Solution for Chapter 10

(compiled by Xinkai Wu)

A. 10.3 Order of magnitude estimates
(i) Steel wire [C.Y.Mou/90]

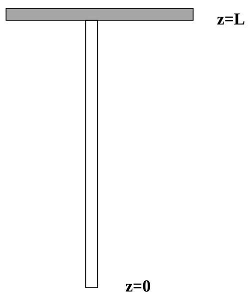

Figure 1: Steel wire

The weight of the wire creates stress inside,

$$T_{zz,z} + \rho g = 0 \Rightarrow T_{zz} = \rho g z$$

The maximum stress is at $z = L$, $T_{zz}^{max} = \rho g L = \epsilon E$, where ϵ is the strain and E is the Young's modulus.

Typically the wire would break if the strain $\epsilon > 10^{-3}$. Hence, the maximum length of a wire is :

$$L = \frac{T_{zz}^{max}}{\rho g} = \frac{\epsilon E}{\rho g} \approx \frac{10^{-3} E_{steel}}{\rho g}$$

Plugging in $E = 210GPa$, $\rho = 7.8\frac{g}{cm^3}$ for steel we get $L \approx 3 \times 10^3 m$,i.e. for a steel wire to break under its own weight it would have to be several kilometers long.

(ii) Non-spherical asteroid

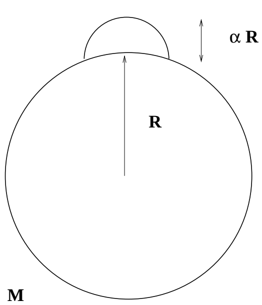

Figure 2: Non-spherical asteroid

Consider first an asteroid of mass M that has a roughly spherical shape with radius R and deviation from sphericity of order αR. Then the typical stress due to self-gravity is

$$T \approx \frac{F}{S} \approx \frac{GM}{R^2} \times \rho\,(\alpha R)^3 \times \frac{1}{(\alpha R)^2} \approx \alpha G \rho^2 R^2$$

If the stress exceeds the elastic limit the asteroid will deform under it becoming more spherical. The maximum size for a non-spherical asteroid is then given by

$$R_{max} = \sqrt{\frac{T_{max}}{\alpha G}} \times \frac{1}{\rho}$$

For non-spherical asteroids, we can take $\alpha \approx 1$. Further, taking $\rho = \rho_{Earth} = 5g/cm^3$, typical maximum strains $\epsilon \approx 10^{-3}$ and maximum stress $T_{max} = \epsilon E \approx 10^8 Pa$, it's then easy to find

$$R_{max} \approx 200km$$

We conclude that the biggest non-spherical asteroids are several hundred kilometers in size.

(iii) Hellium Balloon [N. Niorris/85]

First, let's recognize that the best geometry for the tank is spherical. It maximizes the volume to surface area ratio, and therefore means the lightest tank for a given volume of helium.

Further suppose that the tank has an inner radius of R and thin walls of thickness d. Then the mass of the tank is $M_t = 4\pi\rho_t R^2 d = 3\rho_t V_t d/R$, where V_t is the tank volume.

Treating the helium in the tank and in the balloon as an ideal gas so $PV/T = K_{Boltzman} \times$ (number of Helium molecules), we can write

$$V_b = \frac{P_t V_t}{T_t} \times \frac{T_b}{P_b}$$

The lifting bouyant force on the balloon is $F_b = V_b(\rho_{air} - \rho_{He})g \approx V_b \rho_{air} g$, with g being the acceleration of gravity (assuming that the gas in the balloon is at the same T and P as the surrounding air).

The balloon will lift the tank if the ratio $\kappa = \frac{F_b}{M_t g}$ is greater than one.

Using the expressions above:

$$\kappa = \frac{P_t}{P_b}\,\frac{T_b}{T_t}\left(\frac{R}{3d}\right)\frac{\rho_{air}}{\rho_t}$$

Obviously, the more compressed the gas in the tank is, the bigger the bouyant force. The stress in the tank material is of order $\sigma = P_t R/d$ and so, the maximum pressure the tank can hold is

$$P_t^{max} \approx 10^{-3} E \frac{d}{R}$$

and so

$$\kappa = 0.3 \times 10^{-3} \frac{E}{P_b} \frac{T_b}{T_t} \frac{\rho_{air}}{\rho_t}$$

Using $E = 2 \times 10^6 atm$ for steel, $\frac{\rho_{air}}{\rho_t} = 1.7 \times 10^{-4}$, we get

$$\kappa \approx 0.1 \times \frac{T_b}{T_t}$$

Therefore if the helium in the tank was initially at room temperature it won't be possible to lift the tank with a balloon. It may become feasible if the helium in the tank is cooled to a very low temperature, then is warmed when released into the balloon.

B.

Parts of 10.1 and 10.2: Cylindrical coordinates [by Xinkai Wu and Kip Thorne/02]

10.1a

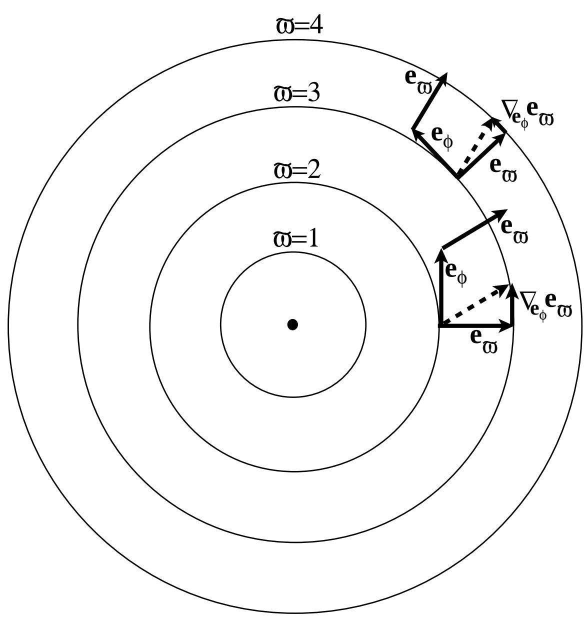

Figure 3: Cylindrical coordinates

It should be evident that $\mathbf{e}_z$ is the same everywere (its length is one everywhere and it always points in the same direction, so $\nabla_k \mathbf{e}_z = 0$ for all k. It should also be evident that all the basis vectors are unchanged as we move in the z direction; each of them maintains its direction and length: $\nabla_z \mathbf{e}_j = 0$ for all j. Therefore, we need only consider the behavior of $\mathbf{e}_\phi$ and $\mathbf{e}_\varpi$ in the ϖ-ϕ plane.

Figure 3 shows the computation of $\nabla_\phi \mathbf{e}_\varpi \equiv \nabla_{\mathbf{e}_\phi} \mathbf{e}_\varpi$ at two different locations in the ϖ-ϕ plane. In each case we examine $\mathbf{e}_\varpi$ at the tail and the tip of $\mathbf{e}_\phi$. We drag $\mathbf{e}_\varpi$ from the tip back to the tail (obtaining the dashed vector), then take its difference from the tail value of $\mathbf{e}_\varpi$ to obtain $\nabla_\phi \mathbf{e}_\varpi \equiv \nabla_{\mathbf{e}_\phi} \mathbf{e}_\varpi$. From the diagrams, it should be evident that $\nabla_\phi \mathbf{e}_\varpi$ always points in the ϕ direction and it has a length that is shorter at larger radii, scaling in fact as $1/\varpi$. In other words:

$$\nabla_\phi \mathbf{e}_\varpi = \frac{\mathbf{e}_\phi}{\varpi}$$

which gives, by the definition $\nabla_k \mathbf{e}_j = \Gamma_{ijk} \mathbf{e}_i$, $\Gamma_{\phi\varpi\phi} = \frac{1}{\varpi}$. By a similar construction one can deduce that

$$\nabla_\phi \mathbf{e}_\phi = \frac{-\mathbf{e}_\varpi}{\varpi}$$

which gives, by the definition $\nabla_k \mathbf{e}_j = \Gamma_{ijk} \mathbf{e}_i$, $\Gamma_{\varpi\phi\phi} = -\frac{1}{\varpi}$ — a value that can also be deduced from the antisymmetry of the connection coefficients on their first two indices, $\Gamma_{\varpi\phi\phi} = -\Gamma_{\phi\varpi\phi}$.

When we move in the ϖ direction rather than the ϕ direction, the basis vectors $\mathbf{e}_\varpi$ and $\mathbf{e}_\phi$ remain unchanged, so $\Gamma_{jk\varpi} = 0$. Therefore the only nonzero connection coefficients are $\Gamma_{\varpi\phi\phi} = -\Gamma_{\phi\varpi\phi} = 1/\varpi$.

The pictorial derivation in Fig. 3 does not fully capture the meaning of the derivative in $\nabla_{\mathbf{e}_\phi} \mathbf{e}_\varpi$. The derivative is really defined by the usual limiting process where one takes differences not at the tip and tail of $\mathbf{e}_\phi$ but rather at the tail and at some distance $\epsilon \ll 1$ up the vector e_ϕ from tail toward tip; and one then divides the difference by ϵ and takes the limit as $\epsilon \to 0$. However, the cruder pictorial derivation in Fig. 3, which ignores the limiting process, is easier to do quickly and illustrates quite clearly what is going on.

10.2

$$\begin{aligned} \nabla \cdot \xi = \xi_{i;i} &= \xi_{i,i} + \Gamma_{iji} \xi_j \\ &= \frac{\partial \xi_\varpi}{\partial \varpi} + \frac{1}{\varpi} \frac{\partial \xi_\phi}{\partial \phi} + \frac{\partial \xi_z}{\partial z} + \Gamma_{\phi\varpi\phi} \xi_\varpi \\ &= \frac{\partial \xi_\varpi}{\partial \varpi} + \frac{1}{\varpi} \frac{\partial \xi_\phi}{\partial \phi} + \frac{\partial \xi_z}{\partial z} + \frac{\xi_\varpi}{\varpi} \end{aligned}$$

10.5 Torsion pendulum [by Xinkai Wu/02]

We'll use cylindrical coordinates (ϖ, ϕ, z), with the fixed end of the wire at $z = 0$ and the end with masses attached at $z = l$. Also we ignore the mass of the wire itself.

(i) The balance of vertical elastic force in the wire and the masses' weight gives the longitudinal strain:

$$\epsilon = \frac{3mg}{\pi a^2 E}$$

(ii) Take a cross section of the wire at z. The restoring torque there due to the elastic force is

$$M = \int_0^a \varpi T_{\phi z} 2\pi\varpi d\varpi$$

while we know that

$$T_{\phi z} = -2\mu\Sigma_{\phi z} = -\mu\frac{\partial \xi_\phi}{\partial z}$$

In the last step of the above equation we've used the expression for $\Sigma_{\phi z}$ in Box. 10.2 and the fact there's no ϕ-dependence due to cylindrical symmetry.

It's not hard to guess that $\xi_\phi = \varpi\frac{z}{l}\phi_0$ with ϕ_0 being the angular displacement at $z = l$. This just corresponds to a rigid rotation of the fibre, in the plane at height z, through an angle $\delta\phi = \frac{z}{l}\phi_0$. (You can also verify explicitly that $\xi_r = 0; \xi_\phi = \varpi\frac{z}{l}\phi_0; \xi_z = z\epsilon$ satisfies the required balance of force inside the wire $\nabla\cdot\mathbf{T} = 0$). This gives for the restoring torque

$$M = -\frac{\pi\mu a^4}{2l}\phi_0$$

The moment of inertia is $I = 3mb^2$. And then the equation of motion $M = I\ddot{\phi}_0$ gives us the period

$$P = 2\pi\left(\frac{6mlb^2}{\pi\mu a^4}\right)^{1/2}$$

Using $\epsilon = \frac{3mg}{\pi a^2 E}$ to eliminate m in favor of ϵ we find

$$P = 2\pi\left(\frac{l}{g}\right)^{1/2}\left(\frac{2b^2E\epsilon}{a^2\mu}\right)^{1/2}$$

(iii) Inverting the above expression we get

$$\frac{lb^2}{a^2} = \frac{P^2 g\mu}{8\pi^2 E\epsilon}$$

Take $P = 1day$ and consider steel wire with $E = 210GPa$, $\mu = 81GPa$, $\epsilon \approx 10^{-3}$, we get

$$\frac{lb^2}{a^2} \approx 4\times 10^{11}m$$

This is a very hard parameter value to achieve, though not totally impossible. For example, one could set $l \approx 10m$, $a \approx 0.01mm = 10$ microns, and $b \approx 2m$. In practice, experimenters performing very high precision mechanical experiments typically use torsion-pendulum periods of order an hour rather than a day.

C.

10.9 Elastica [by Xinkai Wu/02]

(i) Consider the part of the wire between one end and the point a distance z' from this end. The total force (the force applied at the end and the stress force applied by the rest of the wire) exerted on this part must vanish. The $\mathbf{e}_z$ component of this immediately gives

$$Fcos\theta = \int T_{z'z'}dx'dy'$$

while the $\mathbf{e}_x$ component gives

$$Fsin\theta = \int T_{x'z'}dx'dy'$$

Now consider the infinitesimal segment between z' and $z' + dz'$. The total torque exerted on this segment is

$$Fsin\theta dz' + M(z') - M(z + dz')$$

where $M = \int x'T_{z'z'}dx'dy'$ (and by using $T_{z'z'} = -E\xi_{z',z'} = -Ex'\frac{d\theta}{dz'}$ and performing the integral, one gets $M = -D\frac{d\theta}{dz'}$ with the flexural rigidity $D = \frac{Eba^3}{12}$).

This total torque must vanish, which gives

$$Fsin\theta = \frac{dM}{dz'}$$

Combining the above results immediately gives

$$\frac{d^2\theta}{dz'^2} = -\frac{Fsin\theta}{D} = -\frac{sin\theta}{l^2}$$

where we have defined the characteristic length $l \equiv \sqrt{\frac{D}{F}}$. For rubber, $E = 0.002GPa$, and if we take $a = 1cm$, $b = 0.5cm$, and apply a force $F = 10N$, we get $l \approx 9mm$.

(ii) Mathematica gives a solution

$$\theta(z') = 2 \cdot am(\frac{1}{\sqrt{2}}\frac{z'}{l}|2)$$

where $am(u|m)$ is the inverse function of the elliptic integral of the first kind $F(\phi|m)$, namely $\phi = am(u|m) \Leftrightarrow u = F(\phi|m)$. (Recall that $F(\phi|m) = \int_0^\phi \frac{1}{\sqrt{1-msin^2t}}dt$).

(iii) Now to get the shape of the wire we need to find $z(x)$.

In the previous part we obtained $\theta(z')$, whose inverse is just

$$z' = \sqrt{2}l\int_0^{\theta/2}\frac{1}{\sqrt{1-2sin^2t}}dt = \sqrt{2}l\int_0^{\theta/2}\frac{1}{\sqrt{cos2t}}dt$$

(note that this solution has $\theta(z'=0)=0.$) so we get

$$dz' = \frac{l}{\sqrt{2}}\frac{1}{\sqrt{cos\theta}}d\theta$$

therefore we have

$$\begin{aligned} cos\theta &= \frac{dx}{dz'} = \sqrt{2cos\theta}\frac{1}{l}\frac{dx}{d\theta} \\ sin\theta &= \frac{dz}{dz'} = \sqrt{2cos\theta}\frac{1}{l}\frac{dz}{d\theta} \end{aligned}$$

Integrating the second equation using the initial condition $\theta(z=0)=0$ gives

$$cos\theta = \left(\frac{z}{\sqrt{2}} - 1\right)^2$$

Using this result we get

$$\frac{dx}{dz} = \frac{dx/d\theta}{dz/d\theta} = \frac{cos\theta}{sin\theta} = \frac{\left(\frac{z}{\sqrt{2}} - 1\right)^2}{\pm\sqrt{1 - \left(\frac{z}{\sqrt{2}} - 1\right)^4}}$$

Integrating the above equation numerically using Mathematica, we get the plot for $x(z),\ z \in [0, 2\sqrt{2}l]$(which corresponds to positive θ), see Fig. 4

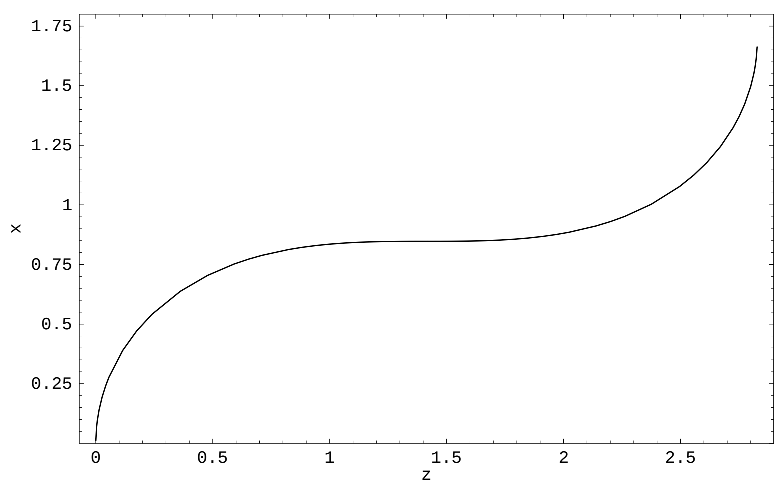

Figure 4: Elastica $x(z)$ in units of l; $z \in [0, 2\sqrt{2}l]$

As x continues increasing, z will decrease(which corresponds to negative θ), and we see that $z(x)$ is a periodic function as that in (b) of Fig. 10.7 of the text, with period $\approx 2 \times 1.7l$, and the crest height is $z_{max} = 2\sqrt{2}l$.

(iv) If anyone of you have a slender wire good for this experiment and would like to give me a demonstration, I'd be very glad to see it!

10.10 Foucault pendulum [by Xinkai Wu/00]

(i) Balance of forces on the mass along the wire gives

$$\mathbf{F} = (mgcos\theta_0 + ml\dot{\theta}_0^2)\mathbf{e}_{z'}$$

where the second term in the above expression is the centripetal force. Using energy conservation

$$\frac{1}{2}m\left(l\dot{\theta}_0\right)^2 = mgl\left(cos\theta_0 - cos\theta_0^{max}\right)$$

we can write the second term as

$$ml\dot{\theta}_0^2 = 2mg\left(cos\theta_0 - cos\theta_0^{max}\right)$$

Therefore

$$F = mgcos\theta_0 + 2mg\left(cos\theta_0 - cos\theta_0^{max}\right) = mg\left(3cos\theta_0 - 2cos\theta_0^{max}\right)$$

For small amplitudes, $cos\theta_0 \approx 1$ and the centripetal force can be neglected, therefore $\mathbf{F} \approx mg\mathbf{e}_{z'}$.

(ii) See Fig. 5. We have

$$\theta(z') = \theta_0 - \phi(z')$$

where $\theta(z')$ is the angle between the tangent of the wire at z' and the vertical direction, and $\phi(z')$ is the angle between the tangent of the wire at z' and the direction of $\mathbf{F}$.

Then using the result of Ex. 10.9, we have

$$\frac{d^2\phi}{dz'^2} = \frac{Fsin\phi}{D} \approx \frac{F\phi}{D}$$

note the relative minus sign on the r.h.s. of the above equation w.r.t. to that in Ex. 10.9: this is a consequence of the fact that now $\mathbf{F}$ is pulling, instead of compressing the wire.

$$\frac{F}{D} = \frac{T_{z'z'}ab}{Ea^3b/12} = \frac{E\epsilon ab}{Ea^3b/12} = \frac{12\epsilon}{a^2} = \frac{1}{\lambda^2}$$

where $\epsilon \equiv \xi_{z',z'}$ and $\lambda \equiv \frac{a}{(12\epsilon)^{1/2}}$.

Solving this differential equation and only keeping the decaying solution (because as z' becomes large, $\theta(z') \to \theta_0$), one finds

$$\phi = const \cdot e^{-z'/\lambda}$$

and therefore

$$\theta(z') = \theta_0 - const \cdot e^{-z'/\lambda}$$

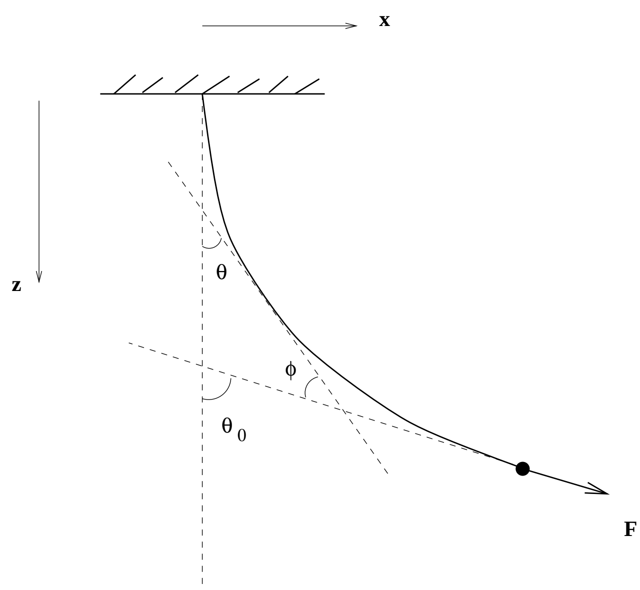

Figure 5: Foucault pendulum

The constant is fixed by the boundary condition $\theta(z'=0)=0$ and we get

$$\theta(z') = \theta_0 \left(1 - e^{-z'/\lambda}\right)$$

(iii)

$$\frac{dx}{dz} = tan\theta \approx \theta = \theta_0 \left(1 - e^{-z'/\lambda}\right) \approx \theta_0 \left(1 - e^{-z/\lambda}\right)$$

where in the last step we've used $z' \approx z$. And the boundary condition is $x(z=0)=0$. This gives

$$x(z) = \left[z - \lambda \left(1 - e^{-z/\lambda}\right)\right] \theta_0$$

Using the fact that $l >> \lambda$, we get

$$x(l) \approx (l-\lambda)\theta_0$$

The equation of motion for the mass is:

$$m\ddot{x}(l) = m(l-\lambda)\ddot{\theta}_0 = -F sin\theta_0 \approx -mg\ddot{\theta}_0$$
$$\text{namely, } \ddot{\theta}_0 + \frac{g}{l-\lambda}\theta_0 = 0$$

where we've used $F \approx mg$ and $sin\theta_0 \approx \theta_0$.

This e.o.m. gives a period

$$P = 2\pi \left(\frac{l-\lambda}{g}\right)^{1/2}$$

(iv) now

$$\delta P = 2\pi \frac{1}{2\sqrt{g(l-\lambda)}} \delta\lambda$$

$$\text{with } \delta\lambda = \frac{b-a}{(12\epsilon)^{1/2}}$$

$$\Rightarrow \frac{\delta P}{P} = \left(\frac{b-a}{l-\lambda}\right)\left(\frac{1}{48\epsilon}\right)^{1/2}$$

$$\approx \left(\frac{b-a}{l}\right)\left(\frac{1}{48\epsilon}\right)^{1/2}$$

D.

10.12 Paraboloidal mirror [[by Alexei Dvoretskii/00]

(i) The equation for a paraboloid with focal length f is

$$z = \frac{r^2}{4f}$$

and for a sphere of radius R,

$$z = R - \sqrt{R^2 - r^2}$$

Choosing $R = 2f$ and expanding for $\frac{r}{R} << 1$, we get

$$z = \frac{r^2}{4f} + \frac{r^4}{64f^3}$$

The vertical displacement of the mirror is therefore $\eta(r) = \frac{r^4}{64f^3}$

(ii) Because of the cylindrical symmetry the laplacian has the simple form

$$\nabla^2 = \frac{1}{r}\frac{\partial}{\partial r} r \frac{\partial}{\partial r}$$

The pressure that must be applied is then given by

$$F = D\nabla^2\nabla^2\eta = \frac{D}{f^3}$$

(ii) The total force is

$$F\pi R^2 = \pi R^2 \frac{D}{f^3} = NS_{zr}$$

Therefore the force applied at each lever is

$$S_{zr} = \frac{\pi D R^2}{N f^3}$$

The associated bending torque is just

$$M = S_{zr} R = \frac{\pi D R^3}{N f^3}$$

(iv) The radial displacement is found from

$$\xi_r = -z\frac{\partial \eta}{\partial r} = -\frac{r^3 z}{16 f^3}$$

(v) The associated expansion and strain are

$$\begin{aligned}
\Theta &= \frac{1}{r}\frac{\partial}{\partial r} r\xi_r = \frac{-r^2 z}{4 f^3} \\
\Sigma_{rr} &= \frac{2\partial\xi_r}{3\partial r} - \frac{\xi_r}{3r} = -\frac{5r^2 z}{48 f^3} \\
\Sigma_{\phi\phi} &= \frac{2\xi_r}{3r} - \frac{\partial\xi_r}{3\partial r} = -\frac{r^2 z}{48 f^3} \\
\Sigma_{zz} &= -\frac{\partial\xi_r}{3\partial r} - \frac{\xi_r}{3r} = \frac{r^2 z}{12 f^3}
\end{aligned}$$

Using

$$\mathbf{T} = -K\Theta\mathbf{g} - 2\mu\mathbf{\Sigma}$$

we see that the maximum stress is T_{rr} at the rim

$$T_{max} = \frac{R^2 h}{8 f^3}\left(K + \frac{5}{6}\mu\right)$$

After straightforward manipulation

$$T_{max} = \frac{3-2\nu}{256(1-2\nu)(1+\nu)} E \left(\frac{2R}{f}\right)^3 \frac{h}{R} = \frac{(3-2\nu)R^2 h E}{32(1-2\nu)(1+\nu) f^3}$$

Now, if the mirror is not to break then it should be that $T_{max} \leq 10^{-4}E$. $\nu = 0.25$ for glass. Then there's a limit on how "fast" and thick a mirror can be at the same time

$$\left(\frac{2R}{f}\right)^3 \frac{h}{R} \leq 1.5 \times 10^{-2}$$

Ex. 10.4 Fracture of a pipe [by Xinkai Wu/02]

(i) This part is straightforward, and similiar to (actually simpler than) what you did in Ex. 10.7, so we omit the details here.

(ii) Cylindrical symmetry together with translational symmetry along the z direction implies that $\xi_\phi = 0$, $\xi_z = 0$, and there is no ϕ or z dependence in the only nonzero displacement ξ_r. (note that in this problem, we use r instead of ϖ to denote the radius)

One easily finds the strain tensor:

$$S_{rr} = \frac{\partial \xi_r}{\partial r}$$
$$S_{\phi\phi} = \frac{\xi_r}{r}$$

and all other components vanish.

From the above strain tensor one finds the expansion and the non-vanishing components of the shear:

$$\begin{aligned}
\Theta &= \frac{\partial \xi_r}{\partial r} + \frac{\xi_r}{r} \\
\Sigma_{rr} &= \frac{2}{3}\frac{\partial \xi_r}{\partial r} - \frac{1}{3}\frac{\xi_r}{r} \\
\Sigma_{\phi\phi} &= \frac{-1}{3}\frac{\partial \xi_r}{\partial r} + \frac{2}{3}\frac{\xi_r}{r} \\
\Sigma_{zz} &= -\frac{1}{3}\left(\frac{\partial \xi_r}{\partial r} + \frac{\xi_r}{r}\right)
\end{aligned}$$

(the shear tensor can also be obtained using the formulas in Box 10.2)

Eq. (10.35) gives

$$f_i = K\Theta_{;i} + 2\mu\Sigma_{ij;j}$$

Let's compute $\Sigma_{ij;j}$:

$$\Sigma_{ij;j} = \Sigma_{ir,r} + \Gamma_{i\phi\phi}\Sigma_{\phi\phi} + \Gamma_{\phi r\phi}\Sigma_{ir}$$

this gives the only non-vanishing component of $\Sigma_{ij;j}$

$$\begin{aligned}
\Sigma_{rj;j} &= \frac{\partial \Sigma_{rr}}{\partial r} - \frac{\Sigma_{\phi\phi}}{r} + \frac{\Sigma_{rr}}{r} \\
&= \frac{2}{3}\frac{\partial}{\partial r}\left(\frac{\partial \xi_r}{\partial r} + \frac{\xi_r}{r}\right) \\
&= \frac{2}{3}\frac{\partial \Theta}{\partial r}
\end{aligned}$$

Therefore

$$f_r = \left(K + \frac{4\mu}{3}\right)\frac{\partial \Theta}{\partial r}$$

and

$$f_r = 0 \Rightarrow \frac{\partial \Theta}{\partial r} = 0$$

namely Θ is a constant, independent of the location r.

This gives us the equation

$$\frac{d\xi_r}{dr} + \frac{\xi_r}{r} = constant$$

solving which gives

$$\xi_r(r) = Ar + \frac{B}{r}$$

with A and B are constants to be fixed by boundary condition.

(iii) Using

$$T_{ij} = -K\Theta g_{ij} - 2\mu\Sigma_{ij}$$

we find that the off-diagonal T_{ij}'s vanish, and

$$T_{rr} = \left(-2K - \frac{2\mu}{3}\right) A + 2\mu \frac{B}{r^2}$$

which combined with the boundary condition $T_{rr}(R_1) = p$, $T_{rr}(R_2) = 0$ fixes the constants

$$A = \frac{p}{2K + \frac{2\mu}{3}} \frac{R_1^2}{R_2^2 - R_1^2}, \quad B = \frac{p}{2\mu} \frac{R_1^2 R_2^2}{R_2^2 - R_1^2}$$

thus

$$T_{rr} = p\frac{R_1^2}{R_2^2 - R_1^2}\left(\frac{R_2^2}{r^2} - 1\right)$$

also we find

$$\begin{aligned} T_{\phi\phi} &= -p\frac{R_1^2}{R_2^2 - R_1^2}\left(\frac{R_2^2}{r^2} + 1\right) \\ T_{zz} &= \left(-K + \frac{2\mu}{3}\right)\frac{p}{K + \frac{\mu}{3}} \frac{R_1^2}{R_2^2 - R_1^2} = -\nu\frac{2pR_1^2}{R_2^2 - R_1^2} \end{aligned}$$

where we've used the eq. (10.44) for ν. Note that T_{zz} is independent of r.

And we find the force inside the pipe walls along the z direction to be

$$F_z = T_{zz}\pi\left(R_2^2 - R_1^2\right) = -2\pi\nu p R_1^2$$

(iv) the skew angle is given by

$$\begin{aligned} \phi &= -2\frac{\partial \xi_r}{\partial r} = 2p\frac{R_1^2}{R_2^2 - R_1^2}\frac{1}{\mu}\left(\frac{R_2^2}{2r^2} - \frac{1}{\frac{2K}{\mu} + \frac{2}{3}}\right) \\ &= 2p\frac{R_1^2}{R_2^2 - R_1^2}\frac{1}{\mu}\left(\frac{R_2^2}{2r^2} - \frac{1}{5}\right) \end{aligned}$$

where to get the last line we've used the fact that $K/\mu = \frac{2(1+\nu)}{3(1-2\nu)}$ and $\nu = 0.3$ for our case.

So we see that the skew angle takes its maximus value at $r = R_1$

$$\phi_{max} = \frac{2p}{\mu} \frac{\frac{\lambda}{2} - \frac{1}{5}}{\lambda - 1}$$

where we've defined $\lambda \equiv \frac{R_2^2}{R_1^2}$.

As λ increases, ϕ_{max} decreases, thus we find the smallest allowable λ

$$\lambda = \frac{\frac{1}{5} - \frac{\mu\phi_{max}}{2p}}{\frac{1}{2} - \frac{\mu\phi_{max}}{2p}}$$

Plugging in the numbers we find

$$\lambda = 1.21$$

and the minimum wall thickness

$$d = R_2 \left(1 - \frac{1}{\sqrt{\lambda}}\right) = 2.73mm$$