# CONTROLLABILITY AND OBSERVABILITY

In this article, we will consider dynamic systems $\Sigma$ described by a set of first order differential or difference equations, together with a set of algebraic equations:

$$\sigma x = f(x,u) \text{ where } (\sigma x)(t) := \begin{cases} \frac{d}{dt}x(t),\ t \in \mathbb{R} \\ x(t+1),\ t \in \mathbb{Z} \end{cases} \tag{1}$$

$$y = h(x,u) \tag{2}$$

Here, $u$ is the input or excitation function, and its values $u(t)$ belong to a Euclidean space $\mathbb{R}^m$. The state at time $t$ is $x(t)$ and, depending on the context, may belong to a finite or an infinite dimensional space. The output or measurement function $y$ takes values $y(t) \in \mathbb{R}^p$. When $u(t) \equiv 0$, one interprets the equation $\sigma x = f(x, 0)$ as describing the evolution of the system in the absence of inputs. Observe that, in contrast to the classical study of dynamic systems, where inputs (or forcing functions) are fixed, the object of control theory is to study the effect of different input functions on the system $\Sigma$, that is, on the solutions of the difference or differential Eq. (1).

In the sequel questions of state controllability and state observability will be investigated. Roughly speaking, the former questions have to do with the extent to which the choice of $u$ can influence the evolution of $x$, while the latter questions are concerned with the extent to which by observing $y$ and $u$ one may deduce the evolution of $x$. We will illustrate these concepts for three different cases: (1) linear finite-dimensional systems, (2) nonlinear systems, and (3) linear infinite dimensional systems.

We remark that one could also study a more general type of dynamic system, namely one given by implicit equations on a set of abstract variables $w$:

$$f(\sigma^n w, \cdots, \sigma w, w) = 0 \tag{3}$$

Two examples are

$$w = \begin{pmatrix} u \\ y \end{pmatrix} \quad \text{or} \quad w = \begin{pmatrix} u \\ x \\ y \end{pmatrix}$$

where $u$, $x$, and $y$ are, as stated, the input, state, and output, respectively. Let $\mathcal{B}$ denote the *behavior* of this system, that is, the set of all time trajectories $w$ which satisfy Eq. (3). This leads to the so-called behavioral theory of dynamic systems (1). Let $w_-$ and $w_+$ denote, respectively, the past and future of a given trajectory with respect to some fixed time $t_0$ (with $w$ restricted to $t \leq t_0$ and $t \geq t_0$). The system described by Eq. (3) is said to be *controllable,* if for any $w_1$, $w_2$ belonging to $\mathcal{B}$, any trajectory composed of the concatenation of $(w_1)_-$ and $(w_2)_+$ belongs to $\mathcal{B}$. This approach has actually been worked out for the case where $f$ is a linear and time-invariant function of its arguments. This provides a generalization of state controllability for the case where states are not available, and without distinguishing between inputs and outputs. It turns out that this concept is indeed the generalization of the concept of state controllability when Eq. (3) is Eq. (1). This will be discussed briefly in a later section of this article.

For reasons of brevity and space, the proofs of the various results are omitted in the sections that follow. The interested reader is referred to the original sources.

## LINEAR FINITE-DIMENSIONAL SYSTEMS

The system described by the state equations Eq. (1) is linear, if $f(x, u) = Fx + Gu$:

$$\sigma x = Fx + Gu \tag{4}$$

where as in Eq. (1), $\sigma$ denotes the derivative operator for continuous-time systems and the (backwards) shift operator for discrete-time systems. The input $u(t)$ and state $x(t)$ of the system at time $t$ belong, respectively, to the input space $U \cong \mathbb{R}^m$ and state space $X \cong \mathbb{R}^n$. Moreover,

$$G: U \to X, \quad F: X \to X$$

are linear maps; the first is called the input map, while the second describes the dynamics or internal evolution of the system.

The output Eq. (2), for both discrete- and continuous-time linear systems, is composed of a set of linear algebraic equations

$$y = Hx + Ju \tag{5}$$

where $y(t)$ is the *output* (response) at time $t$, and belongs to the output space $Y \cong \mathbb{R}^p$; furthermore:

$$H: X \to Y, \quad J: U \to Y$$

are linear maps; $H$ is called the output map. It describes how the dynamics affect the output, while $J$ describes how the input affects the output directly (i.e., without passing through the state). For simplicity of exposition, we will assume that the systems considered are time-invariant, that is to say, there is no explicit dependence of the system matrices on time. Thus the term *linear* will be used in this section to denote a linear, time-invariant, continuous- or discrete-time system which is finite-dimensional. Linearity means: $U$, $X$, $Y$ are linear spaces, and $F$, $G$, $H$, $J$ are linear maps; finite-dimensional means: $U$, $X$, $Y$ are all finite dimensional; time-invariant means: $F$, $G$, $H$, $J$ do not depend on time; their matrix representations are constant $n \times n$, $n \times m$, $p \times n$, $p \times m$ matrices. We are now ready to give

***Definition 1.*** (a) A linear system in state space description is a quadruple of linear maps (matrices)

$$\sum := \left(\begin{array}{c|c} F & G \\ \hline H & J \end{array}\right), F \in \mathbb{R}^{n \times n},\ G \in \mathbb{R}^{n \times m},\ H \in \mathbb{R}^{p \times n},\ J \in \mathbb{R}^{p \times m} \tag{6}$$

The dimension of the system is defined as the dimension of the associated state space:

$$\dim \sum = n \tag{7}$$

(b) $\Sigma$ is called (asymptotically) stable if the eigenvalues of $F$ have negative real parts or lie strictly inside the unit disk, depending on whether $\Sigma$ is a continuous-time or a discrete-time system.

Let $\phi(u; x_0; t)$ denote the solution of the state Eq. (4), that is, the state of the system at time $t$ attained starting from the initial state $x_0$ at time $t_0$, under the influence of the input $u$. For the continuous-time, time-invariant state equations,

$$\phi(u; x_0; t) = e^{F(t-t_0)}x_0 + \int_{t_0}^{t} e^{F(t-\tau)}Gu(\tau)\,d\tau,\ t \geq t_0 \tag{8}$$

while for the discrete-time state equations,

$$\phi(u; x_0; t) = F^{t-t_0}x_0 + \sum_{j=t_0}^{t-1} F^{t-1-j}Gu(j),\ t \geq t_0 \tag{9}$$

For both discrete- and continuous-time systems the output is given by:

$$\begin{aligned} y(t) &= H\phi(u; x(0); t) + Ju(t) \\ &= H\phi(0; x_0; t) + H\phi(u; 0; t) + Ju(t) \end{aligned} \tag{10}$$

If we transform the state under a linear change of coordinates, the corresponding matrices describing the system will change. In particular, if the new state is

$$\tilde{x} := Tx,\ \det T \neq 0$$

then Eqs. (4) and (5) when expressed in terms of the new state $\tilde{x}$, will become

$$\sigma\tilde{x} = \underbrace{TFT^{-1}}_{\tilde{F}}\tilde{x} + \underbrace{TG}_{\tilde{G}}u, \quad y = \underbrace{HT^{-1}}_{\tilde{H}}\tilde{x} + Ju$$

where $J$ remains unchanged. The corresponding triples are called *equivalent.*

***Remark.*** The material that follows was first introduced by R. E. Kalman (2), (3); see also Refs. 4 and 5. For a more recent treatment, we refer to the book by Sontag (6); see also Refs. 7,8, and 9.

## The State Controllability Problem

There are two fundamental concepts associated with the state controllability problem: reachability and controllability. These concepts allow us to answer questions concerning the extent to which the state of the system $x$ can be manipulated through the input $u$.

We will first discuss the concept of state reachability for linear systems. The related concept of controllability will also be discussed. Both concepts involve only the state equations. Consequently, for this subsection, $H$ and $J$ will be ignored.

***Definition 2.*** Given

$$\sum = \left(\begin{array}{c|c} F & G \\ \hline & \end{array}\right)$$

$F \in \mathbb{R}^{n \times n}$, $G \in \mathbb{R}^{n \times m}$, a state $\bar{x} \in X$ is reachable from the zero state iff there exist an input function $\bar{u}(t)$ and a time $\overline{T} < \infty$, such that

$$\bar{x} = \phi(\bar{u}; 0; \overline{T})$$

The reachable subspace $X^{\text{reach}} \subset X$ of $\Sigma$ is the set which contains all reachable states of $\Sigma$. We will call the system $\Sigma$ (completely) reachable iff $X^{\text{reach}} = X$. Furthermore

$$\mathscr{R}_n(F, G) := [G \quad FG \quad F^2G \quad \cdots \quad F^{n-1}G] \tag{11}$$

will be called the reachability matrix of $\Sigma$.

A useful concept is that of the reachability grammian. Complex conjugation and transposition will be denoted by *.

***Definition 3.*** The finite reachability Grammian at time $t < \infty$ is defined as follows. For continuous-time systems:

$$\mathscr{P}(t) := \int_0^t e^{F\tau}GG^*e^{F^*\tau}\,d\tau,\ t > 0 \tag{12}$$

while for discrete-time systems

$$\mathscr{P}(t) := \mathscr{R}_t(F, G)\mathscr{R}_t^*(F, G) = \sum_{k=0}^{t-1} F^kGG^*(F^*)^k,\ t > 0 \tag{13}$$

We will make use of the following input

$$w_{\xi,T}(t) := G^*e^{F^*(T-t)}\xi,\ \xi \in \mathbb{R}^n,\ t, T \in \mathbb{R},\ 0 \leq t \leq T \tag{14}$$

for the continuous-time case, and

$$w_{\xi,T}(t) := G^*(F^*)^{T-t}\xi,\ \xi \in \mathbb{R}^n,\ t, T \in \mathbb{Z},\ 0 \le t \le T \tag{15}$$

for the discrete-time case.

A concept which is closely related to reachability is that of controllability. Here, instead of driving the zero state to a desired state, a given nonzero state is steered to the zero state. More precisely we have:

***Definition 4.*** Given

$$\Sigma = \left(\begin{array}{c|c} F & G \\ \hline & \end{array}\right)$$

a (nonzero) state $\bar{x} \in X$ is controllable to the zero state if and only if there exist an input function $\bar{u}(t)$ and a time $\bar{T} < \infty$, such that

$$\phi(\bar{u}; \bar{x}; \bar{T}) = 0$$

The controllable subspace $X^{\text{contr}}$ of $\Sigma$ is the set of all controllable states. The system $\Sigma$ is (completely) controllable if and only if $X^{\text{contr}} = X$.

The fundamental result concerning reachability is the following

**Theorem 1.** Given

$$\Sigma = \left(\begin{array}{c|c} F & G \\ \hline & \end{array}\right)$$

for both the continuous- and discrete-time case, $X^{\text{reach}}$ is a linear subspace of $X$, given by the formula

$$X^{\text{reach}} = \text{im}\,\mathscr{R}_n(F, G) \tag{16}$$

where im$\mathscr{R}_n$ denotes the image (span of the columns) of $\mathscr{R}_n$.

**Corollary 1.** (a) $FX^{\text{reach}} \subset X^{\text{reach}}$. (b) $\Sigma$ is (completely) reachable if, and only if, rank $\mathscr{R}_n(F, G) = n$. (c) Reachability is basis independent.

In general, reachability is an analytic concept. The previous theorem, however, shows that for linear, finite-dimensional, time-invariant systems, reachability reduces to an algebraic concept depending only on properties of $F$, $G$ and in particular on the rank of the reachability matrix $\mathscr{R}_n(F, G)$, but independent of time and the input function. It is also worthwhile to note that Eq. (16) is valid for both continuous- and discrete-time systems. This, together with a similar result on observability [see Eq. (23)], has as a consequence the fact that many tools for studying linear systems are algebraic. It should be noticed however that the physical significance of $F$ and $G$ is different for the discrete- and continuous-time cases; if for instance we discretize the continuous-time system

$$\frac{d}{dt}x(t) = F_{\text{cont}}x(t) + G_{\text{cont}}u(t),\ \text{to}$$
$$x(t+1) = F_{\text{discr}}x(t) + G_{\text{discr}}u(t)$$

then $F_{\text{discr}} = e^{F_{\text{cont}}}$.

**Proposition 1.** The reachability Grammians have the following properties: (a) $\mathscr{P}(t) = \mathscr{P}^*(t) \geq 0$, and (b) their columns span the reachability subspace, that is

$$\text{im}\,\mathscr{P}(t) = \text{im}\,\mathscr{R}_n(F, G)$$

This relationship holds for continuous-time systems for all $t > 0$, and for discrete-time systems (at least for $t \geq n$).

**Corollary 2**

$$\Sigma = \left(\begin{array}{c|c} F & G \\ \hline & \end{array}\right)$$

is reachable if and only if $\mathscr{P}(t)$ is positive definite for some $t > 0$.

The energy $\|f\|$ of the vector function $f$, defined on an interval $\mathscr{I} \subset \mathbb{R}$ or $\mathbb{Z}$, is defined as

$$\|f\|^2 := \langle f, f\rangle := \begin{cases} \sum_{\tau\in\mathscr{I}} f^*(\tau)f(\tau),\ t \in \mathbb{Z} \\ \int_{\tau\in\mathscr{I}} f^*(\tau)f(\tau)\,d\tau,\ t \in \mathbb{R} \end{cases}$$

The input function $w_{\xi,T}$ defined by Eqs. (14) and (15) has the following property:

**Proposition 2.** Given the reachable state $x \in X^{\text{reach}}$, let $\hat{u}$ be any input function which reaches $x$ at time $T$, that is, $\phi(\hat{u}; 0; T) = x$. There exists $\xi \in \mathbb{R}^n$ satisfying:

$$x = \mathscr{P}(T)\xi \tag{17}$$

It follows that $w_{\xi,T}$ defined by Eqs. (14) and (15) reaches $x$ at time $T$; moreover this is the minimum energy input which achieves this:

$$\|\hat{u}\| \geq \|w_{\xi,T}\| \tag{18}$$

The minimum energy required to reach the state $x$ at time $T$ is equal to the energy of the input function $w_{\xi,T}$. If the system is reachable this energy is equal to:

$$\|w_{\xi,T}\| = \sqrt{x^*\mathscr{P}(T)^{-1}x} \tag{19}$$

From the previous considerations the length of time needed to reach a given reachable state can be derived.

**Proposition 3.** Given is

$$\Sigma = \left(\begin{array}{c|c} F & G \\ \hline & \end{array}\right)$$

(a) For discrete-time systems, every reachable state can be reached in at most $n$ time-steps. (b) For continuous-time systems, every reachable state can be reached in any arbitrary positive length of time.

The second part of the proposition shows that ideally, in continuous-time linear systems, every reachable state can be reached arbitrarily fast. In a practical situation, the extent to which this is not possible gives a measure of how significant the nonlinearities of the system are. We conclude this subsection by stating a result on various equivalent conditions for reachability.

**Theorem 2 Reachability Conditions.** The following are equivalent:

1.

$$\Sigma = \left(\begin{array}{c|c} F & G \\ \hline & \end{array}\right)$$

$F \in \mathbb{R}^{nxn}$, $G \in \mathbb{R}^{n\times m}$ is reachable.

2. The rank of the reachability matrix is full: rank $\mathscr{R}_n(F, G) = n$.
3. The reachability Grammian is positive definite, that is, $\mathscr{P}(t) > 0$, for some $t > 0$.
4. No left eigenvector $v^*$ of $F$ is in the left kernel of $G$:

$$v^*F = \lambda v^* \Rightarrow v^*G \neq 0$$

5. $\text{rank}(\mu I_n - F \;\; G) = n$, for all $\mu \in \mathbb{C}$
6. The polynomial matrices $sI - F$ and $G$ are left coprime.

The fourth and fifth conditions in this theorem are known as the PHB or Popov–Hautus–Belevich tests for reachability. The equivalence of the fifth and sixth conditions is a straightforward consequence of the theory of polynomial matrices; it will not be discussed in this article.

***Remark.*** Reachability is a generic property. This means intuitively that almost every $n \times n$, $n \times m$ pair of matrices $F$, $G$ satisfies

$$\text{rank}\, \mathscr{R}_n(F, G) = n$$

Put in a different way, in the space of all $n \times n$, $n \times m$ pairs of matrices, the unreachable pairs form a hypersurface of measure zero.

The next theorem shows that for continuous-time systems, the concepts of reachability and controllability are equivalent while for discrete-time systems the latter is weaker. This is easily seen by considering the system with state equation: $x(t + 1) = 0$. Clearly, for this system all states are controllable, while none is reachable. Often for this reason, only the notion of reachability is used.

**Theorem 3.** Given is

$$\Sigma = \left(\begin{array}{c|c} F & G \\ \hline & \end{array}\right)$$

(a) For continuous-time systems $X^{\text{contr}} = X^{\text{reach}}$. (b) For discrete-time systems $X^{\text{reach}}. \subset X^{\text{contr}}$; in particular $X^{\text{contr}} = X^{\text{reach}} + \ker F^n$.

***Remark.*** It follows from the previous results that for any two states $x_1, x_2 \in X^{\text{reach}}$ there exist $u_{12}$, $T_{12}$ such that $x_1 = \phi(u_{12}; x_2; T_{12})$. Since $x_2$ is reachable it is also controllable; thus there exist $u_2$, $T_2$ such that $\phi(u_2; x_2; T_2) = 0$. Finally, the reachability of $x_1$ implies the existence of $u_1$, $T_1$ such that $x_1 = \phi(u_1; 0; T_1)$. The function $u_{12}$ is then the concatenation of $u_2$ with $u_1$, while $T_{12} = T_1 + T_2$. In general, if $x_1$, $x_2$ are not reachable, there is a trajectory passing through the two points if, and only if,

$$x_2 - f(F, T)x_1 \in X^{\text{reach}}, \quad \text{for some } T,$$

where $f(F, T) = e^{FT}$ for continuous-time systems and $f(F, T) = F^T$ for discrete-time systems. This shows that if we start from a reachable state $x_1 \neq 0$ the states that can be attained are also within the reachable subspace.

### The State Observation Problem

In order to be able to modify the dynamics of a system, very often the state $x$ needs to be available. Typically however the state variables are inaccessible and only certain linear combinations $y$, given by the output Eqs. (5), are known. Thus, the problem of reconstructing the state $x(T)$ from observations $y(\tau)$, where $\tau$ is in some appropriate interval, arises. If $\tau \in [T, T + t]$, we have the *state observation problem,* while if $\tau \in [T - t, T]$ we have the *state reconstruction problem.*

We will first discuss the observation problem. Without loss of generality we will assume that $T = 0$. Recall Eqs. (8), (9), and (10). Since the input $u$ is known, the latter two terms in Eq. (10) are also known. Therefore, in determining $x(0)$ we may assume without loss of generality that $u(\cdot) \equiv 0$. Thus, the observation problem reduces to the following: given $H\phi(0; x(0); t)$ for $t \geq 0$ or $t \leq 0$, find $x(0)$. Since $G$ and $J$ are irrelevant, for this subsection

$$\Sigma = \left(\begin{array}{c|c} F & \\ \hline H & \end{array}\right), \; F \in \mathbb{R}^{n\times m}, \; H \in \mathbb{R}^{pxn}$$

***Definition 5.*** A state $\bar{x} \in X$ is unobservable iff $y(t) = H\phi(0; \bar{x}; t) = 0$, for all $t \geq 0$, that is, iff $\bar{x}$ is indistinguishable from the zero state for all $t \geq 0$. The unobservable subspace $X^{\text{unobs}}$ of $X$ is the set of all unobservable states of $\Sigma$. $\Sigma$ is (completely) observable iff $X^{\text{unobs}} = 0$. The observability matrix of $\Sigma$ is

$$\mathscr{O}_n(H, F) := \begin{bmatrix} H \\ HF \\ HF^2 \\ \vdots \\ HF^{n-1} \end{bmatrix} \tag{20}$$

***Definition 6.*** Let

$$\Sigma = \left(\begin{array}{c|c} F & \\ \hline H & \end{array}\right)$$

The finite observability Grammians at time $t < \infty$ are:

$$\mathscr{Q}(t) := \int_0^t e^{F^*\tau} H^* H e^{F\tau} d\tau, \; t > 0 \tag{21}$$

$$\mathscr{Q}(t) := \mathscr{O}_t^*(H, F)\mathscr{O}_t(H, F), t > 0 \tag{22}$$

for continuous- and discrete-time systems, respectively.

***Definition 7.*** A state $\bar{x} \in X$ is unreconstructible iff $y(t) = H\phi(0; \bar{x}; t) = 0$, for all $t \le 0$, that is, iff $\bar{x}$ is indistinguishable from the zero state for all $t \le 0$. The unreconstructible subspace $X^{\text{unrecon}}$ of $X$ is the set of all unreconstructible states of $\Sigma$. $\Sigma$ is (completely) reconstructible iff $X^{\text{unrecon}} = 0$.

We are now ready to state the main theorem

**Theorem 4.** Given

$$\Sigma = \left(\begin{array}{c|c} F & \\ \hline H & \end{array}\right)$$

for both continuous- and discrete-time systems, $X^{\text{unobs}}$ is a linear subspace of $X$ given by

$$X^{\text{unobs}} = \ker \mathcal{O}_n(H, F) = \{x \in X : HF^{i-1}x = 0,\ i > 0\} \tag{23}$$

As an immediate consequence of the last formula we have

**Corollary 3.** (a) The unobservable subspace $X^{\text{unobs}}$ is $F$-invariant. (b) $\Sigma$ is observable iff, rank $\mathcal{O}_n(H, F) = n$. (c) Observability is basis independent.

***Remark.*** Given $y(t)$, $t \ge 0$, let $Y_0$ denote the following $np \times 1$ vector:

$$Y_0 := (y^*(0)\, y^*(1) \cdots y^*(n-1))^*,\ t \in \mathbb{Z}$$
$$Y_0 := (y^*(0)\, Dy^*(0) \cdots D^{n-1}y^*(0))^*,\ t \in \mathbb{R}$$

where $D := d/dt$. The observability problem reduces to the solution of the linear set of equations

$$\mathcal{O}_n(H, F)x(0) = Y_0$$

This set of equations is solvable for all initial conditions $x(0)$, that is, it has a unique solution iff, $\Sigma$ is observable. Otherwise $x(0)$ can only be determined modulo $X^{\text{unobs}}$, that is, up to an arbitrary linear combination of unobservable states.

It readily follows that $\ker \mathcal{Q}(t) = \ker \mathcal{O}_n(H, F)$. As in the case of reachability, this relationship holds for continuous-time systems for $t > 0$ and for discrete-time systems, at least for $t \ge n$. The energy of the output function $y$ at time $T$, generated from the initial state $x$ will be denoted by $\|y\|$. In terms of the observability Grammian this energy can be expressed as

$$\|y\| = \sqrt{x^* \mathcal{Q}(T) x} \tag{24}$$

We now briefly turn our attention to the reconstructibility problem. The main result which follows shows that while for continuous-time systems the concepts of observability and reconstructibility are equivalent, for discrete-time systems the latter is weaker. For this reason, the concept of observability is used most of the time.

**Proposition 4.** Given is

$$\Sigma = \left(\begin{array}{c|c} F & \\ \hline H & \end{array}\right)$$

(a) For $t \in \mathbb{R}$, $X^{\text{unrecon}} = X^{\text{unobs}}$. (b) For $t \in \mathbb{Z}$, $X^{\text{unrecon}} \supset X^{\text{unobs}}$, in particular, $X^{\text{unrecon}} = X^{\text{unobs}} \cap \operatorname{im} F^n$.

### The Duality Principle in Linear Systems

The dual of a linear system

$$\Sigma = \left(\begin{array}{c|c} F & G \\ \hline H & J \end{array}\right)$$

is defined as follows. Let $U^*$, $X^*$, $Y^*$ be the dual spaces of the input $U$, state $X$, output $Y$ spaces of $\Sigma$. Let

$$F^* : X^* \to X^*,\ G^* : X^* \to U^*,\ H^* : Y^* \to X^*,\ J^* : Y^* \to U^*$$

be the dual maps to $F$, $G$, $H$, $J$. The dual system $\Sigma^*$ of $\Sigma$ is

$$\Sigma^* := \left(\begin{array}{c|c} F^* & H^* \\ \hline G^* & J^* \end{array}\right) \in \mathbb{R}^{(n+m)\times(n+p)} \tag{25}$$

that is, the input map is given by $H^*$, the output map by $G^*$, and the dynamics are given by $F^*$. Correspondingly the input, state, and output spaces of $\Sigma^*$ are $Y^*$, $X^*$, $U^*$. The matrix representations of $F^*$, $H^*$, $G^*$, $J^*$ are the complex conjugate transposes of $F$, $H$, $G$, $J$, respectively, computed in appropriate dual bases. One may think of the dual system $\Sigma^*$ as the system $\Sigma$ where the role of the inputs and the outputs has been interchanged, or the flow of causality has been reversed. The main result is the duality principle.

**Theorem 5.** The orthogonal complement of the reachable subspace of $\Sigma$ is equal to the unobservable subspace of its dual $\Sigma^*$:

$$(X_{\Sigma}^{\text{reach}})^{\perp} = X_{\Sigma^*}^{\text{unobs}}$$

**Corollary 4.** The system $\Sigma$ is reachable iff its dual $\Sigma^*$ is observable.

It can also be shown that controllability and reconstructibility are dual concepts. We conclude this subsection by stating the dual to theorem 2.

**Theorem 6.** Observability conditions. The following are equivalent:

1. $$\Sigma = \left(\begin{array}{c|c} F & \\ \hline H & \end{array}\right)$$
   $H \in \mathbb{R}^{p\times n}$, $F \in \mathbb{R}^{n\times n}$ is observable.
2. The rank of the observability matrix is full: rank $\mathcal{O}_n(H, F) = n$.
3. The observability Grammian is positive definite $\mathcal{Q}(t) > 0$, for some $t > 0$.
4. No right eigenvector $v$ of $F$ is in the right kernel of $H$:
   $$Fv = \lambda v \Rightarrow Hv \neq 0$$
5. $$\operatorname{rank} \begin{pmatrix} \mu I_n - F \\ H \end{pmatrix} = n$$
   for all $\mu \in \mathbb{C}$

6. The polynomial matrices $sI - F$ and $H$ are right coprime.

### Canonical Forms

A nonreachable system can be decomposed in a canonical way into two subsystems; one whose states are all reachable and a second whose states are all unreachable. The precise result is stated next.

**Lemma 1.** Reachable canonical decomposition. Given is

$$\Sigma = \left(\begin{array}{c|c} F & G \\ \hline & \end{array}\right)$$

There exists a basis in $X$ such that $F$, $G$ have the following matrix representations:

$$\left(\begin{array}{c|c} F & G \\ \hline & \end{array}\right) = \left(\begin{array}{cc|c} F_r & F_{r\bar{r}} & G_r \\ 0 & F_{\bar{r}} & 0 \\ \hline & & \end{array}\right) \tag{26}$$

where the subsystem

$$\Sigma_r := \left(\begin{array}{c|c} F_r & G_r \\ \hline & \end{array}\right)$$

is reachable.

Thus every system

$$\Sigma = \left(\begin{array}{c|c} F & G \\ \hline & \end{array}\right)$$

can be decomposed in a subsystem

$$\Sigma_r = \left(\begin{array}{c|c} F_r & G_r \\ \hline & \end{array}\right)$$

which is reachable, and in a subsystem

$$\Sigma_{\bar{r}} = \left(\begin{array}{c|c} F_{\bar{r}} & 0 \\ \hline & \end{array}\right)$$

which is completely unreachable, that is, it cannot be influenced by outside forces. The interaction between $\Sigma_r$ and $\Sigma_{\bar{r}}$ is given by $F_{r\bar{r}}$. Since $F_{\bar{r}r} = 0$, it follows that the unreachable subsystem $\Sigma_{\bar{r}}$ influences the reachable subsystem $\Sigma_r$ but not vice versa. It should be noticed that although the direct complement $X'$ of $X^{\text{reach}}$ is not unique, the form of the reachable decomposition of Eq. (26) is unique.

Since by duality

$$\left(\begin{array}{c|c} F & G \\ \hline & \end{array}\right)$$

is reachable if, and only if,

$$\left(\begin{array}{c|c} F^* & \\ \hline G^* & \end{array}\right)$$

is observable, we obtain the following results.

**Lemma 2** Observable canonical decomposition. Given is

$$\Sigma = \left(\begin{array}{c|c} F & \\ \hline H & \end{array}\right)$$

There exists a basis in $X$ such that $F$, $H$ have the following matrix representations

$$\left(\begin{array}{c|c} F & \\ \hline H & \end{array}\right) = \left(\begin{array}{cc|c} F_{\bar{o}} & F_{\bar{o}o} & \\ 0 & F_o & \\ \hline 0 & H_o & \end{array}\right)$$

where

$$\Sigma_o = \left(\begin{array}{c|c} F_o & \\ \hline H_o & \end{array}\right)$$

is observable.

The reachable and observable canonical decompositions given in lemmas 1 and 2 can be combined to obtain the following decomposition of the triple $(H, F, G)$:

**Lemma 3.** Reachable-observable canonical decomposition. Given

$$\Sigma = \left(\begin{array}{c|c} F & G \\ \hline H & \end{array}\right)$$

there exists a basis in $X$ such that $F$, $G$, and $H$ have the following matrix representations

$$\Sigma = \left(\begin{array}{c|c} F & G \\ \hline H & \end{array}\right) = \left(\begin{array}{cccc|c} F_{r\bar{o}} & F_{12} & F_{13} & F_{14} & G_{r\bar{o}} \\ 0 & F_{ro} & 0 & F_{24} & G_{ro} \\ 0 & 0 & F_{\bar{r}\bar{o}} & F_{34} & 0 \\ 0 & 0 & 0 & F_{\bar{r}o} & 0 \\ \hline 0 & H_{ro} & 0 & H_{\bar{r}o} & \end{array}\right) \tag{27}$$

where the triple

$$\Sigma_{ro} := \left(\begin{array}{c|c} F_{ro} & G_{ro} \\ \hline H_{ro} & \end{array}\right)$$

is both reachable and observable.

A concept related to, but weaker than reachability, is that of stabilizability. Its dual is detectability.

***Definition 8.*** The pair

$$\left(\begin{array}{c|c} F & G \\ \hline & \end{array}\right)$$

is stabilizable iff in the reachable canonical decomposition, $F_{\bar{r}}$ is stable, that is, all its eigenvalues have either negative real parts or are inside the unit disk, depending on whether we are dealing with continuous- or discrete-time systems.

$$\left(\begin{array}{c|c} F & \\ \hline H & \end{array}\right)$$

is detectable iff in the observable canonical decomposition, $F_o$ is stable.

### The Infinite Grammians

Consider a continuous-time linear system

$$\Sigma_c = \left(\begin{array}{c|c} F & G \\ \hline H & \end{array}\right)$$

which is stable, that is, all eigenvalues of $F$ have negative real parts. In this case both Eqs. (12) as well as (21) are defined for $t = \infty$;

$$\mathscr{P} := \int_0^\infty e^{F\tau} GG^* e^{F^*\tau} d\tau, \quad \mathscr{Q} := \int_0^\infty e^{F^*\tau} H^* H e^{F\tau} d\tau \tag{28}$$

$\mathscr{P}$, $\mathscr{Q}$ are the infinite reachability and infinite observability grammians associated with $\Sigma_c$. These grammians satisfy the following linear matrix equations, called *Lyapunov equations.*

**Proposition 5.** Given the stable, continuous-time system $\Sigma_c$ as stated, the associated infinite grammians $\mathscr{P}$, $\mathscr{Q}$ satisfy the continuous-time Lyapunov equations

$$F\mathscr{P} + \mathscr{P}F^* + GG^* = 0, \quad F^*\mathscr{Q} + \mathscr{Q}F + H^*H = 0 \tag{29}$$

If the discrete-time system

$$\Sigma_d = \left(\begin{array}{c|c} F & G \\ \hline H & \end{array}\right)$$

is stable, that is, all eigenvalues of $F$ are inside the unit disk, the grammian Eqs. (13) as well as (22) are defined for $t = \infty$

$$\mathscr{P} := \sum_{t>0} F^{t-1} GG^* (F^*)^{t-1}, \quad \mathscr{Q} := \sum_{t>0} (F^*)^{t-1} H^* H F^{t-1} \tag{30}$$

Notice that $\mathscr{P}$ can be written as $\mathscr{P} = GG^* + F\mathscr{P}F^*$; moreover $\mathscr{Q} = H^*H + F^*\mathscr{Q}F$. These are the so-called *discrete Lyapunov* or *Stein* equations:

**Proposition 6.** Given the stable, discrete-time system $\Sigma_d$ as stated, the associated infinite grammians $\mathscr{P}$, $\mathscr{Q}$ satisfy the discrete-time Lyapunov equations

$$F\mathscr{P}F^* + GG^* = \mathscr{P}, \quad F^*\mathscr{Q}F + H^*H = \mathscr{Q} \tag{31}$$

Recall Eqs. (18), (19), and (24), valid for both discrete- and continuous-time systems. From the definition of the grammians follows that:

$$\mathscr{P}(t_2) \geq \mathscr{P}(t_1), \; \mathscr{Q}(t_2) \geq \mathscr{Q}(t_1), \; t_2 \geq t_1 \tag{32}$$

irrespective of whether we are dealing with discrete- or continuous-time systems. Hence from Eq. (19) it follows that the minimal energy $\mathscr{E}_r$, required for the transfer of state 0 to $x$, is obtained as the alloted time $T$ tends to infinity. Assuming reachability, this minimal energy is:

$$\mathscr{E}_r^2 = x^* \mathscr{P}^{-1} x \tag{33}$$

Similarly, the largest observation energy $\mathscr{E}_o$, produced by the state $x$ is also obtained for an infinite observation interval, and is equal to:

$$\mathscr{E}_o^2 = x^* \mathscr{Q} x \tag{34}$$

We summarize these results in the following proposition which is important in the theory of balanced representations and Hankel-norm model reduction.

**Lemma 4.** Let $\mathscr{P}$ and $\mathscr{Q}$ denote the infinite grammians of a linear stable system.

(a) The minimal energy required to steer the state of the system from 0 to $x$ is given by Eq. (33).
(b) The maximal energy produced by observing the output of the system whose initial state is $x$ is given by Eq. (34).
(c) The states which are difficult to reach, that is, require large amounts of energy, are in the span of those eigenvectors of $\mathscr{P}$ which correspond to large eigenvalues. Furthermore, the states which are difficult to observe, that is, produce small observation energy, are in the span of those eigenvectors of $\mathscr{Q}$ which correspond to small eigenvalues.
(d) The eigenvalues of the product of the reachability and of the observability grammians are input-output invariants called Hankel singular values of $\Sigma$.

### Controllability in the Behavioral Framework

A dynamical system in the classical framework is a mapping which transforms inputs $u$ into outputs $y$. In many cases however, the distinction between inputs and outputs is not a priori clear. Consider for example the RLC (Resistor–Inductor–Capacitor) network presented in the next section, and suppose that we are interested in the relationship between the current through the resistor $R_C$ and the voltage across the capacitor $C$. Is the voltage causing the current or vice versa? Other than the often encountered inherent difficulty in distinguishing between inputs and outputs, it is desirable to have a framework in which the different representations of a given system (for example: input-output and input-state-output) are treated in a unified way.

The need for a framework at a more abstract level than is provided by the input–output framework gave rise to the behavioral framework. For a tutorial account see Refs. (1) and (10). The variables considered are the external or manifest variables $w$ and (possibly) a second set of variables, the so-called latent variables $a$. The manifest variables consist of $u$ and $y$, without distinguishing between them. In the behavioral theory, a dynamical system is defined as a collection $\mathscr{B}$ of trajectories $w$. This set $\mathscr{B}$, called the *behavior* of the system, is the primary object of study for system and control theoretic issues.

In this section we will provide an overview of controllability in the behavioral framework. For further details and proofs, see Polderman and Willems (1).

The trajectories $w$ composing the behavior $\mathscr{B}$ are most often represented as solutions of appropriate equations, called *behavioral equations;* these are equations providing relationships between $w$ and $\sigma w$, where $\sigma$ is defined by Eq. (1). The most important type of such equations are the annihilating behavioral equations. An important special case of such behavioral equations are: state variable (SV) equations; SV equations, in addition to $\mathbf{w}$, make use of the latent variables $a = \mathbf{x}$, which can be assigned the property of state, and are called state variables. Another is the input–output equation representation, which thus appears as a special case of a more general system representation.

For linear, time-invariant systems, annihilating behavioral equations representing $\Sigma$ have the following form. Let $\mathbb{R}[s]$ denote the ring of polynomials in the indeterminate $s$ with coefficients in $\mathbb{R}$, and $\mathbb{R}^{n_1 \times n_2}[s]$ denote the $n_1 \times n_2$ polynomial matrices. The resulting equation has the form:

$$R(\sigma)w = M(\sigma)a,\ R \in \mathbb{R}^{p\times q}[s],\ M \in \mathbb{R}^{p\times r}[s] \tag{35}$$

It relates

$$\begin{pmatrix} w \\ a \end{pmatrix} \in \mathcal{B}$$

to its shifts or derivatives. This equation can be written explicitly in terms of the coefficient matrices of $R$ and $M$. Let

$$R(s) := \sum_{i=0}^{\ell_1} R_i s^i,\ R_i \in \mathbb{R}^{p\times q},\ M(s) := \sum_{i=0}^{\ell_2} M_i s^i,\ M_i \in \mathbb{R}^{p\times r}$$

Equation (35) becomes:

$$\sum_{i=0}^{\ell_1} R_i(\sigma^i w)(t) = \sum_{i=0}^{\ell_2} M_i(\sigma^i a)(t)$$

Since the differential or difference operator $[R(\sigma) \quad -M(\sigma)]$ annihilates all trajectories

$$\begin{pmatrix} w \\ a \end{pmatrix} \in \mathcal{B}$$

Eq. (35) is referred to as an annihilating behavioral equation. The special case of SV Eq. (4) is described by $w = u$, $a = x$, $R(\sigma) = G$ and $M(s) = \sigma I - F$; while that of input/output equations is described by

$$w = \begin{pmatrix} u \\ y \end{pmatrix}$$

$a$ is nonexistent, and $R(\sigma) = [N(\sigma) \quad -D(\sigma)]$, where $D$ is a square, nonsingular polynomial matrix of size $p$.

A further important aspect at which the behavioral formalism departs from, and generalizes, the classical formalism is that related to controllability; controllability becomes namely an attribute of the *system* (i.e., of a collection of trajectories) as opposed to an attribute of a *system representation* (i.e., of equations generating these trajectories).

Roughly speaking, a system is *controllable* if its behavior has the property: whatever the past history (trajectory), it can always be steered to any desired future trajectory. More precisely, a dynamical system with behavior $\mathcal{B}$ is said to be controllable, if for any $\mathbf{w}_1, \mathbf{w}_2 \in \mathcal{B}$, there exists a $t' > 0$ and a $\mathbf{w} \in \mathcal{B}$ such that

$$\mathbf{w}(t) = \begin{cases} \mathbf{w}_1(t) \text{ for } t < 0 \\ \mathbf{w}_2(t) \text{ for } t > t' \end{cases}$$

In terms of annihilating behavioral equation representations such as Eq. (35), the corresponding system is controllable if, and only if, the rank of the (constant) matrix

$$\operatorname{rank}\,[R(\lambda) \ -M(\lambda)] = \text{constant},\ \forall \lambda \in \mathbb{C} \tag{36}$$

From this follows the condition for controllability of the SV and input/output models. There holds, respectively:

$$\operatorname{rank}\,[\mu I - F\ G] = \text{constant} = n,$$
$$\operatorname{rank}\,[N(\mu)\ D(\mu)] = \text{constant} = p,\ \forall \mu \in \mathbb{C}$$

Notice that the first condition stated here is the same as condition 5 of theorem 2. Thus the behavioral definition of controllability provides a generalization of the classic concept. Furthermore the second condition provides a way of defining controllability without the definition of state.

### Examples

1. The previous issues are illustrated by means of two examples. First, consider an RLC circuit composed of the parallel connection of two branches: the first branch is composed of an inductor $L$ in series with a resistor $R_L$; the second is composed of a capacitor $C$ in series with a resistor $R_C$. The driving force $u$ is a voltage source applied to the two branches. Let the state variables $x_1$, $x_2$ be the current through the inductor, the voltage across the capacitor respectively. The state equations are

$$\left.\begin{aligned} \frac{d}{dt}x_1 &= -\frac{R_L}{L}x_1 + \frac{1}{L}u \\ \frac{d}{dt}x_2 &= -\frac{1}{R_C C}x_2 + \frac{1}{R_C C}u \end{aligned}\right\} \Rightarrow F = \begin{pmatrix} -\frac{R_L}{L} & 0 \\ 0 & -\frac{1}{R_C C} \end{pmatrix}$$

$$G = \begin{pmatrix} \frac{1}{L} \\ \frac{1}{R_C C} \end{pmatrix}$$

Since this is a continuous-time system, reachability and controllability are equivalent notions. The reachability matrix is

$$\mathcal{R}_2 = [G \quad FG] = \begin{pmatrix} \frac{1}{L} & -\frac{R_L}{L^2} \\ \frac{1}{R_C C} & -\frac{1}{R_C^2 C^2} \end{pmatrix}$$

Thus, reachability of this system is equivalent with the nonsingularity of $\mathcal{R}_2$, that is,

$$\det \mathcal{R}_2 = \frac{1}{R_C C L}\left(\frac{R_L}{L} - \frac{1}{R_C C}\right) \neq 0$$

It readily follows that the reachability Grammian is

$$\mathcal{P}(T) = \begin{pmatrix} \frac{g_1^2}{2\tau_L}(1 - e^{-2\tau_L T}) & \frac{g_1 g_2}{\tau_L + \tau_C}(1 - e^{-(\tau_L+\tau_C)T}) \\ \frac{g_1 g_2}{\tau_L + \tau_C}(1 - e^{-(\tau_L+\tau_C)T}) & \frac{g_2^2}{2\tau_C}(1 - e^{-2\tau_C T}) \end{pmatrix}$$

where

$$\tau_L = \frac{R_L}{L},\ \tau_C = \frac{1}{R_C C},\ g_1 = \frac{1}{L},\ g_2 = \tau_C$$

Hence the infinite reachability grammian is

$$\mathcal{P} = \begin{pmatrix} \frac{g_1^2}{2\tau_L} & \frac{g_1 g_2}{\tau_L + \tau_C} \\ \frac{g_1 g_2}{\tau_L + \tau_C} & \frac{g_2^2}{2\tau_C} \end{pmatrix}$$

and it can be verified that $F\mathcal{P} + \mathcal{P}F^* + GG^* = 0$.

Assume now that the variable observed $y$ is the sum of the voltages across the capacitor and across the resistor $R_L$:

$$y = R_L x_1 + x_2 \Rightarrow H = [R_L \quad 1]$$

The observability matrix is

$$\mathcal{O}_2 = \begin{pmatrix} R_L & 1 \\ -\frac{R_L^2}{L} & -\frac{1}{R_C C} \end{pmatrix} \Rightarrow \det \mathcal{O}_2 = R_L R_C L C \det \mathcal{R}_2$$

Thus reachability and observability are lost simultaneously. If this happens, then one can reach any given state $x_1$ and $x_2 = L/R_L\, x_1$; while only the linear combination $R_L x_1 + x_2$ can be deduced, but not $x_1$ or $x_2$ individually.

2. Consider the system given in input/output form by the equation

$$\frac{d}{dt}y = \frac{d}{dt}u,\ u(t), y(t) \in \mathbb{R}$$

We will show that this system is not controllable.

This equation can be rewritten as $d/dt\ v = 0$, where $v = y - u$. All trajectories composing the behavior of this system are constants. But a trajectory defined by $v(t) = c_1$, $t < T$, and $v(t) = c_2$, $t \geq T$, $c_1 \neq c_2$, does not belong to the behavior, as its derivative is not zero. Hence, since the trajectories of this system are not concatenable, the conclusion follows.

## NONLINEAR FINITE-DIMENSIONAL SYSTEMS

We turn attention now to nonlinear systems. Both continuous- and discrete-time systems will be discussed.

### Controllability: Continuous-Time

In continuous time, we consider systems of differential equations of the following general form:

$$\frac{d}{dt}x(t) = f(x(t), u(t)) \tag{37}$$

where $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ is a vector function which specifies, for the current state variables $x(t) \in \mathbb{R}^n$ and the current values of the control variables $u(t) \in \mathbb{R}^m$, the direction of instantaneous movement. For each fixed vector $u \in \mathbb{R}^m$, $f(\cdot, u)$ is thought of as a vector field. Linear (time-invariant) systems are a particular case, namely those systems for which $f$ is a linear map,

$$f(x, u) = Fx + Gu$$

for some matrices $F$ of size $n \times n$ and some matrix $G$ of size $n \times m$.

**An Example.** A simplified model of a front-wheel drive automobile uses a four-dimensional state space. The coordinates of the state

$$x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \\ \varphi \\ \theta \end{pmatrix}$$

denote, respectively, the position of the center of the front axle (coordinates $(x_1, x_2)$), the orientation of the car (angle $\varphi$, measured counterclockwise from the positive $x$-axis), and the angle of the front wheels relative to the orientation of the car ($\theta$, also counterclockwise); see Figure 1.

As controls, we take two-dimensional vectors $u = \text{col}\,(u_1, u_2)$, whose coordinates are proportional to the steering wheel velocity ($u_1$) and the engine speed ($u_2$) at each instant. Thus, a control $u_2(t) \equiv 0$ corresponds to a pure steering move, while one with $u_1(t) \equiv 0$ models a pure driving move in which the steering wheel is fixed in one position. In general, a control is a function $u(t)$ which indicates, at each time $t$, the current steering velocity and engine speed.

Using elementary trigonometry, the following equations are obtained (choosing units so that the distance between the

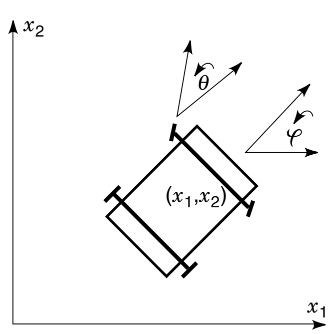


**Figure 1.** Four-dimensional car model.

front and rear axles is unity):

$$\frac{d}{dt}x = u_1 \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} + u_2 \begin{pmatrix} \cos(\varphi+\theta) \\ \sin(\varphi+\theta) \\ \sin\theta \\ 0 \end{pmatrix} \tag{38}$$

This is of the form of Eq. (37), where $f:\mathbb{R}^4 \times \mathbb{R}^2 \to \mathbb{R}^4$.

Note that, in practice, the angle $\theta$ would be restricted to some maximal interval $(-\theta_0, \theta_0)$. For simplicity of exposition, we do not impose this constraint. Similarly, the orientation angle $\varphi$ only makes sense as a number modulo $2\pi$, that is, angles differing by $2\pi$ correspond to the same physical orientation. Nonlinear control theory is usually developed in far more generality than we do here. The more general formalism allows states to evolve in general differentiable manifolds, instead of insisting, as we do here, in Euclidean state spaces. Thus, for instance, a more natural state space than $\mathbb{R}^4$ would be, for this example, $\mathbb{R}^2 \times \mathbb{S}^1 \times (-\theta_0, \theta_0)$, that is, the angle $\varphi$ is thought of as an element of the unit circle. Analogously, we assume here that the controls may attain arbitrary values in $\mathbb{R}^2$; of course, a more realistic model would also incorporate constraints on their magnitude.

**Technical Assumptions.** We will assume, for simplicity of exposition, that the function $f$ is real-analytic. This means that $f(x, u)$ can be expressed, around each point $(x, u) \in \mathbb{R}^n \times \mathbb{R}^m$, as a locally convergent power series. Analyticity is a condition which is satisfied in models derived from physical principles. The assumption of analyticity allows stating results in an elegant necessary and sufficient, rather than merely sufficient, manner. A control (or input) function is by definition of Lebesgue-measurable essentially bounded function $u(\cdot)$ defined on some interval of the form $[0, T]$ and taking values in $\mathbb{R}^m$ (the reader may substitute "piecewise continuous function" without much loss of generality). We let Inp be the set of such controls.

**Accessibility.** Consider any state $\xi \in \mathbb{R}^n$. For each control $u:[0, T] \to \mathbb{R}^m$, we consider the solution of the initial problem $d/dt\, x(t) = f(x(t), u(t))$, $x(0) = \xi$. If this solution is well-defined on the interval $t \in [0, T]$, we denote the final state $x(T)$ as $\phi(\xi, u)$. [If the solution does not exist on the entire interval, we do not define $\phi(\xi, u)$]. The reachable set from $\xi$, denoted as $\mathscr{R}(\xi)$, is by definition the set of states reachable from the origin, that is, the set of states of the form

$$\{\phi(\xi, u) \in \mathbb{R}^n \,|\, u \in \text{Inp}\}$$

***Definition.*** The system Eq. (37) is accessible from the state $\xi$ if the reachable set $\mathscr{R}(\xi)$ contains an open subset of $\mathbb{R}^n$ (that is, it has a nonempty interior).

For linear systems $d/dt\, x = Fx + Gu$, and the zero initial state $\xi = 0$,

$$\mathscr{R}(0) = \text{im}\,(G, FG, \ldots, F^{n-1}G) \tag{39}$$

(where "im" indicates the span of the columns of the matrix) is a linear subspace. so accessibility from 0 is equivalent to complete controllability, that is, $\mathscr{R}(0) = \mathbb{R}^n$. However, for nonlinear systems, the accessibility property is weaker. For example, consider the system with equation $d/dt\, x = u^2$ (having dimension $n = 1$ and input space also of dimension $m = 1$). Take any initial state $\xi$ (for instance, $\xi = 0$). Clearly, $\mathscr{R}(\xi) = [\xi, +\infty)$ [because $d/dt\, x(t) \geq 0$, no matter which control function is applied]. Thus the system is accessible from every state. Observe that, as illustrated by this example, accessibility does not mean that $\xi$ must be in the interior of the set $\mathscr{R}(\xi)$ (local controllability), even for an equilibrium state.

The reason that the accessibility question is studied is that it is far easier to characterize than controllability.

**Accessibility Rank Condition.** Given any two vector fields $f$ and $g$, one can associate the new vector field

$$[f, g]$$

defined by the formula

$$[f, g](x) := g_*[x]f(x) - f_*[x]g(x)$$

where, in general, the notation $h_*[x]$ means the Jacobian of a vector field $h$, evaluated at the point $x$. This is called the *Lie bracket* of $f$ and $g$.

The Lie bracket of $f$ and $g$ can be interpreted in terms of certain trajectories that arise from integrating $f$ and $g$, as follows. We let $e^{th}\xi$ denote the solution at time $t$ (possibly negative) of the differential equation $d/dt\, x = h(x)$ with initial value $x(0) = \xi$. (When the differential equation is linear, i.e., $d/dt\, x = Fx$ and $F$ is a matrix, $e^{th}\xi$ is precisely the same as $e^{tF}\xi$, where $e^{tF}$ is the exponential of the matrix $F$. In general, for nonlinear differential equations, $e^{th}\xi$ is merely a convenient notation for the flow associated to the vector field $h$.) Then, for any two vector fields $f$ and $g$,

$$e^{-tg}e^{-tf}e^{tg}e^{tf}\xi = e^{t^2[f,g]}\xi + o(t^2)(\xi) \tag{40}$$

as $t \to 0$, as a simple computation shows. Therefore, one may understand the Lie bracket of $f$ and $g$ as the infinitesimal direction that results from following solutions of $f$ and $g$ in positive time, followed by $f$ and $g$ in negative time. Another way to state this fact is by introducing the curve

$$\gamma(t) := e^{-\sqrt{t}g}e^{-\sqrt{t}f}e^{\sqrt{t}g}e^{\sqrt{t}f}\xi$$

Observe that $\gamma(0) = \xi$ and that the values of $\gamma(t)$ are all in the set of points $S$ attainable by positive and negative time solutions of the differential equations corresponding to $f$ and $g$; the above expansion implies that $d/dt\, \gamma(0) = [f, g](\xi)$, that is, there is a curve in $S$ whose tangent is the Lie bracket of the two vector fields. Thus, Lie brackets provide new directions of infinitesimal movement in addition to those corresponding to $f$ and $g$ themselves (and their linear combinations).

Given now a system by Eq. (37), we consider, for each possible control value $u \in \mathbb{R}^m$, the following vector field:

$$f_u : \mathbb{R}^n \to \mathbb{R}^n : x \mapsto f(x, u)$$

The accessibility Lie algebra $\mathscr{L}$ associated to the system given by Eq. (37) is the linear span of the set of all vector fields that can be obtained, starting with the $f_u$'s and taking all possible iterated Lie brackets of them. For instance, if $u_1, u_2, u_3, u_4$ are any four control values, the vector field

$$[[f_{u_1}, [f_{u_2}, f_{u_3}]], [f_{u_3}, f_{u_4}]]$$

is in $\mathscr{L}$. The system Eq. (37) satisfies the accessibility rank condition at the state $\xi$ if the vector space

$$\mathscr{L}(\xi) := \{X(\xi), X \in \mathscr{L}\} \subseteq \mathbb{R}^n$$

has the maximal possible dimension, that is, $n$.

There is a special case, of interest because it appears very often in applications, especially to mechanical systems. This is the class consisting of systems for which $f(x, u)$ is affine in $u$. That is, the equations can be written as

$$\frac{d}{dt}x = g_0(x) + \sum_{i=1}^{m} u_i g_i(x) \tag{41}$$

for some vector fields $g_i$'s. It is easy to verify that, for such systems, $\mathscr{L}$ is the Lie algebra generated by taking all possible iterated Lie brackets starting from the $g_i$'s.

For example, consider a linear system

$$\frac{d}{dt}x = Fx + Gu$$

Here $g_0(x) = Fx$ is a linear vector field and the $g_i(x)$'s are the constant vector fields defined by the $m$ columns of the matrix $G$. It then follows that, for each state $\xi$, $\mathscr{L}(\xi)$ is the span of the vector $F\xi$ together with the columns of $G$, $FG$, . . ., $F^{n-1}G$. In particular, for $\xi = 0$, one has that $\mathscr{L}(0)$ is the same as the right-hand side of Eq. (39). Seen in that context, the following result, which is valid in general, is not surprising:

**Theorem.** The system Eq. (37) is accessible from $\xi$ if and only if the accessibility rank condition holds at $\xi$.

There is a subclass of systems for which far stronger conclusions can be drawn. This subclass includes all purely kinematic mechanical models. It is the class of affine systems without drift, that is, systems affine in $u$ (as in Eq. 41, but for which, in addition, $g_0 \equiv 0$). We say that a system is completely controllable if $\mathscr{R}(\xi) = \mathbb{R}^n$ for every $\xi \in \mathbb{R}^n$, that is to say, every state can be steered to every other state by means of an appropriate control action.

**Theorem.** A system affine without drift is completely controllable if and only if the accessibility rank condition holds at every state.

This characterization of controllability for systems without drift belongs more properly to classical Lie theory and differential geometry. As far as control theory is concerned, most interesting questions concern more general classes of systems as well as the design of explicit algorithms for controllability, sometimes imposing optimality constraints.

**The Car Example.** In the notations for systems affine in controls, we have the vector fields $g_1$ and $g_2$, which we call (following (11)) “steer” and “drive” respectively:

$$g_1 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \quad g_2 = \begin{pmatrix} \cos(\varphi+\theta) \\ \sin(\varphi+\theta) \\ \sin\theta \\ 0 \end{pmatrix}$$

Computing some brackets, we get the two new vector fields [again, we borrow our terminology from (11)]:

$$wriggle := [steer, drive] = \begin{pmatrix} -\sin(\varphi+\theta) \\ \cos(\varphi+\theta) \\ \cos\theta \\ 0 \end{pmatrix}$$

and

$$slide := [wriggle, drive] = \begin{pmatrix} -\sin\varphi \\ \cos\varphi \\ 0 \\ 0 \end{pmatrix}$$

(The bracket [wriggle, steer] equals drive, so it is redundant, in so far as checking the accessibility rank condition is concerned.) It turns out that these four brackets are enough to satisfy the accessibility test. Indeed, one computes

$$\det(\text{steer, drive, wriggle, slide}) \equiv 1$$

so there is accessibility from every state. Moreover, since this system is affine without drift, it is completely controllable. (Of course, it is quite obvious from physical reasoning, for this example, that complete controllability holds.)

Consider in particular the problem of accessibility starting from the special state $\xi = 0$ (corresponding to the problem of exiting from a “parallel parked” spot). For $\phi = \theta = 0$, wriggle is the vector (0, 1, 1, 0), a mix of sliding in the $x_2$ direction and a rotation, and slide is the vector (0, 1, 0, 0) corresponding to sliding in the $x_2$ direction. This means that one can in principle implement infinitesimally both of these motions. The wriggling motion is, based on the characterization of Lie brackets mentioned earlier, the one that arises, in a limiting sense, from fast repetitions of the following sequence of four basic actions:

$$steer - drive - reverse\ steer - reverse\ drive\ (*)$$

This is, essentially, what one does in order to get out of a tight parking space. Observe that wriggle(0) equals the sum of slide and rotate [a pure rotation, col(0, 0, 1, 0)]. Interestingly enough, one could also approximate the pure sliding motion in the $x_2$ direction: *wriggle, drive, reverse wriggle, reverse drive, repeat* corresponds to the last vector field.

Note that the term $t^2$ in Eq. (40) explains why many iterations of basic motions (*) are required in order to obtain a displacement in the wriggling direction: the order of magnitude $t^2$ of a displacement in time $t$ is much smaller than $t$.

***Remark.*** If the right-hand side $f$ in Eq. (37) is assumed merely to be infinitely differentiable, instead of analytic, the accessibility rank condition is still sufficient for accessibility, but it is not a necessary condition. Consider for instance the system on $\mathbb{R}^2$, with $\mathbb{R}^2$ also as control space, and having equations as follows:

$$\frac{d}{dt}x = u_1\begin{pmatrix}1\\0\end{pmatrix} + u_2\begin{pmatrix}0\\ \alpha(x_1)\end{pmatrix}$$

where $\alpha$ is the function defined by

$$\alpha(x) = e^{-1/x^2}$$

for $x > 0$, and $\alpha(x) \equiv 0$ for $x \leq 0$. This system is easily shown to be accessible—in fact, it is completely controllable (any state can be steered to any other state)—but the accessibility rank condition does not hold.

### Controllability: Discrete-Time

We next consider discrete time systems. These are described by difference equations analogous to those for Eq. (37):

$$x(t+1) = f(x(t), u(t)) \tag{42}$$

where $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ is a function. This function now plays the role of specifying the state at time $t + 1$, provided that the state at time $t$ was $x(t)$ and the control vector $u(t)$ was applied at that instant.

We again suppose that the function $f$ is real-analytic. Now the set of controls, denoted again Inp, is the set of all possible sequences $u(0), \ldots, u(T)$ consisting of vectors in $\mathbb{R}^m$. An additional assumption which we make for the discrete time system of Eq. (42) is that it is *invertible,* meaning that the map

$$f(\cdot, u)$$

is a diffeomorphism for each fixed $u$; in other words, this map is bijective and has a nonsingular differential at each point. Imposing invertibility simplifies matters considerably, and is a natural condition for equations that arise from the sampling of continuous time systems, which is one of the main ways in which discrete time systems appear in practice.

Accessibility is defined as in the continuous-time case, using the analogous definition of $\mathcal{R}(\xi)$. We discuss only the special case $\xi = 0$ (the general case is a bit more complicated), assuming that this state is in equilibrium for the system, that is,

$$f(0, 0) = 0$$

There is an analogue of the accessibility rank condition for discrete time systems, and this is discussed next.

The notation $f_u$ is as stated earlier, and in particular $f_0$ is the map $f(\cdot, 0)$. Recall that in the discrete case one assumes invertibility, so that the inverse maps $f_u^{-1}$ are well-defined and again analytic. For each $i = 1, \ldots, m$ and each $u \in \mathbb{R}^m$ let

$$X_{u,i}(x) := \left.\frac{\partial}{\partial\epsilon}\right|_{\epsilon=0} f_u \circ f_{u+\epsilon e_i}^{-1}(x)$$

where $e_i$ denotes the $i$th coordinate vector, and more generally for all $u$, $i$ and each integer $k \geq 0$ let

$$(Ad_0^k X_{u,i})(x) := \left.\frac{\partial}{\partial\epsilon}\right|_{\epsilon=0} f_0^k \circ f_u \circ f_{u+\epsilon e_i}^{-1} \circ f_0^{-k}(x)$$

The accessibility Lie algebra is now defined in terms of iterated Lie brackets of these vector fields, and the accessibility rank condition is defined in terms of this, analogously to the continuous time case. The main fact is, then, as follows.

**Theorem.** The system Eq. (36) is accessible from zero if and only if the accessibility rank condition holds at zero.

As in the continuous-time case, for linear (discrete time) systems, the condition reduces to the usual reachability test. The vectors $Ad_0^k X_{u,i}$ are in fact all of the type $F^k G u$, for vectors $u \in \mathbb{R}^m$.

### Accessibility and Controllability of Linearized Systems

It is easy to prove that, for both continuous and discrete time systems, if the linearization about an equilibrium point $\xi$ is controllable as a linear system, then the accessibility condition holds, and, in fact, the system is locally controllable, that is, $\xi$ is in the interior of $\mathcal{R}(\xi)$; see for example, Ref. 6. For instance, each state near zero can be reached from zero, for the system

$$\begin{aligned}\frac{d}{dt}x_1 &= x_1^2 x_2 + \sin x_1\\ \frac{d}{dt}x_1 &= -x_1 e^{x_2} + u\cos x_1\end{aligned}$$

because, up to first order around $\xi = 0$ one has $x_1^2 x_2 + \sin x_1 = x_1 + h_1$, $-x_1 e^{x_2} + u \cos x_1 = -x_1 + u + h_1$, where $h_1$ and $h_2$ are higher-order terms in states and controls, which means that the linearization at the equilibrium $\xi = 0$ is the linear system with matrices

$$F = \begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}, \quad G = \begin{pmatrix}0\\1\end{pmatrix}$$

which is controllable. This is only a sufficient condition. the system $d/dt\ x = u^3$, in dimension one, is clearly (even) completely controllable, but its linearization at $\xi = 0$ gives the noncontrollable linear system $d/dt\ x = 0$. A necessary and sufficient condition does exist linking accessibility and linear controllability, but it is more subtle. It is illustrated next, for simplicity, only for discrete-time systems. In continuous time, an analogous result holds, but it is slightly more complicated to state and prove (12).

Observe that accessibility from $\xi$ corresponds to the requirement that the union of the images of the composed maps

$$f_k(\xi, \cdot) : (\mathbb{R}^m)^k \to \mathbb{R}^n\ k \geq 0$$

cover an open subset, where we are denoting

$$f_k(x, (u_1, \ldots, u_k)) := f(f(\ldots f(f(x, u_1), u_2), \ldots, u_{k-1}), u_k)$$

for every state $x$ and sequence of controls $u_1, \ldots u_k$. A simple argument, based on a standard result in analysis (Sard's theorem) gives that accessibility is equivalent to the following property: there exists some positive integer $k$ and some sequence of controls $u_1, \ldots u_k$ so that the Jacobian of $f_k(\xi, \cdot)$ evaluated at that input sequence,

$$f_k(\xi, \cdot)_*[u_1, \ldots u_k],$$

has rank $n$. Consequently, accessibility is equivalent to accessibility in time exactly $k$. This Jacobian condition can be restated as follows: Consider the linearization of the system Eq. (42) along the trajectory

$$x_1 = \xi, x_2 = f(x_1, u_1), x_3 = f(x_2, u_2), \ldots$$

that is, the linear time-varying system

$$x(t+1) = F_t x(t) + G_t u(t)$$

with

$$F_t = \frac{\partial}{\partial x} f[x_t, u_t], \quad G_t = \frac{\partial}{\partial u} f[x_t, u_t]$$

Then, accessibility is equivalent to the existence of some sequence of controls $u_1, \ldots u_k$ for which this linearization is controllable as a time-varying linear system.

## Observability: Continuous-Time

We present a brief outline of a nonlinear observability test, for the special case of continuous-time systems affine in Eq. (41), with an output map $h: \mathbb{R}^n \to \mathbb{R}^p$ added to the system description. Two states $\xi$ and $\zeta$ are *distinguishable* by input/output experiments if there is at least some input which, when applied to the system in initial state $\xi$, gives a different output than when applied to the system in state $\zeta$. An *observable* system is one with the property that every pair of distinct states is distinguishable. Thus an observable system is one for which, at least in principle, it is possible to distinguish between internal states by means of input/output measurements alone.

Consider the vector space spanned by the set of all functions of the type

$$L_{g_{i_1}} \ldots L_{g_{i_k}} h_j(x) \tag{43}$$

over all possible sequences $i_1, \ldots, i_k$, $k \geq 0$, out of $\{0, \ldots, m\}$ and all $j = 1, \ldots, p$, where $L_g\alpha = \nabla\alpha.g$ for any function $\alpha$ and any vector field $g$ ($\nabla f$ denotes the gradient of $f$). This is called the *observation space,* which we denote as $\mathcal{O}$, associated to the system. We say that two states $x_1$ and $x_2$ are *separated* by $\mathcal{O}$ if there exists some $\alpha \in \mathcal{O}$ such that $\alpha(x_1) \neq \alpha(x_2)$.

One can prove that if two states are separated by $\mathcal{O}$ then they are distinguishable. A sketch of the argument is as follows. Assume that $\xi_1$ is indistinguishable from $\xi_2$ and consider a piecewise constant control which is equal to $u^1$ on $[0, t_1)$, equal to $u^2$ on $[t_1, t_1 + t_2)$, . . ., and equal to $u^k$ on $[t_1 + \cdots + t_{k-1}, t_1 + \cdots + t_k)$. By indistinguishability, we know that the resulting output at time $t = t_1 + \cdots + t_k$ is equal for both. In general, we denote the $j$th coordinate of this output value by

$$h_j(t_1, t_2, \ldots, t_k, u^1, u^2, \ldots, u^k, \xi) \tag{44}$$

if the initial state is $\xi$. It follows that the derivatives with respect to the $t_i$'s of this output are also equal, for $\xi_1$ and $\xi_2$, and for every such piecewise constant control. One may prove by induction that

$$\left.\frac{\partial^k}{\partial t_1 \ldots \partial t_k}\right|_{t_1=t_2=\ldots=0} h_j(t_1, t_2, \ldots, t_k, u^1, u^2, \ldots, u^k, \xi) = L_{X_1} L_{X_2} \ldots L_{X_k} h_j(\xi)$$

where $X_l(x) = g_0(x) + \sum_{i=1}^m u_i^l g_i(x)$. This expression is a multilinear function of the $u_i^l$'s, and a further derivation with respect to these control value coordinates shows that the generators in Eq. (43) must coincide at $x_1$ and $x_2$. It turns out that (for analytic vector fields, as considered in this exposition), separability by $\mathcal{O}$ is necessary as well as sufficient, because Eq. (44) can be expressed as a power series in terms of the generators of Eq. (43). Thus, observability is equivalent to separation by the functions in $\mathcal{O}$.

The *observability rank condition* at a state $\xi$ is the condition that the dimension of the span of

$$\{\nabla L_{g_{i_1}} \ldots L_{g_{i_k}} h_j(\xi) | i_1, \ldots, i_k \in \{0, \ldots, m\}, j = 1, \ldots, p\} \subseteq \mathbb{R}^n$$

be $n$. An application of the implicit function theorem shows that this is sufficient for the distinguishability of states near $\xi$.

## Remarks

The early 1970s saw the beginnings of the systematic study of controllability and observability questions for continuous time nonlinear systems. Building upon previous work (13,14) on partial differential equations, the papers (15), (16), and (17), among others, provided many of the basic accessibility and controllability results. In discrete time, one of the early papers was (18). For more details on accessibility at an expository level, see for instance Refs. 19, 20, 21, or 6 in continuous time, and 22 in discrete time. These references should also be consulted for justifications of all statements given here without proof. For affine systems without drift, for which the accessibility rank condition completely characterizes controllability, Lie techniques can be used to provide efficient algorithms for constructing controls; see for instance Ref. 23.

Similarly, there are useful controllability algorithms available for special classes of systems such as so-called "flat systems," and for systems exhibiting special symmetries which arise from mechanical constraints; see, e.g., Ref. 24, and references therein.

A complete characterization of controllability, as opposed to the weaker accessibility property, has eluded solution, even though substantial progress has been made (see for instance Ref. 25 and the references there). One way to understand the difficulty inherent in checking controllability is by formulating the problem in terms of computational complexity. In Ref. 26, it is shown that, for wide classes of systems, testing for controllability is an NP-hard problem (hence most likely impossible to ever be amenable to an efficiently computable characterization); this contrasts with accessibility, which, for the same classes of systems, can be checked in polynomial time.

A different type of "linearization" is also of interest in control theory. Instead of merely taking first-order approximations, one searches for changes of state and input variables which render a nonlinear system linear. For example, take the nonlinear system $\dot{x} = x^2 + u$. This system becomes linear if we consider the new input variable $v = x^2 + u$, since the equations become $\dot{x} = v$. Observe that this linearized system may be stabilized globally by means of the feedback $v = -x$. (In terms of the original variables, this means that we picked the feedback law $u = -x^2 - x$.) In general, finding linearizing transformations is not as obvious as in this example, of course; much theory has been devoted to the search for Lie-algebraic conditions which guarantee that a system can be so transformed, or as one says, *feedback linearized.* This line of work started with Brockett in the late 1970s, and major results were obtained by Jakubczyk, Respondek, Hunt, Su, and Meyer, in the early 1980s. See Ref. 6 for an elementary introduction to the subject. For some related recent developments, see Ref. 27.

## LINEAR INFINITE-DIMENSIONAL SYSTEMS

Distributed parameter systems are those where the state variable is spatially dependent. Typical examples are systems described by the heat equation, wave equation, beam equations, or delay-differential equations. Due to such a dependence on spatial variables, the state space generally becomes infinite-dimensional. The term "distributed parameter systems" is thus often used synonymously for systems with infinite-dimensional state space. In what follows, we shall also employ this convention, and discuss controllability/observability of infinite-dimensional systems.

A formal generalization of the finite-dimensional (continuous-time) definitions would lead to the following state equations:

$$\frac{d}{dt}x(t) = Fx(t) + Gu(t) \tag{45}$$

$$y(t) = Hx(t) + Ju(t) \tag{46}$$

where the input values $u(t)$, state $x(t)$, and output values $y(t)$ are elements of, for instance, Hilbert spaces. For example, if we consider the heat equation

$$\frac{\partial x}{\partial t}(t,\xi) = \frac{\partial^2 x}{\partial \xi^2}(t,\xi) + G(\xi)u(t)$$

with state space $L^2[0, 1]$ with boundary condition $x(t, 0) = x(t, 1) = 0$, then $F$ is set to be the differential operator $\partial^2/\partial\xi^2$ with domain

$$\mathscr{D}(F) := \{x \in H^2(0,1) : x(0) = x(1) = 0\}$$

Here $H^2(0, 1)$, which is the natural state space for these equations, is the space of functions in $L^2[0, 1]$ whose second-order derivatives again belong to $L^2[0, 1]$. The important point to notice is that the $F$ operator is in general, as in this example, not defined on the whole space, and is discontinuous (unbounded). Thus, some care has to be taken when extending the finite-dimensional framework. We need to assume that $F$ is the infinitesimal generator of a strongly continuous semigroup $S(t)$, which plays a role analogous to that played by $e^{Ft}$ in the finite-dimensional case. Typically, as in the previous example, $F$ is defined only on a domain $\mathscr{D}(F)$ which is a dense subspace of the state space $X$.

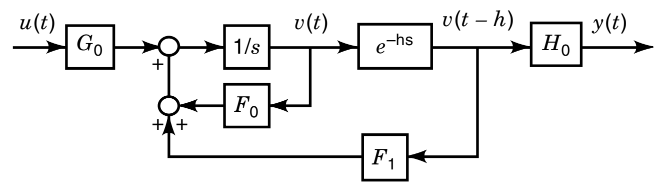


**Figure 2.** Retarded delay-differential system.

Another example often encountered in practice is provided by delay systems. Consider a retarded delay-differential system (see Fig. 2):

$$\begin{aligned}\frac{d}{dt}v(t) &= F_0 v(t) + F_1 v(t-h) + G_0 u(t)\\ y(t) &= H_0 v(t-h), \quad v(t) \in \mathscr{R}^n\end{aligned} \tag{47}$$

This system is described by the functional differential equation

$$\begin{aligned}\frac{d}{dt}\begin{bmatrix} v(t) \\ z_t(\cdot)\end{bmatrix} &= \begin{bmatrix} F_0 & F_1 E_{-h} \\ 0 & \dfrac{\partial}{\partial\theta}\end{bmatrix}\begin{bmatrix} v(t) \\ z_t(\theta)\end{bmatrix} + \begin{bmatrix} G_0 \\ 0\end{bmatrix} u(t)\\ &=: Fx(t) + Gu(t)\end{aligned} \tag{48}$$

$$y(t) = [0 \quad H_0 E_{-h}]\begin{bmatrix} x(t) \\ z_t(\cdot)\end{bmatrix}, \quad z_t(\cdot) \in (L^2[-h,0])^n \tag{49}$$

where $x(t) := [v(t)\ z_t(\cdot)]'$, and $z_t(\cdot)$ is the state in the delay element $e^{-hs}$ which is related to $v(t)$ via $z_t(\theta) = v(t + \theta)$. $E_{-h}$ denotes the point evaluation operator $E_{-h}z(\cdot) := z(-h)$. The domain of $F$ in Eq. (48) is $\{[v\ z(\cdot)]' \in \mathscr{R}^n \times (H^2(-h, 0))^n : z(0) = v\}$. Here the input operator $G$ is bounded but the output operator $C = [0\ C_0 E_{-h}]$ is not. (Point evaluation in $L^2$ cannot be continuous.)

We thus consider the abstract system Eqs. (45), (46) with $x(t)$ in a Hilbert space $X$, with input/output variables $u(t)$ and $y(t)$ being $\mathbb{R}^m$- and $\mathbb{R}^p$-valued. The operators $G : \mathbb{R}^m \to X$ and $H : X \to \mathbb{R}^p$ are often assumed to be bounded. However, there are many systems that do not satisfy this property: $G$ may not take values in $X$ or $H$ may be only densely defined. The system described by Eqs. (48), (49) is an example of the latter case. For brevity of exposition, however, we will not be very rigorous about this point in what follows. Also, since $J$ plays no role in controllability/observability, we will assume $J = 0$. Denote this system by $\Sigma = (X, F, G, H)$.

The solution of Eq. (45) can be written as

$$x(t) = S(t)x_0 + \int_0^t S(t-\tau)Gu(\tau)\,d\tau$$

We assume that $u$ is square integrable. Such a solution is often called the mild solution.

### Controllability and Observability

The technical problems that arise due to the infinite-dimensionality include the following:

- In general, canonical (controllable and observable) realizations are not unique
- Controllability need not guarantee stabilizability; similarly, observability need not imply existence of an observer

The problems involved are mostly topological, and this is one of the reasons why there are several nonequivalent controllability/observability notions. We start by introducing controllability concepts.

In analogy with the finite-dimensional case, one wants to say that a system is *controllable* (or *reachable*) if every state $x \in X$ can be reached from the origin by suitable application of an input. For distributed parameter systems, however, this does not usually occur due to the infinite-dimensionality of the state space.

For each $T > 0$, define a map $\Phi(T) : L^2[0, T] \to X$ by

$$\Phi(T)u := \int_0^T S(T-\tau)Gu(\tau)d\tau \tag{50}$$

We say that a state $x$ is *controllable from zero* in time $T$ if $x$ belongs to the range of $\Phi(T)$. The system $\Sigma$ is simply said to be *controllable from zero* if

$$x \in \bigcup_{T>0} \text{range } \Phi(T)$$

The system $\Sigma$ is said to be *exactly controllable (reachable)* if

$$X = \bigcup_{T>0} \text{range } \Phi(T) \tag{51}$$

Unlike the finite-dimensional case, exact controllability does not occur very often. In fact, when the $G$ operator is bounded (and takes values in $X$), it has finite rank, and hence it can be shown that $\Phi(T)$ is a compact operator. Thus the right-hand side of Eq. (51) cannot be equal to $X$ as the union of images of compact operators. (Of course, this argument does not apply when $G$ does not take values in $X$; extended study has been made on systems with boundary controls; see Refs. 28–30.)

We are thus interested in a less restrictive condition of approximate controllability (reachability). This requires that, in place of Eq. (51),

$$X = \overline{\bigcup_{T>0} \text{range } \Phi(T)} \tag{52}$$

Here $\overline{M}$ denotes the closure of $M$ in $X$. This means that any state $x \in X$ can be approximated arbitrarily closely by controllable elements. When $X$ is a finite-dimensional space, approximate controllability coincides with standard controllability.

We now give definitions for several notions of observability. Fix $T > 0$. Define $\Psi(T) : X \to L^2[0, T]$ by

$$\Psi(T) : x \mapsto HS(t)x, \quad 0 \le t \le T$$

Because of the strong continuity of $S(t)$, this mapping is well defined if $H$ is a bounded operator. If $H$ is unbounded, we generally require that this mapping be defined on a dense subspace of $X$ and is indeed extendable to $X$ as a continuous mapping. While $H$ is often unbounded, $\Psi(T)$ still happens to be continuous in many examples [for example, for delay systems such as Eqs. (48), (49)].

The system $\Sigma$ is said to be *observable in bounded time* $T > 0$ if $\Psi(T)$ is a one-to-one mapping, that is, if $\Psi(T)x = 0$ (almost everywhere) implies $x = 0$. It is *observable* if $\Psi(T)x = 0$ for all $T > 0$ occur only if $x = 0$. (This concept is often called *approximate observability,* indicating its duality to approximate controllability.) To state the observability condition differently, define the *observability map*

$$\Psi : X \to L^2_{\text{loc}}[0, \infty) : x \mapsto HS(t)x, \quad t \ge 0 \tag{53}$$

where $L^2_{\text{loc}}[0, \infty)$ is the space of locally square integrable functions. Then $\Sigma$ is observable if and only if $\Psi$ is one-to-one.

For finite-dimensional systems these observability notions imply the following consequences:

- Observability always implies observability in bounded time
- Initial states can be determined continuously from output data

The latter is an important key to constructing observers and state estimators. For infinite-dimensional systems, this property does not hold, and we define a topological notion of observability as follows.

The system $\Sigma$ is said to be *topologically observable* if $\Psi$ is continuously invertible when its codomain is restricted to im $\Psi$. It is *topologically observable in bounded time* $T > 0$ if the same holds of $\Psi(T)$. (For the same reason as in the case of approximate observability, some authors adopt the terminology *exact observability,* again indicating its duality to exact controllability.)

Topological observability requires that the initial state determination be well posed. The system given by Eqs. (48) and (49) is topologically observable. This property is also crucial in proving uniqueness of canonical realizations. That is, if we understand that a canonical realization is one that is approximately controllable and topologically observable, then there is essentially only one canonical realization for a given input/output behavior (31).

### Duality

Controllability and observability are dual concepts to each other. Reversing the time axis, we see that the controllable states are those in the image of the mapping

$$\Phi : \bigcup_{T>0} L^2[0, T] \to X : u \mapsto \int_0^\infty S(t)Gu(t)\,dt \tag{54}$$

The adjoint of $\Phi$ is easily computable (at least formally) as

$$\Phi^* : X \to L^2_{\text{loc}}[0, \infty) : x \mapsto G^*S^*(t)x, \quad t \ge 0 \tag{55}$$

Define $\Sigma^* := (X, F^*, H^*, G^*)$ as the *dual system* of $\Sigma$. Then the mapping of Eq. (55) is the observability map of $\Sigma^*$. Since a bounded linear $f : X \to Y$ has dense image if and only if its adjoint satisfies ker $f^* = \{0\}$, we can say that $\Sigma$ is approximately controllable if and only if $\Sigma^*$ is observable. Similarly,

$\Sigma$ is observable if and only if $\Sigma^*$ is approximately controllable. Also, $\Sigma$ is topologically observable if and only if the adjoint mapping of $\Psi$ in Eq. (53) is a surjective mapping.

### Related Concepts and Controllability Criteria

When $\Sigma$ is exponentially stable in the sense that there exist $M, \alpha > 0$ such that

$$\|S(t)\| \leq Me^{-\alpha t}$$

then it is possible to relate controllability/observability to Lyapunov equations. In this case it is possible to extend the domain of $\Phi$ and restrict the codomain of $\Psi$ to $L^2[0, \infty)$. Define

$$\mathscr{P} := \Phi\Phi^* = \int_0^\infty S(t)GG^*S^*(t)\,dt$$

$$\mathscr{Q} := \Psi\Psi^* = \int_0^\infty S^*(t)H^*HS(t)\,dt$$

Then $\Sigma$ is approximately controllable if and only if $\mathscr{P}$ is positive definite, and observable if and only if $\mathscr{Q}$ is positive definite. $\mathscr{P}$ and $\mathscr{Q}$ are called *controllability and observability Grammians.* Actually, when $\Sigma$ is observable, $\mathscr{Q}$ is a unique self-adjoint solution to the Lyapunov equation

$$\langle \mathscr{Q}x_1, Fx_2\rangle + \langle Fx_1, \mathscr{Q}x_2\rangle = -\langle Hx_1, Hx_2\rangle, \quad x_1, x_2 \in \mathscr{D}(F)$$

A similar statement holds for controllability.

*Null controllability* refers to the property that every state can be steered back to the origin by application of a suitable input. Its dual concept is the *final state observability* (or *reconstructibility*). Similar theory is possible for these properties.

Closely related concepts are those of stabilizability and detectability. However, in the infinite-dimensional context, controllability need not imply stabilizability. The same can be said of observability and detectability. With a finite-rank input term one may not have enough control freedom to stabilize possibly infinitely many unstable poles. Stabilizability often requires that there be only finitely many unstable poles.

One may also wish to say that a system is *spectrally controllable* if all its finite-dimensional modal subsystems are controllable (in the usual sense for finite-dimensional systems). Some care must be exercised to be clear about the meaning of modal subsystems. Even if there exists a decomposition to modal subsystems, some modes may not be finite-dimensional.

For systems where eigenfunction expansion is possible, one can say that the system is approximately controllable if its eigenfunctions are complete and it is spectrally controllable. Systems with self-adjoint $F$ operators are easy examples. Then several rank conditions are possible in testing controllability of each subsystem. Another class of systems that satisfy this condition is that of delay differential systems. For example, the retarded system of Eqs. (48), (49) is approximately controllable if and only if

1. $\text{rank}[\lambda I - F_0 - F_1e^{-h\lambda} \quad G_0] = n$ for every $\lambda \in \mathbb{C}$ and
2. $\text{rank}[F_1\ G_0] = n$

The first condition guarantees spectral controllability and the second guarantees that the system can be made eigenfunction complete with suitable feedback. Since feedback does not alter controllability, this latter condition preserves controllability.

Similar spectral tests are possible for systems with spectral $F$ operators. For example, consider the case where $F$ is the self-adjoint operator defined by a spectral representation (32):

$$Fx = \sum_{n=1}^\infty \lambda_n \sum_{j=1}^{r_n} (x, \phi_{n_j})\phi_{n_j}$$

where $\{\lambda_n : n = 1, 2, \ldots\}$ are distinct real numbers listed in decreasing order, $\{\phi_{n_j}, j = 1, 2, \ldots, r_n, n = 1, 2, \ldots\}$ is an orthonormal basis in $X$, and $(\cdot\,,\,\cdot)$ denotes the inner product. Let $G$ be given by

$$Gu := \sum_{i=1}^m g_i u_i, \quad g_i \in X$$

Then $(F, G, H)$ ($H$ is irrelevant here and hence not defined) is approximately controllable if and only if

$$\text{rank}\begin{pmatrix} (g_1, \phi_{n_1}) & \cdots & (g_m, \phi_{n_1}) \\ \vdots & & \vdots \\ (g_1, \phi_{n_{r_n}}) & \cdots & (g_m, \phi_{n_{r_n}}) \end{pmatrix} = r_n$$

### Remarks

Controllability/observability questions for distributed parameter systems were first addressed by Fattorini (33). Since then numerous papers have appeared, and the literature is too vast to be listed or surveyed here. For developments up to 1978, consult Russel's survey (34). The recent textbook by Curtain and Zwart (32) gives an extensive set of further references. For abstract operator settings of linear systems, see Refs. 35, 36, and 37. There is also a vast amount of literature concerning controllability of systems described by partial differential equations. For the developments along this line, consult Refs. 28, 29, and 30. We discuss here some limited references that deal with further related subjects.

Controllability/observability for delay systems has received considerable attention: see the work by Datko, Delfour, Langenhop, Kamen, Manitius, Mitter, O'Connor, Pandolfi, Salamon, Triggiani, Yamamoto, and others [references cited in (32)]. The spectral condition for approximate controllability of retarded delay systems cited previously is due to the work of Manitius and Triggiani (38,39). It is also extended to various situations including neutral systems; see, for example, Refs. 40, 41, and 42.

Controllability and observability are also important in realization theory. It was shown by Baras, Brockett, and Fuhrmann (43) that an approximately controllable and observable realization need not be unique. One needs a stronger condition to guarantee uniqueness; the results cited in Ref. 31 are one example.

As with finite-dimensional systems, controllability and observability are closely related to the question of coprimeness of factorizations of transfer functions. For example, a factorization is (left) coprime (in an appropriately defined sense) if a certain system associated with this factorization is approximately controllable (see, for example, Ref. 44). However, in contrast to the finite-dimensional context, there is a large variety of freedom in choosing the algebra over which the factorization is considered. Typical examples are the Callier-Desoer algebra, the space of $H^{\infty}$ functions, and the algebra of distributions with compact support. Each different choice leads to a different theory of realization/stabilization/controller parameterization as is the case with the finite-dimensional theory, but with much wider freedom. Each has its advantage in different contexts, reflecting the variety of distributed parameter systems. The theories can therefore hardly be expected to be complete.

Another topic that is in close connection with controllability/observability is the existence and uniqueness of solutions to Riccati equations. This topic is also related with $H^{\infty}$ control theory and under active research. The current trend is in spectral factorization in an abstract functional equation setting. To this end, various questions concerning the well-posedness of system equations and transfer functions arise and are currently under study.

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**CONTROL, LIGHTING.** See Lighting control.

**CONTROL, NYQUIST CRITERION.** See Nyquist criterion, diagrams, and stability.

**CONTROL OF EXCITATION.** See Excitation control in power systems.

**CONTROL OF INDUSTRIAL SYSTEMS.** See Industrial control.

**CONTROL OF JET ENGINES.** See Jet engine control, implementations.

**CONTROL OF PERIODIC SYSTEMS.** See Periodic control.

**CONTROL, OSCILLATORY.** See Open-loop oscillatory control.

**CONTROL, POWER SYSTEM.** See Power system control; Power system transients.

**CONTROL, PROCESS.** See Process control.

**CONTROL, RELAY.** See Relay control.

**CONTROL, ROBUST.** See Robust control; Robust control analysis.

**CONTROL SYSTEM ANALYSIS.** See Interval analysis for circuits; Servomechanisms.