfrac{2}{5} \times \tfrac{1}{2}$$

$$\therefore \quad k = \tfrac{1}{4}$$

Exercise 2A

1 In the diagram, $\overrightarrow{OA} = \boldsymbol{a}$ and $\overrightarrow{OB} = \boldsymbol{b}$.



a Find in terms of $\boldsymbol{a}$ and $\boldsymbol{b}$:

i $\overrightarrow{OC}$ **ii** $\overrightarrow{OE}$ **iii** $\overrightarrow{OD}$

iv $\overrightarrow{DC}$ **v** $\overrightarrow{DE}$

b If $|\boldsymbol{a}| = 1$ and $|\boldsymbol{b}| = 2$, find:

i $|\overrightarrow{OC}|$ **ii** $|\overrightarrow{OE}|$ **iii** $|\overrightarrow{OD}|$

2 Using a scale of 1 cm = 20 km/h, draw vectors to represent:

a a car travelling south at 60 km/h

b a car travelling north at 80 km/h

3 If the magnitude of $\boldsymbol{a} = 3$, find the magnitude of:

a $2\boldsymbol{a}$ **b** $\frac{3}{2}\boldsymbol{a}$ **c** $-\frac{1}{2}\boldsymbol{a}$

4 $OA' = A'A'' = A''A''' = A'''A$

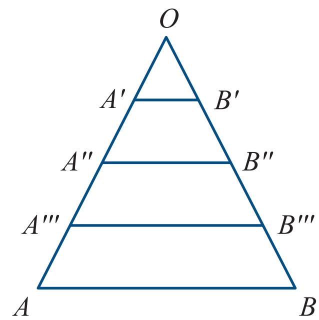
$OB' = B'B'' = B''B''' = B'''B$

If $\overrightarrow{OA} = \boldsymbol{a}$ *and* $\overrightarrow{OB} = \boldsymbol{b}$, find in terms of $\boldsymbol{a}$ and $\boldsymbol{b}$:

a **i** $\overrightarrow{OA'}$ **ii** $\overrightarrow{OB'}$ **iii** $\overrightarrow{A'B'}$

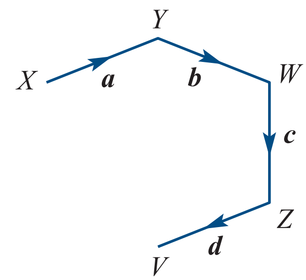
iv $\overrightarrow{AB}$

b **i** $\overrightarrow{OA''}$ **ii** $\overrightarrow{OB''}$ **iii** $\overrightarrow{A''B''}$



5 Find in the terms of $\boldsymbol{a}$, $\boldsymbol{b}$, $\boldsymbol{c}$ and $\boldsymbol{d}$:

a $\overrightarrow{XW}$ **b** $\overrightarrow{VX}$ **c** $\overrightarrow{ZY}$

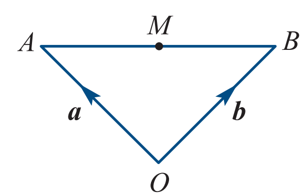


6 The position vectors of two points A and B are $\boldsymbol{a}$ and $\boldsymbol{b}$. Find:

a $\overrightarrow{AB}$

b $\overrightarrow{AM}$ where M is the midpoint of AB

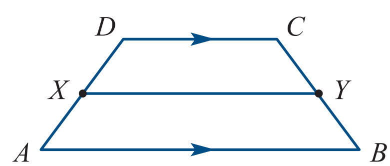
c $\overrightarrow{OM}$



7 $ABCD$ is a trapezium with AB parallel to DC. X and Y are midpoints of AD and BC respectively.

a Express $\overrightarrow{XY}$ in terms of $\boldsymbol{a}$ and $\boldsymbol{b}$ where $\overrightarrow{AB} = \boldsymbol{a}$ and $\overrightarrow{DC} = \boldsymbol{b}$.

b Show that XY is parallel to AB.

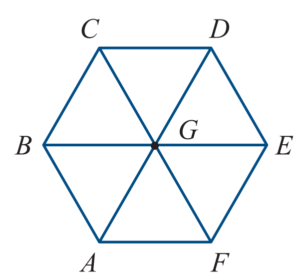


8 $ABCDEF$ is a regular hexagon, centre G. The position vectors of A, B and C relative to an origin O are $\boldsymbol{a}$, $\boldsymbol{b}$ and $\boldsymbol{c}$ respectively.

Express:

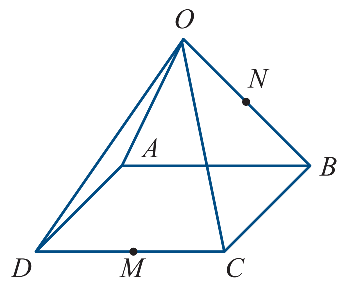
a $\overrightarrow{OG}$ in terms of $\boldsymbol{a}$, $\boldsymbol{b}$ and $\boldsymbol{c}$

b $\overrightarrow{CD}$ in terms of $\boldsymbol{a}$, $\boldsymbol{b}$ and $\boldsymbol{c}$



9 *OABCD* is a right square pyramid.

$\overrightarrow{OA} = \boldsymbol{a}$, $\overrightarrow{OB} = \boldsymbol{b}$, $\overrightarrow{OC} = \boldsymbol{c}$, and $\overrightarrow{OD} = \boldsymbol{d}$



a **i** Find $\overrightarrow{AB}$ in terms of $\boldsymbol{a}$ and $\boldsymbol{b}$.

ii Find $\overrightarrow{DC}$ in terms of $\boldsymbol{c}$ and $\boldsymbol{d}$.

iii Use the fact that $\overrightarrow{AB} = \overrightarrow{DC}$ to find a relationship between $\boldsymbol{a}$, $\boldsymbol{b}$, $\boldsymbol{c}$ and $\boldsymbol{d}$.

b **i** Find $\overrightarrow{BC}$ in terms of $\boldsymbol{b}$ and $\boldsymbol{c}$.

ii Let M be the midpoint of DC and N the midpoint of OB. Find $\overrightarrow{MN}$ in terms of $\boldsymbol{a}$, $\boldsymbol{b}$ and $\boldsymbol{c}$.

10 Determine whether the following sets of vectors are linearly dependent.

a $\boldsymbol{a} = \begin{bmatrix} 4 \\ 1 \\ 3 \end{bmatrix}$, $\boldsymbol{b} = \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}$ and $\boldsymbol{c} = \begin{bmatrix} -4 \\ 2 \\ 6 \end{bmatrix}$

b $\boldsymbol{a} = \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix}$, $\boldsymbol{b} = \begin{bmatrix} 4 \\ 2 \\ 1 \end{bmatrix}$ and $\boldsymbol{c} = \begin{bmatrix} 6 \\ 3 \\ 4 \end{bmatrix}$

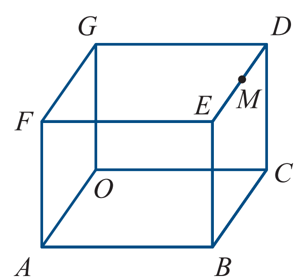
c $\boldsymbol{a} = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$, $\boldsymbol{b} = \begin{bmatrix} 3 \\ -1 \\ 4 \end{bmatrix}$ and $\boldsymbol{c} = \begin{bmatrix} 3 \\ -5 \\ 11 \end{bmatrix}$

11 In the following, $\boldsymbol{a}$ and $\boldsymbol{b}$ are non-zero and non-parallel vectors.

a If $k\boldsymbol{a} + l\boldsymbol{b} = 3\boldsymbol{a} + (1 - l)\boldsymbol{b}$, find the values of l and k.

b If $2(l - 1)\boldsymbol{a} + \left(1 - \dfrac{l}{5}\right)\boldsymbol{b} = \dfrac{-4}{5}k\boldsymbol{a} + 3\boldsymbol{b}$ find the values of l and k.

12 In the cuboid shown $\overrightarrow{OG} = \boldsymbol{g}$, $\overrightarrow{OC} = \boldsymbol{c}$ and $\overrightarrow{OA} = \boldsymbol{a}$. M is the midpoint of ED. Find each of the following in terms of $\boldsymbol{a}$, $\boldsymbol{g}$ and $\boldsymbol{c}$:



a $\overrightarrow{EF}$ **b** $\overrightarrow{AB}$ **c** $\overrightarrow{EM}$

d $\overrightarrow{OM}$ **e** $\overrightarrow{AM}$

13 P, Q and R are points with position vectors $2\boldsymbol{a} - \boldsymbol{b}$, $3\boldsymbol{a} + \boldsymbol{b}$ and $\boldsymbol{a} + 4\boldsymbol{b}$ respectively relative to an origin O where $\boldsymbol{a}$ and $\boldsymbol{b}$ are non-zero, non-parallel vectors. Given that S is the point on OP produced such that $\overrightarrow{OS} = k\overrightarrow{OP}$ and $\overrightarrow{RS} = m\overrightarrow{RQ}$

a Express $\overrightarrow{OS}$ in terms of:

i k, $\boldsymbol{a}$ and $\boldsymbol{b}$

ii m, $\boldsymbol{a}$ and $\boldsymbol{b}$

b Hence evaluate k and m.

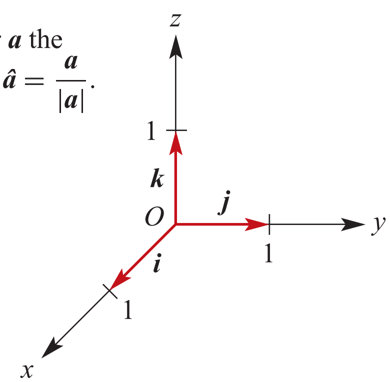
14 The position vectors of points A and B, relative to an origin O, are $\boldsymbol{a}$ and $\boldsymbol{b}$ respectively where $\boldsymbol{a}$ and $\boldsymbol{b}$ are non-zero, non-parallel vectors. The point P is such that $\overrightarrow{OP} = 4\overrightarrow{OB}$. The midpoint of AB is the point Q. The point R is such that $\overrightarrow{OR} = \frac{8}{5}\overrightarrow{OQ}$.

- **a** Find in terms of $\boldsymbol{a}$ and $\boldsymbol{b}$:
  - **i** $\overrightarrow{OQ}$ **ii** $\overrightarrow{OR}$ **iii** $\overrightarrow{AR}$ **iv** $\overrightarrow{RP}$
- **b** Show that R lies on AP and state the ratio $AR : RP$.
- **c** Given that the point S is such that $\overrightarrow{OS} = \lambda\overrightarrow{OQ}$, find the value of λ such that PS is parallel to BA.

15 Let $\boldsymbol{a} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ and $\boldsymbol{b} = \begin{bmatrix} 1 \\ -3 \end{bmatrix}$. Find the value of x and y for which:

- **a** $x\boldsymbol{a} = (y - 1)\boldsymbol{b}$
- **b** $(2 - x)\boldsymbol{a} = 3\boldsymbol{a} + (7 - 3y)\boldsymbol{b}$
- **c** $(5 + 2x)(\boldsymbol{a} + \boldsymbol{b}) = y(3\boldsymbol{a} + 2\boldsymbol{b})$

2.2 Resolution of a vector into rectangular components



A **unit vector** is a vector of magnitude 1. For a given vector $\boldsymbol{a}$ the unit vector with the same direction as $\boldsymbol{a}$ is denoted by $\hat{\boldsymbol{a}}$ and $\hat{\boldsymbol{a}} = \dfrac{\boldsymbol{a}}{|\boldsymbol{a}|}$.

$\boldsymbol{i}$ is a unit vector in the positive direction of the x axis.
$\boldsymbol{j}$ and $\boldsymbol{k}$ are unit vectors in the direction of the y and z axes respectively.

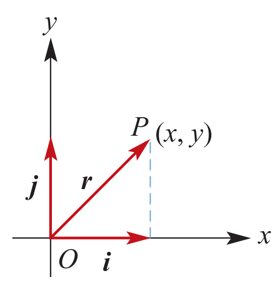
$\boldsymbol{i} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\boldsymbol{j} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ for vectors in two dimensions.

$\boldsymbol{i} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $\boldsymbol{j} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ and $\boldsymbol{k} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ for vectors in three dimensions. It is evident that $\boldsymbol{i}$, $\boldsymbol{j}$ and $\boldsymbol{k}$ are linearly independent.

All vectors in two or three dimensions can be expressed uniquely as a sum of multiples of $\boldsymbol{i}$, $\boldsymbol{j}$ and $\boldsymbol{k}$.

e.g. $$\boldsymbol{r} = \begin{bmatrix} r_1 \\ r_2 \\ r_3 \end{bmatrix} = \begin{bmatrix} r_1 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ r_2 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ r_3 \end{bmatrix} = r_1\boldsymbol{i} + r_2\boldsymbol{j} + r_3\boldsymbol{k}$$

For two dimensions

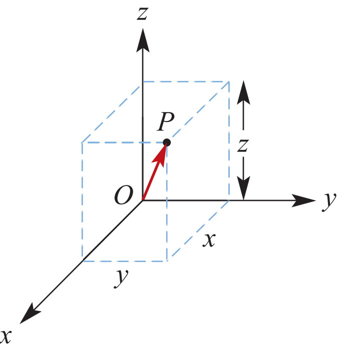


Here $\boldsymbol{r} = x\boldsymbol{i} + y\boldsymbol{j}$

Note $|\boldsymbol{r}| = \sqrt{x^2 + y^2}$

The coordinates of P are (x, y)

For three dimensions



$$\overrightarrow{OP} = x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k}$$

and $|\overrightarrow{OP}| = \sqrt{x^2 + y^2 + z^2}$

The coordinates of P are (x, y, z)

The basic operations on vectors in $\boldsymbol{i}, \boldsymbol{j}$ and $\boldsymbol{k}$ notation can be summarised as follows.

Let $\boldsymbol{a} = a_1\boldsymbol{i} + a_2\boldsymbol{j} + a_3\boldsymbol{k}$, and $\boldsymbol{b} = b_1\boldsymbol{i} + b_2\boldsymbol{j} + b_3\boldsymbol{k}$

Then $\boldsymbol{a} + \boldsymbol{b} = (a_1 + b_1)\boldsymbol{i} + (a_2 + b_2)\boldsymbol{j} + (a_3 + b_3)\boldsymbol{k}$

$\boldsymbol{a} - \boldsymbol{b} = (a_1 - b_1)\boldsymbol{i} + (a_2 - b_2)\boldsymbol{j} + (a_3 - b_3)\boldsymbol{k}$

and $m\boldsymbol{a} = ma_1\boldsymbol{i} + ma_2\boldsymbol{j} + ma_3\boldsymbol{k}$ for scalar m

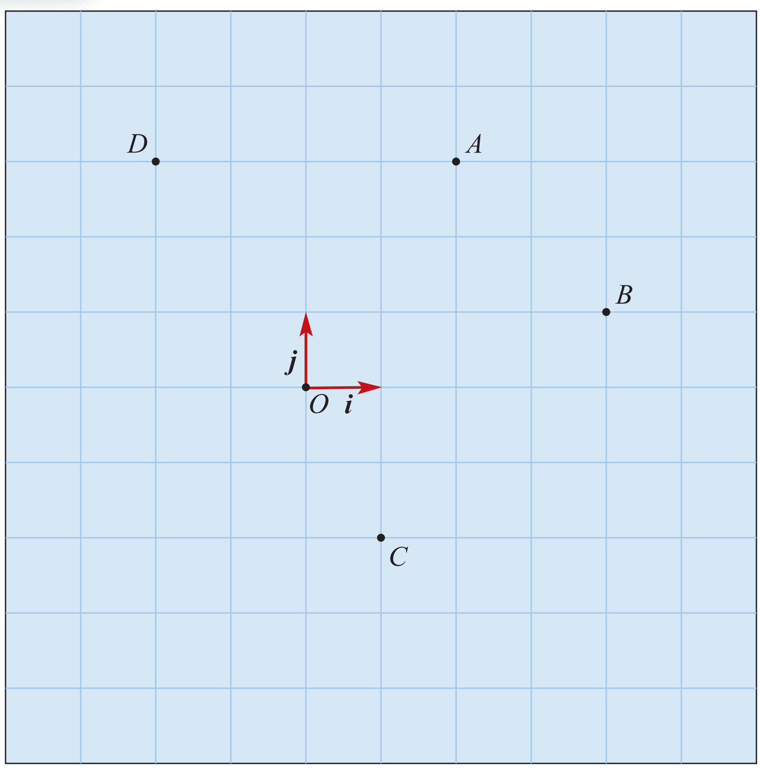
Equivalence

If $\boldsymbol{a} = \boldsymbol{b}$ then $a_1 = b_1$, $a_2 = b_2$ and $a_3 = b_3$

Magnitude

$|\boldsymbol{a}| = \sqrt{a_1^2 + a_2^2 + a_3^2}$

Example 8



a Using the vectors $\boldsymbol{i}$ and $\boldsymbol{j}$, give the vectors:
i $\overrightarrow{OA}$ **ii** $\overrightarrow{OB}$ **iii** $\overrightarrow{OC}$ **iv** $\overrightarrow{OD}$

b Using the vectors $\boldsymbol{i}$ and $\boldsymbol{j}$, give the vectors:
i $\overrightarrow{AB}$ **ii** $\overrightarrow{BC}$

c Find the magnitude of the vectors:
i $\overrightarrow{AB}$ **ii** $\overrightarrow{BC}$

Solution

a **i** $\overrightarrow{OA} = 2\boldsymbol{i} + 3\boldsymbol{j}$ **ii** $\overrightarrow{OB} = 4\boldsymbol{i} + \boldsymbol{j}$

iii $\overrightarrow{OC} = \boldsymbol{i} - 2\boldsymbol{j}$ **iv** $\overrightarrow{OD} = -2\boldsymbol{i} + 3\boldsymbol{j}$

b **i**
$$\begin{aligned}\overrightarrow{AB} &= \overrightarrow{AO} + \overrightarrow{OB}\\ &= -2\boldsymbol{i} - 3\boldsymbol{j} + 4\boldsymbol{i} + \boldsymbol{j}\\ &= 2\boldsymbol{i} - 2\boldsymbol{j}\end{aligned}$$

ii
$$\begin{aligned}\overrightarrow{BC} &= \overrightarrow{BO} + \overrightarrow{OC}\\ &= -4\boldsymbol{i} - \boldsymbol{j} + \boldsymbol{i} - 2\boldsymbol{j}\\ &= -3\boldsymbol{i} - 3\boldsymbol{j}\end{aligned}$$

c **i**
$$\begin{aligned}|\overrightarrow{AB}| &= \sqrt{4+4}\\ &= \sqrt{8}\\ &= 2\sqrt{2}\end{aligned}$$

ii
$$\begin{aligned}|\overrightarrow{BC}| &= \sqrt{(-3)^2 + (-3)^2}\\ &= \sqrt{18}\\ &= 3\sqrt{2}\end{aligned}$$

Example 9

Let $\boldsymbol{a} = \boldsymbol{i} + 2\boldsymbol{j} - \boldsymbol{k}$, $\boldsymbol{b} = 3\boldsymbol{i} - 2\boldsymbol{k}$ and $\boldsymbol{c} = 2\boldsymbol{i} + \boldsymbol{j} + \boldsymbol{k}$.
Find: **a** $\boldsymbol{a} + \boldsymbol{b}$ **b** $\boldsymbol{a} - 2\boldsymbol{b}$ **c** $\boldsymbol{a} + \boldsymbol{b} + \boldsymbol{c}$ **d** $|\boldsymbol{a}|$

Solution

a $\boldsymbol{a} + \boldsymbol{b} = (\boldsymbol{i} + 2\boldsymbol{j} - \boldsymbol{k}) + (3\boldsymbol{i} - 2\boldsymbol{k}) = 4\boldsymbol{i} + 2\boldsymbol{j} - 3\boldsymbol{k}$

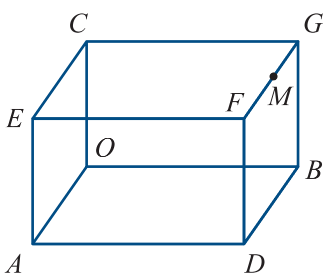
b $\boldsymbol{a} - 2\boldsymbol{b} = (\boldsymbol{i} + 2\boldsymbol{j} - \boldsymbol{k}) - 2(3\boldsymbol{i} - 2\boldsymbol{k}) = -5\boldsymbol{i} + 2\boldsymbol{j} + 3\boldsymbol{k}$

c $\boldsymbol{a} + \boldsymbol{b} + \boldsymbol{c} = (\boldsymbol{i} + 2\boldsymbol{j} - \boldsymbol{k}) + (3\boldsymbol{i} - 2\boldsymbol{k}) + (2\boldsymbol{i} + \boldsymbol{j} + \boldsymbol{k}) = 6\boldsymbol{i} + 3\boldsymbol{j} - 2\boldsymbol{k}$

d $|\boldsymbol{a}| = \sqrt{1^2 + 2^2 + (-1)^2} = \sqrt{6}$

Example 10

A cuboid is labelled as shown.
$\overrightarrow{OA} = 3\boldsymbol{i}$, $\overrightarrow{OB} = 5\boldsymbol{j}$, $\overrightarrow{OC} = 4\boldsymbol{k}$



a Find in terms of $\boldsymbol{i}$, $\boldsymbol{j}$ and $\boldsymbol{k}$:
i $\overrightarrow{DB}$ **ii** $\overrightarrow{OD}$ **iii** $\overrightarrow{DF}$ **iv** $\overrightarrow{OF}$

b Find $|\overrightarrow{OF}|$

c If M is the midpoint of FG find
i $\overrightarrow{OM}$ **ii** $|\overrightarrow{OM}|$

Solution

a **i** $\overrightarrow{DB} = \overrightarrow{AO} = -\overrightarrow{OA} = -3\boldsymbol{i}$

ii $\overrightarrow{OD} = \overrightarrow{OB} + \overrightarrow{BD} = 5\boldsymbol{j} + \overrightarrow{OA}$
$= 5\boldsymbol{j} + 3\boldsymbol{i}$
$= 3\boldsymbol{i} + 5\boldsymbol{j}$

iii $\overrightarrow{DF} = \overrightarrow{OC} = 4\boldsymbol{k}$

iv $\overrightarrow{OF} = \overrightarrow{OD} + \overrightarrow{DF}$
$= 3\boldsymbol{i} + 5\boldsymbol{j} + 4\boldsymbol{k}$

b $|\overrightarrow{OF}| = \sqrt{9 + 25 + 16}$
$= \sqrt{50}$
$= 5\sqrt{2}$

c **i** $\overrightarrow{OM} = \overrightarrow{OD} + \overrightarrow{DF} + \overrightarrow{FM}$
$= 3\boldsymbol{i} + 5\boldsymbol{j} + 4\boldsymbol{k} + \frac{1}{2}(-\overrightarrow{GF})$
$= 3\boldsymbol{i} + 5\boldsymbol{j} + 4\boldsymbol{k} + \frac{1}{2}(-3\boldsymbol{i})$
$= \frac{3}{2}\boldsymbol{i} + 5\boldsymbol{j} + 4\boldsymbol{k}$

ii $|\overrightarrow{OM}| = \sqrt{\frac{9}{4} + 25 + 16}$
$= \frac{1}{2}\sqrt{9 + 100 + 64}$
$= \frac{1}{2}\sqrt{173}$

Example 11

If $\boldsymbol{a} = x\boldsymbol{i} + 3\boldsymbol{j}$ and $\boldsymbol{b} = 8\boldsymbol{i} + 2y\boldsymbol{j}$ and $\boldsymbol{a} + \boldsymbol{b} = -2\boldsymbol{i} + 4\boldsymbol{j}$, find the values of x and y.

Solution

$\boldsymbol{a} + \boldsymbol{b} = (x + 8)\boldsymbol{i} + (2y + 3)\boldsymbol{j} = -2\boldsymbol{i} + 4\boldsymbol{j}$

$\therefore$ $x + 8 = -2$ and $2y + 3 = 4$

i.e. $x = -10$ and $y = \frac{1}{2}$

Example 12

a Show that the vectors $\boldsymbol{a} = 8\boldsymbol{i} + 7\boldsymbol{j} + 3\boldsymbol{k}$, $\boldsymbol{b} = \boldsymbol{i} - \boldsymbol{j} + 3\boldsymbol{k}$ and $\boldsymbol{c} = 2\boldsymbol{i} + 3\boldsymbol{j} - \boldsymbol{k}$ are linearly dependent.

b Show that the vectors $\boldsymbol{a} = 8\boldsymbol{i} + 7\boldsymbol{j} + 3\boldsymbol{k}$, $\boldsymbol{b} = \boldsymbol{i} - \boldsymbol{j} + 3\boldsymbol{k}$ and $\boldsymbol{c} = 2\boldsymbol{i} + 3\boldsymbol{j} + \boldsymbol{k}$ are linearly independent.

Solution

a Vectors $\boldsymbol{b}$ and $\boldsymbol{c}$ are obviously not parallel. Constants k and l are found for $\boldsymbol{a} = k\boldsymbol{b} + l\boldsymbol{c}$.

Consider

$$8\boldsymbol{i} + 7\boldsymbol{j} + 3\boldsymbol{k} = k(\boldsymbol{i} - \boldsymbol{j} + 3\boldsymbol{k}) + l(2\boldsymbol{i} + 3\boldsymbol{j} - \boldsymbol{k})$$

This implies

$$8 = k + 2l \quad \boxed{1} \qquad 7 = -k + 3l \quad \boxed{2} \qquad 3 = 3k - l \quad \boxed{3}$$

Add $\boxed{1}$ and $\boxed{2}$

$$15 = 5l$$

which implies

$$l = 3$$

Substitute in $\boxed{1}$ to show $k = 2$
This must be verified for $\boxed{3}$
When $k = 2$ and $l = 3$, $3k - l = 3 \times 2 - 3 = 3$
Therefore

$$\boldsymbol{a} = 2\boldsymbol{b} + 3\boldsymbol{c} \quad \text{or} \quad \boldsymbol{a} - 2\boldsymbol{b} - 3\boldsymbol{c} = 0$$

Vectors $\boldsymbol{a}$, $\boldsymbol{b}$ and $\boldsymbol{c}$ are linearly dependent.

b The equation $\boxed{3}$ becomes

$$3 = 3k + l$$

Equations $\boxed{1}$ and $\boxed{2}$ are unchanged.
However, $3k + l = 9$ when values of k and l are substituted.
No solutions exist.
Therefore the three vectors are linearly independent.

Example 13

Let $A = (2, -3)$, $B = (1, 4)$ $C = (-1, -3)$. O is the origin. Find:

a **i** $\overrightarrow{OA}$ **ii** $\overrightarrow{AB}$ **iii** $\overrightarrow{BC}$

b F such that $\overrightarrow{OF} = \frac{1}{2}\overrightarrow{OA}$

c G such that $\overrightarrow{AG} = 3\overrightarrow{BC}$

Solution

a **i** $\overrightarrow{OA} = 2\boldsymbol{i} - 3\boldsymbol{j}$

ii
$$\begin{aligned}\overrightarrow{AB} &= \overrightarrow{AO} + \overrightarrow{OB}\\ &= -2\boldsymbol{i} + 3\boldsymbol{j} + \boldsymbol{i} + 4\boldsymbol{j}\\ &= -\boldsymbol{i} + 7\boldsymbol{j}\end{aligned}$$

iii
$$\begin{aligned}\overrightarrow{BC} &= \overrightarrow{BO} + \overrightarrow{OC}\\ &= -\boldsymbol{i} - 4\boldsymbol{j} + -\boldsymbol{i} - 3\boldsymbol{j}\\ &= -2\boldsymbol{i} - 7\boldsymbol{j}\end{aligned}$$

b
$$\begin{aligned}\overrightarrow{OF} = \tfrac{1}{2}\overrightarrow{OA} &= \tfrac{1}{2}(2\boldsymbol{i} - 3\boldsymbol{j})\\ &= \boldsymbol{i} - \frac{3}{2}\boldsymbol{j}\end{aligned}$$

$\therefore \quad F = (1, -1.5)$

c $\overrightarrow{AG} = 3\overrightarrow{BC} = 3(-2\boldsymbol{i} - 7\boldsymbol{j}) = -6\boldsymbol{i} - 21\boldsymbol{j}$

$$\begin{aligned}\therefore \quad \overrightarrow{OG} &= \overrightarrow{OA} + \overrightarrow{AG}\\ &= 2\boldsymbol{i} - 3\boldsymbol{j} - 6\boldsymbol{i} - 21\boldsymbol{j}\\ &= -4\boldsymbol{i} - 24\boldsymbol{j}\end{aligned}$$

$\therefore \quad G = (-4, -24)$

Example 14

Let $A = (2, -4, 5)$ and $B = (5, 1, 7)$. Find M, the midpoint of AB.

Solution

$\overrightarrow{OA} = 2\boldsymbol{i} - 4\boldsymbol{j} + 5\boldsymbol{k}$ and $\overrightarrow{OB} = 5\boldsymbol{i} + \boldsymbol{j} + 7\boldsymbol{k}$

Therefore $\overrightarrow{AB} = \overrightarrow{AO} + \overrightarrow{OB}$

$= -2\boldsymbol{i} + 4\boldsymbol{j} - 5\boldsymbol{k} + 5\boldsymbol{i} + \boldsymbol{j} + 7\boldsymbol{k}$

$= 3\boldsymbol{i} + 5\boldsymbol{j} + 2\boldsymbol{k}$

and $\overrightarrow{AM} = \frac{1}{2}(3\boldsymbol{i} + 5\boldsymbol{j} + 2\boldsymbol{k})$

$\therefore$ $\overrightarrow{OM} = \overrightarrow{OA} + \overrightarrow{AM} = 2\boldsymbol{i} - 4\boldsymbol{j} + 5\boldsymbol{k} + \frac{3}{2}\boldsymbol{i} + \frac{5}{2}\boldsymbol{j} + \boldsymbol{k}$

$= \frac{7}{2}\boldsymbol{i} - \frac{3}{2}\boldsymbol{j} + 6\boldsymbol{k}$

and $M = \left(\frac{7}{2}, \frac{-3}{2}, 6\right)$

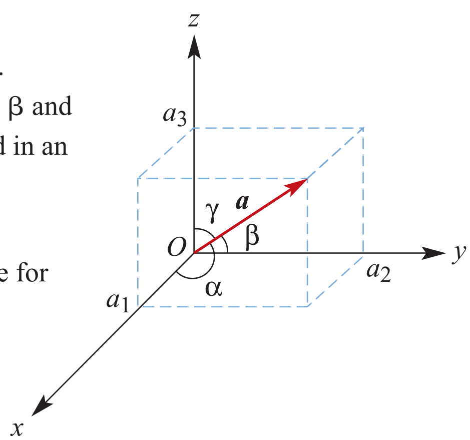
Angles made by a vector with the axes

The *direction* of a vector can be given by the angles which the vector makes with the $\boldsymbol{i}$, $\boldsymbol{j}$ and $\boldsymbol{k}$ directions.

If the vector $\boldsymbol{a} = a_1\boldsymbol{i} + a_2\boldsymbol{j} + a_3\boldsymbol{k}$ makes angles α, β and γ respectively with the positive directions (measured in an anticlockwise sense) of the x, y and z axes, then

$\cos\alpha = \dfrac{a_1}{|\boldsymbol{a}|}$ $\quad \cos\beta = \dfrac{a_2}{|\boldsymbol{a}|}$ $\quad \cos\gamma = \dfrac{a_3}{|\boldsymbol{a}|}$

The derivation of these results is left as an exercise for the reader.



Example 15

$\boldsymbol{a} = 2\boldsymbol{i} - \boldsymbol{j}$ and $\boldsymbol{b} = \boldsymbol{i} + 4\boldsymbol{j} - 3\boldsymbol{k}$

For each of the vectors above, find:

a its magnitude **b** the angle the vector makes with the y axis

Solution

a $|\boldsymbol{a}| = \sqrt{2^2 + (-1)^2} = \sqrt{5}$ $\quad |\boldsymbol{b}| = \sqrt{1^2 + 4^2 + (-3)^2} = \sqrt{26}$

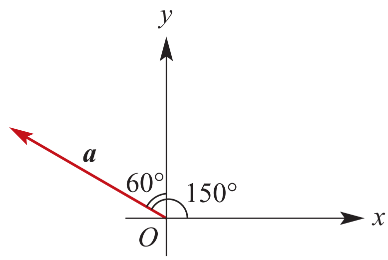
b The angle that $\boldsymbol{a}$ makes with the y axis is $\cos^{-1}\left(\dfrac{-1}{\sqrt{5}}\right) = 116.57°$ correct to two decimal places.

The angle that $\boldsymbol{b}$ makes with the y axis is $\cos^{-1}\left(\dfrac{4}{\sqrt{26}}\right) = 38.33°$ correct to two decimal places.

Example 16

A position vector in two dimensions has magnitude 5 and its direction, measured anticlockwise from the x axis, is 150°. Express this vector in $\boldsymbol{i} - \boldsymbol{j}$ form.

Solution



Let the vector be $\boldsymbol{a} = a_1\boldsymbol{i} + a_2\boldsymbol{j}$

$\boldsymbol{a}$ makes an angle of 150° with the x axis and 60° with the y axis.

$$\therefore \quad \cos 150^\circ = \frac{a_1}{|\boldsymbol{a}|} \text{ and } \cos 60^\circ = \frac{a_2}{|\boldsymbol{a}|}$$

as $|\boldsymbol{a}| = 5$ then $a_1 = |\boldsymbol{a}| \cos 150^\circ = \dfrac{-5\sqrt{3}}{2}$

$$a_2 = |\boldsymbol{a}| \cos 60^\circ = 2.5$$

$$\therefore \quad \boldsymbol{a} = \frac{-5\sqrt{3}}{2}\boldsymbol{i} + 2.5\boldsymbol{j}$$

Example 17

Let $\boldsymbol{i}$ be a unit vector in the east direction and $\boldsymbol{j}$ be a unit vector in the north direction.

a Show that the unit vector in the direction N60°W is $-\dfrac{\sqrt{3}}{2}\boldsymbol{i} + \dfrac{1}{2}\boldsymbol{j}$.

b If a car drives 3 km in a direction N60°W, find the position vector of the car with respect to its starting point.

c The car then drives 6.5 km due north. Find:

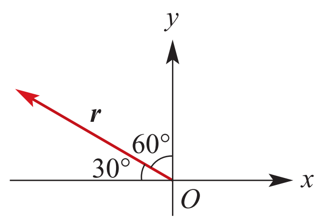
i the position vector of the car

ii the distance of the car from the starting point

iii the bearing of the car from the starting point

Solution

a Let $\boldsymbol{r}$ denote the unit vector in the direction N60°W.



Then $\boldsymbol{r} = -\cos 30^\circ\boldsymbol{i} + \cos 60^\circ\boldsymbol{j}$

$$= -\frac{\sqrt{3}}{2}\boldsymbol{i} + \frac{1}{2}\boldsymbol{j}$$

Note: $|\boldsymbol{r}| = 1$

b The position vector $= 3\boldsymbol{r}$

$$= 3\left(-\frac{\sqrt{3}}{2}\boldsymbol{i} + \frac{1}{2}\boldsymbol{j}\right)$$

$$= -\frac{3\sqrt{3}}{2}\boldsymbol{i} + \frac{3}{2}\boldsymbol{j}$$

c **i** Let $\boldsymbol{r}'$ denote the position vector.

$$\boldsymbol{r}' = -\frac{3\sqrt{3}}{2}\boldsymbol{i} + \frac{3}{2}\boldsymbol{j} + \frac{13}{2}\boldsymbol{j}$$
$$= -\frac{3\sqrt{3}}{2}\boldsymbol{i} + 8\boldsymbol{j}$$

ii and $|\boldsymbol{r}'| = \sqrt{\frac{9 \times 3}{4} + 64}$

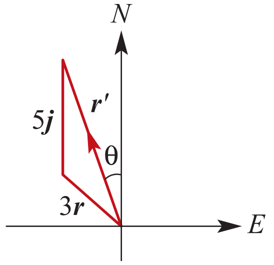
$$= \sqrt{\frac{27 + 256}{4}}$$
$$= \frac{1}{2}\sqrt{283}$$

iii $\boldsymbol{r}' = -\frac{3\sqrt{3}}{2}\boldsymbol{i} + 8\boldsymbol{j}$

$$\therefore \tan\theta = \frac{3\sqrt{3}}{16}$$

and $\theta = \tan^{-1}\left(\frac{3\sqrt{3}}{16}\right)$

≈ 18 where θ is in degrees

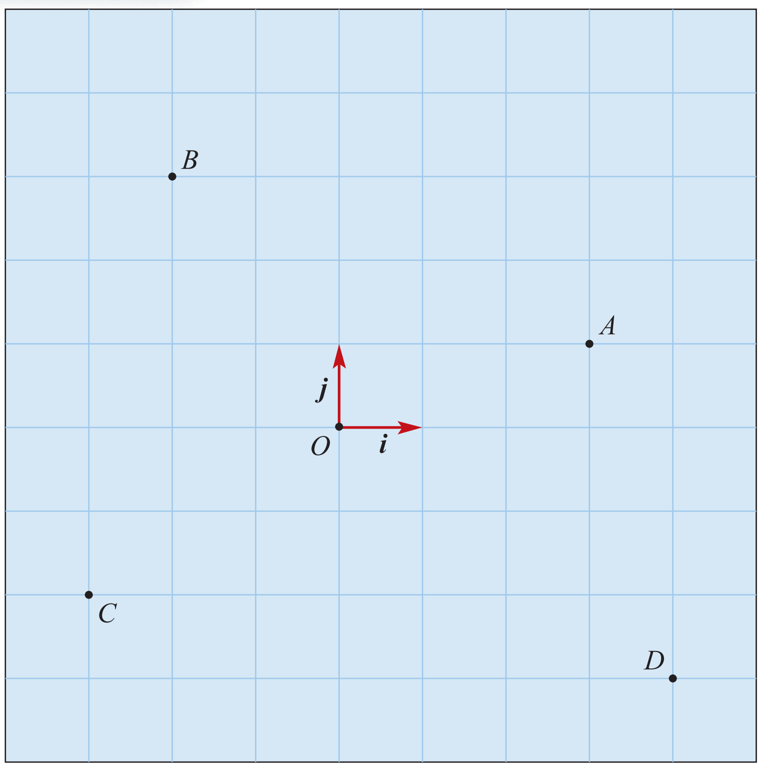


The bearing is 342° (bearing is given to the nearest degree).

Exercise 2B

Skillsheet TEACHER

1



a Give each of the following vectors using $\boldsymbol{i} - \boldsymbol{j}$ notation:

i $\overrightarrow{OA}$ **ii** $\overrightarrow{OB}$ **iii** $\overrightarrow{OC}$ **iv** $\overrightarrow{OD}$

b Find each of the following:

i $\overrightarrow{AB}$ **ii** $\overrightarrow{CD}$ **iii** $\overrightarrow{DA}$

c Find the magnitude of each of the following:

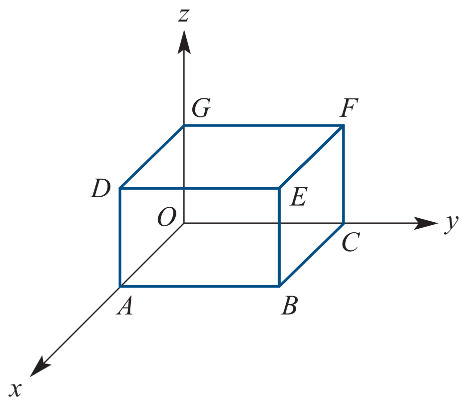
i $\overrightarrow{OA}$ **ii** $\overrightarrow{AB}$ **iii** $\overrightarrow{DA}$

2 $\boldsymbol{a} = 2\boldsymbol{i} + 2\boldsymbol{j} - \boldsymbol{k} \quad \boldsymbol{b} = -\boldsymbol{i} + 2\boldsymbol{j} + \boldsymbol{k} \quad \boldsymbol{c} = 4\boldsymbol{k}$

Find: **a** $\boldsymbol{a} + \boldsymbol{b}$ **b** $2\boldsymbol{a} + \boldsymbol{c}$ **c** $\boldsymbol{a} + 2\boldsymbol{b} - \boldsymbol{c}$

d $\boldsymbol{c} - 4\boldsymbol{a}$ **e** $|\boldsymbol{b}|$ **f** $|\boldsymbol{c}|$

3 *OABCDEFG* is a cuboid set on cartesian axes.
$\overrightarrow{OA} = 5\boldsymbol{i}$, $\overrightarrow{OC} = 2\boldsymbol{j}$ and $\overrightarrow{OG} = 3\boldsymbol{k}$



a Find: **i** $\overrightarrow{BC}$ **ii** $\overrightarrow{CF}$

iii $\overrightarrow{AB}$ **iv** $\overrightarrow{OD}$

v $\overrightarrow{OE}$ **vi** $\overrightarrow{GE}$

vii $\overrightarrow{EC}$ **viii** $\overrightarrow{DB}$

ix $\overrightarrow{DC}$ **x** $\overrightarrow{BG}$

xi $\overrightarrow{GB}$ **xii** $\overrightarrow{FA}$

b Evaluate: **i** $|\overrightarrow{OD}|$ **ii** $|\overrightarrow{OE}|$ **iii** $|\overrightarrow{GE}|$

c *M* is the midpoint of *CB*. Find:

i $\overrightarrow{CM}$ **ii** $\overrightarrow{OM}$ **iii** $\overrightarrow{DM}$

d *N* is a point on *FG*, such that $\overrightarrow{FN} = 2\overrightarrow{NG}$. Find:

i $\overrightarrow{FN}$ **ii** $\overrightarrow{GN}$ **iii** $\overrightarrow{ON}$ **iv** $\overrightarrow{NA}$ **v** $\overrightarrow{NM}$

e Evaluate: **i** $|\overrightarrow{NM}|$ **ii** $|\overrightarrow{DM}|$ **iii** $|\overrightarrow{AN}|$

4 Find the values of *x* and *y* if:

i $\boldsymbol{a} = 4\boldsymbol{i} - \boldsymbol{j}, \boldsymbol{b} = x\boldsymbol{i} + 3y\boldsymbol{j}, \boldsymbol{a} + \boldsymbol{b} = 7\boldsymbol{i} - 2\boldsymbol{j}$

ii $\boldsymbol{a} = x\boldsymbol{i} + 3\boldsymbol{j}, \boldsymbol{b} = -2\boldsymbol{i} + 5y\boldsymbol{j}, \boldsymbol{a} - \boldsymbol{b} = 6\boldsymbol{i} + \boldsymbol{j}$

iii $\boldsymbol{a} = 6\boldsymbol{i} + y\boldsymbol{j}, \boldsymbol{b} = x\boldsymbol{i} - 4\boldsymbol{j}, \boldsymbol{a} + 2\boldsymbol{b} = 3\boldsymbol{i} - \boldsymbol{j}$

5 *A*, *B*, *C* and *D* are points defined respectively by the position vectors.
$\boldsymbol{a} = \boldsymbol{i} + 3\boldsymbol{j} - 2\boldsymbol{k}$, $\boldsymbol{b} = 5\boldsymbol{i} + \boldsymbol{j} - 6\boldsymbol{k}$, $\boldsymbol{c} = 5\boldsymbol{j} + 3\boldsymbol{k}$ and $\boldsymbol{d} = 2\boldsymbol{i} + 4\boldsymbol{j} + \boldsymbol{k}$

a Find: **i** $\overrightarrow{AB}$ **ii** $\overrightarrow{BC}$ **iii** $\overrightarrow{CD}$ **iv** $\overrightarrow{DA}$

b Evaluate: **i** $|\overrightarrow{AC}|$ **ii** $|\overrightarrow{BD}|$

c Find the two parallel vectors in **a**.

6 *A* and *B* are points defined respectively by position vectors $\boldsymbol{a} = \boldsymbol{i} + \boldsymbol{j} - 5\boldsymbol{k}$ and $\boldsymbol{b} = 3\boldsymbol{i} - 2\boldsymbol{j} - \boldsymbol{k}$. *M* is a point on *AB* such that *AM* : *MB* = 4 : 1.

a Find: **i** $\overrightarrow{AB}$ **ii** $\overrightarrow{AM}$ **iii** $\overrightarrow{OM}$

b Find the coordinates of *M*.

7 **a** Show that the vectors $\boldsymbol{a} = 8\boldsymbol{i} + 5\boldsymbol{j} + 2\boldsymbol{k}$, $\boldsymbol{b} = 4\boldsymbol{i} - 3\boldsymbol{j} + \boldsymbol{k}$ and $\boldsymbol{c} = 2\boldsymbol{i} - \boldsymbol{j} + \frac{1}{2}\boldsymbol{k}$ are linearly dependent.

b Show that the vectors $\boldsymbol{a} = 8\boldsymbol{i} + 5\boldsymbol{j} + 2\boldsymbol{k}$, $\boldsymbol{b} = 4\boldsymbol{i} - 3\boldsymbol{j} + \boldsymbol{k}$ and $\boldsymbol{c} = 2\boldsymbol{i} - \boldsymbol{j} + 2\boldsymbol{k}$ are linearly independent.

8 The vectors $\boldsymbol{a} = 2\boldsymbol{i} - 3\boldsymbol{j} + \boldsymbol{k}$, $\boldsymbol{b} = 4\boldsymbol{i} + 3\boldsymbol{j} - 2\boldsymbol{k}$ and $\boldsymbol{c} = 2\boldsymbol{i} - 4\boldsymbol{j} + x\boldsymbol{k}$ are linearly dependent. Find the value of *x*.

9 $A = (2, 1)$, $B = (1, -3)$, $C(-5, 2)$ and $D = (3, 5)$, and O is the origin.

a Find: **i** $\overrightarrow{OA}$ **ii** $\overrightarrow{AB}$ **iii** $\overrightarrow{BC}$ **iv** $\overrightarrow{BD}$

b Show that $\overrightarrow{AB}$ and $\overrightarrow{BD}$ are parallel.

c What can be said of the points A, B and D?

10 Let $A = (1, 4, -4)$, $B = (2, 3, 1)$, $C = (0, -1, 4)$ and $D = (4, 5, 6)$.

a Find: **i** $\overrightarrow{OB}$ **ii** $\overrightarrow{AC}$ **iii** $\overrightarrow{BD}$ **iv** $\overrightarrow{CD}$

b Show that $\overrightarrow{OB}$ and $\overrightarrow{CD}$ are parallel.

11 Let $A = (1, 4, -2)$, $B = (3, 3, 0)$, $C = (2, 5, 3)$ and $D = (0, 6, 1)$.

a Find: **i** $\overrightarrow{AB}$ **ii** $\overrightarrow{BC}$ **iii** $\overrightarrow{CD}$ **iv** $\overrightarrow{DA}$

b Describe the quadrilateral $ABCD$.

12 Let $A = (5, 1)$, $B = (0, 4)$, $C = (-1, 0)$. Find:

a D, such that $\overrightarrow{AB} = \overrightarrow{CD}$

b E, such that $\overrightarrow{AE} = -\overrightarrow{BC}$

c G, such that $\overrightarrow{AB} = 2\overrightarrow{GC}$

13 $ABCD$ is a parallelogram. $A = (2, 1)$, $B = (-5, 4)$, $C = (1, 7)$ and $D = (x, y)$.

a Find: **i** $\overrightarrow{BC}$ **ii** $\overrightarrow{AD}$ (in terms of x and y)

b Hence find the coordinates of D.

14 **a** Let $A = (1, 4, 3)$ and $B = (2, -1, 5)$. Find M, the midpoint of AB.

b Use a similar method to find M, the midpoint of XY where X and Y have coordinates (x_1, y_1, z_1) and (x_2, y_2, z_2) respectively.

15 Let $A = (5, 4, 1)$ and $B = (3, 1, -4)$. M is a point on AB, such that $AM = 4MB$. Find M.

16 Let $A = (4, -3)$ and $B = (7, 1)$. Find N, such that $\overrightarrow{AN} = 3\overrightarrow{BN}$.

17 Find the point P on the line $x - 6y = 11$ such that $\overrightarrow{OP}$ is parallel to the vector $3\boldsymbol{i} + \boldsymbol{j}$.

18 A, B, C and D are points which represent the position vectors $\boldsymbol{a}$, $\boldsymbol{b}$, $\boldsymbol{c}$ and $\boldsymbol{d}$ respectively. Show that, if $ABCD$ is a parallelogram, then $\boldsymbol{a} + \boldsymbol{c} = \boldsymbol{b} + \boldsymbol{d}$.

19 $\boldsymbol{a} = 2\boldsymbol{i} + 2\boldsymbol{j}$, $\boldsymbol{b} = 3\boldsymbol{i} - \boldsymbol{j}$ and $\boldsymbol{c} = 4\boldsymbol{i} + 5\boldsymbol{j}$

Find:

a **i** $\frac{1}{2}a$ **ii** $\boldsymbol{b} - \boldsymbol{c}$ **iii** $3\boldsymbol{b} - \boldsymbol{a} - 2\boldsymbol{c}$

b values k and l such that $k\boldsymbol{a} + l\boldsymbol{b} = \boldsymbol{c}$

20 $\boldsymbol{a} = 5\boldsymbol{i} + \boldsymbol{j} - 4\boldsymbol{k}$, $\boldsymbol{b} = 8\boldsymbol{i} - 2\boldsymbol{j} + \boldsymbol{k}$ and $\boldsymbol{c} = \boldsymbol{i} - 7\boldsymbol{j} + 6\boldsymbol{k}$

Find:

a **i** $2\boldsymbol{a} - \boldsymbol{b}$ **ii** $\boldsymbol{a} + \boldsymbol{b} + \boldsymbol{c}$ **iii** $0.5\boldsymbol{a} + 0.4\boldsymbol{b}$

b values for k and l such that $k\boldsymbol{a} + l\boldsymbol{b} = \boldsymbol{c}$

21 Let $\boldsymbol{a} = 5\boldsymbol{i} + 2\boldsymbol{j}$, $\boldsymbol{b} = 2\boldsymbol{i} - 3\boldsymbol{j}$, $\boldsymbol{c} = 2\boldsymbol{i} + \boldsymbol{j} + \boldsymbol{k}$ and $\boldsymbol{d} = -\boldsymbol{i} + 4\boldsymbol{j} + 2\boldsymbol{k}$

a Find: **i** $|\boldsymbol{a}|$ **ii** $|\boldsymbol{b}|$ **iii** $|\boldsymbol{a} + 2\boldsymbol{b}|$ **iv** $|\boldsymbol{c} - \boldsymbol{d}|$

b Find correct to two decimal places the angle which each of the following vectors makes with the positive direction of the x axis:

i $\boldsymbol{a}$ **ii** $\boldsymbol{a} + 2\boldsymbol{b}$ **iii** $\boldsymbol{c} - \boldsymbol{d}$

22 In the table below, the magnitudes of vectors in two dimensions and the angle they each make with the x axis (measured anticlockwise) are given. Express each of the vectors in $\boldsymbol{i} - \boldsymbol{j}$ form, correct to two decimal places.

	Magnitude	Angle
a	10	110°
b	8.5	250°
c	6	40°
d	5	300°

23 In the table below, the magnitudes of vectors in three dimensions and the angle they each make with the x, y and z axes are given, correct to two decimal places. Express each of the vectors in $\boldsymbol{i} - \boldsymbol{j} - \boldsymbol{k}$ form, correct to two decimal places.

	Magnitude	Angle with x axis	Angle with y axis	Angle with z axis
a	10	130°	80°	41.75°
b	8	50°	54.52°	120°
c	7	28.93°	110°	110°
d	12	121.43°	35.5°	75.2°

24 Show that, if a vector in three dimensions makes angles α, β and γ respectively with the x, y and z axes, then $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$.

25 A, B and C are points defined respectively by position vectors $\boldsymbol{a} = -2\boldsymbol{i} + \boldsymbol{j} + 5\boldsymbol{k}$, $\boldsymbol{b} = 2\boldsymbol{j} + 3\boldsymbol{k}$, and $\boldsymbol{c} = -2\boldsymbol{i} + 4\boldsymbol{j} + 5\boldsymbol{k}$.

a Show that ΔABC is isosceles.

b Find $\overrightarrow{OM}$ where M is the midpoint of BC.

c Find $\overrightarrow{AM}$.

d Find the area of the ΔABC.

26 *OABCV* is a square-based right pyramid. *V* is the vertex. *M* is the point of intersection of base diagonals *OB* and *AC*. $\overrightarrow{OA} = 5\boldsymbol{i}$, $\overrightarrow{OC} = 5\boldsymbol{j}$ and $\overrightarrow{MV} = 3\boldsymbol{k}$. Find each of the following:

a $\overrightarrow{OB}$ **b** $\overrightarrow{OM}$ **c** $\overrightarrow{OV}$ **d** $\overrightarrow{BV}$ **e** $|\overrightarrow{OV}|$

27 *A* and *B* are points defined respectively by position vectors $\boldsymbol{a}$ and $\boldsymbol{b}$. *M* and *N* are midpoints of *OA* and *OB* respectively, where *O* is the origin.

a Show that $\overrightarrow{MN} = \frac{1}{2}\overrightarrow{AB}$.

b Hence describe the geometrical relationships between the two line segments *MN* and *AB*.

28 Let $\boldsymbol{i}$ be a unit vector in the east direction and $\boldsymbol{j}$ be the unit vector in the north direction. A runner sets off on a bearing of 120°.

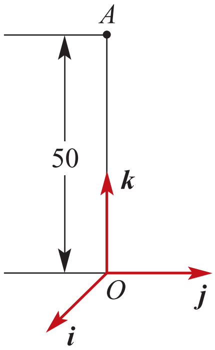
a Find a unit vector in this direction.

b The runner covers 3 km. Find the position of the runner with respect to her starting point.

c The runner now turns and runs for 5 km in a northerly direction. Find the position of the runner with respect to her original starting point.

d Find the distance of the runner from her starting point.

29 A hang-glider jumps from a 50 m cliff.



a Give the position vector of point A with respect to *O*.

b After a short period of time the hang-glider has position *B* given by $(-80\boldsymbol{i} + 20\boldsymbol{j} + 40\boldsymbol{k})$ metres.

i Find the vector $\overrightarrow{AB}$.

ii Find the magnitude of $\overrightarrow{AB}$.

c The hang-glider then moves 600 m in the $\boldsymbol{j}$ direction and 60 m in the $\boldsymbol{k}$ direction. Give the new position vector of the hang-glider.

30 A light plane takes off (from a point which will be considered as the origin) so that its position after a short period of time is given by $\boldsymbol{r}_1 = 1.5\boldsymbol{i} + 2\boldsymbol{j} + 0.9\boldsymbol{k}$ where $\boldsymbol{i}$ is a unit vector in the east direction, $\boldsymbol{j}$ is a unit vector in the north direction and measurements are in kilometres.

a Find the distance of the plane from the origin.

b The position of a second light aircraft at the same time is given by $\boldsymbol{r}_2 = 2\boldsymbol{i} + 3\boldsymbol{i} + 0.8\boldsymbol{k}$. Find:

i $\boldsymbol{r}_1 - \boldsymbol{r}_2$

ii the distance between the two aircraft

c Give a unit vector which would describe the direction in which the first aircraft must fly to pass over the origin at a height of 900 m.

31 Jan starts at a point *O* and walks on level ground 200 metres in a north-westerly direction to *P*. She then walks 50 metres due north to *Q*, which is at the bottom of a building. Jan then climbs to *T*, the top of the building, which is 30 metres vertically above *Q*.

Let $\boldsymbol{i}$, $\boldsymbol{j}$ and $\boldsymbol{k}$ be unit vectors in the east, north and vertically upwards directions respectively. Express each of the following in terms of $\boldsymbol{i}$, $\boldsymbol{j}$ and $\boldsymbol{k}$:

a $\overrightarrow{OP}$ **b** $\overrightarrow{PQ}$ **c** $\overrightarrow{OQ}$ **d** $\overrightarrow{QT}$ **e** $\overrightarrow{OT}$

32 A ship leaves a port and sails on a bearing of NE for 100 km to a point P. $\boldsymbol{i}$ and $\boldsymbol{j}$ are unit vectors in the east and north directions respectively.

a Find the position vector of point P.

b B is a point on the shore with position vector $\overrightarrow{OB} = 100\boldsymbol{i}$. Find:

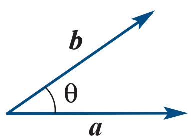
i $\overrightarrow{BP}$ **ii** the bearing of P from B

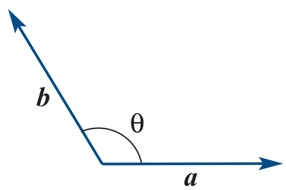
2.3 Scalar (or dot) product of vectors

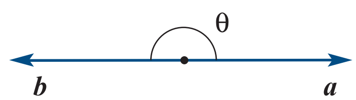
In previous sections linear combinations of vectors have been considered. In this section a product of vectors will be defined which will be shown to be useful. The product is called the **scalar** or **dot product**.

Let $\boldsymbol{a}$ and $\boldsymbol{b}$ be two non-zero vectors. When $\boldsymbol{a}$ and $\boldsymbol{b}$ are placed so that their initial points coincide, the angle, θ, *between* $\boldsymbol{a}$ and $\boldsymbol{b}$ is chosen as shown in the diagrams below.

Note $0 \le \theta \le \pi$.







If $\boldsymbol{a}$ and $\boldsymbol{b}$ are non-zero vectors, the scalar product or dot product, $\boldsymbol{a}.\boldsymbol{b}$, is defined by

$$\boldsymbol{a}.\boldsymbol{b} = |\boldsymbol{a}|\,|\boldsymbol{b}|\cos\theta$$

Note: If $\boldsymbol{a} = \boldsymbol{0}$ or $\boldsymbol{b} = \boldsymbol{0}$ then $\boldsymbol{a}.\boldsymbol{b} = 0$

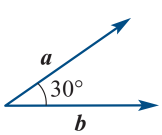
Example 18

a If $|\boldsymbol{a}| = 4$ and $|\boldsymbol{b}| = 5$, and the angle between $\boldsymbol{a}$ and $\boldsymbol{b}$ is 30°, find $\boldsymbol{a}.\boldsymbol{b}$.

b If $|\boldsymbol{a}| = 4$ and $|\boldsymbol{b}| = 5$, and the angle between $\boldsymbol{a}$ and $\boldsymbol{b}$ is 150°, find $\boldsymbol{a}.\boldsymbol{b}$.

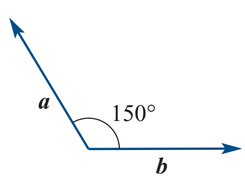
Solution

a



$$\boldsymbol{a}.\boldsymbol{b} = 4 \times 5 \times \cos 30^\circ$$
$$= 20 \times \frac{\sqrt{3}}{2}$$
$$= 10\sqrt{3}$$

b



$$\boldsymbol{a}.\boldsymbol{b} = 4 \times 5 \times \cos(150^\circ)$$
$$= 20 \times \frac{-\sqrt{3}}{2}$$
$$= -10\sqrt{3}$$

Properties of the scalar product

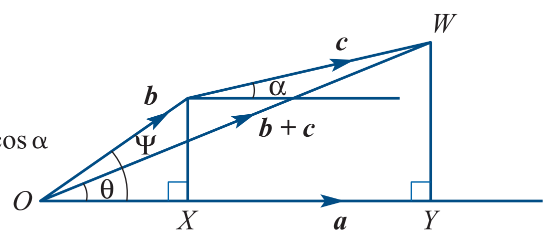
The following are properties of the scalar product:

- $\boldsymbol{a}.\boldsymbol{b} = \boldsymbol{b}.\boldsymbol{a}$
- $k(\boldsymbol{a}.\boldsymbol{b}) = (k\boldsymbol{a}).\boldsymbol{b} = \boldsymbol{a}.(k\boldsymbol{b})$
- $\boldsymbol{a}.\boldsymbol{0} = 0$
- $\boldsymbol{a}.(\boldsymbol{b} + \boldsymbol{c}) = \boldsymbol{a}.\boldsymbol{b} + \boldsymbol{a}.\boldsymbol{c}$
- $\boldsymbol{a}.\boldsymbol{b} = 0$ implies $\boldsymbol{a}$ is perpendicular to $\boldsymbol{b}$ or $\boldsymbol{a} = \boldsymbol{0}$ or $\boldsymbol{b} = \boldsymbol{0}$
- $\boldsymbol{a}.\boldsymbol{a} = |\boldsymbol{a}|^2$
- $\boldsymbol{a}.\boldsymbol{b} = |\boldsymbol{a}|\,|\boldsymbol{b}|$ if $\boldsymbol{a}$ and $\boldsymbol{b}$ are parallel and in the same direction
  $= -|\boldsymbol{a}|\,|\boldsymbol{b}|$ if $\boldsymbol{a}$ and $\boldsymbol{b}$ are in opposite directions

The last four properties require demonstration.

Let $\boldsymbol{a}$, $\boldsymbol{b}$ and $\boldsymbol{c}$ be coplanar.

$$\begin{aligned}
\boldsymbol{a}.(\boldsymbol{b}+\boldsymbol{c}) &= |\boldsymbol{a}|\,|\boldsymbol{b}+\boldsymbol{c}|\cos\theta \\
&= |\boldsymbol{a}|\,OW\cos\theta \\
&= |\boldsymbol{a}|\,OY \\
\boldsymbol{a}.\boldsymbol{b}+\boldsymbol{a}.\boldsymbol{c} &= |\boldsymbol{a}|\,|\boldsymbol{b}|\cos\psi + |\boldsymbol{a}|\,|\boldsymbol{c}|\cos\alpha \\
&= |\boldsymbol{a}|\,(OX+XY) \\
&= |\boldsymbol{a}|\,OY \\
\therefore \quad \boldsymbol{a}.(\boldsymbol{b}+\boldsymbol{c}) &= \boldsymbol{a}.\boldsymbol{b}+\boldsymbol{a}.\boldsymbol{c}
\end{aligned}$$

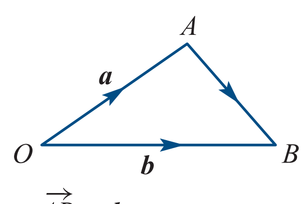


If $\boldsymbol{a}.\boldsymbol{b} = 0$, $|\boldsymbol{a}|\,|\boldsymbol{b}|\cos\theta = 0$ which implies $|\boldsymbol{a}| = 0$ or $|\boldsymbol{b}| = 0$ or $\cos\theta = 0$. If $\cos\theta = 0$ then $\theta = \frac{\pi}{2}$. That is, $\boldsymbol{a}$ and $\boldsymbol{b}$ are perpendicular vectors. Conversely, if $\boldsymbol{a}$ and $\boldsymbol{b}$ are perpendicular, $\boldsymbol{a}.\boldsymbol{b} = 0$.

For parallel vectors $\theta = 0$ and $\boldsymbol{a}.\boldsymbol{b} = |\boldsymbol{a}|\,|\boldsymbol{b}|$, or $\theta = \pi$ and $\boldsymbol{a}.\boldsymbol{b} = -|\boldsymbol{a}|\,|\boldsymbol{b}|$. In particular, if $\boldsymbol{a} = \boldsymbol{b}$, $\boldsymbol{a}.\boldsymbol{a} = |\boldsymbol{a}|^2$.

Scalar product in $\boldsymbol{i} - \boldsymbol{j} - \boldsymbol{k}$ form

Let $\boldsymbol{a} = a_1\boldsymbol{i} + a_2\boldsymbol{j} + a_3\boldsymbol{k}$ and $\boldsymbol{b} = b_1\boldsymbol{i} + b_2\boldsymbol{j} + b_3\boldsymbol{k}$



$\overrightarrow{AB} = \boldsymbol{b} - \boldsymbol{a}$

Using the cosine rule yields

$$\begin{aligned}
|\overrightarrow{AB}|^2 &= |\boldsymbol{a}|^2 + |\boldsymbol{b}|^2 - 2\,|\boldsymbol{a}|\,|\boldsymbol{b}|\cos\theta \\
\therefore \quad |\boldsymbol{b}-\boldsymbol{a}|^2 &= |\boldsymbol{a}|^2 + |\boldsymbol{b}|^2 - 2\,|\boldsymbol{a}|\,|\boldsymbol{b}|\cos\theta
\end{aligned}$$

Rearranging yields

$$\boldsymbol{a}.\boldsymbol{b} = |\boldsymbol{a}|\,|\boldsymbol{b}|\cos\theta = \tfrac{1}{2}(|\boldsymbol{a}|^2 + |\boldsymbol{b}|^2 - |\boldsymbol{b}-\boldsymbol{a}|^2)$$

But
$$|\boldsymbol{a}|^2 = a_1^2 + a_2^2 + a_3^2$$
$$|\boldsymbol{b}|^2 = b_1^2 + b_2^2 + b_3^2$$
$$|\boldsymbol{b} - \boldsymbol{a}|^2 = (b_1 - a_1)^2 + (b_2 - a_2)^2 + (b_3 - a_3)^2$$

Hence
$$\boldsymbol{a}.\boldsymbol{b} = a_1b_1 + a_2b_2 + a_3b_3$$

Note that for the unit vectors $\boldsymbol{i}, \boldsymbol{j}$ and $\boldsymbol{k}$

$$\boldsymbol{i}.\boldsymbol{i} = \boldsymbol{j}.\boldsymbol{j} = \boldsymbol{k}.\boldsymbol{k} = 1$$

and
$$\boldsymbol{i}.\boldsymbol{j} = \boldsymbol{i}.\boldsymbol{k} = \boldsymbol{j}.\boldsymbol{k} = 0$$

For vectors $\boldsymbol{a} = a_1\boldsymbol{i} + a_2\boldsymbol{j} + a_3\boldsymbol{k}$ and $\boldsymbol{b} = b_1\boldsymbol{i} + b_2\boldsymbol{j} + b_3\boldsymbol{k}$,
$$\boldsymbol{a}.\boldsymbol{b} = a_1b_1 + a_2b_2 + a_3b_3$$

Finding the magnitude of the angle between two vectors

The angle between two vectors can be found by using the two forms of the scalar product.

Now $\boldsymbol{a}.\boldsymbol{b} = |\boldsymbol{a}|\,|\boldsymbol{b}| \cos\theta$ and $\boldsymbol{a}.\boldsymbol{b} = a_1b_1 + a_2b_2 + a_3b_3$

Therefore $\dfrac{\boldsymbol{a}.\boldsymbol{b}}{|\boldsymbol{a}|\,|\boldsymbol{b}|} = \cos\theta$

and
$$\frac{a_1b_1 + a_2b_2 + a_3b_3}{|\boldsymbol{a}|\,|\boldsymbol{b}|} = \cos\theta$$

Example 19

A, B and C are points defined by the position vectors $\boldsymbol{a}$, $\boldsymbol{b}$ and $\boldsymbol{c}$ respectively where $\boldsymbol{a} = \boldsymbol{i} + 3\boldsymbol{j} - \boldsymbol{k}$, $\boldsymbol{b} = 2\boldsymbol{i} + \boldsymbol{j}$ and $\boldsymbol{c} = \boldsymbol{i} - 2\boldsymbol{j} - 2\boldsymbol{k}$. Find the magnitude of $\angle ABC$, correct to one decimal place.

Solution

$\angle ABC$ is the angle between vectors $\overrightarrow{BA}$ and $\overrightarrow{BC}$.

$$\overrightarrow{BA} = \boldsymbol{a} - \boldsymbol{b} = -\boldsymbol{i} + 2\boldsymbol{j} - \boldsymbol{k}$$
$$\overrightarrow{BC} = \boldsymbol{c} - \boldsymbol{b} = -\boldsymbol{i} - 3\boldsymbol{j} - 2\boldsymbol{k}$$

Applying the scalar product:

$$\overrightarrow{BA}.\overrightarrow{BC} = |\overrightarrow{BA}|\,|\overrightarrow{BC}| \cos(\angle ABC)$$

and
$$\overrightarrow{BA}.\overrightarrow{BC} = 1 - 6 + 2 = -3$$
$$|\overrightarrow{BA}| = \sqrt{1+4+1} = \sqrt{6}$$
$$\overrightarrow{BC} = \sqrt{1+9+4} = \sqrt{14}$$

$\therefore$
$$-3 = \sqrt{6}\sqrt{14}\cos(\angle ABC)$$
$$\cos(\angle ABC) = \frac{-3}{\sqrt{6}\sqrt{14}}$$

$\angle ABC = 109.1^\circ$ correct to one decimal place or 1.9 radians correct to one decimal place.

Example 20

a Simplify $\boldsymbol{a}.(\boldsymbol{b}+\boldsymbol{c}) - \boldsymbol{b}.(\boldsymbol{a}-\boldsymbol{c})$.

b Expand the following: **i** $(\boldsymbol{a}+\boldsymbol{b}).(\boldsymbol{a}+\boldsymbol{b})$ **ii** $(\boldsymbol{a}+\boldsymbol{b}).(\boldsymbol{a}-\boldsymbol{b})$.

Solution

a $$\begin{aligned}\boldsymbol{a}.(\boldsymbol{b}+\boldsymbol{c}) - \boldsymbol{b}.(\boldsymbol{a}-\boldsymbol{c}) &= \boldsymbol{a}.\boldsymbol{b}+\boldsymbol{a}.\boldsymbol{c}-\boldsymbol{b}.\boldsymbol{a}+\boldsymbol{b}.\boldsymbol{c}\\ &= \boldsymbol{a}.\boldsymbol{c}+\boldsymbol{b}.\boldsymbol{c}\end{aligned}$$

b **i** $$\begin{aligned}(\boldsymbol{a}+\boldsymbol{b}).(\boldsymbol{a}+\boldsymbol{b}) &= \boldsymbol{a}.\boldsymbol{a}+\boldsymbol{a}.\boldsymbol{b}+\boldsymbol{b}.\boldsymbol{a}+\boldsymbol{b}.\boldsymbol{b}\\ &= \boldsymbol{a}.\boldsymbol{a}+2\boldsymbol{a}.\boldsymbol{b}+\boldsymbol{b}.\boldsymbol{b}\end{aligned}$$

ii $$\begin{aligned}(\boldsymbol{a}+\boldsymbol{b}).(\boldsymbol{a}-\boldsymbol{b}) &= \boldsymbol{a}.\boldsymbol{a}-\boldsymbol{a}.\boldsymbol{b}+\boldsymbol{b}.\boldsymbol{a}-\boldsymbol{b}.\boldsymbol{b}\\ &= \boldsymbol{a}.\boldsymbol{a}-\boldsymbol{b}.\boldsymbol{b}\end{aligned}$$

Example 21

Solve the equation $(\boldsymbol{i}+\boldsymbol{j}-\boldsymbol{k}).(3\boldsymbol{i}-x\boldsymbol{j}+2\boldsymbol{k}) = 4$ for x.

Solution

$$(\boldsymbol{i}+\boldsymbol{j}-\boldsymbol{k}).(3\boldsymbol{i}-x\boldsymbol{j}+2\boldsymbol{k}) = 4$$

implies

$$3 - x - 2 = 4$$

i.e. $$1 - x = 4$$

which implies

$$x = -3$$

Exercise 2C

1 Let $\boldsymbol{a} = \boldsymbol{i} - 4\boldsymbol{j} + 7\boldsymbol{k}$, $\boldsymbol{b} = 2\boldsymbol{i} + 3\boldsymbol{j} + 3\boldsymbol{k}$ and $\boldsymbol{c} = -\boldsymbol{i} - 2\boldsymbol{j} + \boldsymbol{k}$.
Find:

a $\boldsymbol{a}.\boldsymbol{a}$ **b** $\boldsymbol{b}.\boldsymbol{b}$ **c** $\boldsymbol{c}.\boldsymbol{c}$ **d** $\boldsymbol{a}.\boldsymbol{b}$ **e** $\boldsymbol{a}.(\boldsymbol{b}+\boldsymbol{c})$
f $(\boldsymbol{a}+\boldsymbol{b}).(\boldsymbol{a}+\boldsymbol{c})$ **g** $(\boldsymbol{a}+2\boldsymbol{b}).(3\boldsymbol{c}-\boldsymbol{b})$

2 Let $\boldsymbol{a} = 2\boldsymbol{i} - \boldsymbol{j} + 3\boldsymbol{k}$, $\boldsymbol{b} = 3\boldsymbol{i} - 2\boldsymbol{k}$ and $\boldsymbol{c} = -\boldsymbol{i} + 3\boldsymbol{j} - \boldsymbol{k}$.
Find: **a** $\boldsymbol{a}.\boldsymbol{a}$ **b** $\boldsymbol{b}.\boldsymbol{b}$ **c** $\boldsymbol{a}.\boldsymbol{b}$ **d** $\boldsymbol{a}.\boldsymbol{c}$ **e** $\boldsymbol{a}.(\boldsymbol{a}+\boldsymbol{b})$

3 Expand and simplify:

a $(\boldsymbol{a}+2\boldsymbol{b}).(\boldsymbol{a}+2\boldsymbol{b})$ **b** $|\boldsymbol{a}+\boldsymbol{b}|^2 - |\boldsymbol{a}-\boldsymbol{b}|^2$
c $\boldsymbol{a}.(\boldsymbol{a}+\boldsymbol{b}) - \boldsymbol{b}.(\boldsymbol{a}+\boldsymbol{b})$ **d** $\dfrac{\boldsymbol{a}.(\boldsymbol{a}+\boldsymbol{b}) - \boldsymbol{a}.\boldsymbol{b}}{|\boldsymbol{a}|}$

4 A and B are points defined respectively by the position vectors $\boldsymbol{a} = \boldsymbol{i} + 2\boldsymbol{j} - \boldsymbol{k}$ and $\boldsymbol{b} = -\boldsymbol{i} + \boldsymbol{j} - 3\boldsymbol{k}$.
Find: **a** $\overrightarrow{AB}$ **b** $|\overrightarrow{AB}|$
c the magnitude of the angle between vectors $\overrightarrow{AB}$ and $\boldsymbol{a}$

5 C and D are points defined respectively by position vectors $\boldsymbol{c}$ and $\boldsymbol{d}$.
If $|\boldsymbol{c}| = 5$, $|\boldsymbol{d}| = 7$ and $\boldsymbol{c}.\boldsymbol{d} = 4$, find $|\overrightarrow{CD}|$.

6 $OABC$ is a rhombus. $\overrightarrow{OA} = \boldsymbol{a}$ and $\overrightarrow{OC} = \boldsymbol{c}$.

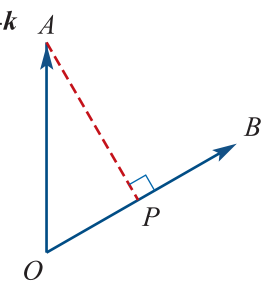
- **a** Express the following vectors in terms of $\boldsymbol{a}$ and $\boldsymbol{c}$:
  i $\overrightarrow{AB}$ **ii** $\overrightarrow{OB}$ **iii** $\overrightarrow{AC}$
- **b** Find $\overrightarrow{OB}.\overrightarrow{AC}$.
- **c** Prove that the diagonals of a rhombus intersect at right angles.

7 From the following list, find three pairs of perpendicular vectors:
$\boldsymbol{a} = \boldsymbol{i} + 3\boldsymbol{j} - \boldsymbol{k}$
$\boldsymbol{b} = -4\boldsymbol{i} + \boldsymbol{j} + 2\boldsymbol{k}$
$\boldsymbol{c} = -2\boldsymbol{i} - 2\boldsymbol{j} - 3\boldsymbol{k}$
$\boldsymbol{d} = -\boldsymbol{i} + \boldsymbol{j} + \boldsymbol{k}$
$\boldsymbol{e} = 2\boldsymbol{i} - \boldsymbol{j} - \boldsymbol{k}$
$\boldsymbol{f} = -\boldsymbol{i} + 4\boldsymbol{j} - 5\boldsymbol{k}$

8 Solve each of the following equations:

- **a** $(\boldsymbol{i} + 2\boldsymbol{j} - 3\boldsymbol{k}).(5\boldsymbol{i} + x\boldsymbol{j} + \boldsymbol{k}) = -6$
- **b** $(x\boldsymbol{i} + 7\boldsymbol{j} - \boldsymbol{k}).(-4\boldsymbol{i} + x\boldsymbol{j} + 5\boldsymbol{k}) = 10$
- **c** $(x\boldsymbol{i} + 5\boldsymbol{k}).(-2\boldsymbol{i} - 3\boldsymbol{j} + 3\boldsymbol{k}) = x$
- **d** $x(2\boldsymbol{i} + 3\boldsymbol{j} + \boldsymbol{k}).(\boldsymbol{i} + \boldsymbol{j} + x\boldsymbol{k}) = 6$

9 Points A and B are defined by the position vectors $\boldsymbol{a} = \boldsymbol{i} + 4\boldsymbol{j} - 4\boldsymbol{k}$ and $\boldsymbol{b} = 2\boldsymbol{i} + 5\boldsymbol{j} - \boldsymbol{k}$.
P is point on OB such that AP is perpendicular to OB.
Let $\overrightarrow{OP} = q\boldsymbol{b}$ for a constant q.



- **a** Express $\overrightarrow{AP}$ in terms of q, $\boldsymbol{a}$ and $\boldsymbol{b}$.
- **b** Use the fact that $\overrightarrow{AP}.\overrightarrow{OB} = 0$ to find the value of q.
- **c** Find the coordinates of the point P.

10 $x\boldsymbol{i} + 2\boldsymbol{j} + y\boldsymbol{k}$ is perpendicular to vectors $\boldsymbol{i} + \boldsymbol{j} + \boldsymbol{k}$ and $4\boldsymbol{i} + \boldsymbol{j} + 2\boldsymbol{k}$. Find x and y.

11 Find the angle, in radians, between each of the following pairs of vectors, correct to three significant figures:

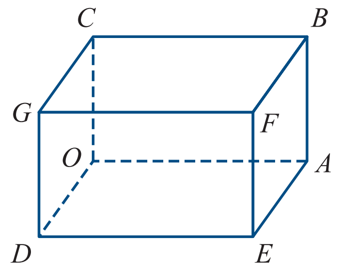
- **a** $\boldsymbol{i} + 2\boldsymbol{j} - \boldsymbol{k}$ and $\boldsymbol{i} - 4\boldsymbol{j} + \boldsymbol{k}$
- **b** $-2\boldsymbol{i} + \boldsymbol{j} + 3\boldsymbol{k}$ and $-2\boldsymbol{i} - 2\boldsymbol{j} + \boldsymbol{k}$
- **c** $2\boldsymbol{i} - \boldsymbol{j} - 3\boldsymbol{k}$ and $4\boldsymbol{i} - 2\boldsymbol{k}$
- **d** $7\boldsymbol{i} + \boldsymbol{k}$ and $-\boldsymbol{i} + \boldsymbol{j} - 3\boldsymbol{k}$

12 $\boldsymbol{a}$ and $\boldsymbol{b}$ are non-zero vectors, such that $\boldsymbol{a}.\boldsymbol{b} = 0$. Use the scalar product to show that $\boldsymbol{a}$ and $\boldsymbol{b}$ are perpendicular vectors.

For questions 13 to 17, find the angles in degrees correct to two decimal places.

13 A and B are points defined by the positions vectors $\boldsymbol{a} = \boldsymbol{i} + \boldsymbol{j} + \boldsymbol{k}$ and $\boldsymbol{b} = 2\boldsymbol{i} + \boldsymbol{j} - \boldsymbol{k}$ respectively. M is the midpoint of AB.
Find: **a** $\overrightarrow{OM}$ **b** $\angle AOM$ **c** $\angle BMO$

14 *OABCDEFG* is a cuboid set on axes at *O*, such that $\overrightarrow{OD} = \boldsymbol{i}$, $\overrightarrow{OA} = 3\boldsymbol{j}$ and $\overrightarrow{OC} = 2\boldsymbol{k}$.
Find:



a **i** $\overrightarrow{GB}$ **ii** $\overrightarrow{GE}$

b $\angle BGE$

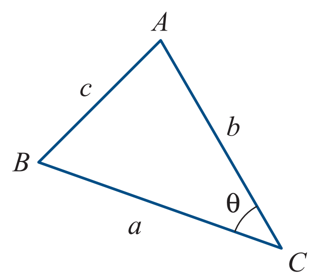
c the angle between diagonals $\overrightarrow{CE}$ and $\overrightarrow{GA}$

15 *A*, *B* and *C* are points defined by the position vectors 4$\boldsymbol{i}$, 5$\boldsymbol{j}$ and $-2\boldsymbol{i} + 7\boldsymbol{k}$ respectively. *M* and *N* are the midpoints of $\overrightarrow{AB}$ and $\overrightarrow{AC}$ respectively.
Find:

a **i** $\overrightarrow{OM}$ **ii** $\overrightarrow{ON}$

b $\angle MON$ **c** $\angle MOC$

16 *ABC* is a triangle with dimensions as shown in the figure. $\angle ACB = \theta$



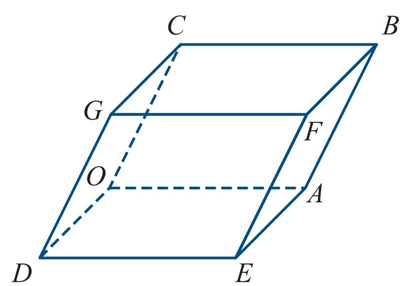
a Express $\overrightarrow{AB}$ in terms of $\overrightarrow{CA}$ and $\overrightarrow{CB}$.

b Use the scalar product $\overrightarrow{AB}.\overrightarrow{AB}$ to prove the cosine rule which states that: $c^2 = a^2 + b^2 - 2ab \cos \theta$.

17 A parallelepiped is an oblique prism that has a parallelogram cross-section. It has three pairs of parallel and congruent faces. *OABCDEFG* is a parallelepiped with

$\overrightarrow{OA} = 3\boldsymbol{j}$, $\overrightarrow{OC} = -\boldsymbol{i} + \boldsymbol{j} + 2\boldsymbol{k}$ and $\overrightarrow{OD} = 2\boldsymbol{i} - \boldsymbol{j}$

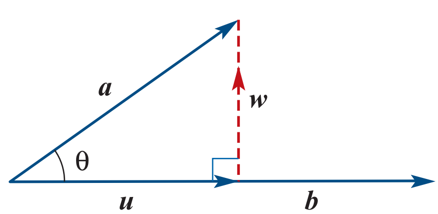
Show that the diagonals $\overrightarrow{DB}$ and $\overrightarrow{CE}$ bisect each other, and find the acute angle between them.



2.4 Vector resolutes

It is often useful to decompose a vector $\boldsymbol{a}$ into a sum of two terms, one parallel to a given vector $\boldsymbol{b}$ and the other perpendicular to $\boldsymbol{b}$.

From the diagram it can be seen that $\boldsymbol{a} = \boldsymbol{u} + \boldsymbol{w}$



where $\boldsymbol{u} = k\boldsymbol{b}$

Therefore $\boldsymbol{w} = \boldsymbol{a} - \boldsymbol{u} = \boldsymbol{a} - k\boldsymbol{b}$

For $\boldsymbol{w}$ to be perpendicular to $\boldsymbol{b}$, $\boldsymbol{w}.\boldsymbol{b} = 0$

Therefore $(\boldsymbol{a} - k\boldsymbol{b}).\boldsymbol{b} = 0$

$$\boldsymbol{a}.\boldsymbol{b} - k\boldsymbol{b}.\boldsymbol{b} = 0 \text{ where } k = \frac{\boldsymbol{a}.\boldsymbol{b}}{\boldsymbol{b}.\boldsymbol{b}}$$

Hence $\boldsymbol{u} = \dfrac{\boldsymbol{a}.\boldsymbol{b}}{\boldsymbol{b}.\boldsymbol{b}}\boldsymbol{b}$

$\boldsymbol{u}$ is called the **vector resolute** of $\boldsymbol{a}$ in the direction of $\boldsymbol{b}$ and exists in many alternative forms.

$$\boldsymbol{u} = \frac{\boldsymbol{a}.\boldsymbol{b}}{\boldsymbol{b}.\boldsymbol{b}}\boldsymbol{b} = \frac{\boldsymbol{a}.\boldsymbol{b}}{|\boldsymbol{b}|^2}\boldsymbol{b} = \left(\boldsymbol{a}.\frac{\boldsymbol{b}}{|\boldsymbol{b}|}\right)\left(\frac{\boldsymbol{b}}{|\boldsymbol{b}|}\right) = (\boldsymbol{a}.\hat{\boldsymbol{b}})\hat{\boldsymbol{b}}$$

$(\boldsymbol{a}.\hat{\boldsymbol{b}})$ or $\dfrac{\boldsymbol{a}.\boldsymbol{b}}{|\boldsymbol{b}|}$, the 'signed' length of the vector resolute $\boldsymbol{u}$ is called the **scalar resolute** of $\boldsymbol{a}$ in the direction of $\boldsymbol{b}$.

Note that $\boldsymbol{w} = \boldsymbol{a} - \boldsymbol{u} = \boldsymbol{a} - \dfrac{\boldsymbol{a}.\boldsymbol{b}}{\boldsymbol{b}.\boldsymbol{b}}\boldsymbol{b}$ and expressing $\boldsymbol{a}$ as a sum of the two components, the first parallel to $\boldsymbol{b}$ and the second perpendicular to $\boldsymbol{b}$, is

$$\boldsymbol{a} = \frac{\boldsymbol{a}.\boldsymbol{b}}{\boldsymbol{b}.\boldsymbol{b}}\boldsymbol{b} + \left(\boldsymbol{a} - \frac{\boldsymbol{a}.\boldsymbol{b}}{\boldsymbol{b}.\boldsymbol{b}}\boldsymbol{b}\right)$$

This is sometimes described as resolving the vector $\boldsymbol{a}$ into **rectangular components**, one in the direction of $\boldsymbol{b}$ and one perpendicular to $\boldsymbol{b}$.

Example 22

Let $\boldsymbol{a} = \boldsymbol{i} + 3\boldsymbol{j} - \boldsymbol{k}$ and $\boldsymbol{b} = \boldsymbol{i} - \boldsymbol{j} + 2\boldsymbol{k}$

Find the vector resolute of:

a $\boldsymbol{a}$ in the direction of $\boldsymbol{b}$

b $\boldsymbol{b}$ in the direction of $\boldsymbol{a}$

Solution

a $\boldsymbol{a}.\boldsymbol{b} = 1 - 3 - 2 = -4$

$\boldsymbol{b}.\boldsymbol{b} = 1 + 1 + 4 = 6$

$\therefore$ the vector resolute of $\boldsymbol{a}$ in the direction of $\boldsymbol{b}$ $= \dfrac{\boldsymbol{a}.\boldsymbol{b}}{\boldsymbol{b}.\boldsymbol{b}}\boldsymbol{b}$

$= -\frac{4}{6}(\boldsymbol{i} - \boldsymbol{j} + 2\boldsymbol{k})$

$= -\frac{2}{3}(\boldsymbol{i} - \boldsymbol{j} + 2\boldsymbol{k})$

b $\boldsymbol{a}.\boldsymbol{a} = 1 + 9 + 1 = 11$

$\therefore$ the vector resolute of $\boldsymbol{b}$ in the direction of $\boldsymbol{a}$ $= \dfrac{\boldsymbol{a}.\boldsymbol{b}}{\boldsymbol{a}.\boldsymbol{a}}\boldsymbol{a}$

$= \frac{-4}{11}(\boldsymbol{i} + 3\boldsymbol{j} - \boldsymbol{k})$

Example 23

Find the scalar resolute of $\boldsymbol{a} = 2\boldsymbol{i} + 2\boldsymbol{j} - \boldsymbol{k}$ in the direction of $\boldsymbol{b} = -\boldsymbol{i} + 3\boldsymbol{k}$.

Solution

$\boldsymbol{a}.\boldsymbol{b} = -2 - 3 = -5$

$|\boldsymbol{b}| = \sqrt{1 + 9} = \sqrt{10}$

The scalar resolute of $\boldsymbol{a}$ in the direction of $\boldsymbol{b}$ is $\dfrac{\boldsymbol{a}.\boldsymbol{b}}{|\boldsymbol{b}|} = \dfrac{-5}{\sqrt{10}} = \dfrac{-\sqrt{10}}{2}$

Example 24

Find the resolution of $\boldsymbol{i} + 3\boldsymbol{j} - \boldsymbol{k}$ into rectangular components one of which is parallel to $2\boldsymbol{i} - 2\boldsymbol{j} - \boldsymbol{k}$.

Solution

Let $\boldsymbol{a} = \boldsymbol{i} + 3\boldsymbol{j} - \boldsymbol{k}$ and $\boldsymbol{b} = 2\boldsymbol{i} - 2\boldsymbol{j} - \boldsymbol{k}$.

The vector resolute of $\boldsymbol{a}$ in the direction of $\boldsymbol{b}$ is calculated from $\dfrac{\boldsymbol{a}.\boldsymbol{b}}{\boldsymbol{b}.\boldsymbol{b}}\boldsymbol{b}$.

$$\boldsymbol{a}.\boldsymbol{b} = 2 - 6 + 1 = -3 \quad \boldsymbol{b}.\boldsymbol{b} = 4 + 4 + 1 = 9$$

$$\begin{aligned}\therefore \text{ the vector resolute } &= \tfrac{-3}{9}(2\boldsymbol{i} - 2\boldsymbol{j} - \boldsymbol{k}) \\ &= -\tfrac{1}{3}(2\boldsymbol{i} - 2\boldsymbol{j} - \boldsymbol{k})\end{aligned}$$

$$\begin{aligned}\text{The perpendicular component } &= \boldsymbol{a} - \left[-\tfrac{1}{3}(2\boldsymbol{i} - 2\boldsymbol{j} - \boldsymbol{k})\right] \\ &= (\boldsymbol{i} + 3\boldsymbol{j} - \boldsymbol{k}) + \tfrac{1}{3}[2\boldsymbol{i} - 2\boldsymbol{j} - \boldsymbol{k}] \\ &= \tfrac{5}{3}\boldsymbol{i} + \tfrac{7}{3}\boldsymbol{j} - \tfrac{4}{3}\boldsymbol{k} \\ &= \tfrac{1}{3}[5\boldsymbol{i} + 7\boldsymbol{j} - 4\boldsymbol{k}]\end{aligned}$$

Note: It is worthwhile verifying that the second vector is indeed perpendicular to $\boldsymbol{b}$, in order to detect any numerical error.

$$(5\boldsymbol{i} + 7\boldsymbol{j} - 4\boldsymbol{k}).(2\boldsymbol{i} - 2\boldsymbol{j} - \boldsymbol{k}) = 10 - 14 + 4 = 0$$

$$\therefore \boldsymbol{i} + 3\boldsymbol{j} - \boldsymbol{k} = -\frac{1}{3}(2\boldsymbol{i} - 2\boldsymbol{j} - \boldsymbol{k}) + \frac{1}{3}(5\boldsymbol{i} + 7\boldsymbol{j} - 4\boldsymbol{k})$$

Exercise 2D

Skillsheet TEACHER

1 A and B are defined by the position vectors $\boldsymbol{a} = \boldsymbol{i} + 3\boldsymbol{j} - \boldsymbol{k}$ and $\boldsymbol{b} = \boldsymbol{i} + 2\boldsymbol{j} + 2\boldsymbol{k}$. Let $\boldsymbol{c} = \overrightarrow{AB}$

Find: **a** $\hat{\boldsymbol{a}}$ **b** $\hat{\boldsymbol{b}}$ **c** $\hat{\boldsymbol{c}}$

2 Let $\boldsymbol{a} = 3\boldsymbol{i} + 4\boldsymbol{j} - \boldsymbol{k}$ and $\boldsymbol{b} = \boldsymbol{i} - \boldsymbol{j} - \boldsymbol{k}$.

a Find: **i** $\hat{\boldsymbol{a}}$ **ii** $|\boldsymbol{b}|$

b Find the vector with the same magnitude as $\boldsymbol{b}$ and with the same direction as $\boldsymbol{a}$.

3 A and B are defined by the position vectors $\boldsymbol{a} = 2\boldsymbol{i} - 2\boldsymbol{j} - \boldsymbol{k}$ and $\boldsymbol{b} = 3\boldsymbol{i} + 4\boldsymbol{k}$.

a Find: **i** $\hat{\boldsymbol{a}}$ **ii** $\hat{\boldsymbol{b}}$

b Find the unit vector which bisects $\angle AOB$.

4 For each of the following pairs of vectors find the vector resolute of $\boldsymbol{a}$ in the direction of $\boldsymbol{b}$.

a $\boldsymbol{a} = \boldsymbol{i} + 3\boldsymbol{j}$ and $\boldsymbol{b} = \boldsymbol{i} - 4\boldsymbol{j} + \boldsymbol{k}$ **b** $\boldsymbol{a} = \boldsymbol{i} - 3\boldsymbol{k}$ and $\boldsymbol{b} = \boldsymbol{i} - 4\boldsymbol{j} + \boldsymbol{k}$

c $\boldsymbol{a} = 4\boldsymbol{i} - \boldsymbol{j} + 3\boldsymbol{k}$ and $\boldsymbol{b} = 4\boldsymbol{i} - \boldsymbol{k}$

5 For each of following pairs of vectors, find the scalar resolute of the first vector in the direction of the second vector.

a $\boldsymbol{a} = 2\boldsymbol{i} + \boldsymbol{j}$ and $\boldsymbol{b} = \boldsymbol{i}$
b $\boldsymbol{a} = 3\boldsymbol{i} + \boldsymbol{j} - 3\boldsymbol{k}$ and $\boldsymbol{c} = \boldsymbol{i} - 2\boldsymbol{j}$
c $\boldsymbol{b} = 2\boldsymbol{j} + \boldsymbol{k}$ and $\boldsymbol{a} = 2\boldsymbol{i} + \sqrt{3}\boldsymbol{j}$
d $\boldsymbol{b} = \boldsymbol{i} - \sqrt{5}\boldsymbol{j}$ and $\boldsymbol{c} = -\boldsymbol{i} + 4\boldsymbol{j}$

6 For each of the following pairs of vectors, find the resolution of the vector $\boldsymbol{a}$ in rectangular components, one of which is parallel to $\boldsymbol{b}$.

a $\boldsymbol{a} = 2\boldsymbol{i} + \boldsymbol{j} + \boldsymbol{k}$ $\quad \boldsymbol{b} = 5\boldsymbol{i} - \boldsymbol{k}$
b $\boldsymbol{a} = 3\boldsymbol{i} + \boldsymbol{j}$ $\quad \boldsymbol{b} = \boldsymbol{i} + \boldsymbol{k}$
c $\boldsymbol{a} = -\boldsymbol{i} + \boldsymbol{j} + \boldsymbol{k}$ $\quad \boldsymbol{b} = 2\boldsymbol{i} + 2\boldsymbol{j} - \boldsymbol{k}$

7 A and B are points defined by the position vectors $\boldsymbol{a} = \boldsymbol{i} + 3\boldsymbol{j} - \boldsymbol{k}$ and $\boldsymbol{b} = \boldsymbol{j} + \boldsymbol{k}$ respectively. Find:

a the vector resolute of $\boldsymbol{a}$ in the direction of $\boldsymbol{b}$
b a unit vector through A perpendicular to OB

8 A and B are points defined by the position vectors $\boldsymbol{a} = 4\boldsymbol{i} + \boldsymbol{j}$ and $\boldsymbol{b} = \boldsymbol{i} - \boldsymbol{j} - \boldsymbol{k}$ respectively. Find:

a the vector resolute of $\boldsymbol{a}$ in the direction of $\boldsymbol{b}$
b the vector component of $\boldsymbol{a}$ perpendicular to $\boldsymbol{b}$
c the shortest distance from A to OB

9 A, B and C are points defined by the position vectors $\boldsymbol{a} = \boldsymbol{i} + 2\boldsymbol{j} + \boldsymbol{k}$, $\boldsymbol{b} = 2\boldsymbol{i} + \boldsymbol{j} - \boldsymbol{k}$ and $\boldsymbol{c} = 2\boldsymbol{i} - 3\boldsymbol{j} + \boldsymbol{k}$. Find:

a **i** $\overrightarrow{AB}$ **ii** $\overrightarrow{AC}$
b the vector resolute of $\overrightarrow{AB}$ in the direction of $\overrightarrow{AC}$
c the shortest distance from B to line AC
d the area of triangle ABC

10 **a** Verify that $\boldsymbol{a} = \boldsymbol{i} - 3\boldsymbol{j} - 2\boldsymbol{k}$ and $\boldsymbol{b} = 5\boldsymbol{i} + \boldsymbol{j} + \boldsymbol{k}$ are two vectors perpendicular to each other.

b If $\boldsymbol{c} = 2\boldsymbol{i} - \boldsymbol{k}$, find:

i $\boldsymbol{d}$, the vector resolute of $\boldsymbol{c}$ in the direction of $\boldsymbol{a}$
ii $\boldsymbol{e}$, the vector resolute of $\boldsymbol{c}$ in the direction of $\boldsymbol{b}$

c Find $\boldsymbol{f}$, such that $\boldsymbol{c} = \boldsymbol{d} + \boldsymbol{e} + \boldsymbol{f}$.
d Hence show that $\boldsymbol{f}$ is perpendicular to both vectors $\boldsymbol{a}$ and $\boldsymbol{b}$.

2.5 Vector proof

Vectors have been used to establish geometric results in the earlier sections of this chapter. In this section the application of vectors to geometric proof is considered. Some results which will be useful in geometric proof are as follows:

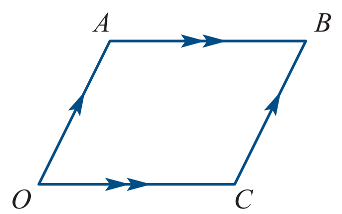
- $n\boldsymbol{a}$ where n $\in R^+$ is a vector in the same direction as $\boldsymbol{a}$ but of magnitude $n|\boldsymbol{a}|$
- $-n\boldsymbol{a}$ where $n \in R^+$ is a vector in the opposite direction to $\boldsymbol{a}$ and of magnitude $n|\boldsymbol{a}|$

- If $\boldsymbol{a}$ is parallel to $\boldsymbol{b}$ there is a non-zero real number n such that $\boldsymbol{b} = n\boldsymbol{a}$
  Conversely if $\boldsymbol{b} = n\boldsymbol{a}$ for some $n \in R^+ \setminus \{0\}$, $\boldsymbol{b}$ is parallel to $\boldsymbol{a}$
- If $\boldsymbol{a}$ is parallel to $\boldsymbol{b}$ and there is at least one point in common then $\boldsymbol{a}$ and $\boldsymbol{b}$ lie on the same straight line. For example if $\overrightarrow{AB} = k\overrightarrow{BC}$ for some $k \in R \setminus \{0\}$, then A, B and C are collinear
- If $\boldsymbol{a.b} = 0$ then $\boldsymbol{a}$ is perpendicular to $\boldsymbol{b}$ or $\boldsymbol{a} = \boldsymbol{0}$ or $\boldsymbol{b} = \boldsymbol{0}$
  Conversely if $\boldsymbol{a}$ is perpendicular to $\boldsymbol{b}$ then $\boldsymbol{a.b} = 0$
- $\boldsymbol{a.a} = |\boldsymbol{a}|^2$

Example 25

Prove that the diagonals of a rhombus are perpendicular.

Solution



$OABC$ is a rhombus.

Let $\overrightarrow{OA} = \boldsymbol{a}$ and $\overrightarrow{OC} = \boldsymbol{c}$

The diagonals of the rhombus are OB and AC.

$$\begin{aligned}\text{Now } \overrightarrow{OB} &= \overrightarrow{OC} + \overrightarrow{CB}\\ &= \overrightarrow{OC} + \overrightarrow{OA}\\ &= \boldsymbol{c} + \boldsymbol{a}\\ \text{and } \overrightarrow{AC} &= \overrightarrow{AO} + \overrightarrow{OC}\\ &= -\boldsymbol{a} + \boldsymbol{c}\end{aligned}$$

Consider the scalar product of $\overrightarrow{AC}$ and $\overrightarrow{OB}$.

$$\begin{aligned}\overrightarrow{AC}.\overrightarrow{OB} &= (\boldsymbol{c} + \boldsymbol{a}).(\boldsymbol{c} - \boldsymbol{a})\\ &= \boldsymbol{c.c} - \boldsymbol{a.a}\\ &= |\boldsymbol{c}|^2 - |\boldsymbol{a}|^2\end{aligned}$$

A rhombus has all sides of equal length.

Therefore $|\boldsymbol{c}| = |\boldsymbol{a}|$

Hence

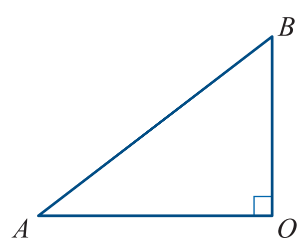
$$\overrightarrow{AC}.\overrightarrow{OB} = |\boldsymbol{c}|^2 - |\boldsymbol{a}|^2 = 0$$

This implies that AC is perpendicular to OB.

Example 26

Prove Pythagoras' theorem using vectors.

Solution



AOB is a right-angled triangle with right angle at O.

Let $\overrightarrow{OA} = \boldsymbol{a}$ and $\overrightarrow{OB} = \boldsymbol{b}$

Then $\overrightarrow{AB} = \overrightarrow{AO} + \overrightarrow{OB} = -\boldsymbol{a} + \boldsymbol{b}$

Let $\overrightarrow{AB} = \boldsymbol{c}$

Then $\quad c = -a + b$

and $\quad c.c = (b - a).(b - a)$

$\quad = b.b - 2a.b + a.a$

But $\quad a.b = 0$ as OB is perpendicular to OA

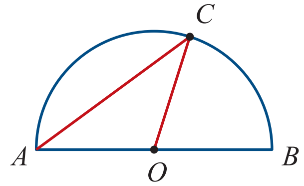
$\therefore \quad c.c = b.b + a.a$

i.e. $\quad |c|^2 = |b|^2 + |a|^2$

Example 27

Prove that the angle subtended by a diameter in a circle is a right angle.

Solution



Let O be the centre of the circle and AB a diameter $|\overrightarrow{OA}| = |\overrightarrow{OB}| = |\overrightarrow{OC}| = r$, the radius of the circle

Let $\overrightarrow{OA} = a$, and $\overrightarrow{OC} = c$. Then $\overrightarrow{OB} = -a$ and $\overrightarrow{AC} = \overrightarrow{AO} + \overrightarrow{OC}$ and $\overrightarrow{BC} = \overrightarrow{BO} + \overrightarrow{OC}$

$$\begin{aligned}\overrightarrow{AC}.\overrightarrow{BC} &= (-a + c).(a + c)\\ &= -a.a + c.c\\ &= -|a|^2 + |c|^2\end{aligned}$$

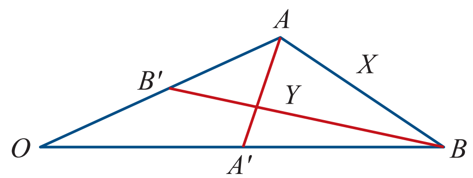
But $\quad |a| = |c|$

$\therefore \quad \overrightarrow{AC}.\overrightarrow{BC} = 0$ and hence $AC \perp BC$

Example 28

Prove that the medians of a triangle are concurrent.

Solution



For triangle OAB

A', B' and X are the midpoints of OB, OA and AB respectively.

Let Y be the point of intersection of the medians, AA' and BB'

Let $\overrightarrow{OA} = a$ and $\overrightarrow{OB} = b$

It will first be shown

$$AY : YA' = BY : YB' = 2 : 1$$

Let $\quad \overrightarrow{AY} = \lambda\overrightarrow{AA'}$ and $\overrightarrow{BY} = \mu\overrightarrow{BB'}$

Now $\quad \overrightarrow{AA'} = \overrightarrow{AO} + \frac{1}{2}\overrightarrow{OB} \qquad$ and $\overrightarrow{BB'} = \overrightarrow{BO} + \frac{1}{2}\overrightarrow{OA}$

$\quad = -a + \frac{1}{2}b \qquad\qquad = -b + \frac{1}{2}a$

$\therefore \quad \overrightarrow{AY} = \lambda\left(-\boldsymbol{a} + \tfrac{1}{2}\boldsymbol{b}\right) \qquad \therefore \overrightarrow{BY} = \mu\left(-\boldsymbol{b} + \tfrac{1}{2}\boldsymbol{a}\right)$

But $\overrightarrow{BY}$ can also be obtained as shown.

$$\begin{aligned}\overrightarrow{BY} &= \overrightarrow{BA} + \overrightarrow{AY}\\ &= \overrightarrow{BO} + \overrightarrow{OA} + \overrightarrow{AY}\\ &= -\boldsymbol{b} + \boldsymbol{a} + \lambda\left(-\boldsymbol{a} + \tfrac{1}{2}\boldsymbol{b}\right)\end{aligned}$$

$$\therefore \quad -\mu\boldsymbol{b} + \frac{\mu}{2}\boldsymbol{a} = (1-\lambda)\boldsymbol{a} + \left(\tfrac{\lambda}{2} - 1\right)\boldsymbol{b}$$

As $\boldsymbol{a}$ and $\boldsymbol{b}$ are independent vectors,

$$\therefore \quad \frac{\mu}{2} = 1 - \lambda \quad \boxed{1} \qquad \text{and} \quad -\mu = \frac{\lambda}{2} - 1 \quad \boxed{2}$$

Multiply $\boxed{1}$ by 2 and add to $\boxed{2}$

$$0 = 2 - 2\lambda + \frac{\lambda}{2} - 1$$

$$\therefore \quad 1 = \frac{3\lambda}{2} \text{ and } \lambda = \frac{2}{3}$$

Substitute in $\boxed{1}$ to find $\mu = \dfrac{2}{3}$

$$\therefore \quad \overrightarrow{AY} : YA' = BY : YB' = 2 : 1$$

Now consider the position vector of Y.

$$\begin{aligned}\overrightarrow{OY} &= \overrightarrow{OA} + \overrightarrow{AY}\\ &= \boldsymbol{a} + v\tfrac{2}{3}\left(-\boldsymbol{a} + \tfrac{1}{2}\boldsymbol{b}\right)\\ &= \tfrac{1}{3}\boldsymbol{a} + \tfrac{1}{3}\boldsymbol{b}\end{aligned}$$

and

$$\begin{aligned}\overrightarrow{OX} &= \overrightarrow{OA} + \tfrac{1}{2}(\overrightarrow{AB})\\ &= \boldsymbol{a} + \tfrac{1}{2}(\boldsymbol{b} - \boldsymbol{a})\\ &= \tfrac{1}{2}(\boldsymbol{b} + \boldsymbol{a})\end{aligned}$$

$$\therefore \quad \overrightarrow{OY} = \tfrac{2}{3} \times \tfrac{1}{2}(\boldsymbol{b} + \boldsymbol{a}) = \tfrac{2}{3}\overrightarrow{OX}$$

i.e. O, Y and X are collinear and Y divides OX in the ratio 2 : 1
The medians are concurrent at Y.

Exercise 2E

1 Prove that the diagonals of a parallelogram bisect each other.

2 Prove that if the midpoints of the sides of a rectangle are joined then a rhombus is formed.

3 Prove that if the midpoints of the sides of a square are joined then another square is formed.

4 Prove that the median to the base of an isosceles traingle is perpendicular to the base.

5 Prove the cosine rule for any triangle.

6 Prove that if the diagonals of a parallelogram are of equal length then the parallelogram is a rectangle.

7 Prove that the midpoint of the hypotenuse of a right-angled triangle is equidistant from the three vertices of the triangle.

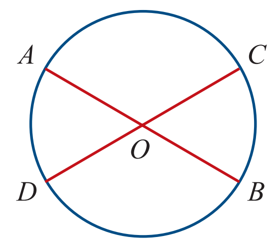
8 Prove that the sum of the squares of the lengths of the diagonals of any parallelogram is equal to the sum of the squares of the lengths of the sides.

9 Prove that if the midpoints of the sides of a quadrilateral are joined then a parallelogram is formed.

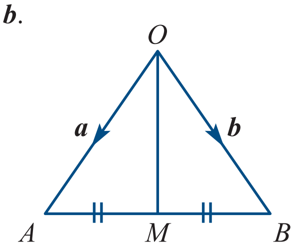
10 $ABCD$ is a parallelogram. M is the midpoint of AB. P is the point of trisection of MD nearer to M. Prove that A, P and C are collinear and that P is a point of trisection of AC.

11 $ABCD$ is a parallelogram with $\overrightarrow{AB} = \boldsymbol{a}$ and $\overrightarrow{AD} = \boldsymbol{b}$. The point P lies on AD and is such that $AP : PD = 1 : 2$ and the point Q lies on BD and is such that $BQ : QD = 2 : 1$. Show that PQ is parallel to AC.

12 AB and CD are diameters of a circle with centre O. Prove that $ACBD$ is a rectangle.



13 In triangle AOB, M is the midpoint of AB, $\overrightarrow{OA} = \boldsymbol{a}$ and $\overrightarrow{OB} = \boldsymbol{b}$.

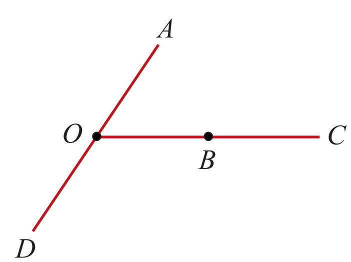


a Find:
  - **i** $\overrightarrow{AM}$ in terms of $\boldsymbol{a}$ and $\boldsymbol{b}$
  - **ii** $\overrightarrow{OM}$ in terms of $\boldsymbol{a}$ and $\boldsymbol{b}$

b Find $\overrightarrow{AM}.\overrightarrow{AM} + \overrightarrow{OM}.\overrightarrow{OM}$

c Hence prove $OA^2 + OB^2 = 2OM^2 + 2AM^2$

14 In the figure, O is the midpoint of AD and B is the midpoint of OC.
If $\overrightarrow{OA} = \boldsymbol{a}$, $\overrightarrow{OB} = \boldsymbol{b}$ and if P is a point such that $\overrightarrow{OP} = \frac{1}{3}(\boldsymbol{a} + 4\boldsymbol{b})$:



a prove that A, P and C are collinear

b prove that D, B and P are collinear

c find $DB : BP$

15 In the triangle OAB, $\overrightarrow{OA} = \boldsymbol{a}$ and $\overrightarrow{OB} = \boldsymbol{b}$. P is a point on AB so that the length of AP is twice the length of BP. Q is a point such that $\overrightarrow{OQ} = 3\overrightarrow{OP}$.

a Find each of the following in terms of $\boldsymbol{a}$ and $\boldsymbol{b}$:

i $\overrightarrow{OP}$ **ii** $\overrightarrow{OQ}$ **iii** $\overrightarrow{AQ}$

b Hence show that $\overrightarrow{AQ}$ is parallel to $\overrightarrow{OB}$.

16 $ORST$ is a parallelogram, U is the midpoint of RS and V is the midpoint of ST. Relative to the origin O, $\boldsymbol{r}$, $\boldsymbol{s}$, $\boldsymbol{t}$, $\boldsymbol{u}$ and $\boldsymbol{v}$ are the position vectors of points R, S, T, U and V respectively.

a Express $\boldsymbol{s}$ in terms of $\boldsymbol{r}$ and $\boldsymbol{t}$. **b** Express $\boldsymbol{v}$ in terms of $\boldsymbol{s}$ and $\boldsymbol{t}$.

c Hence or otherwise, show that $4(\boldsymbol{u} + \boldsymbol{v}) = 3(\boldsymbol{r} + \boldsymbol{s} + \boldsymbol{t})$.

17 Relative to an origin, points A, B, C, D and E shown in the diagram have position vectors

$\boldsymbol{a} = \boldsymbol{i} + 11\boldsymbol{j}$

$\boldsymbol{b} = 2\boldsymbol{i} + 8\boldsymbol{j}$

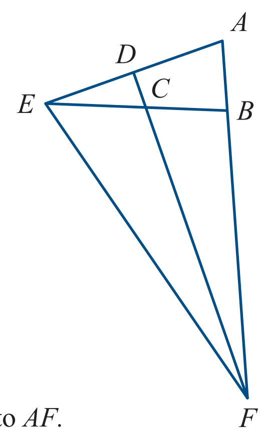
$\boldsymbol{c} = -\boldsymbol{i} + 7\boldsymbol{j}$

$\boldsymbol{d} = -2\boldsymbol{i} + 8\boldsymbol{j}$

$\boldsymbol{e} = -4\boldsymbol{i} + 6\boldsymbol{j}$

respectively.

The lines AB and DC intersect at F as shown.



a Show that E lies on the lines DA and BC.

b Find $\overrightarrow{AB}$ and $\overrightarrow{DC}$.

c Find the position vector of the point F.

d Show that FD is perpendicular to EA and EB is perpendicular to AF.

e Find the position vector of the centre of the circle through E, D, B and F.

18 Coplanar points A, B, C, D and E have position vectors $\boldsymbol{a}$, $\boldsymbol{b}$, $\boldsymbol{c}$, $\boldsymbol{d}$ and $\boldsymbol{e}$ respectively relative to an origin O. A is the midpoint of OB and E divides AC in the ratio 1 : 2. If $\boldsymbol{e} = \frac{1}{3}\boldsymbol{d}$, show that $OCDB$ is a parallelogram.

19 The points A and B have position vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ respectively, relative to an origin O. The point P divides the line segment OA in the ratio 1 : 3 and the point R divides the line segment AB in the ratio 1 : 2. Given that $PRBQ$ is a parallelogram, determine the position of Q.

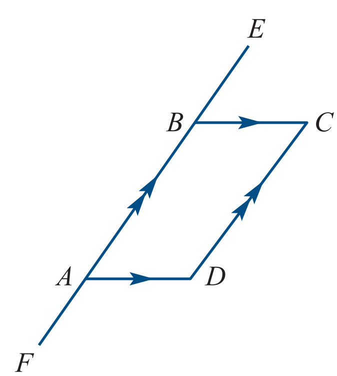
20 $ABCD$ is a parallelogram. AB is produced to E and BA produced to F such that $BE = AF = BC$. Lines EC and FD are produced to meet at X.

a Prove that the lines EX and FX meet at right angles.

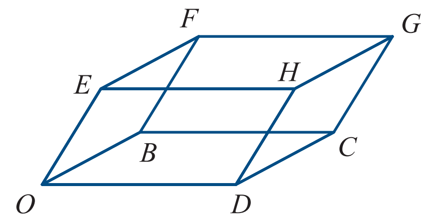
b If $\overrightarrow{EX} = \lambda\overrightarrow{EC}$ and $\overrightarrow{FX} = \mu\overrightarrow{FD}$ and $|\overrightarrow{AB}| = k\,|\overrightarrow{BC}|$ find the values of λ and μ in terms of k.

c Find the values of λ and μ if $ABCD$ is a rhombus.

d If $|\overrightarrow{EX}| = |\overrightarrow{FX}|$ prove that $ABCD$ is a rectangle.

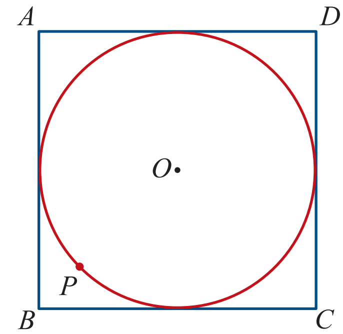


21 $OBCDEFGH$ is a parallelepiped. Let $\overrightarrow{OB} = \boldsymbol{b}$, $\overrightarrow{OD} = \boldsymbol{d}$ and $\overrightarrow{OE} = \boldsymbol{e}$.



- **a** Express each of the vectors $\overrightarrow{OG}$, $\overrightarrow{DF}$, $\overrightarrow{BH}$ and $\overrightarrow{CE}$ in terms of $\boldsymbol{b}$, $\boldsymbol{d}$ and $\boldsymbol{e}$.
- **b** Find $|\overrightarrow{OG}|^2$, $|\overrightarrow{DF}|^2$, $|\overrightarrow{BH}|^2$ and $|\overrightarrow{CE}|^2$ in terms of $\boldsymbol{b}$, $\boldsymbol{d}$, and $\boldsymbol{e}$.
- **c** Show that $|\overrightarrow{OG}|^2 + |\overrightarrow{DF}|^2 + |\overrightarrow{BH}|^2 + |\overrightarrow{CE}|^2 = 4(|\boldsymbol{b}|^2 + |\boldsymbol{d}|^2 + |\boldsymbol{e}|^2)$.

22 In the figure, O is the centre of the circle, radius r.
The circle is inscribed in a square $ABCD$.
P is any point on the circumference of the circle.

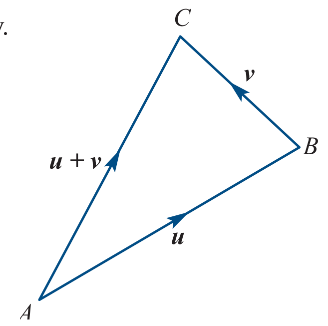


- **a** Show that $\overrightarrow{AP}.\overrightarrow{AP} = 3r^2 - 2\overrightarrow{OP}.\overrightarrow{OA}$
- **b** Hence find $AP^2 + BP^2 + CP^2 + DP^2$ in terms of r.

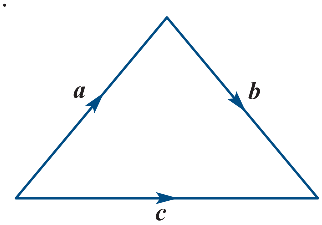
MC TEST D&D TEST

Chapter summary

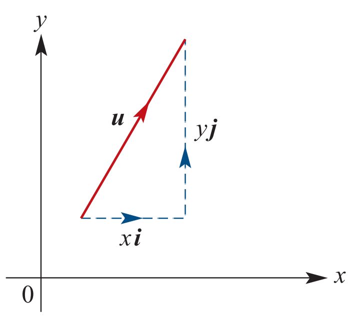
- A **vector** is a set of equivalent **directed line** segments.
- A directed line segment from a point A to a point B is denoted by $\overrightarrow{AB}$.
- In two dimensions, a vector can be represented by a column of numbers, e.g. $\begin{bmatrix} 2 \\ 3 \end{bmatrix}$ is the vector '2 across' and '3 up'.
- The sum $\boldsymbol{u} + \boldsymbol{v}$ can be shown diagrammatically.



- $\boldsymbol{u} + \boldsymbol{v} = \boldsymbol{v} + \boldsymbol{u}$.
- If $\boldsymbol{u} = \begin{bmatrix} a \\ b \end{bmatrix}$ and $\boldsymbol{v} = \begin{bmatrix} c \\ d \end{bmatrix}$ then $\boldsymbol{u} + \boldsymbol{v} = \begin{bmatrix} a+c \\ b+d \end{bmatrix}$.
- The vector $k\boldsymbol{u}$, $k \in R^+$, has the same direction as $\boldsymbol{u}$ but its length is multiplied by a factor k.
- The vector $-\boldsymbol{v}$ is in the opposite direction to $\boldsymbol{v}$ but has the same length.
- $\boldsymbol{u} - \boldsymbol{v} = \boldsymbol{u} + (-\boldsymbol{v})$.
- Two non-zero vectors $\boldsymbol{u}$ and $\boldsymbol{v}$ are said to be parallel if there exists $k \in R \setminus \{0\}$ such that $\boldsymbol{u} = k\boldsymbol{v}$.
- For a point A, the position vector of A is $\overrightarrow{OA}$ where O is the origin.
- The sum $\boldsymbol{a} + \boldsymbol{b} = \boldsymbol{c}$ can be drawn as a triangle of vectors.

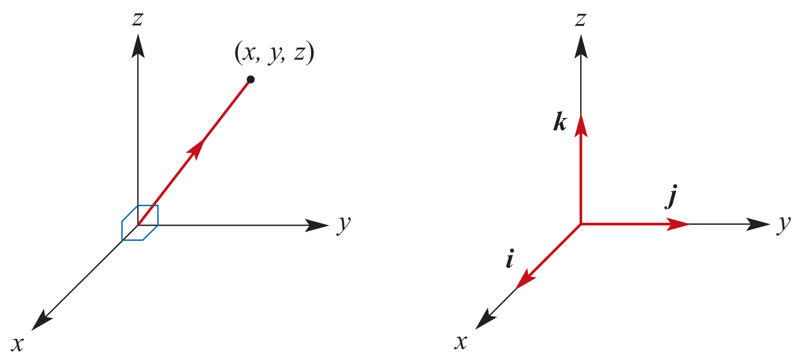


- In two dimensions a vector $\boldsymbol{u}$ can be expressed as the sum of two vectors $x\boldsymbol{i}$ and $y\boldsymbol{j}$, where $\boldsymbol{i}$ is the unit vector in the positive direction of the x axis and $\boldsymbol{j}$ is the unit vector in the positive direction of the y axis.



Review

- In three dimensions a vector $\boldsymbol{u}$ can be written as $\boldsymbol{u} = x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k}$, where $\boldsymbol{i}, \boldsymbol{j}$ and $\boldsymbol{k}$ are unit vectors as shown.



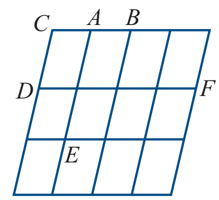
- The magnitude of vector $\boldsymbol{u} = x\boldsymbol{i} + y\boldsymbol{j}$ is denoted by $|\boldsymbol{u}|$ or u, and $|\boldsymbol{u}| = \sqrt{x^2 + y^2}$.
- The magnitude of vector $\boldsymbol{u} = x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k}$ is given by $|\boldsymbol{u}| = \sqrt{x^2 + y^2 + z^2}$.
- If $\boldsymbol{a} = \boldsymbol{b}$, where $\boldsymbol{a} = a_1\boldsymbol{i} + a_2\boldsymbol{j} + a_3\boldsymbol{k}$ and $\boldsymbol{b} = b_1\boldsymbol{i} + b_2\boldsymbol{j} + b_3\boldsymbol{k}$, then $a_1 = b_1$, $a_2 = b_2$ and $a_3 = b_3$.
- $|\boldsymbol{a}| = \sqrt{\boldsymbol{a}.\boldsymbol{a}} = a$.
- The unit vector in the direction of $\boldsymbol{a}$ is $\dfrac{\boldsymbol{a}}{|\boldsymbol{a}|}$ and is denoted by $\hat{\boldsymbol{a}}$.
- If the vector $\boldsymbol{a} = a_1\boldsymbol{i} + a_2\boldsymbol{j} + a_3\boldsymbol{k}$ makes angles α, β and γ measured anticlockwise from the positive directions of the x, y and z axes respectively, then $\cos\alpha = \dfrac{a_1}{|\boldsymbol{a}|}$, $\cos\beta = \dfrac{a_2}{|\boldsymbol{a}|}$ and $\cos\gamma = \dfrac{a_3}{|\boldsymbol{a}|}$.
- The scalar product (or dot product) is given by $\boldsymbol{a}.\boldsymbol{b} = |\boldsymbol{a}|\,|\boldsymbol{b}|\cos\theta$ where θ is the angle between the directions of $\boldsymbol{a}$ and $\boldsymbol{b}$.
- If $\boldsymbol{a} = a_1\boldsymbol{i} + a_2\boldsymbol{j} + a_3\boldsymbol{k}$ and $\boldsymbol{b} = b_1\boldsymbol{i} + b_2\boldsymbol{j} + b_3\boldsymbol{k}$, then the scalar product is given by $\boldsymbol{a}.\boldsymbol{b} = a_1b_1 + a_2b_2 + a_3b_3$.
- Non-zero vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ are perpendicular if and only if $\boldsymbol{a}.\boldsymbol{b} = 0$.
- The magnitude of the angle θ between two vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ can be found using $\cos\theta = \dfrac{\boldsymbol{a}.\boldsymbol{b}}{|\boldsymbol{a}|\,|\boldsymbol{b}|}$ or $\cos\theta = \dfrac{a_1b_1 + a_2b_2 + a_3b_3}{|\boldsymbol{a}|\,|\boldsymbol{b}|}$.
- The resolution of a vector into rectangular components expresses the vector as a sum of two vectors; one in a stated direction and the other perpendicular to that direction.
- The vector resolute of $\boldsymbol{a}$ in the direction of $\boldsymbol{b}$ is the projection of $\boldsymbol{a}$ on a line parallel to $\boldsymbol{b}$.
- The **vector resolute** of $\boldsymbol{a}$ in the direction of $\boldsymbol{b}$ is $\dfrac{\boldsymbol{a}.\boldsymbol{b}}{\boldsymbol{b}.\boldsymbol{b}}\boldsymbol{b}$.
- The **scalar resolute** of $\boldsymbol{a}$ in the direction of $\boldsymbol{b}$ is $\dfrac{\boldsymbol{a}.\boldsymbol{b}}{|\boldsymbol{b}|}$.
- A set of vectors is said to be linearly dependent if one of its members can be expressed as a linear combination of the other members.
- A set of vectors is said to be linearly independent if it is not linearly dependent.

Multiple-choice questions

1 If $\overrightarrow{OA} = \boldsymbol{a} + 2\boldsymbol{b}$ and $\overrightarrow{AB} = \boldsymbol{a} - \boldsymbol{b}$ then $\overrightarrow{OB}$ in terms of $\boldsymbol{a}$ and $\boldsymbol{b}$ is equal to:

A $\boldsymbol{b}$ **B** $3\boldsymbol{b}$ **C** $2\boldsymbol{a} + \boldsymbol{b}$ **D** $2\boldsymbol{a} + 3\boldsymbol{b}$ **E** $3\boldsymbol{a} + \boldsymbol{b}$

2 The grid shown is made up of identical parallelograms. $\overrightarrow{AB} = \boldsymbol{a}$ and $\overrightarrow{CD} = \boldsymbol{c}$. The vector $\overrightarrow{EF}$ in terms of $\boldsymbol{a}$ and $\boldsymbol{c}$ is equal to:



A $\boldsymbol{a} + 3\boldsymbol{c}$ **B** $-3\boldsymbol{a} + \boldsymbol{c}$ **C** $-3\boldsymbol{a} - \boldsymbol{c}$
D $3\boldsymbol{a} - \boldsymbol{c}$ **E** $3\boldsymbol{a} + \boldsymbol{c}$

3 $ABCD$ is a parallelogram. $\overrightarrow{AB} = \boldsymbol{u}$ and $\overrightarrow{BC} = \boldsymbol{v}$. M is the midpoint of $\overrightarrow{AB}$. Vector $\overrightarrow{DM}$ expressed in terms of $\boldsymbol{u}$ and $\boldsymbol{v}$ is equal to:

A $\frac{1}{2}\boldsymbol{u} + \boldsymbol{v}$ **B** $\frac{1}{2}\boldsymbol{u} - \boldsymbol{v}$ **C** $\boldsymbol{u} + \frac{1}{2}\boldsymbol{v}$ **D** $\boldsymbol{u} - \frac{1}{2}\boldsymbol{v}$ **E** $\frac{3}{2}\boldsymbol{u} - \boldsymbol{v}$

4 $A = (3, 6)$ and $B = (11, 1)$. The vector $\overrightarrow{AB}$ in terms of $\boldsymbol{i}$ and $\boldsymbol{j}$ is equal to:

A $3\boldsymbol{i} + 6\boldsymbol{j}$ **B** $8\boldsymbol{i} - 5\boldsymbol{j}$ **C** $8\boldsymbol{i} + 5\boldsymbol{j}$ **D** $14\boldsymbol{i} + 7\boldsymbol{j}$ **E** $14\boldsymbol{i} - 7\boldsymbol{j}$

5 The angle between the vector $2\boldsymbol{i} + \boldsymbol{j} - \sqrt{2}\boldsymbol{k}$ and $5\boldsymbol{i} + 8\boldsymbol{j}$ is approximately:

A $0.72°$ **B** $0.77°$ **C** $43.85°$ **D** $46.15°$ **E** $88.34°$

6 Let OAB be a triangle such that $\overrightarrow{AO} \cdot \overrightarrow{AB} = \overrightarrow{BO} \cdot \overrightarrow{BA}$ and $|\overrightarrow{AB}| \neq |\overrightarrow{OB}|$. Then triangle OAB must be:

A scalene **B** equilateral **C** isosceles **D** right-angled **E** obtuse

7 If $\boldsymbol{a}$ and $\boldsymbol{b}$ are non-zero, non-parallel vectors such that $x(\boldsymbol{a} + \boldsymbol{b}) = 2y\boldsymbol{a} + (y + 3)\boldsymbol{b}$ then the values of x and y are:

A $x = 3, y = 6$ **B** $x = -6, y = -3$ **C** $x = -2, y = -1$
D $x = 2, y = 1$ **E** $x = 6, y = 3$

8 If A and B are points defined respectively by the position vectors $\boldsymbol{a} = \boldsymbol{i} + \boldsymbol{j}$ and $\boldsymbol{b} = 5\boldsymbol{i} - 2\boldsymbol{j} + 2\boldsymbol{k}$ then $|\overrightarrow{AB}|$ is equal to:

A 29 **B** $\sqrt{11}$ **C** 11 **D** $\sqrt{21}$ **E** $\sqrt{29}$

9 Let $\boldsymbol{x} = 3\boldsymbol{i} - 2\boldsymbol{j} + 4\boldsymbol{k}$ and $\boldsymbol{y} = -5\boldsymbol{i} + \boldsymbol{j} + \boldsymbol{k}$. The scalar resolute of $\boldsymbol{x}$ in the direction of $\boldsymbol{y}$ is:

A $\dfrac{21}{\sqrt{27}}$ **B** $\dfrac{-13\sqrt{23}}{23}$ **C** $\dfrac{-13\sqrt{29}}{29}$ **D** $\dfrac{-13\sqrt{27}}{27}$ **E** $\dfrac{-13\sqrt{21}}{21}$

10 Let $ABCD$ be a rectangle such that $|\overrightarrow{BC}| = 3|\overrightarrow{AB}|$. If $\overrightarrow{AB} = \boldsymbol{a}$ then $|\overrightarrow{AC}|$ in terms of $|\boldsymbol{a}|$ is equal to:

A $2|\boldsymbol{a}|$ **B** $\sqrt{10}|\boldsymbol{a}|$ **C** $4|\boldsymbol{a}|$ **D** $10|\boldsymbol{a}|$ **E** $3|\boldsymbol{a}|$

Short-answer questions (technology-free)

1 $ABCD$ is a parallelogram. A, B and C are represented by the position vectors $\boldsymbol{i} + 2\boldsymbol{j} - \boldsymbol{k}$, $2\boldsymbol{i} + \boldsymbol{j} - 2\boldsymbol{k}$ and $4\boldsymbol{i} - \boldsymbol{k}$ respectively. Find:

a $\overrightarrow{AD}$ **b** the cosine of $\angle BAD$

Review

2 A, B and C are points defined by position vectors $2\boldsymbol{i} - \boldsymbol{j} - 4\boldsymbol{k}$, $-\boldsymbol{i} + \boldsymbol{j} + 2\boldsymbol{k}$ and $\boldsymbol{i} - 3\boldsymbol{j} - 2\boldsymbol{k}$ respectively. M is a point of line segment AB, such that $|\overrightarrow{AM}| = |\overrightarrow{AC}|$.

a Find:

i $\overrightarrow{AM}$ ii the position vector which represents N, the midpoint of CM

b Hence, show that $\overrightarrow{AN} \perp \overrightarrow{CM}$.

3 Let $\boldsymbol{a} = 4\boldsymbol{i} + 3\boldsymbol{j} - \boldsymbol{k}$, $\boldsymbol{b} = 2\boldsymbol{i} - \boldsymbol{j} + x\boldsymbol{k}$ and $\boldsymbol{c} = y\boldsymbol{i} + z\boldsymbol{j} - 2\boldsymbol{k}$. Find:

a x, such that $\boldsymbol{a}$ and $\boldsymbol{b}$ are perpendicular to each other

b y and z, such that $\boldsymbol{a}$, $\boldsymbol{b}$ and $\boldsymbol{c}$ are mutually perpendicular

4 Let $\boldsymbol{a} = \boldsymbol{i} - 2\boldsymbol{j} + 2\boldsymbol{k}$ and let $\boldsymbol{b}$ be a vector, such that the vector resolute of $\boldsymbol{a}$ in the direction of $\boldsymbol{b}$ is $\hat{\boldsymbol{b}}$.

a Find the cosine of the angle between the directions of $\boldsymbol{a}$ and $\boldsymbol{b}$.

b Find $|\boldsymbol{b}|$ if the vector resolute of $\boldsymbol{b}$ in the direction of $\boldsymbol{a}$ is $2\hat{\boldsymbol{a}}$.

5 Let $\boldsymbol{a} = 3\boldsymbol{i} - 6\boldsymbol{j} + 4\boldsymbol{k}$ and $\boldsymbol{b} = 2\boldsymbol{i} + \boldsymbol{j} - 2\boldsymbol{k}$.

a Find $\boldsymbol{c}$, the vector component of $\boldsymbol{a}$ perpendicular to $\boldsymbol{b}$.

b Find $\boldsymbol{d}$, the vector resolute of $\boldsymbol{c}$ in the direction of $\boldsymbol{a}$.

c Hence, show that $|\boldsymbol{a}|\,|\boldsymbol{d}| = |\boldsymbol{c}|^2$.

6 A and B are points defined by $\boldsymbol{a} = 2\boldsymbol{i} + 3\boldsymbol{j} - 4\boldsymbol{k}$ and $\boldsymbol{b} = 2\boldsymbol{i} - \boldsymbol{j} + 2\boldsymbol{k}$. C is a point defined by the position vector $\boldsymbol{c} = 2\boldsymbol{i} + (1 + 3t)\boldsymbol{j} + (-1 + 2t)\boldsymbol{k}$.

a Find, in terms of t:

i $\overrightarrow{CA}$ ii $\overrightarrow{CB}$

b Find the values of t for which $\angle BCA = 90°$.

7 $OABC$ is a parallelogram. A and C are defined by $\boldsymbol{a} = 2\boldsymbol{i} + 2\boldsymbol{j} - \boldsymbol{k}$ and $\boldsymbol{c} = 2\boldsymbol{i} - 6\boldsymbol{j} - 3\boldsymbol{k}$.

a Find:

i $|\boldsymbol{a} - \boldsymbol{c}|$ ii $|\boldsymbol{a} + \boldsymbol{c}|$ iii $(\boldsymbol{a} - \boldsymbol{c}).(\boldsymbol{a} + \boldsymbol{c})$

b Hence, find the magnitude of the acute angle between the diagonals of the parallelogram.

8 $OABC$ is a trapezium with $\overrightarrow{OC} = 2\overrightarrow{AB}$. If $\overrightarrow{OA} = 2\boldsymbol{i} - \boldsymbol{j} - 3\boldsymbol{k}$ and $\overrightarrow{OC} = 6\boldsymbol{i} - 3\boldsymbol{j} + 2\boldsymbol{k}$, find:

a $\overrightarrow{AB}$ b $\overrightarrow{BC}$ c the cosine of $\angle BAC$

9 The position vectors of A and B relative to an origin, O, are $6\boldsymbol{i} + 4\boldsymbol{j}$ and $3\boldsymbol{i} + p\boldsymbol{j}$ respectively.

a Express $\overrightarrow{AO}.\overrightarrow{AB}$ in terms of p.

b Find the value of p for which $\overrightarrow{AO}$ is perpendicular to $\overrightarrow{AB}$.

c Find the cosine of $\angle OAB$ when $p = 6$.

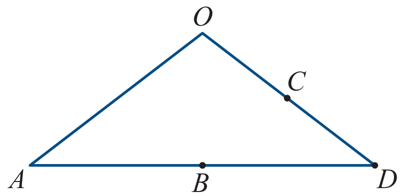
10 The points A, B and C have position vectors $\boldsymbol{p} + \boldsymbol{q}$, $3\boldsymbol{p} - 2\boldsymbol{q}$ and $6\boldsymbol{p} + m\boldsymbol{q}$ respectively relative to an origin O where p and q are non-parallel, non-zero vectors. Find the value of m for which A, B and C are collinear.

11 If $\boldsymbol{r} = 3\boldsymbol{i} + 3\boldsymbol{j} - 6\boldsymbol{k}$, $\boldsymbol{s} = \boldsymbol{i} - 7\boldsymbol{j} + 6\boldsymbol{k}$ and $\boldsymbol{t} = -2\boldsymbol{i} - 5\boldsymbol{j} + 2\boldsymbol{k}$, find the values of λ and μ such that the vector $\boldsymbol{r} + \lambda\boldsymbol{s} + \mu\boldsymbol{t}$ is parallel to the x axis.

12 Show that the points $A(4, 3, 0)$, $B(5, 2, 3)$, $C(4, -1, 3)$ and $D(2, 1, -3)$ form a trapezium and state the ratio of the parallel sides.

13 If $\boldsymbol{a} = 2\boldsymbol{i} - \boldsymbol{j} + 6\boldsymbol{k}$ and $\boldsymbol{b} = \boldsymbol{i} - \boldsymbol{j} - \boldsymbol{k}$ show that $\boldsymbol{a} + \boldsymbol{b}$ is perpendicular to $\boldsymbol{b}$ and find the cosine of the angle between the vectors $\boldsymbol{a} + \boldsymbol{b}$ and $\boldsymbol{a} - \boldsymbol{b}$.

14 Relative to O, the position vectors of A, B and C are $\boldsymbol{a}$, $\boldsymbol{b}$ and $\boldsymbol{c}$. B and C are the midpoints of AD and OD respectively.



a Find $\overrightarrow{OD}$ and $\overrightarrow{AD}$ in terms of $\boldsymbol{a}$ and $\boldsymbol{c}$.

b Find $\boldsymbol{b}$ in terms of $\boldsymbol{a}$ and $\boldsymbol{c}$.

c E is a point on OA produced such that $\overrightarrow{OE} = 4\,\overrightarrow{AE}$. If $\overrightarrow{CB} = k\,\overrightarrow{AE}$, find the value of k.

15 O, A and B are the points with coordinates $(0, 0)$, $(3, 4)$ and $(4, -6)$ respectively.

a C is the point such that $\overrightarrow{OA} = \overrightarrow{OC} + \overrightarrow{OB}$. Find the coordinates of C.

b D is the point $(1, 24)$ and $\overrightarrow{OD} = h\overrightarrow{OA} + k\overrightarrow{OB}$. Find the values of h and k.

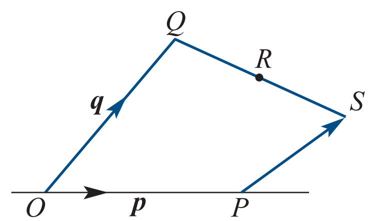
16 $\overrightarrow{OQ} = \boldsymbol{q}$ and $\overrightarrow{OP} = \boldsymbol{p}$

$\overrightarrow{OR} = \frac{1}{3}\boldsymbol{p} + k\boldsymbol{q}$

$\overrightarrow{OS} = h\boldsymbol{p} + \frac{1}{2}\boldsymbol{q}$

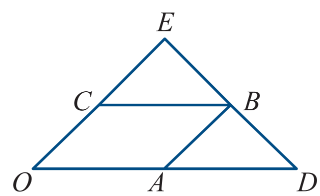
R is the midpoint of QS.

Find h and k.



17 ABC is a right-angled triangle with the right angle at B. If $\overrightarrow{AC} = 2\boldsymbol{i} + 4\boldsymbol{j}$ and $\overrightarrow{AB}$ is parallel to $\boldsymbol{i} + \boldsymbol{j}$, find $\overrightarrow{AB}$.

18 In this diagram, $OABC$ is a parallelogram with $\overrightarrow{OA} = 2\overrightarrow{AD}$. Let $\overrightarrow{AD} = \boldsymbol{a}$ and $\overrightarrow{OC} = \boldsymbol{c}$.



a Express $\overrightarrow{DB}$ in terms of $\boldsymbol{a}$ and $\boldsymbol{c}$.

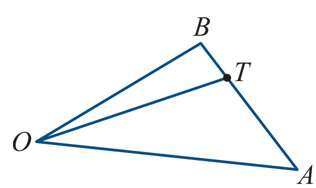
b Use a vector method to prove $\overrightarrow{OE} = 3\overrightarrow{OC}$.

19 For quadrilateral $OABC$, D is the point of trisection of OC nearer O and E is the point of trisection of AB nearer A. If $\overrightarrow{OA} = \boldsymbol{a}$, $\overrightarrow{OB} = \boldsymbol{b}$ and $\overrightarrow{OC} = \boldsymbol{c}$:

a Find: **i** $\overrightarrow{OD}$ **ii** $\overrightarrow{OE}$ **iii** $\overrightarrow{DE}$

b Hence prove that $3\overrightarrow{DE} = 2\overrightarrow{OA} + \overrightarrow{CB}$.

20 In triangle OAB, $\overrightarrow{OA} = \boldsymbol{a}$, $\overrightarrow{OB} = \boldsymbol{b}$ and T is a point on AB such that $AT = 3TB$.



a Find $\overrightarrow{OT}$ in terms of $\boldsymbol{a}$ and $\boldsymbol{b}$.

b If M is a point such that $\overrightarrow{OM} = \lambda\overrightarrow{OT}$, $\lambda > 1$, find:

i $\overrightarrow{BM}$ in terms of $\boldsymbol{a}$, $\boldsymbol{b}$ and λ **ii** λ, if $\overrightarrow{BM}$ is parallel to $\overrightarrow{OA}$

Extended-response questions

1 A spider builds a web in a garden. The position vectors of the ends A and B relative to an origin O of a strand of the web are given by $\overrightarrow{OA} = 2\boldsymbol{i} + 3\boldsymbol{j} + \boldsymbol{k}$ and $\overrightarrow{OB} = 3\boldsymbol{i} + 4\boldsymbol{j} + 2\boldsymbol{k}$ respectively.

a Find: **i** $\overrightarrow{AB}$ **ii** the length of the strand

b A small insect is at point C where $\overrightarrow{OC} = 2.5\boldsymbol{i} + 4\boldsymbol{j} + 1.5\boldsymbol{k}$. Unluckily it flies in a straight line and hits the strand of web between A and B. Let Q be the point where the insect hits the strand where $\overrightarrow{AQ} = \lambda\overrightarrow{AB}$.

i Find $\overrightarrow{CQ}$ in terms of λ.

ii If the insect hits the strand at right angles, find the value of λ and the vector $\overrightarrow{OQ}$.

c Another web strand MN has end points M and N with position vectors $\overrightarrow{OM} = 4\boldsymbol{i} + 2\boldsymbol{j} - \boldsymbol{k}$ and $\overrightarrow{ON} = 6\boldsymbol{i} + 10\boldsymbol{j} + 9\boldsymbol{k}$. The spider decides to continue AB to join MN. Find the position vector of the point of contact.

2 The position vectors of points A and B relative to an origin O are $2\boldsymbol{i} + 3\boldsymbol{j} + \boldsymbol{k}$ and $3\boldsymbol{i} - 2\boldsymbol{j} + \boldsymbol{k}$.

a Find: **i** $|\overrightarrow{OA}|$ and $|\overrightarrow{OB}|$ **ii** $\overrightarrow{AB}$

b X is the midpoint of line segment AB.

i Find $\overrightarrow{OX}$. **ii** Show that $\overrightarrow{OX}$ is perpendicular to $\overrightarrow{AB}$.

c Find the position vector of a point C such that $OACB$ is a parallelogram.

d Show that the diagonal OC is perpendicular to the diagonal AB by considering $\overrightarrow{OC}.\overrightarrow{AB}$.

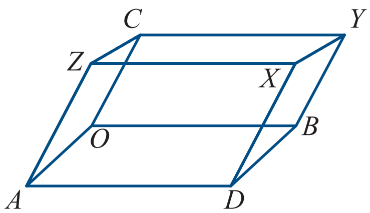
e **i** Find a vector of magnitude $\sqrt{195}$ which is perpendicular to both $\overrightarrow{OA}$ and $\overrightarrow{OB}$.

ii Show that this vector is perpendicular to $\overrightarrow{AB}$ and $\overrightarrow{OC}$.

iii Comment on the relationship between the vector found in **e(i)** and the parallelogram $OACB$.

3 Points A, B and C have position vectors

$$\overrightarrow{OA} = 5\boldsymbol{i}$$
$$\overrightarrow{OB} = \boldsymbol{i} + 3\boldsymbol{k}$$
and $$\overrightarrow{OC} = \boldsymbol{i} + 4\boldsymbol{j}$$



The parallelepiped has OA, OB and OC as three edges and remaining vertices X, Y, Z and D as shown in the diagram.

a Write down the position vectors of X, Y, Z and D in terms of $\boldsymbol{i}$, $\boldsymbol{j}$ and $\boldsymbol{k}$ and calculate the length of OD and OY.

b Calculate the size of angle OZY.

c The point P divides CZ in the ratio $\lambda : 1$, i.e. $CP : PZ = \lambda : 1$.

i Give the position vector of P.

ii Find λ if $\overrightarrow{OP}$ is perpendicular to $\overrightarrow{CZ}$.

4 ABC is a triangle as shown in the diagram below. P, Q and R are the midpoints of the sides BC, CA and AB respectively. O is the point of intersection of the perpendicular bisectors of CA and AB.

Let $\overrightarrow{OA} = \boldsymbol{a}$, $\overrightarrow{OB} = \boldsymbol{b}$ and $\overrightarrow{OC} = \boldsymbol{c}$.

a Express each of the following in terms of $\boldsymbol{a}$, $\boldsymbol{b}$ and $\boldsymbol{c}$:

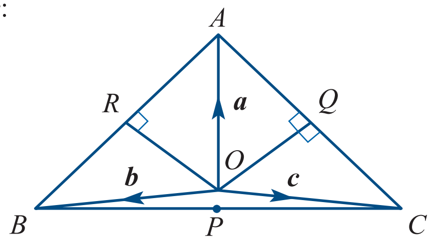
i $\overrightarrow{AB}$ **ii** $\overrightarrow{BC}$ **iii** $\overrightarrow{CA}$

iv $\overrightarrow{OP}$ **v** $\overrightarrow{OQ}$ **vi** $\overrightarrow{OR}$

b Prove that OP is perpendicular to BC.

c Hence prove that the perpendicular bisectors of the sides of a triangle are concurrent.

d Prove that $|\boldsymbol{a}| = |\boldsymbol{b}| = |\boldsymbol{c}|$.



5 The position vectors of two points B and C relative to an origin O are denoted by $\boldsymbol{b}$ and $\boldsymbol{c}$ respectively.

a Find the position vector of L, the point on BC between B and C such that $BL : LC = 2 : 1$, in terms of $\boldsymbol{b}$ and $\boldsymbol{c}$.

b If $\boldsymbol{a}$ is the position vector of a point A such that O is the midpoint of AL, prove that $3\boldsymbol{a} + \boldsymbol{b} + 2\boldsymbol{c} = \boldsymbol{0}$.

c If M is the point on CA between C and A such that $CM : MA = 3 : 2$ then:

i prove that B, O and M are collinear

ii find the ratio $BO : OM$.

d N is the point in AB such that C, O and N are collinear. Find the ratio $AN : NB$.

6 OAB is an isosceles triangle with $OA = OB$. Let $\overrightarrow{OA} = \boldsymbol{a}$ and $\overrightarrow{OB} = \boldsymbol{b}$.

a D is the midpoint of AB and E is a point on OB. Find in terms of $\boldsymbol{a}$ and $\boldsymbol{b}$:

i $\overrightarrow{OD}$

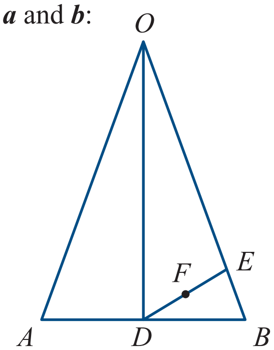
ii $\overrightarrow{DE}$ if $\overrightarrow{OE} = \lambda\overrightarrow{OB}$.

b If DE is perpendicular to OB show that $\lambda = \frac{1}{2}\dfrac{(\boldsymbol{a.b} + \boldsymbol{b.b})}{\boldsymbol{b.b}}$.

c DE is perpendicular to OB and $\lambda = \frac{5}{6}$.

i Show that $\cos\theta = \frac{2}{3}$ where θ is a magnitude of $\angle AOB$.

ii Let F be the midpoint of DE. Show that OF is perpendicular to AE.

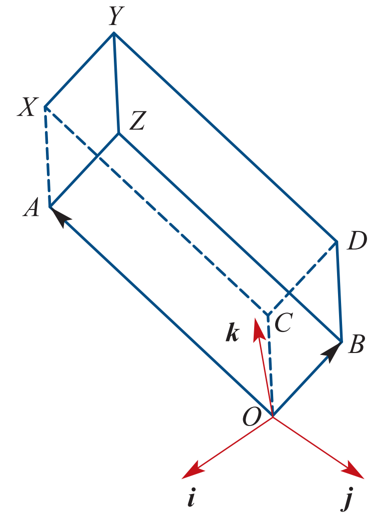


7 A cuboid is positioned on level ground so that it rests on one of its vertices, O. The vectors $\boldsymbol{i}$ and $\boldsymbol{j}$ are on the ground.

$\overrightarrow{OA} = 3\boldsymbol{i} - 12\boldsymbol{j} + 3\boldsymbol{k}$

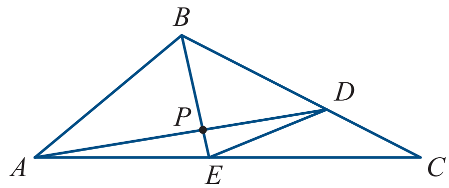
$\overrightarrow{OB} = 2\boldsymbol{i} + a\boldsymbol{j} + 2\boldsymbol{k}$

$\overrightarrow{OC} = x\boldsymbol{i} + y\boldsymbol{j} + 2\boldsymbol{k}$



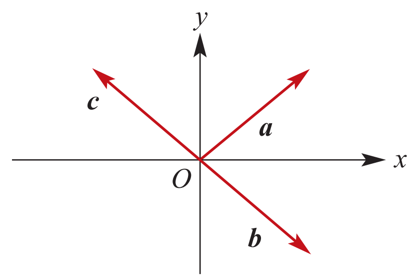
- **a** **i** Find $\overrightarrow{OA}.\overrightarrow{OB}$ in terms of a.
  - **ii** Find a.
- **b** **i** Use the fact that $\overrightarrow{OA}$ is perpendicular to $\overrightarrow{OC}$ to write an equation relating x and y.
  - **ii** Find the values of x and y.
- **c** Find the position vectors:
  i $\overrightarrow{OD}$ **ii** $\overrightarrow{OX}$ **iii** $\overrightarrow{OY}$
- **d** State the height of points X and Y above the ground.

8 D is a point on BC such that $\frac{BD}{DC} = 3$ and E is a point on AC such that $\frac{AE}{EC} = \frac{3}{2}$.
Let P be the point of intersection of AD and BE.
Let $\overrightarrow{BA} = \boldsymbol{a}$ and $\overrightarrow{BC} = \boldsymbol{c}$.



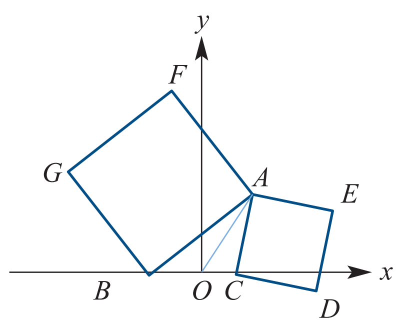
- **a** Find:
  - **i** $\overrightarrow{BD}$ in terms of $\boldsymbol{c}$
  - **ii** $\overrightarrow{BE}$ in terms of $\boldsymbol{a}$ and $\boldsymbol{c}$
  - **iii** $\overrightarrow{AD}$ in terms of $\boldsymbol{a}$ and $\boldsymbol{c}$
- **b** Let $\overrightarrow{BP} = \mu\overrightarrow{BE}$ and $\overrightarrow{AP} = \lambda\overrightarrow{AD}$. Find λ and μ.

9 **a** $\boldsymbol{a} = p\boldsymbol{i} + q\boldsymbol{j}$. The vector $\boldsymbol{b}$ is obtained by rotating $\boldsymbol{a}$ clockwise through 90° about the origin. The vector $\boldsymbol{c}$ is obtained by rotating $\boldsymbol{a}$ anticlockwise through 90° about the origin.
Find $\boldsymbol{b}$ and $\boldsymbol{c}$ in terms of p, q, $\boldsymbol{i}$ and $\boldsymbol{j}$.



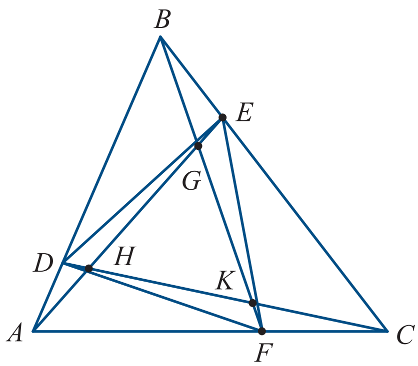
b In the diagram opposite, $OB = OC = 1$, and $ABGF$ and $AEDC$ are squares.
Let $\overrightarrow{OA} = x\boldsymbol{i} + y\boldsymbol{j}$.

- **i** Find $\overrightarrow{AB}$ and $\overrightarrow{AC}$ in terms of x, y, $\boldsymbol{i}$ and $\boldsymbol{j}$.
- **ii** Use the results of **a** to find $\overrightarrow{AE}$ and $\overrightarrow{AF}$ in terms of x, y, $\boldsymbol{i}$ and $\boldsymbol{j}$.



c **i** Prove that $\overrightarrow{OA}$ is perpendicular to $\overrightarrow{EF}$.
ii Prove that $|\overrightarrow{EF}| = 2|\overrightarrow{OA}|$.

10 Triangle ABC is equilateral. Also $AD = BE = CF$.



a Let $\boldsymbol{u}$, $\boldsymbol{v}$ and $\boldsymbol{w}$ be unit vectors in the direction of $\overrightarrow{AB}$, $\overrightarrow{BC}$ and $\overrightarrow{CA}$ respectively.

Let $AB = m\boldsymbol{u}$ and $AD = n\boldsymbol{u}$.

i Find $\overrightarrow{BC}$, $\overrightarrow{BE}$, $\overrightarrow{CA}$ and $\overrightarrow{CF}$.

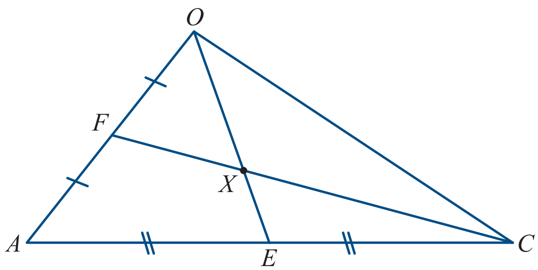
ii Find $|\overrightarrow{AE}|$ and $|\overrightarrow{FB}|$ in terms of m and n.

b Show that $\overrightarrow{AE}.\overrightarrow{FB} = \frac{1}{2}(m^2 - mn + n^2)$.

c Show that triangle GHK is equilateral.

(G is the point of intersection of BF and AE.
H is the point of intersection of AE and CD.
K is the point of intersection of CD and BF.)

11 AOC is a triangle. The medians CF and OE intersect at X.



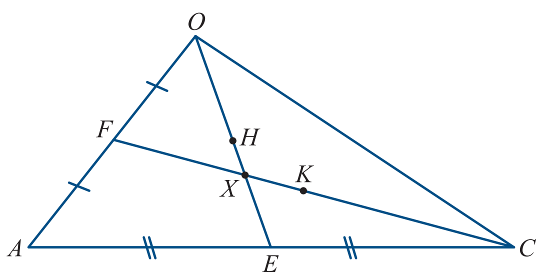
Let $\overrightarrow{OA} = \boldsymbol{a}$ and $\overrightarrow{OC} = \boldsymbol{c}$.

a Find $\overrightarrow{CF}$ and $\overrightarrow{OE}$ in terms of $\boldsymbol{a}$ and $\boldsymbol{c}$.

b **i** If $\overrightarrow{OE}$ is perpendicular to $\overrightarrow{AC}$, prove that triangle OAC is isosceles.

ii If furthermore $\overrightarrow{CF}$ is perpendicular to $\overrightarrow{OA}$ find the magnitude of angle AOC, and hence prove that triangle AOC is equilateral.

c H and K are the midpoints of OE and CF respectively.



i Show that $\overrightarrow{HK} = \lambda\boldsymbol{c}$ for some $\lambda \in R \setminus \{0\}$ and $\overrightarrow{FE} = \mu\boldsymbol{c}$ for some $\mu \in R \setminus \{0\}$.

ii Give reasons why triangle HXK is similar to triangle EXF (vector method not required).

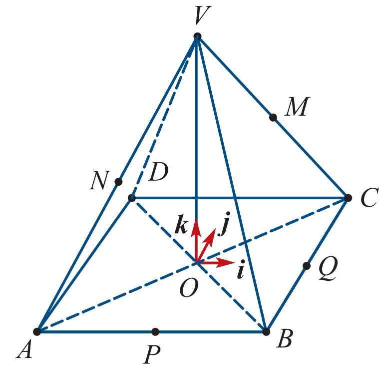
iii Hence prove that $OX : XE = 2 : 1$.

12 $VABCD$ is a square-based pyramid and O is the centre of the base (see diagram on the next page). $\boldsymbol{i}$, $\boldsymbol{j}$ and $\boldsymbol{k}$ are unit vectors in the direction of AB, BC and OV respectively. The point O is to be taken as the origin for position vectors.

$AB = BC = CD = DA = 4$ cm.

$OV = 2h$ cm where h is a positive real number.

P, Q, M and N are the midpoints of AB, BC, VC and VA respectively.



a Find the position vectors of A, B, C and D relative to O.

b Find vectors $\overrightarrow{PM}$ and $\overrightarrow{QN}$, in terms of h.

c Find the position vector $\overrightarrow{OX}$, where X the point of intersection of QN and PM.

d If OX is perpendicular to VB:

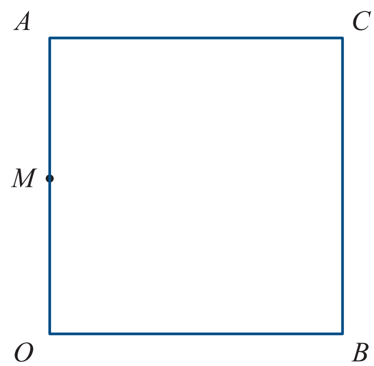
  i find the value of h

  ii calculate the acute angle between PM and QN giving your answer correct to the nearest degree.

e **i** Prove that $NMQP$ is a rectangle.

  ii Find h if $NMPQ$ is a square.

13 $OACB$ is a square with $\overrightarrow{OA} = a\boldsymbol{j}$ and $\overrightarrow{OB} = a\boldsymbol{i}$.
M is the midpoint of OA.



a Find in terms of a:

  i $\overrightarrow{OM}$

  ii $\overrightarrow{MC}$.

b P is a point on MC such that $\overrightarrow{MP} = \lambda\overrightarrow{MC}$.
Find $\overrightarrow{MP}$, $\overrightarrow{BP}$ and $\overrightarrow{OP}$ in terms of λ and a.

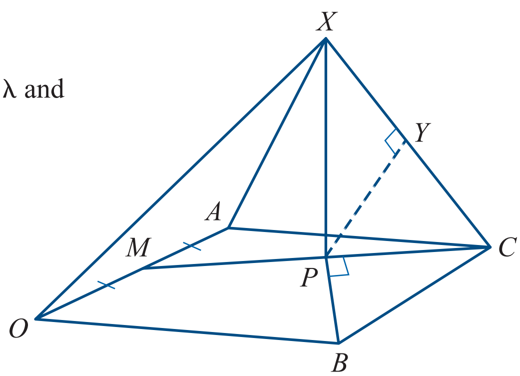
c If BP is perpendicular to MC:

  i find the value of λ and also find $|\overrightarrow{BP}|$, $|\overrightarrow{OP}|$ and $|\overrightarrow{OB}|$

  ii and, if $\theta = \angle PBO$, evaluate $\cos\theta$.

d If $|\overrightarrow{OP}| = |\overrightarrow{OB}|$ find the possible values of λ and illustrate these two cases carefully.

e In the diagram $\overrightarrow{OA} = a\boldsymbol{j}$, $\overrightarrow{OB} = a\boldsymbol{i}$.
BP is perpendicular to MC.
M is the midpoint of OA. $\overrightarrow{PX} = a\boldsymbol{k}$.
Y is a point on XC such that PY is perpendicular to XC.
Find $\overrightarrow{OY}$.



C H A P T E R

3

Circular functions

Objectives

- To understand the **reciprocal circular functions** cosec, sec and cot
- To understand and apply the identities $\sec^2\theta = 1 + \tan^2\theta$ and $\operatorname{cosec}^2\theta = 1 + \cot^2\theta$
- To understand and apply the **compound angle formulas**
- To understand and apply the **double angle formulas**
- To understand the **restricted circular functions** and their inverses $\sin^{-1}$, $\cos^{-1}$ and $\tan^{-1}$
- To understand the graphs of the **inverse functions** $\sin^{-1}$, $\cos^{-1}$ and $\tan^{-1}$
- To solve equations involving circular functions

The sine, cosine and tangent functions are discussed in some detail in section 1.1. Several new circular functions are introduced in this chapter.

3.1 The reciprocal circular functions

The cosecant function, $y = \operatorname{cosec}\theta$

The cosecant function is defined as:

$$\operatorname{cosec}\theta = \frac{1}{\sin\theta} \quad \text{provided } \sin\theta \neq 0$$

Since $\sin\theta = 0$ when $\theta = \pi n, n \in Z$, the domain of $\operatorname{cosec}\theta = R \setminus \{\theta : \theta = \pi n, n \in Z\}$.

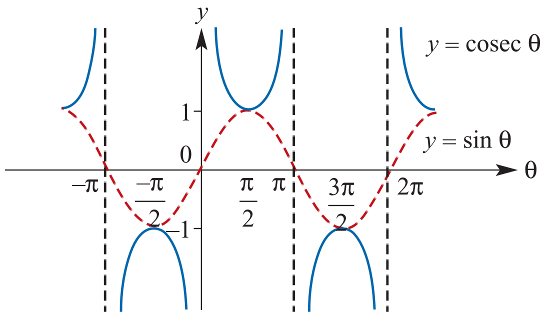
The graph of cosec θ is derived from the graph of sin θ.

The range of sin θ is $[-1, 1]$ therefore the range of cosec θ is $R \setminus (-1, 1)$.

The graph of $y = \sin\theta$ has turning points at $\theta = \frac{\pi}{2} + \pi n, n \in Z$ as does the graph of $y = \operatorname{cosec}\theta$.

$\sin\theta = 0$ at $\theta = \pi n, n \in Z$. These values of θ will be vertical asymptotes for $y = \operatorname{cosec}\theta$.

A sketch of the graph of $f: R \setminus \{\theta : \theta = \pi n, n \in Z\} \to R, f(\theta) = \operatorname{cosec}\theta$ is shown on the next page. The graph of $y = \sin\theta$ is shown on the same set of axes.



The secant function, $y = \sec\theta$

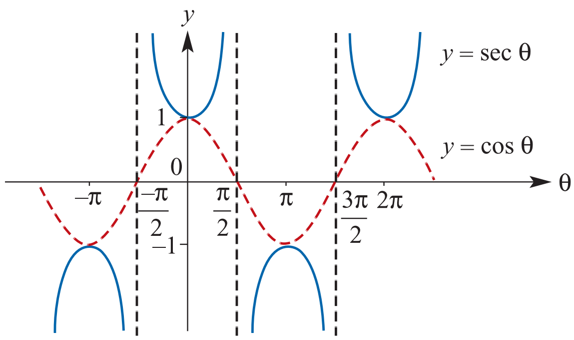
The secant function is defined as:

$$\sec\theta = \frac{1}{\cos\theta} \quad \text{provided } \cos\theta \neq 0$$

As the graph of $y = \cos\theta$ is a translation of the graph of $y = \sin\theta$, the graph of $y = \sec\theta$ must be a translation of the graph of $y = \operatorname{cosec}\theta$, $\frac{\pi}{2}$ units in the negative direction of the x axis.

The domain of $\sec\theta = R \setminus \left\{\theta : \theta = \frac{\pi}{2} + \pi n, n \in Z\right\}$.

A sketch of the graph of $f: R \setminus \left\{\theta : \theta = \frac{\pi}{2} + \pi n, n \in Z\right\} \to R$, $f(\theta) = \sec\theta$ is shown. The graph of $y = \cos\theta$ is shown on the same set of axes.



The range of $\sec\theta$ is $R \setminus (-1, 1)$.

$\sec\theta$ has turning points at $\theta = \pi n, n \in Z$, and has asymptotes at $\theta = \frac{\pi}{2} + \pi n, n \in Z$.

The cotangent function, $y = \cot\theta$

The cotangent function is defined as:

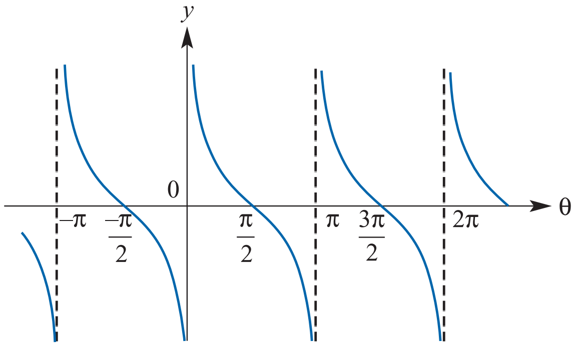
$$\cot\theta = \frac{\cos\theta}{\sin\theta} \quad \text{provided } \sin\theta \neq 0$$

$\therefore \quad \cot\theta = \tan\left(\frac{\pi}{2} - \theta\right)$ (the complementary properties of cot and tan are listed later in this section)

$$= -\tan\left(\pi - \left(\frac{\pi}{2} - \theta\right)\right)$$

$$= -\tan\left(\theta + \frac{\pi}{2}\right)$$

The graph of $\cot\theta$ is a translation of $\tan\theta$, $\frac{\pi}{2}$ units in the negative direction of the x axis, and a reflection in the θ axis. A sketch of the graph of $f: R \setminus \{\theta : \theta = \pi n, n \in Z\} \to R$, $f(\theta) = \cot\theta$ is shown opposite.



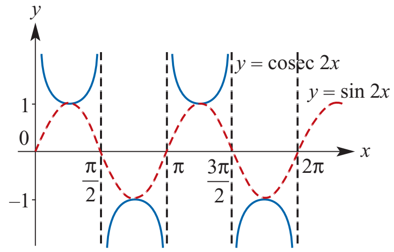
Example 1

Sketch the graph of each of the following over the interval $[0, 2\pi]$.

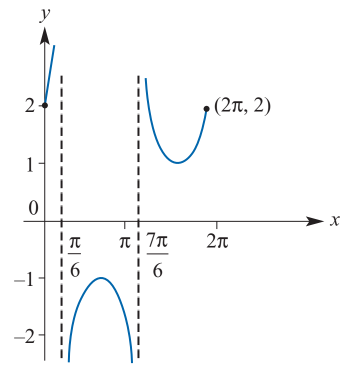
a $y = \text{cosec}(2x)$ **b** $y = \sec\left(x + \frac{\pi}{3}\right)$ **c** $y = \cot\left(x - \frac{\pi}{4}\right)$

Solution

a The graph of $y = \text{cosec}(2x)$ is obtained from the graph of $y = \text{cosec}\, x$ by a dilation of factor $\frac{1}{2}$ from the y axis. The graph of $y = \sin(2x)$ is also shown.



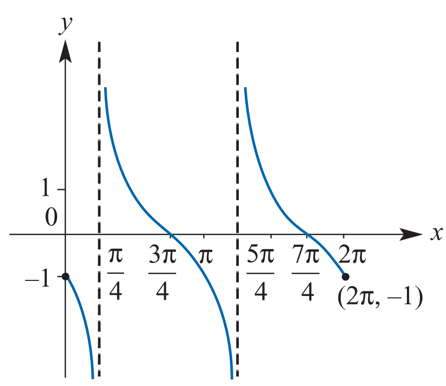
b The graph of $y = \sec\left(x + \frac{\pi}{3}\right)$ is a translation of the graph of $y = \sec x$, $\frac{\pi}{3}$ units in the negative direction of the x axis. The y-axis intercept is $\sec\left(\frac{\pi}{3}\right) = 2$. The asymptotes are at $x = \frac{\pi}{6}$ and $x = \frac{7\pi}{6}$.



c The graph of $y = \cot\left(x - \frac{\pi}{4}\right)$ is a translation of the graph of $y = \cot x$, $\frac{\pi}{4}$ units in the positive direction of the x axis. The y-axis intercept is $\cot\left(-\frac{\pi}{4}\right) = -1$.

The asymptotes are at $x = \frac{\pi}{4}$ and $x = \frac{5\pi}{4}$.

The x-axis intercepts are at $\frac{3\pi}{4}$ and $\frac{7\pi}{4}$.

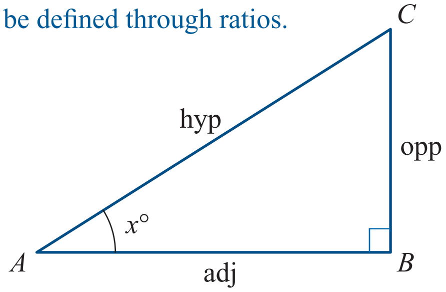


For right-angled triangles, the reciprocal functions can be defined through ratios.

$$\text{cosec}\, x^\circ = \frac{\text{hyp}}{\text{opp}}$$

$$\sec x^\circ = \frac{\text{hyp}}{\text{adj}}$$

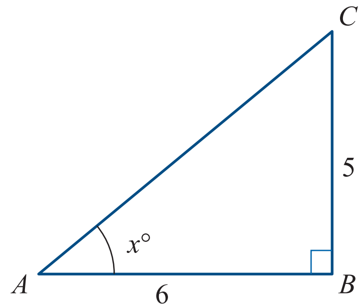
and $$\cot x^\circ = \frac{\text{adj}}{\text{opp}}$$



Example 2

In triangle ABC, $\angle ABC = 90^\circ$ and $\angle CAB = x^\circ$.
$AB = 6$ cm and $BC = 5$ cm. Find:

a AC **b** the trigonometric ratios related to x°



Solution

a By Pythagoras' theorem, $AC^2 = 5^2 + 6^2 = 61$

$\therefore \quad AC = \sqrt{61}$ cm

b $\sin x^\circ = \dfrac{5}{\sqrt{61}}$ $\cos x^\circ = \dfrac{6}{\sqrt{61}}$ $\tan x^\circ = \dfrac{5}{6}$

$\text{cosec}\, x^\circ = \dfrac{\sqrt{61}}{5}$ $\sec x^\circ = \dfrac{\sqrt{61}}{6}$ $\cot x^\circ = \dfrac{6}{5}$

Useful properties

The symmetry properties established for sine, cosine and tangent can be used to establish the following results:

$$\sec(\pi - x) = -\sec x \qquad \sec(\pi + x) = -\sec x$$
$$\text{cosec}(\pi - x) = \text{cosec}\, x \qquad \text{cosec}(\pi + x) = -\text{cosec}\, x$$
$$\cot(\pi - x) = -\cot x \qquad \cot(\pi + x) = \cot x$$
$$\sec(2\pi - x) = \sec x \qquad \sec(-x) = \sec x$$
$$\text{cosec}(2\pi - x) = -\text{cosec}\, x \qquad \text{cosec}(-x) = -\text{cosec}\, x$$
$$\cot(2\pi - x) = -\cot x \qquad \cot(-x) = -\cot x$$

The complementary properties are also useful.

$$\cot\left(\frac{\pi}{2} - x\right) = \tan x \qquad \sec\left(\frac{\pi}{2} - x\right) = \text{cosec}\, x$$
$$\text{cosec}\left(\frac{\pi}{2} - x\right) = \sec x \qquad \tan\left(\frac{\pi}{2} - x\right) = \cot x$$

Example 3

Find the exact value of each of the following:

a $\sec\left(\dfrac{11\pi}{4}\right)$ **b** $\text{cosec}\left(\dfrac{-23\pi}{4}\right)$ **c** $\cot\left(\dfrac{11\pi}{3}\right)$

Solution

a $$\sec\left(\frac{11\pi}{4}\right) = \sec\left(2\pi + \frac{3\pi}{4}\right) = \sec\left(\frac{3\pi}{4}\right) = \frac{1}{\cos\left(\frac{3\pi}{4}\right)} = \frac{1}{-\frac{1}{\sqrt{2}}} = -\sqrt{2}$$

b $$\operatorname{cosec}\left(\frac{-23\pi}{4}\right) = \operatorname{cosec}\left(-6\pi + \frac{\pi}{4}\right) = \operatorname{cosec}\left(\frac{\pi}{4}\right) = \frac{1}{\frac{1}{\sqrt{2}}} = \sqrt{2}$$

c $$\cot\left(\frac{11\pi}{3}\right) = \cot\left(4\pi - \frac{\pi}{3}\right) = \cot\left(-\frac{\pi}{3}\right) = -\cot\left(\frac{\pi}{3}\right) = -\frac{1}{\tan\frac{\pi}{3}} = -\frac{1}{\sqrt{3}}$$

Two new identities

It was shown earlier that, for all values of x, $\sin^2 x + \cos^2 x = 1$. From this identity the following identities can be derived:

$1 + \cot^2 x = \operatorname{cosec}^2 x$	provided $\sin x \neq 0$
$1 + \tan^2 x = \sec^2 x$	provided $\cos x \neq 0$

The first of these identities is obtained by dividing each term of the original identity by $\sin^2 x$:

i.e. $$\frac{\sin^2 x}{\sin^2 x} + \frac{\cos^2 x}{\sin^2 x} = \frac{1}{\sin^2 x}$$

which implies $1 + \cot^2 x = \operatorname{cosec}^2 x$

The derivation of the second identity is left as an exercise for the reader.

Example 4

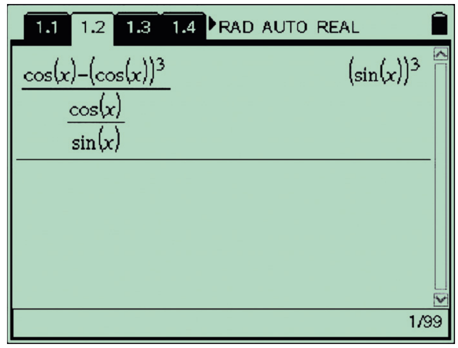
Simplify the expression $\dfrac{\cos x - \cos^3 x}{\cot x}$.

Solution

$$\frac{\cos x - \cos^3 x}{\cot x} = \frac{\cos x(1 - \cos^2 x)}{\cot x} = \cos x \times \sin^2 x \times \frac{\sin x}{\cos x} = \sin^3 x$$

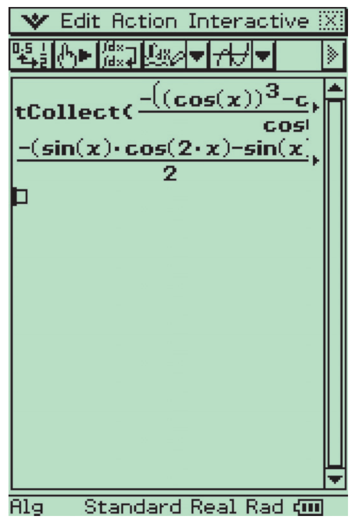
Using a TI-Nspire calculator

The expression can be entered either by using fraction templates or as (cos(x) − (cos(x))^3)/(cos(x)/ sin(x)). Then press **Enter** to simplify it.



Using a Casio ClassPad calculator

From the **Interactive** menu choose **Transformation** and then **tCollect**. Note that an equivalent form is obtained. Substitute $\cos(2x) = 1 - 2\sin^2(x)$ to obtain the given result.



Example 5

If $\tan x = 2,\ x \in \left[0, \dfrac{\pi}{2}\right]$, find:

a $\sec x$ **b** $\cos x$ **c** $\sin x$ **d** $\operatorname{cosec} x$

Solution

a
$$\sec^2 x = \tan^2 x + 1$$
$$= 4 + 1$$
$$\therefore \sec x = \pm\sqrt{5}$$
As $x \in \left[0, \dfrac{\pi}{2}\right]$, $\sec x = \sqrt{5}$

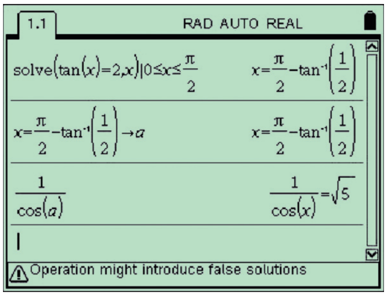
b $\cos x = \dfrac{1}{\sec x} = \dfrac{\sqrt{5}}{5}$

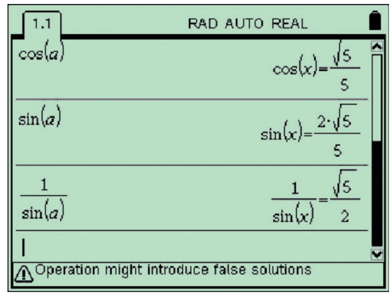
c $\sin x = \tan x \times \cos x = \dfrac{2\sqrt{5}}{5}$

d $\operatorname{cosec} x = \dfrac{1}{\sin x} = \dfrac{\sqrt{5}}{2}$

Using a TI-Nspire calculator

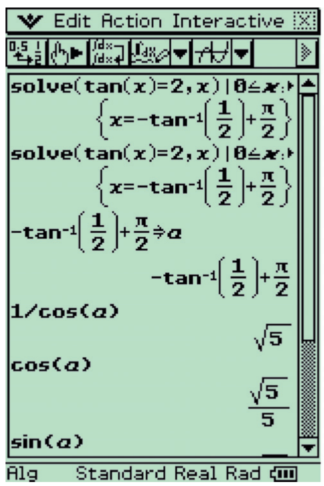
Choose **solve** from the **Algebra** menu and complete as shown.
Store the value to a.
Obtain the results as shown.

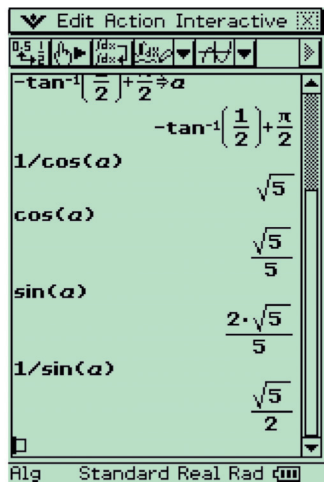




Using a Casio ClassPad calculator

Enter $\tan(x) = 2 \mid 0 \le x \le \frac{\pi}{2}$ and shade $\tan(x) = 2$. Then choose **Interactive**, then **Equation/Inequality** and then **solve**. Store the value to a.
Obtain the results as shown.





Exercise 3A

1 Sketch the graph of each of the following, over the interval $[0, 2\pi]$:

a $y = \operatorname{cosec}\left(x + \frac{\pi}{4}\right)$ **b** $y = \sec\left(x - \frac{\pi}{6}\right)$ **c** $y = \cot\left(x + \frac{\pi}{3}\right)$

d $y = \sec\left(x + \frac{2\pi}{3}\right)$ **e** $y = \operatorname{cosec}\left(x - \frac{\pi}{2}\right)$ **f** $y = \cot\left(x - \frac{3\pi}{4}\right)$

2 Sketch the graph of each of the following, over the interval $[0, \pi]$:

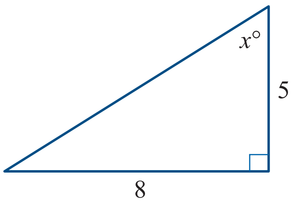
a $y = \sec 2x$ **b** $y = \operatorname{cosec}(3x)$ **c** $y = \cot(4x)$

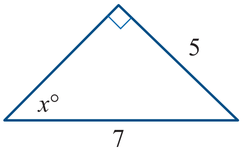
d $y = \operatorname{cosec}\left(2x + \frac{\pi}{2}\right)$ **e** $y = \sec(2x + \pi)$ **f** $y = \cot\left(2x - \frac{\pi}{3}\right)$

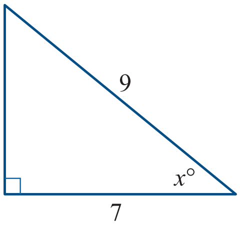
3 Sketch the graph of each of the following, over the interval $[-\pi, \pi]$:

a $y = \sec\left(2x - \frac{\pi}{2}\right)$ **b** $y = \operatorname{cosec}\left(2x + \frac{\pi}{3}\right)$ **c** $y = \cot\left(2x - \frac{2\pi}{3}\right)$

4 Find the trigonometric ratios $\cot x°$, $\sec x°$ and $\operatorname{cosec} x°$ for each of the following triangles:

a 

b 

c 

5 Find the exact value of each of the following:

a $\sin\frac{2\pi}{3}$ **b** $\cos\frac{3\pi}{4}$ **c** $\tan\frac{-\pi}{4}$ **d** $\operatorname{cosec}\frac{\pi}{6}$

e $\sec\frac{\pi}{4}$ **f** $\cot\frac{-\pi}{6}$ **g** $\sin\frac{5\pi}{4}$ **h** $\tan\frac{5\pi}{6}$

i $\sec\frac{-\pi}{3}$ **j** $\operatorname{cosec}\frac{3\pi}{4}$ **k** $\cot\frac{9\pi}{4}$ **l** $\cos\frac{-7\pi}{3}$

6 Simplify each of the following expressions:

a $\sec^2 x - \tan^2 x$ **b** $\cot^2 x - \operatorname{cosec}^2 x$ **c** $\frac{\tan^2 x + 1}{\tan^2 x}$

d $\frac{\sin^2 x}{\cos x} + \cos x$ **e** $\sin^4 x - \cos^4 x$ **f** $\tan^3 x + \tan x$

7 If $\tan x = -4$, $x \in \left[\frac{-\pi}{2}, 0\right]$, find:

a $\sec x$ **b** $\cos x$ **c** $\operatorname{cosec} x$

8 If $\cot x = 3$, $x \in \left[\pi, \frac{3\pi}{2}\right]$, find:

a $\operatorname{cosec} x$ **b** $\sin x$ **c** $\sec x$

9 If $\sec x = 10$, $x \in \left[\frac{-\pi}{2}, 0\right]$, find:

a $\tan x$ **b** $\sin x$

10 If $\operatorname{cosec} x = -6$, $x \in \left[\frac{3\pi}{2}, 2\pi\right]$, find:

a $\cot x$ **b** $\cos x$

11 $\sin x° = 0.5$, $90 < x < 180$. Find:

a $\cos x°$ **b** $\cot x°$ **c** $\operatorname{cosec} x°$

12 $\operatorname{cosec} x° = -3$, $180 < x < 270$. Find:

a $\sin x°$ **b** $\cos x°$ **c** $\sec x°$

13 $\cos x° = -0.7$, $0 < x < 180$. Find:

a $\sin x°$ **b** $\tan x°$ **c** $\cot x°$

14 $\sec x° = 5$, $180 < x < 360$. Find:

a $\cos x°$ **b** $\sin x°$ **c** $\cot x°$

15 Simplify each of the following expressions:

a $\sec^2\theta + \operatorname{cosec}^2\theta - \sec^2\theta\operatorname{cosec}^2\theta$ **b** $(\sec\theta - \cos\theta)(\operatorname{cosec}\theta - \sin\theta)$

c $(1 - \cos^2\theta)(1 + \cot^2\theta)$ **d** $\dfrac{\sec^2\theta - \operatorname{cosec}^2\theta}{\tan^2\theta - \cot^2\theta}$

16 If $x = \sec\theta - \tan\theta$, prove that $x + \dfrac{1}{x} = 2\sec\theta$ and also find a simple expression for $x - \dfrac{1}{x}$ in terms of θ.

3.2 Compound and double angle formulas

Compound angle formulas

The following identities are called the compound angle formulas.

$$\cos(x - y) = \cos x\cos y + \sin x\sin y$$
$$\cos(x + y) = \cos x\cos y - \sin x\sin y$$
$$\sin(x - y) = \sin x\cos y - \cos x\sin y$$
$$\sin(x + y) = \sin x\cos y + \cos x\sin y$$
$$\tan(x - y) = \frac{\tan x - \tan y}{1 + \tan x\tan y}$$
$$\tan(x + y) = \frac{\tan x + \tan y}{1 - \tan x\tan y}$$

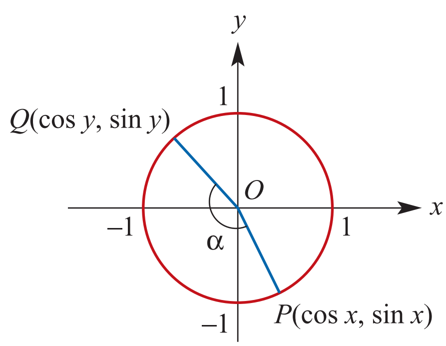
A proof of the first identity is given below and the other identities can be derived from that result.

Proof

$$\cos(x - y) = \cos x\cos y + \sin x\sin y$$

Consider angles x and y, $x > y$, measured counter-clockwise, and the corresponding points P $(\cos x, \sin x)$ and Q $(\cos y, \sin y)$ respectively.

Let α be the angle measured anticlockwise from OQ to OP. Then $x - y = \alpha + 2\pi k$ for some integer constant k.



Two cases need to be considered.

Case 1

$$0 < \alpha \le \pi$$

Using vectors,

$$\overrightarrow{OP} = \cos x\,\boldsymbol{i} + \sin x\,\boldsymbol{j} \quad \text{and } |\overrightarrow{OP}| = 1$$
$$\overrightarrow{OQ} = \cos y\,\boldsymbol{i} + \sin y\,\boldsymbol{j} \quad \text{and } |\overrightarrow{OQ}| = 1$$

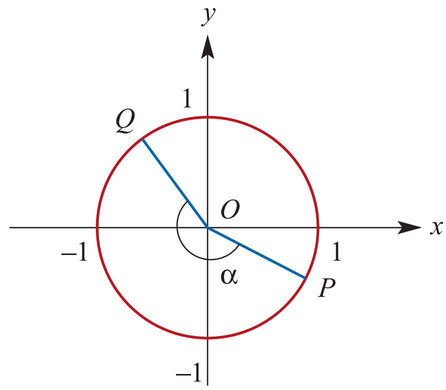
We apply $\boldsymbol{a}.\boldsymbol{b} = |\boldsymbol{a}|\,|\boldsymbol{b}|\cos\theta$

to obtain $\overrightarrow{OP}.\overrightarrow{OQ} = \cos\alpha$

Case 2

$$\pi < \alpha \leq 2\pi$$

In the diagram opposite:



$$\begin{aligned}\overrightarrow{OP}.\overrightarrow{OQ} &= \cos(2\pi - \alpha)\\ &= \cos(-\alpha)\\ &= \cos\alpha\end{aligned}$$

∴ in all cases:

$$\overrightarrow{OP}.\overrightarrow{OQ} = \cos\alpha$$

Now $\overrightarrow{OP}.\overrightarrow{OQ} = \cos x \cos y + \sin x \sin y$ (using dot product)

and $\cos(x - y) = \cos(\alpha + 2\pi k)$ where k is an integer

$$= \cos\alpha$$

∴ $\cos(x - y) = \cos x \cos y + \sin x \sin y$

The derivation of other identities

$$\begin{aligned}\cos(x + y) &= \cos[x - (-y)]\\ &= \cos x \cos(-y) + \sin x \sin(-y)\\ &= \cos x \cos y - \sin x \sin y\end{aligned}$$

$$\begin{aligned}\sin(x - y) &= \cos\left(\frac{\pi}{2} - x + y\right)\\ &= \cos\left(\frac{\pi}{2} - x\right)\cos y - \sin\left(\frac{\pi}{2} - x\right)\sin y\\ &= \sin x \cos y - \cos x \sin y\end{aligned}$$

$$\begin{aligned}\tan(x - y) &= \frac{\sin(x - y)}{\cos(x - y)}\\ &= \frac{\sin x \cos y - \cos x \sin y}{\cos x \cos y + \sin x \sin y}\end{aligned}$$

Dividing top and bottom by $\cos x \cos y$ we have:

$$\begin{aligned}\tan(x - y) &= \frac{\dfrac{\sin x}{\cos x}\dfrac{\cos y}{\cos y} - \dfrac{\cos x}{\cos x}\dfrac{\sin y}{\cos y}}{1 + \dfrac{\sin x}{\cos x}\dfrac{\sin y}{\cos y}}\\ &= \frac{\tan x - \tan y}{1 + \tan x \tan y}\end{aligned}$$

The derivation of the remaining two identities is left as an exercise for the reader.

Example 6

a Use $\frac{5\pi}{12} = \frac{\pi}{6} + \frac{\pi}{4}$ to evaluate $\sin \frac{5\pi}{12}$.

b Use $\frac{\pi}{12} = \frac{\pi}{3} - \frac{\pi}{4}$ to evaluate $\cos \frac{\pi}{12}$.

Solution

a
$$\begin{aligned}\sin\left(\frac{5\pi}{12}\right) &= \sin\left(\frac{\pi}{6} + \frac{\pi}{4}\right)\\ &= \sin\frac{\pi}{6}\cos\frac{\pi}{4} + \cos\frac{\pi}{6}\sin\frac{\pi}{4}\\ &= \frac{1}{2} \times \frac{\sqrt{2}}{2} + \frac{\sqrt{3}}{2} \times \frac{\sqrt{2}}{2}\\ &= \frac{\sqrt{2}}{4}(1 + \sqrt{3})\end{aligned}$$

b
$$\begin{aligned}\cos\frac{\pi}{12} &= \cos\left(\frac{\pi}{3} - \frac{\pi}{4}\right)\\ &= \cos\frac{\pi}{3}\cos\frac{\pi}{4} + \sin\frac{\pi}{3}\sin\frac{\pi}{4}\\ &= \frac{1}{2} \times \frac{\sqrt{2}}{2} + \frac{\sqrt{3}}{2} \times \frac{\sqrt{2}}{2}\\ &= \frac{\sqrt{2}}{4}(1 + \sqrt{3})\end{aligned}$$

Example 7

$\sin x = 0.2,\ x \in \left[0, \frac{\pi}{2}\right]$ and $\cos y = -0.4,\ y \in \left[\pi, \frac{3\pi}{2}\right]$. Find $\sin(x + y)$.

Solution

$\sin x = 0.2$

$\therefore \quad \cos x = \pm\sqrt{1 - 0.2^2} = \pm\sqrt{0.96}$

As $x \in \left[0, \frac{\pi}{2}\right]$, $\cos x = \sqrt{0.96} = \frac{2\sqrt{6}}{5}$

$\cos y = -0.4 \ \therefore \sin y = \pm\sqrt{1 - (-0.4)^2} = \pm\sqrt{0.84}$

As $y \in \left[\pi, \frac{3\pi}{2}\right]$, $\sin y = -\sqrt{0.84} = -\frac{\sqrt{21}}{5}$

$$\begin{aligned}\sin(x + y) &= \sin x \cos y + \cos x \sin y\\ &= 0.2 \times -0.4 + \frac{2\sqrt{6}}{5} \times -\frac{\sqrt{21}}{5}\\ &= -0.08 - \frac{2}{25} \times 3\sqrt{14}\\ &= -\frac{2}{25}[1 + 3\sqrt{14}]\end{aligned}$$

Using a TI-Nspire calculator

First solve the equation

$\sin(x) = 0.2 \mid 0 \leq x \leq \dfrac{\pi}{2}$

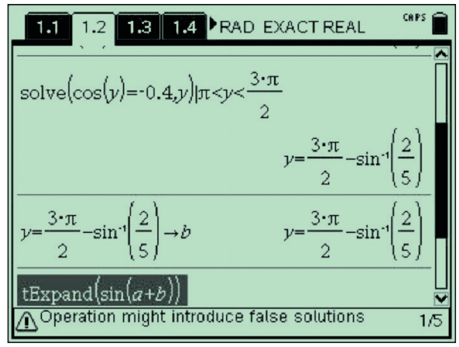
and store the result to a.

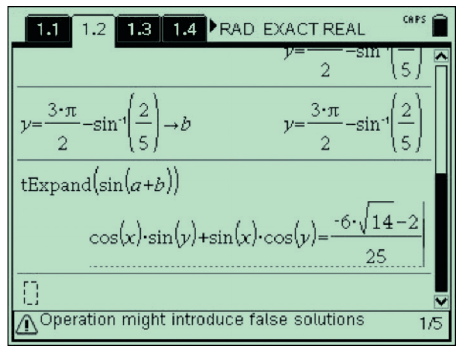
Then solve the equation

$\cos(y) = -0.4 \mid \pi \leq y \leq \dfrac{3\pi}{2}$

and store the result to b.

From the **Algebra** menu choose **Trigonometry** and then **Expand** (menu 9 1).





Using a Casio ClassPad calculator

Solve the equation

$\sin(x) = 0.2 \mid 0 \leq x \leq \dfrac{\pi}{2}$

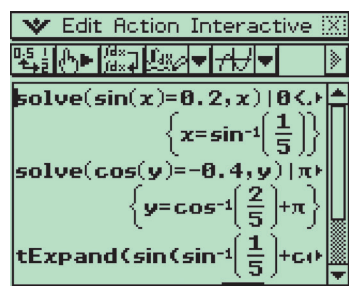
and then solve the equation

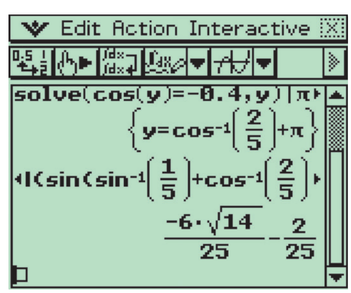
$\cos(y) = -0.4 \mid \pi \leq y \leq \dfrac{3\pi}{2}$.

Paste the results to form the expression

$\sin\left(\sin^{-1}\left(\dfrac{1}{5}\right)\right) + \cos\left(\cos^{-1}\left(\dfrac{2}{5}\right) + \pi\right)$.

From the **Interactive** menu choose **Transformation** and then **tExpand**.





Double angle formulas

The double angle formulas are:

$$\begin{aligned}\sin 2x &= 2\sin x\cos x\\ \cos 2x &= \cos^2 x - \sin^2 x\\ &= 1 - 2\sin^2 x\\ &= 2\cos^2 x - 1\\ \tan 2x &= \frac{2\tan x}{1-\tan^2 x}\end{aligned}$$

These formulas can be derived from the compound angle formulas by substituting y for x:

$$\begin{aligned}\cos(x+y) &= \cos x\cos y - \sin x\sin y\\ \therefore\quad \cos(x+x) &= \cos x\cos x - \sin x\sin x\\ \cos 2x &= \cos^2 x - \sin^2 x\end{aligned}$$

The two other equivalent forms of $\cos 2x$ can be obtained by applying the Pythagorean identity $\sin^2 x + \cos^2 x = 1$

e.g. $\quad \cos^2 x - \sin^2 x = (1 - \sin^2 x) - \sin^2 x$

$= 1 - 2\sin^2 x$

also $\quad = \cos^2 x - (1 - \cos^2 x)$

$= 2\cos^2 x - 1$

Example 8

If $\sin\alpha = 0.6$, $\alpha \in \left[\frac{\pi}{2}, \pi\right]$, find $\sin 2\alpha$.

Solution

$\sin\alpha = 0.6 \quad \therefore \cos\alpha = \pm\sqrt{1 - 0.6^2}$

$= \pm 0.8$

As $\alpha \in \left[\frac{\pi}{2}, \pi\right]$, $\quad \cos\alpha = -0.8$

$$\begin{aligned}\sin 2\alpha &= 2\sin\alpha\cos\alpha\\ &= 2\times 0.6\times -0.8\\ &= -0.96\end{aligned}$$

Example 9

If $\cos\alpha = 0.7$, $\alpha \in \left[\frac{3\pi}{2}, 2\pi\right]$, find $\sin\frac{\alpha}{2}$.

Solution

$$\cos 2x = 1 - 2\sin^2 x$$

$$\therefore \quad \cos\alpha = 1 - 2\sin^2\left(\frac{\alpha}{2}\right)$$

$$2\sin^2\left(\frac{\alpha}{2}\right) = 1 - 0.7$$

$$= 0.3$$

$$\sin\frac{\alpha}{2} = \pm\sqrt{0.15}$$

$$\alpha \in \left[\frac{3\pi}{2}, 2\pi\right], \frac{\alpha}{2} \in \left[\frac{3\pi}{4}, \pi\right]$$

$$\therefore \quad \sin\frac{\alpha}{2} \text{ is positive}$$

$$\sin\frac{\alpha}{2} = \sqrt{0.15} = \frac{\sqrt{15}}{10}$$

Exercise 3B

Skillsheet TEACHER

1 Use the compound angle formulas to expand each of the following:

a $\sin(2x - 5y)$ **b** $\cos(x^2 + y)$ **c** $\tan(x + (y + z))$

2 Simplify each of the following:

a $\sin x \cos 2y - \cos x \sin 2y$ **b** $\cos 3x \cos 2x + \sin 3x \sin 2x$

c $\dfrac{\tan A - \tan(A - B)}{1 + \tan A \tan(A - B)}$ **d** $\sin(A + B)\cos(A - B) + \cos(A + B)\sin(A - B)$

e $\cos y \cos(-2y) - \sin y \sin(-2y)$

3 **a** Expand $\sin(x + 2x)$. **b** Hence, express $\sin 3x$ in terms of $\sin x$.

4 **a** Expand $\cos(x + 2x)$. **b** Hence, express $\cos 3x$ in terms of $\cos x$.

5 Use the compound angle formulas and appropriate angles to find the exact value of each of the following:

a $\sin\dfrac{\pi}{12}$ **b** $\tan\dfrac{5\pi}{12}$ **c** $\cos\dfrac{7\pi}{12}$ **d** $\tan\dfrac{\pi}{12}$

6 Let $\sin x = 0.6$, $x \in \left[\dfrac{\pi}{2}, \pi\right]$ and $\tan y = 2.4$, $y \in \left[0, \dfrac{\pi}{2}\right]$. Find the exact value of each of the following:

a $\cos x$ **b** $\sec y$ **c** $\cos y$

d $\sin y$ **e** $\tan x$ **f** $\cos(x - y)$

g $\sin(x - y)$ **h** $\tan(x + y)$ **i** $\tan(x + 2y)$

7 Let $\cos x = -0.7$, $x \in \left[\pi, \dfrac{3\pi}{2}\right]$ and $\sin y = 0.4$, $y \in \left[0, \dfrac{\pi}{2}\right]$. Find the value of each of the following, correct to two decimal places:

a $\sin x$ **b** $\cos y$ **c** $\tan(x - y)$ **d** $\cos(x + y)$

8 Simplify each of the following:

a $\frac{1}{2}\sin x\cos x$ **b** $\sin^2 x - \cos^2 x$ **c** $\dfrac{\tan x}{1-\tan^2 x}$

d $\dfrac{\sin^4 x - \cos^4 x}{\cos 2x}$ **e** $\dfrac{4\sin^3 x - 2\sin x}{\cos x\cos 2x}$ **f** $\dfrac{4\sin^2 x - 4\sin^4 x}{\sin 2x}$

9 Let $\sin x = -0.8,\ x \in \left[\pi, \dfrac{3\pi}{2}\right]$. Find:

a $\sin 2x$ **b** $\cos 2x$ **c** $\tan 2x$

10 Let $\tan x = 3, x \in \left[0, \dfrac{\pi}{2}\right]$. Find:

a $\tan 2x$ **b** $\tan 3x$

11 Use the double angle formula for $\tan 2x$, and the fact that $\tan\dfrac{\pi}{4} = 1$, to find the exact value of $\tan\dfrac{\pi}{8}$.

12 Let $\sin x = -0.75, x \in \left[\pi, \dfrac{3\pi}{2}\right]$. Find correct to two decimal places:

a $\cos x$ **b** $\sin\frac{1}{2}x$

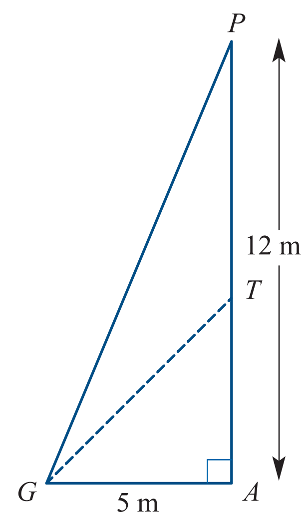
13 If $\cos x = 0.9, x \in \left[0, \dfrac{\pi}{2}\right]$, find $\cos\frac{1}{2}x$ correct to two decimal places.

14 In a right-angled triangle GAP, $AP = 12$ m and $GA = 5$ m. T is a point on AP, such that $\angle AGT = \angle TGP = x^c$. Without using a calculator, find the exact values of the following:

a $\tan 2x$

b $\tan x$, by using the double angle formula

c AT



3.3 Inverses of circular functions

All six circular functions discussed earlier are periodic and are, therefore, many-to-one functions. The inverse of these functions cannot, therefore, be functions. However, by restricting the domain so that the circular functions are one-to-one functions, the inverse circular functions can be defined.

The inverse sine function, $y = \sin^{-1} x$

When the domain for the function sin is restricted to the interval $\left[-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right]$, it is a one-to-one function and an inverse function exists. Other intervals defined through consecutive turning points of the graph (e.g. $\left[\frac{\pi}{2}, \frac{3\pi}{2}\right]$ or $\left[\frac{3\pi}{2}, \frac{5\pi}{2}\right]$) could have been used for the domain, but $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ is the standard convention.

The inverse of the restricted sine function is denoted by $\sin^{-1}$ and is defined by:

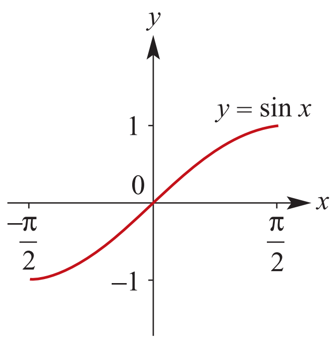
$$\sin^{-1}: [-1, 1] \to R,\ \sin^{-1} x = y \text{ where } \sin y = x,\ y \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$$

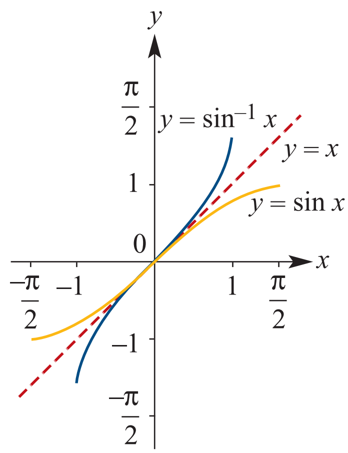
Note: The domain of $\sin^{-1}$ = range of the restricted sine function = $[-1, 1]$.

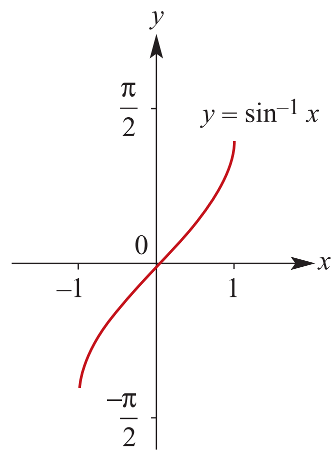
The range of $\sin^{-1}$ = domain of the restricted sine function = $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

By the property of inverse functions, $\sin(\sin^{-1} x) = x$ and, for $x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, $\sin^{-1}(\sin x) = x$.

The graph of $y = \sin^{-1}(x)$ is obtained from the graph of $y = \sin(x)$, $x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, through a reflection in the line $y = x$.







$\sin^{-1}$ is also denoted by arsin, arcsin or asin.

The inverse cosine function, $y = \cos^{-1} x$

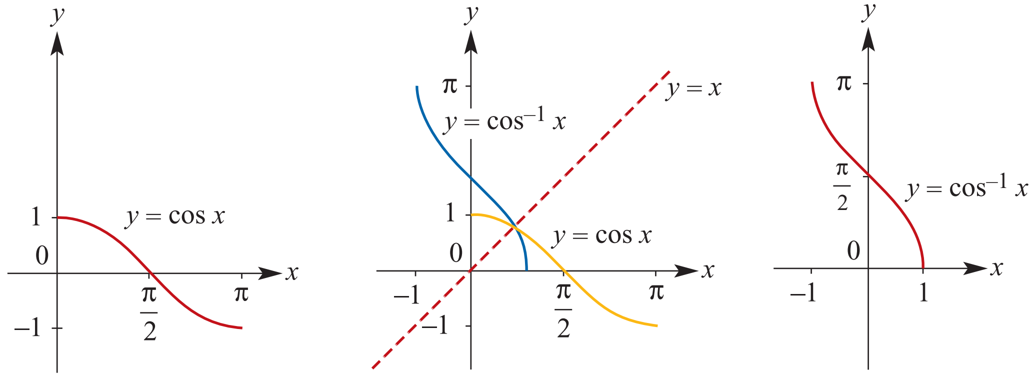
$\cos^{-1} x$ is defined in a similar way to the inverse sine function.

The standard domain of the restricted cosine function is $[0, \pi]$, although other intervals would also produce an inverse function.

$\cos^{-1} x$, the inverse function of the restricted cosine function, is defined as follows:

$$\cos^{-1}: [-1, 1] \to R,\ \cos^{-1} x = y \text{ where } \cos y = x,\ y \in [0, \pi]$$

The graph of the restricted cosine function, its reflection in the line $y = x$ to produce $y = \cos^{-1} x$, and the graph of $y = \cos^{-1} x$ are shown in the three figures below.



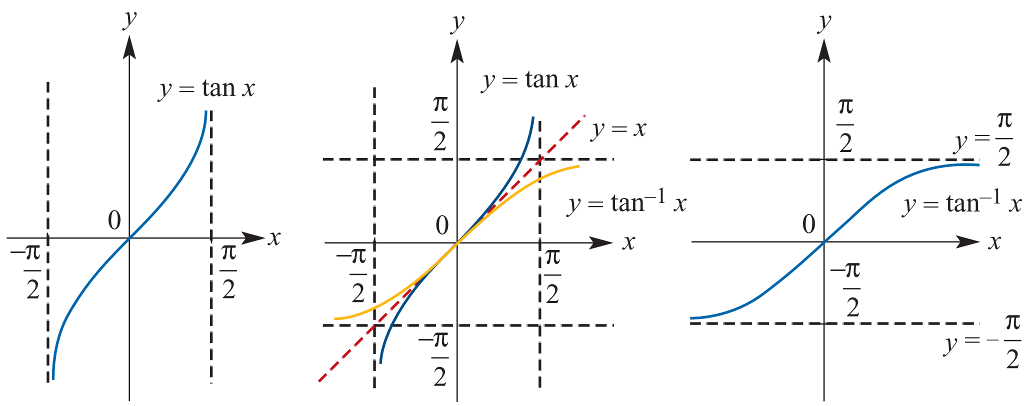
Note: The domain of $\cos^{-1} x$ is $[-1, 1]$ and the range is $[0, \pi]$.
Also, $\cos(\cos^{-1} x) = x$ and, for $x \in [0, \pi]$, $\cos^{-1}(\cos x) = x$.
$\cos^{-1}$ is also denoted by arcos or acos.

The inverse tangent function, $y = \tan^{-1} x$

If the domain of the tangent function is restricted to $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, a one-to-one function is formed and the inverse function exists.

$$\tan^{-1}: R \to R, \tan^{-1} x = y \text{ where } \tan y = x, y \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$$

The graph of the restricted tangent function, its reflection in the line $y = x$ to produce $y = \tan^{-1} x$, and the graph of $y = \tan^{-1} x$ are shown in the three figures below.



Note: The domain of $\tan^{-1} x$ is R and its range is $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.
Also $\tan(\tan^{-1} x) = x$ and, for $x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, $\tan^{-1}(\tan x) = x$.
$\tan^{-1}$ is also denoted by artan, arctan or atan.

Example 10

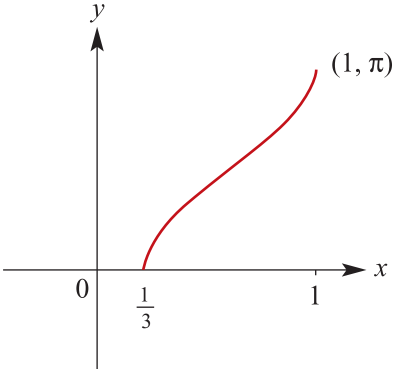
Sketch the graph of each of the following for their maximal domain:

a $y = \cos^{-1}(2 - 3x)$ **b** $y = \tan^{-1}(x + 2) + \dfrac{\pi}{2}$

Solution

a For the function to be defined

$$-1 \le 2 - 3x \le 1$$
$$\Leftrightarrow \quad -3 \le -3x \le -1$$
$$\Leftrightarrow \quad \tfrac{1}{3} \le x \le 1$$



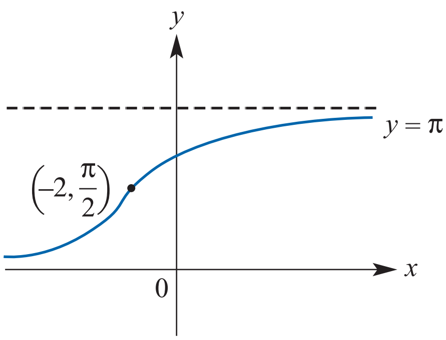
i.e. the implied domain is $\left[\frac{1}{3}, 1\right]$

Note: $y = \cos^{-1}\left[-3\left(x - \frac{2}{3}\right)\right]$

The graph is obtained from the graph of $y = \cos^{-1}(x)$ by the following sequence of transformations:

- a dilation from the y axis of factor $\frac{1}{3}$
- a reflection in the y axis
- a translation of $\frac{2}{3}$ units in the positive direction of the x axis

b The domain of $\tan^{-1}$ is R. The graph of $y = \tan^{-1}(x + 2) + \dfrac{\pi}{2}$ is obtained from the graph of $y = \tan^{-1}(x)$ by a translation of 2 units in the negative direction of the x axis and $\dfrac{\pi}{2}$ units in the positive direction of the y axis.



Example 11

a Evaluate $\sin^{-1}\left(-\dfrac{\sqrt{3}}{2}\right)$

b Simplify:

i $\sin^{-1}\left(\sin\dfrac{\pi}{6}\right)$ **ii** $\sin^{-1}\left(\sin\dfrac{5\pi}{6}\right)$

iii $\sin^{-1}\left(\cos\dfrac{\pi}{3}\right)$ **iv** $\sin\left(\cos^{-1}\dfrac{\sqrt{2}}{2}\right)$

Solution

a Evaluating $\sin^{-1}\left(-\frac{\sqrt{3}}{2}\right)$ is equivalent to solving the equation $\sin x = \frac{-\sqrt{3}}{2}$

$$\sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}$$

$$\therefore \quad \sin\left(-\frac{\pi}{3}\right) = -\frac{\sqrt{3}}{2}$$

$$\therefore \quad \sin^{-1}\left(-\frac{\sqrt{3}}{2}\right) = -\frac{\pi}{3}$$

b **i** Since $\frac{\pi}{6} \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, by definition $\sin^{-1}\left(\sin\frac{\pi}{6}\right) = \frac{\pi}{6}$

ii $$\sin^{-1}\left(\sin\frac{5\pi}{6}\right) = \sin^{-1}\left[\sin\left(\pi - \frac{5\pi}{6}\right)\right] = \sin^{-1}\left(\sin\frac{\pi}{6}\right) = \frac{\pi}{6}$$

iii $$\sin^{-1}\left(\cos\frac{\pi}{3}\right) = \sin^{-1}\left(\sin\left(\frac{\pi}{2} - \frac{\pi}{3}\right)\right) = \sin^{-1}\left(\sin\frac{\pi}{6}\right) = \frac{\pi}{6}$$

iv $$\sin\left(\cos^{-1}\frac{\sqrt{2}}{2}\right) = \sin\frac{\pi}{4} = \frac{\sqrt{2}}{2}$$

Example 12

Find the implied domain and range of:

a $y = \sin^{-1}(2x - 1)$ **b** $y = 3\cos^{-1}(2 - 2x)$

Solution

a For $\sin^{-1}(2x - 1)$ to be defined

$$-1 \le 2x - 1 \le 1$$

$$\Leftrightarrow \quad 0 \le 2x \le 2$$

$$\Leftrightarrow \quad 0 \le x \le 1$$

$\therefore$ the implied domain is $[0, 1]$

The range is $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

b For $3\cos^{-1}(2 - 2x)$ to be defined

$$-1 \le 2 - 2x \le 1$$

$$\Leftrightarrow \quad -3 \le -2x \le -1$$

$$\Leftrightarrow \quad \frac{1}{2} \le x \le \frac{3}{2}$$

$\therefore$ the implied domain is $\left[\frac{1}{2}, \frac{3}{2}\right]$

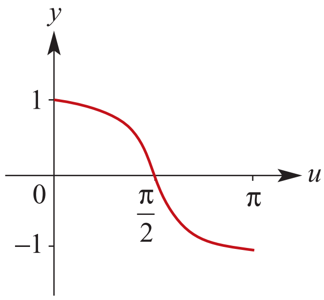
The range is $[0, 3\pi]$.

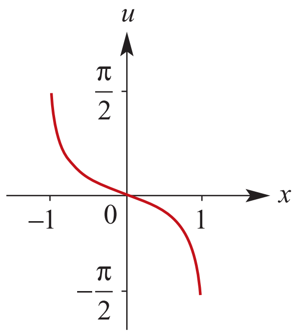
Example 13

Find the implied domain and range of $y = \cos(-\sin^{-1} x)$, where cos has restricted domain $[0, \pi]$.

Solution

Let $y = \cos u$, $u \in [0, \pi]$ where $u = -\sin^{-1} x$





From the graphs it can be seen that the range of $u = \left[\frac{-\pi}{2}, \frac{\pi}{2}\right]$, but for the composite function to exist the values of u must be a subset of $[0, \pi]$, the domain of $\cos u$. Hence the values of u for this composite function to exist, and hence the domain of cos, must lie in the interval $\left[0, \frac{\pi}{2}\right]$.

i.e. $\quad 0 \le u \le \frac{\pi}{2}$

$\therefore \quad 0 \le -\sin^{-1} x \le \frac{\pi}{2} \quad$ since $u = -\sin^{-1} x$

$\therefore \quad -\frac{\pi}{2} \le \sin^{-1} x \le 0$

$\therefore \quad -1 \le x \le 0$

Hence, the domain of $\cos(-\sin^{-1} x)$ is $[-1, 0]$.
The range of $\cos(-\sin^{-1} x)$ is $[0, 1]$.

Exercise 3C

1 Sketch the graphs of each of the following, stating clearly the implied domain and range each time:

a $y = \tan^{-1}(x - 1)$ **b** $y = \cos^{-1}(x + 1)$

c $y = 2\sin^{-1}\left(x + \frac{1}{2}\right)$ **d** $y = 2\tan^{-1}(x) + \frac{\pi}{2}$

e $y = \cos^{-1}(2x)$ **f** $y = \frac{1}{2}\sin^{-1}(3x) + \frac{\pi}{4}$

2 Evaluate each of the following:

a $\sin^{-1} 1$ **b** $\sin^{-1}\left(\frac{-\sqrt{2}}{2}\right)$ **c** $\sin^{-1} 0.5$

d $\cos^{-1}\left(\frac{-\sqrt{3}}{2}\right)$ **e** $\cos^{-1} 0.5$ **f** $\tan^{-1} 1$

g $\tan^{-1}(-\sqrt{3})$ **h** $\tan^{-1}\left(\frac{\sqrt{3}}{3}\right)$ **i** $\cos^{-1}(-1)$

3 Simplify:

a $\sin(\cos^{-1} 0.5)$ **b** $\sin^{-1}\left(\cos\frac{5\pi}{6}\right)$ **c** $\tan\left[\sin^{-1}\left(-\frac{\sqrt{2}}{2}\right)\right]$

d $\cos(\tan^{-1} 1)$ **e** $\tan^{-1}\left(\sin\frac{5\pi}{2}\right)$ **f** $\tan(\cos^{-1} 0.5)$

g $\cos^{-1}\left(\cos\frac{7\pi}{3}\right)$ **h** $\sin^{-1}\left[\sin\left(-\frac{2\pi}{3}\right)\right]$ **i** $\tan^{-1}\left(\tan\frac{11\pi}{4}\right)$

j $\cos^{-1}\left[\sin\left(-\frac{\pi}{3}\right)\right]$ **k** $\cos^{-1}\left[\tan\left(-\frac{\pi}{4}\right)\right]$ **l** $\sin^{-1}\left[\cos\left(-\frac{3\pi}{4}\right)\right]$

4 Let $f: \left[\frac{\pi}{2}, \frac{3\pi}{2}\right] \to R,\ f(x) = \sin x$.

a Define f^{-1}, clearly stating its domain and its range.

b Evaluate:

i $f\left(\frac{\pi}{2}\right)$ **ii** $f\left(\frac{3\pi}{4}\right)$ **iii** $f\left(\frac{7\pi}{6}\right)$

iv $f^{-1}(-1)$ **v** $f^{-1}(0)$ **vi** $f^{-1}(0.5)$

5 Given that the domain of $\sin x$, $\cos x$ and $\tan x$ are restricted to $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, $[0, \pi]$ and $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ respectively, define the implied domain and range of each of the following where y is equal to:

a $\sin^{-1}(2 - x)$ **b** $\sin\left(x + \frac{\pi}{4}\right)$ **c** $\sin^{-1}(2x + 4)$

d $\sin\left(3x - \frac{\pi}{3}\right)$ **e** $\cos\left(x - \frac{\pi}{6}\right)$ **f** $\cos^{-1}(x + 1)$

g $\cos^{-1}(x^2)$ **h** $\cos\left(2x + \frac{2\pi}{3}\right)$ **i** $\tan^{-1}(x^2)$

j $\tan\left(2x - \frac{\pi}{2}\right)$ **k** $\tan^{-1}(2x + 1)$ **l** $\tan(x^2)$

6 Simplify each of the following expressions, in an exact form:

a $\cos\left(\sin^{-1}\frac{4}{5}\right)$ **b** $\tan\left(\cos^{-1}\frac{5}{13}\right)$ **c** $\cos\left(\tan^{-1}\frac{7}{24}\right)$

d $\tan\left(\sin^{-1}\frac{40}{41}\right)$ **e** $\tan\left(\cos^{-1}\frac{1}{2}\right)$ **f** $\sin\left(\cos^{-1}\frac{2}{3}\right)$

g $\sin(\tan^{-1}(-2))$ **h** $\cos\left(\sin^{-1}\frac{3}{7}\right)$ **i** $\sin(\tan^{-1} 0.7)$

7 Let $\sin\alpha = \frac{3}{5}$ and $\sin\beta = \frac{5}{13}$, $\alpha \in \left[0, \frac{\pi}{2}\right]$ and $\beta \in \left[0, \frac{\pi}{2}\right]$

a Find: **i** $\cos\alpha$ **ii** $\cos\beta$

b Use a compound angle formula to show that:

i $\sin^{-1}\frac{3}{5} - \sin^{-1}\frac{5}{13} = \sin^{-1}\frac{16}{65}$ **ii** $\sin^{-1}\frac{3}{5} + \sin^{-1}\frac{5}{13} = \cos^{-1}\frac{33}{65}$

8 Given that the domain of $\sin x$ and $\cos x$ are restricted to $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ and $[0, \pi]$ respectively, define the implied domain and range of each of the following where y is equal to:

a $\sin^{-1}(\cos x)$ **b** $\cos(\sin^{-1} x)$ **c** $\cos^{-1}(\sin 2x)$
d $\sin(-\cos^{-1} x)$ **e** $\cos(2\sin^{-1} x)$ **f** $\tan^{-1}(\cos x)$
g $\cos(\tan^{-1} x)$ **h** $\sin(\tan^{-1} x)$

9 **a** Use a compound angle formula to show that $\tan^{-1} 3 - \tan^{-1}\frac{1}{2} = \frac{\pi}{4}$.

b Hence, show that $\tan^{-1} x - \tan^{-1}\left(\frac{x-1}{x+1}\right) = \frac{\pi}{4}$, $x > -1$.

10 Given that the domain of $\sin x$ and $\cos x$ are restricted to $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ and $[0, \pi]$ respectively, explain why each of these expressions cannot be evaluated:

a $\cos[\sin^{-1}(-0.5)]$ **b** $\sin[\cos^{-1}(-0.2)]$ **c** $\cos[\tan^{-1}(-1)]$

3.4 Solution of equations

In section 1.1, the solution of equations involving sine, cosine and tangent was discussed. In this section, equations that involve the reciprocal circular functions and the use of the double angle formulas are introduced. Equations that are not able to be solved by analytic methods are also considered.

Example 14

Find x, such that $\sec x = 2$, in the interval $[0, 2\pi]$.

Solution

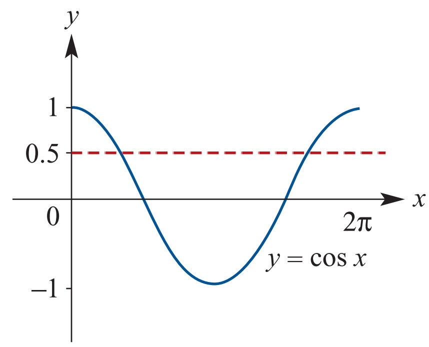
$$\sec x = 2$$

$$\therefore \quad \cos x = \tfrac{1}{2}$$

The values of x which exist in $[0, 2\pi]$ are

$$x = \frac{\pi}{3} \text{ and } 2\pi - \frac{\pi}{3}$$

$$x = \frac{\pi}{3} \text{ and } \frac{5\pi}{3}$$



Example 15

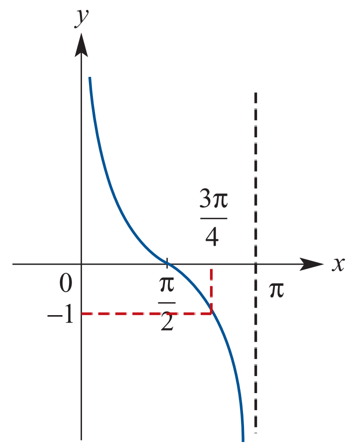
Find all the values of x for which $\cot x = -1$.

Solution

The period of $\cot x$ is π. In the interval $[0, \pi]$ the solution of $\cot x = -1$ is $x = \frac{3\pi}{4}$

$\therefore$ the solutions of the equation are

$$x = \frac{3\pi}{4} + n\pi, n \in Z$$



Example 16

Find x, such that $\operatorname{cosec}\left(2x - \frac{\pi}{3}\right) = \frac{-2\sqrt{3}}{3}$, for $x \in [0, 2\pi]$.

Solution

$$\operatorname{cosec}\left(2x - \frac{\pi}{3}\right) = \frac{-2\sqrt{3}}{3}$$

implies $\sin\left(2x - \frac{\pi}{3}\right) = \frac{-3}{2\sqrt{3}} = \frac{-\sqrt{3}}{2}$

Let $\theta = 2x - \frac{\pi}{3}$ where $\theta \in \left[-\frac{\pi}{3}, \frac{11\pi}{3}\right]$

then $\sin(\theta) = \frac{-\sqrt{3}}{2}$

$\therefore$ $\theta = -\frac{\pi}{3}, \frac{4\pi}{3}, \frac{5\pi}{3}, \frac{10\pi}{3}, \frac{11\pi}{3}$

$\therefore$ $2x - \frac{\pi}{3} = -\frac{\pi}{3}, \frac{4\pi}{3}, \frac{5\pi}{3}, \frac{10\pi}{3}, \frac{11\pi}{3}$

$\therefore$ $2x = 0, \frac{5\pi}{3}, 2\pi, \frac{11\pi}{3}, 4\pi$

$\therefore$ $x = 0, \frac{5\pi}{6}, \pi, \frac{11\pi}{6}, 2\pi$

Example 17

Solve each of the following for $x \in [0, 2\pi]$.

a $\sin(4x) = \sin(2x)$ **b** $\cos x = \sin \frac{x}{2}$

Solution

a $\sin(4x) = \sin(2x)$

$\therefore$ $2\sin(2x)\cos(2x) = \sin(2x)$ where $2x \in [0, 4\pi]$

implies $\sin(2x)(2\cos(2x) - 1) = 0$

$$\therefore \quad \sin(2x) = 0 \quad \text{or} \quad 2\cos(2x) - 1 = 0$$

$$\therefore \quad \cos(2x) = \tfrac{1}{2}$$

$$\therefore \quad 2x = 0, \pi, 2\pi, 3\pi, 4\pi \quad \text{or} \quad 2x = \frac{\pi}{3}, \frac{5\pi}{3}, \frac{7\pi}{3}, \frac{11\pi}{3}$$

$$\therefore \quad x = 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}, 2\pi \quad \text{or} \quad x = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{7\pi}{6}, \frac{11\pi}{6}$$

$$\text{i.e.} \quad x = 0, \frac{\pi}{6}, \frac{\pi}{2}, \frac{5\pi}{6}, \pi, \frac{7\pi}{6}, \frac{3\pi}{2}, \frac{11\pi}{6}, 2\pi$$

b

$$\cos x = \sin \frac{x}{2}$$

$$\therefore \quad 1 - 2\sin^2 \frac{x}{2} = \sin \frac{x}{2} \quad \text{where } \frac{x}{2} \in [0, \pi]$$

$$\therefore \quad 2\sin^2 \frac{x}{2} + \sin \frac{x}{2} - 1 = 0$$

Let $a = \sin \dfrac{x}{2}$

$$2a^2 + a - 1 = 0$$

$$\therefore \quad (2a - 1)(a + 1) = 0$$

$$\therefore \quad 2a - 1 = 0 \quad \text{or} \quad a + 1 = 0$$

$$\therefore \quad a = \tfrac{1}{2} \quad \text{or} \quad a = -1$$

$$\therefore \quad a = \tfrac{1}{2} \quad \text{since } a \in [0, 1]$$

$$\therefore \quad \sin \frac{x}{2} = \tfrac{1}{2}$$

$$\therefore \quad \frac{x}{2} = \frac{\pi}{6}, \frac{5\pi}{6}$$

$$\therefore \quad x = \frac{\pi}{3}, \frac{5\pi}{3}$$

Example 18

Find the maximum and minimum values of:

a $\sin^2(2x) + 2\sin(2x) + 2$

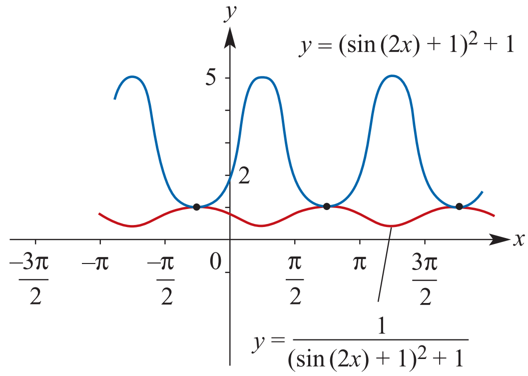
b $\dfrac{1}{\sin^2(2x) + 2\sin(2x) + 2}$

Solution

a Let $a = \sin(2x)$

$$\begin{aligned} \text{Then } \sin^2(2x) + 2\sin(2x) + 2 &= a^2 + 2a + 2 \\ &= (a + 1)^2 + 1 \\ &= (\sin(2x) + 1)^2 + 1 \end{aligned}$$

Now $-1 \leq \sin(2x) \leq 1$ and therefore the maximum value is 5 and the minimum value 1.



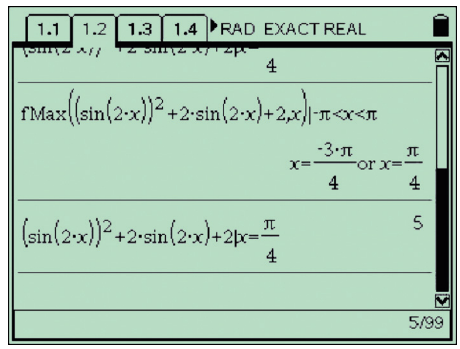
Using a TI-Nspire calculator

From the **Calculus** menu choose **Function Maximum** (menu 4 7).

The restriction is chosen to give particular solutions.

The maximum value is 5.

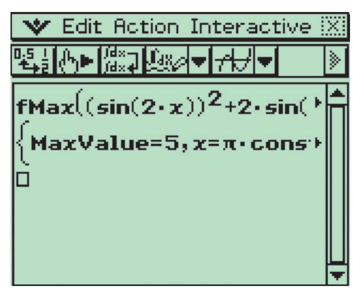
Function Minimum (menu 4 6) can be used to find a value of x for which the minimum occurs.



Using a Casio ClassPad calculator

From the **Interactive** menu choose **Calculus** and then **fMax**.

The minimum value can be found by choosing **Calculus** and then **fMin**.



b Note that $\sin^2(2x) + 2\sin(2x) + 2 \geq 0$ for all x. Thus its reciprocal also has this property.

The local maximum for the original function yields a local minimum for the reciprocal.

The local minimum for the original function yields a local maximum for the reciprocal.

$\therefore$ maximum value is 1 and the minimum value is $\frac{1}{5}$

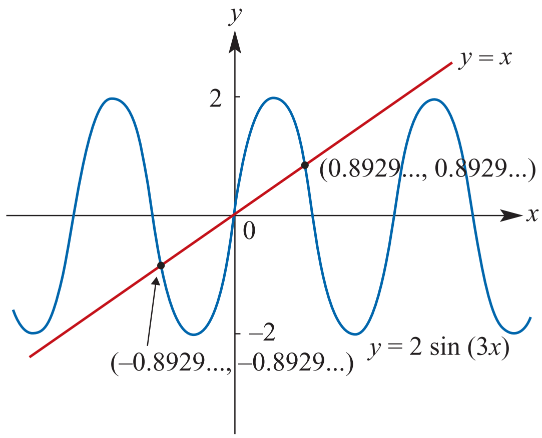
There are many equations involving circular functions which are not solvable by analytic techniques. A CAS calculator can be used for the solution of such equations.

Example 19

Find the solution of the equation $2 \sin(3x) = x$ correct to three decimal places.

Solution

The graphs of $y = 2\sin(3x)$ and $y = x$ are plotted with a graphics calculator. The solutions are $x = 0$, $x \approx 0.893$ and $x \approx -0.893$



Exercise 3D

1 Solve each of the following equations for $x \in [0, 2\pi]$:

a $\operatorname{cosec} x = -2$

b $\operatorname{cosec}\left(x - \frac{\pi}{4}\right) = -2$

c $3 \sec x = 2\sqrt{3}$

d $\operatorname{cosec}(2x) + 1 = 2$

e $\cot x = -\sqrt{3}$

f $\cot\left(2x - \frac{\pi}{3}\right) = -1$

2 Solve each of the following equations, giving solutions in the interval $[0, 2\pi]$:

a $\sin x = 0.5$

b $\cos x = \frac{-\sqrt{3}}{2}$

c $\tan x = \sqrt{3}$

d $\cot x = -1$

e $\sec x = 2$

f $\operatorname{cosec} x = -\sqrt{2}$

3 Find all the solutions of each of the following equations:

a $\sin x = \frac{\sqrt{2}}{2}$

b $\sec x = 1$

c $\cot x = \sqrt{3}$

4 Solve each of the following, in the interval $[-\pi, \pi]$, giving the answers correct to two decimal places:

a $\sec x = 2.5$

b $\operatorname{cosec} x = -5$

c $\cot x = 0.6$

5 Solve each of the following equations for $x \in [0, 2\pi]$:

a $\cos^2 x - \cos x \sin x = 0$

b $\sin 2x = \sin x$

c $\sin 2x = \cos x$

d $\sin 8x = \cos 4x$

e $\cos 2x = \cos x$

f $\cos 2x = \sin x$

g $\sec^2 x + \tan x = 1$

h $\tan x\,(1 + \cot x) = 0$

i $\cot x + 3 \tan x = 5 \operatorname{cosec} x$

j $\sin x + \cos x = 1$

6 Find the maximum and minimum values of each of the following:

a $2 + \sin\theta$ **b** $\dfrac{1}{2 + \sin\theta}$ **c** $\sin^2\theta + 4$

d $\dfrac{1}{\sin^2\theta + 4}$ **e** $\cos^2\theta + 2\cos\theta$ **f** $\cos^2\theta + 2\cos\theta + 6$

7 Using a CAS calculator, find the coordinates of the points of intersection for the graphs of the following pairs of functions. (Give values correct to two decimal places.)

a $y = 2x$ $y = 3\sin(2x)$ **b** $y = x$ $y = 2\sin(2x)$

c $y = 3 - x$ $y = \cos x$ **d** $y = x$ $y = \tan x$ $x \in [0, 2\pi]$

8 Let $\cos x = a$, $a \neq -1$, $x \in [0, 2\pi]$. If q is one of the solutions, find, in terms of q, the second solution.

9 Let $\sin\alpha = a$ where $\alpha \in \left[0, \dfrac{\pi}{2}\right]$. Find, in terms of α, two values of x in the range $[0, 2\pi]$ which satisfy each of the following equations:

a $\sin x = -a$ **b** $\cos x = a$

10 Let $\sec\beta = b$ where $\beta \in \left[\dfrac{\pi}{2}, \pi\right]$. Find, in terms of β, two values of x in the range $[-\pi, \pi]$ which satisfy each of the following equations:

a $\sec x = -b$ **b** $\operatorname{cosec} x = b$

11 Let $\tan\gamma = c$ where $\gamma \in \left[\pi, \dfrac{3\pi}{2}\right]$. Find in terms of γ, two values of x in the range $[0, 2\pi]$ which satisfy each of the following equations:

a $\tan x = -c$ **b** $\cot x = c$

12 Solve, correct to two decimal places, the equation $\sin^2\theta = \dfrac{\theta}{\pi}$ for $\theta \in [0, \pi]$.

13 Find the value of x, correct to two decimal places, such that $\tan^{-1} x = 4x - 5$.

14 A curve on a light rail track is an arc of a circle of length 300 m and the straight line joining the two ends of the curve is 270 m long.

a Show that, if the arc subtends an angle of $2\theta^\circ$ at the centre of the circle, θ is a solution of the equation $\sin\theta^\circ = \dfrac{\pi}{200}\theta^\circ$.

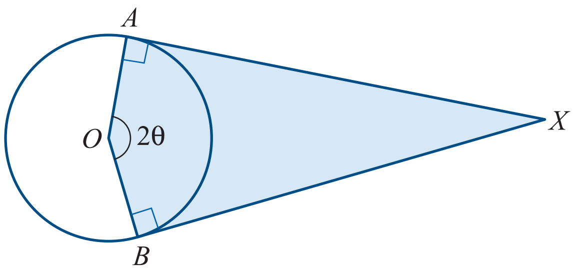
b Solve, correct to two decimal places, the equation for θ.

15 Solve, correct to two decimal places, the equation $\tan x = \dfrac{1}{x}$ for $x \in [0, \pi]$.

16 The area of a segment of a circle is given by the equation $A = \frac{1}{2}r^2(\theta - \sin\theta)$, where θ is the angle subtended at the centre of the circle.

If the radius of the circle is 6 cm and the area of the segment is 18 cm^2, find the value of θ correct to two decimal places.

17 Two tangents are drawn from a point so that the area of the shaded region is equal to the area of the remaining region of the circle.



$\angle AOB = 2\theta$

a Show that θ satisfies the equation $\tan \theta = \pi - \theta$.

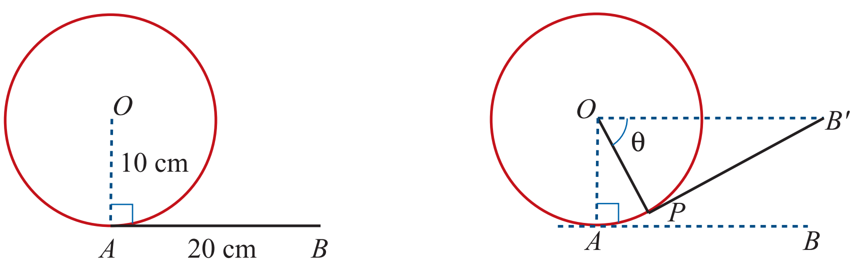
b Solve for θ, giving the answer correct to three decimal places.

18 Two particles A and B move in a straight line. At time t their positions relative to a point O are given by

$$x_A = 0.5 \sin t \quad \text{and} \quad x_B = 0.25t^2 + 0.05t$$

Find the times at which their positions are the same, and give this position (distances are measured in cm and time in seconds).

19 A string is wound around a disc and a horizontal length of the string AB is 20 cm long. The radius of the disc is 10 cm. The string is then moved so that the end of the string, B', is moved to a point at the same level as O, the centre of the circle. $B'P$ is a tangent to the circle.



a Show that θ satisfies the equation $\dfrac{\pi}{2} - \theta + \tan \theta = 2$.

b Find the value of θ, correct to two decimal places, which satisfies this equation.

MC TEST D&D TEST

Chapter summary

- The reciprocal circular functions cosec x, sec x and cot x are defined as follows:

$$\operatorname{cosec} x = \frac{1}{\sin x} \qquad \sin x \neq 0$$

$$\sec x = \frac{1}{\cos x} \qquad \cos x \neq 0$$

$$\cot x = \frac{\cos x}{\sin x} \qquad \sin x \neq 0$$

- Useful symmetry properties for the reciprocal circular functions are:

$$\sec(\pi - x) = -\sec x \qquad \sec(\pi + x) = -\sec x$$

$$\operatorname{cosec}(\pi - x) = \operatorname{cosec} x \qquad \operatorname{cosec}(\pi + x) = -\operatorname{cosec} x$$

$$\cot(\pi - x) = -\cot x \qquad \cot(\pi + x) = \cot x$$

$$\sec(2\pi - x) = \sec x \qquad \sec(-x) = \sec x$$

$$\operatorname{cosec}(2\pi - x) = -\operatorname{cosec} x \qquad \operatorname{cosec}(-x) = -\operatorname{cosec} x$$

$$\cot(2\pi - x) = -\cot x \qquad \cot(-x) = -\cot x$$

- Useful complementary properties for the reciprocal circular functions are:

$$\sec\left(\frac{\pi}{2} - x\right) = \operatorname{cosec} x \qquad \cot\left(\frac{\pi}{2} - x\right) = \tan x$$

$$\operatorname{cosec}\left(\frac{\pi}{2} - x\right) = \sec x \qquad \tan\left(\frac{\pi}{2} - x\right) = \cot x$$

- The Pythagorean identities derived from Pythagoras' theorem are:

$$\sin^2 x + \cos^2 x = 1$$

$$\cot^2 x + 1 = \operatorname{cosec}^2 x$$

$$\tan^2 x + 1 = \sec^2 x$$

- The compound angle formulas express circular functions of sums and differences of two angles (variables) in terms of circular functions of each of the angles:

$$\cos(x - y) = \cos x \cos y + \sin x \sin y$$

$$\cos(x + y) = \cos x \cos y - \sin x \sin y$$

$$\sin(x - y) = \sin x \cos y - \cos x \sin y$$

$$\sin(x + y) = \sin x \cos y + \cos x \sin y$$

$$\tan(x - y) = \frac{\tan x - \tan y}{1 + \tan x \tan y}$$

$$\tan(x + y) = \frac{\tan x + \tan y}{1 - \tan x \tan y}$$

Review

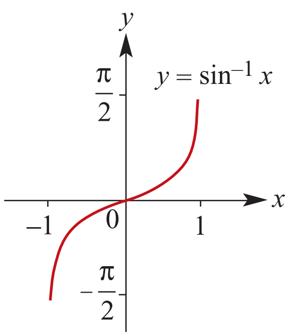
- The double angle formulas are derived from the compound angle formulas:

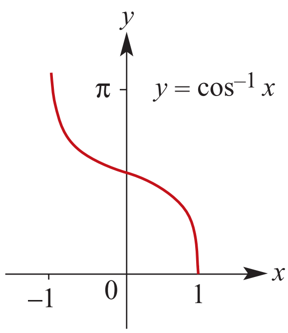
$$\begin{aligned}\cos 2x &= \cos^2 x - \sin^2 x\\ &= 2\cos^2 x - 1\\ &= 1 - 2\sin^2 x\\ \sin 2x &= 2\sin x\cos x\\ \tan 2x &= \frac{2\tan x}{1-\tan^2 x}\end{aligned}$$

- One-to-one inverse circular functions are defined as follows:

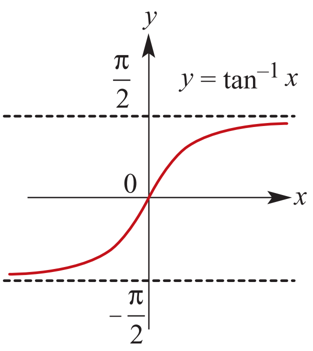
$\sin^{-1}: [-1, 1] \to R$, $\sin^{-1} x = y$
where $\sin y = x$, $y \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

$\cos^{-1}: [-1, 1] \to R$, $\cos^{-1} x = y$
where $\cos y = x$, $y \in [0, \pi]$





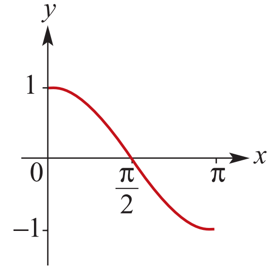
$\tan^{-1}: R \to R$, $\tan^{-1} x = y$
where $\tan y = x$, $y \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$



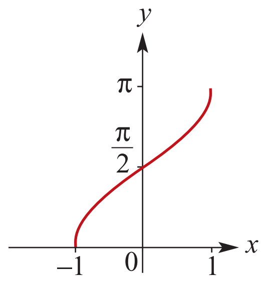
Multiple-choice questions

1 Which of the following is the graph of the function $y = \cos^{-1}(x)$?

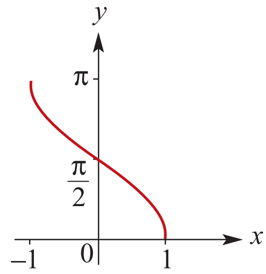
A



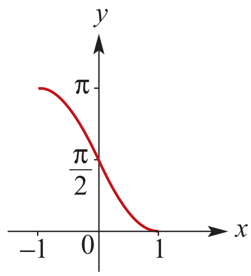
B



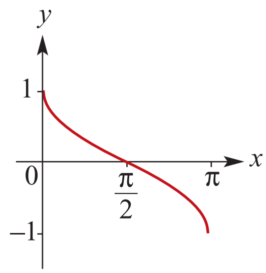
C



D



E



2 If $\cos x = \frac{-2}{3}$ and $2\pi < x < 3\pi$ then the exact value of $\sin x$ is:

A $2\pi + \frac{\sqrt{5}}{3}$ **B** $2\pi - \frac{\sqrt{5}}{3}$ **C** $\frac{\sqrt{5}}{3}$ **D** $\frac{-\sqrt{5}}{3}$ **E** $\frac{5}{9}$

3 Given that $\cos(x) = \frac{-1}{10}$ and $x \in \left(\frac{\pi}{2}, \pi\right)$, the value of $\cot(x)$ is:

A $\frac{10}{3\sqrt{11}}$ **B** $3\sqrt{11}$ **C** $-3\sqrt{11}$ **D** $\frac{\sqrt{11}}{33}$ **E** $\frac{-\sqrt{11}}{33}$

4 The graph of the function $y = 2 + \sec(3x)$, for $x \in \left(-\frac{\pi}{6}, \frac{7\pi}{6}\right)$, has stationary points at:

A $x = \frac{\pi}{3}, \pi$ **B** $x = \frac{\pi}{6}, \frac{\pi}{2}, \frac{5\pi}{6}$ **C** $x = \frac{\pi}{2}$
D $x = 0, \frac{\pi}{3}, \frac{2\pi}{3}, \pi$ **E** $x = 0, \frac{2\pi}{3}$

5 If $\sin x = -\frac{1}{3}$, the possible values of $\cos x$ are:

A $\frac{-2\sqrt{2}}{3}, \frac{2\sqrt{2}}{3}$ **B** $\frac{-2}{3}, \frac{2}{3}$ **C** $\frac{-8}{9}, \frac{8}{9}$ **D** $\frac{-\sqrt{2}}{3}, \frac{\sqrt{2}}{3}$ **E** $\frac{-1}{2}, \frac{1}{2}$

6 The maximum domain of $\cos^{-1}(1 - 5x)$ is given by:

A $\left[0, \frac{2}{5}\right]$ **B** $\left[\frac{1-\pi}{5}, \frac{1}{5}\right]$ **C** $[-1, 1]$ **D** $\left(0, \frac{2}{5}\right)$ **E** $\left[-\frac{1}{5}, \frac{1}{5}\right]$

7 $(1 + \tan x)^2 + (1 - \tan x)^2$ is equal to:

A $2 + \tan x + 2\tan(2x)$ **B** 2 **C** $-4\tan x$ **D** $2 + \tan(2x)$ **E** $2\sec^2 x$

8 The number of solutions of $\cos^2(3x) = \frac{1}{4}$, given that $0 \le x \le \pi$, is:

A 1 **B** 2 **C** 3 **D** 6 **E** 9

9 $\dfrac{\tan(2\theta)}{1 + \sec(2\theta)}$ equals:

A $\tan(2\theta)$ **B** $\tan(2\theta) + 1$ **C** $\tan\theta + 1$ **D** $\sin(2\theta)$ **E** $\tan\theta$

10 For $\frac{\pi}{2} < A < \pi$ and $0 < B < \frac{\pi}{2}$ with $\sin A = t$ and $\cos B = t$, $\cos(B + A)$ is equal to:

A 0 **B** $\sqrt{1 - t^2}$ **C** $2t^2 - 1$ **D** $1 - 2t^2$ **E** $-2t\sqrt{1 - t^2}$

Review

Short-answer questions (technology-free)

1 If θ is an acute angle and $\cos\theta = \frac{4}{5}$, find:

a $\cos 2\theta$
b $\sin 2\theta$
c $\tan 2\theta$
d $\operatorname{cosec}\theta$
e $\cot\theta$

2 Solve each of the following equations for $-\pi < x \le 2\pi$:

a $\sin 2x = \sin x$
b $\cos x - 1 = \cos 2x$
c $\sin 2x = 2\cos x$
d $\sin^2 x \cos^3 x = \cos x$
e $\sin^2 x - \frac{1}{2}\sin x - \frac{1}{2} = 0$
f $2\cos^2 x - 3\cos x + 1 = 0$

3 Solve each of the following equations for θ, $0 \le \theta \le 2\pi$, giving exact answers:

a $2 - \sin\theta = \cos^2\theta + 7\sin^2\theta$
b $\sec 2\theta = 2$
c $\frac{1}{2}(5\cos\theta - 3\sin\theta) = \sin\theta$
d $\sec\theta = 2\cos\theta$

4 Find the exact value of each of the following:

a $\sin\left(\frac{5\pi}{3}\right)$
b $\operatorname{cosec}\left(-\frac{5\pi}{3}\right)$
c $\sec\left(\frac{7\pi}{3}\right)$
d $\operatorname{cosec}\left(\frac{5\pi}{6}\right)$
e $\cot\left(-\frac{3\pi}{4}\right)$
f $\cot\left(-\frac{\pi}{6}\right)$

5 Given that $\tan\alpha = p$, where α is an acute angle, find each of the following in terms of p:

a $\tan(-\alpha)$
b $\tan(\pi - \alpha)$
c $\tan\left(\frac{\pi}{2} - \alpha\right)$
d $\tan\left(\frac{3\pi}{2} + \alpha\right)$
e $\tan(2\pi - \alpha)$

6 Find:

a $\sin^{-1}\left(\frac{\sqrt{3}}{2}\right)$
b $\cos\left[\cos^{-1}\left(\frac{1}{2}\right)\right]$
c $\cos^{-1}\left[\cos\left(\frac{2\pi}{3}\right)\right]$
d $\cos^{-1}\left[\cos\left(\frac{4\pi}{3}\right)\right]$
e $\cos\left[\sin^{-1}\left(-\frac{1}{2}\right)\right]$
f $\cos[\tan^{-1}(-1)]$

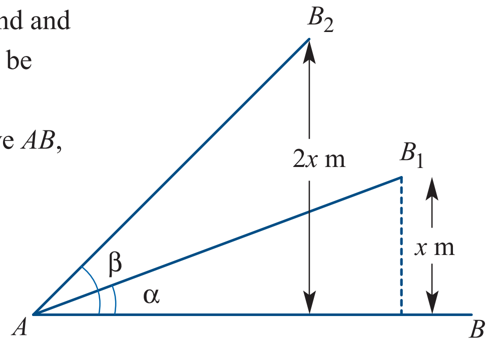
7 Sketch the graph of each of the following, stating the maximal domain and range each time:

a $y = 2\tan^{-1} x$
b $y = \sin^{-1}(3 - x)$
c $y = 3\cos^{-1}(2x + 1)$
d $y = -\cos^{-1}(2 - x)$
e $y = 2\tan^{-1}(1 - x)$

Review

Extended-response questions

1 A horizontal rod 1 m long is hinged at A at one end and rests on a support B at the other end. The rod can be rotated about A with the other end taking the two positions B_1 and B_2 which are x m and $2x$ m above AB, $x < 0.5$. Let $\angle BAB_1 = \alpha$ and $\angle BAB_2 = \beta$.



a Find, in terms of x, each of the following:

i $\sin \alpha$ **ii** $\cos \alpha$
iii $\tan \alpha$ **iv** $\sin \beta$
v $\cos \beta$ **vi** $\tan \beta$

b Using the results of **a**, find:

i $\sin(\beta - \alpha)$ **ii** $\cos(\beta - \alpha)$ **iii** $\tan(\beta - \alpha)$
iv $\tan(2\alpha)$ **v** $\sin(2\alpha)$ **vi** $\cos(2\alpha)$

c If $x = 0.3$, find the magnitude of $\angle B_2AB_1$ and 2α, correct to two decimal places.

CAS CALCULATOR

2 **a** On the one set of axes sketch the graphs of:

i $y = \operatorname{cosec}(x)$ $x \in (0, \pi) \cup (\pi, 2\pi)$
ii $y = \cot(x)$ $x \in (0, \pi) \cup (\pi, 2\pi)$
iii $y = \operatorname{cosec}(x) - \cot(x)$ $x \in (0, \pi) \cup (\pi, 2\pi)$

b **i** Show that $\operatorname{cosec} x - \cot x > 0$ for all $x \in (0, \pi)$ and hence that $\operatorname{cosec} x > \cot x$ for all $x \in (0, \pi)$.

ii Show that $\operatorname{cosec} x - \cot x < 0$ for all $x \in (\pi, 2\pi)$ and hence that $\operatorname{cosec} x < \cot x$ for all $x \in (\pi, 2\pi)$.

c On separate axes sketch the graph of $y = \cot\left(\frac{x}{2}\right)$ for $x \in (0, 2\pi)$ and $y = \operatorname{cosec}(x) + \cot(x)$ for $x \in (0, 2\pi)$.

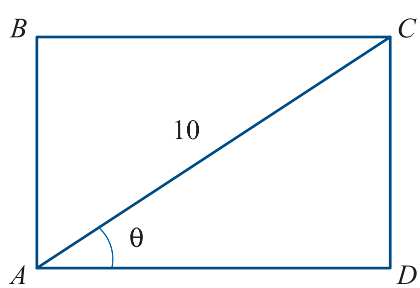
d **i** Prove that $\operatorname{cosec} \theta + \cot \theta = \cot\left(\frac{\theta}{2}\right)$ where $\sin \theta \neq 0$.

ii Use this result to find $\cot \frac{\pi}{8}$ and $\cot \frac{\pi}{12}$.

iii Use the result $1 + \cot^2\left(\frac{\pi}{8}\right) = \operatorname{cosec}^2\left(\frac{\pi}{8}\right)$ to find the exact value of $\sin\left(\frac{\pi}{8}\right)$.

e Use the result of **d** to show that $\operatorname{cosec}(\theta) + \operatorname{cosec}(2\theta) + \operatorname{cosec}(4\theta)$ can be expressed as the difference of two cotangents.

3 **a** $ABCD$ is a rectangle with diagonal AC of length 10 units.



i Find the area of the rectangle in terms of θ.

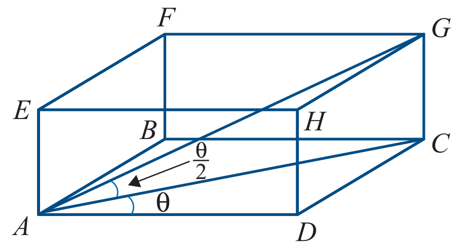
ii Sketch the graph of R against θ where R is the area of the rectangle in square units, $\theta \in \left(0, \frac{\pi}{2}\right)$.

iii Find the maximum value of R (do not use calculus).

iv Find the value of θ for which this maximum occurs.

b $ABCDEFGH$ is a cuboid.

$\angle GAC = \frac{\theta}{2}$, $\angle CAD = \theta$,
$AC = 10$



i Show that the volume, V, of the cuboid is given by $V = 1000 \cos\theta \sin\theta \tan\frac{\theta}{2}$.

ii Find the values of a and b such that
$V = a\sin^2\frac{\theta}{2} + b\sin^4\frac{\theta}{2}$.

iii Let $p = \sin^2\frac{\theta}{2}$, and express V as a quadratic in p.

iv Find the possible values of p for $0 < \theta < \frac{\pi}{2}$.

v Sketch the graphs of V against θ and V against p with the help of a CAS calculator.

vi Find the maximum volume of the cuboid and the values of p and θ for which this occurs. (Determine the maximum through the quadratic found in **b iii**.)

c If for the cuboid $\angle CAD = \theta$ and $\angle GAC = \theta$:

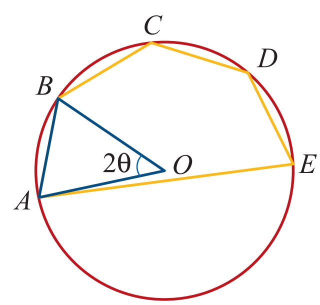
i find V in terms of θ

ii sketch the graph of V against θ

iii discuss the relationship between V and θ using the graph of **c ii**.

4 $ABCDE$ is a pentagon inscribed in a circle.

$AB = BC = CD = DE = 1$ and $\angle BOA = 2\theta$.
O is the centre of the circle.
Let $AE = p$



a Show $p = \dfrac{\sin 4\theta}{\sin\theta}$.

b Express p as a function of $\cos\theta$. Let $x = \cos\theta$.

c **i** If $p = \sqrt{3}$ show that $8x^3 - 4x - \sqrt{3} = 0$.

ii Show that $\frac{\sqrt{3}}{2}$ is a solution to the equation and that it is the only real solution.

iii Find the value of θ for which $p = \sqrt{3}$.

iv Find the radius of the circle.

d Using a CAS calculator sketch the graph of p against θ for $\theta \in \left(0, \frac{\pi}{4}\right]$.

e If $A = E$, find the value of θ.

f **i** If $AE = 1$, show that $8x^3 - 4x - 1 = 0$.

ii Hence show $\frac{1}{4}(\sqrt{5} + 1) = \cos\left(\frac{\pi}{5}\right)$.

Review

5 **a** **i** Prove that $\tan x + \cot x = 2\ \text{cosec}(2x)$ for $\sin 2x \neq 0$.

ii Solve the equation $\tan x = \cot x$ for x.

iii On the one set of axes, sketch the graphs of $y = \tan x$, $y = \cot x$ and $y = 2\ \text{cosec}(2x)$ for $x \in (0, 2\pi)$.

b **i** Prove that $\cot(2x) + \tan x = \text{cosec}(2x)$ for $\sin(2x) \neq 0$.

ii Solve the equation $\cot(2x) = \tan x$.

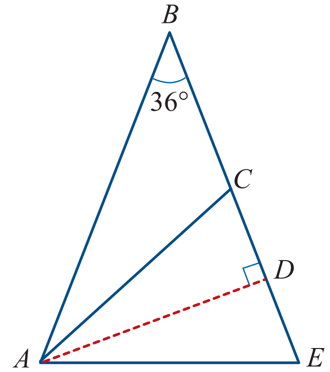
iii On the one set of axes, sketch the graphs of $y = \cot(2x)$, $y = \tan x$ and $y = \text{cosec}(2x)$ for $x \in [0, 2\pi]$.

c **i** Prove that $\cot(mx) + \tan(nx) = \dfrac{\cos[(m-n)x]}{\sin(mx)\cos(nx)}$ $\quad m, n \in Z$.

ii Hence show that $\cot(6x) + \tan(3x) = \text{cosec}(6x)$.

6 Triangle ABE is isosceles with $AB = BE$ and triangle ACE is isosceles with $AC = AE$.

$AE = 1$



a **i** Find the magnitude of $\angle BAE$, $\angle AEC$ and $\angle ACE$.

ii Hence find the magnitude of $\angle BAC$.

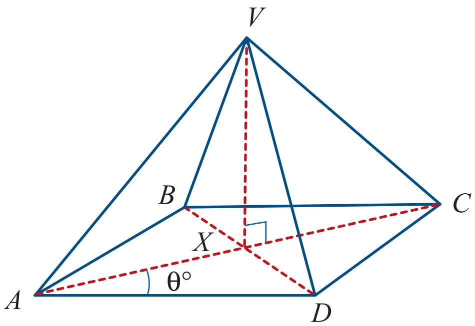
b Show that $BD = 1 + \sin 18°$.

c Use triangle ABD to prove that $\cos 36° = \dfrac{1 + \sin 18°}{1 + 2\ \sin 18°}$.

d Hence show that $4\sin^2 18° + 2\sin 18° - 1 = 0$.

e Find $\sin 18°$ in exact form.

7 $VABCD$ is a right pyramid with diagonal length $AC = 10$. $ABCD$ is a rectangle.



a **i** If $\angle CAD = \theta°$ and $\angle VAX = \theta°$ show that the volume, V, of the pyramid is given by

$$V = \frac{500}{3}\sin^2\theta$$

ii Sketch the graph of V against θ for $\theta \in (0, 90)$.

iii Comment on the graph.

b If the magnitude of angle CAD is $\theta°$ and the magnitude of angle VAX is $\dfrac{\theta°}{2}$:

i show that the volume, V, of the pyramid is given by

$$V = \frac{1000}{3}\sin^2\frac{\theta}{2}\left(1 - 2\ \sin^2\frac{\theta}{2}\right)$$

ii state the maximal domain of the function $V(\theta)$.

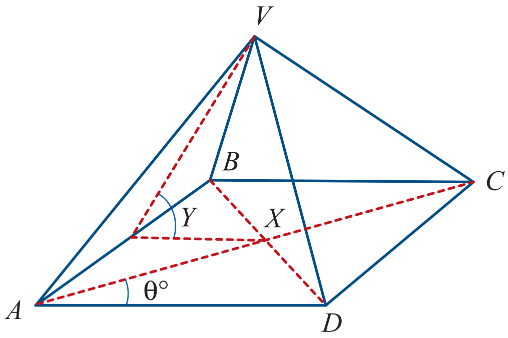
iii Let $a = \sin^2\left(\dfrac{\theta}{2}\right)$ and write V as a quadratic function with variable a.

iv Hence find the maximum value of V and the value of θ for which this occurs.

v Sketch the graph of V against θ for the domain established in **ii**.

Review

8 $VABCD$ is a right pyramid with diagonal length $AC = 10$. $ABCD$ is a rectangle. $\angle CAD = \theta°$ $AY = BY$



a If $\angle VYX = \theta°$ find:

i an expression for the volume of the pyramid in terms of θ

ii the maximum volume and the value of θ for which this occurs.

b If $\angle VYX = \frac{\theta°}{2}$:

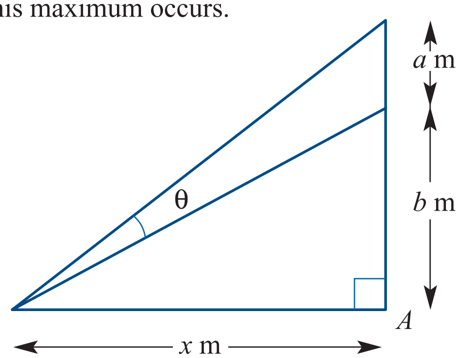
i show that $V = \frac{500}{3}\cos^2\theta(1 - \cos\theta)$

ii state the implied domain for the function.

CAS CALCULATOR

c Let $a = \cos\theta$. Then $V = \frac{500}{3}a^2(1 - a)$. Use a CAS calculator to find the maximum value of V and the values of a and θ for which this maximum occurs.

9 A camera is in a position x m from a point A. A body a metres in length is projected vertically upwards from A. When the body has moved b metres vertically up:



a show $\theta = \tan^{-1}\left(\frac{a+b}{x}\right) - \tan^{-1}\left(\frac{b}{x}\right)$

b use the result of **a** to show

$$\tan\theta = \frac{ax}{x^2 + ba + b^2}$$

c If $\theta = \frac{\pi}{4}$ find: **i** x in terms of a and b **ii** x if $a = 2(1 + \sqrt{2})$ and $b = 1$

d If $a = 2(1 + \sqrt{2})$, $b = 1$ and $x = 1$, find an approximate value of θ.

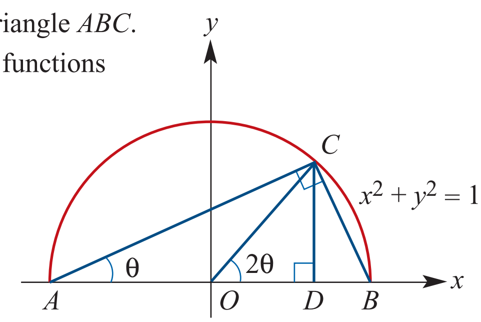
CAS CALCULATOR

e Using a CAS calculator, plot the graphs of θ against b and $\tan\theta$ against b for constant values of a and x as indicated below:

i $a = 1, x = 5$ **ii** $a = 1, x = 10$ **iii** $a = 1, x = 20$

f Comment on these graphs.

10 Points A, B and C lie on a circle with centre O and radius 1 as shown.



a Give reasons why triangle ACD is similar to triangle ABC.

b Give the coordinates of C in terms of circular functions applied to 2θ.

c **i** Find CA in terms of θ from triangle ABC.

ii Find CB in terms of θ from triangle ABC.

d Use the results of **b** and **c** to show $\sin(2\theta) = 2\sin\theta\cos\theta$.

e Use the results of **b** and **c** to show $\cos(2\theta) = 2\cos^2\theta - 1$.

C H A P T E R

4

Complex numbers

Objectives

- To understand the **imaginary number** i
- To understand the set of **complex numbers** C
- To understand the **real-value functions**, Re(z) and Im(z), of complex numbers
- To represent complex numbers graphically on an **Argand diagram**
- To understand the rules which **define equality**, **addition**, **subtraction** and **multiplication** of complex numbers
- To understand the concept of the **complex conjugate**
- To understand the operation of **division** of complex numbers
- To understand the **modulus–argument** form of a complex number and the basic operations on complex numbers in that form
- To understand the **geometrical** significance of multiplication and division of complex numbers in the modulus–argument form
- To understand and apply **De Moivre's theorem**
- To understand the rules of equations defined by the **fundamental theorem of algebra**
- To **factorise** low degree polynomial expressions over C
- To **solve** low degree polynomial equations over C
- To represent **relations** and **regions** in the complex plane

In earlier work in mathematics it was assumed that an equation of the form $x^2 = -1$ had no solutions. However, mathematics of the 18th century used the imaginary number i with the property $i^2 = -1$.

i is defined as $i = \sqrt{-1}$ and the equation $x^2 = -1$ has two solutions, i and $-i$.

The number found applications in many fields of mathematics. It also allowed the theory of equations to develop fully, culminating in the **fundamental theorem of algebra** which is stated in one form as follows.

In the field of complex numbers, every polynomial equation of the form

$$a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0 = 0 \text{ where } a_0, a_1, \ldots, a_n \in C, a_n \neq 0$$

has exactly n roots, some of which may be repeated.

4.1 The set of complex numbers, C

A **complex number** is an expression of the form $a + bi$, where a and b are real numbers.
C is the set of complex numbers, i.e. $C = \{a + bi: a, b \in R\}$.

The letter often used to denote a complex pronumeral is z. Therefore, $z \in C$ implies $z = a + bi$ where $a, b \in R$.

- If $a = 0$, z is said to be **imaginary**.
- If $b = 0$, z is **real**.

Real numbers and imaginary numbers are subsets of C.

Functions of the complex number

Let $z = a + bi$.
$\text{Re}(z)$ is a function which defines the real component of z, i.e. $\text{Re}(z) = a$.
$\text{Im}(z)$ is a function which defines the value of the imaginary component of z, i.e. $\text{Im}(z) = b$.

Note: $\text{Re}(z)$ and $\text{Im}(z)$ are both real-value functions of z, i.e. $\text{Re}: C \to R$ and $\text{Im}: C \to R$.

Example 1

Let $z = 4 - 5i$. Find:

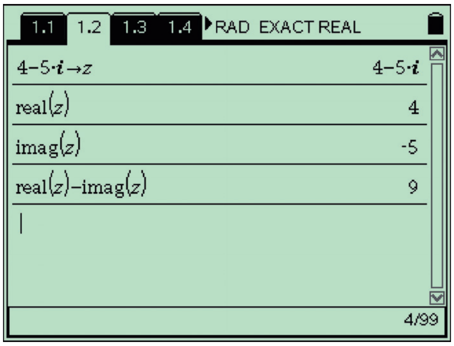
a $\text{Re}(z)$ **b** $\text{Im}(z)$ **c** $\text{Re}(z) - \text{Im}(z)$

Solution

a $\text{Re}(z) = 4$ **b** $\text{Im}(z) = -5$ **c** $\text{Re}(z) - \text{Im}(z) = 4 - (-5) = 9$

Using a TI-Nspire calculator

Enter as shown in the first line. Use the i key. From the **Number** menu choose **Complex Number Tools** and then **Real Part** (menu 2 9 2). **Complex Number Tools** and then **Imaginary Part** gives the second required command (menu 2 9 3).

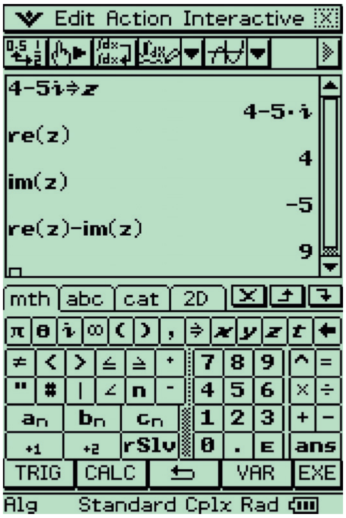


Using a Casio ClassPad calculator

From the **Interactive** menu choose **Complex** and then **re**.

Complex and then **im** gives the second required command.

Make sure the calculator is in Complex mode (**Cplx** at the bottom right of the screen).



Example 2

a Represent $\sqrt{-5}$ as an imaginary number.

b Simplify $2\sqrt{-9} + 4i$.

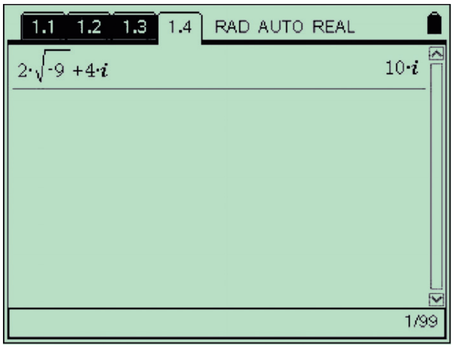
Solution

a
$$\begin{aligned}\sqrt{-5} &= \sqrt{5 \times -1}\\ &= \sqrt{5} \times \sqrt{-1}\\ &= \sqrt{5}i \quad \text{or} \quad i\sqrt{5}\end{aligned}$$

b
$$\begin{aligned}2\sqrt{-9} + 4i &= 2\sqrt{9 \times -1} + 4i\\ &= 2 \times 3 \times i + 4i\\ &= 6i + 4i\\ &= 10i\end{aligned}$$

Using a TI-Nspire calculator

Enter the expression and press **Enter**.



Using a Casio ClassPad calculator

Make sure the calculator is in Complex mode (**Cplx** at the bottom of the screen).
Enter the expression and press **Enter**.

Equal complex numbers

Let $z_1, z_2 \in C$

Then $z_1 = z_2$ if and only if $\text{Re}(z_1) = \text{Re}(z_2)$ and $\text{Im}(z_1) = \text{Im}(z_2)$

If $z_1 = a + bi$ and $z_2 = c + di$

then $z_1 = z_2$ if and only if $a = c$ and $b = d$

Example 3

Solve the equation $(2a - 3) + 2bi = 5 + 6i$, $a \in R$ and $b \in R$.

Solution

$$(2a - 3) + 2bi = 5 + 6i$$

$$\therefore \quad 2a - 3 = 5 \quad \text{and} \quad 2b = 6$$

$$a = 4 \quad \text{and} \quad b = 3$$

Operations on complex numbers

The operations of addition, subtraction and multiplication by a scalar can be defined in the following way.

Let $z_1 = a + bi$ and $z_2 = c + di$.

Addition

$$z_1 + z_2 = (a + bi) + (c + di) = (a + c) + (b + d)i$$

Subtraction

$$z_1 - z_2 = (a + bi) - (c + di) = (a - c) + (b - d)i$$

Multiplication by a scalar

$$kz_1 = k(a + bi) = ka + kbi, k \in R$$

Example 4

Let $z_1 = 2 - 3i$ and $z_2 = 1 + 4i$. Simplify:

a $z_1 + z_2$ **b** $z_1 - z_2$ **c** $3z_1 - 2z_2$

Solution

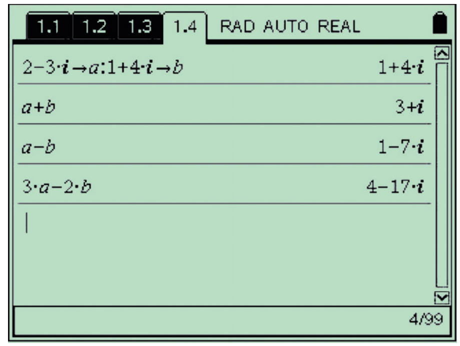
a $z_1 + z_2 = (2 - 3i) + (1 + 4i) = 3 + i$

b $z_1 - z_2 = (2 - 3i) - (1 + 4i) = 1 - 7i$

c $3z_1 - 2z_2 = 3(2 - 3i) - 2(1 + 4i) = (6 - 9i) - (2 + 8i) = 4 - 17i$

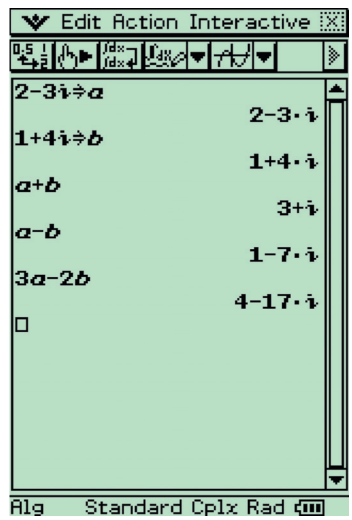
Using a TI-Nspire calculator

Enter the expressions as shown.



Using a Casio ClassPad calculator

Make sure the calculator is in Complex mode (**Cplx** at the bottom of the screen). Enter the expressions as shown.

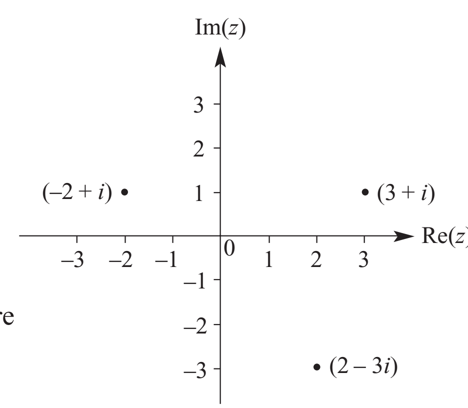


Argand diagrams

An **Argand diagram** is a geometrical representation of the set of complex numbers. In a vector sense, a complex number has two dimensions: the real part and the imaginary part. Therefore a plane is required to represent C.

An Argand diagram is drawn with two perpendicular axes. For $z \in C$, the horizontal axis represents Re(z), and the vertical axis represents Im(z).

Each point on an Argand diagram represents a complex number. The complex number $a + bi$ is situated at the point (a, b) on the equivalent cartesian axes as shown by the examples in the figure opposite. The number written as $a + bi$ is called the cartesian form of the complex number.

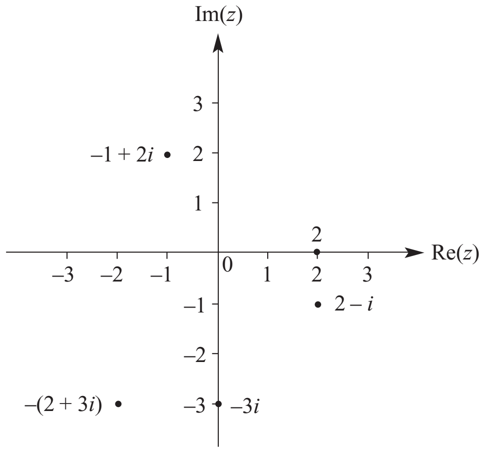


Example 5

Represent the following points on an Argand diagram:

a 2 **b** $-3i$ **c** $2-i$ **d** $-(2+3i)$ **e** $-1+2i$

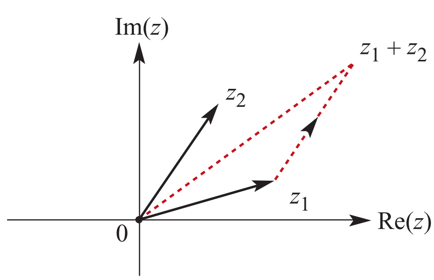
Solution

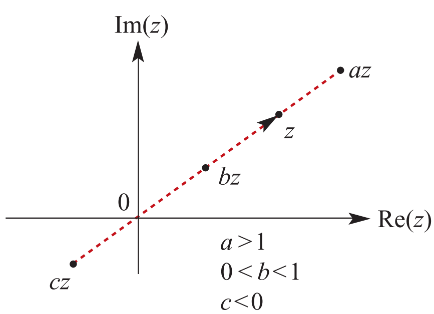


Geometrical representation of the basic operations on complex numbers

The addition of two complex numbers is similar to a vector sum and follows the triangle of vectors rule.

The multiplication by a scalar follows vector properties of parallel position vectors.





The subtraction $z_1 - z_2$ is represented by the sum $z_1 + (-z_2)$.

Example 6

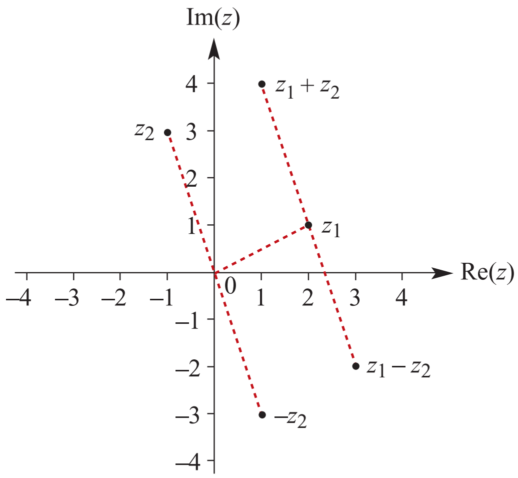
Let $z_1 = 2 + i$ and $z_2 = -1 + 3i$.

Represent z_1, z_2, $z_1 + z_2$ and $z_1 - z_2$ on an Argand diagram and verify that the complex number sum and difference follow the vector triangle properties.

Solution

$$z_1 + z_2 = (2 + i) + (-1 + 3i) = 1 + 4i$$

$$z_1 - z_2 = (2 + i) - (-1 + 3i) = 3 - 2i$$



Multiplication of complex numbers

Multiplication of complex numbers is defined as follows.

Let $z_1 = a + bi$ and $z_2 = c + di$

Then $z_1 z_2 = (ac - bd) + (ad + bc)i$

The reason for this definition can be shown by implementing the distributive law in C.

i.e.
$$\begin{aligned} z_1 \times z_2 = z_1 z_2 &= (a + bi)(c + di) \\ &= ac + adi + bci + bdi^2 \quad \text{(using the distributive law)} \\ &= ac + adi + bci - bd \quad (\text{as } i^2 = -1) \end{aligned}$$

Therefore $z_1 z_2 = (ac - bd) + (ad + bc)i$

Example 7

Simplify:

a $(2 + 3i)(1 - 5i)$ **b** $3i(5 - 2i)$ **c** i^3

Solution

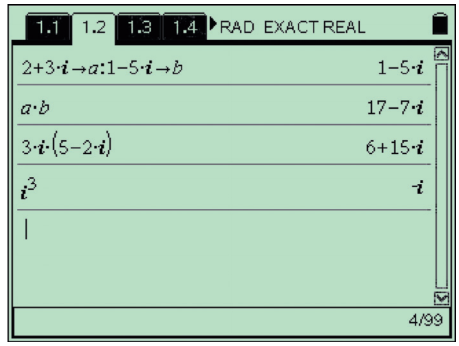
a
$$\begin{aligned} (2 + 3i)(1 - 5i) &= 2 - 10i + 3i - 15i^2 \\ &= 2 - 10i + 3i + 15 \\ &= 17 - 7i \end{aligned}$$

b
$$\begin{aligned} 3i(5 - 2i) &= 15i - 6i^2 \\ &= 15i + 6 \\ &= 6 + 15i \end{aligned}$$

c
$$\begin{aligned} i^3 &= i \times i^2 \\ &= -i \end{aligned}$$

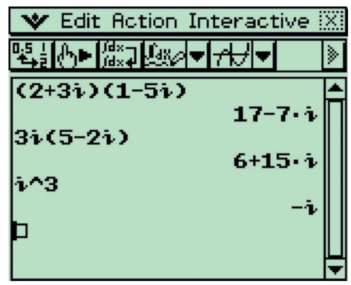
Using a TI-Nspire calculator

Complete as shown.



Using a Casio ClassPad calculator

Complete as shown.



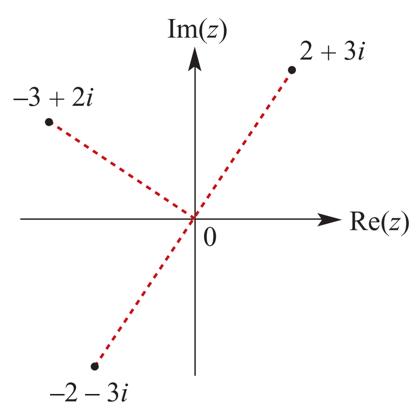
Geometrical significance of multiplication of a complex number by *i*

When the complex number $2 + 3i$ is multiplied by -1 the result is $-2 - 3i$. This can be considered to be achieved through a 180° rotation of the complex number $2 + 3i$ about the origin.

When the complex number $2 + 3i$ is multiplied by i

i.e.
$$\begin{aligned} i(2 + 3i) &= 2i + 3i^2 \\ &= 2i - 3 \\ &= -3 + 2i \end{aligned}$$

the result can be seen as a rotation of the complex number $2 + 3i$ in an anticlockwise direction 90° about the origin.



If $-3 + 2i$ is multiplied by i the result is $-2 - 3i$. This is achieved through a further rotation of the complex number $-3 + 2i$ in an anticlockwise direction 90° about the origin.

Exercise 4A

1 Simplify each of the following:

a $\sqrt{-25}$ **b** $\sqrt{-27}$ **c** $2i - 7i$
d $5\sqrt{-16} - 7i$ **e** $\sqrt{-8} + \sqrt{-18}$ **f** $i\sqrt{-12}$
g $i(2 + i)$ **h** $\text{Im}(2\sqrt{-4})$ **i** $\text{Re}(5\sqrt{-49})$

2 Let $z_1 = 2 - i$, $z_2 = 3 + 2i$ and $z_3 = -1 + 3i$. Find:

a $z_1 + z_2$ **b** $z_1 + z_2 + z_3$ **c** $2z_1 - z_3$
d $3 - z_3$ **e** $4i - z_2 + z_1$ **f** $\text{Re}(z_1)$
g $\text{Im}(z_2)$ **h** $\text{Im}(z_3 - z_2)$ **i** $\text{Re}(z_2) - i\,\text{Im}(z_2)$

3 Solve the following equations for real values x and y:

a $x + iy = 5$ **b** $x + iy = 2i$
c $x = iy$ **d** $x + iy = (2 + 3i) + 7(1 - i)$
e $2x + 3 + 8i = -1 + (2 - 3y)i$ **f** $x + iy = (2y + 1) + (x - 7)i$

4 Represent each of the following complex numbers on an Argand diagram:

a $-4i$ **b** -3 **c** $2(1 + i)$
d $3 - i$ **e** $-(3 + 2i)$ **f** $-2 + 3i$

5 Let $z_1 = 1 + 2i$ and $z_2 = 2 - i$.

a Represent the following complex numbers on an Argand diagram:
i z_1 **ii** z_2 **iii** $2z_1 + z_2$ **iv** $z_1 - z_2$
b Verify that parts **iii** and **iv** follow the vector triangle properties.

6 Simplify each of the following:

a $(5 - i)(2 + i)$ **b** $(4 - 7i)(3 + 5i)$ **c** $(2 + 3i)(2 - 3i)$
d $(1 + 3i)^2$ **e** $(2 - i)^2$ **f** $(1 + i)^3$
g i^4 **h** $i^{11}(6 + 5i)$ **i** i^{70}

7 Solve each of the following equations for real values x and y:

a $2x + (y + 4)i = (3 + 2i)(2 - i)$ **b** $(x + yi)(3 + 2i) = -16 + 11i$
c $(x + 2i)^2 = 5 - 12i$ **d** $(x + iy)^2 = -18i$
e $i(2x - 3yi) = 6(1 + i)$

8 **a** Represent each of the following complex numbers on an Argand diagram:
i $1 + i$ **ii** $(1 + i)^2$ **iii** $(1 + i)^3$ **iv** $(1 + i)^4$
b Describe any geometrical pattern observed in the position of these complex numbers.

9 Let $z_1 = 2 + 3i$ and $z_2 = -1 + 2i$. P, Q and R are the points defined on an Argand diagram by z_1, z_2, and $z_2 - z_1$ respectively.

a Show that $\overrightarrow{PQ} = \overrightarrow{OR}$. **b** Hence find QP.

4.2 The complex conjugate and division

The complex conjugate

For $z = a + bi$ the **modulus** of z, denoted by $|z|$, is defined by $|z| = \sqrt{a^2 + b^2}$. This is the distance of the complex number from the origin. The modulus is also known as the **magnitude** or **absolute value** of z.

For example, if $z_1 = 3 + 4i$ and $z_2 = -3 + 4i$, then $|z_1| = \sqrt{3^2 + 4^2} = 5$ and $|z_2| = \sqrt{(-3)^2 + 4^2} = 5$. Both z_1 and z_2 are a distance of 5 units from the origin.

The importance of the concept of modulus is shown in the following section. In this section it serves as a useful notation.

Let $z \in C$, then the complex conjugate of z, denoted by $\bar{z}$ (read as 'z bar'), is defined by:

$$\bar{z} = \text{Re}(z) - \text{Im}(z)i$$

In particular, if $z = a + bi$ then $\bar{z} = a - bi$.

The conjugate of z plays an important role in complex number theory, as it is the only complex number which will produce a real number under addition and multiplication with z, i.e. $z + \bar{z}$ and $z\bar{z}$ are both real numbers.

Let $\quad z = a + bi$

Then $\quad \bar{z} = a - bi$

$$\begin{aligned} z + \bar{z} &= a + bi + a - bi \\ &= 2a \\ &= 2\,\text{Re}(z) \end{aligned} \qquad \begin{aligned} z\bar{z} &= (a + bi)(a - bi) \\ &= a^2 + abi - abi - b^2i^2 \\ &= a^2 + b^2 \\ &= |z|^2 \end{aligned}$$

The uniqueness of the complex conjugate can be demonstrated as follows:

Let $\quad \bar{z} = c + di \quad c \in R,$ and $d \in R$

Then $\quad z + \bar{z} = (a + bi) + (c + di)$

$$= (a + c) + (b + d)i$$

As $z + \bar{z}$ is real, $b + d = 0$ and $d = -b$

$$\begin{aligned} z\bar{z} &= (a + bi)(c + di) \\ &= (ac - bd) + (ad + bc)i \end{aligned}$$

As $z\bar{z}$ is real, $ad + bc = 0$, and as $d = -b$

$$\begin{aligned} ad + bc &= -ab + bc \\ &= b(-a + c) \end{aligned}$$

Therefore $b(-a + c) = 0$ which implies that $c = a$.

$\bar{z}$ with these properties can only exist in the form:

$$\bar{z} = a - bi$$

The complex conjugate has the following properties:

$$\overline{z_1 + z_2} = \overline{z_1} + \overline{z_2}, \qquad \overline{z_1 z_2} = \overline{z_1}\,\overline{z_2}$$

These can be verified as an exercise by the reader in question 3 of Exercise 4B.

Example 8

Find the complex conjugate of each of the following:

a 2 **b** $3i$ **c** $-1 - 5i$

Solution

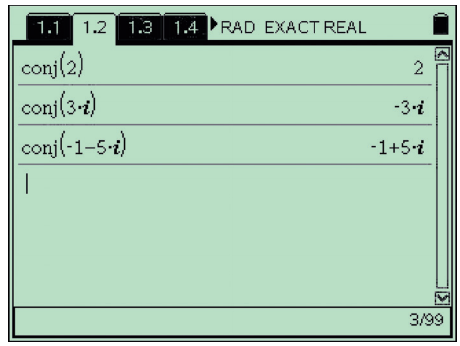
a The complex conjugate of 2 is 2. **b** The complex conjugate of $3i$ is $-3i$.

c The complex conjugate of $-1 - 5i$ is $-1 + 5i$.

Using a TI-Nspire calculator

Use the ⓘ key.

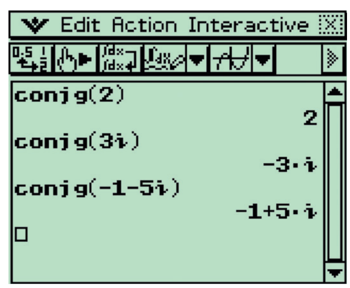
From the **Number** menu choose **Complex Number Tools** and then **Complex Conjugate** (menu 2 9 1).



Using a Casio ClassPad calculator

Make sure the calculator is in Complex mode (**Cplx** at the bottom of the screen).

From the **Interactive** menu choose **Complex** and then **conj**.



Division of complex numbers

Let $z_1 \in C$, and $z_2 \in C$.

Consider the problem $z_1 \div z_2$, or simply $\frac{z_1}{z_2}$, $z_2 \neq 0$. In order to obtain a complex number in the form $a + bi$ as the answer, it is necessary to perform a process similar to the rationalisation of surds to remove the complex number (in particular the imaginary part) from the denominator.

This is done by multiplying the numerator and the denominator by the conjugate of z_2. The expression can then be reduced to a single complex number:

$$\frac{z_1}{z_2} = \frac{z_1 \times \bar{z}_2}{z_2 \times \bar{z}_2} = \frac{z_1 \bar{z}_2}{|z_2|^2}$$

Note that $z^{-1} = \frac{1}{z} = \frac{1}{z} \times \frac{\bar{z}}{\bar{z}} = \frac{\bar{z}}{z\bar{z}} = \frac{\bar{z}}{|z|^2}$

Example 9

a Write each of the following in the form $a + bi$ where $a, b \in R$.

i $\dfrac{1}{3 - 2i}$ **ii** $\dfrac{4 + i}{3 - 2i}$

b Simplify $\dfrac{(1 + 2i)^2}{i(1 + 3i)}$.

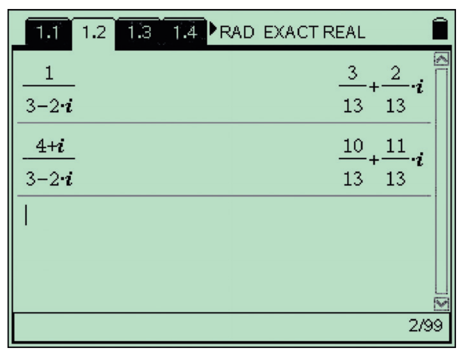
Solution

a i
$$\frac{1}{3 - 2i} = \frac{1}{3 - 2i} \times \frac{3 + 2i}{3 + 2i}$$
$$= \frac{3 + 2i}{3^2 - (2i)^2}$$
$$= \frac{3 + 2i}{13}$$
$$= \frac{3}{13} + \frac{2}{13}i$$

ii
$$\frac{4 + i}{3 - 2i} = \frac{4 + i}{3 - 2i} \times \frac{3 + 2i}{3 + 2i}$$
$$= \frac{(4 + i)(3 + 2i)}{3^2 + 2^2}$$
$$= \frac{12 + 8i + 3i - 2}{13}$$
$$= \frac{10}{13} + \frac{11}{13}i$$

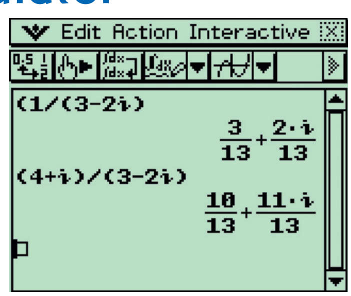
Using a TI-Nspire calculator

Complete as shown.



Using a Casio ClassPad calculator

Complete as shown.



b $\dfrac{(1+2i)^2}{i(1+3i)} = \dfrac{1+4i-4}{-3+i}$

$= \dfrac{-3+4i}{-3+i} \times \dfrac{-3-i}{-3-i}$

$= \dfrac{9+3i-12i+4}{(-3)^2-i^2}$

$= \dfrac{13-9i}{10}$

$= \dfrac{13}{10} - \dfrac{9}{10}i$

Exercise 4B

1 Find the complex conjugate of the following complex numbers:

a $\sqrt{3}$ **b** $8i$ **c** $4-3i$

d $-(1+2i)$ **e** $4+2i$ **f** $-3-2i$

2 Simplify each of the following, giving your answer in the form $a+bi$:

a $\dfrac{2+3i}{3-2i}$ **b** $\dfrac{i}{-1+3i}$ **c** $\dfrac{-4-3i}{i}$

d $\dfrac{3+7i}{1+2i}$ **e** $\dfrac{\sqrt{3}+i}{-1-i}$ **f** $\dfrac{17}{4-i}$

3 Let $z = a+bi$ and $w = c+di$. Show that:

a $\overline{z+w} = \bar{z}+\bar{w}$ **b** $\overline{zw} = \bar{z}\bar{w}$ **c** $\overline{\left(\dfrac{z}{w}\right)} = \dfrac{\bar{z}}{\bar{w}}$

4 Let $z = 2-i$. Simplify the following:

a $z(z+1)$ **b** $\overline{z+4}$ **c** $\overline{z-2i}$

d $\dfrac{z-1}{z+1}$ **e** $(z-i)^2$ **f** $(z+1+2i)^2$

5 For $z = a+ib$, write each of the following in terms of a and b:

a $z\bar{z}$ **b** $\dfrac{z}{|z|^2}$ **c** $z+\bar{z}$

d $z-\bar{z}$ **e** $\dfrac{z}{\bar{z}}$ **f** $\dfrac{\bar{z}}{z}$

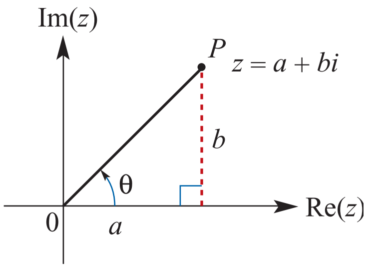
4.3 The modulus–argument (or polar) form of a complex number

In the preceding sections, complex numbers have been expressed in cartesian form. The modulus–argument (or polar) form is an alternative way of representing a complex number.

Consider the complex number $z = a + bi$, represented by the point P on an Argand diagram.

This point can also be represented by the position vector $\overrightarrow{OP}$ and can, therefore, be defined precisely in terms of its magnitude and direction.

The modulus and argument are the complex number equivalents of magnitude (or absolute value) and direction.



The modulus of z

$|z|$, the modulus of z, has been defined and is the distance from the origin of the point represented by z. By Pythagoras' theorem,

$$|z| = \sqrt{(\text{Re}(z))^2 + (\text{Im}(z))^2} \quad \text{or} \quad |z| = \sqrt{a^2 + b^2}$$

The argument of z

arg(z), the argument of z, is the angle measured (positive anticlockwise and negative clockwise) from the positive direction of the Re(z) axis to $\overrightarrow{OP}$.

In the diagram above,

$$\arg(z) = \theta \text{ where } \sin\theta = \frac{\text{Im}(z)}{|z|} \text{ and } \cos\theta = \frac{\text{Re}(z)}{|z|}$$

arg(z) is not defined uniquely, e.g. $\arg(i) = \ldots, \frac{\pi}{2}, \frac{5\pi}{2}, \frac{9\pi}{2} \ldots$ which is written $\arg(i) = \frac{\pi}{2} + 2\pi n,\ n \in Z$

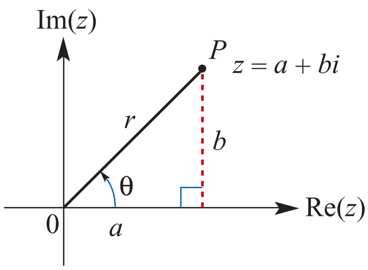
The Argument of z

Arg(z), the Argument of z (with an upper case A), is the single value of arg z in the interval $(-\pi, \pi]$.

$$\text{Arg}(z) = \theta \text{ where } \sin\theta = \frac{\text{Im}(z)}{|z|}, \cos\theta = \frac{\text{Re}(z)}{|z|} \text{ and } \theta \in (-\pi, \pi]$$

Polar form

Every complex number, z, can be represented in polar form by the ordered pair $[r, \theta]$, where r is the modulus of z and θ is an argument of z.



The related algebraic equivalent of $[r, \theta]$ can be derived as follows:

$$\begin{aligned} z &= a + bi \\ &= r\left(\frac{a+bi}{r}\right) \\ &= r\left(\frac{a}{r} + i\frac{b}{r}\right) \end{aligned}$$

Therefore $\quad z = r(\cos\theta + i\sin\theta) \qquad$ since $\cos\theta = \dfrac{a}{r}$ and $\sin\theta = \dfrac{b}{r}$

This is usually written as $z = r\operatorname{cis}\theta \qquad$ where $\operatorname{cis}\theta = \cos\theta + \boldsymbol{i}\sin\theta$

Example 10

a Find the modulus and Argument of each of the following complex numbers:

i 4 **ii** $-2i$ **iii** $1+i$ **iv** $4-3i$

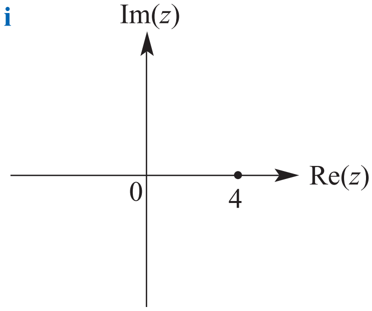
b Find the argument of $-1-i$ in the interval $[0, 2\pi]$.

c Express $-\sqrt{3}+i$ in the form $r\operatorname{cis}\theta$ where $\theta = \operatorname{Arg}(-\sqrt{3}+i)$.

d Express $2\operatorname{cis}\dfrac{-3\pi}{4}$ in the form $a+bi$.

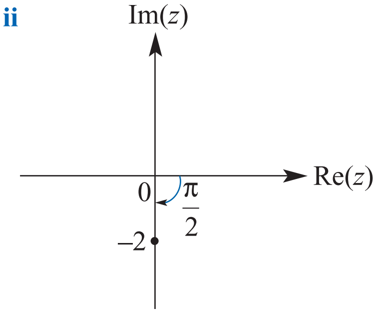
Solution

a i



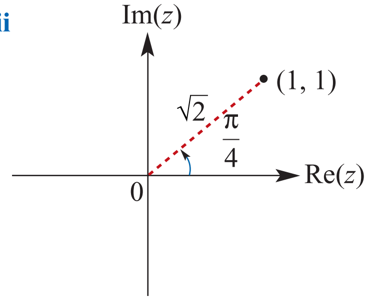
$|4| = 4$ and $\operatorname{Arg}(4) = 0$

ii



$|-2i| = 2$ and $\operatorname{Arg}(-2i) = -\dfrac{\pi}{2}$

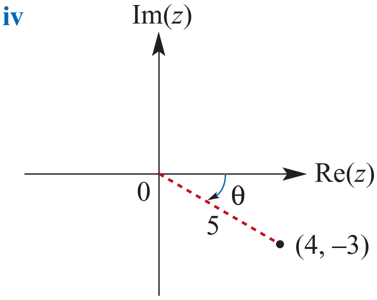
iii



$$\begin{aligned} |1+i| &= \sqrt{1^2+1^2} \\ &= \sqrt{2} \end{aligned}$$

and $\operatorname{Arg}(1+i) = \dfrac{\pi}{4}$

iv

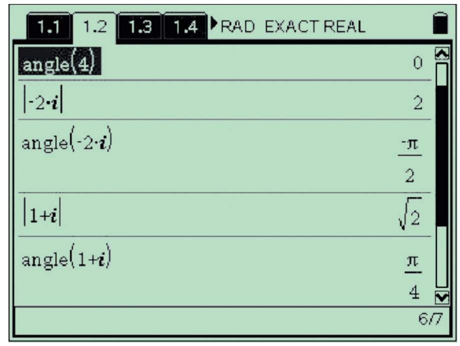


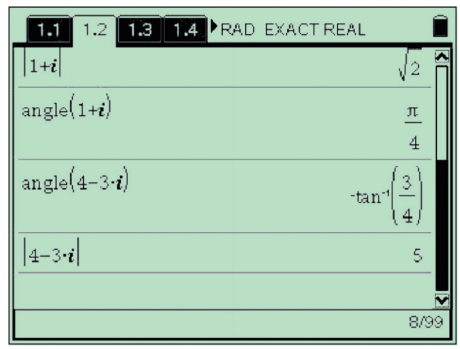
$$\begin{aligned} |4-3i| &= \sqrt{4^2+(-3)^2} \\ &= 5 \end{aligned}$$

and $\operatorname{Arg}(4-3i) = -\tan^{-1}\dfrac{3}{4}$

≈ -0.64 radians

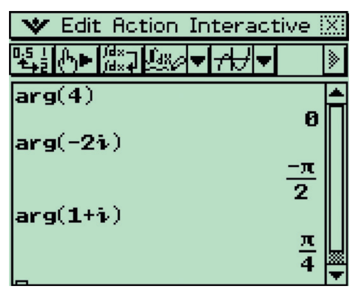
Using a TI-Nspire calculator

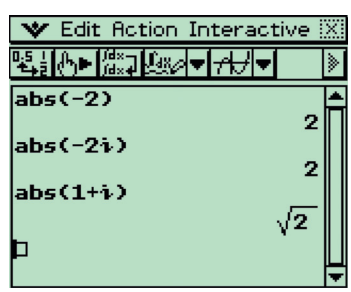




Use the (i) key.
From the **Number** menu choose **Complex Number Tools** and then **Magnitude** (menu 2 9 5).
The modulus can also be obtained through a template.
The argument is obtained from the **Number** menu. Choose **Complex Number Tools** and then **Polar Angle** (menu 2 9 4).

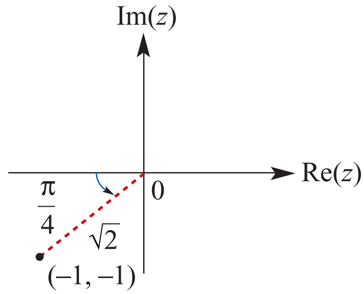
Using a Casio ClassPad calculator





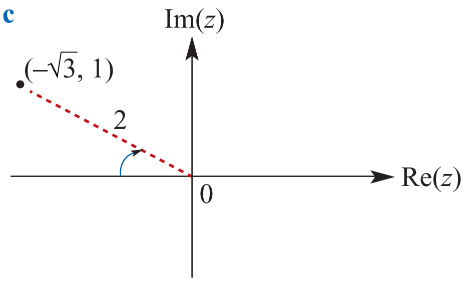
From the **Interactive** menu choose **Complex** and then **arg**.
Make sure the calculator is in Complex mode (**Cplx** at the bottom of the screen).
The modulus is obtained by choosing **mth** and selecting the absolute value template $|x|$.

b



In the interval $[0, 2\pi]$, $\arg(-1 - i) = \dfrac{5\pi}{4}$

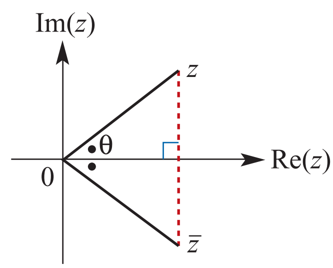
c



$$r = |-\sqrt{3} + i|$$
$$= \sqrt{(\sqrt{3})^2 + 1^2} = 2$$
$$\theta = \operatorname{Arg}\left(-\sqrt{3} + i\right) = \frac{5\pi}{6}$$
Therefore $-\sqrt{3} + i = 2\,\text{cis}\dfrac{5\pi}{6}$

d
$$2\,\text{cis}\frac{-3\pi}{4} = 2\left(\cos\frac{-3\pi}{4} + i\sin\frac{-3\pi}{4}\right)$$
$$= 2\left(-\cos\frac{\pi}{4} - i\sin\frac{\pi}{4}\right)$$
$$= 2\left(-\frac{1}{\sqrt{2}} - i\frac{1}{\sqrt{2}}\right)$$
$$= -\sqrt{2} - i\sqrt{2}$$

The complex conjugate of the modulus–argument form



It is easy to show that the complex conjugate, $\bar{z}$, is a reflection of the point z in the Re(z) axis.

Therefore, if $z = r\,\text{cis}\,\theta$, $\bar{z} = r\,\text{cis}\,(-\theta)$

Exercise 4C

1 Find the modulus and Argument of each of the following complex numbers:

a -3 **b** $5i$ **c** $i - 1$
d $\sqrt{3} + i$ **e** $2 - 2\sqrt{3}i$ **f** $(2 - 2\sqrt{3}i)^2$

2 Find the Argument of each of the following, correct to two decimal places:

a $5 + 12i$ **b** $-8 + 15i$ **c** $-4 - 3i$
d $1 - i\sqrt{2}$ **e** $\sqrt{2} + i\sqrt{3}$ **f** $-(3 + 7i)$

3 Find the argument of each of the following in the interval stated:

a $1 - i\sqrt{3}$ in $[0, 2\pi]$ **b** $-7i$ in $[0, 2\pi]$ **c** $-3 + i\sqrt{3}$ in $[0, 2\pi]$

d $\sqrt{2} + i\sqrt{2}$ in $[0, 2\pi]$ **e** $\sqrt{3} + i$ in $[-2\pi, 0]$ **f** $2i$ in $[-2\pi, 0]$

4 Convert each of the following arguments into Arguments:

a $\frac{5\pi}{4}$ **b** $\frac{17\pi}{6}$ **c** $\frac{-15\pi}{8}$ **d** $\frac{-5\pi}{2}$

5 Convert the following complex numbers from the cartesian form $a + bi$ into the form $r \text{ cis } \theta$, where $\theta = \text{Arg}(a + bi)$:

a $-1 - i$ **b** $\frac{1}{2} - \frac{\sqrt{3}}{2}i$ **c** $\sqrt{3} - i\sqrt{3}$

d $\frac{1}{\sqrt{3}} + \frac{1}{3}i$ **e** $\sqrt{6} - i\sqrt{2}$ **f** $-2\sqrt{3} + 2i$

6 Convert each of the following complex numbers to the form $a + bi$:

a $2 \text{ cis} \frac{3\pi}{4}$ **b** $5 \text{ cis} \frac{-\pi}{3}$ **c** $2\sqrt{2} \text{ cis} \frac{\pi}{4}$

d $3 \text{ cis} \frac{-5\pi}{6}$ **e** $6 \text{ cis} \frac{\pi}{2}$ **f** $4 \text{ cis } \pi$

7 Let $z = \text{cis } \theta$. Show that:

a $|z| = 1$ **b** $\frac{1}{z} = \text{cis}(-\theta)$

8 Find the conjugate of each of the following:

a $2 \text{ cis}\left(\frac{3\pi}{4}\right)$ **b** $7 \text{ cis}\left(\frac{-2\pi}{3}\right)$ **c** $-3 \text{ cis}\left(\frac{2\pi}{3}\right)$ **d** $5 \text{ cis}\left(\frac{-\pi}{4}\right)$

4.4 Basic operations on complex numbers in the modulus–argument form

Addition and subtraction

There is no simple way of adding and subtracting complex numbers given in the form $r \text{ cis } \theta$. Complex numbers need to be expressed in the form $a + bi$ before such operations can be executed.

Example 11

Simplify $2 \text{ cis } \frac{\pi}{3} + 3 \text{ cis } \frac{2\pi}{3}$.

Solution

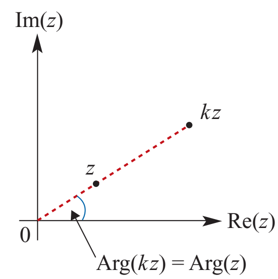
$$2 \text{ cis } \frac{\pi}{3} = 2\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right) = 1 + i\sqrt{3}$$

$$3 \text{ cis } \frac{2\pi}{3} = 3\left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right) = 3\left(-\frac{1}{2} + \frac{\sqrt{3}}{2}i\right) = -\frac{3}{2} + \frac{3\sqrt{3}}{2}i$$

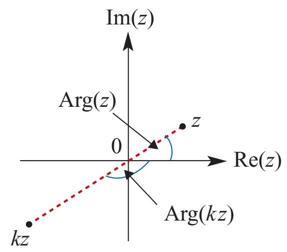
$$2 \text{ cis } \frac{\pi}{3} + 3 \text{ cis } \frac{2\pi}{3} = 1 + i\sqrt{3} + \left(-\frac{3}{2} + \frac{3\sqrt{3}}{2}i\right) = -\frac{1}{2} + \frac{5\sqrt{3}}{2}i$$

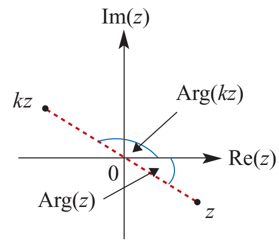
Multiplication by a scalar

If $k > 0$, then $\text{Arg}(kz) = \text{Arg}(z)$



If $k < 0$, then $\text{Arg}(kz) = \begin{cases} \text{Arg}(z) + \pi, & -\pi < \text{Arg}(z) \le 0 \\ \text{Arg}(z) - \pi, & 0 < \text{Arg}(z) \le \pi \end{cases}$





Multiplication of complex numbers

Let $z_1 = r_1 \text{ cis } \theta_1$ and $z_2 = r_2 \text{ cis } \theta_2$

then
$$\begin{aligned} z_1 z_2 &= r_1 \text{ cis } \theta_1 \times r_2 \text{ cis } \theta_2 \\ &= r_1 r_2(\cos\theta_1 + \sin\theta_1 i)(\cos\theta_2 + \sin\theta_2 i) \\ &= r_1 r_2(\cos\theta_1\cos\theta_2 + \cos\theta_1\sin\theta_2 i + \sin\theta_1\cos\theta_2 i - \sin\theta_1\sin\theta_2) \\ &= r_1 r_2[(\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2) + (\cos\theta_1\sin\theta_2 + \sin\theta_1\cos\theta_2)i] \end{aligned}$$

The reader should recall, from Chapter 3, that:

$$\sin(\theta_1 + \theta_2) = \sin\theta_1 \cos\theta_2 + \cos\theta_1 \sin\theta_2$$

and $\cos(\theta_1 + \theta_2) = \cos\theta_1 \cos\theta_2 - \sin\theta_1 \sin\theta_2$

Therefore $z_1 z_2 = r_1 r_2(\cos(\theta_1 + \theta_2) + \sin(\theta_1 + \theta_2)i)$

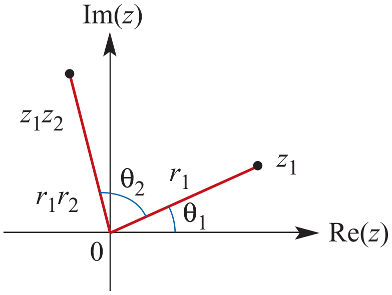
$$= r_1 r_2 \text{ cis } (\theta_1 + \theta_2)$$

$$r_1 \text{ cis } \theta_1 \times r_2 \text{ cis } \theta_2 = r_1 r_2 \text{ cis}(\theta_1 + \theta_2)$$

Some useful properties of the modulus and Argument of z with regard to multiplication of complex numbers are listed here.

$|z_1 z_2| = |z_1||z_2|$
$\text{Arg}(z_1 z_2) = \text{Arg}(z_1) + \text{Arg}(z_2) + 2k\pi$, where $k = 0, 1$ or -1

Geometrical significance of complex number multiplication



The modulus of the product of two complex numbers is the product of their moduli.

The argument of the product of two complex numbers is the sum of their arguments.

Geometrically, the effect of multiplying a complex number z_1 by the complex number $z_2 = r_2 \text{ cis } \theta_2$ is to produce an enlargement of Oz_1, where O is the origin, by a factor r_2 and an anticlockwise turn through an angle θ_2 about the origin.

If $r_2 = 1$, then only the turning effect will take place.

Let $z = \text{cis } \theta$. Multiplication by z^2 is, in effect, the same as a multiplication by z followed by another multiplication by z. The effect is a turn of θ followed by a turn of θ. The end result is an anticlockwise turn of 2θ. This is shown by multiplication of $z \times z$.

$$z^2 = z \times z = \text{cis } \theta \times \text{cis } \theta = \text{cis}(\theta + \theta) \quad \text{by the multiplication rule}$$
$$= \text{cis } 2\theta$$

Division of complex numbers

$$\frac{z_1}{z_2} = \frac{r_1 \text{ cis } \theta_1}{r_2 \text{ cis } \theta_2} = \frac{r_1}{r_2} \text{cis } \theta_1 \text{cis}(-\theta_2) \qquad \text{as } \frac{1}{\text{cis } \theta_2} = \text{cis}(-\theta_2)$$
$$= \frac{r_1}{r_2} \text{cis}(\theta_1 - \theta_2)$$

The reader should recognise the geometrical significance of division.

Some useful properties of the modulus and Argument of z with regard to division of complex numbers are listed here.

$$\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}$$

$$\text{Arg}\left(\frac{z_1}{z_2}\right) = \text{Arg}(z_1) - \text{Arg}(z_2) + 2k\pi \text{ where } k = 0 \text{ or } 1 \text{ or } -1$$

$$\text{Arg}\left(\frac{1}{z}\right) = -\text{Arg}(z)$$

Example 12

Simplify:

a $2 \text{ cis } \frac{\pi}{3} \times \sqrt{3} \text{ cis } \frac{3\pi}{4}$

b $\dfrac{2 \text{ cis } \frac{2\pi}{3}}{4 \text{ cis } \frac{\pi}{5}}$

Solution

a $$2 \text{ cis} \frac{\pi}{3} \times \sqrt{3} \text{ cis} \frac{3\pi}{4} = 2\sqrt{3} \text{ cis} \left(\frac{\pi}{3} + \frac{3\pi}{4}\right)$$
$$= 2\sqrt{3} \text{ cis} \frac{13\pi}{12}$$
$$= 2\sqrt{3} \text{ cis} \left(-\frac{11\pi}{12}\right)$$

Note: A solution of the argument in the range $(-\pi, \pi]$, i.e. the Argument, is preferred unless otherwise stated.

b $$\frac{2 \text{ cis} \frac{2\pi}{3}}{4 \text{ cis} \frac{\pi}{5}} = \frac{1}{2} \text{cis} \left(\frac{2\pi}{3} - \frac{\pi}{5}\right)$$
$$= \frac{1}{2} \text{ cis} \frac{7\pi}{15}$$

De Moivre's theorem

De Moivre's theorem allows us to readily simplify expressions of the form z^n when z is expressed in polar form. De Moivre's theorem states that:

$$(\text{cis } \theta)^n = \text{cis}(n\theta), \ n \in Z$$

This result is usually proved by induction but can be explained by simple argument.

Let $z = \text{cis } \theta$

$z^2 = \text{cis } \theta \times \text{cis } \theta = \text{cis } 2\theta$ by the multiplication rule

$z^3 = z^2 \times \text{cis } \theta = \text{cis } 3\theta$

$z^4 = z^3 \times \text{cis } \theta = \text{cis } 4\theta$, etc.

and, generalising for positive integers, $(\text{cis } \theta)^n = \text{cis}(n\theta)$

For negative integers, again with $z = \text{cis}\,\theta$, consider the following:

$$z^{-1} = \frac{1}{z} = \bar{z} = \text{cis}\,(-\theta)$$

$$\begin{aligned} z^{-p} &= (z^{-1})^p = (\text{cis}\,(-\theta))^p \text{ where } p \in N \\ &= \text{cis}(-p\theta) \text{ from the result obtained above} \end{aligned}$$

Therefore $(\text{cis}\,\theta)^n = \text{cis}(n\theta)$ for $n \in Z$

and $(r\ \text{cis}(\theta))^n = r^n\ \text{cis}(n\theta)$ for $n \in Z$

Example 13

Simplify:

a $\left(\text{cis}\dfrac{\pi}{3}\right)^9$

b $\dfrac{\text{cis}\dfrac{7\pi}{4}}{\left(\text{cis}\dfrac{\pi}{3}\right)^7}$

Solution

a $$\begin{aligned} \text{cis}\left(\frac{\pi}{3}\right)^9 &= \text{cis}\left(\frac{\pi}{3} \times 9\right) \\ &= \text{cis}\,3\pi \\ &= \text{cis}\,\pi \\ &= \cos\pi + i\sin\pi \\ &= -1 \end{aligned}$$

b $$\begin{aligned} \frac{\text{cis}\dfrac{7\pi}{4}}{\left(\text{cis}\dfrac{\pi}{3}\right)^7} &= \text{cis}\frac{7\pi}{4}\left(\text{cis}\frac{\pi}{3}\right)^{-7} \\ &= \text{cis}\frac{7\pi}{4}\text{cis}\frac{-7\pi}{3} \\ &= \text{cis}\left(\frac{7\pi}{4} - \frac{7\pi}{3}\right) \\ &= \text{cis}\frac{-7\pi}{12} \end{aligned}$$

Example 14

Simplify $\dfrac{(1+i)^3}{(1-\sqrt{3}i)^5}$

Solution

$$1 + i = \sqrt{2}\,\text{cis}\,\frac{\pi}{4}$$

$$1 - \sqrt{3}i = 2\,\text{cis}\frac{-\pi}{3}$$

$$\begin{aligned} \therefore \quad \frac{(1+i)^3}{(1-\sqrt{3}i)^5} &= \frac{\left(\sqrt{2}\,\text{cis}\dfrac{\pi}{4}\right)^3}{\left(2\,\text{cis}\dfrac{-\pi}{3}\right)^5} \\ &= \frac{2\sqrt{2}\left(\text{cis}\dfrac{\pi}{4}\right)^3}{32\left(\text{cis}\dfrac{-\pi}{3}\right)^5} \end{aligned}$$

$$= \frac{2\sqrt{2}\operatorname{cis}\frac{3\pi}{4}}{32\operatorname{cis}\frac{-5\pi}{3}} \qquad \text{by De Moivre's theorem}$$

$$= \frac{\sqrt{2}}{16}\operatorname{cis}\left(\frac{3\pi}{4} - \left(\frac{-5\pi}{3}\right)\right)$$

$$= \frac{\sqrt{2}}{16}\operatorname{cis}\frac{29\pi}{12}$$

$$= \frac{\sqrt{2}}{16}\operatorname{cis}\frac{5\pi}{12}$$

Exercise 4D

1 Simplify each of the following:

a $4\operatorname{cis}\frac{2\pi}{3} \times 3\operatorname{cis}\frac{3\pi}{4}$ **b** $\dfrac{\sqrt{2}\operatorname{cis}\frac{\pi}{2}}{\sqrt{8}\operatorname{cis}\frac{5\pi}{6}}$ **c** $\frac{1}{2}\operatorname{cis}\frac{-2\pi}{5} \times \frac{7}{3}\operatorname{cis}\frac{\pi}{3}$

d $\dfrac{4\operatorname{cis}\frac{-\pi}{4}}{\frac{1}{2}\operatorname{cis}\frac{7\pi}{10}}$ **e** $2\operatorname{cis}\frac{5\pi}{6} \times \left(\sqrt{2}\operatorname{cis}\frac{7\pi}{8}\right)^4$ **f** $\dfrac{1}{\left(\frac{3}{2}\operatorname{cis}\frac{5\pi}{8}\right)^3}$

g $\dfrac{4\operatorname{cis}\frac{2\pi}{3}}{32\operatorname{cis}\frac{-\pi}{3}}$ **h** $\left(\operatorname{cis}\frac{\pi}{6}\right)^8 \times \left(\sqrt{3}\operatorname{cis}\frac{\pi}{4}\right)^6$ **i** $\left(\frac{1}{2}\operatorname{cis}\frac{\pi}{2}\right)^{-5}$

j $\left(2\operatorname{cis}\frac{3\pi}{2} \times 3\operatorname{cis}\frac{\pi}{6}\right)^3$ **k** $\left(\frac{1}{2}\operatorname{cis}\frac{\pi}{8}\right)^{-6} \times \left(4\operatorname{cis}\frac{\pi}{3}\right)^2$ **l** $\dfrac{\left(6\operatorname{cis}\frac{2\pi}{5}\right)^3}{\left(\frac{1}{2}\operatorname{cis}\frac{-\pi}{4}\right)^{-5}}$

2 For each of the following find $\operatorname{Arg}(z_1 z_2)$ and $\operatorname{Arg}(z_1) + \operatorname{Arg}(z_2)$ and comment on their relationship:

a $z_1 = \operatorname{cis}\frac{\pi}{4}$ and $z_2 = \operatorname{cis}\frac{\pi}{3}$

b $z_1 = \operatorname{cis}\frac{-2\pi}{3}$ and $z_2 = \operatorname{cis}\frac{-3\pi}{4}$

c $z_1 = \operatorname{cis}\frac{2\pi}{3}$ and $z_2 = \operatorname{cis}\frac{\pi}{2}$

3 Show that if $\frac{-\pi}{2} < \operatorname{Arg}(z_1) < \frac{\pi}{2}$ and $\frac{-\pi}{2} < \operatorname{Arg}(z_2) < \frac{\pi}{2}$ then $\operatorname{Arg}(z_1 z_2) = \operatorname{Arg}(z_1) + \operatorname{Arg}(z_2)$ and $\operatorname{Arg}\left(\frac{z_1}{z_2}\right) = \operatorname{Arg}(z_1) - \operatorname{Arg}(z_2)$.

4 For $z = 1 + i$, find:

a $\operatorname{Arg} z$ **b** $\operatorname{Arg} -z$ **c** $\operatorname{Arg} \frac{1}{z}$

5 **a** Show that $\sin\theta + \cos\theta\, i = \operatorname{cis}\left(\frac{\pi}{2} - \theta\right)$.

b Simplify each of the following:

i $(\sin\theta + \cos\theta i)^7$ **ii** $(\sin\theta + \cos\theta i)(\cos\theta + \sin\theta i)$

iii $(\sin\theta + \cos\theta i)^{-4}$ **iv** $(\sin\theta + \cos\theta i)(\sin\phi + \cos\phi i)$

6 **a** Show that $\cos\theta - \sin\theta\, i = \operatorname{cis}(-\theta)$.

b Simplify each of the following:

i $(\cos\theta - \sin\theta\, i)^5$ **ii** $(\cos\theta - \sin\theta\, i)^{-3}$

iii $(\cos\theta - \sin\theta\, i)(\cos\theta + \sin\theta\, i)$ **iv** $(\cos\theta - \sin\theta\, i)(\sin\theta + \cos\theta\, i)$

7 **a** Show that $\sin\theta - \cos\theta i = \operatorname{cis}\left(\theta - \frac{\pi}{2}\right)$.

b Simplify each of the following:

i $(\sin\theta - \cos\theta\, i)^6$ **ii** $(\sin\theta - \cos\theta\, i)^{-2}$

iii $(\sin\theta - \cos\theta\, i)^2(\cos\theta - \sin\theta\, i)$ **iv** $\dfrac{\sin\theta - \cos\theta i}{\cos\theta + \sin\theta i}$

8 **a** Express each of the following in the modulus–argument form, $0 < \theta < \frac{\pi}{2}$:

i $1 + \tan\theta\, i$ **ii** $1 + \cot\theta\, i$ **iii** $\dfrac{1}{\sin\theta} + \dfrac{1}{\cos\theta} i$

b Hence simplify each of the following:

i $(1 + \tan\theta\, i)^2$ **ii** $(1 + \cot\theta\, i)^{-3}$ **iii** $\dfrac{1}{\sin\theta} - \dfrac{1}{\cos\theta} i$

9 Simplify each of the following, giving your answer in the modulus–Argument form:

a $(1 + \sqrt{3}i)^6$ **b** $(1 - i)^{-5}$ **c** $i(\sqrt{3} - i)^7$

d $(-3 + \sqrt{3}i)^{-3}$ **e** $\dfrac{(1 + \sqrt{3}i)^3}{i(1 - i)^5}$ **f** $\dfrac{(-1 + \sqrt{3}i)^4(-\sqrt{2} - \sqrt{2}i)^3}{\sqrt{3} - 3i}$

g $(-1 + i)^5\left(\frac{1}{2}\operatorname{cis}\frac{\pi}{4}\right)^3$ **h** $\dfrac{\left(\operatorname{cis}\frac{2\pi}{5}\right)^3}{(1 - \sqrt{3}i)^2}$ **i** $\left((1 - i)\operatorname{cis}\frac{2\pi}{3}\right)^7$

4.5 Factorisation of polynomials in *C*

The **fundamental theorem of algebra** has been attributed to Gauss (1799) and is stated as follows:

> Every expression $P(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0$, $a_n \neq 0$, where n is a natural number and the coefficients are complex numbers, has at least one linear factor in the complex number system.

A linear factor in the complex number system is an expression of the form $(z - \alpha_1)$ where $\alpha_1 \in C$.

By the factor theorem, $P(z) = (z - \alpha_1)Q(z)$, where $Q(z)$ is polynomial expression similar to $P(z)$ but with degree $(n - 1)$.

By applying the fundamental theorem of algebra repeatedly, it can be shown that:

A polynomial of degree n can be factorised into n linear factors in C
i.e. $P(z) = a_n(z - \alpha_1)(z - \alpha_2)(z - \alpha_3)\ldots(z - \alpha_n)$ where $\alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_n \in C$

The conjugate factor theorem

Let $P(z) = a_n z^n + a_{n-1}z^{n-1} + \cdots + a_1 z + a_0$, $a_n \neq 0$, where n is a natural number and a_n, $a_{n-1}, \ldots, a_1, a_0$ are real numbers.
By the fundamental theorem of algebra, $P(z)$ can be factorised into n linear factors in C.
Further, the conjugate factor theorem applies and is stated as follows:

If the coefficients of $p(z)$ are real numbers, then the complex roots occur in **conjugate pairs**.
i.e. if $(z - \alpha_1)$ is a factor, so is $(z - \bar{\alpha}_1)$

Factorisation of quadratics by completing the square

Example 15

Factorise:

a $z^2 + 9$ **b** $z^2 + z + 3$ **c** $2z^2 - z + 1$ **d** $2z^2 - 2(3 - i)\,z + 4 - 3i$

Solution

a This expression is changed into the difference of two squares by using the imaginary number i.
$z^2 - 9i^2 = (z + 3i)(z - 3i)$

b Let $P(z) = z^2 + z + 3$. By completing the square on the z terms:

$$P(z) = \left(z^2 + z + \frac{1}{4}\right) + 3 - \frac{1}{4}$$
$$= \left(z + \frac{1}{2}\right)^2 + \frac{11}{4}$$
$$= \left(z + \frac{1}{2}\right)^2 - \frac{11}{4}i^2$$
$$= \left(z + \frac{1}{2} + \frac{\sqrt{11}}{2}i\right)\left(z + \frac{1}{2} - \frac{\sqrt{11}}{2}i\right)$$

c Let $P(z) = 2z^2 - z + 1$

$$= 2\left(z^2 - \tfrac{1}{2}z + \tfrac{1}{2}\right)$$
$$= 2\left[\left(z^2 - \tfrac{1}{2}z + \tfrac{1}{16}\right) + \tfrac{1}{2} - \tfrac{1}{16}\right]$$
$$= 2\left[\left(z - \tfrac{1}{4}\right)^2 + \tfrac{7}{16}\right]$$
$$= 2\left[\left(z - \tfrac{1}{4}\right)^2 - \tfrac{7}{16}i^2\right]$$
$$= 2\left(z - \frac{1}{4} + \frac{\sqrt{7}}{4}i\right)\left(z - \frac{1}{4} - \frac{\sqrt{7}}{4}i\right)$$

d Let $P(z) = 2z^2 - 2(3 - i)z + 4 - 3i$

$$= 2\left(z^2 - (3 - i)z + \frac{4 - 3i}{2}\right)$$

$$= 2\left[z^2 - (3 - i)z + \left(\frac{3 - i}{2}\right)^2 + \frac{4 - 3i}{2} - \left(\frac{3 - i}{2}\right)^2\right]$$

$$= 2\left(z - \frac{3 - i}{2}\right)^2 + 4 - 3i - \frac{(3 - i)^2}{2}$$

$$= 2\left(z - \frac{3 - i}{2}\right)^2 + \frac{8 - 6i - 9 + 6i + 1}{2}$$

$$= 2\left(z - \frac{3 - i}{2}\right)^2$$

Note the conjugate pairs of complex linear factors for **a**, **b** and **c**, but not **d**.

Factorisation of cubic polynomials

A cubic polynomial has three linear factors. If the coefficients are real, at least one factor must be real (as complex factors occur in pairs). The usual method of solution is to find the real linear factor by the factor theorem, and then complete the square on the resulting quadratic factor.

Example 16

Factorise:

a $P(z) = z^3 + z^2 + 4$ **b** $z^3 - iz^2 - 4z + 4i$

Solution

a Possible factors are $\pm 1, \pm 2$ (only -1 and -2 need to be tried).

$$P(-1) = -1 + 1 + 4 \neq 0$$

$$P(-2) = -8 + 4 + 4 = 0$$

$\therefore$ $(z + 2)$ is a factor.

$\therefore$ $P(z) = (z + 2)(z^2 - z + 2)$ by division

$$z^2 - z + 2 = \left(z^2 - z + \tfrac{1}{4}\right) + 2 - \tfrac{1}{4}$$

$$= \left(z - \tfrac{1}{2}\right)^2 - \tfrac{7}{4}i^2$$

$$= \left(z - \frac{1}{2} + \frac{\sqrt{7}}{2}i\right)\left(z - \frac{1}{2} - \frac{\sqrt{7}}{2}i\right)$$

$\therefore$ $P(z) = (z + 2)\left(z - \frac{1}{2} + \frac{\sqrt{7}}{2}i\right)\left(z - \frac{1}{2} - \frac{\sqrt{7}}{2}i\right)$

Note the conjugate pair of complex linear factors.

b Factorise by grouping.

$$\begin{aligned} z^3 - iz^2 - 4z + 4i &= z^2(z-i) - 4(z-i) \\ &= (z-i)(z^2-4) \\ &= (z-i)(z-2)(z+2) \end{aligned}$$

Factorisation of higher degree polynomials

Polynomials of the form $z^4 - a^4$ and $z^6 - a^6$ are considered in the following example.

Example 17

Factorise:

a $P(z) = z^4 - 16$ **b** $P(z) = z^6 - 1$

Solution

a Let $P(z) = z^4 - 16$

$$\begin{aligned} &= (z^2+4)(z^2-4) \quad \text{difference of two squares} \\ &= (z+2i)(z-2i)(z+2)(z-2) \end{aligned}$$

b Let

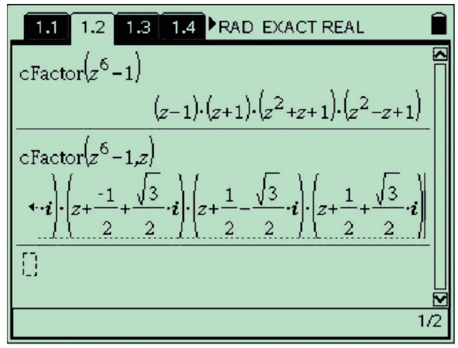
$$\begin{aligned} P(z) &= z^6 - 1 \\ &= (z^3+1)(z^3-1) \\ z^3 + 1 &= (z+1)(z^2 - z + 1) \\ &= (z+1)\left[\left(z^2 - z + \frac{1}{4}\right) + 1 - \frac{1}{4}\right] \\ &= (z+1)\left[\left(z - \frac{1}{2}\right)^2 - \frac{3}{4}i^2\right] \\ &= (z+1)\left(z - \frac{1}{2} + \frac{\sqrt{3}}{2}i\right)\left(z - \frac{1}{2} - \frac{\sqrt{3}}{2}i\right) \\ z^3 - 1 &= (z-1)(z^2 + z + 1) \\ &= (z-1)\left(z + \frac{1}{2} + \frac{\sqrt{3}}{2}i\right)\left(z + \frac{1}{2} - \frac{\sqrt{3}}{2}i\right) \quad \text{by a similar method} \end{aligned}$$

$$\begin{aligned} \therefore \quad z^6 - 1 &= (z+1)(z-1)\left(z - \frac{1}{2} + \frac{\sqrt{3}}{2}i\right)\left(z - \frac{1}{2} - \frac{\sqrt{3}}{2}i\right) \\ &\quad \left(z + \frac{1}{2} + \frac{\sqrt{3}}{2}i\right)\left(z + \frac{1}{2} - \frac{\sqrt{3}}{2}i\right) \end{aligned}$$

Using a TI-Nspire calculator

From the **Algebra** menu choose **Complex** and then **Factor** (menu 3 A 2).

The first operation factors to give integer coefficients and the second fully factorises over the complex numbers.



Using a Casio ClassPad calculator

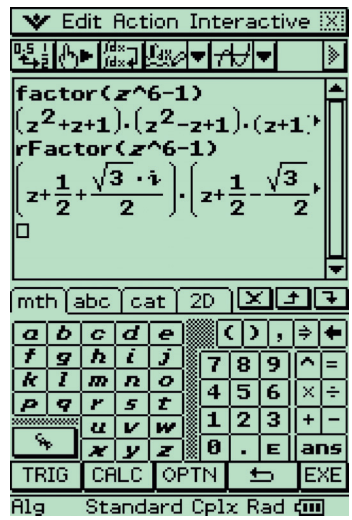
Make sure that the calculator is in Complex mode (**Cplx** at the bottom of the screen).

Enter the expression $z^6 - 1$. Shade the expression.

Choose **Transformation** and then **factor** from the **Interactive** menu to obtain $\left(z^2 + z + 1\right)\left(z^2 - z + 1\right)(z + 1)(z - 1)$. This is the factorisation over the reals.

Enter the expression $z^6 - 1$. Shade the expression.

Choose **Transformation** and then **rfactor** from the **Interactive** menu to factor over the complex field.



Example 18

a Use the factor theorem to show that $z - 1 + \sqrt{2}i$ is a factor of $P(z) = z^3 - 3z^2 + 5z - 3$.

b Find the other linear factors of $P(z)$.

Solution

a If $z - 1 + \sqrt{2}i$, which can be written as $z - (1 - \sqrt{2}i)$, is a factor of $P(z)$, then $P(1 - \sqrt{2}i) = 0$.

Now $P(1 - \sqrt{2}i) = (1 - \sqrt{2}i)^3 - 3(1 - \sqrt{2}i)^2 + 5(1 - \sqrt{2}i) - 3$

$= 0$ (The previous line can be manually expanded and simplified, or keyed into a CAS calculator to evaluate.)

Therefore $z - 1 + \sqrt{2}i$ is a factor of $P(z)$.

b Since the coefficients of $P(z)$ are real, complex linear factors occur in conjugate pairs, so $z - (1 + \sqrt{2}i)$ is also a factor. To find the third linear factor, first multiply the two complex factors together.

$$\begin{aligned}[z - (1 - \sqrt{2}i)][z - (1 + \sqrt{2}i)] &= z^2 - (1 - \sqrt{2}i)z - (1 + \sqrt{2}i)z + (1 - \sqrt{2}i)(1 + \sqrt{2}i) \\ &= z^2 - (1 - \sqrt{2}i + 1 + \sqrt{2}i)z + 1 + 2 \\ &= z^2 - 2z + 3\end{aligned}$$

Divide the quadratic factor into $P(z)$

$$\begin{array}{r@{}l} & \quad z - 1 \\ z^2 - 2z + 3 & \overline{)\, z^3 - 3z^2 + 5z - 3} \\ & \underline{z^3 - 2z^2 + 3z} \quad - \\ & \quad -z^2 + 2z - 3 \\ & \quad \underline{-z^2 + 2z - 3} - \\ & \qquad\qquad 0 \end{array}$$

Therefore the linear factors of $P(z) = z^3 - 3z^2 + 5z - 3$ are $z - 1 + \sqrt{2}i$, $z - 1 - \sqrt{2}i$ and $z - 1$.

Exercise 4E

1 Factorise each of the following into linear factors over C:

a $z^2 + 16$ **b** $z^2 + 5$ **c** $z^2 + 2z + 5$

d $z^2 - 3z + 4$ **e** $2z^2 - 8z + 9$ **f** $3z^2 + 6z + 4$

g $3z^2 + 2z + 2$ **h** $2z^2 - z + 3$ **i** $z^3 - 4z^2 - 4z - 5$

j $z^3 - z^2 - z + 10$ **k** $3z^3 - 13z^2 + 5z - 4$ **l** $2z^3 + 3z^2 - 4z + 15$

m $z^4 - 81$ **n** $z^6 - 64$ **o** $z^3 - (2 - i)z^2 + z - 2 + i$

2 For each of the following, factorise the first expression into linear factors over C, given that the second expression is one of the linear factors:

a $z^3 + (1 - i)z^2 + (1 - i)z - i$, $\quad z - i$

b $z^3 - (2 - i)z^2 - (1 + 2i)z - i$, $\quad z + i$

c $z^3 - (2 + 2i)z^2 - (3 - 4i)z + 6i$, $\quad z - 2i$

d $2z^3 + (1 - 2i)z^2 - (5 + i)z + 5i$, $\quad z - i$

3 **a** Use the factor theorem to show that $(z - 1 - i)$ is a linear factor of $P(z) = z^3 + 4z^2 - 10z + 12$.

b Write down another complex linear factor of $P(z)$.

c Hence, find all the linear factors of $P(z)$ over C.

4 a Use the factor theorem to show that $(z + 2 - i)$ is a linear factor of $P(z) = 2z^3 + 9z^2 + 14z + 5$.
b Write down another complex linear factor of $P(z)$.
c Hence, find all the linear factors of $P(z)$ over C.

5 a Use the factor theorem to show that $(z - 1 + 3i)$ is a linear factor of $P(z) = z^4 + 8z^2 + 16z + 20$.
b Write down another complex linear factor of $P(z)$.
c Hence, find all the linear factors of $P(z)$ over C.

6 Find the value of p in the each of the following given that:
a $(z + 2)$ is a factor of $z^3 + 3z^2 + pz + 12$
b $(z - i)$ is a factor of $z^3 + pz^2 + z - 4$
c $(z + 1 - i)$ is a factor of $2z^3 + z^2 - 2z + p$

4.6 Solution of polynomial equations

When a polynomial equation is reordered into the form $P(z) = 0$, where $P(z)$ is a polynomial, then the equation can be solved by factorising $P(z)$ and extracting a solution from each factor.

If $P(z) = (z - \alpha_1)(z - \alpha_2) \ldots (z - \alpha_n)$ then the solutions (or roots, or zeros) of $P(z) = 0$ are $\alpha_1, \alpha_2, \ldots, \alpha_n$

Note: The special case where $P(z) = z^n - a$, $a \in C$, will be considered separately in section 4.7.

A formula for quadratics

A quadratic equation of the form $az^2 + bz + c = 0$ can be solved quickly by using the quadratic formula:

$$z = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

This formula is obtained by applying the method of completing the square on the expression $az^2 + bz + c$.

Example 19

Solve the equation $z^2 = 2z - 5$.

Solution

The equation is rearranged to the form:

$$z^2 - 2z + 5 = 0$$

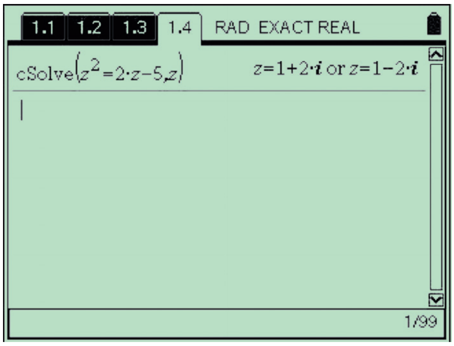
The formula is applied to obtain:

$$\begin{aligned} z &= \frac{-(-2) \pm \sqrt{(-2)^2 - 4 \times 1 \times 5}}{2 \times 1} \\ &= \frac{2 \pm \sqrt{4 - 20}}{2} \\ &= \frac{2 \pm \sqrt{-16}}{2} \\ &= \frac{2 \pm 4i}{2} \\ &= 1 \pm 2i \end{aligned}$$

The solutions are $1 + 2i$, $1 - 2i$.

Using a TI-Nspire calculator

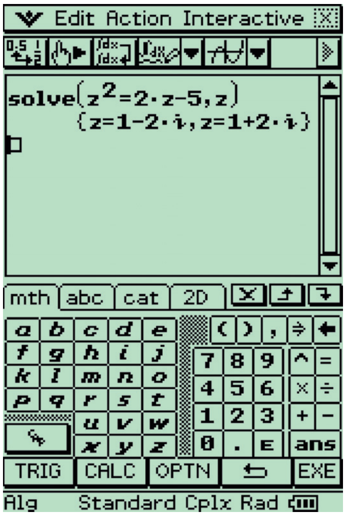
From the **Algebra** menu choose **Complex** and then **Solve** (menu 3 A 1).



Using a Casio ClassPad calculator

Enter the equation $z^2 = 2z - 5$. Shade the equation. Make sure the calculator is in Complex mode (**Cplx** at the bottom of the screen).

Choose **Equation/Inequality** and then **solve** from the **Interactive** menu. Make sure that the variable is z.



Example 20

Solve each of the following equations over C:

a $z^2 + 64 = 0$ **b** $z^3 + 3z^2 + 7z + 5 = 0$

c $z^3 - iz^2 - 4z + 4i = 0$ **d** $2z^2 - 2(3 - i)z + 4 - 3i = 0$

Solution

a $z^2 + 64 = 0$

$(z + 8i)(z - 8i) = 0$

$z = -8i$ or $z = 8i$

b Let $P(z) = z^3 + 3z^2 + 7z + 5$

$P(-1) = 0$

Therefore, by the factor theorem, $z + 1$ is a factor and

$$\begin{aligned} P(z) &= (z+1)(z^2+2z+5) \\ &= (z+1)(z^2+2z+1+4) \\ &= (z+1)[(z+1)^2-(2i)^2] \\ &= (z+1)(z+1-2i)(z+1+2i) \end{aligned}$$

$\therefore$ $P(z) = 0$ implies

$z = -1$ or $z = -1 + 2i$ or $z = -1 - 2i$

c $z^3 - iz^2 - 4z + 4i = 0$

In Example 16b it was found that $z^3 - iz^2 - 4z + 4i = (z - i)(z - 2)(z + 2)$.

Therefore $z^3 - iz^2 - 4z + 4i = 0$

becomes $(z - i)(z - 2)(z + 2) = 0$

$\therefore z = i$ or $z = 2$ or $z = -2$

d For $2z^2 - 2(3 - i)z + 4 - 3i = 0$ use the quadratic formula.

Here $a = 2$, $b = -2(3 - i)$ and $c = 4 - 3i$

$$\begin{aligned} \therefore \quad z &= \frac{2(3-i) \pm \sqrt{4(3-i)^2 - 8(4-3i)}}{4} \\ &= \frac{2(3-i) \pm 2\sqrt{9 - 6i - 1 - 8 + 6i}}{4} \\ &= \frac{2(3-i)}{4} \\ &= \frac{3-i}{2} \end{aligned}$$

Exercise 4F

1 Solve each of the following equations over C:

a $x^2 + 25 = 0$
b $x^2 + 8 = 0$
c $x^2 - 4x + 5 = 0$
d $3x^2 + 7x + 5 = 0$
e $x^2 = 2x - 3$
f $5x^2 + 1 = 3x$
g $x^3 + x^2 - 6x - 18 = 0$
h $x^3 - 6x^2 + 11x - 30 = 0$
i $2x^3 + 3x^2 = 11x^2 - 6x - 16$
j $x^4 + x^2 = 2x^3 + 36$

2 Let $z^2 + az + b = 0$, where a and b are real numbers. Find a and b if one of the solutions is:

a $2i$
b $3 + 2i$
c $-1 + 3i$

3 **a** $1 + 3i$ is a solution of the equation $3z^3 - 7z^2 + 32z - 10 = 0$. Find the other solutions of the equation.

b $-2 - i$ is a solution of the equation $z^4 - 5z^2 + 4z + 30 = 0$. Find the other solutions of the equation.

4 For a cubic polynomial $P(x)$, with real coefficients, $P(2 + i) = 0$, $P(1) = 0$ and $P(0) = 10$. Express $P(x)$ in the form $P(x) = ax^3 + bx^2 + cx + d$ and solve the equation $P(x) = 0$.

5 If $z = 1 + i$ is a zero of the polynomial $z^3 + az^2 + bz + 10 - 6i$, find the constants a and b given that they are real.

6 The polynomial $P(z) = 2z^3 + az^2 + bz + 5$, where a and b are real, has $2 - i$ as one of its zeros.

a Find a quadratic factor of $P(z)$ and hence calculate the real constants a and b.
b Determine the solutions to the equation $P(z) = 0$.

7 For the polynomial $P(z) = az^4 + az^2 - 2z + d$ where a and d are real:

a evaluate $P(1 + i)$
b given that $P(1 + i) = 0$ find the values of a and d
c show that $P(z)$ can be written as the product of two quadratic factors and hence solve the equation $P(z) = 0$

8 The solutions of the quadratic equation $z^2 + pz + q = 0$ are $1 + i$ and $4 + 3i$. Find the complex numbers p and q.

9 Given that $1 - i$ is a solution of the equation $z^3 - 4z^2 + 6z - 4 = 0$, find the other two solutions.

10 Solve each of the following for z:

a $z^2 - (6 + 2i)z + (8 + 6i) = 0$
b $z^3 - 2iz^2 - 6z + 12i = 0$
c $z^3 - z^2 + 6z - 6 = 0$
d $z^3 - z^2 + 2z - 8 = 0$
e $6z^2 - 3\sqrt{2}\,z + 6 = 0$
f $z^3 + 2z^2 + 9z = 0$

4.7 Using De Moivre's theorem to solve equations of the form $z^n = a$ where $a \in C$

Equations of the form $z^n = a, a \in C$ are often solved by using De Moivre's theorem.

Let $\quad a = r_1 \text{ cis } \phi$

and let $\quad z = r \text{ cis } \theta$

$\therefore \quad (r \text{ cis } \theta)^n = r_1 \text{ cis } \phi$

By De Moivre's theorem, $r^n \text{ cis }(n\theta) = r_1 \text{ cis } \phi$

Compare modulus and argument:

$$r^n = r_1 \qquad \text{cis}(n\theta) = \text{cis}\phi$$
$$r = \sqrt[n]{r_1} \qquad n\theta = \phi + 2k\pi, k \in Z$$
$$\theta = \frac{1}{n}(\phi + 2k\pi), k \in Z$$

This will provide all the solutions of the equation.

Example 21

Solve $z^3 = 1$.

Solution

Let $\quad z = r \text{ cis } \theta$

$\therefore \quad (r \text{ cis } \theta)^3 = 1 \text{ cis } 0$

$\therefore \quad r^3 \text{ cis } 3\theta = 1 \text{ cis } 0$

$\therefore \quad r^3 = 1 \text{ and } 3\theta = 0 + 2\pi k, k \in Z$

$\therefore \quad r = 1 \text{ and } \theta = \frac{2\pi k}{3}, \; k \in Z$

$\therefore \quad$ the solutions are in the form $z = \text{cis}\left(\frac{2\pi k}{3}\right), \; k \in Z$

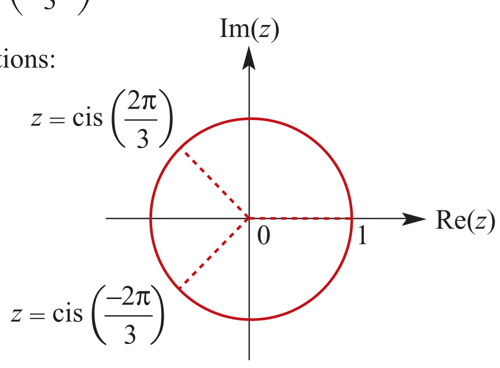
Considering the Arguments of these solutions:

$k = 0 \quad z = \text{cis } 0 = 1$

$k = 1 \quad z = \text{cis } \frac{2\pi}{3}$

$k = 2 \quad z = \text{cis } \frac{4\pi}{3} = \text{cis}\left(-\frac{2\pi}{3}\right)$

$k = 3 \quad z = \text{cis } 2\pi = 1$



The solutions begin to repeat. The three solutions are 1, $\text{cis } \frac{2\pi}{3}$ and $\text{cis}\left(-\frac{2\pi}{3}\right)$.

The solutions are shown to lie on the unit circle at intervals of $\frac{2\pi}{3}$ around the circle.

Example 22

Solve $z^2 = 1 + i$.

Solution

Let $z = r \text{ cis } \theta$. Also $1 + i = \sqrt{2} \text{ cis } \frac{\pi}{4}$.

$\therefore \quad (r \text{ cis } \theta)^2 = \sqrt{2} \text{ cis } \frac{\pi}{4}$

i.e. $(r^2 \text{ cis } 2\theta) = 2^{\frac{1}{2}} \text{ cis } \frac{\pi}{4}$

$\therefore \quad r = 2^{\frac{1}{4}}$ and $2\theta = \frac{\pi}{4} + 2\pi k$ where $k \in Z$

$\therefore \quad z = 2^{\frac{1}{4}} \text{ cis } \left[\frac{1}{2}\left(\frac{\pi}{4} + 2\pi k\right)\right]$ where $k \in Z$

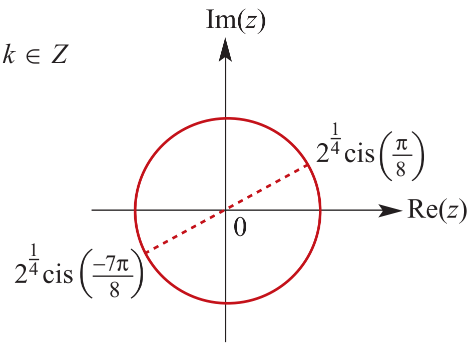
$= 2^{\frac{1}{4}} \text{ cis } \left(\frac{\pi}{8} + \pi k\right)$ where $k \in Z$

For solutions

$k = 0 \qquad z = 2^{\frac{1}{4}} \text{ cis } \left(\frac{\pi}{8}\right)$

$k = 1 \qquad z = 2^{\frac{1}{4}} \text{ cis } \left(\frac{9\pi}{8}\right)$

$= 2^{\frac{1}{4}} \text{ cis } \left(\frac{-7\pi}{8}\right)$



Note that for equations of the form $z^3 = a$, where $a \in R$, there are three solutions. Since $a \in R$, two of the solutions will be conjugate to each other and the third must be a real number.

In general if z_1 is a solution of $z^2 = a$, $a \in C$, then the other solution is $z_2 = -z_1$.

The solutions of any equation of the form $z^n = a$ lie on a circle with centre the origin and radius $|a|^{\frac{1}{n}}$.

Furthermore, the solutions lie on the circle at intervals of $\frac{2\pi}{n}$. This observation can be used to find all solutions if one is known.

The following example shows an alternative method for solving equations of the form $z^2 = a$, $a \in C$.

Example 23

Solve $z^2 = 5 + 12i$ using $z = a + bi$, where $a, b \in R$, and hence factorise $z^2 - 5 - 12i$.

Solution

Since $z = a + bi$, $\quad z^2 = (a + bi)^2$

$$= a^2 + 2abi + b^2 i^2$$

$$= (a^2 - b^2) + 2abi$$

So $z^2 = 5 + 12i$ becomes $(a^2 - b^2) + 2abi = 5 + 12i$

Equating coefficients $\quad a^2 - b^2 = 5 \quad$ and $\quad 2ab = 12$

$$a^2 - \left(\frac{6}{a}\right)^2 = 5 \qquad\qquad b = \frac{6}{a}$$

$$a^2 - \frac{36}{a^2} = 5$$

$$a^4 - 36 = 5a^2$$

$$a^4 - 5a^2 - 36 = 0$$

$$(a^2 - 9)(a^2 + 4) = 0$$

Therefore $\quad a^2 - 9 = 0$

$$(a + 3)(a - 3) = 0$$

$$a = -3 \text{ or } a = 3$$

When $a = -3$, $b = -2$ and when $a = 3$, $b = 2$.
So the solutions to the equation $z^2 = 5 + 12i$ are $z = -3 - 2i$ or $z = 3 + 2i$.
Hence the factors of the expression $z^2 - 5 - 12i$ are $z + 3 + 2i$ and $z - 3 - 2i$.

Exercise 4G

1 Solve each of the following equations over C, and show the solutions for each on an Argand diagram:

a $z^2 + 1 = 0$ **b** $z^3 = 27i$ **c** $z^2 = 1 + \sqrt{3}i$
d $z^2 = 1 - \sqrt{3}i$ **e** $z^3 = i$ **f** $z^3 + i = 0$

2 Find all the cube roots of the following complex numbers:

a $4\sqrt{2} - 4\sqrt{2}i$ **b** $-4\sqrt{2} + 4\sqrt{2}i$ **c** $-4\sqrt{3} - 4i$
d $4\sqrt{3} - 4i$ **e** $-125i$ **f** $-1 + i$

3 Let $z = a + ib$ where $a, b \in R$ and $z^2 = 3 + 4i$.

a Find equations in terms of a and b by equating real and imaginary parts.
b Find the values of a and b and hence the square roots of $3 + 4i$.

4 Find by the methods of question **3** the square roots of each of the following:

a $-15 - 8i$ **b** $24 + 7i$ **c** $-3 + 4i$ **d** $-7 + 24i$

5 Find the solutions of the equation $z^4 - 2z^2 + 4 = 0$ in polar form.

6 Find the solutions to the equation $z^2 - i = 0$ in cartesian form and hence factorise $z^2 - i$.

7 Find the solutions to the equation $z^8 + 1 = 0$ in polar form and hence factorise $z^8 + 1$.

8 **a** Find the square roots of $1 + i$ by:

i cartesian methods **ii** De Moivre's theorem

b Hence find exact values of $\cos \frac{\pi}{8}$ and $\sin \frac{\pi}{8}$.

4.8 Relations and regions of the complex plane

Particular sets of points of the complex plane can be described by placing restrictions on z. For example, $\{z: \text{Re}(z) = 6\}$ is the straight line parallel to the imaginary axis with each point on the line having real value 6.

A set of points which satisfies a given condition is called a **locus** (plural loci). A solid line is used for a boundary which is included in the locus. A dotted line is used for a boundary which is not included in the locus.

Example 24

Sketch the subset, S, of the complex plane where $S = \{z : |z - 1| = 2\}$ on an Argand diagram.

Solution

Method 1

Let $z = x + iy$

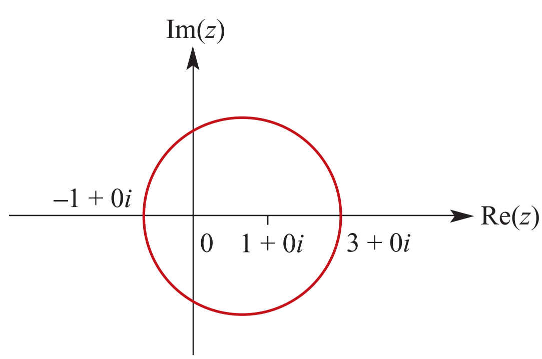
$$\begin{aligned} \text{Then} \quad & |z - 1| = 2 \\ \text{implies} \quad & |x + iy - 1| = 2 \\ \therefore \quad & |(x - 1) + iy| = 2 \\ \therefore \quad & \sqrt{(x - 1)^2 + y^2} = 2 \\ \text{and} \quad & (x - 1)^2 + y^2 = 4 \end{aligned}$$

This demonstrates that S is represented by a circle of radius 2 and centre $1 + 0i$.

Method 2 (geometric)

If z_1 and z_2 are complex numbers $|z_1 - z_2|$ is the distance between the points on the complex plane corresponding to z_1 and z_2.

Hence $\{z : |z - 1| = 2\}$ is the set of all points for which the distance from $1 + 0i$ is 2, i.e. S is represented by a circle of centre $1 + 0i$ and radius 2.



Example 25

Sketch the subset of the complex plane defined by each of the following relations:

a $\text{Arg}(z) = \frac{\pi}{3}$ **b** $\text{Arg}(z + 3) = \frac{-\pi}{3}$ **c** $\text{Arg}(z) \le \frac{\pi}{3}$

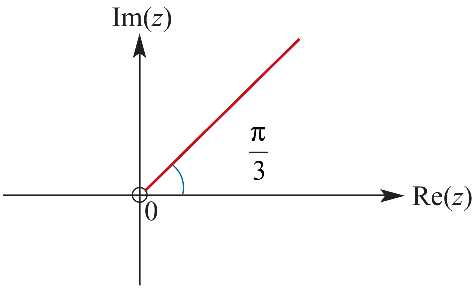
Solution

a $\text{Arg}(z) = \frac{\pi}{3}$ defines a ray or a half line.

Note: (0, 0) is not included.

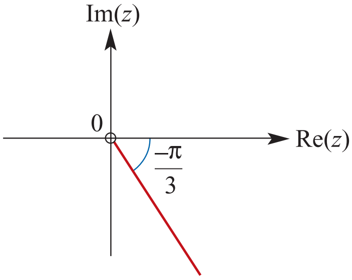
If $z = x + iy$, $\frac{y}{x} = \tan\left(\frac{\pi}{3}\right)$

and $y = \sqrt{3}x$ and $x > 0$



b $\text{Arg}(z + 3) = -\frac{\pi}{3}$

First the graph of $\text{Arg}(z) = -\frac{\pi}{3}$ is drawn.



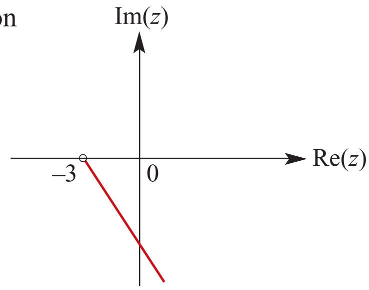
Then draw the final graph through a translation of 3 units to the left.

If $\quad z = x + iy$

$$\text{Arg}((x + 3) + iy) = -\frac{\pi}{3}$$

$$\therefore \quad \frac{y}{x+3} = \tan\left(-\frac{\pi}{3}\right)$$

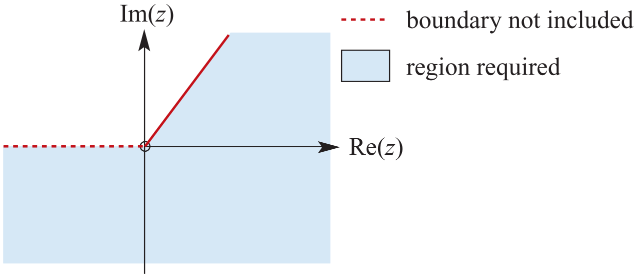
$$\therefore \quad y = -\sqrt{3}x - 3\sqrt{3} \text{ and } x > -3$$



c $\text{Arg}(z) \leq \frac{\pi}{3}$

Note: $-\pi < \text{Arg}(z) \leq \pi$ in general.

Hence $\text{Arg}(z) \leq \frac{\pi}{3}$ implies $-\pi < \text{Arg}(z) \leq \frac{\pi}{3}$



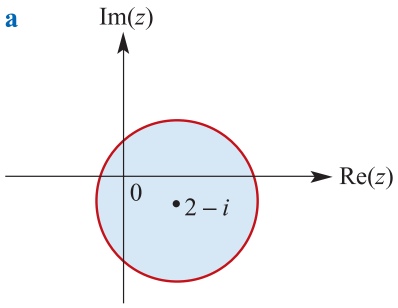
Example 26

Sketch the regions corresponding to each of the following:

a $|z - (2 - i)| \leq 3$ **b** $\{z: 2 < |z| \leq 4\} \cap \left\{z: \frac{\pi}{6} < \text{Arg}(z) \leq \frac{\pi}{3}\right\}$

Solution

a

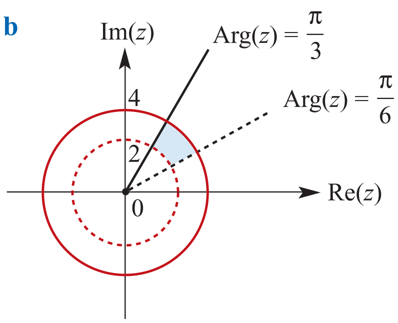


A disc of radius 3 and centre $2 - i$ is defined.

Region required

Note: the cartesian expression to describe this disc is $(x-2)^2 + (y+1)^2 \le 3$

b



Region required

Example 27

Find the locus defined by $|z+3| = 2|z-i|$.

Solution

$$|z+3| = 2|z-i|. \text{ Let } z = x+iy.$$

This implies

$$|(x+3)+yi| = 2|(x+iy)-i|$$

$$\therefore \quad \sqrt{(x+3)^2+y^2} = 2\sqrt{x^2+(y-1)^2}$$

Squaring both sides gives

$$x^2+6x+9+y^2 = 4(x^2+y^2-2y+1)$$

$$\therefore \quad 0 = 3x^2+3y^2-6x-8y-5$$

$$\therefore \quad 5 = 3(x^2-2x)+3\left(y^2-\frac{8}{3}y\right)$$

$$\therefore \quad 5 = 3(x^2-2x+1)+3\left(y^2-\frac{8}{3}y+\frac{16}{9}\right)-\frac{25}{3}$$

$$\therefore \quad \frac{40}{3} = 3(x-1)^2+3\left(y-\frac{4}{3}\right)^2$$

$$\therefore \quad \frac{40}{9} = (x-1)^2+\left(y-\frac{4}{3}\right)^2$$

The locus is a circle with centre $\left(1, \frac{4}{3}\right)$ and radius $\frac{2\sqrt{10}}{3}$

Note that for $a, b \in C$ and $k \in R^+ \backslash \{1\}$, equations of the form $|z - a| = k|z - b|$ define a circle.

Example 28

Find the locus defined by $|z - 2| - |z + 2| = 3$.

Solution

$|z - 2| - |z + 2| = 3$. Let $z = x + iy$.

This implies $|x + iy - 2| - |x + iy + 2| = 3$

$$\therefore \quad \sqrt{(x-2)^2 + y^2} - \sqrt{(x+2)^2 + y^2} = 3$$

$$\therefore \quad \sqrt{(x-2)^2 + y^2} = 3 + \sqrt{(x+2)^2 + y^2}$$

$$\therefore \quad (x-2)^2 + y^2 = 9 + 6\sqrt{(x+2)^2 + y^2} + (x+2)^2 + y^2$$

$$\therefore \quad x^2 - 4x + 4 + y^2 = 9 + 6\sqrt{(x+2)^2 + y^2} + x^2 + 4x + 4 + y^2$$

$$\therefore \quad -8x - 9 = 6\sqrt{(x+2)^2 + y^2}$$

Note: This implies $-8x - 9 > 0$. Therefore $x < -\frac{9}{8}$.

Squaring both sides yields

$$64x^2 + 144x + 81 = 36(x^2 + 4x + 4 + y^2)$$

$$28x^2 - 36y^2 = 63$$

$$\frac{x^2}{36} - \frac{y^2}{28} = \frac{1}{16} \qquad \text{i.e.} \qquad \frac{x^2}{9} - \frac{y^2}{7} = \frac{1}{4}$$

The locus is a hyperbola with asymptotes $y = \pm\frac{\sqrt{7}x}{3}$ and $x \leq -\frac{3}{2}$.

Exercise 4H

Skillsheet TEACHER

1 Illustrate each of the following relations on an Argand diagram:

a $2\,\text{Im}(z) = \text{Re}(z)$ **b** $\text{Im}(z) + \text{Re}(z) = 1$ **c** $|z - 2| = 3$
d $|z - i| = 4$ **e** $|z - (1 + \sqrt{3}i)| = 2$ **f** $|z - (1 - i)| = 6$
g $|z - 1| + |z + 1| = 3$ **h** $|z - 6| - |z + 6| = 3$

2 Sketch the subset, S, of the complex plane $S = \{z : |z - 1| \leq 2\}$.

3 Sketch in the complex plane $\{z : z = i\bar{z}\}$.

4 Describe the relation defined by $\{z : |z - 1| = |z + 1|\}$.

5 Prove that for any complex number z, $3|z - 1|^2 = |z + 1|^2$ if and only if $|z - 2|^2 = 3$. Hence sketch the region $S = \{z : \sqrt{3}\,|z - 1| = |z + 1|\}$.

6 Sketch each of the following:

a $\{z: |z - i| > 1\}$ **b** $\{z: |z + i| \le 2\}$

c $\{z: \text{Re}(z) \ge 0\}$ **d** $\{z: \text{Im}(z) = -2\}$

e $\{z: 2\,\text{Re}(z) + \text{Im}(z) \le 0\}$ **f** $\{z: z + \bar{z} = 5\}$

g $\{z: z\bar{z} = 5\}$ **h** $\{z: \text{Re}(z) > 2 \text{ and } \text{Im}(z) \ge 1\}$

i $\{z: \text{Re}(z^2) = \text{Im}(z)\}$ **j** $\left\{z: \text{Arg}(z - i) = \dfrac{\pi}{3}\right\}$

k $\{z: |z + 2i| = 2|z - i|\}$ **l** $\{z: 2 \le |z| \le 3\} \cap \left\{z: \dfrac{\pi}{4} < \text{Arg}(z) \le \dfrac{3\pi}{4}\right\}$

m $\{z: |z + 3| + |z - 3| = 8\}$

7 Sketch the subset S of the complex plane where $S = \{z: \text{Re}(z) \le 1\} \cap \{z: 0 \le \text{Im}(z) \le 3\}$.

8 Sketch the region in the complex plane for which $\text{Re}(z) \ge 0$ and $|z + 2i| \le 1$.

9 Sketch the locus defined by $|z - 2 + 3i| \le 2$.

10 On the Argand plane, sketch each of the curves or regions whose equations are as follows:

a $\left|\dfrac{z - 2}{z}\right| = 1$ **b** $\left|\dfrac{z - 1 - i}{z}\right| = 1$

11 If the real part of $\dfrac{z + 1}{z - 1}$ is zero, find the locus of points representing z in the complex plane.

12 Given that z satisfies the equation $2|z - 2| = |z - 6i|$, show that z is represented by a point on a circle and find the centre and radius of the circle.

13 If, with an Argand diagram with origin O, the point P represents z and Q represents $\dfrac{1}{\bar{z}}$, prove that O, P and Q are collinear and find the ratio $OP : OQ$ in terms of $|z|$.

14 Find the locus of points described by each of the following relations:

a $|z - (1 + i)| = 1$ **b** $|z - 2| = |z + 2i|$

c $\text{Arg}(z - 1) = \dfrac{\pi}{2}$ **d** $\text{Arg}(z + i) = \dfrac{\pi}{4}$

15 Let $w = 2z$. Describe the locus of w if z describes a circle with centre (1, 2) and radius 3.

16 **a** Find the roots of the equation $z^2 + 2z + 4 = 0$.

b Show that the roots satisfy:

i $|z| = 2$ **ii** $|z - 1| = \sqrt{7}$ **iii** $z + \bar{z} = -2$

c On a single diagram, sketch the loci defined by the equations in **b**.

MC TEST D&D TEST

Review

Chapter summary

- i is an **imaginary** number with the property $i^2 = -1$.
- C, the set of complex numbers, is defined by $C = \{a + bi: a, b \in R\}$.
- Real numbers and imaginary numbers are subsets of C.
- Re(z) is the real component of z.
  Im(z) is the value of the imaginary component of z.
- $z_1 = z_2$ if and only if $\text{Re}(z_1) = \text{Re}(z_2)$ and $\text{Im}(z_1) = \text{Im}(z_2)$.
- Let $z_1 = a + bi$ and $z_2 = c + di$, then $z_1 + z_2 = (a + c) + (b + d)i$.
- Let $z_1 = a + bi$ and $z_2 = c + di$, then $z_1 - z_2 = (a - c) + (b - d)i$.
- Let $z_1 = a + bi$ and $z_2 = c + di$, then $z_1 z_2 = (ac - bd) + (ad + bc)i$.
- The **Argand diagram** is a geometrical representation of C.
- The **modulus** of z, $|z|$, is the distance from the origin of the point represented by z, and is given by $|z| = \sqrt{(\text{Re}(z))^2 + (\text{Im}(z))^2}$ or $|z| = \sqrt{a^2 + b^2}$.
- The **argument** of z, arg(z), is an angle measured anticlockwise about the origin from the positive direction of the x axis to the line joining the origin to z.
- The **Argument** of z, Arg(z), is arg(z) expressed as an angle in the interval $(-\pi, \pi]$.
- The modulus–argument form of the complex number z is given as:

$$z = r(\cos\theta + i\sin\theta) \text{ or } z = r \text{ cis } \theta, \text{ where } r = |z|, \cos\theta = \frac{\text{Re}(z)}{|z|} \text{ and } \sin\theta = \frac{\text{Im}(z)}{|z|}.$$

- The complex conjugate of z is denoted by $\bar{z}$, where $\bar{z} = \text{Re}(z) - \text{Im}(z)i$, and $z\bar{z}$ and $z + \bar{z}$ are both real numbers. If $z = a + bi$, then $\bar{z} = a - bi$. If $z = r \text{ cis } \theta$, then $\bar{z} = r \text{ cis } (-\theta)$.
- The division of complex numbers: $\dfrac{z_1}{z_2} = \dfrac{z_1\bar{z}_2}{|z_2|^2}$.
- Multiplication and division of the modulus–argument form:

Let $\quad z_1 = r_1 \text{ cis } \theta_1, z_2 = r_2 \text{ cis } \theta_2$

Then $\quad z_1 z_2 = r_1 r_2 \text{ cis}(\theta_1 + \theta_2) \quad$ and $\quad \dfrac{z_1}{z_2} = \dfrac{r_1}{r_2} \text{ cis}(\theta_1 - \theta_2), r_2 \neq 0.$

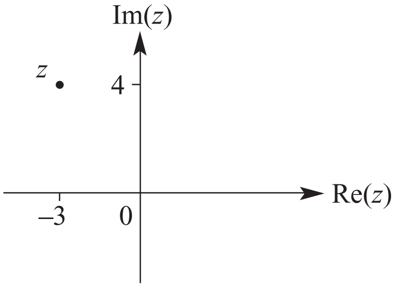
- De Moivre's theorem: $(r \text{ cis } \theta)^n = r^n \text{ cis } n\theta, n \in Z$.
- The fundamental theorem of algebra states that every polynomial with complex coefficients has at least one linear factor in the complex number system.
- The conjugate factor theorem states that if the polynomial has real coefficients, then the complex roots occur in conjugate pairs.
- If $P(z)$ is a polynomial of degree n, then $P(z) = 0$ has n solutions in C, some of which may be repeated.
- If z_1 is a solution of $z^2 = a, a \in C$, then the other solution is $z_2 = -z_1$.
- The solutions of any equation of the form $z^n = a, a \in C$, lie on a circle with centre the origin and radius $|a|^{\frac{1}{n}}$. Furthermore, the solutions lie on the circle at intervals of $\dfrac{2\pi}{n}$.

- Subsets of the complex plane can be defined by suitable complex algebraic descriptions.
- Note that for $a, b \in C$ and $k \in R^+ \backslash \{1\}$, equations of the form $|z - a| = k|z - b|$, define a circle.

Multiple-choice questions

1 If $z_1 = 5 \text{ cis } \left(\frac{\pi}{3}\right)$ and $z_2 = 2 \text{ cis } \left(\frac{3\pi}{4}\right)$, then $z_1 z_2$ is equal to:
A $7 \text{ cis } \left(\frac{\pi^2}{4}\right)$ **B** $7 \text{ cis } \left(\frac{13\pi}{12}\right)$ **C** $10 \text{ cis } \left(\frac{\pi}{4}\right)$ **D** $10 \text{ cis } \left(\frac{\pi^2}{4}\right)$ **E** $10 \text{ cis } \left(\frac{-11\pi}{12}\right)$

2 The complex number z shown in the diagram is best represented by:
A 5 cis (0.93) **B** 5 cis (126.87)
C 5 cis (2.21) **D** 25 cis (126.87)
E 25 cis (2.21)



3 If $(x + iy)^2 = -32i$ for real values of x and y, then:
A $x = 4, y = 4$ **B** $x = -4, y = 4$ **C** $x = 4, y = -4$
D $x = 4, y = -4$ and $x = -4, y = 4$ **E** $x = 4, y = 4$ and $x = -4, y = -4$

4 If $u = 1 - i$, then $\dfrac{1}{3 - u}$ is equal to:
A $\frac{2}{3} + \frac{1}{3}i$ **B** $\frac{2}{5} + \frac{1}{5}i$ **C** $\frac{2}{3} - \frac{1}{3}i$ **D** $-\frac{2}{5} + \frac{1}{5}i$ **E** $\frac{2}{5} - \frac{1}{5}i$

5 The linear factors of $z^2 + 6z + 10$ over C are:
A $(z + 3 + i)^2$ **B** $(z + 3 - i)^2$ **C** $(z + 3 + i)(z - 3 + i)$
D $(z + 3 - i)(z + 3 + i)$ **E** $(z + 3 + i)(z - 3 - i)$

6 The roots of the equation $z^3 + 8i = 0$ are:
A $\sqrt{3} - i, -2i, 2i$ **B** $\sqrt{3} - i, -\sqrt{3} - i, 2i$ **C** $-\sqrt{3} - i, -2, -2i$
D $-\sqrt{3} - i, \sqrt{3} - i, -2i$ **E** $\sqrt{3} - i, -8i, 2i$

7 $\frac{\sqrt{6}}{2}(1 + i)$ is expressed in polar form as:
A $\sqrt{3} \text{ cis } \left(-\frac{\pi}{4}\right)$ **B** $\sqrt{3} \text{ cis } \left(-\frac{7\pi}{4}\right)$ **C** $-\sqrt{3} \text{ cis } \left(-\frac{\pi}{4}\right)$
D $-\sqrt{3} \text{ cis } \left(-\frac{7\pi}{4}\right)$ **E** $\sqrt{3} \text{ cis } \left(\frac{7\pi}{4}\right)$

8 If $z = 1 + i$ is one solution of an equation of the form $z^4 = a$, where $a \in C$, the other solutions are:
A $-1, 1, 0$ **B** $-1, 1, 1 - i$ **C** $-1 + i, -1 - i, 1 - i$
D $-1 + i, -1 - i, 1$ **E** $-1 + i, -1 - i, -1$

9 The square roots of $-2 - 2\sqrt{3}i$ in polar form are:
A $2 \text{ cis } \left(-\frac{2\pi}{3}\right), 2 \text{ cis } \left(\frac{\pi}{3}\right)$ **B** $2 \text{ cis } \left(-\frac{\pi}{3}\right), 2 \text{ cis } \left(\frac{2\pi}{3}\right)$ **C** $4 \text{ cis } \left(-\frac{2\pi}{3}\right), 4 \text{ cis } \left(\frac{\pi}{3}\right)$
D $4 \text{ cis } \left(-\frac{\pi}{3}\right), 4 \text{ cis } \left(\frac{2\pi}{3}\right)$ **E** $4 \text{ cis } \left(-\frac{\pi}{3}\right), 4 \text{ cis } \left(\frac{\pi}{3}\right)$

Review

10 The roots of the equation $2x^2 + 6x + 7 = 0$ are α and β. The value of $|\alpha - \beta|$ is:

A $\sqrt{5}$ **B** $2\sqrt{5}$ **C** $4\sqrt{5}$ **D** $\frac{\sqrt{10}}{2}$ **E** $\frac{\sqrt{5}}{10}$

Short-answer questions (technology-free)

1 Express each of the following in the form $a + ib$, $a \in R$ and $b \in R$:

a $3 + 2i + 5 - 7i$ **b** i^3 **c** $(3 - 2i)(5 + 7i)$

d $(3 - 2i)(3 + 2i)$ **e** $\dfrac{2}{3 - 2i}$ **f** $\dfrac{5 - i}{2 + i}$

g $\dfrac{3i}{2 + i}$ **h** $(1 - 3i)^2$ **i** $\dfrac{(5 + 2i)^2}{3 - i}$

2 Solve each of the following equations for z:

a $(z - 2)^2 + 9 = 0$ **b** $\dfrac{z - 2i}{z + (3 - 2i)} = 2$ **c** $z^2 + 6z + 12 = 0$

d $z^4 + 81 = 0$ **e** $z^3 - 27 = 0$ **f** $8z^3 + 27 = 0$

3 **a** Show that $2 - i$ is a root of the equation $z^3 - 2z^2 - 3z + 10 = 0$. Hence, solve the equation for z.

b Show that $3 - 2i$ is a root of the equation $x^3 - 5x^2 + 7x + 13 = 0$. Hence, solve the equation.

c Show that $1 + i$ is a root of the equation $z^3 - 4z^2 + 6z - 4 = 0$ and, hence, find the other roots of this equation.

4 Express each of the following polynomials as a product of linear factors:

a $2x^2 + 3x + 2$ **b** $x^3 - x^2 + x - 1$ **c** $x^3 + 2x^2 - 4x - 8$

5 $(a + ib)^2 = 3 - 4i$. Find the possible values of a and b, where $a, b \in R$.

6 Pair each of the transformations on the left with the appropriate operation on the complex numbers given on the right:

a reflection in the x axis **i** multiply by -1

b rotation anticlockwise by 90° about O **ii** multiply by i

c rotation through 180° about O **iii** multiply by $-i$

d rotation anticlockwise about O through 270° **iv** take the conjugate

7 $(a + ib)^2 = -24 - 10i$. Find the possible values of a and b where $a, b \in R$.

8 Find the values of a and b if $f(z) = z^2 + az + b$ and $f(-1 - 2i) = 0$, $a \in R$, and $b \in R$.

9 Express $\dfrac{1}{1 + i\sqrt{3}}$ in the form $r \operatorname{cis} \theta$, where $r > 0$ and $-\pi < \theta \leq \pi$.

10 On an Argand diagram, O is the origin and P represents the point $3 + i$. The point Q represents $a + bi$, where a and b are both positive. If triangle OPQ is equilateral, find a and b.

11 Let $z = 1 - i$. Find:

a $2\bar{z}$ **b** $\frac{1}{z}$ **c** $|z^7|$ **d** $\text{Arg}(z^7)$

12 Let $w = 1 + i$ and $z = 1 - i\sqrt{3}$.

a Write down:

i $|w|$ **ii** $|z|$ **iii** $\text{Arg}\ w$ **iv** $\text{Arg}\ z$

b Hence, write down $\left|\frac{w}{z}\right|$ and $\text{Arg}(wz)$.

13 Express $(\sqrt{3} + i)$ in polar form. Hence, find $(\sqrt{3} + i)^7$ and express the answer in cartesian form.

14 Find all real values of r for which ri is a solution of the equation $z^4 - 2z^3 + 11z^2 - 18z + 18 = 0$.
Hence, determine all the solutions of the equation.

15 Express $(1 - i)^9$ in cartesian form.

16 Find the real numbers k such that $z = ki$ is a root of the equation $z^3 + (2 + i)z^2 + (2 + 2i)z + 4 = 0$.
Hence, or otherwise, find the three roots of the equation.

17 **a** Find the three linear factors of $z^3 - 2z + 4$.

b What is the remainder when $z^3 - 2z + 4$ is divided by $z - 3$?

18 If a and b are complex numbers, $\text{Im}(a) = 2$, $\text{Re}(b) = -1$ and $a + b = -ab$, find a and b.

19 **a** Express $S = \{z: |z - (1 + i)| \le 1\}$ in cartesian form.

b Sketch S on an Argand diagram.

20 Describe $\{z: |z + i| = |z - i|\}$.

21 Let $S = \left\{z: z = 2\ \text{cis}\ \theta,\ 0 \le \theta \le \frac{\pi}{2}\right\}$. Sketch:

a S **b** $T = \{w: w = z^2, z \in S\}$ **c** $U = \left\{v: v = \frac{2}{z}, z \in S\right\}$

22 Find the centre of the circle which passes through the points $-2i$, 1 and $2 - i$.

23 On an Argand diagram, A and B represent the complex numbers $a = 5 + 2i$ and $b = 8 + 6i$.

a Find $i(a - b)$ and show that it can be represented by a vector perpendicular to $\overrightarrow{AB}$ and of the same length as $\overrightarrow{AB}$.

b Hence, find complex numbers c and d represented by C and D such that $ABCD$ is a square.

24 Solve each of the following for $z \in C$:

a $z^3 = -8$ **b** $z^2 = 2 + 2\sqrt{3}i$

25 **a** Factorise $x^6 - 1$ for R. **b** Factorise $x^6 - 1$ for C.

c Determine all the sixth roots of unity.

26 If z is any complex number with a non-zero imaginary part, simplify each of the following:

a $\left|\frac{\bar{z}}{z}\right|$ **b** $\frac{i(\text{Re}(z) - z)}{\text{Im}(z)}$ **c** $\text{Arg}\, z + \text{Arg}\left(\frac{1}{z}\right)$

27 If $\text{Arg}\, z = \frac{\pi}{4}$ and $\text{Arg}(z - 3) = \frac{\pi}{2}$ find $\text{Arg}(z - 6i)$.

28 **a** If $\text{Arg}(z + 2) = \frac{\pi}{2}$ and $\text{Arg}(z) = \frac{2\pi}{3}$, find z.

b If $\text{Arg}(z - 3) = \frac{-3\pi}{4}$ and $\text{Arg}(z + 3) = \frac{-\pi}{2}$, find z.

29 A complex number z satisfies the inequality $|z + 2 - 2\sqrt{3}i| \leq 2$.

a Sketch the corresponding region representing possible values of z.

b **i** Find the least possible value of $|z|$.

ii Find the greatest possible value of $\text{Arg}\, z$.

Extended-response questions

1 Let $z = 4 \text{ cis}\left(\frac{5\pi}{6}\right)$ and $w = \sqrt{2} \text{ cis}\left(\frac{\pi}{4}\right)$.

a Find $|z^7|$ and $\text{Arg}(z^7)$.

b Show z^7 on an Argand diagram.

c Express $\frac{z}{w}$ in the form $r \text{ cis}\, \theta$.

d Express z and w in cartesian form and hence express $\frac{z}{w}$ in cartesian form.

e Use the results of **d** to find an exact value for $\tan\left(\frac{7\pi}{12}\right)$ in the form $a + \sqrt{b}$ where a and b are rational.

f Use the result of **e** to find the exact value of $\tan\left(\frac{7\pi}{6}\right)$.

2 Let $v = 2 + i$ and $P(z) = z^3 - 7z^2 + 17z - 15$.

a Show by substitution that $P(2 + i) = 0$.

b Find the other two roots of the equation $P(z) = 0$.

c Let $\boldsymbol{i}$ be a unit vector in the positive Re(z) direction and let $\boldsymbol{j}$ be a unit vector in the positive Im(z) direction.

Let A be the point on the Argand diagram corresponding to $v = 2 + i$.

Let B be the point on the Argand diagram corresponding to $1 - 2i$.

Show that $\overrightarrow{OA}$ is perpendicular to $\overrightarrow{OB}$.

d Find a polynomial with real coefficients and with roots 3, $1 - 2i$ and $2 + i$.

3 **a** Find the exact solutions in C for the equation $z^2 - 2\sqrt{3}z + 4 = 0$, writing your solutions in cartesian form.

b **i** Plot the two solutions from **a** on an Argand diagram.

ii Find the equation of the circle, with centre the origin, which passes through these two points.

iii Find the value of $a \in Z$ such that the circle passes through $(0, \pm a)$.

iv Let $Q(z) = (z^2 + 4)(z^2 - 2\sqrt{3}z + 4)$. Find the polynomial $P(z)$ such that $Q(z)\,P(z) = z^6 + 64$ and explain the significance of the result.

4 **a** Express $-4\sqrt{3} - 4i$ in exact polar form.

b Find the cube roots of $-4\sqrt{3} - 4i$.

c Carefully plot the three roots of $-4\sqrt{3} - 4i$ on an Argand diagram.

d **i** Show that the cubic equation $z^3 - 3\sqrt{3}iz^2 - 9z + 3\sqrt{3}i = -4\sqrt{3} - 4i$ can be written in the form $(z - w)^3 = -4\sqrt{3} - 4i$ where w is a complex number.

ii Hence find in exact cartesian form the solutions of the equation $z^3 - 3\sqrt{3}iz^2 - 9z + (3\sqrt{3} + 4)i + 4\sqrt{3} = 0$.

5 The points X, Y and Z correspond to the numbers $4\sqrt{3} + 2i$, $5\sqrt{3} + i$ and $6\sqrt{3} + 4i$.

a Find the vector $\overrightarrow{XY}$ and the vector $\overrightarrow{XZ}$.

b Let z_1 and z_2 be the complex numbers corresponding to the vectors $\overrightarrow{XY}$ and $\overrightarrow{XZ}$. Find z_3 such that $z_2 = z_3\,z_1$.

c By writing z_3 in modulus–argument form, show that XYZ is half an equilateral triangle XWZ and give the complex number to which W corresponds.

d The triangle XYZ is rotated through an angle of $\frac{\pi}{3}$ anticlockwise about Y. Find the new position of X.

6 Let $z = x + iy$ where x and y are real and let $\text{Re}(z)$ and $\text{Im}(z)$ denote the real and imaginary parts of z respectively.

a Sketch the region T in the complex plane which is obtained by reflecting

$$S = \{z\colon \text{Re}(z) \le 2\} \cap \{z\colon \text{Im}(z) < 2\} \cap \left\{z\colon \frac{\pi}{6} < \text{Arg}(z) < \frac{\pi}{3}\right\}$$

in the line defined by $|z + i| = |z - 1|$.

b Describe the region T by using set notation in a similar way to that used in **a** to describe S.

7 Find the set of real values k, $k \ne -1$, for which the roots of the equation $x^2 + 4x - 1 + k(x^2 + 2x + 1) = 0$ are:

a real and distinct **b** real and equal **c** complex with positive real part

8 A regular hexagon $LMNPQR$ has its centre at the origin O and its vertex L at the point $z = 4$.

a Indicate in a diagram the region in the hexagon in which the inequalities $|z| \ge 2$ and $\frac{-\pi}{3} \le \arg z \le \frac{\pi}{3}$ are satisfied.

b Find in the form $|z - c| = a$ the equation of the circle through O, M and R.

c Find the complex numbers corresponding to the points N and Q.

d The hexagon is rotated clockwise about the origin through an angle of 45°. Express in the form $r \text{ cis } \theta$ the complex numbers corresponding to the new positions of N and Q.

9 **a** If $z = \cos\theta + i\sin\theta$, prove that $\dfrac{1+z}{1-z} = i\cot\dfrac{\theta}{2}$.

b On an Argand diagram O, A, Z, P, Q represent the complex numbers $0, 1, z, 1+z$ and $1-z$. Show these points on a diagram.

c Prove that the magnitude of $\angle POQ = \dfrac{\pi}{2}$. Find in terms of θ the ratio $\dfrac{|OP|}{|OQ|}$.

10 **a** A complex number $z = a + ib$ is such that $|z| = 1$. Show that $\dfrac{1}{z} = \bar{z}$.

b Let $z_1 = \dfrac{1}{2} - \dfrac{\sqrt{3}}{2}i$ and $z_2 = \dfrac{\sqrt{3}}{2} + \dfrac{1}{2}i$. If $z_3 = \dfrac{1}{z_1} + \dfrac{1}{z_2}$, find z_3 in polar form.

c On a diagram, show the points z_1, z_2 and z_3 and $z_4 = \dfrac{1}{z_3}$.

11 **a** Let $P(z) = z^3 + 3pz + q$. It is known that $P(z) = (z-k)^2(z-a)$.

i Show that $p = -k^2$. **ii** Find q in terms of k. **iii** Show that $4p^3 + q^2 = 0$.

b $h(z) = z^3 - 6iz + 4 - 4i$. It is known that $h(z) = (z-b)^2(z-c)$. Find the values of b and c.

12 **a** Let z be a complex number with $|z| = 6$. Let A be the point representing z. Let B be the point representing $(1+i)z$.

i Find $|(1+i)z|$. **ii** Find $|(1+i)z - z|$.

iii Prove that OAB is an isosceles right-angled triangle.

b Let z_1 and z_2 be non-zero complex numbers satisfying $z_1^2 - 2z_1z_2 + 2z_2^2 = 0$. If $z_1 = \alpha\, z_2$:

i show that $\alpha = 1 + i$ or $1 - i$

ii for each of these values of α describe the geometrical nature of the triangle whose vertices are the origin and the points representing z_1 and z_2.

13 **a** Let $z = -12 + 5i$. Find:

i $|z|$ **ii** $\text{Arg}(z)$ correct to two decimal places in degrees.

b Let $w^2 = -12 + 5i$ and $\alpha = \text{Arg}(w^2)$.

i Write $\cos\alpha$ and $\sin\alpha$ in exact form.

ii Using the result $r^2(\cos 2\theta + i\sin 2\theta) = |w^2|(\cos\alpha + i\sin\alpha)$ write r, $\cos 2\theta$ and $\sin 2\theta$ in exact form.

iii Use the result of **ii** to find $\sin\theta$ and $\cos\theta$.

iv Find the two values of w.

c Use a cartesian method to find w.

d Find the square roots of $12 + 5i$ and comment on their relationship with the square roots of $-12 + 5i$.

14 **a** Find the locus defined by the relation $2z\bar{z} + 3z + 3\bar{z} - 10 = 0$.

b Find the locus defined by the relation $2z\bar{z} + (3+i)z + (3-i)\bar{z} - 10 = 0$.

c Find the locus defined by the relation $\alpha z\bar{z} + \beta z + \beta\bar{z} + \gamma = 0$ where α, β and γ are real.

d Find the locus defined by the relation $\alpha z\bar{z} + \beta z + \bar{\beta}\bar{z} + \gamma = 0$ where $\alpha, \gamma \in R$ and $\beta \in C$.

15 **a** Expand $(\cos\theta + i\sin\theta)^5$.

b By De Moivre's theorem $(\text{cis}\,\theta)^5 = \text{cis}\,(5\theta)$. Use this result and the result of **a** to show that:

i $\cos 5\theta = 16\cos^5\theta - 20\cos^3\theta + 5\cos\theta$

ii $\dfrac{\sin 5\theta}{\sin\theta} = 16\cos^4\theta - 12\cos^2\theta + 1$ if $\sin\theta \neq 0$

16 **a** If $\bar{z}$ denotes the complex conjugate of the number $z = x + iy$, find the cartesian equation of the line given by $(1 + i)z + (1 - i)\bar{z} = -2$.

Sketch on an Argand diagram $\left\{z: (1 + i)z + (1 - i)\bar{z} = -2,\ \text{Arg}\, z \leq \dfrac{\pi}{2}\right\}$.

b Let $S = \{z: |z - (2\sqrt{2} + i2\sqrt{2})| \leq 2\}$.

i Sketch S on an Argand diagram.

ii If z belongs to S, find the maximum and minimum values of $|z|$.

iii If z belongs to S, find the maximum and minimum values of $\text{Arg}(z)$.

17 The roots of the equation $z^2 + 2z + 4 = 0$ are denoted by α and β.

a Find α and β in modulus–argument form.

b Show that $\alpha^3 = \beta^3$.

c Find a quadratic equation for which the roots are $\alpha + \beta$ and $\alpha - \beta$.

d Find the exact value of $\alpha\bar{\beta} + \beta\bar{\alpha}$.

18 **a** Let $w = 2\,\text{cis}\,\theta$ and $z = w + \dfrac{1}{w}$.

i Find z in terms of θ.

ii Show that z lies on the ellipse with equation $\dfrac{x^2}{25} + \dfrac{y^2}{9} = \dfrac{1}{4}$.

iii Show that $|z - 2|^2 = \left(\dfrac{5}{2} - 2\cos\theta\right)^2$.

iv Show that $|z - 2| + |z + 2| = 5$.

b Let $w = 2i\,\text{cis}\,\theta$ and $z = w - \dfrac{1}{w}$.

i Find z in terms of θ.

ii Show that z lies on the ellipse with equation $\dfrac{y^2}{25} + \dfrac{x^2}{9} = \dfrac{1}{4}$.

iii Show that $|z - 2i| + |z + 2i| = 5$.

CHAPTER

5

Revision

Revision of Chapters 1 to 4

5.1 Multiple-choice questions

1 If $\sin x = -\frac{1}{5}$ where $\pi \le x \le \frac{3\pi}{2}$ then $\tan x$ equals:

A $\frac{\sqrt{6}}{12}$ **B** $\frac{1}{24}$ **C** $\frac{1}{4}$ **D** $-\frac{1}{24}$ **E** $\frac{-\sqrt{6}}{12}$

2 If $\cos x = a$ where $\frac{\pi}{2} \le x \le \pi$ then $\sin(x + \pi)$ equals:

A $1 - a$ **B** $a - 1$ **C** $\sqrt{1 - a^2}$ **D** $-\sqrt{1 - a^2}$ **E** $1 + a$

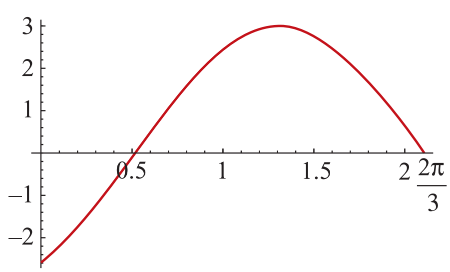
3 The equation $\sin\left(2x + \frac{\pi}{3}\right) = -\frac{\sqrt{3}}{2}$, $-\pi \le x \le \pi$ has:

A 0 solutions **B** 1 solution **C** 2 solutions
D 3 solutions **E** 4 solutions

4 The solutions of $\tan^2 x = 3$, $0 \le x \le 2\pi$ are:

A $\frac{\pi}{3}$ only **B** $\frac{\pi}{3}$ or $\frac{4\pi}{3}$ only **C** $\frac{\pi}{6}$ only
D $\frac{\pi}{6}$ or $\frac{7\pi}{6}$ only **E** $\frac{\pi}{3}$ or $\frac{2\pi}{3}$ or $\frac{4\pi}{3}$ or $\frac{5\pi}{3}$

5 The graph shown is a section of the graph of $y = f(x)$, $0 \le x \le \frac{2\pi}{3}$.
The rule for $f(x)$ is:



A $2\sin\left(3x - \frac{\pi}{3}\right)$ **B** $2\sin\left(3x - \frac{\pi}{6}\right)$
C $3\cos\left(2x - \frac{\pi}{3}\right)$ **D** $2\cos\left(3x + \frac{\pi}{3}\right)$
E $3\sin\left(2x - \frac{\pi}{3}\right)$

6 The y-axis intercept of the graph of $y = 3\tan\left(2x + \frac{5\pi}{6}\right)$ is:

A $\left(0, -\frac{\sqrt{3}}{2}\right)$ **B** $\left(0, -\frac{\sqrt{2}}{2}\right)$ **C** $(0, -\sqrt{3})$

D $(0, \sqrt{2})$ **E** $\left(0, -\frac{\sqrt{3}}{3}\right)$

7 The x-axis intercept of the graph of $y = -2\cos\left(\pi - \frac{x}{3}\right)$, $0 \le x \le 2\pi$ is:

A $\frac{4\pi}{3}$ **B** $\frac{5\pi}{3}$ **C** $\frac{7\pi}{6}$ **D** $\frac{3\pi}{2}$ **E** $\frac{5\pi}{4}$

8 The asymptotes of the graph of $y = 2\tan\left(3x - \frac{\pi}{3}\right)$, $-\frac{\pi}{2} \le x \le \frac{\pi}{2}$, are located at $x =$

A $\pm\frac{\pi}{2}$ **B** $\frac{5\pi}{9}, \frac{\pi}{9}, \frac{7\pi}{9}$ **C** $\frac{5\pi}{18}, \frac{\pi}{18}, \frac{7\pi}{18}$

D $\frac{5\pi}{12}, \frac{\pi}{12}, \frac{7\pi}{12}$ **E** $\frac{13\pi}{18}, \frac{5\pi}{18}$

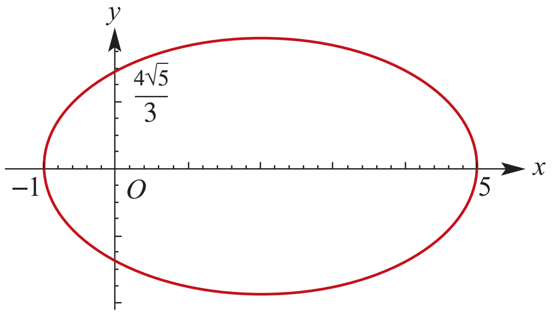
9 The equations for the asymptotes of the hyperbola with equation $\frac{(x+1)^2}{9} - \frac{(y-2)^2}{16} = 1$ are:

A $y = \frac{3}{4}x + \frac{8}{3}$ and $y = \frac{3}{4}x + \frac{2}{3}$ **B** $y = \frac{3}{4}x + \frac{10}{3}$ and $y = \frac{3}{4}x + \frac{2}{3}$

C $y = \frac{4}{3}x + \frac{10}{3}$ and $y = -\frac{4}{3}x + \frac{2}{3}$ **D** $y = \frac{4}{3}x + \frac{10}{3}$ and $y = -\frac{4}{3}x + \frac{10}{3}$

E $y = \frac{3}{4}x - \frac{10}{3}$ and $y = -\frac{3}{4}x + \frac{2}{3}$

10 The equation of the ellipse shown (centre is on x axis) is:



A $\frac{(x+2)^2}{9} + \frac{y^2}{16} = 1$

B $\frac{(x-2)^2}{9} + \frac{y^2}{16} = 1$

C $\frac{(x+2)^2}{3} + \frac{y^2}{4} = 1$

D $\frac{(x-2)^2}{3} + \frac{y^2}{4} = 1$

E $\frac{(x-2)^2}{9} - \frac{y^2}{16} = 1$

11 The equation of the circle which has a diameter with endpoints at $(4, -2)$ and $(-2, -2)$ is:

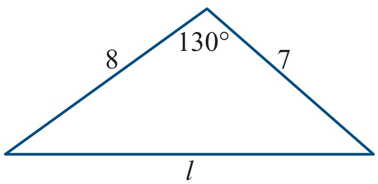
A $(x-1)^2 + (y-2)^2 = 3$ **B** $(x-1)^2 + (y+2)^2 = 3$

C $(x+1)^2 + (y-2)^2 = 6$ **D** $(x-1)^2 + (y+2)^2 = 9$

E $(x-1)^2 + (y+2)^2 = 6$

Revision

12 Which one of the following equations is a correct one for calculating l?



A $l^2 = 49 + 64 + 2 \times 7 \times 8 \cos 50^\circ$

B $l^2 = 49 + 64 + 2 \times 7 \times 8 \cos 130^\circ$

C $\dfrac{l}{\sin 130^\circ} = \dfrac{8}{\sin 25^\circ}$

D $\dfrac{l}{\sin 130^\circ} = \dfrac{7}{\sin 25^\circ}$

E $l^2 = 49 + 64 - 2 \times 7 \times 8 \cos 50^\circ$

13 The coordinates of the x-axis intercepts of the graph of the ellipse with the equation: $\dfrac{x^2}{9} + \dfrac{y^2}{25} = 1$ are

A $(-3, -5)$ and $(3, 5)$ **B** $(-5, -3)$ and $(5, 3)$ **C** $(0, -3)$ and $(0, 3)$
D $(-3, 0)$ and $(3, 0)$ **E** $(3, 0)$ and $(5, 0)$

14 A circle defined by equation $x^2 + y^2 - 6x + 8y = 0$ has centre:

A $(2, 4)$ **B** $(-5, 9)$ **C** $(4, -3)$ **D** $(3, -4)$ **E** $(6, -8)$

15 If the line $x = k$ is a tangent to the circle with equation $(x - 1)^2 + (y + 2)^2 = 1$, then k is equal to:

A 1 or -2 **B** 1 or 3 **C** -1 or -3 **D** 0 or -2 **E** 0 or 2

16 The curve with equation $x^2 - 2x = y^2$ is:

A an ellipse with centre $(1, 0)$ **B** a hyperbola with centre $(1, 0)$
C a circle with centre $(1, 0)$ **D** an ellipse with centre $(-1, 0)$
E a hyperbola with centre $(-1, 0)$

17 If $\boldsymbol{a} = 2\boldsymbol{i} + 3\boldsymbol{j} - 4\boldsymbol{k}$, $\boldsymbol{b} = -\boldsymbol{i} + 2\boldsymbol{j} - 2\boldsymbol{k}$ and $\boldsymbol{c} = -3\boldsymbol{j} + 4\boldsymbol{k}$, then $\boldsymbol{a} - 2\boldsymbol{b} - \boldsymbol{c}$ equals:

A $3\boldsymbol{i} + 10\boldsymbol{j} - 12\boldsymbol{k}$ **B** $-3\boldsymbol{i} + 7\boldsymbol{j} - 12\boldsymbol{k}$ **C** $4\boldsymbol{i} + 2\boldsymbol{j} - 4\boldsymbol{k}$
D $-4\boldsymbol{j} + 4\boldsymbol{k}$ **E** $2\boldsymbol{j} - 4\boldsymbol{k}$

18 A vector of magnitude 6 and with direction opposite to $\boldsymbol{i} - 2\boldsymbol{j} + 2\boldsymbol{k}$ is:

A $6\boldsymbol{i} - 12\boldsymbol{j} + 12\boldsymbol{k}$ **B** $-6\boldsymbol{i} + 12\boldsymbol{j} - 2\boldsymbol{k}$ **C** $-3\boldsymbol{i} + 6\boldsymbol{j} - 6\boldsymbol{k}$
D $-2\boldsymbol{i} + 4\boldsymbol{j} - 4\boldsymbol{k}$ **E** $\frac{2}{3}\boldsymbol{i} - \frac{4}{3}\boldsymbol{j} + \frac{4}{3}\boldsymbol{k}$

19 If $\boldsymbol{a} = 2\boldsymbol{i} - 3\boldsymbol{j} - \boldsymbol{k}$ and $\boldsymbol{b} = -2\boldsymbol{i} + 3\boldsymbol{j} - 6\boldsymbol{k}$, then the vector resolute of $\boldsymbol{a}$ in the direction of $\boldsymbol{b}$ is:

A $7(-2\boldsymbol{i} + 3\boldsymbol{j} - 6\boldsymbol{k})$ **B** $\frac{1}{7}(2\boldsymbol{i} - 3\boldsymbol{j} + 6\boldsymbol{k})$ **C** $-\frac{1}{7}(2\boldsymbol{i} - 3\boldsymbol{j} - \boldsymbol{k})$
D $\frac{-7}{11}(2\boldsymbol{i} - 3\boldsymbol{j} - \boldsymbol{k})$ **E** $-\frac{19}{49}(-2\boldsymbol{i} + 3\boldsymbol{j} - 6\boldsymbol{k})$

20 If $\boldsymbol{a} = 3\boldsymbol{i} - 5\boldsymbol{j} + \boldsymbol{k}$ then a vector which is not perpendicular to $\boldsymbol{a}$ is:

A $\frac{1}{35}(3\boldsymbol{i} - 5\boldsymbol{j} + \boldsymbol{k})$ **B** $2\boldsymbol{i} + \boldsymbol{j} - \boldsymbol{k}$ **C** $\boldsymbol{i} - \boldsymbol{j} - 8\boldsymbol{k}$
D $-3\boldsymbol{i} + 5\boldsymbol{j} + 34\boldsymbol{k}$ **E** $\frac{1}{9}(-3\boldsymbol{i} - 2\boldsymbol{j} - \boldsymbol{k})$

21 The magnitude of vector $\boldsymbol{a} = \boldsymbol{i} - 3\boldsymbol{j} + 5\boldsymbol{k}$ is:

A 3 **B** $\sqrt{17}$ **C** 35 **D** 17 **E** $\sqrt{35}$

22 If $\boldsymbol{u} = 2\boldsymbol{i} - \sqrt{2}\boldsymbol{j} + \boldsymbol{k}$ and $\boldsymbol{v} = \boldsymbol{i} + \sqrt{2}\boldsymbol{j} - \boldsymbol{k}$, then the angle between the direction of $\boldsymbol{u}$ and $\boldsymbol{v}$, correct to two decimal places, is:

A $92.05°$ **B** $87.95°$ **C** $79.11°$ **D** $100.89°$ **E** $180°$

23 Let $\boldsymbol{u} = 2\boldsymbol{i} - a\boldsymbol{j} - \boldsymbol{k}$ and $\boldsymbol{v} = 3\boldsymbol{i} + 2\boldsymbol{j} - b\boldsymbol{k}$. $\boldsymbol{u}$ and $\boldsymbol{v}$ are perpendicular to each other when:

A $a = 2$ and $b = -1$ **B** $a = -2$ and $b = 10$ **C** $a = \frac{1}{2}$ and $b = -5$
D $a = 0$ and $b = 0$ **E** $a = -1$ and $b = 5$

24 Let $\boldsymbol{u} = \boldsymbol{i} + a\boldsymbol{j} - 4\boldsymbol{k}$ and $\boldsymbol{v} = b\boldsymbol{i} - 2\boldsymbol{j} + 3\boldsymbol{k}$. $\boldsymbol{u}$ and $\boldsymbol{v}$ are parallel to each other when:

A $a = -2$ and $b = 1$ **B** $a = -\frac{8}{3}$ and $b = -\frac{3}{4}$ **C** $a = -\frac{3}{2}$ and $b = \frac{-3}{4}$
D $a = -\frac{8}{3}$ and $b = -\frac{4}{3}$ **E** none of these

25 Let $\boldsymbol{a} = \boldsymbol{i} - 5\boldsymbol{j} + \boldsymbol{k}$ and $\boldsymbol{b} = 2\boldsymbol{i} - \boldsymbol{j} + 2\boldsymbol{k}$. Then the vector component of $\boldsymbol{a}$ perpendicular to $\boldsymbol{b}$ is:

A $-\boldsymbol{i} - 4\boldsymbol{j} - \boldsymbol{k}$ **B** $\boldsymbol{i} + 4\boldsymbol{j} + \boldsymbol{k}$ **C** $-5\boldsymbol{i} + \boldsymbol{j} - 5\boldsymbol{k}$

D $5\boldsymbol{i} - \boldsymbol{j} + 5\boldsymbol{k}$ **E** $\frac{5}{3}\boldsymbol{i} + \frac{2}{3}\boldsymbol{j} + \frac{5}{3}\boldsymbol{k}$

26 A, B and C are points such that $\overrightarrow{AB}.\overrightarrow{BC} = 0$. If so, the following statement must be true:

A Either $\overrightarrow{AB}$ or $\overrightarrow{BC}$ is a zero vector.
B $|\overrightarrow{AB}| = |\overrightarrow{BC}|$.
C The vector resolute of $\overrightarrow{AC}$ in the direction of $\overrightarrow{AB}$ is $\overrightarrow{AB}$.
D The vector resolute of $\overrightarrow{AB}$ in the direction of $\overrightarrow{AC}$ is $\overrightarrow{AC}$.
E A, B and C are collinear.

27 If $\boldsymbol{u} = \boldsymbol{i} - \boldsymbol{j} - \boldsymbol{k}$ and $\boldsymbol{v} = 4\boldsymbol{i} + 12\boldsymbol{j} - 3\boldsymbol{k}$, then $\boldsymbol{u}.\boldsymbol{v}$ equals:

A $4\boldsymbol{i} - 12\boldsymbol{j} + 3\boldsymbol{k}$ **B** $5\boldsymbol{i} + 11\boldsymbol{j} - 4\boldsymbol{k}$ **C** -5

D 19 **E** $\dfrac{5}{13}$

28 If $\boldsymbol{a} = 3\boldsymbol{i} + 2\boldsymbol{j} - \boldsymbol{k}$ and $\boldsymbol{b} = 6\boldsymbol{i} - 3\boldsymbol{j} + 2\boldsymbol{k}$, then the scalar resolute of $\boldsymbol{a}$ in the direction of $\boldsymbol{b}$ is:

A $\dfrac{10}{49}(6\boldsymbol{i} - 3\boldsymbol{j} - 2\boldsymbol{k})$ **B** $\dfrac{10}{7}$ **C** $2\boldsymbol{i} - \dfrac{3}{2}\boldsymbol{j} - 2\boldsymbol{k}$

D $\dfrac{10}{49}$ **E** $\dfrac{\sqrt{10}}{7}$

29 The unit vector in the direction of $\boldsymbol{a} - \boldsymbol{b}$, where $\boldsymbol{a} = 3\boldsymbol{i} - 5\boldsymbol{j} - 2\boldsymbol{k}$ and $\boldsymbol{b} = 2\boldsymbol{i} - 3\boldsymbol{j} - 4\boldsymbol{k}$, is:

A $\boldsymbol{i} - 2\boldsymbol{j} + 2\boldsymbol{k}$ **B** $\dfrac{1}{\sqrt{65}}(5\boldsymbol{i} - 2\boldsymbol{j} - 6\boldsymbol{k})$ **C** $\frac{1}{3}(\boldsymbol{i} - 2\boldsymbol{j} + 2\boldsymbol{k})$

D $\frac{1}{9}(\boldsymbol{i} - 2\boldsymbol{j} + 2\boldsymbol{k})$ **E** $\frac{1}{3}(-\boldsymbol{i} + 2\boldsymbol{j} - 2\boldsymbol{k})$

30 $(2\boldsymbol{i} + 3\boldsymbol{j} + \boldsymbol{k}).(\boldsymbol{i} - 4\boldsymbol{j} + \boldsymbol{k})$ equals:

A $2\boldsymbol{i} - 12\boldsymbol{j} + \boldsymbol{k}$ **B** 9 **C** -9
D $9\boldsymbol{i}$ **E** $-9\boldsymbol{i}$

31 If the points P, Q and R are collinear with $\overrightarrow{OP} = 3\boldsymbol{i} + \boldsymbol{j} - \boldsymbol{k}$, $\overrightarrow{OQ} = \boldsymbol{i} - 2\boldsymbol{j} + \boldsymbol{k}$ and $\overrightarrow{OR} = 2\boldsymbol{i} + p\boldsymbol{j} + q\boldsymbol{k}$ then:

A $p = -3$ and $q = 2$ **B** $p = -\frac{7}{2}$ and $q = 2$ **C** $p = -\frac{1}{2}$ and $q = 0$
D $p = 3$ and $q = -2$ **E** $p = -\frac{1}{2}$ and $q = 2$

32 Given that $\tan \alpha = \frac{3}{4}$ and $\tan \beta = \frac{4}{3}$, where α and β are both acute, then $\sin(\alpha + \beta)$ equals:

A $\frac{7}{5}$ **B** $\frac{24}{25}$ **C** $\frac{7}{25}$
D 0 **E** 1

33 Given that $\boldsymbol{a} = 3\boldsymbol{i} + 4\boldsymbol{j}$, $\boldsymbol{b} = 2\boldsymbol{i} - \boldsymbol{j}$, $\boldsymbol{x} = \boldsymbol{i} + 5\boldsymbol{j}$ and $\boldsymbol{x} = s\boldsymbol{a} + t\boldsymbol{b}$, then the scalars s and t are given by:

A $s = -1$ and $t = -1$ **B** $s = -1$ and $t = 1$ **C** $s = 1$ and $t = -1$
D $s = 1$ and $t = 1$ **E** $s = \sqrt{5}$ and $t = 5$

34 Given that $\overrightarrow{OP} = \boldsymbol{p}$, $\overrightarrow{OQ} = \boldsymbol{q}$ and the points O, P and Q are not collinear, which one of the following points, whose position vectors are given, is not collinear with P and Q?

A $\frac{1}{2}\boldsymbol{p} + \frac{1}{2}\boldsymbol{q}$ **B** $3\boldsymbol{p} - 2\boldsymbol{q}$ **C** $\boldsymbol{p} - \boldsymbol{q}$
D $\frac{1}{3}\boldsymbol{p} + \frac{2}{3}\boldsymbol{q}$ **E** $2\boldsymbol{p} - \boldsymbol{q}$

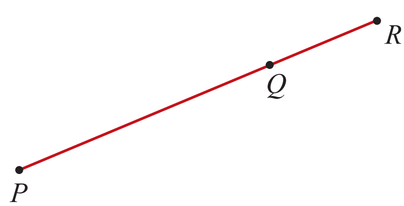
35 $\cos^2 \theta + 3 \sin^2 \theta$ equals:

A $2 + \cos \theta$ **B** $3 - 2 \cos 2\theta$ **C** $2 - \cos \theta$
D $2 \cos 2\theta - 1$ **E** none of these

36 $\cos^{-1}\left(-\frac{\sqrt{3}}{2}\right) - \sin^{-1}\left(-\frac{\sqrt{3}}{2}\right)$ equals:

A $-\frac{5\pi}{6}$ **B** $-\frac{\pi}{2}$ **C** $-\frac{\pi}{6}$
D $\frac{\pi}{2}$ **E** $\frac{7\pi}{6}$

37 PQR is a straight line and $PQ = 2QR$, $\overrightarrow{OQ} = 3\boldsymbol{i} - 2\boldsymbol{j}$ and $\overrightarrow{OR} = \boldsymbol{i} + 3\boldsymbol{j}$. Therefore $\overrightarrow{OP}$ equals:



A $-\boldsymbol{i} + 8\boldsymbol{j}$ **B** $7\boldsymbol{i} - 12\boldsymbol{j}$
C $4\boldsymbol{i} - 10\boldsymbol{j}$ **D** $-4\boldsymbol{i} + 10\boldsymbol{j}$
E $-7\boldsymbol{i} + 12\boldsymbol{j}$

38 $\overrightarrow{OP} = 2\boldsymbol{i} - 2\boldsymbol{j} + \boldsymbol{k}$, $\overrightarrow{PQ} = 2\boldsymbol{i} + 2\boldsymbol{j} - \boldsymbol{k}$. Therefore $|\overrightarrow{OQ}|$ equals:

A $2\sqrt{5}$ **B** $3\sqrt{2}$ **C** 6 **D** 9 **E** 4

39 $z_1 = 2 - i$ and $z_2 = 3 + 4i$, therefore $\left|\frac{z_2}{z_1}\right|^2$ equals:

A $\sqrt{5}$ **B** 5 **C** $\frac{125}{9}$ **D** $\left(\frac{2+11i}{5}\right)^2$ **E** $\left(\frac{10+5i}{5}\right)^2$

40 If $z = -1 - i\sqrt{3}$ then Arg z equals:

A $-\frac{2\pi}{3}$ **B** $-\frac{5\pi}{6}$ **C** $\frac{2\pi}{3}$ **D** $\frac{5\pi}{6}$ **E** $-\frac{\pi}{3}$

41 The value(s) of p for which the vectors $p\boldsymbol{i} + 2\boldsymbol{j} - 3p\boldsymbol{k}$ and $p\boldsymbol{i} + \boldsymbol{k}$ are perpendicular is (are):

A 0 only **B** 3 only **C** 0 and 3 **D** 1 and 2 **E** 1 only

42 One root of the equation $z^3 - 5z^2 + 17z - 13 = 0$ is $2 + 3i$. The other roots are:

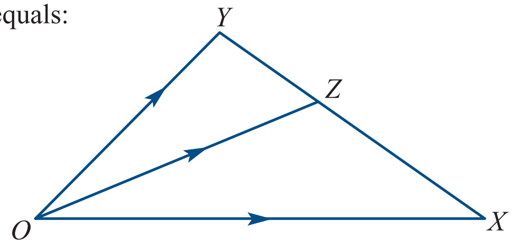
A $-2 - 3i$ and 1 **B** $2 - 3i$ and 1 **C** $-2 + 3i$ and -1
D $2 - 3i$ and -1 **E** $-2 + 3i$ and 1

43 The value of $\frac{(\cos 60^\circ + i \sin 60^\circ)^4}{(\cos 30^\circ + i \sin 30^\circ)^2}$ is:

A -1 **B** i **C** $-i$ **D** $\frac{1}{2} - \frac{\sqrt{3}}{2}i$ **E** $\frac{\sqrt{3}}{2} - \frac{1}{2}i$

44 Given that $3\overrightarrow{OX} + 4\overrightarrow{OY} = 7\overrightarrow{OZ}$, then $\frac{XZ}{ZY}$ equals:

A $\frac{3}{5}$ **B** $\frac{3}{4}$ **C** 1
D $\frac{4}{3}$ **E** $\frac{5}{3}$



45 $\cos\left((\tan^{-1}(1) + \sin^{-1}\left(\frac{1}{\sqrt{2}}\right)\right)$ equals:

A $\frac{\pi}{2}$ **B** 1 **C** 0 **D** $-\frac{1}{\sqrt{2}}$ **E** $-\frac{\sqrt{3}}{2}$

46 If $x + iy = \frac{1}{3+4i}$, where x and y are real, then:

A $x = \frac{3}{25}$ and $y = \frac{-4}{25}$ **B** $x = \frac{3}{25}$ and $y = \frac{4}{25}$ **C** $x = \frac{-3}{7}$ and $y = \frac{4}{7}$
D $x = \frac{1}{3}$ and $y = \frac{1}{4}$ **E** $x = 3$ and $y = -4$

47 Let $\boldsymbol{a} = 2\boldsymbol{i} + 3\boldsymbol{j} + 4\boldsymbol{k}$ and $\boldsymbol{b} = \boldsymbol{i} + p\boldsymbol{j} + \boldsymbol{k}$. If $\boldsymbol{a}$ and $\boldsymbol{b}$ are perpendicular, p equals:

A $-\frac{7}{3}$ **B** -2 **C** $-\frac{5}{3}$ **D** 2 **E** $\frac{7}{3}$

48 If $z = \frac{1}{1-i}$, $r = |z|$ and $\theta = \text{Arg } z$, then:

A $r = 2$ and $\theta = \frac{\pi}{4}$ **B** $r = \frac{1}{2}$ and $\theta = \frac{\pi}{4}$ **C** $r = \sqrt{2}$ and $\theta = -\frac{\pi}{4}$
D $r = \frac{1}{\sqrt{2}}$ and $\theta = -\frac{\pi}{4}$ **E** $r = \frac{1}{\sqrt{2}}$ and $\theta = \frac{\pi}{4}$

49 If $\cos x = \frac{-3}{5}$ and $\pi < x < \frac{3\pi}{2}$ then $\tan x$ is:

A $\frac{4}{3}$ **B** $\frac{3}{4}$ **C** $-\frac{4}{5}$ **D** $-\frac{3}{5}$ **E** $\frac{9}{25}$

50 The value of $\sin^{-1}\left(-\frac{\sqrt{3}}{2}\right)$ is:

A $\frac{5\pi}{6}$ **B** $\frac{2\pi}{3}$ **C** $-\frac{\pi}{3}$ **D** $-\frac{\pi}{6}$ **E** $\frac{7\pi}{6}$

51 The maximal domain of $f(x) = \sin^{-1}(2x - 1)$ is:

A $[-1, 1]$ **B** $(-1, 1)$ **C** $(0, 1)$ **D** $[0, 1]$ **E** $[-1, 0]$

52 If $u = 3 \text{ cis } \frac{\pi}{4}$ and $v = 2 \text{ cis } \frac{\pi}{2}$, then uv is equal to:

A $\text{cis } \frac{7\pi}{4}$ **B** $6 \text{ cis } \frac{\pi^2}{8}$ **C** $6 \text{ cis}^2 \frac{\pi^2}{8}$ **D** $5 \text{ cis } \frac{3\pi}{4}$ **E** $6 \text{ cis } \frac{3\pi}{4}$

53 The exact value of $\sin\left[\cos^{-1}\left(-\frac{1}{2}\right)\right]$ is:

A $\frac{\sqrt{3}}{2}$ **B** $-\frac{1}{2}$ **C** 1 **D** $-\frac{\sqrt{3}}{2}$ **E** $\frac{1}{\sqrt{5}}$

54 The modulus of $12 - 5i$ is:

A 119 **B** 7 **C** 13 **D** $\sqrt{119}$ **E** $\sqrt{7}$

55 When $\sqrt{3} - i$ is divided by $-1 - i$ the modulus and Argument of the quotient are, respectively:

A $2\sqrt{2}$ and $\frac{7\pi}{12}$ **B** $\sqrt{2}$ and $\frac{-11\pi}{12}$ **C** $\sqrt{2}$ and $\frac{7\pi}{12}$

D $2\sqrt{2}$ and $\frac{-11\pi}{12}$ **E** $\sqrt{2}$ and $\frac{11\pi}{12}$

56 The equation $x^2 + 3x + 1 = 0$ has:

A no roots **B** two imaginary roots **C** two complex roots

D two real roots **E** one real and one complex root

57 The product of the complex numbers $\frac{1 - i}{\sqrt{2}}$ and $\frac{\sqrt{3} + i}{2}$ has Argument:

A $-\frac{5\pi}{12}$ **B** $-\frac{\pi}{12}$ **C** $\frac{\pi}{12}$ **D** $\frac{5\pi}{12}$ **E** none of these

58 If $\tan\theta = \frac{1}{3}$, then $\tan 2\theta$ equals:

A $\frac{3}{5}$ **B** $\frac{2}{3}$ **C** $\frac{3}{4}$ **D** $\frac{4}{5}$ **E** $\frac{4}{3}$

59 Which one of the following five expressions is not identical to any of the others?

A $\cos^4\theta - \sin^4\theta$ **B** $1 + \cos\theta$ **C** $\cos 2\theta$

D $2\cos^2\frac{\theta}{2}$ **E** $1 - \cos\theta$

60 The modulus of $1 + \cos 2\theta + i \sin 2\theta$ where $0 < \theta < \frac{\pi}{2}$ is:

A $4\cos^2\theta$ **B** $4\sin^2\theta$ **C** $2\cos\theta$ **D** $2\sin\theta$ **E** none of these

61 An expression for the argument of $1 + \cos\theta + i\sin\theta$ is:

A $2\cos\frac{\theta}{2}$ **B** $2\sin\frac{\theta}{2}$ **C** θ **D** $\frac{\theta}{2}$ **E** $\frac{\pi}{2} - \frac{\theta}{2}$

62 A quadratic equation with roots $2 + 3i$ and $2 - 3i$ is:

A $x^2 + 4x + 13 = 0$ **B** $x^2 - 4x + 13 = 0$ **C** $x^2 + 4x - 13 = 0$
D $x^2 + 4x - 5 = 0$ **E** $x^2 - 4x - 5 = 0$

63 If $\tan^{-1}\frac{1}{2} + \tan^{-1}\frac{1}{3} = \tan^{-1} x$, then x is:

A 1 **B** $\frac{5}{6}$ **C** $\frac{5}{7}$ **D** $\frac{1}{5}$ **E** $\frac{1}{7}$

64 Which one of the following five expressions is not identical to any of the others?

A $\tan\theta + \cot\theta$ **B** $\text{cosec}^2\theta - \cot^2\theta$ **C** 1
D $\text{cosec}\,\theta\cot\theta$ **E** $2\,\text{cosec}\,2\theta$

65 The subset of the complex plane defined by the relation $|z - 2| - |z + 2| = 1$ is:

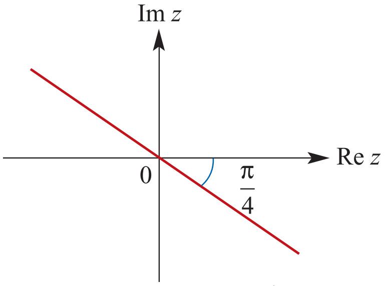
A a circle **B** an ellipse **C** a straight line
D the empty set **E** a hyperbola

66 The subset of the complex plane defined by the relation $|z - (2 - i)| = 6$ is:

A a circle with centre at $-2 + i$ and radius 6
B a circle with centre at $2 - i$ and radius 6
C a circle with centre at $2 - i$ and radius 36
D a circle with centre at $-2 + i$ and radius 36
E a circle with centre at $-2 - i$ and radius 36

67 The graph shown can be represented by the set:

A $\left\{z : \text{Arg } z = \frac{\pi}{4}\right\}$ **B** $\left\{z : \text{Arg } z = -\frac{\pi}{4}\right\}$
C $\left\{z : \text{Arg } z = \frac{7\pi}{4}\right\}$ **D** $\{z : \text{Im } z + \text{Re } z = 0\}$
E $\{z : \text{Im } z - \text{Re } z = 0\}$



68 The subset of the complex plane defined by the relation $|z - 2| - |z - 2i| = 0$ is:

A a circle **B** an ellipse **C** a straight line
D the empty set **E** a hyperbola

69 Which one of the following subsets of the complex plane is **not** a circle?

A $\{z : |z| = 2\}$ **B** $\{z : |z - i| = 2\}$ **C** $\{z : z\bar{z} + 2\text{Re}(iz) = 0\}$
D $\{z : |z - 1| = 2\}$ **E** $\{z : |z| = 2i\}$

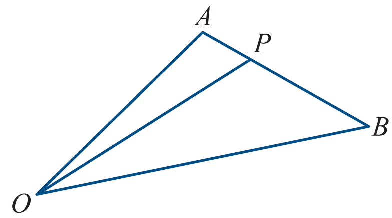
70 Which one of the following subsets of the complex plane is **not** a line?

A $\{z : \text{Im}(z) = 0\}$ **B** $\{z : \text{Im}(z) + \text{Re}(z) = 1\}$ **C** $\{z : z + \bar{z} = 4\}$
D $\left\{z : \text{Arg}(z) = \frac{\pi}{4}\right\}$ **E** $\{z : \text{Re}(z) = \text{Im}(z)\}$

71 Points P, Q, R and M are such that $\overrightarrow{PQ} = 5\boldsymbol{i}$, $\overrightarrow{PR} = \boldsymbol{i} + \boldsymbol{j} + 2\boldsymbol{k}$ and $\overrightarrow{RM}$ is parallel to $\overrightarrow{PQ}$ so that $\overrightarrow{RM} = \lambda\boldsymbol{i}$, where λ is a constant. The value of λ for which angle RQM is a right angle is:

A 0 **B** $\frac{19}{4}$ **C** $\frac{21}{4}$ **D** 10 **E** 6

72 In this diagram $\overrightarrow{OA} = 6\boldsymbol{i} - \boldsymbol{j} + 8\boldsymbol{k}$, $\overrightarrow{OB} = -3\boldsymbol{i} + 4\boldsymbol{j} - 2\boldsymbol{k}$ and $AP : PB = 1 : 2$.
The vector $\overrightarrow{OP}$ is equal to:



A $\frac{7}{3}\boldsymbol{j} + \frac{4}{3}\boldsymbol{k}$ **B** $3\boldsymbol{i} + \frac{7}{3}\boldsymbol{j} + \frac{4}{3}\boldsymbol{k}$
C $3\boldsymbol{j} + 4\boldsymbol{k}$ **D** $3\boldsymbol{i} + \frac{2}{3}\boldsymbol{j} + \frac{14}{3}\boldsymbol{k}$
E none of these

73 In an Argand diagram O is the origin, P the point (2, 1) and Q the point (1, 2). If P represents the complex number z and Q the complex number α then α represents the point:

A $\bar{z}$ **B** $i\bar{z}$ **C** $-\bar{z}$ **D** $-i\bar{z}$ **E** $z\bar{z}$

74 In an Argand diagram the points which represents the complex numbers z, $-\bar{z}$, z^{-1} and $-\overline{z^{-1}}$ necessarily lie at the vertices of a:

A square **B** rectangle **C** parallelogram
D rhombus **E** trapezium

5.2 Extended-response questions

1 **a** Points A, B and P are collinear with B between A and P. The points A, B and P have position vectors $\boldsymbol{a}$, $\boldsymbol{b}$ and $\boldsymbol{r}$ respectively relative to an origin O. If $\overrightarrow{AP} = \frac{3}{2}\overrightarrow{AB}$:

i express $\overrightarrow{AP}$ in terms of $\boldsymbol{a}$ and $\boldsymbol{b}$ **ii** express $\boldsymbol{r}$ in terms of $\boldsymbol{a}$ and $\boldsymbol{b}$.

b The points A, B and C have position vectors $\boldsymbol{i}$, $2\boldsymbol{i} + 2\boldsymbol{j}$ and $4\boldsymbol{i} + \boldsymbol{j}$ respectively.

i Find $\overrightarrow{AB}$ and $\overrightarrow{BC}$.
ii Show that $\overrightarrow{AB}$ and $\overrightarrow{BC}$ have equal magnitudes.
iii Show that AB and BC are perpendicular.
iv Find the position vector of D such that $ABCD$ is a square.

c The triangle OAB is such that O is the origin, $\overrightarrow{OA} = 8\boldsymbol{i}$ and $\overrightarrow{OB} = 10\boldsymbol{j}$. The point P with position vector $\overrightarrow{OP} = x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k}$ is equidistant from O, A and B and is at a distance of 2 above the triangle. Find x, y and z.

2 **a** Let $S_1 = \{z : |z| \le 2\}$ and $T_1 = \{z : \text{Im } z + \text{Re } z \ge 4\}$.

i On the same diagram, sketch S_1 and T_1, clearly indicating which boundary points are included.

ii Let $d = |z_1 - z_2|$ where $z_1 \in S_1$ and $z_2 \in T_1$. Find the minimum value of d.

b Let $S_2 = \{z : |z - 1 - i| \le 1\}$ and $T_2 = \{z : |z - 2 - i| \le |z - i|\}$.

i On the same diagram, sketch S_2 and T_2, clearly indicating which boundaries are included.

ii If z belongs to $S_2 \cap T_2$ find the maximum and minimum values of $|z|$.

3 *OACB* is a trapezium with *OB* parallel to *AC* and $AC = 2OB$. *D* is a point of trisection of *OC* nearer to *O*.

a If $\overrightarrow{OA} = \boldsymbol{a}$ and $\overrightarrow{OB} = \boldsymbol{b}$ find in terms of $\boldsymbol{a}$ and $\boldsymbol{b}$:

i $\overrightarrow{BC}$ **ii** $\overrightarrow{BD}$ **iii** $\overrightarrow{DA}$

b Hence prove that *A*, *D* and *B* are collinear.

4 **a** If $\boldsymbol{a} = \boldsymbol{i} - 2\boldsymbol{j} + 2\boldsymbol{k}$ and $\boldsymbol{b} = 12\boldsymbol{j} - 5\boldsymbol{k}$ find:

i the magnitude of the angle between $\boldsymbol{a}$ and $\boldsymbol{b}$ to the nearest degree

ii the vector resolute of $\boldsymbol{b}$ perpendicular to $\boldsymbol{a}$

iii real numbers x, y and z such that $x\boldsymbol{a} + y\boldsymbol{b} = 3\boldsymbol{i} - 30\boldsymbol{j} + z\boldsymbol{k}$

b In triangle *OAB*, $\overrightarrow{OA} = \boldsymbol{a}$ and $\overrightarrow{OB} = \boldsymbol{b}$. *P* is the point of trisection of *AB* nearer to *B* and $OQ = 1.5OP$.

i Find an expression for $\overrightarrow{AQ}$ in terms of $\boldsymbol{a}$ and $\boldsymbol{b}$. **ii** Show that $\overrightarrow{OA}$ is parallel to $\overrightarrow{BQ}$.

5 **a** Show that if $2a + b - c = 0$ and $a - 4b - 2c = 0$, then $a : b : c = 2 : -1 : 3$.

b It is given that position vector $x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k}$ is perpendicular to vector $2\boldsymbol{i} + \boldsymbol{j} - 3\boldsymbol{k}$ and $\boldsymbol{i} - \boldsymbol{j} - \boldsymbol{k}$. Establish two equations in x, y and z, and find the ratio $x : y : z$.

c Hence, or otherwise, find any vector $\boldsymbol{v}$ which is perpendicular to $2\boldsymbol{i} + \boldsymbol{j} - 3\boldsymbol{k}$ and $\boldsymbol{i} - \boldsymbol{j} - \boldsymbol{k}$.

d Show that the vector $4\boldsymbol{i} + 5\boldsymbol{j} - 7\boldsymbol{k}$ is also perpendicular to vector $\boldsymbol{v}$.

e Find the values of s and t such that $4\boldsymbol{i} + 5\boldsymbol{j} - 7\boldsymbol{k}$ can be expressed in the form $s(2\boldsymbol{i} + \boldsymbol{j} - 3\boldsymbol{k}) + t(\boldsymbol{i} - \boldsymbol{j} - \boldsymbol{k})$.

f Show that any vector $\boldsymbol{r} = t(2\boldsymbol{i} + \boldsymbol{j} - 3\boldsymbol{k}) + s(\boldsymbol{i} - \boldsymbol{j} - \boldsymbol{k})$ is perpendicular to vector $\boldsymbol{v}$ ($t \in R$ and $s \in R$).

6 Consider a triangle with vertices *O*, *A* and *B* where $\overrightarrow{OA} = \boldsymbol{a}$ and $\overrightarrow{OB} = \boldsymbol{b}$. Let θ be the angle between vectors $\boldsymbol{a}$ and $\boldsymbol{b}$.

a Express $\cos\theta$ in terms of vectors $\boldsymbol{a}$ and $\boldsymbol{b}$.

b Hence, express $\sin\theta$ in terms of vectors $\boldsymbol{a}$ and $\boldsymbol{b}$.

c Use the formula for the area of a triangle $\left(\text{area} = \frac{1}{2}ab\sin C\right)$ to show that the area of triangle $OAB = \frac{1}{2}\sqrt{(\boldsymbol{a}.\boldsymbol{a})(\boldsymbol{b}.\boldsymbol{b}) - (\boldsymbol{a}.\boldsymbol{b})^2}$.

7 In the quadrilateral *ABCD*, *X* and *Y* are midpoints of the diagonals *AC* and *BD* respectively.

a Show that $\overrightarrow{BA} + \overrightarrow{BC} = 2\overrightarrow{BX}$. **b** Show that $\overrightarrow{BA} + \overrightarrow{BC} + \overrightarrow{DA} + \overrightarrow{DC} = 4\overrightarrow{YX}$.

8 The position vectors of the vertices of a triangle ABC referred to a given origin, O, are $\boldsymbol{a}$, $\boldsymbol{b}$ and $\boldsymbol{c}$, P is a point on AB such that $AP : PB = 1 : 2$, Q is a point on AC such that $AQ : QC = 2 : 1$, and R is a point on PQ such that $PR : RQ = 2 : 1$.

a Prove that $\overrightarrow{OR} = \frac{4}{9}\boldsymbol{a} + \frac{1}{9}\boldsymbol{b} + \frac{4}{9}\boldsymbol{c}$.

b Let M be the midpoint of AC. Prove that R lies on the median BM.

c Find $BR : RM$.

9 The points A and B have position vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ respectively, referred to an origin O. The point C lies on AB between A and B, and is such that $AC : CB = 2 : 1$, and D is the midpoint of OC. The line AD produced meets OB at E.

a Find, in terms of $\boldsymbol{a}$ and $\boldsymbol{b}$:
i $\overrightarrow{OC}$ **ii** $\overrightarrow{AD}$

b Find the ratios:
i $OE : EB$ **ii** $AE : ED$

10 The position vectors of the vertices A, B and C of a triangle relative to an origin, O, are $\boldsymbol{a}$, $\boldsymbol{b}$ and $\boldsymbol{c}$ respectively. The side BC is produced to D so that $BC = CD$. X is defined as the point dividing the side AB in the ratio 2 : 1, and Y as the point dividing AC in the ratio 4 : 1, i.e. $AX : XB = 2 : 1$ and $AY : YC = 4 : 1$.

a Express, in terms of $\boldsymbol{a}$, $\boldsymbol{b}$ and $\boldsymbol{c}$:
i $\overrightarrow{OD}$ **ii** $\overrightarrow{OX}$ **iii** $\overrightarrow{OY}$

b Show that D, X and Y are collinear.

11 A, B, C and D are points with position vectors $\boldsymbol{j} + 2\boldsymbol{k}$, $-\boldsymbol{i} - \boldsymbol{j}$, $4\boldsymbol{i} + \boldsymbol{k}$ and $3\boldsymbol{i} + \boldsymbol{j} + 2\boldsymbol{k}$ respectively.

a Prove that the triangle ABC is right-angled.

b Prove that the triangle ABD is isosceles.

c Show that $\overrightarrow{BD}$ passes through the midpoint, E, of AC and find the ratio $BE : ED$.

12 **a** For $\alpha = 1 - \sqrt{3}i$, write the product of $z - \alpha$, and $z - \overline{\alpha}$ as a quadratic expression in z, with real coefficients, where $\overline{\alpha}$ denotes the complex conjugate of α.

b **i** Express α in modulus–argument form. **ii** Find α^2 and α^3.

iii Show that α is a root of $z^3 - z^2 + 2z + 4 = 0$ and find all three roots of this equation.

c On an Argand diagram, plot the points corresponding to the three roots. Let A be the point in the first quadrant and B the point on the real axis. Let C be the point corresponding to the third root.

i Find the lengths AB and CB. **ii** Describe the triangle ABC.

13 **a** If $z = 1 + i\sqrt{2}$, express $p = z + \dfrac{1}{z}$ and $q = z - \dfrac{1}{z}$ in the form $a + ib$.

b On an Argand diagram, P and Q are the points which represent p and q respectively. O is the origin, M is the midpoint of PQ, and G is the point on OM such that $OG = \frac{2}{3}OM$. Denote vectors $\overrightarrow{OP}$ and $\overrightarrow{OQ}$ by $\boldsymbol{a}$ and $\boldsymbol{b}$ respectively.

Find each of the following vectors in terms of $\boldsymbol{a}$ and $\boldsymbol{b}$:
i $\overrightarrow{PQ}$ **ii** $\overrightarrow{OM}$ **iii** $\overrightarrow{OG}$ **iv** $\overrightarrow{GP}$ **v** $\overrightarrow{GQ}$

c Prove that angle PGQ is a right angle.

14 **a** Find the linear factors of $z^2 + 4$.

b Express $z^4 + 4$ as a pair of quadratic factors in C.

c Show that: **i** $(1 + i)^2 = 2i$ **ii** $(1 - i)^2 = -2i$

d Use the results of **c** to factorise $z^4 + 4$ into linear factors.

e Hence, factorise $z^4 + 4$ into two quadratic factors with real coefficients.

15 **a** Let $z_1 = 1 + 3i$ and $z_2 = 2 - i$. Show that $|z_1 - z_2|$ is the distance between the points z_1 and z_2 on an Argand diagram.

b Describe the locus of z on an Argand diagram, such that $|z - (2 - i)| = \sqrt{5}$.

c Describe the locus of z, such that $|z - (1 + 3i)| = |z - (2 - i)|$.

16 Let $z = 2 + i$.

a Express z^3 in the form $x + iy$, where x and y are integers.

b Let the polar form of $z = 2 + i$ be $r(\cos \alpha + i \sin \alpha)$. Using the polar form of z^3, but without evaluating α, find the value of:

i $\cos 3\alpha$ **ii** $\sin 3\alpha$

17 The cube roots of unity are often denoted by 1, w and w^2, where $w = -\frac{1}{2} + \frac{1}{2}\sqrt{3}i$ and $w^2 = -\frac{1}{2} - \frac{1}{2}\sqrt{3}i$.

a **i** Illustrate these three numbers on an Argand diagram.

ii Show that $(w^2)^2 = w$.

b By factorising $z^3 - 1$, show that $w^2 + w + 1 = 0$.

c Evaluate: **i** $(1 + w)(1 + w^2)$ **ii** $(1 + w^2)^3$

d Form the quadratic equation whose roots are:

i $2 + w$ and $2 + w^2$ **ii** $3w - w^2$ and $3w^2 - w$

e Find the possible values of the expression $1 + w^n + w^{2n}$.

18 **a** Let $z^5 - 1 = (z - 1)\,P(z)$ where $P(z)$ is a polynomial. Find $P(z)$ by division.

b Show that $z = \text{cis}\,\frac{2\pi}{5}$ is a solution of the equation $z^5 - 1 = 0$.

c Hence, find another complex solution to the equation $z^5 - 1 = 0$.

d Find all the complex solutions of $z^5 - 1 = 0$.

e Hence, factorise $P(z)$ as a product of two quadratic polynomials with real coefficients.

19 **a** A relation between two complex variables, w and z, is given by the equation $w = \dfrac{az + b}{z + c}$ where $a \in R$, $b \in R$ and $c \in R$. Given that $w = 3i$ when $z = -3i$ and $w = 1 - 4i$ when $z = 1 + 4i$, find the values of a, b and c.

b Let $z = x + iy$. Show that, if $w = \bar{z}$, then z lies on a circle of centre (4, 0) and state the radius of this circle.

20 **a** Use De Moivre's theorem to show that $(1 + i\tan\theta)^5 = \dfrac{\text{cis}\,5\theta}{\cos^5\theta}$.

b Hence, find expressions for $\cos 5\theta$ and $\sin 5\theta$ in terms of $\tan\theta$ and $\cos\theta$.

c Show that $\tan 5\theta = \dfrac{5t - 10t^3 + t^5}{1 - 10t^2 + 5t^4}$ where $t = \tan\theta$.

d Use the result of **c** and an appropriate substitution to show that $\tan\frac{\pi}{5} = (5 - 2\sqrt{5})^{\frac{1}{2}}$.

21 **a** Express in terms of θ the roots α and β of the equation $z + z^{-1} = 2\cos\theta$.

b If P and Q are points on the Argand diagram representing $\alpha^n + \beta^n$ and $\alpha^n - \beta^n$ respectively, show that PQ is of constant length.

22 **a** On the same set of axes, sketch the following functions:

i $f(x) = \cos x, \quad -\pi < x < \pi$ **ii** $g(x) = \tan^{-1}x, \quad -\sqrt{\pi} < x < \sqrt{\pi}$

b Find, correct to two decimal places:

i $\tan^{-1}\dfrac{\pi}{4}$ **ii** $\cos 1$

c Hence, show that the graphs of $y = f(x)$ and $y = g(x)$ intersect in the interval $\left[\dfrac{\pi}{4}, 1\right]$.

d Using a CAS calculator, find the solution of $f(x) = g(x)$ correct to two decimal places.

e Show that $f(x) = g(x)$ has no other real solutions.

23 **a** On the same set of axes, sketch the following functions:

i $f(x) = \sin x \quad -\dfrac{\pi}{2} < x < \dfrac{\pi}{2}$ **ii** $g(x) = \cos^{-1} x \quad -1 < x < 1$

b Find, correct to two decimal places:

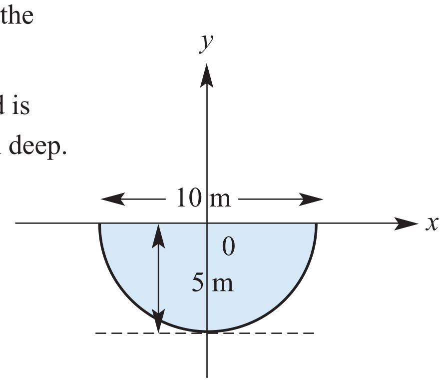
i $\sin 0.5$ **ii** $\cos^{-1}\dfrac{\pi}{4}$

c Hence, show that the graphs of $y = f(x)$ and $y = g(x)$ intersect in the interval $\left[0.5, \dfrac{\pi}{4}\right]$.

d Using a CAS calculator, find the coordinates of the point(s) of intersection of the graphs, correct to three decimal places.

24 The cross-section of a water channel is defined by the function $f(x) = a \sec\left(\dfrac{\pi}{15}x\right) + d$.

The top of the channel is level with the ground and is 10 m wide. At its deepest point, the channel is 5 m deep.



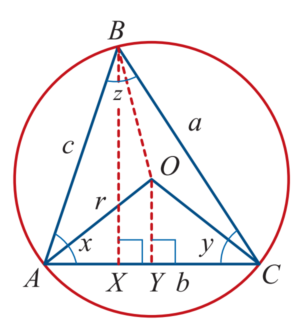
a Find a and d.

b Find, correct to two decimal places:

i the depth of the water when the width of the water surface is 7 m

ii the width of the water surface when the water is 2.5 m deep

25 Triangle ABC has circumcircle centre O.
BX is perpendicular to AC.
OY is perpendicular to AC.
$\angle BAC = x^\circ \quad \angle BCA = y^\circ \quad \angle ABC = z^\circ$



a **i** Find AX in terms of c and x.

ii Find CX in terms of a and y.

iii Use the results of **i** and **ii** to find AC.

b **i** Find the magnitude of $\angle AOC$ in terms of z and hence the magnitude of $\angle AOY$ in terms of z.

ii Find AC in terms of z and length OA.

c Show $\sin(x + y) = \sin z$.

d Let r be the radius of the circumcircle. If $r = \frac{1}{2}$, show that:

i $\sin x = a$ **ii** $\sin y = c$

e Use the results of the above to show $\sin(x + y) = \sin x \cos y + \sin y \cos x$.

26 Let S and T be the subsets of the complex plane

$$S = \left\{z : \sqrt{2} \le |z| \le 3, \frac{\pi}{2} < \text{Arg}\, z \le \frac{3\pi}{4}\right\}$$

$$T = \{z : z\bar{z} + 2\,\text{Re}(iz) \le 0\}$$

a Sketch S on an Argand diagram.

b Find $\{z : z \in S \text{ and } z = x + iy \text{ where } x \text{ and } y \text{ are integers}\}$.

c On a separate diagram sketch $S \cap T$.

27 **a** $A = \left\{z : \text{Arg}\, z = \frac{\pi}{4}\right\}$ and $B = \left\{z : \text{Arg}(z - 4) = \frac{3\pi}{4}\right\}$. Sketch A and B on the same Argand diagram clearly labelling $A \cap B$.

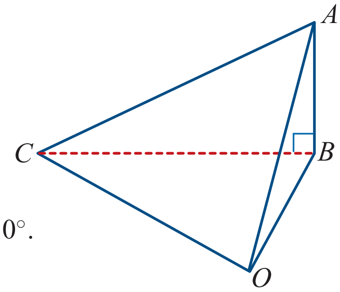
b $C = \left\{z : \left|\frac{z - \bar{z}}{z + \bar{z}}\right| \le 1\right\}$ and $D = \{z : z^2 + (\bar{z})^2 \le 2\}$. Sketch $C \cap D$ on the same Argand diagram.

28 In the tetrahedron shown, $\overrightarrow{OB} = \boldsymbol{i}$, $\overrightarrow{OC} = -\boldsymbol{i} + 3\boldsymbol{j}$ and $\overrightarrow{BA} = \sqrt{\lambda}\boldsymbol{k}$.

a Express $\overrightarrow{OA}$ and $\overrightarrow{CA}$ in terms of $\boldsymbol{i}$, $\boldsymbol{j}$, $\boldsymbol{k}$ and $\sqrt{\lambda}$.

b Find the magnitude of angle CBO to the nearest degree.

c Find the value of λ, if the magnitude of angle OAC is 30°.



29 **a** $ABCD$ is a tetrahedron in which AB is perpendicular to CD and AD is perpendicular to BC. Prove that AC is perpendicular to BD. Let $\boldsymbol{a}$, $\boldsymbol{b}$, $\boldsymbol{c}$ and $\boldsymbol{d}$ be the position vectors of the four vertices.

b Let $ABCD$ be a regular tetrahedron. The intersection point of the perpendicular bisectors of the edges of a triangle is called the circumcentre of the triangle. Let X, Y, Z and W be the circumcentres of faces ABC, ACD, ABD and BCD respectively. The vectors $\boldsymbol{a}$, $\boldsymbol{b}$, $\boldsymbol{c}$ and $\boldsymbol{d}$ are the position vectors of the four vertices.

i Find the position vectors of X, Y, Z and W.

ii Find the vectors $\overrightarrow{DX}$, $\overrightarrow{BY}$, $\overrightarrow{CZ}$ and $\overrightarrow{AW}$.

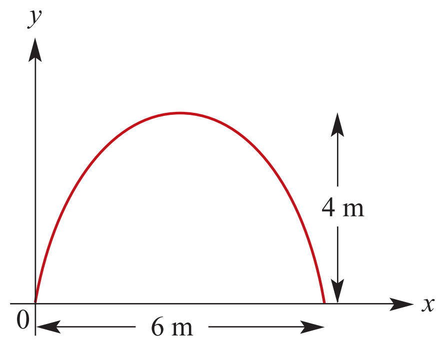
iii Let P be a point on $\overrightarrow{DX}$ such that $\overrightarrow{DP} = \frac{3}{4}\overrightarrow{DX}$. Find the position vector of P.

iv Hence find the position vectors of points Q, R and S which lie on the vectors $\overrightarrow{BY}$, $\overrightarrow{CZ}$ and $\overrightarrow{AW}$ respectively such that $\overrightarrow{BQ} = \frac{3}{4}\overrightarrow{BY}$, $\overrightarrow{CR} = \frac{3}{4}\overrightarrow{CZ}$ and $\overrightarrow{AS} = \frac{3}{4}\overrightarrow{AW}$.

v Explain the geometric significance of results **iii** and **iv**.

Revision

30 An archway is designed using a section of the graph for $y = \sec x$. The arch appears as shown.



The archway is designed using a function of the form

$$g : [0, 6] \to R,$$

$$g(x) = a\sec(bx + c) + d$$

The graph of g is a transformation of the graph of

$$f : \left[\frac{-\pi}{3}, \frac{\pi}{3}\right] \to R,\ f(x) = \sec x$$

Find the values of a, b, c and d.

CHAPTER

6

Differentiation and rational functions

Objectives

- To review differentiation
- To use the rule $\frac{dy}{dx} = \frac{1}{\frac{dx}{dy}}$ to obtain the derivative of a function of the form $x = f(y)$
- To find the **derivatives** of the **inverse circular functions**
- To define the **second derivative** of a function
- To define and investigate points of **inflexion**
- To apply the chain rule to problems involving **related rates**
- To sketch graphs of **simple rational functions**

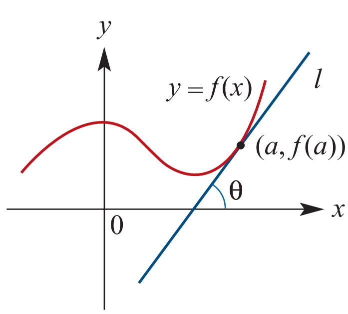
6.1 A review

The derivative of a function f is denoted by f' and the rule for f' is defined by

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$f'(x)$ is also known as the gradient function. If $(a, f(a))$ is a point on the graph of $y = f(x)$ then the gradient of the graph at that point is $f'(a)$.

If l is the tangent to the graph of $y = f(x)$ at the point $(a, f(a))$ and l makes an angle of θ with the positive direction of the x axis as shown, then



$$f'(a) = \text{gradient of } l = \tan\theta$$

In Mathematical Methods, rules for the derivatives of the following functions were presented and are stated on the next page.

$f(x)$	$f'(x)$	
a	0	where a is a constant
x^n	nx^{n-1}	$n \in R \setminus \{0\}$
$\sin x$	$\cos x$	
$\cos x$	$-\sin x$	
e^x	e^x	
$\log_e x$	$\frac{1}{x}$	$x > 0$

CAS calculators have a differentiation facility. This has been discussed in Mathematical Methods Units 3 and 4 (CAS).

The following techniques have been introduced in Mathematical Methods.

The product rule

For $f(x) = g(x)\,h(x)$

$$f'(x) = g'(x)\,h(x) + g(x)\,h'(x)$$

or for $y = uv$ where u and v are functions of x

$$\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$$

The quotient rule

For $f(x) = \dfrac{g(x)}{h(x)}$

$$f'(x) = \frac{g'(x)\,h(x) - g(x)\,h'(x)}{[h(x)]^2}$$

or for $y = \dfrac{u}{v}$

$$\frac{dy}{dx} = \frac{v\dfrac{du}{dx} - u\dfrac{dv}{dx}}{v^2}$$

The chain rule

For $f(x) = h(g(x))$

$$f'(x) = h'(g(x))\,g'(x)$$

or $\dfrac{dy}{dx} = \dfrac{dy}{du}\,\dfrac{du}{dx}$ where $u = f(x)$

Example 1

Differentiate each of the following with respect to x:

a $\sqrt{x}\sin x$ **b** $\dfrac{x^2}{\sin x}$ **c** $\cos(x^2 + 1)$

Solution

a Let $f(x) = \sqrt{x}\sin x$

Then by the product rule $f'(x) = \frac{1}{2}x^{-\frac{1}{2}}\sin x + x^{\frac{1}{2}}\cos x,\ x \neq 0$

$$= \frac{\sqrt{x}\sin x}{2x} + \sqrt{x}\cos x$$

b Let $h(x) = \dfrac{x^2}{\sin x}$

Then by the quotient rule $h'(x) = \dfrac{2x\sin x - x^2\cos x}{\sin^2 x}$

c Let $y = \cos(x^2 + 1)$ and $u = x^2 + 1$. Then $y = \cos u$

By the chain rule $\dfrac{dy}{dx} = \dfrac{dy}{du}\dfrac{du}{dx}$

$$= -\sin u.2x$$

$$= -2x\sin(x^2 + 1)$$

The derivative of $f(x) = \tan(kx)$

Since $f(x) = \tan(kx) = \dfrac{\sin(kx)}{\cos(kx)}$

the quotient rule yields $f'(x) = \dfrac{k\cos(kx)\cos(kx) + k\sin(kx)\sin(kx)}{\cos^2(kx)}$

$$= \frac{k(\cos^2(kx) + \sin^2(kx))}{\cos^2(kx)}$$

$$= k\sec^2(kx)$$

i.e. for $f(x) = \tan(kx)$

$$f'(x) = k\sec^2(kx)$$

Example 2

Differentiate each of the following with respect to x:

a $\tan(5x^2 + 3)$ **b** $\tan^3 x$ **c** $\sec^2(3x)$

Solution

a Let $f(x) = \tan(5x^2 + 3)$

By the chain rule with $g(x) = 5x^2 + 3$ and $f'(x) = h'(g(x))g'(x)$

$$f'(x) = \sec^2(5x^2 + 3) \times 10x$$

$$= 10x\sec^2(5x^2 + 3)$$

b Let $f(x) = \tan^3 x = (\tan x)^3$

By the chain rule with $g(x) = \tan x$ and $f'(x) = h'(g(x))g'(x)$

$$f'(x) = 3(\tan x)^2 \times \sec^2 x$$

$$= 3\tan^2 x\sec^2 x$$

c Let $f(x) = \sec^2(3x)$

$$= \tan^2(3x) + 1 \quad \text{(using the Pythagorean identity)}$$

$$= (\tan(3x))^2 + 1$$

By the chain rule with $g(x) = \tan(3x)$ and $f'(x) = h'(g(x))\,g'(x)$

$$f'(x) = 2(\tan(3x)) \times g'(x)$$

The chain rule is also required to find $g'(x) = \sec^2(3x) \times 3$

$$\therefore \quad f'(x) = 6\tan(3x)\sec^2(3x)$$

Or, consider $y = (\tan(3x))^2 + 1$, let $u = \tan 3x$. Then $y = u^2 + 1$

and $\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$

$= 2u\ 3\sec^2(3x)$

$= 6\tan(3x)\sec^2(3x)$

Operator notation

Sometimes it is appropriate to use notation which emphasises that differentiation is an operation on an expression. For $f(x)$ the derivative can be denoted by $\frac{d}{dx}(f(x))$.

Example 3

Find **a** $\frac{d}{dx}(x^2 + 2x + 3)$ **b** $\frac{d}{dx}(e^{x^2})$ **c** $\frac{d}{dz}(\sin^2(z))$

Solution

a $\frac{d}{dx}(x^2 + 2x + 3) = 2x + 2$

b Let $y = e^{x^2}$ and $u = x^2$

Then $y = e^u$. The chain rule gives

$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$

$= e^u.2x$

$= 2xe^{x^2}$

i.e. $\frac{d}{dx}(e^{x^2}) = 2xe^{x^2}$

c Let $y = \sin^2(z)$

Let $u = \sin z$, then $y = u^2$

$\frac{dy}{dz} = \frac{dy}{du}\frac{du}{dz}$

$= 2u.\cos z$

$= 2\sin z\cos z$

$= \sin(2z)$

Example 4

Find **a** $\frac{d(\log_e|x|)}{dx}$ $x \neq 0$ **b** $\frac{d(\log_e|\sec x|)}{dx}$ $x \neq \frac{(2k+1)\pi}{2}$ $k \in Z$

Solution

a Let $y = \log_e|x|$

If $x > 0$, $y = \log_e x$ and $\frac{dy}{dx} = \frac{1}{x}$

If $x < 0$, then $y = \log_e(-x)$ and using the chain rule

$\frac{dy}{dx} = -1 \times \frac{1}{-x} = \frac{1}{x}$

Hence $\frac{d(\log_e|x|)}{dx} = \frac{1}{x}, x \neq 0$

b Let $y = \log_e|\sec x|$

$= \log_e\left|\frac{1}{\cos x}\right|$

$= \log_e\left(\frac{1}{|\cos x|}\right)$

$= -\log_e(|\cos x|)$

Let $u = \cos x$

Then $y = -\log_e|u|$ and the chain rule gives

$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$

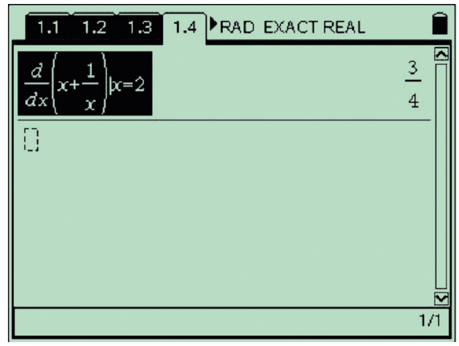
$= -\frac{1}{u}(-\sin x)$

$= -\frac{1}{\cos x}(-\sin x)$

$= \tan x$

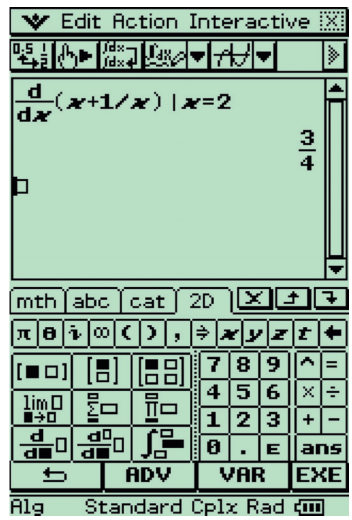
Using a TI-Nspire calculator

From the **Algebra** menu choose **Calculus** and then **Derivative** (menu 4 1).
The symbol | is available on key ①.
The derivative template can also be directly obtained through the Template menu of the catalog. Press ctrl and then the template key.



Using a Casio ClassPad calculator

Choose the **2D** menu and then **CALC** to see the keyboard with the differentiation template. This is shown on the screen right.



Exercise 6A

1 Find the derivative of each of the following with respect to x:

a $x^5 \sin x$ **b** $e^x \tan x$ **c** $\sqrt{x}\cos x$ **d** $e^x \cos x$

e $x^3 e^x$ **f** $\sin x \cos x$ **g** $x^4 \tan x$ **h** $\tan x \log_e x$

i $\sin x \tan x$ **j** $\sqrt{x}\tan x$

2 Find the derivative of each of the following using the quotient rule:

a $\dfrac{x}{\log_e x}$ **b** $\dfrac{\sqrt{x}}{\tan x}$ **c** $\dfrac{e^x}{\tan x}$ **d** $\dfrac{\tan x}{\log_e x}$

e $\dfrac{\sin x}{x^2}$ **f** $\dfrac{\tan x}{\cos x}$ **g** $\dfrac{\cos x}{e^x}$ **h** $\dfrac{\cos x}{\sin x}\ (=\cot x)$

3 Find the derivative of each of the following using the chain rule:

a $\tan(x^2 + 1)$ **b** $\sin^2 x$ **c** $e^{\tan x}$ **d** $\tan^5 x$
e $\sin(\sqrt{x})$ **f** $\sqrt{\tan x}$ **g** $\cos\left(\frac{1}{x}\right)$ **h** $\sec^2 x$
i $\tan\left(\frac{x}{4}\right)$ **j** $\cot x$ $\left(\text{use } \cot x = \tan\left(\frac{\pi}{2} - x\right)\right)$

4 Use appropriate techniques to find the derivative of each of the following:

a $\tan(kx)$ $k \in R$ **b** $e^{\tan(2x)}$ **c** $\tan^2(3x)$ **d** $\log_e(x)\, e^{\sin x}$
e $\sin^3(x^2)$ **f** $\dfrac{e^{3x+1}}{\cos x}$ **g** $e^{3x} \tan(2x)$ **h** $\sqrt{x} \tan \sqrt{x}$
i $\dfrac{\tan^2 x}{(x+1)^3}$ **j** $\sec^2(5x^2)$

5 Find $\dfrac{dy}{dx}$ for each of the following:

a $y = (x - 1)^5$ **b** $y = \log_e(4x)$ **c** $y = e^x \tan(3x)$
d $y = e^{\cos x}$ **e** $y = \cos^3(4x)$ **f** $y = (\sin x + 1)^4$
g $y = \sin(2x) \cos x$ **h** $y = \dfrac{x^2 + 1}{x}$ **i** $y = \dfrac{x^3}{\sin x}$
j $y = \dfrac{1}{x \log_e x}$

6 For each of the following determine the derivative:

a $\dfrac{d}{dx}(x^3)$ **b** $\dfrac{d}{dy}(2y^2 + 10y)$ **c** $\dfrac{d}{dz}(\cos^2(z))$
d $\dfrac{d}{dx}\left(e^{\sin^2 x}\right)$ **e** $\dfrac{d}{dz}(1 - \tan^2 z)$ **f** $\dfrac{d}{dy}(\operatorname{cosec}^2 y)$

7 For each of the following determine the derivative:

a $\log_e|2x + 1|$ **b** $\log_e|-2x + 1|$ **c** $\log_e|\sin x|$
d $\log_e|\sec x + \tan x|$ **e** $\log_e|\operatorname{cosec} x + \tan x|$ **f** $\log_e\left|\tan \frac{1}{2}x\right|$
g $\log_e|\operatorname{cosec} x - \cot x|$ **h** $\log_e|x + \sqrt{x^2 - 4}|$ **i** $\log_e|x + \sqrt{x^2 + 4}|$

8 Find the gradient of the graph of $y = f(x)$ where $f(x) = \tan\left(\frac{x}{2}\right)$ at the point where:

a $x = 0$ **b** $x = \dfrac{\pi}{3}$ **c** $x = \dfrac{\pi}{2}$

9 Let $f\colon \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \to R,\ f(x) = \tan x$. Find:

a the coordinates of the points on the graph where the gradient is 4
b the equation of the tangent at these points

10 Let $f\colon \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \to R,\ f(x) = \tan x - 8 \sin x$.

a Find:
  i the stationary points on the graph of $y = f(x)$
  ii the nature of each of the stationary points
b Sketch the graph of $y = f(x)$.

11 Let $f: \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \to R$, $f(x) = e^x \sin x$. Find:

a the gradient of $y = f(x)$ when $x = \frac{\pi}{4}$

b the coordinates of the point where the gradient is zero

12 Let $f: \left(-\frac{\pi}{4}, \frac{\pi}{4}\right) \to R$, $f(x) = \tan(2x)$. A tangent to the graph of $y = f(x)$ where $x = a$ makes an angle of 70° with the positive direction of the x axis. Find the possible values of a.

13 Let $f(x) = \sec\left(\frac{x}{4}\right)$.

a Find $f'(x)$. **b** Find $f'(\pi)$.

c Find the equation of the tangent of $y = f(x)$ at the point where $x = \pi$.

6.2 Derivatives of $x = f(y)$

From the chain rule:

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$

For the special case where $y = x$, the following result is obtained:

$$\frac{dx}{dx} = \frac{dx}{du} \times \frac{du}{dx}$$

Since $\frac{dx}{dx} = 1$, $\frac{dx}{du} \times \frac{du}{dx} = 1$ (provided neither derivative is zero).

This result is restated in the standard form by replacing u by y in the formula:

i.e. $$\frac{dx}{dy} \times \frac{dy}{dx} = 1$$

Therefore $$\frac{dy}{dx} = \frac{1}{\frac{dx}{dy}} \quad \text{provided } \frac{dx}{dy} \neq 0$$

Example 5

Given $x = y^3$, find $\frac{dy}{dx}$.

Solution

Using the rule $\frac{dx}{dy} = 3y^2$

hence $\frac{dy}{dx} = \frac{1}{3y^2}, y \neq 0$

If this is compared with an alternative method, the power of the rule can be appreciated. The alternative approach is:

$$x = y^3$$

gives $$y = \sqrt[3]{x}$$

$$= x^{\frac{1}{3}}$$

Hence $$\frac{dy}{dx} = \frac{1}{3}x^{-\frac{2}{3}}$$

$\therefore$ $$\frac{dy}{dx} = \frac{1}{3\sqrt[3]{x^2}}, x \neq 0$$

Note that $$\frac{1}{3y^2} = \frac{1}{3\sqrt[3]{x^2}}$$

While the derivative expressed in terms of x is the familiar form, it is no less powerful when it is found in terms of y.

Example 6

Find the gradient of the curve $x = y^2 - 4y$ at the point where $y = 3$.

Solution

$$x = y^2 - 4y$$

$\therefore$ $$\frac{dx}{dy} = 2y - 4$$

and $$\frac{dy}{dx} = \frac{1}{2y - 4}, y \neq 2$$

Hence the gradient at $y = 3$ is $\frac{1}{2}$.

Example 7

Find the gradient of the curve $x = y^2 - 4y$ at $x = 5$.

Solution

$$x = y^2 - 4y$$

$\therefore$ $$\frac{dy}{dx} = \frac{1}{2y - 4}$$ (from Example 6)

Substituting $x = 5$ into $x = y^2 - 4y$ yields

$$5 = y^2 - 4y$$

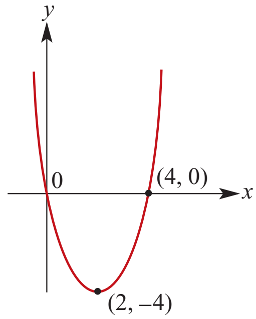
$\therefore$ $$y^2 - 4y - 5 = 0$$

$\therefore$ $$(y - 5)(y + 1) = 0$$

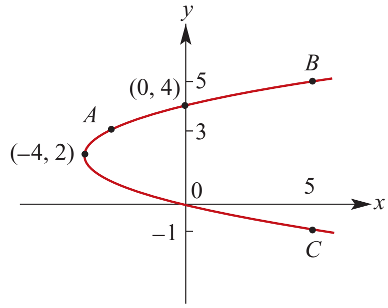
$\therefore$ $$y = 5 \text{ or } y = -1$$

Hence $\frac{dy}{dx} = \frac{1}{6}$ or $\frac{-1}{6}$ by substitution into the derivative.

To explain the two answers here, consider the graph of $x = y^2 - 4y$.



Graph of $y = x^2 - 4x$



Graph of $x = y^2 - 4y$

Note that $x = y^2 - 4y$ is the inverse of $y = x^2 - 4x$
i.e. $x = y^2 - 4y$ is the reflection of the graph of $y = x^2 - 4x$ in the line with equation $y = x$
When $x = 5$, there are two points, B and C, on the graph of $x = y^2 - 4y$

$$\text{At } B,\ y = 5 \text{ and } \frac{dy}{dx} = \frac{1}{6}$$

$$\text{At } C,\ y = -1 \text{ and } \frac{dy}{dx} = -\frac{1}{6}$$

Using a TI-Nspire calculator

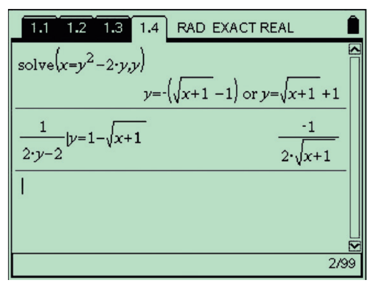
Enter solve($x = y^2 - 2y$, y).

$$\frac{dx}{dy} = 2y - 2$$

and therefore

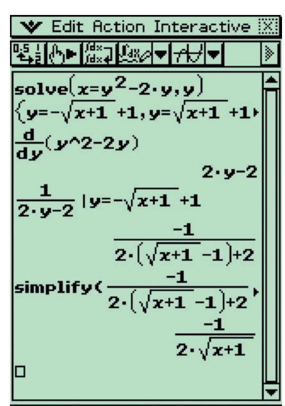
$$\frac{dy}{dx} = \frac{1}{2y - 2}$$

Complete as shown.
Now replace y with the expression in x.



Using a Casio ClassPad calculator

First solve for y, solve($x = y^2 - 2y$, y).
The result is shown on the screen.
Choose the **2D** menu and then **CALC** to see the keyboard with the differentiation template. Complete as shown.
Now substitute for y to find the derivative in terms of x.



Exercise 6B

Skillsheet TEACHER

1 Using $\frac{dy}{dx} = \frac{1}{\frac{dx}{dy}}$, find $\frac{dy}{dx}$ in each of the following:

a $x = 2y + 6$ **b** $x = y^2$ **c** $x = (2y - 1)^2$ **d** $x = e^y$

e $x = \sin 5y$ **f** $x = \log_e y$ **g** $x = \tan y$ **h** $x = y^3 + y - 2$

i $x = \frac{y - 1}{y}$ **j** $x = ye^y$

2 Find the gradient of the curves defined by each of the following, at the given value:

a $x = y^3$ at $y = \frac{1}{8}$ **b** $x = y^3$ at $x = \frac{1}{8}$

c $x = e^{4y}$ at $y = 0$ **d** $x = e^{4y}$ at $x = \frac{1}{4}$

e $x = (1 - 2y)^2$ at $y = 1$ **f** $x = (1 - 2y)^2$ at $x = 4$

g $x = \cos 2y$ at $y = \frac{\pi}{6}$ **h** $x = \cos 2y$ at $x = 0$

3 In each of the following, express $\frac{dy}{dx}$ in terms of y:

a $x = (2y - 1)^3$ **b** $x = e^{2y+1}$

c $x = \log_e(2y - 1)$ **d** $x = \log_e 2y - 1$

4 By first making y the subject for each of the relations of question **3**, find $\frac{dy}{dx}$ for each expression in terms of x.

5 Find the equations of the tangents to the curve with equation $x = 2 - 3y^2$ at the points where $x = -1$.

6 **a** Find the coordinates of the points of intersection of the graphs of the relations $x = y^2 - 4y$ and $y = x - 6$.

b Find the coordinates of the point at which the tangent to the graph of $x = y^2 - 4y$ is parallel to the line $y = x - 6$.

c Find the coordinates of the point at which the tangent to the graph $x = y^2 - 4y$ is perpendicular to the line $y = x - 6$.

7 **a** Show that the graphs of $x = y^2 - y$ and $y = \frac{1}{2}x + 1$ intersect where $x = 2$ and find the coordinates of this point.

b Find correct to two decimal places, the angle between the line $y = \frac{1}{2}x + 1$ and the tangent to the graph of $x = y^2 - y$ at the point of intersection found in **a** (i.e. where $x = 2$).

6.3 Derivatives of inverse circular functions

The result $\frac{dy}{dx} = \frac{1}{\frac{dx}{dy}}$ which was established in the previous section is used to find the rule for the derivative of the inverse of a function for which the rule for the derivative of the function is known.

For example, for the function with rule $y = \log_e x$ the equivalent function is $x = e^y$. It is known that $\frac{dx}{dy} = e^y$ and using the result $\frac{dy}{dx} = \frac{1}{\frac{dx}{dy}}$ yields $\frac{dy}{dx} = \frac{1}{e^y}$, but $x = e^y$ and therefore $\frac{dy}{dx} = \frac{1}{x}$.

Derivative of $\sin^{-1}(x)$

Let $y = \sin^{-1}(x)$ where $x \in [-1, 1]$ and $y \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

Then the equivalent form is $x = \sin y$

Therefore $\frac{dx}{dy} = \cos y$

Therefore $\frac{dy}{dx} = \frac{1}{\cos y}$ and $\cos y > 0$ for $y \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

The Pythagorean identity is used to express $\frac{dy}{dx}$ in terms of x

$$\sin^2 y + \cos^2 y = 1$$

implies $\quad \cos^2 y = 1 - \sin^2 y$

and therefore $\quad \cos y = \pm\sqrt{1 - \sin^2 y}$

but $y \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ and therefore $\cos y > 0$

Thus
$$\begin{aligned}\cos y &= \sqrt{1 - \sin^2 y}\\ &= \sqrt{1 - x^2}\end{aligned}$$

and
$$\begin{aligned}\frac{dy}{dx} &= \frac{1}{\cos y}\\ &= \frac{1}{\sqrt{1 - x^2}} \text{ for } x \in (-1, 1)\end{aligned}$$

Hence for $f(x) = \sin^{-1}(x)$, $f'(x) = \frac{1}{\sqrt{1 - x^2}}$ for $x \in (-1, 1)$

Derivative of $f(x) = \cos^{-1}(x)$

A similar technique is applied.

Let $y = \cos^{-1}(x)$, $x \in [-1, 1]$ and $y \in [0, \pi]$

Then the equivalent function is $x = \cos y$

It follows that $\quad \frac{dx}{dy} = -\sin y$

Therefore $\quad \frac{dy}{dx} = \frac{-1}{\sin y}$ and $\sin y \neq 0$ for $y \in (0, \pi)$

Using the Pythagorean identity yields

$$\sin y = \pm\sqrt{1 - \cos^2 y}$$

but as $y \in (0, \pi)$, $\sin y > 0$

$\therefore \quad \sin y = \sqrt{1 - \cos^2 y}$

Substitution yields

$$\frac{dy}{dx} = \frac{-1}{\sqrt{1-\cos^2 y}}$$

$$= \frac{-1}{\sqrt{1-x^2}} \qquad \text{since } x = \cos y$$

Hence for $f(x) = \cos^{-1}(x)$, $f'(x) = \dfrac{-1}{\sqrt{1-x^2}}$, $x \in (-1, 1)$

Derivative of $f(x) = \tan^{-1}(x)$

Let $y = \tan^{-1} x \quad x \in R \quad y \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

Then $x = \tan y$

$$\therefore \quad \frac{dx}{dy} = \sec^2 y$$

and $$\frac{dy}{dx} = \frac{1}{\sec^2 y}$$

Using the Pythagorean identity $\sec^2 y = \tan^2 y + 1$, a simple substitution produces

$$\frac{dy}{dx} = \frac{1}{\tan^2 y + 1}$$

$$= \frac{1}{x^2+1} \qquad \text{since } x = \tan y$$

Hence for $f(x) = \tan^{-1}(x)$, $f'(x) = \dfrac{1}{x^2+1}$ where $x \in R$

For $a > 0$ the following results can be obtained using the chain rule.

$f: (-a, a) \to R$, $\quad f(x) = \sin^{-1}\dfrac{x}{a}$, $\quad f'(x) = \dfrac{1}{\sqrt{a^2-x^2}}$

$f: (-a, a) \to R$, $\quad f(x) = \cos^{-1}\dfrac{x}{a}$, $\quad f'(x) = \dfrac{-1}{\sqrt{a^2-x^2}}$

$f: R \to R$, $\quad f(x) = \tan^{-1}\dfrac{x}{a}$, $\quad f'(x) = \dfrac{a}{a^2+x^2}$

For the derivative of $f: (-a, a) \to R$, $\quad f(x) = \sin^{-1}\dfrac{x}{a}$

The derivation of the first of these results is shown in the following manner.

Let $y = \sin^{-1}\left(\frac{x}{a}\right)$

Then $\frac{dy}{dx} = \frac{1}{\sqrt{1 - \left(\frac{x}{a}\right)^2}} \times \frac{1}{a}$ (chain rule)

$$= \frac{1}{\sqrt{a^2\left(1 - \left(\frac{x}{a}\right)^2\right)}}$$

$$= \frac{1}{\sqrt{a^2 - x^2}}$$

The derivation of the other two functions is left as an exercise for the reader.

Example 8

Differentiate each of the following with respect to x:

a $\sin^{-1}\left(\frac{x}{3}\right)$ **b** $\cos^{-1}(4x)$

c $\tan^{-1}\left(\frac{2x}{3}\right)$ **d** $\sin^{-1}(x^2 - 1)$

Solution

a The rule can be applied to find that if $y = \sin^{-1}\left(\frac{x}{3}\right)$ then $\frac{dy}{dx} = \frac{1}{\sqrt{9 - x^2}}$

b Let $y = \cos^{-1}(4x)$ and $u = 4x$

then $\frac{dy}{dx} = \frac{-1}{\sqrt{1 - u^2}} \times 4 = \frac{-4}{\sqrt{1 - 16x^2}}$ (chain rule)

c Let $y = \tan^{-1}\left(\frac{2x}{3}\right)$ and $u = \frac{2x}{3}$

then $\frac{dy}{dx} = \frac{1}{1 + u^2} \times \frac{2}{3}$ (chain rule)

$$= \frac{1}{1 + \left(\frac{2x}{3}\right)^2} \times \frac{2}{3}$$

$$= \frac{9}{4x^2 + 9} \times \frac{2}{3}$$

$$= \frac{6}{4x^2 + 9}$$

d Let $y = \sin^{-1}(x^2 - 1)$ and $u = x^2 - 1$

then $\dfrac{dy}{dx} = \dfrac{1}{\sqrt{1-u^2}} \times 2x$ (chain rule)

$$= \frac{2x}{\sqrt{1-(x^2-1)^2}}$$

$$= \frac{2x}{\sqrt{1-(x^4-2x^2+1)}}$$

$$= \frac{2x}{\sqrt{2x^2-x^4}}$$

$$= \frac{2x}{\sqrt{x^2}\sqrt{2-x^2}}$$

$$= \frac{2x}{|x|\sqrt{2-x^2}}$$

$$= \frac{2}{\sqrt{2-x^2}} \text{ if } 0 < x < \sqrt{2} \text{ and } \frac{-2}{\sqrt{2-x^2}} \text{ if } -\sqrt{2} < x < 0$$

Exercise 6C

Skillsheet TEACHER

1 Find the derivative of each of the following with respect to x:

a $\sin^{-1}\left(\dfrac{x}{2}\right)$ **b** $\cos^{-1}\left(\dfrac{x}{4}\right)$ **c** $\tan^{-1}\left(\dfrac{x}{3}\right)$ **d** $\sin^{-1}(3x)$

e $\cos^{-1}(2x)$ **f** $\tan^{-1}(5x)$ **g** $\sin^{-1}\left(\dfrac{3x}{4}\right)$ **h** $\cos^{-1}\left(\dfrac{3x}{2}\right)$

i $\tan^{-1}\left(\dfrac{2x}{5}\right)$ **j** $\sin^{-1}(0.2x)$

2 Find the derivative of each of the following with respect to x:

a $\sin^{-1}(x+1)$ **b** $\cos^{-1}(2x+1)$ **c** $\tan^{-1}(x+2)$

d $\sin^{-1}(4-x)$ **e** $\cos^{-1}(1-3x)$ **f** $3\tan^{-1}(1-2x)$

g $2\sin^{-1}\left(\dfrac{3x+1}{2}\right)$ **h** $-4\cos^{-1}\left(\dfrac{5x-3}{2}\right)$ **i** $5\tan^{-1}\left(\dfrac{1-x}{2}\right)$

j $-\sin^{-1}(x^2)$

3 Find the derivative of each of the following with respect to x:

a $y = \cos^{-1}\left(\dfrac{3}{x}\right)$ where $x > 3$ **b** $y = \sin^{-1}\left(\dfrac{5}{x}\right)$ where $x > 5$

c $y = \cos^{-1}\left(\dfrac{3}{2x}\right)$ where $x > \dfrac{3}{2}$

4 For a positive constant a find the derivative of each of the following:

a $\sin^{-1}(ax)$ **b** $\cos^{-1}(ax)$ **c** $\tan^{-1}(ax)$

5 Let $f(x) = 3\sin^{-1}\left(\frac{x}{2}\right)$.

a Find: **i** the maximal domain of f **ii** the range of f

b Find the derivative of $f(x)$ and state the domain for which the derivative exists.

c Sketch the graph of $y = f'(x)$ labelling the turning points and the asymptotes.

6 Let $f(x) = 4\cos^{-1}(3x)$.

a Find: **i** the maximal domain of f **ii** the range of f

b Find the derivative of $f(x)$ and state the domain for which the derivative exists.

c Sketch the graph of $y = f'(x)$, labelling the turning points and the asymptotes.

7 Let $f(x) = 2\tan^{-1}\left(\frac{x+1}{2}\right)$.

a Find: **i** the maximal domain of f **ii** the range of f

b Find the derivative of $f(x)$.

c Sketch the graph of $y = f'(x)$ labelling the turning points and the asymptotes.

8 Differentiate each of the following with respect to x:

a $(\sin^{-1} x)^2$ **b** $\sin^{-1} x + \cos^{-1} x$ **c** $\sin(\cos^{-1} x)$

d $\cos(\sin^{-1} x)$ **e** $e^{\sin^{-1} x}$ **f** $\tan^{-1}(e^x)$

9 Find, correct to two decimal places where necessary, the gradient of the graphs of the following functions at the value of x indicated:

a $f(x) = \sin^{-1}\left(\frac{x}{3}\right)$, $x = 1$ **b** $f(x) = 2\cos^{-1}(3x)$, $x = 0.1$

c $f(x) = 3\tan^{-1}(2x + 1)$, $x = 1$

10 For each of the following, find the value(s) of a from the given information:

a $f(x) = 2\sin^{-1} x$, $f'(a) = 4$ **b** $f(x) = 3\cos^{-1}\left(\frac{x}{2}\right)$, $f'(a) = -10$

c $f(x) = \tan^{-1}(3x)$, $f'(a) = 0.5$ **d** $f(x) = \sin^{-1}\left(\frac{x+1}{2}\right)$, $f'(a) = 20$

e $f(x) = 2\cos^{-1}\left(\frac{2x}{3}\right)$, $f'(a) = -8$ **f** $f(x) = 4\tan^{-1}(2x - 1)$, $f'(a) = 1$

11 Find in the form $y = mx + c$ the equation of the tangent to the graph of:

a $y = \sin^{-1}(2x)$ at $x = \frac{1}{4}$ **b** $y = \tan^{-1}(2x)$ at $x = \frac{1}{2}$

c $y = \cos^{-1}(3x)$ at $x = \frac{1}{6}$ **d** $y = \cos^{-1}(3x)$ at $x = \frac{1}{2\sqrt{3}}$

12 For the function with rule $f(x) = \cos^{-1}\left(\frac{6}{x}\right)$:

a find the maximal domain **b** find $f'(x)$ and show $f'(x) > 0$ for $x > 6$

c sketch the graph of $y = f(x)$ and label end points and asymptotes

6.4 Second derivatives

As previously introduced for the function f with rule $f(x)$ the derivative is denoted by f' and has rule $f'(x)$. This notation is extended to taking the derivative of the derivative as f'' and $f''(x)$ for the function and rule respectively. In Leibnitz's notation the second derivative is denoted by $\frac{d^2y}{dx^2}$. This new function is also known as the **second derivative**.

Consider the function g with rule $g(x) = 2x^3 - 4x^2$. Let $y = g(x)$. The derivative has the rule given by $g'(x) = 6x^2 - 8x$ or $\frac{dy}{dx} = 6x^2 - 8x$ and $g''(x) = \frac{d^2y}{dx^2} = 12x - 8$.

Example 9

Find the second derivative of each of the following with respect to x:

a $f(x) = 6x^4 - 4x^3 + 4x$ **b** $y = e^x \sin x$

Solution

a For $f(x) = 6x^4 - 4x^3 + 4x$

$f'(x) = 24x^3 - 12x^2 + 4$ and $f''(x) = 72x^2 - 24x$

b For $y = e^x \sin x$

$$\frac{dy}{dx} = e^x \sin x + e^x \cos x \qquad \text{(by the product rule)}$$

$$\frac{d^2y}{dx^2} = e^x \sin x + e^x \cos x + e^x \cos x - e^x \sin x$$

$$= 2e^x \cos x$$

Example 10

If $f(x) = e^{2x}$ find $f''(0)$.

Solution

$$f(x) = e^{2x}$$

$\therefore$ $f'(x) = 2e^{2x}$

and $f''(x) = 4e^{2x}$

$\therefore$ $f''(0) = 4e^0$

$$= 4$$

Example 11

If $y = \cos 2x$, find a simple expression for $\left(\frac{dy}{dx}\right)^2 + \frac{1}{4}\left(\frac{d^2y}{dx^2}\right)^2$.

Solution

$$y = \cos 2x$$

$$\therefore \quad \frac{dy}{dx} = -2\sin 2x$$

$$\text{and} \quad \frac{d^2y}{dx^2} = -4\cos 2x$$

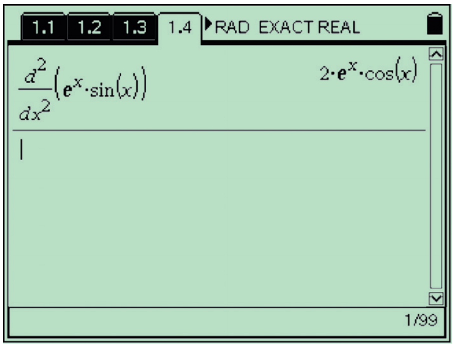
$$\begin{aligned}\therefore \quad \left(\frac{dy}{dx}\right)^2 + \frac{1}{4}\left(\frac{d^2y}{dx^2}\right)^2 &= (-2\sin 2x)^2 + \tfrac{1}{4}(-4\cos 2x)^2 \\ &= 4\sin^2 2x + \tfrac{1}{4}(16\cos^2 2x) \\ &= 4\sin^2 2x + 4\cos^2 2x \\ &= 4(\sin^2 2x + \cos^2 2x) \\ &= 4\end{aligned}$$

Using a TI-Nspire calculator

With a CAS calculator you can find the second derivative directly.

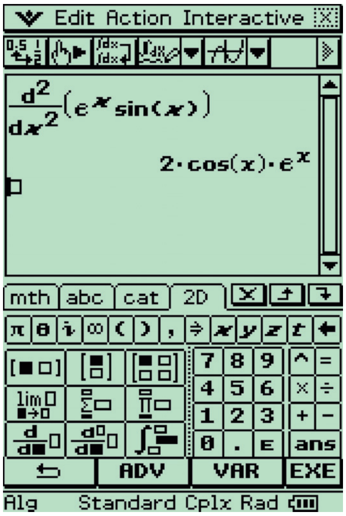
Press ctrl and then [calculus key] to obtain a differentiation template, $\frac{d^n}{d\square}(\square)$.

Complete as shown.



Using a Casio ClassPad calculator

Choose the **2D** menu and then **CALC** to see the keyboard with the differentiation template. Complete as shown.



Exercise 6D

1 Find the second derivative of each of the following:

a $2x + 5$ **b** x^8 **c** $\sqrt{x}$ **d** $(2x + 1)^4$

e $\sin x$ **f** $\cos x$ **g** e^x **h** $\log_e x$

i $\dfrac{1}{x+1}$ **j** $\tan x$

2 Find the second derivative of the following:

a $\sqrt{x^5}$ **b** $(x^2 + 3)^4$ **c** $\sin \dfrac{x}{2}$ **d** $3\cos(4x + 1)$

e $\frac{1}{2}e^{2x+1}$ **f** $\log_e(2x + 1)$ **g** $3\tan(x - 4)$ **h** $4\sin^{-1}(x)$

i $\tan^{-1}(x)$ **j** $2(1 - 3x)^5$

3 Find $f''(x)$ if $f(x)$ is equal to:

a $6e^{3-2x}$ **b** $-8e^{-0.5x^2}$ **c** $e^{\log_e x}$ **d** $\log_e(\sin x)$

e $3\sin^{-1}\left(\dfrac{x}{4}\right)$ **f** $\cos^{-1}(3x)$ **g** $2\tan^{-1}\left(\dfrac{2x}{3}\right)$ **h** $\dfrac{1}{\sqrt{1-x}}$

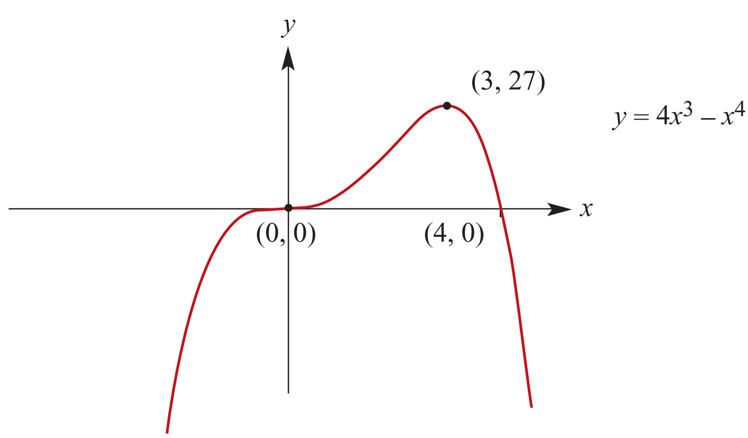
i $5\sin(3 - x)$ **j** $\tan(1 - 3x)$ **k** $\sec\left(\dfrac{x}{3}\right)$ **l** $\operatorname{cosec}\left(\dfrac{x}{4}\right)$

4 Find $f''(0)$ if $f(x)$ is equal to:

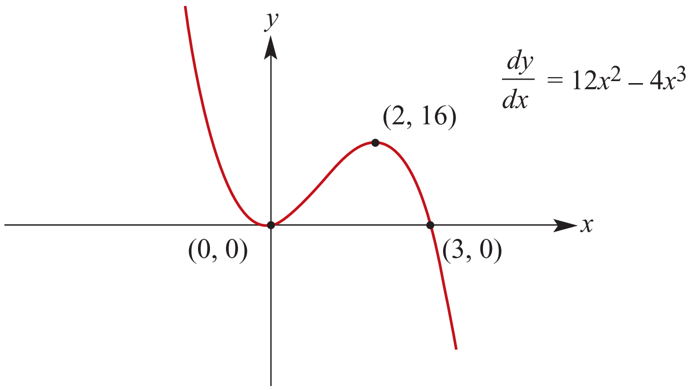
a $e^{\sin x}$ **b** $e^{-\frac{1}{2}x^2}$ **c** $\sqrt{1 - x^2}$ **d** $\tan^{-1}\left(\dfrac{1}{x-1}\right)$

6.5 Points of inflexion

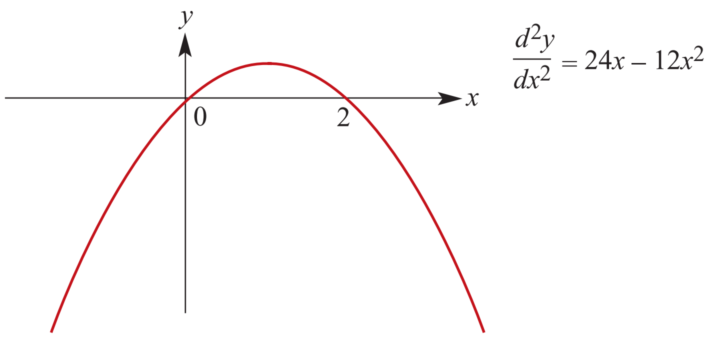
The graph of $y = 4x^3 - x^4$ is shown in the diagram below.



There is a local maximum at (3, 27) and a stationary point of inflexion at (0, 0). These have been determined by considering the derivative function $\frac{dy}{dx} = 12x^2 - 4x^3$. The graph of the derivative function is shown below.

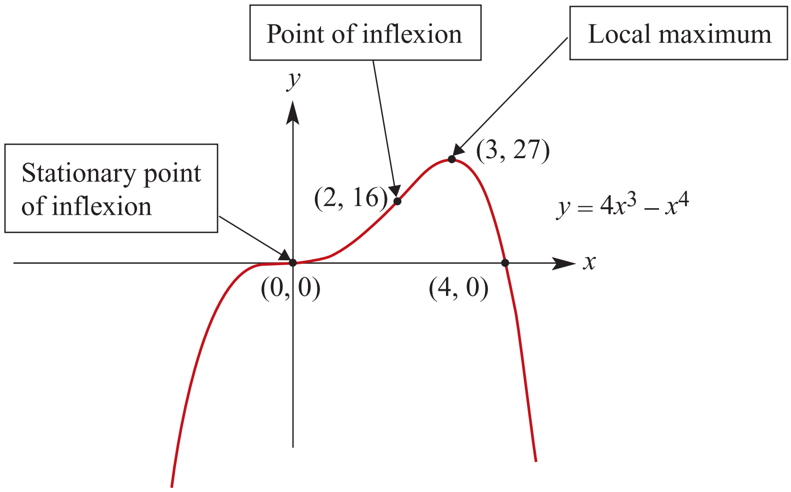


Note that the local maximum and the stationary point of inflexion of the original graph correspond to the x-axis intercepts of the second graph. That is, $\frac{dy}{dx} = 0$ for the values of x at the stationary points of the original graph. Also it can be seen that the gradient of the original graph is positive for $x < 0$ and $0 < x < 3$ and negative for $x > 3$. Further information can be obtained by considering the graph of the second derivative.



The graph of the second derivative reveals that at the point on the original graph where $x = 0$ and $x = 2$ there are important changes in the gradient. At the point where $x = 0$ the gradient changes from decreasing (positive) to increasing (positive). At the point where $x = 2$ the gradient changes from increasing (positive) to decreasing (positive). The point with coordinates (0, 0) is also a stationary point and is known as a **stationary point of inflexion**. The point where $x = 2$ is called a **point of inflexion**. In this case it corresponds to a local maximum of the derivative graph. The gradient increases on the interval (0, 2) and then decreases on the interval (2, 3).

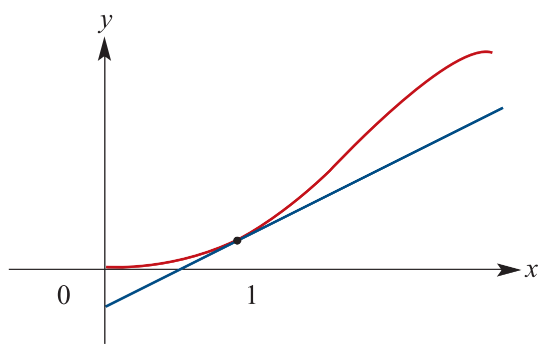
The point (2, 16) is the point of maximum gradient of the original graph for the interval (0, 3).



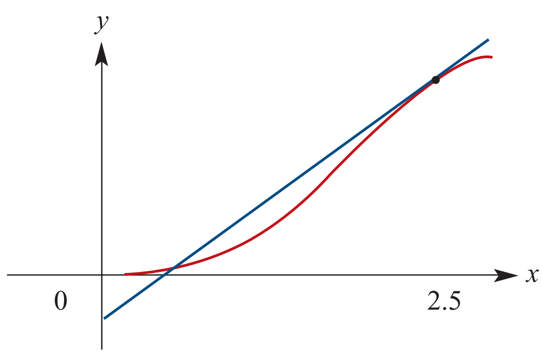
A closer look at the graph for the interval (0, 3) and, in particular, the behaviour of the tangents to the graphs in this interval will reveal more.

The tangents at $x = 1$, 2 and 2.5 have equations $y = 8x - 5$, $y = 16x - 16$ and $y = \frac{25}{2}x - \frac{125}{16}$ respectively. The graphs opposite illustrate the behaviour.

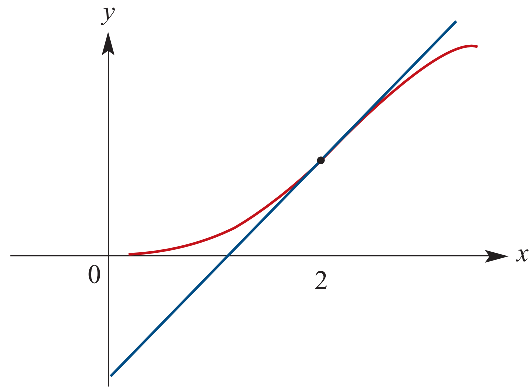
The first diagram shows a section of the graph of $y = 4x^3 - x^4$ and its tangent at $x = 1$. The tangent lies below the graph in the immediate neighbourhood of where $x = 1$. For the interval (0, 2) the gradient of the graph is positive and increasing.



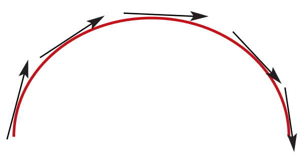
The second diagram shows a section of the graph of $y = 4x^3 - x^4$ and its tangent at $x = 2.5$. The tangent lies above the graph in the immediate neighbourhood of where $x = 2.5$. For the interval (2, 3) the gradient of the graph is positive and decreasing.



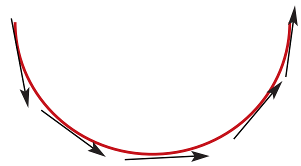
The third diagram shows a section of the graph of $y = 4x^3 - x^4$ and its tangent at $x = 2$. The tangent crosses the graph at the point (2, 16). At $x = 2$ the gradient of the graph is positive and changes from increasing to decreasing.



For the graph of the function $f: (0, 3) \to R, f(x) = 4x^3 - x^4$, for each point on the graph in the interval (0, 2) the tangent lies below the graph and the graph is said to be **concave up**. At the point (2, 16) the tangent passes through the graph and the point is said to be a **point of inflexion**. For each point on the graph in the interval (2, 3) the tangent lies above the graph and the graph is said to be **concave down**.



Concave down for an interval
The tangent is above the curve at each point and the derivative function is decreasing
i.e. $f''(x) < 0$



Concave up for an interval
The tangent is below the curve at each point and the derivative function is increasing
i.e. $f''(x) > 0$

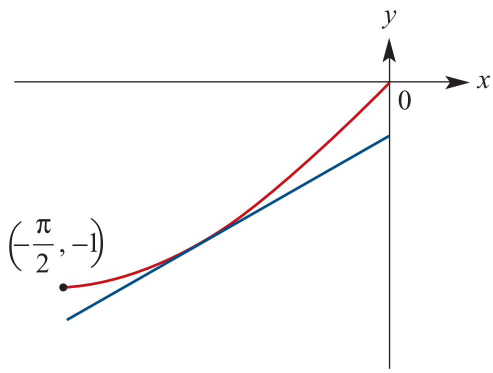
At a point of inflexion the tangent will pass through the curve. A point of inflexion is where the curve changes from concave down to concave up or concave up to concave down. At a point of inflexion of the curve of $y = f(x)$, where f' and f'' both exist, the second derivative has value zero, but the converse does not hold. For example, the second derivative of $y = x^4$ has value 0 when $x = 0$ but there is a local minimum at $x = 0$.

In summary, a point of inflexion of a graph occurs when $x = x_o$ if $f''(x_o) = 0$, and $f''(x_o + \in)$ and $f''(x_o - \in)$ have different signs.

For $f: \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \to R,\ f(x) = \sin x$, $f'(x) = \cos x$ and $f''(x) = -\sin x$. Hence $f'(x) = 0$ where $x = \frac{\pi}{2}$ and $x = -\frac{\pi}{2}$ and $f''(x) = 0$ where $x = 0$.

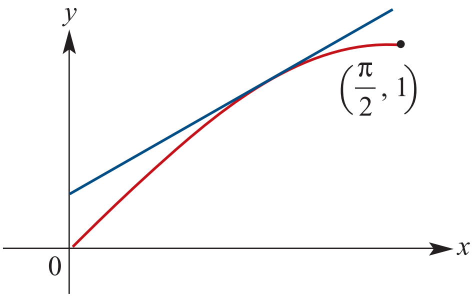
In the interval $\left(-\frac{\pi}{2}, 0\right)$, $f'(x) > 0$ and $f''(x) > 0$. (Graph shown opposite.)

Note the tangents to the curve lie below the curve and it is said to be concave up.

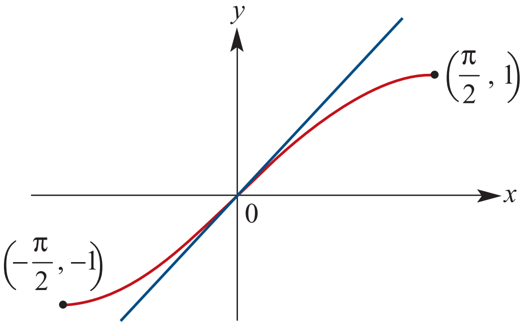


In the interval $\left(0, \frac{\pi}{2}\right)$, $f'(x) > 0$ and $f''(x) < 0$.

Note the tangents to the curve lie above the curve and it is said to be concave down.



Where $x = 0$ the tangent $y = x$, passes through the graph.
There is a point of inflexion at the origin.
This is also the point of maximum gradient in the interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.



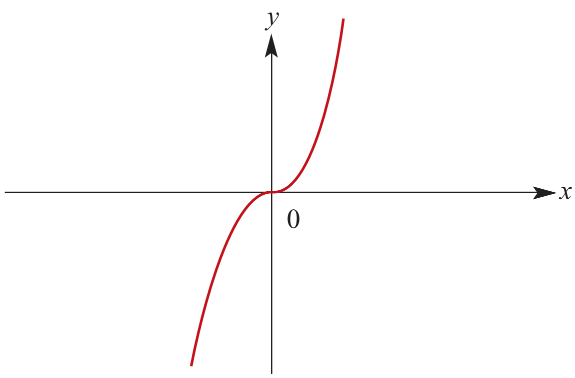
Example 12

For each of the following functions, find the coordinates of the points of inflexion of the curve and state the intervals where the curve is concave up.

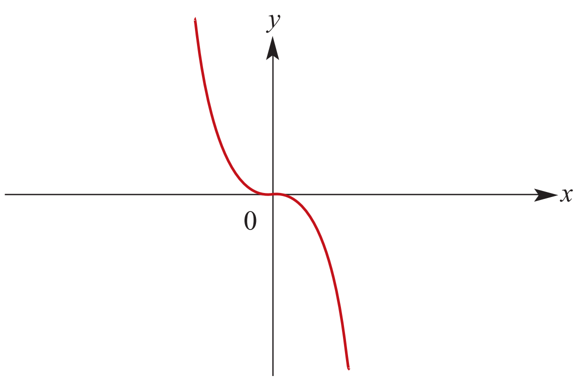
a $f(x) = x^3$ **b** $f(x) = -x^3$ **c** $f(x) = x^3 - 3x^2 + 1$ **d** $f(x) = \dfrac{1}{x^2 - 4}$

Solution

a There is a stationary point of inflexion at (0, 0).
The curve is concave up for the interval $(0, \infty)$.
The second derivative is positive on this interval.
The tangent at $x = 0$ is the line with equation $y = 0$.



b There is a stationary point of inflexion at (0, 0).
The curve is concave up for the interval $(-\infty, 0)$.
The second derivative is positive on this interval.
The tangent at $x = 0$ is the line with equation $y = 0$.



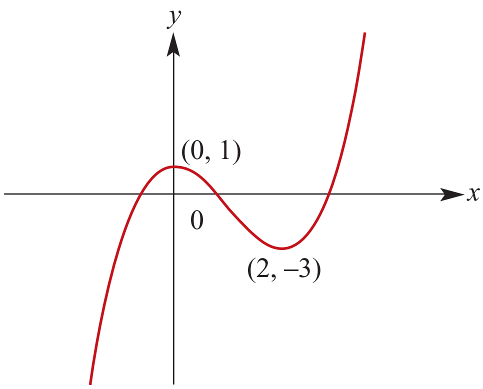
c $\frac{dy}{dx} = 3x^2 - 6x$ and $\frac{d^2y}{dx^2} = 6x - 6$

There is local maximum at the point with coordinates (0, 1) and a local minimum at the point with coordinates (2, −3).

There is a point of inflexion at the point with coordinates (1, −1).

The curve is concave up in the interval (1, ∞).

The second derivative is positive on this interval.



d $\frac{dy}{dx} = \frac{-2x}{(x^2-4)^2}$ and

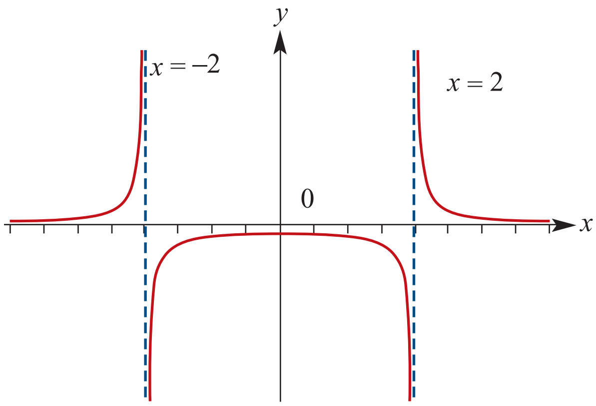
$\frac{d^2y}{dx^2} = \frac{2(3x^2+4)}{(x^2-4)^3}$

There is a local maximum at the point with coordinates $\left(0, \frac{-1}{4}\right)$.

$\frac{d^2y}{dx^2} > 0$ where $x^2 - 4 > 0$

i.e. where $x > 2$ or $x < -2$.

There is no point of inflexion as $\frac{d^2y}{dx^2} \neq 0$ for all x in the domain.



Sketching graphs

The sketching of graphs can be enhanced with the use of the second derivative.

Example 13

Sketch the graph of the function f: $R^+ \to R$, $f(x) = \frac{6}{x} - 6 + 3\log_e x$ showing all key features.

Solution

The derivative function has rule $f'(x) = \frac{3}{x} - \frac{6}{x^2} = \frac{3x-6}{x^2}$ and the second derivative function has rule $f''(x) = \frac{12}{x^3} - \frac{3}{x^2} = \frac{12-3x}{x^3}$.

For stationary points

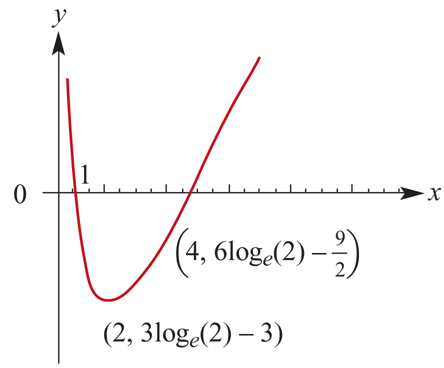
$f'(x) = 0$ implies $x = 2$ and $f'(1) = -3 < 0$ and $f'(3) = \frac{1}{3} > 0$. Hence there is a local minimum at the point with coordinates $(2, 3\log_e(2) - 3)$.

For points of inflexion

$f''(x) = 0$ implies $x = 4$. Also note that $f''(3) = \frac{1}{9}$ and $f''(5) = -\frac{3}{125}$.

Hence there is a point of inflexion at $\left(4, 6\log_e(2) - \frac{9}{2}\right)$. In the interval $(2, 4)$, $f''(x) > 0$, i.e. gradient is increasing and in the interval $(4, \infty)$, $f''(x) < 0$, i.e. gradient is decreasing.

The point of inflexion is the point of maximum gradient in the interval $(2, \infty)$.

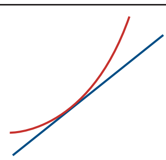
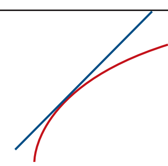
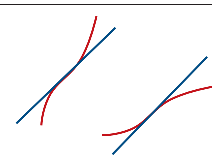
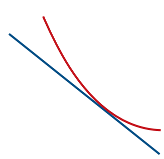
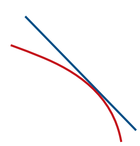
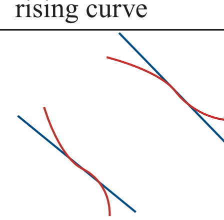
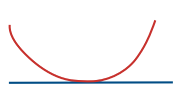
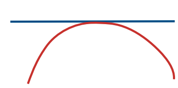
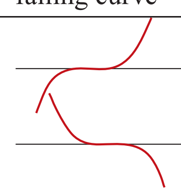


The x-axis intercepts of the graph occur at $x = 1$ and approximately 4.92.

Use of second derivative to determine local maxima and minima

It can be seen from the above that for the graph of $y = f(x)$:

- If $f'(a) = 0$ and $f''(a) > 0$ then the point $(a, f(a))$ is a local minimum as the curve is concave up.
- If $f'(a) = 0$ and $f''(a) < 0$ then the point $(a, f(a))$ is a local maximum as the curve is concave down.
- If $f''(a) = 0$ then further investigation is necessary.

	$\frac{d^2y}{dx^2} > 0$	$\frac{d^2y}{dx^2} < 0$	$\frac{d^2y}{dx^2} = 0$ and point of inflexion
$\frac{dy}{dx} > 0$	 Curve rising and concave upwards	 Curve rising and concave downwards	 Point of inflexion on rising curve
$\frac{dy}{dx} < 0$	 Curve falling and concave upwards	 Curve falling and concave downwards	 Point of inflexion on falling curve
$\frac{dy}{dx} = 0$	 Local minimum	 Local maximum	 Stationary points of inflexion

Example 14

For the function with rule $f(x) = e^{x^2}$

a Find:

i $f'(x)$ **ii** $f''(x)$

b For $x \in [-1, 1]$ sketch the graphs of $y = f(x)$, $y = f'(x)$ and $y = f''(x)$ on the one set of axes. (Use a calculator to help.)

c Solve the equation $f'(x) = 0$.

d Show that $f''(x) > 0$ for all x.

e State the intervals for which:

i $f(x)$ is increasing **ii** $f(x)$ is decreasing

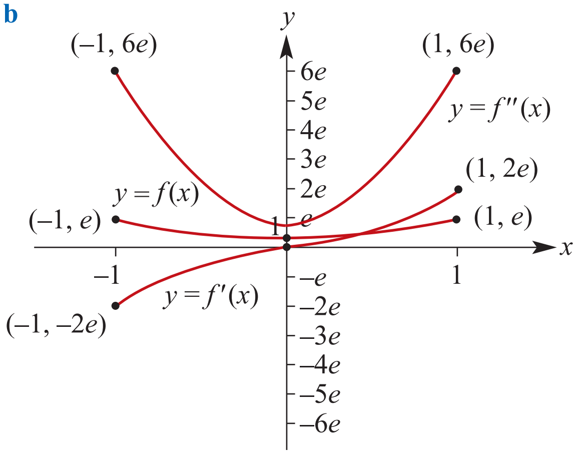
f Compare with the graphs of $y = f(x)$, $y = f'(x)$ and $y = f''(x)$ where $f(x) = x^2 + 1$.

Solution

a **i** For $f(x) = e^{x^2}$ the chain rule for differentiation gives $f'(x) = 2xe^{x^2}$

ii Apply the product rule to obtain $f''(x) = 2e^{x^2} + 4x^2e^{x^2}$

b



c $f'(x) = 0$ implies $2xe^{x^2} = 0$. Thus $x = 0$.

d $f''(x) = e^{x^2}(2 + 4x^2) > 0$ for all x as $e^{x^2} > 0$ and $2 + 4x^2 > 0$ for all x.

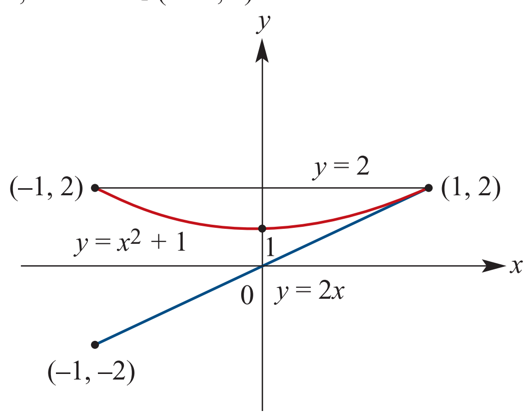
e **i** $f(x)$ is increasing when $f'(x) > 0$, i.e. for $x \in (0, \infty)$.

ii $f(x)$ is decreasing when $f'(x) < 0$, i.e. for $x \in (-\infty, 0)$.

f The graphs of $y = f(x)$, $y = f'(x)$ and $y = f''(x)$ appear as shown.

$$f(x) = x^2 + 1$$
$$f'(x) = 2x$$
$$f''(x) = 2$$

Similar scales to **b** have been used.



Similar features

- $f'(x) > 0$ for $x > 0$
- $f'(x) < 0$ for $x < 0$
- The graphs of $y = x^2 + 1$ and $y = e^{x^2}$ are symmetric about the y axis.

Differences

The second derivatives reveal that the gradient of $y = e^{x^2}$ is increasing rapidly for $x > 0$ while the gradient of $y = x^2$ is increasing at a constant rate.

Example 15

On the one set of axes sketch the graphs of $y = f(x)$, $y = f'(x)$ and $y = f''(x)$ for $x \in [0, 10]$ where $f(x) = x^2(10 - x)$. Find the value of x for which the gradient of $y = f(x)$ is a maximum and find this maximum gradient.

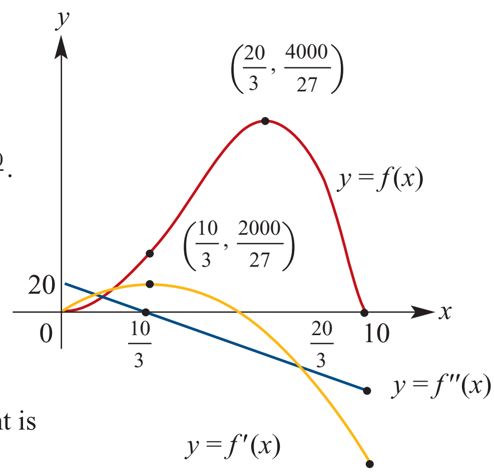
Solution

$f(x) = x^2(10 - x)$ implies
$f(x) = 10x^2 - x^3$.
Therefore $f'(x) = 20x - 3x^2$ and
$f''(x) = 20 - 6x$.
$f'(x) = 0$ implies $x(20 - 3x) = 0$. There are stationary points at $x = 0$ and $x = \frac{20}{3}$.

The coordinates of the stationary points are $(0, 0)$ and $\left(\frac{20}{3}, \frac{4000}{27}\right)$.

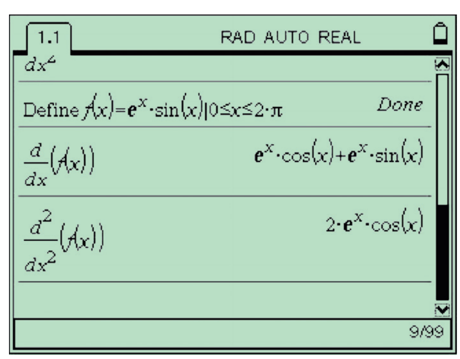
$f''(x) = 0$ implies $x = \dfrac{10}{3}$. Note that the maximum gradient occurs at a point of inflexion.

Note that the rate of change of gradient is decreasing throughout the interval.
The maximum gradient of $y = f(x)$ occurs at the point $\left(\frac{10}{3}, \frac{2000}{27}\right)$. The maximum gradient is $\frac{100}{3}$.

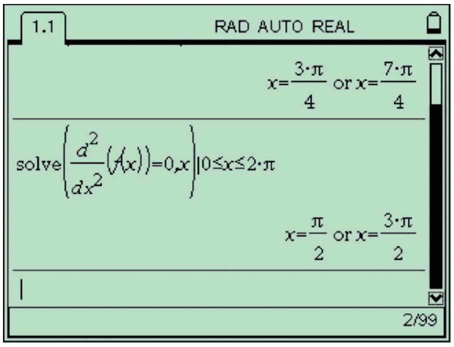


Using a TI-Nspire calculator

Define $f(x) = e^x \sin(x) \mid 0 \le x \le 2\pi$.
Press ctrl and then [template key] to obtain the differentiation template, $\frac{d}{d\square}(\square)$ and complete as shown.
Repeat to find the second derivative with the template. Press ctrl and then [template key] to obtain the differentiation template, $\frac{d^n}{d\square}(\square)$.



Find the turning points by solving the equation $\frac{d}{dx}(f(x)) = 0$ for x and $0 \le x \le 2\pi$.



Find the points of inflexion by solving the equation $\frac{d^2}{dx^2}(f(x)) = 0$ for x and $0 \le x \le 2\pi$.

Substitute to find the second coordinates.

Stationary points:

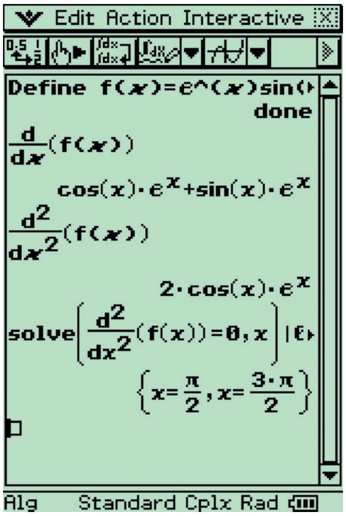
$$\left(\frac{7\pi}{4}, \frac{1}{2}e^{-\frac{7\pi}{4}}\sqrt{2}\right), \left(\frac{3\pi}{4}, \frac{1}{2}e^{-\frac{3\pi}{4}}\sqrt{2}\right).$$

Points of inflexion:

$$\left(\frac{3\pi}{2}, e^{-\frac{3\pi}{2}}\right), \left(\frac{\pi}{2}, e^{-\frac{\pi}{2}}\right).$$

Using a Casio ClassPad calculator

Choose the **2D** menu and then **CALC** to see the keyboard with the differentiation template.



Define $f(x) = e^x \sin x$.

Find the turning points by solving the equation.

Find $\frac{d}{dx}(f(x))$ and $\frac{d^2}{dx^2}(f(x))$.

$\frac{d}{dx}(f(x)) = 0$ for x and $0 \le x \le 2\pi$.

Find the points of inflexion by solving the equation $\frac{d^2}{dx^2}(f(x)) = 0$ for x and $0 \le x \le 2\pi$.

Substitute to find the second coordinates.

Stationary points:

$$\left(\frac{7\pi}{4}, \frac{1}{2}e^{-\frac{7\pi}{4}}\sqrt{2}\right), \left(\frac{3\pi}{4}, \frac{1}{2}e^{-\frac{3\pi}{4}}\sqrt{2}\right).$$

Points of inflexion:

$$\left(\frac{3\pi}{2}, e^{-\frac{3\pi}{2}}\right), \left(\frac{\pi}{2}, e^{-\frac{\pi}{2}}\right).$$

Exercise 6E

1 Sketch a small portion of a continuous curve around a point $x = a$ having the property:

a $\dfrac{dy}{dx} > 0$ when $x = a$ and $\dfrac{d^2y}{dx^2} > 0$ when $x = a$

b $\dfrac{dy}{dx} < 0$ when $x = a$ and $\dfrac{d^2y}{dx^2} < 0$ when $x = a$

c $\dfrac{dy}{dx} > 0$ when $x = a$ and $\dfrac{d^2y}{dx^2} < 0$ when $x = a$

d $\dfrac{dy}{dx} < 0$ when $x = a$ and $\dfrac{d^2y}{dx^2} > 0$ when $x = a$

2 Let $f: R \to R$, $f(x) = 2x^3 + 6x^2 - 12$.

a Find: i $f'(x)$ ii $f''(x)$

b Use $f''(x)$ to find the coordinate of the point on the graph of $f(x) = 2x^3 + 6x^2 - 12$ where the gradient is a minimum (the point of inflexion).

3 Repeat question **2** for each of the following functions:

a $f: [0, 2\pi] \to R$, $f(x) = \sin x$ b $f: R \to R$, $f(x) = xe^x$

4 The graph of $y = f(x)$ has a local minimum at $x = a$ and no other stationary point 'close' to a.

a For a small value h, $h > 0$, what can be said of the values of:

i $f'(a - h)$? ii $f'(a)$? iii $f'(a + h)$?

b What can be said about the gradient of $y = f'(x)$ for the interval $x \in [a - h, a + h]$?

c What can be said of the value of $f''(a)$?

d Verify your observation by calculating the value of $f''(0)$ for each of the following functions:

i $f(x) = x^2$ ii $f(x) = -\cos x$ iii $f(x) = x^4$

e Can $f''(a)$ ever be less than zero if $f(x)$ has a local minimum at $x = a$?

5 Investigate the condition of $f''(a)$ if $f(x)$ has a local maximum at $x = a$.

6 For $f: [0, 20] \to R$, $f(x) = \dfrac{x^2}{10}(20 - x)$:

a Sketch the graphs of $y = f(x)$, $y = f'(x)$ and $y = f''(x)$ on the one set of axes for $x \in [0, 20]$.

b Find the value of x for which the gradient is a maximum and indicate the corresponding point on the graph of $y = f(x)$.

c Describe the rate of change of gradient for the function.

d At which point in the interval is the graph 'steepest'?

7 For f: $[0, 10] \rightarrow R$, $f(x) = x(10 - x)e^x$:

a Find $f'(x)$ and $f''(x)$.

b Sketch the graphs of $y = f(x)$ and $y = f''(x)$ on the one set of axes for $x \in [0, 10]$.

c Find the value of x for which the gradient of the graph of $y = f(x)$ is a maximum and indicate this point on the graph of $y = f(x)$.

8 For the graph of the function with rule $y = \dfrac{1}{1 + x + x^2}$:

a Find the coordinates of the points of inflexion.

b Find the coordinates of the point of intersection of the tangents at the points of inflexion.

9 Find the coordinates of the points of inflexion of the graph of $y = x - \sin x$ for $x \in [0, 4\pi]$.

10 For each of the following functions find the values of x for which the graph of the function has a point of inflexion:

a $y = \sin x$ **b** $y = \tan x$ **c** $y = \sin^{-1}(x)$ **d** $y = \sin(2x)$

11 Show that the parabola with equation $y = ax^2 + bx + c$ has no points of inflexion.

12 For the curve with equation $y = 2x^3 - 9x^2 + 12x + 8$ find the values of x for which:

a $\dfrac{dy}{dx} < 0$ and $\dfrac{d^2y}{dx^2} > 0$ **b** $\dfrac{dy}{dx} < 0$ and $\dfrac{d^2y}{dx^2} < 0$

13 For the following functions, determine the coordinates of any points of inflexion and the gradient of the graph at these points.

a $y = x^3 - 6x$ **b** $y = x^4 - 6x^2 + 4$ **c** $y = 3 - 10x^3 + 10x^4 - 3x^5$

d $y = (x^2 - 1)(x^2 + 1)$ **e** $y = \dfrac{x + 1}{x - 1}$ **f** $y = x\sqrt{x + 1}$

g $y = \dfrac{2x}{x^2 + 1}$ **h** $y = \sin^{-1} x$ **i** $y = \dfrac{x - 2}{(x + 2)^2}$

14 Determine the values of x for which the graph of $y = e^{-x} \sin x$ has:

a stationary points **b** points of inflexion

15 Given that $f(x) = x^3 + bx^2 + cx$, and $b^2 > 3c$, prove that:

a the graph of f has two stationary points

b the graph of f has one point of inflexion

c the point of inflexion is the midpoint of the interval joining the stationary points

16 For the function with rule $f(x) = 2x^2 \log_e(x)$ find:

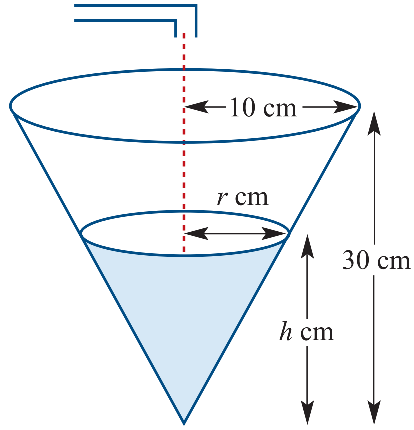
a $f'(x)$

b $f''(x)$

c the stationary points and the points of inflexion of the graph of $y = f(x)$

6.6 Related rates

Consider the situation of a right circular cone being filled from a tap.



At time t seconds:

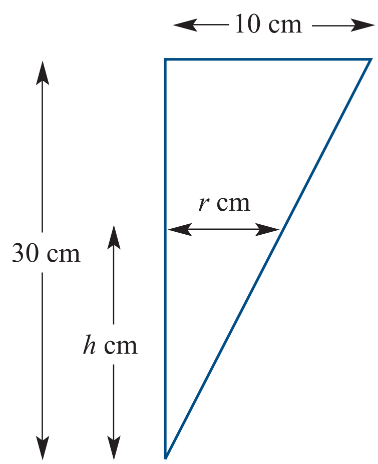
- there are V cm^3 of water in the cone
- the height of the water in the cone is h cm
- the radius of the circular water surface is r cm

V, h and r change as the water flows in.

- $\frac{dV}{dt}$ is the rate of change of volume with respect to time
- $\frac{dh}{dt}$ is the rate of change of height with respect to time
- $\frac{dr}{dt}$ is the rate of change of radius with respect to time

It is clear that these rates are related to each other. The chain rule is used to establish these relationships.

For example, if the height of the cone is 30 cm and the radius of the cone is 10 cm, similar triangles yield



$$\frac{r}{h} = \frac{10}{30}$$

and $$h = 3r$$

Then the chain rule is used

$$\frac{dh}{dt} = \frac{dh}{dr}\frac{dr}{dt} = 3.\frac{dr}{dt}$$

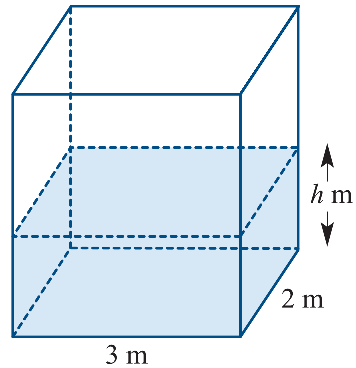
The volume of a cone is given in general by $V = \frac{1}{3}\pi r^2 h$ and in this case $V = \pi r^3$ since $h = 3r$. Therefore by using the chain rule again

$$\frac{dV}{dt} = \frac{dV}{dr}\frac{dr}{dt} = 3\pi r^2.\frac{dr}{dt}$$

The relationships between the rates have been established.

Example 16

A rectangular prism is being filled at a rate of 0.00042 m^3/s. Find the rate at which the height is increasing.



Solution

Let t be the time in seconds after the prism begins to fill. Let V be the volume of water at time t and h m be the height of the water at time t.

It is given that $\frac{dV}{dt} = 0.00042 \text{ m}^3/\text{s}$

Also $V = 6h$

The rate at which h is increasing is given by $\frac{dh}{dt}$

From the chain rule, $\frac{dh}{dt} = \frac{dh}{dV}\frac{dV}{dt}$

Since $V = 6h$, $\frac{dV}{dh} = 6$ and $\frac{dh}{dV} = \frac{1}{6}$

$$\text{Thus } \frac{dh}{dt} = \frac{1}{6} \times 0.00042$$
$$= 0.00007 \text{ m/s}$$

i.e. the height is increasing at a rate of 0.00007 m/s.

Example 17

As Steven's ice block melts, it forms a circular puddle on the floor. The radius of the puddle increases at a rate of 3 cm/min. When its radius is 2 cm find the rate at which the area of the puddle is increasing.

Solution

The area, A, of a circle is given by $A = \pi r^2$, where r is the radius of the circle.

The rate of increase of the radius $= \frac{dr}{dt} = 3$ cm/min

The rate of increase of the area, $\frac{dA}{dt}$, is found by using the chain rule,

$$\frac{dA}{dt} = \frac{dA}{dr}\frac{dr}{dt}$$
$$= 2\pi r \times 3$$
$$= 6\pi r$$

When $r = 2$, $\frac{dA}{dt} = 12\pi$

$\therefore$ the area of the puddle is increasing at 12π cm^2/min.

Example 18

A metal cube is being heated so that the side length is increasing at the rate of 0.02 cm per hour. Calculate the rate at which the volume is increasing when the side is 5 cm.

Solution

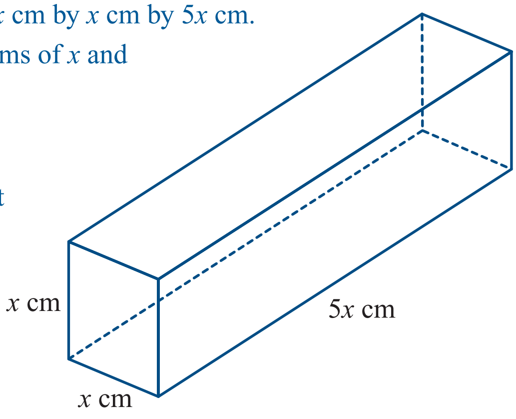
Let x be the length of a side of the cube.
Then the volume $V = x^3$
The rate of increase of x with respect to t is given by $\frac{dx}{dt} = 0.02$ cm/hour
The rate of increase of V with respect to t, $\frac{dV}{dt}$, is calculated from the chain rule

$$\begin{aligned}\frac{dV}{dt} &= \frac{dV}{dx}\frac{dx}{dt}\\ &= 3x^2 \times 0.02\\ &= 0.06x^2\end{aligned}$$

When $x = 5$, the volume of the cube is increasing at a rate of 1.5 cm^3/hour.

Example 19

The diagram shows a rectangular block of ice x cm by x cm by $5x$ cm.



a Express the total surface area, A cm^2, in terms of x and write down an expression for $\frac{dA}{dx}$.

b Given that the ice is melting so that the total surface area is decreasing at a constant rate of 4 cm^2/s, calculate the rate of decrease of x when $x = 2$.

Solution

a
$$\begin{aligned}A &= 4 \times 5x^2 + 2 \times x^2\\ &= 22x^2\\ \frac{dA}{dx} &= 44x\end{aligned}$$

b The surface area is decreasing and so $\frac{dA}{dt} = -4$

By the chain rule,
$$\begin{aligned}\frac{dx}{dt} &= \frac{dx}{dA}\frac{dA}{dt}\\ &= \frac{1}{44x} \times -4\\ &= \frac{-1}{11x}\end{aligned}$$

When $x = 2$, $\dfrac{dx}{dt} = \dfrac{-1}{22}$

The rate of change of the length of the edges are $-\frac{1}{22}$ cm/s, $-\frac{1}{22}$ cm/s and $-\frac{5}{22}$ cm/s.

The negative sign indicates that the lengths are decreasing.

Example 20

A curve has parametric equations $x = 2t - \log_e(2t)$ and $y = t^2 - \log_e(t^2)$.

a Find $\dfrac{dy}{dt}$ and $\dfrac{dx}{dt}$.

b Find $\dfrac{dy}{dx}$.

Solution

a $x = 2t - \log_e(2t)$ and $\dfrac{dx}{dt} = 2 - \dfrac{1}{t} = \dfrac{2t-1}{t}$

$y = t^2 - \log_e(t^2)$ and $\dfrac{dy}{dt} = 2t - \dfrac{2}{t} = \dfrac{2t^2-2}{t}$

b Therefore $\dfrac{dy}{dx} = \dfrac{dy}{dt} \times \dfrac{dt}{dx} = \dfrac{2t^2-2}{t} \times \dfrac{t}{2t-1}$

$$= \frac{2t^2-2}{2t-1}$$

Exercise 6F

1 The radius of a spherical balloon is 2.5 m and its volume is increasing at a rate of 0.1 m^3/min. At what rate are:

a the radius and

b the surface area increasing?

2 When a wine glass is filled to a depth of x cm it contains V cm^3 of wine where $V = 4x^{\frac{3}{2}}$. If the depth is 9 cm and wine is being poured into the glass at 10 cm^3/s, at what rate is the depth changing?

3 Variables x and y are connected by the equation $y = 2x^2 + 5x + 2$. Given that x is increasing at the rate of 3 units/s, find the rate of increase of y with respect to time when $x = 2$.

4 If a hemispherical bowl of radius 6 cm contains water to a depth of x cm, the volume, V cm^3, of the water is given by $V = \frac{1}{3}\pi x^2(18 - x)$. Water is poured into the bowl at a rate of 3 cm^3/s. Find the rate at which the water level is rising when the depth is 2 cm.

5 Variables p and v are linked by the equation $pv = 1500$. Given that p is increasing at the rate of 2 units per minute, find the rate of decrease of v at the instant when $p = 60$.

6 A circular metal disc is being heated so that the radius is increasing at the rate of 0.01 cm per hour. Find the rate at which the area is increasing when the radius is 4 cm.

7 The area of a circle is increasing at the rate of 4 cm^2 per second. At what rate is the circumference increasing at the instant when the radius is 8 cm?

8 A curve has parametric equations $x = \frac{1}{1+t^2}$ and $y = \frac{t}{1+t^2}$.

a Find $\frac{dy}{dt}$ and $\frac{dx}{dt}$

b Find $\frac{dy}{dx}$

9 A curve has parametric equations $x = 2t + \sin 2t$ and $y = \cos 2t$. Find $\frac{dy}{dx}$.

10 A curve has parametric equations $x = t - \cos t$ and $y = \sin t$. Find the equation to the tangent to the curve when $t = \frac{\pi}{6}$.

11 A point moves along the curve $y = x^2$ in such a way that its velocity parallel to the x axis is a constant 2 cm/s (i.e. $\frac{dx}{dt} = 2$). Find its velocity parallel to the y axis (i.e. $\frac{dy}{dt}$) when:

a $x = 3$ **b** $y = 16$

12 Variables x and y are related by the equation $y = \frac{2x - 6}{x}$. Given that $x = f(t)$ and $y = g(t)$, where f and g are functions of time, find $f'(t)$ when $y = 1$ given that $g'(t) = 0.4$.

13 A particle moves along the curve $y = 10\cos^{-1}\left(\frac{x-5}{5}\right)$ in such a way that its velocity parallel to the x axis is a constant 3 cm/s. Find its velocity parallel to the y axis when:

a $x = 6$ **b** $y = \frac{10\pi}{3}$

14 The radius r cm of a sphere is increasing at a constant rate of 2 cm/s. Find, in terms of π, the rate at which the volume is increasing at the instant when the volume is 36π cm^3.

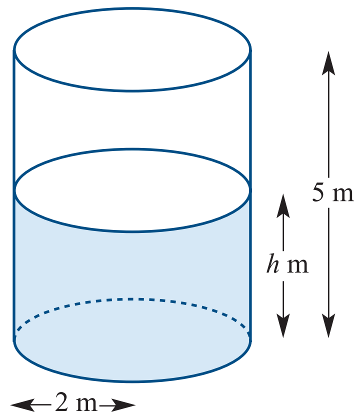
15 Liquid is poured into a container at a rate of 12 cm^3/s. The volume of liquid in the container is V cm^3 where $V = \frac{1}{2}(h^2 + 4h)$ and h is the height of the liquid in the container. Find, when $V = 16$:

a the value of h **b** the rate at which h is increasing

16 The area of an ink blot, which is always circular in shape, is increasing at a rate of 3.5 cm^2/s. Find the rate of increase of the radius when the radius is 3 cm.

17 A tank in the shape of a prism has constant cross-sectional area A cm^2. The amount of water in the tank at time t seconds is V cm^3 and the height of the water is h cm. Find the relationship between $\frac{dV}{dt}$ and $\frac{dh}{dt}$.

18 A cylindrical tank 5 m high with base radius 2 m is initially full of water. Water flows out through a hole at the bottom of the tank at the rate of $\sqrt{h}$ m^3/hour, where h metres is the depth of the water remaining in the tank after t hours.



Find:

a $\dfrac{dh}{dt}$

b **i** $\dfrac{dV}{dt}$ when $V = 10\pi$ m^3

ii $\dfrac{dh}{dt}$ when $V = 10\pi$ m^3

19 For the curve defined by the parametric equations $x = 2\cos t$ and $y = \sin t$, find the equation of the tangent to the curve at the point:

a $\left(\sqrt{2}, \dfrac{\sqrt{2}}{2}\right)$

b $(2\cos t, \sin t)$, where t is any real number

20 For the curve defined by the parametric equations $x = 2\sec\theta$ and $y = \tan\theta$, find the equation of:

a the tangent at the point where $\theta = \dfrac{\pi}{4}$

b the normal at the point where $\theta = \dfrac{\pi}{4}$

c the tangent at the point $(2\sec\theta, \tan\theta)$

6.7 Graphs of some rational functions

Rational functions have a rule of the form:

$$f(x) = \frac{P(x)}{Q(x)}$$

where $P(x)$ and $Q(x)$ are polynomials. There is a huge variety of different types of curves in this particular family of functions. The Specialist Mathematics course considers only those that can be constructed from 'simple' polynomials. Methods for sketching graphs of rational functions include:

- adding the y coordinates (ordinates) of two simple graphs
- taking the reciprocals of the y coordinates (ordinates) of the simple graph

The graph of the rectangular hyperbola $y = \dfrac{1}{x}$ is known from previous work. This is an example of a rational function.

In this case $P(x) = 1$ and $Q(x) = x$

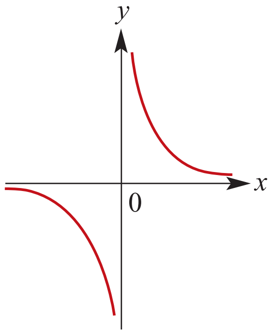
The y axis is the **vertical asymptote** and the x axis is the **horizontal asymptote** for the graph.

As $x \to 0$ from the positive side $y \to \infty$ and as $x \to 0$ from the negative side $y \to -\infty$

As $x \to \infty, y \to 0^+$ and as $x \to -\infty, y \to 0^-$

The maximal domain of $y = \frac{1}{x}$ is $R\backslash\{0\}$

A **non-vertical asymptote** is a line or curve which the function approaches as $x \to \pm\infty$



Adding ordinates

In this process, the key points to look for are:

- when both graphs have the same ordinate, then the y coordinate of the resultant graph will be double this
- when both graphs have the opposite ordinate, then the resultant graph y coordinate will be zero (an x-axis intercept)
- when one of the two ordinates is zero the resulting ordinate is equal to the other ordinate.

Further examples of rational functions considered in this course are:

$$f_1(x) = \frac{x^3 + x - 2}{x} \qquad g_1(x) = \frac{x^2 - 10x + 9}{x} \qquad h_1(x) = \frac{x^4 + 2}{x^2}$$

The following are also rational functions, but are not in a form used for the definition of rational functions:

$$f(x) = 1 + \frac{1}{x} \qquad g(x) = x - \frac{1}{x^2} \qquad h(x) = x - 1 + \frac{1}{x + 2}$$

These can be rewritten in the form $\frac{P(x)}{Q(x)}$ as shown here:

$$f(x) = 1 + \frac{1}{x} = \frac{x}{x} + \frac{1}{x} = \frac{x + 1}{x}$$

$$g(x) = x - \frac{1}{x^2} = \frac{x^3}{x^2} - \frac{1}{x^2} = \frac{x^3 - 1}{x^2}$$

$$h(x) = x - 1 + \frac{1}{x + 2} = \frac{(x - 1)(x + 2)}{x + 2} + \frac{1}{x + 2} = \frac{x^2 + x - 1}{x + 2}$$

For graphing it is also useful to write functions in the alternative form, i.e. with a division performed if possible.

For example:

$$\frac{8x^2 - 3x + 2}{x} = \frac{8x^2}{x} - \frac{3x}{x} + \frac{2}{x} = 8x - 3 + \frac{2}{x}$$

$$\frac{8x^4 + 1}{x^2} = \frac{8x^4}{x^2} + \frac{1}{x^2} = 8x^2 + \frac{1}{x^2}$$

$$\frac{4x - 3}{x + 1} = 4 - \frac{7}{x + 1}$$

Example 21

Sketch the graph of $f: R\backslash\{0\} \to R,\ f(x) = \dfrac{x^2+1}{x}$

Solution

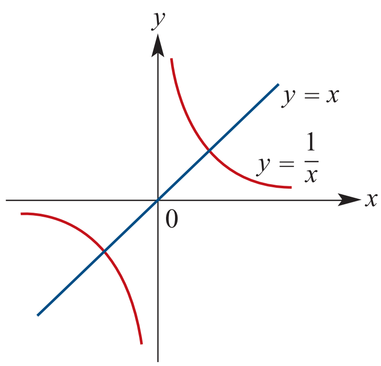
Vertical asymptote has equation $x = 0$, i.e. the y axis.

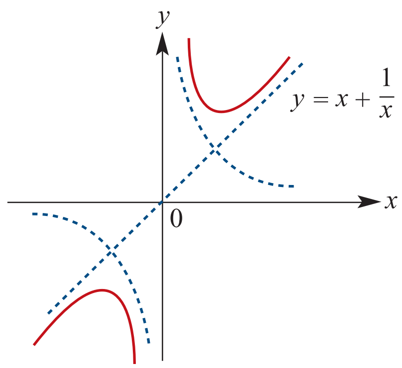
Dividing through gives

$$\frac{x^2+1}{x} = \frac{x^2}{x} + \frac{1}{x} = x + \frac{1}{x}$$

As $x \to \pm\infty$, $\dfrac{1}{x} \to 0$. Therefore the graph of $y = f(x)$ will approach the graph of $y = x$ as $x \to \pm\infty$. The equation of the **non-vertical asymptote** is $y = x$.

The graph of $y = f(x)$ is obtained by the addition of the y coordinates of the functions $y = x$ and $y = \dfrac{1}{x}$.





Other important features of a sketch graph are:

- the axes intercepts
- the turning points

The first is determined by evaluating $f(0)$ to find the y-axis intercepts and solving the equation $f(x) = 0$ to find the x-axis intercepts. In this example the domain is $R\backslash\{0\}$ and there are no solutions for the equation $\dfrac{x^2+1}{x} = 0$.

Calculus is used to determine the turning points.

Consider $f(x) = x + \dfrac{1}{x}$

Then $f'(x) = 1 - \dfrac{1}{x^2}$

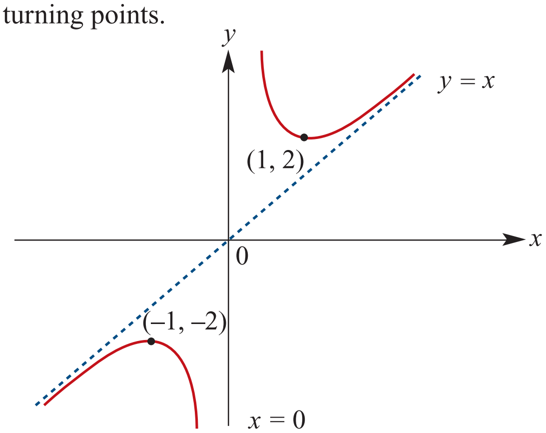
and $f'(x) = 0$ implies

$x^2 = 1$

i.e. $x = \pm 1$

Also $f(1) = 2$ and $f(-1) = -2$

The final sketch of the graph of $y = f(x)$ is as shown



Example 22

Sketch the graph of $f: R\backslash\{0\} \to R, \quad f(x) = \dfrac{x^4+2}{x^2}$

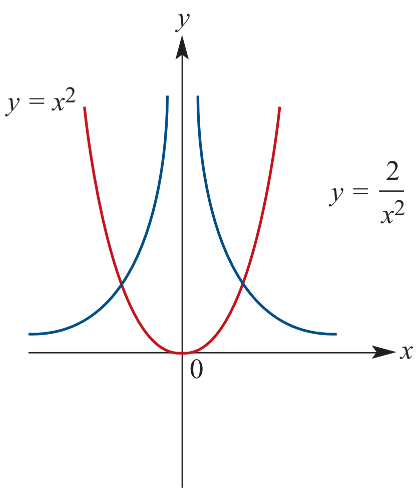
Solution

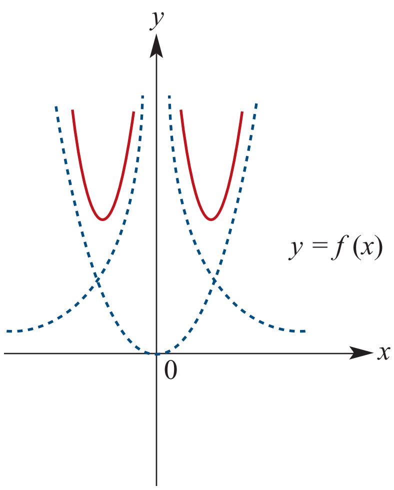
Vertical asymptote has equation $x = 0$

Dividing through gives $f(x) = x^2 + \dfrac{2}{x^2}$

The **non-vertical asymptote** is $y = x^2$

By addition of ordinates $y = f(x)$ can be sketched.





There are no **axes intercepts**.

To determine the coordinates of the **turning points**, $f(x)$ is differentiated.

$$f(x) = x^2 + 2x^{-2}$$

$$\therefore \quad f'(x) = 2x - 4x^{-3}$$

when $f'(x) = 0$, $2x - \dfrac{4}{x^3} = 0$

$\therefore \quad 2x^4 - 4 = 0$

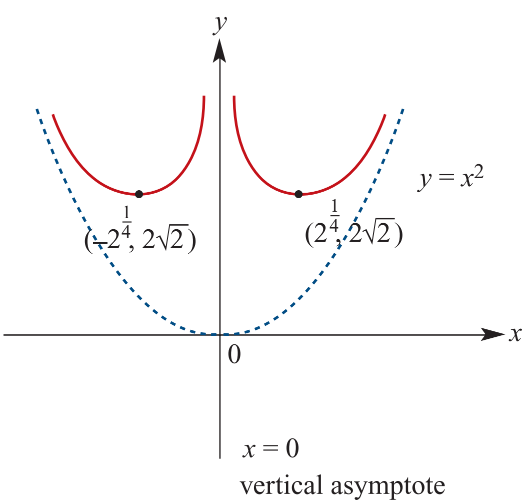
which implies $x = \pm 2^{\frac{1}{4}}$

$f(2^{\frac{1}{4}}) = \dfrac{2+2}{\sqrt{2}} = \dfrac{4}{\sqrt{2}} = 2\sqrt{2}$

and $f(-2^{\frac{1}{4}}) = 2\sqrt{2}$

The turning points have coordinates $(2^{\frac{1}{4}}, 2\sqrt{2})$ and $(-2^{\frac{1}{4}}, 2\sqrt{2})$

The final sketch is as shown.



Example 23

Sketch the graph of $y = \dfrac{x^3 + 2}{x}, x \neq 0$

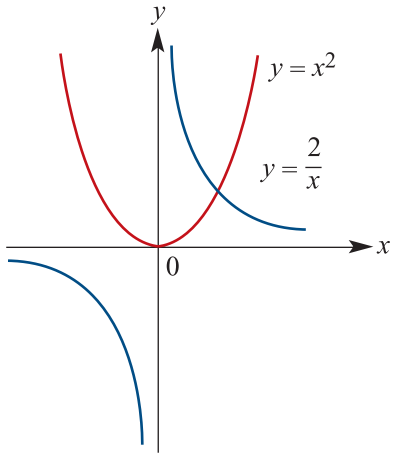
Solution

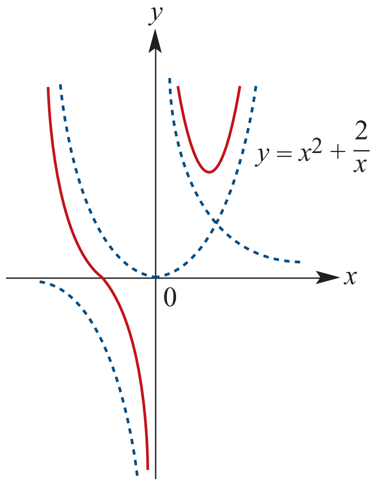
The **vertical asymptote** has equation $x = 0$

Divide through to obtain $y = x^2 + \dfrac{2}{x}$

The **non-vertical asymptote** has equation $y = x^2$

By addition of ordinates the graph can be sketched.





For the **axes intercepts**, consider $y = 0$, which imples $x^3 + 2 = 0$

i.e. $\qquad x = -\sqrt[3]{2}$

For the **turning points**

$$y = x^2 + 2x^{-1} \text{ implies } \frac{dy}{dx} = 2x - 2x^{-2}$$

and $\quad \dfrac{dy}{dx} = 0 \quad$ implies $x - \dfrac{1}{x^2} = 0$

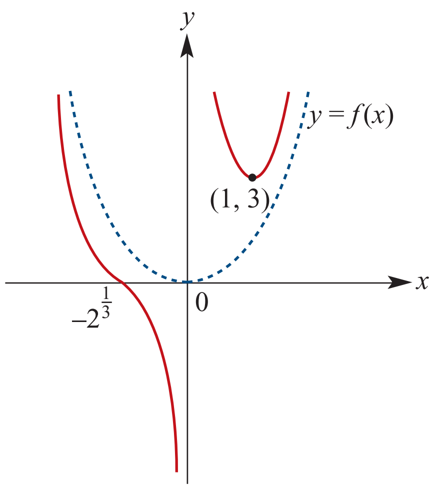
i.e. $x^3 = 1$

which implies $\; x = 1$

$f(1) = 3$

$\therefore$ the turning point has coordinates (1, 3)

The final sketch is shown.



Reciprocal of ordinates

This is the second method for sketching graphs of rational functions. In this situation, functions of the form $f(x) = \dfrac{P(x)}{Q(x)}$ will be considered where $P(x) = 1$ and $Q(x)$ is a quadratic function.

Example 24

Sketch the graph of $f: R\backslash\{0, 4\} \to R,\ f(x) = \dfrac{1}{x^2 - 4x}$

Solution

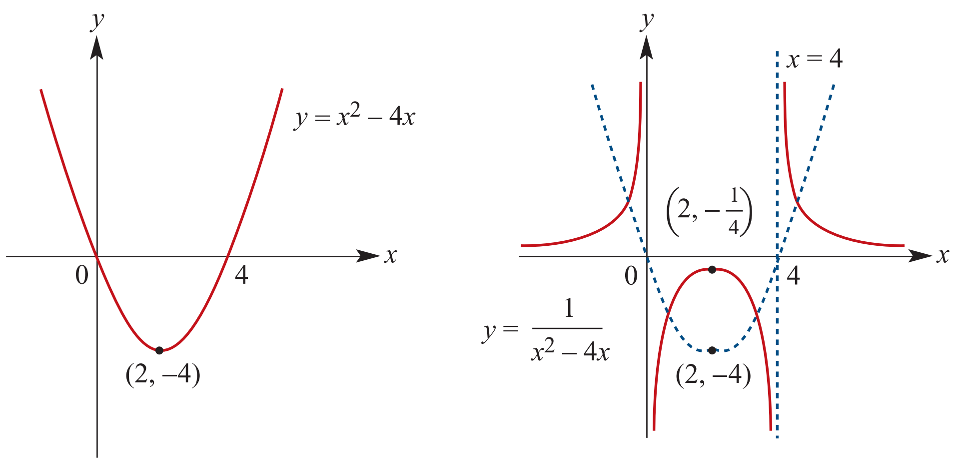
$$f(x) = \frac{1}{x^2 - 4x} = \frac{1}{x(x-4)}$$

Vertical asymptotes have equations $x = 0$ and $x = 4$

The **non-vertical** asymptote is $y = 0$ as $y \to 0$ as $x \to \pm\infty$

The graph is produced by first sketching the graph of $y = Q(x)$

In this case $y = x^2 - 4x$



Summary of properties of reciprocal functions

- x-axis intercepts of the function determine the equation of the asymptotes for the reciprocal of the function.
- The reciprocal of a positive number is positive. The reciprocal of a negative number is negative.
- A graph and its reciprocal will intersect at a point if the y coordinate of the point is 1 or -1.
- Local maximums of the function produce local minimums for the reciprocal.
- Local minimums of the function produce local maximums for the reciprocal.
- If $g(x) = \dfrac{1}{f(x)}$ then $g'(x) = -\dfrac{f'(x)}{f(x)^2}$. Therefore at a given point the gradient of the reciprocal function is opposite in sign to the original function.

Exercise 6G

1 Sketch graphs of each of the following, labelling all intercepts with the axes, turning points and asymptotes:

a $y = \dfrac{1}{x^2 - 2x}$ **b** $y = \dfrac{x^4 + 1}{x^2}$ **c** $y = \dfrac{1}{(x-1)^2 + 1}$

d $y = \dfrac{x^2 - 1}{x}$ **e** $y = \dfrac{x^3 - 1}{x^2}$ **f** $y = \dfrac{x^2 + x + 1}{x}$

g $y = \dfrac{4x^3 - 8}{x}$ **h** $y = \dfrac{1}{x^2 + 1}$ **i** $y = \dfrac{1}{x^2 - 1}$

j $y = \dfrac{x^2}{x^2 + 1} \left(= 1 - \dfrac{1}{x^2 + 1} \right)$ **k** $y = \dfrac{1}{x^2 - x - 2}$ **l** $y = \dfrac{1}{4 + 3x - x^2}$

2 Sketch the graphs of each of the following, labelling all intercepts with the axes, turning points and asymptotes:

a $f(x) = \dfrac{1}{9 - x^2}$ **b** $g(x) = \dfrac{1}{(x-2)(3-x)}$ **c** $h(x) = \dfrac{1}{x^2 + 2x + 4}$

d $f(x) = \dfrac{1}{x^2 + 2x + 1}$ **e** $g(x) = x^2 + \dfrac{1}{x^2} + 2$

3 The equation of a curve is $y = 4x + \dfrac{1}{x}$. Find:

a the coordinates of the turning points

b the equation of the tangent to the curve at the point where $x = 2$

4 Find the x coordinates of the points on the curve $y = \dfrac{x^2 - 1}{x}$ at which the gradient of the curve is 5.

5 Find the gradient of the curve $y = \dfrac{2x - 4}{x^2}$ at the point where the curve crosses the x axis.

6 For the curve $y = x - 5 + \dfrac{4}{x}$, find:

a the coordinates of the points of intersection with the axes

b the equations of all asymptotes **c** the coordinates of all turning points

Use this information to sketch the curve.

7 If x is positive, find the least value of $x + \dfrac{4}{x^2}$.

8 For positive values of x, sketch the graph of $y = x + \dfrac{4}{x}$, and find the least value of y.

9 **a** Find the coordinates of the stationary points of the curve $y = \dfrac{(x-3)^2}{x}$ and determine the nature of each point.

b Sketch the graph of $y = \dfrac{(x-3)^2}{x}$.

10 **a** Find the coordinates of the turning point(s) of the curve $y = 8x + \dfrac{1}{2x^2}$ and determine the nature of each point.

b Sketch the graph of $y = 8x + \dfrac{1}{2x^2}$.

6.8 A summary of differentiation

It is appropriate at this stage to review the techniques of differentiation of Specialist Mathematics. The derivatives of the standard functions also need to be reviewed in preparation for the chapters on antidifferentiation.

The differentiation techniques

The function	The derivative
$f(x)$	$f'(x)$
$a\,f(x),\ a \in R$	$a\,f'(x)$
$f(x) + g(x)$	$f'(x) + g'(x)$
$f(x)\,g(x)$	$f'(x)\,g(x) + f(x)\,g'(x)$
$\dfrac{f(x)}{g(x)}$	$\dfrac{g(x)f'(x) - g'(x)f(x)}{(g(x))^2}$
$f(g(x))$	$f'(g(x))\,g'(x)$

The standard functions

$f(x)$	$f'(x)$
x^n	nx^{n-1}
$(ax + b)^n$	$an\,(ax + b)^{n-1}$
$\sin(ax)$	$a\cos(ax)$
$\cos(ax)$	$-a\sin(ax)$
$\tan(ax)$	$a\sec^2(ax)$
e^{ax}	ae^{ax}
$\log_e(ax)$	$\dfrac{1}{x}$
$\sin^{-1}\left(\dfrac{x}{a}\right)$	$\dfrac{1}{\sqrt{a^2 - x^2}}$
$\cos^{-1}\left(\dfrac{x}{a}\right)$	$-\dfrac{1}{\sqrt{a^2 - x^2}}$
$\tan^{-1}\left(\dfrac{x}{a}\right)$	$\dfrac{a}{a^2 + x^2}$

- $f'(x)$ and $\dfrac{dy}{dx}$ are the first derivatives of $f(x)$ and y respectively.
- $f''(x)$ and $\dfrac{d^2y}{dx^2}$ are the second derivatives of $f(x)$ and y respectively.
- Furthermore, the chain rule can be written as $\dfrac{dy}{dx} = \dfrac{dy}{du} \times \dfrac{du}{dx}$ and an important result from the chain rule is $\dfrac{dy}{dx} = \dfrac{1}{\frac{dx}{dy}}$.

Exercise 6H

1 Find the second derivative of each of the following:

a x^{10} **b** $(2x + 5)^8$ **c** $\sin(2x)$

d $\cos\left(\dfrac{x}{3}\right)$ **e** $\tan\left(\dfrac{3x}{2}\right)$ **f** e^{-4x}

g $\log_e(6x)$ **h** $\sin^{-1}\left(\dfrac{x}{4}\right)$ **i** $\cos^{-1}(2x)$

j $\tan^{-1}\left(\dfrac{x}{2}\right)$

2 Find the first derivative of each of the following:

a $(1 - 4x^2)^3$ **b** $\dfrac{1}{\sqrt{2 - x}}$ **c** $\sin(\cos x)$

d $\cos(\log_e x)$ **e** $\tan\left(\dfrac{1}{x}\right)$ **f** $e^{\cos x}$

g $\log_e(4 - 3x)$ **h** $\sin^{-1}(1 - x)$ **i** $\cos^{-1}(2x + 1)$

j $\tan^{-1}(x + 1)$

3 Find $\dfrac{dy}{dx}$ in each of the following:

a $y = \dfrac{\log_e x}{x}$ **b** $y = \dfrac{x^2 + 2}{x^2 + 1}$ **c** $y = 1 - \tan^{-1}(1 - x)$

d $y = \log_e \dfrac{e^x}{e^x + 1}$ **e** $x = \sqrt{\sin y + \cos y}$ **f** $y = \log_e(x + \sqrt{1 + x^2})$

g $y = \sin^{-1} e^x$ **h** $y = \dfrac{\sin x}{e^x + 1}$

4 **a** If $y = ax + \dfrac{b}{x}$ find:

i $\dfrac{dy}{dx}$ **ii** $\dfrac{d^2y}{dx^2}$

b Show that $x^2\dfrac{d^2y}{dx^2} + x\dfrac{dy}{dx} = y$.

5 **a** If $y = \sin(2x) + 3\cos(2x)$ find:

i $\dfrac{dy}{dx}$ **ii** $\dfrac{d^2y}{dx^2}$

b Hence show that $\dfrac{d^2y}{dx^2} + 4y = 0$

6.9 Implicit differentiation

The rules for curves of ellipses, circles and many other curves are not expressible in the form $y = f(x)$ or $x = f(y)$. Equations such as $x^2 + y^2 = 1$ and $\frac{x^2}{9} + \frac{(y-3)^2}{4} = 1$ are said to be implicit equations. In this section a method for finding $\frac{dy}{dx}$ for such relations is introduced. The technique is called **implicit differentiation**.

In Example 5, in section 6.2, $\frac{dy}{dx} = \frac{1}{\frac{dx}{dy}}$ was used to find $\frac{dy}{dx}$ from the relation $x = y^3$.

By calculating $\frac{dx}{dy} = 3y^2$

the result $\frac{dy}{dx} = \frac{1}{3y^2}$ was obtained.

Implicit differentiation takes account of the fact that if two algebraic expressions are always equal, then the value of each expression must change in an identical way as one of the variables changes.

i.e. if p and q are expressions in x and y and $p = q$, for all x and y

then $\frac{dp}{dx} = \frac{dq}{dx}$ and $\frac{dp}{dy} = \frac{dq}{dy}$

Now for $x = y^3$, differentiate each side with respect to x.

i.e. $\frac{d}{dx}(x) = \frac{d}{dx}(y^3)$

By a simple substitution and the application of the chain rule a result can be obtained for the right-hand side of the equation.

Let $u = y^3$

Then $\frac{d}{dx}(y^3) = \frac{du}{dx} = \frac{du}{dy} \times \frac{dy}{dx}$ (by the chain rule)

Now $u = y^3$ implies $\frac{du}{dy} = 3y^2$

$\therefore \quad \frac{d}{dx}(y^3) = 3y^2 \times \frac{dy}{dx}$

Hence returning to the original expression $\frac{d}{dx}(x) = 3y^2 \times \frac{dy}{dx}$, and thus

$$1 = 3y^2 \times \frac{dy}{dx}$$

and $\frac{dy}{dx} = \frac{1}{3y^2}$, provided $y \neq 0$

Example 25

Find $\frac{dy}{dx}$ by implicit differentiation for each of the following:

a $x^3 = y^2$ **b** $xy = 2x + 1$

Solution

a By differentiating both sides with respect to x,

$$3x^2 = \frac{d}{dx}(y^2)$$
$$= 2y\frac{dy}{dx}$$

Therefore $\quad \frac{dy}{dx} = \frac{3x^2}{2y}$

b Differentiating both sides with respect to x gives

$$\frac{d}{dx}(xy) = 2$$

The product rule applies on the left-hand side

$$y + x\frac{dy}{dx} = 2$$

$$\therefore \quad \frac{dy}{dx} = \frac{2 - y}{x}$$

Example 26

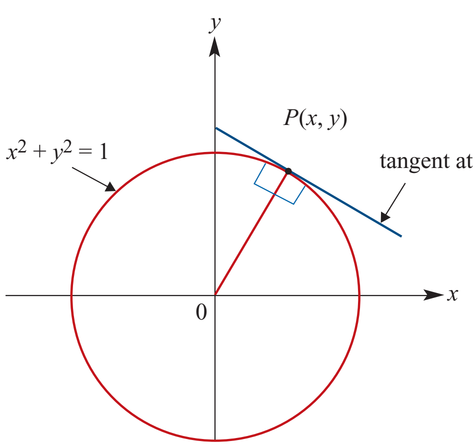
Find $\frac{dy}{dx}$ if $x^2 + y^2 = 1$.

Solution

Now $\quad x^2 + y^2 = 1$

leads to $\quad y = \pm\sqrt{1 - x^2}$

or $\quad x = \pm\sqrt{1 - y^2}$



So y is not a function of x, and x is not a function of y. Implicit differentiation should be used.

The derivative can also be found geometrically, since the graph of $x^2 + y^2 = 1$ is the unit circle.

The gradient of OP using $\frac{\text{rise}}{\text{run}}$ is $\frac{y}{x}$, $x \neq 0$. Since the radius is perpendicular to the tangent for a circle, gradient of tangent $= -\frac{x}{y}$, $y \neq 0$. In other words: $\frac{dy}{dx} = -\frac{x}{y}$.

Clearly, from the graph, when $y = 0$ the tangents are parallel to the y axis, hence $\frac{dy}{dx}$ is not defined here.

Now for implicit differentiation:

$$x^2 + y^2 = 1$$

Differentiate both sides with respect to x.

$$2x + 2y\frac{dy}{dx} = 0$$

$$\therefore \quad 2y\frac{dy}{dx} = -2x$$

$$\therefore \quad \frac{dy}{dx} = -\frac{x}{y},\ y \neq 0$$

Example 27

Given $xy - y - x^2 = 0$, find $\frac{dy}{dx}$.

Solution

Express y as a function of x.

$$xy - y - x^2 = 0$$

$$\therefore \quad xy - y = x^2$$

$$\therefore \quad y(x-1) = x^2$$

$$\therefore \quad y = \frac{x^2}{x-1}$$

$$\therefore \quad y = x + 1 + \frac{1}{x-1}$$

$$\therefore \quad \frac{dy}{dx} = 1 - \frac{1}{(x-1)^2},\ x \neq 1$$

$$= \frac{(x-1)^2 - 1}{(x-1)^2},\ x \neq 1$$

$$= \frac{x^2 - 2x}{(x-1)^2},\ x \neq 1$$

Consider the implicit differentiation alternative.

$$xy - y - x^2 = 0$$

$$\therefore \quad \frac{d}{dx}(xy) - \frac{dy}{dx} - \frac{d}{dx}(x^2) = \frac{d}{dx}(0) \quad \text{(differentiate both sides with respect to } x\text{)}$$

$$\left(x.\frac{dy}{dx} + y.1\right) - \frac{dy}{dx} - 2x = 0 \quad \text{(product rule)}$$

$$x\frac{dy}{dx} - \frac{dy}{dx} = 2x - y$$

$$\frac{dy}{dx}(x-1) = 2x - y$$

$$\therefore \quad \frac{dy}{dx} = \frac{2x - y}{x - 1},\ x \neq 1$$

This can be checked, by substitution of $y = \frac{x^2}{x-1}$, to confirm that the results are identical.

Example 28

Find $\frac{dy}{dx}$ if $2x^2 - 2xy + y^2 = 5$.

Solution

In this case, neither x nor y can be expressed as a function and so implicit differentiation must be used.

$$2x^2 - 2xy + y^2 = 5$$

$$\frac{d}{dx}(2x^2) - \frac{d}{dx}(2xy) + \frac{d}{dx}(y^2) = \frac{d}{dx}(5)$$

$$4x - \left(2x.\frac{dy}{dx} + y.2\right) + 2y\frac{dy}{dx} = 0 \quad \text{(using product rule and chain rule)}$$

$$4x - 2x\frac{dy}{dx} - 2y + 2y\frac{dy}{dx} = 0$$

$$2y\frac{dy}{dx} - 2x\frac{dy}{dx} = 2y - 4x$$

$$\frac{dy}{dx}(2y - 2x) = 2y - 4x$$

$$\therefore \quad \frac{dy}{dx} = \frac{2y - 4x}{2y - 2x} \quad x \neq y$$

$$= \frac{y - 2x}{y - x} \quad \text{if } x \neq y$$

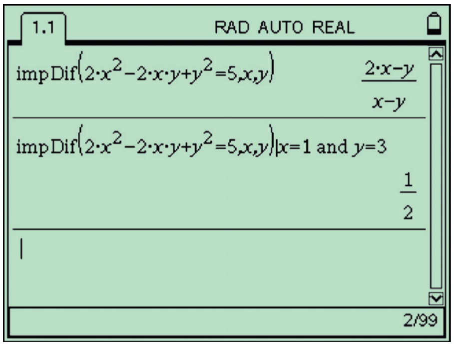
Using a TI-Nspire calculator

From the **Calculus** menu choose **Implicit Differentiation** and complete as shown ((menu) (4) (D)).

The process gives $\frac{dy}{dx}$ in terms of x and y.

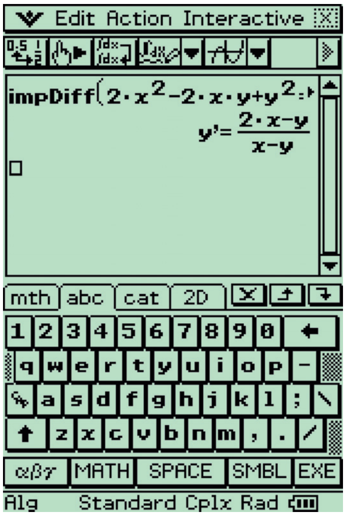
If the positions of x and y are interchanged the result is $\frac{dx}{dy}$.

The gradient at the point (1, 3) is found.



Using a Casio ClassPad calculator

Enter $2x^2 - 2xy + y^2 = 5$ and select it.
From the **Interactive** menu choose **Calculation** and then **ImpDiff**.
Complete with x as the independent variable (**Inde var**) and y as the dependent variable (**Depe var**).



Example 29

Draw the graph of $\cos x + \cos y = \frac{1}{2}$ (only consider x in the interval $[-\pi, \pi]$).

Solution

First note that $\cos(-x) = \cos(x)$ and $\cos(-y) = \cos(y)$. Therefore the graph is symmetrical about the y axis and symmetrical about the x axis. Also it is symmetrical about the line $y = x$. Also $\cos(2\pi + x) = \cos x$ and therefore the graph is periodic with period 2π.

Rearrange the expression to give $\cos y = \frac{1}{2} - \cos x$, and as $\cos y \leq 1$, $\frac{1}{2} - \cos x \leq 1$ and thus $\cos x \geq -\frac{1}{2}$. This implies that the domain is the union of all intervals of the form $\left[\frac{-2\pi}{3} + 2k\pi, \frac{2\pi}{3} + 2k\pi\right]$ where k is an integer. The range is also this union of intervals. **From now, only x and y values in the interval $\left[\frac{-2\pi}{3}, \frac{2\pi}{3}\right]$ will be considered.**

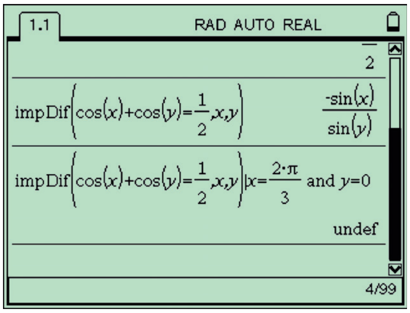
Using a TI-Nspire calculator

Use implicit differentiation

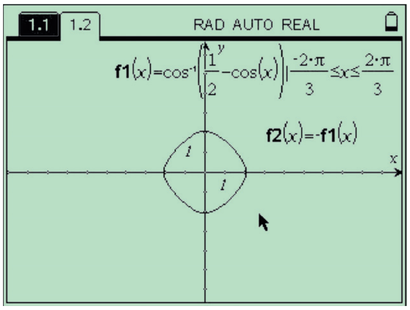
$-\sin x - \sin y\dfrac{dy}{dx} = 0.$

Hence $\dfrac{dy}{dx} = \dfrac{-\sin x}{\sin y}$ provided $y \neq 0$.

When $x = \pm\dfrac{2\pi}{3}$, $\cos y = 1$ and $y = 0$ and $\dfrac{dy}{dx}$ is undefined.



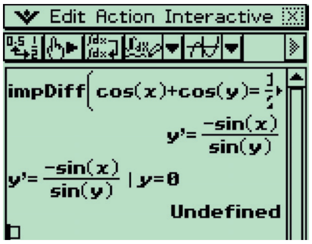
When $x = 0$, $y = \pm\frac{2\pi}{3}$ and $\frac{dy}{dx} = 0$.
The graph is drawn using two functions as shown.



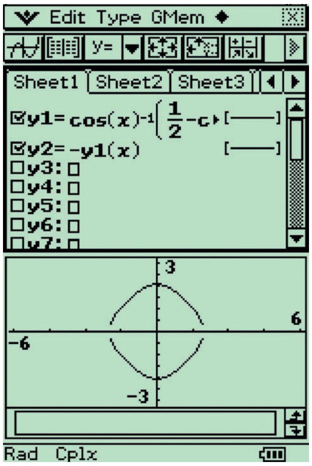
Using a Casio ClassPad calculator

Use implicit differentiation. Select the **Interactive** menu, then **Calculation** and then **ImpDiff**.

$\frac{dy}{dx}$ is not defined when $y = 0$.



Choose **Graph and Table** from the menu. From the **Type** menu select **ParamType**. Enter the equations as shown on the screen and click the graph icon in the icon bar at the top of the screen.



Exercise 6I

1 In each of the following, find $\frac{dy}{dx}$ using implicit differentiation:

a $x^2 - 2y = 3$ **b** $x^2y = 1$ **c** $x^3 + y^3 = 1$
d $y^3 = x^2$ **e** $x - \sqrt{y} = 2$ **f** $xy - 2x + 3y = 0$
g $y^2 = 4ax$ **h** $4x + y^2 - 2y - 2 = 0$

2 Find $\frac{dy}{dx}$ in each of the following:

a $(x + 2)^2 - y^2 = 4$ **b** $\frac{1}{x} + \frac{1}{y} = 1$ **c** $y = (x + y)^2$
d $x^2 - xy + y^2 = 1$ **e** $y = x^2e^y$ **f** $\sin y = \cos^2 x$
g $\sin(x - y) = \sin x - \sin y$ **h** $y^5 - x\sin y + 3y^2 = 1$

3 Find the equation of the tangent at the indicated point in each of the following:

a $y^2 = 8x$ at $(2, -4)$

b $x^2 - 9y^2 = 9$ at $\left(5, \frac{4}{3}\right)$

c $xy - y^2 = 1$ at $\left(\frac{17}{4}, 4\right)$

d $\dfrac{x^2}{16} + \dfrac{y^2}{9} = 1$ at $(0, -3)$

4 Find $\dfrac{dy}{dx}$ in terms of x and y given that $\log_e(y) = \log_e(x)$.

5 Find the gradient of the curve $x^3 + y^3 = 9$ at point $(1, 2)$.

6 A curve is defined by the equation $x^3 + y^3 + 3xy - 1 = 0$. Find the gradient of the curve at the point $(2,-1)$.

7 Given that $\tan x + \tan y = 3$, find the value of $\dfrac{dy}{dx}$ when $x = \dfrac{\pi}{4}$.

8 Find the gradient at the point $(1, -3)$ of the curve with equation $y^2 + xy - 2x^2 = 4$.

9 For the curve with equation $x^3 + y^3 = 28$:

a obtain an expression for $\dfrac{dy}{dx}$

b show that $\dfrac{dy}{dx}$ cannot be positive

c calculate the value of $\dfrac{dy}{dx}$ when $x = 1$.

10 The equation of a curve is $2x^2 + 8xy + 5y^2 = -3$. Find the equation of the two tangents which are parallel to the x axis.

11 The equation of a curve C is $x^3 + xy + 2y^3 = k$ where k is a constant.

a Find $\dfrac{dy}{dx}$ in terms of x and y.

C does have a tangent parallel to the y axis.

b Show that the y coordinate at the point of contact satisfies $216y^6 + 4y^3 + k = 0$.

c Hence show that $k \le \frac{1}{54}$.

d Find the possible value(s) of k in the case where $x = -6$ is a tangent to C.

12 The equation of a curve is $x^2 - 2xy + 2y^2 = 4$.

a Find an expression for $\dfrac{dy}{dx}$ in terms of x and y.

b Find the coordinates of each point on the curve at which the tangents are parallel to the x axis.

13 For the curve with equation $y^2 + x^3 = 1$:

a Find $\dfrac{dy}{dx}$ in terms of x and y.

b Find the coordinates of the points where $\dfrac{dy}{dx} = 0$.

c Find the coordinates of the points where $\dfrac{dx}{dy} = 0$.

d Describe the behaviour as $x \to -\infty$.

e Find the relation in terms of y.

f Find the coordinates of the points of inflection of the curve.

g Use a calculator to help you sketch the graph of $y^2 + x^3 = 1$.

MC TEST D&D TEST

Chapter summary

$f(x)$	$f'(x)$
$\tan x$	$\sec^2 x$
$\sin^{-1} x$	$\frac{1}{\sqrt{1-x^2}}$
$\cos^{-1} x$	$-\frac{1}{\sqrt{1-x^2}}$
$\tan^{-1} x$	$\frac{1}{1+x^2}$

- If $y = f(x)$

$$\frac{dy}{dx} = f'(x)$$

$$\frac{d^2y}{dx^2} = f''(x)$$

Summary properties of rational functions

- Rational functions are of the form:

$$f(x) = \frac{a(x)}{b(x)} \text{ where } a(x) \text{ and } b(x) \text{ are polynomials and } b(x) \neq 0$$

$$= q(x) + \frac{r(x)}{b(x)} \text{ (quotient/remainder form).}$$

- Asymptotes occur when $b(x) = 0$ and $y = q(x)$.
- x-axis intercepts occur when $a(x) = 0$.
- y-axis intercepts occur at $f(0) = \frac{a(0)}{b(0)}$, provided $b(0) \neq 0$.
- Turning points occur when $f'(x) = 0$.
- If $f(x) = \frac{1}{b(x)}$, use reciprocals of ordinates of the graph of $y = b(x)$ to sketch the graph of $y = f(x)$.

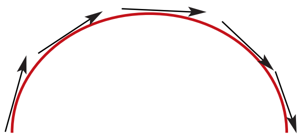
Summary of properties of reciprocal functions

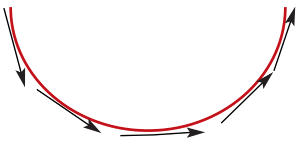
- x-axis intercepts of the function define the equation of the asymptotes for the reciprocal of the function.
- The reciprocal of a positive number is positive. The reciprocal of a negative number is negative.
- A graph and its reciprocal will intersect if the y coordinate of a point on the graph is 1 or -1.
- Local maximums of the function produce local minimums for the reciprocal.
- Local minimums of the function produce local maximums for the reciprocal.

Review

- If $g(x) = \frac{1}{f(x)}$ then $g'(x) = -\frac{f'(x)}{f(x)^2}$. Therefore at a given point the gradient of the reciprocal function is opposite in sign to the original function.
- If $f(x) = q(x) + \frac{r(x)}{b(x)}$ use addition of ordinates of $y = q(x)$ and $y = \frac{r(x)}{b(x)}$ to sketch the graph.

Use of the second derivative to describe behaviour of graphs





- **Concave down for an interval**
The tangent is above the curve for each point and the derivative function is decreasing.
i.e. $f''(x) < 0$

- **Concave up for an interval**
The tangent is below the curve for each point and the derivative function is increasing.
i.e. $f''(x) > 0$

- At a point of inflexion the tangent will pass through the curve. A **point of inflexion** is where the **curve** changes from concave down to concave up or concave up to concave down. At a point of inflexion the second derivative (if it exists) has value zero but the converse does not hold.

Implicit differentiation

- The equations of many curves cannot be defined by a rule of the form $y = f(x)$ or $x = f(y)$. Equations such as $x^2 + y^2 = 1$ and $\frac{x^2}{4} - \frac{y^2}{9} = 1$ are said to be implicitly defined. The method for finding the gradient at a point on a curve defined in this way is called implicit differentiation.
- Using operator notation:

$$\frac{d}{dx}(x^2 + y^2) = 2x + 2y\frac{dy}{dx} \quad \text{(use of chain rule)}$$

$$\frac{d}{dx}(x^2y^2) = 2xy^2 + x^2\frac{d}{dx}(y^2)$$

$$= 2xy^2 + 2yx^2\frac{dy}{dx} \quad \text{(use of product rule)}$$

Multiple-choice questions

1 The equation of the tangent of $x^2 + y^2 = 1$ at the point with coordinates $\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$ is:

A $y = -x$ **B** $y = -x + 2\sqrt{2}$ **C** $y = -x + 1$
D $y = -2\sqrt{x} + 2$ **E** $y = -x + \sqrt{2}$

2 If $f(x) = 2x^2 + 3x - 20$, then the graph of $y = \dfrac{1}{f(x)}$ has:

A x-axis intercepts at $x = \frac{5}{2}$ and $x = -4$
B asymptotes at $x = \frac{5}{2}$ and $x = 4$
C asymptotes at $x = -\frac{5}{2}$ and $x = 4$
D a local minimum at the point $\left(\frac{-3}{4}, \frac{-169}{8}\right)$
E a local maximum at the point $\left(\frac{-3}{4}, \frac{-8}{169}\right)$

3 The coordinates of the points of inflexion of $y = \sin x$ for $x \in [0, 2\pi]$ are:

A $\left(\frac{\pi}{2}, 1\right)$ and $\left(-\frac{\pi}{2}, -1\right)$ **B** $(\pi, 0)$ **C** $(\pi, 0)$, $(0, 0)$ and $(2\pi, 0)$
D $(1, 0)$ **E** $\left(\frac{\pi}{4}, \frac{1}{\sqrt{2}}\right)$, $\left(\frac{3\pi}{4}, \frac{1}{\sqrt{2}}\right)$ and $\left(\frac{5\pi}{4}, -\frac{1}{\sqrt{2}}\right)$

4 Let $g(x) = e^{-x} f(x)$ where f is a differentiable function for all the real numbers. There is a point of inflexion on the graph of $y = g(x)$ at $(a, g(a))$. An expression for $f''(a)$ in terms of $f'(a)$ and $f(a)$ is:

A $f''(a) = f(a) + f'(a)$ **B** $f''(a) = 2f(a)f'(a)$ **C** $f''(a) = 2f(a) + f'(a)$
D $f''(a) = \dfrac{f'(a)}{f(a)}$ **E** $f''(a) = 2f'(a) - f(a)$

5 Given that $x = t^2$ and $y = t^3$ then $\dfrac{dx}{dy}$ is equal to:

A $\dfrac{1}{t}$ **B** $\dfrac{2}{3t}$ **C** $\dfrac{3t}{2}$ **D** $\dfrac{2t}{3}$ **E** $\dfrac{3}{2t}$

6 If $y = \cos^{-1}\left(\dfrac{4}{x}\right)$ and $x > 4$ then $\dfrac{dy}{dx}$ is equal to:

A $\dfrac{-1}{\sqrt{16 - x^2}}$ **B** $\dfrac{-4}{\sqrt{1 - 16x^2}}$ **C** $\dfrac{-4x}{\sqrt{x^2 - 16}}$ **D** $\dfrac{4}{x\sqrt{x^2 - 16}}$ **E** $\dfrac{4}{\sqrt{x^2 - 16}}$

7 The coordinates of the turning point of the graph with equation $y = x^2 + \dfrac{54}{x}$ are:

A $(3, 0)$ **B** $(-3, 27)$ **C** $(3, 27)$ **D** $(-3, 0)$ **E** $(3, 2)$

8 For $y = \sin^{-1}\left(\dfrac{x}{2}\right)$, $x \in [0, 1]$ and $y \in [0, \frac{\pi}{2}]$, $\dfrac{d^2y}{dx^2}$ is equal to:

A $\cos^{-1}\left(\dfrac{x}{2}\right)$ **B** $\dfrac{x}{(4 - x^2)^{3/2}}$ **C** $\dfrac{-x}{\sqrt{4 - x^2}}$
D $\dfrac{-x}{\sqrt{4 - x^2}(4 - x^2)}$ **E** $\dfrac{-1}{\sqrt{4 - x^2}}$

Review

9 Which of the following statements is false for the graph of the function with $y = \cos^{-1}(x)$, for $x \in [-1, 1]$ and $y \in [0, \pi]$?

A The gradient of the graph is always negative.
B There is a point of inflexion for the graph at the point $\left(0, \frac{\pi}{2}\right)$.
C The gradient of the graph has a minimum value of -1.
D The gradient of the graph is undefined at the point $(-1, \pi)$.
E At $x = \frac{1}{2}$, $y = \frac{\pi}{3}$.

10 If $y = \tan^{-1}\left(\frac{1}{3x}\right)$, then $\frac{dy}{dx}$ is equal to:

A $\frac{1}{3(1 + x^2)}$ **B** $\frac{-1}{3(1 + x^2)}$ **C** $\frac{1}{3(1 + 9x^2)}$ **D** $\frac{-3}{9x^2 + 1}$ **E** $\frac{9x^2}{(9x^2 + 1)}$

Short-answer questions (technology-free)

1 Find $\frac{dy}{dx}$ if:

a $y = x \tan x$ **b** $y = \tan(\tan^{-1} x)$ **c** $y = \cos(\sin^{-1} x)$ **d** $y = \sin^{-1}(2x - 1)$

2 Find $f''(x)$ if:

a $f(x) = \tan x$ **b** $f(x) = \log_e(\tan x)$ **c** $f(x) = x \sin^{-1} x$ **d** $f(x) = \sin(e^x)$

3 For each of the following, state the coordinates of the point(s) of inflexion:

a $y = x^3 - 8x^2$ **b** $y = \sin^{-1}(x - 2)$ **c** $y = \log_e(x) + \frac{1}{x}$

4 **a** Sketch the graphs of $f: \left[\pi, \frac{3\pi}{2}\right] \to R$, $f(x) = \sin x$ and f^{-1} on the same set of axes.

b Find the derivative of f^{-1}.

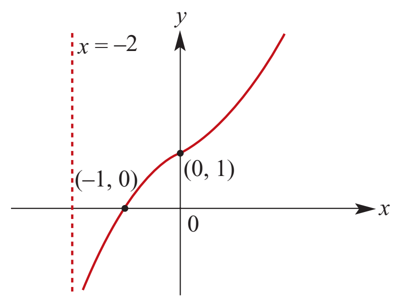
c Find the coordinates of the point on the graph of f^{-1} where the tangent has a gradient of -2.

5 This is the graph of $y = f(x)$.

Sketch the graphs of:

a $y = \frac{1}{f(x)}$

b $y = f^{-1}(x)$



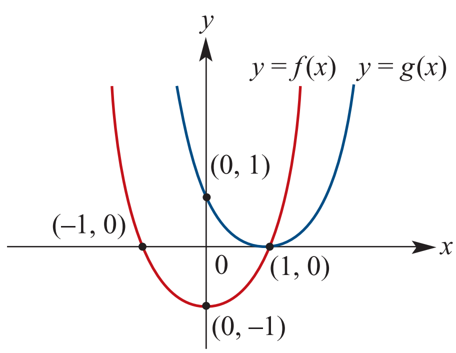
6 These are the graphs of $y = f(x)$ and $y = g(x)$ where f and g are quadratic functions.

a Sketch the graphs of:

i $y = f(x) + g(x)$

ii $y = \frac{1}{f(x) + g(x)}$

iii $y = \frac{1}{f(x)} + \frac{1}{g(x)}$



b Use the points given to determine the rules $y = f(x)$ and $y = g(x)$.

c Hence determine, in simplest form, the rules:

i $y = f(x) + g(x)$ **ii** $y = \dfrac{1}{f(x) + g(x)}$ **iii** $y = \dfrac{1}{f(x)} + \dfrac{1}{g(x)}$

7 Find $\dfrac{dy}{dx}$ by implicit differentiation.

a $x^2 + 2xy + y^2 = 1$ **b** $x^2 + 2x + y^2 + 6y = 10$

c $\frac{2}{x} + \frac{1}{y} = 4$ **d** $(x + 1)^2 + (y - 3)^2 = 1$

8 A point moves along the curve $y = x^3$ in such a way that its velocity parallel to the x axis a constant 3 cm/s. Find its velocity parallel to the y axis when:

a $x = 6$ **b** $y = 8$

Extended-response questions

1 The radius, r cm, and the height, h cm, of a solid circular cylinder vary in such a way that the volume of the cylinder is always 250π cm^3.

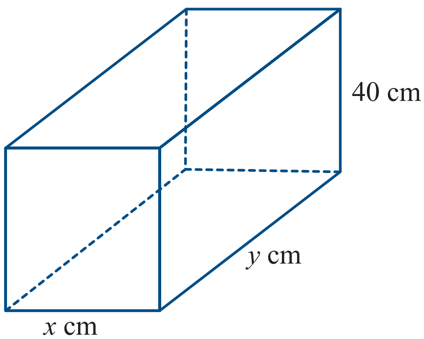
a Show that the total surface area, A cm^2, of the cylinder is given by $A = 2\pi r^2 + \dfrac{500\pi}{r}$.

b **i** Sketch the graph of A against r for $r > 0$.

ii State the equation of asymptotes and the coordinates of the stationary points.

c What is the minimum total surface area?

2 **a** A box with a volume of 1000 cm^3 is to be made in the shape of a rectangular prism. It has a fixed height of 40 cm. A cm^2 is the total surface area and the other dimensions are x cm and y cm as shown in the diagram. Express A in terms of x.

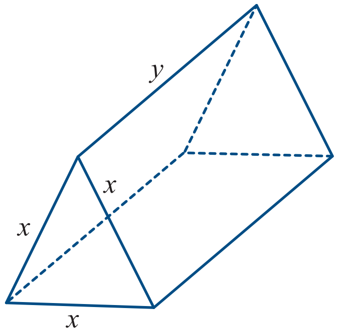


b Sketch the graph of A against x.

c Find the minimum surface area of the box and the dimensions of the box in this situation.

d Find the minimum surface area of the box and the dimensions of the box if the height of the box is k cm (k being a constant) while the volume remains 1000 cm^3.

3 This diagram shows a solid triangular prism with edge lengths as shown. All measurements are in cm. The volume is 2000 cm^3. The surface area is A cm^2.



a Express A in terms of x and y.

b Establish a relationship between x and y.

c Hence, express A in terms of x.

d Sketch the graph of A against x.

e Hence, determine the minimum surface area of the prism.

Review

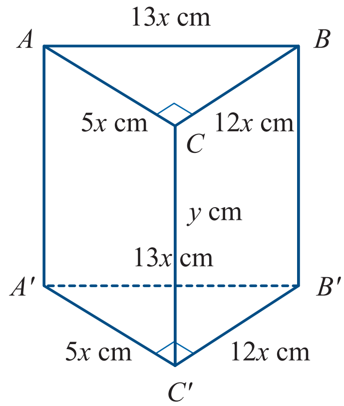
4 **a** Sketch the graph of g: $[0, 5] \to R$, where $g(x) = 4 - \dfrac{8}{2 + x^2}$.

b **i** Find $g'(x)$. **ii** Find $g''(x)$.

c For what value of x is the gradient of the graph of $y = g(x)$ a maximum?

d Sketch the graph of g: $[-5, 5] \to R$, where $g(x) = 4 - \dfrac{8}{2 + x^2}$.

5 The triangular prism as shown in the diagram has a right-angled triangle as its cross-section. The right angle is at C, C' for the ends of the prism. The volume of the prism is 3000 cm^3. The dimensions of the solid are shown on the diagram. Assume the volume remains constant and x varies.



a **i** Find y in terms of x.

ii Find the total surface area, S cm^2, in terms of x.

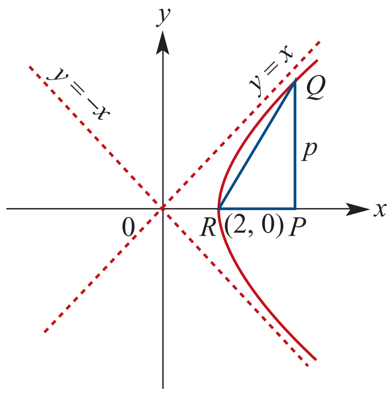
CAS CALCULATOR

iii Sketch the graph of S against x for $x > 0$. Clearly label the asymptotes and the coordinates of the turning point.

b Given that the length of x is increasing at a constant rate of 0.5 m/s, find the rate at which S is increasing when $x = 9$.

c Find the values of x for which the surface area is 2000 cm^2, correct to two decimal places.

6 The diagram shows part of the curve $x^2 - y^2 = 4$. PQ is parallel to the y axis and R is the point (2, 0). The length of PQ is p.



a Find the area, A, of triangle PQR in terms of p.

b **i** Find $\dfrac{dA}{dp}$.

CAS CALCULATOR

ii Use your CAS calculator to help sketch the graph of A against p.

iii Find the value of p for which $A = 50$ (correct to two decimal places).

iv Prove that $\dfrac{dA}{dp} \geq 0$ for all p.

c Q moves along the curve and P along the x axis so that PQ is always parallel to the y axis and p is increasing at a rate of 0.2 units per second. Find the rate at which A is increasing, correct to three decimal places when:

i $p = 2.5$ **ii** $p = 4$ **iii** $p = 50$ **iv** $p = 80$.

(Use calculus to obtain the rate.)

7 Consider the family of cubic functions, i.e. f: $R \to R$, $f(x) = ax^3 + bx^2 + cx + d$.

a Find $f'(x)$.

b Find $f''(x)$.

(*cont.*)

c Under what conditions does this family have no turning points?

d **i** Find the x coordinate of the point where the gradient of this family is a local minimum or maximum.

ii State the conditions for the gradient to be a local maximum.

e If $a = 1$, find the x coordinate of the point where there is a stationary point of $y = f'(x)$.

f For $y = x^3 + bx^2 + cx$ find:

i the relationship between b and c if there is only one x-axis intercept

ii the relationship between b and c if there are two turning points but only one x-axis intercept.

8 The equation of a curve is $y = \dfrac{1 - x^2}{1 + x^2}$.

a **i** Show that $\dfrac{dy}{dx} = \dfrac{-4x}{(1 + x^2)^2}$. **ii** Find $\dfrac{d^2y}{dx^2}$.

b Sketch the graph of $y = \dfrac{1 - x^2}{1 + x^2}$. Label the turning point and give the equation of the asymptote.

CAS CALCULATOR

c With the aid of a CAS calculator, sketch the graphs of $y = \dfrac{1 - x^2}{1 + x^2}$, $y = \dfrac{dy}{dx}$ and $y = \dfrac{d^2y}{dx^2}$ for $x \in [-2, 2]$.

d The curve with equation $y = \dfrac{1 - x^2}{1 + x^2}$ crosses the x axis at A and B and the y axis at C.

i Find the equation of the tangents at A and B.

ii Show that they intersect at C.

9 **a** Consider $f(x) = \tan^{-1}(x) + \tan^{-1}\left(\dfrac{1}{x}\right) \quad x \neq 0$

i Find $f'(x)$. **ii** If $x > 0$, find $f(x)$. **iii** If $x < 0$, find $f(x)$.

b Let $y = \cot x$ where $x \in (0, \pi)$.

i Find $\dfrac{dy}{dx}$. **ii** Find $\dfrac{dy}{dx}$ in terms of y.

c Find the derivative with respect to x of the function $y = \cot^{-1} x$ where $y \in (0, \pi)$ and $x \in R$.

d Find the derivative with respect to x of $\cot(x) + \tan(x)$ where $x \in \left(0, \dfrac{\pi}{2}\right)$.

10 The volume, V litres, of water in a pool at time t minutes is given by the rule $V = -3000\pi\,(\log_e(1 - h) + h)$ where h metres is the depth of water in the pool at time t minutes.

a **i** Find $\dfrac{dV}{dh}$ in terms of h.

ii Sketch the graph of $\dfrac{dV}{dh}$ against h for $0 \leq h \leq 0.9$.

(cont.)

b The maximum depth of the pool is 90 cm.

i Find the maximum volume of the pool to the nearest litre.

ii Sketch the graphs of $y = -3000\pi \log_e(1 - x)$, $y = -3000\pi x$ and use addition of ordinates to sketch the graph of V against h for $0 \leq h \leq 0.9$.

c If water is being poured into the pool at 15 litres/min, find the rate at which the depth of the water is increasing when $h = 0.2$, correct to two significant figures.

11 Consider the function $f: R^+ \to R$, where $f(x) = \dfrac{8}{x^2} - 32 + 16\log_e(2x)$.

a Find $f'(x)$.

b Find $f''(x)$.

c Find the exact coordinates of any stationary points of the graph of $y = f(x)$.

d Find the exact value of x for which there is a point of inflexion.

e State the interval for x for which $f'(x) > 0$.

f Find, correct to two decimal places, any x intercepts other than $x = 0.5$.

g Sketch the graph of $y = f(x)$.

12 An ellipse is described by the parametric equations $x = 3\cos\theta$ and $y = 2\sin\theta$.

a Show that the tangent to the ellipse at the point $P(3\cos\theta, 2\sin\theta)$ is $2x\cos\theta + 3y\sin\theta = 6$.

b The tangent to the ellipse at the point $P(3\cos\theta, 2\sin\theta)$ meets the line with equation $x = 3$ at a point T.

i Find the coordinates of the point T.

ii A is the point with coordinates $(-3, 0)$ and O is the origin. Prove that OT is parallel to AP.

c The tangent to the ellipse at the point $P(3\cos\theta, 2\sin\theta)$ meets the x axis at Q and the y axis at R.

i Find the midpoint M of the line segment QR in terms of θ.

ii Find the locus of M as θ varies.

d $W(-3\sin\theta, 2\cos\theta)$ and $P(3\cos\theta, 2\sin\theta)$ are points on the ellipse.

i Find the equation of the tangent to the ellipse at W.

ii Find the coordinates of the point, Z, the point of intersection of the tangents at P and W in terms of θ.

iii Find the locus of Z as θ varies.

13 An ellipse has equation $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$. The tangent at a point $P(a\cos\theta, b\sin\theta)$ intersects the axes at points M and N and O is the origin.

a Find the area of triangle OMN in terms of a, b and θ.

b Find the values of θ for which the area of triangle OMN is a minimum and state this minimum area in terms of a and b.

14 A hyperbola is described by the parametric equations $x = a \sec \theta$ and $y = b \tan \theta$.

a Show that the equation of the tangent at the point $P(a \sec \theta, b \tan \theta)$ can be written as $\frac{x}{a} \sec \theta - \frac{y}{b} \tan \theta = 1$.

b Find the coordinates of the points of intersection, Q and R, of the tangent with the asymptotes $y = \pm\frac{bx}{a}$ of the hyperbola.

c Find the coordinates of the midpoint of the line segment QR.

15 A section of an ellipse is described by the parametric equations $x = 2 \cos \theta$ and $y = \sin \theta$ where $0 < \theta < \frac{\pi}{2}$. The normal to the ellipse at the point $P(2 \cos \theta, \sin \theta)$ meets the x axis at Q and the y axis at R.

a Find the area of triangle OQR where O is the origin in terms of θ.

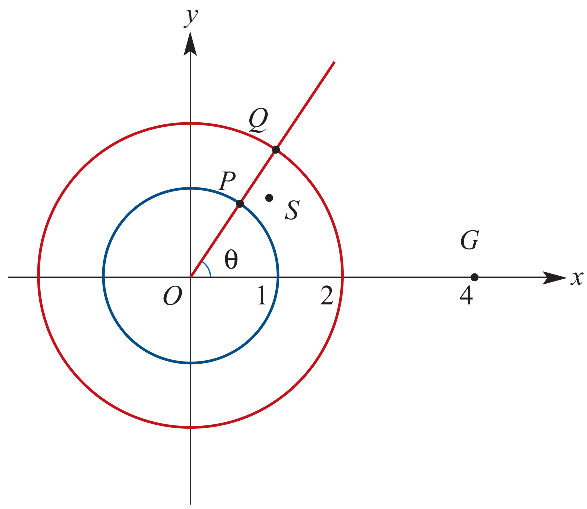
b Find the maximum value of this area and the value of θ for which this occurs.

c Find the midpoint, M, of the line segment QR in terms of θ.

d Find the locus of the point M as θ varies.

16 An electronic game appears on a flat screen, part of which is shown in the diagram. Concentric circles of radii one unit and two units appear on the screen.

Points P and Q move around the circles so that O, P and Q are collinear and OP makes an angle of θ with the x axis. A spaceship S moves around between the two circles and a gun is on the x axis at G, 4 units from O. The player turns the gun and tries to hit the spaceship.



The spaceship moves so that at any time it is at a point (x, y), where x equals the value of the x coordinate of Q and y equals the y coordinate of P.

a Find the cartesian equation of the path C of S.

b Show that the equation of the tangent to C at the point (u, v) on C is $y = \frac{-u}{4v}x + \frac{1}{v}$.

c Show that in order to aim at the spaceship at any point in it path, the player needs to turn the gun through an angle of at most 2α, where $\tan \alpha = \frac{1}{6}\sqrt{3}$.

CHAPTER 7

Antidifferentiation

Objectives

- To review **antidifferentiation by rule**
- To consider the relationship between the graph of a function and the graph of its antiderivative
- To apply the technique of **substitution** to integration
- To apply **trigonometric identities** to integration
- To apply **partial fractions** to integration

7.1 Antidifferentiation

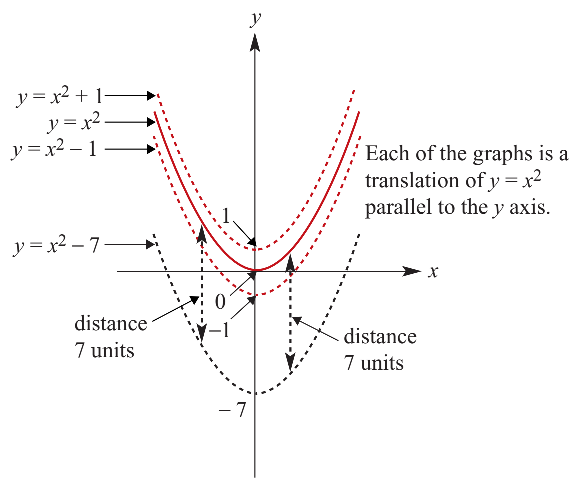
The derivative of x^2 with respect to x is $2x$. Conversely, given that an unknown expression has derivative $2x$ it is clear that the unknown expression could be x^2. The process of finding a function from its derivative is called **antidifferentiation**.

Consider the polynomial functions $f(x) = x^2 + 1$ and $g(x) = x^2 - 7$

Then $f'(x) = 2x$ and $g'(x) = 2x$

i.e. the two different functions have the same derived function.

Both $x^2 + 1$ and $x^2 - 7$ are said to be antiderivatives of $2x$. If two functions have the same derivative on an interval then they differ by a constant. If two functions have the same derived function then the graph of one function is obtained by a translation parallel to the y axis of the other. The diagram illustrates several antiderivatives of $2x$.



The general antiderivative of $2x$ is $x^2 + c$ where c is an arbitrary real number. The notation of Leibniz is used to state this with symbols.

$$\int 2x\,dx = x^2 + c$$

This is read as 'the general antiderivative of $2x$ with respect to x is equal to x^2 plus c' or 'the *indefinite* integral of $2x$ with respect to x is $x^2 + c$'.

To be more precise, the indefinite integral is the set of all antiderivatives and to emphasise this, it is written as shown.

$$\int 2x\,dx = \{f(x): f'(x) = 2x\} = \{x^2 + c: c \in R\}$$

The set notation is not commonly used but it should be clearly understood that there is not a unique antiderivative for a given function. The set notation is not used in this text but it is advisable to keep it in mind when considering further results.

In general, F and f are functions such that:

If $F'(x) = f(x)$

$\int f(x)\,dx = F(x) + c$, where c is an arbitrary real number

Note that $F(x)$ is an antiderivative of $f(x)$.

In *Essential Mathematical Methods 3 and 4* the following rules for integration were established:

$f(x)$	$\int f(x)dx$
$x^n, n \neq -1$	$\dfrac{x^{n+1}}{n+1} + c$
$x^{-1}, x > 0$	$\log_e x + c$
$x^{-1} = \dfrac{1}{x}, x \neq 0$	$\log_e \lvert x\rvert + c$
$\dfrac{1}{ax+b}, x \neq \dfrac{-b}{a}$	$\dfrac{1}{a}\log_e \lvert ax+b\rvert + c$
e^{kx+d}	$\dfrac{1}{k}e^{kx+d} + c$
$\sin(ax+b)$	$-\dfrac{1}{a}\cos(ax+b) + c$
$\cos(ax+b)$	$\dfrac{1}{a}\sin(ax+b) + c$
$(ax+b)^n$	$\dfrac{1}{a(n+1)}(ax+b)^{n+1} + c$

The definite integral

$$\int_a^b f(x)\,dx = F(b) - F(a) \text{ where } F \text{ is any antiderivative of } f$$

and $\int_a^b f(x)\,dx$ is called the **definite integral** from a to b.

The number a is called the **lower limit** of integration and b is called the **upper limit** of integration. The function f is called the **integrand**.

Example 1

Find antiderivatives of each of the following:

a $\sin\left(3x - \frac{\pi}{4}\right)$ **b** e^{3x+4} **c** $6x^3 - \frac{2}{x^2}$

Solution

a $\sin\left(3x - \frac{\pi}{4}\right)$ is of the form $\sin(ax + b)$

From the table $\int \sin(ax + b)\,dx = \frac{-1}{a}\cos(ax + b) + c$

$$\therefore \int \sin\left(3x - \frac{\pi}{4}\right) dx = \frac{-1}{3}\cos\left(3x - \frac{\pi}{4}\right) + c$$

b e^{3x+4} is of the form e^{kx+d}

$$\int e^{kx+d}\,dx = \frac{1}{k}e^{kx+d} + c$$

$$\therefore \int e^{3x+4}\,dx = \frac{1}{3}e^{3x+4} + c$$

c
$$\begin{aligned}\int 6x^3 - \frac{2}{x^2}\,dx &= \int 6x^3 - 2x^{-2}\,dx\\ &= \frac{6x^4}{4} + 2x^{-1} + c\\ &= \frac{3}{2}x^4 + \frac{2}{x} + c\end{aligned}$$

Example 2

Evaluate each of the following integrals:

a $\int_0^{\frac{\pi}{2}} \cos(3x)\,dx$ **b** $\int_0^1 e^{2x} - e^x\,dx$ **c** $\int_0^{\frac{\pi}{8}} \sec^2(2x)\,dx$ **d** $\int_0^1 \sqrt{2x+1}\,dx$

Solution

a
$$\begin{aligned}\int_0^{\frac{\pi}{2}} \cos(3x)\,dx &= \left[\frac{1}{3}\sin(3x)\right]_0^{\frac{\pi}{2}}\\ &= \frac{1}{3}\left(\sin\left(\frac{3\pi}{2}\right) - \sin(0)\right)\\ &= \frac{1}{3}(-1 - 0)\\ &= -\frac{1}{3}\end{aligned}$$

b $$\int_0^1 e^{2x} - e^x = \left[\frac{1}{2}e^{2x} - e^x\right]_0^1$$
$$= \frac{1}{2}e^2 - e^1 - \left(\frac{1}{2}e^0 - e^0\right)$$
$$= \frac{e^2}{2} - e - \left(\frac{1}{2} - 1\right)$$
$$= \frac{e^2}{2} - e + \frac{1}{2}$$

c From Chapter 6 it was found that if $f(x) = \tan(ax + b)$ then $f'(x) = a\sec^2(ax + b)$.

Hence $\int \sec^2(ax + b)\,dx = \frac{1}{a}\tan(ax + b) + c$

$$\therefore \quad \int_0^{\frac{\pi}{8}} \sec^2(2x)\,dx = \left[\tfrac{1}{2}\tan(2x)\right]_0^{\frac{\pi}{8}}$$
$$= \tfrac{1}{2}\left(\tan\frac{\pi}{4} - \tan 0\right)$$
$$= \tfrac{1}{2}(1 - 0)$$
$$= \tfrac{1}{2}$$

d $$\int_0^1 \sqrt{2x+1}\,dx = \int_0^1 (2x+1)^{\frac{1}{2}}\,dx$$
$$= \left[\frac{1}{2 \times \frac{3}{2}}(2x+1)^{\frac{3}{2}}\right]_0^1$$
$$= \tfrac{1}{3}\left[(2+1)^{\frac{3}{2}} - 1^{\frac{3}{2}}\right]$$
$$= \tfrac{1}{3}\left(3^{\frac{3}{2}} - 1\right)$$
$$= \tfrac{1}{3}(3\sqrt{3} - 1)$$

In the previous chapter it was shown that the derivative of $\log_e(|x|) = \frac{1}{x}$.

And by the chain rule the derivative of $\log_e(|ax + b|) = \frac{a}{ax + b}$.

Hence an antiderivative of $\frac{1}{ax + b} = \frac{1}{a}\log_e(|ax + b|)$.

Example 3

a Find an antiderivative of $\frac{1}{4x + 2}$

b Evaluate $\int_0^1 \frac{1}{4x + 2}\,dx$

c Evaluate $\int_{-2}^{-1} \frac{1}{4x + 2}\,dx$

Solution

a $\frac{1}{4x+2}$ is of the form $\frac{1}{ax+b}$ and $\int \frac{1}{ax+b}\,dx = \frac{1}{a}\log_e(|ax+b|)+c$

$$\therefore \int \frac{1}{4x+2}\,dx = \frac{1}{4}\log_e(|4x+2|)+c$$

b $$\int_0^1 \frac{1}{4x+2}\,dx = \left[\frac{1}{4}\log_e|4x+2|\right]_0^1$$
$$= \frac{1}{4}(\log_e(6)-\log_e(2))$$
$$= \frac{1}{4}\log_e(3)$$

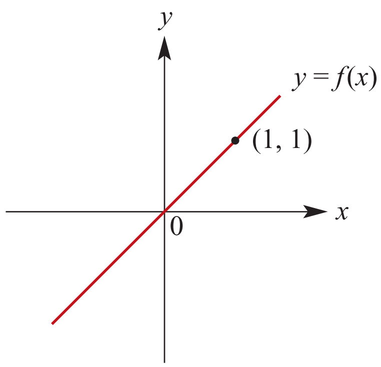
c $$\int_{-2}^{-1} \frac{1}{4x+2}\,dx = \left[\frac{1}{4}\log_e|4x+2|\right]_{-2}^{-1}$$
$$= \frac{1}{4}(\log_e(|-2|)-\log_e(|-6|))$$
$$= \frac{1}{4}\log_e\left(\frac{1}{3}\right)$$
$$= -\frac{1}{4}\log_e(3)$$

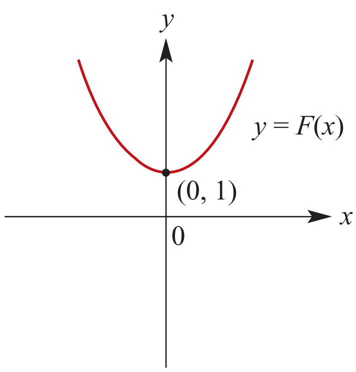
Graphs of functions and their antiderivatives

In the following, F and f are functions such that $F'(x) = f(x)$, i.e. $F(x)$ is an antiderivative of f.

Example 4

Consider the following graphs of $y = f(x)$ and $y = F(x)$.





Find: **a** $f(x)$ **b** $F(x)$

Solution

a $f(x) = mx$ and as $f(1) = 1$, $m = 1$

$\therefore \quad f(x) = x$

b $F(x) = \dfrac{x^2}{2} + c$ (by antidifferentiation)

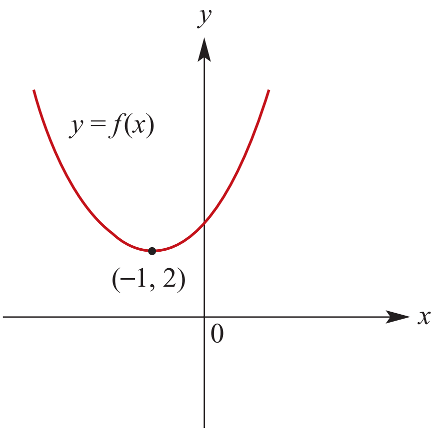
But $F(0) = 1$ and therefore $c = 1$

i.e. $F(x) = \dfrac{x^2}{2} + 1$

Note: The graph of $y = f(x)$ is the gradient graph for the graph of $y = F(x)$.
It has been shown that there are infinitely many graphs defined by $\int f(x)\,dx$.

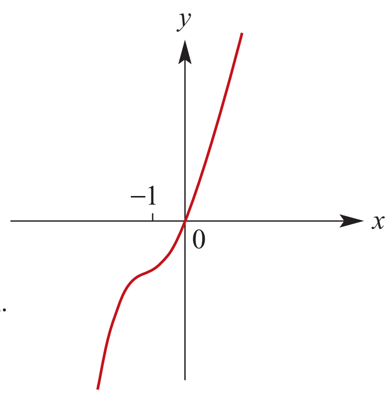
Example 5

The graph of $y = f(x)$ is as shown.
Sketch the graph of $y = F(x)$ given that $F(0) = 0$.



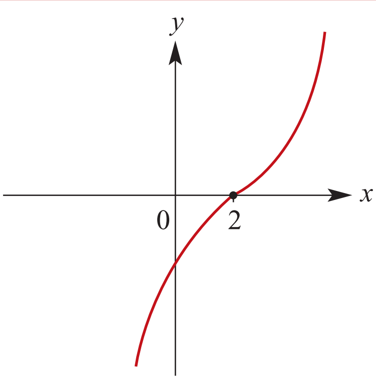
Solution

$y = f(x)$ is the gradient graph of $y = F(x)$. Therefore the gradient of $y = F(x)$ is always positive. The minimum gradient is 2 and this occurs when $x = -1$. There is a line of symmetry $x = -1$ which indicates equal gradients for x values equidistant from $x = -1$. Also $F(0) = 0$. A possible graph is shown.



Example 6

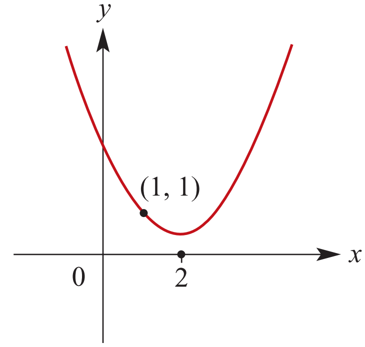
The graph of $y = f(x)$ is as shown. Sketch the graph of $y = F(x)$ given that $F(1) = 1$.



Solution

$y = f(x)$ is the gradient graph of $y = F(x)$. Therefore the gradient of $y = F(x)$ is positive for $x > 2$, negative for $x < 2$ and zero for $x = 2$.

A possible graph is shown.



Exercise 7A

1 Find an antiderivative of each of the following:

a $\sin\left(2x + \frac{\pi}{4}\right)$ **b** $\cos(\pi x)$ **c** $\sin\left(\frac{2\pi x}{3}\right)$ **d** e^{3x+1}

e $e^{5(x+4)}$ **f** $\frac{1}{3x - 2}$ **g** $\frac{3}{2x^2}$ **h** $6x^3 - 2x^2 + 4x + 1$

i $\frac{2x + 1}{x + 3}$

2 Evaluate each of the following integrals:

a $\int_{-1}^{1} e^x - e^{-x}\,dx$ **b** $\int_{0}^{2} 3x^2 + 2x + 4\,dx$ **c** $\int_{0}^{1} \frac{1}{3x + 2}\,dx$

d $\int_{0}^{\frac{\pi}{2}} \sin 2x\,dx$ **e** $\int_{2}^{3} \frac{3}{x^3}\,dx$ **f** $\int_{0}^{\frac{\pi}{4}} \cos(x) + 2x\,dx$

g $\int_{0}^{1} e^{3x} + x\,dx$ **h** $\int_{0}^{\frac{\pi}{2}} \cos(4x)\,dx$ **i** $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin \tfrac{1}{2}x\,dx$

j $\int_{0}^{\frac{\pi}{4}} \sec^2 x\,dx$ **k** $\int_{-3}^{-1} \frac{1}{3x - 2}\,dx$ **l** $\int_{-1}^{0} \frac{1}{4 - 3x}\,dx$

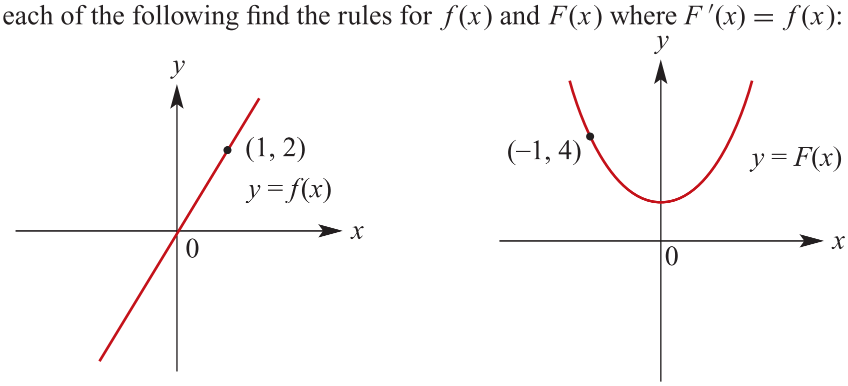
3 Find an antiderivative of each of the following:

a $(3x + 2)^5$ **b** $\sqrt{3x + 2}$ **c** $\frac{1}{(3x + 2)^2}$

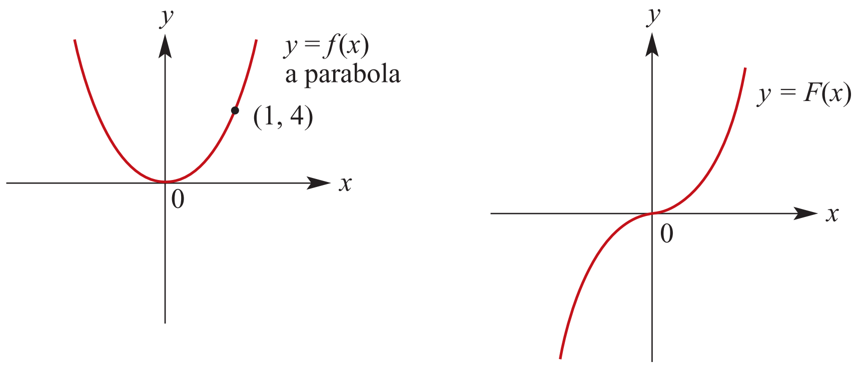
d $\frac{3x + 1}{x + 1}$ **e** $\cos \frac{3x}{2}$ **f** $(5x - 1)^{1/3}$

4 For each of the following find the rules for $f(x)$ and $F(x)$ where $F'(x) = f(x)$:

a

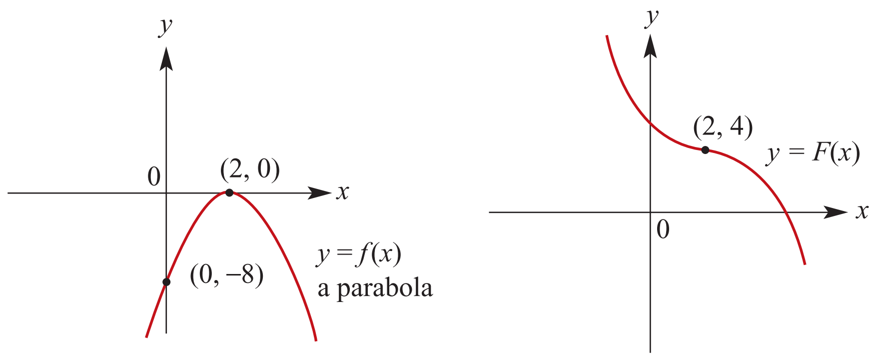


b



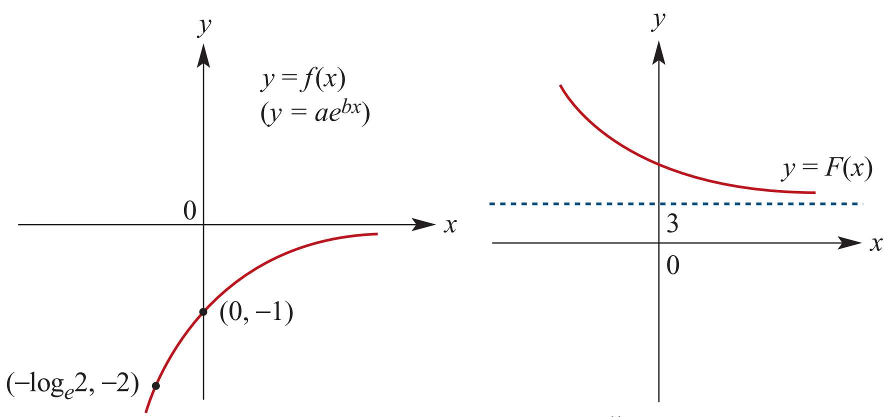
y
y = f(x)
a parabola
(1, 4)
0
x
y
y = F(x)
0
x

c



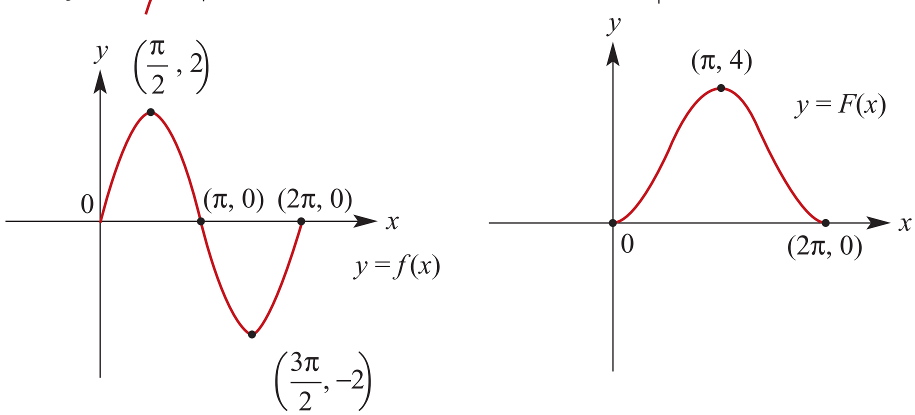
y
0
(2, 0)
x
(0, −8)
y = f(x)
a parabola
y
(2, 4)
y = F(x)
0
x

d



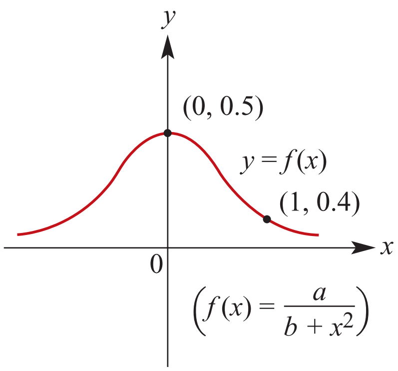
y
y = f(x)
(y = ae^bx)
0
x
(0, −1)
(−log_e 2, −2)
y
y = F(x)
3
0
x

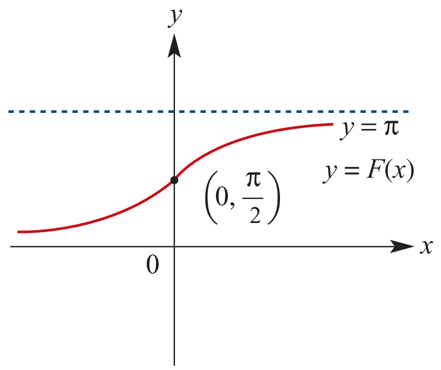
e



y
(π/2, 2)
0
(π, 0)
(2π, 0)
x
y = f(x)
(3π/2, −2)
y
(π, 4)
y = F(x)
0
(2π, 0)
x

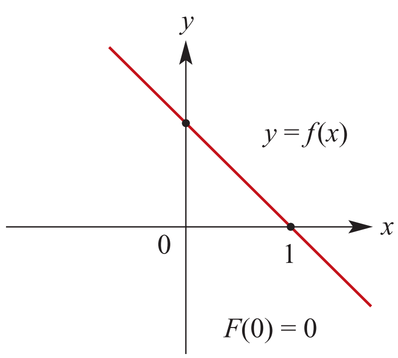
f



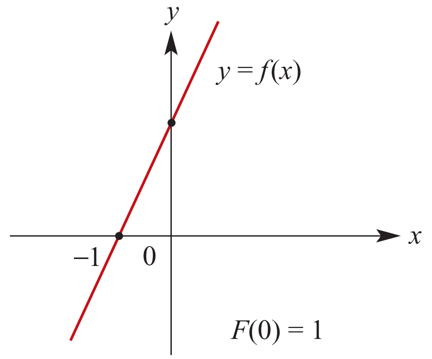


5 For each of the following the graph of $y = f(x)$ is shown. Sketch the graph of $y = F(x)$ for each of them where $F'(x) = f(x)$ and the value of $F(0)$ is given:

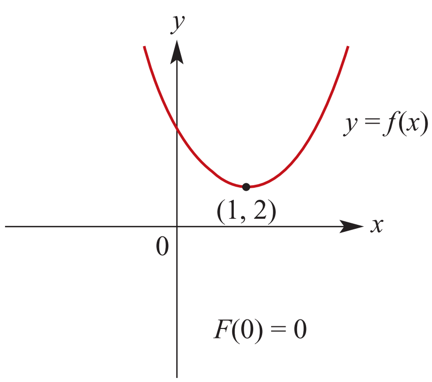
a



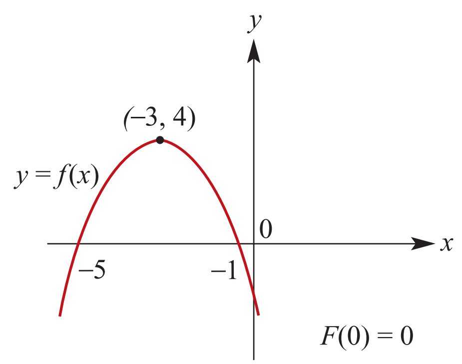
b



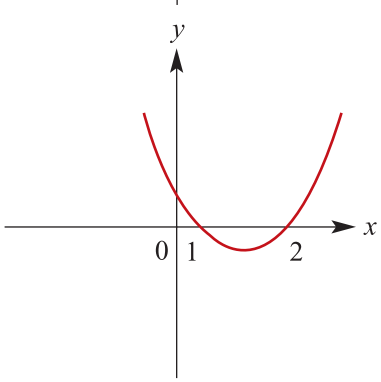
c



d

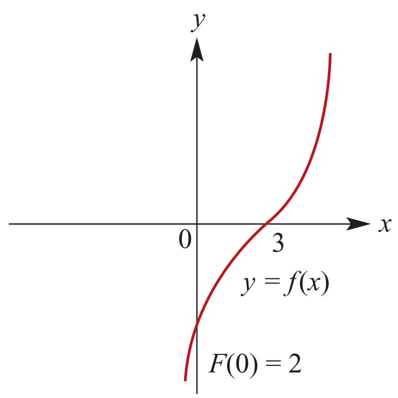


e



$F(0) = 0$

f



7.2 Antiderivatives involving inverse circular functions

In Chapter 6, the following rules for differentiation of inverse circular functions were established.

$$f: (-a, a) \to R,\ f(x) = \sin^{-1}\frac{x}{a},\ f'(x) = \frac{1}{\sqrt{a^2 - x^2}}$$

$$f: (-a, a) \to R,\ f(x) = \cos^{-1}\frac{x}{a},\ f'(x) = \frac{-1}{\sqrt{a^2 - x^2}}$$

$$f: R \to R,\ f(x) = \tan^{-1}\frac{x}{a},\ f'(x) = \frac{a}{a^2 + x^2}$$

From these results, the following can be stated:

$$\int \frac{1}{\sqrt{a^2 - x^2}}\,dx = \sin^{-1}\frac{x}{a} + c_1 \qquad x \in (-a, a)$$

$$\int \frac{-1}{\sqrt{a^2 - x^2}}\,dx = \cos^{-1}\frac{x}{a} + c_2 \qquad x \in (-a, a)$$

$$\int \frac{a}{a^2 + x^2}\,dx = \tan^{-1}\frac{x}{a} + c_3 \qquad x \in R$$

It can be noted from this that $\sin^{-1}\frac{x}{a} + \cos^{-1}\frac{x}{a} = -(c_1 + c_2)$ and it can be shown that $\sin^{-1}\frac{x}{a} + \cos^{-1}\frac{x}{a} = \frac{\pi}{2}$ for all $x \in (-a,\ a)$.

Example 7

Find an antiderivative of each of the following:

a $\dfrac{1}{\sqrt{9 - x^2}}$ **b** $\dfrac{1}{\sqrt{9 - 4x^2}}$ **c** $\dfrac{1}{9 + 4x^2}$

Solution

a $$\int \frac{1}{\sqrt{9 - x^2}}\,dx = \sin^{-1}\left(\frac{x}{3}\right) + c$$

b $$\int \frac{1}{\sqrt{9 - 4x^2}}\,dx = \int \frac{1}{2\sqrt{\frac{9}{4} - x^2}}\,dx$$
$$= \frac{1}{2}\int \frac{1}{\sqrt{\frac{9}{4} - x^2}}\,dx$$
$$= \frac{1}{2}\sin^{-1}\left(\frac{2x}{3}\right) + c$$

c $$\int \frac{1}{9 + 4x^2}\,dx = \int \frac{1}{4\left(\frac{9}{4} + x^2\right)}\,dx$$
$$= \frac{2}{3}\int \frac{\frac{3}{2}}{4\left(\frac{9}{4} + x^2\right)}\,dx$$
$$= \frac{1}{6}\int \frac{\frac{3}{2}}{\frac{9}{4} + x^2}\,dx$$
$$= \frac{1}{6}\tan^{-1}\left(\frac{2x}{3}\right) + c$$

Example 8

Evaluate each of the following definite integrals:

a $\displaystyle\int_0^1 \frac{1}{\sqrt{4-x^2}}\,dx$ **b** $\displaystyle\int_0^2 \frac{1}{4+x^2}\,dx$ **c** $\displaystyle\int_0^1 \frac{3}{\sqrt{9-4x^2}}\,dx$

Solution

a
$$\int_0^1 \frac{1}{\sqrt{4-x^2}}\,dx = \left[\sin^{-1}\left(\frac{x}{2}\right)\right]_0^1 = \sin^{-1}\left(\frac{1}{2}\right) - \sin^{-1} 0 = \frac{\pi}{6}$$

b
$$\int_0^2 \frac{1}{4+x^2}\,dx = \frac{1}{2}\int_0^2 \frac{2}{4+x^2}\,dx = \frac{1}{2}\left[\tan^{-1}\left(\frac{x}{2}\right)\right]_0^2 = \frac{1}{2}(\tan^{-1} 1 - \tan^{-1} 0) = \frac{\pi}{8}$$

c
$$\int_0^1 \frac{3}{\sqrt{9-4x^2}}\,dx = \int_0^1 \frac{3}{2\sqrt{\frac{9}{4}-x^2}}\,dx$$
$$= \frac{3}{2}\int_0^1 \frac{1}{\sqrt{\frac{9}{4}-x^2}}\,dx$$
$$= \frac{3}{2}\left[\sin^{-1}\left(\frac{2x}{3}\right)\right]_0^1$$
$$= \frac{3}{2}\sin^{-1}\frac{2}{3}$$
$$\approx 1.095$$

Exercise 7B

1 Evaluate each of the following integrals:

a $\displaystyle\int \frac{dx}{\sqrt{9-x^2}}$ **b** $\displaystyle\int \frac{dx}{5+x^2}$ **c** $\displaystyle\int \frac{dt}{1+t^2}$

d $\displaystyle\int \frac{5}{\sqrt{5-x^2}}\,dx$ **e** $\displaystyle\int \frac{3}{16+x^2}\,dx$ **f** $\displaystyle\int \frac{dx}{\sqrt{16-4x^2}}$

g $\displaystyle\int \frac{10}{\sqrt{10-t^2}}\,dt$ **h** $\displaystyle\int \frac{dt}{9+16t^2}$ **i** $\displaystyle\int \frac{dx}{\sqrt{5-2x^2}}$

j $\displaystyle\int \frac{7}{3+y^2}\,dy$

2 Evaluate each of the following:

a $\displaystyle\int_0^1 \frac{2}{1+x^2}\,dx$ **b** $\displaystyle\int_0^{\frac{1}{2}} \frac{3}{\sqrt{1-x^2}}\,dx$ **c** $\displaystyle\int_0^1 \frac{5}{\sqrt{4-x^2}}\,dx$

d $\displaystyle\int_0^5 \frac{6}{25+x^2}\,dx$ **e** $\displaystyle\int_0^{\frac{3}{2}} \frac{3}{9+4x^2}\,dx$ **f** $\displaystyle\int_0^2 \frac{dx}{8+2x^2}$

g $\displaystyle\int_0^{\frac{3}{2}} \frac{dx}{\sqrt{9-x^2}}$ **h** $\displaystyle\int_0^{\frac{3\sqrt{2}}{4}} \frac{dx}{\sqrt{9-4x^2}}$ **i** $\displaystyle\int_0^{\frac{1}{3}} \frac{3}{\sqrt{1-9y^2}}\,dy$

j $\displaystyle\int_0^{2} \frac{dx}{1+3x^2}$

7.3 Integration by substitution

In this section, the technique of substitution is introduced. The substitution will result in one of the forms for integrands reviewed in section 7.1.

First consider the following example.

Example 9

Differentiate each of the following with respect to x:

a $(2x^2+1)^5$ **b** $\cos^3 x$ **c** e^{3x^2}

Solution

a Let $y = (2x^2+1)^5$.

Let $u = 2x^2 + 1$. Then $y = u^5$, $\frac{dy}{du} = 5u^4$ and $\frac{du}{dx} = 4x$.

By the chain rule for differentiation:

$$\begin{aligned}\frac{dy}{dx} &= \frac{dy}{du}\frac{du}{dx}\\ &= 5u^4 \cdot 4x\\ &= 20u^4x\\ &= 20x(2x^2+1)^4\end{aligned}$$

b Let $y = \cos^3 x$. Let $u = \cos x$. Then $y = u^3$, $\frac{dy}{du} = 3u^2$ and $\frac{du}{dx} = -\sin x$.

By the chain rule for differentiation:

$$\begin{aligned}\frac{dy}{dx} &= \frac{dy}{du}\frac{du}{dx}\\ &= 3u^2 \cdot (-\sin x)\\ &= 3\cos^2 x(-\sin x)\\ &= -3\cos^2 x \sin x\end{aligned}$$

c Let $y = e^{3x^2}$. Let $u = 3x^2$. Then $y = e^u$, $\frac{du}{dx} = 6x$ and $\frac{dy}{du} = e^u$.

By the chain rule for differentiation:

$$\begin{aligned}\frac{dy}{dx} &= \frac{dy}{du}\frac{du}{dx}\\ &= e^u \cdot 6x\\ &= 6xe^{3x^2}\end{aligned}$$

Example 9 suggests that a 'converse' of the chain rule can be used to obtain a method for antidifferentiating functions of a particular form.

From Example 9a $\int 20x(2x^2+1)^4\,dx = (2x^2+1)^5 + c$

is of the form $\int 5h'(x)[h(x)]^4\,dx = [h(x)]^5 + c$ where $h(x) = (2x^2+1)$

From Example 9b $\int -3\cos^2 x \sin x\,dx = \cos^3 x + c$

is of the form $\int 3h'(x)[h(x)]^2\,dx = [h(x)]^3 + c$ where $h(x) = \cos x$

For Example 9c $\int 6xe^{3x^2}\,dx = e^{3x^2} + c$

is of the form $\int h'(x)e^{h(x)}\,dx = e^{h(x)} + c$ where $h(x) = 3x^2$

This suggests a method that can be used for integration.

e.g. $$\int 2x(x^2+1)^5\,dx = \frac{(x^2+1)^6}{6} + c \quad [h(x) = x^2+1]$$

$$\int \cos x \sin^2 x\,dx = \frac{\sin^3 x}{3} + c \quad [h(x) = \sin x]$$

A formalisation of this idea provides a method of integration for functions of this form.
Let $y = \int f(u)\,du$ where $u = g(x)$.
The chain rule for differentiation gives:

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$$

$$= f(u)\cdot\frac{du}{dx}$$

$$\therefore\ y = \int f(u)\frac{du}{dx}\,dx$$

and this implies:

$$\int f(u)\frac{du}{dx}\,dx = \int f(u)\,du$$

This method is called the **change of variable rule** or the **method of substitution**.

Example 10

Find antiderivatives of each of the following:

a $\sin x\cos^2 x$ **b** $5x^2(x^3-1)^{\frac{1}{2}}$ **c** $3xe^{x^2}$

Solution

a $\int \sin x \cos^2 x \, dx$

Let $u = \cos x$. Then $f(u) = u^2$ and $\frac{du}{dx} = -\sin x$ and thus:

$$
\begin{aligned}
-\int \cos^2 x(-\sin\ x)\, dx &= -\int f(u)\frac{du}{dx}\, dx \\
&= -\int f(u)\, du \\
&= -\int u^2\, du \\
&= \frac{-u^3}{3} + c \\
&= \frac{-\cos^3 x}{3} + c
\end{aligned}
$$

b $\int 5x^2(x^3 - 1)^{\frac{1}{2}}\, dx$

Let $u = x^3 - 1$. Then $f(u) = u^{\frac{1}{2}}$ and $\frac{du}{dx} = 3x^2$.

$$
\begin{aligned}
\therefore \quad \int 5x^2(x^3 - 1)^{\frac{1}{2}}\, dx &= \frac{5}{3}\int (x^3 - 1)^{\frac{1}{2}} 3x^2\, dx \\
&= \frac{5}{3}\int u^{\frac{1}{2}} \frac{du}{dx}\, dx \\
&= \frac{5}{3}\left(\frac{2}{3}u^{\frac{3}{2}}\right) + c \\
&= \frac{10}{9}u^{\frac{3}{2}} + c \\
&= \frac{10}{9}(x^3 - 1)^{\frac{3}{2}} + c
\end{aligned}
$$

c $\int 3xe^{x^2}\, dx$

Let $u = x^2$. Then $f(u) = e^u$ and $\frac{du}{dx} = 2x$.

$$
\begin{aligned}
\therefore \quad \int 3xe^{x^2}\, dx &= \tfrac{3}{2}\int e^u \cdot 2x\, dx \\
&= \tfrac{3}{2}\int f(u)\frac{du}{dx}\, dx \\
&= \tfrac{3}{2}\int e^u\, du \\
&= \tfrac{3}{2}e^u + c \\
&= \tfrac{3}{2}e^{x^2} + c
\end{aligned}
$$

Example 11

Find antiderivatives of each of the following:

a $\dfrac{2}{x^2 + 2x + 6}$ **b** $\dfrac{3}{(9 - 4x - x^2)^{\frac{1}{2}}}$

Solution

a $$\int \frac{2}{x^2 + 2x + 6}\,dx = \int \frac{2}{x^2 + 2x + 1 + 5}\,dx \quad \text{(completing the square)}$$

$$= \int \frac{2}{(x+1)^2 + 5}\,dx$$

Let $u = x + 1$. Then $\dfrac{du}{dx} = 1$ and hence

$$\int \frac{2}{(x+1)^2 + 5}\,dx = \int \frac{2}{u^2 + 5}\,du$$

$$= \frac{2}{\sqrt{5}} \int \frac{\sqrt{5}}{u^2 + 5}\,du$$

$$= \frac{2}{\sqrt{5}} \tan^{-1} \frac{u}{\sqrt{5}} + c$$

$$= \frac{2}{\sqrt{5}} \tan^{-1} \left(\frac{x+1}{\sqrt{5}}\right) + c$$

b $$\int \frac{3}{[-(x^2 + 4x - 9)]^{\frac{1}{2}}}\,dx = \int \frac{3}{(-[(x^2 + 4x + 4) - 9 - 4])^{\frac{1}{2}}}\,dx$$

$$= \int \frac{3}{(-[(x+2)^2 - 13])^{\frac{1}{2}}}\,dx$$

$$= \int \frac{3}{[13 - (x+2)^2]^{\frac{1}{2}}}\,dx$$

Let $u = x + 2$. Then $\dfrac{du}{dx} = 1$

and $$\int \frac{3}{[13 - (x+2)^2]^{\frac{1}{2}}}\,dx = \int \frac{3}{(13 - u^2)^{\frac{1}{2}}}\,du$$

$$= 3 \sin^{-1} \frac{u}{\sqrt{13}} + c$$

$$= 3 \sin^{-1} \frac{(x+2)}{\sqrt{13}} + c$$

Linear substitutions

The antiderivative of expressions such as:

$$(2x+3)\sqrt{3x-4} \qquad \frac{2x+5}{\sqrt{3x-4}} \qquad \frac{2x+4}{(x+2)^2} \qquad (2x+4)(x+3)^{20} \qquad x^2\sqrt{3x-1}$$

can be found using a linear substitution.

Example 12

Find an antiderivative of each of the following:

a $(2x+1)\sqrt{x+4}$ **b** $\dfrac{2x+1}{(1-2x)^2}$ **c** $x^2\sqrt{3x-1}$

Solution

a $\displaystyle\int (2x+1)\sqrt{x+4}\,dx$

Let $u = x+4$. Then $\dfrac{du}{dx} = 1$ and $x = u-4$

$$\begin{aligned}\therefore \int (2x+1)\sqrt{x+4}\,dx &= \int [2(u-4)+1]u^{\frac{1}{2}}\,du \\ &= \int (2u-7)u^{\frac{1}{2}}\,du \\ &= \int 2u^{\frac{3}{2}} - 7u^{\frac{1}{2}}\,du \\ &= 2\left(\tfrac{2}{5}u^{\frac{5}{2}}\right) - 7\left(\tfrac{2}{3}u^{\frac{3}{2}}\right) + c \\ &= \tfrac{4}{5}(x+4)^{\frac{5}{2}} - \tfrac{14}{3}(x+4)^{\frac{3}{2}} + c\end{aligned}$$

b $\displaystyle\int \frac{2x+1}{(1-2x)^2}\,dx$

Let $u = 1-2x$. Then $\dfrac{du}{dx} = -2$ and $2x = 1-u$

$$\begin{aligned}\therefore \int \frac{2x+1}{(1-2x)^2}\,dx &= -\tfrac{1}{2}\int \frac{2-u}{u^2}(-2)\,dx \\ &= -\tfrac{1}{2}\int \frac{2-u}{u^2}\frac{du}{dx}\,dx \\ &= -\tfrac{1}{2}\int 2u^{-2} - u^{-1}\,du \\ &= -\tfrac{1}{2}(-2u^{-1} - \log_e|u|) + c \\ &= u^{-1} + \tfrac{1}{2}\log_e|u| + c \\ &= \frac{1}{1-2x} + \tfrac{1}{2}\log_e|1-2x| + c\end{aligned}$$

c $\int x^2\sqrt{3x-1}\,dx$

Let $u = 3x - 1$. Then $\dfrac{du}{dx} = 3$ and $x = \dfrac{u+1}{3}$. Therefore $x^2 = \dfrac{(u+1)^2}{9}$.

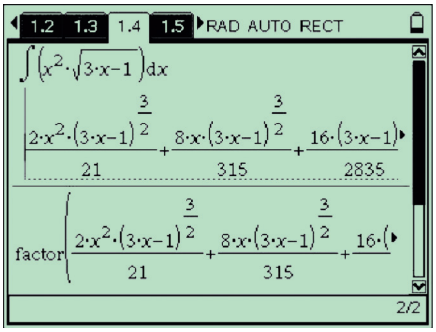
$$\begin{aligned}
\int x^2\sqrt{3x-1}\,dx &= \int \frac{(u+1)^2}{9}\sqrt{u} \\
&= \tfrac{1}{27}\int (u+1)^2 u^{\frac{1}{2}}(3)\,dx \\
&= \tfrac{1}{27}\int (u^2+2u+1)u^{\frac{1}{2}}\,\frac{du}{dx}\,dx \\
&= \tfrac{1}{27}\int \left(u^{\frac{5}{2}} + 2u^{\frac{3}{2}} + u^{\frac{1}{2}}\right) du \\
&= \tfrac{1}{27}\left(\tfrac{2}{7}u^{\frac{7}{2}} + \tfrac{4}{5}u^{\frac{5}{2}} + \tfrac{2}{3}u^{\frac{3}{2}}\right) + c \\
&= \tfrac{2}{27}u^{\frac{3}{2}}\left(\tfrac{1}{7}u^2 + \tfrac{24}{5}u + \tfrac{1}{3}\right) + c \\
&= \tfrac{2}{2835}(3x-1)^{\frac{3}{2}}\,[15(3x-1)^2 + 42(3x-1) + 35)] + c \\
&= \tfrac{2}{2835}(3x-1)^{\frac{3}{2}}\,(135x^2 + 36x + 8) + c
\end{aligned}$$

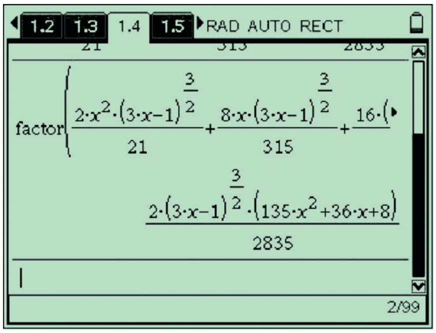
Using a TI-Nspire calculator

From the **Calculus** menu choose **Integral** (menu 4 2).

The derivative template can also be directly obtained through the Template menu of the catalog. Press ctrl and then [template key].

Use **factor** from the **Algebra** menu to obtain the required form.

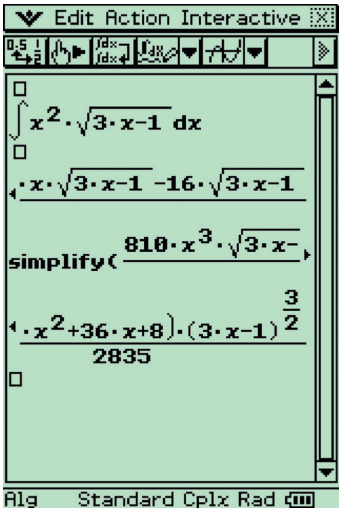




Using a Casio ClassPad calculator

Enter $\sqrt{x^2\,(3x-1)}$ and select it. Then choose **Interactive**, then **Calculation** and then $\int$. Make sure that **Indefinite integral** is selected and x is the variable.

Now apply **simplify** to the resulting expression.



Exercise 7C

1 Find each of the following:

a $\int 2x(x^2+1)^3\,dx$ **b** $\int \dfrac{x}{(x^2+1)^2}\,dx$ **c** $\int \cos x \sin^3 x\,dx$ **d** $\int \dfrac{\cos x}{\sin^2 x}\,dx$

e $\int (2x+1)^5\,dx$ **f** $\int 5x\sqrt{9+x^2}\,dx$ **g** $\int x(x^2-3)^5\,dx$ **h** $\int x(x^2-3)^5\,dx$

i $\int \dfrac{2}{(3x+1)^3}\,dx$ **j** $\int \dfrac{1}{\sqrt{1+x}}\,dx$ **k** $\int (x^2-2x)(x^3-3x^2+1)^4\,dx$

l $\int \dfrac{3x}{x^2+1}\,dx$ **m** $\int \dfrac{3x}{2-x^2}\,dx$

2 Find antiderivatives of each of the following:

a $\dfrac{1}{x^2+2x+2}$ **b** $\dfrac{1}{x^2-x+1}$ **c** $\dfrac{1}{\sqrt{21-4x-x^2}}$

d $\dfrac{1}{\sqrt{10x-x^2-24}}$ **e** $\dfrac{1}{\sqrt{40-x^2-6x}}$ **f** $\dfrac{1}{3x^2+6x+7}$

3 Find an antiderivative of each of the following:

a $x\sqrt{2x+3}$ **b** $x\sqrt{1-x}$ **c** $\dfrac{6x}{(3x-7)^{\frac{1}{2}}}$ **d** $(2x+1)\sqrt{3x-1}$

e $\dfrac{2x-1}{(x-1)^2}$ **f** $(x+3)\sqrt{3x+1}$ **g** $(x+2)(x+3)^{\frac{1}{3}}$ **h** $\dfrac{5x-1}{(2x+1)^2}$

i $x^2\sqrt{x-1}$ **j** $\dfrac{x^2}{\sqrt{x-1}}$

7.4 Definite integrals by substitution

Example 13

Evaluate $\int_0^4 3x\sqrt{x^2+9}\,dx$.

Solution

Let $u = x^2 + 9$, so that $\frac{du}{dx} = 2x$.

Then $\int 3x\sqrt{x^2+9}\,dx = \frac{3}{2}\int \sqrt{x^2+9}\cdot 2x\,dx$

$$= \frac{3}{2}\int u^{\frac{1}{2}}\frac{du}{dx}\,dx$$

$$= \frac{3}{2}\int u^{\frac{1}{2}}\,du$$

$$= \frac{3}{2}\left(\frac{2}{3}u^{\frac{3}{2}}\right) + c$$

$$= u^{\frac{3}{2}} + c = (x^2+9)^{\frac{3}{2}} + c$$

$\therefore \quad \int_0^4 3x\sqrt{x^2+9}\,dx = \left[(x^2+9)^{\frac{3}{2}}\right]_0^4$

$$= 25^{\frac{3}{2}} - 9^{\frac{3}{2}}$$

$$= 125 - 27$$

$$= 98$$

In a definite integral which involves the change of variable rule it is not necessary to return to an expression in x if the values of u corresponding to each of the limits of x are found.

Thus $\quad x = 0 \quad$ implies $u = 9$

and $\quad x = 4 \quad$ implies $u = 25$

and the integral can be evaluated as $\frac{3}{2}\int_9^{25} u^{\frac{1}{2}}\,du = \frac{3}{2}\left[\frac{2}{3}u^{\frac{3}{2}}\right]_9^{25}$

$$= 125 - 27$$

$$= 98$$

Example 14

Evaluate the following:

a $\int_0^{\frac{\pi}{2}} \cos^3 x\,dx$ **b** $\int_0^1 2x^2 e^{x^3}\,dx$

Solution

a $\int_0^{\frac{\pi}{2}} \cos^3 x\,dx = \int_0^{\frac{\pi}{2}} \cos x(\cos^2 x)\,dx = \int_0^{\frac{\pi}{2}} \cos x(1 - \sin^2 x)\,dx$

Let $u = \sin x$. Then $\frac{du}{dx} = \cos x$.

When $x = \frac{\pi}{2}$, $u = 1$ and when $x = 0, u = 0$

$$\therefore \text{ integral becomes } \int_0^1 (1 - u^2)\,du = \left[u - \frac{u^3}{3}\right]_0^1$$
$$= 1 - \tfrac{1}{3}$$
$$= \tfrac{2}{3}$$

b $\int_0^1 2x^2 e^{x^3}\,dx$

Let $u = x^3$. Then $\frac{du}{dx} = 3x^2$. When $x = 1, u = 1$ and when $x = 0, u = 0$

$$\therefore\ \tfrac{2}{3}\int_0^1 e^{x^3}\cdot(3x^2)\,dx = \tfrac{2}{3}\int_0^1 e^u\,du$$
$$= \tfrac{2}{3}[e^u]_0^1$$
$$= \tfrac{2}{3}(e^1 - e^0)$$
$$= \tfrac{2}{3}(e - 1)$$

Exercise 7D

Skillsheet TEACHER

1 Evaluate the following definite integrals:

a $\int_0^3 x\sqrt{x^2 + 16}\,dx$ **b** $\int_0^{\frac{\pi}{4}} \cos x \sin^3 x\,dx$

c $\int_0^{\frac{\pi}{2}} \sin x \cos^2 x\,dx$ **d** $\int_3^4 x(x - 3)^{17}\,dx$

e $\int_0^1 x\sqrt{1 - x}\,dx$ **f** $\int_e^{e^2} \frac{1}{x \log_e x}\,dx$

g $\int_0^4 \frac{1}{\sqrt{3x + 4}}\,dx$ **h** $\int_{-1}^1 \frac{e^x}{e^x + 1}\,dx$

i $\int_0^{\frac{\pi}{4}} \frac{\sin x}{\cos^3 x}\,dx$ **j** $\int_0^1 \frac{2x + 3}{x^2 + 3x + 4}\,dx$

k $\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{\cos x}{\sin x}\,dx$ **l** $\int_{-4}^{-3} \frac{2x}{1 - x^2}\,dx$

m $\int_{-2}^{-1} \frac{e^x}{1 - e^x}\,dx$

7.5 Use of trigonometric identities for integration

Products of sines and cosines

Integrals of the form $\int \sin^m x \cos^n x\,dx$, where m and n are non-negative integers, can be considered in the following cases:

A m is odd **B** m is even and n is odd **C** m and n are both even.

Case A:

If m is odd, write $m = 2k + 1$. Then $\sin^{2k+1} x$ can be written $(\sin^2 x)^k \sin x = (1 - \cos^2 x)^k \sin x$. The substitution $u = \cos x$ can now be made.

Case B:

If m is even and n is odd, write $n = 2k + 1$. Then $\cos^{2k+1} x$ can be written $(\cos^2 x)^k \cos x = (1 - \sin^2 x)^k \cos x$ and the substitution $u = \sin x$ can now be made.

Case C:

If m and n are both even, the identities $\sin^2 x = \dfrac{1 - \cos 2x}{2}$ or $\cos^2 x = \dfrac{1 + \cos 2x}{2}$ or $\sin 2x = 2 \sin x \cos x$ can be used.

Also note that $\int \sec^2 kx\,dx = \dfrac{1}{k} \tan kx + c$. The identity $1 + \tan^2 x = \sec^2 x$ is employed in the following example.

Example 15

Find:

a $\int \cos^2 x\,dx$ **b** $\int \tan^2 x\,dx$ **c** $\int \sin 2x \cos 2x\,dx$

d $\int \cos^4 x\,dx$ **e** $\int \sin^3 x \cos^2 x\,dx$

Solution

a Use the identity $\cos 2x = 2\cos^2 x - 1$.

By rearrangement of the identity:

$$\cos^2 x = \frac{\cos 2x + 1}{2}$$

and thus:

$$\begin{aligned}\int \cos^2 x\,dx &= \int \frac{\cos 2x + 1}{2}\,dx \\ &= \tfrac{1}{2} \int \cos 2x + 1\,dx \\ &= \tfrac{1}{2} \left(\tfrac{1}{2} \sin 2x + x\right) + c \\ &= \tfrac{1}{4} \sin 2x + \frac{x}{2} + c\end{aligned}$$

b Use the identity $1 + \tan^2 x = \sec^2 x$.

By rearrangement of the identity, $\tan^2 x = \sec^2 x - 1$

$$\therefore \quad \int \tan^2 x\,dx = \int \sec^2 x - 1\,dx$$
$$= \tan x - x + c$$

c Use the identity $\sin 2\theta = 2\sin\theta\cos\theta$.

Let $\theta = 2x$. Then $\sin 4x = 2\sin 2x\cos 2x$ and $\sin 2x\cos 2x = \frac{1}{2}\sin 4x$

$$\therefore \quad \int \sin 2x\cos 2x\,dx = \frac{1}{2}\int \sin 4x\,dx$$
$$= \frac{1}{2}\left(-\frac{1}{4}\cos 4x\right) + c$$
$$= -\frac{1}{8}\cos 4x + c$$

d $$\cos^4 x = (\cos^2 x)^2 = \left(\frac{\cos 2x + 1}{2}\right)^2$$
$$= \frac{1}{4}(\cos^2 2x + 2\cos 2x + 1)$$

As $\cos 4x = 2\cos^2 2x - 1$

$$\cos^4 x = \frac{1}{4}\left[\left(\frac{\cos 4x + 1}{2}\right) + 2\cos 2x + 1\right]$$
$$= \frac{1}{8}\cos 4x + \frac{1}{2}\cos 2x + \frac{3}{8}$$
$$\therefore \quad \int \cos^4 x\,dx = \int \frac{1}{8}\cos 4x + \frac{1}{2}\cos 2x + \frac{3}{8}\,dx$$
$$= \frac{1}{32}\sin 4x + \frac{1}{4}\sin 2x + \frac{3}{8}x + c$$

e $$\int \sin^3 x\cos^2 x\,dx = \int \sin x(\sin^2 x)\cos^2 x\,dx$$
$$= \int \sin x(1 - \cos^2 x)\cos^2 x\,dx$$

Let $u = \cos x$. Then $\frac{du}{dx} = -\sin x$

$$= -\int (-\sin x)(1 - u^2)(u^2)\,dx$$
$$= -\int (1 - u^2)u^2\frac{du}{dx}\,dx$$
$$= -\int u^2 - u^4\,dx$$
$$= -\left(\frac{u^3}{3} - \frac{u^5}{5}\right) + c$$
$$= \frac{\cos^5 x}{5} - \frac{\cos^3 x}{3} + c$$

Exercise 7E

Skillsheet TEACHER

1 Find an antiderivative of the following:

a $\sin^2 x$ **b** $\sin^4 x$ **c** $2\tan^2 x$ **d** $2\sin 3x \cos 3x$
e $\sin^2 2x$ **f** $\tan^2 2x$ **g** $\sin^2 x \cos^2 x$ **h** $\cos^2 x - \sin^2 x$
i $\cot^2 x$ **j** $\cos^3 2x$

2 Find an antiderivative of the following:

a $\sec^2 x$ **b** $\sec^2(2x)$ **c** $\sec^2\left(\frac{1}{2}x\right)$ **d** $\sec^2(kx)$
e $\tan^2(3x)$ **f** $1 - \tan^2 x$ **g** $\tan^2 x - \sec^2 x$ **h** $\operatorname{cosec}^2\left(x - \frac{\pi}{2}\right)$

3 Evaluate the following definite integrals:

a $\int_0^{\frac{\pi}{2}} \sin^2 x \, dx$ **b** $\int_0^{\frac{\pi}{4}} \tan^3 x \, dx$ **c** $\int_0^{\frac{\pi}{2}} \sin^2 x \cos x \, dx$
d $\int_0^{\frac{\pi}{4}} \cos^4 x \, dx$ **e** $\int_0^{\pi} \sin^3 x \, dx$ **f** $\int_0^{\frac{\pi}{2}} \sin^2 2x \, dx$
g $\int_0^{\frac{\pi}{3}} \sin^2 x \cos^2 x \, dx$ **h** $\int_0^{1} \sin^2 x + \cos^2 x \, dx$

4 Find an antiderivative of each of the following:

a $\cos^3 x$ **b** $\sin^3 \frac{x}{4}$ **c** $\cos^2(4\pi x)$ **d** $7\cos^7 t$
e $\cos^3(5x)$ **f** $8\sin^4 x$ **g** $\sin^2 x \cos^4 x$ **h** $\cos^5 x$

7.6 Partial fractions

Example 16

Express $\dfrac{5}{2x+1} - \dfrac{3}{x-1}$ as a single fraction.

Solution

$$\begin{aligned}\frac{5}{2x+1} - \frac{3}{x-1} &= \frac{5(x-1) - 3(2x+1)}{(2x+1)(x-1)} \\ &= \frac{5x - 5 - 6x - 3}{(2x+1)(x-1)} \\ &= \frac{-x-8}{(2x+1)(x-1)}\end{aligned}$$

Sometimes it is essential to reverse the process, i.e. to take an algebraic fraction such as $\dfrac{-x-8}{(2x+1)(x-1)}$ and express it as the sum or difference of several partial fractions.

e.g. Find A and B such that: $\dfrac{9x+1}{(x-3)(x+1)} \equiv \dfrac{A}{x-3} + \dfrac{B}{x+1}$

The symbol $\equiv$ which means 'is identically equal to' signifies that the two expressions are equal for all allowable values of x.

Expressing the right-hand side with a common denominator:

$$\frac{9x+1}{(x-3)(x+1)} \equiv \frac{A(x+1)+B(x-3)}{(x-3)(x+1)}$$

which means:

$$9x+1 \equiv A(x+1)+B(x-3)$$

Two polynomials of degree n are equal if they coincide for more than n values.

Therefore, if values of A and B are found such that the polynomials $9x+1$ and $A(x+1)+B(x-3)$ are equal for $x=-1$ and $x=3$, they must be equal for all values of x for these values of A and B. The values that have been chosen are for convenience. The result can be achieved through a choice of any two values for x.

- When $x=-1$, $9x+1=-8$ and $A(x+1)+B(x-3)=-4B$.
  For the two polynomials to be equal $-4B=-8$ and therefore $B=2$
- When $x=3$, $9x+1=28$ and $A(x+1)+B(x-3)=4A$
  For the two polynomials to be equal $4A=28$ and therefore $A=7$
  That is, $A=7$ and $B=2$

This implies: $\dfrac{9x+1}{(x-3)(x+1)} \equiv \dfrac{7}{x-3}+\dfrac{2}{x+1}$

This technique holds for an algebraic fraction with non-repeated linear factors in the denominator,

i.e. $$\frac{a_{n-1}x^{n-1}+a_{n-2}x^{n-2}+\cdots+a_0}{(x-x_1)(x-x_2)(x-x_3)\ldots(x-x_n)} \equiv \frac{A_1}{x-x_1}+\frac{A_2}{x-x_2}+\cdots+\frac{A_n}{x-x_n}$$

Example 17

Find $\displaystyle\int \frac{9x+1}{(x-3)(x+1)}\,dx$.

Solution

Using the partial fractions found above:

$$\frac{9x+1}{(x-3)(x+1)} \equiv \frac{7}{x-3}+\frac{2}{x+1}$$

and thus:
$$\begin{aligned}\int \frac{9x+1}{(x-3)(x+1)}\,dx &= \int \frac{7}{x-3}\,dx+\int \frac{2}{x+1}\,dx \\ &= 7\log_e|x-3|+2\log_e|x+1|+c\end{aligned}$$

Using the logarithm rules:

$$\int \frac{9x+1}{(x-3)(x+1)}\,dx = \log_e|x-3|^7\,|x+1|^2+c$$

Special case 1

If the degree of the denominator is less than or equal to the degree of the numerator, then division must take place first.

Example 18

Express $\frac{x^5+2}{x^2-1}$ as a partial fraction and hence find $\int \frac{x^5+2}{x^2-1}\,dx$.

Solution

Dividing through:

$$\begin{array}{r l} & x^3 + x \\ x^2 - 1\,\big) & \overline{x^5 + 2} \\ & \underline{x^5 - x^3} \\ & \quad x^3 \\ & \quad \underline{x^3 - x} \\ & \qquad x + 2 \end{array}$$

$$\therefore \qquad \frac{x^5+2}{x^2-1} = x^3 + x + \frac{x+2}{x^2-1}$$

Expressing $\frac{x+2}{x^2-1} = \frac{x+2}{(x-1)(x+1)}$ as partial fractions:

$$\frac{x^5+2}{x^2-1} = x^3 + x - \frac{1}{2(x+1)} + \frac{3}{2(x-1)}$$

Therefore:

$$\begin{aligned}\int \frac{x^5+2}{x^2-1}\,dx &= \int x^3 + x - \frac{1}{2(x+1)} + \frac{3}{2(x-1)}\,dx \\ &= \frac{x^4}{4} + \frac{x^2}{2} - \frac{1}{2}\log_e|x+1| + \frac{3}{2}\log_e|x-1| \\ &= x^4 + \frac{x^2}{2} + \frac{1}{2}\log_e\left(\frac{|x-1|^3}{|x+1|}\right)\end{aligned}$$

Special case 2

When the denominator is a perfect square quadratic, the following technique can be used.

Example 19

Express $\frac{3x+1}{(x+2)^2}$ in partial fractions and hence find $\int \frac{3x+1}{(x+2)^2}\,dx$.

Solution

Write $\dfrac{3x+1}{(x+2)^2} = \dfrac{a}{x+2} + \dfrac{b}{(x+2)^2}$

Then $3x + 1 \equiv a(x+2) + b$

When $x = -2$

$-5 = b$

when $x = 0$

$1 = 2a + b$

$\therefore$ $a = 3$

$\therefore$ $\dfrac{3x+1}{(x+2)^2} = \dfrac{3}{x+2} - \dfrac{5}{(x+2)^2}$

and $\displaystyle\int \frac{3x+1}{(x+2)^2}\,dx = \int \frac{3}{x+2} - \frac{5}{(x+2)^2}\,dx$

$= 3\log_e|x+2| + \dfrac{5}{x+2} + c$

Exercise 7F

1 Decompose each of the following into partial fractions and find their antiderivatives:

a $\dfrac{7}{(x-2)(x+5)}$ **b** $\dfrac{x+3}{x^2-3x+2}$ **c** $\dfrac{2x+1}{(x+1)(x-1)}$

d $\dfrac{2x^2}{x^2-1}$ **e** $\dfrac{2x+1}{x^2+4x+4}$ **f** $\dfrac{4x-2}{(x-2)(x+4)}$

2 Find an antiderivative of the following:

a $\dfrac{2x-3}{x^2-5x+6}$ **b** $\dfrac{5x+1}{(x-1)(x+2)}$ **c** $\dfrac{x^3-2x^2-3x+9}{x^2-4}$

d $\dfrac{4x+10}{x^2+5x+4}$ **e** $\dfrac{x^3+x^2-3x+3}{x+2}$ **f** $\dfrac{x^3+3}{x^2-x}$

3 Evaluate the following:

a $\displaystyle\int_1^2 \frac{1}{x(x+1)}\,dx$ **b** $\displaystyle\int_0^1 \frac{1}{(x+1)(x+2)}\,dx$ **c** $\displaystyle\int_2^3 \frac{x-2}{(x-1)(x+2)}\,dx$

d $\displaystyle\int_0^1 \frac{x^2}{x^2+3x+2}\,dx$ **e** $\displaystyle\int_2^3 \frac{x+7}{(x+3)(x-1)}\,dx$ **f** $\displaystyle\int_2^3 \frac{2x+6}{(x-1)^2}\,dx$

g $\displaystyle\int_2^3 \frac{x+2}{x(x+4)}\,dx$ **h** $\displaystyle\int_0^1 \frac{1-4x}{3+x-2x^2}\,dx$ **i** $\displaystyle\int_1^2 \frac{1}{x(x-4)}\,dx$

j $\displaystyle\int_{-3}^{-2} \frac{1-4x}{(x+6)(x+1)}\,dx$

7.7 Further techniques and miscellaneous exercises

In this section, the different techniques are used and arranged so that a choice must be made of the most suitable one for a particular problem. Often there is more than one appropriate technique.

The relationship between a function and its derivative is also exploited. This is illustrated in the following example.

Example 20

a Find the derivative of $\sin^{-1}(x) + x\sqrt{(1-x^2)}$.

b Hence evaluate $\int_0^{\frac{1}{2}} \sqrt{(1-x^2)}\,dx$.

Solution

a Let $y = \sin^{-1}(x) + x\sqrt{1-x^2}$

$$\frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}} + \left(\sqrt{1-x^2} + \frac{(-x)x}{\sqrt{1-x^2}}\right) \text{(using the product rule for } x\sqrt{1-x^2}\text{)}$$

$$= \frac{1}{\sqrt{1-x^2}} + \frac{1-x^2-x^2}{\sqrt{1-x^2}}$$

$$= \frac{2(1-x^2)}{\sqrt{1-x^2}}$$

$$= 2\sqrt{1-x^2}$$

b From **a** $\int 2\sqrt{1-x^2}\,dx = \sin^{-1}(x) + x\sqrt{1-x^2} + c$

$$\therefore \int_0^{\frac{1}{2}} 2\sqrt{1-x^2}\,dx = [\sin^{-1}(x) + x\sqrt{1-x^2}]_0^{\frac{1}{2}}$$

$$\therefore \int_0^{\frac{1}{2}} \sqrt{1-x^2}\,dx = \tfrac{1}{2}\left[\sin^{-1}\left(\tfrac{1}{2}\right) + \tfrac{1}{2}\sqrt{1-\left(\tfrac{1}{2}\right)^2} - (\sin^{-1}(0)+0)\right]$$

$$= \frac{1}{2}\left[\frac{\pi}{6} + \frac{1}{2} \times \frac{\sqrt{3}}{2}\right]$$

$$= \frac{\pi}{12} + \frac{\sqrt{3}}{8}$$

Example 21

a Find the derivative of xe^{2x}.

b Hence find $\int xe^{2x}\,dx$.

Solution

a Let $y = xe^{2x}$

$$\therefore \frac{dy}{dx} = e^{2x} + 2xe^{2x}$$

b $\therefore \quad \int e^{2x} + 2xe^{2x}\, dx = xe^{2x} + c_1$

$\therefore \quad 2\int xe^{2x}\, dx = -\int e^{2x}\, dx + xe^{2x} + c_1$

$\therefore \quad 2\int xe^{2x} dx = -\frac{1}{2}e^{2x} + c_2 + xe^{2x} + c_1$

$\therefore \quad \int xe^{2x}\, dx = \frac{1}{2}\left(-\frac{1}{2}e^{2x} + xe^{2x}\right) + \frac{c_2 + c_1}{2}$

Let $\quad c = \frac{c_2 + c_1}{2}$

$\therefore \quad \int xe^{2x}\, dx = -\frac{1}{4}e^{2x} + \frac{1}{2}xe^{2x} + c$

Exercise 7G

Skillsheet TEACHER

1 If $\int_0^1 \frac{1}{(x+1)(x+2)}\, dx = \log_e p$, find p.

2 Evaluate $\int_0^{\frac{\pi}{6}} \sin^2 x \cos x\, dx$.

3 Evaluate $\int_0^1 \frac{e^{2x}}{1+e^x}\, dx$.

4 Evaluate $\int_0^{\frac{\pi}{3}} \sin^3 x \cos x\, dx$.

5 Evaluate $\int_3^4 \frac{x}{(x-2)(x+1)}\, dx$.

6 Find c if $\int_0^{\frac{\pi}{6}} \frac{\cos x\, dx}{1+\sin x} = \log_e c$.

7 Find an antiderivative of $\sin 3x \cos^5 3x$.

8 If $\int_4^6 \frac{2}{x^2-4}\, dx = \log_e p$, find p.

9 If $\int_5^6 \frac{3}{x^2-5x+4}\, dx = \log_e p$, find p.

10 Find an antiderivative of:

a $\frac{\cos x}{\sin^3 x}$ **b** $x(4x^2+1)^{\frac{3}{2}}$ **c** $\sin^2 x \cos^3 x$ **d** $\frac{e^x}{e^{2x} - 2e^x + 1}$

11 Evaluate $\int_0^3 \frac{x}{\sqrt{25-x^2}}\, dx$.

12 Find an antiderivative of each of the following:

a $\frac{1}{(x+1)^2+4}$ **b** $\frac{1}{\sqrt{1-9x^2}}$ **c** $\frac{1}{\sqrt{1-4x^2}}$ **d** $\frac{1}{(2x+1)^2+9}$

13 Let $f: (1, \infty) \to R$ where $f(x) = \sin^{-1}\left(\frac{1}{\sqrt{x}}\right)$.

a Find $f'(x)$.

b Using the result of **a** find $\int_2^4 \frac{1}{x\sqrt{x-1}}\, dx$.

14 For each of the following, use an appropriate substitution to find an expression for the antiderivative in terms of $f(x)$:

a $\displaystyle\int f'(x)[f(x)]^2\,dx$ **b** $\displaystyle\int \frac{f'(x)}{[f(x)]^2}\,dx$

c $\displaystyle\int \frac{f'(x)}{f(x)}\,dx, f(x) > 0$ **d** $\displaystyle\int f'(x)\sin[f(x)]\,dx$

15 If $y = x\sqrt{4-x}$, find $\dfrac{dy}{dx}$ and simplify. Hence evaluate $\displaystyle\int_0^2 \frac{8-3x}{\sqrt{4-x}}\,dx$.

16 Find constants a, b and c, such that: $\dfrac{2x^3-11x^2+20x-13}{(x-2)^2} \equiv ax+b+\dfrac{c}{(x-2)^2}$.

Hence find $\displaystyle\int \frac{2x^3-11x^2+20x-13}{(x-2)^2}\,dx$.

17 Evaluate each of the following:

a $\displaystyle\int_0^{\frac{\pi}{4}} \sin^2 2x\,dx$ **b** $\displaystyle\int_{-1}^{0} (14-2x)\sqrt{x^2-14x+1}\,dx$ **c** $\displaystyle 9\int_{-\frac{1}{3}\pi}^{\frac{1}{3}\pi} \frac{\sin x}{\sqrt{\cos x}}\,dx$

d $\displaystyle\int_e^{e^2} \frac{dx}{x\log_e x}$ **e** $\displaystyle\int_0^{\frac{\pi}{4}} \tan^2 x\,dx$ **f** $\displaystyle\int_0^{\frac{\pi}{2}} \frac{\sin x}{2+\cos x}\,dx$

18 Find $\displaystyle\int \sin x\cos x\,dx$ using:

a the substitution $u = \sin x$

b the identity $\sin(2x) = 2\sin x\cos x$

19 **a** If $y = \log_e(x+\sqrt{x^2+1})$ find $\dfrac{dy}{dx}$. Hence find $\displaystyle\int \frac{1}{\sqrt{x^2+1}}\,dx$.

b If $y = \log_e(x+\sqrt{x^2-1})$ find $\dfrac{dy}{dx}$. Hence show that $\displaystyle\int_2^7 \frac{1}{\sqrt{x^2-1}}\,dx = \log_e(2+\sqrt{3})$.

20 Find an antiderivative of each of the following:

a $\dfrac{1}{4+x^2}$ **b** $\dfrac{1}{4-x^2}$ **c** $\dfrac{4+x^2}{x}$ **d** $\dfrac{x}{4+x^2}$

e $\dfrac{x^2}{4+x^2}$ **f** $\dfrac{1}{1+4x^2}$ **g** $x\sqrt{4+x^2}$ **h** $x\sqrt{4+x}$

i $\dfrac{1}{\sqrt{4-x}}$ **j** $\dfrac{1}{\sqrt{4-x^2}}$ **k** $\dfrac{x}{\sqrt{4-x}}$ **l** $\dfrac{x}{\sqrt{4-x^2}}$

21 **a** If $y = x\cos x$ find $\dfrac{dy}{dx}$ and hence find $\displaystyle\int x\sin x\,dx$.

b Hence evaluate $\displaystyle\int_0^{\pi} (x-\pi)\sin x\,dx$.

22 Find constants c and d such that $\displaystyle\int_2^3 \frac{x^3-x+2}{x^2-1}\,dx = c+\log_e d$.

23 **a** Differentiate $f(x) = \sin(x)\cos^{n-1}(x)$.

b Hence verify that $n\int \cos^n x\,dx = \sin(x)\cos^{n-1}(x) + (n-1)\int \cos^{n-2}(x)\,dx$.

c Hence evaluate:

i $\int_0^{\frac{\pi}{2}} \cos^4 x\,dx$ **ii** $\int_0^{\frac{\pi}{2}} \cos^6 x\,dx$

iii $\int_0^{\frac{\pi}{2}} \cos^4 x \sin^2 x\,dx$ **iv** $\int_0^{\frac{\pi}{4}} \sec^4(x)\,dx$

24 Evaluate:

a $\int \dfrac{x\,dx}{(x+1)^n}$ **b** $\int_1^2 x(x-1)^n\,dx$

25 **a** Evaluate $\int_0^1 (1+ax)^2\,dx$.

b For what value of a is the value of this integral a minimum?

26 **a** Differentiate $\dfrac{a\sin x - b\cos x}{a\cos x + b\sin x}$ with respect to x.

b Hence evaluate $\int_0^{\frac{\pi}{2}} \dfrac{dx}{(a\cos x + b\sin x)^2}$.

27 Let $U_n = \int_0^{\frac{\pi}{4}} \tan^n x\,dx$ where $n \in Z$ and $n > 1$.

a Express $U_n + U_{n-2}$ in terms of n. **b** Hence show that $U_6 = \dfrac{13}{15} - \dfrac{\pi}{4}$.

28 **a** Simplify $\dfrac{1}{1+\tan x} + \dfrac{1}{1+\cot x}$.

b Let $\phi = \dfrac{\pi}{2} - \theta$. Show that $\int_0^{\frac{\pi}{2}} \dfrac{d\theta}{1+\tan\theta} = \int_0^{\frac{\pi}{2}} \dfrac{d\phi}{1+\cot\phi}$.

c Use these results to evaluate $\int_0^{\frac{\pi}{2}} \dfrac{d\theta}{1+\tan\theta}$.

MC TEST D&D TEST

Review

Chapter summary

- $$\int \frac{1}{\sqrt{a^2 - x^2}}\,dx = \sin^{-1}\left(\frac{x}{a}\right) + c_1$$
  $$\int \frac{-1}{\sqrt{a^2 - x^2}}\,dx = \cos^{-1}\left(\frac{x}{a}\right) + c_2$$
  $$\int \frac{a}{a^2 + x^2}\,dx = \tan^{-1}\left(\frac{x}{a}\right) + c_3$$

- **Change of variable rule** or **method of substitution**

  $$\int f(u)\frac{du}{dx}\,dx = \int f(u)\,du, \text{ where } u \text{ is a function of } x$$

- **Linear substitution**
  The antiderivative of expressions, such as $(2x+3)\sqrt{3x-4}$, $\dfrac{2x+4}{\sqrt{3x-5}}$ and $\dfrac{2x+5}{(x+2)^2}$, can be found using linear subsitution.

  Consider $\int f(x)g(ax+b)\,dx$

  Let $u = ax + b$

  Then $x = \dfrac{u-b}{a}$

  and
  $$\int f(x)g(ax+b)\,dx = \int f\left(\frac{u-b}{a}\right)g(u)\,dx$$
  $$= \frac{1}{a}\int f\left(\frac{u-b}{a}\right)g(u)\,du$$

- Definite integration involving the change of variable rule

  Let $u = g(x)$

  Then $$\int_a^b f(u)\frac{du}{dx}\,dx = \int_{g(a)}^{g(b)} f(u)\,du$$

- Use of **trigonometric identities**

  $\sin 2x = 2\sin x\cos x$

  $\cos 2x = 2\cos^2 x - 1$

  $= 1 - 2\sin^2 x$

  $= \cos^2 x - \sin^2 x$

  $1 + \tan^2 x = \sec^2 x$

- Partial fractions

  For the algebraic fraction $\dfrac{ax+b}{(x-x_1)(x-x_2)}$

  $$\frac{ax+b}{(x-x_1)(x-x_2)} = \frac{A_1}{(x-x_1)} + \frac{A_2}{x-x_2}$$

- For the algebraic fraction $\dfrac{ax+b}{(x-x_1)^2}$

  $$\frac{ax+b}{(x-x_1)^2} = \frac{A}{(x-x_1)^2} + \frac{B}{(x-x_1)}$$

- If $P(x)$ and $Q(x)$ are polynomials in x and the degree of $P(x)$ is greater than or equal to the degree of $Q(x)$, then the algebraic fraction $\dfrac{P(x)}{Q(x)}$ should be written in the form

  $$\frac{P(x)}{Q(x)} = H(x) + \frac{R(x)}{Q(x)}$$

  where the degree of $R(x)$ is less than the degree of $Q(x)$ before partial fractions are considered.

Multiple-choice questions

1 An antiderivative of $x\sqrt{4-x}$ is:

A $(4-x)^{\frac{1}{2}} - \frac{x}{2}(4-x)^{\frac{-1}{2}}$ **B** $\frac{2x}{3}(4-x)^{\frac{3}{2}}$ **C** $\frac{x^2}{3}(4-x)^{\frac{3}{2}}$

D $\frac{8}{3}(4-x)^{\frac{3}{2}} - \frac{2}{5}(4-x)^{\frac{5}{2}}$ **E** $\frac{2}{5}(4-x)^{\frac{5}{2}} - \frac{8}{3}(4-x)^{\frac{3}{2}}$

2 $\displaystyle\int_0^m \tan x \sec^2 x \, dx = \frac{3}{2}$ where $m \in (0, \frac{\pi}{2})$. The value of m is:

A 0.5 **B** 1 **C** $\frac{\pi}{3}$ **D** $\frac{\pi}{6}$ **E** $\frac{\pi}{8}$

3 An antiderivative of $\tan(2x)$ is:

A $\frac{1}{2}\sec^2(2x)$ **B** $\frac{1}{2}\log_e(\cos(2x))$ **C** $\frac{1}{2}\log_e(\sec(2x))$

D $\frac{1}{2}\log_e(\sin(2x))$ **E** $\frac{1}{2}\tan^2(2x)$

4 $\displaystyle\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \frac{\sin(2x)}{2+\cos(2x)} dx$ is equal to:

A $\frac{1}{\sqrt{2}}$ **B** $\log_e\left(\frac{1}{\sqrt{2}}\right)$ **C** $\log_e 2$ **D** $\frac{1}{2}\log_e 2$ **E** 1

5 $\displaystyle\int_0^{\frac{\pi}{3}} \sin x \cos^3 x \, dx$ written as an integral with respect to u, where $u = \cos x$, is:

A $\displaystyle\int_{\frac{1}{2}}^{1} u^3 du$ **B** $\displaystyle\int_0^{\frac{\pi}{3}} u^3 du$ **C** $\displaystyle\int_1^{\frac{1}{2}} u^3\sqrt{1-u^2} du$

D $\displaystyle\int_{\frac{1}{2}}^{0} u^3\sqrt{1-u^2} du$ **E** $\displaystyle\int_1^{\frac{1}{2}} u^3 du$

Review

6 The value of $\int_0^2 \cos^2 x - \sin^2 x \, dx$ correct to four decimal places is:

A -0.0348 **B** 0.0349 **C** -0.3784 **D** 2.0000 **E** 0.3784

7 An antiderivative of $\dfrac{2}{\sqrt{1-16x^2}}$ is:

A $\sin^{-1}\left(\frac{x}{4}\right)$ **B** $\frac{1}{2}\sin^{-1}\left(\frac{x}{4}\right)$ **C** $\sin^{-1}(4x)$ **D** $\frac{1}{2}\sin^{-1}(4x)$ **E** $\frac{1}{8}\sin^{-1}(4x)$

8 An antiderivative of $\dfrac{1}{9+4x^2}$ is:

A $\frac{1}{9}\tan^{-1}\left(\frac{2x}{9}\right)$ **B** $\frac{1}{3}\tan^{-1}\left(\frac{2x}{3}\right)$ **C** $\frac{1}{6}\tan^{-1}\left(\frac{2x}{3}\right)$ **D** $9\tan^{-1}\left(\frac{2x}{9}\right)$ **E** $\frac{3}{2}\tan^{-1}\left(\frac{2x}{3}\right)$

9 If $\dfrac{d(xf(x))}{dx} = xf'(x) + f(x)$ and $xf'(x) = \frac{1}{1+x^2}$ then an antiderivative of $f(x)$ is:

A $xf(x) - \tan^{-1}(x)$ **B** $\log_e(x^2+1)$ **C** $\frac{1}{2x}\log_e(x^2+1)$

D $f(x) - \tan^{-1}(x)$ **E** $\tan^{-1}(x)$

10 If $F'(x) = f(x)$, then an antiderivative of $3f(3-2x)$ is:

A $\frac{3}{2}F(3-2x)$ **B** $\frac{-3}{4}(3-2x)^2$ **C** $\frac{3}{4}(3-2x)^2$

D $\frac{-3}{2}F(3-2x)$ **E** $\frac{-3}{2}f(3-2x)$

Short-answer questions (technology-free)

1 Find an antiderivative of each of the following:

a $\cos^3 2x$ **b** $\dfrac{2x+3}{4x^2+1}$ **c** $\dfrac{1}{1-4x^2}$ **d** $\dfrac{x}{\sqrt{1-4x^2}}$

e $\dfrac{x^2}{1-4x^2}$ **f** $x\sqrt{1-2x^2}$ **g** $\sin^2\left(x-\dfrac{\pi}{3}\right)$ **h** $\dfrac{x}{\sqrt{x^2-2}}$

i $\sin^2 3x$ **j** $\sin^3 2x$ **k** $x\sqrt{x+1}$ **l** $\dfrac{1}{1+\cos 2x}$

m $\dfrac{e^{3x}+1}{e^{3x+1}}$ **n** $\dfrac{x}{x^2-1}$ **o** $\sin^2 x\cos^2 x$ **p** $\dfrac{x^2}{1+x}$

2 Evaluate the following integrals:

a $\int_0^{\frac{1}{2}} x(1-x^2)^{\frac{1}{2}}\,dx$ **b** $\int_0^{\frac{1}{2}} (1-x^2)^{-1}\,dx$ **c** $\int_0^{\frac{1}{2}} x(1+x^2)^{\frac{1}{2}}\,dx$

d $\int_1^2 \dfrac{1}{6x+x^2}\,dx$ **e** $\int_0^1 \dfrac{2x^2+3x+2}{x^2+3x+2}\,dx$ **f** $\int_0^1 \dfrac{dx}{\sqrt{4-3x}}$

g $\int_0^1 \dfrac{dx}{\sqrt{4-x^2}}$ **h** $\int_0^{\frac{\pi}{2}} \sin^2 2x\,dx$ **i** $\int_{-\pi}^{\pi} \sin^2 x\cos^2 x\,dx$

j $\int_0^{\frac{\pi}{2}} \sin^2 2x\cos^2 2x\,dx$ **k** $\int_0^{\frac{\pi}{4}} \dfrac{2\cos x - \sin x}{2\sin x + \cos x}\,dx$ **l** $\int_{-1}^2 x^2\sqrt{x^3+1}\,dx$

3 Show that:

$$\frac{x}{x^2+2x+3} = \frac{1}{2}\left(\frac{2x+2}{x^2+2x+3}\right) - \frac{1}{x^2+2x+3}$$

Hence find:

$$\int \frac{x}{x^2+2x+3}\,dx$$

4 **a** Differentiate $\sin^{-1}\sqrt{x}$ and hence find $\int \frac{1}{\sqrt{x(1-x)}}\,dx$.

b Differentiate $\sin^{-1}(x^2)$ and hence find $\int \frac{2x}{\sqrt{1-x^4}}\,dx$.

5 **a** Find $\frac{d}{dx}(x\sin^{-1}x)$ and hence find $\int \sin^{-1}x\,dx$.

b Find $\frac{d}{dx}(x\log_e x)$ and hence find $\int \log_e x\,dx$.

c Find $\frac{d}{dx}(x\tan^{-1}x)$ and hence find $\int \tan^{-1}x\,dx$.

6 Find an antiderivative of each of the following:

a $\sin 2x\cos 2x$ **b** $x^2(x^3+1)^2$ **c** $\frac{\cos\theta}{(3+2\sin\theta)^2}$ **d** xe^{1-x^2}

e $\tan^2(x+3)$ **f** $\frac{2x}{\sqrt{6+2x^2}}$ **g** $\tan^2 x\sec^2 x$ **h** $\sec^3 x\tan x$

i $\tan^2 3x$

7 Evaluate the following:

a $\int_0^{\frac{\pi}{2}} \sin^5 x\,dx$ **b** $\int_1^8 (13-5x)^{\frac{1}{3}}\,dx$ **c** $\int_0^{\frac{\pi}{8}} \sec^2 2x\,dx$

d $\int_1^2 (3-y)^{\frac{1}{2}}\,dy$ **e** $\int_0^{\pi} \sin^2 x\,dx$ **f** $\int_{-3}^{-1} \frac{x^2+1}{x^3+3x}\,dx$

8 Find the derivative of $\left(x^2+\frac{1}{x}\right)^{\frac{1}{2}}$ and hence evaluate $\int_{-1}^{2} \frac{2x-x^{-2}}{\sqrt{x^2+\frac{1}{x}}}\,dx$.

CHAPTER

8

Applications of integration

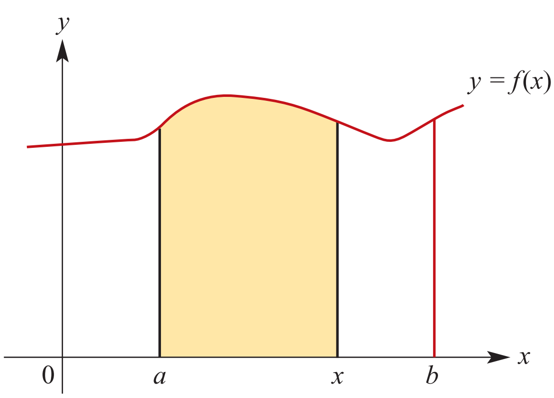
Objectives

- To **integrate** functions required for Specialist Mathematics
- To determine the area between two curves
- To use a graphics calculator to evaluate definite integrals
- To investigate the relationship between the graph of a function and the graphs of its antiderivatives
- To determine volumes of **solids of revolutions**

8.1 Areas of regions

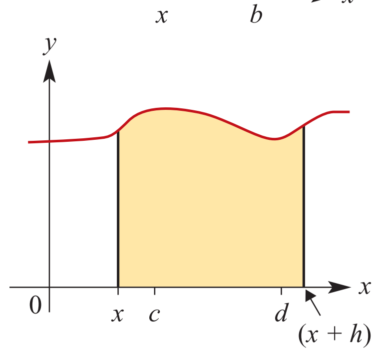
Let $f: [a, b] \to R$ be a continuous function such that $f(x) \geq 0$ for all $x \in [a, b]$

Define geometrically the function $F(x)$ by saying that it is the measure of the area under the curve between a and x. Thus $F(a) = 0$. It will be shown that $F'(x) = f(x)$.



By the definition of $F(x)$, $F(x + h) - F(x)$ is the area between x and $x + h$.

For each interval $[x, x + h]$ there exists a point c in $[x, x + h]$ such that $f(c) \geq f(z)$ for all $z \in [x, x + h]$ and a point d in the same interval such that $f(d) \leq f(z)$ for all $z \in [x, x + h]$.



Thus $\quad f(d) \leq f(z) \leq f(c)$ for all $z \in [x, x + h]$.

Therefore $\quad hf(d) \leq F(x + h) - F(x) \leq hf(c)$

i.e. the shaded region has an area less than the area of the rectangle with base h and height $f(c)$ and an area greater than the area of the rectangle with base h and height $f(d)$.

Dividing by h $\quad f(d) \le \dfrac{F(x+h) - F(x)}{h} \le f(c)$

As $h \to 0$, both $f(c)$ and $f(d)$ approach $f(x)$. Therefore it has been shown that

$$F'(x) = f(x) \qquad \left(F'(x) = \lim_{h \to 0} \frac{F(x+h) - F(x)}{h} \right)$$

It is known that if $G(x)$ is an antiderivative of $f(x)$ then $F(x) = G(x) + k$, where k is a constant.

Let $x = a$.

Then $0 = F(a) = G(a) + k \quad$ i.e. $k = -G(a)$

Thus $F(x) = G(x) - G(a)$ and letting $x = b$ yields

$F(b) = G(b) - G(a)$

> The area under the curve $y = f(x)$, $f(x) \ge 0$, between a and b is $G(b) - G(a)$ where $G(x)$ is an antiderivative of $f(x)$.

A similar argument could be used if $f(x) < 0$ for all $x \in [a, b]$, but in this case F must be taken to be the negative of the area under the curve,

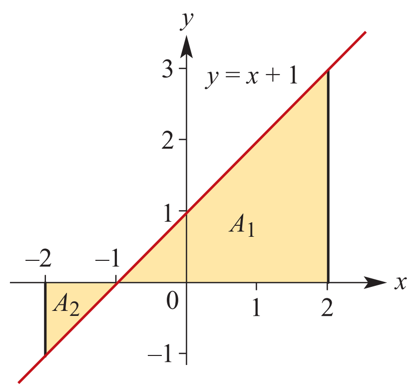
i.e. $\quad -f(c) \le \dfrac{F(x+h) - F(x)}{h} \le -f(d)$ and therefore $F'(x) = -f(x)$

Signed area

For $f(x) = x + 1$

$A_1 = \frac{1}{2} \times 3 \times 3 = 4\frac{1}{2}$ (area of a triangle)

$A_2 = \frac{1}{2} \times 1 \times 1 = \frac{1}{2}$

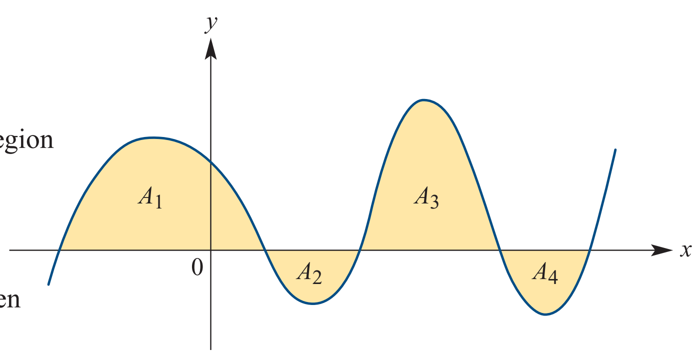


The total area $= A_1 + A_2 = 5$

The **signed area** $= A_1 - A_2 = 4$

Regions below the x axis have *negative signed areas*.

Regions above the x axis have *positive signed areas*.



The total area of the shaded region is $A_1 + A_2 + A_3 + A_4$.

The signed area of the shaded region is $A_1 - A_2 + A_3 - A_4$.

Denote the signed area between the curve $y = f(x)$, the lines $x = a$ and $x = b$ and the x axis by $\int_a^b f(x)\, dx$.

If $f(x) \geq 0$ for $x \in [a, b]$ then $\int_a^b f(x)\, dx$ is the area enclosed between the x axis and the curve $y = f(x)$ and the lines $x = a$ and $x = b$.

Thus $\int_a^b f(x)\, dx = G(b) - G(a)$ where G is an antiderivative of f.

This result holds generally, and is known as the **fundamental theorem of integral calculus**.

Let f be a continuous function on the interval $[a, b]$. Then:

$$\int_a^b f(x)\, dx = G(b) - G(a) \text{ where } G \text{ is any antiderivative of } f$$

To facilitate setting out, write $G(b) - G(a) = [G(x)]_a^b$

Note that if $f(x) \geq 0$ for all $x \in [a, b]$ then the area between b and a is given by

$$\int_a^b f(x)\, dx$$

and if $f(x) \leq 0$ for $x \in [a, b]$ then the area between b and a is given by

$$-\int_a^b f(x)\, dx$$

In general, the area is $\left|\int_a^b f(x)\, dx\right|$ if $f(x) \leq 0$ for $x \in [a, b]$ or $f(x) \geq 0$ for all $x \in [a, b]$.

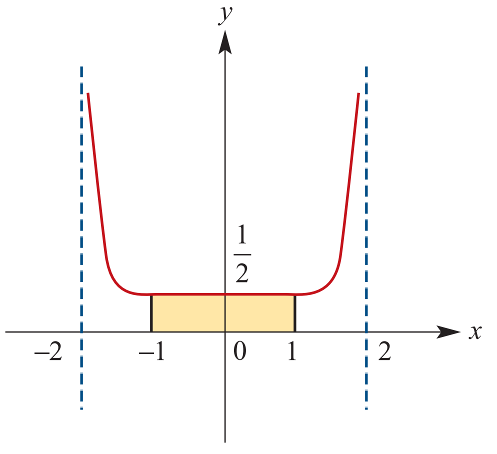
In the following, the graphs of some of the functions introduced in earlier chapters, and the areas of regions defined through these functions, are considered.

It may be desirable to use a graphing package or a CAS calculator to help with the graphing in this section.

Example 1

The graph of $y = \dfrac{1}{\sqrt{4 - x^2}}$ is shown.

Find the area of the shaded region.



Solution

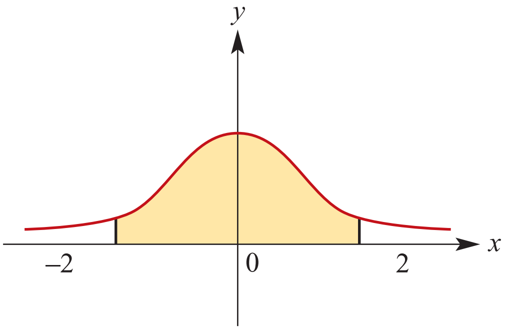
$$\text{Area} = \int_{-1}^{1} \frac{1}{\sqrt{4 - x^2}}\, dx$$

By symmetry:

$$\begin{aligned}\text{Area} &= 2\int_0^1 \frac{1}{\sqrt{4 - x^2}}\, dx \\ &= 2\left[\sin^{-1}\frac{x}{2}\right]_0^1 \\ &= 2\sin^{-1}\frac{1}{2} \\ &= 2 \times \frac{\pi}{6} \\ &= \frac{\pi}{3}\end{aligned}$$

Example 2

The graph of $y = \dfrac{6}{4+x^2}$ is shown. Find the area of the shaded region.



Solution

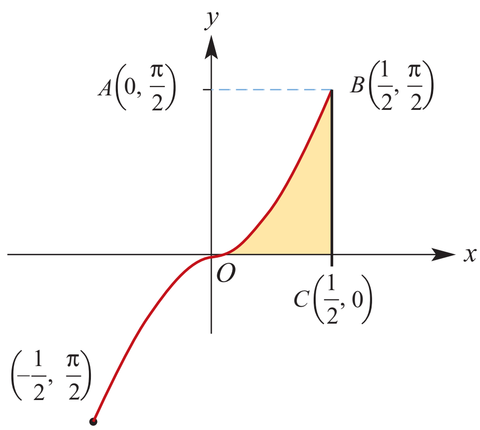
$$\begin{aligned}\text{Area} &= 6\int_{-2}^{2} \frac{1}{4+x^2}\,dx\\ &= \frac{6}{2}\int_{-2}^{2}\frac{2}{4+x^2}\,dx\\ &= 6\int_{0}^{2}\frac{2}{4+x^2}\,dx \text{ (by symmetry)}\\ &= 6\left[\tan^{-1}\frac{x}{2}\right]_0^2\\ &= 6\tan^{-1}1\\ &= 6\times\frac{\pi}{4}\\ &= \frac{3\pi}{2}\end{aligned}$$

Example 3

Sketch the graph of $f\colon \left[-\frac{1}{2}, \frac{1}{2}\right] \to R$, $f(x) = \sin^{-1} 2x$. Shade the region defined by the inequalities $y \le f(x)$, $0 \le x \le \frac{1}{2}$, $y \ge 0$, and find the area of this region.

Solution

$$\text{Area} = \int_0^{\frac{1}{2}} \sin^{-1} 2x\,dx$$



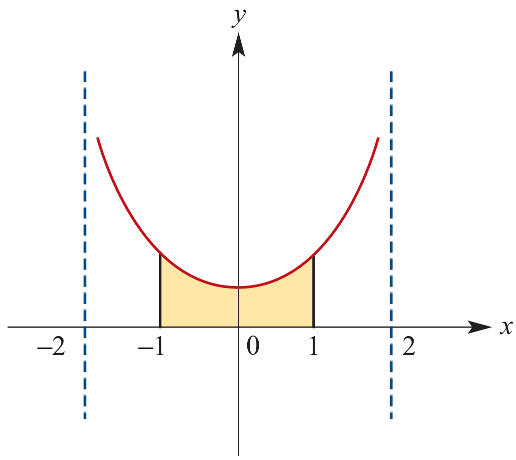
Note: The antiderivative of $\sin^{-1}$ is not required for this course, but the area can still be determined as follows:

$$\begin{aligned}\text{Area} &= \text{area of rectangle}\\ &\quad OABC - \int_0^{\frac{\pi}{2}} \frac{1}{2}\sin y\,dy\\ &= \frac{\pi}{4} - \frac{1}{2}[-\cos y]_0^{\frac{\pi}{2}}\\ &= \frac{\pi}{4} - \frac{1}{2}\end{aligned}$$

Example 4

Sketch the graph of $y = \dfrac{1}{4-x^2}$. Shade the region for the area determined by $\displaystyle\int_{-1}^{1} \frac{1}{4-x^2}\,dx$ and find this area.

Solution



$$\text{Area} = \int_{-1}^{1} \frac{1}{4 - x^2}\,dx$$

$$= \frac{1}{4}\int_{-1}^{1} \frac{1}{2 - x} + \frac{1}{2 + x}\,dx$$

By symmetry:

$$\text{Area} = \frac{1}{2}\int_{0}^{1} \frac{1}{2 - x} + \frac{1}{2 + x}\,dx$$

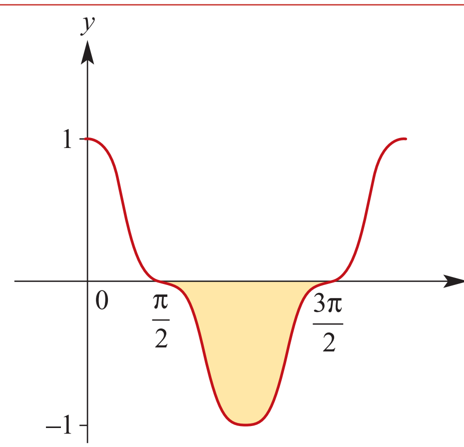
$$= \frac{1}{2}\left[\log_e\left(\frac{2 + x}{2 - x}\right)\right]_0^1$$

$$= \tfrac{1}{2}[\log_e 3 - \log_e 1]$$

$$= \tfrac{1}{2}\log_e 3$$

Example 5

The graph of $y = \cos^3 x$ is shown. Find the area of the shaded region.



Solution

$$\text{Area} = -\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \cos^3 x\,dx$$

$$= -\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \cos x \cos^2 x\,dx$$

$$= -\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \cos x(1 - \sin^2 x)\,dx$$

Let $u = \sin x$. Then $\dfrac{du}{dx} = \cos x$.

When $x = \dfrac{\pi}{2}$, $u = 1$. When $x = \dfrac{3\pi}{2}$, $u = -1$

$$\therefore \quad \text{Area} = -\int_{1}^{-1} (1 - u^2)\,du$$

$$= -\left[u - \frac{u^3}{3}\right]_1^{-1}$$

$$= -\left[-1 + \tfrac{1}{3} - \left(1 - \tfrac{1}{3}\right)\right]$$

$$= \tfrac{4}{3}$$

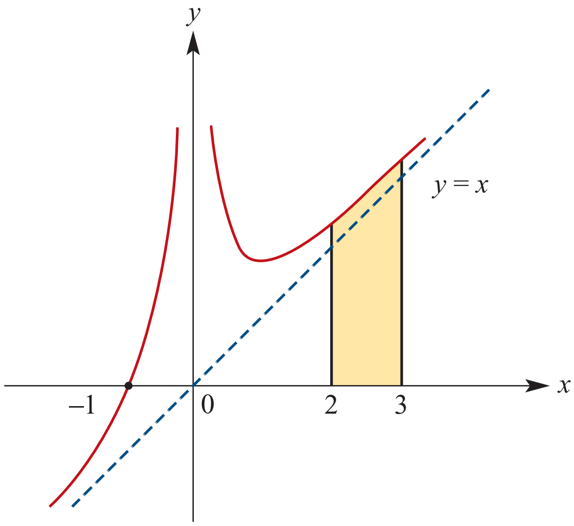
Exercise 8A

1 Sketch the graph of $f\colon \left(-\frac{3}{2}, \frac{3}{2}\right) \to R$, $f(x) = \dfrac{1}{\sqrt{9 - 4x^2}}$ and find the area of the region defined by the inequalities $0 \le y \le f(x)$ and $-1 \le x \le 1$.

2 Sketch the graph of $f\colon R \to R,\ f(x) = \dfrac{9}{4 + x^2}$ and find the area of the region defined by the inequalities $0 \le y \le f(x)$ and $-2 \le x \le 2$.

3 Sketch the graph of $g\colon R \setminus \{-3, 3\} \to R,\ g(x) = \dfrac{4}{9 - x^2}$ and find the area of the region with $-2 \le x \le 2$ and $0 \le y \le g(x)$.

4 The graph of $f(x) = x + \dfrac{1}{x^2}$ is as shown. Find the area of the shaded region.



5 Sketch the graph of $f(x) = x + \dfrac{2}{x}$. Shade the region for which the area is determined by the integral $\displaystyle\int_1^2 f(x)\,dx$ and evaluate this integral.

6 For each of the following:

i sketch the appropriate graph and shade the required region

ii evaluate the integral

a $\displaystyle\int_0^1 \tan^{-1} x\,dx$ **b** $\displaystyle\int_0^{\frac{1}{2}} \cos^{-1} 2x\,dx$ **c** $\displaystyle\int_{-\frac{1}{2}}^{\frac{1}{2}} \cos^{-1} 2x\,dx$

d $\displaystyle\int_0^1 2\sin^{-1} x\,dx$ **e** $\displaystyle\int_0^2 \sin^{-1}\frac{x}{2}\,dx$ **f** $\displaystyle\int_{-1}^2 \sin^{-1}\frac{x}{2}\,dx$

7 For the curve with equation $y = -1 + \dfrac{2}{x^2 + 1}$ find:

a the coordinates of its turning point **b** the equation of its asymptote

c the area enclosed by the curve and the x axis

8 For the graph of $y = x - \dfrac{4}{x + 3}$:

a find the coordinates of the intercepts with the axes

b find the equations of all asymptotes

c sketch the graph

d find the area bounded by the curve, the x axis and the line $x = 8$

9 **a** State the implied domain of the function g, with rule $g(x) = \dfrac{1}{(1-x)(x-2)}$.

b Sketch the graph of $y = g(x)$, indicating the equation of any asymptotes and the coordinates of the turning points.

c State the range of g.

d Find the area of the region bounded by the graph of $y = g(x)$, the x axis and the lines $x = 4$ and $x = 3$.

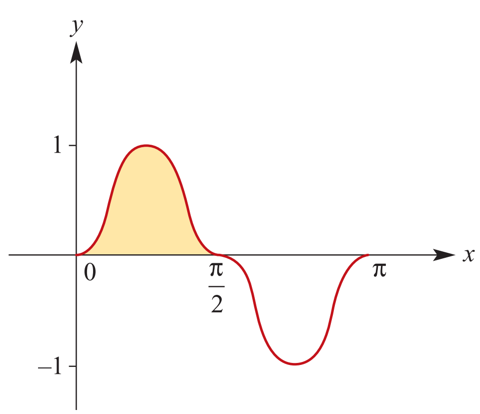
10 Sketch the graph of $f\colon (-1, 1) \to R$, $f(x) = \dfrac{-3}{\sqrt{1-x^2}}$ and evaluate the integral $\displaystyle\int_0^{\frac{1}{2}} \frac{-3}{\sqrt{1-x^2}}\, dx$.

11 Find the area of the region enclosed by the curve $y = \dfrac{1}{\sqrt{4-x^2}}$, the lines $x = 1$ and $x = \sqrt{2}$, and the x axis.

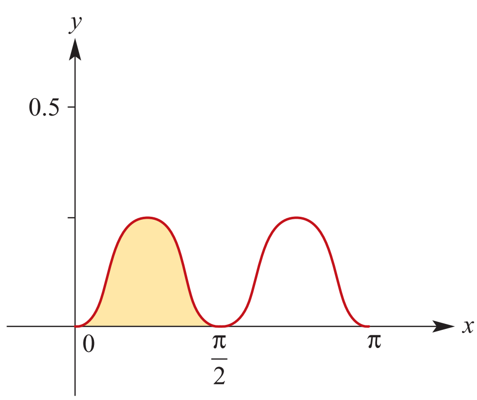
12 Sketch the curve with equation $y = \tan^{-1} x$. Find the area enclosed between this curve, the line $x = \sqrt{3}$ and the x axis.

13 Find the area between the curve $y = \dfrac{2\log_e x}{x}$ and the x axis from $x = 1$ to $x = e$.

14 The graph of $y = \sin^3 2x$ for $x \in [0, \pi]$ is as shown. Find the area of the shaded region.



15 The graph of $y = \sin x \cos^2 x$ for $x \in [0, \pi]$ is as shown. Find the area of the shaded region.



16 Sketch the curve with equation $y = \dfrac{2x}{x+3}$ showing clearly how the curve approaches its asymptotes.

On your diagram, shade the finite region bounded by the curve and the lines $x = 0$, $x = 3$ and $y = 2$. Find the area of this region.

17 **a** Show that the curve $y = \dfrac{3}{(2x+1)(1-x)}$ has only one turning point.

b Find the coordinates of this point and determine its nature.

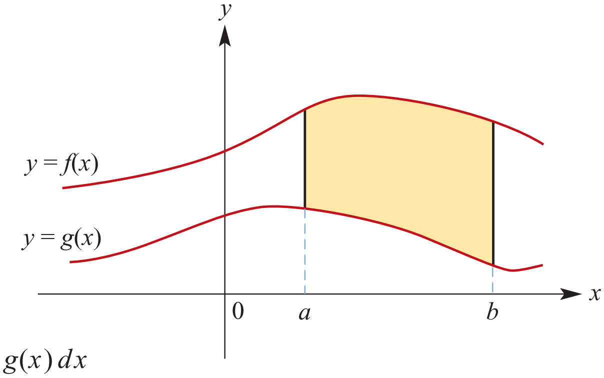
c Sketch the curve.

d Find the area of the region enclosed by the curve and the line $y = 3$.

8.2 Area of a region between two curves

Let f and g be continuous on $[a, b]$ such that $f(u) \geq g(u)$ for $u \in [a, b]$.

Then the area of the region bounded by the curves and the lines $x = a$ and $x = b$ can be calculated through the evaluation of



$$\int_a^b f(x)\,dx - \int_a^b g(x)\,dx = \int_a^b f(x) - g(x)\,dx$$

Example 6

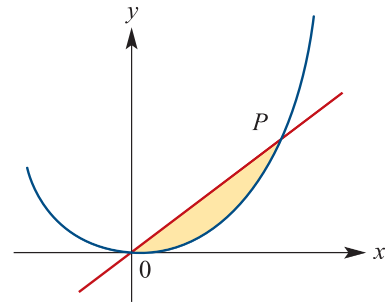
Find the area of the region bounded by the parabola $y = x^2$ and the line $y = 2x$.

Solution

First find the coordinates of the point P.

For $2x = x^2$, $x(x - 2) = 0$ which implies $x = 0$ or $x = 2$.

Therefore, the coordinates of P are (2, 4).



$$\begin{aligned}\text{The required area} &= \int_0^2 2x - x^2\,dx \\ &= \left[x^2 - \frac{x^3}{3}\right]_0^2 \\ &= 4 - \tfrac{8}{3} \\ &= \tfrac{4}{3}\end{aligned}$$

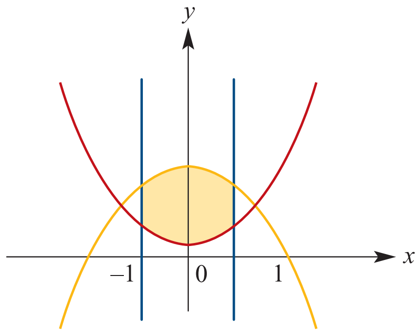
The area is $\frac{4}{3}$ square units.

Example 7

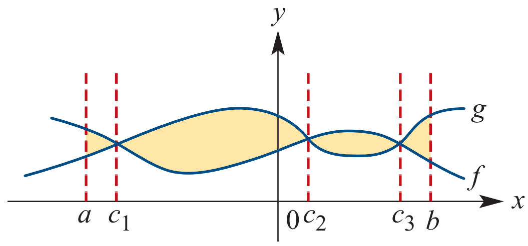
Calculate the area of the region enclosed between the curves with equations $y = x^2 + 1$ and $y = 4 - x^2$, and the lines $x = 1$ and $x = -1$.

Solution

$$\begin{aligned}\text{Required area} &= \int_{-1}^{1} 4 - x^2 - (x^2 + 1)\,dx \\ &= \int_{-1}^{1} (3 - 2x^2)\,dx \\ &= \left[3x - \frac{2x^3}{3}\right]_{-1}^{1} \\ &= \frac{14}{3}\end{aligned}$$



In the two examples considered so far in this section, the graph of one function is 'above' the graph of the other for all of the interval considered. What happens when the graphs cross?



To find the area of the shaded region we must consider the intervals

$[a, c_1]$, $[c_1, c_2]$, $[c_2, c_3]$ and $[c_3, b]$

and the area is given by:

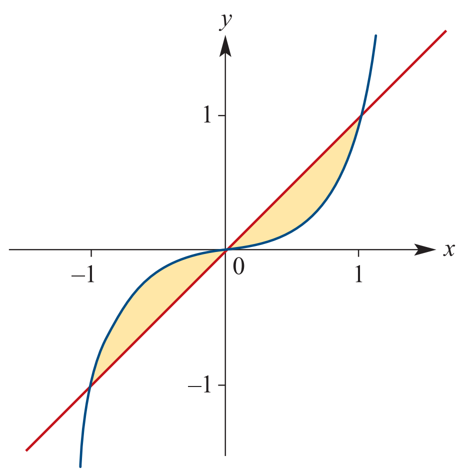
$$\int_a^{c_1} f(x) - g(x)\,dx + \int_{c_1}^{c_2} g(x) - f(x)\,dx + \int_{c_2}^{c_3} f(x) - g(x)\,dx + \int_{c_3}^{b} g(x) - f(x)\,dx$$

The modulus function could also be used here:

$$\int_a^{c_1} |f(x) - g(x)|dx + \int_{c_1}^{c_2} |f(x) - g(x)|dx + \int_{c_2}^{c_3} |f(x) - g(x)|dx + \int_{c_3}^{b} |f(x) - g(x)|dx$$

Example 8

Find the area of the region enclosed by the graphs of $f(x) = x^3$ and $g(x) = x$ as shown in the figure.



Solution

The graphs intersect where $f(x) = g(x)$

$$x^3 = x$$

$$\therefore \quad x^3 - x = 0$$

$$\therefore \quad x(x^2 - 1) = 0$$

$$\therefore \quad x = 0 \text{ or } x = \pm 1$$

$f(x) \geq g(x)$ for $-1 \leq x \leq 0$ and $f(x) \leq g(x)$ for $0 \leq x \leq 1$.
Thus the area is given by:

$$\int_{-1}^{0} f(x) - g(x)\,dx + \int_0^1 g(x) - f(x)\,dx = \int_{-1}^{0} x^3 - x\,dx + \int_0^1 x - x^3 dx$$

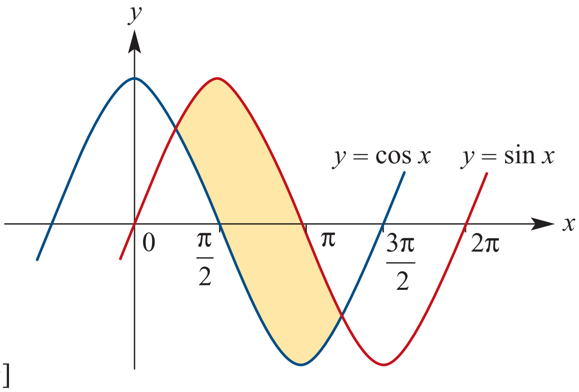
$$= \left[\frac{x^4}{4} - \frac{x^2}{2}\right]_{-1}^{0} + \left[\frac{x^2}{2} - \frac{x^4}{4}\right]_0^1$$

$$= -\left(-\tfrac{1}{4}\right) + \tfrac{1}{4} = \tfrac{1}{2}$$

Note that the result can also be obtained by observing the symmetry of the graph, finding the area of the region where both x and y are non-negative, and multiplying by 2.

Example 9

Find the area of the shaded region of the graph.



Solution

To find the coordinates of the point of intersection:

$\sin x = \cos x$, $\tan x = 1$,

$\therefore \quad x = \dfrac{\pi}{4}$ or $\dfrac{5\pi}{4}$ for $x \in [0, 2\pi]$

$$\text{Area} = \int_{\frac{\pi}{4}}^{\frac{5\pi}{4}} (\sin x - \cos x)\,dx$$

$$= [-\cos x - \sin x]_{\frac{\pi}{4}}^{\frac{5\pi}{4}}$$

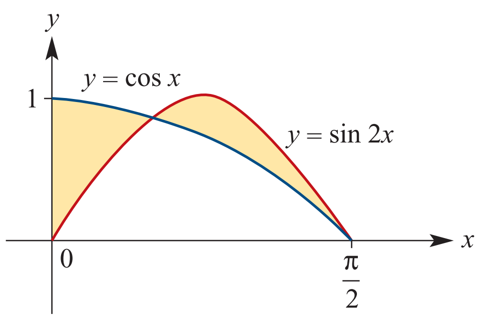
$$= -\cos\frac{5\pi}{4} - \sin\frac{5\pi}{4} - \left(-\cos\frac{\pi}{4} - \sin\frac{\pi}{4}\right)$$

$$= \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} = \frac{4}{\sqrt{2}} = 2\sqrt{2}$$

$\therefore$ Area of shaded region $= 2\sqrt{2}$ square units.

Example 10

Find the area of the shaded region.



Solution

The points of intersection are determined first:

$$\cos x = \sin 2x$$

implies:

$$\cos x = 2\sin x\cos x$$

Therefore $\quad \cos x(2\sin x - 1) = 0$

and $\quad \cos x = 0$ or $\sin x = \frac{1}{2}$

$\therefore \quad x = \dfrac{\pi}{2}$ or $x = \dfrac{\pi}{6}$ for $x \in \left[0, \dfrac{\pi}{2}\right]$

The area of the shaded region $= \displaystyle\int_0^{\frac{\pi}{6}} \cos x - \sin 2x\,dx + \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \sin 2x - \cos x\,dx$

$$= \left[\sin x + \tfrac{1}{2}\cos 2x\right]_0^{\frac{\pi}{6}} + \left[-\tfrac{1}{2}\cos 2x - \sin x\right]_{\frac{\pi}{6}}^{\frac{\pi}{2}}$$

$$= \left(\tfrac{1}{2} + \tfrac{1}{4} - \tfrac{1}{2}\right) + \left(\tfrac{1}{2} - 1 - \left(-\tfrac{1}{4} - \tfrac{1}{2}\right)\right)$$

$$= \tfrac{1}{4} - \tfrac{1}{2} + \tfrac{1}{4} + \tfrac{1}{2}$$

$$= \tfrac{1}{2}$$

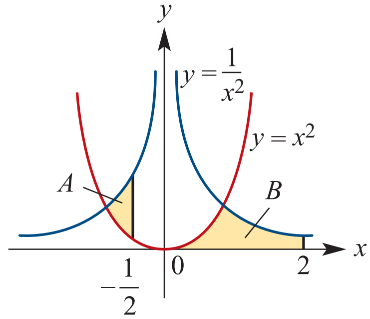
Exercise 8B

1 Find the coordinates of the points of intersection of the two curves with equations $y = x^2 - 2x$ and $y = -x^2 + 8x - 12$. Find the area of the region enclosed between the two curves.

2 Find the area of the region enclosed by the graphs of $y = -x^2$ and $y = x^2 - 2x$.

3 Find the area of region:

a A **b** B

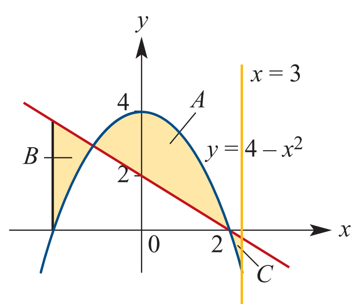


4 For $f: R \to R$, $f(x) = x^2 - 4$, sketch the graphs of $y = f(x)$ and $y = \frac{16}{f(x)}$ on the one set of axes and find the area of the region bounded by the two graphs and the lines $x = 1$ and $x = -1$.

5 The area of the region bounded by $y = \frac{12}{x}$, $x = 1$ and $x = a$ is 24. Find the value of a.

6 Find the area of region:

a A **b** B **c** C



7 Find the area of the regions enclosed by the lines and curves for each of the following. Draw a sketch graph and shade the appropriate region for each example.

a $y = 2 \sin x$ and $y = \sin 2x$ $\quad 0 \le x \le \pi$

b $y = \sin 2x$ and $y = \cos x$ $\quad \frac{-\pi}{2} \le x \le \frac{\pi}{2}$

c $y = \sqrt{x}$, $y = 6 - x$ and $y = 1$

d $y = \frac{2}{1 + x^2}$ and $y = 1$

e $y = \sin^{-1} x$, $x = \frac{1}{2}$ and $y = 0$

f $y = \cos 2x$, $y = 1 - \sin x$ $\quad 0 \le x \le \pi$

g $y = \frac{1}{3}(x^2 + 1)$ and $y = \frac{3}{x^2 + 1}$

8 Evaluate each of the following. (Draw the appropriate graph first.)

a $\int_1^e \log_e x \, dx$ (Hint: $y = \log_e x \Leftrightarrow x = e^y$. Find the area between the curve and the y axis first.)

b $\int_{\frac{1}{2}}^{1} \log_e 2x \, dx$

9 For $f: R \to R,\ f(x) = xe^x$:

a find the derivative of f

b find $\{x: f'(x) = 0\}$

c sketch the graph of $y = f(x)$

d find the equation of the tangent to this curve at $x = -1$

e find the area of the region bounded by this tangent, the curve and the y axis

10 P is the point with coordinates (1, 1) on the curve with equation $y = 1 + \log_e x$.

a Find the equation of the normal to the curve at P.

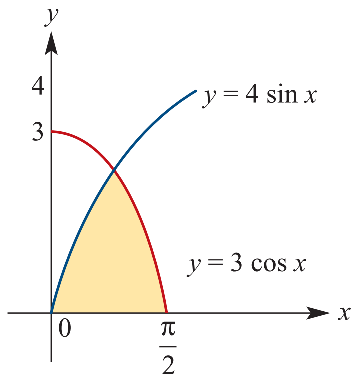
b Find the area of the region enclosed by the normal, the curve and the x axis.

11 **a** Find the coordinates of the points of intersection of the curves whose equations are $y = (x - 1)(x - 2)$ and $y = \dfrac{3(x - 1)}{x}$.

b Sketch the two curves on the one set of axes.

c Find the area of that region bounded by the two curves for which x lies between 1 and 3.

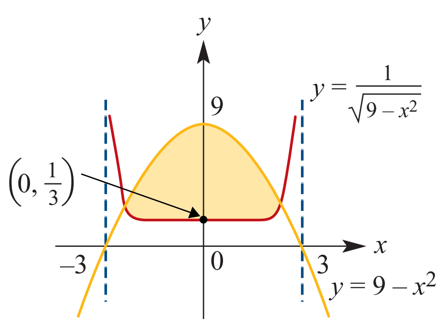
12 Show that the area of the shaded region is 2.



13 The graphs of $y = 9 - x^2$ and $y = \dfrac{1}{\sqrt{9 - x^2}}$ are as shown. Find:

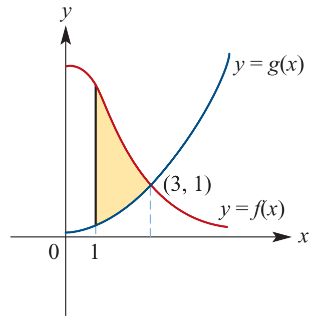
a the coordinates of the points of intersection of the two graphs

b the area of the shaded region



14 Find the area enclosed by the graphs of $y = x^2$ and $y = x + 2$.

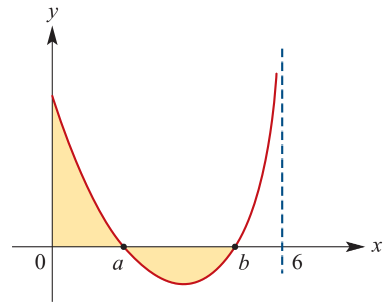
15 Consider the functions $f(x) = \dfrac{10}{1+x^2}$ for $x \geq 0$ and $g(x) = e^{x-3}$ for $x \geq 0$. The graphs of $y = f(x)$ and $y = g(x)$ intersect at the point (3, 1). Find, correct to three decimal places, the area of the region enclosed by the two graphs and the line with equation $x = 1$.



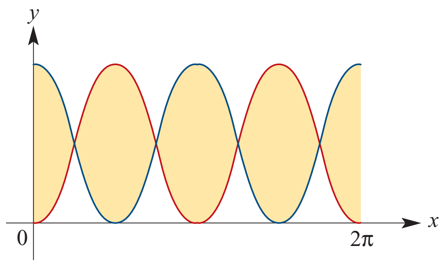
16 The graph of the function f: [0, 6] → R, where $f(x) = \dfrac{8\sqrt{5}}{\sqrt{36-x^2}} - x$, is shown.

a Find the values of a and b.

b Find the total area of the shaded regions.



17 The graphs of $y = \cos^2 x$ and $y = \sin^2 x$ are shown for $0 \leq x \leq 2\pi$. Find the total area of the shaded regions.



8.3 Integration using a CAS calculator

In Chapter 7, methods of integration by rule were discussed. In the first two examples of this section we consider the use of a CAS calculator in evaluating definite integrals.

It is often not possible to determine the antiderivative of a given function by rule. In this section, numerical evaluation of definite integrals will also be discussed.

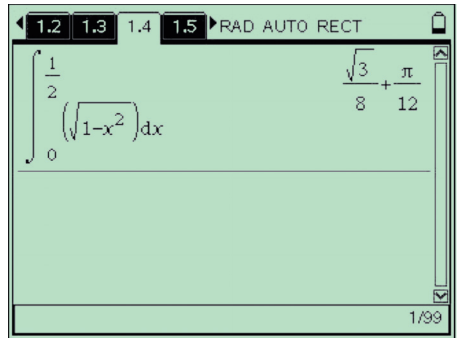
Example 11

Use a CAS calculator to evaluate $\int_0^{\frac{1}{2}} \sqrt{1-x^2}\,dx$.

Solution

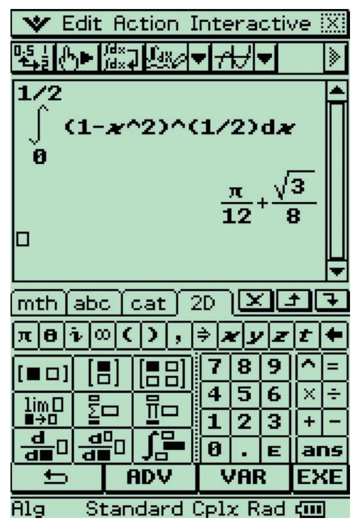
Using a TI-Nspire calculator

From the **Calculus** menu choose **Integral** (menu 4 2).
The integration template can also be directly obtained through the Template menu of the catalog. Press ctrl and then the template key.



Using a Casio ClassPad calculator

From the keyboard choose **2D**, then **CALC** and then the integral template ∫□ **d**□.
Complete as shown.



Example 12

Find the area of the region bounded by the graphs of $y = \log_e x$ and $x = 2$ and the x axis by:

a using a CAS calculator to evaluate $\int_1^2 \log_e x\,dx$.

b using the inverse function.

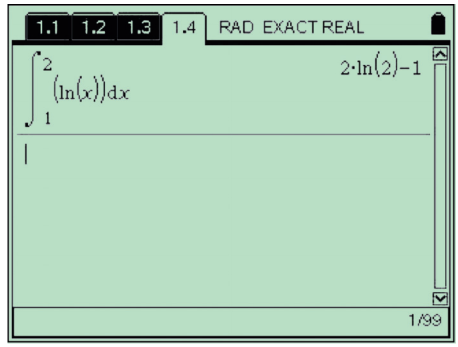
Solution

a

Using a TI-Nspire calculator

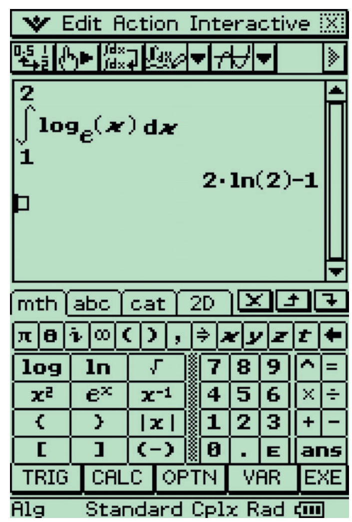
From the **Calculus** menu choose **Integral** ((menu) (4) (2)).

The integration template can also be directly obtained through the Template menu of the catalog. Press (ctrl) and then (template key).



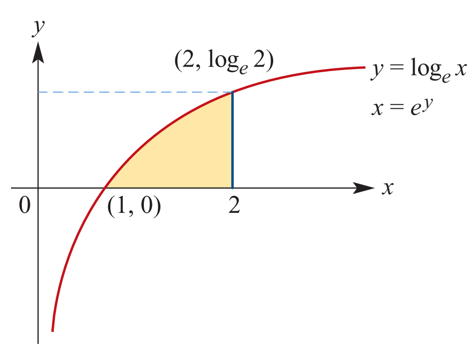
Using a Casio ClassPad calculator

Use the 2D template for the integral and complete as shown.



b It is clear that

$$\int_1^2 \log_e x\,dx = 2\log_e 2 - \int_0^{\log_e 2} e^y\,dy$$
$$= 2\log_e 2 - (e^{\log_e 2} - e^0)$$
$$= 2\log_e 2 - (2 - 1)$$
$$= 2\log_e 2 - 1$$



The area is $(2\log_e 2 - 1)$ square units.

Checking: $2\log_e 2 - 1 \approx 0.386\,294\,361\,1$.

Example 13

Find, correct to four decimal places, the area of the region bounded by the graphs of $y = \frac{1}{\sqrt{2\pi}}e^{\frac{-x^2}{2}}$ and $x = 1$ and the x and y axes.

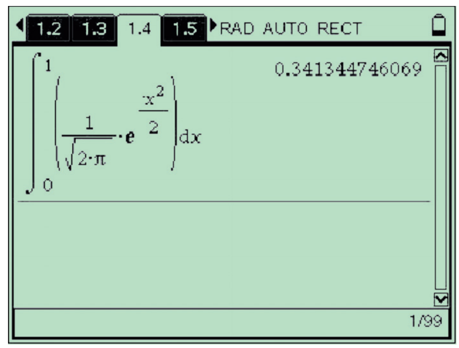
Solution

Using a TI-Nspire calculator

From the **Calculus** menu choose **Integral** (menu 4 2).

Note that the calculator has been set to Auto.

The integration template can also be directly obtained through the Template menu of the catalog. Press ctrl and then [catalog key].

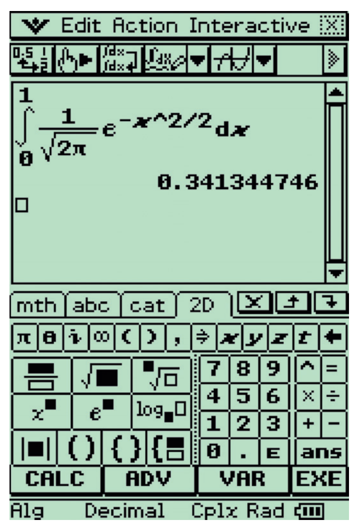


Using a Casio ClassPad calculator

Make sure the calculator is in Decimal mode (**Decimal** at the bottom of the screen).

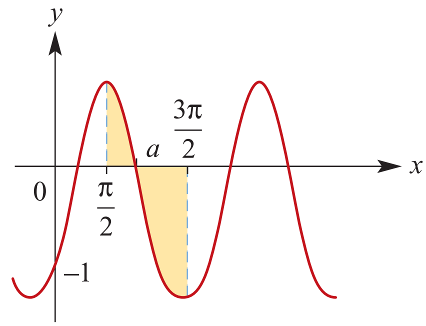
From the keyboard choose **2D**, then **CALC** and then the integral template $\int \square\, \mathbf{d}\square$.

Complete as shown.



Example 14

The graph of $y = e^{\sin x} - 2$ is as shown. Using a CAS calculator, find the area of the shaded regions.



Solution

Using a CAS calculator first find a.
It is found to be approximately 2.3757465.

$$\begin{aligned}\text{Then the required area} &= \int_{\frac{\pi}{2}}^{a} (e^{\sin x} - 2)\,dx - \int_{a}^{\frac{3\pi}{2}} e^{\sin x} - 2\,dx \\ &= 0.369\,213\,61\ldots + 2.674\,936\ldots \\ &= 3.044\,149\,61\ldots\end{aligned}$$

The area is approximately 3.044 square units.

Using the fundamental theorem of calculus

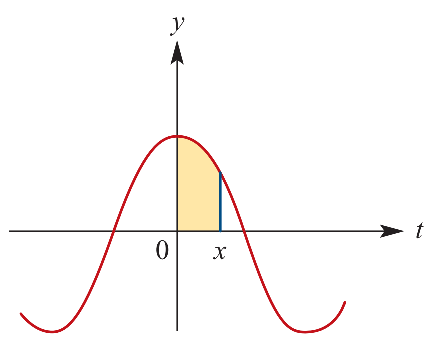
In section 8.1 the fundamental theorem of calculus was introduced. From this the following was observed. For a function f which is able to be integrated there exists a function F such that $F'(x) = f(x)$ and $\int_a^b f(x)\,dx = F(b) - F(a)$. The function F is not unique, but differs by a constant from any other function which satisfies this property.

Using a dummy variable t we can define the function F: $F(x) - F(a) = \int_a^x f(t)\,dt$ which implies $F(x) = F(a) + \int_a^x f(t)\,dt$

Note that $F'(x) = \frac{d}{dx}\left(\int_a^x f(t)\,dt\right) = f(x)$ has a derivative at every point on $[a, b]$.

Example 15

Consider the function defined as $F(x) = \int_0^x \cos t\,dt$.
Find the rule for $F(x)$.



Solution

$$\begin{aligned}F(x) = [\sin t]_0^x &= \sin x - \sin 0 \\ &= \sin x\end{aligned}$$

Example 16

Plot the graph of the function with rule $f(x) = \int_0^x t^2\,dt$.

Solution

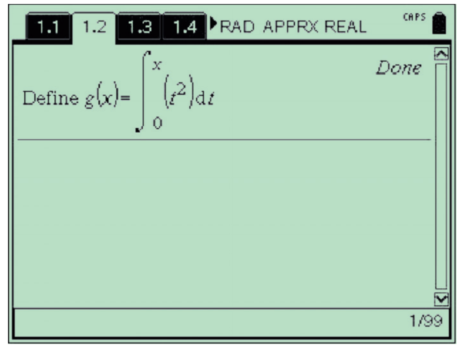
Using a TI-Nspire calculator

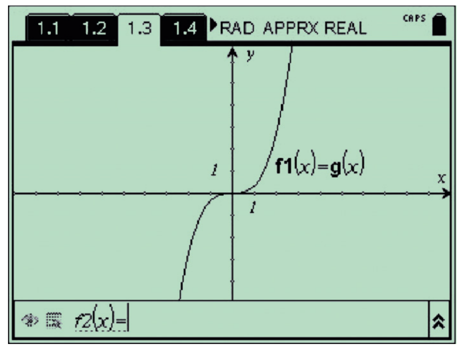
From the **Calculus** menu choose **Integral** (menu 4 2).

The integration template can also be directly obtained through the Template menu of the catalog. Press ctrl and then the template key.

Define $g(x) = \int_0^x t^2 dt$.

Choose a **Graphs & Geometry** application and enter $f1(x) = g(x)$.



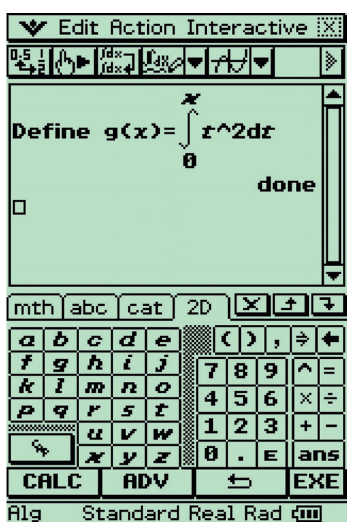


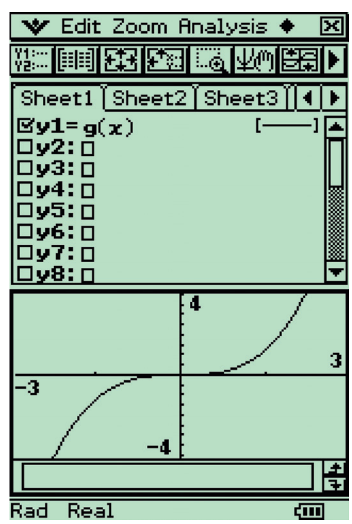
Using a Casio ClassPad calculator

Choose **Define** from the **Interactive** menu.

Use the 2D template for integration.

Graph $g(x)$ as shown.





Example 17

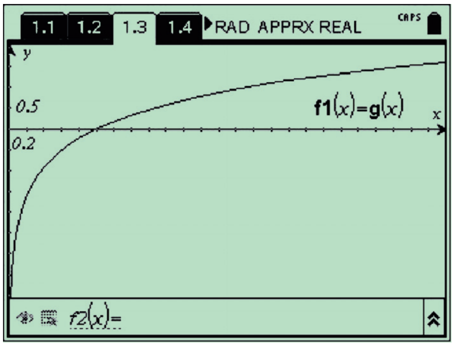
Plot the graph of $F(x) = \int_1^x \frac{1}{t}\,dt$ for $x > 1$.

Solution

Using a TI-Nspire calculator

Define $g(x) = \int_1^x \frac{1}{t}dt$.

Choose a **Graphs & Geometry** application and enter $f1(x) = g(x)$.

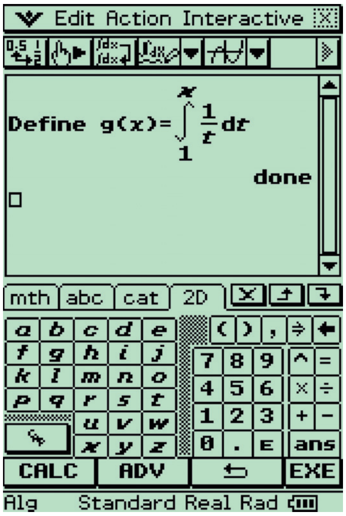


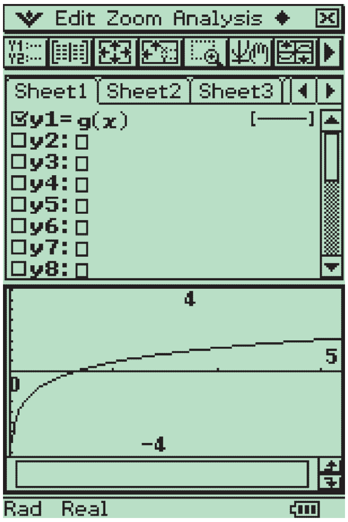
Using a Casio ClassPad calculator

Choose **Define** from the **Interactive** menu.

Use the 2D template for integration.

Graph $g(x)$ as shown.





Example 18

a Use a CAS calculator to find an approximate value of $\int_0^{\frac{\pi}{3}} \cos\left(x^2\right) dx$.

b Use a CAS calculator to plot the graph of $\int_0^x \cos\left(t^2\right) dt$ for $-\frac{\pi}{4} \le x \le \pi$.

Solution

a

Using a TI-Nspire calculator

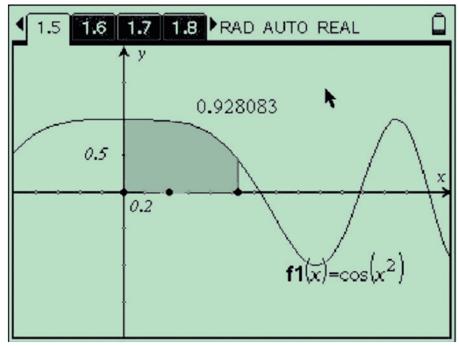
Two methods are illustrated on the screens shown right.

Method 1

In a **Graphs & Geometry** application enter $f1(x) = \cos(x^2)$ and plot the graph for $-\frac{\pi}{4} \le x \le \pi$.

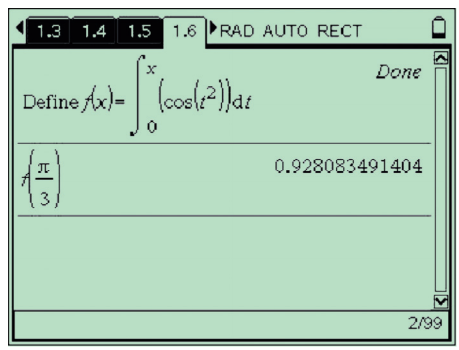
From the **Measurement** menu choose **Integral** (menu 7 5).

Select the graph, enter the lower limit 0 and press **Enter**, and enter the upper limit $\frac{\pi}{3}$ and press **Enter**. The result is as shown.



Method 2

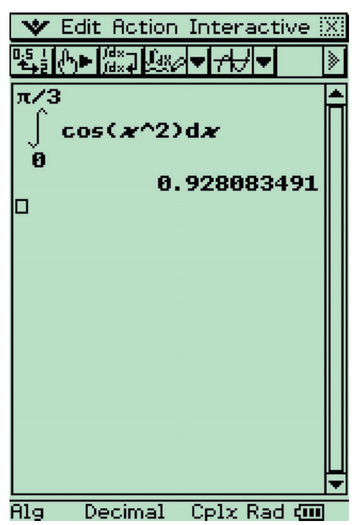
Define the function (menu 1 1) as shown and evaluate for $x = \frac{\pi}{3}$.



Using a Casio ClassPad calculator

Use the **2D** template for the integral and complete as shown.

Notice that the calculator is in Decimal mode (**Decimal** at the bottom of the screen).



b

Using a TI-Nspire calculator

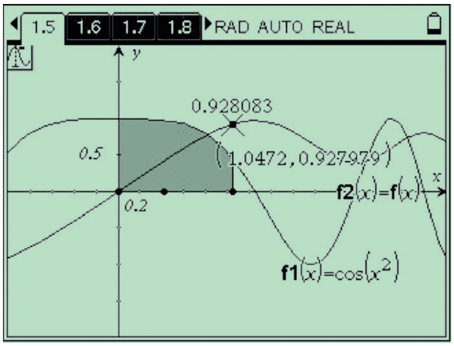
Use the same **Graphs & Geometry** application used in part **a**.

In part **a** $f(x)$ has been defined as $\int_0^x \cos(t^2)\,dt$.

Enter $f2(x) = f(x)$ to obtain the graph of $f(x) = \int_0^x \cos(t^2)\,dt$.

From the **Trace** menu choose **Graph Trace** (menu 5 1).

Type $\frac{\pi}{3}$.



Exercise 8C

1 For each of the following evaluate the integral:

i using a CAS calculator (approximation)

ii exactly by using calculus

a $\int_1^2 \frac{x^2+4}{4}\,dx$ **b** $\int_0^2 \frac{x^4}{x^2+4}\,dx$ **c** $\int_0^{\frac{\pi}{3}} \sin^4 x \cos x\,dx$

d $\int_0^{\frac{1}{2}} \frac{2x+1}{(x-1)^2}\,dx$ **e** $\int_1^2 \frac{1}{\sqrt{2x}}\,dx$ **f** $\int_0^1 xe^{x^2}\,dx$

2 Using a CAS calculator, evaluate each of the following correct to two decimal places:

a $\int_0^2 e^{\sin x}\,dx$ **b** $\int_0^{\pi} x \sin x\,dx$

c $\int_1^3 (\log_e x)^2\,dx$ **d** $\int_{-1}^1 \cos(e^x)\,dx$

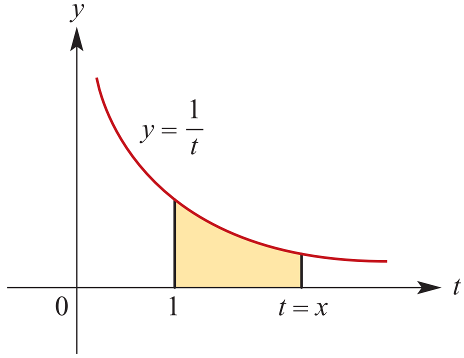
e $\int_{-1}^2 \frac{e^x}{e^x+e^{-x}}\,dx$ **f** $\int_0^2 \frac{x}{x^4+1}\,dx$

g $\int_1^2 x \log_e x\,dx$ **h** $\int_{-1}^1 x^2 e^x\,dx$

i $\int_0^1 \sqrt{1+x^4}\,dx$ **j** $\int_0^{\frac{\pi}{2}} \sin(x^2)\,dx$

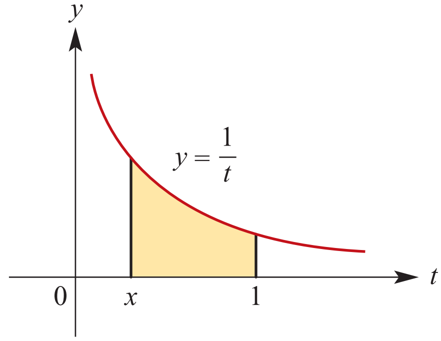
3 In each of the following, the rule of the function is defined as an area function. Find $f(x)$ in each case.

a



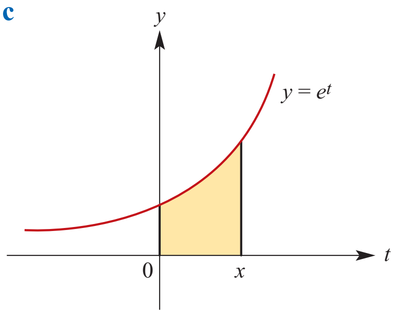
$$f(x) = \int_1^x \frac{1}{t}\,dt,\ x > 1$$

b



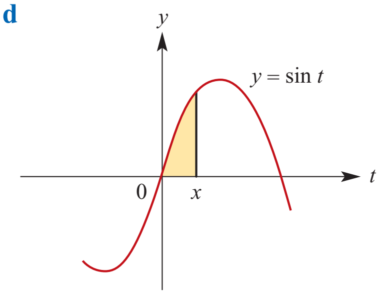
$$f(x) = \int_x^1 \frac{1}{t}\,dt,\ 0 < x < 1$$

c



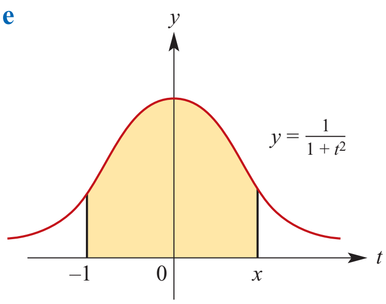
$$f(x) = \int_0^x e^t\,dt,\ x \in R$$

d



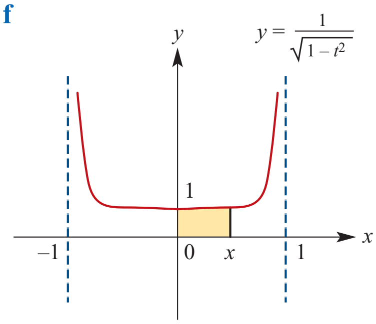
$$f(x) = \int_0^x \sin t\,dt,\ x \in R$$

e



$$f(x) = \int_{-1}^x \frac{1}{1+t^2}\,dt,\ x \in R$$

f



$$f(x) = \int_0^x \frac{1}{\sqrt{1-t^2}}\,dt,\ -1 < x < 1$$

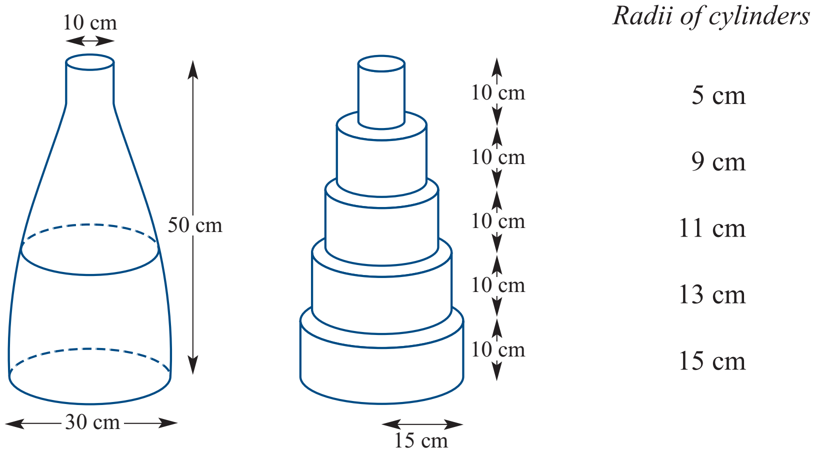
4 Use a CAS calculator to plot the graph of the following:

a $f(x) = \int_0^x \tan^{-1} t \, dt$ **b** $f(x) = \int_0^x e^{t^2} \, dt$ **c** $f(x) = \int_0^x \sin^{-1} t \, dt$

d $f(x) = \int_0^x \sin(t^2) \, dt$ **e** $f(x) = \int_1^x \frac{\sin t}{t} \, dt, x > 1$

8.4 Volumes of solids of revolution

A large glass flask has a shape as illustrated in the figure below. In order to find its approximate volume, consider the flask as a series of cylinders.

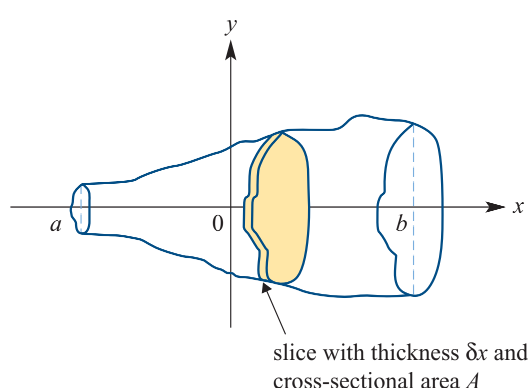


$$\therefore \quad \text{volume of the flask} \approx \pi(15^2 + 13^2 + 11^2 + 9^2 + 5^2) \times 10$$
$$\approx 19\,509.29 \text{ cm}^3$$
$$\approx 19 \text{ litres}$$

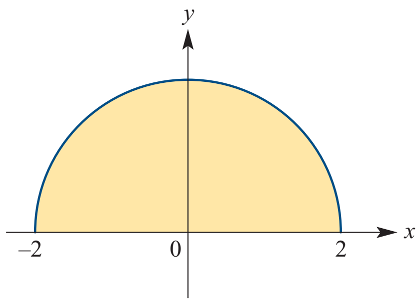
This estimate can be improved by taking more cylinders to obtain a better approximation.

In *Essential Mathematical Methods 3 and 4* it was shown that areas defined by well-behaved functions can be determined by the limit of a sum. This can also be done for volumes.

The volume of a typical thin slice is $A\delta x$ and the approximate total volume is $\sum_{x=a}^{x=b} A\delta x$.

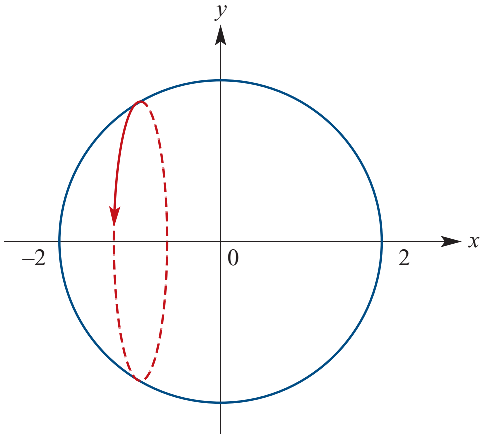


Consider the curve with equation $y = \sqrt{4 - x^2}$.



If the shaded region is rotated around the x axis, it will form a sphere of radius 2.

Divide the interval [–2, 2] into n subintervals $[x_{i-1}, x_i]$ with $x_0 = -2$ and $x_n = 2$.



The volume of a typical slice, a cylinder, is approximately $\pi[y(c_i)]^2\,(x_i - x_{i-1})$ where $y(c_i)$ is the value of y when $x = c_i$ and $c_i \in [x_{i-1}, x_i]$. The total volume will be approximated by the sum of the volumes of these slices and as n, the number of subdivisions, gets larger and larger:

$$V = \lim_{n\to\infty} \sum_{i=1}^{n} \pi[y(c_i)]^2(x_i - x_{i-1})$$

It has been seen that the limit of such a sum is an integral and therefore:

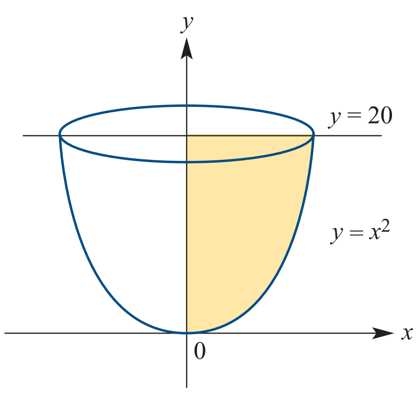
$$V = \int_{-2}^{2} \pi[y(x)]^2\,dx$$

where $y(x)$ is the value of y at x or, in function notation, for $f(x) = \sqrt{4 - x^2}$

$$\begin{aligned} V &= \int_{-2}^{2} \pi[f(x)]^2\,dx \\ &= \int_{-2}^{2} \pi(4 - x^2)\,dx \\ &= \pi\left[4x - \frac{x^3}{3}\right]_{-2}^{2} \\ &= \pi\left(8 - \frac{8}{3} - \left(-8 + \frac{8}{3}\right)\right) \\ &= \pi\left(16 - \frac{16}{3}\right) \\ &= \frac{32\pi}{3} \end{aligned}$$

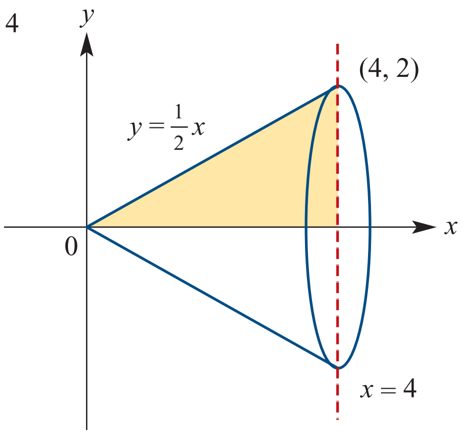
The solid formed by rotating a region about a line is called a **solid of revolution**.

If the region between the graph of $y = x^2$, the line $y = 20$ and the y axis is rotated around the y axis, a solid in the shape of the top of a wine glass is produced.



If the region between the line $y = \frac{1}{2}x$, the line $x = 4$ and the x axis is rotated around the x axis, a solid in the shape of a cone is produced.

The volume, V, of the cone is given by:



$$\begin{aligned} V &= \int_0^4 \pi y^2\,dx \\ &= \int_0^4 \pi\left(\tfrac{1}{2}x\right)^2 dx \\ &= \frac{\pi}{4}\left[\frac{x^3}{3}\right]_0^4 \\ &= \frac{\pi}{4}\left[\frac{64}{3}\right] \\ &= \frac{16\pi}{3} \end{aligned}$$

In general, for **rotation about the x axis**, if the region to be rotated is bounded by the curve with equation $y = f(x)$ and the lines $x = a$ and $x = b$ and the x axis, then:

$$V = \int_{x=a}^{x=b} \pi y^2\,dx = \int_a^b \pi[f(x)]^2\,dx$$

For **rotation about the y axis**, if the region is bound by the curve with equation $x = f(y)$ and the lines $y = a$ and $y = b$ then

$$V = \pi\int_{y=b}^{y=a} x^2\,dy$$

Example 19

Find the volume of the solid of revolution formed by rotating the curve $y = x^3$ about:

a the x axis for $0 \le x \le 1$

b the y axis for $0 \le y \le 1$

Solution

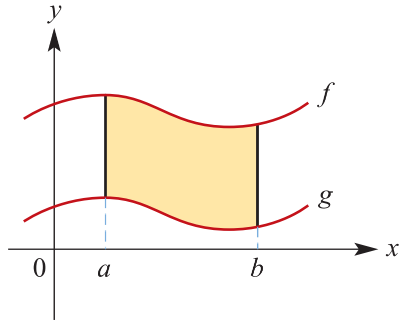
a
$$\begin{aligned} V &= \pi\int_0^1 y^2\,dx \\ &= \pi\int_0^1 x^6\,dx \\ &= \pi\left[\frac{x^7}{7}\right]_0^1 \\ &= \frac{\pi}{7} \end{aligned}$$

b
$$\begin{aligned} V &= \pi\int_0^1 x^2\,dy \\ &= \pi\int_0^1 y^{\frac{2}{3}}\,dy \\ &= \pi\left[\frac{3}{5}y^{\frac{5}{3}}\right]_0^1 \\ &= \frac{3\pi}{5} \end{aligned}$$

Regions not bounded by the x axis

If the shaded region is rotated around the x axis then the volume, V, is given by:

$$V = \pi\int_a^b [f(x)]^2 - [g(x)]^2\,dx$$



Example 20

Find the volume of the solid of revolution when the region bounded by the graphs of $y = 2e^{2x}$, $y = 1$, $x = 0$ and $x = 1$ is rotated around the x axis.

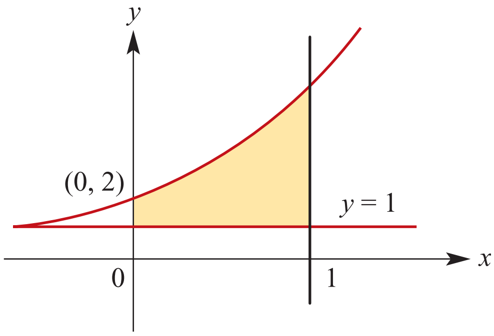
Solution

The volume $V = \pi\int_0^1 4e^{4x} - 1\,dx$

$= \pi[e^{4x} - x]_0^1$

$= \pi(e^4 - 1 - (1))$

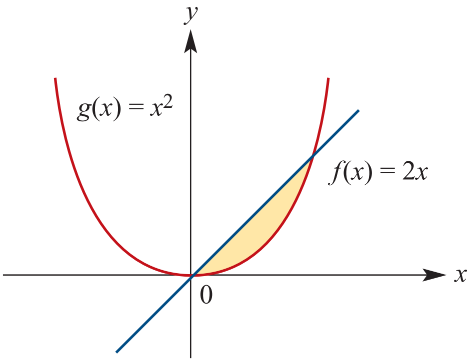
$= \pi(e^4 - 2)$



Note: $f(x) = 2e^{2x}$, $g(x) = 1$

Example 21

The shaded region is rotated around the x axis. Find the volume of the resulting solid.



Solution

The graphs meet where $2x = x^2$, i.e. the graphs meet at the points with coordinates (0, 0) and (2, 4).

$$\therefore \text{Volume} = \pi\int_0^2 [f(x)]^2 - [g(x)]^2\,dx$$

$$= \pi\int_0^2 4x^2 - x^4\,dx$$

$$= \pi\left[\frac{4x^3}{3} - \frac{x^5}{5}\right]_0^2$$

$$= \pi\left[\frac{32}{3} - \frac{32}{5}\right]$$

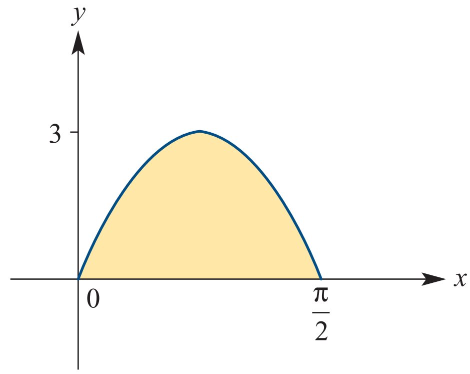
$$= \frac{64\pi}{15}$$

Example 22

A solid is formed when the region bounded by the x axis and the graph of $y = 3\sin 2x$, $0 \le x \le \frac{\pi}{2}$ is rotated around the x axis. Find the volume of this solid.

Solution

$$V = \pi \int_0^{\frac{\pi}{2}} (3\sin 2x)^2\,dx$$
$$= \pi \int_0^{\frac{\pi}{2}} 9\sin^2 2x\,dx$$
$$= 9\pi \int_0^{\frac{\pi}{2}} \sin^2 2x\,dx$$
$$= 9\pi \int_0^{\frac{\pi}{2}} \frac{1}{2}(1 - \cos 4x)\,dx$$
$$= \tfrac{9}{2}\pi \int_0^{\frac{\pi}{2}} (1 - \cos 4x)\,dx$$
$$= \tfrac{9}{2}\pi \left[x - \tfrac{1}{4}\sin 4x\right]_0^{\frac{\pi}{2}}$$
$$= \tfrac{9}{2}\pi \left[\frac{\pi}{2}\right]$$
$$= \frac{9\pi^2}{4}$$

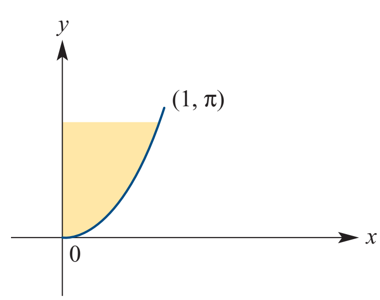


Example 23

The curve $y = 2\sin^{-1} x$, $0 \le x \le 1$ is rotated around the y axis to form a solid of revolution. Find the volume of this solid.

Solution

$$V = \pi \int_0^{\pi} \sin^2 \frac{y}{2}\,dy$$
$$= \frac{\pi}{2} \int_0^{\pi} (1 - \cos y)\,dy$$
$$= \frac{\pi}{2}[y - \sin y]_0^{\pi}$$
$$= \frac{\pi^2}{2}$$



Exercise 8D

1 Find the volume of the solid of revolution when the regions bounded by the given curves, the x axis and the given lines is rotated about the x axis:

a $f(x) = \sqrt{x}, x = 4$ **b** $f(x) = 2x + 1, x = 0, x = 4$

c $f(x) = 2x - 1, x = 4$

d $f(x) = \sin x, x = \frac{\pi}{2}$ $\left(\text{for } 0 \le x \le \frac{\pi}{2}\right)$

e $f(x) = e^x, x = 0, x = 2$

f $f(x) = \sqrt{9 - x^2}, -3 \le x \le 3$

2 Find the area of the region bounded by the x axis and the curve whose equation is $y = 4 - x^2$. Also find the volume of the solid formed when this region is rotated about the y axis.

3 The hyperbola $x^2 - y^2 = 1$ is rotated around the x axis to form a surface of revolution. Find the volume of the solid enclosed by this surface between $x = 1$ and $x = \sqrt{3}$.

4 The region, for which $x \ge 0$, bounded by the curves $y = \cos x$ and $y = \sin x$ and the y axis, is rotated around the x axis forming a solid of revolution. By using the identity $\cos 2x = \cos^2 x - \sin^2 x$, obtain a volume for this solid.

5 Find the volumes of the solids generated by rotating about the x axis each of the regions bounded by the following curves and lines:

a $y = \frac{1}{x},\ y = 0,\ x = 1,\ x = 4$

b $y = x^2 + 1, y = 0, x = 0, x = 1$

c $y = \sqrt{x}, y = 0, x = 2$

d $y = \sqrt{a^2 - x^2}, y = 0$

e $y = \sqrt{9 - x^2}, y = 0$

f $y = \sqrt{9 - x^2}, y = 0, x = 0,$ given $x \ge 0$

6 The region bounded by the line $y = 5$ and the curve $y = x^2 + 1$ is rotated about the x axis. Find the volume generated.

7 The region enclosed by $y = \frac{4}{x^2}, x = 4, x = 1$ and the x axis is rotated about the x axis. Find the volume generated.

8 The region enclosed by $y = x^2$ and $y^2 = x$ is rotated about the x axis. Find the volume generated.

9 A region is bounded by the curve $y = \sqrt{6 - x}$, the straight line $y = x$ and the positive x axis. Find the volume of the solid of revolution formed by rotating this figure about the x axis.

10 The region bounded by the x axis, the line $x = \frac{\pi}{2}$ and the curve $y = \tan \frac{x}{2}$ is rotated about the x axis. Prove that the volume of the solid of revolution is $\frac{\pi}{2}(4 - \pi)$. (Hint: Use the result that $\tan^2 \frac{x}{2} = \sec^2 \frac{x}{2} - 1$)

11 Sketch the graphs of $y = \sin x$ and $y = \sin 2x$ for $0 \le x \le \frac{\pi}{2}$ and show that the area of the region bounded by these graphs is $\frac{1}{4}$ square unit, and the volume formed by rotating this region about the x axis is $\frac{3}{16}\pi\sqrt{3}$ cubic units.

12 Let V be the volume of the solid that results when the region enclosed by $y = \frac{1}{x}$, $y = 0, x = 4$ and $x = b$, where $0 < b < 4$, is rotated about the x axis. Find the value of b for which $V = 3\pi$.

13 Find the volume of the solid generated when the region enclosed by $y = \sqrt{3x + 1}$, $y = \sqrt{3x}$, $y = 0$ and $x = 1$ is rotated about the x axis.

14 Find the volumes of the solids formed when the following regions are rotated around the y axis:

a $x^2 = 4y^2 + 4$ for $0 \le y \le 1$ **b** $y = \log_e(2 - x)$ for $0 \le y \le 2$

15 **a** Find the area of the region bounded by the curve $y = e^x$, the tangent at the point $(1, e)$ and the y axis.

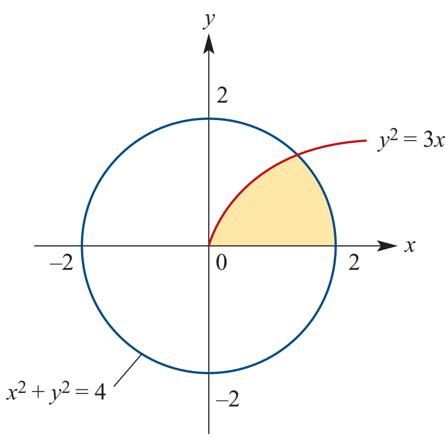
b Find the volume of the solid formed by rotating this region through a complete revolution about the x axis.

16 The region defined by the inequalities $y \ge x^2 - 2x + 4$ and $y \le 4$ is rotated about the line $y = 4$. Find the volume generated.

17 The area enclosed by $y = \cos \frac{x}{2}$ and the x axis, for $0 \le x \le \pi$, is rotated about the x axis. Find the volume generated.

18 Find the volume generated by revolving the region enclosed between the parabola $y = 3x - x^2$ and the line $y = 2$ about the x axis.

19 The shaded region is rotated around the x axis to form a solid of revolution. Find the volume of this solid.



20 The region enclosed between the curve $y = e^x - 1$, the x axis and the line $x = \log_e 2$ is rotated around the x axis to form a solid of revolution. Find the volume of this solid.

21 Show that the volume of the solid of revolution formed by rotating about the x axis the region bounded by the curve $y = e^{-2x}$ and the lines $x = 0$, $y = 0$ and $x = \log_e 2$ is $\frac{15\pi}{64}$.

22 Find the volume of the solid generated by revolving about the x axis the region bounded by $y = 2\tan x$, $x = \frac{-\pi}{4}$, $x = \frac{\pi}{4}$ and $y = 0$.

23 The region bounded by the parabola $y^2 = 4(1 - x)$ and the y axis is rotated about:

a the x axis **b** the y axis

Prove that the volumes of the solids formed are in the ratio 15 : 16.

24 The region bounded by the graph of $y = \dfrac{1}{\sqrt{x^2 + 9}}$, the x axis, the y axis and the line $x = 4$ is rotated about:

a the x axis **b** the y axis

Find the volumes of the solid formed in each case.

25 A bucket is defined by rotating the curve with equation

$y = 40 \log_e\left(\dfrac{x - 20}{10}\right)$, $0 \leq y \leq 40$ about the y axis.

If x and y are measured in centimetres, find the maximum volume of liquid that the bucket could hold. Give the answer to the nearest cm^3.

26 An ellipse has equation $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$. Find the volume of the solid generated when the region bounded by the ellipse is rotated about:

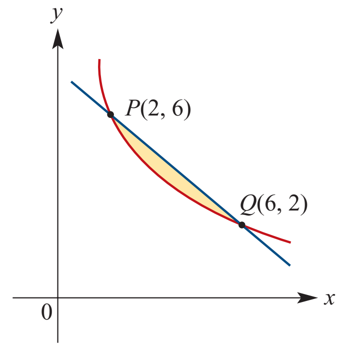
a the x axis **b** the y axis

27 The diagram shows part of the curve with equation $y = \dfrac{12}{x}$. Points $P(2, 6)$ and $Q(6, 2)$ lie on the curve. Find:

a the equation of line PQ

b the volume obtained when the shaded region is rotated about:

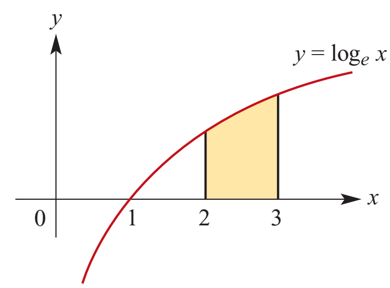
i the x axis **ii** the y axis



28 **a** Sketch the graph of $y = 2x + \dfrac{9}{x}$.

b Find the volume generated when the region bounded by the curve $y = 2x + \dfrac{9}{x}$ and the lines $x = 1$ and $x = 3$ is rotated about the x axis.

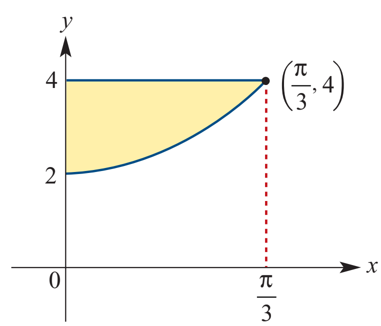
29 The region shown is rotated about the x axis to form a solid of revolution. Find the volume of the solid, correct to three decimal places.



30 The graphs of $y = 2 \sec x$ and $y = 4$ are shown for $0 \leq x \leq \dfrac{\pi}{3}$. The shaded region is rotated about the x axis to form a solid of revolution. Calculate the exact volume of this solid.

MC TEST

D&D TEST

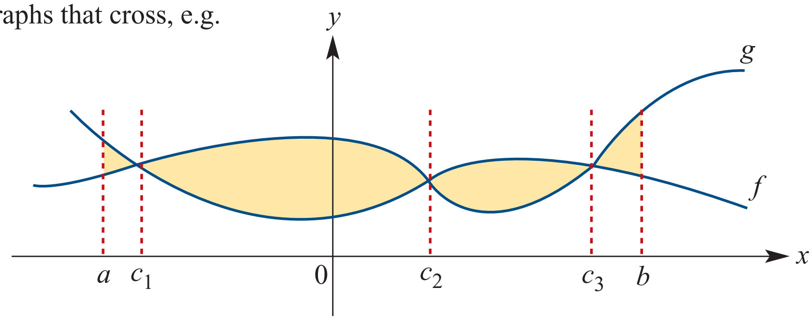


Chapter summary

- Let f and g be continuous on $[a, b]$, such that $f(x) \geq g(x)$ for $x \in [a, b]$. Then the area of the region bounded by the curves and the lines $x = a$ and $x = b$ can be calculated by the evaluation of:

$$\int_a^b (f(x) - g(x))\,dx.$$

- For graphs that cross, e.g.



find the area of the shaded region, by considering the intervals:

$[a, c_1]$, $[c_1, c_2]$, $[c_2, c_3]$, and $[c_3, b]$

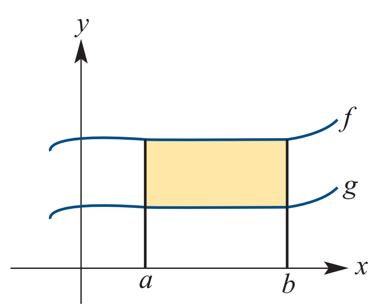
and the area is given by

$$\int_a^{c_1} f(x) - g(x)\,dx + \int_{c_1}^{c_2} g(x) - f(x)\,dx + \int_{c_2}^{c_3} f(x) - g(x)\,dx$$
$$+ \int_{c_3}^{b} g(x) - f(x)\,dx.$$

- In general, for rotation around the x axis, if the region to be rotated about the x axis is bounded by the curve with equation $y = f(x)$ and the lines $x = a$ and $x = b$ and the x axis, then the volume, V, is determined by:

$$V = \int_a^b \pi y^2\,dx = \int_a^b \pi (f(x))^2\,dx.$$

- Regions not bounded by the x axis, but rotated about the x axis, e.g.



If the region is rotated about x axis, the volume, V, is given by:

$$V = \pi \int_a^b (f(x))^2 - (g(x))^2\,dx.$$

- The fundamental theorem of calculus gives that if f is a function that can be integrated on a interval $[a, b]$, then there exists a function F such that $F'(x) = f(x)$ and

$$\int_a^b f(x)\,dx = F(b) - F(a).$$

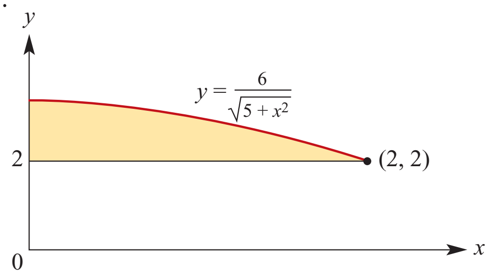
The function F can be defined as $F(x) = \int_a^x f(t)\,dt + F(a)$.

Multiple-choice questions

1 The volume of the solid of revolution formed when the region bounded by the axes, the line $x = 1$ and the curve with equation $y = \dfrac{1}{\sqrt{4 - x^2}}$ is rotated about the x axis is:

A $\frac{\pi^2}{6}$ **B** $\frac{\pi^2}{3}$ **C** $\frac{\pi}{4}\log_e(3)$ **D** $\pi\sqrt{3}\log_e(3)$ **E** $\frac{2\pi^2}{3}$

2 The shaded region shown below is enclosed by the curve $y = \dfrac{6}{\sqrt{5 + x^2}}$, the straight line $y = 2$ and the y axis. The region is rotated about the x axis to form a solid of revolution. The volume of this solid, in cubic units, is given by:



A $\pi \displaystyle\int_0^2 \left(\frac{6}{\sqrt{5 + x^2}} - 2\right)^2 dx$

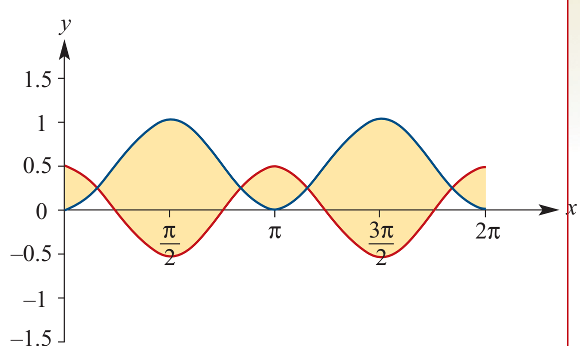
B $6\pi \tan^{-1}\left(\frac{2}{5}\right)$

C $\dfrac{36\pi\sqrt{5}\tan^{-1}\left(\frac{2\sqrt{5}}{5}\right)}{5}$

D $\pi \displaystyle\int_0^2 \left(\frac{6}{\sqrt{5 + x^2}}\right)^2 - 4\,dx$

E 36π

3 The graphs of $y = \sin^2 x$ and $y = \frac{1}{2}\cos(2x)$ are shown in the diagram. The total area of the shaded regions is equal to:



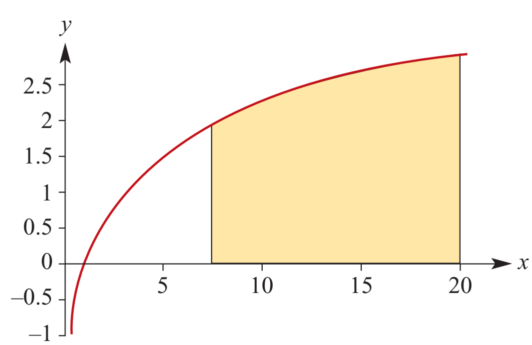
A $\displaystyle\int_0^{2\pi} \sin^2 x - \frac{1}{2}\cos(2x)\,dx$

B $4\displaystyle\int_0^{\frac{\pi}{6}} \frac{1}{2}\cos(2x) - \sin^2 x\,dx + 2\int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} \sin^2 x - \frac{1}{2}\cos(2x)\,dx$

C 3.14

D π **E** $\frac{\sqrt{3}}{2} + \frac{\pi}{3}$

4 The shaded region in the diagram is bounded by the lines $x = e^2$ and $x = e^3$, the x axis and the graph of $y = \log_e(x)$. The volume of the solid of revolution formed by rotating this region about the x axis is equal to:



A $\pi \displaystyle\int_2^3 e^{2x}\,dx$

B $\pi \displaystyle\int_7^{20} (\log_e(x))^2\,dx$

C $\pi \displaystyle\int_{e^2}^{e^3} (\log_e(x))^2\,dx$

D $\pi(e^3 - e^2)$

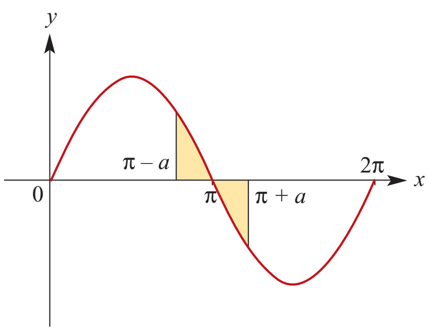
E $\pi^2 \displaystyle\int_{e^2}^{e^3} (\log_e(x))^2\,dx$

Review

5 The graph represents the function $y = \sin x$ where $0 \le x \le 2\pi$. The total area of the shaded regions is:

A $1 - \cos a$ **B** $-2\sin a$ **C** $2(1 - \cos a)$

D 0 **E** $-2(1 - \cos a)$



6 The area of the region enclosed between the curve with equation $y = \sin^3 x$, the x axis and the line with equation $x = a$ where $0 < a < \frac{\pi}{2}$ is:

A $3\cos^2(a)$ **B** $\frac{2}{3} - \frac{1}{3}\sin^3 a$ **C** $(\frac{2}{3} - \frac{1}{3}\sin^2 a)\cos a + \frac{2}{3}$

D $\frac{1}{3}\cos^3 a \sin a$ **E** $\frac{2}{3} - \cos a + \frac{1}{3}\cos^3 a$

7 The shaded region shown is rotated around the x axis to form a solid of revolution. The volume of the solid of revolution is:

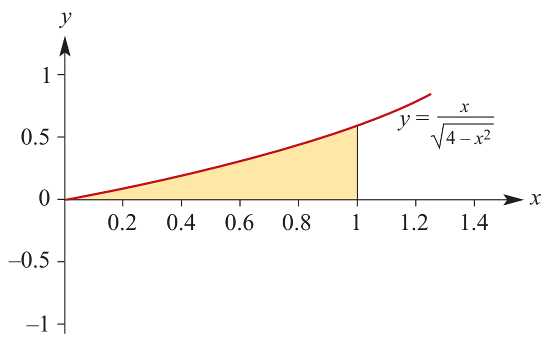
A $1 - \log_e(\frac{1}{3})$

B $\pi(\log_e(3) - 1)$

C 0.099

D $\pi(-1 + \log_e(\frac{1}{3}))$

E 0.1



8 The shaded region shown in the diagram is rotated around the x axis to form a solid of revolution. $f'(x) > 0$ and $f''(x) > 0$ for all $x \in [a, b]$ and the volume of the solid of revolution is V cubic units. Which of the following statements is false?

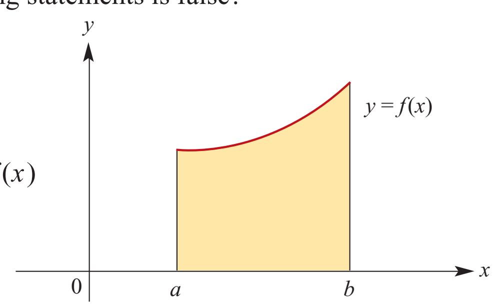
A $V < \pi[f(b)]^2(b - a)$

B $V > \pi[f(a)]^2(b - a)$

C $V = \pi\int_a^b [f(x)]^2 dx$

D $V = \pi([F(b)]^2 - [F(a)]^2)$ where $F'(x) = f(x)$

E $V < \pi([f(b)]^2 b - [f(a)]^2 a]$



9 The area of the region bounded by the curve $y = \cos(\frac{x}{2})$, the x axis and the lines $x = 0$ and $x = \pi$ is:

A 0 **B** 1 **C** 2 **D** π **E** 4

10 The region bounded by the coordinate axes and the graph of $y = \cos x$, for $0 \le x \le \frac{\pi}{2}$, is rotated about the y axis to form a solid of revolution. The volume of the solid is given by:

A $\pi\int_0^1 \cos^2 x\, dx$ **B** $\pi\int_0^{\frac{\pi}{2}} \cos^2 x\, dx$ **C** $\pi\int_0^1 \cos^{-1} y\, dx$

D $\pi\int_0^{\frac{\pi}{2}} (\cos^{-1} y)^2\, dy$ **E** $\pi\int_0^1 (\cos^{-1} y)^2\, dy$

Short-answer questions (technology-free)

1 Calculate the area of the region enclosed by the graph of $y = \frac{x}{\sqrt{x-2}}$ and the line $y = 3$.

2 a If $y = 1 - \cos x$, find the value of $\int_0^{\frac{\pi}{2}} y\,dx$. On a sketch graph, indicate the area represented by the integral.

b Hence find $\int_0^1 x\,dy$.

3 Find the volumes of revolution of each of the following (rotation is about the x axis):

a $y = \sec x$ between $x = 0$ and $x = \frac{\pi}{4}$

b $y = \sin x$ between $x = 0$ and $x = \frac{\pi}{4}$

c $y = \cos x$ between $x = 0$ and $x = \frac{\pi}{4}$

d the region between $y = x^2$ and $y = 4x$

e $y = \sqrt{1+x}$ between $x = 0$ and $x = 8$

4 Find the volume generated when the region bounded by the curve $y = 1 + \sqrt{x}$, the x axis and the lines $x = 1$ and $x = 4$ is rotated about the x axis.

5 The region S in the first quadrant of the x–y plane is bounded by the axes, the line $x = 3$ and the curve $y = \sqrt{1+x^2}$. Find the volume of the solid formed when S is rotated:

a about the x axis

b about the y axis.

6 Sketch the graph of $y = \sec x$ for $x \in \left(\frac{-\pi}{2}, \frac{\pi}{2}\right)$. Find the volume of the solid of revolution obtained by rotating this curve about the x axis for $x \in \left[\frac{-\pi}{4}, \frac{\pi}{4}\right]$.

7 a Find the coordinates of the points of intersection of the graphs of $y^2 = 8x$ and $y = 2x$.

b Find the volume of the solid formed when the area enclosed by these graphs is rotated about the x axis.

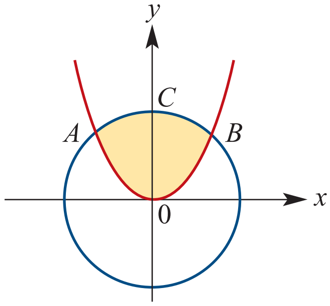
8 a On the one set of axes, sketch the graphs of $y = 1 - x^2$ and $y = x - x^3 = x(1 - x^2)$. (Turning points of the second graph do not have to be determined.)

b Find the area of the region enclosed between the two graphs.

9 The curves $y = x^2$ and $x^2 + y^2 = 2$ meet at the points A and B.

a Find the coordinates of A, B and C.

b Find the volume of the solid of revolution formed by rotating the region about the x axis.



10 a Sketch the graph of $y = 2x - x^2$ for $y \geq 0$.

b Find the area of the region enclosed between this curve and the x axis.

c Find the volume of the solid of revolution formed by rotating this region about the x axis.

Review

11 **a** Let the curve $f: [0, b] \to R, f(x) = x^2$ be rotated:

  i around the x axis to define a solid of revolution and find the volume of this solid in terms of b (region rotated is between the curve and the x axis)

  ii around the y axis to define a solid of revolution and find the volume of this solid in terms of b (region rotated is between the curve and the y axis).

b For what value of b are the two volumes equal?

12 **a** Sketch the graph of $\left\{(x, y): y = \dfrac{1}{4x^2 + 1}\right\}$.

b Find $\dfrac{dy}{dx}$ and hence find the equation of the tangent to this curve at $x = \frac{1}{2}$.

c Find the area of the region bounded by the curve and the tangent to the curve at $x = \frac{1}{2}$.

13 If $f: R \to R, f(x) = x$ and $g: R\backslash\{0\} \to R, g(x) = \dfrac{9}{x}$:

a sketch, on the same set of axes, the graphs of $f + g$ and $f - g$

b find the area of the region bounded by the two graphs sketched in **a** and the lines $x = 1$ and $x = 3$.

14 Sketch the graph of $\left\{(x, y): y = x - 5 + \dfrac{4}{x}\right\}$.

Find the area of the region bounded by this curve and the x axis.

15 Sketch the graph of $\left\{(x, y): y = \dfrac{1}{2 + x - x^2}\right\}$.

Find the area of the region bounded by this graph and the line $y = \frac{1}{2}$.

Extended-response questions

1 **a** Sketch the curve whose equation is $y = 1 - \dfrac{1}{x + 2}$.

b Find the area of the region bounded by the x axis, the curve and the lines $x = 0$ and $x = 2$.

c Find the volume of the solid of revolution formed when this region is rotated around the x axis.

2 Let $f: R \to R, f(x) = x \tan^{-1} x$.

a Find $f'(x)$. **b** Hence find $\displaystyle\int_0^1 \tan^{-1} x \, dx$.

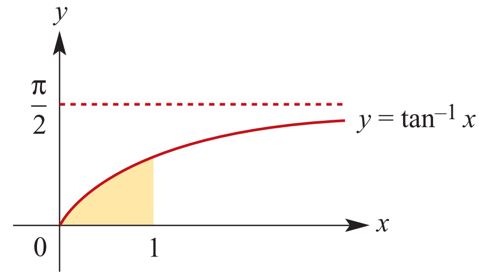
c Use the result of **b** to find the area of the region bounded by $y = \tan^{-1} x$, $y = \dfrac{\pi}{4}$ and the y axis.

d Let $g: R \to R, g(x) = (\tan^{-1} x)^2$.

  i Find $g'(x)$. **ii** Show that $g'(x) > 0$ for $x > 0$.

  iii Sketch the graph of $g: R \to R, g(x) = (\tan^{-1} x)^2$.

e Find the volume of the solid of revolution formed when the shaded region shown is rotated around the y axis.



3 **a** **i** Differentiate $x \log_e x$ and hence find $\int \log_e x \, dx$.

ii Differentiate $x(\log_e x)^2$ and hence find $\int (\log_e x)^2 \, dx$.

b Sketch the graph of f: $[-2, 2] \to R$,

$$f(x) = \begin{cases} e^x & x \in [0, 2] \\ e^{-x} & x \in [-2, 0] \end{cases}$$

c The interior of a wine glass is formed by rotating the curve $y = e^x$ from $x = 0$ to $x = 2$ about the y axis. If the units are in centimetres find, correct to two significant figures, the volume of liquid that the glass contains when full.

4 A bowl is modelled by rotating the curve $y = x^2$ for $0 \le x \le 1$ around the y axis.

a Find the volume of the bowl.

b If liquid is poured into the bowl at a rate of R units of volume per second, find the rate of increase of the depth of liquid in the bowl when the depth is $\frac{1}{4}$.

(Hint: Use the chain rule: $\dfrac{dv}{dt} = \dfrac{dv}{dy}\dfrac{dy}{dt}$.)

c **i** Find the volume of liquid in the bowl when the depth of liquid is $\frac{1}{2}$ a unit.

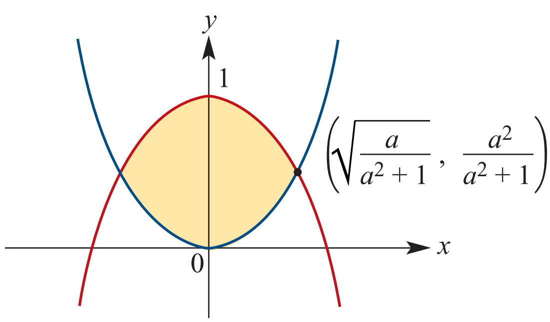
ii Find the depth of liquid in the bowl when it is half full.

5 **a** Show that the area enclosed between the two curves $y = ax^2$ and $y = 1 - \dfrac{x^2}{a}$ is $\dfrac{4}{3}\sqrt{\dfrac{a}{a^2+1}}$ where $a > 0$.

b **i** Find the value of a which gives a maximum area.

ii Find the maximum area.

c Find the volume of the solid formed when the region bounded by these curves is rotated about the y axis.



6 **a** On the same set of axes, sketch the graphs of $y = 3\sec^2 x$ and $y = 16\sin^2 x$ for $0 \le x \le \dfrac{\pi}{4}$.

b Find the coordinates of the point of intersection of these two curves.

c Find the area of the region bounded by the two curves and the y axis.

7 Let $f: (1, \infty) \to R$, be such that:

$$f'(x) = \frac{1}{x - a}, \text{ where } a \text{ is a positive constant}$$
$$f(2) = 1$$
$$f(1 + e^{-1}) = 0.$$

a Find a and use it to determine $f(x)$.

b Sketch the graph of f.

c If f^{-1} is the inverse of f, show that $f^{-1}(x) = 1 + e^{x-1}$. Give the domain and range of f^{-1}.

d Find the area of the region enclosed by $y = f^{-1}(x)$, the x axis, the y axis and the line $x = 1$.

e Find $\int_{1+e^{-1}}^{2} f(x)\, dx$.

8 Let $f: [0, a] \to R$, where $f(x) = 3 \cos \frac{1}{2}x$.

a Find the largest value of a for which f has an inverse function, f^{-1}.

b **i** State the domain and range of f^{-1}. **ii** Find $f^{-1}(x)$.

iii Sketch the graph of f^{-1}.

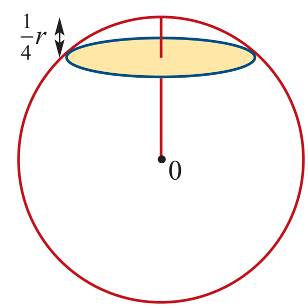
c Find the gradient of the curve $y = f^{-1}(x)$ at the point where the curve crosses the y axis.

d Let the volume of the solid of revolution formed by rotating the curve $y = f(x)$ between $x = 0$ and $x = \pi$ about the x axis be V_1. Let the volume of the solid of revolution formed by rotating the curve $y = f^{-1}(x)$ between $y = 0$ and $y = \pi$ about the y axis be V_2. Find V_1 and, hence, V_2.

9 The curves $cy^2 = x^3$ and $y^2 = ax$ (where $a > 0$ and $c > 0$) intersect at the origin, O, and at a point P in the first quadrant. The areas of the regions enclosed by the curves OP, the x axis and the ordinate through P are A_1 and A_2 respectively for the two curves. The volumes of the two solids formed by rotating these regions about the x axis are V_1 and V_2 respectively. Show that $A_1 : A_2 = 3 : 5$ and $V_1 : V_2 = 1 : 2$.

10 **a** Find the area of the circle formed when a sphere is cut by a plane at a distance y from the centre, where $y < r$.

b By integration, prove that the volume of a 'cap' of height $\frac{1}{4}r$ cut from the top of the sphere, as shown in the diagram, is $\frac{11\pi r^3}{192}$.



11 Consider the section of a hyperbola with $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ and $a \le x \le 2a$, $(a > 0)$.
Find the volume of the solid formed when the area of the region bounded by the hyperbola and the line with equation $x = 2a$ is rotated about the:

a x axis **b** y axis.

12 **a** Show that the line with equation $y = \dfrac{3x}{2}$ does not meet the curve with equation $y = \dfrac{1}{\sqrt{1-x^2}}$.

b Find the area of the region bounded by the curve with equation $y = \dfrac{1}{\sqrt{1-x^2}}$ and the lines $y = \dfrac{3x}{2}$, $x = 0$ and $x = \frac{1}{2}$.

c Find the volume of the solid of revolution formed by rotating the region defined in **b** about the x axis. The answer is to be expressed in the form $\pi(a + \log_e b)$.

13 A model for a bowl is formed by rotating a section of the graph of a cubic function $f(x) = ax^3 + bx^2 + cx + d$ around the x axis to form a solid of revolution. The cubic is chosen to pass through the points with coordinates (0, 0), (5, 1), (10, 2.5) and (30, 10).

a **i** Write down the four simultaneous equations that can be used to determine the coefficients a, b, c and d.

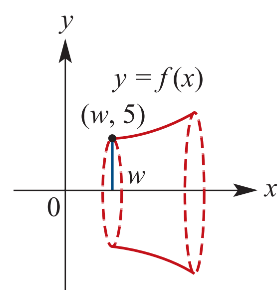
ii Using a CAS calculator or otherwise, find the values of a, b, c and d (exact values should be stated).

b Find the area of the region enclosed by the curve and the line $x = 30$.

c **i** Write the expression that can be used to determine the volume of the solid of revolution when the section of the curve $0 \le x \le 30$ is rotated around the x axis.

ii Use a CAS calculator to determine this volume.

CAS CALCULATOR

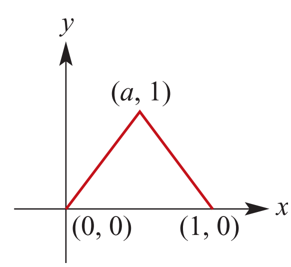


d The bowl is unstable as designed by the cubic. The designer is very fond of the cubic and modifies it so that the base has radius 5 units. Using a CAS calculator:

i find the value of w where $f(w) = 5$

ii find the new volume, correct to four significant figures.

e A mathematician looks at the design and suggests it may be more pleasing to the eye if the base is chosen to occur at a point where $x = p$ and $f''(p) = 0$. Find the values of coordinates of the point $(p, f(p))$.



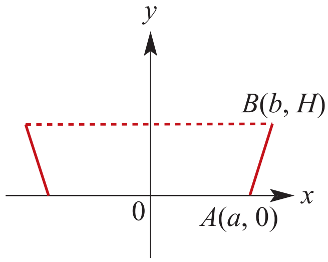
14 **a** For $0 \le a \le 1$, let T_a be the triangle whose vertices are (0, 0) (1, 0) and $(a, 1)$.

Find the volume of the solid of revolution when T_a is rotated about the x axis.

b For $0 \le k \le 1$, let T_k be the triangle whose vertices are (0, 0), $(k, 0)$ and $(0, \sqrt{1-k^2})$. T_k is rotated around the x axis. What value of k gives the maximum volume? What is the maximum volume?

Review

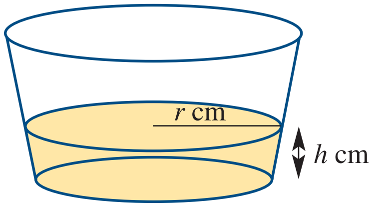
15 A model of a bowl is formed by rotating the line segment AB about the y axis to form a solid of revolution.



a Find the volume, V cm^3, of the bowl in terms of a, b, and H (units are centimetres).

b If the bowl is filled with water to a height $\frac{H}{2}$, find the volume of water.

c Find an expression for the volume of water in the bowl when the radius of the water surface is r cm (the constants a, b and H are to be used).



d **i** Find $\frac{dV}{dr}$.

ii Find an expression for the depth of the water, h cm, in terms of r.

e If $a = 10$, $b = 20$ and $H = 20$:

i find $\frac{dV}{dr}$ in terms of r

ii and if water is being poured into the bowl at 3 cm^3/s, find $\frac{dr}{dt}$ and $\frac{dh}{dt}$ when $r = 12$.

CHAPTER

9

Differential equations

Objectives

- To **verify** a solution for a differential equation
- To apply techniques to **solve** differential equations of the form $\frac{dy}{dx} = f(x)$ and $\frac{d^2y}{dx^2} = f(x)$
- To apply techniques to solve differential equations of the form $\frac{dy}{dx} = g(y)$
- To **construct** differential equations from a given situation
- To solve differential equations using a CAS calculator
- To use **Euler's method** as a numerical method for obtaining approximate solutions for a given differential equation
- To construct a slope field for a given differential equation

9.1 An introduction to differential equations

Differential equations are equations which involve at least one derivative.

$\frac{dx}{dt} = \cos t, \quad \frac{d^2x}{dt^2} - 4x = t, \quad \frac{dy}{dx} = \frac{y}{y+1}$ are all examples of differential equations.

Such equations are used to describe many scientific and engineering principles and their study is a major branch of mathematics. For Specialist Mathematics, only a limited variety of differential equations will be considered.

The following notation is used to denote the y value for a given x value:

$y(0) = 4$ will mean that when $x = 0$, $y = 4$.

Solution of differential equations

A differential equation contains derivatives of a particular function or variable. Its solution is a clear definition of the function or relation without any derivatives included.

e.g. $\frac{dx}{dt} = \cos t$ then $x = \int \cos t \, dt$ i.e. $x = \sin t + c$

$x = \sin t + c$ is the **general solution** of the differential equation $\frac{dx}{dt} = \cos t$.

This example displays the main features of such solutions. Solutions of differential equations are the result of an integral, and therefore produce a family of functions.

To obtain a **particular solution**, further information is required and is usually given as an ordered pair belonging to the function or relation. (For equations with second derivatives, two items of information are required.)

Example 1

a Verify that $y = Ae^x - x - 1$ is a solution of the differential equation $\frac{dy}{dx} = x + y$.

b Hence find the particular solution of the differential equation given that $y(0) = 3$.

Solution

a Solutions of differential equations are verified by substitution.

If $y = Ae^x - x - 1$ then $\frac{dy}{dx} = Ae^x - 1$

Given $\frac{dy}{dx} = x + y$

LHS $= Ae^x - 1$ RHS $= x + Ae^x - x - 1 = Ae^x - 1$

$\therefore \quad y = Ae^x - x - 1$ is a solution of $\frac{dy}{dx} = x + y$.

b $y(0) = 3$ means that when $x = 0$, $y = 3$. Substituting in the solution verified in **a**

$3 = Ae^0 - 0 - 1$

i.e. $3 = A - 1$

$\therefore \quad A = 4$

$\therefore \quad y = 4e^x - x - 1$ is the particular solution.

Example 2

Verify that $y = e^{2x}$ is a solution of the differential equation $\frac{d^2y}{dx^2} + \frac{dy}{dx} - 6y = 0$.

Solution

Now $\quad y = e^{2x}$

$\therefore \quad \frac{dy}{dx} = 2e^{2x}$

$\therefore \quad \frac{d^2y}{dx^2} = 4e^{2x}$

Now consider the differential equation.

$$\begin{aligned} \text{LHS} &= \frac{d^2y}{dx^2} + \frac{dy}{dx} - 6y \\ &= 4e^{2x} + 2e^{2x} - 6e^{2x} \text{ (from above)} \\ &= 0 \\ &= \text{RHS} \end{aligned}$$

Example 3

Verify that $y = ae^{2x} + be^{-3x}$ is a solution of the differential equation $\dfrac{d^2y}{dx^2} + \dfrac{dy}{dx} - 6y = 0$.

Solution

Now $\quad y = ae^{2x} + be^{-3x}$

$$\therefore \quad \frac{dy}{dx} = 2ae^{2x} - 3be^{-3x}$$

$$\therefore \quad \frac{d^2y}{dx^2} = 4ae^{2x} + 9be^{-3x}$$

$$\begin{aligned}
\text{LHS} &= \frac{d^2y}{dx^2} + \frac{dy}{dx} - 6y \\
&= (4ae^{2x} + 9be^{-3x}) + (2ae^{2x} - 3be^{-3x}) - (6ae^{2x} - 6be^{-3x}) \\
&= 4ae^{2x} + 9be^{-3x} + 2ae^{2x} - 3be^{-3x} - 6ae^{2x} + 6be^{-3x} \\
&= 0 \\
&= \text{RHS}
\end{aligned}$$

Example 4

Find the constants a and b if $y = e^{4x}(2x + 1)$ is a solution of the differential equation $\dfrac{d^2y}{dx^2} - a\dfrac{dy}{dx} + by = 0$.

Solution

If $y = e^{4x}(2x + 1)$

$$\begin{aligned}
\text{Then} \quad \frac{dy}{dx} &= 4e^{4x}(2x + 1) + 2e^{4x} \\
&= 2e^{4x}(4x + 2 + 1) \\
&= 2e^{4x}(4x + 3)
\end{aligned}$$

$$\begin{aligned}
\text{Thus} \quad \frac{d^2y}{dx^2} &= 8e^{4x}(4x + 3) + 4 \times 2e^{4x} \\
&= 8e^{4x}(4x + 3 + 1) \\
&= 8e^{4x}(4x + 4) \\
&= 32e^{4x}(x + 1)
\end{aligned}$$

Thus if $\dfrac{d^2y}{dx^2} - a\dfrac{dy}{dx} + by = 0$

$$32e^{4x}(x + 1) - 2ae^{4x}(4x + 3) + be^{4x}(2x + 1) = 0$$

Divide through by e^{4x} $\quad (e^{4x} > 0)$

$$32x + 32 - 8ax - 6a + 2bx + b = 0$$

i.e. $\quad (32 - 8a + 2b)x + (b - 6a + 32) = 0$

Thus

$$32 - 8a + 2b = 0 \quad \boxed{1}$$

and $$32 - 6a + b = 0 \quad \boxed{2}$$

Multiply $\boxed{2}$ by 2 and subtract from $\boxed{1}$

This yields

$$-32 + 4a = 0$$

$\therefore$ $a = 8$ and $b = 16$

Exercise 9A

1 For each of the differential equations given below, verify that the given function or relation is a solution of the differential equation. Hence find the particular solution from the given information.

	Differential equation	Function or relation	Added information
a	$\frac{dy}{dt} = 2y + 4$	$y = Ae^{2t} - 2$	$y(0) = 2$
b	$\frac{dy}{dx} = \log_e \lvert x \rvert$	$y = x \log_e \lvert x \rvert - x + c$	$y(1) = 3$
c	$\frac{dy}{dx} = \frac{1}{y}$	$y = \sqrt{2x + c}$	$y(1) = 9$
d	$\frac{dy}{dx} = \frac{y+1}{y}$	$y - \log_e \lvert y + 1 \rvert = x + c$	$y(3) = 0$
e	$\frac{d^2y}{dx^2} = 6x^2$	$y = \frac{x^4}{2} + Ax + B$	$y(0) = 2,\ y(1) = 2$
f	$\frac{d^2y}{dx^2} = 4y$	$y = Ae^{2x} + Be^{-2x}$	$y(0) = 3,\ y(\log_e 2) = 9$
g	$\frac{d^2x}{dt^2} + 9x = 18$	$x = A\sin(3t) + B\cos(3t) + 2$	$x(0) = 4,\ x\left(\frac{\pi}{2}\right) = -1$

2 Verify that the given function is a solution of the given differential equation in each of the following:

a $\frac{dy}{dx} = 2y,\ y = 4e^{2x}$

b $\frac{dy}{dx} = -4xy^2,\ y = \frac{1}{2x^2}$

c $\frac{dy}{dx} = 1 + \frac{y}{x},\ y = x\log_e|x| + x$

d $\frac{dy}{dx} = \frac{2x}{y^2},\ y = \sqrt[3]{3x^2 + 27}$

e $\frac{d^2y}{dx^2} - \frac{dy}{dx} - 6y = 0,\ y = e^{-2x} + e^{3x}$

f $\frac{d^2y}{dx^2} - 8\frac{dy}{dx} + 16y = 0,\ y = e^{4x}(x + 1)$

g $\frac{d^2y}{dx^2} = -n^2y,\ y = a\sin nx$

h $\frac{d^2y}{dx^2} = n^2y,\ y = e^{nx} + e^{-nx}$

i $\frac{dy}{dx} = \frac{1 + y^2}{1 + x^2},\ y = \frac{x + 1}{1 - x}$

j $y\frac{d^2y}{dx^2} = 2\left(\frac{dy}{dx}\right)^2,\ y = \frac{4}{x + 1}$

3 If $\frac{dx}{dy}$ is inversely proportional to y, and $y = 2$ when $x = 0$ and $y = 4$ when $x = 2$, find y when $x = 3$.

4 If the differential equation $x^2\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} - 10y = 0$ has a solution $y = ax^n$, find the possible values of n.

5 Find the constants a, b and c if $y = a + bx + cx^2$ is a solution of the differential equation $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + 4y = 4x^2$.

6 Find the constants a and b if $x = t(a\cos 2t + b\sin 2t)$ is a solution of the differential equation $\frac{d^2x}{dt^2} + 4x = 2\cos 2t$.

7 Find the constants a, b, c and d if $y = ax^3 + bx^2 + cx + d$ is a solution to the differential equation $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + y = x^3$.

9.2 Solution of differential equations of the forms $\frac{dy}{dx} = f(x)$ and $\frac{d^2y}{dx^2} = f(x)$

Differential equations of the form $\frac{dy}{dx} = f(x)$

Differential equations of the form $\frac{dy}{dx} = f(x)$ present the simplest category of differential equations. Their solution can be obtained if the antiderivative of $f(x)$ can be found.

$$\text{If } \frac{dy}{dx} = f(x), \text{ then } y = \int f(x)\,dx$$

Example 5

Find the general solution of each of the following:

a $\frac{dy}{dx} = x^4 - 3x^2 + 2$ **b** $\frac{dy}{dt} = \sin 2t$

c $\frac{dx}{dt} = e^{-3t} + \frac{1}{t}$ **d** $\frac{dx}{dy} = \frac{1}{1+y^2}$

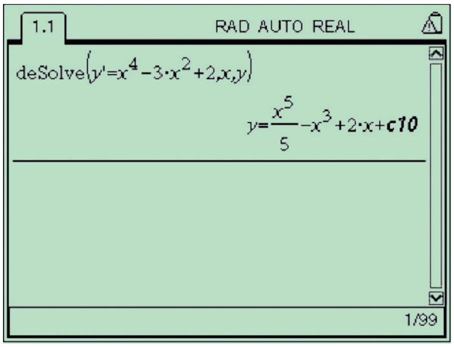
Solution

a $\frac{dy}{dx} = x^4 - 3x^2 + 2$ $\quad\therefore\ y = \int x^4 - 3x^2 + 2\,dx$

$\therefore y = \frac{x^5}{5} - x^3 + 2x + c$ is the general solution.

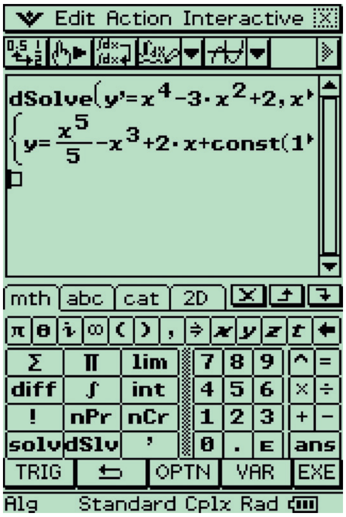
Using a TI-Nspire calculator

Choose **Differential Equation Solver** from the **Calculus** menu (menu 4 C). Complete as shown.



Using a Casio ClassPad calculator

Choose **Transformation**, then **Advanced** and then **dsolve**.
The differentiation symbol (′) is found on the **CALC** keyboard under **mth** as shown in the diagram.



b $\dfrac{dy}{dt} = \sin 2t \qquad \therefore\ y = \displaystyle\int \sin 2t\, dt$

$\therefore y = -\frac{1}{2}\cos 2t + c$ is the general solution.

c $\dfrac{dx}{dt} = e^{-3t} + \dfrac{1}{t} \qquad \therefore\ x = \displaystyle\int e^{-3t} + \frac{1}{t}\, dt$

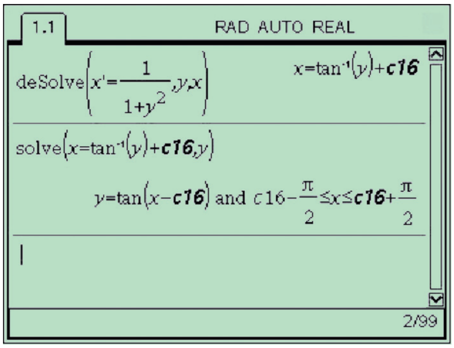
$\therefore x = -\frac{1}{3}e^{-3t} + \log_e|t| + c$ is the general solution.

d $\dfrac{dx}{dy} = \dfrac{1}{1+y^2} \qquad \therefore x = \displaystyle\int \frac{1}{1+y^2}\, dy$

$\therefore x = \tan^{-1} y + c$ or $y = \tan(x - c)$ is the general solution.

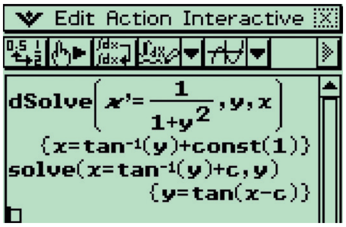
Using a TI-Nspire calculator

Choose **Differential Equation Solver** from the **Calculus** menu (menu 4 C).
Complete as shown.
Notice that this differential equation is of the form $\dfrac{dx}{dy} = f(y)$ and the roles of x and y are reversed.
Use **solve** to find y in terms of x.



Using Casio ClassPad calculator

Choose **Transformation**, then **Advanced** and then **dSolve**.
The differentiation symbol (′) is found on the **CALC** keyboard under **mth**.
Use **solve** to find y in terms of x.



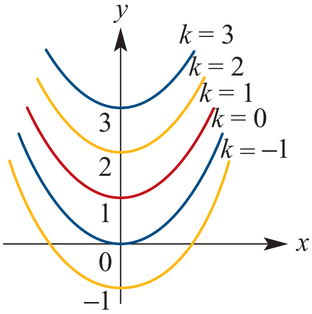
Families of curves

Solving differential equations requires finding an equation that connects the variables, but does not contain a derivative. There are no specific values for the variables. By solving differential equations, it is possible to determine what function or functions might model a particular situation or physical law.

If $\dfrac{dy}{dx} = x$, then it follows that $y = \frac{1}{2}x^2 + k$, where k is a constant.

From this it can be seen that the **general solution** of the differential equation $\dfrac{dy}{dx} = x$ can be given as $y = \frac{1}{2}x^2 + k$.

If different values of the constant k are taken, a family of curves can be obtained. The graphs show that this differential equation represents a family of curves $y = \frac{1}{2}x^2 + k$.



For **particular solutions** to a differential equation, a particular curve from the family can be distinguished by selecting a specific point of the plane through which the curve passes.

For instance, the solution of $\frac{dy}{dx} = x$ for which $y = 2$ when $x = 4$ can be thought of as the solution curve of the differential equation which passes through the point (4, 2).

From above:

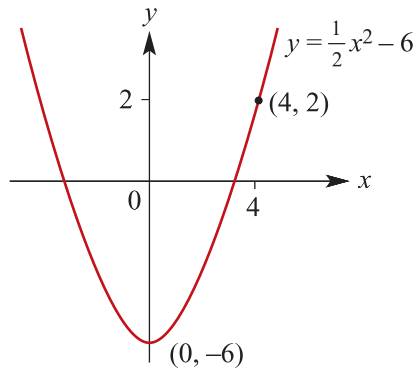
$$y = \tfrac{1}{2}x^2 + k$$

and $\therefore 2 = \tfrac{1}{2} \times 16 + k$

$\therefore 2 = 8 + k$

$k = -6$

and thus the solution is $y = \tfrac{1}{2}x^2 - 6$.



Example 6

a Find the family of curves whose gradient is e^{2x}.

b Find the equation of the curve whose gradient is e^{2x} and which passes through (0, 3).

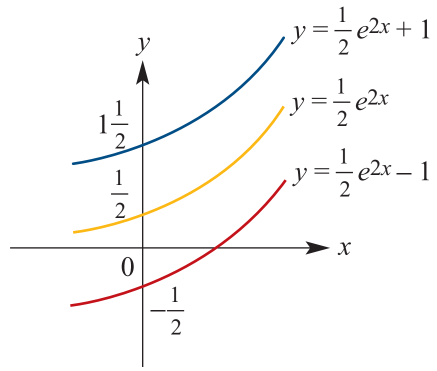
Solution

a $\frac{dy}{dx} = e^{2x}$

$\therefore \quad y = \int e^{2x}\,dx$

$= \tfrac{1}{2}e^{2x} + c$

This represents a family of curves since c can take any real number value. The diagram shows some of these curves.



b Substituting $x = 0, y = 3$ in the equation $y = \tfrac{1}{2}e^{2x} + c$ we have

$$3 = \tfrac{1}{2}e^0 + c$$

$\therefore \quad c = \tfrac{5}{2}$

$\therefore$ The equation is $y = \tfrac{1}{2}e^{2x} + \tfrac{5}{2}$.

Differential equations of the form $\frac{d^2y}{dx^2} = f(x)$

These differential equations are similar to those discussed above, with antidifferentiation being applied twice.

Let $p = \frac{dy}{dx}$

Then $\frac{d^2y}{dx^2} = \frac{dp}{dx} = f(x)$

The technique involves finding p as the solution of the differential equation $\frac{dp}{dx} = f(x)$.

p is then substituted in the differential equation $\frac{dy}{dx} = p$ which is then solved.

Example 7

Find the general solution of the following:

a $\dfrac{d^2y}{dx^2} = 10x^3 - 3x + 4$

b $\dfrac{d^2y}{dx^2} = \cos(3x)$

c $\dfrac{d^2y}{dx^2} = e^{-x}$

d $\dfrac{d^2y}{dx^2} = \dfrac{1}{\sqrt{x+1}}$

Solution

a Let $p = \dfrac{dy}{dx}$

Then $\dfrac{dp}{dx} = 10x^3 - 3x + 4$

$\therefore \quad p = \dfrac{5x^4}{2} - \dfrac{3x^2}{2} + 4x + c$

$\therefore \quad \dfrac{dy}{dx} = \dfrac{5x^4}{2} - \dfrac{3x^2}{2} + 4x + c$

$y = \dfrac{x^5}{2} - \dfrac{x^3}{2} + 2x^2 + cx + d$ where $c, d \in R$

b Let $p = \dfrac{dy}{dx}$

Then $\dfrac{dp}{dx} = \cos(3x)$

$\therefore \quad p = \int \cos(3x)\, dx$

$= \frac{1}{3}\sin(3x) + c$

$\therefore \quad \dfrac{dy}{dx} = \frac{1}{3}\sin(3x) + c$ and $y = \int \left(\frac{1}{3}\sin(3x) + c\right) dx$

$\therefore \quad y = \frac{-1}{9}\cos(3x) + cx + d$ where c and d are real constants.

c The p substitution can be omitted.

$\dfrac{d^2y}{dx^2} = e^{-x}$

$\therefore \quad \dfrac{dy}{dx} = \int e^{-x}\, dx = -e^{-x} + c$

$\therefore \quad y = \int -e^{-x} + c\, dx = e^{-x} + cx + d$ where $c, d \in R$.

d $\dfrac{d^2y}{dx} = \dfrac{1}{\sqrt{x+1}}$

$\therefore \quad \dfrac{dy}{dx} = \int \dfrac{1}{\sqrt{x+1}}\, dx = \int (x+1)^{-\frac{1}{2}}\, dx = 2(x+1)^{\frac{1}{2}} + c$

$\therefore \quad y = \int 2(x+1)^{\frac{1}{2}} + c\, dx = \frac{4}{3}(x+1)^{\frac{3}{2}} + cx + d$ where $c, d \in R$.

Example 8

Given the differential equation $\frac{d^2y}{dx^2} = \cos^2 x$ find:

a the general solution

b the solution, given that $\frac{dy}{dx} = 0$ when $x = 0$ and $y(0) = -\frac{1}{8}$

Solution

a Now $\frac{d^2y}{dx^2} = \cos^2 x$

$\therefore \quad \frac{dy}{dx} = \int \cos^2 x \, dx$

To proceed in this case the trigonometric identity $\cos 2x = 2\cos^2 x - 1$ is used.

$\therefore \quad \cos^2 x = \frac{1}{2}(\cos 2x + 1)$

So $\quad \frac{dy}{dx} = \frac{1}{2}\int \cos 2x + 1 \, dx$

$\therefore \quad \frac{dy}{dx} = \frac{1}{2}\left(\frac{1}{2}\sin 2x + x\right) + c$

$\therefore \quad \frac{dy}{dx} = \frac{1}{4}\sin 2x + \frac{1}{2}x + c$

$\therefore \quad y = \int \left(\frac{1}{4}\sin 2x + \frac{1}{2}x + c\right) dx$

$\therefore \quad y = -\frac{1}{8}\cos 2x + \frac{1}{4}x^2 + cx + d$ is the general solution.

b Using $\frac{dy}{dx} = 0$ when $x = 0$

and $\quad \frac{dy}{dx} = \frac{1}{4}\sin 2x + \frac{1}{2}x + c$ from **a**, substitute and find:

$0 = \frac{1}{4}\sin 0 + 0 + c$

$\therefore \quad c = 0$

So $\quad \frac{dy}{dx} = \frac{1}{4}\sin 2x + \frac{1}{2}x$

$\therefore \quad y = -\frac{1}{8}\cos 2x + \frac{1}{4}x^2 + d$

Now using $y(0) = -\frac{1}{8}$ substitute and find:

$-\frac{1}{8} = -\frac{1}{8}\cos 0 + 0 + d$

$\therefore \quad d = 0$

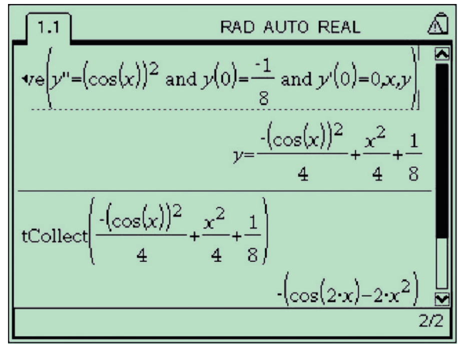
So $\quad y = -\frac{1}{8}\cos 2x + \frac{1}{4}x^2$ is the solution.

Using a TI-Nspire calculator

Choose **Calculus** and then **Differential Equation Solver** to obtain **deSolve** ((menu) 4 C).

Enter **deSolve** ($y'' = (\cos(x))^2$ and $y(0) = \frac{-1}{8}$ and $y'(0) = 0, x, y$).

Choose **Algebra**, then **Trigonometry** and then **Collect** to obtain **tCollect** ((menu) 3 9 2).



Using a Casio ClassPad calculator

Enter $y'' = (\cos(x))^2$ and shade.

Select **Interactive**, then **Advanced**, then **dSolve** and then **Include condition**.

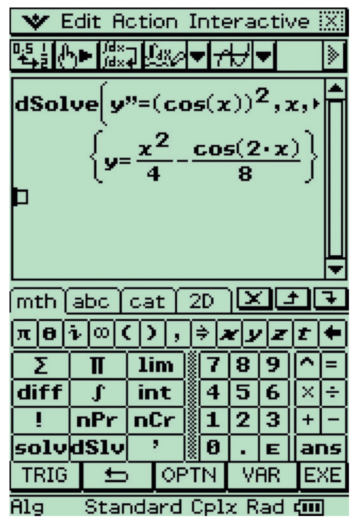
Equation: $y'' = (\cos(x))^2$

Inde var: x

Depe var: y

Condition: $y'(0) = 0$

Condition: $y(0) = -\frac{1}{8}$



Exercise 9B

1 Find the general solution of the following differential equations:

a $\frac{dy}{dx} = x^2 - 3x + 2$

b $\frac{dy}{dx} = \frac{x^2 + 3x - 1}{x}$

c $\frac{dy}{dx} = (2x + 1)^3$

d $\frac{dy}{dx} = \frac{1}{\sqrt{x}}$

e $\frac{dy}{dt} = \frac{1}{2t - 1}$

f $\frac{dy}{dt} = \sin(3t - 2)$

g $\frac{dy}{dt} = \tan(2t)$

h $\frac{dx}{dy} = e^{-3y}$

i $\frac{dx}{dy} = \frac{1}{\sqrt{4 - y^2}}$

j $\frac{dx}{dy} = -\frac{1}{(1 - y)^2}$

2 Find the general solution of the following differential equations:

a $\frac{d^2y}{dx^2} = 5x^3$ **b** $\frac{d^2y}{dx^2} = \sqrt{1-x}$ **c** $\frac{d^2y}{dx^2} = \sin\left(2x + \frac{\pi}{4}\right)$

d $\frac{d^2y}{dx^2} = e^{\frac{x}{2}}$ **e** $\frac{d^2y}{dx^2} = \frac{1}{\cos^2 x}$ **f** $\frac{d^2y}{dx^2} = \frac{1}{(x+1)^2}$

3 Find the solution for the following differential equations:

a $\frac{dy}{dx} = \frac{1}{x^2}$, given that $y = \frac{3}{4}$ when $x = 4$

b $\frac{dy}{dx} = e^{-x}$, given that $y(0) = 0$

c $\frac{dy}{dx} = \frac{x^2-4}{x}$, given that $y = \frac{3}{2}$ when $x = 1$

d $\frac{dy}{dx} = \frac{x}{x^2-4}$, given that $y(2\sqrt{2}) = \log_e 2$

e $\frac{dy}{dx} = x\sqrt{x^2-4}$, given that $y = \frac{1}{4\sqrt{3}}$ when $x = 4$

f $\frac{dy}{dx} = \frac{1}{\sqrt{4-x^2}}$, given that $y(1) = \frac{\pi}{3}$

g $\frac{dy}{dx} = \frac{1}{4-x^2}$, given that $y = 2$ when $x = 0$

h $\frac{dy}{dx} = \frac{1}{4+x^2}$, given that $y(2) = \frac{3\pi}{8}$

i $\frac{dy}{dx} = x\sqrt{4-x}$, given that $y = -\frac{8}{15}$ when $x = 0$

j $\frac{dy}{dx} = \frac{e^x}{e^x+1}$, given that $y(0) = 0$

4 Find the solution for the following differential equations:

a $\frac{d^2y}{dx^2} = e^{-x} - e^x$ given that $y(0) = 0$ and $\frac{dy}{dx} = 0$ when $x = 0$

b $\frac{d^2y}{dx^2} = 2 - 12x$, given that when $x = 0$, $y = 0$ and $\frac{dy}{dx} = 0$

c $\frac{d^2y}{dx^2} = 2 - \sin 2x$, given that when $x = 0$, $y = -1$ and $\frac{dy}{dx} = \frac{1}{2}$

d $\frac{d^2y}{dx^2} = 1 - \frac{1}{x^2}$, given that $y(1) = \frac{3}{2}$ and $\frac{dy}{dx} = 0$ when $x = 1$

e $\frac{d^2y}{dx^2} = \frac{2x}{(1+x^2)^2}$, given that, when $x = 0$, $\frac{dy}{dx} = 0$ and, when $x = 1$, $y = 1$

f $\frac{d^2y}{dx^2} = 24(2x+1)$, given that $y(-1) = -2$ and $\frac{dy}{dx} = 6$ when $x = -1$

g $\frac{d^2y}{dx^2} = \frac{x}{(4-x^2)^{\frac{3}{2}}}$, given that, when $x = 0$, $\frac{dy}{dx} = \frac{1}{2}$ and, when $x = -2$, $y = -\frac{\pi}{2}$

5 Find the family of curves defined by the following differential equations:

a $\dfrac{dy}{dx} = 3x + 4$ **b** $\dfrac{d^2y}{dx^2} = -2x$ **c** $\dfrac{dy}{dx} = \dfrac{1}{x-3}$

6 Find the equation of the curve defined by the following:

a $\dfrac{dy}{dx} = 2 - e^{-x} \quad y(0) = 1$ **b** $\dfrac{dy}{dx} = x + \sin 2x \quad y(0) = 4$

c $\dfrac{dy}{dx} = \dfrac{1}{2-x} \quad y(3) = 2$

9.3 The solution of differential equations of the form $\dfrac{dy}{dx} = f(y)$

The identity $\dfrac{dx}{dy} = \dfrac{1}{\frac{dy}{dx}}$ is used to convert the differential equation $\dfrac{dy}{dx} = f(y)$ to $\dfrac{dx}{dy} = \dfrac{1}{f(y)}$.

If $\dfrac{dy}{dx} = f(y)$ then $x = \int \frac{1}{f(y)}\, dy$

Example 9

Find the general solution of each of the following differential equations:

a $\dfrac{dy}{dx} = 2y + 1, y > -\frac{1}{2}$ **b** $\dfrac{dy}{dx} = e^{2y}$

c $\dfrac{dy}{dx} = \sqrt{1 - y^2}, y \in (-1, 1)$ **d** $\dfrac{dy}{dx} = 1 - y^2, -1 < y < 1$

Solution

a For $\dfrac{dy}{dx} = 2y + 1$

$\dfrac{dx}{dy} = \dfrac{1}{2y+1}$ and $x = \displaystyle\int \frac{1}{2y+1}\, dy$

Therefore $x = \frac{1}{2}\log_e|2y + 1| + k \quad k \in R$

$= \frac{1}{2}\log_e(2y + 1) + k$ as $y > -\frac{1}{2}$

$\therefore \quad 2(x - k) = \log_e(2y + 1)$

and $\quad 2y + 1 = e^{2(x-k)}$

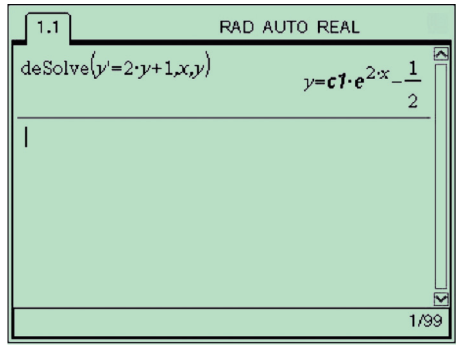
i.e. $\quad y = \frac{1}{2}(e^{2(x-k)} - 1)$

This can also be written as $y = \frac{1}{2}(Ae^{2x} - 1)$ where $A = e^{-2k}$.

Note: For $y < -\frac{1}{2}$, the general solution is given by $y = -\frac{1}{2}(Ae^{2x} + 1)$ where $A = e^{-2k}$.

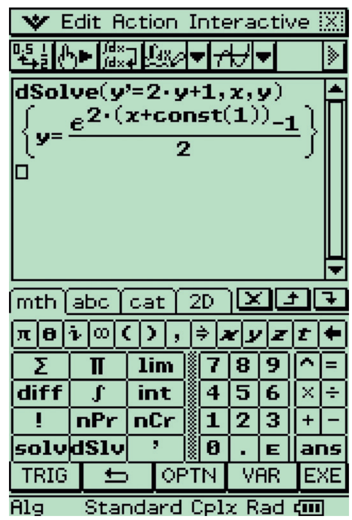
Using a TI-Nspire calculator

Choose **Differential Equation Solver** from the **Calculus** menu (menu 4 C). Complete as shown.



Using a Casio ClassPad calculator

Choose Transformation, then **Advanced** and then **dSolve**.
The differentiation symbol (′) is found on the **CALC** keyboard under **mth** as shown in the diagram.



b For $\frac{dy}{dx} = e^{2y}$

$$\frac{dx}{dy} = e^{-2y} \text{ and } x = \int e^{-2y}\,dy$$

Therefore $\quad x = -\frac{1}{2}e^{-2y} + c$

$\therefore \quad -2(x - c) = e^{-2y}$

and $\quad -2y = \log_e(-2(x - c))$

$\therefore \quad y = -\frac{1}{2}\log_e(-2(x - c))$

$\quad = -\frac{1}{2}\log_e(2c - 2x),\ x < c$

c For $\frac{dy}{dx} = \sqrt{1 - y^2}$

$$\frac{dx}{dy} = \frac{1}{\sqrt{1 - y^2}} \text{ and } x = \int \frac{1}{\sqrt{1 - y^2}}\,dy$$

Therefore $x = \sin^{-1} y + c$

and $\quad y = \sin(x - c)$

d For $\dfrac{dy}{dx} = 1 - y^2$

$\dfrac{dx}{dy} = \dfrac{1}{1 - y^2}$ and $x = \displaystyle\int \dfrac{1}{1 - y^2}\, dy$

$$\therefore \quad x = \int \frac{1}{2(1 - y)} + \frac{1}{2(1 + y)}\, dy$$

$$= -\tfrac{1}{2}\log_e(1 - y) + \tfrac{1}{2}\log_e(1 + y) + c \quad (-1 < y < 1)$$

$$\therefore \quad x - c = \tfrac{1}{2}\log_e\left(\frac{1 + y}{1 - y}\right)$$

$$\therefore \quad e^{2(x - c)} = \frac{1 + y}{1 - y}$$

Let $\quad e^{-2c} = A$

Then $(1 - y)e^{2x}A = 1 + y$

Rearranging $Ae^{2x} - 1 = y(1 + Ae^{2x})$

$$y = \frac{Ae^{2x} - 1}{Ae^{2x} + 1}$$

Exercise 9C

1 Find the general solution of the following differential equations:

a $\dfrac{dy}{dx} = 3y - 5$ **b** $\dfrac{dy}{dx} = 1 - 2y$ **c** $\dfrac{dy}{dx} = e^{2y-1}$

d $\dfrac{dy}{dx} = \cos^2 y$ **e** $\dfrac{dy}{dx} = \cot y$ **f** $\dfrac{dy}{dx} = y^2 - 1$

g $\dfrac{dy}{dx} = 1 + y^2$ **h** $\dfrac{dy}{dx} = \dfrac{1}{5y^2 + 2y}$ **i** $\dfrac{dy}{dx} = \sqrt{y}$

j $\dfrac{dy}{dx} = y^2 + 4y$

2 Find the solution for the following differential equations:

a $\dfrac{dy}{dx} = y$, given that $y = e$ when $x = 0$

b $\dfrac{dy}{dx} = y + 1$, given that $y(4) = 0$

c $\dfrac{dy}{dx} = 2y$, given that $y = 1$ when $x = 1$

d $\dfrac{dy}{dx} = 2y + 1$, given that $y(0) = -1$

e $\dfrac{dy}{dx} = \dfrac{e^y}{e^y + 1}$, given that $y = 0$ when $x = 0$

f $\dfrac{dy}{dx} = \sqrt{9 - y^2}$, given that $y(0) = 3$

g $\dfrac{dy}{dx} = 9 - y^2$, given that $y = 0$ when $x = \dfrac{7}{6}$

h $\dfrac{dy}{dx} = 1 + 9y^2$, given that $y\left(\dfrac{-\pi}{12}\right) = -\dfrac{1}{3}$

i $\dfrac{dy}{dx} = \dfrac{y^2 + 2y}{2}$, given that $y = -4$ when $x = 0$

3 Find the equation for the family of curves for the following:

a $\dfrac{dy}{dx} = \dfrac{1}{y^2}$

b $\dfrac{dy}{dx} = 2y - 1$

9.4 Applications of differential equations

Many differential equations arise from scientific or business situations and are usually constructed from observations and data obtained from experiment.

Two results from science which are described by a differential equation are:

- **Newton's law of cooling** – the rate at which a body cools is proportional to the difference between its temperature and that of its immediate surroundings.
- **Radioactive decay** – the rate at which a radioactive substance decays is proportional to the mass of the substance remaining.

These two results will be further discussed in worked examples in this chapter.

Example 10

t	0	1	2	3	4
$\dfrac{dx}{dt}$	0	2	8	18	32

The above table represents the observed rate of change of a variable x with respect to time t.

a Construct the differential equation which applies to this situation.

b Solve the differential equation to find x, given that $x = 2$ when $t = 0$.

Solution

a From the table it can be established that $\dfrac{dx}{dt} = 2t^2$.

b Hence $x = \displaystyle\int 2t^2\, dt = \dfrac{2t^3}{3} + c$

When $t = 0, x = 2$ $\quad\therefore\quad$ $2 = 0 + c$ and $c = 2$

Therefore $x = \dfrac{2t^3}{3} + 2$

Differential equations can be constructed from statements as exemplified in the following.

Example 11

The population of a city is P at time t years from a certain date. The population increases at a rate that is proportional to the square root of the population at that time. Construct and solve the appropriate differential equation and sketch the population–time graph.

Solution

Remembering that the derivative is a rate, we have $\frac{dP}{dt} \propto \sqrt{P}$.

So $\frac{dP}{dt} = k\sqrt{P}$ where k is the constant of variation.

Since the population increases, $k > 0$.

So $\frac{dP}{dt} = k\sqrt{P}$, $k > 0$ is the appropriate differential equation.

Since there are no initial conditions or values given here, only a general solution for this differential equation can be found.

Now:

$$\frac{dP}{dt} = k\sqrt{P} \quad \left(\text{This is of the form } \frac{dy}{dx} = f(y)\right)$$

$$\therefore \quad \frac{dt}{dP} = \frac{1}{k\sqrt{P}}$$

$$\therefore \quad t = \frac{1}{k}\int P^{-\frac{1}{2}}\,dP$$

$$\therefore \quad t = \frac{1}{k}\,2P^{\frac{1}{2}} + c \quad c \in R$$

$$\therefore \quad t = \frac{2}{k}\sqrt{P} + c \text{ is the general solution of the differential equation.}$$

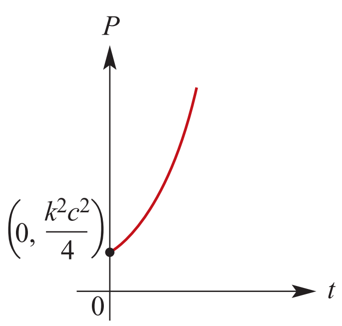
It is easier to sketch the graph of P against t if P is the subject of the formula.

Now: $$t = \frac{2}{k}\sqrt{P} + c$$

$$\therefore \quad t - c = \frac{2}{k}\sqrt{P}$$

$$\therefore \quad \frac{k}{2}(t - c) = \sqrt{P}$$

$$\therefore \quad P = \frac{k^2}{4}(t - c)^2$$



The graph is a section of the parabola $P = \frac{k^2}{4}(t - c)^2$ with vertex at $(c, 0)$.

Example 12

Another city, with population P at time t years after a certain date, has a population that increases at a rate proportional to the population at that time. Construct and solve the appropriate differential equation and sketch the population–time graph.

Solution

Here $\frac{dP}{dt} \propto P$

$\therefore \quad \frac{dP}{dt} = kP,\ k > 0$ is the appropriate differential equation.

Now $\quad \frac{dP}{dt} = kP$

$\therefore \quad \frac{dt}{dP} = \frac{1}{kP}$

$\therefore \quad t = \frac{1}{k}\int \frac{1}{P}\,dP$

$\therefore \quad t = \frac{1}{k}\log_e P + c$

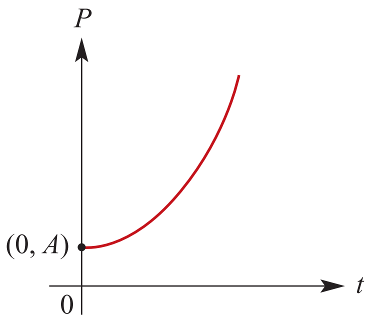
This is the solution.

Rearranging to make P the subject:

$k(t - c) = \log_e P$

$\therefore \quad e^{k(t-c)} = P$

$\therefore \quad P = Ae^{kt}$, where $A = e^{-kc}$



The graph is a section of the exponential curve $P = Ae^{kt}$

Example 13

Suppose that a tank containing liquid has a vent at the top and an outlet at the bottom through which the liquid drains. Torricelli's law states that if, at time t seconds after the opening of the outlet, the depth of the liquid is k m and the surface area of the liquid is A m^2 then:

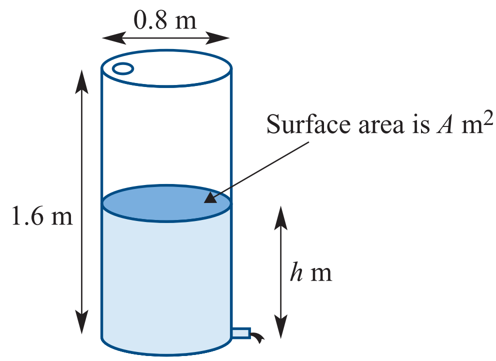
$$\frac{dh}{dt} = \frac{-k\sqrt{h}}{A} \text{ where } k > 0$$

(k actually depends on factors such as the viscosity of the liquid and the cross-sectional area of the outlet.)

Use Torricelli's law for a tank that is cylindrical, initially full, and that has height 1.6 m and radius length 0.4 m. Use $k = 0.025$. Construct the appropriate differential equation, solve it and find how many seconds it will take the tank to empty.

Solution

A diagram should be drawn.



Now $\dfrac{dh}{dt} = \dfrac{-0.025\sqrt{h}}{\pi \times 0.4^2}$

since the surface area is a circle with constant area $\pi \times 0.4^2$

So $\dfrac{dh}{dt} = \dfrac{-0.025\sqrt{h}}{0.16\pi}$

$\therefore \quad \dfrac{dh}{dt} = \dfrac{-5\sqrt{h}}{32\pi}$ is the appropriate differential equation

Now $\dfrac{dt}{dh} = \dfrac{-32\pi}{5}.\, h^{-\frac{1}{2}}$

$\therefore \quad t = \dfrac{-32\pi}{5} \displaystyle\int h^{-\frac{1}{2}}\, dh$

$\therefore \quad t = \dfrac{-32\pi}{5}.\, 2h^{\frac{1}{2}} + c$

$\therefore \quad t = \dfrac{-64\pi}{5}\sqrt{h} + c$

Use the given condition that the tank was initially full, i.e. when $t = 0$, $h = 1.6$.

By substitution: $0 = \dfrac{-64\pi}{5}\sqrt{1.6} + c$

$\therefore \quad c = \dfrac{64\pi}{5}\sqrt{1.6}$

So the particular solution for this differential equation is:

$$t = \frac{-64\pi}{5}\sqrt{h} + \frac{64\pi}{5}\sqrt{1.6}$$

$$\therefore \quad t = \frac{-64\pi}{5}(\sqrt{h} - \sqrt{1.6})$$

Now, for the tank to be empty, $h = 0$.

By substitution: $t = \dfrac{64\pi}{5}(\sqrt{1.6})$

$\therefore \quad t \approx 50.9$

It will take approximately 51 seconds to empty this tank.

The following example uses Newton's law of cooling.

Example 14

An iron bar is placed in a room which has a temperature of 20°C. The iron bar has a temperature of 80°C initially. It cools to 70°C in five minutes. Let T be the temperature of the bar at time t minutes.

a Construct a differential equation.

b Solve this differential equation.

c Sketch the graph of T against t.

d How long does it take the bar to cool to 40°C?

Solution

a Newton's law of cooling yields $\frac{dt}{dt} = -k(T - 20)$ where $k \in R$

Note the use of the negative sign as the temperature is decreasing.

b $\frac{dt}{dT} = -\frac{1}{k(T - 20)}$

$\therefore t = -\frac{1}{k}\log_e(T - 20) + c \quad T > 20$

When $t = 0$, $T = 80$

$$\therefore \quad 0 = -\frac{1}{k}\log_e(80 - 20) + c$$

$$\therefore \quad c = \frac{1}{k}\log_e 60$$

$$\text{and} \quad t = \frac{1}{k}\log_e\left(\frac{60}{T - 20}\right)$$

When $t = 5$, $T = 70$

$$\therefore \quad \frac{1}{k} = \frac{5}{\log_e\left(\frac{6}{5}\right)} \text{ and } t = \frac{5}{\log_e\left(\frac{6}{5}\right)}\log_e\left(\frac{60}{T - 20}\right)$$

For T to be the subject rearrange as shown:

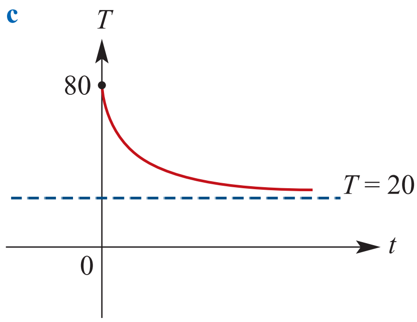
$$\left(\frac{t}{5}\right)\log_e\left(\frac{6}{5}\right) = \log_e\left(\frac{60}{T - 20}\right)$$

$$\therefore \quad \log_e\left(\frac{6}{5}\right)^{\frac{t}{5}} = \log_e\left(\frac{60}{T - 20}\right)$$

$$\therefore \quad \left(\frac{6}{5}\right)^{\frac{t}{5}} = \frac{60}{T - 20}$$

and $T = 20 + 60\left(\frac{5}{6}\right)^{\frac{t}{5}}$

c



d Use the form $t = \frac{5}{\log_e\left(\frac{6}{5}\right)} \log_e\left(\frac{60}{T - 20}\right)$

When $T = 40$, $t = \frac{5}{\log_e\left(\frac{6}{5}\right)} \log_e(3)$

$= 30.1284\ldots$

The bar reaches a temperature of 40° after 30.1 minutes.

Difference of rates

Consider each of the following situations:

- an object being heated while at the same time being subjected to some cooling process
- a population increasing due to births but at the same time diminishing due to deaths
- a liquid being poured into a container while at the same time the liquid is flowing out of the container.

In these situations

the rate of change = rate of increase – rate of decrease

For example if water is flowing into a container at 8 litres per minute while at the same time water is flowing out of the container at 6 litres per minute, the rate of change = (8 − 6) litres per minute. If V litres is the volume of water in the container at time t, $\frac{dV}{dt} = 2$.

Example 15

A certain radioactive isotope decomposes at a rate that is proportional to the mass m kg present at any time t years. The rate of decomposition is $2m$ kg per year. The isotope is formed as a byproduct from a nuclear reactor at a constant rate of 0.5 kg per year. None of the isotope was present initially.

a Construct a differential equation.

b Solve the differential equation.

c Sketch the graph of m against t.

d How much isotope is there after two years?

Solution

a $\frac{dm}{dt} = 0.5 - 2m = \frac{1 - 4m}{2}$

b $\dfrac{dt}{dm} = \dfrac{2}{1-4m}$

This implies $t = -\frac{2}{4}\log_e|1-4m| + c$

$= -\frac{1}{2}\log_e(1-4m) + c,\ \text{since } 0.5 - 2m > 0$

When $t = 0$, $m = 0$ and therefore $c = 0$

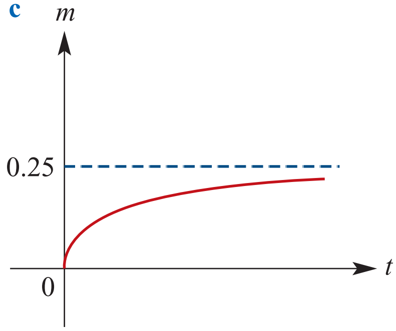
$\therefore\quad -2t = \log_e(1-4m)$

and $\quad e^{-2t} = 1 - 4m$

$\therefore\quad 4m = 1 - e^{-2t}$

i.e. $\quad m = \frac{1}{4}(1 - e^{-2t})$

c



d When $t = 2$

$m = \frac{1}{4}(1 - e^{-4})$

$= 0.245\ldots$

After two years there are 0.245 kg of the isotope.

Example 16

Pure oxygen is pumped into a 50-litre tank of air at 5 litres per minute. The oxygen is well mixed with the air in the tank. The mixture is removed at the same rate.

a Construct a differential equation given that plain air contains 23% oxygen.

b After how many minutes does the mixture have 50% oxygen?

Solution

a Let Q litres be the volume of oxygen at time t minutes.
When $t = 0$, $Q = 50 \times 0.23 = 11.5$

$$\frac{dQ}{dt} = \text{rate of inflow} - \text{rate of outflow}$$

$$= 5 - \frac{Q}{50} \times 5$$

i.e. $$\frac{dQ}{dt} = \frac{50 - Q}{10}$$

b $\dfrac{dt}{dQ} = \dfrac{10}{50 - Q}$

$\therefore \quad t = -10\log_e|50 - Q| + c$

$\therefore \quad\;\; = -10\log_e(50 - Q) + c, \text{ as } Q < 50$

when $t = 0$, $Q = 11.5$

$\therefore \quad c = 10\log_e(38.5)$

$t = 10\log_e\left(\dfrac{77}{2(50 - Q)}\right)$

If there is 50% oxygen $Q = 25$

$\therefore \quad t = 10\log_e\left(\dfrac{77}{2 \times 25}\right)$

$= 10\log_e\left(\frac{77}{50}\right)$

$= 4.317\ldots$

The tank contains 50% oxygen after 4 minutes and 19.07 seconds.

Exercise 9D

1 Each of the following tables gives results of an experiment where rates of change were found to be linear functions of t, i.e. $\dfrac{dx}{dt} = at + b$.
For each of the tables, set up a differential equation and solve it, given the additional information.

a

t	0	1	2	3
$\dfrac{dx}{dt}$	1	3	5	7

and $x(0) = 3$

b

t	0	1	2	3
$\dfrac{dx}{dt}$	−1	2	5	8

and $x(1) = 1$

c

t	0	1	2	3
$dx\ dt$	8	6	4	2

and $x(2) = -3$

2 For each of the following, construct a differential equation but do not attempt to solve it.

a A family of curves has gradient at any point (x, y), $y \neq 0$, which is the reciprocal of the y coordinate.

b A family of curves has gradient at any point (x, y), $y \neq 0$, which is the square of the reciprocal of the y coordinate.

c The rate of increase of a population of size N at time t years is inversely proportional to the square of the population.

d A particle moving in a straight line is x m from a fixed point O after t seconds. The rate at which the particle is moving is inversely proportional to the distance from O.

e A radioactive substance decays according to the rule that the rate of decay of the substance is proportional to the mass of substance remaining. Let m kg be the mass of the substance at time t minutes.

f The gradient of the normal to a curve at any point (x, y) is three times the gradient of the line joining the same point to the origin.

3 A city with population P, at time t years after a certain date, has a population which increases at a rate proportional to the population at that time.

a **i** Set up a differential equation to describe this situation.
ii Solve to obtain a general solution.

b If the initial population was 1000 and after two years the population had risen to 1100:
i find the population after five years
ii sketch a graph of P against t

4 An island has a population of rabbits of size P, t years after 1 Jan 2000. Due to a virus the population is decreasing at a rate proportional to the square root of the population at that time.

a **i** Set up a differential equation to describe this situation.
ii Solve to obtain a general solution.

b If the initial population was 15 000 and the population decreased to 13 500 after five years:
i find the population after 10 years
ii sketch a graph of P against t

5 A city has population P at time t years from a certain date. The population increases at a rate inversely proportional to the population at that time.

a **i** Set up a differential equation to describe this situation.
ii Solve to obtain a general solution.

b Initially the population was 1 000 000 but after four years it had risen to 1 100 000.
i Find an expression for the population in terms of t.
ii Sketch the graph of P against t.

6 A curve has the property that its gradient at any point is one-tenth of the y coordinate at that point. It passes through the point (0, 10). Find the equation of the curve.

7 If the thermostat in an electric heater failed, the rate of increase of temperature, $\frac{d\theta}{dt}$, would be 0.01θ degrees per minute where θ is in degrees Kelvin (K°) and t is in minutes. If a heater was switched on at a room temperature of 300 K° and the thermostat did not function, what would the temperature of the heater be after 10 minutes?

8 The rate of decay of a radioactive substance is proportional to the amount Q of matter present at any time, t. The differential equation for this situation is $\frac{dQ}{dt} = -kQ$ where k is a constant. If $Q = 50$ when $t = 0$ and $Q = 25$ when $t = 10$, find the time taken for Q to reach 10.

9 The rate of decay of a substance is $km(k \in R^+)$ where m *is* the mass of the substance remaining. Show that the half life, the time in which the amount of the original substance remaining is halved, of the substance is $\frac{1}{k}\log_e 2$.

10 The concentration, x grams per litre, of salt in a solution at any time, t minutes, is given by $\frac{dx}{dt} = \frac{20 - 3x}{30}$.

a If the initial concentration was 2 grams per litre, solve the differential equation, giving x in terms of t.

b Find the time taken, to the nearest minute, for the salt concentration to rise to 6 grams per litre.

11 If $\frac{dy}{dx} = 10 - \frac{y}{10}$ and $y = 10$ when $x = 0$, find y in terms of x. Sketch the graph of the equation for $x \geq 0$.

12 The number, n, of bacteria in a colony grows according to the law $\frac{dn}{dt} = kn$, where k is a positive constant. If the number increases from 4000 to 8000 in four days, find, to the nearest hundred, the number of bacteria after three days more.

13 A town had a population of 10 000 in 1990 and 12 000 in 2000. If the population is N at a time t years after 1990, find the predicted population in the year 2010 assuming:

a $\frac{dN}{dt} \propto N$ **b** $\frac{dN}{dt} \propto \frac{1}{N}$ **c** $\frac{dN}{dt} \propto \sqrt{N}$

14 For each of the following construct a differential equation but do not attempt to solve it.

a Water is flowing into a tank at a rate of 0.3 m^3 per hour. At the same time water is flowing out through a hole in the bottom of the tank at a rate of $0.2\sqrt{V}$ m^3 per hour where V m^3 is the volume of the water in the tank at time t hours. (Find an expression for $\frac{dV}{dt}$.)

b A tank initially contains 200 L of pure water. A salt solution containing 5 kg of salt per litre is added at the rate of 10 litres per minute, and the mixed solution is drained simultaneously at the rate of 12 litres per minute. There is m kg of salt in the tank after t minutes. (Find an expression for $\frac{dm}{dt}$.)

c A partly filled tank contains 200 litres of water in which 1500 grams of salt have been dissolved. Water is poured into the tank at a rate of 6 litres/minute. The mixture, which is kept uniform by stirring, leaves the tank through a hole at a rate of 5 litres/minute. There are x grams of salt in the tank after t minutes. (Find an expression for $\frac{dx}{dt}$.)

15 A tank holds 100 litres of water in which 20 kg of sugar was dissolved. Water runs into the tank at the rate of one litre per minute. The solution is continually stirred and at the same time the solution is being pumped out at one litre per minute. At time t minutes there are m kg of sugar in the solution.

a At what rate is the sugar being removed at time t minutes?
b Set up a differential equation to represent this situation.
c Solve the differential equation.
d Sketch the graph of m against t.

16 A tank holds 100 litres of pure water. A sugar solution containing 0.25 kg per litre is being run into the tank at the rate of one litre/minute. The liquid in the tank is continuously stirred and, at the same time, liquid from the tank is being pumped out at the rate of one litre per minute. After t minutes there are m kg of sugar dissolved in the solution.

a At what rate is the sugar being added to the solution at time t?
b At what rate is the sugar being removed from the tank at time t?
c Construct a differential equation to represent this situation.
d Solve this differential equation.
e Find the time taken for the concentration in the tank to reach 0.1 kg per litre.
f Sketch the graph of m against t.

17 A laboratory tank contains 100 litres (L) of a 20% serum solution (i.e. 20% of the contents is pure serum and 80% is distilled water). A 10% serum solution is then pumped in at the rate of 2 L/min, and an amount of the solution currently in the tank is drawn off at the same rate.

a Set up a differential equation to show the relation between x and t, where x L is the amount of pure serum in the tank at time t min.
b How long will it take for there to be an 18% solution in the tank? (Assume that, at all times, the contents of the tank form a uniform solution.)

18 A tank initially contains 400 litres of water in which are dissolved 10 kg of salt. A salt solution of concentration 0.2 kg/L is poured into the tank at the rate of 2 litres per minute; the mixture, which is kept uniform by stirring, flows out at the rate of 2 litres per minute.

a If the mass of salt in the tank after t minutes is x kg, set up and solve the differential equation for x in terms of t.
b If, instead, the mixture flows out at 1 litre per minute, set up, but do not solve, the differential equation for the mass of salt in the tank.

19 A vat contains 100 litres of water. A sugar solution with a concentration of 0.5 kg of sugar per litre is pumped into the vat at 10 litres per minute. The solution is thoroughly mixed in the vat and solution is drawn off at 10 litres per minute. If there is x kg of sugar in solution at any time t minutes, set up, and solve, the differential equation for x.

20 A tank contains 20 litres of water in which 10 kg of salt is dissolved. Pure water is poured in at a rate of 2 litres per minute, mixing occurs uniformly (owing to stirring) and the water is released at 2 litres per minute. The mass of salt in the tank is x kg at time t minutes.

- **a** Construct a differential equation representing this information, expressing $\frac{dx}{dt}$ as a function of x.
- **b** Solve the differential equation.
- **c** Sketch the mass–time graph.
- **d** How long will it take the original mass of salt to be halved?

21 A country's population N at time t years after 1 Jan 2000 changes according to the differential equation $\frac{dN}{dt} = 0.1\text{ N} - 5000$.
(Five thousand people leave the country every year and there is a 10% growth rate.)

- **a** Given that the population was 5 000 000 at the start of 2000 find N in terms of t.
- **b** In which year will the country have a population of 10 million?

9.5 Differential equations with related rates

In Chapter 6 the concept of related rates was introduced. This is a useful technique for constructing and solving differential equations in a variety of situations.

Example 17

For the variables y, x and t, it is known that $\frac{dx}{dt} = \tan t$ and $y = 3x$.

- **a** Find $\frac{dy}{dt}$ as a function of t.
- **b** Find the solution of the resulting differential equation.

Solution

a By the chain rule $\frac{dy}{dt} = \frac{dy}{dx}\frac{dx}{dt}$

It is given that $y = 3x$ and $\frac{dx}{dt} = \tan t$

since $\frac{dy}{dt} = \frac{dy}{dx}\frac{dx}{dt}$

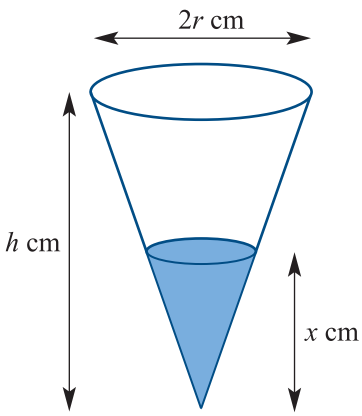
$$\frac{dy}{dt} = 3\tan t$$

b $\dfrac{dy}{dt} = \dfrac{3\sin t}{\cos t}$

Let $u = \cos t$ then $\dfrac{du}{dt} = -\sin t$

$$\therefore \quad y = -3\int \frac{1}{u}\,du$$
$$= -3\log_e|u| + c$$
$$\therefore \quad y = -3\log_e|\cos t| + c$$

Example 18



An inverted cone has height h cm and radius length r cm. It is being filled with water which is flowing from a tap at k litres/minute. The depth of water in the cone is x cm at time t minutes. Construct an appropriate differential equation for $\dfrac{dx}{dt}$ and solve it, given that initially the cone was empty.

Solution

Let V cm^3 be the volume at time t minutes.
The rate given in the information provided is:

$$\frac{dV}{dt} = 1000\,k, k > 0$$

(k litres/minute = $1000\,k$ cm^3/min)

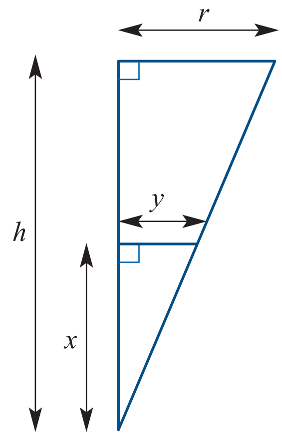
A differential equation for $\dfrac{dx}{dt}$ will be obtained from the following application of the chain rule

$$\frac{dx}{dt} = \frac{dx}{dV}\frac{dV}{dt} \quad \boxed{1}$$

To find $\dfrac{dx}{dV}$, a relationship between x and V needs to be established. The formula for the volume of a cone gives:

$$V = \tfrac{1}{3}\pi y^2 x \quad \boxed{2}$$

where y cm is the radius length of the surface when the depth is x cm.



By similar triangles:

$$\frac{y}{r} = \frac{x}{h}$$

$$\therefore \quad y = \frac{rx}{h}$$

So $\quad V = \frac{1}{3}\pi\frac{r^2x^2}{h^2}\cdot x \qquad$ (substitution into $\boxed{2}$)

$$\therefore \quad V = \frac{\pi r^2}{3h^2}\cdot x^3$$

$\therefore \quad \frac{dV}{dx} = \frac{\pi r^2}{h^2}\cdot x^2 \qquad$ (by differentiation)

$$\therefore \quad \frac{dx}{dV} = \frac{h^2}{\pi r^2}\cdot\frac{1}{x^2}$$

So $\quad \frac{dx}{dt} = \frac{h^2}{\pi r^2}\cdot\frac{1}{x^2}\cdot 1000\,k \qquad$ (substitution into $\boxed{1}$)

$$\therefore \quad \frac{dx}{dt} = \frac{1000\,kh^2}{\pi r^2}\cdot\frac{1}{x^2},\ k > 0$$

To solve this differential equation

$$\frac{dt}{dx} = \frac{\pi r^2}{1000\,kh^2}\cdot x^2$$

$$\therefore \quad t = \frac{\pi r^2}{1000\,kh^2}\int x^2\,dx$$

$$= \frac{\pi r^2}{1000\,kh^2}\frac{x^3}{3} + c$$

$$\therefore \quad t = \frac{\pi r^2}{3000\,kh^2}x^3 + c$$

Now at $t = 0$, $x = 0$ (initially, the cone was empty)

So $\quad c = 0$

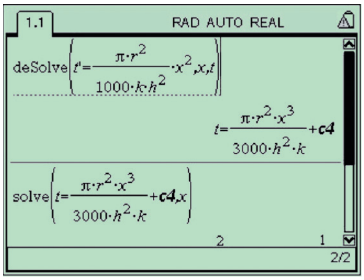
$$\therefore \quad t = \frac{\pi r^2 x^3}{3000\,kh^2}$$

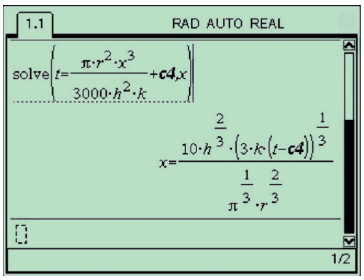
$$\therefore \quad \frac{3000\,kh^2 t}{\pi r^2} = x^3$$

$\therefore \quad x = \sqrt[3]{\frac{3000\,kh^2 t}{\pi r^2}}$ is the solution of the above differential equation.

Using a TI-Nspire calculator

Choose **Differential Equation Solver** from the **Calculus** menu (menu 4 C). Complete as shown. Use **solve** to make x the subject. Note that c4 = 0.



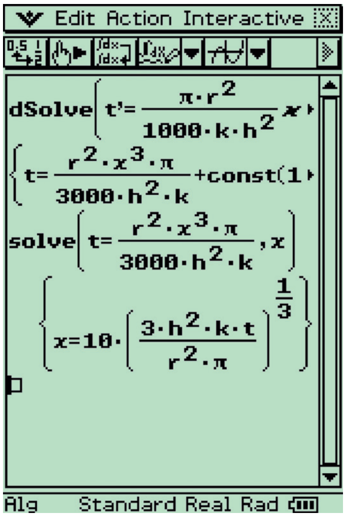


Using a Casio ClassPad calculator

Choose Transformation, then **Advanced** and then **dSolve**.

The differentiation symbol (′) is found on the **CALC** keyboard under **mth**.

Use **solve** to make x the subject.



Exercise 9E

1 Construct but do not solve a differential equation for each of the following. The derivative of the differential equation is stated.

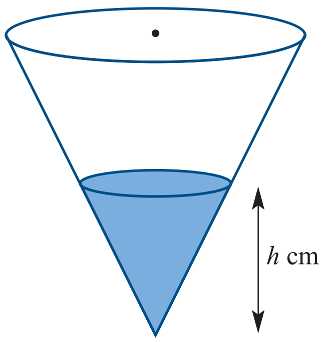
a An inverted cone with depth 50 cm and radius 25 cm is initially full. Water drains out at 0.5 litres per minute. The depth of water in the cone is h cm at time t minutes. (Find an expression for $\frac{dh}{dt}$.)

b A tank with a flat bottom and vertical sides has a constant horizontal cross-section of A square metres. The tank has a tap in the bottom through which water is leaving at a rate $c\sqrt{h}$ cubic metres per minute, where h metres is the height of the water in the tank, and c is a constant. Water is being poured in at a rate of Q cubic metres per minute. (Find an expression for $\frac{dh}{dt}$.)

c Water is flowing at a constant rate of 0.3 m^3 per hour into a tank. At the same time, water is flowing out through a hole in the bottom of the tank at the rate of $0.2\sqrt{V}$ m^3 per hour where V m^3 is the volume of the water in the tank at time t hours. It is known that $V = 6\pi h$ where hm is the height of the water at time t. (Find an expression for $\frac{dh}{dt}$.)

d A cylindrical tank 4 m high with base radius 1.5 m is initially full of water. Water starts flowing out through a hole at the bottom of the tank at the rate of $\sqrt{h}$ m^3/hour, where h m is the depth of water remaining in the tank after t hours. (Find an expression for $\frac{dh}{dt}$.)

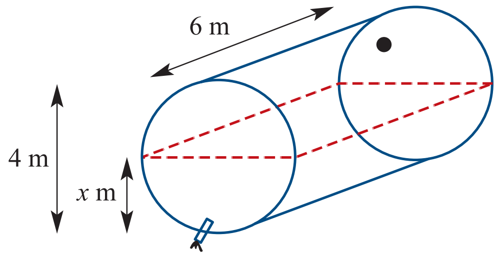
2 A conical tank has a radius length at the top equal to its height. Water, initially with a depth of 25 cm, leaks out through a hole in the bottom of the tank at the rate of $5\sqrt{h}$ cm^3/min where the depth is h cm at time t minutes.



a Construct a differential equation expressing $\frac{dh}{dt}$ as a function of h, and solve it.

b Hence find how long it will take for the tank to empty.

3 A cylindrical tank is lying on its side, as shown in the figure. The tank has a hole in the top, and another in the bottom so that the water in the tank leaks out. If the depth of water is x m at time t minutes and $\frac{dx}{dt} = \frac{-0.025\sqrt{x}}{A}$ where A m^2 is the surface area of the water at t minutes:



a construct the differential equation expressing $\frac{dx}{dt}$ as a function of x only

b solve the differential equation given that initially the tank was full

c find how long it will take to empty the tank

4 A spherical drop of water evaporates so that the volume remaining is V mm^3 and the surface area is A mm^2 when the radius is r mm at time t seconds. Given that $\frac{dV}{dt} = -2A^2$:

a construct the differential equation expressing $\frac{dr}{dt}$ as a function of r

b solve the differential equation given that the initial radius was 2 mm

c sketch the surface area–time graph and the radius–time graph

5 A water tank of uniform cross-sectional area A cm^2 is being filled by a pipe which supplies Q litres of water every minute. The tank has a small hole in its base through which water leaks at a rate of kh litres every minute where h cm is the depth of water in the tank at time t minutes. Initially the depth of the water is h_0.

a Construct the differential equation expressing $\frac{dh}{dt}$ as a function of h.

b Solve the differential equation if $Q > kh_0$.

c Find the time taken for the depth to reach $\frac{Q + kh_0}{2k}$.

9.6 Methods of solving a differential equation using a definite integral

There are many situations when solving differential equations where an exact answer is not required. Indeed, in some situations it may not even be possible to antidifferentiate the function involved. A definite integral can be numerically evaluated to solve such differential equations.

Consider the general case of finding y when $x = b$ given $\frac{dy}{dx} = f(x)$ and $y = d$ when $x = a$.

From $\frac{dy}{dx} = f(x)$

$y = F(x) + c$ (by antidifferentiating, where $F(x)$ is an antiderivative of $f(x)$)

$d = F(a) + c$ as $y = d$ when $x = a$)

so $c = d - F(a)$

$\therefore$ $y = F(x) - F(a) + d$

when $x = b$

$$y = F(b) - F(a) + d$$

$\therefore$ $$y = \int_a^b f(x)\,dx + d$$

This idea is very useful for solving a differential equation that cannot be antidifferentiated.

Example 19

For the differential equation $\frac{dy}{dx} = x^2 + 2$ and given $y = 7$ when $x = 1$, find y when $x = 3$.

Solution

Algebraic

$$\frac{dy}{dx} = x^2 + 2$$

$\therefore$ $y = \frac{x^3}{3} + 2x + c$

$\therefore$ $y = 7$ when $x = 1$

so $7 = \frac{1}{3} + 2 + c$

$\therefore$ $c = \frac{14}{3}$

$\therefore$ $y = \frac{x^3}{3} + 2x + \frac{14}{3}$

when $x = 3$

$y = \frac{1}{3} \times 3^3 + 2 \times 3 + \frac{14}{3}$

$\therefore$ $y = \frac{59}{3}$

Using a definite integral

When $x = 3$

$$y = \int_1^3 (x^2 + 2)\,dx + 7$$

This can be formalised through the fundamental theorem of calculus, introduced in section 8.1. Here it was stated that:

$$\int_a^b f(x)\,dx = G(b) - G(a), \text{ where } G \text{ is an antiderivative of } f$$

The following can be stated:

$$\int_a^b f'(x)\,dx = f(b) - f(a) \qquad (1)$$

$$\int_a^x f'(x)\,dx = f(x) - f(a) \qquad (2)$$

$$\int_a^x f'(t)\,dt = f(x), \quad \text{if } f(a) = 0 \qquad (3)$$

For example, consider $\int_1^x \frac{1}{t}\,dt$. Since this integrand has been met before we can recognise that, in fact:

$$\int_1^x \frac{1}{t}\,dt = \log_e|x|$$

In differential equation form, it can be written as:

> Given $\frac{dy}{dx} = \frac{1}{x}$ and at $x = 1, y = 0$, the solution at $x = a$ is $\log_e|a|$

Particular solutions can be found by defining $f(x) = \int_1^x \frac{1}{t}dt$. Then $f(2) = 0.69315$.

Example 20

Solve the differential equation $\frac{dy}{dx} = \cos x$ at $x = \frac{\pi}{4}$, given that at $x = 0$, $y = 0$, using a definite integral.

Solution

Now $\frac{dy}{dx} = \cos x$

Therefore $y = \int_0^{\frac{\pi}{4}} \cos t\,dt$

$$= [\sin t]_0^{\frac{\pi}{4}}$$

$$= \sin\frac{\pi}{4}$$

$$= \frac{\sqrt{2}}{2}$$

Example 21

Solve the differential equation $\frac{dy}{dx} = x^2 - 2x$ at $x = 4$ given that $y = 0$ at $x = 3$, using a definite integral.

Solution

$$\frac{dy}{dx} = x^2 - 2x$$

By calculus methods, $y = \int_3^4 (x^2 - 2x)\,dx$

$$= \left[\frac{x^3}{3} - x^2\right]_3^4$$

$$= \tfrac{16}{3} \text{ or } 5\tfrac{1}{3}$$

Example 22

Solve the differential equation $\frac{dy}{dx} = \frac{e^{\frac{-x^2}{2}}}{\sqrt{2\pi}}$ at $x = 1$, given that at $x = 0$, $y = 0.5$, giving your answer correct to four decimal places.

Solution

Calculus methods are not available for this differential equation and since an approximate answer is acceptable, the use of a CAS calculator is appropriate.

In this problem, since $f(0) \neq 0$, the fundamental theorem of calculus indicates that

$$\int_0^x \frac{e^{\frac{-t^2}{2}}}{\sqrt{2\pi}}\,dt = f(x) - f(0)$$

and since $f(0) = 0.5$ $\quad \int_0^x \frac{e^{\frac{-t^2}{2}}}{\sqrt{2\pi}}\,dt = f(x) - 0.5$

Therefore $\quad f(x) = \int_0^x \frac{e^{\frac{-t^2}{2}}}{\sqrt{2\pi}}\,dt + 0.5$

So here $\quad f(1) = \int_0^1 \frac{e^{\frac{-t^2}{2}}}{\sqrt{2\pi}}\,dt + 0.5$

The required answer is 0.8413, correct to four decimal places.

Exercise 9F

1 In each of the following use a calculator to find values correct to four decimal places:

a $\frac{dy}{dx} = \sqrt{\cos x}$ and $y = 1$ when $x = 0$. Find y when $x = \frac{\pi}{4}$.

b $\frac{dy}{dx} = \frac{1}{\sqrt{\cos x}}$ and $y = 1$ when $x = 0$. Find y when $x = \frac{\pi}{4}$.

c $\frac{dy}{dx} = \log_e x^2$ and $y = 2$ when $x = 1$. Find y when $x = e$.

d $\frac{dy}{dx} = \sqrt{\log_e x}$ and $y = 2$ when $x = 1$. Find y when $x = e$.

2 Using a CAS calculator, find the general solution of each of the following:

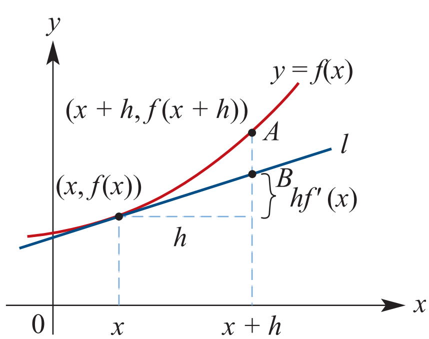
a $\frac{dy}{dx} = \tan^{-1} x$ **b** $\frac{dy}{dx} = x^3 \log_e x$ **c** $\frac{dy}{dx} = e^{3x} \sin 2x$

d $\frac{dy}{dx} = e^{3x} \cos 2x$ **e** $\frac{dy}{dx} = 2^x$

9.7 Euler's method for the solution of a differential equation

Euler's method uses the linear approximation method from calculus introduced in Mathematical Methods

i.e. $$\frac{f(x+h) - f(x)}{h} \approx f'(x) \quad \text{for } h \text{ small}$$

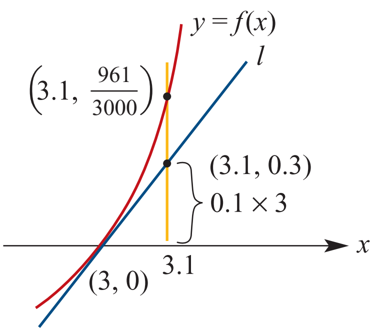


The rearrangement gives $f(x + h) \approx hf'(x) + f(x)$.

This is shown on the diagram. l is a tangent to $y = f(x)$ at the point with coordinates $(x, f(x))$. This gives an approximation to the function in that the y coordinate of B is an approximation of the y coordinate of A on the graph of $y = f(x)$.

Consider the differential equation:

$$\frac{dy}{dx} = x^2 - 2x \text{ and } y(3) = 0$$



The graph shown is a section of the solution curve for the differential equation.

$$\left(\text{It can be shown that } y = f(x) = \frac{x^3}{3} - x^2\right)$$

In this case $h = 0.1$, and $f(x + h) \approx hf'(x) + f(x)$ gives $f(3.1) \approx 0.1 \times 3 + 0$

$= 0.3$

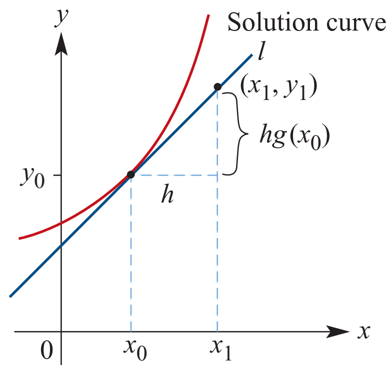
The actual value of $f(3.1)$ is $\dfrac{961}{3000} \approx 0.32$.

This process can be repeated to generate a sequence of coordinates.

Consider the differential equation

$$\frac{dy}{dx} = g(x) \text{ with } y(x_0) = y_0$$

Then $y_1 = y_0 + hg(x_0)$ and $x_1 = x_0 + h$.

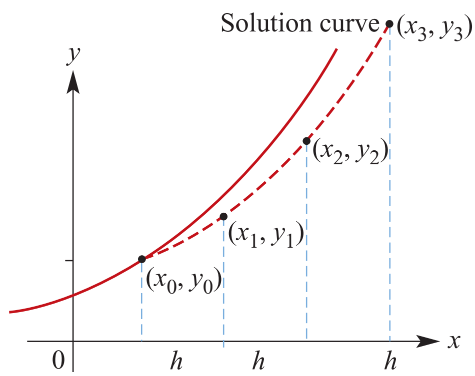


The process is now applied repeatedly to approximate the value of the function for $x_2, x_3, \ldots$

The result is

$$y_2 = y_1 + hg(x_1) \quad \text{and} \quad x_2 = x_1 + h$$
$$y_3 = y_2 + hg(x_2) \quad \text{and} \quad x_3 = x_2 + h$$

etc.



This iterative process leads to the statement of **Euler's formula** as follows:

If $\dfrac{dy}{dx} = g(x)$, $x_0 = a$ and $y_0 = b$, then $x_{n+1} = x_n + h$ and $y_{n+1} = y_n + hg(x_n)$

The accuracy of this formula, and the associated process, can be checked against the values obtained through the solution of the differential equation, where the result is known.

For the example $\dfrac{dy}{dx} = x^2 - 2x$ with $y(3) = 0$ and $h = 0.1$, the values of (x_i, y_i) for $0 \le i \le 10$ are shown in the following table.

i	x_i	y_i	$g(x_i)$
0	3	0	3 (initial values)
1	3.1	0.3	3.41
2	3.2	0.641	3.84
3	3.3	1.025	4.29
4	3.4	1.454	4.76
5	3.5	1.93	5.25
6	3.6	2.455	5.76
7	3.7	3.031	6.29
8	3.8	3.66	6.84
9	3.9	4.344	7.41
10	4.0	5.085	

When the y values are compared with the values of $f(x) = \dfrac{x^3}{3} - x^2$

i	0	1	2	3	4	5	6	7	8	9	10
y_i	0	0.3	0.641	1.025	1.454	1.93	2.455	3.031	3.66	4.344	5.085
$f(x_i)$	0	0.320	0.683	1.089	1.541	2.042	2.592	3.194	3.851	4.563	5.333

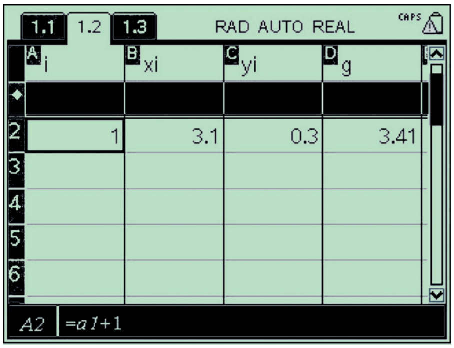
As can be seen, the values obtained are reasonably close to the values obtained through the function $f(x) = \dfrac{x^3}{3} - x^2$ which is the solution to the differential equation. A smaller value for the step would yield a better approximation. For example, for $h = 0.01$ the value of $f(x)$ for $x = 4$ is 5.3085. The percentage error for $h = 0.1$ for $x = 4$ is 4.65% but for $h = 0.01$ the error is 0.46%.

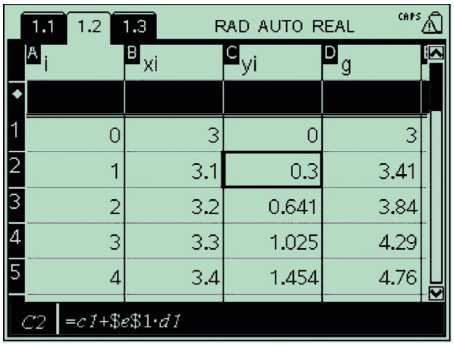
Using a spreadsheet for Euler's method

We use a spreadsheet to solve the equation $\dfrac{dy}{dx} = x^2 - 2x$.

Using a TI-Nspire calculator

Choose a **Lists & Spreadsheets** application. Label the columns as shown.
Enter 0 in A1, 3 in B1, 0 in C1 and $=\text{b}1^2-2\text{b}1$ in D1.
Fill down in Column D to row 16. To do this select cell D1 and then select from the menu **Data** and then **Fill Down**.
Use the arrow keys to go down to D16 and press **Enter**.
Then in A2 enter =a1+1, in B2 enter =b1+0.1 and in C2 enter =c1+0.1*d1.
Select A2, B2 and C2 and fill down to row 16.
The result is as shown.





Using a Casio ClassPad calculator

Label the columns as shown.

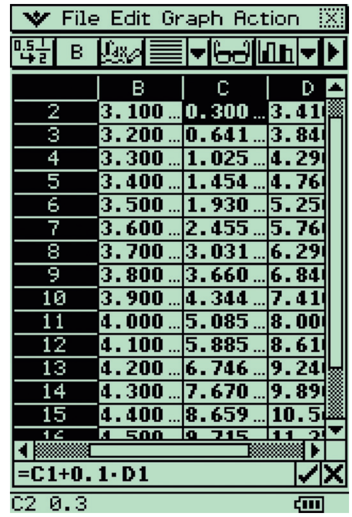
Enter 0 in A1, 3 in B1, 0 in C1 and $= \text{b1}^2 - 2\text{b1}$ in D1.

Fill down in column D to row 16.

Then in A2 enter =a1+1, in B2 enter =b1+0.1 and in C2 enter =c1+0.1*d1.

Select A2, B2 and C2 and fill down to row 16.

The result is as shown.



Euler's method program for a TI-Nspire calculator

A program may be used to achieve the same results.

Step 1 On a calculator page, define the function $g(x)$ as given.

Step 2 From a calculator page choose **Functions & Programs** and then **Program editor** and **New**. Name the program Euler. Complete as shown.

Define euler($x1$, $y1$, c, h) = prgm

local n, x

$x1 \rightarrow x$

$x1 \rightarrow a1[1]$

$y1 \rightarrow b1[1]$

$0 \rightarrow n$

DISP "x y"

DISP $x1$, " ", $y1$

While $x < c$

$n + 1 \rightarrow n$

$b1[n] + h^*g(x) \rightarrow b1[n+1]$

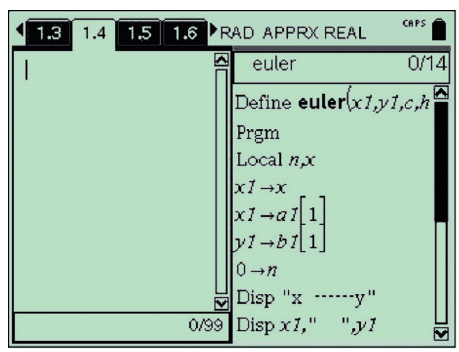
$a1[n] + h \rightarrow a1[n+1]$

$a1[n+1] \rightarrow x$

DISP $a1[n+1]$," ", $b1[n+1]$

Endwhile

EndPrgm



and then choose **Close**.

Step 3 Type in euler($x1, y1, c, h$), replacing the parameters with the values given. The results will be produced on the same page. If a tabulated version is preferred, go to MENU and select **Lists & Spreadsheets**. Name the first two columns $a1$ and $b1$. The results will appear in the spreadsheet.

Example 23

Use a spreadsheet on a CAS calculator to find solutions of the differential equation $\frac{dy}{dx} = e^{\sin x}$ given $y = 1$ when $x = 0$ using step sizes of:

a 0.1 **b** 0.01

Solution

Using a TI-Nspire calculator

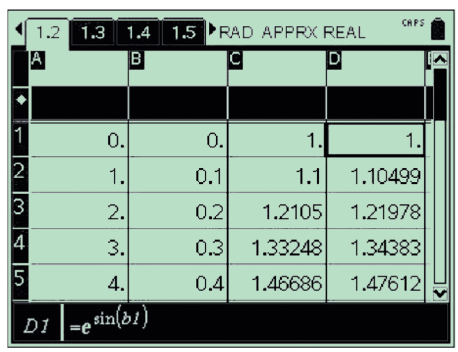
Choose a **Lists & Spreadsheets** application. Label the columns as shown.

Enter 0 in A1, 0 in B1, 1 in C1 and $=e^{\sin(b1)}$ in D1.

Fill down in Column D to row 15. To do this select D1 and then select from the menu **Data** and then **Fill Down**.

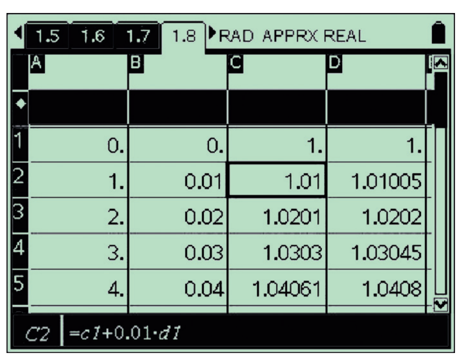
Use the arrow keys to go down to D15 and press **Enter**.

a In A2 enter =a1+1, in B2 enter =b1+0.1 and in C2 enter =c1+0.1*d1.
Select A2, B2 and C2 and fill down to row 15.
The result is as shown.



	A	B	C	D
1	0.	0.	1.	1.
2	1.	0.1	1.1	1.10499
3	2.	0.2	1.2105	1.21978
4	3.	0.3	1.33248	1.34383
5	4.	0.4	1.46686	1.47612

b In B2 enter =b1+0.01 and in C2 enter =c1+0.01*d1.
Select A2, B2 and C2 and fill down to row 15.
The result is as shown.



	A	B	C	D
1	0.	0.	1.	1.
2	1.	0.01	1.01	1.01005
3	2.	0.02	1.0201	1.0202
4	3.	0.03	1.0303	1.03045
5	4.	0.04	1.04061	1.0408

Using a Casio ClassPad calculator

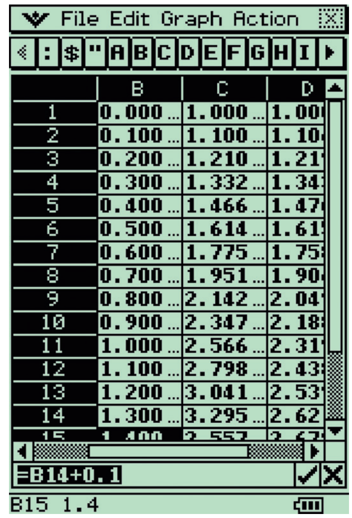
Label the columns as shown.

Enter 0 in A1, 0 in B1, 1 in C1 and $= e^{\sin(b1)}$ in D1.

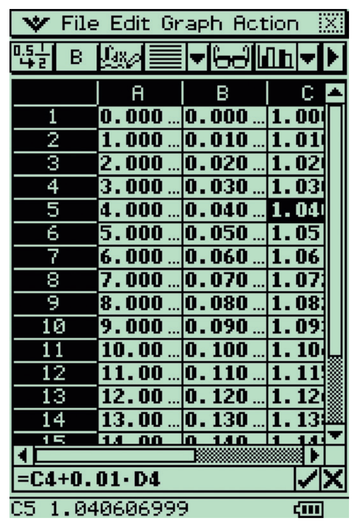
Fill down in Column D to row 15.

Then in A2 enter $=$a1$+$1.

a In B2 enter $=$b1$+$0.1 and in C2 enter $=$c1$+$0.1*d1.
Select A2, B2 and C2 and fill down to row 15.
The result is as shown.



b In B2 enter $=$ b1$+$0.01 and in C2 enter $=$ c1$+$0.01*d1.
Select A2, B2 and C2 and fill down to row 15.
The result is as shown.



Note the significant improvement in accuracy with the smaller step size in the outputs displayed above.

Euler's method with a step size of 0.1 has resulted in a percentage error of approximately 4.7%, whereas using the step size of 0.01 the resultant percentage error is as low as 0.5%.

Exercise 9G

1 Use Euler's formula to find the y_n value indicated, using the given h value, giving each answer correct to 4 decimal places, in each of the following:

a $\frac{dy}{dx} = \cos x$, given $y_0 = y(0) = 1$, find y_3, using $h = 0.1$

b $\frac{dy}{dx} = \frac{1}{x^2}$, given $y_0 = y(1) = 0$, find y_4, using $h = 0.01$

c $\frac{dy}{dx} = \sqrt{x}$, given $y_0 = y(1) = 1$, find y_3, using $h = 0.1$

d $\frac{dy}{dx} = \frac{1}{x^2 + 3x + 2}$, given $y_0 = y(0) = 0$, find y_3, using $h = 0.01$

2 Solve each of the following differential equations, using:

i a calculus method

ii the Euler program above or a spreadsheet, with a step size of 0.01.

a $\frac{dy}{dx} = \cos x$, given $y(0) = 1$, find $y(1)$ **b** $\frac{dy}{dx} = \frac{1}{x^2}$, given $y(1) = 0$, find $y(2)$

c $\frac{dy}{dx} = \sqrt{x}$, given $y(1) = 1$, find $y(2)$ **d** $\frac{dy}{dx} = \frac{1}{x^2 + 3x + 2}$, given $y(0) = 0$, find $y(2)$.

3 Solve the differential equation $\frac{dy}{dx} = \sec^2 x$, at $x = 1$, given $y = 2$ when $x = 0$, using

a a calculus method

b the Euler program or a spreadsheet with step size:

i 0.1 **ii** 0.05 **iii** 0.01

4 Use Euler's method with steps of 0.01 to find an approximate value of y at $x = 0.5$ if $\frac{dy}{dx} = \cos^{-1} x$ and $y = 0$ when $x = 0$.

5 Use Euler's method with steps of size 0.1 to find an approximate value of y at $x = 3$ if $\frac{dy}{dx} = \sin(\sqrt{x})$ and $y = 0$ when $x = 0$.

6 Use Euler's method with steps of size 0.01 to find an approximate value of y at $x = 0.3$ if $\frac{dy}{dx} = \frac{1}{\cos(x^2)}$ and $y = 0$ when $x = 0$.

7 The graph for the standard normal distribution is given by the function with rule $f(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$.

$$\Pr(Z < z) = \int_{-\infty}^{z} f(x)\,dx$$
$$= \tfrac{1}{2} + \int_{0}^{z} f(x)\,dx$$

Let $y = \Pr(Z \le z)$

Then $\frac{dy}{dx} = f(x)$ with $y(0) = \frac{1}{2}$.

a Use Euler's method with step size of 0.1 to find an approximation for $\Pr(Z \le z)$ where $z = 0, 0.1, 0.2, \ldots, 0.9, 1$.

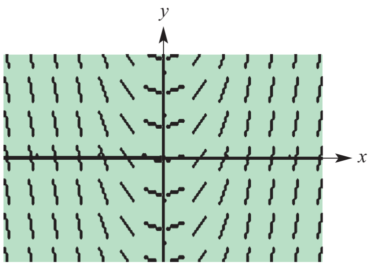
b Compare the values obtained in **a** with the probabilities determined from the CAS calculator.

c Use a step size of 0.01 to obtain an approximation for:
i $\Pr(Z \leq 0.5)$ **ii** $\Pr(Z \leq 1)$

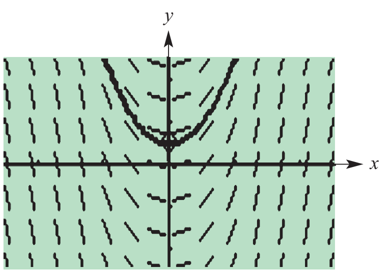
9.8 Direction (slope) field for a differential equation

A slope (direction) field of a differential equation, $\frac{dy}{dx} = f(x)$, assigns to each point $P(x, y)$ in the plane, with x in the domain of f, the number which is the slope (gradient) of the solution curve through P.

For example, for the differential equation $\frac{dy}{dx} = 2x$, a gradient value is assigned for each $P(x, y)$. For the points (1, 3) and (1, 5) the gradient value is 2. For (−2, 5) and (−2, −2) the gradient value is −4. A slope field can, of course, be represented in a graph.

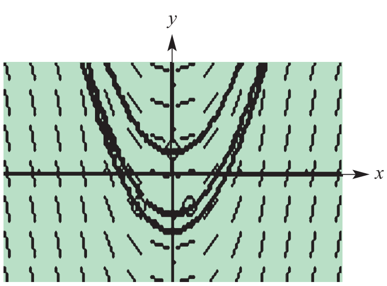


Computer packages are available to create these slope fields. CAS calculators also have this facility. When initial conditions are added a particular solution curve can be drawn. For example, in the graph shown the solution curve for the differential equation $\frac{dy}{dx} = 2x$, with $y = 2$ when $x = 0$, is shown superimposed on the slope field.



Changing the initial conditions of course changes the particular solution.

In the diagram below, the solution curves for when $x = 0$, $y = 2$, when $x = 1$, $y = -3$ and when $x = -2$, $y = -3$ are shown.



Example 24

a Use a CAS calculator to plot the slope field for the differential equation $\frac{dy}{dt} = y$.

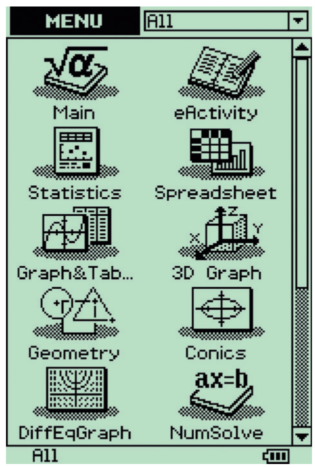
b On the plot of the slope field, plot the graphs of the particular solutions for:

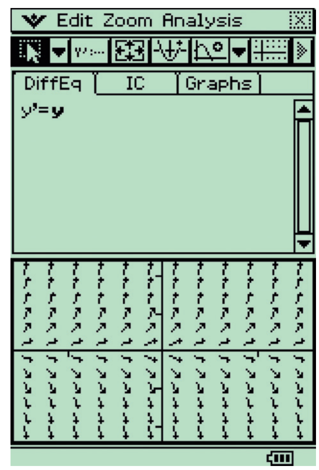
i $y = 2$ when $t = 0$ **ii** $y = -3$ when $t = 1$

Solution

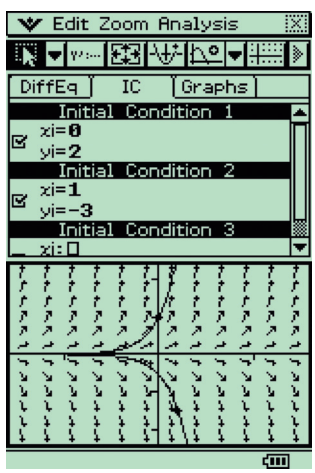
Using a Casio ClassPad calculator

a From the Main menu choose **DiffEqGraph**.
Enter $y' = y$ to achieve the window shown.





b i and **b ii**
Complete the initial conditions as shown.



It is to be noted that the TI-Nspire may contain a 'Plot Differential Equation CAS' facility in **My Document**.

The differential equation $\frac{dy}{dt} = y$ is solved analytically in the usual manner.

$\frac{dt}{dy} = \frac{1}{y}$ and $t = \log_e|y| + c$, which implies $|y| = e^{t-c}$ or $|y| = Ae^t$.

If $y = 2$ when $t = 0$, $2 = A$ and $y = 2e^t$.

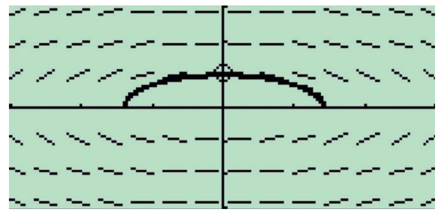
If $y = -3$ when $t = 1$, $3 = Ae$ and $A = \frac{3}{e}$ which implies $|y| = 3e^{t-1}$. This implies $y = -3\,e^{t-1}$ as $y < 0$.

Example 25

Use a CAS calculator to draw the slope field for the differential equation $\frac{dy}{dx} = -\frac{x}{2y}$ and show the solution for the initial condition $x = 0, y = 1$.

Solution

The graph of the solution as displayed in the GRAPH window is shown right.



Note that the upper section of the ellipse is displayed. The complete ellipse is displayed if a second initial condition, $x = 0, y = -1$, is given.

The ellipse is $\frac{x^2}{2} + y^2 = 1$.

Exercise 9H

1 See page 554 of Chapter 15 for multiple-choice questions on slope fields.

2 Sketch a slope field graph and the solution curve for the given initial conditions, using $-3 \le x \le 3$ and $-3 \le y \le 3$ for each of the following differential equations. Use calculus to solve the differential equation in each case.

a $\frac{dy}{dx} = 3x^2$, given $y = 0$ when $x = 1$.

b $\frac{dy}{dx} = \sin x$, given $y = 0$ when $x = 0$. (Make sure that Radian mode is set for this problem.)

c $\frac{dy}{dx} = e^{-2x}$, given $y = 1$ when $x = 0$.

d $\frac{dy}{dx} = y^2$, given $y = 1$ when $x = 1$.

e $\frac{dy}{dx} = y^2$, given $y = -1$ when $x = 1$.

f $\frac{dy}{dx} = y(y - 1)$, given $y = -1$ when $x = 0$.

g $\frac{dy}{dx} = y(y - 1)$, given $y = 2$ when $x = 0$.

h $\frac{dy}{dx} = \tan x$, given $y = 0$ when $x = 0$.

3 Sketch a slope field graph and the solution curve for the given initial conditions, using $-3 \le x \le 3$ and $-3 \le y \le 3$ for each of the following differential equations, but do not attempt to solve by calculus methods.

a $\frac{dy}{dx} = -\frac{x}{y}$, where at $x = 0, y = \pm 1$.

b $\frac{dy}{dx} = -\frac{x}{y}$, where at $x = \frac{1}{2}, y = \frac{\sqrt{3}}{2}$.

Chapter summary

- A differential equation is an equation that contains at least one derivative.
- The solution of a differential equation is a function that satisfies the differential equation when it and its derivatives are substituted. The general solution is the family of functions that satisfies the differential equation.
- If the differential equation has the form:

$$\frac{dy}{dx} = f(x)$$

Then $y = \int f(x)\,dx$

$\therefore \quad y = F(x) + c, \text{ where } F'(x) = f(x)$

- If the differential equation has the form:

$$\frac{dy}{dx} = f(y)$$

Then $\dfrac{dx}{dy} = \dfrac{1}{f(y)}$

$\therefore \quad x = \int \dfrac{1}{f(y)}\,dy$

$\therefore \quad y = F(y) + c, \text{ where } F'(y) = \dfrac{1}{f(y)}$

- If the differential equation has the form:

$$\frac{d^2y}{dx^2} = f(x)$$

Then
$$\begin{aligned}\frac{dy}{dx} &= \int f(x)\,dx \\ &= F(x) + c, \text{ where } F'(x) = f(x)\end{aligned}$$

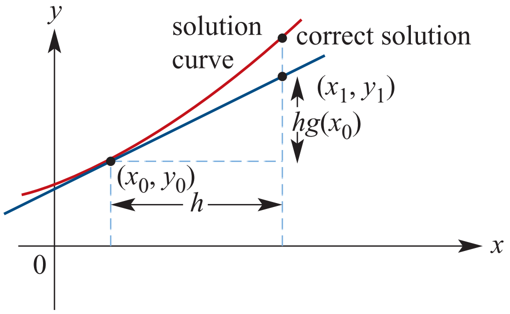
$$\therefore \quad \begin{aligned} y &= \int F(x) + c\,dx \\ &= G(x) + cx + d, \text{ where } G'(x) = F(x)\end{aligned}$$

- Euler's method

For $\dfrac{dy}{dx} = g(x)$ and $y = y_0$ when $x = x_0$

$y_1 = y_0 + hg(x_0)$

$x_1 = x_0 + h$



Review

This process can be applied repeatedly to approximate the value of the function at $x_2, x_3, \ldots$

$y_2 = y_1 + hg(x_1)$ and $x_2 = x_1 + h$

$y_3 = y_2 + hg(x_2)$ and $x_3 = x_2 + h$

$\vdots$

$y_n + 1 = y_n + hg(x_n)$ and $x_n + 1 = x_n + h$

- A slope (direction) field of a differential equation, $\frac{dy}{dx} = f(x)$, assigns to each point $P(x, y)$ in the plane, with x in the domain of f, the number which is the slope (gradient) of the solution curve through P.

Multiple-choice questions

1 The acceleration, a m/s^2, of an object moving in a line at time t seconds is given by $a = \sin(2t)$. If the object has an initial velocity of 4 m/s then v is equal to:

A $2\cos(2t) + 4$ **B** $2\cos(2t) + 2$ **C** $\int_0^t \sin(2x)dx + 4$

D $-\frac{1}{2}\cos(2t) + 4$ **E** $\int_0^t \sin(2x)dx - 4$

2 If $f'(x) = x^2 - 1$ and $f(1) = 3$, an approximate value of $f(1.4)$ using Euler's method with step size of 0.2, is:

A 3.88 **B** 3.688 **C** $3.\dot{6}$ **D** 3.088 **E** 3

3 Euler's method, with a step size of 0.1, is used to approximate the solution of the differential equation $\frac{dy}{dx} = x\log_e x$ with $y(2) = 2$. When $x = 2.2$, the value obtained for y is closest to:

A 2.314 **B** 2.294 **C** 2.291 **D** 2.287 **E** 2.277

4 Given $\frac{dy}{dx} = \frac{2-y}{4}$ and $x = 3$ when $y = 1$, then when $y = \frac{1}{2}$, x is equal to:

A $\int_1^{\frac{1}{2}} \frac{4}{2-t}dt + 3$ **B** $\int_3^{\frac{1}{2}} \frac{4}{2-t}dt + 1$ **C** $\int_1^{\frac{1}{2}} \frac{2-t}{4}dt + 3$

D $\int_3^{\frac{1}{2}} \frac{2-t}{4}dt + 1$ **E** $\int_1^{\frac{1}{2}} \frac{2-y}{4}dy + 3$

5 If $\frac{dy}{dx} = \frac{2x+1}{4}$ and $y = 0$ when $x = 2$, then y is equal to:

A $\frac{1}{4}(x^2 + x) + \frac{1}{2}$ **B** $\frac{x(x+1)}{4}$ **C** $\frac{1}{4}(x^2 + x) + 2$

D $\frac{1}{4}(x^2 + x - 1)$ **E** $\frac{1}{4}(x^2 + x - 6)$

Review

6 If $\frac{dy}{dx} = \frac{1}{5}(y-1)^2$ and $y = 0$ when $x = 0$, then y is equal to:

A $\frac{5}{1-x} - 5$ **B** $1 + \frac{5}{x+5}$ **C** $\frac{x}{x+5}$

D $\frac{5}{x+5} - 1$ **E** $1 - \frac{5}{x}$

7 The solution of the differential equation $\frac{dy}{dx} = e^{-x^2}$ where $y = 4$ when $x = 1$ is:

A $y = \int_1^4 e^{-x^2}\,dx$ **B** $y = \int_1^4 e^{-x^2}\,dx + 4$ **C** $y = \int_1^x e^{-u^2}\,du - 4$

D $y = \int_1^x e^{-u^2}\,du + 4$ **E** $y = \int_4^x e^{-u^2}\,du + 1$

8 For which one of the following differential equations is $y = 2xe^{2x}$ a solution?

A $\frac{dy}{dx} - 2y = 0$ **B** $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} = 0$ **C** $\frac{dy}{dx} + 2y\frac{dy}{dx} = 0$

D $\frac{d^2y}{dx^2} - 4y = e^{2x}$ **E** $\frac{d^2y}{dx^2} - 4y = 8e^{2x}$

9 Water is leaking from an initially full container with a depth of 40 cm. The volume, V cm^3, of water in the container is given by $V = \pi(5h^2 + 225h)$ where h is the depth of the water at time t minutes. If water leaks out at the rate of $\frac{5\sqrt{h}}{2h+45}$ cm^3/min, then the rate of change of the depth, in cm/min, is:

A $\frac{-\sqrt{h}}{\pi(2h+45)^2}$ **B** $5\pi(2h+45)$ **C** $\frac{\sqrt{h}}{\pi(2h+45)^2}$

D $\frac{1}{5\pi(2h+45)}$ **E** $\frac{-1}{5\pi(2h+45)}$

10 The solution of the differential equation $\frac{dy}{dx} = y$, where $y = 2$ when $x = 0$, is:

A $y = e^{2x}$ **B** $y = e^{\frac{x}{2}}$ **C** $y = 2e^x$ **D** $y = \frac{1}{2}e^x$ **E** $y = \log_e \frac{x}{2}$

Short-answer questions (technology-free)

1 Find the general solution of each of the following differential equations:

a $\frac{dy}{dx} = \frac{x^2+1}{x^2},\ x > 0$

b $\frac{1}{y}.\frac{dy}{dx} = 10,\ y > 0$

c $\frac{d^2y}{dt^2} = \frac{1}{2}(\sin 3t + \cos 2t),\ t \geq 0$

d $\frac{d^2y}{dx^2} = \frac{e^{-x} + e^x}{e^{2x}}$

e $\frac{dy}{dx} = \frac{3-y}{2},\ y < 3$

f $\frac{dy}{dx} = \frac{3-x}{2}$

2 Find the solution of the following differential equations under the stated conditions:

a $\frac{dy}{dx} = \pi\cos(2\pi x)$, if $y = -1$ when $x = \frac{5}{2}$

b $\frac{dy}{dx} = \cot 2x$, if $y = 0$ when $x = \frac{\pi}{4}$

c $\dfrac{dy}{dx} = \dfrac{1 + x^2}{x}$, if $y = 0$ when $x = 1$

d $\dfrac{dy}{dx} = \dfrac{x}{1 + x^2}$, if $y(0) = 1$

e $6\dfrac{dy}{dx} = -3y$, if $y = e^{-1}$ when $x = 2$

f $\dfrac{d^2x}{dt^2} = -10$, if $\dfrac{dx}{dt} = 4$ when $x = 0$ and $x = 0$ when $t = 4$

3 **a** If $y = x \sin x$ is a solution of the differential equation $x^2\dfrac{d^2y}{dx^2} - kx\dfrac{dy}{dx} + (x^2 - m)y = 0$, find k and m.

b Prove that $y = xe^{2x}$ is a solution of the differential equation

$$\frac{d^2y}{dx^2} - \frac{dy}{dx} - 3e^{2x} = 2xe^{2x}.$$

4 If a hemispherical bowl of radius 6 cm contains water to a depth of x cm, the volume V cm^3 is given by $V = \dfrac{\pi}{3}x^2(18 - x)$. If water is poured into the bowl at the rate of 3 cm^3/s, construct the differential equation expressing $\dfrac{dx}{dt}$ as a function of x.

5 The area of a circle is A cm^2 and the circumference is C cm at time t s. If the area is increasing at the rate of 4 cm^2/s, construct the differential equation expressing $\dfrac{dC}{dt}$ as a function of C.

6 Some students put three kilograms of soap powder into a water fountain. The soap powder totally dissolved in the 1000 litres of water, thus forming a solution in the fountain. When the soap solution was discovered, clean water was run into the fountain at the rate of 40 litres per minute. The clean water and the solution in the fountain mixed instantaneously and the excess mixture was removed immediately at a rate of 40 litres per minute. If S kilograms was the amount of soap powder in the fountain t minutes after the soap solution was discovered, construct and solve the differential equation to fit this situation.

7 A population is size x is decreasing according to the law $\dfrac{dx}{dt} = -\dfrac{x}{100}$ where t denotes the time in days. If initially the population is of size x_0 find, to the nearest day, how long it takes for the size of the population to be halved.

8 A metal rod, that is initially at a temperature of 10°C, is placed in a warm room. After t minutes, the temperature, θ°C, of the rod is such that $\dfrac{d\theta}{dt} = \dfrac{30 - \theta}{20}$.

a Solve this differential equation, expressing θ in terms of t.

b Calculate the temperature of the rod after one hour has elapsed, giving the answer correct to the nearest degree.

c Find the time taken for the temperature of the rod to rise to 20°C, giving the answer correct to the nearest minute.

9 A fire broke out in a forest and, at the moment of detection, covered an area of 0.5 hectares. From an aerial surveillance it was estimated that the fire was spreading at a rate of increase in area of two per cent per hour. If the area of the fire at time t hours is denoted by A hectares:

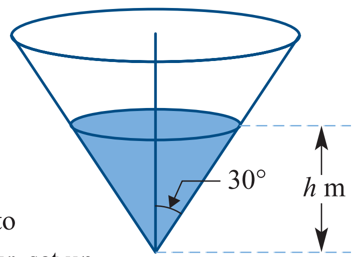
a Write down the differential equation that relates $\frac{dA}{dt}$ and A.

b What would be the area of the fire 10 hours after it is first detected?

c When would the fire cover an area of three hectares (to the nearest quarter-hour)?

10 A flexible beam is supported at its ends, which are at the same horizontal level and at a distance L apart. The deflection, y, of the beam, measured downwards from the horizontal through the supports, satisfies the differential equation $16\frac{d^2y}{dx^2} = L - 3x, 0 \le x \le L$, where x is the horizontal distance from one end. Find where the deflection has its greatest magnitude, and also the value of this magnitude.

11 A vessel in the shape of a right circular cone has a vertical axis and a semi-vertex angle of 30°. There is a small hole at the vertex so that liquid leaks out at the rate of $0.05\sqrt{h}$ cubic metres per hour, where h metres is the depth of liquid in the vessel at time t hours. Given that the liquid is poured into this vessel at a constant rate of two cubic metres per hour, set up, but do not attempt to solve, a differential equation for h.



Extended-response questions

1 The percentage of radioactive carbon-14 in living matter decays, from the time of death, at a rate proportional to the percentage present.

a If x% is present t years after death:

i construct an appropriate differential equation

ii solve the differential equation given that carbon-14 has a half life of 5760 years, i.e. 50% of the original amount will remain after 5760 years.

b Carbon-14 was taken from a tree buried by volcanic ash and was found to contain 45.1% of the amount of carbon-14 present in living timber. How long ago did the eruption occur?

c Sketch the graph of x against t.

2 Two chemicals, A and B, are put together in a solution where they react to form a compound, X. The rate of increase of the mass, x kg, of X is proportional to the product of the masses of *unreacted* A and B present at time t minutes. It takes 1 kg of A and 3 kg of B to form 4 kg of X. Initially 2 kg of A and 3 kg of B are put together in solution. One kg of X forms in one minute.

a Set up the appropriate differential equation expressing $\frac{dx}{dt}$ as a function of x.

b Solve the differential equation.

c Find the time taken to form 2 kg of X.

d Find the mass of X formed after two minutes.

3 Newton's law of cooling states that the rate of cooling of a body is proportional to the excess of its temperature above that of its surroundings. The body has a temperature of T°C at time t minutes while the temperature of the surroundings is a constant T_S°C.

a Construct a differential equation expressing $\frac{dt}{dt}$ as a function of T.

b A teacher pours a cup of coffee at lunchtime. The lunchroom is at a constant temperature of 22°C, while the coffee is initially 72°C. The coffee becomes undrinkable (too cold) when its temperature drops below 50°C. After five minutes the temperature has fallen to 65°C. Find, correct to one decimal place:

i the length of time, after it was poured, that the coffee remains drinkable

ii the temperature of the coffee at the end of 30 minutes.

4 On a cattle station there were p head of cattle at time t years after 1 January 2005. The population naturally increases at a rate proportional to p. Every year 1000 head of cattle are withdrawn from the herd.

a Show that $\frac{dp}{dt} = kp - 1000$ where k is a constant.

b If the herd initially had 5000 head of cattle, find an expression for t in terms of k and p.

c The population increased to 6000 head of cattle after five years.

i Show that $5k = \log_e\left(\frac{6k-1}{5k-1}\right)$.

ii Use a CAS calculator to find an approximation for the value of k.

d Sketch a graph of p against t.

5 In the main lake of a trout farm the trout population is N at time t days after 1 January 2005. The number of fish harvested on a particular day is proportional to the number of fish in the lake at that time. Every day 100 trout are added to the lake.

a Construct a differential equation with $\frac{dN}{dt}$ in terms of N and k where k is a constant.

b Originally the trout population was 1000. Find an expression for t in terms of k and N.

c The trout population decreases to 700 after 10 days. Use a CAS calculator to find an approximation for the value of k.

d Sketch a graph of N against t.

e If the procedure at the farm remains unchanged, find the eventual trout population in the lake.

6 A thin horizontal beam, AB, of length L cm, is bent under a load so that the deflection y cm, at a point x cm from the end A, satisfies the differential equation

$$\frac{d^2y}{dx^2} = \frac{9}{40L^2}(3x - L),\ 0 \le x \le L.$$

Given that the deflection of the beam and its inclination to the horizontal are both zero at A find:

a where the maximum deflection occurs

b the magnitude of the maximum deflection.

7 The water in a hot water tank cools at a rate which is proportional to $(T - T_0)^\circ$ where T°C is the temperature of the water at time t minutes and T_0 the temperature of the surrounding air. When T is 60 the water is cooling at 1°C per minute.

When switched on, the heater supplies sufficient heat to raise the temperature by 2°C each minute (neglecting heat loss by cooling). If $T = 20$ when the heater is switched on and $T_0 = 20$:

- **a** construct a differential equation for $\frac{dT}{dt}$ as a function of T (heating and cooling are both taking place)
- **b** solve the differential equation
- **c** find the temperature of the water 30 minutes after turning it on
- **d** sketch the graph of T against t.

8 **a** The rate of growth of a population of iguanas on an island is given *by* $\frac{dW}{dt} = 0.04\ W$ where W is the number of iguana alive after t years. Initially there were 350 iguanas.

- **i** Solve the differential equation.
- **ii** Sketch the graph of W against t.
- **iii** Give the value of W to the nearest integer when $t = 50$.

b If $\frac{dW}{dt} = kW$ and there are 350 iguanas initially, find the value of k if the population is to remain constant.

c A more realistic rate of growth for the iguanas is determined by the differential equation $\frac{dW}{dt} = (0.04 - 0.000\,05\ W)W$. Initially there were 350 iguanas.

- **i** Solve the differential equation.
- **ii** Sketch the graph of W against t.
- **iii** Find the population after 50 years.

9 A hospital patient is receiving a drug at a constant rate of R milligrams per hour, through a drip. At time t hours the amount of the drug in the patient is x milligrams. The rate of loss of the drug from the patient is proportional to x.

a When $t = 0, x = 0$:

- **i** show that $\frac{dx}{dt} = R - kx$ where k is a positive constant
- **ii** find an expression for x in terms of t, k and R.

b If $R = 50$ and $k = 0.05$:

- **i** sketch the graph of x against t
- **ii** find the time taken for there to be 200 mg in the patient, correct to two decimal places.

c When the patient contains 200 mg of the drug, the drip is disconnected.

- **i** Assuming that the rate of loss remains the same, find the time taken for the amount of the drug in the patient to fall to 100 mg, correct to two decimal places.
- **ii** Sketch the graph of x against t showing the rise to 200 mg and the fall to 100 mg.

CHAPTER

10

Kinematics

Objectives

- To model **rectilinear motion** and use **calculus** to solve problems involving motion in a line with **constant** and **variable acceleration** (dependent on time)
- To use **graphical methods** to solve rectilinear motion problems
- To use techniques of solving differential equations to solve problems of the form

  $v = f(x)$ and $a = f(v)$

  where x, v and a represent position, velocity and acceleration respectively
- To use techniques of solving differential equations to solve problems of the form $a = f(x)$ and other special cases

Kinematics is the study of motion without reference to the cause of motion. In this chapter, motion in a straight line is considered. Such motion is called **rectilinear motion**. Only the motion of a particle is considered, but in fact the theory can be applied to a body of any size by assuming that all forces that act upon the body, causing it to move, act through a single point. Hence the motion of a car or train can be considered in the same way as the motion of a dimensionless particle.

It is important to make a distinction between vector and scalar quantities when studying motion. Quantities such as displacement, velocity and acceleration must be specified by both magnitude and direction. They are vector quantities. Distance, speed and time, on the other hand, are specified by their magnitude only. They are scalar quantities.

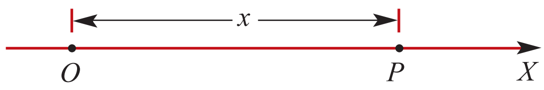
Since only movement in a straight line is being considered, the direction of all vector quantities is simply specified by the sign of the numerical value.

10.1 Position, velocity and acceleration

In the following, the relationship between displacement, velocity and acceleration is discussed.

Position and displacement

The **position** of a particle moving in a straight line is determined by its distance from a fixed point O on the line, called the origin, and whether it is to the right or left of O. Conventionally, the direction to the right of the origin is considered to be positive.



Consider a particle which starts at O and begins to move. The position of a particle P is determined by a number, x. If the unit is metres and if $x = -3$, the position of P is 3 m to the left of O, while if $x = 3$, the position of P is 3 m to the right of O.

The **displacement** is defined as the change in position of the particle. Sometimes there is a rule which enables the position, at any instant, to be calculated. In this case x is redefined as a function of t. Hence $x(t)$ is the displacement relative to O at time t. Specification of a displacement function, together with the physical idealisation of a real situation, constitutes a **mathematical model** of the situation.

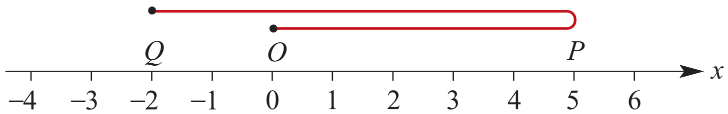
An example of a mathematical model is the following:

A stone is dropped from the top of a vertical cliff 45 metres high.

Assume that the stone is a particle travelling in a straight line. Let $x(t)$ be position of the particle measured downwards from O the top of the cliff, t seconds after the particle is dropped. If air resistance is neglected, an approximate model for the displacement is $x(t) = 5t^2$ for $0 \leq t \leq 3$.

It is important to distinguish between the **scalar** quantity **distance**, and the **vector** quantity **displacement**.

Consider a particle that starts at O and moves firstly five units to the right to point P, and then seven units to the left to point Q.



Its final displacement (or position measured from O), is $x = -2$. However the **distance** the particle has moved is 12 units.

Velocity

Rate of change has been considered in Mathematical Methods.

The **velocity** of a particle is defined as the rate of change of its position with respect to time. Both the **average rate of change**, the change in position over a period of time, and the **instantaneous rate of change**, which specifies the rate of change at a given instant in time, can be considered.

If a particle moves from x_1 at time t_1 to x_2 at time t_2, then its **average velocity** $= \dfrac{x_2 - x_1}{t_2 - t_1}$.

Velocity can be positive, negative or zero. If the velocity is positive the particle is moving to the right and if it is negative the direction of motion is to the left. A velocity of zero means the particle is instantaneously at rest.

The instantaneous rate of change of position with respect to time is the instantaneous velocity. If the position, x, of the particle at time, t, is given as a function of t, then the velocity of the particle at time t is determined by differentiating the rule for position with respect to time.

Common units of velocity (and speed) are:

$$\begin{aligned} 1 \text{ metre per second} &= 1 \text{ m/s} \\ 1 \text{ centimetre per second} &= 1 \text{ cm/s} \\ 1 \text{ kilometre per hour} &= 1 \text{ km/h} \end{aligned}$$

The first and third units are connected in the following way:

$$\begin{aligned} 1 \text{ km/h} &= 1000 \text{ m/h} \\ &= \frac{1000}{60 \times 60} \text{ m/s} \\ &= \tfrac{5}{18} \text{ m/s} \\ \therefore \quad 1 \text{ m/s} &= \tfrac{18}{5} \text{ km/h} \end{aligned}$$

Note the distinction between velocity and speed.

Speed is the magnitude of the velocity.

Average speed for a time interval $[t_1, t_2]$ is equal to $\dfrac{\text{distance travelled}}{t_2 - t_1}$.

Instantaneous velocity $v = \dfrac{dx}{dt}$ where x is a function of time.

Velocity is also denoted by $\dot{x}$ or $\dot{x}(t)$.

Example 1

A particle moves in a straight line so that its position x cm relative to O at time t seconds is given by $x = 3t - t^3$, $t \geq 0$. Find:

a its initial position

b its position at $t = 2$

c its initial velocity

d its velocity when $t = 2$

e its speed when $t = 2$

f when and where the velocity is zero

Solution

a When $t = 0$, $x = 0$. The particle is initially at O.

b When $t = 2$, $x = 3 \times 2 - 2^3$

$$\begin{aligned} &= 6 - 8 \\ &= -2 \end{aligned}$$

The particle is 2 cm to the left of O.

c For $x = 3t - t^3$, the velocity $v = \dfrac{dx}{dt} = 3 - 3t^2$.

When $t = 0$, $v(3) = 3 - 3 \times 0 = 3$.

The velocity is 3 cm/s. The particle is initially moving to the right.

d When $t = 2$, $v = 3 - 3 \times 4 = -9$.

The velocity is -9 cm/s. The particle is moving to the left.

e The speed is the magnitude of the velocity.

When $t = 2$, the speed is 9 cm/s.

f $v = 0$ implies $\quad 3 - 3t^2 = 3(1 - t^2) = 0$

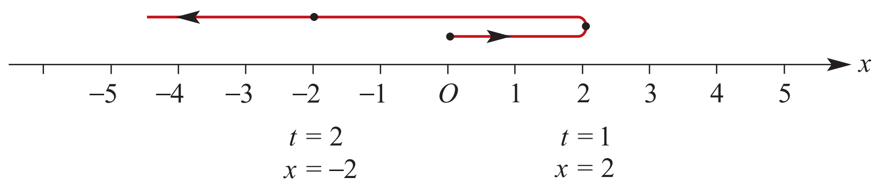
$\therefore \quad t = 1$ or $t = -1$

but $t \geq 0$

The particle is instantaneously at rest when $t = 1$.

When $t = 1$, $x(1) = 3 \times 1 - 1 = 2$.

The motion of the particle can now be demonstrated on a number line.



Example 2

The motion of a particle moving along a straight line is defined by $x(t) = t^2 - t$, $t \geq 0$ where x m is the position of the particle relative to O at time t. Find:

a the average velocity of the particle in the first three seconds

b the distance travelled by the particle in the first three seconds

c the average speed of the particle in the first three seconds

Solution

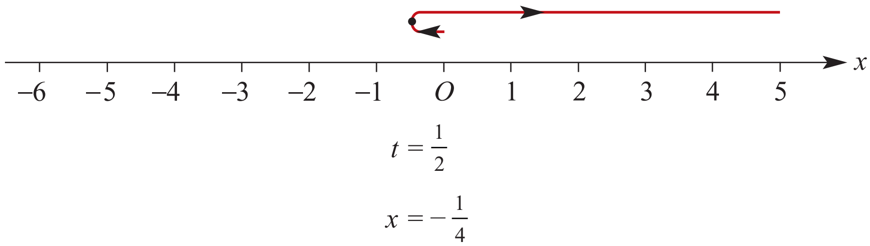
a
$$\text{average velocity} = \frac{x(3) - x(0)}{3} = \frac{6 - 3}{3} = 2 \text{ m/s}$$

b To find the distance travelled in the first three seconds it is useful to show the motion of the particle on a number line. The critical points are where it starts and when and where it changes direction.

The particle starts at the origin. The turning point(s) can be found by finding when the instantaneous velocity is zero.

$v = \dfrac{dx}{dt} = 2t - 1$ so that, $v = 0$ when $t = \frac{1}{2}$. The particle changes direction when $t = \frac{1}{2}$ and $x = t^2 - t = \left(\frac{1}{2}\right)^2 - \frac{1}{2} = \frac{-1}{4}$.

When $0 \le t < \frac{1}{2}$, v is negative and when $t > \frac{1}{2}$, v is positive. Drawn on a number line, the motion of the particle is as shown.



From the number line, the particle travels a distance of $\frac{1}{4}$ m in the first half second. It then changes direction. When $t = 3$, the particle is at the position $x = 3^2 - 3 = 6$ m to the right of O, so the particle has travelled distance of $6 + \frac{1}{4} = 6\frac{1}{4}$ m from when it changed direction.

Thus, the total distance travelled by the particle in the first three seconds is $\frac{1}{4} + 6\frac{1}{4} = 6\frac{1}{2}$ m.

c average speed $= \dfrac{\text{distance travelled}}{t_2 - t_1}$

$$= 6\tfrac{1}{2} \div 3$$

$$= \tfrac{13}{2} \div 3$$

$$= \tfrac{13}{6} \text{ m/s}$$

Acceleration

The **acceleration** of a particle is defined as the rate of change of its velocity with respect to time.

Average acceleration for the time interval $[t_1, t_2]$ is defined by $\dfrac{v_2 - v_1}{t_2 - t_1}$ where v_2 is the velocity at time t_2 and v_1 is the velocity at time t_1.

Instantaneous acceleration $a = \dfrac{dv}{dt} = \dfrac{d}{dt}\left(\dfrac{dx}{dt}\right) = \dfrac{d^2x}{dt^2}$.

For kinematics, the second derivative $\dfrac{d^2x}{dt^2}$ is denoted by $\ddot{x}(t)$.

Acceleration can be positive, negative or zero. Zero acceleration means the particle is moving at a constant velocity. Note that the direction of motion and the acceleration need not coincide. For example, a particle may have a positive velocity indicating it is moving to the right, but a negative acceleration indicating it is slowing down. Also, although a particle may be instantaneously at rest, its acceleration at that instant need not be zero. If acceleration has the same sign as velocity then the particle is ‘speeding up’. If the sign is opposite, the particle is ‘slowing down’.

The most commonly used units for acceleration include cm/s^2 and m/s^2.

Example 3

The position of an object, travelling in a horizontal line, is x metres from O at time t seconds, such that $x = -4t^2 + 8t + 12, \quad t \geq 0$.

a Sketch the position–time graph showing key features.

b Find the velocity at time t seconds and sketch the velocity–time graph.

c Find the acceleration at time t seconds and sketch the acceleration–time graph.

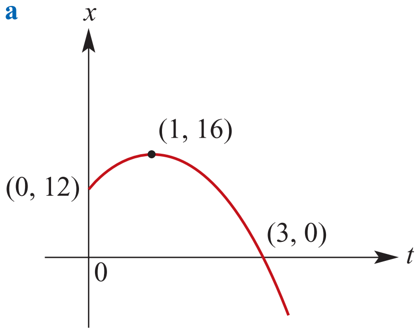
d Represent the motion of the object on a number line for $0 \leq t \leq 4$ seconds.

e Find the displacement of the object in the third second.

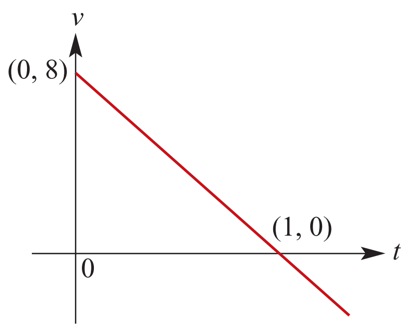
f Find the distance travelled in the first three seconds.

Solution

a

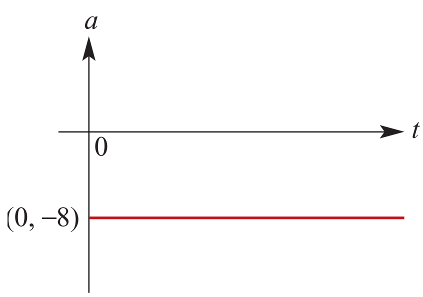


b $v = \dfrac{dx}{dt} = -8t + 8$, for $t \geq 0$



When $t \in [0, 1)$ velocity is positive.
When $t > 1$ the velocity is negative.

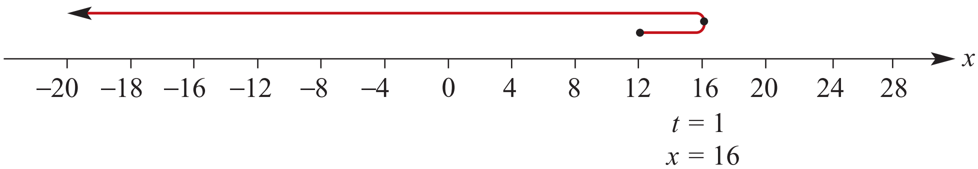
c $a = \dfrac{dv}{dt} = -8$



The acceleration is -8 m/s^2.
The direction of the acceleration is always to the left.

d Starting point: when $t = 0$, $x = 12$. Turning point: when $v = -8t + 8 = 0$, $t = 1$ and $x = 16$.

When $t > 1$, $\frac{dx}{dt} < 0$ and when $t < 1$, $\frac{dx}{dt} > 0$; i.e. when $0 \leq t < 1$ the particle is moving to the right and when $t > 1$ the particle is moving to the left.



e The displacement of the object in the third second is given by

$$x(3) - x(2) = 0 - 12$$
$$= -12$$

The displacement is 12 metres to the left.

f From the displacement time graph in **a** and the path illustrated in **d** it can be seen that the distance travelled in the first three seconds $= 4 + 16$

$$= 20 \text{ m}$$

Example 4

An object moves in a horizontal line so that the position, x m, from a fixed point at time t seconds is given by $x = -t^3 + 3t + 2 \quad t \geq 0$. Find:

a when the position is zero, and the velocity and acceleration at that time

b when the velocity is zero, and the position and acceleration at that time

c when the acceleration is zero, and the position and velocity at that time

d the distance travelled in the first three seconds.

Solution

a $x = 0$ when $-t^3 + 3t + 2 = 0$

$$t^3 - 3t - 2 = 0$$
$$(t - 2)(t + 1)^2 = 0$$
$$\therefore \quad t = 2 \text{ since } t \geq 0$$

Now $x = -t^3 + 3t + 2$

$\therefore \quad v = \dot{x} = -3t^2 + 3$

$\therefore \quad a = \ddot{x} = -6t \quad$ (the acceleration is variable in this case)

At $t = 2$, $v = -3 \times 2^2 + 3 = -9$

At $t = 2$, $a = -6 \times 2 = -12$

When the position is zero, velocity is -9 m/s and acceleration is -12 m/s^2.

b $v = 0$ when $-3t^2 + 3 = 0$

$\therefore \quad t^2 = 1$

$\therefore \quad t = 1$ since $t \geq 0$

At $t = 1$, $x = -1^3 + 3 \times 1 + 2 = 4$

At $t = 1$, $a = -6 \times 1 = -6$

When the object is at rest, the position is 4 m and the acceleration is -6 m/s^2.

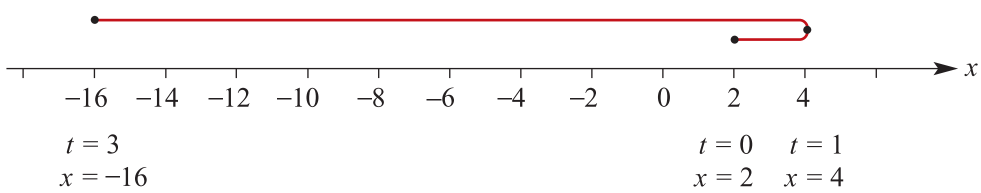
c $a = 0$ when $-6t = 0$

$\therefore \quad t = 0$

At $t = 0$, $x = 2$ and $v = 3$

The object has zero acceleration when the displacement is 2 m and the velocity is 3 m/s.

d The path of the object is illustrated below.



$$\text{Distance travelled} = 2 + 4 + 16$$
$$= 22 \text{ metres}$$

Example 5

A cricket ball projected vertically upwards experiences a gravitational acceleration of 9.8 m/s^2. If the initial speed of the cricket ball is 25 m/s, find:

a its speed after two second

b its height after two seconds

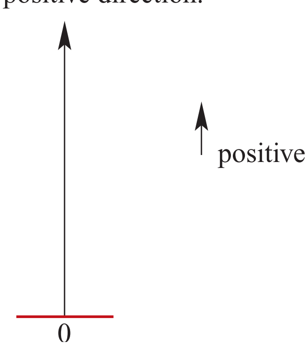
c the greatest height

d the time taken to come back to earth.

Solution

A frame of reference is required. The path of the cricket ball is considered as a vertical straight line with origin O at ground level. (The ball is considered to be projected from ground level.) Vertically up is taken as the positive direction.

Given this, $a = -9.8$ and $v(0) = 25$.



a $a = \dfrac{dv}{dt} = -9.8$

$v = \displaystyle\int \frac{dv}{dt}\, dt = \int -9.8\, dt = -9.8t + c$

When $t = 0$, $v = 25$ and therefore $c = 25$

i.e. $v = -9.8t + 25$

When $t = 2$, $v = -9.8 \times 2 + 25 = 5.4$

The speed of the cricket ball is 5.4 m/s after two seconds.

b $\frac{dx}{dt} = -9.8t + 25$

Therefore $x = \int (-9.8t + 25)\,dt = -4.9t^2 + 25t + d$

When $t = 0$, $x = 0$ and therefore $d = 0$

Hence $x = -4.9t^2 + 25t$

When $t = 2$, $x = 50 - 19.6 = 30.4$

After two seconds, the ball is 30.4 m above ground.

c The greatest height is reached when the object is instantaneously at rest.

i.e. when $v = -9.8t + 25 = 0$

which implies $t = \frac{25}{9.8}$

When $t = \frac{25}{9.8}$ $\quad x = 25 \times \frac{25}{9.8} - 4.9 \times \left(\frac{25}{9.8}\right)^2$

$= 31.89$ correct to two decimal places.

The greatest height reached is 31.89 m.

d The cricket ball reaches the ground again when $x = 0$

i.e. when $x = 25t - 4.9t^2 = 0$

$t(25 - 4.9t) = 0$

$\therefore \quad t = 0$ or $t = \frac{25}{4.9}$

$t = 0$ indicates the initial position of the ball.

$t = \frac{25}{4.9} \approx 5.1$ when the ball returns.

The ball returns to ground level after 5.1 seconds.

Example 6

A particle moving in a straight line has its acceleration in m/s^2 at time t seconds given by $\frac{d^2y}{dt^2} = \cos(\pi t)$. The initial velocity is 3 m/s and the initial position is given by $y = 6$. Find the position y m at time t seconds.

Solution

$$\frac{d^2y}{dt^2} = \cos(\pi t)$$

implies $\frac{dy}{dt} = \int \frac{d^2y}{dt^2}\,dt = \frac{1}{\pi}\sin(\pi t) + c$

When $t = 0$, $\frac{dy}{dt} = 3$. Thus $c = 3$

$$\therefore \quad \frac{dy}{dt} = \frac{1}{\pi}\sin(\pi t) + 3$$

Antidifferentiating again gives

$$y = \frac{-1}{\pi^2}\cos(\pi t) + 3t + d$$

When $t = 0$, $y = 6$

$$\therefore \quad 6 = \frac{-1}{\pi^2} + d \quad \text{and} \quad d = \frac{1}{\pi^2} + 6$$

$$\text{i.e.} \quad y = \frac{-1}{\pi^2}\cos(\pi t) + 3t + \frac{1}{\pi^2} + 6$$

Example 7

A particle travels in a line so that its velocity, v m/s, at time t seconds is given by

$v = 2\cos\left(\frac{1}{2}t - \frac{\pi}{4}\right), \quad t \geq 0.$

The initial position of the particle is $-2\sqrt{2}$ m from O.

a Determine:

- **i** the particle's initial velocity
- **ii** the maximum and minimum velocities
- **iii** for the interval, $0 \leq t \leq 4\pi$, the times when the particle is instantaneously at rest
- **iv** the period of the motion

Use this information to sketch the graph of velocity against time.

b Determine:

- **i** the particle's displacement at time t
- **ii** the particle's maximum and minimum displacement
- **iii** when the particle first passes through the origin
- **iv** the particle's velocity in terms of its displacement

c Determine, the particle's:

- **i** acceleration at time t
- **ii** maximum and minimum acceleration
- **iii** acceleration in terms of displacement
- **iv** acceleration in terms of velocity

d Use the information obtained in answering **a–c** to describe the motion of the particle.

Solution

a **i** $v = 2\cos\left(\frac{1}{2}t - \frac{\pi}{4}\right)$

At $t = 0$, $v = 2\cos\left(-\frac{\pi}{4}\right)$

$$= 2 \cdot \frac{\sqrt{2}}{2}$$

$$= \sqrt{2}$$

ii By inspection, $v_{max} = 2$ m/s $\quad v_{min} = -2$ m/s

iii $v = 0$ when $\cos\left(\frac{1}{2}t - \frac{\pi}{4}\right) = 0$

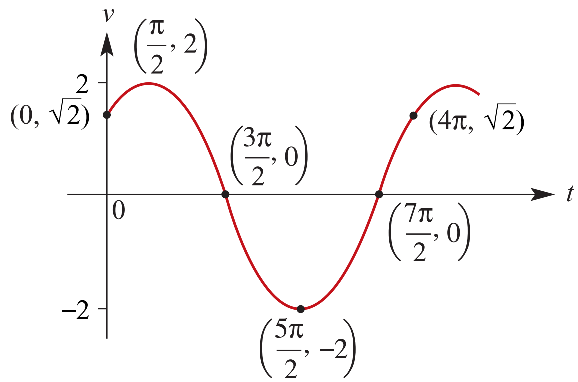
$$\therefore \quad \frac{1}{2}t - \frac{\pi}{4} = \frac{\pi}{2}, \frac{3\pi}{2}, \ldots$$

$$\therefore \quad \frac{1}{2}t = \frac{3\pi}{4}, \frac{7\pi}{4}, \ldots$$

$$\therefore \quad t = \frac{3\pi}{2}, \frac{7\pi}{2}, \ldots$$

The velocity is zero when $t = \frac{3\pi}{2}$ and $t = \frac{7\pi}{2}$ seconds for $0 \leq t \leq 4\pi$.

iv The period of $v = 2\cos\left(\frac{1}{2}t - \frac{\pi}{4}\right)$ is $\frac{2\pi}{\left(\frac{1}{2}\right)} = 4\pi$ seconds.



b **i** $x = \int \frac{dx}{dt}\,dt = \int 2\cos\left(\frac{1}{2}t - \frac{\pi}{4}\right)dt$

Let $u = \frac{1}{2}t - \frac{\pi}{4}$ and $\frac{du}{dt} = \frac{1}{2}$

Then
$$\begin{aligned} x &= 2\int 2\cos u\,\frac{du}{dt}\,dt \\ &= 4\int \cos u\,du \\ &= 4\sin u + c \\ &= 4\sin\left(\frac{1}{2}t - \frac{\pi}{4}\right) + c \end{aligned}$$

Now at $t = 0$, $x = -2\sqrt{2}$

Substituting

$$-2\sqrt{2} = 4\sin\left(\frac{-\pi}{4}\right) + c$$

$$\therefore \quad -2\sqrt{2} = 4 \cdot -\frac{\sqrt{2}}{2} + c$$

$$\therefore \quad c = 0$$

Hence $x = 4\sin\left(\frac{1}{2}t - \frac{\pi}{4}\right)$

ii By inspection, $x_{max} = 4$ m $\quad x_{min} = -4$ m

iii When the particle passes through the origin, $x = 0$ or $0 = 4\sin\left(\frac{1}{2}t - \frac{\pi}{4}\right)$

$$\begin{aligned} \therefore \quad \sin\left(\frac{1}{2}t - \frac{\pi}{4}\right) &= 0 \\ \therefore \quad \frac{1}{2}t - \frac{\pi}{4} &= 0, \pi, 2\pi, \ldots \\ \therefore \quad \frac{1}{2}t &= \frac{\pi}{4}, \frac{5\pi}{4}, \frac{9\pi}{4}, \ldots \\ \therefore \quad t &= \frac{\pi}{2}, \frac{5\pi}{2}, \frac{9\pi}{2}, \ldots \end{aligned}$$

Thus the particle first passes through the origin when $t = \frac{\pi}{2}$ seconds.

iv $v = 2\cos\left(\frac{1}{2}t - \frac{\pi}{4}\right)$ and $x = 4\ \sin\left(\frac{1}{2}t - \frac{\pi}{4}\right)$

Now $\cos^2\left(\frac{1}{2}t - \frac{\pi}{4}\right) + \sin^2\left(\frac{1}{2}t - \frac{\pi}{4}\right) = 1$

$$\therefore \quad \cos\left(\frac{1}{2}t - \frac{\pi}{4}\right) = \pm\sqrt{1 - \sin^2\left(\frac{1}{2}t - \frac{\pi}{4}\right)} = \pm\sqrt{1 - \frac{x^2}{16}}$$

$$= \pm\frac{1}{4}\sqrt{16 - x^2}$$

$$\therefore \quad v = 2\cos\left(\frac{1}{2}t - \frac{\pi}{4}\right)$$

$$= \pm 2 \times \tfrac{1}{4}\sqrt{16 - x^2}$$

$$= \pm\tfrac{1}{2}\sqrt{16 - x^2}$$

When $x = 0$, $t = \frac{\pi}{2}$ and $v = 2\cos\left(\frac{1}{2} \cdot \frac{\pi}{2} - \frac{\pi}{4}\right) = 2$ so, $v = \frac{1}{2}\sqrt{16 - x^2}$ for $-4 \leq x \leq 4$.

c **i** $a = \frac{dv}{dt} = \frac{d}{dt}\left[2\cos\left(\frac{1}{2}t - \frac{\pi}{4}\right)\right]$

$\therefore \quad a = -\sin\left(\frac{1}{2}t - \frac{\pi}{4}\right)$ (chain rule)

At $t = 0$, $a = -\sin\left(-\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}$

ii By inspection, $a_{max} = 1\ \text{m/s}^2$ $\quad a_{min} = -1\ \text{m/s}^2$

iii $a = -\sin\left(\frac{1}{2}t - \frac{\pi}{4}\right)$ and $x = 4\sin\left(\frac{1}{2}t - \frac{\pi}{4}\right)$

$\therefore \quad a = -\frac{x}{4}$

iv $a = -\sin\left(\frac{1}{2}t - \frac{\pi}{4}\right)$ and $v = 2\cos\left(\frac{1}{2}t - \frac{\pi}{4}\right)$

$$\therefore \quad a = -\left[\pm\sqrt{1 - \cos^2\left(\frac{1}{2}t - \frac{\pi}{4}\right)}\right]$$

$$= \mp\sqrt{1 - \frac{v^2}{4}}$$

$$\therefore \quad a = \mp\tfrac{1}{2}\sqrt{4 - v^2}$$

When $v = 0$, $t = \frac{\pi}{2}$ and $a = \frac{\sqrt{2}}{2}$ so, $a = \frac{1}{2}\sqrt{4 - v^2}$ for $-2 \leq v \leq 2$.

d Using the above information, it can be seen that the particle oscillates between positions ± 4 m from O, taking 4π seconds to repeat each cycle. The particle is at rest at times when it is at maximum or minimum displacement, i.e. at $t = \frac{3\pi}{2}, \frac{7\pi}{2}, \ldots$

The particle's acceleration oscillates between ± 1 m/s^2, while its velocity oscillates between ± 2 m/s.

Maximum and minimum acceleration occurs at positions of minimum and maximum displacement. This is where the particle is instantaneously at rest.

Exercise 10A

1 The position of an object travelling in a horizontal line is x metres from a point O on the line at time t seconds. The position is described by $x = 3t - t^2$, $t \geq 0$.

- **a** Find the position of the object at times $t = 0, 1, 2, 3, 4$ and illustrate the motion of the particle on a number line.
- **b** Find the displacement of the particle in the fifth second.
- **c** Find the average velocity in the first four seconds.
- **d** Find the relation between velocity v m/s and t.
- **e** Find the velocity of the particle when $t = 2.5$.
- **f** Find when and where the particle changes direction.
- **g** Find the distance travelled in the first four seconds.
- **h** Find the particle's average speed for the first four seconds.

2 A particle travels in a straight line through a fixed point O. Its distance, x metres, from O is given by $x = t^3 - 9t^2 + 24t$, $t \geq 0$, where t is the time in seconds after passing O. Find:

- **a** the values of t for which the velocity is instantaneously zero
- **b** the acceleration when $t = 5$
- **c** the average velocity of the particle during the first two seconds
- **d** the average speed of the particle during the first four seconds

3 A particle moves in a straight line so that its distance x m from a fixed point O on the line is given by $x = t(t - 3)^2$, where t is the time in seconds after passing O. Find:

- **a** the velocity of the particle after two seconds
- **b** the values of t when the particle is instantaneously at rest
- **c** the acceleration of the particle after four seconds

4 A particle moving in a straight line has position given by $x = 2t^3 - 4t^2 - 100$. Find the time(s) when the particle has zero velocity.

5 A particle moving in a straight line passes through a fixed point O. Its velocity, v m/s, t seconds after passing O is given by $v = 4 + 3t - t^2$. Find:

- **a** the maximum value of v
- **b** the distance of the particle from O when $t = 4$

6 A particle moves in a straight line so that, t seconds after passing through a fixed point O, its velocity, v m/s, is given by $v = 3t^2 - 30t + 72$. Find:

a the initial acceleration of the particle
b the values of t when the particle is instantaneously at rest
c the distance moved by the particle during the interval between these two values
d the total distance moved by the particle between $t = 0$ and $t = 7$

7 A particle moving in a straight line passes through a fixed point O, with a velocity of 8 m/s. Its acceleration, a m/s^2, t seconds after passing O is given by $a = 12 - 6t$. Find:

a the velocity of the particle when $t = 2$
b the displacement of the particle from O when $t = 2$

8 A particle moving in a straight line passes through a fixed point O on the line, with a velocity of 30 m/s. The acceleration, a m/s^2, of the particle, t seconds after passing O, is given by $a = 13 - 6t$. Find:

a the velocity of the particle three seconds after passing O
b the time taken to reach the maximum distance from O in the direction of the initial motion
c the value of this maximum distance

9 An object travels in a line such that its velocity v m/s at time t seconds is given by $v = \cos\left(\frac{1}{2}t\right)$, $t \in [0, 4\pi]$. Given that the initial position of the object is 0.5:

a find an expression for the position x of the object in terms of t
b sketch the position–time graph of the object, indicating clearly the values of t at which the object is instantaneously at rest
c find an expression for the acceleration a m/s^2 of the object in terms of t
d Find an expression (excluding t) for:
  i displacement in terms of acceleration
  ii displacement in terms of velocity
  iii velocity in terms of acceleration

10 A particle moves horizontally in a line so that its position, x m, from O at time t seconds is given by $x = t^3 - \dfrac{15}{2}t^2 + 12t + 10$. Find:

a when and where the particle has zero velocity
b the average velocity during the third second
c the velocity at $t = 2$
d the distance travelled in the first two seconds
e the closest the particle comes to O

11 An object moves in a line so that, at time t seconds, the acceleration $\ddot{x}$ m/s^2 is given by $\ddot{x} = 2\sin\frac{1}{2}t$. The initial velocity is 1 m/s. Find:

a the maximum velocity
b the time taken for the object to first reach the maximum velocity

12 An object is dropped down a well. It takes two seconds to reach the bottom. During its fall, the object travels under a gravitational acceleration of 9.8 m/s^2.

a Find an expression in terms of t for:

i the velocity v m/s **ii** the position x m, measured from the top of the well

b Find the depth of the well.

c At what speed does the object hit the bottom of the well?

13 From a balloon ascending with a velocity of 10 m/s a stone was dropped and reached the ground in 12 seconds. Given that the gravitational acceleration is 9.8 m/s^2, find:

a the height of the balloon when the stone was dropped

b the greatest height reached by the stone

14 An object moves in a line with acceleration $\ddot{x}$ m/s^2 given by $\ddot{x} = \dfrac{1}{(2t+3)^2}$. If the object starts from rest at the origin, find the position–time relationship.

15 A particle moves in a line with acceleration $\ddot{x}$ m/s^2 given by $\ddot{x} = \dfrac{2t}{(1+t^2)^2}$. If the initial velocity is 0.5 m/s find the distance travelled in the first $\sqrt{3}$ seconds.

16 An object moves in a line with velocity $\dot{x}$ m/s given by $\dot{x} = \dfrac{t}{1+t^2}$. The object starts from the origin. Find:

a the initial velocity

b the maximum velocity

c the distance travelled in the third second

d the position–time relationship

e the acceleration–time relationship

f the average acceleration over the third second

g the minimum acceleration

17 An object moves in a horizontal line so that the displacement, x m, from a fixed point at t seconds is given by $x = 2 + \sqrt{t+1}$. Find when the acceleration is -0.016 m/s^2.

18 A particle moves in a straight line so that the displacement, x metres, of the particle from a fixed origin at time t seconds is given by $x = 2\sin t + \cos t$, $t \geq 0$. Find the first value of t for which the particle is instantaneously at rest.

19 A particle moving in a straight line has its acceleration in m/s^2 at time t seconds given by $\dfrac{d^2x}{dt^2} = 8 - e^{-t}$. If the initial velocity is 3 m/s find the velocity when $t = 2$.

10.2 Constant acceleration

When considering motion of a particle due to a constant force, e.g. gravity, the acceleration is constant. There are a number of rules that can be established by considering the case where acceleration remains constant or uniform.

Given that $\dfrac{dv}{dt} = a$

by antidifferentiating

$$v = at + c \text{ where } c \text{ is the initial velocity.}$$

Using the symbol u for initial velocity

$$v = u + at \qquad \boxed{1}$$

Now given that $\dfrac{dx}{dt} = v$

by antidifferentiating a second time

$$x = ut + \frac{1}{2}at^2 + d, \text{ where } d \text{ is the initial position.}$$

If $s = x - d$ is considered as the change in position of the particle from its starting point, i.e. the particle's displacement from its initial position, then

$$s = ut + \frac{1}{2}at^2 \qquad \boxed{2}$$

Transforming the formula $v = u + at$ so that t is the subject gives

$$t = \frac{v - u}{a}$$

By substitution in

$$\begin{aligned} s &= ut + \frac{1}{2}at^2 \\ &= \frac{u(v-u)}{a} + \frac{a(v-u)^2}{2a^2} \end{aligned}$$

$$\begin{aligned} \therefore \qquad 2as &= 2u(v-u) + (v-u)^2 \\ &= 2uv - 2u^2 + v^2 - 2uv + u^2 \\ &= v^2 - u^2 \end{aligned}$$

i.e. $$v^2 = u^2 + 2as \qquad \boxed{3}$$

Also distance travelled = average velocity × time.

i.e. $$s = \tfrac{1}{2}(u + v)t \qquad \boxed{4}$$

These four formulas are very useful, but it must be remembered that they only apply when dealing with **constant** acceleration.

When approaching problems involving constant acceleration it is a good idea to list the quantities given, establish which quantity or quantities are required, and then use the appropriate formula. Ensure that all quantities are converted to compatible units.

Example 8

A body is moving in a straight line with uniform acceleration and an initial velocity of 12 m/s. After five seconds its velocity is 20 m/s. Find:

a the acceleration

b the distance travelled in this time

c the time taken to travel a distance of 200 m

Solution

Given $u = 12$, $v = 20$, $t = 5$

a Find a: using $v = u + at$

$$20 = 12 + 5a$$
$$a = 1.6$$

The acceleration is 1.6 m/s^2.

b Find s: using $s = ut + \frac{1}{2}at^2$

$$= 12(5) + \tfrac{1}{2}(1.6)5^2$$
$$= 80$$

The distance travelled is 80 m.

c Using the formula $s = ut + \frac{1}{2}at^2$ gives

$$200 = 12t + \tfrac{1}{2} \times 1.6 \times t^2$$
$$= 12t + \tfrac{4}{5}t^2$$
$$1000 = 60t + 4t^2$$
$$250 = 15t + t^2$$

i.e. $t^2 + 15t - 250 = 0$

$(t - 100)(t + 25) = 0$

$\therefore$ $t = 10$ or $t = -25$

As $t \geq 0$, $t = 10$ is the acceptable solution.

The body takes 10 seconds to travel a distance of 200 m.

Example 9

A body is moving in a straight line with uniform acceleration and an initial velocity of 12 m/s. If the body stops after 20 metres, find the acceleration of the body.

Solution

Use $v^2 = u^2 + 2as$ as time is not a required or known variable.

$u = 12$, $v = 0$, $s = 20$

$\therefore$ $0 = 144 + 2 \times a \times 20$

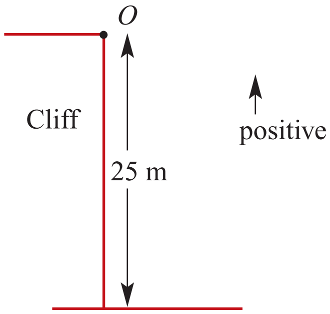
$\therefore$ $\dfrac{-144}{40} = a$

The acceleration is $\frac{-18}{5}$ m/s^2.

Example 10

A stone is thrown vertically upwards from the top of a cliff which is 25 m high. The velocity of projection of the stone is 22 m/s. Find the time it takes to reach the base of the cliff (give answer correct to two decimal places).

Solution



Take the origin at the top of the cliff.

$s = -25$, $u = 22$ and $a = -9.8$

Use $s = ut + \frac{1}{2}at^2$

$-25 = 22t + \frac{1}{2} \times -9.8 \times t^2$

$\therefore \quad 4.9t^2 - 22t - 25 = 0$

Using the quadratic formula

$$t = \frac{22 \pm \sqrt{22^2 - 4 \times -25 \times 4.9}}{2 \times 4.9}$$

$\therefore t = 5.429 \ldots$ or $t = -0.9396$

But $t \geq 0$. Therefore it takes 5.43 seconds for the stone to reach the bottom of the cliff.

Exercise 10B

Skillsheet TEACHER

1 A body with constant acceleration starts with velocity 15 m/s. At the end of the eleventh second its velocity is 48 m/s. What is its acceleration?

2 A car accelerates uniformly from 5 km/h to 41 km/h in 10 seconds. Express this acceleration in:

a km/h^2 **b** m/s^2

3 A body starts from a fixed point O with initial velocity −10 m/s and uniform acceleration 4 m/s^2. Find:

a the displacement of the particle from O after six seconds
b the velocity of the particle after six seconds
c the time when the velocity is zero
d the distance travelled in the first six seconds

4 **a** A stone is thrown vertically upwards from ground level at 21 m/s.
i What is its height above the ground after two seconds?
ii What is the maximum height reached by the stone?
b If the stone is thrown vertically upwards from a cliff 17.5 m high at 21 m/s:
i how long will it take to strike the ground at the base of the cliff?
ii what is the velocity of the stone when it hits the ground?

5 A basketball is thrown vertically upwards with a velocity of 14 m/s. Find:

a the time taken by the ball to reach its maximum height
b the greatest height reached by the ball
c the time taken for the ball to return to the point from which it is thrown

6 A car sliding on ice is decelerating at the rate of 0.1 m/s^2. Initially the car is travelling at 20 m/s. Find:

a the time taken before it comes to rest
b the distance travelled before it comes to rest

7 An object is dropped from a point 100 m above the ground. The acceleration due to gravity is 9.8 m/s^2. Find:

a the time taken by the object to reach the ground
b the velocity at which the object hits the ground

8 An object is projected vertically upwards from a point 50 m above ground level (acceleration due to gravity is 9.8 m/s^2). If the initial velocity is 10 m/s, find:

a the time taken by the object to reach the ground (give answer correct to two decimal places)
b the velocity at that point

9 A book is pushed across a table and is subjected to a retardation of 0.8 m/s^2 due to friction (retardation is acceleration opposite in direction to motion). If the initial speed of the book is 1 m/s, find:

a the time taken for the book to stop **b** the distance over which the book slides

10 A box is pushed across a bench and is subjected to a constant retardation, a m/s^2, due to friction. The initial speed of the box is 1.2 m/s and the box travels 3.2 m before stopping. Find:

a the value of a **b** the time taken by the box before it comes to rest

11 A particle travels in a straight line with a constant velocity of 4 m/s for 12 seconds. It is then subjected to a constant acceleration in the opposite direction for 20 seconds which returns the particle to its original position. Find:

a the acceleration of the particle
b the time the particle is travelling back towards its original position

12 A child slides from rest down a slide 4 m long. The child undergoes constant acceleration and reaches the end of the slide travelling at 2 m/s. Find:

a the time taken to go down the slide
b the acceleration which the child experiences

10.3 Velocity–time graphs

Velocity–time graphs are valuable for the consideration of rectilinear motion. These graphs present information about

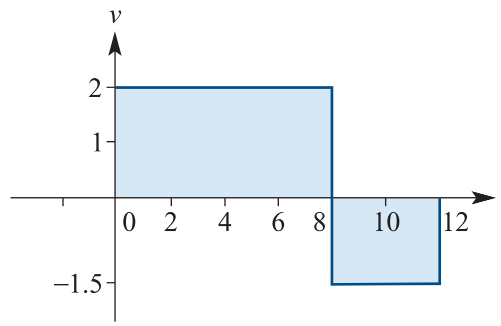
- acceleration (gradient)
- velocity (ordinates)
- displacement (signed area or definite integral)
- distance travelled (area ‘under’ the curve)

Example 11

A man walks east for eight seconds at 2 m/s and then west for four seconds at 1.5 m/s. Sketch the velocity–time graph for this journey and find the displacement from the start of the walk and the total distance travelled.

Solution

The velocity–time graph is as shown:



The distance travelled to the east $= 8 \times 2 = 16$ m

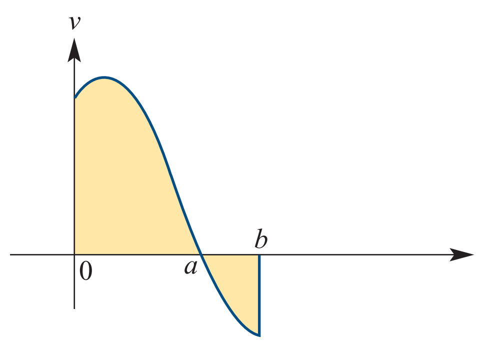
The distance travelled to the west $= 4 \times 1.5 = 6$ m

Displacement from the start $= 8 \times 2 + 4 \times (-1.5) = 10$ m (signed area)

Distance travelled $= 8 \times 2 + 4 \times 1.5 = 22$ m (total area)

Consider the motion of a particle moving in a straight line with its motion described by the velocity–time graph shown opposite.

For the interval of time $[0, a)$ the velocity is positive. The area shaded for $t \in [0, a)$ represents the distance travelled in the first a seconds. It is also the displacement from the start of the motion.



For the interval, $(a, b]$ the velocity is negative (the particle is travelling in the opposite direction) and the area between the graph and the t axis represents the distance travelled for that interval of time.

Using integral notation to describe the areas yields the following:

The distance travelled for the interval of time $[0, a]$ $= \int_0^a v(t)\,dt$

The distance travelled for the interval of time $[a, b]$ $= -\int_a^b v(t)\,dt$

The total distance travelled for the interval of time $[0, b]$ $= \int_0^a v(t)\,dt - \int_a^b v(t)\,dt$

The displacement of the particle for the interval of time $[a, b]$ $= \int_0^b v(t)\,dt$

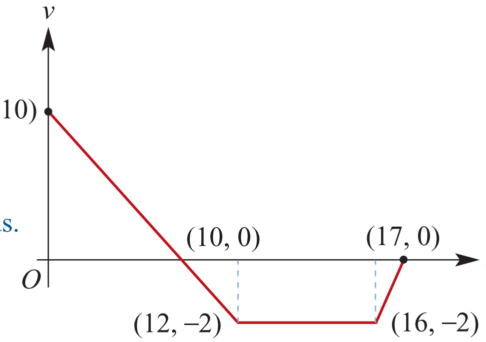
Example 12

The graph describes the motion of a particle.

a Describe the motion.

b Find the distance travelled.

Velocity is measured in m/s and time in seconds.



Solution

a The particle decelerates uniformly from an initial velocity of 10 m/s. After 10 seconds, it is instantaneously at rest before changing direction and continuing to decelerate uniformly for a further two seconds until its velocity reaches –2 m/s. It then continues to travel in the same direction with a uniform velocity of −2 m/s for a further four seconds. Finally, it accelerates uniformly until it comes to rest after 17 seconds.

b $\text{Distance travelled} = \frac{1}{2} \times 10 \times 10 + \frac{1}{2} \times 2 \times 2 + 4 \times 2 + \frac{1}{2} \times 1 \times 2$

$= 61 \text{ m}$

Example 13

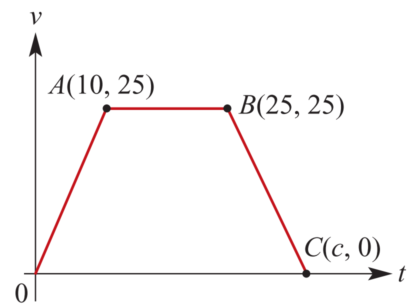
A car travels from rest for 10 seconds, with uniform acceleration, until it reaches a speed of 90 km/h. It then travels with this constant speed for 15 seconds and finally decelerates at a uniform 5 m/s^2 until it stops. Calculate the distance travelled from start to finish.

Solution

The speed given is first changed to standard units.

$$\begin{aligned} 90 \text{ km/h} &= 90\,000 \text{ m/h} \\ &= \frac{90\,000}{3600} \text{ m/s} \\ &= 25 \text{ m/s} \end{aligned}$$

Now sketch a velocity–time graph showing the given information.



The gradient of BC is -5 (deceleration).

$$\therefore \quad \text{gradient} = \frac{25}{25 - c} = -5$$

$$\therefore \quad 25 = -5(25 - c)$$

$$\therefore \quad 25 = -125 + 5c$$

$$\therefore \quad c = 30$$

Now calculate the distance travelled, using the area of trapezium $OABC$.

$$\begin{aligned} \text{Area} &= \tfrac{1}{2} \times 25(30 + 15) \\ &= 562.5 \end{aligned}$$

The total distance travelled in 562.5 metres.

Example 14

An object travels in a line. It decelerates uniformly from 0 m/s^2 to −5 m/s^2 in 15 seconds. If the initial velocity was 24 m/s, find:

a the velocity at the end of the 15 seconds

b the distance travelled in the 15 seconds.

Solution

a It is appropriate here to start with the acceleration–time graph.

The figure shows the uniform deceleration from 0 m/s^2 to −5 m/s^2 in 15 seconds.

Algebraically $a = mt + c$

but $\quad m = \frac{-5}{15} = -\frac{1}{3}$ and $c = 0$

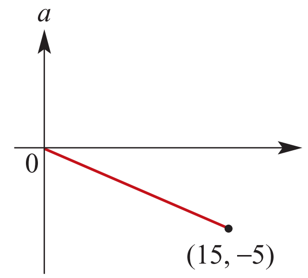
Hence $\quad a = -\frac{1}{3}t$

and $\quad v = -\frac{1}{6}t^2 + d$

But, at $t = 0 \;\; v = 24$

and therefore $d = 24$

and $\quad v = -\frac{1}{6}t^2 + 24$



To sketch the velocity–time graph, first find the t-axis intercepts:

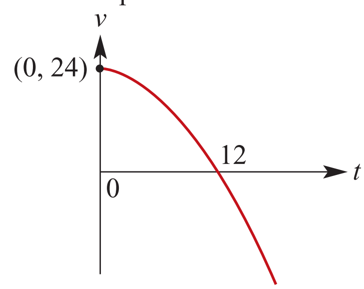
$$-\tfrac{1}{6}t^2 + 24 = 0$$

$$\therefore \quad t^2 = 144$$

$$\therefore \quad t = 12, \text{ since } t \geq 0$$

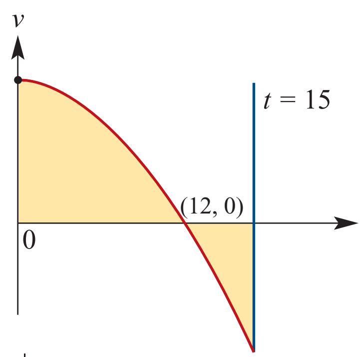
Now, at $t = 15$, $v = -\frac{1}{6} \times 15^2 + 24$

$$= -13.5$$



So, velocity at 15 seconds is −13.5 m/s.

b The distance travelled is given by the area of the shaded region.



$$\text{Area} = \int_0^{12} -\tfrac{1}{6}t^2 + 24\,dt + \left|\int_{12}^{15} \left(-\tfrac{1}{6}t^2 + 24\right) dt\right|$$

$$= 192 + |-19.5|$$

$$= 211.5$$

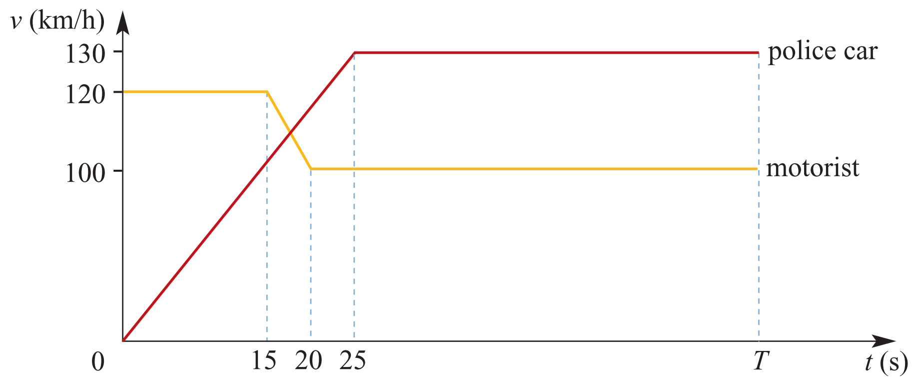
The distance travelled in 15 seconds is 211.5 metres.

Example 15

A motorist is travelling at a constant speed of 120 km/h when he passes a stationary police car. He continues at that speed for another 15 seconds before uniformly decelerating to 100 km/h in five seconds and then continues with constant velocity. The police car takes off after the motorist the instant it passes. It accelerates uniformly for 25 seconds by which time it has reached 130 km/h. It continues at that speed until it catches up to the motorist. When does the police car catch up to the motorist and how far has it travelled in that time?

Solution

We start by representing the information on a velocity–time graph.



The distance travelled by the motorist and the police car when they meet will be the same so the areas under each of the velocity–time graphs will be equal. This fact can be used to find T, the time taken for the police car to catch up to the motorist. For the motorist, the distance travelled in metres after T seconds

$$\begin{aligned} &= \left(120 \times 15 + \tfrac{1}{2}(120 + 100) \times 5 + 100(T - 20)\right)\tfrac{5}{18} \\ &= (1800 + 550 + 100T - 2000)\tfrac{5}{18} \\ &= (100T + 350)\tfrac{5}{18} \end{aligned}$$

Note: The factor $\frac{5}{18}$ changes velocities from km/h to m/s.

Police car: $\left(\tfrac{1}{2} \times 25 \times 130 + 130(T - 25)\right) \times \tfrac{5}{18} = (1625 + 130T - 3250)\tfrac{5}{18}$

$$= (130T - 1625)\tfrac{5}{18}$$

When the police car catches the motorist, $100T + 350 = 130T - 1625$

$$30T = 1975$$

$$T = \frac{395}{6}$$

The police car catches the motorist after 65.83 seconds.

$$\therefore \quad \text{distance} = (100T + 350)\tfrac{5}{18} \text{ where } T = \tfrac{395}{6}$$

$$= \frac{52\,000}{27} \text{ m}$$

$$\text{distance} \approx 1.926 \text{ km}$$

The police car has travelled 1.926 km when it catches the motorist.

Exercise 10C

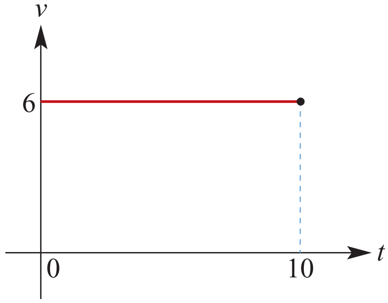
Skillsheet TEACHER

1 Each of the following graphs describes the motion of a particle. For each of them:

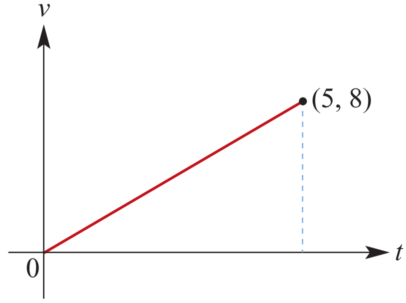
i describe the motion **ii** find the distance travelled

Velocity is measured in m/s and time in seconds.

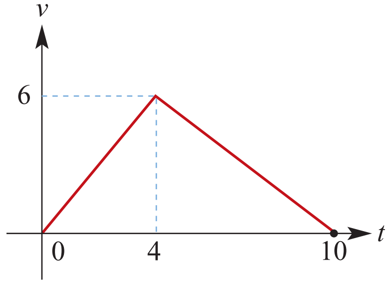
a



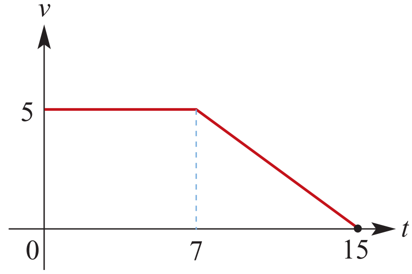
b



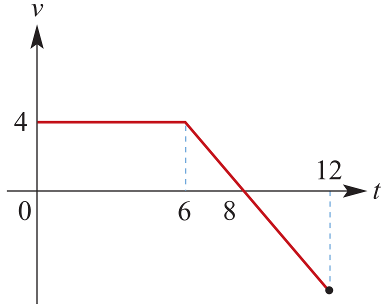
c



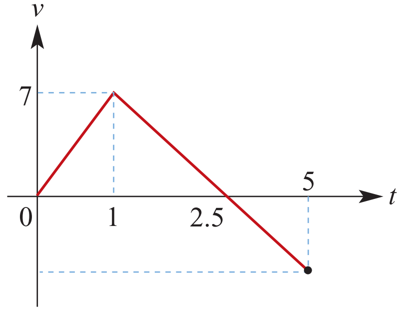
d



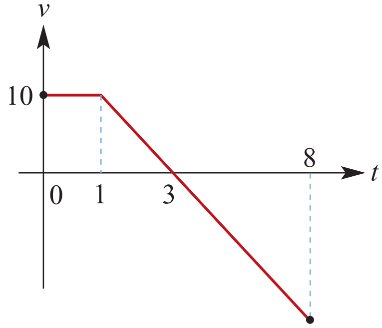
e



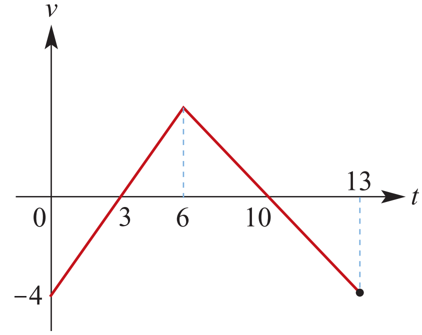
f



g



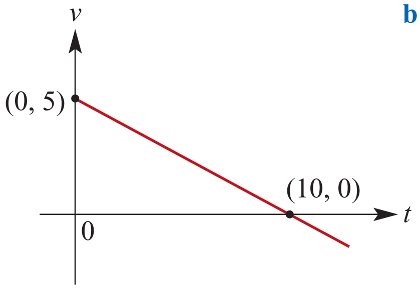
h



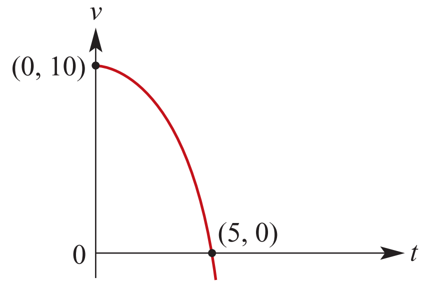
2 In each of the following velocity–time graphs the object starts from the origin and moves in a line. In each case, find the relationship between time and:

i velocity **ii** acceleration **iii** position

a

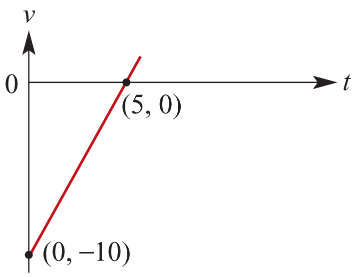


b

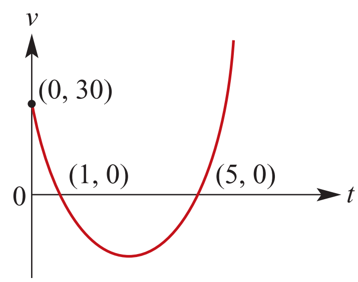


Given that $v = at^2 + b$, a and b constant

c

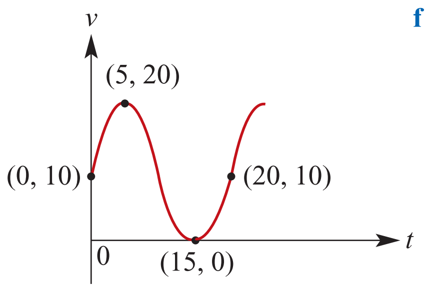


d



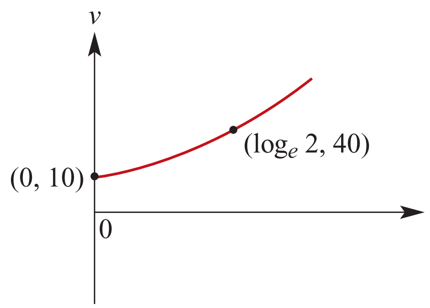
This is a curve of the form $v = at^2 + bt + c$

e



This is a curve of the form $v = a \sin bt + c$

f



This is a curve of the form $v = ae^{bt}$

3 A particle moves in a straight line with a constant velocity of 20 m/s for 10 seconds. It is then subjected to a constant acceleration of 5 m/s^2 in the opposite direction for T seconds at which time the particle is back to its original position.

a Sketch the velocity–time graph representing the motion.

b Find how long it takes the particle to return to its original position.

4 An object travels in a line starting from rest. It accelerates uniformly for three seconds until it reaches a speed of 14 m/s. It then travels at this speed for 10 seconds. Finally, it decelerates uniformly to rest in 4 seconds. Sketch a velocity–time graph and find the total distance travelled.

5 Two tram stops, A and B, are 500 metres apart. A tram starts from A and travels with acceleration a m/s^2 to a certain point. It then decelerates at $4a$ m/s^2 until it stops at B. The total time taken is two minutes. Sketch a velocity–time graph. Find the value of a and the maximum speed reached by the tram.

6 The maximum rate at which a bus can accelerate or decelerate is 2 m/s^2. It has a maximum speed of 60 km/h. Find the shortest time the bus can take to travel between two bus stops 1 km apart on a straight stretch of road.

7 A car being tested on a straight level road starts from rest and accelerates uniformly to 90 km/h. It travels at this speed for a time then comes to rest with a uniform retardation of 1.25 m/s^2. The total distance travelled is 525 metres and the total time is 36 seconds. Find the time taken in the acceleration phase and how far the car travels at 90 km/h.

8 Cars A and B are stationary on a straight road, standing side by side. Car A moves off with an acceleration of 1 m/s^2, which is maintained for twenty seconds, after which it moves at constant speed. Twenty seconds after A starts, B sets off with constant acceleration of 2 m/s^2 until it draws level with A. Find the time taken and the distance travelled by B to catch A.

9 An object is travelling in a line with an initial velocity of 6 m/s. The deceleration changes uniformly from -1 m/s^2 to -3 m/s^2 over one second. If this deceleration continues until the object comes to rest, find:

a the time taken **b** the distance travelled.

10 A stationary police motorcycle is passed by a car travelling at 72 km/h. The motorcycle starts in pursuit three seconds later. Moving with constant acceleration for a distance of 300 metres, it reaches a speed of 108 km/h, which it maintains. Find the time it takes the motorcyclist to catch the car.

11 Two cars A and B, each moving with constant acceleration, are travelling in the same direction along the parallel lanes of a divided road. When A passes B, the speeds are 64 and 48 km/h respectively. Three minutes later, B passes A, travelling at 96 km/h. Find:

a the distance travelled by A and B at this instant (since they first passed) and the speed of A

b the instant at which both are moving with the same speed, and the distance between them at this time

12 A particle, starting from rest, falls vertically with acceleration $\ddot{y}$ m/s^2 at time t seconds, given by $\ddot{y} = ke^{-t}$ where $k < 0$.

a Find the velocity–time relationship and sketch the velocity time graph.

b Briefly describe the motion.

10.4 Differential equations of the form $v = f(x)$ and $a = f(v)$

When information about the motion of an object is given in one of the forms $v = f(x)$ or $a = f(v)$, techniques used for solving differential equations can be applied to obtain other information about the motion.

Example 16

The velocity of a particle moving along a straight line is inversely proportional to its position. If the particle is initially 1 m from point O and 2 m from point O after one second:

a find an expression for the position of the particle t seconds after starting the motion

b find an expression for the velocity, v, of the particle in terms of t.

Solution

a The information can be written as

$$v = \frac{k}{x}, \text{ for } k \in R^+ \quad x(0) = 1 \text{ and } x(1) = 2$$

$v = \frac{k}{x}$ can be written as $\frac{dx}{dt} = \frac{k}{x}$

Thus $\frac{dt}{dx} = \frac{x}{k}$

and $t = \int \frac{x}{k}\,dx = \frac{x^2}{2k} + c$

Since $x(0) = 1$, $0 = \frac{1}{2k} + c$ $\boxed{1}$

Since $x(1) = 2$, $1 = \frac{4}{2k} + c$ $\boxed{2}$

Subtracting $\boxed{1}$ from $\boxed{2}$ yields

$$1 = \frac{3}{2k} \text{ and therefore } k = \frac{3}{2}$$

Substituting in $\boxed{1}$ yields $c = \frac{-1}{2k} = \frac{-1}{3}$

Therefore $t = \frac{x^2}{3} - \frac{1}{3}$

$\therefore \quad x^2 = 3t + 1$

and $\quad x = \pm\sqrt{3t+1}$

But when $t = 0$, $x = 1$ and therefore $x = \sqrt{3t+1}$

b If $\quad x = \sqrt{3t+1}$

then $\quad v = \frac{dx}{dt} = 3 \times \frac{1}{2} \times \frac{1}{\sqrt{3t+1}}$

i.e. $\quad v = \frac{3}{2\sqrt{3t+1}}$

Example 17

The acceleration a, of an object moving along a line is given by $a = -(v+1)^2$ where v is the velocity of the object at time t. Also $v(0) = 10$ and $x(0) = 0$ (x is the position of the particle at time t). Find:

a an expression for the velocity of the object in terms of t

b an expression for the position of the object in terms of t

Solution

a $a = -(v+1)^2$ can be written as $\dfrac{dv}{dt} = -(v+1)^2$

and in the form $\dfrac{dt}{dv} = \dfrac{-1}{(v+1)^2}$

Therefore $\quad t = -\displaystyle\int \frac{1}{(v+1)^2}\,dv$

i.e. $\quad t = \dfrac{1}{v+1} + c$

Since $v(0) = 10$, $\quad 0 = \dfrac{1}{11} + c$

$\therefore \quad c = \dfrac{-1}{11}$ and $t = \dfrac{1}{v+1} - \dfrac{1}{11}$

This can be rearranged to show

$$v = \frac{10 - 11t}{1 + 11t} = -1 + \frac{11}{11t + 1}$$

b If $\quad v = -1 + \dfrac{11}{11t+1}$

i.e. $\quad \dfrac{dx}{dt} = -1 + \dfrac{11}{11t+1}$

$\therefore \quad x = \displaystyle\int -1 + \frac{11}{11t+1}\,dt$

Hence $x = -t + \log_e|1 + 11t| + c$

Since $x(0) = 0$, $c = 0$ and $x = \log_e|1 + 11t| - t$

Example 18

A body moves in a straight line with an initial velocity of 25 m/s so that its acceleration a m/s^2 is given by $a = -k(50 - v)$, where k is a positive constant, and v m/s is its velocity at a given instant. Find v in terms of t and sketch the velocity–time graph for the motion. (The motion stops when the body is instantaneously at rest for the first time.)

Solution

For $a = -k(50 - v)$

$$\frac{dv}{dt} = -k(50 - v)$$

$$\therefore \quad \frac{dt}{dv} = \frac{1}{-k(50 - v)}$$

$$\therefore \quad t = -\frac{1}{k}\int \frac{1}{50 - v}\,dv$$

$$\therefore \quad t = -\frac{1}{k}(-\log_e|50 - v|) + c$$

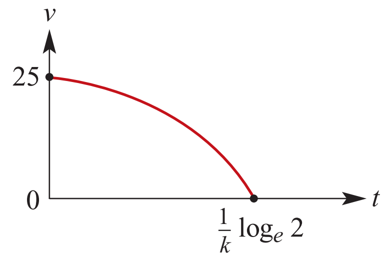
When $t = 0$, $v = 25$

Therefore $c = -\frac{1}{k}\log_e(25)$

and $t = \frac{1}{k}\log_e\left(\frac{50 - v}{25}\right)$ if $v < 50$

$$\therefore \quad e^{kt} = \frac{50 - v}{25}$$

and $v = 50 - 25e^{kt}$



Exercise 10D

1 A particle moves in a line so that the velocity, $\dot{x}$ m/s, is given by $\dot{x} = \frac{1}{2x - 4}$, $x > 2$. If $x = 3$ when $t = 0$, find:

a the position at 24 seconds **b** the distance travelled in the first 24 seconds

2 A particle moves in a line so that the velocity v m/s is given by $v = 1 + e^{-2x}$.

a Find the position, x m, in terms of time t seconds ($t \geq 0$), given that $x = 0$ when $t = 0$.
b Hence find a when $t = \log_e 5$.

3 A particle moves in a line so that the acceleration, a m/s^2, when the velocity is v m/s is given by $a = 3 + v$. If, initially, the object is at rest at the origin find:

a v in term of t **b** a in terms of t **c** x in terms of t

4 An object falls from rest such that the acceleration a m/s^2 is given by: $a = \text{g} - kv$, $k > 0$. Find:

a an expression for the velocity, v m/s, at time t seconds
b the terminal velocity, i.e. the limiting velocity as $t \to \infty$

5 A body is projected along a horizontal surface. Its deceleration is $0.3(v^2 + 1)$ where v m/s is the velocity of the body at time t seconds. If the initial velocity is $\sqrt{3}$ m/s find:

a an expression for v in terms of t
b an expression for x m, the displacement of the body from its original position in terms of t

6 The velocity v m/s and the acceleration a m/s^2 of a body t seconds after it is dropped from rest are related by $a = \frac{450 - v}{50}$ for $v < 450$.
Express v in terms of t.

7 The brakes are applied in a car travelling in a straight line. The acceleration a m/s^2 of the car is given by $a = -0.4\sqrt{225 - v^2}$. If the initial velocity of the car was 12 m/s, find an expression for v the velocity of the car in terms of t, the time after the brakes were first applied.

8 An object moves in a straight line such that its velocity is directly proportional to x m, its displacement from a fixed point O on the line. The object starts 5 m from O with a velocity of 2 m/s.

a Express x in terms of t, where t is the time after the object leaves the point O.
b Find the position of the object after 10 seconds.

9 The velocity v m/s and the acceleration a m/s^2 of a body t seconds after it is dropped from rest are related by the equation $a = \frac{1}{50}(500 - v)$, $0 \leq v < 500$.

a Express t in terms of v. **b** Express v in terms of t.

10 A particle is travelling in a horizontal straight line. The initial velocity of the particle is u and the acceleration is given by $-k(2u - v)$ where v is the velocity of the particle at any instant and k is a positive constant. Find the time taken for the particle to come to rest.

11 A boat is moving at 8 m/s. When the engine stops its acceleration is given by $\frac{dv}{dt} = \frac{-1}{5}v$.
Express v in terms of t and find the velocity when $t = 4$.

12 A particle, initially at a point O, slows down under the influence of an acceleration a m/s^2, such that $a = -kv^2$, where v m/s is the velocity of the particle at any instant. Its initial velocity is 30 m/s and its initial acceleration is -20 m/s^2. Find:

a its velocity at time t seconds **b** its position, relative to a point O, where $t = 10$

10.5 Other expressions for acceleration

In the earlier sections of this chapter acceleration has been written as $\frac{dv}{dt}$ and $\frac{d^2x}{dt^2}$. In this section two further expressions for acceleration are given. Using the chain rule:

$$a = \frac{dv}{dt} = \frac{dv}{dx}\frac{dx}{dt} = \frac{dv}{dx}v$$

$$\text{i.e.} \quad a = v\frac{dv}{dx}$$

Also

$$\frac{d\left(\frac{1}{2}v^2\right)}{dx} = \frac{d}{dv}\left(\frac{1}{2}v^2\right)\frac{dv}{dx} = v\frac{dv}{dx} = a$$

i.e. $a = \dfrac{d\left(\frac{1}{2}v^2\right)}{dx}$

If $a = f(v)$ and initial conditions are given in terms of x and v, use the form $a = v\dfrac{dv}{dx}$.

If $a = f(x)$ and initial conditions are given in terms of v and x, use the form $a = \dfrac{d\left(\frac{1}{2}v^2\right)}{dx}$.

Example 19

An object travels in a line so that the velocity, v m/s, is given by $v^2 = 4 - x^2$. Find the acceleration at $x = 1$.

Solution

Now $\quad v^2 = 4 - x^2$

$\therefore \quad \frac{1}{2}v^2 = 2 - \frac{1}{2}x^2$

Now $\quad a = \dfrac{d}{dx}\left(\frac{1}{2}v^2\right)$

$\therefore \quad a = -x$

So at $x = 1$, $a = -1$

Acceleration at $x = 1$ is -1 m/s^2.

Example 20

An object moves in a line so that the acceleration, $\ddot{x}$ m/s^2, is given by $\ddot{x} = 1 + v$. Its velocity at the origin is 1 m/s. Find the position of the object when its velocity is 2 m/s.

Solution

$$\ddot{x} = 1 + v$$

Since the initial conditions involve v and x, it is appropriate to use $\ddot{x} = v\dfrac{dv}{dx}$.

Now $\quad v\dfrac{dv}{dx} = 1 + v$

$\therefore \quad \dfrac{dv}{dx} = \dfrac{1+v}{v}$

and $\quad \dfrac{dx}{dv} = \dfrac{v}{1+v}$

$\therefore \quad x = \displaystyle\int \frac{v}{1+v}\,dv$

$\quad\quad = \displaystyle\int 1 - \frac{1}{1+v}\,dv$

$\therefore \quad x = v - \log_e|1+v| + c$

Now $\quad v = 1$ when $x = 0$

$\therefore \quad 0 = 1 - \log_e 2 + c$

$c = \log_e 2 - 1$

$x = v - \log_e |1 + v| + \log_e 2 - 1$

$\therefore \quad x = v + \log_e \left(\frac{2}{1+v}\right) - 1$ as $v > 0$

Now, when $\quad v = 2$

$x = 2 + \log_e \frac{2}{3} - 1$

$\therefore \quad x = 1 + \log_e \frac{2}{3}$

≈ 0.59

So, when velocity is 2 m/s, position is 0.59 m.

Example 21

An object is moving in a straight line. The acceleration a m/s^2 is described by the relation $a = -\sqrt{x}$ where x is the position of the particle with respect to an origin O. Find a relation between v and x which describes the motion, given that $v = 2$ m/s when the particle is at the origin.

Solution

Using $\quad a = \frac{d}{dx}\left(\frac{1}{2}v^2\right)$

gives $\quad \frac{d}{dx}\left(\frac{1}{2}v^2\right) = -\sqrt{x} = -x^{\frac{1}{2}}$

$\therefore \quad \frac{1}{2}v^2 = \frac{-2x^{\frac{3}{2}}}{3} + c$

When $\quad x = 0, v = 2$

Therefore $\quad c = 2$ and $\frac{1}{2}v^2 = 2 - \frac{2x^{\frac{3}{2}}}{3}$

$\therefore \quad v^2 = \frac{4}{3}\left(3 - x^{\frac{3}{2}}\right)$

Example 22

An object falls from a hovering helicopter over the ocean 1000 m above sea level. Find the velocity of the object when it hits the water:

a neglecting air resistance

b assuming air resistance is $0.2v^2$.

Solution

a An appropriate starting point is $\ddot{y} = -9.8$.

Now, since the initial conditions give $v = 0$ at $y = 1000$ and v is required at $y = 0$, use:

$$\ddot{y} = \frac{d}{dy}\left(\tfrac{1}{2}v^2\right)$$

Thus $$\frac{d}{dy}\left(\tfrac{1}{2}v^2\right) = -9.8$$

$$\therefore \quad \tfrac{1}{2}v^2 = -9.8y + c$$

but using $v = 0$ at $y = 1000$

$$0 = -9.8 \times 1000 + c$$

$$\therefore \quad c = 9800$$

$$\therefore \quad \tfrac{1}{2}v^2 = -9.8y + 9800$$

$$\therefore \quad v^2 = -19.6y + 19\,600$$

Since the object is falling, $v < 0$

and therefore $v = -\sqrt{19\,600 - 19.6y}$

At sea level, $y = 0$ and therefore

$$v = -\sqrt{19\,600}$$
$$= -140$$

Hence velocity at sea level is -140 m/s (or a speed of 504 km/h).

b In this case $\ddot{y} = -9.8 + 0.2v^2$

$$\therefore \quad \ddot{y} = \frac{v^2 - 49}{5}$$

Because of the initial conditions given, use:

$$\ddot{y} = v\frac{dv}{dy}$$

$$\therefore \quad v\frac{dv}{dy} = \frac{v^2 - 49}{5}$$

$$\frac{dv}{dy} = \frac{v^2 - 49}{5v}$$

$$\frac{dy}{dv} = \frac{5v}{v^2 - 49}$$

$$y = \int \frac{5v}{v^2 - 49}\,dv$$

$$= \frac{5}{2}\int \frac{2v}{v^2 - 49}\,dv$$

$$\therefore \quad y = \tfrac{5}{2}\log_e|v^2 - 49| + c$$

Now, when $v = 0$, $y = 1000$

$$\therefore \quad 1000 = \tfrac{5}{2}\log_e(49) + c$$

$$\therefore \quad c = 1000 - \tfrac{5}{2}\log_e 49$$

$$\begin{aligned}
\text{and} \qquad y &= \tfrac{5}{2}\log_e|49 - v^2| + 1000 - \tfrac{5}{2}\log_e 49 \\
&= \tfrac{5}{2}(\log_e|49 - v^2| - \log_e 49) + 1000 \\
&= \tfrac{5}{2}\log_e\left|\frac{49 - v^2}{49}\right| + 1000 \\
&= \tfrac{5}{2}\log_e\left|1 - \frac{v^2}{49}\right| + 1000 \\
\therefore \qquad y - 1000 &= \tfrac{5}{2}\log_e\left|1 - \frac{v^2}{49}\right| \\
\therefore \qquad \tfrac{2}{5}(y - 1000) &= \log_e\left|1 - \frac{v^2}{49}\right|
\end{aligned}$$

Assume $-7 < v < 7$ then

$$\begin{aligned}
e^{2(y-1000)/5} &= 1 - \frac{v^2}{49} \\
\therefore \qquad \frac{v^2}{49} &= 1 - e^{2(y-1000)/5} \\
\therefore \qquad v^2 &= 49\left(1 - e^{2(y-1000)/5}\right)
\end{aligned}$$

but the object is falling and thus $v < 0$

$$v = -7\sqrt{1 - e^{2(y-1000)/5}}$$

At sea level, $y = 0$

and $\quad v = -7\sqrt{1 - e^{-400}}$

The object has a velocity of approximately -7 m/s at sea level or a speed of approximately 25.2 km/h.

If $v < -7$ then $v^2 = 49(1 + e^{2(y-1000)/5})$ and the initial conditions are not satisfied.

Exercise 10E

1 In each of the following examples, a particle moves in a horizontal line so that, at time t seconds, the displacement is x m from O, the velocity is v m/s and the acceleration is a m/s^2.

- **a** If $a = -x$ and $v = 0$ at $x = 4$, find v at $x = 0$.
- **b** If $a = 2 - v$ and $v = 0$ when $t = 0$, find t when $v = -2$.
- **c** If $a = 2 - v$ and $v = 0$ when $x = 0$ find x when $v = -2$.

2 The motion of a particle is in a horizontal line so that, at time t seconds, the displacement is x m, the velocity is v m/s and the acceleration is a m/s^2.

- **a** If $a = -v^3$ and $v = 1$ when $x = 0$, find v in terms of x.
- **b** If $v = x + 1$ and $x = 0$ when $t = 0$, find:
  - **i** x in terms of t
  - **ii** a in terms of t
  - **iii** a in terms of v

3 An object is projected vertically upwards from the ground with an initial velocity of 100 m/s. Assuming that the acceleration, a m/s^2, is given by $a = -g - 0.2v^2$, find x in terms of v. Hence find the maximum height reached.

4 A particle moves in a horizontal line so that the velocity, v m/s, is given by $v = 2\sqrt{1 - x^2}$. Find:

a the position, x m, in terms of time t seconds, given that, when $t = 0$, $x = 1$

b the acceleration, a m/s^2, in terms of x

5 The acceleration, a m/s^2, of an object travelling in a line, for which $v = 0$ when $x = 0$ and $t = 0$, is given by:

a $a = \dfrac{1}{1+t}$ **b** $a = \dfrac{1}{1+x}, x > -1$ **c** $a = \dfrac{1}{1+v}$

Solve for v in each case.

6 A particle moves in a straight line from a position of rest at a fixed origin O. Its velocity is v when its displacement from O is x. If its acceleration is $(2 + x)^{-2}$, find v in terms of x.

7 A particle moves in a straight line, and, at time t its displacement from a fixed origin is x and its speed v.

a If its acceleration is $1 + 2x$ and $v = 2$ when $x = 0$, find v when $x = 2$.

b If its acceleration is $2 - v$ and $v = 0$ when $x = 0$, find the position at which $v = 1$.

8 A particle is projected vertically upwards. The speed of projection is 50 m/s. The acceleration of the particle a m/s^2 is given by $a = -\frac{1}{5}(v^2 + 50)$ where v m/s is the velocity of the particle when it is x m above the point of projection. Find:

a the height reached by the particle **b** the time taken to reach this highest point

Chapter summary

- If the position of an object at time t is denoted by $x(t)$ then:
  - $\dot{x}(t)$ is the velocity at time t
  - $\ddot{x}(t)$ is the acceleration at time t
  - the magnitude of $\dot{x}(t)$ is the speed at time t
- Velocity–time graphs can be used to find the position of the object and the distance travelled.
- The average velocity over $[t_1, t_2]$ is $\dfrac{x(t_2) - x(t_1)}{t_2 - t_1}$
- Acceleration $\dfrac{d^2x}{dt^2} = \dfrac{dv}{dt} = v\dfrac{dv}{dx} = \dfrac{d}{dx}\left(\tfrac{1}{2}v^2\right)$

Multiple-choice questions

1 A particle moves in a straight line so that its position x cm from a fixed point O at time t seconds ($t \geq 0$) is given by $x = t^3 - 9t^2 + 24t - 1$. The position (in cm) of the particle at $t = 3$ is:

A 17 **B** 16 **C** 24 **D** −17 **E** 8

2 A particle moves in a straight line so that its position x cm from a fixed point O at time t seconds ($t \geq 0$) is given by $x = t^3 - 9t^2 + 24t - 1$. The average speed (in cm/s) of the particle in the first two seconds is:

A 0 **B** −12 **C** 10 **D** −10 **E** 9.5

3 A body is projected up from the ground with a velocity of 30 m/s. Its acceleration due to gravity is -10 m/s^2. The body's velocity in m/s at $t = 2$ is:

A 10 **B** −10 **C** 0 **D** 20 **E** −20

4 A car accelerating uniformly from rest reaches a speed of 50 km/h in five seconds. The car's acceleration during the five seconds is:

A 10 km/s^2 **B** 10 m/s^2 **C** 2.78 m/s^2 **D** $\frac{25}{9}$ m/s^2 **E** $\frac{25}{3}$ m/s^2

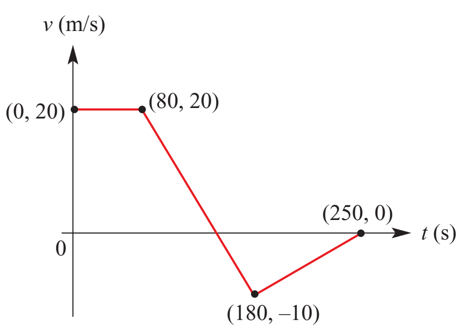
5 A particle moves in a straight line so at time t, $t \geq 0$, its velocity v is given by $v = 5 - \frac{2}{t+2}$. The initial acceleration of the particle is:

A 0 **B** $\frac{1}{2}$ **C** 1 **D** 2 **E** 4

6 The velocity–time graph shown describes the motion of a particle. The time (in seconds) when the velocity of the particle is first zero is closest to:

A 0 **B** 125 **C** 147

D 150 **E** 250



Review

7 The displacement, x metres, from the origin of a particle travelling in a straight line is given by $x = 2t^3 - 10t^2 - 44t + 112$. In the interval $0 \le t \le 10$ the number of times the particle passes through the origin is:

A 0 **B** 1 **C** 2 **D** 3 **E** 4

8 The acceleration a m/s^2 of an object whose displacement is x metres from the origin is given by $a = -x$, $-\sqrt{3} \le x \le \sqrt{3}$. If the object starts from the origin with a velocity of $\sqrt{3}$ m/s, then its velocity v when displacement is x metres from the origin is give by:

A $-\sqrt{3 - x^2}$ **B** $\sqrt{3 - x^2}$ **C** $\pm\sqrt{3 - x^2}$

D $-\sqrt{x^2 - 3}$ **E** $\sqrt{x^2 - 3}$

9 The displacement, x metres, from the origin of a particle travelling in a straight line is given by $x = 2 - 2\cos\left(\frac{3\pi}{4}t - \frac{\pi}{2}\right)$. The velocity (in m/s) at time $t = \frac{8}{3}$ seconds is:

A $-\frac{3\pi}{2}$ **B** $-\frac{3\pi}{4}$ **C** 0 **D** $\frac{3\pi}{4}$ **E** $\frac{3\pi}{2}$

10 An object starting at the origin has a velocity given by $v = 10\sin(\pi t)$. The distance the object travels in the first 1.6 seconds of its motion, correct to two decimal places, is:

A 1.60 **B** 2.20 **C** 4.17 **D** 6.37 **E** 10.53

Short-answer questions (technology-free)

1 The position, x metres, at time t seconds ($t \ge 0$) of a particle moving in a straight line is given by $x = t^2 - 7t + 10$. Find:

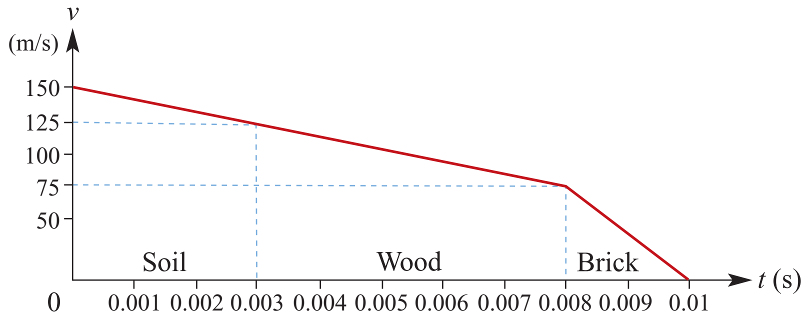
a when its velocity equals zero
b its acceleration at this time
c the distance travelled in the first five seconds
d when and where its velocity is -2m/s.

2 A body moves in a straight line so that its acceleration (a m/s^2) after time t seconds ($t \ge 0$) is given by $a = 2t - 3$. If the initial position of the body is 2 m to the right of a point O and its velocity is 3 m/s, find the particle's position and velocity after 10 seconds.

3 Two tram stops are 800 m apart. A tram starts at rest from the first stop and accelerates at a constant acceleration of a m/s^2 for a certain time and then decelerates at a constant rate of $2a$ m/s^2, before coming to rest at the second stop. The time taken to travel between the stops is 1 min 40 seconds. Find:

a the maximum speed reached by the tram in km/h
b the time at which the brakes are applied
c the value of a.

4 The velocity–time graph shows the journey of a bullet fired into the wall of a practice range made up of three successive layers of soil, wood and brick.



Calculate:

a the deceleration of the bullet as it passes through the soil

b the thickness of the layer of soil

c the deceleration of the bullet as it passes through the wood

d the thickness of the layer of wood

e the deceleraton of the bullet passing through the brick

f the depth penetrated by the bullet into the layer of brick.

5 A helicopter climbs vertically from the top of a 110-metre tall building, so that its height in metres above the ground after t seconds is given by $h = 110 + 55t - 5.5t^2$. Calculate:

a the average velocity of the helicopter from $t = 0$ to $t = 2$

b its instantaneous velocity at time t

c its instantaneous velocity at time $t = 1$

d the time at which the helicopter's velocity is zero

e the maximum height reached above the ground.

6 A particle moves in a straight line so that its displacement, x metres, from a point O on that line, after t seconds, is given by $x = \sqrt{9 - t^2}$, $0 \leq t < 3$.

a When is the displacement $\sqrt{5}$?

b Find expressions for the velocity and acceleration of the particle at time t.

c Find the maximum magnitude of the displacement from O.

d When is the velocity zero?

7 A particle moving in a straight line passes through a fixed point O, with a velocity of 8 m/s. Its acceleration, a m/s^2, t seconds after passing O, is given by $a = 12 – 6t$. Find:

a the velocity of the particle when $t = 2$

b the displacement of the particle from O when $t = 2$.

8 A particle travels at 12 m/s for five seconds. It then accelerates uniformly for the next eight seconds to a velocity of x m/s, and then decelerates uniformly to rest during the next three seconds. Sketch a velocity–time graph.

Given that the total distance travelled is 218 m, calculate:

a the value of x **b** the average velocity.

Review

9 A ball is thrown vertically upwards from ground level with an initial velocity of 35 m/s. (g m/s^2 is the acceleration due to gravity.) Find:

a the velocity in terms of g and direction of motion of the ball after:

i three seconds ii five seconds

b the total distance, in terms of g, travelled by the ball when it reaches the ground again

c the velocity with which the ball strikes the ground.

10 A car is uniformly accelerated from rest at a set of traffic lights until it reaches a speed of 10 m/s in five seconds. It then continues to move at the same constant speed of 10 m/s for six seconds before the car's brakes uniformly retard it at 5 m/s^2 until it comes to rest at a second set of traffic lights. Draw a velocity–time graph of the car's journey and calculate the distance between the two sets of traffic lights.

11 A particle moves in a straight line so that its position x relative to a fixed point O in the line at any time t is given by $x = 4\log_e(t - 1)$, $t \geq 2$. Find expressions for the velocity and acceleration at any time t.

12 A golf ball is putted across a level putting green with an initial velocity of 8 m/s. Owing to friction, the velocity decreases at the rate of 2 m/s^2. How far will the golf ball roll?

13 A missile is fired vertically upwards from a point on the ground, level with the foot of a tower 64 m high. The missile is level with the top of the tower 0.8 seconds after being fired. (g m/s^2 is the acceleration due to gravity.) Find in terms of g:

a the initial velocity of the missile

b the time taken to reach its greatest height

c the greatest height

d the length of time, for which the missile is higher than the top of the tower.

Extended-response questions

CAS CALCULATOR

1 A stone initially at rest is released and falls vertically. Its velocity v m/s at time t seconds satisfies $5\dfrac{dv}{dt} + v = 50$.

a Find the acceleration of the particle when $t = 0$.

b Find v in terms of t.

c i Sketch the graph of v against t.

ii Find the value of t for which $v = 47.5$. (Give answer correct to two decimal places.)

d Let x m be the distance fallen after t seconds.

i Find x in terms of t. ii Sketch the graph of x against t ($t \geq 0$).

iii After how many seconds has the stone fallen eight metres? (Give answer correct to two decimal places.)

CAS CALCULATOR

2 A particle is moving along a straight line and, t seconds after it passes a point O on the line, its velocity is v m/s, where $v = A - \log_e(t + B)$ where A and B are positive constants.

a If $A = 1$ and $B = 0.5$:

i sketch the graph of v against t

ii find the position of the particle when $t = 3$ (correct to two decimal places)

iii find the distance travelled by the particle in the three seconds after passing O (correct to two decimal places).

b If when $t = 10$ the acceleration of the particle is $\frac{-1}{20}$ m/s^2 and the particle comes to rest when $t = 100$, find the exact value of B and the value of A correct to two decimal places.

CAS CALCULATOR

3 The velocity v km/h of a train which moves along a straight track from station A, from which it starts at rest, to station B at which it next stops, is given by $v = kt[1 - \sin(\pi t)]$, where t hours is the time measured from when the train left A and k is a positive constant.

a Find the time that the train takes to travel from A to B.

b **i** Find an expression for the acceleration at time t.

ii Find the interval of time for which the velocity is increasing. (Give answer correct to two decimal places.)

c Given that the distance from A to B is 20 km, find to three significant figures, the value of k.

4 In a tall building, two lifts simultaneously pass the 40th floor, each travelling downwards at 24 m/s. One lift immediately slows down with a constant retardation of $\frac{6}{7}$ m/s^2. The other continues for six seconds at 24 m/s and then slows down with a retardation of $\frac{1}{3}(t - 6)$ m/s^2, where t seconds is the time that has elapsed since passing the 40th floor. Find the difference between the heights of the lifts when both have come to rest.

CAS CALCULATOR

5 A particle A moves along a horizontal line so that its displacement, x m, from a fixed point O, t seconds after the motion has begun, is given by $x = 28 + 4t - 5t^2 - t^3$.

a Find:

i the velocity of A in terms of t

ii the acceleration of A in terms of t

iii the value of t for which the velocity is zero (correct to two decimal places)

iv the times when the particle is 28 m to the right of O (correct to two decimal places)

v the time when the particle is 28 m to the left of O (correct to two decimal places).

b A second particle B moves along the same line as A. It starts from O at the same time that X begins to move. The initial velocity of B is 2 m/s and its acceleration at time t is $(2 - 6t)$ m/s^2.

i Find the position of B at time t.

ii Find the time at which A and B collide.

iii At the time of collision are they going in the same direction?

6 A particle moves in a straight line. At time t seconds its displacement x cm from a fixed point O in the line is given by $x = 5\cos\left(\frac{\pi}{4}t + \frac{\pi}{3}\right)$.

a Find:

i the velocity in terms of t

ii the acceleration in terms of t.

b Find:

i the velocity in terms of x

ii the acceleration in terms of x.

Review

c Find the speed of the particle when $x = -2.5$ correct to one decimal place.

d Find the acceleration when $t = 0$ correct to two decimal places.

e Find:

i the maximum distance of the particle from O

ii the maximum speed of the particle

iii the maximum magnitude of acceleration for the particle.

7 The motion of a bullet through a special shield is modelled by the equation $a = -30(v + 110)^2$, $v \geq 0$ where a m/s^2 is its acceleration and v m/s its velocity t seconds after impact. When $t = 0$, $v = 300$.

CAS CALCULATOR

a Find v in terms of t.

b Sketch the graph of v against t.

c Let x m be the penetration into the shield at time t seconds.

i Find x in terms of t

ii Find x in terms of v.

iii Find how far the bullet penetrates the shield before coming to rest.

d Another model for the bullet's motion is $a = -30(v^2 + 11\,000)$, $v \geq 0$. Given that $t = 0$, $v = 300$:

i find t in terms of v

ii find v in terms of t

iii sketch the graph of v against t

iv find the distance travelled by the bullet in the first 0.0001 seconds after impact.

8 A motorist is travelling at 25 m/s along a straight road and passes a policeman on a motorcycle. Four seconds after the motorist passes the policeman, the policeman starts in pursuit. The policeman's motion for the first six seconds is described by

$$v(t) = \left(\frac{-3}{10}\right)\left(t^3 - 21t^2 + \frac{364}{3}t - \frac{1281}{6}\right) \quad 4 \leq t \leq 10$$

where $v(t)$ is his speed in metres per second t seconds after the motorist has passed.
At the end of six seconds he reaches a speed of v m/s which he maintains until he overtakes the motorist.

a Find the value of v.

b **i** Find $\dfrac{dv}{dt}$ for $4 \leq t \leq 10$.

ii Find the time when the policeman's acceleration is a maximum.

c On the same set of axes sketch the velocity–time graphs for the motorist and the policeman.

d **i** How far has the policeman travelled when he reaches his maximum speed at $t = 10$?

ii Write down an expression for the distance travelled by the policeman for $t \geq 0$.

e For what value of t does the policeman draw level with the motorist? (Give answer correct to two decimal places.)

9 Two cyclists, A and B, pass a starting post together (but at different velocities) and race along a straight road. They are able to pass each other. At time t hours after they pass the post their velocities (in km/h) are given by:

$$V_A = \begin{cases} 9 - t^2 & 0 \le t \le 3 \\ 2t - 6 & t > 3 \end{cases}$$

$$V_B = 8 \quad t \ge 0.$$

- **a** On the one set of axes, draw the velocity–time graphs of the two cyclists.
- **b** Find the times at which the two cyclists have the same velocity.
- **c** Find the time in hours, correct to one decimal place, when:
  - **i** A passes B
  - **ii** B passes A.

10 Two particles, P and Q, move along the same straight line path and can overtake one another. Their velocities are $V_P = 2 - t + \frac{1}{4}t^2$ and $V_Q = \frac{3}{4} + \frac{1}{2}t$ respectively at time t, for $t \ge 0$.

- **a** **i** Find the times when the velocities of P and Q are the same.
  - **ii** On the same diagram sketch velocity–time graphs to represent the motion of P and the motion of Q.
- **b** If the particles start from the same point at time $t = 0$:
  - **i** find the time when P and Q next meet again (correct to one decimal place)
  - **ii** state the times during which P is further than Q from the starting point (correct to one decimal place).

11 Annabelle and Cuthbert are ants on a picnic table. Annabelle falls off the edge of a table at point X. She falls 1.2 m to the ground. (Assume $g = 9.8$ throughout this question.)

- **a** Assuming that Annabelle's acceleration down is g m/s^2 find:
  - **i** Annabelle's velocity when she hits the ground, correct to two decimal places
  - **ii** the time it takes, correct to two decimal places, for Annabelle to hit the ground.
- **b** Assume now that Annabelle's acceleration is slowed by air resistance and is given by $(g - t)$ m/s^2 where t is the time in seconds after leaving the table.
  - **i** Find Annabelle's velocity v m/s at time t.
  - **ii** Find Annabelle's position x m from X at time t.
  - **iii** Find the time in seconds, correct to two decimal places, when Annabelle hits the ground.
- **c** When Cuthbert reaches the edge of the table he observes Annabelle groaning on the ground below. He decides that action must be taken and fashions a parachute from a small piece of potato chip. He jumps from the table and his acceleration is $\frac{g}{2}$ m/s^2 down.
  - **i** Find an expression for x, the distance in metres Cuthbert is from the ground at time t.
  - **ii** Unfortunately Annabelle is very dizzy and on seeing Cuthbert coming down jumps vertically with joy. Her initial velocity is 1.4 m/s up and her acceleration is g m/s^2 down. She jumps 0.45 seconds after Cuthbert leaves the top of the table.
    How far above the ground, to the nearest cm, do the two ants collide?

12 On a straight road a car starts from rest with an acceleration of 2 m/s^2 and travels until it reaches a velocity of 6 m/s. The car then travels with constant velocity for 10 seconds before the brakes cause a deceleration of $(v + 2)$ m/s^2 until it comes to rest, where v m/s is the velocity of the car.

- **a** For how long is the car accelerating?
- **b** Find an expression for the velocity v of the car in terms of t, the time in seconds after it starts.
- **c** Find the total time taken for the motion of the car, to the nearest tenth of a second.
- **d** Draw a velocity–time graph of the motion.
- **e** Find the total distance travelled by the car to the nearest tenth of a metre.

13 A particle is first observed at time $t = 0$ and its position at this point is taken as its initial position. The particle moves in a straight line so that its velocity, v, at time t is given by:

$$v = \begin{cases} 3 - (t - 1)^2 & \text{for} \quad 0 \le t \le 2 \\ 6 - 2t & \text{for} \quad t > 2 \end{cases}$$

- **a** Draw the velocity time graph for $t \ge 0$
- **b** Find the distance travelled by the particle from its initial position until it first comes to rest.
- **c** If the particle returns to its original position at $t = T$, calculate T correct to two decimal places.

CHAPTER

11

Revision

Revision of Chapters 6 to 10

11.1 Multiple-choice questions

1 The graph of the function $f(x) = \dfrac{x^2 + x + 2}{x}$ has asymptotes:

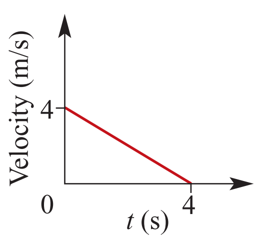
A $y = x$ and $y = x^2 + x + 2$
B $y = x$ and $y = x + 1$
C $x = 0$ and $y = x^2 + x + 2$
D $x = 0$ and $y = x + 1$
E $y = \dfrac{2}{x}$ and $y = x + 1$

2 One solution to the differential equation $\dfrac{d^2y}{dx^2} = 2\cos x + 1$ is:

A $-4\cos x + x$
B $2\sin x + x + 1$
C $-\dfrac{1}{4}\cos 2x + \dfrac{x^2}{2} + x$
D $-2\cos x + \dfrac{x^2}{2} + x$
E $2\cos x + \dfrac{x^2}{2} + x$

3 The graph shows an object which, in four seconds, covers a distance of:

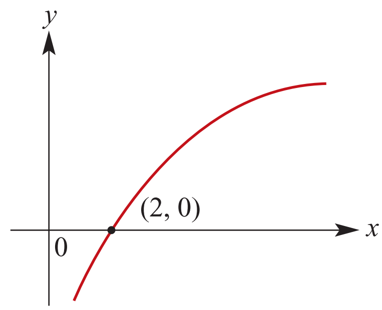
A 1 m **B** 8 m **C** 16 m
D −8 m **E** 4 m



4 The equation of the curve that passes through the point (2, 3) and whose tangent at each point (a, b) is perpendicular to the tangent of $y = 2x^3$ at $(a, 2a^3)$, can be found by using the differential equation:

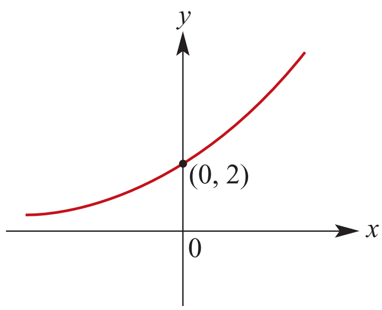
A $\dfrac{dy}{dx} = 2x^3$
B $\dfrac{dy}{dx} = -\dfrac{1}{6x^2}$
C $\dfrac{dy}{dx} = -6x^2$
D $\dfrac{dy}{dx} = \dfrac{2}{x} + c$
E $\dfrac{dy}{dx} = -\dfrac{1}{2x^3}$

5 The graph of $y = f(x)$ is shown here.

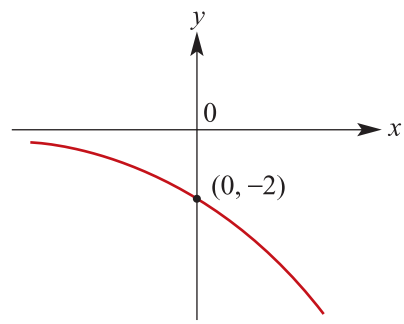


Which one of the following best represents the graph of $y = \dfrac{1}{f(x)}$?

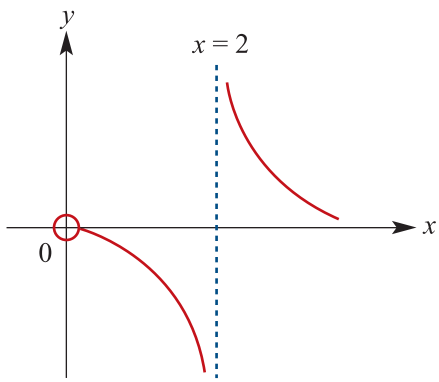
A



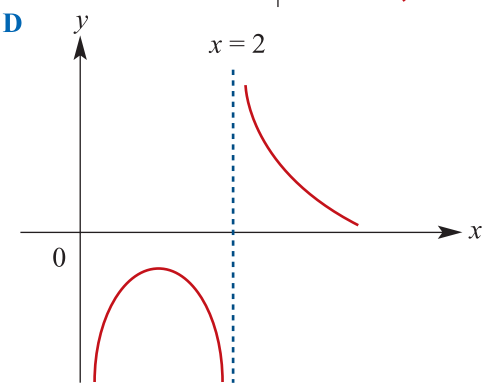
B



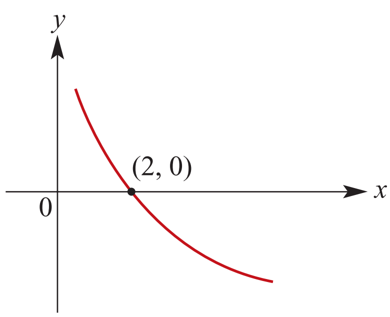
C



D



E

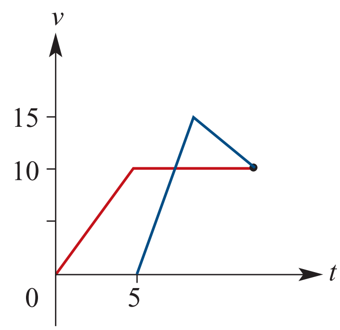


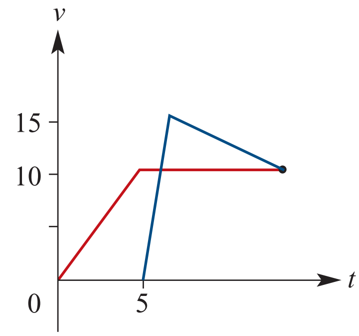
6 The equation of a curve which passes through the point (1, 1) and whose gradient at any point is twice the reciprocal of the x-coordinate can be found by solving the differential equation with the given boundary condition:

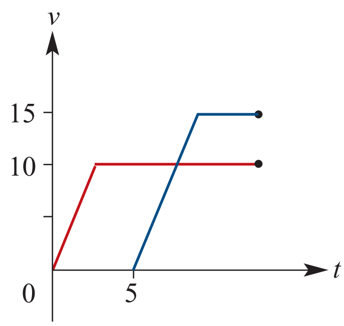
A $x\dfrac{dy}{dx} = 2, x(1) = 1$ **B** $\dfrac{d^2y}{dx^2} = \dfrac{x}{2}, x(1) = 1$ **C** $y\dfrac{dy}{dx} = 2, x(1) = 1$

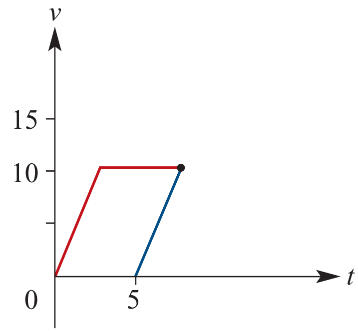
D $\dfrac{dy}{dx} = x$ **E** $\dfrac{1}{2}\dfrac{dy}{dx} = x$

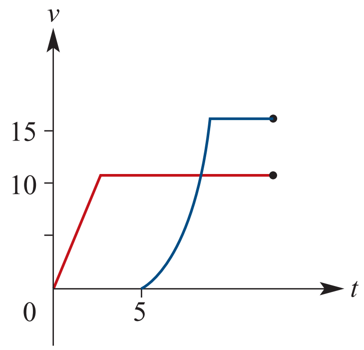
7 Car P leaves a garage, accelerates at a constant rate to a speed of 10 m/s and continues at that speed. Car Q leaves the garage five seconds later, accelerates at the same rate as car P to a speed of 15 m/s and continues at that speed until it hits the back of car P. Which one of the following pairs of graphs represents the motion of these cars?

A


B


C


D


E


8 A container initially holds 20 litres (L) of water. A salt solution of constant concentration 3 g/L is poured into the container at a rate of 2 L/min. The mixture is kept uniform by stirring and flows out at the rate of 2 L/min. If Q g is the amount of salt in the container t minutes after pouring begins, then Q satisfies the equation:

A $\dfrac{dQ}{dt} = \dfrac{Q}{10}$ **B** $\dfrac{dQ}{dt} = Q$ **C** $\dfrac{dQ}{dt} = 6 - \dfrac{Q}{10}$

D $\dfrac{dQ}{dt} = 6 - \dfrac{Q}{10 + t}$ **E** $\dfrac{dQ}{dt} = 6 - \dfrac{Q}{20}$

Revision

9 If $\dfrac{dy}{dx} = 2 - x + \dfrac{1}{x^3}$ then

A $y = 2x - \dfrac{x^2}{2} + \dfrac{1}{2}x^2 + c$

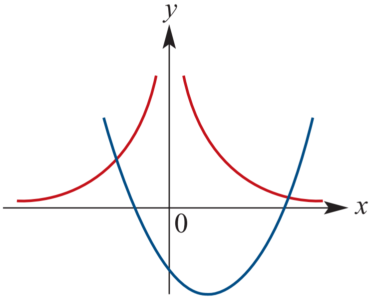
B $y = -1 - \dfrac{3}{x^4} + c$

C $y = 2x - \dfrac{x^2}{2} - \dfrac{1}{2x^2} + c$

D $y = -\dfrac{x^2}{2} - \dfrac{3}{x^4} + c$

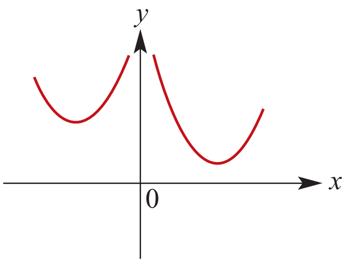
E $y = -1 - \dfrac{1}{2x^2}$

10 The graphs of $y = f(x)$ and $y = g(x)$ are shown here.

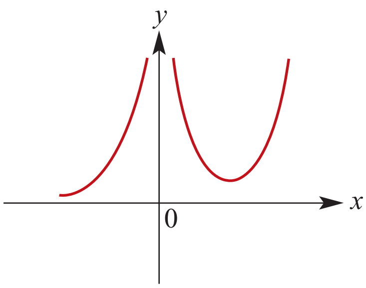


Which of the following (A, B, C, D or E) best represents the graph of $y = f(x) + g(x)$?

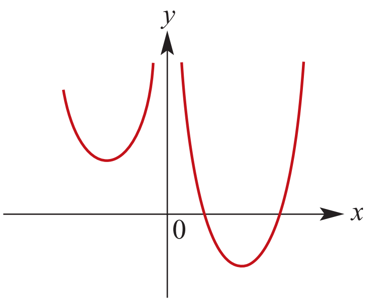
A



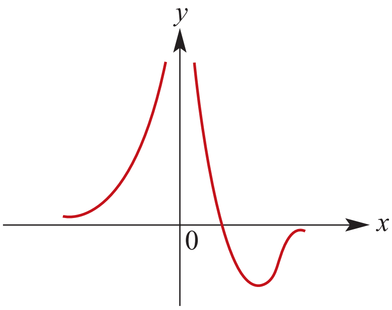
B



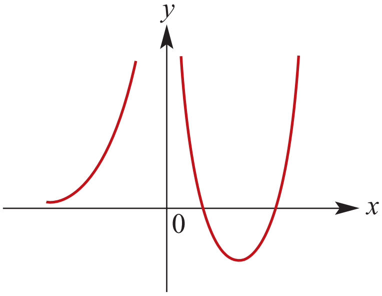
C



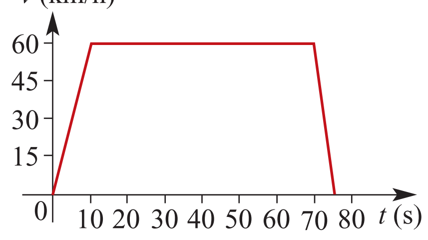
D

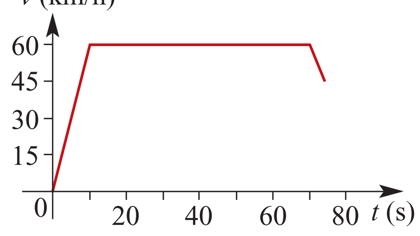


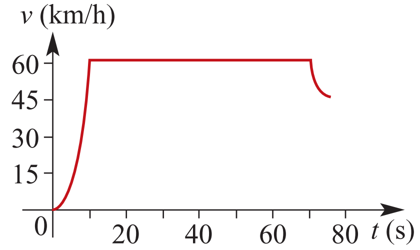
E

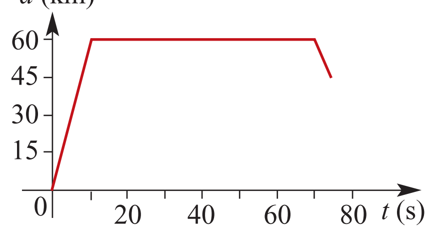


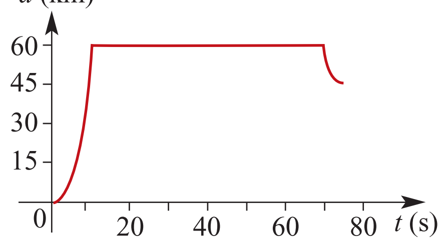
11 A car starts from rest and accelerates for 10 seconds at a constant rate until it reaches a speed of 60 km/h. It travels at constant speed for one minute and then decelerates for five seconds at constant rate until it reaches a speed of 45 km/h. Which one of the following best represents the car's journey?

A


B


C


D


E


12 The equation of the particular member of the family of curves defined by $\frac{dy}{dx} = 3x^2 + 1$, which passes through the point (1, 3), is:

A $y = 6x$ **B** $y = x^3 + x^2 + 1$ **C** $y = x^3 + x + 1$
D $y = x^3 + x + 3$ **E** $y = \frac{x^3}{3} + x$

13 One solution of the differential equation $\frac{d^2y}{dx^2} = e^{3x}$ is:

A $3e^{3x}$ **B** $\frac{1}{3}e^{3x}$ **C** $\frac{1}{3}e^{3x} + x$
D $9e^{3x} + x$ **E** $\frac{1}{9}e^{3x} + x$

14 A body initially travelling at 12 m/s is subject to a constant deceleration of 4 m/s^2. The time taken to come to rest (t seconds) and the distance travelled before it comes to rest (s metres) is:

A $t = 3, s = 24$ **B** $t = 3, s = 18$ **C** $t = 3, s = 8$
D $t = 4, s = 18$ **E** $t = 4, s = 8$

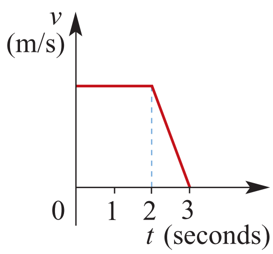
15 If $y = 1 - \sin(\cos^{-1} x)$ then $\dfrac{dy}{dx}$ equals:

A $\dfrac{x}{\sqrt{1-x^2}}$ **B** $-x$ **C** $\cos\sqrt{1-x^2}$

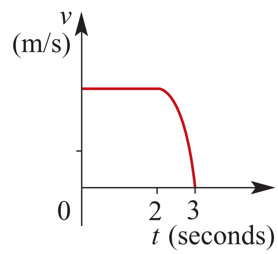
D $-\cos\sqrt{1-x^2}$ **E** $-\cos(\cos^{-1} x)$

16 A bead moves along a straight wire with a constant velocity for two seconds and then its speed decreases at a constant rate to zero. The velocity–time graph illustrating this could be:

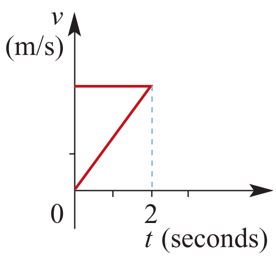
A



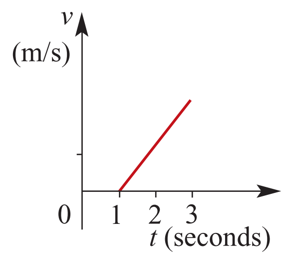
B



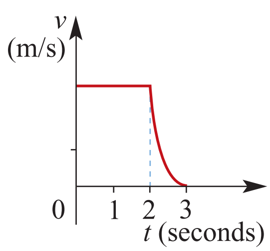
C



D



E



17 If $x = 2\sin^2 y$, then $\dfrac{dy}{dx}$ equals

A $4\sin y$ **B** $\frac{1}{2}\operatorname{cosec} 2y$ **C** $4\sqrt{\dfrac{x}{2}}$

D $2\sqrt{2x}$ **E** $\frac{1}{2}\sin^{-1} 2y$

18 The rate of decay of a radioactive substance is proportional to the amount, x, of the substance present at any time given by the differential equation $\dfrac{dx}{dt} = -kx$ where k is a constant. If initially $x = 20$, and $x = 5$ when $t = 20$, the time for x to decay to 2, correct to two decimal places is:

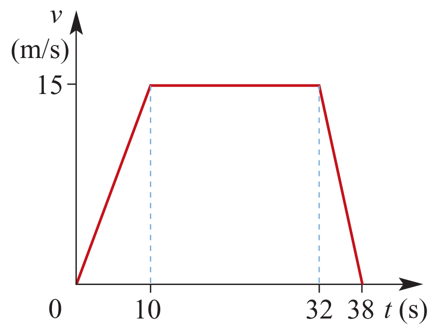
A 22.33 **B** 10.98 **C** 50 **D** 30.22 **E** 33.22

19 $\displaystyle\int_0^{\frac{\pi}{3}} \tan^2 x \sec^2 x \, dx$ equals:

A $\sqrt{3}$ **B** 3 **C** $\dfrac{\pi^3}{81}$ **D** $\dfrac{\pi^2}{9}$ **E** none of these

20 The velocity–time graph shows the motion of a tram between two stops. The distance between the stops in metres is:

A 300 **B** 360 **C** 405
D 450 **E** 570



21 If $\ddot{y} = e^x + e^{-2x}$ given that $y = 0$ and $\dot{y} = \frac{1}{2}$ when $x = 0$, then:

A $y = e^x + \frac{1}{4}e^{-2x} - \frac{5}{4}$ **B** $y = e^x + e^{-2x} - \frac{1}{2}$ **C** $y = e^x + e^{-2x}$
D $y = e^x + e^{-2x} + \frac{1}{2}$ **E** $y = e^x + e^{-2x} + \frac{5}{4}x - \frac{5}{4}$

22 If $\dfrac{dy}{dx} = 2y + 1$ and $y = 3$ when $x = 0$, then:

A $y = \dfrac{7e^{2x} - 1}{2}$ **B** $y = \dfrac{1}{2}\log_e(2x + 1)$ **C** $y = y^2 + y + 1$
D $y = e^{2x}$ **E** $y = \dfrac{2e^{2x} + 1}{7}$

23 A rock falls from the top of a cliff 45 metres high ($g = -10$ m/s^2). The rock's speed just before it hits the ground, in m/s, is:

A 5 **B** 10 **C** 20 **D** 30 **E** 40

24 The velocity, v metres per second, of a particle at time t seconds is given by the equation $v = t - t^2$, $t \geq 30$. The acceleration at time $t = 5$, in m/s^2, is:

A −20 **B** −9 **C** 11 **D** 1 **E** 9

25 $\displaystyle\int_0^{\sqrt{3}} \frac{2x + 3}{9 + x^2}\,dx$ is closest to:

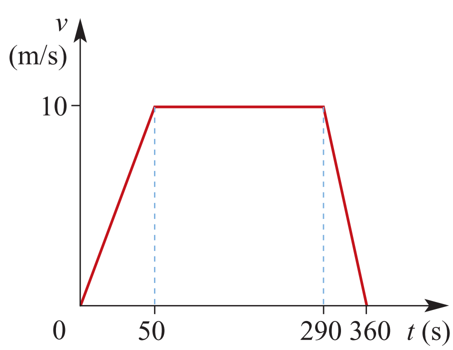
A 0.7 **B** 0.8 **C** 0.9 **D** 1.0 **E** 1.1

26 Given that if $y = x\tan^{-1} x$, then $\dfrac{dy}{dx} = \dfrac{x}{1 + x^2} + \tan^{-1} x$, it follows that an antiderivative of $\tan^{-1} x$ is:

A $x\tan^{-1} x$ **B** $x\tan^{-1} x - \dfrac{x}{1 + x^2}$ **C** $x\tan^{-1} x - \log_e\sqrt{1 + x^2}$
D $\dfrac{1}{1 + x^2} + \dfrac{1}{x}\tan^{-1} x$ **E** $\dfrac{x}{1 + x^2}$

27 The velocity–time graph shows the motion of a train moving between two stations. The distance between the stations in metres is:

A 2500 **B** 2900
C 3000 **D** 3400
E 5800



28 If $\dfrac{dy}{dx} = x^2 + x$ and $x = -3$ when $y = -\frac{1}{2}$, then:

A $y = \frac{1}{3}x^3 + \frac{1}{2}x^2 - 4$ **B** $y = \frac{1}{3}x^3 - \frac{1}{2}x^2 + 4$ **C** $y = -\frac{1}{3}x^3 + \frac{1}{2}x^2 - 4$
D $y = \frac{1}{3}x^3 + \frac{1}{2}x^2 + 4$ **E** $y = -\frac{1}{3}x^3 + \frac{1}{2}x^2 + 4$

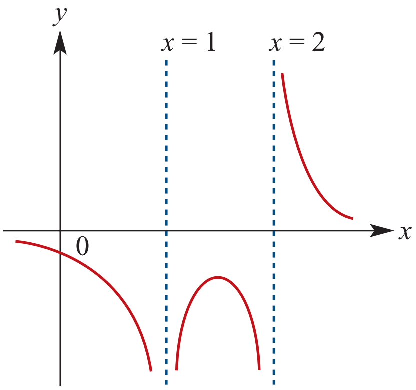
29 The equation of the particular member of the family of curves defined by $\dfrac{dy}{dx} = 1 - e^{-x}$ that passes through the point (0, 6) is:

A $y = x - e^{-x} + 5$ **B** $y = x + e^{-x} + 5$ **C** $y = x + e^{-x} + 7$
D $y = x + e^{-x} + 6$ **E** $y = x - e^{-x} + 6$

30 If $y = \sin^{-1}\sqrt{1-x}$ then $\dfrac{dy}{dx}$ equals:

A $\cos^{-1}\sqrt{1+x}$ **B** $\dfrac{1}{\sqrt{x}}$ **C** $\dfrac{1}{\sqrt{1-x}}$
D $\sqrt{\dfrac{1-x}{x}}$ **E** $-\dfrac{1}{2\sqrt{x(1-x)}}$

31 This is the graph of:



A $y = \dfrac{1}{(x-1)(x-2)}$

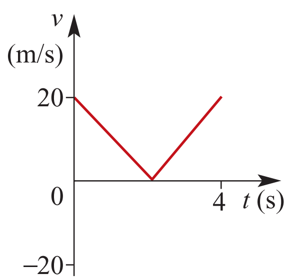
B $y = \dfrac{x}{(x-1)(x-2)}$

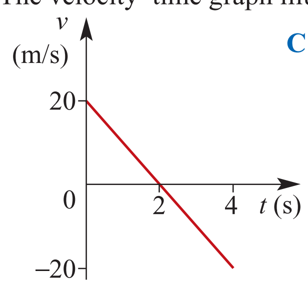
C $y = \dfrac{(x-1)(x-2)}{x}$

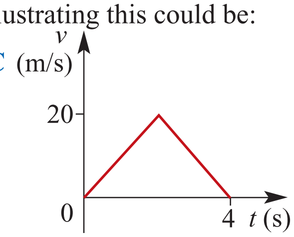
D $y = \dfrac{1}{(x-2)(x-1)^2}$

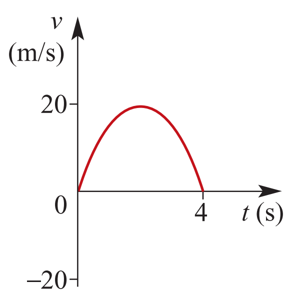
E $y = \dfrac{1}{(x-1)(x-2)^2}$

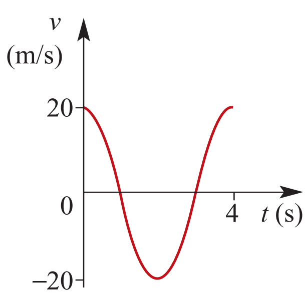
32 A particle is projected vertically upwards from ground level with a velocity of 20 m/s and returns to the point of projection. The velocity–time graph illustrating this could be:

A


B


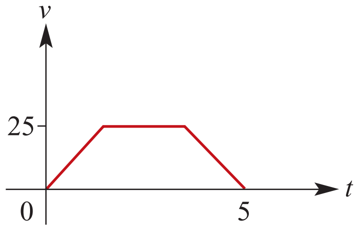
C


D


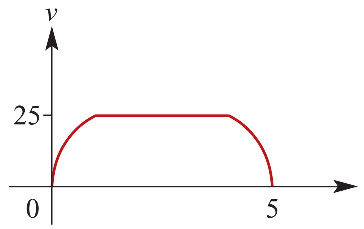
E


33 A car departs from a checkpoint, accelerating initally at 5 m/s^2 but with the rate of acceleration decreasing until a maximum speed of 25 m/s is reached. It continues at 25 m/s for some time, then slows with constant deceleration until it comes to rest. Which one of the following graphs best represents the motion of the car?

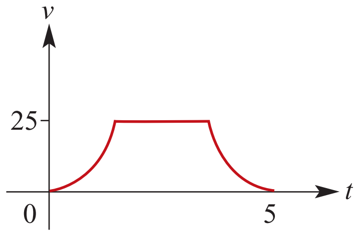
A



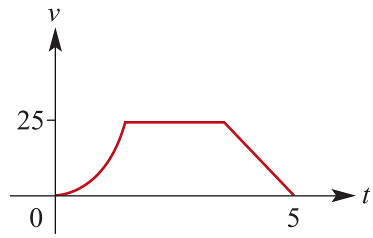
B



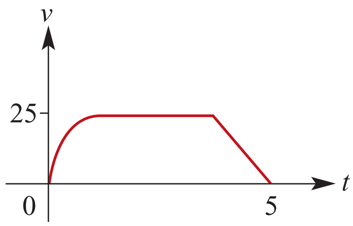
C



D



E



34 The values of m for which $y = e^{mx}$ satisfy the differential equation $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} - 3y = 0$ are:

A $m = 1, m = 2$ **B** $m = 3, m = -1$ **C** $m = -2, m = 3$

D $m = \pm 1$ **E** $m = -3, m = 3$

35 Which one of the following differential equations is satisfied by $y = e^{3x}$ for all values of x?

A $\frac{d^2y}{dx^2} + 9y = 0$ **B** $\frac{d^2y}{dx^2} - 9y = 0$ **C** $\frac{d^2y}{dx^2} + \frac{y}{9} = 0$

D $\frac{d^2y}{dx^2} - 27y = 0$ **E** $\frac{d^2y}{dx^2} - 8y = 0$

36 A particle has initial velocity 3 m/s and its acceleration t seconds later is $(6t^2 + 5t - 3)$ m/s^2. After two seconds, its velocity in m/s is:

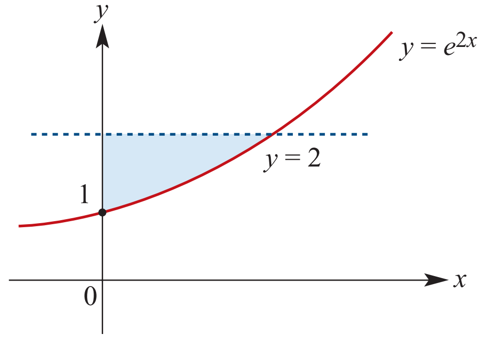
A 15 **B** 18 **C** 21 **D** 27 **E** 23

37 A particle starts from rest at a point O, and moves in a straight line so that after t seconds its velocity, v, is given by $v = 4 \sin 2t$. At this time, its displacement from O is given by:

A $s = 8 \cos 2t$ **B** $s = 2 \cos 2t$ **C** $s = -2 \cos 2t$

D $s = 8 \cos 2t - 8$ **E** $s = 2 - 2 \cos 2t$

38 The volume of the solid of revolution when the shaded region of the diagram is rotated about the y axis is given by:



A $\pi \int_0^{\frac{1}{2}\log_e 2} e^{2x}\,dx$

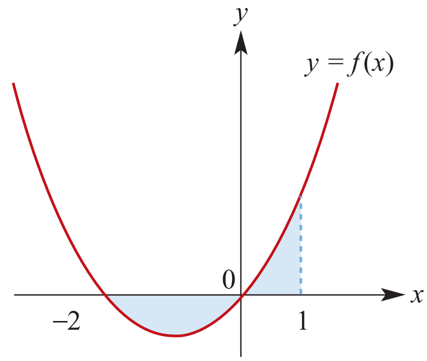
B $\pi \int_0^{2} \frac{1}{2}\log_e y\,dy$

C $\pi \left(\log_e 2 - \int_0^{\frac{1}{2}\log_e 2} e^{2x}\,dx\right)$

D $\pi \int_0^{2} \frac{1}{4}(\log_e y)^2\,dy - \frac{\pi}{2}$

E $\pi \int_1^{2} \frac{1}{4}(\log_e y)^2\,dy$

39 The area of the shaded region in the graph is:



A $\int_0^{1} f(x)\,dx + \int_0^{-2} f(x)\,dx$

B $\int_{-2}^{1} f(x)\,dx$

C $\int_{-2}^{0} f(x)\,dx + \int_0^{1} f(x)\,dx$

D $-\int_0^{1} f(x)\,dx + \int_0^{-2} f(x)\,dx$

E $-\int_0^{-2} f(x)\,dx + \int_0^{1} f(x)\,dx$

40 An arrangement of the integrals:

$P = \int_0^{\frac{\pi}{2}} \sin^2 x\,dx \qquad Q = \int_0^{\frac{\pi}{4}} \cos^2 x\,dx \qquad R = \int_0^{\frac{\pi}{4}} \sin^2 x\,dx$

in ascending order of magnitude is:

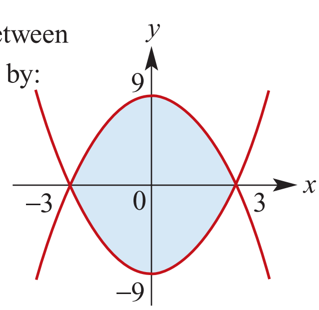
A P, R, Q **B** Q, P, R **C** R, Q, P **D** R, P, Q **E** Q, R, P

41 The value of $\int_0^{1} \frac{e^{2x}}{e^{2x}+1}\,dx$ is:

A $\frac{1}{2}(e^2+1)$ **B** $\frac{1}{2}\log_e(e^2-1)$ **C** $\frac{1}{2}\log_e\left(\frac{e^2+1}{2}\right)$

D $\log_e\left(\frac{e^2+1}{2}\right)$ **E** $2\log_e\left(\frac{e^2+1}{2}\right)$

42 In the diagram on the right, the area of the region enclosed between the graphs with equations $y = x^2 - 9$ and $y = 9 - x^2$ is given by:

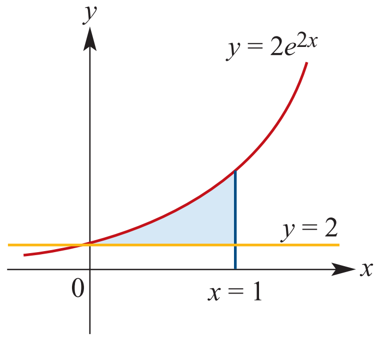


A $\int_{-3}^{3} 2x^2 - 18\,dx$ **B** $\int_{-3}^{3} 18 - 2x^2\,dx$

C 0 **D** $\int_{-9}^{9} 2x^2 - 18\,dx$

E $\int_{-9}^{9} 18 - 2x^2\,dx$

43 The volume of the solid of revolution when the shaded region of this graph is rotated about the x axis is given by:



A $\pi \int_0^1 4e^{4x} - 4\,dx$

B $\pi \int_0^1 e^{2x} - 4\,dx$

C $\pi \int_0^1 (2e^{2x} - 2)^2\,dx$

D $\pi \int_2^{2e} 1\,dy$

E $\pi \int_0^1 4 - 4e^{2x}\,dx$

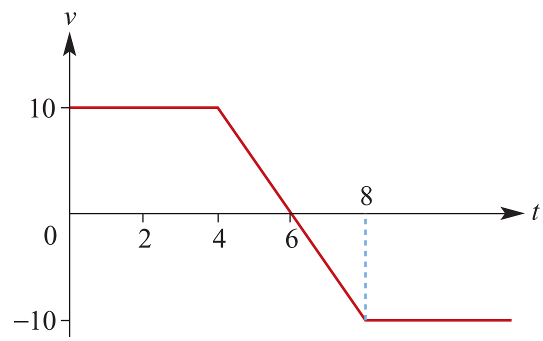
44 A body moves in a straight line so that its acceleration is m/s^2 at time t seconds is given by $\dfrac{d^2x}{dt^2} = 4 - e^{-t}$. If the body's initial velocity is 3 m/s, then when $t = 2$ its velocity, in m/s, is:

A e^{-2} **B** $2 + e^{-2}$ **C** $8 + e^{-2}$ **D** $10 + e^{-2}$ **E** $12 + e^{-2}$

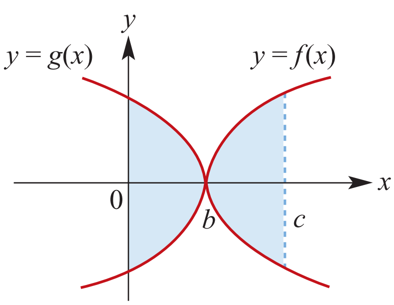
45 A particle moves with velocity v m/s. The distance travelled in metres by the particle in the first eight seconds is:

A 40 **B** 50 **C** 60

D 70 **E** 80



46 The area of the region shaded in the graph is equal to:



A $\int_0^c f(x) - g(x)\,dx$

B $\int_b^c f(x) - g(x)\,dx + \int_0^b f(x) - g(x)\,dx$

C $\int_b^c f(x) - g(x)\,dx + \int_b^0 f(x) - g(x)\,dx$

D $\int_b^c f(x)\,dx + \int_0^b g(x)\,dx$

E $\int_0^c f(x) - g(x)\,dx$

47 An antiderivative of $\cos(3x + 1)$ is

A $-3\sin(3x + 1)$ **B** $-\frac{1}{3}\cos(3x + 1)$ **C** $3\cos(3x + 1)$

D $-\frac{1}{3}\sin(3x + 1)$ **E** $\frac{1}{3}\sin(3x + 1)$

Revision

48 $\int_0^{\frac{\pi}{4}} \tan x \sec^2 x\, dx$ is equal to:

A $\int_0^1 u\, du$ **B** $\int_0^{\frac{\pi}{4}} u^2\, du$ **C** $-\int_0^1 u^2\, du$

D $\int_0^1 \sqrt{1-u^2}\, du$ **E** $\int_0^{\frac{\pi}{4}} \frac{u^2}{2}\, du$

49 The value of $\int_0^2 2e^{2x}\, dx$ is:

A e^4 **B** $e^4 - 1$ **C** $4e^4$ **D** $\frac{1}{2}e^4$ **E** $1 - e^4$

50 An antiderivative of $\frac{\sin x}{\cos^2 x}$ is:

A $\sec x$ **B** $\tan x \cos x$ **C** $\tan^2 x$ **D** $\cot x \sec x$ **E** $\sec^2 x$

51 A partial fraction expansion of $\frac{1}{(2x+6)(x-4)}$ shows that it has an antiderivative $\frac{a}{2}\log_e(2x+6) + b\log_e(x-4)$ where:

A $a = -\frac{1}{7}, b = \frac{1}{14}$ **B** $a = 1, b = 1$ **C** $a = \frac{1}{2}, b = \frac{1}{2}$

D $a = -1, b = -1$ **E** $a = \frac{1}{11}, b = \frac{1}{7}$

52 $\int_0^1 x\sqrt{2x+1}\, dx$ is equal to:

A $\frac{1}{2}\int_0^1 (u-1)\sqrt{u}\, du$ **B** $\int_0^1 u\sqrt{u}\, du$ **C** $\frac{1}{4}\int_1^3 \sqrt{u}\, du$

D $2\int_1^3 \sqrt{u}\, du$ **E** $\frac{1}{4}\int_1^3 u^{\frac{3}{2}} - u^{\frac{1}{2}}\, du$

53 If $\int_0^{\frac{\pi}{6}} \sin^n x \cos x\, dx = \frac{1}{64}$, then n equals:

A 6 **B** 5 **C** 4 **D** 3 **E** 7

54 Of the integrals $\int_0^{\pi} \sin^3\theta \cos^3\theta\, d\theta$, $\int_0^2 t^3(4-t^2)^2\, dt$, $\int_0^{\pi} x^2 \cos x\, dx$, one is negative, one is positive and one is zero. Without evaluating them, determine which is the correct order of signs.

A $- \; 0 \; +$ **B** $+ \; - \; 0$ **C** $+ \; 0 \; -$ **D** $0 \; - \; +$ **E** $0 \; + \; -$

55 $\int_0^{\frac{\pi}{4}} \cos 2x\, dx$ is equal to:

A $\frac{1}{2}\int_0^{\frac{\pi}{2}} \sin 2x\, dx$ **B** $\frac{1}{2}\int_0^{\frac{\pi}{2}} \cos 2x\, dx$ **C** $\int_{-\frac{\pi}{4}}^0 \sin 2x\, dx$

D $\frac{1}{2}\int_0^{\frac{\pi}{2}} \sin 4x\, dx$ **E** $\frac{1}{2}\int_0^{\frac{\pi}{2}} \cos 4x\, dx$

56 $\int_{-a}^{a} \tan x \, dx$ can be evaluated if a equals:

A $\frac{\pi}{2}$ **B** $\frac{3\pi}{2}$ **C** $\frac{\pi}{4}$ **D** π **E** $\frac{-3\pi}{2}$

57 An antiderivative of $\frac{2x}{\sqrt{x^2-1}}$ is:

A $2\sqrt{x^2-1}$ **B** $\frac{x^2}{\sqrt{x^2-1}}$ **C** $2x\sqrt{x^2-1}$

D $\frac{2}{\sqrt{x^2-1}}$ **E** $\frac{2}{x\sqrt{x^2-1}}$

58 If $\frac{3}{(x-1)(2x+1)} \equiv \frac{A}{x-1} + \frac{B}{2x+1}$ then:

A $A = 4, B = 3$ **B** $A = 1, B = 4$ **C** $A = 1, B = -2$

D $A = 3, B = 3$ **E** $A = 2, B = 4$

59 $\int \tan x \, dx$ is equal to:

A $\sec^2 x + c$ **B** $\log_e(\cos x) + c$ **C** $\log_e(\sec x) + c$

D $\log_e(\sin x) + c$ **E** $\frac{1}{2}\tan^2 x + c$

60 The volume of the solid of revolution formed by rotating the region bounded by the curve $y = 2\sin x - 1$ and the lines with equations $x = 0$, $x = \frac{\pi}{4}$ and $y = 0$ about the x axis is given by:

A $\int_0^{\frac{\pi}{2}} \pi^2(2\sin x - 1)^2 \, dx$ **B** $\int_0^{\frac{\pi}{4}} \pi(4\sin^2 x - 1) \, dx$ **C** $\int_0^{\frac{\pi}{4}} \pi(1 - 2\sin x)^2 \, dx$

D $\int_0^{\frac{\pi}{4}} (2\sin x - 1)^2 \, dx$ **E** $\int_0^{\frac{\pi}{4}} \pi(2\sin x - 1) \, dx$

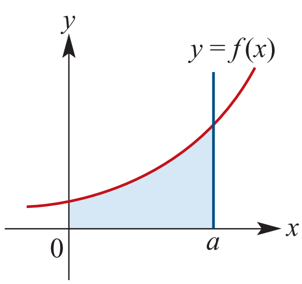
61 The area of the region bounded by the graphs of the function $f\colon \left[0, \frac{\pi}{2}\right) \to R$, $f(x) = \sin x$ and $g\colon \left[0, \frac{\pi}{2}\right) \to R$, $g(x) = \sin 2x$ is:

A $\int_0^{\frac{\pi}{2}} \sin x - x\sin(2x) \, dx$ **B** $\int_0^{\frac{\pi}{3}} \sin 2x - \sin x \, dx$

C $\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \sin 2x - \sin x \, dx$ **D** $\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \sin x - \sin 2x \, dx$

E $\int_{\frac{\pi}{3}}^{\frac{\pi}{4}} \sin 2x - \sin x \, dx$

62 The shaded region is bounded by the curve $y = f(x)$, the coordinate axes and the line $x = a$. Which one of the following statements is false?



A The area of the shaded region is $\int_0^a f(x) \, dx$.

Revision

B The volume of the solid of revolution formed by rotating the region about the x axis is $\pi\int_0^a (f(x))^2\,dx$.

C The volume of the solid of revolution formed by rotating the region about the y axis is $\pi\int_{f(0)}^{f(a)} x^2\,dy$.

D The area of the shaded region is greater than $af(0)$.

E The area of the shaded region is less than $af(a)$.

63 $\displaystyle\int \frac{1}{\sqrt{9-4x^2}}\,dx$ equals:

A $\frac{1}{3}\sin^{-1}\left(\frac{3x}{2}\right)+c$ **B** $\frac{1}{3}\sin^{-1}\left(\frac{2x}{3}\right)+c$ **C** $\frac{1}{2}\sin^{-1}\left(\frac{3x}{2}\right)+c$

D $\frac{1}{2}\sin^{-1}\left(\frac{2x}{3}\right)+c$ **E** $\sin^{-1}\left(\frac{2x}{3}\right)+c$

64 $\displaystyle\int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{1}{(1-x)^2}\,dx$ equals:

A $\frac{4}{3}$ **B** $-\frac{4}{3}$ **C** 1 **D** $\log_e 3$ **E** $-\log_e 3$

65 $\displaystyle\int \frac{1}{\sqrt{\frac{1}{9}-x^2}}\,dx$ equals:

A $\sin^{-1}\left(\frac{x}{3}\right)+c$ **B** $\sin^{-1}(3x)+c$ **C** $\sin^{-1}\left(\frac{3}{x}\right)+c$

D $\frac{3}{2}\log_e\left(\frac{1}{9}-x^2\right)+c$ **E** $\frac{3}{2}\log_e\left(\frac{1+3x}{1-3x}\right)+c$

66 $\displaystyle\int \frac{dx}{9+4x^2}$ is:

A $\frac{1}{9}\tan^{-1}\frac{2x}{9}+c$ **B** $\frac{1}{3}\tan^{-1}\frac{2x}{3}+c$ **C** $\frac{1}{6}\tan^{-1}\frac{2x}{3}+c$

D $9\tan^{-1}\frac{2x}{9}+c$ **E** $\frac{3}{2}\tan^{-1}\frac{2x}{3}+c$

67 $\dfrac{d}{d\theta}(\sec^3\theta)$ is:

A $3\sec^3\theta\tan\theta$ **B** $3\sec^2\theta$ **C** $3\sec^2\theta\tan\theta$

D $3\sec^2\theta\tan^2\theta$ **E** $3\sec\theta\tan^2\theta$

68 If $\int \sin^2 4x\cos 4x\,dx = k\sin^3 4x + c$ then k is:

A $\frac{1}{12}$ **B** $\frac{1}{4}$ **C** $\frac{1}{3}$ **D** $-\frac{1}{4}$ **E** $-\frac{1}{3}$

69 $\dfrac{x+7}{x^2-x-6}$ written as partial fractions is:

A $\dfrac{2}{x-3}-\dfrac{1}{x+2}$ **B** $\dfrac{2}{x+2}-\dfrac{1}{x-3}$ **C** $\dfrac{9}{5(x-2)}-\dfrac{9}{5(x+3)}$

D $\dfrac{4}{5(x-2)}-\dfrac{9}{5(x+3)}$ **E** $\dfrac{1}{x+2}+\dfrac{2}{x-3}$

70 For $y = \sin^{-1}(3x)$, $\dfrac{dy}{dx}$ equals:

A $-\dfrac{3\cos 3x}{\sin^2 3x}$ **B** $3\cos^{-1}(3x)$ **C** $\dfrac{1}{\sqrt{1-9x^2}}$

D $\dfrac{3}{\sqrt{1-9x^2}}$ **E** $\dfrac{1}{3\sqrt{1-9x^2}}$

71 $\dfrac{d}{dx}[\log_e(\tan x)]$ equals

A $\log_e(\sec^2 x)$ **B** $\cot x$ **C** $\dfrac{2}{\sin 2x}$

D $\dfrac{1}{\sin 2x}$ **E** $\sec x$

72 The general solution of the differential equation $\dfrac{dy}{dx} + y = 1$ (P being an arbitrary constant) is

A $2x + (1 - y)^2 = P$ **B** $2x - (1 - y)^2 = P$ **C** $y = 1 + Pe^x$

D $y = 1 + Pe^{-x}$ **E** $y = Pe^{-x} - 1$

73 $\displaystyle\int \frac{x^2}{(x^3+1)^{\frac{1}{2}}}\,dx$ equals

A $\frac{1}{3}\log_e(x^3+1)^{\frac{1}{2}} + c$ **B** $\frac{2}{3}\log_e(x^3+1)^{\frac{1}{2}} + c$ **C** $\frac{2}{3}(x^3+1)^{\frac{1}{2}} + c$

D $\frac{1}{6}(x^3+1)^{\frac{1}{2}} + c$ **E** $\frac{1}{3}(x^3+1)^{\frac{1}{2}} + c$

74 Air leaks from a spherical balloon at a constant rate of 2 m^3/s. When the radius of the balloon is 5 m, the rate, in m^2/s, at which the surface area is decreasing is

A $\frac{4}{5}$ **B** $\frac{8}{5}$ **C** $\frac{1}{50}\pi$ **D** $\frac{1}{100}\pi$ **E** none of these

75 $\displaystyle\int_0^{\frac{\sqrt{3}}{2}} \frac{x}{\sqrt{1-x^2}}\,dx$ equals

A $\frac{1}{4}$ **B** $\frac{1}{2}$ **C** 1 **D** $\dfrac{\pi}{3}$ **E** $-\frac{1}{2}$

76 When $-1 < x < 1$, $\displaystyle\int \frac{1}{1-x^2}\,dx$ equals

A $\frac{1}{2}\log_e\left(\dfrac{1+x}{1-x}\right) + c$ **B** $\frac{1}{2}\log_e\left(\dfrac{1-x}{1+x}\right) + c$ **C** $\log_e\left(\dfrac{1+x}{1-x}\right) + c$

D $\frac{1}{2}\log_e[(1-x)(1+x)] + c$ **E** $\log_e[(1-x)(1+x)] + c$

77 At a certain instant, a sphere is of radius 10 cm and the radius is increasing at a rate of 2 cm/s. The rate of increase in cm^3/s of the volume of the sphere is

A 80π **B** $\dfrac{800\pi}{3}$ **C** 400π **D** 800π **E** $\dfrac{8000\pi}{3}$

78 $\dfrac{d}{d\theta}[\log_e(\sec\theta + \tan\theta)]$ equals

A $\sec\theta$ **B** $\sec^2\theta$ **C** $\sec\theta\tan\theta$ **D** $\cot\theta - \tan\theta$ **E** $\tan\theta$

79 A particle is moving along Ox so that at time t, $x = 3\cos 2t$. When $t = \dfrac{\pi}{2}$, the acceleration of the particle is the direction Ox is

A -12 **B** -6 **C** 0 **D** 6 **E** 12

11.2 Extended-response questions

1 A bowl can be described as the solid of revolution formed by rotating the graph of $y = \frac{1}{4}x^2$ around the y axis for $0 \le y \le 25$.

a Find the volume of the bowl.

b The bowl is filled with water and then, at time $t = 0$, the water begins to run out of a small hole in the bottom. The rate at which the water runs out is proportional to the depth, h, of the water at time t. Let V denote the volume of water at time t.

i Show that $\dfrac{dh}{dt} = \dfrac{-k}{4\pi}$, $k > 0$.

ii Given that the bowl is empty after 30 seconds, find the value of k.

iii Find h in terms of t.

iv Find V in terms of t.

c Sketch the graph of:

i V against h **ii** V against t

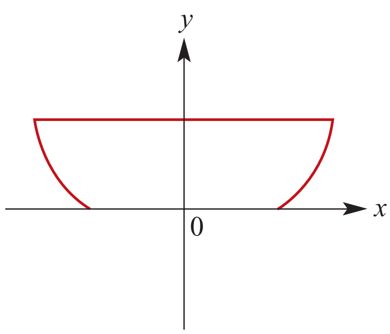
2 **a** Sketch the curve whose equation is $y + 3 = \dfrac{6}{x - 1}$.

b Find the coordinates of the points where the line $y + 3x = 9$ intersects the curve.

c Find the area of the region enclosed between the curve and the line.

d Find the equations of two tangents to the curve that are parallel to the line.

3 The vertical cross-section of a bucket is shown in this diagram. The sides are arcs of a parabola with the y axis as the central axis and the horizontal cross-sections are circular.



The depth is 36 cm, the base radius length is 10 cm and the radius length of the top is 20 cm.

a Prove that the parabolic sides are arcs of the parabola $y = 0.12x^2 - 12$.

b Prove that the bucket holds 9π litres when full.

c Water starts leaking from the bucket, initially full, at the rate given by $\dfrac{dv}{dt} = \dfrac{-\sqrt{h}}{A}$ where, at time t seconds, the depth is h cm, the surface area is A cm^2 and the volume is v cm^3.

Prove that $\dfrac{dv}{dt} = \dfrac{-3\sqrt{h}}{25\pi(h + 12)}$.

d Show that $v = \pi \displaystyle\int_0^h \left(\frac{25y}{3} + 100\right) dy$.

e Hence construct a differential equation expressing:

i $\dfrac{dv}{dh}$ as a function of h

ii $\dfrac{dh}{dt}$ as a function of h

f Hence find the time taken for the bucket to empty.

4 A hemispherical bowl can be described as the solid of revolution generated by rotating $x^2 + y^2 = a^2$ about the y axis for $-a \leq y \leq 0$. The bowl is filled with water. At time $t = 0$, water starts running out of a small hole in the bottom of the bowl, so that the depth of water in the bowl at time t is h cm. The rate which the volume is decreasing is proportional to h. (All length units are cm.)

a **i** Show that, when the depth of water is h, the volume, V cm^3, of water remaining is $V = \pi\left(ah^2 - \frac{1}{3}h^3\right)$ where $0 < h \leq a$.

ii If $a = 10$, find the depth of water in the hemisphere if the volume is one litre.

b Show that there is a positive constant k, such that $\pi(2ah - h^2)\dfrac{dh}{dt} = -kh$.

c Given that the bowl is empty after time T show that $k = \dfrac{3\pi a^2}{2T}$.

d If $T = 30$ and $a = 10$, find k (correct to three significant figures).

e Sketch the graph of:

i $\dfrac{dV}{dt}$ against h for $0 \leq h \leq a$ **ii** $\dfrac{dh}{dt}$ against h for $0 \leq h \leq a$

f Find the rate of change of the depth with respect to time when:

i $h = \dfrac{a}{2}$ **ii** $h = \dfrac{a}{4}$

g If $a = 10$ and $T = 30$, find the rate of change for depth with respect to time when there is one litre of water in the hemisphere.

5 Consider the function with rule $f(x) = \dfrac{1}{ax^2 + bx + c}$ where $a \neq 0$.

a Find $f'(x)$.

b State the coordinates of the turning point, and state the nature of this turning point if:

i $a > 0$ **ii** $a < 0$

c **i** If $b^2 - 4ac < 0$ and $a > 0$, sketch the graph of $y = f(x)$ stating the equations of all asymptotes.

ii If $b^2 - 4ac < 0$ and $a < 0$, sketch the graph of $y = f(x)$, stating the equations of all asymptotes.

d If $b^2 - 4ac = 0$, sketch the graph of $y = f(x)$ for:

i $a > 0$ **ii** $a < 0$

e If $b^2 - 4ac > 0$ and $a > 0$, sketch the graph of $y = f(x)$ stating the equations of all asymptotes.

6 Consider the family of curves with equation $y = ax^2 + \dfrac{b}{x^2}$, where $a, b \in R^+$.

a Find $\dfrac{dy}{dx}$.

b State the coordinates of the turning points of a member of this family in terms of a and b, and state the nature of each.

Revision

c Consider the family $y = ax^2 + \frac{1}{x^2}$. Show that the coordinates of the turning points are $\left(\frac{1}{\sqrt[4]{a}}, 2\sqrt{a}\right)$ and $\left(\frac{-1}{\sqrt[4]{a}}, 2\sqrt{a}\right)$.

7 Let $f: [0, 4\pi] \to R, f(x) = e^{-x} \sin x$.

a Find $\{x: f'(x) = 0\}$.

b Determine the ratio $f(a + 2\pi) : f(a)$.

c Determine the coordinates of all stationary points for $x \in [0, 4\pi]$ and state their nature.

d Differentiate $-\frac{1}{2}e^{-x}(\cos x + \sin x)$ and, hence, evaluate $\int_0^{\pi} e^{-x} \sin x \, dx$.

e Use the results of **b** above to determine $\int_{2\pi}^{3\pi} f(x)\, dx$.

8 **a** Evaluate $\int_0^{\frac{\pi}{4}} \tan^4 \theta \sec^2 \theta \, d\theta$.

b Hence show that $\int_0^{\frac{\pi}{4}} \tan^6 \theta \, d\theta = \frac{1}{5} - \int_0^{\frac{\pi}{4}} \tan^4 \theta \, d\theta$.

c Deduce that $\int_0^{\frac{\pi}{4}} \tan^6 \theta \, d\theta = \frac{13}{15} - \frac{\pi}{4}$.

9 A disease spreads through a population. At time t, p is the proportion of the population who have the disease. The rate of change of p is proportional to the product of p and the proportion $1 - p$ who do not have the disease. When $t = 0$, $p = \frac{1}{10}$ and, when $t = 2$, $p = \frac{1}{5}$.

a **i** Show that $t = \frac{1}{k} \log_e \left(\frac{9p}{1 - p}\right)$ where $k = \log_e \left(\frac{3}{2}\right)$.

ii Hence show that $\frac{9p}{1 - p} = \left(\frac{3}{2}\right)^t$.

b Find p when $t = 4$.

c Find p in terms of t.

d Find the values of t for which $p > \frac{1}{2}$.

e Sketch the graph of p against t.

10 A car moves along a straight level road. Its speed, v, is related to its displacement, x, by the differential equation $v\frac{dv}{dx} = \frac{p}{v} - kv^2$, where p and k are constants.

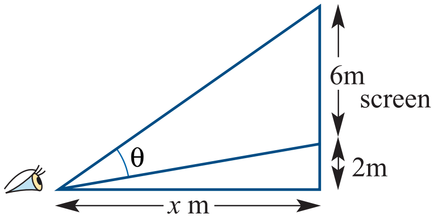
a If $v = 0$ when $x = 0$ show that $v^3 = \frac{1}{k}(p - pe^{-3kx})$.

b Find $\lim_{x \to \infty} v$.

11 A projection screen is six metres in height and has its lower edge two metres above the eye level of an observer. The angle between the lines of sight of the upper and lower edges of the screen is θ.

Let x m be the horizontal distance from an observer to the screen.

CAS CALCULATOR



a Find θ in terms of x.

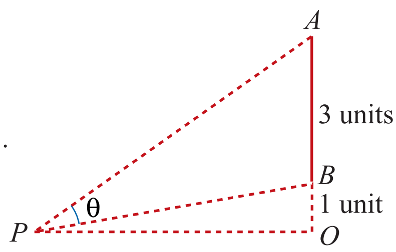
b Find $\dfrac{d\theta}{dx}$.

c What values can θ take?

d Sketch the graph of θ against x.

e If $1 \leq x \leq 25$, find the minimum value of θ.

12 A vertical rod AB of length three units is held with its lower end, B, at a distance one unit vertically above a point O. The angle subtended by AB at a variable point P on the horizontal plane through O is θ.



a Show that $\theta = \tan^{-1} x - \tan^{-1} \dfrac{x}{4}$ where $x = OP$.

b Prove that:

i θ is a maximum when $x = 2$

ii the maximum value of θ is $\tan^{-1} \frac{3}{4}$.

CAS CALCULATOR

13 An open rectangular tank is to have a square base. The capacity of the tank is to be 4000 m.^{3} Let x m be the length of an edge of the square base and A m^2 be the amount of sheet metal used to construct the tank.

a Show that $A = x^2 + \dfrac{16\,000}{x}$.

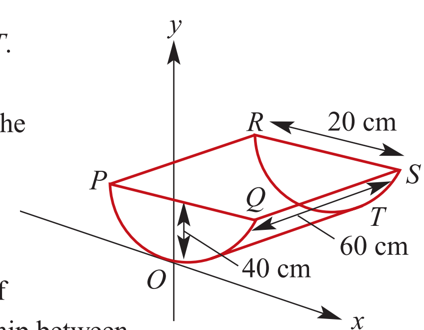
b Sketch the graph of A against x.

c Find, correct to two decimal places, the value(s) of x for which 2500 m^2 of sheet metal is used.

d Find the value of x for which A is a minimum.

14 A closed rectangular box is made of very thin sheet metal and its length is three times its width. If the volume of the box is 288 cm^3, show that its surface area, $A(x)$, is given by $A(x) = \dfrac{768}{x} + 6x^2$ where x is the width of the box. Find the minimum surface area of the box.

15 This container has an open rectangular horizontal top, $PQSR$, and parallel vertical ends, PQO and RST. The ends are parabolic in shape. The x axis and y axis intersect at O, with the x axis horizontal and the y axis the line of symmetry of the end PQO. The dimensions are shown on the diagram.



a Find the equation of the parabolic arc QOP.

b If water is poured into the container to a depth of y cm, with a volume of V cm^3, find the relationship between V and y.

c Calculate the depth, to the nearest mm, when the container is half full.

d Water is poured into the empty container so that the depth is y cm at time t seconds. If the water is poured in at the rate of 60 cm^3/s, construct a differential equation expressing $\frac{dy}{dt}$ as a function of y and solve it.

e Calculate, to the nearest second:

  i how long it will take the water to reach a depth of 20 cm

  ii how much longer it will take for the container to be completely full.

16 Moving in the same direction along parallel tracks, objects A and B pass the point O simultaneously with speeds of 20 m/s and 10 m/s respectively. From then on, the deceleration of A is $\frac{1}{400}v^3$ m/s^2 and, for B, the deceleration is $\frac{1}{100}v^2$ m/s^2 when the speeds are v m/s.

a Find the speeds of A and B at time t seconds after passing O.

b Find the position of A and B at time t seconds after passing O.

c Use a CAS calculator to plot the graphs of the position of objects A and B.

d Use a CAS calculator to find, to the nearest second, when the objects pass.

17 A stone, initially at rest, is released and falls vertically. Its velocity, v m/s, at time t s after release is determined by the differential equation $5\frac{dv}{dt} + v = 50$.

a Find an expression for v in terms of t.

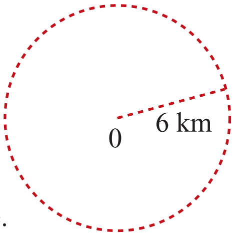
b Find v when $t = 47.5$.

c Sketch the graph of v against t.

d **i** Let x be displacement from the point of release at time t. Find an expression for x terms of t.

  ii Find x when $t = 6$.

18 O is the centre of a city. The city has a populaton of 600 000. All of the population lies within 6 km of the city centre. The number of people who live within r km $(0 \le r \le 6)$ of the city centre is given by $\int_0^r 2\pi k(6-x)^{\frac{1}{2}}x^2\,dx$.



a Find the value of k, correct to three significant figures.

b Find the number of people who live within 3 km of the city centre.

19 The rate of change of a population, y, is given by $\frac{dy}{dt} = \frac{2y(N-y)}{N}$, where $N > 0$ is a constant and when $t = 0$, $y = \frac{N}{4}$.

a Find y in terms of t and $\frac{dy}{dt}$ in terms of t.

b What limiting value does the population size approach for large values of t?

c Explain why the population is always increasing.

d What is the population when the population is increasing most rapidly?

e For $N = 10^6$:

i sketch the graph of $\frac{dy}{dt}$ against y

ii at what time is the population increasing most rapidly?

20 An object projected vertically upwards from the surface of the Earth, neglecting air resistance, experiences an acceleration of a m/s^2 at a point x m from the centre of the Earth. This acceleration is given by $a = \frac{-gR^2}{x^2}$, where g m/s^2 is the acceleration due to gravity and R m is the radius length of the Earth.

a Given that $g = 9.8$, $R = 6.4 \times 10^6$, and that the object has an upwards velocity of u m/s at the Earth's surface:

i express v^2 in terms of x, using $a = \frac{d}{dx}\left(\frac{1}{2}v^2\right)$

ii use the result of **a** to find the position of the object when it has zero velocity

iii for what value of u does the result in part **ii** not exist?

b The minimum value of u for which the object does not fall back to Earth is called the escape velocity. Determine the escape velocity in km/h.

21 Define $f(x) = \frac{e^x - e^{-x}}{e^x + e^{-x}}$.

a Find $f(0)$.

b Find $\lim_{x \to \infty} f(x)$.

c Find $\lim_{x \to -\infty} f(x)$.

d Find $f'(x)$.

e Sketch the graph of f.

f Find $f^{-1}(x)$.

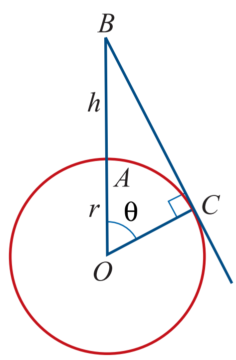
g If $g(x) = f^{-1}(x)$, find $g'(x)$.

h Sketch the graph of g' and prove that the area measure of the region bounded by the graph of $y = g'(x)$, the x axis, the y axis and the line $x = \frac{1}{2}$ is $\log_e \sqrt{3}$.

22 The diagram shows a plane circular section through O, the centre of the Earth (which is assumed to be stationary for the purpose of this problem).

From the point A on the surface, a rocket is launched vertically upwards and, after t hours, it is at B which is h kilometres above A. C is the horizon as seen from B, and the length of AC is y kilometres.

The angle AOC is θ radians. The radius of the Earth is r kilometres.



a **i** Express y in terms of r and θ.

ii Express $\cos\theta$ in terms of r and h.

b Suppose that after t hours the vertical velocity of the rocket is $\frac{dh}{dt} = r\sin t$, $t \in [0, \pi)$. Assume $r = 6000$.

i Find $\frac{dy}{d\theta}$ and $\frac{dy}{dt}$.

ii How high is the rocket when $t = \frac{\pi}{2}$?

iii Find $\frac{dy}{dt}$ when $t = \frac{\pi}{2}$.

23 **a** Differentiate $f(x) = e^{-x}x^n$, and hence prove that
$$\int e^{-x}x^n\,dx = n\int e^{-x}x^{n-1}\,dx - e^{-x}x^n.$$

b $g\colon R^+ \to R$, $g(n) = \int_0^\infty e^{-x}x^n\,dx$

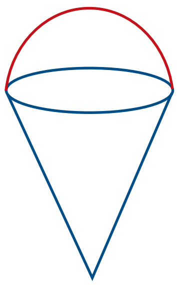
Note: $\int_a^\infty f(x)\,dx = \lim_{b\to\infty}\int_a^b f(x)\,dx$

i Show that $g(0) = 1$.

ii Using the answer to **a**, show that $g(n) = ng(n-1)$.

iii Using your answers to **b(i)** and **b(ii)**, show that $g(n) = n!$ for $n = 0, 1, 2, 3, \ldots$

24 A large weather balloon is in the shape of a hemisphere surmounting a cone, as shown in this diagram. When inflated, the height of the cone is twice the radius length of the hemisphere. The shapes and conditions are true as long as the radius of the hemisphere is at least two metres. At time t minutes, the radius length of the hemisphere is r metres and the volume of the balloon is V m^3, $r \geq 2$. The balloon has been inflated so that the radius length is 10 m, and it is ready to be released when a leak develops. The gas leaks out at the rate of t^2 m^3/min.



a Find the relationship between V and r.

b Construct a differential equation of the form $f(r)\dfrac{dr}{dt} = g(t)$.

c By antidifferentiating both sides of this differential equation with respect to t, and using the fact that the initial radius length is 10 m, solve the differential equation.

d Find how long it will take for the radius length to reduce to two metres.

CHAPTER

12

Vector functions

Objectives

- To sketch the graphs of plane curves specified by a **vector equation**
- To understand the concept of **position vectors** as a function of time
- To represent the path of a particle in two and three dimensions as a **vector function**
- To differentiate and antidifferentiate a vector function
- To use **vector calculus** to analyse the motion of a particle along a curve

In Chapter 2, vectors were introduced and applied to physical and geometric situations.

In order to describe the motion of a particle in space, its motion can be described by giving its position vector with respect to an origin in terms of a variable t. The variable in this situation is referred to as a **parameter**. This idea has been used in section 1.8 where parametric equations were introduced to describe circles, ellipses and hyperbolas.

In two dimensions, the position vector can be described through the use of two functions. The position vector at time t is given by

$$\boldsymbol{r}(t) = x(t)\boldsymbol{i} + y(t)\boldsymbol{j}.$$

$\boldsymbol{r}(t)$ is a **vector function**.

Consider the vector $\boldsymbol{r} = (3 + t)\boldsymbol{i} + (1 - 2t)\boldsymbol{j} \quad t \in R$.

$\boldsymbol{r}$ represents a family of vectors defined by different values of t.

If t represents the time variable then $\boldsymbol{r}$ is a vector function of time.

i.e. $\boldsymbol{r}(t) = (3 + t)\boldsymbol{i} + (1 - 2t)\boldsymbol{j} \quad t \in R$, and when $t = 2$, $\boldsymbol{r}(2) = 5\boldsymbol{i} - 3\boldsymbol{j}$.

Further, if $\boldsymbol{r}(t)$ represents the position of a particle with respect to time, then the graph of the endpoints of $\boldsymbol{r}(t)$ will represent the path of the particle in the cartesian plane.

A table of values for a range of values of t is given below.

t	-3	-2	-1	0	1	2	3
$\boldsymbol{r}(t)$	$7\boldsymbol{j}$	$\boldsymbol{i} + 5\boldsymbol{j}$	$2\boldsymbol{i} + 3\boldsymbol{j}$	$3\boldsymbol{i} + \boldsymbol{j}$	$4\boldsymbol{i} - \boldsymbol{j}$	$5\boldsymbol{i} - 3\boldsymbol{j}$	$6\boldsymbol{i} - 5\boldsymbol{j}$

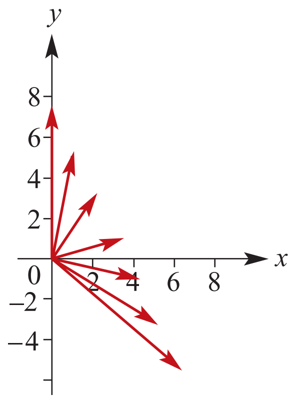


Figure A

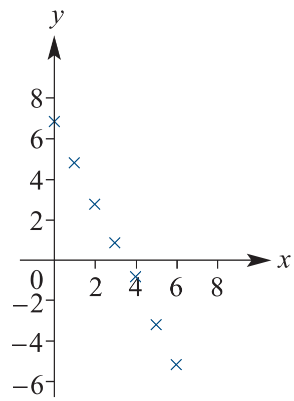


Figure B

These position vectors can be represented in the cartesian plane as shown in Figure A.

The graph of the position vectors is not helpful, but when only the endpoints are plotted (Figure B) the pattern of the path is more obvious.

It is easy to show that the points lie on the graph of $y = 7 - 2x$.

Let (x, y) represent any point on the graph.

$\therefore \quad \boldsymbol{r}(t) = x\boldsymbol{i} + y\boldsymbol{j}$ as a position vector.

$\therefore \quad x\boldsymbol{i} + y\boldsymbol{j} = (3 + t)\boldsymbol{i} + (1 - 2t)\boldsymbol{j}$

$\therefore \quad (x - 3 - t)\boldsymbol{i} = (1 - 2t - y)\boldsymbol{j}$

$\therefore \quad x - 3 - t = 0 \quad \boxed{1}$

and $\quad 1 - 2t - y = 0 \quad \boxed{2}$

The parameter t can be eliminated from the equation as follows:

From $\boxed{1}$, $t = x - 3$

Substituting in $\boxed{2}$ $\quad 1 - 2(x - 3) - y = 0$

which implies $\quad y = 7 - 2x$

$y = 7 - 2x$ is the cartesian equation for $\boldsymbol{r} = (3 + t)\boldsymbol{i} + (1 - 2t)\boldsymbol{j}, \quad t \in R$.

12.1 Vector equations

In the above, a method for describing the path of a particle has been shown. This method can be extended to describing graphs in two dimensions.

For example, consider the cartesian equation $y = x^2$. The graph can also be described by a vector equation using a parameter t, which does not necessarily represent time.

Consider the vector equation $\boldsymbol{r}(t) = t\boldsymbol{i} + t^2\boldsymbol{j}, \quad t \in R$.

Using similar reasoning to the above, if $x\boldsymbol{i} + y\boldsymbol{j} = t\boldsymbol{i} + t^2\boldsymbol{j}$ then $x = t$ and $y = t^2$ and eliminating t yields $y = x^2$.

The representation is not unique. It is clear that $\boldsymbol{r}(t) = t^3\boldsymbol{i} + t^6\boldsymbol{j}, t \in R$, also represents the graph determined by the cartesian equation $y = x^2$. Note that if these vector equations are used to describe the motion of particles, the paths are the same but the particles are at different locations at a given time (with the exception of $t = 0$ and $t = 1$). Also note that $\boldsymbol{r}(t) = t^2\boldsymbol{i} + t^4\boldsymbol{j}$, $t \in R$, only represents the equation $y = x^2$ for $x \geq 0$.

In this section, the graphs defined by vector equations will be considered but not related to the motion of a particle.

Example 1

Find the cartesian equation for the graphs represented by the following vector equations:

a $\boldsymbol{r}(t) = (2 - t)\boldsymbol{i} + (3 + t^2)\boldsymbol{j} \quad t \in R$ **b** $\boldsymbol{r}(t) = (1 - \cos t)\boldsymbol{i} + \sin t\boldsymbol{j} \quad t \in R$

Solution

a Let (x, y) be any point on the curve $\boldsymbol{r}(t) = (2 - t)\boldsymbol{i} + (3 + t^2)\boldsymbol{j}$.

Then $x = 2 - t \quad \boxed{1}$

$y = 3 + t^2 \quad \boxed{2}$

From $\boxed{1}$ $t = 2 - x$

Substituting in $\boxed{2}$ $y = 3 + (2 - x)^2$

$y = x^2 - 4x + 7 \quad x \in R$

b $\boldsymbol{r}(t) = (1 - \cos t)\boldsymbol{i} + \sin t\boldsymbol{j} \quad t \in R$

Let (x, y) be any point on the curve.

$\therefore \; x = 1 - \cos t \quad \boxed{3}$

$y = \sin t \quad \boxed{4}$

In $\boxed{3}$ $\cos t = 1 - x$

In $\boxed{4}$ $y^2 = \sin^2 t = 1 - \cos^2 t = 1 - (1 - x)^2$

$= -x^2 + 2x$

$\therefore \quad y^2 = -x^2 + 2x$ is the cartesian equation.

In Example 1b, the **domain** of the corresponding cartesian relation can be determined by the range of the function $x(t) = 1 - \cos t$. The range of this function is $[0, 2]$. The **range** of the cartesian relation can be determined by the range of the function $y(t) = \sin(t)$. The range of this function is $[-1, 1]$.

Therefore the domain of $y^2 = -x^2 + 2x$ is $[0, 2]$ and the range of $y^2 = -x^2 + 2x$ is $[-1, 1]$.

It can also be seen that the cartesian equation is $(x - 1)^2 + y^2 = 1$, i.e. it is the circle with centre $(1, 0)$ and radius 1.

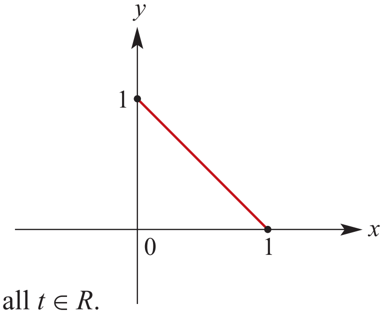
Example 2

Find the cartesian equation of each of the following. State the domain and range and sketch the graph of each of the relations:

a $\boldsymbol{r}(t) = \cos^2 t\boldsymbol{i} + \sin^2 t\boldsymbol{j}, t \in R$ **b** $\boldsymbol{r}(t) = t\boldsymbol{i} + (1 - t)\boldsymbol{j}, t \in R$

Solution

a Let (x, y) be any point on the curve $\boldsymbol{r}(t) = \cos^2 t\boldsymbol{i} + \sin^2 t\boldsymbol{j}, t \in R$.



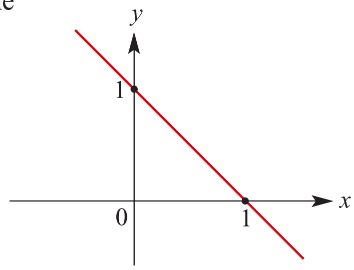
$\therefore \quad x = \cos^2 t$ and $y = \sin^2 t$

$\therefore \quad 1 - x = 1 - \cos^2 t = \sin^2 t = y$

i.e. $\quad y = 1 - x$

Also $0 \le \cos^2 t \le 1$ and $0 \le \sin^2 t \le 1$ for all $t \in R$. Hence the domain of the relation is $[0, 1]$ and the range is $[0, 1]$.

b Let (x, y) be any point on the curve defined by $\boldsymbol{r}(t) = t\boldsymbol{i} + (1 - t)\boldsymbol{j}$.



$x = t$ and $y = 1 - t$

$\therefore \quad y = 1 - x$

and the domain = range = R

Example 3

For each of the following, state the cartesian equation, the domain and range of the corresponding cartesian relation and sketch each of the graphs:

a $\boldsymbol{r}(\lambda) = 2\sec(\lambda)\boldsymbol{i} + \tan(\lambda)\boldsymbol{j}$ **b** $\boldsymbol{r}(\lambda) = (1 - 2\cos(\lambda))\boldsymbol{i} + 3\sin(\lambda)\boldsymbol{j}$

Solution

a Let (x, y) be any point on the curve.

Then $\quad x = 2\sec(\lambda)$ and $y = \tan(\lambda)$

Note: $\lambda \in R \backslash \left\{(2n+1)\frac{\pi}{2}, n \in Z\right\}$

Hence $\quad x^2 = 4\sec^2(\lambda)$ and $y^2 = \tan^2(\lambda)$

$\therefore \quad \frac{x^2}{4} = \sec^2(\lambda)$ and $y^2 = \tan^2(\lambda)$

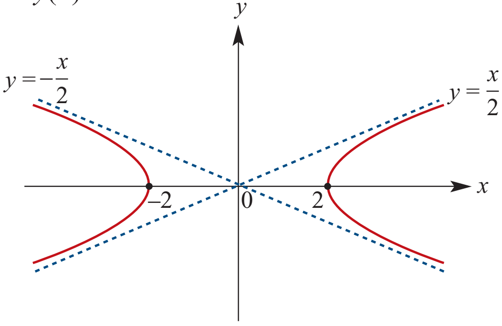
But $\quad \sec^2(\lambda) - \tan^2(\lambda) = 1$

$\therefore \quad \frac{x^2}{4} - y^2 = 1$

The domain of the relation = range of $x(\lambda) = (-\infty, -2] \cup [2, \infty)$.

The range of the relation = range of $y(\lambda) = R$.

The relation represents a hyperbola with centre the origin and asymptotes $y = \pm\frac{x}{2}$.



Note: The graph is produced for $\lambda \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \frac{3\pi}{2}\right)$ and repeated infinitely.

b $\quad r(\lambda) = (1 - 2\cos(\lambda))\boldsymbol{i} + 3\sin(\lambda)\boldsymbol{j}$

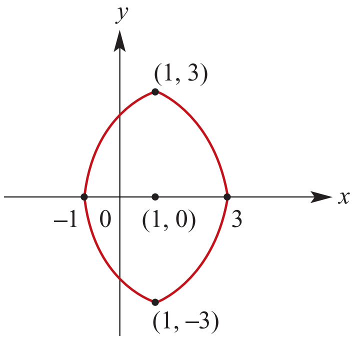
Let $\quad x = 1 - 2\cos(\lambda)$ and $y = 3\sin(\lambda)$

Then $\dfrac{x-1}{-2} = \cos(\lambda)$ and $\dfrac{y}{3} = \sin(\lambda)$

Squaring each and adding yields

$$\frac{(x-1)^2}{4} + \frac{y^2}{9} = \cos^2(\lambda) + \sin^2(\lambda) = 1$$

The relation represents an ellipse with centre (1, 0). The domain of the relation is $[-1, 3]$ and the range is $[-3, 3]$.



Exercise 12A

1 Find the cartesian equation which corresponds to each of the following vector equations. State the domain and range of the cartesian relation.

a $r(t) = t\boldsymbol{i} + 2t\boldsymbol{j} \quad t \in R$

b $r(t) = 2\boldsymbol{i} + 5t\boldsymbol{j} \quad t \in R$

c $r(t) = -t\boldsymbol{i} + 7\boldsymbol{j} \quad t \in R$

d $r(t) = (2 - t)\boldsymbol{i} + (t + 7)\boldsymbol{j} \quad t \in R$

e $r(t) = t^2\boldsymbol{i} + (2 - 3t)\boldsymbol{j} \quad t \in R$

f $r(t) = (t - 3)\boldsymbol{i} + (t^3 + 1)\boldsymbol{j} \quad t \in R$

g $r(t) = (2t + 1)\boldsymbol{i} + 3^t\boldsymbol{j} \quad t \in R$

h $r(t) = \left(t - \dfrac{\pi}{2}\right)\boldsymbol{i} + \cos 2t\boldsymbol{j} \quad t \in R$

i $r(t) = \dfrac{1}{t+4}\boldsymbol{i} + (t^2 + 1)\boldsymbol{j} \quad t \neq -4$

j $r(t) = \dfrac{1}{t}\boldsymbol{i} + \dfrac{1}{t+1}\boldsymbol{j} \quad t \neq 0, -1$

2 Find the cartesian relation which corresponds to each of the following. State the domain and range of each of the relations and sketch the graph of each.

a $r(t) = 2\cos(t)\boldsymbol{i} + 3\sin(t)\boldsymbol{j} \quad t \in R$

b $r(t) = 2\cos^2(t)\boldsymbol{i} + 3\sin^2(t)\boldsymbol{j} \quad t \in R$

c $r(t) = t\boldsymbol{i} + 3t^2\boldsymbol{j} \quad t \geq 0$

d $r(t) = t^3\boldsymbol{i} + 3t^2\boldsymbol{j} \quad t \geq 0$

e $r(\lambda) = \cos(\lambda)\boldsymbol{i} + \sin(\lambda)\boldsymbol{j} \quad \lambda \in \left[0, \dfrac{\pi}{2}\right]$

f $r(\lambda) = 3\sec(\lambda)\boldsymbol{i} + 2\tan(\lambda)\boldsymbol{j} \quad \lambda \in \left(0, \dfrac{\pi}{2}\right)$

g $r(t) = 4\cos(2t)\boldsymbol{i} + 4\sin(2t)\boldsymbol{j} \quad t \in \left[0, \dfrac{\pi}{2}\right]$

h $r(\lambda) = 3\sec^2(\lambda)\boldsymbol{i} + 2\tan^2(\lambda)\boldsymbol{j} \quad \lambda \in \left(-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right)$

i $r(t) = (3 - t)\boldsymbol{i} + (5t^2 + 6t)\boldsymbol{j} \quad t \in R$

3 Find a vector equation which corresponds to each of the following. Note that the answers given are the 'natural choice' but your answer could be different.

a $y = 3 - 2x$ **b** $x^2 + y^2 = 4$ **c** $(x - 1)^2 + y^2 = 4$

d $x^2 - y^2 = 4$ **e** $y = (x - 3)^2 + 2(x - 3)$ **f** $2x^2 + 3y^2 = 12$

4 A circle of radius 5 has its centre at the point C with position vector $2\boldsymbol{i} + 6\boldsymbol{j}$ relative to the origin O. A general point P on the circle has position $\boldsymbol{r}$ relative to O. The angle between $\boldsymbol{i}$ and $\overrightarrow{CP}$ measured in anticlockwise sense from $\boldsymbol{i}$ to $\overrightarrow{CP}$ is denoted by θ.

a Give the vector equation for P. **b** Give the cartesian equation for P.

12.2 Position vectors as a function of time

Consider a particle travelling along a circular path, in an anticlockwise direction, with radius length of one unit and centre at O. The **cartesian equation** of the path is:

$$\{(x, y): x^2 + y^2 = 1\}$$

If the particle starts at (1, 0) when $t = 0$, then this path can be represented by: $\{(x, y): x = \cos t, y = \sin t, t \geq 0\}$.

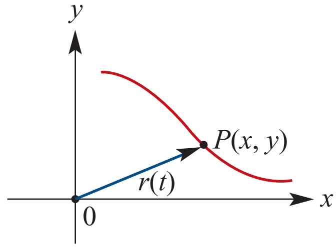
This is called the **parametric form** of the path. It can also be expressed in vector form as:

$$\boldsymbol{r}(t) = \cos t\boldsymbol{i} + \sin t\boldsymbol{j}$$

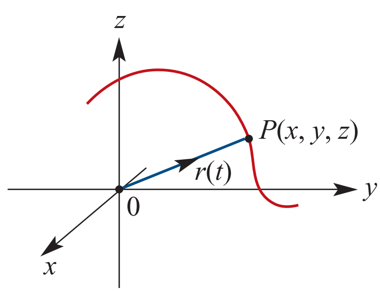
$\boldsymbol{r}(t)$ is called the position vector of the particle at time t.

The graph of a vector function is the set of points determined by the function $\boldsymbol{r}(t)$ as t varies.

In two dimensions, the x axis and y axis are used.



In three dimensions, three mutually perpendicular axes are used. It is best to consider the x axis and y axis as in the horizontal plane and the z axis as vertical and through the point of intersection of the x axis and y axis.



If $t \geq 0$, $\boldsymbol{r}(0)$ is the vector specifying the initial position of the particle. Note that the direction of motion of the particle can be determined through the consideration of increasing t. The vector equation $\boldsymbol{r}(t) = \cos t\boldsymbol{i} + \sin t\boldsymbol{j}$ indicates that the particle starts at the point (1, 0) or the point with position vector $\boldsymbol{i}$. Further, as t increases, it can be seen that the particle is travelling anticlockwise.

Note that the vector equation gives more information about the motion of the particle than the cartesian equation.

The vector equation $\boldsymbol{r}(t) = \cos(t)\boldsymbol{i} + \sin(t)\boldsymbol{j}$ indicates that:

- at time $t = 0$, the particle is at (1, 0)
- the particle moves on the curve with equation $x^2 + y^2 = 1$
- the particle moves in an anticlockwise direction
- the particle moves around the circle with a period of 2π, i.e. it takes 2π units of time to complete one circle

The vector equation $\boldsymbol{r}(t) = \cos(2\pi t)\boldsymbol{i} + \sin(2\pi t)\boldsymbol{j}$ describes a particle moving anticlockwise around a circle with equation $x^2 + y^2 = 1$ but this time the period is one unit of time.

The vector equation $\boldsymbol{r}(t) = -\cos(2\pi t)\boldsymbol{i} + \sin(2\pi t)\boldsymbol{j}$ again describes a particle moving around the circle but with the following features:

- at time $t = 0$, the particle is at $(-1, 0)$
- the particle moves on the curve with equation $x^2 + y^2 = 1$
- the particle moves in a clockwise direction
- the particle moves around the circle with a period of one unit

Example 4

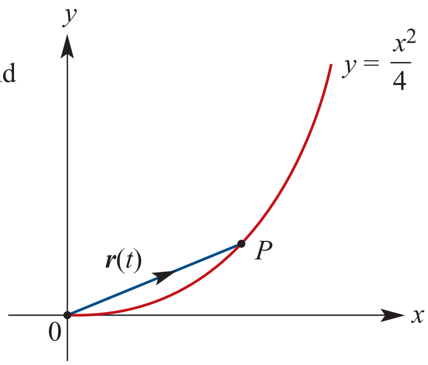
Sketch the path of a particle where the position at time t is given by $\boldsymbol{r}(t) = 2t\boldsymbol{i} + t^2\boldsymbol{j}$, $t \geq 0$.

Solution

Now $x = 2t$ and $y = t^2$ which implies $t = \frac{x}{2}$ and $y = \left(\frac{x}{2}\right)^2$.

The cartesian form is $y = \frac{x^2}{4}$ and $x \geq 0$.

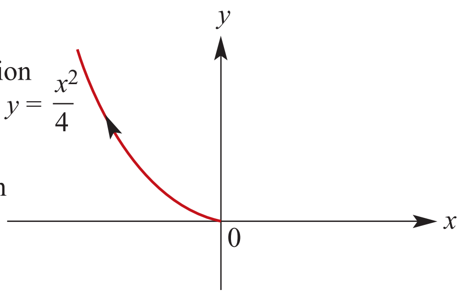
Now, since $\boldsymbol{r}(0) = \boldsymbol{0}$ and $\boldsymbol{r}(1) = 2\boldsymbol{i} + \boldsymbol{j}$, it can be seen that the particle starts at the origin and moves along the parabola $y = \frac{x^2}{4}$ with $x \geq 0$.



Note that the equation $\boldsymbol{r}(t) = t\boldsymbol{i} + \frac{t^2}{4}\boldsymbol{j}$ gives the same cartesian path, but the rate at which the particle moves along the path is different.

If $\boldsymbol{r}(t) = -t\boldsymbol{i} + \frac{t^2}{4}\boldsymbol{j}$ then again the cartesian equation is $y = \frac{x^2}{4}$ but $x \leq 0$.

Hence the motion is along the curve shown and in the direction indicated.



When a particle moves on a curve in a plane (two dimensions):

$$\boldsymbol{r}(t) = x(t)\boldsymbol{i} + y(t)\boldsymbol{j}$$

is the appropriate form of the vector function specifying position.

For three dimensional motion:

$$\boldsymbol{r}(t) = x(t)\boldsymbol{i} + y(t)\boldsymbol{j} + z(t)\boldsymbol{k}$$

is the appropriate form of the vector function specifying position.

Example 5

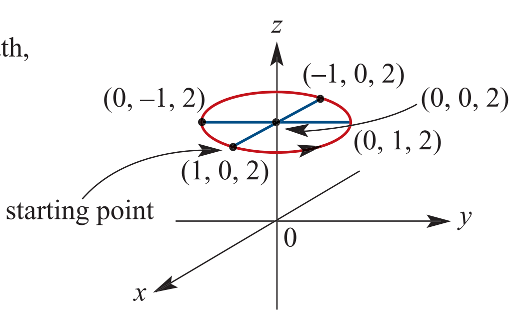
An object moves along a path where the position vector is given by
$\boldsymbol{r}(t) = \cos t\boldsymbol{i} + \sin t\boldsymbol{j} + 2\boldsymbol{k},\ t \geq 0$.
Describe the motion of the object.

Solution

Being unfamiliar with the graphs of relations in three dimensions, it is probably best to determine a number of position vectors (points) and try to visualise joining the dots.

t	$\boldsymbol{r}(t)$	Point
0	$\boldsymbol{i} + 2\boldsymbol{k}$	$(1, 0, 2)$
$\frac{\pi}{2}$	$\boldsymbol{j} + 2\boldsymbol{k}$	$(0, 1, 2)$
π	$-\boldsymbol{i} + 2\boldsymbol{k}$	$(-1, 0, 2)$
$\frac{3\pi}{2}$	$-\boldsymbol{j} + 2\boldsymbol{k}$	$(0, -1, 2)$
2π	$\boldsymbol{i} + 2\boldsymbol{k}$	$(1, 0, 2)$
	etc.	

The object is moving along a circular path, centred on (0, 0, 2) with radius length 1, starting at (1, 0, 2) and moving anticlockwise when viewed from above, always a distance of two ‘above’ the $x-y$ plane (horizontal plane).



Example 6

The motion of two particles is given by the vector equations $\boldsymbol{r}_1(t) = (2t - 3)\boldsymbol{i} + (t^2 + 10)\boldsymbol{j}$ and $\boldsymbol{r}_2(t) = (t + 2)\boldsymbol{i} + 7t\boldsymbol{j}$, where $t \geq 0$. Find:

a the point at which the particles collide

b the points at which the two paths cross

c the distance between the particles when $t = 1$.

Solution

a The two particles collide when they share the same position at the same time.

$$\therefore \quad \boldsymbol{r}_1(t) = \boldsymbol{r}_2(t)$$
$$(2t-3)\boldsymbol{i} + (t^2+10)\boldsymbol{j} = (t+2)\boldsymbol{i} + 7t\boldsymbol{j}$$
$$\therefore \quad 2t-3 = t+2 \qquad \boxed{1}$$
$$\text{and } t^2+10 = 7t \qquad \boxed{2}$$

From $\boxed{1}$ $t = 5$

In $\boxed{2}$ $\quad t^2 - 7t + 10 = 0$

$$(t-5)(t-2) = 0$$
$$\therefore \quad t = 5, 2$$

$\therefore$ The particles will share the same common point when $t = 5$, i.e. they will collide at the point (7, 35).

b At the points where the paths cross, the two paths share common points which may occur at different times for each particle. The vector equations are, therefore, redefined to differentiate between the two time variables.

$$\boldsymbol{r}_1(t) = (2t-3)\boldsymbol{i} + (t^2+10)\boldsymbol{j}$$
$$\boldsymbol{r}_2(s) = (s+2)\boldsymbol{i} + 7s\boldsymbol{j}$$

When the paths cross:

$$2t-3 = s+2 \qquad \boxed{3}$$
$$\text{and } t^2+10 = 7s \qquad \boxed{4}$$

Solving these equations simultaneously:

$\boxed{3}$ becomes $\quad s = 2t-5$

Substituting in $\boxed{4}$ $\quad t^2+10 = 7(2t-5)$

$$\therefore \quad t^2 - 14t + 45 = 0$$
$$(t-9)(t-5) = 0$$
$$t = 5, 9$$

The corresponding values for s are 5 and 13.

These values can be substituted back into the vector equations to obtain the points at which the paths cross, i.e. (7, 35) and (15, 91).

c When $t = 1$ $\quad \boldsymbol{r}_1(1) = -\boldsymbol{i} + 11\boldsymbol{j}$

$$\boldsymbol{r}_2(1) = 3\boldsymbol{i} + 7\boldsymbol{j}$$

The vector representing the displacement between the two particles after one second is $\boldsymbol{r}_1(1) - \boldsymbol{r}_2(1) = -4\boldsymbol{i} + 4\boldsymbol{j}$

The distance between the two particles $= \sqrt{(-4)^2 + 4^2} = 4\sqrt{2}$ units

Exercise 12B

1 The path of a particle with respect to a particular point is described as a function of time, t, by the vector equation $\boldsymbol{r}(t) = \cos t\boldsymbol{i} + \sin t\boldsymbol{j}$, $t \geq 0$.

a Find the cartesian equation of the path.

b Sketch the path of the particle.

c Find the times at which the particle crosses the y axis.

2 Repeat question **1** for paths described by the following vector equations:

a $\boldsymbol{r}(t) = (t^2 - 9)\boldsymbol{i} + 8t\boldsymbol{j}$, where $t \geq 0$

b $\boldsymbol{r}(t) = (t+1)\boldsymbol{i} + \dfrac{1}{t+2}\boldsymbol{j}$, $t > -2$

c $\boldsymbol{r}(t) = \dfrac{t-1}{t+1}\boldsymbol{i} + \dfrac{2}{t+1}\boldsymbol{j}$, $t > -1$

3 The paths of two particles with respect to time t are described by the vector equations $\boldsymbol{r}_1(t) = (3t-5)\boldsymbol{i} + (8-t^2)\boldsymbol{j}$ and $\boldsymbol{r}_2(t) = (3-t)\boldsymbol{i} + 2t\boldsymbol{j}$ where $t \geq 0$. Find:

a the point at which the two particles collide

b the points at which the two paths cross

c the distance between the two particles when $t = 3$

4 Repeat question **3** for paths defined by the following vector equations, where $t \geq 0$:

a $\boldsymbol{r}_1(t) = (2t^2+4)\boldsymbol{i} + (t-2)\boldsymbol{j}$

b $\boldsymbol{r}_2(t) = 9t\boldsymbol{i} + 3(t-1)\boldsymbol{j}$

5 The path of a particle defined as a function of time t is given by the vector equation $\boldsymbol{r}(t) = (1+t)\boldsymbol{i} + (3t+2)\boldsymbol{j}$ where $t \geq 0$. Find:

a the distance of the particle from the origin when $t = 3$

b the times at which the distance of the particle from the origin is one unit

6 $\boldsymbol{r}(t) = t\boldsymbol{i} + 2t\boldsymbol{j} - 3\boldsymbol{k}$ where $t \geq 0$, is the vector equation representing the motion of a particle with respect to time t. Find:

a the position, A, of the particle when $t = 3$

b the distance of the particle from the origin when $t = 3$

c the position, B, of the particle when $t = 4$

d the displacement of the particle in the fourth second in vector form

7 $\boldsymbol{r}(t) = (t+1)\boldsymbol{i} + (3-t)\boldsymbol{j} + 2t\boldsymbol{k}$, where $t \geq 0$ is the vector equation representing the motion of a particle with respect to time t. Find:

a the position of the particle when $t = 2$

b the distance of the particle from the point $(4, -1, 1)$ when $t = 2$

8 $\boldsymbol{r}(t) = at^2\boldsymbol{i} + (b-t)\boldsymbol{j}$ is the vector equation representing the motion of a particle with respect to time t. When $t = 3$, the position of the particle is $(6, 4)$. Find a and b.

9 A particle travels in a path such that the position vector $\boldsymbol{r}(t)$ at time t is given by $\boldsymbol{r}(t) = 3\cos t\boldsymbol{i} + 2\sin t\boldsymbol{j}$, $t \geq 0$.

a Express this vector function as a cartesian relation.

b Find the initial position of the particle.

c If the positive y axis points north and the positive x axis points east, find correct to two decimal places, the bearing of the point P, the position of the particle at $t = \frac{3\pi}{4}$ from:

i the origin **ii** the initial position

10 An object moves so that the position vector $\boldsymbol{r}(t)$ at time t is given by $\boldsymbol{r}(t) = e^t\boldsymbol{i} + e^{-t}\boldsymbol{j}$, $t \geq 0$.

a Express this vector function as a cartesian relation.

b Find the initial position of the object.

c Sketch the graph of the path travelled by the object, indicating the direction of motion.

11 An object is moving so that its position $\boldsymbol{r}$ at time t is given by $\boldsymbol{r}(t) = (e^t + e^{-t})\boldsymbol{i} + (e^t - e^{-t})\boldsymbol{j}$, $t \geq 0$.

a Find the initial position of the object. **b** Find the position at $t = \log_e 2$.

c Find the cartesian equation of the path.

12 An object is projected so that its position, $\boldsymbol{r}$, at time t is given by $\boldsymbol{r}(t) = 100\,t\boldsymbol{i} + (100\sqrt{3}\,t - 5t^2)\boldsymbol{j}$, $0 \leq t \leq 20\sqrt{3}$.

a Find the initial and final positions of the object.

b Find the cartesian form of the path.

c Sketch the graph of the path, indicating the direction of motion.

13 Two particles, A and B, have position vectors $\boldsymbol{r}_A(t)$ and $\boldsymbol{r}_B(t)$ respectively at time t, given by $\boldsymbol{r}_A(t) = 6t^2\boldsymbol{i} + (2t^3 - 18t)\boldsymbol{j}$ and $\boldsymbol{r}_B(t) = (13t - 6)\boldsymbol{i} + (3t^2 - 27)\boldsymbol{j}$ where $t \geq 0$. Find where and when the particles collide.

14 The motion of a particle is described by the vector equation $\boldsymbol{r}(t) = (1 - 2\cos 2t)\boldsymbol{i} + (3 - 5\sin 2t)\boldsymbol{j}$; $t \geq 0$. Find:

a the cartesian equation of the path

b the position at

i $t = 0$ **ii** $t = \frac{\pi}{4}$ **iii** $t = \frac{\pi}{2}$

c the time taken by the particle to return to its initial position

d the direction of motion along the curve.

15 The motion of a particle is described by the vector equation $\boldsymbol{r}(t) = 3\cos t\boldsymbol{i} + 3\sin t\boldsymbol{j} + \boldsymbol{k}$, $t \geq 0$. Describe the motion of the particle.

16 The motion of a particle is described by the vector equation $\boldsymbol{r}(t) = t\boldsymbol{i} + 3t\boldsymbol{j} + t\boldsymbol{k}$, $t \geq 0$. Describe the motion of the particle.

17 For each of the following vector equations:

i find the cartesian equation of the body's path **ii** sketch the path

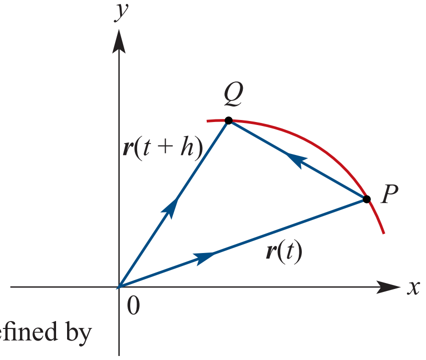
iii describe the motion of the body.

a $\boldsymbol{r}(t) = \cos^2(3\pi t)\boldsymbol{i} + 2\cos^2(3\pi t)\boldsymbol{j},\ t \geq 0$

b $\boldsymbol{r}(t) = \cos(2\pi t)\boldsymbol{i} + \cos(4\pi t)\boldsymbol{j},\ t \geq 0$

c $\boldsymbol{r}(t) = e^t\boldsymbol{i} + e^{-2t}\boldsymbol{j},\ t \geq 0$

12.3 Vector calculus



Consider the curve defined by $\boldsymbol{r}(t)$ with P and Q having position vectors $\boldsymbol{r}(t)$ and $\boldsymbol{r}(t+h)$ respectively.

Now $\overrightarrow{PQ} = \boldsymbol{r}(t+h) - \boldsymbol{r}(t)$.

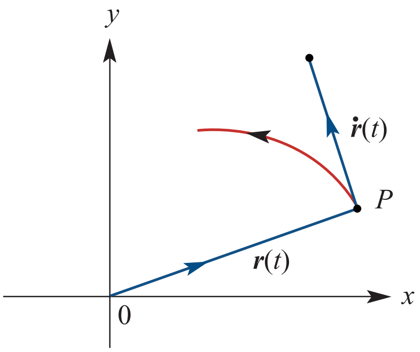
It follows that:

$$\frac{1}{h}(\boldsymbol{r}(t+h) - \boldsymbol{r}(t))$$

is a vector parallel to $\overrightarrow{PQ}$.

As $h \to 0$, Q approaches P along the curve. The derivative of $\boldsymbol{r}$ with respect to t is denoted by $\dot{\boldsymbol{r}}$ and is defined by

$$\dot{\boldsymbol{r}}(t) = \lim_{h\to 0}\frac{\boldsymbol{r}(t+h) - \boldsymbol{r}(t)}{h}$$



$\dot{\boldsymbol{r}}(t)$ is a vector along the tangent at P in the direction of increasing t.

The derivative of a vector function $\boldsymbol{r}(t)$ is denoted by $\frac{d\boldsymbol{r}}{dt}$ or $\dot{\boldsymbol{r}}$ and sometimes as $\boldsymbol{r}'(t)$.

Consider:

$$\dot{\boldsymbol{r}}(t) = \lim_{h\to 0}\frac{\boldsymbol{r}(t+h) - \boldsymbol{r}(t)}{h}$$

where $\boldsymbol{r}(t) = x(t)\boldsymbol{i} + y(t)\boldsymbol{j}$

Now
$$\begin{aligned}\dot{\boldsymbol{r}}(t) &= \lim_{h\to 0}\frac{\boldsymbol{r}(t+h) - \boldsymbol{r}(t)}{h}\\ &= \lim_{h\to 0}\frac{(x(t+h)\boldsymbol{i} + y(t+h)\boldsymbol{j}) - (x(t)\boldsymbol{i} + y(t)\boldsymbol{j})}{h}\\ &= \lim_{h\to 0}\frac{x(t+h)\boldsymbol{i} - x(t)\boldsymbol{i}}{h} + \lim_{h\to 0}\frac{y(t+h)\boldsymbol{j} - y(t)\boldsymbol{j}}{h}\\ &= \lim_{h\to 0}\frac{x(t+h) - x(t)}{h}\boldsymbol{i} + \lim_{h\to 0}\frac{y(t+h) - y(t)}{h}\boldsymbol{j}\end{aligned}$$

Hence $\dot{\boldsymbol{r}}(t) = \frac{d\boldsymbol{r}}{dt} = \frac{dx}{dt}\boldsymbol{i} + \frac{dy}{dt}\boldsymbol{j}$

$\ddot{\boldsymbol{r}}(t) = \frac{d^2x}{dt^2}\boldsymbol{i} + \frac{d^2y}{dt^2}\boldsymbol{j}$ is the second derivative of $\boldsymbol{r}(t)$

This can be extended to three-dimensional vector functions.

For $\boldsymbol{r}(t) = x(t)\boldsymbol{i} + y(t)\boldsymbol{j} + z(t)\boldsymbol{k}$

$$\dot{\boldsymbol{r}}(t) = \frac{dx}{dt}\boldsymbol{i} + \frac{dy}{dt}\boldsymbol{j} + \frac{dz}{dt}\boldsymbol{k}$$

and $$\ddot{\boldsymbol{r}}(t) = \frac{d^2x}{dt^2}\boldsymbol{i} + \frac{d^2y}{dt^2}\boldsymbol{j} + \frac{d^2z}{dt^2}\boldsymbol{k}$$

Example 7

Find $\dot{\boldsymbol{r}}(t)$ and $\ddot{\boldsymbol{r}}(t)$ if $\boldsymbol{r}(t) = 20t\boldsymbol{i} + (15t - 5t^2)\boldsymbol{j}$.

Solution

$$\dot{\boldsymbol{r}}(t) = 20\boldsymbol{i} + (15 - 10t)\boldsymbol{j}$$

and $\ddot{\boldsymbol{r}}(t) = -10\boldsymbol{j}$

Example 8

Find $\dot{\boldsymbol{r}}(t)$ and $\ddot{\boldsymbol{r}}(t)$ if $\boldsymbol{r}(t) = \cos t\boldsymbol{i} - \sin t\boldsymbol{j} + 5t\boldsymbol{k}$.

Solution

$$\dot{\boldsymbol{r}}(t) = -\sin t\boldsymbol{i} - \cos t\boldsymbol{j} + 5\boldsymbol{k}$$

and $\ddot{\boldsymbol{r}}(t) = -\cos t\boldsymbol{i} + \sin t\boldsymbol{j}$

Example 9

If $\boldsymbol{r}(t) = t\boldsymbol{i} + ((t-1)^3 + 1)\boldsymbol{j}$ find $\dot{\boldsymbol{r}}(\alpha)$ and $\ddot{\boldsymbol{r}}(\alpha)$ where $\boldsymbol{r}(\alpha) = \boldsymbol{i} + \boldsymbol{j}$.

Solution

If $\quad \boldsymbol{r}(t) = t\boldsymbol{i} + ((t-1)^3 + 1)\boldsymbol{j}$

$\quad \dot{\boldsymbol{r}}(t) = \boldsymbol{i} + 3(t-1)^2\boldsymbol{j}$

and $\quad \ddot{\boldsymbol{r}}(t) = 6(t-1)\boldsymbol{j}$

Since $\quad \boldsymbol{r}(\alpha) = \boldsymbol{i} + \boldsymbol{j} = \alpha\boldsymbol{i} + ((\alpha - 1)^3 + 1)\boldsymbol{j}$

$\quad \alpha = 1$

and $\quad \dot{\boldsymbol{r}}(1) = \boldsymbol{i}$ and $\ddot{\boldsymbol{r}}(1) = \boldsymbol{0}$

Example 10

If $\boldsymbol{r}(t) = e^t\boldsymbol{i} + ((e^t - 1)^3 + 1)\boldsymbol{j}$, find $\dot{\boldsymbol{r}}(\alpha)$ and $\ddot{\boldsymbol{r}}(\alpha)$ where $\boldsymbol{r}(\alpha) = \boldsymbol{i} + \boldsymbol{j}$.

Solution

Now $\quad \boldsymbol{r}(t) = e^t\boldsymbol{i} + ((e^t - 1)^3 + 1)\boldsymbol{j}$

gives: $\quad \dot{\boldsymbol{r}}(t) = e^t\boldsymbol{i} + 3e^t(e^t - 1)^2\boldsymbol{j}$

and $\quad \ddot{\boldsymbol{r}}(t) = e^t\boldsymbol{i} + (6e^{2t}(e^t - 1) + 3e^t(e^t - 1)^2)\boldsymbol{j}$

If $\quad \boldsymbol{r}(\alpha) = \boldsymbol{i} + \boldsymbol{j} = e^\alpha\boldsymbol{i} + ((e^\alpha - 1)^3 + 1)\boldsymbol{j}$

then $\quad \alpha = 0$

and $\quad \dot{\boldsymbol{r}}(0) = \boldsymbol{i}$ and $\ddot{\boldsymbol{r}}(0) = \boldsymbol{i}$

Example 11

A curve is described by the vector equation $\boldsymbol{r}(t) = 2\cos t\boldsymbol{i} + 3\sin t\boldsymbol{j}$.

a Find:

i $\dot{\boldsymbol{r}}(t)$ **ii** $\ddot{\boldsymbol{r}}(t)$

b Find the gradient of the curve at the point (x, y) where $x = 2\cos t$ and $y = 3\sin t$.

Solution

a **i** $\dot{\boldsymbol{r}}(t) = -2\sin t\boldsymbol{i} + 3\cos t\boldsymbol{j}$

ii $\ddot{\boldsymbol{r}}(t) = -2\cos t\boldsymbol{i} - 3\sin t\boldsymbol{j} = -\boldsymbol{r}(t)$

b Related rates can be used to find $\frac{dy}{dx}$.

$$\frac{dy}{dx} = \frac{dy}{dt}\frac{dt}{dx} \text{ and } \frac{dx}{dt} = -2\sin t,\ \frac{dy}{dt} = 3\cos t$$

$$\therefore \quad \frac{dy}{dx} = 3\cos t.\frac{1}{-2\sin t} = \frac{-3}{2}\cot(t)$$

Note that the gradient is undefined when $\sin t = 0$.

Example 12

A curve is described by the vector equation $\boldsymbol{r}(t) = \sec(t)\boldsymbol{i} + \tan(t)\boldsymbol{j}$ with $t \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. Find the gradient of the curve at the point (x, y) where $x = \sec(t)$ and $y = \tan(t)$. Find the gradient of the curve where $t = \frac{\pi}{4}$.

Solution

$$x = \sec(t) = \frac{1}{\cos(t)} = (\cos t)^{-1} \text{ and } y = \tan(t)$$

$$\frac{dx}{dt} = -(\cos t)^{-2}(-\sin(t)) \qquad \frac{dy}{dt} = \sec^2(t)$$

$$= \frac{\sin(t)}{\cos^2(t)}$$

$$= \tan(t)\sec(t)$$

$$\therefore \quad \frac{dy}{dx} = \frac{dy}{dt}\frac{dt}{dx} = \sec^2(t).\frac{1}{\tan(t)\sec(t)} = \sec(t)\cot(t) = \frac{1}{\sin(t)}$$

$$\text{When} \quad t = \frac{\pi}{4},\ \frac{dy}{dx} = \frac{1}{\sin\left(\frac{\pi}{4}\right)} = \sqrt{2}$$

Antidifferentiation

Consider
$$\int \boldsymbol{r}(t)\,dt = \int x(t)\boldsymbol{i} + y(t)\boldsymbol{j} + z(t)\boldsymbol{k}\,dt$$
$$= \int x(t)\,dt\,\boldsymbol{i} + \int y(t)\,dt\,\boldsymbol{j} + \int z(t)\,dt\,\boldsymbol{k}$$
$$= X(t)\boldsymbol{i} + Y(t)\boldsymbol{j} + Z(t)\boldsymbol{k} + \boldsymbol{c}$$

where $\boldsymbol{c}$ is a constant vector and $\frac{dX}{dt} = x(t),\ \frac{dY}{dt} = y(t),\ \frac{dZ}{dt} = z(t)$

Note that $\frac{dc}{dt} = 0$

Example 13

Given that $\ddot{\boldsymbol{r}}(t) = 10\boldsymbol{i} - 12\boldsymbol{k}$, find:

a $\dot{\boldsymbol{r}}(t)$ if $\dot{\boldsymbol{r}}(0) = 30\boldsymbol{i} - 20\boldsymbol{j} + 10\boldsymbol{k}$ **b** $\boldsymbol{r}(t)$ if $\boldsymbol{r}(0) = 0\boldsymbol{i} + 0\boldsymbol{j} + 2\boldsymbol{k}$

Solution

a $\dot{\boldsymbol{r}}(t) = 10t\boldsymbol{i} - 12t\boldsymbol{k} + \boldsymbol{c}_1$ where $\boldsymbol{c}_1$ is a constant vector.

$\dot{\boldsymbol{r}}(0) = 30\boldsymbol{i} - 20\boldsymbol{j} + 10\boldsymbol{k}$

Therefore $\boldsymbol{c}_1 = 30\boldsymbol{i} - 20\boldsymbol{j} + 10\boldsymbol{k}$

and $\dot{\boldsymbol{r}}(t) = 10t\boldsymbol{i} - 12t\boldsymbol{k} + 30\boldsymbol{i} - 20\boldsymbol{j} + 10\boldsymbol{k}$

$= (10t + 30)\boldsymbol{i} - 20\boldsymbol{j} + (10 - 12t)\boldsymbol{k}$

b $\boldsymbol{r}(t) = (5t^2 + 30t)\boldsymbol{i} - 20t\boldsymbol{j} + (10t - 6t^2)\boldsymbol{k} + \boldsymbol{c}_2$ where $\boldsymbol{c}_2$ is a constant vector.

$\boldsymbol{r}(0) = 0\boldsymbol{i} + 0\boldsymbol{j} + 2\boldsymbol{k}$ and therefore $\boldsymbol{c}_2 = 2\boldsymbol{k}$

Hence $\boldsymbol{r}(t) = (5t^2 + 30t)\boldsymbol{i} - 20t\boldsymbol{j} + (10t - 6t^2 + 2)\boldsymbol{k}$

Example 14

Given $\ddot{\boldsymbol{r}}(t) = -9.8\boldsymbol{j}$, $\boldsymbol{r}(0) = \boldsymbol{0}$ and $\dot{\boldsymbol{r}}(0) = 30\boldsymbol{i} + 40\boldsymbol{j}$, find $\boldsymbol{r}(t)$.

Solution

$\ddot{\boldsymbol{r}}(\mathbf{t}) = -9.8\boldsymbol{j}$

$\therefore \quad \dot{\boldsymbol{r}}(t) = \int -9.8\, dt\, \boldsymbol{j}$

$= -9.8t\boldsymbol{j} + \boldsymbol{c}_1$

but $\dot{\boldsymbol{r}}(0) = 30\boldsymbol{i} + 40\boldsymbol{j}$

$\therefore \quad \boldsymbol{c}_1 = 30\boldsymbol{i} + 40\boldsymbol{j}$

$\therefore \quad \dot{\boldsymbol{r}}(t) = 30\boldsymbol{i} + (40 - 9.8t)\boldsymbol{j}$

Hence $\boldsymbol{r}(t) = \int 30\, dt\, \boldsymbol{i} + \int (40 - 9.8t)\, dt\, \boldsymbol{j}$

$= 30t\boldsymbol{i} + (40t - 4.9t^2)\boldsymbol{j} + \boldsymbol{c}_2$

Now $\boldsymbol{r}(0) = \boldsymbol{0}$ and therefore $\boldsymbol{c}_2 = \boldsymbol{0}$

$\therefore \quad \boldsymbol{r}(t) = 30t\boldsymbol{i} + (40t - 4.9t^2)\boldsymbol{j}$

Exercise 12C

1 Find $\dot{\boldsymbol{r}}(t)$ and $\ddot{\boldsymbol{r}}(t)$ in each of the following:

a $\boldsymbol{r}(t) = e^t\boldsymbol{i} + e^{-t}\boldsymbol{j}$

b $\boldsymbol{r}(t) = t\boldsymbol{i} + t^2\boldsymbol{j}$

c $\boldsymbol{r}(t) = \frac{1}{2}t\boldsymbol{i} + t^2\boldsymbol{j}$

d $\boldsymbol{r}(t) = 16t\boldsymbol{i} - 4(4t - 1)^2\boldsymbol{j}$

e $\boldsymbol{r}(t) = \sin(t)\boldsymbol{i} + \cos(t)\boldsymbol{j}$

f $\boldsymbol{r}(t) = (3 + 2t)\boldsymbol{i} + 5t\boldsymbol{j}$

g $\boldsymbol{r}(t) = 100t\boldsymbol{i} + (100\sqrt{3}t - 4.9t^2)\boldsymbol{j}$

h $\boldsymbol{r}(t) = \tan(t)\boldsymbol{i} + \cos^2(t)\boldsymbol{j}$

2 Sketch graphs of each of the following for $t \geq 0$ and find, $\boldsymbol{r}(t_0)$, $\dot{\boldsymbol{r}}(t_0)$ and $\ddot{\boldsymbol{r}}(t_0)$ for the given t_0:

a $\boldsymbol{r}(t) = e^t\boldsymbol{i} + e^{-t}\boldsymbol{j},\ t_0 = 0$

b $\boldsymbol{r}(t) = t\boldsymbol{i} + t^2\boldsymbol{j},\ t_0 = 1$

c $\boldsymbol{r}(t) = \sin(t)\boldsymbol{i} + \cos(t)\boldsymbol{j},\ t_0 = \dfrac{\pi}{6}$

d $\boldsymbol{r}(t) = 16t\boldsymbol{i} - 4(4t-1)^2\boldsymbol{j},\ t_0 = 1$

e $\boldsymbol{r}(t) = \dfrac{1}{t+1}\boldsymbol{i} + (t+1)^2\boldsymbol{j},\ t_0 = 1$

3 Find the gradient at the point on the curve determined by the given value of t for each of the following:

a $\boldsymbol{r}(t) = \cos(t)\boldsymbol{i} + \sin(t)\boldsymbol{j},\ t = \dfrac{\pi}{4}$

b $\boldsymbol{r}(t) = \sin(t)\boldsymbol{i} + \cos(t)\boldsymbol{j},\ t = \dfrac{\pi}{2}$

c $\boldsymbol{r}(t) = e^t\boldsymbol{i} + e^{-2t}\boldsymbol{j},\ t = 1$

d $\boldsymbol{r}(t) = 2t^2\boldsymbol{i} + 4t\boldsymbol{j},\ t = 2$

e $\boldsymbol{r}(t) = (t+2)\boldsymbol{i} + (t^2 - 2t)\boldsymbol{j},\ t = 3$

f $\boldsymbol{r}(t) = \cos(\pi t)\boldsymbol{i} + \cos(2\pi t)\boldsymbol{j},\ t = \frac{1}{4}$

4 Find $\boldsymbol{r}(t)$ in each of the following:

a $\dot{\boldsymbol{r}}(t) = 4\boldsymbol{i} + 3\boldsymbol{j}$ and $\boldsymbol{r}(0) = \boldsymbol{i} - \boldsymbol{j}$

b $\dot{\boldsymbol{r}}(t) = 2t\boldsymbol{i} + 2\boldsymbol{j} - 3t^2\boldsymbol{k}$ and $\boldsymbol{r}(0) = \boldsymbol{i} - \boldsymbol{j}$

c $\dot{\boldsymbol{r}}(t) = e^{2t}\boldsymbol{i} + 2e^{0.5t}\boldsymbol{j}$ where $\boldsymbol{r}(0) = \dfrac{1}{2}\boldsymbol{i}$

d $\ddot{\boldsymbol{r}}(t) = \boldsymbol{i} + 2t\boldsymbol{j}$ where $\dot{\boldsymbol{r}}(0) = \boldsymbol{i}$ and $\boldsymbol{r}(0) = \boldsymbol{0}$

e $\ddot{\boldsymbol{r}}(t) = \sin(2t)\boldsymbol{i} - \cos\left(\frac{1}{2}t\right)\boldsymbol{j}$ where $\dot{\boldsymbol{r}}(0) = -\frac{1}{2}\boldsymbol{i}$ and $\boldsymbol{r}(0) = 4\boldsymbol{j}$

5 The position of a particle at time t is given by $\boldsymbol{r}(t) = \sin(t)\boldsymbol{i} + t\boldsymbol{j} + \cos(t)\boldsymbol{k}$ where $t \geq 0$. Prove that $\dot{\boldsymbol{r}}(t)$ and $\ddot{\boldsymbol{r}}(t)$ are always perpendicular.

6 The position of a particle at time t is given by $\boldsymbol{r}(t) = 2t\boldsymbol{i} + 16t^2(3-t)\boldsymbol{j}$ where $t \geq 0$. Find:

a when $\dot{\boldsymbol{r}}(t)$ and $\ddot{\boldsymbol{r}}(t)$ are perpendicular

b the pairs of perpendicular vectors $\dot{\boldsymbol{r}}(t)$ and $\ddot{\boldsymbol{r}}(t)$

7 Given that a particle has position $\boldsymbol{r}(t)$ at time t determined by $\boldsymbol{r}(t) = at\boldsymbol{i} + \dfrac{a^2t^2}{4}\boldsymbol{j},\ a > 0$ and $t \geq 0$:

a sketch the graph of the path of the particle

b find when the magnitude of the angle between $\dot{\boldsymbol{r}}(t)$ and $\ddot{\boldsymbol{r}}(t)$ is $45°$

8 Given that a particle has position $\boldsymbol{r}(t)$ at time t specified by $\boldsymbol{r}(t) = 2t\boldsymbol{i} + (t^2 - 4)\boldsymbol{j}$ where $t \geq 0$:

a sketch the graph of the path of the particle

b find the magnitude of the angle between $\dot{\boldsymbol{r}}(t)$ and $\ddot{\boldsymbol{r}}(t)$ at $t = 1$

c find when the magnitude of the angle between $\dot{\boldsymbol{r}}(t)$ and $\ddot{\boldsymbol{r}}(t)$ is $30°$

9 Given $\boldsymbol{r} = 3t\boldsymbol{i} + \frac{1}{3}t^3\boldsymbol{j} + t^3\boldsymbol{k}$, find:

a $\dot{\boldsymbol{r}}$ **b** $|\dot{\boldsymbol{r}}|$ **c** $\ddot{\boldsymbol{r}}$ **d** $|\ddot{\boldsymbol{r}}|$ **e** t when $|\ddot{\boldsymbol{r}}| = 16$

10 Given $\boldsymbol{r} = (V\cos\alpha)t\boldsymbol{i} + \left((V\sin\alpha)t - \frac{1}{2}gt^2\right)\boldsymbol{j}$, where $t \geq 0$ specifies the position of an object at time t, find:

a $\dot{\boldsymbol{r}}$ **b** $\ddot{\boldsymbol{r}}$ **c** when $\dot{\boldsymbol{r}}$ and $\ddot{\boldsymbol{r}}$ are perpendicular

d the position of the object when $\dot{\boldsymbol{r}}$ and $\ddot{\boldsymbol{r}}$ are perpendicular

12.4 Velocity and acceleration for motion along a curve

Velocity

Since velocity has previously been defined as rate of change of position (displacement), it follows that $\boldsymbol{v}(t)$, the velocity at time t, is given by $\dot{\boldsymbol{r}}(t)$. The velocity vector gives the direction of motion at time t.

Acceleration

Since acceleration has previously been defined as rate of change of velocity, it follows that $\boldsymbol{a}(t)$, the acceleration at time t, is given by $\ddot{\boldsymbol{r}}(t) = \dot{v}(t)$.

Speed

From work on kinematics, it is known that speed is the magnitude of velocity, so it follows that, at time t, speed $= |\dot{\boldsymbol{r}}(t)|$

Distance travelled

$|\boldsymbol{r}(t_1) - \boldsymbol{r}(t_0)|$ can be found to find the (shortest) distance between two different points on the curve, but not the distance travelled along the curve. Length of a section of a curve is not part of this course.

Example 15

The position of an object is $\boldsymbol{r}(t)$ metres at time t where $\boldsymbol{r}(t) = e^t\boldsymbol{i} + \frac{2}{9}e^{2t}\boldsymbol{j}$, where $t \geq 0$. Find, at time t:

a the velocity vector **b** the acceleration vector **c** the speed.

Solution

a $\boldsymbol{v}(t) = \dot{\boldsymbol{r}}(t) = e^t\boldsymbol{i} + \frac{4}{9}e^{2t}\boldsymbol{j}$

b $\boldsymbol{a}(t) = \ddot{\boldsymbol{r}}(t) = e^t\boldsymbol{i} + \frac{8}{9}e^{2t}\boldsymbol{j}$

c At time t, speed $= |\boldsymbol{v}(t)| = \sqrt{(e^t)^2 + \left(\frac{4}{9}e^{2t}\right)^2}$ m/s

$$= \sqrt{e^{2t} + \tfrac{16}{81}e^{4t}} \text{ m/s}$$

Example 16

The position vectors, at time t, where $t \geq 0$, of particles A and B are given respectively by:

$$\boldsymbol{r}_A(t) = (t^3 - 9t + 8)\boldsymbol{i} + t^2\boldsymbol{j}$$

$$\boldsymbol{r}_B(t) = (2 - t^2)\boldsymbol{i} + (3t - 2)\boldsymbol{j}$$

Prove that A and B collide while travelling at the same speed but at right angles to each other.

Solution

When the particles collide, they must be at the same position at the same time. Now, the same *position* means:

$$(t^3 - 9t + 8)\boldsymbol{i} + t^2\boldsymbol{j} = (2 - t^2)\boldsymbol{i} + (3t - 2)\boldsymbol{j}$$

$$\therefore \begin{cases} t^3 - 9t + 8 = 2 - t^2 & \boxed{1} \\ t^2 = 3t - 2 & \boxed{2} \end{cases}$$

$$\therefore \text{ From } \boxed{1} \qquad t^3 + t^2 - 9t + 6 = 0 \qquad \boxed{3}$$

$$\therefore \text{ From } \boxed{2} \qquad t^2 - 3t + 2 = 0 \qquad \boxed{4}$$

Equation $\boxed{4}$ is simpler to solve:

$$(t - 2)(t - 1) = 0$$

$$\therefore \quad t = 2, \text{ or } t = 1$$

Now check in $\boxed{3}$ $\quad t = 1 \quad \text{LHS} = 1 + 1 - 9 + 6 \neq 0$

$\quad t = 2 \quad \text{LHS} = 8 + 4 - 18 + 6 = 0$

The particles collide when $t = 2$.
Now consider the speeds when $t = 2$.

$$\dot{\boldsymbol{r}}_A(t) = (3t^2 - 9)\boldsymbol{i} + 2t\boldsymbol{j}$$

$$\therefore \quad \dot{\boldsymbol{r}}_A(2) = 3\boldsymbol{i} + 4\boldsymbol{j}$$

The speed of particle $A = \sqrt{3^2 + 4^2}$
$= 5$

Now $\quad \dot{\boldsymbol{r}}_B(t) = -2t\boldsymbol{i} + 3\boldsymbol{j}$

$$\therefore \quad \ddot{\boldsymbol{r}}_B(2) = -4\boldsymbol{i} + 3\boldsymbol{j}$$

The speed of particle $B = \sqrt{(-4)^2 + 3^2}$
$= 5$

The speeds of the particles are equal at the time of collision.
Consider the scalar product of the velocity vectors for A and B at time $t = 2$.

$$\begin{aligned} \dot{\boldsymbol{r}}_A(2).\dot{\boldsymbol{r}}_B(2) &= (3\boldsymbol{i} + 4\boldsymbol{j}).(-4\boldsymbol{i} + 3\boldsymbol{j}) \\ &= -12 + 12 \\ &= 0 \end{aligned}$$

Hence the velocities are perpendicular at $t = 2$.
The particles are travelling at right angles at the time of collision.

Example 17

The position vector of a particle at time t is given by $\boldsymbol{r}(t) = (2t - t^2)\boldsymbol{i} + (t^2 - 3t)\boldsymbol{j} + 2t\boldsymbol{k}$ where $t \geq 0$. Find:

a the velocity of the particle at time t

b the speed of the particle at time t

c the minimum speed of the particle.

Solution

a $\dot{\boldsymbol{r}}(t) = (2 - 2t)\boldsymbol{i} + (2t - 3)\boldsymbol{j} + 2\boldsymbol{k}$

b $\text{Speed} = |\dot{\boldsymbol{r}}(t)| = \sqrt{4 - 8t + 4t^2 + 4t^2 - 12t + 9 + 4}$

$$= \sqrt{8t^2 - 20t + 17}$$

c Minimum speed occurs when $8t^2 - 20t + 17$ is a minimum.

$$\begin{aligned} 8t^2 - 20t + 17 &= 8\left[t^2 - \frac{5t}{2} + \frac{17}{8}\right] \\ &= 8\left[t^2 - \frac{5t}{2} + \frac{25}{16} + \frac{17}{8} - \frac{25}{16}\right] \\ &= 8\left[\left(t - \frac{5}{4}\right)^2 + \frac{9}{16}\right] \\ &= 8\left(t - \frac{5}{4}\right)^2 + \frac{9}{2} \end{aligned}$$

$\therefore$ minimum speed $= \sqrt{\frac{9}{2}} = \frac{3}{\sqrt{2}} = \frac{3\sqrt{2}}{2}$

This occurs when $t = \frac{5}{4}$

Example 18

The position vector of a particle at time t is given by $\boldsymbol{r}(t) = 2\sin(2t)\boldsymbol{i} + \cos(2t)\boldsymbol{j} + 2t\boldsymbol{k}$ where $t \geq 0$. Find:

a the velocity at time t

b the speed of the particle at time t

c the maximum speed

d the minimum speed.

Solution

a $\dot{\boldsymbol{r}}(t) = 4\cos(2t)\boldsymbol{i} - 2\sin(2t)\boldsymbol{j} + 2\boldsymbol{k}$

b $\text{Speed} = |\dot{\boldsymbol{r}}(t)| = \sqrt{16\cos^2(2t) + 4\sin^2(2t) + 4}$

$$= \sqrt{12\cos^2(2t) + 8}$$

c Maximum speed $= \sqrt{20} = 2\sqrt{5}$ when $\cos(2t) = 1$

d Minimum speed $= \sqrt{8} = 2\sqrt{2}$ when $\cos(2t) = 0$

Example 19

The position of a projectile at time t is given by $\boldsymbol{r}(t) = 400t\boldsymbol{i} + (500t - 5t^2)\boldsymbol{j}$ where $t \geq 0$ and where $\boldsymbol{i}$ is a unit vector in a horizontal direction and $\boldsymbol{j}$ is a unit vector vertically up. The projectile is fired from a point on the ground. Find:

a the time taken to reach the ground again

b the speed at which the projectile hits the ground

c the maximum height of the projectile

d the initial speed of the projectile.

Solution

a When the $\boldsymbol{j}$ component of $\boldsymbol{r}$ is equal to zero the projectile is at ground level.

$$500t - 5t^2 = 0$$

i.e. $$5t(100 - t) = 0$$

$$t = 0 \text{ or } t = 100$$

The projectile reaches the ground again for $t = 100$.

b $\dot{\boldsymbol{r}}(t) = 400\boldsymbol{i} + (500 - 10t)\boldsymbol{j}$

The velocity of the projectile when it hits the ground again is

$\dot{\boldsymbol{r}}(100) = 400\boldsymbol{i} - 500\boldsymbol{j}$

Speed when $t = 100$ is $$|\dot{\boldsymbol{r}}(100)| = \sqrt{160000 + 250000} = \sqrt{410000} = 100\sqrt{41}$$

The projectile hits the ground with a speed $100\sqrt{41}$.

c The projectile reaches its maximum height when the $\boldsymbol{j}$ component of the velocity is equal to zero.

When $500 - 10t = 0$, i.e. when $t = 50$.

$\therefore$ Maximum height $= 500 \times 50 - 5 \times 50^2 = 12500$

d The initial velocity $\dot{\boldsymbol{r}}(0) = 400\boldsymbol{i} + 500\boldsymbol{j}$

$\therefore$ initial speed $= \sqrt{400^2 + 500^2} = 100\sqrt{41}$

Exercise 12D

Skillsheet TEACHER

All distances are measured in metres and time in seconds.

1 The position of a particle at time t is given by $\boldsymbol{r}(t) = t^2\boldsymbol{i} - (1 + 2t)\boldsymbol{j}$ where $t \geq 0$. Find:

a the velocity at time t

b the acceleration at time t

c the average velocity for the first two seconds, i.e. $\dfrac{\boldsymbol{r}(2) - \boldsymbol{r}(0)}{2}$

2 The acceleration of a particle at time t is given by $\ddot{\boldsymbol{r}}(t) = -g\boldsymbol{j}$ where $g = 9.8$. Find:

a the velocity at time t if $\dot{\boldsymbol{r}}(0) = 2\boldsymbol{i} + 6\boldsymbol{j}$

b the displacement at time t if $\boldsymbol{r}(0) = 0\boldsymbol{i} + 6\boldsymbol{j}$

3 The velocity of a particle at time t is given by $\dot{\boldsymbol{r}}(t) = 3\boldsymbol{i} + 2t\boldsymbol{j} + (1 - 4t)\boldsymbol{k}$ where $t \geq 0$

a Find the acceleration of the particle at time t.

b Find the position of the particle at time t if initially the particle is at $\boldsymbol{j} + \boldsymbol{k}$.

c Find an expression for the speed at time t.

d **i** Find the time at which the minimum speed occurs.

ii Find this minimum speed.

4 The acceleration of a particle at time t is given by $\ddot{\boldsymbol{r}} = 10\boldsymbol{i} - g\boldsymbol{k}$ where $g = 9.8$. Find:

a the velocity of the particle at time t if $\dot{\boldsymbol{r}}(0) = 20\boldsymbol{i} - 20\boldsymbol{j} + 40\boldsymbol{k}$

b the displacement of the particle at time t given that $\boldsymbol{r}(0) = 0\boldsymbol{i} + 0\boldsymbol{j} + 0\boldsymbol{k}$

5 The position of an object at time t is given by $\boldsymbol{r}(t) = 5\cos(1 + t^2)\boldsymbol{i} + 5\sin(1 + t^2)\boldsymbol{j}$. Find the speed of the object at time t.

6 The position of a particle $\boldsymbol{r}(t)$ at time t seconds is given by $\boldsymbol{r}(t) = 2t\boldsymbol{i} + (t^2 - 4)\boldsymbol{j}$. Find the magnitude of the angle between the velocity and acceleration vectors at $t = 1$.

7 The position vector of a particle is given by $\boldsymbol{r}(t) = 12\sqrt{t}\boldsymbol{i} + t^{\frac{3}{2}}\boldsymbol{j}$ where $t \geq 0$. Find the minimum speed of the particle, and its position when it has this speed.

8 The position $\boldsymbol{r}(t)$ of a projectile at time $t(t \geq 0)$ is given by $\boldsymbol{r}(t) = 400t\boldsymbol{i} + (300t - 4.9t^2)\boldsymbol{j}$. If the projectile is initially at ground level, find:

a the time taken to reach the ground

b the speed at which the object hits the ground

c the maximum height reached

d the initial speed of the object

e initial angle of projection from the horizontal

9 The acceleration of a particle at time t is given by $\ddot{\boldsymbol{r}}(t) = -3(\sin(3t)\boldsymbol{i} + \cos(3t)\boldsymbol{j})$.

a Find the position vector $\boldsymbol{r}(t)$, given that $\dot{\boldsymbol{r}}(0) = \boldsymbol{i}$ and $\boldsymbol{r}(0) = -3\boldsymbol{i} + 3\boldsymbol{j}$.

b Show that the path of the particle is circular, and state the position of its centre.

c Show that the acceleration is always perpendicular to the velocity.

10 A particle moves so that its position vector at time t is given by $\boldsymbol{r}(t) = 2\cos(t)\boldsymbol{i} + 4\sin(t)\boldsymbol{j} + 2t\boldsymbol{k}$. Find the maximum and minimum speeds of the particle.

11 The velocity vector of a particle at time t seconds is given by $\boldsymbol{v}(t) = (2t + 1)^2\boldsymbol{i} + \dfrac{1}{\sqrt{2t + 1}}\boldsymbol{j}$. Find:

a the magnitude and direction of the acceleration after one second

b the displacement vector at time t seconds if the particle is initially at O

12 The acceleration of a particle moving in the x–y plane is $-g\boldsymbol{j}$. The particle is initially at O with velocity given by $V\cos(\alpha)\boldsymbol{i} + V\sin(\alpha)\boldsymbol{j}$ for some positive real number α.

a Find $\boldsymbol{r}(t)$, the position vector at time t.

b Prove that the particle follows a path with cartesian equation

$$y = x\tan\alpha - \frac{gx^2}{2V^2}\sec^2\alpha.$$

13 Particles A and B move in the x–y plane with constant velocities.
$\dot{\boldsymbol{r}}_A(t) = \boldsymbol{i} + 2\boldsymbol{j}$ and $\dot{\boldsymbol{r}}_B(t) = 2\boldsymbol{i} + 3\boldsymbol{j}$
Also $\boldsymbol{r}_A(2) = 3\boldsymbol{i} + 4\boldsymbol{j}$ and $\boldsymbol{r}_B(3) = \boldsymbol{i} + 3\boldsymbol{j}$
Prove that the particles collide, finding:

a the time of collision **b** the position vector of the point of collision

14 A body moves horizontally along a straight line in a direction N$\alpha°$W with a constant speed of 20 m/s. If $\boldsymbol{i}$ is a horizontal unit vector due east and $\boldsymbol{j}$ is a horizontal unit vector due north and if $\tan \alpha° = \frac{4}{3}$, find:

a the velocity of the body at time t
b the position of the body after five seconds

15 The position vector of a particle at time t is given by $\boldsymbol{r} = 4\sin(2t)\boldsymbol{i} + 4\cos(2t)\boldsymbol{j}$, $t \ge 0$. Find:

a the velocity at time t **b** the speed at time t
c the acceleration in terms of $\boldsymbol{r}$

16 The velocity of a particle is given by $\dot{\boldsymbol{r}}(t) = (2t - 5)\boldsymbol{i}$, $t \ge 0$. Initially the position of the particle relative to an origin O is $-2\boldsymbol{i} + 2\boldsymbol{j}$.

a Find the position of the particle at time t.
b Find the position of the particle when it is instantaneously at rest.
c Find the cartesian equation of the path followed by the particle.

17 A particle has path defined by $\boldsymbol{r}(t) = 6\sec(t)\boldsymbol{i} + 4\tan(t)\boldsymbol{j}$, $t \ge 0$.

a Find the cartesian equation of the path of the particle.
b Find the velocity of the particle at time t.

18 A particle moves so that its position vector at time t is given by $\boldsymbol{r}(t) = 4\cos(t)\boldsymbol{i} + 3\sin(t)\boldsymbol{j}$, $0 \le t \le 2\pi$.

a Find the cartesian equation of the path of the particle and sketch the path.
b **i** Find when the velocity of the particle is perpendicular to its position vector.
ii Find the position vector of the particle at each of these times.
c **i** Find the speed of the particle at time t.
ii Write the speed in terms of $\cos^2 t$.
iii State the maximum and minimum speeds of the particle.

Review

Chapter summary

- The position of a particle at time t can be described by a vector equation:
  $\boldsymbol{r}(t) = f(t)\boldsymbol{i} + g(t)\boldsymbol{j} + h(t)\boldsymbol{k}$
- The velocity of a particle at time t is:
  $\dot{\boldsymbol{r}}(t) = f'(t)\boldsymbol{i} + g'(t)\boldsymbol{j} + h'(t)\boldsymbol{k}$
- The acceleration of a particle at time t is:
  $\ddot{\boldsymbol{r}}(t) = f''(t)\boldsymbol{i} + g''(t)\boldsymbol{j} + h''(t)\boldsymbol{k}$
- The velocity vector $\dot{\boldsymbol{r}}(t)$ has the direction of the motion of the particle at time t.

Multiple-choice questions

1 A particle moves in a plane such that at time t its position is $2t^2\,\boldsymbol{i} + (3t - 1)\boldsymbol{j}$. Its acceleration at time t is given by:

A $4t\boldsymbol{i} + 3\boldsymbol{j}$ **B** $\frac{2}{3}t^3\boldsymbol{i} + \left(\frac{3t^2}{2} - t\right)\boldsymbol{j}$ **C** $4\boldsymbol{i} + 3\boldsymbol{j}$ **D** $0\boldsymbol{i} + 0\boldsymbol{j}$ **E** $4\boldsymbol{i} + 0\boldsymbol{j}$

2 The position vector of a particle at time t, $t \geq 0$ is given by $\boldsymbol{r} = (\sin(3t))\boldsymbol{i} - (2\cos t)\boldsymbol{j}$. The speed of the particle when $t = \pi$ is:

A 2 **B** $2\sqrt{2}$ **C** $\sqrt{5}$ **D** 0 **E** 3

3 A particle moves with constant velocity $5\boldsymbol{i} - 4\boldsymbol{j} + 2\boldsymbol{k}$. Its initial position is $3\boldsymbol{i} - 6\boldsymbol{k}$. Its position vector at time t is given by:

A $(3t + 5)\boldsymbol{i} - 4\boldsymbol{j} + (2 - 6t)\boldsymbol{k}$ **B** $(5t + 3)\boldsymbol{i} - 4t\boldsymbol{j} + (2t - 6)\boldsymbol{k}$
C $5t\boldsymbol{i} - 4t\boldsymbol{j} + 2t\boldsymbol{k}$ **D** $-5t\boldsymbol{i} - 4t\boldsymbol{j} + 2t\boldsymbol{k}$ **E** $(5t - 3)\boldsymbol{i} + (2t - 6)\boldsymbol{k}$

4 A particle moves with its position vector defined with respect to time t by the position vector $\boldsymbol{r}(t) = (2t^3 - 1)\boldsymbol{i} + (2t^2 + 3)\boldsymbol{j} + 6t\boldsymbol{k}$. The acceleration when $t = 1$ is given by:

A $6\boldsymbol{i} + 4\boldsymbol{j} + 6\boldsymbol{k}$ **B** $12\boldsymbol{i} + 4\boldsymbol{j} + 6\boldsymbol{k}$ **C** $12\boldsymbol{i}$ **D** $2\sqrt{10}$ **E** $12\boldsymbol{i} + 4\boldsymbol{j}$

5 The position of a particle at time t seconds is given by the vector $\boldsymbol{r}(t) = (t^2 - 4t)(\boldsymbol{i} - \boldsymbol{j} + \boldsymbol{k})$ measured in metres from a fixed point. The distance in metres travelled in the first four seconds is:

A 0 **B** $4\sqrt{3}$ **C** $8\sqrt{3}$ **D** 4 **E** $\sqrt{3}$

6 The initial position, velocity and constant acceleration of a particle are given by $3\boldsymbol{i}$, $2\boldsymbol{j}$ and $2\boldsymbol{i} - \boldsymbol{j}$ respectively. The position vector of the particle at time t is given by:

A $(2\boldsymbol{i} - \boldsymbol{j})t + 3\boldsymbol{i}$ **B** $t^2\boldsymbol{i} - \frac{t^2}{2}\boldsymbol{j}$ **C** $(t^2 + 3)\boldsymbol{i} + (2t - \frac{t^2}{2})\boldsymbol{j}$
D $3\boldsymbol{i} + 2t\boldsymbol{j}$ **E** $(2\boldsymbol{i} - \boldsymbol{j})\frac{t^2}{2}$

7 The position of a particle at time $t = 0$ is given by $\boldsymbol{r}(0) = \boldsymbol{i} - 5\boldsymbol{j} + 2\boldsymbol{k}$. The position of the particle at time $t = 3$ is $\boldsymbol{r}(3) = 7\boldsymbol{i} + 7\boldsymbol{j} - 4\boldsymbol{k}$. The average velocity for the interval [0, 3] is:

A $\frac{1}{3}(8\boldsymbol{i} + 2\boldsymbol{j} - 2\boldsymbol{k})$ **B** $\frac{1}{3}(21\boldsymbol{i} + 21\boldsymbol{j} - 12\boldsymbol{k})$ **C** $2\boldsymbol{i} + 4\boldsymbol{j} - 2\boldsymbol{k}$
D $\boldsymbol{i} + 2\boldsymbol{j} - \boldsymbol{k}$ **E** $2\boldsymbol{i} - \boldsymbol{j} + \boldsymbol{k}$

Review

8 A particle is moving so its velocity vector at time t is $\dot{\boldsymbol{r}}(t) = 2t\boldsymbol{i} + 3\boldsymbol{j}$ where $\boldsymbol{r}(t)$ is the position vector at time t. If $\boldsymbol{r}(0) = 3\boldsymbol{i} + \boldsymbol{j}$ then $\boldsymbol{r}(t)$ is equal to:

A $2\boldsymbol{i}$ **B** $(3t+1)\boldsymbol{i} + (3t^2+1)\boldsymbol{j}$ **C** $2t^2\boldsymbol{i} + 3t\boldsymbol{j} + 3\boldsymbol{i} + \boldsymbol{j}$
D $5\boldsymbol{i} + 3\boldsymbol{j}$ **E** $(t^2+3)\boldsymbol{i} + (3t+1)\boldsymbol{j}$

9 The velocity of a particle is given by the vector $\dot{\boldsymbol{r}}(t) = t\boldsymbol{i} + e^t\boldsymbol{j}$. At time $t = 0$, the position of the particle is given by $\boldsymbol{r}(0) = 3\boldsymbol{i}$. The position of the particle at time t is given by:

A $\boldsymbol{r}(t) = \frac{1}{2}t^2\boldsymbol{i} + e^t\boldsymbol{j}$ **B** $\boldsymbol{r}(t) = \frac{1}{2}(t^2+3)\boldsymbol{i} + e^t\boldsymbol{j}$ **C** $\boldsymbol{r}(t) = \left(\frac{1}{2}t^2+3\right)\boldsymbol{i} + (e^t-1)\boldsymbol{j}$
D $\boldsymbol{r}(t) = \left(\frac{1}{2}t^2+3\right)\boldsymbol{i} + e^t\boldsymbol{j}$ **E** $\boldsymbol{r}(t) = \frac{1}{2}(t^2+3)\boldsymbol{i} + (e^t-1)\boldsymbol{j}$

10 A curve is described by the vector equation $\boldsymbol{r}(t) = 2\cos(\pi t)\boldsymbol{i} + 3\sin(\pi t)\boldsymbol{j}$. Superimposed on a set of cartesian axes, the gradient of the curve at the point $(\sqrt{3}, 1.5)$ is:

A $\frac{-\sqrt{3}}{2}$ **B** $-(\pi\boldsymbol{i} + 3\sqrt{3}\pi\boldsymbol{j})$ **C** $\pi\boldsymbol{i} + 3\sqrt{3}\pi\boldsymbol{j}$ **D** $\frac{-3\sqrt{3}}{2}\pi$ **E** $\frac{-3\sqrt{3}}{2}$

Short-answer questions (technology-free)

1 The displacement, $\boldsymbol{r}(t)$ (metres), at time t (seconds), of a particle moving in a plane is given by $\boldsymbol{r}(t) = 2t\boldsymbol{i} + (t^2-4)\boldsymbol{j}$.

a Find the velocity and acceleration when $t = 2$.
b Find the cartesian equation of the path.

2 Find the velocity and acceleration vectors of the displacement vectors:

a $\boldsymbol{r} = 2t^2\boldsymbol{i} + 4t\boldsymbol{j} + 8\boldsymbol{k}$ **b** $\boldsymbol{r} = 4\sin t\boldsymbol{i} + 4\cos t\boldsymbol{j} + t^2\boldsymbol{k}$

3 At time t, a particle has coordinates $(6t, t^2+4)$. Find the unit vector along the tangent to the path when $t = 4$.

4 The position vector of a particle is given by $\boldsymbol{r}(t) = 10\sin 2t\boldsymbol{i} + 5\cos 2t\boldsymbol{j}$.

a Find its position vector when $t = \frac{\pi}{6}$.
b Find the cosine of the angle between its direction of motion at $t = 0$ and $t = \frac{\pi}{6}$.

5 Find the unit tangent vector of the curve $\boldsymbol{r} = (\cos t + t\sin t)\boldsymbol{i} + (\sin t - t\cos t)\boldsymbol{j}$.

6 A particle moves on a curve with equation $\boldsymbol{r} = 5(\cos\theta\boldsymbol{i} + \sin\theta\boldsymbol{j})$. Find:

a the velocity at time t **b** the speed at time t
c the acceleration at time t **d** $\dot{\boldsymbol{r}}.\ddot{\boldsymbol{r}}$, and comment

7 Particles A and B move with velocities $\boldsymbol{V}_A = \cos t\boldsymbol{i} + \sin t\boldsymbol{j}$ and $\boldsymbol{V}_B = \sin t\boldsymbol{i} + \cos t\boldsymbol{j}$ respectively. At time $t = 0$ the position vectors of A and B are $\boldsymbol{r}_A = \boldsymbol{i}$ and $\boldsymbol{r}_B = \boldsymbol{j}$. Prove that the particles collide, finding the time of collision.

8 The position vector of a particle at any time, t, is given by $\boldsymbol{r} = (1 + \sin t)\boldsymbol{i} + (1 - \cos t)\boldsymbol{j}$.

a Show that the magnitudes of the velocity and acceleration are constants.
b Find the cartesian equation of the path described by the particle.
c Find the first instant that the displacement is perpendicular to the velocity.

9 The velocities of two particles A and B are given by $\boldsymbol{V}_A = 2\boldsymbol{i} + 3\boldsymbol{j}$ and $\boldsymbol{V}_B = 3\boldsymbol{i} - 4\boldsymbol{j}$. The initial position vector of A is given by $\boldsymbol{r}_A = \boldsymbol{i} - \boldsymbol{j}$. If the particles collide after 3 seconds, find the initial position vector of particle B.

10 A particle starts from point $(\boldsymbol{i} - 2\boldsymbol{j})$ and travels with a velocity given by $t\boldsymbol{i} + \boldsymbol{j}$, t seconds from the start. A second particle travels in the same plane and its position vector is given by $\boldsymbol{r} = (s-4)\boldsymbol{i} + 3\boldsymbol{j}$, s seconds after it started.

- **a** Find an expression for the position of the first particle.
- **b** Find the point at which their paths cross.
- **c** If the particles actually collide, find the time between the two starting times.

11 A particle travels with constant acceleration, given by $\ddot{\boldsymbol{r}}(t) = \boldsymbol{i} + 2\boldsymbol{j}$. Two seconds after starting, the particle passes through the point $\boldsymbol{i}$, travelling at a velocity of $2\boldsymbol{i} - \boldsymbol{j}$. Find:

- **a** an expression for the velocity of the particle at time t
- **b** an expression for its position
- **c** the initial position and velocity of the particle.

12 Two particles travel with constant acceleration given by $\ddot{\boldsymbol{r}}_1(t) = \boldsymbol{i} - \boldsymbol{j}$ and $\ddot{\boldsymbol{r}}_2(t) = 2\boldsymbol{i} + \boldsymbol{j}$. The initial velocity of the second particle is $-4\boldsymbol{i}$ and that of the first particle is $k\boldsymbol{j}$.

- **a** Find an expression for:
  - **i** the velocity of the second particle
  - **ii** the velocity of the first particle.
- **b** At one instant both particles have the same velocity. Find:
  - **i** the time elapsed before that instant
  - **ii** the value of k
  - **iii** the common velocity.

13 The position of an object is given by $\boldsymbol{r}(t) = e^t\boldsymbol{i} + 4e^{2t}\boldsymbol{j}$, $t \geq 0$.

- **a** Show that the path of the particle is given by the function $f: [1, \infty) \to R, f(x) = 4x^2$.
- **b** Find:
  - **i** the velocity vector at time t
  - **ii** the initial velocity.
  - **iii** the time at which the velocity is parallel to vector $\boldsymbol{i} + 12\boldsymbol{j}$.

14 The velocity of a particle is given by $\dot{\boldsymbol{r}}(t) = (t-3)\boldsymbol{j}$, $t > 0$.

- **a** Show that the path of this particle is linear.
- **b** Initially the position of the particle is $2\boldsymbol{i} + \boldsymbol{j}$.
  - **i** Find the cartesian equation of the path followed by the particle.
  - **ii** Find the point at which the particle is momentarily at rest.

Extended-response questions

1 Two particles P and Q are moving in a horizontal plane. The particles are moving with velocities $9\boldsymbol{i} + 6\boldsymbol{j}$ m/s and $5\boldsymbol{i} + 4\boldsymbol{j}$ m/s.

- **a** Determine the speeds of the particles.
- **b** At time $t = 4$, P and Q have position vectors $\boldsymbol{r}_p(4) = 96\boldsymbol{i} + 44\boldsymbol{j}$ and $\boldsymbol{r}_Q(4) = 100\boldsymbol{i} + 96\boldsymbol{j}$. (Distances are measured in metres.)
  - **i** Find the position vectors of P and Q at time $t = 0$.
  - **ii** Find the vector $\overrightarrow{PQ}$ at time t.
- **c** Find the time at which P and Q are nearest to each other and the magnitude of $\overrightarrow{PQ}$ at this instant.

2 Two particles A and B move in the plane. The velocity of A is $(-3\boldsymbol{i} + 29\boldsymbol{j})$ m/s while that of B is $v(\boldsymbol{i} + 7\boldsymbol{j})$ m/s where v is a constant. (All distances are measured in metres.)

a Find the vector $\overrightarrow{AB}$ at time t seconds given that when $t = 0$, $\overrightarrow{AB} = -56\boldsymbol{i} + 8\boldsymbol{j}$.

b Find the value of v so that the particles collide.

c If $v = 3$:

i find $\overrightarrow{AB}$ **ii** find the time when the particles are closest.

3 A child is sitting still in some long grass watching a bee. The bee flies at a constant speed in a straight line from its beehive to a flower and reaches the flower three seconds later. The position vector of the beehive relative to the child is $10\boldsymbol{i} + 2\boldsymbol{j} + 6\boldsymbol{k}$ and the position vector of the flower relative to the child is $7\boldsymbol{i} + 8\boldsymbol{j}$ where all the distances are measured in metres.

a If B is the position of the beehive and F the position of the flower, find $\overrightarrow{BF}$.

b Find the distance BF.

c Find the speed of the bee.

d Find the velocity of the bee.

e Find the time when the bee is closest to the child and its distance from the child at this time.

4 Initially a motor boat is at a point J at the end of a jetty and a police boat is at a point P. The position vector of P relative to J is $400\boldsymbol{i} - 600\boldsymbol{j}$. The motor boat leaves the point J and travels with constant velocity $6\boldsymbol{i}$. At the same time, the police boat leaves its position at P and travels with constant velocity $u(8\boldsymbol{i} + 6\boldsymbol{j})$ where u is a real number. All distances are measured in metres and all times are measured in seconds.

a If the police boat meets the motor boat after t seconds find:

i the value of t **ii** the value of u

iii the speed of the police boat **iv** the position of the point where they meet.

b Find the time when the police boat was closest to J and its distance from J at this time.

5 A particle A is at rest on a smooth horizontal table at a point whose position vector relative to an origin O is $-\boldsymbol{i} + 2\boldsymbol{j}$. B is a point on the table such that $\overrightarrow{OB} = 2\boldsymbol{i} + \boldsymbol{j}$. (All distances are measured in metres and time in seconds.) At time $t = 0$ the particle is projected along the table with velocity $(6\boldsymbol{i} + 3\boldsymbol{j})$ m/s.

a Determine:

i $\overrightarrow{OA}$ at time t **ii** $\overrightarrow{BA}$ at time t.

b Find the time when $|\overrightarrow{BA}| = 5$.

c Using the time found in **b**:

i find a unit vector $\boldsymbol{c}$ along $\overrightarrow{BA}$

ii find a unit vector $\boldsymbol{d}$ perpendicular to $\overrightarrow{BA}$ (Hint: The vector $y\boldsymbol{i} - x\boldsymbol{j}$ is perpendicular to $x\boldsymbol{i} + y\boldsymbol{j}$)

iii express $6\boldsymbol{i} + 3\boldsymbol{j}$ in the form $p\boldsymbol{c} + q\boldsymbol{d}$.

6 **a** Sketch the graph of the cartesian relation corresponding to the vector equation $\boldsymbol{r}(\theta) = \cos(\theta)\boldsymbol{i} - \sin(\theta)\boldsymbol{j}$ where $0 < \theta < \dfrac{\pi}{2}$.

b A particle P describes a circle of radius 16 cm about the origin. It completes the circle every π seconds. At $t = 0$, P is at the point $(16, 0)$ and is moving in a clockwise direction. It can be shown that $\overrightarrow{OP} = a\cos(nt)\boldsymbol{i} + b\sin(nt)\boldsymbol{j}$.

Find the values of:

i a **ii** b **iii** n

iv State the velocity and acceleration of P at time t.

c A second particle Q has position vector given by $\overrightarrow{OQ} = 8\sin(t)\boldsymbol{i} + 8\cos(t)\boldsymbol{j}$ (measurements are in cm).

Obtain an expression for:

i $\overrightarrow{PQ}$ **ii** $|\overrightarrow{PQ}|^2$

d Find the minimum distance between P and Q.

7 At time t a particle has velocity $\boldsymbol{v} = (2\cos t)\boldsymbol{i} - (4\sin t\cos t)\boldsymbol{j}$, $t \geq 0$ and at time $t = 0$ it is at a point which has position vector $3\boldsymbol{j}$.

a Find the position of the particle at time t.

b Find the position of the particle when it first comes to rest.

c **i** Find the cartesian equation of the path of the particle.

ii Sketch the path of the particle.

d Express $|v|^2$ in terms of $\cos t$ and without using calculus, find the maximum speed of the particle.

e Give the time at which the particle is at rest for the second time.

f **i** Show that the distance d of the particle from the origin at time t is given by $d^2 = \cos^2(2t) + 2\cos(2t) + 6$.

ii Find the time(s) at which the particle is closest to the origin.

8 A golfer hits a ball from a point referred to as the origin with a velocity of $a\boldsymbol{i} + b\boldsymbol{j} + 20\boldsymbol{k}$ where $\boldsymbol{i}, \boldsymbol{j}$ and $\boldsymbol{k}$ are unit vectors horizontally forward, horizontally to the right and vertically upwards respectively. After being hit, the ball is subject to an acceleration $2\boldsymbol{j} - 10\boldsymbol{k}$.
(All distances are measured in metres and all times in seconds.)

Find:

a the velocity of the ball at time t

b the position vector of the ball at the time t

c the time of flight to the ball

d the values of a and b if the golfer wishes to hit a *direct* hole in one where the position vector of the hole is $100\boldsymbol{i}$

e the angle of projection of the ball if a hole in one is achieved.

Review

9 Particles A and B move in a cartesian plane so that at any time $t \geq 0$, their position vectors, are

$\boldsymbol{r}_A = 2t\boldsymbol{i} + t\boldsymbol{j}$

$\boldsymbol{r}_B = (4 - 4\sin(\alpha t))\boldsymbol{i} + 4\cos(\alpha t)\boldsymbol{j}$ where α is a positive constant.

a Find the speed of B in terms of α.

b Find the cartesian equations of the paths of A and B.

c On the same set of axes, sketch the paths of A and B and indicate the direction of travel.

d Find coordinates of the points where the paths of A and B cross.

e Find the least value of α correct to two decimal places for which A and B will collide.

10 With respect to an origin O, particles P and Q have variable position vectors $\boldsymbol{p}$ and $\boldsymbol{q}$ respectively given by $\boldsymbol{p}(t) = (\cos t)\boldsymbol{i} + (\sin t)\boldsymbol{j} - \boldsymbol{k}$ and $\boldsymbol{q}(t) = (\cos 2t)\boldsymbol{i} - (\sin 2t)\boldsymbol{j} + \frac{1}{2}\boldsymbol{k}$ where $0 \leq t \leq 2\pi$.

a **i** For $\boldsymbol{p}(t)$, describe the path.

ii Find the distance of particle P from the origin at time t.

iii Find the velocity of particle P at time t.

iv Show that the vector $(\cos t)\boldsymbol{i} + (\sin t)\boldsymbol{j}$ is perpendicular to the velocity vector for any value of t.

v Find the acceleration, $\ddot{\boldsymbol{p}}(t)$, at time t.

b **i** Find the vector $\overrightarrow{PQ}$ at time t.

ii Show that the distance between P and Q at time t is $\sqrt{\frac{17}{4} - 2\cos 3t}$.

iii Find the maximum distance between the particles.

iv Find the times at which this maximum occurs.

v Find the minimum distance between the particles.

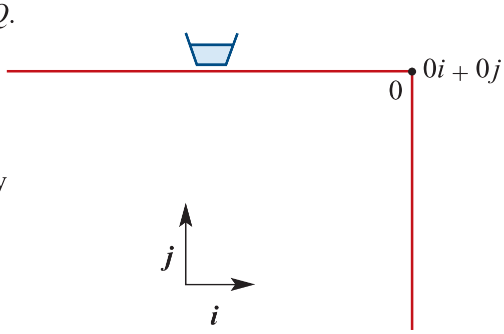
vi Find the times at which this minimum occurs.

c **i** Show that $\boldsymbol{p}(t) \cdot \boldsymbol{q}(t) = \cos(3t) - \frac{1}{2}$.

ii Find an expression for $\cos(POQ)$.

iii Find the greatest magnitude of angle POQ.

11 The bartender slides a glass along a bar for a customer to collect. Unfortunately, the customer has turned to speak to a friend. The glass slides over the edge of the bar with a horizontal velocity of 2 m/s. Assume that air resistance is negligible and the acceleration due to gravity is 9.8 m/s^2 in a downward direction.



a **i** Give the acceleration of the glass as a vector expression.

ii Give the vector expression for the velocity of the glass at time t seconds, where t is measured from when the glass leaves the bar.

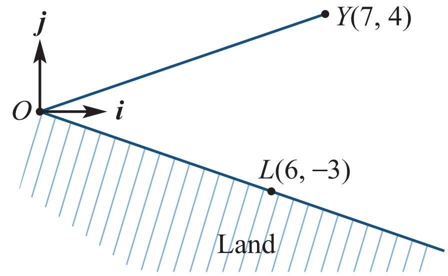
iii Give the position of the glass with respect to the edge of the bar O at time t seconds.

b It is 0.8 m from O to the floor directly below. Find:

i the time it takes to hit the floor

ii the horizontal distance from the bar where the glass hits the floor.

12 A yacht is returning to its marina at O. At noon the yacht is at Y. The yacht takes a straight line course to O. L is the position of a navigation sign on the shore. Coordinates represent distances east and north of the marina measured in kilometres.



a **i** Write down the position vector of the navigation sign L.

ii Find a unit vector in the direction of $\overrightarrow{OL}$.

b Find the vector resolute of $\overrightarrow{OY}$ in the direction of $\overrightarrow{OL}$ and hence find the coordinates of the point on shore closest to the yacht at noon.

c The yacht sails towards O. The position vector at time t hours after 12:00 is given by $\boldsymbol{r}(t) = \left(7 - \frac{7}{2}t\right)\boldsymbol{i} + (4 - 2t)\boldsymbol{j}$.

i Find an expression for $\overrightarrow{LP}$ where P is the position of the yacht at time t.

ii Find the time when the yacht is closest to the navigation sign.

iii Find the closest distance between the sign and the yacht.

CHAPTER

13

Dynamics

Objectives

- To understand and use **definitions** of:
  - mass
  - weight
  - force
  - resultant force
  - momentum
- To apply **Newton's three laws of motion**
- To understand and utilise the **sliding friction force**:

  $F = \mu R$

  where μ is the coefficient of friction
- To consider the case of **equilibrium**, i.e. when acceleration is zero
- To apply vector function techniques

The aim of theoretical dynamics is to provide a quantitative prediction of the motion of objects. In other words, to construct a mathematical model for **motion**. The practical applications of such models are obvious. In this chapter, motion is considered only in a straight line.

The Greeks were the first to record a theoretical basis for the subject. Archimedes (3rd century BC) wrote on the subject and this study was advanced by many others. Many of the great mathematicians of the 17th to 19th centuries worked on the subject. These include Isaac Newton (1642–1727), whose work provides the material for much of this chapter, Leonhard Euler (1707–1783) and Joseph Lagrange (1736–1813).

Measurements

The description of motion is dependent on the measurement of length, time and mass. In this chapter, the principal unit of:

- length will be the metre
- mass will be the kilogram
- time will be the second.

Other units will occur, but it is often advisable to convert these to metres, kilograms and seconds. This system of units is called the mks system.

It is important to note that the mass of an object is the amount of matter it contains. The measurements of mass of an object does not depend on its position. In mathematics, 'mass' and 'weight' do *not* have the same meaning.

Vectors and scalars

In Chapters 2 and 12, vectors and scalars were considered. Time, length and mass are **scalar** quantities, while displacement, velocity and acceleration are **vector** quantities.

Particle

In this chapter, a **particle model** is used. This means that an object is considered as a point. This can be done when the size (dimension) of the object can be neglected in comparison with other lengths in the problem being considered, or when rotational motion effects can be ignored.

13.1 Force

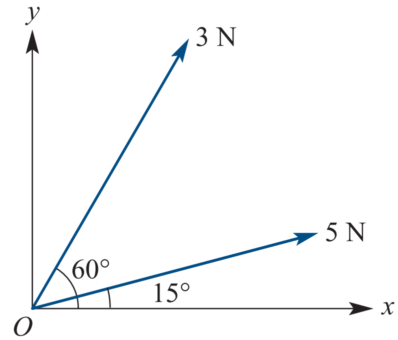
'Force' is a word in common usage and most people have an intuitive idea of its meaning. When a piano or some other object is pushed across the floor, this is done by exerting some force on the piano. A body falls because of the gravitational force exerted on it by the Earth. A discussion of different types of forces follows in the next section.

One unit of force is the **kilogram weight** (kg wt). If a body has mass of one kilogram then the gravitational force acting on this body is one kilogram weight. Another unit of force is the **newton** (N). One kilogram weight = g newtons, where g is the acceleration of the particle owing to gravity. The significance of this unit will be discussed in the following section.

Force is a vector quantity. The vector sum of the forces acting at a point is called the **resultant force**.

Example 1

Find the magnitude and direction of the resultant force of the forces 3 N and 5 N acting on a particle at O as shown in this diagram.



Solution

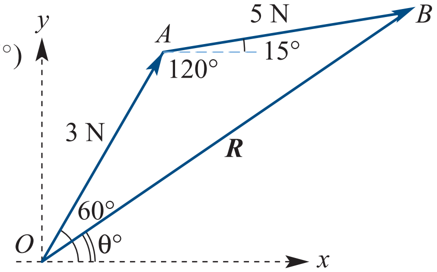
Method 1

The resultant force, $\boldsymbol{R}$, is given by the vector sum.

The angle OAB has magnitude 135°.

Using the cosine rule

$$|\boldsymbol{R}|^2 = 3^2 + 5^2 - 2 \times 3 \times 5\cos(135^\circ)$$
$$= 9 + 25 - 30\cos(135^\circ)$$
$$= 34 + 30 \times \frac{1}{\sqrt{2}}$$
$$= 34 + 15\sqrt{2}$$
$$\therefore \quad |\boldsymbol{R}| \approx 7.43\text{ N}$$



The magnitude of the resultant force is 7.43 N (correct to two decimal places).

In order to describe the direction of the vector, the angle θ° between the vector and the positive direction of the x axis will be found.

Let $\angle AOB = (60 - \theta)^\circ$.

Then $$\frac{|\boldsymbol{R}|}{\sin 135^\circ} = \frac{5}{\sin(60 - \theta)}$$

$$\therefore \quad \sin(60 - \theta) = \frac{5\sin(135^\circ)}{|\boldsymbol{R}|}$$
$$= 0.4758\ldots$$
$$\therefore \quad \theta = 31.587\ldots$$
$$\theta = 31^\circ 35' \text{ to the nearest minute.}$$

Method 2

The problem can also be completed by expressing each of the vectors in $\boldsymbol{i} - \boldsymbol{j}$ notation.

The vector of magnitude 3 N in component form is:

$3\cos 60^\circ \boldsymbol{i} + 3\sin 60^\circ \boldsymbol{j}$

The vector of magnitude 5 N in component form is:

$5\cos 15^\circ \boldsymbol{i} + 5\sin 15^\circ \boldsymbol{j}$

The sum $= (6.3296\ldots)\boldsymbol{i} + (3.8921\ldots)\boldsymbol{j}$

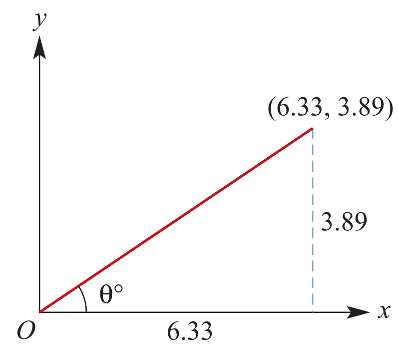
The magnitude of the resultant is = 7.43 N, correct to two decimal places.

Determine the direction:

Note: $\tan\theta^\circ = \frac{3.8921\ldots}{6.3296\ldots}$

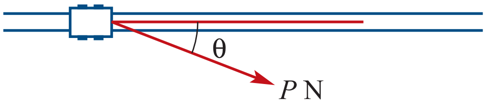
$\theta = 31.5879\ldots$

The resultant force is 7.43 N acting in the direction 31° 35′ anticlockwise from the x axis.



Resolution of a force in a given direction

In the previous example, each of the forces was written as a vector in $\boldsymbol{i} - \boldsymbol{j}$ form, i.e. the resolutes of each force in the $\boldsymbol{i}$ direction and $\boldsymbol{j}$ direction were found. A discussion of the physical interpretation of resolution in a given direction follows.

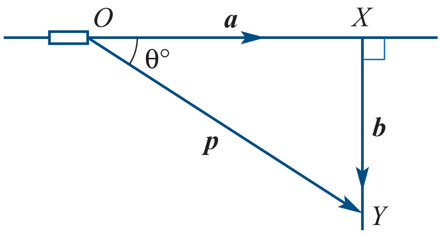


Consider a model railway trolley, set on smooth straight tracks, pulled by a force of magnitude P N along a horizontal string which makes an angle θ with the direction of the track. (The plan view is shown in the diagram.)

When $\theta = 0°$, the trolley moves along the track. As θ increases, the trolley will still move along the track but the same force will have a decreasing effect on the motion of the trolley, i.e. the acceleration of the trolley will be less. When $\theta = 90°$, the trolley will stay in equilibrium, i.e. if at rest it will not move, unless the force is strong enough to cause it to topple sideways.

This example demonstrates the significance of the resolution of a force in a given direction. These and other experimental observations can be reduced to the following fact.

A force acting on a body has an influence in directions other than its line of action, except the direction perpendicular to its line of action.



Let the force of P N be represented by the vector $\boldsymbol{p}$.

Let $\boldsymbol{a}$ be the resolute of $\boldsymbol{p}$ in the $\overrightarrow{OX}$ direction and $\boldsymbol{b}$ be the perpendicular resolute. From the triangle of vectors it can be seen that $\boldsymbol{p} = \boldsymbol{a} + \boldsymbol{b}$. As the force represented by $\boldsymbol{b}$ does not influence the movement of the trolley along the track, the net effect of $\boldsymbol{P}$ on the movement of the trolley in the direction of the track is $\boldsymbol{a}$. The force represented by $\boldsymbol{a}$ is the resolved part (or component) of the force P in the direction of $\overrightarrow{OX}$.

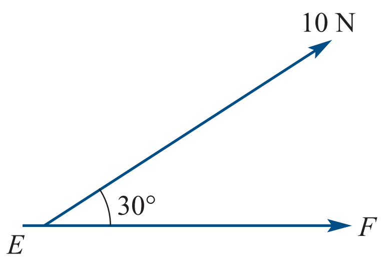
The following summarises this:

The resolved part of a force of P N in a direction which makes an angle θ with its own line of action is a force of magnitude $P \cos \theta$.

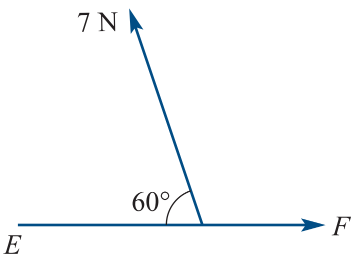
Example 2

Find the resolved part of the following forces in the direction of $\overrightarrow{EF}$.

a



b



Solution

a The resolved part is $10 \cos 30^\circ \text{ N} = 5\sqrt{3}\text{ N}$

b The resolved part is $7 \cos 120^\circ \text{ N} = -3.5\text{ N}$

Example 3

Find the component of the force $\boldsymbol{F} = (3\boldsymbol{i} + 2\boldsymbol{j})$ N in the direction of the vector $2\boldsymbol{i} - \boldsymbol{j}$.

Solution

Let $\boldsymbol{a} = 2\boldsymbol{i} - \boldsymbol{j}$. Then the unit vector in the direction of $\boldsymbol{a}$ is $\hat{\boldsymbol{a}} = \frac{1}{\sqrt{5}}(2\boldsymbol{i} - \boldsymbol{j})$.

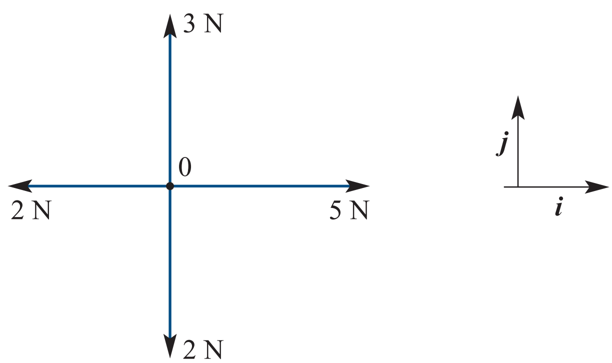
$$\boldsymbol{F}.\hat{\boldsymbol{a}} = (3\boldsymbol{i} + 2\boldsymbol{j}) \cdot \frac{1}{\sqrt{5}}(2\boldsymbol{i} - \boldsymbol{j}) = \frac{1}{\sqrt{5}}(6 - 2) = \frac{4}{\sqrt{5}} = \frac{4\sqrt{5}}{\sqrt{5}}$$

and $$(\boldsymbol{F}.\hat{\boldsymbol{a}}).\hat{\boldsymbol{a}} = \frac{4\sqrt{5}}{5} \times \frac{1}{\sqrt{5}}[(2\boldsymbol{i} - \boldsymbol{j})] = \frac{4}{5}[(2\boldsymbol{i} - \boldsymbol{j})]$$

The component of $\boldsymbol{F}$ in the direction of $2\boldsymbol{i} - \boldsymbol{j}$ is $\frac{4}{5}(2\boldsymbol{i} - \boldsymbol{j})$ N.

Example 4

a Express the resultant force acting in $\boldsymbol{i} - \boldsymbol{j}$ form.



b Give the magnitude of the resultant force and the direction the resultant force makes with the $\boldsymbol{i}$ direction.

Solution

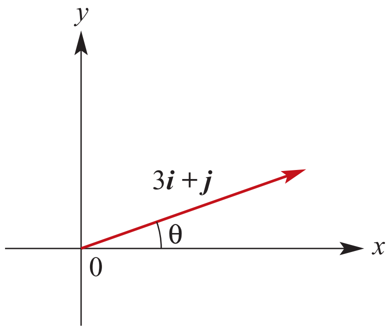
a Resultant force $= (5 - 2)\boldsymbol{i} + (3 - 2)\boldsymbol{j}$
$= (3\boldsymbol{i} + \boldsymbol{j})$ N

b The magnitude of the force $= \sqrt{3^2 + 1^2}$
$= \sqrt{10}$

The angle with $\boldsymbol{i}$ direction is given by

$$\tan\theta = \frac{1}{3}$$

i.e. $\theta = 18^\circ\ 26'$

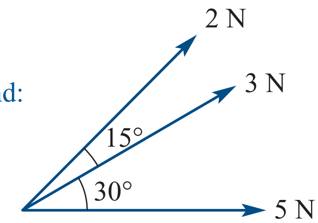


Example 5

Forces of 5 N, 3 N, 2 N act at a point as shown in the diagram. Find:

a the magnitude of the resultant of these forces

b the direction of resultant force with respect to the 5 N force

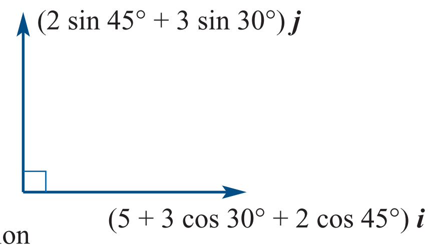


Solution

a Let $\boldsymbol{i}$ be in the direction of the 5 N force.
The sum of the resolved parts in the direction of the 5 N force

$= (5 + 3\cos 30° + 2\cos 45°)\boldsymbol{i}$ N

$= 9.01\boldsymbol{i}$ N correct to two decimal places



The sum of the resolved parts in the direction perpendicular to the 5 N force

$= (2\sin 45° + 3\sin 30°)\boldsymbol{j}$ N

$= 2.91\boldsymbol{j}$ N correct to two decimal places

By vector methods, the magnitude of the resultant force $= \sqrt{9.01^2 + 2.91^2}$

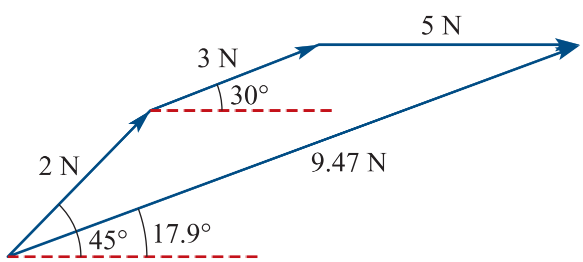
$= 9.47$ N

b Let θ be the angle that the resultant force makes with the 5 N force.

Then, $\tan\theta = \dfrac{2.91}{9.01}$

$\therefore \quad \theta = 17.9°$

The resultant force of 9.47 N is inclined at an angle of 17.9° to the 5 N force.
The vector diagram for the resultant is shown here.



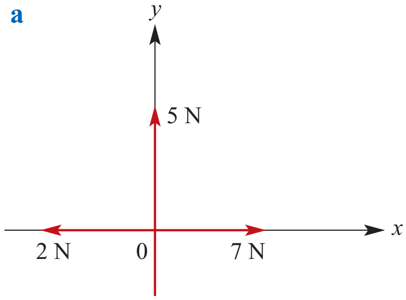
Exercise 13A

1 In the following $\boldsymbol{i}$ is a unit vector in the positive direction of the x axis and $\boldsymbol{j}$ is a unit vector in the positive direction of the y axis.
In each of the following find:

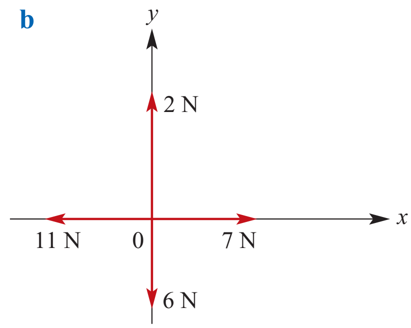
i the resultant force using $\boldsymbol{i} - \boldsymbol{j}$ notation

ii the magnitude and direction of the resultant force. (The angle is measured anticlockwise from the $\boldsymbol{i}$ direction.)

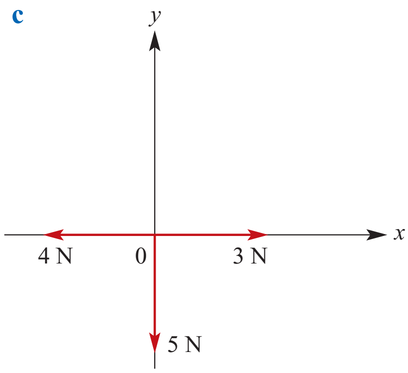
a



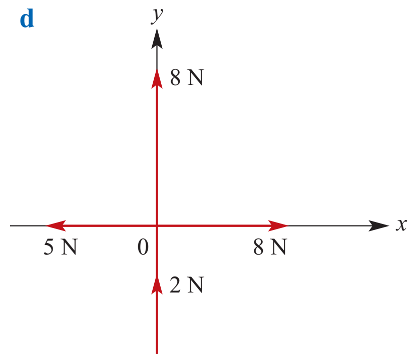
b



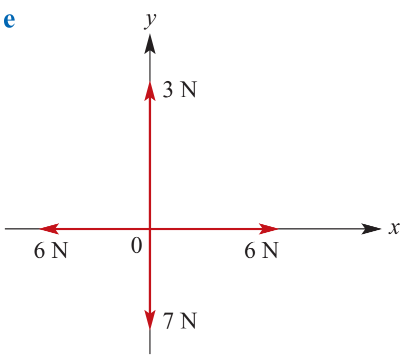
c



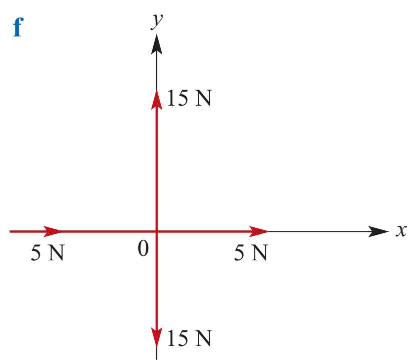
d



e

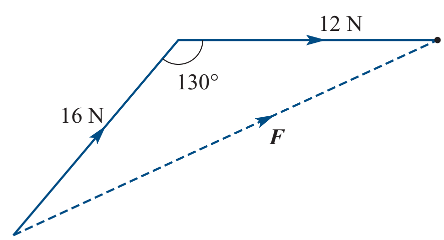


f

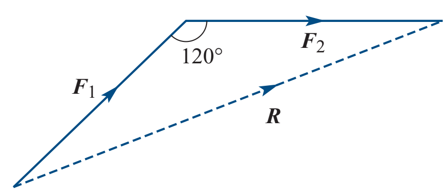


2 The forces $\boldsymbol{F}_1 = (3\boldsymbol{i} + 2\boldsymbol{j})$ N, $\boldsymbol{F}_2 = (6\boldsymbol{i} - 4\boldsymbol{j})$ N and $\boldsymbol{F}_3 = (2\boldsymbol{i} - \boldsymbol{j})$ N act on a particle. Find the resultant force acting on the particle.

3 Find the magnitude of $\boldsymbol{F}$, the resultant force of the 16 N and 12 N forces.

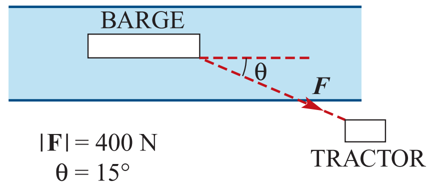


4 $\boldsymbol{R} = \boldsymbol{F}_1 + \boldsymbol{F}_2$
$|\boldsymbol{R}| = 16$ N and $|\boldsymbol{F}_1| = 9$ N
Find $|\boldsymbol{F}_2|$.



5 $\boldsymbol{F}_1 + \boldsymbol{F}_2 + \boldsymbol{F}_3 = \boldsymbol{F}$ and $\boldsymbol{F} = (3\boldsymbol{i} - 2\boldsymbol{j} + \boldsymbol{k})$ N, $\boldsymbol{F}_1 = (2\boldsymbol{i} - \boldsymbol{j} + \boldsymbol{k})$ N and $\boldsymbol{F}_2 = (3\boldsymbol{i} - \boldsymbol{j} - \boldsymbol{k})$ N. Find $\boldsymbol{F}_3$.

6 A tractor is pulling a barge along a canal with a force of 400 N. The barge is moving parallel to the bank. Find the component of $\boldsymbol{F}$ in the direction of motion.

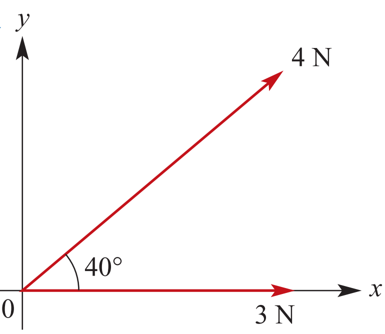


7 In the following $\boldsymbol{i}$ is a unit vector in the positive direction of the x axis and $\boldsymbol{j}$ is a unit vector in the positive direction of the y axis. For each of the following find:

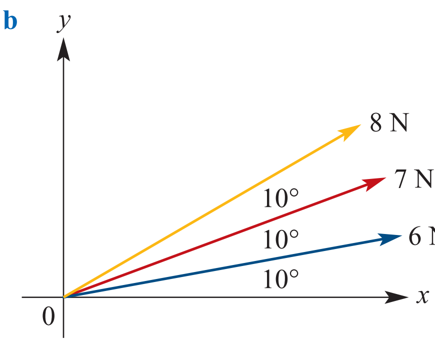
i the resultant force using $\boldsymbol{i} - \boldsymbol{j}$ notation

ii the magnitude and direction of the resultant force. (The angle is measured anticlockwise from the $\boldsymbol{i}$ direction.)

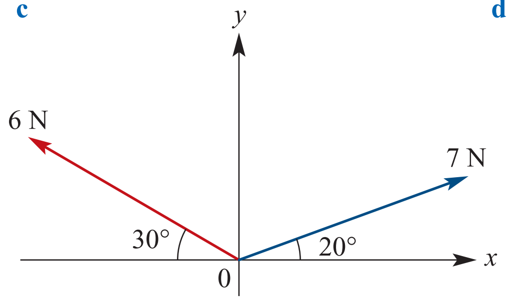
a



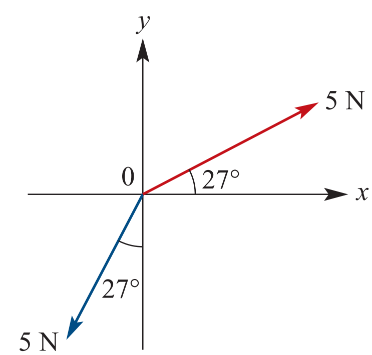
b



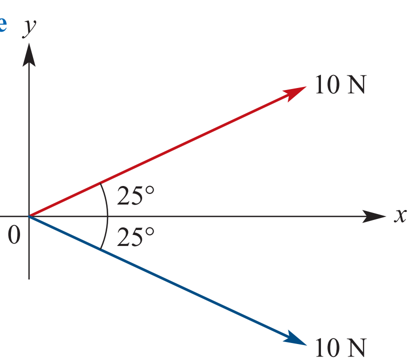
c



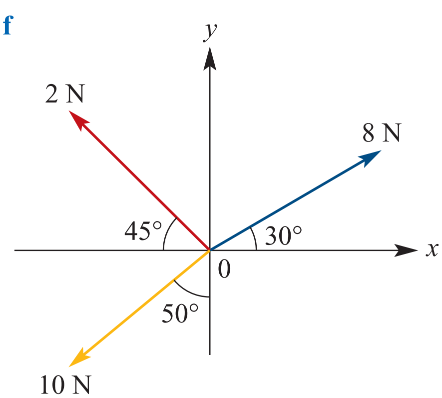
d



e



f

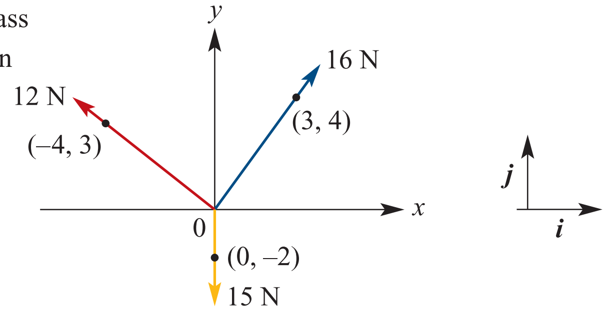


8 Find the resultant force for each of the diagrams **a**, **c** and **e** of question **7** using triangles of forces.

9 Three forces acting at the origin pass through coordinate points as shown in the diagram right.

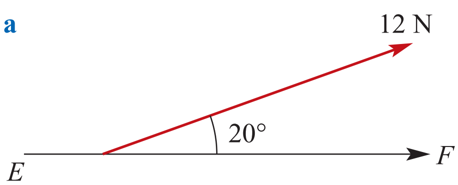
a Find the resultant force.

b Find the magnitude and direction of the resultant force.

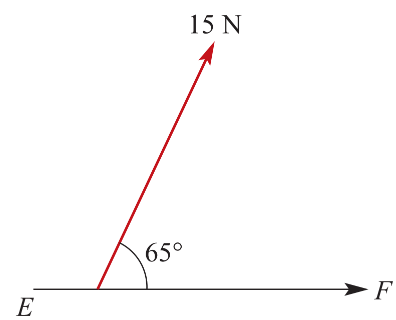


10 Find the resolved part of each of the following forces in the direction of $\overrightarrow{EF}$:

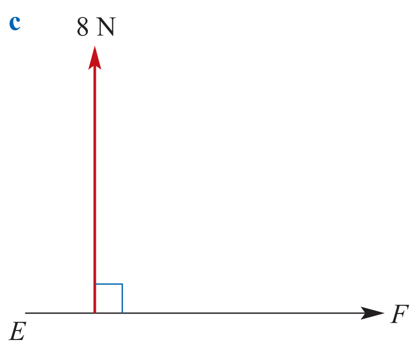
a



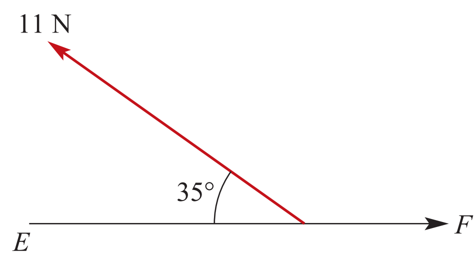
b



c



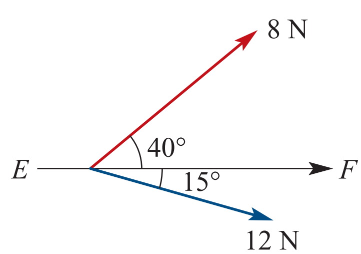
d



11 Two forces act on a particle as shown in the diagram.

a Find the sum of the resolved parts of the forces in the direction of $\overrightarrow{EF}$.

b Find the sum of the resolved parts of the forces in the direction of 8 N force.



12 **a** Find the component of the force $(7\boldsymbol{i} + 3\boldsymbol{j})$ N in the direction of the vector $2\boldsymbol{i} - \boldsymbol{j}$.

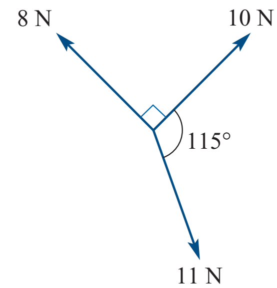
b Find the component of the force $(2\boldsymbol{i} - 3\boldsymbol{j})$ N in the direction of the vector $3\boldsymbol{i} + 4\boldsymbol{j}$.

13 Three forces act on a particle as shown in the diagram. Find the sum of the resolved parts of the forces in the direction of:

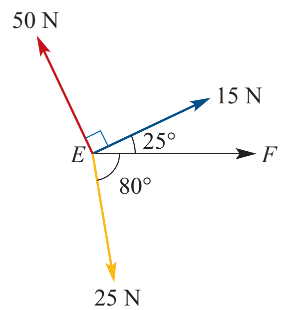
a the 8 N force

b the 10 N force

c the 11 N force



14 Find the sum of the resolved parts of the forces in the direction of $\overrightarrow{EF}$ in this diagram.

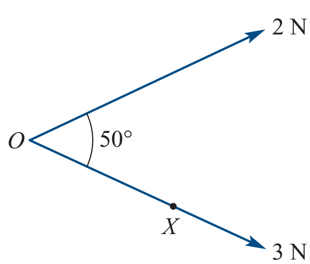


15 A frame is in the shape of a right-angled triangle ABC, where $AB = 6.5$ m, $BC = 6$ m and $AC = 2.5$ m. A force of 10 N acts along $\overrightarrow{BC}$ and a force of 24 N acts along $\overrightarrow{BA}$. Find the sum of the resolved parts of the two forces in the direction of:

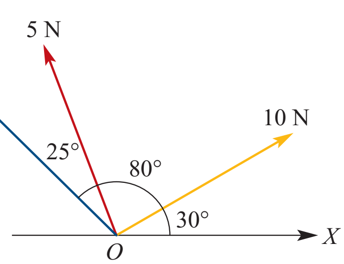
a $\overrightarrow{BC}$ **b** $\overrightarrow{BA}$

16 Find the magnitude and direction with respect to $\overrightarrow{OX}$ of the resultant of the following forces:

a



b

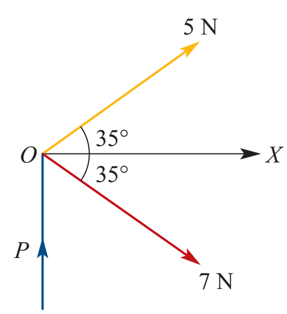


17 Find the magnitude of the resultant of two forces of 7 N and 10 N acting at an angle of 50° to each other.

18 The angles between the forces of magnitude 8 N, 10 N and P N are, respectively, 60° and 90°. The resultant acts along the 10 N force. Find:

a P **b** the magnitude of the resultant

19 Three forces 5 N, 7 N and P N act on a particle at O. Find the value of P that will produce a resultant force along $\overrightarrow{OX}$ if the line of action of P N force is perpendicular to OX.



13.2 Newton's laws of motion

Momentum

The **momentum** of a particle is defined as the product of its mass and velocity.

momentum = mass × velocity

Let $\boldsymbol{v}$ be the velocity of the particle, m the mass and $\boldsymbol{P}$ the momentum. Momentum, $\boldsymbol{P}$, is a vector quantity. It has the same direction as the velocity.

$$\boldsymbol{P} = m\boldsymbol{v}$$

The units of momentum are kg m/s or kg ms^{-1}.

The momentum of a body of mass 3 kg moving at 2 m/s, is 6 kg m/s = 6 kg ms^{-1}.

Momentum can be considered as the fundamental quantity of motion.

Example 6

a Find the momentum of a particle of mass 6 kg moving with a velocity $(3\boldsymbol{i} + 4\boldsymbol{j})$ m/s.

b Find the momentum of a 12 kg particle moving with a velocity of 8 m/s in an easterly direction.

Solution

a momentum $P = 6(3\boldsymbol{i} + 4\boldsymbol{j})$ kg m/s

b momentum = 96 kg m/s in an easterly direction

The **change of momentum** is central to Newton's second law of motion. Its importance is introduced through the following example.

Example 7

Find the change in momentum of a ball of mass 0.5 kg if the velocity changes from 5 m/s to 2 m/s. The ball is moving in the one direction in a straight line.

Solution

Initial momentum of the ball = (0.5×5) kg m/s = 2.5 kg m/s

Momentum after 3 seconds = (0.5×2) kg m/s = 1 kg m/s

Change in momentum = $1 - 2.5 = -1.5$ kg m/s

Newton used this idea of change of momentum to give a formal definition of force and this is stated in his second law of motion given on the next page. In the example, the resistance force has changed the velocity from 5 m/s to 2 m/s. The rate of change of momentum with respect to time is used to define force.

'Dynamics' is based on Newton's laws of motion which are stated on the next page using modern language.

Newton's first law of motion

A particle remains stationary, or in uniform straight line motion (i.e. in a straight line with constant velocity) unless acted on by some overall external force, i.e. if the resultant force is zero.

Newton's second law of motion

A particle acted on by forces whose resultant is not zero will move in such a way that the rate of change of its momentum with respect to time will at any instant be proportional to the resultant force.

Newton's third law of motion

If one particle, A, exerts a force on a second particle, B, then B exerts a collinear force of equal magnitude and opposite direction on A.

Weight

The gravitational force per unit mass due to the Earth is g newtons per kilogram. It varies from place to place on the Earth's surface, having a value of 9.8321 at the poles and 9.7799 at the equator.

In this book, the value 9.8 will be assumed for g, unless otherwise stated.

A mass of m kg on the earth's surface has a force of m kg wt or mg newtons acting on it. This force is known as the **weight**.

Implications of Newton's first law of motion

- A force is needed to start an object moving (or to stop it) but, once moving, the object will continue at a constant velocity without any force being needed.
- When a body is at rest or in uniform straight line motion, then any forces acting must balance.
- When motion is changing, i.e. in speed or direction, then the forces cannot balance—that is, the resultant force is non-zero.

Implications of Newton's second law of motion

Let $\boldsymbol{F}$ represent the resultant force exerted on an object of mass m kg moving at a velocity $\boldsymbol{v}$ m/s in a straight line.

Then Newton's second law gives:

$$\boldsymbol{F} = k\frac{d}{dt}(m\boldsymbol{v})$$

Assuming the mass is a constant:

$$\boldsymbol{F} = km\frac{d}{dt}(\boldsymbol{v}) = km\boldsymbol{a}$$

The newton is the unit of force chosen so that the constant k can be taken to be 1 when the acceleration is measured in m/s^2 (ms^{-2}) and the mass in kilograms.

One newton is the force which causes a change of momentum of 1 kg m/s per second. The formula can be written:

$$\boldsymbol{F} = m\boldsymbol{a}$$

Note that the direction of the acceleration and the resultant force are the same.

Implications of Newton's third law

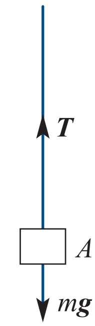
An alternative wording of Newton's third law is:

If one body exerts a force on another (action force), then the second body exerts a force (reaction force) equal in magnitude but opposite in direction to the first.

It is important to note that the action and reaction forces, which always occur in pairs, act on different bodies. If they were to act on the same body, there would never be accelerated motion because the resultant force on every body would be zero.

For example:

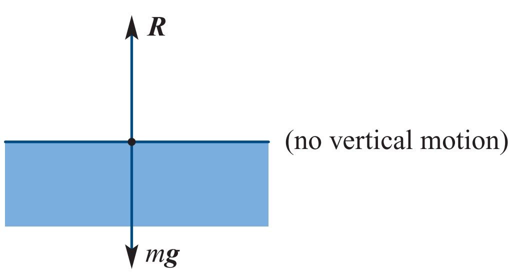
- If a person kicks a door, the door 'accelerates' open because of the force exerted by the person. At the same time, the door exerts a force on the foot of the person which 'decelerates' the foot.
- For a particle hanging from a string the forces $\boldsymbol{T}$ and $m\boldsymbol{g}$ both act on the particle A. They are not necessarily equal and opposite forces. In fact they are equal only if the acceleration of the particle is zero (Newton's second law). $\boldsymbol{T}$ and $m\boldsymbol{g}$ are not an action–reaction pair of Newton's third law as they both act on the one particle.



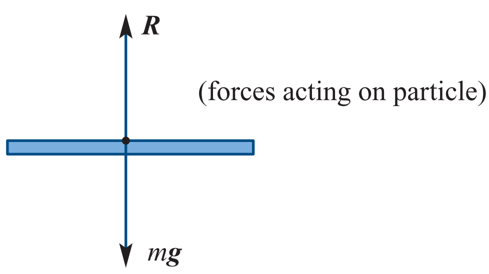
- For a man pulling horizontally on a rope with a force $\boldsymbol{F}$ the rope exerts a force of $-\boldsymbol{F}$ on the man.

Normal reaction force

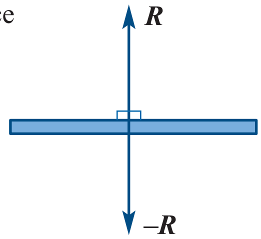
If a particle lies on a surface and exerts a force on the surface then the surface exerts a force $\boldsymbol{R}$ N on the particle. If the surface is smooth, this force is taken to act at right angles to the surface and is called the normal reaction force. In such a situation $\boldsymbol{R} = m\boldsymbol{g}$.



If the particle is on a platform which is being accelerated upwards at an acceleration of $\boldsymbol{a}$ m/s^2, Newton's second law gives $\boldsymbol{R} - m\boldsymbol{g} = m\boldsymbol{a}$.

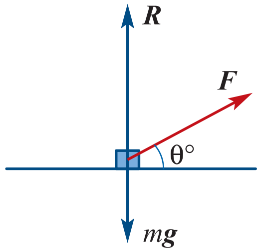


For the force $\boldsymbol{R}$ of the platform there is a force of $-\boldsymbol{R}$ of the particle acting on the platform.



(forces acting on platform)

The forces $\boldsymbol{R}$ and $-\boldsymbol{R}$ are a pair of forces as described by Newton's third law.



(no vertical motion)

If a particle of mass m kg lies on a smooth surface and a force $\boldsymbol{F}$ N acts at an angle of θ° to the horizontal then $|\boldsymbol{R}| = m|\mathbf{g}| - |\boldsymbol{F}|\sin\theta$.

Sliding friction

By experiment, it has been shown that the frictional force $\boldsymbol{F}_R$ of a particle moving on a surface is given by:

$$\boldsymbol{F}_R = \mu\boldsymbol{R}$$

where $\boldsymbol{R}$ is the normal reaction force and μ is the **coefficient of friction**. In a sense, the coefficient of friction is a measure of the roughness of the surfaces of contact.

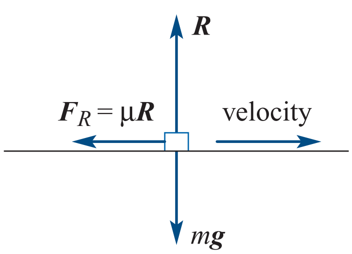
For example:

Surfaces	*Coefficient of friction*
Rubber tyre on dry road	approaches 1
Two wooden surfaces	0.3 to 0.5
Two metal surfaces	0.1 to 0.2

If the surface is taken to be smooth, $\mu = 0$.

The frictional force acts in the opposite direction to the velocity of the particle.

Note: Henceforth vector quantities will not always be presented in bold type. This should not lead to any confusion as all motion is linear.



Example 8

A stone of mass 16 grams is acted on by a force of 0.6 N. What will be its acceleration?

Solution

The formula $\boldsymbol{F} = m\boldsymbol{a}$ will be used.

$$16 \text{ g} = 0.016 \text{ kg}$$
$$0.016a = 0.6$$
$$\therefore \quad a = 37.5$$

The acceleration is 37.5 m/s^2.

Example 9

Three forces $\boldsymbol{F}_1$, $\boldsymbol{F}_2$ and $\boldsymbol{F}_3$ act on a particle of mass 2 kg. $\boldsymbol{F}_1 = (2\boldsymbol{i} - 3\boldsymbol{j})$ N and $\boldsymbol{F}_2 = (4\boldsymbol{i} + 2\boldsymbol{j})$ N. The acceleration of the particle is $4\boldsymbol{i}$ m/s^2. Find $\boldsymbol{F}_3$

Solution

Newton's second law of motion gives

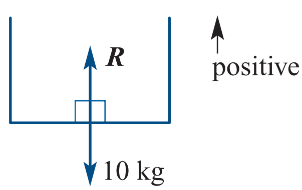
$$(\boldsymbol{F}_1 + \boldsymbol{F}_2 + \boldsymbol{F}_3) = 2 \times 4\boldsymbol{i}$$
$$\therefore \quad 2\boldsymbol{i} - 3\boldsymbol{j} + 4\boldsymbol{j} + 2\boldsymbol{j} + \boldsymbol{F}_3 = 8\boldsymbol{i}$$
$$\text{i.e.} \quad 6\boldsymbol{i} - \boldsymbol{j} + \boldsymbol{F}_3 = 8\boldsymbol{i}$$

This implies $\boldsymbol{F}_3 = (2\boldsymbol{i} + \boldsymbol{j})$ N

Example 10

A box is on the floor of a lift accelerating upwards at 2.5 m/s^2. The mass of the box is 10 kg. Find the reaction of the floor of the lift on the box.

Solution



Let $\boldsymbol{R}$ be the reaction of the floor on the box.

Newton's second law of motion gives

$$|\boldsymbol{R}| - 10 \text{ g} = 10 \times 2.5$$
$$\therefore \quad |\boldsymbol{R}| = 10 \text{ g} + 25$$
$$= 98 + 25$$
$$= 123 \text{ N}$$

The reaction of the floor of the lift on the box is 123 N.

Example 11

An ice hockey puck of mass 150 grams loses speed from 26 m/s to 24 m/s over a distance of 35 m. Find the uniform force which causes this change in velocity. How much further could the puck travel?

Solution

The retarding force is uniform. Therefore, $a = k$, where k is a constant.

Using acceleration $= \dfrac{d}{dx}\left(\frac{1}{2}v^2\right)$

$\frac{1}{2}v^2 = kx + c$ (**Note:** the formula for constant acceleration $v^2 = u^2 + 2as$ could also have been applied)

When $t = 0$, $v = 26$ and $x = 0$ and therefore $c = \dfrac{26^2}{2}$

When $x = 35$, $v = 24$

$$\therefore \quad \frac{24^2}{2} = 35k + \frac{26^2}{2}$$

$$\therefore \quad k = \frac{-10}{7}$$

$\therefore$ The uniform force that is acting is $F = \frac{-10}{7} \times 0.15 = -\frac{3}{14}$ N

When $v = 0$,

$$kx + \frac{26^2}{2} = 0$$

$$\therefore \quad x = \frac{-26^2}{2} \times \frac{-7}{10}$$

$$= 236.6 \text{ m}$$

i.e. The puck would travel a further 201.6 m before coming to rest.

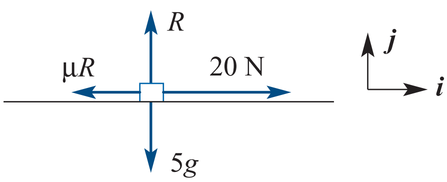
Example 12

A body of mass five kilograms at rest on a rough horizontal plane is pushed by a horizontal force of 20 N for five seconds.

a If $\mu = 0.3$, how far does the body travel in this time?

b How much further will it move after the force is removed?

Solution



a Resolve vertically: i.e. in the $\boldsymbol{j}$ direction.

$$(R - 5g)\boldsymbol{j} = 0 \quad \text{i.e. } R = 5g$$

Resolve horizontally: i.e. in the $\boldsymbol{i}$ direction.

$$(20 - \mu R)\boldsymbol{i} = 5\boldsymbol{a}$$

i.e. $(20 - 1.5g)\boldsymbol{i} = 5\boldsymbol{a}$

$$\boldsymbol{a} = (4 - 0.3g)\boldsymbol{i} = 1.06\boldsymbol{i}$$

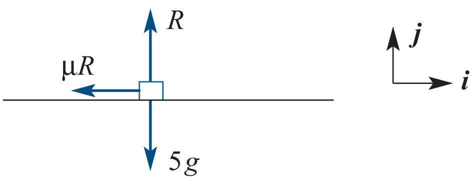
Therefore velocity after five seconds $= 1.06 \times 5 = 5.3$ m/s.

The distance the body travels $= \frac{1}{2} \times a \times t^2 \qquad \left(s = ut + \frac{1}{2}at^2\right)$

$= \frac{1}{2} \times 1.06 \times 25$

$= 13.25$ metres

b From ***a***: $R = 5g$



$$\therefore \quad (-0.3 \times 5g)\boldsymbol{i} = 5\boldsymbol{a}$$
$$-0.3g = a$$
$$-2.94 = a$$

Using $v^2 = u^2 + 2as$

when $v = 0$

$$0 = (5.3)^2 - 2 \times 2.94 \times s$$
$$s = \frac{(5.3)^2}{2 \times 2.94}$$
$$\approx 4.78 \text{ metres}$$

The body will come to rest after 4.78 metres.

Exercise 13B

1 Find the momentum of each of the following:

a a mass of 2 kg moving with a velocity of 5 m/s
b a mass of 300 g moving with a velocity of 3 cm/s
c a mass of 1 tonne moving with a velocity of 30 km/h
d a mass of 6 kg moving with a velocity of 10 m/s
e a mass of 3 tonnes moving with a velocity of 50 km/h

2 **a** Find the momentum of a particle of mass 10 kg moving with a velocity of $(\boldsymbol{i} + \boldsymbol{j})$ m/s.
b **i** Find the momentum of a particle of mass 10 kg moving with a velocity of $(5\boldsymbol{i} + 12\boldsymbol{j})$ m/s.
ii Find the magnitude of this momentum.

3 Find the change in momentum when a body of mass 10 kg moving in a straight line changes its velocity in three seconds from:

a 6 m/s to 3 m/s **b** 6 m/s to 10 m/s **c** −6 m/s to 3 m/s

4 Find the weight, in newtons, of each of the following:

a a 5 kg bag of potatoes **b** a tractor of mass 3 tonnes
c a tennis ball of mass 60 g

5 **a** A body of mass 8 kg is moving with an acceleration of 4 m/s^2 in a straight line. Find the resultant force acting on the body.
b A body of mass 10 kg is moving in a straight line. The resultant force acting on the body is 5 N. Find the magnitude of the acceleration of the body.

6 **a** A force of 10 N acts on a particle of mass m kg and produces an acceleration of 2.5 m/s^2. Find the value of m.
b A force of F N acts on a particle of 2 kg and produces an acceleration of 3.5 m/s^2. Find the value of F.

7 What size mass would be accelerated upwards at 1.2 m/s^2 by an vertical force of 96 N?

8 A parachutist of mass 75 kg, whose parachute only partly opens, accelerates downwards at 1 m/s^2. What upward force must her parachute be providing?

9 In a lift that is accelerating upwards at 2 m/s^2 a spring balance shows the apparent weight of a body to be 2.5 kg wt. What would be the reading if the lift were at rest?

10 An electron of mass 9×10^{-31} kg in a magnetic field has at a given instant, an acceleration of 6×10^{16} m/s^2. Find the resultant force on the electron at that instant.

11 A force of $(2\boldsymbol{i} + 10\boldsymbol{j})$ N acts on a body of mass 2 kg. Find the acceleration of the body.

12 A particle of mass 10 kg is acted on by two forces $(8\boldsymbol{i} + 2\boldsymbol{j})$ N and $(2\boldsymbol{i} - 6\boldsymbol{j})$ N. Find the acceleration of the body.

13 In a lift that is accelerating downwards at 1 m/s^2, a spring balance shows the apparent weight of the body to be 2.5 kg wt. What would be the reading if the lift was:

a at rest? **b** accelerating upwards at 2 m/s^2?

14 A truck of mass 25 tonnes is travelling at 50 km/h when its brakes are applied. What constant force is required to bring it to rest in 10 seconds?

15 A box of mass 10 kg lies on the horizontal floor of a lift which is accelerating upwards at 1.5 m/s^2. Find the reaction, in newtons, of the lift floor on the box.

16 A particle of mass 16 kg is acted on by three forces $\boldsymbol{F}_1$, $\boldsymbol{F}_2$, and $\boldsymbol{F}_3$ in newtons. If $\boldsymbol{F}_1 = -10\boldsymbol{i} - 15\boldsymbol{j}$ and $\boldsymbol{F}_2 = 16\boldsymbol{j}$ and the acceleration of the particle is $0.6\boldsymbol{i}$ m/s^2, find $\boldsymbol{F}_3$.

17 A particle of mass 5 kg is observed to be travelling in a straight line at a speed of 5 m/s. Three seconds later the particle's speed is 8 m/s in the same direction. Find the magnitude of the constant force which could produce this change in speed.

18 A particle of mass 4 kg is subjected to forces of $8\boldsymbol{i} + 12\boldsymbol{j}$ newtons and $6\boldsymbol{i} - 4\boldsymbol{j}$ newtons. Find the acceleration of the particle.

19 A reindeer is hauling a heavy sled of mass 300 kg across a rough surface. The reindeer exerts a horizontal force of 600 N on the sled while the resistance to the sled's motion is 550 N. If the sled is initially at rest, find the velocity of the sled after three seconds.

20 A lift operator of mass 85 kg stands in a lift which is accelerating downwards at 2 m/s^2. Find the reaction force of the lift floor on the operator.

21 A body of mass 10 kg on a rough horizontal table (coefficient of friction 0.2) is acted on by a horizontal force of magnitude 4 kg wt. Find:

a the acceleration of the body

b the velocity of the body after 10 seconds, if it starts from rest.

22 The engine of a train of mass 200 tonnes exerts a force of 8000 kg wt, and the total air and rail resistance is 20 kg wt/tonne. How long will it take the train on level ground to acquire a speed of 30 km/h from rest?

23 One man can push a wardrobe of mass 250 kg with an acceleration of magnitude 0.15 m/s^2. With help from another man pushing just as hard (i.e. with the same force), the wardrobe accelerates at 0.4 m/s^2. How hard is each man pushing and what is the resistance to sliding?

24 What force is necessary to accelerate a train of mass 200 tonnes at 0.2 m/s^2 against a resistance of 20 000 N? What will be the acceleration if the train free-wheels against the same resistance?

25 A body of mass 10 kg is being pulled across a rough, horizontal surface by a force of magnitude 10 N. If the body is moving with constant velocity find the coefficient of friction between the body and the surface.

26 A puck of mass 0.1 kg is sliding in straight line on an ice-rink. The coefficient of friction between the puck and the ice is 0.025.

a Find the resistive force owing to friction.
b Find the speed of the puck after 20 seconds if its initial speed is 10 m/s.

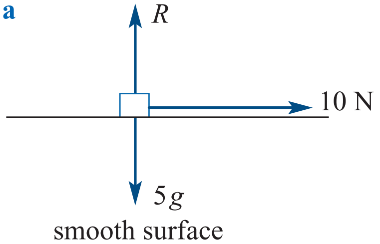
27 A block of 4 kg will move at a constant velocity when pushed along a table by a horizontal force of 24 N. Find the coefficient of friction between the block and the table.

28 A load of 200 kg is being raised by a cable. Find the tension in the cable when:

a the load is lifted at a steady speed of 2 m/s
b the load is lifted with an upward acceleration of 0.5 m/s^2

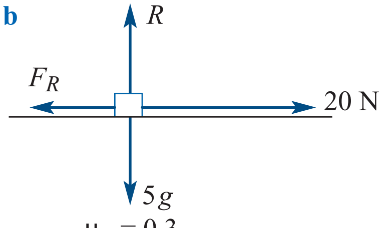
29 Find the acceleration of a 5 kg mass for each of the following situations. (The body moves in a straight line across the surfaces.)

a



smooth surface

b

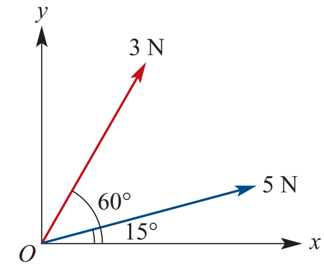


$\mu_R = 0.3$

13.3 Resolution of forces and inclined planes

Example 13

A particle at O is acted on by forces of magnitude 3 N and 5 N respectively. The resultant force was determined in Example 1. If the particle has mass 1 kg, find the acceleration and state the direction of the acceleration.



Solution

The equation $\boldsymbol{F} = m\boldsymbol{a}$ is used.

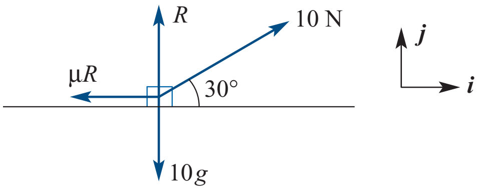
Note: $\boldsymbol{F} = 6.33\boldsymbol{i} + 3.89\boldsymbol{j}$

$\therefore \quad \boldsymbol{a} = 6.33\boldsymbol{i} + 3.89\boldsymbol{j}$

The direction of the acceleration is the same as the direction of the force, i.e. at 31.57° anticlockwise from the x axis.

Example 14

A block of mass 10 kg is pulled along a horizontal plane by a force of 10 N inclined at 30° to the plane. The coefficient of friction between the block and the plane is 0.05. Find the acceleration of the block.



Solution

Resolving in the $\boldsymbol{j}$ direction:

$$(R + 10\cos 60^\circ - 10g)\boldsymbol{j} = \boldsymbol{0}$$

$$\therefore \quad R = 10(g - \cos 60^\circ)$$

$$= 10\left(g - \tfrac{1}{2}\right)$$

Resolving in the $\boldsymbol{i}$ direction:

$$(10\cos 30^\circ - \mu R)\boldsymbol{i} = 10\boldsymbol{a}$$

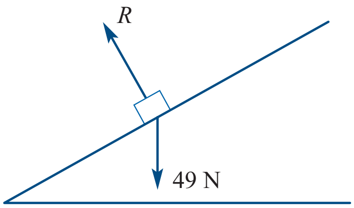
$$\therefore \quad \cos 30^\circ - 0.05\left(g - \tfrac{1}{2}\right) = a$$

$$\therefore \quad a \approx 0.4 \text{ m/s}^2$$

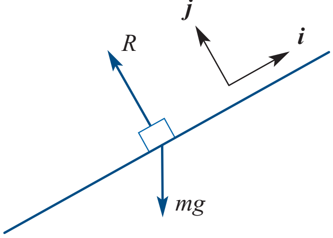
i.e. The acceleration of the block is approximately 0.4 m/s^2.

Normal reaction forces for inclined planes

For a weight of 5 kg on a plane inclined to the horizontal, the normal reaction is at right angles to the plane.

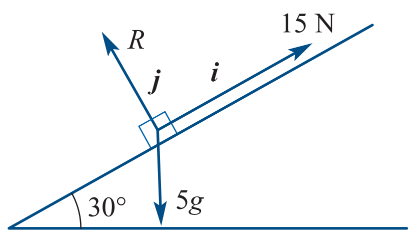


When dealing with problems involving particles moving on planes inclined to the horizontal, it is often advantageous to choose the direction up the plane to be $\boldsymbol{i}$ and the direction perpendicular from the plane to be $\boldsymbol{j}$.



Example 15

A particle of mass 5 kg lies on a plane inclined at 30° to the horizontal. There is a force of 15 N, acting up the plane, that resists motion. Find the acceleration of the particle down the incline and the reaction force R.



Solution

Resolving in the $\boldsymbol{i}$ direction:

$$15 + 5g\cos 120^\circ = 15 - \frac{49}{2} = \frac{-19}{2}$$

For the $\boldsymbol{i}$ direction:

$$\frac{-19}{2}\boldsymbol{i} = 5\boldsymbol{a}$$

$$\therefore \quad \boldsymbol{a} = -1.9\boldsymbol{i}$$

The acceleration is 1.9 m/s^2 down the plane.

Resolving in the $\boldsymbol{j}$ direction

$$R + 5g\cos 150^\circ = R - 5\frac{\sqrt{3}}{2}g = 0$$

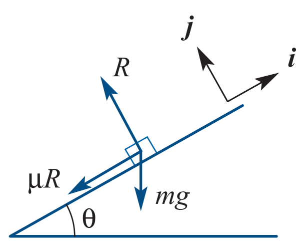
i.e. $$R = 5\frac{\sqrt{3}}{2}g$$

Example 16

A slope is inclined at an angle θ to the horizontal where $\tan\theta = \frac{4}{3}$. A particle is projected from the foot of the slope up a line of greatest slope with a speed of V m/s and comes instantaneously to rest after travelling 6m. If the coefficient of friction between the particle and the slope is $\frac{1}{2}$, calculate:

a the value of V **b** the speed of the particle when it returns to its starting point

Solution



a (**Note:** Friction acts in the opposite direction to motion.)

Resolving in the $\boldsymbol{j}$ direction:

$$mg\cos\theta = R$$

As $\tan\theta = \frac{4}{3}$, $\cos\theta = \frac{3}{5}$

$$\therefore \quad \frac{3mg}{5} = R$$

Resolving parallel to the plane (i.e. in the $\boldsymbol{i}$ direction)

$-\mu R - mg\sin\theta = ma$ (Newton's second law)

As $\tan\theta = \frac{4}{3}$, $\sin\theta = \frac{4}{5}$

$$\therefore \quad -\frac{1}{2} \times \frac{3mg}{5} - \frac{4mg}{5} = ma$$

$$\text{i.e.} \quad -\frac{3g}{10} - \frac{8g}{10} = a$$

$$\frac{-11g}{10} = a$$

i.e. The acceleration is $\frac{-11g}{10}$ m/s^2.

Using the equation of motion $v^2 = V^2 + 2as$ it comes to rest when $s = 6$, $v = 0$.

$$\therefore \quad 0 = V^2 - \frac{66g}{5}$$

$$\text{i.e.} \quad V^2 = \frac{66g}{5}$$

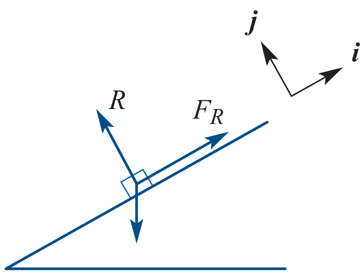
$$\therefore \quad V = \sqrt{\frac{66g}{5}}$$ (**Note:** V is positive, as it is projected up the plane.)

$$V = 11.37$$

The initial velocity is 11.37 m/s.

b **Note:** Friction now acts up the plane.

Resolving in the $\boldsymbol{i}$ direction:



$$\mu R - mg\sin\theta = ma$$

$$\frac{1}{2} \times \frac{3mg}{5} - \frac{4mg}{5} = ma$$

$$\text{i.e.} \quad a = -\frac{g}{2}$$

Using $v^2 = u^2 + 2as$ once again:

$$v^2 = \frac{-g}{2} \times 2 \times -6$$

$$= 6g$$

$$\therefore \quad v = 7.67 \text{ m/s}$$

Exercise 13C

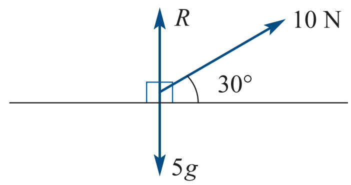
1 A particle slides down a smooth slope of 45°. What is its acceleration?

2 A particle, of mass m kilograms, slides down a slope of 45°. If the coefficient of friction of the surfaces involved is μ, find the acceleration.

3 A 60 kg woman skis down a slope that makes an angle of 60° with the horizontal. The woman has an acceleration of 8 m/s^2. What is the magnitude of the resistive force?

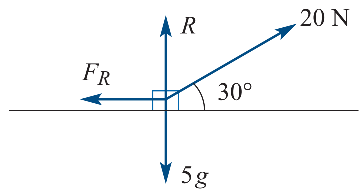
4 Find the acceleration of a 5 kg mass for each of the following situations.

a



smooth surface

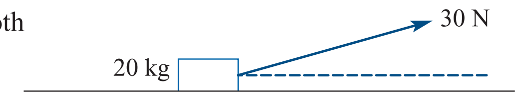
b



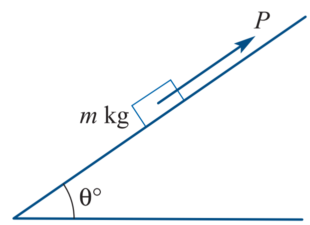
$\mu_R = 0.3$

5 A block of mass 2 kg lies on a rough horizontal table, with a coefficient of friction of $\frac{1}{2}$. Find the magnitude of the force on the block which, when acting at 45° upwards from the horizontal, produces in the block a horizontal acceleration of $\frac{g}{4}$ m/s^2.

6 A box of mass 20 kg is pulled along a smooth horizontal table by a force of 30 newtons acting at an angle of 30° to the horizontal. Find the magnitude of the normal reaction of the table on the box.



7 A particle of mass m kg is being accelerated up a rough inclined plane, with coefficient of friction μ at a m/s^2 by a force of P newtons acting parallel to the plane. The plane is inclined at an angle of $\theta°$ to the horizontal. Find a in terms of P, θ, m, μ and g.



8 A particle is projected up a smooth plane inclined at 30° to the horizontal. $\boldsymbol{i}$ is a unit vector up the plane. Find the acceleration of the particle.

9 A particle slides from rest down a rough plane inclined at 60° to the horizontal. Given that the coefficient of friction between the particle and the plane is 0.8, find the speed of the particle after it has travelled 5 m.

10 A body is projected up an incline of 20° with a velocity of 10 m/s. If the coefficient of friction between the body and the plane is 0.25, find the distance it goes up the plane and the velocity with which it returns to its starting point.

11 A particle of mass m kg slides down a smooth inclined plane x metres long, inclined at $\theta°$ to the horizontal, where $\tan\theta = \frac{4}{3}$.

- **a** With what speed does the particle reach the bottom of the plane?
- **b** At the bottom, it slides over a rough horizontal surface (coefficient of friction 0.3). How far will it travel along this surface?

12 A body mass of M kg is pulled along a rough horizontal plane (coefficient of friction μ) by a constant force of F newtons, at an inclination of θ. Find the acceleration of the body if:

- **a** θ is upwards from the horizontal
- **b** θ is downwards from the horizontal

13 A car of mass one tonne coasts down a slope of 1 in 20 $\left(\sin\theta = \frac{1}{20}\right)$ at constant speed. The car can ascend the same slope with a maximum acceleration of 1 m/s^2. Find:

a the total resistance to the motion (assumed constant)

b the driving force exerted by the engine when the maximum acceleration is reached

14 A particle of mass 0.5 kg is projected up the line of greatest slope of a rough plane inclined at an angle θ to the horizontal, where $\sin\theta = \frac{3}{5}$. Given that the speed of projection is 6 m/s, and that the coefficient of friction between the particle and the plane is $\frac{3}{8}$, calculate:

a the distance travelled up the plane when the speed has fallen to 4 m/s

b the speed of the particle when it returns to its point of projection

15 A body of mass 5 kg is placed on a smooth horizontal plane and is acted upon by the following horizontal forces:

- a force of 8 N in a direction 330°
- a force of 10 N in a direction 090°
- a force of P N in a direction of 180°

Given that the magnitude of the acceleration of the body is 2 m/s^2, calculate the value of P correct to two decimal places.

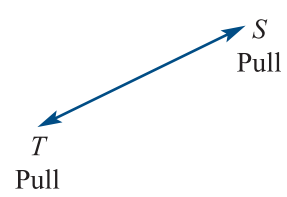
16 A particle of mass 5 kg is being pulled up a slope inclined at 30° to the horizontal. The pulling force, F newtons, acts parallel to the slope, as does the resistance with a magnitude one-fifth of the magnitude of the normal reaction.

a Find the value of F, such that the acceleration is 1.5 m/s^2 up the slope.

b Also find the magnitude of the acceleration if this pulling force now acts at an angle of 30° to the slope (i.e. at 60° to the horizontal).

13.4 Connected particles

Consider a light rope, being pulled from each end. The light rope is considered to have zero mass.



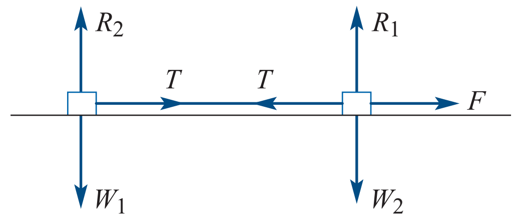
Apply Newton's second law of motion:

$$\boldsymbol{T} + \boldsymbol{S} = 0 \times \boldsymbol{a} \quad \text{(as the mass of the rope is negligible)}$$

$$\therefore \quad \boldsymbol{T} = -\boldsymbol{S}$$

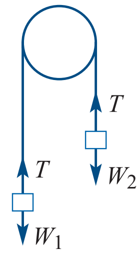
The following are examples of connected particles. Diagrams are given and the forces shown.

Two particles connected by a taut rope moving on a smooth plane.



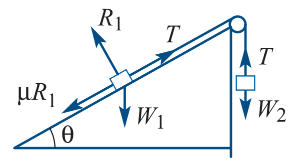
A smooth light pulley (i.e. the weight of the pulley is considered negligible and the friction between rope and pulley is negligible).

The tension in both sections of the rope can be assumed to be equal.

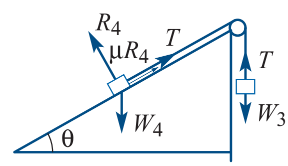


The tension in the string is of equal magnitude in both sections.

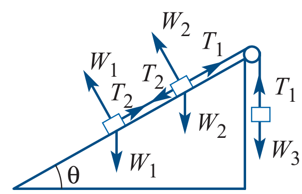
The inclined plane is rough. The body on the inclined plane is accelerating *up* the plane.



The body is accelerating *down* the inclined plane.



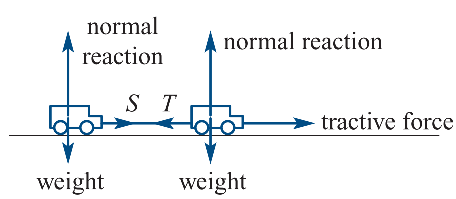
Two masses on an inclined smooth plane.
In general, $T_2 \neq T_1$.



Example 17

A car of mass one tonne tows another car of mass 0.75 tonnes, with a light tow rope. If the towing car exerts a tractive force of magnitude 3000 newtons and the resistance to motion can be neglected, find the acceleration of the two cars and the tension in the rope.

Solution



Note: $\boldsymbol{S} + \boldsymbol{T} = 0$

i.e. $\boldsymbol{S} = -\boldsymbol{T}$

Apply Newton's second law to both cars.

$$3000 = (750 + 1000)a$$

$$\therefore \quad a = \frac{3000}{1750} = \frac{12}{7} = 1\tfrac{5}{7} \text{ m/s}^2$$

For the second car, applying Newton's second law:

$$S = 750 \times \frac{12}{7} \approx 1285.71$$

The tension in the rope is 1286 newtons, to the nearest unit.

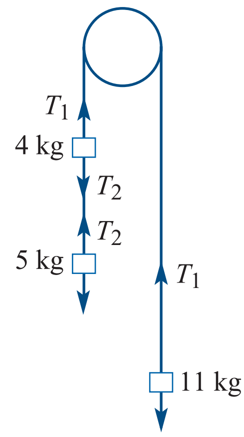
Example 18

The diagram right shows three masses of 4 kg, 5 kg and 11 kg, connected by light inextensible strings, one of which passes over a smooth fixed pulley. The system is released from rest. Calculate:

a the acceleration of the masses

b the tension in the string joining the 4 kg mass to the 11 kg mass

c the tension in the string joining the 4 kg mass to the 5 kg mass



Solution

a For the 11 kg mass, Newton's second law yields:

$$11g - T_1 = 11a \qquad \boxed{1}$$

For the 5 kg mass:

$$T_2 - 5g = 5a \qquad \boxed{2}$$

For the 4 kg mass:

$$T_1 - T_2 - 4g = 4a \qquad \boxed{3}$$

Add $\boxed{1}$ and $\boxed{3}$:

$$7g - T_2 = 15a \qquad \boxed{4}$$

Add $\boxed{2}$ and $\boxed{4}$:

$$2g = 20a$$

$$\therefore \quad a = 0.1g$$

The acceleration of the system is $0.1g$ m/s^2.

b From $\boxed{1}$: $11g - T_1 = 1.1g$

$$\therefore \quad T_1 = 9.9g$$

The tension in the rope between the 11 kg and 4 kg masses is $9.9g$ newtons.

c From $\boxed{4}$: $7g - T_2 = 15 \times 0.1g$

$$\therefore \quad T_2 = 5.5g$$

The tension in the rope between the 4 kg and 5 kg masses is $5.5g$ newtons.

Exercise 13D

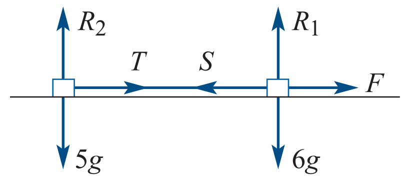
Skillsheet TEACHER

1 Two masses 8 kg and 10 kg are suspended by a light inextensible string over a smooth pulley.

a Find the tension in the string.

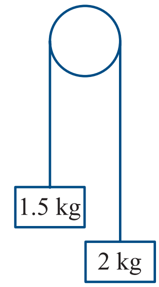
b Find the acceleration of the system.

2 Two particles of mass 6 kg and 5 kg are pulled along a smooth horizontal plane. The forces are as shown.
If the magnitude of F is 10 newtons find:



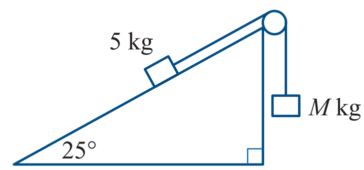
a the acceleration of the system **b** T and S

3 A mass of 1.5 kg is connected to a mass of 2 kg by a light inelastic string which passes over a smooth pulley as shown. Find:



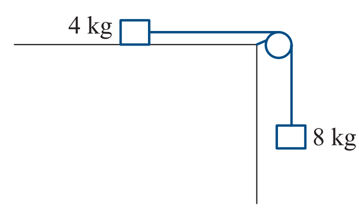
a the tension in the string
b the acceleration of the system

4 The diagram shows a smooth plane inclined at an angle of 25° to the horizontal. At the top of the plane there is a smooth pulley over which passes a taut, light string. On the end of the string is attached a block of mass 5 kg lying on the plane. The other end is attached to a block of mass M kg hanging vertically. If the mass of M kg is moving downwards with an acceleration of 1 m/s^2, find:



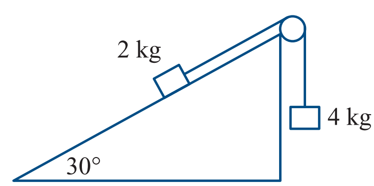
a M **b** the tension in the string

5 The diagram shows a particle of mass 4 kg on a smooth horizontal table. The particle is connected by a light inelastic string which passes over a smooth pulley to a particle of mass 8 kg which hangs vertically. Find:



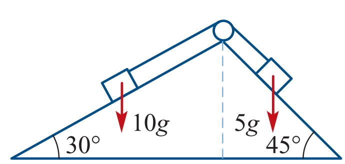
a the acceleration of the system **b** the tension in the string

6 A mass of 2 kg, resting on a smooth plane inclined at 30° to the horizontal, is connected to a mass of 4 kg by a light inelastic string which passes over a smooth pulley as shown in the diagram. Find:



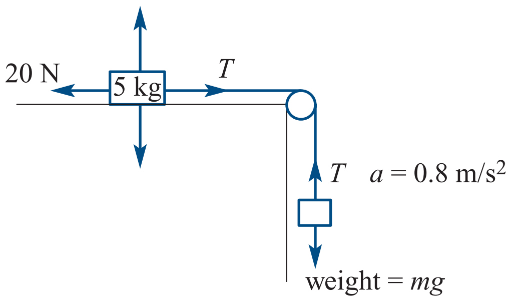
a the tension in the string
b the acceleration of the system

7 Two masses of 10 kg and 5 kg are placed on smooth inclines of 30° and 45°, placed back to back. The masses are connected by a light string over a smooth pulley at the top of the plane.



a Find the acceleration of the system.
b Find the tension in the string.

8 In the situation shown in the diagram, what mass m kg is required in order to give the system an acceleration of 0.8 m/s^2?

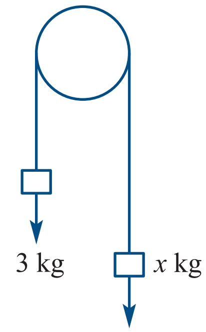


9 A truck of mass 10 tonnes pulls a trailer of mass 5 tonnes with an acceleration of magnitude 2 m/s^2. The truck exerts a tractive force of magnitude 40 000 N. If the trailer has resistance to motion of 750 N:

a what is the tension in the coupling?

b what is the resistance to motion of the truck?

10 Two particles of respective mass 3 kg and x kg $(x > 3)$ are connected by a light inextensible string passing over a smooth fixed pulley. The system is released from rest while the hanging portion of the string is taut and vertical. Given that the tension in the string is 37.5 N, calculate the value of x.

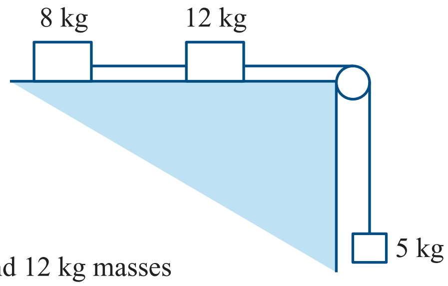


11 An engine of mass 40 tonnes is pulling a truck of mass 8000 kg up a plane inclined at $\theta°$ to the horizontal where $\sin\theta = \frac{1}{8}$. If the tractive force exerted by the engine is 60 000 N, calculate:

a the acceleration of the engine

b the tension in the coupling between the engine and the truck

12 The diagram shows masses of 8 kg and 12 kg lying on a smooth horizontal table and joined, by a light inextensible string, to a mass of 5 kg hanging freely. This string passes over a smooth pulley at the edge of the table.
The system is released from rest. Find:



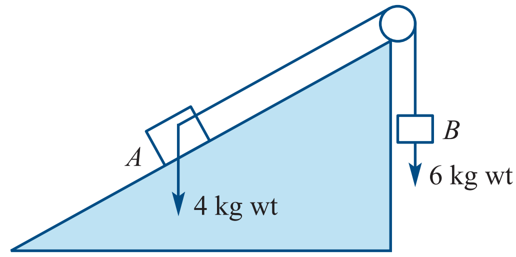
a the tension in the string connecting the 8 kg and 12 kg masses

b the tension in the string connecting the 12 kg and 5 kg masses

c the acceleration of the system

13 A hanging mass of 200 g drags a mass of 500 g along a rough table three metres from rest in three seconds. What is the coefficient of friction?

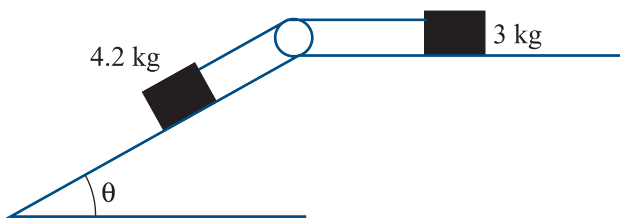
14 Two bodies, A and B, of mass 4 kg and 6 kg respectively, are connected by a light string passing over a smooth pulley. A rests on a rough plane inclined at 30° to the horizontal. When the bodies are released from rest, B moves downward with an acceleration of 1 m/s^2.



a Calculate the value of μ, the coefficient of friction between A and the inclined plane.

b Find the tension in the string connecting A and B.

15 A particle of mass 3 kg is rests on rough, horizontal surface. The particle is attached by a light inextensible string, passing over a smooth fixed pulley, to a particle of mass 4.2 kg on a smooth plane inclined at an angle of $\theta°$ to the horizontal, where $\sin\theta = 0.6$.



When the system is released from rest, each particle moves with an acceleration 2 m/s^2. Calculate:

a the tension in the string

b the coefficient of friction between the horizontal surface and the particle of mass 3 kg

13.5 Variable forces

In the previous sections of this chapter, constant force has been considered. In this section variable force is considered. In Chapter 10 the following expressions for acceleration were established:

$$a = \frac{dv}{dt} = \frac{d^2x}{dt^2} = v\frac{dv}{dx} = \frac{d\left(\frac{1}{2}v^2\right)}{dx}$$

where x, v, and a are, respectively, the displacement, velocity and acceleration at time t.

Example 19

A body of mass 5 kg, initially at rest, is acted on by a force of $F = (6 - t)^2$ newtons where $0 \le t \le 6$ (seconds). Find the speed of the body after six seconds and the distance travelled.

Solution

Newton's second law of motion gives:

$$F = ma$$

$\therefore \quad (6-t)^2 = 5a$

and $\quad a = \frac{1}{5}(6-t)^2$

Hence $\quad \frac{dv}{dt} = \frac{1}{5}(6-t)^2$

and
$$v = \frac{1}{5}\int (6-t)^2\,dt$$
$$= \frac{1}{5} \times -1 \times \frac{(6-t)^3}{3} + c$$
$$= -\frac{1}{15}(6-t)^3 + c$$

When $t = 0$ and $v = 0$, $c = \frac{216}{15}$

$\therefore \quad v = -\frac{1}{15}(6-t)^3 + \frac{216}{15}$

When $t = 6$, $v = \frac{216}{15}$, i.e. the velocity after 6 seconds is $\frac{216}{15}$ m/s.

Integrating again with respect to t gives:

$$x = +\frac{1}{60}(6-t)^4 + \frac{216}{15}t + d$$

when $t = 0$, $x = 0$ and, therefore, $d = -21.6$

Hence when $t = 6$, $x = \frac{216}{15} \times 6 - 21.6 = 64.8$

The distance travelled is 64.8 m.

Example 20

A particle of mass 3 units moves in a straight line and, at time t, its displacement from a fixed origin is x and its speed is v.

a If the resultant force is $9\cos t$, and $v = 2$ and $x = 0$ when $t = 0$, find x in terms of t.

b If the resultant force is $3 + 6x$, and $v = 2$ when $x = 0$, find v when $x = 2$.

Solution

a Using Newton's second law of motion:

$$F = ma$$

gives: $9\cos t = 3a$

$\therefore \quad a = 3\cos t$

i.e. $\quad \frac{dv}{dt} = 3\cos t$

$\therefore \quad v = 3\sin t + c$

When $t = 0$, $v = 2$:

$\therefore \quad v = 3\sin t + 2$

i.e. $\dfrac{dx}{dt} = 3\sin t + 2$

$\therefore \quad x = -3\cos t + 2t + d$

When $t = 0$, $x = 0$ and, therefore, $d = 3$

Hence $x = 3 - 3\cos t + 2t$.

b Using Newton's second law of motion:

$$3 + 6x = 3a$$

i.e. $1 + 2x = a$

$\therefore \quad \dfrac{d\left(\frac{1}{2}v^2\right)}{dx} = 1 + 2x$

$\therefore \quad \frac{1}{2}v^2 = x + x^2 + c$

When $x = 0$, $v = 2$. Therefore $c = 2$.

$\therefore \quad \frac{1}{2}v^2 = x + x^2 + 2$

When $x = 2$, $\frac{1}{2}v^2 = 2 + 4 + 2$

$\therefore \quad v = \pm 4$

Exercise 13E

1 A body of mass 10 kg initially at rest is acted on by a force of $F = (10 - t)^2$ newtons at time t seconds where $0 \leq t \leq 10$. Find the speed of the body after ten seconds and the distance travelled.

2 A particle, of mass 5 kg, moves in a straight line and, at time t seconds, its displacement from a fixed origin is x m and its speed is v m/s.

a If the resultant force acting is $10 \sin t$ and $v = 4$ and $x = 0$ when $t = 0$, find x in terms of t.

b If the resultant force acting is $10 + 5x$ and $v = 4$ when $x = 0$ find v when $x = 4$.

c If the resultant force acting is $10\cos^2 t$ and $v = 0$ and $x = 0$ when $t = 0$, find x in terms of t.

3 A body of mass 6 kg, moving initially with a speed of 10 m/s, is acted on by a force $F = \dfrac{100}{(t+5)^2}$ N. Find the speed reached after 10 seconds and the distance travelled in this time.

4 A particle of unit mass is acted on by a force of magnitude $1 - \sin\left(\dfrac{t}{4}\right)$ with $0 \leq t \leq 2\pi$. If the particle is initially at rest, find an expression for the distance covered at time t.

5 A particle of unit mass is acted on by a force of magnitude $1 - \cos \frac{1}{2}t, 0 \le t \le \frac{\pi}{2}$. If the particle is initially at rest, find an expression for:

a the velocity at time t

b the displacement at time t

6 A particle of mass 4 kg is acted on by a resultant force whose direction is constant and whose magnitude at time t seconds is $(12t - 3t^2)$ N. If the particle has an initial velocity of 2 m/s in the direction of the force, find the velocity at the end of four seconds.

7 A particle of mass 1 kg on a smooth horizontal plane is acted on by a horizontal force $\frac{t}{t+1}$ N at time t seconds after it starts from rest. Find its velocity after 10 seconds.

8 A body of mass 0.5 kg is acted on by a resultant force $e^{-\frac{t}{2}}$ N at time t seconds after the body is at rest.

a It the body starts from rest, find the velocity, v m/s, at time t seconds.

b Sketch the velocity versus time graph.

c If the body moves under the given force for 30 seconds, find the distance travelled.

9 A body of mass 10 units is accelerated from rest by a force F whose magnitude at time t is given by:

$$F(t) = \begin{cases} 14 - 2t & 0 \le t \le 5 \\ 100t^{-2} & t > 5 \end{cases}$$

Find:

a the speed of the body when $t = 10$

b the distance travelled in this time

10 A body of mass m kg moving with velocity of u m/s ($u > 0$) is acted on by a resultant force kv N (in its initial direction) where v m/s is its velocity at time t s, ($k \in R^+$). Find the distance travelled after t seconds.

11 A particle of mass m is projected along a horizontal line from O with speed V. It is acted on by a resistance kv when the speed is v. Find the velocity after the particle has travelled a distance x.

12 A particle of mass m kg at rest on a horizontal plane is acted on by a constant horizontal force b N. The total resistance to motion is cv N where v m/s is the velocity and c is a constant value. Find the velocity at time t seconds and the terminal velocity.

13 A body of mass m is projected vertically upwards with speed u. Air resistance is equal to k times the square of the speed where k is a constant. Find the maximum height reached and the speed when next at the point of projection.

14 A particle of mass 0.2 kg moving on the positive x axis has position x metres and velocity v m/s at time t seconds. At time $t = 0$, $v = 0$ and $x = 1$, the particle moves under the action of a force of magnitude $\frac{4}{x}$ N in the positive direction of the x axis. Show that $v = \sqrt{40 \log_e x}$.

15 A particle P of unit mass moves on the positive x axis. At time t, the velocity of the particle is v and the force F acting on the particle is given by

$$F = \begin{cases} \dfrac{50}{25+v} & \text{for } 0 \le t \le 50 \\ \dfrac{-v^2}{1000} & \text{for } t > 50 \end{cases}$$

Initially the particle is at rest at the origin O.

a Show that $v = 50$ when $t = 50$.

b Find the distance of P from O when $v = 50$.

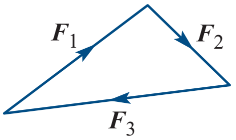
c Find the distance of P from O when $v = 25$ and $t > 50$.

13.6 Equilibrium

In this section the case of the resultant force, acting on a particle, being zero is considered. i.e. if $\boldsymbol{F}$ is the resultant force, $\boldsymbol{F} = \mathbf{0}$. The particle is then said to be in **equilibrium**. The particle has zero acceleration. If the particle is at rest it remains at rest and if it is moving it will continue to move with constant velocity.

Triangle of forces

If three forces $\boldsymbol{F}_1$, $\boldsymbol{F}_2$ and $\boldsymbol{F}_3$ act on a particle such that the resultant force is zero, the situation can be represented by the vector diagram shown.



This represents the vector equation $\boldsymbol{F}_1 + \boldsymbol{F}_2 + \boldsymbol{F}_3 = \mathbf{0}$.
This can of course be generalised to any number of vectors by using a suitable polygon.

Example 21

Forces of magnitude 2 N, 4 N and 5 N act on a particle in equilibrium.

a Sketch a triangle of forces to represent the three forces.

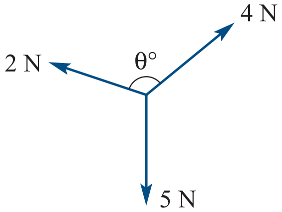
b Find the angle between the 2 N and 4 N forces, correct to two decimal places.

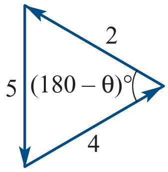
Solution

Let θ be the angle between the 2 N and 4 N forces.

Then in the triangle of forces the angle between the forces 2 N and 4 N is 180 − θ

∴ by the cosine rule we have:





$$25 = 4 + 16 - 2 \times 2 \times 4\cos(180 - \theta)$$

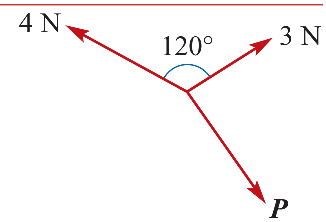
$$\cos(180 - \theta) = -\tfrac{5}{16}$$

$$(180 - \theta)^\circ = 108.21^\circ \text{ (to two decimal places)}$$

$$\therefore \quad \theta = 71.79^\circ$$

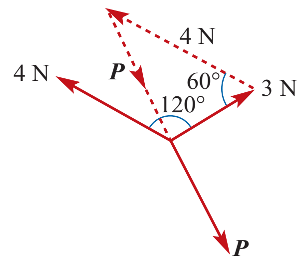
The angle between the 2 N and the 4 N forces is 71.79° correct to two decimal places.

Example 22



Forces of magnitude 3 N, 4 N and P N act on a particle which is in equilibrium as shown in the diagram. Find the magnitude of $\boldsymbol{P}$.

Solution



Complete the triangle of forces as shown.

The cosine rule gives:

$$|\boldsymbol{P}|^2 = 4^2 + 3^2 - 2 \times 4 \times 3 \cos 60^\circ$$

$$|\boldsymbol{P}|^2 = 16 + 9 - 24 \times \frac{1}{2}$$

$$= 16 + 9 - 12$$

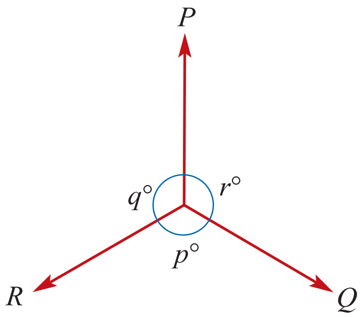
$$= 13$$

$$\therefore \quad |\boldsymbol{P}| = \sqrt{13}\text{ N}$$

Lami's theorem

Lami's theorem is a trigonometrically based identity which simplifies problems involving three forces acting on a particle in equilibrium when the angles between the forces are known.

Let P N, Q N and R N be forces acting on a particle in equilibrium making angles as shown in the diagram.

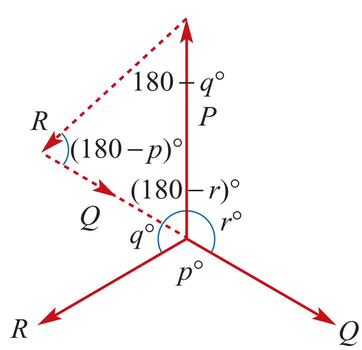


If the particle is in equilibrium then

$$\frac{P}{\sin p^\circ} = \frac{Q}{\sin q^\circ} = \frac{R}{\sin r^\circ}$$

Complete the triangle of forces as shown.

The sine rule now gives

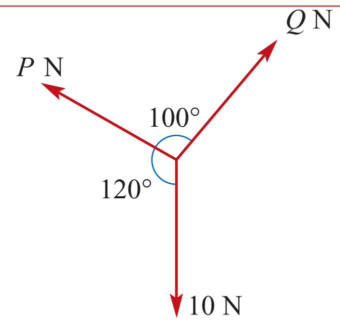


$$\frac{P}{\sin(180 - p)^\circ} = \frac{Q}{\sin(180 - q)^\circ} = \frac{R}{\sin(180 - r)^\circ}$$

i.e.

$$\frac{P}{\sin p^\circ} = \frac{Q}{\sin q^\circ} = \frac{R}{\sin r^\circ}$$

Example 23



Find P and Q in the system of forces in equilibrium as shown in the diagram.

Solution

Applying Lami's theorem we have:

$$\frac{10}{\sin 100^\circ} = \frac{Q}{\sin 120^\circ} = \frac{P}{\sin 140^\circ}$$

$$\therefore \quad Q = \frac{10 \sin 120^\circ}{\sin 100^\circ} = 8.79 \text{ (correct to two decimal places)}$$

$$P = \frac{10 \sin 140^\circ}{\sin 100^\circ} = 6.53 \text{ (correct to two decimal places)}$$

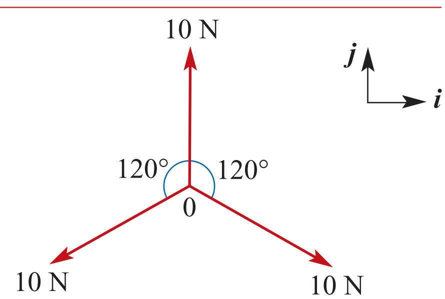
Resolution of forces

If $\boldsymbol{F}_1 = a_1\boldsymbol{i} + b_1\boldsymbol{j}$, $\boldsymbol{F}_2 = a_2\boldsymbol{i} + b_2\boldsymbol{j}$ are $\boldsymbol{F}_3 = a_3\boldsymbol{i} + b_3\boldsymbol{j}$ are forces such that $\boldsymbol{F}_1 + \boldsymbol{F}_2 + \boldsymbol{F}_3 = \boldsymbol{0}$ then

$$a_1 + a_2 + a_3 = 0 \text{ and } b_1 + b_2 + b_3 = 0$$

i.e. for coplanar forces it is sufficient to show that the sum of the resolved parts for each of two perpendicular directions is zero.

Example 24



Three forces of 10 N act on a particle as shown in the diagram. Show that the particle is in equilibrium by resolution in the $\boldsymbol{i}$ and $\boldsymbol{j}$ directions.

Solution

The sum of the resolved parts of the forces in the $\boldsymbol{j}$ direction:

$$10 + 10\cos 120^\circ + 10\cos 120^\circ = 10 - 5 - 5 = 0$$

The sum of the resolved parts of the forces in the $\boldsymbol{i}$ direction:

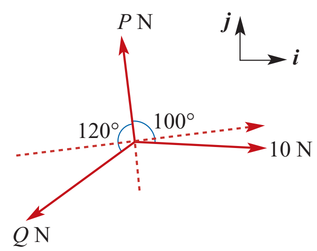
$$10\cos 90^\circ + 10\cos 30^\circ + 10\cos 150^\circ = 0 + 10 \times \frac{\sqrt{3}}{2} + 10 \times -\frac{\sqrt{3}}{2} = 0$$

Therefore the particle is in equilibrium.

Example 25

The angles between three forces of magnitude 10 N, P N and Q N acting on a particle are 100° and 120° respectively. Find P and Q, given that the system is in equilibrium.

Solution



Choose to resolve in directions along and perpendicular to the line of action of the force P N.

In the $\boldsymbol{j}$ direction, the sum of the resolved parts in newtons is:

$$10\cos 100^\circ + P + Q\cos 120^\circ = 0 \quad \boxed{1}$$

In the $\boldsymbol{i}$ direction (this produces an equation without P) the sum of the resolved parts in newtons is:

$$10\cos 10^\circ + Q\cos 150^\circ = 0 \quad \boxed{2}$$

From $\boxed{2}$ $\quad Q = \dfrac{-10\cos 10^\circ}{\cos 150^\circ} = 11.37$ (to two decimal places)

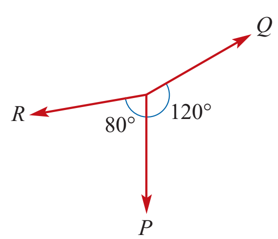
From $\boxed{1}$ $P = -10\cos 100^\circ - Q\cos 120^\circ = 7.42$ (to two decimal places)

Exercise 13F

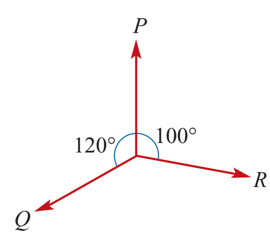
Complete questions 1–4 using a triangle of forces.

1 For each of the following situations of a particle in equilibrium, sketch the corresponding triangle of vectors:

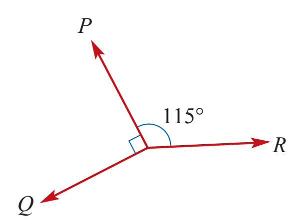
a



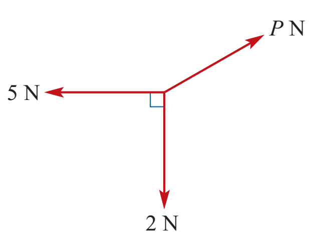
b



c



2 Forces of 2 N and 5 N act on a particle, as shown in the diagram. A force of P N acts such that the particle is in equilibrium.

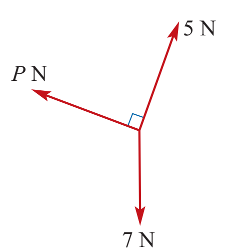


a Sketch a triangle of forces to represent the forces 2 N, 5 N and P N.

b Find P.

c Find the angle that the force P N makes with the force of 5 N.

3 Forces of 7 N, 5 N and P N act on a particle in equilibrium, as shown in the diagram.

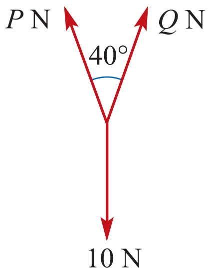


a Sketch a triangle of forces to represent the forces 7 N, 5 N and P N.

b Find P.

c Find the angle between the forces of 5 N and 7 N.

4 Forces of 10 N, P N and Q N act on a particle in equilibrium, as shown in the diagram.



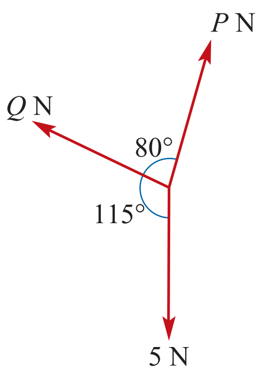
a Sketch a triangle of forces to represent the forces 10 N, P N and Q N.

b If $P = Q$, find P.

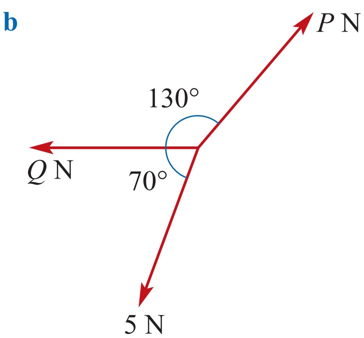
Complete questions 5–7 using Lami's theorem.

5 Find P and Q in each of the following systems of forces in equilibrium.

a



b



6 Two forces of 10 N and a third force of P N act on a body in equilibrium. The angle between the lines of action of the 10 N forces is 50°. Find P.

7 A particle of mass 5 kg hangs from a fixed point O by a light inextensible string. It is pulled aside by a force P N that makes an angle of 100° with the downward vertical and rests in equilibrium with the string inclined at 60° to the vertical. Find P.

Complete questions 8–11 using resolution of forces.

8 The angles between the forces of magnitude 10 N, 5 N and $5\sqrt{3}$ N acting on a particle are 120° and 90° respectively. Show that the particle is in equilibrium.

9 Two equal forces of 10 N act on a particle. The angle between the two forces is 50°.

a State the direction of the resultant of the two forces with respect to the forces.

b Find the magnitude of the resultant of the two forces.

c Find the magnitude and direction of the single force which, when applied, will hold the particle in equilibrium.

10 The angles between three forces, P N, Q N and 23 N, acting on a particle in equilibrium are respectively 80° and 145°. Find P and Q.

11 The angles between four forces, 10 N, 15 N, P N and Q N acting on a particle in equilibrium, are respectively 90°, 120° and 90°. Find P and Q.

12 Forces of 8 N, 16 N and 10 N act on a particle in equilibrium.

a Sketch a triangle of forces to represent the three forces.

b Find the angle between the 8 N and 16 N forces.

13 The angles between the three forces 3 N, 5 N and P N acting on a body in equilibrium are respectively 100° and θ°.

a Sketch a triangle of forces to represent the three forces.

b Find P by using the cosine rule. **c** Hence find θ.

14 A particle of mass 2 kg hangs from a fixed point, O, by a light inextensible string of length 2.5 m. It is pulled aside a horizontal distance of 2 m by a force P N inclined at an angle of 75° with the downward vertical, and rests in equilibrium. Find P and the tension of the string.

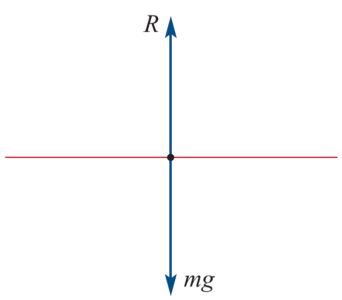
15 A particle of mass 5 kg is suspended by two strings of lengths 5 cm and 12 cm respectively, attached at two points at the same horizontal level and 13 cm apart. Find the tension in the shorter string.

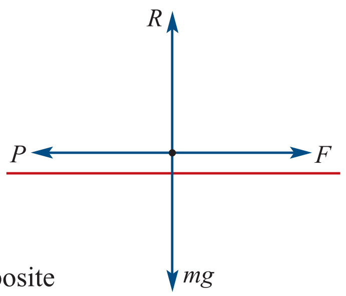
16 The angle between two forces, 10 N and P N, acting on a particle is 50°. A third force of magnitude 12 N holds the particle in equilibrium.

a Find the angle between the third force and the 10 N force. (Hint: find the resolution of the forces in a direction perpendicular to the P N force.)

b Hence find P.

13.7 Friction and equilibrium





In section 13.2, sliding friction was considered. The friction force is the reaction force encountered when forces are applied that can result in making one body move relative to another body with which it is in contact.

When a body rests on a rough horizontal surface, the normal reaction force opposes the weight but there is no friction.

When a small horizontal variable force of P N is applied to the particle lying on a rough plane surface, a friction force of F N acts along the common surface in order to keep the body in equilibrium, i.e. $F = P$.

As P increases in magnitude, F increases similarly in the opposite direction, up to a certain value after which the body begins to slide. The critical value is called the **limiting friction force** (or sliding friction force).

It can be shown experimentally that the limiting friction force F_{max} between two surfaces varies directly as the normal reaction force, R.

$$F_{max} = \mu R$$

where μ is the coefficient of friction between the two given surfaces.

For particles in equilibrium, the friction force obeys the inequality

$$0 \leq F \leq \mu R$$

Example 26

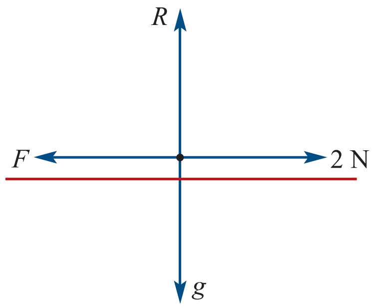
A particle of mass 1 kg rests on a rough horizontal table. The coefficient of friction between the particle and the table is 0.3.

a Find the friction force acting on the particle when a horizontal force of 2 N is applied to the particle.

b The horizontal force is now increased to 5 N. Find the friction force now acting on the particle.

c Another particle of mass 1 kg is placed on top of the particle and the horizontal force of 5 N is applied. Find the friction force acting on the particle.

Solution



a Let R N be the normal reaction force and F N be the friction force.
Resolving perpendicular to the plane gives

$g - R = 0$ which implies $R = g$

Hence the maximum friction force $F_{max} = 0.3g$.
This is larger than the force applied. Therefore the particle is in equilibrium and the friction force $= 2$ N (only large enough to annul the force being applied).

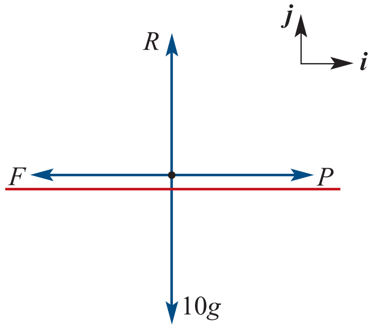
b $F_{max} = 0.3g$, as in part **a**.
F_{max} is less than the magnitude of the applied force. There will be a resultant force when the forces are resolved horizontally. Hence the body will start to slide and the friction force $= 0.3g$ N.

c In this case $R = 2g$ therefore $F_{max} = 0.6g$.
This is larger than the 5 N force being applied. Therefore the particle is in equilibrium and the friction force $= 5$ N.

Example 27

A particle of mass 10 kg rests on a rough horizontal table. The coefficient of friction between the particle and the table is 0.4. Find the greatest horizontal force that can be applied without causing the particle to move.

Solution



Weight $= 10g$ N
Let P N be the horizontally applied force while the particle is in equilibrium. Resolving forces in two directions:

$$\boldsymbol{j}\text{ direction:}\quad R - 10g = 0$$
$$\therefore \quad R = 10g$$

$\boldsymbol{i}$ direction: $\quad P - F = 0$

using the relation $\quad F \leq \mu R$

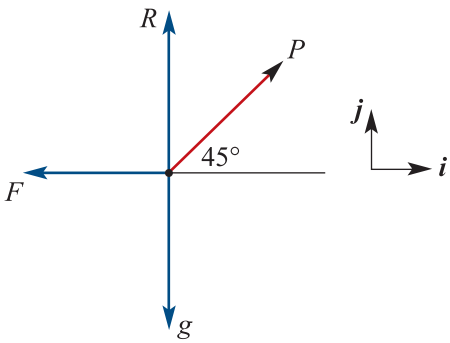
$$P = F \leq 0.4R$$

Hence the greatest force is $4g$ N (or 4 kg wt).

Example 28

A particle of mass 1 kg rests on a rough horizontal table. It is attached to a string and a force, P, is applied along the string which makes an angle of 45° with the plane. Find the largest value of P that can be applied before the particle moves. The coefficient of friction between the particle and the table is 0.2. (Assume the particle stays on the surface.)

Solution



If the particle is in limiting equilibrium, then the friction force $F = F_{max}$

Resolving in the $\boldsymbol{j}$ direction

$$R + P\cos 45° - g = 0 \quad \boxed{1}$$

Resolving in the $\boldsymbol{i}$ direction

$$-F_{max} + P\cos 45° = 0 \quad \boxed{2}$$

However $F_{max} = \mu R = 0.2R$

From $\boxed{1}$: $\quad R + \dfrac{P\sqrt{2}}{2} = g \quad \boxed{3}$

From $\boxed{2}$: $\quad -0.2R + \dfrac{P\sqrt{2}}{2} = 0 \quad \boxed{4}$

Solving simultaneously:

$\boxed{3} - \boxed{4}$ implies $1.2R = g$

$$R = \frac{g}{1.2}$$

Substituting back in $\boxed{4}$:

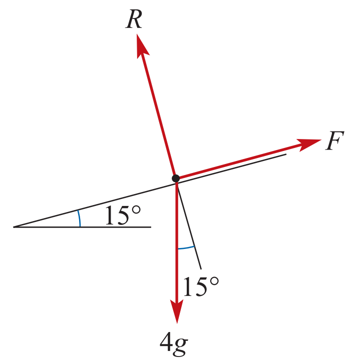
$$\frac{\sqrt{2}P}{2} = 0.2R$$

$$= \frac{g}{6}$$

$$\therefore \quad P = \frac{\sqrt{2}g}{6}$$

The largest value of P is $\dfrac{\sqrt{2}g}{6}$ newtons.

Example 29



A particle of mass 4 kg rests on a rough plane inclined at an angle of 15° to the horizontal. Find the minimum value for μ, the coefficient of friction that will prevent the particle from sliding down the plane.

Solution

Method 1

As the body is in equilibrium, the forces are resolved in two directions, along the line of possible motion and perpendicular to it. (**Note:** These directions are chosen because two of the forces already act in these directions.)

Resolving down the plane:

$$4g\cos 75^\circ - F = 0$$

Resolving perpendicular to the plane:

$$R - 4g\cos 15^\circ = 0$$

$\therefore$ $R = 4g\cos 15^\circ$

and: $F = 4g\cos 75^\circ$

As: $F \le \mu R$

$\therefore$ $\mu \ge \dfrac{F}{R}$

$\therefore$ the minimum value for μ is:

$$\frac{F}{R} = \frac{4g\cos 75^\circ}{4g\cos 15^\circ} = \frac{\sin 15^\circ}{\cos 15^\circ} = \tan 15^\circ$$

Method 2

Apply Lami's theorem to obtain:

$$\frac{R}{\sin 105^\circ} = \frac{F}{\sin 165^\circ} = \frac{4g}{\sin 90^\circ}$$

Using the first pair:

$$\frac{R}{\sin 75^\circ} = \frac{F}{\sin 15^\circ}$$

and with a similar reasoning as in the first method the minimum value for μ is given by

$$\frac{F}{R} = \frac{\sin 15^\circ}{\sin 75^\circ} = \tan 15^\circ$$

Exercise 13G

1 An object of mass 10 kg rests on rough horizontal ground. The coefficient of friction between the object and the ground is 0.3. A horizontal force P N is applied to the object. Find the friction force if:

a $P = 10$ **b** $P = 30$ **c** $P = 40$

2 A particle of mass 1.2 kg rests on a rough horizontal table. The coefficient of friction between the particle and the table is 0.2. A force P N is applied to the particle along a string attached to it. The string is inclined at an angle of 40° to the table. Find the friction force, correct to two decimal places, if:

a $P = 2$ **b** $P = 3$

3 A particle of mass 100 kg rests on rough horizontal ground. If the coefficient of friction is 0.3, find the greatest force that can be applied without moving the particle if:

a a horizontal force is applied
b a pulling force inclined at 60° to the horizontal is applied
c a pushing force inclined at 60° to the horizontal is applied

4 A particle of weight 5 N rests in equilibrium on a rough plane inclined at 30° to the horizontal. Find the frictional force exerted by the plane on the particle.

5 A particle of weight 1 kg wt is placed on a rough plane inclined at an angle of 20° to the horizonntal. Find the values of μ, the coefficient of friction, for which the particle will slide down the plane.

6 A rough plane ($\mu = 0.4$) is inclined at angle θ to the horizontal. A particle of weight w kg wt is placed on the plane and is on the point of sliding down the plane. Find θ.

7 A particle weighing 3 kg wt rests in limiting equilibrium on a rough plane inclined at an angle of 25° to the horizontal. Find:

a the coefficient of friction between the particle and the plane
b the least force which when applied up the plane will cause the particle to move

8 A mass of 24 kg is on the point of motion down a rough inclined plane when supported by a force of 100 N parallel to the plane. If the magnitude of the force is increased to 120 N, the mass is on the point of moving up the plane. Find:

a the friction force **b** the inclination of the plane to the horizontal
c the coefficient of friction

13.8 Vector functions

The equation derived from Newton's second law, i.e. $\boldsymbol{F} = m\boldsymbol{a}$, is a vector equation. In this section, vector function notation is used in dynamics problems. The emphasis is on motion in a straight line.

Example 30

Forces $\boldsymbol{F}_1 = 2\boldsymbol{i} + 3\boldsymbol{j}$ and $\boldsymbol{F}_2 = 3\boldsymbol{i} - 4\boldsymbol{j}$ act on a particle, of mass 2 kg, at rest. Find:

a the acceleration of the particle

b the position of the particle at time t given that initially it is at the point $3\boldsymbol{i} + 2\boldsymbol{j}$

c the cartesian equation of the path of the particle

Solution

a The resultant force acting on the particle

$$\begin{aligned}\boldsymbol{F} &= \boldsymbol{F}_1 + \boldsymbol{F}_2\\ &= 2\boldsymbol{i} + 3\boldsymbol{j} + 3\boldsymbol{i} - 4\boldsymbol{j}\\ &= 5\boldsymbol{i} - \boldsymbol{j}\end{aligned}$$

By Newton's second law

$$\boldsymbol{F} = m\boldsymbol{a}$$

i.e. $\quad 5\boldsymbol{i} - \boldsymbol{j} = 2\boldsymbol{a}$

$\therefore \quad \boldsymbol{a} = \frac{5}{2}\boldsymbol{i} - \frac{1}{2}\boldsymbol{j}$

b Let $\boldsymbol{v}$ be the velocity at time t

$$\frac{d\boldsymbol{v}}{dt} = \frac{5}{2}\boldsymbol{i} - \frac{1}{2}\boldsymbol{j}$$

which implies

$$\boldsymbol{v} = \frac{5}{2}t\boldsymbol{i} - \frac{1}{2}t\boldsymbol{j} + \boldsymbol{c}$$

and as $v = 0$ when $t = 0$ and $\boldsymbol{c} = \boldsymbol{0}$

$\therefore \quad \boldsymbol{v} = \frac{5}{2}t\boldsymbol{i} - \frac{1}{2}t\boldsymbol{j}$

i.e. the velocity at time t is $\boldsymbol{v} = \frac{5}{2}t\boldsymbol{i} - \frac{1}{2}t\boldsymbol{j}$

Let $\boldsymbol{r}$ be the position at time t

Then $\quad \dfrac{d\boldsymbol{r}}{dt} = \frac{5}{2}t\boldsymbol{i} - \frac{1}{2}t\boldsymbol{j}$

and $\quad \boldsymbol{r} = \frac{5}{4}t^2\boldsymbol{i} - \frac{1}{4}t^2\boldsymbol{j} + \boldsymbol{d}$

When $t = 0$, $\boldsymbol{r} = 3\boldsymbol{i} + 2\boldsymbol{j}$

$\therefore \quad \boldsymbol{d} = 3\boldsymbol{i} + 2\boldsymbol{j}$

and $\quad \boldsymbol{r} = \left(3 + \frac{5}{4}t^2\right)\boldsymbol{i} + \left(2 - \frac{1}{4}t^2\right)\boldsymbol{j}$

c For $\boldsymbol{r}(t) = x(t)\boldsymbol{i} + y(t)\boldsymbol{j}$

$$x(t) = 3 + \frac{5}{4}t^2 \text{ and } y(t) = 2 - \frac{1}{4}t^2$$

$$\begin{aligned}\therefore \quad t^2 = \frac{4}{5}(x-3) \text{ and } y &= 2 - \frac{1}{4}\left(\frac{4}{5}(x-3)\right)\\ &= 2 - \frac{1}{5}(x-3)\\ &= 2 - \frac{1}{5}x + \frac{3}{5}\\ &= \frac{13}{5} - \frac{1}{5}x\end{aligned}$$

Motion is in a straight line, and the straight line has equation $y = \frac{13}{5} - \frac{1}{5}x$.

Example 31

At time t the position of a particle of mass 3 kg is given by $\boldsymbol{r}(t) = 3t^3\boldsymbol{i} + 6(t^3 + 1)\boldsymbol{j}$. Find:

a the initial position of the particle

b the cartesian equation describing the path of the particle

c the resultant force acting on the particle at time $t = 1$

Solution

a when $t = 0$, $\boldsymbol{r}(0) = 0\boldsymbol{i} + 6\boldsymbol{j}$

b For $\boldsymbol{r}(t) = x(t)\boldsymbol{i} + y(t)\boldsymbol{j}$

$$x = 3t^3 \text{ and } y = 6t^3 + 6$$

Therefore $t^3 = \dfrac{x}{3}$ and $y = 2x + 6$

The cartesian equation of the path of the particle is $y = 2x + 6$.

c $\boldsymbol{r}(t) = 3t^3\boldsymbol{i} + 6(t^3 + 1)\boldsymbol{j}$

implies $\dot{\boldsymbol{r}}(t) = 9t^2\boldsymbol{i} + 18t^2\boldsymbol{j}$

and $\ddot{\boldsymbol{r}}(t) = 18t\boldsymbol{i} + 36t\boldsymbol{j}$

when $t = 1$, $\ddot{\boldsymbol{r}}(1) = 18\boldsymbol{i} + 36\boldsymbol{j}$

From Newton's second law of motion, $\boldsymbol{F} = m\ddot{\boldsymbol{r}}$ and the resultant force $\boldsymbol{F}$ at time $t = 1$ is $54\boldsymbol{i} + 108\boldsymbol{j}$.

Exercise 13H

Skillsheet TEACHER

1 Forces $\boldsymbol{F}_1 = 2\boldsymbol{i}$ N and $\boldsymbol{F}_2 = -3\boldsymbol{j}$ N act on a particle of mass 1 kg which is initially at rest. Find:

a the acceleration of the particle

b the magnitude of the acceleration

c the velocity of the particle at time t seconds

d the speed of the particle after one second of motion

e the direction of motion (measured anticlockwise from the direction of $\boldsymbol{i}$)

2 A force of $4\boldsymbol{i} + 6\boldsymbol{j}$ acts on a particle of mass 2 kg. If the particle is initially at rest at the point with position vector $0\boldsymbol{i} + 0\boldsymbol{j}$ find:

a the acceleration of the particle

b the velocity of the particle at time t

c the position of the particle at time t

d the cartesian equation of the path of the particle

3 At time t the position of a particle of mass 2 kg is given by $\boldsymbol{r}(t) = 5t^2\boldsymbol{i} + 2(t^2 + 4)\boldsymbol{j}$. Find:

a the initial position of the particle

b the cartesian equation describing the path of the particle

c the resultant force acting on the particle at time t

4 At time t the position of a particle of mass 5 kg is given by $\boldsymbol{r}(t) = 5(5 - t^2)\boldsymbol{i} + 5(t^2 + 2)\boldsymbol{j}$. Find:

a the initial position of the particle

b the cartesian equation describing the path of the particle

c the resultant force acting on the particle at time t

5 Forces $2\boldsymbol{i} + \boldsymbol{j}$ and $\boldsymbol{i} - 2\boldsymbol{j}$ act on a particle of mass 2 kg. The forces are measured in newtons. Find:

a the acceleration of the particle

b the velocity of the particle at time t if it was originally at the point with position vector $2\boldsymbol{i} - 2\boldsymbol{j}$ and at rest

c the position of the particle at time t

6 A body of mass 10 kg changes velocity uniformly from $(3\boldsymbol{i} + \boldsymbol{j})$ m/s to $(27\boldsymbol{i} + 9\boldsymbol{j})$ m/s in three seconds.

a Find a vector expression for the acceleration of the body.

b **i** Find a vector expression for the constant resultant force acting on the body.

ii Find the magnitude of the force.

7 The position of a particle of mass 2 kg at time t is given by $\boldsymbol{r}(t) = 2t^2\boldsymbol{i} + (t^2 + 6)\boldsymbol{j}$.

a Find the cartesian equation of the path of the particle.

b Find the velocity of the particle at time t.

c At what time is the speed of the particle $16\sqrt{5}$ m/s?

d Find the resultant force acting on the particle at time t.

8 A particle of mass 10 kg moving with velocity $3\boldsymbol{i} + 5\boldsymbol{j}$ metres per second is acted on by a force of $\frac{1}{10}(15\boldsymbol{i} + 25\boldsymbol{j})$ newtons. Find:

a the acceleration of the particle

b the velocity at time t

c the position of the particle when $t = 6$ if initially it is at the point with position vector $0\boldsymbol{i} + 0\boldsymbol{j}$

d the cartesian equation of the path of the particle

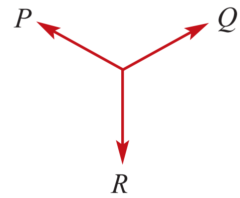
9 A particle is moving along a path which can be described by the cartesian equation $y = 3x$. If the speed of the particle in the positive x direction is 5 m/s, what is the speed of the particle in the positive y direction? Find the speed of the particle.

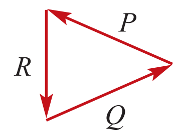
MC TEST D&D TEST

Review

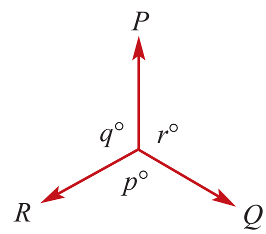
Chapter summary

- The units of force used are the kilogram weight and the newton.
  1 kg wt = g N, where g is the acceleration due to gravity.
- The vector sum of the forces acting at a point is called the **resultant force**.
- The **momentum** of a particle is the product of its mass and velocity, i.e.
  momentum = mass × velocity.
  The units of momentum are kg m/s or kg m s^{-1}.
- A force acting on a body has an influence in directions other than its line of action, except in the direction perpendicular to its line of action.
- The **resolved** part of a force, P N, in a direction that makes an angle θ with its own line of action, is a force of magnitude $P \cos \theta$.
- If a force is resolved in two perpendicular directions, the vector sum of the resolved parts is equal to the force itself.
- **Newton's second law of motion**
  $F = ma$, where the unit of force is the newton, mass is measured in kilograms and the unit of acceleration is m/s^2.
- The frictional force F_R of a particle moving on a surface is given by:
  $F_R = \mu R$
  where R is the normal reaction force and μ is the coefficient of friction.
- A particle is in equilibrium if the resultant force acting on the particle is of zero magnitude.
- If F_R is the frictional force acting on a particle:
  $F_R \leq \mu R$
  where R is the normal reaction force and μ is the coefficient of friction.
- If the particle is just on the point of moving $F_R = \mu R$ (limiting friction).
- **The triangle of forces**
  If the forces P, Q and R are the only forces acting on a particle, and the particle is in equilibrium, then these forces can be represented in magnitude and direction by a triangle of forces.





- **Lami's theorem**
  If the particle is in equilibrium:
  $$\frac{P}{\sin p^\circ} = \frac{Q}{\sin q^\circ} = \frac{R}{\sin r^\circ}$$



Review

Multiple-choice questions

1 The velocity of a body of mass 3 kg has a horizontal component of magnitude 6 m/s, and a vertical component of magnitude 8 m/s. The momentum of the body has a magnitude, in kg m/s, of:

A 6 **B** 18 **C** 24 **D** 30 **E** 42

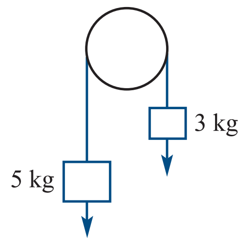
2 A block of mass 10 kg rests on the floor of a lift which is accelerating upward at 4 m/s^2. The magnitude of the reaction force of the floor of the lift on the block (acceleration due to gravity, $g = 9.8$ m/s^2) is:

A 104 N **B** 96 N **C** 60 N **D** 30 N **E** 138 N

3 Two perpendicular forces have magnitudes 8 N and 6 N. The magnitude of the resultant force is:

A 14 N **B** 10 N **C** $2\sqrt{7}$ N **D** 2 N **E** 100 N

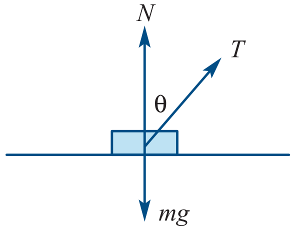
4 A 5 kg mass and a 3 kg mass are connected by a string passing over a smooth pulley as shown. The magnitude of the acceleration of the system is:



A 0.25 m/s^2 **B** $\frac{g}{4}$ m/s^2 **C** $\frac{g}{2}$ m/s^2

D 0.5 m/s^2 **E** 8 gm/s^2

5 A body of mass m kg is being pulled along a smooth horizontal table by means of a string inclined at $\theta°$ to the vertical. The diagram below indicates the forces acting on the body. Which one of the following statements is true?



A $N - mg = 0$

B $N + T\sin\theta - mg = 0$

C $N - T\sin\theta - mg = 0$

D $N + T\cos\theta - mg = 0$

E $N - T\cos\theta - mg = 0$

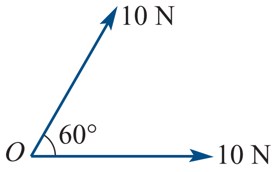
6 A boy slides down a smooth slide with an inclination to the horizontal of $\theta°$ where $\sin\theta = \frac{4}{5}$. The boy's acceleration down the slide in m/s^2, where g m/s^2 is the acceleration due to gravity, is:

A $\frac{3g}{5}$ **B** $\frac{4g}{5}$ **C** $40g$ **D** $\frac{200g}{3}$ **E** $30g$

7 Two forces of magnitude 10 N act on a particle at O as shown. The magnitude of the resultant force in newtons is:

A 20 **B** $10\sqrt{3}$ **C** 0

D 10 **E** 5

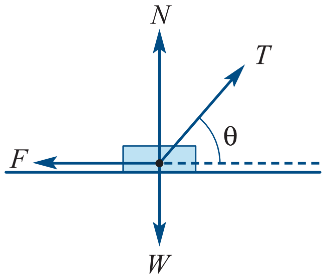


8 The diagram below shows the set of forces acting on a body as it moves with constant velocity across a rough horizontal surface. W newtons is the weight force, N newtons is the normal reaction of the surface on the body and F newtons is the frictional force. T newtons is the tension in a string attached to the body and inclined at an angle θ to the horizontal. The coefficient of friction between the body and the surface is given by:

A $\dfrac{T \sin\theta}{W - T\cos\theta}$ **B** $\dfrac{T \cos\theta}{W - T\sin\theta}$

C $\dfrac{T \cos\theta}{W}$ **D** $\dfrac{T \cos\theta}{W + T\sin\theta}$

E $\dfrac{W - T \sin\theta}{T\cos\theta}$



9 The external resultant force on a body is zero. Which one of the following statements cannot be true?

A The body has constant momentum. **B** The body is moving in a circle.
C The body is moving in a straight line. **D** The body is moving with constant velocity.
E The body is not moving.

10 A particle of mass 9 kg, pulled along a rough horizontal surface by a horizontal force of 54 newtons, is moving with an acceleration of 2 m/s^2. The coefficient of sliding friction between the body and the surface is closest to:

A 0.08 **B** 0.33 **C** 0.41 **D** 0.82 **E** 2.45

Short-answer questions (technology-free)

1 A man of mass 75 kg is in a lift of mass 500 kg that is accelerating upwards at 2 m/s^2. Find:
  a the force exerted by the floor on the man
  b the total tension in the cables raising the lift.

2 Masses of 3 kg and 5 kg are at the ends of a light string that passes over a smooth fixed peg. Calculate:
  a the acceleration of the bodies **b** the tension in the string.

3 Prove that the acceleration of a skier down a slope of angle θ has magnitude $g(\sin\theta - \mu\cos\theta)$ where μ is the coefficient of friction.

4 A block of mass 10 kg is pulled along a horizontal surface by a horizontal force of 100 N. The coefficient of friction between the block and the surface is 0.4.
  a Find the acceleration of the block.
  b If a second block, also of mass 10 kg, is placed on top of the first one, what will be the new acceleration?

5 A particle of mass 5 kg, starting from rest, moves in a straight line under the action of a force which after t seconds is $\dfrac{20}{(t+1)^2}$ newtons. Find:

a the acceleration at time t **b** the velocity at time t

c the displacement from its starting point at time t.

6 A car of mass 1 tonne, travelling at 60 km/h on a level road, has its speed reduced to 24 km/h in 5 seconds when the brakes are applied. Find the total retarding force (assumed constant).

7 A body of mass m rests on a plane of inclination θ in limiting equilibrium. Find the coefficient of friction. If the inclination is increased to ϕ, find the acceleration down the plane.

8 A car of mass 1000 kg coasts down a slope of 1 in 20, i.e. $\sin\theta = \frac{1}{20}$, at constant speed. The car can ascend the same slope with a maximum acceleration of 1 m/s^2. Find:

a the total resistance (assumed constant)

b the driving force exerted by the engine when the maximum acceleration is reached.

9 A parcel rests on a stationary conveyor belt that slopes at an inclination of 30° to the horizontal. The belt begins to move upwards with an increasing acceleration, and when this reaches $\frac{g}{4}$ m/s^2, the parcel begins to slip.

a Calculate the coefficient of friction between the parcel and the belt.

b If, before the parcel begins to slip, the belt is suddenly stopped when its speed reaches 7 m/s, how far will the parcel slide along the belt?

10 A rope will break when its tension exceeds 400 kg wt.

a Calculate the greatest acceleration with which a particle of mass 320 kg can be hauled upwards.

b Show how the rope might be used to lower a particle of mass 480 kg without breaking.

11 A particle of mass 3 kg, moves in a straight line and, at time t, its displacement from a fixed origin is x and its speed is v. If the resultant force is $3 + 6x$, and $v = 2$ when $x = 0$, find v when $x = 2$.

12 A particle of mass 3 kg, moving in a straight line, has initial velocity $\boldsymbol{v} = \boldsymbol{i} + 2\boldsymbol{j}$. It is acted on by a force $\boldsymbol{F} = 3\boldsymbol{i} + 6\boldsymbol{j}$ newtons.

a Find the acceleration at time t.

b **i** Find the velocity at time t. **ii** Find the speed at time t.

c Find the position of the particle at time t if, initially, the particle is at the origin.

d Find the equation of the straight line in which the particle is moving.

13 A train that is moving with uniform acceleration is observed to take 20 s and 30 s to travel successive half kilometres. How much farther will it travel before coming to rest if the acceleration remains constant?

14 What force, in newtons, will give a stationary mass of 9000 kg a horizontal velocity of 15 m/s in 1 minute?

15 A train travelling uniformly on the level at the rate of 20 m/s begins an ascent with an angle of elevation of θ° such that $\sin\theta^\circ = \frac{3}{50}$. The force exerted by the engine is constant throughout, and the resistant force due to friction, etc. is also constant. How far up the incline will the train travel before coming to rest?

16 A body of mass m kg is placed in a lift that is moving with an upward acceleration of f m/s. Find the reaction force of the lift on the body.

17 A 0.05 kg bullet travelling at 200 m/s will penetrate 10 cm into a fixed block of wood. Find the velocity with which it would emerge if fired through a fixed board 5 cm thick, the resistance being supposed uniform and to have the same value in both cases.

18 In a lift accelerated upwards at a m/s^2, a spring balance indicates a particle to have a weight of 10 kg wt. When the lift is accelerated downwards at the rate of $2a$ m/s^2, the mass of the particle appears to be 7 kg wt. Find:

a the weight of the particle **b** the upward acceleration.

19 Two particles A and B, of masses m_1 kg and m_2 kg respectively ($m_1 > m_2$), are connected by a light inextensible string passing over a small smooth fixed pulley. Find:

a the resulting motion of A **b** the tension force in the string.

20 A particle, A, of mass m_2 kg is placed on a smooth horizontal table, and connected by a light inextensible string, passing over a small smooth pulley at the edge of the table, to a particle of mass m_1 kg hanging freely. Find:

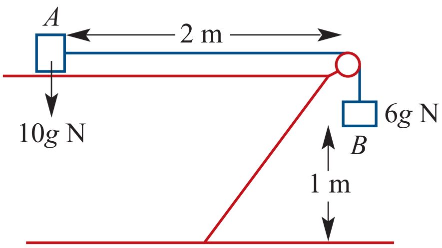
a the resulting motion of A **b** the tension force in the string.

21 A particle, A, of mass m_2 kg is placed on the surface of a smooth plane inclined at an angle α to the horizontal. It is connected by a light in extensible string, passing over a small smooth pulley at the top of the plane, to a particle of mass m_1 kg hanging freely ($m_1 > m_2$). Find:

a the resultant motion of A **b** the tension in the string.

22 A particle of mass m kg slides down a rough inclined plane of inclination α. μ is the coefficient of friction. Find the acceleration of the particle.

23 A particle, A, of mass 10 kg, resting on a smooth horizontal table, is connected by a light string passing over a smooth pulley situated at the edge of the table, to a particle, B, of mass 6 kg hanging freely. A is 2 m from the edge and B is 1 m from the ground. Find:



a the acceleration of particle B

b the tension force in the string

c the resultant force exerted on the pulley by the string

d the time taken for B to reach the ground

e the time taken for A to reach the edge.

Review

24 A particle, A, of mass 10 kg is placed on the surface of a smooth plane inclined at an angle α to the horizontal. It is connected by a light inextensible string passing over a small smooth pulley at the top of the plane to a particle, B, of mass 3 kg hanging freely. Given that $\alpha = 60°$, find:

a the acceleration of A **b** the tension force in the string.

25 A particle, A, of mass 5 kg rests on a rough horizontal table and is connected by a light string over a smooth pulley to a particle, B, of mass 3 kg hanging freely 1 m from the ground. The coefficient of friction between particle A and the table is 0.2. Find:

a the acceleration of particle B **b** the velocity of A as B reaches the ground

c the further distance travelled by A before it comes to rest (assume that A is placed far enough from the edge initially).

26 Show that the magnitude of the resultant of two forces each equal to P N, and inclined at any angle of 120°, is also equal to P N.

27 A particle of mass 10 kg rests in limiting equilibrium on a rough plane inclined at an angle of 30° with the horizontal. The plane is raised until its slope is 60°. Find the magnitude of the force parallel to the plane required to support the body.

28 A particle of mass 5 kg hangs from a fixed point O by a light inextensible string. It is pulled aside by a horizontal force P N and rests in equilibrium with the string inclined at 60° to the vertical. Find P.

29 A particle of mass 2 kg hangs from a fixed point O by a light inextensible string of length 2.5 m. It is pulled aside a horizontal distance of 2 m by a horizontal force P N and rests in equilibrium. Find P and the tension of the string.

30 A particle of mass 5 kg is suspended by two strings of lengths 5 cm and 12 cm respectively, attached at two points at the same horizontal level and 13 cm apart. Find the tension in each of the strings.

Extended-response questions

1 A buoy of mass 4 kg is held five metres below the surface of the water by a vertical cable. There is an upward buoyancy force of 42 N acting on the buoy.

a Find the tension in the cable.

b Suddenly the cable breaks. Find the acceleration of the buoy while it is still in the water.

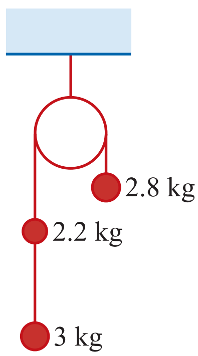
c The buoy maintains this constant acceleration while it is still in the water. Find:

i the time taken for it to reach the surface

ii the velocity of the buoy at this time.

d Ignoring air resistance, find the height above water level that the buoy will reach.

2 Masses of 2.8 kg, 2.2 kg and 3 kg are connected by light inextensible strings, one of which passes over a smooth fixed pulley as shown in the diagram.



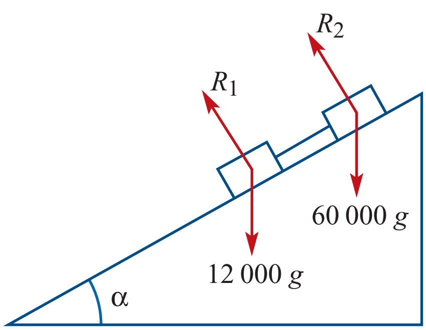
a If the system is released from rest calculate:

i the acceleration of the masses

ii the tension in the string joining the 2.2 kg and 3 kg masses.

b If after $1\frac{1}{2}$ seconds the string joining the 2.2 kg and 3 kg masses breaks, calculate the further distance the 2.2 kg mass falls before coming instantaneously to rest.

3 An engine of mass 60 000 kg is pulling a truck of mass 12 000 kg at constant speed up a slope inclined at α to the horizontal where $\sin\alpha = \frac{1}{200}$. Respective resistances are 50 N per 1000 kg for the engine and 30 N per 1000 kg for the truck.



a Calculate:

i the tractive force exerted by the engine

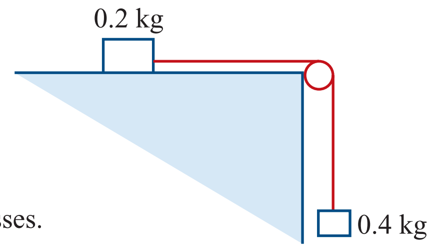
ii the tension in the coupling between the engine and the truck.

b If the engine and the truck were accelerating at 0.1 m/s^2 up the slope find:

i the tractive force exerted by the engine

ii the tension in the coupling between the engine and the truck.

4 A mass of 400 g, hanging vertically, drags a mass of 200 g across a horizontal table. The coefficient of friction is 0.4



a Find:

i the acceleration of the system

ii the tension in the string connecting the two masses.

b If the falling weight strikes the floor after moving 150 cm, how far will the mass on the table move afterwards? (Assume that there is enough table surface for the mass to continue on the table until it stops.)

5 The total resistance on a train with the brakes applied is $(a + bv^2)$ per unit mass where v is its velocity.

a **i** Show that $\dfrac{dv}{dx} = -\dfrac{(a + bv^2)}{v}$, where x is the distance travelled from when the brakes were first applied.

ii If u is the velocity of the train when the brakes were first applied show that the train comes to rest when $x = \dfrac{1}{2b}\log_e\left(1 + \dfrac{bu^2}{a}\right)$.

b **i** Show that the train stops when $t = \dfrac{1}{\sqrt{ab}}\tan^{-1}\left(\dfrac{\sqrt{b}u}{\sqrt{a}}\right)$.

ii Find the time it takes for the train to stop for $b = 0.005$, $a = 2$ and $u = 25$.

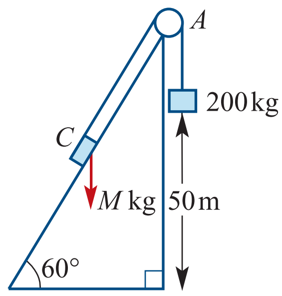
6 A particle of mass m kg falls vertically from rest in a medium in which the resistance is $0.02\,mv^2$ when the velocity is v m/s.

a Find the distance, x m, which the particle has fallen in terms of v.

b Find v in terms of x.

c Sketch the graph of v against x.

7 The diagram shows a crate of mass M kg on a rough inclined slope and a block of mass 200 kg hanging vertically 50 m above the ground. The crate and the block are joined by a light inelastic rope which passes over a smooth pulley at A.



a If $M = 200$, and C is on the point of moving up the plane find the coefficient of friction μ between C and the slope.

b Find the values of M for which the crate will remain stationary.

c Let $M = 150$

i Find the acceleration of the system.

ii Find the tension in the rope.

iii If after two seconds of motion the string breaks, find the speed of the 200 kg weight when it hits the ground.

8 The velocity v m/s of a vehicle, moving in a straight line, is $125(1 - e^{-0.1t})$ m/s at time t. The mass of the vehicle is 250 kg.

a Find the acceleration of the vehicle at time t.

b The resultant force acting on the vehicle is $(P - 20v)$ N where P is the driving force and $20v$ the resistance force.

i Find P in terms of t.

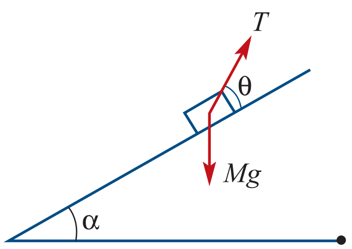
ii Find P in terms of v.

iii Find P when $v = 20$.

iv Find P when $t = 30$.

c Sketch the graph of P against t.

9 A particle of mass M kg is being pulled up a rough inclined plane at constant speed by a force of T N as shown. The coefficient of friction is 0.1.



a Find the normal reaction force R in terms of M, T, θ and α.

b Find T in terms of θ, α and M.

c If $\sin\alpha = \frac{4}{5}$ and $M = 10$

i find T in terms of θ

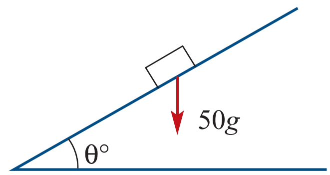
ii find the value of θ which minimises T

iii state this minimum value of T.

d If the body is now accelerating up the plane at 2 m/s^2, find the value of θ which minimises T.

Note: $(\sin\alpha = \frac{4}{5}$ and $M = 10)$

10 A particle of mass 50 kg slides down a rough plane inclined at θ° to the horizontal. The coefficient of friction between the particle and the plane is 0.1. The length of the plane is 10 m and $\sin\theta = \frac{5}{13}$.

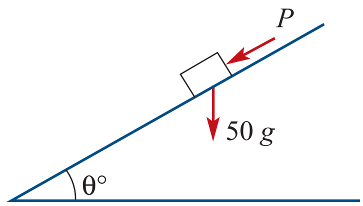


a Find the values of:
  i N, the normal reaction force
  ii F, the friction force.

b Find the acceleration of the particle down the plane.

c If the particle starts at the top of the plane and slides down, find:
  i the speed of the particle at the bottom of the plane
  ii the time it takes to reach the bottom of the plane.

d If an extra pushing force P N acts on the particle and $P = 300 - 250t$ and acts parallel to the line of greatest slope of the plane, find:
  i the acceleration of the particle at time t
  ii the time it takes to reach the bottom of the slope from the top of the slope (t is measured from when motion starts).



CAS CALCULATOR

CHAPTER

14

Revision

Revision of Chapters 12 and 13

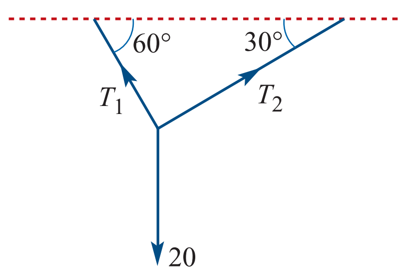
14.1 Multiple-choice questions

1 The velocity, V, of a body is given by $V = (x - 2)^2$, where x is the position of the body at time t. The acceleration of the body at time t is given by:

A $2(x - 2)$ **B** $\frac{(x-2)^2}{t}$ **C** $2(x - 2)^3$

D $x^2 - 4x + 4$ **E** $x - 4$

2 A particle of weight 20 kg wt is supported by two wires attached to a horizontal beam. The tensions in the wires are T_1 kg wt and T_2 kg wt.

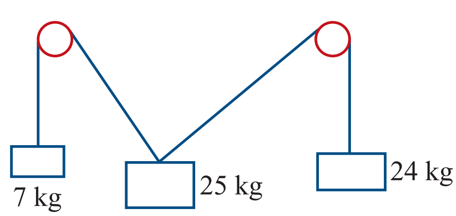


Which one of the following statements is not true?

A $\frac{T_1}{\sin 60^\circ} = \frac{T_2}{\sin 30^\circ}$ **B** $T_2 = 20 \sin 30^\circ$ **C** $T_1 = 20 \cos 30^\circ$

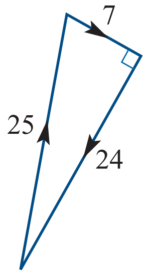
D $T_1 \cos 60^\circ = T_2 \cos 30^\circ$ **E** $T_1 \cos 60^\circ + T_2 \cos 30^\circ = 20$

3 The diagram shows three masses, in equilibrium, connected by strings over smooth fixed pulleys.

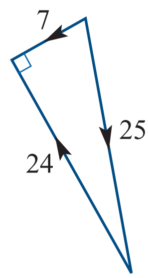


Which one of the following force diagrams is an accurate representation of the forces acting on the 25 kg mass?

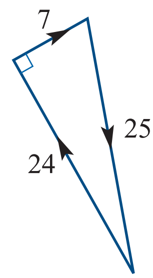
A



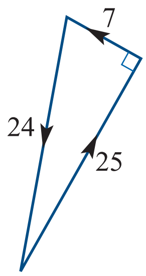
B



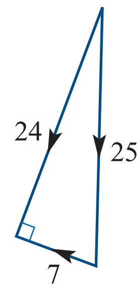
C



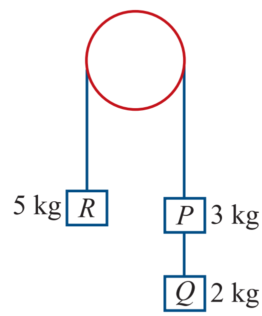
D



E



4 The system shown rests in equilibrium, with the string passing over a smooth pulley. The other parts of the string are vertical. When the string connecting P and Q is cut, the acceleration of R is of magnitude:



A $\frac{g}{4}$ **B** g **C** $\frac{15g}{4}$
D $4g$ **E** none of these

5 A particle, P, of unit mass moves under a resisting force $-kv$, where k is a positive constant and v is the velocity of P. No other forces act on P, which has a velocity V at time $t = 0$. At time t, the velocity of the particle is:

A Ve^{kt} **B** $\left(\frac{V}{k}\right)e^{kt}$ **C** Ve^{-kt} **D** $\left(\frac{V}{k}\right)e^{-kt}$ **E** $V(1 - kt)$

6 A particle of mass m lies on a horizontal platform that is being accelerated upwards with an acceleration f. The force exerted by the platform on the particle is:

A $m(f - g)$ **B** $m(g + f)$ **C** $m(g - f)$ **D** $\frac{mf}{g}$ **E** mf

7 A particle of mass 10 kg is subject to forces of $3\boldsymbol{i}$ newtons and $4\boldsymbol{j}$ newtons. The acceleration of the particle is described by the vector:

A $5\boldsymbol{i}$ **B** $0.3\boldsymbol{i} + 0.4\boldsymbol{j}$ **C** $5\left(\frac{1}{\sqrt{2}}\boldsymbol{i} + \frac{1}{\sqrt{2}}\boldsymbol{j}\right)$
D $5\boldsymbol{j}$ **E** $3\boldsymbol{i} + 4\boldsymbol{j}$

8 A particle moves in the x–y plane so that its position vector $\boldsymbol{r}$ at time t seconds is given by $\boldsymbol{r} = 2t^2\boldsymbol{i} + t^3\boldsymbol{j}$ metres. When $t = 1$, the speed in m/s of the particle is:

A $\frac{3}{4}$ **B** $\sqrt{5}$ **C** 5 **D** 7 **E** 25

Revision

9 A body falls, under gravity, against a resistance of kv^2 per unit mass, where v is the speed and k is a constant. After time t the body has fallen a distance s. Which of the following equations describes the motion?

A $v\dfrac{dv}{ds} = g - kv^2$ **B** $v\dfrac{dv}{dt} = g + kv^2$ **C** $\dfrac{d^2s}{dt^2} = g + kv^2$
D $v\dfrac{dv}{ds} = -(g + kv^2)$ **E** $\dfrac{dv}{dt} = -g + kv^2$

10 A particle moves in the x–y plane so that its position vector $\boldsymbol{r}$ at time t seconds is given by $\boldsymbol{r} = \sin 2t\boldsymbol{i} + e^{-t}\boldsymbol{j}$ metres. When $t = 0$, the speed in m/s of the particle is:

A 1 **B** 3 **C** $\sqrt{3}$ **D** $\sqrt{5}$ **E** 5

11 A block of weight w slides down a fixed slope of angle θ where $\tan\theta = \frac{3}{4}$. The coefficient of friction is $\frac{1}{2}$. The horizontal component of the resultant force acting on the block is:

A 0 **B** $\dfrac{w}{4}$ **C** $\dfrac{2w}{5}$ **D** $\dfrac{6w}{25}$ **E** $\dfrac{4w}{25}$

12 A particle starts at rest at a point O, and moves in a straight line so that after t seconds its velocity v is given by $v = 4\sin 2t$. At this time the displacement, s, from O is given by:

A $s = 8\cos 2t$ **B** $s = 2\cos 2t$ **C** $s = -2\cos 2t$
D $s = 8\cos 2t - 8$ **E** $s = 2 - 2\cos 2t$

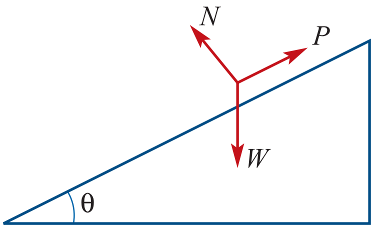
13 A boy of mass 60 kg slides down a frictionless slope that is inclined at $\theta°$ to the horizontal, where $\sin\theta = \frac{4}{5}$. The boy's acceleration down the slide, in m/s^2, is:

A $\frac{3}{5}g$ **B** $\frac{4}{5}g$ **C** $36g$ **D** $48g$ **E** g

14 The position of a particle at time $t = 0$ is given by $\boldsymbol{r}(0) = 2\boldsymbol{i} + 5\boldsymbol{j} + 2\boldsymbol{k}$. The position of the particle at time $t = 2$ is $\boldsymbol{r}(2) = 4\boldsymbol{i} - \boldsymbol{j} + 4\boldsymbol{k}$. The average velocity for the interval [0, 2] is:

A $\frac{1}{2}(6\boldsymbol{i} + 4\boldsymbol{j} + 6\boldsymbol{k})$ **B** $\boldsymbol{i} - 3\boldsymbol{j} + \boldsymbol{k}$ **C** $24\boldsymbol{i} + \boldsymbol{k}$
D $\boldsymbol{i} - 2\boldsymbol{j} + 3\boldsymbol{k}$ **E** $\frac{1}{2}\boldsymbol{i} - \frac{3}{2}\boldsymbol{j} + \frac{1}{2}\boldsymbol{k}$

15 The diagram shows a particle of weight W on an inclined plane. The normal force exerted by the plane is N, and the friction force is F. The force P just prevents the particle from sliding down the plane. Which one of the following is true?



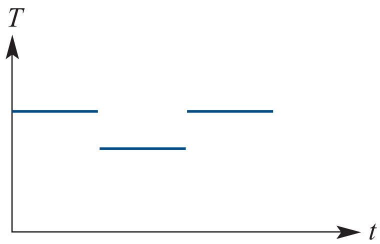
A $P = W\sin\theta - F$ **B** $P = F + W\sin\theta$
C $P = F$ **D** $N = W\sin\theta$ **E** $W = N\cos\theta$

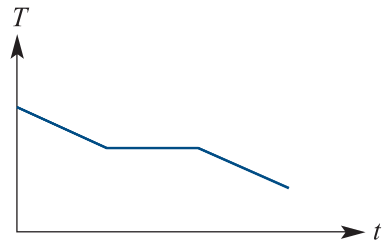
16 The acceleration of a particle at time t is given by the vector equation $\ddot{\boldsymbol{x}}(t) = 2\boldsymbol{i} + t\boldsymbol{j}$. If the velocity of the particle at time $t = 0$ is described by the vector $2\boldsymbol{i}$, the velocity at time t is:

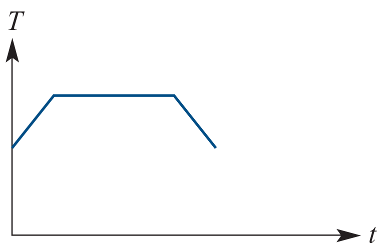
A $\dot{\boldsymbol{x}}(t) = 2t\boldsymbol{i} + \dfrac{t^2}{2}\boldsymbol{j}$ **B** $\dot{\boldsymbol{x}}(t) = (2t + 2)\boldsymbol{i} + \left(\dfrac{t^2}{2}\right)\boldsymbol{j}$ **C** $\dot{\boldsymbol{x}}(t) = 2\boldsymbol{i} + (2\boldsymbol{i} + t\boldsymbol{j})t$

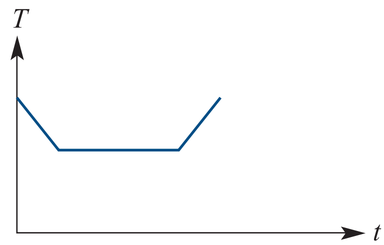
D $\dot{\boldsymbol{x}}(t) = 2(2\boldsymbol{i} + t\boldsymbol{j})$ **E** $\dot{\boldsymbol{x}}(t) = 2 + (2t)\boldsymbol{i} + \left(\dfrac{t^2}{2}\right)\boldsymbol{j}$

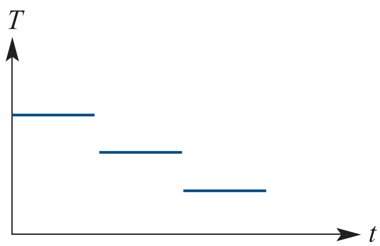
17 A mass is hanging in a lift, being suspended by a light inextensible string. The lift ascends, first moving with uniform acceleration, then with uniform speed, and finally retarding to rest with a retardation of the same magnitude as the acceleration. Given that the tension, T, is greater than zero throughout, which of the following is the graph that best represents T against t?

A


B


C


D


E


18 A mass of 20 kg is supported at rest on a sloping ramp inclined at 30° to the horizontal by a force of 4 g newtons acting up the sloping ramp and parallel to it. The frictional force acting on the mass is:

A $10g$ newtons down the plane
B $10g$ newtons up the plane
C g newtons up the plane
D $6g$ newtons down the plane
E $6g$ newtons up the plane

19 A particle is moving so its velocity vector at time t is $\dot{\boldsymbol{r}}(t) = 2t\boldsymbol{i} + 3\boldsymbol{j}$ where $\boldsymbol{r}(t)$ is the position vector of the particle at time t.
If $\boldsymbol{r}(0) = 3\boldsymbol{i} + \boldsymbol{j}$ then $\boldsymbol{r}(t)$ is equal to:

A $2\boldsymbol{i}$
B $5\boldsymbol{i} + 3\boldsymbol{j}$
C $(3t + 1)\boldsymbol{i} + (3t^2 + 1)\boldsymbol{j}$
D $(t^2 + 3)\boldsymbol{i} + (3t + 1)\boldsymbol{j}$
E $2t^2\boldsymbol{i} + 3t\boldsymbol{j} + 3\boldsymbol{i} + \boldsymbol{j}$

20 A particle of mass 5 kg is subjected to forces of $3\boldsymbol{i}$ newtons and $4\boldsymbol{j}$ newtons. The magnitude of the particle's acceleration is equal to:

A 1 m/s^2
B 7 m/s^2
C 1.2 m/s
D −1.2 m/s
E 5 m/s^2

21 A body is in equilibrium under the action of forces $\boldsymbol{F}_1$, $\boldsymbol{F}_2$ and $\boldsymbol{F}_3$, where $\boldsymbol{F}_1 = 3\boldsymbol{i} + 2\boldsymbol{j} + \boldsymbol{k}$ and $\boldsymbol{F}_2 = \boldsymbol{i} - 2\boldsymbol{j}$. The force $\boldsymbol{F}_3$ is:

A $4\boldsymbol{i} + \boldsymbol{k}$ **B** $2\boldsymbol{i} + 4\boldsymbol{j} + \boldsymbol{k}$ **C** $3\boldsymbol{i} + 4\boldsymbol{j} + \boldsymbol{k}$
D $\boldsymbol{i}$ **E** $-4\boldsymbol{i} - \boldsymbol{k}$

22 A particle of mass m kg lies on a plane inclined at $\theta°$ to the horizontal. The particle is at rest but just about to move down the plane. The frictional force in newtons is:

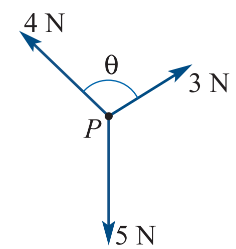
A $m \cos\theta$ **B** $mg \sin\theta$ **C** $mg \tan\theta$ **D** mg **E** $mg \cos\theta$

23 A block of wood of mass 4 kg rests on a rough horizontal table, the coefficient of friction is 0.4. The least force that will cause the block to move when applied horizontally is:

A 15.68 kg wt **B** 1.6 kg **C** 1.6 newtons
D 1.6 kg wt **E** 15.68 kg

24 The particle P is in equilibrium under the action of forces as shown in the diagram.
The magnitude of angle θ is:

A $\sin^{-1}\left(\frac{3}{5}\right)$ **B** $\cos^{-1}\left(\frac{3}{5}\right)$ **C** $\tan^{-1}\left(\frac{3}{5}\right)$
D $90°$ **E** $\tan^{-1}\left(\frac{4}{3}\right)$



25 A particle of mass 5 kg is acted upon by two forces of 0.3 kg wt and 0.4 kg wt, at right angles to each other. The magnitude of the acceleration of the particle is:

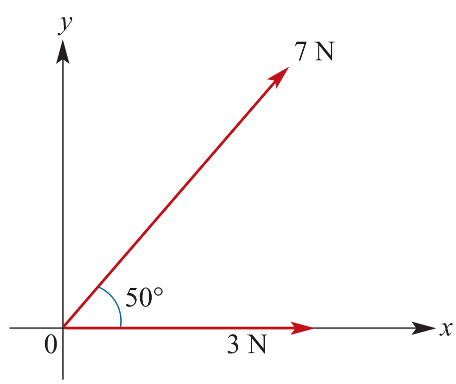
A $0.1\ \text{m/s}^2$ **B** $10\ \text{m/s}^2$ **C** $0.14\ \text{m/s}^2$ **D** $\frac{50}{7}\ \text{m/s}^2$ **E** $0.98\ \text{m/s}^2$

26 A particle of mass 8 kg, travelling at a constant velocity of 20 m/s, is acted upon by a force of 5 N. The magnitude of the resulting acceleration is:

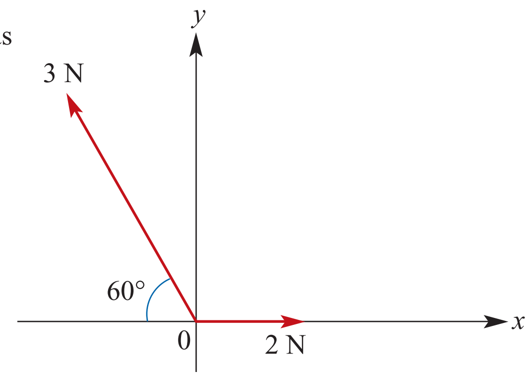
A $32\ \text{m/s}^2$ **B** $\dfrac{5}{8g}\ \text{m/s}^2$ **C** $\dfrac{5}{8}\ \text{m/s}^2$ **D** $1.6\ \text{m/s}^2$
E cannot be found

27 Two forces 7 N and 3 N act at a point as shown in the diagram. The magnitude of the resultant force is:

A 10 N **B** $7 + 3\cos 50°$
C $3 + 7\cos 50°$ **D** $10\cos 25°$
E none of these



28 Two forces of 3 N and 2 N act at a point as shown in the diagram. The resultant of these forces makes an angle θ with the positive direction *x* axis. Which one of the following is true?

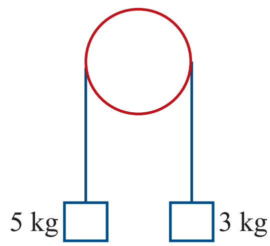


A $\cos\theta = \frac{2}{3}$ **B** $\theta = 60°$
C $\tan\theta = 3\sqrt{3}$ **D** $\theta = 90°$
E $\sin\theta = \dfrac{6\sqrt{3}}{\sqrt{107}}$

29 A particle of mass *m* kg slides down a rough plane inclined at an angle θ to the horizontal. The coefficient of friction between the particle and the plane is μ. The magnitude of the resultant of all the forces acting on the particle is:

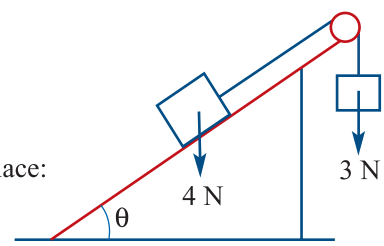
A $mg - \mu$ **B** $mg\sin\theta - \mu$ **C** $mg(\cos\theta - \sin\theta)$
D $mg(\sin\theta - \mu\cos\theta)$ **E** $mg(\cos\theta - \mu\sin\theta)$

30 Two particles of 5 kg and 3 kg are connected by a string that passes over a smooth pulley, and then are released. The magnitude of the acceleration of the particles is:



A 1 m/s^2 **B** $\frac{1}{4}$ m/s^2 **C** g m/s^2
D $\dfrac{g}{4}$ m/s^2 **E** 0

31 A particle of weight 4 N is held in equilibrium on a smooth slope by a string that passes over a smooth pulley and is tied to a suspended particle of weight 3 N. The angle θ, correct to one decimal place:



A is 48.6° **B** is 41.4° **C** is 36.9°
D is 53.1° **E** does not exist

32 A particle has its position in metres from a given point at time *t* seconds defined by the vector $\boldsymbol{r}(t) = 4t\boldsymbol{i} - \frac{1}{3}t^2\boldsymbol{j}$. The average speed of the particle in the third second is:

A 4 m/s **B** $3\frac{2}{3}$ m/s **C** $4\frac{1}{3}$ m/s **D** $6\frac{2}{3}$ m/s **E** 9 m/s

33 The position of a particle at time *t* seconds is given by the vector $\boldsymbol{r}(t) = (t^2 - 2t)(\boldsymbol{i} - 2\boldsymbol{j} + 2\boldsymbol{k})$ measured in metres from a fixed point. The distance travelled by the particle in the first two seconds is:

A 0 m **B** 2 m **C** −2 m **D** 6 m **E** 10 m

34 The position of a particle at time *t* seconds is given by the vector $\boldsymbol{r}(t) = \left(\frac{1}{3}t^3 - 4t^2 + 15t\right)\boldsymbol{i} + \left(t^3 - \frac{15}{2}t^2\right)\boldsymbol{j}$. When the particle is instantaneously at rest, the acceleration vector of the particle is given by:

A $15\boldsymbol{i}$ **B** $-18\boldsymbol{j}$ **C** $2\boldsymbol{i} + 15\boldsymbol{j}$ **D** $-8\boldsymbol{i} - 15\boldsymbol{j}$ **E** $-2\boldsymbol{i} + 3\boldsymbol{j}$

35 A particle moves, with its position defined with respect to time t by the position vector $\boldsymbol{r}(t) = (3t^3 - t)\boldsymbol{i} + (2t^2 + 1)\boldsymbol{j} + 5t\boldsymbol{k}$. When $t = \frac{1}{2}$, the magnitude of the acceleration is given by:

A 12 **B** 17 **C** $4\sqrt{3}$ **D** $4\sqrt{5}$ **E** none of these

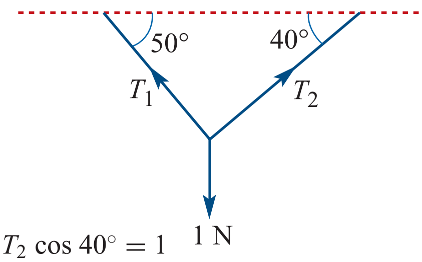
36 The velocity of a particle is given by the vector $\dot{\boldsymbol{r}}(t) = \sin t\boldsymbol{i} + \cos 2t\boldsymbol{j}$. At time $t = 0$, the position of the particle is given by the vector $6\boldsymbol{i} - 4\boldsymbol{j}$. The position of the particle at time t is given by:

A $(6 - \cos t)\boldsymbol{i} + \left(\frac{1}{2}\sin 2t + 4\right)\boldsymbol{j}$ **B** $(5 - \cos t)\boldsymbol{i} + \left(\frac{1}{2}\sin 2t - 3\right)\boldsymbol{j}$
C $(5 + \cos t)\boldsymbol{i} + (2\sin 2t - 4)\boldsymbol{j}$ **D** $(6 + \cos t)\boldsymbol{i} + (2\sin 2t - 4)\boldsymbol{j}$
E $(7 - \cos t)\boldsymbol{i} + \left(\frac{1}{2}\sin 2t - 4\right)\boldsymbol{j}$

37 The initial position, velocity and constant acceleration of a particle are given by $2\boldsymbol{i}$, $3\boldsymbol{j}$ and $\boldsymbol{i} - \boldsymbol{j}$ respectively. The position of the particle at time t is given by:

A $(4 + t)\boldsymbol{i} + \left(3 - \dfrac{t^2}{2}\right)\boldsymbol{j}$ **B** $2\boldsymbol{i} + 3t\boldsymbol{j}$ **C** $2t\boldsymbol{i} + 3t\boldsymbol{j}$
D $\left(2 + \dfrac{t^2}{2}\right)\boldsymbol{i} + \dfrac{t}{2}(6 - t)\boldsymbol{j}$ **E** $(2 + t)\boldsymbol{i} + (3 - t)\boldsymbol{j}$

38 A particle of weight 1 N is supported by two wires attached to a horizontal beam. The tensions in the wires are T_1 N and T_2 N. Which of the following statements is *not* true?



A $\dfrac{T_1}{\sin 50^\circ} = \dfrac{T_2}{\sin 40^\circ}$ **B** $T_1 = \sin 50^\circ$
C $T_2 = \cos 50^\circ$ **D** $T_1 \cos 50^\circ + T_2 \cos 40^\circ = 1$
E $T_1 \cos 50^\circ = T_2 \cos 40^\circ$

39 A particle of mass 5 kg has its momentum defined by the vector $30\boldsymbol{i} - 15\boldsymbol{j} + 10\boldsymbol{k}$ kg m/s. The magnitude of the velocity of the particle is:

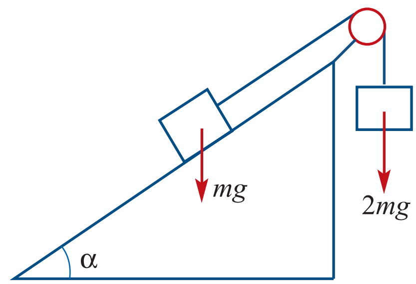
A 25 m/s **B** 5 m/s **C** 7 m/s **D** $\sqrt{31}$ m/s **E** 11 m/s

14.2 Extended-response questions

1 The position vector of a particle at time t seconds is given by $\boldsymbol{r}_1(t) = 2t\boldsymbol{i} - (t^2 + 2)\boldsymbol{j}$ where distances are measured in metres.

a What is the average velocity of the particle for the interval [0, 10]?
b By differentiation, find the velocity at time t.
c In what direction is the particle moving when $t = 3$?
d When is the particle moving with minimum speed?
e At what time is the particle moving at the average velocity for the first 10 seconds?
f A second particle has its position at time t given by $\boldsymbol{r} = (t^3 - 4)\boldsymbol{i} - 3t\boldsymbol{j}$. Are the two particles coincident at any time, t?

2 A particle of mass m is on a rough plane, inclined at an angle α to the horizontal. The particle is connected by a light inextensible string that passes over a smooth pulley at the top of the plane to another particle of mass $2m$ that hangs vertically.



a Find the coefficient of friction if the lighter particle is on the verge of moving up the plane.

b **i** If another particle of mass 3m is attached to the particle hanging vertically, find the acceleration of the particles.

ii Find the time for the particle to go two metres up the slope (starting from rest).

3 The acceleration $\ddot{\boldsymbol{r}}(t)$ m/s^2 of a particle at time t seconds is given by $\ddot{\boldsymbol{r}}(t) = -3\,(\sin 3t\boldsymbol{i} + \cos 3t\boldsymbol{j})$.

a Find the position vector, $\boldsymbol{r}(t)$, given that $\dot{\boldsymbol{r}}(0) = \boldsymbol{i}$ and $\boldsymbol{r}(0) = -3\boldsymbol{i} + 3\boldsymbol{j}$.

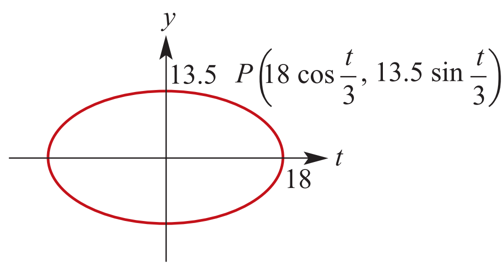
b Show that the path of the particle is a circle and state the position vector of its centre.

c Show that the acceleration is always perpendicular to the velocity.

4 An iceskater describes an elliptic path. His position at time t is given by

$\boldsymbol{r} = 18\cos\left(\frac{t}{3}\right)\boldsymbol{i} + 13.5\sin\left(\frac{t}{3}\right)\boldsymbol{j}$.

When $t = 0$, $\boldsymbol{r} = 18\boldsymbol{i}$.



a How long does the skater take to go around the path once?

b Find:

i the velocity of the iceskater at $t = 2\pi$

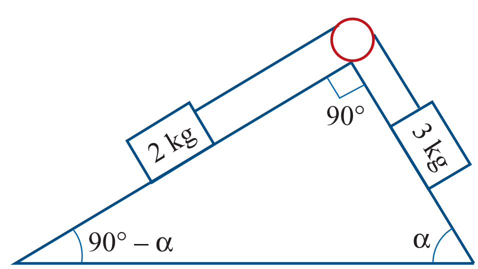
ii the acceleration of the iceskater at $t = 2\pi$.

c **i** Find an expression for the speed of the skater at time t.

ii At what time is his speed greatest?

d Prove that acceleration $\ddot{\boldsymbol{r}} = k\boldsymbol{r}$ and, hence, find when the acceleration has a maximum magnitude.

5 The diagram shows a block of mass 3 kg resting on a rough plane, inclined at an angle α (where $\tan\alpha = \frac{4}{3}$) to the horizontal. This block is connected by a light inextensible string that passes over a smooth pulley to a block of mass 2 kg resting on an equally rough plane inclined at an angle of $(90° - \alpha)$ to the horizontal. Both parts of the string lie in a vertical plane that meets each of the inclined planes in a line of greatest slope.



a If the 3-kg block is on the point of sliding down the plane, show that μ, the coefficient of friction between the blocks and the planes, is $\frac{6}{17}$.

b Calculate the least additional mass that must be attached to the 2-kg block in order that the 3-kg block should be on the point of sliding up the plane.

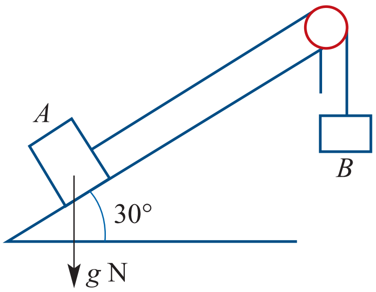
c If an 8-kg mass is added to the 2-kg mass, find the acceleration of the system and the tension in the string.

6 **a** The velocity vector of a particle, P, at time t is $\dot{\boldsymbol{r}}_1(t) = 3\cos 2t\boldsymbol{i} + 4\sin 2t\boldsymbol{j}$ where $\boldsymbol{r}_1(t)$ is the displacement from O at time t. Find:

- **i** $\boldsymbol{r}_1(t)$, given that $\boldsymbol{r}_1(0) = -2\boldsymbol{j}$
- **ii** the acceleration vector at time t
- **iii** the times when the displacement and velocity vectors are perpendicular
- **iv** the cartesian equation of the path.

b At time t, a second particle Q has displacement vector (relative to O) of $\boldsymbol{r}_2(t) = \frac{3}{2}\sin 2t\boldsymbol{i} + 2\cos 2t\boldsymbol{j} + (a - t)\boldsymbol{k}$. Find the possible values of a in order for the particles to collide.

7 A particle, A, of mass 1 kg is placed on a smooth plane inclined at 30°. It is attached by a light inelastic string to a particle B of mass 1 kg. The string passes over a smooth pulley and the particle B hangs 1 m from the floor.



The particles are released from rest. Find:

a the magnitude of the acceleration of the particles

b the tension in the string during this first phase of the motion

c the magnitude of the velocities of the particles when particle B hits the ground

d the time taken before the string is taut again, assuming that there is room on the plane for A to continue travelling up the plane.

8 **a** Two particles of respective masses 1.2 kg and 1.3 kg are connected by a light inextensible string that passes over a fixed light smooth pulley. The system is released from rest with string taut and the straight parts of the string vertical.

- **i** Calculate the acceleration of each particle.
- **ii** Calculate the tension in the string.
- **iii** Calculate the velocity of the 1.2-kg mass after 2 seconds have elapsed, and the distance it has travelled.

b When two seconds have elapsed after the system starts from rest, the lighter particle picks up a mass of 1 kg that was given the same velocity as the lighter particle just before being picked up.

- **i** Calculate the further time that elapses before the system comes instantaneously to rest.
- **ii** Calculate the total distance that the lighter particle has moved.

9 An aircraft takes off from the end of a runway in a southerly direction and climbs at an angle of $\tan^{-1}\left(\frac{1}{2}\right)$ to the horizontal at a speed of $225\sqrt{5}$ km/h.

a Show that t seconds after take-off, the position vector $\boldsymbol{r}$ of the aircraft with respect to the end of the runway is given by $\boldsymbol{r}_1 = \dfrac{t}{16}(2\boldsymbol{i} + \boldsymbol{k})$, where $\boldsymbol{i}$, $\boldsymbol{j}$ and $\boldsymbol{k}$ represent vectors of length 1 km in directions south, east, and vertically upwards respectively.

b At time $t = 0$, a second aircraft, flying horizontally south-west at $720\sqrt{2}$ km/h, has position vector $-1.2\boldsymbol{i} + 3.2\boldsymbol{j} + \boldsymbol{k}$.

i Find its position vector $\boldsymbol{r}_2$ at time t in terms of $\boldsymbol{i}$, $\boldsymbol{j}$ and $\boldsymbol{k}$.

ii Show that there will be a collision and state the time at which it will occur.

10 A particle moves in a straight line, starting from point A. Its motion is assumed to be with constant retardation. During the first, second and third seconds of its motion it covers distances of 70 m, 60 m and 50 m respectively, measured in the same sense.

a **i** Verify that these distances are consistent with the assumption that the particle is moving with constant retardation.

ii Find the retardation and an expression for the displacement of the particle.

b If the particle comes instantaneously to rest at B, find distance AB.

c At the same instant that the first particle leaves A, a second particle leaves B with an initial velocity of 75 m/s and travels with constant acceleration towards A. It meets the first particle at a point C, $1\frac{1}{2}$ seconds after leaving B.

i Find distance BC.

ii Show that the acceleration of the second particle is 60 m/s^2.

11 A particle travels on a path given by the cartesian equation $y = x^2 + 2x$.

a Show that one possible vector representing the position of the particle is $\boldsymbol{r}(t) = (t - 1)\boldsymbol{i} + (t^2 - 1)\boldsymbol{j}$.

b Show that $\boldsymbol{r}(t) = (e^{-t} - 1)\boldsymbol{i} + (e^{-2t} - 1)\boldsymbol{j}$ is also a possible representation of the position of the vector.

c Two particles travel simultaneously. The position of the particles are given by $\boldsymbol{r}_1(t) = (t - 1)\boldsymbol{i} + (t^2 - 1)\boldsymbol{j}$ and $\boldsymbol{r}_2(t) = (e^{-t} - 1)\boldsymbol{i} + (e^{-2t} - 1)\boldsymbol{j}$ respectively.

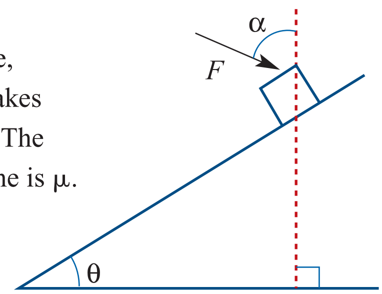
CAS CALCULATOR

i Find the initial position of the two particles.

ii Show that the particles travel in opposite direction along the path defined by $y = x^2 + 2x$.

iii Find, correct to two decimal places, the point at which the two particles collide.

12 A particle of mass m is placed on a plane inclined at an angle θ to the horizontal. An external variable force, F, is applied to the particle so that its line of action makes an angle α with the vertical, as shown in the diagram. The coefficient of friction between the particle and the plane is μ.



a Show that, if $\mu < \tan\theta$ and $F = 0$, the particle will slide down the plane.

Assume that $\mu < \tan\theta$ *for the rest of this problem.*

b Given $\alpha = \theta$, find:
  - **i** the reaction force between the plane and the particle
  - **ii** the maximum friction force between the plane and the particle
  - **iii** the minimum value of F that will prevent the particle from moving.

c Given $\alpha = \dfrac{\pi}{2}$, find:
  - **i** the reaction force between the plane and the particle
  - **ii** the minimum value of F that will prevent the particle from moving down the plane
  - **iii** the minimum value of F that will just move the particle up the plane.

d Consider $\alpha \leq \theta$.
  - **i** Show that the component of F normal to the plane is $F\cos(\theta - \alpha)$.
  - **ii** Find an expression for the minimum value of F that will prevent the particle from moving down the plane.
  - **iii** Hence, show that in this case $\alpha > \theta - \tan^{-1}\mu$.

13 A particle is fired from the top of a cliff h m above sea level with an initial velocity V and inclined at an angle α above the horizontal. Let $\boldsymbol{i}$ and $\boldsymbol{j}$ define the horizontal and upward vertical vectors in the plane of the particle's path.

a Define:
  - **i** the initial position vector of the particle
  - **ii** the particle's initial velocity.

b The acceleration vector of the particle under gravity is given by $\boldsymbol{a} = -g\boldsymbol{j}$. Find:
  - **i** the velocity vector of the particle t seconds after it is projected
  - **ii** the corresponding position vector.

c Use the velocity vector to find the time at which the projectile reaches its highest point.

d Show that the time at which the particle hits the sea is given by the formula:

$$t = \frac{V\sin\alpha t + \sqrt{V^2\sin^2\alpha + 2gh}}{g}.$$

14 A lift that has mass 1000 kg when empty is carrying a man of mass 80 kg. The lift is descending with a downward acceleration of 1 m/s^2.

a Calculate:
  - **i** the tension in the lift cable
  - **ii** the vertical force exerted on the man by the floor of the lift.

b The man drops a coin from a height of 2 m. Calculate the time taken for it to hit the floor of the lift.

c The lift is designed so that during any journey, the magnitude of the acceleration reaches, but does not exceed, 1 m/s^2. Safety regulations do not allow the lift cable to bear a tension greater than 20 000 N. Making reasonable assumptions, suggest the number of people that the lift should be licensed to carry. (Hint: the maximum tension in the lift cable occurs when the lift is accelerating upwards.)

15 Two trains, T_1 and T_2, are moving on perpendicular tracks that cross at the point O. Relative to O, the position vectors of T_1 and T_2 at time t are $\boldsymbol{r}_1$ and $\boldsymbol{r}_2$ respectively, where $\boldsymbol{r}_1 = Vt\boldsymbol{i}$ and $\boldsymbol{r}_2 = 2V(t - t_0)\boldsymbol{j}$ and where V and t_0 are positive constants.

a **i** Which train goes through O first?

ii How much later does the other train go through O?

b **i** Show that the trains are closest together when $t = \dfrac{4t_0}{5}$.

ii Calculate their distance apart at this time.

iii Draw a diagram to show the positions of the trains at this time. Also show the directions in which they are moving.

16 A particle of mass m moves from rest through a distance, d, under a horizontal force, F, on a rough horizonal plane with coefficient of friction μ. It then collides with another particle of mass $2m$, at rest.

a Find the velocity with which the first particle hits the second (in terms of F, m, d and μ).

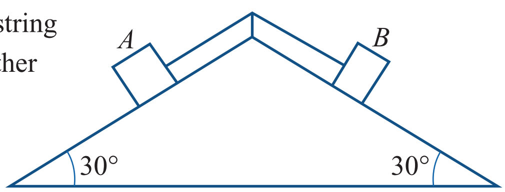
b The two particles adhere to each other. The combined mass moves a further distance d under friction alone.

i Find the retardation of the two particles.

ii Find the initial velocity of the two particles.

c Find F in terms of m and μ.

17 Two particles, A and B, of mass m and $0.7m$ respectively, are linked by a light inelastic string that passes over a smooth pulley, one on either side of a wedge (as shown in the diagram). The coefficients of friction between each of the particles A and B, and the wedge are μ and 2μ respectively. The string is under tension.



a **i** Show that A must be in limiting equilibrium.

ii Use the resolved forces at A to find an expression for the tension in the string.

b B is also in limiting equilibrium. Find:

i the friction force at B

ii the tension force in the string (by considering forces on B).

c Find μ.

d The particles are now placed on the same side of the wedge, with the string taut and with A below B, and then released. Find:

i the acceleration of the particles **ii** the tension in the string.

e Describe what would happen in **d** if B was placed below A initially.

18 A ball is projected against a wall that rebounds the ball in its plane of flight. If the ball has a velocity $a\boldsymbol{i} + b\boldsymbol{j}$ just before hitting the wall, its velocity of rebound is defined as $-0.8a\boldsymbol{i} + b\boldsymbol{j}$. The ball is projected from ground level, and its position vector before hitting the wall is defined by $\boldsymbol{r}(t) = 10t\boldsymbol{i} + t(10\sqrt{3} - 4.9t)\boldsymbol{j}$, $t \geq 0$.

- **a** Find:
  - **i** the initial position vector of the ball
  - **ii** the initial velocity vector of the ball and, hence, the magnitude of the velocity and direction (to be stated as an angle of elevation)
  - **iii** an expression for the acceleration of the ball.
- **b** The wall is at a horizontal distance x from the point of projection. Find in terms of x:
  - **i** the time taken by the ball to reach the wall
  - **ii** the position vector of the ball at impact
  - **iii** the velocity of the ball immediately before impact with the wall
  - **iv** the velocity of the ball immediately after impact.
- **c** Let the second part of the flight of the ball be defined in terms of t_1, a time variable, where $t_1 = 0$ at impact. Assuming that the ball is under the same acceleration vector, find in terms of x and t_1:
  - **i** a new velocity vector of the rebound
  - **ii** a new position vector of the rebound.
- **d** Find the time taken for the ball to hit the ground after the rebound.
- **e** Find the value of x (correct to two decimal places) for which the ball will return to its initial position.

19 An aeroplane takes off from an airport and, with respect to a given frame of reference, its path with respect to the variable time t is described by the vector $\boldsymbol{r}(t) = (5 - 3t)\boldsymbol{i} + 2t\boldsymbol{j} + t\boldsymbol{k}$, $t \geq 0$, where $t = 0$ seconds at the time of take-off.

- **a** Find the position vector that represents the position of the plane at take-off.
- **b** Find:
  - **i** the position of the plane at times t_1 and t_2
  - **ii** the vector which defines the displacement between these two positions in terms of t_1 and t_2 ($t_2 > t_1$).
- **c** Hence, show that the plane is travelling along a straight line and state a position vector parallel to the flight.
- **d** A road on the ground is defined by the vector $\boldsymbol{r}_1(s) = s\boldsymbol{i}$, $s \leq 0$.
  - **i** Find the magnitude of the acute angle between the path of the plane and the road, correct to two decimal places.
  - **ii** Hence, or otherwise, find the shortest distance from the plane to the road six seconds after take-off, correct to two decimal places.

20 The vector $\boldsymbol{r}_1(t) = (2 - t)\boldsymbol{i} + (2t + 1)\boldsymbol{j}$ represents the path of a particle with respect to time t measured in seconds.

a Find the cartesian equation that describes the path of the particle (assume $t \geq 0$).

b **i** Rearrange the above function in the form $\boldsymbol{r}_1(t) = \boldsymbol{a} + t\boldsymbol{b}$, where $\boldsymbol{a}$ and $\boldsymbol{b}$ are vectors.

ii Describe the vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ geometrically with respect to the path of the particle.

c A second particle which started at the same time as the first particle travels along a path that is represented by $\boldsymbol{r}_2(t) = \boldsymbol{c} + t\,(2\boldsymbol{i} + \boldsymbol{j})$, $t \geq 0$. The particles collide after 5 seconds.

i Find $\boldsymbol{c}$. **ii** Find the distance between the two starting points.

21 The paths of two aeroplanes in an aerial display are simultaneously defined by the vectors:

$$\boldsymbol{r}_1(t) = (16 - 3t)\boldsymbol{i} + t\boldsymbol{j} + (3 + 2t)\boldsymbol{k}$$

and $$\boldsymbol{r}_2(t) = (3 + 2t)\boldsymbol{i} + (1 + t)\boldsymbol{j} + (11 - t)\boldsymbol{k}$$

where t represents the time in minutes. Find:

a the position of the first plane after one minute

b the unit vectors parallel to the flights of each of the two planes

c the acute angle between their lines of flight, correct to two decimal places

d the point at which their two paths cross

e the vector which represents the displacement between the two planes after t seconds

f the shortest distance between the two planes during their flight.

22 A hiker starts from a point defined by the position vector $-7\boldsymbol{i} + 2\boldsymbol{j}$ and travels at the rate of 6 km/h along a line parallel to the vector $4\boldsymbol{i} + 3\boldsymbol{j}$. The units in the frame of reference are in kilometres.

a Find the vector which represents the displacement of the hiker in one hour.

b Find, in terms of position vectors, the position of the hiker after:

i 1 hour **ii** 2 hours **iii** t hours.

c The path of a cyclist along a straight road is defined simultaneously by the vector equation $\boldsymbol{b}(t) = (7t - 4)\boldsymbol{i} + (9t - 1)\boldsymbol{j}$.

i Find the position of the hiker when she reaches the road.

ii Find the time taken by the hiker to reach the road.

iii Find, in terms of t, the distance between the hiker and the rider t seconds after the start.

iv Find the shortest distance between the hiker and the rider, correct to two decimal places.

Glossary

$\frac{dx}{dy}$: [p. *202*] $\frac{dx}{dy} = \frac{1}{\frac{dy}{dx}}$

A

acceleration: [p. *392*] The **acceleration** of a particle is defined as the rate of change of its velocity with respect to time.

acceleration, average: [p. *392*] The **average acceleration** of a particle for the time interval $[t_1, t_2]$ is defined by $\frac{v_2 - v_1}{t_2 - t_1}$ where v_2 is the velocity at time t_2 and v_1 is the velocity at time t_1.

acceleration, instantaneous: [p. *392*]
$$a = \frac{dv}{dt} = \frac{d}{dt}\left(\frac{dx}{dt}\right) = \frac{d^2x}{dt^2} = v\frac{dv}{dx} = \frac{d\left(\frac{1}{2}v^2\right)}{dx}$$

addition of complex numbers: [p. *144*] If $z_1 = a + bi$ and $z_2 = c + di$, then $z_1 + z_2 = (a + c) + (b + d)i$.

addition of vectors: [pp. *58, 69*] Let $\boldsymbol{a} = a_1\boldsymbol{i} + a_2\boldsymbol{j} + a_3\boldsymbol{k}$, and $\boldsymbol{b} = b_1\boldsymbol{i} + b_2\boldsymbol{j} + b_3\boldsymbol{k}$. Then $\boldsymbol{a} + \boldsymbol{b} = (a_1 + b_1)\boldsymbol{i} + (a_2 + b_2)\boldsymbol{j} + (a_3 + b_3)\boldsymbol{k}$.

amplitude of circular functions: [p. *4*] The distance between the mean position and the maximum position, e.g. the graph of $y = a \sin x$ has an amplitude of $|a|$.

angle between two vectors: [p. *82*]

$\cos\theta = \frac{\boldsymbol{a}\cdot\boldsymbol{b}}{|\boldsymbol{a}||\boldsymbol{b}|}$, where θ is the angle between the vectors $\boldsymbol{a}$ and $\boldsymbol{b}$.

$\cos\theta = \frac{a_1b_1 + a_2b_2 + a_3b_3}{|\boldsymbol{a}||\boldsymbol{b}|}$, for vectors $\boldsymbol{a} = a_1\boldsymbol{i} + a_2\boldsymbol{j} + a_3\boldsymbol{k}$ and $\boldsymbol{b} = b_1\boldsymbol{i} + b_2\boldsymbol{j} + b_3\boldsymbol{k}$.

angles between a vector and the *i*, *j* and *k* directions: [p. *73*] For vector $\boldsymbol{a} = a_1\boldsymbol{i} + a_2\boldsymbol{j} + a_3\boldsymbol{k}$:

$\cos\alpha = \frac{a_1}{|\boldsymbol{a}|}$, where α is the angle between the vector $\boldsymbol{a}$ and the $\boldsymbol{i}$ direction

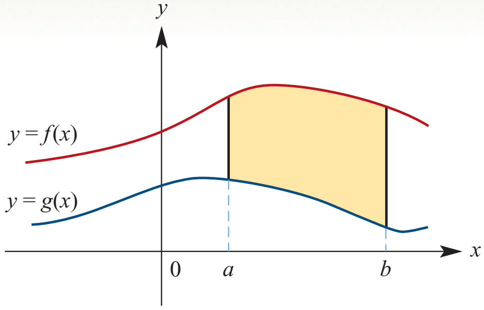
$\cos\beta = \frac{a_2}{|\boldsymbol{a}|}$, where β is the angle between the vector $\boldsymbol{a}$ and the $\boldsymbol{j}$ direction

$\cos\gamma = \frac{a_3}{|\boldsymbol{a}|}$, where γ is the angle between the vector $\boldsymbol{a}$ and the $\boldsymbol{k}$ direction.

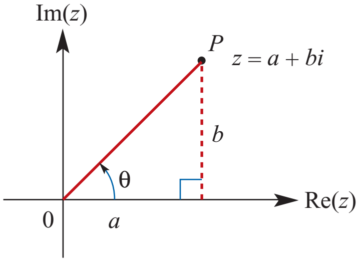
antiderivative of vector function: [p. *466*]
$\int \boldsymbol{r}(t)\,dt = X(t)\boldsymbol{i} + Y(t)\boldsymbol{j} + Z(t)\boldsymbol{k} + \boldsymbol{c}$
where $\boldsymbol{r}(t)$ is a vector function of t, $\boldsymbol{c}$ is a constant vector and $\frac{dX}{dt} = x(t)$, $\frac{dY}{dt} = y(t)$, $\frac{dZ}{dt} = z(t)$

antidifferentiation (or integration): [p. *265*] The process of finding a function from its derivative.

area of a region between two curves: [p. *305*]
$\int_a^b f(x)\,dx - \int_a^b g(x)\,dx = \int_a^b f(x) - g(x)\,dx$, where $f(x) \geq g(x)$ for $x \in [a, b]$



Argand diagram: [p. *145*] A geometrical representation of the set of complex numbers.



argument of a complex number, arg (*z*): [p. *154*]
$\arg(z) = \theta$, where $\sin\theta = \frac{\text{Im}(z)}{|z|}$ and $\cos\theta = \frac{\text{Re}(z)}{|z|}$
arg (*z*) is not defined uniquely.

Argument of a complex number, Arg (*z*): [p. *154*] The single value of arg (*z*) in the interval $(-\pi, \pi]$

argument, properties of: [p. *160*] The argument of the product of two complex numbers is the sum of their arguments.
i.e. $\arg(z_1z_2) = \arg(z_1) + \arg(z_2)$

Argument, properties of: [pp. *160–1*]
- $\text{Arg}(z_1z_2) = \text{Arg}(z_1) + \text{Arg}(z_2) + 2k\pi$ where $k = 0, 1$ or -1
- $\text{Arg}\left(\frac{z_1}{z_2}\right) = \text{Arg}(z_1) - \text{Arg}(z_2) + 2k\pi$ where $k = 0, 1$ or -1
- $\text{Arg}\left(\frac{1}{z}\right) = -\text{Arg}(z)$

arithmetic sequence: [p. *25*] A sequence in which each successive term is found by adding a constant value to the previous term, e.g., 2, 5, 8, 11, . . .

An arithmetic sequence can be defined by a difference equation of the form:

$t_n = t_{n-1} + d$, where d is the common difference.

The nth term of the sequence can be found using:

$t_n = a + (n-1)d$, where $a = t_1$

arithmetic series: [p. *25*] The sum of the terms in an arithmetic sequence.

The sum of the first n terms, S_n, is given by the rule:

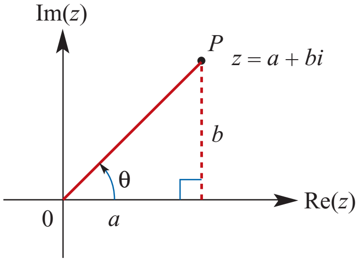
$S_n = \frac{n}{2}[2a + (n-1)d]$, where $a = t_1$ and $d = t_n - t_{n-1}$

C

C: [p. *142*] The set of complex numbers, i.e. $C = \{a + bi : a, b \in R\}$.

cartesian equation: [p. *25*] An equation connecting two variables, often called x and y.

cartesian form of a complex number: [p. *142*] A complex number expressed in the form $a + bi$, represented by the ordered pair (x, y), where x is the real part of z and y is the imaginary part of z.



chain rule: [p. *206*] For $f(x) = h(g(x))$, $f'(x) = h'(g(x))g'(x)$:

$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$ where $u = f(x)$

circle, general cartesian equation of: [p. *32*] $(x-h)^2 + (y-k)^2 = r^2$.
The **centre** of the circle is the point (h, k) and the **radius** is r.

circular functions: [pp. *2, 105*] The sine, cosine, tangent, cosecant, secant and cotangent functions.

cis θ: [p. *155*] $\cos\theta + i\sin\theta$

coefficient of friction, μ: [p. *519*] A constant which determines the resistance to motion between two surfaces in contact.

common difference, d: [p. *25*] The difference between two consecutive terms of an arithmetic sequence, i.e. $d = t_n - t_{n-1}$

common ratio, r: [p. *26*] The quotient of two consecutive terms of a geometric sequence, i.e. $r = \frac{t_n}{t_{n-1}}$

complex conjugate, $\bar{z}$: [pp. *150, 157*]
If $z = a + bi$, then $\bar{z} = a - bi$.
If $z = r\,\text{cis}\,\theta$, then $\bar{z} = r\,\text{cis}(-\theta)$.

complex conjugate, properties of: [p. *150*] Let $z = a + bi$, then $\bar{z} = a - bi$.
- $z + \bar{z} = 2\,\text{Re}(z)$
- $z\bar{z} = |z|^2$
- $\overline{z_1 + z_2} = \overline{z_1} + \overline{z_2}$
- $\overline{z_1z_2} = \overline{z_1}\,\overline{z_2}$

complex number: [p. *142*] An expression of the form $a + bi$, where a and b are real numbers.

compound angle formulas: [p. *113*]
- $\cos(x-y) = \cos x\cos y + \sin x\sin y$
- $\cos(x+y) = \cos x\cos y - \sin x\sin y$
- $\sin(x-y) = \sin x\cos y - \cos x\sin y$
- $\sin(x+y) = \sin x\cos y + \cos x\sin y$
- $\tan(x-y) = \frac{\tan x - \tan y}{1 + \tan x\tan y}$
- $\tan(x+y) = \frac{\tan x + \tan y}{1 - \tan x\tan y}$

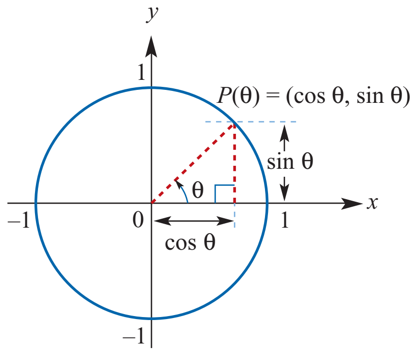
conjugate factor theorem: [p. *165*] If the coefficients of $P(z) = a_nz^n + a_{n-1}z^{n-1} + \cdots + a_1z + a_0$, $a_n \neq 0$, where n is a natural number and $a_n, a_{n-1}, \ldots, a_1, a_0$ are real numbers, then the complex roots occur in **conjugate pairs**, i.e. if $(z - \alpha_1)$ is a factor, so is $(z - \bar{\alpha}_1)$

constant acceleration (or kinematics) formulas: [p. *403*]

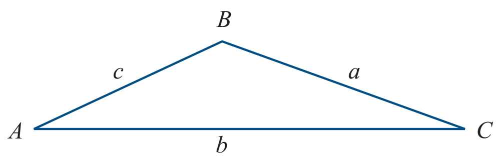
$v = u + at$
$s = ut + \frac{1}{2}at^2$
$v^2 = u^2 + 2as$
$s = \frac{1}{2}(u + v)t$

cosecant function: [p. *105*] $\text{cosec}\,\theta = \frac{1}{\sin\theta}$, provided $\sin\theta \neq 0$

cosine function: [p. *2*] Cosine θ, or cos θ, defined as the x coordinate of the point P on the unit circle where OP forms an angle of θ radians measured anticlockwise from the positive ray of the x axis.



cosine rule: [p. *17*] For triangle *ABC*



$a^2 = b^2 + c^2 - 2bc \cos A$ or, equivalently,

$\cos A = \dfrac{b^2 + c^2 - a^2}{2bc}$

The cosine rule is used to find unknown quantities in a triangle when either two sides and an included angle are given, or three sides are given.

cotangent function: [p. *106*] $\cot \theta = \dfrac{\cos \theta}{\sin \theta}$, provided $\sin \theta \neq 0$

D

De Moivre's theorem: [p. *161*]
$(r \operatorname{cis}(\theta))^n = r^n \operatorname{cis}(n\theta)$ for $n \in Z$

definite integral: [p. *268*] The **definite integral** from a to b is written $\int_a^b f(x)\,dx$, where:

$\int_a^b f(x)dx = F(b) - F(a)$, and F is any antiderivative of f.

The number a is called the lower limit of integration and b is called the upper limit of integration. The function f is called the integrand.

derivative function (or gradient function): [p. *205*] The derivative of a function f is denoted by f' and the rule for f' is defined by

$f'(x) = \lim\limits_{h \to 0} \dfrac{f(x+h) - f(x)}{h}$

derivative of inverse cosine function: [p. *215*] For $f(x) = \cos^{-1}\left(\dfrac{x}{a}\right)$, $f'(x) = \dfrac{-1}{\sqrt{a^2 - x^2}}$ for $x \in (-a, a)$.

derivative of inverse sine function: [p. *215*] For $f(x) = \sin^{-1}\left(\dfrac{x}{a}\right)$, $f'(x) = \dfrac{1}{\sqrt{a^2 - x^2}}$ for $x \in (-a, a)$.

derivative of inverse tangent function: [p. *216*] For $f(x) = \tan^{-1}\left(\dfrac{x}{a}\right)$, $f'(x) = \dfrac{a}{a^2 + x^2}$ for $x \in R$.

derivative of tangent function: [p. *207*] For $f(\theta) = \tan k\theta$, the derivative f' is given by $f'(\theta) = k \sec^2 k\theta$.

derivative of vector functions: [p. *464*] For

$\boldsymbol{r}(t) = x(t)\boldsymbol{i} + y(t)\boldsymbol{j} + z(t)\boldsymbol{k}$

$\dot{\boldsymbol{r}}(t) = \dfrac{dx}{dt}\boldsymbol{i} + \dfrac{dy}{dt}\boldsymbol{j} + \dfrac{dz}{dt}\boldsymbol{k}$

and $\ddot{\boldsymbol{r}}(t) = \dfrac{d^2x}{dt^2}\boldsymbol{i} + \dfrac{d^2y}{dt^2}\boldsymbol{j} + \dfrac{d^2z}{dt^2}\boldsymbol{k}$

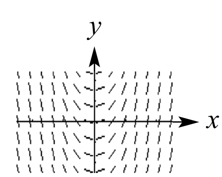
difference equation (or iterative rule): [p. *24*] A rule which enables each subsequent term of a sequence to be found using the previous term or terms, e.g. $t_1 = 1, t_n = t_{n-1} + 2$.

differential equations: [p. *337*] Equations which involve at least one derivative,
e.g. $\dfrac{dx}{dt} = \cos t$, $\dfrac{d^2x}{dt^2} - 4x = t$, $\dfrac{dy}{dx} = \dfrac{y}{y+1}$

differential equations, general solution of: [p. *337*] $x = \sin t + c$ is the **general solution** of the differential equation $\dfrac{dx}{dt} = \cos t$.

differential equations, particular solution of: [p. *338*] $x = \sin t$ is the **particular solution** of the differential equation $\dfrac{dx}{dt} = \cos t$, given $x(0) = 0$.

direction field (or slope field) of a differential equation: [p. *378*] The direction field of a differential equation, $\dfrac{dy}{dx} = f(x)$, assigns to each point $P(x, y)$ in the plane, with x in the domain of f, the number which is the slope (gradient) of the solution curve through P. A slope field can be represented in a graph.



displacement: [p. *389*] The **displacement** of a particle moving in a straight line is defined as the change in position of the particle.

division of complex numbers, z_1 and z_2: [pp. *151, 160*] For cartesian form: $\dfrac{z_1}{z_2} = \dfrac{z_1\bar{z}_2}{|z_2|^2}$ and $z^{-1} = \dfrac{\bar{z}}{|z|^2}$

For polar form: $\dfrac{z_1}{z_2} = \dfrac{r_1}{r_2} \operatorname{cis}(\theta_1 - \theta_2)$ and

$z^{-1} = \dfrac{1}{r} \operatorname{cis}(-\theta)$

dot product (or scalar product), *a.b*: [p. *81*]
$\boldsymbol{a}.\boldsymbol{b} = |\boldsymbol{a}||\boldsymbol{b}| \cos \theta$

For vectors $\boldsymbol{a} = a_1\boldsymbol{i} + a_2\boldsymbol{j} + a_3\boldsymbol{k}$ and $\boldsymbol{b} = b_1\boldsymbol{i} + b_2\boldsymbol{j} + b_3\boldsymbol{k}$, $\boldsymbol{a}.\boldsymbol{b} = a_1b_1 + a_2b_2 + a_3b_3$

double angle formulas: [p. *117*]

- $\sin 2x = 2\sin x\cos x$
- $\cos 2x = \cos^2 x - \sin^2 x$
  $= 1 - 2\sin^2 x$
  $= 2\cos^2 x - 1$
- $\tan 2x = \dfrac{2\tan x}{1 - \tan^2 x}$

E

ellipse, general cartesian equation of: [p. *35*]

$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$$

The **centre** of the ellipse is the point (h, k), the axis of the ellipse parallel to the x axis is of length $2a$ units, and the axis of the ellipse parallel to the y axis is of length $2b$ units.

equal complex numbers: [p. *144*] If $z_1 = a + bi$ and $z_2 = c + di$, then $z_1 = z_2$ if and only if $a = c$ and $b = d$.

equilibrium: [p. *514*] A particle is said to be in equilibrium if the resultant force acting on it is zero, i.e. if $\boldsymbol{F} = \boldsymbol{0}$. In this case the particle has zero acceleration. If the particle is at rest it remains at rest and if it is moving it will continue to move with constant velocity.

equivalence of vectors: [p. *69*] Let $\boldsymbol{a} = a_1\boldsymbol{i} + a_2\boldsymbol{j} + a_3\boldsymbol{k}$, and $\boldsymbol{b} = b_1\boldsymbol{i} + b_2\boldsymbol{j} + b_3\boldsymbol{k}$. If $\boldsymbol{a} = \boldsymbol{b}$ then $a_1 = b_1$, $a_2 = b_2$ and $a_3 = b_3$.

Euler's formula: [p. *372*] If $\dfrac{dy}{dx} = g(x)$, $x_0 = a$ and $y_0 = b$, then $x_{n+1} = x_n + h$ and $y_{n+1} = y_n + h\,g(x_n)$.

Euler's method: [p. *371*] Euler's method uses the linear approximation method from calculus to solve differential equations.

F

fundamental theorem of algebra: [p. *164*] In the field of complex numbers, every polynomial equation of the form

$a_nx^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0 = 0$, where $a_0, a_1, \ldots, a_n \in C$, $a_n \neq 0$

has exactly n roots, some of which may be repeated.

fundamental theorem of integral calculus: [p. *300*] $\int_a^b f(x)dx = G(b) - G(a)$ where G is an antiderivative of f.

G

g: [p. *483*] The acceleration of a particle owing to gravity. Close to Earth's surface, the value of g is approximately 9.8 m/s^2.

general antiderivative (or indefinite integral): [p. *265*] The set of all antiderivatives for a given function, e.g. $\int 2x\,dx = x^2 + c$.

In general, F and f are functions such that, if $F(x)$ is an antiderivative of $f(x)$, then $F'(x) = f(x)$ and $\int f(x)dx = F(x) + c$, where c is an arbitrary real number.

geometric convergent series: [p. *27*] A geometric series with a common ratio $-1 < r < 1$ which will approach a limiting value as successive terms are added to it, i.e. as

$n \to \infty$, $S_n \to \dfrac{a}{1-r}$, where $a = t_1$ and $r = \dfrac{t_n}{t_{n-1}}$

geometric sequence: [p. *26*] A sequence in which each successive term is found by multiplying the previous term by a fixed value, e.g., 2, 6, 18, 54, . . .

A geometric sequence can be defined by an iterative equation of the form:

$t_n = rt_{n-1}$, where r is the common ratio.

The nth term of the sequence can be found using:

$t_n = ar^{n-1}$, where $a = t_1$

geometric series: [p. *26*] The sum of the terms in a geometric sequence.

The sum of the first n terms, S_n, is given by the rule:

$S_n = \dfrac{a(r^n - 1)}{r - 1}$, where $a = t_1$ and $r = \dfrac{t_n}{t_{n-1}}$

gradient function (or derivative function): [p. *205*] The derivative of a function f is denoted by f' and the rule for f' is defined by

$$f'(x) = \lim_{h\to 0}\frac{f(x+h) - f(x)}{h}$$

H

hyperbola, general cartesian equation of: [p. *38*]

$\dfrac{(x-h)^2}{a^2} - \dfrac{(y-k)^2}{b^2} = 1$ or

$\dfrac{(y-k)^2}{b^2} - \dfrac{(x-h)^2}{a^2} = 1$

The **centre** of the hyperbola is the point (h, k), and the equations of the asymptotes are:

$$y - k = \pm\frac{b}{a}(x - h)$$

I

identities: [p. *109*]

$$\begin{aligned} 1 + \tan^2 x &= \sec^2 x \\ \sin 2\theta &= 2 \sin\theta \cos\theta \\ \cos 2x &= 2\cos^2 x - 1 \\ &= 1 - 2\sin^2 x \\ &= \cos^2 x - \sin^2 x \end{aligned}$$

imaginary part of a complex number: [p. *142*] Im(z) is a function which defines the value of the imaginary component of $z = a + bi$, i.e. Im(z) = b

indefinite integral (or general antiderivative): [p. *265*] The set of all antiderivatives for a given function, e.g. $\int 2x\,dx = x^2 + c$.

In general, F and f are functions such that, if $F(x)$ is an antiderivative of $f(x)$, then:

$F'(x) = f(x)$ and $\int f(x)dx = F(x) + c$, where c is an arbitrary real number.

infinite geometric series (or sum to infinity), S_∞: [p. *27*] $S_\infty = \dfrac{a}{1-r}$, where $a = t_1$ and $r = \dfrac{t_n}{t_{n-1}}$

integrand: [p. *266*] In the expression $\int_a^b f(x)\,dx$, the function to be integrated, f, is called the integrand.

integration (or antidifferentiation): [p. *264*] The process of finding a function from its derivative.

inverse cosine function: [p. *120*] $\cos^{-1}: [-1, 1] \to R$, $\cos^{-1} x = y$, where $\cos y = x$, $y \in [0, \pi]$

inverse sine function: [p. *119*] $\sin^{-1}: [-1, 1] \to R$, $\sin^{-1} x = y$, where $\sin y = x$, $y \in \left[-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right]$

inverse tangent function: [p. *121*] $\tan^{-1}: R \to R$, $\tan^{-1} x = y$, where $\tan y = x$, $y \in \left(-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right)$

iterative rule (or difference equation): [p. *24*] A rule which enables each subsequent term of a sequence to be found using the previous term or terms, e.g. $t_1 = 1$, $t_n = t_{n-1} + 2$.

K

kilogram weight, kg wt: [p. *483*] A unit of force. If a body has mass of one kilogram then the gravitational force acting on this body is one kilogram weight.

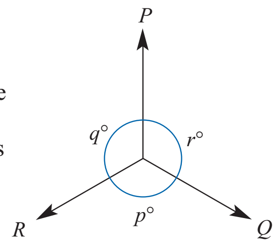
kinematics (under constant acceleration) formulas: [p. *403*]

$$\begin{aligned} v &= u + at \\ s &= ut + \tfrac{1}{2}at^2 \\ v^2 &= u^2 + 2as \\ s &= \tfrac{1}{2}(u + v)t \end{aligned}$$

L

Lami's theorem: [p. *515*] Lami's theorem is a trigonometrically based identity which simplifies problems involving three forces acting on a particle in equilibrium when the angles between the forces are known.

$$\frac{P}{\sin p^\circ} = \frac{Q}{\sin q^\circ} = \frac{R}{\sin r^\circ}$$



limiting equilibrium: [p. *519*] A particle in equilibrium on the point of motion.

limiting (or sliding) friction: [p. *519*] The frictional force $\boldsymbol{F}_{\max}$ (or $\boldsymbol{F}_R$) of a particle **moving** or **on the point of moving** on a surface is given by:

$$\boldsymbol{F}_{\max} = \mu \boldsymbol{R}$$

where $\boldsymbol{R}$ is the normal reaction force and μ is the coefficient of friction.

linear approximation formula: [p. *371*]

$$f(x + h) \approx f(x) + hf'(x)$$

linear dependence: [p. *63*] A set of vectors is said to be **linearly dependent** if one of its members can be expressed as a linear combination of one or more of the other vectors. For example, the set of vectors $\boldsymbol{a}$, $\boldsymbol{b}$ and $\boldsymbol{c}$ are linearly dependent if there exist real numbers k, l and m, not all zero, such that $k\boldsymbol{a} + l\boldsymbol{b} + m\boldsymbol{c} = 0$.

Generally any set of 3 or more two-dimensional vectors will be linearly dependent.

linear independence: [p. *63*] A set of vectors is said to be **linearly independent** if it is not linearly dependent. The vectors $\boldsymbol{a}$, $\boldsymbol{b}$ and $\boldsymbol{c}$ are linearly independent if the solution of the equation $k\boldsymbol{a} + l\boldsymbol{b} + m\boldsymbol{c} = \boldsymbol{0}$ is uniquely represented by $k = l = m = 0$.

local maximum stationary point: [p. *228*] If $f'(a) = 0$ and $f''(a) < 0$ then the point $(a, f(a))$ is a local maximum as the curve is concave down.

local minimum stationary point: [p. *228*] If $f'(a) = 0$ and $f''(a) > 0$ then the point $(a, f(a))$ is a local minimum as the curve is concave up.

locus (plural loci): [p. *177*] A set of points which satisfies a given condition, e.g., the locus which satisfies the equation $|z+3| = 2|z-i|$ is a circle with centre $(1, \frac{4}{3})$ and radius $\frac{2\sqrt{10}}{3}$.

lower limit of integration: [p. *266*] In the expression $\int_a^b f(x)\,dx$, the number a is called the lower limit of integration.

M

magnitude of a vector: [p. *57*] The length of a directed line segment corresponding to the vector. If $\overrightarrow{AB}$ is represented by the vector $x\boldsymbol{i} + y\boldsymbol{j}$, then the magnitude, $|\overrightarrow{AB}|$, is equal to $\sqrt{x^2+y^2}$.

If $\overrightarrow{AB}$ is represented by the vector $x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k}$, then the magnitude, $|\overrightarrow{AB}|$, is equal to $\sqrt{x^2+y^2+z^2}$.

mass: [p. *483*] The mass of an object is the amount of matter it contains. Mass is not the same as weight.

maximum friction: [p. *520*] The frictional force, $\boldsymbol{F}_R$, satisfies $\boldsymbol{0} \le \boldsymbol{F}_R \le \mu\boldsymbol{R}$, where μ is the coefficient of friction and $\boldsymbol{R}$ is the normal reaction force. Friction force has a maximum value, $\boldsymbol{F}_{\max} = \mu\boldsymbol{R}$.

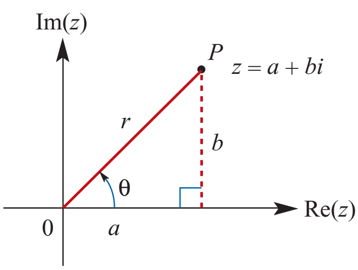
modulus of a complex number, $|z|$: [p. *150*] The distance of the complex number from the origin, also known as the magnitude or absolute value of z.

If $z_1 = a + bi$, then $|z| = \sqrt{a^2+b^2}$

modulus, properties of: [p. *160*]

- The modulus of the product of two complex numbers is the product of their moduli, i.e. $|z_1z_2| = |z_1||z_2|$
- The modulus of the quotient of two complex numbers is the quotient of their moduli, i.e. $\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}$

modulus–argument (or polar) form of a complex number: [pp. *153–4*] A complex number expressed in the form r cis θ, represented by the ordered pair $[r, \theta]$, where r is the modulus of z and θ is an argument of z.



momentum: [p. *492*] The **momentum** of a particle is defined as the product of its mass and velocity.

Momentum can be considered as the fundamental quantity of motion.

multiplication of a complex number by a real number: [p. *159*] If $z = a + bi$, then $kz = ka + kbi$, $k \in R$.
If $z = r$ cis θ, then

$$kz = \begin{cases} kr \text{ cis } \theta & k > 0 \\ kr \text{ cis } (\theta + \pi) & k < 0 \text{ and } -\pi < \theta < 0 \\ kr \text{ cis } (\theta - \pi) & k < 0 \text{ and } 0 < \theta \le \pi \end{cases}$$

multiplication of a complex number by i: [p. *148*] Geometrically, a 90° rotation of the complex number about the origin in an anticlockwise direction, i.e. if $z_1 = a + bi$, then $iz_1 = i(a+bi) = -b + ai$

multiplication of a vector by a scalar: [p. *58*] If $\boldsymbol{a} = a_1\boldsymbol{i} + a_2\boldsymbol{j} + a_3\boldsymbol{k}$, then $m\boldsymbol{a} = ma_1\boldsymbol{i} + ma_2\boldsymbol{j} + ma_3\boldsymbol{k}$, $m \in R$.

multiplication of complex numbers: [pp. *147, 159*] If $z_1 = a + bi$ and $z_2 = c + di$, then $z_1z_2 = (ac - bd) + (ad + bc)i$.

If $z_1 = r_1$ cis θ_1 and $z_2 = r_2$ cis θ_2, then $z_1z_2 = r_1r_2$ cis $(\theta_1 + \theta_2)$.

Geometrically, the effect of multiplying z_1 by z_2 is to produce an enlargement of Oz_1, where O is the origin, by a factor r_2 and an anticlockwise turn through an angle θ_2 about the origin.

N

newton, N: [p. *483*] A unit of force.
One newton $= 1$ kgm/s^2.

Newton's first law of motion: [p. *493*] A particle remains stationary, or in uniform straight line motion (i.e. in a straight line with constant velocity), if the resultant force is zero.

Newton's law of cooling: [p. *352*] The rate at which a body cools is proportional to the difference between its temperature and that of its immediate surroundings.

Newton's second law of motion: [p. *493*] A particle acted on by forces whose resultant is not zero will move in such a way that the rate of change of its momentum with respect to time will at any instant be proportional to the resultant force, i.e. $\boldsymbol{F} = m\boldsymbol{a}$.

Newton's third law of motion: [p. *493*] If one particle, A, exerts a force on a second particle, B, then B exerts a collinear force of equal magnitude and opposite direction on A.

normal reaction force: [p. *494*] If a particle lies on a smooth surface, it exerts a force on the surface perpendicular to the surface. The surface then exerts a force $\boldsymbol{R}$ N on the particle, which acts at right angles to the surface and is called the normal reaction force.

O

operator notation for differentiation: [p. *208*] A notation which emphasises that differentiation is an operation on an expression, e.g.

$$\frac{d}{dx}(x^2 + 5x + 3) = 2x + 5$$

P

parametric equations: [p. *43*] A pair of equations expressing x and y in terms of a third variable (or parameter). The following pairs of parametric equations define, respectively, a circle, ellipse and hyperbola, each with centre the origin:

$x = a\cos t$ and $y = a\sin t$ (radius $= a$)
$x = a\cos t$ and $y = b\sin t$
$x = a\sec t$ and $y = b\tan t$

particle model: [p. *483*] This means that an object is considered as a point. This can be done when the size (dimension) of the object can be neglected in comparison with other lengths in the problem being considered, or when rotational motion effects can be ignored.

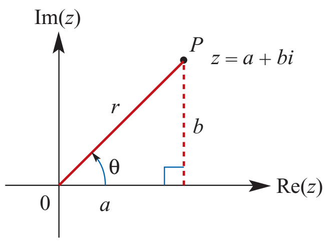
period of a function: [p. *4*] The period of a function f with domain R is a positive number a (the smallest such number) such that $f(x + a) = f(x)$, e.g. the period of the sine function is 2π as $\sin(x + 2\pi) = \sin x$.

For functions of the form $y = a\cos(nx + \xi) + c$ or $y = a\sin(nx + \xi) + c$ the period is given by $\frac{2\pi}{n}$.

For functions of the form $y = a\tan(nx + \xi) + c$ the period is given by $\frac{\pi}{n}$.

point of inflexion (or point of inflection): [p. *223*] A **point of inflexion** is where the curve changes from concave down to concave up, or concave up to concave down. At a point of inflexion the second derivative has value zero. A point of inflexion of a graph occurs at $x = x_0$ if $f''(x_0) = 0$ and $f''(x_0 + \varepsilon)$ and $f''(x_0 - \varepsilon)$ have different signs.

polar (or modulus-argument) form of a complex number: [pp. *153–4*] A complex number expressed in the form $r\operatorname{cis}\theta$, represented by the ordered pair $[r, \theta]$, where r is the modulus of z and θ is an argument of z.



position: [p. *366*] The **position** of a particle moving in a straight line is determined by its distance from a fixed point O on the line, called the origin, and whether it is to the right or left of O. Conventionally the direction to the right of the origin is considered to be positive.

position vector: [p. *61*] A position vector, $\overrightarrow{OP}$, indicates the position in space of the point P relative to the origin O.

product rule: [p. *206*] For

$f(x) = g(x)h(x)$, $f'(x) = g'(x)h(x) + g(x)h'(x)$.

For $y = uv$, where u and v are functions of x,

$$\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}.$$

Pythagoras' theorem: [p. *13*] For a right-angled triangle, the square of the hypotenuse is equal to the sum of the squares of the other two sides, i.e. $(\text{hyp})^2 = (\text{opp})^2 + (\text{adj})^2$

Pythagorean identity: [p. *6*] $\cos^2\theta + \sin^2\theta = 1$

Q

quadratic formula: [p. *170*] An equation of the form $az^2 + bz + c = 0$ may be solved using the quadratic formula:

$$z = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

quotient rule: [p. *206*] For $f(x) = \frac{g(x)}{h(x)}$,

$$f'(x) = \frac{g'(x)h(x) - g(x)h'(x)}{[h(x)]^2}$$

For $y = \frac{u}{v}$, where u and v are functions of x,

$$\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

R

radian: [p. *3*] One radian (written 1^c) is the angle **subtended** at the centre of the unit circle by an arc of length 1 unit.

radioactive decay: [p. *352*] The rate at which a radioactive substance decays is proportional to the mass of the substance remaining.

rational functions: [p. *239*] Functions which have a rule of the form:

$$f(x) = \frac{P(x)}{Q(x)},$$

where $P(x)$ and $Q(x)$ are polynomials

real part of a complex number: [p. *142*] $\text{Re}(z)$ is a function which defines the real component of $z = a + bi$, i.e. $\text{Re}(z) = a$

reciprocal circular functions: [p. *105*] The cosecant, secant and cotangent functions.

reciprocal functions: [p. *244*] $f(x)$ is the reciprocal function of $P(x)$ if $f(x) = \dfrac{1}{P(x)}$

reciprocal functions, properties of: [p. *244*]

- x-axis intercepts of the function determine the position of the asymptotes for the reciprocal of the function
- the reciprocal of a positive number is positive. The reciprocal of a negative number is negative
- a graph and its reciprocal will intersect if the y coordinate is 1 or -1
- local maximums of the function produce local minimums for the reciprocal
- local minimums of the function produce local maximums for the reciprocal
- If $g(x) = \dfrac{1}{f(x)}$ then $g'(x) = -\dfrac{f'(x)}{f(x)^2}$. Therefore at any given point the gradient of the reciprocal function is opposite in sign to the original function.

related rates: [p. *234*] In the chain rule $\dfrac{dy}{dt} = \dfrac{dy}{dx}\dfrac{dx}{dt}$, $\dfrac{dy}{dx}$ and $\dfrac{dx}{dt}$ are related rates.

restricted cosine function: [p. *120*]

$f: [0, \pi] \to R,\ f(x) = \cos x$

restricted sine function: [p. *120*]

$f: \left[-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right] \to R,\ f(x) = \sin x$

restricted tangent function: [p. *121*]

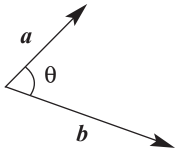
$f: \left(-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right) \to R,\ f(x) = \tan x$

resultant force: [p. *483*] The vector sum of the forces acting at a point.

S

scalar product (or dot product), *a.b*: [p. *80*]

$\boldsymbol{a}.\boldsymbol{b} = |\boldsymbol{a}||\boldsymbol{b}|\cos\theta$.



For vectors $\boldsymbol{a} = a_1\boldsymbol{i} + a_2\boldsymbol{j} + a_3\boldsymbol{k}$ and $\boldsymbol{b} = b_1\boldsymbol{i} + b_2\boldsymbol{j} + b_3\boldsymbol{k}$, $\boldsymbol{a}.\boldsymbol{b} = a_1b_1 + a_2b_2 + a_3b_3$

scalar product, properties of: [p. *81*]

- $\boldsymbol{a}.\boldsymbol{b} = \boldsymbol{b}.\boldsymbol{a}$
- $k(\boldsymbol{a}.\boldsymbol{b}) = (k\boldsymbol{a}).\boldsymbol{b} = \boldsymbol{a}.(k\boldsymbol{b})$
- $\boldsymbol{a}.\boldsymbol{0} = 0$
- $\boldsymbol{a}.(\boldsymbol{b}+\boldsymbol{c}) = \boldsymbol{a}.\boldsymbol{b} + \boldsymbol{a}.\boldsymbol{c}$
- $\boldsymbol{a}.\boldsymbol{b} = 0$ implies $\boldsymbol{a}$ is perpendicular to $\boldsymbol{b}$ or $\boldsymbol{a} = \boldsymbol{0}$ or $\boldsymbol{b} = \boldsymbol{0}$
- $\boldsymbol{a}.\boldsymbol{a} = |\boldsymbol{a}|^2$
- $\boldsymbol{a}.\boldsymbol{b} = |\boldsymbol{a}||\boldsymbol{b}|$ if $\boldsymbol{a}$ and $\boldsymbol{b}$ are parallel and in the same direction
  $= -|\boldsymbol{a}||\boldsymbol{b}|$ if $\boldsymbol{a}$ and $\boldsymbol{b}$ are in opposite directions.

scalar quantity: [p. *389*] A quantity determined only by its magnitude, e.g. distance, time, length, mass.

scalar resolute of *a* in the direction of *b*: [p. *86*]

$(\boldsymbol{a}.\hat{\boldsymbol{b}})$ or $\dfrac{\boldsymbol{a}.\boldsymbol{b}}{|\boldsymbol{b}|}$, the 'signed' length of the vector resolute of $\boldsymbol{a}$ in the direction of $\boldsymbol{b}$.

secant function: [p. *106*]

$\sec\theta = \dfrac{1}{\cos\theta}$ provided $\cos\theta \neq 0$

second derivative: [p. *220*] The second derivative of a function f with rule $f(x)$ is denoted by f'' with rule $f''(x)$. In Leibnitz' notation the second derivative is denoted by $\dfrac{d^2y}{dx^2}$.

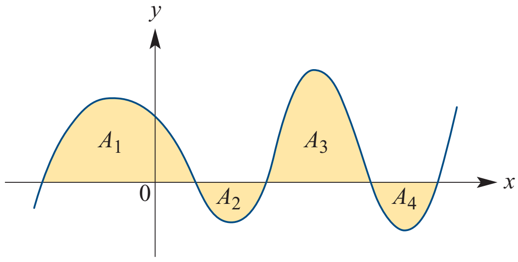
sequence: [p. *24*] The following are examples of sequences of numbers:

A	$1, 3, 5, 7, 9, \ldots$	i.e. $t_1 = 1$, $t_2 = 3$, $t_3 = 5, \ldots$
B	$0.1, 0.11, 0.111, 0.1111, \ldots$	i.e. $t_1 = 0.1$, $t_2 = 0.11$, $t_3 = 0.111, \ldots$
C	$\frac{1}{3}, \frac{1}{9}, \frac{1}{27}, \frac{1}{81}, \ldots$	i.e. $t_1 = \frac{1}{3}$, $t_2 = \frac{1}{9}$, $t_3 = \frac{1}{27}, \ldots$
D	$10, 7, 4, 1, -2, \ldots$	i.e. $t_1 = 10$, $t_2 = 7$, $t_3 = 4, \ldots$
E	$0.6, 1.7, 2.8, 3.9, \ldots$	i.e. $t_1 = 0.6$, $t_2 = 1.7$, $t_3 = 2.8, \ldots$

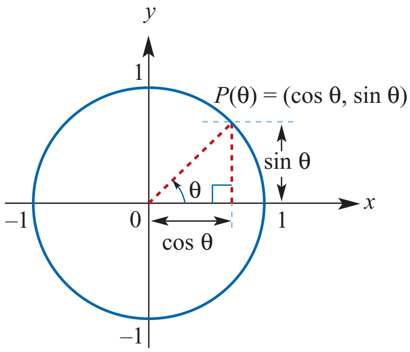
Note each sequence is an ordered set of numbers.

series: [p. *24*] The sum of the terms in a sequence.

signed area: [p. *299*] The signed area of the shaded region is $A_1 - A_2 + A_3 - A_4$

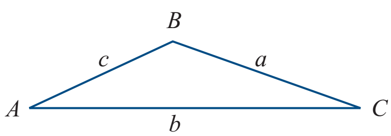


sine: [p. *2*] Sine θ, or sin θ, defined as the y coordinate of the point P on the unit circle where OP forms an angle of θ radians measured anticlockwise from the positive ray of the x axis.



sine rule: [p. *15*] For triangle ABC

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$



The sine rule is used to find unknown quantities in a triangle when either one side and two angles are given, or two sides and a non-included angle are given.

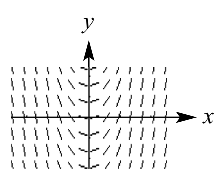
sliding (or limiting) friction: [p. *495*] The frictional force $\boldsymbol{F}_R$ (or $\boldsymbol{F}_{\max}$) of a particle **moving** or **on the point of moving** on a surface is given by:

$$\boldsymbol{F}_R = \mu \boldsymbol{R}$$

where $\boldsymbol{R}$ is the normal reaction force and μ is the coefficient of friction.

Friction acts in the opposite direction to the velocity of the particle.

slope field (or direction field) of a differential equation: [p. *378*] The slope field of a differential equation, $\frac{dy}{dx} = f(x)$, assigns to each point $P(x, y)$ in the plane, with x in the domain of f, the number which is the slope (gradient) of the solution curve through P. A slope field can be represented in a graph.



solid of revolution: [p. *321*] The solid formed by rotating a region about a line.

speed: [p. *390*] The magnitude of velocity.

speed, average: [p. *390*] The **average speed** of a particle for a time interval $[t_1, t_2]$ is equal to

$$\frac{\text{distance travelled}}{t_2 - t_1}$$

subtraction of complex numbers: [p. *144*]

If $\quad z_1 = a + bi$ and $z_2 = c + di$

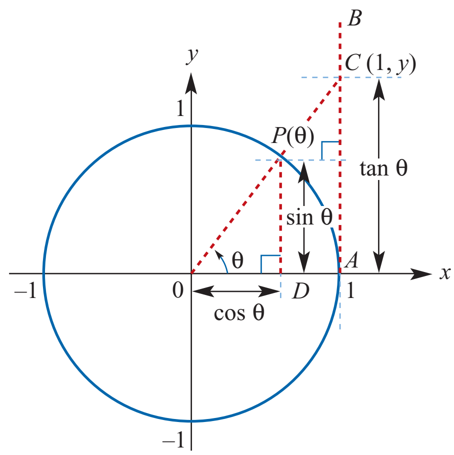
then $\quad z_1 - z_2 = (a + bi) - (c + di)$
$\qquad\qquad\quad = (a - c) + (b - d)i.$

subtraction of vectors: [p. *59*] Let $\boldsymbol{a} = a_1\boldsymbol{i} + a_2\boldsymbol{j} + a_3\boldsymbol{k}$, and $\boldsymbol{b} = b_1\boldsymbol{i} + b_2\boldsymbol{j} + b_3\boldsymbol{k}$. Then $\boldsymbol{a} - \boldsymbol{b} = (a_1 - b_1)\boldsymbol{i} + (a_2 - b_2)\boldsymbol{j} + (a_3 - b_3)\boldsymbol{k}$.

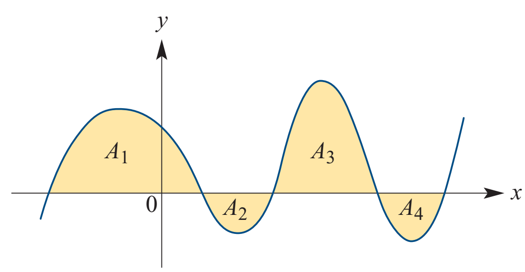
sum to infinity (or infinite geometric series), S_∞: [p. *27*] $S_\infty = \frac{a}{1 - r}$, where $a = t_1$ and $r = \frac{t_n}{t_{n-1}}$

T

tangent function: [p. *2*] If a tangent to the unit circle, at A, is drawn then the y coordinate of C, the point of intersection of the extension of OP and the tangent is called tangent θ, or tan θ.



total area: [p. *299*] The total area of the shaded region is $A_1 + A_2 + A_3 + A_4$



U

unit vector: [p. *68*] A vector of magnitude 1. For a given vector $\boldsymbol{a}$ the unit vector with the same direction as $\boldsymbol{a}$ is denoted by $\hat{\boldsymbol{a}}$ and $\hat{\boldsymbol{a}} = \frac{\boldsymbol{a}}{|\boldsymbol{a}|}$.

$\boldsymbol{i}, \boldsymbol{j}$ and $\boldsymbol{k}$ are unit vectors in the positive directions of the x, y and z axes respectively.

upper limit of integration: [p. *266*] In the expression $\int_a^b f(x)\,dx$, the number b is called the upper limit of integration.

V

vector function (or vector equation): [p. *453*] A vector that is a function of a variable or parameter, e.g. $\boldsymbol{r}(t) = x(t)\boldsymbol{i} + y(t)\boldsymbol{j}$, where $x(t)$ and $y(t)$ are parametric equations.

vector quantity: [p. *389*] A quantity determined by its magnitude and direction, e.g. displacement, velocity, acceleration, force.

vector resolute of *a* in the direction of *b*: [p. *85*]

$$\frac{\boldsymbol{a}.\boldsymbol{b}}{\boldsymbol{b}.\boldsymbol{b}}\boldsymbol{b} = \frac{\boldsymbol{a}.\boldsymbol{b}}{|\boldsymbol{b}|^2}\boldsymbol{b} = (\boldsymbol{a}.\hat{\boldsymbol{b}}).\hat{\boldsymbol{b}}$$

vector resolute of *a* perpendicular to *b*: [p. *85*]

$$\left(\boldsymbol{a} - \frac{\boldsymbol{a}.\boldsymbol{b}}{\boldsymbol{b}.\boldsymbol{b}}\boldsymbol{b}\right)$$

vectors, properties of: [p. *61*]

- $\boldsymbol{a}+\boldsymbol{b}=\boldsymbol{b}+\boldsymbol{a}$ — commutative law for vector addition
- $(\boldsymbol{a}+\boldsymbol{b})+\boldsymbol{c}=\boldsymbol{a}+(\boldsymbol{b}+\boldsymbol{c})$ — associative law for vector addition
- $\boldsymbol{a}+\boldsymbol{0}=\boldsymbol{a}$ — zero vector
- $\boldsymbol{a}+-\boldsymbol{a}=\boldsymbol{0}$ — $-\boldsymbol{a}$ is the opposite or inverse vector
- $m(\boldsymbol{a}+\boldsymbol{b})=m\boldsymbol{a}+m\boldsymbol{b}$ — distributive law where $m \in R$
- $\boldsymbol{a}$ is parallel to $\boldsymbol{b}$ if there exists $k \in R\backslash\{0\}$ such that $\boldsymbol{a} = k\boldsymbol{b}$

velocity: [p. *389*] The **velocity** of a particle is defined as the rate of change of its position with respect to time.

velocity, average: [p. *389*] The **average velocity** of a particle for the time interval $[t_1, t_2]$ is given by $\frac{x_2 - x_1}{t_2 - t_1}$, where x_1 is the position of the particle at t_1, and x_2 is the position of the particle at t_2.

velocity, instantaneous: [p. *390*] The **instantaneous velocity** of a particle, $v = \frac{dx}{dt}$ where x is a function of time, specifies the rate of change at a given instant in time.

velocity–time graphs: [p. *393*] These graphs present information about

- acceleration (gradient)
- velocity (ordinates)
- displacement (signed area or definite integral)
- distance travelled (area 'under' curve)

volume of solids of revolution: [p. *322*] For **rotation about the *x* axis**, if the region to be rotated is bounded by the curve with equation $y = f(x)$ and the lines $x = a$ and $x = b$ and the x axis, then:

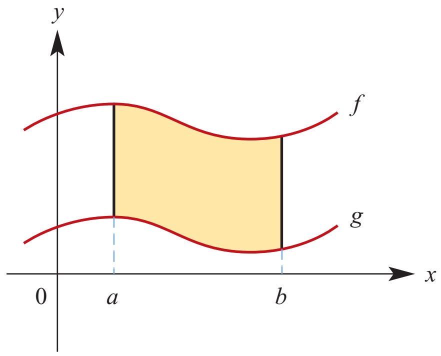
$$V = \int_{x=a}^{x=b} \pi y^2 dx = \int_a^b \pi(f(x))^2 dx$$

For **rotation about the *y* axis**, if the region is bound by the curve with equation $x = f(y)$ and the lines $y = a$ and $y = b$ then:

$$V = \int_{y=b}^{y=a} \pi x^2 dy$$

For **regions not bounded by the *x* axis**, if the shaded region in the diagram is rotated around the x axis then the volume, V, is given by:

$$V = \pi \int_a^b [f(x)]^2 - [g(x)]^2 dx$$



W

weight: [p. *493*] Any mass of m kg, on the Earth's surface, has a force of m kg wt or mg newtons acting on it directed towards the centre of the earth. This force is known as the **weight**.

Z

zero vector, 0: [p. *59*] For three dimensions, the zero vector is $0\boldsymbol{i} + 0\boldsymbol{j} + 0\boldsymbol{k}$.

Answers

Chapter 1

Exercise 1A

1 a i 4π ii 3π iii $-\frac{5\pi}{2}$

iv $\frac{\pi}{12}$ v $-\frac{\pi}{18}$ vi $-\frac{7\pi}{4}$

b i $225°$ ii $-120°$ iii $105°$

iv $-330°$ v $260°$ vi $-165°$

2 a i 0.12^c ii -1.75^c iii -0.44^c

iv 0.89^c v 3.60^c vi -7.16^c

b i $97.40°$ ii $-49.85°$ iii $160.43°$

iv $5.73°$ v $-171.89°$ vi $-509.93°$

3 a $\frac{\sqrt{3}}{2}$ **b** $-\frac{\sqrt{2}}{2}$ **c** $\frac{1}{2}$

d $-\frac{\sqrt{2}}{2}$ **e** $\frac{\sqrt{2}}{2}$ **f** $-\frac{\sqrt{3}}{2}$

g $-\frac{\sqrt{3}}{2}$ **h** $-\frac{\sqrt{3}}{2}$ **i** $\frac{1}{2}$

4 a $\frac{\sqrt{2}}{2}$ **b** $\frac{1}{2}$ **c** $\frac{\sqrt{3}}{2}$

d $-\frac{1}{2}$ **e** $\frac{\sqrt{2}}{2}$ **f** $\frac{\sqrt{3}}{2}$

5 a $-\frac{\sqrt{3}}{2}$ **b** $-\frac{\sqrt{3}}{3}$

6 a $-\frac{\sqrt{51}}{10}$ **b** $\frac{\sqrt{51}}{7}$

7 a $-\frac{\sqrt{3}}{2}$ **b** $\frac{\sqrt{3}}{3}$

8 a $\frac{\sqrt{91}}{10}$ **b** $\frac{-3\sqrt{91}}{91}$

9 a $\frac{4\pi}{3}, \frac{5\pi}{3}$ **b** $\frac{\pi}{6}, \frac{\pi}{3}, \frac{7\pi}{6}, \frac{4\pi}{3}$

c $\frac{\pi}{3}, \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{5\pi}{3}$ **d** $\frac{5\pi}{6}, \frac{3\pi}{2}$

e $0, \frac{\pi}{3}, \pi, \frac{4\pi}{3}, 2\pi$ **f** $\frac{\pi}{2}, \frac{2\pi}{3}, \frac{3\pi}{2}, \frac{5\pi}{3}$

10 a 1 **b** $\sqrt{3}$ **c** $\frac{\sqrt{3}}{3}$ **d** $\sqrt{3}$

11 a $\frac{-\sqrt{17}}{17}$ **b** $\frac{-4\sqrt{17}}{17}$ **c** $\frac{-1}{4}$ **d** $\frac{-1}{4}$

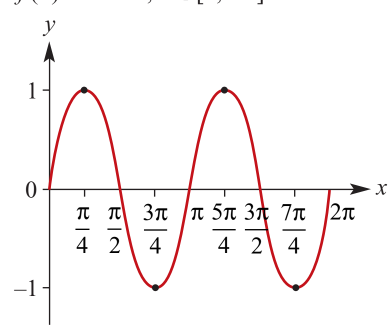
12 a $\frac{\sqrt{21}}{7}$ **b** $\frac{-2\sqrt{7}}{7}$ **c** $\frac{\sqrt{3}}{2}$ **d** $\frac{-\sqrt{3}}{2}$

13 a $\frac{2\pi}{3}, \frac{5\pi}{3}$

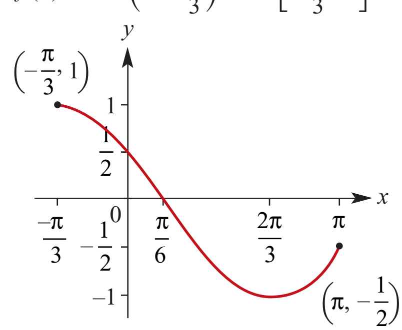
b $\frac{\pi}{9}, \frac{4\pi}{9}, \frac{7\pi}{9}, \frac{10\pi}{9}, \frac{13\pi}{9}, \frac{16\pi}{9}$

c $\frac{3\pi}{2}$ **d** $\frac{\pi}{8}, \frac{5\pi}{8}, \frac{9\pi}{8}, \frac{13\pi}{8}$

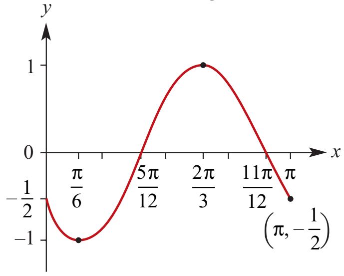
14 a $f(x) = \sin 2x, x \in [0, 2\pi]$



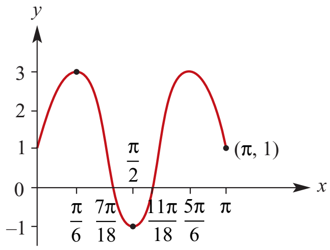
b $f(x) = \cos\left(x + \frac{\pi}{3}\right), x \in \left[-\frac{\pi}{3}, \pi\right]$



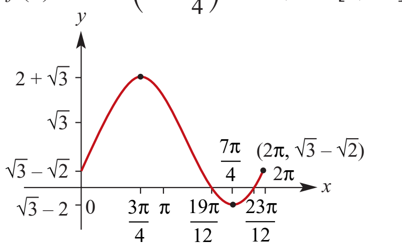
c $f(x) = \cos\left(2\left(x + \frac{\pi}{3}\right)\right), x \in [0, \pi]$



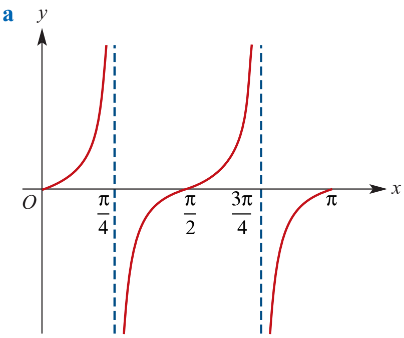
d $f(x) = 2\sin(3x) + 1, x \in [0, \pi]$



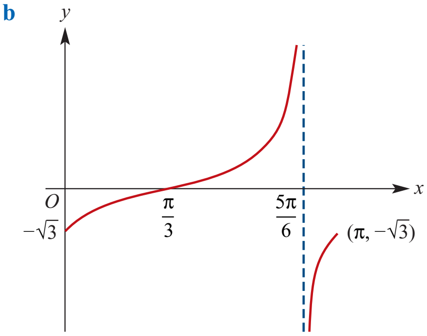
e $f(x) = 2\sin\left(x - \frac{\pi}{4}\right) + \sqrt{3}, x \in [0, 2\pi]$



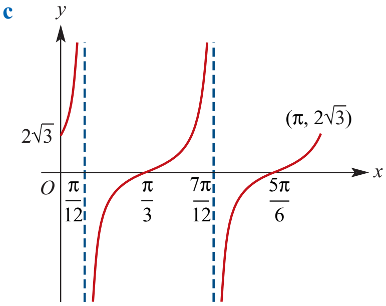
15 a



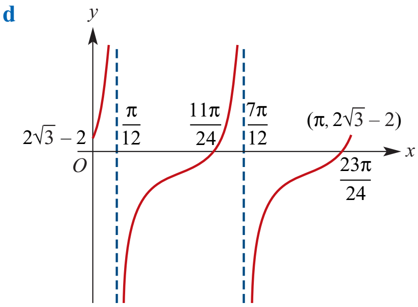
b



c



d



Exercise 1B

1 a $\tan x° = \frac{8}{5}$, $\cos x° = \frac{5\sqrt{89}}{89}$, $\sin x° = \frac{8\sqrt{89}}{89}$

b $\tan x° = \frac{5\sqrt{6}}{12}$, $\cos x° = \frac{2\sqrt{6}}{7}$, $\sin x° = \frac{5}{7}$

c $\tan x° = \frac{4\sqrt{2}}{7}$, $\cos x° = \frac{7}{9}$, $\sin x° = \frac{4\sqrt{2}}{9}$

2 a 6 **b** $6\sqrt{2}$ **c** $\frac{20\sqrt{3}}{3}$

3 a $a = \sqrt{26}$

b $a = \sqrt{5}, b = \sqrt{6}, c = \sqrt{7}$ **c** $a = 1$

d $a = 2, b = \sqrt{3}, c = \sqrt{3}, d = \sqrt{6}$

4 a $a = \sqrt{2}, w = \frac{3 - \sqrt{3}}{2}, x = \frac{1 + \sqrt{3}}{2}$,

$y = \frac{\sqrt{3} - 1}{2}, z = 15$

b $\sin(15°) = \frac{\sqrt{6} - \sqrt{2}}{4}$,

$\cos(15°) = \frac{\sqrt{2} + \sqrt{6}}{4}$, $\tan(15°) = 2 - \sqrt{3}$

c $\sin(75°) = \frac{\sqrt{2} + \sqrt{6}}{4}$,

$\cos(75°) = \frac{\sqrt{6} - \sqrt{2}}{4}$,

$\tan(75°) = \frac{1}{2 - \sqrt{3}} = 2 + \sqrt{3}$

Exercise 1C

1 a 11.67 cm **b** 9.62 cm
2 a 7.15 cm **b** 50.43°
3 16.71 cm
4 a 58.08°, 121.92° **b** 10.01 cm, 4.09 cm

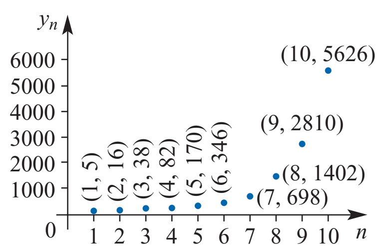
5 a 6.71 cm
b 121.33° (acute angle produces inconsistent triangle)
6 $6\sqrt{6}$ cm **7** $\sqrt{7}$ cm **8** 30.10
9 $5\sqrt{3} \pm \sqrt{39}$ **10 a** 54.90 **b** 100.95

Exercise 1D

1 $a = 82, x = 30, y = 30, z = 82$
2 a 75° **b** 62° **c** 100° **d** 43°
3 $a = 40, b = 90, c = 50$
4 a 150° **b** 15°
5 $a = 69, b = 47, c = 75, d = 28, e = 36$
6 $a - b + c + 180$ **7** $x = 80, y = 140$
8 $a = 60, b = 80, c = 60, d = 40$
9 $x = 70, y = 110$ **10** $x = 30, y = 60$

Exercise 1E

1 $t_2 = 17, t_3 = 50, t_8 = 12\,029$
2 $y_2 = 16$
$y_3 = 38$
$y_{10} = 5626$

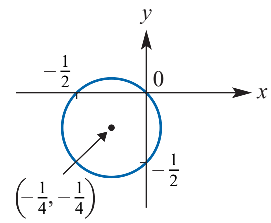


3 1, 1, 2, 3, 5, 8, 13, 21, 34, 55
4 210 **5** $\frac{3}{4}$
6 a 20 **b** $\frac{4}{5}$ **c** $\frac{4^{10}}{5^7}$
7 −9840 **8** $a(2 + \sqrt{2})$
9 a $4\left[1 - \left(\frac{3}{4}\right)^{10}\right]$
b i $-2 < x < 2$ **ii** $\pm 2^{\frac{9}{10}}$
10 a $\frac{1}{1 - \sin\theta}$
b $\frac{\pi}{6} + 2k\pi, \frac{5\pi}{6} + 2k\pi, k \in Z$

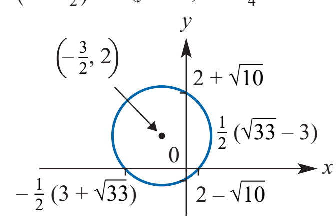
Exercise 1F

1 a $(x - 2)^2 + (y - 3)^2 = 1$
b $(x + 3)^2 + (y - 4)^2 = 25$
c $x^2 + (y + 5)^2 = 25$
d $(x - 3)^2 + y^2 = 2$
2 a centre (−2, 3) radius 1
b centre (1, 2) radius 2
c centre $\left(\frac{3}{2}, 0\right)$ radius $\frac{3}{2}$
d centre (−2, 5) radius 2

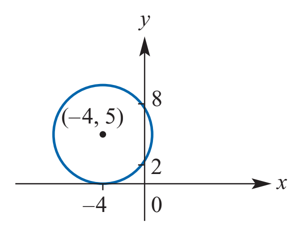
3 a $\left(x + \frac{1}{4}\right)^2 + \left(y + \frac{1}{4}\right)^2 = \frac{1}{8}$



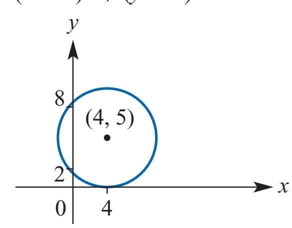
b $\left(x + \frac{3}{2}\right)^2 + (y - 2)^2 = \frac{49}{4}$



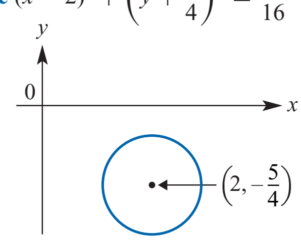
c $(x + 4)^2 + (y - 5)^2 = 25$



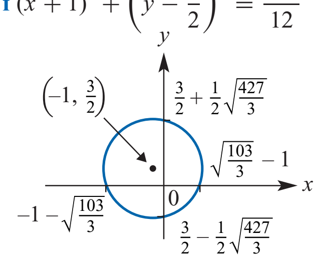
d $(x - 4)^2 + (y - 5)^2 = 25$



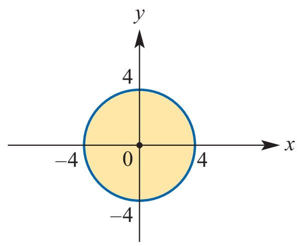
e $(x - 2)^2 + \left(y + \frac{5}{4}\right)^2 = \frac{9}{16}$



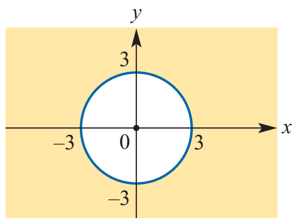
f $(x + 1)^2 + \left(y - \frac{3}{2}\right)^2 = \frac{439}{12}$



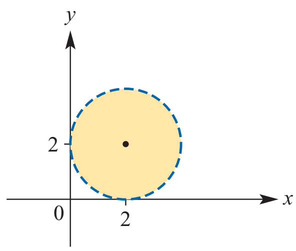
4 a $x^2 + y^2 \le 16$



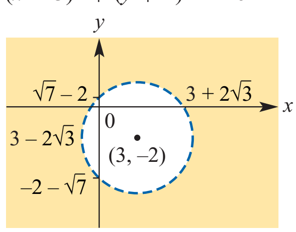
b $x^2 + y^2 \ge 9$



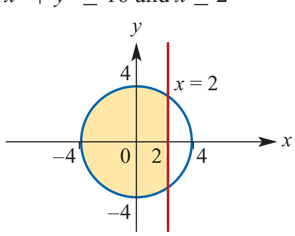
c $(x - 2)^2 + (y - 2)^2 < 4$



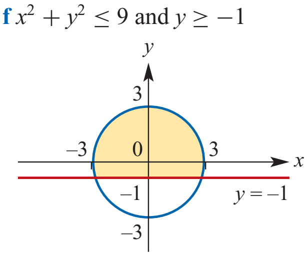
d $(x - 3)^2 + (y + 2)^2 > 16$



e $x^2 + y^2 \le 16$ and $x \le 2$



f $x^2 + y^2 \le 9$ and $y \ge -1$



5 centre (5, 3), radius $\sqrt{10}$

6 $(x - 2)^2 + (y + 3)^2 = 9$

7 $(x - 5)^2 + (y - 4)^2 = 13$

8 $4x^2 + 4y^2 - 60x - 76y + 536 = 0$ has centre $\left(\frac{15}{2}, \frac{19}{2}\right)$ and radius $\frac{5\sqrt{2}}{2}$

$x^2 + y^2 - 10x - 14y + 49 = 0$ has centre (5, 7) and radius 5

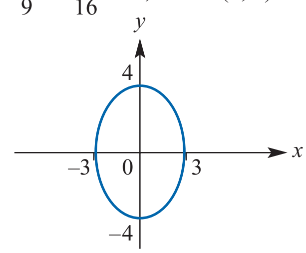
Points of intersection are (5, 12) and (10, 7)

9 a $\left(\frac{5\sqrt{2}}{2}, \frac{5\sqrt{2}}{2}\right), \left(\frac{-5\sqrt{2}}{2}, \frac{-5\sqrt{2}}{2}\right)$

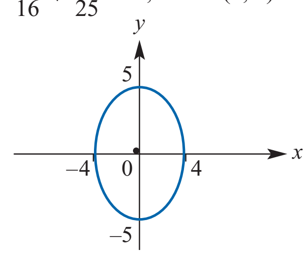
b $(\sqrt{5}, 2\sqrt{5}), (-\sqrt{5}, -2\sqrt{5})$

Exercise 1G

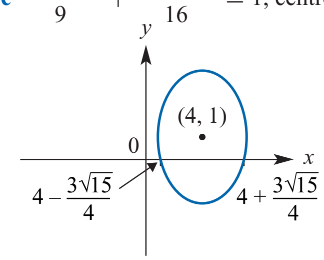
1 a $\frac{x^2}{9} + \frac{y^2}{16} = 1$, centre (0, 0)



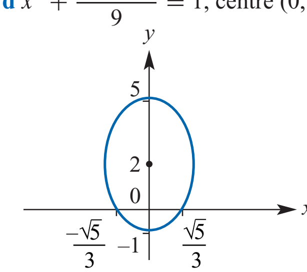
b $\frac{x^2}{16} + \frac{y^2}{25} = 1$, centre (0, 0)



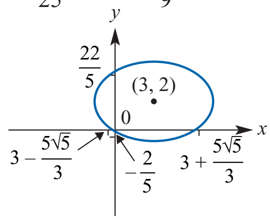
c $\frac{(x - 4)^2}{9} + \frac{(y - 1)^2}{16} = 1$, centre (4, 1)



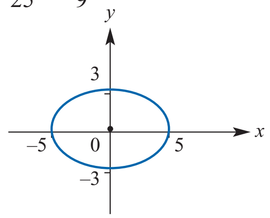
d $x^2 + \frac{(y - 2)^2}{9} = 1$, centre (0, 2)



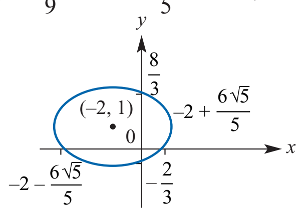
e $\dfrac{(x-3)^2}{25} + \dfrac{(y-2)^2}{9} = 1$, centre (3, 2)



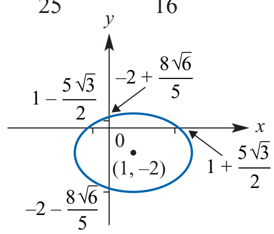
f $\dfrac{x^2}{25} + \dfrac{y^2}{9} = 1$, centre (0, 0)



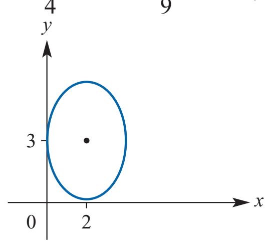
g $\dfrac{(x+2)^2}{9} + \dfrac{(y-1)^2}{5} = 1$, centre (−2, 1)



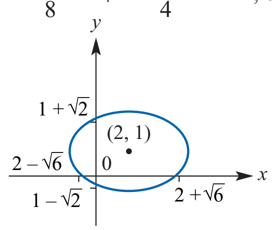
h $\dfrac{(x-1)^2}{25} + \dfrac{(y+2)^2}{16} = 1$, centre (1, −2)



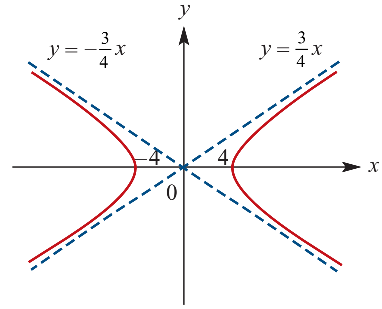
i $\dfrac{(x-2)^2}{4} + \dfrac{(y-3)^2}{9} = 1$, centre (2, 3)



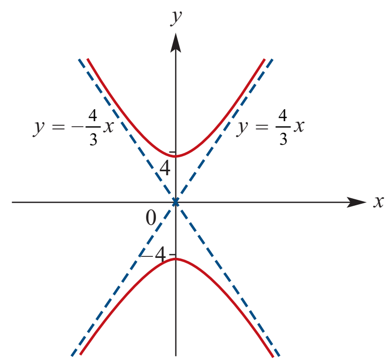
j $\dfrac{(x-2)^2}{8} + \dfrac{(y-1)^2}{4} = 1$, centre (2, 1)



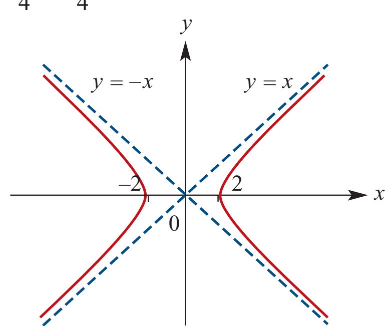
2 a $\dfrac{x^2}{16} - \dfrac{y^2}{9} = 1$, asymptotes $y = \pm\frac{3}{4}x$



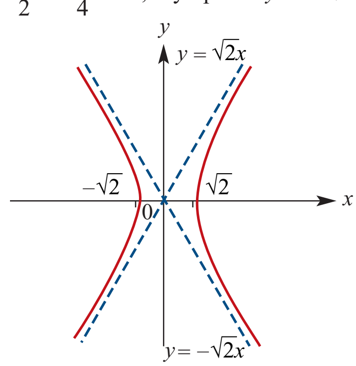
b $\dfrac{y^2}{16} - \dfrac{x^2}{9} = 1$, asymptotes $y = \pm\frac{4}{3}x$



c $\dfrac{x^2}{4} - \dfrac{y^2}{4} = 1$, asymptotes $y = \pm x$

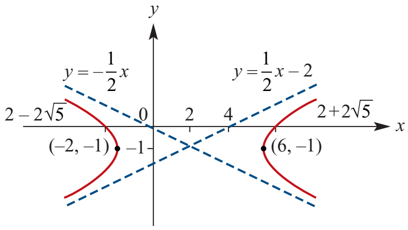


d $\dfrac{x^2}{2} - \dfrac{y^2}{4} = 1$, asymptotes $y = \pm\sqrt{2}x$



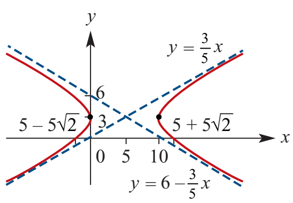
e $\dfrac{(x-2)^2}{16} - \dfrac{(y+1)^2}{4} = 1$, asymptotes

$y = \frac{1}{2}x - 2 \qquad y = -\frac{1}{2}x$



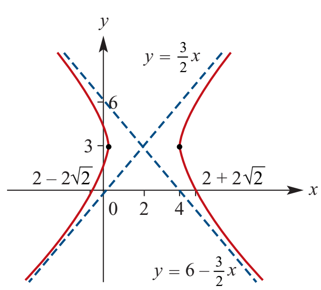
f $\dfrac{(x-5)^2}{25} - \dfrac{(y-3)^2}{9} = 1$, asymptotes

$y = \frac{3}{5}x \qquad y = 6 - \frac{3}{5}x$



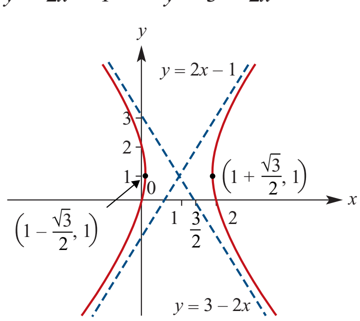
g $\dfrac{(x-2)^2}{4} - \dfrac{(y-3)^2}{9} = 1$, asymptotes

$y = \frac{3}{2}x \qquad y = 6 - \frac{3}{2}x$



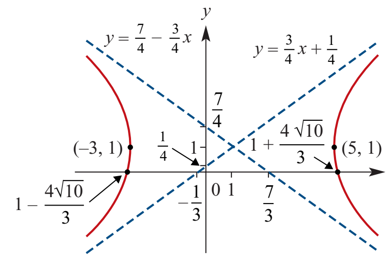
h $\dfrac{4(x-1)^2}{3} - \dfrac{(y-1)^2}{3} = 1$, asymptotets

$y = 2x - 1 \qquad y = 3 - 2x$



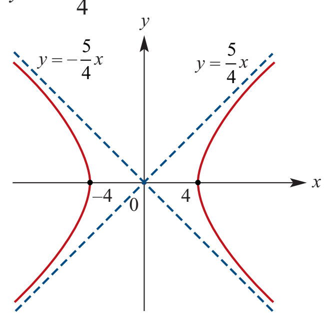
i $\dfrac{(x-1)^2}{16} - \dfrac{(y-1)^2}{9} = 1$, asymptotes

$y = \frac{3}{4}x + \frac{1}{4} \qquad y = \frac{7}{4} - \frac{3}{4}x$



j $\dfrac{x^2}{16} - \dfrac{y^2}{25} = 1$, asymptotes

$y = \pm\dfrac{5}{4}x$



3 a $\left(\dfrac{2\sqrt{3}}{3}, \dfrac{\sqrt{3}}{3}\right), \left(\dfrac{-2\sqrt{3}}{3}, \dfrac{-\sqrt{3}}{3}\right)$

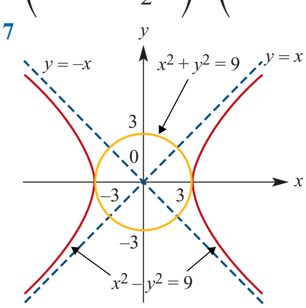
b $\left(\sqrt{2}, \dfrac{\sqrt{2}}{2}\right), \left(-\sqrt{2}, \dfrac{-\sqrt{2}}{2}\right)$

5 $\left(\dfrac{-6\sqrt{13}}{13}, \dfrac{-6\sqrt{13}}{13}\right), \left(\dfrac{6\sqrt{13}}{13}, \dfrac{6\sqrt{13}}{13}\right),$

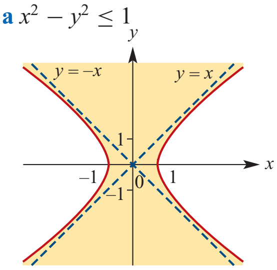
$\left(\dfrac{-6\sqrt{13}}{13}, \dfrac{6\sqrt{13}}{13}\right), \left(\dfrac{6\sqrt{13}}{13}, \dfrac{-6\sqrt{13}}{13}\right)$

6 $\left(-2\sqrt{2}, \dfrac{-5\sqrt{2}}{2}\right), \left(2\sqrt{2}, \dfrac{5\sqrt{2}}{2}\right)$

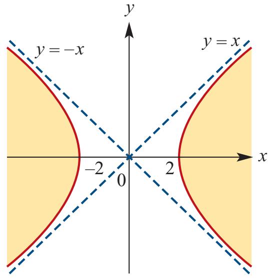
7



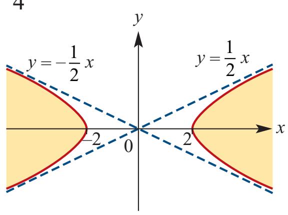
8 a $x^2 - y^2 \le 1$



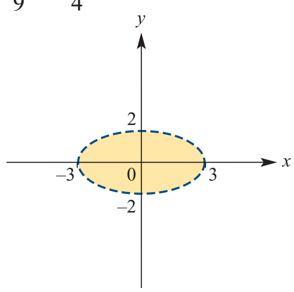
b $\frac{x^2}{4} - \frac{y^2}{4} \geq 1$



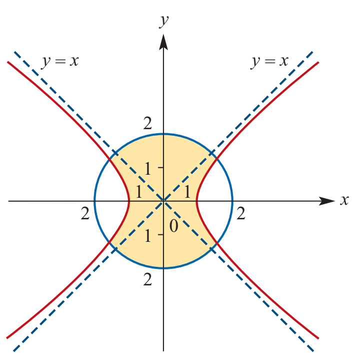
c $\frac{x^2}{4} - y^2 \geq 1$



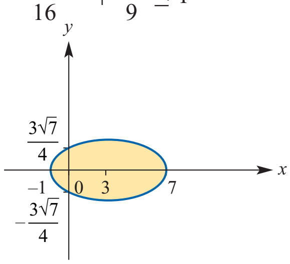
d $\frac{x^2}{9} + \frac{y^2}{4} < 1$



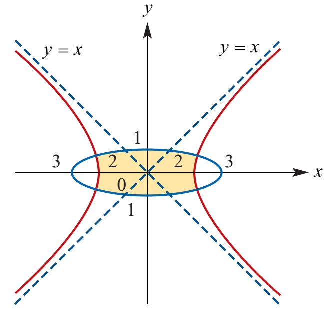
e $x^2 - y^2 \leq 1$ and $x^2 + y^2 \leq 4$



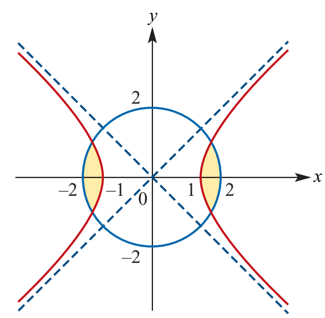
f $\frac{(x-3)^2}{16} + \frac{y^2}{9} \leq 1$



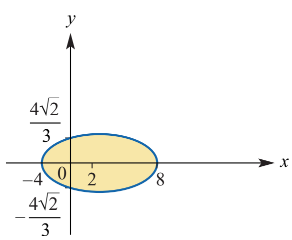
g $\frac{x^2}{4} - \frac{y^2}{4} \leq 1$ and $\frac{x^2}{9} + y^2 \leq 1$



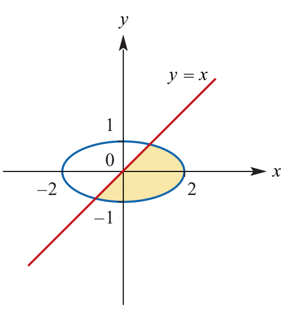
h $x^2 - y^2 > 1$ and $x^2 + y^2 = 4$



i $\frac{(x-2)^2}{36} + \frac{y^2}{4} \leq 1$



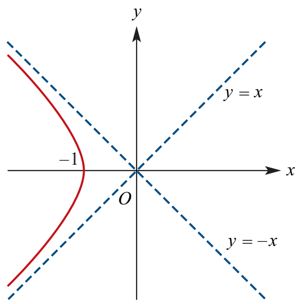
j $\frac{x^2}{4} + y^2 \leq 1$ and $y \leq x$



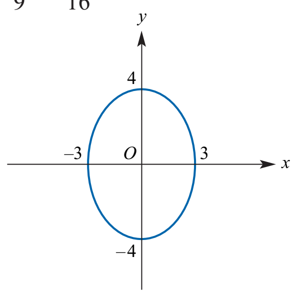
Exercise 1H

1 $x^2 + y^2 = 4$ dom $= [-2, 2]$ ran $= [-2, 2]$

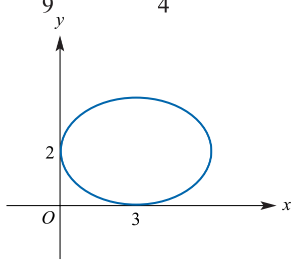
2 a $x^2 - y^2 = 1 \quad x \in (-\infty, -1]$



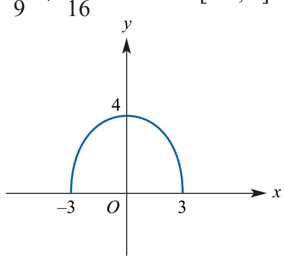
b $\frac{x^2}{9} + \frac{y^2}{16} = 1$



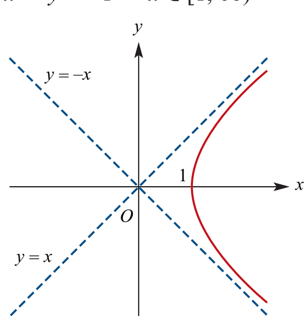
c $\frac{(x-3)^2}{9} + \frac{(y-2)^2}{4} = 1$



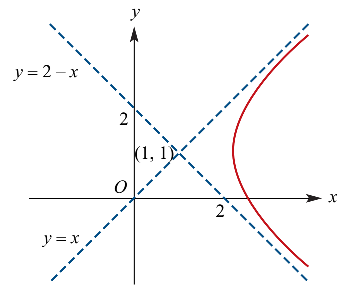
d $\frac{x^2}{9} + \frac{y^2}{16} = 1 \quad x \in [-3, 3] \quad y \in [0, 4]$



e $x^2 - y^2 = 1 \quad x \in [1, \infty)$



f $(x-1)^2 - (y-1)^2 = 1 \quad x \in [2, \infty)$



3 a $x = 4\cos t \quad y = 4\sin t$
b $x = 3\sec t \quad y = 2\tan t$
c $x = 3\cos t + 1 \quad y = 3\sin t - 2$
d $x = 9\cos t + 1 \quad y = 6\sin t - 3$

4 $a = 1, b = 2, c = 3, d = 2$

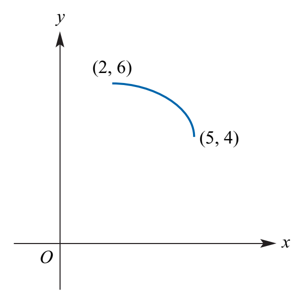
5 $x = 4\cos t, y = 3\sin t$

6 a $x = 2\cos t, y = 6\sin t$ **b** $\frac{x^2}{4} + \frac{y^2}{36} = 1$

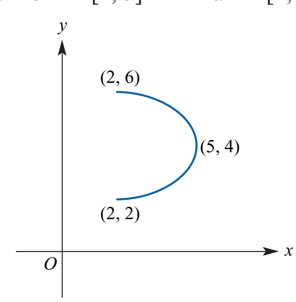
7 a $x = -2\cos\frac{t}{2}, y = 2 + 3\sin\frac{t}{2}$

b $\frac{x^2}{4} + \frac{(y-2)^2}{9} = 1$

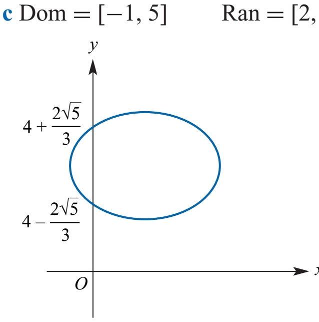
8 a Dom = [2, 5] Ran = [4, 6]



b Dom = [2, 5] Ran = [2, 6]



c Dom = [−1, 5] Ran = [2, 6]



Multiple-choice questions

1 B **2** D **3** C **4** A **5** C
6 C **7** B **8** C **9** D **10** D

Short-answer questions (technology-free)

1 $f_n = 5^n$ **2** $\frac{10}{\cos\alpha}$ cm

3 $\frac{(x+2)^2}{4} + \frac{(y-3)^2}{16} = 1$

4 $\frac{7}{\sqrt{113}}$ **5** $\frac{9}{2}$

6 a $\sqrt{34}$ cm **b** $2\tan^{-1}\left(\frac{5}{3}\right)$

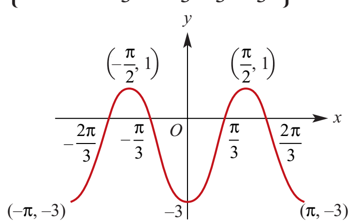
7 a $\frac{\sqrt{2}}{2}$ **b** $-\frac{4}{5}$ **c** $210°$ is one possible answer

8 a x **b** $\sqrt{a(a+b)}$

9 $\tan^{-1}(3\sqrt{2})$

10 a $\left\{x : x = -\frac{2\pi}{3}, -\frac{\pi}{3}, \frac{\pi}{3}, \frac{2\pi}{3}\right\}$

b



c $\left[-\pi, \frac{-2\pi}{3}\right) \cup \left(\frac{-\pi}{3}, \frac{\pi}{3}\right) \cup \left(\frac{2\pi}{3}, \pi\right]$

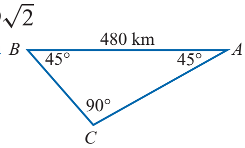
11 a $90°$ **b** $45°$ and $\tan^{-1}\left(\frac{3}{4}\right)$

c $\tan^{-1}\left(\frac{5}{4}\right)$

12 a $3\sqrt{97}$ nautical miles **b** $5\sqrt{97}$ nautical miles

13 $9\sqrt{2}$

14 a



b $240\sqrt{2}$ km
c $480\sqrt{2}$ km

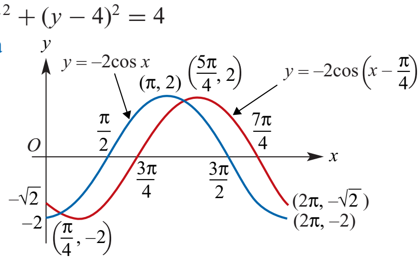
15 $y = 3x + 2$, $y = -3x + 2$

16 $\frac{(x-4)^2}{9} + (y+6)^2 = 1$

17 a 60 **b** $a = 30$, $b = 30$, $c = 120$, $d = 60$

18 $x^2 + (y-4)^2 = 4$

19 a



b $\left\{x : x = \frac{3\pi}{4}, \frac{7\pi}{4}\right\}$

c $\left\{x : 0 \le x \le \frac{\pi}{2}\right\} \cup \left\{x : \frac{3\pi}{2} \le x \le 2\pi\right\}$

20 a $\frac{\pi}{6}, \frac{5\pi}{6}$ **b** $\frac{\pi}{6}, \frac{11\pi}{6}$ **c** $\frac{\pi}{4}, \frac{5\pi}{4}$

21 $a = 1, c = 2, b = d = 3$

22 a $50°$ **b** $50°$ **c** $40°$

23 Centre $(-4, 6)$, radius 7

24 $(\pm 9, 0)$ $(0, \pm 3)$

25 a i $n = 7p + 7$
ii $5n = 70p^2 + 147p + 77$

26 a $t_n = 3^{n-1}$ **b** 3^{190}

Extended-response questions

1 a 10.2 km **b** 049°
c i 11.08 km **ii** 031°
d 11.93 km

2 a i $[-\sqrt{2}, \sqrt{2}]$
ii $[-3-\sqrt{5}, -3+\sqrt{5}]$ **iii** $(0, -3)$

b 2, 3, 1, 2 **c** $\left(\frac{37}{13}, \frac{11}{13}\right)$ **d** $\left(0, \frac{48}{13}\right)$

e $\left(x - \frac{1}{2}\right)^2 + \left(y - \frac{35}{26}\right)^2 = \frac{3890}{676}$

3 e $\frac{3}{4}$, undefined

f $y = 4$ and $y = \frac{-4}{3}x + \frac{20}{3}$

4 a $y = (\tan\theta)x$
b $(-a\cos\theta, -a\sin\theta)$
c $y - a\sin\theta = -\frac{\cos\theta}{\sin\theta}(x - a\cos\theta)$
d $A\left(\frac{a}{\cos\theta}, 0\right)$; $B\left(0, \frac{a}{\sin\theta}\right)$
e Area $= \frac{a^2}{2\sin\theta\cos\theta} = \frac{a^2}{\sin 2\theta}$
Minimum when $\theta = \frac{\pi}{4}$

5 a $y = \frac{-\sqrt{3}}{3}x + \frac{2\sqrt{3}a}{3}$; $y = \frac{\sqrt{3}}{3}x - \frac{2\sqrt{3}a}{3}$
b $x^2 + y^2 = 4a^2$

6 a 100°, 15°, 65° **b** 2.63 km, 4.56 km
c 346° **d** 14.18 km

Chapter 2

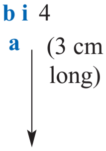
Exercise 2A

1 a i $2\boldsymbol{b}$ **ii** $4\boldsymbol{a}$ **iii** $2\boldsymbol{a} + \frac{3}{2}\boldsymbol{b}$

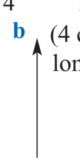
iv $\frac{1}{2}\boldsymbol{b} - 2\boldsymbol{a}$ **v** $2\boldsymbol{a} - \frac{3}{2}\boldsymbol{b}$

b i 4 **ii** 4 **iii** $\sqrt{13}$

2 a



b



3 a 6 **b** $\frac{9}{2}$ **c** $\frac{3}{2}$

4 a i $\frac{1}{4}\boldsymbol{a}$ ii $\frac{1}{4}\boldsymbol{b}$
iii $\frac{1}{4}(\boldsymbol{b}-\boldsymbol{a})$ iv $\boldsymbol{b}-\boldsymbol{a}$
b i $\frac{1}{2}\boldsymbol{a}$ ii $\frac{1}{2}\boldsymbol{b}$ iii $\frac{1}{2}(\boldsymbol{b}-\boldsymbol{a})$
5 a $\boldsymbol{a}+\boldsymbol{b}$ b $-(\boldsymbol{a}+\boldsymbol{b}+\boldsymbol{c}+\boldsymbol{d})$
c $-(\boldsymbol{b}+\boldsymbol{c})$
6 a $\boldsymbol{b}-\boldsymbol{a}$ b $\frac{1}{2}(\boldsymbol{b}-\boldsymbol{a})$ c $\frac{1}{2}(\boldsymbol{a}+\boldsymbol{b})$
7 a $\frac{1}{2}(\boldsymbol{a}+\boldsymbol{b})$
8 a $\boldsymbol{a}+\boldsymbol{c}-\boldsymbol{b}$ b $\boldsymbol{a}+\boldsymbol{c}-2\boldsymbol{b}$
9 a i $\boldsymbol{b}-\boldsymbol{a}$ ii $\boldsymbol{c}-\boldsymbol{d}$
iii $\boldsymbol{b}-\boldsymbol{a}=\boldsymbol{c}-\boldsymbol{d}$
b i $\boldsymbol{c}-\boldsymbol{b}$ ii $-\frac{1}{2}\boldsymbol{a}+\boldsymbol{b}-\boldsymbol{c}$
10 a not linearly dependent
b not linearly dependent
c linearly dependent
11 a $k=3, l=\frac{1}{2}$ b $k=\frac{55}{2}, l=-10$
12 a $-\boldsymbol{c}$ b $\boldsymbol{c}$ c $-\frac{1}{2}\boldsymbol{a}$
d $\boldsymbol{c}+\boldsymbol{g}+\frac{1}{2}\boldsymbol{a}$ e $\boldsymbol{c}+\boldsymbol{g}-\frac{1}{2}\boldsymbol{a}$
13 a i $k(2\boldsymbol{a}-\boldsymbol{b})$
ii $(2m+1)\boldsymbol{a}+(4-3m)\boldsymbol{b}$
b $k=\frac{11}{4}, m=\frac{9}{4}$
14 a i $\frac{1}{2}(\boldsymbol{a}+\boldsymbol{b})$ ii $\frac{4}{5}(\boldsymbol{a}+\boldsymbol{b})$
iii $\frac{1}{5}(4\boldsymbol{b}-\boldsymbol{a})$ iv $\frac{4}{5}(4\boldsymbol{b}-\boldsymbol{a})$
b $\overrightarrow{RP}=4\overrightarrow{AR}$, 1 : 4 c 4
15 a $x=0$ $y=1$ b $x=-1$ $y=\frac{7}{3}$
c $x=-\frac{5}{2}$ $y=0$

Exercise 2B

1 a i $3\boldsymbol{i}+\boldsymbol{j}$ ii $-2\boldsymbol{i}+3\boldsymbol{j}$
iii $-3\boldsymbol{i}-2\boldsymbol{j}$ iv $4\boldsymbol{i}-3\boldsymbol{j}$
b i $-5\boldsymbol{i}+2\boldsymbol{j}$ ii $7\boldsymbol{i}-\boldsymbol{j}$ iii $-\boldsymbol{i}+4\boldsymbol{j}$
c i $\sqrt{10}$ ii $\sqrt{29}$ iii $\sqrt{17}$
2 a $\boldsymbol{i}+4\boldsymbol{j}$ b $4\boldsymbol{i}+4\boldsymbol{j}+2\boldsymbol{k}$
c $6\boldsymbol{j}-3\boldsymbol{k}$ d $-8\boldsymbol{i}-8\boldsymbol{j}+8\boldsymbol{k}$
e $\sqrt{6}$ f 4
3 a i $-5\boldsymbol{i}$ ii $3\boldsymbol{k}$
iii $2\boldsymbol{j}$ iv $5\boldsymbol{i}+3\boldsymbol{k}$
v $5\boldsymbol{i}+2\boldsymbol{j}+3\boldsymbol{k}$ vi $5\boldsymbol{i}+2\boldsymbol{j}$
vii $-5\boldsymbol{i}-3\boldsymbol{k}$ viii $2\boldsymbol{j}-3\boldsymbol{k}$
ix $-5\boldsymbol{i}+2\boldsymbol{j}-3\boldsymbol{k}$ x $-5\boldsymbol{i}-2\boldsymbol{j}+3\boldsymbol{k}$
xi $5\boldsymbol{i}+2\boldsymbol{j}-3\boldsymbol{k}$ xii $5\boldsymbol{i}-2\boldsymbol{j}-3\boldsymbol{k}$
b i $\sqrt{34}$ ii $\sqrt{38}$ iii $\sqrt{29}$
c i $\frac{5}{2}\boldsymbol{i}$ ii $\frac{5}{2}\boldsymbol{i}+2\boldsymbol{j}$
iii $\frac{-5}{2}\boldsymbol{i}+2\boldsymbol{j}-3\boldsymbol{k}$
d i $\frac{-4}{3}\boldsymbol{j}$ ii $\frac{2}{3}\boldsymbol{j}$
iii $\frac{2}{3}\boldsymbol{j}+3\boldsymbol{k}$ iv $5\boldsymbol{i}-\frac{2}{3}\boldsymbol{j}-3\boldsymbol{k}$
v $\frac{5}{2}\boldsymbol{i}+\frac{4}{3}\boldsymbol{j}-3\boldsymbol{k}$
e i $\frac{\sqrt{613}}{6}$ ii $\frac{\sqrt{77}}{2}$ iii $\frac{\sqrt{310}}{3}$

4 a $x=3, y=-\frac{1}{3}$ b $x=4, y=\frac{2}{5}$
c $x=-\frac{3}{2}, y=7$
5 a i $4\boldsymbol{i}-2\boldsymbol{j}-4\boldsymbol{k}$ ii $-5\boldsymbol{i}+4\boldsymbol{j}+9\boldsymbol{k}$
iii $2\boldsymbol{i}-\boldsymbol{j}-2\boldsymbol{k}$ iv $-\boldsymbol{i}-\boldsymbol{j}-3\boldsymbol{k}$
b i $\sqrt{30}$ ii $\sqrt{67}$
c $\overrightarrow{AB}, \overrightarrow{CD}$
6 a i $2\boldsymbol{i}-3\boldsymbol{j}+4\boldsymbol{k}$ ii $\frac{4}{5}(2\boldsymbol{i}-3\boldsymbol{j}+4\boldsymbol{k})$
iii $\frac{1}{5}(13\boldsymbol{i}-7\boldsymbol{j}-9\boldsymbol{k})$
b $\left(\frac{13}{5}, \frac{-7}{5}, \frac{-9}{5}\right)$
8 $\frac{13}{9}$
9 a i $\overrightarrow{OA}=2\boldsymbol{i}+\boldsymbol{j}$ ii $\overrightarrow{AB}=-\boldsymbol{i}-4\boldsymbol{j}$
iii $\overrightarrow{BC}=-6\boldsymbol{i}+5\boldsymbol{j}$ iv $\overrightarrow{BD}=2\boldsymbol{i}+8\boldsymbol{j}$
b $\overrightarrow{BD}=-2\overrightarrow{AB}$
c Points A, B and D are collinear
10 a i $\overrightarrow{OB}=2\boldsymbol{i}+3\boldsymbol{j}+\boldsymbol{k}$
ii $\overrightarrow{AC}=-\boldsymbol{i}-5\boldsymbol{j}+8\boldsymbol{k}$
iii $\overrightarrow{BD}=2\boldsymbol{i}+2\boldsymbol{j}+5\boldsymbol{k}$
iv $\overrightarrow{CD}=4\boldsymbol{i}+6\boldsymbol{j}+2\boldsymbol{k}$
b $\overrightarrow{CD}=2(2\boldsymbol{i}+3\boldsymbol{j}+\boldsymbol{k})=2\overrightarrow{OB}$
11 a i $\overrightarrow{AB}=2\boldsymbol{i}-\boldsymbol{j}+2\boldsymbol{k}$
ii $\overrightarrow{BC}=-\boldsymbol{i}+2\boldsymbol{j}+3\boldsymbol{k}$
iii $\overrightarrow{CD}=-2\boldsymbol{i}+\boldsymbol{j}-2\boldsymbol{k}$
iv $\overrightarrow{DA}=\boldsymbol{i}-2\boldsymbol{j}-3\boldsymbol{k}$
b parallelogram
12 a $(-6, 3)$ b $(6, 5)$ c $\left(\frac{3}{2}, \frac{-3}{2}\right)$
13 a i $\overrightarrow{BC}=6\boldsymbol{i}+3\boldsymbol{j}$
ii $\overrightarrow{AD}=(x-2)\boldsymbol{i}+(y-1)\boldsymbol{j}$
b $(8, 4)$
14 a $(1.5, 1.5, 4)$
b $\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}, \frac{z_1+z_2}{2}\right)$
15 $\left(\frac{17}{5}, \frac{8}{5}, -3\right)$ 16 $\left(\frac{17}{2}, 3\right)$
17 $\left(-11, \frac{-11}{3}\right)$
19 a i $\boldsymbol{i}+\boldsymbol{j}$ ii $-\boldsymbol{i}-6\boldsymbol{j}$ iii $-\boldsymbol{i}-15\boldsymbol{j}$
b $k=\frac{19}{8}, l=\frac{-1}{4}$
20 a i $2\boldsymbol{i}+4\boldsymbol{j}-9\boldsymbol{k}$ ii $14\boldsymbol{i}-8\boldsymbol{j}+3\boldsymbol{k}$
iii $5.7\boldsymbol{i}-0.3\boldsymbol{j}-1.6\boldsymbol{k}$
b There are no values for k and l such that $k\boldsymbol{a}+l\boldsymbol{b}=\boldsymbol{c}$
21 a i $\sqrt{29}$ ii $\sqrt{13}$
iii $\sqrt{97}$ iv $\sqrt{19}$
b i 21.80° anticlockwise
ii 23.96° clockwise
iii 46.51°
22 a $-3.42\boldsymbol{i}+9.40\boldsymbol{j}$ b $-2.91\boldsymbol{i}-7.99\boldsymbol{j}$
c $4.60\boldsymbol{i}+3.86\boldsymbol{j}$ d $2.50\boldsymbol{i}-4.33\boldsymbol{j}$

23 a $-6.43\boldsymbol{i} + 1.74\boldsymbol{j} + 7.46\boldsymbol{k}$
b $5.14\boldsymbol{i} + 4.64\boldsymbol{j} - 4\boldsymbol{k}$
c $6.13\boldsymbol{i} - 2.39\boldsymbol{j} - 2.39\boldsymbol{k}$
d $-6.26\boldsymbol{i} + 9.77\boldsymbol{j} + 3.07\boldsymbol{k}$

25 a $|\overrightarrow{AB}| = |\overrightarrow{AC}| = 3$
b $\overrightarrow{OM} = -\boldsymbol{i} + 3\boldsymbol{j} + 4\boldsymbol{k}$
c $\overrightarrow{AM} = \boldsymbol{i} + 2\boldsymbol{j} - \boldsymbol{k}$ **d** $3\sqrt{2}$

26 a $5\boldsymbol{i} + 5\boldsymbol{j}$ **b** $\frac{1}{2}(5\boldsymbol{i} + 5\boldsymbol{j})$
c $\frac{5}{2}\boldsymbol{i} + \frac{5}{2}\boldsymbol{j} + 3\boldsymbol{k}$ **d** $\frac{-5}{2}\boldsymbol{i} - \frac{5}{2}\boldsymbol{j} + 3\boldsymbol{k}$
e $\frac{\sqrt{86}}{2}$

27 a $\overrightarrow{MN} = \frac{1}{2}\boldsymbol{b} - \frac{1}{2}\boldsymbol{a}$
b $\overrightarrow{MN} \,||\, \overrightarrow{AB}$, $MN = \frac{1}{2}AB$

28 a $\frac{\sqrt{3}}{2}\boldsymbol{i} - \frac{1}{2}\boldsymbol{j}$ **b** $\frac{3\sqrt{3}}{2}\boldsymbol{i} - \frac{3}{2}\boldsymbol{j}$
c $\frac{3\sqrt{3}}{2}\boldsymbol{i} + \frac{7}{2}\boldsymbol{j}$ **d** $\sqrt{19}$ km

29 a $\overrightarrow{OA} = 50\boldsymbol{k}$
b i $-80\boldsymbol{i} + 20\boldsymbol{j} - 10\boldsymbol{k}$ **ii** $10\sqrt{69}$ m
c $-80\boldsymbol{i} + 620\boldsymbol{j} + 100\boldsymbol{k}$

30 a 2.66 km
b i $-0.5\boldsymbol{i} - \boldsymbol{j} + 0.1\boldsymbol{k}$ **ii** 1.12 km
c $-0.6\boldsymbol{i} - 0.8\boldsymbol{j}$

31 a $-100\sqrt{2}\boldsymbol{i} + 100\sqrt{2}\boldsymbol{j}$ **b** $50\boldsymbol{j}$
c $-100\sqrt{2}\boldsymbol{i} + (50 + 100\sqrt{2})\boldsymbol{j}$ **d** $30\boldsymbol{k}$
e $-100\sqrt{2}\boldsymbol{i} + (50 + 100\sqrt{2})\boldsymbol{j} + 30\boldsymbol{k}$

32 a $\overrightarrow{OP} = 50\sqrt{2}\boldsymbol{i} + 50\sqrt{2}\boldsymbol{j}$
b i $(50\sqrt{2} - 100)\boldsymbol{i} + 50\sqrt{2}\boldsymbol{j}$ **ii** $337.5°$

Exercise 2C

1 a 66 **b** 22 **c** 6 **d** 11 **e** 25
f 86 **g** −43

2 a 14 **b** 13 **c** 0 **d** −8 **e** 14

3 a $\boldsymbol{a}.\boldsymbol{a} + 4\boldsymbol{a}.\boldsymbol{b} + 4\boldsymbol{b}.\boldsymbol{b}$ **b** $4\boldsymbol{a}.\boldsymbol{b}$
c $\boldsymbol{a}.\boldsymbol{a} - \boldsymbol{b}.\boldsymbol{b}$ **d** $|\boldsymbol{a}|$

4 a $\overrightarrow{AB} = -2\boldsymbol{i} - \boldsymbol{j} - 2\boldsymbol{k}$ **b** $|\overrightarrow{AB}| = 3$ **c** $105.8°$

5 $\sqrt{66}$

6 a i $\boldsymbol{c}$ **ii** $\boldsymbol{a} + \boldsymbol{c}$ **iii** $\boldsymbol{c} - \boldsymbol{a}$
b $\overrightarrow{OB}.\overrightarrow{AC} = \boldsymbol{c}.\boldsymbol{c} - \boldsymbol{a}.\boldsymbol{a} = 0$

7 $\boldsymbol{d}$ and $\boldsymbol{f}$, $\boldsymbol{a}$ and $\boldsymbol{e}$, $\boldsymbol{b}$ and $\boldsymbol{c}$

8 a −4 **b** 5 **c** 5 **d** −6 or 1

9 a $\overrightarrow{AP} = -\boldsymbol{a} + q\boldsymbol{b}$ **b** $q = \frac{13}{15}$
c $\left(\frac{26}{15}, \frac{13}{3}, \frac{-13}{15}\right)$

10 $x = 1$; $y = -3$

11 a 2.45 **b** 1.11 **c** 0.580 **d** 2.01

13 a $\overrightarrow{OM} = \frac{3}{2}\boldsymbol{i} + \boldsymbol{j}$ **b** $36.81°$ **c** $111.85°$

14 a i $-\boldsymbol{i} + 3\boldsymbol{j}$ **ii** $3\boldsymbol{j} - 2\boldsymbol{k}$
b $37.87°$ **c** $31.00°$

15 a $\overrightarrow{OM} = \frac{1}{2}(4\boldsymbol{i} + 5\boldsymbol{j})$; $\overrightarrow{ON} = \frac{1}{2}(2\boldsymbol{i} + 7\boldsymbol{k})$
b $80.12°$ **c** $99.88°$

16 a $\overrightarrow{AB} = \overrightarrow{CB} - \overrightarrow{CA}$

17 $69.71°$

Exercise 2D

1 a $\frac{\sqrt{11}}{11}(\boldsymbol{i} + 3\boldsymbol{j} - \boldsymbol{k})$ **b** $\frac{1}{3}(\boldsymbol{i} + 2\boldsymbol{j} + 2\boldsymbol{k})$
c $\frac{\sqrt{10}}{10}(-\boldsymbol{j} + 3\boldsymbol{k})$

2 a i $\frac{\sqrt{26}}{26}(3\boldsymbol{i} + 4\boldsymbol{j} - \boldsymbol{k})$ **ii** $\sqrt{3}$
b $\frac{\sqrt{78}}{26}(3\boldsymbol{i} + 4\boldsymbol{j} - \boldsymbol{k})$

3 a i $\hat{\boldsymbol{a}} = \frac{1}{3}(2\boldsymbol{i} - 2\boldsymbol{j} - \boldsymbol{k})$
ii $\hat{\boldsymbol{b}} = \frac{1}{5}(3\boldsymbol{i} + 4\boldsymbol{k})$
b $\frac{\sqrt{510}}{510}(19\boldsymbol{i} - 10\boldsymbol{j} + 7\boldsymbol{k})$

4 a $\frac{-11}{18}(\boldsymbol{i} - 4\boldsymbol{j} + \boldsymbol{k})$ **b** $\frac{-1}{9}(\boldsymbol{i} - 4\boldsymbol{j} + \boldsymbol{k})$
c $\frac{13}{17}(4\boldsymbol{i} - \boldsymbol{k})$

5 a 2 **b** $\frac{\sqrt{5}}{5}$ **c** $\frac{2\sqrt{21}}{7}$
d $\frac{-(1 + 4\sqrt{5})\sqrt{17}}{17}$

6 a $\frac{9}{26}(5\boldsymbol{i} - \boldsymbol{k})$; $\frac{1}{26}(7\boldsymbol{i} + 26\boldsymbol{j} + 35\boldsymbol{k})$
b $\frac{3}{2}(\boldsymbol{i} + \boldsymbol{k})$; $\frac{3}{2}\boldsymbol{i} + \boldsymbol{j} - \frac{3}{2}\boldsymbol{k}$
c $-\frac{1}{9}(2\boldsymbol{i} + 2\boldsymbol{j} - \boldsymbol{k})$; $\frac{-7}{9}\boldsymbol{i} + \frac{11}{9}\boldsymbol{j} + \frac{8}{9}\boldsymbol{k}$

7 a $\boldsymbol{j} + \boldsymbol{k}$ **b** $\frac{1}{3}(\boldsymbol{i} + 2\boldsymbol{j} - 2\boldsymbol{k})$

8 a $\boldsymbol{i} - \boldsymbol{j} - \boldsymbol{k}$ **b** $3\boldsymbol{i} + 2\boldsymbol{j} + \boldsymbol{k}$ **c** $\sqrt{14}$

9 a i $\boldsymbol{i} - \boldsymbol{j} - 2\boldsymbol{k}$ **ii** $\boldsymbol{i} - 5\boldsymbol{j}$
b $\frac{3}{13}(\boldsymbol{i} - 5\boldsymbol{j})$ **c** $\frac{2}{13}\sqrt{195}$ **d** $\sqrt{30}$

10 b i $\frac{2}{7}(\boldsymbol{i} - 3\boldsymbol{j} - 2\boldsymbol{k})$ **ii** $\frac{1}{3}(5\boldsymbol{i} + \boldsymbol{j} + \boldsymbol{k})$
c $\frac{1}{21}(\boldsymbol{i} + 11\boldsymbol{j} - 16\boldsymbol{k})$

Exercise 2E

13 a i $\frac{1}{2}(\boldsymbol{b} - \boldsymbol{a})$ **ii** $\frac{1}{2}(\boldsymbol{a} + \boldsymbol{b})$
b $\frac{1}{2}(\boldsymbol{a}.\boldsymbol{a} + \boldsymbol{b}.\boldsymbol{b})$

14 c 3 : 1

15 a i $\frac{1}{3}(\boldsymbol{a} + 2\boldsymbol{b})$ **ii** $\boldsymbol{a} + 2\boldsymbol{b}$ **iii** $2\boldsymbol{b}$

16 a $\boldsymbol{s} = \boldsymbol{r} + \boldsymbol{t}$
b $\boldsymbol{u} = \frac{1}{2}(\boldsymbol{r} + \boldsymbol{s})$, $\boldsymbol{v} = \frac{1}{2}(\boldsymbol{s} + \boldsymbol{t})$

17 b $\overrightarrow{AB} = \boldsymbol{i} - 3\boldsymbol{j}$, $\overrightarrow{DC} = \boldsymbol{i} - \boldsymbol{j}$ **c** $4\boldsymbol{i} + 2\boldsymbol{j}$
e $4\boldsymbol{j}$

19 $\frac{2}{3}\boldsymbol{b} - \frac{5}{12}\boldsymbol{a}$

20 b $\lambda = \frac{k+2}{2}$, $\mu = \frac{k+2}{2}$ **c** $\lambda = \frac{3}{2}$, $\mu = \frac{3}{2}$

21 a $\overrightarrow{OG} = \boldsymbol{b} + \boldsymbol{d} + \boldsymbol{e}$, $\overrightarrow{DF} = \boldsymbol{b} - \boldsymbol{d} + \boldsymbol{e}$, $\overrightarrow{BH} = -\boldsymbol{b} + \boldsymbol{d} + \boldsymbol{e}$, $\overrightarrow{CE} = -\boldsymbol{b} - \boldsymbol{d} + \boldsymbol{e}$
b $|\overrightarrow{OG}|^2 = |\boldsymbol{b}|^2 + |\boldsymbol{d}|^2 + |\boldsymbol{e}|^2 + 2(\boldsymbol{b}.\boldsymbol{d} + \boldsymbol{b}.\boldsymbol{e} + \boldsymbol{d}.\boldsymbol{e})$, $|\overrightarrow{DF}|^2 = |\boldsymbol{b}|^2 + |\boldsymbol{d}|^2 + |\boldsymbol{e}|^2 + 2(-\boldsymbol{b}.\boldsymbol{d} + \boldsymbol{b}.\boldsymbol{e} - \boldsymbol{d}.\boldsymbol{e})|\overrightarrow{BH}|^2 = |\boldsymbol{b}|^2 + |\boldsymbol{d}|^2 + |\boldsymbol{e}|^2 + 2(-\boldsymbol{b}.\boldsymbol{d} - \boldsymbol{b}.\boldsymbol{e} + \boldsymbol{d}.\boldsymbol{e})$, $|\overrightarrow{CE}|^2 = |\boldsymbol{b}|^2 + |\boldsymbol{d}|^2 + |\boldsymbol{e}|^2 + 2(\boldsymbol{b}.\boldsymbol{d} - \boldsymbol{b}.\boldsymbol{e} - \boldsymbol{d}.\boldsymbol{e})$

22 b $12r^2$

Multiple-choice questions

1 C **2** D **3** B **4** B **5** C
6 C **7** E **8** E **9** D **10** B

Short-answer questions (technology-free)

1 a $2\boldsymbol{i} - \boldsymbol{j} + \boldsymbol{k}$ **b** $\frac{\sqrt{2}}{3}$
2 a $\frac{3}{7}(-3\boldsymbol{i} + 2\boldsymbol{j} + 6\boldsymbol{k})$ **b** $\frac{1}{7}(6\boldsymbol{i} - 11\boldsymbol{j} - 12\boldsymbol{k})$
3 a $x = 5$ **b** $y = 2.8$ $z = -4.4$
4 a $\cos\theta = \frac{1}{3}$ **b** 6
5 a $\frac{1}{9}(43\boldsymbol{i} - 46\boldsymbol{j} + 20\boldsymbol{k})$
b $\frac{485}{549}(3\boldsymbol{i} - 6\boldsymbol{j} + 4\boldsymbol{k})$
6 a i $(2 - 3t)\boldsymbol{j} + (-3 - 2t)\boldsymbol{k}$
ii $(-2 - 3t)\boldsymbol{j} + (3 - 2t)\boldsymbol{k}$
b ± 1
7 a i $2\sqrt{17}$ **ii** $4\sqrt{3}$ **iii** -40
b $\cos^{-1}\left(\frac{5\sqrt{51}}{51}\right)$
8 a $3\boldsymbol{i} - \frac{3}{2}\boldsymbol{j} + \boldsymbol{k}$ **b** $\boldsymbol{i} - \frac{1}{2}\boldsymbol{j} + 4\boldsymbol{k}$ **c** $\frac{8\sqrt{5}}{21}$
9 a $34 - 4p$ **b** 8.5 **c** $\frac{5}{13}$
10 -6.5 **11** $\lambda = \frac{3}{2}$, $\mu = -\frac{3}{2}$
12 $AB \parallel DC$, $AB : CD = 1 : 2$
13 $\frac{\sqrt{19}}{5}$
14 a $2\boldsymbol{c}$, $2\boldsymbol{c} - \boldsymbol{a}$ **b** $\frac{1}{2}\boldsymbol{a} + \boldsymbol{c}$ **c** 1.5
15 a $(-1, 10)$ **b** $h = 3$, $\boldsymbol{k} = -2$
17 $3(\boldsymbol{i} + \boldsymbol{j})$ **16** $h = \frac{2}{3}$, $\boldsymbol{k} = \frac{3}{4}$
18 a $\boldsymbol{c} - \boldsymbol{a}$
19 a i $\frac{1}{3}\boldsymbol{c}$ **ii** $\frac{2}{3}\boldsymbol{a} - \frac{1}{3}\boldsymbol{b}$
iii $\frac{2}{3}\boldsymbol{a} + \frac{1}{3}\boldsymbol{b} - \frac{1}{3}\boldsymbol{c}$
20 a $\frac{1}{4}\boldsymbol{a} + \frac{3}{4}\boldsymbol{b}$
b i $\frac{\lambda}{4}\boldsymbol{a} + \left(\frac{3\lambda}{4} - 1\right)\boldsymbol{b}$ **ii** $\frac{4}{3}$

Extended-response questions

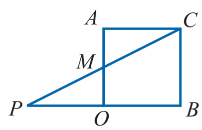
1 a i $\boldsymbol{i} + \boldsymbol{j} + \boldsymbol{k}$ **ii** $\sqrt{3}$
b i $(\lambda - 0.5)\boldsymbol{i} + (\lambda - 1)\boldsymbol{j} + (\lambda - 0.5)\boldsymbol{k}$
ii $\lambda = \frac{2}{3}$, $\overrightarrow{OQ}$ $\frac{1}{3}(8\boldsymbol{i} + 11\boldsymbol{j} + 5\boldsymbol{k})$
c $5\boldsymbol{i} + 6\boldsymbol{j} + 4\boldsymbol{k}$

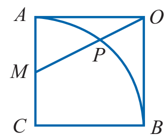
2 a i $|\overrightarrow{OA}| = \sqrt{14}$, $|\overrightarrow{OB}| = \sqrt{14}$
ii $\boldsymbol{i} - 5\boldsymbol{j}$
b i $\frac{1}{2}(5\boldsymbol{i} + \boldsymbol{j} + 2\boldsymbol{k})$ **c** $5\boldsymbol{i} + \boldsymbol{j} + 2\boldsymbol{k}$
e i $5\boldsymbol{i} + \boldsymbol{j} - 13\boldsymbol{k}$ or $-5\boldsymbol{i} - \boldsymbol{j} + 13\boldsymbol{k}$
iii The vector is perpendicular to the plane containing $OACB$.
3 a $\overrightarrow{OX} = 7\boldsymbol{i} + 4\boldsymbol{j} + 3\boldsymbol{k}$, $\overrightarrow{OY} = 2\boldsymbol{i} + 4\boldsymbol{j} + 3\boldsymbol{k}$, $\overrightarrow{OZ} = 6\boldsymbol{i} + 4\boldsymbol{j}$, $\overrightarrow{OD} = 6\boldsymbol{i} + 3\boldsymbol{k}$, $|\overrightarrow{OD}| = 3\sqrt{5}$, $|\overrightarrow{OY}| = \sqrt{29}$
b $48.27°$
c i $\left(\frac{5\lambda}{\lambda + 1} + 1\right)\boldsymbol{i} + 4\boldsymbol{j}$ **ii** $-\frac{1}{6}$
4 a i $\boldsymbol{b} - \boldsymbol{a}$ **ii** $\boldsymbol{c} - \boldsymbol{b}$ **iii** $\boldsymbol{a} - \boldsymbol{c}$
iv $\frac{1}{2}(\boldsymbol{b} + \boldsymbol{c})$ **v** $\frac{1}{2}(\boldsymbol{a} + \boldsymbol{c})$ **vi** $\frac{1}{2}(\boldsymbol{a} + \boldsymbol{b})$
5 a $\frac{1}{3}\boldsymbol{b} + \frac{2}{3}\boldsymbol{c}$ **c ii** 5 : 1 **d** 1 : 3
6 a i $\frac{1}{2}(\boldsymbol{a} + \boldsymbol{b})$ **ii** $-\frac{1}{2}\boldsymbol{a} + \left(\lambda - \frac{1}{2}\right)\boldsymbol{b}$
7 a i $12(1 - a)$ **ii** 1
b i $x - 4y + 2 = 0$ **ii** $x = -2$, $y = 0$
c i $\boldsymbol{j} + 4\boldsymbol{k}$ **ii** $\boldsymbol{i} - 12\boldsymbol{j} + 5\boldsymbol{k}$
iii $3\boldsymbol{i} - 11\boldsymbol{j} + 7\boldsymbol{k}$
d X is 5 units, and Y is 7 units, above the ground.
8 a i $\frac{3}{4}\boldsymbol{c}$ **ii** $\frac{2}{5}\boldsymbol{a} + \frac{3}{5}\boldsymbol{c}$ **iii** $-\boldsymbol{a} + \frac{3}{4}\boldsymbol{c}$
b $\mu = \frac{5}{6}$, $\lambda = \frac{2}{3}$
9 a $\boldsymbol{b} = q\boldsymbol{i} - p\boldsymbol{j}$, $\boldsymbol{c} = -q\boldsymbol{i} + p\boldsymbol{j}$
b i $\overrightarrow{AB} = -(x + 1)\boldsymbol{i} - y\boldsymbol{j}$, $\overrightarrow{AC} = (1 - x)\boldsymbol{i} - y\boldsymbol{j}$
ii $\overrightarrow{AE} = y\boldsymbol{i} + (1 - x)\boldsymbol{j}$, $\overrightarrow{AF} = -y\boldsymbol{i} + (x + 1)\boldsymbol{j}$
10 a i $\overrightarrow{BC} = m\boldsymbol{v}$, $\overrightarrow{BE} = n\boldsymbol{v}$, $\overrightarrow{CA} = m\boldsymbol{w}$, $\overrightarrow{CF} = n\boldsymbol{w}$
ii $|\overrightarrow{AE}| = \sqrt{m^2 - mn + n^2}$, $|\overrightarrow{FB}| = \sqrt{m^2 - mn + n^2}$
11 a $\overrightarrow{CF} = \frac{1}{2}\boldsymbol{a} - \boldsymbol{c}$, $\overrightarrow{OE} = \frac{1}{2}(\boldsymbol{a} + \boldsymbol{c})$
b ii $60°$
c ii HX is parallel to EX, KX is parallel to FX, and HK is parallel to EF
12 a $\overrightarrow{OA} = -2(\boldsymbol{i} + \boldsymbol{j})$, $\overrightarrow{OB} = 2(\boldsymbol{i} - \boldsymbol{j})$, $\overrightarrow{OC} = 2(\boldsymbol{i} + \boldsymbol{j})$, $\overrightarrow{OD} = -2(\boldsymbol{i} - \boldsymbol{j})$
b $\overrightarrow{PM} = \boldsymbol{i} + 3\boldsymbol{j} + h\boldsymbol{k}$, $\overrightarrow{QN} = -3\boldsymbol{i} - \boldsymbol{j} + h\boldsymbol{k}$
c $\overrightarrow{OX} = \frac{1}{2}\boldsymbol{i} - \frac{1}{2}\boldsymbol{j} + \frac{h}{2}\boldsymbol{k}$
d i $\sqrt{2}$ **ii** $71°$ **e ii** $\sqrt{6}$
13 a i $\overrightarrow{OM} = \frac{a}{2}\boldsymbol{j}$ **ii** $\overrightarrow{MC} = a\boldsymbol{i} + \frac{a}{2}\boldsymbol{j}$
b $\overrightarrow{MP} = a\lambda\boldsymbol{i} + \frac{a\lambda}{2}\boldsymbol{j}$, $\overrightarrow{BP} = a(\lambda - 1)\boldsymbol{i} + \frac{a}{2}(\lambda + 1)\boldsymbol{j}$, $\overrightarrow{OP} = a\lambda\boldsymbol{i} + \frac{a}{2}(\lambda + 1)\boldsymbol{j}$

c **i** $\lambda = \frac{3}{5}$, $|\overrightarrow{BP}| = \frac{2\sqrt{5}a}{5}$, $|\overrightarrow{OP}| = a$, $|\overrightarrow{OB}| = a$

ii $\frac{\sqrt{5}}{5}$

d $\lambda = -1$ $\qquad$ $\lambda = \frac{3}{5}$



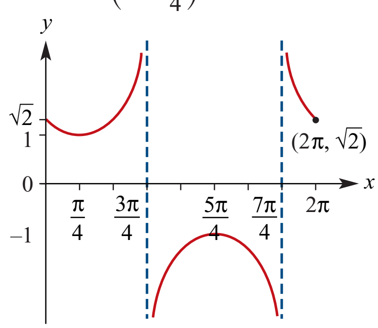


e $\overrightarrow{OY} = \frac{14}{15}a\boldsymbol{i} + \frac{29}{30}a\boldsymbol{j} + \frac{1}{6}a\boldsymbol{k}$

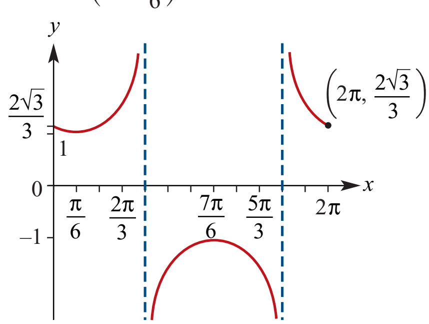
Chapter 3

Exercise 3A

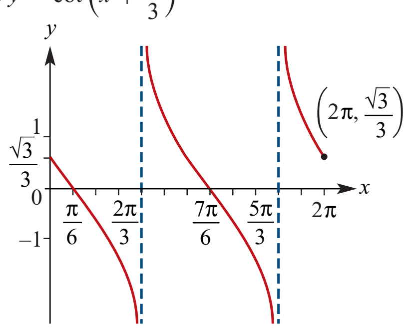
1 **a** $y = \text{cosec}\left(x + \frac{\pi}{4}\right)$



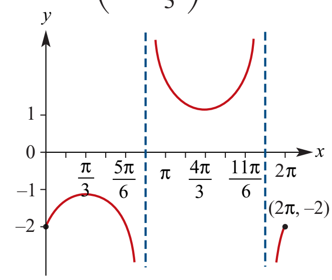
b $y = \sec\left(x - \frac{\pi}{6}\right)$



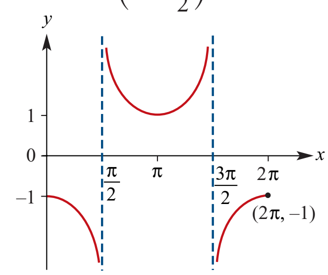
c $y = \cot\left(x + \frac{\pi}{3}\right)$



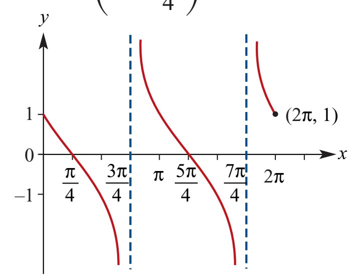
d $y = \sec\left(x + \frac{2\pi}{3}\right)$



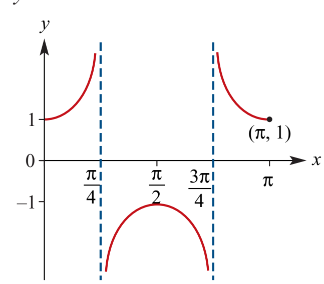
e $y = \text{cosec}\left(x - \frac{\pi}{2}\right)$



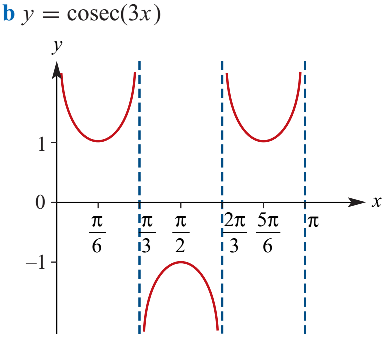
f $y = \cot\left(x - \frac{3\pi}{4}\right)$



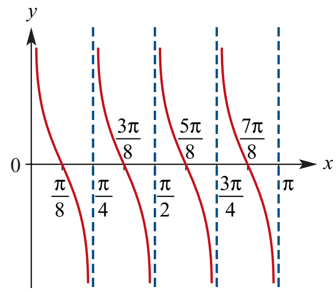
2 **a** $y = \sec 2x$



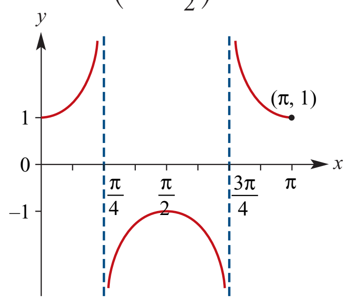
b $y = \text{cosec}(3x)$



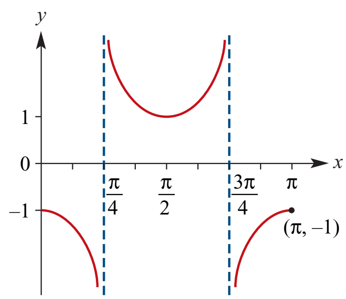
c $y = \cot(4x)$



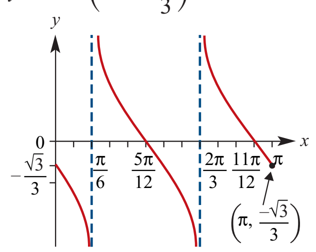
d $y = \operatorname{cosec}\left(2x + \frac{\pi}{2}\right)$



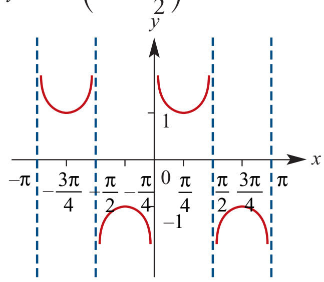
e $y = \sec(2x - \pi)$



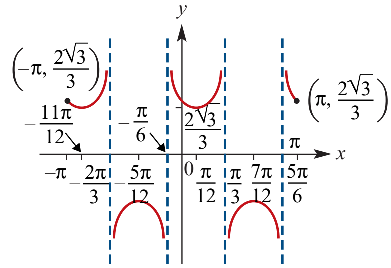
f $y = \cot\left(2x - \frac{\pi}{3}\right)$



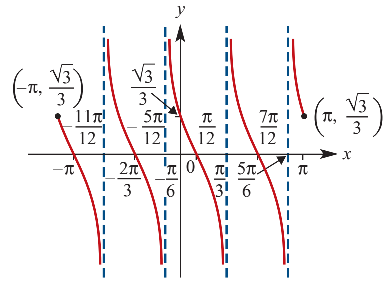
3 a $y = \sec\left(2x - \frac{\pi}{2}\right)$



b $y = \operatorname{cosec}\left(2x + \frac{\pi}{3}\right)$



c $y = \cot\left(2x - \frac{2\pi}{3}\right)$



4 a $\cot x = \frac{5}{8}$, $\sec x = \frac{\sqrt{89}}{5}$, $\operatorname{cosec} x = \frac{\sqrt{89}}{8}$

b $\cot x = \frac{2\sqrt{6}}{5}$, $\sec x = \frac{7\sqrt{6}}{12}$, $\operatorname{cosec} x = \frac{7}{5}$

c $\cot x = \frac{7\sqrt{2}}{8}$, $\sec x = \frac{9}{7}$, $\operatorname{cosec} x = \frac{9\sqrt{2}}{8}$

5 a $\frac{\sqrt{3}}{2}$ **b** $-\frac{\sqrt{2}}{2}$ **c** -1 **d** 2

e $\sqrt{2}$ **f** $-\sqrt{3}$ **g** $-\frac{\sqrt{2}}{2}$ **h** $-\frac{\sqrt{3}}{3}$

i 2 **j** $\sqrt{2}$ **k** 1 **l** $\frac{1}{2}$

6 a 1 **b** -1 **c** $\operatorname{cosec}^2 x$ **d** $\sec x$

e $\sin^2 x - \cos^2 x = -\cos 2x$

f $\tan x \sec^2 x$

7 a $\sqrt{17}$ **b** $\frac{\sqrt{17}}{17}$ **c** $-\frac{\sqrt{17}}{4}$

8 a $-\sqrt{10}$ **b** $-\frac{\sqrt{10}}{10}$ **c** $-\frac{\sqrt{10}}{3}$

9 a $-3\sqrt{11}$ **b** $\frac{-3\sqrt{11}}{10}$

10 a $-\sqrt{35}$ **b** $\frac{\sqrt{35}}{6}$

11 a $\frac{-\sqrt{3}}{2}$ **b** $-\sqrt{3}$ **c** 2

12 a $-\frac{1}{3}$ **b** $-\frac{2\sqrt{2}}{3}$ **c** $-\frac{3\sqrt{2}}{4}$

13 a $\frac{\sqrt{51}}{10}$ **b** $-\frac{\sqrt{51}}{7}$ **c** $-\frac{7\sqrt{51}}{51}$

14 a 0.2 **b** $-\frac{2\sqrt{6}}{5}$ **c** $-\frac{\sqrt{6}}{12}$

15 a 0 **b** $\frac{1}{2}\sin 2\theta$ **c** 1 **d** 1

16 $x - \dfrac{1}{x} = -2\tan\theta$

Exercise 3B

1 a $\sin 2x \cos 5y - \cos 2x \sin 5y$

b $\cos x^2 \cos y - \sin x^2 \sin y$

c $\dfrac{\tan x + \tan y + \tan z - \tan x \tan y \tan z}{1 - \tan x \tan y - \tan x \tan z - \tan y \tan z}$

2 a $\sin(x - 2y)$ **b** $\cos x$ **c** $\tan B$
d $\sin 2A$ **e** $\cos y$

3 a $\sin x \cos 2x + \cos x \sin 2x$
b $3\sin x - 4\sin^3 x$

4 a $\cos x \cos 2x - \sin x \sin 2x$
b $4\cos^3 x - 3\cos x$

5 a $\dfrac{\sqrt{2}(\sqrt{3}-1)}{4}$ **b** $2 + \sqrt{3}$
c $\dfrac{\sqrt{2}}{4}(1 - \sqrt{3})$ **d** $2 - \sqrt{3}$

6 a -0.8 **b** 2.6 **c** $\frac{5}{13}$
d $\frac{12}{13}$ **e** -0.75 **f** $\frac{16}{65}$
g $\frac{63}{65}$ **h** $\frac{33}{56}$ **i** $\frac{-837}{116}$

7 a $\dfrac{-\sqrt{51}}{10}$ **b** $\dfrac{\sqrt{21}}{5}$
c 0.40 **d** -0.36

8 a $\frac{1}{4}\sin 2x$ **b** $-\cos 2x$ **c** $\frac{1}{2}\tan 2x$
d -1 **e** $-2\tan x$ **f** $\sin 2x$

9 a 0.96 **b** -0.28 **c** $-\frac{24}{7}$

10 a $-\frac{3}{4}$ **b** $\frac{9}{13}$

11 $\sqrt{2} - 1$

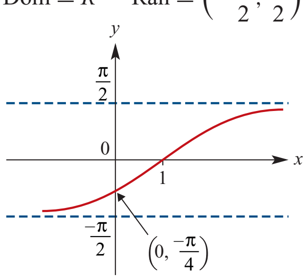
12 a -0.66 **b** 0.91

13 0.97

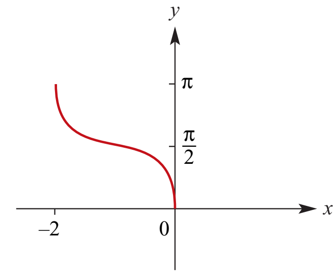
14 a $\frac{12}{5}$ **b** $\frac{2}{3}$ **c** $3\frac{1}{3}$ m

Exercise 3C

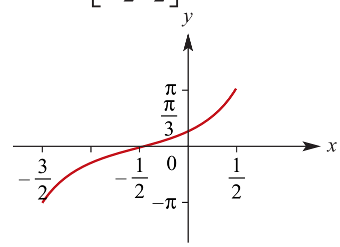
1 a Dom $= R$ Ran $= \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$



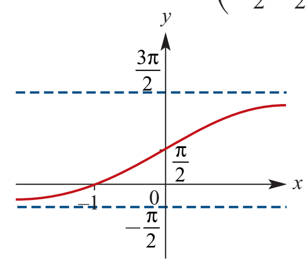
b Dom $= [-2, 0]$ Ran $= [0, \pi]$



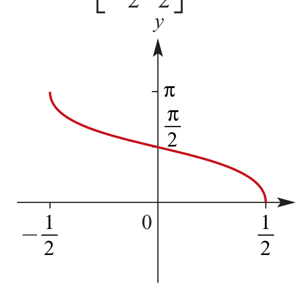
c Dom $= \left[-\frac{3}{2}, \frac{1}{2}\right]$ Ran $= [-\pi, \pi]$



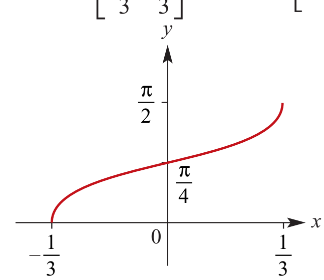
d Dom $= R$ Ran $= \left(-\frac{\pi}{2}, \frac{3\pi}{2}\right)$



e Dom $= \left[-\frac{1}{2}, \frac{1}{2}\right]$ Ran $= [0, \pi]$



f Dom $= \left[\frac{-1}{3}, \frac{1}{3}\right]$ Ran $= \left[0, \frac{\pi}{2}\right]$



Answers

2 a $\frac{\pi}{2}$ **b** $-\frac{\pi}{4}$ **c** $\frac{\pi}{6}$ **d** $\frac{5\pi}{6}$ **e** $\frac{\pi}{3}$
f $\frac{\pi}{4}$ **g** $-\frac{\pi}{3}$ **h** $\frac{\pi}{6}$ **i** π

3 a $\frac{\sqrt{3}}{2}$ **b** $-\frac{\pi}{3}$ **c** -1 **d** $\frac{\sqrt{2}}{2}$
e $\frac{\pi}{4}$ **f** $\sqrt{3}$ **g** $\frac{\pi}{3}$ **h** $-\frac{\pi}{3}$
i $-\frac{\pi}{4}$ **j** $\frac{5\pi}{6}$ **k** π **l** $-\frac{\pi}{4}$

4 a $f^{-1}: [-1, 1] \to R$, $f^{-1}(x) = y$ where $\sin y = x \quad y \in \left[\frac{\pi}{2}, \frac{3\pi}{2}\right]$
$(f^{-1}(x) = \pi - \sin^{-1}(x))$
b i 1 **ii** $\frac{\sqrt{2}}{2}$ **iii** $-\frac{1}{2}$
iv $\frac{3\pi}{2}$ **v** π **vi** $\frac{5\pi}{6}$

5 a $[1, 3]$; $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ **b** $\left[\frac{-3\pi}{4}, \frac{\pi}{4}\right]$; $[-1, 1]$
c $\left[\frac{-5}{2}, \frac{-3}{2}\right]$, $\left[\frac{-\pi}{2}, \frac{\pi}{2}\right]$
d $\left[-\frac{\pi}{18}, \frac{5\pi}{18}\right]$; $[-1, 1]$
e $\left[\frac{\pi}{6}, \frac{7\pi}{6}\right]$; $[-1, 1]$ **f** $[-2, 0]$, $[0, \pi]$
g $[-1, 1]$; $\left[0, \frac{\pi}{2}\right]$ **h** $\left[-\frac{\pi}{3}, \frac{\pi}{6}\right]$; $[-1, 1]$
i R; $\left[0, \frac{\pi}{2}\right)$ **j** $\left(0, \frac{\pi}{2}\right)$; R
k R; $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$
l $\left(\frac{-\sqrt{2\pi}}{2}, \frac{\sqrt{2\pi}}{2}\right)$; $R^+ \cup \{0\}$

6 a $\frac{3}{5}$ **b** $\frac{12}{5}$ **c** $\frac{24}{25}$
d $\frac{40}{9}$ **e** $\sqrt{3}$ **f** $\frac{\sqrt{5}}{3}$
g $\frac{-2\sqrt{5}}{5}$ **h** $\frac{2\sqrt{10}}{7}$ **i** $\frac{7\sqrt{149}}{149}$

7 a i $\frac{4}{5}$ **ii** $\frac{12}{13}$

8 a $[0, \pi]$, $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ **b** $[0, 1]$, $[0, 1]$
c $\left[-\frac{\pi}{4}, \frac{\pi}{4}\right]$, $[0, \pi]$ **d** $[0, 1]$, $[-1, 0]$
e $[0, 1]$, $[-1, 1]$ **f** $[0, \pi]$, $\left[-\frac{\pi}{4}, \frac{\pi}{4}\right]$
g $R^+ \cup \{0\}$, $(0,1]$ **h** R, $(-1, 1)$

Exercise 3D

1 a $\frac{7\pi}{6}, \frac{11\pi}{6}$ **b** $\frac{\pi}{12}, \frac{17\pi}{12}$
c $\frac{\pi}{6}, \frac{11\pi}{6}$ **d** $\frac{\pi}{4}, \frac{5\pi}{4}$
e $\frac{5\pi}{6}, \frac{11\pi}{6}$ **f** $\frac{\pi}{24}, \frac{13\pi}{24}, \frac{25\pi}{24}, \frac{37\pi}{24}$

2 a $\frac{\pi}{6}, \frac{5\pi}{6}$ **b** $\frac{5\pi}{6}, \frac{7\pi}{6}$ **c** $\frac{\pi}{3}, \frac{4\pi}{3}$
d $\frac{3\pi}{4}, \frac{7\pi}{4}$ **e** $\frac{\pi}{3}, \frac{5\pi}{3}$ **f** $\frac{5\pi}{4}, \frac{7\pi}{4}$

3 a $\frac{\pi}{4} + 2\pi n, \frac{3\pi}{4} + 2\pi n, n \in Z$
b $2\pi n, n \in Z$ **c** $\frac{\pi}{6} + \pi n, n \in Z$

4 a ± 1.16 **b** $-0.20, -2.94$ **c** $1.03, -2.11$

5 a $\frac{\pi}{4}, \frac{\pi}{2}, \frac{5\pi}{4}, \frac{3\pi}{2}$ **b** $0, \frac{\pi}{3}, \pi, \frac{5\pi}{3}, 2\pi$
c $\frac{\pi}{6}, \frac{\pi}{2}, \frac{5\pi}{6}, \frac{3\pi}{2}$
d $\frac{\pi}{24}, \frac{\pi}{8}, \frac{5\pi}{24}, \frac{3\pi}{8}, \frac{13\pi}{24}, \frac{5\pi}{8}, \frac{17\pi}{24}, \frac{7\pi}{8}, \frac{25\pi}{24}, \frac{9\pi}{8}, \frac{29\pi}{24}, \frac{11\pi}{8}, \frac{37\pi}{24}, \frac{13\pi}{8}, \frac{41\pi}{24}, \frac{15\pi}{8}$
e $0, \frac{2\pi}{3}, \frac{4\pi}{3}, 2\pi$ **f** $\frac{\pi}{6}, \frac{5\pi}{6}, \frac{3\pi}{2}$
g $0, \frac{3\pi}{4}, \pi, \frac{7\pi}{4}, 2\pi$ **h** $\frac{3\pi}{4}, \frac{7\pi}{4}$
i $\frac{\pi}{3}, \frac{5\pi}{3}$ **j** $0, \frac{\pi}{2}, 2\pi$

6 a Maximum $= 3$, minimum $= 1$
b Maximum $= 1$, minimum $= \frac{1}{3}$
c Maximum $= 5$, minimum $= 4$
d Maximum $= \frac{1}{4}$, minimum $= \frac{1}{5}$
e Maximum $= 3$, minimum $= -1$
f Maximum $= 9$, minimum $= 5$

7 a $(-1.14, -2.28)$, $(0, 0)$, $(1.14, 2.28)$
b $(-1.24, -1.24)$, $(0, 0)$, $(1.24, 1.24)$
c $(3.79, -0.79)$ **d** $(0, 0)$, $(4.49, 4.49)$

8 $2\pi - q$

9 a $\pi + \alpha, 2\pi - \alpha$ **b** $\frac{\pi}{2} - \alpha, \frac{3\pi}{2} + \alpha$

10 a $\pi - \beta, \beta - \pi$ **b** $\frac{\pi}{2} - \beta, \beta - \frac{3\pi}{2}$

11 a $2\pi - \gamma, 3\pi - \gamma$ **b** $\frac{3\pi}{2} - \gamma, \frac{5\pi}{2} - \gamma$

12 0, 0.33, 2.16 **13** 1.50
14 b 45.07 **15** 0.86 **16** 1.93
17 b 1.113
18 When $t = 0$, $x_A = x_B = 0$;
when $t = 1.29$, $x_A = x_B = 0.48$
19 b 0.94

Multiple-choice questions

1 C **2** C **3** E **4** D **5** A
6 A **7** E **8** D **9** E **10** E

Short-answer questions (technology-free)

1 a $\frac{7}{25}$ **b** $\frac{24}{25}$ **c** $\frac{24}{7}$ **d** $\frac{5}{3}$ **e** $\frac{4}{3}$

2 a $0, \pi, 2\pi, \frac{\pi}{3}, -\frac{\pi}{3}, \frac{5\pi}{3}$

b $-\frac{\pi}{2}, \frac{\pi}{2}, \frac{3\pi}{2}, \frac{\pi}{3}, -\frac{\pi}{3}, \frac{5\pi}{3}$

c $-\frac{\pi}{2}, \frac{\pi}{2}, \frac{3\pi}{2}$ **d** $-\frac{\pi}{2}, \frac{\pi}{2}, \frac{3\pi}{2}$

e $\frac{\pi}{2}, -\frac{\pi}{6}, \frac{7\pi}{6}, \frac{11\pi}{6}, -\frac{5\pi}{6}$

f $0, 2\pi, \frac{\pi}{3}, -\frac{\pi}{3}, \frac{5\pi}{3}$

3 a $\frac{7\pi}{6}, \frac{11\pi}{6}, \sin^{-1}\left(\frac{1}{3}\right), \pi - \sin^{-1}\left(\frac{1}{3}\right)$

b $\frac{\pi}{6}, \frac{5\pi}{6}, \frac{7\pi}{6}, \frac{11\pi}{6}$

c $\frac{\pi}{4}, \frac{5\pi}{4}$ **d** $\frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$

4 a $-\frac{\sqrt{3}}{2}$ **b** $\frac{2\sqrt{3}}{3}$ **c** 2

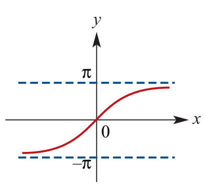
d 2 **e** 1 **f** $-\sqrt{3}$

5 a $-p$ **b** $-p$ **c** $\frac{1}{p}$ **d** $-\frac{1}{p}$ **e** $-p$

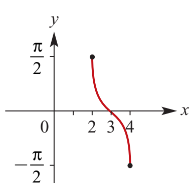
6 a $\frac{\pi}{3}$ **b** $\frac{1}{2}$ **c** $\frac{2\pi}{3}$

d $\frac{2\pi}{3}$ **e** $\frac{\sqrt{3}}{2}$ **f** $\frac{\sqrt{2}}{2}$

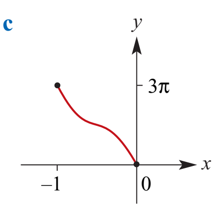
7 a



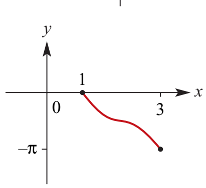
b



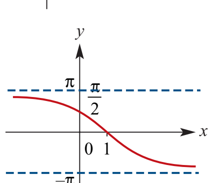
c



d



e



Extended-response questions

1 a **i** x **ii** $\sqrt{1-x^2}$ **iii** $\frac{x}{\sqrt{1-x^2}}$

iv $2x$ **v** $\sqrt{1-4x^2}$ **vi** $\frac{2x}{\sqrt{1-4x^2}}$

b **i** $2x\sqrt{1-x^2} - x\sqrt{1-4x^2}$

ii $\sqrt{(1-4x^2)(1-x^2)} + 2x^2$

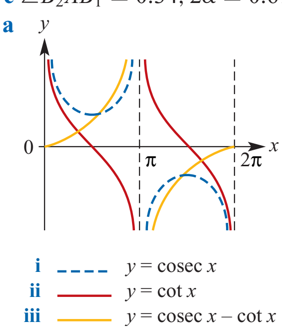
iii $\frac{2x\sqrt{1-x^2} - x\sqrt{1-4x^2}}{\sqrt{(1-4x^2)(1-x^2)} + 2x^2}$

iv $\frac{2x\sqrt{1-x^2}}{1-2x^2}$ **v** $2x\sqrt{1-x^2}$

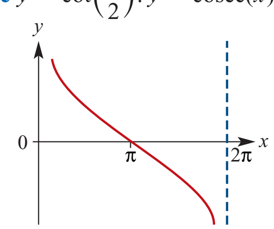
vi $1-2x^2$

c $\angle B_2AB_1 = 0.34, 2\alpha = 0.61$

2 a



c $y = \cot\left(\frac{x}{2}\right)$: $y = \operatorname{cosec}(x) + \cot(x)$



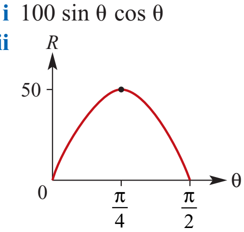
d **ii** $\cot\frac{\pi}{8} = \sqrt{2}+1$, $\cot\frac{\pi}{12} = 2+\sqrt{3}$

iii $\frac{1}{\sqrt{4+2\sqrt{2}}}$

e $\cot\frac{\theta}{2} - \cot 4\theta$

3 a **i** $100 \sin\theta \cos\theta$

ii



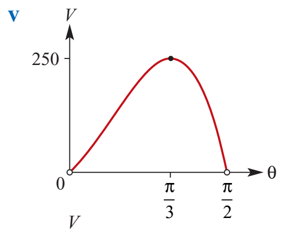
iii 50 **iv** $\frac{\pi}{4}$

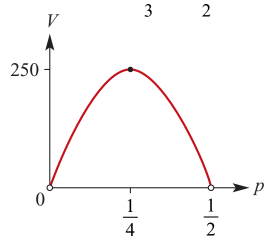
b **ii** $a = 2000, b = -4000$

iii $V = 2000p - 4000p^2$

iv $0 < p < \frac{1}{2}$

v

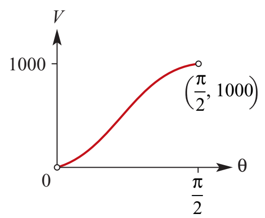




vi Maximum volume = 250 when $p = \frac{1}{4}, \theta = \frac{\pi}{3}$

c i $V = 1000 \sin^2 \theta, 0 < \theta < \frac{\pi}{2}$

ii

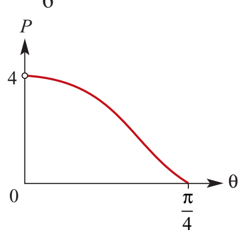


iii V is an increasing function. As the angle θ gets larger, so does the volume of the cuboid. $0 < \theta < \frac{\pi}{2}$ for the cuboid to exist.

4 b $p = 8 \cos^3 \theta - 4 \cos \theta$

c iii $\frac{\pi}{6}$ **iv** 1

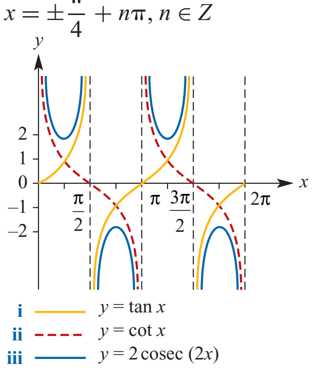
d



e $\frac{\pi}{4}$

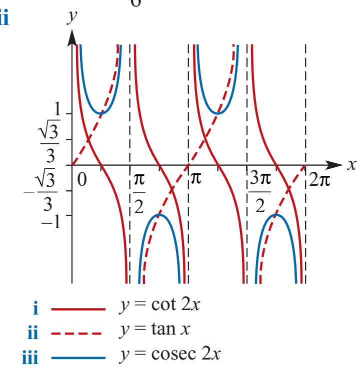
5 a ii $x = \pm\frac{\pi}{4} + n\pi, n \in Z$

iii



b ii $x = n\pi \pm \frac{\pi}{6}, n \in Z$

iii

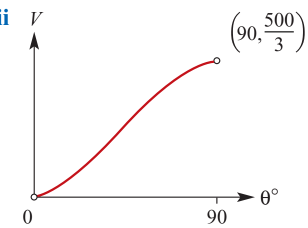


6 a i $\angle BAE = 72°, \angle AEC = 72°, \angle ACE = 72°$

ii 36°

e $\frac{\sqrt{5} - 1}{4}$

7 a ii



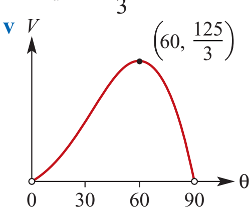
iii V is an increasing function. As the angle θ gets larger, so does the volume of the pyramid. $0 < \theta < 90$ for the pyramid to exist.

b ii $\theta \in (0, 90)$

iii $V = -\frac{2000}{3}a^2 + \frac{1000}{3}a$

iv $V_{max} = \frac{125}{3}$ when $\theta = 60$

v



8 a i $V = \frac{500}{3} \cos \theta° \sin^2 \theta°$

ii $V_{max} = 64.15$ when $\theta = 54.74$

b ii $\theta \in (0, 90)$

iii $V_{max} = 24.69$ when $a = 0.67$

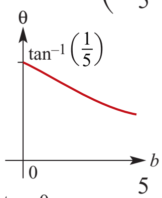
$\theta = 48.19$

9 c i $x = \frac{a \pm \sqrt{a^2 - 4b(a + b)}}{2}$

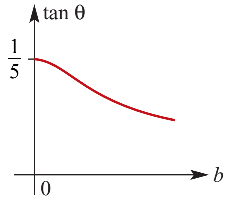
ii $1 + \sqrt{2}$

d 0.62

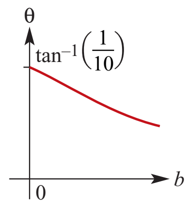
e **i** $\theta = \tan^{-1}\left(\frac{b+1}{5}\right) - \tan^{-1}\left(\frac{b}{5}\right)$



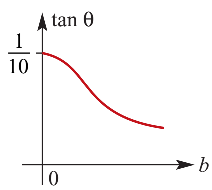
$\tan\theta = \dfrac{5}{25 + b + b^2}$



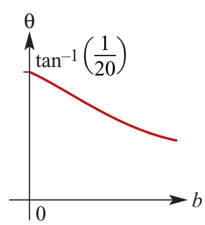
ii $\theta = \tan^{-1}\left(\frac{b+1}{10}\right) - \tan^{-1}\left(\frac{b}{10}\right)$



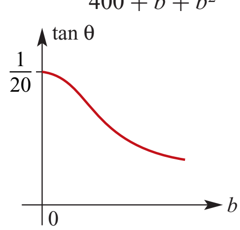
$\tan\theta = \dfrac{10}{100 + b + b^2}$



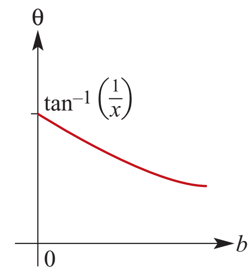
iii $\theta = \tan^{-1}\left(\frac{b+1}{20}\right) - \tan^{-1}\left(\frac{b}{20}\right)$



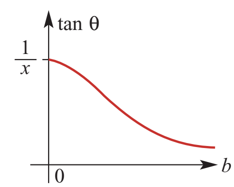
$\tan\theta = \dfrac{20}{400 + b + b^2}$



iv In general, the graph of $\theta = \tan^{-1}\left(\frac{b+1}{x}\right) - \tan^{-1}\left(\frac{b}{x}\right)$ has the b-axis as a horizontal asymptote. The domain is $[0, \infty)$ and the range is $\left(0, \tan^{-1}\left(\frac{1}{x}\right)\right)$. The θ-axis intercept is $\tan^{-1}\left(\frac{1}{x}\right)$. The function is decreasing as b increases.



In general, the graph of $\tan\theta = \dfrac{x}{x^2 + b + b^2}$ has the b axis as a horizontal asymptote. The tan θ axis intercept is $\frac{1}{x}$. The domain is $[0, \infty)$ and the range is $\left(0, \frac{1}{x}\right]$. The function is decreasing as b increases.



10 **a** Each has a right angle, and $\angle CAD$ is common to both triangles.
b $(\cos 2\theta, \sin 2\theta)$
c **i** $2\cos\theta$ **ii** $2\sin\theta$

Chapter 4

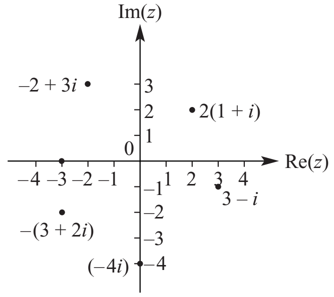
Exercise 4A

1 **a** $5i$ **b** $3\sqrt{3}i$ **c** $-5i$
d $13i$ **e** $5\sqrt{2}i$ **f** $-2\sqrt{3}$
g $-1 + 2i$ **h** 4 **i** 0

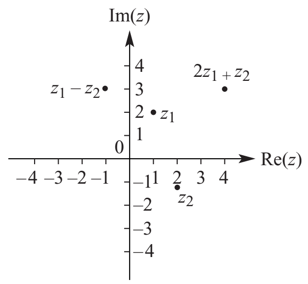
2 **a** $5 + i$ **b** $4 + 4i$ **c** $5 - 5i$
d $4 - 3i$ **e** $-1 + i$ **f** 2
g 2 **h** 1 **i** $3 - 2i$

3 **a** $x = 5, y = 0$ **b** $x = 0, y = 2$
c $x = 0, y = 0$ **d** $x = 9, y = -4$
e $x = -2, y = -2$ **f** $x = 13, y = 6$

4



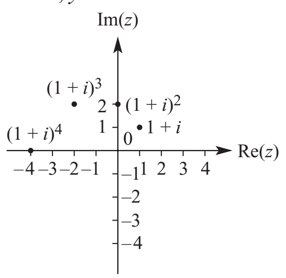
5



6 a $11 + 3i$ **b** $47 - i$ **c** 13
d $-8 + 6i$ **e** $3 - 4i$ **f** $-2 + 2i$
g 1 **h** $5 - 6i$ **i** -1

7 a $x = 4, y = -3$ **b** $x = -2, y = 5$
c $x = -3$
d $x = 3, y = -3$, or $x = -3, y = 3$
e $x = 3, y = 2$

8 a



b Anticlockwise turn $\frac{\pi}{4}$ about the origin.
Distance from origin increases by factor $\sqrt{2}$

9 a $\overrightarrow{PQ} = \begin{bmatrix} -3 \\ -1 \end{bmatrix} = \overrightarrow{OR}$ **b** $|\overrightarrow{QP}| = \sqrt{10}$

Exercise 4B

1 a $\sqrt{3}$ **b** $-8i$ **c** $4 + 3i$
d $-1 + 2i$ **e** $4 - 2i$ **f** $-3 + 2i$

2 a i **b** $\frac{3}{10} - \frac{1}{10}i$ **c** $-3 + 4i$
d $\frac{17}{5} + \frac{1}{5}i$ **e** $\frac{-1 - \sqrt{3}}{2} + \frac{\sqrt{3} - 1}{2}i$
f $4 + i$

4 a $5 - 5i$ **b** $6 + i$ **c** $2 + 3i$
d $\frac{2 - i}{5}$ **e** $-8i$ **f** $8 + 6i$

5 a $a^2 + b^2$ **b** $\frac{a}{a^2 + b^2} + \frac{b}{a^2 + b^2}i$ **c** $2a$
d $2bi$ **e** $\frac{a^2 - b^2}{a^2 + b^2} + \frac{2ab}{a^2 + b^2}i$
f $\frac{a^2 - b^2}{a^2 + b^2} - \frac{2ab}{a^2 + b^2}i$

Exercise 4C

1 a $3; \pi$ **b** $5; \frac{\pi}{2}$ **c** $\sqrt{2}; \frac{3\pi}{4}$
d $2; \frac{\pi}{6}$ **e** $4; -\frac{\pi}{3}$ **f** $16; -\frac{2\pi}{3}$

2 a 1.18 **b** 2.06 **c** −2.50
d −0.96 **e** 0.89 **f** −1.98

3 a $\frac{5\pi}{3}$ **b** $\frac{3\pi}{2}$ **c** $\frac{5\pi}{6}$
d $\frac{\pi}{4}$ **e** $-\frac{11\pi}{6}$ **f** $-\frac{3\pi}{2}$

4 a $-\frac{3\pi}{4}$ **b** $\frac{5\pi}{6}$ **c** $\frac{\pi}{8}$ **d** $-\frac{\pi}{2}$

5 a $\sqrt{2}\text{ cis}\left(-\frac{3\pi}{4}\right)$ **b** $\text{cis}\left(-\frac{\pi}{3}\right)$
c $\sqrt{6}\text{ cis}\left(-\frac{\pi}{4}\right)$ **d** $\frac{2}{3}\text{ cis }\frac{\pi}{6}$
e $2\sqrt{2}\text{ cis}\left(-\frac{\pi}{6}\right)$ **f** $4\text{ cis }\frac{5\pi}{6}$

6 a $-\sqrt{2} + \sqrt{2}i$ **b** $\frac{5}{2} - \frac{5\sqrt{3}}{2}i$
c $2 + 2i$ **d** $\frac{3\sqrt{3}}{2} - \frac{3}{2}i$
e $6i$ **f** -4

8 a $2\text{ cis}\left(-\frac{3\pi}{4}\right)$ **b** $7\text{ cis }\frac{2\pi}{3}$
c $3\text{ cis }\frac{\pi}{3}$ **d** $5\text{ cis }\frac{\pi}{4}$

Exercise 4D

1 a $12\text{ cis}\left(-\frac{7\pi}{12}\right)$ **b** $\frac{1}{2}\text{ cis}\left(-\frac{\pi}{3}\right)$
c $\frac{7}{6}\text{ cis}\left(-\frac{\pi}{15}\right)$ **d** $8\text{ cis}\left(-\frac{19\pi}{20}\right)$
e $8\text{ cis }\frac{\pi}{3}$ **f** $\frac{8}{27}\text{ cis }\frac{\pi}{8}$ **g** $-\frac{1}{8}$
h $27\text{ cis }\frac{5\pi}{6}$ **i** $-32i$ **j** -216
k $1024\text{ cis }-\frac{\pi}{12}$ **l** $\frac{27}{4}\text{ cis}\left(-\frac{\pi}{20}\right)$

2 a $\text{Arg}(z_1z_2) = \frac{7\pi}{12}$
$\text{Arg}(z_1) + \text{Arg}(z_2) = \frac{7\pi}{12}$
$\text{Arg}(z_1z_2) = \text{Arg}(z_1) + \text{Arg}(z_2)$

b $\text{Arg}(z_1z_2) = \frac{7\pi}{12}$

$\text{Arg}(z_1) + \text{Arg}(z_2) = \frac{-17\pi}{12}$

$\text{Arg}(z_1z_2) = \text{Arg}(z_1) + \text{Arg}(z_2) + 2\pi$

c $\text{Arg}(z_1z_2) = \frac{-5\pi}{6}$

$\text{Arg}(z_1) + \text{Arg}(z_2) = \frac{7\pi}{6}$

$\text{Arg}(z_1z_2) = \text{Arg}(z_1) + \text{Arg}(z_2) - 2\pi$

4 a $\frac{\pi}{4}$ **b** $\frac{-3\pi}{4}$ **c** $\frac{-\pi}{4}$

5 b i $\text{cis}\left(\frac{3\pi}{2} - 7\theta\right)$ **ii** i **iii** $\text{cis}\, 4\theta$

iv $\text{cis}(\pi - \theta - \phi)$

6 b i $\text{cis}(-5\theta)$ **ii** $\text{cis}\, 3\theta$

iii 1 **iv** $\text{cis}\left(\frac{\pi}{2} - 2\theta\right)$

7 b i $\text{cis}(6\theta - 3\pi)$ **ii** $\text{cis}(\pi - 2\theta)$

iii $\text{cis}(\theta - \pi)$ **iv** $-i$

8 a i $\sec\theta\,\text{cis}\,\theta$ **ii** $\text{cosec}\,\theta\,\text{cis}\left(\frac{\pi}{2} - \theta\right)$

iii $\frac{1}{\sin\theta\cos\theta}\text{cis}(\theta) = \text{cosec}\,\theta\sec\theta\,\text{cis}(\theta)$

b i $\sec^2\theta\,\text{cis}\,2\theta$

ii $\sin^3\theta\,\text{cis}\left(3\theta - \frac{3\pi}{2}\right)$

iii $\frac{1}{\sin\theta\cos\theta}\text{cis}(-\theta) = \text{cosec}\,\theta\sec\theta\,\text{cis}(-\theta)$

9 a 64 **b** $\frac{\sqrt{2}}{8}\text{cis}\left(\frac{-3\pi}{4}\right)$

c $128\,\text{cis}\left(\frac{-2\pi}{3}\right)$ **d** $\frac{-\sqrt{3}i}{72}$

e $\sqrt{2}\,\text{cis}\left(\frac{-\pi}{4}\right)$ **f** $\frac{64\sqrt{3}}{3}\text{cis}\left(\frac{3\pi}{4}\right)$

g $\frac{\sqrt{2}}{2}i$ **h** $\frac{1}{4}\text{cis}\left(-\frac{2\pi}{15}\right)$

i $8\sqrt{2}\,\text{cis}\,\frac{11\pi}{12}$

Exercise 4E

1 a $(z + 4i)(z - 4i)$ **b** $(z + \sqrt{5}i)(z - \sqrt{5}i)$

c $(z + 1 + 2i)(z + 1 - 2i)$

d $\left(z - \frac{3}{2} + \frac{\sqrt{7}}{2}i\right)\left(z - \frac{3}{2} - \frac{\sqrt{7}}{2}i\right)$

e $2\left(z - 2 + \frac{\sqrt{2}}{2}i\right)\left(z - 2 - \frac{\sqrt{2}}{2}i\right)$

f $3\left(z + 1 + \frac{\sqrt{3}}{3}i\right)\left(z + 1 - \frac{\sqrt{3}}{3}i\right)$

g $3\left(z + \frac{1}{3} + \frac{\sqrt{5}}{3}i\right)\left(z + \frac{1}{3} - \frac{\sqrt{5}}{3}i\right)$

h $2\left(z - \frac{1}{4} + \frac{\sqrt{23}}{4}i\right)\left(z - \frac{1}{4} - \frac{\sqrt{23}}{4}i\right)$

i $(z - 5)\left(z + \frac{1}{2} + \frac{\sqrt{3}}{2}i\right)\left(z + \frac{1}{2} - \frac{\sqrt{3}}{2}i\right)$

j $(z+2)\left(z - \frac{3}{2} + \frac{\sqrt{11}}{2}i\right)\left(z - \frac{3}{2} - \frac{\sqrt{11}}{2}i\right)$

k $3(z-4)\left(z - \frac{1}{6} + \frac{\sqrt{11}}{6}i\right)\left(z - \frac{1}{6} - \frac{\sqrt{11}}{6}i\right)$

l $2(z+3)\left(z - \frac{3}{4} + \frac{\sqrt{31}}{4}i\right)\left(z - \frac{1}{4} - \frac{\sqrt{31}}{4}i\right)$

m $(z + 3)(z - 3)(z + 3i)(z - 3i)$

n $(z + 2)(z - 2)(z - 1 + \sqrt{3}i)(z - 1 - \sqrt{3}i)$
$(z + 1 + \sqrt{3}i)(z + 1 - \sqrt{3}i)$

o $(z + i)(z - i)(z - 2 + i)$

2 a $(z - i)\left(z + \frac{1}{2} + \frac{\sqrt{3}}{2}i\right)\left(z + \frac{1}{2} - \frac{\sqrt{3}}{2}i\right)$

b $(z + i)(z - 1 + \sqrt{2})(z - 1 - \sqrt{2})$

c $(z - 2i)(z - 3)(z + 1)$

d $2(z - i)\left(z + \frac{1}{4} + \frac{\sqrt{41}}{4}\right)\left(z + \frac{1}{4} - \frac{\sqrt{41}}{4}\right)$

3 b $z - 1 + i$

c $(z + 6)(z - 1 + i)(z - 1 - i)$

4 b $z + 2 + i$

c $(2z + 1)(z + 2 + i)(z + 2 - i)$

5 b $z - 1 - 3i$

c $(z - 1 + 3i)(z - 1 - 3i)(z + 1 + i)(z + 1 - i)$

6 a 8 **b** -4 **c** -6

Exercise 4F

1 a $\pm 5i$ **b** $\pm 2\sqrt{2}i$ **c** $2 \pm i$

d $\frac{-7 \pm \sqrt{11}i}{6}$ **e** $1 \pm \sqrt{2}i$

f $\frac{3 \pm \sqrt{11}i}{10}$ **g** $3, -2 \pm \sqrt{2}i$

h $5, \frac{1 \pm \sqrt{23}i}{2}$ **i** $-1, \frac{5 \pm \sqrt{7}i}{2}$

j $-2, 3, \frac{1 \pm \sqrt{23}i}{2}$

2 a $a = 0, b = 4$ **b** $a = -6, b = 13$

c $a = 2, b = 10$

3 a $1 - 3i, \frac{1}{3}$ **b** $-2 + i, 2 \pm \sqrt{2}i$

4 $P(x) = -2x^3 + 10x^2 - 18x + 10$;
$x = 1$ or $x = 2 \pm i$

5 $a = 6, b = -8$

6 a $z^2 - 4z + 5, a = -7, b = 6$

b $z = 2 \pm i$ or $z = \frac{-1}{2}$

7 a $P(1 + i) = (-4a + d - 2) + 2(a - 1)i$

b $a = 1, d = 6$

c $z = 1 \pm i$ or $z = -1 \pm i\sqrt{2}$

8 $p = -(5 + 4i)$, $q = 1 + 7i$

9 $z = 1 + i$ or $z = 2$

10 a $z = 3 + i$

b $z = 2i$ or $\pm\sqrt{6}$

c $z = 1$ or $\pm i\sqrt{6}$

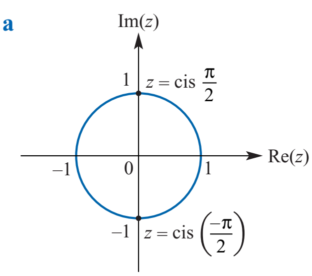
d $z = 2$ or $-\dfrac{1}{2} \pm i\dfrac{\sqrt{15}}{2}$

e $z = \dfrac{\sqrt{2}}{4} \pm i\dfrac{\sqrt{14}}{4}$

f $z = 0$ or $-1 \pm 2\sqrt{2}i$

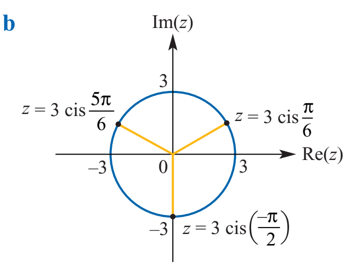
Exercise 4G

1 a



Solutions: $z = i$ or $z = -i$

b

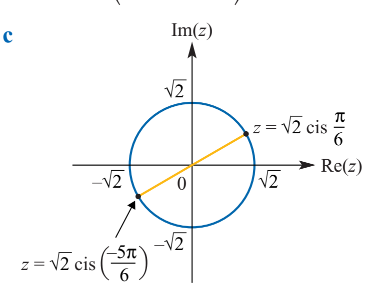


Solutions:

$z = 3\left(\dfrac{\sqrt{3}}{2} + \dfrac{1}{2}i\right)$

or $z = 3\left(\dfrac{-\sqrt{3}}{2} + \dfrac{1}{2}i\right)$ or $z = -3i$

c

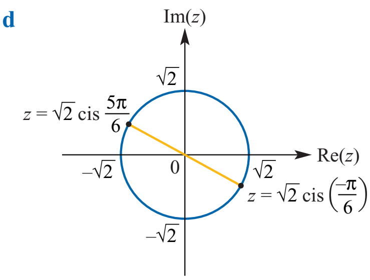


Solutions:

$z = \sqrt{2}\left(\dfrac{\sqrt{3}}{2} + \dfrac{1}{2}i\right)$ or

$z = \sqrt{2}\left(-\dfrac{\sqrt{3}}{2} - \dfrac{1}{2}i\right)$

d

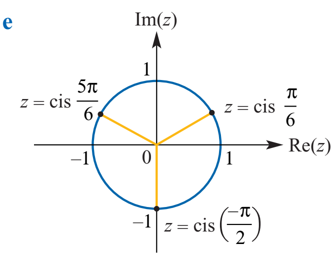


Solutions:

$z = \sqrt{2}\left(\dfrac{\sqrt{3}}{2} - \dfrac{1}{2}i\right)$

or $z = \sqrt{2}\left(-\dfrac{\sqrt{3}}{2} + \dfrac{1}{2}i\right)$

e

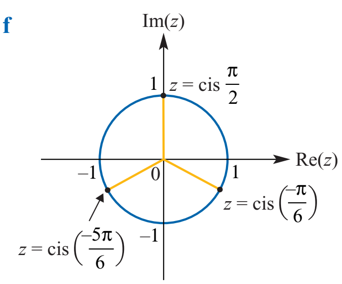


Solutions:

$z = \dfrac{\sqrt{3}}{2} + \dfrac{1}{2}i$ or $z = -\dfrac{\sqrt{3}}{2} + \dfrac{1}{2}i$ or

$z = -i$

f



Solutions: $z = \dfrac{\sqrt{3}}{2} - \dfrac{1}{2}i$

or $z = i$ or $z = -\dfrac{\sqrt{3}}{2} - \dfrac{1}{2}i$

2 a $2 \text{ cis}\left(\dfrac{-\pi}{12}\right), 2 \text{ cis } \dfrac{7\pi}{12}, 2 \text{ cis}\left(\dfrac{-3\pi}{4}\right)$

b $2 \text{ cis } \dfrac{\pi}{4}, 2 \text{ cis } \dfrac{11\pi}{12}, 2 \text{ cis}\left(\dfrac{-5\pi}{12}\right)$

c $2 \text{ cis}\left(\dfrac{-5\pi}{18}\right), 2 \text{ cis } \dfrac{7\pi}{18}, 2 \text{ cis}\left(\dfrac{-17\pi}{18}\right)$

d $2 \text{ cis}\left(\dfrac{-\pi}{18}\right), 2 \text{ cis } \dfrac{11\pi}{18}, 2 \text{ cis}\left(\dfrac{-13\pi}{18}\right)$

e $5 \text{ cis}\left(\dfrac{-\pi}{6}\right), 5 \text{ cis } \dfrac{\pi}{2}, 5 \text{ cis}\left(\dfrac{-5\pi}{6}\right)$

f $2^{\frac{1}{6}} \text{ cis } \dfrac{\pi}{4}, 2^{\frac{1}{6}} \text{ cis } \dfrac{11\pi}{12}, 2^{\frac{1}{6}} \text{ cis}\left(\dfrac{-5\pi}{12}\right)$

3 a $a^2 - b^2 = 3, 2ab = 4$

b $a = \pm 2, b = \pm 1$

The square roots of $3 + 4i$ are $\pm(2 + i)$

4 a $\pm(1 - 4i)$ **b** $\pm\frac{\sqrt{2}}{2}(7 + i)$

c $\pm(1 + 2i)$ **d** $\pm(3 + 4i)$

5 $\sqrt{2}\text{ cis }\frac{\pi}{6}, \sqrt{2}\text{ cis}\left(\frac{-5\pi}{6}\right), \sqrt{2}\text{ cis}\left(\frac{-\pi}{6}\right),$
$\sqrt{2}\text{ cis }\frac{5\pi}{6}$

6 Solutions are $z = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i$ or $\frac{-\sqrt{2}}{2} - \frac{\sqrt{2}}{2}i$; factors are $z - \frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}i$ and $z + \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i$

7 Solutions are $z = \text{cis }\frac{\pi}{8}, \text{cis }\frac{3\pi}{8}, \text{cis }\frac{5\pi}{8},$
$\text{cis }\frac{7\pi}{8}, \text{cis }\frac{9\pi}{8}, \text{cis }\frac{11\pi}{8}, \text{cis }\frac{13\pi}{8}$ or
$\text{cis }\frac{15\pi}{8}$; factors are $z - \text{cis }\frac{\pi}{8}, z - \text{cis }\frac{3\pi}{8},$
$z - \text{cis }\frac{5\pi}{8}, z - \text{cis }\frac{7\pi}{8}, z - \text{cis }\frac{9\pi}{8},$
$z - \text{cis }\frac{11\pi}{8}, z - \text{cis }\frac{13\pi}{8}$ and $z - \text{cis }\frac{15\pi}{8}$

8 a $\pm\left(\sqrt{\frac{1 + \sqrt{2}}{2}} + \sqrt{\frac{\sqrt{2} - 1}{2}}\,i\right)$

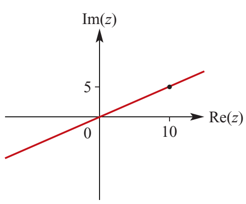
b $2^{\frac{1}{4}}\text{cis}\frac{\pi}{8}, 2^{\frac{1}{4}}\text{cis}\left(\frac{-7\pi}{8}\right)$

c $\cos\frac{\pi}{8} = \frac{(2 + \sqrt{2})^{\frac{1}{2}}}{2},$

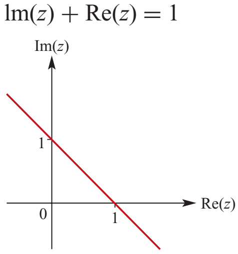
$\sin\frac{\pi}{8} = \frac{(2 - \sqrt{2})^{\frac{1}{2}}}{2}$

Exercise 4H

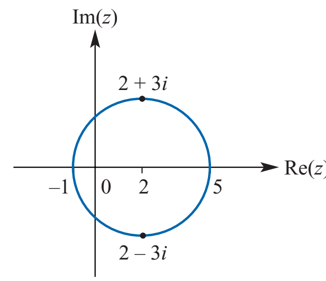
1 a $2\,\text{Im}(z) = \text{Re}(z)$



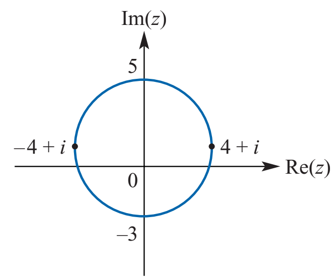
b $\text{Im}(z) + \text{Re}(z) = 1$



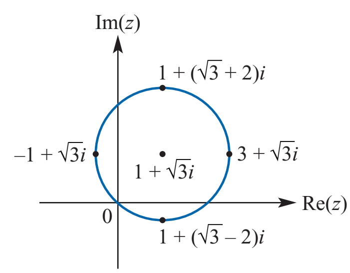
c $|z - 2| = 3$



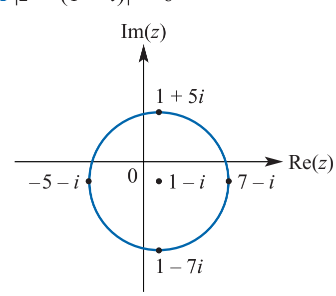
d $|z - i| = 4$



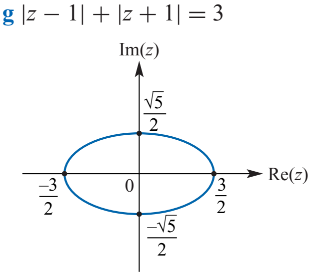
e $|z - (1 + \sqrt{3}i)| = 2$



f $|z - (1 - i)| = 6$

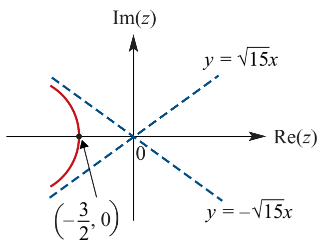


g $|z - 1| + |z + 1| = 3$

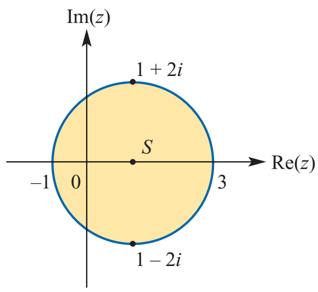


Answers

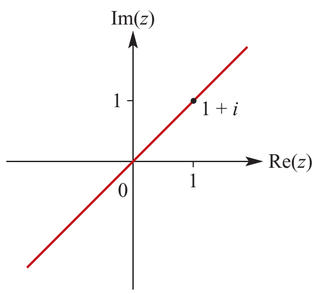
h $|z-6|-|z+6|=3$



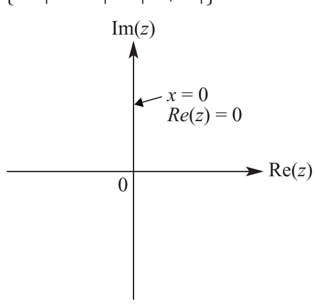
2 $S=\{z:|z-1|\le 2\}$



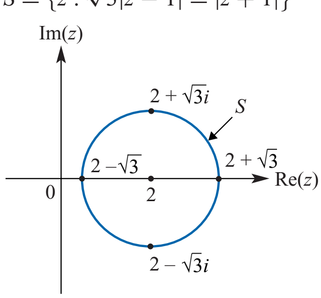
3 $\{z : z = i\bar{z}\}$



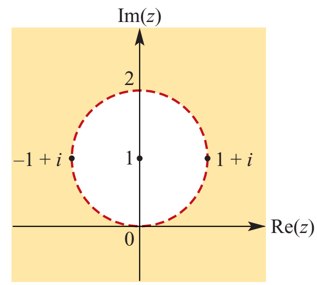
4 $\{z:|z-1|=|z+1|\}$



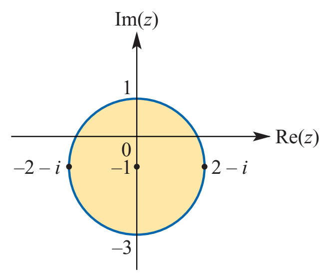
5 $S=\{z:\sqrt{3}|z-1|=|z+1|\}$



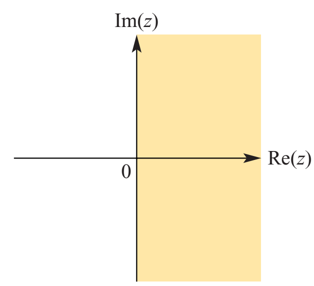
6 a $\{z:|z-i|>1\}$



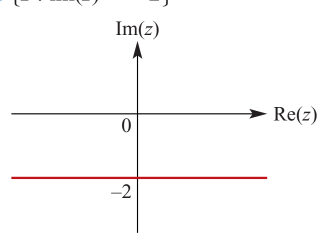
b $\{z:|z+i|\le 2\}$



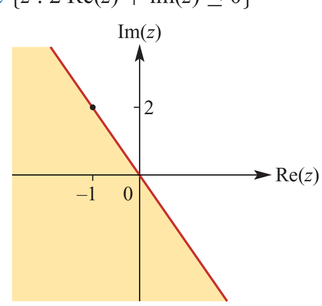
c $\{z \mid \text{Re}(z)\ge 0\}$



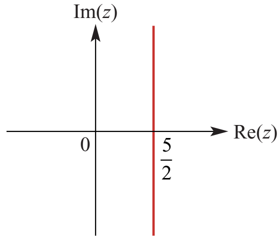
d $\{z:\text{Im}(z)=-2\}$



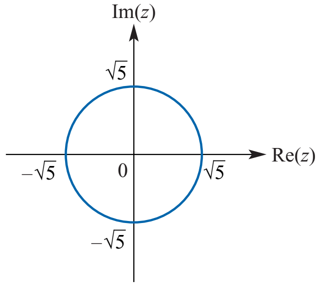
e $\{z:2\,\text{Re}(z)+\text{Im}(z)\le 0\}$



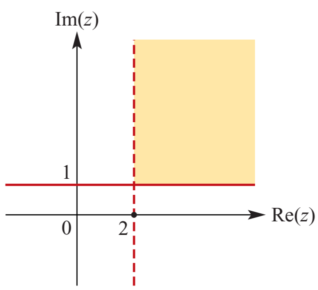
f $\{z : z + \bar{z} = 5\}$



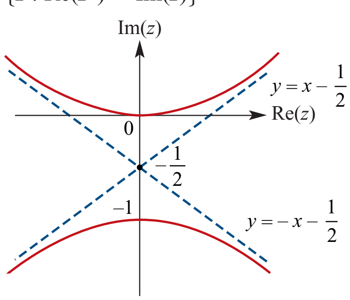
g $\{z : z\bar{z} = 5\}$



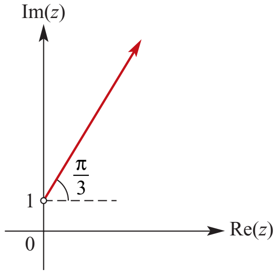
h $\{z : \text{Re}(z) > 2 \text{ and } \text{Im}(z) \geq 1\}$



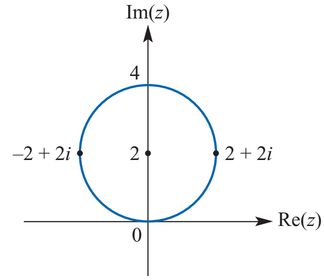
i $\{z : \text{Re}(z^2) = \text{Im}(z)\}$



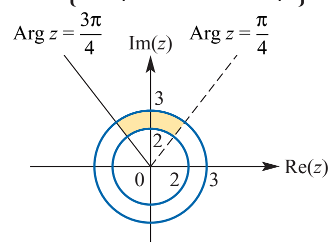
j $\left\{z : \text{Arg}(z - i) = \dfrac{\pi}{3}\right\}$



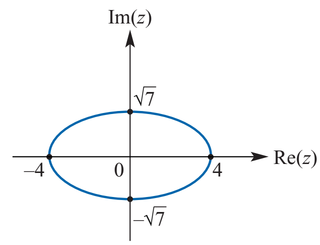
k $\{z : |z + 2i| = 2|z - i|\}$



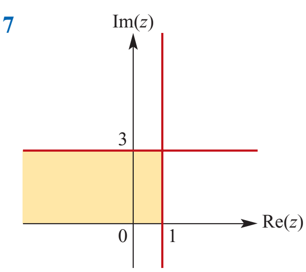
l $\{z : 2 \leq |z| \leq 3\}$
$\cap \left\{z : \dfrac{\pi}{4} < \text{Arg}(z) \leq \dfrac{3\pi}{4}\right\}$



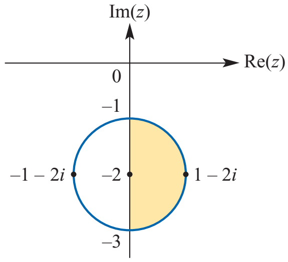
m $\{z : |z + 3| + |z - 3| = 8\}$



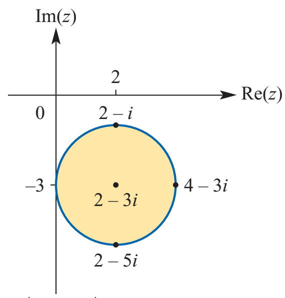
7



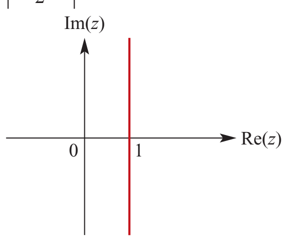
8 $\text{Re}(z) \geq 0$ and $|z + 2i| \leq 1$



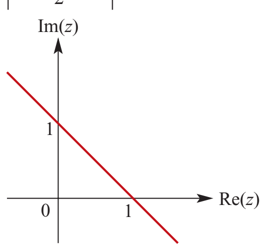
9 $|z - 2 + 3i| \le 2$



10 a $\left|\dfrac{z-2}{z}\right| = 1$



b $\left|\dfrac{z-1-i}{z}\right| = 1$



11 $x^2 + y^2 = 1$

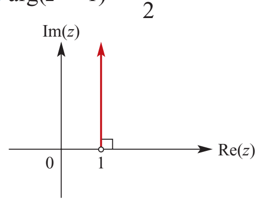
12 The circle has centre $\left(\frac{8}{3}, -2\right)$ and radius $\dfrac{4\sqrt{10}}{3}$

13 $|z|^2 : 1$

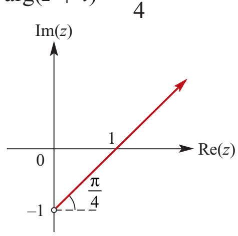
14 a A circle of centre (1, 1) and radius 1.

b $y = -x$

c $\arg(z - 1) = \dfrac{\pi}{2}$



d $\arg(z + i) = \dfrac{\pi}{4}$



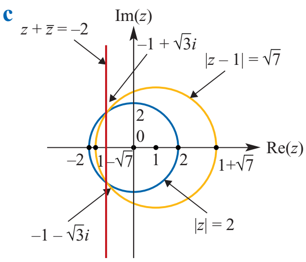
15 A circle with centre (2, 4) and radius 6.

16 a $z = -1 \pm \sqrt{3}i$

b i $|z| = 2$ **ii** $|z - 1| = \sqrt{7}$

iii $z + \bar{z} = -2$

c



Multiple-choice questions

1 E	**2** C	**3** D	**4** E	**5** D
6 B	**7** B	**8** C	**9** B	**10** A

Short-answer questions (technology-free)

1 a $8 - 5i$ **b** $-i$ **c** $29 + 11i$

d 13 **e** $\dfrac{6}{13} + \dfrac{4}{13}i$ **f** $\dfrac{9}{5} - \dfrac{7}{5}i$

g $\dfrac{3}{5} + \dfrac{6}{5}i$ **h** $-8 - 6i$ **i** $\dfrac{43}{10} + \dfrac{81}{10}i$

2 a $2 \pm 3i$ **b** $-6 + 2i$ **c** $-3 \pm \sqrt{3}i$

d $\dfrac{3}{\sqrt{2}}(1 \pm i),\ \dfrac{3}{\sqrt{2}}(-1 \pm i)$

e $3,\ \dfrac{3}{2}(-1 \pm \sqrt{3}i)$ or $3 \operatorname{cis}\left(\pm\dfrac{2\pi}{3}\right)$

f $-\dfrac{3}{2},\ \dfrac{3}{4}(1 \pm \sqrt{3}i)$ or $\dfrac{3}{2}\operatorname{cis}\left(\pm\dfrac{\pi}{3}\right)$

3 a $2 - i, 2 + i, -2$ **b** $3 - 2i, 3 + 2i, -1$

c $1 + i, 1 - i, 2$

4 a $2\left(x + \dfrac{3}{4} + \dfrac{\sqrt{7}}{4}i\right)\left(x + \dfrac{3}{4} - \dfrac{\sqrt{7}}{4}i\right)$

b $(x - 1)(x + i)(x - i)$ **c** $(x + 2)^2(x - 2)$

5 2 and −1, −2 and 1

6 a and **iv**, **b** and **ii**, **c** and **i**, **d** and **iii**

7 −1 and 5, 1 and −5

8 $a = 2, b = 5$ **9** $\frac{1}{2}\operatorname{cis}\left(-\frac{\pi}{3}\right)$

10 $a = \dfrac{3}{2} - \dfrac{\sqrt{3}}{2},\ b = \dfrac{1}{2} + \dfrac{3\sqrt{3}}{2}$

11 a $2 + 2i$ **b** $\dfrac{1}{2}(1 + i)$

c $8\sqrt{2}$ **d** $\dfrac{\pi}{4}$

12 a i $\sqrt{2}$ **ii** 2 **iii** $\dfrac{\pi}{4}$ **iv** $-\dfrac{\pi}{3}$

b $\dfrac{\sqrt{2}}{2}, -\dfrac{\pi}{12}$

13 $2 \operatorname{cis} \frac{\pi}{6}, -64\sqrt{3} - 64i$

14 $\pm 3, \pm 3i, 1 \pm i$

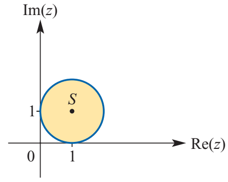
15 $16 - 16i$ **16** $-2i, i, -2, k = -2$ or 1

17 a $(z + 2)(z - 1 + i)(z - 1 - i)$ **b** 25

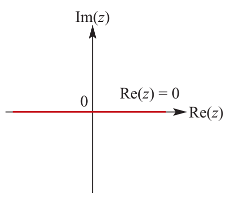
18 $-1 + 2i, -1 - \frac{1}{2}i$

19 a $(x - 1)^2 + (y - 1)^2 \leq 1$

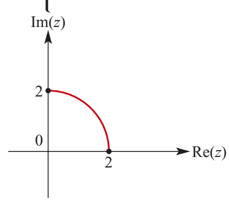
b



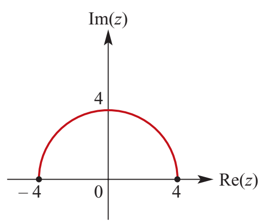
20 $\{z : |z + i| = |z - i|\}$



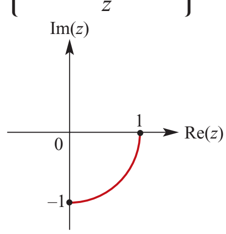
21 a $S : \left\{z : z = 2 \operatorname{cis} \theta, 0 \leq \theta \leq \frac{\pi}{2}\right\}$



b $\{w : w = z^2, z \in S\}$



c $\left\{v : v = \frac{2}{z}, z \in S\right\}$



22 $\left(\frac{5}{6}, \frac{-7}{6}\right)$

23 a $4 - 3i$

b $c = 12 + 3i, d = 9 - i$
or $c = 4 + 9i, d = 1 + 5i$

24 a $2 \operatorname{cis} \frac{\pi}{3}$, $2 \operatorname{cis} \pi$ and $2 \operatorname{cis}\left(-\frac{\pi}{3}\right)$

b $2 \operatorname{cis} \frac{\pi}{6}$ and $2 \operatorname{cis}\left(-\frac{5\pi}{6}\right)$

25 a $(x + 1)(x - 1)(x^2 - x + 1)(x^2 + x + 1)$

b $(x + 1)(x - 1)\left(x - \frac{1}{2} + \frac{\sqrt{3}}{2}i\right)$
$\times \left(x - \frac{1}{2} - \frac{\sqrt{3}}{2}i\right)\left(x + \frac{1}{2} + \frac{\sqrt{3}}{2}i\right)$
$\times \left(x + \frac{1}{2} - \frac{\sqrt{3}}{2}i\right)$

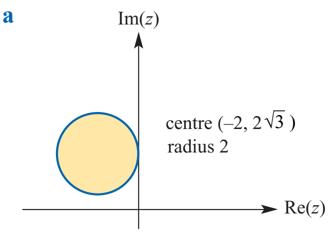
c $-1, 1, \frac{1}{2} \pm \frac{\sqrt{3}}{2}i$, or $-\frac{1}{2} \pm \frac{\sqrt{3}}{2}i$

26 a 1 **b** 1 **c** 0

27 $\frac{-\pi}{4}$

28 a $-2 + 2\sqrt{3}i$ **b** $-3 - 6i$

29 a

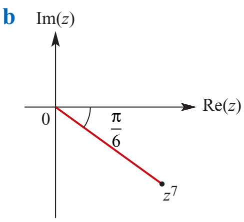


b i 2 **ii** $\frac{5\pi}{6}$

Extended-response questions

1 a $|z^7| = 16\,384$ $\operatorname{Arg}(z^7) = \frac{-\pi}{6}$

b



c $2\sqrt{2} \operatorname{cis} \frac{7\pi}{12}$

d $z = -2\sqrt{3} + 2i, w = 1 + i,$
$\frac{z}{w} = (1 - \sqrt{3}) + (1 + \sqrt{3})i$

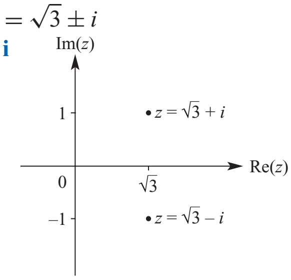
e $-2 - \sqrt{3}$ **f** $\frac{\sqrt{3}}{3}$

2 b $3, 2 - i$

d $z^5 - 9z^4 + 36z^3 - 84z^2 + 115z - 75$

3 a $z = \sqrt{3} \pm i$

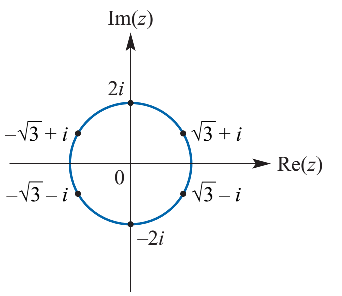
b i



ii $x^2 + y^2 = 4$

iii $a = 2$

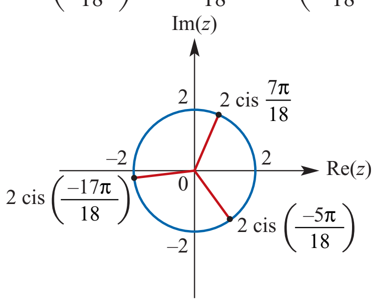
iv $P(z) = z^2 + 2\sqrt{3}z + 4$
The solutions to the equation $z^6 + 64 = 0$ are equally spaced on an Argand diagram around the circumference of the circle $x^2 + y^2 = 4$, and represent the sixth roots of -64. Three of the solutions are the conjugates of the other three solutions.



4 a $8 \operatorname{cis}\left(\frac{-5\pi}{6}\right)$

b $2 \operatorname{cis}\left(\frac{-5\pi}{18}\right), 2 \operatorname{cis} \frac{7\pi}{18}, 2 \operatorname{cis}\left(\frac{-17\pi}{18}\right)$

c



d i $(z - \sqrt{3}i)^3 = -4\sqrt{3} - 4i$

ii $z = 2\cos\left(\frac{-5\pi}{18}\right) + \left(2\sin\left(\frac{-5\pi}{18}\right) + \sqrt{3}\right)i$

$z = 2\cos\frac{7\pi}{18} + \left(2\sin\left(\frac{7\pi}{18}\right) + \sqrt{3}\right)i$

$z = 2\cos\left(\frac{-17\pi}{18}\right) + \left(2\sin\left(\frac{-17\pi}{18}\right) + \sqrt{3}\right)i$

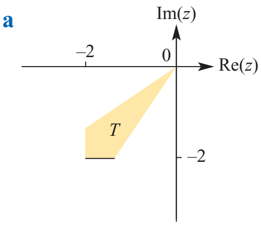
5 a $\overrightarrow{XY} = \sqrt{3}\boldsymbol{i} - \boldsymbol{j}, \overrightarrow{XZ} = 2\sqrt{3}\boldsymbol{i} + 2\boldsymbol{j}$

b $z_3 = 1 + \sqrt{3}i$

c $z_3 = 2 \operatorname{cis} \frac{\pi}{3}$, W corresponds to $6\sqrt{3}$.

d $(4\sqrt{3}, 0)$

6 a

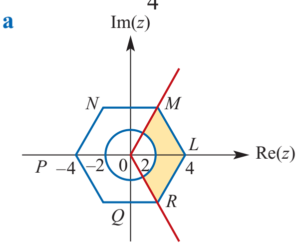


b $T = \{z : \operatorname{Re}(z) > -2\} \cap \{z : \operatorname{Im}(z) \geq -2\} \cap \left\{z : \frac{-5\pi}{6} < \operatorname{Arg} z < \frac{-2\pi}{3}\right\}$

7 a $k > -\frac{5}{4}$ **b** $k = -\frac{5}{4}$

c $-2 < k < -\frac{5}{4}$

8 a



b $|z - 4| = 4$

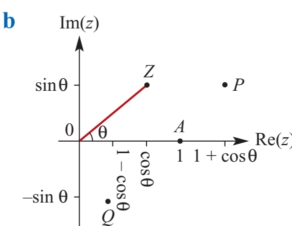
c N is the point $4 \operatorname{cis} \frac{2\pi}{3}$

Q is the point $4 \operatorname{cis}\left(\frac{-2\pi}{3}\right)$

d The new position of N is $4 \operatorname{cis} \frac{5\pi}{12}$

The new position of Q is $4 \operatorname{cis}\left(\frac{-11\pi}{12}\right)$

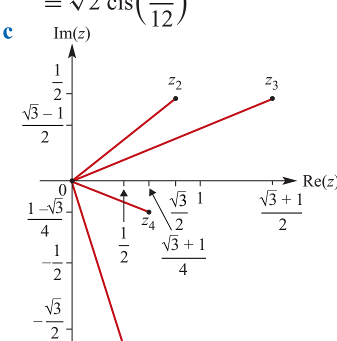
9 b



c $\operatorname{cosec} \theta + \cot \theta = \cot \frac{\theta}{2}$

10 b $z_3 = \sqrt{2} \operatorname{cis}(\tan^{-1}(2 - \sqrt{3}))$
$= \sqrt{2} \operatorname{cis}\left(\frac{\pi}{12}\right)$

c



11 a ii $q = 2k^3$ **b** $b = -1 - i, \boldsymbol{c} = 2 + 2i$

12 a i $6\sqrt{2}$ **ii** 6 **b ii** Isosceles

13 a i 13 **ii** $157.38^\circ = 2.75^c$

b i $\cos\alpha = \dfrac{-12}{13}$, $\sin\alpha = \dfrac{5}{13}$

ii $r = \sqrt{13}$, $\cos 2\theta = \dfrac{-12}{13}$, $\sin 2\theta = \dfrac{5}{13}$

iii $\sin\theta = \pm\dfrac{5\sqrt{26}}{26}$, $\cos\theta = \pm\dfrac{\sqrt{26}}{26}$

iv $w = \pm\dfrac{\sqrt{2}}{2}(1 + 5i)$

d $\pm\dfrac{\sqrt{2}}{2}(5 + i)$, a reflection of the square roots of $-12 + 5i$ in the line $\mathrm{Re}(z) = \mathrm{Im}(z)$.

14 a $\left(x + \dfrac{3}{2}\right)^2 + y^2 = \dfrac{29}{4}$

b $\left(x + \dfrac{3}{2}\right)^2 + \left(y - \dfrac{1}{2}\right)^2 = \dfrac{15}{2}$

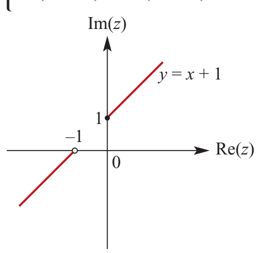
c $\left(x + \dfrac{\beta}{\alpha}\right)^2 + \gamma^2 = \dfrac{\beta^2 - \alpha\gamma}{\alpha^2}$

d $\left(x + \dfrac{a}{\alpha}\right)^2 + \left(y - \dfrac{b}{\alpha}\right)^2 = \dfrac{a^2 + b^2 - \alpha\gamma}{\alpha^2}$

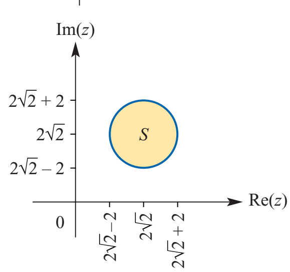
where $\beta = a + bi$

15 a $(\cos^5\theta - 10\cos^3\theta\sin^2\theta + 5\cos\theta\sin^4\theta) + (5\cos^4\theta\sin\theta - 10\cos^2\theta\sin^3\theta + \sin^5\theta)i$

16 a $\left\{z: (1 + i)z + (1 - i)\bar{z} = -2, \mathrm{Arg}\, z \le \dfrac{\pi}{2}\right\}$



b i



ii $|z|_{max} = 6$, $|z|_{min} = 2$

iii $\mathrm{Arg}(z)_{max} = 75^\circ = \dfrac{5\pi}{12}$,

$\mathrm{Arg}(z)_{min} = 15^\circ = \dfrac{\pi}{12}$

17 a $2\,\mathrm{cis}\left(\pm\dfrac{2\pi}{3}\right)$

c $z^2 + (2 - 2\sqrt{3}i)\,z - 4\sqrt{3}i = 0$ or $z^2 + (2 + 2\sqrt{3}i)z + 4\sqrt{3}i = 0$ **d** -4

18 a i $z = 2\,\mathrm{cis}\,\theta + \frac{1}{2}\mathrm{cis}\,(-\theta)$

b i $z = 2i\,\mathrm{cis}\,\theta - \frac{1}{2}i\,\mathrm{cis}\,(-\theta)$

Chapter 5

5.1 Multiple-choice questions

1 A	**2** D	**3** E	**4** E
5 E	**6** C	**7** D	**8** C
9 C	**10** B	**11** D	**12** A
13 D	**14** D	**15** E	**16** B
17 C	**18** D	**19** B	**20** A
21 E	**22** D	**23** C	**24** E
25 A	**26** C	**27** C	**28** B
29 C	**30** C	**31** C	**32** E
33 C	**34** C	**35** E	**36** E
37 B	**38** E	**39** B	**40** A
41 C	**42** B	**43** A	**44** D
45 C	**46** A	**47** B	**48** E
49 A	**50** C	**51** D	**52** E
53 A	**54** C	**55** C	**56** D
57 B	**58** C	**59** E	**60** C
61 D	**62** B	**63** A	**64** D
65 E	**66** B	**67** D	**68** C
69 E	**70** D	**71** C	**72** D
73 B	**74** E		

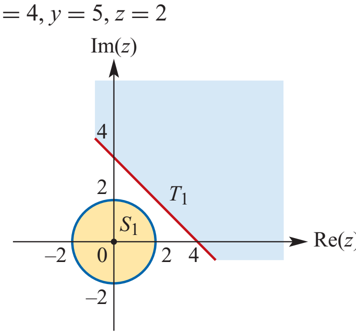
5.2 Extended-response questions

1 a i $\dfrac{3}{2}(\boldsymbol{b} - \boldsymbol{a})$ **ii** $\dfrac{1}{2}(3\boldsymbol{b} - \boldsymbol{a})$

b i $\overrightarrow{AB} = \boldsymbol{i} + 2\boldsymbol{j}$, $\overrightarrow{BC} = 2\boldsymbol{i} - \boldsymbol{j}$ **iv** $3\boldsymbol{i} - \boldsymbol{j}$

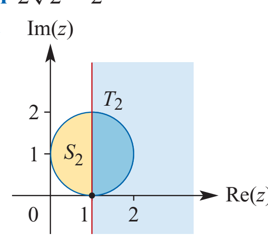
c $x = 4$, $y = 5$, $z = 2$

2 a i



ii $2\sqrt{2} - 2$

b i



ii The maximum and minimum values of $|z|$ are $\sqrt{2} + 1$ and 1 respectively

3 a i $\boldsymbol{a} + \boldsymbol{b}$ **ii** $\frac{1}{3}(\boldsymbol{a} - \boldsymbol{b})$ **iii** $\frac{2}{3}(\boldsymbol{a} - \boldsymbol{b})$

b $\overrightarrow{DA} = 2\overrightarrow{BD}$

4 a i 151° **ii** $\frac{1}{9}(34\boldsymbol{i} + 40\boldsymbol{j} + 23\boldsymbol{k})$

iii $x = 3, y = -2, z = 16$

b i $\boldsymbol{b} - \frac{1}{2}\boldsymbol{a}$ **ii** $\overrightarrow{OA} = 2\overrightarrow{BQ}$

5 b 4 : 1 : 3 **c** $4\boldsymbol{i} + \boldsymbol{j} + 3\boldsymbol{k}$

e $s = 3, t = -2$

6 a $\dfrac{\boldsymbol{a}.\boldsymbol{b}}{|\boldsymbol{a}||\boldsymbol{b}|}$ **b** $\dfrac{\sqrt{(\boldsymbol{a}.\boldsymbol{a})(\boldsymbol{b}.\boldsymbol{b}) - (\boldsymbol{a}.\boldsymbol{b})^2}}{|\boldsymbol{a}||\boldsymbol{b}|}$

8 c 8 : 1

9 a i $\frac{1}{3}(\boldsymbol{a} + 2\boldsymbol{b})$ **ii** $\frac{1}{6}(2\boldsymbol{b} - 5\boldsymbol{a})$

b i 2 : 3 **ii** 6 : 1

10 a i $2\boldsymbol{c} - \boldsymbol{b}$ **ii** $\frac{1}{3}(\boldsymbol{a} + 2\boldsymbol{b})$

iii $\frac{1}{5}(\boldsymbol{a} + 4\boldsymbol{c})$

11 c 3 : 1

12 a $z^2 - 2z + 4$

b i $2 \operatorname{cis}\left(-\frac{\pi}{3}\right)$ **ii** $4 \operatorname{cis}\left(-\frac{2\pi}{3}\right), -8$

iii $1 \pm \sqrt{3}i, -1$

c i $\sqrt{7}, \sqrt{7}$ **ii** isosceles

13 a $p = \frac{1}{3}(4 + 2\sqrt{2}i), q = \frac{1}{3}(2 + 4\sqrt{2}i)$

i $\boldsymbol{b} - \boldsymbol{a}$ **ii** $\frac{1}{2}(\boldsymbol{a} + \boldsymbol{b})$

iii $\frac{1}{3}(\boldsymbol{a} + \boldsymbol{b})$ **iv** $\frac{1}{3}(2\boldsymbol{a} - \boldsymbol{b})$

v $\frac{1}{3}(2\boldsymbol{b} - \boldsymbol{a})$

14 a $(z + 2i)(z - 2i)$ **b** $(z^2 + 2i)(z^2 - 2i)$

d $(z - (1 + i))(z + (1 + i))(z - (1 - i))(z + (1 - i))$

e $(z^2 - 2z + 2)(z^2 + 2z + 2)$

15 b circle centre $2 - i$ and radius $\sqrt{5}$

c perpendicular bisector of line joining $1 + 3i$ and $2 - i$

16 a $2 + 11i$

b i $\dfrac{2\sqrt{5}}{25}$ **ii** $\dfrac{11\sqrt{5}}{25}$

17 c i 1 **ii** -1

d i $z^2 - 3z + 3 = 0$

ii $z^2 + 2z + 13 = 0$

e 0, 3

18 a $z^4 + z^3 + z^2 + z + 1$ **c** $\operatorname{cis}\left(-\frac{2\pi}{5}\right)$

d $\operatorname{cis}\left(\pm\frac{2\pi}{5}\right), \operatorname{cis}\left(\pm\frac{4\pi}{5}\right), 1$

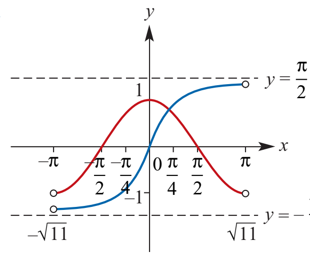
e $\left(z^2 - 2\cos\frac{2\pi}{5}z + 1\right)\left(z^2 - 2\cos\frac{4\pi}{5}z + 1\right)$

19 a 4, 9, -4 **b** 5

20 b $\cos 5\theta = \cos^5\theta\,(1 - 10\tan^2\theta + 5\tan^4\theta)$, $\sin 5\theta = \cos^5\theta\,(5\tan\theta - 10\tan^3\theta + \tan^5\theta)$

21 a $\operatorname{cis}(\pm\theta)$

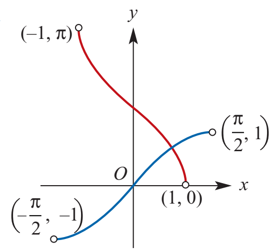
22 a



b i 0.67 **ii** 0.54

d 0.82

23 a



b i 0.48 **ii** 0.67

d (0.768, 0.695)

24 a $a = 5, d = -10$

b i 1.73 metres **ii** 8.03 metres

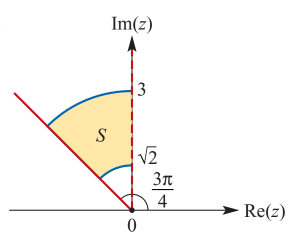
25 a i $c \cos x$ **ii** $a \cos y$

iii $c \cos x + a \cos y$

b i $\angle AOC = 2z, \angle AOY = z$

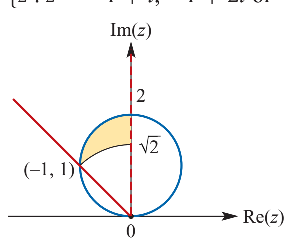
ii $2AO \sin z$

26 a

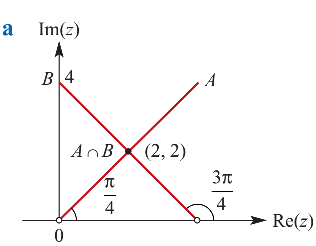


b $\{z : z = -1 + i, -1 + 2i \text{ or } -2 + 2i\}$

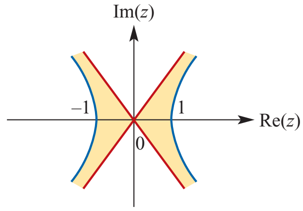
c



27 a



b



28 a $\overrightarrow{OA} = \boldsymbol{i} + \sqrt{\lambda}\boldsymbol{k}$, $\overrightarrow{CA} = 2\boldsymbol{i} - 3\boldsymbol{j} + \sqrt{\lambda}\boldsymbol{k}$
b $56°$ **c** $13 + 8\sqrt{3}$ since $\lambda > 0$

29 b i $\overrightarrow{OX} = \frac{1}{3}(\boldsymbol{a} + \boldsymbol{b} + \boldsymbol{c})$
$\overrightarrow{OY} = \frac{1}{3}(\boldsymbol{a} + \boldsymbol{c} + \boldsymbol{d})$
$\overrightarrow{OZ} = \frac{1}{3}(\boldsymbol{a} + \boldsymbol{b} + \boldsymbol{d})$
$\overrightarrow{OW} = \frac{1}{3}(\boldsymbol{b} + \boldsymbol{c} + \boldsymbol{d})$
ii $\overrightarrow{DX} = \frac{1}{3}(\boldsymbol{a} + \boldsymbol{b} + \boldsymbol{c}) - \boldsymbol{d}$
$\overrightarrow{BY} = \frac{1}{3}(\boldsymbol{a} + \boldsymbol{c} + \boldsymbol{d}) - \boldsymbol{b}$
$\overrightarrow{CZ} = \frac{1}{3}(\boldsymbol{a} + \boldsymbol{b} + \boldsymbol{d}) - \boldsymbol{c}$
$\overrightarrow{AW} = \frac{1}{3}(\boldsymbol{b} + \boldsymbol{c} + \boldsymbol{d}) - \boldsymbol{a}$
iii $\overrightarrow{OP} = \frac{1}{4}(\boldsymbol{a} + \boldsymbol{b} + \boldsymbol{c} + \boldsymbol{d})$
iv $\overrightarrow{OQ} = \overrightarrow{OR} = \overrightarrow{OS} = \frac{1}{4}(\boldsymbol{a} + \boldsymbol{b} + \boldsymbol{c} + \boldsymbol{d})$
v $Q = R = S = P$ which is the centre of the sphere which circumscribes the tetrahedron

30 $a = -4$, $b = \dfrac{\pi}{9}$, $c = \dfrac{-\pi}{3}$, $d = 8$

Chapter 6

Exercise 6A

1 a $x^4(5\sin x + x\cos x)$
b $e^x(\tan x + \sec^2 x)$
c $\sqrt{x}\left(\dfrac{\cos x}{2x} - \sin x\right)$
d $e^x(\text{cox}\, x - \sin x)$
e $x^2 e^x(3 + x)$
f $\cos^2 x - \sin^2 x = \cos 2x$
g $x^3(4\tan x + x\sec^2 x)$
h $\sec^2 x \log_e x + \dfrac{\tan x}{x}$
i $\sin x(1 + \sec^2 x)$
j $\sqrt{x}\left(\dfrac{\tan x}{2x} + \sec^2 x\right)$

2 a $\dfrac{\log_e x - 1}{(\log_e x)^2}$ **b** $\sqrt{x}\left(\dfrac{\cot x}{2x} - \text{cosec}^2 x\right)$
c $e^x(\cot x - \text{cosec}^2 x)$
d $\dfrac{\sec^2 x}{\log_e x} - \dfrac{\tan x}{x(\log_e x)^2}$
e $\dfrac{\cos x}{x^2} - \dfrac{2\sin x}{x^3}$
f $\sec x(\sec^2 x + \tan^2 x)$
g $\dfrac{-(\sin x + \cos x)}{e^x}$
h $-\text{cosec}^2 x$

3 a $2x\sec^2(x^2 + 1)$ **b** $\sin 2x$
c $e^{\tan x}\sec^2 x$ **d** $5\tan^4 x \sec^2 x$
e $\dfrac{\sqrt{x}\cos(\sqrt{x})}{2x}$ **f** $\frac{1}{2}\sec^2 x\sqrt{\cot x}$
g $\dfrac{\sin\left(\dfrac{1}{x}\right)}{x^2}$ **h** $2\tan x\sec^2 x$
i $\frac{1}{4}\sec^2\left(\dfrac{x}{4}\right)$ **j** $-\text{cosec}^2 x$

4 a $k\sec^2(kx)$ **b** $2\sec^2(2x)e^{\tan(2x)}$
c $6\tan(3x)\sec^2(3x)$
d $e^{\sin x}\left(\dfrac{1}{x} + \log_e x\cos x\right)$
e $6x\sin^2(x^2)\cos(x^2)$
f $e^{3x+1}\sec^2 x(3\cos x + \sin x)$
g $e^{3x}(3\tan(2x) + 2\sec^2(2x))$
h $\dfrac{\sqrt{x}\tan(\sqrt{x})}{2x} + \dfrac{\sec^2(\sqrt{x})}{2}$
i $\dfrac{2(x+1)\tan x\sec^2 x - 3\tan^2 x}{(x+1)^4}$
j $20x\sec^3(5x^2)\sin(5x^2)$

5 a $5(x-1)^4$ **b** $\dfrac{1}{x}$
c $e^x(3\sec^2(3x) + \tan(3x))$ **d** $-\sin x\, e^{\cos x}$
e $-12\cos^2(4x)\sin(4x)$
f $4\cos x(\sin x + 1)^3$
g $-\sin x\sin(2x) + 2\cos(2x)\cos x$
h $1 - \dfrac{1}{x^2}$ **i** $\dfrac{x^2(3\sin x - x\cos x)}{\sin^2 x}$
j $\dfrac{-(1 + \log_e x)}{(x\log_e x)^2}$

6 a $3x^2$ **b** $4y + 10$
c $-\sin(2z)$ **d** $\sin(2x)e^{\sin^2 x}$
e $-2\tan z\sec^2 z$ **f** $-2\cos y\,\text{cosec}^3 y$

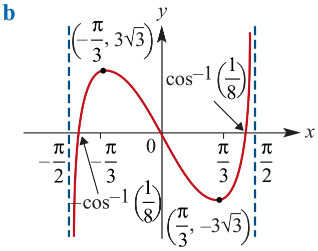
7 a $\dfrac{2}{2x+1}$ **b** $\dfrac{2}{2x-1}$
c $\cot x$ **d** $\sec x$
e $\dfrac{\sin^2 x - \cos^3 x}{\sin x\cos x(\cos x - \sin^2 x)}$
f $\text{cosec}\, x$ **g** $\text{cosec}(x)$
h $\dfrac{1}{\sqrt{x^2 - 4}}$, $x \neq -2, 2$ **i** $\dfrac{1}{\sqrt{x^2 + 4}}$

8 a $\dfrac{1}{2}$ **b** $\dfrac{2}{3}$ **c** 1

9 a $\left(-\dfrac{\pi}{3}, -\sqrt{3}\right)\left(\dfrac{\pi}{3}, \sqrt{3}\right)$
b $y = 4x - \dfrac{4\pi}{3} + \sqrt{3}$;
$y = 4x + \dfrac{4\pi}{3} - \sqrt{3}$

10 a i $\left(\dfrac{-\pi}{3}, 3\sqrt{3}\right), \left(\dfrac{\pi}{3}, -3\sqrt{3}\right)$
ii $\left(-\dfrac{\pi}{3}, 3\sqrt{3}\right)$ is a local maximum
$\left(\dfrac{\pi}{3}, -3\sqrt{3}\right)$ is a local minimum

b



11 a $\sqrt{2}e^{\frac{\pi}{4}}$ **b** $\left(\frac{-\pi}{4}, -\frac{\sqrt{2}}{2}e^{\left(\frac{-\pi}{4}\right)}\right)$

12 $\pm\frac{1}{2}\cos^{-1}\left(\frac{\sqrt{2\tan\frac{7\pi}{18}}}{\tan\frac{7\pi}{18}}\right)$

13 a $\frac{1}{4}\sin\left(\frac{x}{4}\right)\sec^2\left(\frac{x}{4}\right)$ **b** $\frac{\sqrt{2}}{4}$

c $y = \frac{\sqrt{2}}{4}(x - \pi + 4)$

Exercise 6B

1 a $\frac{1}{2}$ **b** $\frac{1}{2y}$ **c** $\frac{1}{4(2y-1)}$ **d** e^{-y}

e $\frac{1}{5\cos(5y)}$ **f** y **g** $\cos^2 y$

h $\frac{1}{3y^2+1}$ **i** y^2 **j** $\frac{1}{e^y(y+1)}$

2 a $\frac{64}{3}$ **b** $\frac{4}{3}$ **c** $\frac{1}{4}$ **d** 1

e $\frac{1}{4}$ **f** $\pm\frac{1}{8}$ **g** $-\frac{\sqrt{3}}{3}$ **h** $\pm\frac{1}{2}$

3 a $\frac{1}{6(2y-1)^2}$ **b** $\frac{1}{2e^{2y+1}}$

c $\frac{1}{2}(2y-1)$ **d** y

4 a $\frac{1}{6\sqrt[3]{x^2}}$ **b** $\frac{1}{2x}$ **c** $\frac{1}{2}e^x$ **d** $\frac{1}{2}e^{x+1}$

5 $y = \frac{1}{6}x - \frac{5}{6}, y = -\frac{1}{6}x + \frac{5}{6}$

6 a (5, −1), (12, 6) **b** $\left(-\frac{15}{4}, \frac{5}{2}\right)$

c $\left(-\frac{15}{4}, \frac{3}{2}\right)$

7 a (2, 2) **b** 8.13°

Exercise 6C

1 a $\frac{1}{\sqrt{4-x^2}}, x \in (-2, 2)$

b $\frac{-1}{\sqrt{16-x^2}}, x \in (-4, 4)$ **c** $\frac{3}{9+x^2}$

d $\frac{3}{\sqrt{1-9x^2}}, x \in \left(-\frac{1}{3}, \frac{1}{3}\right)$

e $\frac{-2}{\sqrt{1-4x^2}}, x \in \left(-\frac{1}{2}, \frac{1}{2}\right)$

f $\frac{5}{1+25x^2}$

g $\frac{3}{\sqrt{16-9x^2}}, x \in \left(-\frac{4}{3}, \frac{4}{3}\right)$

h $\frac{-3}{\sqrt{4-9x^2}}, x \in \left(-\frac{2}{3}, \frac{2}{3}\right)$

i $\frac{10}{25+4x^2}$

j $\frac{1}{\sqrt{25-x^2}}, x \in (-5, 5)$

2 a $\frac{1}{\sqrt{-x(x+2)}}, x \in (-2, 0)$

b $\frac{-1}{\sqrt{-x(x+1)}}, x \in (-1, 0)$

c $\frac{1}{x^2+4x+5}$

d $\frac{-1}{\sqrt{-x^2+8x-15}}, x \in (3, 5)$

e $\frac{3}{\sqrt{6x-9x^2}}, x \in \left(0, \frac{2}{3}\right)$

f $\frac{-3}{2x^2-2x+1}$

g $\frac{6}{\sqrt{-3(3x^2+2x-1)}}, x \in \left(-1, \frac{1}{3}\right)$

h $\frac{20}{\sqrt{-5(5x^2-6x+1)}}, x \in \left(\frac{1}{5}, 1\right)$

i $\frac{-10}{x^2-2x+5}$

j $\frac{-2x}{\sqrt{1-x^4}}, x \in (-1, 1)$

3 a $\frac{3}{x\sqrt{x^2-9}}$ **b** $\frac{-5}{x\sqrt{x^2-25}}$

c $\frac{3}{x\sqrt{4x^2-9}}$

4 a $\frac{a}{\sqrt{1-a^2x^2}}, x \in \left(-\frac{1}{a}, \frac{1}{a}\right)$

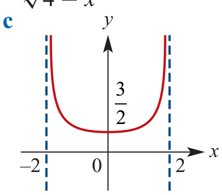
b $\frac{-a}{\sqrt{1-a^2x^2}}, x \in \left(-\frac{1}{a}, \frac{1}{a}\right)$

c $\frac{a}{1+a^2x^2}$

5 a i [−2, 2] **ii** $\left[-\frac{3\pi}{2}, \frac{3\pi}{2}\right]$

b $\frac{3}{\sqrt{4-x^2}}, x \in (-2, 2)$

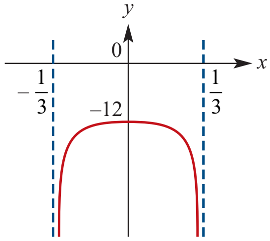
c



6 a i $\left[-\frac{1}{3}, \frac{1}{3}\right]$ **ii** $[0, 4\pi]$

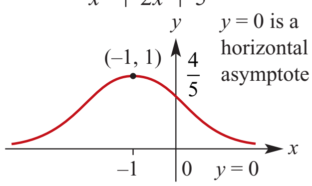
b $f'(x) = \dfrac{-12}{\sqrt{1-9x^2}}$, domain is $\left(-\frac{1}{3}, \frac{1}{3}\right)$

c



7 a i R **ii** $(-\pi, \pi)$

b $f'(x) = \dfrac{4}{x^2 + 2x + 5}$



8 a $f'(x) = \dfrac{2\sin^{-1} x}{\sqrt{1-x^2}}, x \in (-1, 1)$

b $f'(x) = 0, x \in (-1, 1)$

c $f'(x) = \dfrac{-x}{\sqrt{1-x^2}}, x \in (-1, 1)$

d $f'(x) = \dfrac{-x}{\sqrt{1-x^2}}, x \in (-1, 1)$

e $f'(x) = \dfrac{e^{\sin^{-1} x}}{\sqrt{1-x^2}}, x \in (-1, 1)$

f $f'(x) = \dfrac{e^x}{1+e^{2x}}$

9 a 0.35 **b** −6.29 **c** $\frac{3}{5}$

10 a $\pm\dfrac{\sqrt{3}}{2}$ **b** $\pm\dfrac{\sqrt{391}}{10}$

c $\pm\dfrac{\sqrt{5}}{3}$ **d** $-1 \pm \dfrac{\sqrt{1599}}{20}$

e $\pm\dfrac{\sqrt{35}}{4}$ **f** $\dfrac{1}{2}(1 \pm \sqrt{7})$

11 a $y = \dfrac{4\sqrt{3}}{3}x - \dfrac{\sqrt{3}}{3} + \dfrac{\pi}{6}$

b $y = x - \dfrac{1}{2} + \dfrac{\pi}{4}$

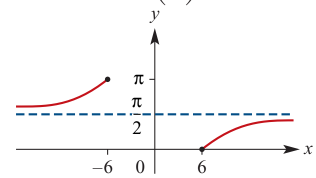
c $y = -2\sqrt{3}x + \dfrac{\sqrt{3}+\pi}{3}$

d $y = -6x + \sqrt{3} + \dfrac{\pi}{6}$

12 a $\{x : x \le -6\} \cup \{x : x \ge 6\}$

b $f'(x) = \dfrac{6}{x\sqrt{x^2-36}}, x < -6$ or $x > 6$

c $f(x) = \cos^{-1}\left(\dfrac{6}{x}\right)$



Exercise 6D

1 a $f''(x) = 0$ **b** $f''(x) = 56x^6$

c $f''(x) = \dfrac{-1}{4\sqrt{x^3}}$ **d** $f''(x) = 48(2x+1)^2$

e $f''(x) = -\sin x$ **f** $f''(x) = -\cos x$

g $f''(x) = e^x$ **h** $f''(x) = \dfrac{-1}{x^2}$

i $f''(x) = \dfrac{2}{(x+1)^3}$

j $f''(x) = 2\sin x \sec^3 x$

2 a $\dfrac{d^2y}{dx^2} = \dfrac{15\sqrt{x}}{4}$

b $\dfrac{d^2y}{dx^2} = 8(x^2+3)^2(7x^2+3)$

c $\dfrac{d^2y}{dx^2} = -\dfrac{1}{4}\sin\dfrac{x}{2}$

d $\dfrac{d^2y}{dx^2} = -48\cos(4x+1)$

e $\dfrac{d^2y}{dx^2} = 2e^{2x+1}$ **f** $\dfrac{d^2y}{dx^2} = \dfrac{-4}{(2x+1)^2}$

g $\dfrac{d^2y}{dx^2} = 6\sin(x-4)\sec^3(x-4)$

h $\dfrac{d^2y}{dx^2} = \dfrac{4x}{\sqrt{(1-x^2)^3}}$

i $\dfrac{d^2y}{dx^2} = \dfrac{-2x}{(1+x^2)^2}$

j $\dfrac{d^2y}{dx^2} = 360(1-3x)^3$

3 a $f''(x) = 24e^{3-2x}$

b $f''(x) = 8e^{-0.5x^2}(1-x^2)$

c $f''(x) = 0$ **d** $f''(x) = -\operatorname{cosec}^2 x$

e $f''(x) = \dfrac{3x}{\sqrt{(16-x^2)^3}}$

f $f''(x) = \dfrac{-27x}{\sqrt{(1-9x^2)^3}}$

g $f''(x) = \dfrac{-96x}{(9+4x^2)^2}$

h $f''(x) = \dfrac{3}{4\sqrt{(1-x)^5}}$

i $f''(x) = -5\sin(3-x)$

j $f''(x) = 18\sin(1-3x)\sec^3(1-3x)$

k $f''(x) = \dfrac{1}{9}\sec\left(\dfrac{x}{3}\right)\left(2\tan^2\left(\dfrac{x}{3}\right)+1\right)$

l $f''(x) = \dfrac{1+\cos^2\left(\frac{x}{4}\right)}{16\sin^3\left(\frac{x}{4}\right)}$

4 a 1 **b** −1 **c** −1 **d** $-\dfrac{1}{2}$

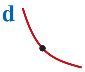
Exercise 6E

1 a

b

c

d



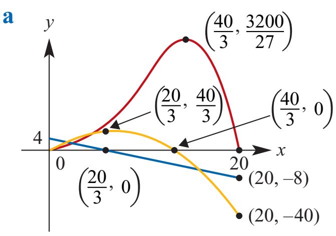
2 a i $f'(x) = 6x^2 + 12x$
ii $f''(x) = 12x + 12$
b $(-1, -8)$

3 a $f'(x) = \cos x$; $f''(x) = -\sin x$; $(\pi, 0)$
b $f'(x) = e^x(x+1)$; $f''(x) = e^x(x+2)$; $(-2, -2e^{-2})$

4 a i $f'(a-h) < 0$ **ii** $f'(a) = 0$
iii $f'(a+h) > 0$
b Non-negative **c** $f''(a) \geq 0$
d i $f''(0) = 2$ **ii** $f''(0) = 1$
iii $f''(0) = 0$
e No

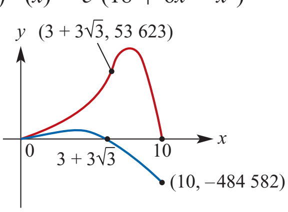
5 $f'(a-h) > 0, f'(a) = 0, f'(a+h) < 0,$ $f''(a) \leq 0$

6 a



b $x = \frac{20}{3}, \left(\frac{20}{3}, \frac{1600}{27}\right)$
c decreasing for $[0, 20]$
d $\left(\frac{20}{3}, \frac{1600}{27}\right)$

7 a $f'(x) = e^x(10 + 8x - x^2)$
$f''(x) = e^x(18 + 6x - x^2)$
b



c $3 + 3\sqrt{3}$ $(3 + 3\sqrt{3}, 53\ 623)$

8 a $(-1, 1), (0, 1)$ **b** $\left(-\frac{1}{2}, \frac{3}{2}\right)$

9 $(0, 0), (\pi, \pi), (2\pi, 2\pi), (3\pi, 3\pi), (4\pi, 4\pi)$

10 a $x = k\pi, k \in Z$ **b** $x = k\pi, k \in Z$
c $x = 0$ **d** $x = \frac{1}{2}k\pi, k \in Z$

12 a $\left(\frac{3}{2}, 2\right)$ **b** $\left(1, \frac{3}{2}\right)$

13 a $(0, 0), -6$
b $(-1, -1), 8$ & $(1, -1), -8$
c $(0, 3), 0$
d No points of inflexion
e No points of inflexion
f No points of inflexion
g $\left(-\sqrt{3}, \frac{-\sqrt{3}}{2}\right), \frac{-1}{4}$ & $(0, 0)$, 2 & $\left(\sqrt{3}, \frac{\sqrt{3}}{2}\right), \frac{-1}{4}$
h $(0, 0), 1$ **i** $\left(10, \frac{1}{18}\right), \frac{-1}{432}$

14 a $x = \frac{\pi}{4} + k\pi, k \in Z$
b $x = \frac{\pi}{2} + k\pi, k \in Z$

16 a $f'(x) = 2x(1 + 2\log_e x)$
b $f''(x) = 2(3 + 2\log_e x)$
c Stationary point at $\left(e^{-\frac{1}{2}}, -e^{-1}\right)$, point of inflexion at $\left(e^{-\frac{3}{2}}, -3e^{-3}\right)$

Exercise 6F

1 a $\frac{dr}{dt} \approx 0.00127$ m/min
b $\frac{dA}{dt} = .08$ m^2/min

2 $\frac{dx}{dt} \approx 0.56$ cm/s **3** $\frac{dy}{dt} = 39$ units/s

4 $\frac{dx}{dt} = \frac{3}{20\pi} \approx 0.048$ cm/s

5 $\frac{dv}{dt} = -\frac{5}{6}$ units/min

6 $\frac{dA}{dt} = 0.08\pi \approx 0.25$ cm^2/h

7 $\frac{dc}{dt} = \frac{1}{2}$ cm/s

8 a $\frac{dy}{dt} = \frac{1 - t^2}{(1 + t^2)^2}$ $\frac{dx}{dt} = \frac{-2t}{(1 + t^2)^2}$
b $\frac{dy}{dx} = \frac{t^2 - 1}{2t}$

9 $\frac{dy}{dx} = \frac{-\sin 2t}{1 + \cos 2t} = -\tan t$

10 $y = \frac{\sqrt{3}}{3}x - \frac{\pi\sqrt{3}}{18} + 1$

11 a $\frac{dy}{dt} = 12$ cm/s **b** $\frac{dy}{dt} = \pm 16$ cm/s

12 2.4

13 a $\frac{-5\sqrt{6}}{2}$ cm/s **b** $-4\sqrt{3}$ cm/s

14 72π cm^3/s

15 a 4 **b** 2 cm/s

16 $\frac{7}{12\pi}$ cm/s **17** $\frac{dV}{dt} = A\frac{dh}{dt}$

18 a $\frac{dh}{dt} = -\frac{\sqrt{h}}{4\pi}$
b i $\frac{dV}{dt} = -\frac{\sqrt{10}}{2}$ m^3/h

ii $\dfrac{dh}{dt} = -\dfrac{\sqrt{10}}{8\pi}$ m/h

19 a $y = -\dfrac{1}{2}x + \sqrt{2}$

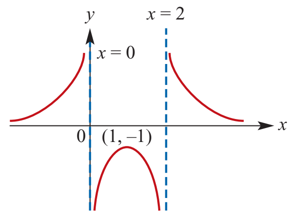
b $y = \dfrac{-\cos t}{2\sin t}x + \dfrac{1}{\sin t}$

20 a $y = \dfrac{\sqrt{2}}{2}x - 1$ **b** $y = -\sqrt{2}x + 5$

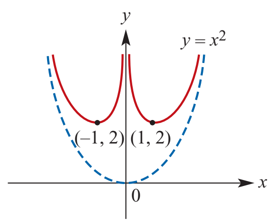
c $y = \dfrac{1}{2\sin\theta}x - \dfrac{\cos\theta}{\sin\theta}$

Exercise 6G

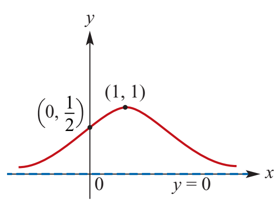
1 a



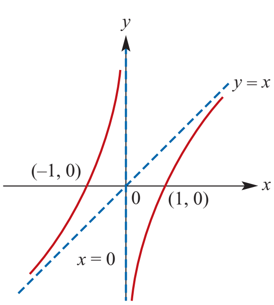
b



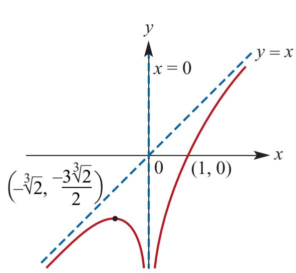
c



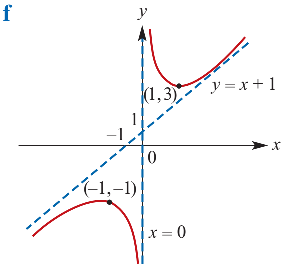
d



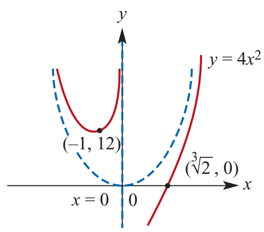
e



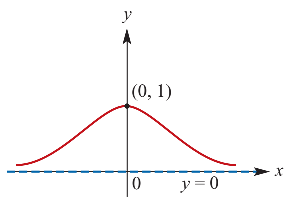
f



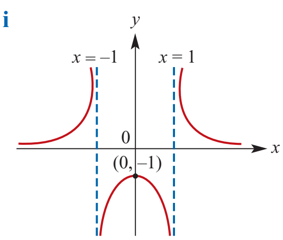
g



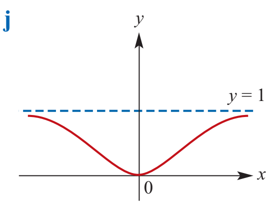
h



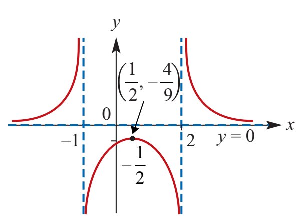
i



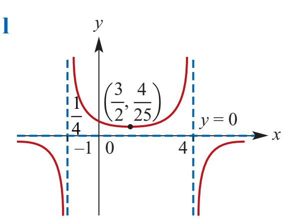
j



k

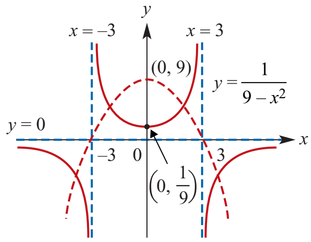


l

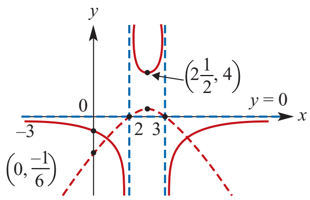


Answers

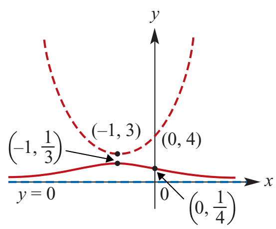
2 a



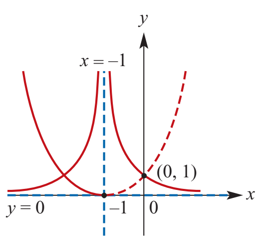
b



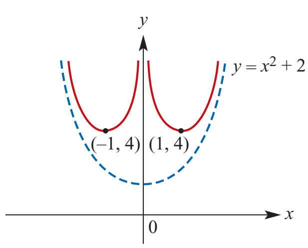
c



d



e



3 a $\left(\frac{1}{2}, 4\right)$(min); $\left(-\frac{1}{2}, -4\right)$(max)

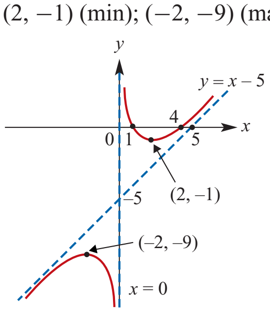
b $y = \frac{15}{4}x + 1$

4 $x = \pm\frac{1}{2}$

5 Gradient $= \frac{1}{2}$

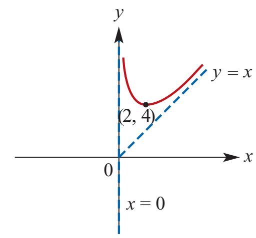
6 a (1, 0); (4, 0) **b** $x = 0, y = x - 5$

c (2, −1) (min); (−2, −9) (max)



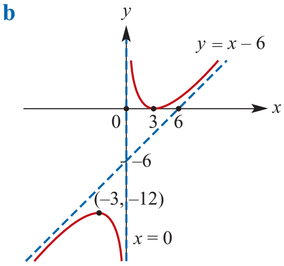
7 Least value = 3

8 Least value = 4



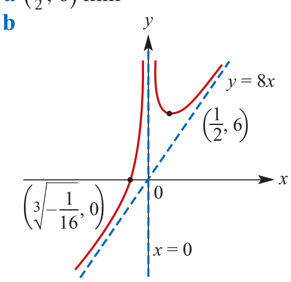
9 a (3, 0) min; (−3, −12) max

b



10 a $\left(\frac{1}{2}, 6\right)$ min

b



Exercise 6H

1 a $f''(x) = 90x^8$

b $f''(x) = 224(2x + 5)^6$

c $f''(x) = -4\sin(2x)$

d $f''(x) = -\frac{1}{9}\cos\left(\frac{x}{3}\right)$

e $f''(x) = \frac{9}{2}\sin\left(\frac{3x}{2}\right)\sec^3\left(\frac{3x}{2}\right)$

f $f''(x) = 16e^{-4x}$

g $f''(x) = \dfrac{-1}{x^2}$

h $f''(x) = \dfrac{x}{\sqrt{(16 - x^2)^3}}$

i $f''(x) = \dfrac{-8x}{\sqrt{(1 - 4x^2)^3}}$

j $f''(x) = \dfrac{-4x}{(4 + x^2)^2}$

2 a $\dfrac{dy}{dx} = -24x(1 - 4x^2)^2$

b $\dfrac{dy}{dx} = \dfrac{1}{2\sqrt{(2 - x)^3}}$

c $\dfrac{dy}{dx} = -\sin x \cos(\cos x)$

d $\dfrac{dy}{dx} = \dfrac{-\sin(\log_e x)}{x}$

e $\dfrac{dy}{dx} = \dfrac{-\sec^2\left(\frac{1}{x}\right)}{x^2}$

f $\dfrac{dy}{dx} = -\sin x e^{\cos x}$

g $\dfrac{dy}{dx} = \dfrac{3}{3x-4}$ **h** $\dfrac{dy}{dx} = \dfrac{-1}{\sqrt{x(2-x)}}$

i $\dfrac{dy}{dx} = \dfrac{-2}{\sqrt{-4x(x+1)}}$

j $\dfrac{dy}{dx} = \dfrac{1}{x^2+2x+2}$

3 a $\dfrac{1-\log_e x}{x^2}$ **b** $\dfrac{-2x}{(x^2+1)^2}$

c $\dfrac{1}{x^2-2x+2}$ **d** $\dfrac{1}{e^x+1}$

e $\dfrac{2\sqrt{\sin y + \cos y}}{\cos y - \sin y}$ **f** $\dfrac{1}{\sqrt{1+x^2}}$

g $\dfrac{e^x}{\sqrt{1-e^{2x}}}$ **h** $\dfrac{e^x(\cos x - \sin x) + \cos x}{(e^x+1)^2}$

4 a i $a - \dfrac{b}{x^2}$ **ii** $\dfrac{2b}{x^3}$

5 a i $2\cos(2x) - 6\sin(2x)$

ii $-4\sin(2x) - 12\cos(2x)$

Exercise 6I

1 a x **b** $-\dfrac{2y}{x}$ **c** $\dfrac{-x^2}{y^2}$ **d** $\dfrac{2x}{3y^2}$

e $2\sqrt{y}$ **f** $\dfrac{2-y}{x+3}$ **g** $\dfrac{2a}{y}$ **h** $\dfrac{2}{1-y}$

2 a $\dfrac{x+2}{y}$ **b** $\dfrac{-y^2}{x^2}$

c $\dfrac{2(x+y)}{1-2(x+y)}$ **d** $\dfrac{y-2x}{2y-x}$

e $\dfrac{e^y.2x}{1-x^2.e^y}$ **f** $\dfrac{-\sin 2x}{\cos y}$

g $\dfrac{\cos x - \cos(x-y)}{\cos y - \cos(x-y)}$

h $\dfrac{\sin y}{5y^4 - x\cos y + 6y}$

3 a $x + y = -2$ **b** $5x - 12y = 9$

c $16x - 15y = 8$ **d** $y = -3$

4 $\dfrac{dy}{dx} = \dfrac{y}{x}$ **5** $\dfrac{-1}{4}$ **6** -1 **7** $\dfrac{-2}{5}$ **8** $\dfrac{-7}{5}$

9 a $\dfrac{dy}{dx} = \dfrac{-x^2}{y^2}$ **c** $\dfrac{-1}{9}$

10 $y = -1, y = 1$

11 a $\dfrac{dy}{dx} = \dfrac{-(3x^2+y)}{x+6y^2}$ **d** -220 or -212

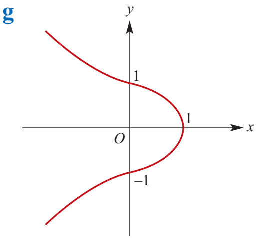
12 a $\dfrac{dy}{dx} = \dfrac{y-x}{2y-x}$ **b** $(-2, -2), (2, 2)$

13 a $\dfrac{dy}{dx} = \dfrac{-3x^2}{2y}$ **b** $(0, -1), (0, 1)$

c $(1, 0)$ **e** $y = \pm\sqrt{1-x^3}$

f $(0, -1), (0, 1)$

g



Multiple-choice questions

1 E **2** E **3** B **4** E **5** B
6 D **7** C **8** B **9** C **10** D

Short-answer questions (technology-free)

1 a $\tan x + x\sec^2 x$

b $\dfrac{1}{1+x^2}\sec^2(\tan^{-1} x) = 1$

c $\dfrac{-x}{\sqrt{1-x^2}}$ **d** $\dfrac{1}{\sqrt{x-x^2}}$

2 a $2\sec^2 x \tan x$

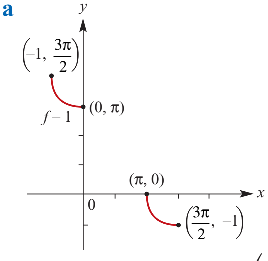
b $\dfrac{\sec^2 x - 2}{\sin^2 x} + \sec^2 x = -4\cot 2x \operatorname{cosec}^2 x = -\operatorname{cosec}^2 x$

c $\dfrac{2-x^2}{(1-x^2)^{\frac{3}{2}}}$ **d** $e^x(\cos e^x - e^x \sin e^x)$

3 a $\left(\dfrac{8}{3}, \dfrac{-1024}{27}\right)$ **b** $(2, 0)$

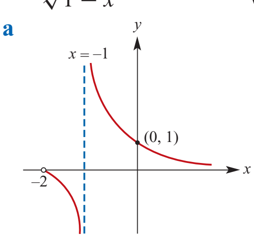
c $\left(2, \log_e(2) + \dfrac{1}{2}\right)$

4 a

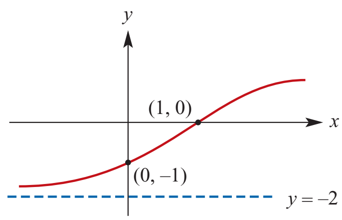


b $-\dfrac{1}{\sqrt{1-x^2}}$ **c** $\left(-\dfrac{\sqrt{3}}{2}, \dfrac{4\pi}{3}\right)$

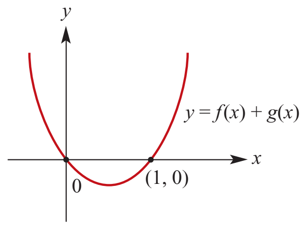
5 a



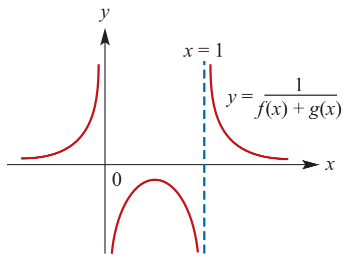
b



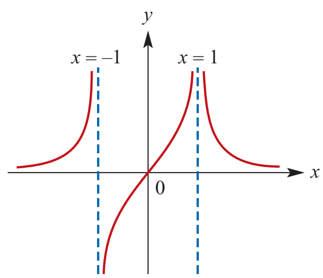
6 a i



ii



iii



b $f(x) = x^2 - 1$
$g(x) = (x - 1)^2$

c **i** $f(x) + g(x) = 2x^2 - 2x$

ii $\dfrac{1}{f(x) + g(x)} = \dfrac{1}{2x^2 - 2x}$

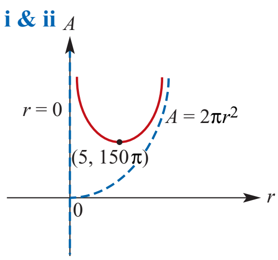
iii $\dfrac{1}{f(x)} + \dfrac{1}{g(x)} = \dfrac{2x}{(x - 1)^2(x + 1)}$

7 a -1 **b** $\dfrac{-(x + 1)}{y + 3}$ **c** $\dfrac{-2y^2}{x^2}$ **d** $\dfrac{-(x + 1)}{y - 3}$

8 a 324 m/s **b** 36 m/s

Extended-response questions

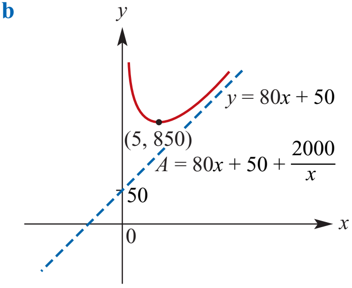
1 b i & ii



c 150π cm^2

2 a $80x + 50 + \dfrac{2000}{x}$

b



c minimum surface area = 850 cm^2
$x = 5, y = 5$

d minimum surface area

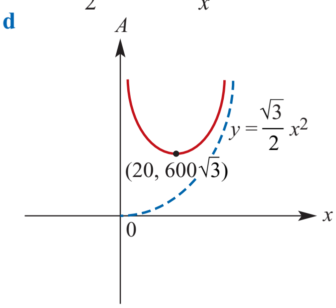
$= \dfrac{2000}{k} + 40\sqrt{10k}$ cm^2;

$x = y = \dfrac{10\sqrt{10k}}{k}$

3 a $A = \dfrac{\sqrt{3}}{2}x^2 + 3xy$ **b** $y = \dfrac{8000\sqrt{3}}{3x^2}$

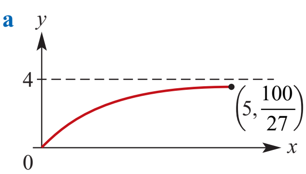
c $A = \dfrac{\sqrt{3}}{2}x^2 + \dfrac{8000\sqrt{3}}{x}$

d



e minimum surface area = $600\sqrt{3}$ cm^2

4 a

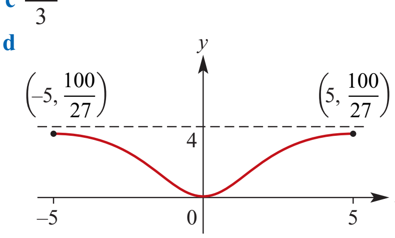


b **i** $\dfrac{16x}{(2 + x^2)^2}$

ii $\dfrac{16}{(2 + x^2)^2}\left(1 - \dfrac{4x^2}{2 + x^2}\right)$

c $\dfrac{\sqrt{6}}{3}$

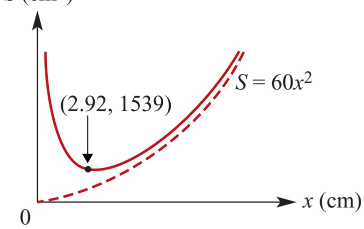
d



5 a i $y = \dfrac{100}{x^2}$

ii $S = 60x^2 + \dfrac{3000}{x}$

iii



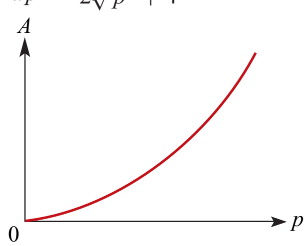
b $521\frac{13}{27}$ cm²/s

c 1.63 cm or 4.78 cm

6 a $A = \dfrac{p\sqrt{p^2+4}}{2} - p$

b i $\dfrac{dA}{dp} = \dfrac{p^2}{2\sqrt{p^2+4}} + \dfrac{\sqrt{p^2+4}}{2} - 1$

ii



iii 10.95

c i 0.315 sq. units/s

ii 0.605 sq. units/s

iii 9.800 sq. units/s

iv 15.800 sq. units/s

7 a $3ax^2 + 2bx + c$

b $6ax + 2b$

c $b^2 \le 3ac$

d i $x = -\dfrac{b}{3a}$

ii max $a < 0$, min $a > 0$

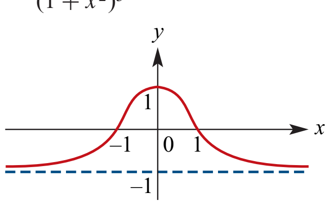
e $-\dfrac{b}{3}$

f i $b^2 < 4c$

ii $3c < b^2 < 4c$

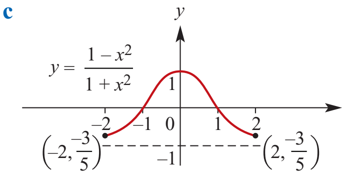
8 a ii $\dfrac{4(3x^2-1)}{(1+x^2)^3}$

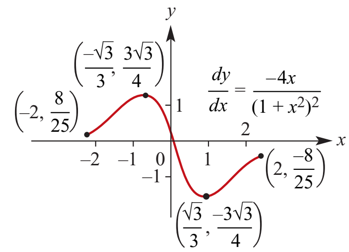
b

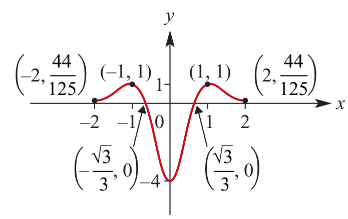


Horizontal asymptote at $y = -1$

c







$\dfrac{d^2y}{dx^2} = \dfrac{4(3x^2-1)}{(1+x^2)^3}$

d i $y = x + 1,\ y = -x + 1$

9 a i $f'(x) = 0$ **ii** $f(x) = \dfrac{\pi}{2}$

iii $f(x) = \dfrac{-\pi}{2}$

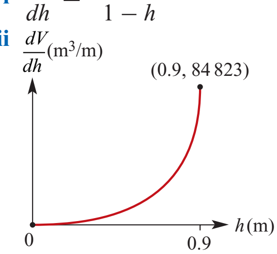
b i $\dfrac{dy}{dx} = -\text{cosec}^2\, x$

ii $\dfrac{dy}{dx} = -(1 + y^2)$

c $\dfrac{-1}{1+x^2}$ **d** $-\text{cosec}^2\, x + \sec^2 x$

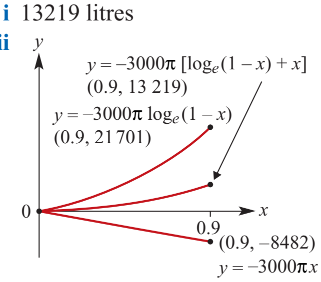
10 a i $\dfrac{dV}{dh} = \dfrac{3000\pi h}{1-h}$

ii



b i 13219 litres

ii



c 0.0064 m/min

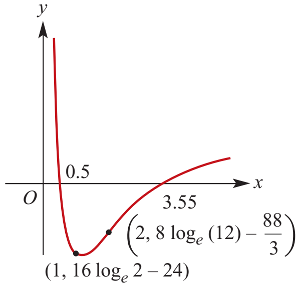
11 a $f'(x) = -\frac{16}{x^3} + \frac{16}{x}$

b $f''(x) = \frac{48}{x^4} - \frac{16}{x^2}$

c $(1, 16\log_e 2 - 24)$ **d** $x = \sqrt{3}$

e $(1, \infty)$ **f** $x = 3.55$

g



12 b i $\left(3, \frac{2 - 2\cos\theta}{\sin\theta}\right)$

c i $M = \left(\frac{3}{2\cos\theta}, \frac{1}{\sin\theta}\right)$

ii $\frac{9}{4x^2} + \frac{1}{y^2} = 1$

d i $y = \frac{2\sin\theta}{3\cos\theta}x + \frac{6}{3\cos\theta}$

ii $Z = (3(\cos\theta - \sin\theta), 2(\cos\theta + \sin\theta))$

iii $(2x + 3y)^2 + (3y - 2x)^2 = 144$

13 a $\left|\frac{ab}{\sin 2\theta}\right|$

b $\theta = (2n+1)\frac{\pi}{4}, n \in Z$; minimum area $= ab$

14 b $Q = \left(\frac{a}{\sec\theta - \tan\theta}, \frac{b}{\sec\theta - \tan\theta}\right)$;
$R = \left(\frac{a}{\sec\theta + \tan\theta}, \frac{-b}{\sec\theta + \tan\theta}\right)$

c Midpoint $= (a\sec\theta, b\tan\theta)$

15 a $\frac{9\sin\theta\cos\theta}{4}$

b Maximum area $= \frac{9}{8}$ when $\theta = \frac{\pi}{4}$

c $M = \left(\frac{3\cos\theta}{4}, \frac{3\sin\theta}{2}\right)$

d $\frac{16x^2}{9} + \frac{4y^2}{9} = 1$

16 a $\frac{x^2}{4} + y^2 = 1$

Chapter 7

Exercise 7A

1 a $-\frac{1}{2}\cos\left(2x + \frac{\pi}{4}\right)$ **b** $\frac{1}{\pi}\sin(\pi x)$

c $-\frac{3}{2\pi}\cos\left(\frac{2\pi x}{3}\right)$ **d** $\frac{1}{3}e^{3x+1}$

e $\frac{1}{5}e^{5(x+4)}$ **f** $\frac{1}{3}\log_e|3x - 2|$

g $-\frac{3}{2x}$ **h** $\frac{3}{2}x^4 - \frac{2}{3}x^3 + 2x^2 + x$

i $2x - 5\log_e|x + 3|$

2 a 0 **b** 20 **c** $\frac{1}{3}\log_e\frac{5}{2} \approx 0.305$

d 1 **e** $\frac{5}{24}$ **f** $\frac{1}{\sqrt{2}} + \frac{\pi^2}{16} \approx 1.324$

g $\frac{e^3}{3} + \frac{1}{6}$ **h** 0 **i** 0

j 1 **k** $\frac{1}{3}\log_e\frac{5}{11}$ **l** $\frac{1}{3}\log_e\frac{7}{4}$

3 a $\frac{(3x+2)^6}{18}$ **b** $\frac{2}{9}(3x+2)^{3/2}$

c $-\frac{1}{3(3x+2)}$ **d** $3x - 2\log_e|x + 1|$

e $\frac{2}{3}\sin\left(\frac{3}{2}x\right)$ **f** $\frac{3}{20}(5x - 1)^{4/3}$

4 a $f(x) = 2x, F(x) = x^2 + 3$

b $f(x) = 4x^2, F(x) = \frac{4}{3}x^3$

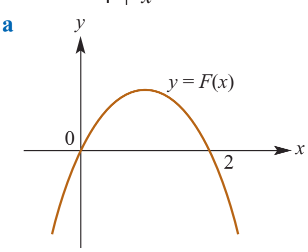
c $f(x) = -2x^2 + 8x - 8,$
$F(x) = -\frac{2}{3}x^3 + 4x^2 - 8x + \frac{28}{3}$

d $f(x) = -e^{-x}, F(x) = e^{-x} + 3$

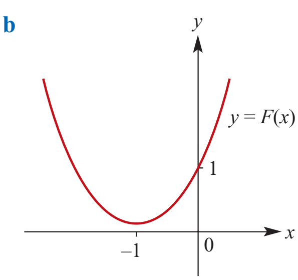
e $f(x) = 2\sin x$
$F(x) = 2 - 2\cos x$

f $f(x) = \frac{2}{4 + x^2}, F(x) = \tan^{-1}\left(\frac{x}{2}\right) + \frac{\pi}{2}$

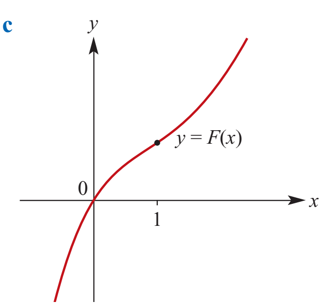
5 a



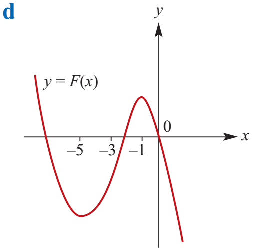
b



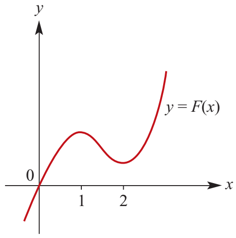
c



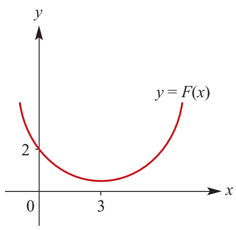
d



e



f



Exercise 7B

1 a $\sin^{-1}\left(\frac{x}{3}\right)+c$ **b** $\frac{\sqrt{5}}{5}\tan^{-1}\left(\frac{x\sqrt{5}}{5}\right)+c$

c $\tan^{-1}(t)+c$ **d** $5\sin^{-1}\left(\frac{x}{\sqrt{5}}\right)+c$

e $\frac{3}{4}\tan^{-1}\left(\frac{x}{4}\right)+c$ **f** $\frac{1}{2}\sin^{-1}\left(\frac{x}{2}\right)+c$

g $10\sin^{-1}\left(\frac{t\sqrt{10}}{10}\right)+c$

h $\frac{1}{12}\tan^{-1}\left(\frac{4t}{3}\right)+c$

i $\frac{\sqrt{2}}{2}\sin^{-1}\left(\frac{x\sqrt{10}}{5}\right)+c$

j $\frac{7\sqrt{3}}{3}\tan^{-1}\frac{y\sqrt{3}}{3}+c$

2 a $\frac{\pi}{2}$ **b** $\frac{\pi}{2}$ **c** $\frac{5\pi}{6}$ **d** $\frac{3\pi}{10}$

e $\frac{\pi}{8}$ **f** $\frac{\pi}{16}$ **g** $\frac{\pi}{6}$ **h** $\frac{\pi}{8}$

i $\frac{\pi}{2}$ **j** $\frac{\sqrt{3}}{3}\tan^{-1}2\sqrt{3}\approx 0.745$

Exercise 7C

1 a $\frac{(x^2+1)^4}{4}+c$ **b** $-\frac{1}{2(x^2+1)}+c$

c $\frac{1}{4}\sin^4 x+c$ **d** $-\frac{1}{\sin x}+c$

e $\frac{1}{12}(2x+1)^6+c$ **f** $\frac{5}{3}(9+x^2)^{3/2}+c$

g $\frac{1}{12}(x^2-3)^6+c$ **h** $-\frac{1}{4(x^2+2x)^2}+c$

i $-\frac{1}{3(3x+1)^2}+c$ **j** $2\sqrt{1+x}+c$

k $\frac{1}{15}(x^3-3x^2+1)^5+c$

l $\frac{3}{2}\log_e(x^2+1)+c$

m $-\frac{3}{2}\log_e|2-x^2|+c$

2 a $\tan^{-1}(x+1)+c$

b $\frac{2\sqrt{3}}{3}\tan^{-1}\frac{\sqrt{3}(2x-1)}{3}+c$

c $\sin^{-1}\left(\frac{x+2}{5}\right)+c$

d $\sin^{-1}(x-5)+c$ **e** $\sin^{-1}\left(\frac{x+3}{7}\right)+c$

f $\frac{\sqrt{3}}{6}\tan^{-1}\left(\frac{(x+1)\sqrt{3}}{2}\right)+c$

3 a $-\frac{1}{2}(2x+3)^{3/2}+\frac{1}{10}(2x+3)^{5/2}+c$

b $\frac{2(1-x)^{5/2}}{5}-\frac{2(1-x)^{3/2}}{3}+c$

c $\frac{4}{9}(3x-7)^{3/2}+\frac{28}{3}(3x-7)^{1/2}+c$

d $\frac{4}{45}(3x-1)^{5/2}+\frac{10}{27}(3x-1)^{3/2}+c$

e $2\log_e|x-1|-\frac{1}{x-1}+c$

f $\frac{2}{45}(3x+1)^{5/2}+\frac{16}{27}(3x+1)^{3/2}+c$

g $\frac{3}{7}(x+3)^{7/3}-\frac{3(x+3)^{4/3}}{4}+c$

h $\frac{5}{4}\log_e|2x+1|+\frac{7}{4(2x+1)}+c$

i $\frac{2}{105}(x-1)^{\frac{3}{2}}(15x^2+12x+8)$

j $\frac{2\sqrt{x-1}}{15}(3x^2+4x+8)+c$

Exercise 7D

1 a $\frac{61}{3}$ **b** $\frac{1}{16}$ **c** $\frac{1}{3}$ **d** $\frac{25}{114}$ **e** $\frac{4}{15}$

f $\log_e 2$ **g** $\frac{4}{3}$ **h** 1 **i** $\frac{1}{2}$ **j** $\log_e 2$

k $\log_e\frac{\sqrt{6}}{2}$ **l** $\log_e\frac{15}{8}$

m $\log_e\frac{e+1}{e}=\log_e(e+1)-1$

Exercise 7E

1 a $\frac{1}{2}x-\frac{1}{4}\sin 2x+c$

b $\frac{1}{32}\sin 4x-\frac{1}{4}\sin 2x+\frac{3}{8}x+c$

c $2\tan x-2x+c$ **d** $-\frac{1}{6}\cos 6x+c$

e $\frac{1}{2}x-\frac{1}{8}\sin 4x+c$ **f** $\frac{1}{2}\tan 2x-x+c$

g $\frac{1}{8}x-\frac{1}{32}\sin 4x+c$ **h** $\frac{1}{2}\sin 2x+c$

i $-\cot x-x+c$

j $\frac{1}{2}\sin 2x-\frac{1}{6}\sin^3 2x+c$

2 a $\tan x\ (c=0)$ **b** $\frac{1}{2}\tan(2x)(c=0)$

c $2\tan\left(\frac{1}{2}x\right)$ $(c = 0)$
d $\frac{1}{k}\tan(kx)(c = 0)$
e $\frac{1}{3}\tan(3x) - x(c = 0)$
f $2x - \tan x$ $(c = 0)$ **g** $-x$ $(c = 0)$
h $\tan x$ $(c = 0)$

3 a $\frac{\pi}{4}$ **b** $\frac{1}{2} + \log_e\frac{\sqrt{2}}{2} = \frac{1}{2} - \frac{1}{2}\log_e 2$
c $\frac{1}{3}$ **d** $\frac{1}{4} + \frac{3}{32}\pi \approx 0.545$
e $\frac{4}{3}$ **f** $\frac{\pi}{4}$
g $\frac{\pi}{24} + \frac{\sqrt{3}}{64} \approx 0.158$ **h** 1

4 a $\sin x - \frac{\sin^3 x}{3} + c$
b $\frac{4}{3}\cos^3\frac{x}{4} - 4\cos\left(\frac{x}{4}\right) + c$
c $\frac{1}{2}x + \frac{1}{16\pi}\sin(8\pi x) + c$
d $7\sin t\left(\cos^2 t + \frac{3}{5}\sin^4 t - \frac{\sin^6 t}{7}\right) + c$
e $\frac{1}{5}\sin 5x - \frac{\sin^3 5x}{15} + c$
f $3x - 2\sin 2x + \frac{1}{4}\sin 4x + c$
g $\frac{1}{48}\sin^3 2x - \frac{1}{64}\sin 4x + \frac{x}{16} + c$
h $\sin x - \frac{2\sin^3 x}{3} + \frac{\sin^5 x}{5} + c$

Exercise 7F

1 a $\log_e\left|\frac{x-2}{x+5}\right| + c$
b $\log_e\left|\frac{(x-2)^5}{(x-1)^4}\right| + c$
c $\frac{1}{2}\log_e|(x+1)(x-1)^3| + c$
d $2x + \log_e\left|\frac{x-1}{x+1}\right| + c$
e $2\log_e|x+2| + \frac{3}{x+2} + c$
f $\log_e|(x-2)(x+4)^3| + c$

2 a $\log_e\left|\frac{(x-3)^3}{x-2}\right| + c$
b $\log_e|(x-1)^2(x+2)^3| + c$
c $\frac{x^2}{2} - 2x + \log_e|(x+2)^{1/4}(x-2)^{3/4}| + c$
d $\log_e((x+1)^2(x+4)^2) + c$
e $\frac{x^3}{3} - \frac{x^2}{2} - x + 5\log_e|x+2| + c$
f $\frac{x^2}{2} + x + \log_e\left|\frac{(x-1)^4}{x^3}\right| + c$

3 a $\log_e\frac{4}{3}$ **b** $\log_e\frac{4}{3}$ **c** $\frac{1}{3}\log_e\frac{625}{512}$
d $1 + \log_e\frac{32}{81}$ **e** $\log_e\frac{10}{3}$ **f** $\log_e 4 + 4$
g $\log_e\left(\frac{7}{4}\right)^{1/2} \approx 0.28$ **h** $\log_e\frac{2}{3}$
i $\frac{1}{4}\log_e\frac{1}{3}$ **j** $5\log_e\frac{3}{4} - \log_e 2$

Exercise 7G

1 $p = \frac{4}{3}$ **2** $\frac{1}{24}$ **3** $e - 1 - \log_e\left(\frac{1+e}{2}\right)$
4 $\frac{9}{64}$ **5** $\frac{1}{3}\log_e(5) \approx 0.536$ **6** $c = \frac{3}{2}$
7 $-\frac{1}{18}\cos^6 3x + c$ **8** $\left(\frac{3}{2}\right)^{1/2}$ **9** $p = \frac{8}{5}$

10 a $-\frac{1}{2\sin^2 x} + c$ **b** $\frac{1}{20}(4x^2+1)^{5/2} + c$
c $\frac{1}{3}\sin^3 x - \frac{1}{5}\sin^5 x + c$ **d** $\frac{1}{1-e^x} + c$

11 1

12 a $\frac{1}{2}\tan^{-1}\frac{x+1}{2} + c$ **b** $\frac{1}{3}\sin^{-1}3x + c$
c $\frac{1}{2}\sin^{-1}2x + c$ **d** $\frac{1}{6}\tan^{-1}\frac{2x+1}{3} + c$

13 a $-\frac{1}{2x\sqrt{x-1}}$ **b** $\frac{\pi}{6}$

14 a $\frac{1}{3}[f(x)]^3 + c$ **b** $-\frac{1}{f(x)} + c$
c $\log_e(f(x)) + c$ **d** $-\cos[f(x)] + c$

15 $\frac{dy}{dx} = \frac{8-3x}{2\sqrt{4-x}}; 4\sqrt{2}$

16 $a = 2, b = -3, c = -1; x^2 - 3x + \frac{1}{x-2} + c$

17 a $\frac{\pi}{8}$ **b** 42 **c** 0 **d** $\log_e 2 \approx 0.693$
e $1 - \frac{\pi}{4} \approx 0.215$ **f** $\log_e\frac{3}{2} \approx 0.405$

18 a $\frac{1}{2}\sin^2 x + c$ **b** $-\frac{1}{4}\cos 2x + c$

19 a $\frac{dy}{dx} = \frac{1}{\sqrt{x^2+1}}$;
$\int\frac{1}{\sqrt{x^2+1}}\,dx = \log_e|x + \sqrt{x^2+1}| + c$
b $\frac{dy}{dx} = \frac{1}{\sqrt{x^2-1}}$

20 a $\frac{1}{2}\tan^{-1}\frac{x}{2}, c = 0$ **b** $\frac{1}{4}\log_e\left|\frac{x+2}{2-x}\right|, c = 0$
c $4\log_e|x| + \frac{1}{2}x^2, c = 0$
d $\frac{1}{2}\log_e(4+x^2), c = 0$
e $x - 2\tan^{-1}\frac{x}{2}, c = 0$
f $\frac{1}{2}\tan^{-1}2x, c = 0$
g $\frac{1}{3}(4+x^2)^{3/2}, c = 0$
h $\frac{2}{5}(x+4)^{5/2} - \frac{8}{3}(x+4)^{3/2}, c = 0$
i $-2\sqrt{4-x}, c = 0$
j $\sin^{-1}\frac{x}{2}, c = 0$
k $-8\sqrt{4-x} + \frac{2}{3}(4-x)^{3/2}, c = 0$
l $-\sqrt{4-x^2}, c = 0$
m $2\sin^{-1}\left(\frac{x}{2}\right) + \frac{x\sqrt{4-x^2}}{2}$

21 a $\frac{dy}{dx} = \cos x - x\sin x$;
$\int x\sin x\,dx = \sin x - x\cos x + c$
b $-\pi$

22 $c = \frac{5}{2}, d = \frac{3}{2}$

23 a $f'(x) = -(n-1)\sin^2 x \cos^{n-2} x + \cos^n x$

c i $\frac{3\pi}{16}$ **ii** $\frac{5\pi}{32}$

iii $\frac{\pi}{32}$ **iv** $\frac{4}{3}$

24 a $\frac{1}{2-n}(x+1)^{2-n} - \frac{1}{1-n}(x+1)^{1-n} + c$

b $\frac{1}{n+2} + \frac{1}{n+1}$

25 a $\frac{1}{3}a^2 + a + 1$ **b** $-\frac{3}{2}$

26 a $\frac{a^2+b^2}{(a\cos x + b\sin x)^2}$ **b** $\frac{1}{ab}$

27 a $U_n + U_{n-2} = \frac{1}{n-1}$

28 a 1 **c** $\frac{\pi}{4}$

Multiple-choice questions

1 E **2** C **3** C **4** D **5** A
6 C **7** D **8** C **9** A **10** D

Short-answer questions (technology-free)

1 a $\frac{1}{6}\sin 2x(3 - \sin^2 2x)$

b $\frac{1}{4}(\log_e(4x^2+1) + 6\tan^{-1} 2x)$

c $\frac{1}{4}\log_e\left|\frac{1+2x}{1-2x}\right|$ **d** $-\frac{1}{4}\sqrt{1-4x^2}$

e $-\frac{1}{4}x + \frac{1}{16}\log_e\left|\frac{1+2x}{1-2x}\right|$

f $-\frac{1}{6}(1-2x^2)^{\frac{3}{2}}$

g $\frac{1}{2}x - \frac{1}{4}\sin\left(2x - \frac{2\pi}{3}\right)$

h $(x^2-2)^{\frac{1}{2}}$ **i** $\frac{1}{2}x - \frac{1}{12}\sin 6x$

j $\frac{1}{6}\cos 2x(\cos^2 2x - 3)$

k $2(x+1)^{\frac{3}{2}}\left(\frac{1}{5}(x+1) - \frac{1}{3}\right)$

l $\frac{1}{2}\tan x$ **m** $\frac{x}{e} - \frac{1}{3e^{3x+1}}$

n $\frac{1}{2}\log_e|x^2-1|$ **o** $\frac{x}{8} - \frac{\sin 4x}{32}$

p $\frac{1}{2}x^2 - x + \log_e|1+x|$

2 a $\frac{1}{3} - \frac{\sqrt{3}}{8}$ **b** $\frac{1}{2}\log_e 3$

c $\frac{1}{3}\left(\frac{5\sqrt{5}}{8} - 1\right)$ **d** $\frac{1}{6}\log_e\frac{7}{4}$

e $2 + \log_e\frac{32}{81}$ **f** $\frac{2}{3}$

g $\frac{\pi}{6}$ **h** $\frac{\pi}{4}$ **i** $\frac{\pi}{4}$

j $\frac{\pi}{16}$ **k** $\log_e\frac{3\sqrt{2}}{2}$ **l** 6

3 $\frac{1}{2}\log_e|x^2+2x+3| - \frac{\sqrt{2}}{2}\tan^{-1}\frac{\sqrt{2}(x+1)}{2} + c$

4 a $\frac{1}{2\sqrt{x(1-x)}}, 2\sin^{-1}\sqrt{x} + c$

b $\frac{2x}{\sqrt{1-x^4}}, \sin^{-1}(x^2) + c$

5 a $\sin^{-1} x + \frac{x}{\sqrt{1-x^2}}$,
$x\sin^{-1} x + \sqrt{1-x^2} + c$

b $\log_e|x| + 1, x\log_e|x| - x + c$

c $\tan^{-1} x + \frac{x}{1+x^2}$,
$x\tan^{-1} x - \frac{1}{2}\log_e(1+x^2) + c$

6 a $-\frac{1}{8}\cos 4x$ **b** $\frac{1}{9}(x^3+1)^3$

c $\frac{-1}{2(3+2\sin\theta)}$ **d** $-\frac{1}{2}e^{1-x^2}$

e $\tan(x+3) - x$ **f** $\sqrt{6+2x^2}$

g $\frac{1}{3}\tan^3 x$ **h** $\frac{1}{3\cos^3 x}$

i $\frac{1}{3}\tan 3x - x$

7 a $\frac{8}{15}$ **b** $-\frac{39}{4}$ **c** $\frac{1}{2}$

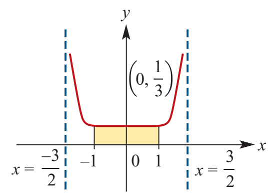
d $\frac{2}{3}(2\sqrt{2}-1)$ **e** $\frac{\pi}{2}$ **f** $\frac{1}{3}\log_e\frac{1}{9}$

8 $\frac{1}{2}\left(x^2 + \frac{1}{x}\right)^{\frac{-1}{2}} \cdot (2x - x^{-2}), 3\sqrt{2}$

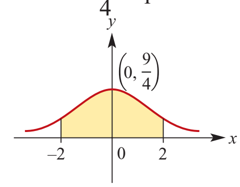
Chapter 8

Exercise 8A

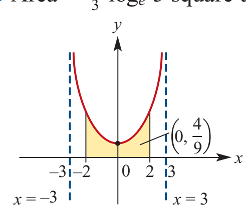
1 Area $= \sin^{-1}\frac{2}{3}$ square units



2 Area $= \frac{9\pi}{4}$ square units

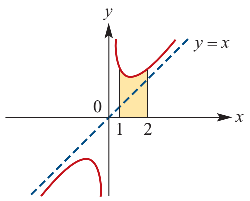


3 Area $= \frac{4}{3}\log_e 5$ square units

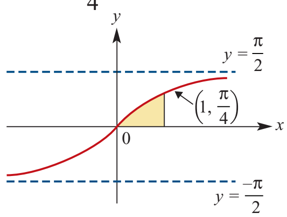


4 Area = $2\frac{2}{3}$ square units

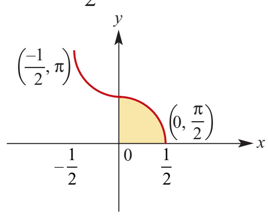
5 Area = $\frac{3}{2} + 2\log_e 2$ square units



6 a Area = $\frac{\pi}{4} - \log_e\sqrt{2}$ square units

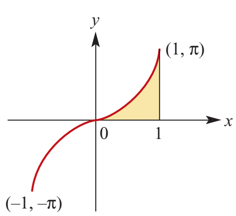


b Area = $\frac{1}{2}$ square unit

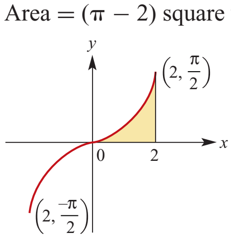


c Area = $\frac{\pi}{2}$ square units

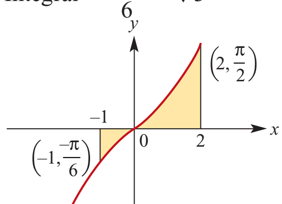
d Area = $(\pi - 2)$ square units



e Area = $(\pi - 2)$ square units



f Integral = $\frac{5\pi}{6} - \sqrt{3}$

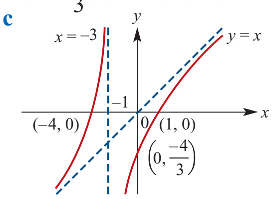


7 a (0, 1) **b** $y = -1$

c $\pi - 2$ square units

8 a $(0, -\frac{4}{3})$ $(-4, 0)$ $(1, 0)$ **b** $y = x; x = -3$

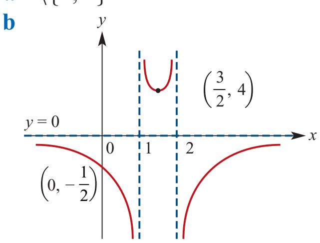
c



d Area = $31\frac{1}{2} + 4\ \log_e \frac{4}{11}$ square units

9 a $R\backslash\{1, 2\}$

b

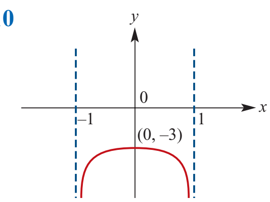


c Range of $g = R^- \cup [4, \infty)$

d Area = $-\log_e \frac{3}{4}$

$= \log_e \frac{4}{3}$ square units

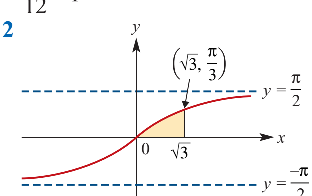
10



$$\int_0^{\frac{1}{2}} \frac{-3}{\sqrt{1-x^2}}\,dx = -\frac{\pi}{2}$$

11 $\frac{\pi}{12}$ square units

12

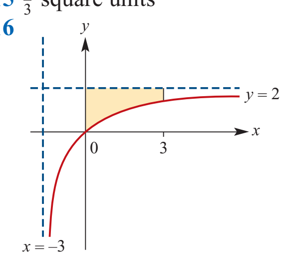


Area = $\frac{\pi\sqrt{3}}{3} - \log_e 2$ square units

13 1 square unit **14** $\frac{2}{3}$ square units

15 $\frac{1}{3}$ square units

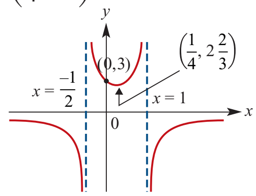
16



Area = $6\log_e 2$ square units

17 b $\left(\frac{1}{4}, 2\frac{2}{3}\right)$ local minimum.

c



d $\frac{3}{2} - \log_e 4$ square units

Exercise 8B

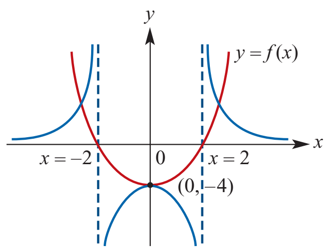
1 (3, 3) (2, 0); $\frac{1}{3}$ square units

2 $\frac{1}{3}$ square units.

3 a $\frac{17}{24}$ square units

b $\frac{5}{6}$ square units

4



Area $= 8 \log_e 3 - \frac{22}{3}$ square units

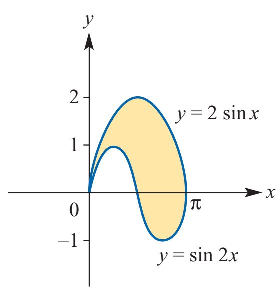
5 $a = e^2$

6 a $4\frac{1}{2}$ square units

b $\frac{11}{6}$ square units

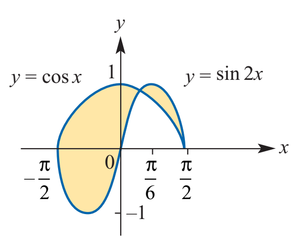
c $\frac{11}{6}$ square units

7 a



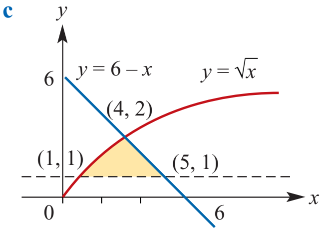
Area = 4 square units

b



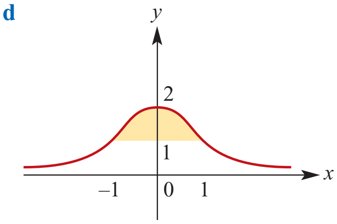
Area $= 2\frac{1}{2}$ square units

c



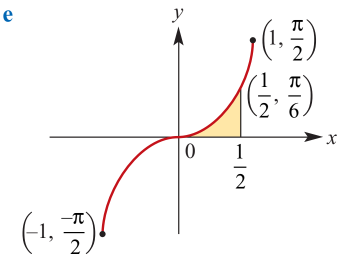
Area $= 2\frac{1}{6}$ square units

d



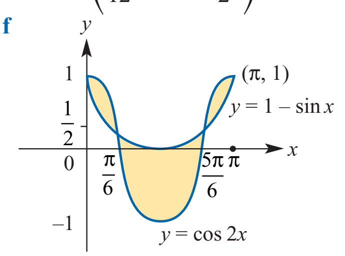
Area $= (\pi - 2)$ square units

e



Area $= \left(\frac{\pi}{12} - 1 + \frac{\sqrt{3}}{2}\right)$ square units

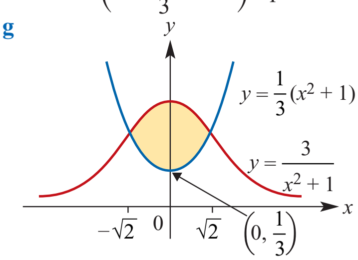
f



$$\text{Area} = \left(2 - \frac{\pi}{3} - \frac{\sqrt{3}}{2}\right) + \left(\frac{2\pi}{3} - \frac{\sqrt{3}}{2}\right) \text{ square units}$$

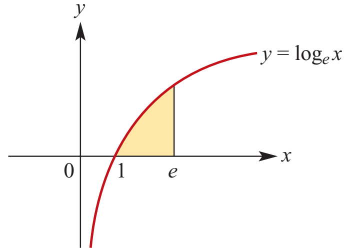
$$= \left(2 + \frac{\pi}{3} - \sqrt{3}\right) \text{ square units}$$

g



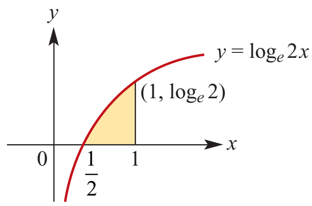
Area ≈ 4.161 square units

8 a



Area = 1 square unit

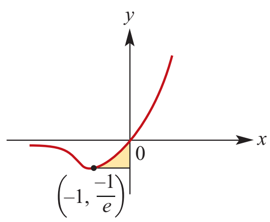
b



Area $= \left(\log_e 2 - \frac{1}{2}\right)$ square units

9 a $f'(x) = e^x + xe^x$ **b** $x = -1$

c



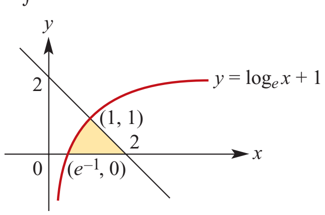
d $y = -\dfrac{1}{e}$

e Area $= \left(\dfrac{3}{e} - 1\right)$ square units

(Note: As $f'(x) = e^x + xe^x$

$\int xe^x\,dx = xe^x - e^x + c$)

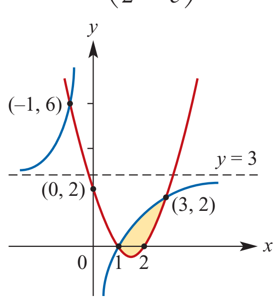
10



a $y = 2 - x$

b Area $= \left(\dfrac{1}{2} + \dfrac{1}{e}\right)$ square units

11



Area $= \left(\frac{16}{3} - 3\log_e 3\right)$ square units

13 a $(-2\sqrt{2}, 1), (2\sqrt{2}, 1)$ **b** 33.36

14 $\dfrac{9}{2}$ **15** 3.772

16 a $a = 4, b = 2\sqrt{5}$ **b** 5.06 **17** 4

Exercise 8C

1 a i 1.583 **ii** $\dfrac{19}{12}$

b i 0.950 **ii** $2\pi - \frac{16}{3}$

c i 0.097 **ii** $\dfrac{9\sqrt{3}}{160}$

d i 1.614 **ii** $3 + 2\log_e \frac{1}{2}$

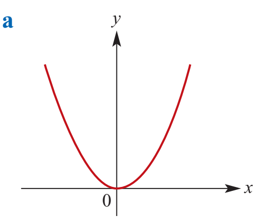
e i 0.586 **ii** $2 - \sqrt{2}$

f i 0.859 **ii** $\frac{1}{2}(e - 1)$

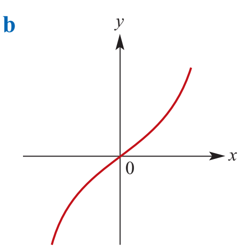
2 a 4.24 **b** 3.14 **c** 1.03 **d** 0.67
e 1.95 **f** 0.66 **g** 0.64 **h** 0.88
i 1.09 **j** 0.83

3 a $\log_e x$ **b** $-\log_e x$ **c** $e^x - 1$
d $1 - \cos x$ **e** $\tan^{-1}(x) + \dfrac{\pi}{4}$
f $\sin^{-1}(x)$

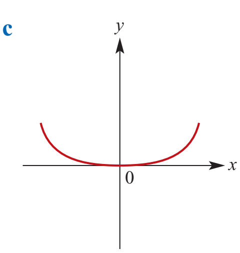
4 a



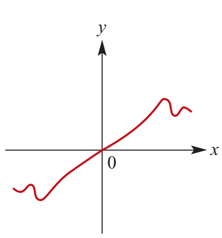
b

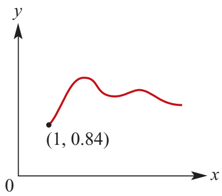


c



d



e


Exercise 8D

1 a 8π cubic units
  b $\frac{364\pi}{3}$ cubic units
  c $\frac{343\pi}{6}$ cubic units
  d $\frac{\pi^2}{4}$ cubic units
  e $\frac{\pi}{2}(e^4 - 1)$ cubic units
  f 36π cubic units

2 Area $= \frac{32}{3}$ square units
  Volume $= 8\pi$ cubic units

3 $\frac{2\pi}{3}$ cubic units

4 $\frac{\pi}{2}$ cubic units

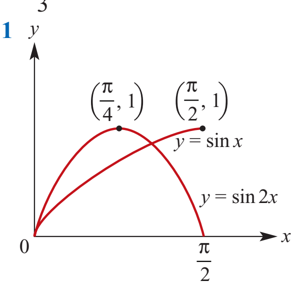
5 a $\frac{3\pi}{4}$ cubic units
  b $\frac{28\pi}{15}$ cubic units
  c 2π cubic units
  d $\frac{4\pi a^3}{3}$ cubic units
  e 36π cubic units
  f 18π cubic units

6 $\frac{1088\pi}{15}$ cubic units

7 $\frac{21\pi}{4}$ cubic units

8 $\frac{3\pi}{10}$ cubic units

9 $\frac{32\pi}{3}$ cubic units

11


12 $\frac{4}{13}$　　13 $\frac{7\pi}{6}$

14 a $\frac{16\pi}{3}$ cubic units
  b $\pi\left(\frac{e^4}{2} - 4e^2 + \frac{23}{2}\right)$ cubic units

15 a $\frac{e}{2} - 1$ square units
  b $\frac{\pi}{6}(e^2 - 3)$ cubic units

16 $\frac{16\pi}{15}$ cubic units　　17 $\frac{\pi^2}{2}$ cubic units

18 $\frac{7\pi}{10}$ cubic units　　19 $\frac{19\pi}{6}$ cubic units

20 $\pi\left(\log_e 2 - \frac{1}{2}\right)$ cubic units

22 $(8\pi - 2\pi^2)$ cubic units

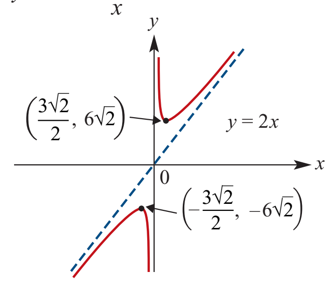
24 a $\frac{\pi}{3}\tan^{-1}\frac{4}{3}$　　b 4π

25 176 779 cm^3

26 a $\frac{4\pi ab^2}{3}$　　b $\frac{4\pi a^2 b}{3}$

27 a $x + y = 8$
  b i $\frac{64\pi}{3}$　　ii $\frac{64\pi}{3}$

28 a $y = 2x + \frac{9}{x}$



  b $\frac{482\pi}{3}$

29 2.642 cubic units

30 $4\pi\left(\frac{4\pi}{3} - \sqrt{3}\right)$

Multiple-choice questions

1 C	2 D	3 B	4 C	5 C
6 E	7 B	8 D	9 C	10 E

Short-answer questions (technology-free)

1 $\frac{1}{3}$

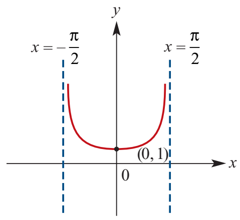
2 a $\frac{\pi}{2} - 1$　　b 1

3 a π　　b $\frac{\pi}{8}(\pi - 2)$　　c $\frac{\pi}{8}(\pi + 2)$
  d $\frac{2048\pi}{15}$　　e 40π

4 $\frac{119\pi}{6}$

5 a 12π　　b $\frac{20\sqrt{10}\pi}{3} - \frac{2\pi}{3}$

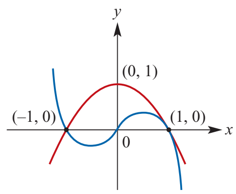
6



Volume $= 2\pi$

7 a (0, 0), (2, 4) **b** $\frac{16\pi}{3}$

8 a

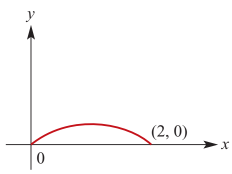


b $\frac{4}{3}$

9 a $A = (-1, 1), B = (1, 1), C = (0, \sqrt{2})$

b $\frac{44\pi}{15}$

10 a

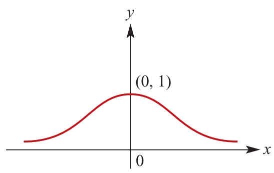


b $\frac{4}{3}$ **c** $\frac{16\pi}{15}$

11 a i $\frac{\pi b^5}{5}$ **ii** $\frac{\pi b^4}{2}$

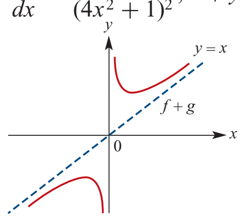
b $b = 2.5$

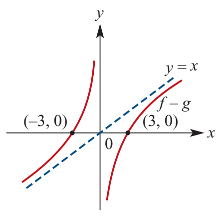
12 a



b $\frac{dy}{dx} = \frac{-8x}{(4x^2+1)^2}, x + y = 1$ **c** $\frac{\pi - 3}{8}$

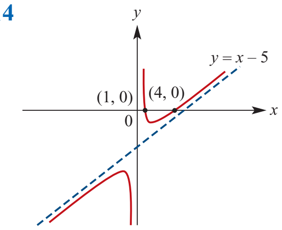
13 a





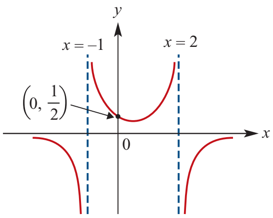
b $18 \log_e 3$

14



Area $= 7.5 - 4 \log_e 4$

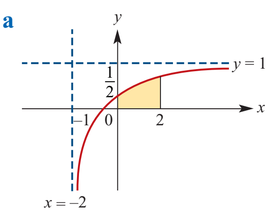
15



Area $= \frac{1}{2} - \frac{1}{3} \log_e 4$

Extended-response questions

1 a



b $2 - \log_e 2$ square units

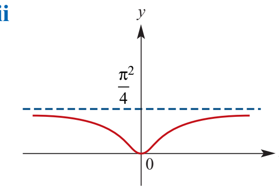
c $2\pi \left(\frac{9}{8} - \log_e 2\right)$ cubic units

2 a $f'(x) = \frac{x}{1+x^2} + \tan^{-1} x$

b $\frac{\pi}{4} - \frac{1}{2} \log_e 2$ **c** $\frac{1}{2} \log_e 2$ square units

d i $g'(x) = \frac{2 \tan^{-1} x}{1 + x^2}$

iii

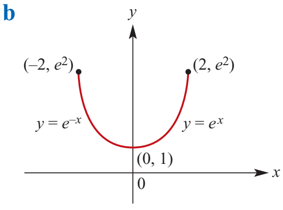


e $\pi \left(\frac{\pi}{2} - 1\right)$ cubic units

3 a i $\log_e x + 1; x \log_e x - x + c$

ii $(\log_e x)^2 + 2 \log_e x;$
$x(\log_e x)^2 - 2x \log_e x + 2x + c$

b



c $V = 2\pi(e^2 - 1)\text{ cm}^3$
$\approx 40\text{ cm}^3$

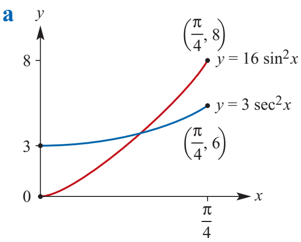
4 a $\dfrac{\pi}{2}$ cubic units **b** $\dfrac{4R}{\pi}$ units per second

c i $\dfrac{\pi}{8}$ cubic units **ii** $\dfrac{\sqrt{2}}{2}$ units

5 b i $a = 1$ **ii** $\dfrac{2\sqrt{2}}{3}$

c $\dfrac{\pi a}{2(a^2+1)}$

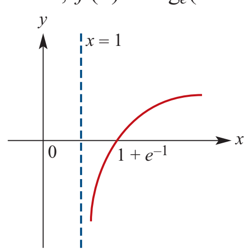
6 a



b $\left(\dfrac{\pi}{6}, 4\right)$ **c** $3\sqrt{3} - \dfrac{4\pi}{3}$

7 a $a = 1$; $f(x) = \log_e(x-1)+1$

b



c domain of $f^{-1} = R$,
range of $f^{-1} = (1, \infty)$

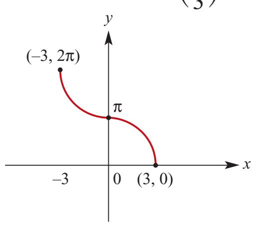
d $2 - e^{-1}$ **e** $\displaystyle\int_{1+e^{-1}}^{2} f(x)\,dx = e^{-1}$

8 a $a = 2\pi$

b i domain of $f^{-1} = [-3, 3]$
range of $f^{-1} = [0, 2\pi]$

ii $f^{-1}(x) = 2\cos^{-1}\left(\dfrac{x}{3}\right)$

iii



c gradient $= -\dfrac{2}{3}$

d $V_1 = V_2 = \dfrac{9\pi^2}{2}$ cubic units

10 a Area $= \pi(r^2 - y^2)$

11 a $\dfrac{4\pi ab^2}{3}$ cubic units **b** $4\sqrt{3}\pi a^2 b$ cubic units

12 b $\dfrac{\pi}{6} - \dfrac{3}{16}$

c $\dfrac{\pi}{2}\left(\dfrac{-3}{16} + \log_e 3\right) = \pi\left(\dfrac{-3}{32} + \log_e(\sqrt{3})\right)$

13 a i $d = 0$
$125a + 25b + 5c = 1$
$1000a + 100b + 10c = 2.5$
$27\,000a + 900b + 30c = 10$

ii $a = \dfrac{-7}{30000}$, $b = \dfrac{27}{2000}$, $c = \dfrac{83}{600}$,
$d = 0$

b $\dfrac{273}{2}$

c i $V = \dfrac{\pi}{900\,000\,000} \times \displaystyle\int_0^{30} (-7x^3 + 405x^2 + 4150x)^2\,dx$

ii $\dfrac{362\,083\pi}{400}$

d i $w = 16.729335$

ii $\dfrac{1\,978\,810\,99\pi}{2\,500\,00} = 2487$ (to 4 significant figures)

e $\left(\dfrac{135}{7}, \dfrac{1179}{196}\right)$

14 a $\dfrac{\pi}{3}$

b Maximum volume of $\dfrac{2\pi\sqrt{3}}{27}$ cubic units
when $k = \dfrac{\sqrt{3}}{3}$

15 a $\dfrac{\pi H}{3}(a^2 + ab + b^2)\text{ cm}^3$

b $\dfrac{\pi H}{24}(7a^2 + 4ab + b^2)\text{ cm}^3$

c $V = \dfrac{\pi H(r^3 - a^3)}{3(b-a)}$

d i $\dfrac{dV}{dr} = \dfrac{\pi H r^2}{b-a}$ **ii** $h = \dfrac{H(r-a)}{b-a}$

e i $\dfrac{dV}{dr} = 2\pi r^2$

ii $\dfrac{dr}{dt} = \dfrac{1}{96\pi}$, $\dfrac{dh}{dt} = \dfrac{1}{48\pi}$

Chapter 9

Exercise 9A

1 a $y = 4e^{2t} - 2$ **b** $y = x\log_e|x| - x + 4$

c $y = \sqrt{2x + 79}$ **d** $y - \log_e|y+1| = x - 3$

e $y = \dfrac{1}{2}x^4 - \dfrac{1}{2}x + 2$

f $y = \dfrac{11}{5}e^{2x} + \dfrac{4}{5}e^{-2x}$

g $x = 3\sin 3t + 2\cos 3t + 2$

3 $4\sqrt{2}$ **4** $-2, 5$ **5** $a = 0, b = -1, c = 1$

6 $a = 0, b = \frac{1}{2}$

7 $a = 1, b = -6, c = 18, d = -24$

Exercise 9B

1 a $y = \frac{1}{3}x^3 - \frac{3}{2}x^2 + 2x + c$

b $y = \frac{1}{2}x^2 + 3x - \log_e|x| + c$
c $y = 2x^4 + 4x^3 + 3x^2 + x + c$
d $y = 2\sqrt{x} + c$ **e** $y = \frac{1}{2}\log_e|2t - 1| + c$
f $y = -\frac{1}{3}\cos(3t - 2) + c$
g $y = -\frac{1}{2}\log_e|\cos(2t)| + c$
h $x = -\frac{1}{3}e^{-3y} + c$ **i** $x = \sin^{-1}\frac{y}{2} + c$
j $x = \frac{1}{y - 1} + c$

2 a $y = \frac{1}{4}x^5 + cx + d$
b $y = \frac{4}{15}(1 - x)^{\frac{5}{2}} + cx + d$
c $y = -\frac{1}{4}\sin\left(2x + \frac{\pi}{4}\right) + cx + d$
d $y = 4e^{\frac{x}{2}} + cx + d$
e $y = -\log_e|\cos x| + cx + d$
f $y = -\log_e|x + 1| + cx + d$

3 a $y = \frac{x - 1}{x}$ **b** $y = 1 - e^{-x}$
c $y = \frac{1}{2}x^2 - 4\log_e x + 1$
d $y = \frac{1}{2}\log_e|x^2 - 4|$
e $y = \frac{1}{3}(x^2 - 4)^{\frac{3}{2}} - \frac{95\sqrt{3}}{12}$
f $y = \sin^{-1}\frac{x}{2} + \frac{\pi}{6}$
g $y = \frac{1}{4}\log_e\left|\frac{2 + x}{2 - x}\right| + 2$
h $y = \frac{1}{2}\tan^{-1}\frac{x}{2} + \frac{\pi}{4}$
i $y = \frac{2}{5}(4 - x)^{\frac{5}{2}} - \frac{8}{3}(4 - x)^{\frac{3}{2}} + 8$
j $y = \log_e\left(\frac{e^x + 1}{2}\right)$

4 a $y = e^{-x} - e^x + 2x$
b $y = x^2 - 2x^3$
c $y = x^2 + \frac{1}{4}\sin 2x - 1$
d $y = \frac{1}{2}x^2 - 2x + \log_e|x| + 3$
e $y = x - \tan^{-1}x + \frac{\pi}{4}$
f $y = 8x^3 + 12x^2 + 6x$
g $y = \sin^{-1}\frac{x}{2}$

5 a $y = \frac{3}{2}x^2 + 4x + c$
b $y = -\frac{1}{3}x^3 + cx + d$
c $y = \log_e|x - 3| + c$

6 a $y = 2x + e^{-x}$
b $y = \frac{1}{2}x^2 - \frac{1}{2}\cos 2x + \frac{9}{2}$
c $y = 2 - \log_e|2 - x|$

Exercise 9C

1 a $y = \frac{1}{3}(Ae^{3x} + 5)$ **b** $y = \frac{1}{2}(Ae^{-2x} + 1)$
c $y = \frac{1}{2} - \frac{1}{2}\log_e|2c - 2x|$
d $y = \tan^{-1}(x - c)$ **e** $y = \cos^{-1}(e^{c-x})$
f $y = \frac{1 + Ae^{2x}}{1 - Ae^{2x}}$ **g** $y = \tan(x - c)$
h $x = \frac{5}{3}y^3 + y^2 + c$ **i** $y = \frac{1}{4}(x - c)^2$
j $y = \frac{4Ae^{4x}}{1 - Ae^{4x}}$

2 a $y = e^{x+1}$ **b** $y = e^{x-4} - 1$
c $y = e^{2x-2}$ **d** $y = -\frac{1}{2}(e^{2x} + 1)$
e $x = y - e^{-y} + 1$
f $y = 3\cos x, -\pi < x < 0$
g $y = \frac{3(e^{6x-7} - 1)}{e^{6x-7} + 1}$
h $y = \frac{1}{3}\tan 3x, -\frac{\pi}{6} < x < \frac{\pi}{6}$
i $y = \frac{4}{e^{-x} - 2}$

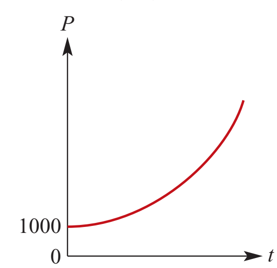
3 a $y = [3(x - c)]^{\frac{1}{3}}$ **b** $y = \frac{1}{2}(Ae^{2x} + 1)$

Exercise 9D

1 a $\frac{dx}{dt} = 2t + 1, x = t^2 + t + 3$
b $\frac{dx}{dt} = 3t - 1, x = \frac{3}{2}t^2 - t + \frac{1}{2}$
c $\frac{dx}{dt} = -2t + 8, x = -t^2 + 8t - 15$

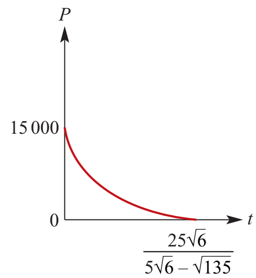
2 a $\frac{dy}{dx} = \frac{1}{y}, y \neq 0$ **b** $\frac{dy}{dx} = \frac{1}{y^2}, y \neq 0$
c $\frac{dN}{dt} = \frac{k}{N^2}, N \neq 0, k > 0$
d $\frac{dx}{dt} = \frac{k}{x}, x \neq 0, k > 0$
e $\frac{dm}{dt} = km, k < 0$ **f** $\frac{dy}{dx} = \frac{-x}{3y}, y \neq 0$

3 a i $\frac{dP}{dt} = kP$
ii $t = \frac{1}{k}\log_e P + c, P > 0$
b i 1269
ii $P = 1000(1.1)^{\frac{t}{2}}, t \geq 0$



4 a i $\frac{dP}{dt} = k\sqrt{P}, k < 0, P > 0$
ii $t = \frac{2\sqrt{P}}{k} + c, k < 0$
b i 12079

ii

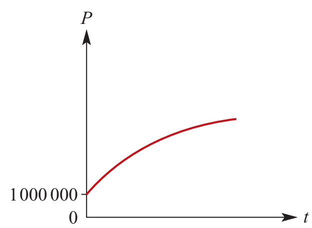


5 a i $\frac{dP}{dt} = \frac{k}{P}, k > 0, P > 0$

ii $t = \frac{1}{2k}P^2 + c$

b i $P = 50\,000\sqrt{21t + 400}, t \geq 0$

ii



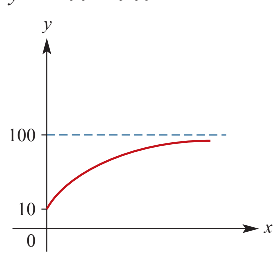
6 $y = 10e^{\frac{x}{10}}$

7 $\theta = 331.55°$ K

8 23.22

10 a $x = \frac{1}{3}\left(20 - 14e^{\frac{-t}{10}}\right)$ **b** 19 minutes

11 $y = 100 - 90e^{\frac{-x}{10}}$



12 13 500

13 a 14 400 **b** 13 711 **c** 14 182

14 a $\frac{dV}{dt} = 0.3 - 0.2\sqrt{V}, V > 0$

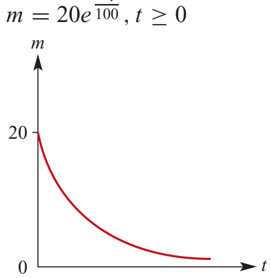
b $\frac{dm}{dt} = 50 - \frac{6m}{100 - t}, 0 \leq t < 100$

c $\frac{dx}{dt} = \frac{-5x}{200 + t}, t \geq 0$

15 a $\frac{m}{100}$ kg/min **b** $\frac{dm}{dt} = \frac{-m}{100}$

c $m = 20e^{\frac{-t}{100}}, t \geq 0$

d



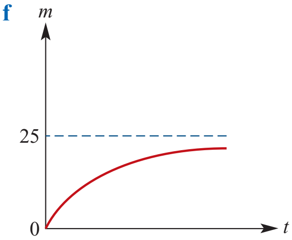
16 a 0.25 kg/min **b** $\frac{m}{100}$ kg/min

c $\frac{dm}{dt} = 0.25 - \frac{m}{100}$

d $m = 25\left(1 - e^{\frac{-t}{100}}\right), t \geq 0$

e 51 minutes

f



17 a $\frac{dx}{dt} = \frac{10 - x}{50}$ **b** 11.16 minutes

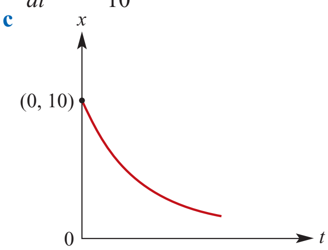
18 a $\frac{dx}{dt} = \frac{80 - x}{200}; x = 80 - 70e^{\frac{-t}{200}}$

b $\frac{dx}{dt} = 0.4 - \frac{x}{400 + t}$

19 $\frac{dx}{dt} = 5 - \frac{x}{10}; x = 50\left(1 - e^{\frac{-t}{10}}\right), t \geq 0$

20 a $\frac{dx}{dt} = -\frac{x}{10}$ **b** $x = 10e^{\frac{-t}{10}}$

c



d $10 \log_e 2 \approx 6.93$ min

21 a $N = 50\,000\left(99e^{\frac{t}{10}} + 1\right), t \geq 0$

b At the end of 1996

Exercise 9E

1 a $\frac{dh}{dt} = \frac{-2000}{\pi h^2}, h > 0$

b $\frac{dh}{dt} = \frac{1}{A}(Q - c\sqrt{h}), h > 0$

c $\frac{dh}{dt} = \frac{3 - 2\sqrt{V}}{60\pi}, V > 0$

d $\frac{dh}{dt} = \frac{-4\sqrt{h}}{9\pi}, h > 0$

2 a $t = -\frac{2\pi}{25}h^{5/2} + 250\pi$

b 13 hrs 5 mins

3 a $\frac{dx}{dt} = -\frac{1}{480\sqrt{4 - x}}$

b $t = 320(4 - x)^{\frac{3}{2}}$ **c** 42 hrs 40 min

4 a $\frac{dr}{dt} = -8\pi r^2$ **b** $r = \frac{2}{16\pi t + 1}$

Answers

5 a $\frac{dh}{dt} = \frac{1000}{A}(Q - kh), h > 0$

b $t = \frac{A}{1000k}\log_e\left(\frac{Q - kh_0}{Q - kh}\right), Q > kh_0$

c $\left(\frac{A}{1000k}\log_e 2\right)$ minutes

Exercise 9F

1 a 1.7443 **b** 1.8309 **c** 4 **d** 3.2556

2 a $y = -\frac{1}{2}\log_e(x^2 + 1) + x\tan^{-1} x + c$

b $y = \frac{1}{4}x^4\log_e|x| - \frac{x^4}{16} + c$

c $y = e^{3x}\left(\frac{3}{13}\sin 2x - \frac{2}{13}\cos 2x\right) + c$

d $y = e^{3x}\left(\frac{3}{13}\cos 2x + \frac{2}{13}\sin 2x\right) + c$

e $y = \frac{2^x}{\log_e(2)} + c$

Exercise 9G

1 a $y_3 \approx 1.2975$ **b** $y_4 \approx 0.0388$

c $y_3 \approx 1.3144$ **d** $y_3 \approx 0.0148$

2 a i 1.8415

ii Euler 1.8438

b i 0.5

ii Euler 0.5038

c i 2.2190

ii Euler 2.2169

d i 0.4055

ii Euler 0.4076

3 a $\tan(1) + 2$

b i 3.444969502 **ii** 3.498989223

iii 3.545369041

4 0.66019008 **5** 2.474287

6 0.30022359

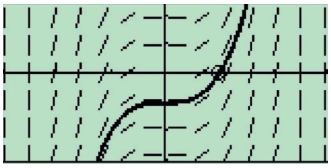
7 a, b

z	$\Pr(Z < z)$ Euler's method	from tables
0	0.5	0.5
0.1	0.53989423	0.53983
0.2	0.57958948	0.57926
0.3	0.61869375	0.61791
0.4	0.65683253	0.65542
0.5	0.69365955	0.69146
0.6	0.72886608	0.72575
0.7	0.76218854	0.75804
0.8	0.79341393	0.78814
0.9	0.82238309	0.81594
1	0.84899161	0.84134

c i 0.69169538 **ii** 0.84212759

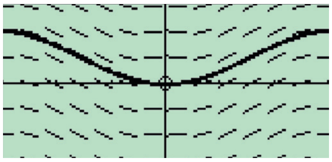
Exercise 9H

2 a



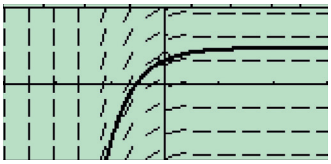
$y = x^3 - 1$

b



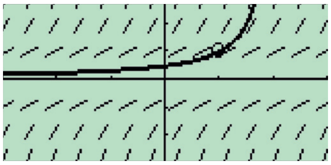
$y = 1 - \cos x$

c



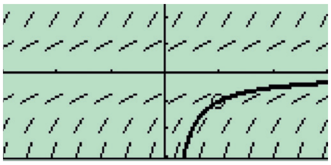
$y = \frac{1}{2}(3 - e^{-2x})$

d



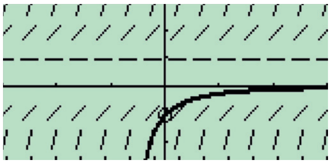
$y = \frac{1}{2 - x}, x < 2$

e



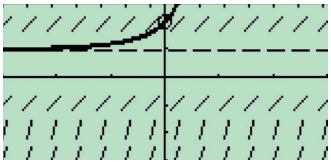
$y = -\frac{1}{x}, x > 0$

f



$y = \frac{1}{1 - 2e^x}$

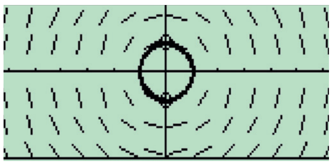
g

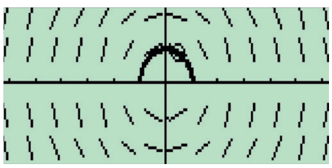


$y = \frac{2}{2 - e^x}$

h


$y = -\log_e(\cos x)$

3 a


b


Multiple-choice questions

1 C **2** D **3** B **4** A **5** E
6 C **7** D **8** E **9** A **10** C

Short-answer questions (technology-free)

1 a $y = x - \frac{1}{x} + c$ **b** $y = e^{10x+c}$

c $y = -\frac{1}{2}\left(\frac{\sin 3t}{9} + \frac{\cos 2t}{4}\right) + at + b$

d $y = \frac{e^{-3x}}{9} + e^{-x} + ax + b$

e $y = 3 - e^{-\frac{x}{2}+c}$ **f** $y = \frac{3x}{2} - \frac{1}{4}x^2 + c$

2 a $y = \frac{1}{2}\sin(2\pi x) - 1$

b $y = \frac{1}{2}\log_e|\sin 2x|$

c $y = \log_e|x| + \frac{1}{2}x^2 - \frac{1}{2}$

d $y = \frac{1}{2}\log_e(1 + x^2) + 1$

e $y = e^{-\frac{x}{2}}$ **f** $x = 64 + 4t - 5t^2$

3 a $k = 2, m = -2$

4 $\frac{dx}{dt} = \frac{3}{\pi x(12 - x)}$

5 $\frac{dC}{dt} = \frac{8\pi}{C}$

6 $\frac{dS}{dt} = -\frac{S}{25}; S = 3e^{\frac{-t}{25}}$

7 $100\log_e 2$ days

8 a $\theta = 30 - 20e^{\frac{-t}{20}}$ **b** 29° **c** 14 mins

9 a $\frac{dA}{dt} = 0.02A$ **b** $0.5\,e^{0.2}$ ha **c** $89\frac{1}{2}$ h

10 $x = \frac{2L}{3}$, maximum deflection $= \frac{L^3}{216}$

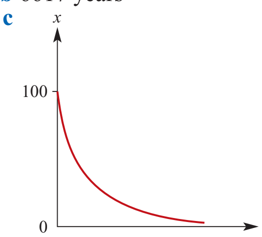
11 $\frac{dh}{dt} = \frac{6 - 0.15\sqrt{h}}{\pi h^2}$

Extended-response questions

1 a i $\frac{dx}{dt} = -kx, k > 0$

ii $x = 100e^{\frac{-t\log_e 2}{5760}}, t \geq 0$

b 6617 years

c


2 a $\frac{dx}{dt} = \frac{3k}{16}(8 - x)(4 - x)$

b $t = \frac{1}{\log_e\left(\frac{7}{6}\right)}\log_e\frac{8 - x}{8 - 2x}$

c 2 min 38 seconds **d** $\frac{52}{31}$ kg

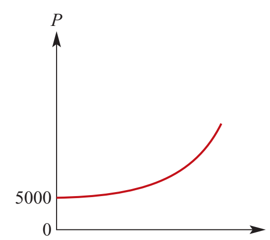
3 a $\frac{dT}{dt} = k(T - T_s), k < 0$

b i 19.2 mins **ii** 42.2° C

4 b $t = \frac{1}{k}\log_e\left(\frac{kp - 1000}{5000k - 1000}\right), kp > 1000$

c iii 0.22

d $p = \frac{1}{k}[e^{kt}(5000k - 1000) + 1000]$

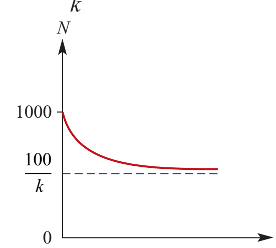


5 a $\frac{dN}{dt} = 100 - kN, k > 0$

b $t = \frac{1}{k}\log_e\left(\frac{100 - 1000k}{100 - kN}\right)$

c 0.16

d $N = \frac{1}{k}(100 - e^{-kt}(100 - 1000k))$



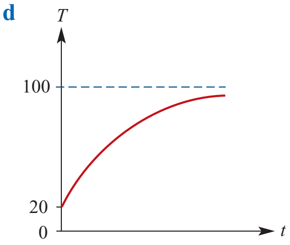
e $\dfrac{100}{k}$

6 a $\dfrac{2L}{3}$ **b** $\dfrac{L}{60}$

7 a $\dfrac{dT}{dt} = \dfrac{100 - T}{40}$ **b** $T = 100 - 80e^{\frac{-t}{40}}$

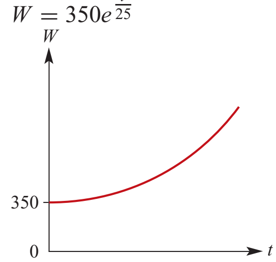
c 62.2° C

d



8 a i $t = 25\log_e\left(\dfrac{W}{350}\right),\ W > 0$

ii $W = 350e^{\frac{t}{25}}$



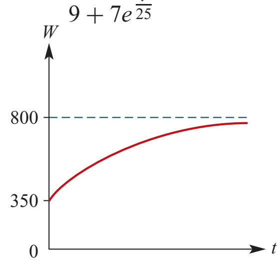
iii 2586

b 0

c i $t = 25\log_e\left[\dfrac{9W}{7(800 - W)}\right]$,

$0 < W < 800$

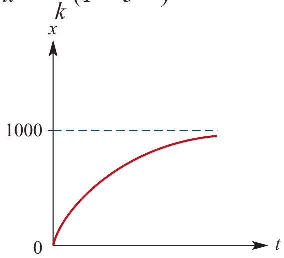
ii $W = \dfrac{5600e^{\frac{t}{25}}}{9 + 7e^{\frac{t}{25}}}$



iii 681

9 a ii $x = \dfrac{R}{k}(1 - e^{-kt})$

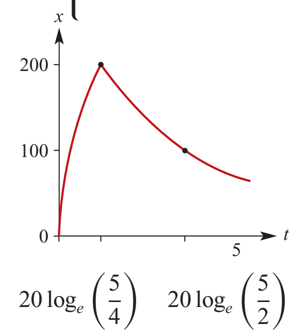
b i



ii 4.46 hours

c i 13.86 hours after the drip is disconnected

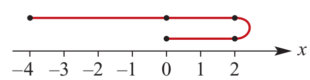
ii $x = \begin{cases} 1000\left(1 - e^{\frac{-t}{20}}\right) & 0 \le t \le 20\log_e\left(\frac{5}{4}\right) \\ 250e^{-\frac{t}{20}} & t > 20\log_e\left(\frac{5}{4}\right) \end{cases}$



Chapter 10

Exercise 10A

1 a $t = 0, x = 0;\ t = 1, x = 2;\ t = 2, x = 2;$
$t = 3, x = 0;\ t = 4, x = -4$



b −6m **c** −1 m/s **d** $v = 3 - 2t$

e −2 m/s **f** $x = \frac{9}{4},\ t = \frac{3}{2}$ **g** $\frac{17}{2}$ m

h $\frac{17}{8}$ m/s

2 a 2, 4 **b** 12 m/s^2 **c** 10 m/s **d** 6 m/s

3 a −3 m/s **b** 1, 3 **c** 12 m/s^2

4 0, $\dfrac{4}{3}$

5 a $\frac{25}{4}$ m/s **b** $\frac{56}{3}$ m

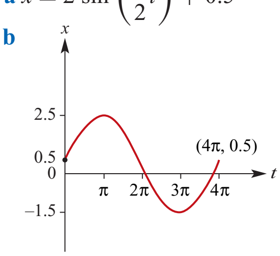
6 a −30 m/s^2 **b** 4, 6 **c** 4 m **d** 120 m

7 a 20 m/s **b** 32 m

8 a 42 m/s **b** 6 s **c** 198 m

9 a $x = 2\sin\left(\dfrac{1}{2}t\right) + 0.5$

b



The object is instantaneously at rest at $t = \pi$ and 3π

c $a = -\frac{1}{2}\sin\left(\frac{1}{2}t\right)$

d i $x = -4a + 0.5$

ii $x = 2\sqrt{1 - v^2} + 0.5$

iii $v = \sqrt{1 - 4a^2}$

10 a 1 s and 15.5 m; 4 s and 2 m

b −6.5 m/s **c** −6 m/s **d** 9 m **e** 2 m

11 a 9 m/s **b** 2π s

12 a i $v = 9.8t$ **ii** $x = 4.9t^2$

b 19.6 m **c** 19.6 m/s

13 **a** 585.6 m **b** 590.70 m

14 $x = \frac{1}{6}t - \frac{1}{4}\log_e\left(\frac{2t+3}{3}\right)$

15 $\left(\frac{3\sqrt{3}}{2} - \frac{\pi}{3}\right)$ m

16 **a** 0 m/s **b** $\frac{1}{2}$ m/s **c** $\frac{1}{2}\log_e 2$ m

d $x = \frac{1}{2}\log_e(1+t^2)$ **e** $\ddot{x} = \frac{1-t^2}{(1+t^2)^2}$

f -0.1 m/s^2 **g** $-\frac{1}{8}$ m/s^2

17 5.25 s 18 1.1 s 19 18.14 m/s

Exercise 10B

1 3 m/s^2

2 **a** 12960 km/h^2 **b** 1 m/s^2

3 **a** 12 m **b** 14 m/s **c** 2.5 s **d** 37 m

4 **a i** 22.4 m **ii** 22.5 m

b i 5 s **ii** -28 m/s

5 **a** $\frac{10}{7}$ s **b** 10 m **c** $\frac{20}{7}$ s

6 **a** 200 s **b** 2 km

7 **a** $\frac{10\sqrt{10}}{7}$ s **b** $14\sqrt{10}$ m/s

8 **a** 4.37 s **b** $-6\sqrt{30}$ m/s

9 **a** 1.25 s **b** 62.5 cm

10 **a** 0.23 **b** $5\frac{1}{3}$ s

11 **a** -0.64 m/s^2 **b** $\frac{55}{4}$ s

12 **a** 4 s **b** $\frac{1}{2}$ m/s^2

Exercise 10C

1 **a ii** 60 m **b ii** 20 m

c ii 30 m **d ii** 55 m

e ii 44 m **f ii** $\frac{70}{3}$ m

g ii $\frac{165}{2}$ m **h ii** 24.5 m

2 **a i** $v = -\frac{1}{2}t + 5$ **ii** $a = -\frac{1}{2}$

iii $x = -\frac{t^2}{4} + 5t$

b i $v = -\frac{2}{5}t^2 + 10$ **ii** $a = -\frac{4}{5}t$

iii $x = -\frac{2}{15}t^3 + 10t$

c i $v = 2t - 10$ **ii** $a = 2$

iii $x = t^2 - 10t$

d i $v = 6(t-1)(t-5)$

ii $a = 12(t-3)$

iii $x = 2(t^3 - 9t^2 + 15t)$

e i $v = 10\sin\frac{\pi}{10}t + 10$

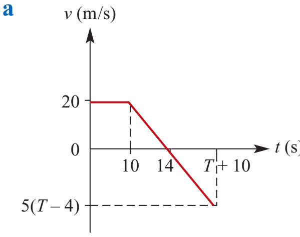
ii $a = \pi\cos\frac{\pi}{10}t$

iii $x = 10\left(t + \frac{10}{\pi} - \frac{10}{\pi}\cos\frac{\pi}{10}t\right)$

f i $v = 10e^{2t}$ **ii** $a = 20e^{2t}$

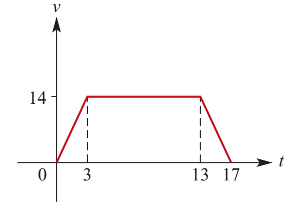
iii $x = 5e^{2t} - 5$

3 **a**

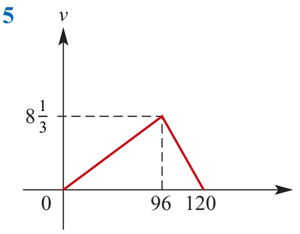


b 23.80 s

4 189 m



5



$a = \frac{25}{288}$ $\dot{x}_{max} = 8\frac{1}{3}$ m/s

6 $68\frac{1}{3}$ s 7 10 s, 150 m

8 $10(3+\sqrt{3})$ s, $200(2+\sqrt{3})$ m

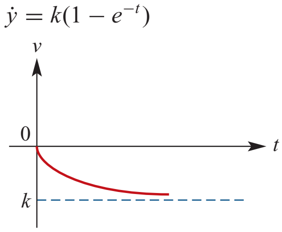
9 **a** 2 s **b** $7\frac{1}{3}$ m

10 36 s

11 **a** 3600 m, 80 km/h

b 90 s after *A* passed *B*, 200 m

12 **a** $\dot{y} = k(1 - e^{-t})$



b limiting velocity of k m/s

Exercise 10D

1 **a** 7 m **b** 4 m

2 **a** $x = \frac{1}{2}\log_e(2e^{2t} - 1)$ **b** $\frac{-100}{2401}$

3 a $v = 3(e^t - 1)$ **b** $a = 3e^t$ **c** $x = 3(e^t - t - 1)$

4 a $v = \frac{g}{k}(1 - e^{-kt})$ **b** $\frac{g}{k}$

5 a $v = \tan\left(\frac{\pi}{3} - \frac{3t}{10}\right)$

b $x = \frac{10}{3}\log_e\left[2\cos\left(\frac{\pi}{3} - \frac{3t}{10}\right)\right]$

6 $v = 450\left(1 - e^{\frac{-t}{50}}\right)$

7 $v = 15\cos\left[\cos^{-1}\frac{4}{5} + \frac{2t}{5}\right]$

8 a $x = 5e^{\frac{2t}{5}}$ **b** 273 m

9 a $t = 50\log_e\left(\frac{500}{500 - v}\right)$

b $v = 500\left(1 - e^{\frac{-t}{50}}\right)$

10 $\frac{1}{k}\log_e 2$ **11** $v = 8e^{\frac{-t}{5}}$; 3.59 m/s

12 a $v = \frac{90}{2t + 3}$ **b** 91.66 m

Exercise 10E

1 a $v = \pm 4$ m/s **b** $t = -\log_e 2$ **c** $x = 2(1 - \log_e 2)$

2 a $v = \frac{1}{x + 1}$

b i $x = e^t - 1$ **ii** $a = e^t$ **iii** $a = v$

3 $x = \frac{-5}{2}\log_e\left(\frac{g + 0.2v^2}{g + 2000}\right)$;

$x_{max} = \frac{5}{2}\log_e\left(\frac{g + 2000}{g}\right)$

4 a $x = \cos 2t$ **b** $a = -4x$

5 a $v = \log_e(1 + t)$ **b** $v^2 = 2\log_e(1 + x)$

c $v = \sqrt{2t + 1} - 1$

6 $v^2 = \frac{x}{2 + x}$

7 a 4 **b** $2\log_e 2 - 1$

8 a 9.83 m **b** 1.01 s

Multiple-choice questions

1 A **2** C **3** A **4** D **5** B
6 C **7** C **8** C **9** A **10** E

Short-answer questions (technology-free)

1 a after 3.5 seconds **b** 2 m/s^2 **c** 14.5 m
d when $t = 2.5$ s and the particle is 1.25 m to the left of O

2 $x = 215\frac{1}{3}$, $v = 73$

3 a 57.6 km/h **b** 1 minute after $6\frac{2}{3}$ seconds **c** 0.24

4 a $\frac{25000}{3}$ m/s^2 **b** 0.4125 m **c** 10000 m/s^2 **d** 0.5 m **e** 37500 m/s^2 **f** 0.075 m

5 a 44 m/s **b** $v = 55 - 11t$ **c** 44 m/s **d** 5 s **e** 247.5 m

6 a 2 s **b** $v = \frac{-t}{\sqrt{9 - t^2}}$, $a = \frac{-9}{(9 - t^2)^{\frac{3}{2}}}$ **c** 3 m **d** $t = 0$

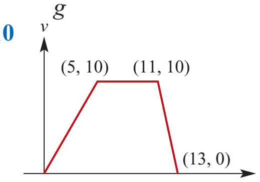
7 a 20 m/s **b** 32 m

8 a $x = 20$ **b** $\frac{109}{8}$ m/s

9 a i $v = 35 - 3g$ up **ii** $v = 5g - 35$ down

b $\frac{35^2}{g}$ metres **c** −35 m/s

10



Distance = 95 m

11 $v = \frac{4}{t - 1}$, $a = -\frac{4}{(t - 1)^2}$

12 16 m

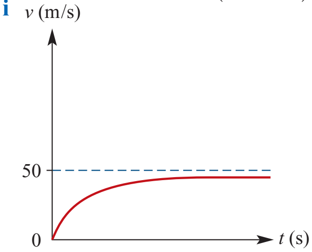
13 a $80 + 0.4g$ m/s **b** $\frac{80 + 0.4g}{g}$ s

c $\frac{(80 + 0.4g)^2}{2g}$ m **d** $\frac{2(80 - 0.4g)}{g}$ s

Extended-response questions

1 a 10 m/s^2 **b** $v = 50\left(1 - e^{\frac{-t}{5}}\right)$

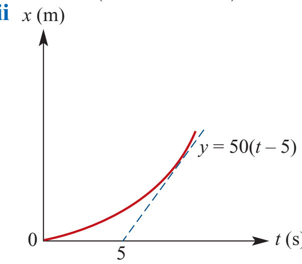
c i



ii 14.98

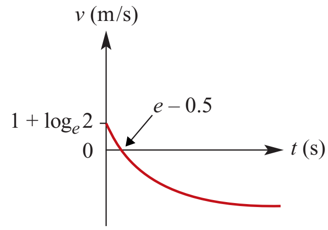
d i $x = 50\left(t + 5e^{\frac{-t}{5}} - 5\right)$

ii



iii 1.32 s

2 a i



ii 1.27 m **iii** 1.47 m

b $B = 10, A = 4.70$

3 a 30 min

b i $a = -k(\sin \pi t + \pi t \cos \pi t - 1)$

ii 0.18 h (from 0 to 0.18 h)

c 845

4 0 m

5 a i $v = 4 - 10t - 3t^2$

ii $a = -10 - 6t$ **iii** 0.36

iv $t = 0$ or 0.70 **v** $t = 2.92$

b i $x = t^2 - t^3 + 2t$ **ii** $\frac{7}{3}$ s **iii** Yes

6 a i $v = -\frac{5\pi}{4}\sin\left(\frac{\pi}{4}t + \frac{\pi}{3}\right)$

ii $a = -\frac{5\pi^2}{16}\cos\left(\frac{\pi}{4}t + \frac{\pi}{3}\right)$

b i $v = -\frac{\pi}{4}\sqrt{25 - x^2}$

ii $a = -\frac{\pi^2 x}{16}$

c 3.4 cm/s

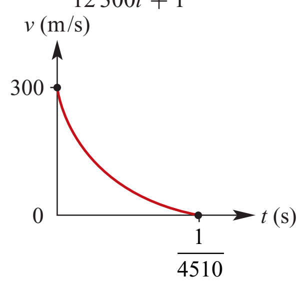
d -1.54 cm/s^2

e i 5 cm **ii** $\frac{5\pi}{4}$ cm/s

iii $\frac{5\pi^2}{16}$ cm/s^2

7 a $v = \frac{300(1 - 4510t)}{12\,300t + 1}, \; 0 \le t \le \frac{1}{4510}$

b



c i $x = -110t + \frac{1}{30}\log_e(12\,300t + 1)$

ii $x = \frac{1}{30}\left[\log_e\left(\frac{410}{v + 110}\right) - \frac{110}{v + 110} + \frac{11}{41}\right]$

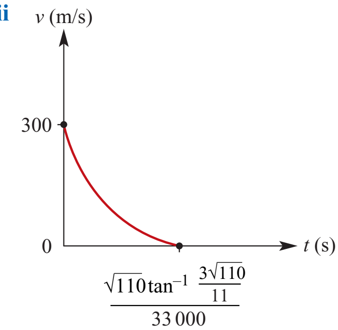
iii 19 mm

d i $t = \frac{\sqrt{110}}{33\,000} \times \left[\tan^{-1}\frac{3\sqrt{110}}{11} - \tan^{-1}\frac{v\sqrt{110}}{1100}\right]$

ii $v = 10\sqrt{110}\tan\left[\tan^{-1}\left(\frac{3\sqrt{110}}{11}\right) - 300\sqrt{110}t\right]$,

$0 \le t \le \frac{\sqrt{110}\tan^{-1}\frac{3\sqrt{110}}{11}}{33\,000}$

iii



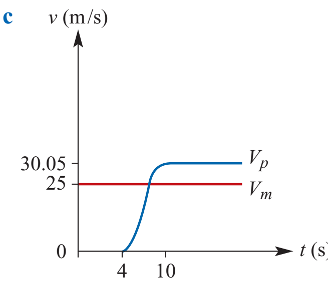
iv 20 mm

8 a 30.05

b i $\frac{dv}{dt} = \frac{-3}{10}\left(3t^2 - 42t + \frac{364}{3}\right)$, $4 \le t \le 10$

ii $t = 7$ (Chasing for 3 s)

c

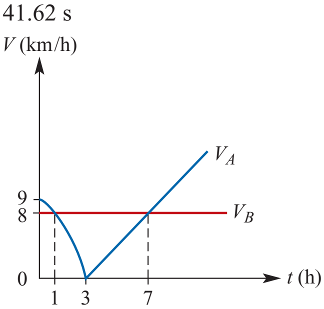


d i 90.3 m

ii $x_p = \begin{cases} 0, & 0 \le t < 4 \\ -\frac{3}{40}t^4 + \frac{21}{10}t^3 - \frac{91}{5}t^2 - \frac{1281}{20}t - \frac{401}{5}, & 4 \le t \le 10 \\ \frac{601}{20}t - \frac{1051}{5}, & t > 10 \end{cases}$

e 41.62 s

9 a



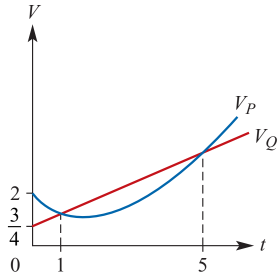
b $t = 1$ or 7

c i 11.7 h **ii** 1.7 h

10 a i $t = 1$ or 5

Answers

ii



b i 2.2

ii $0 < t < 2.2, t > 6.8$

11 a i 4.85 m/s **ii** 0.49 s

b i $v = 9.8t - \frac{1}{2}t^2$

ii $x = 4.9t^2 - \frac{1}{6}t^3$

iii 0.50 s

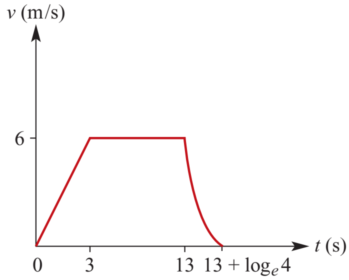
c i $x = 1.2 - 2.45t^2$ **ii** 6 cm

12 a 3 s

b $v = \begin{cases} 2t, & 0 \le t \le 3 \\ 6, & 3 < t \le 13 \\ 8e^{13-t} - 2, & 13 < t \le 13 + \log_e 4 \end{cases}$

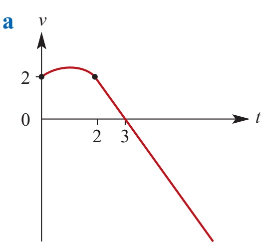
c 14.4 s

d



e 72.2 m

13 a



b $\frac{19}{3}$ **c** 5.52

Chapter 11

11.1 Multiple-choice questions

1 D	**2** D	**3** B	**4** B	**5** C
6 A	**7** C	**8** C	**9** C	**10** C
11 B	**12** C	**13** E	**14** B	**15** A
16 A	**17** B	**18** E	**19** A	**20** D
21 A	**22** A	**23** D	**24** B	**25** B
26 C	**27** C	**28** D	**29** B	**30** E
31 D	**32** B	**33** E	**34** B	**35** B
36 E	**37** E	**38** E	**39** A	**40** C
41 C	**42** B	**43** A	**44** D	**45** C
46 C	**47** E	**48** A	**49** B	**50** A
51 A	**52** E	**53** D	**54** E	**55** A
56 C	**57** A	**58** C	**59** C	**60** C
61 B	**62** C	**63** D	**64** A	**65** B
66 C	**67** A	**68** A	**69** A	**70** D
71 C	**72** D	**73** C	**74** A	**75** B
76 A	**77** D	**78** A	**79** E	

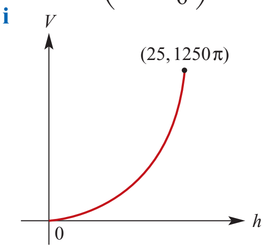
11.2 Extended-response questions

1 a 1250π

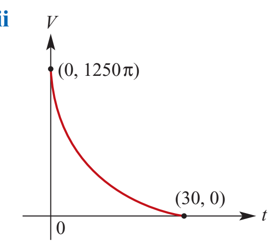
b ii $\frac{10\pi}{3}$ **iii** $h = -\frac{5t}{6} + 25$

iv $V = 2\pi\left(25 - \frac{5t}{6}\right)^2$

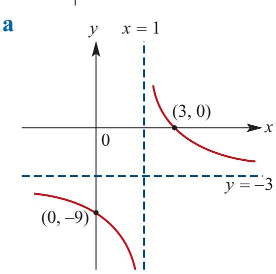
c i



ii



2 a



b (2, 3), (3, 0) **c** $4.5 - \log_e 64$

d $y = -3x + 6\sqrt{2}$

$y = -3x - 6\sqrt{2}$

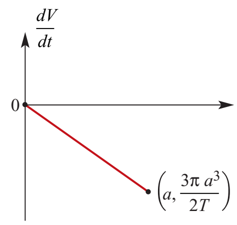
3 e i $\frac{dv}{dh} = \pi\left(\frac{25h}{3} + 100\right)$

ii $\frac{dh}{dt} = \frac{-9\sqrt{h}}{625\pi^2(h + 12)^2}$

f 65 days 19 hours

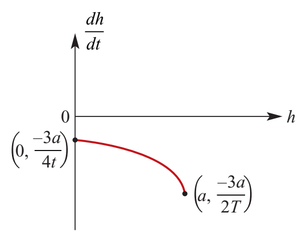
4 a ii 6.355 cm **d** 15.7

e i



$\left(\dfrac{dV}{dt} = \dfrac{-3\pi a^2}{2T}h\right)$

ii



$\left(\dfrac{dh}{dt} = \dfrac{-3a^2}{2T(2a-h)}\right)$

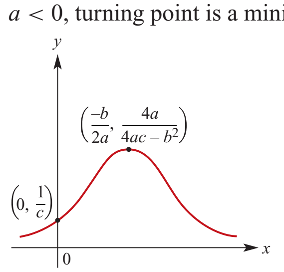
f i $-\dfrac{a}{T}$ **ii** $-\dfrac{6a}{7T}$ **g** -0.37

5 a $\dfrac{-2ax-b}{(ax^2+bx+c)^2}$ **b** $\left(-\dfrac{b}{2a}, \dfrac{4a}{4ac-b^2}\right)$

b i $a > 0$, turning point is a maximum.

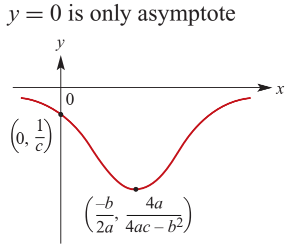
ii $a < 0$, turning point is a minimum.

c i



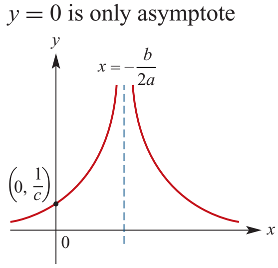
$y = 0$ is only asymptote

ii



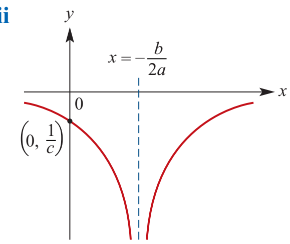
$y = 0$ is only asymptote

d i



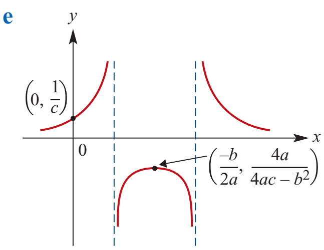
asymptote at $y = 0$

ii



asymptote at $y = 0$

e



Asymptotes at $y = 0$

$x = \dfrac{-b \pm \sqrt{b^2-4ac}}{2a}$

6 a $\dfrac{dy}{dx} = 2ax - \dfrac{2b}{x^3}$

b $\left(\dfrac{\sqrt[4]{a^3b}}{a}, 2\sqrt{ab}\right)$ and $\left(\dfrac{-\sqrt[4]{a^3b}}{a}, 2\sqrt{ab}\right)$

(Both are minimum if $a > 0$, $b > 0$)

7 a $\left\{\dfrac{\pi}{4}, \dfrac{5\pi}{4}, \dfrac{9\pi}{4}, \dfrac{13\pi}{4}\right\}$

b $e^{-2\pi}$

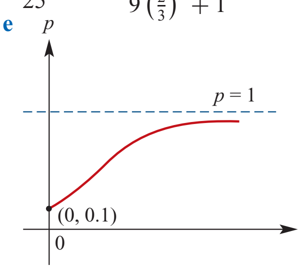
c Maximum at $\left(\dfrac{\pi}{4}, \dfrac{\sqrt{2}}{2}e^{\frac{-\pi}{4}}\right)$ and at $\left(\dfrac{9\pi}{4}, \dfrac{\sqrt{2}}{2}e^{\frac{-9\pi}{4}}\right)$, minimum at $\left(\dfrac{5\pi}{4}, -\dfrac{\sqrt{2}}{2}e^{\frac{-5\pi}{4}}\right)$ and at $\left(\dfrac{13\pi}{4}, -\dfrac{\sqrt{2}}{2}e^{\frac{-13\pi}{4}}\right)$

d $\dfrac{1+e^{\pi}}{2e^{\pi}}$ **e** $\dfrac{1+e^{\pi}}{2e^{3\pi}}$

8 a $\dfrac{1}{5}$

9 b $\dfrac{9}{25}$ **c** $\dfrac{1}{9\left(\frac{2}{3}\right)^t+1}$ **d** $t > 5.419$

e



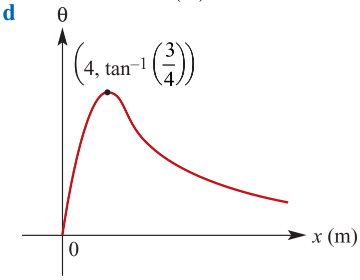
10 b $\dfrac{\sqrt[3]{k^2 p}}{k}$

11 a $\theta = \tan^{-1}\dfrac{8}{x} - \tan^{-1}\dfrac{2}{x}, x > 0$

b $\dfrac{d\theta}{dx} = \dfrac{-8}{x^2+64} + \dfrac{2}{x^2+4}$

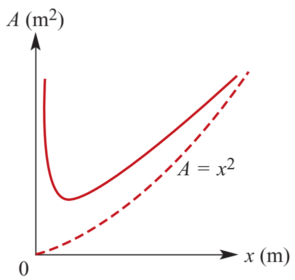
c $0 < \theta < \tan^{-1}\left(\dfrac{3}{4}\right)$

d



e 0.23

13 b



c $x = 6.51$ or 46.43 **d** 20

14 288 cm^2

15 a $y = \dfrac{2}{5}x^2$ **b** $v = 40\sqrt{10}y^{\frac{3}{2}}$

c 252 mm **d** $\dfrac{dy}{dt} = \dfrac{\sqrt{10y}}{10y}, t = \dfrac{2\sqrt{10}}{3}y^{\frac{3}{2}}$

e i 3 mins 9 seconds

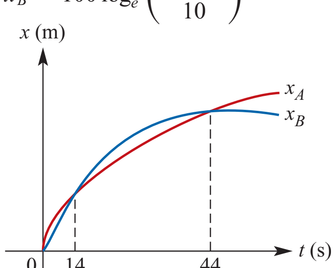
ii 5 mins 45 seconds

16 a $v_A = \dfrac{20}{\sqrt{2t+1}}, v_B = \dfrac{100}{t+10}$

b $x_A = 20(\sqrt{2t+1} - 1)$,

$x_B = 100\log_e\left(\dfrac{t+10}{10}\right)$

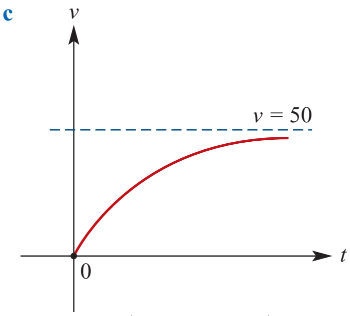
c



d 14 s and 44 s

17 a $v = 50 - 50e^{\frac{-t}{5}}$ **b** 49.9963

c



d i $x = 50\left(t + 5e^{\frac{-t}{5}} - 5\right)$

ii 125.2986

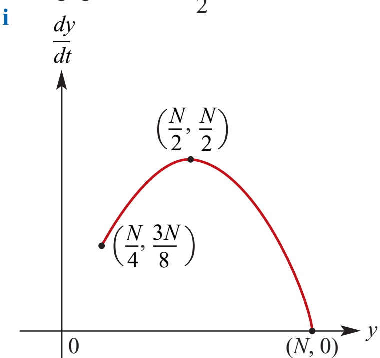
18 a 1180 **b** 129 332

19 a $y = \dfrac{Ne^{2t}}{3+e^{2t}}, \dfrac{dy}{dt} = \dfrac{6Ne^{2t}}{(3+e^{2t})^2}$

b N **c** $\dfrac{dy}{dt} > 0$ for all t

d When population is $\dfrac{N}{2}$

e i



ii At $t = \frac{1}{2}\log_e 3$ or $t \approx 0.549306$

20 a i $v^2 = \dfrac{2gR^2}{x} + u^2 - 2gR$

ii $x = \dfrac{2gR^2}{2gR - u^2}$

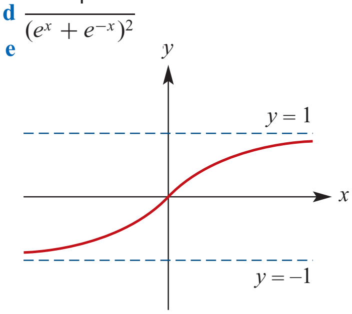
iii $u \geq \sqrt{2gR}$

b 40 320 km/h

21 a 0 **b** 1 **c** −1

d $\dfrac{4}{(e^x + e^{-x})^2}$

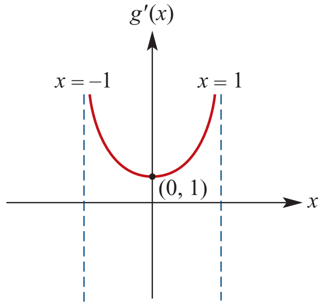
e



f $f^{-1}(x) = \frac{1}{2}\log_e\dfrac{1+x}{1-x}; -1 < x < 1$

g $\dfrac{1}{1-x^2}$

h



22 a i $y = 2r\sin\frac{1}{2}\theta$

ii $\cos\theta = \dfrac{r}{r+h}$

b i $\dfrac{dy}{d\theta} = r\cos\frac{1}{2}\theta$

$$\frac{dy}{dt} = \frac{r\cos\frac{1}{2}\theta\cos^2\theta\sin t}{\sin\theta}$$

ii 6000 km

iii 1500 km/h

24 a $V = \dfrac{4}{3}\pi r^3$

b $4\pi r^2\dfrac{dr}{dt} = -t^2$

c $r = \sqrt[3]{\dfrac{4000\pi - t^3}{4\pi}}$

d 23.2 mins

Chapter 12

Exercise 12A

1 a $y = 2x$
Domain is R, range is R

b $x = 2$
Domain is $\{x : x = 2\}$, range is R

c $y = 7$
Domain is R, range is $\{y : y = 7\}$

d $y = 9 - x$
Domain is R, range is R

e $x = \frac{1}{9}(2-y)^2$
Domain is $[0, \infty)$, range is R

f $y = (x+3)^3 + 1$
Domain is R, range is R

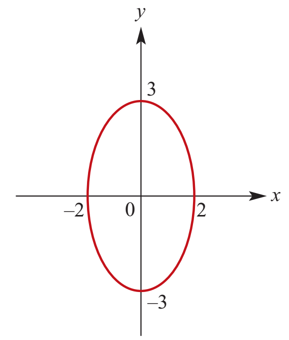
g $y = 3^{\left(\frac{x-1}{2}\right)}$
Domain is R, range is $(0, \infty)$

h $y = \cos(2x + \pi) = -\cos 2x$
Domain is R, range is $[-1, 1]$

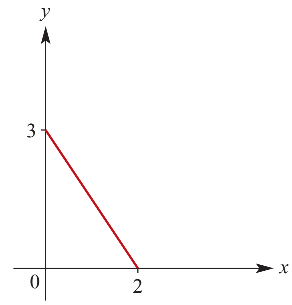
i $y = \left(\dfrac{1}{x} - 4\right)^2 + 1$
Domain is $R\backslash\{0\}$, range is $[1, \infty)$

j $y = \dfrac{x}{1+x}$, $x \neq 0, -1$
Domain is $R\backslash\{-1, 0\}$, range is $R\backslash\{0, 1\}$

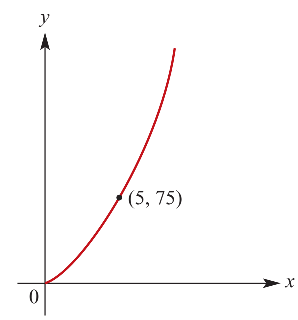
2 a $\dfrac{x^2}{4} + \dfrac{y^2}{9} = 1$, domain $[-2, 2]$
range $[-3, 3]$



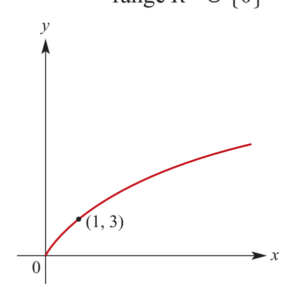
b $3x + 2y = 6$, domain $[0, 2]$
range $[0, 3]$



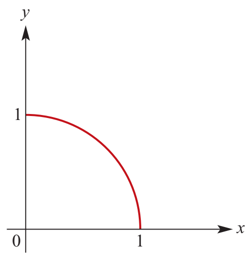
c $y = 3x^2$, domain $R^+ \cup \{0\}$
range $R^+ \cup \{0\}$



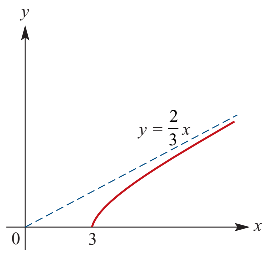
d $y = 3x^{2/3}$, domain $R^+ \cup \{0\}$
range $R^+ \cup \{0\}$



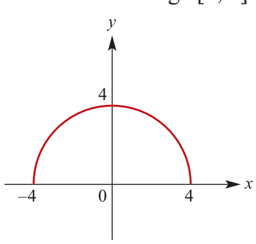
e $x^2 + y^2 = 1$, domain $[0, 1]$
range $[0, 1]$



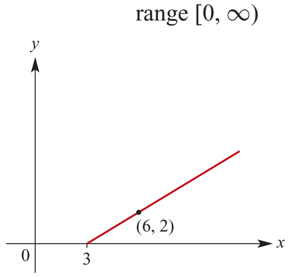
f $\dfrac{x^2}{9} - \dfrac{y^2}{4} = 1$, domain $(3, \infty)$
range $(0, \infty)$



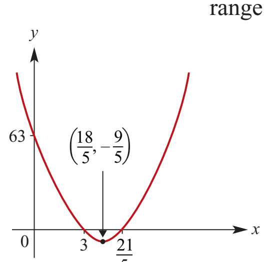
g $x^2 + y^2 = 16$, domain $[-4, 4]$
range $[0, 4]$



h $3y = 2x - 6$, domain $[3, \infty)$
range $[0, \infty)$



i $y = 5x^2 - 36x + 63$, domain R
range $\left[-\frac{9}{5}, \infty\right)$

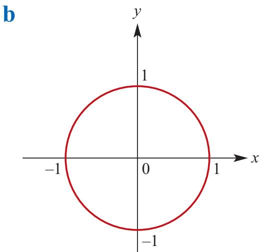


3 a $\boldsymbol{r}(t) = t\boldsymbol{i} + (3 - 2t)\boldsymbol{j}, t \in R$
b $\boldsymbol{r}(t) = 2\cos t\boldsymbol{i} + 2\sin t\boldsymbol{j}, t \in R$
c $\boldsymbol{r}(t) = (2\cos t + 1)\boldsymbol{i} + 2\sin t\boldsymbol{j}, t \in R$
d $\boldsymbol{r}(t) = 2\sec t\boldsymbol{i} + 2\tan t\boldsymbol{j}, t \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$
e $\boldsymbol{r}(t) = t\boldsymbol{i} + ((t-3)^2 + 2(t-3))\boldsymbol{j}, t \in R$
f $\boldsymbol{r}(t) = \sqrt{6}\cos t\boldsymbol{i} + 2\sin t\boldsymbol{j}, t \in R$

4 a $\boldsymbol{r}(\theta) = (2 + 5\cos\theta)\boldsymbol{i} + (6 + 5\sin\theta)\boldsymbol{j}$
b $(x-2)^2 + (y-6)^2 = 25$

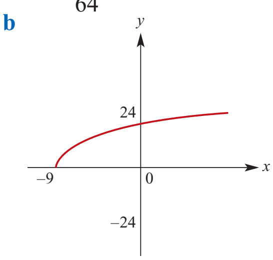
Exercise 12B

1 a $x^2 + y^2 = 1$
b



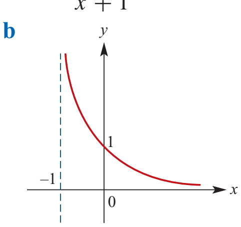
c $\dfrac{\pi}{2}, \dfrac{3\pi}{2}, \dfrac{5\pi}{2}, \ldots$ i.e., $(2n-1)\dfrac{\pi}{2}, n \in N$

2 i a $x = \dfrac{y^2}{64} - 9$
b



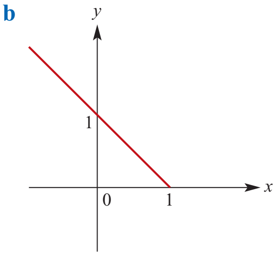
c 3

ii a $y = \dfrac{1}{x+1}, x > -1$
b



c $t = -1$

iii a $y = 1 - x, x < 1$
b



c $t = 1$

3 a $\boldsymbol{r} = \boldsymbol{i} + 4\boldsymbol{j}$ is the position vector; (1, 4) the coordinates of the point
b (1, 4) and (7, −8) **c** $\sqrt{65}$

4 a $\boldsymbol{r} = \frac{9}{2}\boldsymbol{i} - \frac{3}{2}\boldsymbol{j}$ $\left(\frac{9}{2}, \frac{-3}{2}\right)$
b (6, −1) and $\left(\frac{9}{2}, \frac{-3}{2}\right)$ **c** $5\sqrt{2}$

5 a $\sqrt{137}$ **b** $t = \frac{-2}{5}$ and $t = -1$

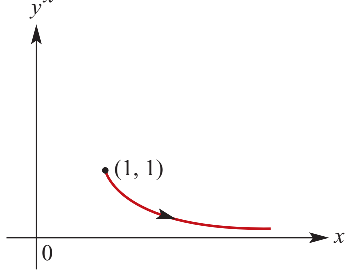
6 a $3\boldsymbol{i} + 6\boldsymbol{j} - 3\boldsymbol{k}$ **b** $3\sqrt{6}$
c $4\boldsymbol{i} + 8\boldsymbol{j} - 3\boldsymbol{k}$ **d** $\boldsymbol{i} + 2\boldsymbol{j}$

7 a $3\boldsymbol{i} + \boldsymbol{j} + 4\boldsymbol{k}$ **b** $\sqrt{14}$

8 $a = \frac{2}{3}, b = 7$

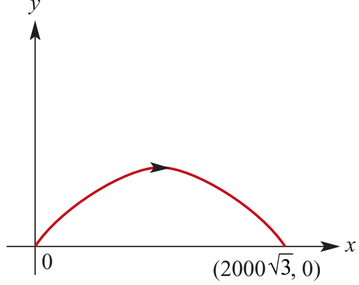
9 a $\frac{x^2}{9} + \frac{y^2}{4} = 1$ **b** $3\boldsymbol{i}$
c i 303.69° **ii** 285.44°

10 a $y = \frac{1}{x}$; $x \geq 1$ if $t \geq 0$ **b** $\boldsymbol{i} + \boldsymbol{j}$
c



11 a $\boldsymbol{r}(0) = 2\boldsymbol{i}$ **b** $\frac{5}{2}\boldsymbol{i} + \frac{3}{2}\boldsymbol{j}$
c $x^2 - y^2 = 4$

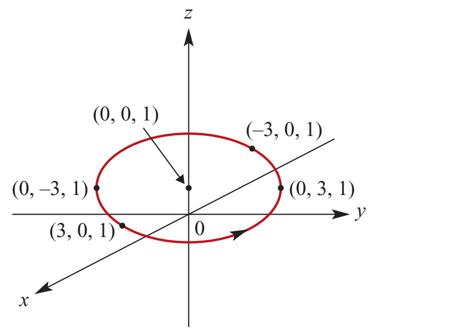
12 a 0
$\boldsymbol{r}(20\sqrt{3}) = 2000\sqrt{3}\boldsymbol{i}$
b $y = \sqrt{3}x - \frac{x^2}{2000}, 0 \leq x \leq 2000\sqrt{3}$
c



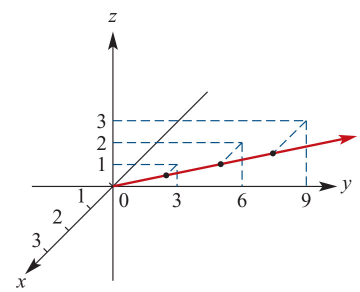
13 collide when $t = \frac{3}{2}$; $\boldsymbol{r}\left(\frac{3}{2}\right) = \frac{27}{2}\boldsymbol{i} - \frac{81}{4}\boldsymbol{j}$

14 a $\frac{(x-1)^2}{4} + \frac{(y-3)^2}{25} = 1$
b i (−1, 3) **ii** (1, −2) **iii** (3, 3)
c π units of time **d** anticlockwise

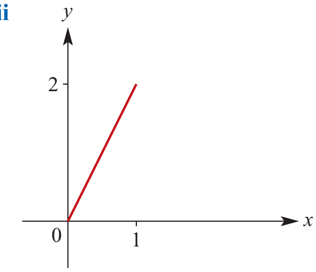
15 The particle is moving along a circular path, centred on (0, 0, 1) with radius length 3, starting at (3, 0, 1) and moving anticlockwise, always a distance of 1 'above' the x−y plane.
It takes 2π units of time to complete one circle.



16 The particle is moving along a linear path, starting at (0, 0, 0) and moving 'forward' one, 'across' three, and 'up' one at each step.

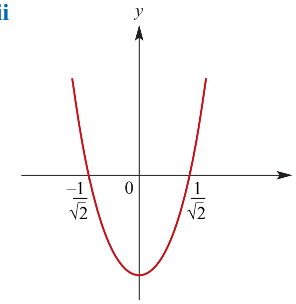


17 a i $y = 2x, 0 \leq x \leq 1$
ii



iii The particle starts at (1, 2) and moves along a linear path towards the origin. When it reaches (0, 0) it reverses direction and heads towards (1, 2). It continues indefinitely in this pattern. It takes $\frac{2\pi}{6\pi} = \frac{1}{3}$ units of time to complete one cycle.

b i $y = 2x^2 - 1, -1 \leq x \leq 1$
ii

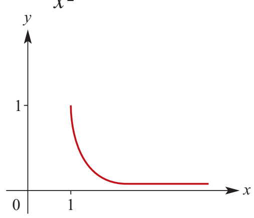


Answers

iii The particle is moving along a parabolic path, starting at (1, 1) and reversing direction at (−1, 1). It takes $\frac{2\pi}{2\pi} = 1$ unit of time for one cycle.

c i $y = \frac{1}{x^2}, \geq 1$

ii



iii The particle is moving along an exponentially decaying path, starting at (1, 1) and moving to the 'right' indefinitely.

Exercise 12C

1 a $\dot{\boldsymbol{r}}(t) = e^t\boldsymbol{i} - e^{-t}\boldsymbol{j}$ $\quad \ddot{\boldsymbol{r}}(t) = e^t\boldsymbol{i} + e^{-t}\boldsymbol{j}$

b $\dot{\boldsymbol{r}}(t) = \boldsymbol{i} + 2t\boldsymbol{j}$ $\quad \ddot{\boldsymbol{r}}(t) = 2\boldsymbol{j}$

c $\dot{\boldsymbol{r}}(t) = \frac{1}{2}i + 2t\boldsymbol{j}$ $\quad \ddot{\boldsymbol{r}}(t) = 2\boldsymbol{j}$

d $\dot{\boldsymbol{r}}(t) = 16\boldsymbol{i} - 32(4t - 1)\boldsymbol{j}$ $\quad \ddot{\boldsymbol{r}}(t) = -128\boldsymbol{j}$

e $\dot{\boldsymbol{r}}(t) = \cos t\boldsymbol{i} - \sin t\boldsymbol{j}$

$\ddot{\boldsymbol{r}}(t) = -\sin t\boldsymbol{i} - \cos t\boldsymbol{j}$

f $\dot{\boldsymbol{r}}(t) = 2\boldsymbol{i} + 5\boldsymbol{j}$ $\quad \ddot{\boldsymbol{r}}(t) = \boldsymbol{0}$

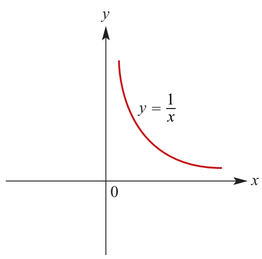
g $\dot{\boldsymbol{r}}(t) = 100\boldsymbol{i} + (100\sqrt{3} - 9.8t)\boldsymbol{j}$

$\ddot{\boldsymbol{r}}(t) = -9.8\boldsymbol{j}$

h $\dot{\boldsymbol{r}}(t) = \sec^2 t\boldsymbol{i} - \sin 2t\boldsymbol{j}$

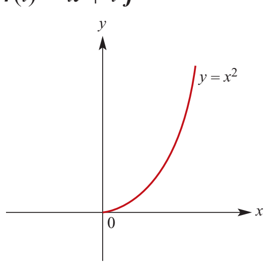
$\ddot{\boldsymbol{r}}(t) = (2\sec^2 t.\tan t)\,\boldsymbol{i} - 2\cos 2t\boldsymbol{j}$

2 a $\boldsymbol{r}(t) = e^t\boldsymbol{i} + e^{-t}\boldsymbol{j}$



$\boldsymbol{r}(0) = \boldsymbol{i} + \boldsymbol{j}$ $\quad \dot{\boldsymbol{r}}(0) = \boldsymbol{i} - \boldsymbol{j}$ $\quad \ddot{\boldsymbol{r}}(0) = \boldsymbol{i} + \boldsymbol{j}$

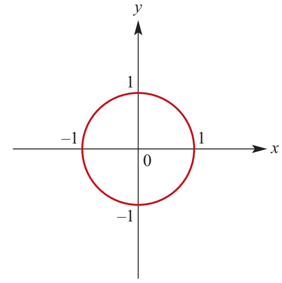
b $\boldsymbol{r}(t) = t\boldsymbol{i} + t^2\boldsymbol{j}$



$\boldsymbol{r}(1) = \boldsymbol{i} + \boldsymbol{j}$ $\quad \dot{\boldsymbol{r}}(1) = \boldsymbol{i} + 2\boldsymbol{j}$

$\ddot{\boldsymbol{r}}(1) = 2\boldsymbol{j}$

c $\boldsymbol{r}(t) = \sin t\boldsymbol{i} + \cos t\boldsymbol{j}$

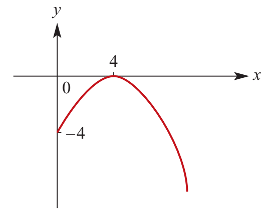


$\boldsymbol{r}\left(\frac{\pi}{6}\right) = \frac{1}{2}\boldsymbol{i} + \frac{\sqrt{3}}{2}\boldsymbol{j}$

$\dot{\boldsymbol{r}}\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2}\boldsymbol{i} - \frac{1}{2}\boldsymbol{j}$

$\ddot{\boldsymbol{r}}\left(\frac{\pi}{6}\right) = -\frac{1}{2}\boldsymbol{i} - \frac{\sqrt{3}}{2}\boldsymbol{j}$

d $\boldsymbol{r}(t) = 16t\boldsymbol{i} - 4(4t - 1)^2\boldsymbol{j}$

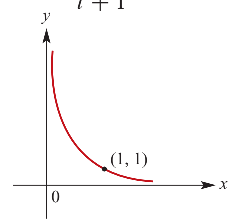


$\boldsymbol{r}(1) = 16\boldsymbol{i} - 36\boldsymbol{j}$

$\dot{\boldsymbol{r}}(1) = 16\boldsymbol{i} - 96\boldsymbol{j}$

$\ddot{\boldsymbol{r}}(1) = -128\boldsymbol{j}$

e $\boldsymbol{r}(t) = \frac{1}{t+1}\boldsymbol{i} + (t+1)^2\boldsymbol{j}$



$\boldsymbol{r}(1) = \frac{1}{2}\boldsymbol{i} + 4\boldsymbol{j}$

$\dot{\boldsymbol{r}}(1) = -\frac{1}{4}\boldsymbol{i} + 4\boldsymbol{j}$

$\ddot{\boldsymbol{r}}(1) = \frac{1}{4}\boldsymbol{i} + 2\boldsymbol{j}$

3 a −1   **b** undefined   **c** $-2e^{-3}$

d $\frac{1}{2}$ **e** 4   **f** $2\sqrt{2}$

4 a $\boldsymbol{r}(t) = (4t + 1)\boldsymbol{i} + (3t - 1)\boldsymbol{j}$

b $\boldsymbol{r}(t) = (t^2 + 1)\boldsymbol{i} + (2t - 1)\boldsymbol{j} - t^3\boldsymbol{k}$

c $\boldsymbol{r}(t) = \frac{1}{2}e^{2t}\boldsymbol{i} + 4(e^{0.5t} - 1)\boldsymbol{j}$

d $\boldsymbol{r}(t) = \left(\frac{t^2 + 2t}{2}\right)\boldsymbol{i} + \frac{1}{3}t^3\boldsymbol{j}$

e $\boldsymbol{r}(t) = -\frac{1}{4}\sin 2t\boldsymbol{i} + 4\cos\frac{1}{2}t\boldsymbol{j}$

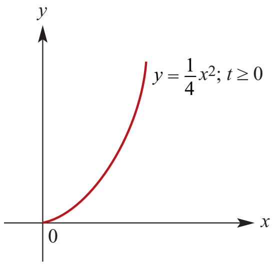
6 a $\dot{\boldsymbol{r}}(t)$ and $\ddot{\boldsymbol{r}}(t)$ are perpendicular when $t = 0, 2$

b Pairs of perpendicular vectors.

$\dot{\boldsymbol{r}}(0)$ and $\ddot{\boldsymbol{r}}(0)$ i.e., $2\boldsymbol{i}$ and $96\boldsymbol{j}$

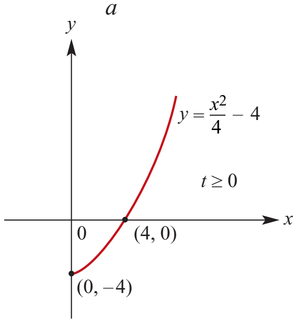
$\dot{\boldsymbol{r}}(2)$ and $\ddot{\boldsymbol{r}}(2)$ i.e., $2\boldsymbol{i}$ and $-96\boldsymbol{j}$

7 a



b When $t = \frac{2}{a}$

8 a



b 45° **c** At $\sqrt{3}$ seconds

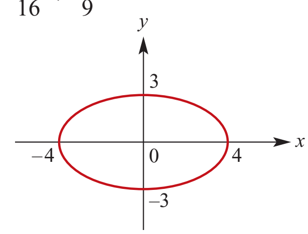
9 a $\dot{\boldsymbol{r}} = 3\boldsymbol{i} + t^2\boldsymbol{j} + 3t^2\boldsymbol{k}$ **b** $|\dot{\boldsymbol{r}}| = \sqrt{9 + 10t^4}$
c $\ddot{\boldsymbol{r}} = 2t\boldsymbol{j} + 6t\boldsymbol{k}$ **d** $|\ddot{\boldsymbol{r}}| = 2\sqrt{10}t$
e $t = \frac{4\sqrt{10}}{5}$

10 a $\dot{\boldsymbol{r}} = V\cos\alpha\boldsymbol{i} + (V\sin\alpha - gt)\boldsymbol{j}$
b $\ddot{\boldsymbol{r}} = -g\boldsymbol{j}$ **c** $t = \frac{V\sin\alpha}{g}$
d $\boldsymbol{r} = \frac{V^2\sin 2\alpha}{2g}\boldsymbol{i} + \frac{V^2\sin^2\alpha}{2g}\boldsymbol{j}$

Exercise 12D

1 a $2t\boldsymbol{i} - 2\boldsymbol{j}$ **b** $2\boldsymbol{i}$ **c** $2\boldsymbol{i} - 2\boldsymbol{j}$
2 a $2\boldsymbol{i} + (6 - 9.8t)\boldsymbol{j}$
b $2t\boldsymbol{i} + (6t - 4.9t^2 + 6)\boldsymbol{j}$
3 a $2\boldsymbol{j} - 4\boldsymbol{k}$
b $3t\boldsymbol{i} + (t^2 + 1)\boldsymbol{j} + (t - 2t^2 + 1)\boldsymbol{k}$
c $\sqrt{20t^2 - 8t + 10}$
d i $\frac{1}{5}$ seconds **ii** $\frac{1}{5}\sqrt{230}$ m/s
4 a $(10t + 20)\boldsymbol{i} - 20\boldsymbol{j} + (40 - 9.8t)\boldsymbol{k}$
b $(5t^2 + 20t)\boldsymbol{i} - 20t\boldsymbol{j} + (40t - 4.9t^2)\boldsymbol{k}$
5 Speed $= 10t$
6 45°
7 minimum speed $= 3\sqrt{2}$; position $= 24\boldsymbol{i} + 8\boldsymbol{j}$
8 a $t = 61\frac{11}{49}$ s **b** 500 m/s **c** $\frac{225\,000}{49}$ m
d 500 m/s **e** $\theta = 36.87°$
9 a $\boldsymbol{r}(t) = \left(\frac{1}{3}\sin 3t - 3\right)\boldsymbol{i} + \left(\frac{1}{3}\cos 3t + \frac{8}{3}\right)\boldsymbol{j}$
b $(x + 3)^2 + \left(y - \frac{8}{3}\right)^2 = \frac{1}{9}$;
centre $= \left(-3, \frac{8}{3}\right)$

10 max speed $= 2\sqrt{5}$; min speed $= 2\sqrt{2}$
11 a magnitude $= \frac{\sqrt{11667}}{9}$ m/s^2,
direction $= \frac{1}{\sqrt{11667}}(108\boldsymbol{i} - \sqrt{3}\boldsymbol{j})$
b $\boldsymbol{r}(t) = \left(\frac{4t^3}{3} + 2t^2 + t\right)\boldsymbol{i} + (\sqrt{2t+1} - 1)\boldsymbol{j}$
12 a $\boldsymbol{r}(t) = V\cos\alpha t\boldsymbol{i} + \left(V\sin\alpha t - \frac{gt^2}{2}\right)\boldsymbol{j}$
13 a time of collision $t = 6$
b position vector $\boldsymbol{r}(6) = 7\boldsymbol{i} + 12\boldsymbol{j}$
14 a $-16\boldsymbol{i} + 12\boldsymbol{j}$ **b** $-80\boldsymbol{i} + 60\boldsymbol{j}$
15 a $8\cos 2t\boldsymbol{i} - 8\sin 2t\boldsymbol{j}$, $t \geq 0$
b 8 **c** $-4\boldsymbol{r}$
16 a $(t^2 - 5t - 2)\boldsymbol{i} + 2\boldsymbol{j}$
b $-\frac{33}{4}\boldsymbol{i} + 2\boldsymbol{j}$ **c** $y = 2$ with $x \geq -8.25$
17 a $\frac{x^2}{36} - \frac{y^2}{16} = 1$
b $6\tan t\sec t\boldsymbol{i} + 4\sec^2 t\boldsymbol{j}$, $t \geq 0$
18 a $\frac{x^2}{16} + \frac{y^2}{9} = 1$



b i $t = 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}, 2\pi$
ii $\boldsymbol{r}(0) = 4\boldsymbol{i}$, $\boldsymbol{r}\left(\frac{\pi}{2}\right) = 3\boldsymbol{j}$, $\boldsymbol{r}(\pi) = -4\boldsymbol{i}$,
$\boldsymbol{r}\left(\frac{3\pi}{2}\right) = -3\boldsymbol{j}$, $\boldsymbol{r}(2\pi) = 4\boldsymbol{i}$
c i $\sqrt{9 + 7\sin^2 t}$ **ii** $\sqrt{16 - 7\cos^2 t}$
iii The maximum and minimum speeds are 4 and 3 respectively.

Exercise 12E

1 a $\boldsymbol{i} + 7\boldsymbol{j}$ **b** $\frac{1}{4}\boldsymbol{i} + 7\boldsymbol{j} + \frac{3}{2}\boldsymbol{k}$
c $\left(1 - \frac{\sqrt{3}}{2}\right)\boldsymbol{i} + \frac{1}{2}\boldsymbol{j} + \frac{\sqrt{3}}{3}\boldsymbol{k}$
d $2\boldsymbol{i} + \frac{2}{3}\boldsymbol{j}$ **e** $\boldsymbol{i} + \frac{1}{2}\boldsymbol{j}$
2 a $\sqrt{17}$ **b** $\frac{9\pi}{2}$ **c** 2π
d $\frac{15\pi}{2}$ **e** $\frac{3}{2}$

Multiple-choice questions

1 E	2 E	3 B	4 E	5 C
6 C	7 C	8 E	9 C	10 E

Short-answer questions (technology-free)

1 a $2\boldsymbol{i} + 4\boldsymbol{j}, 2\boldsymbol{j}$ **b** $4y = x^2 - 16$

2 a $\dot{\boldsymbol{r}}(t) = 4t\boldsymbol{i} + 4\boldsymbol{j}, \ddot{\boldsymbol{r}}(t) = 4\boldsymbol{i}$

b $\dot{\boldsymbol{r}}(t) = 4\cos t\boldsymbol{i} - 4\sin t\boldsymbol{j} + 2t\boldsymbol{k}$

$\ddot{\boldsymbol{r}}(t) = -4\sin t\boldsymbol{i} - 4\cos t\boldsymbol{j} + 2\boldsymbol{k}$

3 $0.6\boldsymbol{i} + 0.8\boldsymbol{j}$

4 a $5\sqrt{3}\boldsymbol{i} + \frac{5}{2}\boldsymbol{j}$

b $\frac{2\sqrt{7}}{7}$

5 $\hat{\boldsymbol{r}} = \cos t\boldsymbol{i} + \sin t\boldsymbol{j}$

6 a $5(-\sin\theta\boldsymbol{i} + \cos\theta\boldsymbol{j})$

b 5 **c** $-5(\cos\theta\boldsymbol{i} + \sin\theta\boldsymbol{j})$

d 0, acceleration at right angles to velocity

7 $\frac{3\pi}{4}$ s

8 a $|\dot{\boldsymbol{r}}| = 1, |\ddot{\boldsymbol{r}}| = 1$

b $(x-1)^2 + (y-1)^2 = 1$ **c** $\frac{3\pi}{4}$

9 $-2\boldsymbol{i} + 20\boldsymbol{j}$

10 a $\boldsymbol{r} = \left(\frac{t^2}{2} + 1\right)\boldsymbol{i} + (t-2)\boldsymbol{j}$

b (13.5, 3) **c** 12.5 s

11 a $\dot{\boldsymbol{r}} = t\boldsymbol{i} + (2t-5)\boldsymbol{j}$

b $\boldsymbol{r} = \left(\frac{t^2}{2} - 1\right)\boldsymbol{i} + (t^2 - 5t + 6)\boldsymbol{j}$

c $-\boldsymbol{i} + 6\boldsymbol{j}, -5\boldsymbol{j}$

12 a i $\dot{\boldsymbol{r}}_2(t) = (2t-4)\boldsymbol{i} + t\boldsymbol{j}$

ii $\dot{\boldsymbol{r}}_1(t) = t\boldsymbol{i} + (k-t)\boldsymbol{j}$

b i 4 **ii** 8 **iii** $4(\boldsymbol{i} + \boldsymbol{j})$

13 b i $\dot{\boldsymbol{r}}(t) = e^t\boldsymbol{i} + 8e^{2t}\boldsymbol{j}$

ii $\boldsymbol{i} + 8\boldsymbol{j}$ **iii** $\log_e 1.5$

14 b i $x = 2$ for $y \geq -3.5$ **ii** $(2, -3.5)$

Extended-response questions

1 a The speeds of particles P and Q are $3\sqrt{13}$ and $\sqrt{41}$ m/s respectively.

b i The position vector of P at time $t = 0$ is $60\boldsymbol{i} + 20\boldsymbol{j}$

The position vector of Q at time $t = 0$ is $80\boldsymbol{i} + 80\boldsymbol{j}$

ii $\overrightarrow{PQ} = (20 - 4t)\boldsymbol{i} + (60 - 2t)\boldsymbol{j}$

c 10 seconds, $20\sqrt{5}$ metres

2 a $\overrightarrow{AB} = ((v+3)t - 56)\boldsymbol{i} + ((7v - 29)t + 8)\boldsymbol{j}$

b 4

c i $\overrightarrow{AB} = (6t - 56)\boldsymbol{i} + (8 - 8t)\boldsymbol{j}$

ii 4 seconds

3 a $\overrightarrow{BF} = -3\boldsymbol{i} + 6\boldsymbol{j} - 6\boldsymbol{k}$ **b** 9 m

c 3 m/s **d** $(-\boldsymbol{i} + 2\boldsymbol{j} - 2\boldsymbol{k})$ m/s

e 2 seconds, $2\sqrt{26}$ metres

4 a i 200 seconds **ii** $\frac{1}{2}$

iii 5 m/s **iv** (1200, 0)

b 8 seconds, 720 metres

5 a i $\overrightarrow{OA} = (6t - 1)\boldsymbol{i} + (3t + 2)\boldsymbol{j}$

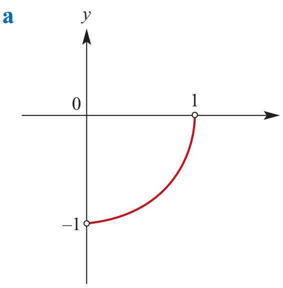
ii $\overrightarrow{BA} = (6t - 3)\boldsymbol{i} + (3t + 1)\boldsymbol{j}$

b One second

c i $\boldsymbol{c} = \frac{1}{5}(3\boldsymbol{i} + 4\boldsymbol{j})$ **ii** $\boldsymbol{d} = \frac{1}{5}(4\boldsymbol{i} - 3\boldsymbol{j})$

iii $6\boldsymbol{c} + 3\boldsymbol{d}$

6 a



b i $a = 16$ **ii** $b = -16$ **iii** $n = 2$

iv $\boldsymbol{v}(t) = -32\sin(2t)\boldsymbol{i} - 32\cos(2t)\boldsymbol{j}$

$\boldsymbol{a}(t) = -4(16\cos(2t)\boldsymbol{i} - 16\sin(2t)\boldsymbol{j})$

c i $\overrightarrow{PQ} = 8((\sin t - 2\cos(2t))\boldsymbol{i} + (\cos t + 2\sin(2t))\boldsymbol{j})$

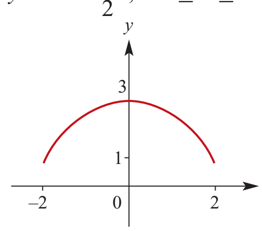
ii $|\overrightarrow{PQ}|^2 = 64(5 + 4\sin t)$

d 8 cm

7 a $2\sin t\boldsymbol{i} + (\cos 2t + 2)\boldsymbol{j}, t \geq 0$ **b** $2\boldsymbol{i} + \boldsymbol{j}$

c i $y = 3 - \frac{x^2}{2}, -2 \leq x \leq 2$

ii



d $|\boldsymbol{v}|^2 = -16\cos^4 t + 20\cos^2 t$, maximum speed is $\frac{5}{2}$

e $\frac{3\pi}{2}$ **f ii** $t = \frac{\pi}{2} + k\pi, k \in Z$

8 a $a\boldsymbol{i} + (b + 2t)\boldsymbol{j} + (20 - 10t)\boldsymbol{k}$

b $at\boldsymbol{i} + (bt + t^2)\boldsymbol{j} + (20t - 5t^2)\boldsymbol{k}$

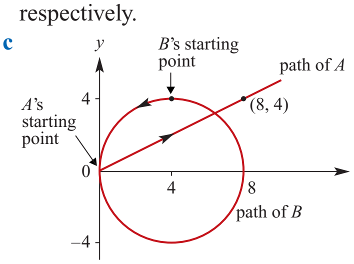
c 4 seconds **d** $a = 25, b = -4$

e 38.3°

9 a 4α

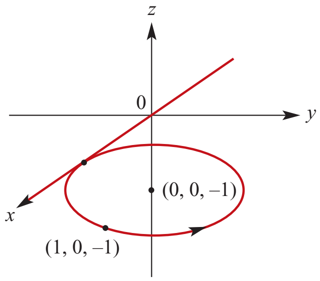
b $y = \frac{x}{2}, x \geq 0$ and $(x-4)^2 + y^2 = 16$ are the cartesian equations of the paths of A and B respectively.

c



d $(0, 0), \left(\frac{32}{5}, \frac{16}{5}\right)$ **e** 1.76

10 a i The particle P is moving along a circular path centred on (0, 0, −1) with radius length one. The particle starts at (1, 0, −1) and moves "anticlockwise" a distance of one "below" the x–y (horizontal) plane. The particle finishes at (1, 0, −1) after one revolution.



ii $\sqrt{2}$
iii $-\sin t\boldsymbol{i} + \cos t\boldsymbol{j},\ 0 \le t \le 2\pi$
v $\ddot{\boldsymbol{p}}(t) = -\cos t\boldsymbol{i} - \sin t\boldsymbol{j},\ 0 \le t \le 2\pi$

b i $\overrightarrow{PQ} = (\cos 2t - \cos t)\boldsymbol{i} + (-\sin t - \sin 2t)\boldsymbol{j} + \frac{3}{2}\boldsymbol{k}$
iii $\frac{5}{2}$ iv $\frac{\pi}{3}, \pi, \frac{5\pi}{3}$
v $\frac{3}{2}$ vi $0, \frac{2\pi}{3}, \frac{4\pi}{3}, 2\pi$

c ii $\frac{\sqrt{10}}{5}\left(\cos 3t - \frac{1}{2}\right)$ iii 162°

11 a i $-9.8\boldsymbol{j}$ ii $2\boldsymbol{i} - 9.8t\boldsymbol{j}$
iii $2t\boldsymbol{i} - 4.9t^2\boldsymbol{j}$
b i $\frac{2\sqrt{2}}{7}$ seconds ii $\frac{4\sqrt{2}}{7}$ metres

12 a i $6\boldsymbol{i} - 3\boldsymbol{j}$ ii $\frac{\sqrt{5}}{5}(2\boldsymbol{i} - \boldsymbol{j})$
b $4\boldsymbol{i} - 2\boldsymbol{j}$, (4, −2)
c i $\overrightarrow{LP} = \left(1 - \frac{7t}{2}\right)\boldsymbol{i} + (7 - 2t)\boldsymbol{j}$
ii 1.05 p.m. iii $\frac{9\sqrt{65}}{13}$ kilometres

Chapter 13

Exercise 13A

1 a i $\boldsymbol{r} = 5\boldsymbol{i} + 5\boldsymbol{j}$
ii $5\sqrt{2} \approx 7.07$ N; 45°
b i $\boldsymbol{r} = -4\boldsymbol{i} - 4\boldsymbol{j}$
ii $4\sqrt{2} \approx 5.66$ N; 225°
c i $\boldsymbol{r} = -\boldsymbol{i} - 5\boldsymbol{j}$
ii $\sqrt{26} \approx 5.10$ N; 258.7°
d i $\boldsymbol{r} = 3\boldsymbol{i} + 10\boldsymbol{j}$
ii $\sqrt{109} \approx 10.44$ N; 73.3°
e i $\boldsymbol{r} = -4\boldsymbol{j}$ ii 4 N; 270°
f i $\boldsymbol{r} = 10\boldsymbol{i}$ ii 10 N; 0°

2 $\boldsymbol{R} = (11\boldsymbol{i} - 3\boldsymbol{j})$ N

3 25.43 N

4 $\frac{\sqrt{781} - 9}{2} \approx 9.5$ N

5 $\boldsymbol{F}_3 = -2\boldsymbol{i} + \boldsymbol{k}$

6 386 N

7 a i $6.064\boldsymbol{i} + 2.57\boldsymbol{j}$
ii 6.59 N; 22°59′
b i $19.41\boldsymbol{i} + 7.44\boldsymbol{j}$
ii 20.79 N; 20°57′
c i $1.382\boldsymbol{i} + 5.394\boldsymbol{j}$
ii 5.57 N; 75°38′
d i $2.19\boldsymbol{i} - 2.19\boldsymbol{j}$ ii 3.09 N; 315°
e i $18.13\boldsymbol{i}$ ii 18.13 N; 0°
f i $-2.15\boldsymbol{i} - 1.01\boldsymbol{j}$
ii 2.37 N; 205°17′

8 a 6.59 N; 22°59′ c 5.57 N; 75°38′
e 18.13 N; 0°

9 a $5\boldsymbol{j}$ b 5 N; 90°

10 a 11.28 N b 6.34 N
c 0 N d −9.01 N

11 a 17.72 N b 14.88 N

12 a $\frac{11}{5}(2\boldsymbol{i} - \boldsymbol{j})$ b $\frac{-6}{25}(3\boldsymbol{i} + 4\boldsymbol{j})$

13 a −1.97 N b 5.35 N c −0.48 N

14 −3.20 N

15 a 32.15 N b 33.23 N

16 a 4.55 N; 19°40′ b 12.42 N; 63°31′

17 15.46 N

18 a 6.93 N b 14 N

19 1.15 N

Exercise 13B

1 a 10 kg m/s b 0.009 kg m/s
c $8333\frac{1}{3}$ kg m/s d 60 kg m/s
e $41666\frac{2}{3}$ kg m/s

2 a $10(\boldsymbol{i} + \boldsymbol{j})$ kg m/s
b i $10(5\boldsymbol{i} + 12\boldsymbol{j})$ kg m/s
ii 130 kg m/s

3 a −30 kg m/s b 40 kg m/s
c 90 kg m/s

4 a $5g \approx 49$ N b $3000g \approx 29400$ N
c $0.06g \approx 0.588$ N

5 a 32 N b $\frac{1}{2}$ m/s^2

6 a 4 kg b 7 N

7 $\frac{96}{(1.2 + g)} \approx 8.73$ kg

8 660 N

9 2.076 kg wt

10 5.4×10^{-14} N

11 a $\boldsymbol{a} = \boldsymbol{i} + 5\boldsymbol{j}$

12 $\boldsymbol{a} = \boldsymbol{i} - \frac{2}{5}\boldsymbol{j}$

13 a 2.78 kg wt b 3.35 kg wt

14 $-34\ 722\frac{2}{9}$ N

15 113 N

16 $\mathbf{F}_3 = 19.6\boldsymbol{i} - \boldsymbol{j}$

17 5 N
18 $\boldsymbol{a} = \frac{7}{2}\boldsymbol{i} + 2\boldsymbol{j}$
19 $\frac{1}{2}$ m/s
20 663 N
21 a $\frac{g}{5} \approx 1.96$ m/s^2 **b** 19.6 m/s
22 42.517 s
23 Pushing Force = 62.5 N; Resistance = 25 N
24 60000 N; -0.1 m/s^2
25 $\frac{5}{49}$
26 a 0.0245 N **b** 5.1 m/s
27 0.612
28 a $200g \approx 1960$ N **b** 2060 N
29 a 2 m/s^2 **b** 1.06 m/s^2

Exercise 13C

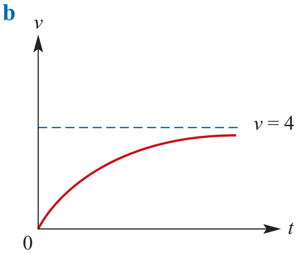
1 $g \cos 45°$ m/s$^2 \approx 6.93$ m/s^2
2 $\dfrac{\sqrt{2}(1-\mu)g}{2}$ m/s^2
3 29.223
4 a $\sqrt{3}$ m/s^2 **b** 1.124 m/s^2
5 $\sqrt{2}g$ N
6 181 N
7 $a = \dfrac{P}{m} - \mu g \cos\theta - g\sin\theta$
8 $\boldsymbol{a} = -\dfrac{g}{2}\boldsymbol{i}$
9 6.76 m/s
10 8.84 m; 4.31 m/s
11 a $\dfrac{\sqrt{40gx}}{5}$ m/s **b** $\dfrac{8x}{3}$ m
12 a $a = \dfrac{F}{M}(\cos\theta + \mu\sin\theta) - \mu g$
b $a = \dfrac{F}{M}(\cos\theta - \mu\sin\theta) - \mu g$
13 a 490 N **b** 1980 N
14 a $\dfrac{100}{9g} \approx 1.13$ m **b** $2\sqrt{3} \approx 3.46$ m/s
15 $(8 + 4\sqrt{3})$ N
16 a 40.49 N **b** 1.22 m/s^2

Exercise 13D

1 a $\dfrac{80g}{9}$ N ≈ 87.1 N
b $\dfrac{g}{9}$ m/s$^2 \approx 1.09$ m/s^2
2 a $\dfrac{10}{11} \approx 0.91$ m/s^2
b $S = T = \dfrac{50}{11}$ N ≈ 4.55 N
3 a 16.8 N **b** 1.4 m/s^2
4 a 2.92 **b** 25.71 N
5 a $\dfrac{98}{15}$ m/s^2 **b** $26\frac{2}{15}$ N
6 a 19.6 N **b** 4.9 m/s^2
7 a 0.96 m/s^2 **b** 39.4 N
8 2.67 kg
9 a 10 750 N **b** 9250 N
10 5.28 kg
11 a 0.025 m/s^2 **b** 10 000 N
12 a $\dfrac{8g}{5} \approx 15.6$ N **b** $4g \approx 39.2$ N
c $\dfrac{g}{5} \approx 1.96$ m/s^2
13 0.305
14 a $\mu = 0.86$ **b** 52.8 N
15 a 16.296 N **b** $\mu = 0.35$

Exercise 13E

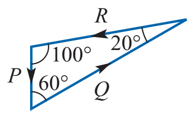
1 $33\frac{1}{3}$ m/s; 250 m
2 a $x = 6t - 2\sin t$ **b** $4\sqrt{3}$
c $x = \dfrac{1}{4}(2t^2 - \cos(2t) + 1)$
3 $\dfrac{110}{9}$ m/s; $\left(\dfrac{400}{3} - \dfrac{50}{3}\log_e 3\right)$ m
4 $x = \dfrac{t^2}{2} + 16\sin\left(\dfrac{t}{4}\right) - 4t$
5 a $\dot{x} = t - 2\sin\left(\frac{1}{2}t\right)$
b $x = \dfrac{t^2}{2} + 4\cos\left(\frac{1}{2}t\right) - 4$
6 10 m/s
7 $10 - \log_e 11 \approx 7.6$ m/s
8 a $v = 4(1 - e^{-0.5t})$ m/s
b



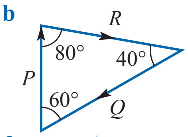
c 112 m (approx)
9 a 5.5 m/s **b** $\dfrac{275}{6} - 10\log_e 2$
10 $\dfrac{um}{k}\left(e^{\frac{kt}{m}} - 1\right)$ metres
11 $V - \dfrac{k}{m}x$
12 $\dfrac{b}{c}(1 - e^{-ct/m})$ m/s; $\dfrac{b}{c}$ m/s
13 maximum height reached $= \dfrac{m}{2k}\log_e\left(1 + \dfrac{ku^2}{mg}\right)$;
speed $= u\sqrt{\dfrac{mg}{ku^2 + mg}}$
15 b $\dfrac{4375}{3}$
c $1000\log_e 2 + \dfrac{4375}{3} \approx 2151.48$

Exercise 13F

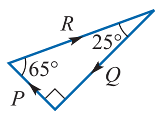
1 a



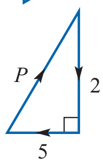
b



c



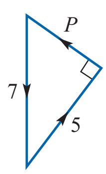
2 a



b $P = \sqrt{29}$ N

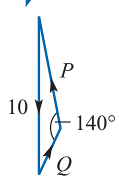
c $180° - \tan^{-1}\left(\dfrac{2}{5}\right) = 158.20°$

3 a



b $2\sqrt{6}$ **c** 135.58°

4 a



b 5.32

5 a $P = 4.60$ N; $Q = 1.31$ N

b $P = 6.13$ N, $Q = 2.23$ N

6 18.13 N

7 66.02 N

9 a along the bisector of the angle between the forces (rhombus property)

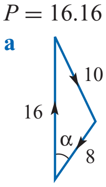
b 18.13 N

c 18.13 N making an angle of 155° with each of the 10 N forces

10 $P = 13.40$ N, $Q = 16.51$ N

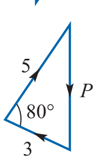
11 $P = 16.16$ N, $Q = 7.99$ N

12 a



b 149.25°

13 a



b 5.37 **c** 146.59°

14 $P = 42.09$; tension $= 50.82$ N

15 45.23 N

16 a 169.67° **b** 2.81 N

Exercise 13G

1 a 10 N **b** $3g$ N **c** $3g$ N

2 a 1.53 N **b** 1.97 N

3 a $30g$ N **b** 386.94 N

c 1224.02 N

4 2.5 N

5 $\mu < \tan 20°$

6 21.80°

7 a 0.47 **b** 24.85 N

8 a 10 N **b** 27.88° **c** 0.05

Exercise 13H

1 a $2\boldsymbol{i} - 3\boldsymbol{j}$ **b** $\sqrt{13}$

c $\boldsymbol{v} = 2t\boldsymbol{i} - 3t\boldsymbol{j}$ **d** $|\boldsymbol{v}| = \sqrt{13}$

e 303.69°

2 a $2\boldsymbol{i} + 3\boldsymbol{j}$ **b** $2t\boldsymbol{i} + 3t\boldsymbol{j}$

c $t^2\boldsymbol{i} + \dfrac{3t^2}{2}\boldsymbol{j}$ **d** $y = \dfrac{3x}{2}; x \geq 0$

3 a $\boldsymbol{r}(0) = 8\boldsymbol{j}$ **b** $y = \dfrac{2x}{5} + 8; x \geq 0$

c $\boldsymbol{F} = 20\boldsymbol{i} + 8\boldsymbol{j}$ N

4 a $\boldsymbol{r}(0) = 25\boldsymbol{i} + 10\boldsymbol{j}$ **b** $y = 35 - x; x \leq 25$

c $-50\boldsymbol{i} + 50\boldsymbol{j}$ N

5 a $\left(\frac{3}{2}\boldsymbol{i} - \frac{1}{2}\boldsymbol{j}\right)$ m/s^2 **b** $\left(\frac{3}{2}t\boldsymbol{i} - \frac{1}{2}t\boldsymbol{j}\right)$ m/s

c $\left(\frac{3}{4}t^2 + 2\right)\boldsymbol{i} - \left(\frac{1}{4}t^2 + 2\right)\boldsymbol{j}$

6 a $\left(8\boldsymbol{i} + \frac{8}{3}\boldsymbol{j}\right)$ m/s^2

b i $\left(80\boldsymbol{i} + \frac{80}{3}\boldsymbol{j}\right)$ N **ii** $\dfrac{80\sqrt{10}}{3}$ N

7 a $y = \dfrac{x}{2} + 6, x \geq 0$

b $\dot{\boldsymbol{r}}(t) = 4t\boldsymbol{i} + 2t\boldsymbol{j}$

c $t = 8$ **d** $8\boldsymbol{i} + 4\boldsymbol{j}$ N

8 a $0.15\boldsymbol{i} + 0.25\boldsymbol{j}$ m/s^2

b $(3 + 0.15t)\boldsymbol{i} + (5 + 0.25t)\boldsymbol{j}$ m/s

c $20.7\boldsymbol{i} + 34.5\boldsymbol{j}$

d $y = \dfrac{5}{3}x, x \geq -30$

9 15 m/s; $5\sqrt{10}$ m/s

Multiple-choice questions

1 D	**2** E	**3** B	**4** B	**5** D
6 B	**7** B	**8** B	**9** B	**10** C

Short-answer questions (technology-free)

1 a 885 N **b** 6785 N

2 a $\dfrac{g}{4}$ m/s^2 **b** $\dfrac{15g}{4}$ N

4 a $(10 - 0.4g)$ m/s^2 b $(5 - 0.4g)$ m/s^2

5 a $\dfrac{4}{(t+1)^2}$ m/s^2 b $\dfrac{4t}{t+1}$ m/s

c $[4t - 4\log_e(t+1)]$ m

6 2000 N

7 $\dfrac{g}{\cos\theta}\sin(\phi - \theta)$

8 a 490 N b 1980 N

9 a $\dfrac{\sqrt{3}}{2}$ b 2 m

10 a $\dfrac{g}{4}$ m/s^2

b particle lowered with $a \geq \dfrac{g}{6}$

11 4 m/s

12 a $(\boldsymbol{i} + 2\boldsymbol{j})$ m/s^2

b i $(t+1)(\boldsymbol{i} + 2\boldsymbol{j})$ m/s

ii $\sqrt{5}\,(t+1)$ m/s

c $\left(\dfrac{t^2}{2} + t\right)(\boldsymbol{i} + 2\boldsymbol{j})$ m

d $y = 2x, x \geq 0$

13 $204\frac{1}{6}$ m 14 2250 N

15 $\dfrac{10000}{3g}$ 16 $m(g + f)$ N

17 $100\sqrt{2}$ m/s

18 a 9 kg wt b $\dfrac{g}{9}$ m/s^2

19 a down with acceleration $\dfrac{(m_1 - m_2)g}{m_1 + m_2}$ m/s^2

b $\dfrac{2m_1m_2\,g}{m_1 + m_2}$ N

20 a moves with acceleration $\dfrac{m_1 g}{m_1 + m_2}$ m/s^2

b $\dfrac{m_1 m_2\, g}{m_1 + m_2}$ N

21 a moves up plane with acceleration

$\dfrac{(m_1 - m_2\ \sin\alpha)g}{m_1 + m_2}$ m/s^2

b $\dfrac{m_1 m_2 g(1 + \sin\alpha)}{m_1 + m_2}$ N

22 $(\sin\alpha - \mu\cos\alpha)g$

23 a $\dfrac{3g}{8}$ m/s^2 b $\dfrac{15g}{4}$ N

c $\dfrac{15g\sqrt{2}}{4}$ N 45° to the horizontal.

d $\dfrac{4}{\sqrt{3g}}$ s e $\dfrac{6}{\sqrt{3g}}$ s

24 a $\dfrac{g(5\sqrt{3} - 3)}{13}$ m/s^2

b $\dfrac{3g(10 + 5\sqrt{3})}{13}$ N

25 a $\dfrac{g}{4}$ m/s^2 b $\dfrac{\sqrt{2g}}{2}$ m/s c $\dfrac{5g}{4}$

27 $\dfrac{10\sqrt{3}g}{3}$ N 28 $5g\sqrt{3}$ N

29 $\dfrac{8g}{3}$ N, $\dfrac{10g}{3}$ N 30 $\dfrac{25g}{13}$ N, $\dfrac{60g}{13}$ N

Extended-response questions

1 a 2.8 N b 0.7 m/s^2

c i $\sqrt{\dfrac{20}{21 - 2g}}$ s ii $\sqrt{5(21 - 2g)}$ m/s

d 0.357 metres

2 a i $0.3\,g$ m/s^2 $= 2.94$ m/s^2

ii $2.1\,g$ N

b 8.26875 m

3 a i 6888 N ii 948 N

b i 14088 N ii 2148 N

4 a i $\dfrac{8g}{15}$ m/s^2 ii $\dfrac{14g}{75}$ N

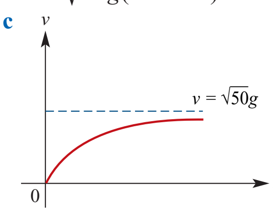
b 0.2 m

5 b ii 8.96

6 a $x = 25\log_e\left(\dfrac{50g}{50g - v^2}\right)$

b $v = \sqrt{50g(1 - e^{\frac{-x}{25}})}$

c



7 a $2 - \sqrt{3}$ b $200 \leq M \leq 100\,(\sqrt{3} + 1)$

c i $\dfrac{g}{7}$ m/s^2 ii $\dfrac{1200g}{7}$ N iii 30.54 m/s

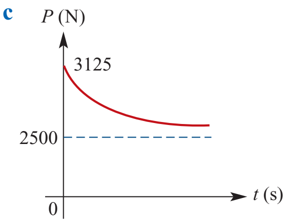
8 a $\dfrac{25}{2}e^{-0.1t}$ m/s^2

b i $625(4 + e^{-0.1t})$ ii $5(625 - v)$

iii 3025 N

iv $625(4 + e^{-3}) \approx 2531.11$ N

c



9 a $R = Mg\cos\alpha - T\sin\theta$

b $T = \dfrac{Mg\sin\alpha + 0.1Mg\cos\alpha}{\cos\theta + 0.1\sin\theta}$

c i $T = \dfrac{8.6g}{\cos\theta + 0.1\sin\theta}$

ii 5°43′ iii $\dfrac{86\sqrt{101}}{101}g$ N

d 5°43′

10 a i $\dfrac{600g}{13}$ N ii $\dfrac{60g}{13}$ N

b $\dfrac{19g}{65}$ m/s^2

c i $\dfrac{14\sqrt{1235}}{65}$ m/s ii 2.64 seconds

d i $(8.86 - 5t)$ m/s^2 ii 1.86 s

Chapter 14

14.1 Multiple-choice questions

1 C	**2** E	**3** C	**4** A
5 C	**6** B	**7** B	**8** C
9 A	**10** D	**11** E	**12** E
13 B	**14** B	**15** A	**16** B
17 E	**18** E	**19** D	**20** A
21 E	**22** B	**23** D	**24** D
25 E	**26** C	**27** E	**28** C
29 D	**30** D	**31** A	**32** C
33 D	**34** C	**35** E	**36** E
37 D	**38** D	**39** C	

14.2 Extended-response questions

1 a $2\boldsymbol{i} - 10\boldsymbol{j}$ m/s **b** $\dot{\boldsymbol{r}}_1(t) = 2\boldsymbol{i} - 2t\boldsymbol{j}$
c $\boldsymbol{i} - 3\boldsymbol{j}$ **d** $t = 0$
e 5 s **f** Yes; $t = 2$

2 a $\dfrac{2 - \sin\alpha}{\cos\alpha}$
b i $\dfrac{g}{2}$ **ii** $\dfrac{2\sqrt{10}}{7}$

3 a $\boldsymbol{r} = \left(\dfrac{\sin 3t}{3} - 3\right)\boldsymbol{i} + \left(\dfrac{8 + \cos 3t}{3}\right)\boldsymbol{j}$
b $-3\boldsymbol{i} + \dfrac{8}{3}\boldsymbol{j}$ **c** $\dot{\boldsymbol{r}}.\ddot{\boldsymbol{r}} = 0$

4 a 6π s
b i $-(3\sqrt{3}\boldsymbol{i} + 2.25\boldsymbol{j})$ **ii** $\boldsymbol{i} - \dfrac{3\sqrt{3}}{4}\boldsymbol{j}$
c i $1.5\sqrt{9 + 7\sin^2\dfrac{t}{3}}$
ii $t = 3\left(\dfrac{\pi}{2} + n\pi\right), n \in N \cup \{0\}$
d $\ddot{\boldsymbol{r}} = -9\boldsymbol{r}, t = 3\pi n, n \in N \cup \{0\}$

5 b $\dfrac{68}{9}$ kg **c** 0.1064 m/s^2, 30.065 N

6 a i $\dfrac{3}{2}\sin 2t\boldsymbol{i} - 2\cos 2t\boldsymbol{j}$
ii $-6\sin 2t\boldsymbol{i} + 8\cos 2t\boldsymbol{j}$
iii $t = \dfrac{\pi n}{4}, n \in N \cup \{0\}$
iv $16x^2 + 9y^2 = 36$
b $a = (2n + 1)\dfrac{\pi}{4}, n \in N \cup \{0\}$

7 a $\dfrac{g}{4}$ **b** $\dfrac{3g}{4}$ N
c $\dfrac{\sqrt{2g}}{2}$ **d** 0.903 s

8 a i $\dfrac{g}{25}$ m/s^2 **ii** $\dfrac{156g}{125}$ N
iii $\dfrac{2g}{25}$ m; $\dfrac{2g}{25}$ m/s
b i $\dfrac{14}{45}$ s **ii** $\dfrac{14g}{1125}$ m $+ \dfrac{2g}{25}$ m $= \dfrac{104g}{1125}$ m

9 b i $\boldsymbol{r}_2 = (0.2t - 1.2)\boldsymbol{i} + (-0.2t + 3.2)\boldsymbol{j} + \boldsymbol{k}$
ii $t = 16$ at $2\boldsymbol{i} + \boldsymbol{k}$

10 a ii 10 m/s^2, $75t - 5t^2$
b 281.25 m **c i** 180 m

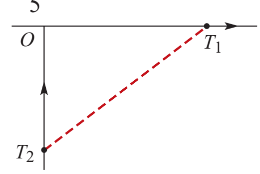
11 c i $-(\boldsymbol{i} + \boldsymbol{j}), \boldsymbol{0}$ **iii** $-0.43\boldsymbol{i} - 0.68\boldsymbol{j}$

12 b i $F + mg\cos\theta$ **ii** $\mu(F + mg\cos\theta)$
iii $\dfrac{mg(\sin\theta - \mu\cos\theta)}{\mu}$
c i $F\sin\theta + mg\cos\theta$
ii $\dfrac{mg(\sin\theta - \mu\cos\theta)}{\mu\sin\theta + \cos\theta}$
iii $\dfrac{mg(\sin\theta + \mu\cos\theta)}{\cos\theta - \mu\sin\theta}$
d ii $\dfrac{mg(\sin\theta - \mu\cos\theta)}{\mu\cos(\theta - \alpha) - \sin(\theta - \alpha)}$

13 a i $h\boldsymbol{j}$ for $0\boldsymbol{i} + 0\boldsymbol{j}$ at the foot of the cliff
ii $V\cos\alpha\boldsymbol{i} + V\sin\alpha\boldsymbol{j}$
b i $V\cos\alpha\boldsymbol{i} + (V\sin\alpha - gt)\boldsymbol{j}$
ii $Vt\cos\alpha\boldsymbol{i} + \left(h + Vt\sin\alpha - \dfrac{gt^2}{2}\right)\boldsymbol{j}$
c $\dfrac{V\sin\alpha}{g}$

14 a i 9504 N **ii** 704 N
b 0.6742 s **c** about 10 persons (852 kg)

15 a i T_1 **ii** t_0
b ii $\dfrac{2\sqrt{5}}{5}Vt_0$
iii



16 a $\dfrac{\sqrt{2dm(F - mg\mu)}}{m}$
b i μg
ii $\dfrac{\sqrt{2dm(F - mg\mu)}}{3m}$
c $F = 10m\mu g$

17 a ii $\dfrac{mg}{2}(1 - \mu\sqrt{3})$
b i $\dfrac{\mu mg\,7\sqrt{3}}{10}$
ii $\dfrac{7mg}{20}(1 + 2\sqrt{3}\mu)$
c $\dfrac{\sqrt{3}}{24}$
d i $\dfrac{7g}{17}$
ii $\dfrac{7mg}{272}$ N
e A will slide into stationary B.

18 a i $0\boldsymbol{i} + 0\boldsymbol{j}$ **ii** $10\boldsymbol{i} + 10\sqrt{3}\boldsymbol{j}$, 20, 60°
iii $-9.8\boldsymbol{j}$

b **i** $\frac{x}{10}$

ii $x\boldsymbol{i} + (x\sqrt{3} - 0.049x^2)\boldsymbol{j}$

iii $10\boldsymbol{i} + (10\sqrt{3} - 0.98x)\boldsymbol{j}$

iv $-8\boldsymbol{i} + (10\sqrt{3} - 0.98x)\boldsymbol{j}$

c **i** $-8\boldsymbol{i} + (10\sqrt{3} - 0.98x - 9.8t_1)\boldsymbol{j}$

ii $\boldsymbol{r} = (x - 8t_1)\boldsymbol{i} + (x\sqrt{3} - 0.049x^2 + t_1(10\sqrt{3} - 0.98x - 4.9t_1))\boldsymbol{j}$

d $\frac{20\sqrt{3} - 0.98x}{9.8}$ **e** 15.71 m

19 **a** $5\boldsymbol{i}$

b **i** $(5 - 3t_1)\boldsymbol{i} + 2t_1\boldsymbol{j} + t_1\boldsymbol{k}$, $(5 - 3t_2)\boldsymbol{i} + 2t_2\boldsymbol{j} + t_2\boldsymbol{k}$,

ii $-3(t_2 - t_1)\boldsymbol{i} + 2(t_2 - t_1)\boldsymbol{j} + (t_2 - t_1)\boldsymbol{k}$

c $-3\boldsymbol{i} + 2\boldsymbol{j} + \boldsymbol{k}$

d **i** 36.70° **ii** 13.42

20 **a** $y = 5 - 2x, x \le 2$

b **i** $\boldsymbol{r}_1(t) = 2\boldsymbol{i} + \boldsymbol{j} + t(-\boldsymbol{i} + 2\boldsymbol{j})$

ii $2\boldsymbol{i} + \boldsymbol{j}$ is the starting position, $(-\boldsymbol{i} + 2\boldsymbol{j})$ represents the velocity

c **i** $-13\boldsymbol{i} + 6\boldsymbol{j}$ **ii** $5\sqrt{10}$

21 **a** $13\boldsymbol{i} + \boldsymbol{j} + 5\boldsymbol{k}$

b $\frac{\sqrt{14}}{14}(-3\boldsymbol{i} + \boldsymbol{j} + 2\boldsymbol{k})$, $\frac{\sqrt{6}}{6}(2\boldsymbol{i} + \boldsymbol{j} - \boldsymbol{k})$

c 40.20° **d** $7\boldsymbol{i} + 3\boldsymbol{j} + 9\boldsymbol{k}$

e $13\boldsymbol{i} - \boldsymbol{j} - 8\boldsymbol{k} + t(-5\boldsymbol{i} + 3\boldsymbol{k})$

f $\frac{\sqrt{1190}}{34}$

22 **a** $\frac{6}{5}(4\boldsymbol{i} + 3\boldsymbol{j})$

b **i** $\frac{1}{5}(-11\boldsymbol{i} + 28\boldsymbol{j})$ **ii** $\frac{1}{5}(13\boldsymbol{i} + 46\boldsymbol{j})$

iii $-7\boldsymbol{i} + 2\boldsymbol{j} + \frac{6}{5}t\,(4\boldsymbol{i} + 3\boldsymbol{j})$

c **i** $\frac{1}{5}(29\boldsymbol{i} + 58\boldsymbol{j})$ **ii** $\frac{8}{3}$ hours

iii $\frac{1}{5}\sqrt{(15 + 11t)^2 + (27t - 15)^2}$

iv 3.91 km

Chapter 15

15.1 Multiple-choice questions

1 B	**2** D	**3** B	**4** E
5 A	**6** D	**7** E	**8** A
9 B	**10** A	**11** B	**12** C
13 C	**14** A	**15** D	**16** D
17 B	**18** D	**19** A	**20** E
21 A	**22** C	**23** D	**24** A
25 A	**26** B	**27** D	**28** E
29 C	**30** E	**31** C	**32** E
33 D	**34** E		

15.2 Extended-response questions

1 **a** $f'(x) = \log_e(x) - 2$ **b** $A = (e^3, 0)$

c $y = x - e^3$ **d** 2:1

2 **a** **i** $\frac{dy}{dx} = \frac{(b^2 - a^2)\cos x}{(b + a\sin x)^2}$

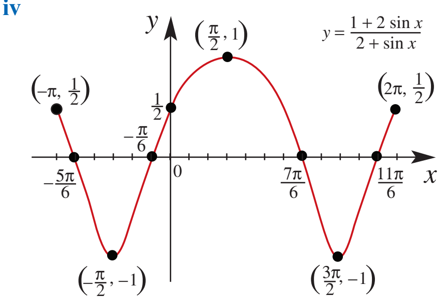
ii 1, −1

b **i** $\left(0, \frac{1}{2}\right)$

ii $\left(\frac{-5\pi}{6}, 0\right)\left(\frac{-\pi}{6}, 0\right)\left(\frac{7\pi}{6}, 0\right), \left(\frac{11\pi}{6}, 0\right)$

iii $\left(-\frac{\pi}{2}, -1\right), \left(\frac{\pi}{2}, 1\right), \left(\frac{3\pi}{2}, -1\right)$

iv



v $2\pi(3 - \sqrt{3})$

3 **a** 2 and $\frac{\pi}{3}$ **b** $[-2, 2]$

c (0, 1) **d** $\left(\frac{5\pi}{6}, 0\right)$ and $\left(\frac{11\pi}{6}, 0\right)$

e $\frac{\pi}{12}, \frac{7\pi}{12}$ **f** $\frac{\log_e(21 + 12\sqrt{3})}{4}$

g $(10\pi + 3\sqrt{3})\frac{\pi}{6}$

4 **a** To show **b** To show **c** 88.01 cm² **d** 306.15 cm²

5 **a** **i** $\int_{10}^{5} \frac{-50}{v(1 + v^2)}dv$

ii $25\log_e\left(\frac{104}{101}\right)$ seconds

b **i** To show

ii $x = 50(\tan^{-1}(10) - \tan^{-1} v)$

iii To show

iv 74 m

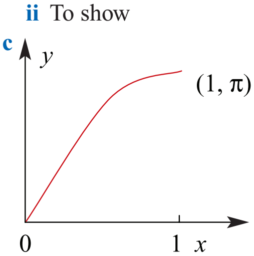
6 **a** **i** $\cos \pi x - \pi x \sin \pi x$

ii $\frac{1}{\pi^2}\sin \pi x - \frac{x}{\pi}\cos(\pi x)$

b **i** $p = \pi$

ii To show

c



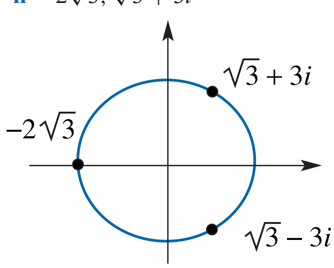
d $\frac{(2\pi^2 + 15)\pi}{6}$

e $k = 2$
f 1.066
g 0.572

7 a **i** $3g - T = 3b,\ T - 2g = 4b$
ii $b = \frac{g}{7},\ T = \frac{18g}{7}$
b **i** $3 - 0.1t$ kg
ii $(3 - 0.1t)g - T_1 = (3 - 0.1t)a$
iii $T_1 - 2g = 4a$
iv $a = \frac{(1 - 0.1t)g}{7 - 0.1t}$
c $\frac{dv}{dt} = \frac{(1 - 0.1t)g}{7 - 0.1t},\ v = gt + 60g\log_e \frac{7 - 0.1t}{7}$
d 18.999
e 93.188 m

8 a $m = \sqrt{3}$
b **i** Points A, O, C must be collinear as $\overrightarrow{AC} = -\overrightarrow{OC}$. Hence AC is a diameter.
ii To show
c **i** To show
ii $2\boldsymbol{i} - \boldsymbol{j} + 2\boldsymbol{k}$ and $\frac{8}{3}\boldsymbol{i} - \frac{1}{3}\boldsymbol{j} + \frac{4}{3}\boldsymbol{k}$
d $\left(\frac{3}{\sqrt{18 - 2\sqrt{3}}}\right)((2 + \sqrt{3})\boldsymbol{i} + (-1 + \sqrt{3})\boldsymbol{j} + (2 - \sqrt{3})\boldsymbol{k})$
e $t = \frac{3}{4},\ k = \frac{1}{2},\ l = \frac{13\sqrt{3}}{12}$
f The particle lies outside the circle

9 a **i** $24\sqrt{3}$
ii $-2\sqrt{3},\ \sqrt{3} + 3i$
b



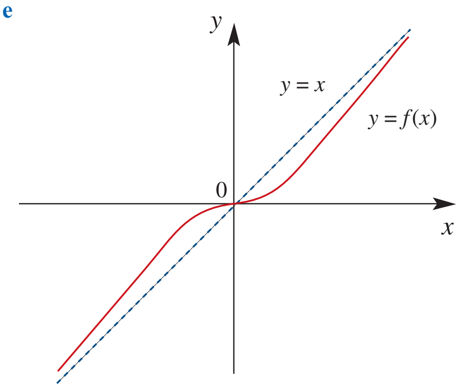
c **i** To show
ii $\sqrt{3} \pm 6i$
iii $\frac{(x - \sqrt{3})^2}{27} + \frac{y^2}{36} = 1$

10 a **i** $\frac{x^2}{9} + \frac{(y + a)^2}{36} = 1$
ii $\pm\frac{\sqrt{36 - a^2}}{2}$
b $f(x) = 2\sqrt{9 - x^2} - a$
c $\sqrt{9 - x^2} - \frac{x^2}{\sqrt{9 - x^2}}$
d **i** $A = 9$ **ii** To show
e $\frac{1}{2}\left(x\sqrt{9 - x^2} + 9\arcsin\left(\frac{x}{3}\right)\right)$
f $18\arcsin\frac{\sqrt{36 - a^2}}{6} - \frac{a}{2}(\sqrt{36 - a^2})$
g 9π
h 144π

11 a $y^2 = 16x^2(1 - x^2)(1 - 2x^2)^2$
b $\frac{dx}{dt} = \cos t,\ \frac{dy}{dt} = 4\cos 4t,\ \frac{dy}{dx} = \frac{4\cos 4t}{\cos t}$
c **i** $\frac{\pi}{8}, \frac{3\pi}{8}, \frac{5\pi}{8}, \frac{7\pi}{8}, \frac{9\pi}{8}, \frac{11\pi}{8}, \frac{13\pi}{8}, \frac{15\pi}{8}$
ii $-\frac{\sqrt{2-\sqrt{2}}}{2}, -\frac{\sqrt{2+\sqrt{2}}}{2}, \frac{\sqrt{2-\sqrt{2}}}{2}, \frac{\sqrt{2+\sqrt{2}}}{2}$
iii $\left(-\frac{\sqrt{2 - \sqrt{2}}}{2}, 1\right), \left(-\frac{\sqrt{2 - \sqrt{2}}}{2}, -1\right),$
$\left(-\frac{\sqrt{2 + \sqrt{2}}}{2}, 1\right), \left(-\frac{\sqrt{2 + \sqrt{2}}}{2}, -1\right),$
$\left(\frac{\sqrt{2 - \sqrt{2}}}{2}, 1\right), \left(\frac{\sqrt{2 - \sqrt{2}}}{2}, -1\right),$
$\left(\frac{\sqrt{2 + \sqrt{2}}}{2}, 1\right), \left(\frac{\sqrt{2 + \sqrt{2}}}{2}, -1\right)$
iv $\frac{dy}{dx} = \pm 4$ when $x = 0$;
$\frac{dy}{dx} = \pm 4\sqrt{2}$ when $x = \pm\frac{1}{\sqrt{2}}$
v To show
d $\frac{16}{15}(\sqrt{2} + 1)$
e $\frac{64\pi}{63}$

12 a $y^2 = \frac{64x^2(25 - x^2)}{25}$
b **i** ± 8 **ii** $\pm\frac{14}{5}$
c **i** $\frac{\pi\sqrt{2}}{12}$ **ii** $\frac{\pi\sqrt{2}}{12}$
d $\frac{800}{3}$
e $\frac{325}{16}$
f $\frac{6400\pi}{3}$

13 a $f'(x) = \frac{x^4 + 3ax^2}{(x^2 + a)^2},\ f''(x) = \frac{6a^2x - 2ax^3}{(x^2 + a)^3}$
b (0, 0) stationary point of inflexion
c $\left(-\sqrt{3a}, \frac{-3\sqrt{3a}}{4}\right), \left(\sqrt{3a}, \frac{3\sqrt{3a}}{4}\right)$
d $y = x$
e



f $a = 1$

14 a $f'(x) = \dfrac{x^4 - 3ax^2}{(x^2-a)^2}$, $f''(x) = \dfrac{6a^2x + 2ax^3}{(x^2-a)^3}$

b $\left(-\sqrt{3a}, \dfrac{-3\sqrt{3a}}{2}\right)$ local maximum,

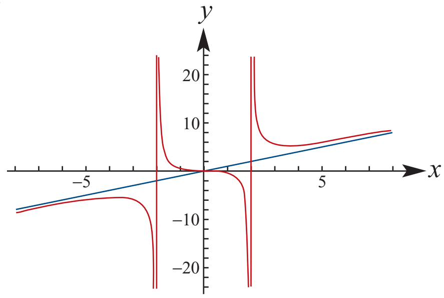
$\left(\sqrt{3a}, \dfrac{3\sqrt{3a}}{2}\right)$ local minimum,

(0,0) stationary point of inflexion

c (0, 0)

d $y = x, x = \sqrt{a}, x = -\sqrt{a}$

e



f $a = 16$

15 a $\dfrac{x}{\sqrt{1-x^2}} + \arcsin(x)$, (0, 0) local minimum

(Note: It is easy to see $f(x) \geq 0$ for all x as x and arcsin (x) have the same sign for all x and $f(x) = 0$ if and only if $x = 0$.)

b $\dfrac{x^2\sqrt{1-x^2} + 2(1-x^2)^{\frac{3}{2}}}{(x^2-1)^2} =$

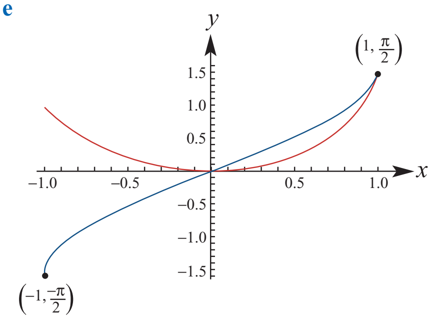
$\dfrac{\sqrt{1-x^2}(2-x^2)}{(x^2-1)^2} \geq 0$ for all $x \in (-1, 1)$.

No points of inflexion.

c $f(x) \geq 0$ for all x as x and arcsin (x) have the same sign for all x

d $x = 0$ and $x = 1$

e



f $\dfrac{3\pi}{8} - 1$

16 a $x = \dfrac{3}{4}\sin 2t$, $y = -\dfrac{1}{2}\cos 2t$

b $\dfrac{16x^2}{9} + 4y^2 = 1$

c $\dfrac{2}{3}\tan 2t$

d $y = -\dfrac{1}{2}\sec 2t$, $x = \dfrac{3}{4}\operatorname{cosec} 2t$

e $|\dfrac{3}{8}\operatorname{cosec} 4t|$, minimum area $= \dfrac{3}{8}$ when

$t = -\dfrac{3\pi}{8}, -\dfrac{\pi}{8}, \dfrac{\pi}{8}, \dfrac{3\pi}{8}, \ldots$

f $x = \dfrac{3}{4}\sin 2t$, $y = \dfrac{3}{4}\cos 2t$ (infinitely many answers of course)

g $\dfrac{3\pi}{16}$

h $\dfrac{5\pi}{16}$

17 a $y^2 = x\left(\dfrac{x}{3} - 1\right)^2$

b $\left(1, \dfrac{2}{3}\right), \left(1, -\dfrac{2}{3}\right)$

c $\dfrac{8\sqrt{3}}{5}$

d $\dfrac{3}{4}\pi$