CONTROL SYSTEM DESIGN, CONTINUOUS-TIME

This article discusses a means of improving performance of existing control systems and of designing new control systems with satisfactory performance. The most common approach to improving the performance of single-input–single-output control systems is to insert a suitable compensator in the system. In this article we are concerned with the design of various types of compensators.

Actual control systems are generally nonlinear. However, if they can be approximated by linear mathematical models, we may use one or more of the well-developed design methods. In a practical sense, the performance specifications given to the particular system suggest which method to use. If the performance specifications are given in terms of transient-response characteristics and/or frequency-domain performance measures, then we have no choice but to use a conventional or classical approach based on the root-locus and/or frequency-response methods. If the performance specifications are given as performance indexes in terms of state variables, then modern control approaches must be used.

The systems that may be designed by a conventional or classical approach are usually limited to single-input–single-output, linear time-invariant systems. The designer seeks to satisfy all performance specifications by means of educated trial-and-error repetition. After a system is designed, the designer checks to see if the designed system satisfies all the performance specifications. If it does not, then he or she repeats the design process by adjusting parameter settings or by changing the system configuration until the given specifications are met. Although the design is based on a trial-and-error procedure, the ingenuity and know-how of the designer will play an important role in a successful design. An experienced designer may be able to design an acceptable system without using many trials.

The primary objective of this article is to present procedures for the design and compensation of single-input–single-output linear time-invariant control systems. Compensation is the modification of the system dynamics to satisfy the given specifications. The methods to the control system design and compensation used in this article are the root-locus method and frequency-response method. These methods are commonly called the classical or conventional methods of control systems design. Note that in designing control systems by the root-locus or frequency-response methods the final result is not unique, because the best or optimal solution may not be precisely defined if the time-domain specifications or frequency-domain specifications are given.

SYSTEM COMPENSATION

Setting the gain is the first step in adjusting the system for satisfactory performance. In many practical cases, however, the adjustment of the gain alone may not provide sufficient alteration of the system behavior to meet the given specifications. As is frequently the case, increasing the gain value will improve the steady-state behavior but will result in poor stability or even instability. It is then necessary to redesign the system by modifying the structure or by incorporating additional devices or components to alter the overall behavior so that the system will behave as desired. A device inserted into the system for the purpose of satisfying the specifications is called a compensator. The compensator compensates for deficit performance of the original system.

In discussing compensators, we frequently use such terminologies as lead network, lag network, and lag-lead network. If a sinusoidal input e_i is applied to the input of a network and the steady-state output e_o (which is also sinusoidal) has a phase lead, then the network is called a lead network. (The amount of phase lead angle is a function of the input frequency.) If the steady-state output e_o has a phase lag, then the network is called a lag network. In a lag-lead network, both phase lag and phase lead occur in the output but in different frequency regions; phase lag occurs in the low-frequency region and phase lead occurs in the high-frequency region. A compensator having a characteristic of a lead network, lag network, or lag-lead network is called a lead compensator, lag compensator, or lag-lead compensator.

In this article we specifically consider the design of lead compensators, lag compensators, and lag-lead compensators.

In such design problems, we place a compensator in series with the unalterable plant transfer function $G(s)$ to obtain desirable behavior. The main problem then involves the judicious choice of the pole(s) and zero(s) of the compensator $G_c(s)$ to alter the root loci or frequency response so that the performance specifications will be met.

In the actual design of a control system, whether to use an electronic, pneumatic, or hydraulic compensator is a matter that must be decided partially based on the nature of the controlled plant. For example, if the controlled plant involves flammable fluid, then we have to choose pneumatic components (both a compensator and an actuator) to avoid the possibility of sparks. If, however, no fire hazard exists, then electronic compensators are most commonly used. In fact, we often transform nonelectrical signals into electrical signals because of the simplicity of transmission, increased accuracy, increased reliability, ease of compensation, and the like.

Lead, Lag, and Lag-Lead Compensation

Lead compensation essentially yields an appreciable improvement in transient response and a small change in steady-state accuracy. It may accentuate high-frequency noise effects. Lag compensation, on the other hand, yields an appreciable improvement in steady-state accuracy at the expense of increasing the transient response time. Lag compensation will suppress the effects of high-frequency noise signals. Lag-lead compensation combines the characteristics of both lead compensation and lag compensation. The use of a lead or lag compensator raises the order of the system by 1 (unless cancellation occurs between the zero of the compensator and a pole of the uncompensated open-loop transfer function). The use of a lag-lead compensator raises the order of the system by 2 [unless cancellation occurs between zero(s) of the lag-lead compensator and pole(s) of the uncompensated open-loop transfer function], which means that the system becomes more complex and it is more difficult to control the transient response behavior. The particular situation determines the type of compensation to be used.

ROOT-LOCUS METHOD

The basic characteristic of the transient response of a closed-loop system is closely related to the location of the closed-loop poles. If the system has a variable loop gain, then the location of the closed-loop poles depends on the value of the loop gain chosen. It is important, therefore, that the designer know how the closed-loop poles move in the s-plane as the loop gain is varied.

From the design viewpoint, in some systems simple gain adjustment may move the closed-loop poles to desired locations. Then the design problem may become the selection of an appropriate gain value. If the gain adjustment alone does not yield a desired result, addition of a compensator to the system will become necessary.

A simple method for finding the roots of the characteristic equation has been developed by W. R. Evans and used extensively in control engineering. This method, called the root-locus method, is one in which the roots of the characteristic equation are plotted for all values of a system parameter. The roots corresponding to a particular value of this parameter can then be located on the resulting graph. Note that the parameter is usually the gain, but any other variable of the open-loop transfer function may be used. Unless otherwise stated, we shall assume that the gain of the open-loop transfer function is the parameter to be varied through all values, from zero to infinity.

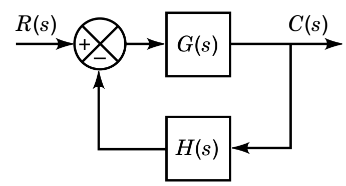


Figure 1. Control system.

Angle and Magnitude Conditions

The basic idea behind the root-locus method is that the values of s that make the transfer function around the loop equal -1 must satisfy the characteristic equation of the system. Consider the system shown in Fig. 1. The closed-loop transfer function is

$$\frac{C(s)}{R(s)} = \frac{G(s)}{1 + G(s)H(s)}$$

The characteristic equation for this closed-loop system is obtained by setting the denominator of the right-hand side of this last equation equal to zero. That is,

$$1 + G(s)H(s) = 0$$

or

$$G(s)H(s) = -1 \tag{1}$$

Here we assume that $G(s)H(s)$ is a ratio of polynomials in s. Since $G(s)H(s)$ is a complex quantity, Eq. (1) can be split into two equations by equating the angles and magnitudes of both sides, respectively, to obtain the following:

Angle condition:

$$\underline{/G(s)H(s)} = \pm 180^\circ(2k + 1) \qquad (k = 0,\ 1,\ 2,\ \ldots)$$

Magnitude condition:

$$|G(s)H(s)| = 1$$

The values of s that fulfill both the angle and magnitude conditions are the roots of the characteristic equation, or the closed-loop poles. A plot of the points of the complex plane satisfying the angle condition alone is the root locus. The roots of the characteristic equation (the closed-loop poles) corresponding to a given value of the gain can be determined from the magnitude condition.

FREQUENCY-RESPONSE METHOD

By the term *frequency response,* we mean the steady-state response of a system to a sinusoidal input. In frequency-response methods, we vary the frequency of the input signal over a certain range and study the resulting response.

The Nyquist stability criterion enables us to investigate both the absolute and relative stabilities of linear closed-loop systems from a knowledge of their open-loop frequency-response characteristics. An advantage of the frequency-response approach is that frequency-response tests are, in general, simple and can be made accurately by use of readily available sinusoidal signal generators and precise measurement equipment. Often the transfer functions of complicated components can be determined experimentally by frequency-response tests. In addition, the frequency-response approach has the advantage that a system may be designed so that the effects of undesirable noise are negligible and that such analysis and design can be extended to certain nonlinear control systems.

Frequency-Response Approach to the Design of Control Systems

It is important to note that in a control system design, transient-response performance is usually most important. In the frequency-response approach, we specify the transient-response performance in an indirect manner. That is, the transient-response performance is specified in terms of the phase margin, gain margin, and resonant peak magnitude (they give a rough estimate of the system damping); the gain crossover frequency, resonant frequency, and bandwidth (they give a rough estimate of the speed of transient response); and static error constants (they give the steady-state accuracy). Although the correlation between the transient response and frequency response is indirect, the frequency-domain specifications can be conveniently met in the Bode diagram approach.

After the open loop has been designed by the frequency-response method, the closed-loop poles and zeros can be determined. The transient-response characteristics must be checked to see whether the designed system satisfies the requirements in the time domain. If it does not, then the compensator must be modified and the analysis repeated until a satisfactory result is obtained.

Design in the frequency domain is simple and straightforward. The frequency-response plot indicates clearly the manner in which the system should be modified, although the exact quantitative prediction of the transient-response characteristics cannot be made. The frequency-response approach can be applied to systems or components whose dynamic characteristics are given in the form of frequency-response data. Note that because of difficulty in deriving the equations governing certain components, such as pneumatic and hydraulic components, the dynamic characteristics of such components are usually determined experimentally through frequency-response tests. The experimentally obtained frequency-response plots can be combined easily with other such plots when the Bode diagram approach is used. Note also that in dealing with high-frequency noise we find that the frequency-response approach is more convenient than other approaches.

A common approach to the design by use of the Bode diagram is that we first adjust the open-loop gain so that the requirement on the steady-state accuracy is met. Then the magnitude and phase curves of the uncompensated open loop (with the open-loop gain just adjusted) is plotted. If the specifications on the phase margin and gain margin are not satisfied, then a suitable compensator that will reshape the open-loop transfer function is determined. Finally, if there are any other requirements to be met, we try to satisfy them, unless some of them are contradictory to the other.

ROOT-LOCUS APPROACH TO THE DESIGN OF CONTROL SYSTEMS

The root-locus approach to design is very powerful when the specifications are given in terms of time-domain quantities, such as the damping ratio and undamped natural frequency of the desired dominant closed-loop poles, maximum overshoot, rise time, and settling time.

Consider a design problem in which the original system either is unstable for all values of gain or is stable but has undesirable transient-response characteristics. In such a case, the reshaping of the root locus is necessary in the broad neighborhood of the $j\omega$ axis and the origin in order that the dominant closed-loop poles be at desired locations in the complex plane. This problem may be solved by inserting an appropriate lead compensator in cascade with the feedforward transfer function.

If it is desired to improve steady-state performance (such as to reduce the error in following the ramp input), insertion of a lag compensator in the feedforward path will do the job. If it is desired to improve both the transient-response and steady-state performance, insertion of a lag-lead compensator will accomplish the job. In what follows we discuss the lead, lag, and lag-lead compensation techniques.

Lead Compensation

The procedure for designing a lead compensator for the system shown in Fig. 2 by the root-locus method may be stated as follows:

1. From the performance specifications, determine the desired location for the dominant closed-loop poles.
2. By drawing the root-locus plot, ascertain whether or not the gain adjustment alone can yield the desired closed-loop poles. If not, calculate the angle deficiency ϕ. This angle must be contributed by the lead compensator if the new root locus is to pass through the desired locations for the dominant closed-loop poles.
3. Assume the lead compensator $G_c(s)$ to be

$$G_c(s) = K_c\alpha\frac{Ts+1}{\alpha Ts+1} = K_c\frac{s+\dfrac{1}{T}}{s+\dfrac{1}{\alpha T}}, \quad (0 < \alpha < 1) \tag{2}$$

where α and T are determined from the angle deficiency. K_c is determined from the requirement of the open-loop gain.

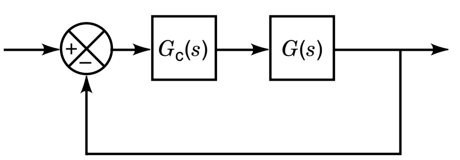


Figure 2. Control system.

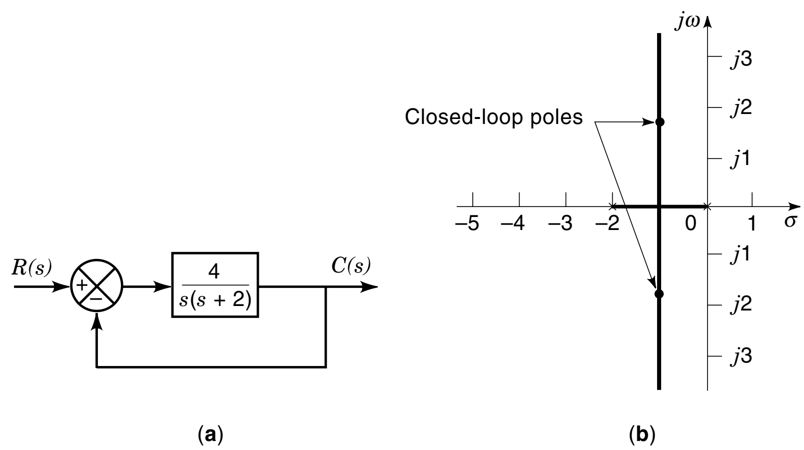


Figure 3. (a) Control system; (b) root-locus plot.

4. If static error constants are not specified, determine the location of the pole and zero of the lead compensator so that the lead compensator will contribute the necessary angle ϕ. If no other requirements are imposed on the system, try to make the value of α as large as possible. A larger value of α generally results in a larger value of K_v, which is desirable. (If a particular static error constant is specified, it is generally simpler to use the frequency-response approach.)
5. Determine the open-loop gain of the compensated system from the magnitude condition.

Once a compensator has been designed, check to see whether all performance specifications have been met. If the compensated system does not meet the performance specifications, then repeat the design procedure by adjusting the compensator pole and zero until all such specifications are met. If a large static error constant is required, cascade a lag network or alter the lead compensator to a lag-lead compensator.

Example 1. Consider the system shown in Fig. 3(a). The feedforward transfer function is

$$G(s) = \frac{4}{s(s+2)}$$

The root-locus plot for this system is shown in Fig. 3(b). The closed-loop poles are located at

$$s = -1 \pm j\sqrt{3}$$

The damping ratio of the closed-loop poles is 0.5. The undamped natural frequency of the closed-loop poles is 2 rad/s. The static velocity error constant is 2 s^{-1}.

It is desired to modify the closed-loop poles so that an undamped natural frequency $\omega_n = 4$ rad/s is obtained, without changing the value of the damping ratio, $\zeta = 0.5$. In the present example, the desired locations of the closed-loop poles are

$$s = -2 \pm j2\sqrt{3}$$

In some cases, after the root loci of the original system have been obtained, the dominant closed-loop poles may be moved to the desired location by simple gain adjustment. This is, however, not the case for the present system. Therefore, we shall insert a lead compensator in the feedforward path.

A general procedure for determining the lead compensator is as follows: First, find the sum of the angles at the desired location of one of the dominant closed-loop poles with the open-loop poles and zeros of the original system, and determine the necessary angle ϕ to be added so that the total sum of the angles is equal to $\pm 180°\,(2k + 1)$. The lead compensator must contribute this angle ϕ. (If the angle ϕ is quite large, then two or more lead networks may be needed rather than a single one.)

If the original system has the open-loop transfer function $G(s)$, then the compensated system will have the open-loop transfer function

$$G_c(s)G(s) = \left(K_c \frac{s + \frac{1}{T}}{s + \frac{1}{\alpha T}} \right) G(s)$$

where

$$G_c(s) = K_c \alpha \frac{Ts + 1}{\alpha Ts + 1} = K_c \frac{s + \frac{1}{T}}{s + \frac{1}{\alpha T}}, \quad (0 < \alpha < 1)$$

Notice that there are many possible values for T and α that will yield the necessary angle contribution at the desired closed-loop poles.

The next step is to determine the locations of the zero and pole of the lead compensator. There are many possibilities for the choice of such locations. (See the comments at the end of this example problem.) In what follows, we shall introduce a procedure to obtain the largest possible value for α. (Note that a larger value of α will produce a larger value of K_v. In most cases, the larger the K_v is, the better the system performance.) First, draw a horizontal line passing through point P, the desired location for one of the dominant closed-loop poles. This is shown as line PA in Fig. 4. Draw also a line connecting point P and the origin. Bisect the angle between the lines PA and PO, as shown in Fig. 4. Draw two lines PC and PD that make angles $\pm\phi/2$ with the bisector PB. The intersections of PC and PD with the negative real axis give the necessary location for the pole and zero of the lead net-

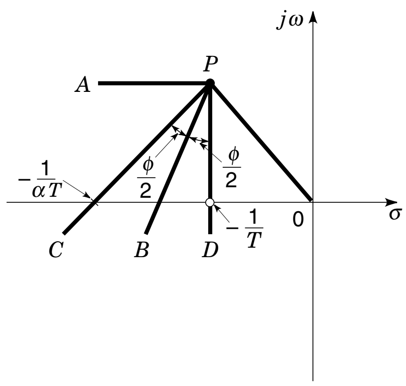


Figure 4. Determination of the pole and zero of a lead network.

work. The compensator thus designed will make point P a point on the root locus of the compensated system. The open-loop gain is determined by use of the magnitude condition.

In the present system, the angle of $G(s)$ at the desired closed-loop pole is

$$\left. \angle \frac{4}{s(s+2)} \right|_{s=-2+j2\sqrt{3}} = -210^\circ$$

Thus, if we need to force the root locus to go through the desired closed-loop pole, the lead compensator must contribute $\phi = 30^\circ$ at this point. By following the foregoing design procedure, we determine the zero and pole of the lead compensator, as shown in Fig. 5, to be

$$\text{Zero at } s = -2.9, \quad \text{Pole at } s = -5.4$$

or

$$T = \frac{1}{2.9} = 0.345, \quad \alpha T = \frac{1}{5.4} = 0.185$$

Thus $\alpha = 0.537$. The open-loop transfer function of the compensated system becomes

$$G_c(s)G(s) = K_c \frac{s+2.9}{s+5.4} \frac{4}{s(s+2)} = \frac{K(s+2.9)}{s(s+2)(s+5.4)}$$

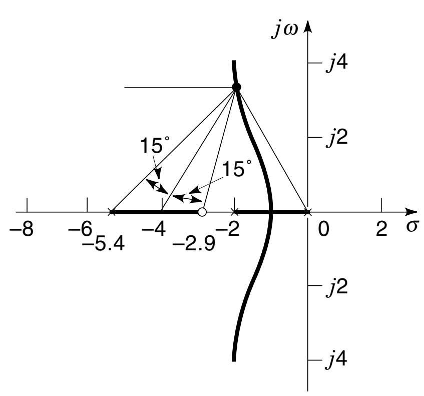


Figure 5. Root-locus plot of the compensated system.

where $K = 4K_c$. The root-locus plot for the compensated system is shown in Fig. 5. The gain K is evaluated from the magnitude condition as follows: Referring to the root-locus plot for the compensated system shown in Fig. 5, the gain K is evaluated from the magnitude condition as

$$\left| \frac{K(s+2.9)}{s(s+2)(s+5.4)} \right|_{s=-2+j2\sqrt{3}} = 1$$

or

$$K = 18.7$$

It follows that

$$G_c(s)G(s) = \frac{18.7(s+2.9)}{s(s+2)(s+5.4)}$$

The constant K_c of the lead compensator is

$$K_c = \frac{18.7}{4} = 4.68$$

Hence, $K_c\alpha = 2.51$. The lead compensator, therefore, has the transfer function

$$G_c(s) = 2.51 \frac{0.345s+1}{0.185s+1} = 4.68 \frac{s+2.9}{s+5.4}$$

The static velocity error constant K_v is obtained from the expression

$$\begin{aligned} K_v &= \lim_{s\to 0} sG_c(s)G(s) \\ &= \lim_{s\to 0} \frac{s18.7(s+2.9)}{s(s+2)(s+5.4)} \\ &= 5.02 s^{-1} \end{aligned}$$

Note that the third closed-loop pole of the designed system is found by dividing the characteristic equation by the known factors as follows:

$$s(s+2)(s+5.4) + 18.7(s+2.9) = (s+2+j2\sqrt{3})(s+2-j2\sqrt{3})(s+3.4)$$

The foregoing compensation method enables us to place the dominant closed-loop poles at the desired points in the complex plane. The third pole at $s = -3.4$ is close to the added zero at $s = -2.9$. Therefore, the effect of this pole on the transient response is relatively small. Since no restriction has been imposed on the nondominant pole and no specification has been given concerning the value of the static velocity error coefficient, we conclude that the present design is satisfactory.

Comments. We may place the zero of the compensator at $s = -2$ and pole at $s = -4$ so that the angle contribution of the lead compensator is 30°. (In this case the zero of the lead compensator will cancel a pole of the plant, resulting in the second-order system, rather than the third-order system as we designed.) It can be seen that the K_v value in this case is $4\ s^{-1}$. Other combinations can be selected that will yield 30°

phase lead. (For different combinations of a zero and pole of the compensator that contribute 30°, the value of α will be different and the value of K_v will also be different.) Although a certain change in the value of K_v can be made by altering the pole-zero location of the lead compensator, if a large increase in the value of K_v is desired, then we must alter the lead compensator to a lag-lead compensator.

Lag Compensation

Consider the case where the system exhibits satisfactory transient-response characteristics but unsatisfactory steady-state characteristics. Compensation in this case essentially consists of increasing the open-loop gain without appreciably changing the transient-response characteristics. This means that the root locus in the neighborhood of the dominant closed-loop poles should not be changed appreciably, but the open-loop gain should be increased as much as needed. This can be accomplished if a lag compensator is put in cascade with the given feedforward transfer function.

The procedure for designing lag compensator for the system shown in Fig. 2 by the root-locus method may be stated as follows:

1. Draw the root-locus plot for the uncompensated system whose open-loop transfer function is $G(s)$. Based on the transient-response specifications, locate the dominant closed-loop poles on the root locus.
2. Assume the transfer function of the lag compensator to be

$$G_c(s) = \hat{K}_c\beta\frac{Ts+1}{\beta Ts+1} = \hat{K}_c\frac{s+\dfrac{1}{T}}{s+\dfrac{1}{\beta T}} \qquad (\beta > 1) \tag{3}$$

   Then the open-loop transfer function of the compensated system becomes $G_c(s)G(s)$.
3. Evaluate the particular static error constant specified in the problem.
4. Determine the amount of increase in the static error constant necessary to satisfy the specifications.
5. Determine the pole and zero of the lag compensator that produce the necessary increase in the particular static error constant without appreciably altering the original root loci. (Note that the ratio of the value of gain required in the specifications and the gain found in the uncompensated system is the required ratio between the distance of the zero from the origin and that of the pole from the origin.)
6. Draw a new root-locus plot for the compensated system. Locate the desired dominant closed-loop poles on the root locus. (If the angle contribution of the lag network is very small—that is, a few degrees—then the original and new root loci are almost identical. Otherwise, there will be a slight discrepancy between them. Then locate, on the new root locus, the desired dominant closed-loop poles based on the transient-response specifications.)
7. Adjust gain $\hat{K}_c$ of the compensator from the magnitude condition so that the dominant closed-loop poles lie at the desired location.

Example 2. Consider the system shown in Fig. 6(a). The root-locus plot for the system is shown in Fig. 6(b). The closed-loop transfer function becomes

$$\frac{C(s)}{R(s)} = \frac{1.06}{s(s+1)(s+2)+1.06}$$
$$= \frac{1.06}{(s+0.3307-j0.5864)(s+0.3307+j0.5864)(s+2.3386)}$$

The dominant closed-loop poles are

$$s = -0.3307 \pm j0.5864$$

The damping ratio of the dominant closed-loop poles is $\zeta = 0.491$. The undamped natural frequency of the dominant closed-loop poles is 0.673 rad/s. The static velocity error constant is 0.53 s^{-1}.

It is desired to increase the static velocity error constant K_v to about 5 s^{-1} without appreciably changing the location of the dominant closed-loop poles. To meet this specification, let us insert a lag compensator as given by Eq. (3) in cascade with the given feedforward transfer function. To increase the static velocity error constant by a factor of about 10, let us choose $\beta = 10$ and place the zero and pole of the lag compen-

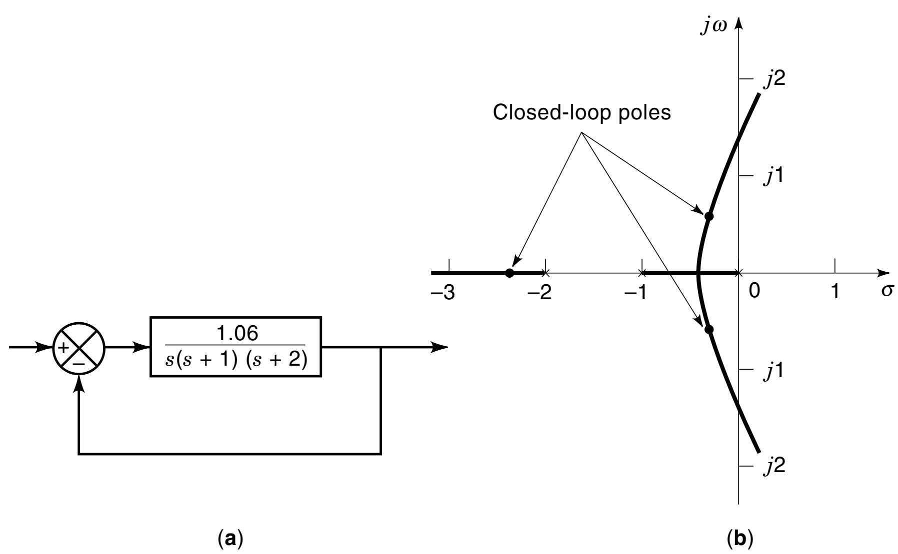


Figure 6. (a) Control system; (b) root-locus plot.

sator at $s = -0.05$ and $s = -0.005$, respectively. The transfer function of the lag compensator becomes

$$G_c(s) = \hat{K}_c \frac{s + 0.05}{s + 0.005}$$

The angle contribution of this lag network near a dominant closed-loop pole is about 4°. Because this angle contribution is not very small, there is a small change in the new root locus near the desired dominant closed-loop poles.

The open-loop transfer function of the compensated system then becomes

$$G_c(s)G(s) = \hat{K}_c \frac{s + 0.05}{s + 0.005} \frac{1.06}{s(s+1)(s+2)}$$
$$= \frac{K(s + 0.05)}{s(s + 0.005)(s+1)(s+2)}$$

where

$$K = 1.06\hat{K}_c$$

The block diagram of the compensated system is shown in Fig. 7(a). The root-locus plot for the compensated system near the dominant closed-loop poles is shown in Fig. 7(b), together with the original root-locus plot. Figure 7(c) shows the root-locus plot of the compensated system near the origin.

If the damping ratio of the new dominant closed-loop poles is kept the same, then the poles are obtained from the new root-locus plot as follows:

$$s_1 = -0.31 + j0.55, \quad s_2 = -0.31 - j0.55$$

The open-loop gain K is

$$K = \left| \frac{s(s + 0.005)(s+1)(s+2)}{s + 0.05} \right|_{s=-0.31+j0.55}$$
$$= 1.0235$$

Then the lag compensator gain $\hat{K}_c$ is determined as

$$\hat{K}_c = \frac{K}{1.06} = \frac{1.0235}{1.06} = 0.9656$$

Thus the transfer function of the designed lag compensator is

$$G_c(s) = 0.9656 \frac{s + 0.05}{s + 0.005} = 9.656 \frac{20s + 1}{200s + 1}$$

Then the compensated system has the following open-loop transfer function:

$$G_1(s) = \frac{1.0235(s + 0.05)}{s(s + 0.005)(s+1)(s+2)}$$
$$= \frac{5.12(20s + 1)}{s(200s + 1)(s+1)(0.5s + 1)}$$

The static velocity error constant K_v is

$$K_v = \lim_{s \to 0} sG_1(s) = 5.12\, s^{-1}$$

In the compensated system, the static velocity error constant has increased to 5.12 s^{-1}, or 5.12/0.53 = 9.66 times the origi-

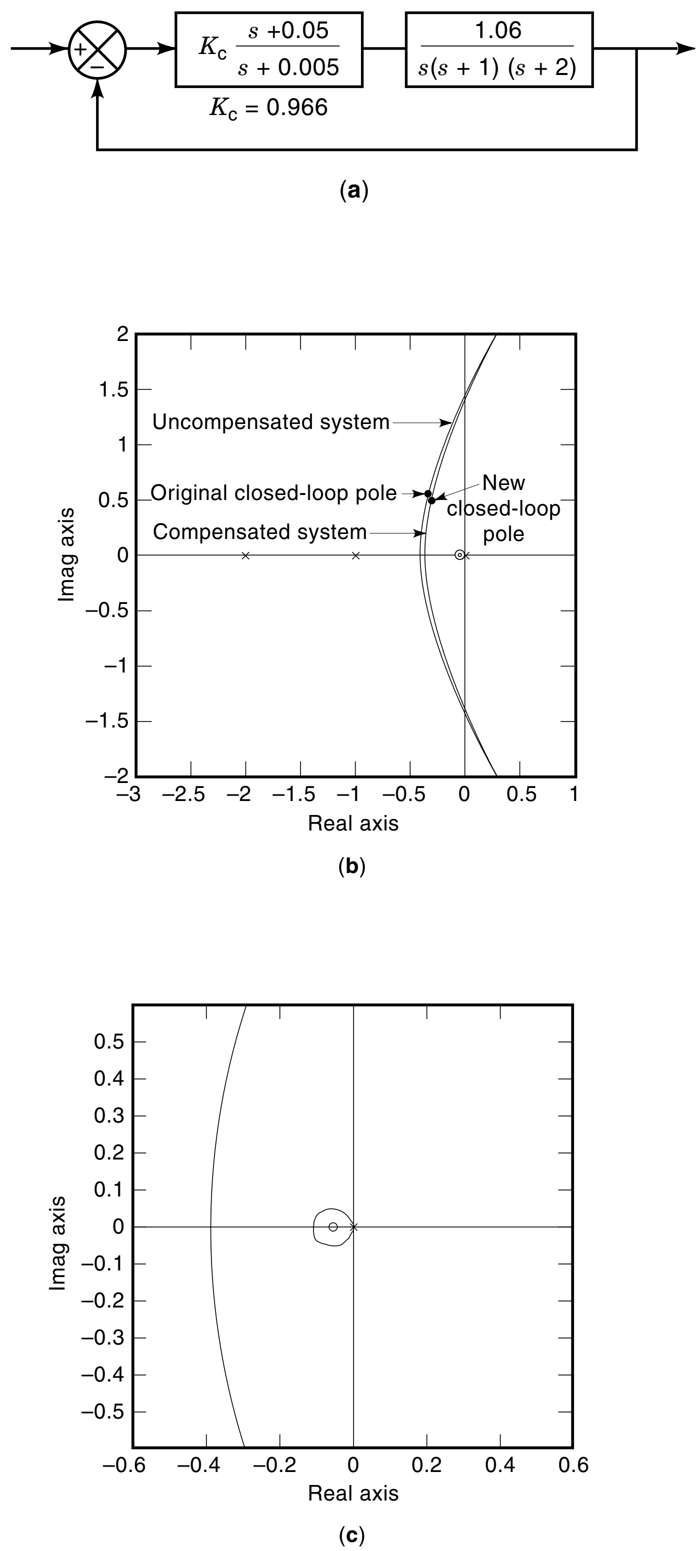


Figure 7. (a) Compensated system; (b) root-locus plots of the compensated system and the uncompensated system; (c) root-locus plot of compensated system near the origin.

nal value. (The steady-state error with ramp inputs has decreased to about 10% of that of the original system.) We have essentially accomplished the design objective of increasing the static velocity error constant to about 5 s^{-1}.

Note that, since the pole and zero of the lag compensator are placed close together and are located very near the origin, their effect on the shape of the original root loci has been small. Except for the presence of a small closed root locus near the origin, the root loci of the compensated and the uncompensated systems are very similar to each other. However, the value of the static velocity error constant of the compensated system is 9.66 times greater than that of the uncompensated system.

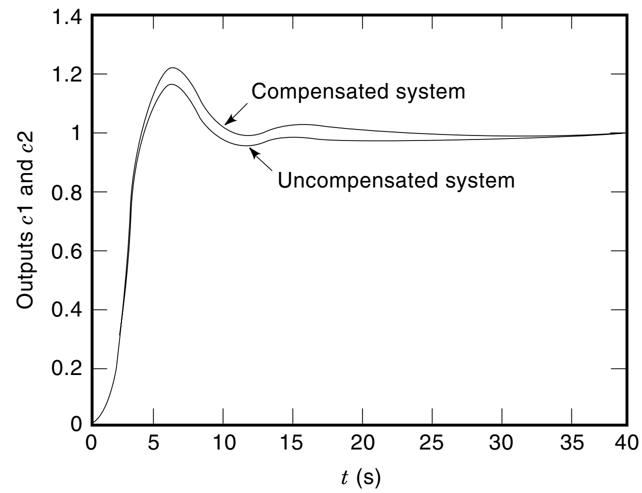
The two other closed-loop poles for the compensated system are found as follows:

$$s_3 = -2.326, \qquad s_4 = -0.0549$$

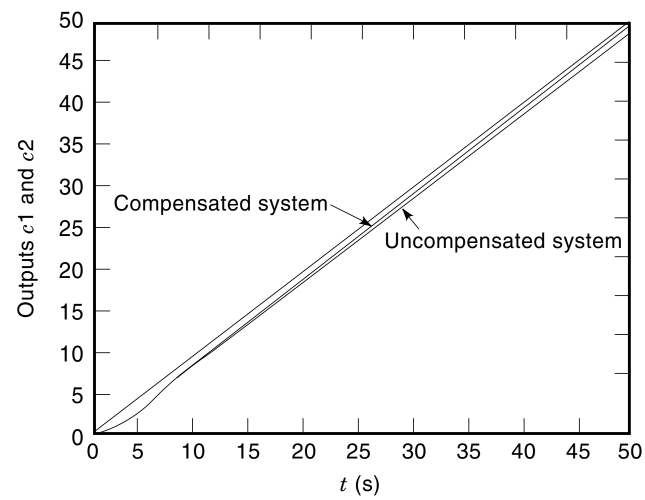
The addition of the lag compensator increases the order of the system from 3 to 4, adding one additional closed-loop pole close to the zero of the lag compensator. (The added closed-loop pole at $s = -0.0549$ is close to the zero at $s = -0.05$.) Such a pair of a zero and pole creates a long tail of small amplitude in the transient response, as we will see later in the unit-step response. Since the pole at $s = -2.326$ is very far from the $j\omega$ axis compared with the dominant closed-loop poles, the effect of this pole on the transient response is also small. Therefore, we may consider the closed-loop poles at $s = -0.31 \pm j0.55$ to be the dominant closed-loop poles.

The undamped natural frequency of the dominant closed-loop poles of the compensated system is 0.631 rad/s. This value is about 6% less than the original value, 0.673 rad/s. This implies that the transient response of the compensated system is slower than that of the original system. The response will take a longer time to settle down. The maximum overshoot in the step response will increase in the compensated system. If such adverse effects can be tolerated, the lag compensation as discussed here presents a satisfactory solution to the given design problem.

Figures 8(a) and 8(b) show the unit-step response curves and unit-ramp response curves, respectively, of the compensated and uncompensated systems.



(a)



(b)

Figure 8. (a) Unit-step response curves for the compensated and uncompensated systems; (b) unit-ramp response curves for both systems.

Lag-Lead Compensation

Lead compensation basically speeds up the response and increases the stability of the system. Lag compensation improves the steady-state accuracy of the system but reduces the speed of the response.

If improvements in both transient response and steady-state response are desired, then both a lead compensator and a lag compensator may be used simultaneously. Rather than introducing both a lead compensator and a lag compensator as separate elements, however, it is economical to use a single lag-lead compensator.

Consider the system shown in Fig. 2. Assume that we use the following lag-lead compensator:

$$G_c(s) = K_c \frac{(T_1 s + 1)(T_2 s + 1)}{\left(\dfrac{T_1}{\beta} s + 1\right)(\beta T_2 s + 1)} = K_c \frac{\left(s + \dfrac{1}{T_1}\right)\left(s + \dfrac{1}{T_2}\right)}{\left(s + \dfrac{\beta}{T_1}\right)\left(s + \dfrac{1}{\beta T_2}\right)} \tag{4}$$

where $\beta > 1$. The design procedure may be stated as follows:

1. From the given performance specifications, determine the desired location for the dominant closed-loop poles.
2. If the static velocity error constant K_v is specified, determine the value of constant K_c from the following equation:

$$\begin{aligned} K_v &= \lim_{s\to 0} s G_c(s) G(s) \\ &= \lim_{s\to 0} s K_c G(s) \end{aligned}$$

3. To have the dominant closed-loop poles at the desired location, calculate the angle contribution ϕ needed from the phase lead portion of the lag-lead compensator.

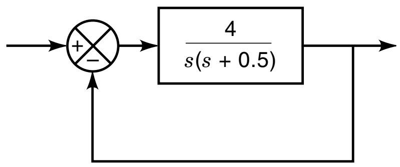


Figure 9. Control system.

4. For the lag-lead compensator, we later choose T_2 sufficiently large so that

$$\left| \frac{s + \dfrac{1}{T_2}}{s_1 + \dfrac{1}{\beta T_2}} \right|$$

is approximately unity, where $s = s_1$ is one of the dominant closed-loop poles. Determine the values of T_1 and β from the magnitude and angle conditions:

$$\left| K_c \left(\frac{s_1 + \dfrac{1}{T_1}}{s_1 + \dfrac{\beta}{T_1}} \right) G(s_1) \right| = 1$$

$$\angle \frac{s_1 + \dfrac{1}{T_1}}{s_1 + \dfrac{\beta}{T_1}} = \phi$$

5. Using the value of β just determined, choose T_2 so that

$$\left| \frac{s_1 + \dfrac{1}{T_2}}{s_1 + \dfrac{1}{\beta T_2}} \right| \doteqdot 1$$

$$-5^\circ < \angle \frac{s_1 + \dfrac{1}{T_2}}{s_1 + \dfrac{1}{\beta T_2}} < 0^\circ$$

The value of βT_2, the largest time constant of the lag-lead compensator, should not be too large to be physically realized.

Example 3. Consider the control system shown in Fig. 9. It is desired to make the damping ratio of the dominant closed-loop poles equal to 0.5 and to increase the undamped natural frequency to 5 rad/s and the static velocity error constant to 80 s^{-1}. Design an appropriate compensator to meet all the design specifications.

Let us use a lag-lead compensator of the form given by Eq. (4). The desired locations for the dominant closed-loop poles are at

$$s = -2.50 \pm j4.33$$

The open-loop transfer function of the compensated system is

$$G_c(s)G(s) = K_c \frac{\left(s + \dfrac{1}{T_1}\right)\left(s + \dfrac{1}{T_2}\right)}{\left(s + \dfrac{\beta}{T_1}\right)\left(s + \dfrac{1}{\beta T_2}\right)} \cdot \frac{4}{s(s+0.5)}$$

Since the requirement on the static velocity error constant K_v is 80 s^{-1}, we have

$$K_v = \lim_{s\to 0} sG_c(s)G(s) = \lim_{s\to 0} K_c \frac{4}{0.5} = 8K_c = 80$$

Thus

$$K_c = 10$$

Noting that

$$\left. \angle \frac{4}{s(s+0.5)} \right|_{s=-2.50+j4.33} = -235^\circ$$

the time constant T_1 and the value of β are determined from

$$\left| \frac{s + \dfrac{1}{T_1}}{s + \dfrac{\beta}{T_1}} \right| \left| \frac{40}{s(s+0.5)} \right|_{s=-2.5+j4.33} = \left| \frac{s + \dfrac{1}{T_1}}{s + \dfrac{\beta}{T_1}} \right| \frac{8}{4.77} = 1$$

$$\left. \angle \frac{s + \dfrac{1}{T_1}}{s + \dfrac{\beta}{T_1}} \right|_{s=-2.5+j4.33} = 55^\circ$$

Referring to Fig. 10, we can easily locate points A and B such that

$$\angle APB = 55^\circ, \quad \frac{\overline{PA}}{\overline{PB}} = \frac{4.77}{8}$$

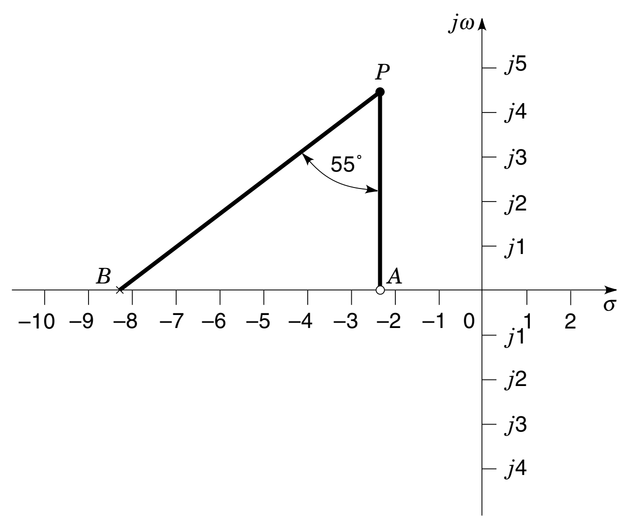


Figure 10. Determination of the desired pole-zero location.

(Use a graphical approach or a trigonometric approach.) The result is

$$\overline{AO} = 2.38, \quad \overline{BO} = 8.34$$

or

$$T_1 = \frac{1}{2.38} = 0.420, \quad \beta = 8.34T_1 = 3.503$$

The phase lead portion of the lag-lead network thus becomes

$$10\left(\frac{s+2.38}{s+8.34}\right)$$

For the phase lag portion, we may choose

$$T_2 = 10$$

Then

$$\frac{1}{\beta T_2} = \frac{1}{3.503 \times 10} = 0.0285$$

Thus, the lag-lead compensator becomes

$$G_c(s) = (10)\left(\frac{s+2.38}{s+8.34}\right)\left(\frac{s+0.1}{s+0.0285}\right)$$

The compensated system will have the open-loop transfer function

$$G_c(s)G(s) = \frac{40(s+2.38)(s+0.1)}{(s+8.34)(s+0.0285)s(s+0.5)}$$

No cancellation occurs in this case, and the compensated system is of fourth order. Because the angle contribution of the phase lag portion of the lag-lead network is quite small, the dominant closed-loop poles are located very near the desired location. In fact, the dominant closed-loop poles are located at $s = -2.4539 \pm j4.3099$. The two other closed-loop poles are located at

$$s = -0.1003, \quad s = -3.8604$$

Since the closed-loop pole at $s = -0.1003$ is very close to a zero at $s = -0.1$, they almost cancel each other. Thus, the effect of this closed-loop pole is very small. The remaining closed-loop pole ($s = -3.8604$) does not quite cancel the zero at $s = -2.4$. The effect of this zero is to cause a larger overshoot in the step response than a similar system without such a zero. The unit-step response curves of the compensated and uncompensated systems are shown in Fig. 11(a). The unit-ramp response curves for both systems are depicted in Fig. 11(b).

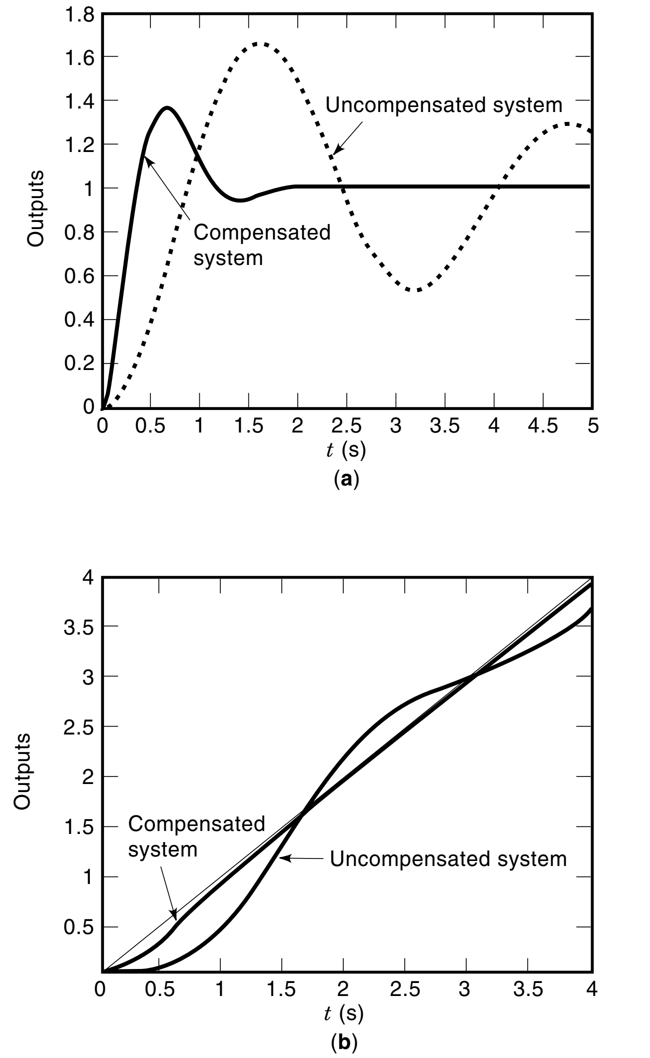


Figure 11. (a) Unit-step response curves for the compensated and uncompensated systems; (b) unit-ramp response curves for both systems.

FREQUENCY-RESPONSE APPROACH TO THE DESIGN OF CONTROL SYSTEMS

Lead Compensation

We shall first examine the frequency characteristics of the lead compensator. Then we present a design technique for the lead compensator by use of the Bode diagram.

Characteristics of Lead Compensators. Consider a lead compensator defined by

$$K_c\alpha\,\frac{j\omega T+1}{j\omega\alpha T+1} \qquad (0 < \alpha < 1)$$

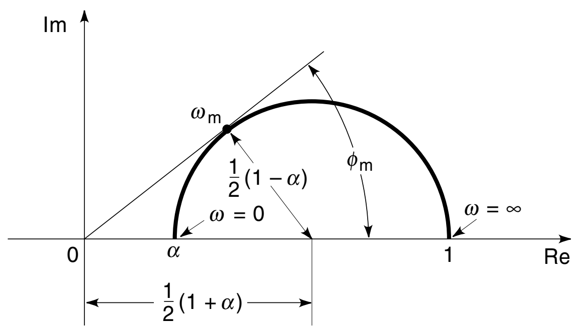


Figure 12. Polar plot of a lead compensator $\alpha(j\omega T + 1)/(j\omega\alpha T + 1)$, where $0 < \alpha < 1$.

Figure 12 shows the polar plot of this compensator with $K_c = 1$. For a given value of α, the angle between the positive real axis and the tangent line drawn from the origin to the semicircle gives the maximum phase lead angle, ϕ_m. We shall call the frequency at the tangent point ω_m. From Fig. 12 the phase angle at $\omega = \omega_m$ is ϕ_m, where

$$\sin \phi_m = \frac{\frac{1-\alpha}{2}}{\frac{1+\alpha}{2}} = \frac{1-\alpha}{1+\alpha} \tag{5}$$

Equation (5) relates the maximum phase lead angle and the value of α.

Figure 13 shows the Bode diagram of a lead compensator when $K_c = 1$ and $\alpha = 0.1$. The corner frequencies for the lead compensator are $\omega = 1/T$ and $\omega = 1/(\alpha T) = 10/T$. By examining Fig. 13, we see that ω_m is the geometric mean of the two corner frequencies, or

$$\log \omega_m = \frac{1}{2}\left(\log \frac{1}{T} + \log \frac{1}{\alpha T}\right)$$

Hence,

$$\omega_m = \frac{1}{\sqrt{\alpha}T} \tag{6}$$

As seen from Fig. 13, the lead compensator is basically a high-pass filter. (The high frequencies are passed, but low frequencies are attenuated.)

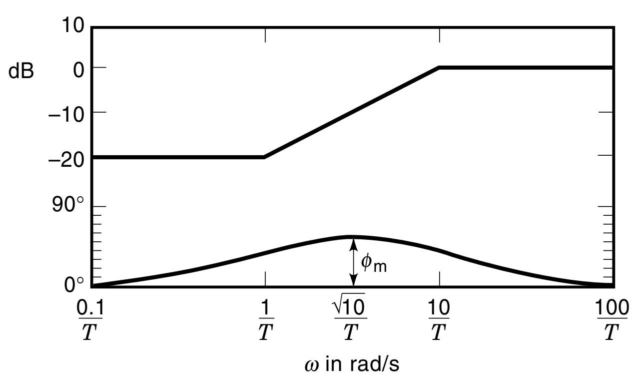


Figure 13. Bode diagram of a lead compensator $\alpha(j\omega T + 1)/(j\omega\alpha T + 1)$, where $\alpha = 0.1$.

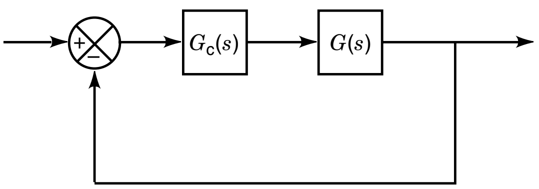


Figure 14. Control system.

Lead Compensation Techniques Based on the Frequency-Response Approach. The primary function of the lead compensator is to reshape the frequency-response curve to provide sufficient phase-lead angle to offset the excessive phase lag associated with the components of the fixed system.

Consider the system shown in Fig. 14. Assume that the performance specifications are given in terms of phase margin, gain margin, static velocity error constants, and so on. The procedure for designing a lead compensator by the frequency-response approach may be stated as follows:

1. Assume the following lead compensator:

$$G_c(s) = K_c\alpha \frac{Ts+1}{\alpha Ts+1} = K_c \frac{s+\frac{1}{T}}{s+\frac{1}{\alpha T}} \qquad (0 < \alpha < 1) \tag{7}$$

Define

$$K_c\alpha = K$$

Then

$$G_c(s) = K\frac{Ts+1}{\alpha Ts+1}$$

The open-loop transfer function of the compensated system is

$$G_c(s)G(s) = K\frac{Ts+1}{\alpha Ts+1}G(s) = \frac{Ts+1}{\alpha Ts+1}KG(s) = \frac{Ts+1}{\alpha Ts+1}G_1(s)$$

where

$$G_1(s) = KG(s)$$

Determine gain K to satisfy the requirement on the given static error constant.

2. Using the gain K thus determined, draw a Bode diagram of $G_1(j\omega)$, the gain-adjusted but uncompensated system. Evaluate the phase margin.
3. Determine the necessary phase lead angle ϕ to be added to the system.
4. Determine the attenuation factor α by use of Eq. (5). Determine the frequency where the magnitude of the uncompensated system $G_1(j\omega)$ is equal to $-20 \log (1/\sqrt{\alpha})$. Select this frequency as the new gain crossover frequency. This frequency corresponds to $\omega_m = 1/(\sqrt{\alpha}T)$, and the maximum phase shift ϕ_m occurs at this frequency.

5. Determine the corner frequencies of the lead compensator as follows:

$$\text{Zero of lead compensator:} \quad \omega = \frac{1}{T}$$

$$\text{Pole of lead compensator:} \quad \omega = \frac{1}{\alpha T}$$

6. Using the value of K determined in step 1 and that of α determined in step 4, calculate constant K_c from

$$K_c = \frac{K}{\alpha}$$

7. Check the gain margin to be sure it is satisfactory. If not, repeat the design process by modifying the pole-zero location of the compensator until a satisfactory result is obtained.

Example 4. Consider the system shown in Fig. 15. The open-loop transfer function is

$$G(s) = \frac{4}{s(s+2)}$$

It is desired to design a compensator for the system so that the static velocity error constant K_v is 20 s^{-1}, the phase margin is at least 50°, and the gain margin is at least 10 dB.

We shall use a lead compensator of the form defined by Eq. (7). Define

$$G_1(s) = KG(s) = \frac{4K}{s(s+2)}$$

where $K = K_c\alpha$.

The first step in the design is to adjust the gain K to meet the steady-state performance specification or to provide the required static velocity error constant. Since this constant is given as 20 s^{-1}, we obtain

$$K_v = \lim_{s\to 0} sG_c(s)G(s) = \lim_{s\to 0} s\frac{Ts+1}{\alpha Ts+1}G_1(s)$$
$$= \lim_{s\to 0}\frac{s4K}{s(s+2)} = 2K = 20$$

or

$$K = 10$$

With $K = 10$, the compensated system will satisfy the steady-state requirement.

We shall next plot the Bode diagram of

$$G_1(j\omega) = \frac{40}{j\omega(j\omega+2)} = \frac{20}{j\omega(0.5j\omega+1)}$$

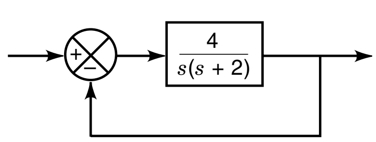


Figure 15. Control system.

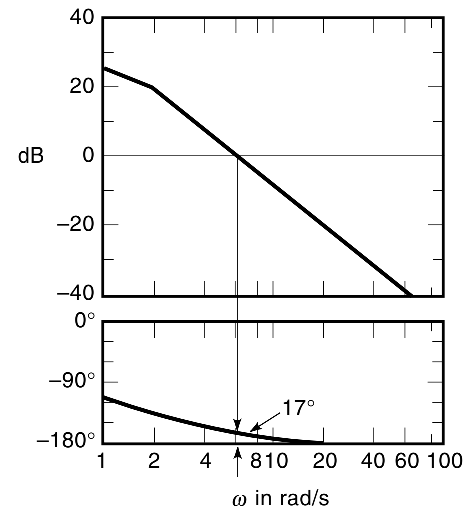


Figure 16. Bode diagram for $G_1(j\omega) = 10G(j\omega) = 40/[j\omega(j\omega + 2)]$.

Figure 16 shows the magnitude and phase angle curves of $G_1(j\omega)$. From this plot, the phase and gain margins of the system are found to be 17° and $+\infty$ dB, respectively. (A phase margin of 17° implies that the system is quite oscillatory. Thus, satisfying the specification on the steady state yields a poor transient-response performance.) The specification calls for a phase margin of at least 50°. We thus find the additional phase lead necessary to satisfy the relative stability requirement is 33°. To achieve a phase margin of 50° without decreasing the value of K, the lead compensator must contribute the required phase angle.

Noting that the addition of a lead compensator modifies the magnitude curve in the Bode diagram, we realize that the gain crossover frequency will be shifted to the right. We must offset the increased phase lag of $G_1(j\omega)$ due to this increase in the gain crossover frequency. Considering the shift of the gain crossover frequency, we may assume that ϕ_m, the maximum phase lead required, is approximately 38°. (This means that 5° has been added to compensate for the shift in the gain crossover frequency.)

Since

$$\sin\phi_m = \frac{1-\alpha}{1+\alpha}$$

$\phi_m = 38°$ corresponds to $\alpha = 0.24$. Once the attenuation factor α has been determined on the basis of the required phase lead angle, the next step is to determine the corner frequencies $\omega = 1/T$ and $\omega = 1/(\alpha T)$ of the lead compensator. To do so, we first note that the maximum phase lead angle ϕ_m occurs at the geometric mean of the two corner frequencies, or $\omega = 1/(\sqrt{\alpha}T)$. [See Eq. (6).] The amount of the modification in the magnitude curve at $\omega = 1/(\sqrt{\alpha}T)$ due to the inclusion of the term $(Ts + 1)/(\alpha Ts + 1)$ is

$$\left|\frac{1+j\omega T}{1+j\omega\alpha T}\right|_{\omega=1/(\sqrt{\alpha}T)} = \left|\frac{1+j\frac{1}{\sqrt{\alpha}}}{1+j\alpha\frac{1}{\sqrt{\alpha}}}\right| = \frac{1}{\sqrt{\alpha}}$$

Note that

$$\frac{1}{\sqrt{\alpha}} = \frac{1}{\sqrt{0.24}} = \frac{1}{0.49} = 6.2\,\text{dB}$$

and $|G_1(j\omega)| = -6.2$ dB corresponds to $\omega = 9$ rad/s. We shall select this frequency to be the new gain crossover frequency ω_c. Noting that this frequency corresponds to $1/(\sqrt{\alpha}T)$, or $\omega_c = 1/(\sqrt{\alpha}T)$, we obtain

$$\frac{1}{T} = \sqrt{\alpha}\omega_c = 4.41$$

and

$$\frac{1}{\alpha T} = \frac{\omega_c}{\sqrt{\alpha}} = 18.4$$

The lead compensator thus determined is

$$G_c(s) = K_c \frac{s+4.41}{s+18.4} = K_c\alpha \frac{0.227s+1}{0.054s+1}$$

where the value of K_c is determined as

$$K_c = \frac{K}{\alpha} = \frac{10}{0.24} = 41.7$$

Thus, the transfer function of the compensator becomes

$$G_c(s) = 41.7 \frac{s+4.41}{s+18.4} = 10 \frac{0.227s+1}{0.054s+1}$$

Note that

$$\frac{G_c(s)}{K} G_1(s) = \frac{G_c(s)}{10} 10G(s) = G_c(s)G(s)$$

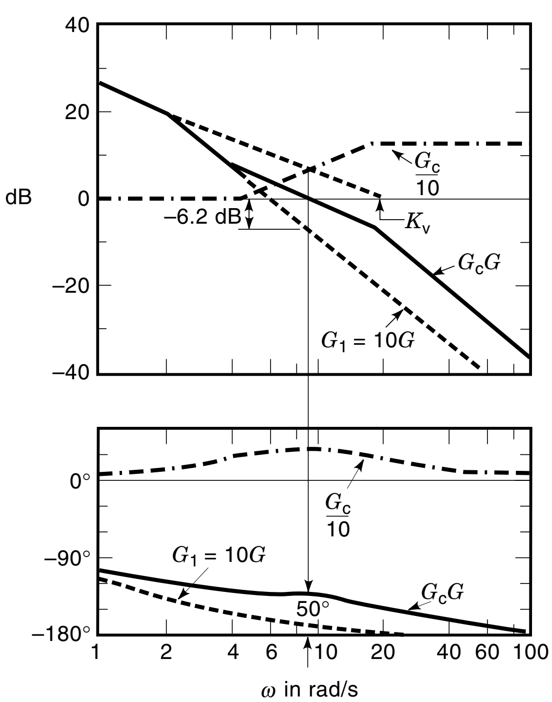


Figure 17. Bode diagram for the compensated system.

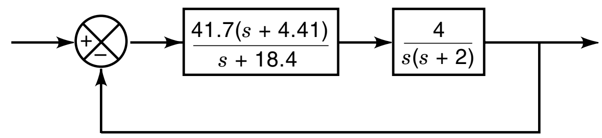


Figure 18. Compensated system.

The magnitude curve and phase-angle curve for $G_c(j\omega)/10$ are shown in Fig. 17. The compensated system has the following open-loop transfer function:

$$G_c(s)G(s) = 41.7 \frac{s+4.41}{s+18.4} \frac{4}{s(s+2)}$$

The solid curves in Fig. 17 show the magnitude curve and phase-angle curve for the compensated system. The lead compensator causes the gain crossover frequency to increase from 6.3 to 9 rad/s. The increase in this frequency means an increase in bandwidth. This implies an increase in the speed of response. The phase and gain margins are seen to be approximately 50° and $+\infty$ dB, respectively. The compensated system shown in Fig. 18 therefore meets both the steady-state and the relative-stability requirements.

Note that for type 1 systems, such as the system just considered, the value of the static velocity error constant K_v is merely the value of the frequency corresponding to the intersection of the extension of the initial −20 dB/decade slope line and the 0 dB line, as shown in Fig. 17.

Figures 19 and 20 show, respectively, the unit-step and unit-ramp responses of both the compensated system and uncompensated system.

Lag Compensation

Characteristics of Lag Compensators. Consider the lag compensator given by Eq. (3). Figure 21 shows a polar plot of the lag compensator. Figure 22 shows a Bode diagram of the compensator, where $K_c = 1$ and $\beta = 10$. The corner frequencies of the lag compensator are at $\omega = 1/T$ and $\omega = 1/(\beta T)$. As seen from Fig. 22, where the values of K_c and β are set equal to 1 and 10, respectively, the magnitude of the lag com-

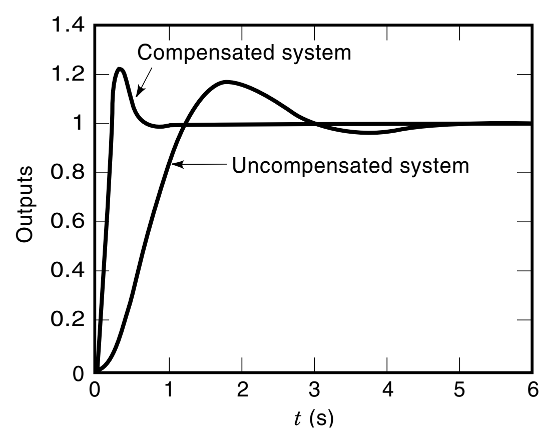


Figure 19. Unit-step response curves of the compensated and uncompensated systems.

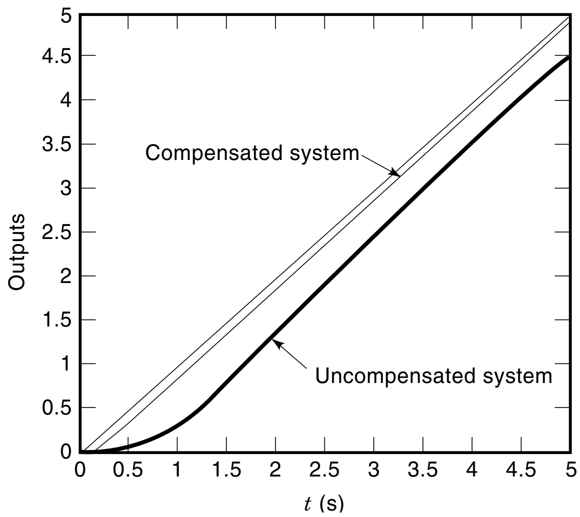


Figure 20. Unit-ramp response curves of the compensated and uncompensated systems.

pensator becomes 10 (or 20 dB) at low frequencies and unity (or 0 dB) at high frequencies. Thus, the lag compensator is essentially a low-pass filter.

Lag Compensation Techniques Based on the Frequency-Response Approach. The primary function of a lag compensator is to provide attenuation in the high-frequency range to give a system sufficient phase margin. The phase lag characteristic is of no consequence in lag compensation.

The procedure for designing lag compensators for the system shown in Fig. 14 by the frequency-response approach may be stated as follows:

1. Assume the following lag compensator:

$$G_c(s) = K_c\beta\,\frac{Ts+1}{\beta Ts+1} = K_c\frac{s+\dfrac{1}{T}}{s+\dfrac{1}{\beta T}} \qquad (\beta > 1)$$

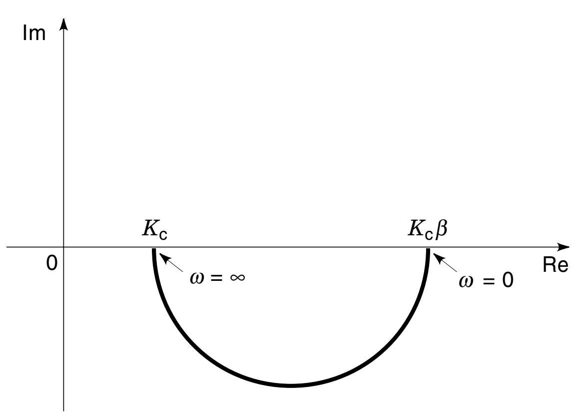


Figure 21. Polar plot of a lag compensator $K_c\beta(j\omega T + 1)/(j\omega\beta T + 1)$.

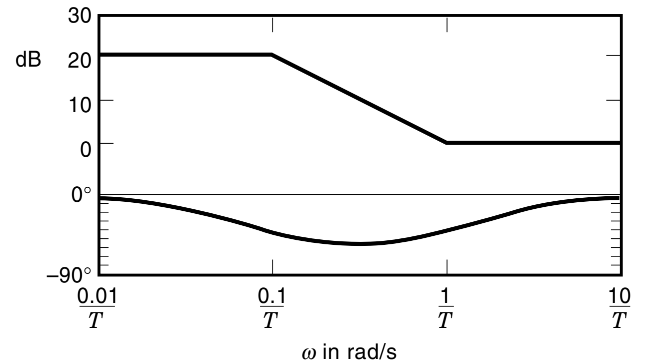


Figure 22. Bode diagram of a lag compensator $\beta(j\omega T + 1)/(j\omega\beta T + 1)$, with $\beta = 10$.

Define

$$K_c\beta = K$$

Then

$$G_c(s) = K\,\frac{Ts+1}{\beta Ts+1}$$

The open-loop transfer function of the compensated system is

$$\begin{aligned} G_c(s)G(s) &= K\,\frac{Ts+1}{\beta Ts+1}\,G(s) = \frac{Ts+1}{\beta Ts+1}\,KG(s) \\ &= \frac{Ts+1}{\beta Ts+1}\,G_1(s) \end{aligned}$$

where

$$G_1(s) = KG(s)$$

Determine gain K to satisfy the requirement on the given static error constant.

2. If the uncompensated system $G_1(j\omega) = KG(j\omega)$ does not satisfy the specifications on the phase and gain margins, then find the frequency point where the phase angle of the open-loop transfer function is equal to $-180°$ plus the required phase margin. The required phase margin is the specified phase margin plus 5° to 12°. (The addition of 5° to 12° compensates for the phase lag of the lag compensator.) Choose this frequency as the new gain crossover frequency.
3. To prevent detrimental effects of phase lag due to the lag compensator, the pole and zero of the lag compensator must be located substantially lower than the new gain crossover frequency. Therefore, choose the corner frequency $\omega = 1/T$ (corresponding to the zero of the lag compensator) 1 octave to 1 decade below the new gain crossover frequency. (If the time constants of the lag compensator do not become too large, the corner frequency $\omega = 1/T$ may be chosen 1 decade below the new gain crossover frequency.)
4. Determine the attenuation necessary to bring the magnitude curve down to 0 dB at the new gain crossover frequency. Noting that this attenuation is $-20\log\beta$,

determine the value of β. Then the other corner frequency (corresponding to the pole of the lag compensator) is determined from $\omega = 1/(\beta T)$.

5. Using the value of K determined in step 1 and that of β determined in step 4, calculate constant K_c from

$$K_c = \frac{K}{\beta}$$

Example 5. Consider the system shown in Fig. 23. The open-loop transfer function is given by

$$G(s) = \frac{1}{s(s+1)(0.5s+1)}$$

It is desired to compensate the system so that the static velocity error constant K_v is 5 s^{-1}, the phase margin is at least 40°, and the gain margin is at least 10 dB.

We shall use a lag compensator of the form

$$G_c(s) = K_c\beta\,\frac{Ts+1}{\beta Ts+1} = K_c\,\frac{s+\dfrac{1}{T}}{s+\dfrac{1}{\beta T}} \qquad (\beta > 1)$$

Define

$$K_c\beta = K$$

Define also

$$G_1(s) = KG(s) = \frac{K}{s(s-1)(0.5s+1)}$$

The first step in the design is to adjust the gain K to meet the required static velocity error constant. Thus,

$$K_v = \lim_{s\to 0} sG_c(s)G(s) = \lim_{s\to 0} s\,\frac{Ts+1}{\beta Ts+1}\,G_1(s) = \lim_{s\to 0} sG_1(s)$$
$$= \lim_{s\to 0} \frac{sK}{s(s+1)(0.5s+1)} = K = 5$$

With $K = 5$, the compensated system satisfies the steady-state performance requirement.

We shall next plot the Bode diagram of

$$G_1(j\omega) = \frac{5}{j\omega(j\omega+1)(0.5j\omega+1)}$$

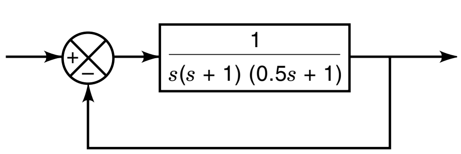


Figure 23. Control system.

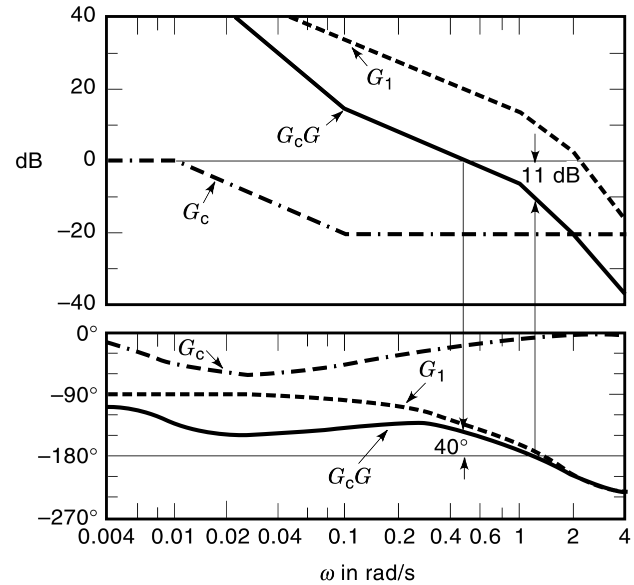


Figure 24. Bode diagrams for the uncompensated system, the compensator, and the compensated system. (G_1: uncompensated system, G_c: compensator, G_cG: compensated system.)

The magnitude curve and phase-angle curve of $G_1(j\omega)$ are shown in Fig. 24. From this plot, the phase margin is found to be −20°, which means that the system is unstable.

Noting that the addition of a lag compensator modifies the phase curve of the Bode diagram, we must allow 5° to 12° to the specified phase margin to compensate for the modification of the phase curve. Since the frequency corresponding to a phase margin of 40° is 0.7 rad/s, the new gain crossover frequency (of the compensated system) must be chosen near this value. To avoid overly large time constants for the lag compensator, we shall choose the corner frequency $\omega = 1/T$ (which corresponds to the zero of the lag compensator) to be 0.1 rad/s. Since this corner frequency is not too far below the new gain crossover frequency, the modification in the phase curve may not be small. Hence, we add about 12° to the given phase margin as an allowance to account for the lag angle introduced by the lag compensator. The required phase margin is now 52°. The phase angle of the uncompensated open-loop transfer function is −128° at about $\omega = 0.5$ rad/s. So we choose the new gain crossover frequency to be 0.5 rad/s. To bring the magnitude curve down to 0 dB at this new gain crossover frequency, the lag compensator must give the necessary attenuation, which in this case is −20 dB. Hence,

$$20\log\frac{1}{\beta} = -20$$

or

$$\beta = 10$$

The other corner frequency $\omega = 1(\beta T)$, which corresponds to the pole of the lag compensator, is then determined as

$$\frac{1}{\beta T} = 0.01 \text{ rad/s}$$

Thus, the transfer function of the lag compensator is

$$G_c(s) = K_c(10)\,\frac{10s+1}{100s+1} = K_c\,\frac{s+\dfrac{1}{10}}{s+\dfrac{1}{100}}$$

Since the gain K was determined to be 5 and β was determined to be 10, we have

$$K_c = \frac{K}{\beta} = \frac{5}{10} = 0.5$$

The open-loop transfer function of the compensated system is

$$G_c(s)G(s) = \frac{5(10s+1)}{s(100s+1)(s+1)(0.5s+1)}$$

The magnitude and phase-angle curves of $G_c(j\omega)G(j\omega)$ are also shown in Fig. 24.

The phase margin of the compensated system is about 40°, which is the required value. The gain margin is about 11 dB, which is quite acceptable. The static velocity error constant is 5 s^{-1}, as required. The compensated system, therefore, satisfies the requirements on both the steady state and the relative stability.

Note that the new gain crossover frequency is decreased from approximately 2 to 0.5 rad/s. This means that the bandwidth of the system is reduced.

Figures 25 and 26 show, respectively, the unit-step and unit-ramp responses of the compensated and uncompensated systems. (The uncompensated system is shown in Fig. 23.)

Lag-Lead Compensation

Lag-Lead Compensation Based on the Frequency-Response Approach. The design of a lag-lead compensator by the frequency-response approach is based on the combination of the design techniques discussed under lead compensation and lag compensation.

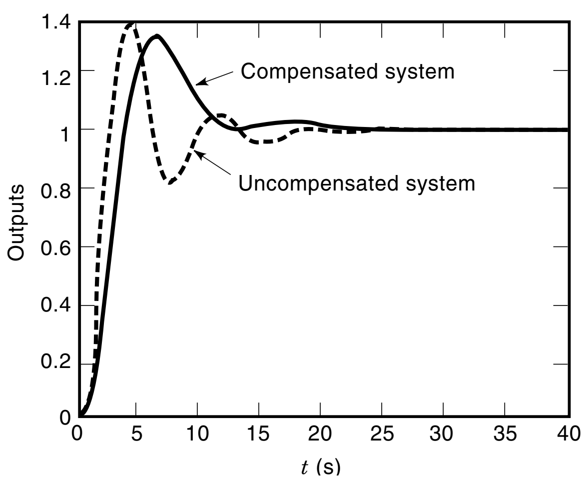


Figure 25. Unit-step response curves for the compensated and uncompensated systems.

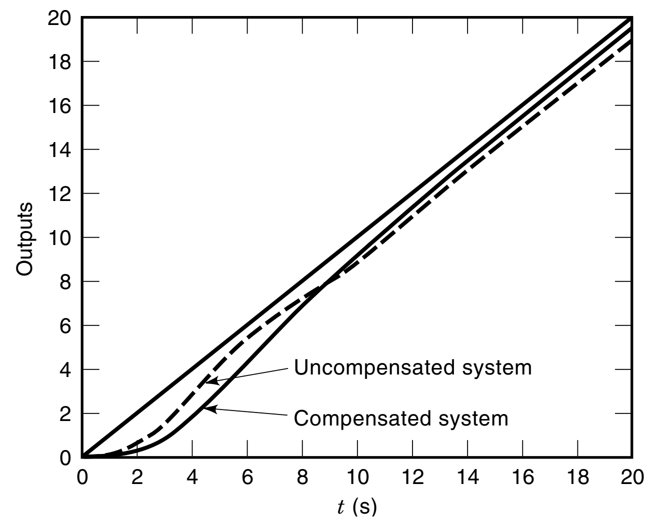


Figure 26. Unit-ramp response curves for the compensated and uncompensated systems.

Let us assume that the lag-lead compensator is of the following form:

$$G_c(s) = K_c\frac{(T_1s+1)(T_2s+1)}{\left(\dfrac{T_1}{\beta}s+1\right)(\beta T_2s+1)} = K_c\frac{\left(s+\dfrac{1}{T_1}\right)\left(s+\dfrac{1}{T_2}\right)}{\left(s+\dfrac{\beta}{T_1}\right)\left(s+\dfrac{1}{\beta T_2}\right)} \tag{8}$$

where $\beta > 1$. The phase lead portion of the lag-lead compensator (the portion involving T_1) alters the frequency-response curve by adding phase lead angle and increasing the phase margin at the gain crossover frequency. The phase lag portion (the portion involving T_2) provides attenuation near and above the gain crossover frequency and thereby allows an increase of gain at the low-frequency range to improve the steady-state performance.

Figure 27 shows a Bode diagram of a lag-lead compensator when $K_c = 1$, $\beta = 10$, and $T_2 = 10T_1$. Notice that the magnitude curve has the value 0 dB at both low-frequency and high-frequency regions.

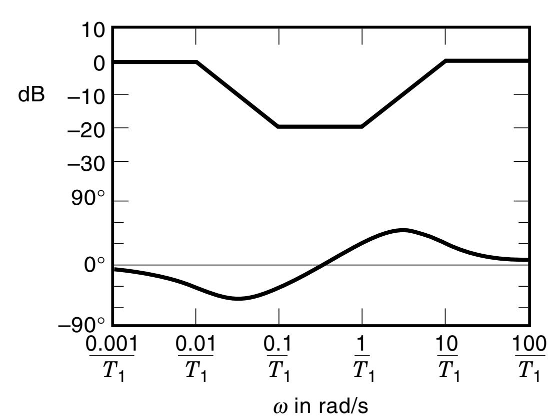


Figure 27. Bode diagram of a lag-lead compensator given by Eq. (8) with $K_c = 1$, $\beta = 10$, and $T_2 = 10T_1$.

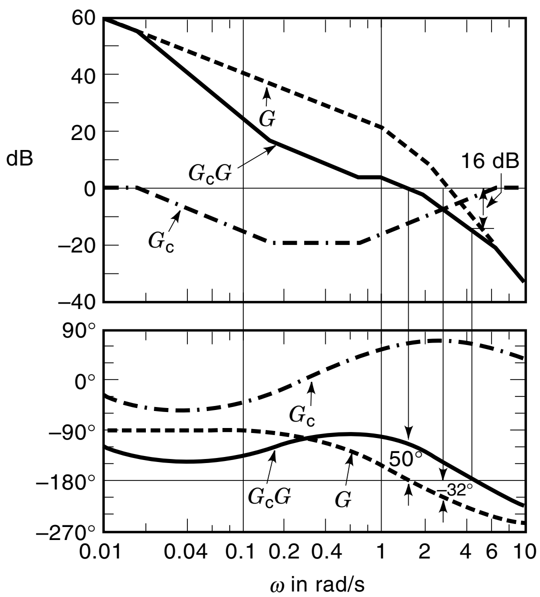


Figure 28. Bode diagrams for the uncompensated system, the compensator, and the compensated system. (G: uncompensated system, G_c: compensator, G_cG: compensated system.)

We shall illustrate the details of the procedure for designing a lag-lead compensator by an example.

Example 6. Consider the unity-feedback system whose open-loop transfer function is

$$G(s) = \frac{K}{s(s+1)(s+2)}$$

It is desired that the static velocity error constant be 10 s^{-1}, the phase margin be 50°, and the gain margin be 10 dB or more.

Assume that we use the lag-lead compensator given by Eq. (8). The open-loop transfer function of the compensated system is $G_c(s)G(s)$. Since the gain K of the plant is adjustable, let us assume that $K_c = 1$. Then $\lim_{s\to 0} G_c(s) = 1$.

From the requirement on the static velocity error constant, we obtain

$$K_v = \lim_{s\to 0} sG_c(s)G(s) = \lim_{s\to 0} sG_c(s)\,\frac{K}{s(s+1)(s+2)} = \frac{K}{2} = 10$$

Hence,

$$K = 20$$

We shall next draw the Bode diagram of the uncompensated system with $K = 20$, as shown in Fig. 28. The phase margin of the uncompensated system is found to be $-32°$, which indicates that the incompensated system is unstable.

The next step in the design of a lag-lead compensator is to choose a new gain crossover frequency. From the phase angle curve for $G(j\omega)$, we notice that $\underline{/G(j\omega)} = -180°$ at $\omega = 1.5$ rad/s. It is convenient to choose the new gain crossover frequency to be 1.5 rad/s so that the phase-lead angle required at $\omega = 1.5$ rad/s is about 50°, which is quite possible by use of a single lag-lead compensator.

Once we choose the gain crossover frequency to be 1.5 rad/s, we can determine the corner frequency of the phase lag portion of the lag-lead compensator. Let us choose the corner frequency $\omega = 1/T_2$ (which corresponds to the zero of the phase-lag portion of the compensator) to be 1 decade below the new gain crossover frequency, or at $\omega = 0.15$ rad/s.

Recall that for the lead compensator the maximum phase lead angle ϕ_m is given by Eq. (5), where α in Eq. (5) is $1/\beta$ in the present case. By substituting $\alpha = 1/\beta$ in Eq. (5), we have

$$\sin\phi_m = \frac{1 - \dfrac{1}{\beta}}{1 + \dfrac{1}{\beta}} = \frac{\beta - 1}{\beta + 1}$$

Notice that $\beta = 10$ corresponds to $\phi_m = 54.9°$. Since we need a 50° phase margin, we may choose $\beta = 10$. (Note that we will be using several degrees less than the maximum angle, 54.9°.) Thus,

$$\beta = 10$$

Then the corner frequency $\omega = 1/\beta T_2$ (which corresponds to the pole of the phase lag portion of the compensator) becomes $\omega = 0.015$ rad/s. The transfer function of the phase lag portion of the lag-lead compensator then becomes

$$\frac{s+0.15}{s+0.015} = 10\left(\frac{6.67s+1}{66.7s+1}\right)$$

The phase lead portion can be determined as follows: Since the new gain crossover frequency is $\omega = 1.5$ rad/s, from Fig. 28, $G(j1.5)$ is found to be 13 dB. Hence, if the lag-lead compensator contributes -13 dB at $\omega = 1.5$ rad/s, then the new gain crossover frequency is as desired. From this requirement, it is possible to draw a straight line of slope 20 dB/decade, passing through the point (-13 dB, 1.5 rad/s). The intersections of this line and the 0 dB line and -20 dB line determine the corner frequencies. Thus, the corner frequencies for the lead portion are $\omega = 0.7$ rad/s and $\omega = 7$ rad/s. Thus, the transfer function of the lead portion of the lag-lead compensator becomes

$$\frac{s+0.7}{s+7} = \frac{1}{10}\left(\frac{1.43s+1}{0.143s+1}\right)$$

Combining the transfer functions of the lag and lead portions of the compensator, we obtain the transfer function of the lag-lead compensator. Since we chose $K_c = 1$, we have

$$G_c(s) = \left(\frac{s+0.7}{s+7}\right)\left(\frac{s+0.15}{s+0.015}\right) = \left(\frac{1.43s+1}{0.143s+1}\right)\left(\frac{6.67s+1}{66.7s+1}\right)$$

The magnitude and phase-angle curves of the lag-lead compensator just designed are shown in Fig. 28. The open-loop transfer function of the compensated system is

$$\begin{aligned} G_c(s)G(s) &= \frac{(s+0.7)(s+0.15)20}{(s+7)(s+0.015)s(s+1)(s+2)} \\ &= \frac{10(1.43s+1)(6.67s+1)}{s(0.143s+1)(66.7s+1)(s+1)(0.5s+1)} \end{aligned} \tag{9}$$

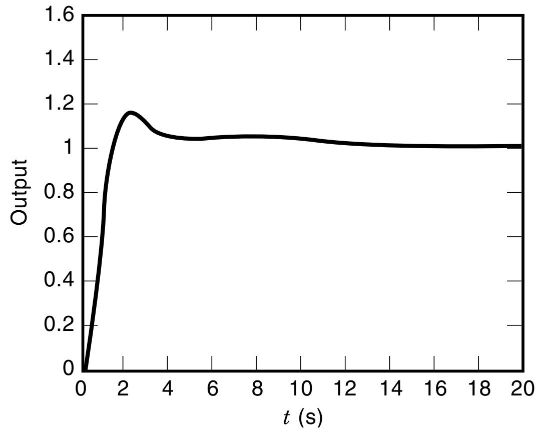


Figure 29. Unit-step response of the compensated system.

The magnitude and phase-angle curves of the system of Eq. (9) are also shown in Fig. 28. The phase margin of the compensated system is 50°, the gain margin is 16 dB, and the static velocity error constant is 10 s^{-1}. All the requirements are therefore met, and the design has been completed.

The unit-step response and unit-ramp response of the designed system are shown in Figs. 29 and 30, respectively.

COMPARISON OF LEAD, LAG, LAG-LEAD COMPENSATION

1. Lead compensation achieves the desired result through the merits of its phase-lead contribution, whereas lag compensation accomplishes the result through the merits of its attenuation property at high frequencies. (In some design problems both lag compensation and lead compensation may satisfy the specifications.)
2. Lead compensation is commonly used for improving stability margins. Lead compensation yields a higher gain crossover frequency than is possible with lag compensation. The higher gain crossover frequency means larger bandwidth. A large bandwidth means reduction in the settling time. The bandwidth of a system with lead compensation is always greater than that with lag compensation. Therefore, if a large bandwidth or fast response is desired, lead compensation should be employed. If, however, noise signals are present, then a large bandwidth may not be desirable, since it makes the system more susceptible to noise signals because of increase in the high-frequency gain.
3. Lead compensation requires an additional increase in gain to offset the attenuation inherent in the lead network. This means that lead compensation will require a larger gain than that required by lag compensation. A larger gain, in most cases, implies larger space, greater weight, and higher cost.
4. Lag compensation reduces the system gain at higher frequencies without reducing the system gain at lower frequencies. Since the system bandwidth is reduced, the system has a slower speed to respond. Because of the reduced high-frequency gain, the total system gain can be increased, and thereby low-frequency gain can be increased and the steady-state accuracy can be improved. Also, any high-frequency noises involved in the system can be attenuated.
5. If both fast responses and good static accuracy are desired, a lag-lead compensator may be employed. By use of the lag-lead compensator, the low-frequency gain can be increased (which means an improvement in steady-state accuracy), while at the same time the system bandwidth and stability margins can be increased.
6. Although a large number of practical compensation tasks can be accomplished with lead, lag, or lag-lead compensators, for complicated systems, simple compensation by use of these compensators may not yield satisfactory results. Then different compensators having different pole-zero configurations must be employed.

MULTI-DEGREES-OF-FREEDOM CONTROL

In the classical design approaches presented in this article, we design control systems such that the response to the reference input is satisfactory. If the control system is subjected to other inputs, such as disturbance input and noise input, it is not possible to design the system such that the responses to the disturbance input and noise input are also satisfactory, in addition to the primary requirement that the response to the reference input is satisfactory. This is because the systems we considered so far simply do not have the freedom to satisfy requirements on the responses to disturbances and noises.

If we wish to design high-performance control systems in the presence of disturbances and sensor noises, we must change the configuration of the control system. This means that we must provide additional degrees of freedom to the control system to handle additional requirements.

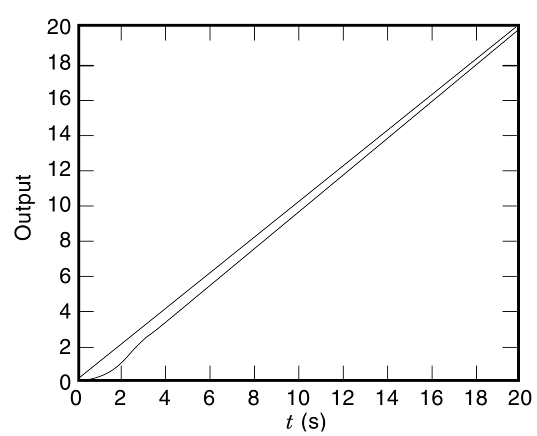


Figure 30. Unit-ramp response of the compensated system.

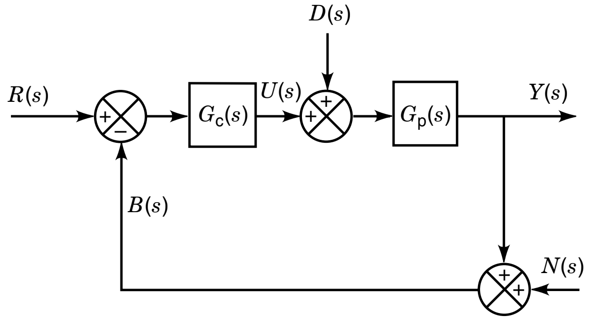


Figure 31. One-degree-of-freedom control system.

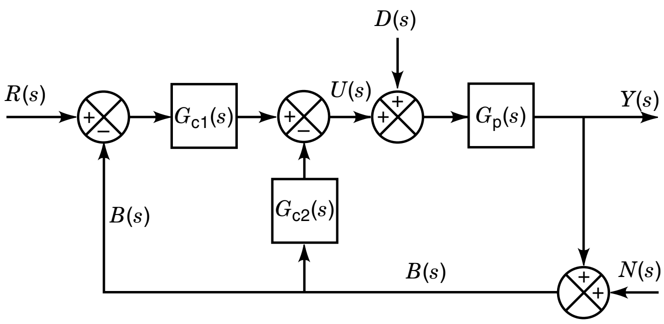


Figure 32. Two-degrees-of-freedom control system.

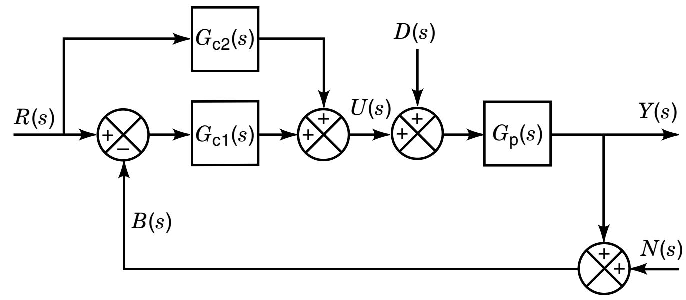


Figure 33. Two-degrees-of-freedom control system.

In what follows we first discuss the single-degree-of-freedom control systems and then discuss the two-degrees-of-freedom control systems. Finally, we present an example of three-degrees-of-freedom control systems that can satisfy the requirements on the responses to the reference input, disturbance input, and noise input.

Single-Degree-of-Freedom Control

Consider the system shown in Fig. 31, where the system is subjected to the disturbance input $D(s)$ and noise input $N(s)$. $G_c(s)$ is the transfer function of the controller and $G_p(s)$ is the transfer function of the plant. We assume that $G_p(s)$ is fixed and unalterable.

For this system, three closed-loop transfer functions $Y(s)/R(s) = G_{yr}$, $Y(s)/D(s) = G_{yd}$, and $Y(s)/N(s) = G_{yn}$ may be derived. They are

$$G_{yr} = \frac{Y(s)}{R(s)} = \frac{G_c G_p}{1 + G_c G_p}$$
$$G_{yd} = \frac{Y(s)}{D(s)} = \frac{G_p}{1 + G_c G_p}$$
$$G_{yn} = \frac{Y(s)}{N(s)} = \frac{G_c G_p}{1 + G_c G_p}$$

[In deriving $Y(s)/R(s)$, we assumed $D(s) = 0$ and $N(s) = 0$. Similar comments apply to the derivations of $Y(s)/D(s)$ and $Y(s)/N(s)$.] The degrees of freedom of the control system refers to how many of these closed-loop transfer functions are independent. In the present case, we have

$$G_{yr} = \frac{G_p - G_{yd}}{G_p}$$
$$G_{yn} = \frac{G_{yd} - G_p}{G_p}$$

Among the three closed-loop transfer functions G_{yr}, G_{yn}, and G_{yd}, if one of them is given, the remaining two are fixed. This means that the system shown in Fig. 31 is a one-degree-of-freedom system.

Two-Degrees-of-Freedom Control

Next consider the system shown in Fig. 32, where $G_p(s)$ is the transfer function of the plant and is assumed to be fixed and unalterable. For this system, closed-loop transfer functions G_{yr}, G_{yn}, and G_{yd} are given, respectively, by

$$G_{yr} = \frac{Y(s)}{R(s)} = \frac{G_{c1} G_p}{1 + (G_{c1} + G_{c2}) G_p}$$
$$G_{yd} = \frac{Y(s)}{D(s)} = \frac{G_p}{1 + (G_{c1} + G_{c2}) G_p}$$
$$G_{yn} = \frac{Y(s)}{N(s)} = -\frac{G_{c1} + G_{c2}) G_p}{1 + (G_{c1} + G_{c2}) G_p}$$

Hence, we have

$$G_{yr} = G_{c1} G_{yd}$$
$$G_{yn} = \frac{G_{yd} - G_p}{G_p}$$

In this case, if G_{yd} is given, then G_{yn} is fixed, but G_{yr} is not fixed, because G_{c1} is independent of G_{yd}. Thus, two closed-loop transfer functions among three closed-loop transfer functions G_{yr}, G_{yd}, and G_{yn} are independent. Hence, this system is a two-degrees-of-freedom control system.

Similarly, the system shown in Fig. 33 is also a two-degrees-of-freedom control system, because for this system

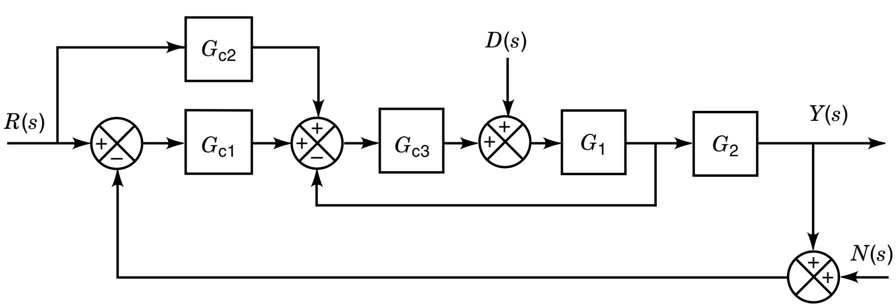


Figure 34. Three-degrees-of-freedom system.

$$G_{\mathrm{yr}} = \frac{Y(s)}{R(s)} = \frac{G_{\mathrm{c1}}G_{\mathrm{p}}}{1+G_{\mathrm{c1}}G_{\mathrm{p}}} + \frac{G_{\mathrm{c2}}G_{\mathrm{p}}}{1+G_{\mathrm{c1}}G_{\mathrm{p}}}$$
$$G_{\mathrm{yd}} = \frac{Y(s)}{D(s)} = \frac{G_{\mathrm{p}}}{1+G_{\mathrm{c1}}G_{\mathrm{p}}}$$
$$G_{\mathrm{yn}} = \frac{Y(s)}{N(s)} = -\frac{G_{\mathrm{c1}}G_{\mathrm{p}}}{1+G_{\mathrm{c1}}G_{\mathrm{p}}}$$

Hence,

$$G_{\mathrm{yr}} = G_{\mathrm{c2}}G_{\mathrm{yd}} + \frac{G_{\mathrm{p}} - G_{\mathrm{yd}}}{G_{\mathrm{p}}}$$
$$G_{\mathrm{yn}} = \frac{G_{\mathrm{yd}} - G_{\mathrm{p}}}{G_{\mathrm{p}}}$$

Clearly, if G_{yd} is given, then G_{yn} is fixed, but G_{yr} is not fixed because G_{c2} is independent of G_{yd}.

Three-Degrees-of-Freedom Control

In the control system shown in Fig. 34, the transfer functions G_{c1}, G_{c2}, and G_{c3} are controllers, and transfer functions G_1 and G_2 are plant transfer functions that are unalterable. It can be shown that this control system is a three-degrees-of-freedom system. If a system has this configuration, then it is possible to design three controllers by use of the root-locus method and/or frequency-response method (or other methods) such that the responses to all three inputs are acceptable.

CONCLUDING COMMENTS

This article has presented easy-to-understand procedures for designing lead compensators, lag compensators, and lag-lead compensators by use of the root-locus method or frequency-response method. The systems are limited to single-input–single-output, linear time-invariant control systems. For such systems various design methods are available in addition to the root-locus method and frequency-response method. Interested readers are referred to specialized books on control systems, as listed in the *Reading List.*

Toward the end of this article we included discussions on multi-degrees-of-freedom control systems for the informed specialist.

Most of the materials presented in this article were taken, with permission, from Katsuhiko Ogata, *Modern Control Engineering 3/e,* © 1997. Prentice Hall, Upper Saddle River, New Jersey.

Reading List

H. W. Bode, *Network Analysis and Feedback Design,* New York: Van Nostrand Reinhold, 1945.

R. C. Dorf, *Modern Control Systems,* 6th ed., Reading, MA: Addison-Wesley, 1992.

W. R. Evans, Graphical analysis of control systems, *AIEE Trans. Part II,* **67**: 547–551, 1948.

W. R. Evans, Control system synthesis by root locus method, *AIEE Trans. Part II,* **69**: 66–69, 1950.

W. R. Evans, The use of zeros and poles for frequency response or transient response, *ASME Trans.,* **76**: 1135–1144, 1954.

G. F. Franklin, J. D. Powell, and A. Emami-Naeini, *Feedback Control of Dynamic Systems,* Reading, MA: Addison-Wesley, 1986.

B. C. Kuo, *Automatic Control Systems,* 6th ed., Upper Saddle River, NJ: Prentice Hall, 1991.

K. Ogata, *Modern Control Engineering,* 3rd ed., Upper Saddle River, NJ: Prentice Hall, 1997.

K. Ogata, *System Dynamics,* 3rd ed., Upper Saddle River, NJ: Prentice Hall, 1998.

K. Ogata, *Solving Control Engineering Problems with MATLAB,* Upper Saddle River, NJ: Prentice Hall, 1994.

C. L. Phillips and R. D. Harbor, *Feedback Control Systems,* Upper Saddle River, NJ: Prentice Hall, 1988.

KATSUHIKO OGATA
University of Minnesota

CONTROL SYSTEM SYNTHESIS. See DELAY SYSTEMS; SERVOMECHANISMS.

CONVERSION, THERMIONIC. See THERMIONIC CONVERSION.

CONVERTERS. See ANALOG-TO-DIGITAL CONVERSION.

CONVERTERS, AC-AC. See AC-AC POWER CONVERTERS.

CONVERTERS, AC-DC. See AC-DC POWER CONVERTERS.

CONVERTERS, DC-AC. See DC-AC POWER CONVERTERS.

CONVERTER TO BOOST VOLTAGE. See SYNCHRONOUS CONVERTER TO BOOST BATTERY VOLTAGE.