Electrostatic Energy

8–1 The electrostatic energy of charges. A uniform sphere

In the study of mechanics, one of the most interesting and useful discoveries was the law of the conservation of energy. The expressions for the kinetic and potential energies of a mechanical system helped us to discover connections between the states of a system at two different times without having to look into the details of what was occurring in between. We wish now to consider the energy of electrostatic systems. In electricity also the principle of the conservation of energy will be useful for discovering a number of interesting things.

The law of the energy of interaction in electrostatics is very simple; we have, in fact, already discussed it. Suppose we have two charges q_1 and q_2 separated by the distance r_{12}. There is some energy in the system, because a certain amount of work was required to bring the charges together. We have already calculated the work done in bringing two charges together from a large distance. It is

$$\frac{q_1 q_2}{4\pi\epsilon_0 r_{12}} . \tag{8.1}$$

We also know, from the principle of superposition, that if we have many charges present, the total force on any charge is the sum of the forces from the others. It follows, therefore, that the total energy of a system of a number of charges is the sum of terms due to the mutual interaction of each pair of charges. If q_i and q_j are any two of the charges and r_{ij} is the distance between them (Fig. 8–1), the energy of that particular pair is

$$\frac{q_i q_j}{4\pi\epsilon_0 r_{ij}} . \tag{8.2}$$

The total electrostatic energy U is the sum of the energies of all possible pairs of charges:

$$U = \sum_{\text{all pairs}} \frac{q_i q_j}{4\pi\epsilon_0 r_{ij}} . \tag{8.3}$$

If we have a distribution of charge specified by a charge density ρ, the sum of Eq. (8.3) is, of course, to be replaced by an integral.

We shall concern ourselves with two aspects of this energy. One is the *application* of the concept of energy to electrostatic problems; the other is the *evaluation* of the energy in different ways. Sometimes it is easier to compute the work done for some special case than to evaluate the sum in Eq. (8.3), or the corresponding integral. As an example, let us calculate the energy required to assemble a sphere of charge with a uniform charge density. The energy is just the work done in gathering the charges together from infinity.

Imagine that we assemble the sphere by building up a succession of thin spherical layers of infinitesimal thickness. At each stage of the process, we gather a small amount of charge and put it in a thin layer from r to $r + dr$. We continue the process until we arrive at the final radius a (Fig. 8–2). If Q_r is the charge of the sphere when it has been built up to the radius r, the work done in bringing a charge dQ to it is

$$dU = \frac{Q_r \, dQ}{4\pi\epsilon_0 r} . \tag{8.4}$$

Review: Chapter 4, Vol. I, *Conservation of Energy*
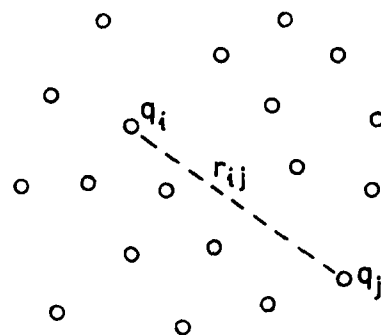Chapters 13 and 14, Vol. I, *Work and Potential Energy*

Fig. 8–1. The electrostatic energy of a system of particles is the sum of the electrostatic energy of each pair.
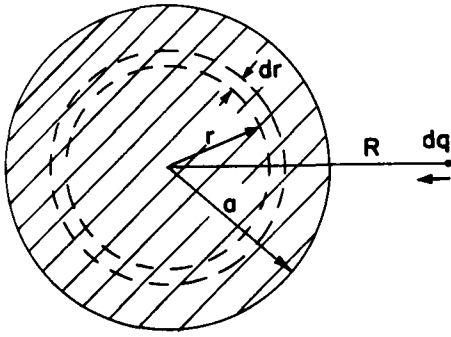
If the density of charge in the sphere is ρ, the charge Q_r is

$$Q_r = \rho \cdot \frac{4}{3}\pi r^3,$$

and the charge dQ is

$$dQ = \rho \cdot 4\pi r^2 \, dr.$$

Equation (8.4) becomes

$$dU = \frac{4\pi\rho^2 r^4 \, dr}{3\epsilon_0}. \tag{8.5}$$

The total energy required to assemble the sphere is the integral of dU from $r = 0$ to $r = a$, or

$$U = \frac{4\pi\rho^2 a^5}{15\epsilon_0}. \tag{8.6}$$

Or if we wish to express the result in terms of the total charge Q of the sphere,

$$U = \frac{3}{5}\frac{Q^2}{4\pi\epsilon_0 a}. \tag{8.7}$$

The energy is proportional to the square of the total charge and inversely proportional to the radius. We can also interpret Eq. (8.7) as saying that the average of $(1/r_{ij})$ for all pairs of points in the sphere is $3/5a$.

8–2 The energy of a condenser. Forces on charged conductors

We consider now the energy required to charge a condenser. If the charge Q has been taken from one of the conductors of a condenser and placed on the other, the potential difference between them is

$$V = \frac{Q}{C}, \tag{8.8}$$

where C is the capacity of the condenser. How much work is done in charging the condenser? Proceeding as for the sphere, we imagine that the condenser has been charged by transferring charge from one plate to the other in small increments dQ. The work required to transfer the charge dQ is

$$dU = V \, dQ.$$

Taking V from Eq. (8.8), we write

$$dU = \frac{Q \, dQ}{C}.$$

Or integrating from zero charge to the final charge Q, we have

$$U = \frac{1}{2}\frac{Q^2}{C}. \tag{8.9}$$

This energy can also be written as

$$U = \tfrac{1}{2}CV^2. \tag{8.10}$$

Recalling that the capacity of a conducting sphere (relative to infinity) is

$$C_{\text{sphere}} = 4\pi\epsilon_0 a,$$

we can immediately get from Eq. (8.9) the energy of a charged sphere,

$$U = \frac{1}{2}\frac{Q^2}{4\pi\epsilon_0 a}. \tag{8.11}$$

Fig. 8–2. The energy of a uniform sphere of charge can be computed by imagining that it is assembled from successive spherical shells.

This, of course, is also the energy of a thin *spherical shell* of total charge Q and is just 5/6 of the energy of a *uniformly charged* sphere, Eq. (8.7).

We now consider applications of the idea of electrostatic energy. Consider the following questions: What is the force between the plates of a condenser? Or what is the torque about some axis of a charged conductor in the presence of another with opposite charge? Such questions are easily answered by using our result Eq. (8.9) for electrostatic energy of a condenser, together with the principle of virtual work (Chapters 4, 13, and 14 of Vol. I).

Let's use this method for determining the force between the plates of a parallel-plate condenser. If we imagine that the spacing of the plates is increased by the small amount Δz, then the mechanical work done from the outside in moving the plates would be

$$\Delta W = F \Delta z, \tag{8.12}$$

where F is the force between the plates. This work must be equal to the change in the electrostatic energy of the condenser.

By Eq. (8.9), the energy of the condenser was originally

$$U = \frac{1}{2} \frac{Q^2}{C}.$$

The change in energy (if we do not let the charge change) is

$$\Delta U = \frac{1}{2} Q^2 \Delta\left(\frac{1}{C}\right). \tag{8.13}$$

Equating (8.12) and (8.13), we have

$$F \Delta z = \frac{Q^2}{2} \Delta\left(\frac{1}{C}\right). \tag{8.14}$$

This can also be written as

$$F \Delta z = -\frac{Q^2}{2C^2} \Delta C. \tag{8.15}$$

The force, of course, results from the attraction of the charges on the plates, but we see that we do not have to worry in detail about how they are distributed; everything we need is taken care of in the capacity C.

It is easy to see how the idea is extended to conductors of any shape, and for other components of the force. In Eq. (8.14), we replace F by the component we are looking for, and we replace Δz by a small displacement in the corresponding direction. Or if we have an electrode with a pivot and we want to know the torque τ, we write the virtual work as

$$\Delta W = \tau \Delta\theta,$$

where $\Delta\theta$ is a small angular displacement. Of course, $\Delta(1/C)$ must be the change in $1/C$ which corresponds to $\Delta\theta$. We could, in this way, find the torque on the movable plates in a variable condenser of the type shown in Fig. 8–3.

Returning to the special case of a parallel-plate condenser, we can use the formula we derived in Chapter 6 for the capacity:

$$\frac{1}{C} = \frac{d}{\epsilon_0 A}, \tag{8.16}$$

where A is the area of each plate. If we increase the separation by Δz,

$$\Delta\left(\frac{1}{C}\right) = \frac{\Delta z}{\epsilon_0 A}.$$

From Eq. (8.14) we get that the force between the plates is

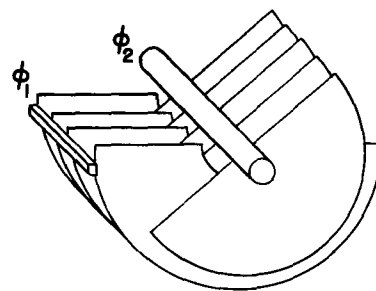$$F = \frac{Q^2}{2\epsilon_0 A}. \tag{8.17}$$

Fig. 8–3. What is the torque on a variable capacitor?

Let's look at Eq. (8.17) a little more closely and see if we can tell how the force arises. If for the charge on one plate we write

$$Q = \sigma A,$$

Eq. (8.17) can be rewritten as

$$F = \frac{1}{2} Q \frac{\sigma}{\epsilon_0}.$$

Or, since the electric field between the plates is

$$E_0 = \frac{\sigma}{\epsilon_0},$$

then

$$F = \tfrac{1}{2} Q E_0. \tag{8.18}$$

One would immediately guess that the force acting on one plate is the charge Q on the plate times the field acting on the charge. But we have a surprising factor of one-half. The reason is that E_0 is not the field *at* the charges. If we imagine that the charge at the surface of the plate occupies a thin layer, as indicated in Fig. 8-4, the field will vary from zero at the inner boundary of the layer to E_0 in the space outside of the plate. The average field acting on the surface charges is $E_0/2$. That is why the factor one-half is in Eq. (8.18).

You should notice that in computing the virtual work we have assumed that the charge on the condenser was constant—that it was not electrically connected to other objects, and so the total charge could not change.

Suppose we had imagined that the condenser was held at a constant potential difference as we made the virtual displacement. Then we should have taken

$$U = \tfrac{1}{2} C V^2$$

and in place of Eq. (8.15) we would have had

$$F \Delta z = \tfrac{1}{2} V^2 \, \Delta C,$$

which gives a force equal in magnitude to the one in Eq. (8.15) (because $V = Q/C$), but with the opposite sign! Surely the force between the condenser plates doesn't reverse in sign as we disconnect it from its charging source. Also, we know that two plates with opposite electrical charges must attract. The principle of virtual work has been incorrectly applied in the second case—we have not taken into account the virtual work done on the charging source. That is, to keep the potential constant at V as the capacity changes, a charge $V \Delta C$ must be supplied by a source of charge. But this charge is supplied at a potential V, so the work done by the electrical system which keeps the potential constant is $V^2 \, \Delta C$. The mechanical work $F \Delta z$ *plus* this electrical work $V^2 \, \Delta C$ together make up the change in the total energy $\tfrac{1}{2} V^2 \, \Delta C$ of the condenser. Therefore $F \Delta z$ is $-\tfrac{1}{2} V^2 \, \Delta C$, as before.

Fig. 8-4. The field at the surface of a conductor varies from zero to $E_0 = \sigma/\epsilon_0$, as one passes through the layer of surface charge.

8-3 The electrostatic energy of an ionic crystal

We now consider an application of the concept of electrostatic energy in atomic physics. We cannot easily measure the forces between atoms, but we are often interested in the energy differences between one atomic arrangement and another, as, for example, the energy of a chemical change. Since atomic forces are basically electrical, chemical energies are in large part just electrostatic energies.

Let's consider, for example, the electrostatic energy of an ionic lattice. An ionic crystal like NaCl consists of positive and negative ions which can be thought of as rigid spheres. They attract electrically until they begin to touch; then there is a repulsive force which goes up very rapidly if we try to push them closer together.

For our first approximation, therefore, we imagine a set of rigid spheres that represent the atoms in a salt crystal. The structure of the lattice has been determined by x-ray diffraction. It is a cubic lattice—like a three-dimensional
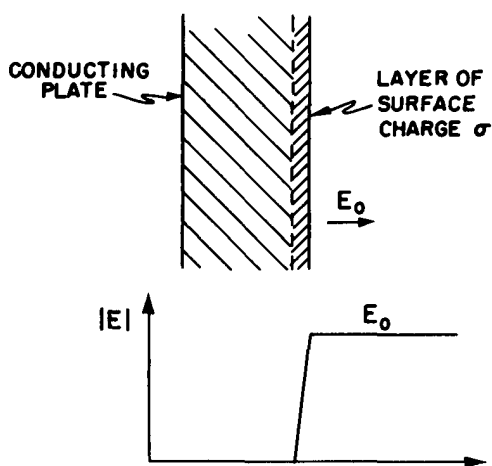
8-4

checkerboard. Figure 8–5 shows a cross-sectional view. The spacing of the ions is 2.81 A ($= 2.81 \times 10^{-8}$ cm).

If our picture of this system is correct, we should be able to check it by asking the following question: How much energy will it take to pull all these ions apart—that is, to separate the crystal completely into ions? This energy should be equal to the heat of vaporization of NaCl plus the energy required to dissociate the molecules into ions. This total energy to separate NaCl to ions is determined experimentally to be 7.92 electron volts per molecule. Using the conversion

$$1 \text{ ev} = 1.602 \times 10^{-19} \text{ joule},$$

and Avogadro's number for the number of molecules in a mole,

$$N_0 = 6.02 \times 10^{23},$$

the energy of vaporization can also be given as

$$W = 7.64 \times 10^5 \text{ joules/mole}.$$

Physical chemists prefer for an energy unit the kilocalorie, which is 4190 joules; so that 1 ev per molecule is 23 kilocalories per mole. A chemist would then say that the dissociation energy of NaCl is

$$W = 183 \text{ kcal/mole}.$$

Can we obtain this chemical energy theoretically by computing how much work it would take to pull apart the crystal? According to our theory, this work is the sum of the potential energies of all the pairs of ions. The easiest way to figure out this sum is to pick out a particular ion and compute its potential energy with each of the other ions. That will give us *twice* the energy per ion, because the energy belongs to the *pairs* of charges. If we want the energy to be associated with one particular ion, we should take half the sum. But we really want the energy *per molecule*, which contains two ions, so that the sum we compute will give directly the energy per molecule.

The energy of an ion with one of its nearest neighbors is e^2/a, where $e^2 = q_e^2/4\pi\epsilon_0$ and a is the center-to-center spacing between ions. (We are considering monovalent ions.) This energy is 5.12 ev, which we already see is going to give us a result of the correct order of magnitude. But it is still a long way from the infinite sum of terms we need.

Let's begin by summing all the terms from the ions along a straight line. Considering that the ion marked Na in Fig. 8–5 is our special ion, we shall consider first those ions on a horizontal line with it. There are two nearest Cl ions with negative charges, each at the distance a. Then there are two positive ions at the distance $2a$, etc. Calling the energy of this sum U_1, we write

$$U_1 = \frac{e^2}{a}\left(-\frac{2}{1} + \frac{2}{2} - \frac{2}{3} + \frac{2}{4} + \cdots\right)$$
$$= -\frac{2e^2}{a}\left(1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots\right). \tag{8.19}$$

The series converges slowly, so it is difficult to evaluate numerically, but it is known to be equal to ln 2. So

$$U_1 = -\frac{2e^2}{a}\ln 2 = -1.386\frac{e^2}{a}. \tag{8.20}$$

Now consider the next adjacent line of ions above. The nearest is negative and at the distance a. Then there are two positives at the distance $\sqrt{2}\,a$. The next pair are at the distance $\sqrt{5}\,a$, the next at $\sqrt{10}\,a$, and so on. So for the whole line we get the series

$$\frac{e^2}{a}\left(-\frac{1}{1} + \frac{2}{\sqrt{2}} - \frac{2}{\sqrt{5}} + \frac{2}{\sqrt{10}} \cdots\right). \tag{8.21}$$
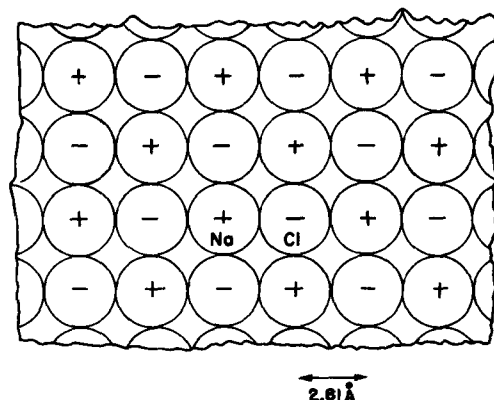

Fig. 8–5. Cross section of a salt crystal on an atomic scale. The checkerboard arrangement of Na and Cl ions is the same in the two cross sections perpendicular to the one shown. (See Vol. I, Fig. 1–7.)

There are *four* such lines: above, below, in front, and in back. Then there are the four lines which are the nearest lines on diagonals, and on and on.

If you work patiently through for all the lines, and then take the sum, you find that the grand total is

$$U = 1.747 \frac{e^2}{a},$$

which is just somewhat more than what we obtained in (8.20) for the first line. Using $e^2/a = 5.12$ ev, we get

$$U = 8.94 \text{ ev}.$$

Our answer is about 10% above the experimentally observed energy. It shows that our idea that the whole lattice is héld together by electrical Coulomb forces is fundamentally correct. This is the first time that we have obtained a specific property of a macroscopic substance from a knowledge of atomic physics. We will do much more later. The subject that tries to understand the behavior of bulk matter in terms of the laws of atomic behavior is called *solid-state physics*.

Now what about the error in our calculation? Why is it not exactly right? It is because of the repulsion between the ions at close distances. They are not perfectly rigid spheres, so when they are close together they are partly squashed. They are not very soft, so they squash only a little bit. Some energy, however, is used in deforming them, and when the ions are pulled apart this energy is released. The actual energy needed to pull the ions apart is a little less than the energy that we calculated; the repulsion helps in overcoming the electrostatic attraction.

Is there any way we can make an allowance for this contribution? We could if we knew the law of the repulsive force. We are not ready to analyze the details of this repulsive mechanism, but we can get some idea of its characteristics from some large-scale measurements. From a measurement of the *compressibility* of the whole crystal, it is possible to obtain a quantitative idea of the law of repulsion between the ions and therefore of its contribution to the energy. In this way it has been found that this contribution must be 1/9.4 of the contribution from the electrostatic attraction and, of course, of opposite sign. If we subtract this contribution from the pure electrostatic energy, we obtain 7.99 ev for the dissociation energy per molecule. It is much closer to the observed result of 7.92 ev, but still not in perfect agreement. There is one more thing we haven't taken into account: we have made no allowance for the kinetic energy of the crystal vibrations. If a correction is made for this effect, very good agreement with the experimental number is obtained. The ideas are then correct; the major contribution to the energy of a crystal like NaCl is electrostatic.

8–4 Electrostatic energy in nuclei

We will now take up another example of electrostatic energy in atomic physics, the electrical energy of atomic nuclei. Before we do this we will have to discuss some .properties of the main forces (called nuclear forces) that hold the protons and neutrons together in a nucleus. In the early days of the discovery of nuclei—and of the neutrons and protons that make them up—it was hoped that the law of the strong, nonelectrical part of the force between, say, a proton and another proton would have some simple law, like the inverse square law of electricity. For once one had determined this law of force, and the corresponding ones between a proton and a neutron, and a neutron and a neutron, it would be possible to describe theoretically the complete behavior of these particles in nuclei. Therefore a big program was started for the study of the scattering of protons, in the hope of finding the law of force between them; but after thirty years of effort, nothing simple has emerged. A considerable knowledge of the force between proton and proton has been accumulated, but we find that the force is as complicated as it can possibly be.

What we mean by "as complicated as it can be" is that the force depends on as many things as it possibly can.

First, the force is not a simple function of the distance between the two protons. At large distances there is an attraction, but at closer distances there is a repulsion. The distance dependence is a complicated function, still imperfectly known.

Second, the force depends on the orientation of the protons' spin. The protons have a spin, and any two interacting protons may be spinning with their angular momenta in the same direction or in opposite directions. And the force is different when the spins are parallel from what it is when they are antiparallel, as in (a) and (b) of Fig. 8–6. The difference is quite large; it is not a small effect.

Third, the force is considerably different when the separation of the two protons is in the direction *parallel* to their spins, as in (c) and (d) of Fig. 8–6, than it is when the separation is in a direction *perpendicular* to the spins, as in (a) and (b).

Fourth, the force depends, as it does in magnetism, on the velocity of the protons, only much more strongly than in magnetism. And this velocity-dependent force is not a relativistic effect; it is strong even at speeds much less than the speed of light. Furthermore, this part of the force depends on other things besides the magnitude of the velocity. For instance, when a proton is moving near another proton, the force is different when the orbital motion has the same direction of rotation as the spin, as in (e) of Fig. 8–6, than when it has the opposite direction of rotation, as in (f). This is called the "spin orbit" part of the force.

The force between a proton and a neutron and between a neutron and a neutron are also equally complicated. To this day we do not know the machinery behind these forces—that is to say, any simple way of understanding them.

There is, however, one important way in which the nucleon forces are *simpler* than they could be. That is that the *nuclear* force between two neutrons is the same as the force between a proton and a neutron, which is the same as the force between two protons! If, in any nuclear situation, we replace a proton by a neutron (or vice versa), the *nuclear interactions* are not changed. The "fundamental reason" for this equality is not known, but it is an example of an important principle that can be extended also to the interaction laws of other strongly interacting particles— such as the π-mesons and the "strange" particles.

This fact is nicely illustrated by the locations of the energy levels in similar nuclei. Consider a nucleus like B^{11} (boron-eleven), which is composed of five protons and six neutrons. In the nucleus the eleven particles interact with one another in a most complicated dance. Now, there is one configuration of all the possible interactions which has the lowest possible energy; this is the normal state of the nucleus, and is called the *ground state*. If the nucleus is disturbed (for example, by being struck by a high-energy proton or other particle) it can be put into any number of other configurations, called *excited states*, each of which will have a characteristic energy that is higher than that of the ground state. In nuclear physics research, such as is carried on with Van de Graaff generator (for example, in Caltech's Kellogg and Sloan Laboratories), the energies and other properties of these excited states are determined by experiment. The energies of the fifteen lowest known excited states of B^{11} are shown in a one-dimensional graph on the left half of Fig. 8–7. The lowest horizontal line represents the ground state. The first excited state has an energy 2.14 Mev higher than the ground state, the next an energy 4.46 Mev higher than the ground state, and so on. The study of nuclear physics attempts to find an explanation for this rather complicated pattern of energies; there is as yet, however, no complete general theory of such nuclear energy levels.

If we replace one of the neutrons in B^{11} with a proton, we have the nucleus of an isotope of carbon, C^{11}. The energies of the lowest sixteen excited states of C^{11} have also been measured; they are shown in the right half of Fig. 8–7. (The broken lines indicate levels for which the experimental information is questionable.)

Looking at Fig. 8–7, we see a striking similarity between the pattern of the energy levels in the two nuclei. The first excited states are about 2 Mev above the ground states. There is a large gap of about 2.3 Mev to the second excited state, then a small jump of only 0.5 Mev to the third level. Again, between the fourth and fifth levels, a big jump; but between the fifth and sixth a tiny separation of the
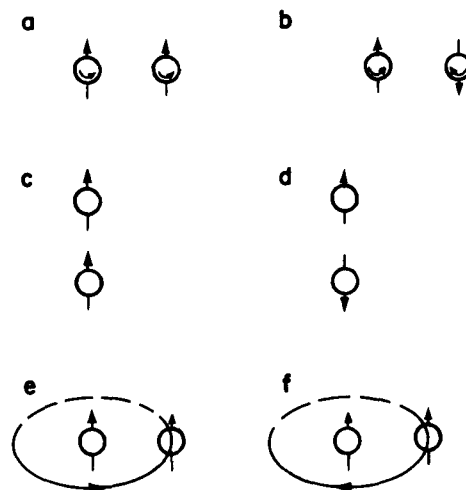
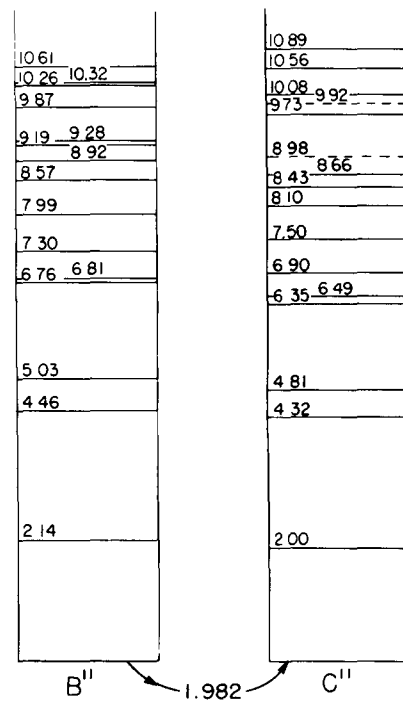Fig. 8–6. The force between two protons depends on every possible parameter.

Fig. 8–7. The energy levels of B^{11} and C^{11} (energies in Mev). The ground state of C^{11} is 1.982 Mev higher than that of B^{11}.

order of 0.1 Mev. And so on. After about the tenth level, the correspondence seems to become lost, but can still be seen if the levels are labeled with their other defining characteristics—for instance, their angular momentum and what they do to lose their extra energy.

The striking similarity of the pattern of the energy levels of B^{11} and C^{11} is surely not just a coincidence. It must reveal some physical law. It shows, in fact, that even in the complicated situation in a nucleus, replacing a neutron by a proton makes very little change. This can mean only that the neutron-neutron and proton-proton forces must be nearly identical. Only then would we expect the nuclear configurations with five protons and six neutrons to be the same as with six protons and five neutrons.

Notice that the properties of these two nuclei tell us nothing about the neutron-proton force; there are the same number of neutron-proton combinations in both nuclei. But if we compare two other nuclei, such as C^{14}, which has six protons and eight neutrons, with N^{14}, which has seven of each, we find a similar correspondence of energy levels. So we can conclude that the p-p, n-n, and p-n forces are identical in all their complexities. There is an unexpected principle in the laws of nuclear forces. Even though the force between each pair of nuclear particles is very complicated, the force between the three possible different pairs is the same.

But there are some small differences. The levels do not correspond exactly; also, the ground state of C^{11} has an absolute energy (its mass) which is higher than the ground state of B^{11} by 1.982 Mev. All the other levels are also higher in absolute energy by this same amount. So the forces are not exactly equal. But we know very well that the *complete* forces are not exactly equal; there is an *electrical* force between two protons because each has a positive charge, while between two neutrons there is no such electrical force. Can we perhaps explain the differences between B^{11} and C^{11} by the fact that the electrical interaction of the protons is different in the two cases? Perhaps even the remaining minor differences in the levels are caused by electrical effects? Since the nuclear forces are so much stronger than the electrical force, electrical effects would have only a small perturbing effect on the energies of the levels.

In order to check this idea, or rather to find out what the consequences of this idea are, we first consider the difference in the ground-state energies of the two nuclei. To take a very simple model, we suppose that the nuclei are spheres of radius r (to be determined), containing Z protons. If we consider that a nucleus is like a sphere with uniform charge density, we would expect the electrostatic energy (from Eq. 8.7) to be

$$U = \frac{3}{5} \frac{(Zq_e)^2}{4\pi\epsilon_0 r},$$ (8.22)

where q_e is the elementary charge of the proton. Since Z is five for B^{11} and six for C^{11}, their electrostatic energies would be different.

With such a small number of protons, however, Eq. (8.22) is not quite correct. If we compute the electrical energy between all pairs of protons, considered as points which we assume to be nearly uniformly distributed throughout the sphere, we find that in Eq. (8.22) the quantity Z^2 should be replaced by $Z(Z-1)$, so the energy is

$$U = \frac{3}{5} \frac{Z(Z-1)q_e^2}{4\pi\epsilon_0 r} = \frac{3}{5} \frac{Z(Z-1)e^2}{r}.$$ (8.23)

If we knew the nuclear radius r, we could use (8.23) to find the electrostatic energy difference between B^{11} and C^{11}. But let's do the opposite; let's instead use the observed energy difference to compute the radius, assuming that the energy difference is all electrostatic in origin.

That is, however, not quite right. The energy difference of 1.982 Mev between the ground states of B^{11} and C^{11} includes the rest energies—that is, the energy mc^2—of all the particles. In going from B^{11} to C^{11}, we replace a neutron by a proton, which has less mass. So part of the energy difference is the difference in the rest energies of a neutron and a proton, which is 0.784 Mev. The difference,

to be accounted for by electrostatic energy, is thus more than 1.982 Mev; it is

$$1.982 + 0.784 = 2.786 \text{ Mev}.$$

Using this energy in Eq. (8.23), for the radius of either B^{11} or C^{11} we find

$$r = 3.12 \times 10^{-13} \text{ cm}. \tag{8.24}$$

Does this number have any meaning? To see whether it does, we should compare it with some other determination of the radius of these nuclei. For example, we can make another measurement of the radius of a nucleus by seeing how it scatters fast particles. From such measurements it has been found, in fact, that the *density* of matter in all nuclei is nearly the same, i.e., their volumes are proportional to the number of particles they contain. If we let A be the number of protons and neutrons in a nucleus (a number very nearly proportional to its mass), it is found that its radius is given by

$$r = A^{1/3} r_0, \tag{8.25}$$

where

$$r_0 = 1.2 \times 10^{-13} \text{ cm}. \tag{8.26}$$

From these measurements we find that the radius of a B^{11} (or a C^{11}) nucleus is expected to be

$$r = (1.2 \times 10^{-13})(11)^{1/3} = 2.7 \times 10^{-13} \text{ cm}.$$

Comparing this result with (8.24), we see that our assumptions that the energy difference between B^{11} and C^{11} is electrostatic is fairly good; the discrepancy is only about 15% (not bad for our first nuclear computation!).

The reason for the discrepancy is probably the following. According to the current understanding of nuclei, an even number of nuclear particles—in the case of B^{11}, five neutrons together with five protons—makes a kind of *core;* when one more particle is added to this core, it revolves around on the outside to make a new spherical nucleus, rather than being absorbed. If this is so, we should have taken a different electrostatic energy for the additional proton. We should have taken the excess energy of C^{11} over B^{11} to be just

$$\frac{Z_B q_e^2}{4\pi\epsilon_0 a},$$

which is the energy needed to add one more proton to the outside of the core. This number is just 5/6 of what Eq. (8.23) predicts, so the new prediction for the radius is 5/6 of (8.24), which is in much closer agreement with what is directly measured.

We can draw two conclusions from this agreement. One is that the electrical laws appear to be working at dimensions as small as 10^{-13} cm. The other is that we have verified the remarkable coincidence that the nonelectrical part of the forces between proton and proton, neutron and neutron, and proton and neutron are all equal.

8–5 Energy in the electrostatic field

We now consider other methods of calculating electrostatic energy. They can all be derived from the basic relation Eq. (8.3), the sum, over all pairs of charges, of the mutual energies of each charge-pair. First we wish to write an expression for the energy of a charge distribution. As usual, we consider that each volume element dV contains the element of charge $\rho \, dV$. Then Eq. (8.3) should be written

$$U = \frac{1}{2} \int_{\substack{\text{all} \\ \text{space}}} \frac{\rho(1)\rho(2)}{4\pi\epsilon_0 r_{12}} \, dV_1 \, dV_2. \tag{8.27}$$

Notice the factor $\frac{1}{2}$, which is introduced because in the double integral over dV_1 and dV_2 we have counted all pairs of charge elements twice. (There is no convenient way of writing an integral that keeps track of the pairs so that each pair is counted only once.) Next we notice that the integral over dV_2 in (8.27) is just the potential at (1). That is,

$$\int \frac{\rho(2)}{4\pi\epsilon_0 r_{12}} \, dV_2 = \phi(1),$$

so that (8.27) can be written as

$$U = \frac{1}{2} \int \rho(1)\phi(1) \, dV_1.$$

Or, since the point (2) no longer appears, we can simply write

$$U = \frac{1}{2} \int \rho\phi \, dV. \qquad (8.28)$$

This equation can be interpreted as follows. The potential energy of the charge $\rho \, dV$ is the product of this charge and the potential at the same point. The total energy is therefore the integral over $\phi\rho \, dV$. But there is again the factor $\frac{1}{2}$. It is still required because we are counting energies twice. The mutual energies of two charges is the charge of one times the potential at it due to the other. *Or*, it can be taken as the second charge times the potential at it from the first. Thus for two point charges we would write

$$U = q_1\phi(1) = q_1 \frac{q_2}{4\pi\epsilon_0 r_{12}}$$

or

$$U = q_2\phi(2) = q_2 \frac{q_1}{4\pi\epsilon_0 r_{12}}.$$

Notice that we could also write

$$U = \frac{1}{2}[q_1\phi(1) + q_2\phi(2)]. \qquad (8.29)$$

The integral in (8.28) corresponds to the sum of both terms in the brackets of (8.29). That is why we need the factor $\frac{1}{2}$.

An interesting question is: Where is the electrostatic energy located? One might also ask: Who cares? What is the meaning of such a question? If there is a pair of interacting charges, the combination has a certain energy. Do we need to say that the energy is located at one of the charges or the other, or at both, or in between? These questions may not make sense because we really know only that the total energy is conserved. The idea that the energy is located *somewhere* is not necessary.

Yet suppose that it *did* make sense to say, in general, that energy is located at a certain place, as it does for heat energy. We might then *extend* our principle of the conservation of energy with the idea that if the energy in a given volume changes, we should be able to account for the change by the flow of energy into or out of that volume. You realize that our early statement of the principle of the conservation of energy is still perfectly all right if some energy disappears at one place and appears somewhere else far away without anything passing (that is, without any special phenomena occurring) in the space between. We are, therefore, now discussing an extension of the idea of the conservation of energy. We might call it a principle of the *local* conservation of energy. Such a principle would say that the energy in any given volume changes only by the amount that flows into or out of the volume. It is indeed possible that energy is conserved locally in such a way. If it is, we would have a much more detailed law than the simple statement of the conservation of total energy. It does turn out that in nature *energy is conserved locally*. We can find formulas for where the energy is located and how it travels from place to place.

There is also a *physical* reason why it is imperative that we be able to say where energy is located. According to the theory of gravitation, all mass is a source

of gravitational attraction. We also know, by $E = mc^2$, that mass and energy are equivalent. All energy is, therefore, a source of gravitational force. If we could not locate the energy, we could not locate all the mass. We would not be able to say where the sources of the gravitational field are located. The theory of gravitation would be incomplete.

If we restrict ourselves to electrostatics there is really no way to tell where the energy is located. The complete Maxwell equations of electrodynamics give us much more information (although even then the answer is, strictly speaking, not unique.) We will therefore discuss this question in detail again in a later chapter. We will give you now only the result for the particular case of electrostatics. The energy is located in space, where the electric field is. This seems reasonable because we know that when charges are accelerated they radiate electric fields. We would like to say that when light or radiowaves travel from one point to another, they carry their energy with them. But there are no charges in the waves. So we would like to locate the energy where the electromagnetic field is and not at the charges from which it came. We thus describe the energy, not in terms of the charges, but in terms of the fields they produce. We can, in fact, show that Eq. (8.28) is *numerically* equal to

$$U = \frac{\epsilon_0}{2} \int \mathbf{E} \cdot \mathbf{E} \, dV. \tag{8.30}$$

We can then interpret this formula as saying that when an electric field is present, there is located in space an energy whose *density* (energy per unit volume) is

$$u = \frac{\epsilon_0}{2} \mathbf{E} \cdot \mathbf{E} = \frac{\epsilon_0 E^2}{2}. \tag{8.31}$$

This idea is illustrated in Fig. 8-8.

To show that Eq. (8.30) is consistent with our laws of electrostatics, we begin by introducing into Eq. (8.28) the relation between ρ and ϕ that we obtained in Chapter 6:

$$\rho = -\epsilon_0 \nabla^2 \phi.$$

We get

$$U = -\frac{\epsilon_0}{2} \int \phi \, \nabla^2 \phi \, dV. \tag{8.32}$$

Writing out the components of the integrand, we see that

$$
\begin{aligned}
\phi \nabla^2 \phi &= \phi \left(\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2} \right) \\
&= \frac{\partial}{\partial x} \left(\phi \frac{\partial \phi}{\partial x} \right) - \left(\frac{\partial \phi}{\partial x} \right)^2 + \frac{\partial}{\partial y} \left(\phi \frac{\partial \phi}{\partial y} \right) - \left(\frac{\partial \phi}{\partial y} \right)^2 + \frac{\partial}{\partial z} \left(\phi \frac{\partial \phi}{\partial z} \right) - \left(\frac{\partial \phi}{\partial z} \right)^2 \\
&= \nabla \cdot (\phi \nabla \phi) - (\nabla \phi) \cdot (\nabla \phi). \tag{8.33}
\end{aligned}
$$

Fig. 8–8. Each volume element $dV = dx \, dy \, dz$ in an electric field contains the energy $(\epsilon_0/2)E^2 \, dV$.

Our energy integral is then

$$U = \frac{\epsilon_0}{2} \int (\nabla \phi) \cdot (\nabla \phi) \, dV - \frac{\epsilon_0}{2} \int \nabla \cdot (\phi \nabla \phi) \, dV.$$

We can use Gauss' theorem to change the second integral into a surface integral:

$$\int_{\text{vol.}} \nabla \cdot (\phi \nabla \phi) \, dV = \int_{\text{surface}} (\phi \nabla \phi) \cdot \mathbf{n} \, da. \tag{8.34}$$

We evaluate the surface integral in the case that the surface goes to infinity (so the volume integrals become integrals over all space), supposing that all the charges are located within some finite distance. The simple way to proceed is to take a spherical surface of enormous radius R whose center is at the origin of coordinates. We know that when we are very far away from all charges, ϕ varies as $1/R$ and $\nabla \phi$ as $1/R^2$. (Both will decrease even faster with R if there the net
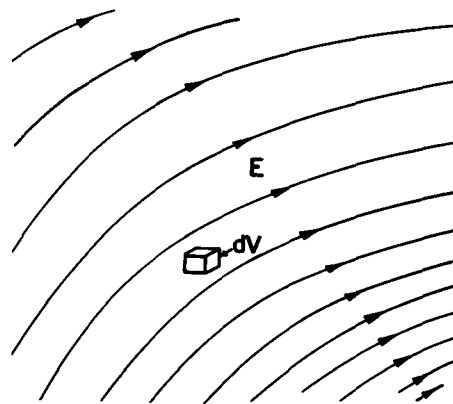
charge in the distribution is zero.) Since the surface area of the large sphere increases as R^2, we see that the surface integral falls off as $(1/R)(1/R^2)R^2 = (1/R)$ as the radius of the sphere increases. So if we include all space in our integration $(R \to \infty)$, the surface integral goes to zero and we have that

$$U = \frac{\epsilon_0}{2} \int\limits_{\substack{\text{all} \\ \text{space}}} (\nabla\phi) \cdot (\nabla\phi)\, dV = \frac{\epsilon_0}{2} \int\limits_{\substack{\text{all} \\ \text{space}}} E \cdot E\, dV. \tag{8.35}$$

We see that it is possible for us to represent the energy of any charge distribution as being the integral over an energy density located in the field.

8–6 The energy of a point charge

Our new relation, Eq. (8.35), says that even a single point charge q will have some electrostatic energy. In this case, the electric field is given by

$$E = \frac{q}{4\pi\epsilon_0 r^2}.$$

So the energy density at the distance r from the charge is

$$\frac{\epsilon_0 E^2}{2} = \frac{q^2}{32\pi^2\epsilon_0 r^4}.$$

We can take for an element of volume a spherical shell of thickness dr and area $4\pi r^2$. The total energy is

$$U = \int\limits_{r=0}^{\infty} \frac{q^2}{8\pi\epsilon_0 r^2}\, dr = -\frac{q^2}{8\pi\epsilon_0} \frac{1}{r} \bigg|_{r=0}^{r=\infty}. \tag{8.36}$$

Now the limit at $r = \infty$ gives no difficulty. But for a point charge we are supposed to integrate down to $r = 0$, which gives an infinite integral. Equation (8.35) says that there is an infinite amount of energy in the field of a point charge, although we began with the idea that there was energy only *between* point charges. In our original energy formula for a collection of point charges (Eq. 8.3), we did not include any interaction energy of a charge with itself. What has happened is that when we went over to a continuous distribution of charge in Eq. (8.27), we counted the energy of interaction of every *infinitesimal* charge with all other infinitesimal charges. The same account is included in Eq. (8.35), so when we apply it to a *finite* point charge, we are including the energy it would take to assemble that charge from infinitesimal parts. You will notice, in fact, that we would also get the result in Eq. (8.36) if we used our expression (8.11) for the energy of a charged sphere and let the radius tend toward zero.

We must conclude that the idea of locating the energy in the field is inconsistent with the assumption of the existence of point charges. One way out of the difficulty would be to say that elementary charges, such as an electron, are not points but are really small distributions of charge. Alternatively, we could say that there is something wrong in our theory of electricity at very small distances, or with the idea of the local conservation of energy. There are difficulties with either point of view. These difficulties have never been overcome; they exist to this day. Sometime later, when we have discussed some additional ideas, such as the momentum in an electromagnetic field, we will give a more complete account of these fundamental difficulties in our understanding of nature.