J. Webster (ed.), *Wiley Encyclopedia of Electrical and Electronics Engineering*

INPUT-OUTPUT STABILITY

The earliest mathematical studies of control systems focused solely on the input-output stability of systems as described in the works of Black (1), Bode (2), and Nyquist (3). In fact, most of the classical control work was limited to the input-output study of single-input–single-output linear and mostly time-invariant systems. Notions of input-output stability for nonlinear systems were later advanced by Sandberg (4), Zames, (5,6) Safonov (7), and others. The first book dedicated completely to the subject was by Willems (8) in 1971, followed shortly by that of Desoer and Vidyasagar in 1975 (9). With the popularity of state-space methods, Lyapunov-stability concepts became the preferred analysis and design tools for nonlinear systems until the 1980s when researchers became interested again in the input-output behavior of systems. The relationships between the input-output and Lyapunov-stability concepts were developed in Refs. 10 and 11 and culminated in the various versions of the Kalman-Yakubovich-Popov (*KYP*) lemma (12). The current studies in input-output systems are highly dynamic with the introduction of new concepts such as input-to-state stability (13,14,15), the interaction with geometric nonlinear control (16), applications to robust control (16,17,18), research in the adaptive control for linear (19) and nonlinear systems (17), the interest in various mechanical and electric systems (20), and the publication of various theoretical and applied books (18,20,21). It is now clear that the two points of view (state-space and input-output) are complementary and that many deep relationships between the various stability concepts are yet to be explored.

In this article we concentrate our discussion on continuous-time systems and survey the classical as well as the more recent results in the input-output approach. The major results included in this article are taken mainly from Refs. 9,18,22,23,24, and the reader is referred to those books for most proofs. An excellent 1995 chapter on input-output stability from a distance-separation point of view appeared in Ref. 25. Related concepts for discrete-time systems were presented in Ref. 9 and recently revived by Byrnes and co-workers (26,27), but will not be discussed here. In addition, while we mention absolute stability as an important application area of input-output concepts, we refer the reader to the article on Absolute Stability in this encyclopedia for details.

The article starts with a collection of basic definitions followed by the general results on the basic concepts of input-output stability and results for testing input-output stability and its relationship with Lyapunov stability. Next, we discuss the stability of interconnected systems and present the small-gain and passivity results. Related concepts such as absolute stability, dissipativity, and input-to-state and input-to-output stability are then reviewed, followed by our conclusions. Various technical definitions are presented in the appendices. We have attempted to include the main references on the subject of input-output stability, striving to be current and relevant rather than encyclopedic.

Basic Concepts

The general ideas of input-output stability involve the relative "size" of signals as they are processed by dynamical systems. We will thus begin by providing mathematical measures of the size of signals and of the effect that a particular system has on that size.

2 INPUT-OUTPUT STABILITY

In order to introduce the mathematical notions of input-output stability, we need some preliminary definitions of signal spaces. (A detailed treatment of *measure*, *signal spaces*, and *signal norms* is beyond the scope of this article. The reader is referred, for example, to Ref. 28.) The $\mathcal{L}^m{}_p$ set, with $p \in [1, +\infty)$, consists of all functions u: $[0, +\infty) \to \mathbb{R}^m$ that are *Lebesgue* measurable [i.e., functions that are the limits (except for a set of measure zero) of a sequence of piecewise constant functions], such that

$$\int_0^{+\infty} \|u(t)\|_q^p dt < +\infty \tag{1}$$

where $\|v\|_q$ denotes the q-norm of the vector $v = (v_1\ v_2\ \cdots\ v_m)^T$, defined as

$$\|v\|_q = \begin{cases} (|v_1|^q + |v_2|^q + \cdots + |v_m|^q)^{1/q}, & q \in [1, +\infty) \\ \max\{|v_1|, |v_2|, \cdots, |v_m|\}, & q = +\infty \end{cases}$$

Remark 1. For the *finite-dimensional* set $\mathbb{R}^m$ all q-norms are equivalent, in the sense that for all k, $h \in [1, +\infty]$ there exist positive constants c_1 and c_2 such that

$$c_1\|x\|_k \le \|x\|_h \le c_2\|x\|_k$$

for all $x \in \mathbb{R}^m$. This is why, when defining the $\mathcal{L}_p{}^m$ set, we do not need to specify q, as $u(t) \in \mathbb{R}^m$ for any fixed $t \in [0, \infty)$.

The $\mathcal{L}^m{}_\infty$ set consists of all the functions u: $[0, +\infty) \to \mathbb{R}^m$ that are measurable and essentially bounded on $[0, +\infty)$ (where *essentially bounded* means that the function is bounded except on a set of measure zero). For a function belonging to $\mathcal{L}^m{}_p$, $p \in [1, +\infty)$, we introduce the norm

$$\|u(\cdot)\|_{\mathcal{L}_p} = \left(\int_0^{+\infty} \|u(t)\|_q^p dt\right)^{1/p} \tag{2}$$

whereas for a function belonging to $\mathcal{L}^m{}_\infty$ we introduce the norm

$$\|u(\cdot)\|_{\mathcal{L}_\infty} = \sup_{t\ge 0} \|u(t)\|_q \tag{3}$$

Note that at the left-hand side of definitions (2) and (3), we should specify the dependence of the norm on q, but we omit it to avoid cumbersome notation; the choice of q will be clear from the context.

For the $\mathcal{L}^m{}_2$ set, the norm is usually defined by setting $q = 2$ in Eq. (2), obtaining

$$\|u(\cdot)\|_{\mathcal{L}_2} = \left(\int_0^{+\infty} u(t)^T u(t) dt\right)^{1/2} \tag{4}$$

The $\mathcal{L}^m{}_2$ set with the norm (4) is of particular interest, since in some contexts the norm (4) is proportional to the *energy* of the signal u, as is the case, for instance, for which $u(t)$ is the voltage across a resistance R. Then the total energy delivered to the resistance would be found by integrating the instantaneous power $u(t)^2/R$. As usual, in the following we shall denote the $\mathcal{L}^1{}_p$ set, $p \in [1, +\infty)$, by $\mathcal{L}_p$.

Remark 2. In contrast with finite-dimensional spaces (see Remark 1), for the *infinite-dimensional* set $\mathcal{L}^m$ (the set of all measurable functions u: $[0, \infty) \rightarrow \mathbb{R}^m$) the p-norms may be nonequivalent. For instance,

- $u(t) = 1/(1+t)^\alpha$ with $0 < \alpha < 1$ belongs to $\mathcal{L}_\infty$ and to $\mathcal{L}_p$ with $p < 1/\alpha$, but does not belong to $\mathcal{L}_p$ for $p \geq 1/\alpha$.
- $u(t) = \sin(t)$ belongs to $\mathcal{L}_\infty$ but does not belong to $\mathcal{L}_p$ for any $p \in [1, +\infty)$.

In this article, we are interested in studying the *stability properties* of a generic operator G that maps a signal space$\mathcal{U}$ into another signal space$\mathcal{Y}$,

$$y = G(u) \tag{5}$$

We shall define the operator G in Eq. (5) as being $\mathcal{L}_p$-stable if for any $u(\cdot) \in \mathcal{L}^m{}_p$ the output $y(\cdot)$ belongs to $\mathcal{L}^r{}_p$. To give a formal definition of $\mathcal{L}_p$ stability and to provide a broad definition of "unstable systems," we first need to define precisely the domain and the image of the operator G, which in general are not limited to $\mathcal{L}^m{}_p$ and $\mathcal{L}^r{}_p$, respectively (for example, we may be interested in considering unbounded inputs and/or outputs). For this reason, we need to introduce some *extended* $\mathcal{L}^n{}_p$ spaces. First, let us define the *truncation* $u_T(t)$ of a signal $u(t)$ in the following way:

$$u_T(t) = \begin{cases} u(t), & 0 \leq t \leq T \\ 0, & t > T \end{cases}$$

The extended set $\mathcal{L}^m{}_{pe}$, with $p \in [1, +\infty]$, consists of all the functions u: $[0, +\infty) \rightarrow \mathbb{R}^m$ such that

$$u_T(t) \in \mathcal{L}_p^m \quad \forall\, T \geq 0$$

Introducing this extended space, we can treat unbounded signals. For instance, $u(t) = t$ does not belong to $\mathcal{L}_p$ for any $p \in [1, +\infty]$ but its truncation

$$u_T(t) = \begin{cases} u(t), & 0 \leq t \leq T \\ 0, & t > T \end{cases}$$

belongs to $\mathcal{L}_p$, with $p \in [1, +\infty]$, for every finite T. Therefore $u(t) = t$ belongs to $\mathcal{L}_{pe}$ for every $p \in [1, +\infty]$.

Finally we end the section with the definition of *causality* of an operator G. When we later specify an operator in terms of state-space differential equations, causality is an intrinsic property of the operator: the output y at time t depends on the initial conditions and on the values of the input u up to time t. On the other hand, when dealing with a generic operator as in Eq. (5), we need to enforce causality. An operator G: $\mathcal{L}^m{}_{pe} \rightarrow \mathcal{L}^r{}_{pe}$ is said to be *causal* if the value of the output at time t depends only on the values of the input up to time t as defined next.

Definition 1 (causality). *An operator* G: $\mathcal{L}^m{}_{pe} \rightarrow \mathcal{L}^r{}_{pe}$ *is said to be causal if* $(G(u))_T = (G(u_T))_T\ \forall\, T \geq 0, \forall\, u \in \mathcal{L}^m{}_{pe}$.

We are now ready to present the first input-output stability concept.

$\mathcal{L}_p$ Stability

Now we give the definition of $\mathcal{L}_p$-stable systems that transform $\mathcal{L}_p$ input signals into $\mathcal{L}_p$ output signals.

Definition 2. *An operator G: $\mathcal{L}^m{}_{pe} \to \mathcal{L}^r{}_{pe}$ is*

- *$\mathcal{L}_p$-stable if*

$$G(u) \in \mathcal{L}_p^r \quad \forall u \in \mathcal{L}_p^m \tag{6}$$

- *Finite-gain $\mathcal{L}_p$-stable if there exist finite constants γ_p and β_p such that*

$$\|G(u)\|_{\mathcal{L}_p} \leq \gamma_p \|u\|_{\mathcal{L}_p} + \beta_p \quad \forall u \in \mathcal{L}_p^m \tag{7}$$

- *Finite-gain $\mathcal{L}_p$-stable with zero bias if there exists a finite constant γ_p such that*

$$\|G(u)\|_{\mathcal{L}_p} \leq \gamma_p \|u\|_{\mathcal{L}_p} \quad \forall u \in \mathcal{L}_p^m \tag{8}$$

For finite-gain $\mathcal{L}_p$-stable systems, the smallest scalar γ_p for which there is a β_p such that relation (7) is satisfied (when such γ_p exist) is called the *gain* of the system. Similarly, for finite-gain $\mathcal{L}_p$-stable systems with zero bias, the smallest scalar γ_p such that relation (8) is satisfied (when such γ_p exist), is called the *gain with zero bias* of the system.

Regarding finite-gain $\mathcal{L}_p$ stability for a causal operator G, we have the following result.

Lemma 1. *(Ref. 29)*

Let G: $\mathcal{L}^m{}_{pe} \to \mathcal{L}^r{}_{pe}$ be a causal finite-gain $\mathcal{L}_p$-stable operator with constants γ_p and β_p. Then

$$\|(G(u))_T\|_{\mathcal{L}_p} \leq \gamma_p \|u_T\|_{\mathcal{L}_p} + \beta_p \quad \forall\, T \geq 0, \forall u \in \mathcal{L}_{pe}^m$$

Proof. Since G is causal $(G(u))_T = (G(u_T))_T$. Moreover if $u \in \mathcal{L}^m{}_{pe}$, then $u_T \in \mathcal{L}^m{}_p$ for all $T \geq 0$; hence $G(u_T) \in \mathcal{L}^m{}_p$. Finally, for a generic function x, $\|x_T\|_{\mathcal{L}p} \leq \|x\|_{\mathcal{L}p}$. Therefore the following inequalities hold:

$$\|(G(u))_T\|_{\mathcal{L}_p} = \|(G(u_T))_T\|_{\mathcal{L}_p} \leq \|G(u_T)\|_{\mathcal{L}_p} \leq \gamma_p \|u_T\|_{\mathcal{L}_p} + \beta_p$$

For a causal, finite-gain $\mathcal{L}_p$-stable operator with zero bias, it can be shown in the same way that

$$\|(G(u))_T\|_{\mathcal{L}_p} \leq \gamma_p \|u_T\|_{\mathcal{L}_p} \quad \forall\, T \geq 0, \forall\, u \in \mathcal{L}_{pe}^m$$

Remark 3. Recalling the definition of the $\mathcal{L}^m{}_\infty$ set with the norm (3), $\mathcal{L}_\infty$ stability is in fact what is normally termed bounded-input, bounded-output (*BIBO*) or *external* stability. It is usually defined in reference to a system specified through a state-space representation, and guarantees that if the input to the system is essentially bounded, then its output is also essentially bounded.

Example: This example illustrates an $\mathcal{L}_\infty$-stable and a finite-gain $\mathcal{L}_\infty$-stable operator.

- Let us consider the function $g\colon \mathbb{R} \to \mathbb{R}$ defined by $g(u) = u^k$ with $k \in (1, +\infty)$. Correspondingly, define the operator G: $\mathcal{L}_{\infty e} \to \mathcal{L}_{\infty e}$ that assigns to every input $u(t)$, $t \geq 0$ the output $g(u(t))$. We will show that G is $\mathcal{L}_\infty$ stable. Indeed set $\|u\|_{\mathcal{L}\infty} = c < +\infty$; then we have

$$\|g(u)\|_{\mathcal{L}_\infty} = \|u^k\|_{\mathcal{L}_\infty} = \sup_{t\geq 0} |u(t)^k| = \sup_{t\geq 0} |u(t)|^k = c^k < +\infty$$

Therefore G is a BIBO operator. It is not a finite-gain $\mathcal{L}_\infty$-stable operator; however, since we cannot find *fixed* scalars γ_∞ and β_∞ such that $c^k \leq \gamma_\infty c + \beta_\infty$ holds for all $c \in [0, +\infty)$.

- By similar arguments as above, it is easy to check that G associated with the function $g(u) = u^k$ with $k \in (0, 1)$ is a finite-gain $\mathcal{L}_\infty$-stable operator.

Example: There are some special cases when the *gain* of an operator can be numerically or explicitly found; one of these cases is the $\mathcal{L}_2$ gain for linear systems. Let us consider a linear time-varying (*LTV*) system in the form (for the sake of simplicity, we assume no feedthrough term)

$$\dot{x} = A(t)x + B(t)u, \qquad x(0) = 0 \tag{9a}$$

$$y = C(t)x \tag{9b}$$

where $A(t)$, $B(t)$, and $C(t)$ are piecewise continuous and bounded.
We assume that the unforced system $\dot{x} = A(t)x$ is exponentially stable (see Appendix 1 for the definition of exponential stability and Appendix 2 for a necessary and sufficient condition for exponential stability of LTV systems). Let G be the input-output operator mapping u to y; then the $\mathcal{L}_2$ gain of G (which is also called *energy* gain and induced-operator norm of G) is defined by

$$\gamma_2 = \sup_{\substack{u \in \mathcal{L}_2 \\ \|u\|_{\mathcal{L}_2} \neq 0}} \frac{\|y\|_{\mathcal{L}_2}}{\|u\|_{\mathcal{L}_2}} \tag{10}$$

For a given $\gamma > 0$, we find (see Ref. 30) that $\gamma_2 \leq \gamma$ if and only if there exists an $\epsilon > 0$ such that the Riccati differential equation

$$\dot{P}(t) + A^T(t)P(t) + P(t)A(t) - \frac{1}{\gamma^2}P(t)B(t)B^T(t)P(t) + \epsilon I = 0 \tag{11}$$

admits a positive definite solution $P(t)$ for which a symmetric matrix-valued function $P(t)$ is said to be positive definite if there exists a positive α such that $x^T P(t)x \geq \alpha x^2$ for all $x \in \mathbb{R}^n$ and $t \geq 0$. Therefore by conducting a binary search over γ, the $\mathcal{L}_2$ gain of G can be computed up to the desired precision.
The time-invariant case is even simpler, since we only have to deal with the algebraic version of Eq. (11). In this case there is actually another way of computing the $\mathcal{L}_2$ norm; let $\hat{G}(s)$ denote, as usual, the transfer matrix $\hat{G}(s) = C(sI - A)^{-1}B$. It is possible to show that the $\mathcal{L}_2$ gain (10) is related to the transfer matrix $\hat{G}(s)$; in fact it is given by

$$\gamma_2 = \sup_{\omega \in \mathbb{R}} \|\hat{G}(j\omega)\|_2 = \sup_{\omega \in \mathbb{R}} \sigma_{\max}[\hat{G}(j\omega)] \tag{12}$$

where $\sigma_{\max}(A) := \sqrt{\lambda_{\max}(A^*A)}$ is the maximum singular value of the matrix A, and A^* denotes the conjugate transpose of a matrix A. The norm (12) is known in literature as the $\mathcal{H}_\infty$ norm of $\hat{G}(j\omega)$ and is denoted by $\|\hat{G}\|_\infty$.

So far, we have considered operators whose domain is the whole space $\mathcal{L}^m{}_p$. The following example motivates a *local* version of Definition 1, concerning input signals that lie in a *subset* of $\mathcal{L}^m{}_p$.

Example: Let us consider the function g: $(-1, 1) \to \mathbb{R}$ defined by $g(u) = 1/(1-u^2)$. As in Example 1 define the associated operator G: $\mathcal{L}_{\infty e} \to \mathcal{L}_{\infty e}$ that assigns to every input $u(t)$, $t \geq 0$, the output $g(u(t))$. Since the function is defined only when the input signal is such that

$$|u(t)| < 1 \quad \forall\, t \geq 0$$

the G is *not* an $\mathcal{L}_\infty$-stable operator according to Definition 1. However, let $|u| \le c < 1$; then

$$|y| \le \left(\frac{c}{1-c^2}\right)|u| + 1$$

and so

$$\|y\|_{\mathcal{L}_\infty} \le \left(\frac{c}{1-c^2}\right)\|u\|_{\mathcal{L}_\infty} + 1$$

which implies that G is an $\mathcal{L}_\infty$-stable operator in a new sense made clear in the next definition.

The following definition is an extension of one originally presented in Ref. 31.

Definition 3. *An operator G: $\mathcal{L}^m{}_{pe} \to \mathcal{L}^r{}_{pe}$ is small-signal $\mathcal{L}_p$-stable if there exists a positive constant r such that Eq. (6) is satisfied for all $u \in \mathcal{L}^m{}_p$ with $\sup_t \|u(t)\| < r$. Similarly, G is a small-signal finite-gain $\mathcal{L}_p$-stable [operator small-signal finite-gain $\mathcal{L}_p$-stable operator with zero bias] if there exists a positive constant r such that inequality (7) [inequality (8)] is satisfied for all $u \in \mathcal{L}^m{}_p$ with $\sup_t \|u(t)\| < r$.*

Remark 4. In Definition 3, we do not need to specify a particular norm to evaluate $\sup_t \|u(t)\|$, since we are dealing with a norm in $\mathbb{R}^m$ and norms are equivalent in *finite*-dimensional spaces (see Remark 1). From Eq. (3) it is clear that if $\|u(t)\|$ is uniformly bounded then so is the *signal* norm $\|u(\cdot)\|_{\mathcal{L}\infty}$. This is not true in general as it is easy to construct examples in which the signal norm $\|u(\cdot)\|_{\mathcal{L}p}$, with $p \in [1, +\infty)$, is arbitrarily large even if $\|u(t)\|$ is uniformly bounded.

Sufficient conditions for $\mathcal{L}_p$ stability. In the previous section we have provided the definition of $\mathcal{L}_p$ stability for a given input-output operator. An important question then arises: How do we check whether an input-output operator is $\mathcal{L}_p$-stable or not?

To answer this question we should not focus, as done so far, on generic input-output operators; in this section we assume that the operators under consideration are specified in terms of a state-space representation of a dynamical system in the form

$$\dot{x} = f(t, x, u) \qquad x(o) = x_0 \tag{13a}$$

$$y = h(t, x, u) \tag{13b}$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, $y \in \mathbb{R}^r$, and

$$f: [0, +\infty) \times D \times D_u \to \mathbb{R}^n$$
$$h: [0, +\infty) \times D \times D_u \to \mathbb{R}^r$$

with $D := \{x \in \mathbb{R}^n: \|x\| < r\}$, $D_u := \{u \in {}^m: \|u\| < r_u\}$, where r and r_u are positive numbers.

It is important to note that Eqs. (13a) define an input-output operator for any given initial state x_0. In other words, system (13) defines an entire class of input-output operators, each obtained in relation to a particular x_0. We should always distinguish between the concept of a *system* and that of an *operator*. In many textbooks, when looking at the input-output behavior of a system, it is assumed that the initial condition is zero, and therefore the system and the corresponding operator are the same object. As opposed to our treatment, the discussion of Ref. 25 maintains the distinction between the input-output concepts and the state-space description.

In the remainder of this article, we assume that a state-space description of the dynamical system is given unless otherwise specified. This will allow us to provide in this section a sufficient condition for $\mathcal{L}_p$ stability and, at the same time, to establish the first connection between Lyapunov (internal) stability and $\mathcal{L}_p$ stability. Assume that $x = 0$ is an equilibrium point for system (13a) with $u = 0$, that is,

$$f(t, 0, 0) = 0$$

We shall see that, if $x = 0$ is an exponentially stable equilibrium point (see Appendix A for the definition) and some other additional technical assumptions hold, the corresponding input-output operator is $\mathcal{L}_p$-stable for any x_0 as described in the following theorem.

Theorem 1. (Corollary 6.1 of Ref. 23)
Assume that

(1) $x = 0$ *is an exponentially stable equilibrium point for system* (13a) *under the input* $u = 0$;
(2) f *is continuously differentiable and the Jacobian matrices* $\partial f/\partial x$ and $\partial f/\partial u$ *are bounded, uniformly in* t
(3) h *is of Lipschitz form with respect to* x *and* u, *that is, there exist positive constants* η_1, η_2 *such that*

$$\|h(t, x, u)\| \leq \eta_1\|x\| + \eta_2\|u\|.$$

for all $(t, x, u) \in [0, +\infty) \times D \times D_u$.

Then there exists a constant $r_0 > 0$ *such that for each* x_0 *satisfying* $\|x_0\| < r_0$ *the operator defined by system* (13) *with initial condition* x_0, *is a small-signal finite-gain* $\mathcal{L}_p$*-stable operator for each* $p \in [1, +\infty]$.

If all the assumptions hold globally, with $D = \mathbb{R}^n$ *and* $D_u = \mathbb{R}^m$, *then for each* $x_0 \in \mathbb{R}^n$ *the operator defined by system* (13) *with initial condition* x_0 *is a finite-gain* $\mathcal{L}_p$*-stable operator for each* $p \in [1, +\infty]$.

Note that a linear system

$$\dot{x} = A(t)x + B(t)u, \qquad x(t_0) = x_0 \tag{14a}$$

$$y = C(t)x + D(t)u \tag{14b}$$

always satisfies assumption ii and iii of Theorem 1 globally if $A(\cdot)$ and $B(\cdot)$ are continuously differentiable (actually this hypothesis can be relaxed to piecewise continuity) and uniformly bounded and $C(\cdot)$ and $D(\cdot)$ are uniformly bounded; moreover, the exponential stability of $x = 0$ of system (14a) with $u = 0$ is always global. Therefore we can state the following corollary of Theorem 1.

Corollary 1. *Consider the linear system*

$$\dot{x} = A(t)x + B(t)u, \qquad x(t_0) = x_0 \tag{15a}$$

$$y = C(t)x + D(t)u \tag{15b}$$

where $A(\cdot)$ and $B(\cdot)$ are continuously differentiable and uniformly bounded and $C(\cdot)$ and $D(\cdot)$ are uniformly bounded. Assume that the equilibrium point $x = 0$ under $u = 0$ of Eq. (15a) is exponentially stable; then for each $x_0 \in \mathbb{R}^n$ the operator defined by system (15) with initial condition x_0 is finite gain $\mathcal{L}_p$-stable for each $p \in [1, +\infty]$.

Recall that the $\mathcal{L}_2$ gain of the operator associated with the linear system (14) for $x_0 = 0$ can be computed according to the procedure detailed in Remark 4. Finally, a sufficient condition for exponential stability is given in Appendix 2.

Relations between Lyapunov stability and $\mathcal{L}_p$ stability. So far in this section we have shown the following (Theorem 1):

$$\text{Exponential stability of } x = 0 \text{ and Technical assumptions on } f,\ h \Longrightarrow \mathcal{L}_p \text{ stability}$$

This represents the first connection between Lyapunov and $\mathcal{L}_p$ stabilities.

The remainder of this section is devoted to find the reverse connection between Lyapunov and $\mathcal{L}_p$ stabilities. It is, however, difficult to find a general result in the spirit of Theorem 1. Following the guidelines of Ref. 29 we shall restrict ourselves to time-invariant systems and focus on attractivity rather than exponential stability. Roughly speaking, the next theorem will show the following result:

$$\mathcal{L}_2 \text{ stability and Reachability and observability} \Longrightarrow \text{attractivity}$$

Theorem 2. *Assume that*

(1) *system* (13) *is time invariant, reachable, and uniformly observable and*
(2) *the input-output operator defined by system* (13) *with initial condition* $x(0) = 0$ *is a small-signal* $\mathcal{L}_2$*-stable operator*.

Then $x = 0$ *is an attractive equilibrium point for system* (13).

Moreover, if system (13) *is globally reachable and the input-output operator is* $\mathcal{L}_2$*-stable*, $x = 0$ *is a globally attractive equilibrium point*.

For the definitions of reachability, uniform observability, and attractivity see Appendix 1. A LTI system in the form

$$\dot{x} = Ax + Bu \tag{16a}$$

$$y = Cx + Du \tag{16b}$$

is globally reachable if and only if it is reachable and is uniformly observable if and only if it is observable. Moreover small-signal $\mathcal{L}_2$ stability implies $\mathcal{L}_2$ stability, and attractivity implies exponential stability. Therefore we can derive the following corollary of Theorem 2.

Corollary 2. *Assume that*

(1) system (16) is reachable and observable and
(2) The input-output operator defined by system (16) with initial condition x(0) = 0 is $\mathcal{L}_2$*-stable.*

Then x = 0 is an exponentially stable equilibrium point for system (16).

Interconnected Systems

One of the main applications of the formalism of input-output stability is the study of the stability of interconnected systems, without explicit knowledge of the internal dynamics of the composite subsystems. Let us consider the feedback interconnection of Fig. 1, where G_1: $\mathcal{L}^m{}_{pe} \rightarrow \mathcal{L}^r{}_{pe}$ and G_2: $\mathcal{L}^r{}_{pe} \rightarrow \mathcal{L}^m{}_{pe}$. Input-output stability allows us to investigate how the signals propagate through this scheme. Before presenting the main

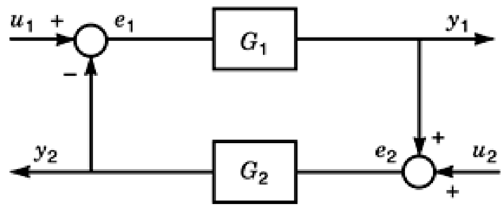

Fig. 1. Basic feedback system.

results, we need to introduce the concept of *well-posedness* of the feedback interconnection. Well-posedness guarantees that for each choice of $u_1 \in \mathcal{L}^m{}_{pe}$ and $u_2 \in \mathcal{L}^r{}_{pe}$ there exist unique solutions $e_1, y_2 \in \mathcal{L}^m{}_{pe}$ and $e_2, y_1 \in \mathcal{L}^r{}_{pe}$ that satisfy the loop equations

$$e_1 = u_1 - G_2(e_2), \qquad y_1 = G_1(e_1)$$
$$e_2 = u_2 + G_1(e_1), \qquad y_2 = G_2(e_2)$$

Small-Gain Theorems. **Theorem 3**. *Consider the feedback system of Fig.* 1. *Suppose that* G_1: $\mathcal{L}^m{}_{pe} \to \mathcal{L}^r{}_{pe}$ *and* G_2: $\mathcal{L}^r{}_{pe} \to \mathcal{L}^m{}_{pe}$ *are causal finite-gain* $\mathcal{L}_p$*-stable operators with constants* γ_{p1}, β_{p1} *and* γ_{p2}, β_{p2}, *respectively. Moreover, suppose that the feedback interconnection is well-posed. Then the feedback system of Fig.* 1 *is a finite-gain* $\mathcal{L}_p$*-stable system if*

$$\gamma_{p1}\gamma_{p2} < 1$$

and

$$\|e_1\|_{\mathcal{L}_p} \leq \frac{1}{1-\gamma_{p1}\gamma_{p2}}(\|u_1\|_{\mathcal{L}_p} + \gamma_{p2}\|u_2\|_{\mathcal{L}_p} + \gamma_{p2}\beta_{p1} + \beta_{p2}) \tag{17a}$$

$$\|e_2\|_{\mathcal{L}_p} \leq \frac{1}{1-\gamma_{p1}\gamma_{p2}}(\|u_2\|_{\mathcal{L}_p} + \gamma_{p1}\|u_1\|_{\mathcal{L}_p} + \gamma_{p1}\beta_{p2} + \beta_{p1}) \tag{17b}$$

$$\|y_1\|_{\mathcal{L}_p} \leq \frac{1}{1-\gamma_{p1}\gamma_{p2}}(\gamma_{p1}\|u_1\|_{\mathcal{L}_p} + \gamma_{p1}\gamma_{p2}\|u_2\|_{\mathcal{L}_p} + \beta_{p1} + \gamma_{p1}\beta_{p2}) \tag{17c}$$

$$\|y_2\|_{\mathcal{L}_p} \leq \frac{1}{1-\gamma_{p1}\gamma_{p2}}(\gamma_{p2}\|u_2\|_{\mathcal{L}_p} + \gamma_{p1}\gamma_{p2}\|u_1\|_{\mathcal{L}_p} + \beta_{p2} + \gamma_{p2}\beta_{p1}) \tag{17d}$$

Proof. Consider inputs $u_1 \in \mathcal{L}^m{}_p$ and $u_2 \in \mathcal{L}^r{}_p$. Since the closed-loop system is well-posed, there exist unique solutions e_1, e_2, y_1, y_2. With respect to e_1 and e_2 we have

$$\|e_1\|_{\mathcal{L}_p} \leq \|u_1\|_{\mathcal{L}_p} + \|G_2 e_2\|_{\mathcal{L}_p}$$
$$\|e_2\|_{\mathcal{L}_p} \leq \|u_2\|_{\mathcal{L}_p} + \|G_1 e_1\|_{\mathcal{L}_p}$$

Since G_1 and G_2 are causal finite-gain $\mathcal{L}_p$-stable operators, we find that (see Lemma 1)

$$\|(e_1)_T\|_{\mathcal{L}_p} \le \|(u_1)_T\|_{\mathcal{L}_p} + \gamma_{p2}\|(e_2)_T\|_{\mathcal{L}_p} + \beta_{p2} \qquad \forall\, T \ge 0 \tag{18a}$$

$$\|(e_2)_T\|_{\mathcal{L}_p} \le \|(u_2)_T\|_{\mathcal{L}_p} + \gamma_{p1}\|(e_1)_T\|_{\mathcal{L}_p} + \beta_{p1} \qquad \forall\, T \ge 0 \tag{18b}$$

After some trivial manipulations, recalling that $\gamma_{p1}\,\gamma_{p2} < 1$ by assumption, Eqs. (18a) become

$$\|(e_1)_T\|_{\mathcal{L}_p} \le \frac{1}{1-\gamma_{p1}\gamma_{p2}}[\|(u_1)_T\|_{\mathcal{L}_p} + \gamma_{p2}\|(u_2)_T\|_{\mathcal{L}_p} + \gamma_{p2}\beta_{p1} + \beta_{p2}] \qquad \forall\, T \ge 0 \tag{19a}$$

$$\|(e_2)_T\|_{\mathcal{L}_p} \le \frac{1}{1-\gamma_{p1}\gamma_{p2}}[\|(u_2)_T\|_{\mathcal{L}_p} + \gamma_{p1}\|(u_1)_T\|_{\mathcal{L}_p} + \gamma_{p1}\beta_{p2} + \beta_{p1}] \qquad \forall\, T \ge 0 \tag{19b}$$

Since for a generic function x, $\|x_T\|_{\mathcal{L}p} \le \|x\|_{\mathcal{L}p}$, we have

$$\|(e_1)_T\|_{\mathcal{L}_p} \le \frac{1}{1-\gamma_{p1}\gamma_{p2}}(\|u_1\|_{\mathcal{L}_p} + \gamma_{p2}\|u_2\|_{\mathcal{L}_p} + \gamma_{p2}\beta_{p1} + \beta_{p2}) \quad \forall\, T \ge 0 \tag{20a}$$

$$\|(e_2)_T\|_{\mathcal{L}_p} \le \frac{1}{1-\gamma_{p1}\gamma_{p2}}(\|u_2\|_{\mathcal{L}_p} + \gamma_{p1}\|u_1\|_{\mathcal{L}_p} + \gamma_{p1}\beta_{p2} + \beta_{p1}) \quad \forall\, T \ge 0 \tag{20b}$$

Now, the right-hand sides of inequalities (20) are *independent* of T. Therefore it can be easily shown that e_1 and e_2 belong to $\mathcal{L}^m{}_p$ and $\mathcal{L}^r{}_p$, respectively, and that

$$\|(e_1)\|_{\mathcal{L}_p} \le \frac{1}{1-\gamma_{p1}\gamma_{p2}}(\|u_1\|_{\mathcal{L}_p} + \gamma_{p2}\|u_2\|_{\mathcal{L}_p} + \gamma_{p2}\beta_{p1} + \beta_{p2}) \tag{21a}$$

$$\|(e_2)\|_{\mathcal{L}_p} \le \frac{1}{1-\gamma_{p1}\gamma_{p2}}(\|u_2\|_{\mathcal{L}_p} + \gamma_{p1}\|u_1\|_{\mathcal{L}_p} + \gamma_{p1}\beta_{p2} + \beta_{p1}) \tag{21b}$$

In a similar way it can be shown that y_1 and y_2 belong to $\mathcal{L}^r{}_p$ and to $\mathcal{L}^m{}_p$, respectively, and that inequalities (17c) and (17d) hold.

The work of Safonov (7) exploited the general input-output concepts in order to study the robustness of closed-loop systems. His results and variations were later exploited in $\mathcal{H}_\infty$ robust control analysis and design. The small-gain theorem is thus extremely useful in studying the robustness of a closed-loop system, when a *nominal* system is subject to a *perturbation* as shown in Fig. 2. The next example shows one of these applications.

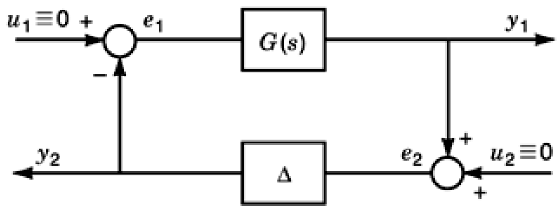

Fig. 2. Feedback robustness loop.

Example: Let us consider the feedback scheme of Fig. 2 where $G(s)$ and $\Delta(s)$ are asymptotically stable transfer matrices. The transfer matrix $G(s)$ represents the nominal system, whereas $\Delta(s)$ is a model of the uncertainty. Let u_1 and u_2 belong to $\mathcal{L}_2$. In Example 2 we have seen how to compute the $\mathcal{L}_2$ gains of $G(s)$ and $\Delta(s)$; let

$$\gamma_{2_G} = \|G\|_\infty, \qquad \gamma_{2_\Delta} = \|\Delta\|_\infty$$

The small-gain theorem tells us that if

$$\gamma_{2_G}\gamma_{2_\Delta} < 1$$

then the closed-loop system is a finite-gain $\mathcal{L}_2$-stable system. In other words, it gives us an estimate of how large the perturbation $\Delta(s)$ can be, in terms of its $\mathcal{H}_\infty$ norm, preserving the closed-loop $\mathcal{L}_2$ stability.

There are various versions of the small-gain theorem, a sample of which is the incremental small-gain theorem below, which needs a preliminary definition.

Definition 4. *An operator G: $\mathcal{L}^m{}_{pe} \rightarrow \mathcal{L}^r{}_{pe}$ is said to be an incrementally finite-gain stable operator if*

(1) $G(u) \in \mathcal{L}^r{}_p$ when $u \equiv 0$ and
(2) there exists a constant γ such that

$$||(G(u) - G(v))_T||_{\mathcal{L}_p} \leq \gamma||(u - v)_T||_{\mathcal{L}_p}$$

for all $T > 0$ and for all $u, v \in \mathcal{L}^m{}_{pe}$.

Theorem 4. *Consider the interconnected system of Fig.* 1. *Let both G_1 and G_2 be incrementally finite-gain stable operators with respective gains* γ_1 and γ_2. *Then, the feedback interconnection is well-posed and incrementally finite-gain stable from $u = [u_1\ u_2]$ to $y = [y_1\ y_2]$ if*

$$\gamma_1\gamma_2 < 1$$

Passivity Theorems. One of the main related concepts to the input-output stability concepts discussed so far is the concept of passive systems. In a way, while $\mathcal{L}_p$ stability deals with the effect the system has on the size of signals, passivity results deal with the effect the system has on the "energy" of signals. We start with few definitions and follow with the main results for interconnected systems.

Definition 5. *We say that a system G: $\mathcal{L}^m{}_{2e} \rightarrow \mathcal{L}^r{}_{2e}$ is*

(1) Passive if there exists $\beta \in \mathbb{R}$ such that for all T and all $u \in \mathcal{L}^m{}_{2e}$

$$\int_0^T u^T(t)(G(u(t)))^T dt \geq \beta$$

(2) Strictly passive if there exists $\alpha > 0$ and $\beta \in \mathbb{R}$ such that for all T and all $u \in \mathcal{L}^m{}_{2e}$

$$\int_0^T u^T(t)(G(u(t)))^T dt \geq \alpha ||u_T||^2 + \beta$$

Theorem 5. *Consider the interconnected system of Fig.* 1. *Assume that the systems G_1 and G_2 satisfy*

$$\begin{array}{lll} \int_0^T e_1^T(t)y_1(t)dt \geq \epsilon_1 & \int_0^T e_1^T(t)e_1(t)dt + \delta_1 & \int_0^T y_1^T(t)y_1(t)dt \\ \int_0^T e_2^T(t)y_2(t)dt \geq \epsilon_2 & \int_0^T e_2^T(t)e_2(t)dt + \delta_2 & \int_0^T y_2^T(t)y_2(t)dt \end{array}$$

The closed-loop system is $\mathcal{L}_2$ finite-gain stable if

$$0 < \delta_1 + \epsilon_2 \quad \text{and} \quad 0 < \delta_2 + \epsilon_1$$

Note that this theorem does not require both systems G_1 and G_2 to be passive, as long as one of the two systems is passive enough. If on the other hand, one of the two systems is passive and the other is strictly passive, the previous theorem simplifies to the following.

Theorem 6. *Consider the interconnected system of Fig.* 1, *and let $u_2 = 0$. Then, the closed-loop system is finite-gain $\mathcal{L}_2$ stable if one of the following conditions holds*:

- *G_1 is passive and G_2 is strictly passive*
- *G_1 is strictly passive and G_2 is passive.*

In the special case of affine in the control systems, the passivity of one system is equivalent to the $\mathcal{L}_2$-stability of a related system. This is detailed in the following theorem:

Theorem 7. (Proposition 3.2.12 of Ref. 18)
Let

$$\dot{x} = f(x) + g(x)u \tag{22a}$$

$$y = h(x) \tag{22b}$$

and

$$\dot{x} = [f(x) - g(x)h(x)] + g(x)v \tag{23a}$$

$$z = 2h(x) - v \tag{23b}$$

where $u = \frac{1}{2}(v - z)$ and $y = \frac{1}{2}(v + z)$. Then, system (22) is passive $\Longleftrightarrow$ system (23) has $\mathcal{L}_2$ gain ≤ 1.

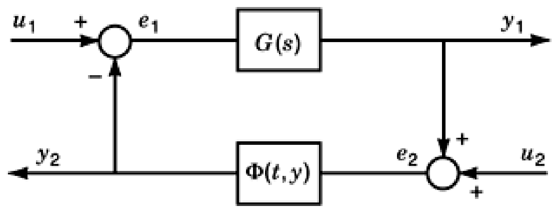

Fig. 3. The Lur'e problem.

Related Stability Concepts And Applications

In this section, we review various concepts that are related to input-output stability and discuss some of their applications.

Dissipativity. In 1972, Willems (10,11) introduced the notion of dissipativity in an attempt to further unify input-output and Lyapunov-stability concepts. The notion of dissipativity is a generalization of passivity and captures the concept that a system will dissipate energy if the sum of the energy it stores and the energy it dissipates to the environment is less than the total energy that the environment supplies to it. This is a manifestation of the second law of thermodynamics and is the case of most physical systems that transform some form of energy to another, but also lose some in the process. In the following, $\mathbb{R}^+ = [0, \infty)$.

Definition 6. *The system (13) is dissipative with respect to the supply rate* $w(u, y)$*:* $\mathbb{R}^m \times {}^r \rightarrow$ *if and only if there exists a storage function* V*:* $\mathbb{R}^n \rightarrow {}^+$ *such that*

$$V(x(T)) \le V(x(0)) + \int_0^T w(u(t), y(t))\, dt$$

for all u, all $T \ge 0$*, and all* $x(0) \in \mathbb{R}^n$.

Note that passivity can actually be defined as a special case of dissipativity by letting $w(u, y) = u^{\mathrm{T}}y$ (therefore, the system is square and $m = r$). We can also define other types of passivity as follows: the system is an input-strictly-passive (*ISP*) system if it is dissipative with supply rate $w(u, y) = u^{\mathrm{T}}y - \delta_u \|u\|^2$, $\delta_u > 0$, and it is an output-strictly-passive (*OSP*) system if it is dissipative with supply rate $w(u, y) = u^T y - \delta_y \|y\|^2$, $\delta_y > 0$. The system is a state-strictly-passive (*SSP*) system if $w(u, y) = u^{\mathrm{T}}y - \delta_x \psi(x)$, $\delta_x > 0$ and $\psi(x)$ is a positive-definite function of x. Note that an OSP system is necessarily $\mathcal{L}_2$ stable (18,23). In addition, one can guarantee the $\mathcal{L}_2$ stability of a system by making sure it is dissipative with the particular supply rate $w(u, y) = \frac{1}{2}\, \gamma^2 \|u\|^2 - \|y\|^2$ for some positive γ, which then becomes an upper bound on the $\mathcal{L}_2$ gain of the system (18,23).

The Linear Case and the KYP Lemma. One of the more important applications of the input-output approach is in the solution of the so-called Lur'e problem shown in Fig. 3. The details of this approach are detailed in another chapter of this encyclopedia and only a few comments are included here for completeness.

The basic question asked by Lur'e is to find conditions in the single-input–single-output case on the linear system $G(s)$ such that when the nonlinear block $\Phi(t, y)$ is static (i.e., a non-time-varying function of y only), the closed-loop system is stable (32). Popov provided graphical, frequency-domain criterion for the absolute stability problem when the nonlinear block $\Phi(y)$ is time invariant (33,34). Yakubovich (35) and Kalman (36) introduced different versions of the so-called positive-real or Kalman-Yakubovich-Popov (KYP) lemma to relate Popov's criterion to the existence of a special Lyapunov function. This then provides another connection between input-output stability concepts and Lyapunov concepts. Anderson then (12,22) extended the KYP lemma to the multi-input–multi-output case. The KYP lemma has found various applications in adaptive control (19) and has recently been generalized to the case in which the linear block $G(s)$ is replaced by a nonlinear but affine nonlinear system $\dot{x} = f(x) + g(x)u$, $y = h(x)$ (16,18).

In the linear case, passivity concepts may be related to the concept of positive-realness, already introduced in the study of electrical networks (22). In fact, consider a stable, square, LTI system with minimal state-space realization

$$\dot{x} = Ax + Bu, \quad x(0) = x_0 \tag{24a}$$

$$y = Cx + Du \tag{24b}$$

where $u, y \in \mathbb{R}^m$, $x \in \mathbb{R}^n$, and let the transfer function be

$$H(s) = C(sI - A)^{-1}B + D. \tag{25}$$

Since the state-space realization is minimal, then (A, B) is controllable and (C, A) is observable. Recall that (A, B) is controllable if and only if rank $(\mathcal{C}) = n$ and (C, A) is observable if and only if rank $(\mathcal{O}) = n$ where $\mathcal{C} = [B \; AB \cdots A^{n-1}B]$ and $\mathcal{O}^T = [C^T \; A^TC^T \; (A^{n-1})^TC^T]$.

Definition 7. *Let H(s) be a proper m × m rational transfer matrix. Then*

- *H(s) is positive real (PR) if*

  (1) *No element of H(s) has a pole in Re[s] > 0*
  (2) *Any pole of an element of H(s) on the jω axis must be simple and the associated residue matrix is positive semidefinite Hermitian, and*
  (3) *For all real ω for which jω is not a pole of an element of H(s), $Z(j\omega) + Z^T(-j\omega)$ is positive semidefinite.*
  (4) *H(s) is said to be strictly positive real (SPR) if H(s − ϵ) is PR for some ϵ > 0.*

For variations on this definition, the reader should consult Ref. 37, where various PR concepts are discussed.

Lemma 2. *Let H(s) be an m × m transfer matrix (25) where*

(1) A is stable
(2) (A, B) is controllable and (C, A) is observable

Then, the transfer function H(s) is SPR if and only if there exist a positive definite symmetric matrix P, matrices W and L, and a real ϵ > 0 such that

$$A^TP + PA = -L^TL - \epsilon P \tag{26a}$$

$$B^TP = C - W^TL \tag{26b}$$

$$W^TW = D + D^T \tag{26c}$$

Note that if $D = 0$, that is, $H(s)$ is strictly proper, then Eqs. (26a) simplify to the more familiar

$$\begin{aligned} A^T P + PA &= -L^T L \\ B^T P &= C \end{aligned}$$

Lemma 3. *H(s) is PR if and only if it is dissipative with storage function* $V(x) = x^T Px$.

Hill and Moylan (38) and others expanded the dissipativity notions in order to explain the KYP lemmas. The KYP lemmas have applications to adaptive control for linear systems (19) and a generalization to nonlinear systems (17). Connections between passivity and stability are provided in the next lemma.

Lemma 4. *(Lemma 10.6 in Ref. 23)*
Given the autonomous system

$$\dot{x} = f(x, u), \quad y = h(x) \tag{27}$$

then the following holds true.

(1) If Eq. (27) is passive with a positive-definite storage function V(x), then the origin of Eq. (27) with zero input is stable.
(2) If the system is OSP, then it is a finite-gain $\mathcal{L}_2$-stable system.
(3) If the system is OSP with a positive-definite storage function V(x) and zero-state observable (see Appendix 1), then the origin of Eq. (27) with zero input is asymptotically stable.
(4) If the system is SSP with a positive-definite storage function V(x), then the origin of Eq. (27) with zero input is asymptotically stable.

Passification via Feedback. In recent years, the input-output approach has gained new footing as a design tool for nonlinear control systems. One of the main applications of such an approach has been to use feedback in order to render a closed-loop system passive or strictly passive (or the passification of an open-loop system). The main motivation for such designs is of course that a passive system will tolerate large-magnitude uncertainties as long as the uncertainties are passive (see Theorem 5). References 16 and 17 contain a large number of results on the passification of nonlinear systems. Roughly speaking, all designs require that the open-loop system be of minimum phase and of a relative degree one in order for it to be made passive using static output feedback. Such concepts have been generalized to a large class of nonlinear systems. As mentioned previously, and following the early concepts (9), there has been much recent work on the discrete-time versions of the input-output stability concepts including the passification designs in Refs. 26 and 27.

Input-to-State and Input-to-Output Stability. In a series of papers (13,14,15), Sontag and co-workers have advanced the notion of input-to-state stability to study the behavior of state-space systems when the input is bounded. Roughly, the input-to-state stability concepts guarantee that the state $x(t)$ is bounded for any bounded input $u(t)$, which may be an external disturbance or a tracking signal. This idea is in some ways a more restrictive version of the input-output concepts unless $y = x$ and is more tightly coupled to the Lyapunov-stability concepts. In what follows, we deal with system (13), or with its autonomous version:

$$\dot{x} = f(x, u), \quad x(0) = x_0 \tag{28a}$$

$$y = h(x, u) \tag{28b}$$

Definition 8. *The system (13a) is said to be locally input-to-state stable (ISS) if there exists a class* $\mathcal{KL}$ *function* α*, a class* $\mathcal{K}$ *function* β *(see Appendix 1 for the definitions of such functions), and positive constants* k_1 *and* k_2 *such that for any initial state* $x(t_0)$ *with* $\|x(t_0)\| < k_1$ *and any input* $u(t)$ *with* $\sup_t \geq t_0 \; \|u(t)\| < k_2$*, the solution* $x(t)$ *exists and*

$$\|x(t)\| \leq \alpha(\|x(t_0)\|, t - t_0) + \beta\left(\sup_{t_0 \leq \tau \leq t} \|u(\tau)\|\right) \quad \text{for all} \quad t \geq t_0 \geq 0 \tag{29}$$

The system is said to be ISS stable if the preceding requirement holds globally (i.e., if $D = \mathbb{R}^n$ *and* $D_u = \mathbb{R}^m$*) for any bounded input* $u(t)$ *and any initial condition* $x(t_0)$*.*

Theorem 5.4. (Lemmas 5.4 and 5.5 (Ref. 23

Let $f(t, x, u)$ *be continuously differentiable and of global Lipschitz form in* (x, u) uniformly in t. *Then, if the system (13a) has a globally exponentially stable equilibrium point at* $x = 0$, *it is ISS. In the case for which the system is autonomous,* $f(x, u)$ *in* Eq. (28a) *is continuously differentiable, and the origin is an asymptotically stable equilibrium point of* Eq. (28a), *then* Eq. (28a) *is ISS*.

Definition 9. *The system (13) is locally input-to-output stable if there exists a class* $\mathcal{KL}$ *function* α*, a class* $\mathcal{K}$ *function* β*, and positive constants* k_1 *and* k_2 *such that for any initial condition* $x(t_0)$ *such that* $\|x(t_0)\| < k_1$ *and any input* $u(t)$ *such that* $\sup_t \geq t_0 \; \|u(t)\| < k_2$*, and for any* $t \geq t_0 \geq 0$*, the following holds true.*

(1) The solution $x(t)$ *exists.*
(2)

$$\|y(t)\| \leq \alpha(\|x(t_0)\|, t - t_0) + \beta\left(\sup_{t_0 \leq \tau \leq t} \|u(\tau)\|\right) \tag{30}$$

The system (13) is said to be input-to-output stable (IOS) if $D = \mathbb{R}^n$*,* $D_u = \mathbb{R}^m$*, and Eq. (30) holds for any initial state* $x(t_0)$ *and any bounded input* $u(t)$*.*

Note that while this is similar to the $\mathcal{L}_p$-stability concepts presented previously, it is actually more general as the α function need not be linear and the β function need not be a constant.

Theorem 9. (Theorem 6.3 of Ref. 23)

Let $f(t, x, u)$ *be piecewise continuous in* t *and of local Lipschitz form in* (x, u)*, and let* h *be piecewise continuous in* t *and continuous in* (x, u)*. Assume that the system (13) is ISS, and that there exists class* $\mathcal{K}$ *functions* α_1 *and* α_2 *such that*

$$\|h(t, x, u)\| \leq \alpha_1(\|x\|) + \alpha_2(\|u\|) \tag{31}$$

Then the system (13) is locally IOS. If all assumptions hold globally and Eq. (13a) *is ISS, then it is IOS*.

Conclusions

In this article we have attempted to summarize various concepts of input-output stability for nonlinear dynamical systems, focusing on the continuous-time case. We have presented the basic input-output concepts but also some extensions and their applications to stability robustness analysis and design and to the adaptive control of linear and nonlinear systems, as mentioned previously.

It is now clear that the connections between Lyapunov stability and input-output stability are strong and may be exploited for further design. On the other hand, it is clear that the input-output approach remains a versatile tool. This approach allows us to be able to determine the stability of the closed-loop system although we have have little knowledge of the internal dynamics of the open-loop system and its uncertainties. This is clearly an advantage when dealing with uncertain systems as the dynamics of the systems and its uncertainties may be unknown. One of the limitations of the input-output approach, however, is that it remains limited to the study of stability while other objectives such as the optimization of some performance indices remain beyond the reach of these techniques. We end this article by mentioning that prior to its introduction to feedback systems and control, the input-output approach was part of operator theory and functional analysis. Finally, the input-output approach has been applied to various areas such as communications (39,40) and to the study of neural network stability (41).

Appendix 1: Definitions

Definition 1.1 (Lipschitz functions). *A function $f(x)$ is said to be of local Lipschitz form on a domain $D \subset \mathbb{R}^n$ if for each point $x_0 \in D$ there exist a neighborhood D_0 of the point x_0 and a constant L_0 such that*

$$\| f(x) - f(y)\| \le L_0 \|x - y\| \qquad \forall\, x, y \in D_0 \tag{1.1}$$

If equation (32) holds for all $x \in D$ with the same constant L, then f is said to be of Lipschitz form on D. If f is of Lipschitz form on $\mathbb{R}^n$ than it is said to be of global Lipschitz form.

Definition 1.1 can be extended to the case of $f(t, x)$ provided that the Lipschitz condition holds *uniformly* on t for a given time interval.

Definition 1.2. *(Function of class $\mathcal{K}$.*

A continuous function α: $[0, +\infty) \to [0, +\infty)$ is said to be of class $\mathcal{K}$ if it is strictly increasing and $\alpha(0) = 0$.

Definition 1.3. *(Function of class $\mathcal{K}_\infty$).*

A function α: $[0, +\infty) \to [0, +\infty)$ is said to be of class $\mathcal{K}_\infty$ if it is of class $\mathcal{K}$ and $\alpha(s) \to \infty$ as $s \to \infty$.

Definition 1.4. *(Function of class $\mathcal{KL}$).*

A function α: $[0, +\infty) \times [0, +\infty) \to [0, +\infty)$ is said to be of class $\mathcal{KL}$ if for each fixed t the mapping $\alpha(s, t)$ is of class $\mathcal{K}$ and for each fixed s the mapping $\alpha(s, t)$ is decreasing with respect to t and $\alpha(s, t) \to 0$ as $t \to \infty$.

Example:

- The function $\alpha(s)$: $[0, +\infty) \to s/(s+1)$ is of class $\mathcal{K}$ since $\alpha'(s) = 1/(s+1)^2 > 0$, but it is not of class $\mathcal{K}_\infty$ since $\lim_{s\to\infty} \alpha(s) = 1 < +\infty$.
- The function $\alpha(s)$: $[0, +\infty) \to s$ is of class $\mathcal{K}_\infty$ since $\alpha'(s) = 1 > 0$ and $\lim_{s\to\infty} \alpha(s) = +\infty$.
- The function $\alpha(s, t)$: $[0, +\infty) \times [0, +\infty) \to se^{-t}$ is of class $\mathcal{KL}$. Indeed it is strictly increasing in s, since

$$\frac{\partial \alpha}{\partial s} = e^{-t} > 0 \qquad \forall \quad t \ge 0$$

strictly decreasing in t, since

$$\frac{\partial \alpha}{\partial t} = -se^{-t} < 0 \qquad \forall \quad t \geq 0$$

and $\alpha(s, t) \to 0$ as $t \to \infty$.

Definition 1.5 (State transition matrix). *Given the LTV system*

$$\dot{x} = A(t)x, \qquad t \geq 0$$

with $x \in \mathbb{R}^n$ and A(t) piecewise continuous, the state transition matrix $\Phi(\dot{c},\dot{c})$: $\mathbb{R}^+ \times \mathbb{R}+ \to \mathbb{R}^{n\times n}$, $(t, t_0) \to \Phi(t, t_0)$, is defined as the unique solution of the matrix differential equation

$$\frac{\partial}{\partial t}\Phi(t, t_0) = A(t)\Phi(t, t_0), \qquad \Phi(t_0, t_0) = I$$

Consider the zero-input system

$$\dot{x} = f(x), \qquad x(0) = x_0 \tag{1.2}$$

where $f(\cdot)$: $D \subseteq \mathbb{R}^n \to \mathbb{R}^n$ is of local Lipschitz form on D.

Definition 1.6 (Attractivity). *Consider the zero-input system (1.2) and denote by $s(t, x_0)$ the solution starting from x_0 at time $t = 0$. Assume that $x = 0$ is an equilibrium point of system (1.2); then $x = 0$ is attractive if there exists a domain $D_a \subseteq D$, $0 \in D_a$ such that*

$$\lim_{t\to\infty} \|s(t, x)\| = 0 \qquad \forall\, x \in D_a$$

The equilibrium point $x = 0$ is globally attractive if $D_a = \mathbb{R}^n$.

Definition 1.7 (Stability and Asymptotic Stability). *Consider the zero-input system (1.2) and denote by $s(t, x_0)$ the solution starting from x_0 at time $t = 0$. Assume that $x = 0$ is an equilibrium point of system (1.2); then $x = 0$ is*

- *stable if, for each $\epsilon > 0$, there exists a $\delta = \delta(\epsilon)$ such that*

$$\|x_0\| < \delta \Rightarrow \|s(t, x_0)\| < \epsilon \qquad \forall\, t \geq 0$$

- *unstable if it is not stable.*
- *asymptotically stable if it is stable and δ can be chosen such that $x = 0$ is attractive on the domain $D = \{x \in \mathbb{R}^n,\ \|x\| < \delta\}$.*
- *globally asymptotically stable if it is stable and globally attractive.*

Now consider the zero-input system

$$\dot{x} = f(t, x), \qquad x(t_0) = x_0 \tag{1.3}$$

where $f(\cdot,\cdot)$: $[0, +\infty) \times D \to \mathbb{R}^n$, $D = \{x \in \mathbb{R}^n,\ \|x\| < r\}$, and $t_0 \geq 0$.

Definition 1.8 (Exponential stability). *Consider the zero-input system (1.3) and assume that $x = 0$ is an equilibrium point. Then $x = 0$ is exponentially stable if there exist positive numbers K, γ, c such that*

$$\|x(t)\| \leq k\|x_0\|e^{-\gamma(t-t_0)} \qquad \forall\, t \geq t_0 \geq 0, \qquad \forall\, x_0 : \|x_0\| < c \tag{1.4}$$

The equilibrium $x = 0$ is globally exponentially stable if the above condition is verified for any initial state.

Definition 1.9 (Zero-State Observability). *The system (27) is said to be zero-state observable from the output y if for all initial conditions, $y(t) \equiv 0 \Rightarrow x(t) \equiv 0$. The system is zero-state detectable if for all initial conditions $y(t) \equiv 0 \Rightarrow \lim_{t\to\infty} x(t) = 0$.*

Definition 1.10 (Reachability). *Consider the system*

$$\dot{x} = f(t, x, u) \tag{1.5}$$

and denote by $s(t, x_0, u)$ the solution starting from x_0 at time $t = 0$ under the input u. Then system (1.5) is said to be reachable if there exists a class $\mathcal{K}$ function α and a set $D := \{x \in \mathbb{R}^n: \|x\| < r\}$, such that for all $x \in D$ there exists a time $t*$ and an input $u*$ such that $\|u*\|_{\mathcal{L}\infty} \leq \alpha(\|x\|)$ and $s(t, 0, u*) = x$. The system is said to be globally reachable if all the assumptions hold for all $x \in \mathbb{R}^n$.*

Definition 1.11 (Uniform observability). *Consider the system*

$$\dot{x} = f(t, x, u) \tag{1.6a}$$

$$y = h(t, x, u) \tag{1.6b}$$

and denote by $s(t, x_0, u)$ the solution starting from x_0 at time $t = 0$ under the input u. Then system (1.6) is said to be uniformly observable if there exists a class $\mathcal{K}$ function α such that for all x,

$$\|h(\cdot, s(\cdot, x, 0), 0)\|_{\mathcal{L}_2} \geq \alpha(\|x\|)$$

Appendix 2: Sufficient Conditions For Exponential Stability

Theorem 2.1 (Sufficient condition for exponential stability). *Let $x = 0$ be an equilibrium point of system (1.3) and assume there exists a continuously differentiable Lyapunov function $V(\cdot,\cdot)$: $(t, x) \in [0, +\infty) \times D \to \mathbb{R}$ satisfying*

$$k_1\|x\|^\gamma \quad \leq \quad V(t, x) \leq k_2\|x\|^\gamma \tag{2.1a}$$

$$\dot{V}(t, x) \quad = \quad \frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} g(t, x) \leq -k_3\|x\|^\gamma \tag{2.1b}$$

for some positive constants k_1, k_2, k_3, and γ. Then the equilibrium $x = 0$ is exponentially stable. Moreover, if all the assumptions hold globally, $x = 0$ is globally exponentially stable.

Theorem 2.2 (Exponential stability of linear systems). *The equilibrium point* $x = 0$ *of the LTV system*

$$\dot{x} = A(t)x \tag{2.2}$$

is exponentially stable if and only if there exist positive constants α *and* k *such that*

$$\|\Phi(t, t_0)\| \leq ke^{-\alpha(t-t_0)} \qquad \forall\ (t, t_0) \in \mathbb{R}^+ \times \mathcal{R}^+, \qquad t \geq t_0$$

where $\Phi(t, t_0)$ *is the state transition matrix of system* (2.2).

Note that the concept of exponential stability is equivalent, for linear systems, to that of *uniform asymptotic stability* (see Ref. 23 for the definition). This equivalence is no longer true for nonlinear systems, where the concept of exponential stability is stronger than that of uniform asymptotic stability. Finally, note that for LTI systems a necessary and sufficient condition for exponential stability is the Hurwitz character of the A matrix, that is, all its eigenvalues should have negative real part.

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