18

Angular Momentum

18–1 Electric dipole radiation

In the last chapter we developed the idea of the conservation of angular momentum in quantum mechanics, and showed how it might be used to predict the angular distribution of the proton from the disintegration of the Λ-particle. We want now to give you a number of other, similar, illustrations of the consequences of momentum conservation in atomic systems. Our first example is the radiation of light from an atom. The conservation of angular momentum (among other things) will determine the polarization and angular distribution of the emitted photons.

Suppose we have an atom which is in an excited state of definite angular momentum—say with a spin of one—and it makes a transition to a state of angular momentum zero at a lower energy, emitting a photon. The problem is to figure out the angular distribution and polarization of the photons. (This problem is almost exactly the same as the Λ^0 disintegration, except that we have spin-one instead of spin one-half particles.) Since the upper state of the atom is spin one, there are three possibilities for its z-component of angular momentum. The value of m could be $+1$, or 0, or -1. We will take $m = +1$ for our example. Once you see how it goes, you can work out the other cases. We suppose that the atom is sitting with its angular momentum along the $+z$-axis—as in Fig. 18–1(a)—and ask with what amplitude it will emit right circularly polarized light upward along the z-axis, so that the atom ends up with zero angular momentum—as shown in part (b) of the figure. Well, we don't know the answer to that. But we do know that right circularly polarized light has one unit of angular momentum about its direction of propagation. So after the photon is emitted, the situation would have to be as shown in Fig. 18–1(b)—the atom is left with zero angular momentum

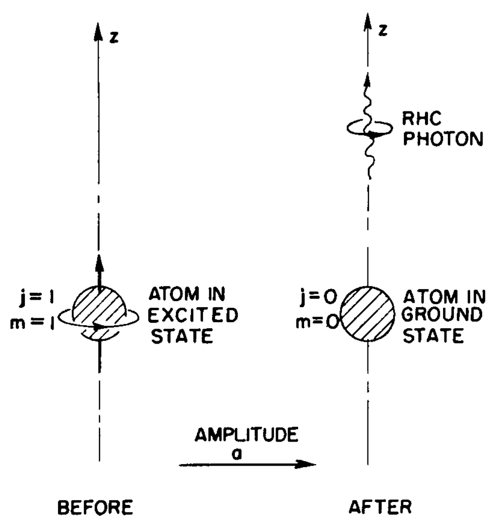

Fig. 18–1. An atom with $m = +1$ emits a RHC photon along the $+z$-axis.

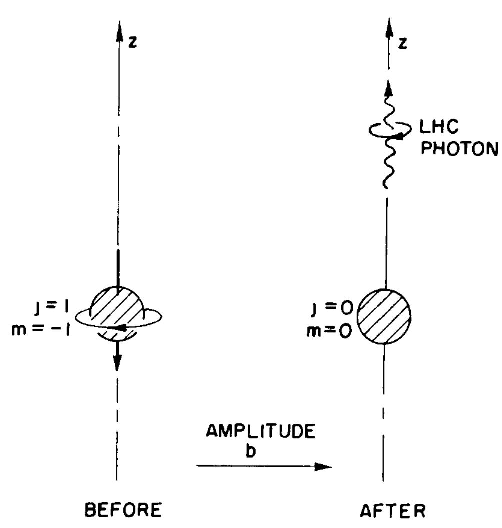

Fig. 18–2. An atom with $m = -1$ emits a LHC photon along the $+z$-axis.

about the z-axis, since we have assumed an atom whose lower state is spin zero. We will let a stand for the amplitude for such an event. More precisely, we let a be the amplitude to emit a photon into a certain small solid angle $\Delta\Omega$, centered on the z-axis, during a time dt. Notice that the amplitude to emit a LHC photon in the same direction is zero. The net angular momentum about the z-axis would be -1 for such a photon and zero for the atom for a total of -1, which would not conserve angular momentum.

Similarly, if the spin of the atom is initially "down" (-1 along the z-axis), it can emit only a LHC polarized photon in the direction of the $+z$-axis, as shown in Fig. 18–2. We will let b stand for the amplitude for this event—meaning again the amplitude that the photon goes into a certain solid angle $\Delta\Omega$. On the other hand, if the atom is in the $m = 0$ state, it cannot emit a photon in the $+z$-direction at all, because a photon can have only the angular momentum $+1$ or -1 along its direction of motion.

(a)

(b)

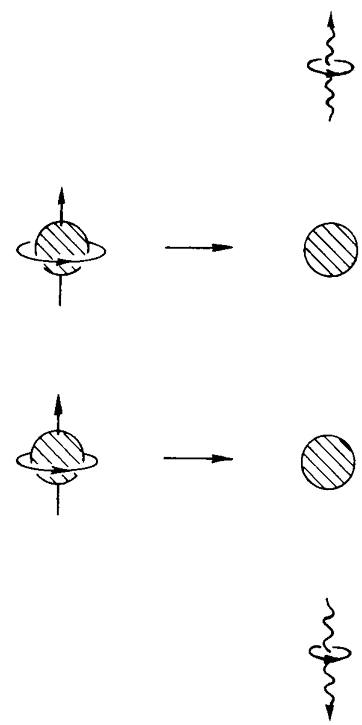

Fig. 18–3. If the process of (a) is transformed by an inversion through the center of the atom, it appears as in (b).

Next, we can show that b is related to a. Suppose we perform an inversion of the situation in Fig. 18–1, which means that we should imagine what the system would look like if we were to move each part of the system to an equivalent point on the opposite side of the origin. This does *not* mean that we should reflect the angular momentum vectors, because they are artificial. We should, rather, invert the actual character of the motion that would correspond to such an angular momentum. In Fig. 18–3(a) and (b) we show what the process of Fig. 18–1 looks like before and after an inversion with respect to the center of the atom. Notice that the sense of rotation of the atom is unchanged.† In the inverted system of Fig. 18–3(b) we have an atom with $m = +1$ emitting a LHC photon downward.

If we now rotate the system of Fig. 18–3(b) by 180° about the x- or y-axis, it becomes identical to Fig. 18–2. The combination of the inversion and rotation turns the second process into the first. Using Table 17–2, we see that a rotation of 180° about the y-axis just throws an $m = -1$ state into an $m = +1$ state, so the amplitude b must be equal to the amplitude *a except for a possible sign change due to the inversion.* The sign change in the inversion will depend on the parities of the initial and final state of the atom.

In atomic processes, parity is conserved, so the parity of the whole system must be the same before and after the photon emission. What happens will depend on whether the parities of the initial and final states of the atom are even or odd—the angular distribution of the radiation will be different for different cases. We will take the common case of *odd* parity for the initial state and *even* parity for the final state; it will give what is called "electric dipole radiation." (If the initial and final states have the same parity we say there is "magnetic dipole radiation," which has the character of the radiation from an oscillating current in a loop.) If the parity of the initial state is odd, its amplitude reverses its sign in the inversion which takes the system from (a) to (b) of Fig. 18–3. The final state of the atom has even parity, so its amplitude doesn't change sign. If the reaction is going to conserve parity, the amplitude b must be equal to a in magnitude but of the opposite sign.

We conclude that if the amplitude is a that an $m = +1$ state will emit a photon upward, then for the assumed parities of the initial and final states the amplitude that an $m = -1$ state will emit a LHC photon upward is $-a$.‡

We have all we need to know to find the amplitude for a photon to be emitted at any angle θ with respect to the z-axis. Suppose we have an atom originally polarized with $m = +1$. We can resolve this state into $+1$, 0, and -1 states with respect to a new z'-axis in the direction of the photon emission. The amplitudes for these three states are just the ones given in the lower half of Table 17–2.

† When we change x, y, z into $-x$, $-y$, $-z$, you might think that all vectors get reversed. That is true for *polar* vectors like displacements and velocities, but *not* for an *axial* vector like angular momentum—or any vector which is derived from a cross product of two polar vectors. Axial vectors have the same components after an inversion.

‡ Some of you may object to the argument we have just made, on the basis that the final states we have been considering do not have a definite parity. You will find in Added Note 2 at the end of this chapter another demonstration, which you may prefer.

The amplitude that a RHC photon is emitted in the direction θ is then a times the amplitude to have $m = +1$ in that direction, namely,

$$a\langle + \mid R_y(\theta) \mid + \rangle = \frac{a}{2}(1 + \cos\theta). \tag{18.1}$$

The amplitude that a LHC photon is emitted in the same direction is $-a$ times the amplitude to have $m = -1$ in the new direction. Using Table 17–2, it is

$$-a\langle - \mid R_y(\theta) \mid + \rangle = \frac{a}{-2}(1 - \cos\theta). \tag{18.2}$$

If you are interested in other polarizations you can find out the amplitude for them from the superposition of these two amplitudes To get the intensity of any component as a function of angle, you must, of course, take the absolute square of the amplitudes.

18–2 Light scattering

Let's use these results to solve a somewhat more complicated problem—but also one which is somewhat more real. We suppose that the same atoms are sitting in their ground state ($j = 0$), and *scatter* an incoming beam of light. Let's say that the light is going initially in the $+z$-direction, so that we have photons coming up to the atom *from* the $-z$-direction, as shown in Fig. 18–4(a). We can consider the scattering of light as a two-step process: The photon is absorbed, and then is re-emitted. If we start with a RHC photon as in Fig. 18–4(a), and angular momentum is conserved, the atom will be in an $m = +1$ state after the absorption—as shown in Fig. 18–4(b). We call the amplitude for this process c. The atom can then emit a RHC photon in the direction θ—as in Fig. 18–4(c). The total amplitude that a RHC photon is scattered in the direction θ is just c times (18.1). Let's call this scattering amplitude $\langle R' \mid S \mid R \rangle$; we have

$$\langle R' \mid S \mid R \rangle = \frac{ac}{2}(1 + \cos\theta). \tag{18.3}$$

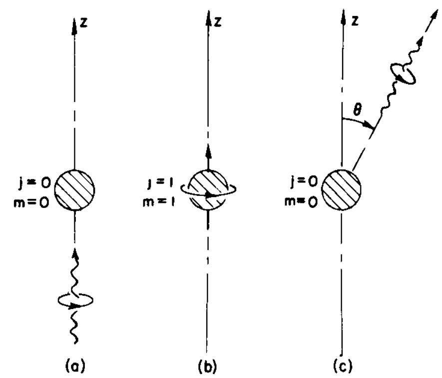

Fig. 18–4. The scattering of light by an atom seen as a two-step process.

There is also an amplitude that a RHC photon will be absorbed and that a LHC photon will be emitted. The product of the two amplitudes is the amplitude $\langle L' \mid S \mid R \rangle$ that a RHC photon is scattered as a LHC photon. Using (18.2), we have

$$\langle L' \mid S \mid R \rangle = -\frac{ac}{2}(1 - \cos\theta). \tag{18.4}$$

Now let's ask about what happens if a LHC photon comes in. When it is absorbed, the atom will go into an $m = -1$ state. By the same kind of arguments we used in the preceding section, we can show that this amplitude must be $-c$. The amplitude that an atom in the $m = -1$ state will emit a RHC photon at the angle θ is a times the amplitude $\langle + \mid R_y(\theta) \mid - \rangle$, which is $\frac{1}{2}(1 - \cos\theta)$. So we have

$$\langle R' \mid S \mid L \rangle = -\frac{ac}{2}(1 - \cos\theta). \tag{18.5}$$

Finally, the amplitude for a LHC photon to be scattered as a LHC photon is

$$\langle L' \mid S \mid L \rangle = \frac{ac}{2}(1 + \cos\theta). \tag{18.6}$$

(There are two minus signs which cancel.)

If we make a measurement of the scattered *intensity* for any given combination of circular polarizations it will be proportional to the square of one of our four amplitudes. For instance, with an incoming beam of RHC light the intensity of the RHC light in the scattered radiation will vary as $(1 + \cos\theta)^2$.

That's all very well, but suppose we start out with *linearly* polarized light. What then? If we have x-polarized light, it can be represented as a superposition of RHC and LHC light. We write (see Section 11–4)

$$|\,x\rangle = \frac{1}{\sqrt{2}}\,(|\,R\rangle + |\,L\rangle). \tag{18.7}$$

Or, if we have y-polarized light, we would have

$$|\,y\rangle = -\frac{i}{\sqrt{2}}\,(|\,R\rangle - |\,L\rangle). \tag{18.8}$$

Now what do you want to know? Do you want the amplitude that an x-polarized photon will scatter into a RHC photon at the angle θ? You can get it by the usual rule for combining amplitudes. First, multiply (18.7) by $\langle R'\,|\,S$ to get

$$\langle R'\,|\,S\,|\,x\rangle = \frac{1}{\sqrt{2}}\,(\langle R'\,|\,S\,|\,R\rangle + \langle R'\,|\,S\,|\,L\rangle), \tag{18.9}$$

and then use (18.3) and (18.5) for the two amplitudes. You get

$$\langle R'\,|\,S\,|\,x\rangle = \frac{ac}{\sqrt{2}}\cos\theta. \tag{18.10}$$

If you wanted the amplitude that an x-photon would scatter into a LHC photon, you would get

$$\langle L'\,|\,S\,|\,x\rangle = \frac{ac}{\sqrt{2}}\cos\theta. \tag{18.11}$$

Finally, suppose you wanted to know the amplitude that an x-polarized photon will scatter while keeping its x-polarization. What you want is $\langle x'\,|\,S\,|\,x\rangle$. This can be written as

$$\langle x'\,|\,S\,|\,x\rangle = \langle x'\,|\,R'\rangle\langle R'\,|\,S\,|\,x\rangle + \langle x'\,|\,L'\rangle\langle L'\,|\,S\,|\,x\rangle. \tag{18.12}$$

If you then use the relations

$$|\,R'\rangle = \frac{1}{\sqrt{2}}\,(|\,x'\rangle + i\,|\,y'\rangle), \tag{18.13}$$

$$|\,L'\rangle = \frac{1}{\sqrt{2}}\,(|\,x'\rangle - i\,|\,y'\rangle), \tag{18.14}$$

it follows that

$$\langle x'\,|\,R'\rangle = \frac{1}{\sqrt{2}}, \tag{18.15}$$

$$\langle x'\,|\,L'\rangle = \frac{1}{\sqrt{2}}\cdot \tag{18.16}$$

So you get that

$$\langle x'\,|\,S\,|\,x\rangle = ac\cos\theta. \tag{18.17}$$

The answer is that a beam of x-polarized light will be scattered at the direction θ (in the xz-plane) with an *intensity* proportional to $\cos^2\theta$. If you ask about y-polarized light, you find that

$$\langle y'|\,S\,|\,x\rangle = 0. \tag{18.18}$$

So the scattered light is completely polarized in the x-direction.

Now we notice something interesting. The results (18.17) and (18.18) correspond exactly to the classical theory of light scattering we gave in Vol. 1, Section 32–6, where we imagined that the electron was bound to the atom by a linear restoring force—so that it acted like a classical oscillator. Perhaps you are thinking: "It's so much easier in the classical theory; if it gives the right answer why bother with the quantum theory?" For one thing, we have considered so far only the special—though common—case of an atom with a $j = 1$ excited state and a $j = 0$ ground state. If the excited state had spin two, you would get a different result. Also, there is no reason why the model of an electron attached to a spring and driven by an oscillating electric field should work for a single photon. But we have found that it does in fact work, and that the polarization and intensities come out right. So in a certain sense we are bringing the whole course around to the real truth. Whereas we have, in Vol. I, done the theory of the index of refraction, and of light scattering, by the classical theory, we have now shown that the quantum theory gives the same result for the most common case. In effect we have now done the polarization of sky light, for instance, by quantum mechanical arguments, which is the only truly legitimate way.

It should be, of course, that all the classical theories which work are supported ultimately by legitimate quantum arguments. Naturally, those things which we have spent a great deal of time in explaining to you were selected from just those parts of classical physics which still maintain validity in quantum mechanics. You'll notice that we did not discuss in great detail any model of the atom which has electrons going around in orbits. That's because such a model doesn't give results which agree with the quantum mechanics. But the electron on a spring—which is not, in a sense, at all the way an atom "looks"—does work, and so we used that model for the theory of the index of refraction.

18–3 The annihilation of positronium

We would like next to take an example which is very pretty. It is quite interesting and, although somewhat complicated, we hope not too much so. Our example is the system called *positronium*, which is an "atom" made up of an electron and a positron—a bound state of an e^+ and an e^-. It is like a hydrogen atom, except that a positron replaces the proton. This object has—like the hydrogen atom—many states. Also like the hydrogen, the ground state is split into a "hyperfine structure" by the interaction of the magnetic moments. The spins of the electron and positron are each one-half, and they can be either parallel or antiparallel to any given axis. (In the ground state there is no other angular momentum due to orbital motion.) So there are four states: three are the substates of a spin-one system, all with the same energy; and one is a state of spin zero with a different energy. The energy splitting is, however, much larger than the 1420 megacycles of hydrogen because the positron magnetic moment is so much stronger—1000 times stronger—than the proton moment.

The most important difference, however, is that positronium cannot last forever. The position is the antiparticle of the electron; they can annihilate each other. The two particles disappear completely—converting their rest energy into radiation, which appears as γ-rays (photons). In the disintegration, two particles with a finite rest mass go into two or more objects which have zero rest mass.†

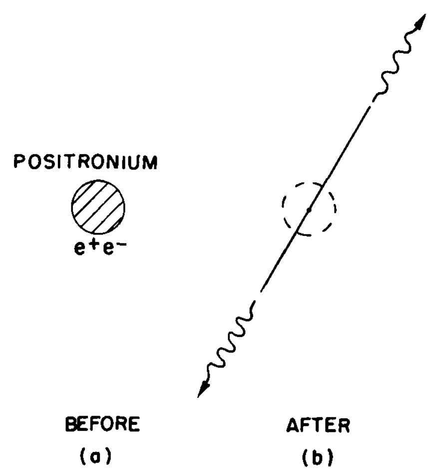

Fig. 18–5. The two-photon annihilation of positronium.

We begin by analyzing the disintegration of the spin-zero state of the positronium. It disintegrates into two γ-rays with a lifetime of about 10^{-10} second. Initially, we have a positron and an electron close together and with spins antiparallel, making the positronium system. After the disintegration there are two photons going out with equal and opposite momenta (Fig. 18–5). The momenta must be equal and opposite, because the total momentum after the disintegration must be zero, as it was before, if we are taking the case of annihilation at rest. If the positronium is not at rest, we can ride with it, solve the problem, and then transform everything back to the lab system. (See, we can do anything now; we have all the tools.)

First, we note that the angular distribution is not very interesting. Since the initial state has spin zero, it has no special axis—it is symmetric under all rotations. The final state must then also be symmetric under all rotations. That means that all angles for the disintegration are equally likely—the amplitude is the same for a photon to go in any direction. Of course, once we find *one* of the photons in some direction the *other* must be opposite.

† In the deeper understanding of the world today, we do not have an easy way to distinguish whether the energy of a photon is less "matter" than the energy of an electron, because as you remember all the particles behave very similarly. The only distinction is that the photon has zero rest mass.

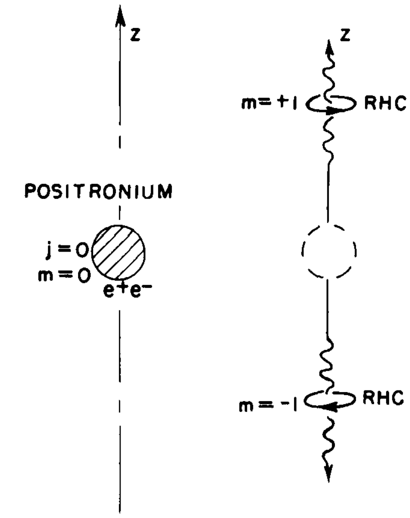

Fig. 18-6. One possibility for positronium annihilation along the z-axis.

The only remaining question, which we now want to look at, is about the polarization of the photons. Let's call the directions of motion of the two photons the plus and minus z-axes. We can use any representations we want for the polarization states of the photons; we will choose for our description right and left circular polarization—always with respect to the directions of motion. Right away, we can see that if the photon going upward is RHC, then angular momentum will be conserved if the downward going photon is also RHC. Each will carry $+1$ unit of angular momentum *with respect to its momentum direction*, which means plus and minus one unit about the z-axis. The total will be zero, and the angular momentum after the disintegration will be the same as before. See Fig. 18–6.

The same arguments show that if the upward going photon is RHC, the downward cannot be LHC. Then the final state would have two units of angular momentum. This is not permitted if the initial state has spin zero. Note that such a final state is also not possible for the other positronium ground state of spin one, because it can have a maximum of one unit of angular momentum in any direction.

Now we want to show that two-photon annihilation is not possible at all from the spin-one state. You might think that if we took the $j = 1$, $m = 0$ state—which has zero angular momentum about the z-axis—it should be like the spin-zero state, and could disintegrate into two RHC photons. Certainly, the disintegration sketched in Fig. 18–7(a) conserves angular momentum about the z-axis. But now look what happens if we rotate this system around the y-axis by 180°; we get the picture shown in Fig. 18–7(b). It is exactly the same as in part (a) of the figure. All we have done is interchange the two photons. Now photons are Bose particles; if we interchange them, the amplitude has the same sign, so the amplitude for the disintegration in part (b) must be the same as in part (a). But we have assumed that the initial object is spin one. And when we rotate a spin-one object in a state with $m = 0$ by 180° about the y-axis, its amplitudes change sign (see Table 17–2 for $\theta = \pi$). So the amplitudes for (a) and (b) in Fig. 18–7 should have opposite signs; the spin-one state *cannot disintegrate into two photons.*

When positronium is formed you would expect it to end up in the spin-zero state 1/4 of the time and in the spin-one state (with $m = -1, 0$, or $+1$)3/4 of the time. So 1/4 of the time you would get two-photon annihilations. The other 3/4

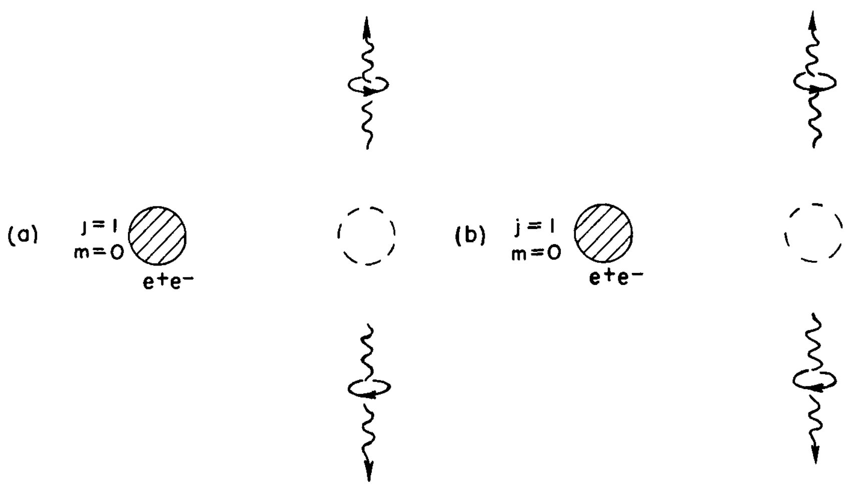

Fig. 18–7. For the $j = 1$ state of positronium, the process (a) and its 180° rotation about y (b) are exactly the same.

† Note that we always analyze the angular momentum about the direction of motion of the particle. If we were to ask about the angular momentum about any other axis, we would have to worry about the possibility of "orbital" angular momentum—from a $\boldsymbol{p} \times \boldsymbol{r}$ term. For instance, we can't say that the photons leave exactly from the center of the positronium. They could leave like two things shot out from the rim of a spinning wheel. We don't have to worry about such possibilities when we take our axis along the direction of motion.

of the time there can be no two-photon annihilations. There is still an annihilation, but it has to go with *three* photons. It is harder for it to do that and the lifetime is 1000 times longer—about 10^{-7} second. This is what is observed experimentally. We will not go into any more of the details of the spin-one annihilation.

So far we have that if we only worry about angular momentum, the spin-zero state of the positronium can go into two RHC photons. There is also another possibility: it can go into two LHC photons as shown in Fig. 18–8. The next question is, what is the relation between the amplitudes for these two possible decay modes? We can find out from the conservation of parity.

To do that, however, we need to know the parity of the positronium. Now theoretical physicists have shown in a way that is not easy to explain that the parity of the electron and the positron—its antiparticle—must be opposite, so that the spin-zero ground state of positronium must be odd. We will just assume that it is odd, and since we will get agreement with experiment, we can take that as sufficient proof.

Let's see then what happens if we make an inversion of the process in Fig. 18–6. When we do that, the two photons reverse directions *and* polarizations. The inverted picture looks just like Fig. 18–8. Assuming that the parity of the positronium is odd, the amplitudes for the two processes in Figs. 18–6 and 18–8 must have the opposite sign. Let's let $|\,R_1R_2\rangle$ stand for the final state of Fig. 18–6 in which both photons are RHC, and let $|\,L_1L_2\rangle$ stand for the final state of Fig. 18–8, in which both photons are LHC. The true final state—let's call it $|\,F\rangle$—must be

$$|\,F\rangle = |\,R_1R_2\rangle - |\,L_1L_2\rangle. \tag{18.19}$$

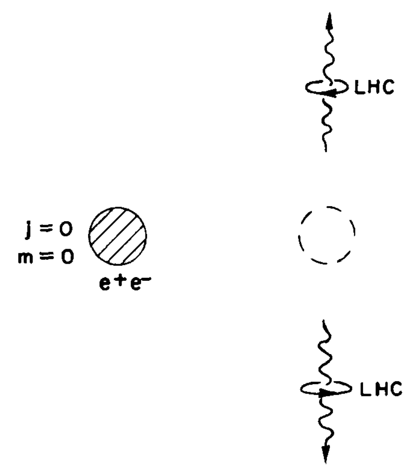

Fig. 18–8. Another possible process for positronium annihilation.

Then an inversion changes the R's into L's and gives the state

$$P\,|\,F\rangle = |\,L_1L_2\rangle - |\,R_1R_2\rangle = -\,|\,F\rangle, \tag{18.20}$$

which is the negative of (18.19). So the final state $|\,F\rangle$ has negative parity, which is the same as the initial spin-zero state of the positronium. This is the only final state that conserves both angular momentum and parity. There is some amplitude that the disintegration into this state will occur, which we don't need to worry about now, however, since we are only interested in questions about the polarization.

What does the final state of (18.19) mean physically? One thing it means is the following: If we observe the two photons in two detectors which can be set to count separately the RHC or LHC photons, we will always see two RHC photons together, or two LHC photons together. That is, if you stand on one side of the positronium and someone else stands on the opposite side, you can measure the polarization and tell the other guy what polarization he will get. You have a 50–50 chance of catching a RHC photon or a LHC photon; whichever one you get, you can predict that he will get the same.

Since there is a 50–50 chance for RHC or LHC polarization, it sounds as though it might be like linear polarization. Let's ask what happens if we observe the photon in counters that accept only linearly polarized light. For γ-rays it is not as easy to measure the polarization as it is for light; there is no polarizer which works well for such short wavelengths. But let's imagine that there is, to make the discussion easier. Suppose that you have a counter that only accepts light with x-polarization, and that there is a guy on the other side that also looks for linear polarized light with, say, y-polarization. What is the chance you will pick up the two photons from an annihilation? What we need to ask is the amplitude that $|\,F\rangle$ will be in the state $|\,x_1y_2\rangle$. In other words, we want the amplitude

$$\langle x_1y_2\,|\,F\rangle,$$

which is, of course, just

$$\langle x_1y_2\,|\,R_1R_2\rangle - \langle x_1y_2\,|\,L_1L_2\rangle. \tag{18.21}$$

Now although we are working with two-particle amplitudes for the two photons, we can handle them just as we did the single particle amplitudes, since each particle acts independently of the other. That means that the amplitude $\langle x_1 y_2 \mid R_1 R_2 \rangle$ is just the product of the two independent amplitudes $\langle x_1 \mid R_1 \rangle$ and $\langle y_2 \mid R_2 \rangle$. Using Table 17–3, these two amplitudes are $1/\sqrt{2}$ and $i/\sqrt{2}$, so

$$\langle x_1 y_2 \mid R_1 R_2 \rangle = +\frac{i}{2}\cdot$$

Similarly, we find that

$$\langle x_1 y_2 \mid L_1 L_2 \rangle = -\frac{i}{2}\cdot$$

Subtracting these two amplitudes according to (18.21), we get that

$$\langle x_1 y_2 \mid F \rangle = +i. \tag{18.22}$$

So there is a *unit* probability† that *if* you get a photon in your x-polarized detector, the other guy will get a photon in his y-polarized detector.

Now suppose that the other guy sets his counter for x-polarization the same as yours. He would never get a count when you got one. If you work it through, you will find that

$$\langle x_1 x_2 \mid F \rangle = 0. \tag{18.23}$$

It will, naturally, also work out that if you set your counter for y-polarization he will get coincident counts only if he is set for x-polarization.

Now this all leads to an interesting situation. Suppose you were to set up something like a piece of calcite which separated the photons into x-polarized and y-polarized beams, and put a counter in each beam. Let's call one the x-counter and the other the y-counter. If the guy on the other side does the same thing, you can always tell him which beam his photon is going to go into. Whenever you and he get simultaneous counts, you can see which of your detectors caught the photon and then tell him which of his counters had a photon. Let's say that in a certain disintegration you find that a photon went into your x-counter; you can tell him that he must have had a count in his y-counter.

Now many people who learn quantum mechanics in the usual (old-fashioned) way find this disturbing. They would like to think that once the photons are emitted it goes along as a wave with a definite character. They would think that since "any given photon" has some "amplitude" to be x-polarized or to be y-polarized, there should be some chance of picking it up in either the x- or y-counter and that this chance shouldn't depend on what some other person finds out about a completely different photon. They argue that "someone else making a measurement shouldn't be able to change the probability that I will find something." Our quantum mechanics says, however, that by making a measurement on photon number one, you *can* predict precisely what the polarization of photon number two is going to be when it is detected. This point was never accepted by Einstein, and he worried about it a great deal—it became known as the "Einstein-Podalsky-Rosen paradox." But when the situation is described as we have done it here, there doesn't seem to be any paradox at all; it comes out quite naturally that what is measured in one place is correlated with what is measured somewhere else. The argument that the result is paradoxical runs something like this:

(1) If you have a counter which tells you whether your photon is RHC or LHC, you can predict exactly what kind of a photon (RHC or LHC) he will find.
(2) The photons he receives must, therefore, each be purely RHC or purely LHC, some of one kind and some of the other.
(3) Surely you cannot alter the physical nature of *his* photons by changing the kind of observation you make on *your* photons. No matter what measurements you make on yours, his must still be either RHC or LHC.

† We have not normalized our amplitudes, or multiplied them by the amplitude for the disintegration into any particular final state, but we can see that this result is correct because we get zero probability when we look at the other alternative—see Eq. (18.23).

(4) Now suppose he changes his apparatus to split his photons into two linearly polarized beams with a piece of calcite so that all of his photons go either into an x-polarized beam or into a y-polarized beam. There is absolutely no way, according to quantum mechanics, to tell into which beam any particular RHC photon will go. There is a 50% probability it will go into the x-beam and a 50% probability it will go into the y-beam. And the same goes for a LHC photon.

(5) Since each photon is RHC or LHC—according to (2) and (3)—each one must have a 50–50 chance of going into the x-beam or the y-beam and there is no way to predict which way it will go.

(6) Yet the theory predicts that if *you* see your photon go through an x-polarizer you can predict *with certainty* that his photon will go into his y-polarized beam. This is in contradiction to (5) so there is a paradox.

Nature apparently doesn't see the "paradox," however, because experiment shows that the prediction in (6) is, in fact, true. We have already discussed the key to this "paradox" in our very first lecture on quantum mechanical behavior in Chapter 35, Vol. I. In the argument above, steps (1), (2), (4), and (6) are all correct, but (3), and its consequence (5), are wrong; they are not a true description of nature. Argument (3) says that by *your* measurement (seeing a RHC or a LHC photon) you can determine which of two alternative events occurs for him (seeing a RHC or a LHC photon), *and* that even if you do *not* make your measurement you can still say that his event will occur either by one alternative or the other. But it was precisely the point of Chapter 35, Vol. I, to point out right at the beginning that this is not so in Nature. *Her* way requires a description in terms of interfering amplitudes, one amplitude for each alternative. A measurement of which alternative actually occurs destroys the interference, but if a measurement is *not* made you cannot still say that "one alternative or the other is still occurring."

If you could determine for each one of your photons whether it was RHC and LHC, and *also* whether it was x-polarized (all for the same photon) there would indeed be a paradox. But you cannot do that—it is an example of the uncertainty principle.

Do you still think there is a "paradox"? Make sure that it is, in fact, a paradox about the behavior of Nature, by setting up an imaginary experiment for which the theory of quantum mechanics would predict inconsistent results via two different arguments. Otherwise the "paradox" is only a conflict between reality and your feeling of what reality "ought to be."

Do you think that it is *not* a "paradox," but that it is still very peculiar? On that we can all agree. It is what makes physics fascinating.

18–4 Rotation matrix for any spin

By now you can see, we hope, how important the idea of the angular momentum is in understanding atomic processes. So far, we have considered only systems with spins—or "total angular momentum"—of zero, one-half, or one. There are, of course, atomic systems with higher angular momenta. For analyzing such systems we would need to have tables of rotation amplitudes like those in Section 17–6. That is, we would need the matrix of amplitudes for spin $\frac{3}{2}$, 2, $\frac{5}{2}$, 3, etc. Although we will not work out these tables in detail, we would like to show you how it is done, so that you can do it if you ever need to.

As we have seen earlier, any system which has the spin or "total angular momentum" j can exist in any one of $(2j + 1)$ states for which the z-component of angular momentum can have any one of the discrete values in the sequence j, $j - 1, j - 2, \dots, -(j - 1), -j$ (all in units of $\hbar$). Calling the z-component of angular momentum of any particular state $m\hbar$, we can define a particular angular momentum state by giving the numerical values of the two "angular momentum quantum numbers" j and m. We can indicate such a state by the state vector $|\,j, m\rangle$. In the case of a spin one-half particle, the two states are then $|\,\frac{1}{2}, \frac{1}{2}\rangle$ and $|\,\frac{1}{2}, -\frac{1}{2}\rangle$; or for a spin-one system, the states would be written in this notation as $|\,1, +1\rangle$, $|\,1, 0\rangle$, $|\,1, -1\rangle$. A spin-zero particle has, of course, only the one state $|\,0, 0\rangle$.

Now we want to know what happens when we project the general state $|\,j, m\rangle$ into a representation with respect to a rotated set of axes. First, we know that j is a number which characterizes *the system*, so it doesn't change. If we rotate the axes, all we do is get a mixture of the various m-values for the same j. In general, there will be some amplitude that in the rotated frame the system will be in the state $|\,j, m'\rangle$, where m' gives the new z-component of angular momentum. So what we want are all the matrix elements $\langle j, m' \mid R \mid j, m\rangle$ for various rotations. We already know what happens if we rotate by an angle ϕ about the z-axis. The new state is just the old one multiplied by $e^{im\phi}$—it still has the same m-value. We can write this by

$$R_z(\phi)\,|\,j, m\rangle = e^{im\phi}\,|\,j, m\rangle. \tag{18.24}$$

Or, if you prefer,

$$\langle j, m' \mid R_z(\phi) \mid j, m\rangle = \delta_{m,m'}e^{im\phi} \tag{18.25}$$

(where $\delta_{m,m'}$ is 1 if $m' = m$, or zero otherwise).

For a rotation about any other axis there will be a mixing of the various m-states. We could, of course, try to work out the matrix elements for an arbitrary rotation described by the Euler angles β, α, and γ. But it is easier to remember that the most general such rotation can be made up of the three rotations $R_z(\gamma)$, $R_y(\alpha)$, $R_z(\beta)$; so if we know the matrix elements for a rotation about the y-axis, we will have all we need.

How can we find the rotation matrix for a rotation by the angle θ about the y-axis for a particle of spin j? We can't tell you how to do it in a basic way (with what we have had). We did it for spin one-half by a complicated symmetry argument. We then did it for spin one by taking the special case of a spin-one system which was made up of two spin one-half particles. If you will go along with us and accept the fact that in the general case the answers depend only on the spin j, and are independent of how the inner guts of the object of spin j are put together, we can extend the spin-one argument to an arbitrary spin. We can, for example, cook up an artificial system of spin $\frac{3}{2}$ out of three spin one-half objects. We can even avoid complications by imagining that they are all distinct particles—like a proton, an electron, and a muon. By transforming each spin one-half object, we can see what happens to the whole system—remembering that the three amplitudes are multiplied for the combined state. Let's see how it goes in this case.

Suppose we take the three spin one-half objects all with spins "up"; we can indicate this state by $|+++\rangle$. If we look at this system in a frame rotated about the z-axis by the angle ϕ, each plus stays a plus, but gets multiplied by $e^{i\phi/2}$. We have three such factors, so

$$R_z(\phi)\,|+++\rangle = e^{i(3\phi/2)}\,|+++\rangle. \tag{18.26}$$

Evidently the state $|+++\rangle$ is just what we mean by the $m = +\frac{3}{2}$ state, or the state $|\,\frac{3}{2}, +\frac{3}{2}\rangle$.

If we now rotate this system about the y-axis, each of the spin one-half objects will have some amplitude to be plus or to be minus, so the system will now be a mixture of the *eight* possible combinations $|+++\rangle$, $|++-\rangle$, $|+-+\rangle$, $|-++\rangle$, $|+--\rangle$, $|-+-\rangle$, $|--+\rangle$, or $|---\rangle$. It is clear, however, that these can be broken up into four sets, each set corresponding to a particular value of m. First, we have $|+++\rangle$, for which $m = \frac{3}{2}$. Then there are the three states $|++-\rangle$, $|+-+\rangle$, and $|-++\rangle$—each with two plusses and one minus. Since each spin one-half object has the same chance of coming out minus under the rotation, the amounts of each of these three combinations should be equal. So let's take the combination

$$\frac{1}{\sqrt{3}}\,\{|++-\rangle + |+-+\rangle + |-++\rangle\} \tag{18.27}$$

with the factor $1/\sqrt{3}$ put in to normalize the state. If we rotate this state about the z-axis, we get a factor $e^{i\phi/2}$ for each plus, and $e^{-i\phi/2}$ for each minus. Each term in (18.27) is multiplied by $e^{i\phi/2}$, so there is the common factor $e^{i\phi/2}$. This one "$-$" pieces. For instance,

$$\begin{aligned}|+\,+\,-\rangle = {} & a^2c\,|+'\,+'\,+'\rangle + a^2d\,|+'\,+'\,-'\rangle + abc\,|+'\,-'\,+'\rangle \\ & + bac\,|-'\,+'\,+'\rangle + abd\,|+'\,-'\,-'\rangle + bad\,|-'\,+'\,-'\rangle \\ & + b^2c\,|-'\,-'\,+'\rangle + b^2d\,|-'\,-'\,-'\rangle. \end{aligned} \tag{18.33}$$

Adding two similar expressions for $|+\,-\,+\rangle$ and $|-\,+\,+\rangle$ and dividing by $\sqrt{3}$, we find

$$\begin{aligned}|\tfrac{3}{2},+\tfrac{1}{2},S\rangle = {} & \sqrt{3}\,a^2c\,|\tfrac{3}{2},+\tfrac{3}{2},T\rangle \\ & +(a^2d + 2abc)\,|\tfrac{3}{2},+\tfrac{1}{2},T\rangle \\ & +(2bad + b^2c)\,|\tfrac{3}{2},-\tfrac{1}{2},T\rangle \\ & +\sqrt{3}\,b^2d\,|\tfrac{3}{2},-\tfrac{3}{2},T\rangle. \end{aligned} \tag{18.34}$$

Continuing the process we find all the elements $\langle jT\,|\,iS\rangle$ of the transformation matrix as given in Table 18–2. The first column comes from Eq. (18.32); the second from (18.34). The last two columns were worked out in the same way.

Table 18–2

Rotation matrix for a spin $\frac{3}{2}$ particle

(The coefficients a, b, c, and d are given in Table 12–4.)

$\langle jT\,\|\,iS\rangle$	$\|\tfrac{3}{2},+\tfrac{3}{2},S\rangle$	$\|\tfrac{3}{2},+\tfrac{1}{2},S\rangle$	$\|\tfrac{3}{2},-\tfrac{1}{2},S\rangle$	$\|\tfrac{3}{2},-\tfrac{3}{2},S\rangle$
$\langle\tfrac{3}{2},+\tfrac{3}{2},T\,\|$	a^3	$\sqrt{3}\,a^2c$	$\sqrt{3}\,ac^2$	c^3
$\langle\tfrac{3}{2},+\tfrac{1}{2},T\,\|$	$\sqrt{3}\,a^2b$	$a^2d + 2abc$	$c^2b + 2dac$	$\sqrt{3}\,c^2d$
$\langle\tfrac{3}{2},-\tfrac{1}{2},T\,\|$	$\sqrt{3}\,ab^2$	$2bad + b^2c$	$2cdb + d^2a$	$\sqrt{3}\,cd^2$
$\langle\tfrac{3}{2},-\tfrac{3}{2},T\,\|$	b^3	$\sqrt{3}\,b^2d$	$\sqrt{3}\,bd^2$	d^3

Now suppose the T-frame were rotated with respect to S by the angle θ about their y-axes. Then a, b, c, and d have the values [see (12.54)] $a = d = \cos\theta/2$, and $c = -b = \sin\theta/2$. Using these values in Table 18–2 we get the forms which correspond to the second part of Table 17–2, but now for a spin $\frac{3}{2}$ system.

The arguments we have just gone through are readily generalized to a system of any spin j. The states $|j, m\rangle$ can be put together from $2j$ particles, each of spin one-half. (There are $j + m$ of them in the $|+\rangle$ state and $j - m$ in the $|-\rangle$ state.) Sums are taken over all the possible ways this can be done, and the state is normalized by multiplying by a suitable constant. Those of you who are mathematically inclined may be able to show that the following result comes out†:

$$\begin{aligned}\langle j, m'\,|\,R_y(\theta)\,|\,j, m\rangle = {} & [(j+m)!(j-m)!(j+m')!(j-m')!]^{1/2} \\ & \times \sum_k \frac{(-1)^k(\cos\theta/2)^{2j+m'-m-2k}(\sin\theta/2)^{m-m'+2k}}{(m-m'+k)!(j+m'-k)!(j-m-k)!k!}, \end{aligned} \tag{18.35}$$

where k is to go over all values which give terms ≥ 0 in all the factorials.

This is quite a messy formula, but with it you can check Table 17–2 for $j = 1$ and prepare tables of your own for larger j. Several special matrix elements are of extra importance and have been given special names. For example the matrix elements for $m = m' = 0$ and integral j are known as the Legendre polynomials and are called $P_j(\cos\theta)$:

$$\langle j, 0\,|\,R_y(\theta)\,|\,j, 0\rangle = P_j(\cos\theta). \tag{18.36}$$

† If you want details, they are given in an appendix to this chapter.

The first few of these polynomials are:

$$P_0(\cos\theta) = 1, \tag{18.37}$$

$$P_1(\cos\theta) = \cos\theta, \tag{18.38}$$

$$P_2(\cos\theta) = \tfrac{1}{2}(3\cos^2\theta - 1), \tag{18.39}$$

$$P_3(\cos\theta) = \tfrac{1}{2}(5\cos^3\theta - 3\cos\theta). \tag{18.40}$$

18–5 Measuring a nuclear spin

We would like to show you one example of the application of the coefficients we have just described. It has to do with a recent, interesting experiment which you will now be able to understand. Some physicists wanted to find out the spin of a certain excited state of the Ne^{20} nucleus. To do this, they bombarded a carbon target with a beam of accelerated carbon ions, and produced the desired excited state of Ne^{20}—called Ne^{20*}—in the reaction

$$C^{12} + C^{12} \to Ne^{20*} + \alpha_1,$$

where α_1 is the α-particle, or He^4. Several of the excited states of Ne^{20} produced this way are unstable and disintegrate in the reaction

$$Ne^{20*} \to O^{16} + \alpha_2.$$

So experimentally there are two α-particles which come out of the reaction. We call them α_1 and α_2; since they come off with different energies, they can be distinguished from each other. Also, by picking a particular energy for α_1 we can pick out any particular excited state of the Ne^{20}.

The experiment was set up as shown in Fig. 18–9. A beam of 16-Mev carbon ions was directed onto a thin foil of carbon. The first α-particle was counted in a silicon diffused junction detector marked α_1—set to accept α-particles of the proper energy moving in the forward direction (with respect to the incident C^{12} beam). The second α-particle was picked up in the counter α_2 at the angle θ with respect to α_1. The counting rate of coincidence signals from α_1 and α_2 were measured as a function of the angle θ.

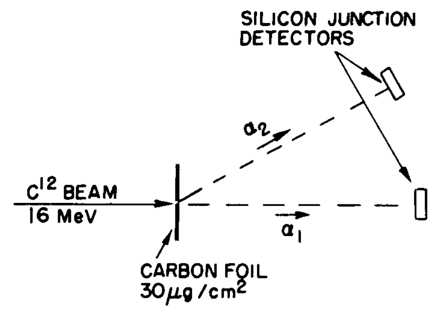

Fig. 18–9. Experimental arrangement for determining the spin of certain states of Ne^{20}.

The idea of the experiment is the following. First, you need to know that the spins of C^{12}, O^{16}, and the α-particle are all zero. If we call the direction of motion of the initial C^{12} the $+z$-direction, then we know that the Ne^{20*} must have zero angular momentum about the z-axis. None of the other particles has any spin; the C^{12} arrives along the z-axis and the α_1 leaves along the z-axis so they can't have any angular momentum about it. So whatever the spin j of the Ne^{20*} is, we know that it is in the state $|j, 0\rangle$. Now what will happen when the Ne^{20*} disintegrates into an O^{16} and the second α-particle? Well, the α-particle is picked up in the counter α_2 and to conserve momentum the O^{16} must go off in the opposite direction.† *About the new axis* through α_2, there can be no component of angular momentum. The final state has zero angular momentum about the new axis, so the Ne^{20*} can disintegrate this way only if it has some amplitude to have m' equal to zero, where m' is the quantum number of the component of angular momentum about the new axis. In fact, the probability of observing α_2 at the angle θ is just the square of the amplitude (or matrix element)

$$\langle j, 0 \,|\, R_y(\theta) \,|\, j, 0\rangle. \tag{18.41}$$

To find the spin of the Ne^{20*} state in question, the intensity of the second α-particle was plotted as a function of angle and compared with the theoretical

† We can neglect the recoil given to the Ne^{20*} in the first collision. Or better still, we can calculate what it is and make a correction for it.

curves for various values of j. As we said in the last section, the amplitudes $\langle j, 0 \mid R_y(\theta) \mid j, 0\rangle$ are just the functions P_j (cos θ). So the possible angular distributions are curves of $[P_j (\cos\theta)]^2$. The experimental results are shown in Fig. 18–10 for two of the excited states. You can see that the angular distribution for the 5.80-Mev state fits very well the curve for $[P_1 (\cos\theta)]^2$, and so it must be a spin-one state. The data for the 5.63-Mev state, on the other hand, are quite different; they fit the curve $[P_3 (\cos\theta)]^2$. The state has a spin of 3.

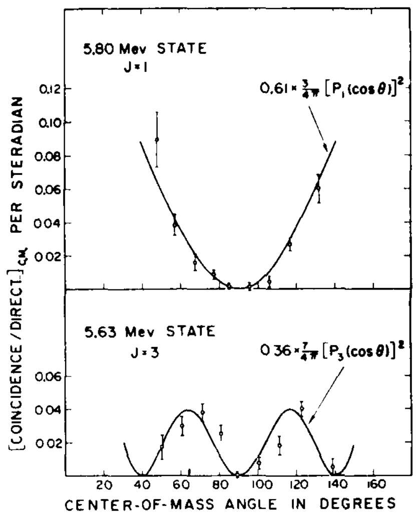

Fig. 18–10. Experimental results for the angular distribution of the α-particles from two excited states of Ne^{20} produced in the setup of Fig. 18–9. [From J. A. Kuehner, *Physical Review*, Vol. 125, p. 1653, 1962.]

From this experiment we have been able to find out the angular momentum of two of the excited states of Ne^{20*}. This information can then be used for trying to understand what the configuration of protons and neutrons is inside this nucleus—one more piece of information about the mysterious nuclear forces.

18–6 Composition of angular momentum

When we studied the hyperfine structure of the hydrogen atom in Chapter 12 we had to work out the internal states of a system composed of two particles—the electron and the proton—each with a spin of one-half. We found that the four possible spin states of such a system could be put together into two groups—a group with one energy that looked to the external world like a spin-one particle, and one remaining state that behaved like a particle of zero spin. That is, putting together two spin one-half particles we can form a system whose "total spin" is one, or zero. In this section we want to discuss in more general terms the spin states of a *system* which is made up of two particles of arbitrary spin. It is another important problem about angular momentum in quantum mechanical systems.

Let's first rewrite the results of Chapter 12 for the hydrogen atom in a form that will be easier to extend to the more general case. We began with two particles which we will now call particle a (the electron) and particle b (the proton). Particle a had the spin j_a $(=\frac{1}{2})$, and its z-component of angular momentum m_a could have one of several values (actually 2, namely $m_a = +\frac{1}{2}$ or $m_a = -\frac{1}{2}$). Similarly, the spin state of particle b is described by its spin j_b and its z-component of angular momentum m_b. Various combinations of the spin states of the two particles could be formed. For instance, we could have particle a with $m_a = \frac{1}{2}$ and particle b with $m_b = -\frac{1}{2}$, to make a state $\mid a, +\frac{1}{2}; b, -\frac{1}{2}\rangle$. In general, the combined states formed a system whose "system spin," or "total spin," or "total angular momentum" J could be 1, or 0. And the system could have a z-component of angular momentum M, which was $+1$, 0, or -1 when $J = 1$, or 0 when $J = 0$. In this new language we can rewrite the formulas in (12.41) and (12.42) as shown in Table 18–3.

In the table the left-hand column describes the compound state in terms of its total angular momentum J and the z-component M. The right-hand column shows how these states are made up in terms of the m-values of the two particles a and b.

We want now to generalize this result to states made up of two objects a and b of arbitrary spins j_a and j_b. We start by considering an example for which $j_a = \frac{1}{2}$

Table 18–3

Composition of angular momenta for two spin $\frac{1}{2}$ particles $(j_a = \frac{1}{2}, j_b = \frac{1}{2})$

$$\mid J = 1, M = +1\rangle = \mid a,+\tfrac{1}{2}; b,+\tfrac{1}{2}\rangle$$

$$\mid J = 1, M = 0\rangle = \frac{1}{\sqrt{2}}\{\mid a,+\tfrac{1}{2}; b,-\tfrac{1}{2}\rangle + \mid a,-\tfrac{1}{2}; b,+\tfrac{1}{2}\rangle\}$$

$$\mid J = 1, M = -1\rangle = \mid a,-\tfrac{1}{2}; b,-\tfrac{1}{2}\rangle$$

$$\mid J = 0, M = 0\rangle = \frac{1}{\sqrt{2}}\{\mid a,+\tfrac{1}{2}; b,-\tfrac{1}{2}\rangle - \mid a,-\tfrac{1}{2}; b,+\tfrac{1}{2}\rangle\}$$

and $j_b = 1$, namely, the deuterium atom in which particle a is an electron (e) and particle b is the nucleus—a deuteron (d). We have then that $j_a = j_e = \frac{1}{2}$. The deuteron is formed of one proton and one neutron in a state whose total spin is one, so $j_b = j_d = 1$. We want to discuss the hyperfine states of deuterium—just the way we did for hydrogen. Since the deuteron has three possible states $m_b = m_d = +1, 0, -1$, and the electron has two, $m_a = m_e = +\frac{1}{2}, -\frac{1}{2}$, there are *six* possible states as follows (using the notation $|\,\mathrm{e}, m_e; \mathrm{d}, m_d\rangle$):

$$\begin{aligned}
&|\,\mathrm{e},+\tfrac{1}{2};\mathrm{d},+1\rangle,\\
&|\,\mathrm{e},+\tfrac{1}{2};\mathrm{d},0\rangle;\ |\,\mathrm{e},-\tfrac{1}{2};\mathrm{d},+1\rangle,\\
&|\,\mathrm{e},+\tfrac{1}{2};\mathrm{d},-1\rangle;\ |\mathrm{e},-\tfrac{1}{2};\mathrm{d},0\rangle,\\
&|\,\mathrm{e},-\tfrac{1}{2};\mathrm{d},-1\rangle.
\end{aligned} \tag{18.42}$$

You will notice that we have grouped the states according to the values of the sum of m_e and m_d—arranged in descending order.

Now we ask: What happens to these states if we project into a different coordinate system? If the new system is just rotated about the z-axis by the angle ϕ, then the state $|\,\mathrm{e}, m_e; \mathrm{d}, m_d\rangle$ gets multiplied by

$$e^{im_e\phi}e^{im_d\phi} = e^{i(m_e+m_d)\phi}. \tag{18.43}$$

(The state may be thought of as the product $|\,\mathrm{e}, m_e\rangle\,|\,\mathrm{d}, m_d\rangle$, and each state vector contributes independently its own exponential factor.) The factor (18.43) is of the form $e^{iM\phi}$, so the state $|\,\mathrm{e}, m_e; \mathrm{d}, m_d\rangle$ has a z-component of angular momentum equal to

$$M = m_e + m_d. \tag{18.44}$$

The z-component of the total angular momentum is the sum of the z-components of angular momentum of the parts.

In the list of (18.42), therefore, the state in the top line has $M = +\frac{3}{2}$, the two in the second line have $M = +\frac{1}{2}$, the next two have $M = -\frac{1}{2}$, and the last state has $M = -\frac{3}{2}$. We see immediately one possibility for the spin J of the combined state (the total angular momentum) must be $\frac{3}{2}$, and this will require four states with $M = +\frac{3}{2}, +\frac{1}{2}, -\frac{1}{2}$, and $-\frac{3}{2}$.

There is only one candidate for $M = \frac{3}{2}$, so we know already that

$$|\,J = \tfrac{3}{2}, M = +\tfrac{3}{2}\rangle = |\,\mathrm{e},+\tfrac{1}{2};\mathrm{d},+1\rangle. \tag{18.45}$$

But what is the state $|\,J = \frac{3}{2}, M = \frac{1}{2}\rangle$? We have two candidates in the second line of (18.42), and, in fact, any linear combination of them would also have $M = \frac{1}{2}$. So, in general, we must expect to find that

$$|\,J = \tfrac{3}{2}, M = +\tfrac{1}{2}\rangle = \alpha\,|\,\mathrm{e},+\tfrac{1}{2};\mathrm{d},0\rangle + \beta\,|\,\mathrm{e},-\tfrac{1}{2};\mathrm{d},+1\rangle, \tag{18.46}$$

where α and β are two numbers. They are called the *Clebsch-Gordon coefficients.* Our next problem is to find out what they are.

We can find out easily if we just remember that the deuteron is made up of a neutron and a proton, and write the deuteron states out more explicitly using the rules of Table 18–3. If we do that, the states listed in (18.42) then look as shown in Table 18–4.

We want to form the four states of $J = \frac{3}{2}$, using the states in the table. But we already know the answer, because in Table 18–1 we have states of spin $\frac{3}{2}$ formed from three spin one-half particles. The first state in Table 18–1 has $|\,J = \frac{3}{2}, M = +\frac{3}{2}\rangle$ and it is $|\,+\,+\,+\rangle$, which—in our present notation—is the same as $|\,\mathrm{e},+\frac{1}{2};\mathrm{n},+\frac{1}{2},\mathrm{p},+\frac{1}{2}\rangle$, or the first state in Table 18–4. But this state is also the same as the first in the list of (18.42), confirming our statement in (18.45). The second line of Table 18–1 says—changing to our present notation—that

$$\begin{aligned}
|\,J = \tfrac{3}{2}; M = +\tfrac{1}{2}\rangle = \frac{1}{\sqrt{3}}\,\{&|\,\mathrm{e},+\tfrac{1}{2};\mathrm{n},+\tfrac{1}{2};\mathrm{p},-\tfrac{1}{2}\rangle\\
&+|\,\mathrm{e},+\tfrac{1}{2};\mathrm{n},-\tfrac{1}{2};\mathrm{p},+\tfrac{1}{2}\rangle + |\,\mathrm{e},-\tfrac{1}{2};\mathrm{n},+\tfrac{1}{2};\mathrm{p},+\tfrac{1}{2}\rangle\}.
\end{aligned} \tag{18.47}$$

Table 18–4

Angular momentum states for a deuterium atom

$m = \frac{3}{2}$ $\lvert e,+\frac{1}{2}; d,+1\rangle = \lvert e,+\frac{1}{2}; n,+\frac{1}{2}; p,+\frac{1}{2}\rangle$
$m = \frac{1}{2}$ $\lvert e,+\frac{1}{2}; d,0\rangle = \frac{1}{\sqrt{2}}\{\lvert e,+\frac{1}{2}; n,+\frac{1}{2}; p,-\frac{1}{2}\rangle + \lvert e,+\frac{1}{2}; n,-\frac{1}{2}; p,+\frac{1}{2}\rangle\}$ $\lvert e,-\frac{1}{2}; d,+1\rangle = \lvert e,-\frac{1}{2}; n,+\frac{1}{2}; p,+\frac{1}{2}\rangle$
$m = -\frac{1}{2}$ $\lvert e,+\frac{1}{2}; d,-1\rangle = \lvert e,+\frac{1}{2}; n,-\frac{1}{2}; p,-\frac{1}{2}\rangle$ $\lvert e,-\frac{1}{2}; d,0\rangle = \frac{1}{\sqrt{2}}\{\lvert e,-\frac{1}{2}; n,+\frac{1}{2}; p,-\frac{1}{2}\rangle + \lvert e,-\frac{1}{2}; n,-\frac{1}{2}; p,+\frac{1}{2}\rangle\}$
$m = -\frac{3}{2}$ $\lvert e,-\frac{1}{2}; d,-1\rangle = \lvert e,-\frac{1}{2}; n,-\frac{1}{2}; p,-\frac{1}{2}\rangle$

The right side can evidently be put together from the two entries in the second line of Table 18–4 by taking $\sqrt{2/3}$ of the first term with $\sqrt{1/3}$ of the second. That is, Eq. (18–47) is equivalent to

$$\lvert J = \tfrac{3}{2}, M = \tfrac{1}{2}\rangle = \sqrt{2/3}\,\lvert e,+\tfrac{1}{2}; d,0\rangle + \sqrt{1/3}\,\lvert e,-\tfrac{1}{2}; d,1\rangle. \qquad (18.48)$$

We have found our two Clebsch-Gordon coefficients α and β in Eq. (18.46):

$$\alpha = \sqrt{2/3}, \qquad \beta = \sqrt{1/3}. \qquad (18.49)$$

Following the same procedure we can find that

$$\lvert J = \tfrac{3}{2}, M = -\tfrac{1}{2}\rangle = \sqrt{1/3}\,\lvert e,+\tfrac{1}{2}; d,-1\rangle + \sqrt{2/3}\,\lvert e,-\tfrac{1}{2}; d,0\rangle. \qquad (18.50)$$

And, also, of course,

$$\lvert J = \tfrac{3}{2}, M = -\tfrac{3}{2}\rangle = \lvert e,-\tfrac{1}{2}; d,-1\rangle. \qquad (18.51)$$

These are the rules for the composition of spin 1 and spin $\frac{1}{2}$ to make a total $J = \frac{3}{2}$. We summarize (18.45), (18.48), and (18.50) in Table 18–5.

We have, however, only four states here while the system we are considering has six possible states. Of the two states in the second line of (18.42) we have used only one linear combination to form $\lvert J = \frac{3}{2}, M = +\frac{1}{2}\rangle$. There is another linear combination orthogonal to the one we have taken which also has $M = +\frac{1}{2}$, namely

$$\sqrt{1/3}\,\lvert e,+\tfrac{1}{2}; d,0\rangle - \sqrt{2/3}\,\lvert e,-\tfrac{1}{2}; d,+1\rangle. \qquad (18.52)$$

Table 18–5

The $J = \frac{3}{2}$ states of the deuterium atom

$\lvert J = \frac{3}{2}, M = +\frac{3}{2}\rangle = \lvert e,+\frac{1}{2}; d,+1\rangle$
$\lvert J = \frac{3}{2}, M = +\frac{1}{2}\rangle = \sqrt{2/3}\,\lvert e,+\frac{1}{2}; d,0\rangle + \sqrt{1/3}\,\lvert e,-\frac{1}{2}; d,1\rangle$
$\lvert J = \frac{3}{2}, M = -\frac{1}{2}\rangle = \sqrt{1/3}\,\lvert e,+\frac{1}{2}; d,-1\rangle + \sqrt{2/3}\,\lvert e,-\frac{1}{2}; d,0\rangle$
$\lvert J = \frac{3}{2}, M = -\frac{3}{2}\rangle = \lvert e,-\frac{1}{2}; d,-1\rangle$

Similarly, the two states in the third line of (18.42) can be combined to give two orthogonal states, each with $M = -\frac{1}{2}$. The one orthogonal to (18.52) is

$$\sqrt{2/3}\,|\,\mathrm{e},+\tfrac{1}{2};\,\mathrm{d},-1\rangle - \sqrt{1/3}\,|\,\mathrm{e},-\tfrac{1}{2};\,\mathrm{d},0\rangle. \tag{18.53}$$

These are the two remaining states. They have $M = m_e + m_d = \pm\frac{1}{2}$; and must be the two states corresponding to $J = \frac{1}{2}$. So we have

$$\begin{aligned} |\,J = \tfrac{1}{2}, M = \tfrac{1}{2}\rangle &= \sqrt{1/3}\,|\,\mathrm{e},+\tfrac{1}{2};\,\mathrm{d},0\rangle - \sqrt{2/3}\,|\,\mathrm{e},-\tfrac{1}{2};\,\mathrm{d},+1\rangle,\\ |\,J = \tfrac{1}{2}, M = -\tfrac{1}{2}\rangle &= \sqrt{2/3}\,|\,\mathrm{e},+\tfrac{1}{2};\,\mathrm{d},-1\rangle - \sqrt{1/3}\,|\,\mathrm{e},-\tfrac{1}{2};\,\mathrm{d},0\rangle. \end{aligned} \tag{18.54}$$

We can verify that these two states do indeed behave like the states of a spin one-half object by writing out the deuterium parts in terms of the neutron and proton states—using Table 18–3. The first state in (18.53) is

$$\begin{aligned} \sqrt{1/6}\{|\,\mathrm{e},+\tfrac{1}{2};\,\mathrm{n},+\tfrac{1}{2};\,\mathrm{p},-\tfrac{1}{2}\rangle + |\,\mathrm{e},+\tfrac{1}{2};\,\mathrm{n},-\tfrac{1}{2};\,\mathrm{p},+\tfrac{1}{2}\rangle\}\\ - \sqrt{2/3}\,|\,\mathrm{e},-\tfrac{1}{2};\,\mathrm{n},+\tfrac{1}{2};\,\mathrm{p},+\tfrac{1}{2}\rangle, \end{aligned} \tag{18.55}$$

which can also be written

$$\begin{aligned} &\sqrt{1/3}[\sqrt{1/2}\;\{|\,\mathrm{e},+\tfrac{1}{2};\,\mathrm{n},+\tfrac{1}{2};\,\mathrm{p},-\tfrac{1}{2}\rangle - |\,\mathrm{e},-\tfrac{1}{2};\,\mathrm{n},+\tfrac{1}{2};\,\mathrm{p},+\tfrac{1}{2}\rangle\}\\ &+ \sqrt{1/2}\;\{|\,\mathrm{e},+\tfrac{1}{2};\,\mathrm{n},-\tfrac{1}{2};\,\mathrm{p},+\tfrac{1}{2}\rangle - |\,\mathrm{e},-\tfrac{1}{2}:\,\mathrm{n},+\tfrac{1}{2};\,\mathrm{p},+\tfrac{1}{2}\rangle\}]. \end{aligned} \tag{18.56}$$

Now look at the terms in the first curly brackets, and think of the e and p taken together. Together they form a spin-zero state (see the bottom line of Table 18–3), and contribute no angular momentum. Only the neutron is left, so the whole of the *first* curly bracket of (18.56) behaves under rotations like a neutron, namely as a state with $J = \frac{1}{2}$, $M = +\frac{1}{2}$. Following the same reasoning, we see that in the *second* curly bracket of (18.56) the electron and neutron team up to produce zero angular momentum, and only the proton contribution—with $m_p = \frac{1}{2}$—is left. The terms behave like an object with $J = \frac{1}{2}$, $M = +\frac{1}{2}$. So the whole expression of (18.56) transforms like $|\,J = +\frac{1}{2}, M = +\frac{1}{2}\rangle$ as it should. The $M = -\frac{1}{2}$ state which corresponds to (18.57) can be written down (by changing the proper $+\frac{1}{2}$'s to $-\frac{1}{2}$'s) to get

$$\begin{aligned} &\sqrt{1/3}[\sqrt{1/2}\;\{|\,\mathrm{e},+\tfrac{1}{2};\,\mathrm{n},-\tfrac{1}{2};\,\mathrm{p},-\tfrac{1}{2}\rangle - |\,\mathrm{e},-\tfrac{1}{2};\,\mathrm{n},-\tfrac{1}{2};\,\mathrm{p},+\tfrac{1}{2}\rangle\}\\ &+ \sqrt{1/2}\;\{|\,\mathrm{e},+\tfrac{1}{2};\,\mathrm{n},-\tfrac{1}{2};\,\mathrm{p},-\tfrac{1}{2}\rangle - |\,\mathrm{e},-\tfrac{1}{2};\,\mathrm{n},+\tfrac{1}{2};\,\mathrm{p},-\tfrac{1}{2}\rangle\}]\cdot \end{aligned} \tag{18.57}$$

You can easily check that this is equal to the second line of (18.54), as it should be if the two terms of that pair are to be the two states of a spin one-half system. So our results are confirmed. A deuteron and an electron can exist in six spin states, four of which act like the states of a spin $\frac{3}{2}$ object (Table 18–5) and two of which act like an object of spin one-half (18.54).

The results of Table 18–5 and of Eq. (18.54) were obtained by making use of the fact that the deuteron is made up of a neutron and a proton. The truth of the equations does not depend on that special circumstance. For *any* spin-one object put together with any spin one-half object the composition laws (and the coefficients) are the same. The set of equations in Table 18–5 means that if the coordinates are rotated about, say, the y-axis—so that the states of the spin one-half particle and of the spin-one particle change according to Table 18–1 and Table 18–2—the linear combinations on the right-hand side will change in the proper way for a spin $\frac{3}{2}$ object. Under the same rotation the states of (18.54) will change as the states of a spin one-half object. The results depend only on the

Table 18–6

Composition of a spin one-half particle ($j_a = \frac{1}{2}$) and a spin-one particle ($j_b = 1$).

$\lvert J = \frac{3}{2}, M = \frac{3}{2}\rangle$	$=$	$\lvert a,+\frac{1}{2}; b,+1\rangle$
$\lvert J = \frac{3}{2}, M = \frac{1}{2}\rangle$	$=$	$\sqrt{2/3}\,\lvert a,+\frac{1}{2}; b,0\rangle + \sqrt{1/3}\,\lvert a,-\frac{1}{2}; b,+1\rangle$
$\lvert J = \frac{3}{2}, M = -\frac{1}{2}\rangle$	$=$	$\sqrt{1/3}\,\lvert a,+\frac{1}{2}; b,-1\rangle + \sqrt{2/3}\,\lvert a,-\frac{1}{2}; b,0\rangle$
$\lvert J = \frac{3}{2}, M = -\frac{3}{2}\rangle$	$=$	$\lvert a,-\frac{1}{2}; b,-1\rangle$
$\lvert J = \frac{1}{2}, M = +\frac{1}{2}\rangle$	$=$	$\sqrt{1/3}\,\lvert a,+\frac{1}{2}; b,0\rangle - \sqrt{2/3}\,\lvert a,-\frac{1}{2}; b,+1\rangle$
$\lvert J = \frac{1}{2}, M = -\frac{1}{2}\rangle$	$=$	$\sqrt{2/3}\,\lvert a,+\frac{1}{2}; b,-1\rangle - \sqrt{1/3}\,\lvert a,-\frac{1}{2}; b,0\rangle$

rotation properties (that is, the spin states) of the two original particles but not in any way on the origins of their angular momenta. We have only made use of this fact to work out the formulas by choosing a special case in which one of the component parts is itself made up of two spin one-half particles in a symmetric state. We have put all our results together in Table 18–6, changing the notation "e" and "d" to "*a*" and "*b*" to emphasize the generality of the conclusions.

Suppose we have the general problem of finding the states which can be formed when two objects of arbitrary spins are combined. Say one has j_a (so its z-component m_a runs over the $2j_a + 1$ values from $-j_a$ to $+j_a$) and the other has j_b (with z-component m_b running over the values from $-j_b$ to $+j_b$). The combined states are $\lvert a, m_a; b, m_a\rangle$, and there are $(2j_a + 1)(2j_b + 1)$ different ones. Now what states of total spin J can be found?

The total z-component of angular momentum M is equal to $m_a + m_b$, and the states can all be listed according to M [as in (18.42)]. The largest M is unique; it corresponds to $m_a = j_a$ and $m_b = j_b$, and is, therefore, just $j_a + j_b$. That means that the largest total spin J is also equal to the sum $j_a + j_b$:

$$J = (M)_{\max} = j_a + j_b.$$

For the first M value smaller than $(M)_{\max}$, there are two states (either m_a or m_b is one unit less than its maximum). They must contribute one state to the set that goes with $J = j_a + j_b$, and the one left over will belong to a new set with $J = j_a + j_b - 1$. The next M-value—the third from the top of the list—can be formed in *three* ways. (From $m_a = j_a - 2$, $m_b = j_b$; from $m_a = j_a - 1$, $m_b = j_b - 1$; and from $m_a = j_a$, $m_b = j_b - 2$.) Two of these belong to groups already started above; the third tells us that states of $J = j_a + j_b - 2$ must also be included. This argument continues until we reach a stage where in our list we can no longer go one more step down in one of the m's to make new states.

Let j_b be the smaller of j_a and j_b (if they are equal take either one); then only $2j_b$ values of J are required—going in integer steps from $j_a + j_b$ down to $j_a - j_b$. That is, when two objects of spin j_a and j_b are combined, the system can have a total angular momentum J equal to any one of the values

$$J = \begin{cases} j_a + j_b \\ j_a + j_b - 1 \\ j_a + j_b - 2 \\ \vdots \\ \lvert j_a - j_b\rvert. \end{cases} \tag{18.58}$$

(By writing $\lvert j_a - j_b\rvert$ instead of $j_a - j_b$ we can avoid the extra admonition that $j_a \geq j_b$.)

For *each* of these J values there are the $2J + 1$ states of different M-values—with M going from $+J$ to $-J$. Each of these is formed from linear combinations of the original states $\lvert a, m_a; b, m_b\rangle$ with appropriate factors—the Clebsch-Gordon coefficients for each particular term. We can consider that these coefficients give the "amount" of the state $|j_a, m_a; j_b, m_b\rangle$ which appears in the state $|J, M\rangle$. So each of the Clebsch-Gordon coefficients has, if you wish, *six* indices identifying its position in the formulas like those of Tables 18–3 and 18–6. That is, calling these coefficients $C(J, M; j_a, m_a; j_b, m_b)$, we could express the equality of the second line of Table 18–6 by writing

$$C(\tfrac{3}{2},+\tfrac{1}{2};\tfrac{1}{2},+\tfrac{1}{2};1,0) = \sqrt{2/3},$$
$$C(\tfrac{3}{2},+\tfrac{1}{2};\tfrac{1}{2},-\tfrac{1}{2};1,+1) = \sqrt{1/3}.$$

We will not calculate here the coefficients for any other special cases.† You can, however, find tables in many books. You might wish to try another special case for yourself. The next one to do would be the composition of two spin-one particles. We give just the final result in Table 18–7.

These laws of the composition of angular momenta are very important in particle physics—where they have innumerable applications. Unfortunately, we have no time to look at more examples here.

Table 18–7

Composition of two spin-one particles ($j_a = 1, j_b = 1$)

$\lvert J = 2, M = +2\rangle = \lvert a,+1; b,+1\rangle$
$\lvert J = 2, M = +1\rangle = \frac{1}{\sqrt{2}}\lvert a,+1; b,0\rangle + \frac{1}{\sqrt{2}}\lvert a,0; b,+1\rangle$
$\lvert J = 2, M = 0\rangle = \frac{1}{\sqrt{6}}\lvert a,+1; b,-1\rangle + \frac{1}{\sqrt{6}}\lvert a,-1; b,+1\rangle + \frac{2}{\sqrt{6}}\lvert a,0; b,0\rangle$
$\lvert J = 2, M = -1\rangle = \frac{1}{\sqrt{2}}\lvert a,0; b,-1\rangle + \frac{1}{\sqrt{2}}\lvert a,-1; b,0\rangle$
$\lvert J = 2, M = -2\rangle = \lvert a,-1; b,-1\rangle$
$\lvert J = 1, M = +1\rangle = \frac{1}{\sqrt{2}}\lvert a,+1; b,0\rangle - \frac{1}{\sqrt{2}}\lvert a,0; b,+1\rangle$
$\lvert J = 1, M = 0\rangle = \frac{1}{\sqrt{2}}\lvert a,+1; b,-1\rangle - \frac{1}{\sqrt{2}}\lvert a,-1; b,+1\rangle$
$\lvert J = 1, M = -1\rangle = \frac{1}{\sqrt{2}}\lvert a,0; b,-1\rangle - \frac{1}{\sqrt{2}}\lvert a,-1; b,0\rangle$
$\lvert J = 0, M = 0\rangle = \frac{1}{\sqrt{3}}\{\lvert a,+1; b,-1\rangle + \lvert a,-1; b,+1\rangle - \lvert a,0; b,0\rangle\}$

Added Note 1: Derivation of the rotation matrix‡

For those who would like to see the details, we work out here the general rotation matrix for a system with spin (total angular momentum) j. It is really not very important to work out the general case; once you have the idea, you can find the general results in tables in many books. On the other hand, after coming this far you might like to see that you can indeed understand even the very complicated formulas of quantum mechanics, such as Eq. (18.35), that come into the description of angular momentum.

† A large part of the work is done now that we have the general rotation matrix Eq. (18.35).

‡ The material of this appendix was originally included in the body of the lecture. We now feel that it is unnecessary to include such a detailed treatment of the general case.

We extend the arguments of Section 18–4 to a system with spin j, which we consider to be made up of $2j$ spin one-half objects. The state with $m = j$ would be $| + + + \cdots + \rangle$ (with j plus signs). For $m = j - 1$, there will be $2j$ terms like $| + + \cdots + + - \rangle$, $| + + \cdots + - + \rangle$, and so on. Let's consider the general case in which there are r plusses and s minuses—with $r + s = 2j$. Under a rotation about the z-axis each of the r plusses will contribute $e^{+i\phi/2}$. The result is a phase change of $i(r/2 - s/2)\phi$. You see that

$$m = \frac{r - s}{2}. \tag{18.59}$$

Just as for $J = \frac{3}{2}$, each state of definite m must be the linear combination with plus signs of all the states with the same r and s—that is, states corresponding to every possible arrangement which has r plusses and s minuses. We assume that you can figure out that there are $(r + s)!/r!s!$ such arrangements. To normalize each state, we should divide the sum by the square root of this number. We can write

$$\left[\frac{(r+s)!}{r!s!}\right]^{-1/2} \{|\underbrace{+ + + \cdots + +}_{r}\underbrace{- - - \cdots - -}_{s}\rangle$$

$$+ \text{ (all rearrangements of order)}\} = | j, m\rangle \tag{18.60}$$

with

$$j = \frac{r + s}{2}, \qquad m = \frac{r - s}{2}. \tag{18.61}$$

It will help our work if we now go to still another notation. Once we have defined the states by Eq. (18.60), the two numbers r and s define a state just as well as j and m. It will help us keep track of things if we write

$$| j, m\rangle = | {}^{r}_{s}\rangle, \tag{18.62}$$

where, using the equalities of (18.67)

$$r = j + m, \qquad s = j - m.$$

Next, we would like to write Eq. (18.60) with a new *special notation* as

$$| j, m\rangle = | {}^{r}_{s}\rangle = \left[\frac{(r+s)!}{r!s!}\right]^{+1/2} \{| +\rangle^{r} | -\rangle^{s}\}_{\text{perm}}. \tag{18.63}$$

Note that we have changed the exponent of the factor in front to *plus* $\frac{1}{2}$. We do that because there are just $N = (r + s)!/r!s!$ terms inside the curly brackets. Comparing (18.63) with (18.60) it is clear that

$$\{| +\rangle^{r} | -\rangle^{s}\}_{\text{perm}}$$

is just a shorthand way of writing

$$\frac{\{| + + \cdots - - \rangle + \text{all rearrangements}\}}{N},$$

where N is the number of different terms in the bracket. The reason that this notation is convenient is that each time we make a rotation, all of the plus signs contribute the same factor, so we get this factor to the rth power. Similarly, all together the s minus terms contribute a factor to the sth power no matter what the sequence of the terms is.

Now suppose we rotate our system by the angle θ about the y-axis. What we want is $R_y(\theta) | {}^{r}_{s}\rangle$. When $R_y(\theta)$ operates on each $| +\rangle$ it gives

$$R_y(\theta) | +\rangle = | +\rangle C + | -\rangle S, \tag{18.64}$$

where $C = \cos\theta/2$ and $S = \sin\theta/2$. When $R_y(\theta)$ operates on each $| -\rangle$ it gives

$$R_y(\theta) | -\rangle = | -\rangle C - | +\rangle S.$$

So what we want is

$$
\begin{aligned}
R_y(\theta)\,|\,{}^r_s\rangle &= \left[\frac{(r+s)!}{r!s!}\right]^{1/2} R_y(\theta)\{|+\rangle^r\,|-\rangle^s\}_{\text{perm}} \\
&= \left[\frac{(r+s)!}{r!s!}\right]^{1/2} \{(R_y(\theta)\,|+\rangle)^r(R_y(\theta)\,|-\rangle)^s\}_{\text{perm}} \\
&= \left[\frac{(r+s)!}{r!s!}\right]^{1/2} \{|+\rangle C + |-\rangle S)^r(|-\rangle C - |+\rangle S^s\}_{\text{perm}}. \qquad (18.65)
\end{aligned}
$$

Now each binomial has to be expanded out to its appropriate power and the two expressions multiplied together. There will be terms with $|+\rangle$ to all powers from zero to $(r+s)$. Let's look at all of the terms which have $|+\rangle$ to the r' power. They will appear always multiplied with $|-\rangle$ to the s' power, where $s' = 2j - r'$. Suppose we collect all such terms. For each permutation they will have some numerical coefficient involving the factors of the binomial expansion as well as the factors C and S. Suppose we call that factor $A_{r'}$. Then Eq. (18.65) will look like

$$
R_y(\theta)\,|\,{}^r_s\rangle = \sum_{r'=0}^{r+s} \{A_{r'}\,|+\rangle^{r'}\,|-\rangle^{s'}\}_{\text{perm}}. \qquad (18.66)
$$

Now let's say that we divide $A_{r'}$ by the factor $[(r'+s')!/r'!s'!]^{1/2}$ and call the quotient $B_{r'}$. Equation (18.66) is then equivalent to

$$
R_y(\theta)\,|\,{}^r_s\rangle = \sum_{r'=0}^{r+s} B_{r'}\left[\frac{r'+s'}{r'!s'!}\right]^{1/2} \{|+\rangle^{r'}\,|-\rangle^{s'}\}_{\text{perm}}. \qquad (18.67)
$$

(We could just say that this equation defines $B_{r'}$ by the requirement that (18.67) gives the same expression that appears in (18.65).)

With this definition of $B_{r'}$ the remaining factors on the right-hand side of Eq. (18.67) are just the states $|\,{}^{r'}_{s'}\rangle$. So we have that

$$
R_y(\theta)\,|\,{}^r_s\rangle = \sum_{r'=0}^{r+s} B_{r'}\,|\,{}^{r'}_{s'}\rangle, \qquad (18.68)
$$

with s' always equal to $r + s - r'$. This means, of course, that the coefficients $B_{r'}$ are just the matrix elements we want, namely

$$
\langle\,{}^{r'}_{s'}\,|\,R_y(\theta)\,|\,{}^r_s\rangle = B_{r'}. \qquad (18.69)
$$

Now we just have to push through the algebra to find the various $B_{r'}$. Comparing (18.39) with (18.37)—and remembering that $r' + s' = r + s$—we see that $B_{r'}$ is just the coefficient of $a^{r'}b^{s'}$ in the following expression:

$$
\left(\frac{r'!s'!}{r!s!}\right)^{1/2} (aC + bS)^r(bC - aS)^s. \qquad (18.70)
$$

It is now only a dirty job to make the expansions by the binomial theorem, and collect the terms with the given power of a and b. If you work it all out, you find that the coefficient of $a^{r'}b^{s'}$ in (18.70) is

$$
\left[\frac{r'!s'!}{r!s!}\right]^{1/2} \sum_k (-1)^k S^{r-r'+2k} C^{s+r'-2k} \cdot \frac{r!}{(r-r'+k)!(r'-k)!} \cdot \frac{s!}{(s-k)!k!}. \qquad (18.71)
$$

The sum is to be taken over all integers k which give terms of zero or greater in the factorials. This expression is then the matrix element we wanted.

Finally, we can return to our original notation in terms of j, m, and m' using

$$
r = j + m, \qquad r' = j + m', \qquad s = j - m, \qquad s' = j - m'.
$$

Making these substitutions, we get Eq. (18.34) in Section 18–4.

Added Note 2: Conservation of parity in photon emission

In Section 1 of this chapter we considered the emission of light by an atom that goes from an excited state of spin 1 to a ground state of spin 0. If the excited state has its spin up ($m = +1$), it can emit a RHC photon along the $+z$-axis or a LHC photon along the $-z$-axis. Let's call these two states of the photon $|\,R_{\text{up}}\rangle$ and $|\,L_{\text{dn}}\rangle$. Neither of these states has a definite parity. Letting $\hat{P}$ be the parity operator, $\hat{P}\,|\,R_{\text{up}}\rangle = |\,L_{\text{dn}}\rangle$ and $\hat{P}\,|\,L_{\text{dn}}\rangle = |\,R_{\text{up}}\rangle$.

What about our earlier proof that an atom in a state of definite energy must have a definite parity, and our statement that parity is conserved in atomic processes? Shouldn't the final state in this problem (the state after the emission of a photon) have a definite parity? It *does* if we consider the *complete* final state which contains amplitudes for the emission photons into all sorts of angles. In Section 1 we chose to consider only a part of the complete final state.

If we wish we can look only at final states that do have a definite parity. For example, consider a final state $|\,\psi_F\rangle$ which has some amplitude α to be a RHC photon going along $+z$ and some amplitude β to be a LHC photon going along $-z$. We can write

$$|\,\psi_F\rangle = \alpha\,|\,R_{\text{up}}\rangle + \beta\,|\,L_{\text{dn}}\rangle. \tag{18.72}$$

The parity operation on this state gives

$$\hat{P}\,|\,\psi_F\rangle = \alpha\,|\,L_{\text{dn}}\rangle + \beta\,|\,R_{\text{up}}\rangle. \tag{18.73}$$

This state will be $\pm\,|\,\psi_F\rangle$ if $\beta = \alpha$ or if $\beta = -\alpha$. So a final state of even parity is

$$|\,\psi_F^+\rangle = \alpha\,\{R_{\text{up}}\rangle + |\,L_{\text{dn}}\rangle\}, \tag{18.74}$$

and a state of odd parity is

$$|\,\psi_F^-\rangle = \alpha\,\{|\,R_{\text{up}}\rangle - |\,L_{\text{dn}}\rangle\}. \tag{18.75}$$

Next, we wish to consider the decay of an excited state of odd parity to a ground state of even parity. If parity is to be conserved, the final state of the photon must have odd parity. It must be the state in (18.75). If the amplitude to find $|\,R_{\text{up}}\rangle$ is α, the amplitude to find $|\,L_{\text{dn}}\rangle$ is $-\alpha$.

Now notice what happens when we perform a rotation of 180° about the y-axis. The initial excited state of the atom becomes an $m = -1$ state (with no change in sign, according to Table 17-2). And the rotation of the final state gives

$$R_y(180^\circ)\,|\,\psi_F^-\rangle = \alpha\,\{|\,R_{\text{dn}}\rangle - |\,L_{\text{up}}\rangle\}. \tag{18.76}$$

Comparing this equation with (18.75), you see that for the assumed parity of the final state, the amplitude to get a LHC photon along $+z$ from the $m = -1$ initial state is the negative of the amplitude to get a RHC photon from the $m = +1$ initial state. This agrees with the result we found in Section 1.