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E. B. Dynkin

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Diffusions, Superdiffusions and Partial Differential Equations

E. B. Dynkin

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ABSTRACT. The subject of this book is connections between linear and semilinear differential equations and the corresponding Markov processes called diffusions and superdiffusions. Most of the book is devoted to a systematic presentation of the results obtained by the author and his collaborators since 1988. Many results obtained originally by using superdiffusions are extended in the book to more general equations by applying a combination of diffusions with purely analytic methods. Almost all chapters involve a mixture of probability and analysis.

For researchers and graduate students working in probability theory and theory of partial differential equations.

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Preface

Interactions between the theory of partial differential equations of elliptic and parabolic types and the theory of stochastic processes are beneficial for, both, probability theory and analysis. At the beginning, mostly analytic results were used by probabilists. More recently, the analysts (and physicists) took inspiration from the probabilistic approach. Of course, the development of analysis, in general, and of theory of partial differential equations, in particular, was motivated to a great extent by the problems in physics. A difference between physics and probability is that the latter provides not only an intuition but also rigorous mathematical tools for proving theorems.

The subject of this book is connections between linear and semilinear differential equations and the corresponding Markov processes called diffusions and superdiffusions. A diffusion is a model of a random motion of a single particle. It is characterized by a second order elliptic differential operator L. A special case is the Brownian motion corresponding to the Laplacian Δ. A superdiffusion describes a random evolution of a cloud of particles. It is closely related to equations involving an operator $Lu - \psi(u)$. Here ψ belongs to a class of functions which contains, in particular $\psi(u) = u^\alpha$ with $\alpha > 1$. Fundamental contributions to the analytic theory of equations

$$Lu = \psi(u) \tag{0.1}$$

and

$$\dot{u} + Lu = \psi(u) \tag{0.2}$$

were made by Keller, Osserman, Brezis and Strauss, Loewner and Nirenberg, Brezis and Véron, Baras and Pierre, Marcus and Véron.

A relation between the equation (0.1) and superdiffusions was established, first, by S. Watanabe. Dawson and Perkins obtained deep results on the path behavior of the super-Brownian motion. For applying a superdiffusion to partial differential equations it is insufficient to consider the mass distribution of a random cloud at fixed times t. A model of a superdiffusion as a system of exit measures from time-space open sets was developed in [**Dyn91c**], [**Dyn92**], [**Dyn93**]. In particular, a branching property and a Markov property of such system were established and used to investigate boundary value problems for semilinear equations. In the present book we deduce the entire theory of superdiffusion from these properties.

We use a combination of probabilistic and analytic tools to investigate positive solutions of equations (0.1) and (0.2). In particular, we study removable singularities of such solutions and a characterization of a solution by its trace on the boundary. These problems were investigated recently by a number of authors. Marcus and Véron used purely analytic methods. Le Gall, Dynkin and Kuznetsov combined probabilistic and analytic approach. Le Gall invented a new powerful probabilistic tool — a path-valued Markov process called the Brownian snake. In his pioneering work he used this tool to describe all solutions of the equation $\Delta u = u^2$ in a bounded smooth planar domain.

Most of the book is devoted to a systematic presentation (in a more general setting, with simplified proofs) of the results obtained since 1988 in a series of papers of Dynkin and Dynkin and Kuznetsov. Many results obtained originally by using superdiffusions are extended in the book to more general equations by applying a combination of diffusions with purely analytic methods. Almost all chapters involve a mixture of probability and analysis. Exceptions are Chapters **7** and **9** where the probability prevails and Chapter **13** where it is absent. Independently of the rest of the book, Chapter **7** can serve as an introduction to the Martin boundary theory for diffusions based on Hunt's ideas. A contribution to the theory of Markov processes is also a new form of the strong Markov property in a time inhomogeneous setting.

The theory of parabolic partial differential equations has a lot of similarities with the theory of elliptic equations. Many results on elliptic equations can be easily deduced from the results on parabolic equations. On the other hand, the analytic technique needed in the parabolic setting is more complicated and the most results are easier to describe in the elliptic case.

We consider a parabolic setting in Part 1 of the book. This is necessary for constructing our principal probabilistic model — branching exit Markov systems. Superprocesses (including superdiffusions) are treated as a special case of such systems. We discuss connections between linear parabolic differential equations and diffusions and between semilinear parabolic equations and superdiffusions. (Diffusions and superdiffusions in Part 1 are time inhomogeneous processes.)

In Part 2 we deal with elliptic differential equations and with time-homogeneous diffusions and superdiffusions. We apply, when it is possible, the results of Part 1. The most of Part 2 is devoted to the characterization of positive solutions of equation (0.1) by their traces on the boundary and to the study of the boundary singularities of such solutions (both, from analytic and probabilistic points of view). Parabolic counterparts of these results are less complete. Some references to them can be found in bibliographical notes in which we describe the relation of the material presented in each chapter to the literature on the subject.

Chapter 1 is an informal introduction where we present some of the basic ideas and tools used in the rest of the book. We consider an elliptic setting and, to simplify the presentation, we restrict ourselves to a particular case of the Laplacian Δ (for L) and to the Brownian and super-Brownian motions instead of general diffusions and superdiffusions.

In the concluding chapter, we give a brief description of some results not included into the book. In particular, we describe briefly Le Gall's approach to superprocesses via random snakes (path-valued Markov processes). For a systematic presentation of this approach we refer to [**Le 99a**]. We do not touch some other important recent directions in the theory of measure-valued processes: the Fleming-Viot model, interactive measure-valued models... We refer on these subjects to Lecture Notes of Dawson [**Daw93**] and Perkins [**Per01**]. A wide range of topics is covered (mostly, in an expository form) in "An introduction to Superprocesses" by Etheridge [**Eth00**].

Appendix A and Appendix B contain a survey of basic facts about Markov processes, martingales and elliptic differential equations. A few open problems are suggested in the Epilogue.

I am grateful to S. E. Kuznetsov for many discussions which lead to the clarification of a number of points in the presentation. I am indebted to him for providing me his notes on relations between removable boundary singularities and the Poisson capacity. (They were used in the work on Chapter 13.) I am also indebted to P. J. Fitzsimmons for the notes on his approach to the construction of superprocesses (used in Chapter 4) and to J.-F. Le Gall whose comments helped to fill some gaps in the expository part of the book.

I take this opportunity to thank experts on PDEs who gladly advised me on the literature in their field. Especially important was the assistance of N.V. Krylov and V. G. Maz'ya.

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CHAPTER 1

Introduction

1. Brownian and super-Brownian motions and differential equations

1.1. Brownian motion and Laplace equation. Let D be a bounded domain in $\mathbb{R}^d$ with smooth boundary ∂D and let f be a continuous function on ∂D. Then there exists a unique function u of class C^2 such that

$$\begin{aligned} \Delta u &= 0 \quad \text{in} \quad D, \\ u &= f \quad \text{on} \quad \partial D. \end{aligned} \tag{1.1}$$

It is called the solution of the Dirichlet problem for the Laplace equation in D with the boundary value f. A probabilistic approach to this problem can be traced to the classical work [**CFL28**] of Courant, Friedrichs and Lewy published in 1928. The authors replaced the Laplacian Δ by its lattice approximation and they represented the solution of the corresponding boundary value problem in terms of the random walk on the lattice. Suppose that a particle starts from a site x in D and moves in one step from a site x to any of $2d$ nearest neighbor sites with equal probabilities. Let τ be its first exit time from D and ξ_τ be its location at time τ. Then the solution of the Dirichlet problem on the lattice is given by the formula

$$u(x) = \Pi_x f(\xi_\tau) = \int f(\xi_{\tau(\omega)}(\omega))\Pi_x(d\omega), \tag{1.2}$$

where Π_x is the probability distribution in the path space Ω corresponding to the initial point x. The solution of the problem (1.1) can be obtained by the passage to the limit as the lattice mesh and the duration of each step tend to 0 in a certain relation.

In fact, this passage to the limit yields a measure Π_x on the space of continuous paths. The stochastic process $\xi = (\xi_t, \Pi_x)$ is called the Brownian motion and formula (1.2) gives an explicit solution of the problem (1.1) in terms of the Brownian motion ξ. This result is due to Kakutani [**Kak44a**], [**Kak44b**].

1.2. Semilinear equations. Partial differential equations involving a nonlinear operator $\Delta u - \psi(u)$ appeared in meteorology (Emden, 1897), theory of atomic spectra (Thomas-Fermi, 1920s) and astrophysics (Chandrasekhar, 1937). [1]

Since the 1960s, geometers have been interested in these equations in connection with the Yamabe problem: which two functions represent scalar curvature of two Riemannian metrics related by a conformal mapping.

The equation

$$\Delta u = \psi(u) \tag{1.3}$$

[1] See the bibliography in [**Vér96**].

was investigated under various conditions on the function ψ. All these conditions hold for the family

$$\psi(u) = u^{\alpha}, \alpha > 1. \tag{1.4}$$

For a wide class of ψ, the problem

$$\begin{aligned} \Delta u &= \psi(u) \quad \text{in} \quad D, \\ u &= f \quad \text{on} \quad \partial D, \end{aligned} \tag{1.5}$$

has a unique solution under the same conditions on D and f as the classical problem (1.1). However, analysts discovered a number of new phenomena related to this equation. In 1957 Keller [**Kel57a**] and Osserman [**Oss57**] found that all positive solutions of (1.3) are uniformly locally bounded. The most work was devoted to the case of ψ given by (1.4). In 1974, Loewner and Nirenberg [**LN74**] proved that, in an arbitrary domain D, there exists the maximal solution. This solution tends to ∞ at ∂D if D is bounded and ∂D is smooth. [2] In 1980 Brezis and Véron [**BV80**] showed that the maximal solution in the punctured space $\mathbb{R}^d \backslash \{0\}$ is trivial if

$$d \geq \kappa_{\alpha} = \frac{2\alpha}{\alpha - 1}$$

and it is equal to

$$q \mid x \mid^{-2/(\alpha-1)}$$

with

$$q = [2(\alpha-1)^{-1}(\kappa_{\alpha} - d)]^{1/(\alpha-1)}$$

if $d < \kappa_{\alpha}$.

1.3. Super-Brownian approach to semilinear equations. A probabilistic formula (1.2) for solving the problem (1.1) involves the value of f at a random point ξ_{τ} on the boundary. The problem (1.5) can be approached by introducing, instead, a random measure X_D on ∂D and by taking the integral $\langle f, X_D \rangle$ of f with respect to X_D. The probability law P_{μ} of X_D depends on an initial measure μ and the role similar to that of (1.2) is played by the formula

$$u(x) = -\log P_x e^{-\langle f, X_D \rangle}. \tag{1.6}$$

Here P_x stands for P_{μ} corresponding to the initial state $\mu = \delta_x$ (unit mass concentrated at x). We call (X_D, P_{μ}) the exit measure from D. Heuristically, we can think of a random cloud for which X_D is the mass distribution on an absorbing barrier placed on ∂D.

We consider families of exit measures which we call branching exit Markov (shortly, BEM) systems because their principal characteristics are a branching property and a Markov property. The BEM system used in formula (1.6) is called the super-Brownian motion. In the next section we explain how it can be obtained by a passage to the limit from discrete BEM systems. Before that, we give, as the first application of (1.6), an expression for a solution exploding on the boundary. Note that, if $X_D \neq 0$, then $e^{-c\langle 1, X_D \rangle} \to 0$ as $c \to +\infty$ and, if $X_D = 0$, then $e^{-c\langle 1, X_D \rangle} = 1$ for all c. Therefore a solution tending to ∞ at ∂D can be expressed by the formula

$$u(x) = -\log \; P_x\{X_D = 0\}. \tag{1.7}$$

[2] They considered, in connection with a geometric problem, a special case $\alpha = \frac{d+2}{d-2}$.

The fact that u is finite is equivalent to the property $P_x\{X_D = 0\} > 0$, i.e., the cloud is extinct in D with positive probability.

1.4. Super-Brownian motion. We start from a system of Brownian particles which die at random times leaving a random number of offspring N with the generating function $Ez^N = \varphi(z)$.

The following picture [3] explains the construction of the exit measure (X_D, P_μ). We have here a particle system started by two particles located at points x_1, x_2 of D. At the death time, the first particle creates two children who survives until they reach ∂D at points y_1, y_2. Of three children of the second particle, one hits ∂D at point y_3, one dies childless and one has two children. Only one child reaches the boundary (at point y_4).

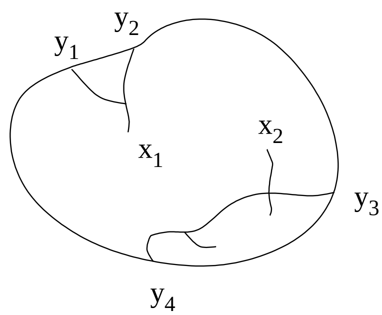

FIGURE 1

The initial and exit measures are given by the formulae

$$\mu = \sum \delta_{x_i} \qquad\qquad X_D = \sum \delta_{y_i}$$

where δ_c is the unit mass concentrated at c.

This way we arrive at a family X of integer-valued random measures (X_D, P_μ) where D is an arbitrary bounded open set and μ is an arbitrary integer-valued measure. Since particles do not interact, we have

$$P_\mu e^{-\langle f, X_D\rangle} = e^{-\langle u,\mu\rangle} \tag{1.8}$$

where

$$u(x) = -\log P_x e^{-\langle f, X_D\rangle}. \tag{1.9}$$

We call this relation the branching property. We also have the following Markov property: for every $C \in \mathcal{F}_{\supset D}$ and for every μ,

$$P_\mu\{C \mid \mathcal{F}_{\subset D}\} = P_{X_D}(C) \quad P_\mu\text{-a.s.} \tag{1.10}$$

Here $\mathcal{F}_{\subset D}$ and $\mathcal{F}_{\supset D}$ are the σ-algebras generated by $X_{D'}, D' \subset D$ and by $X_{D''}, D'' \supset D$.

If the mass of each particle is equal to β, then the initial measure and the exit measures take values $0, \beta, 2\beta, \dots$. We pass to the limit as β and the expected life time of particles tend to 0 and the initial number of particles tends to infinity. In

[3]Of course, this is only a scheme. Path of the Brownian motion are very irregular which is not reflected in our picture.

the limit, we get an initial measure on D and an exit measure on ∂D which are not discrete. We denote them again μ and X_D. The branching property and the Markov property are preserved under this passage to the limit and we get a BEM system (X_D, P_μ) where D is an arbitrary bounded open set and μ is an arbitrary finite measure.

A function ψ obtained by a passage to the limit from φ belongs to a class Ψ_0 which contains $\psi(u) = u^\alpha$ with $1 < \alpha \leq 2$ but not with $\alpha > 2$. The probability distribution of the random measure (X_D, P_μ) is described by (1.8)–(1.9) and u is a solution of the integral equation

$$u(x) + \Pi_x \int_0^\tau \psi[u(\xi_s)]ds = \Pi_x f(\xi_\tau). \tag{1.11}$$

If ∂D is smooth and f is continuous, then (1.11) implies (1.5). Hence, (1.6) is a solution of the problem (1.5).

Formulae (1.8) and (1.11) determine the probability distribution of X_D for a fixed D. Joint probability distributions of $X_{D_1}, \cdots, X_{D_n}$ can be defined recursively for every n by using the branching and Markov properties.

The following equations, similar to (1.8) and (1.11), describe the mass distribution X_t at time t:

$$P_\mu \exp\langle -f, X_t\rangle = \exp\langle -u_t, \mu\rangle, \tag{1.12}$$

$$u_t(x) + \Pi_x \int_0^t \psi[u_{t-s}(\xi_s)]ds = \Pi_x f(\xi_t). \tag{1.13}$$

We cover both sets of equations, (1.8), (1.11) and (1.12), (1.13), by considering exit measures (X_Q, P_μ) for open subsets Q of the time-space $S = \mathbb{R} \times \mathbb{R}^d$ and measures μ on S. They satisfy the equations

$$P_\mu \exp\langle -f, X_Q\rangle = \exp\langle -u, \mu\rangle, \tag{1.14}$$

$$u(r, x) + \Pi_{r,x} \int_r^\tau \psi[u(s, \xi_s)]ds = \Pi_{r,x} f(\tau, \xi_\tau) \tag{1.15}$$

where $\tau = \inf\{t : (t, \xi_t) \notin Q\}$ is the first exit time from Q. Note that $X_t = X_{S_{<t}}$ where $S_{<t} = (-\infty, t) \times \mathbb{R}^d$. If $Q = (-\infty, t) \times D$ where D is a bounded smooth domain [4] and if f is bounded and continuous, then u is a solution of a parabolic equation

$$\dot{u} + \frac{1}{2}\Delta u = \psi(u) \quad \text{in } Q \tag{1.16}$$

such that $u = f$ on ∂Q.

The maximal solution of the equation (1.3) can be described through the range of X. This is the minimal closed set $\mathcal{R}$ which contains the support $\operatorname{supp} X_t$ for all t. (It contains, a.s., $\operatorname{supp} X_D$ for each D.) [5] For every open set D, a maximal solution in D is given by

$$u(x) = -\log \ P_x\{\mathcal{R} \subset D\}. \tag{1.17}$$

---

[4]The name "smooth" is used for domains of class $C^{2,\lambda}$ (see section **6**.1.3).

[5]Writing "a.s." means P_μ-a.s. for all μ [or Π_μ-a.s. for all μ in the case of a Brownian motion].

2. Exceptional sets in analysis and probability

2.1. Capacities. The most important class of exceptional sets in analysis are sets of Lebesgue measure 0. The next important class are sets of capacity 0. A capacity is a function $C(B) \geq 0$ defined on all Borel sets. [6] It is not necessarily additive but it is monotone increasing and continuous with respect to the monotone increasing limits. For every B, $C(B)$ is equal to the supremum of $C(K)$ over all compact sets $K \subset B$ and it is equal to the infimum of $C(O)$ over all open sets $O \supset B$. [A more systematic presentation of Choquet's capacities is given in section **10**.3.2]

To every random closed set $(F(\omega), P)$ there corresponds a capacity

$$C(B) = P\{F \cap B \neq \emptyset\}. \tag{2.1}$$

Another remarkable class of capacities correspond to pairs $(k, \|\cdot\|)$ where $k(x, y)$ is a function on the product space $E \times \tilde{E}$ and $\|\cdot\|$ is a norm in a space of functions on E. The most important are the uniform norm

$$\|f\| = \sup_x |f(x)| \tag{2.2}$$

and the $L^\alpha(m)$-norms

$$\|f\|_\alpha = \left[\int |f(x)|^\alpha \ m(dx)\right]^{1/\alpha} \tag{2.3}$$

where $1 \leq \alpha < \infty$ and m is a measure on E. We assume that E and $\tilde{E}$ are nice metric spaces and that $k(x, y)$ is positive valued, lower semicontinuous in x and measurable in y. To every measure ν on $\tilde{E}$ there corresponds a function

$$K\nu(x) = \int_{\tilde{E}} k(x, y) \ \nu(dy) \tag{2.4}$$

on E. The capacity corresponding to $(k, \|\cdot\|)$ is defined on subsets B of $\tilde{E}$ by the formula

$$C(B) = \sup\{\nu(B) : \nu \quad \text{is concentrated on } B \text{ and } \|K\nu\| \leq 1\}. \tag{2.5}$$

Our primary interest is not in capacities themselves but rather in the classes of sets on which they vanish, and we say that two capacities are of the same type if these classes coincide.

2.2. Exceptional sets for the Brownian motion. The Brownian motion ξ in a domain D killed at the first exit time τ from D has a transition density $p_t(x, y)$. If D is bounded, then

$$g(x, y) = \int_0^\infty p_t(x, y)dt \tag{2.6}$$

is finite for $x \neq y$. We call $g(x, y)$ Green's function. The Green's capacity corresponds to the kernel $g(x, y)$ and the uniform norm (2.2). [7] It is of the same type as the capacity corresponding to

$$g_1(x, y) = \int_0^\infty e^{-t} p_t(x, y) \ dt \tag{2.7}$$

[6] And even on a larger class of analytic sets

[7] If $d > 1$, then $g(x, x) = \infty$. Therefore Green's capacity of a single point is equal to 0.

(and to the uniform norm). In the case $d \geq 3$, it is also of the same type as the capacity corresponding to the kernel $|x-y|^{2-d}$.

For a bounded set D, $\tau < \infty$ a.s. The range $\mathcal{R}$ of ξ is a continuous image of a compact set $[0, \tau]$ and therefore, for every $x \in D$, $(\mathcal{R}, \Pi_x)$ is a random closed set. Consider the corresponding capacity

$$C_x(B) = \Pi_x\{\mathcal{R} \cap B \neq \emptyset\}. \tag{2.8}$$

A set B is called polar for ξ if $C_x(B) = 0$ for all $x \in D \setminus B$. This is equivalent to the condition

$$\Pi_x\{\xi_t \in B \quad \text{for some} \quad t\} = 0 \quad \text{for all } x \in D \setminus B \tag{2.9}$$

[in other words, a.s., ξ does not hit B]. It is well-known (see, e.g., [**Doo84**]) that a set B is polar if and only if its Green's capacity is equal to 0. This gives an analytic characterization of the class of polar sets.

2.3. Exceptional sets for the super-Brownian motion. We say that a set B is polar for X if it is not hit by the range of X, that is if

$$P_x\{\mathcal{R} \cap B \neq \emptyset\} = 0 \text{ for all } \ x \notin B. \tag{2.10}$$

In other words, B is polar, if, for all $x \notin B$, $\mathrm{Cap}^x(B) = 0$ where Cap^x is the capacity associated with a random closed set $(\mathcal{R}, P_x)$. It was proved in [**Dyn91c**] that all capacities Cap^x are of the same type as the capacity determined by the kernel (2.7) and the norm (2.3) (assuming that ψ is given by (1.4)).

It is clear from (1.17) that a closed set B is polar for X if and only if equation (1.3) has only a trivial solution $u = 0$ in $\mathbb{R}^d \setminus B$. By the analytic result described in section 1.2, a single point is polar if and only if $d \geq \kappa_\alpha$.

2.4. Exceptional boundary sets. Suppose that D is a bounded smooth domain. Denote by $\gamma(dy)$ the normalized surface area on ∂D. [8] If τ is the first exit time of the Brownian motion ξ from D, then, for every Borel (or analytic) subset Γ of ∂D,

$$\Pi_x\{\xi_\tau \in \Gamma\} = \int_\Gamma k(x,y)\gamma(dy), \quad x \in D \tag{2.11}$$

where $k(x, y)$ is a strictly positive continuous function on $D \times \partial D$ called the Poisson kernel. Note that $\Pi_x\{\xi_\tau \in \Gamma\} = 0$ if and only if $\gamma(\Gamma) = 0$. In other words, the capacity corresponding to a random closed set $(\{\xi_\tau\}, \Pi_x)$ is of the same type as the measure γ.

A class of exceptional boundary sets related to the super-Brownian motion X is more interesting. It can be defined probabilistically in terms of the range $\mathcal{R}_D$ of X in D — the minimal closed subset supporting $X_{D'}$ for all $D' \subset D$. Or it can be introduced analytically via the capacity CP_α corresponding to the Poisson kernel k and the $L^\alpha(m)$-norm

$$\|f\|_\alpha = \left[\int_D |f(x)|^\alpha m(dx)\right]^{1/\alpha}. \tag{2.12}$$

Here $m(dx) = \mathrm{dist}(x, \partial D)dx$. It is proved in Chapter **13** that

$$P_x\{\mathcal{R}_D \cap \Gamma \neq \emptyset\} = 0 \quad \text{for all } \ x \in D, \tag{2.13}$$

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[8] This is a measure on ∂D determined by the Riemannian metric induced on ∂D by the Euclidean metric in $\mathbb{R}^d$. An explicit expression for γ is given in section **6**.1.8.

if and only if $CP_\alpha(\Gamma) = 0$. We call sets Γ with these properties polar boundary sets. The class of such sets can be also characterized by the condition: $\nu(\Gamma) = 0$ for all $\nu \in \mathcal{N}_1$. Here $\mathcal{N}_1$ is a certain set of finite measures on ∂D introduced in Chapter **8**.

We also establish a close relation between polar boundary sets of the super-Brownian motion and removable boundary singularities for positive solutions of the equation

$$\Delta u = u^\alpha. \tag{2.14}$$

Namely, we prove that a closed subset Γ of ∂D is polar if and only if it is a removable boundary singularity for (2.14) which means: every positive solution in D equal to 0 on $\partial D \setminus \Gamma$ is identically equal to 0.

3. Positive solutions and their boundary traces

3.1. One of our principal objectives is to describe the class $\mathcal{U}(D)$ of all positive solutions of the equation

$$\Delta u = \psi(u) \tag{3.1}$$

in an arbitrary domain D. One of the first results in this direction was obtained by Brezis and Véron who proved that, in the case of ψ given by (1.4) and $D = \mathbb{R}^d \setminus \{0\}$, $\mathcal{U}(D)$ contains only a trivial solution $u = 0$ if $d \geq \kappa_\alpha$ (see section 1.2). If $3 \leq d < \kappa_\alpha$, then $\mathcal{U}(D)$ consists of the maximal solution described in section 1.2 and the one-parameter family $v_c, 0 \leq c < \infty$ such that

$$v_c(x)|x|^{d-2} \to c \text{ as } \ x \to 0. \tag{3.2}$$

All positive solutions of the linear equation $\Delta u = 0$ in an arbitrary domain D (that is all positive harmonic functions) were described by Martin. We present a probabilistic version of the Martin boundary theory in Chapter **7**. We start the investigation of the class $\mathcal{U}(D)$ in Chapter **8** by introducing a subclass $\mathcal{U}_1(D)$ of moderate solutions which are closely related to harmonic functions. Moderate solutions are used as a tool to define, for an arbitrary solution its trace on the boundary. There are two versions of this definition: the rough trace determines a solution only in the case of $\alpha < (d+1)/(d-1)$. The fine trace is a more complete characteristic. It determines uniquely every σ-moderate solution, that is a solution which is the limit of an increasing sequence of moderate solutions. It remains an open problem if there exist solutions which are not σ-moderate.

3.2. Positive harmonic functions in a bounded smooth domain. We denote by $\mathcal{H}(D)$ the class of all positive harmonic functions in a domain D. If D is bounded and smooth, then every $h \in \mathcal{H}(D)$ has a unique representation

$$h(x) = \int_{\partial D} k(x,y)\nu(dy) \tag{3.3}$$

where k is the Poisson kernel and ν is a finite measure on ∂D. We call ν the boundary trace of h and we write $\nu = \operatorname{tr} h$. Formula (3.3) establishes a 1-1 correspondence between $\mathcal{H}(D)$ and the set $\mathcal{M}(\partial D)$ of all finite measures on ∂D.

The constant 1 belongs to $\mathcal{H}(D)$ and its trace is the normalized surface area γ (cf. section 2.4). The trace of an arbitrary bounded $h \in \mathcal{H}(D)$ is a measure absolutely continuous with respect to γ, and the formula

$$h(x) = \int_{\partial D} k(x,y)f(y)\gamma(dy) \tag{3.4}$$

defines a 1-1 correspondence between bounded $h \in \mathcal{H}(D)$ and classes of γ-equivalent bounded positive Borel functions on ∂D.

It follows from (2.11) that (3.4) is equivalent to (1.2). If f is continuous, then (3.4) is a solution of the Dirichlet problem (1.1). For an arbitrary bounded Borel function f, (3.4) can be considered as a generalized solution of the problem (1.1) because, a.s., $h(\xi_t) \to f(\xi_\tau)$ as $t \uparrow \tau$. An analytic counterpart to this statement is Fatou's boundary limit theorem: for γ-almost all $c \in \partial D$, $h(x) \to f(c)$ as $x \to c \in \partial D$ non tangentially.

It is natural to interpret the measure ν in (3.3) as a weak boundary value of h. In other words, h given by (3.3) can be considered as a solution of a generalized Dirichlet problem

$$\begin{aligned} \Delta h &= 0 \quad \text{in } D, \\ h &= \nu \quad \text{on } \partial D. \end{aligned} \tag{3.5}$$

3.3. Positive harmonic functions in an arbitrary domain. Martin boundary. Let D be an arbitrary domain and let $g(x,y)$ be given by (2.6). If D is bounded, then $g(x,y) < \infty$ for $x \neq y$. The same is true for a wide class of unbounded domains. If this is the case, we choose a point $c \in D$ and put

$$k(x,y) = \frac{g(x,y)}{g(c,y)}.$$

It is possible to imbed D into a compact metric space $\hat{D} = D \cup \Gamma$ and to extend $k(x,y)$ to $y \in \Gamma$ in such a way that $y_n \to y \in \Gamma$ if and only if $k(x,y_n) \to k(x,y)$ for all $x \in D$. Set Γ is called the Martin boundary of D. There exists a Borel subset Γ' of Γ such that every $h \in \mathcal{H}(D)$ has a unique representation

$$h(x) = \int_{\Gamma'} k(x,y)\ \nu(dy) \tag{3.6}$$

where $\nu \in \mathcal{M}(\Gamma')$. We write $\nu = \operatorname{tr} h$ and we denote by γ the trace of $h = 1$.

There exists, a.s., a limit of ξ_t in $\hat{D}$ as $t \uparrow \tau$. It belongs to Γ'. We denote it $\xi_{\tau-}$. The trace of a bounded harmonic function has a form $f d\gamma$ and, a.s., $h(\xi_t) \to f(\xi_{\tau-})$ as $t \uparrow \tau$.

3.4. Moderate solutions. We say that a solution u of (3.1) is moderate and we write $u \in \mathcal{U}_1(D)$ if there exists $h \in \mathcal{H}(D)$ such that $u \le h$. In Chapter **8** we prove that formula

$$u(x) + \int_D g(x,y)\psi(y)\ dy = h(x) \tag{3.7}$$

establishes a 1-1 correspondence between $\mathcal{U}_1(D)$ and a subclass $\mathcal{H}_1(D)$ of $\mathcal{H}(D)$. Moreover, h is the minimal harmonic function dominating u and u is the maximal element of $\mathcal{U}(D)$ dominated by h. The class $\mathcal{H}_1(D)$ can be characterized by the condition: $h \in \mathcal{H}_1(D)$ if and only if the trace of h does not charge exceptional boundary set (described in section 2.4). If u corresponds to h and if $\operatorname{tr} h = \nu$, then u can be considered as a solution of a generalized Dirichlet problem

$$\begin{aligned} \Delta u &= \psi(u) \quad \text{in } D, \\ u &= \nu \quad \text{on } \partial D \end{aligned} \tag{3.8}$$

(cf. (3.5)). It is natural to call ν the boundary trace of a moderate solution u.

3.5. Rough trace. For every $u \in \mathcal{U}(D)$ and for every closed subset B of ∂D we define the sweeping $Q_B(u)$ of u to B. In the case of a smooth domain, $Q_B(u)$ is the maximal solution dominated by u and equal to 0 on $\partial D \setminus B$. [The definition is more complicated in the case of an arbitrary domain.]

The rough trace of u is a pair (Γ, ν) where Γ is a closed subset of ∂D and ν is a Radon measure on $O = \partial D \setminus \Gamma$. Namely, Γ is the minimal closed set such that $Q_B(u)$ is moderate for all B disjoint from Γ. The measure ν is determined by the condition: the restriction of ν to every $B \subset O$ is equal to the trace of the moderate solution $Q_B(u)$.

The main results about the rough trace presented in Chapter **10** are:

A. Characterization of all pairs (Γ, ν) which are traces. [The principal condition is that $\nu(B) = 0$ for all exceptional boundary sets.]

B. Existence of the maximal solution with a given trace and an explicit probabilistic formula for this solution.

Le Gall's example (presented in section 3.5 of Chapter 10) shows that, in general, infinitely many solutions can have the same rough trace.

3.6. Fine trace. Again this is a pair (Γ, ν) where Γ is a subset of ∂D and ν is a measure on $O = \partial D \setminus \Gamma$. However Γ is not necessarily closed and ν is not necessarily Radon measure.

Roughly speaking, the set Γ consists of points of the boundary near which u rapidly tends to infinity. A precise definition can be formulated, both, in analytic and probabilistic terms. Here we sketch a probabilistic approach based on the concept of a Brownian motion in D conditioned to exit from D at a given point y of the boundary. This stochastic process is described by a measure Π_x^y on the space of continuous paths which start at point $x \in D$ and which are at y at the fist exit time τ from D. Let f be a positive Borel function in D. We say that y is a point of rapid growth of f if, for every x,

$$\int_0^\tau f(\xi_s)\, ds = \infty \quad \Pi_x^y\text{-a.s.}$$

We say that y is a singular point of a solution u if it is a point of rapid growth of function $\psi'(u)$. We define Γ as the set of all singular points of u. To define the measure ν, we consider all moderate solutions $v \le u$ with the trace not charging Γ. ν is the minimal measure such that, for every such v, $\operatorname{tr} v \le \nu$. We prove that:

A. A pair (Γ, ν) is a trace if and only if ν does not charge exceptional boundary sets and if Γ contains all singular points of the following two solutions:

$$u^* = \sup\{\text{moderate } v \text{ with the trace dominated by } \nu\},$$

$$u_\Gamma = \sup\{\text{moderate } v \text{ with the trace concentrated on } \Gamma\}$$

B. Among the solutions with a given trace, there exists a minimal solution and this solution is σ-moderate. [9]

C. A σ-moderate solution is determined uniquely by its trace.

The solutions in Le Gall's example are uniquely characterized by their fine traces.

[9] See the definition in section 3.1.

Part 1

Parabolic equations and branching exit Markov systems

CHAPTER 2

Linear parabolic equations and diffusions

We introduce diffusions by using analytic results on fundamental solutions of parabolic differential equations. A probabilistic approach to boundary value problems is based on the Perron method in PDEs. A central role is played by Poisson's and Green's operators which we define in terms of diffusions. Fundamental concepts of regular boundary points and of regular domains are also defined in probabilistic terms.

1. Fundamental solution of a parabolic equation

1.1. Operator L**.** We work with functions $u(r,x), r \in \mathbb{R}, x \in E = \mathbb{R}^d$ on $(d+1)$-dimensional Euclidean space $S = \mathbb{R} \times E$. The first coordinate of a point $z \in S$ is interpreted as a time parameter. We write $\dot{u}$ for $\frac{\partial u}{\partial r}$ and $\mathcal{D}_i u$ for $\frac{\partial u}{\partial x_i}$ where $x_1, \dots, x_d$ are coordinates of x. Put $\mathcal{D}_{ij} = \mathcal{D}_i \mathcal{D}_j$.

Operator L is defined by the formula

$$Lu(r,x) = \sum_{i,j=1}^{d} a_{ij}(r,x)\mathcal{D}_{ij}u(r,x) + \sum_{i=1}^{d} b_i(r,x)\mathcal{D}_i u(r,x) \tag{1.1}$$

where $a_{ij} = a_{ji}$. We assume that:

1.1.A. There exists a constant $\kappa > 0$ such that

$$\sum a_{ij}(r,x)t_i t_j \geq \kappa \sum t_i^2 \quad \text{for all } (r,x) \in S, t_1, \dots, t_d \in \mathbb{R}.$$

[κ is called the *ellipticity coefficient* of L .]

1.1.B. a_{ij} and b_i are bounded continuous and satisfy a Hölder's type condition: there exist constants $0 < \lambda < 1$ and $\Lambda > 0$ such that

$$|a_{ij}(r,x) - a_{ij}(s,y)| \leq \Lambda(|x-y|^\lambda + |r-s|^{\lambda/2}), \tag{1.2}$$

$$|b_i(r,x) - b_i(r,y)| \leq \Lambda|x-y|^\lambda \tag{1.3}$$

for all $r, s \in \mathbb{R}, x, y \in E$.

For every interval I, we denote by S_I or $S(I)$ the slab $I \times E$. We write $S_{<t}$ for S_I with $I = (-\infty, t)$ and we write $Q_{<t}$ for the intersection of Q with $S_{<t}$. . Writing $U \Subset Q$ means that Q and U are open subsets of S, U is bounded and its closure $\bar{U}$ is contained in Q. We use the name a *sequence exhausting* Q for a sequence of open sets $Q_n \uparrow Q$ such that $Q_n \Subset Q_{n+1}$ for all n.

1.2. Equation $\dot{u} + Lu = 0$. We investigate equation [1]

$$\dot{u} + Lu = 0 \quad \text{in } Q. \tag{1.4}$$

Speaking about solutions of (1.4), we assume that the partial derivatives $\dot{u}$, $\mathcal{D}_i u, i = 1, \dots, d$ and $\mathcal{D}_{ij} u, i, j = 1, \dots, d$ are continuous. We denote $\mathbb{C}^2(Q)$ the class of functions with this property.

Another class of functions plays a special role – continuous functions on Q that are locally Hölder continuous in x uniformly in r. More precisely, we put $u \in \mathbb{C}^\lambda(Q)$ if $u(r, x)$ is continuous on Q and if, for every compact $\Gamma \subset Q$, there exists a constant Λ_Γ such that

$$|u(r, x) - u(r, y)| \le \Lambda_\Gamma |x - y|^\lambda \quad \text{for all } (r, x), (r, y) \in \Gamma.$$

[λ (called Hölder's exponent) satisfies the condition $0 < \lambda < 1$.]

1.3. Fundamental solution. The following results are proved in the theory of partial differential equations (see Chapter **1** in [**Fri64**] and section 4 in [**IKO62**]).

THEOREM 1.1. *There exists a unique continuous function $p(r, x; t, y)$ on the set $\{r < t, x, y \in E\}$ with the properties:*

1.3.A. For every (t, y), the function $u(r, x) = p(r, x; t, y)$ is a solution of

$$\dot{u} + Lu = 0 \quad \textit{in } S_{<t}. \tag{1.5}$$

1.3.B. For every $t_1 < t_2$ and every $\delta > 0$, the function $p(r, x; t, y)$ is bounded on the set $\{t_1 < r < t < t_2, t - r + |y - x| \ge \delta\}$.

1.3.C. If φ is continuous at a and bounded, then

$$\int_E p(r, x; t, y)\varphi(y)\, dy \to \varphi(a) \quad \textit{as } r \uparrow t, x \to a.$$

Function p is strictly positive and

$$\int_E p(r, x; t, y)\, dy = 1 \quad \textit{for all } r < t \quad \textit{and all } x; \tag{1.6}$$

$$\int_E p(r, x; s, y) p(s, y; t, z)\, dy = p(r, x; t, z) \quad \textit{for all } r < s < t \quad \textit{and all } x, z. \tag{1.7}$$

Function $p(r, x; t, y)$ is called a *fundamental solution* of equation (1.5).

We say that a function f is *exp-bounded* on B if $\sup_B |f(r, x)| e^{-\beta |x|^2} < \infty$ for every $\beta > 0$. (Clearly, all bounded functions are exp-bounded.)

We use the following properties of a fundamental solution.

1.3.1. If κ is the ellipticity coefficient of L, then, for every $\beta < \kappa$,

$$p(r, x; t, y) \le C(t - r)^{-d/2} \exp\left[\frac{-\beta |y - x|^2}{2(t - r)}\right] \quad \text{for all } t_1 < r < t < t_2, x, y \in E \tag{1.8}$$

where the constant C depends on t_1, t_2 and β.

[1] This equation can be reduced by the time reversal $r \to -r$ to the equation $\dot{u} = Lu$ which is usually considered in the literature on partial differential equations. The form (1.4) is more appropriate from a probabilistic point of view.

1.3.2. If φ is an exp-bounded function on $S_t = \{t\} \times E$, then

$$u(r,x) = \int_E p(r,x;t,y)\varphi(y)\, dy \tag{1.9}$$

is exp-bounded on $S[t',t)$ for every finite interval $[t',t)$ and it satisfies equation $\dot{u} + Lu = 0$ in $S_{<t}$. If, in addition, φ is continuous, then, for every $t' < t$, u is a unique exp-bounded solution of the problem

$$\begin{aligned} \dot{u} + Lu = 0 \quad &\text{in } S(t',t), \\ u = \varphi \quad &\text{on } S_t. \end{aligned} \tag{1.10}$$

[Writing $u = \varphi$ at $\tilde{z} \in \partial Q$ means $u(z) \to \varphi(\tilde{z})$ as $z \in Q$ tends to $\tilde{z}$.]

1.3.3. If ρ is a bounded Borel function on $S(t',t)$ and if

$$v(r,x) = \int_r^t ds \int_E p(r,x;s,y)\rho(s,y)\, dy, \tag{1.11}$$

then $\mathcal{D}_i v$ are continuous on $S(t',t)$ [and therefore $v \in \mathbb{C}^\lambda[S(t',t)]$]. If, in addition, $\rho \in \mathbb{C}^\lambda[S(t',t)]$, then v is a unique bounded solution of the problem

$$\begin{aligned} \dot{v} + Lv = -\rho \quad &\text{in } S(t',t), \\ v = 0 \quad &\text{on } S_t. \end{aligned} \tag{1.12}$$

2. Diffusions

2.1. Continuous strong Markov processes. Here we describe a class of Markov processes which contains all diffusions in a d-dimensional Euclidean space E. [2] Imagine a particle moving at random in E. Suppose that the motion starts at time r at a point x and denote ξ_t the state at time $t \geq r$. The probability that ξ_t belongs to a set B depends on r and x and we assume that it is equal to $\int_B p(r,x;t,y)\, dy$. Moreover, we assume that, for every $n = 1, 2, \dots$ and for all $r < t_1 < \dots < t_n$ and all Borel sets $B_1, \dots, B_n$,

(2.1) Probability of the event$\{\xi_{t_1} \in B_1, \dots, \xi_{t_n} \in B_n\}$

$$= \int_{B_1} dy_1 \dots \int_{B_n} dy_n\; p(r,x;t_1,y_1)p(t_1,y_1;t_2,y_2)\dots p(t_{n-1},y_{n-1};t_n,y_n).$$

If the conditions (1.6)-(1.7) are satisfied, then the results of computation with different n do no contradict each other and, by a Kolmogorov's theorem [3] there exists a probability measure $\Pi_{r,x}$ on the space of all paths in E starting at time r which satisfies (2.1). We say that $p(r,x;t,y)$ is the *transition density* of the stochastic process $(\xi_t, \Pi_{r,x})$. Sometimes the measures $\Pi_{r,x}$ can be defined on the space of all continuous paths. For instance, this is possible for

$$p(r,x;t,y) = [2\pi(t-r)]^{-d/2} \exp\left[-\frac{|x-y|^2}{2(t-r)}\right]. \tag{2.2}$$

The corresponding continuous process is called the *Brownian motion* . Diffusions also have continuous paths. (Their transition densities will be defined in section 2.2.)

[2] Basic facts on Markov processes are presented more systematically in the Appendix A.

[3] See [**Kol33**], Section III.4. Two proofs of Kolmogorov's theorem are presented in [**Bil95**].

We denote by Ω_r the space of all continuous paths $\omega(t), t \in [r, \infty)$ in E. To deal with a single space Ω, we introduce an extra point $\flat$ and we put $\omega(t) = \flat$ for $\omega \in \Omega_r$ and $t < r$. We consider ξ_t as a function on Ω, namely, $\xi_t(\omega) = \omega(t)$. The birth time α is a function on Ω equal to r on Ω_r. Measure $\Pi_{r,x}$ is concentrated on the set $\{\alpha = r, \xi_\alpha = x\}$. For every interval I, the σ-algebra $\mathcal{F}(I)$ generated by $\xi_s, s \in I$ can be viewed as the class of all events determined by the behavior of the path during I. Note that $\{\alpha \leq t\} = \{\xi_t \in E\}$ belongs to $\mathcal{F}(I)$ for all I which contain t. We use an abbreviation $\mathcal{F}_{\geq t} = \mathcal{F}[t, \infty)$.

Every process $(\xi_t, \Pi_{r,x})$ satisfies the following condition (which is called the *Markov property*): events observable before and after time t are conditionally independent given ξ_t. More precisely, if $r < t$, $A \in \mathcal{F}[r, t]$ and $B \in \mathcal{F}_{\geq t}$, then

$$\Pi_{r,x}(AB) = \int_A \Pi_{t,\xi_t}(B)\Pi_{r,x}(d\omega). \tag{2.3}$$

To simplify notation we write z for (r, x) and η_t for (t, ξ_t). Formula (2.3) implies that for all $X \in \mathcal{F}[r, t]$ and every $Y \in \mathcal{F}_{\geq t}$, [4]

$$\Pi_z(XY) = \Pi_z(X\Pi_{\eta_t} Y). \tag{2.4}$$

Diffusions satisfy a stronger condition called the *strong Markov property*. Roughly speaking, it means that (2.4) can be extended to all stopping times τ. The definition of stopping times and their properties are discussed in the Appendix A. An important class of stopping times are the first exit times. The *first exit time from an open set* Q is defined by the formula

$$\tau(Q) = \inf\{t \geq \alpha : \eta_t \notin Q\}. \tag{2.5}$$

[We put $\tau(Q) = \infty$ if $\eta_t \in Q$ for all $t \geq \alpha$.]We say that X is a pre-τ random variable if $X1_{\tau \leq t} \in \mathcal{F}_{\leq t}$ for all t.

In the Appendix A we give a general formulation of the strong Markov property, we prove it for a wide class of Markov processes which includes all diffusions and we deduce from it propositions 2.1.A–2.1.C — the only implications which we need in this book.

2.1.A. Let ρ be a positive Borel function on S. For every stopping time τ and every pre-τ $X \geq 0$,

$$\Pi_z X \int_\tau^\infty \rho(\eta_s)\, ds = \Pi_z X G\rho(\eta_\tau) \tag{2.6}$$

where

$$G\rho(z) = \Pi_z \int_\alpha^\infty \rho(\eta_s)\, ds. \tag{2.7}$$

2.1.B. Let τ' be the first exit time from an open set Q'. Then for every stopping time $\tau \leq \tau'$, for every pre-τ $X \geq 0$ and for every Borel function $f \geq 0$,

$$\Pi_z X 1_{Q'}(\eta_\tau) 1_{\tau' < \infty} f(\eta_{\tau'}) = \Pi_z X 1_{\tau < \infty} 1_{Q'}(\eta_\tau) K_{Q'} f(\eta_\tau) \tag{2.8}$$

where

$$K_{Q'} f(z) = \Pi_z 1_{\tau' < \infty} f(\eta_{\tau'}). \tag{2.9}$$

[4]Writing $X \in \mathcal{F}$ means that $X \geq 0$ and X is measurable with respect to a σ-algebra $\mathcal{F}$. It can be proved (by using Theorem 1.1 in the Appendix A) that every $X \in \mathcal{F}_{\leq t}$ coincide $\Pi_{r,x}$-a.e. with a $\mathcal{F}[r, t]$-measurable function. Therefore (2.4) holds for all $X \in \mathcal{F}_{\leq t}$.

[The value of η_∞ is not defined. Instead of introducing in (2.8) and (2.9) factors $1_{\tau'<\infty}$ and $1_{\tau<\infty}$, we can agree to put $f(\eta_\infty) = 0$.]

2.1.C. Suppose that V is an open subset of $S \times S$ and τ is a stopping time. Put

$$\sigma_t = \inf\{u \geq \alpha : u > t, (\eta_t, \eta_u) \notin V\}. \tag{2.10}$$

If $\sigma_\tau < \infty$ Π_z-a.s. for all z, then, for every pre-τ function $X \geq 0$ and every Borel function $f \geq 0$,

$$\Pi_z X f(\eta_{\sigma_\tau}) = \Pi_z X F(\eta_\tau) \tag{2.11}$$

where

$$F(t, y) = \Pi_{t,y} f(\eta_{\sigma_t}). \tag{2.12}$$

2.2. L-diffusion. An L-diffusion is a continuous strong Markov process with transition density $p(r, x; t, y)$ which is a fundamental solution of (1.5). The existence of such a process is proved in Chapter 5 of [**Dyn65**].

Note that

$$\Pi_{r,x}\varphi(\xi_t) = \int_E p(r, x; t, y)\varphi(y)\, dy. \tag{2.13}$$

It follows from Fubini's theorem that

$$\Pi_{r,x} \int_r^t \rho(s, \xi_s)\, ds = \int_r^t ds \int_E p(r, x; s, y)\rho(s, y)\, dy. \tag{2.14}$$

Therefore, under the conditions on φ and ρ formulated in 1.3.2–1.3.3,

$$u(r, x) = \Pi_{r,x}\varphi(\xi_t) \tag{2.15}$$

is a solution of the problem (1.10) and

$$v(r, x) = \Pi_{r,x} \int_r^t \rho(s, \xi_s)\, ds \tag{2.16}$$

is a solution of the problem (1.12).

2.3. Martingales associated with L-diffusions. Martingales are one of new tools contributed to analysis by probability theory. [5] The following theorem establishes a link between martingales and parabolic differential equations.

THEOREM 2.1. *Suppose $f \in \mathbb{C}^2(S(t_1, t_2))$ is exp-bounded on $S(t_1, t_2)$ and that $\rho = \dot{f} + Lf$ is bounded and belongs to $\mathbb{C}^\lambda(S(t_1, t_2))$. Then, for every $r \in (t_1, t_2)$ and every $x \in E$,*

$$Y_t = f(\eta_t) - \int_r^t \rho(\eta_s)\, ds, \qquad t \in (r, t_2) \tag{2.17}$$

is a martingale with respect to $\mathcal{F}[r, t]$ and $\Pi_{r,x}$.

PROOF. First, we prove that, for all $r < t$ and all x,

$$w(r, x) = \Pi_{r,x}[f(\eta_t) - f(\eta_r) - \int_r^t \rho(\eta_s)\, ds] \tag{2.18}$$

is equal to 0. Indeed, $w = u - f - v$ where u is defined by (2.15) with $\varphi(x) = f(t, x)$ and v is defined by (2.16). It follows from 1.3.2–1.3.3, that w is an exp-bounded

[5] See the Appendix A for basic facts on martingales.

solution of problem (1.10) with $\varphi = 0$. Such a solution is unique and therefore $w = 0$.

For every t, Y_t is measurable relative to $\mathcal{F}[r,t]$ and $\Pi_{r,x}|Y_t| < \infty$. We need to prove that, for all $r \le t' < t$ and for every bounded $\mathcal{F}[r,t']$-measurable X,

$$\Pi_{r,x} X(Y_t - Y_{t'}) = 0. \tag{2.19}$$

Note that

$$Y_t - Y_{t'} = f(\eta_t) - f(\eta_{t'}) - \int_{t'}^{t} \rho(\eta_s)\, ds$$

is $\mathcal{F}_{\ge t'}$-measurable. By the Markov property (2.4),

$$\Pi_{r,x} X(Y_t - Y_{t'}) = \Pi_{r,x} X \Pi_{t',\xi_{t'}}(Y_t - Y_{t'})$$

and (2.19) holds because, by (2.18), $\Pi_{t',y}(Y_t - Y_{t'}) = 0$ for all y. □

Corollary. *Suppose that $U \Subset Q$ and let τ be the first exit time from U. If $f \in \mathbb{C}^2(Q)$ and if $\rho = \dot f + Lf \in \mathbb{C}^\lambda(Q)$, then*

$$\Pi_{r,x} f(\eta_\tau) = f(r,x) + \Pi_{r,x} \int_r^\tau (\dot f + Lf)(\eta_s)\, ds. \tag{2.20}$$

Proof. Since U is bounded, it is contained in S_I for some finite interval I. There exists a bounded function of class $\mathbb{C}^2(S_I)$ which coincides with f on $\bar U$ and therefore we can assume that f is defined on S_I and that it satisfies the conditions of Theorem 2.1. The martingale Y_t given by (2.17) is continuous and τ is bounded $\Pi_{r,x}$-a.s. By Theorem 4.1 in the Appendix A, $\Pi_{r,x} Y_\tau = \Pi_{r,x} Y_r$ which implies (2.20). □

3. Poisson operators and parabolic functions

3.1. Poisson operators. The *Poisson operator* corresponding to an open set Q is defined by the formula

$$K_Q f(z) = \Pi_z 1_{\tau<\infty} f(\eta_\tau) \tag{3.1}$$

where τ is the first exit time from Q (cf. formula (2.9)). Note that $K_Q f = f$ on Q^c. It follows from **2**.2.1.B that, for every $U \Subset Q$ and every $f \ge 0$,

$$K_U K_Q f = K_Q f. \tag{3.2}$$

3.2. Parabolic functions. We say that a continuous function u in Q is *parabolic* if, for every open set $U \Subset Q$,

$$K_U u = u \quad \text{in } U. \tag{3.3}$$

The following lemma is an immediate implication of Corollary to Theorem 2.1.

Lemma 3.1. *Every solution u of the equation*

$$\dot u + Lu = 0 \quad \textit{in } Q \tag{3.4}$$

is a parabolic function in Q.

We say that a Borel subset $\mathcal{T}$ of ∂Q is *total* if, for all $z \in Q$,

$$\Pi_z\{\tau < \infty, \eta_\tau \in \mathcal{T}\} = 1.$$

In particular, ∂Q is total if and only if $\Pi_z\{\tau = \infty\} = 0$ for all $z \in Q$. [This condition holds, for instance, if $Q \subset S_{<t}$ with a finite t.] If ∂Q is not total, then there exist no total subsets of ∂Q.

LEMMA 3.2. *Suppose $\mathcal{T}$ is a total subset of ∂Q. If u is bounded and continuous on $Q \cup \mathcal{T}$ and if it is parabolic in Q, then*

$$K_Q u = u \quad \text{in } Q. \tag{3.5}$$

PROOF. Consider a sequence Q_n exhausting Q. The sequence $\tau_n = \tau(Q_n)$ is monotone increasing. Denote its limit by σ. For almost all ω, $\sigma \le \tau < \infty$ and $\eta_\sigma \in \mathcal{T}$. Therefore $\sigma = \tau(Q)$. We get (3.5) by passing to the limit in the equation $u(z) = \Pi_z u(\eta_{\tau_n})$. □

LEMMA 3.3. *Suppose that parabolic functions u_n converge to u at every point of Q. If u_n are locally uniformly bounded, then u is also parabolic.*

PROOF. If $U \Subset Q$, then u_n are uniformly bounded on $\bar{U}$. By passing to the limit in the equation $K_U u_n = u_n$, we get $K_U u = u$. □

3.3. Poisson operator corresponding to a cell. Subsets of S of the form $C = (a_0, b_0) \times (a_1, b_1) \times \cdots \times (a_d, b_d)$ are called (open) *cells*. Points of ∂C with the first coordinate equal to a_0 form the bottom B of C. Clearly, $\mathcal{T} = \partial C \setminus B$ is a total subset. We denote it $\partial_r C$. A basic result proved in every book on parabolic equations [6] implies that, if f is a bounded continuous function on $\partial_r C$, then there exists a continuous function u on $C \cup \partial_r C$ such that

$$\begin{aligned} \dot{u} + Lu = 0 &\quad \text{in } C, \\ u = f &\quad \text{on } \partial_r C. \end{aligned} \tag{3.6}$$

It follows from Lemmas 3.1 and 3.2 that $u = K_C f$. Note that K_C is continuous with respect to the bounded convergence. It follows from the multiplicative systems theorem (Theorem 1.1 in the Appendix A) that these two properties characterize K_C. This provides a purely analytic definition of K_C.

A particular class of cells is defined by the formula

$$C(z, \beta) = \{z' : \tilde{d}(z, z') < \beta\}$$

where

$$\tilde{d}(z, z') = \max_i |x_i - x'_i| \quad \text{for } z = (x_0, \dots, x_d), z' = (x'_0, \dots, x'_d).$$

3.4. Superparabolic and subparabolic functions. A lower semicontinuous function u is called *superparabolic* if, for every open set $U \Subset Q$,

$$K_U u \le u \quad \text{in } U. \tag{3.7}$$

A function u is called *subparabolic* if $-u$ is superparabolic.

LEMMA 3.4. *Suppose that u is a bounded below lower semicontinuous function in Q and that (3.7) holds for every cell $C \Subset Q$. Then u is superparabolic and, moreover, (3.7) holds for all $U \subset Q$.*

PROOF. For $U = S$, the relation (3.7) is satisfied because its left side is 0. If $U \neq S$, then

$$d(z) = \inf_{z' \in \partial U} \tilde{d}(z, z') < \infty$$

for all z. Put

$$V = \{(z, \tilde{z}) : \tilde{d}(z, \tilde{z}) < \frac{1}{2} d(z)\}$$

[6] See, e.g., Chapter 3, section 4 in [**Fri64**] or Chapter V, section 2 in [**Lie96**].

and consider the function σ_t defined by the formula (2.10). The stopping times

$$\tau_0 = \alpha, \tau_{n+1} = \sigma_{\tau_n} \quad \text{for } n \geq 0$$

are finite and $\tau_0 \leq \tau_1 \leq \cdots \leq \tau_n \leq \ldots$. It follows from 2.1.C that $\Pi_z u(\eta_{\tau_{n+1}}) = \Pi_z F(\eta_{\tau_n})$ where $F(z) = \Pi_z u(\eta_{\tau_1})$. If (3.7) is satisfied for cells, then $F(z) \leq u(z)$. Hence, $\Pi_z u(\eta_{\tau_{n+1}}) \leq \Pi_z u(\eta_{\tau_n})$ and, by induction, $\Pi_z u(\eta_{\tau_n}) \leq \Pi_z u(\eta_{\tau_0}) = u(z)$. We have $\tilde{d}(\eta_{\tau_{n+1}}, \eta_{\tau_n}) = d(\eta_{\tau_n})/2$. If τ is the limit of τ_n, then, on the set $\{\tau < \infty\}$, $\eta_{\tau_n} \to \eta_\tau$ and therefore $0 = \tilde{d}(\eta_\tau, \eta_\tau) = d(\eta_\tau)/2$. We conclude that $\{\tau < \infty\} \subset \{\eta_\tau \in \partial U\} \subset \{\tau = \tau(U)\}$. By the definition of the lower semicontinuity, on the set $\{\tau < \infty\}$, $u(\eta_\tau) \leq \liminf u(\eta_{\tau_n})$. Therefore, by Fatou's lemma,

$$\Pi_z 1_{\tau<\infty} u(\eta_\tau) \leq \Pi_z 1_{\tau<\infty} \liminf u(\eta_{\tau_n}) \leq \liminf \Pi_z 1_{\tau<\infty} u(\eta_{\tau_n}) \leq u(z).$$

□

LEMMA 3.5. *Suppose that w is superparabolic in Q and bounded below. Let $\mathcal{T}$ be a total subset of ∂Q. If, for every $\tilde{z} \in \mathcal{T}$,*

$$\liminf w(z) \geq 0 \quad \text{as } z \to \tilde{z}, \tag{3.8}$$

then $w \geq 0$ in Q.

PROOF. Let $\tau = \tau(Q)$. It follows from Lemma 3.4 that $\Pi_z 1_{\tau<\infty} w(\eta_\tau) \leq w(z)$. Condition (3.8) implies that $w(\eta_\tau) \geq 0$ Π_z-a.s. on $\{\tau < \infty\}$. Hence $w(z) \geq 0$. □

3.5. The Perron solution. The following analytic results are proved, for instance, in [**Lie96**], Chapter III, section 4. [7] Let f be a bounded Borel function on ∂Q. A bounded below superparabolic function w is in the upper Perron class U_+ for f, if

$$\liminf_{z \to \tilde{z}} w(z) \geq f(\tilde{z}) \quad \text{for all } \tilde{z} \in \partial Q. \tag{3.9}$$

Analogously, a bounded above subparabolic function v is in the lower Perron class U_- for f, if

$$\limsup_{z \to \tilde{z}} w(z) \leq f(\tilde{z}) \quad \text{for all } \tilde{z} \in \partial Q.$$

It follows from Lemma 3.5 that $v \leq w$ for every $v \in U_-$ and every $w \in U_+$. Since f is bounded, all sufficiently big constants are in U_+ and all sufficiently small constants are in U_-. Therefore functions of class U_- are uniformly bounded from above and functions of class U_+ are uniformly bounded from below.

It is proved that the infimum u of all functions $w \in U_+$ coincides with the supremum of all functions $v \in U_-$. Moreover, u is a solution of the equation (3.4). It is called the *Perron solution corresponding to f*.

THEOREM 3.1. *If Q is a bounded open set and f is a bounded Borel function on ∂Q, then $u = K_Q f$ is the Perron solution corresponding to f.*

PROOF. Let $w \in U_+$ and let τ be the first exit time from Q. Consider a sequence Q_n exhausting Q and the corresponding first exit times τ_n. We have $w(z) \geq \Pi_z w(\eta_{\tau_n})$ and, by condition (3.9) and Fatou's lemma,

$$w(z) \geq \liminf \Pi_z w(\eta_{\tau_n}) \geq \Pi_z \liminf w(\eta_{\tau_n}) \geq \Pi_z f(\eta_\tau).$$

Similarly, if $v \in U_-$, then $v(z) \leq \Pi_z f(\eta_\tau)$. □

[7] See also [**Doo84**], 1.XVIII.1. There only the case $L = \Delta$ is considered but the arguments can be modified to cover a general L.

COROLLARY. *A function u is parabolic in Q if and only if it is a solution of* (3.4).

3.6. Smooth superparabolic functions. The improved Maximum principle. We wish to prove:

3.6.A. If $u \in \mathbb{C}^2(Q)$ and if

$$\dot{u} + Lu \leq 0 \quad \text{in } Q, \tag{3.10}$$

then U is superparabolic in Q.

This follows immediately from (2.20) if $\dot{u} + Lu \in \mathbb{C}^\lambda(Q)$. To eliminate this restriction, we use:

3.6.B. Suppose that C is a cell and $u \in \mathbb{C}^2(C)$ satisfies the conditions

$$\dot{u} + Lu \geq 0 \quad \text{in } C,$$

$$\limsup u(z) \leq 0 \quad \text{as } z \to \tilde{z} \quad \text{for all } \tilde{z} \in \partial_r C.$$

Then $u \leq 0$ in C.

[This proposition is proved in any book on parabolic PDEs (for instance, in Chapter 2 of [**Fri64**] or in Chapter II of [**Lie96**]).]

To prove 3.6.A we consider an arbitrary cell $C \Subset Q$. As we know, $v = K_C u$ is a solution of the problem (3.6) with f equal to the restriction of u to $\partial_r C$. Therefore $w = v - u$ satisfies conditions $\dot{w} + Lw \geq 0$ in C and $w(z) \to 0$ as $z \to \tilde{z} \in \partial_r C$. By 3.6.B, $w \leq 0$ in C. Hence, $K_C u \leq u$. By Lemma 3.4, u is superparabolic.

3.6.C. [The improved maximum principle.] Let $\mathcal{T}$ be a total subset of ∂Q. If $v \in \mathbb{C}^2(Q)$ is bounded above and satisfies the condition

$$\dot{v} + Lv \geq 0 \quad \text{in } Q \tag{3.11}$$

and if, for every $\tilde{z} \in \mathcal{T}$,

$$\limsup v(z) \leq 0 \quad \text{as } z \to \tilde{z}, \tag{3.12}$$

then $v \leq 0$ in Q.

Indeed, by 3.6.A, $u = -v$ is superparabolic and, by Lemma 3.5, $u \geq 0$.

3.7. Superparabolic functions and supermartingales.

PROPOSITION 3.1. *Suppose u is a positive lower semicontinuous superparabolic function in Q and $\tau = \tau(Q)$. Then, for every r, x,*

$$X_t = 1_{t<\tau} u(\eta_t)$$

is a supermartingale on $[r, \infty)$ relative to $\mathcal{F}[r, t]$ and $\Pi_{r,x}$.

PROOF. Note that $\sigma = \tau \wedge t$ is the first exit time from $Q \cap S_{<t}$. Since $\sigma < \infty$, by Lemma 3.4, for every $s < t$,

$$\Pi_{s,x} u(\eta_\sigma) \leq u(s, x).$$

Since $\{\sigma < \tau\} = \{\sigma = t\}$, we have

$$\Pi_{s,x} X_t = \Pi_{s,x} 1_{\sigma=t} u(\eta_t) = \Pi_{s,x} 1_{\sigma=t} u(\eta_\sigma) \leq u(s, x).$$

Let τ_r be the first after r exit time from Q. If $r < s$, then $\{\tau_r > s\} \in \mathcal{F}[r,s]$. Clearly, $\tau_r = \tau$ $\Pi_{r,x}$-a.s. If $A \in \mathcal{F}[r,s]$, then $\{A, s < \tau_r\} \in \mathcal{F}[r,s]$ and, by the Markov property (2.4),

$$\int_{A,s<\tau_r} X_t \, d\Pi_{r,x} = \int_{A,s<\tau_r} \Pi_{\eta_s} X_t \, d\Pi_{r,x} \leq \int_{A,s<\tau_r} u(\eta_s) \, d\Pi_{r,x}.$$

For $s < t$, $X_t 1_{s<\tau_r} = X_t$ $\Pi_{r,x}$-a.s. and therefore $\int_A X_t \, d\Pi_{r,x} \leq \int_A X_s \, d\Pi_{r,x}$. Since X_t is $\mathcal{F}[r,t]$-measurable and $\Pi_{r,x}$-integrable, it is a supermartingale. □

4. Regular part of the boundary

4.1. Regular points. A point $\tilde{z} = (\tilde{r}, \tilde{x})$ of ∂Q is called *regular* if, for every $t > \tilde{r}$,

$$\Pi_{\tilde{z}}\{\eta_s \in Q \quad \text{for all } s \in (\tilde{r}, t)\} = 0. \tag{4.1}$$

.

THEOREM 4.1. *Let τ be the first exit time from Q. If a point $\tilde{z} = (\tilde{r}, \tilde{x}) \in \partial Q$ is regular then, for every $t > \tilde{r}$,*

$$\Pi_z\{\tau > t\} \to 0 \quad \text{as } z \in Q \quad \text{tends to } \tilde{z}. \tag{4.2}$$

PROOF. 1°. Fix t and put, for every $r \leq s < t$, $A(s,t) = \{\eta_u \in Q \quad \text{for } u \in (s,t)\}$ and $q_s^r(x) = \Pi_{r,x} A(s,t)$. Note that $q_r^r(x) = \Pi_{r,x}\{\tau > t\}$ for $(r,x) \in Q$. Therefore the conditions (4.1) and (4.2) are equivalent to the conditions $q_{\tilde{r}}^{\tilde{r}}(\tilde{x}) = 0$ and $q_r^r(x) \to 0$ as $(r,x) \to (\tilde{r}, \tilde{x})$.

2°. By the Markov property of ξ, for all $r \leq s < t$,

$$q_s^r(x) = \Pi_{r,x} \Pi_{s,\xi_s} A(s,t) = \int_E p(r,x;s,y) q_s^s(y) \, dy.$$

3°. It follows from 2° and 1.3.2 that $q_s^r(x)$ is continuous in (r,x) for $r < s$. Therefore, for every $\varepsilon > 0$, there exists a neighborhood U of $(\tilde{r}, \tilde{x})$ such that

$$|q_s^r(x) - q_s^{\tilde{r}}(\tilde{x})| < \varepsilon \quad \text{for all } (r,x) \in U.$$

4°. Clearly, $q_s^r(x) \downarrow q_r^r(x)$ as $s \downarrow r$.

5°. Suppose $\tilde{r} < t$. If $(\tilde{r}, \tilde{x})$ is regular, then, by 1°, $q_{\tilde{r}}^{\tilde{r}}(\tilde{x}) = 0$ and, by 4°, for every $\varepsilon > 0$, there exists $s \in (\tilde{r}, t)$ such that $q_s^{\tilde{r}}(\tilde{x}) < \varepsilon$. By 3°, if $(r,x) \in U$, then

$$q_r^r(x) \leq q_s^r(x) \leq q_s^{\tilde{r}}(\tilde{x}) + |q_s^r(x) - q_s^{\tilde{r}}(\tilde{x})| < 2\varepsilon.$$

By 1°, this implies (4.2). □

REMARK. The converse to Theorem 4.1 is also true: (4.2) implies (4.1). We do not use this fact. In an elliptic setting, it is proved in Chapter 13 of [**Dyn65**].

The role of condition (4.2) is highlighted by the following theorem:

THEOREM 4.2. *If (4.2) holds at $\tilde{z} \in \partial Q$ and if a bounded function f on ∂Q is continuous at $\tilde{z}$, then*

$$K_Q f(z) \to f(\tilde{z}) \quad \text{as } z \to \tilde{z}. \tag{4.3}$$

Informally, we have the following implications:

$$\begin{aligned}(4.4)\quad \{z=(r,x)\in Q \quad &\text{is close to } \tilde z=(\tilde r,\tilde x)\}\Longrightarrow\{\tau \quad \text{is close to } r\}\\ &\Longrightarrow \{\eta_\tau \quad \text{is close to } z \quad \text{and therefore close to } \tilde z\}\\ &\Longrightarrow \{f(\eta_\tau) \quad \text{is close to } f(\tilde z)\}\Longrightarrow \{\Pi_z f(\eta_\tau) \quad \text{is close to } f(\tilde z)\}.\end{aligned}$$

A rigorous proof is based on the following lemma.

LEMMA 4.1. *Fix $t\in\mathbb{R}$ and put*

$$D_r=\sup_{r<s<t}|\eta_s-\eta_r|. \tag{4.5}$$

For every $\varepsilon>0$, there exits $\delta>0$ such that

$$\Pi_{r,x}\{D_r>\varepsilon\}<\varepsilon \tag{4.6}$$

for all x and all $r\in(t-\delta,t)$.

PROOF. If $\xi_s=(\xi_s^1,\dots,\xi_s^d)$, then

$$D_r\le t-r+\sum_1^d D_r^i$$

where

$$\mathcal{D}_r^i=\sup_{r<s<t}|\xi_s^i-\xi_r^i|.$$

To prove the lemma it is sufficient to show that, for every $\varepsilon>0$, there exists $\delta>0$ such that

$$\Pi_{r,x}\{D_r^i>\varepsilon\}<\varepsilon \tag{4.7}$$

for all x and all $r\in(t-\delta,t)$. Each function $f_i(r,x)=x_i$ is exp-bounded on S and $\rho_i=\dot f_i+Lf_i=b_i$ (a coefficient in (1.1)). The conditions of Theorem 2.1 hold for f_i on every interval (t_1,t_2). Therefore, for every $r\in(t_1,t_2)$,

$$Y_s^i=\xi_s^i-\int_r^s\rho_i(\eta_u)\,du,\quad s\in(r,t_2)$$

is a martingale relative to $\mathcal{F}[r,s],\Pi_{r,x}$. Choose (t_1,t_2) which contains $[r,t]$. By Kolmogorov's inequality (see section 4.4 in the Appendix A)

$$\Pi_{r,x}\{\sup_{r<s<t}|Y_s^i-Y_r^i|>\delta\}\le\delta^{-2}\Pi_{r,x}|Y_t^i-Y_r^i|^2. \tag{4.8}$$

If $|\rho_i|\le c$, then

$$D_r^i\le\sup_{r<s<t}|Y_s^i-Y_r^i|+c(t-r). \tag{4.9}$$

On the other hand, since $(A+B)^2\le 2A^2+2B^2$ for all A,B, we have

$$\Pi_{r,x}|Y_t^i-Y_r^i|^2\le 2\Pi_{r,x}|\xi_t^i-\xi_r^i|^2+2c^2(t-r)^2. \tag{4.10}$$

The bound (1.8) implies that

(4.11)

$$\Pi_{r,x}(\xi_t^i-\xi_r^i)^2=\int_E p(r,x;t,y)(y_i-x_i)^2\,dy\le\int_E p(r,x;t,y)|y-x|^2\,dy\le C(t-r)$$

where C is a constant [depending on t_1,t_2]. The bound (4.7) follows from (4.8)–(4.11). □

Proof of Theorem 4.2. Let $z = (r, x), \tilde{z} = (\tilde{r}, \tilde{x})$. For every $t > \tilde{r}$,

$$\Pi_{r,x}\{|\eta_\tau - z| > \varepsilon\} \leq \Pi_{r,x}\{\tau \geq t\} + \Pi_{r,x}\{D_r > \varepsilon\}.$$

By Lemma 4.1, for every $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\Pi_{r,x}\{D_r \geq \varepsilon\} < \varepsilon$$

for all x and all $r \in (t - \delta, t)$. Choose $t \in (\tilde{r}, \tilde{r} + \delta/2)$. Then every $r \in (\tilde{r} - \delta/2, \tilde{r})$ belongs to $(t - \delta, t)$.

By Theorem 4.1,

$$\Pi_{r,x}\{\tau \geq t\} < \varepsilon$$

in a neighborhood U of $\tilde{z}$. Suppose that $r \in (t - \delta, t)$ and $z \in U$. Then

$$\Pi_{r,x}\{|\eta_\tau - z| > \varepsilon\} \leq 2\varepsilon.$$

Let V be the intersection of U with the ε-neighborhood of $\tilde{z}$. If $z \in V$, then

$$\Pi_{r,x}\{|\eta_\tau - \tilde{z}| > 2\varepsilon\} \leq \Pi_{r,x}\{|\eta_\tau - z| > \varepsilon\} \leq 2\varepsilon.$$

Suppose that N is an upper bound for $|f|$ and let $|f(z) - f(\tilde{z})| < \varepsilon$ for $z \in V$. Then

$$\Pi_{r,x}|f(\eta_\tau) - f(\tilde{z})| \leq 2N\Pi_{r,x}\{|\eta_\tau - \tilde{z}| > \delta\} + \varepsilon \leq (2N + 1)\varepsilon \quad \text{in } V$$

which implies (4.3). □

Theorem 4.3. *Suppose Q is bounded and all points of a total subset $\mathcal{T}$ of ∂Q are regular. If a function f is bounded and continuous on $\mathcal{T}$, then $u = K_Q f$ is a unique bounded solution of the problem*

$$\begin{aligned} \dot{u} + Lu &= 0 \quad \textit{in } Q, \\ u &= f \quad \textit{on } \mathcal{T}. \end{aligned} \tag{4.12}$$

Proof. Put $f = 0$ on $\partial Q \setminus \mathcal{T}$. By Theorems 3.1 and 4.2, $u = K_Q f$ is a solution of the problem (4.12). Clearly, u is bounded. For an arbitrary bounded solution v of (4.12), by Lemmas 3.1 and 3.2, $v = K_Q v = K_Q f = u$. □

We say that a function u is a *barrier* at $\tilde{z}$ if there exists a neighborhood U of $\tilde{z}$ such that $u \in \mathbb{C}^2(U) \cap C(\bar{U})$ and

$$\dot{u} + Lu \leq 0 \quad \text{in } Q \cap U, u(\tilde{z}) = 0, \quad u > 0 \quad \text{on } \bar{U} \cap \bar{Q} \quad \text{except } \tilde{z}. \tag{4.13}$$

Lemma 4.2. *The condition* (4.2) *holds if there exists a barrier u at $\tilde{z}$.* [8]

Proof. Put $V = Q \cap U$. For every $t > \tilde{r}$, the infimum β of u on the set $\partial V \cap S_{\geq t}$ is strictly positive. Let $\tau = \tau(V)$. By Chebyshev's inequality and (3.1),

$$\Pi_z\{\tau > t\} \leq \Pi_z\{u(\eta_\tau) \geq \beta\} \leq \Pi_z u(\eta_\tau)/\beta = K_V u(z)/\beta. \tag{4.14}$$

By 3.6.A, u is superparabolic in V. Denote by f the restriction of u to ∂V. Clearly, u belongs to the upper Perron class for f. By Theorem 3.1, $K_V f$ is the corresponding Perron solution. Hence, $K_V u = K_V f \leq u$, and (4.14) implies (4.2). □

By constructing a suitable barrier, we prove that (4.2) holds if $\tilde{z}$ can be touched from outside by a ball. More precisely, we have the following test.

[8] The existence of a barrier is also a necessary condition for the regularity of $\tilde{z}$. (See, e.g., [**Lie96**], Lemma 3.23 or [**Dyn65**], Theorem 13.6.)

THEOREM 4.4. *The property (4.2) holds at $\tilde{z} = (\tilde{r}, \tilde{x}) \in \partial Q$ if there exists $z' = (r', x')$ with $x' \neq \tilde{x}$ such that $|z - z'| > |\tilde{z} - z'|$ for all $z \in \bar{Q}$ sufficiently close to $\tilde{z}$ and different from $\tilde{z}$. In other words, $\tilde{z}$ is the only common point of three sets: $\bar{Q}$, a closed ball centered at z' and a neighborhood of $\tilde{z}$.*

PROOF. We claim that, if $\varepsilon = |\tilde{z} - z'|$ and if p is sufficiently large, then

$$u(z) = \varepsilon^{-2p} - |z - z'|^{-2p},$$

is a barrier at $\tilde{z}$. Clearly, $u(\tilde{z}) = 0$. There exists a neighborhood U of $\tilde{z}$ such that, for all $z \in \bar{U} \cap \bar{Q}$, $|z - z'| > \varepsilon$ and therefore $u(z) > 0$. We have

$$\dot{u} + Lu = A[-(p+1)B + C]$$

where

$$A = 2p|z - z'|^{-2(p+2)}, \quad B = \sum a_{ij}(x_i - x'_i)(x_j - x'_j),$$
$$C = |z - z'|^2 \sum [a_{ii} + b_i(x_i - x'_i)].$$

Note that $B \geq \kappa |x - x'|^2$ where κ is the ellipticity coefficient of L. Since a_{ij} and b_i are bounded, we see that, for sufficiently large p, $\dot{u} + Lu \leq 0$ in a neighborhood of $\tilde{z}$ assuming that $\tilde{x} \neq x'$. □

4.2. Regular open sets. We denote by $\partial_{reg} Q$ the set of all regular points of ∂Q and by $\partial_r Q$ the set of all interior (relative to ∂Q) points of $\partial_{reg} Q$. We say that Q is *regular* if $\partial_{reg} Q$ contains a total subset of ∂Q. A smaller class of *strongly regular* open sets is defined by the condition: $\partial_r Q$ is total in ∂Q.

For a cell C, $\partial_r C$ coincides with the set introduced in section 3.3. This set is relatively open in ∂C and therefore cells are strongly regular open sets.

Note that the following conditions are equivalent: (a) B is a relatively open subset of A; (b) $B = A \cap O$ where O is an open subset of S; (c) $A = B \cup F$ where F is a closed subset of S. Therefore, if B_i is a relatively open subset of A_i, $i = 1, 2$, then $B_1 \cap B_2$ is relatively open in $A_1 \cap A_2$ and $B_1 \cup B_2$ is relatively open in $A_1 \cup A_2$.

LEMMA 4.3. *If U is strongly regular, then $Q = U \cap Q_1$ is strongly regular for every open set Q_1 such that $\bar{U} \cap \partial Q_1 \subset \partial_r Q_1$.*

PROOF. The boundary ∂Q is the union of three sets $A_1 = \partial U \cap Q_1$, $A_2 = U \cap \partial Q_1$ and $A_3 = \partial U \cap \partial Q_1$. Sets $B_1 = \partial_r U \cap Q_1$, $B_2 = U \cap \partial_r Q_1$ and $B_3 = \partial_r U \cap \partial_r Q_1$ are relatively open in, respectively, A_1, A_2, A_3 and therefore $\mathcal{T} = B_1 \cup B_2 \cup B_3$ is relatively open in ∂Q. Every point of $\mathcal{T}$ is regular in ∂Q. It remains to show that $\mathcal{T}$ is total in ∂Q. Let τ be the first exit time from Q and let $z \in Q$. Since $\partial_r U$ is total in ∂U and $\partial_r U \cap (A_1 \cup A_3) \subset B_1 \cup B_3$, we have $\{\eta_\tau \in A_1 \cup A_3\} \subset \{\eta_\tau \in B_1 \cup B_3\}$ Π_z-a.s. On the other hand, $A_2 \subset \partial_r Q_1$ and therefore $A_2 = B_2$ and $\{\eta_\tau \in A_2\} = \{\eta_\tau \in B_2\}$. □

Now we introduce an important class of simple open sets. We start from closed cells $[a_0, b_0] \times [a_1, b_1] \times \cdots \times [a_d, b_d]$. We call finite unions of closed cells *simple compact sets*. We define a *simple open set* as the collection of all interior points of a simple compact set . The boundary ∂C of a cell $C = [a_0, b_0] \times [a_1, b_1] \times \cdots \times [a_d, b_d]$ consists of $2(d+1)$ d-dimensional faces. We distinguish two horizontal faces: the top $\{b_0\} \times [a_0, b_0] \times [a_1, b_1] \times \cdots \times [a_d, b_d]$ and the bottom $\{a_0\} \times [a_0, b_0] \times [a_1, b_1] \times \cdots \times [a_d, b_d]$. We call the rest side faces.

THEOREM 4.5. *Every simple open set is strongly regular. For an arbitrary open set Q, there exists a sequence of simple open sets exhausting Q.*

In the proof we use the following observations.

4.2.A. Let H be a $(d-1)$-dimensional affine subspace of $\mathbb{R}^d$ [that is the set of $x = (x_1, \dots, x_d)$ such that $a_1x_1 + \dots + a_dx_d = c$ for some constants $a_1, \dots, a_d$ not all equal to 0]. Then for all $r < t, x \in \mathbb{R}^d$, $\Pi_{r,x}\{\xi_t \in H\} = 0$. If H is a $(d-2)$-dimensional affine subspace, then $\Pi_{r,x}\{\xi_t \in H \text{ for some } t > r\} = 0$.

The first part holds because the probability distribution of ξ_t is absolutely continuous with respect to the Lebesgue measure. We leave the second part as an exercise for a reader.

4.2.B. If F is a $(d-1)$-dimensional face of a cell C, then

$$\Pi_{r,x}\{(t, \xi_t) \in F \quad \text{for some } t > r\} = 0 \quad \text{for all } (r, x) \in S.$$

This follows easily from 4.2.A.

Proof of Theorem 4.5. Every compact simple set A can be represented as the union of closed cells $C_1, \dots, C_n$ such that the intersection of every two distinct cells C_i, C_j is either empty or it is a common face of both cells. Let Q be the set of all interior points of A. Note that $\partial Q = \cup_1^N F_k$ where $F_1, \dots, F_N$ are d-dimensional cells which enter the boundary of exactly one of C_i. Clearly, the set F_k^0 of all points of F_k that do not belong to any $(d-1)$-dimensional face of any C_i is open in ∂Q. By 4.2.B, to prove that Q is strongly regular, it is sufficient to show that, for every k, either $\Pi_z\{\eta_\tau \in F_k^0\} = 0$ for all $z \in Q$ or $F_k^0 \subset \partial_{reg} Q$. Clearly, the first case takes place if F_k is the bottom of C_i. If F_k is the top of C_i, then, obviously, $F_k^0 \subset \partial_{reg} Q$. If F_k is a side face, then $F_k^0 \subset \partial_{reg} Q$ by Theorem 4.4.

It remains to construct sets Q_n. It is easy to reduce the general case to the case of a bounded Q. Suppose that Q is bounded. Put $\varepsilon_n = (d+1)^{1/2} 2^{-n}$. Consider a partition of $S = \mathbb{R}^{d+1}$ into cells with vertices in the lattice $2^{-n}\mathbb{Z}^{d+1}$ and take the union A_n of all cells whose ε_n-neighborhood are contained in Q. The set Q_n of all interior points of A^n is a simple open set. Clearly, the sequence Q_n exhaust Q. $\square$

For every two sets A, B, we denote by $d(A, B)$ the infimum of $d(a, b) = |a - b|$ over all $a \in A, b \in B$.

Suppose that Q is an open set and Γ is a closed subset of ∂Q. We say that a sequence of open sets $Q_n \uparrow Q$ is a (Q, Γ)-*sequence* if Q_n are bounded and strongly regular and if

$$\bar{Q}_n \uparrow \bar{Q} \setminus \Gamma; \qquad d(Q_n, Q \setminus Q_{n+1}) > 0. \tag{4.15}$$

LEMMA 4.4. *A (Q, Γ)-sequence exists if Γ contains all irregular points of ∂Q.*

PROOF. By Theorem 4.5, there exists a sequence of strongly regular open sets U_n exhausting $S \setminus \Gamma$. If Γ contains all irregular points of ∂Q, then, by Lemma 4.3, sets $Q_n = U_n \cap Q$ are strongly regular.

Note that $\bar{Q}_n \subset \bar{Q}$ and $\bar{Q}_n \cap \Gamma \subset \bar{U}_n \cap \Gamma \subset U_{n+1} \cap \Gamma = \emptyset$. Hence $\bar{Q}_n \subset \bar{Q} \setminus \Gamma$.

If K is a compact set disjoint from Γ, then $K \subset U_n$ for some n. Let $x \in \bar{Q} \setminus \Gamma$. For sufficiently small $\delta > 0$, $K = \{y : |y - x| \le \delta\}$ is disjoint from Γ. If $x_m \to x$ and $x_m \in Q$, then, for sufficiently large m, $x_m \in U_n \cap Q = Q_n$. Hence $x \in \bar{Q}_n$. This proves the first part of (4.15). The second part holds because $Q_n \subset \bar{U}_n, Q \setminus Q_{n+1} \subset U_{n+1}^c$ and $d(\bar{U}_n, U_{n+1}^c) > 0$. $\square$

5. Green's operators and equation $\dot{u} + Lu = -\rho$

5.1. Parts of a diffusion. A part $\tilde{\xi}$ of a diffusion ξ in an arbitrary open set $Q \subset S$ is obtained by killing ξ at the first exit time τ from Q. More precisely, we consider

$$\begin{aligned} \tilde{\xi}_t &= \xi_t \quad \text{for } t \in [\alpha, \tau), \\ &= \dagger \quad \text{for } t \geq \tau \end{aligned}$$

where † – "the cemetery" – is an extra point (not in E). The state space at time t is the t-section $Q_t = \{x : (t, x) \in Q\}$ of Q. We will show that $\tilde{\xi} = (\tilde{\xi}_t, \Pi_{r,x})$ is a Markov process with the transition density

$$p_Q(r, x; t, y) = p(r, x; t, y) - \Pi_{r,x} p(\tau, \xi_\tau; t, y) \quad \text{for } x \in Q_r, y \in Q_t. \tag{5.1}$$

[We set $p(r, x; t, y) = 0$ for $r \geq t$.]

Theorem 5.1. *For every Borel function $f \geq 0$ on Q,*

$$\Pi_{r,x} 1_{t<\tau} f(t, \xi_t) = \int_{Q_t} p_Q(r, x; t, y) f(t, y)\, dy \quad \textit{for all } x \in Q_r. \tag{5.2}$$

Moreover, for every $n = 1, 2, \ldots$ and for all $r < t_1 < \cdots < t_n$, $x \in Q_r$ and Borel sets $B_1, \ldots, B_n$,

$$\begin{aligned} &\Pi_{r,x}\{\tau > t_n, \xi_{t_1} \in B_1, \ldots, \xi_{t_n} \in B_n\} \\ &= \int_{B_1} dy_1 \ldots \int_{B_n} dy_n p_Q(r, x; t_1, y_1) p_Q(t_1, y_1; t_2, y_2) \ldots p_Q(t_{n-1}, y_{n-1}; t_n, y_n). \end{aligned} \tag{5.3}$$

We have:

$$p_Q(r, x; t, y) \geq 0 \quad \textit{for all } r < t, x \in Q_r, y \in Q_t; \tag{5.4}$$

$$\int_{Q_t} p_Q(r, x; t, y)\, dy \leq 1 \quad \textit{for all } r < t \quad \textit{and all } x \in Q_r; \tag{5.5}$$

$$\begin{aligned} &\int_{Q_s} p_Q(r, x; s, y) p_Q(s, y; t, z)\, dy = p_Q(r, x; t, z) \\ &\qquad \textit{for all } r < s < t \quad \textit{and all } x \in Q_r, z \in Q_t. \end{aligned} \tag{5.6}$$

Proof. If we set $f = 0$ outside Q, then

$$\Pi_{r,x} 1_{\tau=t} f(t, \xi_t) = 0 \tag{5.7}$$

because $f(\tau, \xi_\tau) = 0$. Therefore

$$\Pi_{r,x} 1_{t<\tau} f(t, \xi_t) = u(r, x) - v(r, x) \tag{5.8}$$

where

$$\begin{aligned} u(r, x) &= \Pi_{r,x} f(t, \xi_t), \\ v(r, x) &= \Pi_{r,x} 1_{\tau<t} f(t, \xi_t). \end{aligned}$$

By (1.9) and (2.15),

$$u(s, x) = \int_E p(s, x; t, y) f(t, y) dy. \tag{5.9}$$

By 2.1.B (applied to $\tau' = t$),

$$v(r, x) = \Pi_{r,x} 1_{\tau<t} f(t, \xi_t) = \Pi_{r,x} 1_{\tau<t} F(\tau, \xi_\tau) \tag{5.10}$$

where

$$F(s,w) = \Pi_{s,w} f(t,\xi_t) = \int_E p(s,w;t,y) f(t,y)\, dy.$$

Formula (5.2) follows from (5.8), (5.9), (5.10) and (5.1).

We establish (5.3) by induction by applying (5.2) and the Markov property (2.4).

To prove (5.4), (5.5) and (5.6), we establish that $p_Q(r,x;t,y)$ is continuous in $y \in Q_t$ for every $r < t$ and every $x \in Q_r$. This follows from a similar property of $p(r,x;t,y)$ because $(\tau,\xi_\tau) \in \partial Q$ $\Pi_{r,x}$-a.s. and, by 1.3.B, $p(\tau,\xi_\tau;t,y)$ is uniformly bounded in a neighborhood of each $y \in Q_t$.

Formula (5.2) implies (5.4) and (5.5). To prove (5.6), we note that, for $r < s < t$,

$$\Pi_{r,x}\{\tau > t, \xi_s \in Q_s, \xi_t \in B\} = \Pi_{r,x}\{\tau > t, \xi_t \in B\}$$

By (5.3), this implies that the functions of z in both parts of (5.6) have the same integrals over B. Therefore (5.6) holds for almost all z. It holds for all z because both parts are continuous in z. □

Formula (5.3) is an analog of formula (2.1) for a process with a random death time τ.

5.2. Green's functions. We prove that function p_Q defined by (5.1) has properties similar to 1.3.A–1.3.C. We call it Green's function for operator $\dot{u} + Lu$ in Q.

5.2.A. If $Q_{<t} = S_{<t} \cap Q$ is bounded, then for every $(t,y) \in Q$, function $u(r,x) = p_Q(r,x;t,y)$ is a solution of (3.4) in $Q_{<t}$.

Indeed, for every $(t,y) \in Q$, $u = \tilde{u} - K_Q\tilde{u}$ where $\tilde{u}(r,x) = p(r,x;t,y)$. By 1.3.A, $\tilde{u}$ satisfies (3.4) in $S_{<t}$. By 1.3.B, $\tilde{u}$ is bounded on ∂Q and, by Theorem 3.1 $K_Q\tilde{u}$ satisfies (3.4) in $Q_{<t}$.

5.2.B. For every $t_1 < t_2$ and every $\delta > 0$, function $p_Q(r,x;t,y)$ is bounded on the intersection of Q with the set $\{t_1 < r < t < t_2, t - r + |y - x| \geq \delta\}$.

This follows from 1.3.B because $p_Q \leq p$.

5.2.C. If φ is bounded and continuous at $a \in Q_t$, then

$$\int_{Q_t} p_Q(r,x;t,y)\varphi(y)\, dy \to \varphi(a) \quad \text{as } (r,x) \to (t,a), (r,x) \in Q_{<t}.$$

Proof. By (5.1) and (5.10),

$$0 \leq \int_{Q_t} (p - p_Q)(r,x;t,y)\varphi(y)\, dy = \Pi_{r,x} \int_{Q_t} p(\tau,\xi_\tau;t,y)\varphi(y)\, dy = \Pi_{r,x} 1_{\tau<t}\varphi(\xi_t),$$

and 5.2.C follows from 1.3.C if we prove that

$$u(r,x) = \Pi_{r,x}\{\tau < t\} \to 0 \quad \text{as } (r,x) \to (t,a), (r,x) \in Q_{<t}. \tag{5.11}$$

Note that $u = K_{Q_{<t}} 1_{S_{<t}}$. If $a \in Q_t$, then (t,a) is a regular point of $\partial Q_{<t}$. Since $1_{S_{<t}}$ is continuous and equal to 0 at (t,a), formula (5.11) follows from Theorem 4.2. □

There exists a simple relation between p_Q and $p_{Q'}$ for $Q' \subset Q$:

$$p_{Q'}(r, x; t, y) = p_Q(r, x; t, y) - \Pi_{r,x} p_Q(\tau', \xi_{\tau'}; t, y) \tag{5.12}$$

where $\tau' = \tau(Q')$. Indeed, put $f(r, x) = p(r, x; t, y)$ and denote by $f_Q, f_{Q'}$ the functions obtained in a similar way from $p_Q, p_{Q'}$. By (5.1), $f_Q = f - K_Q f$ and, by (3.2), $K_{Q'} f_Q = K_{Q'} f - K_Q f$. Hence, $f_Q - f_{Q'} = -K_Q f + K_{Q'} f = K_{Q'} f_Q$.

5.3. Green's operators. Green's operator in an arbitrary open set Q is defined by the formula

$$G_Q \rho(r, x) = \Pi_{r,x} \int_r^\tau \rho(s, \xi_s)\, ds \tag{5.13}$$

(cf. (2.7)). By (5.2),

$$G_Q \rho(r, x) = \int_r^\infty ds \int_{Q_s} p_Q(r, x; s, y) \rho(s, y) dy. \tag{5.14}$$

If $Q' \subset Q$, then, by (5.14) and (5.12),

$$G_Q = G_{Q'} + K_{Q'} G_Q. \tag{5.15}$$

5.3.A. Suppose that $Q \subset S_I = I \times E$, I is a finite interval and ρ is bounded. Then function $w = G_Q \rho$ belongs to $\mathbb{C}^\lambda(Q)$. If $\rho \in \mathbb{C}^\lambda(Q)$, then $w \in \mathbb{C}^2(Q)$ and it is a solution of the equation

$$\dot{w} + Lw = -\rho \quad \text{in } Q. \tag{5.16}$$

If ρ is bounded and if $\tilde{z}$ is a regular point of ∂Q, then

$$w(z) \to 0 \quad \text{as } z \to \tilde{z}. \tag{5.17}$$

Proof. Note that $w = v - K_Q v$ where v is given by (1.11). Since $K_Q v$ is parabolic in Q, the first part of 5.3.A follows from 1.3.3.

If $z = (r, x)$ and if N is an upper bound of $|\rho|$, then, for every $\varepsilon > 0$, $|w(z)| \leq N[(t - r)\Pi_z\{\tau > r + \varepsilon\} + \varepsilon]$ and therefore (5.17) follows from Theorem 4.1. □

5.3.B. Let τ be the first exit time from an arbitrary open set Q. If $\rho \geq 0$ and $w = G_Q \rho$ is finite at a point $z \in Q$, then

$$\lim_{t \uparrow \tau} w(\eta_t) = 0 \quad \Pi_z\text{-a.s.} \tag{5.18}$$

Proof. Let $z = (r, x)$. We can assume that $\rho \geq 0$. We prove that $M_t = 1_{t<\tau} w(\eta_t), t \in [r, \infty)$ is a supermartingale relative to $\mathcal{F}[r, t], \Pi_{r,x}$. To this end, we consider a bounded positive $\mathcal{F}[r, t]$-measurable function X and we note that, by the Markov property (2.4),

$$\Pi_{r,x} X 1_{t<\tau} \int_t^\tau \rho(s, \xi_s)\, ds = \Pi_{r,x} X 1_{t<\tau} \Pi_{t,\xi_t} \int_t^\tau \rho(s, \xi_s)\, ds = \Pi_{r,x} X 1_{t<\tau} w(t, \xi_t).$$

Hence $\Pi_{r,x} X M_t \leq \Pi_{r,x} X M_s$ for $r \leq s \leq t$. Since M_t is $\mathcal{F}[r, t]$-measurable and $\Pi_{r,x}$-integrable, our claim is proved.

Since M_t is right continuous, a limit $M_{\tau-}$ as $t \uparrow \tau$ exists $\Pi_{r,x}$-a.s. (see 4.3.C in the Appendix A). Suppose Q_n exhaust Q and let τ_n be the first exit time from Q_n.

By (5.15), $w = G_{Q_n}\rho + K_{Q_n}w$. Since $G_{Q_n}\rho \uparrow G_Q\rho$, we conclude that $K_{Q_n}w \downarrow 0$ and

$$\Pi_{r,x}M_{\tau-} = \Pi_{r,x}\lim w(\tau_n, \xi_{\tau_n}) \le \lim \Pi_{r,x}w(\tau_n, \xi_{\tau_n}) = \lim K_{Q_n}w = 0.$$

□

5.3.C. If Q, ρ and f are bounded and if f is continuous on $\partial_{reg}Q$, then

$$v = G_Q\rho + K_Q f \tag{5.19}$$

is a solution of the problem

$$\begin{aligned} \dot v + Lv &= -\rho \quad \text{in } Q, \\ v &= f \quad \text{on } \partial_{reg}Q. \end{aligned} \tag{5.20}$$

This follows from 5.3.A and Theorems 3.1 and 4.2.

5.3.D. Let ρ be bounded. A function w is a solution of equation (5.16) if and only if w is locally bounded and, for every $U \Subset Q$,

$$w = G_U\rho + K_U w. \tag{5.21}$$

Proof. Suppose that w satisfies (5.16). By Theorem 4.5, for an arbitrary U there exists a sequence of regular open sets $U_n \uparrow U$. Since w is bounded and continuous on $\bar U$, we have $K_{U_n}w \to K_U w$. Also $G_{U_n}\rho \to G_U\rho$. Therefore it is sufficient to prove (5.21) for a regular U.

By 5.3.A and Theorem 4.3, $u = G_U\rho + K_U w - w$ is a solution of the problem

$$\begin{aligned} \dot u + Lu &= 0 \quad \text{in } U, \\ u &= 0 \quad \text{on } \partial_{reg}U. \end{aligned} \tag{5.22}$$

By 3.6.C, $u = 0$.

If w satisfies (5.21) and is bounded on $\bar U$, then the equation (5.16) holds on U by 5.3.A and Theorem 3.1. □

5.3.E. Suppose that solutions w_n of (5.16) converge to w at every point of Q. If w_n are locally uniformly bounded, then w also satisfies (5.16).

This follows from 5.3.D (cf. the proof of Lemma 3.3).

6. Notes

Our treatment of diffusions is in spirit of the book [**Dyn65**]. However, in this book only time-homogeneous case was considered. Inhomogeneous diffusions were covered in [**Dyn93**]. In particular, one can found there a probabilistic formula for the Perron solutions, the improved maximum principle and an approximation of arbitrary domains by simple domains. A concept of strongly regular domains was introduced in [**Dyn98a**]. This class of domains plays a special role in the theory of semilinear partial differential equations (see, Chapter **5**).

A fundamental monograph of Doob [**Doo84**] contains the most complete presentation of the connections between the Brownian motion and classical potential theory related to the Laplace equation. Bibliographical notes in [**Doo84**] should be consulted for the early history of this subject. A special role in the book is played by martingale theory. Much of this theory was created by Doob.

Martingale are the principal tool used by Stroock and Varadhan to develop a new approach to diffusions. A construction of diffusions by solving a martingale problem is presented in their monograph [**SV79**].

A direct construction of the paths of diffusions by solving stochastic differential equations is due to Itô [**Itô51**]. A modern presentation of Itô's calculus and its applications is given in the books of Ikeda and Watanabe [**IW81**] and Rogers and Williams [**RW87**].

CHAPTER 3

Branching exit Markov systems

In this chapter we introduce a general model — BEM systems — which is the basis for the theory of superprocesses and, in particular, superdiffusions to be developed in the next chapters. A BEM system describes a mass distribution of a random cloud started from a distribution μ and frozen at the exit from Q. Mathematically, this is a family X of random measures (X_Q, P_μ) in a space S. The parameter Q takes values in a class of subsets of S, μ is a measure on S, X_Q is a function of $\omega \in \Omega$ and P_μ is a probability measure on Ω. A Markov property is defined with the role of "past" and "future" played by $Q' \subset Q$ and $Q'' \supset Q$. [This definition can be applied to a parameter Q taking values in any partially ordered set.] We consider systems which combine the Markov property with a branching property which means, heuristically, an absence of interaction between any parts of the random cloud described by X.

We start from historical roots of the concept of branching. Then we introduce branching particle systems (they were described on a heuristic level in Chapter **1**) and we use them to motivate a general definition of BEM systems. The transition operators V_Q play a role similar to the role of the transition functions in the theory of Markov processes. We investigate properties of these operators and we show how a BEM system can be constructed starting from a family of operators V_Q. At the end of the chapter some basic properties of BEM systems are proved.

1. Introduction

1.1. Simple models of branching. The first probabilistic model of branching appeared in 1874 in the problem of the family name extinction posed by Francis Galton and solved by H. W. Watson [**WG74**]. Galton's motivation was to evaluate a conjecture that the extinction of prominent families is more likely than the extinction of ordinary ones. He suggested to start from probabilities p_n for a man to have n sons evaluated by the demographical data for the general population. The problem consisted in computation of the probability of extinction after k generations. Watson's solution contained an error but he introduced a tool of fundamental importance for the theory of branching. The principal observation was: if

$$\varphi(z) = \sum_0^\infty p_n z^k$$

is the generating function for the number of sons, then the generating function φ_k for the number of descendants in the k-th generation can be evaluated by the recursive formula

$$\varphi_{k+1} = \varphi(\varphi_k). \tag{1.1}$$

The Galton-Watson model and its modifications found many applications in biology, physics, chemistry... [1]

A model of branching with a continuous time parameter was suggested in 1947 in [**KD47**] (an output of a Kolmogorov's seminar held at Moscow University in 1946-47). Consider a particle system and assume that a single particle produces, during time interval (r,t), $k = 0,1,2,\dots$ particles with probability $p_k(r,t)$. Generating functions

$$\varphi(r,t;z) = \sum_0^\infty p_k(r,t)z^k$$

satisfy the condition

$$\varphi(r,t;z) = \varphi(r,s;\varphi(s,t;z)) \quad \text{for } r < s < t. \tag{1.2}$$

Suppose that

$$p_k(r-h,r) = a_k(r)h + o(h) \quad \text{for } k \neq 1,$$
$$p_1(r-h,r) = 1 + a_1(r)h + o(h)$$

as $h \downarrow 0$ and let

$$\Phi(r;z) = \sum_0^\infty a_k(r)z^k. \tag{1.3}$$

We arrive at a differential equation

$$\frac{\partial\varphi(r,t;z)}{\partial r} + \Phi(r,\varphi(r,t;z)) = 0 \quad \text{for } r < t \tag{1.4}$$

with a boundary condition

$$\Phi(r,t;z) \to z \quad \text{as } r \uparrow t. \tag{1.5}$$

Equation (1.4) and a linear equation $\dot{u} + Lu = 0$ considered in Chapter **2** are particular cases of a semilinear parabolic equation

$$\dot{u} + Lu = \psi(u). \tag{1.6}$$

A probabilistic approach to (1.6) is based on a model which involves both L-diffusion and branching.

1.2. Exit systems associated with branching particle systems. Consider a system of particles moving in E according to the following rules:

(1) The motion of each particle is described by a right continuous strong Markov process ξ.

(2) A particle dies during time interval $(t, t+h)$ with probability $kh + o(h)$, independently on its age.

(3) If a particle dies at time t at point x, then it produces n new particles with probability $p_n(t,x)$.

(4) The only interaction between the particles is that the birth time and place of offspring coincide with the death time and place of their parent.

[Assumption (2) implies that the life time of every particle has an exponential probability distribution with the mean value $1/k$.]

We denote by $P_{r,x}$ the probability law corresponding to a process started at time r by a single particle located at point x. Suppose that particles stop to move

[1] More on an early history of the branching processes can be found in [**Har63**].

and to procreate outside an open subset Q of S. In other words, we observe each particle at the first, in the family history, [2] exit time from Q. The exit measure from Q is defined by the formula

$$X_Q = \delta_{(t_1,y_1)} + \cdots + \delta_{(t_n,y_n)}$$

where $(t_1, y_1), \ldots, (t_n, y_n)$ are the states of frozen particles and $\delta_{(t,y)}$ means the unit measure concentrated at (t, y). We also consider a process started by a finite or infinite sequence of particles that "immigrate" at times r_i at points x_i. There is no interaction between their descendants and therefore the corresponding probability law is the convolution of P_{r_i,x_i}. We denote it P_μ where

$$\mu = \sum \delta_{(r_i,x_i)}$$

is a measure on S describing the immigration. We arrive at a family X of random measures $(X_Q, P_\mu), Q \in \mathbb{O}, \mu \in \mathbb{M}$ where $\mathbb{O}$ is a class of open subsets of S and $\mathbb{M}$ is the class of all integer-valued measures on S. Family X is a special case of a branching exit Markov system. A general definition of such systems is given in the next section.

1.3. Branching exit Markov systems. A *random measure* on a measurable space $(S, \mathcal{B}_S)$ is a pair (X, P) where $X(\omega, B)$ is a kernel [3] from an auxiliary measurable space $(\Omega, \mathcal{F})$ to $(S, \mathcal{B}_S)$ and P is a probability measure on $\mathcal{F}$. We assume that S is a Borel subset of a compact metric space and $\mathcal{B}_S$ is the class of all Borel subsets of S.

Suppose that:

(i) $\mathbb{O}$ is a subset of σ-algebra $\mathcal{B}_S$;

(ii) $\mathbb{M}$ is a class of measures on $(S, \mathcal{B}_S)$ which contains all measures $\delta_y, y \in S$.

(iii) to every $Q \in \mathbb{O}$ and every $\mu \in \mathbb{M}$, there corresponds a random measure (X_Q, P_μ) on $(S, \mathcal{B}_S)$.

Condition (ii) is satisfied, for instance, for the class $\mathcal{M}(S)$ of all finite measures and for the class $\mathbb{N}(S)$ of all integer-valued measures.

We use notation $\langle f, \mu \rangle$ for the integral of f with respect to a measure μ. Denote by $\mathbb{Z}$ the class of functions

$$Z = \exp\{-\sum_1^n \langle f_i, X_{Q_i} \rangle\} \tag{1.7}$$

where $Q_i \in \mathbb{O}$ and f_i are positive measurable functions on S. We say that $X = (X_Q, P_\mu), Q \in \mathbb{O}, \mu \in \mathbb{M}$ is a *branching system* if

1.3.A. For every $Z \in \mathbb{Z}$ and every $\mu \in \mathbb{M}$,

$$P_\mu Z = e^{-\langle u, \mu \rangle} \tag{1.8}$$

where

$$u(y) = -\log P_y Z \tag{1.9}$$

and $P_y = P_{\delta_y}$.

[2]By the family history we mean the path of a particle and all its ancestors. If the family history starts at (r, x), then the probability law of this path is $\Pi_{r,x}$.

[3]A kernel from a measurable space $(E_1, \mathcal{B}_1)$ to a measurable space $(E_2, \mathcal{B}_2)$ is a function $K(x, B)$ such that $K(x, \cdot)$ is a measure on $\mathcal{B}_2$ for every $x \in E_1$ and $K(\cdot, B)$ is an $\mathcal{B}_1$-measurable function for every $B \in \mathcal{B}_2$.

Condition 1.3.A (we call it the *continuous branching property*) implies that

$$P_\mu Z = \prod P_{\mu_n} Z$$

for all $Z \in \mathbb{Z}$ if $\mu_n, n = 1, 2, \dots$ and $\mu = \sum \mu_n$ belong to $\mathbb{M}$.

A family X is called an *exit system* if:

1.3.B. For all $\mu \in \mathbb{M}$ and $Q \in \mathbb{O}$,

$$P_\mu\{X_Q(Q) = 0\} = 1.$$

1.3.C. If $\mu \in \mathbb{M}$ and $\mu(Q) = 0$, then

$$P_\mu\{X_Q = \mu\} = 1.$$

Finally, we say that X is a *branching exit Markov [BEM] system*, if $X_Q \in \mathbb{M}$ for all $Q \in \mathbb{O}$ and if, in addition to 1.3.A–1.3.C, we have:

1.3.D. [Markov property.] Suppose that $Y \geq 0$ is measurable with respect to the σ-algebra $\mathcal{F}_{\subset Q}$ generated by $X_{Q'}, Q' \subset Q$ and $Z \geq 0$ is measurable with respect to the σ-algebra $\mathcal{F}_{\supset Q}$ generated by $X_{Q''}, Q'' \supset Q$.

Then

$$P_\mu(YZ) = P_\mu(Y P_{X_Q} Z). \tag{1.10}$$

It follows from the principles (1)-(4) stated at the beginning of section 1.2 that conditions 1.3.A–1.3.D hold for the systems of random measures associated with branching particle systems. For them $S = \mathbb{R} \times E$, $\mathbb{M} = \mathbb{N}(S)$ and $\mathbb{O}$ is a class of open subsets of S.

1.4. Transition operators. Let $X = (X_Q, P_\mu), Q \in \mathbb{O}, \mu \in \mathbb{M}$ be a family of random measures. Denote by $\mathbb{B}$ the set of all bounded positive $\mathcal{B}_S$-measurable functions. Operators $V_Q, Q \in \mathbb{O}$ acting on $\mathbb{B}$ are called the *transition operators of* X if, for every $\mu \in \mathbb{M}$ and every $Q \in \mathbb{O}$,

$$P_\mu e^{-\langle f, X_Q\rangle} = e^{-\langle V_Q(f), \mu\rangle}. \tag{1.11}$$

If X is a branching system, then (1.11) follows from the formula

$$V_Q(f)(y) = -\log P_y e^{-\langle f, X_Q\rangle} \quad \text{for } f \in \mathbb{B}. \tag{1.12}$$

In this chapter we establish sufficient conditions for operators V_Q to be transition operators of a branching exit Markov system. In the next chapter we study a special class of BEM systems which we call superprocesses.

A link between operators V_Q and a BEM system X is provided by a family of transition operators of higher order $V_{Q_1,\dots,Q_n}$. We call it a $\mathbb{V}$-family.

2. Transition operators and $\mathbb{V}$-families

2.1. Transition operators of higher order. Suppose that

$$P_\mu \exp[-\langle f_1, X_{Q_1}\rangle - \dots - \langle f_n, X_{Q_n}\rangle], \\ = \exp[-\langle V_{Q_1,\dots,Q_n}(f_1, \dots, f_n), \mu\rangle] \tag{2.1}$$

for all $\mu \in \mathbb{M}$, $f_1, \dots, f_n \in \mathbb{B}$ and $Q_1, \dots, Q_n \in \mathbb{O}$. Then we say that operators $V_{Q_1,\dots,Q_n}$ are the *transition operators of order* n *for* X. Condition (2.1) is equivalent to the assumption that X is a branching system and that

$$V_{Q_1,\dots,Q_n}(f_1,\dots,f_n)(y) = -\log P_y \exp[-\langle f_1, X_{Q_1}\rangle - \dots - \langle f_n, X_{Q_n}\rangle], \quad f_1,\dots,f_n \in \mathbb{B}, y \in S. \tag{2.2}$$

[For $n = 1$, formulae (2.1)–(2.2) coincide with (1.11)–(1.12).]

We use the following abbreviations. For every finite subset $I = \{Q_1, \dots, Q_n\}$ of $\mathbb{O}$, we put

$$X_I = \{X_{Q_1}, \dots, X_{Q_n}\}, f_I = \{f_1, \dots, f_n\}, \quad \langle f_I, X_I\rangle = \sum_{i=1}^n \langle f_i, X_{Q_i}\rangle. \tag{2.3}$$

In this notation, formulae (2.2) and (2.1) can be written as

$$V_I(f_I)(y) = -\log P_y e^{-\langle f_I, X_I\rangle} \tag{2.4}$$

and

$$P_\mu e^{-\langle f_I, X_I\rangle} = e^{-\langle V_I(f_I),\mu\rangle}. \tag{2.5}$$

If X satisfies condition 1.3.C, then:

2.1.A. For every $Q_i \in I$, $V_I(f_I) = f_i + V_{I_i}(f_{I_i})$ on Q_i^c where I_i is the set obtained from I by dropping Q_i.

Indeed,

$$\langle f_I, X_I\rangle = \langle f_i, X_{Q_i}\rangle + \langle f_{I_i}, X_{I_i}\rangle$$

and $\langle f_i, X_{Q_i}\rangle = f_i(y)$ P_y-a.s. if $y \in Q_i^c$.

For a branching exit system X, the Markov property 1.3.D is equivalent to:

2.1.B. If $Q \subset Q_i$ for all $Q_i \in I$, then

$$V_Q V_I = V_I. \tag{2.6}$$

Formula (2.6) can be rewritten in the form

$$V_I(f_I) = V_Q[1_{Q^c} V_I(f_I)] \quad \text{for all } f_I. \tag{2.7}$$

Proof. It follows from (2.5) that

$$e^{-\langle V_Q V_I(f_I),\mu\rangle} = P_\mu e^{-\langle V_I(f_I), X_Q\rangle} = P_\mu P_{X_Q} e^{-\langle f_I, X_I\rangle}.$$

If $Q \subset Q_i$ for all $Q_i \in I$, then $\langle f_I, X_I\rangle \in \mathcal{F}_{\supset Q}$ and 1.3.D implies that the right side is equal to

$$P_\mu e^{-\langle f_I, X_I\rangle} = e^{-\langle V_I(f_I),\mu\rangle}.$$

Hence (2.6) follows from 1.3.D. By 1.3.B, for every F, the value of $V_Q(F)$ does not depend on the values of F on Q. Therefore (2.7) and (2.6) are equivalent.

To deduce 1.3.D from 2.1.B, it is sufficient to prove (1.10) for

$$Y = e^{-\langle f_I, X_I\rangle}, \quad Z = e^{-\langle f_{\tilde I}, X_{\tilde I}\rangle}$$

where $I = \{Q_1, \dots, Q_n\}$, $\tilde I = \{\tilde Q_1, \dots, \tilde Q_m\}$ with $Q_i \subset Q \subset \tilde Q_j$. Note that $YZ \in \mathbb{Z}$. By 1.3.A, the same is true for $Y P_{X_Q} Z$. Therefore (1.10) will follow from 1.3.A if we check that it holds for all $\mu = \delta_y$. We use the induction in n. The condition (2.6) implies

$$P_\mu Z = P_\mu P_{X_Q} Z. \tag{2.8}$$

Hence, (1.10) holds for $n = 0$. Suppose it holds for $n-1$. If $y \in Q_i^c$, then, by 1.3.C, $P_y\{Y = e^{-f_i(y)} Y_i\} = 1$ where $Y_i = e^{-\langle f_{I_i}, X_{I_i}\rangle}$, and we have

$$P_y Y Z = e^{-f_i(y)} P_y Y_i Z = e^{-f_i(y)} P_y (Y_i P_{X_Q} Z) = P_y(Y P_{X_Q} Z)$$

by the induction hypothesis. Hence (1.10) holds for δ_y with y not in the intersection Q_I of $Q_i \in I$. For an arbitrary y, by (2.8), $P_y Y Z = P_y P_{X_{Q_I}} Y Z$. By 1.3.B, X_{Q_I} is concentrated, P_y-a.s. on Q_I^c and therefore

$$P_{X_{Q_I}} Y Z = P_{X_{Q_I}} (Y P_{X_Q} Z).$$

We conclude that

$$P_y Y Z = P_y P_{X_{Q_I}} (Y P_{X_Q} Z) = P_y (Y P_{X_Q} Z).$$

□

Transition operators of order n for a BEM system can be expressed through transition operators of order $n-1$ by the formulae

$$V_I(f_I) = f_i + V_{I_i}(f_{I_i}) \quad \text{on } Q_i^c \quad \text{for every } Q_i \in I, \tag{2.9}$$

$$V_I(f_I) = V_{Q_I}[1_{Q_I^c} V_I(f_I)] \quad \text{where } Q_I \quad \text{is the intersection of all } Q_i \in I. \tag{2.10}$$

Formula (2.9) (equivalent to 2.1.A) defines the values of $V_I(f_I)$ on Q_I^c. Formula (2.10) follows from (2.7). It provides an expression for all values of $V_I(f_I)$ through its values on Q_I^c.

Conditions (2.9)–(2.10) can be rewritten in the form

$$V_I = V_{Q_I} \tilde{V}_I \tag{2.11}$$

where

$$\tilde{V}_I(f_I) = \begin{cases} f_i + V_{I_i}(f_{I_i}) & \text{on } Q_i^c, \\ 0 & \text{on } Q_I. \end{cases} \tag{2.12}$$

2.2. Properties of V_Q. We need the following simple lemma.

LEMMA 2.1. *Let Y be a positive random variable and let $0 \le c \le \infty$. If $Pe^{-\lambda Y} \le e^{-\lambda c}$ for all $\lambda > 0$, then $P\{Y \ge c\} = 1$. If, in addition $Pe^{-Y} = e^{-c}$, then $P\{Y = c\} = 1$.*

PROOF. If $c = \infty$, then, P-a.s., $e^{-\lambda Y} = 0$ and therefore $Y = \infty$. If $c < \infty$, then $Pe^{-\lambda(Y-c)} \le 1$ and, by Fatou's lemma, $P\{\lim_{\lambda\to\infty} e^{-\lambda(Y-c)}\} \le 1$. Hence, $P\{Y \ge c\} = 1$. The second part of the lemma follows from the first one. □

THEOREM 2.1. *Transition operators of an arbitrary system of finite random measures X satisfy the condition:*

2.2.A. For all $Q \in \mathbb{O}$,

$$V_Q(f_n) \to 0 \quad \textit{as } f_n \downarrow 0. \tag{2.13}$$

A branching system X is a branching exit system if and only if:

2.2.B.

$$V_Q(f) = V_Q(\tilde{f}) \quad \textit{if } f = \tilde{f} \quad \textit{on } Q^c.$$

2.2.C. For every $Q \in \mathbb{O}$ and every $f \in \mathbb{B}$,

$$V_Q(f) = f \quad \textit{on } Q^c.$$

PROOF. 1°. Property 2.2.A is obvious. It is clear that 1.3.B implies 2.2.B and 1.3.C implies 2.2.C.

2°. If 2.2.B holds, then $V_Q(1_Q) = V_Q(0) = 0$ and therefore $P_y e^{-X_Q(Q)} = 1$ which implies 1.3.B.

3°. It follows from 2.2.C and (1.11) that, if $\mu(Q) = 0$, then, for all $f \in \mathbb{B}$ and all $\lambda > 0$,

$$P_\mu e^{-\lambda\langle f, X_Q\rangle} = e^{-\lambda\langle f,\mu\rangle}$$

and, by Lemma 2.1,

$$\langle f, X_Q\rangle = \langle f, \mu\rangle \quad P_\mu\text{-a.s.} \tag{2.14}$$

Since there exists a countable family of $f \in \mathbb{B}$ which separate measures, 1.3.C follows from (2.14).

□

2.3. $\mathbb{V}$-families. We call a collection of operators V_I a $\mathbb{V}$-*family* if it satisfies conditions (2.9)–(2.10) [equivalent to (2.11)–(2.12)] and 2.2.A. We say that a $\mathbb{V}$-family and a system of random measures correspond to each other if they are connected by formula (2.1).

THEOREM 2.2. *Suppose that operators $V_Q, Q \in \mathbb{O}$ satisfy conditions 2.2.A–2.2.C and the condition*

2.3.A. For all $Q \subset \tilde{Q} \in \mathbb{O}$,

$$V_Q V_{\tilde{Q}} = V_{\tilde{Q}}.$$

Then there exists a $\mathbb{V}$-family $\{V_I\}$ such that $V_I = V_Q$ for $I = \{Q\}$.

PROOF. Denote by $|I|$ the cardinality of I. For $|I| = 1$, operators V_I are defined. Suppose that V_I, subject to conditions (2.9)–(2.10), are already defined for $|I| < n$. For $|I| = n$, we define V_I by (2.9)–(2.10). This is not contradictory because

$$f_i + V_{I_i}(f_{I_i}) = f_j + V_{I_j}(f_{I_j}) = f_i + f_j + V_{I_{ij}}(f_{I_{ij}}) \quad \text{on } Q_i^c \cap Q_j^c.$$

By 2.2.B it is legitimate to define $V_I(f_I)$ on Q_I by (2.10). □

3. From a $\mathbb{V}$-family to a BEM system

3.1. P-matrices and N-matrices. First, we prepare some algebraic and analytic tools.

Suppose that a symmetric $n \times n$ matrix (a_{ij}) satisfies the condition: for all real numbers $t_1, \dots, t_n$,

$$\sum_{i,j=1}^{n} a_{ij} t_i t_j \geq 0.$$

In algebra, such matrices are called positive semidefinite. Some authors (e.g., [**BJR84**]) use the name positive definite. We resolve this controversy by using a short name a P-*matrix*.

We need another class of matrices which are called negative definite in [**BJR84**]. [This is inconsistent with the common usage in algebra where negative definite means $(-1)\times$ positive definite.] We prefer again a short name. We call an $n \times n$ symmetric matrix an N-*matrix* if

$$\sum_{i,j=1}^{n} a_{ij} t_i t_j \le 0$$

for every $n \ge 2$ and all $t_1, \dots, t_n \in \mathbb{R}$ such that $\sum t_i = 0$.

The following property of these classes is obvious:

3.1.A. The classes P and N are closed under entry wise convergence. Moreover, they are convex cones in the following sense: if $(B, \mathcal{B}, \eta)$ is a measure space, if $a_{ij}(b)$ is a P-matrix (N-matrix) for all $b \in B$ and if $a_{ij}(b)$ are η-integrable, then

$$a_{ij} = \int a_{ij}(b)\eta(db)$$

is also a P-matrix (respectively, an N-matrix).

Here are some algebraic properties of both classes.

(i) A matrix (a_{ij}) is a P-matrix if and only if it has a representation

$$a_{ij} = \sum_{k=1}^{m} q_{ik} q_{jk}$$

where $m \le n$.

This follows from the fact that a quadratic form is positive semidefinite if and only if it can be transformed by a linear transformation to the sum of $m \le n$ squares.

(ii) If (a_{ij}) and (b_{ij}) are P-matrices, then so is the matrix $c_{ij} = a_{ij} b_{ij}$.

Indeed, by using (i), we get

$$\sum_{ij} c_{ij} t_i t_j = \sum_{k} \sum_{ij} b_{ij} (q_{ik} t_i)(q_{jk} t_j) \ge 0.$$

(iii) If (a_{ij}) is a P-matrix, then $c_{ij} = e^{a_{ij}}$ is also a P-matrix.

This follows from (ii) and 3.1.A.

(iv) Suppose that a $(n+1) \times (n+1)$ matrix $(a_{ij})_0^n$ and an $n \times n$-matrix $(b_{ij})_1^n$ are connected by the formula

$$b_{ij} = -a_{ij} + a_{i0} + a_{0j} - a_{00}, \quad i, j = 1, \dots, n. \tag{3.1}$$

Then, for all $t_0, \dots, t_n$ such that $t_0 + \dots + t_n = 0$,

$$\sum_{i,j=0}^{n} a_{ij} t_i t_j = - \sum_{i,j=1}^{n} b_{ij} t_i t_j.$$

Therefore (a_{ij}) is an N-matrix if and only if (b_{ij}) is a P-matrix.

Now we can prove the following proposition:

3.1.B. A matrix (a_{ij}) belongs to class N if and only if $c_{ij}(\lambda) = e^{-\lambda a_{ij}}$ is a P-matrix for all $\lambda > 0$.

PROOF. 1°. First, we prove that if (a_{ij}) is an N-matrix, then $(c_{ij}(\lambda))$ is a P-matrix. Clearly, it is sufficient to check this for $\lambda = 1$. Define (b_{ij}) by formula (3.1) and note that

$$c_{ij} = e^{b_{ij}} e^{-a_{i0}} e^{-a_{j0}} e^{a_{00}}.$$

The first factor defines a P-matrix by (iii), and $e^{-a_{i0}} e^{-a_{j0}}$ is a P-matrix by (i). The product is a P-matrix by (ii).

2°. Now suppose that $c_{ij}(\lambda) = e^{-\lambda a_{ij}}$ is a P-matrix for all $\lambda > 0$. Clearly, $[1 - c_{ij}(\lambda)]/\lambda$ is an N-matrix. By passing to the limit as $\lambda \to 0$, we get that (a_{ij}) is in class N. □

3.2. P-functions and N-functions. Suppose that G is a subset of a linear space closed under the addition and the multiplication by constants $a \geq 0$. (We deal with $G = \mathbb{B}$ and, more generally, $G = \mathbb{B}^n$.) We say that a real-valued function on G is a *P-function* if, for every $n \geq 1$ and for all $f_1, \dots, f_n \in G$

$$a_{ij} = u(f_i + f_j) \tag{3.2}$$

is a P-matrix. We call u an *N-function* if the matrix (3.2) is an N-matrix for every $n \geq 2$ and all $f_1, \dots, f_n \in G$.

3.2.A. If u is a P-function, then, for all f,

$$u(f) \geq 0 \quad \text{and } u(f)^2 \leq u(2f)u(0). \tag{3.3}$$

If, in addition, u is bounded, then, for all f,

$$u(f) \leq u(0). \tag{3.4}$$

PROOF. The first inequality in (3.3) holds because a 1×1-matrix $u(f/2 + f/2)$ is a P-matrix. The second inequality is true because the determinant of a 2×2 P-matrix

$$\begin{pmatrix} u(2f) & u(f) \\ u(f) & u(0) \end{pmatrix}$$

is positive. By (3.3), $u(f) = 0$ if $u(0) = 0$. If $u(0) > 0$, then $v(f) = u(f)/u(0)$ satisfies the condition $v(f)^2 \leq v(2f)$ which implies that, for every n,

$$v(f)^{2^n} \leq v(2^n f).$$

If v is bounded, then the sequence $v(f)^{2^n}$ is bounded and therefore $v(f) \leq 1$. □

3.3. Laplace functionals of random measures. Let (X, P) be a random measure on $(S, \mathcal{B}_S)$. The corresponding *Laplace functional* is defined on $f \in \mathbb{B}$ by the formula

$$L(f) = Pe^{-\langle f, X \rangle} = \int_\Omega e^{-\langle f, X(\omega) \rangle} P(d\omega). \tag{3.5}$$

THEOREM 3.1. *A function u on $\mathbb{B}$ is the Laplace functional of a random measure if and only if it is a bounded P-function such that*

$$u(f_n) \to 1 \quad \textit{as } f_n \downarrow 0. \tag{3.6}$$

It is clear that a Laplace functional has all the properties described in the theorem. To prove the converse statement, we use the Krein-Milman theorem on extreme points of convex sets in topological linear spaces $\mathbb{H}$ (see, e.g., [**BJR84**], Section 2.5). Recall that a set K in $\mathbb{H}$ is called convex if it contains, with every $u, v \in K$, a point $pu + qv$ where $p, q > 0, p + q = 1$. A point $\rho \in K$ is called *extreme* if a relation $\rho = pu + qv$ with $u, v \in K, p, q > 0, p + q = 1$ implies that $u = v = \rho$.

The Krein-Milman theorem holds for all locally convex Hausdorff topological linear spaces but we need only a special case formulated in the next proposition:

Proposition 3.1. *Let G be an arbitrary set and let $\mathbb{H} = \mathbb{H}(G)$ be the space of all bounded functions on G endowed with the topology of pointwise convergence.*

Suppose that K is a compact convex subset of $\mathbb{H}$ and that the set K_e of all extreme points of K is closed. Then every $u \in K$ can be represented by the formula

$$u(f) = \int_{K_e} \rho(f)\gamma(d\rho) \tag{3.7}$$

where γ is a probability measure on K_e.

We apply Proposition 3.1 to $\mathbb{H}(\mathbb{B})$ and to the class K of all bounded P-functions u on $\mathbb{B}$ subject to the condition $u(0) = 1$ (we get this condition from (3.6) by taking $f_n = 0$). By 3.2.A, K is contained in the space of all functions from $\mathbb{B}$ to $[0, 1]$ which is compact with respect to the topology of pointwise convergence (see, e.g., [**Kel57b**], Chapter 5, Theorem 13 or [**Kur66**], section 41, Theorem 4). Being its closed subset, K is also compact.

The first step in the proof of Theorem 3.1 is the following: [4]

Proposition 3.2. *A function ρ on $\mathbb{B}$ belongs to K_e if and only if:*

$$\rho(f+g) = \rho(f)\rho(g) \quad \text{for all } f, g, \tag{3.8}$$

$$0 \le \rho(f) \le 1 \quad \text{for all } f \quad \text{and } \rho(0) = 1. \tag{3.9}$$

Proof. Suppose that ρ in K_e. Fix $g \in \mathbb{B}$ and consider a family of functions

$$\rho_\lambda(f) = \rho(f) + \lambda\rho(f+g)$$

where $\lambda \in \mathbb{R}$. Note

$$\sum \rho_\lambda(f_i + f_j)t_i t_j = \hat{\rho}(0) + \lambda\hat{\rho}(g) \tag{3.10}$$

where

$$\hat{\rho}(g) = \sum t_i t_j \rho(f_i + f_j + g).$$

For all s_α and g_α,

$$\sum_{\alpha,\beta} s_\alpha s_\beta \hat{\rho}(g_\alpha + g_\beta) = \sum_{\alpha,i;\beta,j} t_{i\alpha} t_{j\beta} \rho(f_{i\alpha} + f_{j\beta})$$

with $t_{i\alpha} = t_i s_\alpha, f_{i\alpha} = f_i + g_\alpha$. Hence, $\hat{\rho}$ is a P-function. By (3.4), $\hat{\rho}(g) \le \hat{\rho}(0)$ and therefore (3.10) implies that, for $|\lambda| \le 1$, ρ_λ is a P-function . Since $\rho \in K$, it satisfies (3.9). To prove (3.8), we note that $q = [1 - \rho(g)]/2 \ge 0$ and $p = [1 + \rho(g)]/2 \ge q$. If $q > 0$, then $\rho = pu + qv$ where $u = \rho_1/(2p), v = \rho_{-1}/(2q)$. Since ρ is extreme, $\rho = u$. Hence, $\rho(f) = [\rho(f) + \rho(f+g)]/[1 + \rho(g)]$ which implies (3.8). If $q = 0$, then $\rho_{-1}(0) = 0$. Therefore, for all f $\rho_{-1}(f) = 0$ by (3.4), and $\rho(f+g) = \rho(f)$ which also implies (3.8) because $\rho(g) = 1$.

Now suppose that ρ satisfies conditions (3.8)–(3.9). By using (3.8), we check that ρ is a P-function. Suppose that $\rho = pu + qv$ where $u, v \in K, p, q > 0, p + q = 1$. By (3.3), $u(f)^2 \le u(2f), v(f)^2 \le v(2f)$ and therefore

$$pu(f)^2 + qv(f)^2 \le pu(2f) + qv(2f) = \rho(2f) = \rho(f)^2 = [pu(f) + qv(f)]^2. \tag{3.11}$$

Since $\phi(t) = t^2$ is a strictly convex function, (3.11) implies $u(f) = v(f) = \rho(f)$. Hence $\rho \in K_e$. □

[4] In [**BJR84**], functions ρ with the properties (3.8)–(3.9) are called bounded semicharacters.

PROOF OF THEOREM 3.1. Suppose that ρ satisfies conditions (3.8)–(3.9) and, in addition,

$$\rho(f_n) \to 1 \quad \text{as } f_n \downarrow 0. \tag{3.12}$$

For every f and every n, $\rho(f) = \rho(f/n)^n$, and (3.12) implies

$$\rho(f) > 0. \tag{3.13}$$

It follows from (3.8), (3.12) and (3.13) that

$$\nu_\rho(B) = -\log \rho(1_B), \mathcal{B} \in \mathcal{B}_S \tag{3.14}$$

is a finite measure on S and moreover $\langle f, \nu_\rho \rangle$ is measurable in ρ for every $f \in \mathbb{B}$. Denote by K'_e the set of $\rho \in K_e$ which satisfy (3.12). If

$$\gamma(K_e \setminus K'_e) = 0, \tag{3.15}$$

then, by (3.7) and (3.14),

$$u(f) = \int_{K'_e} e^{-\langle f, \nu_\rho \rangle} \gamma(d\rho)$$

and therefore $u(f)$ is the Laplace functional of the random measure (ν_ρ, γ).

In fact, the assumption (3.6) does not imply (3.15). It implies a weaker condition:

$$\text{If } f_n \downarrow 0, \text{ then } \rho(f_n) \to 1 \text{ for } \gamma\text{-almost all } \rho. \tag{3.16}$$

Indeed, by (3.8), $\rho(f_n)$ is an increasing sequence and, if $\rho(f_n) \uparrow \beta(\rho)$, then, by (3.7)

$$\int \beta(\rho) \gamma(d\rho) = \lim \int \rho(f_n) \gamma(d\rho) = \lim u(f_n) = 1.$$

By (3.9), $\beta(\rho) \le 1$. Hence, $\beta(\rho) = 1$ for γ-almost all ρ.

Fortunately, by a result on the regularization of pseudo-kernels ([**Get75**], Proposition 4.1), the property (3.16) is sufficient to define $\nu_\rho \in \mathcal{M}(S)$ such that, for every f, $\langle f, \nu_\rho \rangle$ is measurable in ρ and

$$\langle f, \nu_\rho \rangle = -\log \rho(f) \quad \text{for } \gamma\text{-almost all } \rho.$$

We have

$$u(f) = \int_{K_e} \rho(f) \gamma(d\rho) = \int_{K_e} e^{-\langle f, \nu_\rho \rangle} \gamma(d\rho)$$

and therefore, u is the Laplace functional of the random measure (ν_ρ, γ). □

The probability distribution of a random measure (X, P) is a probability measure $\mathcal{P}$ on the space $\mathcal{M}(S)$ of finite measures on S. The domain of $\mathcal{P}$ is the σ-algebra generated by functions $F_B(\mu) = \mu(B), B \in \mathcal{B}_S$. The Laplace functional of (X, P) can be expressed through $\mathcal{P}$ by the formula

$$L(f) = \int_{\mathcal{M}(S)} e^{-\langle f, \nu \rangle} \ \mathcal{P}(d\nu).$$

The Laplace functional of a probability measure $\mathcal{P}$ on $\mathcal{M}(S)^n$ is defined by the formula

$$L_{\mathcal{P}}(f_1, \dots, f_n) = \int e^{-\langle f_1, \nu_1 \rangle - \dots - \langle f_n, \nu_n \rangle} \mathcal{P}(d\nu_1, \dots, d\nu_n). \tag{3.17}$$

By identifying $\mathcal{M}(S)^n$ with the space of finite measures on the union of n copies of S, we get a multivariant version of Theorem 3.1:

Theorem 3.2. *A functional $u(f_1, \dots, f_n)$ on $\mathbb{B}^n$ is the Laplace functional of a probability measure on $\mathcal{M}(S)^n$ if and only if it is a bounded P-function with the property*

$$u(f_1^i, \dots, f_n^i) \to 1 \quad \text{as } f_1^i \downarrow 0, \dots, f_n^i \downarrow 0. \tag{3.18}$$

3.4. Constructing a BEM system. We say that a function u from G to $\mathbb{B}$ is an N-function if, for every x, $u(f)(x)$ is a real-valued N-function. By 3.1.A, this implies $\langle u(f), \mu\rangle$ is in the class N for all $\mu \in \mathcal{M}(S)$. The class of P-functions from G to $\mathbb{B}$ is defined in a similar way.

Theorem 3.3. *A $\mathbb{V}$-family $\mathbb{V} = \{V_I\}$ corresponds to a BEM system if and only if:*

(a) V_Q satisfy conditions 2.2.A–2.2.C and 2.3.A;
(b) for every I, $V_I(f_I^j) \to 0$ as $f_I^j \downarrow 0$;
(c) for every I, V_I is an N-function.

Proof. If $\mathbb{V}$ corresponds to a BEM system, then (a) follows from Theorem 2.1 and 2.1.B, and (b)–(c) follow from (3.18) and 3.1.B.

Suppose that $\mathbb{V}$ satisfies (a)–(c). If $|I| = n$, then, by 3.1.A, for every $\mu \in \mathcal{M}(S)$, $\langle V_I(f_I), \mu\rangle$ is an N-function and, by 3.1.B, $L_{\mu,I}(f_I) = e^{-\langle V_I(f_I),\mu\rangle}$ is an P-function. By Theorem 3.2, $L_{\mu,I}$ is the Laplace functional of a probability measure on $\mathcal{M}(S)^n$. These measures satisfy consistency conditions and, by Kolmogorov's theorem, they are probability distributions of X_I relative to P_μ for a system X of random measures (X_Q, P_μ). By Theorem 2.1 and 2.1.B, X is a BEM system. □

Theorem 3.4. *Suppose that operators V_Q acting in $\mathbb{B}$ satisfy conditions 2.2.A–2.2.C and 2.3.A. They are the transition operators of a BEM system if, in addition:*

3.4.A. For every n and every N-function U from $\mathbb{B}^n$ to $\mathbb{B}$, $V_Q U$ is also an N-function.

Proof. By Theorem 2.2, V_Q are a part of a $\mathbb{V}$-family $\{V_I\}$. We need only to check that this family satisfies conditions (b)–(c) of Theorem 3.3. We use the induction in $n = |I|$. For $n = 1$, (b) follows from 2.2.A and we get (c) by taking an identity map from $\mathbb{B}$ to $\mathbb{B}$ for U in 3.4.A. Let $\tilde{V}_I$ be given by (2.12). Clearly, if V_{I_i} satisfy (b)–(c), then so does $\tilde{V}_I$. By (2.10) and 3.4.A, the same is true for V_I. By induction, (b)–(c) hold for all I. □

3.5. Passage to the limit. Transition operators not satisfying 3.4.A can be obtained by a passage to the limit. We denote by $\mathbb{B}_c$ the set of all $\mathcal{B}_S$-measurable functions f such that $0 \le f \le c$ and we put $\|f\| = \sup_S |f(y)|$ for every function f on S. Writing $V^k \xrightarrow{u} V$ means that V^k converges to V uniformly on each set $\mathbb{B}_c$.

Theorem 3.5. *Suppose that X^k is a sequence of BEM systems and that V_Q^k are the transition operators of X^k. If $V_Q^k \xrightarrow{u} V_Q$ for every $Q \in \mathbb{O}$ and if V_Q satisfies the Lipschitz condition on every $\mathbb{B}_c$, then V_Q are the transition operators of a BEM system.*

The proof is based on Theorem 3.3 and two lemmas.

Put $\|f\| = \max\{\|f_1\|, \dots, \|f_n\|\}$ for $f = (f_1, \dots, f_n) \in \mathbb{B}^n$. Writing $f \in \mathbb{B}_c^n$ means that $0 \le f_i \le c$ for $i = 1, \dots, n$.

LEMMA 3.1. *Suppose that V^k are operators from $\mathbb{B}$ to $\mathbb{B}$, $V^k \xrightarrow{u} V$ and V satisfies the Lipschitz condition on each $\mathbb{B}_c$. Suppose that $\tilde{V}^k$ are operators from $\mathbb{B}^n$ to $\mathbb{B}$, $\tilde{V}^k \xrightarrow{u} \tilde{V}$ and $\tilde{V}$ satisfies the Lipschitz condition on each $\mathbb{B}^n_c$. Then $V^k\tilde{V}^k \xrightarrow{u} V\tilde{V}$ and $V\tilde{V}$ satisfies the Lipschitz condition on each $\mathbb{B}^n_c$.*

PROOF. We have

$$\begin{aligned} &\|V^k(f) - V(f)\| \le \varepsilon_k(c) \quad \text{for } f \in \mathbb{B}_c, \\ &\|\tilde{V}^k(\tilde{f}) - \tilde{V}(\tilde{f})\| \le \tilde{\varepsilon}_k(c) \quad \text{for } \tilde{f} \in \mathbb{B}^n_c \end{aligned} \tag{3.19}$$

with $\varepsilon_k(c) + \tilde{\varepsilon}_k(c) \to 0$ as $k \to \infty$. There exist constants $a(c)$ and $\tilde{a}(c)$ such that

$$\begin{aligned} &\|V(f) - V(g)\| \le a(c)\|f - g\| \quad \text{for all } f, g \in \mathbb{B}_c, \\ &\|\tilde{V}(\tilde{f}) - \tilde{V}(\tilde{g})\| \le \tilde{a}(c)\|\tilde{f} - \tilde{g}\| \quad \text{for all } \tilde{f}, \tilde{g} \in \mathbb{B}^n_c. \end{aligned} \tag{3.20}$$

By taking $g = \tilde{g} = 0$, we get

$$\|V(f)\| \le ca(c) \quad \text{for } f \in \mathbb{B}_c; \qquad \|\tilde{V}(\tilde{f})\| \le c\tilde{a}(c) \quad \text{for } \tilde{f} \in \mathbb{B}^n_c. \tag{3.21}$$

Note that

$$\|V^k[\tilde{V}^k(\tilde{f})] - V[\tilde{V}(\tilde{f})]\| \le q(k) + h(k)$$

where

$$q(k) = \|V^k[\tilde{V}^k(\tilde{f})] - V[\tilde{V}^k(\tilde{f})]\|$$

and

$$h(k) = \|V[\tilde{V}^k(\tilde{f})] - V[\tilde{V}(\tilde{f})]\|.$$

For all sufficiently large k and for all $\tilde{f} \in \mathbb{B}^n_c$, $\|\tilde{V}^k(\tilde{f}) - \tilde{V}(\tilde{f})\| \le 1$ and, by (3.21), $\|\tilde{V}^k(\tilde{f})\| \le \tilde{c}_1 = c\tilde{a}(c) + 1$. By (3.19), $q(k) \le \varepsilon_k(\tilde{c}_1)$. By (3.20) and (3.19),

$$h(k) \le a(\tilde{c}_1)\|\tilde{V}^k(\tilde{f}) - \tilde{V}(\tilde{f})\| \le a(\tilde{c}_1)\tilde{\varepsilon}_k(c).$$

Therefore $V^k\tilde{V}^k \xrightarrow{u} V\tilde{V}$. We have

$$\|V[\tilde{V}(\tilde{f})] - V[\tilde{V}(\tilde{g})]\| \le a(\tilde{c}_1)\|\tilde{V}(\tilde{f}) - \tilde{V}(\tilde{g})\| \le a(\tilde{c}_1)\tilde{a}(c)\|\tilde{f} - \tilde{g}\|.$$

□

LEMMA 3.2. *Suppose $\mathbb{V}^k$ is a sequence of $\mathbb{V}$-families and let V^k_Q satisfy the conditions of Theorem 3.5. Then*

(i) a limit $V_I(f)$ of $V^k_I(f)$ exists for every $I = (Q_1, \dots, Q_n) \subset \mathbb{O}$ and every $f = (f_1, \dots, f_n) \in \mathbb{B}^n$;

(ii) the convergence is uniform on every set $\mathbb{B}^n_c$;

(iii) $V_I(f)$ satisfies the Lipschitz condition on every $\mathbb{B}^n_c$;

PROOF. By (2.11)–(2.12)

$$V^k_I = V^k_{Q_I}\tilde{V}^k_I \tag{3.22}$$

where

$$\tilde{V}^k_I = \begin{cases} f_i + \tilde{V}^k_{I_i}(f_{I_i}) & \text{on } Q^c_i, \\ 0 & \text{on } Q_I \end{cases} \tag{3.23}$$

and therefore, for all k, m,

$$|\tilde{V}^k_I(f_I) - \tilde{V}^m_I(f_I)| = \begin{cases} |\tilde{V}^k_{I_i}(f_{I_i}) - \tilde{V}^m_{I_i}(f_{I_i})| & \text{on } Q^c_i, \\ 0 & \text{on } Q_I. \end{cases} \tag{3.24}$$

If conditions (i)–(iii) hold for $\tilde{V}^k_{I_i}$, then, by (3.24), they hold for $\tilde{V}^k_I$ and, by Lemma 3.1, they hold for V_I. □

PROOF OF THEOREM 3.5. It is sufficient to prove that operators V_I defined in Lemma 3.2 satisfy the conditions (a)–(c) of Theorem 3.3. The property (ii) implies that (b) and (c) for V_I follow from analogous properties for V^k_I. The same is true for 2.2.A, 2.2.B and 2.2.C. The condition 2.3.A follows from (ii) and Lemma 3.1. □

3.6. Extension of class $\mathbb{M}$. Suppose that $X = (X_Q, P_\mu), Q \in \mathbb{O}, \mu \in \mathbb{M}$ is a branching exit system. We get a new branching exit system by extending class $\mathbb{M}$ to the class $\sigma(\mathbb{M})$ of all measures $\mu = \sum_1^\infty \mu_n$ where $\mu_n \in \mathbb{M}$ and by defining P_μ as the convolution of measures P_{μ_n}. For every $Z \in \mathbb{Z}$,

$$P_\mu Z = \prod P_{\mu_n} Z. \tag{3.25}$$

By using this formula, it is easy to check that 1.3.A holds for the extended system. Condition 1.3.B holds because, if $Y = X_Q(Q)$ and $\mu \in \sigma(\mathbb{M})$, then, for every $\lambda > 0$,

$$P_\mu e^{-\lambda Y} = \prod P_{\mu_n} e^{-\lambda Y}$$

and by tending λ to $+\infty$, we get $P_\mu\{Y = 0\} = \prod P_{\mu_n}\{Y = 0\} = 1$.

By 1.3.C, $P_z e^{-\lambda\langle f, X_Q\rangle} = e^{-\lambda f(z)}$ for $z \notin Q$. If $\mu(Q) = 0$, then, by 1.3.A,

$$P_\mu e^{-\lambda\langle f, X_Q\rangle} = e^{-\lambda\langle f, \mu\rangle}$$

and property 1.3.C follows from Lemma 2.1 [which is true also for infinite measures].

4. Some properties of BEM systems

4.1. CB-property. We say that a *CB-property* holds for a positive measurable function Z and a measure $\mu \in \mathbb{M}$ if

$$\log P_\mu Z = \int \log P_y Z \ \mu(dy). \tag{4.1}$$

Note that this condition is equivalent to

$$P_\mu Z = e^{-\langle u, \mu\rangle} \tag{4.2}$$

where

$$u(y) = -\log P_y Z. \tag{4.3}$$

By 1.3.A, this is true for all $Z \in \mathbb{Z}$ and all $\mu \in \mathbb{M}$. Suppose that $Z_n \downarrow Z$ and $0 \le Z_n \le 1$. Then $P_\mu Z_n \downarrow P_\mu Z$ and $0 \le -\log P_y Z_n \uparrow -\log P_y Z$. By the monotone convergence theorem, the condition (4.1) holds for Z, μ if it holds for Z_n, μ.

Denote by $\mathbb{Y}$ the class of functions

$$Y = \sum_1^\infty \langle f_i, X_{Q_i}\rangle$$

where $Q_1, \dots, Q_n, \dots \in \mathbb{O}$ and $f_1, \dots, f_n, \dots$ are positive $\mathcal{B}_S$-measurable functions.

PROPOSITION 4.1. *If $Y \in \mathbb{Y}$, then the CB-property holds for $\{Y = 0\}$ and all $\mu \in \mathbb{M}$.*

Indeed, if $Y_m = \sum_1^m \langle f_i, X_{Q_i}\rangle$, then $Z^m_n = e^{-nY_m}$ belongs to $\mathbb{Z}$ for all n and $Z^m_n \downarrow 1_{Y_m=0}$ as $n \to \infty$. Hence the CB-property holds for $\{Y_m = 0\}$ and μ. It remains to note that $\{Y_m = 0\} \downarrow \{Y = 0\}$ as $m \to \infty$.

Proposition 4.2. *If* $Y_1, Y_2 \in \mathbb{Y}$ *and if* $\mu(Q^c) = 0$ *then the relation*

$$\{Y_1 = 0\} \subset \{Y_2 = 0\} \tag{4.4}$$

holds P_μ*-a.s. if it holds* P_y*-a.s. for all* $y \in Q$.

Proof. The relation (4.4) is equivalent to $\{Y_1 = 0\} = \{Y_1 + Y_2 = 0\}$ which holds P_μ-a.s. if and only if $P_\mu\{Y_1 = 0\} = P_\mu\{Y_1 + Y_2 = 0\}$. □

4.2. Writing "a.s." means "almost sure with respect to all $P_\mu, \mu \in \mathbb{M}$".

Theorem 4.1. *Suppose that* $X = (X_Q, P_\mu), Q \in \mathbb{O}, \mu \in \mathbb{M}$ *is a BEM system and let* $Q_1 \subset Q_2$ *be elements of* $\mathbb{O}$*. Then:*

4.2.A.

$$\{X_{Q_1} = 0\} \subset \{X_{Q_2} = 0\} \quad a.s..$$

4.2.B. For every $\mu \in \mathbb{M}$ *and every bounded measurable function* f *on* $\mathbb{M} \times \mathbb{M}$,

$$P_\mu f(X_{Q_1}, X_{Q_2}) = P_\mu F(X_{Q_1})$$

where

$$F(\nu) = P_\nu f(\nu, X_{Q_2}).$$

4.2.C. If $0 \le \varphi_1 \le \varphi_2$ *and* $\varphi_2 = 0$ *on* Q_2*, then*

$$\langle \varphi_1, X_{Q_1} \rangle \le \langle \varphi_2, X_{Q_2} \rangle \quad a.s.$$

4.2.D. If $\Gamma \subset Q_2^c$*, then* $X_{Q_1}(\Gamma) \le X_{Q_2}(\Gamma)$ *a.s.*

Proof. By 1.3.D,

$$P_\mu\{X_{Q_1} = 0, X_{Q_2} \neq 0\} = P_\mu 1_{X_{Q_1}=0} P_{X_{Q_1}}\{X_{Q_2} \neq 0\} = 0$$

which implies 4.2.A.

Bounded functions f for which 4.2.B is true form a linear space closed under the bounded convergence. By the Markov property 1.3.D, this space contains all functions $f_1(\nu_1) f_2(\nu_2)$. By the multiplicative systems theorem (see Theorem 1.1 in the Appendix A), it contains all bounded measurable functions.

To prove 4.2.C, we consider

$$F(\nu) = P_\nu\{\langle \varphi_1, \nu \rangle \le \langle \varphi_2, X_{Q_2} \rangle\}.$$

By 4.2.B,

$$P_\mu\{\langle \varphi_1, X_{Q_1} \rangle \le \langle \varphi_2, X_{Q_2} \rangle\} = P_\mu \; F(X_{Q_1}). \tag{4.5}$$

Let ν' be the restriction of ν to Q_2^c. For all $\lambda > 0$, by 1.3.A and 1.3.C,

$$P_\nu e^{-\lambda\langle \varphi_2, X_{Q_2} \rangle} \le P_{\nu'} e^{-\lambda\langle \varphi_2, X_{Q_2} \rangle} = e^{-\lambda\langle \varphi_2, \nu' \rangle} = e^{-\lambda\langle \varphi_2, \nu \rangle} \le e^{-\lambda\langle \varphi_1, \nu \rangle}.$$

By Lemma 2.1, this implies $F(\nu) = 1$ and 4.2.C follows from (4.5).

To get 4.2.D, it is sufficient to apply 4.2.C to $\varphi_1 = \varphi_2 = 1_\Gamma$. □

5. Notes

5.1. Branching particle systems corresponding to a diffusion ξ were studied, first, in [**Sko64**]. Special classes of such systems were investigated earlier in [**Sev58**]. A general theory of branching particle systems was developed in [**INW68**]–[**INW69**].

In earlier papers, a superdiffusion was interpreted as a Markov process X_t in the space of measures. A reacher model based on the concept of exit measures has been introduced in [**Dyn91c**] in the time homogeneous case and in [**Dyn92**] in the time inhomogeneous setting. In [**Dyn93**], an integral equation describing the joint probability distribution of $X_{Q_1}, \dots, X_{Q_n}$ was introduced and solved, and the Markov and the branching properties were proved.

An alternative approach based on the concept of a historical process was developed by Dawson and Perkins in [**DP91**]. [5] A historical process is a family $\hat{X}$ of random measures $(\hat{X}_t, P_\mu)$ on the space $\hat{S}$ of paths in S. Exit measures for X can be expressed in terms of $\hat{X}$ (see [**Dyn91b**]).

5.2. In the presentation of the properties of N-functions and their relations to P-functions, we follow the book [**BJR84**]. Theorem 3.1 is due to Fitzsimmons (see the Appendix in [**Fit88**]).

[5] More on this approach can be found in section **14**.11.

CHAPTER 4

Superprocesses

We define a superprocess as a BEM system with the transition operator satisfying a certain integral equation. We construct superprocesses by two different methods. The first one uses a passage to the limit from branching particle systems. The second method is based on Theorem **3**.3.4.

1. Definition and the first results

1.1. Definition of a superprocess. We denote by $\mathbb{R}_+$ the half-line $[0, \infty)$. We say that a BEM system

$$X = (X_Q, P_\mu), \quad Q \in \mathbb{O}, \mu \in \mathcal{M}(S)$$

is a (ξ, ψ)-*superprocess* if $\mathbb{O}$ is a class of open subsets of $S = \mathbb{R} \times E$, if $\xi = (\xi_t, \Pi_{r,x})$ is a right continuous strong Markov process, $\psi(z, t)$ is a positive function on $S \times \mathbb{R}_+$ and if the transition operators V_Q of X satisfy the condition: for every $f \in \mathbb{B}$, $u = V_Q(f)$ is a solution of the equation

$$u + G_Q \psi(u) = K_Q f. \tag{1.1}$$

Here K_Q is the Poisson operator defined by **2**.(3.1), G_Q is Green's operator defined by **2**.(5.13) and $\psi(u)$ means $\psi(r, x; u(r, x))$.

1.2. Gronwall's lemma and its application. Does equation (1.1) determine uniquely V_Q? The answer is positive for a wide class of sets Q and functions ψ.

Put $Q \in \mathbb{O}_0$ if Q is an open subset of S and if $Q \subset \Delta \times E$ for some finite interval Δ.This is equivalent to the condition:

1.2.A. There exists a constant N such that $\tau(Q) - r \leq N$ for all paths of ξ starting from $(r, x) \in Q$.

We use the following modification of Gronwall's inequality:

Lemma 1.1. *Let τ be the first exit time from $Q \in \mathbb{O}_0$. If a positive bounded function $h(r, x)$ satisfies the condition:*

$$h(r, x) \leq a + q\Pi_{r,x} \int_r^\tau h(s, \xi_s)\, ds \quad \text{in } Q \tag{1.2}$$

for some constants a and q, then

$$h(r, x) \leq a\Pi_{r,x} e^{q(\tau - r)} \quad \text{in } Q. \tag{1.3}$$

Proof. Suppose that $h \leq A$. We prove, by induction, that

$$h(r, x) \leq \Pi_{r,x} Y_n(r) \tag{1.4}$$

where

$$Y_n(r) = a\sum_{k=0}^{n-1} q^k \frac{(\tau - r)^k}{k!} + Aq^n \frac{(\tau - r)^n}{n!}.$$

Clearly, (1.4) holds for $n = 1$. If it is true for n, then, by (1.2),

$$h(r, x) \le a + q\Pi_{r,x} \int_r^\tau \Pi_{s,\xi_s} Y_n(s)\, ds \quad \text{in } Q. \tag{1.5}$$

By the Markov property **2**.(2.3),

$$\Pi_{r,x} 1_{\tau>s} \Pi_{s,\xi_s} Y_n(s) = \Pi_{r,x} 1_{\tau>s} Y_n(s) \quad \text{for all } (r, x) \in Q$$

because $\{\tau > s\} \subset \{\tau = \tau_s\}$ where τ_s is the first after s exit time from Q and τ_s is $\mathcal{F}_{\ge s}$-measurable. Hence, the right side in (1.5) is equal to

$$a + q\Pi_{r,x} \int_r^\tau Y_n(s)\, ds = \Pi_{r,x} Y_{n+1}(r)$$

and (1.4) holds for $n + 1$. Bound (1.3) follows from (1.4) and 1.2.A. □

THEOREM 1.1. *Suppose that $Q \in \mathbb{O}_0$ and $\psi(z, t)$ is locally Lipschitz continuous in t uniformly in z, i.e., for every $c > 0$, there exists a constant $q(c)$ such that*

$$|\psi(z; u_1) - \psi(z; u_2)| \le q(c)|u_1 - u_2| \quad \textit{for all } z \in S, u_1, u_2 \in [0, c]. \tag{1.6}$$

Then equation (1.1) has at most one solution. Moreover, if u satisfies (1.1) and if $\tilde{u} + G_Q\psi(\tilde{u}) = K_Q\tilde{f}$, then

$$\|u - \tilde{u}\| \le e^{q(c)N}\|f - \tilde{f}\| \quad \textit{for all } f, \tilde{f} \in \mathbb{B}_c \tag{1.7}$$

where N is the constant in 1.2.A.

Suppose $\psi(z, 0)$ is bounded and $u_\beta + G_Q\psi(u_\beta) = K_Q f_\beta$. If $f \in \mathbb{B}_c$ and $\|f_\beta - f\| \to 0$ as $\beta \downarrow 0$, then there exists a solution u of (1.1) such that

$$\|u_\beta - u\| \le e^{q(2c)N}\|f_\beta - f\| \quad \textit{for all sufficiently small } \beta. \tag{1.8}$$

PROOF. By (1.1), $\|u\| \le \|f\|, \|\tilde{u}\| \le \|\tilde{f}\|$ and

$$u - \tilde{u} = K_Q(f - \tilde{f}) + G_Q[\psi(\tilde{u}) - \psi(u)].$$

Put $h = |u - \tilde{u}|$. By (1.6), $|\psi(\tilde{u}) - \psi(u)| \le q(c)h$ and therefore

$$h \le \|f - \tilde{f}\| + q(c)G_Q h$$

and (1.7) follows from Gronwall's inequality (1.2).

If $f \in \mathbb{B}_c$, then for all sufficiently small β, $f_\beta \in \mathbb{B}_{2c}$ and, by (1.7), for sufficiently small β and $\tilde{\beta}$,

$$\|u_\beta - u_{\tilde{\beta}}\| \le e^{q(2c)N}\|f_\beta - f_{\tilde{\beta}}\|$$

which implies the existence of the limit $u = \lim u_\beta$ and the bound (1.8). By (1.6), $\psi(u_\beta) \le \psi(0) + 2cq(2c)$ and, by the dominated convergence theorem, u satisfies (1.1). □

REMARK. Similar arguments based on Gronwall's lemma show that, if

$$|\psi(z, t) - \tilde{\psi}(z, t)| \le \varepsilon(c) \quad \text{for all } z \in S, t \in [0, c]$$

and if $\tilde{u} + G_Q\tilde{\psi}(\tilde{u}) = K_Q f$, then

$$\|u - \tilde{u}\| \le Ne^{q(c)N}\varepsilon(c).$$

1.3. BEM systems corresponding to branching particle systems. We return to the branching particle system and the corresponding BEM system $X = (X_Q, P_\mu)$ described in section **3**.1.2. Recall that such a system is determined by a right continuous strong Markov process $\xi = (\xi_t, \Pi_{r,x})$, a set of probabilities $p_n(t,x)$ describing a branching and a parameter k defining the life time probability distribution. If X is an associated BEM system, then

$$V_Q(f) = -\log w$$

where

$$w(r,x) = P_{r,x} e^{-\langle f, X_Q \rangle}. \tag{1.9}$$

We introduce an offspring generating function

$$\varphi(t,x;z) = \sum_0^\infty p_n(t,x) z^n, \quad 0 \le z \le 1.$$

The four principles stated at the beginning of section **3**.1.2 imply

$$w(r,x) = \Pi_{r,x} \left[e^{-k(\tau - r)} e^{-f(\tau, \xi_\tau)} + k \int_r^\tau e^{-k(s-r)} \, ds \, \varphi(s, \xi_s; w(s, \xi_s)) \right] \tag{1.10}$$

where τ is the first exit time of (t, ξ_t) from Q. The first term in the brackets corresponds to the case when the particle started the process is still alive at time τ, and the second term corresponds to the case when it dies at time $s \in (r, \tau)$. [If N is a random measure equal to $n\delta_{(s,y)}$ with probability $p_n(s,y)$, then

$$P_N e^{-\langle f, X_Q \rangle} = \sum_0^\infty p_n(s,y) w(s,y)^n = \varphi(s, y; w(s,y)).]$$

1.4. An integral identity. We simplify equation (1.10) by using the following lemma which has also other important applications.

LEMMA 1.2. *If*

$$w(r,x) = \Pi_{r,x} \left[e^{-k(\tau - r)} u(\tau, \xi_\tau) + \int_r^\tau e^{-k(s-r)} v(s, \xi_s) \, ds \right], \tag{1.11}$$

then

$$w(r,x) + \Pi_{r,x} \int_r^\tau k w(s, \xi_s) \, ds = \Pi_{r,x} \left[u(\tau, \xi_\tau) + \int_r^\tau v(s, \xi_s) \, ds \right]. \tag{1.12}$$

PROOF. Note that

$$H(r,t) = e^{-k(t-r)}$$

satisfies the relation

$$k \int_r^t H(s,t) \, ds = 1 - H(r,t) \tag{1.13}$$

and that

$$w(r,x) = \Pi_{r,x}(Y_r + Z_r) \tag{1.14}$$

where

$$Y_s = H(s, \tau) u(\tau, \xi_\tau),$$

$$Z_s = \int_s^\tau H(s,t) v(t, \xi_t) \, dt.$$

By (1.14) and Fubini's theorem,

$$\Pi_{r,x}\int_r^\tau kw(s,\xi_s)\,ds = \int_r^\infty k\Pi_{r,x}1_{s<\tau}\Pi_{s,\xi_s}(Y_s+Z_s)\,ds.$$

By the Markov property **2**.(2.4),

$$\Pi_{r,x}1_{s<\tau}\Pi_{s,\xi_s}(Y_s+Z_s) = \Pi_{r,x}1_{s<\tau}(Y_s+Z_s)$$

and therefore

$$w(r,x) + \Pi_{r,x}\int_r^\tau kw(s,\xi_s)\,ds = \Pi_{r,x}(I_1+I_2) \tag{1.15}$$

where

$$I_1 = H(r,\tau)u(\tau,\xi_\tau) + k\int_r^\tau Y_s\,ds \quad \text{and } I_2 = \int_r^\tau [H(r,s)v(s,\xi_s)+kZ_s]\,ds.$$

By (1.13) and Fubini's theorem

$$I_1 = u(\tau,\xi_\tau), \qquad I_2 = \int_r^\tau v(t,\xi_t)\,dt,$$

and (1.12) follows from (1.15). □

1.5. Heuristic passage to the limit. By applying Lemma 1.2 to $u(s,x) = e^{-f(s,x)}$ and $v(s,x) = k\varphi(s,x;w(s,x))$, we get the following result:

THEOREM 1.2. *Let V_Q be the transition operators of X. Then for every $f \in \mathbb{B}$, function $v = V_Q(f)$ satisfies the equation*

$$e^{-v(r,x)} = \Pi_{r,x}\left[k\int_r^\tau \Phi(s,\xi_s;e^{-v(s,\xi_s)})\,ds + e^{-f(\tau,\xi_\tau)}\right] \tag{1.16}$$

where

$$\Phi(t,x;z) = \varphi(t,x;z) - z.$$

Assuming that all particles have mass β, we get a transformed system of random measures $X^\beta = (X_Q^\beta, P_\mu^\beta), \mu \in \mathbb{M}^\beta$ where

$$\mathbb{M}^\beta = \beta\mathbb{M}, X_Q^\beta = \beta X_Q, P_\mu^\beta = P_{\frac{\mu}{\beta}}.$$

The transition operators of X^β are related to the transition operators of X by the formula $V_Q^\beta(f) = V_Q(\beta f)/\beta$ and therefore (1.16) implies the following equation for $v_\beta = V_Q^\beta(f)$

$$e^{-\beta v_\beta(r,x)} = \Pi_{r,x}\left[\int_r^\tau k\Phi(s,\xi_s;e^{-\beta v_\beta(s,\xi_s)})\,ds + e^{-\beta f(\tau,\xi_\tau)}\right]. \tag{1.17}$$

Note that (1.17) is equivalent to the equation

$$u_\beta(r,x) + \Pi_{r,x}\int_r^\tau \psi_\beta(s,\xi_s;u_\beta(s,\xi_s))\,ds = \Pi_{r,x}f_\beta(\tau,\xi_\tau) \tag{1.18}$$

where

$$u_\beta = [1-e^{-\beta v_\beta}]/\beta, \quad f_\beta = [1-e^{-\beta f}]/\beta \tag{1.19}$$

and

$$\psi_\beta(r,x;u) = [\varphi_\beta(r,x;1-\beta u) - 1 + \beta u]k_\beta/\beta \quad \text{for } \beta u \le 1. \tag{1.20}$$

[We assume that parameters φ and k depend on β. Since $\beta u_\beta = 1 - e^{-\beta v_\beta} \le 1$, the value $\varphi_\beta(r, x; 1 - \beta u_\beta)$ is defined.] Equation (1.18) can be rewritten in the form

$$u_\beta + G_Q \psi_\beta(u_\beta) = K_Q f_\beta. \tag{1.21}$$

Suppose that $\beta \to 0$. Then $f_\beta \to f$. If $\psi_\beta \to \psi$, then we expect that u_β tends to a limit u which is a solution of the equation (1.1).

2. Superprocesses as limits of branching particle systems

2.1. We use the bounds

$$\begin{aligned} 0 \le 1 - e^{-\lambda} \le 1 \wedge \lambda, \\ 0 \le e^{-\lambda} - 1 + \lambda \le \lambda \wedge \lambda^2 \end{aligned} \tag{2.1}$$

for all $\lambda \ge 0$. Put

$$e(\lambda) = e^{-\lambda} - 1 + \lambda. \tag{2.2}$$

Since, for $u > 0$, $0 < e'(u) = 1 - e^{-u} < 1 \wedge u$, we have

$$|e(\lambda u_2) - e(\lambda u_1)| \le (1 \vee c)(\lambda \wedge \lambda^2)|u_2 - u_1| \quad \text{for all } u_1, u_2 \in [0, c]. \tag{2.3}$$

2.2. A class of superprocesses.

THEOREM 2.1. *A (ξ, ψ)-superprocess exists for every function*

$$\psi(r, x; u) = b(r, x)u^2 + \int_0^\infty (e^{-\lambda u} - 1 + \lambda u) n(r, x; d\lambda) \tag{2.4}$$

where a positive Borel function $b(r, x)$ and a kernel n from $(S, \mathcal{B}_S)$ to $\mathbb{R}_+$ satisfy the condition

$$b(r, x) \quad \textit{and} \quad \int_0^\infty \lambda \wedge \lambda^2 n(r, x; d\lambda) \quad \textit{are bounded.} \tag{2.5}$$

REMARK. The family (2.4) contains the functions

$$\psi(u) = \text{const.}\, u^\alpha, 1 < \alpha < 2 \tag{2.6}$$

that correspond to $b = 0$ and $n(d\lambda) = \text{const.}\, \lambda^{-(1+\alpha)}\, d\lambda$.

Theorem 2.1 can be proved for a wider class of ψ (see [**Dyn93**]). We restrict ourselves by the most important functions.

PROOF. 1°. We choose parameters φ_β, k_β of a branching particle system to make ψ_β given by (1.20) independent of β. To this end we put

$$\begin{aligned} k_\beta &= \frac{\gamma}{\beta}, \\ \varphi_\beta(z; u) &= u + \frac{\beta^2}{\gamma} \psi\left(z; \frac{1-u}{\beta}\right) \quad \text{for } 0 \le u \le 1 \end{aligned} \tag{2.7}$$

where γ is a strictly positive constant. We need to show that φ_β is a generating function. To simplify formulae, we drop arguments z. Clearly, $\varphi_\beta(1) = 1$. We have

$$\varphi_\beta(u) = \sum_0^\infty p_k^\beta u^k$$

where

$$p_0^\beta = \frac{\beta^2}{\gamma}\psi(\frac{1}{\beta}),$$
$$p_1^\beta = \frac{1}{\gamma}[\gamma - 2b - \beta\int_0^\infty \lambda(1-e^{-\lambda/\beta})n(d\lambda)],$$
$$p_2^\beta = \frac{b}{\gamma} + \frac{1}{\gamma}\int_0^\infty e^{-\lambda/\beta}\lambda^2 n(d\lambda),$$
$$p_k^\beta = \frac{\beta^2}{k!\gamma}\int_0^\infty e^{-\lambda/\beta}\left(\frac{\lambda}{\beta}\right)^k n(d\lambda) \quad \text{for } k > 2.$$

p_0^β and p_k^β are positive for all $\beta > 0$ and $k \geq 2$. Function p_1^β is positive for $0 < \beta \leq 1$ if γ is an upper bound of

$$2b + \int_0^\infty \lambda \wedge \lambda^2 n(d\lambda).$$

2°. We claim that there exists a solution u of (1.1) and a function $a(c)$ such that

$$\|u - v_\beta\| \leq \beta a(c) \quad \text{for all } f \in \mathbb{B}_c \quad \text{and all sufficiently small } \beta. \tag{2.8}$$

If A is an upper bound for the functions (2.5), then, by (2.3), ψ satisfies the condition (1.6) with $q(c) = 3A(1 \vee c)$.

Suppose $f \in \mathbb{B}_c$. Then, by (1.19) and (2.1), $f - f_\beta = e(\beta f)/\beta \leq \beta f^2 \leq \beta c^2$ and, by Theorem 1.1, there exists a solution u of (1.1) such that, for sufficiently small β,

$$\|u_\beta - u\| \leq e^{q(2c)N}\beta c^2. \tag{2.9}$$

By (1.19), $v_\beta = -\beta^{-1}\log(1-\beta u_\beta)$ and

$$v_\beta - u_\beta = F_\beta(u_\beta)$$

where $F_\beta(t) = -\beta^{-1}\log(1-\beta t) - t$. Note that $F_\beta(0) = 0$ and, for $0 < \beta t < 1/2$,

$$0 < F'_\beta(t) = \beta t(1-\beta t)^{-1} < 2\beta t$$

which implies $0 < F_\beta(t) < \beta t^2$. We have $0 \leq f_\beta \leq f$ and $u_\beta \leq K_Q f$. Therefore $u_\beta \in \mathbb{B}_c$ and

$$|v_\beta - u_\beta| \leq \beta c^2 \quad \text{for } 0 < \beta < 1/(2c). \tag{2.10}$$

It follows from (2.9) and (2.10) that (2.8) holds with $a(c) = c^2(e^{q(2c)N} + 1)$.

3°. We conclude from 2° that the limit V_Q of operators V_Q^β satisfies the Lipschitz condition on each set $\mathbb{B}_c$ and that $V_Q^\beta \xrightarrow{u} V_Q$. By Theorem **3**.3.5, there exists a BEM system X with transition operators V_Q. Since $u = V_Q(f)$ satisfies (1.1), this is a (ξ,ψ)-superprocess. □

3. Direct construction of superprocesses

3.1. Analytic definition of operators V_Q.

THEOREM 3.1. *Suppose that $Q \in \mathbb{O}_0$ and that ψ satisfies the conditions:*

3.1.A. $\psi(z,0) = 0$ for all z.

3.1.B. ψ is monotone increasing in t, i.e., $\psi(z,t_1) \le \psi(z,t_2)$ for all $z \in S$ and all $t_1 < t_2 \in \mathbb{R}_+$.

3.1.C. ψ is locally Lipschitz continuous in t uniformly in z (i.e., it satisfies (1.6)*).*

Then the equation (1.1) *has a unique solution for every $f \in \mathbb{B}$. We denote it $V_Q(f)$.*

PROOF. [1] By Theorem 1.1, equation (1.1) can have no more than one solution.

Suppose that $f \in \mathbb{B}_c$. Fix a constant $k \ge q(c)$ where $q(c)$ is defined in (1.6) and put, for every $u \ge 0$,

$$T(u) = \Pi_{r,x}\left[e^{-k(\tau-r)}f(\tau,\xi_\tau) + \int_r^\tau e^{-k(s-r)}\Phi(s,\xi_s;u(s,\xi_s))\,ds\right] \tag{3.1}$$

where $\Phi(u) = ku - \psi(u)$. [We do not indicate explicitly the dependence on T of k and f.] The key step is to prove that the sequence

$$\begin{aligned} &u_0 = 0,\\ &u_n = T(u_{n-1}) \quad \text{for } n = 1,2,\dots \end{aligned} \tag{3.2}$$

is monotone increasing and bounded. Clearly, its limit u is a bounded solution of the equation

$$u(r,x) = \Pi_{r,x}\left[e^{-k(\tau-r)}f(\tau,\xi_\tau) + \int_r^\tau e^{-k(s-r)}\Phi(s,\xi_s;u(s,\xi_s))\,ds\right]. \tag{3.3}$$

By Lemma 1.2, (3.3) implies

$$u(r,x) + k\Pi_{r,x}\int_r^\tau u(s,\xi_s)\,ds = \Pi_{r,x}\left[f(\tau,\xi_\tau) + \int_r^\tau \Phi(s,\xi_s;u(s,\xi_s))\,ds\right]$$

which is equivalent to (1.1).

We prove that:

(a) $T(v_1) \le T(v_2)$ if $0 \le v_1 \le v_2 \le c$ in Q;

(b) $T(c) \le c$.

To get (a), we note that, for $0 \le t_1 \le t_2 \le c$,

$$\Phi(t_2) - \Phi(t_1) = k(t_2 - t_1) - [\psi(t_2) - \psi(t_1)] \ge (t_2 - t_1)(k - q(c)) \ge 0.$$

Since $\psi \ge 0$, $\Phi(u) \le ku$ and therefore

$$T(c) \le \Pi_{r,x}[ce^{-k(\tau-r)} + ck\int_r^\tau e^{-k(s-r)}\,ds].$$

Since $e^{-k(\tau-r)} + k\int_r^\tau e^{-k(s-r)}ds = 1$, this implies (b).

By 3.1.A $u_1 = T(0) \ge 0$. By (a) and (b), $u_1 = T(0) \le T(c) \le c$. We use (a) and (b) to prove, by induction in n, that $0 = u_0 \le \dots \le u_n \le c$. □

[1] We use the so called monotone iteration scheme (cf., e.g., [**Sat73**]).

3.2. Properties of V_Q. We claim that:

3.2.A. If $f \le \tilde{f}$, then $V_Q(f) \le V_Q(\tilde{f})$.

3.2.B. If $Q \subset \tilde{Q}$ and if $f = 0$ on $\tilde{Q}$, then $V_Q(f) \le V_{\tilde{Q}}(f)$.

3.2.C. If $f_n \uparrow f$, then $V_Q(f_n) \uparrow V_Q(f)$.

To prove 3.2.A and 3.2.B, we indicate explicitely the dependence of operator (3.1) from k, Q and f and we note that, if $0 \le f \le \tilde{f} \le c$ and if $k > q(c)$, then $T(k, Q, f; u) \le T(k, Q, \tilde{f}; u)$ for every function $0 \le u \le c$. This implies 3.2.A. If $Q \subset \tilde{Q}$, then the first exit time $\tilde{\tau}$ from $\tilde{Q}$ is bigger than or equal to τ. If $\eta_\tau = (\tau, \xi_\tau) \in \tilde{Q}$, then $f(\eta_\tau) = 0$, and if $\eta_\tau \notin \tilde{Q}$, then $\tilde{\tau} = \tau$. In both cases, $e^{-k(\tau - r)} f(\eta_\tau) = e^{-k(\tilde{\tau} - r)} f(\eta_{\tilde{\tau}})$. If $k > q(c)$ and $0 \le u \le c$, then $T(k, \tilde{Q}, f; u) \ge T(k, Q, f; u)$ which implies 3.2.B.

Suppose that $f_n \uparrow f$ and let $u_n = V_Q(f_n)$. By 3.2.A, $u_n \uparrow u$. By passing to the limit in the equation $u_n + G_Q\psi(u_n) = K_Q f_n$, we get $u + G_Q\psi(u) = K_Q f$. Hence $u = V_Q(f)$ which proves 3.2.C.

3.3. An alternative construction of superprocesses. We deduce a slightly weaker version of Theorem 2.1 by a method suggested by Fitzsimmons (see [**Fit88**]).

Theorem 3.2. *A (ξ, ψ)-superprocess exists for function ψ given by* (2.4) *if b and n satisfy condition* (2.5) *and an additional assumption*

$$\sup_z \int_0^\beta \lambda^2 n(z; d\lambda) \to 0 \quad \text{as } \beta \downarrow 0. \tag{3.4}$$

Remark. Condition (2.5) implies pointwise but not the uniform convergence of $\int_0^\beta \lambda^2 n(z; d\lambda)$ to 0 as $\beta \downarrow 0$.

We need the following lemma:

Lemma 3.1. *Suppose that u is a solution of equation* (1.1) *and $f \in \mathbb{B}$. If $Q' \Subset Q$, then*

$$u + G_{Q'}\psi(u) = K_{Q'}u. \tag{3.5}$$

Proof. By **2**.(3.2) and **2**.(5.15), $K_{Q'}K_Q = K_Q$ and $G_Q = G_{Q'} + K_{Q'}G_Q$. The equation (1.1) implies that $G_Q\psi(u) \in \mathbb{B}$. Hence, $K_{Q'}G_Q\psi(u) \in \mathbb{B}$. Therefore

$$u + G_{Q'}\psi(u) = u + G_Q\psi(u) - K_{Q'}G_Q\psi(u) = K_{Q'}(K_Q f - G_Q\psi(u)) = K_{Q'}u.$$

□

Proof of Theorem 3.2. 1°. Operators V_Q defined in Theorem 3.1 satisfy conditions of Theorem **3**.2.1. Indeed, by (1.1), $V_Q(f) \le K_Q f$ which implies **3**.2.2.A. Properties **3**.2.2.B and **3**.2.2.C also follow easily from (1.1). Let us prove **3**.2.3.A. Suppose $Q \subset \tilde{Q} \in \mathbb{O}_0$. By Lemma 3.1, $v = V_{\tilde{Q}}(f)$ satisfies the equation $v + G_Q\psi(v) = K_Q v$. On the other hand, $u = V_Q(v)$ is a solution of the equation $u + G_Q\psi(u) = K_Q v$. The equality $u = v$ follows from Theorem 1.1.

We claim that operators V_Q satisfy condition **3**.3.4.A if:

3.3.A. There exists $k > 0$ such that $ku(f) - \psi(\cdot; u(f))$ is an N-function from $\mathbb{B}$ to $\mathbb{B}$ for every real-valued N-function $u(f)$ on $\mathbb{B}$.

Indeed, let T be the operator defined by (3.1). It follows from 3.3.A that, for all sufficiently large k, $\Phi(u(f))$ belongs to the class N if $u(f)$ is an N-function and, by **3**.3.1.A, operator T preserves the class N. Therefore $V_Q(f)$ which is the limit of $T^n(f)$ has the same property.

By Theorem **3**.3.4, V_Q are the transition operators of a BEM system X and, since $V_Q(f)$ is a solution of (1.1), X is a (ξ, ψ)-superprocess.

2°. Condition 3.3.A holds for ψ given by (2.4) under an extra assumption

$$b = 0, m(z) = \int_0^\infty \lambda n(z, d\lambda) \quad \text{is bounded.} \tag{3.6}$$

Indeed,

$$ku - \psi(u) = \int_0^\infty (1 - e^{-\lambda u}) n(d\lambda) + (k - m)u.$$

If $u \in N$, then $1 - e^{-\lambda u}$ belongs to N by **3**.3.1.B, and $ku - \psi(u)$ is an N-function if $k > m(z)$ for all z.

3°. To eliminate the side condition 3.3.A, we approximate ψ given by (2.4) by functions

$$\psi_\beta(u) = \int_0^\infty (e^{-\lambda u} - 1 + \lambda u)\ n_\beta(d\lambda)$$

where $0 < \beta < 1$ and

$$n_\beta(d\lambda) = 1_{\lambda > \beta} n(d\lambda) + 2b\beta^{-2}\delta_\beta.$$

Note that ψ_β satisfies (2.5). It satisfies (3.6) because

$$\int_0^\infty \lambda n_\beta(d\lambda) \le \beta^{-1} \int_0^\infty \lambda \wedge \lambda^2 n(d\lambda) + 2b/\beta.$$

Let V_Q^β be the transition operators of the (ξ, ψ_β)-superprocess. We demonstrated in the proof of Theorem 2.1 that V_Q satisfies the Lipschitz condition on each set $\mathbb{B}_c$. By Theorem **3**.3.5, to prove the existence of a (ξ, ψ)-superprocess, it is sufficient to show that $V_Q^\beta \xrightarrow{u} V_Q$. We have

$$\psi(u) - \psi_\beta(u) = R_\beta(u)$$

where

$$R_\beta(u) = \int_0^\beta (e^{-\lambda u} - 1 - \lambda u) n(d\lambda) + 2b\beta^{-2}[1 - \beta u + (\beta u)^2/2 - e^{-\beta u}].$$

We use the bound (2.1) and its implication

$$0 \le 1 - \beta u + (\beta u)^2/2 - e^{-\beta u} \le (\beta u)^3 \quad \text{for all } \beta > 0, u > 0 \tag{3.7}$$

and we get

$$|R_\beta(u)| \le u^2 \int_0^\beta \lambda \wedge \lambda^2 n(d\lambda) + 2b\beta u^3.$$

By conditions (2.5) and (3.4), ψ_β converges to ψ uniformly on each set $S \times [0, c]$. It follows from Remark to Theorem 1.1 that operators $V_Q^\beta \xrightarrow{u} V_Q$. □

4. Supplement to the definition of a superprocess

4.1. Extension of parameter sets. We constructed a superprocess as a BEM system with parameter sets $\mathbb{M}_0 = \mathcal{M}(S)$ and $\mathbb{O}_0$. Now we extend $\mathbb{O}_0$ to the class $\mathbb{O}_1$ of all open subsets of S and we extend $\mathbb{M}_0$ to the class $\mathbb{M}_1 = \sigma(\mathbb{M}_0)$. Measure P_μ is defined for $\mu \in \mathbb{M}_1$ by formula **3**.(3.25). For every Q and $k = 1, 2, \dots$, we denote by Q^k the intersection of Q with $(-k, k) \times E$. By **3**.4.2.D,

$$X_{Q^{k+1}}(\Gamma) \geq X_{Q^k}(\Gamma) \quad \text{a.s. for every } \Gamma \subset Q^c. \tag{4.1}$$

Therefore there exists a measure $\hat{X}_Q$ such that

$$\hat{X}_Q(\Gamma) = \lim X_{Q^k}(\Gamma) \quad \text{for } \Gamma \subset Q^c,$$
$$\hat{X}_Q(\Gamma) = 0 \quad \text{for } \Gamma \subset Q$$

[Every X_Q is defined only up to equivalence. We choose versions of X_{Q^k} for all positive integers k in such a way that (4.1) holds for all ω and all k.] Clearly, $\hat{X}_Q$ is a measure of class $\mathbb{M}_1$ and

$$\hat{X}_Q = X_Q \quad P_\mu\text{-a.s. for all } Q \in \mathbb{O}_0, \mu \in \mathbb{M}_1.$$

If $\hat{V}_Q$ is the transition operator of $\hat{X} = (\hat{X}_Q, P_\mu), Q \in \mathbb{O}_1, \mu \in \mathbb{M}_1$, then

$$\begin{aligned} \hat{V}_{Q^k} &= V_{Q^k} \quad \text{for all } k, \\ \hat{V}_{Q^k}(f) &\uparrow \hat{V}_Q(f) \quad \text{for every } f \in \mathbb{B}. \end{aligned} \tag{4.2}$$

By a monotone passage to the limit, we establish that **3**.1.3.A holds for $\hat{X}$ and that **3**.2.2.B, **3**.2.2.C and **3**.2.3.A hold for $\hat{V}_Q$. Hence, $\hat{X}$ is a branching system and, by Theorem **3**.2.1, $\hat{X}$ is a BEM system.

If $\psi(r, x; u)$ is continuous in u and satisfies condition 3.1.B, then, for every $Q \in \mathbb{O}_1$, $u = \hat{V}_Q(f)$ is a solution of (1.1). Indeed, by **3**.2.2.B, $u = \hat{V}_Q(f')$ where $f' = 1_{Q^c} f$. Since $Q^k \in \mathbb{O}_0$, function $u_k = V_{Q^k}(f')$ satisfies the equation

$$u_k(r, x) + \Pi_{r,x} \int_r^{\tau_k} \psi(s, \xi_s; u_k(s, \xi_s)) \, ds = \Pi_{r,x} f'(\tau_k, \xi_{\tau_k})$$

where τ_k is the first exit time from Q^k. For sufficiently large k, it is equal to $\tau \wedge k$ where τ is the first exit time from Q. If $\tau > k$, then $(\tau_k, \xi_{\tau_k}) \in Q$. Therefore

$$u_k(r, x) + \Pi_{r,x} \int_r^{\tau_k} \psi(s, \xi_s; u_k(s, \xi_s)) \, ds = \Pi_{r,x} 1_{\tau \leq k} f(\tau, \xi_\tau) \quad \text{for } r < k.$$

By passing to the limit as $k \to \infty$, we get that u is a solution of (1.1).

4.2. Branching measure-valued Markov processes. To every superprocess $X = (X_Q, P_\mu), Q \in \mathbb{O}_1, \mu \in \mathcal{M}(S)$ there corresponds a measure-valued Markov process $\tilde{X} = (\tilde{X}_t, \tilde{P}_{r,\nu})$. Here $\tilde{X}_t$ is the restriction of $X_{S_{<t}}$ to $S_t = \{t\} \times E$ and $\tilde{P}_{r,\nu} = P_{\delta_r \times \nu}$. Let $\tilde{\mathcal{F}}_\Delta$ stand for the σ-algebra generated by $\tilde{X}_t, t \in \Delta$. Clearly, $\mathcal{F}[r, t] \subset \mathcal{F}_{\subset S_{<t}}$ and $\mathcal{F}_{\geq t} \subset \mathcal{F}_{\supset S_{<t}}$ and the Markov property of $\tilde{X}$ follows from **3**.1.3.D. If $\varphi \in \mathbb{B}(E)$ and if $f(s, x) = \varphi(x)$ for all s, then, for all $r < t$,

$$\tilde{P}_{r,\nu} e^{-\langle f, \tilde{X}_t \rangle} = e^{-\langle u, \nu \rangle}$$

where $u_t = V_{S_{<t}}(f)$ satisfies the equation

$$u_t(r,x) + \Pi_{r,x} \int_r^t \psi(s, \xi_s; u_t(s, \xi_s))\, ds = \Pi_{r,x}\varphi(\xi_t) \quad \text{for } r \le t. \tag{4.3}$$

4.3. More properties of X_Q. Put $Y = \langle f, X_Q \rangle$. By the definition of a superprocess,

$$P_\mu e^{-\lambda Y} = e^{-\langle u_\lambda, \mu \rangle} \tag{4.4}$$

where

$$u_\lambda + G_Q \psi(u_\lambda) = \lambda K_Q f. \tag{4.5}$$

We have:

4.3.1. Measure X_Q is finite, P_μ-a.s., for every open set Q and every $\mu \in \mathcal{M}(S)$.

PROOF. Let $Y = \langle 1, X_Q \rangle$. By (4.4) and (4.5), $P_\mu e^{-\lambda Y} \ge e^{-\lambda \langle 1, \mu \rangle}$ because $u_\lambda \le \lambda K_Q 1 \le \lambda$. By taking $\lambda \to 0$, we get $P_\mu\{Y < \infty\} = 1$. □

4.3.2. For every total subset $\mathcal{T}$ of ∂Q and every μ concentrated on Q,

$$P_\mu\{X_Q(\mathcal{T}^c) = 0\} = 1.$$

Indeed, if $f = 0$ on $\mathcal{T}$, then $K_Q f = 0$ in Q and, by (4.4) and (4.5), $P_\mu e^{-\langle f, X_Q \rangle} = 1$.

We say that a function ψ belongs to class CR [2] if

$$P_\mu \langle f, X_Q \rangle = \langle K_Q f, \mu \rangle \tag{4.6}$$

for every open set Q, every positive Borel function f and every measure $\mu \in \mathcal{M}(S)$.

LEMMA 4.1. *Suppose that ψ satisfies condition 3.1.B and the condition:*

4.3.A. Function $\psi(z,u)/u$ is bounded on every set $Q \times [0, \kappa]$ and it tends to 0 as $u \to 0$.

Then $\psi \in CR$.

PROOF. It is sufficient to prove (4.6) for bounded f vanishing outside a set S_Δ with finite Δ. The general case can be obtained then by a monotone passage to the limit.

By (4.5), $u_\lambda/\lambda \le K_Q f \le \|f\|$ and, by 3.1.B and 4.3.A,

$$\lambda^{-1} G_Q \psi(u_\lambda) \le G_Q[\lambda^{-1}\psi(\lambda \|f\|)] \to 0 \quad \text{as } \lambda \to 0$$

and therefore (4.5) implies

$$\langle u_\lambda, \mu \rangle / \lambda \to \langle K_Q f, \mu \rangle \quad \text{as } \lambda \to 0. \tag{4.7}$$

On the other hand, by (4.4),

$$P_\mu \left(\frac{1 - e^{-\lambda Y}}{\lambda} \right) = \left\langle \frac{1 - e^{-\langle u_\lambda, \mu \rangle}}{\lambda} \right\rangle.$$

By (4.7), the right side tends to $\langle K_Q f, \mu \rangle$. Since the left side tends to $P_\mu Y$, we get (4.6). [3] □

[2] This is an abbreviation for "critical" — the name used often in the literature.

[3] It follows from Fatou's lemma that $P_\mu Y \le \|f\| \langle 1, \mu \rangle < \infty$ and, by (2.1), we can apply the dominated convergence theorem.

In the rest of this chapter we assume that ψ belongs to the class CR.

5. Graph of X

5.1. Random closed sets. Suppose $(\Omega, \mathcal{F})$ is a measurable space, Q is a locally compact metrizable space and $\omega \to F(\omega)$ is a map from Ω to the collection of all closed subsets of Q. Let P be a probability measure on $(\Omega, \mathcal{F})$. We say that (F, P) is a *random closed set (r.c.s.)* if, for every open set U in Q,

$$\{\omega : F(\omega) \cap U = \emptyset\} \in \mathcal{F}^P \tag{5.1}$$

where $\mathcal{F}^P$ is the completion of $\mathcal{F}$ relative to P. Two r.c.s. (F, P) and $(\tilde{F}, P)$ are equivalent if $P\{F = \tilde{F}\} = 1$. Suppose $(F_a, P), a \in A$ is a family of r.c.s. We say that a r.c.s. (F, P) is an *envelope* of (F_a, P) if:

(a) $F_a \subset F$ P-a.s. for every $a \in A$.

(b) If (a) holds for $\tilde{F}$, then $F \subset \tilde{F}$ P-a.s.

An envelope exists for every countable family. For an uncountable family, it exists under certain separability assumptions.

Note that the envelope is determined uniquely up to equivalence and that it does not change if every r.c.s. (F_a, P) is replaced by an equivalent set.

Suppose that (M, P) is a random measure on Q.The support $\mathbb{S}$ of M satisfies condition

$$\{\mathbb{S} \cap U = \emptyset\} = \{M(U) = 0\} \in \mathcal{F} \tag{5.2}$$

for every open set U and therefore $\mathbb{S}(\omega)$ is a r.c.s.

5.2. Definition and construction of graph. In section 5 we consider a (ξ, ψ)-superprocess X corresponding to a continuous strong Markov process ξ. Let $\mathcal{F}$ be the σ-algebra in Ω generated by $X_O(U)$ corresponding to all open sets O, U. The support $\mathbb{S}_O$ of X_O is a closed subset of S. To every open set O and every $\mu \in \mathcal{M}(S)$ there corresponds a r.c.s. $(\mathbb{S}_O, P_\mu)$ in S (defined up to equivalence). We shall prove that, for every Q and every μ, there exists an envelope of the family $(\mathbb{S}_O, P_\mu), O \subset Q$. We call it the *graph* $\mathcal{G}_Q$ *of* X in Q and we denote it $(\mathcal{G}_Q, P_\mu)$. We write $\mathcal{G}$ for $\mathcal{G}_S$.

THEOREM 5.1. *Consider a countable family of open subsets $\{O_1, \dots, O_n, \dots\}$ of Q such that for every open set $O \subset Q$ there exists a subsequence O_{n_k} exhausting O.* [4] *Put $a_n = \langle 1, X_{O_n} \rangle \vee 1$ and denote by $\mathbb{S}$ the support of the measure*

$$Y = \sum \frac{1}{a_n 2^n} X_{O_n}. \tag{5.3}$$

The r.c.s. $(\mathbb{S}, P_\mu)$ is the graph of X in Q.

A key step in the proof of Theorem 5.1 is the following:

LEMMA 5.1. *Suppose that O_k exhaust O. Then, for all $\mu \in \mathcal{M}(S)$ and all open sets U,*

$$P_\mu\{X_{O_k}(U) = 0 \quad \textit{for all } k, \quad X_O(U) \neq 0\} = 0. \tag{5.4}$$

We deduce this result from a relation between exit measures for X and exit points for ξ:

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[4] For instance, take a countable everywhere dense subset Λ of Q. Consider all balls contained in Q centered at points of Λ with rational radii and enumerate all finite unions of these balls.

LEMMA 5.2. *Let $O_1 \subset O_2 \subset \dots \subset O_k \Subset O$ and let τ_i be the first exit time from O_i and τ be the first exit time from O. Fix an open set U and put*

$$A_k = \{X_{O_1}(U) = \dots X_{O_k}(U) = 0\},$$
$$B_k = \{\eta_{\tau_1}, \dots, \eta_{\tau_k} \notin U\}.$$

For every positive Borel function f and every $z \in O$,

$$\int_{A_k} \langle f, X_O\rangle \, dP_z \le \int_{B_k} 1_{\tau<\infty} f(\eta_\tau) \, d\Pi_z. \tag{5.5}$$

PROOF. Formula (5.5) holds for $z \notin O_k$ because in this case $1_{A_k} = 1_{U^c}(z)$ P_z-a.s. and $1_{B_k} = 1_{U^c}(z)$ Π_z-a.s. (For $z \notin U$, we refer to (4.6).)

For $k > 1$, $A_k = A_{k-1} \cap \{X_{Q_k}(U) = 0\}$ and $B_k = B_{k-1} \cap \{\eta_{\tau_k} \notin U\}$. [For $k = 1$, these relations hold if we put $A_0 = \Omega, B_0 = \tilde{\Omega}$.] [5]

For $k = 0$, (5.5) follows from (4.6). If $k \ge 1$, then, by the Markov property **3**.1.3.D,

$$\int_{A_k} \langle f, X_O\rangle dP_z = \int_{A_{k-1}} 1_{X_{O_k}(U)=0} \langle f, X_O\rangle dP_z \tag{5.6}$$
$$= \int_{A_{k-1}} 1_{X_{O_k}(U)=0} P_{X_{O_k}} \langle f, X_O\rangle dP_z.$$

It follows from (4.6) that $P_{X_{O_k}}\langle f, X_O\rangle = \langle g, X_{O_k}\rangle$ where $g(z) = P_z\langle f, X_O\rangle = K_O f(z)$. For every ν, $1_{\nu(U)=0}\langle g, \nu\rangle \le \langle g\, 1_{U^c}, \nu\rangle$ and, by (5.6),

$$\int_{A_k} \langle f, X_O\rangle \, dP_z \le \int_{A_{k-1}} \langle 1_{U^c}\, g, X_{O_k}\rangle \, dP_z. \tag{5.7}$$

Suppose that (5.5) holds for $k-1$. Then the right side in (5.7) is dominated by

$$\int_{B_{k-1}} 1_{\tau_k<\infty} (1_{U^c}\, g)(\eta_{\tau_k}) d\Pi_z = \Pi_z X[1_O K_O(f)](\eta_{\tau_k}) \tag{5.8}$$

where

$$X = 1_{B_{k-1}} 1_{U^c}(\eta_{\tau_k}) 1_{\tau_k<\infty} = 1_{B_k} 1_{\tau_k<\infty}$$

is a pre-τ function. By **2**.2.1.B (applied to the pair $\tau_k \le \tau$ and to $Q' = O$), the right side in (5.8) is equal to

$$\Pi_z X 1_O(\eta_{\tau_k}) 1_{\tau<\infty} f(\eta_\tau) = \Pi_z 1_{B_k} 1_O(\eta_{\tau_k}) 1_{\tau<\infty} f(\eta_\tau),$$

and (5.7) implies (5.5). □

PROOF OF LEMMA 5.1. Put

$$Y_1 = \sum_1^\infty X_{O_k}(U), \quad Y_2 = Y_1 + X_O(U).$$

We need to prove that $\{Y_1 = 0\} \subset \{Y_2 = 0\}$ P_μ-a.s. for all μ. By Proposition **3**.4.1, it is sufficient to prove this relation for $P_z, z \in S$. If $z \notin O$, then $P_z\{Y_1 = Y_2 = 0\} = 1$ for $z \notin U$ and $P_z\{Y_1 = Y_2 = \infty\} = 1$ for $z \in U$. If $z \in O$, then we apply Lemma 5.2 and we pass to the limit in (5.5). Note that $A_k \downarrow A_\infty = \{X_{O_k}(U) = 0 \quad \text{for all } k\}$; and $B_k \downarrow B_\infty = \{\eta_{\tau_k} \notin U \text{ for all } k\}$. Choose a continuous bounded function f such that $f > 0$ on U and $f = 0$ on U^c. Note that $\tau_k \uparrow \tau$. If $\tau < \infty$, then

[5] Parameter ω in $X_O(\omega)$ and in $\xi_t(\omega)$ takes values in two unrelated spaces which we denote Ω and $\tilde{\Omega}$.

$\eta_{\tau_k} \to \eta_\tau$ and $f(\eta_{\tau_k}) \to f(\eta_\tau)$. Hence, $\{B_\infty, \tau < \infty\} \subset \{f(\eta_\tau) = 0\}$. We deduce from (5.5) that

$$P_z 1_{A_\infty} \langle f, X_O \rangle \leq \Pi_z 1_{B_\infty, \tau<\infty} f(\eta_\tau) = 0 \tag{5.9}$$

which implies (5.4). □

Proof of Theorem 5.1. We fix a measure $\mu \in \mathcal{M}(S)$ and we use a short writing a.s. for P_μ-a.s. Let us prove that, for each $O \subset Q$, $\mathbb{S} \supset \mathbb{S}_O$ a.s. Note that, if F_1, F_2 are closed sets, then $F_1 \supset F_2$ if and only if, for every open U,

$$\{F_1 \cap U = \emptyset\} \subset \{F_2 \cap U = \emptyset\}.$$

Therefore we need only to prove that, for every open U,

$$\{\mathbb{S} \cap U = \emptyset\} \subset \{\mathbb{S}_O \cap U = \emptyset\} \quad \text{a.s.} \tag{5.10}$$

Since $\mathbb{S}$ and the measure (5.3) are related by (5.2), it is sufficient to demonstrate that

$$\{Y(U) = 0\} \subset \{X_O(U) = 0\} \quad \text{a.s. for all } O \subset Q. \tag{5.11}$$

Clearly,

$$\{Y(U) = 0\} \subset \{X_{O_n}(U) = 0 \quad \text{for all } n\}. \tag{5.12}$$

Consider a subsequence O_{n_k} of O_n exhausting O. By Lemma 5.1,

$$\{X_{O_{n_k}}(U) = 0 \quad \text{for all } k\} \subset \{X_O(U) = 0\} \quad \text{a.s.}. \tag{5.13}$$

Formula (5.11) follows from (5.12) and (5.13).

Suppose that F is a r.c.s. such that $\mathbb{S}_O \subset F$ a.s. for all $O \subset Q$. Then $X_O(F^c) = 0$ and therefore $Y(F^c) = 0$ a.s. which implies that $\mathbb{S} \subset F$ a.s. □

5.3. Graphs and null sets of exit measures. We write $\mu \in \mathcal{M}(B)$ if $\mu(B^c) = 0$ (i.e., if μ is concentrated on B). We denote by $\mathcal{M}_c(Q)$ the class of measures μ such that the support of μ is compact and is contained in Q.

First we prove the following result.

Theorem 5.2. *Let B be a closed subset of S and let O_n be bounded open sets such that $\bar{O}_{n-1} \subset O_n$ and $O_n \uparrow B^c$. Then*

$$\{X_{O_n} = 0\} \uparrow \{\mathcal{G} \quad \textit{is compact and } \mathcal{G} \cap B = \emptyset\} \quad \textit{a.s.} \tag{5.14}$$

[If measure μ is not specified, then "a.s." means "P_μ-a.s. for all μ".]

The sequence $\{X_{O_n} = 0\}$ is, a.s., monotone increasing by **3**.4.2.A. To prove (5.14), it is sufficient to establish that

$$\{\mathcal{G} \subset \bar{O}_{n-1}\} \subset \{X_{O_n} = 0\} \subset \{\mathcal{G} \subset \bar{O}_n\} \quad \text{a.s.}$$

This is an implication of the following two propositions:

5.3.A. If $\bar{O}_1 \subset O_2 \subset Q$, then

$$\{\mathcal{G}_Q \subset \bar{O}_1\} \subset \{X_{O_2} = 0\} \quad \text{a.s.} \tag{5.15}$$

Proof. By the definition of the graph, X_{O_2} is concentrated, a.s., on $\mathcal{G}_Q$. By **3**.1.3.B, it is concentrated on O_2^c. Since $\bar{O}_1 \cap O_2^c = \emptyset$, we get (5.15). □

5.3.B. For every Q,

$$\{X_Q = 0\} \subset \{\mathcal{G} \subset \bar{Q}\} \quad \text{a.s.} \tag{5.16}$$

PROOF. Let U be an arbitrary open subset of S. By **3**.4.2.D applied to $Q_1 = U, Q_2 = Q \cup U, \Gamma = Q_2^c$, we have $X_U(Q_2^c) \le X_{Q\cup U}(Q_2^c)$ a.s. By **3**.4.2.A,

$$\{X_Q = 0\} \subset \{X_{Q\cup U} = 0\} \quad \text{a.s.}$$

Hence

$$\{X_Q = 0\} \subset \{X_U(Q_2^c) = 0\}.$$

Since X_U is concentrated on U^c, the condition $\{X_Q = 0\}$ implies that X_U is concentrated, a.s., on $Q_2 \cap U^c \subset Q$ and therefore $\mathbb{S}_U \subset \bar{Q}$. □

COROLLARY 5.1. *If Q_n are bounded open sets such that $\bar{Q}_n \subset Q_{n+1}$ and $Q_n \uparrow S$, then*

$$\{X_{Q_n} = 0\} \uparrow \{\mathcal{G} \quad \textit{is compact}\} \quad \textit{a.s.} \tag{5.17}$$

THEOREM 5.3. *Let Γ be a closed subset of ∂Q which contains all irregular points and let*

$$\Omega_\Gamma = \{\mathcal{G}_Q \quad \textit{is compact and } \mathcal{G}_Q \cap \Gamma = \emptyset\}. \tag{5.18}$$

*If Q_n is a (Q,Γ)-sequence [see **2**. Lemma 4.4], then the sequence*

$$A_n = \{X_{Q_n}(Q) = 0\}$$

satisfies the conditions: for every $\mu \in \mathcal{M}(Q_k)$,

$$A_k \subset A_{k+1} \subset \dots \subset A_n \subset \dots \quad P_\mu\textit{-a.s.}, \tag{5.19}$$

$$\bigcup_{n\ge k} A_n = \Omega_\Gamma \quad P_\mu\textit{-a.s.} \tag{5.20}$$

The proof is based on the following three propositions:

5.3.C. If $\mu \in \mathcal{M}(Q)$, then, P_μ-a.s., $X_Q \in \mathcal{M}(\partial Q)$.

PROOF. For every Γ, we have $P_\mu e^{-X_Q(\Gamma)} = e^{-\langle u,\mu\rangle}$ where $u = V_Q(1_\Gamma)$ satisfies equation (1.1). Clearly, $u \le K_Q(1_\Gamma)$. If $\Gamma = \bar{Q}^c$, then $K_Q(1_\Gamma) = 0$ on Q. Hence, $u = 0$ on Q and $X_Q(\Gamma) = 0$ P_μ-a.s. because $P_\mu e^{-X_Q(\Gamma)} = 1$. By **3**.1.3.B, $X_Q(Q) = 0$ a.s. which implies 5.3.C. □

5.3.D. If $Q_1 \subset Q_2$ and if $\mu \in \mathcal{M}(Q_1)$, then

$$\{X_{Q_1}(Q_2) = 0\} \subset \{X_{Q_2}(\bar{Q}_1^c) = 0\} \quad P_\mu\text{-a.s.} \tag{5.21}$$

PROOF. We note that $A = \{X_{Q_1}(Q_2) = 0\} \in \mathcal{F}_{\subset Q_1}$ and $C = \{X_{Q_2}(\bar{Q}_1^c) > 0\} \in \mathcal{F}_{\supset Q_1}$. Therefore, by the Markov property **3**.1.3.D,

$$P_\mu AC = P_\mu\{1_A P_{X_{Q_1}}(C)\}. \tag{5.22}$$

If $\nu(Q_2) = 0$, then $X_{Q_2} = X_{Q_1} = \nu$ P_ν-a.s. and $P_\nu(C) = 1_{\nu(\bar{Q}_1^c)>0}$. If, in addition, $\nu(\bar{Q}_1^c) = 0$, then $P_\nu(C) = 0$. By 5.3.C, X_{Q_1} is concentrated, P_μ-a.s., on $\bar{Q}_1$ and therefore (5.22) implies (5.21). □

5.3.E. If $Q_1 \subset Q_2$ and if $\mu \in \mathcal{M}(Q_1)$, then

$$\{X_{Q_1}(Q_2) = 0\} \subset \{\mathcal{G}_{Q_2} \subset \bar{Q}_1\} \quad P_\mu\text{-a.s.} \tag{5.23}$$

PROOF. It is sufficient to show that the relation

$$\{X_{Q_1}(Q_2) = 0\} \subset \{X_U(\bar{Q}_1^c) = 0\} \tag{5.24}$$

holds P_μ-a.s. for every $U \subset Q_2$. By Proposition **3**.4.2, we need only to establish that (5.24) holds P_z-a.s. for all $z \in Q_1$. It holds for $z \notin U$ because, for such z, $X_U(\bar{Q}_1^c) = \delta_z(\bar{Q}_1^c) = 0$ P_z-a.s. by **3**.1.3.C. If $z \in U_1 = U \cap Q_1$, then, by 5.3.D,

$$\{X_{U_1}(U) = 0\} \subset \{X_U \in \mathcal{M}(\bar{U}_1)\} \subset \{X_U \in \mathcal{M}(\bar{Q}_1)\} P_z - \text{a.s.}$$

It remains to show that, P_z-a.s.,

$$\{X_{Q_1}(Q_2) = 0\} \subset \{X_{U_1}(U) = 0\}. \tag{5.25}$$

Put $\Gamma = U \cap Q_1^c$. By **3**.1.3.B, $X_{U_1}(U) = X_{U_1}(\Gamma)$ a.s. By **3**.4.2.D, $X_{U_1}(\Gamma) \leq X_{Q_1}(\Gamma)$ a.s. because $U_1 \subset Q_1$ and $\Gamma \subset Q_1^c$. Since $\Gamma \subset U \subset Q_2$, we have

$$X_{U_1}(\Gamma) \leq X_{Q_1}(\Gamma) \leq X_{Q_1}(Q_2) \quad \text{a.s.}$$

which implies (5.25). □

PROOF OF THEOREM 5.3. 1°. We claim that, if $\mu \in \mathcal{M}(Q)$, then

$$B_n = \{\mathcal{G}_Q \subset \bar{Q}_n\} \uparrow \Omega_\Gamma \quad P_\mu\text{-a.s.}$$

To prove this, we establish that every compact set $K \subset \bar{Q}$ disjoint from Γ is contained in $\bar{Q}_n$ for all sufficiently large n. Indeed, if this is false, then, for every n, there exists $x_n \in K$ such that $x_n \in \bar{Q} \setminus Q_n$. If $x_{n_i} \to x$, then $x \in K \cap (\bar{Q} \setminus Q_n)$ for all n and therefore $x \in \partial Q \setminus \Gamma$. Since $\partial Q_n \cap Q \uparrow \partial Q \setminus \Gamma$, x belongs to $\partial Q_m \cap \partial Q$ for some m. But then the relation $x_{n_i} \to x$ is in contradiction with the definition of (Q, Γ)-sequence (see **2**.(4.15)).

2°. We fix a measure μ concentrated on Q_k and we drop indications that each of subsequent statements holds P_μ-a.s. Let $n \geq k$. Then

$$A_n \subset \{X_Q(\bar{Q}_n^c) = 0\} \subset A_{n+1}. \tag{5.26}$$

The first part follows from 5.3.D. The second part holds because, by 5.3.C, $X_{Q_{n+1}}$ is concentrated on ∂Q_{n+1}; therefore $A_{n+1} = \{X_{Q_{n+1}}(Q \cap \partial Q_{n+1}) = 0\}$. It remains to note that $\bar{Q}_n^c \supset \partial Q_{n+1} \cap Q$.

3°. By 5.3.E, $A_n \subset B_n$. By the definition of the graph, $B_n \subset \{X_{Q_{n+1}}(\bar{Q}_n^c) = 0\}$ and, by (5.26), $B_n \subset A_{n+1}$. Formulae (5.19) and (5.20) follow from 1°. □

COROLLARY 5.2. *The CB property holds for* Ω_Γ *and* P_μ *if* Γ *is a closed subset of* ∂Q *which contains all irregular points and if* $\mu \in \mathcal{M}_c(Q)$.

This follows from Theorem 5.3 and Proposition **3**.4.1.

6. Notes

6.1. Early history. Various generalizations of Galton-Watson process are presented in books of Harris [**Har63**], Sevast'yanov [**Sev71**], Athreya and Ney [**AN72**] and Jagers [**Jag75**]. Feller [**Fel51**] considered a passage to the limit in the Galton-Watson model and he obtained this way a Markov process on $\mathbb{R}_+$ which, in our terminology, is a (ξ, ψ)-superprocess with a single point space E, $\xi_t = \xi_0$ and $\psi(u) = u^2$. Superprocesses with the same space E but more general ψ were studied by Lamperti [**Lam67**]. Jiřina [**Jiř58**] investigated superprocesses with finite space E. [He called them continuous state branching processes.] The foundations of a general theory of superprocesses were laid by S. Watanabe in [**Wat68**]. Like all his predecessors, he worked with time homogeneous ξ and time independent ψ and he investigated the corresponding time homogeneous branching measure-valued process (X_t, P_μ). He paid a special attention to the quadratic branching $\psi(x,u) = b(x)u^2$. He proved, that in this case X_t is continuous and that it can be obtained by a passage to the limit from branching particle systems. Dawson [**Daw75**] initiated another approach to superprocesses via the Itô stochastic calculus. A series of papers by Dawson and his collaborators was devoted to investigation of the super-Brownian motion with the quadratic branching mechanism. This process is often called the Dawson-Watanabe superprocess. [The name "superprocess" appeared, first in [**Dyn88**].] Dawson's Saint-Flour lecture notes [**Daw93**] contain a survey of the literature on measure-valued processes until 1992.

6.2. General measure-valued branching Markov processes. The branching property for a measure-valued Markov process $X = (X_t, P_{r,\nu})$ can be stated as follows. For every ν and every f,

$$\log P_{r,\nu} e^{-\langle f, X_t \rangle} = \int_E \log P_{(r,x)} e^{-\langle f, X_t \rangle} \; \nu(dx). \tag{6.1}$$

The problem of description of all measure-valued Markov processes with this property has attracted a number of investigators. A survey of the results in this direction is given in section **14**.1.

6.3. Regularity properties, range and graph. Regularity properties of paths of superprocesses were investigated by Fitzsimmons [**Fit88**] in time homogeneous setting. His results were extended to a nonhomogeneous case in [**Kuz94**] (see also [**Dyn89a**]). A set-valued process $\operatorname{supp} X_t$ was investigated in great detail in the case of the Dawson-Watanabe superprocess [**DH79**], [**Per88**], [**Per89**], [**Per90**], [**DIP89**]. (We describe main results of this work in section **14**.3.)

In [**Dyn92**] the graph of X_t was defined as the minimal closed set which contains supports of X_t for all t. The definition and construction of the graph of X in this chapter follows [**Dyn98a**].

A closely related concept of the range of X (see the definition in Chapter 10) was investigated in [**DIP89**] (see also [**Isc88**], [**She97**], [**Dyn91c**]).

6.4. Limit theorems. Convergence of rescaled branching particle system to superprocesses in various settings was studied by a number of authors (see, in particular, [**Daw75**], [**EK86**], Chapter 9, [**Dyn91a**], [**Dyn91b**]). Recently it was discovered that the Dawson-Watanabe process is the limit of other well known particle systems (contact processes, voter models, coalescing random walks...) [**CDP99**], [**DP99**], [**CDP00**], [**BCG01**].

CHAPTER 5

Semilinear parabolic equations and superdiffusions

1. Introduction

In this chapter we investigate semilinear differential equations

$$\dot{u} + Lu = \psi(u) \quad \text{in } Q \tag{1.1}$$

and their connections with superprocesses. [1]

We say that a (ξ, ψ)-superprocess is an (L, ψ)-*superdiffusion*, if ξ is an L-diffusion. If ψ has the form described in Theorem **4**.2.1, then for every L-diffusion ξ, there exists an (L, ψ)-superdiffusion and we can use it for investigating the equation (1.1). (In particular, this is possible for functions $k(z)u^\alpha$ with $1 < \alpha \le 2$). For a wider class of ψ, we use analytic tools and diffusions. A link between (1.1), diffusions and superdiffusions is provided by the integral equation

$$u + G_Q\psi(u) = K_Q f \tag{1.2}$$

(cf. **4**.(1.1)).

We start from the study of relations between (1.1) and (1.2). Then we establish that, under mild conditions on ψ, all solutions of (1.1) are locally uniformly bounded. At the end of the chapter we investigate boundary value problems for (1.1) (with functions on the boundary taking values in $[0, +\infty]$). We construct the minimal solution with prescribed boundary value on a portion of the boundary and the maximal solution vanishing on a given part of the boundary. Both solutions have simple expressions in terms of an (L, ψ)-superdiffusion (in the case when such a superdiffusion exists).

At various stages of our investigation, we impose some of the following assumptions on ψ: [2]

1.A. $\psi(z, 0) = 0$ for all z.

1.B. All the first partials of ψ are continuous.

1.C. ψ is monotone increasing in t.

1.D. ψ is locally Lipschitz continuous in t uniformly in z.

2. Connections between differential and integral equations

2.1. From integral equation (1.2) **to differential equation** (1.1)**.** We use the following results on operators K_Q and G_Q proved in Chapter **2** [see Theorem **2**.3.1, Theorem **2**.4.2 and propositions **2**.5.3.A, **2**.5.3.C].

Suppose that Q is a bounded open set and $f \in \mathbb{B}$. Then:

[1] We consider positive solutions u of (1.1) which belong to class $\mathbb{C}^2(Q)$ defined in section **2**.1.2.

[2] Cf. conditions 3.1.A, 3.1.B, 3.1.C in Chapter **4**

2.1.A. Function $u = K_Q f$ belongs to class $\mathbb{C}^2(Q)$ and

$$\dot{u} + Lu = 0 \quad \text{in } Q.$$

If $\tilde{z}$ is a regular point of ∂Q and if f is continuous at $\tilde{z}$, then

$$u(z) \to f(\tilde{z}) \quad \text{as } z \to \tilde{z}.$$

2.1.B. Function $w = G_Q f$ belongs to $\mathbb{C}^\lambda(Q)$. If $f \in \mathbb{C}^\lambda(Q)$, then $w \in \mathbb{C}^2(Q)$ and it is a solution of the equation

$$\dot{w} + Lw = -f \quad \text{in } Q.$$

If $\tilde{z}$ is a regular point of ∂Q, then

$$w(z) \to 0 \quad \text{as } z \to \tilde{z}.$$

THEOREM 2.1. *Suppose that $f \in \mathbb{B}$. Under the condition 1.B, every solution u of* (1.2) *satisfies* (1.1).

If $\tilde{z} \in \partial_{reg} Q$ and f is continuous at $\tilde{z}$, then

$$u(z) \to f(\tilde{z}) \quad \text{as } z \to \tilde{z}, z \in Q. \tag{2.1}$$

PROOF. By Lemma **4**.3.1, equation (1.2) implies a similar equation in every subdomain of Q. Therefore we can assume that Q is bounded.

Function $h = K_Q f$ is bounded. Since $u \le h$, u is also bounded. Function ψ is bounded on each set $Q \times [0, c]$. Hence, $\rho = \psi(u)$ is bounded. By 2.1.A and 2.1.B, h and $F = G_Q \rho$ belong to $\mathbb{C}^\lambda(Q)$. Therefore $u = h - F \in \mathbb{C}^\lambda(Q)$. By 1.B, $\rho \in \mathbb{C}^\lambda(Q)$. By 2.1.A and 2.1.B, this implies $h, F \in \mathbb{C}^2(Q)$ and therefore u also belongs to $\mathbb{C}^2(Q)$. By using 2.1.A and 2.1.B once more,we prove that u is a solution of (1.1).

The second part of the theorem also follows from 2.1.A and 2.1.B. □

The following result is an immediate implication of Theorem 2.1 and the definition of a superdiffusion.

THEOREM 2.2. *If (X_Q, P_μ) is an (L, ψ)-superdiffusion and if ψ satisfies condition 1.B, then, for every $f \in \mathbb{B}$, function*

$$u(z) = -\log P_z e^{-\langle f, X_Q \rangle} \tag{2.2}$$

is a solution of equation (1.1).

2.2. From differential equation (1.1) **to integral equation** (1.2)**.** Recall the Improved maximum principle **2**.3.6.C:

2.2.1. Suppose that $\mathcal{T}$ is a total subset of ∂Q. If $v \in \mathbb{C}^2(Q)$ is bounded above and if it satisfies conditions

$$\dot{v} + Lv \ge 0 \quad \text{in } Q,$$

$$\limsup v(z) \le 0 \quad \text{as } z \to \tilde{z} \quad \text{for all } \tilde{z} \in \mathcal{T},$$

then $v \le 0$ in Q.

By using this principle, we get:

2.2.A. Suppose Q is bounded, $\mathcal{T} \subset \partial_{reg} Q$ is total in ∂Q and u is a bounded solution of (1.1) in Q. If u is continuous on $Q \cup \mathcal{T}$, then

$$u + G_Q \psi(u) = K_Q u. \tag{2.3}$$

PROOF. By 2.1.A and 2.1.B, $F_1 = u + G_Q \psi(u)$ and $F_2 = K_Q u$ satisfy equation $\dot{F} + LF = 0$ in Q and $F_1 = F_2 = u$ on $\mathcal{T}$. It follows from 2.2.1 that $F_1 = F_2$ in Q. □

2.2.B. If u is a solution of (1.1) in Q, then $V_U(u) = u$ for every $U \Subset Q$.

PROOF. We need to check that

$$u + G_U \psi(u) = K_U u. \tag{2.4}$$

If U is regular, this follows from 2.2.A . In general, we consider a sequence of regular open sets U_n exhausting U (which exists by Theorem **2**.4.5) and we pass to the limit in the equation $u + G_{U_n} \psi(u) = K_{U_n} u$. □

2.2.C. If u is a bounded solution of (1.1) in Q and if $u = f$ on a total subset $\mathcal{T}$ of ∂Q, then u satisfies equation (1.2).

PROOF. We consider a sequence of open sets Q_n exhausting Q. By (2.4), $u + G_{Q_n} \psi(u) = K_{Q_n} u$. If τ_n and τ are the first exit times from Q_n and Q, then $\tau_n \uparrow \tau$ Π_z-a.s. for all $z \in Q$. Hence, $K_{Q_n} u \to K_Q f$ and $G_{Q_n} u \to G_Q u$ which implies (1.2). □

2.3. Comparison principle. The following theorem provides for semilinear equations a tool similar to 2.2.1.

THEOREM 2.3. *Suppose that $\mathcal{T}$ is a total subset of ∂Q and ψ satisfies 1.C. Then $u \le v$ in Q assuming that:*

(a) $u, v \in \mathbb{C}^2(Q)$;

(b) $u - v$ is bounded above and

$$\dot{u} + Lu - \psi(u) \ge \dot{v} + Lv - \psi(v) \quad \textit{in } Q; \tag{2.5}$$

(c) for every $\tilde{z} \in \mathcal{T}$,

$$\limsup [u(z) - v(z)] \le 0 \quad \textit{as } z \to \tilde{z}. \tag{2.6}$$

PROOF. Let $w = u - v$. If the theorem is false, then $\tilde{Q} = \{z : z \in Q, w(z) > 0\}$ is not empty. By (2.5), $\dot{w}(z) + Lw(z) \ge \psi(z, u(z)) - \psi(z, v(z)) \ge 0$ in $\tilde{Q}$. Note that $\tilde{\mathcal{T}} = \partial \tilde{Q} \cap (Q \cup \mathcal{T})$ is a total subset of $\partial \tilde{Q}$. If $\tilde{z} \in \partial \tilde{Q} \cap Q$, then $w(\tilde{z}) = 0$. If $\tilde{z} \in \partial \tilde{Q} \cap \mathcal{T}$, then

$$\limsup w(z) \le 0 \quad \text{as } z \to \tilde{z}, z \in Q$$

by (2.6). We arrive at a contradiction with 2.2.1. □

Suppose $u \in \mathbb{C}^2(Q)$. We say that u is a supersolution of (1.1) if

$$\dot{u} + Lu \le \psi(u) \quad \text{in } Q \tag{2.7}$$

and that it is a subsolution of (1.1) if

$$\dot{u} + Lu \ge \psi(u) \quad \text{in } Q. \tag{2.8}$$

The Comparison principle implies

2.3.A. If u is a subsolution and v is a supersolution in a domain Q and if $u-v$ is bounded above, then (2.6) implies that $u \le v$ in Q.

3. Absolute barriers

3.1. Classes BR and BR_1. A real-valued function $u^0(x)$ is called an *absolute barrier* for the equation (1.1) in Q if it is an upper bound for every bounded positive subsolution of (1.1). Note that, if u_i^0 is an absolute barrier in Q_i for $i = 1, \dots, n$, then $\max u_i^0$ is an absolute barrier in the union of Q_i. If $Q_n \uparrow Q$ and if u_n^0 is an absolute barrier in Q_n, then

$$u^0 = \begin{cases} u_1^0 & \text{on } Q_1, \\ u_n^0 & \text{on } Q_n \setminus Q_{n-1} \quad \text{for } n > 1 \end{cases}$$

is a barrier in Q.

Put $\psi \in BR$ if (1.1) has an absolute barrier in every open set Q. Clearly, class BR contains with every function ψ all functions bigger than ψ. To prove that $\psi \in BR$, it is sufficient to construct an absolute barrier in each cylinder

$$Q = (t_1, t_2) \times D \quad \text{where } D = \{x : |x - x^0| < R\}. \tag{3.1}$$

Moreover, it is sufficient to do this for sufficiently small $d(Q) = t_2 - t_1 + R$. Recall that

$$\mathcal{T} = [(t_1, t_2) \times \partial D] \cup [\{t_2\} \times \bar{D}] \tag{3.2}$$

is a total subset of ∂Q (see section **2**.4.2). Denote by $\mathbb{Q}_\psi$ the class of all cylinders (3.1) with the property: there exists a supersolution u^0 in Q such that

$$u^0(z) \to \infty \quad \text{as } z \to \tilde{z}, z \in Q \tag{3.3}$$

for all $\tilde{z} \in \mathcal{T}$. Put $\psi \in BR_1$ if $\mathbb{Q}_\psi$ is a base of topology in S (i.e., if, for every z and every neighborhood U of z there exists $Q \in \mathbb{Q}_\psi$ such that $z \in Q \subset U$). By Theorem 2.3, a supersolution u^0 with property (3.3) is an absolute barrier in Q and therefore $BR_1 \subset BR$. Later we will see that $BR_1 = BR$.

THEOREM 3.1. *A function $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ belongs to class BR_1 if:*

3.1.A. ψ is convex and $\psi(0+) = 0$, $\psi(u) > 0$ for $u > 0$.

3.1.B. $\int_N^\infty ds \ \left[\int_0^s \psi(u) \ du\right]^{-1/2} < \infty$ for some $N > 0$.

REMARK 3.1. If a function $\psi(u)$ satisfies conditions 3.1.A and 3.1.B, then $\psi(u)/u \to \infty$ as $u \to \infty$. Condition 3.1.A implies that assumption 3.1.B holds for all $N > 0$ if it holds for some $N > 0$.

REMARK 3.2. Function $\psi(u) = ku^\alpha$ satisfies conditions 3.1.A–3.1.B if $\alpha > 1$ and $k > 0$ is a constant. If $k(z)$ is a continuous and strictly positive function, then $\psi(z, u) = k(z)u^\alpha$ belongs to class BR because it is bigger than a function of this class on every cylinder (3.1).

Proof of Theorem 3.1 is based on an inequality $\psi(u + v) \ge \psi(u) + \psi(v)$ which follows from the condition 3.1.A. Namely, we use that

$$u^0(r, x) = u(r) + v(x)$$

is a supersolution with the property (3.3) if

$$\begin{aligned} \dot{u} &\le \psi(u) \quad \text{for } t_1 \le r < t_2, \\ u &= \infty \quad \text{for } r = t_2 \end{aligned} \tag{3.4}$$

and

$$\begin{aligned} Lv &\le \psi(v) \quad \text{in } D, \\ v &= \infty \quad \text{on } \partial D. \end{aligned} \tag{3.5}$$

In an important particular case $\psi(u) = u^\alpha, \alpha > 1$, problems (3.4) and (3.5) can be solved easily by taking

$$u(r) = [(\alpha - 1)(t_2 - r)]^{-1/(\alpha-1)} \tag{3.6}$$

and

$$v(x) = \lambda(R^2 - |x - x^0|^2)^{-2/(\alpha-1)} \tag{3.7}$$

with sufficiently big λ. [3] For a general ψ, we solve problems (3.4) and (3.5) by investigating certain ordinary differential equations. This will be done by proving a series of lemmas in the next section.

3.2. We assume that ψ satisfies conditions 3.1.A–3.1.B.

LEMMA 3.1. *Problem (3.4) has a solution for sufficiently small $t_2 - t_1$.*

PROOF. By 3.1.A, $\psi(s)$ and $\psi(s)/s$ are monotone increasing functions and therefore, for $s > 1$,

$$\int_0^s \psi(u)\, du \le s\psi(s) \le \psi(s)^2/\psi(1).$$

Hence,

$$\sqrt{\psi(1)} \int_N^\infty \psi(u)^{-1} du \le \int_N^\infty \, ds[\int_0^s \psi(u)\, du]^{-1/2} < \infty$$

for $N > 1$. The function

$$F(w) = \int_w^\infty [\psi(s)]^{-1}\, ds, w \in [N, \infty)$$

is continuous, monotone decreasing and satisfies conditions $F(\infty) = 0$ and $F'(w) = -1/\psi(w)$. The inverse function $w(r)$ is a solution of equation $\dot{w} = -\psi(w)$ on the interval $(0, F(N))$ and $w(0+) = \infty$. If $t_2 - t_1 < F(N)$, then $u(r) = w(t_2 - r)$ satisfies equation $\dot{u}(r) = \psi[w(t_2 - r)]$ on (t_1, t_2) and $u(t_2) = \infty$. □

LEMMA 3.2. *Suppose that $0 < R < 1$ and that ϕ is a function of class [4] $C^2[0, R)$. Let*

$$u^0(x) = \phi(\rho) \tag{3.8}$$

where $\rho = |x - x^0|^2$. Then u^0 belongs to $C^2(D)$. If

$$\phi'(t) \quad \text{and } \phi''(t) \ge 0 \quad \text{for all } t, \tag{3.9}$$

---

[3] One can take $\lambda = cR^{2/(\alpha-1)}(1 \vee R)^{1/(\alpha-1)}$ where c is a constant depending only on upper bounds of the coefficients of L in D. [See [**Dyn91c**], pp. 101-102].

[4] We denote, as usual, by C^2 the class of twice continuously differentiable functions.

then

$$Lu^0(x) \le \alpha\phi''(\rho(x)) + \beta\phi'(\rho(x)) \quad \textit{in } D \tag{3.10}$$

where positive constants α, β *depend only on the coefficients* a_{ij}, b_i *of* L *in* D.

PROOF. Put $z_i = x_i - x_i^0$. We have

$$\begin{aligned}\frac{\partial u^0}{\partial x_i} &= 2\phi'(\rho)z_i,\\ \frac{\partial^2 u^0}{\partial x_i \partial x_j} &= 4\phi''(\rho)z_i z_j \quad \text{for } i \ne j,\\ \frac{\partial^2 u^0}{(\partial x_i)^2} &= 4\phi''(\rho)z_i^2 + 2\phi'(\rho).\end{aligned}$$

Note that

$$Lu^0 = A_1\phi''(\rho) + A_2\phi'(\rho)$$

where

$$\begin{aligned}A_1 &= 4\sum a_{ij}z_i z_j,\\ A_2 &= 2\sum a_{ii} + 2\sum b_i z_i.\end{aligned}$$

If $\alpha_1(x)$ is the maximal eigenvalue of the matrix $4a_{ij}(x)$, then

$$4\sum a_{ij}(x)z_i z_j \le \alpha_1(x)|z|^2.$$

Therefore $A_1 \le \alpha$ where α is an upper bound for $\alpha_1(x)$ in D. If β_1 is an upper bound for $(\sum b_i(x)^2)^{1/2}$ in D, then $A_2 \le \beta = 2\beta_1 + \alpha d$. This implies (3.10). □

LEMMA 3.3. *For every* $\varepsilon > 0$, *there exists a constant* $0 < R < \varepsilon$ *and a solution of equation*

$$f''(t) = \psi(f(t)) \quad \textit{for } 0 < t < R \tag{3.11}$$

with the properties

$$f, f', f'' \ge 0 \quad \textit{and } f'(0) = 0, \tag{3.12}$$

$$f(R-) = \infty, \tag{3.13}$$

$$0 < f'(t) \le t\psi(f(t)) \quad \textit{for } 0 < t < R. \tag{3.14}$$

PROOF. We consider all values of R for which there exists a solution f of (3.11) subject to the conditions $f(0) = c, f'(0) = 0$ and we denote by R_c the supremum of such R. A basic theorem on ordinary differential equations implies that $R_c > 0$ for every $c > 0$. We are going to prove that $R_c \to 0$ as $c \to \infty$ and that $f(R_c-) = \infty$.

We have

$$f'(t) = \int_0^t \psi(f(r))\ dr \quad \text{in } (0, R_c) \tag{3.15}$$

and

$$f(t) = c + \int_0^t ds \int_0^s \psi(f(r))\ dr \quad \text{on } (0, R_c). \tag{3.16}$$

It is clear from (3.15) and (3.16) that f satisfies conditions (3.12) and (3.14) on $(0, R_c)$.

Put

$$q(s_0, s) = \int_{s_0}^{s} \psi(r)\ dr. \tag{3.17}$$

Note that

$$\frac{1}{2}f'(r)^2 = \int_0^r f''(t)f'(t)\ dt = \int_0^r \psi(f(t))f'(t)\ dt = q(c, f(r)) \quad \text{for } 0 < r < R_c.$$

Hence,

$$R_c = \int_0^{R_c} df(r)/f'(r) = \frac{1}{\sqrt{2}}\int_0^{R_c} q^{-1/2}(c, f(r))\ df(r) = \frac{1}{\sqrt{2}}\int_c^{f(R_c)} q^{-1/2}(c, s)\ ds. \tag{3.18}$$

Since $\psi(r) \le \psi(r + c)$ for $c \ge 0$, we have $q(0, c) \le q(c, 2c)$ and therefore

$$q(0, s) = q(0, c) + q(c, s) \le 2q(c, s) \quad \text{for } s > 2c.$$

Hence, $q(c, s)^{-1/2} \le \sqrt{2}q(0, s)^{-1/2}$ and

$$R_c \le I(c) + J(c) \tag{3.19}$$

where

$$I(c) = \frac{1}{\sqrt{2}}\int_c^{2c} q(c, s)^{-1/2}\ ds, \quad J(c) = \int_{2c}^{\infty} q(0, s)^{-1/2}\ ds.$$

For $s \in (c, 2c)$, $q(c, s) \ge \psi(c)(s - c)$ and therefore

$$I(c) \le \frac{1}{\sqrt{2}}\int_c^{2c} [\psi(c)(s - c)]^{-1/2}\ ds = \sqrt{\frac{2c}{\psi(c)}} \to 0 \quad \text{as } c \to \infty.$$

By 3.1.B, $J(c)$ also tends to 0 as $c \to \infty$. By (3.19), $R_c \to 0$.

If $f(t-) < \infty$, then, by (3.14), $f'(t-) < \infty$ and a solution of (3.11) can be continued to an interval $(0, t_1)$ with $t_1 > t$. Hence, $t < R_c$. We conclude that $f(R_c-) = \infty$. □

LEMMA 3.4. *For every $\varepsilon \in (0, 1)$, there exists $0 < R < \varepsilon$ such that the problem (3.5) has a solution in $D = \{|x - x^0| < R\}$.*

PROOF. It is sufficient to construct, for some $R < \varepsilon$, a function ϕ of class $C^2[0, R)$ with the property (3.9) such that $\phi(R-) = \infty$ and

$$\alpha\phi'' + \beta\phi' \le \psi(\phi) \quad \text{on } (0, R). \tag{3.20}$$

Indeed, by Lemma 3.2, (3.20) implies that u^0 given by (3.8) is a solution of the problem (3.5).

We apply Lemma 3.3 to function ψ/α with ε replaced by $\lambda\varepsilon$ and we get a constant $R' \in (0, \lambda\varepsilon)$ and a function f on an interval $(0, R')$ subject to the condition (3.12) and the conditions

$$f(R'-) = \infty, f''(t) = \psi(f(t))/\alpha \quad \text{and } 0 < f'(t) \le t\psi(f(t))/\alpha \quad \text{for } 0 < t < R'. \tag{3.21}$$

We define ϕ by the formula $\phi(\rho) = f(\lambda\rho)$ for $0 < \rho < R$ where $R = R'/\lambda < \varepsilon$. Note that left side in (3.20) is equal to

$$\alpha\lambda^2 f''(\lambda\rho) + \lambda\beta f'(\lambda\rho)$$

which, by (3.21), does not exceed $\lambda^2\psi[\phi(\rho)][1 + \beta\varepsilon\alpha^{-1}]$ for $0 < \rho < R$. Condition (3.20) holds if λ is sufficiently small. □

3.3. Passage to the limit.

THEOREM 3.2. *Suppose that $\psi \in BR$ satisfies conditions 1.B and 1.C and that $u_n \to u$ at every point of Q. If u_n are solutions of* (1.1), *then so is u.*

PROOF. Every $z \in Q$ is covered by $U \Subset Q$. By 2.2.B, equation (2.4) holds for u_n. Since $\psi \in BR$, u_n are uniformly bounded in $\bar{U}$. By the dominated convergence theorem, (2.4) holds also for u. By Theorem 2.1, u is a solution of (1.1) in U. □

THEOREM 3.3. *Let $\psi \in BR_1$ satisfy 1.B and 1.C and suppose that solutions u_n of* (1.1) *converge to u at every point of Q. Let $O \subset \partial_{reg} Q$ be relatively open in ∂Q and let f be a continuous function on O. If u_n satisfy the boundary condition*

$$u_n = f \quad \text{on } O, \tag{3.22}$$

then the same condition holds for u.

The proof is based on the following lemma.

LEMMA 3.5. *Suppose that ψ satisfies the conditions of Theorem 3.3 and that O is a relatively open subset of ∂Q. Denote $\mathcal{U}(O, \kappa)$ the class of all positive solutions of* (1.1) *such that, for all $z' \in O$,*

$$\limsup u(z) \leq \kappa \quad \text{as } z \to z'. \tag{3.23}$$

Then, for every $\tilde{z} \in O$, there exist $N < \infty$ and an open cylinder V containing $\tilde{z}$ such that

$$u(z) \leq N \quad \text{for all } u \in \mathcal{U}(O, \kappa) \quad \text{and all } z \in Q \cap \bar{V}. \tag{3.24}$$

PROOF. Since $\psi \in BR_1$, there exists a cylinder $U \in \mathbb{Q}_\psi$ which contains $\tilde{z}$. We can choose this cylinder in such a way that $A = \bar{U} \cap \partial Q \subset O$ and that the boundary of $Q^0 = Q \cap U$ is the union of A and $B = \partial U \cap Q$. Let u^0 be a supersolution in U exploding on the set $\mathcal{T}$ defined by (3.2). We can assume that $u^0 \geq \kappa$ (otherwise we replace it by $u^0 + \kappa$). The set $\mathcal{T}^0 = (\mathcal{T} \cap Q) \cup A$ is total in ∂Q^0. If $u \in \mathcal{U}(O, \kappa)$, then

$$\limsup (u(z) - u^0(z)) \leq 0 \quad \text{as } z \to z' \in \mathcal{T}^0.$$

By the comparison principle, $u \leq u^0$ in Q^0. Let V be a cylinder such that $\tilde{z} \in V$ and $\bar{V} \subset U$. The condition (3.24) holds for V and the maximum N of u^0 on $\bar{V}$. □

PROOF OF THEOREM 3.3. We can assume that $\bar{O} \subset \partial_{reg} Q$ and $f \leq \kappa$ on O for some κ. Let $\tilde{z} \in O$ and let V be the cylinder constructed in Lemma 3.5. If the diameter of V is sufficiently small, then $A_1 = \bar{V} \cap \partial Q \subset O$ and $Q_1 = V \cap Q$ is strongly regular by Lemma **2**.4.3. Put $\bar{f}_n = f$ on A_1 and $\bar{f}_n = u_n$ on $\partial_r Q_1 \setminus A_1$. By 2.2.C,

$$u_n + G_{Q_1} \psi(u_n) = K_{Q_1} \bar{f}_n.$$

By (3.24), $u_n \leq N$ in Q^1. Since $u_n \to u$ in Q^1,

$$u + G_{Q_1} \psi(u) = K_{Q_1} \bar{f}$$

by the dominated convergence theorem. By Theorem 2.1, this implies $u(\tilde{z}) = f(\tilde{z})$. □

4. Operators V_Q

4.1. *In this section we assume that ψ satisfies conditions 1.A, 1.C and 1.D.* The transition operators of a superprocess (X_Q, P_μ) were defined originally by the formula

$$V_Q(f)(z) = -\log P_z e^{-\langle f, X_Q\rangle} \tag{4.1}$$

for $Q \in \mathbb{O}_0$ and $f \in \mathbb{B}$. Function $u = V_Q(f)$ satisfies the equation

$$u + G_Q\psi(u) = K_Q f. \tag{4.2}$$

In section **4**.4.1 both formulae were extended to all open sets Q. By a monotone passage to the limit they can be extended to all measurable functions f with values in $[0, \infty]$.

On the other hand, in Theorem **4**.3.1, we introduced $V_Q(f)$ for $Q \in \mathbb{O}_0, f \in \mathbb{B}$ starting from the equation (4.2) without assuming the existence of a (ξ, ψ)-superprocess. Properties **4**.3.2.A–**4**.3.2.C allow us to define $V_Q(f)$ for all open sets Q and all positive functions f by the formula

$$V_Q(f) = \sup_{k,\ell} V_{Q^k}(f \wedge \ell\, 1_{Q^c}) \tag{4.3}$$

where Q^k is the intersection of Q and $S^k = (-k, k) \times E$. If there exists a (ξ, ψ)-superprocess, then (4.3) is equivalent to the probabilistic formula (4.2).

Note that $V_Q(f) = f$ on Q^c and that $V_Q(f) = V_Q(\tilde{f})$ if $f = \tilde{f}$ on Q^c. Moreover, $V_Q(f) = V_Q(\tilde{f})$ on $\bar{Q}$ if $f = \tilde{f}$ on ∂Q.

We have:

4.1.A. If $f \le \tilde{f}$ on Q^c, then $V_Q(f) \le V_Q(\tilde{f})$. Moreover, $V_Q(f) \le V_Q(\tilde{f})$ on $\bar{Q}$ if $f \le \tilde{f}$ on ∂Q.

4.1.B. If $Q \subset \tilde{Q}$ and if $f = 0$ on $\tilde{Q}$, then $V_Q(f) \le V_{\tilde{Q}}(f)$.

These properties follow immediately from **4**.3.2.A–3.2.B.

4.1.C. If $f_n \uparrow f$, then $V_Q(f_n) \uparrow V_Q(f)$.

Proof. By **4**.3.2.C, for every k,

$$V_{Q^k}(f_n \wedge \ell\, 1_{Q^c}) \uparrow V_{Q^k}(f \wedge \ell\, 1_{Q^c})$$

as $n \to \infty$. By (4.3),

$$\begin{aligned}\sup_n V_Q(f_n) &= \sup_n \sup_{k,\ell} V_{Q^k}(f_n \wedge \ell\, 1_{Q^c}) \\ &= \sup_{k,\ell} \sup_n V_{Q^k}(f_n \wedge \ell\, 1_{Q^c}) = \sup_{k,\ell} V_{Q^k}(f \wedge \ell\, 1_{Q^c}) = V_Q(f).\end{aligned}$$

□

4.1.D. For arbitrary Q and f, $u = V_Q(f)$ satisfies the integral equation (4.2). [5]

Proof. Because of 4.1.C, it is sufficient to prove this for bounded f. Functions $u_n = V_{Q^n}(f 1_{Q^c})$ satisfy equation

$$u_n + G_{Q^n}\psi(u_n) = K_{Q^n}(f 1_{Q^c}). \tag{4.4}$$

[5] Both sides in 4.2 can be infinite in which case the equation is rather useless.

Note that the first exit time from Q^n is equal to $\tau \wedge n$ where τ is the first exit time from Q. If $\tau > n$, then $\eta_{\tau\wedge n} \in Q$.Therefore

$$K_{Q^n}(f 1_{Q^c})(z) = \Pi_z\{f(\eta_\tau), \tau \le n\} \to K_Q(f)(z).$$

We get (4.2) by passing to the limit in (4.4). □

4.1.E. For arbitrary open sets $Q' \Subset Q$,

$$V_{Q'} V_Q = V_Q. \tag{4.5}$$

PROOF. If $f \in \mathbb{B}$, then (4.5) follows from Lemma **4**.3.1. For an arbitrary f and $f_n = f \wedge n$, we have $V_{Q'} V_Q(f_n) = V_Q(f_n)$. By 4.1.C, this implies $V_{Q'} V_Q(f) = V_Q(f)$. □

4.2. We use notation $Q^k = Q \cap S^k$ introduced in the previous section.

PROPOSITION 4.1. *If* $f^m \uparrow f$, *then*

$$V_{Q^k}(f^m 1_{Q^c}) \uparrow V_Q(f) \quad \textit{as } k \uparrow \infty, m \uparrow \infty. \tag{4.6}$$

PROOF. It follows from 4.1.A and 4.1.B that

$$u_{k\ell m} = V_{Q^k}(f^m \wedge \ell 1_{Q^c})$$

increases in k, ℓ and m. Denote its supremum by u. By (4.3),

$$\sup_{k,\ell} u_{k\ell m} = V_Q(f^m)$$

and, by 4.1.C,

$$u = \sup_m V_Q(f^m) = V_Q(f).$$

On the other hand, by 4.1.C,

$$\sup_\ell u_{k\ell m} = V_{Q^k}(f^m \; 1_{Q^c}).$$

Therefore

$$V_Q(f) = u = \sup_{k,m} V_{Q^k}(f^m 1_{Q^c}).$$

□

PROPOSITION 4.2. *If* $f^n \uparrow f$, *then*

$$V_{Q^m}(f^n) \uparrow V_Q(f) \quad \textit{as } m \uparrow \infty, n \uparrow \infty. \tag{4.7}$$

PROOF. By 4.1.A and 4.1.B, $u_{mn} = V_{Q^m}(f^n)$ is increasing in m and n. Note that $Q^n \cap S^k = Q^{n\wedge k}$ and, by (4.3),

$$u_{mn} = \sup_{k,\ell} V_{Q^{m\wedge k}}[f^n \wedge \ell \; 1_{(Q^{m\wedge k})^c}].$$

Therefore

$$\sup_{m,n} V_{Q^m}(f^n) = \sup_{j,\ell,n} V_{Q^j}[f^n \wedge \ell 1_{(Q^j)^c}] = \sup_n V_Q(f^n) = V_Q(f).$$

□

LEMMA 4.1. *Suppose that $\psi \in BR$, Q is an open set and O is a regular* [6] *relatively open subset of ∂Q. Let f be a function with values in $[0,\infty]$ equal to 0 on $S \setminus O$. Suppose that $Q_n \in \mathbb{O}_0$, $Q_n \uparrow Q$, O_n is a relatively open subset of $\partial Q \cap \partial Q_n$ and $O_n \uparrow O$. Let f_n be a bounded function vanishing on $S \setminus O_n$. If $f_n \uparrow f$ on O, then $V_{Q_n}(f_n) \to V_Q(f)$.*

PROOF. 1°. Suppose that $Q \in \mathbb{O}_0$ and $f \in \mathbb{B}$. Then $u = V_Q(f)$ is a unique solution of the equation

$$u + G_Q\psi(u) = K_Q(f) \tag{4.8}$$

and $u_n = V_{Q_n}(f_n)$ is a solution of

$$u_n + G_{Q_n}\psi(u_n) = K_{Q_n}(f_n). \tag{4.9}$$

It follows from 4.1.A and 4.1.B that u_n is an increasing sequence. We claim that $u = \lim u_n$ satisfies (4.8). By the monotone convergence theorem, $G_{Q_n}\psi(u_n) \to G_Q\psi(u)$. To get (4.8) from (4.9), it is sufficient to show that $K_{Q_n}(f_n) \to K_Q(f)$ which will be established if we prove that

$$f_n(\eta_{\tau_n}) \uparrow f(\eta_\tau) \tag{4.10}$$

where τ and τ_n are the first exit times from Q and Q_n. Put $A = \{\eta_\tau \in O\}, A_n = \{\eta_{\tau_n} \in O_n\}$. Since O_n is regular, we have

$$A_n \subset \{\tau_n = \tau\} \cap \{\eta_\tau \in O_n\}.$$

Therefore $A_1 \subset A_2 \subset \cdots \subset A_n \subset \cdots \subset A$. Denote by $\eta(I)$ the image of an interval I under the mapping $t \to \eta_t$. Note that $A = \{\eta[\alpha,\tau] \subset Q \cup O\}$ and $A_n = \{\eta[\alpha,\tau_n] \subset Q \cup O_n\}$. If $\eta[\alpha,\tau] \subset Q \cup O$, then, for some n, $\eta[\alpha,\tau] \subset Q \cup O_n$ and therefore $\eta[\alpha,\tau_n] \subset Q \cup O_n$. Hence $A_n \uparrow A$, and (4.10) holds because $f_n(\eta_{\tau_n}) = f_n(\eta_{\tau_n})1_{A_n} = f_n(\eta_\tau)1_{A_n}$ and $f(\eta_\tau)1_A = f(\eta_\tau)$.

2°. Now we consider the case when $f \in \mathbb{B}$ and Q is an arbitrary open set. Denote by Q^k, Q^k_n, O^k and O^k_n the intersections of Q, Q_n, O and O_n with S^k. Put

$$f^k_n = f_n 1_{O^k_n}, \quad f^k = f 1_{O^k}, \quad V^k_n = V_{Q^k_n}, \quad u^k_n = V^k_n(f^k_n).$$

If $n' \geq n$, then $O_{n'} \supset O_n$ and $f_{n'} = f_n$ on O_n. It follows from 4.1.A and 4.1.B that u^k_n is increasing in n and k. By Proposition 4.1,

$$\sup_n V^k_n(f^k_n) = V_{Q^k}(f^k).$$

By Proposition 4.2,

$$\sup_k V_{Q^k}(f^k) = V_Q(f)$$

and

$$\sup_k V^k_n(f^k_n) = V_{Q_n}(f_n).$$

Hence, $V_Q(f) = \sup_n V_{Q_n}(f_n)$.

3°. If f is unbounded, then, by 2°,

$$V_Q(f \wedge \ell) = \sup_n V_{Q_n}(f_n \wedge \ell)$$

for every ℓ. By 4.1.C,

$$V_Q(f) = \sup_\ell V_Q(f \wedge \ell) = \sup_{\ell,n} V_{Q_n}(f_n \wedge \ell) = \sup_n V_{Q_n}(f_n).$$

[6]This means: all points of O belong to $\partial_{reg}Q$.

□

5. Boundary value problems

5.1. *In the rest of this chapter we assume that ξ is an L-diffusion and that ψ satisfies conditions 1.A–1.D.* [These conditions hold if ψ does not depend on z, and satisfies 3.1.A and 1.B.]

For a bounded strongly regular domain Q and for a continuous function f, $u = V_Q(f)$ can be defined as a unique bounded solution of the boundary value problem

$$\begin{aligned} \dot{u} + Lu &= \psi(u) \quad \text{in } Q, \\ u &= f \quad \text{on } \partial_r Q. \end{aligned} \tag{5.1}$$

Indeed, $u = V_Q(f)$ is bounded because $u \le K_Q f$ and f is bounded. By Theorem 2.1, u satisfies (5.1) and, by the comparison principle (Theorem 2.3), the problem (5.1) has no more than one bounded solution.

A more general boundary value problem is treated in the following theorem.

Theorem 5.1. *Suppose that $\psi \in BR$, Q is an open set and O is a regular relatively open subset of ∂Q.*

If a function $f : \partial Q \to [0, \infty]$ is continuous on O and equal to 0 on $\Gamma = \partial Q \setminus O$, then $u = V_Q(f)$ is the minimal solution of the boundary value problem

$$\begin{aligned} \dot{u} + Lu &= \psi(u) \quad \text{in } Q, \\ u &= f \quad \text{on } O. \end{aligned} \tag{5.2}$$

Proof. By Lemma **2**.4.4, there exists a (Q, Γ)-sequence Q_n. Denote by O_n the set of all $z \in O$ such that $d(z, Q \setminus Q_n) > 0$. Clearly, $O_n \uparrow O$. Functions

$$f_n(z) = \begin{cases} f(z) \wedge [d(z, Q \setminus Q_n)n] & \text{for } z \in O_n, \\ 0 & \text{for } z \notin O_n \end{cases}$$

satisfy conditions

(i) $f_n \ge 0$ and $f_n = 0$ on O_n^c;

(ii) f_n are continuous on ∂Q_n;

(iii) $f_n \uparrow f$;

(iv) for an arbitrary k, $O_n \cap \{f < k\} \subset \{f = f_n\}$ for all sufficiently large n.

By (i)–(ii), the function $u_n = V_{Q_n}(f_n)$ is a solution of the problem

$$\begin{aligned} \dot{u}_n + Lu_n &= \psi(u_n) \quad \text{in } Q_n, \\ u_n &= f_n \quad \text{on } \partial_r Q_n. \end{aligned} \tag{5.3}$$

It follows from 4.1.A and 4.1.B (or from **4**.3.2.A and 3.2.B) that $u_n \le u_{n+1}$ in Q. By Theorem 3.2, $u = \lim u_n$ satisfies the equation $\dot{u} + Lu = \psi(u)$ in Q. By (iv), $f_n = f$ for all sufficiently large n on the set $O_n \cap \{f < k\}$. Since $O_n \subset \partial_r Q_n$, $u_n = f$ on $O_n \cap \{f < k\}$ by (5.3). By Theorem 3.3, $u = f$ on $O \cap \{f < \infty\}$. If $z \in O_k \cap \{f = \infty\}$, then, for every $n \ge k$, z belongs to O_n and, by (5.3),

$$u_n(z) = f_n(z) = d(z, Q \setminus Q_n)n.$$

Since $u \ge u_n$, $u(z) = \infty = f(z)$.

We proved that u is a solution of (5.1). If v is an arbitrary solution of this problem, then $v \ge u_n$ on Q_n by the comparison principle. Hence, $v \ge u$ on Q.

It remains to prove that $u = V_Q(f)$. Since $u_n = V_{Q_n}(f_n)$ by 2.2.C, this is an implication of Lemma 4.1. □

COROLLARY 5.1. *Suppose $Q \subset \tilde{Q}$ and $O = \partial Q \cap \tilde{Q}$ is regular for Q. Then $V_Q(1_{\tilde{Q}}\tilde{u}) \le \tilde{u}$ for every $\tilde{u} \in \mathcal{U}(\tilde{Q})$.*

Indeed, $\tilde{u} \in \mathcal{U}(\tilde{Q})$ satisfies (5.2) with $f = 1_{\tilde{Q}}\tilde{u}$. By Theorem 5.1, $u = V_Q(f)$ is a minimal solution of (5.2). Hence, $u \le \tilde{u}$.

COROLLARY 5.2. *A function ψ belongs to the class BR if and only if it belongs to BR_1.*

We already know (see section 3.1) that $BR_1 \subset BR$. Now let $\psi \in BR$. If Q is a cylinder (3.1) and if $O = \mathcal{T}$ is given by (3.2), then, by Theorem 5.1, the condition (3.3) holds for $u^0 = V_Q(f)$ where $f = \infty$ on O, $f = 0$ on $\partial Q \setminus O$. Hence, $\psi \in BR_1$.

5.2. Minimal absolute barrier. Here is another implication of Theorem 5.1.

THEOREM 5.2. *Let $\psi \in BR$. If Q is strongly regular, then $u = V_Q(\infty \cdot 1_{\partial_r Q})$* [7] *is equal to the supremum of all bounded subsolutions of the equation* (1.1). *[Hence, it is the smallest absolute barrier in Q.]*

PROOF. Let $\tilde{u}$ be the supremum of bounded subsolutions in Q. Functions $u_n = V_Q(n1_{\partial_r Q})$ are bounded and, by Theorem 5.1, they satisfy the equation (1.1). Therefore $u_n \le \tilde{u}$. By 4.1.C, $u_n \uparrow u$. Hence, $u \le \tilde{u}$. On the other hand, if v is any bounded subsolution, then $v \le u$ by Theorem 2.3. Thus $\tilde{u} \le u$. □

REMARK 5.1. Suppose that O is a regular relatively open subset of ∂Q. If X is an (L, ψ)-superdiffusion, then

$$V_Q(\infty \cdot 1_O)(z) = -\log P_z\{X_Q(O) = 0\}. \tag{5.4}$$

If Q is strongly regular, then a minimal absolute barrier for (1.1) is given by the formula

$$u^0(z) = -\log P_z\{X_Q = 0\}. \tag{5.5}$$

Indeed, by (4.1), $V_Q(n1_O)(z) = -\log P_z e^{-nX_Q(O)}$. By passing to the limit as $n \to \infty$, we get (5.4). Theorem 5.2, (5.4) and **4**.4.3.2 imply (5.5).

5.3. Maximal solutions.

THEOREM 5.3. *Suppose that $\psi \in BR$ and Q is an arbitrary open set. If a closed subset Γ of ∂Q contains all irregular points, then there exists a maximal solution w_Q^Γ of the problem*

$$\begin{aligned} \dot{u} + Lu &= \psi(u) \quad \textit{in } Q; \\ u &= 0 \quad \textit{on } \partial Q \setminus \Gamma. \end{aligned} \tag{5.6}$$

If X is an (L, ψ)-superdiffusion, then

$$w_Q^\Gamma(z) = -\log P_z(\Omega_\Gamma) \tag{5.7}$$

where

$$\Omega_\Gamma = \{\mathcal{G}_Q \quad \textit{is compact and } \mathcal{G}_Q \cap \Gamma = \emptyset\}. \tag{5.8}$$

[7]A notation $\infty \cdot \varphi$ is used for the function equal to infinity on the set $\{\varphi > 0\}$ and equal to 0 on the set $\{\varphi = 0\}$.

The maximal solution w_Q of equation (1.1) *is given by the formula*

$$w_Q(z) = -\log P_z\{\mathcal{G}_Q \quad \textit{is compact and } \mathcal{G}_Q \subset Q\}. \tag{5.9}$$

PROOF. 1°. Consider a (Q,Γ)- sequence Q_n and put

$$O_n = \partial_r Q_n \cap Q, f_n = \infty \cdot 1_{O_n}, v_n = V_{Q_n}(f_n).$$

By Theorem 5.1,

$$\begin{aligned} \dot{v}_n + Lv_n &= \psi(v_n) \quad \text{in } Q_n; \\ v_n &= \infty \quad \text{on } O_n. \end{aligned} \tag{5.10}$$

By 4.1.D,

$$v_n + G_{Q_n}\psi(v_n) = K_{Q_n} f_n$$

and therefore $v_n \le K_{Q_n} f_n = 0$ on $B_n = \partial Q_n \setminus O_n$. We claim that

$$v_{n+1} \le v_n \quad \text{on } Q_n. \tag{5.11}$$

To prove this, we use the Comparison principle (Theorem 2.3). Since $\partial_r Q_n$ is total in ∂Q_n, we need only to check that $v_{n+1} \le v_n$ on $\partial_r Q_n$ and that v_{n+1} is bounded on Q_n. The first statement is true because $v_n = \infty$ on O_n and $v_{n+1} = 0$ on B_n. The second one holds because v_{n+1} is continuous on $\bar{Q}_n$. By Theorem 3.2 and Theorem 3.3, $w_Q^\Gamma = \lim v_n$ is a solution of the problem (5.6). If u is an arbitrary solution, then u is bounded on Q_n and $u = 0$ on B_n, $v_n = \infty$ on O_n. By the Comparison principle, $u \le v_n$ in Q_n. Therefore $u \le w_Q^\Gamma$ in Q.

2°. If X is an (L,ψ)-superdiffusion, then, by (5.4), $v_n = -\log P_z\{X_{Q_n}(O_n) = 0\}$. By **4**.4.3.2, X_{Q_n} does not charge, a.s., the set of all irregular points of ∂Q_n. Since all points in $\partial Q_n \cap Q$ not in O_n are irregular, $X_{Q_n}(O_n) = X_{Q_n}(Q)$ a.s. and $v_n(z) = -\log P_z\{X_{Q_n}(Q) = 0\}$. By Theorem **4**.5.3,

$$P_z\{X_{Q_n}(Q) = 0\} \uparrow P_z(\Omega_\Gamma) \tag{5.12}$$

and therefore $v_n \to -\log P_z(\Omega_\Gamma)$.

We get (5.9) by applying (5.7) to $\Gamma = \partial Q$. □

REMARK 5.2. The construction in the proof shows that

$$w_Q^\Gamma = \lim V_{Q_n}(f_n) \tag{5.13}$$

where Q_n is a (Q,Γ)-sequence and

$$f_n = \begin{cases} \infty & \text{on } O_n = \partial_r Q_n \cap Q, \\ 0 & \text{on } \partial Q_n \setminus O_n. \end{cases} \tag{5.14}$$

THEOREM 5.4. *Suppose $\psi \in BR$, $\tilde{Q} \subset Q$ are open sets and Γ is a closed subset of $\partial Q \cap \partial\tilde{Q}$. If Γ contains all irregular points of $\partial\tilde{Q}$, then $w_{\tilde{Q}}^\Gamma \le w_Q^\Gamma$.*

PROOF. If there exists an (L,ψ)-superdiffusion, this follows immediately from (5.7)-(5.8) because $\mathcal{G}_{\tilde{Q}} \subset \mathcal{G}_Q$ a.s.

In the general case, we apply Remark 5.2. Note that, if Q_n is a (Q,Γ)-sequence, then $\tilde{Q}_n = Q_n \cap \tilde{Q}$ is a $(\tilde{Q},\Gamma)$-sequence. Let $\tilde{f}_n$ be given by (5.14) with Q_n and O_n replaced by $\tilde{Q}_n$ and $\tilde{O}_n = \tilde{Q} \cap \partial\tilde{Q}_n$. Then $\tilde{V}_{Q_n}(\tilde{f}_n)$ is the minimal solution of the problem

$$\begin{aligned} \dot{u} + Lu &= \psi(u) \quad \text{in } \tilde{Q}_n, \\ u &= \infty \quad \text{on } \tilde{O}_n. \end{aligned}$$

Since the restriction of $V_{Q_n} f_n$, also satisfies these conditions, we have $\tilde{V}_{Q_n}(\tilde{f}_n) \leq V_{Q_n} f_n$ which implies our theorem. $\square$

6. Notes

Relations between semilinear parabolic equations and superdiffusions were studied in [**Dyn92**] and [**Dyn93**] for the case $\psi(u) = u^\alpha, 1 < \alpha \leq 2$.

Conditions for the existence of absolute barriers for the equation $\Delta u = \psi(u)$ were obtained independently by Keller [**Kel57a**] and Osserman [**Oss57**]. A more general equation $Lu = \psi(u)$ was considered in the Appendix in [**DK98a**]. Theorem 3.1 provides an adaptation of the previous results to a parabolic setting.

The minimal solution of the problem (5.2) and the maximal solution of the problem (5.6) were investigated in [**DK99**] under the assumption of the existence of an (L, ψ)-superdiffusion. [8] The minimal solution in an elliptic setting was studied earlier in [**Dyn97b**]. In section 5 of Chapter **5** we cover a much wider class of functions ψ by combining probabilistic and analytic arguments.

[8] In [**DK99**] a more general operator L with a zero order term was considered.

Part 2

Elliptic equations and diffusions

CHAPTER 6

Linear elliptic equations and diffusions

In section 1 we formulate some fundamental results on elliptic differential equations of the second order. We also give exact references to books where the proofs of these results can be found. Most references are to monographs [**GT98**] and [**Mir70**]. We were not able to find a proof of an important bound for the Poisson kernel in any book and we give such a proof (based on a paper [**Maz75**]) in the Appendix B.

In section 2 we introduce time homogeneous diffusions and we consider their Poisson and Green's operators. (We use mostly a shorter name "homogeneous" instead of "time homogeneous".)

At the end of the chapter we establish a probabilistic formula for a solution of the Dirichlet problem for the equation $Lu(x) = a(x)u(x)$.

1. Basic facts on second order elliptic equations

1.1. Hölder classes. Let D be a domain in $\mathbb{R}^d$. A function f is called Hölder continuous in $\bar{D}$ with Hölder's exponent λ if there exists a constant Λ such that

$$|f(x) - f(y)| \leq \Lambda |x-y|^\lambda \quad \text{for all } x, y \in \bar{D}.$$

It is assumed that $0 < \lambda \leq 1, \Lambda > 0$. Λ is called Hölder's coefficient Hölder coefficient of f. The class of all Hölder continuous functions in $\bar{D}$ with Hölder's exponent λ is denoted by $C^\lambda(\bar{D})$. We put $f \in C^\lambda(D)$ if $f \in C^\lambda(\bar{U})$ for all domains $U \Subset D$. [1] A function f belongs to $C^{1,\lambda}(D)$ if all $\mathcal{D}_i f$ are in $C^\lambda(D)$ and f belongs to $C^{2,\lambda}(D)$ if all $\mathcal{D}_{ij} f \in C^\lambda(D)$ for all i, j. . Notation $C^{1,\lambda}(\bar{D}), C^{2,\lambda}(\bar{D})$ have a similar meaning. Note that $C^{2,\lambda}(D)$ is a subclass of the class $C^2(D)$ of all twice continuously differentiable functions.

Let E and $\tilde{E}$ be open subsets of $\mathbb{R}^d$ and let $\tilde{x} = T(x)$ be a mapping from E onto $\tilde{E}$. Suppose that the coordinates $(\tilde{x}_1, \dots, \tilde{x}_d)$ of $\tilde{x}$ are functions of class $C^{2,\lambda}$ of the coordinates $(x_1, \dots, x_d)$ of x and $(x_1, \dots, x_d)$ are functions of class $C^{2,\lambda}$ of $(\tilde{x}_1, \dots, \tilde{x}_d)$. Then we call T a diffeomorphism of class $C^{2,\lambda}$.

Formula $u(x) = \tilde{u}(\tilde{x})$ where $\tilde{x} = T(x)$ (and $x = T^{-1}(\tilde{x})$) establishes a 1-1 correspondence between functions on E and functions on $\tilde{E}$ and $u \in C^{2,\lambda}(E)$ if and only if $\tilde{u} \in C^{2,\lambda}(\tilde{E})$.

1.2. Operator L. We consider a second order differential operator

$$Lu(x) = \sum_{i,j=1}^{d} a_{ij}(x)\mathcal{D}_{ij}u(x) + \sum_{i=1}^{d} b_i(x)\mathcal{D}_i u(x) \tag{1.1}$$

[1] Some times functions $f \in C^\lambda(D)$ are called locally Hölder continuous and functions $f \in C^\lambda(\bar{D})$ are called uniformly Hölder continuous in D.

in a domain D in $\mathbb{R}^d$. Without loss of generality we can put $a_{ij} = a_{ji}$. We assume that

1.2.A. [2] There exists a constant $\kappa > 0$ such that

$$\sum a_{ij}(x)t_i t_j \geq \kappa \sum t_i^2 \quad \text{for all } x \in D, t_1, \dots, t_d \in \mathbb{R}.$$

1.2.B. All coefficients $a_{ij}(x)$ and $b_i(x)$ are Hölder continuous in $\bar{D}$ with exponent λ and Hölder's coefficient Λ.

From time to time we use the adjoint operator

$$L^*u = \sum_{i,j=1}^{d} \mathcal{D}_{ij}(a_{ij}u) + \sum_{i=1}^{d} \mathcal{D}_i(b_i u) = \sum_{i,j=1}^{d} a_{ij}^* \mathcal{D}_{ij}u + \sum_{i=1}^{d} b_i^* \mathcal{D}_i u + c^* u \tag{1.2}$$

where

$$a_{ij}^* = a_{ij}, \quad b_i^* = b_i + 2\sum_j \mathcal{D}_j a_{ij},$$
$$c* = \sum_i \mathcal{D}_i b_i + \sum_{ij} \mathcal{D}_{ij} a_{ij}.$$

Operator L^* is well defined if, in addition to 1.2.A–1.2.B, we impose the condition:

1.2.C. $a_{ij} \in C^{2,\lambda}(\bar{D})$, $b_i \in C^{1,\lambda}(\bar{D})$.

Moreover L^* has the form (1.1) with an additional term c^*u where $c^* \in C^\lambda(\bar{D})$.

Suppose that T is a diffeomorphism from an open set $E \subset \mathbb{R}^d$ onto an open set $\tilde{E} \subset \mathbb{R}^d$. To every differential operator L in E of the form (1.1) there corresponds an operator $\tilde{L}$ in $\tilde{E}$ of a similar form defined by the formula

$$\tilde{v} = \tilde{L}\tilde{u} \tag{1.3}$$

where $\tilde{u}(\tilde{x}) = u(x)$ and $\tilde{v}(\tilde{x}) = v(x) = Lu(x)$. The coefficients of $\tilde{L}$ and the coefficients of L are connected by the formulae

$$\tilde{a}_{ij}(\tilde{x}) = \sum_{k,\ell} c_k^i c_\ell^j a_{k\ell}(x), \quad \tilde{b}_i(\tilde{x}) = \sum_{k,\ell} c_{k\ell}^i a_{k\ell} + \sum_k c_k^i b_k(x) \tag{1.4}$$

where $c_k^i = \frac{\partial \tilde{x}_i}{\partial x_k}$ and $c_{k\ell}^i = \frac{\partial^2 \tilde{x}_i}{\partial x_k \partial x_\ell}$.

1.3. Straightening of the boundary. We say that D is a *smooth domain* (or a domain of class $C^{2,\lambda}$) if, for every $y \in \partial D$, there exists a ball U_y centered at y and a diffeomorphism ψ_y of class $C^{2,\lambda}$ from U_y onto $\tilde{D} \subset \mathbb{R}^d$ such that $\psi(U_y \cap D) \subset \mathbb{R}^d_+ = \{x \in \mathbb{R}^d : x_d > 0\}$ and $\psi(U_y \cap \partial D) \subset \partial\mathbb{R}^d_+ = \{x \in \mathbb{R}^d : x_d = 0\}$. We say that ψ_y straightens the boundary near y.

Diffeomorphisms ψ_y can be chosen in such a way that: [3]

(a) all operators L_y obtained from L by ψ_y satisfy conditions 1.2.A–1.2.B with constants $\tilde{\kappa}, \tilde{\lambda}, \tilde{\Lambda}$ which depend only on κ, λ and Λ (but not on y);

(b) $\psi_y(y) = 0$ and $L_y = \Delta$ at point y (that is, for the transformed operator, $\tilde{a}_{ij} = \delta_{ij}$).

If diffeomorphisms ψ_y with the properties described above exist only for y in a subset O of ∂D, then we say that O is a *smooth portion* of ∂D.

[2]The property 1.2.A is called uniform ellipticity and κ is called the *ellipticity coefficient* of L .

[3]See [**GT98**], section 6.2 and the proof of Lemma 6.1.

1.4. Maximum principle. We use the following versions of the maximum principle: [4]

1.4.A. Suppose that D is a bounded domain and $a(x) \geq 0$ for all $x \in D$. If $u \in C^2(D)$ satisfies conditions

$$Lu - au \geq 0 \quad \text{in } D, \tag{1.5}$$

and, for all $\tilde{x} \in \partial D$,

$$\limsup u(x) \leq 0 \quad \text{as } x \to \tilde{x}, \tag{1.6}$$

then $u \leq 0$ in D.

1.4.B. Suppose that D is an arbitrary domain, $u \in C^2(D)$ and $Lu \leq 0$. If u attains its minimum at a point $x_0 \in D$, then it is equal to a constant.

1.5. L-harmonic functions. A function $h \in C^2(D)$ is called L-harmonic if it satisfies the equation $Lh = 0$ in D. Classical harmonic functions are Δ-harmonic where Δ is the Laplace operator. We use the shorter name "harmonic functions" if there is no need to refer explicitly to L.

The set $\mathcal{H}(D)$ of all positive harmonic functions in a domain D has the following properties.

1.5.A. (Harnack's inequality) For $U \Subset D$ there exists a constant A such that $h(x_1) \leq Ah(x_2)$ for all $x_1, x_2 \in U$ and all $h \in \mathcal{H}(D)$.

1.5.B. If $h_n \in \mathcal{H}(D)$ and if the values $h_n(c)$ at some point $c \in D$ are bounded, then there exists a subsequence h_{n_k} which converges uniformly in every $U \Subset D$.

For the Laplacian $L = \Delta$ these results can be found in the most textbooks on partial differential equations (e.g., [**Pet54**]). The general case is covered, for instance, in [**Mir70**] and [**Fel30**].

It follows from 1.5.A, 1.5.B and Lemma **2**.3.3 that:

1.5.C. If $h_n \in \mathcal{H}(D)$ converge pointwise to h and if $h(c) < \infty$ for some c, then $h \in \mathcal{H}(D)$ and the convergence is uniform on every $U \Subset D$.

Remark. Under condition 1.2.C, 1.5.C holds also for positive solutions of $L^*u = 0$ (see [**GT98**], section 6.1).

The maximum principle 1.4.B implies:

1.5.D. Every $h \in \mathcal{H}(D)$ either is strictly positive or it vanishes identically.

1.6. Poisson's equation. Poisson's equation

$$Lu = -f \quad \text{in } D \tag{1.7}$$

can be investigated by the Perron method. Here is the way this method is presented in [**GT98**]. [5]

A continuous function u in D is called a supersolution of (1.7) if, for every open ball $U \Subset D$ and every v such $Lv = -f$ in U the inequality $u \geq v$ on ∂U implies that $u \geq v$ in U. A function u is called a subsolution if $-u$ is a supersolution. Suppose

[4] Proofs can be found, for instance, in [**GT98**] (sections 3.1 and 3.2), [**Mir70**] (Chapter 1, section 3) or [**BJS64**] (Part II, Chapter 2).

[5] The case $L = \Delta$ and $f = 0$ is treated in many books including [**Pet54**] and [**Doo84**]. Gilbarg and Trudinger cover also the general case. Theorem 1.1 below is stated as Theorem 6.11 in [**GT98**].

that φ is a function on ∂D. We call $u \in C(\bar{D})$ a superfunction (subfunction) relative to φ if it is a supersolution (subsolution) and $u \geq \varphi$ ($u \leq \varphi$) on ∂D.

THEOREM 1.1. *Suppose that D is a bounded domain, L satisfies conditions 1.2.A–1.2.B, $f \in C^\lambda(D)$ and φ is a bounded Borel function on ∂D. Then there exists a unique solution u of Poisson's equation* (1.7) *such that $u_1 \leq u \leq u_2$ for every subfunction u_1 and every superfunction u_2 (relative to φ). Moreover, $u \in C^{2,\lambda}(D)$.*

We call u the *Perron solution* of (1.7) corresponding to φ.

THEOREM 1.2. *If D is a bounded smooth domain and if φ is continuous then the Perron solution u is a unique solution of the Dirichlet problem*

$$\begin{aligned} Lu &= -f \quad \text{in } D, \\ u &= \varphi \quad \text{on } \partial D. \end{aligned} \tag{1.8}$$

This follows, for instance, from Theorem 6.13 in [**GT98**].

1.7. Green's function. The problem (1.8) can be reduced to two particular cases. In the case when $\varphi = 0$ the solution can be expressed in terms of the Green's function $g(x, y)$ of L in D. This is a function from $D \times D$ to $(0, \infty]$. The following results, due to Girod, are presented in Miranda's monograph [**Mir70**][6] (see section 21 and, in particular, Theorem 21.VI).

THEOREM 1.3. *Suppose that D is a bounded smooth domain, L satisfies conditions 1.2.A–1.2.C and $f \in C^\lambda(D)$. Then the solution of the problem* (1.8) *with $\varphi = 0$ is given by the formula*

$$u(x) = \int_D g(x, y) f(y) dy. \tag{1.9}$$

The Green's function $g(x, y)$ is strictly positive and it has the following properties:

1.7.A. For every $y \in D$, $u(x) = g(x, y)$ is a solution of the problem

$$\begin{aligned} Lu &= 0 \quad \text{in } D \setminus \{y\}, \\ u &= 0 \quad \text{on } \partial D. \end{aligned} \tag{1.10}$$

If $d \geq 2$, then $g(y, y) = \infty$. For every $x \in D$, $v(y) = g(x, y)$ is a solution of the problem

$$\begin{aligned} L^* v &= 0 \quad \text{in } D \setminus \{x\}, \\ v &= 0 \quad \text{on } \partial D. \end{aligned} \tag{1.11}$$

1.7.B. [7] *For all $x, y \in D$,*

$$g(x, y) \leq C\Gamma(x - y) \tag{1.12}$$

where C is a constant depending only on D and L and

$$\Gamma(x) = \begin{cases} |x|^{2-d} & \text{for } d \geq 3, \\ -(\log|x|) \vee 1 & \text{for } d = 2, \\ 1 & \text{for } d = 1. \end{cases} \tag{1.13}$$

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[6] The case $L = \Delta$ is considered also in [**GT98**], Chapter 4 and in [**Doo84**], Part 1, Chapter VII .

[7] Corollary to Theorem 3 in [**Kry67**] implies that 1.7.B holds under very mild conditions on the coefficients of L.

1.8. Poisson kernel. The problem (1.8) in the case of $f = 0$ can be solved in terms of the Poisson kernel $k(x, y)$ [which is a function from $D \times \partial D$ to $(0, \infty)$] and the normalized surface area γ [which is a probability measure on ∂D].

Let Γ be a subset of $\mathbb{R}^d$. We call it a smooth surface if, for every $y_0 \in \Gamma$, there exists $\varepsilon > 0$ such that the intersection of Γ with the ε-neighborhood U_ε of y_0 can be described by parametric equations $y_i = \varphi_i(t_1, \ldots, t_{d-1})$ where $t = (t_1, \ldots, d_{t-1})$ is in an open subset of $\mathbb{R}^{d-1}$ and φ_i are $C^{2,\lambda}$-functions such that the $(d-1) \times d$ matrix $b^i_k = \frac{\partial \varphi_i}{\partial t_k}$ has the rank $d-1$. The boundary ∂D of a smooth domain D is a smooth surface.

The measure γ_0 is defined on the $\Gamma \cap U_\varepsilon$ by the formula

$$\gamma_0(B) = \int\limits_{\varphi^{-1}(B)} \sqrt{D(t)} dt_1 \ldots dt_{d-1} \tag{1.14}$$

where $D(t)$ is the determinant of the matrix

$$d_{ij} = \sum_{k=1}^{d-1} b^i_k b^j_k.$$

Note that $D(t)$ is continuously differentiable and $0 < D(t) < \infty$. The measure γ_0 does not depend on a parameterization of Γ. It is called the surface area. The corresponding normalized surface area γ is equal to γ_0 divided by $\gamma_0(\partial D)$.

The Poisson kernel $k(x, y)$ can be expressed through Green's function $g(x, y)$ by the formula

$$k(x, y) = \sum_1^d q_i(y) \mathcal{D}_{y_i} g(x, y) \tag{1.15}$$

where $n_y = (q_1(y), \ldots, q_d(y))$ is the conormal to ∂D at y. [8] In other words, $k(x, y)$ is the derivative of $g(x, y)$ considered as a function of y in the direction of inward conormal n_y to ∂D at point y.

THEOREM 1.4. *Poisson kernel $k(x, y)$ is continuous in y and it has the following properties:*

1.8.A. If D and L are as in Theorem 1.3 and $\varphi \in C(\partial D)$, then the solution of the problem (1.8) *with $f = 0$ is given by the formula*

$$h(x) = \int_{\partial D} k(x, y) \varphi(y) \ \gamma(dy)$$

where γ is the normalized surface area on ∂D. For every $y \in \partial D$, $h(x) = k(x, y)$ is a solution of the problem

$$\begin{aligned} Lh &= 0 \quad \text{in } D, \\ h &= 0 \quad \text{on } \partial D \setminus \{y\}. \end{aligned} \tag{1.16}$$

1.8.B. For all $x \in D, y \in \partial D$,

$$k(x, y) \le C d(x, \partial D) |x - y|^{-d}. \tag{1.17}$$

These properties are also proved in [**Mir70**]. An exception is the property 1.8.B. We prove it in the Appendix B.

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[8] It is defined as directed inwards vector n_y with $|n_y| = 1$ that is orthogonal to ∂D in the Riemannian metric $\sum a^{i,j}(x) dx_i dx_j$ associated with L. [(a^{ij}) is the inverse to the matrix (a_{ij}).]

2. Time homogeneous diffusions

2.1. Homogeneous L-diffusions. An L-diffusion corresponding to an operator **2**.(1.1) was introduced in section **2**.2.2. For an operator (1.1) with time independent coefficients, this is a homogeneous Markov process (see the definition in section 3.5 of the Appendix A). If φ is continuous and bounded, then

$$u(t,x) = \Pi_x \varphi(\xi_t) = \int_E p_t(x,y)\varphi(y)\ dy \tag{2.1}$$

satisfies conditions

$$\dot{u} = Lu \quad \text{for } t > 0, x \in E \tag{2.2}$$

and

$$u(t,x) \to \varphi(x) \quad \text{as } t \downarrow 0 \tag{2.3}$$

[(2.1) is a particular case of **2**.(2.13) and (2.2)–(2.3) follow from **2**.(1.10)].

2.2. First exit times from a bounded domain.

LEMMA 2.1. *Let τ be the first exit time from a bounded open set D. There exist constants $\gamma < 1$ and C such that, for all $s \geq 0$ and all x,*

$$\Pi_x\{1_{\tau > s}\tau\} \leq C\gamma^s. \tag{2.4}$$

Therefore $\Pi_x \tau \leq C$.

PROOF. Function $F(x) = p_1(x, D^c)$ is continuous in x and strictly positive. Therefore there exists $\beta > 0$ such that $F(x) \geq \beta$ for all $x \in D$. This implies $\Pi_x\{\tau > 1\} \leq \gamma = 1 - \beta$. By (3.12) in the Appendix A, $\{\tau > s\}\theta_s\{\tau > 1\} = \{\tau > s+1\}$ and therefore, by the Markov property (the Appendix A.(3.10)),

$$\Pi_x\{\tau > s+1\} = \Pi_x 1_{\tau > s} \Pi_{\xi_s}\{\tau > 1\} \leq \gamma \Pi_x\{\tau > s\}.$$

Thus $\Pi_x\{\tau > n\} \leq \gamma^n$ and

$$\Pi_x 1_{\tau > s}\tau = \int_s^\infty \Pi_x\{\tau > t\}\ dt \leq \sum_{n > s-1} \gamma^n$$

which implies (2.4). □

2.3. Regular open sets. A point a of ∂D is called *regular* if, for every $\varepsilon > 0$, $\Pi_a\{\xi_s \in D \text{ for all } 0 < s < \varepsilon\} = 0$. We say that an open set D is *regular* if all points of ∂D are regular. This is a stronger definition than one used in the parabolic setting (see section **2**.4.2). It is justified because, in the elliptic setting, every open set can be approximated by regular domains in the present stronger sense. [9]

Theorem **2**.4.4 implies that $a \in \partial D$ is regular if there exists a closed ball A such that a is the only common point of A, $\bar{D}$ and a neighborhood of a. It follows from this criterion that all smooth domains are regular.

[9] Note that, if D is regular in an elliptic setting, then every cylinder $(t_1, t_2) \times D$ is regular in the sense of section **2**.4.2.

2.4. Poisson operator. The Poisson operator corresponding to D is defined by the formula

$$K_D f(x) = \Pi_x 1_{\tau<\infty} f(\xi_\tau) \tag{2.5}$$

where $\tau = \tau(D)$ is the first exit time from D.

The following results are elliptic versions of the results proved in section 3 of chapter **2**:

2.4.A. A continuous function h is L-harmonic in D if and only if $K_U h = h$ for all $U \Subset D$.

2.4.B. If D and φ are bounded, then $h = K_D\varphi$ is the Perron solution of the equation $Lh = 0$ corresponding to φ.

These propositions can be proved by the same arguments as their parabolic counterparts or they can be deduced from the results of Part 1. Indeed, if K_Q are operators defined by **2**.(3.1), then, for a time independent function φ, $K_Q\varphi = K_D\varphi$ with $Q = \mathbb{R} \times D$. To get 2.4.A, we use the following lemma:

LEMMA 2.2. *Let $Q^k = (-k, k) \times D$. If $\Pi_x\tau(D) < \infty$ and if f is bounded and time independent, then*

$$K_{Q^k} f(r, x) \to K_D f(x) \quad \textit{for all } r. \tag{2.6}$$

PROOF. If $x \notin D$, then both parts of (2.6) are equal to $f(x)$. If $x \in D$ and $r \in (-k, k)$, then $(r, x) \in Q^k$ and, $\Pi_{r,x}$-a.s., the first exit time τ_k from Q^k is equal to $\tau \wedge k$ where $\tau = \tau(D)$. Hence, $\Pi_{r,x}$-a.s., $f(\xi_{\tau_k}) \to f(\xi_\tau)$ and (2.6) follows from the dominated convergence theorem. □

We leave it to the reader to get, in a similar way, 2.4.B.

It follows from (3.11) in the Appendix A that, for all $U \Subset D$,

$$K_U K_D = K_D. \tag{2.7}$$

(Cf. **2**.(3.2).)

For bounded smooth D and continuous φ,

$$K_D\varphi(x) = \int_{\partial D} k(x, y)\varphi(y)\gamma(dy) \tag{2.8}$$

where $k(x, y)$ is the Poisson kernel described in section 1.8. Indeed, by 2.4.B and Theorems 1.1–1.3, both sides represent the Perron solution of $Lu = 0$ in D corresponding to φ. Formula (2.8) can be extended to all positive Borel φ.

Theorem **2**.4.2 implies:

2.4.C. If a is a regular point of ∂D and if φ is a bounded function on ∂D which is continuous at a, then

$$K_D\varphi(x) \to \varphi(a) \quad \text{as } x \to a. \tag{2.9}$$

2.5. Green's operators. Green's operator for (ξ_t, Π_x) in D is defined by the formula

$$G_D f(x) = \Pi_x \int_0^\tau f(\xi_s)\, ds \tag{2.10}$$

where $\tau = \tau(D)$.

If G_Q are operators defined by the formula **2**.(5.13), then, for a time independent function f, $G_Q f = G_D f$ for $Q = \mathbb{R} \times D$. Indeed, by (3.12) in the Appendix A,

$$\{r < \tau\} \subset \{\theta_r[1_{s<\tau} f(\xi_s)] = 1_{s<\tau-r} f(\xi_{s+r})\}$$

and therefore

$$\theta_r \int_0^\tau f(\xi_s)\, ds = \int_r^\tau f(\xi_s)\, ds \tag{2.11}$$

on the set $\{r < \tau\}$. For every $(r, x) \in Q$, $\Pi_{r,x}\{r < \tau\} = 1$. The equation $G_D f = G_Q f$ follows from (3.9) in the Appendix A and (2.11).

The relation **2**.(5.15) implies that, if $\tilde{D} \subset D$, then

$$G_D = G_{\tilde{D}} + K_{\tilde{D}} G_D. \tag{2.12}$$

LEMMA 2.3. *Let $\tau = \tau(D)$ and let $Q^k = (-k, k) \times D$. If $G_D|f| < \infty$ for all $x \in D$, then*

$$G_{Q^k} f(r, x) \to G_D f(x) \quad \textit{for all } (r, x). \tag{2.13}$$

The convergence is uniform if f and D are bounded.

PROOF. Let $(r, x) \in Q^k$. If τ is the first exit time from D, then, $\Pi_{r,x}$-a.s., $\tau \wedge k$ is the first exit time from Q^k and therefore $G_{Q^k} f(r, x) = \Pi_{r,x} Y$ where

$$Y = \int_r^{\tau \wedge k} f(\xi_s)\, ds.$$

It follows from (2.11) that

$$Y = \theta_r \int_0^{\tau \wedge (k-r)} f(\xi_s)\, ds \quad \Pi_{r,x}\text{-a.s.}$$

and, by (3.9) in the Appendix A,

$$G_{Q^k} f(r, x) = \Pi_x \int_0^{\tau \wedge (k-r)} f(\xi_s)\, ds.$$

Therefore

$$G_D f(x) - G_{Q_k} f(r, x) = \Pi_x \int_{\tau \wedge (k-r)}^{\tau} f(\xi_s)\, ds. \tag{2.14}$$

The integrand tends to 0 as $k \to \infty$ and it is dominated in the absolute value by $\int_0^\tau |f(\xi_s)|\, ds$. By the dominated convergence theorem, the integral tends to 0.

If f and D are bounded, then the convergence is uniform by (2.14). □

We have:

2.5.A. If f and D are bounded and if c is a regular point of ∂D, then

$$G_D f(x) \to 0 \quad \text{as } x \to a. \tag{2.15}$$

This follows from **2**.(5.17) and Lemma 2.3.

2.5.B. If D is a bounded regular domain and if f is bounded, then $u = G_D f \in C^\lambda(D)$. If $f \in C^\lambda(D)$, then $u \in C^2(D)$, $Lu = -f$ in D and $u = 0$ on ∂D.

Indeed, let Q^k be the domains defined in Lemma 2.3. If f is bounded, then, by **2**.5.3.A, $u_k = G_{Q^k} f$ belong to $\mathbb{C}^\lambda(Q^k)$. By Lemma 2.3, u_k converge uniformly to u which implies the first part of 2.5.B. If $f \in C^\lambda(D)$, then, by **2**.5.3.A, u_k is a solution of the equation $\dot{u}_k + Lu_k = -f$ in Q^k. Therefore the second part of 2.5.B follows from Lemma 2.3, **2**.5.3.E and 2.5.A.

2.5.C. For every bounded smooth D and every bounded or positive Borel function f,

$$G_D f(x) = \int_D g(x,y) f(y)\, dy \tag{2.16}$$

where $g(x,y)$ is the Green's function in D described in Theorem 1.3.

For $f \in C^\lambda(D)$, this follows from 2.5.B and Theorem 1.3. It can be extended to all bounded Borel functions f by the multiplicative systems theorem (Theorem 1.1 in the Appendix A) and to all positive Borel f by a monotone passage to the limit.

2.5.D. Let $g, \tilde{g}, k, \tilde{k}$ be the Green's and Poisson's kernels for two bounded smooth domains $\tilde{D} \subset D$. Then

$$g(x,y) = \tilde{g}(x,y) + \int_C \tilde{k}(x,z)\gamma(dz) g(z,y) \quad \text{for all } x, y \in \tilde{D} \tag{2.17}$$

and

$$k(x,y) = \tilde{k}(x,y) + \int_C \tilde{k}(x,z)\gamma(dz) k(z,y) \quad \text{for all } x \in \tilde{D}, y \in A \tag{2.18}$$

where $A = \partial D \cap \partial \tilde{D}$, $C = D \cap \partial \tilde{D}$ and γ is the normalized surface area on $\partial \tilde{D}$.

Let us prove (2.18) [(2.17) can be proved in a similar way]. Denote by τ and $\tilde{\tau}$ the first exit times from D and $\tilde{D}$. Consider a continuous function $f \geq 0$ on ∂D vanishing off A. Note that $\theta_{\tilde{\tau}} f(\xi_\tau) = f(\xi_\tau)$ on $\{\tilde{\tau} < \tau\}$ and therefore, by (3.11) in the Appendix A, for every $x \in \tilde{D}$,

$$\Pi_x f(\xi_\tau) = \Pi_x 1_{\tilde{\tau} = \tau} f(\xi_\tau) + \Pi_x 1_{\tilde{\tau} < \tau} \Pi_{\xi_{\tilde{\tau}}} f(\xi_\tau). \tag{2.19}$$

Since $\{\tilde{\tau} = \tau\} = \{\xi_{\tilde{\tau}} \in A\}$, the first term is equal to $\int_A \tilde{k}(x,y)\gamma(dy) f(y)$. If $\tilde{\tau} < \tau$, then $\xi_{\tilde{\tau}} \in C$ and therefore the second term is equal

$$\int_C \tilde{k}(x,z)\gamma(dz) \Pi_z f(\xi_\tau) = \int_C \tilde{k}(x,z)\gamma(dz) \int_A k(z,y)\gamma(dy) f(y).$$

The left side in (2.19) is equal to $\int_A k(x,y)\gamma(dy) f(y)$. Therefore, for every $x \in \tilde{D}$, the equation (2.18) holds γ-a.e. on A. It holds everywhere because both sides are continuous in y.

2.6. The Dirichlet problem for Poisson's equation. We have:

2.6.A. If D is bounded and regular, $f \in C^\lambda(D), \varphi \in C(\partial D)$, then $u = G_D f + K_D \varphi$ is a unique solution of the problem

$$\begin{aligned} Lu &= -f \quad \text{in } D, \\ u &= \varphi \quad \text{on } \partial D. \end{aligned} \tag{2.20}$$

Indeed, $Lu = -f$ by 2.4.B and 2.5.B. By 2.4.C and 2.5.A, $u = \varphi$ on ∂D. The uniqueness follows from the maximum principle 1.4.A.

2.6.B. Suppose that D is a bounded regular domain and $f \in C^\lambda(D)$. If u is continuous in $\bar{D}$ and $Lu = -f$ in D, then $u = G_D f + K_D u$.

This follows from 2.6.A because $v = G_D f + K_D u$ is a solution of the problem (2.20) with $\varphi = u$.

Remark. For smooth domains 2.6.A and 2.6.B follow from Theorem 1.3. The equation $Lw = -f$ holds for $w = G_D f$ even if f is not bounded but $G_D|f|$ is finite.

2.7. Green's function for an arbitrary domain. For an arbitrary domain D, we consider a sequence of bounded smooth domains D_n exhausting D. It follows from 2.5.D that $g_{D_n} \le g_{D_{n+1}}$. Therefore there exists a limit

$$g_D(x,y) = \lim g_{D_n}(x,y).$$

Clearly, this limit does not depend on the choice of D_n and we have:

2.7.A. If $\tilde{D} \subset D$, then $g_{\tilde{D}} \le g_D$ in $\tilde{D}$.

2.7.B. If $D_n \uparrow D$, then $g_{D_n} \uparrow g_D$.

It follows from 1.7.A and 1.5.C that, if $g_D(x,y) < \infty$ for some x, y, then $u(x) = g_D(x,y)$ is L-harmonic in $D \setminus \{y\}$ and $v(y) = g_D(x,y)$ is L^*-harmonic in $D \setminus \{x\}$. We say that the domain D is a *Greenian* if, for every $\tilde{D} \Subset D$, the bound (1.12) holds for $x \in D, y \in \tilde{D}$. By 1.7.B all bounded domains are Greenian.

The part of ξ in $Q = \mathbb{R} \times D$ is a homogeneous Markov process on a random time interval $[0, \tau)$ with a stationary transition function $p_t(x,y) = p_Q(0,x;t,y)$ where p_Q is defined by **2**.(5.1). Note that, by (2.10) and **2**.(5.2),

$$G_D f(x) = \int_D dy f(y) \int_0^\infty ds\; p_s(x,y).$$

By comparing this expression with (2.16), we conclude that, for a bounded smooth domain D,

$$g(x,y) = \int_0^\infty p_s(x,y)\; ds. \tag{2.21}$$

This formula can be extended to all domains D.

3. Probabilistic solution of equation $Lu = au$

3.1.

Theorem 3.1. *Suppose ξ is an L-diffusion, τ is the first exit time from a bounded regular domain D, $a \ge 0$ is bounded and belongs to $C^\lambda(D)$. If $\varphi \ge 0$ is a continuous function on ∂D, then*

$$u(x) = \Pi_x \exp\left[-\int_0^\tau a(\xi_s)\; ds\right] \varphi(\xi_\tau) \tag{3.1}$$

is a unique solution of the problem

$$\begin{aligned} Lu &= au \quad \text{in } D,\\ u &= \varphi \quad \text{on } \partial D. \end{aligned} \tag{3.2}$$

It is also a unique solution of the integral equation

$$u + G_D(au) = K_D\varphi. \tag{3.3}$$

PROOF. 1°. If

$$Y_t = \exp\left[-\int_t^\tau a(\xi_s)\ ds\right],$$

then

$$Y_t' = a(\xi_t)Y_t \quad \text{for } t \in (0, \tau)$$

and therefore

$$\int_0^\tau a(\xi_s)Y_s\ ds = 1 - Y_0.$$

We have $\theta_t Y_0 = Y_t, \theta_t\varphi(\xi_\tau) = \varphi(\xi_\tau)$ for $t \in (0, \tau)$ and, by the Markov property,

$$\begin{aligned} G_D(au)(x) &= \Pi_x \int_0^\tau a(\xi_t)\Pi_{\xi_t} Y_0\varphi(\xi_\tau)\ dt \\ &= \Pi_x \int_0^\tau a(\xi_t)Y_t\varphi(\xi_\tau)\ dt = \Pi_x(1 - Y_0)\varphi(\xi_\tau) = K_D\varphi(x) - u(x). \end{aligned}$$

This implies (3.3).

2°. Every solution of (3.3) satisfies (3.2). Indeed, since φ is bounded, $K_D\varphi$ and u are bounded and therefore au is also bounded. By 2.5.B, $v = G_D(au) \in C^\lambda(D)$. By 2.4.B, $w = K_D\varphi \in C^2(D)$ and therefore $u = w - v \in C^\lambda(D)$. We conclude that $u = G_D f + K_D\varphi$ where $f = -au$ and our statement follows from 2.6.A.

3°. The maximum principle 1.4.A implies that the problem (3.2) has no more than one solution. □

3.2. Suppose that $a \geq 0$ belongs to $C^\lambda(E)$. Theorem 3.1 implies the following properties of the equation

$$Lu = au \quad \text{in } E. \tag{3.4}$$

3.2.A. A positive function u is a solution of (3.4) if and only if

$$u + G_D(au) = K_D u \quad \text{for all } D \Subset E. \tag{3.5}$$

The class coincides with the class of continuous u subject to the condition

$$u(x) = \Pi_x \exp[-\int_0^\tau a(\xi_s)ds]u(\xi_\tau) \quad \text{for all } D \Subset E \tag{3.6}$$

where $\tau = \tau(D)$.

To get 3.2.A it is sufficient to apply Theorem 3.1 to $\varphi = u$ on ∂D.

It follows from 3.2.A that:

3.2.B. If solutions u_n of (3.4) are locally uniformly bounded and if they converge pointwise to u, then u is a solution of (3.4).

We also have:

3.2.C. If $u \geq 0$ satisfies (3.4), then either u is strictly positive or it is identically equal to 0.

Indeed, suppose $u(c) = 0$ at $c \in E$ and let $c \in D \Subset E$. It follows from (3.6) that $u(\xi_\tau) = 0$ Π_c-a.s. Hence, $K_D u(c) = 0$. By 1.5.D, $K_D u = 0$ in D, and, by (3.5), $u = 0$ in D.

4. Notes

4.1. Formula (3.1) can be considered as a version of the Feynman-Kac formula. The original version of this formula was proved for $L = \Delta$. A probabilistic approach to boundary value problems for general elliptic operators L was developed in [**Dyn65**]. All propositions in sections 2 and 3 of Chapter **6** can be deduced from the results presented in [**Dyn65**].

4.2. A concept of a Greenian domain was introduced by Doob (see [**Doo84**]) in the case $L = \Delta$. The entire space $\mathbb{R}^d$ is Greenian if $d > 2$. For $d = 2$, D is Greenian if and only if $g_D(x, y) < \infty$ for all $x \neq y \in D$. In terms of the corresponding Brownian motion, D is Greenian if and only if $\Pi_x \tau(D) < \infty$ for all $x \in D$.

For a general elliptic operator L, all bounded domains are Greenian. The class of unbounded Greenian domains depends on L. It is proved in [**LSW63**] that the bound (1.12) for $D = \mathbb{R}^d$ holds if L is of divergence form

$$Lu = \sum_{i,j} \mathcal{D}_i(a_{ij}\mathcal{D}_j u)$$

(with the coefficients a_{ij} subject to very broad conditions). Hence, for such operators all domains $D \subset \mathbb{R}^d$ are Greenian for $d > 2$. This is true also for the Laplace-Beltrami operator on a Riemannian manifold.

4.3. A lower bound

$$k(x, y) \geq Cd(x, \partial D)|x - y|^{-d} \tag{4.1}$$

with $C > 0$ similar to the upper bound (1.17) is also true. For operators L of divergence form this follows , e.g, from Lemma 6 in [**Maz75**].

CHAPTER 7

Positive harmonic functions

We fix a Greenian domain E in $\mathbb{R}^d$, $d \geq 2$ and we use shorter notation $K, G, \mathcal{H}, \dots$ for $K_E, G_E, \mathcal{H}(E), \dots$. Denote by ζ the first exit time from E.

In this chapter we investigate the set $\mathcal{H}$ of all positive harmonic functions in E. If E is smooth, then every $h \in \mathcal{H}$ has a representation

$$h(x) = \int_{\partial E} k(x, y)\nu(dy)$$

where $k(x, y)$ is the Poisson kernel and $\nu \in \mathcal{M}(\partial E)$. Moreover, this formula establishes a 1-1 correspondence between $\mathcal{M}(\partial E)$ and $\mathcal{H}$. A similar correspondence exists in the general case, but instead of ∂E we need to consider a certain subset of the Martin boundary $\hat{\partial} E$ and the Poisson kernel must be replaced by the Martin kernel.

1. Martin boundary

1.1. The Martin kernel. According to Theorem **6**.1.3, $g(x, y) > 0$ for all x, y; since E is a Greenian domain, $g(x, y) < \infty$ for $x \neq y$ and, since $d \geq 2$, $g(x, x) = \infty$. Fix a point $c \in E$. The Martin kernel k is defined by the formula

$$k_y(x) = k(x, y) = \begin{cases} \frac{g(x,y)}{g(c,y)} & \text{for } y \neq c, \\ 0 & \text{for } y = c. \end{cases} \tag{1.1}$$

Denote by C_0 the class of all positive continuous functions on E which vanish outside U for some $U \Subset E$. Put

$$k_y(\phi) = \int_E \phi(x)\, dx\, k(x, y).$$

LEMMA 1.1. *For every $\phi \in C_0$, function $k_y(\phi)$ is continuous and bounded.*

PROOF. Put

$$F(x) = \int_0^\infty e^{-t} p_t(c, x)\, dt.$$

Note that

$$\int_E p_t(c, x) g(x, y)\, dx = \int_t^\infty p_s(c, y)\, ds \leq g(c, y).$$

Therefore

$$\int_E F(x) g(x, y)\, dx \leq \int_0^\infty dt\, e^{-t} g(c, y) = g(c, y)$$

and $\int F(x) k_y(x)\, dx \leq 1$. Since F is continuous and strictly positive, the ratio ϕ/F is bounded on every $U \Subset E$. It vanishes outside some $U \Subset E$. If $\phi/F \leq N_\phi$, then

$$k_y(\phi) \leq N_\phi \int_E F(x) k_y(x)\, dx \leq N_\phi.$$

□

It follows from **6**.1.7.A and (1.1) that k_y is a harmonic function in $E \setminus \{y\}$. Note that $k_y(c) = 1$. Denote by $\mathcal{H}^1$ the class of all $h \in \mathcal{H}$ such that $h(c) = 1$. Suppose that a sequence $y_n \in E$ contains only a finite number of points in every $U \Subset E$. If $k_{y_n} \to h$, then, by **6**.1.5.C, $h \in \mathcal{H}^1$. Let $\tilde{\mathcal{H}}$ stand for the subset of $\mathcal{H}^1$ obtained by this procedure. We will show that every element of $\mathcal{H}^1$ can be obtained as a barycentre of a probability measure on $\tilde{\mathcal{H}}$.

1.2. The Martin boundary. A metrizable compact topological space $\hat{E}$ is called a compactification of E if it contains an everywhere dense open subset homeomorphic to E. It is called the *Martin compactification* if $k(x, y)$ can be extended to a continuous function on $E \times (\hat{E} \setminus \{c\})$ such that $k_{y_1} \neq k_{y_2}$ for every $y_1 \neq y_2 \in \hat{E}$. We use the name the Martin kernel and the notation k also for the extended function. The complement $\hat{\partial} E$ of E in $\hat{E}$ is called the *Martin boundary*. For every $y \in \hat{\partial} E$, $k(\cdot, y)$ belongs to $\mathcal{H}^1$. By **6**.1.4.B, $k(x, y) > 0$ for all $x \in E, y \in \hat{\partial} E$.

To construct a Martin compactification, we consider a countable everywhere dense subset W of C_0 (in the sense of the uniform convergence), we choose constants $c_\phi > 0$ such that

$$\sum_{\phi \in W} c_\phi N_\phi < \infty$$

and we put

$$\hat{d}(y_1, y_2) = \sum_{\phi \in W} |k_{y_1}(\phi) - k_{y_2}(\phi)| c_\phi \quad \text{for } y_1, y_2 \in E.$$

If $\hat{d}(y_1, y_2) = 0$, then $k_{y_1}(\phi) = k_{y_2}(\phi)$ for all $\phi \in W$ and therefore $k_{y_1} = k_{y_2}$. Since $k_x(y) = \infty$ if and only if $x = y$, this implies $y_1 = y_2$. Clearly, $\hat{d}$ is a metric in E. We define $\hat{E}$ as the completion of E with respect to this metric.

LEMMA 1.2. *If $y_n \in E, y \in \hat{E} \setminus E$ and if $\hat{d}(y_n, y) \to 0$, then k_{y_n} converges to an element of $\mathcal{H}^1$*

PROOF. Since $\hat{d}(y_m, y_n) \to 0$ as $m, n \to \infty$, we have $|k_{y_m}(\phi) - k_{y_n}(\phi)| \to 0$ for all $\phi \in W$ and therefore there exist limits $\lim k_{y_n}(\phi) = \ell(\phi) < \infty$. By Fatou's lemma, $\liminf k_{y_n}(x) < \infty$ for some x and, by **6**.1.5.B, k_{y_n} contains a subsequence k_{z_i} which converges uniformly in every $U \Subset E$. The limit h belongs to $\mathcal{H}^1$. To prove the lemma, it is sufficient to show that every subsequence of k_{y_n} contains a subsequence tending to h. [1] We already know that k_{y_n} contains a convergent subsequence k_{z_i}. We need only to check that $\tilde{h} = \lim k_{z_i} = h$. This follows from the equations $\int \tilde{h}\phi = \int h\phi = \ell(\phi)$ for all $\phi \in W$. □

Note that $\lim k_{y_n}$ does not depend on the choice of $y_n \in E$ tending to $y \in \hat{\partial} E$. We denote it k_y.

For every distinct points $y, y' \in \hat{E}$, $k_y \neq k_{y'}$. We already have seen this if $y, y' \in E$. If $y \in E, y' \notin E$, then $k_y(y) = \infty$ and $k_{y'}(y) < \infty$. Finally, if $y, y' \notin E$ and if $\hat{d}(y_n, y) \to 0$, then $\hat{d}(y_n, y') \geq \hat{d}(y, y')/2$ for all sufficiently large n which is impossible if $k_y = k_{y'}$. It remains to prove that $\hat{E}$ is compact and that the

[1] If $k_{y_n}(x)$ does not converge to $h(x)$, then there exists $\varepsilon > 0$ and a subsequence n_k such that $|k_{y_{n_k}}(x) - h(x)| > \varepsilon$ and no subsequence of $k_{y_{n_k}}(x)$ converges to $h(x)$.

metrics $\hat{d}(x,y)$ and $|x-y|$ define the same topology on E. This follows from three propositions:

1.2.A. Every sequence $y_n \in E$ contains a subsequence y_{n_k} such that $\hat{d}(y_{n_k}, y) \to 0$ for some $y \in \hat{E}$.

1.2.B. If $y_n \in E$ and $y_n \to y \in E \setminus \{c\}$, then $k_{y_n}(x) \to k_y(x)$ for all $x \neq y$ and $\hat{d}(y_n, y) \to 0$.

1.2.C. If $y_n \in E, y \in E \setminus \{c\}$ and if $\hat{d}(y_n, y) \to 0$, then $|y_n - y| \to 0$.

To prove 1.2.A, we note that, for every $\phi \in W$, the sequence $k_{y_n}(\phi)$ is bounded by Lemma 1.1 and therefore there exists a subsequence y_{n_i} such that $k_{y_{n_i}}(\phi)$ converges for all $\phi \in W$. This implies the convergence of y_{n_i} in $\hat{E}$.

1.2.B follows from the continuity of $g(x,y)$ in y for every $x \neq y$.

Let us prove 1.2.C. It follows from 1.2.A that the sequence y_n is contained in some $U \Subset E$. It follows from 1.2.B that, if $|y_{n_k} - z| \to 0$ for a subsequence y_{n_k}, then $z = y$. Hence, $|y_n - y| \to 0$.

We conclude that $\hat{E}$ is the Martin compactification of E. For every subset B of $\hat{E}$ we denote by B^{cl} the closure of B in $\hat{E}$.

Remark 1.1. If E is a smooth Greenian domain, then the Martin boundary $\hat{\partial} E$ can be identified with ∂E. The Martin kernel k_M and the Poisson kernel k_P defined by **6**.(1.15) are connected by the formula

$$k_M(x,y) = k_P(x,y)/k_P(c,y). \tag{1.2}$$

Both statements follow from the fact: if $y_n \in E, z \in \partial E$ and if $|y_n - z| \to 0$, then

$$g(x,y_n)/g(c,y_n) \to k_P(x,z)/k_P(c,z). \tag{1.3}$$

By the straightening of the boundary near z (see **6**.1.3), we reduce the general case to the case when there exists a ball U centered at z such that $E \cap U \subset \mathbb{R}^d_+$, $\partial E \cap U \subset \partial \mathbb{R}^d_+$. Since $g(x,y) = 0$ for $y \in \partial E$, we have $g(x,y) = (y_d - z_d) D_{y_d} g(x,z) + o(|y-z|)$. This implies (1.3).

To get an integral representation of any $h \in \mathcal{H}^1$, we use an L-diffusion in E and its transformation related to h.

2. The existence of an exit point $\xi_{\zeta-}$ on the Martin boundary

2.1. L-diffusion in E. A time homogeneous L-diffusion was introduced in section 2 of Chapter **6**. In section **2**.5.1 we defined a part of an L-diffusion in an open subset Q of S. If $Q = \mathbb{R}_+ \times E$, then this is a time-homogeneous Markov process in E over a random time interval $[0, \zeta)$ where ζ is the first exit time of η_t from Q which coincides with the first exit time of ξ_t from E. We put $\xi_t = \dagger$ ("cemetery") for $t \geq \zeta$. If $p_t(x,y)$ is the transition density of ξ, then, for all $x \in E, 0 < t_1 < \cdots < t_n, B_1, \ldots, B_n \in \mathcal{B}_E$,

$$\begin{aligned} \Pi_x\{\xi_{t_1} \in B_1, \ldots, \xi_{t_n} \in B_n\} \\ = \int_{B_1} dy_1 \ldots \int_{B_n} dy_n \; p_{t_1}(x,y_1) p_{t_2-t_1}(y_1,y_2) \ldots p_{t_n - t_{n-1}}(y_{n-1}, y_n) \end{aligned} \tag{2.1}$$

(cf. **2**.(5.3)). [2]

[2] Condition $t_n < \zeta$ can be dropped because it follows from $\{\xi_{t_n} \in B_n\}$.

2.2. Our next goal is to prove the following result.

THEOREM 2.1. *A limit*

$$\xi_{\zeta-} = \lim_{t\uparrow\zeta} \xi_t \tag{2.2}$$

in the topology of $\hat{E}$ exists, Π_c-a.s., and it belongs to $\hat{\partial}E$. For every positive Borel f and every $x \in E$,

$$\Pi_x f(\xi_{\zeta-}) = \Pi_c k(x, \xi_{\zeta-}) f(\xi_{\zeta-}). \tag{2.3}$$

2.3. We say that σ is an *L-time* if $\sigma \leq \zeta$ and $\theta_t \sigma = (\sigma - t)_+$ for all $t > 0$. [3] This condition is satisfied for ζ. It holds also for the *last exit time* σ from any $U \subset E$ defined by the formula

$$\sigma = \sup\{s : \xi_s \in U\}.$$

If σ is an L-time, then, for every $s > 0$,$(\sigma - s)_+$ is also an L-time. We denote by $\mathcal{P}_\sigma$ the class of all positive $\mathcal{F}$-measurable functions Y such that, for every $t > 0$,

$$\{t < \sigma\} \subset \{\theta_t Y = Y\}. \tag{2.4}$$

Heuristically, $\mathcal{P}_\sigma$ is the set of functions determined by $\xi_t, t \geq \sigma$. Note that $\xi_\sigma \in \mathcal{P}_\sigma$ and that $\mathcal{P}_{(\sigma-s)_+} \supset \mathcal{P}_\sigma$ for all $s \geq 0$.

The first step in proving Theorem 2.1 is the following result:

THEOREM 2.2. *If σ is an L-time and if a bounded Y belongs to $\mathcal{P}_\sigma$, then, for every $\phi \in C_0$,*

$$\Pi_c 1_{\sigma > t} k_{\xi_{\sigma-t}}(\phi) Y = \int_E \Pi_x\{1_{\sigma>t} Y\} \phi(x)\, dx. \tag{2.5}$$

PROOF. 1°. We claim that, for every $Y \in \mathcal{P}_\sigma$ and for all $\rho \geq 0, f \geq 0$,

$$\Pi_x \int_0^\infty 1_{t<\sigma} \rho(t) f(\xi_{\sigma-t}) Y\, dt = \int_E g(x,y) f(y) \Pi_y Z\, dy \tag{2.6}$$

where $Z = 1_{\sigma>0} \rho(\sigma) Y$.

Change of variables $s = \sigma - t$ shows that the left side in (2.6) is equal to

$$\Pi_x \int_0^\infty 1_{s<\sigma} \rho(\sigma - s) f(\xi_s) Y\, ds. \tag{2.7}$$

For all $s > 0$, by (2.4),

$$\theta_s [f(\xi_0) Z] = 1_{s<\sigma} \rho(\sigma - s) f(\xi_s) Y.$$

Therefore, by the Markov property of ξ (see (3.10) in the Appendix A),

$$\Pi_x 1_{s<\sigma} \rho(\sigma - s) f(\xi_s) Y = \Pi_x \Pi_{\xi_s} [f(\xi_0) Z] = \int_E p_s(x,y) F(y)\, dy$$

where $F(y) = \Pi_y [f(\xi_0) Z] = f(y) \Pi_y Z$. By **6**.(2.21), this implies that the integral (2.7) is equal to the right side in (2.6).

2°. By applying formula (2.6) to $x = c$, we get

$$\Pi_c \int_0^\infty 1_{t<\sigma} \rho(t) f(\xi_{\sigma-t}) Y\, dt = \int_E g(c,y) f(y) \Pi_y Z\, dy.$$

[3] a_+ is an abbreviation for $a \vee 0$.

If $f(y) = k_y(x)$, then $g(c,y)f(y) = g(x,y)$ and therefore

$$\Pi_c \int_0^\infty 1_{t<\sigma}\rho(t)k_{\xi_{\sigma-t}}(x)Y\ dt = \int_E g(x,y)\Pi_y Z\ dy.$$

Formula (2.6) with $f = 1$ yields

$$\int_E g(x,y)\Pi_y Z\ dy = \Pi_x \int_0^\infty 1_{t<\sigma}\rho(t)Y\ dt.$$

By Fubini's theorem, this implies

$$\int_0^\infty \rho(t)\ dt \Pi_c 1_{t<\sigma} k_{\xi_{\sigma-t}}(\phi)Y = \int_0^\infty \rho(t)\ dt \int_E dx\ \phi(x)\Pi_x 1_{t<\sigma}Y.$$

Since this is true for all $\rho \in C_0$, (2.5) holds for almost all $t \geq 0$. To complete the proof, it is sufficient to show that both parts of (2.5) are right continuous in t.

3°. Since $\phi \in C_0$, it vanishes outside some $U \Subset E$. Since E is a Greenian domain, $k(x,y) \leq C\Gamma(x-y)$ for all $x \in E, y \in U$. Therefore $k_y(\phi)$ is continuous in y on $E \setminus \{c\}$ and $k_{\xi_t}(\phi)$ is continuous in t. Right continuity of the left side in (2.5) follows from Lemma 1.1 and the dominated convergence theorem. The other side is right continuous because $\{\sigma > t_n\} \uparrow \{\sigma > t\}$ as $t_n \downarrow t$. □

2.4. Our next step is:

THEOREM 2.3. *Suppose σ is the last exit time from $U \Subset E$ and let $\phi \in C_0$. If*

$$Z_t = 1_{t<\sigma} k_{\xi_{\sigma-t}}(\phi),$$

then (Z_t, Π_c) is a right continuous supermartingale on the interval $[0,\infty)$ relative to a suitable filtration $\mathcal{A}_t$.

PROOF. We put $A \in \mathcal{A}_s$ if $1_A \in \mathcal{P}_{(\sigma-s)+}$. Clearly, $\mathcal{A}_s$ is a filtration and $Z_t \in \mathcal{A}_t$. Right continuity of Z_t follows from continuity of k_{ξ_t}. If $0 \leq s \leq t$ and if $A \in \mathcal{A}_s$, then $1_A \in \mathcal{P}_\sigma$ and, by (2.5),

$$\int_A Z_t\ d\Pi_c = \Pi_c 1_{\sigma>t} k_{\xi_{\sigma-t}}(\phi)1_A = \int_E \Pi_x\{A, \sigma > t\}\phi(x)\ dx. \tag{2.8}$$

The right side is monotone decreasing in t and therefore

$$\int_A Z_t\ d\Pi_c \leq \int_A Z_s\ d\Pi_c.$$

Hence, (Z_t, Π_c) is a supermartingale relative to $\mathcal{A}_t$. □

2.5. PROOF OF THEOREM 2.1.

1°. By Theorem 2.3, to every $U \Subset E$ there corresponds a positive right continuous supermartingale. Denote it by $Z(U)$. By Theorem 4.2 in the Appendix A, for every $0 \leq a < b$,

$$\Pi_c \mathbb{D}(U) \leq b/b - a) \tag{2.9}$$

where $\mathbb{D}(U) = \mathbb{D}(Z(U), \mathbb{R}_+, [a,b])$ is the number of downcrossings of $[a,b]$ by $Z(U)$.

We claim that $\mathbb{D}(U) \geq \mathbb{U}(U)$ where $\mathbb{U}(U)$ is the number of upcrossings of $[a,b]$ by the process

$$Y_t(U) = 1_{t<\sigma} k_{\xi_t}(\phi).$$

Indeed, $Z_t(U) = 1_{t<\sigma} Y_{\sigma-t}$ for $t \geq 0$. Suppose that $\mathbb{U}(U) \geq n$. Then there exist

$$0 \leq s_1 < s_2 < \cdots < s_{2n-1} < s_{2n} \leq \sigma$$

such that

$$Y_{s_1}(U) \le a, Y_{s_2}(U) \ge b, \dots Y_{s_{2n-1}}(U) \le a, Y_{s_{2n}}(U) \ge b.$$

Let

$$t_1 = \sigma - s_{2n}, t_2 = \sigma - s_{2n-1}, \dots t_{2n-1} = \sigma - s_2, t_{2n} = \sigma - s_1.$$

Then

$$0 \le t_1 < t_2 < \dots < t_{2n-1} < t_{2n} \le \sigma$$

and

$$Z_{t_1}(U) \ge b, Z_{t_2}(U) \le a, \dots Z_{t_{2n-1}}(U) \ge b, Z_{t_{2n}}(U) \le a.$$

Hence, $\mathbb{D}(U) \ge n$.

We conclude from (2.9) that $\Pi_c \mathbb{U}(U) \le b/(b-a)$ for every $U \Subset E$. Therefore $\Pi_c \mathbb{U} \le b/(b-a)$ where $\mathbb{U}$ the number of upcrossings of $[a,b]$ by $1_{t<\zeta} k_{\xi_t}(\phi)$. This implies the existence, Π_c-a.s., of the limit of $k_{\xi_t}(\phi)$ as $t \uparrow \zeta$. Hence, $\hat{d}(\xi_s, \xi_t) \to 0$ as $s, t \uparrow \zeta$ and therefore there exists the limit (2.2).

2°. To prove formula (2.3) we consider a sequence D_n exhausting E and we apply (2.5) to the last exit time σ_n from D_n, to $Y = f(\xi_{\zeta-})$ and to $t = 0$. We get

$$\Pi_c 1_{\sigma_n > 0} k_{\xi_{\sigma_n}}(\phi) Y = \int_E \Pi_x (1_{\sigma_n > 0} Y) \phi(x)\, dx.$$

Passing to the limit yields

$$\Pi_c k_{\xi_{\zeta-}}(\phi) Y = \int_E \Pi_x Y \phi(x)\, dx.$$

Since this is true for all $\phi \in C_0$, formula (2.3) holds for almost all x. Since both parts are harmonic functions in E, it holds for all x. □

3. h-transform

3.1.

LEMMA 3.1. *For every stopping time τ and for every pre-τ positive Y ,*

$$\Pi_x^h Y 1_{\tau<\zeta} = \Pi_x Y h(\xi_\tau). \tag{3.1}$$

PROOF. It is sufficient to prove (3.1) for bounded Y. Every harmonic function is superparabolic, and, by Proposition **2**.3.1, $X_t = 1_{t<\zeta} h(\xi_t)$ is a supermartingale relative to $\mathcal{F}[0,t]$ and Π_x. Consider simple stopping times τ_n approximating τ (see section 2 in the Appendix A). By 4.3.B in the Appendix A, X_{τ_n} are uniformly integrable with respect to Π_x. Therefore, if (3.1) holds for τ_n, it holds also for τ.

We start from the case $\tau = t$ and $Y = 1_{B_1}(\xi_{t_1}) \dots 1_{B_n}(\xi_{t_n})$ where $B_1, \dots, B_n \in \mathcal{B}_E$, $0 \le t_1 \le \dots \le t_n \le t$. In this case (3.1) follows from (**??**). By the multiplicative systems theorem (Theorem 1.1, Appendix A), it holds for all $Y \in \mathcal{F}[0,t]$.

If τ is simple with values $0 \le t_1 < \dots < t_k < \dots$, then we have

$$Y 1_{\tau<\zeta} = \sum_k Y_k 1_{t_k<\zeta}$$

where $Y_k = Y 1_{\tau=t_k} \in \mathcal{F}_{\le t_k}$. Note that

$$\Pi_x^h Y_k 1_{\zeta>t_k} = \Pi_x Y_k h(\xi_{t_k})$$

which implies (3.1). □

REMARK. By applying Lemma 3.1, we can deduce that ξ_t is Π_x^h-a.s. continuous on $[0,\zeta)$ from the fact that it is continuous Π_x-a.s.

4. Integral representation of positive harmonic functions

4.1. The Green's function corresponding to p^h is given by the formula

$$g^h(x,y) = \int_0^\infty p_t^h(x,y)\ dt = \frac{1}{h(x)} g(x,y)h(y) \tag{4.1}$$

and the Martin kernel has an expression

$$k^h(x,y) = \frac{g^h(x,y)}{g^h(c,y)} = \frac{h(c)}{h(x)} k(x,y). \tag{4.2}$$

Since $k_y^h(\phi) = h(c)k_y(\phi/h)$ and since $\phi/h \in C_0$ if $\phi \in C_0$, the Martin compactification $\hat{E}$ constructed starting from p is also a Martin compactification for all h-transforms p^h. Moreover, the arguments in the proof of Theorem 2.1 can be applied to measures $\tilde{\Pi}_x^h = \Pi_x^h/h(x)$ and they yield the existence $\tilde{\Pi}_x^h$-a.s. of the limit $\xi_{\zeta-}$ in the topology of $\hat{E}$ and the formula

$$\tilde{\Pi}_x^h f(\xi_{\zeta-}) = \tilde{\Pi}_c^h k^h(x, \xi_{\zeta-}) f(\xi_{\zeta-}).$$

For $h \in \mathcal{H}_1$, this is equivalent to the formula

$$\Pi_x^h f(\xi_{\zeta-}) = \Pi_c^h k(x, \xi_{\zeta-}) f(\xi_{\zeta-}). \tag{4.3}$$

Put

$$\nu_h(B) = \Pi_c^h \{\xi_{\zeta-} \in B\}. \tag{4.4}$$

We have:

$$\Pi_c^h f(\xi_{\zeta-}) = \int_{\hat{\partial} E} f\ d\nu_h. \tag{4.5}$$

By (4.3)and (4.5),

$$\Pi_x^h f(\xi_{\zeta-}) = \Pi_c^h k(x, \xi_{\zeta-}) f(\xi_{\zeta-}) = \int_{\hat{\partial} E} k(x,y) f(y)\ \nu_h(dy). \tag{4.6}$$

By taking $f = 1$, we get

$$h(x) = \int_{\hat{\partial} E} k(x,y)\ \nu_h(dy). \tag{4.7}$$

This is the Martin integral representation of positive harmonic functions. The next step is to investigate properties of measures ν_h.

4.2. Put

$$\Pi_x^y = \Pi_x^h, \nu_y = \nu_h$$

for $h = k_y$. Since $k_y(c) = 1$, ν_y is a probability measure.

Denote by E' the set of all $y \in \hat{\partial} E$ such that $\nu_y = \delta_y$. For every $x \in E$,

$$E' = \{y : \nu_y(y) = 1\} = \{y : \Pi_x^y \{\xi_{\zeta-} = y\} = k(x,y)\}. \tag{4.8}$$

[If $y \in E'$, then probability measure $\frac{\Pi_x^y}{k(x,y)}$ can be interpreted as the conditional probability distribution of the path given that $\xi_{\zeta-} = y$.]

THEOREM 4.1. *For every $h \in \mathcal{H}$, the measure ν_h is concentrated on E'.*

PROOF. Consider a sequence D_n exhausting E and let τ_n be the first exit time from D_n. Choose two positive continuous functions ϕ and f on $\hat{E}$. It follows from (3.1) and the strong Markov property of ξ (see (3.11) in the Appendix A) that

$$\Pi_c^h \phi(\xi_{\tau_m}) f(\xi_{\tau_n}) = \Pi_c \phi(\xi_{\tau_m}) \Pi^h_{\xi_{\tau_m}} f(\xi_{\tau_n}).$$

By passing to the limit as $n \to \infty$, we obtain

$$\Pi_c^h \phi(\xi_{\tau_m}) f(\xi_{\zeta-}) = \Pi_c \phi(\xi_{\tau_m}) \Pi^h_{\xi_{\tau_m}} f(\xi_{\zeta-}). \tag{4.9}$$

By (4.6), the right side is equal to

$$\Pi_c \phi(\xi_{\tau_m}) \int k(\xi_{\tau_m}, y) f(y) \nu_h(dy) = \int \nu_h(dy) f(y) \Pi_c \phi(\xi_{\tau_m}) k(\xi_{\tau_m}, y).$$

By (3.1),

$$\Pi_c \phi(\xi_{\tau_m}) k(\xi_{\tau_m}, y) = \Pi_c^y \phi(\xi_{\tau_m}).$$

Therefore by (4.9), (4.6) and Fubini's theorem,

$$\Pi_c^h \phi(\xi_{\tau_m}) f(\xi_{\zeta-}) = \Pi_c \phi(\xi_{\tau_m}) \int k(\xi_{\tau_m}, y) f(y) \nu_h(dy)$$
$$= \int \nu_h(dy) f(y) \Pi_c k(\xi_{\tau_m}, y) \phi(\xi_{\tau_m}) = \int \nu_h(dy) f(y) \Pi_c^y \phi(\xi_{\tau_m}).$$

By passing to the limit as $m \to \infty$, we get

$$\Pi_c^h (\phi f)(\xi_{\zeta-}) = \int \nu_h(dy) f(y) \Pi_c^y \phi(\xi_{\zeta-}).$$

By (4.5), this implies

$$\int (\phi f)(y) \ \nu_h(dy) = \int \int \nu_h(dy) f(y) \phi(z) \nu_y(dz).$$

Therefore, for every ϕ,

$$\phi(y) = \int \phi(z) \nu_y(dz) \quad \nu_h\text{-a.s.}.$$

We conclude that $\nu_y = \delta_y$ for ν_h-almost all y which means that ν_h is concentrated on E'. □

4.3. It follows from (4.7) and Theorem 4.1 that every $h \in \mathcal{H}$ has a representation

$$h(x) = \int_{E'} k(x, y) \nu(dy) \tag{4.10}$$

where $\nu = \nu_h$ is a probability measure.

THEOREM 4.2. *A measure ν is determined uniquely by* (4.10).

PROOF. If $f \in C(\hat{E})$, then, by (4.5) and (3.1),

$$\int f \ d\nu_h = \Pi_c^h f(\xi_{\zeta-}) = \lim \Pi_c^h f(\xi_{\tau_n}) = \lim \Pi_c (fh)(\xi_{\tau_n}). \tag{4.11}$$

By (4.10) and (3.1),

$$\Pi_c (fh)(\xi_{\tau_n}) = \int_{E'} \Pi_c f(\xi_{\tau_n}) k(\xi_{\tau_n}, y) \nu(dy) = \int_{E'} \nu(dy) \Pi_c^y f(\xi_{\tau_n}).$$

Hence, (4.11) and (4.5) imply

$$\int_{E'} f(y)\nu_h(dy) = \lim \int_{E'} \nu(dy)\Pi_c^y f(\xi_{\tau_n})$$
$$= \int_{E'} \nu(dy)\Pi_c^y f(\xi_{\zeta-}) = \int_{E'} \nu(dy) \int_{E'} f(z)\nu_y(dz).$$

Since $\nu_y = \delta_y$ for $y \in E'$, we conclude that $\nu_h = \nu$. □

4.4. We say that $h \in \mathcal{H}^1$ is an *extreme element* if the decomposition

$$h = \int \tilde{h}\gamma(d\tilde{h})$$

where γ is a probability measure on $\mathcal{H}^1$ implies that γ is concentrated at a single point h.

THEOREM 4.3. *The set of extreme elements of $\mathcal{H}^1$ coincides with the set $k_y, y \in E'$.*

PROOF. If h is an extreme element of $\mathcal{H}^1$, then the representation (4.10) implies that ν is concentrated at a single point. Hence, $h = k_y$ for some $y \in E'$.

On the other hand, suppose that

$$\int_{\mathcal{H}^1} h\gamma(dh) = k_y.$$

Formula (**??**) implies $\Pi_c^y = \int \Pi_c^h \gamma(dh)$ and, by (4.4),

$$\nu_y = \int_{\mathcal{H}^1} \nu_h \gamma(dh). \tag{4.12}$$

If $y \in E'$, then, by (4.8), $\nu_y(y) = 1$. On the other hand, for all $h \in \mathcal{H}^1$, ν_h is a probability measure and therefore $\nu_h(y) \leq 1$. Formula (4.12) implies that $\nu_h(y) = 1$ for γ-almost all h. Hence, $\nu_h = \delta_y$ for γ-almost all h and, by (4.10), $h = \int k_y d\nu_h = k_y$ for γ-almost all h. □

Theorems 4.1–4.3 imply the following result:

THEOREM 4.4. *Let $\hat{\partial}E$ be the Martin boundary and $k(x, y)$ be the Martin kernel for a domain E. Denote by E' the set of $y \in \hat{\partial}E$ for which $k_y = k(\cdot, y)$ is an extreme element of $\mathcal{H}^1$. Formula* (4.10) *defines a 1-1 mapping K from the set of all finite measures on E' onto $\mathcal{H}$.*

If h and ν are connected by formula (4.10), then we write $h = h_\nu, \nu = \operatorname{tr} h$ and we call ν the *boundary trace of h* . Note that, if $\operatorname{tr} h = \nu$, then, by (4.6),

$$\Pi_x^h\{\xi_{\zeta-} \in B\} = \int_B k(x, y)\nu(dy) \tag{4.13}$$

for every Borel subset of $\hat{\partial}E$.

REMARK 4.1. As we know (see Remark 1.1), $\hat{\partial}E = \partial E$ for a smooth Greenian domain E. We claim that in this case E' also coincides with ∂E. It is sufficient to prove that, if $h(x) = k(x, y), y \in \partial E$, then $\Pi_c^h\{\xi_{\zeta-} \neq y\} = 0$. To this end we show that, for every neighborhood U of y, $\Pi_c^h \varphi(\xi_{\zeta-}) = 0$ for every bounded continuous function φ vanishing in U. Let τ_n be the first exit time from $D_n = \{x \in E : d(x, \partial E) > 1/n\}$. By Lemma 3.1, $\Pi_c^h \varphi(\xi_{\tau_n}) = \Pi_c(\varphi h)(\xi_{\tau_n})$ and therefore

$$\Pi_c^h \varphi(\xi_{\zeta-}) = \lim \Pi_c(\varphi h)(\xi_{\tau_n}). \tag{4.14}$$

If $\xi_{\zeta-} \in U$, then $(\varphi h)(\xi_{\tau_n}) = 0$ for all sufficiently large n. If $\xi_{\zeta-} \notin U$, then, by **6**.(1.16), $\lim h(\xi_{\tau_n}) = h(\xi_{\zeta-}) = 0$. By **6**.1.8.B, φh is bounded. By the dominated convergence theorem, the right side in (4.14) is equal to zero.

5. Extreme elements and the tail σ-algebra

5.1. We prove that $h \in \mathcal{H}^1$ is extreme if and only if $\tilde{\Pi}_x^h(A) = 0$ or 1 for all A in a certain σ-algebra $\mathcal{T}$.

We define this σ-algebra for an arbitrary homogeneous continuous strong Markov process (ξ_t, Π_x) in a domain $E \subset \mathbb{R}^d$. Denote by $\mathcal{E}$ the family of the first exit times τ from all domains $D \Subset E$. Denote by $\mathcal{F}^0$ the minimal σ-algebra in Ω which contains pre-τ sets for all $\tau \in \mathcal{E}$. Put $A \in \mathcal{T}$ if $A \in \mathcal{F}^0$ and if $\theta\tau A = A$ for all $\tau \in \mathcal{E}$. We call $\mathcal{T}$ the *tail σ-algebra*.

We say that a probability measure μ on a σ-algebra $\mathcal{B}$ is trivial if, for every $B \in \mathcal{B}$, $\mu(A) = 0$ or 1. If $\mathcal{B}$ is the Borel σ-algebra in a compact or σ-compact space, then μ is trivial if and only if it is concentrated at a single point. [4]

If a measure $\tilde{\Pi}_x^h$ is trivial on the tail σ-algebra $\mathcal{T}$, then the measure ν_h is trivial on the Borel σ-algebra in E'. Therefore it is concentrated at a point $z \in E'$ and $h = k_z$ is an extreme element of $\mathcal{H}^1$. After some preparations we prove the converse result.

THEOREM 5.1. *If $h \in \mathcal{H}^1$ is extreme, then all measures Π_x^h are trivial on $\mathcal{T}$.*

5.2. Let $h \in \mathcal{H}^1$. We say that a function φ is h-harmonic if

$$\tilde{\Pi}_x^h \varphi(\xi_{\tau_D}) = \varphi(x)$$

for all $D \Subset E$ and all $x \in E$. This is equivalent to the condition $\varphi h \in \mathcal{H}$. By Theorem 4.1,

$$(\varphi h)(x) = \int_{E'} k(x,y)\nu_{\varphi h}(dy). \tag{5.1}$$

We have:

5.2.A. If h is an extreme element of $\mathcal{H}^1$ and if an h-harmonic function φ is bounded, then $\varphi(x) = \varphi(c)$ for all x.

Indeed, the condition $\varphi h \le Nh$ implies $\nu_{\varphi h} \le N\nu_h$ and, by the Radon-Nikodym theorem,

$$\nu_{\varphi h}(dy) = \rho(y)\nu_h(dy). \tag{5.2}$$

Since ν_h is extreme, it is concentrated at $z \in E'$. Hence $h = k_z$ and, by (5.1) and (5.2),

$$\varphi(x)k_z(x) = k_z(x)\rho(z)$$

which implies 5.2.A.

[4] This is true for every topological space such that every cover by open sets contains a countable subcover.

5.2.B. [Strong Markov property of $(\xi_t, \tilde{\Pi}_x^h)$.] If $\tau < \zeta$ and if X is a bounded pre-τ function, then, for every bounded $Y \in \mathcal{F}$,

$$\tilde{\Pi}_x^h(X\theta_\tau Y) = \tilde{\Pi}_x^h(X\tilde{\Pi}_{\xi_\tau}^h Y).$$

PROOF. We can assume that $Y \in \mathcal{F}_{\le t}$. Then $\theta_\tau Y \in \mathcal{F}_{\le \tau+t}$ and, by Lemma 3.1,

$$\Pi_x^h X\theta_\tau Y = \Pi_x XY h(\xi_{\tau+t}) = \Pi_x X\Pi_{\xi_\tau} Y h(\xi_\tau) = \Pi_x X\Pi_{\xi_\tau}^h Y.$$

The right side is equal to $\tilde{\Pi}_x^h(X\tilde{\Pi}_{\xi_\tau}^h Y)$ because $\Pi_x Z = \Pi_x^h Z/h(\xi_\tau)$ for every pre-τ Z. □

5.2.C. If $A \in \mathcal{T}$, then $\varphi(x) = \tilde{\Pi}_x^h(A)$ is h-harmonic. If $\tau_n = \tau_{D_n}$ where D_n exhaust E, then

$$1_A = \lim \varphi(\xi_{\tau_n}) \quad \tilde{\Pi}_x^h\text{-a.s.} \tag{5.3}$$

for all $x \in E$.

PROOF. The first part follows from 5.2.B.

If X is a bounded pre-τ_n function, then, by 5.2.B,

$$\tilde{\Pi}_x^h XY = \tilde{\Pi}_x^h X\theta_{\tau_n} Y = \tilde{\Pi}_x^h X\varphi(\xi_{\tau_n}). \tag{5.4}$$

Therefore $\varphi(\xi_{\tau_n})$ is a martingale relative to $\mathcal{F}_{\le \tau_n}$ and $\tilde{\Pi}_x^h$. By 4.3.A in the Appendix A, there exists, $\tilde{\Pi}_x^h$-a.s., the limit $\tilde{Y}$ of $\varphi(\xi_{\tau_n})$. It follows from (5.4) that $\tilde{\Pi}_x^h XY = \tilde{\Pi}_x^h X\tilde{Y}$ for all $X \in \mathcal{F}_{\le \tau_n}$. By the multiplicative systems theorem, this is true for all bounded $X \in \mathcal{F}^0$ and therefore $Y = \tilde{Y}$ $\tilde{\Pi}_x^h$-a.s. □

5.3. Proof of Theorem 5.1. By 5.2.C, $\varphi(x) = \tilde{\Pi}_x^h(A)$ is h-harmonic. Clearly, $\varphi \le 1$. If h is extreme, then, by 5.2.A, $\varphi(x) = \varphi(c)$. We conclude from 5.2.C that $1_A = \varphi(c)$ $\tilde{\Pi}_x^h$-a.s. Hence $\varphi(c) = 0$ or 1 which implies Theorem 5.1.

6. Notes

6.1. In 1941 Martin [**Mar41**], a young mathematician at the University of Illinois proposed a method of characterizing all positive solutions of the Laplace equation in an arbitrary domain of $\mathbb{R}^d$. He died shortly after his paper appeared and the importance of his results was not immediately appreciated. It seems that Brelot was the first who attracted attention to Martin's paper. A probabilistic interpretation of Martin's ideas was suggested by Doob [**Doo59**] who applied them to harmonic functions associated with discrete Markov chains. A new approach to Martin's theory is due to Hunt. In [**Hun68**] he has shown that the Doob's results can be deduced from the study of the limit behavior of paths of the chain as $t \to \infty$. An improved and simplified presentation of Hunt's theory is given in [**Dyn69a**]. In [**Dyn69b**] and [**Dyn70**] the Martin theory was extended to processes with continuous time parameter and general state space under minimal conditions on the process. [Some results in this direction were obtained earlier by Kunita and Watanabe [**KW65**].] The results in section 5 are due to Doob (see section 2.X.11 in [**Doo84**]).

The papers [**Dyn69b**] and [**Dyn70**] are the basis of Chapter **7**. However, by imposing stronger conditions on the Green's function, we are able to simplify greatly the presentation.

6.2. In this book, we concentrate on diffusions in $\mathbb{R}^d$, but the boundary theory developed in Chapter **7** depends only on the existence of Green's function for ξ and on certain properties of this function. The most important is the bound **6**.(1.12). Instead of this bound, it is sufficient to require that

$$\int_{U_\varepsilon(y)} g(x,y)m(dx) \to 0,$$

$$\int_{U_\varepsilon(x)} g(x,y)m(dy) \to 0$$

uniformly on every compact subset of E as $\varepsilon \to 0$. Here $U_\varepsilon(z)$ is the ε-neighborhood of z, and m is a measure in the definition of the transition density of ξ.

CHAPTER 8

Moderate solutions of $Lu = \psi(u)$

1. Introduction

Our objective is to investigate the set $\mathcal{U}(E)$ of all positive solutions of the equation

$$Lu = \psi(u) \quad \text{in } E. \tag{1.1}$$

We assume that ψ and the coefficients of L do not depend on time and that ψ satisfies conditions:

1.A. For every x, $\psi(x, \cdot)$ is convex and $\psi(x, 0) = 0$, $\psi(x, u) > 0$ for $u > 0$.

1.B. $\psi(x, u)$ is continuously differentiable.

1.C. ψ is locally Lipschitz continuous in u uniformly in x: for every $t \in \mathbb{R}_+$, there exists a constant c_t such that

$$|\psi(x, u_1) - \psi(x, u_2)| \leq c_t |u_1 - u_2| \quad \text{for all } x \in E, u_1, u_2 \in [0, t].$$

[Note that 1.A implies that $\psi(x, u_1) \leq \psi(x, u_2)$ for all $0 \leq u_1 < u_2$.] Additional conditions will be mentioned each time explicitely.

In section 2 we establish some properties of operators V_D. Starting from section 3, we fix a Greenian domain E and we consider an L-diffusion ξ in E [see section **7**.2.1]. It is terminated [sent to a "cemetery" †] at the first exit time ζ from E. If we agree to put $f(\dagger) = 0$ for all functions f defined on E, then the formula **6**.(2.5) takes the form

$$K_D f(x) = \Pi_x f(\xi_\tau) 1_{\tau < \zeta} \tag{1.2}$$

In Chapter **8**, we investigate solutions of (1.1) dominated by harmonic functions.

2. From parabolic to elliptic setting

2.1. Operators V_D. The values of $V_Q(f)$ for all open subsets Q of $\mathbb{R} \times \mathbb{R}^d$ and all Borel functions $f : \mathbb{R} \times \mathbb{R}^d \to [0, \infty]$ were defined in section 4 of Chapter **5**. If $Q = \mathbb{R} \times D$, then V_Q preserves the set of time-independent functions and we denote by V_D its restriction to this set. Operators $K_{\mathbb{R} \times D}$ and $G_{\mathbb{R} \times D}$ induce the Poisson and Green's operators K_D and G_D defined by the formulae **6**.(2.5) and **6**.(2.16). Proposition **5**.4.1.D implies:

2.1.A. For every D and every Borel $f \geq 0$, $u = V_D(f)$ satisfies the integral equation

$$u + G_D \psi(u) = K_D f. \tag{2.1}$$

It follows, respectively, from Theorem **5**.2.1, **5**.4.1.A, **5**.2.2.B, **5**.4.1.E and Corollary **5**.5.1 that:

2.1.B. If f is bounded, then (2.1) implies that $Lu = \psi(u)$ in D. If $\tilde{x} \in \partial D$ is regular and if f is continuous at $\tilde{x}$, then $V_D(f)(x) \to f(\tilde{x})$ as $x \to \tilde{x}$.

2.1.C. If $f_1 \leq f_2$, then $V_D(f_1) \leq V_D(f_2)$.

2.1.D. (Mean value property) If $u \in \mathcal{U}(D)$, then, for every $U \Subset D$, $V_U(u) = u$ and therefore $u + G_U\psi(u) = K_U u$.

2.1.E. If $D' \Subset D$, then $V_{D'}V_D = V_D$.

2.1.F. Suppose $D' \subset D$ and $\partial D' \cap D$ is regular for D'.Then $V_{D'}(1_D u) \leq u$ for every $u \in \mathcal{U}(D)$.

We also have:

2.1.G. If $D_1 \subset D_2 \subset D$ and $\partial D_2 \cap D$ is regular for D_2, then $V_{D_2}(1_D u) \leq V_{D_1}(1_D u)$ for every $u \in \mathcal{U}(D)$.

Indeed, by 2.1.E, 2.1.F and 2.1.C,

$$V_{D_2}(1_D u) = V_{D_1}V_{D_2}(1_D u) \leq V_{D_1}(1_D u). \tag{2.2}$$

Theorem **5**.2.3 implies the following version of the comparison principle:

2.1.H. Suppose D is bounded. Then $u \leq v$ assuming that $u, v \in C^2(D)$,

$$Lu - \psi(u) \geq Lv - \psi(v) \quad \text{in } D \tag{2.3}$$

and, for every $\tilde{x} \in \partial D$,

$$\limsup[u(x) - v(x)] \leq 0 \quad \text{as } x \to \tilde{x}. \tag{2.4}$$

This principle is used in the proof of the next proposition:

2.1.I. Suppose $\psi \in BR$ is monotone increasing in u. If $f : \partial D \to [0, \infty]$ is continuous on a regular relatively open subset O of ∂D and if it vanishes on $\partial D \setminus O$, then $u = V_D(f)$ is the minimal solution of the boundary value problem

$$\begin{aligned} Lu &= \psi(u) \quad \text{in } D, \\ u &= f \quad \text{on } O. \end{aligned} \tag{2.5}$$

Proof. Let $Q^k = (-k, k) \times D$ and let $f_k(r, x) = f(x)$ for $(r, x) \in O_k = (-k, k) \times O$ and $f_k(r, x) = 0$ on $\partial Q^k \setminus O_k$. By Theorem **5**.5.1, $u_k = V_{Q^k}(f_k)$ is a minimal solution of the problem

$$\begin{aligned} \dot{u}_k + Lu_k &= \psi(u_k) \quad \text{in } Q^k, \\ u &= f \quad \text{on } O_k. \end{aligned}$$

Since $u = V_D(f) = \lim u_k$, (2.5) holds by Theorems **5**.3.2 and **5**.3.3. If v is an arbitrary solution of the problem (2.5), then $v \geq u_k$ in Q^k by the comparison principle. Hence $v \geq u$ in D. □

A stronger version of 2.1.B follows from Theorems **5**.3.2 and **5**.3.3:

2.1.J. Suppose that $\psi \in BR$. For every D and every Borel $f \geq 0$, the function $u = V_D(f)$ is a solution of the equation $Lu = \psi(u)$ in D. If $\tilde{x} \in \partial D$ is regular and if f is continuous in a neighborhood of $\tilde{x}$, then $u(x) \to f(\tilde{x})$ as $x \to \tilde{x}$.

To prove this statement, we apply 2.1.B to $f_n = f \wedge n$ and then pass to the limit.

2.2. Subadditivity of V_D.

THEOREM 2.1. *If ψ satisfies condition 1.A, then*

$$V_D(f_1 + f_2) \le V_D(f_1) + V_D(f_2) \tag{2.6}$$

for all $f_1, f_2 \ge 0$.

We prove the theorem by using two lemmas that are also of independent interest.

Consider operators

$$G_D^a f(x) = \Pi_x \int_0^\tau H(s) f(\xi_s) ds \tag{2.7}$$

where

$$H(t) = \exp[-\int_0^t a(\xi_s)\ ds] \tag{2.8}$$

and a is a positive Borel function on D. [This is Green's operator of the L-diffusion in D with the killing rate $a(x)$.] Note that $G_D^0 = G_D$.

LEMMA 2.1. *If $G_D|f| < \infty$, then*

$$G_D^a[f + aG_D f] = G_D f. \tag{2.9}$$

PROOF. It is sufficient to prove (2.9) for $f \ge 0$. Put $G_D = G, G_D^a = \tilde{G}$ and let $F = Gf$. We have

$$\begin{aligned} \tilde{G}(aF)(x) &= \Pi_x \int_0^\tau ds\ H(s) a(\xi_s) \Pi_{\xi_s} \int_0^\tau f(\xi_t) dt \\ &= \int_0^\infty ds\ \Pi_x 1_{s<\tau}\ H(s) a(\xi_s) \Pi_{\xi_s} \int_0^\tau f(\xi_t) dt. \end{aligned} \tag{2.10}$$

By the strong Markov property of ξ ((3.11) in the Appendix A) and Fubini's theorem, the right side is equal to

(2.11)
$$\int_0^\infty ds\ \Pi_x 1_{s<\tau}\ H(s) a(\xi_s) \int_s^\tau f(\xi_t) dt$$
$$= \Pi_x \int_0^\infty \int_0^\infty ds\ dt\ 1_{0<s<t<\tau} H(s) a(\xi_s) f(\xi_t) = \Pi_x \int_0^\tau f(\xi_t) dt \int_0^t H(s) a(\xi_s) ds.$$

Therefore the left side in (2.9) is equal to

$$\Pi_x \int_0^\tau f(\xi_t) Y(t)\ dt \tag{2.12}$$

where $Y(t) = H(t) + \int_0^t H(s) a(\xi_s)\ ds$. Note that $Y'(t) = 0$ and $Y(0) = 1$. Hence, $Y(t) = 1$ and (2.12) is equal to Gf. □

LEMMA 2.2. *If $u, v, \rho \ge 0$ and if*

$$u + G_D\psi(u) + G_D\rho = v + G_D\psi(v) < \infty, \tag{2.13}$$

then $v = u + G_D^a \rho$ and therefore $v \ge u$.

PROOF. Put $w = v - u, f = \psi(v) - \psi(u)$. There exists a function $a \geq 0$ such that $f = aw$. We have $G_D|f| \leq G_D\psi(u) + G_D\psi(v)$. By (2.13), $G_D\psi(u) < \infty$ and $G_D\psi(v) < \infty$. Hence, $G_D|f| < \infty$. By Lemma 2.1, f satisfies (2.9). By (2.13),

$$w + G_D f = G_D\rho \tag{2.14}$$

which implies $f + aG_D f = aG_D\rho$ and

$$G_D^a(f + aG_D f) = G_D^a(aG_D\rho).$$

By (2.9), the left side is equal to $G_D f$ and the right side is equal to $G_D\rho - G_D^a\rho$. Hence, $G_D f = G_D\rho - G_D^a\rho$ and, by (2.14), $w = G_D\rho - G_D f = G_D^a\rho$. □

PROOF OF THEOREM 2.1. By **5**.4.1.C, it is sufficient to prove (2.6) for bounded f_1, f_2. Let $u_i = V_D(f_i), i = 1, 2$, $u = V_D(f_1 + f_2)$ and $v = u_1 + u_2$. By (2.1),

$$u_i + G_D\psi(u_i) = K_D f_i, \qquad u + G_D\psi(u) = K_D f_1 + K_D f_2. \tag{2.15}$$

Condition 1.A implies that $\psi(u_1 + u_2) - \psi(u_1)$ is an increasing function of u_1 and therefore $\rho = \psi(v) - \psi(u_1) - \psi(u_2) \geq 0$. Note that

$$\begin{aligned} v + G_D\psi(v) &= u_1 + u_2 + G_D\psi(u_1) + G_D\psi(u_2) + G_D\rho \\ &= K_D(f_1 + f_2) + G_D\rho = u + G_D\psi(u) + G_D\rho. \end{aligned}$$

It follows from (2.15) that $u_i, \psi(u_i)$ and $G_D\psi(u_i)$ are bounded. Therefore $v \geq u$ by Lemma 2.2 which implies (2.6). □

COROLLARY 2.1. *For every D_1, D_2,*

$$V_{D_1 \cap D_2}(f) \leq V_{D_1}(f) + V_{D_2}(f) \quad \textit{in } D_1 \cap D_2. \tag{2.16}$$

PROOF. Put $D = D_1 \cap D_2$. Note that $\partial D = B_1 \cup B_2$ where $B_1 \subset \partial D_1$, $B_2 \subset \partial D_2$ and $B_1 \cap B_2 = \emptyset$. Put $f_i = 1_{B_i} f$. We have $1_{\partial D} f = f_1 + f_2$. By Theorem 2.1,

$$V_D(1_{\partial D} f) \leq V_D(f_1) + V_D(f_2)$$

which implies (2.16) because $V_D(1_{\partial D} f) = V_D(f)$ and $V_D(f_i) \leq V_{D_i}(f_i) \leq V_{D_i}(f)$ in D by **5**.4.1.A–**5**.4.1.B. □

COROLLARY 2.2. *For every $c \geq 0$, $V_D(cu) \leq 2(c \vee 1)V_D(u)$.*

This follows from 2.1.C for $c < 1$. If $c > 1$ and if $c \leq 2^k < 2c$, then, by 2.1.C and (2.6), $V_D(cu) \leq V_D(2^k u) \leq 2^k V_D(u) \leq 2cV_D(u)$.

2.3. Homogeneous superdiffusions. Suppose that $\xi = (\xi_t, \Pi_x)$ is a homogeneous right continuous strong Markov process in a topological space E and let $\psi(x, u)$ be a positive function on $E \times \mathbb{R}_+$. We say that a BEM system $X = (X_D, P_\mu), D \in \mathbb{O}, \mu \in \mathbb{M}$ is a homogeneous (ξ, ψ)-superprocess if $\mathbb{O}$ is the class of all open subsets of E, $\mathbb{M} = \mathcal{M}(E)$ is the class of all finite measures on E and if, for every $f \in \mathbb{B}(E)$ and all $D \in \mathbb{O}$, $\mu \in \mathbb{M}$,

$$P_\mu e^{-\langle f, X_D\rangle} = e^{-\langle V_D(f), \mu\rangle} \tag{2.17}$$

where $u = V_D(f)$ is a solution of the equation (2.1).

To construct such a process, we start from the superprocess $\hat{X}$ described in section **4**.4.1. We imbed E into $\mathbb{R} \times E$ by identifying $x \in E$ with $(0, x) \in \mathbb{R} \times E$. We define X_D as the projection of $\hat{X}_{\mathbb{R}\times D}$ on E and we put $P_\mu = \hat{P}_{\delta_0 \times \mu}$ for every finite measure μ on E (δ_0 is the unit mass on $\mathbb{R}$ concentrated at 0).

We deal with homogeneous (L, ψ)-superdiffusions, that is with homogeneous (ξ, ψ)-superprocesses corresponding to homogeneous L-diffusions ξ. By Theorem **4**.2.1, an (L, ψ)-superdiffusion exists for

$$\psi(x, u) = b(x)u^2 + \int_0^\infty (e^{-\lambda u} - 1 + \lambda u) n(x, d\lambda) \tag{2.18}$$

if

$$b(x) \quad \text{and} \int_0^\infty \lambda \wedge \lambda^2 n(x, d\lambda) \quad \text{are bounded.} \tag{2.19}$$

The conditions 1.A–1.C hold for functions of the form (2.18) under mild restrictions on b and n.

3. Moderate solutions

3.1. Operators i and j. Fix an arbitrary domain E in $\mathbb{R}^d$ and denote by $\mathcal{H}$ the set of all positive L-harmonic functions in E and by $\mathcal{U}$ the set of all positive solutions of (1.1). We say that $u \in \mathcal{U}$ is a *moderate solution* and we write $u \in \mathcal{U}_1$ if $u \le h$ for some $h \in \mathcal{H}$. We establish a 1-1 correspondence between $\mathcal{U}_1$ and a subclass $\mathcal{H}_1$ of class $\mathcal{H}$. Our arguments are applicable to all continuously differentiable functions $\psi(x, u)$ that are monotone increasing in u.

Fix a sequence D_n exhausting E and put $K_n = K_{D_n}, G_n = G_{D_n}, V_n = V_{D_n}, K = K_E, G = G_E$.

THEOREM 3.1. *For every $u \in \mathcal{U}$, a limit*

$$j(u) = \lim K_n u \tag{3.1}$$

exists and

$$j(u) = u + G\psi(u). \tag{3.2}$$

For every $h \in \mathcal{H}$, there exists a limit

$$i(h) = \lim V_n h. \tag{3.3}$$

j is a 1-1 mapping from $\mathcal{U}_1$ onto a subset $\mathcal{H}_1$ of $\mathcal{H}$ and i is the inverse mapping from $\mathcal{H}_1$ onto $\mathcal{U}_1$. Moreover, $h = j(u)$ is the minimal harmonic function dominating u and u is the maximal solution of (1.1) dominated by h.

The relation

$$u + G\psi(u) = h \tag{3.4}$$

holds for $h \in \mathcal{H}$ and $u \ge 0$ if and only if $h \in \mathcal{H}_1$ and $u = i(h)$.

PROOF. 1°. If $u \in \mathcal{U}$, then, by the mean value property 2.1.D,

$$u + G_n\psi(u) = h_n$$

where $h_n = K_n u$.

Since the sequence $G_n u$ is increasing, the sequence h_n is also increasing and therefore the limit (3.1) exists and it satisfies (3.2). By **6**.1.5.C, it belongs to $\mathcal{H}$ unless it is identically equal to infinity.

2°. If $u \in \mathcal{U}_1$, then $h = j(u)$ belongs to $\mathcal{H}$ and it is the minimal harmonic function dominating u. Indeed, if $u \in \mathcal{U}_1$, then $u \le \tilde{h} \in \mathcal{H}$ and therefore $K_n u \le K_n \tilde{h} = \tilde{h}$ for all n. Hence, $h \le \tilde{h}$.

3°. Suppose that $h \in \mathcal{H}$ and let $u_n = V_n(h)$. Since $K_n h = h$, we have

$$u_n + G_n\psi(u_n) = h$$

and therefore $u_n \le h$ for all n. By the mean value property 2.1.D and the monotonicity 2.1.C of V_n ,

$$u_{n+1} = V_n(u_{n+1}) \le V_n(h) = u_n.$$

This implies the existence of the limit (3.3). Since $u_n \in \mathcal{U}(D_n)$, $u = i(h) \in \mathcal{U}(E)$ by Theorem **5**.3.2. We get (3.4) by a monotone passage to the limit.

If $\tilde{u} \in \mathcal{U}(E)$ is dominated by h, then $\tilde{u} = V_n(\tilde{u}) \le V_n(h)$ and therefore $\tilde{u} \le i(h)$.

4°. Let us prove that $i[j(u)] = u$ for all $u \in \mathcal{U}_1$. Indeed, let $h = j(u)$ and $i(h) = u'$. Functions $u_n = V_n(h)$ satisfy the equation $u_n + G_n\psi(u_n) = h$. By Fatou's lemma and (3.2), $u' + G\psi(u') \le h = u + G\psi(u)$. By (3.2) $u \le h$. Hence, $V_n(h) \ge V_n(u) = u$ and $u' \ge u$. We conclude that $0 \le u' - u \le G\psi(u) - G\psi(u') \le 0$ and therefore $u' = u$.

5°. If $h \in \mathcal{H}_1$, then $h = j(u)$ for some $u \in \mathcal{U}_1$, and, by 4°, $j[i(h)] = j[i(j(u))] = j(u) = h$.

6°. Let $h \in \mathcal{H}$ and $u \ge 0$ satisfy (3.4). Then $G\psi(u) = h - u$. By **6**.2.12, for every $U \Subset E$, $G = G_U + K_U G$. Therefore $G_U\psi(u) + K_U(h - u) = h - u$ and $u + G_U\psi(u) = K_U u$. Since $u \le h$ is bounded in U, $Lu = \psi(u)$ in U by 2.1.B. Therefore $u \in \mathcal{U}$. Clearly, $u \in \mathcal{U}_1$.

By (3.2), $h = j(u)$. Hence, $h \in \mathcal{H}_1$ and $u = i(h)$. □

The equation (3.4) implies that, if $g_E(x, y) = \infty$ for all $x, y \in E$, then the only moderate solution is 0.

3.2. Characterization of class $\mathcal{H}_1$. Let $\mu \in \mathcal{M}(E)$. A sequence of functions f_n is called *uniformly μ-integrable* if, for every $\varepsilon > 0$ there exists N such that

$$I(n, N) = \int_{|f_n|>N} |f_n|\, d\mu < \varepsilon \quad \text{for all } n. \tag{3.5}$$

If this condition is satisfied and if $f_n \to f$ μ-a.e., then $\int f_n\, d\mu \to \int f\, d\mu$. Indeed, $g_n = |f_n - f|$ are uniformly integrable and $g_n \to 0$ μ-a.s. We have $g_n \le g'_{N,n} + g''_{N,n}$ where $g'_{N,n} = 1_{g_n \le N} g_n$ and $g''_{N,n} = 1_{g_n > N} g_n$. For every ε, there exists N such that $\int g''_{N,n}\, d\mu < \varepsilon$ and, for every N, $\int g'_{N,n}\, d\mu \to 0$ as $n \to \infty$ by the dominated convergence theorem.

We prove the converse statement for positive functions f_n.

LEMMA 3.1. *Suppose that $f_n \ge 0$ and $f_n \to f$ μ-a.e. If*

$$\int f_n\, d\mu \to \int f\, d\mu < \infty, \tag{3.6}$$

then f_n are uniformly μ-integrable.

PROOF. Note that

$$I(n, N) = \int_{f_n>N} (f_n - f)\, d\mu + \int_{f_n>N} f\, d\mu \le \alpha(n) + \beta(n, N)$$

where

$$\alpha(n) = \int_E |f_n - f| \; d\mu = \int_{f>f_n} (f - f_n) \; d\mu + \int_{f_n>f} (f_n - f) \; d\mu$$
$$= \int_E (f_n - f) \; d\mu + 2\int_{f>f_n} (f - f_n) \; d\mu \tag{3.7}$$

and

$$\beta(n, N) = \int_{f_n>N} f \; d\mu.$$

If $f_n > N$, then either $f > N - 1$ or $f_n - f > 1$. Therefore

$$\beta(n, N) \le \int_{f>N-1} f \; d\mu + \gamma(n) \tag{3.8}$$

where

$$\gamma(n) = \int_{f_n-f>1} f \; d\mu.$$

By the dominated convergence theorem and by (3.7), $\lim \gamma(n) = \lim \alpha(n) = 0$ and therefore, for every $\varepsilon > 0$ there exists n_0 such that $\alpha(n) + \gamma(n) < \varepsilon/2$ for all $n > n_0$. By (3.8), there exists N_0 such that, for all $N > N_0, n > n_0$, $\beta(n, N) < \varepsilon/2$ and, consequently, $I(n, N) < \varepsilon$. Since all f_n are integrable, there is N_1 such that $I(n, N) < \varepsilon$ for all $N > N_1, n \le n_0$. Hence (3.5) holds for $N > N_0 \vee N_1$. □

THEOREM 3.2. *A function $h \in \mathcal{H}$ belongs to $\mathcal{H}_1$ if and only if, for every x, functions*

$$F^x_n(y) = g_n(x, y)\psi[V_n(h)](y) \tag{3.9}$$

are uniformly integrable (with respect to the Lebesgue measure).

If E is connected, then $h \in \mathcal{H}$ belongs to $\mathcal{H}_1$ if the family F^c_n is uniformly integrable for some c.

PROOF. Since $K_n h = h$, function $u_n = V_n(h)$ satisfies the equation

$$u_n + G_n\psi(u_n) = h. \tag{3.10}$$

Note that

$$G_n\psi(u_n)(x) = \int F^x_n(y) \; dy.$$

If $u = i(h)$, then, by (3.3), $u_n \to u$ and therefore

$$F^x_n(y) \to F^x(y) = g(x, y)\psi[u(y)].$$

Equation (3.10) implies that the functions F^x_n are integrable.

If F^x_n are uniformly integrable, then

$$G_n\psi(u_n) \to G\psi(u),$$

and the equation (3.10) implies (3.4). By Theorem 3.1, $h \in \mathcal{H}_1$.

If $h \in \mathcal{H}_1$ and if $u = i(h)$, then $h = j(u)$, and (3.4) follows from (3.2). Hence, $u_n + G_n\psi(u_n) = u + G\psi(u)$ and $G_n\psi(u_n) \to G\psi(u)$ because $u_n \to u$. Functions (3.9) are uniformly integrable by Lemma 3.1.

If E is connected, then, for every $u \in \mathcal{U}$, $h = u + G\psi(u)$ either belongs to $\mathcal{H}$ or it is infinite in all E. This follows from **6**.1.5.C because, by (3.1)–(3.2), $h = \lim K_n u$. If the family (3.9) is uniformly integrable for some $c \in E$, then $h(c) < \infty$ and therefore $h \in \mathcal{H}_1$. □

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COROLLARY 3.1. *If $h \in \mathcal{H}_1, \tilde{h} \in \mathcal{H}$ and if $\tilde{h} \le h$, then $\tilde{h} \in \mathcal{H}_1$.*

Indeed, by 2.1.C, $V_n(\tilde{h}) \le V_n(h)$ and therefore $\psi[V_n(\tilde{h})] \le \psi[V_n(h)]$.
More can be said on the set $\mathcal{H}_1$ under an additional assumption:

3.2.A. There is a constant a such that

$$\psi(x, 2u) \le a\psi(x, u)$$

for all u and x.

THEOREM 3.3. *If ψ satisfies conditions 1.A and 3.2.A, then $\mathcal{H}_1$ is a convex cone.*

PROOF. It follows from 1.A and (2.6) that

$$\psi[V_n(h_1 + h_2)] \le \psi[V_n(h_1) + V_n(h_2)] \le \frac{1}{2}\psi[2V_n(h_1)] + \frac{1}{2}\psi[2V_n(h_2)].$$

If ψ satisfies 3.2.A, then the right side does not exceed $\frac{a}{2}\{\psi[V_n(h_1)] + \psi[V_n(h_2)]\}$. If the conditions of Theorem 3.2 hold for h_1 and h_2, then they hold for $h_1 + h_2$.

We claim that, for every $c \ge 0$, ch satisfies the conditions of Theorem 3.2 if this is true for h. Clearly, that this is correct for $0 < c < 1$. If $1 \le c < 2^k$, then, by Corollary 2.2, $V_n(ch) \le 2cV_n(h)$ and, by 3.2.A,

$$\psi[V_n(ch)] \le \psi[2cV_n(h)] \le \psi[2^{k+1}V_n(h)] \le a^{k+1}\psi[V_n(h)].$$

□

3.3. Trace of moderate solutions. By Theorem **7**.4.4, every $h \in \mathcal{H}$ has a unique representation $h = K\nu$ where ν is a finite measure on E'. Put $\nu \in \mathcal{N}_1$ if $h \in \mathcal{H}_1$. Corollary 3.1 implies that $\mathcal{N}_1$ contains, with every measure ν, all measures $\tilde{\nu} \le \nu$. We have a 1-1 mapping $\nu \to u_\nu = i(K\nu)$ from $\mathcal{N}_1$ onto $\mathcal{U}_1$. If $u = u_\nu$, then we say that ν is the *boundary trace* of u and we write $\nu = \operatorname{tr} u$.

4. Sweeping of solutions

4.1. Operators Q_B. Suppose E is a domain in $\mathbb{R}^d$ and B is a compact subset of ∂E. We claim that, for every $\varepsilon > 0$, the set

$$D(B, \varepsilon) = \{x \in E : d(x, B) > \varepsilon\}$$

satisfies the condition: all points $a \in O = \partial D(B, \varepsilon) \cap E$ are regular. To prove this, we use the criterion of regularity stated in section **6**.2.3. Note $d(a, B) = \varepsilon$ and there exists a point $b \in B$ such that $d(a, B) = d(a, b)$. Define A as the closed ball centered at $c = (a + b)/2$ of radius $\varepsilon/2$. If $x \in A \cap \bar{D}$, then $d(x, b) \ge d(x, B) \ge \varepsilon$ and $d(x, c) \le \varepsilon/2$. Since $d(b, c) = \varepsilon/2$, we have $d(x, c) + d(c, b) \le d(x, b)$. Hence, x, c, b lie on a straight line, and $x = a$.

We say that a sequence of open sets D_n is a $[E, B]$-*sequence* if

$$D_n \uparrow E, \quad \bar{D}_n \uparrow \bar{E} \setminus B, \quad d(D_n, E \setminus D_{n+1}) > 0 \quad \text{and } \partial D_n \cap E \quad \text{is regular for } D_n$$

To every sequence $\varepsilon_n \downarrow 0$ there corresponds an $[E, B]$-sequence $D_n = D(B, \varepsilon_n)$.

We deal with positive functions on E and we agree to continue them by 0 to E^c. Suppose that D_n is a $[E, B]$-sequence and $u \in \mathcal{U}$. It follows from 2.1.G that $V_{D_n}(u)$

is a monotone decreasing sequence. The *sweeping* $Q_B(u)$ *of* u *to* B is defined by the formula

$$Q_B(u) = \lim V_{D_n}(u). \tag{4.1}$$

It is easy to see that the limit does not depend on the choice of a $[E, B]$-sequence. Theorem **5**.3.2 implies that it belongs to $\mathcal{U}$. Operators Q_B have the following properties:

4.1.A. $Q_B(u_1) \le Q_B(u_2)$ for $u_1 \le u_2$.

4.1.B. $Q_B(u) \le u$.

4.1.C. If $B_1 \subset B_2$, then $Q_{B_1}(u) \le Q_{B_2}(u)$.

4.1.D. For every $B_1, B_2, \quad Q_{B_1 \cup B_2}(u) \le Q_{B_1}(u) + Q_{B_2}(u)$.

4.1.E. $Q_{\partial E}(u) = u$.

4.1.F. If $u \le u_1 + u_2$, then $Q_B(u) \le Q_B(u_1) + Q_B(u_2)$.

Properties 4.1.A and 4.1.B follow, respectively, from 2.1.C and 2.1.F. If $B_1 \subset B_2$, then $D(B_2, \varepsilon) \subset D(B_1, \varepsilon)$ and therefore 2.1.G implies 4.1.C. 4.1.D is an implication of (2.16) and the relation $D(B_1 \cup B_2, \varepsilon) = D(B_1, \varepsilon) \cap D(B_2, \varepsilon)$. 4.1.E holds because, by 2.1.D, $V_{D(\partial E, \varepsilon)}(u) = u$ for all ε. Finally, 4.1.F follows from (2.6) and 2.1.C.

4.2. Extended mean value property. According to the mean value property 2.1.D, if $Lu = \psi(u)$ in E, then, for every $D \Subset E$, $V_D(u) = u$ which is equivalent to the relation

$$u + G_D\psi(u) = K_D u. \tag{4.2}$$

In general, this is not true for $D \subset E$. However we will prove (4.2) under some additional assumptions on u and D.

THEOREM 4.1. *(Extended mean value property.) Let* u *be a moderate solution with trace* ν. *The relation* (4.2) *holds if* $\bar{D} \cap B = \emptyset$ *and if* ν *is concentrated on* B.

PROOF. Consider a sequence $\tilde{D}_n$ exhausting E and put $D_n = \tilde{D}_n \cap D$. Since $D_n \Subset E$, we have

$$u + G_{D_n}\psi(u) = K_{D_n} u. \tag{4.3}$$

Let τ_n and τ be the first exit times from D_n and from D. Clearly, $\tau_n \uparrow \tau$ and therefore $G_{D_n}\psi(u) \uparrow G_D\psi(u)$. We get (4.2) from (4.3) if we prove that

$$K_{D_n} u \to K_D u. \tag{4.4}$$

Since $\tau_n \le \tau$, we have

$$K_{D_n} u(x) = I_n + J_n$$

where

$$I_n = \Pi_x\{u(\xi_{\tau_n}) 1_{\tau_n = \tau}\} = \Pi_x\{u(\xi_\tau) 1_{\tau_n = \tau}\}$$

and

$$J_n = \Pi_x\{u(\xi_{\tau_n}) 1_{\tau_n < \tau}\} = \Pi_x\{(u 1_D)(\xi_{\tau_n})\}.$$

Note that $\{\tau_n = \tau\} \uparrow \{\tau < \zeta\}$ and therefore $I_n \to K_D u$. It remains to show that $J_n \to 0$. By the condition of the theorem, u is dominated by a harmonic function

$$h(x) = \int_B k(x, y)\nu(dy)$$

and, by Lemma **7**.3.1,

$$J_n \le \Pi_x\{(h1_D)(\xi_{\tau_n})\} = \Pi^h_x\{\xi_{\tau_n} \in D\}.$$

Note that

$$\{\xi_{\tau_n} \in D\} \downarrow \{\tau = \zeta, \xi_{\zeta-} \in C\}$$

where $C = \partial E \cap \bar{D}$. Therefore

$$\lim \Pi^h_x\{\xi_{\tau_n} \in D\} \le \Pi^h_x\{\xi_{\zeta-} \in C\}.$$

By **7**.(4.6),

$$\Pi^h_x\{\xi_{\zeta-} \in C\} = \int_C k(x,y)\nu(dy).$$

Since $\nu(C) = 0$, the integral is equal to 0. □

COROLLARY 4.1. *If trace ν of u is concentrated on B, then $Q_B(u) = u$.*

4.3. Trace of $Q_B(u)$.

THEOREM 4.2. *If u is a moderate solution with the trace ν, then $v = Q_B(u)$ is a moderate solution with the trace equal to the restriction ν_B of ν to B.*

PROOF. Put $h(x) = \int_{E'} k(x,y)\nu(dy)$ and $h_B(x) = \int_B k(x,y)\nu(dy)$. Let D_n be a $[E,B]$-sequence and let τ_n be the first exit time from D_n. We claim that

$$\Pi_x h(\xi_{\tau_n}) \to h_B(x). \quad \text{as } n \to \infty. \tag{4.5}$$

Indeed, $\{\tau_n < \zeta\} \downarrow \{\xi_{\zeta-} \in B\}$ and therefore

$$\Pi^h_x\{\tau_n < \zeta\} \downarrow \Pi^h_x\{\xi_{\zeta-} \in B\}. \tag{4.6}$$

By **7**.(4.13), the right side is equal to $h_B(x)$. By Lemma **7**.3.1, $\Pi^h_x\{\tau_n < \zeta\} = \Pi_x h(\xi_{\tau_n})$. Therefore (4.6) implies (4.5).

Note that

$$V_n(u) \le K_n u = \Pi_x u(\xi_{\tau_n}) \le \Pi_x h(\xi_{\tau_n})$$

and, by (4.1) and (4.5), $v = Q_B(u) \le h_B$ which implies that $\operatorname{tr} v \le \nu_B$.

Since $h_B \le h$, $v_B = i(h_B) \le i(h) = u$ and therefore $\operatorname{tr} v_B$ is concentrated on B. By the Corollary 4.1, $v_B = Q_B(v_B) \le Q_B(u) = v$ and $\nu_B = \operatorname{tr} v_B \le \operatorname{tr} v$. □

5. Lattice structure of $\mathcal{U}$

5.1. Operator π. Denote by $C_+(E)$ the class of all positive functions $f \in C(E)$. Put $u \in \mathcal{D}(\pi)$ and $\pi(u) = v$ if $u \in C_+(E)$ and $V_{D_n}(u) \to v$ pointwise for every sequence D_n exhausting E. By 2.1.B and Theorem **5**.3.2, $\pi(u) \in \mathcal{U}$. It follows from 2.1.C that $\pi(u_1) \le \pi(u_2)$ if $u_1 \le u_2$.

Put

$$\mathcal{U}^- = \{u \in C_+(E) : V_D(u) \le u \quad \text{for all } D \Subset E\}$$

and

$$\mathcal{U}^+ = \{u \in C_+(E) : V_D(u) \ge u \quad \text{for all } D \Subset E\}.$$

By 2.1.D, $\mathcal{U} \subset \mathcal{U}^- \cap \mathcal{U}^+$. If $h \in \mathcal{H}$, then, by (2.1), $V_D(h) \le K_D h = h$ for every $D \Subset E$. Hence, $\mathcal{H} \subset \mathcal{U}^-$.

For every sequence D_n exhausting E, we have:

5.1.A. If $u \in \mathcal{U}^-$, then $V_{D_n}(u) \downarrow \pi(u)$ and

$$\pi(u) = \sup\{\tilde{u} \in \mathcal{U} : \tilde{u} \le u\} \le u.$$

Indeed, by 2.1.E, $V_{D_{n+1}} u = V_{D_n} V_{D_{n+1}} u \le V_{D_n} u$. If $\tilde{u} \le u$, then, by 2.1.D, $\tilde{u} = \pi(\tilde{u}) \le \pi(u)$.

Analogously,

5.1.B. If $\psi \in BR$ and if $u \in \mathcal{U}^+$, then $V_{D_n}(u) \uparrow \pi(u)$ and

$$\pi(u) = \inf\{\tilde{u} \in \mathcal{U} : \tilde{u} \ge u\} \ge u.$$

Clearly,

5.1.C. If $u, v \in \mathcal{U}^+$, then $\max\{u, v\} \in \mathcal{U}^+$. If $u, v \in \mathcal{U}^-$, then $\min\{u, v\} \in \mathcal{U}^-$.

It follows from subadditivity of V_D (Theorem 2.1) that:

5.1.D. If $u, v \in \mathcal{U}^-$, then $u + v \in \mathcal{U}^-$.

5.2. Lattices of functions and measures. Let $(L, \prec)$ be a partially ordered set. Writing $v = \operatorname{Sup} C$ means that $u \prec v$ for all $u \in C$ and that $v \prec \tilde{v}$ if $u \prec \tilde{v}$ for all $u \in C$. The $\operatorname{Inf} C$ is defined in a similar way. A partially ordered set is called a lattice if $\operatorname{Sup}\{u, v\}$ and $\operatorname{Inf}\{u, v\}$ exist for every pair $u, v \in L$. These two elements are denoted $u \vee v$ and $u \wedge v$.

A lattice L is called *complete* if $\operatorname{Sup} C$ and $\inf C$ exist for every $C \subset L$.

Examples

1. The set $[0, \infty]$ with the order $\le$ is a complete lattice. The set $[0, \infty)$ is an incomplete lattice.

2. The set $C_+(E)$ of all continuous functions from a topological space E to $[0, \infty]$ with the order $\le$ is a complete lattice. The same is true for the set of all positive Borel functions.

3. The set $\mathcal{M}(E)$ of all finite measures on a measurable space $(E, \mathcal{B})$ is a lattice. Measure $\mu \vee \nu$ can be calculated by the formula $\max\{a, b\} d\gamma$ where $\gamma = \mu + \nu$ and $a = d\mu/d\gamma, b = d\nu/d\gamma$. A similar expression holds for $\mu \wedge \nu$. Clearly, the lattice $\mathcal{M}(E)$ is incomplete.

5.3. Lattice $\mathcal{U}$. *In the rest of this chapter we assume that* $\psi \in BR$. The set $\mathcal{U}$ with the partial order $\le$ is a lattice with $u \vee v = \pi[\max\{u, v\}]$ and $u \wedge v = \pi[\min\{u, v\}]$ (if $u, v \in \mathcal{U}$, then $\max\{u, v\}$ and $\min\{u, v\}$ belong to $\mathcal{D}(\pi)$ by 5.1.C). In addition, we introduce in $\mathcal{U}$ an operation $u \oplus v = \pi(u + v)$.

Theorem 5.1. *The lattice* $\mathcal{U}$ *is complete. Moreover:*

5.3.A. For every $C \subset \mathcal{U}$, *there exists a sequence* $u_n \in C$ *such that* $\operatorname{Sup} C = \operatorname{Sup} u_n$.

5.3.B. If C *is closed under* $\vee$ *and if* $v = \operatorname{Sup} C$, *then there exists a sequence* $u_n \in C$ *such that* $u_n(x) \uparrow v(x)$ *for all* $x \in E$. *We have* $v(x) = \sup_C u(x)$ *for all* $x \in E$.

Proof. By Theorem **5**.3.2, every monotone increasing sequence $u_n \in \mathcal{U}$ converges pointwise to an element u of $\mathcal{U}$. Clearly, $u = \operatorname{Sup} u_n$. If v_n is an arbitrary sequence in $\mathcal{U}$, then $u_n = v_1 \vee \cdots \vee v_n$ is monotone increasing and $\operatorname{Sup} v_n = \operatorname{Sup} u_n$. Therefore $\operatorname{Sup} C_0$ exists for every countable set C_0.

Let $C \subset \mathcal{U}$. For every $x \in E$, $\ell(x) = \sup_C u(x) < \infty$. Select a sequence x_k everywhere dense in E. For every k there exists a sequence $u_{nk} \in C$ such that $u_{nk}(x_k) \to \ell(x_k)$. Consider a countable set $C_0 = \{u_{nk} : n = 1, 2, \dots, , k = 1, 2, \dots\}$ and put $v = \operatorname{Sup} C_0$. We claim that $v = \operatorname{Sup} C$. Indeed, if $u \in C$, then $v(x_k) \geq \ell(x_k) \geq u(x_k)$ for all k and, since u and v are continuous, $v \geq u$. If $\tilde{u} \in \mathcal{U}$ is an upper bound of C, then $\tilde{u} \geq u_{nk}$ for all n, k which implies $\tilde{u} \geq v$. The existence of $\operatorname{Inf} C$ can be established in a similar way.

Our arguments prove also 5.3.A. The first part of 5.3.B follows immediately from 5.3.A. It remains to prove the second part. If C is closed under $\vee$, then there exist $u_n \in C$ such that $v(x) = \lim u_n(x)$ for all x. By the definition of ℓ, $u_n(x) \leq \ell(x)$ for all n and x. Hence, $v(x) \leq \ell(x)$. On the other hand, we have already proved that $\ell(x) \leq v(x)$. Thus $v = \ell$ which implies 5.3.B. □

For any $\mu, \nu \in \mathcal{N}_1$, the relation $u_\mu \leq u_\nu$ is equivalent to the relation $\mu \leq \nu$. Therefore

$$u_{\mu \vee \nu} = u_\mu \vee u_\nu, u_{\mu \wedge \nu} = u_\mu \wedge u_\nu, u_{\mu + \nu} = u_\mu \oplus u_\nu \tag{5.1}$$

and

$$\text{if } \nu_n \uparrow \nu \in \mathcal{N}_1, \quad \text{then } u_{\nu_n} \uparrow u_\nu. \tag{5.2}$$

5.4. Class $\mathcal{N}_0$ and σ-moderate solutions. If $\nu_1 \leq \dots \leq \nu_n \leq \dots$ is an increasing sequence of measures, then $\nu = \lim \nu_n$ is also a measure. We put $\nu \in \mathcal{N}_0$ if $\nu_n \in \mathcal{N}_1$. Note that

5.4.A. A measure $\nu \in \mathcal{N}_0$ belongs to $\mathcal{N}_1$ if and only if $\nu(E) < \infty$.

Since all measures of class $\mathcal{N}_1$ are finite, we need only to demonstrate that, if $\nu_n \in \mathcal{N}_1, \nu_n \uparrow \nu$ and $\nu(E) < \infty$, then $\nu \in \mathcal{N}_1$. By the Radon-Nikodym theorem, $\nu_n(dy) = \rho_n(y)\nu(dy)$. Since $\nu_n \uparrow \nu$, $\rho_n \uparrow 1$ ν-a.e. and

$$h_n(x) = K\nu_n(x) = \int k(x, y)\rho_n(y)\nu(dy) \uparrow h(x) = K\nu(x).$$

By Theorem 3.1, $h_n = u_n + G\psi(u_n)$ where $u_n = i(h_n)$ which implies that $h = u + G\psi(u)$ where $u = \lim u_n$.

5.4.B. If $\nu \in \mathcal{N}_0$ and if $\mu \leq \nu$, then $\mu \in \mathcal{N}_0$.

Indeed, suppose $\nu_n \in \mathcal{N}_1$ and $\nu_n \uparrow \nu$. Then $\mu_n = \nu_n \wedge \mu \in \mathcal{N}_1$ by Corollary 3.1, and $\mu_n \uparrow \mu$.

We say that $u \in \mathcal{U}$ is *σ-moderate* and we write $u \in \mathcal{U}_0$ if there exist moderate solutions u_n such that $u_n \uparrow u$.

LEMMA 5.1. *There exists a monotone mapping $\nu \to u_\nu$ from $\mathcal{N}_0$ onto $\mathcal{U}_0$ such that $u_\nu = i(K\nu)$ for $\nu \in \mathcal{N}_1$ and*

$$u_{\nu_n} \uparrow u_\nu \quad \textit{if } \nu_n \uparrow \nu. \tag{5.3}$$

PROOF. Suppose $\nu_n \uparrow \nu, \nu'_n \uparrow \nu$ and $u_{\nu_n} \uparrow u, u_{\nu'_n} \uparrow u'$ for $\nu_n, \nu'_n \in \mathcal{N}_1$ and put $\nu_{mn} = \nu_m \wedge \nu'_n$. Note that $\nu_{mn} \uparrow \nu_m$ as $n \to \infty$ and, by (5.2), $u_{\nu_{mn}} \uparrow u_{\nu_m}$. Therefore

$$u = \lim_{m \to \infty} u_{\nu_m} = \lim_{m \to \infty} \lim_{n \to \infty} u_{\nu_{mn}} = \sup_m \sup_n u_{\nu_{mn}}$$

Analogously, $u' = \sup_n \sup_m u_{\nu_{mn}}$. Thus $u = u'$. We proved that $u_\nu = \lim u_{\nu_n}$ does not depend on the choice of $\nu_n \uparrow \nu$. Clearly, u_ν defined by this formula satisfies (5.3). □

Note that the map $\nu \to u_\nu$ from $\mathcal{N}_0$ onto $\mathcal{U}_0$ is not 1-1: it can happen that $u_\nu = u_{\nu'}$ for $\nu \neq \nu'$.

Lemma 5.2. *Suppose E is a bounded smooth domain and O is a relatively open subset of ∂E. If $\nu \in \mathcal{N}_0$ and $\nu(O) = 0$, then $u_\nu = 0$ on O.*

Proof. There exist $\nu_n \in \mathcal{N}_1$ such that $u_n = u_{\nu_n} \uparrow u_\nu$. The moderate solution u_n is dominated by a harmonic function

$$h_n(x) = \int_{\partial E \setminus O} k(x,y)\nu_n(dy).$$

It follows from **6**.1.8.B and the dominated convergence theorem that $h_n(x) \to 0$ as $x \to \tilde{x} \in O$. This implies $u_n(x) \to 0$, and $u_\nu(x) \to 0$ by Theorem **5**.3.3. □

5.5. Solutions u_B. Let ν be a measure on E' and let B be a Borel subset of E'. We put $\nu \in \mathcal{N}(B)$ if ν is concentrated on B, i.e., if $\nu(E' \setminus B) = 0$. We put $\mathcal{N}_1(B) = \mathcal{N}_1 \cap \mathcal{N}(B), \mathcal{N}_0(B) = \mathcal{N}_0 \cap \mathcal{N}(B)$. An important role is played by the solutions

$$u_B = \mathrm{Sup}\{u_\nu : \nu \in \mathcal{N}_1(B)\}. \tag{5.4}$$

We claim that:

$$u_{B_1 \cup B_2} \leq u_{B_1} + u_{B_2}. \tag{5.5}$$

Indeed, if $\nu \in \mathcal{N}(B_1 \cup B_2)$, then $\nu = \nu_1 + \nu_2$ with $\nu_1 \in \mathcal{N}(B_1)$ and $\nu_2 \in \mathcal{N}(B_2)$. If $\nu \in \mathcal{N}_1$, then ν_1 and ν_2 belong to $\mathcal{N}_1$ by Corollary 3.1. Since $u_{\nu_1}, u_{\nu_2} \in \mathcal{U}^-$, $u_{\nu_1} + u_{\nu_2}$ in $\mathcal{U}^-$ by 5.1.D. By (5.1) and 5.1.A, $u_\nu = u_{\nu_1} \oplus u_{\nu_2} = \pi[u_{\nu_1} + u_{\nu_2}] \leq u_{\nu_1} + u_{\nu_2}$ which implies (5.5).

We have:

5.5.A. $u_B \geq u_\nu$ for all $\nu \in \mathcal{N}_0(B)$ and $u_B = u_\nu$ for some $\nu \in \mathcal{N}_0(B)$.

[Hence, all u_B are σ-moderate.]

The first part follows from the definition of Sup. The second part is true because, by 5.3.B, there exist $\nu_n \in \mathcal{N}_1(B)$ such that $u_{\nu_n} \uparrow u_B$. If $\nu_n \uparrow \nu$, then by (5.2), $u_{\nu_n} \uparrow u_\nu$. Hence, $u_B = u_\nu$.

Remark. We say that ν is a $(0,\infty)$-measure if $\nu(B) = 0$ or ∞ for all B. To every measure ν there corresponds a $(0,\infty)$-measure $\infty \cdot \nu = \lim_{k\to\infty} k\nu$. It belongs to $\mathcal{N}_0(B)$ if $\nu \in \mathcal{N}_0(B)$. Therefore measure ν in 5.5.A can be chosen to be a $(0,\infty)$-measure.

6. Notes

6.1. Most results presented in this chapter can be found in [**DK96b**], [**DK98c**], [**DK98b**] and [**DK98a**]. The concept of moderate solutions was introduced in [**DK96b**]. Theorem 3.1 was proved there in the case $\psi(u) = u^\alpha, 1 < \alpha \leq 2$. Sweeping operators Q_B (also for $\psi(u) = u^\alpha$) appeared in [**DK98c**], [**DK98b**] [1] as a

[1] In [**DK98b**], operators Q_B are introduced for closed subsets B of the Martin boundary $\hat{\partial} E$ of a Riemannian manifold E.

tool in the theory of the rough boundary trace of a solution (presented in Chapter 10). Lattice properties of $\mathcal{U}$ were used in [**Kuz98c**] and [**DK98a**] to define and investigate the fine trace (see Chapter 11).

6.2. A number of authors studied solutions of $\Delta u = \psi(u)$ tending to infinity as distance to the boundary tends to 0. [2] By 2.1.I, $V_D(\infty)$ is the minimal among these solutions. Very simple arguments show that the equation $\Delta u = u^\alpha$ can not have more than one solution in a star domain D. A domain D is called a star domain if there is a point $x_0 \in D$ such that, for every $y \in \partial D$, it contains the open line segment connecting y and x_0. Suppose that u and v are two large solutions. Without any loss of generality, we can assume that $x_0 = 0$. Put $v_\lambda(x) = \lambda^{\frac{2}{\alpha-1}} v(\lambda x)$ with $\lambda \leq 1$. Note that $\lambda D \subset D$, $\Delta v_\lambda = v_\lambda^\alpha$ in D and $v_\lambda < \infty$ on ∂D. By the comparison principle, $v_\lambda \leq u$. Hence, $v \leq u$. Analogously $u \leq v$ and therefore $u = v$.

The uniqueness of a large solution was proved under various conditions on ψ and D in [**LN74**], [**BM92**]. More general equation $Lu = \psi(u)$ was considered in [**BM95**]. For the equation $\Delta u = u^\alpha$ and very broad class of domains D the uniqueness was proved in [**MV97**].

A Wiener-type criterion for the existence of $u \geq 0$ such that $\Delta u = u^2$ in D and u blows up at a given point of ∂D was given by Dhersin and Le Gall in [**DL97**]. Their criterion implies a complete characterization of the class of domains in which large solutions exist. Obtained by probabilistic methods, this result was much stronger than the conditions known to analysts at that time. Very recently Labutin [**Lab01**] proved, by analytic methods, a similar result for all equations $\Delta u = u^\alpha$ with $\alpha > 1$. Parabolic versions of the results of [**DL97**] were obtained in [**DD99**].

[2] The name "large solutions" or "solutions with blow-up on the boundary" are used in the literature.

CHAPTER 9

Stochastic boundary values of solutions

In this chapter we characterize an arbitrary positive solution of $Lu = \psi(u)$ in an arbitrary domain E in terms of (L, ψ)-superdiffusion X. We define a σ-algebra $\mathcal{F}_\partial = \mathcal{F}_\partial(E)$ describing the class of events observable at the exit of X from E and we introduce a class $\mathfrak{Z}$ of $\mathcal{F}_\partial$-measurable functions which we call boundary linear functionals of X. We establish a 1-1 correspondence between $\mathcal{U}(E)$ and $\mathfrak{Z}$. If Z corresponds to u, then we say that Z is the stochastic boundary value of u and that u is the log-potential of Z. We investigate subclasses $\mathfrak{Z}_1$ and $\mathfrak{Z}_0$ of $\mathfrak{Z}$ corresponding to the class $\mathcal{U}_1$ of moderate solutions and to the class $\mathcal{U}_0$ of σ-moderate solutions. In particular, we get a relation between the stochastic boundary values of u and its minimal harmonic majorant $h = j(u)$. At the end of the chapter we establish a connection between superdiffusions and conditional diffusions. This connection was the original motivation of the theory of the fine trace presented in Chapter **11**.

1. Stochastic boundary values and potentials

1.1. Definition. We fix a domain E in $\mathbb{R}^d$ and we put $\mu \in \mathcal{M}_c$ if μ belongs to $\mathcal{M}(E)$ and is concentrated on a compact subset of E. Writing "a.s." means "P_μ-a.s. for all $\mu \in \mathcal{M}_c$".

Let (X_D, P_μ) be an (L, ψ)-superdiffusion. *We assume that ψ belongs to BR and satisfies conditions **8**.1.A–1.C.* Suppose that u is a Borel function with values in $\mathbb{R}_+$. We say that $Z \geq 0$ is a *stochastic boundary value of* u and we write $Z = \mathrm{SBV}(u)$ if, for every sequence D_n exhausting E,

$$\lim \langle u, X_{D_n} \rangle = Z \quad \text{a.s.} \tag{1.1}$$

Clearly, Z is defined by (1.1) uniquely up to equivalence. [We say that Z_1 and Z_2 are equivalent if $Z_1 = Z_2$ a.s.] [1] We call u the *log-potential* of Z and we write $u = \mathrm{LPT}(Z)$ if

$$u(x) = -\log P_x e^{-Z} \tag{1.2}$$

1.2. Existence of $\mathrm{SBV}(u)$**.**

THEOREM 1.1. *For every $u \in \mathcal{U}^-$, there exists a stochastic boundary value Z. The log-potential of Z is equal to $\pi(u)$. More generally,*

$$P_\mu e^{-Z} = e^{-\langle \pi(u), \mu \rangle} \tag{1.3}$$

for all $\mu \in \mathcal{M}_c$. [2]

[1]It is possible that Z_1 and Z_2 are equivalent but $P_\mu\{Z_1 \neq Z_2\} > 0$ for some $\mu \in \mathcal{M}(E)$.

[2]Convexity of ψ is not used in the proofs of Theorems 1.1 and 1.2.

PROOF. Put $Y_n = e^{-\langle u, X_{D_n}\rangle}$. By the Markov property **3**.1.3.D, for every $A \in \mathcal{F}_{\subset D_n}$,

$$\int_A Y_{n+1}\, dP_\mu = \int_A P_{X_{D_n}} Y_{n+1}\, dP_\mu.$$

By **8**.(2.17),

$$P_{X_{D_n}} Y_{n+1} = e^{-\langle V_{D_{n+1}}(u), X_{D_n}\rangle}$$

and, since $V_{D_{n+1}}(u) \le u$, by the definition of $\mathcal{U}^-$, we have

$$\int_A Y_{n+1}\, dP_\mu \ge \int_A Y_n\, dP_\mu.$$

Hence $(Y_n, \mathcal{F}_{\subset D_n}, P_\mu)$ is a bounded submartingale. By 4.3.A in the Appendix A, this implies the existence, P_μ-a.s., of $\lim\langle u, X_{D_n}\rangle$.

Functions $V_{D_n}(u) \in \mathcal{U}(D_n)$ are uniformly bounded on every $D \Subset E$ and therefore $\langle V_{D_n}(u), \mu\rangle \to \langle \pi(u), \mu\rangle$ for $\mu \in \mathcal{M}_c(E)$. Hence, $P_\mu Y_n = e^{-\langle V_{D_n}(u),\mu\rangle} \to e^{-\langle \pi(u),\mu\rangle}$ and, since $Y_n \to e^{-Z}$, P_μ-a.s., (1.3) holds by the dominated convergence theorem. □

REMARK 1.1. The same arguments are applicable to the case of $u \in \mathcal{U}^+$. Recall that $\pi(u) = u$ for $u \in \mathcal{U}$.

REMARK 1.2. By Jensen's inequality, $P_\mu e^{-Z} \ge e^{-P_\mu Z}$ and therefore (1.3) implies that, for all $\mu \in \mathcal{M}_c$,

$$\langle \pi(u), \mu\rangle \le P_\mu Z.$$

1.3. Linear boundary functionals. Denote by $\mathcal{F}_{\subset E-}$ the minimal σ-algebra which contains $\mathcal{F}_{\subset D}$ for all $D \Subset E$ and by $\mathcal{F}_{\supset E-}$ the intersection of $\mathcal{F}_{\supset D}$ over all $D \Subset E$. Note that, if D_n is a sequence exhausting E, then $\mathcal{F}_{\subset E-}$ is generated by the union of $\mathcal{F}_{\subset D_n}$ and $\mathcal{F}_{\supset E-}$ is the intersection of $\mathcal{F}_{\supset D_n}$.

We define $\mathcal{F}_\partial$ as the completion of the σ-algebra $\mathcal{F}_{\subset E-} \cap \mathcal{F}_{\supset E-}$ with respect to the family of measures $P_\mu, \mu \in \mathcal{M}_c$.

We say that a positive function Z is a *linear boundary functional* [3] if

1.3.A. Z is $\mathcal{F}_\partial$-measurable.

1.3.B. For all $\mu \in \mathcal{M}_c$,

$$-\log P_\mu e^{-Z} = \int [-\log P_x e^{-Z}]\mu(dx).$$

1.3.C. $\quad P_x\{Z < \infty\} > 0 \quad$ for all $x \in E$.

We denote by $\mathfrak{Z}$ the set of all such functionals (two functionals that coincide a.s. are identified).

THEOREM 1.2. *The stochastic boundary value Z of any $u \in \mathcal{U}^-$ belongs to $\mathfrak{Z}$. Let $Z \in \mathfrak{Z}$. Then the log-potential u of Z belongs to $\mathcal{U}$ and Z is the stochastic boundary value of u.*

[3] The word "boundary" refers to condition 1.3.A and the word "linear" refers to 1.3.B. In the terminology introduced in section **3**.4.1, 1.3.B means that the CB-property holds for e^{-Z} and $\mu \in \mathcal{M}_c$.

PROOF. If $u \in \mathcal{U}^-$ and if $Z = \mathrm{SBV}(u)$, then condition 1.3.A follows from (1.1), 1.3.B follows from (1.3). Since $u(x) < \infty$ for all $x \in E$, (1.2) implies 1.3.C.

Now suppose that $Z \in \mathfrak{Z}$ and $u = \mathrm{LPT}(Z)$. By 1.3.C, $u < \infty$. Suppose D_n exhausts E. By 1.3.A and the Markov property,

$$P_\mu\{e^{-Z} \big| \mathcal{F}_{\subset D_n}\} = P_{X_{D_n}} e^{-Z}.$$

If $\mu \in \mathcal{M}_c$ then, P_μ-a.s., X_{D_n} belongs to $\mathcal{M}_c$ (it is concentrated on the union of compact sets $\bar{D}_n$ and the support of μ) and therefore, by 1.3.B and (1.2),

$$P_{X_{D_n}} e^{-Z} = e^{-\langle u, X_{D_n}\rangle}.$$

Therefore

$$e^{-Z} = P_\mu\{e^{-Z} \big| \mathcal{F}_{\subset E-}\} = \lim e^{-\langle u, X_{D_n}\rangle}$$

which implies (1.1).

It follows from (1.1) and (1.2) that $u = \mathrm{LPT}(Z) = \lim u_n$ where

$$u_n(x) = -\log P_x e^{-\langle u, X_{D_n}\rangle}.$$

By **8**.2.1.J, $Lu_n = \psi(u_n)$ in D_n. Theorem **5**.3.2 implies that $u \in \mathcal{U}$. □

1.4. Properties of $\mathfrak{Z}$, LPT and SBV.

THEOREM 1.3. *If $Z_1, Z_2 \in \mathfrak{Z}$, then $Z_1 + Z_2 \in \mathfrak{Z}$ and $\mathrm{LPT}(Z_1+Z_2) \le \mathrm{LPT}(Z_1) + \mathrm{LPT}(Z_2)$. If $Z \in \mathfrak{Z}$, then $cZ \in \mathfrak{Z}$ for all $c \ge 0$.*

If $Z_n \in \mathfrak{Z}$, $Z_n \to Z$ a.s., then $Z \in \mathfrak{Z}$.

PROOF. 1°. If $Z_1, Z_2 \in \mathfrak{Z}$, then $Z = Z_1 + Z_2$ satisfies the condition 1.3.A. To check the condition 1.3.B, we note that, by Theorem 1.2,

$$Z_1 = \lim\langle u_1, X_{D_n}\rangle, \quad Z_2 = \lim\langle u_2, X_{D_n}\rangle \quad \text{a.s.}$$

where $u_1 = \mathrm{LPT}(Z_1), u_2 = \mathrm{LPT}(Z_2)$, and therefore

$$Z = \lim\langle u, X_{D_n}\rangle \quad \text{a.s.}$$

with $u = u_1 + u_2$. For every $\mu \in \mathcal{M}_c$,

$$-\log P_\mu e^{-\langle u, X_{D_n}\rangle} = -\int \log P_x e^{-\langle u, X_{D_n}\rangle}\mu(dx) = \int V_{D_n}(u)(x)\mu(dx). \tag{1.4}$$

Since $P_\mu e^{-\langle u, X_{D_n}\rangle} \to P_\mu e^{-Z}$ and $P_x e^{-\langle u, X_{D_n}\rangle} \to P_x e^{-Z}$, we get 1.3.B by a passage to the limit in (1.4). To justify an application of the dominated convergence theorem to the integral in the right side of (1.4), we use a bound

$$V_{D_n}(u) \le V_{D_n}(u_1) + V_{D_n}(u_2) = u_1 + u_2 \tag{1.5}$$

which follows from Theorem **8**.2.1.

Since $\mathrm{LPT}(Z_1 + Z_2) = \lim V_{D_n}(u_1 + u_2)$, subadditivity of LPT follows from (1.5).

The condition 1.3.C is equivalent to the condition $\mathrm{LPT}(Z) < \infty$. The subadditivity of LPT implies that (1.3.C) holds for Z if it holds for Z_1 and Z_2.

2°. If $Z \in \mathfrak{Z}$, then cZ satisfies 1.3.A and 1.3.C. To check 1.3.B, we use the same arguments as in 1°, but instead of (1.5) we apply a bound for $V_D(cu)$ established in Corollary 2.2 in Chapter **8**.

3°. If $Z_n \to Z$ a.s., then $u_n = \mathrm{LPT}(Z_n) \to u = \mathrm{LPT}(Z)$. Since $\psi \in BR$, solutions u_n are uniformly bounded on every D_n and, if 1.3.B holds for Z_n, then it holds for Z. □

Both mappings (1.1) and (1.2) are monotonic and therefore $\mathfrak{Z}$ is a lattice isomorphic to $\mathcal{U}$. We have:

1.4.A. For every $u_1, u_2 \in \mathcal{U}$,

$$\mathrm{SBV}(u_1 \oplus u_2) = \mathrm{SBV}(u_1) + \mathrm{SBV}(u_2). \tag{1.6}$$

.

Indeed, by **8**.5.1.D, $u_1 + u_2 \in \mathcal{U}^-$. If $Z_i = \mathrm{SBV}(u_i)$, then

$$Z_1 + Z_2 = \mathrm{SBV}(u_1 + u_2)$$

and, by Theorem 1.1, $\pi(u_1 + u_2) = \mathrm{LPT}(Z_1 + Z_2)$ which implies (1.6).

1.4.B. Let $u_n \in \mathcal{U}$ and $Z_n = \mathrm{SBV}(u_n)$. If $u_n \uparrow u$, then $Z_n \uparrow Z = \mathrm{SBV}(u)$.

Indeed, $u = \operatorname{Sup} u_n$ in the lattice $\mathcal{U}$. Therefore $Z = \operatorname{Sup} Z_n$ in $\mathfrak{Z}$, and $Z_n \uparrow Z$ because Z_n is monotone increasing.

1.5. Example. If $Z \in \mathfrak{Z}$, then, by Theorem 1.3,

$$\begin{aligned} Z^0 &= \lim_{\lambda \to 0} \lambda Z, \\ Z^\infty &= \lim_{\lambda \to \infty} \lambda Z \end{aligned} \tag{1.7}$$

also belong to $\mathfrak{Z}$. The corresponding solutions are

$$\begin{aligned} u^0(x) &= -\log P_x\{Z < \infty\}, \\ u^\infty(x) &= -\log P_x\{Z = 0\}. \end{aligned} \tag{1.8}$$

2. Classes $\mathfrak{Z}_1$ and $\mathfrak{Z}_0$

2.1. Stochastic boundary value of a positive harmonic function. We introduce a subclass of class $\mathfrak{Z}$ which is in 1-1 correspondence with the sets $\mathcal{H}_1$ and $\mathcal{N}_1$. Then we extend the correspondence to larger classes $\mathfrak{Z}_0, \mathcal{H}_0$ and $\mathcal{N}_0$.

In the rest of this chapter, we assume (in addition to the conditions stated at the beginning of the chapter) that ψ is in class CR (i.e., it satisfies the condition **4**.(4.6)).

If $h \in \mathcal{H}$, then $h \in \mathcal{U}^-$ and, by Theorem 1.1, there exists a stochastic boundary value $Z = \lim \langle h, X_{D_n} \rangle$. By Theorem 1.2, it belongs to $\mathfrak{Z}$. By **4**.(4.6) and **6**.2.4.A

$$P_\mu \langle h, X_{D_n} \rangle = \langle K_{D_n}(h), \mu \rangle = \langle h, \mu \rangle.$$

It follows from (1.1) and Fatou's lemma that

$$P_\mu Z \leq \langle h, \mu \rangle \tag{2.1}$$

for all $\mu \in \mathcal{M}_c$. If $\langle h, X_{D_n} \rangle$ are uniformly P_μ-integrable, then

$$P_\mu Z = \langle h, \mu \rangle. \tag{2.2}$$

Lemma **8**.3.1 implies that the equality (2.2) holds only if $\langle h, X_{D_n} \rangle$ are uniformly P_μ-integrable.

2.2. Connections between $\mathfrak{Z}_1$ and $\mathcal{H}_1$. We say that h is the *potential of* Z and we write $h = \mathrm{PT}(Z)$ if

$$h(x) = P_x Z. \tag{2.3}$$

Put $Z \in \mathfrak{Z}_1$ if $Z \in \mathfrak{Z}$ and $P_x Z < \infty$ for some $x \in E$.

2.2.A. For every $h \in \mathcal{H}$, $Z = \mathrm{SBV}(h)$ belongs to $\mathfrak{Z}_1$.

Indeed, by (2.1), $P_x Z \le h(x)$.

2.2.B. Let $Z \in \mathfrak{Z}_1, h = \mathrm{PT}(Z), u = \mathrm{LPT}(Z)$. Then $u \in \mathcal{U}_1$ and $h = j(u) \in \mathcal{H}_1$.

PROOF. By Theorem 1.2, $u \in \mathcal{U}$ and $Z = \mathrm{SBV}(u) = \lim \langle u, X_{D_n} \rangle$ a.s. By Fatou's lemma,

$$h(x) = P_x Z \le \liminf h_n(x) \tag{2.4}$$

where $h_n(x) = P_x \langle u, X_{D_n} \rangle$. By **4**.(4.6), $h_n = K_{D_n} u$. By Remark 1.2, $\langle u, \mu \rangle = \langle \pi(u), \mu \rangle \le P_\mu Z$ for every $\mu \in \mathcal{M}_c$. The Markov property implies $h_n(x) = P_x \langle u, X_{D_n} \rangle \le P_x P_{X_{D_n}} Z = P_x Z = h(x)$. Hence,

$$\limsup h_n(x) \le h(x). \tag{2.5}$$

It follows from (2.4) and (2.5) that $h_n \to h$. By **6**.1.5.C, $h \in \mathcal{H}$. According to **8**.(3.1), $h = j(u)$. Since $u \le h$, u belongs to $\mathcal{U}_1$ and therefore $h \in \mathcal{H}_1$. □

2.2.C. If $Z \in \mathfrak{Z}_1$ and if $h = \mathrm{PT}(Z)$, then the equality (2.2) holds for every $\mu \in \mathcal{M}_c$.

PROOF. Put $Z_\lambda = (1 - e^{-\lambda Z})/\lambda$ and note that $Z_\lambda \to Z$ as $\lambda \to 0$ and $0 \le Z_\lambda \le Z$. By (2.1), $P_\mu Z \le \langle h, \mu \rangle < \infty$ and, by the dominated convergence theorem,

$$P_\mu Z_\lambda \to P_\mu Z \quad \text{as } \lambda \to 0. \tag{2.6}$$

Consider a function

$$\Psi_\mu(\lambda) = -\frac{1}{\lambda} \log P_\mu e^{-\lambda Z} = -\frac{1}{\lambda} \log(1 - \lambda P_\mu Z_\lambda).$$

Since $\log(1+t) = t + o(t)$ as $t \to 0$, it follows from (2.6) that $\Psi_\mu(\lambda) \to P_\mu Z$ as $\lambda \to 0$.

By Theorem 1.3, $\lambda Z \in \mathfrak{Z}$ for all $\lambda \ge 0$, and, by 1.3.B,

$$\Psi_\mu(\lambda) = \int \Psi_x(\lambda) \mu(dx).$$

By Jensen's inequality, $P_x e^{-\lambda Z} \ge e^{-\lambda P_x Z}$ and therefore $\Psi_x(\lambda) \le P_x Z$. By the dominated convergence theorem,

$$\lim_{\lambda \to 0} \Psi_\mu(\lambda) = \int \lim_{\lambda \to 0} \Psi_x(\lambda) \mu(dx) = \langle h, \mu \rangle.$$

□

2.2.D. If $h \in \mathcal{H}_1$ and $Z = \mathrm{SBV}(h)$, then $h = \mathrm{PT}(Z)$.

Indeed, $Z \in \mathfrak{Z}_1$ by 2.2.A and therefore, by 2.2.C, $\langle h, X_{D_n} \rangle$ are uniformly integrable. Hence,

$$P_x Z = \lim P_x \langle h, X_{D_n} \rangle = h(x).$$

2.2.E. If $Z \in \mathfrak{Z}_1$ and $h = \mathrm{PT}(Z)$, then $Z = \mathrm{SBV}(h)$.

Indeed, by the Markov property and 2.2.C,

$$P_\mu\{Z|\mathcal{F}_{\subset D_n}\} = P_{X_{D_n}} Z = \langle h, X_{D_n}\rangle$$

and therefore

$$\lim \langle h, X_{D_n}\rangle = P_\mu\{Z|\mathcal{F}_{\subset E-}\} = Z.$$

2.2.F. If $Z \in \mathfrak{Z}_1$ and $h = \mathrm{PT}(Z)$, then $\langle h, X_{D_n}\rangle$ are uniformly P_μ-integrable for every $\mu \in \mathcal{M}_c$.

This follows from Lemma **8**.3.1 because $\langle h, X_{D_n}\rangle \to Z$ by 2.2.E and

$$P_\mu\langle h, X_{D_n}\rangle = P_\mu P_{X_{D_n}} Z = P_\mu Z$$

by 2.2.C and the Markov property.

2.2.G. Each of the following conditions is necessary and sufficient for a positive harmonic function h to belong to $\mathcal{H}_1$:

(a) $\langle h, X_{D_n}\rangle$ are uniformly P_c-integrable for some $c \in E$.

(b) $\langle h, X_{D_n}\rangle$ are uniformly P_μ-integrable for all $\mu \in \mathcal{M}_c$.

PROOF. If $h \in \mathcal{H}$, then, by Theorem 1.2, $Z = \mathrm{SBV}(h) \in \mathfrak{Z}$. Put $\tilde{h} = \mathrm{PT}(Z)$. By (2.1), $\tilde{h} \le h$ and therefore $Z \in \mathfrak{Z}_1$. By 2.2.B, $\tilde{h} \in \mathcal{H}_1$. The function $h_1 = h - \tilde{h}$ belongs to $\mathcal{H}$. If (a) holds, then, by (2.2), $h_1(c) = 0$. By **6**.1.5.D $h_1(x) = 0$ for all $x \in E$, and $h = \tilde{h} \in \mathcal{H}_1$.

On the other hand, if $h \in \mathcal{H}_1$, then, by 2.2.D, $h = \mathrm{PT}(Z)$ where $Z = \mathrm{SBV}(h)$. By 2.2.A, $Z \in \mathfrak{Z}_1$. The property (b) follows from 2.2.F. □

2.3. It follows from 2.2.B, 2.2.D and 2.2.E that PT is a 1-1 mapping from $\mathfrak{Z}_1$ onto $\mathcal{H}_1$ and SBV is the inverse mapping from $\mathcal{H}_1$ onto $\mathfrak{Z}_1$. Both mapping are monotonic and they preserve the addition and the multiplication by positive numbers. One of implications is that $\mathcal{H}_1$ is a convex cone. [4]

2.4. Let $\nu \to u_\nu$ be the mapping from $\mathcal{N}_0$ onto $\mathcal{U}_0$ described in Lemma **8**.5.1. Formula $Z_\nu = \mathrm{SBV}(u_\nu)$ defines a monotone mapping from $\mathcal{N}_0$ onto $\mathfrak{Z}_0$ such that $Z_{\nu_n} \uparrow Z_\nu$ if $\nu_n \uparrow \nu$. If $\nu \in \mathcal{N}_1$, then Z_ν is the element of $\mathfrak{Z}_1$ with potential $K\nu$. Note that, for all $\nu_1, \nu_2, \nu \in \mathcal{N}_1$ and all $\lambda \ge 0$,

$$Z_{\nu_1+\nu_2} = Z_{\nu_1} + Z_{\nu_2}, \quad Z_{\lambda\nu} = \lambda Z_\nu. \tag{2.7}$$

Formulae (2.7) remain valid for $\nu_1, \nu_2, \nu \in \mathcal{N}_0$.

3. A relation between superdiffusions and conditional diffusions

3.1. In Theorem 3.1 we establish a relation between an (L,ψ)-superdiffusion and the h-transform (ξ_t, Π^h_x) of an L-diffusion introduced in section 3.1 of Chapter **7**. First, we prove a lemma that will be also used in Chapter **11**.

[4]This was proved under assumption **8**.3.2.A in Theorem **8**.3.3. Instead of this assumption we use the existence of an (L,ψ)-superdiffusion.

LEMMA 3.1. *Let*

$$u + G_D(au) = h \in \mathcal{H}(D). \tag{3.1}$$

Then

$$u(x) = \Pi_x^h \exp[-\int_0^\tau a(\xi_s)\ ds] \tag{3.2}$$

where τ is the first exit time from D.

PROOF. Note that

$$H(t) = e^{-\int_0^t a(\xi)(t)\ dt}$$

satisfies the equation

$$\int_0^\tau a(\xi_t)H(t)\ dt = 1 - H(\tau). \tag{3.3}$$

By Fubini's theorem and Lemma **7**.3.1,

$$\Pi_x \int_0^\tau H(t)(ah)(\xi_t)\ dt = \Pi_x^h \int_0^\tau H(t)a(\xi_t)\ dt.$$

By Lemma **8**.2.1, (3.1) implies that the left side is equal to $G_D(au) = h - u$. By (3.3), the right side is equal to $h(x) - \Pi_x^h H(\tau)$. □

THEOREM 3.1. *For every $Z \in \mathfrak{Z}$ and every $\nu \in \mathcal{N}_0$,*

$$P_x Z_\nu e^{-Z} = e^{-u(x)} \int_{E'} \Pi_x^y e^{-\Phi(u)} \nu(dy) \tag{3.4}$$

where $u = \mathrm{LPT}(Z)$,

$$\Phi(u) = \int_0^\zeta \psi'[u(\xi_t)]\ dt \tag{3.5}$$

and $\Pi_x^y = \Pi_x^h$ with $h = k_y$.

PROOF. If formula (3.4) holds for ν_n, then it holds for $\nu = \sum \nu_n$. Therefore it is sufficient to prove the theorem for $\nu \in \mathcal{N}_1$. Let $h = K\nu$. It follows from the definition of the measures Π_x^h in section **7**.3.1 that $\Pi_x^h = \int_{E'} \Pi_x^y d\nu$. Therefore formula (3.4) is equivalent to

$$v(x) = \Pi_x^h e^{-\Phi(u)} \tag{3.6}$$

where

$$v(x) = e^{u(x)} P_x Z_\nu e^{-Z}. \tag{3.7}$$

Note that $\mathrm{PT}(Z_\nu) = h$. [This follows from 2.2.B because $h = j(u_\nu)$ where $u_\nu = \mathrm{LPT}(Z_\nu)$.] By 2.2.E, $Z_\nu = \mathrm{SBV}(h)$. By Theorem 1.2, $Z = \mathrm{SBV}(u)$. Hence

$$Z = \lim Z_n \quad \text{a.s.}, \qquad Z_\nu = \lim Y_n \quad \text{a.s.} \tag{3.8}$$

where

$$Z_n = \langle u, X_{D_n} \rangle, \quad Y_n = \langle h, X_{D_n} \rangle.$$

By **8**.2.1.D, $V_{D_n}(u) = u$ and therefore

$$-\log P_x e^{-Z_n} = u(x).$$

By **4**.(4.6),

$$P_x Y_n = K_{D_n} h(x) = h(x).$$

Consider functions

$$u_n^s(x) = -\log P_x e^{-Z_n - sY_n}, \qquad v_n(x) = \frac{d}{ds} u_n^s(x) \Big|_{s=0} .$$

We have $u_n^0 = u$ and

$$P_x Y_n e^{-Z_n} = -\frac{d}{ds}(e^{-u_n^s(x)}) \Big|_{s=0} = v_n(x) e^{-u(x)}. \tag{3.9}$$

By (3.8),

$$P_x Y_n e^{-Z_n} \to P_x Z_\nu e^{-Z}$$

because Y_n are uniformly P_x-integrable by 2.2.G. By (3.9) and (3.7), this implies

$$v_n \to v. \tag{3.10}$$

We have

$$u_n^s + G_{D_n}\psi(u_n^s) = K_{D_n}(u + sh) = K_{D_n}u + sh.$$

By taking the derivatives with respect to s at $s = 0$, we get

$$v_n + G_{D_n}[\psi'(u)v_n] = h. \tag{3.11}$$

By Lemma 3.1, (3.11) implies

$$v_n(x) = \Pi_x^h \exp\left[-\int_0^{\tau_n} \psi'[u(\xi_t)]dt\right] \tag{3.12}$$

where τ_n is the first exit time from D_n. Formula (3.6) follows from (3.10) and (3.12). □

4. Notes

An idea to represent solutions of semilinear differential equations in terms of superdiffusions was inspired by a well known connection between solutions of the Laplace equation and the Brownian motion: if h is bounded and $\Delta h = 0$ in D, then

$$h(x) = \Pi_x \lim_{t \uparrow \tau} h(\xi_t)$$

where τ is the first exit time from D. [This follows immediately from the fact that $h(\xi_t)$ is a bounded martingale.] A representation of solutions of the equation $\dot{u} + Lu = \psi(u)$ through a (L, ψ)-superdiffusion appeared, first, in [**Dyn93**].[5] Most results presented in Chapter **9** can be found (in the case $\psi(u) = u^\alpha$) in [**Dyn98b**].

The connection between superdiffusions and conditional diffusions stated in Theorem 3.1 was established in [**Dyn97a**].

[5] Linear boundary functional are defined in [**Dyn93**] (and [**Dyn98b**]) as $\mathcal{F}_{\supset E-}$-measurable functions but, in the proofs, measurability with respect to a σ-algebra which we call $\mathcal{F}_\partial$ is used.

CHAPTER 10

Rough trace

Now we suppose that the Martin boundary of E coincides with ∂E. [This is true for all bounded smooth domains.] We define a boundary trace of an arbitrary solution by using two tools introduced in Chapter **8**: (a) traces of moderate solutions; (b) sweeping of solutions. With the regard of the function ψ, we assume that it belongs to class BR and satisfies the conditions **8**.1.A–1.C and **8**.3.2.A.

1. Definition and preliminary discussion

1.1. Definition of rough trace. Let E be a bounded smooth domain in $\mathbb{R}^d$. We say that a compact set $B \subset \partial E$ is *moderate for* u if the solution $Q_B(u)$ is moderate. Let ν_B stand for the trace of $Q_B(u)$. By **8**.4.1.D, the union of two moderate sets is moderate. Suppose that B is moderate and let $\tilde{B} \subset B$. By **8**.4.1.C, $\tilde{B}$ is moderate, and by Theorem **8**.4.2, $\nu_{\tilde{B}}$ is the restriction of ν_B to $\tilde{B}$.

A relatively open subset A of ∂E is called *moderate* if all compact subsets of A are moderate. The union O of all moderate open sets is moderate. Clearly, there exists a unique measure ν on O such that its restriction to an arbitrary compact subset B coincides with ν_B. The measure ν has the property: for every compact $B \subset O$ the restriction of ν to B belongs to the class $\mathcal{N}_1$. We denote by $\mathcal{N}_1(O-)$ the class of measures with this property. We call O the *moderate boundary portion* and we call $\Gamma = \partial E \setminus O$ the *special set* for the solution u. We call the pair (Γ, ν) the *rough boundary trace* of u and we denote it by $\operatorname{tr}(u)$.

1.2. Extremal characterization of sweeping. Let B be a compact subset of ∂E and let

$$D(B, \varepsilon) = \{x \in E : d(x, B) > \varepsilon\}$$

(cf. section **8**.4.1). To any sequence $\varepsilon_n \downarrow 0$, there corresponds an $[E, B]$-sequence

$$D_n = \{x \in E : d(x, B) > \varepsilon_n\}$$

and we have

$$Q_B(u) = \lim V_{D_n}(u) \tag{1.1}$$

(cf. **8**.(4.1))

Theorem 1.1. *For every $u \in \mathcal{U}$, function $v = Q_B(u)$ is the maximal element of $\mathcal{U}$ subject to conditions:*

$$v \le u; \quad v = 0 \quad \text{on } \partial E \setminus B. \tag{1.2}$$

Proof. Recall that we put $u = 0$ on E^c and therefore $V_{D_n}(u) = V_{D_n}(f_n)$ where $f_n = u$ on $O_n = \partial D_n \cap E$ and $f_n = 0$ on the rest of ∂D_n. Put $A_n = \{x \in$

$\partial E : d(x,B) > \varepsilon_n\}$. By **8**.2.1.J, $v_n = V_{D_n}(f_n)$ is a solution of the problem

$$\begin{aligned} Lv_n &= \psi(v_n) \quad \text{in } D_n, \\ v_n &= u \quad \text{on } O_n, \\ v_n &= 0 \quad \text{on } A_n. \end{aligned}$$

It follows from Theorems **5**.3.2 and **5**.3.3 that $v = \lim v_n$ belongs to $\mathcal{U}$ and it satisfies (1.2).

Suppose (1.2) holds for $\tilde{u}$. Then $\tilde{u} \le u = v_n$ on O_n and $\tilde{u} = 0 \le v_n$ on $\partial D_n \setminus O_n \subset \partial E \setminus O_n$. By the comparison principle **8**.2.1.H, $\tilde{u} \le v_n$ in D_n and therefore $\tilde{u} \le v$. □

1.3. The maximal solution w_B and the range $\mathcal{R}$.

Theorem 1.2. *Suppose that B is a compact subset of ∂E and that all points of $\partial E \setminus B$ are regular. Then there exists a maximal solution w_B of the problem*

$$\begin{aligned} Lu &= \psi(u) \quad \text{in } E; \\ u &= 0 \quad \text{on } \partial E \setminus B. \end{aligned} \tag{1.3}$$

It can be obtained by the formula

$$w_B = \lim V_{D_n}(f_n) \tag{1.4}$$

where D_n is an $[E,B]$-sequence and

$$f_n = \begin{cases} \infty & \text{on } \partial D_n \cap E, \\ 0 & \text{on } \partial D_n \cap \partial E. \end{cases} \tag{1.5}$$

Proof. By Theorem **5**.5.3, there exists a maximal solution w_Q^Γ of problem **5**.(5.6) for $Q = \mathbb{R} \times E$ and $\Gamma = \mathbb{R} \times B$. If $u(r,x)$ is a solution of **5**.(5.6), then $u_t(r,x) = u(t+r,x)$ is also a solution of this problem. Therefore the maximal solution w_Q^Γ does not depend on r. It is easy to see that $w_B = w_Q^\Gamma$ is the maximal solution of (1.3).

The second part of the theorem follows from Remark **5**.5.2 (which is true even if Q_n are unbounded). □

The *range* $\mathcal{R} = \mathcal{R}_E$ of a (L,ψ)-superdiffusion in E is defined as the envelope of random closed sets $(\mathbb{S}_D, P_\mu), D \subset E$ [$\mathbb{S}_D$ is the support of X_D]. If the graph $\mathcal{G} = \mathcal{G}_Q$ in $Q = \mathbb{R} \times E$ is compact, then $\mathcal{R}$ is its projection from $\bar{Q} = \mathbb{R} \times \bar{E}$ to $\bar{E}$.

Theorem 1.3. *Suppose that E is bounded and regular. If X is an (L,ψ)-superdiffusion, then the maximal solution of the problem* (1.3) *is given by the formula*

$$w_B(x) = -\log P_x\{\mathcal{R} \cap B = \emptyset\}. \tag{1.6}$$

The range $\mathcal{R}$ is compact P_x-a.s. and the maximal solution w of equation $Lu = \psi(u)$ in E is given by the formula

$$w(x) = -\log P_x\{\mathcal{R} \cap \partial E = \emptyset\}. \tag{1.7}$$

Proof. First, we apply Theorem **5**.5.3 to $Q = \mathbb{R} \times E$ and $\Gamma = \emptyset$ (since E is regular, all points of ∂Q are regular). We get that $w(r,x) = -\log P_{r,x}\{\mathcal{G} \text{ is compact}\}$ is the maximal solution of the equation $\dot{u} + Lu = \psi(u)$ in Q such that $u = 0$ on ∂Q. Since w does not depend on time, it satisfies the equation $Lu = \psi(u)$ in E with the boundary condition $u = 0$ on ∂E. By the comparison principle **8**.2.1.H, $w = 0$. Hence, $P_{r,x}\{\mathcal{G}_Q \text{ is compact}\} = 1$.

Now we take $\Gamma = \mathbb{R} \times B$ and we identify $x \in E$ with $(0, x) \in Q$. By **5**.(5.7), $w_Q^\Gamma(x) = -\log P_x\{\mathcal{G} \cap \Gamma = \emptyset\}$ and, since, P_x-a.s., $\mathcal{R}$ is the projection of $\mathcal{G}$ on $\bar{E}$, $w_B = w_Q^\Gamma$ is given by (1.6). We get (1.7) by taking $B = \partial E$. □

Remark 1.1. The CB property holds for $\{\mathcal{R} \cap B = \emptyset\}$ and P_μ assuming that $B \subset \partial E$ is compact and $\mu \in \mathcal{M}_c(E)$.

Indeed, if $\bar{\mu}$ is the image of $\mu \in \mathcal{M}_c(E)$ under the mapping $x \to (0, x)$ from E to $Q = \mathbb{R} \times E$, then $\bar{\mu}$ belongs to $\mathcal{M}_c(Q)$ and, by Corollary **4**.5.2,

$$\log P_{\bar{\mu}}\{\mathcal{G} \cap \Gamma = \emptyset\} = \int_Q \bar{\mu}(dz) \, \log P_z\{\mathcal{G} \cap \Gamma = \emptyset\}.$$

The left side is equal to $\log P_\mu\{\mathcal{R} \cap B = \emptyset\}$ and the right side is equal to

$$\int_E \mu(dx) \, \log P_x\{\mathcal{R} \cap B = \emptyset\}.$$

Remark 1.2. Note that w_B is the log-potential of

$$Z_B = \begin{cases} 0 & \text{if } \ \mathcal{R} \cap B = \emptyset, \\ \infty & \text{if } \ \mathcal{R} \cap B \neq \emptyset. \end{cases} \tag{1.8}$$

We claim that $Z_B \in \mathfrak{Z}$. Indeed, by (1.6),

$$P_x\{Z_B < \infty\} = P_x\{\mathcal{R} \cap B = \emptyset\} = \exp[-w_B(x)] > 0$$

because $w_B(x) < \infty$. Property **9**.1.3.B follows from Remark 1.1. Property **9**.1.3.A can be deduced from the relation $\{\mathcal{R} \cap B = \emptyset\} = \{\mathcal{G}_Q \cap \Gamma = \emptyset\}$ for $\Gamma = \mathbb{R} \times B$ and $Q = \mathbb{R} \times E$ and Theorem **4**.5.3. By Theorem **9**.1.2, $Z_B = \mathrm{SBV}(w_B)$.

We have:

1.3.A. $w_{B_1} \leq w_{B_2}$ if $B_1 \subset B_2$.

1.3.B. $Q_B(u) \leq w_B$ for all $u \in \mathcal{U}$.

1.3.C. $Q_B(w_\Gamma) = 0$ if $B \cap \Gamma = \emptyset$.

1.3.D. $Q_B(w_B) = w_B$.

1.3.E. $w_{B_1 \cup B_2} \leq w_{B_1} + w_{B_2}$.

1.3.F. If $B_n \downarrow B$, then $w_{B_n} \downarrow w_B$.

1.3.A and 1.3.B follow from the maximal property of w_B (recall that $Q_B(u) = 0$ on $\partial E \setminus B$). To prove 1.3.C, we note that, by **8**.4.1.B, $Q_B(w_\Gamma) \leq w_\Gamma$. Since $w_\Gamma = 0$ on $\partial E \setminus \Gamma$ and $Q_B(w_\Gamma) = 0$ on $\partial E \setminus B$, $Q_B(w_\Gamma)$ vanishes on ∂E and, by the comparison principle, it vanishes on E.

Note that $v = w_B$ satisfies conditions $v \leq w_B, v = 0$ on $\partial E \setminus B$ and $Q_B(w_B)$ is a maximal solution with these properties. Hence, $w_B \leq Q_B(w_B)$. Therefore 1.3.D follows from 1.3.B.

To prove 1.3.E, we put $B = B_1 \cup B_2$ and we note that, by 1.3.D, **8**.4.1.D and 1.3.B,

$$w_B = Q_B(w_B) \leq Q_{B_1}(w_B) + Q_{B_2}(w_B) \leq w_{B_1} + w_{B_2}.$$

Let us prove 1.3.F. Function $w_n = w_{B_n}$ is a maximal element of $\mathcal{U}$ vanishing on $O_n = \partial E \setminus B_n$. By 1.3.A $w_n \downarrow v \geq w_B$. By Theorems **5**.3.2–3.3, v is a solution equal to 0 on $O = \partial E \setminus B$. Hence, $v \leq w_B$.

1.4. Removable and polar boundary sets. We say that a compact set $B \subset \partial E$ is *removable* if 0 is the only solution of the problem (1.3). [In the literature, such sets are called removable boundary singularities for solutions of the equation $Lu = \psi(u)$.] Clearly, B is removable if and only if $w_B = 0$. A set A is called *polar* if all its compact subsets are removable. If X is a (L, ψ)-superdiffusion, then, by Theorem 1.3, a compact set B is removable if and only if $P_x\{\mathcal{R} \cap B \neq \emptyset\} = 0$ for every $x \in E$. It follows from 1.3.A and 1.3.E that:

1.4.A. All compact subsets of a removable set are removable and all subsets of a polar set are polar.

1.4.B. The class of all removable sets is closed under the finite unions.

We say that a Borel boundary subset B is *weakly polar* or w-*polar* if $\nu(B) = 0$ for all $\nu \in \mathcal{N}_1$. This name is justified by the following proposition:

1.4.C. All polar Borel sets are w-polar.

PROOF. The class $\mathcal{N}_1$ contains, with every ν, its restriction to any B (see Corollary **8**.3.1). Therefore it is sufficient to show that, if $\nu \in \mathcal{N}_1$ is concentrated on a removable compact set B, then $\nu = 0$. The property **6**.1.8.B of the Poisson kernel $k(x, y)$ implies that $h = K\nu = 0$ on $\partial E \setminus B$. The solution $i(h)$ satisfies the same condition because $i(h) \leq h$. Therefore $i(h) \leq w_B$. If $w_B = 0$, then $i(h) = 0$. Thus $h = 0$ and $\nu = 0$. □

We also have:

1.4.D. If w_B is moderate, then B is removable. If $B \subset \Gamma$ and if $Q_B(w_\Gamma)$ is moderate, then B is removable.

PROOF. The second part follows from the first one because, by 1.3.A, $w_B \leq w_\Gamma$ and, by 1.3.D and the monotonicity of Q_B, $w_B = Q_B(w_B) \leq Q_B(w_\Gamma)$.

Suppose that w_B is moderate. By Theorem **8**.3.1, $w_B = i(h)$ for some $h \in \mathcal{H}_1$. By Theorem **8**.3.3, $2h \in \mathcal{H}_1$. By **8**.(3.3) and Theorem **8**.2.1,

$$i(2h) = \lim V_n(2h) \leq 2 \lim V_n(h) = 2i(h) = 2w_B.$$

Hence, $i(2h) = 0$ on $\partial E \setminus B$ which implies that $i(2h) \leq w_B$. By the monotonicity of j, $2h \leq j(w_B) = h$. Therefore $h = 0$ and $w_B = i(h) = 0$. □

1.4.E. Suppose Γ is removable and let $B_n = \{x \in \partial E : d(x, \Gamma) \geq \varepsilon_n\}$. If $\varepsilon_n \downarrow 0$, then $Q_{B_n}(u) \uparrow u$ for every $u \in \mathcal{U}$.

PROOF. Put $\Gamma_n = \{y \in \partial E : d(y, \Gamma) \leq 2\varepsilon_n\}$. Note that $\Gamma_n \cup B_n = \partial E$ and $\Gamma_n \downarrow \Gamma$. By **8**.4.1.E and **8**.4.1.D, $u = Q_{\partial E}(u) \leq Q_{\Gamma_n}(u) + Q_{B_n}(u)$. By 1.3.B and 1.3.F, $Q_{\Gamma_n}(u) \leq w_{\Gamma_n} \downarrow w_\Gamma = 0$ and therefore 1.4.E follows from **8**.4.1.B. □

2. Characterization of traces

2.1. Properties of the trace. We say that x is an *explosion point of a measure* ν and we write $x \in Ex(\nu)$ if $\nu(U) = \infty$ for every neighborhood U of x. If $B \cap Ex(\nu) = \emptyset$ and if B is compact, then $\nu(B) < \infty$. Note that $O \cap Ex(\nu) = \emptyset$ for every measure $\nu \in \mathcal{N}_1(O-)$.

We say that (Γ, ν) is a *normal pair* if:

(a) Γ is a compact subset of ∂E;

(b) $\nu \in \mathcal{N}_1(O-)$ where $O = \partial E \setminus \Gamma$;

(c) the conditions:

(2.1) $\Lambda \subset \Gamma$ is polar and contains no explosion points of ν, $\Gamma \setminus \Lambda$ is compact−

imply that $\Lambda = \emptyset$.

We will prove that these conditions hold for the trace of an arbitrary solution u. First, we prove a few auxiliary propositions.

2.1.A. If $u \in \mathcal{U}$ vanishes on a compact set $B \in \partial E$, then $Q_B(u) = 0$.

Indeed, by Theorem 1.1, $v = Q_B(u) = 0$ on $\partial E \setminus B$ and $v \le u$. Hence, $v = 0$ on ∂E, and $v = 0$ by the comparison principle **8**.2.1.H.

2.1.B. Suppose that $\mathrm{tr}(u) = (\Gamma, \nu)$. If $u = 0$ on an open subset O_1, then $O_1 \cap \Gamma = \emptyset$ and $\nu(O_1) = 0$.

Indeed, for every compact subset B of O_1, $Q_B(u) = 0$ by 2.1.A and therefore $\nu(B) = 0$.

2.1.C. Let $\mathrm{tr}(u) = (\Gamma, \nu)$. If Γ is removable and ν is finite, then u is moderate.

To prove this, we apply 1.4.E. Let ν_n be the restriction of ν to B_n. We have $u_n = Q_{B_n}(u) \le K\nu_n$ because u_n is a moderate solution with the trace ν_n. By 1.4.E, $u_n \uparrow u$. Since $K\nu_n \uparrow K\nu$, we get that $u \le h$.

Theorem 2.1. *The trace of an arbitrary solution u is a normal pair.*

Proof. 1°. Properties (a) and (b) follow immediately from the definition of the trace. Let us prove (c). Suppose that Λ satisfies the conditions (2.1) and let $\Gamma_0 = \Gamma \setminus \Lambda$. Theorem will be proved if we show that $v = Q_{B_1}(u)$ is moderate for every compact subset B_1 of $O_0 = \partial E \setminus \Gamma_0$. Indeed, this implies $O_0 \subset \partial E \setminus \Gamma$. Therefore $\Gamma_0 \supset \Gamma$ and $\Lambda = \emptyset$.

2°. Let (Γ_1, ν_1) be the trace of v. By 2.1.C, it is sufficient to prove that Γ_1 is polar and ν_1 is finite.

Put $O = \partial E \setminus \Gamma$ and $O_1 = \partial E \setminus \Gamma_1$. By **8**.4.1.B, $v \le u$ and **8**.4.1.A implies

$$O \subset O_1, \quad \Gamma_1 \subset \Gamma, \quad \nu_1 \le \nu \quad \text{on } O. \tag{2.2}$$

By (1.2), $v = 0$ on $\partial E \setminus B_1$ and, by 2.1.B, $\partial E \setminus B_1 \subset \partial E \setminus \Gamma_1$. Hence, $\Gamma_1 \subset B_1$. By (2.2), $\Gamma_1 \subset B_1 \cap \Gamma$.

Note that $B_1 \subset O \cup \Lambda$. Therefore $\Gamma_1 \subset \Lambda$ is polar.

3°. Measure ν_1 is concentrated on B_1. Indeed, if $B \cap B_1 = \emptyset$, then, by 1.3.A and 1.3.B,

$$Q_B(v) = Q_B[Q_{B_1}(u)] \le Q_B(w_{B_1}) = 0.$$

We have

$$\nu_1(O_1) = \nu_1(B_1 \cap \mathcal{O}_1) \le \nu_1(B_1 \cap \Gamma) + \nu_1(B_1 \cap O).$$

Since $B_1 \cap \Gamma$ is polar and $\nu_1 \in \mathcal{N}_1(O_1-)$, the first term is 0 by 1.4.C. Since O and Λ contain no explosion points of ν, $\nu(B_1) < \infty$. Therefore $\nu_1(O_1) < \infty$. □

2.2. Maximal solution with a given trace. Note that $\mathcal{N}_1(O-) \subset \mathcal{N}_0$ and therefore, by section **8**.5.4, to every $\nu \in \mathcal{N}_1(O-)$ there corresponds a σ-moderate solution u_ν. We construct a maximal solution with a given trace by using the operation $\oplus$. We start with the following observations:

2.2.A. Let Γ_1, Γ_2 and Γ be special sets for u_1, u_2 and $u = u_1 \oplus u_2$ and let O_1, O_2, O be the corresponding moderate boundary portions. Then $\Gamma = \Gamma_1 \cup \Gamma_2$ and $O = O_1 \cap O_2$.

Indeed, $u_1 \vee u_2 \leq u \leq u_1 + u_2$. Therefore, by **8**.4.1.A and **8**.4.1.F,

$$Q_B(u_1) \vee Q_B(u_2) \leq Q_B(u) \leq Q_B(u_1) + Q_B(u_2). \tag{2.3}$$

Hence, B is moderate for u if and only if it is moderate for u_1 and for u_2.

2.2.B. If $u = u_1 \oplus u_2$ and $Q_B(u_2) = 0$, then $Q_B(u) = Q_B(u_1)$.

This follows from (2.3).

2.2.C. For every $\nu \in \mathcal{N}_0$ and every compact $B \subset O$, $Q_B(u_\nu) = u_{\nu_B}$ where ν_B is the restriction of ν to B.

Indeed, B is contained in an open subset O_1 of ∂E such that $\bar{O}_1 \subset O$. Let ν_1 and ν_2 be the restrictions of ν to O_1 and to $O \setminus O_1$. By Lemma **8**.5.2, $u_{\nu_2} = 0$ on O_1. By 2.1.A, $Q_B(u_{\nu_2}) = 0$. By 2.2.B, $Q_B(u_\nu) = Q_B(u_{\nu_1})$. Note that the restriction of ν_1 to B coincides with ν_B. Since u_{ν_1} is moderate, $Q_B(u_{\nu_1}) = u_{\nu_B}$ by Theorem **8**.4.2,

THEOREM 2.2. *If (Γ, ν) is a normal pair, then $u = w_\Gamma \oplus u_\nu$ is a solution with the trace (Γ, ν). Moreover, every solution v with the trace (Γ, ν) is dominated by u.*

PROOF. 1°. If $B \subset O = \partial E \setminus \Gamma$, then, by 1.3.C, $Q_B(w_\Gamma) = 0$ and, by 2.2.B and 2.2.C, $Q_B(u) = Q_B(u_\nu) = u_{\nu_B}$.

2°. Denote the trace of u by (Γ_0, ν_0). It follows from 1° that $O \subset O_0 = \partial E \setminus \Gamma_0$ and $\nu = \nu_0$ on O_0. Since ν is concentrated on O, we have $\nu \leq \nu_0$ and therefore $Ex(\nu) \subset Ex(\nu_0) \subset \Gamma_0 \subset \Gamma$. Every compact $B \subset \Lambda = \Gamma \setminus \Gamma_0$ is moderate for u (because $B \subset O_0$) and it is removable by 1.4.D. Thus, Λ is polar. Since $\Lambda \cap Ex(\nu) = \emptyset$ and $\Gamma \setminus \Lambda = \Gamma_0$ is compact, $\Lambda = \emptyset$ by the definition of a normal pair. Hence $(\Gamma, \nu) = (\Gamma_0, \nu_0)$.

3°. Suppose that $\operatorname{tr} v = (\Gamma, \nu)$. Consider compact sets

$$B_n = \{x \in \partial E : d(x, \Gamma) \geq 1/n\}, \quad \Gamma_n = \{x \in \partial E : d(x, \Gamma) \leq 1/n\}.$$

Since $B_n \cup \Gamma_n = \partial E$

$$v = Q_{\partial E}(v) \leq Q_{B_n}(v) + Q_{\Gamma_n}(v) \tag{2.4}$$

by **8**.4.1.E and **8**.4.1.D. Note that $Q_{B_n}(v) = u_{\nu_n} \leq u_\nu$ where ν_n is the restriction of ν to B_n. By 1.3.B $Q_{\Gamma_n}(v) \leq w_{\Gamma_n}$ and, by 1.3.F $w_{\Gamma_n} \downarrow w_\Gamma$. Therefore (2.4) implies that $v \leq u_\nu + w_\Gamma$ and $v \leq u$ because u is the maximal element of $\mathcal{U}$ dominated by $w_\Gamma + u_\nu$. $\square$

REMARK 2.1. By **9**.1.4.A, $\mathrm{SBV}(w_\Gamma \oplus u_\nu) = Z_\Gamma + Z_\nu$ where Z_Γ is defined by (1.8) and $Z_\nu = \mathrm{SBV}(u_\nu)$. Hence, $w_\Gamma \oplus u_\nu = \mathrm{LPT}(Z_\Gamma + Z_\nu)$ which means

$$(w_\Gamma \oplus u_\nu)(x) = -\log \int_{\mathcal{R} \cap \Gamma = \emptyset} e^{-Z_\nu} \, dP_x. \tag{2.5}$$

2.3. Relation to a boundary value problem.

THEOREM 2.3. *Suppose that*

$$Lu = \psi(u) \quad \text{in } E, \qquad u = f \quad \text{on } O \tag{2.6}$$

where O is an open subset of ∂E and f is a continuous function on O. Then $\mathrm{tr}(u) = (\Gamma, \nu)$ *satisfies the conditions:* $\Gamma \cap O = \emptyset$ *and*

$$\nu(B) = \int_B f(y)\gamma(dy) \quad \text{for all } B \subset O \tag{2.7}$$

where γ is the surface area on ∂E. The maximal solution of problem (2.6) *is given by the formula $w_\Gamma \oplus u_\nu$ where $\Gamma = \partial E \setminus O$ and ν is defined by* (2.7).

PROOF. If B is a compact subset of O, then u is bounded in a neighborhood of B. By Theorem 1.1, $v = Q_B(u)$ vanishes on $\partial E \setminus B$ and $v \le u$. Therefore v is bounded and B is moderate. Since O is a moderate open set, $\Gamma \cap O = \emptyset$. Let D_n be an $[E, B]$-sequence. By the extended mean value property (Theorem **8**.4.1)

$$v + G_{D_n}\psi(v) = K_{D_n}v. \tag{2.8}$$

If τ_n is the first exit time from D_n, then $\{\tau_n < \zeta\} \downarrow \{\xi_{\zeta-} \in B\}$ and therefore

$$K_{D_n}v(x) = \Pi_x v(\xi_{\tau_n})1_{\tau_n<\zeta} \downarrow \Pi_x(f1_B)(\xi_{\zeta-}).$$

By **6**.(2.8), the right side is equal to $\int_B k(x,y)f(y)\gamma(dy)$. It follows from (2.8) that $v + G\psi(v) = h$ which implies (2.7). Now suppose that $\Gamma = \partial E \setminus O$. If u' is an arbitrary solution of (2.6) and if (Γ', ν') is its trace, then $\Gamma' \subset \Gamma$ and $\nu' = \nu + \nu_1$ where ν is given by (2.7) and ν_1 is the restriction of ν' to $\Gamma \setminus \Gamma'$. By **8**.(5.1), $u_{\nu'} = u_\nu \oplus u_{\nu_1}$. By Lemma **8**.5.2, $u_{\nu_1} = 0$ on O. By Theorem 2.2, $u' \le w_{\Gamma'} \oplus u_{\nu'} = \tilde{u} \oplus u_\nu$ where $\tilde{u} = w_{\Gamma'} \oplus u_{\nu_1} = 0$ on O. Hence $\tilde{u} \le w_\Gamma$ and $u' \le w_\Gamma \oplus u_\nu$. □

3. Solutions w_B with Borel B

3.1. Assuming the existence of an (L, ψ)-superdiffusion, we construct such solutions by using, as a tool, capacities related to the range. We calculate $\mathrm{tr}(w_B)$ and $\mathrm{tr}(w_B \oplus u_\nu)$ for all $\nu \in \mathcal{N}_0$.

3.2. Choquet capacities. Suppose that E is a separable locally compact metrizable space. Denote by $\mathcal{K}$ the class of all compact sets and by $\mathcal{O}$ the class of all open sets in E. A $[0, +\infty]$-valued function Cap on the collection of all subsets of E is called a *capacity* if:

3.2.A. $\mathrm{Cap}(A) \le \mathrm{Cap}(B)$ if $A \subset B$.

3.2.B. $\mathrm{Cap}(A_n) \uparrow \mathrm{Cap}(A)$ if $A_n \uparrow A$.

3.2.C. $\mathrm{Cap}(K_n) \downarrow \mathrm{Cap}(K)$ if $K_n \downarrow K$ and $K_n \in \mathcal{K}$.

These conditions imply

$$\mathrm{Cap}(B) = \sup\{\mathrm{Cap}(K) : K \subset B, K \in \mathcal{K}\} = \inf\{\mathrm{Cap}(O) : O \supset B, O \in \mathcal{O}\} \tag{3.1}$$

for every Borel set B. [1]

[1] The relation (3.1) is true for a larger class of analytic sets but we do not use this fact.

The following result is due to Choquet [**Cho54**]. Suppose that a function $\mathrm{Cap} : \mathcal{K} \to [0, +\infty]$ satisfies 3.2.A–3.2.C and the following condition:

3.2.D. For every $K_1, K_2 \in \mathcal{K}$,

$$\mathrm{Cap}(K_1 \cup K_2) + \mathrm{Cap}(K_1 \cap K_2) \leq \mathrm{Cap}(K_1) + \mathrm{Cap}(K_2).$$

Then Cap can be extended to a capacity on E.

An important class of capacities related to random closed sets has been studied in the original memoir of Choquet [**Cho54**]. Let (F, P) be a random closed set in E. Put

$$\Lambda_B = \{\omega : F(\omega) \cap B \neq \emptyset\}. \tag{3.2}$$

The definition of a random closed set (see, section **4**.5.1) implies Λ_B belongs to the completion $\mathcal{F}^P$ of $\mathcal{F}$ for all B in $\mathcal{O}$ and $\mathcal{K}$.

Note that

$$\begin{gathered}\Lambda_A \subset \Lambda_B \quad \text{if } A \subset B,\\ \Lambda_{A\cup B} = \Lambda_A \cup \Lambda_B, \quad \Lambda_{A\cap B} \subset \Lambda_A \cap \Lambda_B,\\ \Lambda_{B_n} \uparrow \Lambda_B \quad \text{if } B_n \uparrow B,\\ \Lambda_{K_n} \downarrow \Lambda_K \quad \text{if } K_n \downarrow K \quad \text{and } K_n \in \mathcal{K}.\end{gathered}$$

Therefore the function

$$\mathrm{Cap}(K) = P(\Lambda_K), \quad K \in \mathcal{K} \tag{3.3}$$

satisfies conditions 3.2.A–3.2.D and it can be continued to a capacity on E. It follows from 3.2.B that $\mathrm{Cap}(O) = P(\Lambda_O)$ for all open O. Suppose that B is a Borel set. By (3.1), there exist $K_n \in \mathcal{K}$ and $O_n \in \mathcal{O}$ such that $K_n \subset B \subset O_n$ and $\mathrm{Cap}(O_n) - \mathrm{Cap}(K_n) < 1/n$. Since $P(\Lambda_{K_n}) \leq P(\Lambda_B) \leq P(\Lambda_{O_n})$ and since $P(\Lambda_{O_n}) - P(\Lambda_{K_n}) < 1/n$, we conclude that $\Lambda_B \in \mathcal{F}^P$ and

$$\mathrm{Cap}(B) = P(\Lambda_B). \tag{3.4}$$

3.3. Solutions w_B. Suppose $X = (X_D, P_\mu)$ is an (L, ψ)-superdiffusion and $\mathcal{R}$ is its range in a bounded smooth domain E. Denote by C_μ the capacity on ∂E corresponding to a random set $(\mathcal{R}, P_\mu)$ and put $C_\mu = C_x$ if $\mu = \delta_x$. Formula (3.4) implies

$$C_\mu(B) = P_\mu\{\mathcal{R} \cap B \neq \emptyset\}.$$

By Remark 1.2,

$$\{\mathcal{R} \cap B \neq \emptyset\} \in \mathcal{F}_\partial \tag{3.5}$$

for every compact set $B \subset \partial E$. Since $\mathcal{F}_\partial$ is complete with respect to all measures $\mu \in \mathcal{M}_c$, (3.5) holds for all Borel B. Note that the function w_B defined by (1.6), can be expressed as follows:

$$w_B(x) = -\log[1 - C_x(B)]. \tag{3.6}$$

We use this expression to define w_B for all Borel B. By (3.1), $w_B(x) = \sup\{w_K(x) : K \subset B, K \in \mathcal{K}\}$. Therefore a Borel set B is polar if and only if $w_B(x) = 0$ for all $x \in E$ (which is equivalent to the condition $C_x(B) = 0$ for all $x \in E$).

THEOREM 3.1. *For every Borel subset B of ∂E, w_B belongs to $\mathcal{U}$ and*

$$\langle w_B, \mu\rangle = -\log[1 - C_\mu(B)] = -\log P_\mu\{\mathcal{R} \cap B = \emptyset\} \tag{3.7}$$

for all $\mu \in \mathcal{M}_c$.

PROOF. 1°. First, we prove that, for every $\mu \in \mathcal{M}_c$, there exists an increasing sequence of compact subsets K_n of B such that $C_\mu(K_n) \uparrow C_\mu(B)$ and $w_{K_n}(x) \uparrow w_B(x)$ μ-a.e. To this end we consider, besides C_μ, another capacity C^μ associated with a random closed set $(\mathcal{R}, P^\mu)$ where

$$P^\mu = \int P_x \ \mu(dx).$$

There exists an increasing sequence $K_n \in \mathcal{K}$ such that $C_\mu(K_n) \uparrow C_\mu(B)$ and $C^\mu(K_n) \uparrow C^\mu(B)$. [2] Put

$$\varphi_n(x) = P_x\{\mathcal{R} \cap K_n \neq \emptyset\}, \qquad \varphi(x) = P_x\{\mathcal{R} \cap B \neq \emptyset\}$$

and note that $\varphi_n \uparrow \tilde{\varphi} \leq \varphi$. Since

$$\langle \varphi_n, \mu \rangle = C^\mu(K_n) \uparrow C^\mu(B) = \langle \varphi, \mu \rangle,$$

we have $\tilde{\varphi} = \varphi$ μ-a.e. Therefore $\varphi_n \uparrow \varphi$ μ-a.e. and $w_{K_n} \uparrow w_B$ μ-a.e.

2°. By Remark 1.1,

$$-\log P_\mu\{\mathcal{R} \cap K_n = \emptyset\} = \int \mu(dx)[-\log P_x\{\mathcal{R} \cap K_n = \emptyset\}]$$

and therefore

$$-\log[1 - C_\mu(K_n)] = \langle w_{K_n}, \mu \rangle.$$

By passing to the limit, we get (3.7).

3°. Note that $w_B = \mathrm{LPT}(Z_B)$ where Z_B is defined by (1.8). By Theorem **9**.1.2, to prove that $w_B \in \mathcal{U}$, it is sufficient to show that $Z_B \in \mathfrak{Z}$. By (3.5), Z_B satisfies the condition **9**.1.3.A. Formula (3.7) implies **9**.1.3.B. By (1.7),

$$P_x\{Z_B < \infty\} = P_x\{\mathcal{R} \cap B = \emptyset\} \geq P_x\{\mathcal{R} \cap \partial E = \emptyset\} = e^{-w(x)} > 0$$

and therefore Z_B satisfies **9**.1.3.C. □

3.4. Trace of w_B.

LEMMA 3.1. *Let B be a Borel subset of ∂E. The trace of w_B is equal to* $(\Gamma, 0)$ *where Γ is the smallest compact set such that $B \cap (\partial E \setminus \Gamma)$ is polar.*

PROOF. Suppose $\operatorname{tr} w_B = (\Gamma, \nu)$ and put $A = B \cap O$ where $O = \partial E \setminus \Gamma$. Let K be a compact subset of A. It follows from (3.6) that $w_K \leq w_B$. By 1.3.D and **8**.4.1.A, $w_K = Q_K(w_K) \leq Q_K(w_B)$. Since $K \subset O$, $Q_K(w_B)$ is moderate. Hence w_K is also moderate and K is removable by 1.4.D. We conclude that A is polar.

Now suppose that Γ' is an arbitrary compact set such that $A' = B \cap O'$ is polar (here $O' = \partial E \setminus \Gamma'$). Since $B \subset \Gamma' \cup A'$, we have, by 3.2.D, that $C_x(B) \leq C_x(\Gamma') + C_x(A') = C_x(\Gamma')$. By (3.6), this implies $w_B \leq w_{\Gamma'}$ and therefore $Q_K(w_B) \leq Q_K(w_{\Gamma'})$. We use this inequality to establish: (a) $\Gamma' \supset \Gamma$; (b) $\nu = 0$.

Indeed, if $K \subset O'$, then $Q_K(w_{\Gamma'}) = 0$ by 1.3.C. Hence, $Q_K(w_B) = 0$. We conclude that O' is a moderate open set. Therefore $O' \subset O$. This implies (a). By taking $\Gamma' = \Gamma$, we get that $Q_K(w_B) \leq Q_K(w_\Gamma)$. Hence, for $K \subset O$, $Q_K(w_B) = 0$ which implies (b). □

[2] We can define K_n as the union of K'_n and K''_n such that $C_\mu(K'_n) \uparrow C_\mu(B)$ and $C^\mu(K''_n) \uparrow C^\mu(B)$.

3.5. Le Gall's example. The following theorem, due to Le Gall, shows that, in general, the rough trace does not determine a solution.

THEOREM 3.2. *Suppose that there exists an (L,ψ)-superdiffusion, $\psi \in CR$ and:*

3.5.A. All single-point subsets of ∂E are polar.

Then there exist infinite many solutions with the trace $(\partial E, 0)$.

[We will see in Chapter **13** that, in the case $\psi(u) = u^\alpha, \alpha > 1$ and a bounded smooth domain E, the condition 3.5.A is satisfied if and only if $\alpha \geq \frac{d+1}{d-1}$ (the so-called supercritical case).]

The proof of Theorem 3.2 uses the following lemmas.

LEMMA 3.2. *Let $\psi \in CR$. If $u, v \in \mathcal{U}$ and $u \geq v$, then either $u(x) = v(x)$ for all $x \in E$ or $u(x) > v(x)$ for all $x \in E$.*

PROOF. It is sufficient to prove that if D is a smooth domain such that $D \Subset E$ and if u and v coincide at a point $c \in D$, then $u = v$ in D. By the mean value property **8**.2.1.D,

$$P_c e^{-\langle u, X_D\rangle} = e^{-u(c)}, P_c e^{-\langle v, X_D\rangle} = e^{-v(c)}$$

and therefore $\langle u, X_D\rangle = \langle v, X_D\rangle$ P_c-a.s. By **4**.(4.6),

$$K_D u(c) = P_c\langle u, X_D\rangle = P_c\langle v, X_D\rangle = K_D v(c).$$

We conclude from this equation and formula **6**.(2.8) that $k(c,y)u(y) = k(c,y)v(y)$ for γ-almost all $y \in \partial D$. Since $k(c,y) > 0$ and u, v are continuous on $\bar{D}$, $u = v$ on ∂D. By the comparison principle, $u = v$ in D. □

LEMMA 3.3. *If O is a nonempty relatively open subset of ∂E, then O is not polar. If $\psi \in CR$, then $w_O(x) > 0$ for all $x \in E$.*

PROOF. Let O_1 be a nonempty open subset of O such that $\bar{O}_1 \subset O$. By **8**.2.1.I, there exists a non-zero solution u vanishing on $\partial E \setminus \bar{O}_1$. Since $w_{\bar{O}_1}$ is the maximal solution equal to 0 on $\partial D \setminus \bar{O}_1$, we have $w_{\bar{O}_1} \geq u$ and $w_{\bar{O}_1}(x) > 0$ at some $x \in E$. Hence, $C_x(O) \geq C_x(\bar{O}_1) > 0$ and O is not polar. The second part of the lemma follows from Lemma 3.2. □

LEMMA 3.4. *Under the condition 3.5.A,* $\operatorname{tr}(w_\Lambda) = (\partial E, 0)$ *for each open everywhere dense subset Λ of ∂E.*

PROOF. By Lemma 3.1, $\operatorname{tr}(w_\Lambda) = (\Gamma, 0)$ for some Γ. Let us prove that $\Gamma = \partial E$. Suppose B is a compact subset of $A = O \cap \Lambda$ where $O = \partial E \setminus \Gamma$. Solution $Q_B(w_\Lambda)$ is moderate because $B \subset O$. Since $B \subset \Lambda$, we have $Q_B(w_\Lambda) \geq Q_B(w_B) = w_B$, and w_B is moderate. By 1.4.D, B is removable. Hence, A is polar. Since A is open, it is empty by Lemma 3.3. Hence $\Lambda \subset \Gamma$. Since Λ is everywhere dense, $\Gamma = \partial E$. □

PROOF OF THEOREM 3.2. Let B be a countable everywhere dense subset of ∂E. Fix $x \in E$. Condition 3.5.A implies that $C_x(B) = 0$ and therefore there exists a decreasing sequence of open subsets O_n of ∂E such that $O_n \supset B$ and $C_x(O_n) \to 0$. By Lemma 3.3, $w_{O_n}(x) > 0$. Since $w_{O_n}(x) \to 0$, the sequence w_{O_n} contains infinite many different functions. By Lemma 3.4, each of them has the trace $(\partial E, 0)$. □

3.6. (B, ν)**-solutions.**

LEMMA 3.5. *Let $\nu \in \mathcal{N}_0$. The trace of u_ν is equal to (Γ, μ) where $\Gamma = Ex(\nu)$ and μ coincides with the restriction of ν to $\partial E \setminus \Gamma$.*

PROOF. Let B be a compact subset of ∂E. By 2.2.C, $Q_B(u_\nu) = u_{\nu_B}$ where ν_B is the restriction of ν to B. Therefore B is moderate for u_ν if and only if u_{ν_B} is moderate, that is if and only if $\nu(B) < \infty$. Note that $\partial E \setminus Ex(\nu)$ is the maximal open set such that $\nu(B) < \infty$ for all its compact subsets. Clearly, $\mu = \nu$ on this set. $\square$

Now we calculate the rough trace of $w_B \oplus u_\nu$ for an arbitrary Borel B and arbitrary $\nu \in \mathcal{N}_0$. [We call $w_B \oplus u_\nu$ the (B, ν)-*solution.*]

THEOREM 3.3. *The trace (Γ, μ) of a (B, ν)-solution $u = w_B \oplus u_\nu$ can be described as follows: Γ is the smallest compact set such that $\Gamma \supset Ex(\nu)$ and $B \cap O$ is polar (here $O = \partial E \setminus \Gamma$), and μ is the restriction of ν to O.*

PROOF. By Lemma 3.1, $\mathrm{tr}(w_B) = (\Gamma_1, 0)$ where Γ_1 is the smallest compact set such that $B \cap O_1$ is polar (here $O_1 = \partial E \setminus \Gamma_1$). By Lemma 3.5, $\mathrm{tr}(u_\nu) = (\Gamma_2, \mu)$ where $\Gamma_2 = Ex(\nu)$ and μ is the restriction of ν to $O_2 = \partial E \setminus \Gamma_2$. By 2.2.A, $\Gamma = \Gamma_1 \cup \Gamma_2$ and $O = O_1 \cap O_2$. Hence, $\Gamma \supset Ex(\nu)$ and $B \cap O$ is polar.

Let us prove the minimal property of Γ. If Γ' is compact and $B \cap O'$ (where $O' = \partial E \setminus \Gamma'$) is polar, then, by Lemma 3.1, $\Gamma' \supset \Gamma_1$. If, in addition $\Gamma' \supset Ex(\nu)$, then $\Gamma' \supset \Gamma_1 \cup \Gamma_2 = \Gamma$.

Suppose K is a compact subset of O. Then $K \subset O_1$. Since $\mathrm{tr}(w_B) = (\Gamma_1, 0)$, $Q_K(w_B) = 0$. By 2.2.B, this implies $Q_K(u) = Q_K(u_\nu)$. By Theorem **8**.4.2, $Q_K(u_\nu) = u_{\nu_K}$ where ν_K is the restriction of ν to K. Therefore $\mu = \nu$ on O. $\square$

4. Notes

4.1. A program to describe the set $\mathcal{U}$ of all positive solutions of a semilinear equation

$$Lu = \psi(u), \tag{4.1}$$

(initiated by Dynkin in the earlier 1990s) was a subject of a discussion with H. Brezis during his visit to Cornell University. Brezis suggested that boundary values of $u \in \mathcal{U}$ may be described as measures on the boundary.

A pioneering result in this direction was announced by Le Gall in [**Le 93b**]. He established a 1-1 correspondence between all solutions of equation $\Delta u = u^2$ in the unit disk $D \subset \mathbb{R}^2$ and all pairs (Γ, ν) where Γ is a compact subset of ∂D and ν is a Radon measure on $O = \partial D \setminus \Gamma$. Roughly speaking Γ was defined as the set of points $y \in \partial D$ near which u behaved like the inverse of the squared distance to the boundary. More precisely, $y \in \Gamma$ if

$$\limsup_{x \to y, x \in D} d(x, \partial D)^2 u(x) > 0.$$

The measure ν was defined as a vague limit of measures $1_O(y) u(r, y) \gamma(dy)$ as $r \uparrow 1$ (γ is the Lebesgue measure on ∂D). Le Gall expressed the solution corresponding to (Γ, ν) in terms of the Brownian snake — a path-valued Markov process introduced in his earlier publications (closely related to the super-Brownian motion). In [**Le 97**], the results announced in [**Le 93b**] were proved and extended to all smooth domains in $\mathbb{R}^2$.

4.2. The presentation in Chapter **10** is based on [**DK98c**] and [**DK00**]. In [**DK98c**], the traces of solutions of equation $Lu = u^\alpha$ with $1 < \alpha \leq 2$ in an arbitrary bounded smooth domain in $\mathbb{R}^d$ were investigated . In [**DK00**], more general results covering a large class of functions ψ (not just $\psi(u) = u^\alpha$) were obtained.

In [**DK98b**] the results of [**DK98c**] were extended to a wide class of nonsmooth domains E and even to a certain class of Riemannian manifolds. The main difficulty is that the extremal characterization of the sweeping [Theorem 1.1] is not available. Among tools used to overcome this difficulty was a stochastic version of the comparison principle.

4.3. The equation $\Delta u = u^\alpha$ in the d-dimensional unit ball B with arbitrary $\alpha > 1$ and d was investigated by purely analytic methods by Marcus and Véron. The results were announced in [**MV95**] and proved in [**MV98a**] and [**MV98b**]. [The name "trace" for a pair (Γ, ν) was suggested, first, in [**MV95**].] The case of a subcritical value $\alpha < (d+1)/(d-1)$ was studied in [**MV98a**] and the case of $\alpha \geq (d+1)/(d-1)$ was treated in [**MV98b**]. A principal difference between these cases is that there exist no polar sets except the empty set in the first case and such sets exist in the second case (see section **13**.3.) Marcus and Véron defined the special set Γ [which they call the singular boundary] by the condition: $y \in \partial B$ belongs to Γ if, for every neighborhood U of y in ∂B,

$$\int_U u(ry)\gamma(dy) \to \infty \quad \text{as } r \uparrow 1.$$

Their definition of the measure ν is similar to that of Le Gall. They also observed that a pair (Γ, ν) can be replaced by one outer regular measure [3] μ on ∂B: Γ consists of all explosion points of μ and ν is the restriction of μ to $\partial B \setminus \Gamma$.

In [**MV98a**] Marcus and Véron proved that $\mathrm{tr}(u)$ determines u uniquely for subcritical values of α. Le Gall's example (see Theorem 3.2) shows that this is not true for supercritical values. [A parabolic version of this example was published in [**Le 96**].]

[3] Outer regularity means that, for every Borel A, $\mu(A)$ is equal to the infimum of $\mu(U)$ over all open sets $U \supset A$.

CHAPTER 11

Fine trace

As in the case of the rough trace, the fine trace of $u \in \mathcal{U}$ is a pair (Γ, ν) where Γ is a subset of the boundary and ν is a measure on the portion of the boundary complementary to Γ. However, Γ may not be closed (it is closed in a fine topology) and ν does not need to be a Radon measure (but it is σ-finite). In contrast to the case of the rough trace, there is no special advantage in restricting the theory to smooth domains, and we return to the general setting of Chapters **7**–**9** when E is an arbitrary Greenian domain in $\mathbb{R}^d$, $k(x, y)$ is a Martin kernel and $\hat{\partial E}$ is the Martin boundary corresponding to L.

The first step in the definition of the rough trace was constructing a measure ν on the moderate boundary portion O. The first (and main) part in the theory of the fine trace is the study of the set Γ of all singular points of u. We denote it $SG(u)$. Roughly speaking, $y \in SG(u)$ if $\psi'(u)$ tends to infinity sufficiently fast as $x \to y, x \in E$. The function $\psi'(u)$ plays a key role in the description of the tangent cone to $\mathcal{U}$ at point u. It is also a principle ingredient in the probabilistic formula **9**.(3.4). An important step is an investigation of a curve $u_t = u \oplus u_{t\nu}$ in $\mathcal{U}$. If there exists a (L, ψ)-superdiffusion, then we deduce from **9**.(3.4) an integral equation for u_t (see, Theorem **11**.3.1). For a general ψ, we prove a weaker statement about u_t, but it is sufficient for establishing fundamental properties of $SG(u)$. Relying on these properties, we introduce a fine topology on the Martin boundary. The fine trace is defined by the formula **11**.(7.1). The main results on the fine trace are stated in Theorems **11**.7.1 and **11**.7.2. At the end of the chapter we demonstrate that for the solutions in Le Gall's example (having identical rough traces) the fine traces are distinct.

In this chapter we assume that ψ satisfies conditions **8**.1.A–1.C and **8**.3.2.A and, moreover, that:

0.A. Function $\frac{\partial \psi(x,u)}{\partial u}$ is continuously differentiable.

1. Singularity set $SG(u)$

1.1. Points of rapid growth. We consider the tangent cone to $\mathcal{U}$ at point u which we define as the set of tangent vectors v to all smooth curves u_t in $\mathcal{U}$ with the properties:

(a) $u_0 = u$ and $u_t \in \mathcal{U}$ for $0 \le t < \varepsilon$;

(b) $u_t(x)$ is monotone increasing in t.

Condition (a) implies that $Lu_t = \psi(u_t)$ for $0 \le t < \varepsilon$ and therefore $v(x) = \partial u_t(x)/\partial t\big|_{t=0}$ satisfies a linear equation

$$Lv = av \tag{1.1}$$

where

$$a = \psi'(u).$$

[We use abbreviation $\psi'(u)$ for $\frac{\partial\psi(x,u)}{\partial u}$.] Since ψ is monotone increasing in u, $a(x) \geq 0$.

Condition (b) implies that $v(x) \geq 0$. In the case $a = 0$, we established, in Chapter **7**, an integral representation of all positive solutions of (1.1) through the Martin kernel $k(x, y)$

$$h(x) = \int_{E'} k(x, y)\nu(dy) \tag{1.2}$$

where ν is a finite measure on a Borel subset E' of $\partial\hat{E}$. A similar representation is possible for any $a \geq 0$. However the corresponding kernel $k_a(x, y)$ can vanish identically in x for some points $y \in E'$. This happens if a blows up sufficiently fast. We say that $y \in E'$ is a *point of rapid growth for* a if $k_a(x, y) = 0$ for all $x \in E$. We say that $y \in E'$ is a *singular point of* $u \in \mathcal{U}$ and we write $y \in \mathrm{SG}(u)$ if y is a point of rapid growth of $\psi'(u)$, i.e., if $j_u(x, y) = k_{\psi'(u)}(x, y) = 0$ for all $x \in E$. The rest of boundary points are called *regular points* of u. We denote the set of all regular points by $\mathrm{RG}(u)$.

We give an analytic and a probabilistic construction of the kernel k_a. Based on the latter, we introduce an equivalent definition of singular points y in terms of the behavior of $\psi'(u)$ along Π_x^y-almost all paths.

1.2. Analytic construction of kernel k_a. Suppose $a \geq 0$ is a continuously differentiable function on E, $h \in \mathcal{H}(E)$ and $D \Subset E$ is regular. By Theorem **6**.3.1, the boundary value problem

$$\begin{aligned} Lu - au &= 0 \quad \text{in } D, \\ u &= h \quad \text{on } \partial D \end{aligned} \tag{1.3}$$

is equivalent to the integral equation

$$u + G_D(au) = h \tag{1.4}$$

and each of the problems (1.3) and (1.4) has a unique positive solution. We denote it $K_D^a h$. We have:

1.2.A. $K_D^a h \leq h$ and $K_D^0 h = h$.

1.2.B. If $D \subset \tilde{D} \Subset E$, then $K_D^a h \geq K_{\tilde{D}}^a h$.

Indeed, let $u = K_D^a h$ and $\tilde{u} = K_{\tilde{D}}^a h$. We have $h = \tilde{u} + G_{\tilde{D}}(a\tilde{u}) \geq \tilde{u}$. Therefore $w = \tilde{u} - u \leq 0$ on ∂D because $u = h$ on ∂D. Since $Lw - aw = 0$ in D, the maximum principle **6**.1.4.A implies that $w \leq 0$ in D.

Put $k_y(x) = k(x, y)$ and denote by $k_a^D(x, y)$ the value of $K_D^a k_y$ at point x. Note that, for h given by (1.2),

$$K_D^a h(x) = \int_{E'} k_a^D(x, y)\nu(dy). \tag{1.5}$$

Consider a sequence of regular domains D_n exhausting E. By 1.2.B, the sequence $k_a^{D_n}$ is monotone decreasing, and we denote its limit by $k_a(x, y)$. [The limit does not depend on the choice of D_n.] If h is defined by (1.2), then, by the dominated convergence theorem,

$$\int_{E'} k_a(x,y)\nu(dy) = \lim K^a_{D_n} h(x). \tag{1.6}$$

By **6**.3.2.B, this is a solution of equation **6**.(3.4) and, by **6**.3.2.C, it is either strictly positive or equal to 0 for all x. In particular, this is true for functions $k_a(\cdot,y)$ and therefore $k_a(x,y) > 0$ for all $x \in E$ if y is not a point of rapid growth of a.

1.3. Probabilistic definition of kernel k_a. A probabilistic formula for a solution of equation (1.4) is given by Lemma **9**.3.1. By applying this formula to a sequence of regular domains D_n exhausting E, we get

$$k_a^{D_n}(x,y) = \Pi^y_x \exp[-\int_0^{\tau_n} a(\xi_s)ds] \tag{1.7}$$

where τ_n is the first exit time from D_n. By passing to the limit as $n \to \infty$, we get an expression

$$k_a(x,y) = \Pi^y_x e^{-\Phi_a} \tag{1.8}$$

where

$$\Phi_a = \int_0^\zeta a(\xi_s)\, ds. \tag{1.9}$$

If $a_1 \le a_2$, then $\Phi_{a_1} \le \Phi_{a_2}$ and $k_{a_1} \ge k_{a_2}$. Note that y is a point of rapid growth for a if and only if

$$\Phi_a = \infty \quad \Pi^y_x - \text{a.s.} \tag{1.10}$$

for all $x \in E$.

The set $A = \{\Phi_a = \infty\}$ belongs to the tail σ-algebra $\mathcal{T}$ and, by Theorem **7**.5.1, if condition (1.10) is not satisfied, then for all $x \in E$, $\Pi^y_x(A) = 0$ and therefore $\Phi_a < \infty$ Π^y_x-a.s. One of implications is that a point $y \in E'$ is a point of rapid growth for $a_1 + a_2$ if and only if it is a point of rapid growth either for a_1 or for a_2.

2. Convexity properties of V_D

2.1. Condition **8**.1.A implies that, for all $0 < u_1 < u_2$, $\psi(u_1) < \psi(u_2)$ and the ratio

$$\frac{\psi(u_2) - \psi(u_1)}{u_2 - u_1} \tag{2.1}$$

is monotone increasing in u_1 and in u_2. Note that every function, for which the ratio (2.1) is monotone increasing in u_2 (or in u_1), is convex.

2.2.

THEOREM 2.1. *Suppose that D is a bounded regular domain and ψ satisfies **8**.1.A. Let $u \ge 0, \varphi \ge \tilde\varphi \ge 0$ be continuous functions on ∂D. Then*

$$V_D(u+\varphi) - V_D(u+\tilde\varphi) \le V_D(\varphi) - V_D(\tilde\varphi). \tag{2.2}$$

Function $F(t) = V_D(u + t\varphi), t \ge 0$ is concave.

Proof is based on the following lemma.

LEMMA 2.1. *Let D and ψ be as in Theorem 2.1 and suppose that continuous functions $f_1, f_2, \tilde{f}_1, \tilde{f}_2$ on ∂D and a constant $\lambda \geq 0$ satisfy conditions*

$$0 \leq f_1 \leq \tilde{f}_1 \leq \tilde{f}_2, \qquad 0 \leq f_1 \leq f_2 \leq \tilde{f}_2 \tag{2.3}$$

and

$$\tilde{f}_2 - \tilde{f}_1 \leq \lambda(f_2 - f_1). \tag{2.4}$$

Then

$$V_D(\tilde{f}_2) - V_D(\tilde{f}_1) \leq \lambda[V_D(f_2) - V_D(f_1)]. \tag{2.5}$$

PROOF. Put $u_i = V_D(f_i)$ and $\tilde{u}_i = V_D(\tilde{f}_i)$. By **8**.2.1.C, $v = u_2 - u_1 \geq 0$ and $\tilde{v} = \tilde{u}_2 - \tilde{u}_1 \geq 0$. We have $Lu_1 - \psi(u_1) = L(u_2) - \psi(u_2) = 0$ and therefore $Lv = av$ where

$$a = \frac{\psi(u_2) - \psi(u_1)}{u_2 - u_1} \quad \text{for } u_1 \neq u_2,$$

and $a = 0$ for $u_1 = u_2$. By **6**.3.2.C, $v = 0$ in D or v is strictly positive. The same is true for $\tilde{v}$. Formula (2.5) holds if $\tilde{v} = 0$. If $v = 0$, then $f_1 = u_1 + G_D\psi(u_1) = u_2 + G_D\psi(u_2) = f_2$. By (2.4), $\tilde{f}_1 = \tilde{f}_2$, and we again have (2.5). Therefore we can assume that v and $\tilde{v}$ are strictly positive.

Put

$$a = [\psi(u_2) - \psi(u_1)]/v, \quad \tilde{a} = [\psi(\tilde{u}_2) - \psi(\tilde{u}_1)]/\tilde{v}.$$

By (2.3), $u_1 \leq \tilde{u}_1 \leq \tilde{u}_2$ and $u_1 \leq u_2 \leq \tilde{u}_2$. Since the ratio (2.1) is monotone increasing in u_1 and in u_2, we have $\tilde{a} \geq a$. Put $w = \tilde{v} - \lambda v$. Note that

$$Lw = L\tilde{v} - \lambda Lv = \tilde{a}\tilde{v} - \lambda av = aw + (\tilde{a} - a)\tilde{v} \quad \text{in } D.$$

Hence, $Lw - aw \geq 0$ in D. On ∂D, $u_i = f_i, \tilde{u}_i = \tilde{f}_i$ and, by (2.4), $w \leq 0$. By the maximum principle **6**.1.4.A, $w \leq 0$ in D which implies (2.5). □

2.3. Proof of Theorem 2.1. To prove (2.2), it is sufficient to apply Lemma 2.1 to $f_1 = \tilde{\varphi}, \quad f_2 = \varphi, \quad \tilde{f}_1 = u + \tilde{\varphi}, \quad \tilde{f}_2 = u + \varphi, \quad \lambda = 1.$

By taking

$$f_1 = \tilde{f}_1 = f + s\varphi, \quad f_2 = f + t\varphi, \quad \tilde{f}_2 = f + \tilde{t}\varphi, \quad \lambda = (\tilde{t} - s)/(t - s)$$

in (2.5), we see that $[F(t) - F(s)]/(t - s)$ is monotone decreasing in t and therefore F is concave. □

REMARK. By applying Theorem 2.1 to $\tilde{\varphi} = 0$, we get another proof of subadditivity of V_D but only for bounded regular domains – a restriction not imposed in Theorem **8**.2.1.

3. Functions J_u

3.1. To every $u \in \mathcal{U}$ and every $\nu \in \mathcal{N}_1$ there corresponds a positive function on E given by the formula

$$J_u(\nu)(x) = \int_{E'} j_u(x, y)\nu(dy) \tag{3.1}$$

where $j_u(x, y) = k_a(x, y)$ with $a = \psi'(u)$. Note that $J_u(\nu) = 0$ if and only if ν is concentrated on $\mathrm{SG}(u)$.

By using the probabilistic expression (1.8) for k_a, we get

$$J_u(\nu)(x) = \int_{E'} \Pi_x^y e^{-\Phi(u)}\nu(dy). \tag{3.2}$$

3.2. A 1-1 mapping $\nu \to u_\nu$ from $\mathcal{N}_1$ onto the class $\mathcal{U}_1$ of all moderate solutions was defined in section **8**.3.3. (Formula $\nu = \operatorname{tr} u$ describes the inverse mapping from $\mathcal{U}_1$ to $\mathcal{N}_1$.) By Theorem **8**.3.3, under condition **8**.3.2.A, $t\nu \in \mathcal{N}_1$ for all $t > 0$ if $\nu \in \mathcal{N}_1$.

THEOREM 3.1. *Let* $u_t = u \oplus u_{t\nu}$ *where* $u \in \mathcal{U}, \nu \in \mathcal{N}_1$. *If there exists an* (L, ψ)*-superdiffusion, then*

$$u_t = u + \int_0^t J_{u_s}(\nu) ds. \tag{3.3}$$

PROOF. If $Z_t = \mathrm{SBV}(u_t)$, then, by Theorem **9**.1.1, $u_t = \mathrm{LPT}(Z_t)$ which means

$$e^{-u_t(x)} = P_x e^{-Z_t}. \tag{3.4}$$

Put $Z = \mathrm{SBV}(u), Z_\nu = \mathrm{SBV}(u_\nu), Z_{t\nu} = \mathrm{SBV}(u_{t\nu})$. By **9**.1.4.A and **9**.(2.7), $Z_t = Z + Z_{t\nu} = Z + tZ_\nu$. Therefore (3.4) implies

$$e^{-u_t} du_t/dt = P_x Z_\nu e^{-Z_t}. \tag{3.5}$$

By applying Theorem **9**.3.1 to Z_t, we get

$$P_x Z_\nu e^{-Z_t} = e^{-u_t(x)} J_{u_t}(\nu)(x). \tag{3.6}$$

By (3.5) and (3.6),

$$du_t/dt = e^{u_t} P_x Z_\nu e^{-Z_t} = J_{u_t}(\nu)$$

which implies (3.3). □

3.3. Formula (3.3) plays a central role in the theory of the fine trace. Without assuming the existence of an (L, ψ)-superdiffusion, we prove a weaker form of Theorem 3.1 by using the convexity properties of operators V_D established in section 2.

THEOREM 3.2. *For all* $u \in \mathcal{U}, \nu \in \mathcal{N}_1$ *and* $t \geq 0$,

$$u \oplus u_{t\nu} \leq u + tJ_u(\nu) \tag{3.7}$$

and

$$u_{t\nu} = \int_0^t J_{u_{s\nu}}(\nu)\, ds. \tag{3.8}$$

We use the following lemmas:

LEMMA 3.1. *For every* $\lambda > 0$, *there exists a constant* $C(\lambda)$ *such that*

$$\psi'(\lambda u) \leq C(\lambda)\psi'(u) \quad \text{for all } u \geq 0. \tag{3.9}$$

PROOF. Let $2^n > \lambda + 1$. By **8**.1.A, ψ' is positive and monotone increasing and therefore

$$u\psi'(\lambda u) \leq \int_{\lambda u}^{(\lambda+1)u} \psi'(t)dt \leq \int_0^{2^n u} \psi'(t)dt = \psi(2^n u).$$

By **8**.3.2.A and **8**.1.A,

$$\psi(2^n u) \leq a^n \psi(u) = a^n \int_0^u \psi'(t)dt \leq a^n u\psi'(u).$$

Therefore (3.9) holds with $C(\lambda) = a^n$. □

LEMMA 3.2. *If f is a continuous function on an interval $[a, b]$ and its right derivative $f^+(t) \le \Lambda$ for all $a \le t < b$, then*

$$f(b) - f(a) \le \Lambda(b - a).$$

PROOF. Consider a continuous function $F_\varepsilon(t) = f(t) - f(a) - (\Lambda + \varepsilon)(t - a)$. We have $F_\varepsilon^+(a) < 0$ because $f^+(a) \le \Lambda$. Hence $F_\varepsilon(t) < 0$ in a neighborhood of a. If $F_\varepsilon(b) > 0$, then there exists a point $c \in (a, b)$ such that $F_\varepsilon(c) = 0$ and $F_\varepsilon(t) > 0$ for $t \in (c, b]$. This implies $f(t) - f(c) > (\Lambda + \varepsilon)(t - c)$ and therefore $F_\varepsilon^+(c) \ge \Lambda + \varepsilon$. This contradicts our assumption. Hence $F_\varepsilon(b) \le 0$ for all $\varepsilon > 0$. □

PROOF OF THEOREM 3.2. By Lemma 3.2, to prove (3.7), it is sufficient to show that $u_t = u \oplus u_{t\nu}$ is continuous in t and that its right derivative w_t satisfies the condition

$$w_t \le J_u(\nu). \tag{3.10}$$

Let D_n be a sequence of regular sets exhausting E. Put $V_n = V_{D_n}, K_n = K_{D_n}, G_n = G_{D_n}$ and let $h = K\nu, u_t^n = V_n(u + th)$.

1°. Note that $u_{t\nu} \le K(t\nu) = th$. By **8**.5.1.D, $u + th \in \mathcal{U}^-$, and, by **8**.5.1.A, $u_t^n \ge u \oplus (th) \ge u_t$.

It follows from the monotonicity of V_n (**8**.2.1.C), Theorem 2.1 and the mean value property (**8**.2.1.D) that

$$0 \le u_t^n - V_n(u + u_{th}) \le V_n(th) - V_n(u_{t\nu}) = V_n(th) - u_{t\nu}.$$

As $n \to \infty$, $V_n(u + u_{t\nu}) \to u_t$ and $V_n(th) \to i(th) = u_{t\nu}$ and therefore

$$\lim_{n\to\infty} u_t^n = u_t. \tag{3.11}$$

Note that $V_n(th) \le th$ because $th \in \mathcal{U}_-$ (see **8**.5.1.A). Therefore by (2.2),

$$0 \le u_t^n - u = V_n(u + th) - V_n(u) \le V_n(th) \le th. \tag{3.12}$$

By passing to the limit, we get

$$0 \le u_t - u \le th. \tag{3.13}$$

2°. We claim that $v_t^n = (u_t^n - u)/t$ satisfies the equation

$$v_t^n + G_n(a_t^n v_t^n) = h \tag{3.14}$$

where

$$a_t^n = \begin{cases} [\psi(u_t^n) - \psi(u)]/(u_t^n - u) & \text{if } u_t^n \ne u, \\ \psi'(u) & \text{if } u_t^n = u. \end{cases} \tag{3.15}$$

Indeed,

$$u_t^n + G_n\psi(u_t^n) = K_n(u + th) = K_n u + th \tag{3.16}$$

and, by the mean value property **8**.2.1.D,

$$u + G_n\psi(u) = K_n u. \tag{3.17}$$

Formulae (3.14)–(3.15) follow from (3.16) and (3.17).

3°. By Theorem 2.1, functions u_t^n are concave and, by (3.11) u_t is also concave. By (3.12) and (3.13), u_t^n and u_t are continuous at 0. Hence, they are continuous and have right derivatives for all $t \ge 0$. We denote them w_t^n and w_t. Since w_t is decreasing and since $u_t^n \ge u_t, u_0^n = u_0$, we have

$$w_t \le w_0 \le w_0^n. \tag{3.18}$$

Note that $v_t^n \uparrow w_0^n$ as $t \downarrow 0$ and, by (3.14), $v_t^n \leq h$. We see from (3.15) and (3.12) that $a_t^n \downarrow a = \psi'(u)$ as $t \downarrow 0$. It follows from **8**.1.A that $a_t^n \leq a_1^n \leq \psi'(u_1^n)$ for all $t \in [0,1]$. Since u_1^n and h are bounded in D_n,

$$G_n(a_t^n v_t^n) \to G_n(aw_0^n) \quad \text{as } t \to 0$$

by the dominated convergence theorem, and (3.14) yields

$$w_0^n + G_n(aw_0^n) = h. \tag{3.19}$$

Therefore $w_0^n = K_{D_n}^a h$ and, by (3.18), $w_t \leq K_{D_n}^a h$. By (1.6) and (3.1), $K_{D_n}^a h \to J_u(\nu)$ as $n \to \infty$ which implies (3.10) and (3.7).

4°. To prove (3.8), it is sufficient to show that the right derivative w_s of function $u_s = u_{s\nu}$ satisfies the equation

$$w_s + G(a^s w_s) = h \tag{3.20}$$

where $a^s = \psi'(u_s)$ and $h = K\nu$. Indeed, (3.20) implies that (1.4) (with a replaced by a^s) holds for every $D \Subset E$ and therefore $w_s = K_{D_n}^{a^s} h$. By (1.6), $w_s = J_{u_s}(\nu)$ and we get (3.8) by integrating w_s over $[0,t]$.

Since $u_{th} = i(th)$,

$$u_t + G\psi(u_t) = th \tag{3.21}$$

by **8**.(3.4), and therefore, for every $0 \leq s < t$,

$$v_{st} + G(a_{st} v_{st}) = h \tag{3.22}$$

where $v_{st} = (u_t - u_s)/(t-s)$ and

$$a_{st} = \begin{cases} [\psi(u_t) - \psi(u_s)]/(u_t - u_s) & \text{if } u_t \neq u_s, \\ \psi'(u_s) & \text{if } u_t = u_s. \end{cases}$$

We have

$$v_{st} \to w_s, \quad a_{st} \to a^s \quad \text{as } t \downarrow s.$$

Equation (3.20) follows from (3.22) if we prove that

$$G(a_{st} v_{st}) \to G(a^s w_s) \quad \text{as } t \downarrow s. \tag{3.23}$$

By (3.22) and Fatou's lemma,

$$G(a^s w_s) \leq h < \infty. \tag{3.24}$$

Clearly,

$$v_{st} \leq w_s, \qquad a_{st} \leq \psi'(u_{s+1}) \quad \text{for } s < t < s+1. \tag{3.25}$$

Since u_t is concave and ψ' is monotone increasing, we get $u_{s+1} \leq \lambda_s u_s$ with $\lambda_s = (s+1)/s$ and $\psi'(u_{s+1}) \leq \psi'(\lambda_s u_s)$. It follows from (3.25) and (3.9) that

$$v_{st} a_{st} \leq w_s \psi'(u_{s+1}) \leq w_s \psi'(\lambda_s u_s) \leq C(\lambda_s) w_s a^s \quad \text{for } s < t < s+1. \tag{3.26}$$

Relation (3.23) holds by (3.26), (3.24) and the dominated convergence theorem. □

4. Properties of $SG(u)$

4.1. It is clear from the probabilistic description of k_a that

$$\begin{aligned}\mathrm{SG}(u) &= \{y \in E' : \int_0^\zeta \psi'[u(\xi_s)]\, ds = \infty \quad \Pi_x^y\text{–a.s. for all } x \in E\};\\ \mathrm{RG}(u) &= \{y \in E' : \int_0^\zeta \psi'[u(\xi_s)]\, ds < \infty \quad \Pi_x^y\text{–a.s. for all } x \in E\}.\end{aligned} \tag{4.1}$$

It follows immediately from the definitions of J_u and $\mathrm{SG}(u)$ that:

4.1.A. If $u_1 \le u_2$, then $J_{u_1} \ge J_{u_2}$ and $\mathrm{SG}(u_1) \subset \mathrm{SG}(u_2)$, $\mathrm{RG}(u_1) \supset \mathrm{RG}(u_2)$.

We also have:

4.1.B. If $u \le u_1 + u_2$, then $\mathrm{SG}(u) \subset \mathrm{SG}(u_1) \cup \mathrm{SG}(u_2)$.

Indeed, we can assume that $u_1 \ge u_2$. It follows from **8**.1.A and (3.9) that

$$\psi'(u_1 + u_2) \le \psi'(2u_1) \le c\psi'(u_1) \le c[\psi'(u_1) + \psi'(u_2)]$$

where $c = C(2)$. Therefore every $y \in \mathrm{SG}(u)$ is a point of rapid growth of $\psi'(u_1) + \psi'(u_2)$. Hence, it is a point of rapid growth of $\psi'(u_1)$ or $\psi'(u_2)$.

4.2. We put $\mathrm{SG}(\nu) = \mathrm{SG}(u_\nu), \mathrm{SG}(B) = \mathrm{SG}(u_B)$. The notation $\mathrm{RG}(\nu), \mathrm{RG}(B)$ has a similar meaning.

We have:

4.2.A. If $\nu \in \mathcal{N}_1$ is concentrated on $\mathrm{SG}(u)$, then $u_\nu \le u$.

Indeed, $u_\nu = \pi(u_\nu) \le \pi(u+u_\nu) = u \oplus u_\nu$ because $u_\nu \le u+u_\nu$. Since $J_u(\nu) = 0$, (3.7) implies $u \oplus u_\nu \le u$.

4.2.B. Every $\nu \in \mathcal{N}_1$ is concentrated on $\mathrm{SG}(\infty \cdot \nu)$.

PROOF. Put $u_t = u_{t\nu}$. We need to prove that ν is concentrated on $\mathrm{SG}(u)$ where $u = u_\infty$. For every $t \in \mathbb{R}_+$, $u_t \le u_\infty$ and $J_{u_t} \ge J_u$ by 4.1.A. It follows from (3.8) that $tJ_u(\nu) \le u_t \le u$. By passing to the limit as $t \to \infty$, we conclude that $J_u(\nu) = 0$. □

4.2.C. If $u \in \mathcal{U}_1$, then $\mathrm{SG}(u)$ is w-polar.

PROOF. Let $\tilde{\nu}$ be the restriction of $\nu \in \mathcal{N}_1$ to $\Gamma = \mathrm{SG}(u)$. For every $t \in \mathbb{R}_+$, measure $t\tilde{\nu} \in \mathcal{N}_1(\Gamma)$, and $u_{t\tilde{\nu}} \le u$ by 4.2.A. If $u \in \mathcal{U}_1$, then $u = u_\mu$ for some $\mu \in \mathcal{N}_1$. The inequality $u_{t\tilde{\nu}} \le u_\mu$ implies $t\tilde{\nu} \le \mu$. Hence, $\tilde{\nu} = 0$ and $\nu(\Gamma) = 0$. □

4.2.D. Set $\Lambda = B \cap \mathrm{RG}(B)$ is w-polar for every B.

PROOF. Let $\tilde{\nu}$ be the restriction of $\nu \in \mathcal{N}_1$ to B. The measure $\nu' = \infty \cdot \tilde{\nu}$ belongs to $\mathcal{N}_0(B)$, and the definition of u_B (see **8**.(5.4)) implies that $u_{\nu'} \le u_B$. By 4.1.A, $\mathrm{RG}(u_{\nu'}) \supset \mathrm{RG}(B)$ and therefore $\nu(\Lambda) = \tilde{\nu}(\Lambda) \le \tilde{\nu}[\mathrm{RG}(B)] \le \tilde{\nu}[\mathrm{RG}(u_{\nu'})]$. By 4.2.B, $\tilde{\nu}[\mathrm{RG}(u_{\nu'})] = 0$. Hence, $\nu(\Lambda) = 0$. □

We write $B_1 \sim B_2$ if the symmetric difference $B_1 \Delta B_2$ is w-polar. If this is the case, then $\nu \in \mathcal{N}_1$ is concentrated on B_1 if and only if it is concentrated on B_2 and therefore $u_{B_1} = u_{B_2}$.

4.2.E. If $\Gamma = \mathrm{SG}(u)$, then $u_\Gamma \le u$, $\mathrm{SG}(\Gamma) \subset \Gamma$ and $\Lambda = \Gamma \setminus \mathrm{SG}(\Gamma)$ is w-polar.

PROOF. If $\nu \in \mathcal{N}_1$ is concentrated on Γ, then $u_\nu \leq u$ by 4.2.A. Therefore $u_\Gamma \leq u$. This implies $\mathrm{SG}(\Gamma) = \mathrm{SG}(u_\Gamma) \subset \mathrm{SG}(u) = \Gamma$. Set $\Lambda = \Gamma \cap \mathrm{RG}(\Gamma)$ is w-polar by 4.2.D. $\square$

4.2.F. If $\Gamma = \mathrm{SG}(B)$, then $u_\Gamma = u_B$ and $\mathrm{SG}(\Gamma) = \Gamma$.

Indeed, $u_\Gamma \leq u_B$ by 4.2.E. Note that $B = (B \cap \Gamma) \cup \Lambda'$ where $\Lambda' = B \cap \mathrm{RG}(B)$ is w-polar by 4.2.D. Hence, $u_B = u_{B\cap\Gamma} \leq u_\Gamma$.

4.2.G. For every $u_1, u_2 \in \mathcal{U}$,

$$\mathrm{SG}(u_1 \oplus u_2) = \mathrm{SG}(u_1) \cup \mathrm{SG}(u_2). \tag{4.2}$$

For every Borel B_1, B_2,

$$\mathrm{SG}(B_1 \cup B_2) = \mathrm{SG}(B_1) \cup \mathrm{SG}(B_2). \tag{4.3}$$

PROOF. Since $u_1 \vee u_2 \leq u_1 \oplus u_2 \leq u_1 + u_2$, we get from 4.1.A and 4.1.B that $\mathrm{SG}(u_1) \cup \mathrm{SG}(u_2) \subset \mathrm{SG}(u_1 \oplus u_2) \subset \mathrm{SG}(u_1) \cup \mathrm{SG}(u_2)$ which implies (4.2).

By **8**.(5.5), $u_{B_1\cup B_2} \leq u_{B_1} + u_{B_2}$ and, by the definition of $\oplus$, $u_{B_1\cup B_2} \leq u_{B_1} \oplus u_{B_2}$. Hence, by (4.2), $\mathrm{SG}(B_1 \cup B_2) \subset \mathrm{SG}(B_1) \cup \mathrm{SG}(B_2)$. This implies (4.3) since $\mathrm{SG}(B_1)$ and $\mathrm{SG}(B_2)$ are contained in $\mathrm{SG}(B_1 \cup B_2)$. $\square$

5. Fine topology in E'

5.1. Put $B \in \mathbb{F}_0$ if B is a Borel subset of E' and if $\mathrm{SG}(u_B) \subset B$. Let $B \in \mathbb{F}$ if B is the intersection of a collection of sets of class $\mathbb{F}_0$. It follows from (4.3) that the class $\mathbb{F}_0$ is closed under finite unions . Therefore $\mathbb{F}$ has the same property. Clearly, $\mathbb{F}$ is also closed under intersections. Thus (see, e.g., [**Kur66**], I.5.II) $\mathbb{F}$ is the class of all closed sets for a topology in E'. We call it the *fine topology* or *f-topology*. Elements of $\mathbb{F}$ will be called f-closed sets. For every $B \subset E'$, we denote by B^f the f-closure of B that is the intersection of all f-closed sets $C \supset B$.

Here are some properties of the f-topology.

5.1.A. The set $\mathrm{SG}(u)$ belongs to $\mathbb{F}_0$ for every $u \in \mathcal{U}$.

Indeed, $\mathrm{SG}(u)$ is a Borel set because it consists of $y \in E'$ such that $j_u(x_0, y) = 0$ for a fixed $x_0 \in E$. By 4.2.E, $\mathrm{SG}[\mathrm{SG}(u)] \subset \mathrm{SG}(u)$.

5.1.B. The f-closure B^f of a Borel set B is equal to $B \cup \Gamma$ where $\Gamma = \mathrm{SG}(B)$. Moreover, $B^f \sim \Gamma$.

PROOF. If $B \subset C \in \mathbb{F}_0$, then $\Gamma = \mathrm{SG}(B) \subset \mathrm{SG}(C) \subset C$ and $B \cup \Gamma \subset C$. Hence, $B \cup \Gamma \subset B^f$. On the other hand, $B \cup \Gamma = \Gamma \cup \Lambda$ where $\Lambda = B \cap \mathrm{RG}(B)$ is w-polar by 4.2.D. Hence, $B \cup \Gamma \sim \Gamma, u_{B\cup\Gamma} = u_\Gamma$ and $\mathrm{SG}(B \cup \Gamma) = \mathrm{SG}(\Gamma) \subset \Gamma$ by 4.2.F. We conclude that $B \cup \Gamma \in \mathbb{F}_0$ and therefore $B^f \subset B \cup \Gamma$. $\square$

We define the f-support $\mathrm{Supp}\,\nu$ of ν as the intersection of all sets $B \in \mathbb{F}_0$ such that $\nu(E' \setminus B) = 0$. It is not clear if the $\mathrm{Supp}\,\nu$ is a Borel set. However, for $\nu \in \mathcal{N}_0$, this follows from the next proposition.

5.1.C. For every $\nu \in \mathcal{N}_0$, $\mathrm{Supp}\,\nu = \mathrm{SG}(\infty \cdot \nu) \in \mathbb{F}_0$ and ν is concentrated on $\mathrm{Supp}\,\nu$.

PROOF. Note that $\nu' = \infty \cdot \nu \in \mathcal{N}_0$, that $\operatorname{Supp}\nu = \operatorname{Supp}\nu'$. By 5.1.A, $\Gamma = \mathrm{SG}(\nu') \in \mathbb{F}_0$. If $B \in \mathbb{F}_0$ and if $\nu(E' \setminus B) = 0$, then $u_{\nu'} \le u_B$ by **8**.(5.4) and $\Gamma = \mathrm{SG}(\nu') \subset \mathrm{SG}(B) \subset B$ by 4.1.A. Hence $\Gamma \subset \operatorname{Supp}\nu$. On the other hand, if $\nu_n \uparrow \nu$ and $\nu_n \in \mathcal{N}_1$, then, by 4.2.B, ν_n are concentrated on $\mathrm{SG}(\infty \cdot \nu_n) \subset \mathrm{SG}(\nu') = \Gamma$. Hence ν is concentrated on Γ and $\operatorname{Supp}\nu \subset \Gamma$. □

5.1.D. For every $\nu \in \mathcal{N}_0$, $\mathrm{SG}(\nu) \subset \operatorname{Supp}\nu$

Indeed, by 4.1.A, $\mathrm{SG}(\nu) \subset \mathrm{SG}(\infty \cdot \nu)$ and, by 5.1.C, $SG(\infty \cdot \nu) = \operatorname{Supp}\nu$.

5.1.E. Let $\nu \in \mathcal{N}_0$ and $\Gamma = \mathrm{SG}(\nu)$. If $\operatorname{Supp}\nu \subset \Gamma$, then $u_\nu = u_\Gamma$.

PROOF. If $\mu \in \mathcal{N}_1$ is concentrated on Γ, then $u_\mu \le u_\nu$ by 4.2.A. Hence, $u_\Gamma \le u_\nu$. On the other hand, ν is concentrated on $\operatorname{Supp}\nu$ by 5.1.C. Hence, ν is concentrated on Γ and $u_\nu \le u_\Gamma$ by the definition of u_Γ. □

5.1.F. If $\nu \in \mathcal{N}_0$ is a $(0,\infty)$-measure, then $\mathrm{SG}(\nu) = \operatorname{Supp}\nu$. If, in addition, $u_\nu = u_\Gamma$, then $\operatorname{Supp}\nu = \mathrm{SG}(\Gamma)$.

Since $\infty \cdot \nu = \nu$, the first part follows from 5.1.C. The second part follows from the first one because $\mathrm{SG}(\nu) = \mathrm{SG}(\Gamma)$ if $u_\nu = u_\Gamma$.

5.1.G. If $B_1, B_2 \in \mathbb{F}_0$ and if $u_{B_1} = u_{B_2}$, then $B_1 \sim B_2$.

PROOF. Clearly, $\mathrm{SG}(B_1) = \mathrm{SG}(B_2)$. By 5.1.B, $B_i \sim \mathrm{SG}(B_i)$. Hence $B_1 \sim B_2$. □

6. Auxiliary propositions

6.1. More about operations ⊕ and ∨.

6.1.A. For every $u > \tilde{u}, v > \tilde{v}$ in $\mathcal{U}$,

$$u \oplus v - \tilde{u} \oplus \tilde{v} \le u - \tilde{u} + v - \tilde{v}. \tag{6.1}$$

PROOF. Let D_n be a sequence exhausting E. By Theorem 2.1 and the mean value property, **8**.2.1.D,

$$V_D(u+v) - V_D(u+\tilde{v}) \le V_D(v) - V_D(\tilde{v}) = v - \tilde{v} \quad \text{for all } D \Subset E.$$

By **8**.5.1.D and **8**.5.1.A, this implies $u \oplus v - u \oplus \tilde{v} \le v - \tilde{v}$. Analogously, $u \oplus \tilde{v} - \tilde{u} \oplus \tilde{v} \le u - \tilde{u}$. □

6.1.B. If $u_n, v_n, u, v \in \mathcal{U}$ and if $u_n \uparrow u, v_n \uparrow v$, then $u_n \oplus v_n \uparrow u \oplus v$.

This follows from 6.1.A.

6.1.C. If $\mu, \nu \in \mathcal{N}_0$, then $u_{\mu+\nu} = u_\mu \oplus u_\nu$ and $u_{\mu \vee \nu} = u_\mu \vee u_\nu$.

To prove 6.1.C, we consider $\mu_n, \nu_n \in \mathcal{N}_1$ such that $\mu_n \uparrow \mu, \nu_n \uparrow \nu$. By Lemma **8**.5.1, $u_{\mu_n} \uparrow u_\mu, u_{\nu_n} \uparrow u_\nu$, and 6.1.C follows from **8**.(5.1) and 6.1.B.

6.1.D. For every Borel Γ and every $\nu \in \mathcal{N}_0$,

$$u_\Gamma \oplus u_\nu = u_\Gamma \vee u_\nu.$$

Indeed, by Remark after **8**.5.5.A, $u_\Gamma = u_\mu$ for a $(0,\infty)$-measure μ. For such a measure, $\mu + \nu = \mu \vee \nu$ and $u_\mu \oplus u_\nu = u_{\mu+\nu} = u_{\mu \vee \nu} = u_\mu \vee u_\nu$ by 6.1.C.

6.2. On Σ-finite and σ-finite measures. We say that a measure ν on a measurable space $(S, \mathcal{B}_S)$ is Σ-finite if it can be represented as the sum of a series of finite measures. All measures in $\mathcal{N}_0$ are Σ-finite.

6.2.A. For every Σ-finite measure ν, there exists a finite measure m and a positive function ρ such that $d\nu = \rho dm$.

Proof. Suppose that $\nu = \nu_1 + \dots + \nu_k + \dots$ where ν_k are finite measures. All measures ν_k are absolutely continuous relative to

$$m = \sum_k a_k \nu_k$$

where $a_k = 2^{-k}\nu_k(S)^{-1}$. By the Radon-Nikodym theorem, $d\nu_k = \rho_k dm$ and therefore $d\nu = \rho dm$ where $\rho = \sum_k \rho_k$. □

6.2.B. For every Σ-finite measure ν there exists a partition of S into two disjoint sets S^∞ and S^* such that the restriction ν^∞ of ν to S^∞ is a $(0,\infty)$-measure and the restriction ν^* of ν to S^* is σ-finite. Measures ν^∞ and ν^* are determined uniquely.

Proof. If ρ and m are as in 6.2.A, then the sets $S^\infty = \{\rho = \infty\}$ and $S^* = \{\rho < \infty\}$ satisfy 6.2.B.

Suppose that $\nu = \nu_1 + \nu_2$ where ν_1 is a $(0,\infty)$-measure and ν_2 is σ-finite. Denote by ν_i^∞ and ν_i^* the restriction of ν_i to S^∞ and to S^*. The measures ν_1^* and ν_2^∞ are σ-finite $(0,\infty)$-measures and therefore they are equal to 0. We have

$$\begin{aligned}(6.2)\quad & \nu^* = \nu_1^* + \nu_2^* = \nu_2^*, \quad \nu^\infty = \nu_1^\infty + \nu_2^\infty = \nu_1^\infty; \\ & \nu_1 = \nu_1^\infty + \nu_1^* = \nu_1^\infty, \quad \nu_2 = \nu_2^\infty + \nu_2^* = \nu_2^*\end{aligned}$$

and therefore $\nu_1 = \nu^\infty, \nu_2 = \nu^*$. □

6.2.C. If $\nu_1 \le \nu_2$ are Σ-finite, then there exists a measure γ such that $\nu_2 = \nu_1 + \gamma$.

Proof. By 6.2.A, there exists a finite measure m such that $d\nu_i = \rho_i dm$. Since $\nu_1 \le \nu_2$, we have $\rho_1 \le \rho_2$ m-a.e. Put $\rho = \rho_2 - \rho_1$ on the set $\rho_1 < \infty$ and $\rho = 0$ on its complement. Clearly, $\rho + \rho_1 = \rho_2$ m-a.e. and $\nu_2 = \nu_1 + \gamma$ with $\gamma(B) = \int_B \rho dm$. □

7. Fine trace

7.1. Main results. With every $u \in \mathcal{U}$ we associate

$$\begin{aligned}(7.1)\qquad & \Gamma = \mathrm{SG}(u), \\ & \nu(B) = \sup\{\mu(B) : \mu \in \mathcal{N}_1, \mu(\Gamma) = 0, u_\mu \le u\}.\end{aligned}$$

We call the pair (Γ, ν) the *fine trace of* u and we denote it by $\mathrm{Tr}(u)$. We prove:

Theorem 7.1. *The fine trace of every solution u has the following properties:*

7.1.A. Γ is a Borel f-closed set.

7.1.B. ν is a σ-finite measure of class $\mathcal{N}_0$ such that $\nu(\Gamma) = 0$ and $\mathrm{SG}(u_\nu) \subset \Gamma$.

If $\mathrm{Tr}(u) = (\Gamma, \nu)$, then $u_{\Gamma,\nu} = u_\Gamma \oplus u_\nu$ is the maximal σ-moderate solution dominated by u.

We say that pairs (Γ, ν) and (Γ', ν') are equivalent and we write $(\Gamma, \nu) \sim (\Gamma', \nu')$ if $\nu = \nu'$ and the symmetric difference between Γ and Γ' is w-polar. Clearly, $u_{\Gamma,\nu} = u_{\Gamma',\nu'}$ if $(\Gamma, \nu) \sim (\Gamma', \nu')$.

THEOREM 7.2. *Let (Γ,ν) satisfy conditions 7.1.A-7.1.B. Then the fine trace of $u_{\Gamma,\nu} = u_\Gamma \oplus u_\nu$ is equivalent to (Γ,ν). Moreover, $u_{\Gamma,\nu}$ is the minimal solution with this property and the only one which is σ-moderate.*

7.2. Proof of Theorem 7.1. 1°. Let $\mathrm{Tr}(u) = (\Gamma,\nu)$. Property 7.1.A follows from 5.1.A.

Denote by $\mathcal{L}$ the set of all $\mu \in \mathcal{N}_1$ such that $u_\mu \le u$ and $\mu(\Gamma) = 0$ and apply Theorem **8**.5.1 to $C = \{u_\mu, \mu \in \mathcal{L}\}$. Let $v = \operatorname{Sup} C$. By **8**.5.3.B, there is a sequence $u_n \in C$ such that, for all x, $u_n(x) \uparrow v(x)$ and $v(x) = \sup\{u_\mu(x), \mu \in \mathcal{L}\}$. The sequence $\nu_n = \mathrm{tr}(u_n)$ is increasing and therefore $\nu_n \uparrow \nu' \in \mathcal{N}_0$. By **8**.(5.3), $u_n = u_{\nu_n} \uparrow u_{\nu'}$ and therefore $u_{\nu'} = v \le u$. The condition $\nu_n(\Gamma) = 0$ implies that $\nu'(\Gamma) = 0$.

2°. We claim that, for every B, $\nu'(B)$ is equal to

$$\nu(B) = \sup\{\mu(B) : \mu \in \mathcal{L}\}$$

and therefore ν is a measure of class $\mathcal{N}_0$. The inequality $\nu'(B) \le \nu(B)$ follows from the relation $\nu_n(B) \le \nu(B)$. It remains to prove that

$$\nu(B) \le \nu'(B). \tag{7.2}$$

Let $\mu \in \mathcal{L}$. Consider $\kappa = \mu \vee \nu'$. We have $u_\mu \le u_{\nu'}$ and therefore $u_{\nu'} = u_{\nu'} \vee u_\mu = u_\kappa$ by 6.1.C. Suppose $\nu' \ne \kappa$. By 6.2.C, there exists a measure γ such that $\nu' + \gamma = \kappa$. By **8**.5.4.B, $\gamma \in \mathcal{N}_0$. By 6.1.C, $u_{\nu'} = u_\kappa = u_{\nu'} \oplus u_\gamma$ and therefore $u_{\nu'} = u_{\nu'} \oplus u_{n\gamma} \ge u_{n\gamma}$ for every n. Hence $u_{\infty\cdot\gamma} \le u_{\nu'} \le u$. By 4.1.A, $\mathrm{SG}(\infty\cdot\gamma) \subset \mathrm{SG}(u) = \Gamma$. By 5.1.C, γ is concentrated on $\operatorname{Supp}\gamma = \mathrm{SG}(\infty\cdot\gamma) \subset \Gamma$. Relations $\mu(\Gamma) = \nu'(\Gamma) = 0$ imply that $\kappa(\Gamma) = 0$ and therefore $\gamma(\Gamma) = 0$. We conclude that $\gamma = 0$ and $\nu' = \kappa = \mu \vee \nu'$. Hence, $\nu' \ge \mu$ which implies (7.2).

3°. Let $\nu = \nu^* + \nu^\infty$ be the decomposition described in 6.2.B. By 5.1.C, ν^∞ is concentrated on $\mathrm{SG}(\nu^\infty)$. Since $u_{\nu^\infty} \le u_\nu \le u$, $\mathrm{SG}(\nu^\infty) \subset \Gamma$ by 4.1.A, and ν^∞ is concentrated on Γ. Since $\nu(\Gamma) = 0$, we conclude that $\nu^\infty = 0$ and $\nu = \nu^*$ is σ-finite. This completes the proof of the first part of Theorem 7.1.

4°. By 4.2.E, $u \ge u_\Gamma$ and therefore $u \ge u_\Gamma \vee u_\nu$, which coincides with $u_\Gamma \oplus u_\nu$ by 6.1.D.

5°. Solution u_ν is σ-moderate because $\nu \in \mathcal{N}_0$. Solution u_Γ is also σ-moderate by **8**.(5.4) and **8**.5.3.B. It follows from 6.1.B that $u_{\Gamma,\nu} \in \mathcal{U}_0$.

Let us prove that, if $\tilde{u} \in \mathcal{U}_0$ and if $\mathrm{Tr}(\tilde{u}) = (\Gamma,\nu)$, then $\tilde{u} \le u_{\Gamma,\nu}$. We know that $\tilde{u} = u_\mu$ for some $\mu \in \mathcal{N}_0$. Consider the restrictions μ_1 and μ_2 of μ to Γ and Γ^c. Note that $u_{\mu_1} \le u_\Gamma$ by the definition of u_Γ. Let $\lambda_i \uparrow \mu_2, \lambda_i \in \mathcal{N}_1$. By Lemma **8**.5.1, $u_{\lambda_i} \uparrow u_{\mu_2}$. Since $\lambda_i \in \mathcal{L}$, we have $u_{\lambda_i} \le u_\nu$ and $u_{\mu_2} \le u_\nu$. Therefore $\tilde{u} = u_{\mu_1} \oplus u_{\mu_2} \le u_\Gamma \oplus u_\nu$. □

7.3. Proof of Theorem 7.2. 1°. Let $\mathrm{Tr}(u_{\Gamma,\nu}) = (\Gamma',\nu')$. By 4.2.G, $\Gamma' = \mathrm{SG}(u_\Gamma \oplus u_\nu) = \mathrm{SG}(u_\Gamma) \cup \mathrm{SG}(u_\nu)$ and, by 7.1.A-7.1.B, $\Gamma' \subset \Gamma$. Note that $\Gamma \setminus \Gamma' \subset \Gamma \setminus \mathrm{SG}(u_\Gamma) = \Gamma \cap \mathrm{RG}(\Gamma)$ which is w-polar by 4.2.D. Therefore $\Gamma' \sim \Gamma$. Since $\nu \in \mathcal{N}_0$ does not charge w-polar sets, $\nu(\Gamma') = \nu(\Gamma) = 0$ by 7.1.B.

2°. Since $\nu \in \mathcal{N}_0$, there exist $\nu_n \in \mathcal{N}_1$ such that $\nu_n \uparrow \nu$. We have $u_{\nu_n} \le u_\nu \le u_{\Gamma,\nu}$ and, by (7.1), $\nu_n \le \nu'$. Hence, $\nu \le \nu'$. By 6.2.C, $\nu' = \nu + \gamma$. By Theorem 7.1, $u_{\Gamma,\nu} \ge u_{\Gamma',\nu'}$ (because $\mathrm{Tr}(u_{\Gamma,\nu}) = (\Gamma',\nu')$). Hence,

$$u_{\Gamma,\nu} \ge u_{\nu'} \oplus u_{\Gamma'} = u_{\nu'} \oplus u_\Gamma = u_\nu \oplus u_\gamma \oplus u_\Gamma = u_{\Gamma,\nu} \oplus u_\gamma.$$

We get, by induction, that $u_{\Gamma,\nu} \geq u_{\Gamma,n\gamma}$ for all n. Hence, $u_{\Gamma,\nu} \geq u_{\Gamma,\infty\cdot\gamma} \geq u_{\infty\cdot\gamma}$. By 4.1.A, $\mathrm{SG}(\infty\cdot\gamma) \subset \mathrm{SG}(u_{\Gamma,\nu}) = \Gamma'$. By 5.1.F, a $(0,\infty)$- measure $\infty\cdot\gamma$ is concentrated on $\mathrm{SG}(\infty\cdot\gamma)$. Therefore it is concentrated on Γ' and, since $\gamma \leq \infty\cdot\gamma$, γ does not charge the complement of Γ'. But $\gamma(\Gamma') \leq \nu'(\Gamma') = 0$. Hence $\gamma = 0$, $\nu' = \nu$ and $(\Gamma,\nu) \sim (\Gamma',\nu')$.

3°. Let $\tilde{u}$ be a solution with the fine trace $(\Gamma',\nu') \sim (\Gamma,\nu)$. By Theorem 7.1, $\tilde{u} \geq u_{\Gamma',\nu'} = u_{\Gamma,\nu}$ which implies the minimal property of $u_{\Gamma,\nu}$. If, in addition, $\tilde{u}$ is σ-moderate, then, by Theorem 7.1, $\tilde{u} \leq u_{\Gamma',\nu'} = u_{\Gamma,\nu}$ and therefore $\tilde{u} = u_{\Gamma,\nu}$. □

8. On solutions w_O

8.1. Solutions w_O corresponding to open subsets O of ∂E were used in section **10**.3.5 to define infinite many solutions with the same rough trace. Now we show that these solutions have distinct fine traces. It follows from Theorem 7.2 that this goal will be achieved if we prove that all solutions w_O are σ-moderate. We use as a tool a uniqueness theorem.

8.2. Uniqueness of solutions blowing up at the boundary. It is known that a solution of the problem

$$\begin{aligned} Lu &= \psi(u) \quad \text{in } E,\\ u &= \infty \quad \text{on } \partial E \end{aligned} \tag{8.1}$$

is unique for wide classes of domains E and functions ψ (see [**BM92**], [**MV97**]). We present here a very short proof for bounded star-shaped domains E, the Laplacian $L = \Delta$ and $\psi(u) = u^\alpha$. [The case of a ball and $\alpha = 2$ was considered in [**Isc88**].] Without loss of generality, we can assume that E is star-shaped relative to 0 that is, for every $\lambda > 1$, $E_\lambda = \frac{1}{\lambda}E \subset E$. Note that, if

$$\begin{aligned} \Delta u &= u^\alpha \quad \text{in } E,\\ u &= \infty \quad \text{on } \partial E, \end{aligned} \tag{8.2}$$

then $u_\lambda(x) = \lambda^{2/(\alpha-1)}u(\lambda x)$ satisfies (8.2) in E_λ. Suppose that $\tilde{u}$ is another solution of (8.2). Both u_λ and u satisfy the equation $\Delta u = u^\alpha$ in E_λ and $\tilde{u} < u_\lambda = \infty$ on ∂E_λ. By the Comparison principle **8**.2.1.H, $\tilde{u} \leq u_\lambda$ in E_λ. By taking $\lambda \downarrow 1$, we get that $\tilde{u} \leq u$ in E. Analogously, $u \leq \tilde{u}$.

8.3. Proof that w_O are σ-moderate. Recall that solutions u_B defined by the formula **8**.(5.4) are σ-moderate. Hence, it is sufficient to prove:

THEOREM 8.1. *Suppose that the problem* (8.1) *has a unique solution. Then*

$$w_O = u_O$$

for every open subset O of ∂E.

PROOF. 1°. First, we prove that $u_O \leq w_O$.

By the definition of u_O (see **8**.(5.4)), it is sufficient to show that, for every $\nu \in \mathcal{N}_1(O)$ and every $D \Subset E$, $u_\nu \leq w_O$ in D. We use the following fact established in the first part of the proof of Theorem **10**.3.1: for every Borel set $B \subset \partial E$ and every measure $\mu \in \mathcal{M}_c(E)$, there exists a sequence of compact sets $K_n \subset B$ such that $w_{K_n} \uparrow w_B$ μ-a.e. We apply this fact to $B = O$ and the Lebesgue measure on D. By Theorem **5**.3.2, $v = \lim w_{K_n} \in \mathcal{U}$. Since $v = w_O$ μ-a.e. and both functions are continuous, they coincide on D.

Let $\nu \in \mathcal{N}_1(O)$ and let ν_n be the restriction of ν to K_n. By Lemma **8**.5.2, $u_{\nu_n} = 0$ on $\partial E \setminus K_n$ and, since w_{K_n} is the maximal solution with this property, $u_{\nu_n} \le w_{K_n}$. By Lemma **8**.5.1, $u_{\nu_n} \uparrow u_\nu$. Since $w_{K_n} \uparrow w_O$ in D, we get that $u_\nu \le w_O$ in D.

2°. To prove that $u_O \ge w_O$, it is sufficient to demonstrate that $u_O \ge w_B$ for every closed $B \subset O$.

Consider measure $\nu = \infty \cdot \gamma$ where γ is the surface area on ∂E. It follows from Theorem **5**.5.1 that $u_\nu = \infty$ on ∂E. Since u_ν and $w_{\partial E} \ge u_\nu$ satisfy (8.1), they coincide. The relation $u_\nu = w_{\partial E} \ge w_B$ and properties **8**.4.1.A and **10**.1.3.D of Q_B imply

$$Q_B(u_\nu) \ge Q_B(w_B) = w_B. \tag{8.3}$$

Note that $u_\nu = u_{\nu_1} \oplus u_{\nu_2}$ where ν_1 and ν_2 are the restrictions of ν to O and $\partial E \setminus O$. By **10**.2.1.A, $Q_B(u_{\nu_2}) = 0$ and, by **10**.2.2.B , $Q_B(u_\nu) = Q_B(u_{\nu_1})$. By **8**.4.1.B, $u_{\nu_1} \ge Q_B(u_{\nu_1}) = Q_B(u_\nu)$, and, by (8.3), $u_{\nu_1} \ge w_B$. By **8**.5.5.A, $u_O \ge u_{\nu_1}$ and therefore $u_O \ge w_B$. □

9. Notes

In spring of 1996, in response to a question of Dynkin, Le Gall communicated by e-mail an example of nonuniqueness of solutions with a given (rough) trace. Soon after that Kuznetsov conjectured that this difficulty could be overcame by using a finer topology on the boundary. Such a topology was suggested by Dynkin in [**Dyn97a**]. Its definition included two ingredients:

(a) a set SG(u) of boundary singularities of u described in terms of conditional Brownian motions as in section 1.3;

(b) solutions $w_B(x)$ corresponding to Borel subsets B of the boundary and defined in terms of hitting probabilities of B by the range of the superdiffusion (see section **10**.3.3).

However, when we tried to use this topology, we were not able to make a fundamental step — to prove that SG(w_B) is closed. We fixed this problem by replacing w_B by u_B as defined in section **8**.5.5. (We still do not know if $u_B = w_B$ for all B or not.) This was our motivation for introducing a class of σ-moderate solutions which are determined uniquely by their fine traces. We characterized all pairs (Γ, ν) which are fine traces and we described for each pair (Γ, ν) the minimal solution with the fine trace (Γ, ν) (it is σ-moderate). These results for equation $Lu = u^\alpha$ with $1 < \alpha \le 2$ in a smooth domain E were obtained by Kuznetsov and published in [**Kuz98c**]. In [**DK98a**] we extended them to the general equation $Lu = \psi(u)$ in an arbitrary domain.

The presentation of the fine trace in Chapter **11** is based on [**DK98a**]. In addition, a section is included on traces of solutions w_O.

CHAPTER 12

Martin capacity and classes $\mathcal{N}_1$ and $\mathcal{N}_0$

In this chapter we restrict ourselves to the case of $\psi(u) = u^\alpha$ *with* $\alpha > 1$. The Martin capacity CM_α is one of the Choquet capacities discussed in section **10**.3.2. We prove that a measure ν which charges no null sets of CM_α belongs to the class $\mathcal{N}_1$ if it is finite and it belongs to $\mathcal{N}_0$ if it is Σ-finite. We also prove that $CM_\alpha(B) = 0$ for all w-polar sets B. Clearly, if $\nu \in \mathcal{N}_0$, then $\nu(B) = 0$ for all w-polar B. If the class of null sets of CM_α and the class of w-polar sets coincide, then we get two versions of necessary and sufficient conditions characterizing classes $\mathcal{N}_1$ and $\mathcal{N}_0$. In the next chapter we show that this is true for bounded smooth domains E and $1 < \alpha \leq 2$.

1. Martin capacity

1.1. The Martin capacity is defined on compact subsets B of the Martin boundary $\hat{\partial} E$ by the formula

$$CM_\alpha(B) = \sup\{\nu(B) : \nu \in \mathcal{M}(B), \int_E g(c,x)dx \left[\int_B k(x,y)\nu(dy)\right]^\alpha \leq 1\} \tag{1.1}$$

where k is the Martin kernel, g is Green's function, $\alpha > 1$ and c is the reference point used in the definition **7**.(1.1) of the kernel k.

The capacity (1.1) is a special case of a capacity corresponding to a function $k(x,y)$ from $E \times \tilde{E}$ to $[0,+\infty]$ where E and $\tilde{E}$ are two separable locally compact metrizable spaces and $k(x,y)$ is lower semicontinuous in x and Borelian in y. For every $\alpha > 1$ and every Radon measure m on E, there exists a Choquet capacity given on compact subsets of $\tilde{E}$ by the formula

$$\mathrm{Cap}(B) = \sup\{\nu(B) : \nu \in \mathcal{M}(B), \int_E m(dx)[\int_B k(x,y)\nu(dy)]^\alpha \leq 1\}. \tag{1.2}$$

The existence is proved, for instance, in [**Mey70**] and in [**AH96**], Chapter 2. For every two Borel sets A, B,

$$\mathrm{Cap}(A \cup B) \leq \mathrm{Cap}(A) + \mathrm{Cap}(B). \tag{1.3}$$

[This follows from Proposition 2.3.6 and Theorem 2.5.1 in [**AH96**].]

Formula (1.1) is a particular case of (1.2) when E is a Greenian domain in $\mathbb{R}^d$, $\tilde{E} = \hat{\partial} E$ and

$$m(dx) = g(c,x)\, dx. \tag{1.4}$$

Both (1.2) and (1.1) hold for all Borel sets.

1.2. We consider the linear space $L^\alpha = L^\alpha(E, m)$ where m is defined by (1.4). It consists of functions f such that

$$\|f\|_\alpha^\alpha = \int_E |f(x)|^\alpha \, dm = G(|f|^\alpha)(c) < \infty. \tag{1.5}$$

(Functions which coincide m-a.e. are identified.) Formula (1.1) can be written in a form

$$CM_\alpha(B) = \sup\{\nu(B) : \nu \in \mathcal{M}(B), \|K\nu\|_\alpha \leq 1\} \tag{1.6}$$

where

$$K\nu(x) = \int_B k(x, y)\nu(dy).$$

Since $k(x, y) > 0$ for all $x \in E, y \in \hat{\partial} E$, [1] $CM_\alpha(B) = 0$ if and only if $\|K\nu\|_\alpha = \infty$ for all nontrivial $\nu \in \mathcal{M}(B)$. [2] In particular, a single-point set y_0 is a null set of CM_α if and only if

$$\int_E g(c, x)k(x, y_0)^\alpha \, dx = \infty. \tag{1.7}$$

1.3.

THEOREM 1.1. *Let E be a Greenian domain in $\mathbb{R}^d$. $CM_\alpha(B) = 0$ for all w-polar sets B.*

PROOF. Without any loss of generality we can assume that B is compact. We prove that, if $CM_\alpha(B) > 0$, then there exists a measure $\nu \in \mathcal{N}_1$ such that $\nu(B) > 0$.

If $CM_\alpha(B) > 0$, then there exists a nontrivial measure $\nu \in \mathcal{M}(B)$ such that $G(h^\alpha)(c) < \infty$ where $h = K\nu$. Clearly, $\nu(B) > 0$. Theorem **8**.3.2 implies that $\nu \in \mathcal{N}_1$. Indeed, if K_n, G_n and V_n are operators corresponding to a sequence D_n exhausting E, then $K_n h = h$ (by **6**.(2.7)) and $u_n = V_n(h) \leq h$ (because $u_n + G_n(u_n^\alpha) = K_n h = h$). Functions $F_n^c(y)$ given by **8**.(3.9) are uniformly integrable because they are dominated by an integrable function $g(c, y)h(y)^\alpha$. Therefore $h \in \mathcal{H}_1$ and $\nu \in \mathcal{N}_1$. $\square$

1.4. The main result of this chapter is the following theorem.

THEOREM 1.2. *A measure ν not charging null-sets of capacity CM_α belongs to $\mathcal{N}_1$ if it is finite and it belongs to $\mathcal{N}_0$ if it is Σ-finite.*

The results stated at the beginning of this chapter follow from Theorems 1.1 and 1.2.

To prove Theorem 1.2, we need some preparations.

2. Auxiliary propositions

2.1. Classes $\mathcal{H}^\alpha$ and $\mathcal{N}^\alpha$. Put $\mathcal{H}^\alpha = \mathcal{H} \cap L^\alpha$ and denote by $\mathcal{N}^\alpha$ the set of all finite measures ν on $\hat{\partial} E$ such that $K\nu \in \mathcal{H}^\alpha$. We have:

2.1.A. $\mathcal{H}^\alpha \subset \mathcal{H}_1$.

This follows from Theorem **8**.3.2. Indeed, since $V_n(h) \leq h$, the functions $F_n^c(y)$ defined by **8**.(3.9) are dominated by an integrable function $\varphi(y) = g(c, y)h(y)^\alpha$.

[1] This follows from **6**.1.5.D because $k_y(x) = k(x, y)$ is a harmonic function in E and $k_y(c) = 1$.

[2] We say that ν is nontrivial if $\nu(A) \neq 0$ for some A.

2.1.B. $CM_\alpha(B) = 0$ if and only if $\nu(B) = 0$ for all $\nu \in \mathcal{N}^\alpha$.

Proof. If $\nu \in \mathcal{N}^\alpha$, then its restriction ν_B to B also belongs to $\mathcal{N}^\alpha$. If $CM_\alpha(B) = 0$, then $\nu_B = 0$ because $\|K\nu\|_\alpha < \infty$.

If $CM_\alpha(B) > 0$, then there exists ν, concentrated on B such that $0 < \|K\nu\|_\alpha < \infty$. Clearly, $\nu \in \mathcal{N}^\alpha$ and $\nu(B) > 0$. □

Writing "q.e." (quasi-everywhere) means "everywhere except a set of capacity 0".

2.1.C. Let φ be a Borel function on ∂E. The condition $\langle\varphi, \nu\rangle = 0$ for all $\nu \in \mathcal{N}^\alpha$ is equivalent to the condition $\varphi = 0$ q.e.

Indeed, $\langle\varphi, \nu\rangle = 0$ for all $\nu \in \mathcal{N}^\alpha$ if and only if $\int_B \varphi d\nu = 0$ for all B and all $\nu \in \mathcal{N}^\alpha$ which is equivalent to the condition $\nu\{\varphi \neq 0\} = 0$ for all $\nu \in \mathcal{N}^\alpha$.

2.2. Operator $\hat{K}$ and space $\mathbb{K}$. For every positive Borel function f on E, we set

$$\hat{K}f(y) = \int_E m(dx)f(x)k(x,y), y \in E' \tag{2.1}$$

where m is defined by (1.4).

Put $(f, \tilde{f}) = \int f\tilde{f}\, dm$. If $f \in L_+^{\alpha'}$ [3] where $\alpha' = \alpha/(\alpha - 1)$, then, for every $\nu \in \mathcal{N}^\alpha$, $\langle\hat{K}f, \nu\rangle = (f, K\nu) < \infty$ and therefore $\nu\{\hat{K}f = \infty\} = 0$. By 2.1.C, $\hat{K}f < \infty$ q.e.

For an arbitrary $f \in L^{\alpha'}$, $f_+ = f \vee 0, f_- = (-f) \vee 0$ belong to $L_+^{\alpha'}$. Therefore, q.e., $\hat{K}(f_+)$ and $\hat{K}(f_-)$ are finite and the formula

$$\hat{K}f = \hat{K}(f_+) - \hat{K}(f_-) \tag{2.2}$$

determines $\hat{K}f$ q.e. Note that

$$\langle\hat{K}f, \nu\rangle = (f, K\nu) \quad \text{for all } f \in L^{\alpha'}, \nu \in \mathcal{N}^\alpha. \tag{2.3}$$

Put $f \in \mathbb{L}$ if $f \in L^{\alpha'}$ and $\hat{K}f = 0$ q.e. It follows from (2.3) and 2.1.C that $\mathbb{L} = \{f \in L^{\alpha'} : (f, g) = 0 \text{ for all } g \in \mathcal{H}^\alpha\}$. Therefore $\mathbb{L}$ is a closed subspace of $L^{\alpha'}$ and, since $\mathcal{H}^\alpha$ is a closed subspace of L^α, we have

$$\mathcal{H}^\alpha = \{f \in L^\alpha : (f, g) = 0 \quad \text{for all } g \in \mathbb{L}\}. \tag{2.4}$$

The quotient space $L^{\alpha'}/\mathbb{L}$ is a locally convex linear topological space. We denote by $\mathbb{K}$ its image under the mapping $\hat{K}$ (two functions are identified if they coincide q.e.). We have an 1-1 linear map $\hat{K}$ from $L^{\alpha'}/\mathbb{L}$ onto $\mathbb{K}$. We introduce in $\mathbb{K}$ a topology which makes $\hat{K}$ a homeomorphism. [4] Denote by $\mathbb{K}_+$ the image of $L_+^{\alpha'}$. A linear functional ℓ on $\mathbb{K}$ is called positive if $\ell(\varphi) \geq 0$ for all $\varphi \in \mathbb{K}$.

Lemma 2.1. *Every positive continuous linear functional ℓ on $\mathbb{K}$ has the form*

$$\ell(\varphi) = \langle\varphi, \nu\rangle \tag{2.5}$$

where $\nu \in \mathcal{N}^\alpha$.

---

[3]Writing $f \in L_+^{\alpha'}$ means that $f \in L^{\alpha'}$ and $f \geq 0$ m-a.e.

[4]This topology is defined by the family of subsets of the form $\hat{K}(U)$ where U is an open subset of $L^{\alpha'}/\mathbb{L}$.

PROOF. Formula

$$\tilde{\ell}(f) = \ell(\hat{K}f), \quad f \in L^{\alpha'}$$

defines a positive continuous linear functional on $L^{\alpha'}$. Such a functional has the form $\tilde{\ell}(f) = (h, f)$ where $h \in L^{\alpha}_{+}$. If $f \in \mathbb{L}$, then $(h, f) = \ell(\hat{K}f) = 0$ and, by (2.4), $h \in \mathcal{H}^{\alpha}$. Hence, $h = K\nu$ with ν in $\mathcal{N}^{\alpha}$. If $\varphi = \hat{K}f$ with $f \in L^{\alpha'}$, then

$$\ell(\varphi) = \tilde{\ell}(f) = (f, K\nu) = \langle \hat{K}f, \nu\rangle = \langle \varphi, \nu\rangle$$

by (2.3). □

LEMMA 2.2. *If $f_n \to f$ in $L^{\alpha'}$, then $\hat{K}f_{n_k} \to \hat{K}f$ q.e. for some $n_1 < \dots < n_k < \dots$.*

PROOF. 1°. Suppose $f \in L^{\alpha'}$ and $\hat{K}f \geq 1$ on B. By (2.3) and Hölder's inequality,

$$\nu(B) \leq \langle \hat{K}f, \nu\rangle = (f, K\nu) \leq \|f\|_{\alpha'}\|K\nu\|_{\alpha}$$

for every $\nu \in \mathcal{N}^{\alpha}$. If $\nu \in \mathcal{M}(\hat{\partial}E)$ is not in $\mathcal{N}^{\alpha}$, then this bound for $\nu(B)$ is trivial. Therefore, by (1.6),

$$\begin{aligned} CM_{\alpha}(B) &= \sup\{\nu(B) : \nu(E' \setminus B) = 0, \|K\nu\|_{\alpha} \leq 1\} \\ &\leq \|f\|_{\alpha'} \sup\{\|K\nu\|_{\alpha} : \|K\nu\|_{\alpha} \leq 1\} \leq \|f\|_{\alpha'}. \end{aligned} \tag{2.6}$$

2°. Choose $n_1 < \dots < n_k \dots$ such that $\|f_{n_k} - f\|_{\alpha'} \leq 4^{-k}$. Let $\varphi_k = 2^k|f_{n_k} - f|$ and $B_k = \{\hat{K}(\varphi_k) \geq 1\}$. By (2.6), $CM_{\alpha}(B_k) \leq \|\varphi_k\|_{\alpha'} \leq 2^{-k}$. Put $B^k = B_k \cup B_{k+1} \cup \dots$ and let $A = B^1 \cap B^2 \dots$. It follows from (1.3) and **10**.3.2.B that $CM_{\alpha}(B^k) \leq 2^{-(k-1)}$ and therefore $CM_{\alpha}(A) = 0$. If $x \notin A$, then $\hat{K}(|f_{n_i} - f|) \leq 2^{-i}$ for all sufficiently large i and therefore $\hat{K}f_{n_i} - \hat{K}f \to 0$. □

3. Proof of the main theorem

3.1. Theorem 1.2 will follow if we prove

THEOREM 3.1. *Suppose that ν is a finite measure with the property: $\nu(B) = 0$ if $CM_{\alpha}(B) = 0$. Then there exist measures $\nu_n \in \mathcal{N}^{\alpha}$ such that $\nu_n \uparrow \nu$.*

Indeed, by 2.1.A, $\mathcal{N}^{\alpha} \subset \mathcal{N}_1$. Hence $\nu_n \in \mathcal{N}_1$ and $\nu \in \mathcal{N}_1$ by **8**.5.4.A. If a Σ-finite measure ν does not charge null sets of CM_{α}, then there exist finite measures ν_n with the same property such that $\nu_n \uparrow \nu$. By Theorem 3.1, $\nu_n \in \mathcal{N}_1$. Hence $\nu \in \mathcal{N}_0$.

3.2. Choose a strictly positive function $f_0 \in L^{\alpha'}$ such that $(f_0, K\nu) < \infty$. Put $\varphi_0 = \hat{K}f_0$ and consider a functional

$$p(\varphi) = \int_{E'} \varphi_+ d\nu, \quad \varphi \in \mathbb{K} \tag{3.1}$$

where $\varphi_+ = \varphi \vee 0$. First, we prove Theorem 3.1 by using the following Lemma 3.1. Then we prove Lemma 3.1.

LEMMA 3.1. *For every $\varepsilon > 0$, there exists a continuous linear functional ℓ_{ε} on $\mathbb{K}$ such that*

$$\ell_{\varepsilon}(\varphi) \leq p(\varphi) \quad \textit{for all } \varphi \in \mathbb{K}, \tag{3.2}$$

and

$$\ell_{\varepsilon}(\varphi_0) > p(\varphi_0) - \varepsilon. \tag{3.3}$$

3.3. Proof of Theorem 3.1. It follows from (3.2) and (3.1) that $\ell_\varepsilon(\varphi) \geq 0$ if $\varphi \in \mathbb{K}$ is positive. Indeed, $(-\varphi)_+ = 0$ and therefore $\ell_\varepsilon(-\varphi) \leq p(0) = 0$. By Lemma 2.1, there is a $\eta_\varepsilon \in \mathcal{N}^\alpha$ such that

$$\ell_\varepsilon(\varphi) = \langle \varphi, \eta_\varepsilon \rangle \quad \text{for all } \varphi \in \mathbb{K}. \tag{3.4}$$

We claim that

$$\eta_\varepsilon \leq \nu, \qquad \langle \varphi_0, \nu - \eta_\varepsilon \rangle < \varepsilon. \tag{3.5}$$

Indeed, by (2.3), for every $f \in L_+^{\alpha'}$,

$$(f, K\nu) = \langle \hat{K}f, \nu \rangle = p(\hat{K}f),$$
$$(f, K\eta_\varepsilon) = \langle \hat{K}f, \eta_\varepsilon \rangle = \ell_\varepsilon(\hat{K}f)$$

and therefore, by (3.2), $(f, K\nu) \geq (f, K\eta_\varepsilon)$. Function $h = K\nu - K\eta_\varepsilon$ is harmonic. It is positive because $(f, h) \geq 0$ for all positive $f \in L^{\alpha'}$. Hence, $h = K\gamma_\varepsilon$ for some finite measure γ_ε, and $\nu - \eta_\varepsilon = \gamma_\varepsilon$. This implies the first part of (3.5). The second part follows from (3.3) because $\langle \varphi_0, \nu - \eta_\varepsilon \rangle = p(\varphi_0) - \ell_\varepsilon(\varphi_0)$ by (3.1) and (3.4).

There exists a constant C such that $(a + b)^\alpha \leq C(a^\alpha + b^\alpha)$ for all $a, b \geq 0$. Therefore $\mathcal{H}^\alpha$ and $\mathcal{N}^\alpha$ are closed under addition and

$$\nu^n = \eta_1 + \eta_{\frac{1}{2}} + \cdots + \eta_{\frac{1}{n}} \in \mathcal{N}^\alpha.$$

Since

$$\nu_n = \eta_1 \vee \eta_{\frac{1}{2}} \vee \cdots \vee \eta_{\frac{1}{n}} \leq \nu^n,$$

it also belongs to $\mathcal{N}^\alpha$. By (3.5),

$$\eta_{\frac{1}{n}} \leq \nu_n \leq \nu$$

and

$$\langle \varphi_0, \nu - \nu_n \rangle \leq \langle \varphi_0, \nu - \eta_{\frac{1}{n}} \rangle \leq 1/n.$$

Clearly, $\nu_1 \leq \nu_2 \leq \cdots \leq \nu_n \cdots$ and therefore $\nu_n \uparrow \nu$. □

3.4. Proof of Lemma 3.1. 1°. We use the following version of the Hahn-Banach theorem (see, e.g.,[**DS58**],V.2.12): If B is a closed convex set in a locally convex linear topological space and if $x \notin B$, then there exists a continuous linear functional ℓ such that

$$\sup_{y \in B} \ell(y) < \ell(x) \tag{3.6}$$

(that is, x can be separated from B by a a hyperplane $\ell =$ const.). Suppose that B has the property:

$$\text{If } y \in B, \text{ then } \lambda y \in B \text{ for all } \lambda > 0. \tag{3.7}$$

Then (3.6) implies

$$\sup_{y \in B} \ell(y) = 0 < \ell(x) \tag{3.8}$$

[because $a = 0$ if $\sup_{\lambda > 0} \lambda a = a < \infty$].

2°. This result is applicable to a subset

$$B = \{(\varphi, t) : p(\varphi) \leq t\}$$

of the space $\mathbb{K} \times \mathbb{R}$. Indeed, since p is subadditive and $p(\lambda\varphi) = \lambda p(\varphi)$ for all $\lambda > 0$, the set B is convex and it satisfies (3.7). It remains to prove that B is closed. Suppose that $(\varphi_n, t_n) \in B$ tends to (φ, t) in $\mathbb{K} \times \mathbb{R}$. Then $\varphi_n \to \varphi$ in $\mathbb{K}$ and $t_n \to t$.

By the definition of $\mathbb{K}$ there exist $f_n, f \in L^{\alpha'}$ such that $\varphi_n = \hat{K} f_n, \varphi = \hat{K} f$ and $f_n \to f$ in $L^{\alpha'}$. By Lemma 2.2, a subsequence $\varphi_{n_k} \to \varphi$ q.e. Since ν does not charge null sets of CM_α, $\varphi_{n_k} \to \varphi$ ν-a.e. By Fatou's lemma, $p(\varphi) \leq \liminf p(\varphi_{n_k}) \leq t$. Hence, $(\varphi, t) \in B$.

3°. All continuous linear functionals in $\mathbb{K} \times \mathbb{R}$ have the form

$$\ell(\varphi, t) = \sigma(\varphi) + bt \tag{3.9}$$

where σ is a continuous linear functional on $\mathbb{K}$ and $b \in \mathbb{R}$. Since $x = (\varphi_0, p(\varphi_0) - \varepsilon) \notin B$, conditions (3.8) hold for a functional (3.9) which means

$$\sigma(\varphi) + bt \leq 0 \quad \text{if } p(\varphi) \leq t, \tag{3.10}$$

$$\sigma(\varphi_0) + b[p(\varphi_0) - \varepsilon] > 0. \tag{3.11}$$

Since $p(0) = 0 \leq 1$, condition (3.10) implies that $b \leq 0$. An assumption that $b = 0$ leads to a contradiction: $\sigma(\varphi_0) \leq 0$ by (3.10) and $\sigma(\varphi_0) > 0$ by (3.11). Hence, $b < 0$. Continuous linear functional $\ell(\varphi) = -\sigma(\varphi)/b$ satisfies conditions (3.2)–(3.3). □

4. Notes

4.1. The main Theorem 1.2 was proved in [**DK96b**]. The proof is close to the proof of Theorem 4.1 in [**BP84b**] which the authors attribute to Grune-Rehomme [**GR77**].

By proposition **10**.1.4.C and Theorem 1.1, the class of w-polar sets contains the class of polar sets and is contained in the class of null sets of CM_α. Characterization of classes $\mathcal{N}_1$ and $\mathcal{N}_0$ in terms of this intermediate class is a novelty introduced in the present book.

4.2. Boundary value problems with measures. A number of authors considered boundary value problems of the type

$$\begin{aligned} Lu &= \psi(u) \quad \text{in } E, \\ u &= \nu \quad \text{on } \partial E \end{aligned} \tag{4.1}$$

where E is a smooth domain in $\mathbb{R}^d$ and ν is a finite measure on the boundary. The results of Chapter **12** can be interpreted in terms of an analogous problem in a general setting. Let E be an arbitrary Greenian domain in $\mathbb{R}^d$ and let ν be a finite measure on the Martin boundary $\hat{\partial} E$. By a solution of the boundary value problem

$$\begin{aligned} Lu &= \psi(u) \quad \text{in } E, \\ u &= \nu \quad \text{on } \hat{\partial} E \end{aligned} \tag{4.2}$$

we mean a solution of the integral equation

$$u(x) + \int_E g(x,y)\psi[u(y)]\, dy = \int_{\hat{\partial} E} k(x,y)\nu(dy). \tag{4.3}$$

where $g(x,y)$ is Green's function and $k(x,y)$ is the Martin kernel for L in E (cf. **8**.(3.4)). By Theorem **8**.3.1, the problem (4.2) has a solution if and only if $\nu \in \mathcal{N}_1$ and, by the results of Chapter **12**, the condition "$\nu(B) = 0$ for all w-polar sets B" is necessary and the condition "$\nu(B) = 0$ for all null sets B of CM_α" is sufficient for the existence of such a solution.

CHAPTER 13

Null sets and polar sets

In Chapter **12** we established the following inclusions between three classes of exceptional boundary sets

$$\{polar\ sets\} \subset \{w - polar\ sets\} \subset \{null\ sets\ of\ CM_\alpha\}.$$

Now *we consider the equation*

$$Lu = u^\alpha, \quad 1 < \alpha \le 2$$

*in a bounded smooth domain E and we assume that L satisfies the condition **6**.1.2.C besides **6**.1.2.A–**6**.1.2.B.* We prove that, under these conditions, all three classes coincide.

In our case, the Martin boundary can be identified with ∂E (see Remark **7**.1.1). Formula

$$m(dx) = d(x, \partial E)dx.$$

defines a measure on E which we call the *canonical measure*. We consider capacities Cap defined by the formula **12**.(1.2) with the canonical measure m. We show in section 1 that the class of boundary sets B such that $\mathrm{Cap}(B) = 0$ is the same for a wide variety of kernels $k(x, y)$. One can take the Martin kernel or the Poisson kernel of L. Moreover, in the definition of the Poisson kernel $k(x, y) = \mathcal{D}_{n_y} g(x, y)$ (see section **6**.1.8), the conormals n_y can be replaced by an arbitrary nontangential vector field on the boundary directed inwards; and $g(x, y)$ can be replaced by $g(x, y)q(y)$ where q is a strictly positive differentiable function.

The main result of Chapter **13** is the following theorem:

THEOREM 0.1. *All null sets are polar.*

One of our tools is the straightening of the boundary. To use this tool, we need to investigate the action of diffeomorphisms on the null sets. This is done in section 2. In section 3 we study the cases when there exist no nonempty null sets. In section 4 we demonstrate that Theorem 0.1 can be deduced from its special case Theorem 4.1 and we establish a test of the removability. This test and a dual definition of capacities introduced in section 5 are applied in sections 7 to prove Theorem 0.1. The restriction $\alpha \le 2$ is not used before section 6. [1]

1. Null sets

1.1. Poisson kernel. It follows from Theorems **6**.1.4 and **6**.1.2 that the Poisson kernel $k(x, y)$ is uniquely determined by the condition: for every $\varphi \in C(\partial E)$,

$$h(x) = \int_{\partial E} k(x, y)\varphi(y)\ \gamma(dy) \tag{1.1}$$

[1] It is used in the proof of Theorem 6.1. Probably, Theorem 0.1 is true also for $\alpha > 2$.

is a unique solution of the problem

$$
\begin{aligned}
Lh &= 0 \quad \text{in } E, \\
h &= \varphi \quad \text{on } \partial E.
\end{aligned}
\tag{1.2}
$$

[Here γ is the normalized surface area on ∂E.]

Recall that every positive harmonic function h in E has a unique representation

$$
h(x) = \int_{\partial E} k(x,y)\ \nu(dy) \tag{1.3}
$$

where ν is a finite measure.

We say that two kernels $k(x,y)$ and $\tilde{k}(x,y)$ are equivalent if $\tilde{k}(x,y) = k(x,y)\rho(y)$ where $\rho(y) > 0$. The role of this concept is illuminated by the following lemma:

LEMMA 1.1. *Suppose that $\tilde{k}(\cdot,y)$ is harmonic in E for every $y \in \partial E$. Every positive harmonic function h has a representation*

$$
h(x) = \int_{\partial E} \tilde{k}(x,y)\nu(dy) \tag{1.4}
$$

if and only if $\tilde{k}$ is equivalent to k.

PROOF. The representation (1.4) easily follows from (1.3) if $\tilde{k}(x,y) = k(x,y)\rho(y)$. If (1.4) holds for all positive harmonic functions, then, for every $y \in \partial E$, there exists a measure $\tilde{\nu}_y$ such that

$$
k(x,y) = \int_{\partial E} \tilde{k}(x,z)\nu_y(dz).
$$

Since $k(\cdot,y)$ is an extremal harmonic function, we have $\nu_y = \rho(y)\delta_y$ and $k(x,y) = \tilde{k}(x,y)\rho(y)$. Since $k(x,y) > 0$, ρ is strictly positive. □

It follows from **7**.(1.2) that the Martin kernel is equivalent to the Poisson kernel. If $\tilde{n}_y$ is an arbitrary vector field on ∂E directed inward and if $\tilde{g}(x,y) = g(x,y)q(y)$ with $q > 0$, then the derivative $\tilde{k}(x,y)$ of $\tilde{g}(x,y)$ in the direction of $\tilde{n}_y$ is equivalent to $k(x,y)$.

1.2. Classes $\mathbb{N}(m,k)$. We say that B is an (m,k)-null set and we write $B \in \mathbb{N}(m,k)$ if $\operatorname{Cap}(B) = 0$ where Cap is defined by the formula **12**.(1.2). According to section **12**.1.2, $B \in \mathbb{N}(m,k)$ if and only if

$$
\int_E m(dx) \left[\int_{\partial E} k(x,y)\nu(dy) \right]^\alpha = \infty \tag{1.5}
$$

for every non-trivial $\nu \in \mathcal{M}(B)$.

We have:

1.2.A. If k and $\tilde{k}$ are equivalent, then $\mathbb{N}(m,k) = \mathbb{N}(m,\tilde{k})$.

PROOF. Let Cap and $\widetilde{\operatorname{Cap}}$ be the capacities associated with (k,m) and $(\tilde{k},m)$ and let $\tilde{k}(x,y) = k(x,y)\rho(y)$ with $\rho > 0$. We need to prove that, if $\operatorname{Cap}(B) = 0$, then $\widetilde{\operatorname{Cap}}(B) = 0$. Since $B_n = B \cap \{\rho \geq 1/n\} \uparrow B$, it is sufficient to show that $\widetilde{\operatorname{Cap}}(B_n) = 0$ if $\operatorname{Cap}(B_n) = 0$. Note that $\tilde{K}\nu = K\tilde{\nu}$ where $\tilde{\nu}(dy) = \rho(y)\nu(dy)$. We have $\int_E (\tilde{K}\nu_n)^\alpha\, dm \geq n^{-\alpha} \int_E (K\nu_n)^\alpha\, dm$. Hence, the condition (1.5) holds for $\tilde{k}$ if it holds for k. □

Let m_1 and m_2 be measures on E. We say that m_1 is dominated by m_2 and we write $m_1 \prec m_2$ if $m_1 \le Cm_2$ on the complement of $E^\varepsilon = \{x \in E : d(x, \partial E) > \varepsilon\}$ for some $\varepsilon > 0$ and some C. We have:

1.2.B. If k is given by (1.6) and if $m_1 \prec m_2$, then $\mathbb{N}(m_1, k) \subset \mathbb{N}(m_2, k)$.

Indeed, for every $\nu \in \mathcal{M}(B)$, $K\nu = \int_B k(x,y)\nu(dy)$ is bounded on E^ε. Therefore, the condition (1.5) holds for (m_2, k) if it holds for (m_1, k).

Remark 1.1. Clearly, $\mathbb{N}(m, k_1) \subset \mathbb{N}(m, k_2)$ if k_1/k_2 is bounded. In particular, the bound **6**.1.8.B implies that $\mathbb{N}(m, k_L) \subset \mathbb{N}(m, k)$ for the Poisson kernel k_L of L and

$$k(x,y) = \frac{d(x, \partial E)}{|x-y|^d}. \tag{1.6}$$

Moreover, the bound **6**.(4.1) implies that $\mathbb{N}(m, k_L) = \mathbb{N}(m, k)$ (and therefore $\mathbb{N}(m, k_L)$ does not depend on L).

1.3. Null sets on ∂E. Let L be an elliptic operator in E. We reserve the name *null sets* for the elements of $\mathbb{N}(m, k)$ where m is the canonical measure on E and k is the Poisson kernel of L or any equivalent kernel. [2] Note that the class of null sets contains $\mathbb{N}(m_0, k)$ where

$$m_0(dx) = g(c, x)dx. \tag{1.7}$$

This is an immediate implication of 1.2.B and the following lemma:

Lemma 1.2. *If $\varepsilon > 0$ is sufficiently small, then*

$$g(c,x) \le Cd(x, \partial E) \quad \text{for all } x \in E_\varepsilon = \{x \in E : d(x, \partial E) < \varepsilon\}. \tag{1.8}$$

Proof. If $x \in E_{2\varepsilon}$ and if $\varepsilon < d(c, \partial E)/3$, then $d(c, \partial E) \le d(c, x) + d(x, \partial E) \le d(c, x) + 2d(c, \partial E)/3$. Hence, $d(c, x) \ge d(c, \partial E)/3$. Note that ∂E is a relatively open subset of $\partial E_{2\varepsilon}$. By **6**.1.7.A, $v(x) = g(c, x)$ is harmonic in $E_{2\varepsilon}$, continuous on $\bar{E}_{2\varepsilon}$ and $v = 0$ on ∂E. By Theorem 2.3 in the Appendix B, $v \in C^{2,\lambda}(E_{2\varepsilon} \cup \partial E)$. Hence, all partial derivatives $\mathcal{D}_i v$ are bounded in $\bar{E}_\varepsilon$ which implies (1.8). □

2. Action of diffeomorphisms on null sets

2.1. Change of surface area. Suppose that a smooth surface Γ is given by a parameterization $y = \varphi(t), t \in U$ where U is an open subset of $\mathbb{R}^{d-1}$ and $\varphi \in C^{2,\lambda}(U)$. Formula **6**.(1.14) implies that, if γ is the surface area (or the normalized surface area) on Γ, then, for an arbitrary Borel function $F \ge 0$ on Γ,

$$\int_\Gamma F(y)\gamma(dy) = \int_U F[\varphi(t)]\rho(t)dt \tag{2.1}$$

where ρ is a strictly positive continuous function.

Suppose that Γ is contained in a domain V and let T be a diffeomorphism of class $C^{2,\lambda}$ from V onto $\tilde{V}$. If $y = \varphi(t)$ is a parameterization of Γ, then $\tilde{\varphi}(t) = T[\varphi(t)]$ is a parameterization of a smooth surface $\tilde{\Gamma}$ lying in $\tilde{V}$. We claim that, for every Borel function F on $\tilde{\Gamma}$,

$$\int_{\tilde{\Gamma}} F(\tilde{y})\tilde{\gamma}(d\tilde{y}) = \int_\Gamma F[T(y)]\beta[T(y)]\ \gamma(dy) \tag{2.2}$$

[2] According to Remark 1.1, the class of null sets does not depend on L.

where $\beta(y)$ is a strictly positive continuous function. Indeed, by formula (2.1) applied to $\tilde{\Gamma}$,

$$\int_{\tilde{\Gamma}} F(\tilde{y})\tilde{\gamma}(d\tilde{y}) = \int_U F[\tilde{\varphi}(t)]\tilde{\rho}(t)\, dt$$

where $\tilde{\rho}$ is a strictly positive continuous function. Put

$$\beta(y) = \tilde{\rho}[\tilde{\varphi}^{-1}(y)]/\rho[\tilde{\varphi}^{-1}(y)].$$

Note that $\tilde{\rho}(t) = \rho(t)\beta[\tilde{\varphi}(t)]$ and therefore

$$\int_U F[\tilde{\varphi}(t)]\tilde{\rho}(t)\, dt = \int_U F_1[T(\varphi(t))]\rho(t)\, dt$$

where $F_1(y) = F(y)\beta(y)$. Therefore (2.2) follows from (2.1).

2.2. Transformation of the Poisson kernel.

LEMMA 2.1. *Suppose T is a diffeomorphism from V to $\tilde{V}$ and let E be a smooth domain such that $E \Subset V$. Consider operators L in E and $\tilde{L}$ in $\tilde{E}$ related by the formula* ***6***.(1.4) *and the corresponding Poisson kernels $k(x, y)$ and $\tilde{k}(\tilde{x}, \tilde{y})$. We have*

$$k(x, y) = \tilde{k}(T(x), T(y))\beta(y) \tag{2.3}$$

where β is a continuous strictly positive function.

PROOF. Recall that, for every $\varphi_1 \in C(\partial\tilde{E})$,

$$h_1(\tilde{x}) = \int_{\partial\tilde{E}} \tilde{k}(\tilde{x}, y)\varphi_1(y)\ \tilde{\gamma}(dy) \tag{2.4}$$

is a unique solution of the problem

$$\begin{aligned} \tilde{L}h_1 &= 0 \quad \text{in } \tilde{E}, \\ h_1 &= \varphi_1 \quad \text{on } \partial\tilde{E} \end{aligned} \tag{2.5}$$

[Here $\tilde{\gamma}$ is the normalized surface area on $\partial\tilde{E}$.] If $\varphi(y) = \varphi_1[T(y)]$, then $h_2(x) = h_1[T(x)]$ is a solution of the problem (1.2). Function h defined by (1.1) also satisfies (1.2) and therefore $h = h_2$ which means that

$$\int_{\partial E} k(x, y)\varphi(y)\gamma(dy) = \int_{\partial\tilde{E}} \tilde{k}(T(x), \tilde{y})\varphi_1(\tilde{y})\tilde{\gamma}(d\tilde{y}). \tag{2.6}$$

Now we apply (2.2) to $\Gamma = \partial E$ and $F(\tilde{y}) = \tilde{k}(T(x), \tilde{y})\varphi_1(\tilde{y})$ and we get

$$\int_{\partial\tilde{E}} \tilde{k}(T(x), \tilde{y})\varphi_1(\tilde{y})\tilde{\gamma}(d\tilde{y}) = \int_{\partial E} \tilde{k}(T(x), T(y))\varphi(y)\beta(y)\gamma(dy),$$

and (2.6) implies (2.3). □

2.3. Change of variables. Suppose T is a measurable mapping from E to $\tilde{E}$. The image of $\nu \in \mathcal{M}(E)$ under T is a measure ν_T defined by the formula $\nu_T(B) = \nu[T^{-1}(B)]$. For every Borel function $F \geq 0$ on $\tilde{E}$,

$$\int_{\tilde{E}} F(\tilde{y})\nu_T(d\tilde{y}) = \int_E F[T(y)]\nu(dy). \tag{2.7}$$

Moreover, if E is open and if T is a diffeomorphism of class $C^{2,\lambda}$ from E onto $\tilde{E}$, then

$$\int_{\tilde{E}} F(\tilde{x})\nu_T(d\tilde{x}) = \int_E F[T(x)]|J_T(x)|\, dx \tag{2.8}$$

where $J_T(x)$ is the Jacobian of T [that is the determinant of the matrix $c^i_k = \frac{\partial \tilde{x}_i}{\partial x_k}$].

2.4. Transformation of the canonical measure.

LEMMA 2.2. *Let $E, T, \tilde{E}$ be the same as in Lemma 2.1. If m and $\tilde{m}$ are the canonical measures on E and $\tilde{E}$, then*

$$C^{-1} m_T \le \tilde{m} \le C m_T \tag{2.9}$$

where $C > 0$ is a constant.

PROOF. By (2.7) and (2.8)

$$\int_{\tilde{E}} F(\tilde{x}) m_T(d\tilde{x}) = \int_E F[T(x)] m(dx) = \int_E F[T(x)] d(x, \partial E)\, dx \tag{2.10}$$

and

$$\int_{\tilde{E}} F(\tilde{x}) \tilde{m}(d\tilde{x}) = \int_{\tilde{E}} F(\tilde{x}) d(\tilde{x}, \partial \tilde{E})\, d\tilde{x} = \int_E F[T(x)] d(T(x), \partial \tilde{E}) |J_T(x)|\, dx. \tag{2.11}$$

Since $\bar{E}$ is compact, there exists a constant $C_1 > 0$ such that $C_1^{-1} \le |J_T(x)| \le C_1$ for all $x \in \bar{E}$ and $C_1^{-1} d(x,y) \le d(T(x), T(y)) \le C_1 d(x,y)$ for all $x, y \in \bar{E}$. Therefore formula (2.9) follows from (2.10) and (2.11). □

2.5. Invariance of null sets.

LEMMA 2.3. *Let $E, T, \tilde{E}$ be the same as in Lemmas 2.1 and 2.2. If B is a null set in ∂E, then $T(B)$ is a null set in $\partial \tilde{E}$.*

PROOF. By (2.7) applied to $F(\tilde{y}) = \tilde{k}(T(x), \tilde{y})$ and (2.3),

$$\int_{\partial \tilde{E}} \tilde{k}(Tx, \tilde{y}) \nu_T(d\tilde{y}) = \int_{\partial E} \tilde{k}(T(x), T(y)) \nu(dy) = \int_{\partial E} \beta(y)^{-1} k(x,y) \nu(dy).$$

Therefore, by (2.9) and (2.7),

$$\begin{aligned} \int_{\tilde{E}} \tilde{m}(d\tilde{x}) \left[\int_{\partial \tilde{E}} \tilde{k}(\tilde{x}, \tilde{y}) \nu_T(d\tilde{y}) \right]^\alpha &\ge C \int_{\tilde{E}} m_T(d\tilde{x}) \left[\int_{\partial \tilde{E}} \tilde{k}(\tilde{x}, \tilde{y}) \nu_T(d\tilde{y}) \right]^\alpha \\ &= C \int_E m(dx) \left[\int_{\partial E} \beta(y)^{-1} k(x,y) \nu(dy) \right]^\alpha. \end{aligned}$$

Since β^{-1} is bounded, this implies

$$\int_{\tilde{E}} \tilde{m}(d\tilde{x}) \left[\int_{\partial \tilde{E}} \tilde{k}(\tilde{x}, \tilde{y}) \nu_T(d\tilde{y}) \right]^\alpha \ge C \int_E m(dx) \left[\int_{\partial E} k(x,y) \nu(dy) \right]^\alpha.$$

Since $\nu \to \nu_T$ is a 1-1 mapping from $\mathcal{M}(B)$ onto $\mathcal{M}(T(B))$, we conclude that $T(B)$ is a null set if so is B. □

3. Supercritical and subcritical values of α

3.1. Let $\psi(u) = u^\alpha, \alpha > 1$. We say that a value of α is *supercritical* in dimension d if, for every bounded smooth domain $E \subset \mathbb{R}^d$, all single-point subsets of ∂E are null sets of CM_α. We say that α is *subcritical* if $CM_\alpha(B) > 0$ for all singletons B. [Clearly, this implies that $CM_\alpha(B) > 0$ for every nonempty set B]. We establish that

$$\alpha(d) = \frac{d+1}{d-1}$$

is the supremum of all subcritical α and the infimum of all supercritical α.

3.2. We fix a point c of E and we put

$$Q(x,y_0)=g(c,x)k(x,y_0)^\alpha;\qquad I(y_0)=\int_E Q(x,y_0)\,dx.$$

Note that α is supercritical if and only if

$$I(y_0)=\infty\quad\text{for all } y_0\in\partial E. \tag{3.1}$$

(cf. **12**.(1.7)) and α is subcritical if and only if

$$I(y_0)<\infty\quad\text{for all } y_0\in\partial E. \tag{3.2}$$

We denote by C a *strictly positive* constant. Its value can change from line to line.

3.3.

THEOREM 3.1. *All $\alpha<\alpha(d)$ are subcritical.*

PROOF. Put

$$A=\{x\in E:|x-y_0|<\varepsilon\},\quad B=\{x\in E:|x-y_0|\ge\varepsilon\}.$$

If $x\in A$, then, by (1.8), $g(c,x)\le Cd(x,\partial E)$ and, by **6**.1.8.B,

$$k(x,y_0)\le Cd(x,\partial E)|x-y_0|^{-d}.$$

Hence

$$Q(x,y_0)\le Cd(x,\partial E)^{1+\alpha}|x-y_0|^{-d\alpha}\le C|x-y_0|^{1+\alpha-d\alpha}$$

and $\int_A Q\,dx\le C\int_0^\varepsilon r^{\alpha+d-d\alpha}\,dr$. If $\alpha<\alpha(d)$, then $\alpha+d-d\alpha>-1$ and $\int_A Q\,dx<\infty$.

Since $k(x,y_0)$ is bounded on B,

$$\int_B Q\,dx\le C\int_E g(c,x)\,dx<\infty.$$

Hence the condition (3.2) is satisfied. □

3.4.

THEOREM 3.2. *All $\alpha\ge\alpha(d)$ are supercritical.*

PROOF. 1°. Put

$$E_\varepsilon=\{x\in E:|x-y_0|<\varepsilon\},\qquad O_\varepsilon=\{y\in\partial E:|y-y_0|<\varepsilon\}.$$

By using the straightening the boundary near y_0 (see section **6**.1.3), we reduce the general case to the case when, for sufficiently small ε, $E_\varepsilon\subset\mathbb{R}^d_+=\{x:x_d>0\}$ and $O_{2\varepsilon}\subset\partial\mathbb{R}^d_+=\{x:x_d=0\}$. We can assume that $c\notin E_\varepsilon$. Consider an open subset $U=U_\varepsilon=\{(y,t):y\in\partial E,|y-y_0|<\varepsilon,0<t<2\varepsilon\}$ of the set $\partial E\times\mathbb{R}_+$. Denote by $V=V_\varepsilon$ its image under the mapping $T(y,t)=y+tn_y$ where n_y is the directed inwards unit conormal to ∂E at $y\in\partial E$. If ε is sufficiently small, then $V\subset E$ and T is a diffeomorphism from U onto V. We have,

$$I(y_0)\ge\int_V Q(x)\,dx=\int_U J(y,t)Q[T(y,t)]\,dt\,dy$$

where $Q(x)=Q(x,y_0)$ and J is the Jacobian of T. Since $J\ge C>0$ in U, we get

$$I(y_0)\ge C\int_U Q[T(y,t)]\,dt\,dy. \tag{3.3}$$

2°. We claim that, if ε is sufficiently small, then, for every $(y,t) \in U$,

$$g(c, T(y,t)) \geq Ct; \tag{3.4}$$

$$d(T(y,t), \partial E) \geq Ct; \tag{3.5}$$

$$k(T(y,t), y_0) \geq Ct(|y - y_0| + t)^{-d}. \tag{3.6}$$

First, we prove (3.4). By **6**.1.7.A, the function $h(x) = g(c,x)$ is L^*-harmonic in V, continuous on $\bar{V}$ and equal to 0 on the portion O_ε of ∂V. By Theorem 2.3 in the Appendix B, $h \in C^{2,\lambda}(V \cup O_\varepsilon)$. Put $v(y,t) = h[T(y,t)]$ and let $\dot{v}(t,y) = \frac{\partial v(y,t)}{\partial t}$. Note that $\dot{v}(y,0) = k(c,y)$ is a strictly positive continuous function on $\bar{O}_\varepsilon$. Hence, $\dot{v}(y,0) \geq C > 0$ on O_ε. Since $\dot{v}$ is Hölder continuous, it is bounded from below on U by a strictly positive constant C. This implies (3.4).

For $y \in \bar{O}_\varepsilon$, the coordinates $(z_1, \dots, z_d)$ of n_y are continuous functions of y and $z_d > 0$. Therefore $z_d \geq C > 0$ for all $y \in \bar{O}_\varepsilon$. If ε is sufficiently small, then $d(T(y,t), \partial E) = tz_d \geq Ct$ which implies (3.5).

Since $|n_y| = 1$, we have $|T(y,t) - y_0| \leq |y - y_0| + t$ and the bound (3.6) follows from **6**.(4.1) and (3.5).

3°. By (3.3), (3.4) and (3.6),

$$I(y_0) \geq C \int_{O_\varepsilon} dy \int_0^{2\varepsilon} dt\ t^{\alpha+1}(|y - y_0| + t)^{-d\alpha}.$$

Since $2\varepsilon/|y - y_0| \geq 2$ for $y \in O_\varepsilon$, the change of variables $t = s|y - y_0|$ yields that

$$\int_0^{2\varepsilon} t^{\alpha+1}(|y - y_0| + t)^{-d\alpha}\ dt \geq C|y - y_0|^{\alpha+2-d\alpha}$$

where

$$C = \int_0^2 s^{\alpha+1}(1+s)^{-d\alpha}\ ds > 0.$$

Hence

$$I(y_0) \geq C \int_{O_\varepsilon} |y - y_0|^{\alpha+2-d\alpha}\ dy = C \int_0^\varepsilon r^{\alpha+d-d\alpha}\ dr = \infty$$

for $\alpha \geq \alpha(d)$ which implies (3.1). □

4. Null sets and polar sets

4.1. Theorem **12**.1.1 and the proposition **10**.1.4.C imply that all polar sets are null sets and therefore, to prove Theorem 0.1, we need only to prove that every null set is polar. It is sufficient to prove that every compact null set is removable.

First, we consider a special case which can be treated by a direct computation. Put

$$\mathbb{E} = \{x = (x_1, \dots, x_d) : 0 < x_d < 1\} = \mathbb{R}^{d-1} \times (0,1),$$
$$\partial'\mathbb{E} = \{x : x_d = 0\} = \mathbb{R}^{d-1} \times 0.$$

THEOREM 4.1. *Let $1 < \alpha \leq 2$. Suppose that:*
(a) E is a bounded smooth domain such that $\bar{E} \subset \mathbb{E} \cup \partial'\mathbb{E}$;
(b) B is a compact subset of $\partial'\mathbb{E}$ such that $d(B, \mathbb{E} \setminus E) > 0$;
(c) $k(x,y) = \frac{x_d}{|x-y|^d}$;
(d) $m(dx) = x_d\ dx$.
If B is a (m,k)-null set, then B is removable.

[The function $k(x,y)$ coincides, up to a constant factor, with the Poisson kernel for the Laplacian Δ in the half-space $\{x_d > 0\}$.]

Theorem 0.1 follows from Theorem 4.1 and the arguments presented in the next few sections.

4.2. LC-property of null sets. We prove that a change of ∂E away from $A \subset \partial E$ preserves the family of null sets B that are contained in A. In the next section, we prove an analogous property (we call it *LC-property*) [3] for removable sets.

PROPOSITION 4.1. *Let $\tilde{E} \subset E$ be bounded smooth domains and let B be a compact subset of $\partial E \cap \partial \tilde{E}$ at a strictly positive distance from $E \setminus \tilde{E}$. Then B is a null set of ∂E if and only if it is a null set of $\partial \tilde{E}$.*

PROOF. 1°. Let $C = E \cap \partial \tilde{E}$ and $A = \{x \in \tilde{E} : d(x,B) < d(x,C)\}$. All points of E sufficiently close to B belong to A and, if $x \in A$, then $d(x,\partial E) = d(x,\partial \tilde{E})$. Hence, canonical measures m and $\tilde{m}$ on E and on $\tilde{E}$ coincide on A.

2°. If K and $\tilde{K}$ are the Poisson operators in E and $\tilde{E}$, then, for every $\nu \in \mathcal{M}(B)$, $\tilde{K}\nu \le K\nu$ and, since $K\nu$ is bounded on $E \setminus A$,

$$\int_{\tilde{E}} (\tilde{K}\nu)^\alpha \, d\tilde{m} \le \int_A (K\nu)^\alpha \, dm + C_1 \le \int_E (K\nu)^\alpha \, dm + C_1.$$

It follows from (1.5) that, if B is a null set of $\partial \tilde{E}$, then it is a null set of ∂E.

3°. By **6**.(2.18),

$$K\nu = \tilde{K}\nu + \int_C \tilde{k}(x,z) f(z) \gamma(dz) \tag{4.1}$$

where $f(x) = \int_B k(x,y)\nu(dy)$. By **6**.1.8.B, f is bounded on C and therefore $K\nu \le \tilde{K}\nu + C_2$. Hence,

$$\int_E (K\nu)^\alpha \, dm \le \int_A (K\nu)^\alpha \, dm + C_3 \le \int_{\tilde{E}} (\tilde{K}\nu + C_2)^\alpha \, d\tilde{m} + C_3.$$

By Minkowski's inequality, this implies

$$\left[\int_E (K\nu)^\alpha \, dm\right]^{1/\alpha} \le \left[\int_{\tilde{E}} (\tilde{K}\nu)^\alpha \, d\tilde{m}\right]^{1/\alpha} + C_4$$

Therefore B is a null set of $\partial \tilde{E}$ if it is a null set of ∂E. □

4.3. LC-property of removable sets.

LEMMA 4.1. *Let $f \ge 0$ be a continuous function on ∂E. If $B \subset \partial E$ is removable, then the boundary value problem*

$$\begin{aligned} Lu &= u^\alpha \quad \text{in } E, \\ u &= f \quad \text{on } \partial E \setminus B, \end{aligned} \tag{4.2}$$

cannot have more than one solution

[3] LC stands for "local character".

PROOF. If v is the maximal solution of (4.2) (see Theorem **10**.2.3) and if u is any solution, then $g = v - u \geq 0$ and $g = 0$ on $\partial E \setminus B$. Consider a $[E, B]$-sequence D_n and denote by w_n a solution of the boundary value problem

$$(4.3)\qquad \begin{aligned} Lw_n &= w_n^\alpha \quad \text{in } D_n, \\ w_n &= g \quad \text{on } \partial D_n. \end{aligned}$$

Since $\alpha > 1$, we have $g^\alpha \leq v^\alpha - u^\alpha$ and therefore $Lg = v^\alpha - u^\alpha \geq g^\alpha$ in E. By the comparison principle **8**.2.1.H, $w_n \geq g$. Since $w_{n+1} \geq g = w_n$ on ∂D_n, the comparison principle implies $w_{n+1} \geq w_n$ in D_n. Put $w = \lim w_n$. Clearly, $w \geq g$ and, by Theorems **5**.3.2–3.3,

$$(4.4)\qquad \begin{aligned} Lw &= w^\alpha \quad \text{in } E, \\ w &= g = 0 \quad \text{on } \partial E \setminus B. \end{aligned}$$

By the definition of a removable set, $w = 0$. Therefore $g = 0$ and $v = u$. □

PROPOSITION 4.2. *Let $E, \tilde{E}$ and B be as in Proposition 4.1. Then B is a removable subset of $\partial \tilde{E}$ if and only if it is a removable subset of ∂E.*

PROOF. Consider the maximal solution w_B of the problem **10**.(1.3) and the maximal solution $\tilde{w}_B$ of an analogous problem with E replaced by $\tilde{E}$. We need to prove that $w_B = 0$ if and only if $\tilde{w}_B = 0$.

By Theorem **5**.5.4, $\tilde{w}_B = 0$ if $w_B = 0$. Now suppose that $\tilde{w}_B = 0$. Consider a continuous function f on $\partial \tilde{E}$ which vanishes on $\partial \tilde{E} \cap \partial E$ and is equal to w_B on $\partial \tilde{E} \cap E$. Functions w_B and $V_{\tilde{E}}(f)$ are solutions of the problem

$$(4.5)\qquad \begin{aligned} Lu &= u^\alpha \quad \text{in } \tilde{E}, \\ u &= f \quad \text{on } \partial \tilde{E} \setminus B. \end{aligned}$$

By Lemma 4.1 they coincide. Function $V_{\tilde{E}}(f)$ is bounded in $\tilde{E}$. Hence, w_B is bounded in $\tilde{E}$. Since it vanishes on $\partial E \setminus B$, it is bounded in ∂E and, by the comparison principle, it is moderate. By **10**.1.4.D, $w_B = 0$. □

REMARK. Proofs in section 4.3 are applicable not only to $\psi(u) = u^\alpha$ but also to all functions ψ considered in Chapters **8** and **10**.

4.4. Reduction to the special case. We use a straightening of the boundary described in section **6**.1.3. Clearly, every compact subset of a null set is a null set and, by **12**.(1.3), the union of two null sets is a null set. Similar properties of removable sets follow from **10**.1.3.A and **10**.1.3.E. Therefore Theorem 0.1 holds for $B_1 \cup \cdots \cup B_n$ if it holds for each of these sets.

Suppose that B is a null set on ∂E. Without any loss of generality we can assume that there exists a diffeomorphism T of class $C^{2,\lambda}$ from a ball $U \supset B$ onto a domain $U' \subset \mathbb{R}^d$ such that $T(U \cap E) \subset \mathbb{E}$ and $T(U \cap \partial E) \subset \partial' \mathbb{E}$.

Let W be a smooth subdomain of $V = U \cap E$ such that $B \subset \partial W$ and B is at a positive distance from $V \setminus W$. The images B', V' and W' of B, V, W have analogous properties. By Proposition 4.1, B is a null set on ∂V and, by Lemma 2.3, B' is a null subset of $\partial V'$ and therefore it is also a null subset of $\partial \mathbb{E}$. By Remark 1.1, B' is an (m, k)-null set for m and k defined in Theorem 4.1. If Theorem 4.1 is true, then B' is a removable set on $\partial \mathbb{E}$. By Proposition 4.2, B' is a removable subset of $\partial V'$. It follows easily from the definition of removable sets that this class is invariant under diffeomorphisms and therefore B is a removable subset of ∂V. By applying once more Proposition 4.2, we conclude that B is a removable subset of ∂E.

4.5. Test of removability. To prove Theorem 4.1 we use the following test.

PROPOSITION 4.3. *Suppose E, B, m, k satisfy conditions (a)—(d) of Theorem 4.1. A set $B \subset \partial E$ is removable if there exists an open subset U of E such that $d(B, E \setminus U) > 0$ and*

$$\int_U u(x)^\alpha m(dx) < \infty \tag{4.6}$$

for every solution u of the problem

$$\begin{aligned} Lu &= u^\alpha \quad \text{in } \mathbb{E}, \\ u &= 0 \quad \text{on } \partial'\mathbb{E} \setminus B. \end{aligned} \tag{4.7}$$

PROOF. By **10**.1.4.D, it is sufficient to demonstrate that the maximal solution w_B of the problem (4.2) with $f = 0$ is moderate. By Theorem **5**.5.4, w_B is dominated by the maximal solution u of the problem (4.7) and therefore, by (4.6),

$$\int_U w_B^\alpha dm < \infty. \tag{4.8}$$

Choose $\varepsilon > 0$ such that $U_\varepsilon = \{x \in E : d(x, B) < \varepsilon\} \subset U$ and the bound (1.8) holds in U_ε. Then $g(c, x)$ is bounded in U_ε and, by (4.8),

$$\int_{U_\varepsilon} g(c, x) w_B(x)^\alpha \, dx < \infty.$$

Since w_B is finite and continuous in $\bar{E} \setminus U_\varepsilon$, it is bounded in $E \setminus U_\varepsilon$. Hence,

$$G(w_B^\alpha)(c) \le \int_{U_\varepsilon} g(c, x) w_B(x)^\alpha \, dx + \text{const.} \int_{E \setminus U_\varepsilon} g(c, x) \, dx < \infty$$

and the function $h = w_B + G(w_B^\alpha)$ is finite at point c. By Theorem **8**.3.1, $h = j(w_B)$ is the limit of harmonic functions. Since $h(c) < \infty$, h is harmonic by **6**.1.5.C. Hence, $w_B \le h$ is moderate. □

To prove Theorem 4.1, it is sufficient to construct, for every (m, k)-null set B subject to the condition (b), a set U with properties described in Proposition 4.3. In sections 5 and 6, we prepare necessary tools: dual definitions of capacities and truncating sequences.

5. Dual definitions of capacities

5.1. A capacity corresponding to (m, k) was defined by the formula **12**.(1.2) which is equivalent to the formula

$$\text{Cap}(B) = \sup_{\nu \in \mathcal{M}(B)} \{\nu(B) : \|K\nu\|_\alpha \le 1\} \tag{5.1}$$

where

$$\|f\|_\alpha = \left(\int_E |f|^\alpha \, dm \right)^{1/\alpha}$$

(cf. **12**.(1.6)). Put

$$\text{Cap}'(B) = \inf_{f \in L_+} \{\|f\|_{\alpha'} : \hat{K} f \ge 1 \quad \text{on } B\} \tag{5.2}$$

where $\alpha' = \alpha/(\alpha-1)$, L_+ is the set of all positive Borel functions f on E such that $\|f\|_{\alpha'} < \infty$, and

$$\hat{K}f(y) = \int_E f(x)m(dx)k(x,y). \tag{5.3}$$

Our goal is to prove

THEOREM 5.1. $\mathrm{Cap}(B) = \mathrm{Cap}'(B)$.

The proof is based on von Neumann's minimax theorem which claims that, under certain conditions on $\mathcal{E}, \Phi$ and $\mathcal{M}$, [4]

$$\inf_{\mu\in\mathcal{M}} \sup_{\varphi\in\Phi} \mathcal{E}(\varphi,\mu) = \sup_{\varphi\in\Phi} \inf_{\mu\in\mathcal{M}} \mathcal{E}(\varphi,\mu). \tag{5.4}$$

We apply (5.4) to

$$\mathcal{E}(\varphi,\mu) = \int_E m(dx)\varphi(x) \int_B k(x,y)\mu(dy), \quad \varphi \in \Phi, \mu \in \mathcal{M} \tag{5.5}$$

where Φ is the set of all $\varphi \in L_+$ with $\|\varphi\|_{\alpha'} \le 1$ and $\mathcal{M}$ is the set of all probability measures on B. Put

$$\begin{aligned} I(B) &= \inf_{\mu\in\mathcal{M}} \sup_{\varphi\in\Phi} \mathcal{E}(\varphi,\mu), \\ J(B) &= \sup_{\varphi\in\Phi} \inf_{\mu\in\mathcal{M}} \mathcal{E}(\varphi,\mu). \end{aligned} \tag{5.6}$$

5.2. First, we prove that

$$\mathrm{Cap}(B) = 1/I(B). \tag{5.7}$$

Put $(\varphi_1,\varphi_2) = \int_E \varphi_1\varphi_2\, dm$. For every $\psi \ge 0$,

$$\sup_{\varphi\in\Phi}(\varphi,\psi) = \|\psi\|_\alpha.$$

Indeed $(\varphi_0,\psi) = \|\psi\|_\alpha$ for $\varphi_0 = [\psi/\|\psi\|_\alpha]^{\alpha-1}$ which belongs to Φ and, by Hölder's inequality, $(\varphi,\psi) \le \|\varphi\|_{\alpha'}\|\psi\|_\alpha = \|\psi\|_\alpha$ for every $\varphi \in \Phi$.

Since $\mathcal{E}(\varphi,\mu) = (\varphi, K\mu)$, we have $\sup_{\varphi\in\Phi} \mathcal{E}(\varphi,\mu) = \|K\mu\|_\alpha$ and

$$I(B) = \inf_{\mu\in\mathcal{M}} \|K\mu\|_\alpha.$$

Every $\nu \in \mathcal{M}(B)$ is equal to $\lambda\mu$ where $\lambda \ge 0$ and $\mu \in \mathcal{M}$. Note that $\sup_{\lambda\ge 0}\{\lambda : \lambda\|K\mu\|_\alpha \le 1\} = 1/\|K\mu\|_\alpha$ and therefore

$$\mathrm{Cap}(B) = \sup_{\mu\in\mathcal{M}} \sup_{\lambda\ge 0}\{\lambda : \lambda\|K\mu\|_\alpha \le 1\} = 1/I(B).$$

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[4] At the end of the section we check that these conditions are satisfied in our case.

5.3. Next, we prove that

$$\mathrm{Cap}'(B) = 1/J(B). \tag{5.8}$$

We have $\mathcal{E}(\varphi, \mu) = \langle \hat{K}\varphi, \mu\rangle$. Note that

$$\inf_{\mu \in \mathcal{M}} \langle \hat{K}\varphi, \mu\rangle = \inf_{x \in B} \hat{K}\varphi(x).$$

Therefore

$$J(B) = \sup_{\varphi \in \Phi} \quad \inf_{x \in B} \hat{K}\varphi(x)$$

and

$$1/J(B) = \inf_{\varphi \in \Phi} \quad \sup_{x \in B} 1/\hat{K}\varphi(x).$$

Every $f \in L_+$ can be represented as $\lambda\varphi$ where $\lambda \geq 0$ and $\varphi \in \Phi$. We have

$$\mathrm{Cap}'(B) = \inf_{\varphi \in \Phi} \quad \inf_{\lambda \geq 0} \{\lambda : \lambda \hat{K}\varphi \geq 1 \quad \text{on } B\} = \inf_{\varphi \in \Phi} \quad \sup_{x \in B} 1/\hat{K}\varphi(x) = 1/J(B).$$

Theorem 5.1 follows from (5.4), (5.7) and (5.8).

5.4. The following conditions are sufficient for the formula (5.4) to be true: $\mathcal{M}$ and Φ are convex sets, $\mathcal{M}$ is compact, $\mathcal{E}(\varphi, \mu)$ is an affine function in μ and in φ and it is lower semicontinuous in μ. [5] Recall that a subset A of a linear space is called convex if $pa_1 + qa_2 \in A$ for $a_1, a_2 \in A$, $p, q \geq 0$ and $p + q = 1$. A function $f : A \to [0, \infty)$ is called affine if $f[pa_1 + qa_2] = pf(a_1) + qf(a_2)$ for a_1, a_2, p, q as above. Clearly, the function (5.5) is affine. The set $\mathcal{M}$ of probability measures on a compact set B is compact relative to the weak convergence of measures. It remains to verify that $\mathcal{E}(\varphi, \mu)$ is lower semicontinuous in μ. It follows from Fatou's lemma that, for every $\varphi \in \Phi$, $\hat{K}\varphi$ is a lower semicontinuous function on B. There exists a sequence of continuous functions φ_n such that $\varphi_n(x) \uparrow \hat{K}\varphi(x)$ for all $x \in B$ (see, e.g., [**Rud87**], Chapter 2, Exercise 22). Functions $F_n(\mu) = \langle \varphi_n, \mu\rangle$ are continuous. Since $F_n(\mu) \uparrow \mathcal{E}(\varphi, \mu)$, $\mathcal{E}(\varphi, \mu)$ is lower semicontinuous in μ.

6. Truncating sequences

6.1. Suppose α, E, B, m, k satisfy conditions of Theorem 4.1 and let B be a (m, k)-null set. Our goal is to construct a sequence of functions φ_n, equal to 1 in a neighborhood of B, which tends to 0 in a special manner determined by the value of α. [6]

Denote by $\|f\|_\alpha$ the norm of f in the space $L^\alpha(\mathbb{E}, m)$ where m is the measure described in Theorem 4.1. For every $f \in C^2(\mathbb{E})$, we put

$$\|f\|_{2,\alpha} = \|f\|_\alpha + \sum_{i=1}^{d} \|\mathcal{D}_i f\|_\alpha + \sum_{i,j=1}^{d} \|\mathcal{D}_{ij} f\|_\alpha. \tag{6.1}$$

[This is the norm of f in a weighted Sobolev space.]

[5] Proof of the minimax theorem under weaker conditions can be found in [**AH96**], section 2.4.

[6] Namely, φ_n has to satisfy the conditions 6.7.A-6.7.G with α replaced by α'. The restriction $\alpha \leq 2$ (equivalent to $\alpha' \geq 2$) is needed to check 6.7.D which is deduced from the part 6.2.D of Theorem 6.1. The property 6.2.D is established in the step 8°of the proof. This is the only place where the restriction on α is used directly.

6.2. Operator $\mathcal{J}$. It is convenient to consider elements of $\mathbb{E}$ as pairs (x, r) where $x \in \mathbb{R}^{d-1}$ and $0 < r < 1$. The kernel k introduced in Theorem 4.1 can be represented by the formula

$$k((x, r), y) = q^r(x - y)$$

where

$$q^r(x) = \frac{r}{(\|x\|^2 + r^2)^{d/2}}. \tag{6.2}$$

We use notation $f(x, r)$ and $f^r(x)$ for functions on $\mathbb{E}$.

A special role belongs to an operator acting on functions in $\mathbb{E}$ by the formula

$$(\mathcal{J}f)^t(y) = \int_0^1 r\, dr A(\sqrt{r/t}) \int_{\mathbb{R}^{d-1}} f^r(x) q^r(y - x)\, dx \tag{6.3}$$

where A is an increasing function of class $C^2(\mathbb{R}_+)$ such that $A(s) = 0$ for $s \leq 1$ and $A(s) = 1$ for $s \geq \sqrt{2}$.

Theorem 6.1. *If $\varphi = \mathcal{J}(f)$ and if $f \geq 0$ belongs to L^α, then $\varphi \in C^2(\mathbb{E})$ and*

6.2.A. $\|\varphi\|_{2,\alpha} \leq C\|f\|_\alpha$.

6.2.B. $\|\frac{1}{x_d}\mathcal{D}_d\varphi\|_\alpha \leq C\|f\|_\alpha$.

6.2.C. $\mathcal{D}_d\varphi \leq 0$.

6.2.D. If $\alpha \geq 2$ and if $\|f\|_\alpha < \infty$, then

$$\int_{R^{d-1}} |\mathcal{D}_d\varphi(y, x_d)|^\alpha\, dy \to 0 \quad \text{as } x_d \downarrow 0$$

and, for almost all $y \in \mathbb{R}^{d-1}$,

$$\mathcal{D}_d\varphi(y, x_d) \to 0 \quad \text{as } x_d \downarrow 0.$$

To prove Theorem 6.1, we need some preparations.

6.3. Note that

$$\|f\|_\alpha^\alpha = \int_0^1 \ell_\alpha(f^r) r\, dr \tag{6.4}$$

where

$$\ell_\alpha(f^r) = \int_{\mathbb{R}^{d-1}} |f^r(x)|^\alpha\, dx. \tag{6.5}$$

Suppose that for some j,

$$f^r(x) = r^{j-d} f^1(x/r) \quad \text{for all } x \in \mathbb{R}^{d-1}, r \in \mathbb{R}_+. \tag{6.6}$$

By a change of variables in the integral (6.5), we get

$$r^{1-j}\ell_1(f^r) = \ell_1(f^1). \tag{6.7}$$

The condition (6.6) holds, with $j = 1, 0$ and -1, for functions $q^r, \mathcal{D}_i q^r$ and $\mathcal{D}_{ij} q^r$. Therefore

6.3.A. $\ell_1(q^r), r\ell_1(\mathcal{D}_i q^r), r^2\ell_1(\mathcal{D}_{ij} q^r)$, $i, j = 1, \ldots, d-1$ do not depend on r.

6.4. The convolution $\psi^r = f^r * g^r$ of f^r and g^r is defined by the formula

$$\psi^r(y) = \int_{\mathbb{R}^{d-1}} f^r(x) g^r(y-x)\, dx. \tag{6.8}$$

LEMMA 6.1. *If $\psi^r = f^r * g^r$ and if $\ell_1(g^r)$ does not depend on r, then*

$$\|\psi\|_\alpha \le \ell_1(g^1)\|f\|_\alpha. \tag{6.9}$$

PROOF. Note that

$$|\psi^r(y)| \le \int \varphi^r(x,y)\, dx$$

where $\varphi^r(x,y) = |f^r(y-x)g^r(x)|$. Therefore

$$\ell_\alpha(\psi^r)^{1/\alpha} \le \left[\int [\int \varphi^r(x,y)\, dx]^\alpha\, dy\right]^{1/\alpha}. \tag{6.10}$$

To get a bound for the right side, we use Minkowski's inequality

$$\left[\int [\int \varphi(x,y)\, dx]^\alpha\, dy\right]^{1/\alpha} \le \int [\int \varphi(x,y)^\alpha\, dy]^{1/\alpha}\, dx \tag{6.11}$$

for every $\alpha > 1$ and every $\varphi \ge 0$. (See, e.g, [**HLP34**], section 202 or [**Rud87**], Chapter 8, Exercise 16.) It follows from (6.10) and (6.11) that

$$\ell_\alpha(\psi^r)^{1/\alpha} \le \int \left[\int \varphi^r(x,y)^\alpha\, dy\right]^{1/\alpha} dx = \int \ell_\alpha(f^r)^{1/\alpha}|g^r(x)|\, dx = \ell_\alpha(f^r)^{1/\alpha}\ell_1(g^r).$$

Therefore (6.9) follows from (6.4). □

We denote by C a constant which depends only on α. Its value may change from line to line.

LEMMA 6.2. *Suppose $\beta+\gamma+1 \ge 0, \beta+2 > 0$ and*

$$|F^t|^\alpha \le C t^\beta \int_t^1 r^\gamma |g^r|^\alpha\, dr. \tag{6.12}$$

Then

$$\|F\|_\alpha^\alpha \le C\|g\|_\alpha^\alpha. \tag{6.13}$$

PROOF. By (6.12),

$$\ell_\alpha(F^t) \le C t^\beta \int_t^1 r^\gamma \ell_\alpha(g^r)\, dr$$

and, by (6.4),

$$\begin{aligned} \|F\|_\alpha^\alpha \le C\int_0^1 t^{1+\beta}\, dt \int_t^1 r^\gamma \ell_\alpha(g^r)\, dr &= C\int_0^1 dr\, r^\gamma \ell_\alpha(g^r) \int_0^r t^{1+\beta}\, dt \\ &= \frac{C}{2+\beta}\int_0^1 r^{2+\beta+\gamma}\ell_\alpha(g^r)\, dr. \end{aligned} \tag{6.14}$$

Since $\beta+\gamma+1 \ge 0$, this implies (6.13). □

6.5. Proof of Theorem 6.1.

PROOF. 1°. Put

$$\varphi_i = D_i\varphi, i = 1, \dots, d-1, \quad \varphi_{ij} = \mathcal{D}_{ij}\varphi, i, j = 1, \dots, d-1$$

and

$$\psi^t = f^t * q^t, \quad \psi_i^t = t\mathcal{D}_i\psi^t = f^t * (t\mathcal{D}_i q^t), \quad \psi_{ij}^t = t^2\mathcal{D}_{ij}\psi^t = f^t * (t^2\mathcal{D}_{ij}q^t).$$

By 6.3.A and Lemma 6.1,

$$\|\psi\|_\alpha, \|\psi_i\|_\alpha, \|\psi_{ij}\|_\alpha \le C\|f\|_\alpha. \tag{6.15}$$

2°. If $\varphi = \mathcal{J}f$ and $\psi^r = f^r * q^r$, then, by (6.3),

$$\varphi^t = \int_t^1 r\ dr A(\sqrt{r/t})\psi^r. \tag{6.16}$$

Since A is bounded,

$$|\varphi^t| \le C\int_t^1 r\ dr|\psi^r|. \tag{6.17}$$

By Hölder's inequality,

$$\left|\int_t^1 r\ dr\psi^r\right| \le \left(\int_t^1 r\ dr\right)^{1/\alpha'}\left(\int_t^1 |\psi^r|^\alpha r\ dr\right)^{1/\alpha} \le \left(\int_t^1 |\psi^r|^\alpha r\ dr\right)^{1/\alpha}$$

and therefore

$$|\varphi^t|^\alpha \le C\int_t^1 r\ dr|\psi^r|^\alpha.$$

Hence, (6.12) holds for $\beta = 0, \gamma = 1$ and $g = \psi$, and, by Lemma 6.2 and (6.15),

$$\|\varphi\|_\alpha \le C\|f\|_\alpha. \tag{6.18}$$

3°. Note that

$$\varphi_i^t = \int_t^1 \ dr A(\sqrt{r/t})\psi_i^r$$

and

$$|\varphi_i^t| \le C\int_t^1 \ dr|\psi_i^r|.$$

By Hölder's inequality,

$$|\varphi_i^t|^\alpha \le C\int_t^1 \ dr|\psi_i^r|^\alpha.$$

Hence, (6.12) holds for $\beta = \gamma = 0$ and $g = \psi_i$. By Lemma 6.2 and (6.15),

$$\|\varphi_i\|_\alpha \le C\|f\|_\alpha. \tag{6.19}$$

4°. For every $\varepsilon > 0$,

$$|\varphi_{ij}^t| \le C\int_t^1 r^{-1-\varepsilon}\ dr|r^\varepsilon\psi_{ij}^r|.$$

By Hölder's inequality,

$$|\varphi_{ij}^t|^\alpha \le C\int_t^1 \ dr\ r^{-1-\varepsilon}|r^\varepsilon\psi_{ij}^r|^\alpha\left(\int_t^1 r^{-1-\varepsilon}\ dr\right)^{\alpha-1}.$$

The second factor is not larger than $(\varepsilon t^{\varepsilon})^{1-\alpha}$. By taking $\varepsilon = 1/(\alpha-1)$, we get

$$|\varphi_{ij}^t|^\alpha \le Ct^{-1}\int_t^1 dr|\psi_{ij}^r|^\alpha.$$

The condition (6.12) holds for $\beta = -1, \gamma = 0$ and $g = \psi_{ij}$. By Lemma 6.2 and (6.15),

$$\|\varphi_{ij}\|_\alpha \le C\|f\|_\alpha. \tag{6.20}$$

5°. Put $F^t = \frac{1}{t}\frac{\partial \varphi^t}{\partial t}$. Since $A(1) = 0$ and $A'(s) = 0$ for $s \ge \sqrt{2}$, we have

$$\begin{aligned}\frac{\partial \varphi^t}{\partial t} &= -\frac{1}{2}\int_t^1 A'(\sqrt{r/t})r^{3/2}t^{-3/2}\psi^r\ dr \\ &= -\frac{1}{2}\int_t^{1\wedge(2t)} A'(\sqrt{r/t})r^{3/2}t^{-3/2}\psi^r\ dr.\end{aligned} \tag{6.21}$$

Since $1 \le r/t \le 2$ for $t \le r \le 2t$, we have $t^{-3/2}r^{3/2} \le 2t^{-1/2}r^{1/2}$ and

$$|F^t| \le C\int_t^{1\wedge(2t)} t^{-3/2}r^{1/2}|\psi^r|\ dr.$$

By Hölder's inequality,

$$|F^t|^\alpha \le C\int_t^{1\wedge(2t)} t^{-3/2}r^{1/2}|\psi^r|^\alpha\ dr \tag{6.22}$$

[because $\int_t^{1\wedge(2t)} t^{-3/2}r^{1/2}\ dr \le 2$]. The condition (6.12) holds for $\beta = -3/2, \gamma = 1/2$ and $g = \psi$. By Lemma 6.2 and (6.15),

$$\left\|\frac{1}{t}\frac{\partial \varphi^t}{\partial t}\right\|_\alpha \le C\|f\|_\alpha. \tag{6.23}$$

6°. By (6.21),

$$\frac{\partial \varphi^t}{\partial t} = -\int_t^1 B(\sqrt{r/t})\psi^r\ dr$$

where $B(s) = \frac{1}{2}s^3 A'(s)$. Put $\Phi^t = \frac{\partial^2 \varphi^t}{(\partial t)^2}$. Since $B(1) = 0$ and $B'(s) = 0$ for $s \ge \sqrt{2}$,

$$\Phi^t = \frac{1}{2}\int_t^1 B'(\sqrt{r/t})r^{1/2}t^{-3/2}\psi^r\ dr = \frac{1}{2}t^{-3/2}\int_t^{1\wedge(2t)} B'(\sqrt{r/t})r^{1/2}\psi^r\ dr.$$

Hence

$$|\Phi^t| \le Ct^{-3/2}\int_t^{1\wedge(2t)} r^{1/2}|\psi^r|\ dr.$$

By applying Hölder's inequality, we get that Φ^t satisfies the condition (6.12) with $\beta = -3/2, \gamma = 1/2$ and $g = \psi$. By Lemma 6.2 and (6.15),

$$\left\|\frac{\partial^2 \varphi^t}{(\partial t)^2}\right\|_\alpha \le C\|f\|_\alpha. \tag{6.24}$$

7°.

Since $\mathcal{D}_i\psi^r = \psi_i^r/r$, (6.21) implies

$$\mathcal{D}_i\frac{\partial \varphi^t}{\partial t} = -\frac{1}{2}\int_t^{1\wedge(2t)} A'(\sqrt{r/t})r^{1/2}t^{-3/2}\psi_i^r\ dr.$$

Therefore the condition (6.12) with $\beta = -3/2, \gamma = 1/2$, holds for $\mathcal{D}_i \frac{\partial \varphi^t}{\partial t}$ and ψ_i. By Lemma 6.2 and (6.15),

$$\left\| \mathcal{D}_i \frac{\partial \varphi^t}{\partial t} \right\|_\alpha \leq C\|f\|_\alpha. \tag{6.25}$$

8°. In notation of Theorem 6.1,

$$\varphi^t(y) = \varphi(y, x_d)$$

where

$$t = x_d, \frac{\partial \varphi}{\partial t} = \mathcal{D}_d \varphi, \mathcal{D}_i \frac{\partial \varphi}{\partial t} = \mathcal{D}_i D_d \varphi, \frac{\partial^2 \varphi}{(\partial t)^2} = \mathcal{D}_{dd} \varphi.$$

By changing notation, we get the bound 6.2.B from (6.23). Since $0 < x_d < 1$, we have $\|\mathcal{D}_d \varphi\|_\alpha \leq \|\frac{1}{x_d} \mathcal{D}_d \varphi\|$ and therefore the bound 6.2.A follows from (6.18), (6.19), (6.20), (6.23), (6.24) and (6.25). Condition 6.2.C follows from (6.21) because $A' \geq 0$.

By (6.22),

$$\left| \frac{\partial \varphi^t}{\partial t} \right|^\alpha = |tF^t|^\alpha \leq Ct^{\alpha-2} \int_t^{1 \wedge (2t)} r|\psi^r|^\alpha \, dr \tag{6.26}$$

[because $r^{1/2} \geq t^{1/2}$ for $r \geq t$]. It follows from (6.26) that

$$\ell_\alpha(\frac{\partial \varphi^t}{\partial t}) \leq Ct^{\alpha-2} \int_t^{1 \wedge (2t)} r\ell_\alpha(\psi_r) \, dr$$

and therefore, if $\alpha \geq 2$, then $\int_{\mathbb{R}^{d-1}} (\frac{\partial \varphi^t}{\partial t})^\alpha dx \to 0$ as $t \downarrow 0$. If

$$\int_0^1 r|\psi^r(y)|^\alpha \, dr < \infty,$$

then the right side in (6.26) tends to 0 as $t \downarrow 0$ and therefore $\frac{\partial \varphi^t}{\partial t} \to 0$. This is true for almost all $y \in \mathbb{R}^{d-1}$ because, by (6.4), (6.5) and (6.15),

$$\int_0^1 r \, dr \int_{\mathbb{R}^{d-1}} |\psi^r(y)|^\alpha \, dy = \|\psi\|_\alpha^\alpha \leq C\|f\|_\alpha^\alpha < \infty.$$

□

6.6. Bounds for the norm of $|\nabla f|^2$. In addition to Theorem 6.1, we need an estimate for the norm of

$$|\nabla f|^2 = \sum_1^d (\mathcal{D}_i f)^2. \tag{6.27}$$

LEMMA 6.3. [7] *If $\alpha > 1, u \in C^2(\mathbb{R}), u \geq 0$ and $u = 0$ outside a finite interval, then*

$$\int \frac{|u'|^{2\alpha}}{u^\alpha} dt \leq C_\alpha \int |u''|^\alpha dt \tag{6.28}$$

where the integration is taken over the support of u.

[7] See [**Maz85**], Lemma 8.2.1.

PROOF. Fix $\varepsilon > 0$ and note that

$$\frac{|u'|^{2\alpha}}{(u+\varepsilon)^\alpha} = AB'_\varepsilon$$

where

$$A = |u'|^{2(\alpha-1)}u', \quad B_\varepsilon = \frac{1}{1-\alpha}(u+\varepsilon)^{1-\alpha}. \tag{6.29}$$

Integration by parts yields

$$\int \frac{|u'|^{2\alpha}}{(u+\varepsilon)^\alpha}\,dt = -\int A'B_\varepsilon dt = \frac{2\alpha-1}{\alpha-1}\int FG_\varepsilon\,dt \tag{6.30}$$

where $F = u''$ and $G_\varepsilon = |u'|^{2(\alpha-1)}(u+\varepsilon)^{1-\alpha}$. [To evaluate A', we use a formula

$$(|f|^\alpha f)' = (\alpha+1)|f|^\alpha f'$$

which is true for every differentiable function f and every $\alpha \geq 0$.] By Hölder's inequality,

(6.31)

$$(\int FG_\varepsilon\,dt)^\alpha \leq \int |F|^\alpha dt \left[\int |G_\varepsilon|^{\alpha'}\right]^{\alpha/\alpha'} = \int |u''|^\alpha\,dt \left[\int \frac{|u'|^{2\alpha}}{(u+\varepsilon)^\alpha}\,dt\right]^{\alpha-1}.$$

By using the monotone convergence theorem, we pass to the limit in (6.30) and (6.31) as $\varepsilon \to 0$, and we get (6.28) with $C_\alpha = [\frac{2\alpha-1}{\alpha-1}]^\alpha$. □

LEMMA 6.4. *Let $u \in C^2(0,\infty)$ and let $u \geq 0, u' \leq 0, \lim_{t\to 0} u'(t) = 0$ and $u(s) = 0$ for all sufficiently large t. Then, for every $\alpha > 1$,*

$$\int_0^\infty \frac{|u'|^{2\alpha}}{u^\alpha} t\,dt \leq C_\alpha \int_0^\infty |u''|^\alpha t\,dt. \tag{6.32}$$

REMARK. Existence of $u'(0)$ is not required.

PROOF. Choose $\varepsilon > 0$. Integrating by parts yields

$$\begin{aligned}\int_s^\infty \frac{|u'|^{2\alpha}}{(u+\varepsilon)^\alpha} t\,dt &= -\frac{1}{\alpha-1}\int_s^\infty |u'|^{2(\alpha-1)}u't\,d[(u+\varepsilon)^{1-\alpha}]\\ &= \frac{|u'(s)|^{2(\alpha-1)}}{(\alpha-1)(u(s)+\varepsilon)^{\alpha-1}}u'(s)s + \frac{2\alpha-1}{\alpha-1}\int_s^\infty \frac{|u'|^{2(\alpha-1)}}{(u+\varepsilon)^{\alpha-1}}u''t\,dt\\ &\quad + \frac{1}{\alpha-1}\int_s^\infty \frac{|u'|^{2(\alpha-1)}}{(u+\varepsilon)^{\alpha-1}}u'\,dt\\ &\leq \frac{|u'(s)|^{2(\alpha-1)}}{(\alpha-1)(u(s)+\varepsilon)^{\alpha-1}}u'(s)s + \frac{2\alpha-1}{\alpha-1}\int_s^\infty \frac{|u'|^{2(\alpha-1)}}{(u+\varepsilon)^{\alpha-1}}u''t\,dt.\end{aligned}$$

By the Hölder's inequality,

$$\int_s^\infty \frac{|u'|^{2(\alpha-1)}}{(u+\varepsilon)^{\alpha-1}}u''t\,dt \leq \Big(\int_s^\infty \frac{|u'|^{2\alpha}}{(u+\varepsilon)^\alpha}t\,dt\Big)^{(\alpha-1)/\alpha}\Big(\int_s^\infty |u''|^\alpha t\,dt\Big)^{1/\alpha}.$$

It remains to pass to the limit, first, as $s \to 0$, and then as $\varepsilon \to 0$. □

Denote by $\nabla^2 f$ the matrix of the second partial derivatives $\mathcal{D}_{ij}f$ and put

$$\|\nabla^2 f\|_\alpha = \sum_{i,j} \|\mathcal{D}_{ij}f\|_\alpha. \tag{6.33}$$

THEOREM 6.2. *Suppose $f \in C^2(\mathbb{E})$ is positive, vanishes outside a compact set and satisfies conditions $\mathcal{D}_d f \le 0$ and $\mathcal{D}_d f \to 0$ as $x_d \downarrow 0$ for almost all $x_1, \dots, x_{d-1}$. Then*

$$\| |\nabla f|^2 / f \|_\alpha \le C \| \nabla^2 f \|_\alpha. \tag{6.34}$$

PROOF. It is sufficient to prove that

$$\| (\mathcal{D}_i f)^2 / f \|_\alpha \le C \| \mathcal{D}_{ii} f \|_\alpha \tag{6.35}$$

for every $i = 1, \dots, d$. This follows from (6.28) for $i < d$, and from (6.32) for $i = d$. □

6.7. α-sequences. We say that a sequence of functions $\varphi_n \in C^2(\mathbb{E})$ is an *α-sequence* if:

6.7.A. $\|\varphi_n\|_{2,\alpha} \to 0$;

6.7.B. $\|\frac{1}{x_d} \mathcal{D}_d \varphi_n\|_\alpha \to 0$;

6.7.C. $\mathcal{D}_d \varphi_n \le 0$.

6.7.D. For every n,

$$\int_{R^{d-1}} |\mathcal{D}_d \varphi_n(y, x_d)|^\alpha \, dy \to 0 \quad \text{as } x_d \downarrow 0$$

and, for almost all $y \in \mathbb{R}^{d-1}$,

$$\mathcal{D}_d \varphi_n(y, x_d) \to 0 \quad \text{as } x_d \downarrow 0.$$

6.7.E. $\| |\nabla \varphi_n|^2 \|_\alpha \to 0$.

6.7.F. $0 \le \varphi_n \le 1$.

6.7.G. There exists a bounded subdomain V of $\mathbb{E}$ such that $\varphi_n = 0$ outside V.

LEMMA 6.5. *Suppose that $f_n \ge 0$ and $\|f_n\|_\alpha \to 0$. If $\alpha \ge 2$, then it is possible to choose functions g on $\mathbb{E}$ and h on $\mathbb{R}_+$ in such a way that*

$$\varphi_n = h[g \mathcal{J}(f_n)] \tag{6.36}$$

is an α-sequence.

PROOF. 1°. Put

$$u_n = \mathcal{J}(f_n), \quad v_n = g u_n, \quad \varphi_n = h(v_n).$$

By Theorem 6.1 u_n satisfies conditions 6.7.A–6.7.D.

2°. Suppose that g is positive, twice continuously differentiable and has a compact support. Then g and its first and second partial derivatives are bounded and therefore

$$\begin{aligned} |\mathcal{D}_i v_n| &\le C(|u_n| + |\mathcal{D}_i u_n|), \\ |\mathcal{D}_{ij} v_n| &\le C(|u_n| + |\mathcal{D}_i u_n| + |\mathcal{D}_j u_n| + |\mathcal{D}_{ij} u_n|). \end{aligned} \tag{6.37}$$

Hence, $\|v_n\|_{2,\alpha} \le C \|u_n\|_{2,\alpha}$ and v_n satisfies 6.7.A.

3°. Now we take $g(x, r) = a(r) b(x)$. We assume that $b \in C^2(\mathbb{R}^{d-1})$ is positive and has a compact support and that $a \in C^2(\mathbb{R}_+)$, $a \ge 0$, $a' \le 0$, $a = 1$ near 0 and $a = 0$ for sufficiently large r.

By 2°, v_n satisfies 6.7.A. Clearly, it satisfies 6.7.C. We have

$$\mathcal{D}_d v_n = b(a \mathcal{D}_d u_n + a' u_n).$$

For sufficiently small r, $a' = 0$ and therefore v_n satisfies 6.7.D. Since a'/r is bounded, it satisfies 6.7.B.

4°. We choose $h \in C^2(\mathbb{R}_+)$ such that $h' \geq 0$, $h = 0$ on $[0, 1/4]$ and $h = 1$ on $[3/4, \infty)$. Note that $0 \leq h \leq 1$ and that h', h'' are bounded.

For all $i, j = 1, \ldots, d$,

$$D_i \varphi_n = h'(v_n) \mathcal{D}_i v_n,$$
$$D_{ij} \varphi_n = h'' \mathcal{D}_i v_n D_j v_n + h'(v_n) \mathcal{D}_{ij} v_n.$$

Clearly, φ_n satisfies 6.7.A–6.7.D, and 6.7.F–6.7.G.

Note that

$$(\mathcal{D}_i \varphi_n)^2 \leq N (\mathcal{D}_i v_n)^2 / v_n$$

where N is the supremum of $sh'(s)^2$. Hence

$$\| |\nabla \varphi_n|^2 \|_\alpha \leq N \| |\nabla v_n|^2 / (v_n) \|_\alpha.$$

By Theorem 6.2, the right side is not bigger than $N \| \nabla^2 v_n \|_\alpha$. Since v_n satisfies 6.7.A, φ_n satisfies 6.7.E. □

6.8. Truncating sequences. For every closed subset B of $\mathbb{E} \cup \partial' \mathbb{E}$, we put $\partial' B = \partial B \cap \partial' \mathbb{E}$ and $\partial'' B = \partial B \setminus \partial' B$. We call U an $\mathbb{E}$-*neighborhood of* B if U is open subset of $\mathbb{E}$ and $\bar{B} \subset U \cup \partial' U$. An expression "near B" means in an $\mathbb{E}$-neighborhood of B.

We say that φ_n is a *truncating sequence* for $B \subset \partial' \mathbb{E}$ if it is an α'-sequence and if, for every n, $\varphi_n(x) = 1$ near B.

Theorem 6.3. *Suppose α, E, B, m, k satisfy conditions of Theorem 4.1 and let B be a (m, k)-null set. Then there exists a truncating sequence for B.*

Proof. By the definition of a (m, k)-null set, Theorem 5.1 and formula (5.2), there exists a sequence f_n such that $\|f_n\|_{\alpha'} \to 0$ and $\hat{K} f_n \geq 1$ on B. Clearly, we can assume that $f_n \geq 0$. We claim that the α'-sequence φ_n constructed in Lemma 6.5 is a truncating sequence for B if function $b(x)$ chosen in the proof of Lemma 6.5 (in part 3°), is equal to 1 in a neighborhood of B.

Put $u_n = \mathcal{J}(f_n)$ on $\mathbb{E}$ and $u_n = \hat{K} f_n$ on $\{x_d = 0\}$. It follows from (6.16), (6.3) and Fatou's lemma that u_n are lower semicontinuous and therefore $V_n = \{u_n > 3/4\}$ are open. Note that $B \subset \{\hat{K} f_n \geq 1\} \subset V_n$. Since $g = ab = 1$ in a neighborhood $\tilde{U}$ of B in $\hat{\mathbb{E}}$ and $h = 1$ on $[3/4, \infty)$, we conclude that $\varphi_n = h(g u_n) = 1$ on $U_n = V_n \cap \tilde{U}$. □

7. Proof of the principal results

7.1. A priori estimates for solutions of equation $Lu = u^\alpha$.

Theorem 7.1. *Let B be a compact subset of $\partial' \mathbb{E}$ and let $Lu = u^\alpha$ in $\mathbb{E}$ and $u = 0$ on $\partial' \mathbb{E} \setminus B$. For each $\mathbb{E}$-neighborhood V of B, there exist constants $\delta > 0$ and C such that*

(7.1)
$$u(x, r) \leq Cr, \quad |\nabla u(x, r)| \leq C, \quad |\nabla^2 u(x, r)| \leq \frac{C}{r} \quad \text{for all } (x, r) \in \mathbb{E} \setminus V, r < \delta.$$

Without any loss of generality we can assume that the coefficients of L are bounded in $\mathbb{R}^d$. The proof is based on two lemmas.

LEMMA 7.1. *Let* $0 < \gamma_1 < \gamma_2 < \gamma_3$. *Put* $\rho = |z - z^0|$ *and consider sets*

$$R = \{z : \gamma_1 \le \rho < \gamma_3\}, \quad R' = \{z : \gamma_1 \le \rho < \gamma_2\}. \tag{7.2}$$

There exists a function $v \in C^2(R)$ *such that*

$$Lv \le v^\alpha \quad in \quad R, \tag{7.3}$$

$$v = 0 \quad on \quad \{\rho = \gamma_1\}, \tag{7.4}$$

$$v = \infty \quad on \quad \{\rho = \gamma_3\}, \tag{7.5}$$

$$v(z) \le C(\rho - \gamma_1) \quad on \ R' \tag{7.6}$$

where C *is a constant independent of* z^0.

PROOF. 1°. Let $t_1 < t_2 < t_3$ and let c be an arbitrary constant. We consider a positive solution f of the problem

$$\begin{aligned} f''(t) + cf'(t) = f(t)^\alpha \quad &\text{for } t_1 < t < t_3, \\ f(t_1) = 0, \quad &f(t_3) = \infty. \end{aligned} \tag{7.7}$$

Note that (7.7) is a particular case of **8**.(2.5) corresponding to $D = (t_1, t_3), Lu = u'' + cu', \psi(u) = u^\alpha$ and $O = \partial D$. By **8**.2.1.I, we get a solution by the formula $f = V_D(\varphi)$ where $\varphi(t_1) = 0$, $\varphi(t_3) = \infty$. [8]

Note that

$$f(t) \le N(t - t_1) \qquad \text{on } (t_1, t_2) \tag{7.8}$$

where N is the supremum of $f'(t)$ on $[t_1, t_2]$. Since $f(t_1) = 0$ and $f(t) \ge 0$ for all t, we have $f'(t_1) \ge 0$. Function $F(t) = e^{ct} f'$ satisfies the conditions $F'(t) = e^{ct} f^\alpha \ge 0$ and $F(t_1) \ge 0$ and therefore F and f' are positive for all $t \in [t_1, t_3]$.

2°. If $t_i = \gamma_i^2$ and $\lambda > 0, c$ are constants, then the function

$$v(z) = \lambda f(\rho^2) \tag{7.9}$$

satisfies conditions (7.4) and (7.5). The bound (7.6) with $C = 2\lambda N \gamma_2$, follows from (7.8). It remains to choose λ and c to satisfy (7.3). Note that $Lv(z) = \lambda[Af''(\rho^2) + Bf'(\rho^2)]$ where

$$A = 4 \sum a_{ij}(z_i - z_i^0)(z_j - z_j^0), \qquad B = 2 \sum [a_{ii} + b_i(z_i - z_i^0)].$$

In the annulus R,

$$A \le q, B/A \le c$$

where constants q and c depend only γ_1, γ_3 and on the bounds of the coefficients of L. Since $f, f' \ge 0$, we have

$$Lv - v^\alpha = \lambda A \left(f'' + \frac{B}{A} f' - \frac{\lambda^{\alpha-1}}{A} f^\alpha \right) \le \lambda A \left(f'' + cf' - \frac{\lambda^{\alpha-1}}{q} f^\alpha \right) \tag{7.10}$$

in R. If f is chosen to satisfy (7.7) with the value of c which appears in (7.10) and if $\lambda = q^{1/(\alpha-1)}$, then the right side in (7.10) is equal to $\lambda A(1 - \lambda^{\alpha-1}/q) f^\alpha = 0$ which implies (7.3). □

[8] A solution f can be also constructed directly by starting from the solutions f_k of equation $f'' + cf' = f^\alpha$ subject to the boundary conditions $f_k(t_1) = 0, f_k(t_3) = k$ and by passing to the limit as $k \to \infty$.

LEMMA 7.2. *Suppose that $u > 0$ and $Lu = u^\alpha$ in a bounded domain D. Put $d_x = d(x, \partial D)$. If $M(u) = \sup_D u \le 1$, then, for all $i = 1, \dots, d$,*

$$d_x |\mathcal{D}_i u(x)| \le CM(u) \tag{7.11}$$

and, for all $i, j = 1, \dots, d$,

$$d_x^2 |\mathcal{D}_{ij} u(x)| \le CM(u) \tag{7.12}$$

*where C depends only on the diameter of D and constants κ, λ, Λ in conditions **6**.1.2.A-1.2.B.*

PROOF. We apply Theorem 2.1 (the Appendix B) to u and $f = u^\alpha$. Note that $M(f) = M(u)^\alpha$ and, since $M(u) \le 1$ and $\alpha > 1$,

$$M(u) + M(f) \le CM(u). \tag{7.13}$$

Therefore the bound (2.3) (the Appendix B) implies (7.11).

For all positive a, b, $|a^\alpha - b^\alpha| \le |a - b| \alpha (a \vee b)^{\alpha - 1}$. Therefore $|f(x) - f(y)| \le \alpha |u(x) - u(y)|$ for all $x, y \in D$, and

$$S(f) \le \alpha S(u) \le CM(u).$$

The bound (2.2) (the Appendix B) implies (7.12). □

PROOF OF THEOREM 7.1

1°. Every $\mathbb{E}$-neighborhood of B contains a set $O \times (0, \beta)$ where O is a neighborhood of B in $\partial'\mathbb{E}$ and $\beta > 0$. Therefore it is sufficient to consider $V = O \times (0, \beta)$. If $z = (x^0, r) \in \mathbb{E} \setminus V$ and if $r < \beta$, then $x^0 \notin O$. We apply Lemma 7.1 to $z^0 = (x^0, -1), \gamma_1 = 1, \gamma_2 = 1 + \delta, \gamma_3 = 1 + 2\delta$. Denote by $R(x^0)$ and $R'(x^0)$ the sets described by (7.2). By (7.6),

$$v(z) \le C(|z - z^0| - 1) \quad \text{on} \quad R'(x^0). \tag{7.14}$$

We claim that $u \le v$ in $U = R(x^0) \cap \mathbb{E}$. Indeed, $\partial U = A \cup A^*$ where $A = \partial'\mathbb{E} \cap \bar{R}(x^0), A^* = \partial R(x^0) \cap \mathbb{E}$. If δ is sufficiently small, then $A \cap B = \emptyset$ for all $x^0 \in \partial'\mathbb{E} \setminus O$. Therefore $u = 0 \le v$ on A. Also $u < v = \infty$ on A^*. Since $Lv - v^\alpha \le 0 = Lu - u^\alpha$ in U, the comparison principle **8**.2.1.H implies that $u \le v$ in U. In particular, $u(x^0, r) \le v(x^0, r)$ for $r < \delta$. If $z = (x^0, r)$, then $|z - z^0| = 1 + r$ and, by (7.14), $u(x^0, r) \le Cr$. The first bound in Theorem 7.1 is proved.

2°. The second and the third bounds can be deduced from the first one and Lemma 7.2. Consider a relatively open subset O' of $\partial'\mathbb{E}$ such that $B \subset O' \subset O$ and $d(O', \partial'\mathbb{E} \setminus O) = \gamma > 0$. We proved in 1° that there exist constants $\delta > 0$ and C such that

$$u(x, r) \le Cr \quad \text{for all } x \in \partial'\mathbb{E} \setminus O', 0 < r < \delta.$$

Let $x^0 \in \partial'\mathbb{E} \setminus O$. If $|x - x^0| < \gamma$, then $x \in \partial'\mathbb{E} \setminus O'$ and therefore

$$u(x, r) < Cr \quad \text{for } 0 < r < \delta. \tag{7.15}$$

Suppose that $0 < r^0 < \gamma \wedge (\delta/4)$ and that $2r^0 < \varepsilon < 4r^0$. Let $D = \{(x, r) : |x - x^0| < \gamma, 0 < r < \varepsilon\}$. For $z^0 = (x^0, r^0)$, $d(z^0, \partial D) = r^0$. By (7.15), $M(u) < C\varepsilon$ and, by (7.11), $r^0 |D_i u(z^0)| \le CM(u) \le C^2 \varepsilon \le 4C^2 r^0$ and, by (7.12), $(r^0)^2 |D_{ij} u(z^0)| \le CM(u) \le 4C^2 r^0$. Hence

$$|D_i u(x^0, r^0)| \le 4C^2, |D_{ij} u(x^0, r^0)| \le 4C^2 / r^0$$

for all $x^0 \in \partial'\mathbb{E} \setminus O, 0 < r^0 < \gamma \wedge (\delta/4)$. □

7.2.

LEMMA 7.3. *There exists a diffeomorphism $y = \psi(x)$ of class $C^{2,\lambda}$ from $\mathbb{R}^d$ onto $\mathbb{R}^d$ such that:*

(a) $y_d = x_d$;

(b) $\psi(x) = x$ on $\{x_d = 0\}$.

(c) The coefficients $\tilde{a}_{id}, i < d$ of the operator L transformed by ψ vanish on $\{x_d = 0\}$.

PROOF. 1°. Note that

$$\tilde{a}_{id} = \sum_{k=1,\ell=1}^{d} c_{ik} c_{d\ell} a_{k\ell}$$

where $c_{ik} = \frac{\partial y_i}{\partial x_k}$. Condition (a) implies that $c_{d\ell} = 0$ for $\ell < d$ and $c_{dd} = 1$. By (b), $c_{ik} = \delta_{ik}$ on $\{x_d = 0\}$ for $i < d$ and $k < d$. Therefore

$$\tilde{a}_{id} = a_{id} + c_{id} a_{dd}$$

on $\{x_d = 0\}$ for $i < d$. The condition (c) holds if

$$\frac{\partial y_i}{\partial x_d} = -\frac{a_{id}}{a_{dd}} \quad \text{for } i < d. \tag{7.16}$$

2°. To define ψ, we consider a vector field $f(y)$ on $\mathbb{R}^d$ with components

$$f_i(y) = -\frac{a_{id}(y)}{a_{dd}(y)} \quad \text{for } i < d,$$
$$f_d(y) = 1.$$

Note that a_{dd} is bigger than or equal to the ellipticity coefficient of L and therefore f is bounded. It is continuously differentiable and therefore, for every $x \in \mathbb{R}^d$, the equation

$$\frac{dy}{dt} = f(y) \tag{7.17}$$

has a unique solution $T_t(x)$ which is equal to x at $t = 0$ and this solution is defined for all $t \in \mathbb{R}$. Moreover, for every t, T_t is a diffeomorphism of class $C^{2,\lambda}$ from $\mathbb{R}^d$ onto $\mathbb{R}^d$. The mapping

$$\psi(x_1, \dots, x_d) = T_{x_d}(x_1, \dots, x_{d-1}, 0)$$

also belongs to class $C^{2,\lambda}$ and it satisfies conditions (a) and (b). The formula

$$\phi(x_1, \dots, x_d) = (y_1, \dots, y_d)$$

where $(y_1, \dots, y_{d-1}, 0) = T_{-x_d}(x_1, \dots, x_d)$ and $y_d = x_d$ defines a C^2 mapping inverse to ψ. Hence, ψ is a diffeomorphism from $\mathbb{R}^d$ onto $\mathbb{R}^d$. Clearly, ψ satisfies (7.16) and therefore it satisfies (c). □

7.3. Proof of Theorem 4.1. 1°. Our main tool is the truncating sequence φ_n defined in Theorem 6.3. It satisfies conditions 6.7.A–6.7.G with α replaced by α' and $\varphi_n = 1$ near B. By Proposition 4.3, our theorem will be proved if we define a positive function $p \in C(\mathbb{E})$ such that $p \geq$ const. > 0 in a $\mathbb{E}$-neighborhood of B and

$$\int_{\mathbb{E}} u^\alpha p \, dm < \infty \tag{7.18}$$

for every solution u of the problem (4.7).

By Lemma 7.3, we may assume that L satisfies condition $a_{id} = 0$ on $\partial'\mathbb{E}$ for $i < d$. Put $\beta = 2\alpha'$ and let

$$\mathfrak{A}\varphi(x,r) = 2\sum_1^d a_{id}\mathcal{D}_i\varphi. \tag{7.19}$$

Since $a_{dd} > 0$, a first order differential equation

$$\beta\mathfrak{A}v + (2\sum_1^d \mathcal{D}_j a_{jd} - b_d)v = 0 \tag{7.20}$$

has a unique solution v in an $\mathbb{E}$-neighborhood W of B such that $v = 1/2$ on $\partial'W$. [9] Consider a function p on $\mathbb{E}$ such that $0 \le p \le 1$ and $p = v^\beta$ on W and $p = 0$ off a bounded set.

2°. Put $h_n = (1-\varphi_n)^\beta, g = px_d, w_n = gh_n$. We claim that

$$\int_{\mathbb{E}} w_n Lu\ dx = \int_{\mathbb{E}} uL^* w_n\ dx. \tag{7.21}$$

Consider a bounded $\mathbb{E}$-neighborhood $U \subset W$ of B on which $\varphi_n = 1$ and $\frac{1}{4} < v < \frac{3}{4}$. There exists a bounded smooth $\mathbb{E}$-neighborhood V of B which contains $U \cup \partial''U$ such that $p = 0$ on $\mathbb{E} \setminus V$. Put $V_\varepsilon = V \cap \{x_d > \varepsilon\}$. It is sufficient to prove that

$$I_\varepsilon = \int_{V_\varepsilon} w_n Lu\ dx - \int_{V_\varepsilon} uL^* w_n\ dx \to 0 \quad \text{as } \varepsilon \to 0. \tag{7.22}$$

We apply Green's formula (1.2) (the Appendix B) to V_ε and to the functions u and w_n.

Function w_n and all its partial derivatives vanish on $U \cup \partial''V$ and therefore

$$I_\varepsilon = \int_{A_\varepsilon} \left[\frac{\partial u}{\partial\lambda} w_n - \frac{\partial w_n}{\partial\nu} u\right] \rho\ d\gamma$$

where A_ε is the intersection of $V \setminus U$ with $\{x_d = \varepsilon\}$. Choose the field ν such that $\frac{\partial}{\partial\nu} = \mathcal{D}_d$ on A_ε. For sufficiently small ε, by Theorem 7.1, $u \le C\varepsilon, |\nabla u| \le C$ on A_ε and therefore

$$|I_\varepsilon| \le C\int_{A_\varepsilon} (w_n + \varepsilon|\mathcal{D}_d w_n|)\ d\gamma.$$

Since g and $\mathcal{D}_d g$ are bounded and since $w_n \le \varepsilon$ on A_ε, we have

$$|\mathcal{D}_d w_n| \le |\mathcal{D}_d g| h_n + g|\mathcal{D}_d h_n| \le C(1 + |\mathcal{D}_d h_n|).$$

By Hölder's inequality,

$$\int_{A_\varepsilon} |\mathcal{D}_d h_n|\ d\gamma \le C\left(\int_{A_\varepsilon} |\mathcal{D}_d h_n|^{\alpha'}\ d\gamma\right)^{1/\alpha'}.$$

The condition 6.7.D implies that $I_\varepsilon \to 0$ as $\varepsilon \to 0$.

3°. By 6.7.A, $\|\varphi_n\|_{\alpha'} \to 0$ and therefore there exists a subsequence φ_{n_k} which tends to 0 a.e. Since $h_{n_k} \to 1$ a.e., we get (7.18) by Fatou's lemma if we prove that

$$\sup J_n < \infty \tag{7.23}$$

[9]See, e.g., the Appendix in [**Pet47**].

where

$$J_n = \int_{\mathbb{E}} u^\alpha p h_n dm.$$

Since $u^\alpha = Lu$, (7.21) implies

$$J_n = \int_{\mathbb{E}} u L^*(h_n g)\, dx. \tag{7.24}$$

4°. Put $\gamma = p^{1/\beta}, \gamma_n = \gamma(1-\varphi_n) = (h_n p)^{1/\beta}$. Direct computation yields

$$L^*(h_n g) = L^*(x_d \gamma_n^\beta) = x_d \gamma_n^{\beta-2}(\gamma_n \mathfrak{M}\gamma_n + \mathfrak{N}\gamma_n) + \gamma_n^{\beta-1}\mathfrak{P}\gamma_n \tag{7.25}$$

where

$$\begin{aligned}
\mathfrak{M}f &= \sum_{ij}[\beta a_{ij}\mathcal{D}_{ij}f + 2\beta\mathcal{D}_i a_{ij}\mathcal{D}_j f + f\mathcal{D}_{ij}a_{ij}] - \sum_i [\beta b_i \mathcal{D}_i f + f\mathcal{D}_i b_i],\\
\mathfrak{N}f &= \sum \beta(\beta-1)a_{ij}\mathcal{D}_i f\mathcal{D}_j f,\\
\mathfrak{P}f &= \beta\mathfrak{A}f + (\sum 2\mathcal{D}_j a_{jd} - b_d)f
\end{aligned}$$

with $\mathfrak{A}$ given by (7.19).

Note that $(\beta-2)/\beta = 1/\alpha$ and therefore

$$J_n = \int_{\mathbb{E}} u^\alpha h_n p\, dm = \|u(h_n p)^{1/\alpha}\|_\alpha^\alpha = \|u\gamma_n^{\beta-2}\|_\alpha^\alpha. \tag{7.26}$$

Since $0 \le \gamma, \gamma_n \le 1$ and $\gamma_n = 0$ on $\mathbb{E} \setminus V$, we get from (7.25) that $J_n \le J_n' + J_n''$ where

$$\begin{aligned}
J_n' &= \int_V u\gamma_n^{\beta-2}(|\mathfrak{M}\gamma_n| + |\mathfrak{N}\gamma_n|)\, dm,\\
J_n'' &= \int_V u\gamma_n^{\beta-2}|\mathfrak{P}\gamma_n|\, dx.
\end{aligned} \tag{7.27}$$

By Hölder's inequality and (7.26),

$$J_n' \le J_n^{1/\alpha}(\|\mathfrak{M}\gamma_n\|_{\alpha'} + \|\mathfrak{N}\gamma_n\|_{\alpha'}). \tag{7.28}$$

We have

$$\mathfrak{P}\gamma_n = F_n + G_n$$

where

$$F_n = (1-\varphi_n)\left[\beta\mathfrak{A}\gamma + \gamma\left(\sum 2\mathcal{D}_j a_{jd} - b_d\right)\right], \qquad G_n = -\beta\gamma\mathfrak{A}\varphi_n.$$

Functions F_n are uniformly bounded on V and, by (7.20), $F_n = 0$ on U (because $\gamma = v$ on U). Since u is bounded on $V \setminus U$, we have

$$\int_V u\gamma_n^{\beta-2}|F_n|\, dx = \int_{V\setminus U} u\gamma_n^{\beta-2}|F_n|\, dx \le C. \tag{7.29}$$

On the other hand, by Hölder's inequality and (7.26)

$$\int_V u\gamma_n^{\beta-2}|G_n|\, dx \le \beta\int_V u\gamma_n^{\beta-2}|\mathfrak{A}\varphi_n/x_d|\, dm \le \beta J_n^{1/\alpha}\|\mathfrak{A}\varphi_n/x_d\|_{\alpha'}. \tag{7.30}$$

For $i < d$, $a_{id} = 0$ on $\{x_d = 0\}$ and therefore $|a_{id}| \le Cx_d$ on V. Hence $|\mathfrak{A}\varphi_n/x_d| \le C(1 + |\mathcal{D}_d\varphi_n/x_d|)$ and, by 6.7.B, $\|\mathfrak{A}\varphi_n/x_d\|_{\alpha'} \le C$. We conclude from (7.29) and (7.30) that

$$J_n'' \le C(1 + J_n^{1/\alpha}). \tag{7.31}$$

If we prove that

$$\|\mathfrak{M}\gamma_n\|_{\alpha'} \leq C, \quad \|\mathfrak{N}\gamma_n\|_{\alpha'} \leq C, \tag{7.32}$$

then, by (7.28) and (7.31),

$$J_n \leq C(1 + J_n^{1/\alpha}).$$

After multiplying by $J_n^{-1/\alpha}$, we obtain $J_n^{(\alpha-1)/\alpha} \leq C(J_n^{-1/\alpha} + 1)$. If $J_n > 1$, then $J_n^{(\alpha-1)/\alpha} \leq 2C$ and $J_n \leq (2C)^{\alpha/(\alpha-1)}$. Hence $J_n \leq 1 \vee (2C)^{\alpha/(\alpha-1)}$, and (7.23) is proved.

5°. It remains to establish the bounds (7.32). Note that

$$\begin{aligned} \mathcal{D}_i\gamma_n &= (1-\varphi_n)\mathcal{D}_i\gamma - \gamma\mathcal{D}_i\varphi_n, \\ \mathcal{D}_{ij}\gamma_n &= (1-\varphi_n)\mathcal{D}_{ij}\gamma - \mathcal{D}_i\gamma\mathcal{D}_j\varphi_n - \mathcal{D}_i\varphi_n\mathcal{D}_j\gamma - \gamma\mathcal{D}_{ij}\varphi_n \end{aligned} \tag{7.33}$$

and therefore

$$\mathfrak{M}\gamma_n = \tilde{\mathfrak{M}}\varphi_n + (1-\varphi_n)\mathfrak{M}\gamma \tag{7.34}$$

where $\tilde{\mathfrak{M}}$ is a linear second order differential operator with bounded coefficients. We have $\|\tilde{\mathfrak{M}}\varphi_n\|_{\alpha'} \leq C\|\varphi_n\|_{2,\alpha'}$. By 6.7.A, $\|\varphi_n\|_{2,\alpha'} \to 0$. Therefore

$$\|\mathfrak{M}\gamma_n\|_{\alpha'} \leq \|\tilde{\mathfrak{M}}\varphi_n\|_{\alpha'} + \|(1-\varphi_n)\mathfrak{M}\gamma\|_{\alpha'} \leq \|\tilde{\mathfrak{M}}\varphi_n\|_{\alpha'} + \|\mathfrak{M}\gamma\|_{\alpha'} \leq C. \tag{7.35}$$

To establish a bound for $\|\mathfrak{N}\gamma_n\|_{\alpha'}$, we note that

$$(\mathfrak{N}\gamma_n)^2 \leq C\sum_1^d (\mathcal{D}_i\gamma_n)^2$$

and

$$\begin{aligned} (\mathcal{D}_i\gamma_n)^2 = (1-\varphi_n)^2(\mathcal{D}_i\gamma)^2 - 2\gamma(1-\varphi_n)\mathcal{D}_i\gamma\mathcal{D}_i\varphi_n + \gamma^2(\mathcal{D}_i\varphi_n)^2 \\ \leq C(1 + |\mathcal{D}_i\varphi_n| + (\mathcal{D}_i\varphi_n)^2). \end{aligned}$$

Hence,

$$\|\mathfrak{N}\gamma_n\|_{\alpha'} \leq C(1 + \|\varphi_n\|_{2,\alpha'} + \||\nabla\varphi_n|^2\|_{\alpha'}). \tag{7.36}$$

It follows from 6.7.A and 6.7.E that $\|\mathfrak{N}\gamma_n\|_{\alpha'} \leq C$. □

8. Notes

8.1. Removable boundary singularities. Investigation of such singularities was started by Gmira and Véron [**GV91**] who proved that a single point is a removable boundary singularity for the equation $\Delta u = u^\alpha$ if $d \geq (\alpha+1)/(\alpha-1)$. [They considered also more general equation $\Delta u = \psi(u)$.]

It was conjectured in [**Dyn94**] that a compact set $B \subset \partial E$ is removable if and only if the Poisson capacity $CP_\alpha(B) = 0$. The conjecture agreed with the work of Gmira and Véron and it was supported by some results obtained about that time by Le Gall [**Le 94**] and Sheu [**She94b**]. In [**Le 95**], Le Gall proved the conjecture in the case $L = \Delta, \alpha = 2$ by using the Brownian snake. We borrowed his ideas in [**DK96c**] to extend the result to an arbitrary L and $\psi(u) = u^\alpha$ with $1 < \alpha \leq 2$. If $\alpha > 2$, then there is no (L, α)-superdiffusion. This case was investigated by Marcus and Véron in [**MV98b**] by purely analytic means which do not work for $\alpha \leq 2$. [10]

[10] A gap in [**MV98b**] discovered by Kuznetsov was bridged by the authors using arguments contained in the paper. Details were presented in [**Vér01**] – a paper based on Véron's survey lecture at the USA-Chile Workshop on Nonlinear Analysis.

In [**MV01**] Marcus and Véron covered, by a different method, the full range $\alpha > 1$ (in the case $L = \Delta$).

In Chapter **13**, the case $1 < \alpha \leq 2$ is treated by using a tool (truncating sequences) developed by Kuznetsov in [**Kuz98b**].

In [**DK96c**], [**MV98b**] and [**MV01**], not the Poisson capacities CP_α but the Bessel capacities $\mathrm{Cap}^{\partial}_{2/\alpha,\alpha'}$ are used. [11]

The Bessel capacities belong to the class of capacities defined by formula **12**.(1.2). The definition involves the modified Bessel functions $K_\nu(r)$ of the third kind (called also Macdonald functions). However, the asymptotic expression for K_ν as $r \downarrow 0$ (in the case $\nu > 0$) implies that the class of compact null sets does not change if K_ν is replaced by $r^{-\nu}$. If E is a bounded smooth domain in $\mathbb{R}^d$ and if B is a compact subset of ∂E, then, for $d > \beta + 1$, the condition $\mathrm{Cap}^{\partial}_{\beta,p}(B) = 0$ is equivalent to the condition

$$\int_{\partial E} \gamma(dx) \left[\int_{\partial E} |x-y|^{\beta-d+1} \nu(dy) \right]^{p'} = \infty \tag{8.1}$$

for every nontrivial $\nu \in \mathcal{M}(B)$. [Here γ is the surface area on ∂E.] In particular, for $d \geq 3$, $\mathrm{Cap}^{\partial}_{2/\alpha,\alpha'}(B) = 0$ if and only if

$$\int_{\partial E} \gamma(dx) \left[\int_{\partial E} |x-y|^{2/\alpha-d+1} \nu(dy) \right]^{\alpha} = \infty \tag{8.2}$$

for every $\nu \in \mathcal{M}(B), \nu \neq 0$. [If $d < 3$, then $\mathrm{Cap}^{\partial}_{2/\alpha,\alpha'}(B) > 0$ for all nonempty B.]

8.2. Removable interior singularities. They were studied much earlier than the boundary singularities. Suppose that E is a bounded regular domain in $\mathbb{R}^d$. We say that a compact subset B of E is a removable (interior) singularity if 0 is the only solution of the equation

$$Lu = u^\alpha \tag{8.3}$$

in $E \setminus B$ vanishing on ∂E. Brezis-Véron [**BV80**] proved that a singleton is removable if and only if $d \geq 2\alpha/(\alpha-1)$. Baras and Pierre [**BP84b**] established that the class of removable compact sets B coincides with the class of null sets of the Bessel capacity $\mathrm{Cap}_{2,\alpha'}$. Such sets can be characterized by the condition:

$$\int_E dx \left[\int_E \Gamma(x-y) \nu(dy) \right]^{\alpha} = \infty \tag{8.4}$$

for every nontrivial $\nu \in \mathcal{M}(B)$ (Γ is defined by **6**.(1.13)).

Dynkin [**Dyn91c**] used the result of Baras and Pierre to prove that (in the case $\alpha \leq 2$) B is removable if and only if it is polar (that is, a.s., not hit by the range of an (L,α)-superdiffusion). A simplified presentation of these results can be found in [**DK96c**] (see also [**Kuz00a**]). In [**Kuz00b**] Kuznetsov extended the results on interior singularities to a more general equation $Lu = \psi(u)$ by using the Orlicz capacity associated with ψ.

Suppose that $\mathcal{R}$ is the range of (L,α)-superdiffusion in E. By applying Theorems **10**.1.2 and **10**.1.3 to $\tilde{E} = E \setminus B$ and $\tilde{B} = \partial E$, we conclude that

$$w_B(x) = -\log P_x\{\mathcal{R} \cap B = \emptyset\}$$

[11] The comparison of these papers with Theorem 0.1 of Chapter **13** shows that the null sets for both capacities are identical. A direct proof of this fact for $1 < \alpha \leq 2$ was given in [**DK96c**].

is the maximal solution of the problem

$$Lu = u^\alpha \quad \text{in } E \setminus B,$$
$$u = 0 \quad \text{on } \partial E.$$

Clearly, B is removable if and only if $w_B = 0$ which is equivalent to the condition

$$P_x\{\mathcal{R} \cap B = \emptyset\} = 1 \quad \text{for all } x \in E \setminus B.$$

The first results on polar sets for superdiffusions were obtained by Dawson, Iscoe and Perkins [**DIP89**] and by Perkins [**Per90**]. They investigated the case $L = \Delta, \alpha = 2$ (the Dawson-Watanabe super-Brownian motion) by direct probabilistic method without using any results of analysts. A sufficient condition for polarity in terms of the Hausdorff dimension was established in [**DIP89**]. It was proved in [**Per90**] that all polar sets have capacity 0.

8.3. Critical Hausdorff dimension. To every $\beta > 0$ there corresponds the Hausdorff measure of $B \subset \mathbb{R}^d$ defined by the formula

$$H_\beta(B) = \lim_{\varepsilon \to 0} H_\beta^\varepsilon(B),$$

where

$$H_\beta^\varepsilon(B) = \inf \sum r_i^\beta$$

with infimum taken over all countable coverings of B by open balls $B(x_i, r_i)$ of center x_i and radius $r_i \leq \varepsilon$.

The Hausdorff dimension H-dim B is the supremum of β such that $H_\beta(B) > 0$.

We say that d_c is *the critical dimension* for polarity if every set B with H-dim $B < d_c$ is polar and every set B with H-dim $B > d_c$ is not polar.

The relations between the Hausdorff dimension and the Bessel capacity established in [**Mey70**] (Theorem 20) and in [**AM73**] (Theorems 4.2 and 4.3) allow to get the values of the critical Hausdorff dimension from the results described in sections 5.1 and 5.2. Put

$$\kappa_\alpha = \frac{2\alpha}{\alpha - 1}.$$

For an interior singularity, the critical dimension is equal to $d - \kappa_\alpha$ and, for a boundary singularity, it is equal to $d + 1 - \kappa_\alpha$. In the case $\alpha = 2$, the first value (equal to $d - 4$) follows from the results in [**DIP89**] and [**Per90**]. In [**Dyn91c**], it was proved for $\alpha \leq 2$. It was also established that, if $d = \kappa_\alpha$, then B can be polar or not polar depending on its Carleson logarithmic dimension. Note that, if B is a singleton, then H-dim$B = 0$ and therefore these results are consistent with the results of the previous section.

The critical dimension in the case of a boundary singularity was conjectured in [**Dyn94**]. The conjecture was proved in [**DK98a**]. The case of $L = \Delta, \alpha = 2$ was treated earlier in [**Le 94**] and partial results for general α were obtained in [**She94b**].

8.4. Parabolic setting. In [**Dyn92**] (see also [**Dyn93**]) Dynkin investigated $\mathcal{G}$-polar subsets of $\mathbb{R} \times \mathbb{R}^d$, that is subsets not hit by the graph $\mathcal{G}$ of the (L, α)-superdiffusion. Suppose that Γ is a closed set which does not contain the set $(-\infty, t) \times \mathbb{R}^d$. Then the following conditions on Γ are equivalent:

1. $P_{r,x}\{\mathcal{G} \cap \Gamma = \emptyset\} = 1$ for all $(r, x) \notin \Gamma$.
2. The equation $\dot{u} + Lu = u^\alpha$ has no solutions in Γ^c except 0.
3. $\text{Cap}_\alpha(\Gamma) = 0$.

Here Cap_α is a capacity defined by **12**.(1.2) with E and $\bar{E}$ replaced by $\mathbb{R}\times\mathbb{R}^d$, the Lebesgue measure m and $k(r,x;t,y) = e^{-(t-r)/2}(2\pi(t-r))^{-d/2}e^{-|x-y|^2/2(t-r)}$(the transition density of the Brownian motion with killing rate 1/2). The proof of the implication (3) $\Longrightarrow$ (2) was based on analytic results in [**BP84a**].

Sheu [**She93**] obtained conditions of $\mathcal{G}$-polarity in terms of restricted Hausdorff dimension $\mathcal{R}$-H-dim B. The definition of $\mathcal{R}$-H-dim B is similar to the definition of H-dim B but the balls of radius r are replaced by the sets of the form $[a_1, a_1 + r^2] \times \prod_2^{d+1}[a_i, a_i + r]$. In particular, he proved that a subset B of $\mathbb{R}^{d+1}$ is $\mathcal{G}$-polar if $\mathcal{R}$-H-dim $B < d - 2/(\alpha-1)$ and it is not $\mathcal{G}$-polar if $\mathcal{R}$-H-dim $B > d - 2/(\alpha-1)$.

Boundary singularities in a parabolic setting were studied by Kuznetsov [**Kuz97**], [**Kuz98a**], [**Kuz98b**]. Suppose that E is a bounded smooth domain in $\mathbb{R}^d$ and $\mathcal{G}$ is the graph of an (L,α)-superdiffusion in the cylinder $Q = \mathbb{R}_+ \times E$. Let Γ be a compact subset of the lateral boundary $\mathbb{R}_+ \times \partial E$ of Q. Then the following conditions on Q are equivalent:

(i) $P_{r,x}\{\mathcal{G} \cap \Gamma = \emptyset\} = 1$ for all $(r,x) \in Q$.

(ii) The problem

$$\begin{aligned} \dot{u} + Lu &= u^\alpha \quad \text{in } Q,\\ u &= 0 \quad \text{on } \partial Q \setminus \Gamma,\\ u &\to 0 \quad \text{as } t \to \infty. \end{aligned}$$

has no solutions except 0.

(iii) The parabolic Poisson capacity of Γ is equal to 0.

[The parabolic Poisson capacity is defined by **12**.(1.2) with E replaced by Q, $\tilde{E}$ equal to the lateral boundary of Q, $dm = \rho dt\ dx$ where $\rho(x) = d(x, \partial E)$ and $k(r,x;t,y)$ is proportional to the derivative of the transition density $p(r,x;t,y)$ with respect to y in the direction of inward normal.] In [**Kuz98b**] Kuznetsov proved that the null sets of the parabolic Poisson capacity coincide with the null sets of the Besov capacity $\mathrm{Cap}_{1/\alpha,2/\alpha,\alpha'}$ in notation of [**BIN79**].

Sheu [**She00**] investigated relations between the Poisson capacity and the restricted Hausdorff measure. As an application, he proved that the critical restricted Hausdorf dimension for $\mathcal{G}$-polarity on the lateral boundary is $d - (3-\alpha)/(\alpha-1)$.

CHAPTER 14

Survey of related results

In this chapter we give an exposition of various results closely related to the subject of the book but not covered in its main part.

1. Branching measure-valued processes

1.1. Definition and construction of BMV processes. A branching measure-valued (BMV) process is a Markov process $(X_t, P_{r,\nu})$ in the space $\mathcal{M}(E)$ of finite measures on a measurable space $(E, \mathcal{B})$ subject to the condition: for every positive $\mathcal{B}$-measurable function f and for all $r < t \in \mathbb{R}$ and $\nu \in \mathcal{M}(E)$,

$$\log P_{r,\nu} e^{-\langle f, X_t \rangle} = \int_E \nu(dx) \log P_{r,x} e^{-\langle f, X_t \rangle}. \tag{1.1}$$

(Here $P_{r,x} = P_{r,\delta_x}$.) Such a process was constructed in section **4**.4.2 from a BEM system. [1] Its transition function is determined by the equation

$$P_{r,\nu} e^{-\langle f, X_t \rangle} = e^{-\langle u_t, \nu \rangle} \tag{1.2}$$

where $u_t(r, x)$ is a solution of the integral equation

$$u_t(r, x) + \Pi_{r,x} \int_r^t \psi(s, \xi_s; u(s, \xi_s)) ds = \Pi_{r,x} f(t, \xi_t) \quad \text{for } r < t. \tag{1.3}$$

A wider class of BMV processes was constructed in [**Dyn91a**] by replacing the equation (1.3) by the equation

$$u_t(r, x) + \Pi_{r,x} \int_r^t \psi(s, \xi_s; u(s, \xi_s)) K(ds) = \Pi_{r,x} f(t, \xi_t) \quad \text{for } r < t \tag{1.4}$$

where K is a continuous additive functional of ξ. [2] An example of such a functional is given by the formula

$$K(B) = \int_B \rho(s, \xi_s) \; ds \quad \text{for Borel sets } B \subset \mathbb{R}. \tag{1.5}$$

In general, an additive functional K of ξ is a random measure on $\mathbb{R}$ such that, for every $r < t$ and every μ, $K[(r, t)]$ is measurable relative to the $P_{r,\mu}$-completion of the σ-algebra $\mathcal{F}(r, t)$ generated by $\xi_s, r < s < t$. An additive functional K is continuous if $K(\{s\}) = 0$ for every singleton $\{s\}$. It was proved in [**Dyn91a**] that (under some boundness restrictions on K) the equation (1.4) has a unique solution and the formula (1.2) determines a transition function of a BMV process X. The process

[1] In section **4**.4.2 we considered a time-dependent base space $(E_t, \mathcal{B}_t)$. Now, to simplify the presentation, we assume that all $(E_t, \mathcal{B}_t)$ are identical.

[2] Partial results in the same direction were obtained much earlier by Silverstein [**Sil68**], [**Sil69**].

X is time-homogeneous if ξ is time-homogeneous and K is a time-homogeneous additive functional. [3]

Suppose that ξ is a diffusion in a domain $E \subset \mathbb{R}^d$. Then to every finite measure η vanishing on polar sets [4] of ξ, there corresponds a time-homogeneous continuous additive functional K_η such that, for every positive Borel function $f(s,y)$

$$\Pi_{r,x} \int_{\mathbb{R}} f(s,\xi_s) K_\eta(ds) = \int_{\mathbb{R}} ds \int_E \eta(dy) p(r,x;s,y) f(s,y) \tag{1.6}$$

where p is the transition density of ξ. [5] Single point sets are not polar for one-dimensional diffusions. Therefore for such diffusions there exist continuous additive functionals K_{δ_x} corresponding to the Dirac's measures. (They are called the local times.) A study of superprocesses determined by K_{δ_x} was started by Dawson and Fleischman [**DF94**] who suggested for one of them the name "a super-Brownian motion with a single point catalyst". There was a number of subsequent publications devoted to this kind of superprocesses, among them [**Dyn95**], [**Del96**], [**FG95**].

1.2. Nonlocal branching. More BMV processes can be obtained by replacing a real-valued function ψ by an operator Ψ in the space of positive measurable functions on $\mathbb{R} \times E$. The equation (1.4) takes the form

$$u_t(r,x) + \Pi_{r,x} \int_r^t \Psi(u_t)(s,\xi_s) K(ds) = \Pi_{r,x} f(t,\xi_t) \quad \text{for } r < t. \tag{1.7}$$

The corresponding BMV process can be constructed by a passage to the limit from branching particle systems. However, in contrast to the systems considered in section **3**.1.2, we do not assume that the birthplace of each particle coincides with the deathplace of the parent (it can be random). On the other hand, we still assume that every particle is born at the deathtime of its parent. As a result, in our model, the values of $\Psi(u)$ at time s depend only on the values of u at the same time, that is, $\Psi(u)(s,x) = \Psi^s(u^s)(x)$ where $u^s(x) = u(s,x)$ and Ψ^s is an operator in the space of functions on E.

In [**Dyn93**] a class of Ψ was investigated for which the equation (1.7) has a unique solution and this solution determines a BMV process.

1.3. Structure of general BMV processes. In [**Wat69**] Watanabe described all time-homogeneous BMV processes in a two-point base space E under an additional assumption: for every strictly positive f, the function $P_x e^{-\langle f, X_t \rangle}$ is differentiable in t at $t = 0$. Most of his arguments are applicable to any finite space E. Another proof of a similar result is sketched in [**RS70**].

The structure of BMV processes in a rather general base space E was investigated in [**DKS94**]. [6] The first step is to define a process ξ starting from a BMV process X. The function

$$p(r,x;t,B) = P_{r,x} X_t(B) \tag{1.8}$$

[3] The functional (1.5) is time-homogeneous if $\rho(s,x)$ does not depend on s.

[4] B is a polar set of ξ if no path of ξ hits B with a positive probability.

[5] This follows, for instance, from a general theory of additive functionals developed in [**Dyn75**] and [**Dyn77**].

[6] The results of [**DKS94**] are presented with complete proofs and all prerequisites in [**Dyn94**].

always satisfies the condition 3.1.A in the Appendix A. We assume that it satisfies also the condition 3.1.B. Then p is the transition function of a Markov process ξ in E. We impose regularity assumptions on ξ and X which are commonly used in the theory of Markov processes. [7] In addition, we introduce a restriction slightly stronger than the finiteness of the second moments of the total mass $\langle 1, X_t\rangle$. Under these assumptions we prove that X satisfies the conditions (1.2) and (1.7) and we describe the general form of the operator Ψ. Similar results under weaker restriction on X were proved in [**Led00**].

2. Additive functionals

2.1. Generalized Poisson equation. Suppose that E is a bounded smooth domain in $\mathbb{R}^d$. The classical boundary value problem

$$\begin{aligned} -Lv + v^\alpha &= f \quad \text{in } E,\\ v &= \varphi \quad \text{on } \partial E \end{aligned} \tag{2.1}$$

(with Hölder continuous f and continuous φ) is equivalent to an integral equation

$$v(x) + \int_E g(x,y) v(y)^\alpha \, dy = h(x) \tag{2.2}$$

where

$$h(x) = \int_E g(x,y) f(y) dy + \int_{\partial E} k(x,y)\varphi(y)\gamma(dy), \tag{2.3}$$

$g(x,y)$ is the Green's function, $k(x,y)$ is the Poisson kernel of L in E and $\gamma(dy)$ is the normalized surface area on ∂E. We interpret v as a (generalized) solution of the problem

$$\begin{aligned} -Lv + v^\alpha &= \eta \quad \text{in } E,\\ v &= \nu \quad \text{on } \partial E \end{aligned} \tag{2.4}$$

involving two measures η and ν if the equation (2.2) holds with

$$h(x) = \int_E g(x,y)\eta(dy) + \int_{\partial E} k(x,y)\nu(dy). \tag{2.5}$$

It was proved in [**DK96a**] that the equation (2.2) (with $1 < \alpha \le 2$) has a solution if and only if η does not charge removable sets in E and ν does not charge removable sets in ∂E. Moreover, the solution of (2.2) is defined uniquely on the set $\{h < \infty\}$. [These conditions are still necessary for an arbitrary domain E. Sufficient conditions can be obtained by replacing removable sets by null sets of the Greenian capacity in E and the Martin capacity in ∂E.]

If $\eta = 0$, then h given by (2.5) is either harmonic or infinite. In the first case, a solution u of the integral equation (2.2) is moderate and, according to Chapter **9**, it has a unique representation of the form

$$u(x) = -\log P_x e^{-Z_\nu} \tag{2.6}$$

where Z_ν is a linear boundary functional of an (L,α)-superdiffusion X corresponding to ν. In the second case, the problems (2.2) and (2.1) have not much sense.

In the case $\nu = 0$, the integral equation (2.2) takes the form

$$u + G(u^\alpha) = G\eta \tag{2.7}$$

[7] Namely, we assume that ξ and X are Hunt processes – a subclass of the class of right processes described in the Appendix A.

where

$$G\eta(x) = \int_E g(x,y)\eta(dy). \tag{2.8}$$

If $\eta(dx) = f(x)dx$ with $f \in C^{2,\lambda}(E)$, then a solution of (2.7) can be expressed by the formula

$$u(x) = -\log P_x \exp\left[-\int_0^\infty \langle f, X_s\rangle\, ds\right]. \tag{2.9}$$

In the general case, we have a similar expression

$$u(x) = -\log P_x \exp[-A_\eta(0,\infty)] \tag{2.10}$$

where A_η is a linear additive functional of X (a concept which we introduce in the next section).

2.2. Linear additive functionals of superdiffusions. Let f be a positive Borel function on E. The formula

$$A(B) = \int_B \langle f, X_s\rangle\, ds \tag{2.11}$$

defines a random measure on $(0,\infty)$ with the following properties:

2.2.A. For every open interval I, $A(I)$ is measurable with respect to the σ-algebra $\mathcal{F}(I)$ generated by $X_s, s \in I$.

2.2.B. For every $\mu \in \mathcal{M}(E)$ and every I,

$$P_\mu A(I) = \int_E P_x A(I)\mu(dx). \tag{2.12}$$

Every random measure A with properties 2.2.A–2.2.B is called a linear additive functional (LAF) of X.

Formula (2.11) describes a LAF A_η corresponding to $\eta(dx) = f(x)dx$. A LAF corresponding to an arbitrary measure η not charging removable sets is given by the formula

$$A_\eta(0,t] = \lim_{\lambda\to\infty} \int_0^t \langle f_\lambda, X_s\rangle\, ds \quad \text{in } P_\mu\text{-probability} \tag{2.13}$$

for μ in a sufficiently rich subset $\mathcal{M}^*$ of $\mathcal{M}(E)$. Functions f_λ can be defined, starting from the transition density $p_t(x,y)$ of L-diffusion in E, by the formula

$$f_\lambda(x) = \lambda \int_E g_\lambda(x,y)\eta(dy) \tag{2.14}$$

where

$$g_\lambda(x,y) = \int_0^\infty e^{-\lambda s} p_s(x,y)\, ds. \tag{2.15}$$

[Note that the measures $f_\lambda(x)dx$ converge weakly to η.]

More precisely, A_η defined by formula (2.13) satisfies a weaker versions of 2.2.A–2.2.B. [In 2.2.A, the σ-algebras $\mathcal{F}(I)$ need to be replaced by their completion with respect to the family $P_\mu, \mu \in \mathcal{M}^*$, and (2.12) is satisfied only for $\mu \in \mathcal{M}^*$.]

Formula (2.6) can also be interpreted in terms of linear additive functionals: it is possible to associate with every measure ν on ∂E charging no removable sets a LAF A_ν such that $Z_\nu = A(0,\infty)$. Moreover, a solution of the general problem (2.4) can be expressed by the formula

$$u(x) = -\log P_x \exp[-(A_\eta + A_\nu)(0,\infty)]. \tag{2.16}$$

All these results were proved in [**DK97b**] in a more general time-inhomogeneous setting. The foundation for [**DK97b**] has been laid by a theory of natural linear additive functionals developed in [**DK97a**]. In particular, it was proved in [**DK97a**] that all such functionals have only fixed discontinuities [8] and all time-homogeneous functionals of time-homogeneous superprocesses are continuous.

3. Path properties of the Dawson-Watanabe superprocess

3.1. The Dawson-Watanabe superprocess (or, in other words, the super-Brownian motion with quadratic branching $\psi(u) = u^2$) has been investigated in great detail. For this process $\langle f, X_t\rangle$ is continuous *a.s.* for any bounded Borel function f ([**Rei86**]). [For a general (L,α)-superdiffusion, $\langle f, X_t\rangle$ is right continuous *a.s* for bounded continuous f.] The proof in [**Rei86**] is based on non-standard analysis. A standard proof for a broader class of processes is given in [**Per91**].

3.2. The support process. A set-valued process $K_t = \text{supp}\ X_t$ is studied in [**DIP89**] and [**Per90**]. It is proved that K_t is right continuous with left limits (in the topology induced by the Hausdorff metric in the space of compact sets) and that, for almost all ω,

(i) $K_t \subset K_{t-}$ for all $t > 0$;
(ii) $K_{t-}\backslash K_t$ is empty or a singleton for all $t > 0$;
(iii) $K_t = K_{t-}$ for each fixed $t > 0$.

It is easy to deduct from these results that the graph G of X is the union of all sets $\{t\} \times K_{t-}$, $t > 0$ and $\{0\} \times K_0$.

3.3. Relations between X_t and the Hausdorff measures. These relations were investigated in [**DH79**], [**Per88**], [**Per89**], [**Per90**], [**DIP89**] and [**DP91**].

If $d = 1$, then $X_t(dx) = \rho(t,x)dx$ with a continuous ρ [**Rei86**]. This result was established independently in [**KS88**]. (An earlier result in the same direction was obtained in [**RC86**].)

According to [**DH79**], for any d, the measure X_t is concentrated, a.s., on a random Borel set of Hausdorff dimension not larger than 2. Perkins [**Per89**] has proved that, for $d > 2$, $X_t(dx) = \rho_t(x)\eta(dx)$ where η is the Hausdorff measure corresponding to the function $\varphi(r) = r^2 \log\ \log \frac{1}{r}$. Moreover, *a.s.*, $0 < c_d \le \rho_t \le C_d < \infty$ on K_t for all $t > 0$. These results have been refined in [**DP91**]. It was shown that, for every fixed t, $\rho_t = const$ *a.s* and therefore K_t is a set-valued Markov process.

The case $d = 2$ left open in [**Per89**] and [**DP91**] was settled by Le Gall and Perkins in [**LP95**].

3.4. The fact that the range $\mathcal{R}$ of the Dawson-Watanabe superprocess is, a.s., compact was established, first, in [**Isc88**]. For the same process, necessary conditions for polarity were established in [**Per90**] independently of any results obtained by analysts. However, sufficient conditions (even for $\alpha = 2$) are not yet

[8] That is the set of discontinuities of the function $A(0,t]$ is independent of ω.

proved this way. Sharp results on the Hausdorff measure of supp $\mathcal{R}$ were obtained in [**DIP89**] for $d \geq 5$. The case $d = 4$ is covered in [**Le 99b**].

3.5. In [**Per90**] and [**DP91**] besides the super-Brownian motion also the (ξ, ψ)-superprocesses were investigated corresponding to symmetric stable processes ξ and $\psi(u) = u^2$. In this situation, the topological support K_t is, a.s., either the empty set or the entire state space and it must be replaced by a random Borel set Λ_t supporting X_t.

4. A more general operator L

4.1. In this book we considered (L, ψ)-superdiffusions with L given by **2**.(1.1). Most of the results can be extended to more general operators

$$Lu = L^0 u + cu \tag{4.1}$$

where L^0 has the form **2**.(1.1) and $c(r, x)$ is bounded, continuous in (r, x) and Lipschitz continuous in x uniformly in r. [9] Let $\xi = (\xi_t, \Pi_z)$ be an L^0-diffusion and let

$$H_t^r = \int_r^t c(s, \xi_s)\, ds. \tag{4.2}$$

An (L, ψ)-superprocess $X = (X_Q, P_\mu)$ is a BEM system subject to the condition **4**.(1.1) with operators G_Q and K_Q defined by the formulae

$$G_Q f(r, x) = \Pi_{r,x} \int_r^\tau H_s^r f(s, \xi_s)\, ds, \tag{4.3}$$

and

$$K_Q f(r, x) = \Pi_{r,x} H_\tau^r f(\tau, \xi_\tau) \tag{4.4}$$

(τ is the first exit time from Q). The existence of such a system can be proved, for the same class of function ψ, by the arguments used in Chapter **4** in the case $c = 0$. Most results of Chapters **4** and **5** are established for a general c in [**Dyn98a**] and [**DK99**]. In particular, it is proved that, if O is a regular relatively open subset of ∂Q, then the minimal solution of the problem

$$\begin{aligned} \dot{u} + Lu &= \psi(u) \quad \text{in } Q, \\ u &= \infty \quad \text{on } O \end{aligned} \tag{4.5}$$

is given by the formula

$$u(z) = -\log P_z\{X_Q(O) = 0\}. \tag{4.6}$$

4.2. Main results of [**DK99**] concern the extinction time σ_Q of X in Q. Suppose that ψ given by the formula **4**.(2.4) belongs to BR. If $c \geq 0$, then $\sigma_Q < \infty$ a.s. for all nonempty Q. This is not true for a general c.

To define σ_Q, we consider the restrictions $\tilde{L}$ and $\tilde{\psi}$ of L and ψ to Q. Let $(\tilde{X}_t, \tilde{P}_{r,x})$ be the branching measure-valued process associated with $(\tilde{L}, \tilde{\psi})$-superdiffusion $\tilde{X}$ (see section **4**.4.2). The extinction time σ_Q is defined by the formula

$$\sigma_Q = \sup\{t : \tilde{X}_t \neq 0\}. \tag{4.7}$$

[9] Like $b_i(r, x)$ in the condition **2**.(1.3).

It is proved in [**DK99**] that the event $\{\sigma_Q < \infty\}$ coincides, a.s., with the event $\{\mathcal{G}_Q$ is compact $\}$. [10] If coefficients of L do not depend on time, then the maximal solution of the equation $Lu = \psi(u)$ in a regular domain $D \subset \mathbb{R}^d$ vanishing on ∂D is given by the formula

$$w(x) = -\log\{\mathcal{R} \subset D, \sigma_D < \infty\} \tag{4.8}$$

where $\mathcal{R}$ is the range of $\tilde{X}$. For a bounded smooth domain D and for an operator L of the divergence form, it is proved that $\{\sigma_D < \infty\} = \{X_D(O) < \infty\}$ a.s. for all nonempty relatively open subsets O of ∂D.

5. Equation $Lu = -\psi(u)$

5.1. This equation was investigated by many authors. In [**Dyn00b**] an attempt was made to establish a link between a set $\mathcal{V}(E)$ of all its positive solutions in an open set E and an analogous set $\mathcal{U}(E)$ for equation $Lu = \psi(u)$. Recall that Theorem **8**.3.1 establishes a 1-1 correspondence between $\mathcal{U}_1(E) \subset \mathcal{U}(E)$ and a subclass $\mathcal{H}_1(E)$ of the class $\mathcal{H}(E)$ of all positive L-harmonic functions in E. In [**Dyn00b**], a 1-1 map from $\mathcal{V}^1(E) \subset \mathcal{V}(E)$ onto a set $\mathcal{H}^1(E) \subset \mathcal{H}(E)$ is introduced. Functions $u \in \mathcal{V}^1(E)$ and $h \in \mathcal{H}^1(E)$ correspond to each other if u is the minimal solution of the integral equation

$$u = G_E\psi(u) + h. \tag{5.1}$$

This is equivalent to the condition: h is the maximal L-harmonic function dominated by u. The class $\mathcal{H}^1(E)$ is contained in $\mathcal{H}_1(E)$, and therefore we have a 1-1 correspondence between $\mathcal{V}^1(E)$ and a subclass of $\mathcal{U}_1(E)$.

It is well known that the boundary value problem

$$\begin{aligned} Lu &= -\psi(u) \quad \text{in } E, \\ u &= f \quad \text{on } \partial E \end{aligned} \tag{5.2}$$

can have more than one solution [even in the case of bounded E and f]. However, it cannot have more than one solution of class $\mathcal{V}^1(E)$.

5.2. A connection between the equation $Lu = -u^2$ and superdiffusions was established in [**Dyn00a**]. Suppose that E is bounded regular and $f \geq 0$ is continuous. Then if

$$u(x) = \log P_x e^{\langle f, X_E\rangle} \tag{5.3}$$

is locally bounded, then u belongs to $\mathcal{V}^1(E)$ and it is a solution of the problem (5.2).

The class $\mathcal{U}_1(E)$ (moderate solutions) plays an important role in the study of the class $\mathcal{U}(E)$ of all positive solutions. The role of $\mathcal{V}^1(E)$ in the study of $\mathcal{V}(E)$ is much more limited.

[10] In the case $L = \Delta$ and $Q = \mathbb{R} \times \mathbb{R}^d$, this follows also from earlier results of Sheu [**She94a**], [**She97**].

6. Equilibrium measures for superdiffusions

6.1. A measure m is called an invariant measure for a stationary Markov transition function $p_t(x, dy)$ or, shortly, *p-invariant* if

$$\int m(dx)p_t(x, B) = m(B) \quad \text{for all } \ t, B.$$

If, in addition, it is a probability, then it is called a *p-equilibrium* (or a steady state for p). If p is the transition function of the Brownian motion in $\mathbb{R}^d$ (see **2**.(2.2)), then all the p-invariant measures are described by the formula $m(dx) = \text{const.}\, dx$ and there exist no p-equilibria.

In general, the p-equilibria form a convex cone generated by its extreme elements.

6.2. Let X be a time-homogeneous (L, ψ)-superdiffusion. Denote by p the transition function of an L-diffusion ξ and by $\mathcal{P}$ the transition function of X. If $\psi \in BR$ and if the state space of X is the set of finite measures, then the unit measure at point 0 is the only $\mathcal{P}$-equilibrium. To get a meaningful theory, we need to modify our setting and to construct X as a process in an appropriate space of infinite measures. We fix a measurable function $\rho > 0$ and we introduce a space $\mathcal{M}_\rho = \{\nu : \langle \rho, \nu \rangle < \infty\}$. This space is invariant with respect to the operators

$$\mathbb{P}_t f(x) = \int p_t(x, dy) f(y)$$

if the function $\mathbb{P}_t \rho/\rho$ is bounded for each $t > 0$. Put

$$Wf(x) = \int_0^\infty \psi(V_t f) dt$$

where

$$V_t f(x) = -\log \ P_{\delta_x} \exp\langle -f, X_t \rangle.$$

We say that a p-invariant measure m is dissipative if

$$\lim_{\lambda \downarrow 0} \frac{1}{\lambda} \langle W(\lambda f), m \rangle = 0$$

for some $f > 0$.

The case of $\psi = bu^2$ was treated in [**Dyn89b**]. [11] By Theorem 1.7 in [**Dyn89b**], to every p-invariant measure m there corresponds a unique $\mathcal{P}$-equilibrium measure M_m such that

$$\int_{\mathcal{M}_\rho} M_m(d\nu) e^{-\langle f, \nu \rangle} = e^{-\langle f - Wf, m \rangle}, \quad f \in L^1_+(m).$$

The map $m \to M_m$ establishes a 1-1 correspondence between the set of all dissipative p-invariant measures m and the set of all nontrivial extreme $\mathcal{P}$-equilibrium measures M such that

$$\int_{\mathcal{M}_\rho} M(d\nu) \langle \rho, \nu \rangle < \infty. \tag{6.1}$$

The inverse mapping is given by the formula

$$\int_{\mathcal{M}_\rho} M(d\nu)\nu = m.$$

[11] Only minor modifications are needed to cover other functions ψ of the form **8**.(2.18).

[M_m is concentrated at 0 if $\langle Wf, m\rangle = \langle f, m\rangle < \infty$ for some $f > 0$.]

6.3. A measure m is dissipative if $\psi(u) = u^\alpha$ and

$$\int_0^\infty \langle (\mathbb{P}_t f)^\alpha, m\rangle dt < \infty \tag{6.2}$$

for some $f > 0$.[This follows from an inequality $V_t f \le \mathbb{P}_t f$.]

The condition (6.2) is satisfied for all p-invariant measures $m(dx) = \text{const.}\, dx$ of the Brownian motion if $d > 2/(\alpha - 1)$. Indeed, $p_t(x, dy) = q_t(x - y)dy$ where

$$q_t(x) = (2\pi t)^{-d/2} e^{-|x|^2/2t}$$

and, if $f = q_1$, then $\mathbb{P}_t f = q_{t+1}$ and $\langle (\mathbb{P}_t f)^\alpha, m\rangle = const.(t+1)^{-d(\alpha-1)/2}$.

We conclude that there exists a one parameter family of equilibria for the super-Brownian motion if $d > 2/(\alpha-1)$. In the case $\alpha = 2$, this result was obtained much earlier by Dawson [**Daw77**]. Bramson, Cox and Greven [**BCG93**] have shown that there exist no nontrivial $\mathcal{P}$-equilibria if $\alpha = 2$ and $d \le 2$.

6.4. A closely related subject is the asymptotic behavior of branching particle systems and superprocesses as $t \to \infty$. Many authors contributed to this subject (see, e.g., [**Daw77**], [**Dyn89b**], [**Kal77b**], [**LMW89**], [**DP91**] and references there).

7. Moments of higher order

7.1. Suppose that X is a (ξ, ψ)-superprocess. If ψ satisfies the conditions **4**.3.1.A and **4**.4.3.A, then, by Lemma **4**.4.1 and **4**.(4.6), for all $r < t$,

$$P_{r,\mu}\langle f, X_t\rangle = \langle T_t^r f, \mu\rangle \tag{7.1}$$

where $T_t^r f(r, x) = \Pi_{r,x} f(\xi_t) = \int_E p(r, x; t, dt) f(y)$.

Denote by $\psi_m(r, x; u)$ the mth derivative of $\psi(r, x; u)$ with respect to u evaluated at $u = 0$ and put $q_m(r, x) = (-1)^m \psi_m(r, x; u)$. It is not too hard to prove that, if $r < \min\{t_1, t_2\}$, then

$$\begin{aligned} &P_{r,\mu}\langle f_1, X_{t_1}\rangle\langle f_2, X_{t_2}\rangle \\ &= \int_r^\infty ds \int_E \mu(dx) p(r, x; s, dy) q_2(s, y) T_{t_1}^s f_1(y) T_{t_2}^s f_2(y) + \langle T_{t_1}^r f_1, \mu\rangle\langle T_{t_2}^r f_2, \mu\rangle. \end{aligned} \tag{7.2}$$

To evaluate higher moments, we introduce notation

$$\{\varphi_1 \dots \varphi_m\}(r, x) = \Pi_{r,x} \int_r^\infty ds\ q_m(s, \xi_s)\varphi_1(s, \xi_s) \dots \varphi_m(s, \xi_s) \quad \text{for } m > 1 \tag{7.3}$$

or, in terms of the transition function,

$$\{\varphi_1 \dots \varphi_m\}(r, x) = \int_r^\infty ds \int_E p(r, x; s, dy) q_m(s, y)\varphi_1(s, y) \dots \varphi_m(s, y). \tag{7.4}$$

In addition, we put $\{\varphi_1\} = \varphi_1$. We consider monomials like $\{\{\varphi_3\varphi_2\}\varphi_1\{\varphi_4\varphi_5\}\}$. Permutations of terms inside any group $\{\dots\}$ does not change the result and we do not distinguish monomials obtained from each other by such permutations. There exist one monomial $\{\varphi_1\varphi_2\}$ of degree 2 and four distinguishable monomials of degree 3:

$$\{\varphi_1\varphi_2\varphi_3\}, \{\{\varphi_1\varphi_2\}\varphi_3\}, \{\{\varphi_2\varphi_3\}\varphi_1\}, \{\{\varphi_3\varphi_1\}\varphi_2\}. \tag{7.5}$$

Denote by $W(t_1, f_1; \dots; t_m, f_m)$ the sum of all monomials of degree m of $\varphi_1, \dots, \varphi_m$ where $\varphi_i(r, x) = T^r_{t_i} f_i(r, x)$. [For instance,

$$W(t_1, f_1; t_2, f_2; t_3, f_3) = \{\varphi_1\varphi_2\varphi_3\} + \{\{\varphi_1\varphi_2\}\varphi_3\} + \{\{\varphi_2\varphi_3\}\varphi_1\} + \{\{\varphi_3\varphi_1\}\varphi_2\}].$$

We also consider functions corresponding to subsets $\Lambda = \{i_1, \dots, i_\ell\}$ of $\{1, \dots, m\}$. We denote them by $W(t_\Lambda; f_\Lambda)$. [Abbreviations t_Λ and f_Λ are used for $(t_{i_1}, \dots, t_{i_\ell})$ and $(f_{i_1}, \dots, f_{i_\ell})$.]

Suppose that the m-th derivative of $\psi(r, x; u)$ with respect to u is continuous and bounded. Then, for every $r < \min\{t_1, \dots, t_n\}$ and for an arbitrary measure μ on E,

$$P_{r,\mu}\langle f_1, X_{t_1}\rangle \dots \langle f_m, X_{t_m}\rangle = \sum_{\Lambda_1,\dots,\Lambda_k} \int \prod_{i=1}^{k} W(t_{\Lambda_i}, f_{\Lambda_i})(r, x_i)\mu(dx_i) \tag{7.6}$$

where the sum is taken over all partitions of the set $\{1, \dots, m\}$ into disjoint nonempty subsets $\Lambda_1, \dots, \Lambda_k$, $k = 1, \dots, m$. Formulae (7.1) and (7.2) are particular cases of this expression.

7.2. If ψ is given by formula **4**.(2.4), then

$$q_2(r, x) = 2b(r, x) + \int_0^\infty \lambda^2 n(r, x; d\lambda)$$

and

$$q_m(r, x) = \int_0^\infty \lambda^m n(r, x; d\lambda) \quad \text{for } m > 2.$$

In particular, if $n = 0$, then $q_m = 0$ for all $m > 2$ and therefore only a binary operation $\{\varphi_1\varphi_2\}$ appears in formula (7.6). This case was investigated in [**Dyn88**]. [We used there notation $\varphi_1 * \varphi_2$ for $\{\varphi_1\varphi_2\}$.] In [**Dyn91a**] a formula similar to (7.6) was established for a wider class of superprocesses governed by the integral equation (1.4). For such superprocesses, $K(ds)$ has to be substituted for ds in (7.3).

7.3. To every monomial there corresponds a rooted tree with the leaves labeled by $1, 2, \dots, m$. Here are trees corresponding to the monomials (7.5):

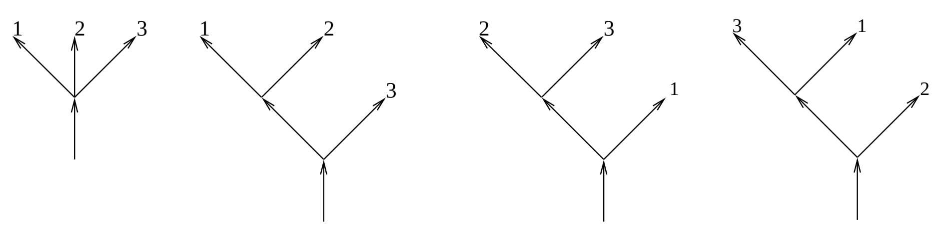

FIGURE 1

A diagram is the union of disjoint rooted trees. It consists of a set V of vertices (or sites) and a set A of arrows. We write $a : v \to v'$ if v is the beginning and v' is the end of an arrow a. We denote by $a_+(v)$ the number of arrows which end at v and by $a_-(v)$ the number of arrows which begin at v. Note that $a_+(v) = 0, a_-(v) = 1$ for roots and $a_+(v) = 1, a_-(v) = 0$ for leaves. We denote the corresponding subsets of V by V_- and V_+. For the rest of vertices, $a_+(v) = 1, a_-(v) > 1$. We denote the set of these vertices by V_0.

Let $\mathbb{D}_m$ be the set of all diagrams with leaves marked by $1, 2, \ldots, m$. We label each site of $D \in \mathbb{D}_m$ by two variables – one real-valued and the other with values in E. Namely, $t_i z_i$ is the label of the leave marked by i, rx_v is the label of a root v and $s_v y_v$ is the label of $v \in V_0$.

For an arrow $a : v \to v'$, we put $p_a = p(s, w; s', dw')$ where sw is the label of v and $s'w'$ is the label of v'. [12] In this notation, the right side in (7.6) is equal to the sum of c_D over all $D \in \mathbb{D}_m$ where

$$c_D = \int \prod_{v \in V_-} \mu(dx_v) \prod_{a \in A} p_a \prod_{v \in V_0} q_{a_-(v)}(s_v, y_v)\ ds_v \prod_{i=1}^{m} f_i(z_i). \tag{7.7}$$

Example. The diagram D corresponding to $\{\varphi_1 \varphi_2\} + \varphi_3$ can be labeled as follows (in contrast to the marking of the leaves, the enumeration of V_- and V_0 is of no importance),

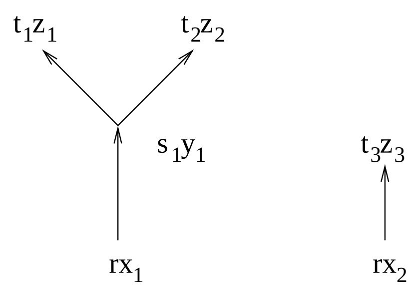

FIGURE 2

and we have

$$c_D = \int \mu(dx_1)\mu(dx_2)p(r, x_1; s_1, dy_1)q_2(s_1, y_1)ds_1 p(s_1, y_1; t_1, dz_1)p(s_1, y_1; t_2, dz_2) \times p(r, x_2; t_3, dz_3)f_1(z_1)f_2(z_2)f_3(z_3).$$

8. Martingale approach to superdiffusions

We already have mentioned (see Notes to Chapter **2**) a method of Stroock and Varadhan for constructing diffusions by solving a martingale problem. It is based on a relation between martingales and partial differential equations stated in Theorem **2**.2.1. A version of this theorem for a time-homogeneous case can be formulated as follows. If (ξ_t, Π_x) is an L-diffusion in $\mathbb{R}^d$, then, for every $f \in C_0^2(\mathbb{R}^d)$

$$Y_t = f(\xi_t) - \int_0^t Lf(\xi_s)\ ds, t \geq 0$$

is a continuous martingale with respect to $\mathcal{F}[0, t]$ and Π_x. The Stroock-Varadhan martingale problem is: for a given operator L, define measures Π_x on the space of continuous paths such that Y_t is a martingale for a wide enough class $\mathcal{D}$ of functions f.

[12] Recall that we set $p(s, x; t, B) = 0$ for $s > t$.

To apply this approach to superdiffusions, it is necessary to introduce an appropriate class $\mathcal{D}$ of functions on the space $\mathcal{M}$ of finite measures and to define on this class an operator $\mathcal{L}$ related to L. For ψ given by the formula **4**.(2.4) with time independent b and n, the operator $\mathcal{L}$ can be defined by the formula

$$\mathcal{L}f(\mu) = F'(\langle\varphi,\mu\rangle)\langle L\varphi,\mu\rangle + F''(\langle\varphi,\mu\rangle)\langle b\varphi^2,\mu\rangle$$
$$+ \int \mu(dx)\int_0^\infty n(x,d\lambda)\{F[\langle\varphi,\mu\rangle + \lambda\varphi(x)] - F(\langle\varphi,\mu\rangle) - F'(\langle\varphi,\mu\rangle)\lambda\varphi(x)\}$$

on functions $f(\mu) = F(\langle\varphi,\mu\rangle)$ where $F \in \mathbb{C}^\infty$ and $\varphi \in C_0^2$. For a solution of this martingale problem we refer to [**KRC91**], [**Fit88**], [**Fit92**], [**Daw93**] and [**Eth00**].

9. Excessive functions for superdiffusions and the corresponding h-transforms

With every stationary transition function $p_t(x,B)$ a class $Exc(p)$ of p-excessive functions is associated. It consists of positive Borel functions h such that

$$\mathbb{P}_t h(x) \le h(x) \quad \text{for all } t, x$$

and

$$\mathbb{P}_t h(x) \to h(x) \quad \text{for all } x \quad \text{as } t \downarrow 0$$

where

$$\mathbb{P}_t h(x) = \int_E p_t(x,dy)h(y).$$

If p is the transition function of a diffusion, then $Exc(p)$ contains all positive L-harmonic functions studied in Chapter **7**. The h-transform introduced in section **7**.4.1 can be applied to any p-excessive function. The corresponding measure $\tilde{\Pi}_x^h$ can be obtained from the measure Π_x by the conditioning on a specific limit behavior of the path as $t \to \infty$ or to the death time of the process.

Let $\mathcal{P}$ be the transition function of a superdiffusion X. An example of a $\mathcal{P}$-excessive function is the total mass $h(\mu) = \langle 1,\mu\rangle$. The corresponding h-transform is a superdiffusion conditioned on non-extinction investigated in [RR89], [EP90], [Eva93] and [Eth93] .

In the case of $\psi(u) = u^2$, a reach class of $\mathcal{P}$-excessive functions was constructed in [**Dyn99**]. Let $p_t(x,B)$ be the transition function of an L- diffusion ξ. Put

$$\mathbb{P}_t^n f(x_1,\dots,x_n) = \int_{E^n} p_t(x_1,dy_1)\dots p_t(x_n,dy_n) f(y_1,\dots,y_n).$$

Suppose that

$$\mathbb{P}_t^n f \le f, \quad \lim_{t\downarrow 0} \mathbb{P}_t^n f = f \quad \text{and} \quad \lim_{t\to\infty} \mathbb{P}_t^n f = 0.$$

By Theorem 3.1 in [**Dyn99**], to every f with these properties there corresponds a $\mathcal{P}$-excessive function

(9.1)

$$h(\mu) = \int_{E^n} f(x_1,\dots,x_n)\mu(dx_1)\dots\mu(dx_n) + \sum_{i=1}^{n-1}\int_{E^i}\varphi_i^n(x_1,\dots,x_i)\mu(dx_1)\dots\mu(dx_i)$$

[functions φ_i^n are determined by f and the transition function p].

It is interesting to compare the corresponding h-transforms with the processes constructed in [SaV99] and [SaV00] by conditioning on the hitting a finite number of specified boundary points by the exit measure from a domain E.

10. Infinite divisibility and the Poisson representation

10.1. Infinitely divisible random measures. Suppose that $(S, \mathcal{B}_S)$ is a measurable space and let $X = (X(\omega), P)$ be a random measure with values in the space $\mathcal{M} = \mathcal{M}(S)$ of all finite measures on S. [13] X is called infinitely divisible if, for every n, there exist independent identically distributed random measures $(X_1, P), \dots, (X_n, P)$ such that $X_1 + \dots + X_n$ has the same probability distribution as X.

10.2. Laplace functionals of infinitely divisible measures. The probability distribution of (X, P) is a measure on $\mathcal{M}(S)$. It is determined uniquely by the Laplace functional

$$L_X(f) = Pe^{-\langle f, X\rangle}, \quad f \in \mathbb{B} \tag{10.1}$$

[Cf. **3**.(3.5).] X is infinitely divisible if and only if, for every n, there exists a random measure (Y, P) such that $L_X = L_Y^n$. It is clear that this condition is satisfied if

$$L_X(f) = \exp\left[-\langle f, m\rangle - \int_{\mathcal{M}} (1 - e^{-\langle f, \nu\rangle})\mathcal{R}(d\nu)\right] \tag{10.2}$$

where m is a measure on S and $\mathcal{R}$ is a measure on $\mathcal{M}$. If $(S, \mathcal{B}_S)$ is a Luzin space, [14] then the Laplace functional of an arbitrary infinitely divisible random measure X has the form (10.2) (see, e.g., [**Kal77a**] or [**Daw93**]). We can assume that $\mathcal{R}\{0\} = 0$.

It follows from (10.1) and (10.2) that, for every constant $\lambda > 0$,

$$\lambda\langle 1, m\rangle + \int_{\mathcal{M}} (1 - e^{-\lambda\langle 1, \nu\rangle})\mathcal{R}(d\nu) \le -\log Pe^{-\lambda\langle 1, X\rangle}.$$

The right side tends to $-\log P\{X = 0\}$ as $\lambda \to \infty$. Therefore if $P\{X = 0\} > 0$, then $m = 0$, $\mathcal{R}(\mathcal{M}) < \infty$ and (10.2) takes the form

$$L_X(f) = \exp[-\int_{\mathcal{M}} (1 - e^{-\langle f, \nu\rangle})\mathcal{R}(d\nu)].\text{[15]} \tag{10.3}$$

10.3. Poisson random measure. An example of an infinitely divisible random measure is provided by the Poisson random measure. Suppose that J is a probability measure on a measurable space $(S, \mathcal{B}_S)$. Then there exists a random measure (X, P) on S with the properties:

10.3.A. $X(B_1), \dots, X(B_n)$ are independent for disjoint $B_1, \dots, B_n$.

10.3.B. $X(B)$ is a Poisson random variable with the mean $J(B)$, i.e.,

$$P\{X(B) = n\} = \frac{1}{n!}J(B)^n e^{-J(B)} \quad \text{for } n = 0, 1, 2, \dots.$$

[13] More general classes $\mathcal{M}$ are also considered. For instance, the class $\mathcal{M} = \{\nu : \langle \rho, \nu\rangle < \infty\}$ for a fixed function $\rho > 0$ is considered in [**Dyn89b**].

[14] That is if there exists a 1-1 mapping from S onto a Borel subset $\tilde{S}$ of a compact metric space such that sets of $\mathcal{B}_S$ corresponds to Borel subsets of $\tilde{S}$.

[15] A finite measure $\mathcal{R}$ not charging 0 is determined uniquely by (10.3).

We call X the *Poisson random measure with intensity* J. [16] This is an integer-valued measure concentrated on a finite set. Its values can be interpreted as configurations in S.

The Laplace functional of the Poisson random measure is given by the formula

$$L_X(f) = \exp[\langle e^{-f} - 1, J\rangle]. \tag{10.4}$$

Clearly, X is infinitely divisible.

If (X, P) is a random measure on S with the Laplace functional (10.3) and if (Y, Q) is the Poisson random measure on $\mathcal{M}(S)$ with intensity $\mathcal{R}$, then, by (10.4), $L_X(f) = L_Y(F)$ where $F(\nu) = \langle f, \nu\rangle$. Therefore (X, P) has the same probability distribution as $(\tilde{X}, Q)$ where $\tilde{X}(B) = \int_{\mathcal{M}} \nu(B)Y(d\nu)$.

10.4. Poisson representation of superprocesses. If $X = (X_Q, P_\mu)$ is a (ξ, ψ)-superprocess [or, more generally, a BEM system], then, by **3**.(1.11), the Laplace functional L_Q^μ of the random measure (X_Q, P_μ) is given by the formula

$$L_Q^\mu(f) = P_\mu e^{-\langle f, X_Q\rangle} = e^{-\langle V_Q(f), \mu\rangle}. \tag{10.5}$$

Hence (X_Q, P_μ) is infinitely divisible. If $\psi \in BR$, then, by **5**.(5.5), $u^0(z) = -\log P_z\{X_Q = 0\}$ is the minimal barrier. Hence, $P_z\{X_Q = 0\} = e^{-u^0(z)} > 0$ and, by (10.3),

$$L_Q^z(f) = \exp\left[-\int_{\mathcal{M}} (1 - e^{-\langle f, \nu\rangle})\mathcal{R}_Q^z(d\nu)\right] \tag{10.6}$$

for a finite measure $\mathcal{R}_Q^z$ which does not charge 0. It follows from the continuous branching property **3**.1.3.A that

$$L_Q^\mu(f) = \exp\left[-\int_{\mathcal{M}} (1 - e^{-\langle f, \nu\rangle})\mathcal{R}_Q^\mu(d\nu)\right] \tag{10.7}$$

where

$$\mathcal{R}_Q^\mu = \int_{\mathcal{M}} \mu(dz)\mathcal{R}_Q^z.$$

Let $(Y, \tilde{P})$ be the Poisson random measure on $\mathcal{M}$ with intensity $\mathcal{R}_Q^\mu$ and let $\tilde{X}_Q^\mu(B) = \int_{\mathcal{M}} Y(d\nu)\nu(B)$. The random measure (X_Q, P_μ) has the same probability distribution as the random measure $(\tilde{X}_Q^\mu, \tilde{P})$.

Note that, by (10.5) and (10.6),

$$V_Q(f)(z) = -\log L_Q^z z(f) = \int_{\mathcal{M}} (1 - e^{-\langle f, \nu\rangle})\mathcal{R}_Q^z(d\nu). \tag{10.8}$$

The Poisson representation (10.7) and a closely related the Poisson cluster representation are among the principal tools used by Dawson and Perkins in [**DP91**] to study the structure of random measures $(X_t, P_{r,x})$ for (ξ, ψ)-superprocesses and the corresponding historical superprocesses (described in section 11).

An intuitive meaning of the normalized measure

$$J_Q^z(d\nu) = \mathcal{R}_Q^z(d\nu)/\mathcal{R}_Q^z(\mathcal{M})$$

---

[16] If J is a finite measure, then $J(B)$ in 10.3.B is to be replaced by $J(B)/J(S)$.

is illuminated by the formula [17]

$$\int_{\mathcal{M}} e^{-\langle f,\nu\rangle} J_Q^z(d\nu) = \lim_{\varepsilon\to 0} P_{\varepsilon\delta_z}\left\{e^{-\langle f,X_Q\rangle}\big|X_Q \neq 0\right\}. \tag{10.9}$$

To prove this formula, we note that, by (10.5) and (10.8), for every $\varepsilon > 0$,

$$P_{\varepsilon\delta_z}\{X_Q = 0\} = \lim_{\lambda\to\infty} P_{\varepsilon\delta_z} e^{-\lambda\langle 1,X_Q\rangle}$$
$$= \lim_{\lambda\to\infty} \exp\left[-\varepsilon\int_{\mathcal{M}} (1-e^{-\lambda\langle 1,\nu\rangle})\mathcal{R}_Q^z(d\nu)\right] = e^{-\varepsilon\mathcal{R}_Q^z(\mathcal{M})}.$$

Therefore

$$P_{\varepsilon\delta_z}\left\{e^{-\langle f,X_Q\rangle}\big|X_Q \neq 0\right\} = \frac{P_{\varepsilon\delta_z}[X_Q\neq 0, e^{-\langle f,X_Q\rangle}]}{P_{\varepsilon\delta_z}[X_Q\neq 0]} = \frac{P_{\varepsilon\delta_z}e^{-\langle f,X_Q\rangle} - P_{\varepsilon\delta_z}[X_Q = 0, e^{-\langle f,X_Q\rangle}]}{1-P_{\varepsilon\delta_z}[X_Q=0]}.$$

The right side tends to

$$1 - \frac{V_Q(f)(z)}{\mathcal{R}_Q^z(\mathcal{M})} = \int_{\mathcal{M}} e^{-\langle f,\nu\rangle} J_Q^z(d\nu)$$

as $\varepsilon \to 0$.

11. Historical superprocesses and snakes

11.1. We start from a branching particle system introduced in section **3**.1.2. An evolution of such a system can be described by the mass distribution X_t at every time t. However, the BMV process (X_t, P_μ) reflects only a small part of information about the system. A part sufficient for applications to partial differential equations is provided by a BEM system (X_Q, P_μ) constructed in section **3**.1.3. A complete picture of the evolution is given by a random tree composed from the paths of all particles. Two ways to encode this picture are provided by historical BMV process $\hat{X}$ and by random snakes $\mathfrak{Z}$.

11.2. Historical superprocesses. The first step in building $\hat{X}$ is an introduction of a historical process $\hat{\xi}$ corresponding to ξ. A state $\hat{\xi}_t$ of $\hat{\xi}$ at time t is the path $\xi_{\le t} = \{\xi_s : s\in[0,t]\}$ of ξ over the time interval $[0,t]$. The historical process is a Markov process with a state space $\mathbb{W}_t$ depending on t. Put $|w| = t, \partial w = w_t$ for $w \in \mathbb{W}_t$. For every $w\in\mathbb{W}_r$, the probability distribution of $\xi_{\le t}, t\ge r$ with respect to $\Pi_{r,w}$ is the same as the probability distribution of a random path which coincides with w up to time r and coincides, after that time, with the path $\xi_{\ge r} = \{\xi_s : s \ge r\}$ of ξ started at time r from w_r.

Let $\mathcal{M}_t$ be the space of all finite measures on $\mathbb{W}_t$. The *historical BMV superprocess* is a Markov process $(X_t, P_{r,\mu})$ in the space $\mathcal{M}_t$ such that, for every r,μ, and every positive measurable function f on $\mathbb{W}_t$,

$$P_{r,\mu}e^{-\langle f,X_t\rangle} = e^{-\langle v^r,\mu\rangle}$$

where

$$v^r(w) + \Pi_{r,w}\int_r^t \psi(v^s)(\xi_{\le s})ds = \Pi_{r,w} f(\xi_{\le t}) \quad \text{for all } r\le t, w\in\mathbb{W}_r.$$

[17]See (3.7) in [**DP91**].

The concept of historical superprocesses is in the center of a monograph [**DP91**] by Dawson and Perkins. We refer also to their expositions [**Daw93**] and [**Per01**]. A slightly different approach to this subject is contained in [**Dyn91b**].

11.3. Snakes. Like a historical process, a snake $\mathfrak{Z}$ is a path-valued process with a time dependent state space. However $|\mathfrak{Z}_t| = \zeta_t$ is random. The dependence of $\mathfrak{Z}$ on ξ is easy to describe if the time parameter t takes values in $\beta\mathbb{Z}_+ = \{0, \beta, 2\beta, \dots\}$. Then $\mathfrak{Z}_{t+1}$ is the restriction of $\mathfrak{Z}_t$ to $[0, \zeta_{t+1}]$ if $\zeta_{t+1} \leq \zeta_t$ and it is an extension of $\mathfrak{Z}_t$ if $\zeta_{t+1} > \zeta_t$, namely, it is defined on the interval $[\zeta_t, \zeta_{t+1}]$ as the path of ξ started at time ζ_t from the end point $\partial\mathfrak{Z}_t$ of $\mathfrak{Z}_t$.

The snake corresponding to a branching particle system is described as follows. We write $b \prec b'$ if b is the parent of b'. The historical path of b is the combination $w^b = w(b_0)w(b_1)\cdots w(b_n)$ of paths of $b = b_n$ and all its ancestors $b_0 \prec b_1 \prec \cdots \prec b_n$.

Denote by P_x the probability law of a branching particle system started at time 0 by a progenitor b_0 located at x. Its total posterity is finite a.s. Order, some way, the offspring of each particle. Enumerate the particles starting from b_0 as follows: b_{n+1} is the first child of b_n which is not among $b_1, \dots, b_{n-1}$ or, if there is no such children, then b_{n+1} is the parent of b_n. The process is terminated at the first return to b_0.

In general, the corresponding snake is not a Markov process. It is Markovian if every particle lives for a constant time β and it produces at the death time offspring of size n with probability

$$p_n = \frac{1}{2^{n+1}}, \quad n = 0, 1, 2, \dots.$$

The corresponding process ζ_t is a simple random walk on $\beta\mathbb{Z}_+$ killed at the first hitting of the origin. Instead of terminating the process at the first return to its initial state, we can continue it indefinitely by replacing ζ_t by the reflecting random walk in $\beta\mathbb{Z}_+$.

The Brownian snake can be obtained, heuristically, by passing to the limit as $\beta \to 0$. Le Gall defined it as a continuous path-valued strong Markov process $(\mathfrak{Z}_t, \mathbb{P}_x)$ with the transition mechanism characterized by several properties including:

(a) $|\mathfrak{Z}_t| = \zeta_t$ is the reflecting Brownian motion in $\mathbb{R}_+$;

(b) $\mathfrak{Z}_{s_1}(t) = \mathfrak{Z}_{s_2}(t)$ if $t \leq \zeta_s$ for all $s \in [s_1, s_2]$. [18]

Le Gall [**Le 91**], [**Le 93a**] proved the existence of the Brownian snake $\mathfrak{Z}$ and expressed, in terms of $\mathfrak{Z}$, the exit measures of the Dawson-Watanabe superprocess (that is (ξ, ψ)-superprocess corresponding to the Brownian motion ξ and $\psi(u) = u^2$). In [**Le 95**], [**Le 97**] he applied this construction to the investigation of positive solutions and removable boundary singularities of equation $\Delta u = u^2$. A systematic presentation of his method is given in [**Le 99a**].

A correspondence between additive functionals of historical superprocesses and additive functionals of snakes was established in [**DK95**].

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[18] Clearly, the transition rules described above in the case of a discrete time parameter t imply (b).

APPENDIX A

Basic facts on Markov processes and martingales

In this appendix we collect basic definitions and results on Markov processes and martingales used in the book. We present the results not in the most general form but just in the setting we need them. As a rule we give no proofs but refer to books where the proofs can be found. An exception is the concept of a strong Markov process in the time-inhomogeneous setting. We introduce it in a form more general and more convenient for applications than anything available in the literature and we prove in detail all related statements.

We start from a theorem on classes of measurable functions which is applied many times in the book.

1. Multiplicative systems theorem

1.1. A family Q of real-valued functions on Ω is called multiplicative if it is closed under multiplication. We say that a sequence of functions $X_n(\omega)$ converges boundedly to $X(\omega)$, if it converges at every ω and if functions $X_n(\omega)$ are uniformly bounded.

THEOREM 1.1. *Let a linear space H of bounded functions contain 1 and be closed under bounded convergence. If H contains a multiplicative family Q, then it contains all bounded functions measurable with respect to the σ-algebra generated by Q.*

We call Theorem 1.1 the *multiplicative systems theorem*. We deduce it from the so-called π-λ-lemma.

1.2. A class $\mathcal{P}$ of subsets of Ω is called a π-system if $\mathcal{P}$ contains $A \cap B$ for every $A, B \in \mathcal{P}$. A class $\mathcal{L}$ is called λ-system if it contains Ω and is closed under the formation of complements and of finite or countable disjoint unions.

LEMMA 1.1. *If a λ-system $\mathcal{L}$ contains a π-system $\mathcal{P}$, then $\mathcal{L}$ contains the σ-algebra generated by $\mathcal{P}$.*

We refer for the proof to [**Dyn60**] (Lemma 1.1) or to [**Bil95**] (Theorem 3.2).

1.3. Proof of the multiplicative systems theorem. Put $A \in \mathcal{L}$ if $1_A \in H$ and denote by $\mathcal{P}$ the family of sets

$$\{\omega : X_1(\omega) \in I_1, \dots, X_m(\omega) \in I_m\} \tag{1.1}$$

where $m = 1, 2, \dots,$ $X_1, \dots, X_m \in Q$ and $I_1, \dots, I_m$ are open intervals. Clearly $\mathcal{L}$ is a λ-system and $\mathcal{P}$ is a π-system.

Let us prove that $\mathcal{P} \subset \mathcal{L}$. It is easy to construct, for every k a sequence of continuous functions f_n^k which converges boundedly to 1_{I_k}. Suppose $|X_k(\omega)| \le c$ for $k = 1, \dots, m$ and for all $\omega \in \Omega$. By the Weierstrass theorem, there exists a polynomial g_n^k such that $|g_n^k(t) - f_n^k(t)| \le \frac{1}{n}$ for all $|t| \le c$. Functions $Y_n(\omega) = g_n^1[X_1(\omega)] \dots g_n^m[X_m(\omega)]$ belong to H and converge boundedly, as $n \to \infty$, to the indicator of the set (1.1). We conclude that the set (1.1) belongs to $\mathcal{L}$.

By the π-λ-lemma, $\mathcal{L}$ contains the σ-algebra generated by $\mathcal{P}$ which is the same as the σ-algebra generated by Q. If X is an arbitrary bounded function measurable with respect to this σ-algebra, then the functions

$$X_n = \sum_k \frac{k}{n} 1_{\frac{k}{n} < X \le \frac{k+1}{n}}$$

belong to H and they converge boundedly to X. Thus $X \in H$. [1] □

2. Stopping times

This is a concept playing a central role, both, in theory of Markov processes and in theory of martingales.

2.1. Suppose that $\mathbb{T}$ is a subset of $\mathbb{R}$ and $(\Omega, \mathcal{A}, P)$ is a probability space. A family of sub-σ-algebras $\mathcal{A}_t, t \in \mathbb{T}$ of $\mathcal{A}$ is called a *filtration* of $(\Omega, \mathcal{A}, P)$ if $\mathcal{A}_s \subset \mathcal{A}_t$ for $s \le t$. We say that a function τ from Ω to $\mathbb{T} \cup \{+\infty\}$ is a *stopping time* if $\{\tau \le t\} \in A_t$ for every $t \in \mathbb{T}$. Subsets $C \in \mathcal{A}$ with the property $C \cap \{\tau \le t\} \in \mathcal{A}_t$ for all $t \in \mathbb{T}$ form a σ-algebra which we denote $\mathcal{A}_\tau$. Functions measurable with respect to $\mathcal{A}_\tau$ are called *pre-τ functions*. Note that $f(\tau)$ is a pre-τ function for every Borel function f. If τ and τ' are stopping times and if $\tau \le \tau'$, then $\mathcal{A}_\tau \subset \mathcal{A}_{\tau'}$.

If $\mathbb{T}$ is finite, then τ is a stopping time if and only if $\{\tau = t\} \in \mathcal{A}_t$ for all $t \in \mathbb{T}$.

We say that a stopping time is *simple* if it takes a finite number of finite values and, possibly, the value $+\infty$. If finite values of τ are equal to $t_1, \dots, t_m$, then τ is a stopping time if and only if $C_i = \{\tau = t_i\} \in \mathcal{A}_{t_i}$ for every i and X is a pre-τ function if and only if $X 1_{C_i}$ is $\mathcal{A}_{t_i}$-measurable for every i.

2.2. The most important are the cases $\mathbb{T} = \mathbb{R}$ and $\mathbb{T} = \mathbb{R}_+$. For every finite subset $\Lambda = \{t_1 < \dots < t_m\}$ of $\mathbb{T}$, we put

$$\varphi_\Lambda(t) = \begin{cases} t_1 & \text{for } t \le t_1, \\ t_i & \text{for } t_{i-1} < t \le t_i, \\ \infty & \text{for } t > t_m. \end{cases} \tag{2.1}$$

If τ is a stopping time, then, for every Λ, $\tau_\Lambda = \varphi_\Lambda(\tau)$ is also a stopping time. If Λ_n is an increasing sequence of finite sets with the union everywhere dense in $\mathbb{T}$, then $\tau_{\Lambda_n} \downarrow \tau$. We call $\tau_n = \tau_{\Lambda_n}$ *simple stopping times approximating* τ.

3. Markov processes

3.1. Markov transition functions and Markov processes. A Markov transition function in a measurable space $(E, \mathcal{B})$ is a function $p(r, x; t, B)$, $r < t \in \mathbb{R}, x \in E, B \in \mathcal{B}$ which is $\mathcal{B}$-measurable in x and which is a measure in B subject to the conditions:

3.1.A. $\int_E p(r, x; t, dy) p(t, y; u, B) = p(r, x; u, B)$ for all $r < t < u, x \in E$ and all $B \in \mathcal{B}$.

[1] Cf. [**Dyn60**] (Lemma 1.2) and [**Mey66**] (Chapter 1, Theorem 20). Theorem 1.1 and Lemma 1.1 are proved also in [**EK86**] (the Appendix, section 4).

3.1.B. $p(r, x; t, E) \le 1$ for all r, x, t.

[These conditions are satisfied for $p(r, x; t, B) = \int_B p(r, x; t, y)\, dy$ where $p(r, x; t, y)$ has the properties **2**.(1.6)–(1.7). They also hold for the function p_Q introduced in **2**.5.1.]

To every Markov transition function, there corresponds a stochastic process $\xi = (\xi_t, \Pi_{r,x})$ such that

$$\Pi_{r,x}\{\xi_{t_1} \in B_1, \dots, \xi_{t_n} \in B_n\} = \int_{B_1} \dots \int_{B_n} p(r, x; t_1, dy_1) p(t_1, y_1; t_2, dy_2) \dots p(t_{n-1}, y_{n-1}; t_n, dy_n). \tag{3.1}$$

It satisfies the condition **2**.(2.4) (the Markov property).

3.2. Right processes. We say that $\xi = (\xi_t, \Pi_{r,x})$ is a *right process* if its transition function has the following property: for every $r < u, x$ and every bounded measurable function f,

$$\Pi_{t,\xi_t} f(\xi_u) = \int_E p(t, \xi_t; u, dy) f(y)$$

is, $\Pi_{r,x}$-a.s., right continuous in t on the interval $[r, u)$. It follows from **2**.1.3.2 that all diffusions are right processes.

3.2.A. If ξ is a right process and if a function $h \ge 0$ satisfies the condition

$$\int_E p(t_1, x; t_2, dy) h(t_2, y) = h(t_1, x) \quad \text{for all } x \in E, t_1 < t_2 \in [r, u),$$

then $h(t, \xi_t)$ is, $\Pi_{r,x}$-a.s. right continuous in t on the interval $[r, u)$. In particular, this is true for $h(t, y) = \Pi_{t,y} Y$ where $Y \in \mathcal{F}_{\ge u}$.

This follows from Theorem 5.1 in [**Dyn73**]. [2]

3.3. Strong Markov property. We introduce a class $\mathcal{R}$ of functions Y on $\mathbb{R} \times \Omega$ (we call them *reconstructable functions*) in three steps. First, we consider the set $\mathcal{R}_0$ of all right continuous in t functions $Y_t(\omega)$ such that $Y_t \in \mathcal{F}_{\ge t}$ for all t. Then we put $Y \in \mathcal{R}_1$ if Y is measurable relative to the σ-algebra in $\mathbb{R} \times \Omega$ generated by $\mathcal{R}_0$. Finally, we put $Y \in \mathcal{R}$ if there exists $\tilde{Y} \in \mathcal{R}_1$ such that $\Pi_{r,x}\{Y \ne \tilde{Y}\} = 0$ for all r, x.

We start with the following lemma.

LEMMA 3.1. *Suppose that $\xi = (\xi_t, \Pi_{r,x})$ is a right Markov process and let $Y_t \ge 0$ be a reconstructable function. If τ is a stopping time* [3] *and if $X \ge 0$ is a pre-τ function, then, for every u,*

$$\Pi_{r,x}(X 1_{\tau < u} Y_u) = \Pi_{r,x}[X 1_{\tau < u} F^u(\tau, \xi_\tau)] \tag{3.2}$$

where

$$F^u(t, y) = \Pi_{t,y} Y_u. \tag{3.3}$$

---

[2] It is used in [**Dyn73**] that the Markov property **2**.(2.4) of ξ holds not only for $X \in \mathcal{F}[r, t]$ but also for all $X \in \mathcal{F}[r, t+)$ which is the intersection of $\mathcal{F}[r, v]$ over $v > t$. This is true for all right processes. [See, e. g., [**Sha88**], Theorem 7.4.viii.]

[3] We consider stopping times relative to the filtration $\mathcal{F}_{\le t}$.

PROOF. First, we assume that τ is simple. By the Markov property of ξ,

$$\Pi_{r,x}(X1_{\tau=t}Y_u) = \Pi_{r,x}[X1_{\tau=t}F^u(t,\xi_t)] = \Pi_{r,x}[X1_{\tau=t}F^u(\tau,\xi_\tau)]$$

for every $t \le u$. This implies (3.2). To extend this relation to an arbitrary stopping time τ we note that, by 3.2.A, $F^u(t,\xi_t)$ is, $\Pi_{r,x}$-a.s., right continuous in t. We apply (3.2) to simple stopping times τ_n approximating τ and we pass to the limit. [First, we assume that X and Y are bounded and we use the dominated convergence theorem. This restriction can be eliminated by a monotone passage to the limit.] □

A Markov process $\xi = (\xi_t, \Pi_{r,x})$ is called strong Markov if it satisfies the following condition (called the strong Markov property): for every stopping time τ, every pre-τ function $X \ge 0$ and every reconstructable function $Y_t \ge 0$,

$$\Pi_{r,x}(X1_{\tau<\infty}Y_\tau) = \Pi_{r,x}[X1_{\tau<\infty}F(\tau,\xi_\tau)] \tag{3.4}$$

where

$$F(t,y) = \Pi_{t,y}Y_t. \tag{3.5}$$

Functions Y_τ in (3.4) can be interpreted as post-τ random variables.

THEOREM 3.1. *All right processes have the strong Markov property.*

PROOF. It is sufficient to prove (3.4) for bounded X and Y_t. Moreover, by the multiplicative systems theorem (Theorem 1.1), it is sufficient to consider right continuous Y_t.

Let τ_n be simple stopping times approximating τ. Note that τ_n are $\mathcal{F}_\tau$-measurable. Therefore $X_n = X1_{\tau_n=t} \in \mathcal{F}_\tau$ and, by Lemma 3.1, for every $\varepsilon > 0$,

$$\Pi_{r,x}(X_n 1_{\tau<t+\varepsilon}Y_{t+\varepsilon}) = \Pi_{r,x}[X_n 1_{\tau<t+\varepsilon}F^{t+\varepsilon}(\tau,\xi_\tau)].$$

Since $\{\tau_n = t\} \subset \{\tau < t+\varepsilon\}$, this implies

$$\Pi_{r,x}(X1_{\tau_n=t}Y_{\tau_n+\varepsilon}) = \Pi_{r,x}[X1_{\tau_n=t}F^{\tau_n+\varepsilon}(\tau,\xi_\tau)].$$

By taking the sum over all $t \in \Lambda_n$, we get

$$\Pi_{r,x}(X1_{\tau_n<\infty}Y_{\tau_n+\varepsilon}) = \Pi_{r,x}[X1_{\tau_n<\infty}F^{\tau_n+\varepsilon}(\tau,\xi_\tau)]. \tag{3.6}$$

Clearly, $F^u(t,y)$ is right continuous in u. Therefore $F^{\tau_n+\varepsilon}(\tau,\xi_\tau) \to F(\tau,\xi_\tau)$ as $\varepsilon \downarrow 0$ and $n \to \infty$. By passing to the limit in (3.6), we get (3.4). □

3.4. Implications of the strong Markov property. Now we prove propositions **2**.2.1.A–**2**.2.1.C.

Proposition **2**.2.1.A follows immediately from the strong Markov property. Indeed, the function

$$Y_t = 1_{\alpha\le t}\int_t^\infty \rho(\eta_s)\,ds$$

is right continuous and $Y_t \in \mathcal{F}_{\ge t}$ for all t. Hence it is reconstructable and **2**.(2.6) follows from (3.4).

We need the following

LEMMA 3.2. *Suppose that ξ is right continuous. If $\psi \in \mathcal{F}_{\ge t}$ and $\psi \ge t$, then $\eta_\psi \in \mathcal{F}_{\ge t}$.*

Proof. Put $\psi_\Lambda = \varphi_\Lambda(\psi)$ for every finite set $\Lambda = \{t_1 < \cdots < t_m\} \subset [t, \infty)$. We have

$$\eta_{\psi_\Lambda} = \sum_2^m 1_{t_{k-1} < \psi \le t_k} \eta_{t_k} \in \mathcal{F}_{\ge t}.$$

If Λ_n is an increasing sequence of finite sets with the union everywhere dense in $[t, \infty)$, then $\psi_{\Lambda_n} \downarrow \psi$ and $\eta_{\psi_{\Lambda_n}} \to \eta_\psi$. □

Theorem 3.2. *Propositions* **2**.*2.1.B and* **2**.*2.1.C hold for every continuous right process.*

Proof. 1°. The first after t exit time from Γ

$$\sigma_t(\Gamma) = \inf\{u \ge \alpha : u > t, \eta_u \notin \Gamma\} \tag{3.7}$$

is right continuous if Γ is closed. [4] Indeed, if $\sigma_t(\Gamma) > t$, then $\sigma_s(\Gamma) = \sigma_t(\Gamma)$ for all $s \in (t, \sigma_t(\Gamma))$. If $\sigma_t(\Gamma) = t$, then, for every $t' > t$ there exists $t'' \in (t, t')$ such that $\eta_{t''} \notin \Gamma$ and therefore $\sigma_s(\Gamma) < t'$ for $s \in (t, t'')$.

2°. We claim that $\sigma_t(\Gamma) \in \mathcal{F}_{\ge t}$. Indeed, $\sigma_t(\Gamma) \ge t$ and, for every $t' > t$, $\{\sigma_t(\Gamma) < t'\}$ is the union of $\{\eta_r \notin \Gamma\}$ over all rational $r \in (t, t')$.

Let $Y_t(\Gamma) = f(\eta_{\sigma_t(\Gamma)})$ if $\sigma_t(\Gamma) < \infty$ and $Y_t(\Gamma) = 0$ otherwise. If f is continuous, then $Y_t(\Gamma)$ is right continuous by 1°. By Lemma 3.2, $Y_t(\Gamma) \in \mathcal{F}_{\ge t}$. Hence, $Y_t(\Gamma)$ is reconstructable.

3°. If closed sets $\Gamma_n \uparrow Q'$, then $\sigma_t(\Gamma_n) \uparrow \sigma_t(Q')$ and $Y_t(\Gamma_n) \to Y_t(Q')$. Therefore $\sigma_t(Q')$ and $Y_t(Q')$ are reconstructable. It follows from the multiplicative systems theorem (Theorem 1.1) that $Y_t = Y_t(Q')$ is reconstructable for all Borel f.

4°. To prove **2**.2.1.B, we note that $\sigma_\tau(Q') = \tau'$ on the set $A = \{\tau < \infty, \eta_\tau \in Q'\}$. We have $Y_\tau = 1_{\tau' < \infty} f(\eta_{\tau'})$ on A. Hence, for all ω,

$$1_{Q'}(\eta_\tau) 1_{\tau' < \infty} f(\eta_{\tau'}) = 1_{Q'}(\eta_\tau) 1_{\tau < \infty} Y_\tau$$

(if $\omega \notin A$, then both sides vanish). By 3°, Y_t is reconstructable. Since $X' = X 1_{\tau < \infty} 1_{Q'}(\eta_\tau)$ is a pre-τ random variable, (3.4) implies

$$\Pi_{r,x} X' 1_{\tau' < \infty} f(\eta_{\tau'}) = \Pi_{r,x} X' 1_{\tau < \infty} Y_\tau = \Pi_{r,x} X' 1_{\tau < \infty} F(\tau, \xi_\tau) \tag{3.8}$$

where

$$F(t, y) = \Pi_{t,y} Y_t.$$

Note that, if $z = (t, y) \in Q'$, then $\Pi_{t,y}$-a.s., $\alpha = t$ and $\sigma_t(Q') = \tau'$. Hence, $F = K_{Q'} f$ on Q' and **2**.(2.8) follows from (3.8).

5°. To prove **2**.2.1.C, we consider σ_t given by formula **2**.(2.10) and $Y_t = f(\eta_{\sigma_t})$. The same arguments as in 1°-3° show that these functions are reconstructable. The left side in **2**.(2.11) is equal to $\Pi_z X Y_\tau$. Since $\tau \le \sigma_\tau < \infty$ Π_z-a.s. for all z, **2**.(2.11) follows from (3.4). □

Remark. The continuity of ξ_t was used in 3° to get convergence of $Y_t(\Gamma_n)$ to $Y_t(Q')$. Such a convergence takes a place for all Hunt's processes. Therefore Theorem 3.2 is valid for a wider class of Hunt's processes.

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[4] We put $\sigma_t = \infty$ if $\eta_u \in \Gamma$ for all $u \ge \alpha$.

3.5. Time homogeneous Markov processes. We introduce, for every $t \geq 0$ a transformation $\theta_t : \omega \to \tilde{\omega}$ in the path space defined by the formula $\tilde{\omega}(s) = \omega(t+s)$. A Markov process $\xi = (\xi_t, \Pi_{r,x})$ is called *time homogeneous* if, for all r, s, t, x, ω, C,

$$\begin{aligned} \xi_t(\theta_s \omega) &= \xi_{t+s}(\omega) \\ \Pi_{r,x}(\theta_s C) &= \Pi_{r+s,x}(C). \end{aligned} \tag{3.9}$$

These relations allow to consider only processes with the birth time $\alpha = 0$ and to deal with (ξ_t, Π_x) where $\Pi_x = \Pi_{0,x}$ and ξ_t is defined for $t \geq 0$.

We put $(\theta_t Y)(\omega) = Y(\theta_t \omega)$. The Markov and strong Markov properties can be formulated as follows. If $Y \in \mathcal{F} = \mathcal{F}_{\geq 0}$, then

$$\Pi_x(X \theta_t Y) = \Pi_x(X \Pi_{\xi_t} Y) \quad \text{for } X \in \mathcal{F}_{\leq t} = \mathcal{F}[0, t] \tag{3.10}$$

and

$$\Pi_x(X 1_{\tau < \infty} \theta_\tau Y) = \Pi_x(X 1_{\tau < \infty} \Pi_{\xi_\tau} Y) \quad \text{for } X \in \mathcal{F}_{\leq \tau}. \tag{3.11}$$

By using (3.9), formula (3.10) can be easily obtained from **2**.(2.4) and formula (3.11) can be deduced from (3.4)–(3.5).

The transition function of a homogeneous Markov process satisfies a condition: $p(r, x; t, B) = p(r + s, x; t + s, B)$ for all r, s, t, B. If the process has a transition density, then this density can be chosen in such a way that $p(r, x; t, y) = p(r + s, x; t + s, y)$ for all r, s, t, x, y. [The Brownian density **2**.(2.2) has this property.] We write $p_t(x, B)$ for $p(0, x; t, B)$ and $p_t(x, y)$ for $p(0, x; t, y)$.

The first exit time of ξ_t from D

$$\tau = \inf\{t : \xi_t \notin D\}$$

coincides with the first exit time of $\eta_t = (t, \xi_t)$ from $Q = \mathbb{R} \times D$. Note that, for every $r \geq 0$,

$$\{\tau > r\} \subset \{\theta_r \tau = \tau - r, \theta_r \xi_\tau = \xi_\tau\}. \tag{3.12}$$

4. Martingales

4.1. Definition. Let $\mathcal{A}_t, t \in \mathbb{T}$ be a filtration of a probability space $(\Omega, \mathcal{A}, P)$. A family of real-valued functions $X_t(\omega)$ is called a martingale (relative to $\mathcal{A}$ and P) if:

(a) X_t is $\mathcal{A}_t$-measurable and P-integrable;

(b) for every $s < t \in \mathbb{T}$ and for every $A \in \mathcal{A}_s$,

$$\int_A X_s \, dP = \int_A X_t \, dP. \tag{4.1}$$

Supermartingales (submartingales) are defined in a similar way but the equality sign in (4.1) needs to be replaced by $\geq$ $(\leq)$. Since X is a submartingale if and only $-X$ is a supermartingale, it is sufficient to investigate only supermartingales. A special role is played by positive supermartingales. For them a value $+\infty$ is permitted and the condition of P-integrability of X_t is dropped.

4.2. Optional stopping.

THEOREM 4.1. *Let σ and τ be two stopping times such that $\sigma \le \tau$. If X_t is a right continuous martingale and if σ and τ are bounded, then*

$$\int_A X_\sigma \, dP = \int_A X_\tau \, dP \tag{4.2}$$

for every $A \in \mathcal{A}_\sigma$.

If X_t is a positive right continuous supermartingale, then

$$\int_A X_\sigma \, dP \ge \int_A X_\tau \, dP \tag{4.3}$$

for every $A \in \mathcal{A}_\sigma$ (even if σ and τ are unbounded and even if they take value $+\infty$).

Proof can be found, e.g., in [**RY91**] (Theorems 3.1 and 3.3 in Chapter II) (see also [**DM87**] (Chapter V, sections 11 and 15)).

Assuming that X_t is right continuous, we get the following implications of Theorem 4.1:

4.2.A. Suppose that X_t is a martingale and $\tau_1 \le \tau_2 \le \dots \tau_n \le \dots$ are bounded stopping times. Then X_{τ_n} is a martingale with respect to the filtration $\mathcal{A}_{\tau_n}$.

4.2.B. Suppose that X_t is a positive supermartingale and $\tau_1 \le \tau_2 \le \dots \tau_n \le \dots$ are arbitrary stopping times. Then X_{τ_n} is a supermartingale with respect to the filtration $\mathcal{A}_{\tau_n}$.

This is proved, for instance, in [**Mey70**] V.T21.

4.2.C. Suppose that X_t is a positive supermartingale and $\tau_1 \ge \tau_2 \ge \dots \tau_n \ge \dots$ are arbitrary stopping times. Then $\hat{X}_{-n} = X_{\tau_n}$ is a supermartingale with respect to the filtration $\hat{\mathcal{A}}_{-n} = \mathcal{A}_{\tau_n}$.

4.3. Downcrossings inequality and its applications. Let f be a function from $\mathbb{T} \subset \mathbb{R}$ to $\mathbb{R}$ and let $0 \le a < b$. Put $n \in \mathcal{N}$ if there exist $0 \le t_1 < t_2 < \dots < t_{2n-1} < t_{2n} \in \mathbb{T}$ such that

$$f_{t_1} \ge b, f_{t_2} \le a, \dots, f_{t_{2n-1}} \ge b, f_{t_{2n}} \le a.$$

The supremum $\mathbb{D}(f, \mathbb{T}, [a,b])$ of $\mathcal{N}$ is called the *number of downcrossings of* $[a,b]$ *by* f. The principal interest of this number is the following: if $\mathbb{D}(f, \mathbb{T}, [a,b]) < \infty$ for all rational $a < b$, then there exist limits

$$f(t+) = \lim_{s \in \mathbb{T}, s \downarrow t} f(s), \qquad f(t-) = \lim_{s \in \mathbb{T}, s \uparrow t} f(s)$$

at all t (possibly, equal to $+\infty$ or $-\infty$).

Here is one of versions of Doob's downcrossings inequality. It can be easily deduced from other versions proved, for instance, in [**RY91**], section II.2, [**Doo84**],Chapter VII, Theorem 3.3, [**DM87**], VI.1.1.

THEOREM 4.2. *If $X_t, t \in \mathbb{T}$ is a right continuous supermartingale, then*

$$(b-a)P\mathbb{D}(X, \mathbb{T}, [a,b]) \le |b| + \sup_{t \in \mathbb{T}} P[(-X_t) \vee 0]$$

This inequality implies:

4.3.A. A supermartingale $X_1, X_2, \dots, X_n, \dots$ bounded from below converges a.s. as $n \to \infty$. The limit is, a.s., finite if $P|X_n| < \infty$ for some n.

4.3.B. If $\dots, X_n, \dots, X_2, X_1$ is a positive supermartingale and if $\sup PX_n < \infty$, then X_n converges a.s. and X_n are uniformly integrable.

4.3.C. If $X_t, t \in \mathbb{R}_+$ is a positive right continuous supermartingale, then, P-a.s., there exist all the limits $X_{t-} = \lim_{s \uparrow t} X_s$ for $t \in \mathbb{R}$ and for $t = +\infty$.

4.4. Kolmogorov's inequality. We use the following form of this inequality: If X_t is a right continuous martingale on an interval I and if $r < t \in I$, then, for every $\delta > 0$,

$$P\{\sup_{r<s<t} |X_s - X_r| > \delta\} \leq \delta^{-2} P|X_t - X_r|^2.$$

This can be easily derived from maximal inequalities presented, for instance, in [**Mey70**] (VI.T1), [**RY91**] (II,(1.6)), [**DM87**] (V.20).

APPENDIX B

Facts on elliptic differential equations

1. Introduction

1.1. We consider a bounded smooth domain D in $\mathbb{R}^d$ $(d \geq 2)$ and an operator L subject to the conditions **6**.1.2.A–**6**.1.2.C. We start with a Green's identity for L. We present the Brandt interior estimates of the difference-quotients and the derivatives for the solutions of the Poisson equation $Lu = f$. We also state a special case of the Schauder boundary estimate which we need to prove the bound

$$k_D(x,y) \leq Cd(x,\partial D)|x-y|^{-d} \tag{1.1}$$

for the Poisson kernel in D stated as 1.8.B in Theorem **6**.1.4. [1]

1.2. Green's formula. We refer to [**Mir70**], Chapter I, section 6 for the following result. If V is a smooth domain and if $u, v \in C^2(V)$, then

$$\int_V (uLv - vL^*u)dx = \int_{\partial V} (\frac{\partial u}{\partial \lambda} v - \frac{\partial v}{\partial \nu} u)\rho \, d\gamma. \tag{1.2}$$

Here γ is the surface area on ∂V, ρ is a bounded positive function depending on L, $\frac{\partial}{\partial \lambda}$ and $\frac{\partial}{\partial \nu}$ are the derivatives in the direction of the vector fields λ and ν on ∂V. One of the fields (say, ν) can be any piecewise smooth field with no vectors tangent to ∂V. The second field is determined by the first one.

2. The Brandt and Schauder estimates

2.1. The Brandt interior estimates. Suppose that $u \in C^2(D)$ and $Lu = f$. Brandt [**Bra69**] developed a method which allows to get estimates of the difference-quotients

$$\Phi(u)(x;y) = \frac{|u(x+y) - u(x-y)|}{2|y|}$$

and

$$\Psi(u)(x;y_1,y_2) = \frac{u(x+y_1+y_2) - u(x+y_1-y_2) - u(x-y_1+y_2) + u(x-y_1-y_2)}{4|y_1||y_2|}$$

in terms of

$$d_x = d(x,\partial D), \qquad M(u) = \sup_D |u(x)|, \qquad q(f) = \sup_D |f(x)|$$

and

$$S(f) = \sup_{\hat{D}} \{d_x^2 \Phi(f)(x,y)\}$$

where $\hat{D} = \{(x,y) : x, x+y, x-y \in D\}$.

[1] Everywhere C means a constant depending only on D and L. Moreover, it depends only on the diameter of D and on constants κ, λ and Λ in the conditions **6**.1.2.A–**6**.1.2.B. (The bound (2.2) depends also on Hölder's exponent λ and Hölder's coefficient Λ of f.)

His method is based on applying the maximum principle to elliptic operators in higher dimensions which he derives from L. One of these operators acts on functions of (x, y) and another on functions of (x, y_1, y_2). [2]

We need the following two results which follow from Theorems 6.6 and 7.2 in [**Bra69**].

THEOREM 2.1. *For every* $(x, y) \in \hat{D}$,

$$d_x \Phi(u)(x, y) \leq C[M(u) + M(f)]. \tag{2.1}$$

If $f \in C^\lambda$, *then, for every* $x \in D$, $|y_1|, |y_2| < d_x/2$,

$$d_x^2 \Psi(u)(x; y_1, y_2) \leq C[M(u) + M(f) + S(f)]. \tag{2.2}$$

It follows from (2.1) that, for all $i = 1, \dots, d$,

$$d_x |\mathcal{D}_i u(x)| \leq C[M(u) + M(f)]. \tag{2.3}$$

2.2. A Schauder boundary estimate. We need a very special case of the Schauder boundary estimate.

THEOREM 2.2. *Let* O *be a flat* [3] *relatively open smooth portion of* ∂D. *Suppose* u *is* L-*harmonic in* D, *continuous on* $\bar{D}$ *and equal to 0 on* O. *If* $x \in D$ *and* $d(x, \partial D \setminus O) \geq r$, *then, for all* $i = 1, \dots, d$,

$$|\mathcal{D}_i u(x)| \leq C r^{-1} M(u). \tag{2.4}$$

This is an implication of Lemma 6.4 in [**GT98**] and the following result which is a particular case of Lemma 6.18 in [**GT98**].

THEOREM 2.3. [4] *If* O *is a smooth portion of* ∂D, *if an* L-*harmonic function* h *is continuous on* $\bar{D}$ *and equal to 0 on* O, *then* $h \in C^{2,\lambda}(D \cup O)$.

3. Upper bound for the Poisson kernel

3.1. To establish the bound (1.1), we consider a diffeomorphism ψ_y straightening the boundary near y (see section **6**.1.3). There exists a diffeomorphism T from $\mathbb{R}^d$ to $\mathbb{R}^d_+$ which coincides with ψ_y for all $x \in D$ sufficiently close to y. Note that

$$C_1^{-1}|x - y| \leq |T(x) - T(y)| \leq C_1 |x - y| \quad \text{for all } x, y \in \bar{D}$$

with a constant $C_1 > 0$. It follows from Lemma **13**.2.1 that the Poisson kernel $\tilde{k}$ for $\tilde{L}$ in $\tilde{D} = T(D)$ satisfies the condition

$$k(x, y) \leq C_2 \tilde{k}(T(x), T(y)) \quad \text{for all } x, y \in D$$

for some C_2. If we prove that

$$\tilde{k}(\tilde{x}, \tilde{y}) \leq C_0 \frac{d(\tilde{x}, \partial\tilde{D})}{|\tilde{x} - \tilde{y}|^d},$$

then the bound (1.1) will hold with $C = C_0 C_1^{d+1} C_2$.

Therefore without any loss of generality we can assume that $D \subset \mathbb{R}^d_+$ and $y = 0$ is a (relatively) interior point of $\partial D \cap \partial \mathbb{R}^d_+$. Note that $k_D(x, 0) \leq k(x, 0)$ where k is the Poisson kernel in $\mathbb{R}^d_+$. [This follows from the maximum principle or from

[2] A simplified construction for the case $L = \Delta$ is presented in [**GT98**] (section 3.4).

[3] For instance, if $O \subset \{x_d = 0\}$.

[4] Both Theorems 2.3 and 2.2 hold for every L subject to the conditions **6**.1.2.A–1.2.B.

the probabilistic interpretation of Poisson kernel.] Therefore it is sufficient to prove that, for every R, there exists a constant C such that

$$k(x,0) \le C x_d |x|^{-d} \tag{3.1}$$

in the ball $\{|x| \le R\}$.

3.2. For every domain $D \subset \mathbb{R}^d_+$ we denote by $\partial' D$ the set of all $y \in \partial D$ at positive distances from $\mathbb{R}^d_+ \setminus D$ and by $\partial'' D$ the complement of $\partial' D$ in ∂D. Put $V_k = \{x \in \mathbb{R}^d_+ : |x| < kr\}$ and $V_{k,\ell} = \{x \in \mathbb{R}^d_+ : kr < |x| < \ell r\}$.

We get the bound (3.1) in four steps. It is sufficient to show that there exists a constant C such that, for every $0 < r < R/4$, (3.1) holds on $V_{4,5}$. Indeed, if $|x| < R$, then $x \in V_{4,5}$ for $r \in (|x|/5, |x|/4)$.

Step 1.

For all $x \in V_{3,6}, y \in V_1$ and all i,

$$\mathcal{D}_{y_i} g(x,y) \le C r^{1-d}. \tag{3.2}$$

Indeed, if $x \in V_{3,6}$, then, by **6**.1.7.A, function $u(y) = g(x,y)$ is L^*-harmonic in V_2, continuous in $\bar{V}_2$ and it vanishes on $\partial' V_2$. If $y \in V_1$, then $d(y, \partial'' V_2) > r$ and, by (2.4),

$$|\mathcal{D}_{y_i} g(x,y)| \le C r^{-1} \sup_{z \in V_2} g(x,z). \tag{3.3}$$

We use the bound **6**.1.7.B. [5] If $x \in V_{3,6}, z \in V_2$, then $|x - z| > r$ and therefore $g(x,z) \le C r^{2-d}$. Therefore (3.2) follows from (3.3).

Step 2.

Put $h(x) = k(x,0)$ and prove that, for all $x \in V_{3,6}$,

$$h(x) \le C r^{1-d}. \tag{3.4}$$

By **6**.(1.15),

$$h(x) = \lim_{y \to 0} \sum_i q_i(y) D_{y_i} g(x,y)$$

where $q_i(y)$ are bounded functions. Therefore (3.4) follows from (3.2).

Step 3.

For all $x \in V_{4,5}$ and all i,

$$\mathcal{D}_i h(x) \le C r^{-d}. \tag{3.5}$$

Indeed, the function $h(x)$ is L-harmonic in $V_{3,6}$, continuous on $\partial V_{3,6}$ and it vanishes on $\partial' V_{3,6}$. If $x \in V_{4,5}$, then $d(x, \partial'' V_3) > r$. By Theorem 2.2,

$$\mathcal{D}_{x_i} h(x) \le C r^{-1} \sup_{x \in V_{3,6}} h(x)$$

and (3.5) follows from (3.4).

Step 4.

For all $x \in V_{4,5}$,

$$h(x) \le C x_d |x|^{-d}. \tag{3.6}$$

To prove this, consider a smooth curve Γ connecting x with a point $x^0 \in \partial' V_{4,5}$. Denote by $x(s)$ the point of Γ at the distance s from x^0 measured along the curve.

---

[5] We assume that $d \ge 3$. The case $d = 2$ can be considered in a similar way.

In particular, $x^0 = x(0)$ and $x = x(t)$ where t is the length of Γ. If $\Gamma \subset V_{4,5}$, then, by (3.5),

$$\left|\frac{dh(x(s))}{ds}\right| \le Cr^{-d}$$

and, since $h(x^0) = 0$, we have

$$|h(x)| \le Ctr^{-d}. \tag{3.7}$$

Curve Γ can be chosen in such a way that $t < Cx_d$ [6] Therefore (3.7) implies (3.6).

[6] For instance, consider a semicircle defined as the intersection of the sphere $\{y : |y| = |x|\}$ with the two-dimensional plane $\{y_1 = x_1, \dots, y_{d-2} = x_{d-2}\})$ and take for Γ a shorter of two arcs into which it is split by the point x.

Epilogue

We conclude the book with a few challenging open problems and possible directions of research.

1. σ-moderate solutions

The description of all positive solutions of the equation $Lu = \psi(u)$ in a domain E is one of central problems investigated in this book. For the class of σ-moderate solutions this problem is solved in Chapter **11**. By Theorem **11**.7.2, such a solution is determined by its fine trace (Γ, ν). By Theorem **11**.7.1, all pairs (Γ, ν) have properties 7.1.A–7.1.B and, by Theorem **11**.7.2, every pair with these properties is equivalent to the fine trace of a σ-moderate solution. These results are proved for a broad class of functions ψ and for an arbitrary Greenian domain E. [They can be extended to the case of differential manifolds E.]

A fundamental question remains open:

Is every solution σ-moderate?

Depending on the answer to this question, the theory of the fine trace provides either a complete solution of our problem or only a step in this direction.

The primary suspect for being non σ-moderate are maximal solutions w_B introduced in Chapter **10**. Proving that they are σ-moderate would be a strong evidence in favor of the conjecture that all solutions are σ-moderate.

In section 8 of Chapter **11** we proved that w_O is σ-moderate if O is relatively open in ∂E. Similar arguments show that w_B are σ-moderate for all Borel B if they are σ-moderate for closed B.

2. Exceptional boundary sets

Three classes of such sets were studied in this book:

(a) Polar sets introduced in section 1.4 of Chapter **10** (in probabilistic terms, these are sets which are not hit, a.s., by the range of a superdiffusion);

(b) w-polar sets that charge no measure of class $\mathcal{N}_1$;

(c) null sets described in Chapter **13**.

Class (a) is the most interesting from a probabilistic point of view, and class (c) is the easiest to deal with by analytic means. In Chapter **12** we proved that, under broad conditions,

$$\{polar\ sets\} \subset \{w - polar\ sets\} \subset \{null\ sets\ of\ CM_\alpha\}.$$

In Chapter **13** we established that all three classes coincide if $\psi(u) = u^\alpha, 1 < \alpha \le 2$ and if a domain E is smooth. The following problems remain open:

Does there exist any domain E in which not all w-polar sets are polar?

Does there exist any Greenian domain E in which not all null sets are w-polar?

3. Exit boundary for a superdiffusion

3.1. In Chapter **7** we established a 1-1 correspondence between positive L-harmonic functions h subject to the condition $h(c) = 1$ and probability measures on the set of extreme elements which we identified with a subset of the Martin boundary.

An exciting problem is to develop a similar theory for superdiffusions. Among other applications this can provide a new approch to the description of all positive solutions of the equation $Lu = \psi(u)$ in an arbitrary domain E. It is natural to start with the case of quadratic branching $\psi(u) = u^2$.

We consider functions H on the space of measures $\mathcal{M}_c = \mathcal{M}_c(E)$ on E with values in $\mathbb{R}_+$. We say that H is X-harmonic and we write $H \in \mathbb{H}(X)$ if, for every $D \Subset E$ and every $\mu \in \mathcal{M}_c$,

$$P_\mu H(X_D) = H(\mu).$$

Let $c \in E$. Denote by $\mathbb{H}(X, c)$ the set of functions $H \in \mathbb{H}(X)$ such that $H(\delta_c) = 1$.

The set Γ of all extreme elements of $\mathbb{H}(X, c)$ is called the exit boundary for X. It is possible to prove that every $H \in \mathbb{H}(X)$ has a unique representation

$$H(\mu) = \int_\Gamma \mathcal{K}_\gamma(\mu) M(d\gamma)$$

where $\mathcal{K}_\gamma$ is an extreme element of $\mathbb{H}(X, c)$ corresponding to $\gamma \in \Gamma$ and M is a finite measure on Γ. [The proof is based on the following result which can be obtained by arguments similar to those in the proof of Theorem 1.1 in [**EP91**]:

For an arbitrary domain E, the class of null sets of the measure

$$\mathcal{P}_E(\mu, C) = P_\mu\{X_E \in C\}$$

is the same for all $\mu \in \mathcal{M}_c$.]

A key problem remains open:

To find all extreme X-harmonic functions.

A number of interesting classes of X-harmonic functions are known. Most of these functions are not extreme. However no function is proved to be extreme. For instance, to every positive L-harmonic function h there corresponds an X-harmonic function $H(\mu) = \langle h, \mu\rangle$. Clearly, H can be extreme only if h is extreme. [7] But are they extreme? If $E = \mathbb{R}^d$, then the function $h = 1$ is extreme for the Brownian motion. The corresponding X-harmonic function is the total mass $h(\mu) = \langle 1, \mu\rangle$. Is it extreme?

3.2. Let $\mathcal{U}$ be the set of all positive solutions of the equation $Lu = \psi(u)$ in E. It is easy to prove that

$$H_u(\mu) = 1 - e^{-\langle u,\mu\rangle}$$

is X-harmonic for every $u \in \mathcal{U}$. Therefore

$$H_u(\mu) = \int_\Gamma \mathcal{K}_\gamma(\mu) M_u(d\gamma)$$

[7] That is if $h = k_y, y \in E'$ (see Theorem **7**.4.4).

where M_u is a finite measure on Γ determined uniquely by u. Put $M = M_w$ where w is the maximal element of $\mathcal{U}$. For all $u \in \mathcal{U}$, $M_u \le M$, and the Radon-Nikodym theorem implies that $M_u(d\gamma) = A_u(\gamma)M(d\gamma)$ where $0 \le A_u \le 1$ M-a.e. We arrive at a conclusion: For every $u \in \mathcal{U}$,

$$H_u(\mu) = \int_\Gamma A_u(\gamma)\mathcal{K}_\gamma(\mu)M(d\gamma).$$

Function A_u is determined uniquely, up to M-equivalence, by u and $0 \le A_u \le 1$ M-a.e.

It is interesting to establish connections between A_u and the boundary trace of u (rough and fine).

3.3. We say that $H \in \mathbb{H}(X)$ is extreme if the conditions $\tilde{H} \le H$, $\tilde{H} \in \mathbb{H}(X)$ imply that $\tilde{H} = \text{const.}\, H$. If $H(\delta_c) > 0$, then H is extreme if and only if $H/H(\delta_c)$ is an extreme element of $\mathbb{H}(X, c)$

Note that $H_{\tilde{u}} \le H_u$ if and only if $\tilde{u} \le u$. On the other hand, if u and au belong to $\mathcal{U}$, then $\psi(au) = a\psi(u)$ and therefore $a = 1$. Hence H_u is not extreme if there exists $\tilde{u} \in \mathcal{U}$, different from 0 and u, such that $\tilde{u} \le u$. This implies: if H_u is extreme, then u cannot be moderate or σ-moderate. Moreover, it cannot dominate any non-zero σ-moderate solution. If the fine trace of u is (Γ, ν), then, by Theorem **11**.7.2, u dominates a σ-moderate solution $u_{\Gamma,\nu}$. Hence, if u is extreme, then $u_{\Gamma,\nu} = 0$ which implies that Γ is w-polar and $\nu = 0$. It is unknown if there exist any non-zero solution with this property.

3.4 An H-transform (X_D, P^H_μ) can be defined for every X-harmonic function H and it can be interpreted as a conditioned superdiffusion. In the case of $H(\mu) = \langle 1, \mu\rangle$, this is the superdiffusion conditioned on non-extinction. Evans [**Eva93**] constructed such a superdiffusion in the terms of an “immortal” Brownian particle throwing off pieces of measures that evolve according to the super-Brownian motion. A similar representation of H-transforms for some other functions H was obtained in [**Ove93**], [**Ove94**],[**SaV99**] and [**SaV00**].

3.5. An H-transform can be also defined for a wider than $\mathbb{H}(X)$ class of X-excessive functions H. These are functions from $\mathcal{M}_c$ to $[0, \infty]$ such that

$$P_\mu H(X_D) \le H(\mu) \quad \text{for all } D \Subset E, \mu \in \mathcal{M}_c$$

and $H(\mu_0) < \infty$ for some μ_0. Examples of X-excessive functions are given by the formula $H_\nu(\mu) = \mathcal{G}(\mu, \nu)$ where $\mathcal{G}$ is Green's function of X (it can be defined for every superdiffusion). If H_ν is not extreme, then ν is a branching point of the superdiffusion. No such points exist for diffusions. Have superdiffusions the same kind of regularity?

A class of X-excessive functions can be defined in terms of the support process $K_t = \operatorname{supp} X_t$ and the extinction time $\sigma = \sup\{t : X_t \ne 0\}$. We have $\sigma < \infty$ P_μ-a.s. for every μ and the properties of K_t described in section **14**.3.2 imply that, if $\sigma > 0$, then $K_{\sigma-}$ is, a.s., a single point. For every open set U, $\varphi_U(\mu) = P_\mu\{\sigma > 0, K_{\sigma-} \in U\}$ is a X-excessive function and so is the function

$$H_U(\mu) = \frac{\varphi_U(\mu)}{\varphi_U(\mu_0)}.$$

Let $y \in E$ and let $U_n = \{x : |x - y| < 1/n\}$. We ask:

Is $H_y(\mu) = \lim h_{U_n}(\mu)/U_n$ extreme?

Note that H_y-transform can be interpreted as the superprocess conditioned to be extinct at point y. If h_y is not extreme, then stronger conditions on the character of extinction are possible. [We mean conditions preserving time-homogeneity of the process. This excludes conditions on the extinction time σ.]

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