NONLINEAR CONTROL SYSTEMS, DESIGN METHODS

A basic problem in control theory may be described as follows: Given a plant, design a control mechanism in such a way that the plant together with the controller meets certain design specifications. The above *regulation problem* arises in numerous situations; for instance, the temperature in a house is regulated by a thermostat to keep the temperature in the house constant, notwithstanding changing external effects such as outdoor temperature, wind, open doors, and so on. Other regulation devices in everyday life are easy to find: washing machines, modern automobiles, and so on.

Probably the first mathematical study on regulation ever published was written by J. C. Maxwell (1831–1870). His paper "On governors" published in the *Proceedings of the Royal Society of London* in 1868 treats the problem of tuning centrifugal governors to achieve fast regulation towards a constant speed, thereby avoiding oscillatory motions ("hunting") of a steam engine.

Clearly, in the past century a lot of theoretical and practical work has been carried out on the regulation problem, and it is certainly beyond the scope of this article to present a historical review of the subject; readers further interested in the subject are referred to Ref. 1. However, one particular type of controllers, the so-called proportional integral differential (PID) controller, originally proposed by N. Minorsky in 1922, deserves separate mentioning.

In a PID controller the control signal is built up as a weighted sum of three terms; a proportional term (proportional to the error between the actual and desired value of the to-be-controlled plant's output) drives the plant's output to the reference. An integral term (of the error) compensates for the steady-state error caused by uncertainties in the plant's model, and a differential term (proportional to the time derivative of the plant's output) speeds up the convergence towards the desired reference. The PID controller has had and still has many applications in technical systems.

More recent methods in nonlinear control theory that should be mentioned are feedback linearization, passivity-based control, and Lyapunov control.

The feedback linearization approach applies to a small class of systems for which it is possible to use a nonlinear control law which, given an appropriate coordinate change, cancels all nonlinearities in the system. The rationale behind this approach is that the resulting closed-loop system is linear, and thus linear control theory is then applicable. A drawback is that this technique may fail if one does not know the plant's model accurately; this uncertainty can lead to instability or, in the best case, to a steady-state error.

An alternative approach is the so-called passivity-based control. This technique applies to a certain class of systems which are "dissipative with respect to a storage function" (2). Passive systems constitute a particular case of dissipative systems for which the storage function happens to be an energy function. Hence, the rationale behind the passivity-based approach is physical: Roughly speaking, a passive system is a system from which one cannot pull out more energy than is fed in. A very simple example of a passive system is a conventional *RLC* network, which dissipates part of the supplied electrical energy in the form of heat. A fundamental property of passive systems is that the interconnection of two passive systems is passive. With this motivation, in passivity-based control one aims at designing passive controllers, so that the closed-loop system have some desired energy properties. In many cases, seeking for passive controllers results in compensating instead of canceling the nonlinearities of the system, which can give considerably more robust results than using feedback linearization.

Modern control theory leans on the so-called *Lyapunov stability* theory which was launched by the Russian mathematician A. M. Lyapunov in his celebrated article (3). The Lyapunov theory consists of a set of mathematical tools, including comparison equations, which help us to analyze the asymptotic behavior of the solutions of a (possibly nonlinear and time-varying) differential equation. The advantage of this theory is that it allows us to know the asymptotic behavior of the solutions without solving the differential equation. The price paid for this is that one must find a suitable Lyapunov function for the system in question, which satisfies some desired properties. More precisely, the Lyapunov function is positive definite while its time derivative is negative definite. This is in general not an easy task.

Thus, the Lyapunov control approach consists of proposing a positive definite Lyapunov function candidate and designing the control input in such a way that its time derivative becomes negative definite. Then, some conclusions about the stability of the system can be drawn.

Although in general, it is very difficult to find a suitable Lyapunov function, often a good start is to use the total energy function (if available) of the system in question. This may motivate us to think that the passivity-based approach and Lyapunov control are very related since, for a physical system, the storage function is the total energy of the system. Nevertheless, it must be pointed out that the passivity-based approach is based upon the *input–output* properties of the system; that is, the system is viewed as an operator which transforms an input into some output, regardless of the internal state of the system. The Lyapunov control approach is based upon the asymptotic behavior of the system's state. Both methods are complementary to one another.

We consider in more detail passivity-based and feedback linearization schemes, and the reader may consult Ref. 4 and

5 for introductory texts on Lyapunov theory. We consider from a mathematical perspective the regulation problem of an important class of nonlinear *passive* systems: the so-called Euler–Lagrange (EL) systems. This class of systems includes various mechanical systems such as the robot manipulators.

Our motivation to illustrate these control techniques by addressing the regulation problem for EL systems is multiple: Not only are EL systems passive (hence passive controllers do a good job for this class), but also the stability theory of Lyapunov was inspired upon the previous work of J. H. Poincaré (1854–1912) on second-order nonlinear systems and, in particular, mechanical systems. Moreover, even though the Lagrangian formulation is most popular in mechanics, it must be remarked that it applies to a wide class of physical systems which are modeled using variational principles. Thus, on one hand, the EL class is fairly wide; hence the available results are applicable to many physical (industrial) systems. On the other hand, the theory is simple enough to be treated in a few pages and to give the reader a general appreciation of the flavor of nonlinear systems control design in this context.

We consider the regulation problem which consists of designing a feedback that enforces asymptotic tracking of the output to be controlled (e.g., the endpoint of a rigid manipulator) towards a given desired trajectory. Provided that an exact model of the plant is given and assuming that full state (that is, joint position and velocity) measurements are available, a tracking controller can easily be designed, like for instance the computed torque controller (6). See, for example, Refs. 7, 8, and 18 for a literature review.

Unfortunately, in many cases, neither an exact model is available, nor are joint velocities measured. The latter problem can be resolved by introducing an observer, a dynamic system which uses only position information (or more general, the measurable output) to reconstruct the velocity signal (or more general, the unmeasured part of the state). Then, the controller is implemented by replacing the velocity by its estimate. It is interesting to note that even though it is well known that the separation principle (see section entitled "Linear Time-Invariant Systems") does not apply for nonlinear systems (specifically, an observer that asymptotically reconstructs the state of a nonlinear system does not guarantee that a given stabilizing state feedback law will remain stable when using the estimated state instead of the true one), the rationale behind this approach is precisely that the estimate will converge to the true signal and, in some particular cases of EL systems, this in turn entails stability of the closed loop.

To cope with a nonexact plant model, one may proceed in different ways, depending on to what extent the model uncertainties appear. In essence, discussion in this article will be limited to the occurrence of some unknown (linearly depending) parameters, such as an unknown mass of the end-tool, or there will be limited discussion of the model structure. In the first case a parameter adapting or a PID controller can be used whereas in the last case we return to a (high-gain) PD controller design. Some of the above-mentioned results for EL plants are discussed in some detail in the following sections and can be found in the literature (8,9). See also Ref. 7 for a comprehensive tutorial on adaptive control of robot manipulators.

One may wonder whether results as announced above for fully actuated systems also extend to other classes of nonlinear systems. Apart from the case where the system is assumed to be linear (see Refs. 10), the observer and observer–controller problems turn out to be difficult in general. At the moment, very few general results are at hand (11); as an illustration in the last section we discuss one specific mathematical example of a single-input single-output nonlinear system.

Notation. In this article, $\|x\| \triangleq \sqrt{x^\top x}$ is the Euclidean norm of $x \in \mathbb{R}^n$. The largest and smallest eigenvalues of a square symmetric matrix K are k_M and k_m, respectively. The extended space of square integrable signals is $\mathcal{L}_{2e}^n \triangleq \{u \in \mathbb{R}^n,\ T > 0 : \int_0^T \|u(t)\|^2\, dt < \infty\}$. We denote $B_\eta = \{x \in \mathbb{R}^n : \|x\| \leq \eta\}$ a ball of radius η, centered at the origin in $\mathbb{R}^n$.

THE REGULATION PROBLEM

We define the general regulation problem for NL systems as follows:

Tracking Control of NL Systems. Consider the system

$$\dot{x} = f(t, x, u) \tag{1}$$

$$y = h(x) \tag{2}$$

where $x \in \mathbb{R}^n$ is the state, $u \in \mathbb{R}^k$ is the control input, $y \in \mathbb{R}^m$ is the output to be controlled, and functions f and h are continuous in their arguments and $t \geq 0$. Given a desired reference output trajectory $y_d(t)$ assume that there exists a (unique) "desired" state $x_d(t)$ and input $u_d(t)$ trajectories which generate $y_d(t)$, that is,

$$\dot{x}_d = f(t, x_d, u_d) \tag{3}$$

$$y_d = h(x_d) \tag{4}$$

Then the tracking control problem consists in finding, if possible, a smooth control law u such that for any bounded initial conditions (t_0, x_0), $x_0 = x(t_0)$,

$$\lim_{t\to\infty} \tilde{y}(t) \triangleq \lim_{t\to\infty} (y(t) - y_d(t)) = 0 \tag{5}$$

Remark 1. It is important to note that we have assumed that there exists $x_d(t)$ and $u_d(t)$ which are solutions of Eq. (3). This point is crucial since in practical applications one should pay attention to the definition of the desired output trajectory, which may not be realizable due to the structure of the system. In this article we do not address the so-called "trajectory generation problem" (see, for instance, Ref. 12) or the inversion problem considered in Ref. 13, but we simply assume that the triple (x_d, u_d, y_d) is given.

Note that in the problem formulation above we have not imposed any specific structure for the control law; however, one can distinguish (at least) the following variants of the tracking control problem depending on the measurements which are available:

1. (State feedback). Assume that the full state is available for measurement; then find, if possible, a control law u of the form
   (a) (Static state feedback) $u = \alpha(t, x, x_d)$, or
   (b) (Dynamic state feedback) $u = \alpha(t, x, x_d, x_c)$, $\dot{x}_c = \psi(t, x, x_d, x_c)$ with x_c being the dynamic compensator state, say $x_c \in \mathbb{R}^l$.

2. (Output feedback). Assume that only an output $z \triangleq k(x)$ is measurable; then find, if possible, a control law u of the form
   (a) (Static output feedback) $u = \alpha(t, z, z, x_d)$,or
   (b) (Dynamic output feedback) $u = \alpha(t, z, x_d, x_c)$, $\dot{x}_c = \psi(t, z, x_d, x_c)$ with, again, $x_c \in \mathbb{R}^l$ being the dynamic compensator state.

LINEAR TIME-INVARIANT SYSTEMS

Before presenting some results on state and output feedback control of nonlinear systems we briefly revisit some facts about linear systems theory. It is well known that the controller–observer design problem for linear systems is, completely solved (10), but the static output feedback problem is only partially understood. Consider the linear time-invariant (LTI) forced system

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx \end{aligned} \tag{6}$$

where $y \in \mathbb{R}^m$, $x \in \mathbb{R}^n$, $u \in \mathbb{R}^k$, and A, B, and C are matrices of suitable dimensions. Assume that we wish to achieve the tracking control for system (6), which in the linear case boils down to solving the tracking problem for $x_d \equiv 0$, with corresponding $y_d \equiv 0$ and $u_d \equiv 0$. In order to do so we use the linear state-feedback controller

$$u = -Kx \tag{7}$$

where K is chosen in a way that the closed-loop system

$$\dot{x} = (A - BK)x \tag{8}$$

is exponentially stable; that is, the matrix $(A - BK)$ must be Hurwitz. Indeed a necessary and sufficient condition for the matrix K to exist is the stabilizability of the system [Eq. (6)].

Now, as we have mentioned we are also interested in the output feedback control problem. To date, even for linear systems the tracking control problem using *static* output feedback is not yet completely solved, and we therefore look at the observer-design problem. For system [Eq. (6)] a linear observer is given by

$$\dot{\hat{x}} = A\hat{x} + Bu + L(y - \hat{y}) \tag{9}$$

$$\hat{y} = C\hat{x} \tag{10}$$

where L is an "output-injection" matrix. Note that the estimation error $\tilde{x} = x - \hat{x}$ dynamics has the form

$$\dot{\tilde{x}} = (A - LC)\tilde{x} \tag{11}$$

It is clear that one can find an L such that Eq. (11) be asymptotically stable—thus $\tilde{x} \to 0$ as $t \to \infty$—if the pair (A, C) is detectable (10).

At this point we have successfully designed an asymptotically stabilizing feedback and a asymptotically converging observer. The natural question which arises now is whether it is possible to use the state estimates in the feedback Eq. (7) as if they were the true ones, that is,

$$u = -K\hat{x} \tag{12}$$

where $\hat{x}$ follows from Eq. (9). To find an answer to this question we write the overall controlled system plus observer:

$$\begin{bmatrix} \dot{x} \\ \dot{\tilde{x}} \end{bmatrix} = \begin{bmatrix} A - BK & BK \\ 0 & A - LC \end{bmatrix} \begin{bmatrix} x \\ \tilde{x} \end{bmatrix} \tag{13}$$

and calculate the roots of its characteristic polynomial—that is, the roots of

$$p(s) = \det \begin{bmatrix} sI - A + BK & -BK \\ 0 & sI - A + LC \end{bmatrix}$$

or, equivalently, the roots of

$$p(s) = \det(sI - A + BK)\det(sI - A + LC)$$

That is, the characteristic polynomial of the overall system is the product of the characteristic polynomials of the observer [Eq. (11)] and the controlled system [Eq. (8)]. Thus one can design the controller and observer separately without caring if the true states are available or if the observer will be used in open or closed loop. This nice property is called the separation principle and, unfortunately, in general it is exclusive to linear systems.

For this reason, we are obliged to explore new techniques to achieve output feedback control for nonlinear systems.

FEEDBACK LINEARIZATION

Consider a single input nonlinear system

$$\dot{x} = f(x) + g(x)u \tag{14}$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}$, and f and g are smooth vector fields on $\mathbb{R}^n$; that is, their derivatives exist and are continuous up to infinite order. In this section we describe the feedback linearization approach for system [Eq. (14)] around an equilibrium point $x_0 \in \mathbb{R}^n$; that is, $f(x_0 = 0$. The local feedback linearization problem for the system [Eq. (14)] is to find—if possible—a static state feedback law

$$u = \alpha(x) + \beta(x)v, \qquad \beta(x_0) \neq 0 \tag{15}$$

and a smooth, local coordinate transformation

$$z = S(x), \qquad S(x_0) = 0 \in \mathbb{R}^n \tag{16}$$

such that the closed-loop system [Eqs. (14)–(15)] in the z coordinates is a controllable system and without loss of generality may be assumed to be in the Brunovski form:

$$\frac{d}{dt}\begin{bmatrix} z_1 \\ \vdots \\ \vdots \\ z_n \end{bmatrix} = \begin{bmatrix} 0 & 1 & & 0 \\ & \ddots & & \\ & & \ddots & 1 \\ 0 & & & 0 \end{bmatrix} \begin{bmatrix} z_1 \\ \vdots \\ \vdots \\ z_n \end{bmatrix} + \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} v \tag{17}$$

The local linearization problem is solvable only under quite restrictive conditions on f and g. In order to see this we will derive a set of sufficient conditions for the solvability of the linearization problem. Assuming there exists a feedback (15) and a coordinate transformation (16) that linearize the closed-loop system, we note the following. Let $z_i = S_i(x)$, $i = 1, \ldots, n$; then we have, using the first equation of Eq. (17)

$$\dot{z}_1 = \frac{\partial S_1(x)}{\partial x}(x) \cdot \dot{x} = \frac{\partial S_1(x)}{\partial x}(x) \cdot f(x) + \frac{\partial S_1(x)}{\partial x}(x) \cdot g(x)u = z_2 = S_2(x) \tag{18}$$

Defining $L_f S(x) = \partial S(x)/\partial x \cdot f(x)$ as the directional or Lie derivative of function $S(x)$ in the direction of f, we obtain from Eq. (18) that

$$S_2(x) = L_f S_1(x) \tag{19}$$

$$0 = L_g S_1(x) \tag{20}$$

In an analogous way we derive, using the ith equation of Eq. (17)

$$S_{i+1}(x) = L_f S_i(x), \qquad i = 1, \ldots, n-1 \tag{21}$$

$$0 = L_g S_i(x), \qquad i = 1, \ldots, n-1 \tag{22}$$

If we introduce the Lie bracket of two vector fields g_1 and g_2 on $\mathbb{R}^n$ denoted by $[g_1, g_2]$ as

$$[g_1, g_2](x) = \frac{\partial g_2}{\partial x}(x) g_1(x) - \frac{\partial g_1}{\partial x}(x) g_2(x) \tag{23}$$

then it follows that the function $S_1(x)$ should satisfy $S_1(x_0) = 0$ and the $n - 1$ conditions

$$L_g S_1(x) = L_{[f,g]} S_1(x) = L_{[f,[f,g]]} S_1(x) = L_{[f,\ldots,[f,g]\ldots]} S_1(x) = 0 \tag{24}$$

In other words, if the local feedback linearization problem is solvable, then there exists a nontrivial function $S_1(x)$ that satisfies the $n - 1$ partial differential equations [Eq. (24)]. The functions α and β in the feedback [Eq. (15)] are given as follows. Since

$$\dot{z}_n = f(x) S_n(x) + L_g S_n(x) u = v \tag{25}$$

we obtain from Eq. (15)

$$\alpha(x) = -(L_g S_n(x))^{-1} L_f S_n(x), \qquad \beta(x) = (L_g S_n(x))^{-1} \tag{26}$$

In general one may not expect that a nontrivial function $S_1(x)$ exists that fulfills Eq. (24). Writing the iterated Lie brackets of the vector fields f and g as $\mathrm{ad}_f^k g = [f, \mathrm{ad}_f^{k-1} g]$, $k = 1, 2, \ldots$, with $\mathrm{ad}_f^0 g = g$, one can derive the following necessary and sufficient conditions for the existence of $S_1(x)$ (see, for example, Refs. 13 and 14).

Theorem 1. Consider the system [Eq. (14)] about an equilibrium point x_0. The local feedback linearization problem is solvable about x_0 if and only if

1. The vector fields $ad_f^i g_i = 0, \ldots, n-1$ are linearly independent.
2. For any two vector fields X_1, X_2 in the set $\{ad_f^i g, i = 0, \ldots, n-2\}$ we have

$$[X_1, X_2](x) = \sum_{1=0}^{n-2} \phi_i(x) ad_f^i g(x)$$

for certain functions $\phi_0, \ldots, \phi_{n-2}$.

Note that the theorem above gives necessary and sufficient conditions for *local* feedback linearizability. For global results, further conditions on f and g are required. In the following example of a flexible joint pendulum the local solution extends to the full state space and thus become global.

Example 1. Consider the model of a robot link with a flexible joint (15,16)

$$\begin{cases} I\ddot{q}_1 + mgl \sin q_1 + k(q_1 - q_2) = 0 \\ J\ddot{q}_2 - k(q_1 - q_2) = u \end{cases} \tag{27}$$

where q_1 is the angle of the link, q_2 is the angle of the motor shaft, and u is the torque applied to the motor shaft. The flexibility is modeled via a torsional spring with constant k, m is the mass of the link, and l is the distance from the motor shaft to the center of mass of the link. J and I are the momenta of inertia of motor and link, respectively. With $x = (q_1, \dot{q}_1, q_2, \dot{q}_2)$ we obtain a system of the form Eq. (14) with

$$f(x) = \begin{bmatrix} x_2 \\ -mgl \sin x_1 - \frac{k}{I}(x_1 - x_3) \\ x_4 \\ \frac{k}{J}(x_1 - x_3) \end{bmatrix} \qquad g(x) = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1/J \end{bmatrix} \tag{28}$$

One may verify that conditions (1) and (2) for feedback linearization are fulfilled about the equilibrium $x_0 = 0$. In order to find the linearizing coordinate change [Eq. (16)] and feedback [Eq. (15)], we first solve Eq. (24). Using Eq. (28), this set of partial differential equations reads as

$$\frac{\partial S_1(x)}{\partial x_2} = \frac{\partial S_1(x)}{\partial x_3} = \frac{\partial S_1(x)}{\partial x_4} = 0$$

having as a (nonunique) nontrivial solution

$$z_1 = S_1(x) = x_1 \tag{29}$$

which via Eq. (21) implies

$$z_2 = S_2(x) = L_f S_1(x) = x_2 \tag{30}$$

$$z_3 = S_3(x) = L_f S_2(x) = -\frac{mgl}{I} \sin x_1 - \frac{k}{I}(x_1 - x_3) \tag{31}$$

$$z_4 = S_4(x) = L_f S_3(x) = -\frac{mgl}{I} x_2 \cos x_1 - \frac{k}{I}(x_2 - x_4) \tag{32}$$

Finally using Eq. (26) we find the linearizing feedback Eq. (15) as

$$\beta(x) = (L_g S_4(x))^{-1} = \frac{IJ}{k}$$
$$\alpha(x) = (L_g S_4(x))^{-1} L_f S_4(x)$$
$$= \frac{IJ}{k}\left[\frac{mgl}{I}\sin x_1\left(x_2^2 + \frac{mgl}{I}\cos x_1 + \frac{k}{I}\right) + \frac{k}{I}(x_1 - x3)\left(\frac{k}{I} + \frac{k}{J} + \frac{mgl}{J}\cos x_1\right)\right]$$

As a matter of fact, the above derivations are globally defined, and in addition Eqs. (29)–(32) have a physical interpretation.

In a similar way, the feedback linearization problem may be stated and solved for multivariable nonlinear systems but it is beyond the scope of this article to go further into this topic; interested readers are referred to Refs. 13 and 14. However, let us mention a simple example: the computed torque controller for rigid joint robot manipulators (EL systems), whose dynamical model is

$$D(q_p)\ddot{q}_p + C(q_p, \dot{q}_p)\dot{q}_p + g(q_p) = u \tag{33}$$

where $q_p \in \mathbb{R}^n$ is the vector of link positions (generalized position coordinates), $D(q_p) = D^\top(q_p) > 0$ is the inertia matrix, $C(q_p, \dot{q}_p)$ is the Coriolis and centrifugal forces matrix, $g(q_p)$ is the gravitational vector, and $u \in \mathbb{R}^n$ is the vector of control torques.

The tracking control problem for system [Eq. (33)] is to make the link position q_p follow a desired trajectory $y_d(t) = q_{pd}(t)$. The computed torque controller is a feedback linearization approach which, since it was proposed in Ref. 6, has become very popular. The control law is given by

$$u = D(q_p)v + C(q_p, \dot{q}_p)\dot{q}_p + g(q_p) \tag{34}$$

where v is a “new” input to be defined. It is easy to see that by substituting Eq. (34) in Eq. (33) we obtain the linear closed-loop dynamics $\ddot{q}_p = v$; then in order to solve the tracking problem for Eq. (33) we choose $v = -K_p(q_p - q_{pd}) - K_d(\dot{q}_p - \dot{q}_{pd}) - \ddot{q}_{pd}$ and we obtain that the closed loop is globally exponentially stable for any positive definite matrices of K_p and K_d.

Note that in this simple example, the feedback linearization—which is global—does not require any change of coordinates like Eq. (17). On the other hand, one may verify that system [Eq. (33)] does fulfill multivariable conditions similar to those given in Theorem 1 that guarantee the feedback linearizability.

Except from several more recent modifications of the computed torque controller (see Ref. 17 and references therein) the computed torque controller [Eq. (34)] requires full state feedback in order to obtain a linear closed-loop system; and in fact, output feedback will never lead to linear dynamics in closed loop. It is therefore attractive to investigate alternative tracking strategies.

PASSIVITY-BASED CONTROL

The Passivity Concept

As we mentioned in the introduction, the physical interpretation of the passivity concept is related to the system’s energy. In order to better understand the passivity concept we should think of a system like a black box which transforms some input into some output. More precisely, we say that a system with input u and output y defines a passive operator $u \mapsto y$ if the following energy balance equation is verified:

$$\underbrace{H(T) - H(0) + \int_0^T \delta_i \|u(t)\|^2 dt}_{\text{stored energy}} + \underbrace{\int_0^T \delta_o \|y(t)\|^2 dt}_{\text{dissipated}} = \underbrace{\int_0^T u(t)y(t)dt}_{\text{supplied}} \tag{35}$$

where $H(T)$ is the total energy of the system at time instant T. Expressed in words, the energy balance equation (35) establishes that one cannot pull more energy out of a passive system than the energy which was fed in. To illustrate this simple idea, consider an ordinary *RLC* (with all elements connected in series) network: In this case, $y \triangleq i$ is the current running through the resistor, while $u \triangleq v$ is the input voltage. Hence, if we look at Eq. (35), the term $\int_0^T \delta_i \|u(t)\|^2\, dt$ corresponds to the (electrical) potential energy stored in the capacitor, while the term $\int_0^T \delta_o \|y(t)\|^2\, dt$ corresponds to the (electrical) potential energy dissipated in the resistor (considering $R = \delta_o$). The energy stored in the inductance corresponds to the (magnetic) kinetic energy which has been considered in the terms $H(T) - H(0)$.

The stored energy in the capacitor plus the term $H(T)$ is called *available storage* and since $H(0) > 0$, it satisfies

$$H(T) + \int_0^T \delta_i \|v(t)\|^2\, dt < \int_0^T v(t)i(t)\, dt - \int_0^T \delta_o \|i(t)\|^2\, dt$$

that is, we can recover less energy than what was fed to the circuit. Formally, the definition of passivity we will use is the following (4):

Definition 1. Let $T > 0$ be any. A system with input $u \in \mathcal{L}_{2e}^n$ and output $y \in \mathcal{L}_{2e}^n$ defines a passive operator $\Sigma : u \mapsto y$ if there exists a $\beta \in \mathbb{R}$ such that

$$\int_0^T u(t)^\top y(t)\, dt \ge \beta \tag{36}$$

The operator Σ is output strictly passive (OSP) if moreover, there exists $\delta_o > 0$ such that

$$\int_0^T u(t)^\top y(t)\, dt \ge \delta_o \int_0^T \|y(t)\|^2\, dt + \beta \tag{37}$$

Finally, Σ is said to be input strictly passive (ISP) if there exists $\delta_i > 0$ such that

$$\int_0^T u(t)^\top y(t)\, dt \ge \delta_i \int_0^T \|u(t)\|^2\, dt + \beta \tag{38}$$

It should be noted that mainly every physical system has some passivity property; this has motivated researchers to use passivity-based control, that is, to exploit the passivity properties of the plant in order to achieve the control task by preserving the passivity in closed loop. The literature on passive systems is very diverse. We will illustrate this technique on a class of passive systems, the Euler–Lagrange systems. It is worth remarking that the robot manipulators belong to this class.

The Lagrangian Formulation

Euler–Lagrange (EL) systems can be characterized by the EL parameters

$$\{T(q,\dot q, V(q), \mathscr{F}(\dot q)\} \tag{39}$$

where $q \in \mathbb{R}^n$ are the generalized coordinates and n corresponds to the number of degrees of freedom of the system. We focus our attention on fully actuated EL systems—that is, systems for which there is a control input available for each generalized coordinate. Moreover, we assume that the kinetic energy function is of the form

$$T(q,\dot q) = \frac{1}{2}\dot q^\top D(q)\dot q \tag{40}$$

where the inertia matrix $D(q)$ satisfies $D(q) = D^\top(q) > 0$. Next, $V(q)$ represents the potential energy which is assumed to be bounded from below; that is, there exists a $c \in \mathbb{R}$ such that $V(q) > c$ for all $q \in \mathbb{R}^n$, and $\mathscr{F}(\dot q) = \frac{1}{2}\dot q^\top R\dot q$ with $R = R^\top > 0$ is the Rayleigh's dissipation function.

EL systems are defined by the EL equations

$$\frac{d}{dt}\left(\frac{\partial \mathscr{L}(q,\dot q)}{\partial \dot q}\right) - \frac{\partial \mathscr{L}(q,\dot q)}{\partial q} = Q \tag{41}$$

where $\mathscr{L}(q, \dot q) \triangleq T(q, \dot q) - V(q)$ is the Lagrangian function. We assume that the external forces, $Q \in \mathbb{R}^n$, are composed only of potential forces (derived from a time-invariant potential $V(q)$) $u \in \mathbb{R}^n$ and dissipative forces $-\partial\mathscr{F}(\dot q)/\partial \dot q$, hence

$$Q = u - \frac{\partial \mathscr{F}(\dot q)}{\partial \dot q} \tag{42}$$

At this point, we find it convenient to partition the vector q as $q \triangleq \text{col}[q_p\ q_c]$ where we call q_p the undamped coordinates and call q_c the damped ones. With this notation we can distinguish two classes of systems: An EL system with parameters [Eq. (39)] is said to be a *fully damped* EL system if ($\alpha > 0$)

$$\dot q^\top \frac{\partial \mathscr{F}(\dot q)}{\partial \dot q} \ge \alpha \|\dot q\|^2 \tag{43}$$

An EL system is *underdamped* if

$$\dot q^\top \frac{\partial \mathscr{F}(\dot q)}{\partial \dot q} \ge \alpha \|\dot q_c\|^2 \tag{44}$$

It is also well known (18) that the Lagrangian equations [Eq. (41)] can be written in the equivalent form [note that Eq. (45) is exactly the same as Eq. (33) with Raleigh dissipation zero]

$$D(q)\ddot q + C(q,\dot q)\dot q + g(q) + \frac{\partial \mathscr{F}(\dot q)}{\partial \dot q} = u \tag{45}$$

where the entries of the matrix $C(q, \dot q)$ are called the "Coriolis and centrifugal forces"; the kjth entry is

$$C_{kj}(q,\dot q) \triangleq \sum_i^n c_{ijk}(q)\dot q_i \tag{46}$$

$$c_{ijk}(q) \triangleq \frac{1}{2}\left(\frac{\partial D_{ik}(q)}{\partial q_j} + \frac{\partial D_{jk}(q)}{\partial q_i} - \frac{\partial D_{ij}(q)}{\partial q_k}\right) \tag{47}$$

With these definitions of matrices $D(q)$ and $C(q, \dot q)$ the following properties hold:

P1. The matrix $D(q)$ is positive definite, and the matrix $N(q, \dot q) = \dot D(q) - 2C(q, \dot q)$ is skew symmetric, that is, $N = -N^\top$. Moreover, there exist some positive constants d_m and d_M such that

$$d_m I < D(q) < d_M I \tag{48}$$

P2. The matrix $C(x, y)$ is bounded in x. Moreover, it is easy to see from Eq. (46) that $C(x, y)$ is linear in y, then for all $z \in \mathbb{R}^n$

$$C(x,y)z = C(x,z)y \tag{49}$$

$$\|C(x,y)\| \le k_c\|y\|, \quad k_c > 0 \tag{50}$$

Furthermore we will focus our attention on those systems for which the following additional property on the potential energy holds:

P3. There exists some positive constants k_g and k_v such that

$$k_g \ge \sup_{q\in\mathbb{R}^n}\left\|\frac{\partial^2 V(q)}{\partial q^2}\right\|, \quad \forall q \in \mathbb{R}^n \tag{51}$$

$$k_v \ge \sup_{q\in\mathbb{R}^n}\left\|\frac{\partial V(q)}{\partial q}\right\|, \quad \forall q \in \mathbb{R}^n \tag{52}$$

It is well known (7) that EL systems have some nice *energy dissipation properties:*

Proposition 1. (Passivity). An EL system defines a *passive operator* from the inputs u to the actuated generalized velocities $\dot q$, with storage function, the total energy function. Moreover, it is output strictly passive if there is a suitable *dissipation*—that is, if $\dot q^\top(\partial\mathscr{F}(\dot q)/\partial\dot q) \ge \delta_o\|\dot q\|^2$ for some $\delta_o > 0$.

Below, we present other properties of EL systems which are related to the stability in the sense of Lyapunov. For the sake of clarity, we distinguish two classes of EL systems, fully damped and underdamped systems.

The proposition below establishes conditions for *internal stability* of fully damped EL systems. After Joseph L. La Grange, the equilibria of a mechanical system correspond to the minima of the potential energy function (see Ref. 19 for a definition). Inspired by this well-known fact, we can further establish the following:

Proposition 2. (GAS with full damping). The equilibria of a fully damped free EL system (i.e., with $u = 0$) are $(q, \dot q) =$

$(\bar{q}, 0)$, where $\bar{q}$ is the solution of

$$\frac{\partial V(q)}{\partial q} = 0 \tag{53}$$

The equilibrium is *unique and stable* if it is a global and unique minimum of the potential energy function $V(q)$ and V is proper (4). Furthermore, this equilibrium is globally asymptotically (GAS) stable if the map defined by the Rayleigh dissipation function is *input strictly passive*.

As far as we know, the first article which establishes sufficient conditions for asymptotic stability of underdamped Lagrangian systems is more than 35 years old (20). In the proposition below we show that global asymptotic stability of a unique equilibrium point can still be ensured even when energy is not dissipated "in all directions," provided that the inertia matrix $D(q)$ has a certain block diagonal structure and the dissipation is suitably propagated.

Proposition 3. (GAS with partial damping). The equilibrium $(\dot{q}, q) = (0, \bar{q})$ of a free $(u = 0)$ underdamped EL system is GAS if the potential energy function is proper and has a global and unique minimum at $q = \bar{q}$, and if

1. $D(q) \triangleq \begin{bmatrix} D_p(q_p) & 0 \\ 0 & D_c(q_c) \end{bmatrix}$, where $D_c(q_c) \in \mathbb{R}^{n_c \times n_c}$.
2. $\dot{q}^\top \dfrac{\partial \mathcal{F}(\dot{q})}{\partial \dot{q}} \ge \alpha \|\dot{q}_c\|^2$ for some $\alpha > 0$.
3. For each q_c, the function $\dfrac{\partial V(q)}{\partial q_c} = 0$ has only isolated zeros in q_p.

Condition (2) establishes that enough damping is present in the coordinates q_c while the other two conditions help to guarantee that the energy dissipation suitably propagates from the damped coordinates to the undamped ones. Hence, one can think of an underdamped EL system as the interconnection of two EL systems. As a matter of fact the feedback interconnection of two EL systems yields an EL system.

Output-Feedback Set-Point Control

In this section we illustrate the passivity-based control approach by addressing the position-feedback set-point control problem of EL systems. The results we will present are based on the so-called energy shaping plus damping injection methodology. Launched in the seminal paper (21), this methodology aims at shaping the potential energy of the plant via a passive controller in such a way that the "new" energy function has a global and unique minimum at the desired equilibrium. It is worth remarking that this methodology was originally proposed in the context of robot control; however, it has been proved useful in the solution of other control problems as it will be clear from this section. Also, it shall be noticed that the passivity property of robot manipulators was first pointed out in Ref. 7.

Motivated by the energy shaping plus damping injection technique of Takegaki and Arimoto, as well as by the properties of Lagrangian systems described in previous sections, it becomes natural to consider El controllers (22) with generalized coordinates $q_c \in \mathbb{R}^{n_c}$ and EL parameters $\{T_c(q_c, \dot{q}_c), V_c(q_c, q_p), \mathcal{F}_c(\dot{q}_c)\}$. That is, the controller is a Lagrangian system with dynamics

$$D_c(q_c)\ddot{q}_c + \dot{D}_c(q_c)\dot{q}_c - \frac{\partial T_c(q_c, \dot{q}_c)}{\partial q_c} + \frac{\partial V_c(q_c, q_p)}{\partial q_c} + \frac{\partial \mathcal{F}_c(\dot{q}_c)}{\partial \dot{q}_c} = 0 \tag{54}$$

Note that the potential energy of the controller depends on the measurable output q_p, and therefore q_p enters into the controller via the term $\partial V_c(q_c, q_p)/\partial q_c$. On the other hand, the feedback interconnection between plant and controller is established by

$$u = -\frac{\partial V_c(q_c, q_p)}{\partial q_p} \tag{55}$$

then the closed-loop system is Lagrangian and its behavior is characterized by EL parameters $\{T(q, \dot{q}), V(q), \mathcal{F}(\dot{q})\}$, where

$$\begin{aligned} T(q, \dot{q}) &\triangleq T_p(q_p, \dot{q}_p) + T_c(q_c, \dot{q}_c), \\ V(q) &\triangleq V_p(q_p) + V_c(q_c, q_p), \\ \mathcal{F}(\dot{q}) &\triangleq \mathcal{F}_p(\dot{q}_p) + \mathcal{F}_c(\dot{q}_c) \end{aligned}$$

The resulting feedback system is the feedback interconnection of the operator $\Sigma_p : u_p \mapsto q_p$, defined by the dynamic equation [Eq. (33)] and the operator $\Sigma_c : q_p \mapsto u_p$, defined by Eqs. (54) and (55).

Note that the EL closed-loop system is damped only through the *controller* coordinates q_c. From the results presented in section entitled "The Lagrangian Formulation" we see that to attain the GAS objective, $V(q)$ must have a global and unique minimum at the desired equilibrium, $q = q_d$, and $\mathcal{F}(\dot{q})$ must satisfy Eq. (44). These conditions are summarized in the proposition below whose proof follows trivially from Proposition 3.

Proposition 4. (Output feedback stabilization) (22).
Consider an EL plant (33) where $u \in \mathbb{R}^m$, $m \le n$, with EL parameters $\{T_p(q_p, \dot{q}_p), V_p(q_p), \mathcal{F}_p(\dot{q}_p)\}$. An EL controller (54), (55) with EL parameters $\{T_c(q_c, \dot{q}_c), V_c(q_c, q_p), \mathcal{F}_c(\dot{q}_c)\}$, where

$$\dot{q}_c^\top \frac{\partial \mathcal{F}_c(\dot{q}_c)}{\partial \dot{q}_c} \ge \alpha \|\dot{q}_c\|^2$$

for some $\alpha > 0$, solves the global output feedback stabilization problem if

1. (Dissipation propagation) For each trajectory such that $q_c \equiv$ const and $\partial V_c(q_c, q_p)/\partial q_c = 0$, we have that $q_p \equiv$ const.
2. (Energy shaping) $\partial V(q)/\partial q = 0$ admits a constant solution $\bar{q}$ such that $q_{pd} = [I_{n_p} \mid 0]\bar{q}$, and $q = \bar{q}$ is a global and unique minimum of $V(q)$, and V is proper. For instance, this is the case if $\partial^2 V(q)/\partial q^2 > I_n \epsilon > 0$, $\epsilon > 0\ \forall\ q \in \mathbb{R}^n$.

A simple example of EL controllers is the dirty derivatives filter, widely used in practical applications:

$$\dot{q}_c = -A(q_c + Bq_p) \tag{56}$$

$$\vartheta = (q_c + Bq_p) \tag{57}$$

where A, B are diagonal positive definite matrices. With an obvious abuse of notation this system lies in the EL class and has the EL parameters:

$$T_c(q_c, \dot{q}_c) = 0, \mathscr{F}_c(\dot{q}_c) = \frac{1}{2}\dot{q}_c^\top B^{-1} A^{-1} \dot{q}_c$$

$$V_c(q_c, q_p) = \frac{1}{2}(q_c + Bq_p)^\top B^{-1}(q_c + Bq_p)$$

The controller above injects the necessary damping to achieve asymptotic stability and its action has the following nice passivity interpretation. First, we recall that the EL plant Eq. (41) defines a passive operator $u \mapsto -\dot{q}_p$. On the other hand, the controller Eq. (54) defines a passive operator $\dot{q}_p \mapsto \partial V_c(q_p, q_c)/\partial q_p$. These properties follow, of course, from the passivity of EL systems established in Proposition 1. It suffices then to choose u as in Eq. (55) to render the closed-loop passive.

CONTROL UNDER MODEL AND PARAMETER UNCERTAINTIES

In all the results presented above, we assumed that we had accurate knowledge about the system's model and its parameters; however, this rarely happens to be the case in practical applications. It is of interest then to use techniques such as *robust* and/or PID control.

PID Control

PID control was originally formulated by Nicholas Minorsky in 1922; since then it has become one of the most applied control techniques in practical applications. In the western literature, the first theoretical stability proof of a PID in closed loop with a robot manipulator is due to Ref. 23. We reformulate below the original contribution of the authors.

Proposition 5. Consider the dynamic model [Eq. (33)] in closed loop with the PID control law

$$u = -K_P \tilde{q}_p - K_D \dot{q}_p + \nu \tag{58}$$

$$\dot{\nu} = -K_I \tilde{q}_p \nu(0) = \nu_0 \in \mathbb{R}^n \tag{59}$$

Then, if $K_P > k_g I$ and K_I is sufficiently small, the closed loop is asymptotically stable.

The proposition above establishes only local asymptotic stability. By looking at the proof (see Ref. 23) of the above result we see that what hampers the global asymptotic stability is the quadratic terms in $\dot{q}_p$ contained in the Coriolis matrix. This motivates us to wonder about the potential of a linear controller designed for a nonlinear plant.

As a matter of fact, with some smart modifications one can design nonlinear PIDs which guarantee global asymptotic stability. As far as we know, the first nonlinear PID controller is due to Ref. 24 [even though Kelly (24) presented his result as an "adaptive" controller, in the sequel it will become clear why we use the "PID" qualifier] which was inspired upon the results of Tomei (25). In order to motivate the nonlinear PID of Ref. 24, let us first treat in more detail the PD-like adaptive control law of Tomei

$$u = -K_p \tilde{q}_p - K_d \dot{\tilde{q}}_p + \Phi(q_p)\hat{\theta} \tag{60}$$

together with the update law

$$\dot{\hat{\theta}} = -\Phi(q_p)^\top \left[\gamma \dot{\tilde{q}}_p + \frac{2\dot{\tilde{q}}_p}{1 + 2\|\tilde{q}_p\|^2}\right] \tag{61}$$

where γ is a suitably defined positive constant. Tomei (25) proved that under this adaptive control law, the position error is globally asymptotically convergent. The normalization used in Eq. (61), probably first introduced by Koditschek (26), helps in guaranteeing the globality, in the un-normalized case being only semiglobally convergent.

Note that the result of Tomei is based on a PD plus gravity cancellation; since the gravity vector is not well known, an adaptive update law must be used. Let us consider that instead of cancelling the gravity term, we compensate it at the desired position, then we will be aiming at estimating the constant vector $\Phi(q_{pd})\hat{\theta}$. More precisely, consider the control law (24)

$$u = -K_P' \tilde{q}_p - K_D \dot{\tilde{q}}_p + \Phi(q_{pd})\hat{\theta} \tag{62}$$

together with the update law

$$\dot{\tilde{\theta}} = \dot{\hat{\theta}} = -\frac{1}{\gamma}\Phi(q_{pd})^\top \left[\dot{q}_p + \frac{\epsilon \tilde{q}_p}{1 + \|\tilde{q}_p\|}\right] \tag{63}$$

where $\epsilon > 0$ is a small constant. Kelly (24) proved that this "adaptive" controller in closed loop with Eq. (33) results in a globally convergent system. As a matter of fact, since the regressor vector $\Phi(q_{pd})$ is *constant*, the update law [Eq. (63)], together with the control input [Eq. (62)], can be implemented as a *nonlinear* PID controller by integrating out the velocities vector from Eq. (63):

$$\hat{\theta} = -\frac{1}{\gamma}\Phi(q_{pd})^\top \left[\tilde{q}_p + \int_0^t \epsilon \frac{\tilde{q}_p}{1 + \|\tilde{q}_p\|}\,dt\right] + \hat{\theta}(0)$$

Note that the choice $K_P = K_P' + K_I$, with $K_I = 1/\gamma\ \Phi(q_{pd})\Phi(q_{pd})^\top$, yields the controller implementation

$$u = -K_P \tilde{q}_p - K_D \dot{q}_p + \nu \tag{64}$$

$$\dot{\nu} = -\epsilon K_I \frac{\tilde{q}_p}{1 + \|\tilde{q}_p\|}, \qquad \nu(0) = \nu_0 \in \mathbb{R}^n \tag{65}$$

Following Ref. 24, one can prove global asymptotic stability of the closed-loop system [Eqs. (33), (64), (65)]. An alternative trick to achieve GAS is the scheme of Arimoto (27), who proposed the following nonlinear PID:

Proposition 6. Consider the dynamic model [Eq. (33)] in closed loop with the PID control law

$$u = -K_P\tilde{q}_p - K_D\dot{q}_p + \nu \tag{66}$$

$$\dot{\nu} = -K_I\,\mathrm{sat}(\tilde{q}_p), \qquad \nu(0) = \nu_0 \in \mathbb{R}^n \tag{67}$$

Then, if $K_P > k_g I$ and if K_I is sufficiently small, the closed loop is asymptotically stable.

It is clear from the proof (see Ref. 27) that the use of a saturation function in the integrator helps to render the system globally asymptotically stable, just as the normalization did in Tomei's and Kelly's schemes.

In the sequel we assume that velocities are not available for measurement. In general it is a difficult problem to design adaptive output feedback controllers for nonlinear systems and achieve global convergence. The result we present below is inspired by the work of Kelly and the key observation that when compensating with the (unknown) gravity vector evaluated at the desired position, one can simply integrate the velocities out of the update law.

Proposition 7. Consider the dynamic model of the EL plant Eq. (33) in closed loop with the PI^{2}D control law

$$u = -K_P\tilde{q}_p - K_D\vartheta + \nu \tag{68}$$

$$\dot{\nu} = -K_I(\tilde{q}_p - \vartheta), \qquad \nu(0) = \nu_0 \in \mathbb{R}^n \tag{69}$$

$$\dot{q}_c = -A(q_c + Bq_p) \tag{70}$$

$$\vartheta = q_c + Bq_p \tag{71}$$

Let K_P, K_I, K_D, $A \triangleq \mathrm{diag}\{a_i\}$, and $B \triangleq \mathrm{diag}\{b_i\}$ be positive definite diagonal matrices with

$$B > \frac{4d_M}{d_m}I \tag{72}$$

$$K_P > (4k_g + 1)I \tag{73}$$

where k_g is defined by Eq. (49), and define $x \triangleq \mathrm{col}[\tilde{q}_p, \dot{q}_p, \vartheta, \nu - g(q_{pd})]$. Then, given any (possibly arbitrarily large) initial condition $\|x(0)\|$, there exist controller gains that ensure $\lim_{t\to\infty} \|x(t)\| = 0$.

In Ref. 28, precise bounds for the controller gains are given, depending on bounds on the plant's parameters.

Note that the PI^{2}D controller is linear and, as in the case of a conventional PID scheme, it only establishes semiglobal asymptotic stability. The technical limitation is the high nonlinearities in the Coriolis matrix. See Ref. 28 for details.

In the sequel we give an interpretation of the nonlinear PID controllers above, as well as the PI^{2}D scheme, from the passivity point of view, or more precisely passifiability—that is, the possibility of rendering a system passive via feedback.

From Proposition 1 we know that the plant's total energy function $T(q_p, \dot{q}_p) + V(q_p)$ qualifies as a storage function for the supply rate $w(u, \dot{q}) = u^\top\dot{q}_p$. From this property, output strict passifiability of the map $u \mapsto \dot{q}_p$ follows taking $u = -K_D\dot{q}_p + u_1$, with u_1 an input which shapes the potential energy.

Other applications, including the present study of PI controllers, require a passifiability property of a map including also q_p besides $\dot{q}_p$, at the output. This can be accomplished with a storage function that includes cross terms. Very recently, Ref. 27 showed, by using a saturation function $\mathrm{sat}(\cdot)$, that the nonlinear PID (66) can be regarded as the feedback interconnection of two passive operators $\Sigma_1 : -z \mapsto \epsilon\,\mathrm{sat}(\tilde{q}_p) + \dot{q}_p$ and $\Sigma_2 : -\epsilon\,\mathrm{sat}(\tilde{q}_p) - \dot{q}_p \mapsto z$; hence the closed loop system is also passive. The same can be proven for the normalized scheme of Kelly [Eqs. (64) and (65)]. As a matter of fact it can be proven that Σ_1 is OSP (17).

Instrumental in proving OSP for Σ_1 is the use of either a normalization or a saturation function; unfortunately in the case of the PI^{2}D controller, these "tricks" do not lead us to OSP actually, and the output strict passifiability property we can establish is only *local*. That is, the property holds only for inputs that restrict the operator's output to remain within a compact subset (29). Nonetheless, this compact subset can be arbitrarily enlarged with high gains, and this explains the semiglobal—instead of global—nature of this result.

Robust Control

We have assumed so far that even though the plant's model is known, some uncertainties over the parameters exist. Nevertheless, in some applications it may happen that we have only a very rough idea of the plants model. For instance, some lightweight robot manipulators with direct-drive motors present highly nonlinear and coupled dynamics for which a model is not known. It is good to know that, at least for a certain class of EL plants, still a high-gain PD control can be used, leading to some robustness satisfactory results. In particular, for the EL plant model we have

$$D(q_p)\ddot{q}_p + C(q_p, \dot{q}_p)\dot{q}_p + g(q_p) + F(\dot{q}_p) + T = u \tag{74}$$

where $F(\dot{q}_p)$ is the vector of frictional torques which satisfies $\|F(\dot{q}_p)\| \le k_{f_1} + k_{f_2}\|\dot{q}_p\|$ for all $\dot{q}_p$ and T is the vector of load disturbances which is bounded as $\|T\| \le k_t$; we have the following result (30):

Proposition 8. Consider the EL plant [Eq. (74)], let $e = q_p - q_{pd}$ be the position error, and let $\hat{e}$ an estimate of it. Consider the control law

$$u = -K_d\dot{\hat{e}} - K_p\hat{e} \tag{75}$$

$$\dot{\hat{e}} = w + L_d(e - \hat{e}) \tag{76}$$

$$\dot{w} = L_p(e - \hat{e}) \tag{77}$$

Then for any set of bounded initial conditions $(t_0, x(t_0))$ we can always find sufficiently large gains K_p, K_d, L_p, and L_d such that the trivial solution of the closed-loop system: $x(t) \triangleq \mathrm{col}[e, \dot{e}, \hat{e}, \dot{\hat{e}} = 0$ is uniformly ultimately bounded. That is, for every bounded initial conditions $(t_0; x(t_0))$ there exist a finite constant $\eta > 0$ and a time instant $t_1(\eta, \|x(t_0)\|)$ such that

$$\|x(t)\| \le \eta, \qquad \forall t \ge t_0 + t_1$$

Moreover, in the limit case, when $L_p, L_d \to \infty$ the origin is asymptotically stable.

A particular case of the result above is presented in Ref. 31 where velocity measurements are used. Like in Ref. 31 the uniform ultimate boundedness result of Proposition 8 is of local nature because the controller gains depend on the initial conditions $x(t_0)$; nevertheless, it is important to remark that the Proposition states that "for *any* set of finite initial conditions there always exist control gains . . ."; that is, the result is semiglobal.

However, even without the knowledge of bounds over the plant's parameters, the closed loop system can be made uniformly ultimately bounded by selecting the control gains sufficiently large. Hence there is no need to quantify these bounds *a priori*.

It is also important to remark that the linear control scheme [Eq. (75)–(77)] allows quick response in an online implementation, due to its simplicity. Since this control scheme completely ignores the system dynamics, however, the conditions of Proposition 8 (see [30] for details) may require that K_d and L_d be large to obtain an acceptable tracking performance. Such high gain implementations are not always desirable in practical applications. For this reason it may be profitable to add model-based compensation terms to the control input, when available. See, for instance, Ref. 17 and references therein.

THIRD-ORDER FEEDBACK LINEARIZABLE SYSTEMS

So far we considered n coupled second-order (fully actuated EL) systems to illustrate different control design methods for nonlinear systems. Even though the class of EL plants includes a large number of physical systems it is interesting to investigate output feedback control of higher-order systems. As a matter of fact, the topic of output feedback control of nonlinear systems is one of the most studied in the literature and very few particular results exist guaranteeing global asymptotic stability; see, for example, a recent study of the semiglobal problem (11).

In this section we illustrate this problem by addressing the partial state feedback control of a complex system, the so-called Rössler system (see, for instance, Ref. 32)

$$\dot{x}_1 = -(x_2 + x_3) \tag{78}$$

$$\dot{x}_2 = x_1 + ax_2 \tag{79}$$

$$\dot{x}_3 = x_3(x_1 - b) + c + u \tag{80}$$

where a, b, and c are positive constants. It can be seen from simulations that the trajectories of this system have a chaotic behavior, for instance if $a = 0.2$, $b = 5$, and $c = 0.2$. A behavior is chaotic if it has a sensitive dependence on initial conditions. By this, we mean that the difference between two solutions of a differential equation with a slight difference in the initial conditions grows exponentially (32).

The motivation to consider the Rössler system is to investigate to what extent the techniques used for second-order systems can be successfully used for third-order feedback linearizable systems.

Note for the Rössler system that if the whole state is supposed to be measured, then it is easy to see that there exist positive gains k_1, k_2, and k_3 such that the feedback linearizing control law

$$u = -x_3(x_1 - b) - c - k_1x_1 - k_2x_2 - k_3x_3 \tag{81}$$

in closed loop with Eq. (78) is GAS. Now, let us suppose that only x_2 and x_3 are measurable. In this particular case, a nonlinear observer for x_1 can be easily designed using the Lyapunov control approach (4). Motivated by the control law Eq. (81), consider the control law $u \triangleq u(x_2, x_3, \hat{x}_1)$:

$$u = -x_3(\hat{x}_1 - b) - c - k_1\hat{x}_1 - k_2x_2 - k_3x_3 \tag{82}$$

where $\hat{x}_1$ is the estimate of x_1. Consider the function

$$V(x, \hat{x}_1) = \frac{1}{2}\left[x_1^2 + x_2^2 + x_3^2 + \tilde{x}_1^2\right] + \epsilon x_2(x_3 - x_1) \tag{83}$$

where $\tilde{x}_1 = x_1 - \hat{x}_1$ is the estimation error and $\epsilon > 0$ is sufficiently small to ensure that $V(x)$ is positive definite and radially unbounded. Then, let $V(x)$ be a Lyapunov function candidate for system Eq. (78) in closed loop with the control law Eq. (82). We proceed now with the design of a reduced observer $\dot{\hat{x}}_1 = f(x_2, x_3, \hat{x}_1)$. First, evaluating the time derivative of $V(x, \hat{x}_1)$ along the trajectories of the closed loop Eqs. (78), (82) we get after some bounding

$$\dot{V}(x, \hat{x}_1) \leq -\gamma_1x_1^2 - \gamma_2x_2^2 - \gamma_2x_3^2 + (x_3 + \epsilon x_2)(k_1 + x_3)\tilde{x}_1 + \tilde{x}_1\dot{\tilde{x}}_1$$

where γ_1, γ_2, and γ_3 are positive constants. Hence, by setting

$$\dot{\tilde{x}}_1 = -(x_3 + \epsilon x_2)(k_1 + x_3)$$

the Lyapunov function becomes negative semidefinite. A simple analysis using the Krasovskii–LaSalle's invariance principle shows that the closed-loop system is GAS. Note, moreover, that the observer can be implemented as

$$\dot{\hat{x}}_1 = (x_3 + \epsilon x_2)(k_1 + x_3) - (x_2 + x_3)$$

without measurement of x_1.

In the case when more variables are unmeasurable, one may think that a similar procedure leads to the design of an observer for the unavailable variables. Unfortunately, this seems not the case when only x_2 or x_3 are considered available for measurement. Moreover, the lack of a physical interpretation for the Rössler system makes this task more difficult. The lesson one can take from this illustrative example is that observer-based schemes become complicated even if the system itself is feedback-linearizable. The lack of (physical) passivity properties hampers the use of passivity-based control.

CONCLUSION

We have briefly illustrated different control design methods for nonlinear systems. We derived necessary and sufficient conditions to solve the local feedback linearization problem and illustrated this approach on the flexible joints robots case.

We focused our attention into a special class of second-order systems, the EL systems. However, the Lagrangian formulation applies to all fields in which variational principles can be used to model the plant in question; hence this class includes a wide number of physical systems such as robot manipulators.

We saw that the EL class has some nice energy properties which can be exploited by using passivity-based control. The goal of this methodology is to design a controller and a dynamic extension to the plant which renders the closed-loop system passive. This approach appeared very useful in solving the set-point control problem.

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See INTERMODULATION MEASUREMENT.