DISCRETE TIME SYSTEMS DESIGN METHODS

This article discusses fundamental concepts in discrete-time control system design. The rapid advancements in digital system technology have radically altered the boundaries of control system design options. Currently, it is routinely practicable to design very complicated digital controllers and to carry out the extensive calculations required for their design. These advances in implementation and design capability can be obtained at low cost because of the widespread availability of inexpensive and powerful digital computers and their related devices.

The focus of discussion is the modern, state space-based design of linear discrete-time control systems with an appreciation for classical viewpoints and methods. To begin, we present an overview of the classical approach to discrete-time tracking system design. The key concepts, involving specification of transient and steady state response requirements, are also much a part of the modern approach.

Two important classes of control systems are the *tracking system* and the *regulator*. A tracking system is one in which the plant outputs are controlled so that they become and remain nearly equal to a set of externally applied reference signals. In a regulator, the objective is to bring the system-tracking outputs near zero in an acceptable manner, often in the face of disturbances. Thus regulation is a special case of tracking, in which the externally applied reference signals are zero.

In the classical approach, whether in the discrete-time domain or the continuous-time domain, the designer begins with a lower-order controller and raises the controller order as necessary to meet the feedback system performance requirements. The digital controller parameters are chosen to give feedback system pole locations that result in acceptable zero-input (transient) response. At the same time, requirements are placed on the overall system's zero-state response components for representative discrete-time reference inputs, such as steps or ramps.

In general, tracking control system design has two basic concerns:

1. Obtaining acceptable zero-input response, that due to initial conditions
2. Obtaining acceptable zero-state system response to reference inputs

In addition, if the plant to be controlled is continuous-time and the controller is discrete-time, a third concern is:

3. Obtaining acceptable between-sample response of the continuous-time plant

Using the superposition theorem, the zero-input response, and the individual zero-state response contributions of each input can be dealt with separately. The first concern of tracking system design is met by selecting a controller that places all of the overall system poles at desired locations inside the unit circle on the complex plane. Having designed a feedback structure to achieve the desired character of zero-input response, additional design freedom can then be used to obtain good tracking of reference inputs.

The first two concerns of discrete-time tracking system design are the subject of this chapter. The third concern, however, is beyond our scope but is covered thoroughly in Ref. 1.

CLASSICAL CONTROL SYSTEM DESIGN METHODS

The tools of classical linear discrete-time control system design, which parallel the tools for continuous-time systems, are

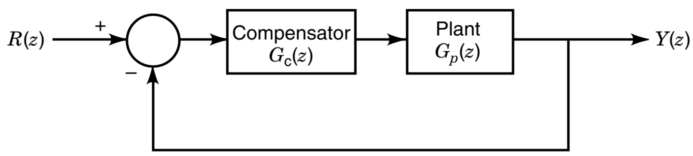

Figure 1. Cascade compensation of a unity feedback system.

the z-transform, stability testing, root locus, and frequency response methods.

The standard classical control system design problem is to determine the transfer function $G_c(z)$ of a compensator that results in a feedback tracking system with prescribed performance requirements. The basic system configuration for this problem is shown in Fig. 1. There are many variations on this basic theme, including situations where the system structure is more involved, where there is a feedback transmittance $H(z)$, and where there are disturbance inputs to be considered. Usually, these disturbances are undesirable inputs that the plant should not track.

The character of a system's zero-input response is determined by its pole locations, so the first concern of tracking system design is met by choosing a compensator $G_c(z)$ that results in acceptable pole locations for the overall transfer function:

$$T(z) = \frac{Y(z)}{R(z)} = \frac{G_c(z)G_p(z)}{1 + G_c(z)G_p(z)}$$

Root locus is an important design tool because, with it, the effects on closed-loop system pole location of varying design parameters are quickly and easily visualized.

The second concern of tracking system design is obtaining acceptable closed-loop zero-state response to reference inputs. For the discrete-time system shown in Fig. 2, the open-loop transfer function may be expressed as

$$KG(z)H(z) = \frac{K(z+\alpha_1)(z+\alpha_2)\ldots(z+\alpha_l)}{(z-1)^n(z+\beta_1)(z+\beta_2)\ldots(z+\beta_m)} = \frac{KN(z)}{(z-1)^n D(z)} \tag{1}$$

If n is nonnegative, the system is said to be *type n*.

The error between the input and the output of the system is

$$E(z) = R(z) - Y(z)H(z)$$

but

$$Y(z) = KE(z)G(z)$$

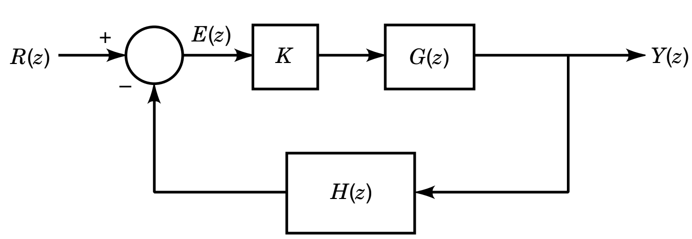

Figure 2. A discrete-time control system.

then

$$T_E(z) = \frac{E(z)}{R(z)} = \frac{1}{1 + KG(z)H(z)}$$

Assuming that all the closed-loop poles of the system are inside the unit circle on the z plane, the steady-state error to a power-of-time input is given by the final value theorem:

$$\operatorname*{limit}_{k\to\infty} e(k) = \operatorname*{limit}_{z\to1}(1 - z^{-1})E(z) = \operatorname*{limit}_{z\to1} \frac{(1-z^{-1})R(z)}{1 + KG(z)H(z)} \tag{2}$$

Similar to continuous-time systems, there are three reference inputs for which steady-state errors are commonly defined. They are the step (position), ramp (velocity), and parabolic (acceleration) inputs. The step input has the form

$$r(kT) = A\,u(kT)$$

or, in the z domain,

$$R(z) = \frac{Az}{z-1}$$

The ramp input is given by

$$r(kT) = AkTu(kT)$$

or

$$R(z) = \frac{ATz}{(z-1)^2}$$

and the parabolic input is given by

$$r(kT) = \frac{1}{2}A(kT)^2u(kT)$$

or

$$R(z) = \frac{T^2}{2}\frac{Az(z+1)}{(z-1)^3}$$

Table 1 summarizes steady-state errors using Eqs. (1) and (2) for various system types for power-of-time inputs.

There are two basic ways of approaching classical discrete-time control. In the sampled data approach, discrete-time signals are represented by continuous-time impulse trains so that all signals in a plant and controller model are continuous-time signals. This was appealing in the early days of digital control when digital concepts were new and most designers had backgrounds that were solidly in continuous-time control. Currently, however, there is little to recommend this complexity. In the conventional approach, which is used here, discrete-time signals are represented as sequences of numbers.

Root Locus Design Methods

We now present an overview of classical discrete-time control system design using an example. Similar to continuous-time systems, a root locus plot consists of a pole–zero plot of the open-loop transfer function of a feedback system, upon which

Table 1. Steady-State Errors to Power-of-Time Inputs

System Type	Steady-State Error to Step Input $R(z) = \frac{Az}{z-1}$	Steady-State Error to Ramp Input $R(z) = \frac{ATz}{(z-1)^2}$	Steady-State Error to Parabolic Input $R(z) = \frac{ATz}{(z-1)^2}$
0	$\dfrac{A}{1+K\frac{N(1)}{D(1)}}$	∞	∞
1	0	$\dfrac{AT}{K\frac{N(1)}{D(1)}}$	∞
2	0	0	$\dfrac{AT^2}{K\frac{N(1)}{D(1)}}$
.	.	.	.
.	.	.	.
.	.	.	.
n	0	0	0

is superimposed the locus of the poles of the closed-loop transfer function as some parameter is varied. For the configuration shown in Fig. 2 where the constant gain K is the parameter of interest, the overall transfer function of this system is

$$T(z) = \frac{KG(z)}{1+KG(z)H(z)}$$

and the poles of the overall system are the roots of

$$1+KG(z)H(z) = 0$$

which depend on the parameter K. The rules for constructing the root locus of discrete-time systems are identical to the rules for plotting the root locus of continuous-time systems. The root locus plot, however, must be interpreted relative to the z plane.

Consider the block diagram of the commercial broadcast videotape-positioning system shown in Fig. 3(a). The transfer function $G(s)$ relates the applied voltage to the drive motor armature and the tape speed at the recording and playback heads. The delay term accounts for the propagation of speed changes along the tape over the distance of physical separation of the tape drive mechanism and the recording and playback heads. The pole term in $G(s)$ represents the dynamics of the motor and tape drive capstan. Tape position is sensed by a recorded signal on the tape itself.

It is desired to design a digital controller that results in zero steady-state error to any step change in desired tape position. Also, the system should have a zero-input (or transient) response that decays to no more than 10% of any initial value within a 1/30 s interval, which is the video frame rate. The sampling period of the controller is chosen to be T = 1/120 s in order to synchronize the tape motion control with the 1/60 s field rate (each frame consists of two fields of the recorded video). As shown in Fig. 3(b), the diagram of Fig. 3(a) has been rearranged to emphasize the discrete-time input

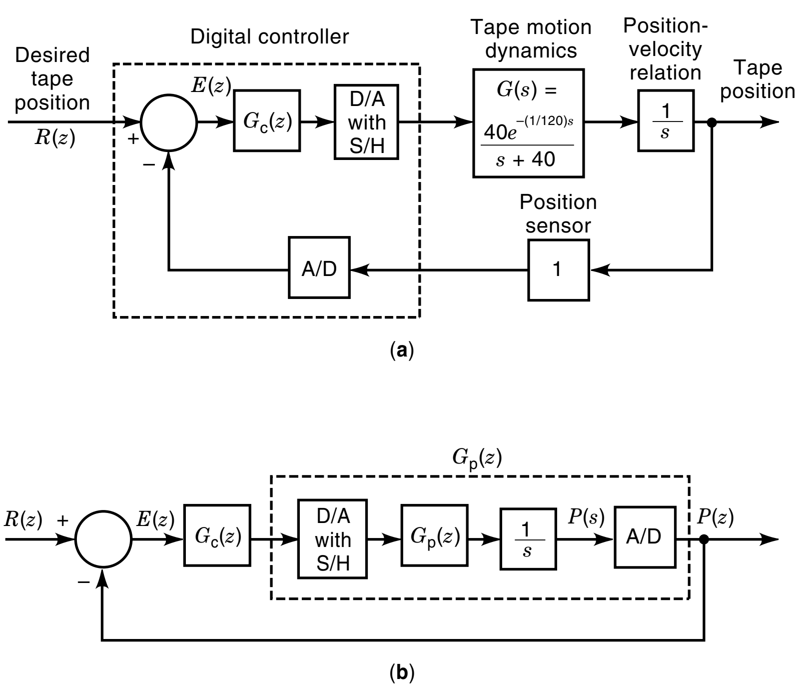

Figure 3. Videotape-positioning system. (a) Block diagram. (b) Relation between discrete-time signals.

$R(z)$ and the discrete-time samples $P(z)$ of the tape position. The open-loop transfer function of the system is

$$
\begin{aligned}
G(z) &= Z\left[\frac{1-e^{-(1/120)s}}{s}\,\frac{40e^{-(1/120)s}}{s+40}\,\frac{1}{s}\right]\\
&= Z\left\{[1-e^{-(1/120)s}]e^{-(1/120)s}\left[\frac{-1/40}{s}+\frac{1}{s^2}+\frac{1/40}{s+40}\right]\right\}\\
&= (1-z^{-1})z^{-1}\left[\frac{-z/40}{z-1}+\frac{z/120}{(z-1)^2}+\frac{z/40}{z-0.72}\right]\\
&= \frac{0.00133(z+0.75)}{z(z-1)(z-0.72)}
\end{aligned}
$$

The position error signal, in terms of the compensator's z-transfer function $G_c(z)$, is given by

$$
\begin{aligned}
E(z) = R(z) - Y(z) &= \left[1-\frac{G_c(z)G_p(z)}{1+G_c(z)G_p(z)}\right]R(z)\\
&= \frac{1}{1+G_c(z)G_p(z)}R(z)
\end{aligned}
$$

For a unit step input sequence we have

$$E(z) = \frac{1}{1+G_c(z)G_p(z)}\left(\frac{z}{z+1}\right)$$

Assuming that the feedback system is stable, we obtain

$$\operatorname*{limit}_{k\to\infty} e(k) = \operatorname*{limit}_{z\to 1}(1-z^{-1})E(z) = \operatorname*{limit}_{z\to 1}\frac{1}{1+G_c(z)G_p(z)}$$

Provided that the compensator does not have a zero at $z = 1$, the system type is 1 and, therefore according to Table 1, the steady-state error to a step input is zero. For the feedback system transient response to decay at least by a factor 1/10 within 1/30 s, the desired closed loop poles must be located such that a decay of at least this amount occurs every 1/120 s steps. This implies that the closed-loop poles must lie within a radius c of the origin on the z plane, where

$$c^4 = 0.1, \qquad c = 0.56$$

Similar to continuous-time systems, one usually begins with the simplest compensator consisting of only a gain K. The feedback system is stable for $0 < K < 95$; but as shown in Fig. 4, this compensator is inadequate because there are always poles at distances from the origin greater than the required $c = 0.56$ regardless of the value of K. As shown in Fig. 5(a), another compensator with z-transfer function

$$G_c(z) = \frac{K(z-0.72)}{z}$$

which cancels the plant pole at $z = 0.72$ is tried. The root locus plot for this system is shown in Fig. 5(b). For $K = 90$, the design is close to meeting the requirements, but it is not quite good enough. However, if the compensator pole is moved from the origin to the left as shown in Fig. 6, the root locus is pulled to the left and the performance requirements are met.

For the compensator with z-transfer function

$$G_c(z) = \frac{150(z-0.72)}{z+0.4} \tag{3}$$

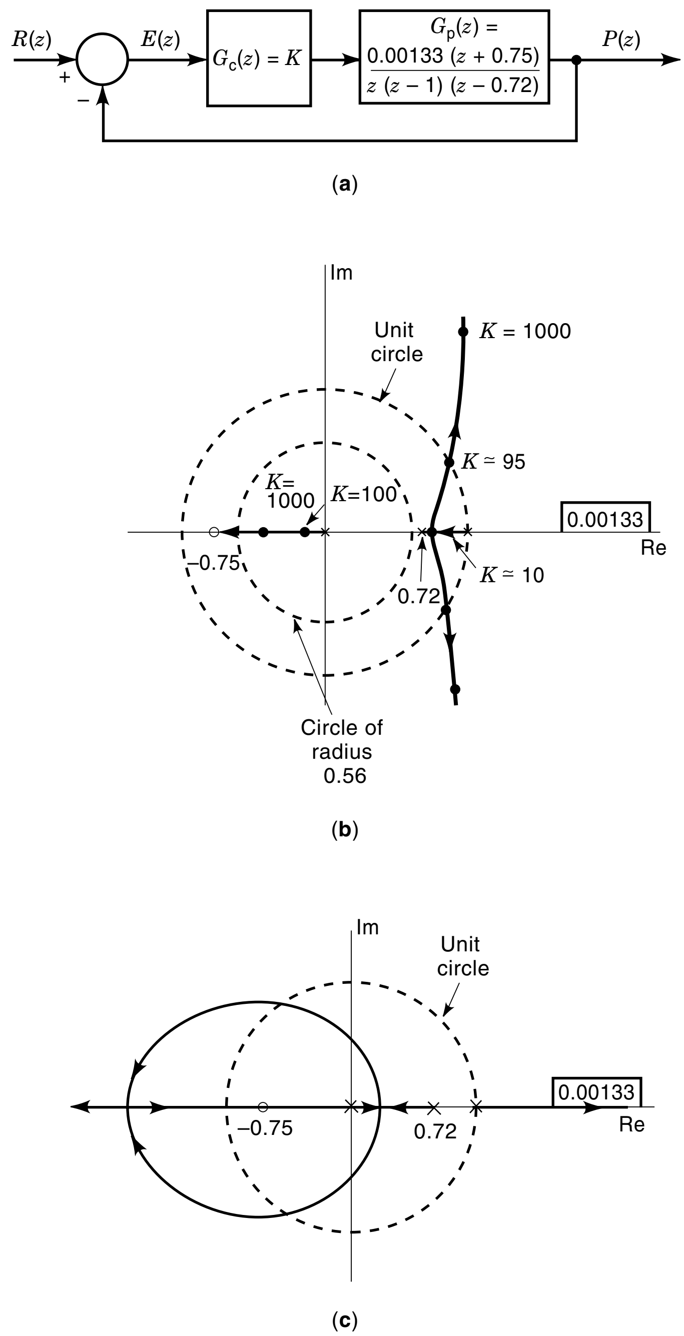

Figure 4. Constant-gain compensator. (a) Block diagram. (b) Root locus for positive K. (c) Root locus for negative gain.

the feedback system z-transfer function is

$$T(z) = \frac{G_c(z)G_p(z)}{1+G_c(z)G_p(z)} = \frac{0.2(z+0.75)}{z^3-0.6z^2-0.2z+0.15}$$

As expected, the steady-state error to a step input is zero:

$$\operatorname*{limit}_{z\to 1}\frac{z^3-0.6z^2-0.4z}{z^3-0.6z^2-0.2z+0.15} = 0$$

The steady-state error to a unit ramp input is

$$\operatorname*{limit}_{z\to 1}\frac{\frac{1}{120}(z^2+0.4z)}{z^3-0.6z^2-0.2z+0.15} = \frac{1}{30}$$

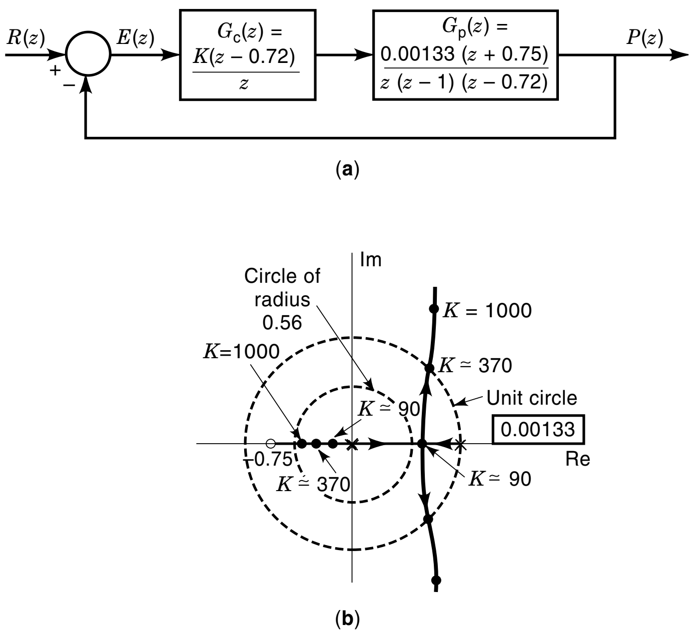

Figure 5. Compensator with zero at $z = 0.72$ and pole at $z = 0$. (a) Block diagram. (b) Root locus for positive K.

For a compensator with a z-transfer function of the form we obtain

$$G_c(z) = \frac{150(z - 0.72)}{z + a}$$

The feedback system has the z-transfer function:

$$T(z) = \frac{G_c(z)G_p(z)}{1 + G_c(z)G_p(z)} = \frac{0.2(z + 0.75)}{z^3 - z^2 + 0.2z + 0.15 + a(z^2 - z)}$$

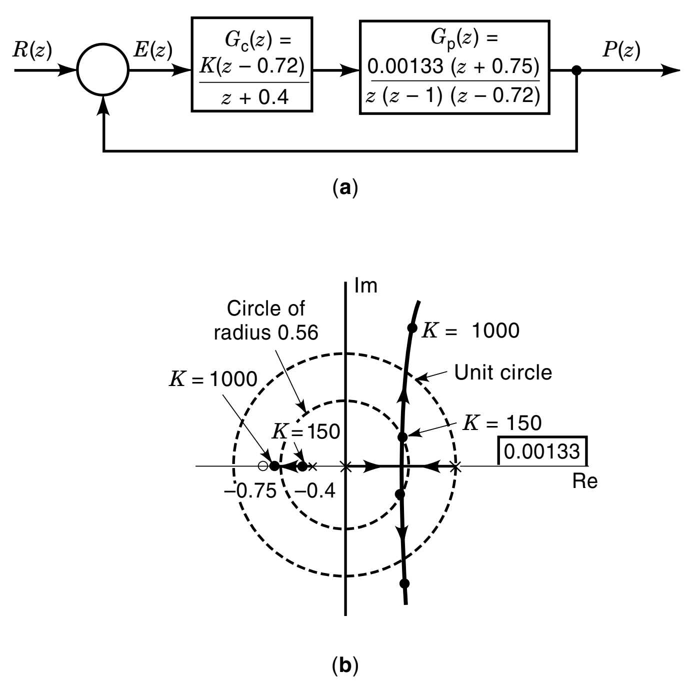

Figure 6. Compensator with zero at $z = 0.72$ and pole at $z = -0.4$. (a) Block diagram. (b) Root locus for positive K.

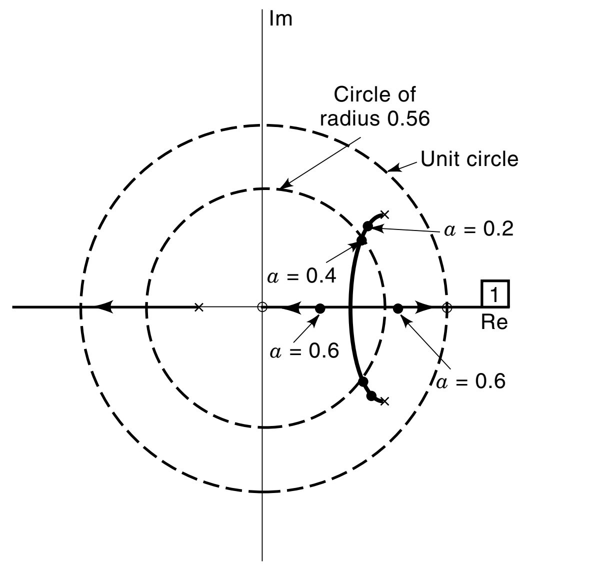

Figure 7. Root locus plot as a function of the compensator pole location.

A root locus plot in terms of positive a is shown in Fig. 7, from which it is seen that choices of a between 0.4 and 0.5 give a controller that meets the performance requirements.

Classical discrete-time control system design is an iterative process just like its continuous-time counterpart. Increasingly complicated controllers are tried until both the steady-state error and transient performance requirements are met. Root locus is an important tool because it easily indicates qualitative closed-loop system pole locations as a function of a parameter. Once feasible controllers are selected, root locus plots are refined to show quantitative results.

Frequency Domain Methods

Frequency response characterizations of systems have long been popular because of the ease and practicality of steady-state sinusoidal response methods. Furthermore, frequency response methods do not require explicit knowledge of system transfer function models.

For the videotape-positioning system, the open loop z-transfer function, which includes the compensator given by Eq. (3), is

$$G_c(z)G_p(z) = \frac{(150)(0.00133)(z + 0.75)}{z(z + 0.4)(z - 1)}$$

Substituting $z = e^{j\omega T}$, we obtain

$$G_c(e^{j\omega T})G_p(e^{j\omega T}) = \frac{0.1995(e^{j\omega T} + 0.75)}{e^{j\omega T}(e^{j\omega T} + 0.4)(e^{j\omega T} - 1)} \tag{4}$$

which has the frequency response plots shown in Fig. 8. At the *phase crossover frequency* (114.2 rad/s) the *gain margin* is about 11.5 dB, and at the *gain crossover frequency* (30 rad/s) the *phase margin* is about 66.5°s.

For ease of generating frequency response plots and to gain greater insight into the design process, the frequency domain methods such as Nyquist, Bode, Nichols, and so on, for discrete-time systems can be developed using the *w transform.* This is because in the w plane the wealth of tools and

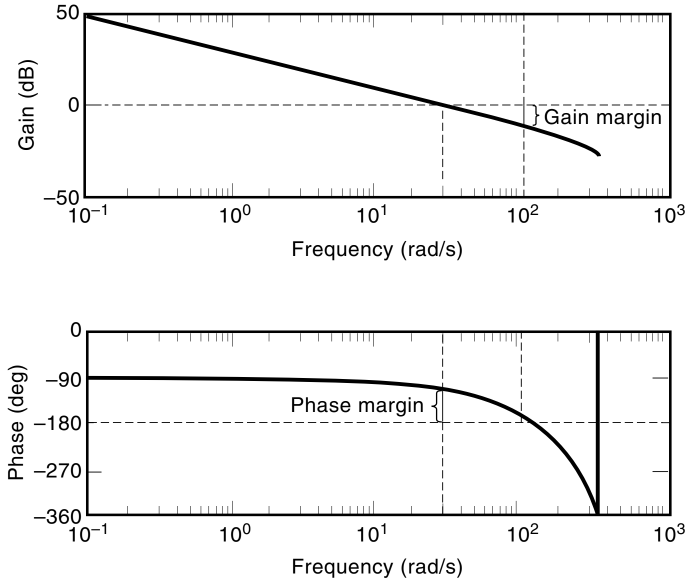

Figure 8. Frequency response plots of the videotape-positioning system.

techniques that were developed for continuous-time systems are directly applicable to discrete-time systems as well. The w transform is given by

$$w = \frac{z-1}{z+1}, \qquad z = \frac{w+1}{1-w}$$

which is a bilinear transformation between the w plane and the z plane.

The general procedure for analyzing and designing discrete-time systems using the w transform is summarized as follows:

1. Replace each z in the open-loop transfer function $G(z)H(z)$ with
$$z = \frac{w+1}{1-w}$$
to obtain $G(w)H(w)$.
2. Substitute $w = j\nu$ into $G(w)H(w)$ and generate frequency response plots in terms of the real frequency ν, such as Nyquist, Bode, Nichols, and so on. The w plane can be thought of as if it were the s plane.
3. Determine the stability margins, crossover frequencies, bandwidth, closed-loop frequency response, or any other desired frequency response characteristics.
4. If it is necessary, design a compensator $G_c(w)$ to satisfy the frequency domain performance requirements.
5. Convert critical frequencies ν in the w plane to frequencies ω in the z domain according to
$$\omega = \frac{2}{T}\tan^{-1}\nu$$
6. Finally, transform the controller $G_c(w)$ to $G_c(z)$ according to the mapping
$$w = \frac{z-1}{z+1}$$

Control system design for discrete-time systems using Bode, Nyquist, or Nichols methods can be found in Refs. 2 and 3. Frequency response methods are most useful in developing models from experimental data, in verifying the performance of a system designed by other methods, and in dealing with those systems and situations in which rational transfer function models are not adequate.

The extension of the classical single-input/single-output control system design methods to the design of complicated feedback structures involving many loops, each of which might include a compensator, is not easy. Put another way, modern control systems require the design of compensators having multiple inputs and multiple outputs. Design *is* iterative, and it can involve considerable trial and error. Therefore when there are many design variables, it is important to deal efficiently with those design decisions that need not be iterative. The powerful methods of state space offer insights about what is possible and what is not. They also provide an excellent framework for general methods of approaching and accomplishing design objectives.

EIGENVALUE PLACEMENT WITH STATE FEEDBACK

Consider a linear, step-invariant nth-order system described by the state equations

$$\boldsymbol{x}(k+1) = \boldsymbol{A}\boldsymbol{x}(k) + \boldsymbol{B}\boldsymbol{u}(k)$$

where $\boldsymbol{x}(k)$ is an n vector, and $\boldsymbol{u}(k)$ is an r vector. When the state $\boldsymbol{x}(k)$ of this system is available and is used for feedback, the input vector $\boldsymbol{u}(k)$ is given the form

$$\boldsymbol{u}(k) = \boldsymbol{E}\boldsymbol{x}(k) + \boldsymbol{\rho}(k)$$

where $\boldsymbol{\rho}(k)$ is a vector of external inputs as shown in Fig. 9, and $\boldsymbol{E}$ is a gain matrix. The state equation for the plant with feedback becomes

$$\boldsymbol{x}(k+1) = (\boldsymbol{A} + \boldsymbol{B}\boldsymbol{E})\boldsymbol{x}(k) + \boldsymbol{B}\boldsymbol{\rho}(k)$$

If the plant is completely controllable, the eigenvalues of the feedback system, those of $\boldsymbol{A} + \boldsymbol{B}\boldsymbol{E}$, can be placed at any desired locations selected by the designer by appropriately choosing the feedback gain matrix $\boldsymbol{E}$. This is to say that the

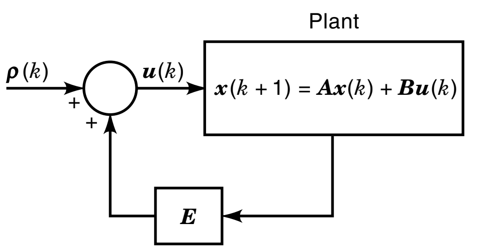

Figure 9. State feedback.

designer can freely choose the character of the overall system's transient performance. When the plant state vector is not available for feedback, as is usually the case, an observer can be designed to estimate the state vector. As we shall see later, the observer state estimate can be used for feedback in place of the state itself.

Eigenvalue Placement for Single-Input Systems

If a single-input plant is in controllable canonical form, finding the feedback gains for arbitrary eigenvalue placement is especially simple because, in that form, each element of the feedback gain vector determines one coefficient of the feedback system's characteristic equation. In general, however, the plant is not in controllable form. One way for calculating the state feedback gain for eigenvalue placement for plants that are not in controllable canonical form is to transform the plant to controllable form, calculate the state feedback gain for the transformed system, and then transform the state feedback gain of the transformed system back to the original system (see Ref. 1).

There are a number of other methods for finding the state feedback gain vector of single-input plants. Two of these methods are summarized below. Additional ones can be found in Refs. 1 and 4–8. The state feedback gain vector is given by *Ackermann's formula:*

$$\boldsymbol{e}^{\dagger} = -\boldsymbol{j}_{\mathbf{n}}^{\dagger}\boldsymbol{M}_{\mathrm{c}}^{-1}\Delta_{\mathrm{c}}(\boldsymbol{A}) \tag{5}$$

where $\boldsymbol{j}_n^{\dagger}$ is the transpose of the nth-unit coordinate vector

$$\boldsymbol{j}_n^{\dagger} = [0 \quad 0 \quad \ldots \quad 0 \quad 1]$$

$\boldsymbol{M}_c$ is the controllability matrix of the system, and $\Delta_c(\boldsymbol{A})$ is the desired characteristic equation with the matrix $\boldsymbol{A}$ substituted for the variable z.

For example, for the completely controllable system

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 3 & 0 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} + \begin{bmatrix} 1 \\ 2 \end{bmatrix} u(k)$$

it is desired to place the feedback system eigenvalues at $z = 0, -0.5$. Then,

$$\Delta_{\mathrm{c}}(z) = z(z+0.5) = z^2 + 0.5z$$

and

$$\Delta_{\mathrm{c}}(\boldsymbol{A}) = \boldsymbol{A}^2 + 0.5\boldsymbol{A}$$

Using Ackermann's formula, the state feedback gain vector is

$$\begin{aligned} \boldsymbol{e}^{\dagger} &= -[0 \quad 1] \begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix}^{-1} \left\{ \begin{bmatrix} -2 & -1 \\ 3 & -3 \end{bmatrix} + 0.5 \begin{bmatrix} 1 & -1 \\ 3 & 0 \end{bmatrix} \right\} \\ &= [-1.5 \quad 0] \end{aligned}$$

Yet another method for calculating $\boldsymbol{E}$ is as follows. If λ_i is an eigenvalue of $(\boldsymbol{A} + \boldsymbol{BE})$, then there exists an eigenvector $\boldsymbol{v}_i$ such that

$$(\boldsymbol{A} + \boldsymbol{BE})\boldsymbol{v}_i = \lambda_i \boldsymbol{v}_i$$

Letting

$$\boldsymbol{E}\boldsymbol{v}_i = \boldsymbol{\delta}_i$$

we obtain

$$\boldsymbol{v}_i = (\lambda_i \boldsymbol{I} - \boldsymbol{A})^{-1}\boldsymbol{B}\boldsymbol{\delta}_i$$

If λ_i is not an eigenvalue of $\boldsymbol{A}$, the inverse matrix exists. If the eigenvalues λ_i are distinct, the eigenvectors are linearly independent. Choosing the $\boldsymbol{\delta}_i$ to give n linearly independent eigenvectors, we obtain

$$\boldsymbol{E}[\boldsymbol{v}_1\boldsymbol{v}_2 \ldots \boldsymbol{v}_n] = \boldsymbol{EV} = \Delta = [\boldsymbol{\delta}_1\boldsymbol{\delta}_2 \cdots \boldsymbol{\delta}_n]$$

and the desired feedback gain matrix is

$$\boldsymbol{E} = \Delta \boldsymbol{V}^{-1} \tag{6}$$

For the previous example, choosing $\boldsymbol{\delta}_1 = 1$ and $\boldsymbol{\delta}_2 = 1$, we obtain

$$\boldsymbol{e}^{\dagger} = [-1.5 \quad 0]$$

which is the same result obtained using Ackermann's formula.

The results of the above development can be extended to situations where $\boldsymbol{A} + \boldsymbol{BE}$ is required to have repeated eigenvalues as discussed in Ref. 1.

Eigenvalue Placement with Multiple Inputs

If the plant for eigenvalue placement has multiple inputs and if it is completely controllable from one of the inputs, then that one input alone can be used for feedback. If the plant is not completely controllable from a single input, a single input can usually be distributed to the multiple ones in such a way that the plant is completely controllable from the single input. For example, for the system

$$\begin{aligned} \begin{bmatrix} x_1(k+1) \\ x_2(k+1) \\ x_3(k+1) \end{bmatrix} &= \begin{bmatrix} -0.5 & 0 & 1 \\ 0 & 0.5 & 2 \\ 1 & -1 & 0 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \\ x_3(k) \end{bmatrix} \\ &\quad + \begin{bmatrix} 1 & 0 \\ 0 & -2 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} u_1(k) \\ u_2(k) \end{bmatrix} \end{aligned}$$

we let

$$u_1(k) = 3\mu(k)$$

and

$$u_2(k) = \mu(k)$$

Thus

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \\ x_3(k+1) \end{bmatrix} = \begin{bmatrix} -0.5 & 0 & 1 \\ 0 & 0.5 & 2 \\ 1 & -1 & 0 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \\ x_3(k) \end{bmatrix} + \begin{bmatrix} 3 \\ -2 \\ 2 \end{bmatrix} \mu(k)$$

which is a controllable single input system. If the desired eigenvalues are located at $z_1 = -0.1$, $z_2 = -0.15$, and $z_3 = 0.1$, Ackermann's formula gives

$$e^{\dagger} = [0.152 \quad 0.0223 \quad 0.2807]$$

and hence the feedback gain matrix for the multiple input system is

$$\boldsymbol{E} = \begin{bmatrix} 0.4559 & 0.0669 & 0.8420 \\ 0.1520 & 0.0223 & 0.2807 \end{bmatrix}$$

Equation (6) can also be applied to multiple input systems. Continuing with the example, if for each eigenvector, we choose $\delta_1 = 3$ and $\delta_2 = 1$, then

$$\boldsymbol{v}_1 = \begin{bmatrix} 6.5871 \\ 4.5506 \\ -0.3652 \end{bmatrix}, \quad \boldsymbol{v}_2 = \begin{bmatrix} 6.9304 \\ 4.8442 \\ -0.5744 \end{bmatrix}, \quad \boldsymbol{v}_3 = \begin{bmatrix} 5.5076 \\ 3.4772 \\ 0.3046 \end{bmatrix}$$

and therefore

$$\boldsymbol{E} = \begin{bmatrix} 3 & 3 & 3 \\ 1 & 1 & 1 \end{bmatrix} \boldsymbol{V}^{-1} = \begin{bmatrix} 0.4559 & 0.0669 & 0.8420 \\ 0.1520 & 0.0223 & 0.2807 \end{bmatrix}$$

which agrees with the previous results.

Eigenvalue Placement with Output Feedback

It is the measurement vector of a plant, not the state vector, that is available for feedback. We now consider what eigenvalue placement can be performed with output feedback alone. With enough linearly independent outputs, the plant state can be recovered from the outputs and inputs and the state feedback results applied. With a single plant input and only a few outputs, the designer's options for placing feedback system eigenvalues could be (and often are) severely limited. Multiple plant inputs can also be used to advantage for eigenvalue placement with output feedback, but it still may not be possible to achieve an acceptable design. If the nth-order plant with state and output equations

$$\boldsymbol{x}(k+1) = \boldsymbol{Ax}(k) + \boldsymbol{Bu}(k)$$
$$\boldsymbol{y}(k) = \boldsymbol{Cx}(k) + \boldsymbol{Du}(k)$$

has n linearly independent outputs, that is, if the output coupling matrix $\boldsymbol{C}$ has n linearly independent rows, then the plant state can be recovered from the plant inputs and the measurement outputs:

$$\boldsymbol{x}(k) = \boldsymbol{C}^{-1}\{\boldsymbol{y}(k) - \boldsymbol{Du}(k)\}$$

The output feedback

$$\boldsymbol{u}(k) = \boldsymbol{Ex}(k) + \boldsymbol{\rho}(k) = \boldsymbol{EC}^{-1}\{\boldsymbol{y}(k) - \boldsymbol{Du}(k)\} + \boldsymbol{\rho}(k)$$

or

$$\boldsymbol{u}(k) = (\boldsymbol{I} + \boldsymbol{EC}^{-1}\boldsymbol{D})^{-1}\boldsymbol{EC}^{-1}\boldsymbol{y}(k) + (\boldsymbol{I} + \boldsymbol{EC}^{-1}\boldsymbol{D})^{-1}\boldsymbol{\rho}(k)$$

will place the feedback system eigenvalues arbitrarily, provided that the matrix $\boldsymbol{I} + \boldsymbol{EC}^{-1}\boldsymbol{D}$ is nonsingular. If it is singular, a small change in the feedback gain matrix $\boldsymbol{E}$ which corresponds to small changes in the desired feedback system eigenvalue locations eliminates the singularity. If the nth-order system has more than n outputs, only n of these can be linearly independent, so excess linearly dependent output equations can simply be ignored when recovering a system's state from its output. To improve a feedback system's reliability and its performance in the presence of noise, one may wish instead to combine linearly dependent outputs with other outputs rather than ignore them.

When the nth-order plant does not have n linearly independent measurement outputs, it still might be possible to select a feedback matrix $\boldsymbol{E}$ in

$$\boldsymbol{u}(k) = \boldsymbol{E}\{\boldsymbol{y}(k) - \boldsymbol{Du}(k)\} + \boldsymbol{\rho}(k) = \boldsymbol{ECx}(k) + \boldsymbol{\rho}(k)$$

to place all of the feedback system eigenvalues, those of $(\boldsymbol{A} + \boldsymbol{BEC})$, acceptably. Generally, however, output feedback alone does not allow arbitrary feedback system eigenvalue placement.

Pole Placement with Feedback Compensation

We now present another viewpoint for placing the feedback system poles using a transfer function approach. Although our discussion is limited to single-input and single-output plants, the results can be generalized to the case of plants with multiple inputs and multiple outputs. Similar to output feedback, pole placement with feedback compensation assumes that the measurement outputs of a plant, not the state vector, are available for feedback.

For an nth-order, linear, step-invariant, discrete-time system described by the transfer function $G(z)$, arbitrary pole placement of the feedback system can be accomplished with an mth-order feedback compensator as shown in Fig. 10.

Let the numerator and denominator polynomials on $G(z)$ be $N_p(z)$ and $D_p(z)$, respectively. Also, let the numerator and denominator of the compensator transfer function $H(z)$ be $N_c(z)$ and $D_c(z)$, respectively. Then, the overall transfer function of the system is

$$T(z) = \frac{G(z)}{1 + G(z)H(z)} = \frac{N_p(z)D_c(z)}{D_p(z)D_c(z) + N_p(z)N_c(z)} = \frac{P(z)}{Q(z)}$$

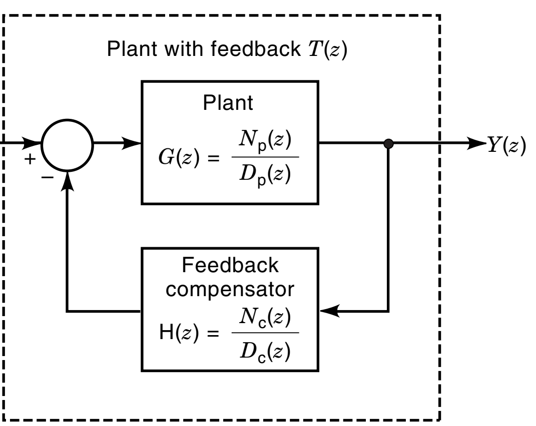

Figure 10. Pole placement with feedback compensation.

which has closed-loop zeros in $P(z)$ that are those of the plant, in $N_p(z)$, together with zeros that are the poles of the feedback compensator, in $D_c(z)$.

For a desired set of poles of $T(z)$, given with an unknown multiplicative constant by the polynomial $Q(z)$, we obtain

$$D_p(z)D_c(z) + N_p(z)N_c(z) = Q(z) \tag{7}$$

The desired polynomial $Q(z)$ has the form

$$Q(z) = \alpha_0(z^{n+m} + \beta_{n+m-1}z^{n+m-1} + \cdots + \beta_1 z + \beta_0)$$

where the β's are known coefficients but the α_0 is unknown. In general, for a solution to exist there must be at least as many unknowns as equations:

$$n + m + 1 \le 2m + 2$$

or

$$m \ge n - 1 \tag{8}$$

where n is the order of the plant and m is the order of the compensator. Equation (8) states that the order of the feedback controller is at least one less than the order of the plant. If the plant transfer function has *coprime* numerator and denominator polynomials (that is, plant pole–zero cancellations have been made), then a solution is guaranteed to exist.

For example, consider the second-order plant

$$G(z) = \frac{(z+1)(z+0.5)}{z(z-1)} = \frac{N_p(z)}{D_p(z)} \tag{9}$$

According to Eq. (8), a first-order feedback compensator of the form

$$H(z) = \frac{\alpha_1 z + \alpha_2}{z + \alpha_3} = \frac{N_c(z)}{D_c(z)}$$

places the three closed-loop poles of the feedback system at any desired location in the z plane by appropriate choice of α_1, α_2, and α_3. Let the desired poles of the plant with feedback be at $z = 0.1$. Then,

$$Q(z) = \alpha_0(z - 0.1)^3 = \alpha_0(z^3 - 0.3z^2 + 0.03z - 0.001) \tag{10}$$

In terms of the compensator coefficients, the characteristic equation of the feedback system is

$$\begin{aligned} D_p(z)D_c(z) + N_p(z)N_c(z) &= z(z-1)(z+\alpha_3) + (z+1)(z+0.5)(\alpha_1 z + \alpha_2) \\ &= (\alpha_1 + 1)z^3 + (1.5\alpha_1 + \alpha_2 + \alpha_3 - 1)z^2 \\ &\quad + (0.5\alpha_1 + 1.5\alpha_2 - \alpha_3)z + 0.5\alpha_2 \end{aligned} \tag{11}$$

Equating coefficients in Eqs. (10) and (11) and solving for the unknowns gives

$$\alpha_0 = 1.325, \quad \alpha_1 = 0.325, \quad \alpha_2 = -0.00265, \quad \alpha_3 = 0.1185$$

Therefore, the compensator

$$H(z) = \frac{0.325z - 0.00265}{z + 0.1185}$$

will place the closed-loop poles where desired.

As far as feedback system pole placement is concerned, a feedback compensator of order $n - 1$ (where n is the order of the plant) can always be designed. It is possible, however, that a lower-order feedback controller may give acceptable feedback pole locations even though those locations are constrained and not completely arbitrary. This is the thrust of classical control system design, in which the increasingly higher-order controllers are tested until satisfactory results are obtained.

For the plant given by Eq. (9), for example, a zeroth-order feedback controller of the form

$$H(z) = K$$

gives overall closed-loop poles at $z = 0.1428$ and $z = 0.5$ for $K = 1/6$, which might be an adequate pole placement design.

QUADRATIC OPTIMAL CONTROL

We have shown in the previous section that provided the plant is completely controllable, a feedback gain matrix $\boldsymbol{E}$ can always be determined so that all of the eigenvalues of the feedback system can be placed arbitrarily. It can be easily shown that for a single-input plant, the feedback gain vector is unique. For multiple-input plants, however, there are many feedback gain matrices that lead to the same set of feedback eigenvalues. The process of selecting an *optimum* feedback gain matrix from among the many possible gain matrices is the subject of this section.

The approach of selecting the optimal gain matrix is termed *optimal regulation,* in which the plant feedback is chosen to minimize a scalar performance measure that weights the control input and the error from zero of the plant state at each step.

Principle of Optimality

The discrete-time, linear-quadratic, optimal control problem is to find the inputs $\boldsymbol{u}(0), \boldsymbol{u}(1), \ldots, \boldsymbol{u}(N-1)$ to the plant with linear state equations

$$\boldsymbol{x}(k+1) \equiv \boldsymbol{A}(k)\boldsymbol{x}(k) + \boldsymbol{B}(k)\boldsymbol{u}(k)$$

such that a scalar quadratic performance measure (or *cost function*)

$$J = \boldsymbol{x}^\dagger(N)\boldsymbol{P}(N)\boldsymbol{x}(N) + \sum_{k=0}^{N-1}[\boldsymbol{x}^\dagger(k)\boldsymbol{Q}(k)\boldsymbol{x}(k) + \boldsymbol{u}^\dagger(k)\boldsymbol{R}(k)\boldsymbol{u}(k)]$$

is minimized. The matrix $\boldsymbol{P}(N)$, the matrices $\boldsymbol{Q}(0), \boldsymbol{Q}(1), \ldots, \boldsymbol{Q}(N-1)$, and the matrices $\boldsymbol{R}(0), \boldsymbol{R}(1), \ldots, \boldsymbol{R}(N-1)$ are each taken to be symmetric because each defines a quadratic form. Each is assumed to be positive semidefinite, which means that the contribution to J by each of the individual terms is never negative.

The solution to the linear-quadratic optimal control problem is obtained by applying the *principle of optimality,* a tech-

Table 2. Procedure for Backward-in-Time Calculation of Optimal Quadratic Regulator Gains

For the plant

$$\boldsymbol{x}(k+1) = \boldsymbol{A}(k)\boldsymbol{x}(k) + \boldsymbol{B}(k)\boldsymbol{u}(k)$$

with state feedback

$$\boldsymbol{u}(k) = \boldsymbol{E}(k)\boldsymbol{x}(k)$$

and performance measure

$$J = \boldsymbol{x}^{\dagger}(N)\boldsymbol{P}(N)\boldsymbol{x}(N) + \sum_{i=0}^{N-1} [\boldsymbol{x}^{\dagger}(i)\boldsymbol{Q}(i)\boldsymbol{x}(i) + \boldsymbol{u}^{\dagger}(i)\boldsymbol{R}(i)\boldsymbol{u}(i)]$$

begin with $i = 1$ and the known $\boldsymbol{P}(N)$

1. $\boldsymbol{E}(N-i) = -[\boldsymbol{B}^{\dagger}(N-i)\boldsymbol{P}(N+1-i)\boldsymbol{B}(N-i) + \boldsymbol{R}(N-i)]^{-1} \times \boldsymbol{B}^{\dagger}(N-i)\boldsymbol{P}(N+1-i)\boldsymbol{A}(N-i)$

2. $\boldsymbol{P}(N-i) = [\boldsymbol{A}(N-i) + \boldsymbol{B}(N-i)\boldsymbol{E}(N-i)]^{\dagger}\boldsymbol{P}(N+1-i) \times [\boldsymbol{A}(N-i) + \boldsymbol{B}(N-i)\boldsymbol{E}(N-i)] + \boldsymbol{E}^{\dagger}(N-i)\boldsymbol{R}(N-i)\boldsymbol{E}(N-i) + \boldsymbol{Q}(N-i)$

3. Increment i and repeat steps 1, 2, and 3 until $\boldsymbol{E}(0)$ and (if desired) $\boldsymbol{P}(0)$ have been calculated.

The minimum performance measure is

$$\min_{\boldsymbol{u}(0),\ldots,\boldsymbol{u}(N-1)} \left\{J\right\} = \boldsymbol{x}^{\dagger}(0)\boldsymbol{P}(0)\boldsymbol{x}(0)$$

nique developed by Richard Bellman in the 1950s in connection with his invention of *dynamic programming*. To apply the principle of optimality, one begins at the next-to-last step $N-1$ and finds the last input $\boldsymbol{u}(N-1)$ that minimizes the cost of control from step $N-1$ to step N, $J(N-1, N)$, as a function of the beginning state for that step, $\boldsymbol{x}(N-1)$. Then the input $\boldsymbol{u}(N-2)$ is found that minimizes $J(N-2, N)$ when $\boldsymbol{u}(N-1)$ is as previously determined. One proceeds in this manner finding one control vector at a time, from the last to the first, as a function of the system's state. This results in a recursive calculation of the optimal feedback gains for the linear-quadratic regulator as given in Table 2. Beginning with known N, $\boldsymbol{P}(N)$, $\boldsymbol{Q}$, and $\boldsymbol{R}$, the last feedback gain matrix $\boldsymbol{E}(N-1)$ is calculated. Using $\boldsymbol{E}(N-1)$, the matrix $\boldsymbol{P}(N-1)$ is computed. Then all of the indices are stepped backward one step and, with $\boldsymbol{P}(N-1)$, the feedback gain matrix $\boldsymbol{E}(N-2)$ is calculated. Using $\boldsymbol{E}(N-2)$, we can calculate $\boldsymbol{P}(N-2)$. The cycle is continued until $\boldsymbol{E}(0)$ is found. A formidable amount of algebraic computation is required; the user should therefore have digital computer aid for all but the lowest-order problems.

Closed-Form Solution for Optimal Gain

When the matrices $\boldsymbol{A}$, $\boldsymbol{B}$, $\boldsymbol{Q}$, and $\boldsymbol{R}$ are constants, it is possible to generate an analytical expression for the optimal gain from which the numerical value can be calculated for any point of time. A closed-form solution for the optimal gain $\boldsymbol{E}(N-1)$ is summarized in Table 3. Derivation of this procedure along with detailed numerical examples can be found in Ref. 1.

Steady-State Regulation

For a completely controllable, step-invariant plant and constant cost weighting matrices $\boldsymbol{Q}$ and $\boldsymbol{R}$, the optimum feedback gains, from $\boldsymbol{E}(N)$ backward, are not changed if the final step N is changed. This is to say that the sequence of gains starting at N and proceeding backward is always the same independent of the value of N. The procedure summarized in Table 3 for calculating the optimal regulator gain can be easily adapted to the steady-state regulator gain by replacing steps 5, 6, and 7 with the following single step 5:

5. Form

$$\boldsymbol{E} = -[\boldsymbol{B}^{\dagger}\boldsymbol{W}_{11}\boldsymbol{W}_{21}^{-1}\boldsymbol{B} + \boldsymbol{R}]^{-1}\boldsymbol{B}^{\dagger}\boldsymbol{W}_{11}\boldsymbol{W}_{21}^{-1}\boldsymbol{A}$$

which gives a steady-state gain matrix.

STEP-INVARIANT DISCRETE-TIME OBSERVER DESIGN

In 1964, David Luenberger of Stanford University put forth the idea of observers, systems that recursively estimate the state of other systems. It was soon realized that observers offer a powerful, unified framework for feedback control system design.

Full-Order Observers

When the plant state vector is not entirely available for feedback, as is usually the case, the state is *estimated* with an observer, and the estimated state can be used in place of the actual state for feedback (see Refs. 9, 10).

For an nth-order step-invariant discrete-time plant,

$$\begin{aligned} \boldsymbol{x}(k+1) &= \boldsymbol{A}\boldsymbol{x}(k) + \boldsymbol{B}\boldsymbol{u}(k) \\ \boldsymbol{y}(k) &= \boldsymbol{C}\boldsymbol{x}(k) + \boldsymbol{D}\boldsymbol{u}(k) \end{aligned} \tag{12}$$

another nth-order system, driven by the inputs and outputs of the plant, of the form

$$\boldsymbol{\xi}(k+1) = \boldsymbol{F}\boldsymbol{\xi}(k) + \boldsymbol{G}\boldsymbol{y}(k) + \boldsymbol{H}\boldsymbol{u}(k) \tag{13}$$

Table 3. Procedure for Calculation of Optimal Regulator Gains

For the nth-order plant

$\boldsymbol{x}(k+1) = \boldsymbol{A}\boldsymbol{x}(k) + \boldsymbol{B}\boldsymbol{u}(k)$

with state feedback

$\boldsymbol{u}(k) = \boldsymbol{E}(k)\boldsymbol{x}(k)$

and performance measure

$$J = \boldsymbol{x}^{\dagger}(N)\boldsymbol{P}(N)\boldsymbol{x}(N) + \sum_{i=0}^{N-1} [\boldsymbol{x}^{\dagger}(i)\boldsymbol{Q}\boldsymbol{x}(i) + \boldsymbol{u}^{\dagger}(i)\boldsymbol{R}\boldsymbol{u}(i)]$$

begin with $i = 1$ and the known $\boldsymbol{P}(N)$

1. Form the matrix

$$\boldsymbol{H} = \begin{bmatrix} \boldsymbol{A}^{\dagger} + \boldsymbol{Q}\boldsymbol{A}^{-1}\boldsymbol{B}\boldsymbol{R}^{-1}\boldsymbol{B}^{\dagger} & \boldsymbol{Q}\boldsymbol{A}^{-1} \\ \boldsymbol{A}^{-1}\boldsymbol{B}\boldsymbol{R}^{-1}\boldsymbol{B}^{\dagger} & \boldsymbol{A}^{-1} \end{bmatrix}$$

2. Find the eigenvalues and the corresponding eigenvectors of $\boldsymbol{H}$
3. Generate the matrix $\boldsymbol{W}$ from eigenvectors such that

$$\boldsymbol{W}^{-1}\boldsymbol{H}\boldsymbol{W} = \boldsymbol{D} = \begin{bmatrix} \boldsymbol{\Lambda} & 0 \\ 0 & \boldsymbol{\Lambda}^{-1} \end{bmatrix}$$

   where $\boldsymbol{\Lambda}$ is the diagonal matrix of eigenvalues outside the unit circle on the z plane
4. Partition $\boldsymbol{W}$ into four $n \times n$ submatrices as

$$\boldsymbol{W} = \begin{bmatrix} \boldsymbol{W}_{11} & \boldsymbol{W}_{12} \\ \boldsymbol{W}_{21} & \boldsymbol{W}_{22} \end{bmatrix}$$

5. Form

$$\boldsymbol{G}(i) = \boldsymbol{\Lambda}^{-i}[\boldsymbol{P}(N)\boldsymbol{W}_{22} - \boldsymbol{W}_{12}]^{-1}[\boldsymbol{W}_{11} - \boldsymbol{P}(N)\boldsymbol{W}_{21}]\boldsymbol{\Lambda}^{-i}$$

6. Form

$$\boldsymbol{P}(N-i) = [\boldsymbol{W}_{11} + \boldsymbol{W}_{12}\boldsymbol{G}(i)][W_{21} + \boldsymbol{W}_{22}\boldsymbol{G}(i)]^{-1}$$

7. Form

$$\boldsymbol{E}(N-i) = -[\boldsymbol{B}^{\dagger}\boldsymbol{P}(N+1-i)\boldsymbol{B} + \boldsymbol{R}]^{-1}\boldsymbol{B}^{\dagger}\boldsymbol{P}(N+1-i)\boldsymbol{A}$$

   where

$$\boldsymbol{P}(N+1-i) = [\boldsymbol{W}_{11} + \boldsymbol{W}_{12}\boldsymbol{G}(i-1)][\boldsymbol{W}_{21} + \boldsymbol{W}_{22}\boldsymbol{G}(i-1)]^{-1}$$

is termed a *full-order state observer* of the plant, provided that the error between the plant state and the observer state,

$$\begin{aligned} \boldsymbol{x}(k+1) - \boldsymbol{\xi}(k+1) &= \boldsymbol{A}\boldsymbol{x}(k) + \boldsymbol{B}\boldsymbol{u}(k) - \boldsymbol{F}\boldsymbol{\xi}(k) - \boldsymbol{G}\boldsymbol{y}(k) - \boldsymbol{H}\boldsymbol{u}(k) \\ &= \boldsymbol{A}\boldsymbol{x}(k) + \boldsymbol{B}\boldsymbol{u}(k) - \boldsymbol{F}\boldsymbol{\xi}(k) - \boldsymbol{G}\boldsymbol{C}\boldsymbol{x}(k) \\ &\quad - \boldsymbol{G}\boldsymbol{D}\boldsymbol{u}(k) - \boldsymbol{H}\boldsymbol{u}(k) \\ &= (\boldsymbol{A} - \boldsymbol{G}\boldsymbol{C})\boldsymbol{x}(k) - \boldsymbol{F}\boldsymbol{\xi}(k) \\ &\quad + (\boldsymbol{B} - \boldsymbol{G}\boldsymbol{D} - \boldsymbol{H})\boldsymbol{u}(k) \end{aligned}$$

is governed by an autonomous equation. When $\boldsymbol{F}$ and $\boldsymbol{G}$ are chosen as

$$\boldsymbol{F} = \boldsymbol{A} - \boldsymbol{G}\boldsymbol{C} \quad (14)$$

$$\boldsymbol{H} = \boldsymbol{B} - \boldsymbol{G}\boldsymbol{D} \quad (15)$$

so that the error satisfies

$$\boldsymbol{x}(k+1) - \boldsymbol{\xi}(k+1) = (\boldsymbol{A} - \boldsymbol{G}\boldsymbol{C})[\boldsymbol{x}(k) - \boldsymbol{\xi}(k)]$$

or

$$\boldsymbol{x}(k) - \boldsymbol{\xi}(k) = (\boldsymbol{A} - \boldsymbol{G}\boldsymbol{C})^{k}[\boldsymbol{x}(0) - \boldsymbol{\xi}(0)] = \boldsymbol{F}^{k}[\boldsymbol{x}(0) - \boldsymbol{\xi}(0)]$$

then the system in Eq. (13) is a full-order state observer of the plant in Eq. (12), if the matrix $\boldsymbol{G}$ can be chosen so that all the eigenvalues of $\boldsymbol{F} = \boldsymbol{A} - \boldsymbol{G}\boldsymbol{C}$ are inside the unit circle in the complex plane. The observer error, then, approaches zero with step regardless of the initial values of $\boldsymbol{x}(0)$ and $\boldsymbol{\xi}(0)$. That is, the observer state $\boldsymbol{\xi}(k)$ will approach the plant state $\boldsymbol{x}(k)$. The full-order observer relations are summarized in Table 4. If all n of the observer eigenvalues (eigenvalues of $\boldsymbol{F}$) are selected to be zero, then the characteristic equation of $\boldsymbol{F}$ is

$$\lambda^{n} = 0$$

and since every matrix satisfies its own characteristic equation, then

$$\boldsymbol{F}^{n} = \boldsymbol{0}$$

At the nth step, the error between the plant state and the observer state is given by

$$\boldsymbol{x}(n) - \boldsymbol{\xi}(n) = \boldsymbol{F}^{n}[\boldsymbol{x}(0) - \boldsymbol{\xi}(0)]$$

so that

$$\boldsymbol{x}(n) = \boldsymbol{\xi}(n)$$

and the observer state equals the plant state. Such an observer is termed *deadbeat.* In subsequent steps, the observer state continues to equal the plant state.

There are several methods for calculating the observer gain matrix $\boldsymbol{g}$ for single-output plants. Similar to the situation with state feedback, if a single-output plant is in observable canonical form, finding the elements of the observer gain vector $\boldsymbol{g}$ for arbitrary eigenvalue placement is simple, because each element of the observer gain vector determines one coefficient of the observer characteristic equation. Usually, however, the plant is not in observable canonical form. One way of designing an observer for a completely observable single-output plant that is not in observable form is to change the

Table 4. Full-Order State Observer Relations

Plant Model

$$\boldsymbol{x}(k+1) = \boldsymbol{A}\boldsymbol{x}(k) + \boldsymbol{B}\boldsymbol{u}(k)$$
$$\boldsymbol{y}(k) = \boldsymbol{C}\boldsymbol{x}(k) + \boldsymbol{D}\boldsymbol{u}(k)$$

Observer

$$\boldsymbol{\xi}(k+1) = \boldsymbol{F}\boldsymbol{\xi}(k) + \boldsymbol{G}\boldsymbol{y}(k) + \boldsymbol{H}\boldsymbol{u}(k)$$

where

$$\boldsymbol{F} = \boldsymbol{A} - \boldsymbol{G}\boldsymbol{C}$$
$$\boldsymbol{H} = \boldsymbol{B} - \boldsymbol{G}\boldsymbol{D}$$

Observer Error

$$\boldsymbol{x}(k+1) - \boldsymbol{\xi}(k+1) = \boldsymbol{F}[\boldsymbol{x}(k) - \boldsymbol{\xi}(k)]$$
$$\boldsymbol{x}(k) - \boldsymbol{\xi}(k) = \boldsymbol{F}^{k}[\boldsymbol{x}(0) - \boldsymbol{\xi}(0)]$$

plant to the observable form, design an observer in that form, and then convert back to the original system realization.

Another method of observer design for single-output plants that does not require transforming the system to observable canonical form is to use Ackermann's formula. Provided that $(\boldsymbol{A}, \boldsymbol{c}^\dagger)$ is completely observable, the eigenvalues of $\boldsymbol{F} = \boldsymbol{A} - \boldsymbol{g}\boldsymbol{c}^\dagger$ can be placed arbitrarily by choice of $\boldsymbol{g}$ according to Ackermann's formula:

$$\boldsymbol{g} = \Delta_0(\boldsymbol{A})\boldsymbol{M}_0^{-1}\boldsymbol{j}_n \tag{16}$$

provided that $(\boldsymbol{A}, \boldsymbol{c}^\dagger)$ is completely observable. In Eq. (16), $\Delta_0(\boldsymbol{A})$ is the desired characteristic equation of the observer eigenvalues with the matrix $\boldsymbol{A}$ substituted for the variable z, $\boldsymbol{M}_0$ is the observability matrix

$$\boldsymbol{M}_0 = \begin{bmatrix} \boldsymbol{c}^\dagger \\ \cdots \\ \boldsymbol{c}^\dagger\boldsymbol{A} \\ \cdots \\ \boldsymbol{c}^\dagger\boldsymbol{A}^2 \\ \cdots \\ \cdot \\ \cdot \\ \cdot \\ \cdots \\ \boldsymbol{c}^\dagger\boldsymbol{A}^n \end{bmatrix}$$

and $\boldsymbol{j}_n$ is the nth-unit coordinate vector

$$\boldsymbol{j}_n = \begin{bmatrix} 0 \\ \cdots \\ 0 \\ \cdots \\ 0 \\ \cdots \\ \cdot \\ \cdot \\ \cdot \\ \cdots \\ 1 \end{bmatrix}$$

Another popular form of a full-order observer which is viewed as an error feedback system can be obtained by expressing the observer Eqs. (13), (14), and (15) in the form

$$\begin{aligned}\boldsymbol{\xi}(k+1) &= (\boldsymbol{A} - \boldsymbol{G}\boldsymbol{C})\xi(k) + \boldsymbol{G}\boldsymbol{y}(k) + (\boldsymbol{B} - \boldsymbol{G}\boldsymbol{D})\boldsymbol{u}(k) \\ &= \boldsymbol{A}\boldsymbol{\xi}(k) + \boldsymbol{B}\boldsymbol{u}(k) + \boldsymbol{G}[\boldsymbol{y}(k) - \boldsymbol{w}(k)]\end{aligned}$$

where

$$\boldsymbol{w}(k) = \boldsymbol{C}\boldsymbol{\xi}(k) + \boldsymbol{D}\boldsymbol{u}(k)$$

Here, the observer consists of a model of the plant driven by the input $\boldsymbol{u}(k)$ and the error between the plant output $\boldsymbol{y}(k)$ and the plant output that is estimated by the model $\boldsymbol{w}(k)$. This form of a full-order observer is similar to the Kalman–Bucy filter (see Ref. 1).

Reduced-Order State Observers

If a completely observable nth-order plant has m linearly independent outputs, a *reduced-order* state observer, of order $n - m$, having an output that observes the plant state can be constructed.

For the plant described by Eq. (12), when an observer's state

$$\boldsymbol{\xi}(k+1) = \boldsymbol{F}\boldsymbol{\xi}(k) + \boldsymbol{G}\boldsymbol{y}(k) + \boldsymbol{H}\boldsymbol{u}(k)$$

estimates a linear combination $\boldsymbol{M}\boldsymbol{x}(k)$ of the plant state rather than the state itself, the error between the observer state and the plant state transformation is given by

$$\begin{aligned}\boldsymbol{M}\boldsymbol{x}(k+1) - \boldsymbol{\xi}(k+1) \\ &= \boldsymbol{M}\boldsymbol{A}\boldsymbol{x}(k) + \boldsymbol{M}\boldsymbol{B}\boldsymbol{u}(k) - \boldsymbol{F}\boldsymbol{\xi}(k) - \boldsymbol{G}\boldsymbol{y}(k) - \boldsymbol{H}\boldsymbol{u}(k) \\ &= (\boldsymbol{M}\boldsymbol{A} - \boldsymbol{G}\boldsymbol{C})\boldsymbol{x}(k) - \boldsymbol{F}\boldsymbol{\xi}(k) + (\boldsymbol{M}\boldsymbol{B} - \boldsymbol{G}\boldsymbol{D} - \boldsymbol{H})\boldsymbol{u}(k)\end{aligned}$$

where $\boldsymbol{M}$ is $m \times n$. For the observer error system to be autonomous, we require

$$\begin{aligned}\boldsymbol{F}\boldsymbol{M} &= \boldsymbol{M}\boldsymbol{A} - \boldsymbol{G}\boldsymbol{C} \\ \boldsymbol{H} &= \boldsymbol{M}\boldsymbol{B} - \boldsymbol{G}\boldsymbol{D}\end{aligned} \tag{17}$$

so that the error is governed by

$$\boldsymbol{M}\boldsymbol{x}(k+1) - \boldsymbol{\xi}(k+1) = \boldsymbol{F}[\boldsymbol{M}\boldsymbol{x}(k) - \boldsymbol{\xi}(k)]$$

For a completely observable plant, the observer gain matrix $\boldsymbol{g}$ can always be selected so that all the eigenvalues of $\boldsymbol{F}$ are inside the unit circle on the complex plane. Then the observer error

$$\boldsymbol{M}\boldsymbol{x}(k) - \boldsymbol{\xi}(k) = \boldsymbol{F}^k[\boldsymbol{M}\boldsymbol{x}(0) - \boldsymbol{\xi}(0)]$$

will approach zero asymptotically with step and then

$$\boldsymbol{\xi}(k) \to \boldsymbol{M}\boldsymbol{x}(k)$$

If the plant outputs, which also involve linear transformation of the plant state, are used in the formulation of a state observer, the dynamic order of the observer can be reduced. For the nth-order plant given by Eq. (12) with the m rows of $\boldsymbol{C}$ linearly independent, an observer of order $n - m$ with n outputs

$$\boldsymbol{W}'(k) = \begin{bmatrix}\boldsymbol{0} \\ \boldsymbol{I}\end{bmatrix}\boldsymbol{\xi}(k) + \begin{bmatrix}\boldsymbol{I} \\ \boldsymbol{0}\end{bmatrix}\mathbf{y}(k) + \begin{bmatrix}\boldsymbol{D} \\ \boldsymbol{0}\end{bmatrix}\boldsymbol{u}(k)$$

observes

$$\boldsymbol{W}'(k) \to \begin{bmatrix}\boldsymbol{C} \\ \boldsymbol{M}\end{bmatrix}\boldsymbol{x}(k) = \boldsymbol{N}\boldsymbol{x}(k)$$

Except in special cases, the rows of $\boldsymbol{M}$ and the rows of $\boldsymbol{C}$ are linearly independent. If they are not so, slightly different observer eigenvalues can be chosen to give linear independence. Therefore, the observer output

$$\boldsymbol{w}(k) = \boldsymbol{N}^{-1}\boldsymbol{w}'(k)$$

observes $\boldsymbol{x}(k)$.

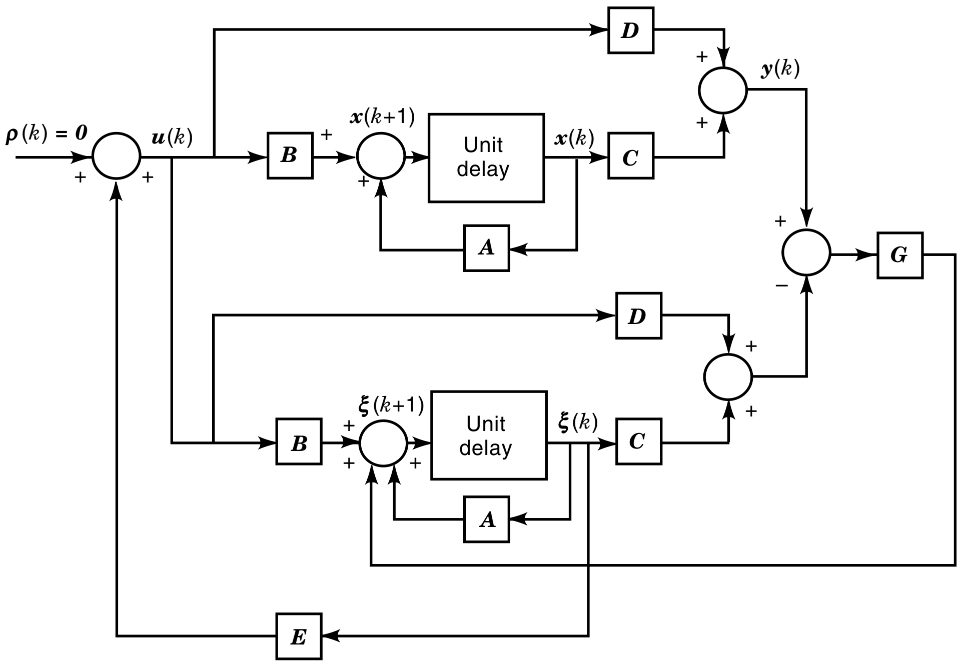

Figure 11. Eigenvalue placement with full-order state observer feedback.

Eigenvalue Placement with Observer Feedback

When observer feedback is used in place of plant state feedback, the eigenvalues of the feedback system are those the plant would have if the state feedback were used and those of the observer. This result is known as the *separation theorem* for observer feedback. For a completely controllable and completely observable plant, an observer of the form

$$\boldsymbol{\xi}(k+1) = \boldsymbol{F}\boldsymbol{\xi}(k) + \boldsymbol{G}\boldsymbol{y}(k) + \boldsymbol{H}\boldsymbol{u}(k) \quad (18)$$

$$\boldsymbol{w}(k) = \boldsymbol{L}\boldsymbol{\xi}(k) + \boldsymbol{N}[\mathbf{y}(k) - \boldsymbol{D}\boldsymbol{u}(k)] \quad (19)$$

with feedback to the plant given by

$$\boldsymbol{u}(k) = \boldsymbol{E}\boldsymbol{w}(k) \quad (20)$$

can be designed such that the overall feedback system eigenvalues are specified by the designer. The design procedure proceeds in two steps. First, the state feedback is designed to place the n-state feedback system eigenvalues at desired locations as if the state vector were accessible. Second, the state feedback is replaced by feedback of an observer estimate of the same linear transformations of the state. As an example of eigenvalue placement with observer feedback, Figure 11 shows eigenvalue placement with full order state observer. The eigenvalues of the overall system are those of the state feedback and those of the full-order observer.

TRACKING SYSTEM DESIGN

The second concern of tracking system design, that of obtaining acceptable zero-state system response to reference inputs, is now discussed. It is assumed that the first concern of tracking system design—namely, satisfactory zero-input response by feedback system eigenvalues placement—has been achieved.

A tracking system in which the plant outputs are controlled so that they become and remain nearly equal to externally applied reference signals $\boldsymbol{r}(k)$ is shown in Fig. 12(a). The outputs $\bar{\boldsymbol{y}}(k)$ are said to *track* or *follow* the reference inputs.

As shown in Fig. 12(b), a linear, step-invariant controller of a multiple-input/multiple-output plant is described by two transfer function matrices: one relating the reference inputs to the plant inputs, and the other relating the output feedback vector to the plant inputs. The feedback compensator is used for shaping the plant's zero-input response by placing the feedback system eigenvalues at desired locations as was discussed in the previous subsections. The input compensator, on the other hand, is designed to achieve good tracking of the reference inputs by the system outputs.

The output of any linear system can always be decomposed into two parts: the zero-input component due to the initial conditions alone, and the zero-state component due to the input alone. That is,

$$\bar{\boldsymbol{y}}(k) = \bar{\boldsymbol{y}}_{\text{zero input}}(k) + \bar{\boldsymbol{y}}_{\text{zero state}}(k)$$

Basically, there are three methods for tracking system design:

1. Ideal tracking system design
2. Response model design
3. Reference model design

Ideal Tracking System Design

In this first method, *ideal tracking* is obtained if the measurement output equals the tracking input:

$$\bar{\boldsymbol{y}}_{\text{zero state}}(k) = \boldsymbol{r}(k)$$

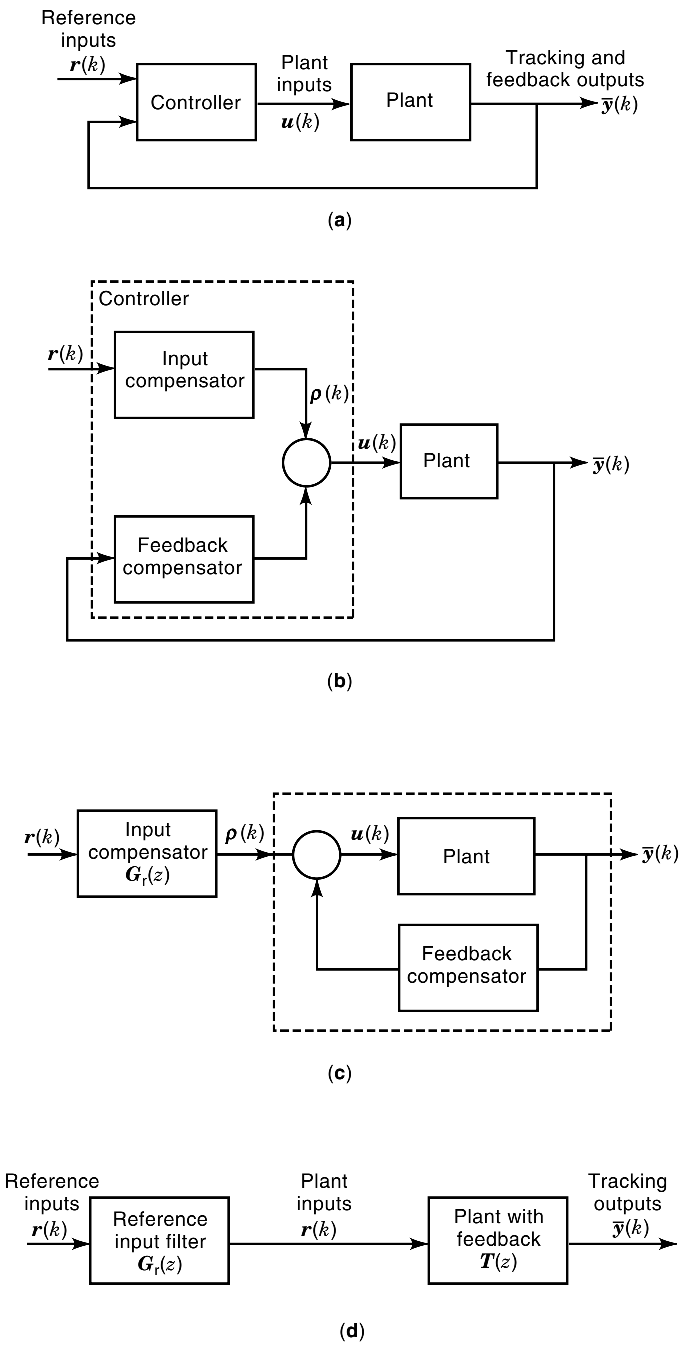

Figure 12. Controlling a multiple-input/multiple-output plant. (a) The output $\bar{y}(k)$ is to track the reference input r(k). (b) A tracking system using the reference inputs and plant outputs. (c) Representing a controller with a feedback compensator and an input compensator. (d) Feedback compensator combined with plant to produce a plant-with-feedback transfer function matrix $T(z)$.

The tracking outputs $\bar{\boldsymbol{y}}(k)$ have initial transient errors due to any nonzero plant initial conditions; after that they are equal to the reference inputs $\boldsymbol{r}(k)$, no matter what these inputs are.

As shown in Fig. 12(c), if the plant with feedback has the z-transfer function matrix $\boldsymbol{T}(z)$ relating the tracking output to the plant inputs, then

$$\bar{\boldsymbol{Y}}(z) = \boldsymbol{T}(z)\boldsymbol{\rho}(z)$$

An input compensator or a *reference input filter,* as shown in Fig. 12(d), with transfer function matrix $\boldsymbol{G}(z)$, for which

$$\boldsymbol{\rho}(z) = \boldsymbol{G}(z)\boldsymbol{R}(z)$$

gives

$$\bar{\boldsymbol{Y}}(z) = \boldsymbol{T}(z)\boldsymbol{G}(z)\boldsymbol{R}(z)$$

Ideal tracking is achieved if

$$\boldsymbol{T}(z)\boldsymbol{G}(z) = \boldsymbol{I}$$

where $\boldsymbol{I}$ is the identity matrix with dimensions equal to the number of reference inputs and tracking outputs. This is to say that ideal tracking is obtained if the reference input filter is an *inverse filter* of the plant with feedback. Reference input filters do not change the eigenvalues of the plant with feedback which are assumed to have been previously placed with output or observer feedback.

When a solution exists, ideal tracking system design achieves *exact* zero-state tracking of any reference input. Because it involves constructing inverse filters, the ideal tracking system design may require unstable or noncausal filters. An ideal tracking solution can also have other undesirable properties, such as unreasonably large gains, high oscillatory plant control inputs, and the necessity of canceling plant poles and zeros when the plant model is not known accurately.

Response Model Design

When ideal tracking is not possible or desirable, the designer can elect to design *response model* tracking, for which

$$\boldsymbol{T}(z)\boldsymbol{G}(z) = \boldsymbol{\Omega}(z)$$

where the response model z-transfer function matrix $\boldsymbol{\Omega}(z)$ characterizes an acceptable relation between the tracking outputs and the reference inputs. Clearly, the price one pays for the added design freedom of a reference model can be degraded tracking performance. However, performance can be improved by increasing the order of the reference input filter. Response model design is a generalization of the classical design technique of imposing requirements for a controller's steady-state response to power-of-time inputs.

The difficulty with the response model design method is in selecting suitable model systems. For example, when two or more reference input signals are to be tracked simultaneously, the response model z-transfer functions to be selected include not only those relating plant tracking outputs and the reference inputs they are to track, but also those relating unwanted coupling between each tracking output and the other reference inputs.

Reference Model Tracking System Design

The awkwardness of the practical response model performance design arises because of the difficulty in relating performance criteria to the z-transfer functions of response models. An alternative design method models the reference input signals $\boldsymbol{r}(k)$ instead of the system response. This method, termed the reference model tracking system design, allows the designer to specify a class of representative reference inputs that are to be tracked exactly, rather than having to specify acceptable response models for all the possible inputs.

In the reference model tracking system design, additional external input signals $\boldsymbol{r}(k)$ to the composite system are applied to the original plant inputs and to the observer state equations so that the feedback system is, instead of being described by Eqs. (18), (19), and (20), described by Eq. (12) and

$$\boldsymbol{\xi}(k+1) = \boldsymbol{F}\boldsymbol{\xi}(k) + \boldsymbol{G}\boldsymbol{y}(k) + \boldsymbol{H}\boldsymbol{u}(k) + \boldsymbol{J}\boldsymbol{r}(k) \tag{21}$$

$$\boldsymbol{w}(k) = \boldsymbol{L}\boldsymbol{\xi}(k) + \boldsymbol{N}[\boldsymbol{y}(k) - \boldsymbol{D}\boldsymbol{u}(k)] \tag{22}$$

with

$$\boldsymbol{u}(k) = \boldsymbol{E}\boldsymbol{w}(k) + \boldsymbol{P}\boldsymbol{r}(k) \tag{23}$$

Then, the overall composite system has the state equations

$$\begin{aligned}\begin{bmatrix}\boldsymbol{x}(k+1)\\ \boldsymbol{\xi}(k+1)\end{bmatrix} &= \begin{bmatrix}\boldsymbol{A}+\boldsymbol{BENC} & \boldsymbol{BEL}\\ \boldsymbol{GC}+\boldsymbol{H'ENC} & \boldsymbol{F}+\boldsymbol{H'EL}\end{bmatrix}\begin{bmatrix}\boldsymbol{x}(k)\\ \boldsymbol{\xi}(k)\end{bmatrix}\\ &\quad + \begin{bmatrix}\boldsymbol{BP}\\ \boldsymbol{H'P}+\boldsymbol{J}\end{bmatrix}\boldsymbol{r}(k)\\ &= \boldsymbol{A'}\boldsymbol{x'}(k) + \boldsymbol{B'}\boldsymbol{r}(k)\end{aligned}$$

and the output equation becomes

$$\begin{aligned}\bar{\boldsymbol{y}}(k) &= [\boldsymbol{C}+\boldsymbol{ENC} \quad \boldsymbol{DEL}]\boldsymbol{x'}(k) + \boldsymbol{DP}\boldsymbol{r}(k)\\ &= \boldsymbol{C'}\boldsymbol{x'}(k) + \boldsymbol{D'}\boldsymbol{r}(k)\end{aligned}$$

where

$$\boldsymbol{H'} = \boldsymbol{H} + \boldsymbol{GD}$$

Examining the composite state coupling matrix $\boldsymbol{A'}$ above shows that the coupling of external inputs $\boldsymbol{r}(k)$ to the feedback system does not affect its eigenvalues. The input coupling matrix $\boldsymbol{B'}$ has matrices $\boldsymbol{P}$ and $\boldsymbol{J}$ which are entirely arbitrary and thus can be selected by the designer. Our objective is to select $\boldsymbol{P}$ and $\boldsymbol{J}$ such that the system output $\bar{\boldsymbol{y}}(k)$ tracks the reference input $\boldsymbol{r}(k)$.

Consider the class of reference signals which are generated by the autonomous state variable model of the form

$$\begin{aligned}\boldsymbol{\sigma}(k+1) &= \boldsymbol{\Psi}\boldsymbol{\sigma}(k)\\ \boldsymbol{r}(k) &= \boldsymbol{\Theta}\boldsymbol{\sigma}(k)\end{aligned} \tag{24}$$

The output of this *reference input model system* may consist of step, ramp, parabolic, exponential, sinusoidal, and other common sequences. For example, the model

$$\begin{bmatrix}\sigma_1(k+1)\\ \sigma_2(k+1)\end{bmatrix} = \begin{bmatrix}2 & 1\\ -1 & 0\end{bmatrix}\begin{bmatrix}\sigma_1(k)\\ \sigma_2(k)\end{bmatrix}$$

$$r(k) = [1 \quad 0]\begin{bmatrix}\sigma_1(k)\\ \sigma_2(k)\end{bmatrix}$$

produces the sum of an arbitrary constant plus an arbitrary ramp:

$$r(k) = \sigma_1(0) + \sigma_1(0)k$$

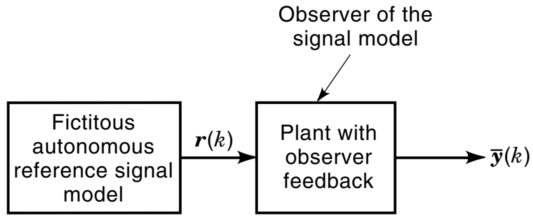

Figure 13. Observing a reference signal model.

In the reference model tracking system design, the concept of an observer is used in a new way; it is the plant with feedback that is an observer of the *fictitious* reference input model system as shown in Fig. 13. When driven by $\boldsymbol{r}(k)$, the state of the composite system observes

$$\boldsymbol{x'}(k) \rightarrow \boldsymbol{M}\boldsymbol{\sigma}(k)$$

where $\boldsymbol{M}$ satisfies, according to Eq. (17),

$$\boldsymbol{M}\boldsymbol{\Psi} - \boldsymbol{A'}\boldsymbol{M} = \boldsymbol{B'}\boldsymbol{\Theta} \tag{25}$$

The plant tracking output $\bar{\boldsymbol{y}}(k)$ observes

$$\bar{\boldsymbol{y}}(k) = \boldsymbol{C'}\boldsymbol{x'}(k) + \boldsymbol{D'}\boldsymbol{r}(k) \rightarrow \boldsymbol{C'}\boldsymbol{M}\boldsymbol{\sigma}(k) + \boldsymbol{D'}\boldsymbol{r}(k)$$

and for

$$\bar{\boldsymbol{y}}(k) \rightarrow \boldsymbol{r}(k)$$

it is necessary that

$$\boldsymbol{C'}\boldsymbol{M}\boldsymbol{\sigma}(k) + \boldsymbol{D'}\boldsymbol{r}(k) = \boldsymbol{r}(k) \tag{26}$$

Equations (25) and (26) constitute a set of linear algebraic equations where the elements of $\boldsymbol{M}$, $\boldsymbol{P}$, and $\boldsymbol{J}$ are unknowns. If, for an initial problem formulation, there is no solution to the equations, one can reduce the order of the reference signal model and/or raise the order of the observer used for plant feedback until an acceptable solution is obtained.

The autonomous reference input model has no physical existence; the actual reference input $\boldsymbol{r}(k)$ likely deviates somewhat from the prediction of the model. The designer deals with representative reference inputs, such as constants and ramps, and, by designing for exact tracking of these, obtains acceptable tracking performance for other reference inputs.

SIMULATION

One of the most important control system design tools is simulation—that is, computer modeling of the plant and controller to verify the properties of a preliminary design and to test its performance under conditions (e.g., noise, disturbances, parameter variations, and nonlinearities) that might be difficult or cumbersome to study analytically. It is usually through simulation that difficulties with between-sample plant response are covered and solved.

When a continuous-time plant is simulated on a digital computer, its response is computed at closely spaced discrete times. It is plotted by joining the closely spaced calculated response values with straight line segments in approximation of continuous curve. A digital computer simulation of discrete-time control of a continuous-time system involves at least two sets of discrete-time calculations. One runs at a high rate for simulation of the continuous-time plant. The other runs at a lower rate (say once every 10 to 50 of the former calculations) to generate new control signals at each discrete control step.

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