Chapter 21

Kinetic Theory of Warm Plasmas

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21.1 Overview

At the end of Chap. 20, we showed how to generalize cold-plasma two-fluid theory so as to accommodate several distinct plasma beams, and thereby we discovered an instability. If the beams are not individually monoenergetic (i.e. cold), as we assumed they were in Chap. 20, but instead have broad velocity dispersions that overlap in velocity space (i.e. if the beams are *warm*), then the approach of Chap. 20 cannot be used, and a more powerful description of the plasma is required. Chapter 20's approach entailed specifying the positions and velocities of specific groups of particles (the "fluids"); this is an example of a *Lagrangian* description. It turns out that the most robust and powerful method for developing the kinetic theory of warm plasmas is an *Eulerian* one in which we specify how many particles are to be found in a fixed volume of one-particle phase space.

In this chapter, using this Eulerian approach, we develop the kinetic theory of plasmas. We begin in Sec. 21.2 by introducing kinetic theory's one-particle *distribution function*, $f(\mathbf{x}, \mathbf{v}, t)$ and recovering its evolution equation (the *Vlasov* equation), which we have met previously in Chap. 2. We then use the Vlasov equation to derive the two-fluid formalism used in Chap. 20 and to deduce some physical approximations that underlie the two-fluid description of plasmas.

In Sec. 21.3 we explore the application of the Vlasov equation to Langmuir waves—the one-dimensional electrostatic modes in an unmagnetized plasma that we met in Chap. 20. Using kinetic theory, we rederive the Bohm-Gross dispersion relation for Langmuir waves, and as a bonus we uncover a physical damping mechanism, called *Landau damping*, that did not and cannot emerge from the two-fluid analysis of Chap. 20. This subtle process leads to the transfer of energy from a wave to those particles that can "surf" or "phase-ride" the wave (i.e. those whose velocity resolved parallel to the wave vector is just slightly lses than the wave's phase speed). We show that Landau damping works because there are usually fewer particles traveling faster than the wave and augmenting its energy density than those traveling slower and extracting energy from it. However, in a collisionless plasma, the particle distributions need not be Maxwellian. In particular, it is possible for a plasma to possess an "inverted" particle distribution with more fast ones than slow ones; then there is a net injection of particle energy into the waves, which creates an instability. In Sec. 21.3 we use kinetic theory to derive a necessary and sufficient criterion for instability due to this cause.

In Sec. 21.4 we examine in greater detail the physics of Landau damping and show that it is an intrinsically nonlinear phenomenon; and we give a semi-quantitative discussion of *Nonlinear Landau damping*, prefatory to a more detailed treatment of some other nonlinear plasma effects in the following chapter.

Although the kinetic-theory, Vlasov description of a plasma that is developed and used in this chapter is a great improvement on the two-fluid description of Chap. 20, it is still an approximation; and some situations require more accurate descriptions. We conclude this chapter by introducing greater accuracy via *N-particle distribution functions*, and as applications we use them (i) to explore the approximations underlying the Vlasov description, and (ii) to explore two-particle correlations that are induced in a plasma by Coulomb interactions and the influence of those correlations on a plasma's equation of state.

21.2 Basic Concepts of Kinetic Theory and its Relationship to Two-Fluid Theory

21.2.1 Distribution Function and Vlasov Equation

In Chap. 2 we introduced the number density of particles in phase space, called the *distribution function* $\mathcal{N}(\mathcal{P}, \vec{p})$. We showed that, this quantity is Lorentz invariant and that it satisfies the Vlasov equation (2.90) and (2.92); and we interpreted this equation as $\mathcal{N}$ being constant along the phase-space trajectory of any freely moving particle.

In order to comply with the conventions of the plasma-physics community, we now change notation in a manner described by Eq. (2.12) and the associated text: We use velocity $\mathbf{v}$ rather than momentum as an independent variable, we denote the distribution function by $f(\mathbf{v}, \mathbf{x}, t)$, and we normalize it so that

$$\int f(\mathbf{v}, \mathbf{x}, t) dV_v = n(\mathbf{x}, t) \tag{21.1}$$

where $n(\mathbf{x}, t)$ is the particle space density at that point in spacetime and $dV_v \equiv dv_x dv_y dv_z$ is the three-dimensional volume element of velocity space. (For simplicity, we shall also restrict ourselves to nonrelativistic speeds; the generalization to relativistic plasma theory is straightforward, though seldom used.)

This one-particle distribution function $f(\mathbf{v}, \mathbf{x}, t)$ and its resulting kinetic theory give a good description of a plasma in the regime of large Debye number, $N_D \gg 1$—which includes almost all plasmas that occur in the universe; cf. Sec. 19.3.2 and Fig. 19.1. The reason is that, when $N_D \gg 1$, we can define $f(\mathbf{v}, \mathbf{x}, t)$ by averaging over a physical-space volume that is large compared to the interparticle spacing and that thus contains many particles, but is still small compared to the Debye length. By such an average—the starting point of kinetic theory—, the electric fields of individual particles are made unimportant, and the Coulomb-interaction-induced correlations between pairs of particles are made unimportant. In the last section of this chapter we shall use a 2-particle distribution function to explore this issue in detail.

In Chap. 2 we showed that in the absence of collisions (a good assumption for plasmas!), the distribution function evolves in accord with the Vlasov equation (2.90), (2.92). We shall now rederive the Vlasov equation beginning with the law of conservation of particles for each species $s = e$ (electrons) and p (protons):

$$\frac{\partial f_s}{\partial t} + \boldsymbol{\nabla} \cdot (f_s \mathbf{v}) + \boldsymbol{\nabla}_v \cdot (f_s \mathbf{a}) \equiv \frac{\partial f_s}{\partial t} + \frac{\partial (f_s v_j)}{\partial x_j} + \frac{\partial (f_s a_j)}{\partial v_j} = 0 \ . \tag{21.2}$$

Here

$$\mathbf{a} = \frac{d\mathbf{v}}{dt} = \frac{q_s}{m_s}(\mathbf{E} + \mathbf{v} \times \mathbf{B}) \tag{21.3}$$

is the electromagnetic acceleration of a particle of species s, which has mass m_s and charge q_s, and $\mathbf{E}$ and $\mathbf{B}$ are the electric and magnetic fields averaged over the same volume as is used in constructing f. Equation (21.2) has the standard form for a conservation law: the time derivative of a density (in this case density of particles in phase space, not just physical space), plus the divergence of a flux (in this case the spatial divergence of the particle flux, $f\mathbf{v} = f d\mathbf{x}/dt$, in the physical part of phase space plus the velocity divergence of the particle flux, $f\mathbf{a} = f d\mathbf{v}/dt$, in velocity space) is equal to zero.

Now $\mathbf{x}, \mathbf{v}$ are independent variables, so that $\boldsymbol{\nabla} \cdot \mathbf{v} = 0$ and $\boldsymbol{\nabla} \mathbf{x} = 0$. In addition, $\mathbf{E}$ and $\mathbf{B}$ are functions of $\mathbf{x}, t$ not $\mathbf{v}$, and the term $\mathbf{v} \times \mathbf{B}$ is perpendicular to $\mathbf{v}$. Therefore,

$$\boldsymbol{\nabla}_v \cdot (\mathbf{E} + \mathbf{v} \times \mathbf{B}) = 0 \ . \tag{21.4}$$

These facts permit us to pull $\mathbf{v}$ and $\mathbf{a}$ out of the derivatives in Eq. (21.2), thereby obtaining

$$\frac{\partial f_s}{\partial t} + (\mathbf{v} \cdot \boldsymbol{\nabla}) f_s + (\mathbf{a} \cdot \boldsymbol{\nabla}_v) f_s \equiv \frac{\partial f_s}{\partial t} + \frac{dx_j}{dt}\frac{\partial f_s}{\partial x_j} + \frac{dv_j}{dt}\frac{\partial f_s}{\partial v_j} = 0 \ . \tag{21.5}$$

We recognize this as the statement that f_s is a constant along the trajectory of a particle in phase space,

$$\frac{df_s}{dt} = 0 \ , \tag{21.6}$$

which is the *Vlasov equation* for species s. Equation (21.6) tells us that, when the space density near a given particle increases, the velocity-space density must decrease, and vice versa. Of course, if we find that other forces or collisions are important in some situation, we can represent them by extra terms added to the right hand side of the Vlasov equation (21.6) in the manner of the Boltzman transport equation (2.96); cf. Sec. 2.8.

So far, we have treated the electromagnetic field as being somehow externally imposed. However, it is actually produced by the net charge and current densities associated with the two particle species. These are expressed in terms of the distribution function by

$$\rho_e = \sum_s q_s \int f_s \, dV_v \ , \quad \mathbf{j} = \sum_s q_s \int f_s \mathbf{v} \, dV_v \ . \tag{21.7}$$

These equations, together with Maxwell's equations and the Vlasov equation (21.5) with $\mathbf{a} = d\mathbf{v}/dt$ given by the Lorentz force law (21.3), form a complete set of equations for the structure and dynamics of a plasma. They constitute the kinetic theory of plasmas.

21.2.2 Relation of Kinetic Theory to Two-Fluid Theory

Before developing techniques to solve the Vlasov equation, we shall first relate it to the two-fluid approach used in the previous chapter. We begin doing so by constructing the moments of the distribution function f_s, defined by

$$\begin{aligned} n_s &= \int f_s \, dV_v \ , \\ \mathbf{u}_s &= \frac{1}{n_s} \int f_s \mathbf{v} \, dV_v \ , \\ \mathsf{P}_s &= m_s \int f_s (\mathbf{v} - \mathbf{u}_s) \otimes (\mathbf{v} - \mathbf{u}_s) \, dV_v \ . \end{aligned} \tag{21.8}$$

These are the density, the mean fluid velocity and the pressure tensor for species s. (Of course, P_s is just the three-dimensional stress tensor T_s [Eqs. (2.46) and (2.53)] evaluated in the rest frame of the fluid.)

By integrating the Vlasov equation (21.5) over velocity space and using

$$\begin{aligned} \int (\mathbf{v} \cdot \boldsymbol{\nabla}) f_s \, dV_v &= \int \boldsymbol{\nabla} \cdot (f_s \mathbf{v}) \, dV_v = \boldsymbol{\nabla} \cdot \int f_s \mathbf{v} \, dV_v \ , \\ \int (\mathbf{a} \cdot \boldsymbol{\nabla}_v) f_s \, dV_v &= \mathbf{a} \cdot \int \boldsymbol{\nabla}_v f_s \, dV_v = 0 \ , \end{aligned} \tag{21.9}$$

together with Eq. (21.8), we obtain the continuity equation

$$\frac{\partial n_s}{\partial t} + \boldsymbol{\nabla} \cdot (n_s \mathbf{u}_s) = 0 \tag{21.10}$$

for each particle species s. [It should not be surprising that the Vlasov equation implies the continuity equation, since the Vlasov equation is equivalent to the conservation of particles in phase space (21.2), while the continuity equation is just the conservation of particles in physical space.]

The continuity equation is the first of the two fundamental equations of two-fluid theory. The second is the equation of motion, i.e. the evolution equation for the fluid velocity $\mathbf{u}_s$. To derive this, we multiply the Vlasov equation (21.5) by the particle velocity $\mathbf{v}$ and then integrate over velocity space, i.e. we compute the Vlasov equation's first moment. The details are a useful exercise for the reader (Ex. 21.1); the result is

$$n_s m_s \left(\frac{\partial \mathbf{u}_s}{\partial t} + (\mathbf{u}_s \cdot \boldsymbol{\nabla}) \mathbf{u}_s \right) = -\boldsymbol{\nabla} \cdot \mathsf{P}_s + n_s q_s (\mathbf{E} + \mathbf{u}_s \times \mathbf{B}) \ , \tag{21.11}$$

which is identical with Eq. (20.14).

A difficulty now presents itself. We can compute the charge and current densities from n_s and $\mathbf{u}_s$ using Eqs. (21.8). However, we do not yet know how to compute the pressure tensor $\mathbf{P}_s$. We could derive an equation for the evolution of $\mathbf{P}_s$ by taking the second moment of the Vlasov equation (i.e. multipling it by $\mathbf{v} \otimes \mathbf{v}$ and integrating over velocity space), but that evolution equation would involve an unknown third moment of f_s on the right hand side, $\mathbf{M}_3 = \int f_s \mathbf{v} \otimes \mathbf{v} \otimes \mathbf{v} \, dV_v$ which is related to the heat-flux tensor. In order to determine the evolution of this $\mathbf{M}_3$, we would have to construct the third moment of the Vlasov equation, which would involve the fourth moment of f_s as a driving term, and so on. Clearly, this procedure will never terminate unless we introduce some additional relationship between the moments. Such a relationship, called a *closure relation*, permits us to build a self-contained theory involving only a finite number of moments.

For the two-fluid theory of Chap. 20, the closure relation that we implicitly used was the same idealization that one makes when regarding a fluid as perfect, namely that the heat-flux tensor vanishes. This idealization is less well justified in a collisionless plasma, with its long long mean free paths, than in a normal gas or liquid with its short mean free paths.

An example of an alternative closure relation is one that is appropriate if radiative processes thermostat the plasma to a particular temperature so T_s =constant; then we can set $\mathbf{P}_s = n_s k_B T_s \mathbf{g} \propto n_s$ where $\mathbf{g}$ is the metric tensor. Clearly, a fluid theory of plasmas can be no more accurate than its closure relation.

21.2.3 Jeans' Theorem.

Let us now turn to the difficult task of finding solutions to the Vlasov equation. There is an elementary (and, after the fact, obvious) method to write down a class of solutions that are often useful. This is based on Jeans' theorem (named after the astronomer who first drew attention to it in the context of stellar dynamics[1]).

Suppose that we know the particle acceleration $\mathbf{a}$ as a function of $\mathbf{v}$, $\mathbf{x}$, and t. (We assume this for pedagogical purposes; it is not necessary for our final conclusion). Then, for any particle with phase space coordinates $(\mathbf{x}_0, \mathbf{v}_0)$ specified at time t_0, we can (at least in principle) compute the particle's future motion, $\mathbf{x} = \mathbf{x}(\mathbf{x}_0, \mathbf{v}_0, t), \mathbf{v} = \mathbf{v}(\mathbf{x}_0, \mathbf{v}_0, t)$. These particle trajectories are the *characteristics* of the Vlasov equation, analogous to the characteristics of the equations for one-dimensional supersonic fluid flow which we studied in Sec. 16.4 (see Fig. 16.6). Now, for many choices of the acceleration $\mathbf{a}(\mathbf{v}, \mathbf{x}, t)$, there are *constants of the motion* also known as *integrals of the motion* that are preserved along the particle trajectories. Simple examples, familiar from elementary mechanics, include the energy (for a time-independent plasma) and the angular momentum (for a spherically symmetric plasma). These integrals can be expressed in terms of the initial coordinates $(\mathbf{x}_0, \mathbf{v}_0)$. If we know n constants of the motion, then only $6 - n$ additional variables need be chosen from $(\mathbf{x}_0, \mathbf{v}_0)$ to specify the motion completely.

Now, the Vlasov equation tells us that f_s is constant along a trajectory in $\mathbf{x} - \mathbf{v}$ space. Therefore, f_s must, in general be expressible as a function of $(\mathbf{x}_0, \mathbf{v}_0)$. Equivalently, it can be rewritten as a function of the n constants of the motion and the remaining $6-n$ initial phase-space coordinates. However, there is no requirement that it actually depend on *all* of these

[1] Jeans (1926)

variables. In particular, any function of the integrals of motion alone that is independent of the remaining initial coordinates will satisfy the Vlasov equation (21.5). This is *Jeans' Theorem.* In words, *functions of constants of the motion take constant values along actual dynamical trajectories in phase space and therefore satisfy the Vlasov equation.*

Of course, a situation may be so complex that no integrals of the particles' equation of motion can be found, in which case, Jeans' theorem is useless. Alternatively, there may be integrals but the initial conditions may be sufficiently complex that extra variables are required to determine f_s. However, it turns out that in a wide variety of applications, particularly those with symmetries such as time independence $\partial f_s/\partial t = 0$, simple functions of simple integrals suffice to describe a plasma's distribution functions.

We have already met and used a variant of Jeans' theorem in our analysis of statistical equilibrium in Sec. 3.4. There the statistical mechanics distribution function ρ was found to depend only on the integrals of the motion.

We have also, unknowingly, used Jeans' theorem in our discussion of Debye screening in a plasma (Sec. 19.3.1). To understand this, let us suppose that we have a single isolated positive charge at rest in a stationary plasma ($\partial f_s/\partial t = 0$), and we want to know the electron distribution function in its vicinity. Let us further suppose that the electron distribution at large distances from the charge is known to be Maxwellian with temperature T, i.e. $f_e(\mathbf{v}, \mathbf{x}, t) \propto \exp(-\frac{1}{2}m_e v^2/k_B T)$. Now, the electrons have an energy integral, $E = \frac{1}{2}m_e v^2 - e\Phi$, where Φ is the electrostatic potential. As Φ becomes constant at large distance from the charge, we can therefore write $f_e \propto \exp(-E/k_B T)$ at large distance. However, the particles near the charge must have traveled there from large distance and so must have this same distribution function. Therefore, close to the charge,

$$f_e \propto e^{-E/k_B T} = e^{-[(m_e v^2/2 - e\Phi)/k_B T]} \,, \tag{21.12}$$

and the electron density is obtained by integration over velocity

$$n_e = \int f_e \, dV_v \propto e^{(e\Phi/k_B T)} \,. \tag{21.13}$$

This is just the Boltzmann distribution that we asserted to be appropriate in Sec. 19.3.1.

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EXERCISES

Exercise 21.1 *Derivation: Two-fluid Equation of Motion*
Derive the two-fluid equation of motion (21.11) by multiplying the Vlasov equation (21.5) by $\mathbf{v}$ and integrating over velocity space.

Exercise 21.2 *Example: Positivity of Distribution Function*
The one-particle distribution function $f(\mathbf{x}, \mathbf{v}, t)$ ought not to become negative if it is to remain physical. Show that this is guaranteed if it initially is everywhere nonnegative and it evolves by the collisionless Vlasov equation.

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21.3 Electrostatic Waves in an Unmagnetized Plasma: Landau Damping

21.3.1 Formal Dispersion Relation

As our principal application of the kinetic theory of plasmas, we shall explore its predictions for the dispersion relations, stability, and damping of longitudinal, electrostatic waves in an unmagnetized plasma—Langmuir waves and ion acoustic waves. When studying these waves in Sec. 20.3 using two-fluid theory, we alluded time and again to properties of the waves that could not be derived by fluid techniques. Our goal, now, is to elucidate those properties using kinetic theory. As we shall see, their origin lies in the plasma's velocity-space dynamics.

Consider an electrostatic wave propagating in the z direction. Such a wave is one dimensional in that the electric field points in the z direction, $\mathbf{E} = E\mathbf{e}_z$, and varies as $e^{i(kz-\omega t)}$ so it depends only on z and not on x or y; the distribution function similarly varies as $e^{i(kz-\omega t)}$ and is independent of x, y; and the Vlasov, Maxwell, and Lorentz force equations produce no coupling of particle velocities v_x, v_y into the z direction. This suggests the introduction of one-dimensional distribution functions, obtained by integration over v_x and v_y:

$$F_s(v, z, t) \equiv \int f_s(v_x, v_y, v = v_z, z, t) dv_x dv_y \ . \tag{21.14}$$

Here and throughout we suppress the subscript z on v_z.

Restricting ourselves to weak waves so nonlinearities can be neglected, we linearize the one-dimensional distribution functions:

$$F_s(v, z, t) \simeq F_{s0}(v) + F_{s1}(v, z, t) \ . \tag{21.15}$$

Here $F_{s0}(v)$ is the distribution function of the unperturbed particles ($s = e$ for electrons and $s = p$ for protons) in the absence of the wave and F_{s1} is the perturbation induced by and linearly proportional to the electric field E. The evolution of F_{s1} is governed by the linear approximation to the Vlasov equation (21.5):

$$\frac{\partial F_{s1}}{\partial t} + v\frac{\partial F_{s1}}{\partial z} + \frac{q_s E}{m_s}\frac{dF_{s0}}{dv} \ . \tag{21.16}$$

We seek a monochromatic, plane-wave solution to this Vlasov equation, so $\partial/\partial t \to -i\omega$ and $\partial/\partial z \to ik$ in Eq. (21.16). Solving the resulting equation for F_{s1}, we obtain an equation for F_{s1} in terms of E:

$$F_{s1} = \frac{-iq_s}{m_s(\omega - kv)}\frac{dF_{s0}}{dv}E \ . \tag{21.17}$$

This equation implies that the charge density associated with the wave is related to the electric field by

$$\rho_e = \sum_s q_s \int_{-\infty}^{+\infty} F_{s1} dv = \left(\sum_s \frac{-iq_s^2}{m_s} \int_{-\infty}^{+\infty} \frac{F'_{s0}\, dv}{\omega - kv} \right) E \ , \tag{21.18}$$

where the prime denotes a derivative with respect to v: $F'_{s0} = dF_{s0}/dv$.

A quick route from here to the waves' dispersion relation is to insert this charge density into Poisson's equation $\boldsymbol{\nabla} \cdot \mathbf{E} = ikE = \rho_e/\epsilon_0$ and note that both sides are proportional to E, so a solution is possible only if

$$1 + \sum_s \frac{q_s^2}{m_s \epsilon_0 k} \int_{-\infty}^{+\infty} \frac{F'_{s0}\, dv}{\omega - kv} = 0 \; . \tag{21.19}$$

An alternative route, which makes contact with the general analysis of waves in a dielectric medium (Sec. 20.2), is developed in Ex. 21.3 below. This route reveals that the dispersion relation is given by the vanishing of the zz component of the dielectric tensor, which we denoted ϵ_3 in Chap. 20 [cf. Eq. (20.49)], and it shows that ϵ_3 is given by expression (21.19):

$$\epsilon_3(\omega, k) = 1 + \sum_s \frac{q_s^2}{m_s \epsilon_0 k} \int_{-\infty}^{+\infty} \frac{F'_{s0}\, dv}{\omega - kv} = 0 \; . \tag{21.20}$$

Since $\epsilon_3 = \epsilon_{zz}$ is the only component of the dielectric tensor that we shall meet in this chapter, we shall simplify notation henceforth by omitting the subscript 3.

The form of the dispersion relation (21.20) suggests that we combine the unperturbed electron and proton distribution functions $F_{e0}(v)$ and $F_{p0}(v)$ to produce a single, unified distribution function

$$F(v) = F_{e0}(v) + \frac{m_e}{m_p} F_{p0}(v) \; , \tag{21.21}$$

in terms of which Eq. (21.20) takes the form

$$\epsilon(\omega, k) = 1 + \frac{e^2}{m_e \epsilon_0 k} \int_{-\infty}^{+\infty} \frac{F'\, dv}{\omega - kv} = 0 \tag{21.22}$$

Note that each proton is weighted less heavily than each electron by a factor $m_e/m_p = 1/1860$ in the unified distribution function (21.21) and the dispersion relation (21.22). This is due to the protons' greater inertia and corresponding weaker response to an applied electric field, and it causes the protons to be of no importance in Langmuir waves (Sec. 21.3.5 below). However, in ion-acoustic waves (Sec. 21.3.6), the protons can play an important role because large numbers of them may move with thermal speeds that are close to the waves' phase velocity and thereby can interact resonantly with the waves.

21.3.2 Two-Stream Instability

As a first application of the general dispersion relation (21.22), we use it to rederive the dispersion relation (20.76) associated with the cold-plasma two-stream instability of Sec. 20.7.

We begin by performing an integration by parts on the general dispersion relation (21.22), obtaining:

$$\frac{e^2}{m_e \epsilon_0} \int_{-\infty}^{+\infty} \frac{F dv}{(\omega - kv)^2} = 1 \; . \tag{21.23}$$

We then presume, as in Sec. 20.7, that the fluid consists of two or more streams of cold particles (protons or electrons) moving in the z direction with different fluid speeds u_1, u_2,

..., so $F(v) = n_1\delta(v - u_1) + n_2\delta(v - u_2) + \ldots$. Here n_j is the number density of particles in stream j if the particles are electrons, and m_e/m_p times the number density if they are protons. Inserting this $F(v)$ into Eq. (21.22) and noting that $n_j e^2/m_e\epsilon_0$ is the squared plasma frequency ω_{pj}^2 of stream j, we obtain the dispersion relation

$$\frac{\omega_{p1}^2}{(\omega - ku_1)^2} + \frac{\omega_{p2}^2}{(\omega - ku_2)^2} + \ldots = 1\,, \tag{21.24}$$

which is identical to the dispersion relation Eq. (20.76) used in our analysis of the two-stream instability.

Clearly the general dispersion relation (21.23) [or equally well (21.22)] provides us with a tool for exploring how the two-stream instability is influenced by a warming of the plasma, i.e. by a spread of particle velocities around the mean, fluid velocity of each stream. We shall explore this in Sec. 21.4 below.

21.3.3 The Landau Contour

The general dispersion relation (21.22) has a troubling feature: for real ω and k its integrand becomes singular at $v = \omega/k =$ (the waves' phase velocity) unless dF/dv vanishes there, which is generically unlikely. This tells us that if, as we shall assume, k is real, then ω cannot be real, except, perhaps, for a non-generic mode whose phase velocity happens to coincide with a velocity for which $dF/dv = 0$.

With ω/k complex, we must face the possibility of some subtlety in how the integral over v in the dispersion relation (21.22) is performed—the possibility that we may have to make v complex in the integral and follow some special route in the complex velocity plane from $v = -\infty$ to $v = +\infty$. Indeed, there is such a subtlety, as Lev Landau has shown.[2] Our simple derivation of the dispersion relation, above, cannot reveal this subtlety—and, indeed, is suspicious, since in going from Eq. (21.16) to (21.17) our derivation entailed dividing by $\omega - kv$ which vanishes when $v = \omega/k$, and dividing by zero is always a suspicious practice. Faced by this conundrum, Landau developed a more sophisticated derivation of the dispersion relation, one based on posing generic initial data for electrostatic waves, then evolving those data forward in time and identifying the plasma's electrostatic modes by their late-time sinusoidal behaviors, and finally reading off the dispersion relation for the modes from the equations for the late-time evolution. We shall sketch a variant of this analysis:

For simplicity, from the outset we restrict ourselves to plane waves propagating in the z direction with some fixed, real wave number k, so the linearized one-dimensional distribution function and the electric field have the forms

$$F_s(v, z, t) = F_{s0}(v) + F_{s1}(v, t)e^{ikz}\,, \quad E(t)e^{ikz}\,. \tag{21.25}$$

At $t = 0$ we pose initial data $F_{s1}(v, 0)$ for the electron and proton velocity distributions; these data determine the initial electric field $E(0)$ via Poisson's equation. We presume that these initial distributions [and also the unperturbed plasma's velocity distribution $F_{s0}(v)$] are analytic functions of velocity v, but aside from this constraint the $F_{s1}(v, 0)$ are generic. (A

[2] Landau, L. D. 1946. *J. Phys. USSR*, **10**, 25.

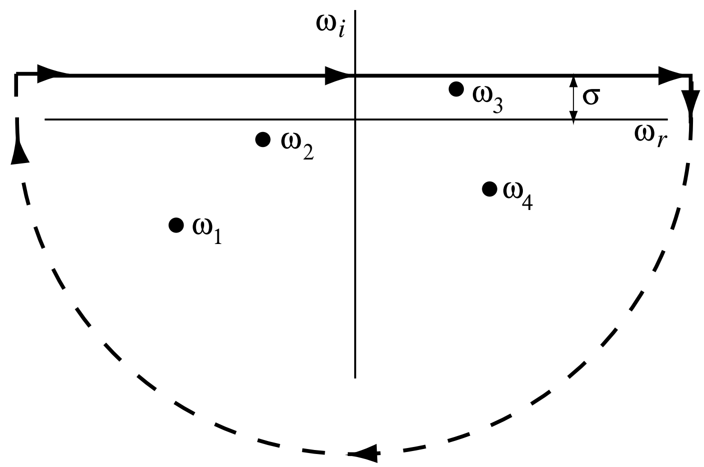

Fig. 21.1: Contour of integration for evaluating $E(t)$ [Eq. (21.26)] as a sum over residues of the integrand's poles—the modes of the plasma.

Maxwellian distribution is analytic, and most any physically reasonable initial distribution can be well approximated by an analytic function.)

We then evolve these initial data forward in time. The ideal tool for such evolution is the Laplace transform, and *not* the Fourier transform. The power of the Laplace transform is much appreciated by engineers, and under-appreciated by many physicists. Those readers who are not intimately familiar with evolution via Laplace transforms should work carefully through Ex. 21.4. That exercise uses Laplace transforms, followed by conversion of the final answer into Fourier language, to derive the following formula for the time-evolving electric field in terms of the initial velocity distributions $F_{s1}(v,0)$:

$$E(t) = -\int_{i\sigma-\infty}^{i\sigma+\infty} \frac{e^{-i\omega t}}{\epsilon(\omega,k)} \left[\sum_s \frac{q_s}{2\pi\epsilon_0 k} \int_{-\infty}^{+\infty} \frac{F_{s1}(v,0)}{\omega - kv} dv \right] d\omega \; . \tag{21.26}$$

Here the integral in frequency space is along the solid horizontal line at the top of Fig. 21.1, with the imaginary part of ω held fixed at $\omega_i = \sigma$ and the real part ω_r varying from $-\infty$ to $+\infty$. The Laplace techniques used to derive this formula are carefully designed to avoid any divergences and any division by zero. This careful design leads to the requirement that the height σ of the integration line above the real frequency axis be *larger* than the e-folding rate $\Im(\omega)$ of the plasma's most rapidly growing mode (or, if none grow, still larger than zero and thus larger than the e-folding rate of the most slowly decaying mode):

$$\sigma > \max_n \Im(\omega_n) \; , \quad \text{and } \sigma > 0 \; , \tag{21.27}$$

where $n = 1, 2, \ldots$ labels the modes and ω_n is the complex frequency of mode n.

Equation (21.26) also entails a velocity integral. In the Laplace-based analysis that leads to this formula, there is never any question about the nature of the velocity v: it is always real, so the integral is over real v running from $-\infty$ to $+\infty$. However, because all the frequencies ω appearing in Eq. (21.26) have $\omega_i = \sigma > 0$, there is no possibility in the velocity integral of any divergence of the integrand.

In Eq. (21.26) for the evolving field, $\epsilon(\omega, k)$ is the same dielectric function as we deduced in our previous analysis (Sec. 21.3.1):

$$\epsilon(\omega, k) = 1 + \frac{e^2}{m_e \epsilon_0 k} \int_{-\infty}^{+\infty} \frac{F' \, dv}{\omega - kv} \,. \tag{21.28}$$

However, here by contrast with there our derivation has dictated unequivocally how to handle the v integration—the same way as in Eq. (21.26): v is strictly real and the only frequencies appearing in the evolution equations have $\omega_i = \sigma > 0$, so the v integral running along the real velocity axis passes under the integrand's pole at $v = \omega/k$ as shown in Fig. 21.2(a).

To read off the modal frequencies from the evolving field $E(t)$ at times $t > 0$, we use techniques from complex-variable theory. It can be shown that, because (by hypothesis) $F_{s1}(v, 0)$ and $F_{s0}(v)$ are analytic functions of v, the integrand of the ω integral in Eq. (21.26) is meromorphic—i.e., when the integrand is analytically continued throughout the complex frequency plane, its only singularities are poles. This permits us to evaluate the frequency integral, at times $t > 0$, by closing the integration contour in the lower-half frequency plane as shown by the dashed curve in Fig. 21.1. Because of the exponential factor $e^{-i\omega t}$, the contribution from the dashed part of the contour vanishes, which means that the integral around the contour is equal to $E(t)$ (the contribution from the solid horizontal part). Complex-variable theory tells us that this integral is given by a sum over the residues R_n of the integrand at the poles (labeled $n = 1, 2, \ldots$):

$$E(t) = 2\pi i \sum_n R_n = \sum_n A_n e^{-i\omega_n t} \,. \tag{21.29}$$

Here ω_n is the frequency at pole n and A_n is $2\pi i R_n$ with its time dependence $e^{-i\omega_n t}$ factored out. *It is evident, then, that each pole of the analytically continued integrand of Eq. (21.26) corresponds to a mode of the plasma and the pole's complex frequency is the mode's frequency.*

Now, for very special choices of the initial data $F_{s1}(v, 0)$, there may be poles in the square-bracket term in Eq. (21.26), but for generic initial data there will be none, and the only poles will be the zeroes of $\epsilon(\omega, k)$. *Therefore, generically, the modes' frequencies are the zeroes of $\epsilon(\omega, k)$—when that function (which was originally defined only along the line $\omega_i = \sigma$) has been analytically extended throughout the complex frequency plane.*

So how do we compute the analytically extended dielectric function $\epsilon(\omega, k)$? Imagine holding k fixed and real, and exploring the (complex) value of ϵ, thought of as a function of ω/k, by moving around the complex ω/k plane (same as complex velocity plane). In particular, imagine computing ϵ from Eq. (21.28) at one point after another along the arrowed path shown in Fig. 21.2. This path begins at an initial location ω/k where $\omega_i/k = \sigma/k > 0$ and travels downward to some other location below the real axis. At the starting point, the discussion following Eq. (21.28) tells us how to handle the velocity integral: just integrate v along the real axis. As ω/k is moved continuously (with k held fixed), $\epsilon(\omega, k)$ being analytic must vary continuously. If, when ω/k crosses the real velocity axis, the integration contour in Eq. (21.28) were to remain on the velocity axis, then the contour would jump over the integral's moving pole $v = \omega/k$, and there would be a discontinuous jump of the function $\epsilon(\omega, k)$ at the moment of crossing, which is not possible. To avoid such a discontinous jump, *it is necessary that the contour of integration be kept below the pole, $v = \omega/k$, as that pole*

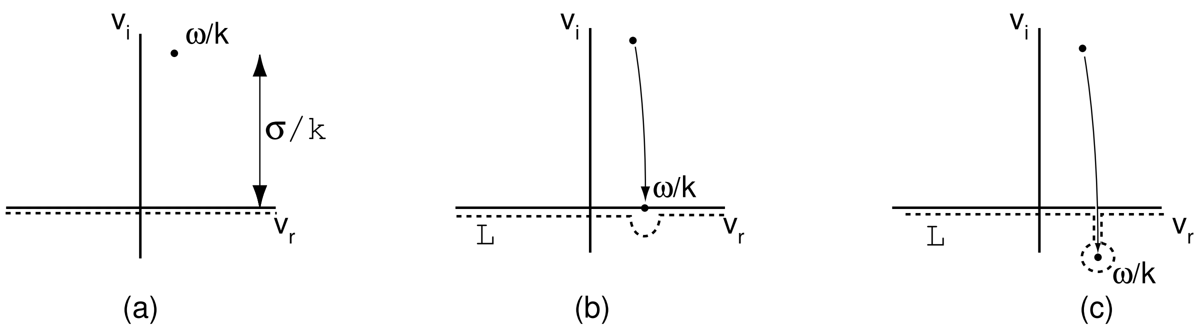

Fig. 21.2: Derivation of the Landau contour $\mathcal{L}$: The dielectric function $\epsilon(\omega, k)$ is originally defined, in Eqs. (21.26) and (21.28), solely for $\omega_i/k = \sigma/k > 0$, the point in diagram (a). Since $\epsilon(\omega, k)$ must be an analytic function of ω at fixed k and thus must vary continuously as ω is continuously changed, the dashed contour of integration in Eq. (21.28) must be kept always below the pole $v = \omega/k$, as shown in (b) and (c).

moves into the lower half velocity plane; cf. Fig. 21.2(b,c). The rule that the integration contour must always pass beneath the pole $v = \omega/k$ as shown in Fig. 21.2 is called the *Landau prescription*; the contour is called the *Landau contour* and is denoted $\mathcal{L}$; and our final formula for the dielectric function (and for its vanishing at the modal frequencies—the dispersion relation) is

$$\epsilon(\omega, k) = 1 + \frac{e^2}{m_e \epsilon_0 k} \int_{\mathcal{L}} \frac{F' dv}{\omega - kv} = 0 \; . \tag{21.30}$$

21.3.4 Dispersion Relation For Weakly Damped or Growing Waves

In most practical situations, electrostatic waves are weakly damped or weakly unstable, i.e. $|\omega_i| \ll \omega_r$, so the amplitude changes little in one wave period. In this case, the dielectric function (21.30) can be evaluated at $\omega = \omega_r + i\omega_i$ using the first term in a Taylor series expansion away from the real axis:

$$\begin{aligned} \epsilon(k, \omega_r + i\omega_i) &\simeq \epsilon(k, \omega_r) + \omega_i \left(\frac{\partial \epsilon_r}{\partial \omega_i} + i \frac{\partial \epsilon_i}{\partial \omega_i} \right)_{\omega_i = 0} \\ &= \epsilon(k, \omega_r) + \omega_i \left(-\frac{\partial \epsilon_i}{\partial \omega_r} + i \frac{\partial \epsilon_r}{\partial \omega_r} \right)_{\omega_i = 0} \\ &\simeq \epsilon(k, \omega_r) + i\omega_i \left(\frac{\partial \epsilon_r}{\partial \omega_r} \right)_{\omega_i = 0} \; . \end{aligned} \tag{21.31}$$

Here ϵ_r and ϵ_i are the real and imaginary parts of ϵ; in going from the first line to the second we have assumed that $\epsilon(k, \omega)$ is an analytic function of ω near the real axis and thence have used the Cauchy-Riemann equations for the derivatives; and in going from the second line to the third we have used the fact that $\epsilon_i \to 0$ when the velocity distribution is one that produces $\omega_i \to 0$ [cf. Eq. (21.33) below], so the middle term on the second line is second order in ω_i and can be neglected.

Equation (21.31) expresses the dielectric function slightly away from the real axis in terms of its value and derivative on and along the real axis. The on-axis value can be computed by breaking the Landau contour depicted in Fig. 21.2(b) into three pieces: two lines from $\pm\infty$ to a small distance δ from the pole, plus a semicircle of radius δ around the pole, and by then taking the limit $\delta \to 0$. The first two terms (the two straight lines) together produce the Cauchy principle value of the integral (denoted $\int_{\rm P}$ below), and the third produces $2\pi i$ times half the residue of the pole at $v = \omega_r/k$:

$$\epsilon(k, \omega_r) = 1 - \frac{e^2}{m_e \epsilon_0 k^2} \left[\int_{\rm P} \frac{F'\,dv}{v - \omega_r/k} dv + i\pi F'(v = \omega_r/k) \right] . \tag{21.32}$$

Inserting this equation and its derivative with respect to ω_r into Eq. (21.31), and setting the result to zero, we obtain

$$\epsilon(k, \omega_r + i\omega_i) \simeq 1 - \frac{e^2}{m_e \epsilon_0 k^2} \left[\int_{\rm P} \frac{F'\,dv}{v - \omega_r/k} + i\pi F'(\omega_r/k) + i\omega_i \frac{\partial}{\partial \omega_r} \int_{\rm P} \frac{F'\,dv}{v - \omega_r/k} \right] = 0 . \tag{21.33}$$

This is the dispersion relation in the limit $|\omega_i| \ll \omega_r$.

Notice that the vanishing of ϵ_r determines the real part of the frequency

$$1 - \frac{e^2}{m_e \epsilon_0 k^2} \int_{\rm P} \frac{F'}{v - \omega_r/k} dv = 0 , \tag{21.34}$$

and the vanishing of ϵ_i determines the imaginary part

$$\omega_i = \frac{\pi F'(\omega_r/k)}{\frac{\partial}{\partial \omega_r} \int_{\rm P} \frac{F'}{v - \omega_r/k} dv} . \tag{21.35}$$

Notice, further, that the sign of ω_i is influenced by the sign of $F' = dF/dv$ at $v = \omega_r/k = V_\phi =$ (the waves' phase velocity). As we shall see, this has a simple physical origin and important physical consequences.

21.3.5 Langmuir Waves and their Landau Damping

We shall now apply the dispersion relation (21.33) to Langmuir waves in a thermalized plasma. Langmuir waves typically move so fast that the slow ions cannot interact with them, so their dispersion relation is influenced significantly only by the electrons. We therefore shall ignore the ions and include only the electrons in $F(v)$. We obtain $F(v)$ by integrating out v_y and v_z in the 3-dimensional Maxwellian velocity distribution (2.39); the result is

$$F = n \left(\frac{m_e}{2\pi k_B T} \right)^{1/2} e^{-(m_e v^2 / 2 k_B T)} , \tag{21.36}$$

where T is the electron temperature.

Now, as we saw in Eq. (21.35), $|\omega_i| \propto |F'(v = \omega_r/k)|$. Physically, this proportionality arises from the manner in which electrons surf on the waves: Those electrons moving slightly faster than the waves' phase velocity $V_\phi = \omega_r/k$ lose energy to the waves, while those moving slightly slower extract energy from the waves; so the bigger the disparity between the number of slightly faster electrons and the number of slightly slower electrons, i.e. the bigger $F'(\omega_r/k)$, the larger will be the damping or growth of wave energy, i.e. the larger will be $|\omega_i|$. It will turn out, quantitatively, that, if the waves' phase velocity ω_r/k is anywhere near the steepest point on the side of the electron velocity distribution, *i.e.* if ω_r/k is of order the electron thermal velocity $\sqrt{k_BT/m_e}$, then the waves will be strong damped, $\omega_i \sim -\omega_r$. Since our dispersion relation (21.36) is valid only when the waves are weakly damped, we must restrict ourseves to waves with $\omega_r/k \gg \sqrt{k_BT/m_e}$ (a physically allowed regime) or to $\omega_r/k \ll \sqrt{k_BT/m_e}$ (a regime that does not occur in Langmuir waves; cf. Fig. 20.1).

Requiring, then, that $\omega_r/k \gg \sqrt{k_BT/m_e}$ and noting that the integral in Eq. (21.33) gets its dominant contribution from velocities $v \lesssim \sqrt{k_BT/m_e}$, we can expand $1/(v-\omega_r/k)$ in the integrand as a power series in vk/ω_r, obtaining

$$
\begin{aligned}
\int_{\rm P} \frac{F'\,dv}{v-\omega_r/k} &= \int_{-\infty}^{\infty} dv F' \left[\frac{k}{\omega_r} + \frac{k^2 v}{\omega_r^2} + \frac{k^3 v^2}{\omega_r^3} + \frac{k^4 v^3}{\omega_r^4} + \dots\right] \\
&= \frac{nk^2}{\omega_r^2} + \frac{3n\langle v^2\rangle k^4}{\omega_r^4} + \dots \\
&= \frac{nk^2}{\omega_r^2}\left(1 + \frac{3k_BTk^2}{m_e\omega_r^2} + \dots\right) \\
&\simeq \frac{nk^2}{\omega_r^2}\left(1 + 3k^2\lambda_D^2\frac{\omega_p^2}{\omega_r^2}\right) \ .
\end{aligned} \tag{21.37}
$$

Substituting Eqs. (21.36) and (21.37) into Eq. (21.33), evaluating the real and imaginary parts, and noting that $\omega_r/k \gg \sqrt{k_BT/m_e} \equiv \omega_p\lambda_D$ implies $k\lambda_D \ll 1$ and $\omega_r \simeq \omega_p$, we obtain

$$
\begin{aligned}
\omega_r &= \omega_p(1+3k^2\lambda_D^2)^{1/2} \\
\omega_i &= -\left(\frac{\pi}{8}\right)^{1/2} \frac{\omega_p}{k^3\lambda_D^3} \exp\left(-\frac{1}{2k^2\lambda_D^2} - \frac{3}{2}\right) \ .
\end{aligned} \tag{21.38}
$$

The real part of this dispersion relation, $\omega_r = \omega_p\sqrt{1+3k^2\lambda_D^2}$, reproduces the Bohm-Gross result that we derived from fluid theory in Sec. 20.3.4 and plotted in Fig. 20.1. The imaginary part reveals the damping of these Langmuir waves by surfing electrons—so-called *Landau damping*. The fluid theory could not predict this Landau damping, because it is a result of the internal dynamics in velocity space, of which the fluid theory is obvlivious.

Notice that, as the waves' wavelength is decreased, i.e. as k increases, the waves' phase velocity decreases toward the electron thermal velocity and the damping becomes stronger, as is expected from our discussion of the number of electrons that can surf on the waves. In the limit $k \to 1/\lambda_D$ (where our dispersion relation has broken down and so is only an order-of-magnitude guide), the dispersion relation predicts that $\omega_r/k \sim \sqrt{k_BT}$ and $\omega_i/\omega_r \sim 1/10$.

In the opposite regime of large wavelength $k\lambda_D \ll 1$ (where our dispersion relation should be quite accurate), the Landau damping is very weak—so weak that ω_i decreases to zero with increasing k faster than any power of k.

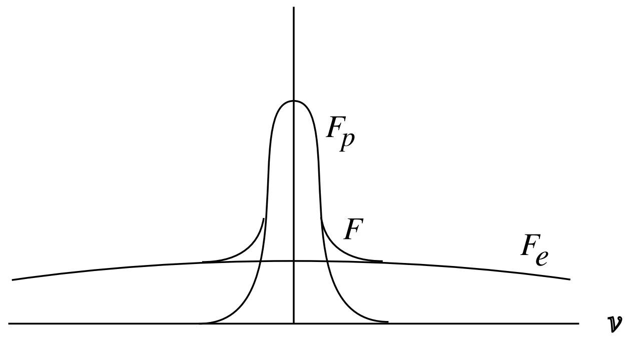

Fig. 21.3: Electron and ion contributions to the net distribution function $F(v)$in a thermalized plasma. When $T_e \sim T_p$, the phase speed of ion acoustic waves is near the left tick mark on the horizontal axis—a speed at which surfing protons have a maximal ability to Landau-damp the waves, and the waves are strongly damped. When $T_e \gg T_p$, the phase speed is near the right tick mark—far out on the tail of the proton velocity distribution so few protons can surf and damp the waves, and near the peak of the electron distribution so the number of electrons moving slightly slower than the waves is nearly the same as the number moving slightly faster and there is little net damping by the electrons. In this case the waves can propagate.

21.3.6 Ion Acoustic Waves and Conditions for their Landau Damping to be Weak

As we saw in Sec. 20.3.4, ion acoustic waves are the analog of ordinary sound waves in a fluid: They occur at low frequencies where the mean (fluid) electron velocity is very nearly locked to the mean (fluid) proton velocity so the electric polarization is small; the restoring force is due to thermal pressure and not to the electrostatic field; and the inertia is provided by the heavy protons. It was asserted in Sec. 20.3.4 that to avoid these waves being strongly damped, the electron temperature must be much higher than the proton temperature, $T_e \gg T_p$. We can now understand this in terms of particle surfing and Landau damping:

Suppose that the electrons and protons have Maxwellian velocity distributions but possibly with different temperatures. Because of their far greater inertia, the protons will have a far smaller mean thermal speed than the electrons, so the net one-dimensional distribution function $F(v) = F_e(v) + (m_e/m_p)F_p(v)$ [Eq. (21.21)] that appears in the kinetic-theory dispersion relation has the form shown in Fig. 21.3. Now, if $T_e \sim T_p$, then the contributions of the electron pressure and proton pressure to the waves' restoring force will be comparable, and the waves' phase velocity will therefore be $\omega_r/k \sim \sqrt{k(T_e + T_p)/m_p} \sim \sqrt{kT_p/m_p}$, which is the speed at which the proton contribution to $F(v)$ has its steepest slope (see the left tick mark on the horizontal axis in Fig. 21.3), so $|F'(v = \omega_r/k)|$ is large. This means there will be large numbers of protons that can surf on the waves and a large disparity between the number moving slightly slower than the waves (which extract energy from the waves) and the number moving slightly faster (which give energy to the waves). The result will be strong Landau damping by the protons.

This strong Landau damping is avoided if $T_e \gg T_p$. Then the waves' phase velocity will be $\omega_r/k \sim \sqrt{kT_e/m_p}$ which is large compared to the proton thermal velocity $\sqrt{kT_p/m_p}$ and so is way out on the tail of the proton velocity distribution where there are very few protons that can surf and damp the waves; see the right tick mark on the horizontal axis in Fig. 21.3. Thus, Landau damping by protons has been shut down by raising the electron temperature. What about Landau damping by electrons? The phase velocity $\omega_r/k \sim \sqrt{kT_e/m_p}$ is small compared to the electron thermal velocity $\sqrt{kT_e/m_e}$, so the waves reside near the peak of the electron velocity distribution, where $F_e(v)$ is large so many electrons can surf with the waves, but $F_e'(v)$ is small so there are nearly equal numbers of faster and slower electrons and the surfing produces little net Landau damping. Thus, $T_e/T_p \gg 1$ leads to successful propagation of ion acoustic waves.

A detailed computation based on the $|\omega_i| \ll \omega_r$ version of our kinetic-theory dispersion relation, Eq. (21.33), makes this physical argument quantitative. The details are carried out in Ex. 21.5 under the assumptions that $T_e \gg T_p$ and $\sqrt{k_BT_p/m_p} \ll \omega_r/k \ll \sqrt{k_BT_e/m_e}$ (corresponding to the above discussion); and the result is:

$$\begin{aligned}\frac{\omega_r}{k} &= \sqrt{\frac{k_BT_e/m_p}{1+k^2\lambda_D^2}}\,, \\ \frac{\omega_i}{\omega_r} &= -\frac{\sqrt{\pi/8}}{(1+k^2\lambda_D^2)^{3/2}}\left[\sqrt{\frac{m_e}{m_p}} + \left(\frac{T_e}{T_p}\right)^{3/2}\exp\left(\frac{-T_e/T_p}{2(1+k^2\lambda_D^2)}\right)\right]\,. \qquad (21.39)\end{aligned}$$

The real part of this dispersion relation was plotted in Fig. 20.1; as is shown there and in the above formula, for $k\lambda_D \ll 1$ the waves' phase speed is $\sqrt{k_BT_e/m_p}$, and the waves are only weakly damped: they can propagate for roughly $\sqrt{m_p/m_e} \sim 43$ periods before damping has a strong effect. This damping remains present when $T_e/T_p \to 0$, so it must be due to surfing electrons. When the wavelength is decreased (k is increased) into the regime $k\lambda_D \gtrsim 1$, the waves' frequency asymptotes toward $\omega_r = \omega_{pp}$, the proton plasma frequency, and the phase velocity decreases, so more protons can surf the waves and the Landau damping increases. Equation (21.39) shows us that the damping becomes very strong when $k\lambda_D \sim \sqrt{T_e/T_p}$, and that this is also the point at which ω_r/k has decreased to the proton thermal velocity $\sqrt{k_BT_p/m_p}$—which is in accord with our physical arguments about proton surfing.

When T_e/T_p is decreased from $\gg 1$ toward unity, the ion damping becomes strong regardless of how small may be k [cf. the second term of ω_i/ω_r in Eq. (21.39)]. This is also in accord with our physical reasoning.

Ion acoustic waves are readily excited at the bow shock where the earth's magnetosphere impacts the solar wind. It is observed that they are not able to propagate very far away from the shock, by contrast with the Alfvén waves, which are much less rapidly damped.

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EXERCISES

Exercise 21.3 *Example: Derivation of Dielectric Tensor for Longitudinal, Electrostatic Waves*

Derive expression (21.20) for the zz component of the dielectric tensor in a plasma excited by a weak electrostatic wave and show that the wave's dispersion relation is $\epsilon_3 = 0$. *Hints:* Notice that the z component of the plasma's electric polarization P_z is related to the charge density by $\nabla \cdot \mathbf{P} = ikP_z = -\rho_e$ [Eq. (20.1)]; combine this with Eq. (21.18) to get a linear relationship between P_z and $E_z = E$; argue that the only nonzero component of the plasma's electric susceptibility is χ_{zz} and deduce its value by comparing the above result with Eq. (20.3); then construct the dielectric tensor ϵ_{ij} from Eq. (20.5) and the algebratized wave operator L_{ij} from Eq. (20.9), and deduce that the dispersion relation $\det||L_{ij}|| = 0$ takes the form $\epsilon_{zz} \equiv \epsilon_3 = 0$, where ϵ_3 is given by Eq. (21.20).

Exercise 21.4 *Example: Landau Contour Deduced Using Laplace Transforms*
Use Laplace-transform techniques to derive Eqs. (21.26)–(21.28) for the time-evolving electric field of electrostatic waves with fixed wave number k and initial velocity perturbations $F_{s1}(v, 0)$. A sketch of the solution follows.

(a) When the initial data are evolved forward in time, they produce $F_1(v, t)$ and $E(t)$. Construct the Laplace transforms of these evolving quantities:[3]

$$\tilde{F}_1(v, p) = \int_0^\infty dt e^{-pt} F_1(v, t) \; , \quad \tilde{E}(p) = \int_0^\infty dt e^{-pt} E(t) \; . \tag{21.40}$$

To ensure that the time integral is convergent, insist that $\Re(p)$ be greater than $p_0 \equiv \max_n \Im(\omega_n) \equiv$ (the e-folding rate of the most strongly growing mode—or, if none grow, then the most weakly damped mode). This is an essential step in the argument leading to the Landau contour. Also, for simplicity of the subsequent analysis, insist that $\Re(p) > 0$.

(b) By constructing the Laplace transform of the one-dimensional Vlasov equation (21.16) and integrating by parts the term involving $\partial F_{s1}/\partial t$, obtain an equation for a linear combination of $\tilde{F}_{s1}(v, p)$ and $\tilde{E}(p)$ in terms of the initial data $F_{s1}(v, t = 0)$. By then combining with the Laplace transform of Poisson's equation, show that

$$\tilde{E}(p) = \frac{1}{\epsilon(ip, k)} \sum_s \frac{q_s}{k\epsilon_0} \int_{-\infty}^{\infty} \frac{F_{s1}(v, 0)}{ip - kv} dv \; . \tag{21.41}$$

Here $\epsilon(ip, k)$ is the dielectric function (21.22) evaluated for frequency $\omega = ip$, with the integral running along the real v axis, and [as we noted in part (a)] $\Re(p)$ must be greater than p_0, the largest ω_i of any mode, and greater than 0. This situation for the dielectric function is the one depicted in Fig. 21.2(a).

(c) Laplace-transform theory tells us that the time-evolving electric field (with wave number k) can be expressed in terms of its Laplace transform (21.41) by

$$E(t) = \int_{\sigma - i\infty}^{\sigma + i\infty} E(p) e^{pt} \frac{dp}{2\pi i} \; , \tag{21.42}$$

where σ is any real number larger than p_0 and larger than 0. Combine this equation with expression (21.41) for $\tilde{E}(p)$, and set $p = i\omega$. Thereby arrive at the desired result, Eq. (21.26).

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[3] For useful insights into the Laplace transform, see, e.g., Sec. 4-2 of Mathews, J. and Walker, R. L., 1964, *Mathematical Methods of Physics*, New York: Benjamin.

Exercise 21.5 *Derivation: Ion Acoustic Dispersion Relation*
Consider a plasma in which the electrons have a Maxwellian velocity distribution with temperature T_e, the protons are Maxwellian with temperature T_p, and $T_p \ll T_e$; and consider a mode in this plasma for which $\sqrt{k_B T_p/m_p} \ll \omega_r/k \ll \sqrt{k_B T_e/m_e}$ (right tick mark in Fig. 21.3). As was argued in the text, for such a mode it is reasonable to expect weak damping, $|\omega_i| \ll \omega_r$. Making approximations based on these "≪" inequalities, show that the dispersion relation (21.33) reduces to Eq. (21.39).

Exercise 21.6 *Problem: Dispersion relations for a non-Maxwellian distribution function.*
Consider a plasma with cold protons [whose velocity distribution can be ignored in $F(v)$] and hot electrons with one dimensional distribution function of the form

$$F(v) = \frac{n v_0}{\pi(v_0^2 + v^2)} \ . \tag{21.43}$$

(a) Derive the dielectric function $\epsilon(\omega, k)$ for this plasma and use it to show that the dispersion relation for Langmuir waves is

$$\omega = \omega_{pe} - ikv_0 \ . \tag{21.44}$$

(b) Compute the dispersion relation for ion acoustic waves assuming that their phase speeds are much less than v_0 but large compared to the cold protons' thermal velocities (so the contribution from proton surfing can be ignored). Your result should be

$$\omega = \frac{kv_0(m_e/m_p)^{1/2}}{[1 + (kv_0/\omega_{pe})^2]^{1/2}} - \frac{ikv_0(m_e/m_p)}{[1 + (kv_0\omega_{pe})^2]^2} \ . \tag{21.45}$$

****

21.4 Stability of Electrostatic Waves in Unmagnetized Plasmas

Equation (21.35) implies that the sign of F' at resonance dictates the sign of the imaginary part of ω. This raises the interesting possibility that distribution functions that increase with velocity over some range of positive v might be unstable to the exponential growth of electrostatic waves. In fact, the criterion for instability turns out to be a little more complex than this [as one might suspect from the fact that the sign of the denominator of Eq. (21.35) is non-obvious], and deriving it is an elegant exercise in complex variable theory.

Let us rewrite our dispersion relation (21.22) (again restricting our attention to real values of k) in the form

$$Z(\omega/k) = k^2 > 0 \ , \tag{21.46}$$

where the complex function $Z(\zeta)$ is given by

$$Z(\zeta) = \frac{e^2}{m_e \epsilon_0} \int_{\mathcal{L}} \frac{F'}{v - \zeta} dv \ . \tag{21.47}$$

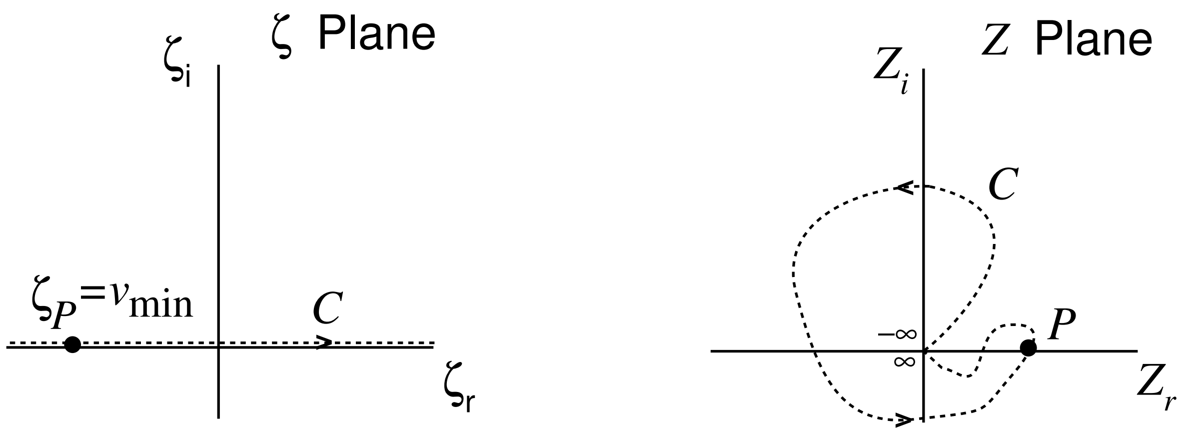

Fig. 21.4: Mapping of the real axis in the complex ζ plane onto the Z plane; cf. Eq. (21.47). This is an example of a *Nyquist diagram*.

Now let us map the real axis in the complex ζ plane onto the Z plane (Fig. 21.4). The points at $(\zeta_r = \pm\infty,\, \zeta_i = 0)$ clearly map onto the origin and so the image of the real ζ axis must be a closed curve. Furthermore, by inspection, the region enclosed by the curve is the image of the upper half plane. (The curve may cross itself, if $\zeta(Z)$ is multi-valued, but this will not affect our argument.) Now, there will be a growing mode ($\zeta_i > 0$) if and only if the curve crosses the positive real Z axis. Let us consider an upward crossing at some point $\mathcal{P}$ as shown in Fig. 21.4 and denote by $\zeta_{\mathcal{P}}$ the corresponding point on the real ζ axis. At this point, the imaginary part of Z vanishes, which implies immediately that $F'(v = \zeta_{\mathcal{P}}) = 0$; cf. Eq. (21.35). Furthermore, as Z_i is increasing at $\mathcal{P}$ (cf. Fig. 21.4), $F''(v = \zeta_{\mathcal{P}}) > 0$. Thus, for instability to the growth of Langmuir waves it is necessary that the one-dimensional distribution function have a minimum at some $v = v_{\rm min} \equiv \zeta_{\mathcal{P}}$. This is not sufficient, as the real part of $Z(\zeta_{\mathcal{P}}) = Z(v_{\rm min})$ must also be positive.

Since $v_{\rm min} \equiv \zeta_{\mathcal{P}}$ is real, $Z_r(v_{\rm min})$ is the Cauchy principal value of the integral (21.47), which we can rewrite as follows using an integration by parts:

$$\begin{aligned} Z_r(v_{\rm min}) &= \int_{\rm P} \frac{F'}{v-\zeta} dv = \int_{\rm P} \frac{d[F(v) - F(v_{min})]/dv}{v - v_{min}} dv \\ &= \int_{\rm P} \frac{[F(v) - F(v_{min})]}{(v - v_{min})^2} dv \\ &\quad + \lim_{\delta \to 0} \left[\frac{F(v_{min} - \delta) - F(v_{min})}{-\delta} - \frac{F(v_{min} + \delta) - F(v_{min})}{\delta} \right] . \end{aligned} \tag{21.48}$$

The second, limit term vanishes since $F'(v_{\rm min}) = 0$, and in the first term we do not need the Cauchy principal value because F is a minimum at $v_{\rm min}$, so our requirement is that

$$Z_r(v_{\rm min}) = \int_{-\infty}^{+\infty} \frac{[F(v) - F(v_{min})]}{(v - v_{min})^2} dv > 0 \, . \tag{21.49}$$

Thus, a sufficient condition for instability is that there exist some velocity $v_{\rm min}$ at which the distribution function $F(v)$ has a minimum, and that in addition the minimum be deep *enough that the integral (21.49) is positive.* This is called the *Penrose criterion* for instability.[4]

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EXERCISES

Exercise 21.7 *Example: Penrose Criterion*
Consider an unmagnetized electron plasma with a one dimensional distribution function

$$F(v) \propto \{[(v-v_0)^2+u^2]^{-1}+[(v+v_0)^2+u^2]^{-1}\}\ , \tag{21.50}$$

where v_0 and u are constants. Show that this distribution function possesses a minimum if $v_0 > 3^{-1/2}u$, but the minimum is not deep enough to cause instability unless $v_0 > u$.

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21.5 Particle Trapping

We now return to the description of Landau damping. Our treatment so far has been essentially linear in the wave amplitude, or equivalently in the perturbation to the distribution function. What happens when the wave amplitude is not infinitesimally small?

Consider a single Langmuir wave mode as observed in a frame moving with the mode's phase velocity. In this frame the electrostatic field oscillates spatially, $E = E_0 \sin kz$, but has no time dependence. Figure 21.5 shows some phase-space orbits of electrons in this oscillatory potential. The solid curves are orbits of particles that move fast enough to avoid being trapped in the potential wells at $kz = 0, 2\pi, 4\pi, \ldots$. The dashed curves are orbits of trapped particles. As seen in another frame, these trapped particles are surfing on the wave, with their velocities performing low-amplitude oscillations around the wave's phase velocity ω/k.

The equation of motion for an electron trapped in the minimum $z = 0$ has the form

$$\begin{aligned}\ddot{z} &= \frac{-eE_0 \sin kz}{m_e} \\ &\simeq -\omega_b^2 z\ ,\end{aligned} \tag{21.51}$$

where we have assumed small-amplitude oscillations and approximated $\sin kz \simeq kz$, and where

$$\omega_b = \left(\frac{eEk}{m_e}\right)^{1/2} \tag{21.52}$$

is known as the *bounce frequency*. Since the potential well is actually anharmonic, the trapped particles will mix in phase quite rapidly.

[4]Penrose, O. 1960, *Phys. Fluids* **3**, 258.

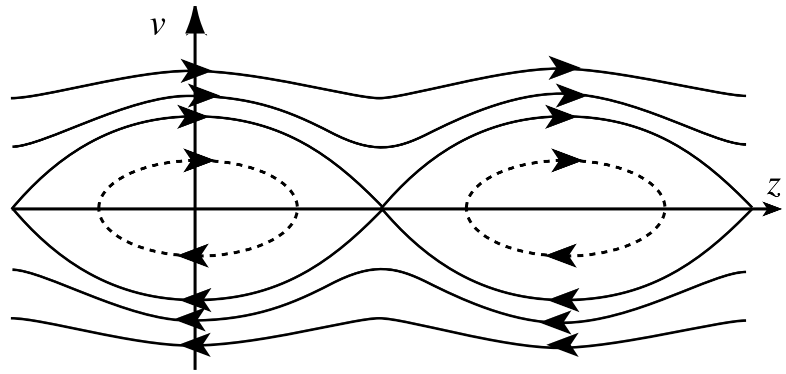

Fig. 21.5: Phase-space Orbits for trapped (dashed lines) and untrapped (solid lines) electrons.

The growth or damping of the wave is characterized by a growth or damping of E_0, and correspondingly by a net acceleration or deceleration of *untrapped* particles, when averaged over a wave cycle. It is this net feeding of energy into and out of the untrapped particles that causes the wave's Landau damping or growth.

Now suppose that the amplitude E_0 of this particular wave mode is changing on a time scale τ due to interactions with the electrons, or possibly (as we shall see in Chap. 22) due to interactions with other waves propagating through the plasma. The potential well will then change on this same timescale and we expect that τ will also be a measure of the maximum length of time a particle can be trapped in the potential well. These nonlinear, wave trapping effects should only be important when the bounce period $\sim \omega_b^{-1}$ is short compared with τ, i.e. when $E \gg m_e/ek\tau^2$.

Electron trapping can cause particles to be bunched together at certain preferred phases of a growing wave. This can have important consequences for the radiative properties of the plasma. Suppose, for example, that the plasma is magnetized. Then the electrons gyrate around the magnetic field and emit cyclotron radiation. If their gyrational phases are random then the total power that they radiate will be the sum of their individual particle powers. However, if N electrons are localized at the same gyrational phase due to being trapped in a potential well of a wave, then, they will radiate like one giant electron with a charge Ne. As the radiated power is proportional to the square of the charge carried by the radiating particle, the total power radiated by the bunched electrons will be N times the power radiated by the same number of unbunched electrons. Very large amplification factors are thereby possible both in the laboratory and in Nature, for example in the Jovian magnetosphere.

This brief discussion suggests that there may be much more richness in plasma waves than is embodied in our dispersion relations with their simple linear growth and decay, even when the wave amplitude is small enough that the particle motion is only slightly perturbed by its interaction with the wave. This motivates us to discuss more systematically nonlinear plasma physics, which is the topic of our next chapter.

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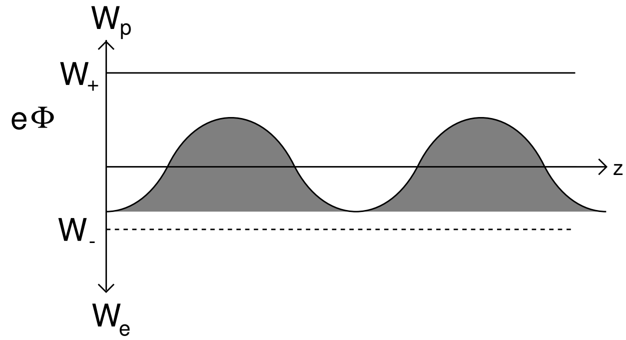

Fig. 21.6: BGK waves. The ordinate is $e\Phi$, where $\Phi(z)$ is the one dimensional electrostatic potential. The proton total energies, W_p, are displayed increasing upward; the electron energies, W_e, increase downward. In this example, corresponding to Challenge 21.8, there are monoenergetic proton (solid line) and electron (dashed line) beams plus a bound distribution of electrons (shaded region) trapped in the potential wells formed by the electrostatic potential.

EXERCISES

Exercise 21.8 *Challenge: BGK Waves*
Consider a steady, one dimensional, large amplitude electrostatic wave in an unmagnetized, proton-electron plasma. Write down the Vlasov equation for each component in a frame moving with the wave, i.e. in which the electrostatic potential is a time-independent function of z, $\Phi = \Phi(z)$, not necessarily precisely sinusoidal.

(a) Use Jeans' theorem to argue that proton and electron distribution functions that are just functions of the energy $W_{p,e} = m_{p,e}v^2/2 \pm e\Phi$ satisfy the Vlasov equation. Then show that Poisson's equation for the potential Φ can be rewritten in the form

$$\frac{1}{2}\left(\frac{d\Phi}{dz}\right)^2 + V(\Phi) = \text{const.} \tag{21.53}$$

where the *potential* V is $-2/\epsilon_0$ times the kinetic energy density of the particles (which depends on Φ).

(b) It is possible to find many self-consistent potential profiles and distribution functions in this manner. These are called BGK waves.[5] Explain how, in principle, one can solve for the self-consistent distribution functions in a large amplitude wave of given potential profile.

(c) Carry out this procedure, assuming that the potential profile is of the form $\Phi(z) = \Phi_0 \cos kz$ with $\Phi_0 > 0$. Assume also that the protons are monoenergetic with $W_p = W_+ > e\Phi_0$ and move along the positive z-direction, and that there are both monoenergetic (with $W_e = W_-$), untrapped electrons (also moving along the positive z-direction), and trapped electrons with distribution $F_e(W_e), -e\Phi_0 \le W_e < e\Phi_0$.

[5] Bernstein, I. B., Greene, J. M. & Kruskal, M. D. 1957 Phys. Rev. 108 546

Show that the density of trapped electrons must vanish at the wave troughs (at $z = (2n+1)\pi/k; n = 0, 1, 2, 3\ldots$). Let the proton density at the troughs be n_{p0}, and assume that there is no net current as well as no net charge density. Show that the total electron density can be written as

$$n_e(z) = \left[\frac{m_e(W_+ - e\Phi_0)}{m_p(W_- + e\Phi)}\right]^{1/2} n_{p0} + \int_{-e\Phi_0}^{e\Phi_0} \frac{dW_e F_e(W_e)}{[2m_e(W_e + e\Phi)]^{1/2}} \; . \tag{21.54}$$

(c) Use Poisson's equation to show that

$$\int_{-\xi_0}^{\xi_0} \frac{dW_e F_e(W_e)}{[2m_e(W_e + \xi)]^{1/2}} = \frac{\epsilon_0 k^2 \Phi}{e^2} + n_{p0}\left[\left(\frac{W_+ - \xi_0}{W_+ - \xi}\right)^{1/2} - \left(\frac{m_e(W_+ - \xi_0)}{m_p(W_- + \xi)}\right)^{1/2}\right] \; , \tag{21.55}$$

where $e\Phi_0 = \xi_0$.

(d) Solve this integral equation for $F_e(W_e)$. (Hint: it is of Abel type.)

(e) Exhibit some solutions graphically.

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21.6 N Particle Distribution Function

Before turning to nonlinear phenomena in plasmas (the next chapter), let us digress briefly and explore ways to study correlations between particles, of which our Vlasov formalism is oblivious.

The Vlasov formalism treats each particle as independent, migrating through phase space in response to the local mean electromagnetic field and somehow unaffected by individual electrostatic interactions with neighboring particles. As we discussed in chapter 19, this is likely to be a good approximation in a collisionless plasma because of the influence of screening—except in the tiny "independent-particle" region of Fig. 19.1. However, we would like to have some means of quantifying this and of deriving an improved formalism that takes into account the correlations between individual particles.

One environment where this may be particularly relevant is the interior of the sun. Here, although the gas is fully ionized, the Debye number is not particularly large (i.e., one is near the independent particle region of Fig. 19.1) and Coulomb corrections to the perfect gas equation of state may be responsible for measurable changes in the internal structure as derived, for example, using helioseismological analysis (cf. Sec. 15.2). In this application our task is simpler than in the general case because the gas will locally be in thermodynamic equilibrium at some temperature T. It turns out that the general case when the plasma departs significantly from thermal equilibrium is extremely hard to treat.

The one-particle distribution function that we use in the Vlasov formalism is the first member of a hierarchy of k-particle distribution functions, $f^{(k)}(\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_k, \mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k, t)$

which equal the probability that particle 1 will be found in a volume of its phase space (which we shall denote by $d\mathbf{x}_1 d\mathbf{v}_1 \equiv dx_1 dx_2 dx_3 dv_{x_1} dv_{x_2} dv_{x_3}$), and that particle 2 will be found in volume $d\mathbf{x}_2 d\mathbf{v}_2$ of its phase space, etc. Our probability interpretation of these distribution functions dictates for $f^{(1)}$ a different normalization than we use in the Vlasov formalism, $f^{(1)} = f/n$ where n is the number density of particles, and dictates that

$$\int f^{(k)} d\mathbf{x}_1 d\mathbf{v}_1 \cdots d\mathbf{x}_k d\mathbf{v}_k = 1. \tag{21.56}$$

It is useful to relate the distribution functions $f^{(k)}$ to the concepts of statistical mechanics, which we developed in Chap. 3. Suppose we have an ensemble of N-electron plasmas and let the probability that a member of this ensemble is in a volume $d^N\mathbf{x} d^N\mathbf{v}$ of $6N$ dimensional phase space be $\tilde{F} d^N\mathbf{x} d^N\mathbf{v}$. ($N$ is a very large number!) $\tilde{F}$ satisfies the Liouville equation

$$\frac{\partial \tilde{F}}{\partial t} + \sum_{i=1}^{N} \left[(\mathbf{v}_i \cdot \boldsymbol{\nabla}_i)\tilde{F} + (\mathbf{a}_i \cdot \boldsymbol{\nabla}_{\mathbf{v}i})\tilde{F} \right] = 0 \tag{21.57}$$

where $\mathbf{a}_i$ is the electromagnetic acceleration of the i'th particle, and $\boldsymbol{\nabla}_i$ and $\boldsymbol{\nabla}_{\mathbf{v}i}$ are gradients with respect to the position and velocity of particle i. We can construct the k-particle "reduced" distribution function from the statistical-mechanics distribution function $\tilde{F}$ by integrating over all but k of the particles:

$$\begin{aligned} &f^{(k)}(\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_k, \mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k, t) \\ &\quad = N^k \int d\mathbf{x}_{k+1} \ldots d\mathbf{x}_N \mathbf{v}_{k+1} \ldots d\mathbf{v}_N \tilde{F}(\mathbf{x}_1, \ldots, \mathbf{x}_N, \mathbf{v}_1, \ldots, \mathbf{v}_N) \end{aligned} \tag{21.58}$$

(Note: k is typically a very small number, by contrast with N; below we shall only be concerned with $k = 1, 2, 3$.) The reason for the prefactor N^k in Eq. (21.58) is that whereas $\tilde{F}$ referred to the probability of finding particle 1 in $d\mathbf{x}_1 d\mathbf{v}_1$, particle 2 in $d\mathbf{x}_2 d\mathbf{v}_2$ etc, the reduced distribution function describes the probability of finding *any* of the N identical, though distinguishable particles in $d\mathbf{x}_1 d\mathbf{v}_1$ and so on. (As long as we are dealing with non-degenerate plasmas we can count the electrons classically.) As $N \gg k$, the number of possible ways we can choose k particles for k locations in phase space is approximately N^k.

For simplicity, suppose that the protons are immobile and form a uniform, neutralizing background of charge so that we need only consider the electron distribution and its correlations. Let us further suppose that the forces associated with *mean* electromagnetic fields produced by *external* charges and currents can be ignored. We can then restrict our attention to direct electron-electron electrostatic interactions. The acceleration is then

$$\mathbf{a}_i = \frac{e}{m_e} \sum_j \boldsymbol{\nabla}_i \Phi_{ij} \ , \tag{21.59}$$

where $\Phi_{ij}(x_{ij}) = -e/4\pi\epsilon_0 x_{ij}$ is the electrostatic potential between two electrons i, j and $x_{ij} = |\mathbf{x}_i - \mathbf{x}_j|$.

We can now derive the so-called BBGKY[6] hierarchy of kinetic equations which relate the k-particle distribution function to integrals over the $k+1$ particle distribution function. The first equation in this hierarchy is given by integrating Liouville's Eq. (21.57) over $d\mathbf{x}_2 \dots d\mathbf{x}_N d\mathbf{v}_2 \dots d\mathbf{v}_N$. If we assume that the distribution function decreases to zero at large distances, then integrals of the type $\int d\mathbf{x}_i \boldsymbol{\nabla}_i etildeF/\partial \mathbf{x}_i$ vanish and the one particle distribution function evolves according to

$$\begin{aligned}\frac{\partial f^{(1)}}{\partial t} + (\mathbf{v}_1 \cdot \boldsymbol{\nabla}) f^{(1)} &= \frac{-eN}{m_e} \int d\mathbf{x}_2 \dots d\mathbf{x}_N d\mathbf{v}_2 \dots d\mathbf{v}_N \Sigma_j \boldsymbol{\nabla}_1 \Phi_{1j} \cdot \boldsymbol{\nabla}_{\mathbf{v}_1} \tilde{F} \\ &= \frac{-eN^2}{m_e} \int d\mathbf{x}_2 \dots d\mathbf{x}_N d\mathbf{v}_2 \dots d\mathbf{v}_N \boldsymbol{\nabla}_1 \Phi_{12} \cdot \boldsymbol{\nabla}_{\mathbf{v}_1} \tilde{F} \\ &= \frac{-e}{m_e} \int d\mathbf{x}_2 d\mathbf{v}_2 \left(\boldsymbol{\nabla}_{\mathbf{v}1} f^{(2)} \cdot \boldsymbol{\nabla}_1 \right) \Phi_{12} \ , \end{aligned} \tag{21.60}$$

where we have again replaced the probability of having *any* particle at a location in phase space by N times the probability of having *one* specific particle there. The left hand side of Eq. (21.60) describes the evolution of independent particles and the right hand side takes account of their pairwise mutual correlation.

The evolution equation for $f^{(2)}$ can similarly be derived by integrating the Liouville equation (21.57) over $d\mathbf{x}_3 \dots d\mathbf{x}_N d\mathbf{v}_3 \dots d\mathbf{v}_N$

$$\begin{aligned}&\frac{\partial f^{(2)}}{\partial t} + (\mathbf{v}_1 \cdot \boldsymbol{\nabla}_1) f^{(2)} + (\mathbf{v}_2 \cdot \boldsymbol{\nabla}_2) f^{(2)} + \frac{e}{m_e} \left[\left(\boldsymbol{\nabla}_{\mathbf{v}1} f^{(2)} \cdot \boldsymbol{\nabla}_1 \right) \Phi_{12} + \left(\boldsymbol{\nabla}_{\mathbf{v}2} f^{(2)} \cdot \boldsymbol{\nabla}_2 \right) \Phi_{12} \right] \\ &\quad = \frac{-e}{m_e} \int d\mathbf{x}_3 d\mathbf{v}_3 \left[\left(\boldsymbol{\nabla}_{\mathbf{v}1} f^{(3)} \cdot \boldsymbol{\nabla}_1 \right) \Phi_{13} + \left(\boldsymbol{\nabla}_{\mathbf{v}2} f^{(3)} \cdot \boldsymbol{\nabla}_2 \right) \Phi_{23} \right] \ , \end{aligned} \tag{21.61}$$

and, in general, allowing for the presence of mean electromagnetic field (in addition to the inter-electron electrostatic field) causing an acceleration $\mathbf{a}^{ext} = -(e/m_e)(\mathbf{E} + \mathbf{v} \times \mathbf{B})$, we obtain the BBGKY hierarchy of kinetic equations

$$\begin{aligned}&\frac{\partial f^{(k)}}{\partial t} + \sum_{i=1}^{k} \left[(\mathbf{v}_i \cdot \boldsymbol{\nabla}_i) f^{(k)} + (\mathbf{a}_i^{ext} \cdot \boldsymbol{\nabla}_{\mathbf{v}i}) f^{(k)} + \frac{e}{m_e} (\boldsymbol{\nabla}_{\mathbf{v}_i} f^{(k)} \cdot \boldsymbol{\nabla}_i) \sum_{j \neq i}^{k} \Phi_{ij} \right] \\ &\quad = \frac{-e}{m_e} \int d\mathbf{x}_{k+1} d\mathbf{v}_{k+1} \sum_{i=1}^{k} \left(\boldsymbol{\nabla}_{\mathbf{v}_i} f^{(k+1)} \cdot \boldsymbol{\nabla}_i \right) \Phi_{ik+1} \ . \end{aligned} \tag{21.62}$$

We see explicitly how we require knowledge of the $(k+1)$-particle distribution function in order to determine the evolution of the k-particle distribution function. The pairwise mutual correlation that appeared in Eq. (21.60) implicitly contains information about triple and higher correlations.

[6]Bogolyubov, N. N. 1962, *Studies in Statistical Mechanics* Vol 1, ed. J. de Boer & G. E. Uhlenbeck, Amsterdam: North Holland; Born, M. & Green, H. S. 1949, *A General Kinetic Theory of Liquids*, Cambridge: Cambridge University Press; Kirkwood, J. G. 1946, J. Chem. Phys. **14**, 18; Yvon, J. 1935, *La Théorie des Fluides et l'Équation d'État*, Actualités Scientifiques et Industrielles, Paris: Hermann.

It is convenient to define the *two point correlation function*, $\xi_{12}(\mathbf{x}_1, \mathbf{v}_1, \mathbf{x}_2, \mathbf{v}_2, t)$ for particles 1,2, by

$$f^{(2)}(1,2) = f_1 f_2 (1 + \xi_{12}) \ , \tag{21.63}$$

where we introduce the notation $f_1 = f^{(1)}(\mathbf{x}_1, \mathbf{v}_1, t)$ and $f_2 = f^{(1)}(\mathbf{x}_2, \mathbf{v}_2, t)$. We now restrict attention to a plasma in thermal equilibrium at temperature T. In this case, f_1, f_2 will be Maxwellian distribution functions, independent of $\mathbf{x}, t$. Now, let us make an *ansatz*, namely that ξ_{12} is just a function of the electrostatic interaction energy between the two electrons and therefore it does not involve the electron velocities. (It is, actually, possible to justify this directly for an equilibrium distribution of particles interacting electrostatically, but we shall make do with showing that our final answer for ξ_{12} is just a function of x_{12}.) As screening should be effective at large distances, we anticipate that $\xi_{12} \to 0$ as $x_{12} \to \infty$.

Now turn to Eq. (21.61), and introduce the simplest imaginable closure relation, $\xi_{12} = 0$. In other words, completely ignore all correlations. We can then replace $f^{(2)}$ by $f_1 f_2$ and perform the integral over $\mathbf{x}_2, \mathbf{v}_2$ to recover the collisionless Vlasov equation (21.5). We therefore see explicitly that particle-particle correlations are indeed ignored in the simple Vlasov approach.

For the 3-particle distribution function, we expect that when electron 1 is distant from both electrons 2,3, that $f^{(3)} \sim f_1 f_2 f_3 (1 + \xi_{23})$, etc. Summing over all three pairs we write,

$$f^{(3)} = f_1 f_2 f_3 (1 + \xi_{23} + \xi_{31} + \xi_{12} + \chi_{123}) \ , \tag{21.64}$$

where χ_{123} is the *three point correlation function* that ought to be significant when all three particles are close together. χ_{123} is, of course determined by the next equation in the BBGKY hierarchy.

We next make the closure relation $\chi_{123} = 0$, that is to say, we ignore the influence of third bodies on pair interactions. This is reasonable because close, *three body* encounters are even less frequent than close *two body* encounters. We can now derive an equation for ξ_{12} by seeking a steady state solution to Eq. (21.61), i.e. a solution with $\partial f^{(2)}/\partial t = 0$. We substitute Eqs. (21.63) and (21.64) into (21.61) (with $\chi_{123} = 0$) to obtain

$$\begin{aligned} & f_1 f_2 \left[(\mathbf{v}_1 \cdot \boldsymbol{\nabla}_1)\xi_{12} + (\mathbf{v}_2 \cdot \boldsymbol{\nabla}_2)\xi_{12} - \frac{e(1+\xi_{12})}{k_B T} \left\{ (\mathbf{v}_1 \cdot \boldsymbol{\nabla}_1)\Phi_{12} + (\mathbf{v}_2 \cdot \boldsymbol{\nabla}_2)\Phi_{12} \right\} \right] \\ & = \frac{e f_1 f_2}{k_B T} \int d\mathbf{x}_3 d\mathbf{v}_3 f_3 \left\{ 1 + \xi_{23} + \xi_{31} + \xi_{12} \right\} \left[(\mathbf{v}_1 \cdot \boldsymbol{\nabla}_1)\Phi_{13} + (\mathbf{v}_2 \cdot \boldsymbol{\nabla}_2)\Phi_{23} \right] \ , \end{aligned} \tag{21.65}$$

where we have used the relation

$$\boldsymbol{\nabla}_{\mathbf{v}1} f_1 = -\frac{m_e \mathbf{v}_1 f_1}{k_B T} \ , \tag{21.66}$$

valid for an unperturbed Maxwellian distribution function. We can rewrite this equation using the relations

$$\boldsymbol{\nabla}_1 \Phi_{12} = -\boldsymbol{\nabla}_2 \Phi_{12} \ , \quad \boldsymbol{\nabla}_1 \xi_{12} = -\boldsymbol{\nabla}_2 \xi_{12} \ , \quad \xi_{12} \ll 1 \ , \quad \int d\mathbf{v}_3 f_3 = n \ , \tag{21.67}$$

to obtain

$$(\mathbf{v}_1 - \mathbf{v}_2)\cdot\boldsymbol{\nabla}_1\left(\xi_{12} - \frac{e\Phi_{12}}{k_BT}\right) = \frac{ne}{k_BT}\int d\mathbf{x}_3(1+\xi_{23}+\xi_{31}+\xi_{12})[(\mathbf{v}_1\cdot\boldsymbol{\nabla}_1)\Phi_{13} + (\mathbf{v}_2\cdot\boldsymbol{\nabla}_2)\Phi_{23}]\ . \tag{21.68}$$

Now, symmetry considerations tell us that

$$\begin{aligned}\int d\mathbf{x}_3(1+\xi_{31})\boldsymbol{\nabla}_1\Phi_{13} &= 0\ ,\\ \int d\mathbf{x}_3(1+\xi_{12})\boldsymbol{\nabla}_2\Phi_{23} &= 0\ ,\end{aligned} \tag{21.69}$$

and, in addition,

$$\begin{aligned}\int d\mathbf{x}_3\xi_{12}\boldsymbol{\nabla}_1\Phi_{13} &= -\xi_{12}\int d\mathbf{x}_3\boldsymbol{\nabla}_3\Phi_{13} = 0\ ,\\ \int d\mathbf{x}_3\xi_{12}\boldsymbol{\nabla}_2\Phi_{23} &= -\xi_{12}\int d\mathbf{x}_3\boldsymbol{\nabla}_3\Phi_{23} = 0.\end{aligned} \tag{21.70}$$

Therefore, we end up with

$$(\mathbf{v}_1 - \mathbf{v}_2)\cdot\boldsymbol{\nabla}_1\left(\xi_{12} - \frac{e\Phi_{12}}{k_BT}\right) = \frac{ne}{k_BT}\int d\mathbf{x}_3[\xi_{23}(\mathbf{v}_1\cdot\boldsymbol{\nabla}_1)\Phi_{31} + \xi_{31}(\mathbf{v}_2\cdot\boldsymbol{\nabla}_2)\Phi_{23}]\ . \tag{21.71}$$

As this equation must be true for arbitrary velocities, we can set $\mathbf{v}_2 = 0$ and obtain

$$\boldsymbol{\nabla}_1(k_BT\xi_{12} - e\Phi_{12}) = ne\int d\mathbf{x}_3\xi_{23}\boldsymbol{\nabla}_1\Phi_{31}\ . \tag{21.72}$$

We take the divergence of Eq. (21.72) and use Poisson's equation, $\boldsymbol{\nabla}_1^2\Phi_{12} = e\delta(\mathbf{x}_{12})/\epsilon_0$, to obtain

$$\boldsymbol{\nabla}_1^2\xi_{12} - \frac{\xi_{12}}{\lambda_D^2} = \frac{e^2}{\epsilon_0 k_BT}\delta(\mathbf{x}_{12})\ , \tag{21.73}$$

where $\lambda_D = (k_BT\epsilon_0/ne^2)^{1/2}$ is the Debye length [Eq. (19.10)]. The solution of Eq. (21.73) is

$$\xi_{12} = \frac{-e^2}{4\pi\epsilon_0 k_BT}\frac{e^{-x_{12}/\lambda_D}}{x_{12}}\ . \tag{21.74}$$

Note that the sign is negative because the electrons repel one another. Note also that, to order of magnitude, $\xi_{12}(\lambda_D) \sim N_D^{-1}$ which is small if the Debye number is much greater than unity. At the mean interparticle spacing, $\xi_{12}(n^{-1/3}) \sim N_D^{-2/3}$. Only for distances $x_{12} \lesssim e^2/\epsilon_0 k_BT$ will the correlation effects become large and our expansion procedure and truncation ($\chi_{123} = 0$) become invalid. This analysis justifies the use of the Vlasov equation when $N_D \gg 1$.

Let us now return to the problem of computing the Coulomb correction to the pressure of an ionized gas. It is easiest to begin by computing the Coulomb correction to the internal energy density. For a one component plasma this is simply given by

$$U_c = \frac{-e}{2}\int d\mathbf{x}_1 n_1 n_2\xi_{12}\Phi_{12}\ , \tag{21.75}$$

where the factor 1/2 compensates for double counting the interactions. Substituting Eq. (21.74) and performing the integral, we obtain

$$U_c = \frac{-ne^2}{8\pi\epsilon_0\lambda_D} \ . \tag{21.76}$$

The pressure can be obtained from this energy density using elementary thermodynamics. From the definition of the free energy, Eq. (4.21), the volume density of Coulomb free energy, F_c is given by integrating

$$U_c = -T^2\left(\frac{\partial(F_c/T)}{\partial T}\right)_n \ . \tag{21.77}$$

From this, we obtain $F_c = -ne^2/12\pi\epsilon_0\lambda_D$. The Coulomb contribution to the pressure is then given given by Eq. (4.27)

$$P_c = n^2\left(\frac{\partial(F_c/n)}{\partial n}\right)_T = \frac{-ne^2}{24\pi\epsilon_0\lambda_D} = \frac{1}{3}U_c \ . \tag{21.78}$$

Therefore, including the Coulomb interaction *decreases* the pressure at a given density and temperature.

We have kept the neutralising protons fixed so far. In a real plasma they are mobile and so the Debye length must be reduced by a factor $2^{-1/2}$ [cf. Eq. (19.9)]. In addition, Eq. (21.75) must be multiplied by a factor 4 to take account of the proton-proton and proton electron interactions. The end result is

$$P_c = \frac{-n^{3/2}e^3}{2^{3/2}3\pi\epsilon_0^{3/2}T^{1/2}} \ , \tag{21.79}$$

where n is still the number density of electrons. Numerically the gas pressure for a perfect electron-proton gas is

$$P = 1.6\times 10^{13}(\rho/1000\text{kg m}^{-3})(T/10^6\text{K})\text{N m}^{-2} \ , \tag{21.80}$$

and the Coulomb correction to this pressure is

$$P_c = -7.3\times 10^{11}(\rho/1000\text{kg m}^{-3})^{3/2}(T/10^6\text{K})^{-1/2}\text{N m}^{-2}. \tag{21.81}$$

In the interior of the sun this is about one percent of the total pressure. In denser, cooler stars, it is significantly larger.

However, for most of the plasmas that one encounters, this analysis suffices to show that the order of magnitude of the two point correlation function ξ_{12} is $\sim N_D^{-1}$ across a Debye sphere and only $\sim N_D^{-2/3}$ at the distance of the mean inter-particle spacing. Only those particles that are undergoing large deflections through angles ~ 1 radian are close enough for $\xi = O(1)$. This is the ultimate justification for treating plasmas as collisionless and using the mean electromagnetic fields in the Vlasov description.

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EXERCISES

Exercise 21.9 *Derivation: BBGKY Hierarchy*
Complete the derivation of Eq. (21.71) from Eq. (21.61).

Exercise 21.10 *Problem: Correlations in a Tokamak Plasma*
For a Tokamak plasma compute, in order of magnitude, the two point correlation function for two electrons separated by

(a) a Debye length,

(b) the mean inter-particle spacing.

Exercise 21.11 *Derivation: Thermodynamic identities*
Verify Eq. (21.77), (21.78).

Exercise 21.12 *Example: Thermodynamics of Coulomb Plasma*
Compute the entropy of a proton-electron plasma in thermal equilibrium at temperature T including the Coulomb correction.

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The references for this chapter are similar to those for Chap. 20. The most useful are probably:

Boyd, T. J. M. and Sanderson, J. J. 1969. *Plasma Dynamics*, Nelson: London.

Krall, N. A. and Trivelpiece, A. W. 1973. *Principles of Plasma Physics*, New York: McGraw-Hill.

Lifshitz, E. M. and Pitaevski, L. P. 1981. *Physical Kinetics*, Oxford: Pergamon.

Schmidt, G. 1966. *Physics of High Temperature Plasmas*, New York: Academic Press.