MULTIVARIABLE SYSTEMS

MULTIVARIABLE LINEAR SYSTEMS

Introduction

With the development of miniaturized, cheap, sensor, and actuator technology, many of today's control problems must coordinate the actions of multiple actuators, based on multiple output signals from diverse sensors. Such systems, illustrated in Fig. 1, are referred to as multi-input, multi-output (MIMO) systems. An important class of MIMO systems is linear MIMO systems, whose relationship between input and output signals is represented by linear transformations.

This article introduces the basic concepts for analyzing linear time-varying and time-invariant MIMO systems for continuous time input and output signals. Extensions of the concepts in this article to discrete time signals are straightforward and are found in the references at the end. The chapter discusses input-output and state-space models of linear MIMO systems and introduces the concepts of controllability and observability for state-space models. It also discusses modal analysis for state-space models of time invariant MIMO systems, MIMO poles and zeros, and singular-value analysis for characterizing the frequency response of linear, time-invariant MIMO systems.

Input-Output Models

Input-output models capture the essential relationships between inputs to a system and the outputs of that system. Instead of focusing on the internal representation and operation of a system, input-output models represent these internal effects implicitly within a transformation from inputs to outputs, as illustrated in Fig. 1. In this section, we review results on input-output models of linear MIMO systems in continuous time.

Consider a system with input u and output y, where the relationship between input and output is denoted by the map

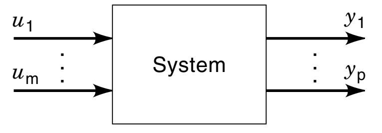

Figure 1. Multi-input, multi-output system.

$y = H(u)$. A system is linear if it satisfies the following properties:

1. *Homogeneity.* For any scalar multiple a, the response to a scaled input is equal to the scaled output response. That is

$$H(au) = aH(u)$$

2. *Superposition.* Given two inputs u_1, u_2, the response to the combined input $u_1 + u_2$ is the sum of the individual output responses. That is

$$H(u_1 + u_2) = H(u_1) + H(u_2)$$

For MIMO continuous time systems, the inputs consist of vector-valued signals $u(t) \in R^m$ and the outputs consist of vector-valued signals $y(t) \in R^p$, both of which are functions of the continuous time parameter t. Assume that such a system is well behaved, in that small changes in the input signals over a finite time result in small changes in the output signal over the same interval. The general form of the input-output description of such systems is given by

$$y(t) = \int_{-\infty}^{\infty} H(t,\tau)u(\tau)\,d\tau \tag{1}$$

In the previous integral equation, $H(t, \tau)$ is the $p \times m$ impulse response or weighting pattern, that is, if the components of $u(t)$ are impulses of the form

$$u_j(t) = \delta(t - t_0), \quad u_k(t) = 0, k \neq j \tag{2}$$

then

$$y_i(t) = h_{ij}(t - t_0), i = 1, \ldots, p \tag{3}$$

where $h_{ij}(t, \tau)$ is the ijth element of $H(t, \tau)$.

Example. Consider a two-input, two-output system described by the following impulse response:

$$H(t,\tau) = \begin{bmatrix} \delta(t-\tau) + 0.5 * e^{-(t-\tau)}u_{-1}(t-\tau) & 0 \\ \delta(t-\tau) + e^{-2(t-\tau)}u_{-1}(t-\tau) & e^{-3(t-\tau)}u_{-1}(t-\tau) \end{bmatrix}$$

where $u_{-1}(t)$ is the unit step function, which is 0 for $t < 0$ and 1 otherwise. The output of the system is given by

$$\begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} = \int_{-\infty}^{\infty} H(t,\tau) \begin{bmatrix} u_1(\tau) \\ u_1(\tau) \end{bmatrix} d\tau$$

In the previous example, note that the impulse response $H(t,\tau)$ is zero for $\tau > t$. Thus, the output $y(t)$ depends only on inputs up to time t. Systems with this special property are known as *causal.*

Definition. The input-output system Eq. (1) is *causal* if

$$H(t,\tau) = 0 \quad \text{for } t < \tau \tag{4}$$

so that

$$y(t) = \int_{-\infty}^{t} H(t,\tau)u(\tau)\,d\tau \tag{5}$$

Another important property of the example is that the impulse response depends only on the difference $t - \tau$. This property is known as *time invariance.*

Definition. The input-output system Eq. (1) is *time-invariant* if

$$H(t,\tau) = H(t-\tau, 0) \equiv H(t-\tau) \tag{6}$$

for all t and τ. The last equality is a slight abuse of notation introduced by convenience. If Eq. (1) is time-invariant, then $y(t)$ is the convolution of $H(t)$ and $u(t)$:

$$y(t) = \int_{-\infty}^{+\infty} H(t-\tau)u(\tau)\,d\tau = \int_{-\infty}^{+\infty} H(\tau)u(t-\tau)\,d\tau \tag{7}$$

Such a system is called a linear time-invariant (LTI) system. An LTI system is causal if and only if $H(t) = 0$ for all $t < 0$.

State-Space Models

In contrast with input-output models, state-space models provide an explicit representation of the internal operations of a system, leading to the transformation between input time functions and output time functions. The *state* of a system at a given time provides a complete summary of the effects of past inputs to the system, which is sufficient to uniquely characterize system output responses to future inputs. This article focuses on linear MIMO systems where the state takes values in a finite-dimensional space of dimension n. Such models are analyzed with concepts from linear algebra and matrix differential equations.

The general form of a state-space model for a linear system with m inputs $u(t) \in R^m$ and p outputs $y(t)$ is given by the matrix differential equation

$$\frac{d}{dt}x(t) = A(t)x(t) + B(t)u(t) \tag{8}$$

$$y(t) = C(t)x(t) + D(t)u(t) \tag{9}$$

with initial condition specified at time t_0 as $x(t_0) = x_0$. In Eqs. (8–9), $x(t) \in R^n$ is the system state at time t, $A(t)$ is the $n \times n$ *system* matrix, $B(t)$ is the $n \times m$ *input* matrix, $C(t)$ is the $p \times n$ *output* matrix, and $D(t)$ is the $p \times m$ *feedthrough* matrix.

Example. Consider the state-space model specified by the following matrices:

$$A = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -3 \end{bmatrix}; \qquad B = \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$C = \begin{bmatrix} 0.5 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix}; \qquad D = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$$

This state-space model is equivalent to the input-output model in the previous subsection, as seen later.

An important property of state-space models is that they generate causal input-output maps, because past inputs affect only future outputs through the value of the current state. To determine the implied relationship between inputs and outputs in Eqs. (8–9), we need expressions for the solution of the vector differential equation in Eq. (8). First consider solutions of the unforced (homogeneous) equation

$$\frac{d}{dt}x(t) = A(t)x(t) \tag{10}$$

starting from arbitrary initial conditions $x(t_0)$ at time t_0. Under the assumption that the matrix $A(t)$ is piecewise continuous, there exists a unique continuous solution of Eq. (10) given by

$$x(t) = \Phi_A(t, t_0)x(t_0) \tag{11}$$

for some $n \times n$ matrix continuous matrix $\Phi_A(t, t_0)$. The matrix $\Phi_A(t, \tau)$, known as the *state transition matrix,* has the following properties:

1. $\Phi_A(t, t) = I$, where I is the $n \times n$ identity matrix.
2. If $A(t) = A$ is time-invariant, then

$$\Phi_A(t, \tau) = e^{A(t-\tau)}$$

   where the matrix exponential is defined by the series

$$e^{At} = \sum_{i=0}^{\infty} \frac{(At)^i}{i!}$$

3. $\Phi_A(t, t_0)$ is the unique continuous solution of the $n \times n$ matrix differential equation

$$\frac{d}{dt}X(t) = A(t)X(t)$$

   with initial condition $X(t_0) = I$.
4. For every t, τ, t_0, the following *compositional* property holds:

$$\Phi_A(t, \tau)\Phi_A(\tau, t_0) = \Phi_A(t, t_0)$$

5. $\frac{\partial}{\partial t}\Phi_A(t, \tau) = A(t)\Phi_A(t, \tau)$.

Given the state transition matrix $\Phi_A(t, \tau)$, the general solution of Eq. (8) is written as

$$x(t) = \Phi_A(t, t_0)x(t_0) + \int_{t_0}^{t} \Phi_A(t, \tau)B(\tau)u(\tau)\, d\tau, \qquad t \geq t_0 \tag{12}$$

It is straightforward to use the properties of $\Phi_A(t, \tau)$ to verify that Eq. (11) satisfies Eq. (8). Using Eq. (9), the input-output relationship of a state-space model is written

$$\begin{aligned} y(t) &= C(t)\Phi(t, t_0)x(t_0) \\ &\quad + \int_{t_0}^{t} C(t)\Phi_A(t, \tau)B(\tau)u(\tau)\, d\tau + D(t)u(t) \end{aligned} \tag{13}$$

If the initial time at $t_0 = -\infty$ and it is assumed that the system starts at rest, the output $y(t)$ becomes

$$y(t) = \int_{-\infty}^{t} C(t)\Phi_A(t, \tau)B(\tau)u(\tau)\, d\tau + D(t)u(t) \tag{14}$$

which is a causal input-output model for the system. Thus, every state-space model leads to a corresponding input-output model for the system. The question of whether input-output models have a finite-dimensional state-space representation is addressed in the next section.

Example (continued). Because of the special diagonal, time-invariant form of the A matrix, it is straightforward to compute the state transition matrix as

$$\Phi_A(t, \tau) = \begin{bmatrix} e^{-(t-\tau)} & 0 & 0 \\ 0 & e^{-2(t-\tau)} & 0 \\ 0 & 0 & e^{-3(t-\tau)} \end{bmatrix}$$

Because C, B do not depend on time, the product $C\Phi_A(t, \tau)B$ is written as

$$C\Phi_A(t, \tau)B = \begin{bmatrix} 0.5e^{-(t-\tau)} & 0 \\ e^{-2(t-\tau)} & e^{-3(t-\tau)} \end{bmatrix}$$

Substituting Eq. (14) yields

$$\begin{aligned} y(t) &= \int_{-\infty}^{t} \begin{bmatrix} 0.5e^{-(t-\tau)} & 0 \\ e^{-2(t-\tau)} & e^{-3(t-\tau)} \end{bmatrix} u(\tau)\, d\tau + \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} u(t) \\ &= \int_{-\infty}^{5} \begin{bmatrix} \delta(t-\tau) + 0.5e^{-(t-\tau)} & 0 \\ \delta(t-\tau) + e^{-2(t-\tau)} & e^{-3(t-\tau)} \end{bmatrix} u(\tau)\, d\tau \end{aligned}$$

which is equivalent to the input-output model in the previous subsection.

State-space models have important qualitative properties which are useful for control design. Of particular interest are the properties of stability, controllability, and observability, discussed in greater length in the articles by Sontag, Freudenberg and Vidyasagar. Some of the relevant concepts are summarized here.

Stability. Given an n-dimensional state-space model with matrices $[A(t), B(t), C(t), D(t)]$, there are two types of stability properties of interest, internal stability and input-output stability. Internal stability is the stability of trajectories of the homogeneous system Eq. (10) and thus involves only the matrix $A(t)$. Different types of internal stability are possible. *Asymptotic* stability corresponds to all solutions to Eq. (10) converging to 0 as $t \to \infty$, and *exponential* stability corresponds to all solutions converging to zero exponentially fast, that is,

$$\|x(t)\| \leq Ke^{-at} \tag{15}$$

for some K, $a > 0$, where the vector norm $\| \ \|$ is the standard Euclidean vector magnitude. Exponential stability is equivalent to

$$\|\Phi_A(t, \tau)\| \leq Me^{-a(t-\tau)}, t \geq \tau \tag{16}$$

where $\|M\|$ is the matrix norm corresponding to the square root of the largest eigenvalue of the matrix M^TM.

Input-output stability refers to the full input-output map in Eq. (13). A MIMO system is said to be *bounded-input/bounded-output* (BIBO) stable if bounded inputs lead to bounded outputs, that is, if $\|u(t)\| \le K_1 < \infty$ for all $t \ge t_0$ implies $\|y(t)\| \le K_2 > 0$ for all $t \ge t_0$. For state-space models, if $B(t)$, $C(t)$, $D(t)$ are bounded, then exponential stability guarantees BIBO stability. However, the converse is not true, as shown by the example below:

$$\frac{d}{dt}x(t) = x(t) + u(t)$$
$$y(t) = u(t)$$

In this example, the state x does not affect the output y. Thus, the system is internally unstable although it is BIBO stable.

Controllability and Observability. The concepts of controllability and observability of state-space models characterize the degree to which inputs and outputs determine the internal state trajectory of a state-space model. This section presents an overview of these concepts for linear, state-space models. For a more detailed exposition of these concepts see CONTROLLABILITY AND OBSERVABILITY.

Consider a linear system with a state-space model whose matrices are $A(t)$, $B(t)$, $C(t)$, $D(t)$, which are assumed to be continuous functions of time. The system is said to be *controllable* on the interval $[t_0, t_1]$ if, given any initial state x_0 at $t = t_0$ and any desired final state $x_1 = t = t_1$, it is possible to specify a continuous input $u(t)$, $t_0 \le t < t_1$ so that if $x(t_0) = x_0$, then $x(t_1) = x_1$. The system is *observable* on the interval $[t_0, t_1]$ if, given knowledge of $u(t)$, $t_0 \le t < t_1$ and $y(t)$, $t_0 \le t \le t_1$, the initial state $x(t_0)$ (and thus the entire state trajectory $x(t)$, $t \in [t_0, t_1]$) is uniquely determined.

Conditions for verifying controllability and observability are determined from the explicit representation of the trajectories of state-space models in Eqs. (11) and (13). Using Eq. (11), controllability is equivalent to finding a control $u(t)$, $t \in [t_0, t_1]$, which solves

$$x(t_1) - \Phi_A(t_1, t_0)x(t_0) = \int_{t_0}^{t_1} \Phi_A(t_1, \tau)B(\tau)u(\tau)\, d\tau \tag{17}$$

for any pair of states $x(t_1)$, $x(t_0)$.

Define the controllability Gramian as the $n \times n$ matrix

$$W_C(t_0, t_1) = \int_{t_0}^{t_1} \Phi_A(t_0, \tau)B(\tau)B^T(\tau)\Phi_A^T(t_0, \tau)\, d\tau \tag{18}$$

The system is controllable on $[t_0, t_1]$ if and only if the matrix $W_C(t_0, t_1)$ is invertible. To establish this, if the inverse exists, then the control $u(t) = -B^T(t)\Phi_A^T(t_0, t)W_C^{-1}(t_0, t_1)[x(t_0) - \Phi_A(t_0, t_1)x(t_1)]$ is continuous and, when substituted in Eq. (17) yields

$$\begin{aligned}
&\int_{t_0}^{t_1} \Phi_A(t_1, \tau)B(\tau)u(\tau)\, d\tau \\
&= -\left[\int_{t_0}^{t_1} \Phi_A(t_1, \tau)B(\tau)B^T(\tau)\Phi_A^T(t_0, \tau)\, d\tau\right] \\
&\quad W_C^{-1}(t_0, t_1)\left[x(t_0) - \Phi_A(t_0, t_1)x(t_1)\right] \\
&= -\Phi_A(t_1, t_0)W_C(t_0, t_1)W_C^{-1}(t_0, t_1)[x(t_0) - \Phi_A(t_0, t_1)x(t_1)] \\
&= x(t_1) - \Phi_A(t_1, t_0)x(t_0)
\end{aligned}$$

Conversely, assume that the system is controllable but that the matrix $W_C(t_0, t_1)$ is not invertible. Then, there must exist a nonzero vector $z \in R^n$ such that $z^TW_C(t_0, t_1)z = 0$, that is,

$$\int_{t_0}^{t_1} z^T\Phi_A(t_0, \tau)B(\tau)B^T(\tau)\Phi_A^T(t_0, \tau)z\, d\tau = 0$$

Since the integrand is nonnegative, it follows that $z^T\Phi_A(t_0, t)B(t) = 0$ for all $t \in [t_0, t_1]$. Controllability implies that a control exists which yields $x(t_1) = 0$ for $x(t_0) = z$. From Eq. (17), this requires that

$$z = -\int_{t_0}^{t_1} \Phi_A(t_0, \tau)B(\tau)u(\tau)\, d\tau$$

Because of the choice of z, it follows that

$$z^Tz = -\int_{t_0}^{t_1} [z^T\Phi_A(t_0, \tau)B(\tau)]u(\tau)\, d\tau = 0$$

implying that $z = 0$.

The controllability Gramian has many properties. It is symmetric, positive-semidefinite for all $t_1 > t_0$, and satisfies the following matrix differential equation:

$$\begin{aligned}
&\frac{d}{dt}W_C(t, t_1) = A(t)W_C(t, t_1) + W_C(t, t_1)A^T(t) - B(t)B^T(t) \\
&W_C(t_1, t_1) = 0
\end{aligned} \tag{19}$$

Direct integration of this matrix equation is preferred to the integral expression for numerically computing the controllability Gramian.

In the special case that the matrices A, B do not depend on time, there is a simple algebraic test for controllability. Define the matrix

$$M_c(A, B) = (B \quad AB \quad A^2B \quad \cdots \quad A^{n-1}B)$$

where n is the dimension of the state. The state-space model is controllable if and only if the rank of $M_c(A, B)$ is n.

Observability is characterized similarly. Using Eq. (13), it is sufficient to consider the case where $u(t) = 0$ for $t \in [t_0, t_1]$, so that the output response is given by

$$y(t) = C(t)\Phi(t, t_0)x(t_0), \quad t > t_0 \tag{20}$$

Define the observability Gramian as the $n \times n$ matrix

$$W_0(t_0, t_1) = \int_{t_0}^{t_1} \Phi^T(\tau, t_0)C^T(\tau)C(\tau)\Phi(\tau, t_0)x\, d\tau$$

The observability Gramian is again symmetric, positive-semidefinite, and satisfies the matrix differential equation

$$\begin{aligned}
&\frac{d}{dt}W_0(t, t_1) = -A^T(t)W_0(t, t_1) - W_0(t, t_1)A(t) - C(t)^TC(t) \\
&W_0(t_1, t_1) = 0
\end{aligned} \tag{21}$$

The system is observable on $[t_0, t_1]$ if and only if $W_0(t_0, t_1)$ is invertible. If the system is observable, then, in the absence of external inputs, the initial condition is given by

$$x(t_0) = W_0^{-1}(t_0, t_1) \int_{t_0}^{t_1} \Phi^T(\tau, t_0) C^T(\tau) y(\tau)\, d\tau$$

In the special case where the matrices A, C are independent of time, observability is determined from the matrix

$$M_o(A, C) = M_c(A^T, C^T)^T$$

The state-space model is observable if and only if the matrix $M_o(A, C)$ has rank n.

State-Space Realization of Input-Output Models

An important question in MIMO systems is determining when a causal input-output model of the form

$$y(t) = \int_{t_0}^{t} G(t, \tau) u(\tau)\, d\tau + D(t) u(t) \tag{22}$$

[where $G(t, \tau)$ does not contain generalized functions such as impulses] is represented by a finite-dimensional state-space model of the form

$$\begin{aligned} \frac{d}{dt} x(t) &= A(t) x(t) + B(t) u(t) \\ y(t) &= C(t) x(t) + D(t) u(t) \end{aligned} \tag{23}$$

where $x(t) \in R^n$, $u(t) \in R^m$, $y(t) \in R^p$ and $A(t)$, $B(t)$, $C(t)$, $D(t)$ are continuous matrices.

The converse of the question is straightforward. State-space models correspond to causal input-output models of the form

$$y(t) = \int_{t_0}^{t} C(t) \Phi_A(t, \tau) B(\tau) u(\tau)\, d\tau + D(t) u(t), \quad t \ge t_0$$

In the problem of realizability, it is straightforward to identify the correspondence between $D(t)$ in Eqs. (22) and (23). Thus, the focus is on identifying matrices $A(t)$, $B(t)$, $C(t)$ in Eq. (23) from $G(t,\tau)$ in Eq. (22). Two variations of this problem are of interest. In the first variation, the function $G(t,\tau)$ is continuous and known for all values of t, τ. This information corresponds closely to that provided by a state-space model, because the term $C(t)\Phi_A(t,\tau)B(\tau)$ is defined for all t and τ and is continuous by the assumptions for a state-space model. In the second variation, the function $G(t,\tau)$ is known only for values $t \ge \tau$, corresponding to causal observations of impulse responses of the system. In this variation, the realization problem is more complex and requires additional smoothness assumptions on $G(t,\tau)$. In this overview, we focus on the first variation where $G(t,\tau)$ is known and continuous for all t, τ. The interested reader should consult (1,2,3) for further details on realization from impulse responses.

Definition. A state-space model Eq. (23) is a *realization* of a weighting pattern $G(t,\tau)$ if, for all t, τ, $G(t,\tau) = C(t)\Phi_A(t,\tau)B(\tau)$.

There are many possible realizations corresponding to a specific weighting pattern $G(t,\tau)$. Any change of basis in the state space results in an equivalent realization. In addition, additional extraneous states can be added which do not affect the input-output behavior of the system. It is important to identify realizations with minimal numbers of states. If a realization with state dimension n exists and no other realization exists with dimension less than n, then realizations with state dimension n are known as *minimal realizations.*

The following result provides an answer to the realization problem:

Theorem (3). There exists a state space realization of dimension n for the weighting pattern $G(t, \tau)$ if and only if there exists a $p \times n$ matrix matrix function $H(t)$ and an $n \times m$ matrix function $F(t)$, both continuous for all t, such that

$$G(t, \tau) = H(t) F(\tau)$$

for all t, τ.

The proof of this theorem is straightforward. The invertibility properties of the state transitional matrix guarantee that, for a state-space model, its input-output relationship is factored as

$$C(t)\Phi(t, \tau) B(\tau) = C(t) \Phi(t, 0) \Phi(0, \tau) B(\tau)$$

so that $H(t) = C(t)\Phi(t, 0)$, $F(t) = \Phi(0, t)B(t)$. Conversely, given $H(t)$, $F(t)$, the state-space model

$$\begin{aligned} \dot{x}(t) &= F(t) u(t) \\ y(t) &= H(t) x(t) \end{aligned} \tag{24}$$

is a realization for $H(t)F(\tau)$, because the state transitional function $\Phi(t, \tau)$ is the identity.

Next, consider the problem of determining whether a state-space realization is a minimal realization. The answer is tied to the concepts of controllability and observability of state-space models discussed in greater depth in CONTROLLABILITY AND OBSERVABILITY.

Theorem (3). Suppose the linear system Eq. (22) is a realization of the weighting pattern $G(t,\tau)$. Then, Eq. (22) is a minimal realization if and only if for some t_0 and $t_f > t_0$, Eq. (22) is controllable and observable on the interval $[t_0, t_f]$.

Another important question is determining the existence of a time-invariant realization for a weighting pattern. The answer is provided in the following result.

Theorem (3). A weighting pattern $G(t,\tau)$ is realizable by a time-invariant linear system Eq. (23) if and only if it is realizable, continuously differentiable in both t and τ, and $G(t,\tau) = G(t - \tau, 0)$ for all t and τ. Under these conditions, there exists a time-invariant minimal realization of $G(t,\tau)$.

Linear Time-Invariant MIMO Systems

The analysis of linear MIMO systems in the time domain is greatly simplified in the time-invariant case by using results from linear algebra. Furthermore, the time invariance property allows applying transform techniques to represent the behavior of the system in the frequency domain, as illustrated in the next subsection. This subsection discusses the analysis of causal LTI MIMO systems described by state-space models in the time domain. Relevant concepts from linear algebra are introduced as needed.

Consider a MIMO LTI system described in state space form as

$$\begin{aligned} \frac{d}{dt}x(t) &= Ax(t) + Bu(t) \\ y(t) &= Cx(t) + Du(t) \end{aligned} \tag{25}$$

where the state $x(t) \in R^n$, the input $u(t) \in R^m$, the output $y(t) \in R^p$, and the matrices A, B, C, D do not depend on time. The algebraic structures of the matrices A, B, C, D completely determine the qualitative behavior of the system, as is evident after some concepts from linear algebra are reviewed.

Eigenvalues and Eigenvectors. Let C^n denote the space of n-dimensional, complex-valued vectors. Consider a vector $v \in C^n$ of dimension n, expressed in terms of its real and imaginary parts as

$$v = \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix} = \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix} + j \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix} = a + jb$$

where a_i, b_i, $i = 1, \ldots, n$ are real-valued. Denote by v^H the *Hermitian* of the vector v, defined as the complex conjugate of the transpose of v:

$$v^H = a^T - jb^T$$

Given two vectors u, $v \in C^n$, the inner product of u and v is defined as the complex-valued scalar product $\langle u, v\rangle = u^H v$. The *Euclidean norm* of a vector $v \in C^n$ is given by

$$\|v\|_2 = \sqrt{v^H v} = \sqrt{\sum_{i=1}^{n} (a_i^2 + b_i^2)}$$

The norm $\|\cdot\|_2$ is used to determine the size of vectors $v \in C^n$ and corresponds to the standard notion of vector length.

Denote by $C^{n\times n}$ the space of $(n \times n)$-dimensional, complex-valued matrices. Let $M \in C^{n\times n}$ be a square matrix. An eigenvalue λ of M is a complex number which is a root of the characteristic polynomial of M:

$$\det(\lambda I - M) = 0$$

Associated with each distinct eigenvalue are nonzero left and right eigenvectors u, $v \in C^n$, which satisfy the linear equations

$$Mv = \lambda v$$

and

$$u^H M = \lambda u^H$$

In the special case where the matrix M is real-valued ($M \in R^{n\times n}$), if λ is an eigenvalue of M with a nonzero imaginary part, then its complex conjugate is also an eigenvalue of M.

Because the previous equations are linear any multiple of an eigenvector is also an eigenvector. Thus, eigenvectors can be scaled to have any nonzero magnitude. In the rest of this chapter, assume that right and left eigenvectors are scaled so that the inner product of eigenvectors for the same eigenvalue is 1, that is, $u_i^H v_i = 1$ for all $i = 1, \ldots, n$. A useful property of eigenvectors is that the right and left eigenvectors corresponding to different eigenvalues are mutually orthogonal. Let u_i and v_i denote, respectively, the left and right eigenvectors corresponding to eigenvalue λ_i. Then,

$$u_i^H v_j = 0$$

whenever $\lambda_i \neq \lambda_j$.

Now consider a matrix $M \in C^{n\times n}$ with n distinct eigenvalues λ_i, $i = 1, \ldots, n$, and, with corresponding left and right eigenvectors u_i, v_i, $i = 1, \ldots, n$. It can be shown that the n eigenvectors v_i form a basis for the space C^n. In this case, the matrix M is represented in terms of these eigenvalues and eigenvectors in a dyadic expansion as

$$M = \sum_{i=1}^{n} \lambda_i v_i u_i^H \tag{26}$$

This expansion is useful for computing powers of M, because

$$M^k = \sum_{i=1}^{n} \lambda_i^k v_i u_i^H$$

resulting from the orthogonality property of left and right eigenvectors corresponding to different eigenvalues.

System Modes. Consider the LTI state-space model in Eq. (25). In the absence of inputs $u(t)$, the response of the system is determined by the initial condition $x(0)$ and the system matrix A, as

$$x(t) = e^{At}x(0)$$

Assume that the matrix A has distinct eigenvalues λ_i, $i = 1, \ldots, n$, with corresponding left and right eigenvectors u_i, v_i, $i = 1, \ldots, n$. A treatment of the general case with repeated eigenvalues is found in (12). Using Eq. (26) in the expansion of e^{At} yields

$$\begin{aligned} e^{At} &= (At)^0 + At + A^2\frac{t^2}{2} + \cdots \\ &= \sum_{i=1}^{n} (\lambda_i t)^0 v_i u_i^H + \sum_{i=1}^{n} (\lambda_i t) v_i u_i^H + \sum_{i=1}^{n} \frac{(\lambda^i t)^2}{2} v_i u_i^H + \cdots \\ &= \sum_{i=1}^{n} e^{\lambda_i t} v_i u_i^H \end{aligned}$$

Thus, the unforced system response is given as

$$x(t) = \sum_{i=1}^{n} e^{\lambda_i t} v_i [u_i^H x(0)]$$

This is interpreted as follows: The initial condition $x(0)$ is decomposed into its contributions along n different system modes using the left eigenvectors. The ith system mode is defined as $e^{\lambda_i t} v_i$ and has its own characteristic exponent λ_i. When the initial condition x_0 corresponds to a right eigenvector v_i, the state response $x(t) = e^{\lambda_i t} v_i$ is focused along the same direction v_i.

The system modes are also used to understand the output response of the system in the presence of input signals $u(t)$. Substituting the dyadic expansion of e^{At} into the output response Eq. (13) yields

$$y(t) = Ce^{At}x(0) + \int_0^t C(t)e^{A(t-\tau)}Bu(\tau)\,d\tau + Du(t)$$
$$= \sum_{i=1}^{n} e^{\lambda_i t}[(Cv_i)(u_i^H x(0)]$$
$$+ \sum_{i=1}^{n}(Cv_i)(u_i^H B)\int_0^t e^{\lambda_i(t-\tau)}u(\tau)\,d\tau + Du(t)$$

The term $u_i^H B$ indicates how the control action affects the ith mode. Similarly, the term Cv_i shows how much the ith mode affects the system output $y(t)$. Thus, modal analysis of LTI systems decomposes the performance of MIMO systems into a superposition of n independent modes which are exited by the input signals and initial condition.

Based on Eq. (27), one can derive intuitive conditions for controllability and observability of LTI systems using the system modes. In particular, note that the ith mode is uncontrollable if $u_i^H B = 0$, because the input has no effect on the ith mode trajectory. Thus, controllability requires that $u_i^H B \neq 0$ for all modes $i = 1, \ldots, n$. Similarly, the ith mode does not affect the output if $Cv_i = 0$. In this case, an initial condition of v_i yields an identical output to an initial condition of 0 and thus is unobservable. Observability requires that $Cv_i \neq 0$ for all modes $i = 1, \ldots, n$.

Example. Consider the state-space model specified by the following matrices:

$$A = \begin{bmatrix} -1 & 1 & 2 \\ 0 & -2 & 1 \\ 0 & 0 & -3 \end{bmatrix}$$
$$B = \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$$
$$C = \begin{bmatrix} 0.5 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$
$$D = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$$

The eigenvalues of A are $\lambda_1 = -1$, $\lambda_2 = -2$, $\lambda_3 = -3$. A set of left and right eigenvectors is given by

$$u_1 = \begin{bmatrix} 1 \\ 1 \\ 1.5 \end{bmatrix}, v_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}; u_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix},$$
$$v_1 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}; u_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, v_1 = \begin{bmatrix} -0.5 \\ -1 \\ 1 \end{bmatrix}$$

Using these modes, it is straightforward to verify that $Cv_i \neq 0$ and $u_i^H B \neq 0$ for $i = 1, \ldots, 3$, which establishes that the system is controllable and observable.

MIMO Transfer Function Matrix. One of the powerful tools of classical single-input, single-output (SISO) control theory is frequency-domain analysis using transform methods for LTI systems. SISO systems are often characterized by their transfer functions relating input signals to output signals in the frequency domain with Laplace transforms. Transform techniques are also applied to LTI MIMO systems to obtain generalizations of system transfer functions to MIMO systems, as follows.

Consider an LTI MIMO system, characterized by the impulse response matrix-valued function $H(t)$, which describes the input-output behavior of the system as

$$y(t) = \int_{-\infty}^{\infty} H(t-\tau)u(\tau)\,d\tau \tag{27}$$

where $H(t)$ includes generalized functions, such as the unit impulse $\delta(t)$.

Let $X(s)$ denote the bilateral Laplace transform of the function $x(t)$:

$$X(s) = \int_{-\infty}^{+\infty} x(t)e^{-st}\,dt \tag{28}$$

For the MIMO LTI system Eq. (27), application of Laplace transforms on both sides yields

$$Y(s) = \mathscr{H}(s)U(s) \tag{29}$$

where $Y(s)$ is the p-dimensional, two-sided Laplace transform of the output $y(t)$, $U(s)$ is the m-dimensional, two-sided Laplace transform of the input $u(t)$, and $\mathscr{H}(s)$ is the $p \times m$ two-sided Laplace transform of the impulse response $H(t)$, called the *system transfer function matrix.* In coordinates, this relationship is given by

$$Y_i(s) = \sum_{k=1}^{m} \mathscr{H}_{ik}(s)U_k(s), \qquad k = 1, \ldots, p$$

where $\mathscr{H}_{ik}(s)$ is the Laplace transform of $H_{ik}(t)$.

For causal LTI systems described in state-space form, as in Eq. (25), the transfer function matrix is obtained directly from the state-space representation. Assume that the system is at rest with no initial conditions. Taking bilateral Laplace transforms of both sides in Eq. (25) yields

$$sX(s) = AX(s) + BU(s)$$

and

$$Y(s) = CX(s) + DU(s)$$

Solving these equations simultaneously,

$$Y(s) = [C(sI - A)^{-1}B + D]U(s)$$

which yields the transfer function matrix $\mathscr{H}(s) = C(sI - A)^{-1}B + D$. Note that, although there can be different state-space models for a given LTI system, the transfer function matrix $\mathscr{H}(s)$ is unique.

There are some special properties of system transfer function matrices of MIMO LTI systems. First, the variable s enters into the expression in the inverse $(sI - A)^{-1}$. If A is an $(n \times n)$ matrix, this means that the entries of $\mathcal{H}(s)$ are rational functions, ratios of polynomials, with denominator degree no greater than n. Furthermore, the numerator degree is no greater than n either and is strictly less than n for all entries unless $D \neq 0$. Transfer function matrices with entries as rational functions with numerator degree less than or equal to denominator degree are known as *proper*. If the numerator degree is strictly less than the denominator degree for each entry, the transfer function matrix is known as *strictly proper*.

Multivariable Poles and Zeros. For SISO LTI systems, the poles and zeros of the system are determined from the transfer function, consisting of a ratio of a numerator polynomial and a denominator polynomial. The roots of the numerator polynomial determine the zero frequencies of the system, frequencies which, if present at the input, are blocked by the system and are thus not present at the output. Similarly, the roots of the denominator polynomial determine the poles that are frequencies appearing at the output in response to initial conditions with no external input.

Although a rich theory exists for generalizing the SISO transfer function decomposition to transfer function matrices of MIMO systems using polynomial matrices and matrix fraction descriptions (see e.g. [12]), the simplest definition of MIMO poles and zeros is given in terms of state-space models. Consider the LTI state-space model of Eq. (25). The poles of the system are the complex frequencies that appear in the output in response to initial conditions. Based on the discussion of the previous subsections, these frequencies are the eigenvalues of the matrix A. This is also seen directly from the transfer function matrix $\mathcal{H}(s) = C(sI - A)^{-1}B + D$. Using the expression for inverting a matrix, it is clear that the denominator of all of the entries in the transfer function matrix is given by $\det(sI - A)$. Thus, the poles correspond to roots of the equation $\det(sI - A) = 0$, which are the eigenvalues of A.

In contrast with multivariable poles, there are several ways in which zeros have been defined for LTI MIMO systems. First consider a system with equal number of inputs and outputs ($m = p$), and assume that the state-space model in Eq. (25) is minimal (thus controllable and observable). Multivariable transmission zeros are defined as complex frequencies where, given a particular nonzero combination of input directions at that frequency and initial conditions, there is no output generated by the system. The formal definition is given here:

Definition. The system Eq. (25) has a *transmission zero* at the complex frequency s_k if there exist complex vectors $u_k \in C^m$, $x_k \in C^n$, one of which is nonzero, such that the system Eq. (25) with initial condition $x(0) = x_k$, and input $u(t) = u_k e^{s_k t}$, $t \geq 0$ has the property that $y(t) = 0$ for all $t > 0$.

The initial condition x_k must be chosen carefully to ensure that the state trajectory does not contain modes other than those of the input $e^{s_k t}$, because those modes are observable (the minimality assumption) and lead to nonzero outputs. Thus, $x(t) = x_k e^{s_k t}$ is a solution for the trajectory of the system. Substituting this solution in the system Eq. (25) with input $u(t) = u_k e^{s_k t}$ gives the following equations after dividing by the term $e^{s_k t}$:

$$\begin{aligned} s_k x_k &= A x_k + B u_k \\ 0 &= C x_k + D x_k \end{aligned}$$

Rearranging these equations as a set of linear equations in the unknowns x_k, u_k yields

$$\begin{pmatrix} s_k I - A & -B \\ -C & -D \end{pmatrix} \begin{pmatrix} x_k \\ u_k \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

This is in the form of a generalized eigenvalue problem. Indeed, under the assumption of a minimal realization, the MIMO transmission zeros are obtained as the roots of the following equation:

$$\det \left[s \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} A & B \\ C & D \end{pmatrix} \right] = 0$$

For a given transmission zero s_k, the generalized eigenvector associated with that transmission zero provides the initial condition and input directions x_k, u_k which yield zero output at that input frequency.

Example. Consider the MIMO LTI system described by state-space matrices

$$A = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -3 \end{bmatrix}$$

$$B = \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$C = \begin{bmatrix} 0.5 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

$$D = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$$

The transfer function matrix for this two-input, two-output system is given by

$$\mathcal{H}(s) = \begin{bmatrix} \frac{s+1.5}{s+1} & 0 \\ \frac{s+3}{s+2} & \frac{1}{s+3} \end{bmatrix}$$

Because A is diagonal, the poles of the system are easily determined as -1, -2, -3. Solving the generalized eigenvalue problem, one obtains two transmission zeros, at frequencies -1.5 and -2. In particular, the input

$$u(t) = \begin{bmatrix} 1 \\ -4.5 \end{bmatrix} e^{-1.5t}$$

with initial condition

$$x(0) = \begin{bmatrix} -2 \\ 2 \\ -3 \end{bmatrix}$$

yields output $y(t) = 0$. Similarly, the input

$$u(t) = \begin{bmatrix} 0 \\ 1 \end{bmatrix} e^{-2t}$$

with initial condition

$$x(0) = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$$

also yields $y(t) = 0$ for $t > 0$. Note the presence of both a zero and a pole at -2, without a pole-zero cancellation. Note also that the zero at -2 is not a zero of any individual transfer function entry in the transfer function matrix.

Now consider the general case where the number of inputs m is different from the number of outputs p and the state space model Eq. (25) is still controllable and observable. If the number of inputs is less than the number of outputs, the appropriate generalization is the concept of a right transmission zero, as defined here:

Definition. The system Eq. (25) has a *right transmission zero* at the complex frequency s_K if there exist complex vectors $u_k \in C^m$, $x_k \in C^n$, both of which are not identically zero, such that

$$\begin{aligned} s_k x_k &= A x_k + B u_k \\ 0 &= C x_k + D x_k \end{aligned}$$

In essence, a right transmission zero is a complex frequency where, for an appropriate input direction and initial condition, the output is identically zero. When the number of inputs m is greater than the number of outputs p, there are additional complications, because there can exist nonzero proper $(m \times 1)$ transfer functions $U(s)$ such that $\mathscr{H}(s)U(s) = 0$ for all s! Instead of declaring every complex frequency a transmission zero, one defines the concept of a left transmission zero, as follows:

Definition. The system Eq. (25) has a *left transmission zero* at the complex frequency s_k if there exist complex vectors $\alpha_k \in C^m$, $\beta_k \in C^n$, both of which are not identically zero, such that

$$\begin{aligned} s_k \alpha_k^T &= \alpha_k^T A + \beta_k^T C \\ 0 &= \alpha_k^T B + B_k^T D \end{aligned}$$

that is, a left transmission zero is a right transmission zero of the state-space model

$$\begin{aligned} \frac{d}{dt} x(t) &= A^T x(t) + C^T u(t) \\ y(t) &= B^T x(t) + D^T u(t) \end{aligned} \tag{30}$$

For square systems, any frequency that is a left transmission zero is also a right transmission zero.

As a final note on this topic, consider a state-space realization that is not minimal, so that it is either unobservable or uncontrollable. Then, any uncontrollable mode of the system λ_k with left eigenvector u_k is a left transmission zero with direction $\alpha_k = 0$, $\beta_k = u_k$. Any unobservable mode of the system with right eigenvector v_k is a right transmission zero with directions $x_k = v_k$, $u_k = 0$. Thus, the presence of unobservable and uncontrollable modes gives rise to transmission zeros in the same directions as the modes of the system, leading to pole-zero cancellations.

Singular Values and MIMO Frequency Response

Consider an LTI MIMO system, specified by its transfer function matrix $\mathscr{H}(s)$. Assume that the system is bounded-input, bounded-output stable, with no initial conditions. The transfer function $\mathscr{H}(s)$ can be interpreted as the complex gain of the linear system in response to bounded inputs of the form e^{st}. That is, if the input is defined as $u(t) = ve^{s_0t}$ for $t \geq 0$ for some complex number s_0 with nonpositive real part and some direction vector $v \in R_m$, the output $y(t)$ is given by

$$y(t) = \mathscr{H}(s_0) v e^{s_0 t}$$

The frequency response of the system is the set of transfer functions $\mathscr{H}(j\omega)$ for all frequencies $\omega \in R$. Thus, the frequency response of the system defines the outputs corresponding to sinusoidal inputs of the form $e^{j\omega t}$.

In single-input, single-output (SISO) systems, the transfer function is a scalar. Thus, the frequency response is characterized by the complex-valued function $\mathscr{H}(j\omega)$, which is represented by a magnitude and phase. In contrast, the frequency response of MIMO systems is a complex, matrix-valued function of the frequency, which has a range of gains, depending on the direction a of the sinusoidal input. To understand how to represent this effect, it is useful to review some concepts of gains for complex-valued matrices.

Complex Matrices and Gains. At a specific frequency, the transfer function matrix $\mathscr{H}(j\omega)$ is a complex-valued matrix of dimension $p \times m$. Denote by $C^{p\times m}$ the space of complex-valued matrices of dimension $p \times m$. Any matrix $M \in C^{p\times m}$, is decomposed into its real and imaginary parts, as

$$M = A + jB$$

where $A, B \in R^{p\times m}$. In a manner similar to a vector, the *Hermitian* of a matrix is defined as the complex conjugate of its transpose, that is,

$$M^H = A^T - jB^T$$

Given a matrix $M \in C^{p\times m}$, the *spectral norm* of the matrix, denoted as $\|M\|_2$, is the maximum amplification of any input unit vector, defined as

$$\|M\|_2 = \max_{\|v\|_2} \|Mv\|_2$$

A complex-valued square matrix is called *Hermitian* if $M^H = M$. A nonsingular, complex-valued matrix is called *unitary* if $M^{-1} = M^H$, which implies $MM^H = M^H M = I$. Hermitian matrices have the property that all of their eigenvalues are real-valued. This can be readily derived by noting that, for an eigenvalue λ with right eigenvector u,

$$(u^H M u)^H = (\lambda u^H u)^H = \lambda^H u^H u$$
$$= u^H M^H u = u^H M u = \lambda u^H u$$

which establishes that λ is equal to its complex conjugate and thus is a real number. Hermitian matrices also have the property that repeated eigenvalues have a full complement of eigenvectors, and thus Hermitian matrices are represented as diagonal matrices with an appropriate change of basis.

The eigenvalues of unitary matrices have unit magnitude, as is readily seen from

$$(u^H M^H)(Mu) = (\lambda^H u^H)(\lambda u) = |\lambda|^2 u^H u$$
$$= u^H (M^{-1} M) u = u^H u$$

Thus, unitary matrices acting on vectors preserve the Euclidean norm. Let $M \in C^{p\times p}$ be unitary and $u \in C^p$ be an arbitrary vector. Then

$$\|Mu\|_2 = \sqrt{(Mu)^H Mu} = \sqrt{u^H (M^H M) u} = \sqrt{u^H u}$$

Now consider an arbitrary matrix $M \in C^{p\times m}$. The square matrices $M^H M$ and MM^H are Hermitian and thus have real-valued eigenvalues. They also have the additional property that the eigenvalues are nonnegative, and the nonzero eigenvalues of MM^H are equal to the nonzero eigenvalues of $M^H M$. Let λ denote an eigenvalue of MM^H with eigenvector v. Then,

$$v^H (MM^H) v = \lambda \|v\|_2^2 = (M^H v)^H M^H v = \|M^H v\|_2^2 \geq 0$$

which shows that $\lambda \geq 0$. If λ is a nonzero eigenvalue of MM^H with eigenvector u, then

$$M^H (MM^H) u = M^H \lambda u = \lambda (M^H u) = (M^H M) M^H u$$

which establishes that λ is also an eigenvalue of $M^H M$ with eigenvector $M^H u$. Note that $M^H u$ must be nonzero if λ is nonzero.

For a general matrix $M \in C^{p\times m}$ with rank k, the *singular values* of M are the k square roots of the nonzero eigenvalues of $M^H M$ or MM^H. Let $\sigma_i(M)$, $i = 1, \ldots, k$ denote the k singular values and $\lambda_i(M^H M)$ denote the corresponding k nonzero eigenvalues of $M^H M$. Then,

$$\sigma_i(M) = \sqrt{\lambda_i(M^H M)} = \sqrt{\lambda_i(MM^H)}, \quad i = 1, \ldots, k$$

Because the k nonzero eigenvalues of $M^H M$ are real and positive, the singular values are also real and positive. Assume that the singular values are ordered in descending order, that is,

$$\sigma_1(M) \geq \sigma_2(M) \geq \ldots \geq \sigma_k(M) > 0$$

The singular-value decomposition of a matrix $M \in C^{p\times m}$ states that there are convenient changes of bases in C^p and C^m, so that the linear transformation M can be visualized as a nearly diagonal transformation. Define the matrix of singular values Σ as

$$\Sigma = \left\{ \begin{matrix} \begin{bmatrix} \sigma_1 & 0 & \cdots & 0 \\ 0 & \sigma_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \sigma_k \end{bmatrix} & 0_{k\times(m-k)} \\ 0_{(p-k)\times k} & 0_{(p-k)\times(m-k)} \end{matrix} \right\}$$

Note that $k \leq \min(m, p)$; if $k = m$ or $k = p$, some of the zero blocks in the above matrix are removed. The singular value decomposition states the following:

Theorem. Given a matrix $M \in C^{p\times m}$, there exist a $p \times p$ unitary matrix U and $m \times m$ unitary matrix V such that

$$M = U\Sigma V^H$$
$$\Sigma = U^H M V$$

The column vectors of U are called the left singular vectors of M, and the column vectors of V are called the right singular vectors of M. Because U and V are unitary, they correspond to an orthogonal change of basis in C^p and C^m, respectively. It is easy to show that the left singular vectors of M are the normalized right eigenvectors of the $p \times p$ matrix MM^H and that the right singular vectors of M are the normalized right eigenvectors of the $m \times m$ matrix $M^H M$. Reliable and efficient numerical techniques for computing singular-value decompositions are available in commercial software packages.

The singular-value decomposition allows us to estimate the gain of the matrix M when acting on an input u of unit Euclidean norm, as follows. Let $y = Mu$ denote the output of M with input u. Using the singular-value decomposition yields

$$y = U\Sigma V^H u = U\Sigma v$$

where v is also a unit norm vector because V is unitary. Then,

$$\|y\|_2 = \|U^H y\|_2 = \|U^H U \Sigma v\|_2 = \|\Sigma v\|_2$$

This establishes that

$$\|M\|_2 \leq \sigma_1(M)$$

If u is the first column of V, the unitary property of V gives

$$V^H u = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

which shows that

$$\|M\|_2 \geq \sigma_1(M)$$

which establishes that the spectral norm of M is equal to the maximum singular value.

It is also possible to establish a lower bound on the gain of the matrix M under the condition that the number of outputs

p is greater than or equal to the number of inputs m. Let $m = \min(p, m)$. When the rank of M is less than m, define the singular values $\sigma_{k+1} = \ldots = \sigma_m = 0$. Then, a similar argument as previous establishes that, for unit norm vectors m,

$$\|Mu\|_2 \geq \sigma_m \geq 0$$

If the rank of M is m, the lower bound is strictly positive. When the number of outputs is less than the number of inputs ($p < m$), M must have a nontrivial null space, and thus the lower bound is always 0.

Singular-Value Representation of MIMO Frequency Response. Now consider the MIMO frequency response of a bounded-input, bounded-output stable system with transfer function matrix $\mathscr{H}(s)$. Assume that the number of inputs m is less than or equal to the number of outputs p. When the input vector is a complex exponential of the form $u(t) = ae^{j\omega t,}$ the output vector is given by

$$y(t) = \mathscr{H}(j\omega)u(t) = \mathscr{H}(j\omega)ae^{j\omega t} = be^{j\omega t}$$

for some complex vector $b = \mathscr{H}(j\omega)a$. A useful characterization of the MIMO frequency response is provided in terms of bounds on the gain of complex matrix $\mathscr{H}(j\omega)$ as a function of frequency.

For each frequency ω, the singular-value decomposition of $\mathscr{H}(j\omega)$ is obtained as

$$\mathscr{H}(j\omega) = U(j\omega)\Sigma(\omega)V^H(j\omega)$$

with nonnegative singular values $\sigma_1(\omega), \ldots, \sigma_m(\omega)$. Assume that a has unit Euclidean norm. Then, for each frequency, the maximum and minimum singular values $\sigma_1(\omega)$, $\sigma_m(\omega)$ are available, and

$$\sigma_1 \geq \|\mathscr{H}(j\omega)a\|_2 \geq \sigma_m$$

A simple way of visualizing the gain of the transfer function matrix is to plot the maximum and minimum singular values, expressed in decibels, in a Bode plot against frequency in radians/second on a semilog scale. Figure 2 illustrates the MIMO frequency response plot for the transfer function matrix

$$H(s) = \begin{bmatrix} \frac{s+1.5}{s+1} & 0 \\ \frac{s+3}{s+2} & \frac{1}{s+3} \end{bmatrix}$$

The information contained in such a Bode plot provides direction-independent information concerning the magnitude of the frequency response at specific frequencies. In particular, at frequencies where the minimum singular value is large, all of the singular values of the system are large, and thus the system has a large gain in all directions. In regions where the maximum singular value is small, all of the singular values are small, and the system has a small gain in all directions.

At other frequencies, it is necessary to use the direction of the input a to determine the magnitude of the frequency response. The maximum and minimum singular values provide bounds on the range of gains which are possible. The unitary matrix $V(j\omega)$ is a change of basis transformation on the input space. Thus, $V^H(j\omega)a$ is an m-dimensional complex vector which is a decomposition of a into components along the right singular vectors which form a new basis in the input space. The magnitude and phase of the frequency response for each singular direction are readily evaluated from the matrices $\Sigma(\omega)$ and $U(j\omega)$.

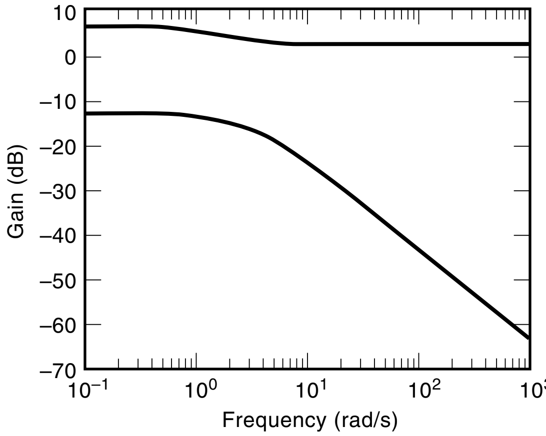

Figure 2. Singular-value MIMO frequency response.

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