Relevant and Substructural Logics

GREG RESTALL[*]

PHILOSOPHY DEPARTMENT, MACQUARIE UNIVERSITY

Greg.Restall@mq.edu.au

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http://www.phil.mq.edu.au/staff/grestall/

Abstract: This is a *history* of relevant and substructural logics, written for the *Handbook of the History and Philosophy of Logic*, edited by Dov Gabbay and John Woods.[1]

1 Introduction

Logics tend to be viewed of in one of two ways — with an eye to *proofs,* or with an eye to *models.*[2] Relevant and substructural logics are no different: you can focus on notions of proof, inference rules and structural features of deduction in these logics, or you can focus on interpretations of the language in other structures.

This essay is structured around the bifurcation between proofs and models: The first section discusses *Proof Theory* of relevant and substructural logics, and the second covers the *Model Theory* of these logics. This order is a natural one for a history of relevant and substructural logics, because much of the initial work — especially in the Anderson–Belnap tradition of relevant logics — *started* by developing proof theory. The model theory of relevant logic came some time later. As we will see, Dunn's algebraic models [76, 77] Urquhart's operational semantics [267, 268] and Routley and Meyer's relational semantics [239, 240, 241] arrived decades after the initial burst of activity from Alan Anderson and Nuel Belnap. The same goes for work on the Lambek calculus: although inspired by a very particular application in linguistic typing, it was developed first proof-theoretically, and only later did model theory come to the fore. Girard's linear logic is a different story: it was discovered though considerations of the categorical models of coherence

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[1]The title, *Relevant and Substructural Logics* is not to be read in the same vein as "apples and oranges" or "Australia and New Zealand." It is more in the vein of "apples and fruit" or "Australia and the Pacific Rim." It is a history of substructural logics with a particular attention to relevant logics, or dually, a history of relevant logics, playing particular attention to their presence in the larger class of substructural logics.

[2]Sometimes you see this described as the distinction between an emphasis on *syntax* or *semantics.* But this is to cut against the grain. On the face of it, rules of proof have as much to do with the *meaning* of connectives as do model-theoretic conditions. The rules interpreting a formal language in a model pay just as much attention to *syntax* as does any proof theory.

spaces. However, as linear logic appears on the scene much later than relevant logic or the Lambek calculus, starting with proof theory does not result in *too* much temporal reversal.

I will end with one smaller section *Loose Ends*, sketching avenues for further work. The major sections, then, are structured thematically, and inside these sections I will endeavour to sketch the core historical lines of development in substructural logics. This, then, will be a *conceptual* history, indicating the linkages, dependencies and development of the content itself. I will be less concerned with identifying who did what and when.[3]

I take it that logic is best learned by *doing it*, and so, I have taken the liberty to sketch the proofs of major results when the techniques used in the proofs us something distinctive about the field. The proofs can be skipped or skimmed without any threat to the continuity of the story. However, to get the full flavour of the history, you should attempt to savour the proofs at leisure.

Let me end this introduction by situating this essay in its larger context and explaining how it differs from other similar introductory books and essays. Other comprehensive introductions such as Dunn's "Relevance Logic and Entailment" [81] and its descendant "Relevance Logic" [94], Read's *Relevant Logic* [224] and Troelstra's *Lectures on Linear Logic* [264] are more narrowly focussed than this essay, concentrating on one or other of the many relevant and substructural logics. The Anderson–Belnap two-volume *Entailment* [10, 11] is a goldmine of historical detail in the tradition of relevance logic, but it contains little about other important traditions in substructural logics.

My *Introduction to Substructural Logics* [234] has a similar scope to this chapter, in that it covers the broad sweep of substructural logics: however, that book is more technical than this essay, as it features many formal results stated and proved in generality. It is also written to introduce the subject purely *thematically* instead of *historically*.

2 Proofs

The discipline of *relevant* logic grew out of an attempt to understand notions of consequence and conditionality where the conclusion of a valid argument is *relevant* to the premises, and where the consequent of a true conditional is *relevant* to the antecedent.

"*Substructural*" is a newer term, due to Schröder-Heister and Došen. They write:

> Our proposal is to call logics that can be obtained . . . by restricting structural rules, *substructural logics*. [250, page 6]

The *structural rules* mentioned here dictate admissible forms of transformations of premises in proofs. Later in this section, we will see how relevant logics are naturally counted as substructural logics, as certain commonly admitted structural rules are to blame for introducing irrelevant consequences into proofs.

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[3]In particular, I will say little about the intellectual ancestry of different results. I will not trace the degree to which researchers in one tradition were influenced by those in another.

Historical priority in the field belongs to the tradition of relevant logic, and it is to the early stirrings of considerations of relevance that we will turn.

2.1 Relevant Implication: Orlov, Moh and Church

Došen has shown us [71] that substructural logic dates back at least to 1928 with I. E. Orlov's axiomatisation of a propositional logic weaker than classical logic [207].[4] Orlov axiomatised this logic in order to "represent relevance between propositions in symbolic form" [71, page 341]. Orlov's propositional logic has this axiomatisation.[5]

◇	$A \rightarrow \sim\sim A$	double negation introduction
◇	$\sim\sim A \rightarrow A$	double negation elimination
◇	$A \rightarrow \sim(A \rightarrow \sim A)$	contraposed *reductio*
◇	$(A \rightarrow B) \rightarrow (\sim B \rightarrow \sim A)$	contraposition
◇	$(A \rightarrow (B \rightarrow C)) \rightarrow (B \rightarrow (A \rightarrow C))$	permutation
◇	$(A \rightarrow B) \rightarrow ((C \rightarrow A) \rightarrow (C \rightarrow B))$	prefixing
◇	$A, A \rightarrow B \Rightarrow B$	*modus ponens*

The axioms and rule here form a traditional Hilbert system. The rule *modus ponens* is written in the form using a turnstile to echo the general definition of logical consequence in a Hilbert system. Given a set X of formulas, and a single formula A, we say that A can be proved from X (which I write "$X \Rightarrow A$") if and only if there is a proof in the Hilbert system with A as the conclusion, and with hypotheses from among the set X. A proof from hypotheses is simply a list of formulas, each of which is either an hypothesis, an axiom, or one which follows from earlier formulas in the list by means of a rule. In Orlov's system, the only rule is *modus ponens*. We will see later that this is not necessarily the most useful notion of logical consequence applicable to relevant and substructural logics. In particular, more interesting results can be proven with consequence relations which do not merely relate *sets* of formulas as premises to a conclusion, but rather relate lists, or other forms of structured collections as premises, to a conclusion. This is because lists or other structures can distinguish the *order* or *quantity* of individual premises, while sets cannot. However, this is all that can simply be done to define consequence relations within the confines of a Hilbert system, so here is where our definition of consequence will start.

These axioms and the rule do not *explicitly* represent any notion of relevance. Instead, we have an axiomatic system governing the behaviour of *implication* and *negation*. The system tells us about *relevance* in virtue of what

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[4]Allen Hazen has shown that in Russell's 1906 paper "The Theory of Implication" his propositional logic (without negation) is free of the structural rule of contraction [133, 243]. Only after negation is introduced can contraction can be proved. However, there seems to be no real sense in which Russell could be pressed in to favour as a proponent of substructural logics, as his aim was not to do without contraction, but to give an axiomatic account of material implication.

[5]The names are mine, and not Orlov's. I have attempted to give each axiom or rule its common name (see for example Anderson and Belnap's *Entailment* [10] for a list of axioms and their names). In this case, "contraposed reductio" is my name, as the axiom $A \rightarrow \sim(A \rightarrow \sim A)$ is a rarely seen axiom, but it is a contraposed form of $(A \rightarrow \sim A) \rightarrow \sim A$, which is commonly known as *reductio*.

it leaves out, rather than what it includes. Neither of the following formulas are provable in Orlov's system:

$$A \rightarrow (B \rightarrow B) \qquad \sim(B \rightarrow B) \rightarrow A$$

This distinguishes his logic from both classical and intuitionistic propositional logic.[6] If the "→" is read as either the material conditional or the conditional of intuitionistic logic, those formulas are provable. However, both of these formulas commit an obvious failure of *relevance*. The consequent of the main conditional need not have anything to do with the antecedent. If when we say "if A then B" we mean that B follows *from* A, then it seems that we have *lied* when we say that "if A then B → B", for B → B (though true enough) need not follow *from* A, if A has nothing to do with B → B. Similarly, A need not follow *from* ∼(B → B) (though ∼(B → B) is false enough) for again, A need not have anything to do with ∼(B → B). If "following *from*" is to respect these intuitions, we need look further afield than classical or intuitionistic propositional logic, for these logics contain those formulas as tautologies. Excising these fallacies of relevance is no straightforward job, for once they go, so must other tautologies, such as these

⋄ $A \rightarrow (B \rightarrow A)$ weakening
⋄ $B \rightarrow (\sim B \rightarrow A)$ ex contradictione quodlibet

from which they can be derived.[7] To do without obvious fallacies of relevance, we must do without these formulas too. And this is exactly what Orlov's system manages to do. His system contains none of these "fallacies of relevance", and this makes his system a *relevant logic*. In Orlov's system, a formula $A \rightarrow B$ is provable only when A and B share a propositional atom. There is no way to prove a conditional in which the antecedent and the consequent have nothing to do with one another. Orlov did not prove this result in his paper. It only came to light more than 30 years later, with more recent work in relevant logic. This more recent work is applicable to Orlov's system, because Orlov has axiomatised the implication and negation fragment of the now well-known relevant logic R.

Orlov's work didn't end with the implication and negation fragment of a relevant propositional logic. He looked at the behaviour of other connectives definable in terms of conjunction and negation. In particular, he showed that defining a conjunction connective

$$A \circ B =_{df} \sim(A \rightarrow \sim B)$$

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[6]Heyting's original text is still a classic introduction to intuitionistic logic, dating from this era [134].

[7]Using substitution and *modus ponens*, and identity. If weakening is an axiom then $(A \rightarrow A) \rightarrow (B \rightarrow (A \rightarrow A))$ is an instance, and hence, by *modus ponens*, with $A \rightarrow A$, we get $B \rightarrow (A \rightarrow A)$.

gives you a connective you can prove to be associative and symmetric and square increasing[8]

$$
\begin{array}{rcl}
(A \circ B) \circ C & \to & A \circ (B \circ C) \\
A \circ (B \circ C) & \to & (A \circ B) \circ C \\
A \circ B & \to & B \circ A \\
A & \to & A \circ A
\end{array}
$$

However, the converse of the "square increasing" postulate

$$A \circ A \to A$$

is not provable, and neither are the stronger versions $A \circ B \to A$ or $B \circ A \to A$. However, for all of that, the connective Orlov defined is quite like a *conjunction*, because it satisfies the following condition:

$$\Rightarrow A \to (B \to C) \text{ if and only if } \Rightarrow A \circ B \to C$$

You can prove a nested conditional if and only if you can prove the corresponding conditional with the two antecedents combined together as one. This is a *residuation* property.[9] It renders the connective ∘ with properties of *conjunction*, for it stands with the implication ∘ in the same way that extensional conjunction and the conditional of intuitionistic or classical logic stand together.[10] Residuation properties such as these will feature a great deal in what follows.

It follows from this residuation property that ∘ cannot have *all* of the properties of extensional conjunction. $A \circ B \to A$ is not provable because if it were, then weakening axiom $A \to (B \to A)$ would also be provable. $B \circ A \to A$ is not provable, because if it were, $B \to (A \to A)$ would be.

In the same vein, Orlov defined a *disjunction* connective

$$A + B =_{df} {\sim}A \to B$$

which can be proved to be associative, symmetric and square *decreasing* ($A + A \to A$) but not square increasing. It follows that these *defined* connectives do not have the full force of the *lattice* disjunction and conjunction present in classical and intuitionistic logic. At the very first example of the study of substructural logics we are that the doorstep of one of the profound insights made clear in this area: the *splitting* of notions identified in stronger logical systems. Had Orlov noticed that one could define conjunction explicitly following the lattice definitions (as is done in intuitionistic logic, where the definitions in terms of negation and implication also fail) then he would have noticed the *split* between the *intensional* notions of conjunction and disjunction, which he defined so clearly, and the *extensional* notions which are distinct. We will see this distinction in more detail and in different contexts as

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[8]Here, and elsewhere, brackets are minimised by use of binding conventions. The general rules are simple: conditional-like connectives such as $\to$ bind less tightly than other two-place operators such as conjunction and disjunction (and fusion ∘ and fission +) which in turn bind less tightly than one place operators. So, ${\sim}A \vee B \to C \wedge D$ is the conditional whose antecedent is the disjunction of ${\sim}A$ with B and whose consequent is the conjunction of C with D.

[9]It ought to remind you of simple arithmetic results: $x = z/y$ if and only if $x \times y = z$; $x = z - y$ if and only if $x + y = z$.

[10]Namely, that $A \wedge B \supset C$ is provable if and only if $A \supset (B \supset C)$.

we continue our story through the decades. In what follows, we will refer to $\circ$ and $+$ so much that we need to give them names. I will follow the literature of *relevant* logic and call them *fusion* and *fission*.

Good ideas have a feature of being independently discovered and rediscovered. The logic R is no different. Moh [253] and Church [56], independently formulated the implication fragment of R in the early 1950's. Moh formulated an axiom system

$\diamond$	$A \to A$	identity
$\diamond$	$(A \to (A \to B)) \to (A \to B)$	contraction
$\diamond$	$A \to ((A \to B) \to B)$	assertion
$\diamond$	$(A \to B) \to ((B \to C) \to (A \to C))$	suffixing

Whereas Church's replaces the *assertion* and *suffixing* with *permutation* and *prefixing*

$\diamond$	$(A \to (B \to C)) \to (B \to (A \to C))$	permutation
$\diamond$	$(A \to B) \to ((C \to A) \to (C \to B))$	prefixing

Showing that these two axiomatisations are equivalent is an enjoyable (but lengthy) exercise in axiom chopping. It is a well-known result that in the presence either prefixing and suffixing, permutation is equivalent to assertion. Similarly, in the presence of either permutation or assertion, prefixing is equivalent to suffixing. (These facts will be more perspicuous when we show how the presence of these axioms correspond to particular structural rules. But this is to get ahead of the story by a number of decades.)

Note that each of the axioms in either Church's or Moh's presentation of R are tautologies of intuitionistic logic. Orlov's logic of relevant implication extends intuitionistic logic when it comes to *negation* (as double negation elimination is present) but when it comes to implication alone, the logic R is *weaker* than intuitionistic logic. As a corollary, Peirce's law

$\diamond$	$((A \to B) \to A) \to A$	Peirce's law

is not provable in R, despite being a classical tautology. The fallacies of relevance are examples of intuitionistic tautologies which are not present in relevant logic. Nothing so far has shown us that adding negation *conservatively extends* the implication fragment of R (in the sense that there is no implicational formula which can be proved with negation which cannot also be proved without it). However, as we will see later, this is, indeed the case. Adding negation does not lead to new implicational theorems.

Church's work on his *weak implication* system closely paralleled his work on the lambda calculus. (As we will see later, the tautologies of this system are exactly the types of the terms in his λI calculus.[11]) Church's work extends that of Orlov by proving a *deduction theorem*. Church showed that if there is a proof with hypotheses A_1 to A_n with conclusion B, then there is either a proof of B from hypotheses A_1 to A_{n-1} (in which case A_n was irrelevant as an hypothesis) or there is a proof of $A_n \to B$ from $A_1, \ldots, A_{n-1}$.

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[11] In which λx can abstract a variable from only those terms *in which the variable x occurs*. As a result, the λ-term $\lambda x.\lambda y.x$, of type $A \to (B \to A)$, is a term of the traditional λ-calculus, but not of the λI calculus.

FACT 1 (CHURCH'S DEDUCTION THEOREM) *In the implicational fragment of the relevant logic R, if $A_1, \ldots, A_n \Rightarrow B$ can be proved in the Hilbert system then either of the following two consequences can also be proved in that system.*

⋄ $A_1, \ldots, A_{n-1} \Rightarrow B$,

⋄ $A_1, \ldots, A_{n-1} \Rightarrow A_n \to B$.

PROOF   The proof follows the traditional proof of the Deduction Theorem for the implicational fragment of either classical or intuitionistic logic. A proof for $A_1, \ldots, A_n \Rightarrow B$ is transformed into a proof for $A_1, \ldots, A_{n-1} \Rightarrow A_n \to B$ by prefixing each step of the proof by "$A_n \to$". The weakening axiom $A \to (B \to A)$ is needed in the traditional result for the step showing that if an hypothesis is not used in the proof, it can be introduced as an antecedent anyway. Weakening is not present in R, and this step is not needed in the proof of Church's result, because he allows a special clause, exempting us from proving $A_n \to B$ when A_n is not actually used in the proof.   □

We will see others later on in our story. This deduction theorem lays some claim to helping explain the way in which the logic R can be said to be *relevant*. The conditional of R respects *use* in proof. To say that $A \to B$ is true is to say not only that B is true whenever A is true (keeping open the option that A might have nothing to do with B). To say that $A \to B$ is true is to say that B follows *from* A. This is not the only kind of deduction theorem applicable to relevant logics. In fact, it is probably not the most satisfactory one, as it fails once the logic is extended to include extensional conjunction. After all, we would like $A, B \Rightarrow A \wedge B$ but we can have neither $A \Rightarrow B \to A \wedge B$ (since that would give the fallacy of relevance $A \Rightarrow B \to A$, in the presence of $A \wedge B \to A$) nor $A \Rightarrow A \wedge B$ (which is classically invalid, and so, relevantly invalid). So, another characterisation of relevance must be found in the presence of conjunction. In just the same way, combining conjunction-like pairing operations in the λI calculus has proved quite difficult [212]. Avron has argued that this difficulty should make us conclude that relevance and extensional connectives cannot live together [13, 14].

Meredith and Prior were also aware of the possibility of looking for logics weaker than classical propositional logic, and that different axioms corresponded to different principles of the λ-calculus (or in Meredith and Prior's case, combinatory logic). Following on from work of Curry and Feys [62, 63], they formalised subsystems of classical logic including what they called BCK (logic without contraction) and BCI (logic without contraction or weakening: which is know known as *linear logic*) [169]. They, with Curry, are the first to explicitly chart the correspondence of propositional axioms with the behaviour of combinators which allow the rearrangement of premises or antecedents.[12]

For a number of years following this pioneering work, the work of Anderson and Belnap continued in this vein, using techniques from other branches of proof theory to explain how the logic R and its cousins respected conditions of relevance and necessity. We will shift our attention now to another of the precursors of Anderson and Belnap's work, one which pays attention to conditions of *necessity* as well as *relevance*.

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[12]It is in their honour that I use Curry's original terminology for the structural rules we will see later: W for contraction, K for weakening, C for commutativity, etc.

2.2 Entailment: Ackermann

Ackermann formulated a logic of *entailment* in the late 1950s [2]. He extended C. I. Lewis' work on systems of entailment to respect relevance and to avoid the paradoxes of strict implication. His favoured system of entailment is a weakening of the system S4 of strict implication designed to avoid the paradoxes. Unlike earlier work on relevant implication, Ackermann's system includes the full complement of sentential connectives.

To motivate the departures that Ackermann's system takes from R, note that the arrow of R is no good at all to model *entailment*. If we want to say that A entails that B, the arrow of R is significantly too strong. Specifically, axioms such as *permutation* and *assertion* must be rejected for the arrow of entailment. To take an example, suppose that A is contingently true. It is an instance of assertion that

$$A \to ((A \to A) \to A)$$

However, even if A is true, it ought not be true that $A \to A$ *entails* A. For $A \to A$ is presumably *necessarily* true. We cannot not have this necessity transferring to the contingent claim A.[13] *Permutation* must go too, as assertion is follows from permuting the identity $(A \to B) \to (A \to B)$. So, a logic of entailment must be weaker than R. However, it need not be *too* much weaker. It is clear that prefixing, suffixing and contraction are not prone to any sort of counterexample along these lines: they can survive into a logic of entailment.

Ackermann's original paper features two different presentations of the system of entailment. The first, Σ', is an ingenious consecution calculus, which is unlike any proof theory which has survived into common use, so unfortunately, I must skim over it here in one paragraph.[14] The system manipulates consecutions of the form $A, B \vdash C$ (to be understood as $A \wedge B \to C$) and $A^*, B \vdash C$ (to be understood as as $A \to (B \to C)$). If you note that the comma in the antecedent place has no uniform interpretation, and that what you have, in effect, is two different premise combining operations. This is, in embryonic form at least, the first explicit case of a dual treatment of both intensional and extensional conjunction in a proof theory that I have found.

Ackermann's other presentation of the logic of entailment is a Hilbert system. The axioms and rules are presented in Figure 1. You can see that many of the axioms considered have already occurred in the study of relevant implication. The innovations appear in both what is omitted (assertion and permutation, as we have seen) and in the full complement of rules for conjunction and disjunction.[15]

To make up for the absence of assertion and permutation, Ackermann adds *restricted permutation*. This rule is not a *permutation* rule (it doesn't permute anything) but it is a *restriction* of the permutation rule to infer $B \to (A \to C)$ from $A \to (B \to C)$. For the restricted rule we conclude $A \to C$ from

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[13]If something is *entailed* by a necessity, it too is necessary. If A entails B then if we cannot have A false, we cannot have B false either.

[14]The interested reader is referred to Ackermann's paper (in German) [2] or to Anderson, Belnap and Dunn's sympathetic summary [11, §44–46] (in English).

[15]The choice of *counterexample* as a thesis connecting implication and negation in place of *reductio* (as in Orlov) is of no matter. The two are equivalent in the presence of contraposition and double negation rules. Showing this is a gentle exercise in axiom-chopping.

Axioms

⋄	$A \rightarrow A$	identity
⋄	$(A \rightarrow B) \rightarrow ((C \rightarrow B) \rightarrow (C \rightarrow A))$	prefixing
⋄	$(A \rightarrow B) \rightarrow ((B \rightarrow C) \rightarrow (A \rightarrow C))$	suffixing
⋄	$(A \rightarrow (A \rightarrow B)) \rightarrow (A \rightarrow B)$	contraction
⋄	$A \wedge B \rightarrow A, A \wedge B \rightarrow B$	conjunction elimination
⋄	$(A \rightarrow B) \wedge (A \rightarrow C) \rightarrow (A \rightarrow B \wedge C)$	conjunction introduction
⋄	$A \rightarrow A \vee B, B \rightarrow A \vee B$	disjunction introduction
⋄	$(A \rightarrow C) \wedge (B \rightarrow C) \rightarrow (A \vee B \rightarrow C)$	disjunction elimination
⋄	$A \wedge (B \vee C) \rightarrow B \vee (A \wedge C)$	distribution
⋄	$(A \rightarrow B) \rightarrow (\sim B \rightarrow \sim A)$	contraposition
⋄	$A \wedge \sim B \rightarrow \sim (A \rightarrow B)$	counterexample
⋄	$A \rightarrow \sim\sim A$	double negation introduction
⋄	$\sim\sim A \rightarrow A$	double negation elimination

Rules

(α)	$A, A \rightarrow B \Rightarrow B$	modus ponens
(β)	$A, B \Rightarrow A \wedge B$	adjunction
(γ)	$A, \sim A \vee B \Rightarrow B$	disjunctive syllogism
(δ)	$A \rightarrow (B \rightarrow C), B \Rightarrow A \rightarrow C$	restricted permutation rule

Figure 1: Ackermann's axiomatisation Π'

$A \rightarrow (B \rightarrow C)$ and B. Clearly this follows from permutation. This restriction allows a restricted form of *assertion* too.

⋄ $(A \rightarrow A') \rightarrow (((A \rightarrow A') \rightarrow B) \rightarrow B)$ restricted assertion

This is an instance of the assertion where the first position A is replaced by the entailment $A \rightarrow A'$. While assertion might not be valid for the logic of entailment, it is valid when the proposition in the first position is itself an entailment.

As Anderson and Belnap point out [11, §8.2], (δ) is not a particularly satisfactory rule. Its status is akin to that of the rule of necessitation in modal logic (from $\Rightarrow A$ to infer $\Rightarrow \Box A$). It does not extend to an *entailment* ($A \rightarrow \Box A$). If it is possible to do without a rule like this, it seems preferable, as it licences transitions in proofs which do not correspond to valid entailments. Anderson and Belnap showed that you can indeed do without (δ) to no ill effect. The system is unchanged when you replace *restricted permutation* by *restricted assertion*.

This is not the only rule of Ackermann's entailment which provokes comment. The rule (γ) (called *disjunctive syllogism*) has had more than its fair share of ink spilled. It suffers the same failing in this system of entailment as does (δ): it does not correspond to a valid entailment. The corresponding entailment $A \wedge (\sim A \vee B) \rightarrow B$ is not provable. I will defer its discussion to Section 2.4, by which time we will have sufficient technology available to prove *theorems* about disjunctive syllogism as well as arguing about its significance.

Ackermann's remaining innovations with this system are at least twofold.

First, we have an thorough treatment of extensional disjunction and conjunction. Ackermann noticed that you need to add distribution of conjunction over disjunction as a separate axiom.[16] The conjunction and disjunction elimination and introduction rules are sufficient to show that conjunction and disjunction are lattice join and meet on propositions ordered by provable entailment. (It is a useful exercise to show that in this system of entailment, you can prove $A \vee \sim A$, $\sim(A \wedge \sim A)$, and that all de Morgan laws connecting negation, conjunction and disjunction hold.)

The final innovation is the treatment of *modality*. Ackermann notes that as in other systems of modal logic which take entailment as primary, it is possible to define the one-place modal operators of necessity, possibility and others in terms of entailment. A traditional choice is to take *impossibility* "$\sqcup$"[17] defined by setting $\sqcup A$ to be $A \to B \wedge \sim B$ for some choice of a contradiction. Clearly this will not do in the case of a *relevant* logic as even though it makes sense to say that if A entails the contradictory $B \wedge \sim B$ then A is impossible, we might have A entailing *some* contradiction (and so, being impossible) without entailing *that* contradiction. It is a fallacy of relevance to take all contradictions to be provably equivalent. No, Ackermann takes another tack, by introducing a new constant f, with some special properties.[18] The intent is to take f to mean "some contradiction is true". Ackermann then posits the following axioms and rules.

$$\diamond \qquad A \wedge \sim A \to f$$
$$\diamond \qquad (A \to f) \to \sim A$$
$$(\epsilon) \qquad A \to B, (A \to B) \wedge C \to f \Rightarrow C \to f$$

Clearly the first two are true, if we interpret f as the disjunction of all contradictions. The last we will not tarry with. It is an idiosyncratic rule, distinctive to Ackermann. More important for our concern is the definition of f. It is a new constant, with new properties which open up once we enter the substructural context. Classically (or intuitionistically) f would behave as $\perp$, a proposition which entails all others. In a substructural logic like R or Ackermann's entailment, f does no such thing. It is true that f is provably false (we can prove $\sim f$, from the axiom $(f \to f) \to \sim f$) but it does not follow that f entails *everything*. Again, a classical notion *splits*: there are two different kinds of *falsehood*. There is the *Ackermann* false constant f, which is the *weakest* provably false proposition, and there is the *Church* false constant $\perp$, which is the *strongest* false proposition, which entails every proposition whatsoever. Classically and intuitionistically, both are equivalent. Here, they come apart.

The two false constants are mirrored by their negations: two *true* constants. The Ackermann true constant t (which is $\sim f$) is the conjunction of all tautologies. The Church true constant $\top$ (which is $\sim \perp$) is the weakest proposition of all, such that $A \to \top$ is true for each A. If we are to define *necessity* by means of a propositional constant, then $t \to A$ is the appropriate choice. For $t \to A$ will hold for all *provable* A. Choosing $\top \to A$ would be much too

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[16] If we have the residuation of conjunction by $\supset$ (intuitionistic or classical material implication) then distribution follows. The algebraic analogue of this result is the thesis that a residuated lattice is distributive.

[17] For *unmöglich*.

[18] Actually, Ackermann uses the symbol "$\wedge$", but it now appears in the literature as "f".

restrictive, as we would only allow as "necessary" propositions which were entailed by all others. Since we do not have $A \vee \sim A \rightarrow B \vee \sim B$, if we want *both* to be necessary, we must be happy with the weaker condition, of being entailed by t.

This choice of true constant to define necessity motivates the choice that Anderson and Belnap used. t must entail each proposition of the form $A \rightarrow A$ (as each is a tautology). Anderson and Belnap showed that $t \rightarrow A$ in Ackermann's system is equivalent to $(A \rightarrow A) \rightarrow A$, and so they use $(A \rightarrow A) \rightarrow A$ as a definition of $\Box A$, and in this way, they showed that it was possible to define the one-place modal operators in the original language alone, without the use of propositional constants at all.[19] It is instructive to work out the details of the behaviour of $\Box$ as we have defined it. Necessity here has properties roughly of S4. In particular, you can prove $\Box A \rightarrow \Box\Box A$ but not $\Diamond A \rightarrow \Box\Diamond A$ in Ackermann's system.[20] (You will note that using this definition of necessity and without (δ) you need to add an axiom to the effect that $\Box A \wedge \Box B \rightarrow \Box(A \wedge B)$,[21] as it cannot be proved from the system as it stands. Defining $\Box A$ as $t \rightarrow A$ does not have this problem.)

2.3 Anderson and Belnap

We have well-and-truly reached beyond Ackermann's work on entailment to that of Alan Anderson and Nuel Belnap. Anderson and Belnap started their exploration of relevance and entailment with Ackermann's work [6, 8], but very soon it became an independent enterprise with a wealth of innovations and techniques from their own hands, and from their students and colleagues (chiefly J. Michael Dunn, Robert K. Meyer, Alasdair Urquhart, Richard Routley (later known as Richard Sylvan) and Kit Fine). Much of this research is reported in the two-volume *Entailment* [10, 11], and in the papers cited therein. There is no way that I can adequately summarise this work in a few pages. However, I can sketch what I take to be some of the most important enduring themes of this tradition.

2.3.1 Fitch Systems

Hilbert systems are not the most only way to present proofs. Other proof theories give us us different insights into a logical system by isolating rules relevant to each different connective. Hilbert systems, with many axioms and few rules, are not so suited to a project of understanding the internal structure of a family of logical systems. It is no surprise that in the relevant logic tradition, a great deal of work was invested toward providing different proof theories which model directly the relationship between premises and conclusions.

The first natural deduction system for R and E (Anderson and Belnap's system of *entailment*) was inspired by Fitch's natural deduction system, in common use in undergraduate and postgraduate logic instruction in the 1950s in the United States [100].[22] A Fitch system is a *linear* presentation of a natural

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[19]Impossibility $\mathsf{U}A$ is then $A \rightarrow \sim(A \rightarrow A)$.

[20]Defining $\Diamond A$ as $\sim\Box\sim A$, i. e., as $\sim((A \rightarrow A) \rightarrow \sim A)$.

[21]The axiom is ungainly when it is written out in full: $((A \rightarrow A) \rightarrow A) \wedge ((B \rightarrow B) \rightarrow B) \rightarrow ((A \wedge B \rightarrow A \wedge B) \rightarrow A \wedge B)$.

[22]That Fitch systems would be used by Anderson and Belnap is to be expected. It is also to be expected that Read [224] and Slaney [256] (from the U. K.) use Lemmon-style natural deduc-

deduction proof, with introduction and elimination rules for each connective, and the use of vertical lines to present subproofs — parts of proofs under hypotheses. Here, for example, is a proof of the relevantly unacceptable *weakening* axiom in a Fitch system for classical (or intuitionistic) logic:

$$
\begin{array}{lll}
1 & A & \text{hyp} \\
2 & \quad B & \text{hyp} \\
3 & \quad A & \text{1 reit} \\
4 & B \to A & \text{2--3 } \to\text{I} \\
5 & A \to (B \to A) & \text{1--4 } \to\text{I}
\end{array}
$$

Each line is numbered to the left, and the annotation to the right indicates the provenance of each formula. A line marked with "hyp" is an hypothesis, and its introduction increases the level of nesting of the proof. In line 4 we have the application of *conditional proof*, or as it is indicated here, *implication introduction* ($\to$I). Since A has been proved under the hypothesis of B, we deduce $B \to A$, discharging that hypothesis. The other distinctive feature of Fitch proofs is the necessity to *reiterate* formulas. If a formula appears outside a nested subproof, it is possible to reiterate it under the assumption, for use inside the subproof.

Now, this proof is defective, if we take $\to$ to indicate relevant implication. There are two possible points of disagreement. One is to question the proof at the point of line 3: perhaps something has gone wrong at the point of *reiterating* A in the subproof. This is not where Anderson and Belnap modify Fitch's system in order to model R.[23] As you can see in the proof of (relevantly acceptable) *assertion* axiom, reiteration of a formula from outside a subproof is unproblematic.

$$
\begin{array}{lll}
1 & A & \text{hyp} \\
2 & \quad A \to B & \text{hyp} \\
3 & \quad A & \text{1 reit} \\
4 & \quad B & \text{2--3 } \to\text{E} \\
5 & (A \to B) \to B & \text{2--4 } \to\text{I} \\
6 & A \to ((A \to B) \to B) & \text{1--5 } \to\text{I}
\end{array}
$$

The difference between the two proofs indicates what has gone wrong in the proof of the weakening axiom. In this proof, we have indeed *used* $A \to B$ in the proof of B from lines 1 to 4. In the earlier "proof", we indeed proved A *under* the assumption of B but we did not *use* B in that proof. The implication introduction in line 4 is fallacious. If I am to pay attention to *use* in proof, I must keep track of it in some way. Anderson and Belnap's innovation is to add *labels* to formulas in proofs. The label is a set of indices, indicating the hypotheses upon which the formula depends. If I introduce an hypothesis A in a proof, I add a new label, a singleton of a new index standing for

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tion [153], modelled after Lemmon's textbook, used in the U. K. for many years. Logicians on continental Europe are much more likely to use Prawitz [214] or Gentzen-style [111, 112] natural deduction systems. This geographic distribution of pedagogical techniques (and its resulting influence on the way *research* is directed, as well as teaching) is remarkably resilient across the decades. The recent publication of Barwise and Etchemendy's popular textbook introducing logic still uses a Fitch system [19]. As far as I am aware, instruction in logic in none of the major centres in Europe or Australia centres on Fitch-style presentation of natural deduction.

[23]Restricting reiteration is the way to give hypothesis generation and conditional introduction *modal* force, as we shall see soon.

that hypothesis. The implication introduction and elimination rules must be amended to take account of labels. For implication elimination, given A_a and $A \to B_b$, I conclude $B_{a \cup b}$, for this instance of B in the proof depends upon everything we needed for A and for $A \to B$. For implication elimination, given a proof of B_a under the hypothesis $A_{\{i\}}$, I can conclude $A \to B_{a-\{i\}}$, *provided that* $i \in a$. With these amended rules, we can annotate the original proof of *assertion* with labels, as follows.

$$
\begin{array}{lll}
1 & A_{\{1\}} & \text{hyp} \\
2 & \quad A \to B_{\{2\}} & \text{hyp} \\
3 & \quad A_{\{1\}} & 1 \text{ reit} \\
4 & \quad B_{\{1,2\}} & 2\text{--}3 \to\text{E} \\
5 & (A \to B) \to B_{\{1\}} & 2\text{--}4 \to\text{I} \\
6 & A \to ((A \to B) \to B) & 1\text{--}5 \to\text{I}
\end{array}
$$

The proof of *weakening*, on the other hand, cannot be annotated with labels satisfying the rules for implication.

$$
\begin{array}{lll}
1 & A_{\{1\}} & \text{hyp} \\
2 & \quad B_{\{2\}} & \text{hyp} \\
3 & \quad A_{\{1\}} & 1 \text{ reit} \\
4 & B \to A_{???} & 2\text{--}3 \to\text{I} \\
5 & A \to (B \to A)_{???} & 1\text{--}4 \to\text{I}
\end{array}
$$

Modifying the system to model entailment is straightforward. As I hinted earlier, if the arrow has a modal force, we do not want unrestricted reiteration. Instead of allowing an arbitrary formula to be reiterated into a subproof, since entertaining an hypothesis now has the force of considering an alternate possibility, we must only allow for reiteration formulas which might indeed hold in those alternate possibilities. Here, the requisite formulas are entailments. Entailments are not only true, but true of necessity, and so, we can reiterate an entailment under the context of an hypothesis, but we cannot reiterate atomic formulas. So, the proof above of *assertion* breaks down at the point at which we wished to reiterate A into the second subproof. The proof of *restricted assertion* will succeed.

$$
\begin{array}{lll}
1 & A \to A'_{\{1\}} & \text{hyp} \\
2 & \quad (A \to A') \to B_{\{2\}} & \text{hyp} \\
3 & \quad A \to A'_{\{1\}} & 1 \text{ reit} \\
4 & \quad B_{\{1,2\}} & 2\text{--}3 \to\text{E} \\
5 & ((A \to A') \to B) \to B_{\{1\}} & 2\text{--}4 \to\text{I} \\
6 & (A \to A') \to (((A \to A') \to B) \to B) & 1\text{--}5 \to\text{I}
\end{array}
$$

This is a permissible proof because we are entitled to reiterate $A \to A'$ at line 3. Even given the assumption that $(A \to A') \to B$, the prior assumption of $A \to A'$ holds in the new context.

Here is a slightly more complex proof in this Fitch system for entailment. (Recall that $\Box A$ is shorthand for $(A \to A) \to A$, for Anderson and Belnap's system of entailment.) This proof shows that in E, the truth of an entailment (here $B \to C$) entails that anything *entailed* by that entailment (here A) is

itself necessary too. The reiterations on lines 4 and 5 are permissible, because $B \to C$ and $(B \to C) \to A$ are both entailments.

1	$B \to C_{\{1\}}$	hyp
2	$(B \to C) \to A_{\{2\}}$	hyp
3	$A \to A_{\{3\}}$	hyp
4	$B \to C_{\{1\}}$	1 reit
5	$(B \to C) \to A_{\{2\}}$	2 reit
6	$A_{\{1,2\}}$	$4, 5 \to$E
7	$A_{\{1,2,3\}}$	$3, 6 \to$E
8	$(A \to A) \to A_{\{1,2\}}$	3–$7 \to$I
9	$((B \to C) \to A) \to \Box A_{\{1\}}$	2–$8 \to$I
10	$(B \to C) \to (((B \to C) \to A) \to \Box A)$	1–$9 \to$I

We say that A follows *relevantly* from B when a proof of with hypothesis $A_{\{i\}}$ concludes in $B_{\{i\}}$. This is written "$A \vdash B$." We say that A is provable by itself when there is a proof of A with no label at all. Then the Hilbert system and the natural deduction system match in the following two senses.

FACT 2 (HILBERT AND FITCH EQUIVALENCE) $\Rightarrow A \to B$ *if and only if* $A \vdash B$. $\Rightarrow A$ *if and only if* $\vdash A$.

PROOF   The proof is by an induction on the complexity of proofs in both directions. To convert a Fitch proof to a Hilbert proof, we replace the hypotheses $A_{\{i\}}$ by the identity $A \to A$, and the arbitrary formula $B_{\{i_1, i_2, \ldots, i_n\}}$ by $A_1 \to (A_2 \to \cdots \to (A_n \to B) \cdots)$ (where A_j is the formula introduced with label A_j). Then you show that the the steps between *these* formulas can be justified in the Hilbert system. Conversely, you simply need to show that each Hilbert axiom is provable in the Fitch system, and that *modus ponens* preserves provability. Neither proof is difficult.   □

Other restrictions on reiteration can be given to this Fitch system in order to model even weaker logics. In particular, Anderson and Belnap examine a system T of *ticket* entailment, whose rationale is the idea that statements of the form $A \to B$ are *rules* but not *facts*. They are to be used as major premises of implication eliminations, but not as minor premises. The restriction on reiteration to get this effect allows you to conclude $B_{a \cup b}$ from A_a and $A \to B_b$, *provided that* $\max(b) \le \max(a)$. The effect of this is to render restricted assertion unprovable, while *identity*, *prefixing*, *suffixing* and *contraction* remain provable (and these axiomatise the calculus T of ticket entailment).[24] (It is an open problem to this day whether the implicational fragment of T is decidable.)

Before considering the extension of this proof theory to deal with the extensional connectives, let me note one curious result in the vicinity of T. The logic TW you get by removing *contraction* from T has a surprising property. Errol Martin has shown that if $A \to B$ and $B \to A$ are provable in TW, then A and B must be the same formula [166].[25]

[24]This is as good a place as any to note that the axiom of *self distribution* $(A \to (B \to C)) \to ((A \to B) \to (A \to C))$ will do instead of *contraction* in any of these axiomatisations.

[25]Martin's proof proceeds via a result showing that the logic given by *prefixing* and *suffixing* (without identity) has *no* instances of identity provable at all. This is required, for $(A \to B) \to ((B \to A) \to (A \to A))$ is an instance of *suffixing*. The system S (for *syllogism*) has interesting properties in its own right, modelling noncircular (non "question begging"?) logic [165].

2.3.2 First Degree Entailment

It is one thing to provide a proof theory for implication or entailment. It is another to combine it with a theory of the other propositional connectives: conjunction, disjunction and negation. Anderson and Belnap's strategy was to first decide the behaviour of conjunction, disjunction and negation, and *then* combine this theory with the theory of entailment or implication. This is gives the structure of the first volume of *Entailment* [10]. The first 100 pages deals with implication alone, the next 50 with implication and negation, the next 80 with the *first degree* fragment (entailments between formulas not including implication) and only at page 231 do we find the formulation of the full system E of entailment.

Anderson and Belnap's work on entailments between truth functions (or what they call *first degree entailment*) dates back to a paper in 1962 [9]. There are many different ways to carve out first degree entailments which are *relevant* from those which are not. For example, *filter* techniques due to von Wright [288], Lewy [155], Geach [109] and Smiley [257] tell us that statements like

$$A \rightarrow B \vee {\sim}B \qquad A \wedge {\sim}A \rightarrow B$$

fail as entailments because there is no atom shared between antecedent and consequent. So far, so good, and their account follows Anderson and Belnap's. However, if this is the *only* criterion to add to classical entailment, we allow through as entailments their analogues:

$$A \rightarrow A \wedge (B \vee {\sim}B) \qquad (A \wedge {\sim}A) \vee B \rightarrow B$$

for the propositional atom A is shared in the first case, and B in the second. Since both of the following classical entailments

$$A \wedge (B \vee {\sim}B) \rightarrow B \vee {\sim}B \qquad A \wedge {\sim}A \rightarrow (A \wedge {\sim}A) \vee B$$

also satisfy the atom-sharing requirement, using variable sharing as the *only* criterion makes us reject the transitivity of entailment. After all, given $A \rightarrow A \wedge (B \vee {\sim}B)$ and given $A \wedge (B \vee {\sim}B) \rightarrow B \vee {\sim}B$, if $\rightarrow$ is transitive, we get $A \rightarrow B \vee {\sim}B$.[26]

Anderson and Belnap respond by noting that if $A \rightarrow B \vee {\sim}B$ is problematic because of relevance, then $A \rightarrow A \wedge (B \vee {\sim}B)$ is at least 50% problematic [10, page 155]. Putting things another way, if to say that A entails $B \wedge C$ is at least to say that A entails B and that A entails C, then we cannot just add a blanket atom-sharing criterion to filter out failures of relevance, for it might apply to one conjunct and not the other. Filter techniques do not work.

Anderson and Belnap characterise valid first degree entailments in a number of ways. The simplest way which does not use any *model* theory is a normal form theorem for first degree entailments. We will use a process of *reduction* to transform arbitrary entailments into *primitive* entailments, which

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[26]Nontransitive accounts of entailment have survived to this day, with more sophistication. Neil Tennant has an idiosyncratic approach to normalisation in logics, arguing for a "relevant logic" which differs from our substructural logics by allowing the validity of $A \vee B, {\sim}A \vdash B$ and $A \vdash A \vee B$, while rejecting $A, {\sim}A \vdash B$ [261, 262]. Tennant's system rejects the unrestricted transitivity of proofs: the 'Cut' which would allow $A, {\sim}A \vdash B$ from the proofs of $A \vee B, {\sim}A \vdash B$ and $A \vdash A \vee B$ is not admissible. Tennant uses normalisation to motivate this system.

we can determine on sight. The first part of the process is to drive negations inside other operators, leaving them only on atoms. We use the de Morgan equivalences and the double negation equivalence to do this.[27]

$$\sim(A \lor B) \leftrightarrow \sim A \land \sim B \qquad \sim(A \land B) \leftrightarrow \sim A \lor \sim B \qquad \sim\sim A \leftrightarrow A$$

(I write "$A \leftrightarrow B$" here a shorthand for "both $A \to B$ and $B \to A$".)

The next process involves pushing conjunctions and disjunctions around. The aim is to make the antecedent of our putative entailment a disjunction of conjunctions, and the consequent a conjunction of disjunctions. We use these distribution facts to this effect.[28]

$$(A \lor B) \land C \leftrightarrow (A \land C) \lor (B \land C) \qquad (A \land B) \lor C \leftrightarrow (A \lor C) \land (B \lor C)$$

With that transformation done, we break the entailment up into primitive entailments in these two kinds of steps:

$$A \lor B \to C \text{ if and only if } A \to C \text{ and } B \to C$$

$$A \to B \land C \text{ if and only if } A \to B \text{ and } A \to C$$

Each of these transformation rules is intended to be unproblematically valid when it comes to relevant entailment. The first batch (the negation conditions) seem unproblematic if negation is truth functional. The second batch (the distribution conditions, together with the associativity, commutativity and idempotence of both disjunction and conjunction) are sometimes questioned[29] but we have been given no reason *yet* to quibble with these as relevant entailments. Finally, the steps to break down entailments from disjunctions and entailments to disjunction are fundamental to the behaviour of conjunction and disjunction as lattice connectives. They are also fundamental to inferential properties of these connectives. $A \lor B$ licences an inference to C (and a *relevant* one, presumably!) if and only if A and B both licence that inference. $B \land C$ follows from A (and *relevantly* presumably!) if and only if B and C both follow from A.

The result of the completed transformation will then be a collection of primitive entailments: each of which is a conjunction of atoms and negated atoms in the antecedent, and a disjunction of atoms and negated atoms in the consequent. Here are some examples of primitive entailments:

$$p \land \sim p \to q \lor \sim q \qquad p \to p \lor \sim p \qquad p \land \sim p \land \sim q \land r \to s \lor \sim s \lor q \lor \sim r$$

Anderson and Belnap's criterion for deciding a primitive entailment is simple. A primitive entailment $A \to B$ is valid if and only if one of the conjuncts in the antecedent also features as a disjunct in the consequent. If there is such an atom, clearly the consequent follows *from* the antecedent. If there is no such

---

[27]We also lean on the fact that we can replace provable equivalents *ad libitum* in formulas. Formally, if we can prove $A \to B$ and $B \to A$ then we can prove $C \to C'$ and $C' \to C$, where C' results from C by changing as many instances of A to B in C as you please. All substructural logics satisfy this condition.

[28]Together with the associativity, commutativity and idempotence of both disjunction and conjunction, which I will not bother to write out formally.

[29]We will see later that linear logic rejects the distribution of conjunction over disjunction.

atom, the consequent may well be true (and perhaps even necessarily so, if an atom and its negation both appear as disjuncts) but its truth does not follow from the truth of the antecedent. This makes some kind of sense: what is it for the consequent to be true? It's for B_1 or B_2 or B_3 ... to be true. (And that's all, as that's all that the consequent says.) If none of these things are given by the antecedent, then the consequent as a whole doesn't follow from the antecedent either.[30]

We can then decide an *arbitrary* first degree entailment by this reduction process. Given an entailment, reduce it to a collection of primitive entailments, and then the original entailment is valid if and only if *each* of the primitive entailments is valid. Let's apply this to the inference of disjunctive syllogism: $(A \lor B) \land \sim A \to B$. Distributing the disjunction over the conjunction in the antecedent, we get $(A \land \sim A) \lor (B \land \sim A) \to B$. This is a valid entailment if and only if $A \land \sim A \to B$ and $B \land \sim A \to B$ both are. The second is, but the first is not. Disjunctive syllogism is therefore rejected by Anderson and Belnap. To accept it as a valid entailment is to accept $A \land \sim A \to B$ as valid. Since this is a fallacy of relevance, so is disjunctive syllogism.

This is one simple characterisation of first degree entailments. Once we start looking at *models* we will see some different models for first degree entailment which give us other straightforward characterisations of the first-degree fragment of R and E. Now, however, we must consider how to graft this account together with the account of implicational logics we have already seen.

2.3.3 *Putting them together*

To add the truth functional connectives to a Hilbert system for R or E, Anderson and Belnap used the axioms due to Ackermann for his system Π'. The *conjunction introduction* and *elimination*, *disjunction introduction* and *elimination* axioms, together with *distribution* and the rule of *adjunction* is sufficient to add the distributive lattice connectives. To add negation, you add the *double negation* axioms and *contraposition*, and *counterexample* (or equivalently, *reductio*). Adding the truth functions to a Hilbert system is straightforward.

It is more interesting to see how to add the connectives to the natural deduction system, because these systems usually afford a degree of separation between different connectives, and they provide a context in which you can see the distinctive behaviour of those connectives. Let's start with negation. Here are the negation rules proposed by Anderson and Belnap:

⋄ (∼I) From $\sim A_\alpha$ proved under the hypothesis $A_{\{k\}}$, deduce $\sim A_{\alpha - \{k\}}$ (if $k \in \alpha$). (This discharges the hypothesis.)

---

[30] I am not here applying the fallacious condition that $B_1 \lor B_2$ follows from A if and only if B_1 follows from A or B_2 follows from A, which is invalid in general. Let A be $B_1 \lor B_2$, for example. But in *that* case we note that B_1 follows from some disjunct of A and B_2 also follows from other disjunct of A. In the atomic case, A can no longer be split up.

Classically the idea to show the entailment $p \to q \lor \sim q$ the idea would be to import the *tautologous* $q \lor \sim q$ into the antecedent, to get $p \land (q \lor \sim q) \to q \lor \sim q$, distribute to get $(p \land q) \lor (p \land \sim q) \to q \lor \sim q$, and split to get both $p \land q \to q \lor \sim q$ (which is valid, by means of q) and $p \land \sim q \to q \lor \sim q$ (which is valid, by means of $\sim q$). With eyes of *relevance* there's no reason to see the appeal for importing $q \lor \sim q$ in the first place.

◇ (Contraposition) From B_a and $\sim B_b$ proved under the hypothesis $A_{\{k\}}$, deduce $\sim A_{a \cup b - \{k\}}$ (if $k \in b$). (This discharges the hypothesis.)

◇ ($\sim\sim$E) From $\sim\sim A_a$ to deduce A_a.

These rules follow directly the axioms of *reductio, contraposition* and *double negation elimination*. They are sufficient to derive all of the desired negation properties of E and R. Here, for example, is a proof of the *reductio* axiom.

$$
\begin{array}{lll}
1 & A \to \sim A_{\{1\}} & \text{hyp} \\
2 & \quad A_{\{2\}} & \text{hyp} \\
3 & \quad A \to \sim A_{\{1\}} & \text{1 reit} \\
4 & \quad \sim A_{\{1,2\}} & \text{2–3 } \to\text{E} \\
5 & \sim A_{\{1\}} & \text{2–4 } \sim\text{I} \\
6 & (A \to \sim A) \to \sim A & \text{1–5 } \to\text{I}
\end{array}
$$

The rules for conjunction are also straightforward.

◇ ($\wedge E_1$) From $A \wedge B_a$ to deduce A_a.

◇ ($\wedge E_2$) From $A \wedge B_a$ to deduce B_a.

◇ ($\wedge I$) From A_a and B_a to deduce $A \wedge B_a$.

These rules mirror the Hilbert axiom conditions (which make $\wedge$ a lattice join). The conjunction entails both conjuncts, and the conjunction is the strongest thing which entails both conjuncts.

We do not have a rule which says that if A depends on *something* and B depends on *something else* then $A \wedge B$ depends on *those things together*, because that would allow us to do too much. If we *did* have a connective (use "&" for this connective for the moment) which satisfied the same *elimination* clause as conjunction, and which satisfied that liberal *introduction* rule, it would allow us to prove the positive paradox in the following way.

$$
\begin{array}{lll}
1 & A_{\{1\}} & \text{hyp} \\
2 & \quad B_{\{2\}} & \text{hyp} \\
3 & \quad A_{\{1\}} & \text{1 reit} \\
4 & \quad A\&B_{\{1,2\}} & \text{2, 3 \&E} \\
5 & \quad A_{\{1,2\}} & \text{4 \&E reit} \\
6 & B \to A_{\{1\}} & \text{2–5 } \sim\text{I} \\
7 & A \to (B \to A) & \text{1–6 } \to\text{I}
\end{array}
$$

If we have a connective with the elimination rules of conjunction (which we surely require, if that connective is to be "and" in the traditional sense) then the liberal rules are too strong. They would allow us to take vacuous excursions through conjunction introductions and elimination, picking up irrelevant indices along the way.

No, the appropriate introduction rule for a conjunction is the restricted one which requires that both conjuncts *already* have the same relevance label. This, somewhat surprisingly, suffices to prove everything we can prove in the Hilbert system. Here, for an example, is the proof of the *conjunction*

introduction Hilbert axiom

1	$(A \to B) \wedge (A \to C)_{\{1\}}$	hyp
2	$A \to B_{\{1\}}$	$1 \wedge E$
3	$A \to C_{\{1\}}$	$1 \wedge E$
4	$\quad A_{\{2\}}$	hyp
5	$\quad A \to B_{\{1\}}$	2 reit
6	$\quad B_{\{1,2\}}$	$4,5 \ \to E$
7	$\quad A \to C_{\{1\}}$	3 reit
8	$\quad C_{\{1,2\}}$	$4,7 \ \to E$
9	$\quad B \wedge C_{\{1,2\}}$	$6,8 \wedge I$
10	$A \to B \wedge C_{\{1\}}$	$4–9 \sim I$
11	$(A \to B) \wedge (A \to C) \to (A \to B \wedge C)$	$1–10 \ \to I$

From these rules, using from the de Morgan equivalence between $A \vee B$ and $\sim(\sim A \wedge \sim B)$ it is possible to derive the following two rules for disjunction.[31] Unfortunately, these rules essentially involve the conditional. There seems to be no way to isolate rules which involve disjunction alone.

⋄ $(\vee I_1)$ From A_a to deduce $A \vee B_a$.

⋄ $(\vee I_2)$ From B_a to deduce $A \vee B_a$.

⋄ $(\vee E)$ From $A \to C_a$ and $B \to C_a$ and from $A \vee B_b$ to deduce $C_{a \cup b}$.

The most disheartening thing about these rules for disjunction (and about the natural deduction system itself) is that they do not suffice. They do not prove the distribution of conjunction over disjunction. Anderson and Belnap had to posit an extra rule.

⋄ (Dist) From $A \wedge (B \vee C)_a$ to deduce $(A \wedge B) \vee C_a$

It follows that this Fitch-style proof theory, while useful for proving things in R or E, and while giving *some* separation of the distinct behaviours of the logical connectives, does not provide pure introduction and elimination rules for each connective. For a proof theory which does *that*, the world would have to wait until the 1970s, and for some independent work of Grigori Minc [195, 197][32] and J. Michael Dunn [78].[33]

The *fusion* connective ∘ plays a minor role in early work in the Anderson–Belnap tradition.[34] They noted that it has some interesting properties in R, but that the residuation connection fails in E if we take $A \circ B$ to be defined as $\sim(A \to \sim B)$. Residuation fails because $\sim(A \to \sim B) \to C$ does not $A \to (B \to C)$ if we cannot permute antecedents of arbitrary conditionals. Since E was their focus, fusion played little role in their early work. *Later*, with Dunn's development of natural algebraic semantics, and with the shift of focus to R, fusion began to play a more central role.

The topic of finding a natural proof theory for relevant implication — and in particular, the place of distribution in such a proof theory — was a recurring theme in logical research in this tradition. The problem is not re-

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[31] See Anderson and Belnap's *Entailment* [10, §23.2] for the details.

[32] Then in Russia, and now at Stanford. He publishes now under the name "Grigori Mints"

[33] A graduate student of Nuel Belnap's.

[34] They call ∘ "fusion" after trying out names such as "cotenability" or "compossibility", connected with the definition as $\sim(A \to \sim B)$.

stricted to Fitch-style systems. Dag Prawitz's 1965 monograph *Natural Deduction* [214], launched Gentzen-style natural deduction systems on to centre stage. At the end of the book, Prawitz remarked that modifying the rules of his system would give you a system of relevant implication. Indeed they do. Almost.

Rules in Prawitz's system are simple. Proofs take the form of a tree. Some rules simply extend trees downward, from one conclusion to another. Others, take two trees and join them into a new tree with a single conclusion.

$$\frac{A \wedge B}{A} \qquad \frac{A \wedge B}{B} \qquad \frac{A \quad B}{A \wedge B} \qquad \frac{A \rightarrow B \quad A}{B}$$

These rules have as *assumptions* any undischarged premises at the top of the tree. To prove things on the basis of no assumptions, you need to use rules which *discharge* them. For example, the implication introduction rule is of this form:

$$\frac{\begin{array}{c}[A]\\ \vdots\\ B\end{array}}{A \rightarrow B}$$

This indicates that at the node for B there is a collection of open assumptions A, and we can derive $A \rightarrow B$, closing those assumptions. Prawitz hypothesised that if you modified his rules to only allow the discharge of assumptions which were actually *used* in a proof, as opposed to allowing vacuous discharge (which is required in the proof of $A \rightarrow (B \rightarrow A)$, for example), you would get a system of relevant logic in the style of Anderson and Belnap. Keeping our attention to implication alone, the answer is correct. His rule modification gives us a simple natural deduction system for R.

However, Prawitz's rules for relevant logic are less straightforward once we attempt to add conjunction. If we keep the rules as stated, then the conjunction rules allow us to prove the positive paradox in exactly the same way as in the case with & in the Fitch system.[35]

$$\frac{\dfrac{\dfrac{\dfrac{A^2 \qquad B^1}{A \wedge B} (\wedge I)}{A} (\wedge E)}{B \rightarrow A} (1, \rightarrow I)}{A \rightarrow (B \rightarrow A)} (2, \rightarrow I)$$

We must do something to the rule for conjunction introduction to ban this proof. The required emendation is to only allow conjunction introduction when the two subproofs have exactly the same open assumptions. A similar emendation is required for disjunction elimination. And *then*, once those patches are applied, it turns out that distribution is no longer provable in the system. (The intuitionistic or classical proof of distribution in Prawitz's system requires either a weakening in of an irrelevant assumption or a banned

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[35]The superscripts and the line numbers pair together assumptions and the points in the proof at which they were discharged.

conjunction or disjunction move.) Prawitz's system is no friendlier to distribution than is Fitch's.

Logics *without* distribution, such as linear logic are popular, in part, because of the difficulty of presenting straightforward proof systems for logics with distribution. In general, proof theories seem more natural or straightforward doing without it. The absence of distribution has also sparked debate. The *naturalness* or otherwise of a proof theory is no argument in and of itself for the *failure* of distribution. See Belnap's "Life in the Undistributed Middle" [32] for more on this point.

2.3.4 Embeddings

One of the most beautiful results of early work on relevant logic is the *embedding* results showing how intuitionistic logic, classical logic and S4 find their home inside R and E [7, 176, 180]. The idea is that we can move to an *irrelevant* conditional by recognising that they might be *enthymemes*. When I say that $A \supset B$ holds ($\supset$ is the intuitionistic conditional), I am not saying that B follows *from* A, I am saying that B follows from A together perhaps with some truth or other. One simple way to say this is to lean on the addition of the Ackermann constant t. We can easily add t to R by way of the following equivalences

$$A \to (t \to A) \qquad (t \to A) \to A$$

These state that a claim is *true* just in case it follows from t.[36] Given t we can define the enthymematic conditional $A \supset B$ as follows. $A \supset B$ is

$$A \wedge t \to B$$

which states that B follows from A together with some truth or other. Now, $A \supset (B \supset A)$ is provable: in fact, the stronger claim $A \to (B \supset A)$ is provable, since it follows directly from the axiom $A \supset (t \supset A)$. But this is no longer paradoxical, since $B \supset A$ does not state that A follows from B. (The proof that you get precisely intuitionistic logic through this embedding is a little trickier than it might seem. You need to revisit the definition of intuitionistic negation (write it "$\neg$" for the moment) in order to show that $A \wedge \neg A \supset B$ holds.[37] The subtleties are to be found in a paper by Meyer [180].)

The same kind of process will help us embed the *strict* conditional of S4 into E. In E, t is not only true but *necessary* (as the necessary propositions are those entailed by t) so the effect of the enthymematic definition in E is to get a strict conditional. The result is the strict conditional of E. If we define $A \Rightarrow B$ as $A \wedge t \to B$ in E, then the $\wedge, \vee, \Rightarrow$ fragment of E is exactly the $\wedge, \vee, \Rightarrow$ fragment of S4 [11, §35].

We can extend the modelling of intuitionistic logic into E if we step further afield. We require not only the propositional atom t, but some more machinery: the machinery of propositional *quantification*. If we add propositional

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[36] The first axiom here is too strong to govern t in the logic E, in which case we replace it by the permuted form $t \to (A \to A)$. The claim t doesn't *entail* all truths. (If it did, then all truths would be provable, since t is provable.) Instead, t entails all *identities*.

[37] You can't just use the negation of relevant logic, because of course we get $A \supset B \vee \sim B$, since $t \to B \vee \sim B$.

quantifiers $\forall p$ and $\exists p$ to E^{38} then intuitionistic and strict implication are defined as follows:

$$A \supset B \quad =_{df} \quad \exists p(p \wedge (p \wedge A \rightarrow B))$$
$$A \Rightarrow B \quad =_{df} \quad \exists p(\Box p \wedge (p \wedge A \rightarrow B))$$

An intuitionistic conditional asserts that there is some *truth*, such that it conjoined with A *entails* B. A strict conditional asserts that there is some *necessary truth*, such that it conjoined with A *entails* B.

Embedding the classical conditional into relevant logic is also possible. The emendation is that not only do we need to show that *weakening* is possible, but contradictions must entail everything: and we want to attempt this without introducing a new negation. The innovation comes from noticing that we can *dualise* the enthymematic construction. Instead of just requiring an extra truth as a conjunct in the antecedent, we can admit an extra *falsehood* as a *disjunct* in the *consequent*. The classical conditional (also written "A $\supset$ B") can be defined like this

$$A \wedge t \rightarrow B \vee f$$

where $f = {\sim}t$. Now we will get $A \wedge {\sim}A \supset B$ since $A \wedge {\sim}A \rightarrow f$.[39] Anderson and Belnap make some sport of material "implication" on the basis of this definition. Note that constructive implication is still genuinely an *implication* with the consequent being what we expect to conclude. A "material" implication is genuinely an implication, but you cannot conclude with the consequent of the original "conditional." No, you can only conclude the consequent with a disjoined $\vee f$.[40]

Arguments about disjoined fs lead quite well into arguments over the law of *disjunctive syllogism*, and these are the focus of our next section.

2.4 Disjunctive Syllogism

We have already seen that Ackermann's system Π' differs from Anderson and Belnap's system E by the presence of the rule (γ). In Ackermann's Π', we can directly infer B from the premises $A \vee B$ and ${\sim}A$. In E, this is not possible: for E a rule of inference from X to B is admitted only when there is some corresponding *entailment* from X (or the conjunction of formulas in X) to A. As disjunctive syllogism in an *entailment*

$$(A \vee B) \wedge {\sim}A \rightarrow B$$

is not present, Anderson and Belnap decided to do without the rule too. This motivates a question. Does dropping the rule (γ) change the set of theorems?

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[38]And the proof theory for propositional quantifiers is not difficult [11, §30–32].

[39]The result can be extended to the embed the whole of S4 into E (rather than only its positive fragment of S4) by setting $A \Rightarrow B =_{df} \exists p(\Box p \wedge (p \wedge A \rightarrow B \vee {\sim}p))$.

[40]I suspect that the situation is not quite so bad for material implication. If one treats acceptance and rejection, assertion and denial with equal priority, and if you take the role of implication as not only warranting the acceptance of the consequent, given the acceptance of the antecedent but also the rejection of the antecedent on the basis of the rejection of the consequent, then the enthymematic definition of the material conditional seems not so bad [236].

Is there anything you can prove with (γ) that you cannot prove without it? Of course there are things you can prove from from *hypotheses*, using (γ) which cannot be proved without it. In Ackermann's system Π' there is a straightforward proof for $A, {\sim}A \Rightarrow B$. In Anderson and Belnap's E there is no such proof. However, this leaves the special case of proofs from no hypotheses. Is it the case that in E, if $\vdash A \vee B$ and $\vdash {\sim}A$ that $\vdash B$ too? This is the question of the *admissibility* of disjunctive syllogism. If disjunctive syllogism is admissible in E then its *theorems* do not differ from the theorems of Ackermann's Π'.

2.4.1 A Proof of the Admissibility of Disjunctive Syllogism

There are four different proofs of the admissibility of disjunctive syllogism for logics such as E and R. The first three proofs [184, 241, 170] are due to Meyer (with help from Dunn on the first, and help from Routley on the second). They all depend on the same first step, which we will describe here as the *way up lemma*. The last proof was obtained by Kripke in 1978 [92]. In this section I will sketch the third of Meyer's proofs, because it will illustrate two techniques which have proved fruitful in the study of relevant and substructural logics. It is worth examining this result in some detail because it shows some of the distinctive techniques in the metatheory of relevant logics.

FACT 3 (DISJUNCTIVE SYLLOGISM IS ADMISSIBLE IN E AND R) *In both* E *and* R, *if* $\vdash A \vee B$ *and* $\vdash {\sim}A$ *then* $\vdash B$.

To present the bare bones of the proof of this result, we need some definitions.

DEFINITION 4 (THEORIES) A set T of formulas is a *theory* if whenever $A, B \in T$ then $A \wedge B \in T$, and if $A \vdash B$ then if $A \in T$ we also have $B \in T$. Theories are closed under conjunction and provable consequence.

Note that theories in relevant logics are rather special. Nonempty theories in *irrelevant* logics contain all theorems, since if $A \in T$ and if B is a *theorem* then so is $A \rightarrow B$ in an irrelevant logic. In relevant logics this is not the case, so theories need not contain all theorems.

Furthermore, since $A \wedge {\sim}A \rightarrow B$ is not a theorem of relevant logics, theories may be inconsistent without being trivial. A theory might contain an inconsistent pair A and ${\sim}A$, and contain its logical consequences, without the theory containing every formula whatsoever.

Finally, consistent and complete theories in classical propositional logic respect all logical connectives. In particular, if $A \vee B$ is a member of a consistent and complete theory, then one of A and B is a member of that theory. For if neither are, then ${\sim}A$ and ${\sim}B$ are members of the theory, and so is ${\sim}(A \vee B)$ (by logical consequence) contradicting $A \vee B$'s membership of the theory. In a logic like R or E it is quite possible for $A \vee B$ and ${\sim}(A \vee B)$ to be members of our theory without the theory becoming trivial. A theory can be complete without respecting disjunction. It turns out that theories which respect disjunction play a very important role, not only in our proof of the admissibility of disjunctive syllogism, but also in the theory of models for substructural logics. So, they deserve their own definition.

DEFINITION 5 (SPECIAL THEORIES) A theory T is said to be *prime* if whenever $A \vee B \in T$ then either $A \in T$ or $B \in T$. A theory T is said to be *regular* (with respect to a particular logic) whenever it contains all of the theorems of that logic.

Now we can sketch the proof of the admissibility of (γ).

PROOF We will argue by *reductio*, showing that there cannot be a case where $A \vee B$ and $\sim A$ are provable but B is not. Focus on B first. If B is not provable, we will show first that there is a *prime theory* containing all of the theorems of the logic but which still avoids B. This stage is the *Way Up*. We may have overshot our mark on the Way Up, as a prime theory containing all theorems will certainly be *complete* (as $C \vee \sim C$ is a theorem in E or R so one of C and $\sim C$ will be present in our theory) but it may not be *consistent*. If we can have a *consistent* complete prime theory containing all theorems but still missing out B we will have found our contradiction, for since this new theory contains all theorems, it contains $A \vee B$ and $\sim A$. By *primeness* it contains either A or it contains B. Containing A is ruled out since it already contains $\sim A$, so containing B is the remaining option.[41] So, the *Way Down* cuts down our original theory into a consistent and complete one. Given the way up and the way down, we will have our result. Disjunctive syllogism is admissible. □

All that remains is to prove both Way Up and Way Down lemmas.

FACT 6 (WAY UP LEMMA) *If $\not\vdash A$, then there is a regular prime theory T such that $A \notin T$.*

This is a special case of the general *pair extension theorem*, which is so useful in relevant and substructural logics that it deserves a separate statement and a sketch of its proof. To introduce this proof, we need a new definition to keep track of formulas which are to appear *in* our theory, and those which are to be kept *out*.

DEFINITION 7 ($\vdash$-PAIRS) An ordered pair $\langle L, R \rangle$ of sets of formulae is said to be a $\vdash$-*pair* if and only if there are *no* formulas $A_1, \ldots, A_n \in L$ and $B_1, \ldots, B_m \in R$ where $A_1 \wedge \cdots \wedge A_n \vdash B_1 \vee \cdots \vee B_m$.

A helpful shorthand will be to write '$\bigwedge A_i \vdash \bigvee B_j$' for the extended conjunctions and disjunctions. A $\vdash$-pair is represents a set of formulas we wish to take to be *true* (those in the left) and those we wish to take to be *false* (those in the right). The process of constructing a prime theory will involve enumerating the entire language and building up a pair, taking as many formulas as possible to be true, but adding some as false whenever we need to. So, we say that a $\vdash$-pair $\langle L', R' \rangle$ *extends* $\langle L, R \rangle$ if and only if $L \supseteq L'$ and $R \supseteq R'$. We write this as "$\langle L, R \rangle \subseteq \langle L', R' \rangle$." The end point of this process will be a *full* pair.

DEFINITION 8 (FULL $\vdash$-PAIRS) A $\vdash$-pair $\langle L, R \rangle$ is a *full $\vdash$-pair* if and only if $L \cup R$ is the entire language.

Full $\vdash$-pairs are important, as they give us prime theories.

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[41]Note here that disjunctive syllogism was used in the language used to present the proof. Much has been made of this in the literature on the significance of disjunctive syllogism [33, 182].

FACT 9 (PRIME THEORIES FROM FULL $\vdash$-PAIRS) *If $\langle L, R \rangle$ is a full $\vdash$-pair, L is a prime theory.*

PROOF We need to verify that L is closed under consequence and conjunction, and that it is prime. First, consequence. Suppose $A \in L$ and that $A \vdash B$. If $B \notin L$, then since $\langle L, R \rangle$ is full, $B \in R$. But then $A \vdash B$ contradicts the condition that $\langle L, R \rangle$ is a $\vdash$-pair.

Second, conjunction. If $A_1, A_2 \in x$, then since $A_1 \wedge A_2 \vdash A_1 \wedge A_2$, and $\langle L, R \rangle$ is a $\vdash$-pair, we must have $A_1 \wedge A_2 \notin y$, and since $\langle L, R \rangle$ is full, $A_1 \wedge A_2 \in L$ as desired.

Third, primeness. If $A_1 \vee A_2 \in L$, then if A_1 and A_2 are both not in L, by fullness, they are both in R, and since $A_1 \vee A_2 \vdash A_1 \vee A_2$, we have another contradiction to the claim that $\langle x, y \rangle$ is a $\vdash$-pair. Hence, one of A_1 and A_2 is in R, as we wished. $\square$

FACT 10 (PAIR EXTENSION THEOREM) *If $\vdash$ is the logical consequence relation of a logic including all distributive lattice properties, then any $\vdash$-pair $\langle L, R \rangle$ is extended by some full $\vdash$-pair $\langle L', R' \rangle$.*

To prove this theorem, we will assume that we have enumerated the language so that every formula in the language is in the list $C_1, C_2, \ldots, C_n, \ldots$ We will to consider each formula one by one, to check to see whether we should throw it in L or in R instead. We assume, in doing this, that our language is *countable*.[42]

PROOF First we show that if $\langle L, R \rangle$ is a $\vdash$-pair, then so is at least one of $\langle L \cup \{C\}, R \rangle$ and $\langle L, R \cup \{C\} \rangle$, for any formula C. Equivalently, we show that if $\langle L \cup \{C\}, R \rangle$ is not a $\vdash$-pair, then the alternative, $\langle L, R \cup \{C\} \rangle$, is. If this were not a $\vdash$-pair either, then there would be some $A \in \bigwedge L$ (the set of all conjunctions of formulae from L) and $B \in \bigvee R$ where $A \vdash B \vee C$. Since $\langle L \cup \{C\}, R \rangle$ is not a $\vdash$-pair, there are also $A' \in \bigwedge L$ and $B' \in \bigvee R$ such that $A' \wedge C \vdash B'$. But then, $A \wedge A' \vdash B \vee C$. But this means that $A \wedge A' \vdash (B \vee C) \wedge A'$. Now by distributive lattice properties, we then get $A \wedge A' \vdash B \vee (A' \wedge C)$. But $A' \wedge C \vdash B'$, so cut, and disjunction properties give us $A \wedge A' \vdash B \vee B'$, contrary to the fact that $\langle L, R \rangle$ is a $\vdash$-pair.

With that fact in hand, we can create our full pair. Define the series of $\vdash$-pairs $\langle L_n, R_n \rangle$ as follows. Let $\langle L_0, R_0 \rangle = \langle L, R \rangle$, and given $\langle L_n, R_n \rangle$ define $\langle L_{n+1}, R_{n+1} \rangle$ in this way.

$$\langle L_{n+1}, R_{n+1} \rangle = \begin{cases} \langle L_n \cup \{C_n\}, R_n \rangle & \text{if } \langle L_n \cup \{C_n\}, R_n \rangle \text{ is a } \vdash\text{-pair,} \\ \langle L_n, R_n \cup \{C_n\} \rangle & \text{otherwise.} \end{cases}$$

Each $\langle L_{n+1}, R_{n+1} \rangle$ is a $\vdash$-pair if its predecessor $\langle L_n, R_n \rangle$ is, for there is always a choice for placing C_n while keeping the result a $\vdash$-pair. So, by induction on n, each $\langle L_n, R_n \rangle$ is a $\vdash$-pair. It follows then then $\langle \bigcup_{n \in \omega} L_n, \bigcup_{n \in \omega} R_n \rangle$, the limit of this process, is also a $\vdash$-pair, and it covers the whole language. (If $\langle \bigcup_{n \in \omega} L_n, \bigcup_{n \in \omega} R_n \rangle$ is not a $\vdash$-pair, then we have some $A_i \in \bigcup L_n$ and some $B_j \in \bigcup R_n$ such that $A_1 \wedge \cdots \wedge A_l \vdash B_1 \vee \cdots \vee B_m$, but if this is the case, then there is some number n where each A_i is in L_n and each B_j is in R_n. It would follow that $\langle L_n, R_n \rangle$ is not a $\vdash$-pair. So, we are done. $\square$

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[42]The general kind of proof works for *well-ordered* languages as well as countable languages.

Belnap proved the Pair Extension Theorem in the early 1970s. Dunn circulated a write-up of it in about 1975, and cited it in some detail in 1976 [80]. Gabbay independently used the result for first-order intuitionistic logic, also in 1976 [106]. The theorem gives us the Way Up Lemma, because if $\not\vdash B$, then $\langle \text{Th}, \{B\} \rangle$ is a $\vdash$-pair, where Th is the set of theorems. Then this pair is extended by a full pair, the left part of which is a regular prime theory, avoiding B.

Now we can complete our story with the proof of the Way Down Lemma.

PROOF We must move from our regular prime theory T to a consistent regular prime theory $T^* \subseteq T$. We need the concept of a "metavaluation." The concept and its use in proving the admissibility (γ) is first found in Meyer's paper from 1976 [170]. A metavaluation is a set of formulas T^* on formulas defined inductively on the construction of formulas as follows:

  ◇ For a propositional atom p, $p \in T^*$ if and only if $p \in T$;

  ◇ $\sim A \in T^*$ iff **(a)** $A \notin T^*$, and **(b)** $\sim A \in T$;

  ◇ $A \wedge B \in T^*$ iff both $A \in T^*$ and $B \in T^*$;

  ◇ $A \vee B \in T^*$ iff either $A \in T^*$ or $B \in T^*$;

  ◇ $A \rightarrow B \in T^*$ iff **(a)** if $A \in T^*$ then $B \in T^*$ and **(b)** $A \rightarrow B \in T$.

Note the difference between the clauses for the *extensional* connectives $\wedge$ and $\vee$ and the *intensional* connectives $\rightarrow$ and $\sim$. The extensional connectives have one-punch rules which match their evaluation with respect to truth tables. The *intensional* connectives are more complicated. They require *both* that the formula is in the original theory and that the extensional condition holds in the new set T^*.

We will prove that T^* is a regular theory. Its *primeness* and *consistency* are already delivered by fiat, from the clauses for $\vee$ and $\sim$. The first step on the way is a simple lemma.

FACT 11 (COMPLETENESS LEMMA) *If* $A \in T^*$ *then* $A \in T$, *and if* $A \notin T^*$ *then* $\neg A \in T$.

It is simplest to prove both parts *together* by induction on the construction of A. As an example, consider the case for implication. The positive part is straightforward: if $A \rightarrow B \in T^*$ then $A \rightarrow B \in T$ by fiat. Now suppose $A \rightarrow B \notin T^*$. Then it follows that either $A \rightarrow B \notin T$ or $A \in T^*$ and $B \in T^*$. In the first case, by the completeness of T, $\sim(A \rightarrow B) \in T$ follows immediately. In the second case, $A \in T^*$ (so by the induction hypothesis, $A \in T$) and $B \notin T^*$ (so by the induction hypothesis, $\sim B \in T$). Since $A, \sim B \vdash \sim(A \rightarrow B)$ in both R and E, we have $\sim(A \rightarrow B) \in T$, as desired.

It is also not too difficult to check that T^* is a regular theory. First, T^* is closed under conjunction (by the conjunction clause) and it is *detached* (closed under *modus ponens*, by the implication clause). To show that it is a regular theory, then, it suffices to show that every axiom of the Hilbert system for R is a member. To give you an idea of how it goes, I shall consider two typical cases.

First we check *suffixing*: $(A \rightarrow B) \rightarrow ((B \rightarrow C) \rightarrow (A \rightarrow C))$. Suppose it isn't in T^*. Since it is a theorem of the logic and thus a member of T, it satisfies the intensional condition and so must fail to satisfy the extensional condition. So $A \rightarrow B \in T^*$ and $(B \rightarrow C) \rightarrow (A \rightarrow C) \notin T^*$. By the Completeness Lemma,

then $A \to B \in T$, and so by *modus ponens* from the suffixing axiom itself, we have that $(B \to C) \to (A \to C) \in T$. So $(B \to C) \to (A \to C)$ satisfies the intensional condition, and so must fail to satisfy the extensional condition: $B \to C \in T^*$ and $A \to C \notin T^*$. By similar reasoning we derive that $A \to C$ must finally fail to satisfy the extensional condition, i.e. $A \in T^*$ and $C \notin T^*$. But since of $A \to B \in T^*$, $B \to C \in T^*$, $A \in T^*$, by the extensional condition, $C \in T^*$, and we have a contradiction.

Second, check *double negation elimination*: $\sim\sim A \to A$. Suppose it isn't in T^*. Again, since it's a theorem of the logic and thus a member of T, if it fails it must fail the extensional condition. So, $\sim\sim A \in T^*$ but $A \notin T^*$. Since $\sim\sim A \in T^*$, by the negation clause, we have both $\sim A \notin T^*$ and $\sim\sim A \in T$. From $\sim\sim A \in T$, using double negation elimination, we get $A \in T$. Using the negation clause again, unpacking $\sim A \notin T^*$, we have either $A \in T^*$ or $\sim A \notin T$. The first possibility clashes with our assumption that $A \notin T^*$. The second possibility, $\sim A \notin T$ clashes again with $A \notin T^*$, using the Completeness Lemma.

The same techniques show that each of the other axioms are also present in T^*. Finally T^* is closed under *modus ponens*, and as a result, T^* is a complete, consistent regular theory, and a subset of T. This completes our proof of the Way Down Lemma. □

Meyer pioneered the use of metavaluations in relevant logic [178, 181]. Metavaluations were also used be Kleene in his study of intuitionistic theories [143, 144], who was in turn inspired by Harrop, who used the technique in the 1950s to prove primeness for intuitionistic logic [128].

There are many different proofs of the admissibility of disjunctive syllogism. Meyer pioneered the technique using metavaluations, and Meyer and Dunn have used other techniques [183, 184]. Friedman and Meyer showed that disjunctive syllogism fails in first-order relevant Peano arithmetic [103], but that it holds when you add an infinitary "omega" rule. Meyer and I have used a different style of metavaluation argument to construct a complete "true" relevant arithmetic [189]. This metavaluation argument treats negation with "one punch" clause: $\sim A \in T^*$ if and only if $A \notin T^*$. In this arithmetic, $0 = 1 \to 0 = 2$ is a theorem, as you can deduce $0 = 2$ from $0 = 1$ by arithmetic means, while $\sim(0 = 2 \to 0 = 1)$ is a theorem, as there is no way, by using multiplication, addition and identity, to deduce $0 = 1$ from $0 = 2$.

2.4.2 *Interpretation*

A great deal of the literature *interpreting* relevant logics has focussed on the status of disjunctive syllogism. The *relevantist* of Belnap and Dunn's essay "Entailment and Disjunctive Syllogism" [33] is a stout hearted person who rejects all use of disjunctive syllogism. Belnap and Dunn explain how difficult it is to maintain this line. Once you learn $A \lor B$ and you learn $\sim A$, it is indeed difficult to admit that you have no reason at all to conclude B. Stephen Read is perhaps the most prominent *relevantist* active today [223, 224]. Read's way of resisting disjunctive syllogism is to argue that in any circumstance in which there is pressure to conclude B from $A \lor B$ and $\sim A$, we have pressure to admit more than $A \lor B$: we have reason to admit $\sim A \to B$, which will licence the conclusion to B.

Some proponents of relevantists reject disjunctive syllogism not merely because it leads to fallacies of relevance, but because it renders non-trivial but inconsistent theories impossible [186, 238]. The *strong* version of this view is that inconsistencies are not only items of non-trivial theories, they are genuine *possibilities* [217]. Such a view is *dialetheism*, the thesis that contradictions are possible. Not all proponents of relevant logics are dialetheists, but dialetheism has provided a strong motivation for research into relevant logics, especially in Australia.[43]

My view on this issue differs from both the relevantist, the dialetheist and the classicalist (who accepts disjunctive syllogism, and hence rejects relevant logic) *pluralistic* [22, 233]. Disjunctive syllogism is indeed inappropriate to apply to the content of inconsistent theories. However it is impossible that the premises of a disjunctive syllogism be true while if the very same time the conclusion is false. Relevant entailment is not the *only* constraint under which truth may be regulated. Relevant entailment is one useful criterion for evaluating reasoning, but it is not the only one. If we are given reason to believe $A \lor B$ and reason to believe $\sim A$, then (provided that these reasons to do not conflict with one another) we have reason to believe B. This reason is not one licensed by relevant consequence, but relevant consequence is not the only sort of licence to which a good inference might aspire.

Debate over disjunctive syllogism has motivated interesting formal work in relevant logics. If you take the lack of disjunctive syllogism to be a fault in relevant logics, you can always add a new negation (say, *Boolean* negation, written '$-$') which satisfies the axioms $A \land -A \to B$ and $A \to B \lor -B$. Then relevant logics are truly systems of modal logic extending classical propositional logic with two modal or intensional operators, $\sim$ (a one-place operator) and $\to$ (a two-place operator). Meyer and Routley have presented alternative axiomatisations of relevant logics which contain Boolean negation '$-$', and the material conditional $A \supset B =_{df} -A \lor B$, as the primary connective [191, 192].

2.5 Lambek Calculus

Lambek worked on his calculus to model the behaviour of syntactic and semantic types in natural languages. He used technique from proof theory [149, 150] (as well as techniques from category theory which we will see later [151]). His techniques built on work of Bar-Hillel [15] and Ajdukiewicz [4] who in turn formalised some insights of Husserl.

The logical systems Lambek studied contain implication connectives and a fusion connective. Fusion in this language is not commutative, so it naturally motivates *two* implication connectives $\to$ and $\leftarrow$.[44] We get two arrow connectives because we may residuate $A \circ B \vdash C$ by isolating A on the an-

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[43]See the Australian entries in the volume "Paraconsistent Logic: Essays on the Inconsistent" [221], for example [46, 49, 173, 219, 254].

[44]Lambek wrote the two implication connectives as "/" for $\to$ and "\" for $\leftarrow$, and fusion as concatenation, but to keep continuity with other sections I will use the notation of arrows and the circle for fusion.

tecedent, or equally, by isolating B.

$$\frac{\dfrac{A \vdash B \to C}{A \circ B \vdash C}}{B \vdash C \leftarrow A}$$

If $A \circ B$ has the same effect as $B \circ A$, then $B \to C$ will have the same effect as $C \leftarrow A$. If $B \circ A$ differs from $A \circ B$ then so will $\to$ and $\leftarrow$.

One way to view Lambek's innovation is to see him as to motivating and develop a substructural logic in which two implications have a natural home. To introduce this system, consider the problem of assigning *types* to strings in some language. We might assign types to primitive expressions in the language, and explain how these cold be combined to form complex expressions. The result of such a task is a *typing judgement* of the form $x \Vdash A$, indicating that the string x has the type A. Here are some example typing judgements.

$$
\begin{array}{rcl}
\texttt{John} & \Vdash & n \\
\texttt{poor} & \Vdash & n \to n \\
\texttt{John works} & \Vdash & s \\
\texttt{works} & \Vdash & s \leftarrow n \\
\texttt{must work} & \Vdash & s \leftarrow n \\
\texttt{work} & \Vdash & i \\
\texttt{must} & \Vdash & i \to (s \leftarrow n) \\
\texttt{John work} & \Vdash & n \circ i
\end{array}
$$

Types can be atomic or complex: they form an algebra of formulas just like those in a propositional logic. Here, the judgement "John $\Vdash n$" says that the string John has the type n (for *name*, or *noun*). The next judgement says that poor has a special *compound* type $n \to n$: it converts names to names. It does this by *composition*. The string poor has the property that when you *prefix* it to a string of type n you get another (compound) string of type n. So, poor John has type n. So does poor Jean, and poor Joan of Arc (if Jean and Joan of Arc have the requisite types).[45] Strings can, of course, be concatenated at the end of other strings too. The string works has type $s \leftarrow n$ because whenever you *suffix* a string of type n with works you get a string of type s (a *sentence*). John works, poor Joan works and poor poor Joan of Arc works are all sentences, according to this grammar.

Typing can be nested arbitrarily. We see that must work has type $s \leftarrow n$ (it acts like works). The word work has type i (*intransitive infinitive*) so must has type $i \to (s \leftarrow n)$. When you concatenate it in front of any string of type i you get a string of type $s \leftarrow n$. So must play and must subscribe to *New Scientist* also have type $s \leftarrow n$, as play and subscribe to *New Scientist* have type i.

Finally, compositions have types even if the results do not have a predifined simple type. John work at least has the type $n \circ i$, as it is a concatenation of a string of type n with a string of type i. The string *must work* also has type $(i \to (s \leftarrow n)) \circ i$, because it is a composition of a string of type $i \to (s \leftarrow n)$

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[45]According to this definition, poor poor John and poor poor poor poor Joan of Arc are also strings of type n.

with a string of type i. Clearly here fusion is not commutative. `John work` has type $n \circ i$, but `work John` does not. As a corollary, $\rightarrow$ and $\leftarrow$ differ. Given the associativity of concatenation of strings, fusion is associative too. Any string of type $A \circ (B \circ C)$ is of type $(A \circ B) \circ C$. We can associate freely in any direction.[46]

Once we have a simple type system like this, typing *inferences* are then possible, on the basis of the interactions of the type-constructors $\rightarrow$, $\leftarrow$ and $\circ$. One of Lambek's innovations was to notice that this type system can be manipulated using a simple Gentzen-style consecution calculus. This calculus manipulates consecutions of the form $A_1, A_2, \ldots, A_n \vdash B$. We read this consecution as asserting that any string which is a concatenation of strings of type $A_1, A_2, \ldots, A_n$ also has type B.[47] A list of types will be treated as a type in my explanations below.[48]

The system is made up of one axiom and a collection of rules. The elementary type axiom is the *identity*.

$$A \vdash A$$

Any string of type A is of type A. The rules introduce type constructors on the left and the right of the turnstile. Here are the rules for the left-to-right arrow.

$$\frac{X, A \vdash B}{X \vdash A \rightarrow B} \, [\rightarrow R] \qquad \frac{X \vdash A \quad Y, B, Z \vdash C}{Y, A \rightarrow B, X, Z \vdash C} \, [\rightarrow L]$$

If you have any string of type X concatenated with a string of type A is a string of type B, then this means that X is of type $A \rightarrow B$. Conversely, if any string of type X is also of type A, and strings of type Y, B, Z are also of type C, then strings of type $Y, A \rightarrow B, X, Z$ are also of type C. Why is this? It is because strings of type $A \rightarrow B, X$ are also of type B, because they are concatenations of a string of type $A \rightarrow B$ to the left of a string of type X (which also has type A). The mirror image of this reasoning motivates the right-to-left conditional rules:

$$\frac{A, X \vdash B}{X \vdash B \leftarrow A} \, [\leftarrow R] \qquad \frac{X \vdash A \quad Y, B, Z \vdash C}{Y, X, B \leftarrow A, Z \vdash C} \, [\leftarrow L]$$

The next rules make fusion the direct object language correlate of the comma in the metalanguage.

$$\frac{X \vdash A \quad Y \vdash B}{X, Y \vdash A \circ B} \, [\circ R] \qquad \frac{X, A, B, Y \vdash C}{X, A \circ B, Y \vdash C} \, [\circ L]$$

Proofs in this system are trees with consecutions at the nodes, and whose leaves are axioms of identity. Each step in the tree is an instance of one or other of the rules. Here is a proof, showing that the *prefixing* axiom holds in

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[46]We can associate *fusion* freely, not the conditionals. $A \rightarrow (B \rightarrow C)$ is not the same type as $(A \rightarrow B) \rightarrow C$, as you can check.

[47]Lambek used the same notation (an arrow) to stand ambiguously for the two relations we mark with $\Vdash$ and $\vdash$ respectively.

[48]The list constructor is the metalinguistic analogue of the fusion connective. Note too that "metalinguistic" here means the metalanguage of the *type* language, which itself is a kind of metalanguage of the language of strings which it types.

rule form.[49]

$$\dfrac{C \vdash C \qquad \dfrac{A \vdash A \qquad B \vdash B}{A \to B, A \vdash B} \, (\to L)}{\dfrac{A \to B, C \to A, C \vdash B}{\dfrac{A \to B, C \to A \vdash C \to B}{A \to B \vdash (C \to A) \to (C \to B)} \, (\to R)} \, (\to R)} \, (\to L)$$

Here is another proof, which combines both implication connectives.

$$\dfrac{C \vdash C \qquad \dfrac{A \vdash A \quad B \vdash B}{A \to B, A \vdash B} \, (\to L)}{\dfrac{C, (A \to B) \leftarrow C, A \vdash B}{\dfrac{(A \to B) \leftarrow C, A \vdash B \leftarrow C}{(A \to B) \leftarrow C \vdash A \to (B \leftarrow C)} \, (\to R)} \, (\leftarrow R)} \, (\leftarrow L)$$

A proof system like this has a number of admirable properties. Most obvious is the clean division of labour in the rules for each connective. Each rule features only the connective being introduced, whether in antecedent (left) or consequent (right) position. Another admirable property is the way that formulas appearing in the premises also appear in the conclusion of a rule (either as entire formulas or as subformulas of other formulas). In *proof search*, there is no need to go looking for other intermediate formulas in the proof of a consecution. These two facts prove simple *conservative extension* results. Adding $\circ$ to the logic of $\leftarrow$ and $\to$ would result in no more provable consecutions in the original language, because a proof of a consecution involving no fusions could not involve any fusions at all.

All of this would be for naught if the deduction system were *incomplete*. If it didn't match its intended interpretation, these beautiful properties would be useless. One important step toward proving that the deduction system is complete is proving that the *cut rule* is admissible. (Recall that a rule is admissible if whenever you can prove the premises you can prove the conclusion: adding it as an extra rule does not increase the stock of provable things.)

$$\dfrac{X \vdash A \qquad Y, A, Z \vdash B}{Y, X, Z \vdash B} \, [\mathrm{Cut}_A]$$

This is not a primitive rule in our calculus, because adding it would destroy the subformula property, and make proof search intolerably unbounded. It ought to be *admissible* because of the intended interpretation of $\vdash$. If $X \vdash A$, every string of type X is also of type A. If $Y, A, Z \vdash B$, then every string which is a concatenation of a Y an A and a Z has type B. So, given a concatenation of a Y and an X and a Z, this is also a type B since the string of type X is a string of type A. The cut rule expresses the transitivity of the "every string of type x is of type y" relation.

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[49]There is no sense at this point in which some type is a *theorem* of the calculus, so we focus on the *consecution* forms of axioms, in which the main arrow is converted into a turnstile.

FACT 12 (CUT IS ADMISSIBLE IN THE LAMBEK CALCULUS) *If* $X \vdash A$ *is provable in the Lambek calculus with the aid of the cut rule, it can also be proved without it.*

PROOF Lambek's proof of the cut admissibility theorem parallels Gentzen's own [111, 112]. You take a proof featuring a *cut* and you push that cut upwards to the top of the tree, where it evaporates. So, given an instance of the cut rule, if the formula featured in the cut is not introduced in the rules above the cut, you *permute* the cut with the other rules. (You show that you could have done the cut *before* applying the other rule, instead of *after.*) Once that is done as much as possible, you have a cut where the cut formula was *introduced* in both premises of the cut. If the formula is *atomic,* then the only way it was introduced was in an axiom, and the instance of cut is irrelevant (it has *evaporated*: cutting $Y, A, Z \vdash B$ with $A \vdash A$ gives us just $Y, A, Z \vdash B$). If the formula is not atomic, you show that you could trade in the cut on that formula with cuts on smaller formulas. Here is an example cut on the implication formula $A \to B$ introduced in both left and right branches.

$$\cfrac{\cfrac{W, A \vdash B}{W \vdash A \to B} \; {\scriptstyle [\to \text{R}]} \qquad \cfrac{X \vdash A \quad Y, B, Z \vdash C}{Y, A \to B, X, Z \vdash C} \; {\scriptstyle [\to \text{L}]}}{Y, W, X, Z \vdash C} \; {\scriptstyle [\text{Cut}_{A \to B}]}$$

We can transform it so that cuts occur on the subformulas of $A \to B$.

$$\cfrac{\cfrac{X \vdash A \quad W, A \vdash B}{W, X \vdash B} \; {\scriptstyle [\text{Cut}_A]} \qquad Y, B, Z \vdash C}{Y, W, X, Z \vdash C} \; {\scriptstyle [\text{Cut}_B]}$$

The cases for the other formulas are just as straightforward. As formulas have only finite complexity, and trees have only finite height, this process terminates. □

The result that cut is admissible gives us a *decision procedure* for the calculus.

FACT 13 (DECIDABILITY OF THE LAMBEK CALCULUS) *The issue of whether or not a consecution* $X \vdash A$ *has a proof is decidable.*

PROOF To check if $X \vdash A$ is provable, consider its possible ancestors in the Gentzen proof system. There are only finitely many ancestors, each corresponding to the decomposition of one of the formulas inside the consecution. (The complex cases are the implication left rules, which give you the option of many different possible places to split the Y in the antecedent $X, A \to B, Y$ or $Y, B \leftarrow A, X$, and the fusion right rule, which gives you the choice of locations to split X in $X \vdash A \circ B$.) The possible ancestors themselves are *simpler* consecutions, with fewer connectives. Decision of consecutions with *no* connectives is trivial ($X \vdash p$ is provable if and only if X *is* p) so we have our algorithm by a recursion. □

This decision procedure for the calculus is *exceedingly* simple. Gentzen's procedure for classical and intuitionistic logic has to deal with the structural rule

of *contraction*:[50]

$$\frac{X(Y, Y) \vdash A}{X(Y) \vdash A} \ [\text{WI}]$$

which states that if a formula is used twice in a proof, it may as well have been used once. This makes proof search chronically more difficult, as some kind of limit must be found on how many consecutions might have appeared as the premises of the consecution we are trying to prove.

Sometimes people refer to the Lambek calculus as a logic *without* structural rules, but this is not the case. The Lambek calculus presumes the *associativity* of concatenation. A proper generalisation of the calculus treats antecedent structures not as *lists* of formulas but as more general *bunches* for which the comma is a genuine ordered-pairing operation. In this case, the antecedent structure $A, (B, C)$ is not the same structure as $(A, B), C$.[51] Lambek's original calculus is properly called Lambek's *associative* calculus. The *non-associative* calculus can no longer prove the prefixing consecution. (Try to follow through the proof in the absence of associativity. It doesn't work.) Of course, given a non-associative calculus, you must modify the rules for the connectives. Instead of the rules with antecedent $X, A, Y \vdash B$ we can have $X(A) \vdash B$, where "$X(A)$" indicates a structure with a designated *instance* of A. The rule for implication on the left becomes, for example

$$\frac{X \vdash A \quad Y(B) \vdash C}{Y(A \to B, X) \vdash C} \ [\to \text{L}]$$

Absence of structural rules also makes other things fail. The structural rule of contraction (W) is required for the *contraction* consecution.[52]

$$\frac{X((Y, Z), Z)) \vdash A}{X(Y, Z) \vdash C} \ [\text{W}] \qquad \frac{\dfrac{A \vdash A \quad \dfrac{A \vdash A \quad B \vdash B}{A \to B, A \vdash B} \ (\to \text{L})}{\dfrac{((A \to (A \to B), A), A) \vdash B}{\dfrac{A \to (A \to B), A \vdash B}{A \to (A \to B) \vdash A \to B} \ (\to \text{R})} \ (\text{W})} \ (\to \text{L})}$$

---

[50]You'll see that the structural rule is stated in generality: contraction operates on arbitrary structures, in arbitrary contexts. This is needed for the cut elimination process. If we could contract only whole formulas, then if we wanted to push a cut past the move from $X(A, A) \vdash B$ to $X(A) \vdash B$, where we are cutting with $C, D \vdash A$, the result would require us to somehow get from $X((C, D), (C, D)) \vdash B$ to $X(C, D) \vdash B$. We cannot do this if cut operates only on *formulas*, and if associativity or commutativity is absent.

[51]Non-associative combination plays an important role in general grammars, according to Lambek [150]. The role of some conversions such as wh- constructions (replacing names by "who", to construct questions, etc.) seem to require a finer analysis of the *phrase* structure of sentences, and seem to motivate a rejection of associativity.

Commutative composition may also have a place in linguistic analysis. Composition of different gestures in sign language may run in parallel, with no natural ordering. This kind of composition might be best modelled as distinct from the temporal ordered composition of different sign units. In this case, we have reason to admit *two* forms of composition, a situation we will see more of later.

[52]Sometimes you see it claimed that (WI) is required for the contraction consecution, this is true in the presence of associativity, but can fail outside that context. The rule (WI) corresponds to the validity of $A \wedge (A \to B) \vdash B$. It does not correspond to the validity of any consecution in the $\to$ only fragment of the language.

The structural rule of weakening (K) is required for the *weakening* axiom,

$$\frac{X(Y) \vdash C}{X(Y, Z) \vdash C} \; [\text{K}] \qquad \frac{\dfrac{A \vdash A}{A, B \vdash A} \; (\text{K})}{A \vdash B \to A} \; (\to \text{R})$$

and the structural rule of *permutation* (C) gives the permutation axiom.

$$\frac{X(Y_1, (Y_2, Z)) \vdash D}{X(Y_2, (Y_1, Z)) \vdash D} \; [\text{C}] \qquad \frac{\dfrac{A \vdash A \quad \dfrac{B \vdash B \quad C \vdash C}{B \to C, B \vdash C} \; (\to \text{L})}{\dfrac{(A \to (B \to C), A), B \vdash C}{\dfrac{(A \to (B \to C), B), A \vdash C}{\dfrac{A \to (B \to C), B \vdash A \to C}{A \to (B \to C) \vdash B \to (A \to C)} \; (\to \text{R})} \; (\to \text{R})} \; (\text{C})} \; (\to \text{L})}{}$$

Finally (for this brief excursus into the effect of structural rules) the *mingle* rule (M) has been of interest to the relevant logic community. It is the converse of WI contraction, and a special instance of weakening (K). It corresponds to the mingle consecution $A \vdash A \to A$, whose addition to R results in the well-behaved system RM. We will consider models of RM in the next section.

$$\frac{X(Y) \vdash C}{X(Y, Y) \vdash C} \; [\text{M}] \qquad \frac{\dfrac{A \vdash A}{A, A \vdash A} \; (\text{M})}{A \vdash A \to A} \; (\to \text{R})$$

There are many different structural rules which feature in different logics for different purposes. Table 1 contains some prominent structural rules. I use the notation $X \Leftarrow Y$ to stand for the structural rule

$$\frac{Z(X) \vdash A}{Z(Y) \vdash A}$$

You can replace Y by X (reading the proof upwards) in any context in an antecedent.

This proliferation of options concerning structural rules leaves us with the issue of how to choose between them. In some cases, such as Lambek's analysis of typing regimes on languages, the domain is explicit enough for the appropriate structural rules to be "read off" the objects being modelled. In the case of finding an appropriate logic of *entailment*, the question is more fraught. Anderson and Belnap's considerations in favour of the logic E are by no means the only choices available for a relevantist. Richard Sylvan's depth relevant program [139, 242] and Brady's constraints of *concept containment* [45, 48] motivate logics much weaker than E. They motivate logics without weakening, commutativity, associativity and contraction.

Let's return to Lambek, after that excursus on structural rules. In one of his early papers, Lambek considered adding conjunction to his calculus with these rules [150].

$$\frac{X \vdash A \quad X \vdash B}{X \vdash A \wedge B} \; [\wedge\text{R}] \qquad \frac{X(A) \vdash C}{X(A \wedge B) \vdash C} \; [\wedge\text{L}_1] \qquad \frac{X(B) \vdash C}{X(A \wedge B) \vdash C} \; [\wedge\text{L}_2]$$

Name	Label	Rule		
Associativity	B	$X, (Y, Z)$	$\Leftarrow$	$(X, Y), Z$
Twisted Associativity	B $'$	$X, (Y, Z)$	$\Leftarrow$	$(Y, X), Z$
Converse Associativity	B^c	$(X, Y), Z$	$\Leftarrow$	$X, (Y, Z)$
Strong Commutativity	C	$(X, Y), Z$	$\Leftarrow$	$(X, Z), Y$
Weak Commutativity	CI	X, Y	$\Leftarrow$	Y, X
Strong Contraction	W	$(X, Y), Y$	$\Leftarrow$	X, Y
Weak Contraction	WI	X, X	$\Leftarrow$	X
Mingle	M	X	$\Leftarrow$	X, X
Weakening	K	X	$\Leftarrow$	X, Y
Commuted Weakening	K $'$	X	$\Leftarrow$	Y, X

Table 1: Structural Rules

Adding disjunction with dual rules is also straightforward.

$$\frac{X \vdash A}{X \vdash A \vee B} [\vee R_1] \qquad \frac{X \vdash B}{X \vdash A \vee B} [\vee R_2] \qquad \frac{X(A) \vdash C \quad X(B) \vdash C}{X(A \vee B) \vdash C} [\wedge R]$$

Conjunctive and disjunctive types have clear interpretations in the calculus of syntactic types. In English, and is promiscuous. It conjoins sentences, names, verbs, and other things. It makes sense to say that it has a *conjunctive type*

$$\text{and} \Vdash ((a_1 \to a_1) \leftarrow a_1) \wedge \cdots \wedge ((a_n \to a_n) \leftarrow a_n)$$

for n types a_i.[53] Similarly, disjunctive types have a simple interpretation.

- $\diamond$ $x \Vdash A \wedge B$ if and only if $x \Vdash A$ and $x \Vdash B$.
- $\diamond$ $x \Vdash A \vee B$ if and only if $x \Vdash A$ or $x \Vdash B$.

Lambek's rules for conjunction and disjunction are satisfied under this interpretation of their behaviour. Lambek's rules are *sound* for this interpretation.

Cut is still admissible with the addition of these rules. It is straightforward to permute cuts past these rules, and to eliminate conjunctions introduced simultaneously by both. However, the addition results in the failure of distribution. The traditional proof of distribution (in Figure 2) requires *both* contraction and weakening. This means that the simple rules for conjunction and disjunction (in the context of this proof theory, including its structural rules) are *incomplete* for the intended interpretation.

2.6 *Kripke's Decidability Technique for* R$[\to, \wedge]$

Lambek's proof theory for the calculus of syntactic types has a close cousin, for the relevant logic R. Within a year of Lambek's publication of his calculus

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[53]*Actually* it makes sense to think of and as having type $\forall p((p \to p) \leftarrow p)$. However, propositionally quantified Lambek calculus is a wide-open field. No-one that I know of has explored this topic, at the time of writing.

$$\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{A \vdash A}{A, B \vdash B}\ (\mathsf{K}) \qquad \cfrac{B \vdash B}{A, B \vdash B}\ (\mathsf{K})}{A, B \vdash A \wedge B}\ (\wedge \mathsf{R})}{A, B \vdash (A \wedge B) \vee (A \wedge C)}\ (\vee \mathsf{R}) \qquad \cfrac{\cfrac{\cfrac{A \vdash A}{A, C \vdash C}\ (\mathsf{K}) \qquad \cfrac{C \vdash C}{A, C \vdash C}\ (\mathsf{K})}{A, C \vdash A \wedge C}\ (\wedge \mathsf{R})}{A, C \vdash (A \wedge B) \vee (A \wedge C)}\ (\vee \mathsf{R})}{A, B \vee C \vdash (A \wedge B) \vee (A \wedge C)}\ (\vee \mathsf{L})}{\cfrac{\cfrac{A, A \wedge (B \vee C) \vdash (A \wedge B) \vee (A \wedge C)}{A \wedge (B \vee C), A \wedge (B \vee C) \vdash (A \wedge B) \vee (A \wedge C)}\ (\wedge \mathsf{L})}{A \wedge (B \vee C) \vdash (A \wedge B) \vee (A \wedge C)}\ (\mathsf{WI})}\ (\wedge \mathsf{L})$$

Figure 2: Proof of Distribution of $\wedge$ over $\vee$

of types, Saul Kripke published a decidability result using a similar Gentzen system for the implication fragments of the relevant logics R and E [147]. Kripke's results extend without much modification to the implication and conjunction fragments of these logics, and less straightforwardly to the implication, negation fragment [35, 27, 10] or to the whole logic *without distribution* [174] (Meyer christened the resulting logic LR for *lattice* R). I will sketch the decidability argument for the implication and conjunction fragment R[$\rightarrow, \wedge$], and then show how LR can be embedded within R[$\rightarrow, \wedge$], rendering it decidable as well.

The technique uses the Gentzen proof system for R[$\rightarrow, \wedge$], which is a version of the Gentzen systems seen in the previous section. It uses the the same rules for $\rightarrow$ and $\wedge$, and it is modified to make it model the logic R. We have the structural rules of associativity and commutativity (which we henceforth ignore, taking antecedents of consecutions to be *multisets* of formulas). We add also the structural rule WI of *contraction*. Cut is eliminable from this system, using standard techniques. However, the decidability of the system is not straightforward, given the presence of the rule WI. WI makes proof-search fiendishly difficult. The main strategy of the decision procedure for R[$\rightarrow, \wedge$] is to limit applications WI in order to prevent a proof search from running on forever in the following way: "Is $p \vdash q$ derivable? Well it is if $p, p \vdash q$ is derivable. Is $p, p \vdash q$ derivable? Well it is if $p, p, p \vdash q$ is ..."

We need one simple notion before this strategy can be explained. We will say that the consecution $X' \vdash A$ is *a contraction* of $X \vdash A$ just in case $X' \vdash A$ can be derived from $X \vdash A$ by (repeated) applications of the the structural rules. (This means *contraction*, in effect, if you take the structures X and X' to be multisets, identifying different permutations and associations of the formulas therein.) Kripke's plan is to drop the WI, replacing it by building into the connective rules a limited amount of contraction.

More precisely, the idea is to allow a contraction of the conclusion of an connective rule only in so far as the same result could not be obtained by first contracting the premises. A little thought shows that this means no change for the rules ($\rightarrow$ R), ($\wedge$L) and ($\wedge$R), and that the following is what is needed

suffice to modify $(\rightarrow \mathrm{L})$.

$$\frac{X \vdash A \quad Y, B \vdash C}{[X, Y, A \rightarrow B] \vdash C} \, [\rightarrow \mathrm{L}']$$

where $[X, Y, A \rightarrow B]$ is any contraction of $X, Y, A \rightarrow B$ such that

⋄ $A \rightarrow B$ occurs only 0, 1, or 2 times fewer than in $X, Y, A \rightarrow B$;

⋄ Any formula other than $A \rightarrow B$ occurs only 0 or 1 time fewer.

It is clear that after modifying the system $\mathrm{R}[\rightarrow, \wedge]$ by building some limited contraction into $(\rightarrow \mathrm{L}')$ in the manner just discussed, the following lemma is provable by an induction on length of derivations:

LEMMA 14 (CURRY'S LEMMA) *If a consecution* $X' \vdash A$ *is a contraction of a consecution* $X \vdash A$ *and* $X \vdash A$ *has a derivation of length* n, *then* $X' \vdash A$ *has a derivation of length no greater than* n.[54]

This shows that the modification of the system leaves the same consecutions derivable (since the lemma shows that the effect of contraction is retained). For the rest of this section we will work in the modified proof system.

Curry's Lemma also has the corollary that every derivable consecution has an *irredundant* derivation: that is, a proof containing no branch with a consecution $X' \vdash A$ below a sequent $X \vdash A$ of which it is a contraction.

Now we can describe the decision procedure. Given a consecution $X \vdash A$, you test for provability by building a proof search tree: you place above $X \vdash A$ all possible premises or pairs of premises from which $X \vdash A$ follows by one of the rules. Even though we have built some contraction into one rule, this will be only a finite number of consecutions. This gives a tree. If a proof of the consecution exists, it will be formed as a subtree of this proof search tree. By Curry's Lemma, the proof search tree can be made irredundant. The tree is also finite, by the following lemma.

LEMMA 15 (KÖNIG'S LEMMA) *A tree with finitely branching tree with branches of finite length is itself finite.*

We have already proved that the tree is finitely branching (each consecution can have only finitely many possible ancestors). The question of the length of the branches remains open, and this is where an Kripke proved an important lemma. To state it we need an idea from Kleene. Two consecutions $X \vdash A$ and $X' \vdash A$ are *cognate* just when exactly the same formulas X as in X'. The class of all consecutions cognate to a given consecution is called a *cognation class*. Now we can state and prove Kripke's lemma.

LEMMA 16 (KRIPKE'S LEMMA) *There is no infinite sequence of cognate consecutions such that no earlier consecution is a contraction of a later consecution in the sequence.*

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[54]The name comes from Anderson and Belnap [10], who note that it is a modification of a lemma due to Curry [61], applicable to classical and intuitionistic Gentzen systems.

This means that the number of cognation classes occurring in any derivation (and hence in each branch) is finite. But Kripke's Lemma also shows that only a finite number of members of each cognation class occur in a branch (this is because we have constructed the complete proof search tree to be irredundant). So every branch is finite, and so both conditions of König's Lemma hold. It follows that the complete proof search tree is finite and so there is a decision procedure. So, a proof of Kripke's Lemma concludes our search for a decision procedure for R$[\wedge, \rightarrow]$.

PROOF   This is not a complete proof of Kripke's Lemma. (The literature contains some clear expositions [10, 35].) The kernel idea can be seen in a picture. As a special case, consider consecutions cognate to $X, Y \vdash A$. Each such consecution can be depicted as a point in the upper right-hand quadrant of the plane, marked with the origin at $(1, 1)$ rather than $(0, 0)$ since $X, Y \vdash A$ is the minimal consecution in the cognation class. So, $X, X, Y, Y, Y, Y \vdash A$ is represented as $(2, 4)$: '2 X units' and '4 Y units'. Now given any initial consecution, for example

$$(\Gamma_0) \quad X, X, X, Y, Y \vdash A$$

you might try to build an irredundant sequence by first inflating the number of Ys (for purposes of keeping on the page we let this be to 5 rather than 3088). But then, you have to decrement number of Xs at least by one. The result is depicted in Figure 3 for the first two members of the sequence Γ_0, Γ_1.
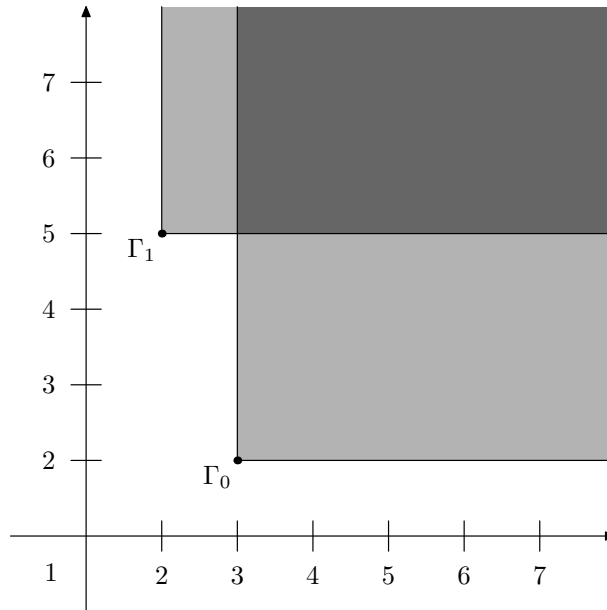

Figure 3: Descending Regions

The purpose of the intersecting lines at each point is to mark off areas (shaded in the diagram) into which no further points of the sequence may be placed. If Γ_2 were placed as indicated at the point $(6, 5)$, it would reduce to Γ_0. This this means that each new point must proceed either one unit closer

to the X axis or one unit closer to the Y axis. After a finite number of choices the consecutions will arrive at one or other of the two axes, and then after a time, you will arrive at the other. At that time, no more additions can be made, keeping the sequence irredundant.

This proof sketch generalises to n-dimensional space, corresponding to an initial consecution with n different antecedent parts. The only difficulty is in drawing the pictures.[55]   □

Extending this result to the whole of R (without distribution) is not difficult. You can amend the proof system to manipulate consecutions with structure on the *right* as well as on the left. (I won't present the modification of the rules here because they are the same as the rules for those connectives in *linear logic* which we will see in a few sections time.) The system will not prove the distribution of conjunction over disjunction, but an explicit decision procedure for the whole logic can be found. This result is due to Meyer, and can be first found in his dissertation [174] from 1966. Meyer also showed how LR can be embedded in R$[\to, \wedge]$ by translation. Meyer's translation is fairly straightforward, but I will not go through the details here.[56] I will sketch a simpler translation which comes from the Vincent Danos' more recent work on linear logic [64, 65], and which is a simple consequence of the soundness and completeness of phase space models. We translate formulas in the language of LR into the language of implication and negation by picking a particular distinguished proposition in the target language and designating that as f. Then we define $\sim$ in the language of R$[\to, \wedge]$ by setting $\sim A$ to be $A \to f$. Then the rest of the translation goes as follows:

$$
\begin{aligned}
p^t &= \sim\sim p \\
(A \wedge B)^t &= \sim\sim(A^t \wedge B^t) \\
(A \vee B)^t &= \sim(\sim A^t \wedge \sim B^t) \\
(A \circ B)^t &= \sim(A^t \to \sim B^t) \\
(A \to B)^t &= A^t \to B^t \\
(\sim A)^t &= \sim A^t
\end{aligned}
$$

I will not go through the proof of the adequacy of this translation, as we will see it when we come to look at phase spaces. However, a direct demonstration of its adequacy (without an argument taking a detour through models) is possible.[57] Given this translation, any decision procedure for R$[\to, \wedge]$ transforms into a decision procedure for the whole of LR.

McRobbie and Belnap [168] have translated the implication negation fragment of the proof theory in an analytic tableau style, and Meyer has extended this to give analytic tableau for linear logic and other systems in the vicinity of

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[55]Meyer discovered that Kripke's Lemma is equivalent to Dickson's Theorem about primes: Given any set of natural numbers all of which are composed out of the first m primes (that is, every member has the form $p_1^{n_1} p_2^{n_2} \ldots p_k^{n_k}$) if no member of this set has a proper divisor in the set, then the set is finite.

[56]The details of the translation can be found elsewhere [81, 94]. The point which makes the translation a little more complex than the translation I use here is the treatment of f and its negation t.

[57]The nicest is due to Danos. Take a proof of $X \vdash Y$ in the calculus for LR and translate it step by step into a proof of $X^t, \sim Y^t \vdash f$. (Here $\sim Y^t$ is the collection of the *negations* of the *translations* of each of the elements of Y.) The translation here is exactly what you need to make the rules correspond (modulo a few applications of Cut).

R [187]. Neither time nor space allows me to consider tableaux for substructural logics, save for this reference.

Some recent work of Alasdair Urquhart has shown that although $R[\to, \wedge]$ is decidable, it has *great* complexity [272, 275]: Given any particular formula, there is no primitive recursive bound on either the time or the space taken by a computation deciding that formula. Urquhart follows some work in linear logic [156] by using the logic to encode the behaviour of a *branching counter machines*. A counter machine has a finite number of *registers* (say, r_i for suitable i) which each hold one non-negative integer, and some finite set of possible *states* (say, q_j for suitable j). Machines are coded with a list of instructions, which enable you to *increment* or *decrement* registers, and test for registers' being zero. A *branching* counter machine dispenses with the test instructions and allows instead for machines to take multiple execution paths, by way of *forking* instructions. The instruction $q_i + r_j q_k$ means "when in q_i, add 1 to register r_j and enter stage q_k," and $q_i - r_j q_k$ means "when in q_i, subtract 1 to register r_j (if it is non-empty) and enter stage q_k," and $q_i f q_j q_k$ is "when in q_i, fork into two paths, one taking state q_j and the other taking q_k."

A machine configuration is a state, together with the values of each register. Urquhart uses the logic LR to simulate the behaviour of a machine. For each register r_i, choose a distinct variable R_i, for each state q_j choose a distinct variable Q_j. The configuration $\langle q_i; n_1, \ldots, n_l \rangle$, where n_i is the value of r_i is the formula

$$Q_i \circ R_1^{n_1} \circ \cdots \circ R_l^{n_l}$$

and the instructions are modelled by sequents in the Gentzen system, as follows:

Instruction	Sequent
$q_i + r_j q_k$	$Q_i \vdash Q_k \circ R_j$
$q_i - r_j q_k$	$Q_i, R_j \vdash Q_k$
$q_i f q_j q_k$	$Q_i \vdash Q_j \vee Q_k$

Given a machine program (a set of instructions) we can consider what is provable from the sequents which code up those instructions. This set of sequents we can call the *theory* of the machine. $Q_i \circ R_1^{n_1} \circ \cdots \circ R_l^{n_l} \vdash Q_j \circ R_1^{m_1} \circ \cdots \circ R_l^{m_l}$ is intended to mean that from state configuration $\langle q_i; n_1, \ldots, n_l \rangle$ all paths will go through configuration $\langle q_j; m_1, \ldots, m_l \rangle$ after some number of steps.

A branching counter machine *accepts* an initial configuration if when run on that configuration, all branches terminate at the final state q_f, with all registers taking the value zero. The corresponding condition in LR will be the provability of

$$Q_i \circ R_1^{n_1} \circ \cdots \circ R_l^{n_l} \vdash Q_m$$

This will *nearly* do to simulate branching counter machines, except for the fact that in LR we have $A \vdash A \circ A$. This means that each of our registers can be incremented as much as you like, provided that they are non-zero to start with. This means that each of our machines need to be equipped with every instruction of the form $q_i > 0 + r_j q_i$, meaning "if in state q_i, add 1 to r_j, provided that it is already nonzero, and remain in state q_i."

Urquhart is able to prove that a configuration is accepted in branching counter machine, if and only if the corresponding sequent is provable from

the theory of that machine. But this is equivalent to a formula

$$\bigwedge \text{Theory}(M) \wedge t \to (Q_1 \to Q_m)$$

in the language of LR. It is then a short step to our complexity result, given the fact that there is no primitive recursive bound on determining acceptability for these machines. Once this is done, the translation of LR into $R_{\to\wedge}$ gives us our complexity result.

Despite this complexity result, Kripke's algorithm has been implemented with quite some success. The theorem prover `Kripke`, written by McRobbie, Thistlewaite and Meyer, implements Kripke's decision procedure, together with some quite intelligent proof-search pruning, by means of finite models. This implementation *works* in many cases [263]. Clearly, work must be done to see whether the horrific complexity of this problem in general can be transferred to results about *average case* complexity.

2.7 Richer Structures Gentzen systems for distribution

Grigori Minc [195, 197] and J. Michael Dunn [78] independently developed a Gentzen-style consecution calculus for relevant logics in the vicinity of R. As we have seen in the Gentzen calculus for $R[\to]$, the distinctive behaviour of implication arises out of the presence or absence of structural rules governing the combination of premises. To find a logic without the paradoxes of implication, we are lead to reject the structural rule of *weakening*. However, the structural rule of weakening is *required* to prove distribution.[58] Dunn and Minc's innovations were to see a way around this apparent impasse. One way to think of the problem is this: consider the proof of distribution in Figure 2 on page 36. Focus on the point at which $(\vee L)$ is applied. The proof moves from $A, B \vdash \cdots$ and $A, C \vdash \cdots$ to $A, B \vee C \vdash \cdots$. It is *this* point at which *some* form of distribution has just been used: we have used the *disjunction* rule inside a *comma* context. This makes disjunction distribute over whatever the comma represents. In the case where comma is the metalinguistic analogue of fusion (as it is in these proof systems) we can prove $A \circ (B \vee C) \vdash (A \circ B) \vee (A \circ C)$. We cannot prove the distribution of extensional conjunction over disjunction simply because there is no structure able to represent conjunction in this proof system.[59] The solution to provide distribution is then to allow a structure to represent *extensional* conjunction, just as there is a structural analogue for *intensional* conjunction.

In a proof system like this, we define structures recursively, allowing both intensional and extensional conjunction.

◇ A formula is a structure.

◇ If X and Y are structures, so is $(X; Y)$. This is the *intensional* combination of X and Y.

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[58]At least, it is required if the proof is going to be anything like the proof of distribution in *standard* Gentzen systems.

[59]That is a simplification. The proof could be dualised, and work in a proof system with single antecedent and multiple consequents, for a dual intuitionistic logic. In this case it is the structure for extensional disjunction which would distribute over the *conjunction* rule. The relevant part of a proof would be the move from $\cdots \vdash A, B$ and $\cdots \vdash A, C$ to $\cdots \vdash A, B \wedge C$, distributing a conjunction over a disjunction again.

◇ If X and Y are structures, so is (X, Y). This is the *extensional* combination of X and Y.

Then *all* of the traditional structural rules (B, C, K, W) are admitted for *extensional* combination, and only a weaker complement (say, omitting K, for relevant logic, or all but associativity, for the Lambek calculus, or some other menu of choices for some other substructural logic) are admitted for *intensional* combination.

The rules for the connectives may remain unchanged (apart from the notational variation ";" for intensional combination, instead of "," which was used up until this point). However, the rules for conjunction may be varied to match those for fusion: we can instead take extensional conjunction to be explicitly paired with extensional combination.

$$\frac{X(A, B) \vdash C}{X(A \wedge B) \vdash C} \, [\wedge L'] \qquad \frac{X \vdash A \quad Y \vdash B}{X, Y \vdash A \wedge B} \, [\wedge R']$$

These rules are admissible, given the original structure-free rules, as these demonstrations show.[60]

$$\frac{\dfrac{X(A, B) \vdash C}{X(A \wedge B, B) \vdash C} \, (\wedge L)}{\dfrac{X(A \wedge B, A \wedge B) \vdash C}{X(A \wedge B) \vdash C} \, (\wedge L)} \, (WI) \qquad \frac{\dfrac{X \vdash A}{X, Y \vdash A} \, (K) \quad \dfrac{\dfrac{Y \vdash B}{Y, X \vdash B} \, (K)}{X, Y \vdash B} \, (C)}{X, Y \vdash A \wedge B} \, (\wedge R)$$

The modified proof theory is sound and complete for the relevant logic R and its neighbours. The cut elimination proof works as before — even with richer structures, the conditions of the cut elimination proof (permutability of cut with other rules, eliminability of matching principal formulas) are still satisfied. The subformula property is also satisfied, and the proof theory

However, the beneficial consequences of a cut-free Gentzen system for a logic — its decidability — is not always available. The difficulty is the presence of contraction for extensional combination. This is not surprising, because as we will see later, R is undecidable. You cannot extract a decision procedure from its Gentzen calculus. However, in the absence of *expansive* rules such as W and WI, a decision procedure can be found, as Steve Giambrone found in the early 1980s [114]. Giambrone's decidability argument for the negation-free fragment of R without contraction (which, we will see, is equivalent to linear logic with distribution added) and also for positive TW. Ross Brady extended this argument in the early 1990s to show that RW and TW are decidable [47]. Brady's technique involved extending the Gentzen system with *signed* formulas, to give straightforward rules for negation without resorting to a multiple consequent calculus.

Other extensions to this proof theory are possible for different applications. Belnap, Dunn and Gupta extended Dunn's original work to model R with an S4-style necessity [34]. I have shown how a system like this one can be used to motivate an extension of the Lambek calculus which is sound and

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[60]The converse proofs, to the effect that $(\wedge L)$ and $(\wedge R)$ are admissible in the presence of $(\wedge L')$ and $(\wedge R')$ are just as simple.

complete for its intended interpretation on conjunction and disjunction on frame models [228] (unlike the structure-free rules which Lambek originally proposed).

The natural deduction analogue of the Gentzen system has been the focus of much attention, too. Read uses the natural deduction system as the basis of his presentation of R in his *Relevant Logic* [224]. Slaney, in an influential article from 1990 [256] gives a philosophical defence of the two different sorts of bunching operators, characterising *extensional* combination of bodies of information as a monotonic *lumping* of information together, while taking *intensional* combination of X with Y (that's X; Y) as the *application* of X to Y. This distinction motivates the rules for implication (X ⊢ A → B iff X; A ⊢ B: A → B follows from X just when whenever you *apply* X to A, the resulting information gives B).

O'Hearn and Pym call this kind of proof theory the logic of *bunched implications* [204], and they use it to model computation.

2.8 Display Logic

Nuel Belnap's *Display Logic* [30] is a neat, uniform method for providing a cut-free consecution calculus for a wide range of formal systems. The central ideas of Belnap's *Display Logic* are simple and elegant. Like other consecution proof theories, the calculus deals with structured collections of formulas, *consecutions*. In display logic, consecutions are of the form X ⊢ Y, where X and Y are *structures*, made up from formulas. Structures are made up of structure-connectives operating on structures, building up structures from smaller structures, in much the same way as formulas are built up by formula-connectives. The base level of structures are the formulas. So far, display logic is of a piece with standard Gentzen systems — in traditional systems structures are simply lists, and in the more *avant garde* systems of Dunn and Minc, structures can be made up of two bunching operators — but in Belnap's work, structures can be even richer. This richness is present so that consecutions can support the *display property*: any substructure of a consecution can be *displayed* to be the entire antecedent or consequent of an equivalent consecution. In general, what is wanted is a way to "unravel" a context like so that we can perform equivalences such as this:

$$X(Y) \vdash Z \text{ is equivalent to } Y \vdash X^{-1}(Z)$$

where the Y inside the structure X(Y) is exposed to view, and the surrounding X(—) context is unravelled. Once you can do this, connective rules are simple, because you can assume that each formula is displayed to be the entire antecedent or consequent of a consecution.

Belnap's original work on display logic was motivated by the problem for finding a natural proof theory for relevant logics and their neighbours. As a result, it is illustrative to see the choices he made in constructing rules to allow the display of substructures. Here are some equivalences present in R and weaker relevant logics.

$$\frac{A \circ B \vdash C}{A \vdash {\sim}B + C} \qquad \frac{\dfrac{A \vdash B + C}{A \circ {\sim}B \vdash C}}{A \vdash C + B} \qquad \frac{\dfrac{A \vdash B}{{\sim}B \vdash {\sim}A}}{{\sim}{\sim}A \vdash B}$$

These equivalences allow us to "get under" the connectives in formulas. Here, the equivalences govern fusion, fission and negation. In traditional Gentzen systems, the "comma" is an overloaded operator, signifying *conjunction* in antecedent position and *disjunction* in consequent position. That is, a consecution of the form $X \vdash Y$ is interpreted as saying something like: "if everything in X is true, something in Y must be true." In substructural logics, this comma (the one which also governs the behaviour of implication) is interpreted as *fusion* on the left, and if it appears on the right at all, as *fission*. Belnap noted that we could get the display property if you add a structural connective for *negation*. If you write this connective with an asterisk, you get the following *display postulates* to parallel the facts we have already seen, governing fusion and fission.

$$\frac{X \circ Y \vdash Z}{X \vdash *Y \circ Z} \qquad \frac{\dfrac{X \vdash Y \circ Z}{X \circ *Y \vdash Z}}{X \vdash Z \circ Y} \qquad \frac{\dfrac{X \vdash Y}{*Y \vdash *X}}{**X \vdash Y}$$

(Belnap uses "$\circ$" for the *structure connective* which is fusion- and fission-like, and I will follow him in this notation.) As before, structures can be interpreted in "antecedent" position or in "consequent" position. However, now we can have "$\circ$" representing fusion on the *right* of the turnstile, or fission on the *left*, because the *negation* operator flips structures from one position to another. Consider the equivalence of $X \circ Y \vdash Z$ with $X \vdash *Y \circ Z$. In the first consecution, the structure Y is on the left of the turnstile, but on the second it is on the right. It must have the same content in both cases[61] which means that the structure connectives inside Y must be interpreted in the same way. With this caveat, the display calculus is a straightforward Gentzen-system with structure connectives allowing both positive and negative information. The rules governing the connectives are straightforward analogs of the traditional rules, with the simplification that we can now assume that principal formulas are the entire antecedent or consequent of the consecutions which introduce them. Here are the conditional rules:

$$\frac{X \circ A \vdash B}{X \vdash A \to B} \qquad \frac{X \vdash A \quad B \vdash Y}{A \to B \vdash *X \circ Y}$$

The display postulates mean that the cut rule appropriate for a display calculus can be stated exceedingly simply:

$$\frac{X \vdash A \quad A \vdash Y}{X \vdash Y} \, [\text{Cut}]$$

There is no need for a stronger rule placing the cut-formula in a context, because we can always assume that the cut formula has been displayed. This is an advance in the proof theories of substructural logics because some the various strengthenings of the cut rule, required to prove the cut-elimination theorem, are not valid in some substructural systems.[62] In his original paper,

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[61] It is justified by the equivalence of $A \circ B \vdash C$ with $A \vdash {\sim}B + C$, and in this case B "means the same thing" in both cases.

[62] The most generous case of *Mix* — from $X \vdash Y(A)$ and $X'(A) \vdash Y'$ to some conclusion, where both $Y(A)$ and $X'(A)$ involve multiple occurrences of A to ve eliminated — seems to have no appropriate valid conclusion in general substructural logics.

Belnap provides a list of eight easily checked conditions. If a display proof theory satisfies these conditions, then Cut is admissible in the system. We need not go in to the detail of these conditions here.[63]

Belnap shows that different logical systems can be given by adding different structural rules governing the display connectives — and that furthermore, the one proof system can have more than one *family* of display connectives. This is the parallel with the Dunn-Minc Gentzen system for logics with distribution. Belnap shows how you can construct proof theories for relevant logics, modal logics, intuitionistic logic, and logics which combine connectives from different families.

The idea of using display postulates to provide proof theories for different connectives is not restricted to Belnap's original family featuring a binary operator $\circ$ and a unary $*$. Wansing [285] extended Belnap's original work showing that a unary structure $\bullet$ with display rules

$$\frac{\bullet X \vdash Y}{X \vdash \bullet Y}$$

would suffice to model normal modal logics. The corresponding connective rules for $\Box$ are

$$\frac{X \vdash \bullet A}{X \vdash \Box A} \, [\Box R] \qquad \frac{A \vdash X}{\Box A \vdash \bullet X} \, [\Box L]$$

This shows that $\Box$ is the object-language correlate of $\bullet$ in *consequent* position.[64] As a result, display logic has been used outside its original substructural setting. Wansing has shown that display logic is a natural home for proof theory for classical modal logics [285, 286], Belnap has extended his calculus to model Girard's linear logic [31],[65] Goré and I have used the display calculus to model substructural logics other than those considered by Belnap [124, 229], and I have extracted some decidability results in the vein of Giambrone and Brady [232].[66]

2.9 *Linear Logic*

Girard, in 1987, introduced *linear logic*, a particular substructural system that allows commuting and reassociating of premises, but no contraction or weakening [117]. Perhaps Girard's major innovation in linear logic is the introduction of the *modalities* — the *exponentials*[67] which allow the recovery of these

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[63]I have generalised Belnap's conditions for the admissibility of cut in such a way as to include traditional consecution systems as well as display logics. It remains unclear if this generalisation will prove useful in practice, but it does seem to be an advantage to not have to prove the cut elimination theorem again and again for each proof system you construct [234, Chapter 6].

[64]Its partner in *antecedent* position is a possibility operator, but the *dual* possibility operator which looks *backwards* down the accessibility relation for necessity. It is tied together to $\Box$ by the display postulates $A \vdash \Box B$ if and only if $\blacklozenge A \vdash B$.

[65]The issue is the treatment of the exponentials.

[66]However, Kracht has shown that in general, decidability results from a display calculus are not to be expected. He has shown that in general, it is *undecidable* whether a given displayed modal logic is decidable [145].

[67]So called because of the equivalence between $!(A \wedge B)$ and $!A \circ !B$, and dually, between $?(A \vee B)$ and $?A + ?B$. This also explains why $\wedge$ and $\vee$ are the *additives* and $\circ$ and $+$ are the *multiplicatives* in the parlance of linear logic.

structural rules in a limited, controlled fashion. Linear logic has a straightforward *resource* interpretation: when premises and conclusions are taken to be resources to be used in proof, then the absence of contraction indicates that resources cannot be duplicated, and the absence of weakening indicates that resources cannot be simply thrown away. Only particular kinds of resources — those marked off by the exponentials — can be treated in this manner. Linear logic has received a great deal of attention in the literature in theoretical computer science.

2.9.1 Gentzen Systems

The most straightforward proof theory for linear logic is a consecution system where consecutions feature structure in the antecedent and the consequent:

$$A \vdash A \qquad \frac{X \vdash Y, A \quad X', A \vdash Y'}{X', X \vdash Y, Y'} \text{ [Cut]}$$

$$\frac{X \vdash A, Y}{X, {\sim}A \vdash Y} \text{ [${\sim}$L]} \qquad \frac{X, A \vdash Y}{X \vdash {\sim}A, Y} \text{ [${\sim}$R]}$$

$$\frac{X, A \vdash Y}{X, A \wedge B \vdash Y} \text{ [$\wedge$L}_1\text{]} \qquad \frac{X, B \vdash Y}{X, A \wedge B \vdash Y} \text{ [$\wedge$L}_2\text{]} \qquad \frac{X \vdash Y, A \quad X \vdash Y, B}{X \vdash Y, A \wedge B} \text{ [$\wedge$R]}$$

$$\frac{X, A, B \vdash Y}{X, A \circ B \vdash Y} \text{ [$\circ$L]} \qquad \frac{X \vdash Y, A \quad X' \vdash B, Y'}{X, X' \vdash Y, A \circ B, Y'} \text{ [$\circ$R]}$$

$$\frac{X, A \vdash Y \quad X, B \vdash Y}{X, A \vee B \vdash Y} \text{ [$\vee$L]} \qquad \frac{X \vdash Y, A}{X \vdash Y, A \vee B} \text{ [$\vee$R}_1\text{]} \qquad \frac{X \vdash Y, B}{X \vdash Y, A \vee B} \text{ [$\vee$R}_2\text{]}$$

$$\frac{X, A \vdash Y \quad B, X' \vdash Y'}{X, A + B, X' \vdash Y, Y'} \text{ [$+$L]} \qquad \frac{X \vdash Y, A, B}{X \vdash Y, A + B} \text{ [$+$R]}$$

$$\frac{X \vdash Y, A \quad X', B \vdash Y'}{X', X, A \to B \vdash Y, Y'} \text{ [$\to$L]} \qquad \frac{X, A \vdash B, Y}{X \vdash A \to B, Y} \text{ [$\to$R]}$$

$$\frac{X \vdash Y}{X, t \vdash Y} \text{ [tL]} \qquad \vdash t \text{ [tR]}$$

$$f \vdash \text{ [fL]} \qquad \frac{X \vdash Y}{X \vdash Y, f} \text{ [fR]}$$

$$X(\bot) \vdash Y \qquad X \vdash Y(\top)$$

$$\frac{X \vdash Y}{X, !A \vdash Y} \text{ [K!]} \qquad \frac{X, A \vdash Y}{X, !A \vdash Y} \text{ [L!]} \qquad \frac{!X \vdash A, ?Y}{!X \vdash !A, ?Y} \text{ [R!]} \qquad \frac{X, !A, !A \vdash Y}{X, !A \vdash Y} \text{ [WI!]}$$

$$\frac{X \vdash Y}{X \vdash ?A, Y} \text{ [K?]} \qquad \frac{X \vdash Y, A}{X \vdash Y, ?A} \text{ [R?]} \qquad \frac{!X, A \vdash ?Y}{!X, ?A \vdash ?Y} \text{ [L?]} \qquad \frac{X \vdash Y, ?A, ?A}{X \vdash Y, ?A} \text{ [WI?]}$$

Girard's notation for the connectives differs from the one we have chosen here. Figure 4 contains a translation manual between the three traditions we have seen so far.

Linear logic has two distinctive features. First, the exponentials, which allow the recovery of structural rules. Girard in fact *discovered* linear logic as a decomposition of the intuitionistic conditional $A \supset B$ into $!A \to B$ in the

Connective	Here	Lambek	Girard
Implication	$A \rightarrow B$	$A \,/\, B$	$A \multimap B$
Converse Implication	$B \leftarrow A$	$B \setminus A$	
Fusion	$A \circ B$	$A \bullet B$	$A \otimes B$
Fission	$A + B$		$A \,\invamp\, B$
Conjunction	$A \wedge B$		$A \,\&\, B$
Disjunction	$A \vee B$		$A \oplus B$
Negation	$\sim A$		$A^{\perp}$
Church Truth	t		$\mathbf{1}$
Church Falsehood	f		$\perp$
Ackermann Truth	$\top$		$\top$
Ackermann Falsehood	$\perp$		$\mathbf{0}$
Of course	$!$		$!$
Why not	$?$		$?$

Figure 4: Translation between notations

models of *coherence spaces*, which we shall see in the next part of this essay. For now, it is enough to get a taste of this decomposition. The linear implication $A \rightarrow B$ indicates that one use of A is sufficient to get one instance of B. The exponential is the operator which licences arbitrary *re-use* of resources. So, an intuitionistic conditional says that the consequent B can be found, using as many instances of A as we need. Here are the proofs of the equivalence between $!(A \wedge B)$ and $!A \circ !B$.

$$
\frac{\dfrac{\dfrac{A \vdash A}{A \wedge B \vdash A}\,{\scriptstyle(L\wedge)}}{\dfrac{!(A \wedge B) \vdash A}{!(A \wedge B) \vdash !A}\,{\scriptstyle(R!)}}\,{\scriptstyle(L!)} \quad \dfrac{\dfrac{\dfrac{B \vdash B}{A \wedge B \vdash A}\,{\scriptstyle(L\wedge)}}{\dfrac{!(A \wedge B) \vdash B}{!(A \wedge B) \vdash !B}\,{\scriptstyle(R!)}}\,{\scriptstyle(L!)}}{\dfrac{!(A \wedge B), !(A \wedge B) \vdash !A \circ !B}{!(A \wedge B) \vdash !A \circ !B}\,{\scriptstyle(WI!)}}\,{\scriptstyle(\circ R)}
$$

$$
\frac{\dfrac{\dfrac{\dfrac{A \vdash A}{!A \vdash A}\,{\scriptstyle(L!)}}{!A, !B \vdash A}\,{\scriptstyle(K!)} \quad \dfrac{\dfrac{B \vdash B}{!B \vdash B}\,{\scriptstyle(L!)}}{!A, !B \vdash B}\,{\scriptstyle(K!)}}{\dfrac{!A, !B \vdash A \wedge B}{\dfrac{!A, !B \vdash !(A \wedge B)}{!A \circ !B \vdash !(A \wedge B)}\,{\scriptstyle(L\circ)}}\,{\scriptstyle(R!)}}}{}\,{\scriptstyle(\circ R)}
$$

Vincent Danos has shown that this modelling of intuitionistic logic can be made *very* intimate [64, 65]. It is possible to translate intuitionistic logic into linear logic in such a way that *all* intuitionistic Gentzen proofs have step-by-step equivalent linear logic proofs of their translations.

Another distinctive feature of linear logic is the pervasive presence of *duality* in the system. The presence of negation means that other connectives can be easily defined in terms of their duals. On the other hand, it is also possible to take negation as the defined connective in the following way: for each atomic formula p pick out a distinguished atomic formula to be $\sim p$. Then de-

fine $\sim A$ for complex formulas as follows:

$$
\begin{array}{rcl cll}
\sim\sim A & \text{is} & A \\
\sim(A \wedge B) & \text{is} & \sim A \vee \sim B & \quad \sim(A \vee B) & \text{is} & \sim A \wedge \sim B \\
\sim\top & \text{is} & \bot & \quad \sim\bot & \text{is} & \top \\
\sim(A \circ B) & \text{is} & \sim A + \sim B & \quad \sim(A + B) & \text{is} & \sim A \circ \sim B \\
\sim t & \text{is} & f & \quad \sim f & \text{is} & t \\
\sim!A & \text{is} & ?\sim A & \quad \sim?A & \text{is} & !\sim A
\end{array}
$$

We also take $A \to B$ to be defined as $\sim A + B$ (or if you like, $\sim(A \circ \sim B)$, which is *literally* the same formula under this new regime). Together with this aspect of duality, we can also transpose consecutions from the multiple left-right variety, to a conclusion only system. We replace the consecution $X \vdash Y$ with the consecution $\vdash \sim X, Y$, where $\sim X$ is the structure containing the negations of all of the formulas in X. Then formulas are introduced only in the right, and we get a much simpler system, with one rule for every connective, as opposed to two.

$$\vdash A, \sim A \qquad \frac{\vdash X, A \quad \vdash \sim A, Y}{\vdash X, Y} \, [\text{Cut}]$$

$$\frac{\vdash X, A \quad \vdash X, B}{\vdash X, A \wedge B} \, [\wedge] \qquad \frac{\vdash X, A}{\vdash X, A \vee B} \, [\vee_1] \qquad \frac{\vdash X, B}{\vdash X, A \vee B} \, [\vee_2]$$

$$\frac{\vdash X, A \quad \vdash B, Y}{\vdash X, A \circ B, Y} \, [\circ] \qquad \frac{\vdash X, A, B}{\vdash X, A + B} \, [+]$$

$$\vdash t \, [\text{t}] \qquad \frac{\vdash X}{\vdash X, f} \, [\text{f}] \qquad \vdash X, \top \, [\top]$$

$$\frac{\vdash ?X, A}{\vdash ?X, !A} \, [!] \qquad \frac{\vdash X, A}{\vdash X, ?A} \, [?] \qquad \frac{\vdash X}{\vdash X, ?A} \, [\text{K?}] \qquad \frac{\vdash X, ?A, ?A}{\vdash X, ?A} \, [\text{WI?}]$$

2.9.2 Proof Nets

[[This section must be added. Relevant citations will be from among [26, 43, 57, 66, 107, 117, 119]]]

2.10 Curry-Howard

Some logicians have found that it is possible to analyse proofs more closely by giving them *names*. After all, if proofs are first-class entities, we will be better-off if we can distinguish different proofs. I can illustrate this by looking at an example from intuitionistic logic. The language for describing proofs in the intuitionistic logic of the conditional and implication is given by the λ-calculus with pairing. A *term* of this calculus is built up from variables $x, y, \ldots$ using the *constructors* $\langle -, - \rangle$, fst $(-)$, snd $(-)$, $\lambda x.M$ and application (which we write as juxtaposition). A *judgement* is a pair $M{:}A$ of a term M and a formula A. Then in proofs in this system we keep tabs on what is going on by building terms up to represent the ongoing proof. We start with the identity rule $x{:}A \vdash x{:}A$. Then for conjunction, we reason as follows:

$$\frac{\Gamma \vdash M{:}A \quad \Gamma \vdash N{:}B}{\Gamma \vdash \langle M, N \rangle{:}A \wedge B} \qquad \frac{\Gamma \vdash M{:}A \wedge B}{\Gamma \vdash \text{fst}\,(M){:}A} \qquad \frac{\Gamma \vdash M{:}A \wedge B}{\Gamma \vdash \text{snd}\,(M){:}B}$$

If M is the proof of A from Γ, and N is the proof of B from Γ, then the pair $\langle M, N \rangle$ is the proof of $A \wedge B$ from Γ. Similarly, if M is a proof of $A \wedge B$ (from Γ) then fst (M) (the "first part" of M) is the proof of A from Γ. Similarly, snd (M) is the proof of B from Γ. For implication, we have these rules:

$$\frac{\Gamma \vdash M{:}A \rightarrow B \quad \Delta \vdash N{:}A}{\Gamma, \Delta \vdash (MN){:}B} \qquad \frac{\Gamma, x{:}A \vdash M{:}B}{\Gamma \vdash \lambda x.M{:}A \rightarrow B}$$

If M is a proof of $A \rightarrow B$, and N is a proof of A, then you get a proof of B by *applying* M to N. So, this proof is (MN). Similarly, if M is a proof of B from $\Gamma, x{:}A$, then a proof of $A \rightarrow B$ is a function from proofs of A to the proof of B. It is of type $\lambda x.M$. We put these together to get names for more complex proofs

$$\frac{\dfrac{x{:}A \rightarrow B \vdash x{:}A \rightarrow B \quad y{:}A \vdash y{:}A}{\dfrac{x{:}A \rightarrow B, y{:}A \vdash (xy){:}B}{\dfrac{y{:}A \vdash \lambda x.(xy){:}(A \rightarrow B) \rightarrow B}{}}}{0 \vdash \lambda y.\lambda x.(xy){:}A \rightarrow ((A \rightarrow B) \rightarrow B)}$$

The term $\lambda y.\lambda x.(xy)$ encodes the shape of the proof. The first step was an application of one assumption on another (the term (xy)). The second was the abstraction of the first assumption (λx), and the last step was the abstraction of the second assumption (λy). The term encodes the proof. There are a number of important features of these terms.

⋄ Terms encoding proofs with no premises are *closed*. They have no free variables.

⋄ More generally, if $\Gamma \vdash M{:}A$ is provable and x is free in M then x appears free in Γ too.

⋄ Proofs encode connective steps, not structural rules. For example, the rule CI or C was used in the proof of $A \rightarrow ((A \rightarrow B) \rightarrow B$. It is not encoded in the term explicitly. Its presence can be seen implicitly by noting that the variables x and y are bound in the opposite order to their appearance.

Once we have a term system, we have *contracting rules*, which give us the behaviour of *proof reduction*.

$$\begin{array}{rcl} \text{fst } \langle M, N \rangle & \rightsquigarrow & M \\ \text{snd } \langle M, N \rangle & \rightsquigarrow & N \\ (\lambda x.M)N & \rightsquigarrow & M[x := N] \end{array}$$

These correspond to cutting the detours out of proofs. For example, consider the reduction

$$\frac{\dfrac{\Gamma \vdash M{:}A \quad \Gamma \vdash N{:}B}{\Gamma \vdash \langle M, N \rangle{:}A \wedge B}}{\Gamma \vdash \text{fst } \langle M, N \rangle{:}A} \quad \rightsquigarrow \quad \Gamma \vdash M{:}A$$

Or a slightly more complex case:

$$\frac{\dfrac{\Gamma, x{:}A \vdash M{:}B}{\Gamma \vdash \lambda x.M{:}A \rightarrow B} \quad \Delta \vdash N{:}A}{\Gamma, \Delta \vdash (\lambda x.M)N{:}B} \quad \rightsquigarrow \quad \Gamma, \Delta \vdash M[x := N]{:}B$$

The term $M[x := N]$ indicates that the assumption(s) marked x in M are replaced by N. This matches the assumption(s) A marked x in $\Gamma, x{:}A$ which are replaced by the Δ in the transformation.

An explanation of the Curry–Howard isomorphism between intuitionistic logic and the types of terms in the λ-calculus is found in Howard's original paper [137]. As we've already heard, Church's original calculus, the λI-calculus, was actually a model for the implicational fragment of R and not intuitionistic logic, as Church's calculus did not allow the binding of variables which were not free in the term in question [54]. You eliminate contraction if you do not allow a λ term to bind more than one instance of a variable at once. Similarly, you eliminate C if you allow variables to be bound only in the order in which they are introduced. Structural rules correspond to restrictions on binding. A helpful account of more recent general work in types and logic is found in Girard, Lafont and Taylor's *Proofs and Types* [121], and Girard's monograph *Proof Theory and Logical Complexity* [118].

Work on the application of the term calculus to substructural logics, which focuses on three aspects. First, on encoding the normalisation results (that cutting detours out of proofs ends, and ends in a canonical "normal" proof). Second, on the appropriate term encoding of the exponentials of linear logic. Work in this area has not yet reached stability. The work of Benton, Bierman, Hyland and de Paiva [38, 39, 40] shows the difficulty present in the area. Thirdly, on showing that the restrictions on λ-abstraction in substructural logics has useful parallels in computation where resources may be consumed by computation. Wadler and colleagues show that this kind of term system has connections with functional programming [160, 278, 279, 281, 280, 282, 283].

2.11 *Structurally Free Logic*

A very recent innovation in the proof theory of substructural logics is the advent of *structurally free logic*. The idea is not new — it comes from a 1976 essay by Bob Meyer [171]. However, the detailed exposition is new, dating from 1997 [42, 41, 93]. The motivating idea is simple. Just as *free logic* is free from existential commitments and any existence claims can be explicitly examined and questioned, so in a *structurally free logic, no* structural rules are present in and of themselves, but structural rules, if applied, are *marked* in a proof as explicit premises. So, structural rules are tagged with a *combinator*, such as these examples:

$$\frac{W(X,(Y,Z)) \vdash A}{W(((\mathbf{B},X),Y),Z) \vdash A}\,[\mathbf{B}] \qquad \frac{W((X,Z),Y)) \vdash A}{W(((\mathbf{C},X),Y),Z) \vdash A}\,[\mathbf{C}] \qquad \frac{W((X,Y),Y)) \vdash A}{W((\mathbf{W},X),Y) \vdash A}\,[\mathbf{W}]$$

These are the combinator versions of the structural rules B (association) C (commutativity) and W (contraction). Now the conclusions not only feature the structures as rearranged: they also feature a combinator marking the action of the structural rule. Proofs in this kind of system then come with "tick-

ets" indicating which kinds of structural rules licence the conclusion:

$$\frac{\dfrac{\dfrac{B \vdash B \quad A \vdash A}{(A \to B), A \vdash B}\ (\to L) \qquad C \vdash C}{(A \to B), (C \to A, C) \vdash B}\ (\to L)}{\dfrac{\dfrac{((\mathbf{B}, A \to B), C \to A), C \vdash B}{(\mathbf{B}, A \to B), C \to A \vdash C \to B}\ (\to R)}{\dfrac{\mathbf{B}, A \to B \vdash (C \to A) \to (C \to B)}{\mathbf{B} \vdash (A \to B) \to ((C \to A) \to (C \to B))}\ (\to R)}\ (\to R)}\ (\mathbf{B})}$$

Any further explanation the workings of this proof system for structurally free logic brings us perilously close to looking at models for combinatory logic and the λ-calculus [16, 172]. I will defer this discussion to the next section, where we broach the question in a broader setting. What on earth counts as a *model* of a substructural logic?

3   Models

Our focus so far has been *syntax* and *proof*. Now we turn our gaze to *interpretation*. Clearly we have not been unconcerned with matters of interpretation thus far. We have paid some attention to the meanings of the connectives when we have examined the kinds of inferential steps appropriate for sentences formed out of these connectives. According to some views, in giving these rules for a connective we have thereby explicated their meanings. According to other views, we have merely cashed out a *consequence* of the meanings of the connectives, meanings which are to be found in some other way.[68] Thankfully, we have no need to adjudicate such a debate here. It is not our place to clarify the ultimate source of meaning. It is, however, our place to consider some of the different kinds of interpretations open to logical systems, and particular, substructural logics.

An interpretation of a language maps the sentences in the language onto some kind of structure. There are many possible kinds of interpretations. Some propositions are true and others are not true. We can interpret a language in the structure {t, f} of *truth values* by setting the interpretation $[\![A]\!]$ of A to be t if A is true, and f otherwise. This interpretation is helpful in the study of logical consequence because of the way it interacts with the traditional propositional connectives. A conjunction is true if and only if both of the conjuncts are true. A disjunction is true if and only if one of the disjuncts is true. A negation is true if and only if the negand is not true. It follows that $[\![A \wedge B]\!]$, $[\![A \vee B]\!]$ are functions of $[\![A]\!]$ and $[\![B]\!]$, in the sense that once the values $[\![A]\!]$ and $[\![B]\!]$ are fixed, the values $[\![A \wedge B]\!]$, $[\![A \vee B]\!]$ are also fixed. The behaviour of the operations of conjunction, disjunction and negation on the set {t, f} of truth values goes some way towards telling us the meanings of those connectives. More than that, it gives us an account of the behaviour of logical

---

[68]This debate is between *truth conditional* [260] versus *inferentialist* [50] accounts of meaning in philosophy of language, *proof theorists* [118, 73] and *model theorists* [25, 136] in mathematical logic, and *operational* and *denotational* semantics in computer science [198].

consequence, as the set of truth values has a natural *order*. We can order the set by saying that f < t, in the sense that t is "more true" than f. An argument is {t, f}-valid if no matter how you interpret the propositions in the argument, the conclusion is never any less true than the premises. Or in this case, you never can interpret the premises as true and the conclusion as false. This is the traditional truth-table conception of validity.

The simple set {t, f} of truth values is not the only domain in which a language can be interpreted. For example, we might think that not all propositions or sentences in the language are truth-valued. We might interpret the language in the structure {t, n, f}, where the true claims are interpreted as t, the false ones as f, and the non-truth-valued sentences are interpreted as n. This path leads one to *many valued logics* [91, 271].

However, one need not interpret the domain of values as *truth values*. For one early example of an alternative sort of domain in which sentences can be interpreted, consider Frege's later philosophy of language. For the Frege of the *Grundgesetze* [102] declarative sentences had a *reference* (*Bedeutung*) and a *sense* (*Sinn*). We can interpret sentences by mapping them onto a domain of senses and by interpreting the connectives as functions on senses. This is another "denotational" semantics for declarative sentences.[69]

Different applications will motivate different sorts of models and domains of semantic values. In the Lambek calculus for syntactic types, the formulas can be mapped onto sets of syntactic strings. In this interpretation, a sentence will be modelled by the set of strings (in the analysed language) which have the type denoted by the sentence.

These last two examples — of possible worlds and of syntactic strings — have similar structures. Formulas are interpreted as *sets* of objects of one kind or other. These are especially interesting models, which we will discuss in detail soon. For now, however, I will focus on the general idea of interpreting logics in structures, for simple *algebras* are the first port of call when it comes to models of substructural logics.

3.1 Algebras

The most direct way to interpret a logic is by a map from the language of the logic into some structure. Such structures are usually equipped with operations to match the connectives in the language. The interpretation of a complex formula is then defined recursively in terms of the operations on the interpretations of the atomic subformulas. All of this is standard. In this section, I will examine just a few structures which have proved to be useful in the study of substructural logics. Then in the next section, I will explain just a few of the theorems which can be proved about substructural logics by using these structures.

---

[69]For a modern interpretation of Frege's ideas, one could consider a sense of a claim to be the set of possible worlds in which it is true. Now for each sentence you have its interpretation as some set of possible worlds. For an account of how this approach might be philosophically productive, see Robert Stalnaker's *Inquiry* [258], David Lewis' *On the Plurality of Worlds* [154].

For recent work which takes Frege's talk of senses at face value (and which motivates a weak substructural logic, to boot) consider the paper "Sense, Entailment and *Modus Ponens*" by Graham Priest [215].

3.1.1 Example Algebras

EXAMPLE 17 (BN4) Perhaps the most simple, yet rich, finite structure used to interpret substructural logics is the four-valued lattice BN4 [79, 28, 29]. It first came to fame as a simple lattice sufficient to interpret first degree entailments. Any valid first degree entailment is valid in this structure (in a sense to be explained soon) and any invalid first degree entailment is invalid in this structure. It is also the source of intuitions in its own right. The behaviour of BN4 is presented in the diagram and tables in Figure 5. The diagram can present the behaviour of conjunction, disjunction, $\top$ and $\bot$. The conjunction of two elements is their greatest lower bound, their disjunction, the least upper bound, $\top$ is the top element and $\bot$ is the bottom element. The operations of negation and implication and fusion are read off the tables.
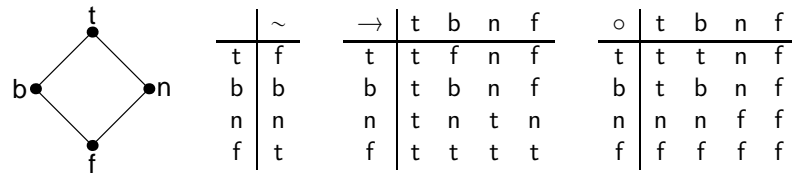

	~		→	t	b	n	f		∘	t	b	n	f
t	f		t	t	f	n	f		t	t	t	n	f
b	b		b	t	b	n	f		b	t	b	n	f
n	n		n	t	n	t	n		n	n	n	f	f
f	t		f	t	t	t	t		f	f	f	f	f

Figure 5: The Algebra BN4

If you think of the values t, b, n, f as the values "true only", "both true and false", "neither true nor false" and "false only" then the negation of a set values is simply the set of the negations of values in that set. Implication is similarly defined. The value "true" is in the set $a \to b$ just when if a is at least "true" then b is at least "true", and if b is at least "false" then so is a. On the other hand, a conditional $a \to b$ is at least "false" if a is at least "true" and b is at least "false." This gives the implication table. The values for fusion table are given by setting $a \circ b$ to be $\sim(a \to \sim b)$.

Given this definition, fusion is commutative and associative, with an identity b. Negation is definable in terms of implication by setting $\sim a$ to be $a \to b$. So in this algebra, the false constant f is modelled by b, as is the true constant t. Fusion is residuated by $\to$, and the lattice is distributive.

In this algebra, the order in the diagram (read from bottom to top, and written "$\leq$") models *entailment*. You can see that $a \wedge b$ always entails a, as the greatest lower bound of a and b (whatever a and b might be) is always lower than, or equal to, a. In just the same way, you can show that all of the entailments of a distributive lattice hold for $\wedge$ and $\vee$, that $a = \sim\sim a$ (and so, double negation elimination and introduction hold) and that the de Morgan laws, such as $\sim(a \vee b) = \sim a \wedge \sim b$ also hold in this structure.

In this lattice, some structural rules fail: WI is not satisfied, as $n \not\leq n \circ n$. The K rule also fails, as $t \circ b \leq b$, and hence we do not have $b \leq t \to b$. However, fusion is symmetric and associative. So BN4 is a model for linear logic (with the addition of distribution), in the sense that if $A \vdash B$ holds in linear logic plus distribution, then for any interpretation $[\![\cdot]\!]$ into BN4, we must have $[\![A]\!] \leq [\![B]\!]$.

However, BN4 is not a model of R, for some contraction related principles fail in BN4. For example, there is an interpretation in which $[\![A \wedge (A \to B)]\!] \not\leq$
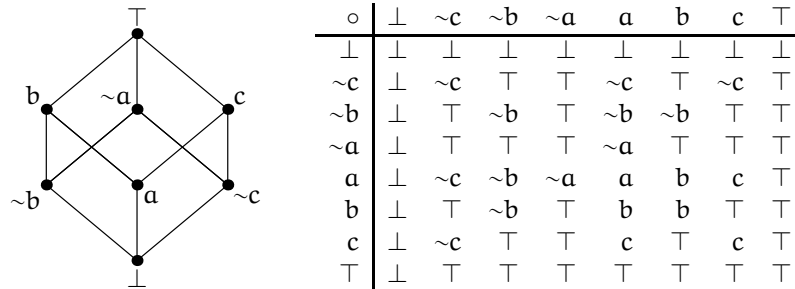
∘	⊥	∼c	∼b	∼a	a	b	c	⊤
⊥	⊥	⊥	⊥	⊥	⊥	⊥	⊥	⊥
∼c	⊥	∼c	⊤	⊤	∼c	⊤	∼c	⊤
∼b	⊥	⊤	∼b	⊤	∼b	∼b	⊤	⊤
∼a	⊥	⊤	⊤	⊤	∼a	⊤	⊤	⊤
a	⊥	∼c	∼b	∼a	a	b	c	⊤
b	⊥	⊤	∼b	⊤	b	b	⊤	⊤
c	⊥	∼c	⊤	⊤	c	⊤	c	⊤
⊤	⊥	⊤	⊤	⊤	⊤	⊤	⊤	⊤

Figure 6: An Eight-Point Lattice for R

$[\![B]\!]$.[70]

This lattice has also been used in the semantics of programming [101]. Interpreting the four values as epistemic states of no information, positive and negative information, and conflicting information, may be of some help in modelling states of information-bearing devices.

EXAMPLE 18 (AN EIGHT POINT MODEL) Consider the structure with the order and fusion table shown in Figure 6. This is a model of R: Fusion is commutative (the table is symmetric about the diagonal), and associative. We have $x \le x \circ x$, so WI holds. The element a is an identity for fusion. Negation is defined by the names of the elements and the fact that $\sim$ is a de Morgan negation. Setting $x \to y = \sim(x \circ \sim y)$ makes $\to$ residuate fusion.

We can use this structure to show that R has the relevance property. Suppose we have two propositions A and B, in the language $\wedge$, $\vee$, $\sim$, $\circ$ and $\to$, such that there is no atom shared between A and B. Construct an evaluation $[\![\circ]\!]$, such that $[\![p]\!]$ is either b or $\sim b$ for any atom p in A, and it is either c or $\sim c$ for any atom p in B.

By induction, we can verify that the value $[\![A]\!]$ is one of b and $\sim b$, and similarly, the value $[\![B]\!]$ is one of c or $\sim c$. Therefore, $[\![A]\!] \not\le [\![B]\!]$, and since this is a model of R, we have $A \not\vdash B$ in R, and hence, $A \not\vdash B$ in any sublogic of R.

EXAMPLE 19 (SUGIHARA MODELS) One can modify BN4 in a number of ways. You can leave out the value b, and get Łukasiewicz's three-valued logic. Extensions to Łukasiewicz's n-valued, and infinitary logics are straightforward too. These systems all invalidate contraction, but validate weakening and the other common contraction-free structural rules. Another way of modifying BN4 is to leave out the value n. This gives us the structure known as RM3, a three-valued algebra useful in the study of relevant logics, because this is a model of R. This simple three-valued model can be generalised to RM_{2n+1} for any n as follows by setting the domain of propositions to be the numbers

$$\{-n, -(n-1), \ldots, -1, 0, 1, \ldots, n-1, n\}$$

---

[70]Hint: set $[\![A]\!] = n$ and $[\![B]\!] = f$. Check for yourself that this is a counterexample.

where we set $\sim a$ to be $-a$, and $\rightarrow$ and fusion are defined as follows:

$$a \rightarrow b = \left\{ \begin{array}{ll} -a \vee b & \text{if } a \leq b \\ -a \wedge b & \text{if } a > b \end{array} \right. \qquad a \cdot b = \left\{ \begin{array}{ll} a \wedge b & \text{if } a \leq -b \\ a \vee b & \text{if } a > -b \end{array} \right.$$

Fusion is commutative (verify by eye) and associative (verify by checking case by case), with identity 0. Note that $a \cdot a = a$, so the logic satisfies both W and M — this is a model for the logic RM discussed earlier.

This model can also be extended by not stopping at $-n$ or n but by including all of $\mathbb{Z}$, the positive and negative integers. This infinite model captures exactly the logic RM in the language $\wedge, \vee, \rightarrow, \circ, \sim, t$. The infinite model has no members fit for either $\top$ or $\bot$, but they can be added as ∞ and $-\infty$ without disturbing the logic of the model.

EXAMPLE 20 (THE INTEGERS) The integers feature in the RM algebra above. The choice of the interpretation of implication in that model is only one of many different ways you could go in this structure. Another is to consider *addition* as a model for fusion. The residual for addition is obvious: it is subtraction. $X \rightarrow Y$ is $Y - X$. This structure is unlike the RM algebra in a number of ways. First, W fails, as $a \not\leq a + a$ whenever a is negative. Second, M fails (and so, K and K$'$ do too) as $a + a \not\leq a$ whenever a is positive. However, C and B are satisfied in this structure, so we have a structure fit for linear logic. In particular, since the structure is totally ordered, we have the distribution of conjunction over disjunction, so we have a model for *distributive* linear logic.

More interestingly, $(x \rightarrow y) \rightarrow y = x$ for each x and y. This does not hold in any boolean algebra or in any other non-trivial structure with $\top$. If $\top$ were present, then $(a \rightarrow \top) \rightarrow \top = a$ but $\top \leq b \rightarrow \top$ for each b (including $a \rightarrow \top$) so $\top \leq a$.

It is possible to define the negation $\sim a$ as $a \rightarrow 0 = -a$. However, other choices are possible. Taking b an arbitrary proposition, we can define $\sim_b a$ as $a \rightarrow b$, and the condition $(x \rightarrow y) \rightarrow y = x$ states, in effect, that $\sim_b \sim_b a = a$. Double negation introduction and elimination holds for *any* negation $\sim_b$ we choose.

This model is a way to invalidate simple consecutions in distributive linear logic. For example, can we prove $A \rightarrow (B \rightarrow C) \vdash (A \rightarrow B) \rightarrow (A \rightarrow C)$? If this holds in our structure we must have $(z - y) - x \leq (z - x) - (y - x)$, but this simplifies to $z \leq (z - x) + 2x = z + x$ (add $x + y$ to both sides). And when $x \geq 0$ we have $z \leq z + x$, but if $x < 0$ this fails. Similar manipulations can be used to invalidate other consecutions. However, some consecutions invalid in distributive linear logic do hold in the integers. We have seen that $(A \rightarrow B) \rightarrow B \vdash A$ already. Another case is $t \vdash (A \rightarrow B) \vee (B \rightarrow A)$. So the integers do not give an exact fit for distributive linear logic.

(Others have been aware that simple "counting" mechanisms can provide a useful filter for issues of validity in substructural logics [37, 148, 237, 210].)

The logic here is known as *abelian logic*: It was introduced by Meyer and Slaney, who show that it is the logic of ordered abelian groups [173].

EXAMPLE 21 (ω UNDER DIVISION) Using number systems as structures gives us rich mathematical tradition upon which we can build. However, the structures we have seen so far are all *totally ordered*: for any x and y either $x \leq y$ or

$y \leq x$. This is not always desirable — it leads to the truth of $(A \rightarrow B) \vee (B \rightarrow A)$. Now some "natural" orderings of numbers are total orders, but others are not. For example, take the positive integers, ordered by *divisibility*. This is a partial ordering — indeed, a lattice ordering — in which join is the lowest common multiple and meet is the greatest common divisor. Fusion has a natural model in multiplication.

The lattice is distributive, as $\gcd(a, \mathrm{lcm}(b, c)) \mid \mathrm{lcm}(\gcd(a, b), \gcd(b, c))$. With fusion modelled as multiplication, 1 is the identity of fusion and we have a distributive lattice-ordered commutative monoid with a unit. Furthermore, the monoid is square-increasing (as $a \mid a^2$), so it models the structural rules of $R[\wedge, \vee, \circ, t]$.

How can you model a conditional residuating fusion? We want

$$xy \mid z \text{ if and only if } x \mid y \rightarrow z$$

If y divides z, then we can set $y \rightarrow z$ to be z/y. For any x you choose, $xy \mid z$ if and only if $x \mid z/y$. However, if y does not divide z, we do not have anything to choose, as 1 is the bottom of the order we have thus far. To get a residual in every instance, we need to add another element to the ordering. It will be the lowest element in the ordering, so we will call it 0 (for reasons which will become more obvious later). We can by fiat determine that $0 \mid x$ for every x in the structure, and that $x \mid 0$ only when $x = 0$. Conjunction and disjunction are as before, with the addition that $0 \wedge x = 0$ and $0 \vee x = x$ for each x. The rule for implication is then:

$$x \rightarrow y = \begin{cases} y/x & \text{if } x \mid y, \\ 0 & \text{otherwise.} \end{cases}$$

Given 0 we need to extend the interpretation of fusion. But this is simple

$$0x = x0 = 0$$

for every x. So defined, the operation is still order-preserving, commutative, square-increasing and with 1 as the identity. This structure is a model for $R[\wedge, \vee, \circ, t, \rightarrow]$. To model the whole of R we need to model a de Morgan negation. That requires an order-inverting involution on the structure. To do this, we need to introduce many more elements in the structure, as no order-inverting involution can be found on what we have here before us: consider the infinite ascending chain

$$0 \mid 1 \mid 2 \mid 4 \mid \cdots \mid 2^n \mid \cdots$$

To negate each element in the series you must get an infinite descending chain. Why? Because we need an involution: $x \mid y$ if and only if $-y \mid -x$, and in particular, if $-x = -y$, then we must have $x = y$. Each element in the inverted chain must be distinct. Alas, there are no such chains in our structure, as every number has only finitely many divisors. So, we need to add more elements to do the job. As our notation has suggested, we will add the negative integers and ∞. The order is given by setting

$$0 \mid x \mid -y \mid \infty$$

for every positive x and y, and in particular, $-x \mid -y$ if and only if $y \mid x$. So you can read '$\mid$' as divides only when it holds between positive integers. Otherwise, it is defined by these clauses. The infinite ascending chain is then mapped onto the infinite descending chain *above it* as follows:

$$0 \mid 1 \mid 2 \mid 4 \mid \cdots \mid 2^n \mid \cdots \mid -2^n \mid \cdots \mid -4 \mid -2 \mid -1 \mid \infty$$

The result is still a distributive lattice order, and conjunction and disjunction are obviously definable as greatest lower bound and least upper bound, respectively. Implication between all pairs of elements is defined as follows:

   $\diamond$ If x is negative and y is positive, $x \to y = 0$.

   $\diamond$ If x is positive and y is negative, then $x \to y = (-x)y$

   $\diamond$ If x and y are both negative, then $x \to y = -y \to -x$

Fusion is then defined by setting $xy = -(x \to -y)$, and you can show that this is commutative, square-increasing and with 1 as the identity.

The lattice is not *complete*, in that not every subset has a least upper or a greatest lower bound: The chain $0 \mid 1 \mid 2 \mid \cdots \mid 2^n \mid \cdots$ has an upper bound (any negative number will do) but no least upper bound.

This structure was first constructed by Meyer [177] in 1970 who used it to establish some formal properties of R. The technique of expanding an algebra to model negation is one we shall see again as an important technique in the metatheory of these logics.

EXAMPLE 22 (ALGEBRAS OF RELATIONS) A generalisation of Boolean algebras due to de Morgan [201] and Peirce [209] and later developed, for example, by Schröder [249], was to consider algebras of binary relations. A *concrete relation algebra* is the set of all subsets of some set $D \times D$ of pairs of elements from a set D under not only the Boolean operations of intersection, union and complementation but also under new operations which exploit the relational structure.

For any two relations R and S their composition is also a relation: $R \cdot S$ is defined by setting $x(R \cdot S)y$ if and only if $(\exists z \in D)(xRz \land zRy)$. This is a model for fusion. Fusion has a left and right identity, 1, the identity relation on D. Furthermore, for any relation R we have its converse, given by setting $x\check{R}y$ if and only if yRx. Note that $(R \cdot S)\check{} = \check{S} \cdot \check{R}$.

We can define left and right residuals for composition directly by the residuation conditions, or we can note that they are definable in terms of the Boolean connectives, fusion and converse. $R \to S = -(-S \cdot \check{R})$ and $S \leftarrow R = -(\check{R} \cdot -S)$.

It is possible to modify the behaviour of these algebras by considering restricted classes of relations. For example, we could look at algebras of reflexive relations. These are odd, in that $1 \leq R$ for each R, so the bottom element of the algebra is also the identity for fusion. These algebras are closed under some of the operations at issue, but not all. The Boolean complement of a reflexive relation is not reflexive, but the conjunction or disjunction of two reflexive relations is.

Another possibility is to consider, for example, equivalence relations [96]. When is the composition of two equivalence relations an equivalence relation? It turns out that $R \cdot S$ is also commutative when $R \cdot S = S \cdot R$. And if this

obtains, then their composition is the least upper bound of the two relations (in the set of equivalence relations). Therefore, a class of commuting equivalence relations forms a lattice, in which fusion is least upper bound. And it is not too hard to show that this lattice generally fails to be distributive, but it is *modular*. It satisfies the modular law

$$a \wedge (b \vee (a \wedge c)) \leq (a \wedge b) \vee (a \wedge c)$$

but not the more general distributive law.

Tarski [259] helped bring relation algebras back to prominence in modern logic, and there is much contemporary research in the area, particularly in Hungary [12]. Vaughan Pratt has also considered them (and *dynamic algebras*, a tractable fragment of relation algebras) as a useful model of computation [213].

3.1.2   General Structures

The algebraic study of models of substructural logics was first explicitly and comprehensively tackled by J. Michael Dunn in his doctoral dissertation from the middle 1960s [75]. The techniques he used are mostly standard ones, adapted to the new context of relevant logics. There had been a long tradition of using finite algebras (also called 'matrices' for obvious reasons) to prove syntactic results about logics, such as the relevance property for R, as we have seen. Section 22 of *Entailment* Volume 1 [10] contains a good discussion of results of this sort. However, it was Dunn's work that first took such structures as a fit object of study in their own right.

For a helpful guide to the state of the art in the 1970s, Helena Rasiowa's *An Algebraic Approach to Non-classical Logics* [222] is a compendium of results in the field. Meyer and Routley's groundbreaking paper "An Algebraic Analysis of Entailment" [190] did a great deal of work showing how a whole host of logics fit together, all with the theme of *residuation* or the connection of fusion with implication. They showed that not only in R but also in other relevant logics, fusion is connected together with implication by the residuation postulate

$$a \circ b \leq c \text{ iff } a \leq b \rightarrow c$$

and that the natural way to ring the changes in the logic is to vary the postulates governing fusion. Here a summary of Dunn's and Meyer and Routley's work on the general theory of algebras for substructural logics.

DEFINITION 23 (POSETS) A *poset* (a partially ordered set) is a set equipped with a binary relation $\leq$ which is reflexive, transitive, and asymmetric. That is, $a \leq a$ for each a, if $a \leq b$ and $b \leq c$ then $a \leq c$, and if both $a \leq b$ and $b \leq a$, then $a = b$.

Posets are the basic structure of an algebra for a logic. The order is *entailment* between propositions in structure. Entailment is asymmetric as we assume that co-entailing propositions are *identical*. This is what makes propositions in this kind of structure differ from sentences in a formal language.

Extensional conjunction and disjunction enrich the poset into a familiar algebraic structure:

DEFINITION 24 (LATTICES) A lattice is a partially ordered set equipped with least upper bound $\vee$ and a greatest lower bound $\wedge$.

A lattice is *distributive* if and only if $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$ holds for each a, b and c.

A lattice is *bounded* if it has greatest and least elements, $\top$ and $\bot$ respectively.

For traditional substructural logics, two more additions are required to this kind of structure. First, negation, and second, fusion and implication. Let's tackle negation first.

DEFINITION 25 (NEGATIONS) A *negation* on a poset is an order inverting operation $\sim$: that is, if $a \leq b$ then $\sim b \leq \sim a$.

A negation on a lattice is *de Morgan* $\sim\sim a = a$ and it satisfies the de Morgan identities $\sim(a \wedge b) = \sim a \vee \sim b$ and $\sim(a \vee b) = \sim a \wedge \sim b$.

A de Morgan negation in a bounded lattice is an *ortho-negation* if $a \wedge \sim a = \bot$ and $a \vee \sim a = \top$.

Note that a de Morgan negation need not be an ortho-negation. (The negations in each of the structures in the previous section are de Morgan but not ortho-negation.) An ortho-negation operation in a distributive lattice is the *Boolean* negation in that structure.[71]

Some very recent work of Dunn has charted even more possibilities for the behaviour of negation. In particular, he has shown that a basic structure in a substructural logic is a *split negation* satisfying the following residuation-like clauses

$$a \leq \neg b \text{ iff } b \leq -a$$

Given this equivalence, both $\neg$ and $-$ are negations, and both satisfy *some* of the de Morgan inequalities but not others [87].[72]

The most interesting operations in algebras for substructural logics are fusion and implication. The simplest way to define them is by residuation.

DEFINITION 26 (RESIDUATED PAIRS AND TRIPLES) $\langle \circ, \rightarrow \rangle$ is a *residuated pair* in a poset if and only if $a \circ b \leq c$ if and only if $a \leq b \rightarrow c$.

$\langle \circ, \rightarrow, \leftarrow \rangle$ is a *residuated triple* in a poset if and only if $a \circ b \leq c$ if and only if $a \leq b \rightarrow c$ if and only if $b \leq c \leftarrow a$.

If $\langle \circ, \rightarrow \rangle$ is a residuated pair then it immediately follows that $\circ$ is *isotonic* in both places with respect to the entailment ordering. That is

$$\text{if } a \leq a' \text{ and } b \leq b' \text{ then } a \circ b \leq a' \circ b'$$

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[71]There may be more than one ortho-negation in a lattice, but there is only one ortho-negation in a distributive lattice.

[72]In particular, $\neg a \wedge \neg b = \neg(a \vee b)$ and $\neg a \vee \neg b \leq \neg(a \wedge b)$ (this latter inequality is satisfied by *any* negation) but the converse can fail: $\neg(a \wedge b) \not\leq \neg a \vee \neg b$. The negation $\neg$ differs from intuitionistic or minimal negation, however, by not necessarily satisfying $a \leq \neg\neg a$. We do have, however, $a \leq \neg - a$ and $a \leq -\neg a$. For an example of a split negation, let $-A$ be $A \rightarrow f$ and let $\sim A$ be $f \leftarrow A$ in the Lambek calculus extended with a false constant f.

Implication, on the other hand, is not isotonic in both places. It is isotonic in the *consequent* place and *antitonic* in the antecedent place. That is we have

$$\text{if } a' \leq a \text{ and } b \leq b' \text{ then } a \to b \leq a' \to b'$$

All of this was noticed by Meyer and Routley in the 1970s and made rigorous (and generalised to arbitrary n-place operations and residuated families) by Dunn in the 1980s and 1990s in his work on *gaggle theory* (from "ggl" for "Generalised Galois Logic": a *Galois connection* is the general phenomenon of which a residuated pair or triple is a special case) [85, 86].[73]

Tonicity is not the only behaviour of fusion and implication present in these models. If the poset is a *lattice* ordering, then tonicity generalises to *distribution*. It is also an elementary consequence of the residuation clause that fusion distributes over disjunction in both places

$$(a \lor a') \circ b = (a \circ b) \lor (a' \circ b) \text{ and } a \circ (b \lor b') = (a \circ b) \lor (a \circ b')$$

and implication distributes over the extensional connectives in a more complicated fashion.

$$(a \lor a') \to b = (a \to b) \land (a' \to b) \text{ and } a \to (b \land b') = (a \to b) \land (a \to b')$$

These sorts of structures are well known, and they appear independently in different disciplines. *Quantales* [203] are but one example. These are lattice-ordered semigroups (so, $\circ$ is *associative*) with arbitrary disjunctions but only finite conjunctions. They appear in both pure mathematics and theoretical computer science. They are discussed a little in Vickers' *Topology via Logic* [277], which is a useful source book of other algebraic constructions and their use in modelling processes and observation. The existence of arbitrary disjunctions means that in a quantale, implication is *definable* from fusion. If you set $a \to b$ as follows

$$a \to b = \bigvee \{x : x \circ a \leq b\}$$

then $\to$ satisfies the residuating condition for fusion.[74] The same definition is possible in the other direction too. If you have a lattice with arbitrary *conjunctions* (and implication distributes over conjunction in the right way) then you can define fusion from implication

$$a \circ b = \bigwedge \{x : a \leq b \to x\}$$

This definition is key to one of the important techniques in understanding the behaviour of fusion and the connections between fusion and implication. For fusion plays no part in the Hilbert systems introducing some substructural logics. Yet it is present in the Gentzen systems (at least as the comma,

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[73]And I have begun to sketch the obvious parallels between gaggle theory and display logic. Residuation is displaying, and isotonicity and antitonicity have connections to antecedent and consequent position in the proof rules for a connective. When you process a fusion, the subformulas remain on the same side of the turnstile as the original formula. On the other hand, when you process an implication, the antecedent swaps sides and the consequent stays put.

[74]The distribution of $\circ$ over the infinitary disjunction is essential here.

if not explicitly) and in these algebras. Does the addition of fusion add anything new to the system in the language of implication? Or is the addition of fusion *conservative*? In the next section I will sketch Meyer's techniques for proving conservative extensions for many substructural logics, by way of algebraic models.

Before that, I must say a little about *truth* in these algebras. In Boolean algebras (for classical logic) and Heyting lattices (for intuitionistic logic) the *truths* in a structure are the formulas which are interpreted as the top element. There is no need for this to be the case in our structures. In the absence of K', we might have $b \not\leq a \to a$. That means that a true conditional (as every identity $a \to a$ is true) need not be the top element of the ordering. So, instead of picking out true propositions as those at the top of the ordering, substructural logics need be more subtle.

DEFINITION 27 (A TRUTH SET) Given an algebra with $\to$, the *truth set* T is the set of all x where $a \to b \leq x$ for some a, b where $a \leq b$.[75]

The truth set is the set of all conditionals true on the basis of logic alone, and anything entailed by those conditionals. A truth set has some nice properties.

FACT 28 (A TRUTH SET IN A LATTICE IS A FILTER) *Any truth set* T *in a lattice is a* filter. *(A filter is a set which is closed under* $\leq$*, and closed under conjunction.*[76]*) If* $x, y \in \mathsf{T}$ *then* $x \wedge y \in \mathsf{T}$. *If* $x \leq x'$ *then* $x' \in \mathsf{T}$ *too.*

PROOF That T is closed upwards is immediate. That T is closed under conjunction, note that if $x, y \in \mathsf{T}$ then $a \to b \leq x$ and $a' \to b' \leq y$ where $a \leq b$ and $a' \leq b'$. Then $a \wedge a' \leq b \vee b'$, and $(a \to b) \wedge (a' \to b') = a \wedge a' \to b \vee b'$, so since $(a \to b) \wedge (a' \to b') \leq x \wedge y$, we have $x \wedge y \in \mathsf{T}$ too. $\square$

If the logic contains t, then the truth set the filter *generated* by t: $\mathsf{T} = \{x : t \leq x\}$. The presence of truth sets in models shows that a logic without t can be conservatively extended by it. Both conservative extension constructions — due to Meyer in the 1970s, are the topic of the next section.

3.1.3 Conservative Extension Theorems

Meyer's conservative extension results follow the one technique [179]. Suppose we have consecution invalid in a logic with a restricted language $A \not\vdash B$. Then (by the soundness and completeness results for propositional structures) there is an algebra $\mathcal{A}$ and an interpretation $[\![\cdot]\!]$ into $\mathcal{A}$ where $[\![A]\!] \not\leq [\![B]\!]$. Then, we *manipulate* $\mathcal{A}$ into a new structure $\mathcal{A}'$, appropriate for the larger language, and in which we have a new interpretation which is still a counterexample to $[\![A]\!] \not\leq [\![B]\!]$. There are two separate techniques Meyer pioneered. One, injecting a structure $\mathcal{A}$ into it's *completion* $\mathcal{A}^*$ (giving us a way to interpret t, $\circ$, and conjunction and disjunction if those are not present), and then, taking a structure $\mathcal{A}$ and pasting on an inverted duplicate $\mathcal{A}^{\mathrm{op}}$, in order to model negation.

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[75] Equivalently, it is the set generated by all identities $a \to a$, since $a \to a \leq a \to b$ if $a \leq b$.

[76] It is the algebraic analogue of a *theory*, which we have already seen.

EXAMPLE 29 (MAPPING $\mathcal{A}$ INTO $\mathcal{A}^*$) The map from a propositional structure into its *completion* is given in the following way. $\mathcal{A}^*$ is defined in a number of alternative ways.

◇ If $\mathcal{A}$ is only a poset or a semilattice (with $\wedge$ but not $\vee$), then $\mathcal{A}^*$ is the set of all downwardly closed sets of $\mathcal{A}$. That is, I is an element of $\mathcal{A}^*$ if and only if whenever $a \in I$ and $b \leq a$ then $b \in I$.

◇ If $\mathcal{A}$ contains disjunction, then $\mathcal{A}^*$ is the set of all *ideals* in $\mathcal{A}$. (Ideals are dual to filters. They are closed downward, and closed under disjunction: if $a, b \in I$ then $a \vee b \in I$.)

◇ If $\mathcal{A}$ contains disjunction and $\perp$, then $\mathcal{A}^*$ is the set of nonempty ideals in $\mathcal{A}$ — every ideal must contain $\perp$, the least element of $\mathcal{A}$.

$\mathcal{A}^*$ is a complete lattice — order is subsethood, the conjunction of a class of elements is their intersection, and the disjunction of a class of elements is the intersection of all elements above each element in that class. It is not difficult to show that it completely distributive if the original lattice contains no counterexample to distributivity. If fusion is present in $\mathcal{A}$ then it is present in $\mathcal{A}^*$ too.

$$I \circ J = \{z : \exists x \in I, y \in J (z \leq x \circ y)\}$$

and other connectives lift in a similar way. The structural rules of $\mathcal{A}$ are preserved in $\mathcal{A}^*$.[77] This shows that $\mathcal{A}^*$ has the nice logical properties of $\mathcal{A}$.

However, since $\mathcal{A}^*$ is a complete lattice, we can do interesting things with it. If $\mathcal{A}$ doesn't contain a truth element t as a left identity for fusion, $\mathcal{A}^*$ still does. Since $\mathcal{A}^*$ is complete, you can set t to be $\bigwedge\{I \to J : I \leq J\}$. Then $t \circ I \leq J$ if and only if $t \leq I \to J$ if and only if $I \leq J$, and so, t is a left identity for fusion.

Furthermore, the map from $\mathcal{A}$ to $\mathcal{A}^*$ which sends a to $\downarrow a = \{x : x \leq a\}$ injects one structure into the other, preserving all of the operations in $\mathcal{A}$. Any consecution with a counterexample in $\mathcal{A}$ will have a counterexample in $\mathcal{A}^*$ too. It shows that linear logic without the additives is conservatively extended by additives which distribute, for example.

EXAMPLE 30 (PASTING $\mathcal{A}$ AND $\mathcal{A}^{\text{op}}$ TOGETHER) Modifying a structure in such a way as to add negation is more difficult. To add a de Morgan negation to a structure, we need an *upside down copy* $\mathcal{A}^{\text{op}}$ of $\mathcal{A}$ so that for negation can be an order inverting map of period two. Following the details of this construction will be a great deal easier if we take $\mathcal{A}$ to include top and bottom elements, so from now I will do so.

Conjunction and disjunction in $\mathcal{A}^{\text{op}}$ can be defined as the de Morgan dual of that in $\mathcal{A}$. So, if $a, b \in \mathcal{A}^{\text{op}}$ then $a \wedge b = -(-a \vee -b)$, where $-$ is the map from $\mathcal{A}$ to $\mathcal{A}^{\text{op}}$ and back. Defining conjunctions and disjunctions of elements between $\mathcal{A}$ and $\mathcal{A}^{\text{op}}$ depends on another decision we need take. If $a \in \mathcal{A}$ and $b \in \mathcal{A}^{\text{op}}$, then we need decide on what we take $a \wedge b$ and $a \vee b$ to be. There are three options for this, each depending on the relevant *positioning* of $\mathcal{A}$ and $\mathcal{A}^{\text{op}}$. Meyer's original choice [179] was to put $\mathcal{A}^{\text{op}}$ *above* $\mathcal{A}$. Then the disjunction of an element form $\mathcal{A}$ with an element from $\mathcal{A}^{\text{op}}$ will be the element from $\mathcal{A}^{\text{op}}$ and the conjunction will be the element from $\mathcal{A}$. This choice (rather than putting $\mathcal{A}^{\text{op}}$ *under* $\mathcal{A}$) is the one to take if you wish to end up with a model for the relevant logic R, for we wish to end up with $t \vdash A \vee -A$. The

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[77]The proof is tedious but straightforward [234, Chapter 9].

element t is in $\mathcal{A}$, and we wish it to be under each $a \vee -a$. But $a \vee -a$ can be *any* element in the top half of the model, so t must be under each element in the top half, so it is either the *bottom element* of the top half of the model (not likely, if any conditional is untrue at all in the original model) or it is in the bottom half.

The *other* choice for ordering the two components — putting $\mathcal{A}^{\mathrm{op}}$ below $\mathcal{A}$ — is required if you wish the original model to satisfy K. Then, t must be $\top$, and since t is in the original model, it must be at the top of the new model, so $\mathcal{A}^{\mathrm{op}}$ can go underneath.

There is one other natural choice for the ordering of $\mathcal{A}$ and $\mathcal{A}^{\mathrm{op}}$, and that is to take them in *parallel*. You can paste together the top elements both models and the bottom elements of both models (or add new top and bottom elements if you prefer) and then take the disjunction of a pair, one from $\mathcal{A}$ and $\mathcal{A}^{\mathrm{op}}$, to be the $\top$ element of the whole structure, and the conjunction of that pair to be the $\bot$ element. This in another natural option, but the resulting lattice is not distributive if $\mathcal{A}$ is not trivial.

To make the resulting structure a model for a logic, you must define $\circ$ and $\rightarrow$ in the whole structure. Most choices are fixed in advance, if fusion is commutative in $\mathcal{A}$. Since we want $a \circ b$ to be $-(a \rightarrow -b)$, and $a \rightarrow b$ to be $-b \rightarrow -a$, we take $a \circ b$ when $a \in \mathcal{A}$ and $b \in \mathcal{A}^{\mathrm{op}}$ to be $-(a \rightarrow -b)$ (and the dual choice when $a \in \mathcal{A}^{\mathrm{op}}$ and $b \in \mathcal{A}$). The remaining choice is for $a \circ b$ where $a, b \in \mathcal{A}^{\mathrm{op}}$. Here, it depends on the relative position of $\mathcal{A}$ and $\mathcal{A}^{\mathrm{op}}$. If we add the new structure on top, take $a \circ b$ to be $\top$. If we add the new structure below, or alongside, take $a \circ b$ to be $\bot$. The new structure satisfies many of the structural rules of the old structure, and as a result, a conservative extension result for logics in the vicinity of R follows [177, 179, 234].

There is one substructural logic for which a conservative extension by negation fails: RM. RM is given by extending R with the mingle rule $A \circ A \rightarrow A$ (or equivalently, $A \vdash A \rightarrow A$). If you add mingle to positive R then the result is still a sublogic of intuitionistic logic, and as a result, total ordering $t \vdash (A \rightarrow B) \vee (B \rightarrow A)$ is not provable. This logic is called RM0. In the presence of negation, however, the addition of mingle brings along with itself the total ordering principle. (This result is due to Meyer and Parks, from 1972 [188].)

3.2 Categories

In propositional structures, we abstract away from the particulars of the languages in which our propositions are expressed to focus on the propositions themselves, ordered under entailment. In propositional structures, propositions are first-class citizens, and *proofs* between propositions fade into the background. If there is a proof from A to B, then $[\![A]\!] \leq [\![B]\!]$. The differences between proofs from A to B are not registered in this algebraic semantics.

Models do not have to be like this. We can consider not only propositions as objects but also proofs as "arrows" between objects. If we have one proof from A to B, we might indicate this as '$f : A \longrightarrow B$,' where f is the proof. We might have another proof $g : B \longrightarrow C$, and then we could *compose* them to construct another proof $gf : A \longrightarrow C$, which runs though f and then g.

Logicians did not have to go to the trouble of inventing structures like this. It turns out that mathematical objects with just these properties have been widely studied for many decades. Categories are important mathematical structures. Category theory is a helpful language for describing constructions which appear in disparate parts of mathematics. This means that category theory is, by its nature, very abstract. This also means that category theory is rich in examples, interesting categories are models of substructural logics. In particular, I will look at one example categorical model of a logic, Girard's model of *coherence spaces*, for linear logic.[78]

To understand the role of categories as models of logic, you need to focus on one particular part of categorical technology: the *adjoint pair*. An adjoint pair is a relationship between two *functors*, and functors are structure preserving maps between categories. Thinking of a category as a model of a logic generalising an algebra, the operators such as fusion, implication and so on are all *functors* from the category to itself (or perhaps, from the category to its *opposite*, which is found by swapping arrows from a to b to go from b to a instead). Operators like fusion, which are isotonic, are really two-place maps from a category to itself, not only sending a pair of category objects to another object (their fusion) but *also* sending arrows $f : a \longrightarrow a'$ and $g : b \longrightarrow b'$ to an arrow $f \circ g : a \circ b \longrightarrow a \circ b'$.

DEFINITION 31 (ADJOINT PAIRS) If $F : \mathcal{C} \longrightarrow \mathcal{D}$ and $G : \mathcal{D} \longrightarrow \mathcal{D}$, and there is a bijection $\phi : \mathrm{Hom}_C(X, F(Y)) \cong \mathrm{Hom}_D(G(X), Y)$, natural in both X and Y,[79] then we say that $\langle F, G, \phi \rangle$ is an *adjunction*. F is the *left adjoint* and G is the *right adjoint*.

EXAMPLE 32 (ADJUNCTION BETWEEN FUSION AND IMPLICATION) In *cartesian closed categories*,[80] product: $— \times B$ is a functor $\mathcal{C} \longrightarrow \mathcal{C}$. Similarly $[B \Rightarrow —]$ is a functor $\mathcal{C} \longrightarrow \mathcal{C}$. These functors form an adjunction. If $f : A \times B \longrightarrow C$, then $\lambda f : A \longrightarrow [B \Rightarrow C]$. Conversely, if $g : A \longrightarrow [B \to C]$, then $\mathrm{ev}(g \times \mathrm{id}_B) : A \times B \longrightarrow C$. This is a bijection $\mathrm{Hom}(A \times B, C) \cong \mathrm{Hom}(A, [B \Rightarrow C])$.

This is the categorical equivalent of the residuation between extensional conjunction and intuitionistic implication. Cartesian closed categories are models of intuitionistic logic [152].

3.2.1 Coherence Spaces

Coherence spaces arise as a model of the λ-calculus, and intuitionistic logic. They provided the first model which gave Girard an insight into the decomposition of intuitionistic implication in terms of *linear* implication and the exponential ! [117, 121].

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[78]In a history like this I can only assume some category theory, and not introduce it myself. Here are some standard references: Mac Lane's *Categories for the Working Mathematician* is a very good introduction to the area [158], very readable, even by those who are not working mathematicians. Barr and Wells' *Category Theory for Computing Science* is also clear, from a perspective of the theory of computation [17]. Chapter 10 of *An Introduction to Substructural Logics* [234] contains *just* the category theory you need to go through the detail of this model. Došen's paper "Deductive Completeness" is a clear introduction focussing on the use of categories in logic [72].

[79]I can't tell you what it is for a bijection to be *natural*, unfortunately. However, $\mathrm{Hom}_C(X, Y)$ is the class of all arrows from X to Y in the category.

[80]I can't tell you what these are, for lack of space.

DEFINITION 33 (COHERENCE SPACES) A *coherence space* is a set $\mathcal{A}$ of sets, satisfying the following two conditions.

⋄ If $a \in \mathcal{A}$ and $b \subseteq a$ then $b \in \mathcal{A}$, and
⋄ If for each $x, y \in a$, $\{x, y\} \in \mathcal{A}$, then $a \in \mathcal{A}$.

But coherence spaces are much better thought of as undirected graphs. We say a *coheres* with b (in $\mathcal{A}$) if $\{x, y\} \in \mathcal{A}$. We write this: '$x \asymp y$ (mod $\mathcal{A}$).' The coherence relation determines the coherence space completely. Coherent sets ($a \in \mathcal{A}$) are cliques in the graph. The coherence relation is reflexive and symmetric, but not, in general, transitive.[81]

Given a coherence space $\mathcal{A}$, we define *coherence relations* as follows:

⋄ $x \frown y$ (mod $\mathcal{A}$) iff $x \asymp y$ (mod $\mathcal{A}$) and $x \neq y$.
⋄ $x \smile y$ (mod $\mathcal{A}$) iff $\{x, y\} \notin \mathcal{A}$.
⋄ $x \asymp\!\!\!\!\frown y$ (mod $\mathcal{A}$) iff it is not the case that $x \frown y$ (mod $\mathcal{A}$).

DEFINITION 34 (PRODUCT, SUM AND NEGATION SPACES) Given spaces $\mathcal{A}$ and $\mathcal{B}$, the coherence spaces $\mathcal{A} \wedge \mathcal{B}$ and $\mathcal{A} \vee \mathcal{B}$ are defined on the disjoint union of the points x of the graph of $\mathcal{A}$ and y of the graph of $\mathcal{B}$, as follows:

$$
\begin{array}{lll}
(0, x) \asymp (0, x') \ (\text{mod } \mathcal{A} \wedge \mathcal{B}) & \text{iff } x \asymp x' \ (\text{mod } \mathcal{A}) \\
(1, y) \asymp (1, y') \ (\text{mod } \mathcal{A} \wedge \mathcal{B}) & \text{iff } y \asymp y' \ (\text{mod } \mathcal{B}) \\
(0, x) \asymp (1, y) \ (\text{mod } \mathcal{A} \wedge \mathcal{B}) & \text{always} \\
\\
(0, x) \asymp (0, x') \ (\text{mod } \mathcal{A} \vee \mathcal{B}) & \text{iff } x \asymp x' \ (\text{mod } \mathcal{A}) \\
(1, y) \asymp (1, y') \ (\text{mod } \mathcal{A} \vee \mathcal{B}) & \text{iff } y \asymp y' \ (\text{mod } \mathcal{B}) \\
(0, x) \asymp (1, y) \ (\text{mod } \mathcal{A} \vee \mathcal{B}) & \text{never}
\end{array}
$$

Given a coherence space $\mathcal{A}$, the coherence space $\sim\!A$ is defined by setting $x \asymp y$ (mod $\sim\!\mathcal{A}$) if and only if $x \asymp\!\!\!\!\frown y$ (mod $\mathcal{A}$). Note that $\sim\!\sim\!\mathcal{A} = \mathcal{A}$. $\mathsf{Sgl} = \{\emptyset, \{*\}\}$, the one-point coherence space. $\mathsf{Emp} = \{\emptyset\}$, the empty coherence space. Note that $\sim\!\mathsf{Sgl} = \mathsf{Sgl}$ and $\sim\!\mathsf{Emp} = \mathsf{Emp}$.

Note that here $\sim\!(\mathcal{A} \wedge \mathcal{B}) = \sim\!\mathcal{A} \vee \sim\!\mathcal{B}$, and $\sim\!(\mathcal{A} \vee \mathcal{B}) = \sim\!\mathcal{A} \wedge \sim\!\mathcal{B}$. Furthermore, $\mathcal{A} \wedge \mathsf{Emp} = \mathsf{Emp} \wedge \mathcal{A} = \mathcal{A} = \mathcal{A} \vee \mathsf{Emp} = \mathsf{Emp} \vee \mathcal{A}$. Emp does the job of both $\top$ and $\bot$ in the category of coherence spaces.[82]

The class of all coherence spaces can be made into a cartesian closed category, if we take the arrows to be *continuous functions*.

DEFINITION 35 (CONTINUOUS FUNCTIONS) $F : \mathcal{A} \longrightarrow \mathcal{B}$ is *continuous* if and only if

⋄ If $a \subseteq b$ then $F(a) \subseteq F(b)$.
⋄ If $S \subseteq \mathcal{A}$ is *directed* (that is, if $a, b \in S$, then $a \cup b \in S$ too) then $F(\bigcup S) = \bigcup \{F(a) : a \in S\}$.

---

[81]Erhard's *hypercoherences* are a generalisation of coherence spaces which are richer than a graph represents [95]. In hypercoherences, a might be a coherent set without $a' \subseteq a$ also being coherent. The category of hypercoherences is also a model of linear logic.

[82]This shows how categories have a kind of flexibility unavailable to posets. In a poset, $\top = \bot$ only if the poset is trivial. In a category, $\top$ and $\bot$ might be identical or isomorphic, without the category structure being trivial. Yes, there will be arrows from every object to every other object, but it is not the case that all objects are isomorphic.

FACT 36 (MINIMAL REPRESENTATIVES) *If* $F : \mathcal{A} \longrightarrow \mathcal{B}$ *is continuous, and if* $a \in \mathcal{A}$ *and* $y \in F(a)$, *then there is a* minimal *finite* $a' \in \mathcal{A}$ *where* $y \in F(a')$.

PROOF If $y \in F(a)$ then $y \in F(a^*)$ for some finite a^*. Pick some *smallest* subset a' of a^* with this property. (This is possible, as a^* is finite.) $\square$

We want to construct a coherence space representing $F : \mathcal{A} \longrightarrow \mathcal{B}$. We start by defining the trace of a function.

$$\text{Trace}(F) = \{(a, y) \in \mathcal{A}_{\text{fin}} \times |\mathcal{B}| : y \in F(a) \text{ and } a \text{ is minimal}\}$$

Note that $\text{Trace}(F) \subseteq \mathcal{A}_{\text{fin}} \times |\mathcal{B}|$ has the following properties.

   ◇ If $(a, y), (a, y') \in \text{Trace}(F)$ then $y \frown y' \pmod{\mathcal{B}}$.
   ◇ If $a' \subseteq a, (a, y), (a', y) \in \text{Trace}(F)$, then $a = a'$.

Conversely, if $\mathfrak{F}$ is any set with these two properties, then define $F_{\mathfrak{F}}$ by setting

$$F_{\mathfrak{F}}(a) = \{y \in |\mathcal{B}| : \exists a' \subseteq a \text{ where } (a', y) \in \mathfrak{F}\}$$

We can represent continuous functions by their traces. In fact, if F is continuous, then $F = F_{\text{Trace}(F)}$. Can we define a coherence relation on traces? Consider the special case where there are *two* minimal representatives, that is, $(a, y), (a', y) \in \text{Trace}(F)$. Under what circumstances are they coherent? Unfortunately, we need more information in order to define a coherence relation — we need a relationship between a and a'. We can show that in a particular class of continuous functions, there is always a *unique* minimal a.

DEFINITION 37 (STABLE FUNCTIONS) $F : \mathcal{A} \longrightarrow \mathcal{B}$ is *stable* if it is continuous, and in addition, whenever $a, a', a \cup a' \in \mathcal{A}$, then $F(a \cap a') = F(a) \cap F(a')$.

With stable functions, we can choose a unique minimal representative a.

FACT 38 (UNIQUE MINIMAL REPRESENTATIVES) $F : \mathcal{A} \longrightarrow \mathcal{B}$ *is stable if and only if for each* $a \in \mathcal{A}$, *where* $y \in F(a)$, *there is a unique minimal* $a' \in \mathcal{A}_{\text{fin}}$ *such that* $y \in F(a')$.

PROOF For left to right, it is straightforward to check that $a' = \bigcap \{a^* \in \mathcal{A} : a^* \subseteq a, \text{ where } y \in F(a^*)\}$ is the required a'. For right to left, monotonicity tells us that $F(a \cap a') \subseteq F(a)$ and $F(a \cap a') \subseteq F(a')$, so $F(a \cap a') \subseteq F(a) \cap F(a')$. Conversely, if $a, a', a \cup a' \in \mathcal{A}$, then if $y \in F(a)$ and $y \in F(a')$, then y is in $F(a'')$ for a unique minimal a''. Therefore $a'' \subseteq a$ and $a'' \subseteq a'$, so $a'' \subseteq a \cap a'$, and hence $y \in F(a'') \subseteq F(a \cap a')$, as desired. $\square$

The next result is simple to verify.

FACT 39 (CHARACTERISING STABLE FUNCTIONS) *If* F *is stable, then whenever* $(a, y), (a', y') \in \text{Trace}(F)$
   ◇ *If* $a \cup a' \in \mathcal{A}$ *then* $y \frown y' \pmod{\mathcal{B}}$.
   ◇ *If* $a \cup a' \notin \mathcal{A}$ *then* $y = y' \pmod{\mathcal{B}}$.
Conversely, if the set $\mathfrak{F}$ *satisfies these conditions, then* $F_{\mathfrak{F}}$ *is stable.* $\square$

Given this, we can define $\mathcal{A} \supset \mathcal{B}$. $|\mathcal{A} \supset \mathcal{B}| = \mathcal{A}_{\text{fin}} \times |\mathcal{B}|$ as follows: $(a, y) \frown (a', y')$ $\pmod{\mathcal{A} \supset \mathcal{B}}$ if and only if

   ◇ If $a \cup a' \in \mathcal{A}$ then $y \frown y' \pmod{\mathcal{B}}$.

◇ If $a \cup a' \in \mathcal{A}$ then $y = y'$ (mod $\mathcal{B}$).

That is, $\mathcal{A} \supset \mathcal{B} = \{\text{Trace}(F) \,|\, F : \mathcal{A} \longrightarrow \mathcal{B}$ is stable$\}$. The category of coherent spaces and stable functions between them is cartesian closed. This construction is *obviously* a two-stage process. It *begs* to be decomposed. We should define a coherence space !A on the set of finite coherent sets of $\mathcal{A}_{\text{fin}}$ as follows:

$$a \frown a' \pmod{!A} \text{ iff } a \cup a' \in \mathcal{A}$$

and define *linear* implication $\mathcal{A} \to \mathcal{B}$ by setting $(x, y) \frown (x', y')$ (mod $\mathcal{A} \to \mathcal{B}$) if and only if

◇ If $x \frown x'$ (mod $\mathcal{A}$) then $y \frown y'$ (mod $\mathcal{B}$).

◇ If $x \frown x'$ (mod $\mathcal{A}$) and $y = y'$ then $x = x'$.

Note that $\mathcal{A} \to \mathcal{B}$ is (isomorphic to) $\sim\mathcal{B} \to \sim\mathcal{A}$. Furthermore, $\mathsf{Sgl} \to \mathcal{A}$ is (isomorphic to) $\mathcal{A}$ and $\mathcal{A} \to \mathsf{Sgl}$ is (isomorphic to) $\sim\mathcal{A}$. The operation $\supset$ stands to stable functions as $\to$ stands to a new kind of function: the *linear* functions.

DEFINITION 40 (LINEAR MAPS) F is a *linear* map if and only if whenever $A \subseteq \mathcal{A}$ is linked (that is, if $a, b \in A$ then $a \cup b \in A$) then $F(\bigcup A) = \bigcup \{F(a) : a \in A\}$.

If F is linear then F is stable (this is straightforward) and in addition, if $x \in F(a)$ then the minimal b where $x \in F(b)$ is a *singleton*. It follows that the trace of F can be simplified. The *linear trace* of a linear map F is defined as follows:

$$\text{Trlin}(F) = \{(x, y) : y \in F(\{x\})\}$$

Therefore, $\mathcal{A} \to \mathcal{B} = \{\text{Trlin}(F) \,|\, F : \mathcal{A} \longrightarrow \mathcal{B}$ is linear$\}$.

Given $\to$, we can see that it is connected by an adjunction to a natural fusion operation. We can define $\mathcal{A} \circ \mathcal{B}$ as follows: $|\mathcal{A} \circ \mathcal{B}| = |\mathcal{A}| \times |\mathcal{B}|$, and $(x, y) \frown (x', y')$ if and only if $x \frown x'$ (mod $\mathcal{A}$) and $y \frown y'$ (mod $\mathcal{B}$).

FACT 41 (THE ADJUNCTION BETWEEN FUSION AND IMPLICATION) *In the category of coherence spaces with linear maps*

$$Hom(\mathcal{A} \circ \mathcal{B}, \mathcal{C}) \cong Hom(\mathcal{A}, \mathcal{B} \to \mathcal{C})$$

is an adjunction for all $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$.

This, with the associativity and commutativity of $\circ$, together with the behaviour of !, shows that the category of coherence spaces and linear maps is a model of linear logic.

Some very recent work (2001) of Schalk and de Paiva's on *poset-valued sets* [248] generalises coherence spaces in an interesting direction. They show that coherence spaces and hypercoherences can be seen as maps from Set $\times$ Set to the algebra RM3.[83] If $x \frown y$ then $f(x, y) = t$, if $x \smile y$ then $f(x, y) = f$, if $x = y$ then $f(x, y) = b$. The logical operators of negation, fusion and implication then lift from the algebra to the coherence spaces. (In other words, if $\mathcal{A}$: Set $\times$ Set $\to$ RM3 is a coherence space, then $\sim\mathcal{A}$ is the map composing $\mathcal{A}$

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[83]They do not recognise that the algebra is already quite studied in the relevant logic literature.

with ~ : RM3 → RM3. The same goes for the other operations.) Different categorical models in the style of coherence spaces can then be given by varying the target algebra. I suspect that using some of the algebras known in the substructural literature will lead to interesting categorical models of linear logic and related systems.

Girard has shown that *Banach spaces* can be used in place of coherent spaces to model linear logic [120]. The norm in a Banach space takes the place of the coherence relation. As we shall see later, it is not the only point at which geometric intuitions have come to play a role in substructural logic.

3.3 Frames

The study of modal logic found new depth and vigour with the advent of possible worlds semantics. As we have seen, algebras are useful models of substructural logics. However, they are so *close* to the proof theory of these logics that they do not provide a great deal of new information, either about the intrinsic properties of the logic in question, or about how it is to be applied. Models in terms of *frames* are one way to extract more information. Perhaps this is because frames are a further step removed from the logic in an important sense. In algebras, each formula in the language is interpreted as an element in the algebra. In frames, each formula is not interpreted as an element in the frame — the elements in the frame lie *underneath* the interpretation of formulas. Formulas are interpreted as *collections* of frame elements. Therefore the interpretations of connectives on a frame are themselves *decomposed*. They are no longer simply *functions* on algebras satisfying specified conditions. Their action on sets of frame elements is factored through their action on individual frame elements. As a result, frame interpretations of logics can be thought to carry more information than algebras. In addition, frame semantics is suggestive of applications of logics. Just as the idea of the interpretation of a proposition as a set of possible worlds, or a set of times or a set of locations has driven the application of different models of modal or temporal logics, so the interpretation of frame semantics for substructural logics has led to their use in diverse applications. But enough of scene-setting. Let's start with a the first attempts to give precise frame semantics for substructural logics. As before, our story starts with the relevant logic R.

3.3.1 Operational Frames

The idea of frame semantics for relevant logics occurred independently to Routley and to Urquhart in the late 1960s and early 1970s. Routley's techniques are more general than Urquhart's, but Urquhart's were published first, and are the simplest to introduce, so we will start with them.

Consider the constraints for developing a frame semantics for a relevant logic. The bare bones of any frame semantics are as follows. A *frame* is a set of objects (call them *points*, though "worlds", "situations", "set-ups" and other names have all been used), and a *model* on that frame is a relation ⊩ which indicates what formulas are true at what points. We read "x ⊩ A" as "A is true at x." Typically, the relation ⊩ is constrained by inductive clauses that

indicate how the truth (or otherwise) of a complex formula at each point is determined by the truth (or otherwise) of its subformulas. Given a particular model, then, we say that A *entails* B on that model if and only if for every point x, if $x \Vdash A$ then $x \Vdash B$. Entailment is preservation of truth at all points in a model.

This is the bare bones of a frame semantics for a logic. Consider how this determines what we can do to interpret relevant implication. It is axiomatic for a relevant logic that the entailment from A to $B \to B$ can fail. In frame terms this means that we must have points in our models in which $B \to B$ can fail. This means that the interpretation of implication must differ from any kind of frame interpretation of conditionals seen. For a *strict* conditional $A \Rightarrow B$ to be true at a world, we need to check all accessible worlds, to see if B is true whenever A is true. As a result, $B \Rightarrow B$ is true at every world. Similarly, for *counterfactual* conditionals $A \mathrel{\square\!\!\rightarrow} B$, we check the *nearby* worlds where A is true, to see if B is true there too. Again, $B \mathrel{\square\!\!\rightarrow} B$ is true, because we check the consequent at the very same points in the model where we have taken the antecedent to be true. Something different must be done for a relevant conditional. At the very least we need to check the value of the consequent somewhere at places other than simply where we have checked the antecedent.

Urquhart's innovation was a natural way to do just this [265, 266, 267, 268]. Consider again what an implication $A \to B$ says. To be committed to $A \to B$ is to be committed to B whenever we gain the information that A. To put it another way, a body of information warrants $A \to B$ if and only if whenever you *update* that information with new information which warrants A, the resulting (perhaps new) body of information warrants B. Putting this idea in technical garb, we get a familiar-looking inductive clause from a frame semantics:

⋄ $x \Vdash A \to B$ if and only if for each y, if $y \Vdash A$ then $x \sqcup y \Vdash B$.

But this inductive clause has a new twist. Unlike the clauses for strict or counterfactual conditionals, in this clause we check the consequent and antecedent at different points in the model structure. The way is open for $B \to B$ to fail.

Let's take some time to examine the detail of this clause. We have a class of points (over which "x" and "y" vary), and a function $\sqcup$ which gives us new points from old. The point $x \sqcup y$ is supposed, on Urquhart's interpretation, to be the body of information given by *combining* x with y. The properties we take combination to have will influence the properties of the conditional. First up, let's consider our old enemy, $A \vdash B \to B$. For this to fail, we need have a point x where $x \nVdash B \to B$, and for this, we need just some y where $y \Vdash B$ and $x \cup y \nVdash B$. This means that combination of bodies of information cannot satisfy this *hereditary* condition:

⋄   If $x \Vdash A$ then $x \sqcup y \Vdash A$ left hereditary condition

Similarly, if we are to have $A \vdash B \to A$ to fail, then combination cannot satisfy the *dual* hereditary condition.

⋄   If $x \Vdash A$ then $y \sqcup x \Vdash A$ right hereditary condition

This means that combination is sometimes *nonmonotonic* in a natural sense. Sometimes when a body of information is combined with another body of information, some of the original body of information might be lost. This is simplest to see in the case motivating the failure of $A \vdash B \to A$. A body of information might tell us that A. However, when we combine it with something which tells us B, we the resulting body of information might no longer warrant A (as A might conflict with B). Combination might not simply result in the *addition* of information. It may well warrant its *revision*.

To model the logic R, combination must satisfy a number of properties:

$$\diamond \quad x \sqcup y = y \sqcup x \qquad\qquad \text{commutativity}$$
$$\diamond \quad (x \sqcup y) \sqcup z = x \sqcup (y \sqcup z) \quad \text{associativity}$$
$$\diamond \quad x \sqcup x = x \qquad\qquad\quad \text{idempotence}$$

Commutativity gives us *assertion*, associativity gives us *prefixing* and *suffixing*, and idempotence gives us *contraction*, as is easily verified. For example, consider assertion: to verify that $A \vdash (A \to B) \to B$, suppose that $x \Vdash A$. To show that $x \Vdash (A \to B) \to B$, take a y where $y \Vdash A \to B$. We wish to show that $x \sqcup y \Vdash B$. By commutativity, $x \sqcup y = y \sqcup x$, and since $y \Vdash A \to B$ and $x \Vdash A$, we can apply the conditional clause at y to give $y \sqcup x \Vdash B$. So, $x \sqcup y \Vdash B$ as desired.

These frame properties, governing the behaviour of $\sqcup$ are very similar in scope to the structural rules governing fusion and intensional combination in different proof theories. This is no surprise, as $\sqcup$ is the frame analogue of fusion. It comes as no surprise, then, that as you vary conditions on $\sqcup$ you can model different substructural logics.

Keeping the analogy afloat, then, we can see how these models might interpret *theoremhood* in our logics. In analogy with the proof theory and algebraic models of our logics, we can see that there are two different grades of truth. It is one thing for a formula to be true everywhere in a model — this corresponds to being entailed by the Church true constant $\top$. It is another thing for it to be a tautology, for it to be entailed by the Ackermann true constant t. Identities are entailed by t. What corresponds to being a tautology in this sense in our models? Clearly being true at every point is ruled out, as identities can fail at different points in a model. Continuing the interpretation of points as bodies of information, if we can have bodies of information which do *not* warrant all of the tautologies of logic, then we need some way of talking about which bodies of information do. The simplest approach (and the one which Urquhart took) is to take a special body of information 0 to stand for "logic." A natural condition to take on 0 is that it is a left identity for composition

$$\diamond \quad 0 \sqcup x = x \quad \text{left identity}$$

In this way, $0 \Vdash A \to B$ if and only if for each x, if $x \Vdash A$ then $x \Vdash B$ — so the conditionals warranted by *logic* correspond to exactly the entailments valid in the frame. The identity point 0 does a good job of modelling logic.

The interpretation of points as bodies of information warrants a simple interpretation of conjunction as well. The usual clause

$\diamond \; x \Vdash A \wedge B$ if and only if $x \Vdash A$ and $x \Vdash B$

is uncontroversial. If a body of information warrants $A \wedge B$ it warrants A and it warrants B, and conversely. Adding this clause to the semantics gives us the conjunction and implication fragment of R (and its neighbours, varying the behaviour of of $\sqcup$).

Intensional conjunction is also straightforward. We can add fusion as the object-language witness of composition:

$\diamond$ $x \Vdash A \circ B$ if and only if for some y, z where $x = y \sqcup z$, $y \Vdash A$ and $z \Vdash B$.

It is instructive to verify that in these models, that $A \circ B \vdash C$ if and only if $A \vdash B \to C$. Residuation between $\to$ and $\circ$ corresponds to the *universal* clause modelling $\to$ interacting with the *existential* clause modelling $\circ$.

Let's now turn to soundness and completeness with respect to these models. To prove soundness of a proof theory with respect to these models, it is required only to show that everything provable in the proof theory holds in the model (either holds at 0 for a Hilbert system, or holds over the entire frame for a proof theory which delivers consecutions). As usual, verifying soundness is a straightforward matter of checking axioms and rules.

There are two different ways to prove the completeness of a proof theory with Urquhart's operational models. Again, as usual, the common technique is to provide a *counterexample* for an unprovable formula (or consecution). Both techniques use a *canonical model* construction, familiar from the worlds semantics for modal logics. Where these constructions differ is in the stuff out of which the points in the model are made. The first, and most general kind of canonical model we can provide for an operational semantics is the *theory* model, in which the points are all of the *theories* of the logic in question.

DEFINITION 42 (THE *Theory* CANONICAL MODEL) The set of points is the set $\mathcal{T}$ of theories. The identity theory is the set L of all of the tautologies of the logic. The composition relation $\sqcup$ is defined as follows:

$$S \sqcup T = \{B : (\exists A)(A \to B \in S \text{ and } A \in T)\}$$

and $S \Vdash A$ if and only if $A \in S$.

To verify that the *theory* canonical model is indeed a canonical model we must show that $\sqcup$ so defined satisfies all of the conditions of a composition relation, and that $\Vdash$ satisfies the recursive conditions of an evaluation relation.

To show that $\sqcup$ satisfies the conditions of composition, you need first show that $\sqcup$ is indeed a *function* on the class of theories: that if S and T are theories, so is $S \sqcup T$. The verification of this fact is elementary. The frame conditions on $\sqcup$ correspond quite neatly to axioms or structural rules.

To show that $\Vdash$ satisfies the recursive conditions, you need show that $A \wedge B \in T$ if and only if $A, B \in T$ (which is an immediate consequence of the definition of a theory) and that $A \to B \in S$ if and only if for each T where $A \in T$, $B \in S \sqcup T$. The verification from left to right is an immediate consequence of the definition of $\sqcup$. The verification from right to left is simplest to prove in the contrapositive: that if $A \to B \notin S$ then there is a T where $A \in T$ and $B \notin S \sqcup T$. Finding such a T is easy here: let $T = \{C : A \vdash C\}$. If $B \in S \sqcup T$

then there is some $C \in T$ where $C \rightarrow B \in S$. Since $A \vdash C$, it follows that $C \rightarrow B \vdash A \rightarrow B$ (by monotonicity of $\rightarrow$) and $A \rightarrow B \in S$ contrary to what we have assumed.

It is possible to extend this kind of completeness proof to show that the condition for fusion models this connective correctly too.

DEFINITION 43 (THE *Finite Set* CANONICAL MODEL) The points are the finite *sets* of formulas. Composition $\sqcup$ is set union. $\{A_1, \ldots, A_n\} \Vdash B$ if and only if

$$\vdash A_1 \rightarrow (\cdots \rightarrow (A_n \rightarrow B))$$

(The permutation axiom shows that the order of presentation in the set is irrelevant in this definition.)

It is not difficult to show that this is indeed a model — that the recursive clause defining $\rightarrow$ is satisfied.

This is a simple model which gives a straightforward counterexample to any invalid argument. If $A \nvdash B$ then $\{A\}$ is the point in the model invalidating the argument: $\{A\} \Vdash A$ and $\{A\} \nVdash B$.

Operational frames are important models of other substructural logics too.

EXAMPLE 44 (LANGUAGE FRAMES) A *language* frame on alphabet $\mathcal{A}$ is the collection of all strings on that alphabet, with $\sqcup$ defined as concatenation.

Language frames are a model of the Lambek calculus. The composition operation $\sqcup$ is associative but not commutative (except in the case where $\mathcal{A}$ is a singleton). It was an open question for many years whether or not the Lambek calculus is complete for Language frames. Mati Pentus showed that it is, using an ingenious (and difficult!) model construction argument pasting chains of partial models together to form a string model [210, 211]. Different frames for the Lambek calculus feature prominently in some recent work on the system and its linguistic applications [200, 202].

EXAMPLE 45 (DOMAIN SPACES) Models of the λ-calculus [1, 126, 251, 252] are models for substructural logics too. Scott's famous model construction involves a topological space D such that D is isomorphic to the space $[D \rightarrow D]$ of continuous functions from D to itself. Each element of D is paired with a function in $[D \rightarrow D]$, so can think of the objects equally well as functions. Therefore, there is a two-place operation of application on the domain. Consider $x(y)$ — the application of x to y. We can assign types to functions in this model by "reading" the model as a frame for a logic. If we set $x \sqcup y$ to be $x(y)$, then this is an operational frame:

$$x \Vdash A \rightarrow B \text{ if and only if for each } y, \text{ where } y \Vdash A, x(y) \Vdash B.$$

In other words, x is of type $A \rightarrow B$ if and only if whenever given an input of type x, the output is of type B. This gives us a plausible notion of function *typing*. For example, $\lambda x.(x + 1)$ will have type *Even* $\rightarrow$ *Odd* and *Odd* $\rightarrow$ *Even*. The function $\lambda x.\lambda y.(2x + y)$ has type $N \rightarrow (Odd \rightarrow Odd)$ (whatever number x is, if y is odd, so is $2x + y$) but it does not have type $Odd \rightarrow (N \rightarrow Odd)$ (if y

is even, the output will be even, not odd). This is an example demonstrating the failure of the permutation-related rule: $A \rightarrow (B \rightarrow C) \vdash B \rightarrow (A \rightarrow B)$.[84]

This is an important model because it motivates the failure of not only commutativity of $\sqcup$ but also *associativity*. There is no sense in which $x(y(z))$ ought be equal to $(x(y))(z)$. Function typing models a very weak substructural logic.

Urquhart considered adding *disjunction* to operational frames, with the natural clause:

$\diamond$ $x \Vdash A \vee B$ if and only if $x \Vdash A$ or $x \Vdash B$

However, this is not as satisfactory as its cousin for conjunction. For one thing, R models extended with this clause validate the following formula

$$(A \rightarrow B \vee C) \wedge (B \rightarrow C) \rightarrow (A \rightarrow C) \tag{1}$$

which is not valid in R.[85] Secondly, and more importantly, the interpretation in terms of *pieces of information* simply doesn't motivate the straightforward clause for disjunction. Pieces of information may well warrant disjunctions without warranting either disjunct. To interpret disjunction in operational models (and to get a logic in the vicinity of R or any of the other logics we are interested in) you can do one of two things. One approach is, taken by Ono [205, 206], Došen [68, 69] and Wansing [284], is to admit some kind of *closure operator* on the frame: $A \vee B$ is true not only at points where A is true and where B is true, but also at some more points, found by *closing* the original set under some operation. Doing this will almost invariably invalidate distribution, and we will look at one example of this kind of semantics in the a couple of section's time, when we come to phase space models for linear logic.

A related method, and one which validates distribution, was discovered by Kit Fine [97, 99] in the mid 1970s. He showed that if you have a *two tiered* collection of points, the whole class S with a special subset P of *prime* points (in analogy with prime theories) which respect disjunction. For points in P, a disjunction is true if and only if a disjunct is. For arbitrary points in S this may fail. For an arbitrary point in S, however, you have a guarantee that there it can be *covered* by a point in P. For each $s \in S$ there is at least one $s' \in P$ where $s \sqsubseteq s'$. This means that disjunctions are at least promissory notes: although a disjunct may not be true given *this* body of information, it is possible for the information to be filled out so that you get one or other disjunct. Then, a disjunction, in a Fine model, is evaluated like this:

$\diamond$ $x \Vdash A \vee B$ if and only if for each $y \in T$ where $x \sqsubseteq y$, $y \Vdash A$ or $y \Vdash B$.

Fine's models will satisfy distribution, and model the positive fragment of R nicely. They do so at the cost of requiring *special* bodies of information, those

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[84]We can use other connectives to expand the type analysis of terms. Conjunction clearly makes sense in this interpretation: $x \Vdash A \wedge B$ if and only if $x \Vdash A$ and $x \Vdash B$. In this way, we have models not only for typing functions with $\rightarrow$ but also with intersection. These are models for the Torino type system $\lambda \cap$ [16, 60, 135, 276].

[85]This is not to say that the operational semantics with this disjunction clause hasn't been investigated. See some interesting papers of Charlewood: [52, 53], following on from a result of Fine [98].

which are prime. They also have the cost of requiring a new notion $\sqsubseteq$, of informational inclusion. This requires a new condition on frames, the hereditary condition, familiar from models for intuitionistic logic:

DEFINITION 46 (HEREDITARY CONDITION) If $x \Vdash p$ and $x \sqsubseteq y$ then $y \Vdash p$ too.

To show that the hereditary condition extends from atomic propositions to all propositions, a further model condition is required to validate the inductive step for the conditional. You need to assume that

$$\text{If } x \sqsubseteq x' \text{ and } y \sqsubseteq y' \text{ then } x \sqcup y \sqsubseteq x' \sqcup y'.$$

Given this clause, we indeed have a model for positive R. The cost has been a complication of the clause for disjunction, the requirement that we have a two-tiered universe of points, and a hereditary condition on points. This is not the only way to model the whole of R. Routley and Meyer, independently of Fine, came to an equally powerful semantics, with a slightly smaller set of primitive notions. Before looking at the Routley–Meyer semantics in the next section, I must say a little about negation in the operational semantics.

How one interprets negation depends to a great extent on the intended interpretation. The *Boolean* clause for negation

$\diamond$ $x \Vdash -A$ if and only if $x \nVdash A$

is marvellously appropriate in string models of the Lambek calculus (a string is of type "not a noun" just when it is not of type "noun") and in function typing (a function like $\lambda x.x^2$ has type $-\text{Even} \to -\text{Even}$: it sends inputs which are not even to outputs which are also not even) but it is terrible when it comes to taking points as bodies of information. There is little reason to think that a body of information x warrants the negation of A just when it fails to warrant A. Bodies of information can be incomplete (warranting neither a claim nor its negation) and they can be inconsistent (warranting — you might think misleadingly — a claim and its negation). Something else too must be done to model negation. Fine had a treatment of negation in his models, but it too appeals essentially to the two-tiered nature of a model, and it is simpler in the Routley–Meyer incarnation.

3.3.2   Routley–Meyer Frames

Routley and Meyer [239, 240, 241, 242] chose to keep the interpretation for disjunction simple, and to generalise the interpretation for implication. The central feature of a Routley–Meyer frame is the *ternary relation* R. The clause for implication then is:

$\diamond$ $x \Vdash A \to B$ if and only if for each y, z where $Rxyz$, if $y \Vdash A$ then $z \Vdash B$.

This is a generalisation of the operational semantics. An operational frame is a Routley–Meyer frame where $Rxyz$ holds if and only if $x \sqcup y = z$. The interpretation of R is similarly a generalisation of that for $\sqcup$. Reading the implication clause "in reverse" (as assigning meaning to R and not to $\to$)[86] we have that $Rxyz$ if and only if the *laws* (or conditionals) in x, applied to the *facts*

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[86]Which, frankly, is exactly what is done in cases of interpreting the accessibility relation in a modal logic as "relative possibility".

(antecedents) in y give outcomes (consequents) true in z. Or more shortly, applying x to y gives an outcome included in z. That this is a genuine *relation* means that applying x to y might give no outcome at all. On the other hand, it we might have $Rxyz$ and $Rxyz'$ for different z and z'. The result of applying x and y is no doubt a body of information, but it might not be a prime body of information. For example we might have $x \Vdash A \to B \lor C$ and $y \Vdash A$. Applying the information in x to that in y will give $B \lor C$, without giving us any guidance on which of B or C it is to be. And this is possible even if x is prime — for in R we don't have the counterintuitive entailment $A \to B \lor C \vdash (A \to B) \lor (A \to C)$, so we have no reason to think that x might contain either $A \to B$ or $A \to C$. So, in this case, we'd have two points z and z' where $Rxyz$ and $Rxyz'$. At z we can have B and at z' we can have C. In this way, we verify $A \to B \lor C$ at x, all the time using prime points. We only have

a semantics for the positive part of R when endow the ternary relation R with some properties. Routley and Meyer's original properties are best stated with the use of some shorthand.

◇ "R^2abcd" is shorthand for $(\exists x)(Rabx \land Rxcd)$.

◇ Given the distinguished point 0, we let "$a \sqsubseteq b$" be shorthand for $R0ab$.

These abbreviations make sense, given the interpretations of the concepts at hand. R^2abcd conjoins application. You apply a to b and get a result in x (for some x) which we then apply to c to get a result in d. One way of thinking of this is applying a to b and applying all of this to c. The inclusion relation is defined by looking at what happens when you apply *logic* to a state. Applying *logic* to a ought result in nothing more than a. So, if $R0ab$ if and only if a is included in b.

Fine has suggested [97] writing $Rabc$ as "$b \sqsubseteq_a c$", and reading it as: according to a, b is contained in c.[87] In this case, $\sqsubseteq$ is $\sqsubseteq_0$, containment from the point of view of *logic*.

Here are the postulates Routley and Meyer gave to make their semantics model the logic R.

◇ (Identity) $R0aa$ for each a.

◇ (Commutativity) If $Rabc$ then $Rbac$.

◇ (Pasch's Postulate) If R^2abcd then R^2acbd.

◇ (Idempotence) $Raaa$ for each a.

◇ (Heredity) If $Rabc$ and $a' \sqsubseteq a$ then $Ra'bc$.

These postulates parallel the postulates for $\sqcup$. Identity and heredity govern the behaviour of 0, making it fit to do the job of t, and to be a place to witness logical truths. Commutativity corresponds to the commutativity of fusion, Pasch's postulate corresponds to B': an equivalent postulate, given commutativity, would be

◇ (Associativity) If R^2abcd then $R^2a(bc)d$.

Where "$R^2a(bc)d$" is read as $(\exists x)(Rbcx \land Raxd)$.

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[87]This is a plausible proposal, provided that you are aware that in models for R, $\sqsubseteq_a$ may fail to be reflexive, as is needed to forma counterexample to $A \vdash B \to B$.

Idempotence does the job of WI. So, we have a match with the postulates for an operational frame. And as with operational frames, ringing the changes with regard to the behaviour of R will result in different logical systems.

Soundness of Routley–Meyer models is a straightforward matter of showing that each provable consecution is valid on each model. (A valid consecution is, as usual, one which is preserved at every point.) To interpret consecutions, you must have an interpretation of fusion, but that is as you would expect.

$\diamond$ $x \Vdash A \circ B$ iff there are y, z where $Ryzx$, $y \Vdash A$ and $z \Vdash B$.

A fusion is true at a point in a model when it is the composition of two points, at which the "fusejuncts" are, respectively, true. Logically true formulas are then always true at 0 in a Routley–Meyer model.

Demonstrating completeness, as always for a semantics like this, is much more involved. As usual, it is a canonical model construction. To construct a canonical model for a logic like R, instead of dealing with all *theories*, as we could with operational models, we must deal in *prime theories*[88] But here, not just any prime theories will do. In these models, each point is closed under consequence as defined at the point 0. This is a fundamental fact about Routley–Meyer models:

FACT 47 (SEMANTIC ENTAILMENT) A *entails* B *in a Routley–Meyer model (that is, for all* x, *if* $x \Vdash A$ *then* $x \Vdash B$*) if and only if* $0 \Vdash A \to B$.

This means that these points are not only prime theories, they are prime 0-theories. They are closed under the "logic of 0." And here, what is going on at 0 may be more than just genuinely "logic" according to the logic in question. In particular, this is the case at R, at least if negation is around. (Bear with the fact that I haven't told you how to interpret negation yet.) For R proves $A \vee \sim A$, and so, by the primeness of the point 0, we will have either $0 \Vdash A$ or $0 \Vdash \sim A$. In any particular model, 0 will validate more than "logic alone". So, to construct a model, we will first construct a prime theory T for the base point 0, and then the other points in the model will be prime T-theories: theories closed under the inferences licensed by T.

DEFINITION 48 (THE *Prime Theory* CANONICAL MODEL ON T) Given a prime, regular theory T, *prime theory* canonical model on T is populated by prime T-theories. The identity point is T itself. The ternary relation R is defined as follows:

$$RUVW \text{ if and only if } \{B : (\exists A)(A \to B \in U \text{ and } A \in V)\} \subseteq W$$

and $U \Vdash A$ if and only if $A \in U$.

Note the similarity of the definition of R here to the definition of $\sqcup$ on the canonical theory model on page 71. Here, R is defined by composition of theories, but the composition of two prime T-theories may not itself be a prime theory, so we resort to the ternary relation.

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[88] Defined at Definition 5 on page 24.

Proving that this is indeed a *model* is a matter of checking all of the clauses. The difficult conditions are the *existential* ones, according to which there is a point in the model with certain properties. An example is one half of the implication clause: if $A \rightarrow B \notin U$, we need to find V, W where $RUVW$, $A \in V$ and $B \notin W$. This is a matter of using Belnap's Pair Extension Theorem[89] twice. First, to construct V you use the pair $\langle \{A\}, \{C : C \rightarrow B \in U\} \rangle$, and extend it to get a prime V. Then, for W you use the pair $\langle U \sqcup V, \{B\} \rangle$. The result will be the two prime theories you wish. The same techniques work for the other difficult clauses. The canonical prime theory model is indeed a model.

This shows the ubiquity of the pair extension theorem in the metatheory of distributive substructural logics. Prime theories play the part here of consistent and complete theories in the metatheory of classical intensional logics.

I have said nothing about the treatment of negation. Routley's innovation is to understand that negation can be modelled by another operator which takes us away from the current point of evaluation. The Boolean clause will not do. The alternative is this:

⋄ $x \Vdash {\sim} A$ if and only if $x^* \nVdash A$

where * is a map of period two on the set of points in a model. That is, $x^{**} = x$. This indeed suffices to make $\sim$ a de Morgan negation on the model. Adding the following condition

⋄ (Contraposition) If $Rxyz$ then Rxz^*y^*

results in the model validating the contraposition axiom.

There has been a great deal of debate centred around the interpretation of the * operator [58, 59, 186].[90] There is no doubt that the * operation is not particularly perspicuous in and of itself. (Being told that it turns set-ups "inside out" is not particularly enlightening.) Instead of pursuing that debate here (which largely burned out), I will merely quote an insight from Belnap and Dunn:

> … we are convinced of the high probability that a mathematical apparatus of such proven formal power will eventually find its concrete applications and its resting place in intuition (think of tensors). [11, page 164]

This, I think, has been borne out in the later development of the Routley–Meyer semantics, and its applications. But to find a plausible interpretation of *, and to understand the semantics more fully, we need to work with it some more. As it stands so far, the Routley–Meyer construction might seem *ad hoc* and fit simply for R and its neighbours. For although you can ring the changes with *some* of the rules (commutativity, associativity, contraction) others seem hopelessly fixed. The models, as they stand, do not seem natural in the way that Kripke models for modal logic do.

This is merely an appearance. Recent work (dating from the 1990s, and chiefly due to Dunn, on gaggle theory) has shown that ternary frames are

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[89]Fact 10 on 25.

[90]Not to be confused with the $*$ of display logic, which simply means "not" in the metatheory of structures.

completely natural models for substructural logics in just the same way as Kripke models interpret normal modal logics [85, 86, 87, 88, 234].

DEFINITION 49 (TERNARY FRAMES) A *ternary frame* is a set with a ternary relation R on that set. The connectives $\wedge, \vee, \top, \perp, \to, \circ, \leftarrow$ can be defined on a ternary frame as follows:

$\diamond$ $x \Vdash A \wedge B$ if and only if $x \Vdash A$ and $x \Vdash B$

$\diamond$ $x \Vdash A \vee B$ if and only if $x \Vdash A$ or $x \Vdash B$

$\diamond$ $x \Vdash \perp$ never

$\diamond$ $x \Vdash \top$ always

$\diamond$ $x \Vdash A \to B$ if and only if for each y, z where $Rxyz$, if $y \Vdash A$ then $z \Vdash B$.

$\diamond$ $x \Vdash A \leftarrow B$ if and only if for each y, z where $Ryxz$, if $y \Vdash A$ then $z \Vdash B$.

$\diamond$ $x \Vdash A \circ B$ iff there are y, z where $Ryzx$, $y \Vdash A$ and $z \Vdash B$.

Many structural rules come with a corresponding conditions on R.[91]

There are no restrictions on R in such an interpretation. It models distributive lattice operations, together with the triple $\langle \to, \circ, \leftarrow \rangle$. A soundness and completeness result, using standard techniques, works for this frame semantics.

Interpreting R is a tricky business, as we have seen. Probably no non-circular definition (one which doesn't appeal to conditionality) will be possible. However, some interesting *explications* of R have been tried in the applied semantics of relevant logics. One answer which has some cachét at present explains R in terms of *situation theory*. If the points in a model are *circumstances* or *situations* of some kind, then R indicates the degree to which situations can carry information about other situations. In particular, $Rabc$ holds just when circumstance a acts as a *information channel* from b to c. There is a significant growing recent literature on the connections between traditional situation theory and the semantics of relevant logics [21, 18, 161, 228, 231].

Some conditions (such as the condition for double negation elimination, which is too complex to discuss here [235]) require talk of a relation of inclusion between points in models. This is not surprising, if in the intended interpretation, points are possibly *incomplete* (think of models for intuitionistic logic) then sometimes the relation of *extension* or *inclusion* might play a role.

DEFINITION 50 (INCLUSION) $\sqsubseteq$ is an *inclusion* relation on a ternary frame if and only if it is reflexive, transitive and asymmetric, and in addition

$\diamond$ For all x, y, z if $Rxyz$ and $x' \sqsubseteq x$, $y' \sqsubseteq y$ and $z \sqsubseteq z'$ then $Rx'y'z'$.[92]

A model on a ternary frame with an inclusion relation must satisfy the hereditary condition on atomic formulas

$\diamond$ If $x \Vdash p$ and $x \sqsubseteq x'$ then $x' \Vdash p$

The clause linking $\sqsubseteq$ and R suffices to prove the hereditary lemma: complex formulas involving $\to, \circ, \leftarrow$ satisfy the hereditary lemma if their constituents do.

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[91]Some, however, require an inclusion relation $\sqsubseteq$, to be defined below.

[92]This is quite a defensible condition as it stands, but it's more general than it needs to be to prove the hereditary lemma for all formulas [234, Chapter 11].

Inclusion as a relation between points in a model, is usually simple to explain given an interpretation of these points. If points are situations, then $a \sqsubseteq b$ just when a is a substitution of b. The situation of my bedroom is a substitution of the situation of my house. An inconsistent circumstance described by the first chapter of some fiction may well be a substitution of an inconsistent circumstance described by the whole book. Given these explications of inclusion, the connection between it and R is plausible. As x shrinks to x', it connects *more* pairs of circumstances, as for a given antecedent circumstance there are *more* possible consequent circumstances. Given x, as y shrinks to y', again, there are more possible consequent circumstances, as y' gives us less information to constrain possible consequents. These explications are probably not *reductions* of the notion, but they go some way to explain their appeal and their use.

Another significant role in models is played by the distinguished point 0 in Routley–Meyer models. This point plays the part of modelling t,

DEFINITION 51 (TRUTH SET) T is a *truth set* in a ternary model with inclusion if and only if.
  $\diamond$ $RTab$ if and only if $a \sqsubseteq b$.
where "$RTab$" stands for $(\exists x)(x \in T \wedge Rxab)$. A truth set is *reduced* if it has the form $\{x : 0 \sqsubseteq x\}$ for some point 0.

A truth set does the job of recording frame consequence. The $\rightarrow$ formulas true at every point in T are exactly the entailments witnessed by the entire model. For some applications (in particular, using frames to prove the admissibility of disjunctive syllogism [239]) reduced truth sets are desirable. A great deal of work has gone in to showing the circumstances in which a logic has a semantics with a reduced truth set [255, 220, 227, 230]. On the other hand, in our intended interpretation, it is by no means obvious.

Most contentious is the interpretation of negation. Some of Dunn's recent work, however, has served to take the sting out of * [87, 233]. Dunn notes that * is a particular case of a more understandable clause for negation:[93]

DEFINITION 52 (COMPATIBILITY) A compatibility relation C on a frame is an arbitrary two-place relation. Negation is interpreted using C as follows:
  $\diamond$ $x \Vdash \sim A$ if and only if for all y, if xCy then $y \nVdash A$.
If the frame uses an inclusion relation, compatibility is related to inclusion as follows:
  $\diamond$ If xCy, $x' \sqsubseteq x$ and $y' \sqsubseteq y$ then $x'Cy'$.

If you think of x and y as compatible just when there are no clashes between them, then these clauses are defensible. A circumstance warrants $\sim A$ just when there's no compatible circumstance in which A holds. So, $\sim A$'s holding in a circumstance just when A is *ruled out* by that circumstance: there is no compatible circumstance in which A. If circumstances are possibly incomplete (they might be compatible with more than just themselves) and if they are possibly inconsistent (not compatible with themselves: it contains

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[93]The expression in terms of *compatibility* is mine. Dunn uses a relation of incompatibility, expressed the symbol "$\perp$", which is already overloaded here.

an internal contradiction) then we have counterexamples to the paradoxes of implication from before. We may have $x \Vdash A \wedge {\sim}A$ (when it is not the case that xCx) or we may have $x \nVdash A \vee {\sim}A$ (when $x \nVdash A$, but xCy where $y \Vdash A$).

But what of the dreaded *? It can be seen to be a special case of C. The behaviour of C is wrapped up by * just if a^* is the unique $\sqsubseteq$-maximum of the set $\{x : aCx\}$ of all points compatible with a. If this set has a unique $\sqsubseteq$-maximum, a^*, then indeed $a \Vdash {\sim}A$ just when $a^* \nVdash A$ [233].

So, the ternary frame semantics can be "deconstructed" into individual components, each of which may be explained and applied in different circumstances. Here are some other examples of ternary frames which have been useful in the study of substructural logics.

EXAMPLE 53 (TWO-DIMENSIONAL FRAMES) Given a set D, we can define a frame on the set $D \times D$ of pairs of D elements, by defining the ternary relation R on $D \times D$, setting $R\langle a,b\rangle\langle c,d\rangle\langle e,f\rangle$ if and only if $b = c$, $e = a$ and $b = f$. In other words, $\langle a,b\rangle$ composes with $\langle b,c\rangle$ to result in $\langle a,c\rangle$, and no other relations hold between pairs. So, the evaluation conditions on these two-dimensional frames reduce as follows:

$\diamond$ $\langle a,b\rangle \Vdash A \circ B$ if and only if for some $c \in D$, $\langle a,c\rangle \Vdash A$ and $\langle c,b\rangle \Vdash B$.

$\diamond$ $\langle a,b\rangle \Vdash A \to B$ if and only if for each $c \in D$ if $\langle b,c\rangle \Vdash A$ then $\langle a,c\rangle \Vdash B$.

$\diamond$ $\langle a,b\rangle \Vdash B \leftarrow A$ if and only if for each $c \in D$ if $\langle c,a\rangle \Vdash A$ then $\langle c,b\rangle \Vdash B$.

In this frame, the ternary R reduces to a *partial* function on pairs. This function is associative but not symmetric, where defined. The point set is flat — there is no natural notion of inclusion to be imposed. This frame has a truth set, but in this case it is not reduced: it is the set $\{\langle a,a\rangle : a \in D\}$.

These models are studied by van Benthem, Došen and Orłowska [37, 70, 208] in the context of substructural logics, and they have blossomed into their own industry, under the suggestive name 'arrow logics' [167]. In these logics, we think of the points $\langle a,b\rangle$ as transitions, or arrows, from a to b.

These are important frames for they are closely related to language frames in a number of respects — the relation R is functional: if $Rxyz$ and $Rxyz'$ then $z = z'$. However, in this case the relation is *partial*. For some x, y there is no z such that $Rxyz$.

EXAMPLE 54 (MITCHELL'S *IE models*) Mitchell's IE models, are models of linear logic with distribution of $\wedge$ over $\vee$ [199]. In these models, points are *pairs* $\langle m,n\rangle$ whose elements are taken from a commutative monoid R of *resources*. As R is a commutative monoid, there is an operation $+$ on R, such that $m + n = n + m$, with an identity 0, such that $m + 0 = m = 0 + m$. We evaluate propositions at points as follows: $\langle m,n\rangle \Vdash A \circ B$ if and only if for some n_1, n_2 where $n = n_1 + n_2$, $\langle m + n_1, n_2\rangle \Vdash A$ and $\langle m + n_2, n_1\rangle \Vdash B$. $\langle m,n\rangle \Vdash {\sim}A$ if and only if $\langle n,m\rangle \nVdash A$. $\langle m,n\rangle \Vdash A \to B$ if and only if for each m_1, m_2 where $m = m_1 + m_2$, if $\langle n + m_2, m_1\rangle \Vdash A$ then $\langle m_2, n + m_1\rangle \Vdash B$. Conjunction, disjunction, $\top$ and $\bot$ are defined in the usual way.

Early antecedents of the frame semantics for substructural logics can be found in Jónsson and Tarski's work on the representation of Boolean algebras with operators [141, 142]. This work presents what amounts to a soundness and completeness result for frames of substructural logics (with Boolean

negation), though it takes a certain amount of hindsight to see it as such. The papers are written very much from the perspective of algebra and representation theory.

An extensive study of the properties of ternary frames is given in Routley, Meyer, Brady and Plumwood's *Relevant Logics and their Rivals* [242]. Gabbay [105] also gave a ternary relational semantics for implication, independently of the tradition of Routley and Meyer. Frames can be viewed from an algebraic point of view. The class of propositions of the frame is a completely distributive lattice under intersection and union, and it is equipped with the appropriate operators, defined by the clauses in the evaluation conditions. For example, the implication clause gives us

$$\alpha \to \beta = \{x : (\forall y, z \in \mathcal{F})(Rxyz \to (y \in \alpha \to z \in \beta))\}$$

Similarly, $\sim\alpha = \{x : (\forall y \in \mathcal{F})(xCy \to y \notin \alpha)\}$, and so on. We will call the resulting propositional structure 'Alg($\mathcal{F}$)' the *algebra* of the frame $\mathcal{F}$. Furthermore, any interpretation $\Vdash$ on a frame gives you an evaluation $v_{\Vdash}$ given by setting $v_{\Vdash}(A)$ to be $[\![A]\!]$. The connections with algebra run even deeper. Our canonical model is constructed out of prime theories in a language. A similar construction can work with the *prime filters* of a propositional structure. Dunn's work on gaggles [85, 86, 87, 88] generalises the results here to operators with arbitrary arity. An n-ary operator is modelled with an $n + 1$-place relation.

Duality theory is the study of the relationship between algebras and their representations in terms of frames. There is an important strand of recent work in the semantics of substructural logic exploring duality theory in this context [129, 130, 132, 247, 274].

Meyer and Mares have important work on the particular case of adding an S4-type necessity for R [164], and they have shown that disjunctive syllogism is admissible in this case, using the frame semantics to prove it. Meyer and Mares have also studied the extensions of these logics with Boolean negation [163, 185].

Study of the frame conditions corresponding to rules brings forward questions of canonicity and correspondence. When is the canonical frame for a logic itself a model of the logic? This is not always the case in modal logics, and also, not always the case in substructural logics. There has been some work in attempting to pin down the class of substructural logics for which canonicity holds [113, 148, 234].

Not all logics have connectives which are amenable to the treatment of accessibility relations. We will see this when we consider ! from linear logic. Another case is the counterfactual conditional. These are more aptly modelled by *neighbourhood frames*. There has been a little work considering how neighbourhood frames can be used in a substructural setting [5, 104, 162].

3.3.3 *Projective Frames, and Undecidability*

R is undecidable. Alasdair Urquhart proved this in his ground breaking papers [269, 270]. The general idea is a straightforward one: encode a known undecidable problem into the language of R. Meyer showed how to do this

in the 1960s, by constructing a simple substructural logic, such that deciding what was a theorem in a that logic would enable you to solve the word problem for free semigroups [175, 193]. That logic was not particularly natural. (It was the Lambek calculus together with just enough contraction to enable you to represent the deducibility problem as a conditional.) The logic was not particularly like R. The insights that helped decide the issue for R came from an unexpected quarter — projective geometry. To see why projective geometry gave the necessary insights, we will first consider a simple case, the undecidability of the system KR. KR is given by adding $A \wedge {\sim}A \to B$ to R. A KR frame is one satisfying the following conditions (given by adding the clause that $a = a^*$ to the conditions for an R frame).[94]

> $R0ab$ iff $a = b$ $Rabc$ iff $Rbac$ iff $Racb$ (total permutation)
> $Raaa$ for each a R^2abcd only if R^2acbd

The clauses for the connectives are standard, with the proviso that $a \Vdash {\sim}A$ iff $a \nVdash A$, since $a = a^*$.

Urquhart's first important insight was that KR frames are surprisingly similar to projective spaces. A *projective space* $\mathcal{P}$ is a set P of points, and a collection L of subsets of P called *lines*, such that any two distinct points are on exactly one line, and any two distinct lines intersect in exactly one point. But we can define projective spaces instead through the ternary relation of collinearity. Given a projective space $\mathcal{P}$, its collinearity relation C is a ternary relation satisfying the condition:

> $Cabc$ iff $a = b = c$, or a, b and c are distinct and they lie on a common line.

If $\mathcal{P}$ is a projective space, then its collinearity relation C satisfies the following conditions,

> $Caaa$ for each a. $Cabc$ iff $Cbac$ iff $Cacb$. C^2abcd only if C^2acbd.

provided that every line has at least four points (this last requirement is necessary to verify the last condition). Conversely, if we have a set with a ternary relation C satisfying these conditions, then the space defined with the original set as points and the sets $l_{ab} = \{c : Cabc\} \cup \{a, b\}$ where $a \neq b$ as lines is a projective space.

Now the similarity with KR frames becomes obvious. If $\mathcal{P}$ is a projective space, the frame $\mathcal{F}(\mathcal{P})$ generated by P is given by adjoining a new point 0, adding the conditions $C0aa$, $Ca0a$, and $Caa0$, and by taking the extended relation C to be the accessibility relation of the frame.

Projective spaces have a naturally associated undecidable problem. The problem arises when considering the *linear subspaces* of projective spaces. A subspace of a projective space is a subset which is also a projective space under its inherited collinearity relation. Given any two linear subspaces $\mathcal{X}$ and $\mathcal{Y}$, the subspace $\mathcal{X} + \mathcal{Y}$ is the set of all points on lines through points in $\mathcal{X}$ and points in $\mathcal{Y}$.

In KR frames there are propositions which play the role of linear subspaces in projective spaces. We need a convention to deal with the extra point 0, and we simply decree that 0 should be *in* every "subspace." Then linear subspaces

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[94]My presentation of these results is indebted to many discussions with Pragati Jain [140].

are equivalent to the *positive idempotents* in a frame. That is, they are the propositions X which are *positive* (so $0 \in X$) and *idempotent* (so $X = X \circ X$). Clearly for any formula A and any KR model $\mathcal{M}$, the extension of A, $\|A\|$ in $\mathcal{M}$ is a positive idempotent iff $0 \Vdash A \wedge (A \leftrightarrow A \circ A)$. It is then not too difficult to show that if A and B are positive idempotents, so are $A \circ B$ and $A \wedge B$, and that t and $\top$ are positive idempotents.

Given a projective space $\mathcal{P}$, the lattice algebra $\langle \mathcal{L}, \cap, + \rangle$ of all linear subspaces of the projective space, under intersection and $+$ is a modular geometric lattice. That is, it is a complete lattice, satisfying these conditions:

Modularity $a \geq c \Rightarrow (\forall b)\big(a \cap (b + c) \leq (a \cap b) + c\big)$

Geometricity Every lattice element is a join of atoms, and if a is an atom and X is a set where $a \leq \Sigma X$ then there's some finite $Y \subseteq X$, where $a \leq \Sigma Y$.

The lattice of linear subspaces of a projective space satisfies these conditions, and that in fact, any modular geometric lattice is isomorphic to the lattice of linear subspaces of some projective space. Furthermore the lattice of positive idempotents of any KR frame is also a modular geometric lattice.

The undecidable problem which Urquhart uses to prove the undecidability of KR is now simple to state. Hutchinson [138] and Lipshitz [157] proved that

FACT 55 (MODULAR LATTICE WORD PROBLEM) *The word problem for a class of modular lattices which includes the subspace lattice of an infinite dimensional projective space is undecidable.*

Given an infinite dimensional projective space in which every line includes at least four points $\mathcal{P}$, the logic of the frame $(\mathcal{P})$ is said to be a *strong* logic. Our undecidability theorem then goes like this:

FACT 56 (UNDECIDABILITY FOR KR) *Any logic between* KR *and a strong logic is undecidable.*

PROOF Consider a modular lattice problem

$$\text{If } v_1 = w_1 \ldots v_n = w_n \text{ then } v = w$$

stated in a language with variables x_i ($i = 1, 2, \ldots$) constants 1 and 0, and the lattice connectives $\cap$ and $+$. Fix a map into the language of KR by setting $x_i^t = p_i$ for variables, $0^t = t$, $1^t = \top$, $(v \cap w)^t = v^t \wedge w^t$ and $(v + w)^t = v^t \circ w^t$. The translation of our modular lattice problem is then the KR formula

$$\big(B \wedge (v_1^t \leftrightarrow w_1^t) \wedge \cdots \wedge (v_n^t \leftrightarrow w_n^t) \wedge t\big) \to (v^t \leftrightarrow w^t)$$

where the formula B is the conjunction of all formulas $p_i \wedge (p_i \leftrightarrow p_i \circ p_i)$ for each p_i appearing in the formulae v_j^t or w_j^t.

We will show that given a particular infinite dimensional projective space (with every line containing at least four points) $\mathcal{P}$, then the word problem is valid in the lattice of linear subspaces of $\mathcal{P}$ if and only if its translation is provable in L, for any logic L intermediate between KR and the logic of the frame $\mathcal{F}(\mathcal{P})$.

If the translation of the word problem is valid in L, then it holds in the frame $\mathcal{F}(\mathcal{P})$. Consider the word problem. If it were invalid, then there would be linear subspaces $x_1, x_2, \ldots$ in the space $\mathcal{P}$ such that each $v_i = w_i$ would be true while $v \neq w$. Construct a model on the frame $\mathcal{F}(\mathcal{P})$ as follows. Let the extension of p_i be the space x_i together with the point 0. It is then simple to show that $0 \Vdash B$, as each p_i is a positive idempotent. In addition, $0 \Vdash t$, and $0 \Vdash v_i^t \leftrightarrow w_i^t$, for the extension of each v_i^t and w_i^t will be the spaces picked out by v_i and w_i (both with the obligatory 0 added). However, we would have $0 \nVdash v^t \leftrightarrow w^t$, since the extensions of v^t and w^t were picked out to differ. This would amount to a counterexample to the translation of the word problem, which we said was valid. As a result, the word problem is valid in the space $\mathcal{P}$. The converse reasoning is straightforward. Deciding the logic would give us a decision for the word problem.   □

Unfortunately, these techniques do not work for systems weaker than KR. The proof that positive idempotents are modular uses essentially the special properties of KR. Not every positive idempotent in R is modular. Nonetheless, the techniques of the proof can be extended to apply to a much wider range of systems. You do not need to restrict your attention to modular lattices to construct an undecidable word problem. But to do that, you need to examine Lipshitz and Hutchinson's proof more carefully. In the rest of this section, I will hint at the structure of Urquhart's undecidability proof for R and other logics. For detail, the reader is urged to consult Urquhart's original paper [270] or my retelling of the proof [234, Chapter 15]

Lipshitz and Hutchinson proved that the word problem for modular lattices was undecidable by embedding into that problem the already known undecidable word problem for semigroups. It is enough to show that a structure can define a free associative binary operation, for then you will have the tools for representing arbitrary semigroup problems. Urquhart showed that this could be done without resorting to the full power of a modular lattice.

It suffices to have an 0-*structure*, and a *modular 4-frame* defined within that 0-structure. An 0-structure is a set equipped with the following structure

- ⋄ It has a semilattice join operator $\sqcap$, defining an order $\leq$;
- ⋄ It has a commutative and associative binary operator $+$;
- ⋄ $x \leq y \Rightarrow x + z \leq y + z$;
- ⋄ $0 + x = x$;
- ⋄ $y \geq 0 \Rightarrow x \sqcap (x + y) = x$;

A 4-frame in a 0-structure is a set $\{a_1, a_2, a_3, a_4\} \cup \{c_{ij} : i \neq j, i,j = 1, \ldots, 4\}$ such that

- ⋄ The a_is are *independent*. If $G, H \subseteq \{a_1, \ldots, a_4\}$ then $(\Sigma G) \sqcap (\Sigma H) = \Sigma(G \cap H)$ (where $\Sigma \emptyset = 0$)
- ⋄ If $G \subseteq \{a_1, \ldots, a_4\}$ then ΣG is modular
- ⋄ $a_i + a_i = a_i$
- ⋄ $c_{ij} = c_{ji}$
- ⋄ $a_i + a_j = a_i + c_{ik}$; $c_{ij} \sqcap a_j = 0$, if $i \neq j$
- ⋄ $(a_i + a_k) \sqcap (c_{ij} + c_{jk}) = c_{ik}$ for distinct i, j, k

Given the 4-frame, we can define a semigroup structure. For each distinct i, j, we define the set L_{ij} to be $\{x : x + a_j = a_i + a_j \text{ and } x \sqcap a_j = 0\}$. Then if $b \in L_{ij}$ and $d \in L_{jk}$ where i, j, k are distinct, we set $b \otimes d = (b + d) \sqcap (a_i + a_k)$. It follows (through some manipulation) that $b \otimes d \in L_{ik}$. Then, we can define a semigroup operation '.' on L_{12} by:

$$x.y = (x \otimes c_{23}) \otimes (c_{31} \otimes y)$$

It is quite an involved operation to show that this is associative. Furthermore, in certain circumstances, the operation is *freely* associative. Given a countably infinite-dimensional vector space $\mathcal{V}$, its lattice of subspaces is a 0-structure, and it is possible to define a modular 4-frame in this lattice of subspaces, such that any countable semigroup is isomorphic to a subsemigroup of L_{12} under the defined associative operation.

   The rest of the work of the undecidability proof involves showing that this construction can be modelled in a logic. Perhaps surprisingly, it can all be done in a weak logic like $\mathrm{TW}[\wedge, \vee, \rightarrow, \top, \bot]$. We can do without negation by defining it implicationally as usual: Pick a distinguished propositional atom f, and by defining $-A$ to be $A \rightarrow f$, t to be $-f$, and $A : B$ to be $-(A \rightarrow -B)$. A is a *regular* proposition iff $- - A \leftrightarrow A$ is provable. The *regular* propositions form an 0-structure, under the assumption of the formula $\Theta = \{R(t, f, \top, \bot), N(t, f, \top, \bot), -\top \leftrightarrow \bot\}$. where $R(A)$ is $- - A \leftrightarrow A$, $N(A)$ is $(t \rightarrow A) \rightarrow A$, and $R(A, B, \ldots)$ is $R(A) \wedge R(B) \wedge \cdots$ and similarly for N. So, we can show that the conditions for an 0-structure hold in the regular propositions, assuming Θ as a premise. To interpret the 0-structure conditions we model $\sqcap$ by $\wedge$, $+$ by $:$ and 0 by t. To model a 4-frame in the 0-structure, Define $K(A)$ to be $R(A) \wedge (A \wedge -A \leftrightarrow \bot) \wedge (A \vee -A \leftrightarrow \top) \wedge (A : - A \leftrightarrow -A) \wedge (A \leftrightarrow A : A)$. Then we can show the following

$$K(A), R(B, C), C \rightarrow A \;\Rightarrow\; A \wedge (B : C) \leftrightarrow (A \wedge B) : C$$

Then the conditions for a 4-frame go as follows: Choose distinct atomic formulas $A_1, \ldots, A_4$ and $C_{12}, \ldots, C_{34}$ to match $a_1, \ldots, a_4$ and $c_{12}, \ldots, c_{34}$. One independence axiom is then

$$(A_1 : A_2 : A_3) \wedge (A_2 : A_3 : A_4) \leftrightarrow (A_2 : A_3)$$

and one modularity condition is

$$K(A_1 : A_3 : A_4)$$

Let Π be the conjunction of the statements expressing that the propositions A_i and C_{ij} form a 4-frame in the 0-structure of regular propositions. In any algebra in which $\Theta \cup \Pi$ is true, the lattice of regular propositions is a 0-structure, and the denotations of the propositions A_i and C_{ij} form a 4-frame. Finally, when coding up a semigroup problem with variables $x_1, x_2, \ldots, x_m$, we will need formulas doing duty for these variables: We need a condition to pick out the fact that p_i (standing for x_i) is in L_{12}. We define $L(p)$ to be $(p : A_2 \leftrightarrow A_1 : A_2) \wedge (p \wedge A_2 \leftrightarrow t)$. Then the semigroup operation on elements of L_{12} can be defined in terms of $\wedge$ and $:$ and the formulas A_i and C_{ij}. We assume that done, and we will take it that there is an operation $\cdot$ on formulas which picks out the operation on L_{12}. Then we have the following:

FACT 57 (DEDUCIBILITY FROM TW TO KR IS UNDECIDABLE) *For any logic between* TW$[\wedge, \vee, \rightarrow, \top, \bot]$ *and* KR, *the Hilbert deducibility problem is undecidable.*

PROOF Take a semigroup problem which is known to be undecidable. It may be presented in the following way

$$\text{If } v_1 = w_1 \ldots v_n = w_n \text{ then } v = w$$

where each term v_i, w_i is a term in the language of semigroups, constructed out of the variables $x_1, x_2, \ldots, x_m$ for some m. The translation of that problem into the language of TW$[\wedge, \vee, \rightarrow, \top, \bot]$ is the deducibility problem

$$\Theta, \Pi, L(p_1, \ldots, p_m), v_1^t \leftrightarrow w_1^t, \ldots, v_n^t \leftrightarrow w_n^t \Rightarrow v^t \leftrightarrow w^t$$

where each the translation u^t of each term u is defined recursively by setting x_i^t to be p_i, and $(u_1.u_2)^t$ to be $u_1^t \cdot u_2^t$.

For any logic between TW and KR the word problem in semigroups is valid if and only if its translation is valid in that logic. If the word problem is valid in the theory of semigroups, its translation must be valid, for given the truth of Θ and Π and $L(p_1, \ldots, p_m)$, the operator $\cdot$ is provably a semigroup operation on the propositions in L_{12} in the algebra of the logic, and the terms v_i and w_i satisfy the semigroup conditions. As a result, v^t and w^t pick out the same propositions, and we have a proof of $v^t \leftrightarrow w^t$.

Conversely, if the word problem is invalid, then it has an interpretation in the semigroup $\mathcal{S}$ defined on L_{12} in the lattice of subspaces of an infinite dimensional vector space. The lattice of subspaces of this vector space is the 0-structure in our countermodel. Consider the argument for KR. There, the subspaces were the positive idempotents in the frame. The other propositions in the frame were *arbitrary* subsets of points. Something similar can work here. On the vector space, consider the subsets of points which are closed under multiplication (that is, if $x \in \alpha$, so is kx, where k is taken from the field of the vector space). This is a De Morgan algebra, defining conjunction and disjunction by means of intersection and union as is usual. Negation is modelled by set difference. The fusion $\alpha \circ \beta$ of two sets of points is the set $\{x + y : x \in \alpha \text{ and } y \in \beta\}$. It is not too difficult to show that this is commutative and associative, and square increasing, when the vector space is in a field of characteristic other than 2, since if $x \in \alpha$ then $x = \frac{1}{2}x + \frac{1}{2}x \in \alpha \circ \alpha$. Then $\alpha \rightarrow \beta$ is simply $-(\alpha \circ -\beta)$. This is an algebraic model for KR, and that the regular propositions in this model are exactly the subspaces of the vector space. It follows that our counterexample in the 0-structure is a counterexample in a model of KR to the translation of the word problem. As a result, the translation is not provable in KR or in any weaker logic. □

This result applies to systems between TW and KR, and it shows that the deducibility problem is undecidable for any of these systems. In the presence of the *modus ponens* axiom $A \wedge (A \rightarrow B) \wedge t \rightarrow B$, this immediately yields the undecidability of the *theoremhood* problem, as the deducibility problem can be rewritten as a single formula.

$$\big(\Theta \wedge \Pi \wedge L(p_1, \ldots, p_m) \wedge (v_1^t \leftrightarrow w_1^t) \wedge \cdots \wedge (v_n^t \leftrightarrow w_n^t) \wedge t\big) \rightarrow (v^t \leftrightarrow w^t)$$

As a result, the theoremhood problem for logics between T and KR is undecidable. In particular, R, E and T are all undecidable.

The restriction to TW is necessary in the theorem. Without the prefixing and suffixing axioms, you cannot show that the lattice of regular propositions is closed under the 'fusion-like' connective ': '.

Before moving on to our next section, let us mention that these geometric methods have been useful not only in proving the undecidability of logics, but also in showing that interpolation fails in R and related logics [273].

3.3.4 *Phase Spaces*

Not all substructural logics are distributive, and not all point models validate distribution. In this section, we will look at phase spaces for linear logic as an example of a frame invalidating distribution. Before launching into the definition (due to Girard [117]) I will set the scene with some historical precedents.

An important idea germane to the representation of non-distributive lattices is that of a Dedekind–MacNeille closure [67, 125, 130, 159, 264].

EXAMPLE 58 (DEDEKIND–MACNEILLE FRAMES) Consider a poset with order $\sqsubseteq$. Define '$y \sqsubseteq \alpha$' for a set α to mean $y \sqsubseteq x$ for each $x \in \alpha$. Then the *closure* $\Gamma\alpha$ of a set α of points can be given as follows:

$$\Gamma\alpha = \{z : \forall y(y \sqsubseteq \alpha \to y \sqsubseteq z)\}$$

Consider the closure operation on the class of all theories of some logic. If α is a set of theories, then suppose $y \sqsubseteq \alpha$ — that is, $y \sqsubseteq x$ for each $x \in \alpha$. This is equivalent to saying that $y \sqsubseteq \bigcap \alpha$: y is no bigger than the intersection of the set of the theories in α, which is itself a theory. So, if $y \sqsubseteq z$, then we must have $\bigcap \alpha \sqsubseteq z$ too. If $x \in \Gamma\alpha$, then anything true in all of α must also be true in x. So in these frames, to model disjunction we require $x \Vdash A \vee B$ if and only if $x \in \Gamma(\llbracket A \rrbracket \cup \llbracket B \rrbracket)$.

Sambin and others have used the notion of a "pretopology" (in our language, a set with a closure operator) not only as a model of substructural logics but also as a constructive generalisation of a topological space [131, 244, 245, 246]. Došen [68, 69], Ono and Komori [205], and Ono [206] have also used given semantics involving a closure operation

This is not the only way to avoid distribution. In a model without a notion of inclusion, we can get by with a *negation* to define a closure operator:

EXAMPLE 59 (GOLDBLATT FRAMES) Consider *orthologic*: an ortho-negation combined lattice logic. Here a frame will most likely appeal to a two-place compatibility relation C to deal with negation. The compatibility relation is reflexive (so $A \wedge {\sim}A \vdash \bot$) and symmetric (so $A \vdash {\sim}{\sim}A$). Robert Goldblatt [122] showed (in 1974) how to deal with disjunction by considering a simple compatibility frame $\langle P, C \rangle$, where P is a set of points (unordered by any inclusion relation) and C is a symmetric, reflexive compatibility relation on P.[95] Conjunction and negation are modelled in the standard way:

⋄ $x \Vdash A \wedge B$ iff $x \Vdash A$ and $x \Vdash B$

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[95]J. L. Bell presents an interesting philosophical analysis of Goldblatt's semantics, in which C is understood as *proximity* [24].

⋄ $x \Vdash {\sim}A$ iff for each y where xCy, $y \nVdash A$

However, as it stands, this semantics does not validate ${\sim}{\sim}A \vdash A$. To add an extra condition on C to validate double negation elimination would result in C being the identity relation and the logic would collapse into classical propositional logic. Goldblatt's insight was to instead restrict the evaluation of propositions on the frame to those propositions for which ${\sim}{\sim}A \vdash A$ is valid. In the process, you reject distribution.

Given a set $\alpha \subseteq P$, let $\alpha^{\sim} = \{y : \forall x(x \in \alpha \to {\sim}yCx)\}$, or equivalently, $\{y : \forall x(yCx \to x \notin \alpha)\}$. Therefore, for any evaluation, $[\![A]\!]^{\sim} = [\![{\sim}A]\!]$. A set $\alpha \subseteq P$ is said to be C-*closed* if and only if $\alpha = \alpha^{\sim\sim}$. The C-closed sets will model our propositions. Since C is symmetric, $\alpha \subseteq \alpha^{\sim\sim}$.

A disjunction is true not only at the points at which either disjunct is true but also at the closure of that set of points. Here, however, it is C-closure at work.

⋄ $x \Vdash A \vee B$ iff $x \in ([\![A]\!] \cup [\![B]\!])^{\sim\sim}$

Girard's *phase spaces* (1987) are a generalisation of Goldblatt's compatibility frames (discovered independently of Goldblatt's work, despite being 10 years later).

EXAMPLE 60 (PHASE SPACES) A *phase space* is a quadruple $\langle P, \cdot, 1, \mathbf{0} \rangle$ in which $\langle P, \cdot, 1 \rangle$ is a commutative monoid, and in which $\mathbf{0}$ is a distinguished subset of P. The elements of P are *phases*, and $\mathbf{0}$ is the set of *orthogonal phases* of P.[96] In a phase space, the binary operator $\cdot$ is used for the ternary relation for implication. Here, $Rxyz$ if and only if $x \cdot y = z$. For any subset $G \subseteq P$, the *dual* $G^{\sim}$ of G is defined as follows:

$$G^{\sim} = \{x \in P : \text{for all } y \in G(x \cdot y \in \mathbf{0})\}$$

In other words, $G^{\sim}$ is the set of all objects which send each element of G (by the monoid operation) to $\mathbf{0}$. For any set G of phases, $G^{\sim\sim}$ is the *closure* of G. It is not too hard to verify that this is indeed a closure operation, by showing the following:

⋄ $G \subseteq G^{\sim\sim}$.
⋄ $G^{\sim\sim\sim} = G^{\sim}$.
⋄ If $G \subseteq H$ then $H^{\sim} \subseteq G^{\sim}$.
⋄ $G = G^{\sim\sim}$ iff $G = H^{\sim}$ for some $H \subseteq P$.

The closed sets are called *facts*. The set of facts can be equipped with a natural monoid operation, $(G \cdot H)^{\sim\sim}$, where $G \cdot H$ is defined in the obvious way as $\{x \cdot y : x \in G \text{ and } y \in H\}$. This operation is residuated by the operation $\to$ defined by setting $G \to H = \{x : \forall y \in G(xy \in H)\}$, which can be shown to equal $(GH^{\sim})^{\sim}$.

For negation, we define xCy to hold if and only if $x \cdot y \notin \mathbf{0}$. C is symmetric, given the commutativity composition, and the negation of a fact G is $G^{\sim}$. The negation of a fact is itself a fact.

It follows that this is a model for linear logic without exponentials. R satisfies the conditions for C and B, as composition is associative and commutative. The set $\mathbf{1} = \{1\}^{\sim\sim}$ is the identity (both left and right) for fusion.

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[96]In the linear logic literature, '$\perp$' is used instead of '$\mathbf{0}$' for the set of orthogonal phases. We use $\perp$ for the bottom element of a lattice, so we will use $\mathbf{0}$ for the set of orthogonal phases.

Phase spaces are a particular kind of closure frame. They are special in a number of ways. Not only is the closure operation defined by negation, and not only are the structural rules B and C satisfied, but the accessibility relation underlying the frame is *functional*. Nevertheless, phase spaces are still a faithful model for linear logic. We have the following theorem.

FACT 61 (SOUNDNESS AND COMPLETENESS IN PHASE SPACES) $X \vdash A$ *is provable in linear logic if and only if* $X \vdash A$ *holds in all phase spaces.*

PROOF   The soundness result is straightforward as usual. For completeness, we construct the canonical phase space out of formulas. The operator $\cdot$ on this frame is fusion. If you wish to think of a ternary relation, think of $RABC$ iff $A = B \circ C$. Then for **0**, we have $\{A : 0 \vdash \sim A\}$. The *false* elements are the set of all formulae whose negations are provable, as you would expect. This is the correct choice, as $G^{\smile} = \{A : \forall B \in G(B \vdash \sim A)\}$, and so, $G^{\smile\smile} = \{A : \forall B \in G^{\smile}(B \vdash \sim A)\} = \{A : \forall B(\forall C \in G \text{ where } C \vdash \sim B)B \vdash \sim A\}$. Verifying the details is no more difficult in this case than in Urquhart's operational models for the conjunction/implication fragment of R. $\square$

The definition of a phase space gives us a nice result. It motivates an embedding of the whole of multiplicative additive linear logic into its $(\to, \wedge, t)$ fragment. You choose f to be some arbitrary proposition, a translation as follows (where we set $\sim A = A \to f$).

$$
\begin{aligned}
p^t &= \sim\sim p \\
(A \wedge B)^t &= \sim\sim(A^t \wedge B^t) \\
(A \vee B)^t &= \sim(\sim A^t \wedge \sim B^t) \\
(A \circ B)^t &= \sim(A^t \to \sim B^t) \\
(A \to B)^t &= A^t \to B^t \\
t^t &= \sim\sim t
\end{aligned}
$$

FACT 62 (EMBEDDING USING $\wedge$, $\to$ AND t) $A \vdash B$ *holds in multiplicative, additive linear logic in if and only if* $A^t \vdash B^t$ *in the* $(\to, \wedge, t)$ *fragment.*

PROOF   First, if $A^t \vdash B^t$ is provable then $A^t \vdash B^t$ is provable in linear logic, and in particular, it is provable when we choose $\sim t$ for f. In this case, A^t is equivalent to A in linear logic, and therefore, $A \vdash B$ is provable.

Conversely, if $A^t \vdash B^t$ does not hold in the $(\to, \wedge, t)$ fragment then in the canonical model (constructed simply out of *theories*) we have a counterexample to $A^t \vdash B^t$. Construct a phase space out of this model. The phases are the theories in the canonical model. The monoid operation is theory fusion, and the set **0** is $\{x : f \in x\}$. It is straightforward to check that any set of the form $[\![\sim\sim A]\!]$ in the original canonical model is a fact in the phase space we are constructing. Construct an interpretation of the language of linear logic by setting $[\![A]\!]$ in the phase space to be $[\![A^t]\!]$ in the canonical model. As the definition of the translation ^{t} mimics the evaluation clauses in a phase space, this is an acceptable phase space evaluation, and it is one which invalidates $A \vdash B$, so this consecution fails in linear logic. $\square$

Note that this construction works in logics other than linear logic. For example, it will work to embed the whole of R without distribution into R[$\rightarrow$, $\wedge$, t], for if the original model satisfies W, so will the phase model for R without distribution.

We will end this section by sketching how to cope with non-normal modal operators, such as ! and ? of linear logic. The difficulty with operators like these is the way the distribution properties of normal operators fail. We do not have !A$\wedge$!B $\vdash$!(A$\wedge$B). So, we cannot use standard accessibility relations. However, something is possible.

DEFINITION 63 (TOPOLINEAR SPACES) A phase space with a set $\mathfrak{F}$ of facts satisfying the following conditions:
  ◇ If X $\subseteq$ $\mathfrak{F}$ then $\bigcap$X $\in$ $\mathfrak{F}$.
  ◇ If F, G $\in$ $\mathfrak{F}$ then F + G $\in$ $\mathfrak{F}$.
  ◇ If F $\in$ $\mathfrak{F}$ then F + F = F.
  ◇ $\bigcap\mathfrak{F} = \mathbf{0}$.
is called a *topolinear space*. G is a *closed fact* iff G $\in$ $\mathfrak{F}$, and G is a *open fact* iff G$^\sim$ $\in$ $\mathfrak{F}$.

Now, given any fact G, the *consideration* of G, ?G, is

$$?G = \bigcap\{F : G \subseteq F \text{ and } F \in \mathfrak{F}\}$$

It is simply the smallest element of $\mathfrak{F}$ containing G. Its dual, the *affirmation* of G, !G is

$$!G = (\bigcup\{H : H \subseteq G \text{ and } H^\sim \in \mathfrak{F}\})^{\sim\sim}$$

These are duals, as you can readily check.

LEMMA 64 (DUALITY OF ! AND ?) *For any fact* G, !(G$^\sim$) = (?G)$^\sim$, *and dually,* ?(G$^\sim$) = (!G)$^\sim$. □

This definition gives us a semantics for the exponentials. The semantics does as we would expect: by construction G $\subseteq$?G, for any fact G, so by duality, !G $\subseteq$ G. Furthermore, ?G is itself a closed fact, so ?G = ??G, and dually, !G = !!G. Similarly, all of the closed facts are fixed points for fission, ?G + ?G = ?G, and by duality, !G $\circ$!G = !G. Finally, $\mathbf{0}$ $\subseteq$?G by construction, so by duality !G $\subseteq$ t, and by the behaviour of t, G $\circ$ t = t gives F $\circ$!G $\subseteq$ F.

Each of these simple verifications shows that the construction of ! and ? satisfies the rules for the exponentials in linear logic. This gives us the first part of the following fact.

FACT 65 (SOUNDNESS AND COMPLETENESS IN TOPOLINEAR SPACES) X $\vdash$ A *is provable in* LL *if and only if* X $\vdash$ A *holds in all topolinear spaces.*

PROOF As we have seen, the rules for the exponentials hold in topolinear spaces. For the converse, we must verify that the canonical topolinear space satisfies the conditions required for a topolinear space. So how should we construct the canonical topolinear space?

We will use the canonical phase space we have seen to construct a set of closed facts. Obviously, each {A : ?B $\vdash$ A} ought to be a closed fact for any choice of B. This cannot be the whole thing, as the intersection of a class

of closed facts is not necessarily a set of the form $\{A : ?B \vdash A\}$. So we add these intersections. For any class of formulae B_i, we will let $\bigcap_i\{A : ?B_i \vdash A\}$ be a closed fact. Once we do this, it is straightforward to check that $?[\![A]\!] = \bigcap\{F : [\![A]\!] \subseteq F$ and $F \in \mathfrak{F}\}$ for any formula A in the canonical model structure. The duality of $?$ and $!$, together with the duality of their defining conditions, ensures that the result for $!$ holds too. □

This kind of closure operation works well to model the exponentials in phase spaces.

4 Loose Ends

Let me end this whirlwind tour through the history of substructural logic by indicating what I take to be some interesting directions for further research.

4.1 Paradox

Untutored intuitions about collections might lead you to believe that for any property, there is a collection of all and only those things which have that property. Formally, you might try this:

$$a \in \{x : \phi(x)\} \dashv\vdash \phi(a)$$

A object a is in the collection $\{x : \phi(x)\}$ of all of the ϕs if and only if a has property ϕ. This is the *naïve membership scheme*. Russell has shown that from naïve membership scheme, paradox follows. Consider the *Russell set* $\{x : x \notin x\}$. As an instance of the general scheme of membership, we have *Russell's paradox*:

$$\{x : x \notin x\} \in \{x : x \notin x\} \leftrightarrow \{x : x \notin x\} \notin \{x : x \notin x\}$$

The Russell set is a member of itself if and only if it is not a member of itself. In many traditional logics (classical or intuitionistic propositional logic, for example) from $p \leftrightarrow \sim p \vdash p \wedge \sim p$, and from this, anything at all.

The mainstream response to Russell's paradox is to calm our enthusiasm for the naïve membership scheme and to hunt around for weaker theories of set membership which are not so extravagant.[97]

However, this is not the only possibility. There is a motivation to consider logics in which we can retain the naïve membership scheme. Clearly, something must be done with the logic of negation, as we wish to retain propositions p such that $p \leftrightarrow \sim p$, without everything following from this. There are generally two options, logics with "gaps" or "gluts," corresponding to the point in the inference from $p \leftrightarrow \sim p$ to $p \wedge \sim p$ to $\bot$ which is taken to fail. A logic allows "gaps" if it the first inference fails, for p would then be "neither true nor false." A logic allows "gluts" if the second inference fails. Plenty of work has been done on both options for a number of years [115, 116, 216, 226]

However, it is not just the logic of negation which must be non-classical in order to retain the naïve membership scheme. Curry's paradox [108, 194].

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[97]There is some interesting work in this area, attempting to admit sets which are self-membered, without paradox [3, 20].

Curry's paradox shows that more must be done, if the logic is to contain implication. Consider $\{x : x \in x \to F\}$, for some false proposition F.

$$\{x : x \in x \to F\} \in \{x : x \in x \to F\} \leftrightarrow (\{x : x \in x \to F\} \in \{x : x \in x \to F\} \to F)$$

This paradox reveals that there is a proposition p such that $p \leftrightarrow (p \to F)$, and as the following deduction shows, it is hard to avoid the inference to F:

$$
\cfrac{
  \cfrac{
    \cfrac{
      \cfrac{
        \cfrac{p \vdash p \to F \quad p \vdash p}{p; p \vdash F}\,(\to E)
      }{p \vdash F}\,[WI]
    }{0 \vdash p \to F}\,(\to I) \quad p \to F \vdash p
  }{\vdash p}\,[Cut]
  \quad
  \cfrac{
    \cfrac{
      \cfrac{p \vdash p \to F \quad p \vdash p}{p; p \vdash F}\,(\to E)
    }{p \vdash F}\,[WI]
  }{}
}{\vdash F}\,[Cut]
$$

As the choice of F is arbitrary, we must attempt to stop this somewhere. A number of people have taken the step of *contraction* as the one to blame [44, 49, 46, 51, 287]. However, contraction is a useful inferential move. It is *required* in mathematical induction. The step to, say, $F5$ from $F0 \wedge (\forall x)(Fx \to Fx + 1)$ requires the use of the premise no less than six times. Doing without contraction seems a little like cutting off one's nose to spite one's face. Can better be done here?

4.2 *Relevant Predication*

Dunn's *Relevant Predication* program is an interesting application of relevant logic to the clarification of philosophical issues [82, 83, 84, 90, 89, 146]. A theory of relevant implication is used to attempt to mark out the distinction between *genuine properties* — say, my height, which is a genuine property of me — and *fake properties* — say, my height, as a fake property of you. I am indeed such that I am under 1.8 metres tall, and *you* are such that I am under 1.8 metres tall. But in the first case I have described how *I* am, and in the second, I haven't described any genuine property of *you*.

   Classical logic is not good at marking out such a distinction, for if Hx stands for 'x is under 1.8 metres tall', and g stands for Greg, and h stands for you, then Hx is true of x iff it is under 1.8 metres tall, and $(Hg \wedge x = x) \vee Hg$ is true of something iff I am under 1.8 metres tall. Why is one a 'real' property and the other not? If we can reason using relevant implication, we can make the following distinction: It is true that if x is Greg then x is under 1.8 metres. However, it is *not* true that if x is you then Greg is under 1.8 metres. At least, it is plausible that this conditional fail, when read as a relevant conditional. This can be cashed out as follows.

DEFINITION 66 (RELEVANT PREDICATION) F is a *relevant property of* a (written $(\rho x Fx)a$) if and only if $(\forall x)(x = a \to Fx)$.

If F is a relevant property of a then Fa holds (quite clearly) and if F and G are relevant properties of a then so is their conjunction, and the disjunction of any relevant property with anything at all is still a relevant property.

Relevant logics excel at telling you what follows from what as a matter of *logic* — this gives us an interesting picture of the logical structure of relevant predication. However, that is only half the story. Applying the *semantics* of relevant logics ought to give us insight into what it is to that a relevant implication is *true*. That task is as yet, undone.

4.2.1 Monism and Pluralism

One debate in philosophical logic has been inspired by work in relevance and substructural logic, and we have already seen a hint of it in the discussion of disjunctive syllogism in Section 2.4. This is a debate between *pluralists* [22, 23] and *monists* [218, 225] with respect to logical consequence. Is there *one* relation of deductive logical consequence (relative, say, to a particular choice of language, if this is a concern), or are there more than one? To make the discussion particular, given a particular instance of the inference of disjunctive syllogism

$$A \vee B, {\sim}A \vdash_? B$$

should the reasoner accept the inference as valid, reject it as invalid, or is there more to be said? In an interpretation which gives a counterexample to this inference, we may have a "point" x such that $x \Vdash A$, $x \Vdash {\sim}A$ and $x \nVdash B$. What are we to say about this?

Monists will say that if the choice of interpretations is correct, then this provides a counterexample to the inference. If the choice of interpretations is not a good one (if the interpretations are a model of a logic but not of the *One True Logic*) then the argument may well still be valid. For example, Priest [218] argues that for an argument to be valid, it must be that in *every* circumstance in which the premises are true so is the conclusion, and the *One True Logic* is one which is sound and complete for the intended interpretation on the actual class of circumstances. Any logical consequence relation other than this either undergenerates by adding extra circumstances (which are alleged counterexamples to *really* valid arguments) or overgenerates by missing some out (which are counterexamples to invalid arguments missed out by the logic which is too strong).

Pluralists about logical consequence, on the other hand will say that a logic (and its attendant interpretations) may give us some information about the inference, but that this may not be the whole story about its validity or otherwise. For example, a pluralist may agree that there are indeed circumstances in which the premises of a disjunctive syllogism are true and the conclusion untrue. However, this choice of circumstances may include special circumstances not always considered: it includes *impossible* circumstances, as one would expect, if we are taking *relevance* seriously. It is natural too, to only consider possible circumstances, and if these are the only circumstances to consider, then disjunctive syllogism ought be considered valid in this new, restricted sense. It is a lesson of relevant logic and its semantics that these are choices which can be made. For a monist, there is one definitive best answer to this choice. For a pluralist, both sides may have competing merits.

Pluralism extends beyond our interpretation of the semantics into our interpretation of proof theory too. Substructural logics have shown us that

there is remarkable robustness in the interpretation of a conditional by means of the residuation clause:

$$\frac{X, A \vdash B}{X \vdash A \rightarrow B}$$

However, the introduction and elimination rules for a conditional laid down by this clause, does *not* determine the meaning of the conditional.[98] These rules only pick out a fixed interpretation in combination with some account of the behaviour of the structural feature of the *comma*.

At the very least, relevant and substructural logics have provided so many new tools for understanding logical consequence that they have put the issue of pluralism on the agenda. Clarifying these options will deepen our understanding of logical consequence.

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