

Introduction to Superstring Theory

John H. Schwarz¹

*California Institute of Technology
Pasadena, CA 91125, USA*

Abstract

These four lectures, addressed to an audience of graduate students in experimental high energy physics, survey some of the basic concepts in string theory. The purpose is to convey a general sense of what string theory is and what it has achieved. Since the characteristic scale of string theory is expected to be close to the Planck scale, the structure of strings probably cannot be probed directly in accelerator experiments. The most accessible experimental implication of superstring theory is supersymmetry at or below the TeV scale.

*Lectures presented at the NATO Advanced Study Institute
on Techniques and Concepts of High Energy Physics
St. Croix, Virgin Islands — June 2000*

¹Work supported in part by the U.S. Dept. of Energy under Grant No. DE-FG03-92-ER40701.

Contents

1	Lecture 1: Overview and Motivation	3
1.1	Supersymmetry	4
1.2	Basic Ideas of String Theory	5
1.3	A Brief History of String Theory	6
1.4	Compactification	7
1.5	Perturbation Theory	8
1.6	The Second Superstring Revolution	8
1.7	The Origins of Gauge Symmetry	10
1.8	Conclusion	11
2	Lecture 2: String Theory Basics	12
2.1	World-Line Description of a Point Particle	12
2.2	World-Volume Actions	13
2.3	Boundary Conditions	15
2.4	Quantization	17
2.5	The Free String Spectrum	18
2.6	The Number of Physical States	21
2.7	The Structure of String Perturbation Theory	22
2.8	Recapitulation	23
3	Lecture 3: Superstrings	23
3.1	The Gauge-Fixed Theory	24
3.2	The R and NS Sectors	25
3.3	The GSO Projection	27
3.4	Type II Superstrings	29
3.5	Anomalies	30
3.6	Heterotic Strings	31
3.7	T Duality	31
4	Lecture 4: From Superstrings to M Theory	32
4.1	M Theory	34
4.2	Type II p -branes	36
4.3	Type IIB Superstring Theory	37
4.4	The D3-Brane and $\mathcal{N} = 4$ Gauge Theory	39
4.5	Conclusion	40

Introduction

Tom Ferbel has presented me with a large challenge: explain string theory to an audience of graduate students in experimental high energy physics. The allotted time is four 75-minute lectures. This should be possible, if the goals are realistic. One goal is to give a general sense of what the subject is about, and why so many theoretical physicists are enthusiastic about it. Perhaps you should regard these lectures as a cultural experience providing a window into the world of abstract theoretical physics. Don't worry if you miss some of the technical details in the second and third lectures. There is only one message in these lectures that is important for experimental research: low-energy supersymmetry is very well motivated theoretically, and it warrants the intense effort that is being made to devise ways of observing it. There are other facts that are nice to know, however. For example, consistency of quantum theory and gravity is a severe restriction, with farreaching consequences.

As will be explained, string theory requires supersymmetry, and therefore string theorists were among the first to discover it. Supersymmetric string theories are called superstring theories. At one time there seemed to be five distinct superstring theories, but it was eventually realized that each of them is actually a special limiting case of a completely unique underlying theory. This theory is not yet fully formulated, and when it is, we might decide that a new name is appropriate. Be that as it may, it is clear that we are exploring an extraordinarily rich structure with many deep connections to various branches of fundamental mathematics and theoretical physics. Whatever the ultimate status of this theory may be, it is clear that these studies have already been a richly rewarding experience.

To fully appreciate the mathematical edifice underlying superstring theory requires an investment of time and effort. Many theorists who make this investment really become hooked by it, and then there is no turning back. Well, hooking you in this way is not my goal, since you are engaged in other important activities; but hopefully these lectures will convey an idea of why many theorists find the subject so enticing. For those who wish to study the subject in more detail, there are two standard textbook presentations [1, 2].

The plan of these lectures is as follows: The first lecture will consist of a general non-technical overview of the subject. It is essentially the current version of my physics colloquium lecture. It will describe some of the basic concepts and issues without technical details. If successful, it will get you sufficiently interested in the subject that you are willing to sit through some of the basic nitty-gritty analysis that explains what we mean by a relativistic string, and how its normal modes are analyzed. Lecture 2 will present the analysis for the bosonic string theory. This is an unrealistic theory, with bosons only, but its study is a pedagogically useful first step. It involves many, but not all, of the issues that arise

for superstrings. In lecture 3 the extension to incorporate fermions and supersymmetry is described. There are two basic formalisms for doing this (called RNS and GS). Due to time limitations, only the first of these will be presented here. The final lecture will survey some of the more recent developments in the field. These include various nonperturbative dualities, the existence of an 11-dimensional limit (called M-theory) and the existence of extended objects of various dimensionalities, called p -branes. As will be explained, a particular class of p -branes, called D-branes, plays an especially important role in modern research.

1 Lecture 1: Overview and Motivation

Many of the major developments in fundamental physics of the past century arose from identifying and overcoming contradictions between existing ideas. For example, the incompatibility of Maxwell's equations and Galilean invariance led Einstein to propose the special theory of relativity. Similarly, the inconsistency of special relativity with Newtonian gravity led him to develop the general theory of relativity. More recently, the reconciliation of special relativity with quantum mechanics led to the development of quantum field theory. We are now facing another crisis of the same character. Namely, general relativity appears to be incompatible with quantum field theory. Any straightforward attempt to "quantize" general relativity leads to a nonrenormalizable theory. In my opinion, this means that the theory is inconsistent and needs to be modified at short distances or high energies. The way that string theory does this is to give up one of the basic assumptions of quantum field theory, the assumption that elementary particles are mathematical points, and instead to develop a quantum field theory of one-dimensional extended objects, called strings. There are very few consistent theories of this type, but superstring theory shows great promise as a unified quantum theory of all fundamental forces including gravity. There is no realistic string theory of elementary particles that could serve as a new standard model, since there is much that is not yet understood. But that, together with a deeper understanding of cosmology, is the goal. This is still a work in progress.

Even though string theory is not yet fully formulated, and we cannot yet give a detailed description of how the standard model of elementary particles should emerge at low energies, there are some general features of the theory that can be identified. These are features that seem to be quite generic irrespective of how various details are resolved. The first, and perhaps most important, is that general relativity is necessarily incorporated in the theory. It gets modified at very short distances/high energies but at ordinary distance and energies it is present in exactly the form proposed by Einstein. This is significant, because it is arising within the framework of a consistent quantum theory. Ordinary quantum field theory does

not allow gravity to exist; string theory requires it! The second general fact is that Yang–Mills gauge theories of the sort that comprise the standard model naturally arise in string theory. We do not understand why the specific $SU(3) \times SU(2) \times U(1)$ gauge theory of the standard model should be preferred, but (anomaly-free) theories of this general type do arise naturally at ordinary energies. The third general feature of string theory solutions is supersymmetry. The mathematical consistency of string theory depends crucially on supersymmetry, and it is very hard to find consistent solutions (quantum vacua) that do not preserve at least a portion of this supersymmetry. This prediction of string theory differs from the other two (general relativity and gauge theories) in that it really is a prediction. It is a generic feature of string theory that has not yet been discovered experimentally.

1.1 Supersymmetry

Even though supersymmetry is a very important part of the story, the discussion here will be very brief, since it will be discussed in detail by other lecturers. There will only be a few general remarks. First, as we have just said, supersymmetry is the major prediction of string theory that could appear at accessible energies, that has not yet been discovered. A variety of arguments, not specific to string theory, suggest that the characteristic energy scale associated to supersymmetry breaking should be related to the electroweak scale, in other words in the range 100 GeV – 1 TeV. The symmetry implies that all known elementary particles should have partner particles, whose masses are in this general range. This means that some of these superpartners should be observable at the CERN Large Hadron Collider (LHC), which will begin operating in the middle part of this decade. There is even a chance that Fermilab Tevatron experiments could find superparticles earlier than that.

In most versions of phenomenological supersymmetry there is a multiplicatively conserved quantum number called R-parity. All known particles have even R-parity, whereas their superpartners have odd R-parity. This implies that the superparticles must be pair-produced in particle collisions. It also implies that the lightest supersymmetry particle (or LSP) should be absolutely stable. It is not known with certainty which particle is the LSP, but one popular guess is that it is a “neutralino.” This is an electrically neutral fermion that is a quantum-mechanical mixture of the partners of the photon, Z^0 , and neutral Higgs particles. Such an LSP would interact very weakly, more or less like a neutrino. It is of considerable interest, since it is an excellent dark matter candidate. Searches for dark matter particles called WIMPS (weakly interacting massive particles) could discover the LSP some day. Current experiments might not have sufficient detector volume to compensate for the exceedingly small cross sections.

There are three unrelated arguments that point to the same mass range for superparticles.

The one we have just been discussing, a neutralino LSP as an important component of dark matter, requires a mass of order 100 GeV. The precise number depends on the mixture that comprises the LSP, what their density is, and a number of other details. A second argument is based on the famous hierarchy problem. This is the fact that standard model radiative corrections tend to renormalize the Higgs mass to a very high scale. The way to prevent this is to extend the standard model to a supersymmetric standard model and to have the supersymmetry be broken at a scale comparable to the Higgs mass, and hence to the electroweak scale. The third argument that gives an estimate of the susy-breaking scale is grand unification. If one accepts the notion that the standard model gauge group is embedded in a larger gauge group such as $SU(5)$ or $SO(10)$, which is broken at a high mass scale, then the three standard model coupling constants should unify at that mass scale. Given the spectrum of particles, one can compute the evolution of the couplings as a function of energy using renormalization group equations. One finds that if one only includes the standard model particles this unification fails quite badly. However, if one also includes all the supersymmetry particles required by the minimal supersymmetric extension of the standard model, then the couplings do unify at an energy of about 2×10^{16} GeV. For this agreement to take place, it is necessary that the masses of the superparticles are less than a few TeV.

There is other support for this picture, such as the ease with which supersymmetric grand unification explains the masses of the top and bottom quarks and electroweak symmetry breaking. Despite all these indications, we cannot be certain that this picture is correct until it is demonstrated experimentally. One could suppose that all this is a giant coincidence, and the correct description of TeV scale physics is based on something entirely different. The only way we can decide for sure is by doing the experiments. As I once told a newspaper reporter, in order to be sure to be quoted: discovery of supersymmetry would be more profound than life on Mars.

1.2 Basic Ideas of String Theory

In conventional quantum field theory the elementary particles are mathematical points, whereas in perturbative string theory the fundamental objects are one-dimensional loops (of zero thickness). Strings have a characteristic length scale, which can be estimated by dimensional analysis. Since string theory is a relativistic quantum theory that includes gravity it must involve the fundamental constants c (the speed of light), \hbar (Planck's constant divided by 2π), and G (Newton's gravitational constant). From these one can form a length,

known as the Planck length

$$\ell_p = \left(\frac{\hbar G}{c^3} \right)^{3/2} = 1.6 \times 10^{-33} \text{ cm}. \quad (1)$$

Similarly, the Planck mass is

$$m_p = \left(\frac{\hbar c}{G} \right)^{1/2} = 1.2 \times 10^{19} \text{ GeV}/c^2. \quad (2)$$

Experiments at energies far below the Planck energy cannot resolve distances as short as the Planck length. Thus, at such energies, strings can be accurately approximated by point particles. From the viewpoint of string theory, this explains why quantum field theory has been so successful.

As a string evolves in time it sweeps out a two-dimensional surface in spacetime, which is called the world sheet of the string. This is the string counterpart of the world line for a point particle. In quantum field theory, analyzed in perturbation theory, contributions to amplitudes are associated to Feynman diagrams, which depict possible configurations of world lines. In particular, interactions correspond to junctions of world lines. Similarly, string theory perturbation theory involves string world sheets of various topologies. A particularly significant fact is that these world sheets are generically smooth. The existence of interaction is a consequence of world-sheet topology rather than a local singularity on the world sheet. This difference from point-particle theories has two important implications. First, in string theory the structure of interactions is uniquely determined by the free theory. There are no arbitrary interactions to be chosen. Second, the ultraviolet divergences of point-particle theories can be traced to the fact that interactions are associated to world-line junctions at specific spacetime points. Because the string world sheet is smooth, string theory amplitudes have no ultraviolet divergences.

1.3 A Brief History of String Theory

String theory arose in the late 1960's out of an attempt to describe the strong nuclear force. The inclusion of fermions led to the discovery of supersymmetric strings — or superstrings — in 1971. The subject fell out of favor around 1973 with the development of QCD, which was quickly recognized to be the correct theory of strong interactions. Also, string theories had various unrealistic features such as extra dimensions and massless particles, neither of which are appropriate for a hadron theory.

Among the massless string states there is one that has spin two. In 1974, it was shown by Scherk and me [3], and independently by Yoneya [4], that this particle interacts like a

graviton, so the theory actually includes general relativity. This led us to propose that string theory should be used for unification rather than for hadrons. This implied, in particular, that the string length scale should be comparable to the Planck length, rather than the size of hadrons (10^{-13} cm) as we had previously assumed.

In the “first superstring revolution,” which took place in 1984–85, there were a number of important developments (described later) that convinced a large segment of the theoretical physics community that this is a worthy area of research. By the time the dust settled in 1985 we had learned that there are five distinct consistent string theories, and that each of them requires spacetime supersymmetry in the ten dimensions (nine spatial dimensions plus time). The theories, which will be described later, are called type I, type IIA, type IIB, $SO(32)$ heterotic, and $E_8 \times E_8$ heterotic.

1.4 Compactification

In the context of the original goal of string theory – to explain hadron physics – extra dimensions are unacceptable. However, in a theory that incorporates general relativity, the geometry of spacetime is determined dynamically. Thus one could imagine that the theory admits consistent quantum solutions in which the six extra spatial dimensions form a compact space, too small to have been observed. The natural first guess is that the size of this space should be comparable to the string scale and the Planck length. Since the equations must be satisfied, the geometry of this six-dimensional space is not arbitrary. A particularly appealing possibility, which is consistent with the equations, is that it forms a type of space called a Calabi–Yau space [5].

Calabi–Yau compactification, in the context of the $E_8 \times E_8$ heterotic string theory, can give a low-energy effective theory that closely resembles a supersymmetric extension of the standard model. There is actually a lot of freedom, because there are very many different Calabi–Yau spaces, and there are other arbitrary choices that can be made. Still, it is interesting that one can come quite close to realistic physics. It is also interesting that the number of quark and lepton families that one obtains is determined by the topology of the Calabi–Yau space. Thus, for suitable choices, one can arrange to end up with exactly three families. People were very excited by the picture in 1985. Nowadays, we tend to make a more sober appraisal that emphasizes all the arbitrariness that is involved, and the things that don’t work exactly right. Still, it would not be surprising if some aspects of this picture survive as part of the story when we understand the right way to describe the real world.

1.5 Perturbation Theory

Until 1995 it was only understood how to formulate string theories in terms of perturbation expansions. Perturbation theory is useful in a quantum theory that has a small dimensionless coupling constant, such as quantum electrodynamics, since it allows one to compute physical quantities as power series expansions in the small parameter. In QED the small parameter is the fine-structure constant $\alpha \sim 1/137$. Since this is quite small, perturbation theory works very well for QED. For a physical quantity $T(\alpha)$, one computes (using Feynman diagrams)

$$T(\alpha) = T_0 + \alpha T_1 + \alpha^2 T_2 + \dots \quad (3)$$

It is the case generically in quantum field theory that expansions of this type are divergent. More specifically, they are asymptotic expansions with zero radius convergence. Nonetheless, they can be numerically useful if the expansion parameter is small. The problem is that there are various non-perturbative contributions (such as instantons) that have the structure

$$T_{NP} \sim e^{-(const./\alpha)}. \quad (4)$$

In a theory such as QCD, there are regimes where perturbation theory is useful (due to asymptotic freedom) and other regimes where it is not. For problems of the latter type, such as computing the hadron spectrum, nonperturbative methods of computation, such as lattice gauge theory, are required.

In the case of string theory the dimensionless string coupling constant, denoted g_s , is determined dynamically by the expectation value of a scalar field called the dilaton. There is no particular reason that this number should be small. So it is unlikely that a realistic vacuum could be analyzed accurately using perturbation theory. More importantly, these theories have many qualitative properties that are inherently nonperturbative. So one needs nonperturbative methods to understand them.

1.6 The Second Superstring Revolution

Around 1995 some amazing and unexpected “dualities” were discovered that provided the first glimpses into nonperturbative features of string theory. These dualities were quickly recognized to have three major implications.

The dualities enabled us to relate all five of the superstring theories to one another. This meant that, in a fundamental sense, they are all equivalent to one another. Another way of saying this is that there is a unique underlying theory, and what we had been calling five theories are better viewed as perturbation expansions of this underlying theory about five different points (in the space of consistent quantum vacua). This was a profoundly satisfying

realization, since we really didn't want five theories of nature. That there is a completely unique theory, without any dimensionless parameters, is the best outcome one could have hoped for. To avoid confusion, it should be emphasized that even though the theory is unique, it is entirely possible that there are many consistent quantum vacua. Classically, the corresponding statement is that a unique equation can admit many solutions. It is a particular solution (or quantum vacuum) that ultimately must describe nature. At least, this is how a particle physicist would say it. If we hope to understand the origin and evolution of the universe, in addition to properties of elementary particles, it would be nice if we could also understand cosmological solutions.

A second crucial discovery was that the theory admits a variety of nonperturbative excitations, called p -branes, in addition to the fundamental strings. The letter p labels the number of spatial dimensions of the excitation. Thus, in this language, a point particle is a 0-brane, a string is a 1-brane, and so forth. The reason that p -branes were not discovered in perturbation theory is that they have tension (or energy density) that diverges as $g_s \rightarrow 0$. Thus they are absent from the perturbative theory.

The third major discovery was that the underlying theory also has an eleven-dimensional solution, which is called M-theory. Later, we will explain how the eleventh dimension arises.

One type of duality is called S duality. (The choice of the letter S is a historical accident of no great significance.) Two string theories (let's call them A and B) are related by S duality if one of them evaluated at strong coupling is equivalent to the other one evaluated at weak coupling. Specifically, for any physical quantity f , one has

$$f_A(g_s) = f_B(1/g_s). \quad (5)$$

Two of the superstring theories — type I and $SO(32)$ heterotic — are related by S duality in this way. The type IIB theory is self-dual. Thus S duality is a symmetry of the IIB theory, and this symmetry is unbroken if $g_s = 1$. Thanks to S duality, the strong-coupling behavior of each of these three theories is determined by a weak-coupling analysis. The remaining two theories, type IIA and $E_8 \times E_8$ heterotic, behave very differently at strong coupling. They grow an eleventh dimension!

Another astonishing duality, which goes by the name of T duality, was discovered several years earlier. It can be understood in perturbation theory, which is why it was found first. But, fortunately, it often continues to be valid even at strong coupling. T duality can relate different compactifications of different theories. For example, suppose theory A has a compact dimension that is a circle of radius R_A and theory B has a compact dimension that is a circle of radius R_B . If these two theories are related by T duality this means that they

are equivalent provided that

$$R_A R_B = (\ell_s)^2, \tag{6}$$

where ℓ_s is the fundamental string length scale. This has the amazing implication that when one of the circles becomes small the other one becomes large. In a later lecture, we will explain how this is possible. T duality relates the two type II theories and the two heterotic theories. There are more complicated examples of the same phenomenon involving compact spaces that are more complicated than a circle, such as tori, K3, Calabi–Yau spaces, etc.

1.7 The Origins of Gauge Symmetry

There are a variety of mechanisms that can give rise to Yang–Mills type gauge symmetries in string theory. Here, we will focus on two basic possibilities: Kaluza–Klein symmetries and brane symmetries.

The basic Kaluza–Klein idea goes back to the 1920’s, though it has been much generalized since then. The idea is to suppose that the 10- or 11-dimensional geometry has a product structure $M \times K$, where M is Minkowski spacetime and K is a compact manifold. Then, if K has symmetries, these appear as gauge symmetries of the effective theory defined on M . The Yang–Mills gauge fields arise as components of the gravitational metric field with one direction along K and the other along M . For example, if the space K is an n -dimensional sphere, the symmetry group is $SO(n+1)$, if it is CP^n — which has $2n$ dimensions — it is $SU(n+1)$, and so forth. Elegant as this may be, it seems unlikely that a realistic K has any such symmetries. Calabi–Yau spaces, for example, do not have any.

A rather more promising way of achieving realistic gauge symmetries is via the brane approach. Here the idea is that a certain class of p -branes (called D-branes) have gauge fields that are restricted to their world volume. This means that the gauge fields are not defined throughout the 10- or 11-dimensional spacetime but only on the $(p+1)$ -dimensional hypersurface defined by the D-branes. This picture suggests that the world we observe might be a D-brane embedded in a higher-dimensional space. In such a scenario, there can be two kinds of extra dimensions: compact dimensions along the brane and compact dimensions perpendicular to the brane.

The traditional viewpoint, which in my opinion is still the best bet, is that all extra dimensions (of both types) have sizes of order 10^{-30} to 10^{-32} cm corresponding to an energy scale of $10^{16} - 10^{18}$ GeV. This makes them inaccessible to direct observation, though their existence would have definite low-energy consequences. However, one can and should ask “what are the experimental limits?” For compact dimensions along the brane, which support gauge fields, the nonobservation of extra dimensions in tests of the standard model implies

a bound of about 1 TeV. The LHC should extend this to about 10 TeV. For compact dimensions “perpendicular to the brane,” which only support excitations with gravitational strength forces, the best bounds come from Cavendish-type experiments, which test the $1/R^2$ structure of the Newton force law at short distances. No deviations have been observed to a distance of about 1 mm, so far. Experiments planned in the near future should extend the limit to about 100 microns. Obviously, observation of any deviation from $1/R^2$ would be a major discovery.

1.8 Conclusion

This introductory lecture has sketched some of the remarkable successes that string theory has achieved over the past 30 years. There are many others that did not fit in this brief survey. Despite all this progress, there are some very important and fundamental questions whose answers are unknown. It seems that whenever a breakthrough occurs, a host of new questions arise, and the ultimate goal still seems a long way off. To convince you that there is a long way to go, let us list some of the most important questions:

- What is the theory? Even though a great deal is known about string theory and M theory, it seems that the optimal formulation of the underlying theory has not yet been found. It might be based on principles that have not yet been formulated.
- We are convinced that supersymmetry is present at high energies and probably at the electroweak scale, too. But we do not know how or why it is broken.
- A very crucial problem concerns the energy density of the vacuum, which is a physical quantity in a gravitational theory. This is characterized by the cosmological constant, which observationally appears to have a small positive value — so that the vacuum energy of the universe is comparable to the energy in matter. In Planck units this is a tiny number ($\Lambda \sim 10^{-120}$). If supersymmetry were unbroken, we could argue that $\Lambda = 0$, but if it is broken at the 1 TeV scale, that would seem to suggest $\Lambda \sim 10^{-60}$, which is very far from the truth. Despite an enormous amount of effort and ingenuity, it is not yet clear how superstring theory will conspire to break supersymmetry at the TeV scale and still give a value for Λ that is much smaller than 10^{-60} . The fact that the desired result is about the square of this might be a useful hint.
- Even though the underlying theory is unique, there seem to be many consistent quantum vacua. We would very much like to formulate a theoretical principle (not based on observation) for choosing among these vacua. It is not known whether the right approach to the answer is cosmological, probabilistic, anthropic, or something else.

2 Lecture 2: String Theory Basics

In this lecture we will describe the world-sheet dynamics of the original bosonic string theory. As we will see this theory has various unrealistic and unsatisfactory properties. Nonetheless it is a useful preliminary before describing supersymmetric strings, because it allows us to introduce many of the key concepts without simultaneously addressing the added complications associated with fermions and supersymmetry.

We will describe string dynamics from a first-quantized point of view. This means that we focus on understanding it from a world-sheet sum-over-histories point of view. This approach is closely tied to perturbation theory analysis. It should be contrasted with “second quantized” string field theory which is based on field operators that create or destroy entire strings. Since the first-quantized point of view may be less familiar to you than second-quantized field theory, let us begin by reviewing how it can be used to describe a massive point particle.

2.1 World-Line Description of a Point Particle

A point particle sweeps out a trajectory (or world line) in spacetime. This can be described by functions $x^\mu(\tau)$ that describe how the world line, parameterized by τ , is embedded in the spacetime, whose coordinates are denoted x^μ . For simplicity, let us assume that the spacetime is flat Minkowski space with a Lorentz metric

$$\eta_{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \quad (7)$$

Then, the Lorentz invariant line element is given by

$$ds^2 = -\eta_{\mu\nu} dx^\mu dx^\nu. \quad (8)$$

In units $\hbar = c = 1$, the action for a particle of mass m is given by

$$S = -m \int ds. \quad (9)$$

This could be generalized to a curved spacetime by replacing $\eta_{\mu\nu}$ by a metric $g_{\mu\nu}(x)$, but we will not do so here. In terms of the embedding functions, $x^\mu(t)$, the action can be rewritten in the form

$$S = -m \int d\tau \sqrt{-\eta_{\mu\nu} \dot{x}^\mu \dot{x}^\nu}, \quad (10)$$

where dots represent τ derivatives. An important property of this action is invariance under local reparametrizations. This is a kind of gauge invariance, whose meaning is that the form of S is unchanged under an arbitrary reparametrization of the world line $\tau \rightarrow \tau(\tilde{\tau})$. Actually, one should require that the function $\tau(\tilde{\tau})$ is smooth and monotonic ($\frac{d\tau}{d\tilde{\tau}} > 0$). The reparametrization invariance is a one-dimensional analog of the four-dimensional general coordinate invariance of general relativity. Mathematicians refer to this kind of symmetry as diffeomorphism invariance.

The reparametrization invariance of S allows us to choose a gauge. A nice choice is the “static gauge”

$$x^0 = \tau. \quad (11)$$

In this gauge (renaming the parameter t) the action becomes

$$S = -m \int \sqrt{1 - v^2} dt, \quad (12)$$

where

$$\vec{v} = \frac{d\vec{x}}{dt}. \quad (13)$$

Requiring this action to be stationary under an arbitrary variation of $\vec{x}(t)$ gives the Euler–Lagrange equations

$$\frac{d\vec{p}}{dt} = 0, \quad (14)$$

where

$$\vec{p} = \frac{\delta S}{\delta \vec{v}} = \frac{m\vec{v}}{\sqrt{1 - v^2}}, \quad (15)$$

which is the usual result. So we see that usual relativistic kinematics follows from the action $S = -m \int ds$.

2.2 World-Volume Actions

We can now generalize the analysis of the massive point particle to a p -brane of tension T_p . The action in this case involves the invariant $(p + 1)$ -dimensional volume and is given by

$$S_p = -T_p \int d\mu_{p+1}, \quad (16)$$

where the invariant volume element is

$$d\mu_{p+1} = \sqrt{-\det(-\eta_{\mu\nu} \partial_\alpha x^\mu \partial_\beta x^\nu)} d^{p+1}\sigma. \quad (17)$$

Here the embedding of the p -brane into d -dimensional spacetime is given by functions $x^\mu(\sigma^\alpha)$. The index $\alpha = 0, \dots, p$ labels the $p + 1$ coordinates σ^α of the p -brane world-volume and the index $\mu = 0, \dots, d - 1$ labels the d coordinates x^μ of the d -dimensional spacetime. We have defined

$$\partial_\alpha x^\mu = \frac{\partial x^\mu}{\partial \sigma^\alpha}. \quad (18)$$

The determinant operation acts on the $(p + 1) \times (p + 1)$ matrix whose rows and columns are labeled by α and β . The tension T_p is interpreted as the mass per unit volume of the p -brane. For a 0-brane, it is just the mass.

Exercise: Show that S_p is reparametrization invariant. In other words, substituting $\sigma^\alpha = \sigma^\alpha(\tilde{\sigma}^\beta)$, it takes the same form when expressed in terms of the coordinates $\tilde{\sigma}^\alpha$.

Let us now specialize to the string, $p = 1$. Evaluating the determinant gives

$$S[x] = -T \int d\sigma d\tau \sqrt{\dot{x}^2 x'^2 - (\dot{x} \cdot x')^2}, \quad (19)$$

where we have defined $\sigma^0 = \tau$, $\sigma^1 = \sigma$, and

$$\dot{x}^\mu = \frac{\partial x^\mu}{\partial \tau}, \quad x'^\mu = \frac{\partial x^\mu}{\partial \sigma}. \quad (20)$$

This action, called the Nambu–Goto action, was first proposed in 1970 [6, 7]. The Nambu–Goto action is equivalent to the action

$$S[x, h] = -\frac{T}{2} \int d^2\sigma \sqrt{-h} h^{\alpha\beta} \eta_{\mu\nu} \partial_\alpha x^\mu \partial_\beta x^\nu, \quad (21)$$

where $h_{\alpha\beta}(\sigma, \tau)$ is the world-sheet metric, $h = \det h_{\alpha\beta}$, and $h^{\alpha\beta}$ is the inverse of $h_{\alpha\beta}$. The Euler–Lagrange equation obtained by varying $h^{\alpha\beta}$ are

$$T_{\alpha\beta} = \partial_\alpha x \cdot \partial_\beta x - \frac{1}{2} h_{\alpha\beta} h^{\gamma\delta} \partial_\gamma x \cdot \partial_\delta x = 0. \quad (22)$$

Exercise: Show that $T_{\alpha\beta} = 0$ can be used to eliminate the world-sheet metric from the action, and that when this is done one recovers the Nambu–Goto action. (Hint: take the determinant of both sides of the equation $\partial_\alpha x \cdot \partial_\beta x = \frac{1}{2} h_{\alpha\beta} h^{\gamma\delta} \partial_\gamma x \cdot \partial_\delta x$.)

In addition to reparametrization invariance, the action $S[x, h]$ has another local symmetry, called conformal invariance (or Weyl invariance). Specifically, it is invariant under the replacement

$$\begin{aligned} h_{\alpha\beta} &\rightarrow \Lambda(\sigma, \tau) h_{\alpha\beta} \\ x^\mu &\rightarrow x^\mu. \end{aligned} \quad (23)$$

This local symmetry is special to the $p = 1$ case (strings).

The two reparametrization invariance symmetries of $S[x, h]$ allow us to choose a gauge in which the three functions $h_{\alpha\beta}$ (this is a symmetric 2×2 matrix) are expressed in terms of just one function. A convenient choice is the “conformally flat gauge”

$$h_{\alpha\beta} = \eta_{\alpha\beta} e^{\phi(\sigma, \tau)}. \quad (24)$$

Here, $\eta_{\alpha\beta}$ denoted the two-dimensional Minkowski metric of a flat world sheet. However, because of the factor e^{ϕ} , $h_{\alpha\beta}$ is only “conformally flat.” Classically, substitution of this gauge choice into $S[x, h]$ leaves the gauge-fixed action

$$S = \frac{T}{2} \int d^2\sigma \eta^{\alpha\beta} \partial_\alpha x \cdot \partial_\beta x. \quad (25)$$

Quantum mechanically, the story is more subtle. Instead of eliminating h via its classical field equations, one should perform a Feynman path integral, using standard machinery to deal with the local symmetries and gauge fixing. When this is done correctly, one finds that in general ϕ does not decouple from the answer. Only for the special case $d = 26$ does the quantum analysis reproduce the formula we have given based on classical reasoning [8]. Otherwise, there are correction terms whose presence can be traced to a conformal anomaly (i.e., a quantum-mechanical breakdown of the conformal invariance).

The gauge-fixed action is quadratic in the x 's. Mathematically, it is the same as a theory of d free scalar fields in two dimensions. The equations of motion obtained by varying x^μ are simply free two-dimensional wave equations:

$$\ddot{x}^\mu - x''^\mu = 0. \quad (26)$$

This is not the whole story, however, because we must also take account of the constraints $T_{\alpha\beta} = 0$. Evaluated in the conformally flat gauge, these constraints are

$$\begin{aligned} T_{01} &= T_{10} = \dot{x} \cdot x' = 0 \\ T_{00} &= T_{11} = \frac{1}{2}(\dot{x}^2 + x'^2) = 0. \end{aligned} \quad (27)$$

Adding and subtracting gives

$$(\dot{x} \pm x')^2 = 0. \quad (28)$$

2.3 Boundary Conditions

To go further, one needs to choose boundary conditions. There are three important types. For a closed string one should impose periodicity in the spatial parameter σ . Choosing its range to be π (as is conventional)

$$x^\mu(\sigma, \tau) = x^\mu(\sigma + \pi, \tau). \quad (29)$$

For an open string (which has two ends), each end can be required to satisfy either Neumann or Dirichlet boundary conditions (for each value of μ).

$$\text{Neumann : } \frac{\partial x^\mu}{\partial \sigma} = 0 \quad \text{at } \sigma = 0 \text{ or } \pi \quad (30)$$

$$\text{Dirichlet : } \frac{\partial x^\mu}{\partial \tau} = 0 \quad \text{at } \sigma = 0 \text{ or } \pi. \quad (31)$$

The Dirichlet condition can be integrated, and then it specifies a spacetime location on which the string ends. The only way this makes sense is if the open string ends on a physical object – it ends on a D-brane. (D stands for Dirichlet.) If all the open-string boundary conditions are Neumann, then the ends of the string can be anywhere in the spacetime. The modern interpretation is that this means that there are spacetime-filling D-branes present.

Let us now consider the closed-string case in more detail. The general solution of the 2d wave equation is given by a sum of “right-movers” and “left-movers”:

$$x^\mu(\sigma, \tau) = x_R^\mu(\tau - \sigma) + x_L^\mu(\tau + \sigma). \quad (32)$$

These should be subject to the following additional conditions:

- $x^\mu(\sigma, \tau)$ is real
- $x^\mu(\sigma + \pi, \tau) = x^\mu(\sigma, \tau)$
- $(x'_L)^2 = (x'_R)^2 = 0$ (These are the $T_{\alpha\beta} = 0$ constraints in eq. (28).)

The first two of these conditions can be solved explicitly in terms of Fourier series:

$$\begin{aligned} x_R^\mu &= \frac{1}{2}x^\mu + \ell_s^2 p^\mu(\tau - \sigma) + \frac{i}{\sqrt{2}}\ell_s \sum_{n \neq 0} \frac{1}{n} \alpha_n^\mu e^{-2in(\tau - \sigma)} \\ x_L^\mu &= \frac{1}{2}x^\mu + \ell_s^2 p^\mu(\tau + \sigma) + \frac{i}{\sqrt{2}}\ell_s \sum_{n \neq 0} \frac{1}{n} \tilde{\alpha}_n^\mu e^{-2in(\tau + \sigma)}, \end{aligned} \quad (33)$$

where the expansion parameters $\alpha_n^\mu, \tilde{\alpha}_n^\mu$ satisfy

$$\alpha_{-n}^\mu = (\alpha_n^\mu)^\dagger, \quad \tilde{\alpha}_{-n}^\mu = (\tilde{\alpha}_n^\mu)^\dagger. \quad (34)$$

The center-of-mass coordinate x^μ and momentum p^μ are also real. The fundamental string length scale ℓ_s is related to the tension T by

$$T = \frac{1}{2\pi\alpha'}, \quad \alpha' = \ell_s^2. \quad (35)$$

The parameter α' is called the universal Regge slope, since the string modes lie on linear parallel Regge trajectories with this slope.

2.4 Quantization

The analysis of closed-string left-moving modes, closed-string right-moving modes, and open-string modes are all very similar. Therefore, to avoid repetition, we will focus on the closed-string right-movers. Starting with the gauge-fixed action in eq.(25), the canonical momentum of the string is

$$p^\mu(\sigma, \tau) = \frac{\delta S}{\delta \dot{x}^\mu} = T \dot{x}^\mu. \quad (36)$$

Canonical quantization (this is just free 2d field theory for scalar fields) gives

$$[p^\mu(\sigma, \tau), x^\nu(\sigma', \tau)] = -i\hbar \eta^{\mu\nu} \delta(\sigma - \sigma'). \quad (37)$$

In terms of the Fourier modes (setting $\hbar = 1$) these become

$$[p^\mu, x^\nu] = -i\eta^{\mu\nu} \quad (38)$$

$$\begin{aligned} [\alpha_m^\mu, \alpha_n^\nu] &= m\delta_{m+n,0}\eta^{\mu\nu}, \\ [\tilde{\alpha}_m^\mu, \tilde{\alpha}_n^\nu] &= m\delta_{m+n,0}\eta^{\mu\nu}, \end{aligned} \quad (39)$$

and all other commutators vanish.

Recall that a quantum-mechanical harmonic oscillator can be described in terms of raising and lowering operators, usually called a^\dagger and a , which satisfy

$$[a, a^\dagger] = 1. \quad (40)$$

We see that, aside from a normalization factor, the expansion coefficients α_{-m}^μ and α_m^μ are raising and lowering operators. There is just one problem. Because $\eta^{00} = -1$, the time components are proportional to oscillators with the wrong sign ($[a, a^\dagger] = -1$). This is potentially very bad, because such oscillators create states of negative norm, which could lead to an inconsistent quantum theory (with negative probabilities, etc.). Fortunately, as we will explain, the $T_{\alpha\beta} = 0$ constraints eliminate the negative-norm states from the physical spectrum.

The classical constraint for the right-moving closed-string modes, $(x'_R)^2 = 0$, has Fourier components

$$L_m = \frac{T}{2} \int_0^\pi e^{-2im\sigma} (x'_R)^2 d\sigma = \frac{1}{2} \sum_{n=-\infty}^{\infty} \alpha_{m-n} \cdot \alpha_n, \quad (41)$$

which are called Virasoro operators. Since α_m^μ does not commute with α_{-m}^μ , L_0 needs to be normal-ordered:

$$L_0 = \frac{1}{2} \alpha_0^2 + \sum_{n=1}^{\infty} \alpha_{-n} \cdot \alpha_n. \quad (42)$$

Here $\alpha_0^\mu = \ell_s p^\mu / \sqrt{2}$, where p^μ is the momentum.

2.5 The Free String Spectrum

Recall that the Hilbert space of a harmonic oscillator is spanned by states $|n\rangle$, $n = 0, 1, 2, \dots$, where the ground state, $|0\rangle$, is annihilated by the lowering operator ($a|0\rangle = 0$) and

$$|n\rangle = \frac{(a^\dagger)^n}{\sqrt{n!}}|0\rangle. \quad (43)$$

Then, for a normalized ground-state ($\langle 0|0\rangle = 1$), one can use $[a, a^\dagger] = 1$ repeatedly to prove that

$$\langle m|n\rangle = \delta_{m,n} \quad (44)$$

and

$$a^\dagger a|n\rangle = n|n\rangle. \quad (45)$$

The string spectrum (of right-movers) is given by the product of an infinite number of harmonic-oscillator Fock spaces, one for each α_n^μ , subject to the Virasoro constraints [9]

$$\begin{aligned} (L_0 - q)|\phi\rangle &= 0 \\ L_n|\phi\rangle &= 0, \quad n > 0. \end{aligned} \quad (46)$$

Here $|\phi\rangle$ denotes a physical state, and q is a constant to be determined. It accounts for the arbitrariness in the normal-ordering prescription used to define L_0 . As we will see, the L_0 equation is a generalization of the Klein–Gordon equation. It contains $p^2 = -\partial \cdot \partial$ plus oscillator terms whose eigenvalue will determine the mass of the state.

It is interesting to work out the algebra of the Virasoro operators L_m , which follows from the oscillator algebra. The result, called the Virasoro algebra, is

$$[L_m, L_n] = (m - n)L_{m+n} + \frac{c}{12}(m^3 - m)\delta_{m+n,0}. \quad (47)$$

The second term on the right-hand side is called the “conformal anomaly term” and the constant c is called the “central charge.”

Exercise: Verify the first term on the right-hand side. For extra credit, verify the second term, showing that each component of x^μ contributes $c = 1$, so that altogether $c = d$.

There are more sophisticated ways to describe the string spectrum (in terms of BRST cohomology), but they are equivalent to the more elementary approach presented here. In the BRST approach, gauge-fixing to the conformal gauge in the quantum theory requires the addition of world-sheet Faddeev-Popov ghosts, which turn out to contribute $c = -26$. Thus the total anomaly of the x^μ and the ghosts cancels for the particular choice $d = 26$, as we

asserted earlier. Moreover, it is also necessary to set the parameter $q = 1$, so that mass-shell condition becomes

$$(L_0 - 1)|\phi\rangle = 0. \quad (48)$$

Since the mathematics of the open-string spectrum is the same as that of closed-string right movers, let us now use the equations we have obtained to study the open string spectrum. (Here we are assuming that the open-string boundary conditions are all Neumann, corresponding to spacetime-filling D-branes.) The mass-shell condition is

$$M^2 = -p^2 = -\frac{1}{2}\alpha_0^2 = N - 1, \quad (49)$$

where

$$N = \sum_{n=1}^{\infty} \alpha_{-n} \cdot \alpha_n = \sum_{n=1}^{\infty} n a_n^\dagger \cdot a_n. \quad (50)$$

The a^\dagger 's and a 's are properly normalized raising and lowering operators. Since each $a^\dagger a$ has eigenvalues $0, 1, 2, \dots$, the possible values of N are also $0, 1, 2, \dots$. The unique way to realize $N = 0$ is for all the oscillators to be in the ground state, which we denote simply by $|0; p^\mu\rangle$, where p^μ is the momentum of the state. This state has $M^2 = -1$, which is a tachyon (p^μ is spacelike). Such a faster-than-light particle is certainly not possible in a consistent quantum theory, because the vacuum would be unstable. However, in perturbation theory (which is the framework we are implicitly considering) this instability is not visible. Since this string theory is only supposed to be a warm-up exercise before considering tachyon-free superstring theories, let us continue without worrying about it.

The first excited state, with $N = 1$, corresponds to $M^2 = 0$. The only way to achieve $N = 1$ is to excite the first oscillator once:

$$|\phi\rangle = \zeta_\mu \alpha_{-1}^\mu |0; p\rangle. \quad (51)$$

Here ζ_μ denotes the polarization vector of a massless spin-one particle. The Virasoro constraint condition $L_1|\phi\rangle = 0$ implies that ζ_μ must satisfy

$$p^\mu \zeta_\mu = 0. \quad (52)$$

This ensures that the spin is transversely polarized, so there are $d-2$ independent polarization states. This agrees with what one finds for a massless Maxwell or Yang-Mills field.

At the next mass level, where $N = 2$ and $M^2 = 1$, the most general possibility has the form

$$|\phi\rangle = (\zeta_\mu \alpha_{-2}^\mu + \lambda_{\mu\nu} \alpha_{-1}^\mu \alpha_{-1}^\nu) |0; p\rangle. \quad (53)$$

However, the constraints $L_1|\phi\rangle = L_2|\phi\rangle = 0$ restrict ζ_μ and $\lambda_{\mu\nu}$. The analysis is interesting, but only the results will be described. If $d > 26$, the physical spectrum contains a negative-norm state, which is not allowed. However, when $d = 26$, this state becomes zero norm and decouples from the theory. This leaves a pure massive “spin two” (symmetric traceless tensor) particle as the only physical state at this mass level.

Let us now turn to the closed-string spectrum. A closed-string state is described as a tensor product of a left-moving state and a right-moving state, subject to the condition that the N value of the left-moving and the right-moving state is the same. The reason for this “level-matching” condition is that we have $(L_0 - 1)|\phi\rangle = (\tilde{L}_0 - 1)|\phi\rangle = 0$. The sum $(L_0 + \tilde{L}_0 - 2)|\phi\rangle$ is interpreted as the mass-shell condition, while the difference $(L_0 - \tilde{L}_0)|\phi\rangle = (N - \tilde{N})|\phi\rangle = 0$ is the level-matching condition.

Using this rule, the closed-string ground state is just

$$|0\rangle \otimes |0\rangle, \quad (54)$$

which represents a spin 0 tachyon with $M^2 = -2$. (The notation no longer displays the momentum p of the state.) Again, this signals an unstable vacuum, but we will not worry about it here. Much more important, and more significant, is the first excited state

$$|\phi\rangle = \zeta_{\mu\nu}(\alpha_{-1}^\mu|0\rangle \otimes \tilde{\alpha}_{-1}^\nu|0\rangle), \quad (55)$$

which has $M^2 = 0$. The Virasoro constraints $L_1|\phi\rangle = \tilde{L}_1|\phi\rangle = 0$ imply that $p^\mu \zeta_{\mu\nu} = 0$. Such a polarization tensor encodes three distinct spin states, each of which plays a fundamental role in string theory. The symmetric part of $\zeta_{\mu\nu}$ encodes a spacetime metric field $g_{\mu\nu}$ (massless spin two) and a scalar dilaton field ϕ (massless spin zero). The $g_{\mu\nu}$ field is the graviton field, and its presence (with the correct gauge invariances) accounts for the fact that the theory contains general relativity, which is a good approximation for $E \ll 1/\ell_s$. Its vacuum value determines the spacetime geometry. Similarly, the value of ϕ determines the string coupling constant ($g_s = \langle e^\phi \rangle$).

$\zeta_{\mu\nu}$ also has an antisymmetric part, which corresponds to a massless antisymmetric tensor gauge field $B_{\mu\nu} = -B_{\nu\mu}$. This field has a gauge transformation of the form

$$\delta B_{\mu\nu} = \partial_\mu \Lambda_\nu - \partial_\nu \Lambda_\mu, \quad (56)$$

(which can be regarded as a generalization of the gauge transformation rule for the Maxwell field: $\delta A_\mu = \partial_\mu \Lambda$). The gauge-invariant field strength (analogous to $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$) is

$$H_{\mu\nu\rho} = \partial_\mu B_{\nu\rho} + \partial_\nu B_{\rho\mu} + \partial_\rho B_{\mu\nu}. \quad (57)$$

The importance of the $B_{\mu\nu}$ field resides in the fact that the fundamental string is a source for $B_{\mu\nu}$, just as a charged particle is a source for the vector potential A_μ . Mathematically, this is expressed by the coupling

$$q \int B_{\mu\nu} dx^\mu \wedge dx^\nu, \quad (58)$$

which generalizes the coupling of a charged particle to a Maxwell field

$$q \int A_\mu dx^\mu \quad (59)$$

in a convenient notation.

2.6 The Number of Physical States

The number of physical states grows rapidly as a function of mass. This can be analyzed quantitatively. For the open string, let us denote the number of physical states with $\alpha' M^2 = n - 1$ by d_n . These numbers are encoded in the generating function

$$G(w) = \sum_{n=0}^{\infty} d_n w^n = \prod_{m=1}^{\infty} (1 - w^m)^{-24}. \quad (60)$$

The exponent 24 reflects the fact that in 26 dimensions, once the Virasoro conditions are taken into account, the spectrum is exactly what one would get from 24 transversely polarized oscillators. It is easy to deduce from this generating function the asymptotic number of states for large n , as a function of n

$$d_n \sim n^{-27/4} e^{4\pi\sqrt{n}}. \quad (61)$$

Exercise: Verify this formula.

This asymptotic degeneracy implies that the finite-temperature partition function

$$\text{tr} (e^{-\beta H}) = \sum_{n=0}^{\infty} d_n e^{-\beta M_n} \quad (62)$$

diverges for $\beta^{-1} = T > T_H$, where T_H is the Hagedorn temperature

$$T_H = \frac{1}{4\pi\sqrt{\alpha'}} = \frac{1}{4\pi\ell_s}. \quad (63)$$

T_H might be the maximum possible temperature or else a critical temperature at which there is a phase transition.

2.7 The Structure of String Perturbation Theory

As we discussed in the first lecture, perturbation theory calculations are carried out by computing Feynman diagrams. Whereas in ordinary quantum field theory Feynman diagrams are webs of world lines, in the case of string theory they are two-dimensional surfaces representing string world sheets. For these purposes, it is convenient to require that the world-sheet geometry is Euclidean (i.e., the world-sheet metric $h_{\alpha\beta}$ is positive definite). The diagrams are classified by their topology, which is very well understood in the case of two-dimensional surfaces. The world-sheet topology is characterized by the number of handles (h), the number of boundaries (b), and whether or not they are orientable. The order of the expansion (i.e., the power of the string coupling constant) is determined by the Euler number of the world sheet M . It is given by $\chi(M) = 2 - 2h - b$. For example, a sphere has $h = b = 0$, and hence $\chi = 2$. A torus has $h = 1, b = 0$, and $\chi = 0$, a cylinder has $h = 0, b = 2$, and $\chi = 0$, and so forth. Surfaces with $\chi = 0$ admit a flat metric.

A scattering amplitude is given by a path integral of the schematic structure

$$\int Dh_{\alpha\beta}(\sigma) Dx^\mu(\sigma) e^{-S[h,x]} \prod_{i=1}^{n_c} \int_M V_{\alpha_i}(\sigma_i) d^2\sigma_i \prod_{j=1}^{n_o} \int_{\partial M} V_{\beta_j}(\sigma_j) d\sigma_j. \quad (64)$$

The action $S[h, x]$ is given in eq. (21). V_{α_i} is a vertex operator that describes emission or absorption of a closed-string state of type α_i from the interior of the string world sheet, and V_{β_j} is a vertex operator that describes emission or absorption of an open-string state of type β_j from the boundary of the string world sheet. There are lots of technical details that are not explained here. In the end, one finds that the conformally inequivalent world sheets of a given topology are described by a finite number of parameters, and thus these amplitudes can be recast as finite-dimensional integrals over these “moduli.” (The momentum integrals are already done.) The dimension of the resulting integral turns out to be

$$N = 3(2h + b - 2) + 2n_c + n_o. \quad (65)$$

As an example consider the amplitude describing elastic scattering of two-open string ground states. In this case $h = 0, b = 1, n_c = 0, n_o = 4$, and therefore $N = 1$. In terms of the usual Mandelstam invariants $s = -(p_1 + p_2)^2$ and $t = -(p_1 - p_4)^2$, the result is

$$A(s, t) = g_s^2 \int_0^1 dx x^{-\alpha(s)-1} (1-x)^{-\alpha(t)-1}, \quad (66)$$

where the Regge trajectory $\alpha(s)$ is

$$\alpha(s) = 1 + \alpha' s. \quad (67)$$

This integral is just the Euler beta function

$$A(s, t) = g_s^2 B(-\alpha(s), -\alpha(t)) = g_s^2 \frac{\Gamma(-\alpha(s))\Gamma(-\alpha(t))}{\Gamma(-\alpha(s) - \alpha(t))}. \quad (68)$$

This is the famous Veneziano amplitude [10], which got the whole business started.

2.8 Recapitulation

This lecture described some of the basic facts of the 26-dimensional bosonic string theory. One significant point that has not yet been made clear is that there are actually a number of distinct theories depending on what kinds of strings one includes

- oriented closed strings only
- oriented closed strings and oriented open strings. In this case one can incorporate $U(n)$ gauge symmetry.
- unoriented closed strings only
- unoriented closed strings and unoriented open strings. In this case one can incorporate $SO(n)$ or $Sp(n)$ gauge symmetry.

As we have mentioned already, all the bosonic string theories are sick as they stand, because (in each case) the closed-string spectrum contains a tachyon. A tachyon means that one is doing perturbation theory about an unstable vacuum. This is analogous to the unbroken symmetry extremum of the Higgs potential in the standard model. In that case, we know that there is a stable minimum, where the Higgs fields acquires a vacuum value. It is conceivable that the closed-string tachyon condenses in an analogous manner, or else there might not be a stable vacuum. Recently, there has been success in demonstrating that open-string tachyons condense at a stable minimum, but the fate of closed-string tachyons is still an open problem.

3 Lecture 3: Superstrings

Among the deficiencies of the bosonic string theory is the fact that there are no fermions. As we will see, the addition of fermions leads quite naturally to supersymmetry and hence superstrings. There are two alternative formalisms that are used to study superstrings. The original one, which grew out of the 1971 papers by Ramond [11] and by Neveu and me [12], is called the RNS formalism. In this approach, the supersymmetry of the two-dimensional

world-sheet theory plays a central role. The second approach, developed by Michael Green and me in the early 1980's [13], emphasizes supersymmetry in the ten-dimensional spacetime. Due to lack of time, only the RNS approach will be presented.

In the RNS formalism, the world-sheet theory is based on the d functions $x^\mu(\sigma, \tau)$ that describe the embedding of the world sheet in the spacetime, just as before. However, in order to supersymmetrize the world-sheet theory, we also introduce d fermionic partner fields $\psi^\mu(\sigma, \tau)$. Note that x^μ transforms as a vector from the spacetime viewpoint, but as d scalar fields from the two-dimensional world-sheet viewpoint. The ψ^μ also transform as a spacetime vector, but as world-sheet spinors. Altogether, x^μ and ψ^μ described d supersymmetry multiplets, one for each value of μ .

The reparametrization invariant world-sheet action described in the preceding lecture can be generalized to have local supersymmetry on the world sheet, as well. (The details of how that works are a bit too involved to describe here.) When one chooses a suitable conformal gauge ($h_{\alpha\beta} = e^\phi \eta_{\alpha\beta}$), together with an appropriate fermionic gauge condition, one ends up with a world-sheet theory that has global supersymmetry supplemented by constraints. The constraints form a super-Virasoro algebra. This means that in addition to the Virasoro constraints of the bosonic string theory, there are fermionic constraints, as well.

3.1 The Gauge-Fixed Theory

The globally supersymmetric world-sheet action that arises in the conformal gauge takes the form

$$S = -\frac{T}{2} \int d^2\sigma (\partial_\alpha x^\mu \partial^\alpha x_\mu - i \bar{\psi}^\mu \rho^\alpha \partial_\alpha \psi_\mu). \quad (69)$$

The first term is exactly the same as in eq. (25) of the bosonic string theory. Recall that it has the structure of d free scalar fields. The second term that has now been added is just d free massless spinor fields, with Dirac-type actions. The notation is that ρ^α are two 2×2 Dirac matrices and $\psi = \begin{pmatrix} \psi_- \\ \psi_+ \end{pmatrix}$ is a two-component Majorana spinor. The Majorana condition simply means that ψ_+ and ψ_- are real in a suitable representation of Dirac algebra. In fact, a convenient choice is one for which

$$\bar{\psi} \rho^\alpha \partial_\alpha \psi = \psi_- \partial_+ \psi_- + \psi_+ \partial_- \psi_+, \quad (70)$$

where ∂_\pm represent derivatives with respect to $\sigma^\pm = \tau \pm \sigma$. In this basis, the equations of motion are simply

$$\partial_+ \psi_-^\mu = \partial_- \psi_+^\mu = 0. \quad (71)$$

Thus ψ_-^μ describes right-movers and ψ_+^μ describes left-movers.

Concentrating on the right-movers ψ_-^μ , the global supersymmetry transformations, which are a symmetry of the gauge-fixed action, are

$$\begin{aligned}\delta x^\mu &= i\epsilon\psi_-^\mu \\ \delta\psi_-^\mu &= -2\partial_-x^\mu\epsilon.\end{aligned}\tag{72}$$

Exercise: Show that this is a symmetry of the action (69).

There is an analogous symmetry for the left-movers. (Accordingly, the world-sheet theory is said to have (1, 1) supersymmetry.) Continuing to focus on the right-movers, the Virasoro constraint is

$$(\partial_-x)^2 + \frac{i}{2}\psi_-^\mu\partial_-\psi_{\mu-} = 0.\tag{73}$$

The first term is what we found in the bosonic string theory, and the second term is an additional fermionic contribution. There is also an associated fermionic constraint

$$\psi_-^\mu\partial_-x_\mu = 0.\tag{74}$$

The Fourier modes of these constraints satisfy the super-Virasoro algebra. There is a second identical super-Virasoro algebra for the left-movers.

As in the bosonic string theory, the Virasoro algebra has conformal anomaly terms proportional to a central charge c . As in that theory, each component of x^μ contributes +1 to the central charge, for a total of d , while (in the BRST quantization approach) the reparametrization symmetry ghosts contribute -26 . But now there are additional contributions. Each component of ψ^μ gives $+1/2$, for a total of $d/2$, and the local supersymmetry ghosts contribute $+11$. Adding all of this up, gives a grand total of $c = \frac{3d}{2} - 15$. Thus, we see that the conformal anomaly cancels for the specific choice $d = 10$. This is the preferred critical dimension for superstrings, just as $d = 26$ is the critical dimension for bosonic strings. For other values the theory has a variety of inconsistencies.

3.2 The R and NS Sectors

Let us now consider boundary conditions for $\psi^\mu(\sigma, \tau)$. (The story for x^μ is exactly as before.) First, let us consider open-string boundary conditions. For the action to be well-defined, it turns out that one must set $\psi_+ = \pm\psi_-$ at the two ends $\sigma = 0, \pi$. An overall sign is a matter of convention, so we can set

$$\psi_+^\mu(0, \tau) = \psi_-^\mu(0, \tau),\tag{75}$$

without loss of generality. But this still leaves two possibilities for the other end, which are called R and NS:

$$\begin{aligned} \text{R} &: \psi_+^\mu(\pi, \tau) = \psi_-^\mu(\pi, \tau) \\ \text{NS} &: \psi_+^\mu(\pi, \tau) = -\psi_-^\mu(\pi, \tau). \end{aligned} \tag{76}$$

Combining these with the equations of motion $\partial_- \psi_+ = \partial_+ \psi_- = 0$, allows us to express the general solutions as Fourier series

$$\begin{aligned} \text{R} : \quad \psi_-^\mu &= \frac{1}{\sqrt{2}} \sum_{n \in \mathbf{Z}} d_n^\mu e^{-in(\tau-\sigma)} \\ \psi_+^\mu &= \frac{1}{\sqrt{2}} \sum_{n \in \mathbf{Z}} d_n^\mu e^{-in(\tau+\sigma)} \\ \text{NS} : \quad \psi_-^\mu &= \frac{1}{\sqrt{2}} \sum_{r \in \mathbf{Z}+1/2} b_r^\mu e^{-ir(\tau-\sigma)} \\ \psi_+^\mu &= \frac{1}{\sqrt{2}} \sum_{r \in \mathbf{Z}+1/2} b_r^\mu e^{-ir(\tau+\sigma)}. \end{aligned} \tag{77}$$

The Majorana condition implies that $d_{-n}^\mu = d_n^{\mu\dagger}$ and $b_{-r}^\mu = b_r^{\mu\dagger}$. Note that the index n takes integer values, whereas the index r takes half-integer values ($\pm\frac{1}{2}, \pm\frac{3}{2}, \dots$). In particular, only the R boundary condition gives a zero mode.

Canonical quantization of the free fermi fields $\psi^\mu(\sigma, \tau)$ is very standard and straightforward. The result can be expressed as anticommutation relations for the coefficients d_m^μ and b_r^μ :

$$\begin{aligned} \text{R} &: \quad \{d_n^\mu, d_n^\nu\} = \eta^{\mu\nu} \delta_{m+n,0} \quad m, n \in \mathbf{Z} \\ \text{NS} &: \quad \{d_r^\mu, d_s^\nu\} = \eta^{\mu\nu} \delta_{r+s,0} \quad r, s \in \mathbf{Z} + \frac{1}{2}. \end{aligned} \tag{78}$$

Thus, in addition to the harmonic oscillator operators α_m^μ that appear as coefficients in mode expansions of x^μ , there are fermionic oscillator operators d_m^μ or b_r^μ that appear as coefficients in mode expansions of ψ^μ . The basic structure $\{b, b^\dagger\} = 1$ is very simple. It describes a two-state system with $b|0\rangle = 0$, and $b^\dagger|0\rangle = |1\rangle$. The b 's or d 's with negative indices can be regarded as raising operators and those with positive indices as lowering operators, just as we did for the α_n^μ .

In the NS sector, the ground state $|0; p\rangle$ satisfies

$$\alpha_m^\mu |0; p\rangle = b_r^\mu |0; p\rangle = 0, \quad m, r > 0 \tag{79}$$

which is a straightforward generalization of how we defined the ground state in the bosonic string theory. All the excited states obtained by acting with the α and b raising operators

are spacetime bosons. We will see later that the ground state, defined as we have done here, is again a tachyon. However, in this theory, as we will also see, there is a way by which this tachyon can (and must) be removed from the physical spectrum.

In the R sector there are zero modes that satisfy the algebra

$$\{d_0^\mu, d_0^\nu\} = \eta^{\mu\nu}. \quad (80)$$

This is the d -dimensional spacetime Dirac algebra. Thus the d_0 's should be regarded as Dirac matrices and all states in the R sector should be spinors in order to furnish representation spaces on which these operators can act. The conclusion, therefore, is that whereas all string states in the NS sector are spacetime bosons, all string states in the R sector are spacetime fermions.

In the closed-string case, the physical states are obtained by tensoring right-movers and left-movers, each of which are mathematically very similar to the open-string spectrum. This means that there are four distinct sectors of closed-string states: NS \otimes NS and R \otimes R describe spacetime bosons, whereas NS \otimes R and R \otimes NS describe spacetime fermions. We will return to explore what this gives later, but first we need to explore the right-movers by themselves in more detail.

The zero mode of the fermionic constraint $\psi^\mu \partial_- x_\mu = 0$ gives a wave equation for (fermionic) strings in the Ramond sector, $F_0|\psi\rangle = 0$, which is called the Dirac–Ramond equation. In terms of the oscillators

$$F_0 = \alpha_0 \cdot d_0 + \sum_{n \neq 0} \alpha_{-n} \cdot d_n. \quad (81)$$

The zero-mode piece of F_0 , $\alpha_0 \cdot d_0$, has been isolated, because it is just the usual Dirac operator, $\gamma^\mu \partial_\mu$, up to normalization. (Recall that $\alpha_{0\mu}$ is proportional to $p_\mu = -i\partial_\mu$, and d_0^μ is proportional to the Dirac matrices γ^μ .) The fermionic ground state $|\psi_0\rangle$, which satisfies

$$\alpha_n^\mu |\psi_0\rangle = d_n^\mu |\psi_0\rangle = 0, \quad n > 0, \quad (82)$$

satisfies the wave equation

$$\alpha_0 \cdot d_0 |\psi_0\rangle = 0, \quad (83)$$

which is precisely the massless Dirac equation. Hence the fermionic ground state is a massless spinor.

3.3 The GSO Projection

In the NS (bosonic) sector the mass formula is

$$M^2 = N - \frac{1}{2}, \quad (84)$$

which is to be compared with the formula $M^2 = N - 1$ of the bosonic string theory. This time the number operator N has contributions from the b oscillators as well as the α oscillators. (The reason that the normal-ordering constant is $-1/2$ instead of -1 works as follows. Each transverse α oscillator contributes $-1/24$ and each transverse b oscillator contributes $-1/48$. The result follows since the bosonic theory has 24 transverse directions and the superstring theory has 8 transverse directions.) Thus the ground state, which has $N = 0$, is now a tachyon with $M^2 = -1/2$.

This is where things stood until the 1976 work of Gliozzi, Scherk, and Olive [14]. They noted that the spectrum admits a consistent truncation (called the GSO projection) which is necessary for the consistency of the interacting theory. In the NS sector, the GSO projection keeps states with an odd number of b -oscillator excitations, and removes states with an even number of b -oscillator excitation. Once this rule is implemented the only possible values of N are half integers, and the spectrum of allowed masses are integral

$$M^2 = 0, 1, 2, \dots \quad (85)$$

In particular, the bosonic ground state is now massless. The spectrum no longer contains a tachyon. The GSO projection also acts on the R sector, where there is an analogous restriction on the d oscillators. This amounts to imposing a chirality projection on the spinors.

Let us look at the massless spectrum of the GSO-projected theory. The ground state boson is now a massless vector, represented by the state $\zeta_\mu b_{-1/2}^\mu |0; p\rangle$, which (as before) has $d - 2 = 8$ physical polarizations. The ground state fermion is a massless Majorana–Weyl fermion which has $\frac{1}{4} \cdot 2^{d/2} = 8$ physical polarizations. Thus there are an equal number of bosons and fermions, as is required for a theory with spacetime supersymmetry. In fact, this is the pair of fields that enter into ten-dimensional super Yang–Mills theory. The claim is that the complete theory now has spacetime supersymmetry.

If there is spacetime supersymmetry, then there should be an equal number of bosons and fermions at every mass level. Let us denote the number of bosonic states with $M^2 = n$ by $d_{NS}(n)$ and the number of fermionic states with $M^2 = n$ by $d_R(n)$. Then we can encode these numbers in generating functions, just as we did for the bosonic string theory

$$f_{NS}(w) = \sum_{n=0}^{\infty} d_{NS}(n)w^n = \frac{1}{2\sqrt{w}} \left(\prod_{m=1}^{\infty} \left(\frac{1+w^{m-1/2}}{1-w^m} \right)^8 - \prod_{m=1}^{\infty} \left(\frac{1-w^{m-1/2}}{1-w^m} \right)^8 \right) \quad (86)$$

$$f_R(w) = \sum_{n=0}^{\infty} d_R(n)w^n = 8 \prod_{m=1}^{\infty} \left(\frac{1+w^m}{1-w^m} \right)^8. \quad (87)$$

The 8's in the exponents refer to the number of transverse directions in ten dimensions. The effect of the GSO projection is the subtraction of the second term in f_{NS} and reduction of coefficient in f_R from 16 to 8. In 1829, Jacobi discovered the formula

$$f_R(w) = f_{NS}(w). \quad (88)$$

(He used a different notation, of course.) For him this relation was an obscure curiosity, but we now see that it provides strong evidence for supersymmetry of this string theory in ten dimensions. A complete proof of supersymmetry for the interacting theory was constructed by Green and me five years after the GSO paper [13].

3.4 Type II Superstrings

We have described the spectrum of bosonic (NS) and fermionic (R) string states. This also gives the spectrum of left-moving and right-moving closed-string modes, so we can form the closed-string spectrum by forming tensor products as before. In particular, the massless right-moving spectrum consists of a vector and a Majorana–Weyl spinor. Thus the massless closed-string spectrum is given by

$$(\text{vector} + \text{MW spinor}) \otimes (\text{vector} + \text{MW spinor}). \quad (89)$$

There are actually two distinct possibilities because two MW spinor can have either opposite chirality or the same chirality.

When the two MW spinors have opposite chirality, the theory is called type IIA superstring theory, and its massless spectrum forms the type IIA supergravity multiplet. This theory is left-right symmetric. In other words, the spectrum is invariant under mirror reflection. This implies that the IIA theory is parity conserving. When the two MW spinors have the same chirality, the resulting type IIB superstring theory is chiral, and hence parity violating. In each case there are two gravitinos, arising from vector \otimes spinor and spinor \otimes vector, which are gauge fields for local supersymmetry. (In four dimensions we would say that the gravitinos have spin 3/2, but that is not an accurate description in ten dimensions.) Thus, since both type II superstring theories have two gravitinos, they have local $\mathcal{N} = 2$ supersymmetry in the ten-dimensional sense. The supersymmetry charges are Majorana–Weyl spinors, which have 16 components, so the type II theories have 32 conserved supercharges. This is the same amount of supersymmetry as what is usually called $\mathcal{N} = 8$ in four dimensions.

The type II superstring theories contain only oriented closed strings (in the absence of D-branes). However, there is another superstring theory, called type I, which can be obtained by a projection of the type IIB theory, that only keeps the diagonal sum of the

two gravitinos. Thus, this theory only has $\mathcal{N} = 1$ supersymmetry (16 supercharges). It is a theory of unoriented closed strings. However, it can be supplemented by unoriented open strings. This introduces a Yang–Mills gauge group, which classically can be $SO(n)$ or $Sp(n)$ for any value of n . Quantum consistency singles out $SO(32)$ as the unique possibility. This restriction can be understood in a number of ways. The way that it was first discovered was by considering anomalies.

3.5 Anomalies

Chiral (parity-violating) gauge theories can be inconsistent due to anomalies. This happens when there is a quantum mechanical breakdown of the gauge symmetry, which is induced by certain one-loop Feynman diagrams. (Sometimes one also considers breaking of global symmetries by anomalies, which does not imply an inconsistency. That is not what we are interested in here.) In the case of four dimensions, the relevant diagrams are triangles, with the chiral fields going around the loop and three gauge fields attached as external lines. In the case of the standard model, the quarks and leptons are chiral and contribute to a variety of possible anomalies. Fortunately, the standard model has just the right content so that all of the gauge anomalies cancel. If one discarded the quark or lepton contributions, it would not work.

In the case of ten-dimensional chiral gauge theories, the potentially anomalous Feynman diagrams are hexagons, with six external gauge fields. The anomalies can be attributed to the massless fields, and therefore they can be analyzed in the low-energy effective field theory. There are several possible cases in ten dimensions:

- $\mathcal{N} = 1$ supersymmetric Yang–Mills theory. This theory has anomalies for every choice of gauge group.
- Type I supergravity. This theory has gravitational anomalies.
- Type IIA supergravity. This theory is non-chiral, and therefore it is trivially anomaly-free.
- Type IIB supergravity. This theory has three chiral fields each of which contributes to several kinds of gravitational anomalies. However, when their contributions are combined, the anomalies all cancel. (This result was obtained by Alvarez–Gaumé and Witten in 1983 [15].)
- Type I supergravity coupled to super Yang–Mills. This theory has both gauge and gravitational anomalies for every choice of Yang-Mills gauge group except $SO(32)$ and

$E_8 \times E_8$. For these two choices, all the anomalies cancel. (This result was obtained by Green and me in 1984 [16].)

As we mentioned earlier, at the classical level one can define type I superstring theory for any orthogonal or symplectic gauge group. Now we see that at the quantum level, the only choice that is consistent is $SO(32)$. For any other choice there are fatal anomalies. The term $SO(32)$ is used here somewhat imprecisely. There are several different Lie groups that have the same Lie algebra. It turns out that the precise Lie group that is appropriate is $\text{Spin}(32)/\mathbf{Z}_2$.

3.6 Heterotic Strings

The two Lie groups that are singled out — $E_8 \times E_8$ and $\text{Spin}(32)/\mathbf{Z}_2$ — have several properties in common. Each of them has dimension = 496 and rank = 16. Moreover, their weight lattices correspond to the only two even self-dual lattices in 16 dimensions. This last fact was the crucial clue that led Gross, Harvey, Martinec, and Rohm [17] to the discovery of the heterotic string soon after the anomaly cancellation result. One hint is the relation $10 + 16 = 26$. The construction of the heterotic string uses the $d = 26$ bosonic string for the left-movers and the $d = 10$ superstring for the right movers. The sixteen extra left-moving dimensions are associated to an even self-dual 16-dimensional lattice. In this way one builds in the $SO(32)$ or $E_8 \times E_8$ gauge symmetry.

Thus, to recapitulate, by 1985 we had five consistent superstring theories, type I (with gauge group $SO(32)$), the two type II theories, and the two heterotic theories. Each is a supersymmetric ten-dimensional theory. The perturbation theory was studied in considerable detail, and while some details may not have been completed, it was clear that each of the five theories has a well-defined, ultraviolet-finite perturbation expansion, satisfying all the usual consistency requirements (unitarity, analyticity, causality, etc.) This was pleasing, though it was somewhat mysterious why there should be five consistent quantum gravity theories. It took another ten years until we understood that these are actually five special quantum vacua of a unique underlying theory.

3.7 T Duality

T duality, an amazing result obtained in the late 1980's, relates one string theory with a circular compact dimension of radius R to another string theory with a circular dimension of radius $1/R$ (in units $\ell_s = 1$). This is very profound, because it indicates a limitation of our usual motions of classical geometry. Strings see geometry differently from point particles. Let us examine how this is possible.

The key to understanding T duality is to consider the kinds of excitations that a string can have in the presence of a circular dimension. One class of excitations, called Kaluza–Klein excitations, is a very general feature of any quantum theory, whether or not based on strings. The idea is that in order for the wave function e^{ipx} to be single valued, the momentum along the circle must be a multiple of $1/R$, $p = n/R$, where n is an integer. From the lower-dimension viewpoint this is interpreted as a contribution $(n/R)^2$ to the square of the mass.

There is a second type of excitation that is special to closed strings. Namely, a closed string can wind m times around the circular dimension, getting caught up on the topology of the space, contributing an energy given by the string tension times the length of the string

$$E_m = 2\pi R \cdot m \cdot T. \quad (90)$$

Putting $T = \frac{1}{2\pi}$ (for $\ell_s = 1$), this is just $E_m = mR$.

The combined energy-squared of the Kaluza–Klein and winding-mode excitations is

$$E^2 = \left(\frac{n}{R}\right)^2 + (mR)^2 + \dots, \quad (91)$$

where the dots represent string oscillator contributions. Under T duality

$$m \leftrightarrow n, \quad R \leftrightarrow 1/R. \quad (92)$$

Together, these interchanges leave the energy invariant. This means that what is interpreted as a Kaluza–Klein excitation in one string theory is interpreted as a winding-mode excitation in the T-dual theory, and the two theories have radii R and $1/R$, respectively. The two principle examples of T-dual pairs are the two type II theories and the two heterotic theories. In the latter case there are additional technicalities that explain how the two gauge groups are related. Basically, when the compactification on a circle to nine dimension is carried out in each case, it is necessary to include effects that we haven't explained (called Wilson lines) to break the gauge groups to $SO(16) \times SO(16)$, which is a common subgroup of $SO(32)$ and $E_8 \times E_8$.

4 Lecture 4: From Superstrings to M Theory

Superstring theory is currently undergoing a period of rapid development in which important advances in understanding are being achieved. The focus in this lecture will be on explaining why there can be an eleven-dimensional vacuum, even though there are only ten dimensions in perturbative superstring theory. The nonperturbative extension of superstring theory that allows for an eleventh dimension has been named *M theory*. The letter M is intended to be

flexible in its interpretation. It could stand for *magic*, *mystery*, or *meta* to reflect our current state of incomplete understanding. Those who think that two-dimensional supermembranes (the M2-brane) are fundamental may regard M as standing for *membrane*. An approach called *Matrix theory* is another possibility. And, of course, some view M theory as the *mother* of all theories.

In the first superstring revolution we identified five distinct superstring theories, each in ten dimensions. Three of them, the type I theory and the two heterotic theories, have $\mathcal{N} = 1$ supersymmetry in the ten-dimensional sense. Since the minimal 10d spinor is simultaneously Majorana and Weyl, this corresponds to 16 conserved supercharges. The other two theories, called type IIA and type IIB, have $\mathcal{N} = 2$ supersymmetry (32 supercharges). In the IIA case the two spinors have opposite handedness so that the spectrum is left-right symmetric (nonchiral). In the IIB case the two spinors have the same handedness and the spectrum is chiral.

In each of these five superstring theories it became clear, and was largely proved, that there are consistent perturbation expansions of on-shell scattering amplitudes. In four of the five cases (heterotic and type II) the fundamental strings are oriented and unbreakable. As a result, these theories have particularly simple perturbation expansions. Specifically, there is a unique Feynman diagram at each order of the loop expansion. The Feynman diagrams depict string world sheets, and therefore they are two-dimensional surfaces. For these four theories the unique L -loop diagram is a closed orientable genus- L Riemann surface, which can be visualized as a sphere with L handles. External (incoming or outgoing) particles are represented by N points (or “punctures”) on the Riemann surface. A given diagram represents a well-defined integral of dimension $6L + 2N - 6$. This integral has no ultraviolet divergences, even though the spectrum contains states of arbitrarily high spin (including a massless graviton). From the viewpoint of point-particle contributions, string and supersymmetry properties are responsible for incredible cancellations. Type I superstrings are unoriented and breakable. As a result, the perturbation expansion is more complicated for this theory, and various world-sheet diagrams at a given order have to be combined properly to cancel divergences and anomalies .

An important discovery that was made between the two superstring revolutions is *T duality*. As we explained earlier, this duality relates two string theories when one spatial dimension forms a circle (denoted S^1). Then the ten-dimensional geometry is $R^9 \times S^1$. T duality identifies this string compactification with one of a second string theory also on $R^9 \times S^1$. If the radii of the circles in the two cases are denoted R_1 and R_2 , then

$$R_1 R_2 = \alpha'. \tag{93}$$

Here $\alpha' = \ell_s^2$ is the universal Regge slope parameter, and ℓ_s is the fundamental string length scale (for both string theories). Note that T duality implies that shrinking the circle to zero in one theory corresponds to decompactification of the dual theory.

The type IIA and IIB theories are T dual, so compactifying the nonchiral IIA theory on a circle of radius R and letting $R \rightarrow 0$ gives the chiral IIB theory in ten dimensions! This means, in particular, that they should not be regarded as distinct theories. The radius R is actually the vacuum value of a scalar field, which arises as an internal component of the 10d metric tensor. Thus the type IIA and type IIB theories in 10d are two limiting points in a continuous moduli space of quantum vacua. The two heterotic theories are also T dual, though there are additional technical details in this case. T duality applied to the type I theory gives a dual description, which is sometimes called type I' or IA.

4.1 M Theory

In the 1970s and 1980s various supersymmetry and supergravity theories were constructed. In particular, supersymmetry representation theory showed that the largest possible space-time dimension for a supergravity theory (with spins ≤ 2) is eleven. Eleven-dimensional supergravity, which has 32 conserved supercharges, was constructed in 1978 by Cremmer, Julia, and Scherk [18]. It has three kinds of fields—the graviton field (with 44 polarizations), the gravitino field (with 128 polarizations), and a three-index gauge field $C_{\mu\nu\rho}$ (with 84 polarizations). These massless particles are referred to collectively as the *supergraviton*. 11d supergravity is nonrenormalizable, and thus it cannot be a fundamental theory. However, we now believe that it is a low-energy effective description of M theory, which is a well-defined quantum theory. This means, in particular, that higher-dimension terms in the effective action for the supergravity fields have uniquely determined coefficients within the M theory setting, even though they are formally infinite (and hence undetermined) within the supergravity context.

Intriguing connections between type IIA string theory and 11d supergravity have been known for a long time, but the precise relationship was only explained in 1995. The field equations of 11d supergravity admit a solution that describes a supermembrane. In other words, this solution has the property that the energy density is concentrated on a two-dimensional surface. A 3d world-volume description of the dynamics of this supermembrane, quite analogous to the 2d world volume actions of superstrings (in the GS formalism [19]), was constructed by Bergshoeff, Sezgin, and Townsend in 1987 [20]. The authors suggested that a consistent 11d quantum theory might be defined in terms of this membrane, in analogy to string theories in ten dimensions. (Most experts now believe that M theory cannot be defined as a supermembrane theory.) Another striking result was that a suitable dimen-

sional reduction of this supermembrane gives the (previously known) type IIA superstring world-volume action. For many years these facts remained unexplained curiosities until they were reconsidered by Townsend [21] and by Witten [22]. The conclusion is that type IIA superstring theory really does have a circular 11th dimension in addition to the previously known ten spacetime dimensions. This fact was not recognized earlier because the appearance of the 11th dimension is a nonperturbative phenomenon, not visible in perturbation theory.

To explain the relation between M theory and type IIA string theory, a good approach is to identify the parameters that characterize each of them and to explain how they are related. Eleven-dimensional supergravity (and hence M theory, too) has no dimensionless parameters. The only parameter is the 11d Newton constant, which raised to a suitable power $(-1/9)$, gives the 11d Planck mass m_p . When M theory is compactified on a circle (so that the spacetime geometry is $R^{10} \times S^1$) another parameter is the radius R of the circle. Now consider the parameters of type IIA superstring theory. They are the string mass scale m_s , introduced earlier, and the dimensionless string coupling constant g_s .

We can identify compactified M theory with type IIA superstring theory by making the following correspondences:

$$m_s^2 = 2\pi R m_p^3 \tag{94}$$

$$g_s = 2\pi R m_s. \tag{95}$$

Using these one can derive $g_s = (2\pi R m_p)^{3/2}$ and $m_s = g_s^{1/3} m_p$. The latter implies that the 11d Planck length is shorter than the string length scale at weak coupling by a factor of $(g_s)^{1/3}$.

Conventional string perturbation theory is an expansion in powers of g_s at fixed m_s . Equation (95) shows that this is equivalent to an expansion about $R = 0$. In particular, the strong coupling limit of type IIA superstring theory corresponds to decompactification of the eleventh dimension, so in a sense M theory is type IIA string theory at infinite coupling. (The $E_8 \times E_8$ heterotic string theory is also eleven-dimensional at strong coupling.) This explains why the eleventh dimension was not discovered in studies of string perturbation theory.

These relations encode some interesting facts. For one thing, the fundamental IIA string actually *is* an M2-brane of M theory with one of its dimensions wrapped around the circular spatial dimension. Denoting the string and membrane tensions (energy per unit volume) by T_{F1} and T_{M2} , one deduces that

$$T_{F1} = 2\pi R T_{M2}. \tag{96}$$

However, $T_{F1} = 2\pi m_s^2$ and $T_{M2} = 2\pi m_p^3$. Combining these relations gives eq. (94).

4.2 Type II p -branes

Type II superstring theories contain a variety of p -brane solutions that preserve half of the 32 supersymmetries. These are solutions in which the energy is concentrated on a p -dimensional spatial hypersurface. (The world volume has $p+1$ dimensions.) The corresponding solutions of supergravity theories were constructed in 1991 by Horowitz and Strominger [23]. A large class of these p -brane excitations are called *D-branes* (or *Dp-branes* when we want to specify the dimension), whose tensions are given by

$$T_{Dp} = 2\pi m_s^{p+1}/g_s. \quad (97)$$

This dependence on the coupling constant is one of the characteristic features of a D-brane. Another characteristic feature of D-branes is that they carry a charge that couples to a gauge field in the RR sector of the theory [24]. The particular RR gauge fields that occur imply that p takes even values in the IIA theory and odd values in the IIB theory.

In particular, the D2-brane of the type IIA theory corresponds to the supermembrane of M theory, but now in a background geometry in which one of the transverse dimensions is a circle. The tensions check, because (using eqs. (94) and (95))

$$T_{D2} = 2\pi m_s^3/g_s = 2\pi m_p^3 = T_{M2}. \quad (98)$$

The mass of the first Kaluza–Klein excitation of the 11d supergraviton is $1/R$. Using eq. (95), we see that this can be identified with the D0-brane. More identifications of this type arise when we consider the magnetic dual of the M theory supermembrane, which is a five-brane, called the M5-brane.² Its tension is $T_{M5} = 2\pi m_p^6$. Wrapping one of its dimensions around the circle gives the D4-brane, with tension

$$T_{D4} = 2\pi R T_{M5} = 2\pi m_s^5/g_s. \quad (99)$$

If, on the other hand, the M5-brane is not wrapped around the circle, one obtains the NS5-brane of the IIA theory with tension

$$T_{NS5} = T_{M5} = 2\pi m_s^6/g_s^2. \quad (100)$$

To summarize, type IIA superstring theory is M theory compactified on a circle of radius $R = g_s \ell_s$. M theory is believed to be a well-defined quantum theory in 11d, which is approximated at low energy by 11d supergravity. Its excitations are the massless supergraviton,

²In general, the magnetic dual of a p -brane in d dimensions is a $(d-p-4)$ -brane.

the M2-brane, and the M5-brane. These account both for the (perturbative) fundamental string of the IIA theory and for many of its nonperturbative excitations. The identities that we have presented here are exact, because they are protected by supersymmetry.

4.3 Type IIB Superstring Theory

Type IIB superstring theory, which is the other maximally supersymmetric string theory with 32 conserved supercharges, is also 10-dimensional, but unlike the IIA theory its two supercharges have the same handedness. At low-energy, type IIB superstring theory is approximated by type IIB supergravity, just as 11d supergravity approximates M theory. In each case the supergravity theory is only well-defined as a classical field theory, but still it can teach us a lot. For example, it can be used to construct p -brane solutions and compute their tensions. Even though such solutions are only approximate, supersymmetry considerations ensure that the tensions, which are related to the kinds of conserved charges the p -branes carry, are exact. Since the IIB spectrum contains massless chiral fields, one should check whether there are anomalies that break the gauge invariances—general coordinate invariance, local Lorentz invariance, and local supersymmetry. In fact, the UV finiteness of the string theory Feynman diagrams ensures that all anomalies must cancel, as was verified from a field theory viewpoint by Alvarez-Gaumé and Witten [15].

Type IIB superstring theory or supergravity contains two scalar fields, the dilation ϕ and an axion χ , which are conveniently combined in a complex field

$$\rho = \chi + ie^{-\phi}. \quad (101)$$

The supergravity approximation has an $SL(2, R)$ symmetry that transforms this field non-linearly:

$$\rho \rightarrow \frac{a\rho + b}{c\rho + d}, \quad (102)$$

where a, b, c, d are real numbers satisfying $ad - bc = 1$. However, in the quantum string theory this symmetry is broken to the discrete subgroup $SL(2, Z)$ [25], which means that a, b, c, d are restricted to be integers. Defining the vacuum value of the ρ field to be

$$\langle \rho \rangle = \frac{\theta}{2\pi} + \frac{i}{g_s}, \quad (103)$$

the $SL(2, Z)$ symmetry transformation $\rho \rightarrow \rho + 1$ implies that θ is an angular coordinate. Moreover, in the special case $\theta = 0$, the symmetry transformation $\rho \rightarrow -1/\rho$ takes $g_s \rightarrow 1/g_s$. This symmetry, called *S duality*, implies that coupling constant g_s is equivalent to coupling constant $1/g_s$, so that, in the case of Type II superstring theory, the weak coupling expansion

and the strong coupling expansion are identical! (An analogous S-duality transformation relates the Type I superstring theory to the $SO(32)$ heterotic string theory.)

Recall that the type IIA and type IIB superstring theories are T dual, meaning that if they are compactified on circles of radii R_A and R_B one obtains equivalent theories for the identification $R_A R_B = \ell_s^2$. Moreover, we saw that the type IIA theory is actually M theory compactified on a circle. The latter fact encodes nonperturbative information. It turns out to be very useful to combine these two facts and to consider the duality between M theory compactified on a torus ($R^9 \times T^2$) and type IIB superstring theory compactified on a circle ($R^9 \times S^1$).

A torus can be described as the complex plane modded out by the equivalence relations $z \sim z + w_1$ and $z \sim z + w_2$. Up to conformal equivalence, the periods w_1 and w_2 can be replaced by 1 and τ , with $\text{Im } \tau > 0$. In this characterization τ and $\tau' = (a\tau + b)/(c\tau + d)$, where a, b, c, d are integers satisfying $ad - bc = 1$, describe equivalent tori. Thus a torus is characterized by a modular parameter τ and an $SL(2, Z)$ modular group. The natural, and correct, conjecture at this point is that one should identify the modular parameter τ of the M theory torus with the parameter ρ that characterizes the type IIB vacuum [26, 27]. Then the duality of M theory and type IIB superstring theory gives a geometrical explanation of the nonperturbative S duality symmetry of the IIB theory: the transformation $\rho \rightarrow -1/\rho$, which sends $g_s \rightarrow 1/g_s$ in the IIB theory, corresponds to interchanging the two cycles of the torus in the M theory description. To complete the story, we should relate the area of the M theory torus (A_M) to the radius of the IIB theory circle (R_B). This is a simple consequence of formulas given above

$$m_p^3 A_M = (2\pi R_B)^{-1}. \quad (104)$$

Thus the limit $R_B \rightarrow 0$, at fixed ρ , corresponds to decompactification of the M theory torus, while preserving its shape. Conversely, the limit $A_M \rightarrow 0$ corresponds to decompactification of the IIB theory circle. The duality can be explored further by matching the various p -branes in 9 dimensions that can be obtained from either the M theory or the IIB theory viewpoints. When this is done, one finds that everything matches nicely and that one deduces various relations among tensions [28].

Another interesting fact about the IIB theory is that it contains an infinite family of strings labeled by a pair of integers (p, q) with no common divisor [26]. The $(1, 0)$ string can be identified as the fundamental IIB string, while the $(0, 1)$ string is the D-string. From this viewpoint, a (p, q) string can be regarded as a bound state of p fundamental strings and q D-strings [29]. These strings have a very simple interpretation in the dual M theory description. They correspond to an M2-brane with one of its cycles wrapped around a (p, q)

cycle of the torus. The minimal length of such a cycle is proportional to $|p + q\tau|$, and thus (using $\tau = \rho$) one finds that the tension of a (p, q) string is given by

$$T_{p,q} = 2\pi|p + q\rho|m_s^2. \quad (105)$$

Imagine that you lived in the 9-dimensional world that is described equivalently as M theory compactified on a torus or as the type IIB superstring theory compactified on a circle. Suppose, moreover, you had very high energy accelerators with which you were going to determine the “true” dimension of spacetime. Would you conclude that 10 or 11 is the correct answer? If either A_M or R_B was very large in Planck units there would be a natural choice, of course. But how could you decide otherwise? The answer is that either viewpoint is equally valid. What determines which choice you make is which of the massless fields you regard as “internal” components of the metric tensor and which ones you regards as matter fields. Fields that are metric components in one description correspond to matter fields in the dual one.

4.4 The D3-Brane and $\mathcal{N} = 4$ Gauge Theory

D-branes have a number of special properties, which make them especially interesting. By definition, they are branes on which strings can end—D stands for *Dirichlet* boundary conditions. The end of a string carries a charge, and the D-brane world-volume theory contains a $U(1)$ gauge field that carries the associated flux. When n D p -branes are coincident, or parallel and nearly coincident, the associated $(p + 1)$ -dimensional world-volume theory is a $U(n)$ gauge theory [29]. The n^2 gauge bosons A_μ^{ij} and their supersymmetry partners arise as the ground states of oriented strings running from the i th D p -brane to the j th D p -brane. The diagonal elements, belonging to the Cartan subalgebra, are massless. The field A_μ^{ij} with $i \neq j$ has a mass proportional to the separation of the i th and j th branes.

The $U(n)$ gauge theory associated with a stack of n D p -branes has maximal supersymmetry (16 supercharges). The low-energy effective theory, when the brane separations are small compared to the string scale, is supersymmetric Yang–Mills theory. These theories can be constructed by dimensional reduction of 10d supersymmetric $U(n)$ gauge theory to $p + 1$ dimensions. A case of particular interest, which we shall now focus on, is $p = 3$. A stack of n D3-branes in type IIB superstring theory has a decoupled $\mathcal{N} = 4$, $d = 4$ $U(n)$ gauge theory associated to it. This gauge theory has a number of special features. For one thing, due to boson–fermion cancellations, there are no UV divergences at any order of perturbation theory. The beta function $\beta(g)$ is identically zero, which implies that the theory is scale invariant. In fact, $\mathcal{N} = 4$, $d = 4$ gauge theories are conformally invariant. The conformal

invariance combines with the supersymmetry to give a superconformal symmetry, which contains 32 fermionic generators. Another important property of $\mathcal{N} = 4$, $d = 4$ gauge theories is an electric-magnetic duality, which extends to an $SL(2, Z)$ group of dualities. Now consider the $\mathcal{N} = 4$ $U(n)$ gauge theory associated to a stack of n D3-branes in type IIB superstring theory. There is an obvious identification, that turns out to be correct. Namely, the $SL(2, Z)$ duality of the gauge theory is induced from that of the ambient type IIB superstring theory. The D3-branes themselves are invariant under $SL(2, Z)$ transformations.

As we have said, a fundamental $(1, 0)$ string can end on a D3-brane. But by applying a suitable $SL(2, Z)$ transformation, this configuration is transformed to one in which a (p, q) string ends on the D3-brane. The charge on the end of this string describes a dyon with electric charge p and magnetic charge q , with respect to the appropriate gauge field. More generally, for a stack of n D3-branes, any pair can be connected by a (p, q) string. The mass is proportional to the length of the string times its tension, which we saw is proportional to $|p+q\rho|$. In this way one sees that the electrically charged particles, described by fundamental fields, belong to infinite $SL(2, Z)$ multiplets. The other states are nonperturbative excitations of the gauge theory. The field configurations that describe them preserve half of the supersymmetry. As a result their masses are given exactly by the considerations described above. An interesting question, whose answer was unknown until recently, is whether $\mathcal{N} = 4$ gauge theories in four dimensions also admit nonperturbative excitations that preserve 1/4 of the supersymmetry. The answer turns out to be that they do, but only if $n \geq 3$. This result has a nice dual description in terms of three-string junctions [30].

4.5 Conclusion

In this lecture we have described some of the interesting advances in understanding superstring theory that have taken place in the past few years. The emphasis has been on the nonperturbative appearance of an eleventh dimension in type IIA superstring theory, as well as its implications when combined with superstring T dualities. In particular, we argued that there should be a consistent quantum vacuum, whose low-energy effective description is given by 11d supergravity.

What we have described makes a convincing self-consistent picture, but it does not constitute a complete formulation of M theory. In the past several years there have been some major advances in that direction, which we will briefly mention here. The first, which goes by the name of *Matrix Theory*, bases a formulation of M theory in flat 11d spacetime in terms of the supersymmetric quantum mechanics of N D0-branes in the large N limit. Matrix Theory has passed all tests that have been carried out, some of which are very nontrivial. The construction has a nice generalization to describe compactification of M theory on a

torus T^n . However, it does not seem to be useful for $n > 5$, and other compactification manifolds are (at best) awkward to handle. Another shortcoming of this approach is that it treats the eleventh dimension differently from the other ones.

Another proposal relating superstring and M theory backgrounds to large N limits of certain field theories has been put forward by Maldacena in 1997 [31] and made more precise by Gubser, Klebanov, and Polyakov [32], and by Witten [33] in 1998. (For a review of this subject, see [34].) In this approach, there is a conjectured duality (*i.e.*, equivalence) between a conformally invariant field theory (CFT) in d dimensions and type IIB superstring theory or M theory on an Anti-de-Sitter space (AdS) in $d + 1$ dimensions. The remaining $9 - d$ or $10 - d$ dimensions form a compact space, the simplest cases being spheres. Three examples with unbroken supersymmetry are $AdS_5 \times S^5$, $AdS_4 \times S^7$, and $AdS_7 \times S^4$. This approach is sometimes referred to as *AdS/CFT duality*. This is an extremely active and very promising subject. It has already taught us a great deal about the large N behavior of various gauge theories. As usual, the easiest theories to study are ones with a lot of supersymmetry, but it appears that in this approach supersymmetry breaking is more accessible than in previous ones. For example, it might someday be possible to construct the QCD string in terms of a dual AdS gravity theory, and use it to carry out numerical calculations of the hadron spectrum. Indeed, there have already been some preliminary steps in this direction.

Despite all of the successes that have been achieved in advancing our understanding of superstring theory and M theory, there clearly is still a long way to go. In particular, despite much effort and several imaginative proposals, we still do not have a convincing mechanism for ensuring the vanishing (or extreme smallness) of the cosmological constant for nonsupersymmetric vacua. Superstring theory is a field with very ambitious goals. The remarkable fact is that they still seem to be realistic. However, it may take a few more revolutions before they are attained.

References

- [1] M.B. Green, J.H. Schwarz, and E. Witten, *Superstring Theory*, in 2 vols., Cambridge Univ. Press, 1987.
- [2] J. Polchinski, *String Theory*, in 2 vols., Cambridge Univ. Press, 1998.
- [3] J. Scherk and J. H. Schwarz, *Nucl. Phys.* **B81** (1974) 118.
- [4] T. Yoneya, *Prog. Theor. Phys.* **51** (1974) 1907.

- [5] P. Candelas, G.T. Horowitz, A. Strominger, and E. Witten, *Nucl. Phys.* **B258** (1985) 46.
- [6] Y. Nambu, Notes prepared for the Copenhagen High Energy Symposium (1970).
- [7] T. Goto, *Prog. Theor. Phys.* **46** (1971) 1560.
- [8] A. M. Polyakov, *Phys. Lett.* **103B** (1981) 207.
- [9] M. Virasoro, *Phys. Rev.* **D1** (1970) 2933.
- [10] G. Veneziano, *Nuovo Cim.* **57A** (1968) 190.
- [11] P. Ramond, *Phys. Rev.* **D3** (1971) 2415.
- [12] A. Neveu and J. H. Schwarz, *Nucl. Phys.* **B31** (1971) 86.
- [13] M. B. Green and J. H. Schwarz, *Nucl. Phys.* **B181** (1981) 502; *Nucl. Phys.* bf B198 (1982) 252; *Phys. Lett.* **109B** (1982) 444.
- [14] F. Gliozzi, J. Scherk, and D. Olive, *Phys. Lett.* **65B** (1976) 282.
- [15] L. Alvarez-Gaumé and E. Witten, *Nucl. Phys.* **B234** (1983) 269.
- [16] M.B. Green and J.H. Schwarz, *Phys. Lett.* **149B** (1984) 117.
- [17] D.J. Gross, J.A. Harvey, E. Martinec, and R. Rohm, *Phys. Rev. Lett.* **54** (1985) 502.
- [18] E. Cremmer, B. Julia, and J. Scherk, *Phys. Lett.* **76B** (1978) 409.
- [19] M.B. Green and J.H. Schwarz, *Phys. Lett.* **136B** (1984) 367.
- [20] E. Bergshoeff, E. Sezgin, and P.K. Townsend, *Phys. Lett.* **B189** (1987) 75.
- [21] P.K. Townsend, *Phys. Lett.* **B350** (1995) 184, hep-th/9501068.
- [22] E. Witten, *Nucl. Phys.* **B443** (1995) 85, hep-th/9503124.
- [23] G.T. Horowitz and A. Strominger, *Nucl. Phys.* **B360** (1991) 197.
- [24] J. Polchinski, *Phys. Rev. Lett.* **75** (1995) 4724, hep-th/9510017.
- [25] C. Hull and P. Townsend, *Nucl. Phys.* **B438** (1995) 109, hep-th/9410167.
- [26] J.H. Schwarz, *Phys. Lett.* **B360** (1995) 13, Erratum: *Phys. Lett.* **B364** (1995) 252, hep-th/9508143.

- [27] P.S. Aspinwall, *Nucl. Phys. Proc. Suppl.* **46** (1996) 30, hep-th/9508154.
- [28] J.H. Schwarz, *Phys. Lett.* **B367** (1996) 97, hep-th/9510086.
- [29] E. Witten, *Nucl. Phys.* **B460** (1996) 335, hep-th/9510135.
- [30] O. Bergman, *Nucl. Phys.* **B525** (1998) 104, hep-th/9712211.
- [31] J. Maldacena, *Adv. Theor. Phys.* **2** (1998) 231, hep-th/9711200.
- [32] S.S. Gubser, I.R. Klebanov, and A.M. Polyakov, *Phys. Lett.* **B428** (1998) 105, hep-th/9802109.
- [33] E. Witten, *Adv. Theor. Math. Phys.* **2** (1998) 253, hep-th/9802150.
- [34] O. Aharony, S.S. Gubser, J. Maldacena, H. Ooguri, and Y. Oz, *Phys. Rept.* **323** (2000) 183.