OPEN-LOOP OSCILLATORY CONTROL

Conventional control systems rely on feedback, feedforward, or a combination of the two. In a feedback control system, the controlled variable is usually compared with a reference variable, and the difference between the two, the error, is used to regulate the system. In a feedforward control system, an appropriate additive control signal is introduced to compensate for disturbances. While feedback and feedforward rely on different principles, both methods require measurements. In a feedback control system, the controlled variable is measured. Likewise, in a feedforward control system the measurement of disturbances is used in the implementation. However, measurements of states or disturbances are often costly, difficult, or even impossible to obtain. In these cases, feedback and feedforward are not feasible means of control.

Consider the particle accelerator originally described in (1) and later discussed in detail in (2). The control objective is to focus a beam of particles along the accelerator. In cyclic accelerators with azimuth symmetrical magnetic fields, the *plant,* a beam of particles, is described by

$$\frac{d^2x}{d\theta^2} + \omega^2(1-n)x = 0$$
$$\frac{d^2z}{d\theta^2} + \omega^2 nz = 0 \tag{1}$$

where x and z are state coordinates representing small betatron oscillations of the beam, ω is the betatron wave number, θ is an independent variable (azimuth), and n is the field index of refraction. For proper operation, the field index of refraction should satisfy $0 < n < 1$. However, for this range of n, the beam focusing is often unacceptable. Feedback is usually not possible due to the difficulty of measuring x and z. Feedforward also has similar measurement difficulties.

In such cases, a natural question is whether control is possible for such 'unmeasureable' systems. For many systems, one alternative is *open-loop oscillatory control,* sometimes referred to as *vibrational control* (not to be confused with *vibration control* where the idea is to *reduce* vibrations). Open-loop oscillatory control is a fairly recently developed control methodology that does not require measurements of states or disturbances. Instead, zero mean periodic excitation is used to modify the plant behavior in such a way that control is achieved as the result of the system's natural response to the excitation. For example, oscillations in the cyclic accelerator can be introduced by appropriately focusing and defocusing sectors of the magnetic lens. This causes a suppression of betatron oscillations and thereby makes the focus beam more acceptable. An early heuristic description of this phenomena was given by Livingston (1), but it was not until 1980 that the heuristically controlled azimuth accelerator was explained in the context of open-loop oscillatory control in (2).

Definition 1. Open-loop Oscillatory Control. The utilization of periodic (or almost periodic) control laws, without the use of measurements in order to induce a desired dynamic response in a system is referred to as open-loop oscillatory control or vibrational control.

The simplicity of open-loop oscillatory control synthesis is offset by the difficulty added in introducing explicit time dependence in the state system models. In order to simplify the analysis, open-loop oscillatory control algorithms may restrict the control action so the controlled system admits a small parameter. One way to obtain a small parameter is to introduce periodic excitation whose frequency is an order of magnitude larger than the highest system natural frequency. The small parameter will then arise as the result of a rescaling of time. For such systems, the time-varying open-loop controlled system can be approximated by the behavior of a time-invarient averaged equation, to which the usual analytical techniques for time-invariant systems may be applied. This result forms the basis of classical averaging theory in applied mathematics and dynamical systems. Within the context of forced mechanical systems and averaging, energy methods and a quantity called the *averaged potential* provide the most direct method of analysis. In the absence of distinct system time or length scales, the local stability of an equilibrium or periodic orbit can be studied by the analysis of the linearized system's first return map, or monodromy matrix, obtained through Floquet theory.

One of the most compelling examples of open-loop oscillatory control is the stabilization of the simple pendulum's inverted equilibrium by high frequency vertical oscillation of the pendulum's suspension point. This discovery is usually attributed to Bogoliubov (3,4) and Kapitsa (5), although earlier references to similar phenomena exist (6). More recent accounts of this stabilization may be found in (7,8,9,10), where the averaged potential is used to study global structure and stability of the periodically forced system. The main result is that for a given forcing amplitude, there is a unique critical forcing frequency at which the inverted equilibrium experiences a pitchfork bifurcation. For forcing frequencies higher than the critical frequency, the inverted equilibrium is stable. Reference 10 considers a generalization of the classic problem, where the periodic forcing is directed along an incline with respect to the horizontal. In this case, the pendulum tends to stabilize in configurations aligned with the direction of the forcing, and in fact this phenomenon holds in the general n-link case as well (11).

Open-loop oscillatory control has been applied to many systems, and new applications continue to emerge. In (12) the technique was applied to exothermic chemical reactions in a continuous stirred tank reactor (CSTR). In this work, it was shown that by modulating the input and exit chemical feed rates of the CSTR, it is possible to operate in stabilized averaged conversion rates that would otherwise be unstable unless expensive feedback is applied. Although, on average, the same amount of input chemical has been used, the stable operating regimes of the CSTR change substantially with the use of an oscillatory control input. In similar work, the results of (12) are analytically extended to include chemical reactions in a CSTR with delayed recycle stream (13,14). Historically, all of this work on oscillatory open-loop control was prompted by the work of periodic operation of chemical reactors using the sometimes heuristic techniques of push-pull, periodic optimization, and asynchronous quenching (12,15).

Experimental applications of open-loop oscillatory control have also included laser illuminated thermochemical systems (16), stabilization of plasma (17), and car parking algorithms (18). In (19), sufficient conditions are given for a periodic process to minimize periodic paths. This approach generalized the result in (20) that showed that periodic paths improve aircraft fuel economy. Other analytic applications of open-loop oscillatory control include rotating chains (21,22), n-link pendula (11), axial compressors (23), and population models (24,25).

In the work by Lehman et al. (13,25,26) the technique of oscillatory open-loop control is developed for systems with time-delays. Bentsman and Hong (27,28) have extended the technique to parabolic partial differential equations (PDEs). The application of open-loop control to delay systems and PDE's shows interesting potential since these types of infinite dimensional systems are often difficult to control when using feedback. Likewise, there has been success in combining the benefits of open-loop oscillations with conventional feedback in order to robustly stabilize systems with zeros in the open-right half plane and systems with decentralized fixed zeros (29–32).

As with all other control algorithms, the important issues of designing open-loop oscillatory control include stability, transient response, and accuracy of the controlled system. Certainly, the most important issue is stability. Many classical results on stability of operating points for systems with oscillatory inputs depend on eigenvalues of the averaged system lying in the left half plane, or equivalently the eigenvalues of the monodromy matrix lying within the unit disk. However, there has been growing interest in the stabilization of systems to which such classical results do not apply. These include the mechanical systems studied in (7,8,10,11,33), where eigenvalues locations are typically symmetric with respect to the imaginary axis. Coron (34) has shown the existence of a time-varying feedback stabilizer for systems whose averaged versions have eigenvalues on the imaginary axis. Additional interest in this design derives from the observation that it provides a method of smooth feedback stabilization for systems which Brockett (35) had previously shown were not stabilizable by smooth, time-invariant feedback.

Stability of a system is concerned with the asymptotic behavior of the system. Often it is important to study trajectories of systems as steady-state behavior is being approached. Analysis of such trajectories when there is an oscillatory control input is a difficult task. The oscillatory control is usually designed to be high frequency. As a result, the controlled system is composed of a fast zero average oscillatory trajectory superimposed on a slow trajectory. Therefore, the designer must attempt to control the slow part of the trajectory and ignore (or filter out) the high frequency component.

One disadvantage of open-loop oscillatory control is its accuracy. It is well known that driving a nonlinear system with a periodic signal generally excites an array of resonances, and under appropriate conditions chaos in the homoclinic tangles of unstable resonances [See (36) for a complete exposition on this topic]. While subharmonic resonances and chaos tend to be suppressed at high forcing frequencies, 1 : 1 resonances (primary resonances, or periodic orbits), whose averages correspond to fixed points of an averaged representation of the dynamics, persist. If a stable 1 : 1 resonance has no association with a fixed point of the time-varying system (i.e., it arises through a bifurcation), it is called a *hovering motion.* These high frequency responses limit the utility of open-loop oscillatory control when control accuracy is important.

PROBLEMS IN OPEN-LOOP OSCILLATORY CONTROL

Classes of Systems

This section considers systems of ordinary differential equations, with inputs, of the form

$$\dot{x} = f(x, u) \tag{2}$$

where $x \in \mathbb{R}^n$ and $u \in \mathbb{R}^m$. The function f: $\mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}^n$ will always be assumed sufficiently continuous so that solutions to Eq. (2) exist. Models of this form describe most of the systems appearing in the recent engineering literature on open-loop oscillatory control, as discussed in detail in (37).

Stabilization

Here we introduce the problem of vibrational stabilization, as found in (24,38). Suppose that (39) has an unstable equilibrium point, x_s when u = *constant* = λ_0, and the goal is to determine a control input $u(t)$ that stabilizes this operating point. In addition, suppose this stabilization is to be performed without any state or disturbance measurements.

For the stabilization problem being considered, the methods of oscillatory open-loop control are as follows. Introduce into Eq. (2) oscillatory inputs according to the law $u(t) = \lambda_0 + \gamma(t)$ where λ_0 is a constant vector and $\gamma(t)$ is a periodic average zero (PAZ) vector, that is, $\gamma(t) = \gamma(t + T)$ with $\int_0^T \gamma(t)dt = 0$. Even though the average value of $u(t)$ remains equal to λ_0, it is hoped that the periodic forcing can impose a stabilizing effect on the system.

In this case, Eq. (2) becomes

$$\frac{dx}{dt} = f(x, \lambda_0 + \gamma(t)) \tag{3}$$

Definition 2. Vibrationally Stabilizable. An equilibrium point x_s of Eq. (2) is said to be vibrationally stabilizable if for any $\delta > 0$ there exists a PAZ vector $\gamma(t)$ such that Eq. (3) has an asymptotically stable periodic solution, $x^*(t)$, characterized by

$$\|\overline{x}^* - x_s\| \leq \delta; \quad \overline{x}^* = \frac{1}{T}\int_0^T x^*(t)\,dt$$

It is often preferable that Eq. (3) has a fixed equilibrium point, x_s. However, this is not usually the case since the right hand side of Eq. (3) is time varying and periodic. Therefore, the technique of vibrational stabilization is to determine vibrations $\gamma(t)$ such that the (possibly unstable) equilibrium point x_s bifurcates into a stable periodic solution whose average is close to x_s.

The engineering aspects of the problem consist of:

1. Finding conditions for the existence of stabilizing periodic inputs
2. Determining which oscillatory inputs, $u(\cdot)$, are physically realizable and
3. Determining the shape (waveform type, amplitude, phase) of the oscillations to be inserted which will ensure the desired response

At this time, it may be useful to explain why it is necessary to use time-varying control inputs as opposed to simply using classical time-invariant open-loop control techniques. Suppose that there is a single-input single-output linear time-invariant (LTI) system with proper transfer function $Y(s)/U(s) = n(s)/d(s)$, where Y and U are the Laplace transform of the output and the input, respectively, and n and d are polynomials in s. If all the roots of $d(s) = 0$ have negative real parts, then open-loop control can be used to arbitrarily place system poles simply by letting $U(s) = d(s)/p(s)$, where $p(s) = 0$ has the desired system pole location and the degree of p is greater or equal to the degree of d. At times, this pole–zero cancellation open-loop control strategy might give desired system performance, especially if there is no need for feedback (perhaps because there are no disturbances).

Unfortunately, though, perfect pole–zero cancellation is not possible. This may not be worrisome if all the roots are in the left-half plane, but when there exists at least one root of $d(s) = 0$ with positive real part, LTI open-loop control cannot stabilize a system. On the other hand, when $u(t)$ is an oscillatory open-loop control input, stabilization is often possible, even when there is a pole in the right-half plane. Indeed, oscillatory open-loop controls have also shown a remarkable robustness to disturbances in many experimental applications (12,37). This is a quality that is absent in LTI open-loop control.

Remark 1. This subsection has attempted to state the problem of stabilization in its broadest terms. There are classes of systems, however, for which discussion of stabilization and stability is problematic. Such systems include conservative systems, or more specifically, Hamiltonian systems. Hamiltonian systems include dissipation-free mechanical systems, and include many electrical and optical systems as well. The primary defect of Hamiltonian systems as far as control theory is concerned is that the strongest stability these systems can possess is neutral stability; that is, eigenvalues/poles on the imaginary axis. For this reason, standard concepts from control theory seldom yield strong stability results. Progress has recently been made in developing techniques for the stability analysis of these systems. The new techniques make use of the system energy, and in the case of periodically forced systems the *averaged potential,* to assess the stability of equilibriums. A technique for the equilibrium and stability analysis of a large class of periodically forced Hamiltonian systems is presented later in this article.

Transient Behavior and Performance

Once a system is determined to be stable, the next issue in evaluating its performance is to determine how quickly the solutions decay to their steady state. This finite time transient behavior is sometimes crucial to system performance. For LTI systems, there are several methods that can be used to obtain estimates for the transient behavior of the output. For example, an estimated output trajectory is obtained from information on the location of the dominant system eigenvalues. Even for nonlinear time-invariant systems, it is common to examine the eigenvalues of a Jacobian linearization in order to examine the rates of decay of solutions.

For systems subject to oscillatory open-loop control, the analysis techniques are not so straightforward. As previously mentioned, the control inputs cause Eq. (3) to be time-varying, and analysis of time-varying systems remains an open area of research. However, since it has been assumed that the control inputs have a special structure, that is, periodic and high-frequency, it will be possible to apply the method of averaging to find approximations of the system transient behavior.

Essentially the problem of controlling the transient behavior of time-varying system Eq. (2) is to

1. Determine control inputs, $\gamma(t)$ in Eq. (3) so that the solutions to Eq. (3) can be approximated by the solutions of a simpler equation
2. Control the transient behavior of the approximate equation

Sometimes this simpler equation turns out to be purely time-invariant and in the form of

$$\frac{dy}{dt} = P(y), \quad w = q(y) \tag{4}$$

where w and y are both vectors in $\mathbb{R}^n$, and w approximates x, the solution to Eq. (3). Often, though, a time-varying output equation is used and the approximate equation becomes

$$\frac{dy}{dt} = P(y), \quad w = h(t, y) \tag{5}$$

where, once again, w approximates x. The oscillatory open-loop control results in a superposition of fast oscillatory trajectories on slow trajectories. The slow dynamics are represented by y, and h can be a fast periodic function. In either of the above two cases, it is hoped to find oscillatory control input $u(t)$ such that the transient performance of w meets desired objectives. Since the state equation of the approximations are time-invariant, the analysis becomes simpler. In fact, even though Eq. (5) is time-varying, it is only the output equation which explicitly depends on t. Therefore, many of the well established tools can be applied directly to the state equation. In particular, when $P(y) = Ay + B$, then the eigenvalues of matrix A help determine the qualitative features of transient behavior.

Steering and Path Planning for Kinematically Nonholonomic Systems

An application of open-loop oscillatory control which lies largely outside the boundaries of this chapter is the use of periodic functions in path generation for so-called kinematically nonholonomic systems. Such systems include wheeled vehicles such as the unicycle, autonomous wheeled robots, and cars with or without trailers. More generally, kinematically nonholonomic systems are systems which possess nonintegrable constraints, and typically the state equations do not include dynamic effects, such as torques, accelerations, and forces. Since this class of problems does not involve the article's central themes of stabilization and improvement of transient performance, only a brief description is given.

Consider the special case of Eq. (2) in which

$$f(x, u) = \sum_{i=1}^{m} u_i g_i(x)$$

A large body of literature has been published on the use of oscillatory inputs designed to force such systems along prescribed paths. The reader is referred to (37) and its references for details of these types of problems. The types of problems encountered in this application include the following:

1. The prescribed endpoint steering problem requires that given any pair of points $x_0, x_1 \in \mathbb{R}^n$, a vector of piecewise analytic control inputs $u(\cdot) = (u_1(\cdot), \ldots, u_m(\cdot))$ is to be determined to steer from some state x_0 at time $t = 0$ to x_1 at time $t = T > 0$.
2. The trajectory approximation steering problem requires that given any sufficiently regular curve $\gamma: [0, T] \to \mathbb{R}^n$, a sequence $[u^j(\cdot)]$ of control input vectors is found such that the corresponding sequence of trajectories converges (uniformly) to γ.

Several authors have suggested constructive methods for periodic controllers in this context, and further details may be found in (37,39–44).

STABILIZATION BY OSCILLATORY CONTROLS: METHODS AND SOLUTIONS

Applications of Classical Averaging Theory

The goal of the open-loop oscillatory control is usually to stabilize an unstable equilibrium x_s of Eq. (2). This is performed

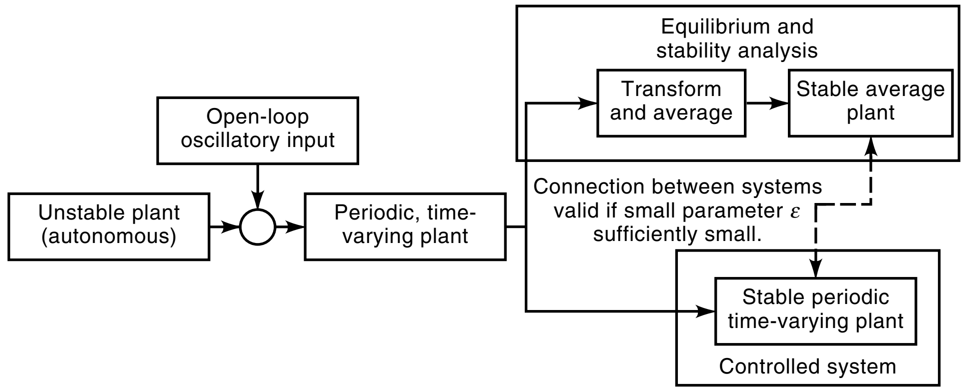

Figure 1. Flow graph of typical open loop oscillatory control design procedure.

by selecting the $\gamma(t)$ in Eq. (3) to be periodic zero average signals, such as sinusoidal inputs or zero average square waves. The frequency of the input is selected to be large, or equivalently, as Fig. 1 shows, the period is small. The periodic system, in the form of Eq. (3) can then be transformed into the form of Eq. (1.1) in Appendix 1, where ϵ turns out to be proportional to the period. At this point, the transformed system can be averaged. If the averaged system has a uniformly asymptotically stable equilibrium point, then this implies that there will be a uniformly asymptotically stable periodic orbit of the transformed time-varying system in the vicinity of the equilibrium point. The final criteria for vibrational stabilization is that the periodic orbit satisfying Eq. (3) remain in the vicinity of x_s (even though a transformation is used prior to averaging). This is the reason for introducing the definition of $\bar{x}^*$, which is the average value of the periodic solution of Eq. (3).

What follows is a step-by-step procedure for the analysis of open-loop oscillatory control laws by the classical method of averaging. A brief introduction to the topic of averaging and references to more comprehensive accountings may be found in Appendix A.1. Many summaries of this procedure detailed in this section can also be found in the literature (e.g. see Refs. 13,24,25,37). The following discussion is based on (37).

Assume that f in Eq. (3) has a special structure so that Eq. (3) can be rewritten as

$$\frac{dx}{dt} = f_1(x(t)) + f_2(x(t), \gamma(t)) \tag{6}$$

where $f_1(x(t)) = f_1(\lambda_0, x(t))$ and the function $f_2(x(t), \gamma(t))$ is linear with respect to its second argument. Additionally, assume that $\gamma(t)$ is periodic of period T $(0 < T \ll 1)$ and of the form $\gamma(t) = \omega\tilde{u}(\omega t)$, where $\omega = 2\pi/T$, and $\tilde{u}(\cdot)$ is some fixed period-2π function. Since the primary interest is high frequency forcing, the usual implication is that the amplitude of $\gamma(t)$ is large. It is possible, however, that $\tilde{u}(\cdot)$ has small amplitude, making the amplitude of $\gamma(t)$ small also.

Then Eq. (6) can be rewritten as

$$\frac{dx}{dt} = f_1(x(t)) + \omega f_2(x(t), \tilde{u}(\omega t)) \tag{7}$$

In order to proceed with the stability analysis, Eq. (7) will be transformed to an ordinary differential equation in standard form Eq. (1.1) in Appendix 1. To make this desired transformation, consider the so called generating equation given as

$$\frac{dx}{dt} = f_2(x(t), \tilde{u}(t))$$

Suppose that this generating equation has a period T general solution $h(t, c)$, for some $\tilde{u}(\cdot)$ and $t \geq t_0$, where $h: \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ and $c \in \mathbb{R}^n$ is uniquely defined for every initial condition $x(t_0) \in \Omega \subset \mathbb{R}^n$.

Introduce into Eq. (7) the Lyapunov substitution $x(t) = h(\omega t, z(t))$ to obtain

$$\frac{dz}{dt} = \left[\frac{\partial h(\omega t, z(t))}{\partial z}\right]^{-1} f_1(h(\omega t, z(t))) \tag{8}$$

If time is rescaled by letting $\tau = \omega t$, $\epsilon = 1/\omega$, then using the standard abuse of notation of letting $z_{\text{new}}(\tau) = z_{\text{old}}(\tau/\omega)$, Eq. (8) becomes

$$\frac{dz}{d\tau} = \epsilon\left[\frac{\partial h(\tau, z(\tau))}{\partial z}\right]^{-1} f_1(h(\tau, z(\tau))) \tag{9}$$

Equation (9) is a periodic differential equation in standard form with normalized period $T = 2\pi$ and averaging can be applied. The averaged equation (autonomous) corresponding to Eq. (9) is given as

$$\frac{dy}{d\tau} = \epsilon P(y(\tau)); \quad P(c) = \frac{1}{2\pi}\int_0^{2\pi}\left[\frac{\partial h(\tau, c)}{\partial c}\right]^{-1} f_1(h(\tau, c))\,d\tau \tag{10}$$

It is now possible to convert the averaged equation back to fast time to obtain

$$\frac{dy}{dt} = P(y(t)) \tag{11}$$

By the theory of averaging, there exists an $\epsilon_0 > 0$ such that for $0 < \epsilon \leq \epsilon_0$, the hyperbolic stability properties of Eqs. (9) and (10) are the same. This also implies that for ω sufficiently large, the hyperbolic stability properties of Eqs. (8) and (11) are also the same. Specifically, if y_s is an asymptotically stable equilibrium point of Eq. (11) (it will also be an asymptotically stable equilibrium point of Eq. (10)), this implies that, for ω sufficiently large, there exists a unique T-periodic solution,

$\psi^*(t)$ satisfying Eq. (8), in the vicinity of y_s that is asymptotically stable also. Furthermore, T is known to be equal to $2\pi/\omega$. Since the transform $x(t) = h(\omega t, z(t))$ is a homeomorphism, there will exist an asymptotically stable T-periodic solution to Eq. (7) given by $x^*(t) = h(\omega t, \psi^*(t))$. Equation (2) is said to be *vibrationally stabilized* provided that $\bar{x}^* = 1/T \int_0^T x^*(t)dt$ remains in the vicinity of x_s.

Example 1. Oscillatory stabilization of scalar differential equations. Consider the scalar linear differential equation

$$x^{(n)} + (a_1 + u_1(t))x^{(n-1)} + \ldots + (a_n + u_n(t))x = 0 \tag{12}$$

In (45), the problem of stabilizing Eq. (12) is studied using zero average periodic control inputs in the form

$$u_i(t) = k_i \omega \sin(\omega t + \phi_i) \quad i = 1, 2, \ldots, n \tag{13}$$

where k_i are constants. Furthermore, the results determined in (45) show that the impact of the control u_1 for stabilization is nonexistent. Hence, assume that $k_1 = 0$.

This system can easily be rewritten in state space form of $\dot{q} = Aq + \sum_{i=1}^{m} u_i(t)B_i q$. However, due to the results determined in (45) there is no need for this. For sufficiently large ω the hyperbolic stability properties of $x_s = 0$ in Eq. (12) are the same as the hyperbolic stability properties of the equilibrium point $y_s = 0$ of the corresponding differential equation with constant coefficients given by

$$y^{(n)} + (a_1 + \sigma_1)y^{(n-1)} + \ldots + (a_n + \sigma_n)y = 0 \tag{14}$$

where

$$\sigma_i = \frac{k_2 k_i}{2}\cos(\phi_2 - \phi_i) \quad i = 1, 2, \ldots, n$$

The impact of the above result is that it presents a calculation formula for the system. Without knowledge of any periodic transformations or mathematical analysis, it is possible to select the gain and phase of each oscillatory control to stabilize the zero equilibrium of Eq. (12) based on the stability properties of Eq. (14), for sufficiently large ω. Since all the coefficients in Eq. (14) are known, the analysis becomes simple.

Some important comments on Eq. (14) need to be made. First, notice that since $\sigma_1 = 0$, this implies that the coefficient of the $n - 1$th derivative in Eq. (14) cannot be changed. This coefficient is equal to the negative of the sum of all system eigenvalues ($= -trace[A]$). Hence, for vibrational stabilization to take place, it must be that $a_1 > 0$. Reference 45 shows this to be a necessary and sufficient condition for scalar differential equations. (In fact, for all systems $\dot{q} = Aq + \sum_{i=1}^{m} u_i(t)B_i q$ with $u_i(t)$ zero average, the $trace[A]$ must always be less than zero for vibrational stabilization to be possible.) This trace condition is never satisfied for linearized versions of the mechanical systems treated in the following section, indicating one direction in which the theory has been considerably extended in recent years. Next, note that σ_2 is always positive, and therefore, the coefficient of the $n - 2$th derivative in Eq. (14) can only be increased. The quantitites σ_i, $i \geq 3$ can be made either positive or negative; however, they depend on k_2. Therefore, oscillatory control must enter through the a_2 coefficient or else all σ_i will be zero and vibrational stabilization will not take place.

Example 2. Oscillatory stabilization of a second-order LTI system in state space. This example is a slight modification of the problem discussed by (46). Consider the second-order system

$$\dot{x} = \left(\begin{pmatrix} 0.6 & 1.3 \\ 0.8 & -1.6 \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} u(t) \right) x \tag{15}$$

where u is the scalar control. It is easy to verify that when $u = 0$ the equilibrium point $x_s = 0$ is unstable (check the eigenvalues of the system).

Suppose $u(t) = \beta\omega \cos(\omega t)$. Then

$$\frac{dx}{dt} = \begin{pmatrix} 0.6 & 1.3 \\ 0.8 & -1.6 \end{pmatrix} x + \begin{pmatrix} 0 & \beta\omega\cos(\omega t) \\ 0 & 0 \end{pmatrix} x \tag{16}$$

which is in the form of Eq. (7). The generating equation is therefore

$$\dot{x} = \begin{pmatrix} 0 & \beta\cos(t) \\ 0 & 0 \end{pmatrix} x$$

which has solution $x_2 = c_2$ and $x_1 = c_1 + \beta \sin(t)c_2$.

Now introduce the substitutions $x_2 = z_2$ and $x_1 = z_1 + \beta \sin(\omega t)z_2$ into Eq. (16) and convert time to $\tau = \omega t$ with $\epsilon \equiv 1/\omega$ to obtain

$$\frac{dz}{d\tau} = \epsilon \begin{pmatrix} 1 & -\beta\sin(\tau) \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0.6 & 1.3 \\ 0.8 & -1.6 \end{pmatrix} \begin{pmatrix} 1 & \beta\sin(\tau) \\ 0 & 1 \end{pmatrix} z(\tau) \tag{17}$$

which is now in a form that averaging can take place. Taking the average of Eq. (17) and converting back to regular time t leads to the equation corresponding to Eq. (11) of

$$\frac{dy}{dt} = \begin{pmatrix} 0.6 & 1.3 - 0.4\beta^2 \\ 0.8 & -1.6 \end{pmatrix} y(t) \tag{18}$$

The eigenvalues of Eq. (18) have negative real part when $\beta > 2.5$. The equilibrium point at zero remains unchanged. Therefore, for sufficiently large ω (equivalently sufficiently small $\epsilon > 0$) and for $\beta > 2.5$ the equilibrium $x_s = 0$ of Eq. (15) is vibrationally stabilized.

Example 3. Oscillatory stabilization of a simple pendulum: Classical Averaging. Consider a simple pendulum consisting of a massless but rigid link of length ℓ to which a tip of mass m and inertia I is attached, and let θ denote the counterclockwise rotation about the vertical hanging configuration. Suppose the hinge point of the pendulum is forced to oscillate vertically, where the elevation of the hinge above some reference height at time t is given by $R(t)$. An illustration of such a system is given in Fig. 2. Accounting for Rayleigh damping $b\dot{\theta}$ and gravitational forces, the pendulum dynamics can be written

$$I\ddot{\theta} + b\dot{\theta} + m\ell\ddot{R}\sin\theta + mg\ell\sin\theta = 0 \tag{19}$$

Suppose $R(t) = \beta \sin \omega t$. Then $\ddot{R}(t) = -\eta\omega \sin \omega t$, where $\eta = \eta(\omega) = \omega\beta$. Writing Eq. (19) as a system of first order equa-

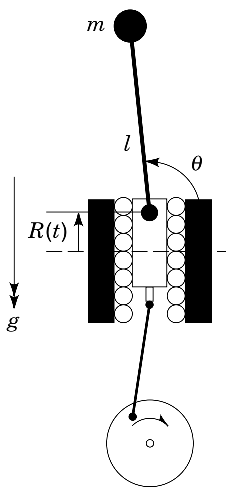

Figure 2. A simple pendulum whose hinge point undergoes vertical motion.

tions where $x_1 = \theta$ and $x_2 = \dot{\theta}$, it is clear that the first order system can be written in the form of Eq. (7). Following the steps detailed in the previous section, the generating equation is found to be

$$\dot{x}_1 = 0,$$
$$\dot{x}_2 = \frac{\eta m \ell}{I} \sin t \sin x_1$$

which has the solution

$$x_1 = c_1 = h_1(t, c),$$
$$x_2 = -\frac{\eta m \ell}{I} \cos t \sin c_1 + c_2 = h_2(t, c)$$

Introducing the transformation

$$x_1 = z_1,$$
$$x_2 = -\frac{\eta m \ell}{I} \cos \omega t \sin z_1 + z_2$$

letting $\tau = \omega t$, and letting $\epsilon = 1/\omega$, Eq. (9) specializes to

$$\dot{z}_1 = \epsilon \left[-\frac{\eta m \ell}{I} \cos \tau \sin z_1 + z_2 \right]$$
$$\dot{z}_2 = \epsilon \left[-\left(\frac{\eta m \ell}{I}\right)^2 \cos^2 \tau \cos z_1 \sin z_1 - \frac{m g \ell}{I} \sin z_1 \right.$$
$$\left. + \frac{\eta m \ell}{I} z_2 \cos \tau \cos z_1 + \frac{\eta m \ell b}{I^2} \cos \tau \sin z_1 - \frac{b}{I} z_2 \right]$$

Therefore the averaged equations, given by Eq. (11), are

$$\dot{y}_1 = y_2,$$
$$\dot{y}_2 = -\frac{1}{2}\left(\frac{\eta m \ell}{I}\right)^2 \cos y_1 \sin y_1 - \frac{m g \ell}{I} \sin y_1 - \frac{b}{I} y_2$$

Notice that the averaging preserves the upper equilibrium, and $x_s = 1/T \int_0^T h(t, y_s) dt$. Therefore, by the previous discussion, if the inverted equilibrium is asymptotically stable for the averaged equation, then for sufficiently large forcing frequencies ω there exists an asymptotically stable periodic orbit near the inverted equilibrium. A simple linearization of the averaged equation reveals that the stability condition for the inverted equilibrium is given by $\omega^2 \beta^2 > 2Ig/m\ell$.

Remark 2. Note that in the absence of dissipative forces that the linearized averaged system will possess eigenvalues either of the form $\lambda_{1,2} = \pm\lambda$ where $\lambda \in \mathbb{R}$, or of the form $\lambda_{1,2} = \pm i\lambda$, where $i\lambda \in \mathbb{C}$. Hence the system is never asymptotically stable in the absence of damping, and stability results in this case are weak. The lack of asymptotic stability is a characteristic of Hamiltonian, and more generally, conservative systems. The averaging technique of the next subsection is more suited to such systems and yields stronger stability results.

Remark 3. Simple nonquantitative experiments demonstrating the stabilization described in this example are not difficult to build and work remarkably well. Such an experiment is shown in Fig. 3. In this experiment, the rotary motion of a dc motor is rectified to periodic linear motion by the mechanism shown in the left frame of Fig. 3. Note that by virtue of the construction, the forcing amplitude is fixed and the forcing frequency can vary. It is observed that when current is applied to the motor, the inverted equilibrium is unstable until the forcing frequency reaches a critical frequency at which the inverted equilibrium experiences a bifurcation which renders it stable, as depicted in the right frame of Fig. 3. The inverted equilibrium is then stable for all higher frequencies.

Remark 4. To this point, the main goal has been to use averaging as a means of studying the local stability properties of periodically excited systems. Under certain conditions, however, the averaged system gives far more information about the global structure of the periodically excited system. As essentially a perturbation technique, averaging theorems as found in (36,47,48) give no clues as to how large the small parameter ϵ can be perturbed off zero before the averaged dynamics fail to describe the forced dynamics. For ϵ sufficiently large, a variety of undesirable nonlinear effects arise, such as subharmonic resonance and stochasticity, which are not captured in any way by the simple averaging of the nonautonomous dynamics. Because of the inherent difficulty of the analysis, theory for the prediction of nonlinear effects in this range has been slow to emerge. A later section briefly illustrates some of the features of periodically excited systems exhibit when ϵ is allowed to vary.

Averaging for Mechanical Systems

Recently, interest has emerged in using high frequency oscillatory forcing to control the dynamics of mechanical systems. The typical applications setting is a controlled Lagrangian system where only some of the degrees of freedom are directly controlled:

$$\frac{d}{dt}\frac{\partial \mathscr{L}}{\partial \dot{q}_1} - \frac{\partial \mathscr{L}}{\partial q_1} = u \tag{20}$$

$$\frac{d}{dt}\frac{\partial \mathscr{L}}{\partial \dot{q}_2} - \frac{\partial \mathscr{L}}{\partial q_2} = 0 \tag{21}$$

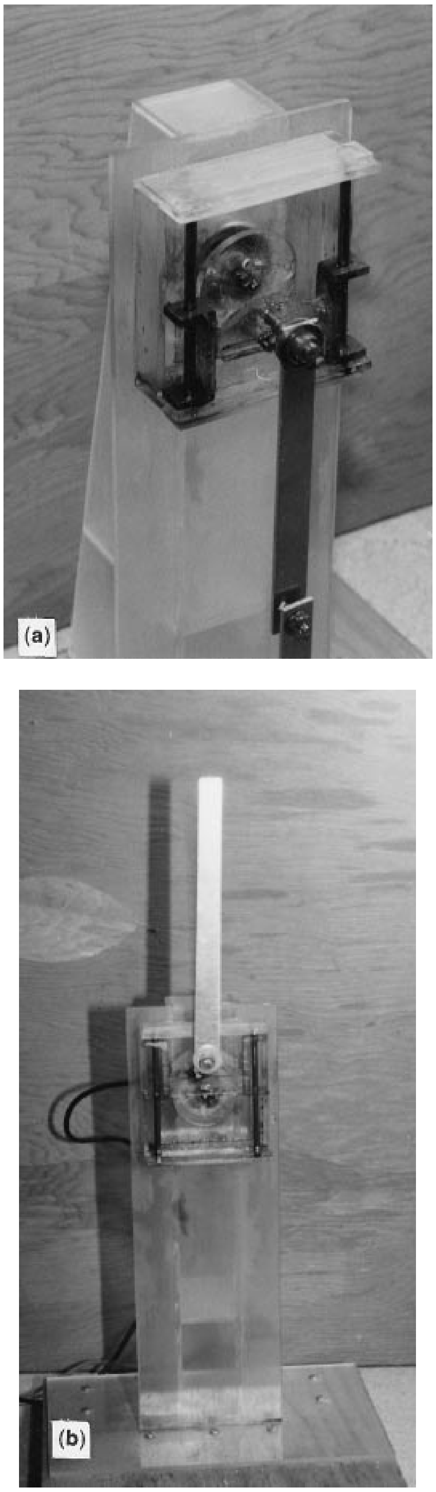

Figure 3. A simple experiment to demonstrate the stabilization of the inverted equilibrium of the vertically forced pendulum. The picture on the left (a) shows the mechanism which rectifies the rotary motion of the dc motor into periodic linear motion. The picture on the right (b) shows the pendulum stabilized in the inverted equilibrium.

where it is assumed *dim* $q_1 = m$ and *dim* $q_2 = n$, and u is an m-vector of controls. (Systems of this form have been called *super-articulated* in the literature, and the reader is referred to (49) for details and references.) Within this class of models, it is further assumed that there is enough control authority to always be able to completely specify any trajectory $q_1(\cdot)$ over an interval of interest. When this is the case, $q_1(\cdot)$, $\dot q_1(\cdot)$, and $\ddot q_1(\cdot)$ are viewed collectively as generalized inputs, and are used to control the dynamics of the configuration variables $q_2(\cdot)$. The starting point may thus be taken to be a (generalized) control system [see Fliess (50) for an introduction to generalized control systems] prescribed by a Lagrangian

$$\mathscr{L}(q,\dot q;x,v) = \frac{1}{2}\dot q^T \mathscr{M}(q,x)\dot q + v^T \mathscr{A}(q,x)\dot q - \mathscr{V}_a(q;x,v) \quad (22)$$

If

$$\mathscr{L}(q_1,\dot q_1;q_2,\dot q_2) = \tfrac{1}{2}(\dot q_1^T,\dot q_2^T)\begin{pmatrix} M_{11} & M_{12} \\ M_{12}^T & M_{22}\end{pmatrix}\begin{pmatrix}\dot q_1 \\ \dot q_2\end{pmatrix} - \mathscr{V}(q_1,q_2)$$

is the Lagrangian associated with Eqs. (20) and (21), then with the identifications $q_1 \to x$, $\dot q_1 \to v$, $q_2 \to q$, $M_{22} \to \mathscr{M}$, $M_{12} \to \mathscr{A}$, and $\mathscr{V}_a(q;x,v) = \mathscr{V}(x,q_\mathscr{V}) - \frac{1}{2}V^T M_{11} V$, the connection between the Lagrangian dynamics prescribed by Eq. (22) and Eq. (21) is clear.

To simplify averaging, perform the usual Legendre transform $\mathscr{H} = p\dot q - \mathscr{L}$, where $p = \partial\mathscr{L}/\partial\dot q$, and write the resulting Hamiltonian in terms of the variables q, p; x, v

$$\mathscr{H}(q,p;x,v) = \frac{1}{2}(p - \mathscr{A}^T v)^T \mathscr{M}^{-1}(p - \mathscr{M}^T v) + \mathscr{V}_a \quad (23)$$

This quantity is not a proper Hamiltonian since in general $\partial\mathscr{H}/\partial t \neq 0$. It is remarkable that if the (generalized) input functions $x(\cdot)$, and $v(\cdot) = \dot x(\cdot)$ are restricted to be periodic and the *simple average* of $\mathscr{H}$ over one period is computed, (i) the resulting quantity $\overline{\mathscr{H}}$ will itself be a proper Hamiltonian, and (ii) in many cases the dynamics associated with $\overline{\mathscr{H}}$ will closely approximate the dynamics of the nonautonomous system prescribed by Eq. (23). Recall that the *simple average* is the time average over one period of $\mathscr{H}(q, p; x(t), v(t))$ where q and p are viewed as variables which do not depend on the time t. The averaged Hamiltonian Eq. (23) can be written

$$\begin{aligned}
\overline{\mathscr{H}}&(q,p) \\
&= \frac{1}{2}p^T\overline{\mathscr{M}^{-1}}p - \overline{v^T\mathscr{A}\mathscr{M}^{-1}}p + \frac{1}{2}\overline{v^T\mathscr{A}\mathscr{M}^{-1}}(\overline{\mathscr{M}^{-1}})^{-1}\overline{\mathscr{M}^{-1}\mathscr{A}^T v} \\
&\quad + \frac{1}{2}\overline{v^T\mathscr{A}\mathscr{M}^{-1}\mathscr{A}v} - \frac{1}{2}\overline{v^T\mathscr{A}\mathscr{M}^{-1}}(\overline{\mathscr{M}^{-1}})^{-1}\overline{\mathscr{M}^{-1}\mathscr{A}^T v} + \overline{\mathscr{V}} \\
&= \underbrace{\frac{1}{2}(\overline{\mathscr{M}^{-1}}p - \overline{\mathscr{M}^{-1}\mathscr{A}^T v})^T(\overline{\mathscr{M}^{-1}})^{-1}(\overline{\mathscr{M}^{-1}}p - \overline{\mathscr{M}^{-1}\mathscr{A}^T v})}_{\text{averaged kinetic energy}} \\
&\quad + \underbrace{\frac{1}{2}\overline{v^T\mathscr{A}\mathscr{M}^{-1}\mathscr{A}^T v} - \frac{1}{2}\overline{v^T\mathscr{A}\mathscr{M}^{-1}}(\overline{\mathscr{M}^{-1}})^{-1}\overline{\mathscr{M}^{-1}\mathscr{A}^T v} + \overline{\mathscr{V}}}_{\text{averaged potential}}
\end{aligned} \quad (24)$$

The *averaged potential* given in Eq. (24) is an energy-like function of the generalized coordinates q which is abbreviated $\mathscr{V}_A(q)$. A complete understanding of the relationship between the dynamics of nonautonomous Hamiltonian systems of Eq. (23) and the appropriate counterparts for averaged Hamiltonian $\mathscr{I}$ of Eq. (24) does not presently exist. There are very broad classes of such systems, however, for which it is possible to prove the validity of the following:

Averaging Principle for Periodically Forced Hamiltonian Systems. The dynamics associated with Eq. (23) under periodic forcing $(x(t), v(t))$ are locally determined in neighborhoods of critical points of the *averaged potential* $\mathscr{V}_A(q)$ as follows:

- If q^* is a strict local minimum of $\mathscr{V}_A(\cdot)$, then provided the frequency of the periodic forcing $(x(\cdot), v(\cdot))$ is sufficiently high, the system will execute motions confined to a neighborhood of q^*.
- If $(q, p) = (q^*, 0)$ is a hyperbolic fixed point of the corresponding averaged system (i.e., the Hamiltonian system determined by Eq. (24)), then there is a corresponding periodic orbit of the forced system such that the asymptotic stability properties of the fixed point $(q^*, 0)$ of the averaged system coincide with the asymptotic stability properties of the periodic orbit for the forced system.

This type of averaging for the analysis of periodically forced mechanical systems has been treated in (7) in the case in which $\mathscr{M}$ and $\mathscr{A}$ in Eq. (22) do not depend explicitly on the variable x. A detailed stability analysis based on Floquet theory appears in (8), but this is restricted to the case in which local minima of the averaged potential correspond to rest point of the nonautonomous dynamics. In the more general case, the motion of systems defined by Eq. (23) are organized around local minima of the averaged potential which are not rest points of Eq. (23). The theory is less well developed for this case, but (33) and (10) analyzes the class of single input systems and presents detailed results on the correspondence between averaged system phase portraits and the corresponding Poincaré maps of the nonautonomous system defined by Eq. (23).

Example 4. Oscillatory stabilization of a simple pendulum: Averaged potential. This example illustrates the use of the averaged potential in analyzing the dynamics of a pendulum whose hinge point is forced to undergo oscillatory linear motion which is not necessarily vertical as in the last example. Suppose (x, y) gives the coordinates of the horizontal and vertical displacement of the hinge point of a pendulum attached to a sliding block which is controlled to execute the oscillatory motion

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} \cos\alpha \\ \sin\alpha \end{pmatrix} \sin\omega t$$

where α prescribes the direction of the oscillatory motion, and ω is the frequency. This system is illustrated in Fig. 4. If, as in the last example, the pendulum has total mass m and inertia I about its hinge point, the motion under this oscillatory forcing is described by a second order differential equation

$$I\ddot{\theta} - m\ell\omega^2\beta\cos(\theta - \alpha)\sin\omega t + mg\ell\,\sin\theta = 0 \qquad (25)$$

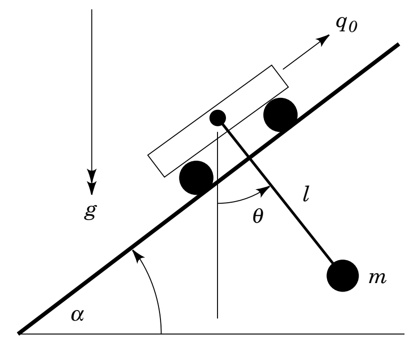

Figure 4. Periodically forced pendulum where the forcing is directed along a line of angle α with respect to the horizontal.

This equation may be derived from a Lagrangian of the form

$$\mathscr{L}(\theta, \dot{\theta}, v) = \frac{1}{2}I\dot{\theta}^2 + m\ell\cos(\theta - \alpha)v(t)\dot{\theta} + mg\ell\cos\theta$$

where $v(t) = \omega\beta\cos\omega t$. The averaged potential for this system is

$$\mathscr{V}_A(\theta) = \frac{(m\ell\omega\beta)^2}{4I}\cos^2(\theta - \alpha) - mg\ell\cos\theta$$

When $\alpha = \pi/2$, the hinge of the pendulum undergoes the vertical oscillation described in the previous example. The averaged potential has two or four critical points in the interval $[0, 2\pi)$ depending on whether or not $\beta^2\omega^2$ is less than or larger than $2Ig/m\ell$. Clearly the equilibrium $\theta = \pi$ is a strict local minimum of the averaged potential if and only if $\beta^2\omega^2 > 2Ig/m\ell$. According to the theory of averaging presented in (7) and (8), the pendulum will execute stable motions confined to a neighborhood of this equilibrium for sufficiently large values of ω. This also recovers the result obtained in the previous example.

Remark 5. This example illustrates nonclassical behavior in the case $\alpha \neq \pi/2$. For this case there will be, for sufficiently large values of ω, strict local minima of the averaged potential which are *not* equilibrium points of the nonautonomous Eq. (25). Nevertheless, the pendulum will still execute motions confined to neighborhoods of such local minima. For details on this type of emergent behavior, see (33) and (10).

Remark 6. The strategy behind the very simple (open loop) control designs associated with the averaged potential (and more generally with systems having oscillatory control inputs) is to produce robustly stable emergent behavior which is related to the critical point structure of the averaged potential. The design method for control laws in this category involves designing the averaged potential functions themselves by means of appropriately chosen inputs. The guiding theory for this approach remains very much under development.

Floquet Theory

Another body of theory used in the study of the stability of equilibriums and periodic orbits of systems controlled by open-loop oscillatory inputs is Floquet theory. As described in Appendix 2, the central idea behind the theory is that the local stability of an equilibrium or periodic orbit may be determined from the eigenvalues of the *monodromy matrix M.* The monodromy matrix represents the growth or decay of solutions of the linearized system, where the linearization is about that equilibrium or periodic orbit. In general, computing the monodromy matrix is not straightforward. The calculation is relatively easy, however, if the linearization of the system state equations is piecewise constant in t. For example, suppose that the linearized time-varying system is

$$\dot{x} = A(t)x$$

where $A(t) = A_1$ on $0 \le t < t'$, and $A(t) = A_2$ on $t' \le t < T$, such that $A_1A_2 = A_2A_1$. Then the monodromy matrix M can be obtained by computing the state transition matrix Φ on the interval $[0, T]$; that is

$$\begin{aligned}\Phi(T,0) &= \Phi(t',0)\Phi(T,t')\\ &= e^{\int_0^{t'} A_1 dt} e^{\int_{t'}^{T} A_2 dt}\\ &= M\end{aligned}$$

While in mechanical problems the assumption of piecewise constant forcing is somewhat nonphysical, such approximations are often useful when the forcing frequency is sufficiently large. Piecewise constant forcing is often not a problem in the analysis electrical and electronic systems, where such forcing is common.

Example 5. Oscillatory stabilization of a simple pendulum: Floquet theory. In the previous examples, it was shown that the inverted pendulum may be stabilized with high frequency vertical oscillation by averaging the time-varying equations and studying the stability of the inverted equilibrium with the averaged system. In this example, stabilization is inferred by linearizing the pendulum dynamics about the inverted equilibrium and studying the eigenvalues of the monodromy matrix. To facilitate the calculation, assume that the pendulum is forced by square wave forcing, that is, in Eq. (19) $\ddot{R}(t) = \omega^2\beta u(\omega t)$ where $u(t) = u(t + 1/\omega)$ is a square wave which switches periodically between $+1$ and -1. Also, assume that the damping term $b = 0$. This assumption simplifies the following calculations and more importantly allows us to show that pendulum stabilization does not require dissipation. Linearizing Eq. (19) about the inverted equilibrium gives rise to the linear system

$$\begin{pmatrix}\dot{\theta}_1\\ \dot{\theta}_2\end{pmatrix} = \begin{pmatrix}0 & 1\\ \dfrac{1}{I}[mg\ell - m\ell\omega^2\beta u(\omega t)] & 0\end{pmatrix}\begin{pmatrix}\theta_1\\ \theta_2\end{pmatrix}$$

Because the input $u(t)$ is piecewise constant, the state transition matrix $\Phi(t, 0)$ over one period of $u(t)$ may be computed as follows:

$$\begin{aligned}&\Phi(0,1/\omega)\\ &= \Psi(0,1/2\omega)\Psi(1/2\omega,1\omega)\\ &= \begin{pmatrix}\cos\dfrac{\Omega_1}{2\omega} & \dfrac{1}{\Omega_1}\sin\dfrac{\Omega_1}{2\omega}\\ -\Omega_1\sin\dfrac{\Omega_1}{2\omega} & \cos\dfrac{\Omega_1}{2\omega}\end{pmatrix}\begin{pmatrix}\cosh\dfrac{\Omega_2}{2\omega} & \dfrac{1}{\Omega_2}\sinh\dfrac{\Omega_2}{2\omega}\\ \Omega_2\sinh\dfrac{\Omega_2}{2\omega} & \cosh\dfrac{\Omega_2}{2\omega}\end{pmatrix}\\ &= \begin{pmatrix}\cos\dfrac{\Omega_1}{2\omega}\cosh\dfrac{\Omega_2}{2\omega} + \dfrac{\Omega_2}{\Omega_1}\sin\dfrac{\Omega_1}{2\omega}\sinh\dfrac{\Omega_1}{2\omega} & \dfrac{1}{\Omega_2}\cos\dfrac{\Omega_1}{2\omega}\sinh\dfrac{\Omega_2}{2\omega} + \dfrac{1}{\Omega_1}\sin\dfrac{\Omega_1}{2\omega}\cosh\dfrac{\Omega_2}{2\omega}\\ -\Omega_1\sin\dfrac{\Omega_1}{2\omega}\cosh\dfrac{\Omega_2}{2\omega} + \Omega_2\cos\dfrac{\Omega_1}{2\omega}\sinh\dfrac{\Omega_2}{2\omega} & \cos\dfrac{\Omega_1}{2\omega}\cosh\dfrac{\Omega_2}{2\omega} - \dfrac{\Omega_1}{\Omega_2}\sin\dfrac{\Omega_1}{2\omega}\sinh\dfrac{\Omega_1}{2\omega}\end{pmatrix}\end{aligned}$$

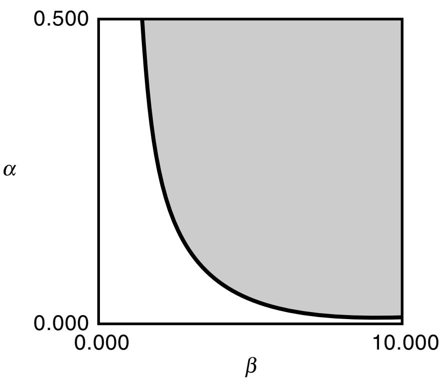

Figure 5. Regions of stability (darkened) and instability (light) for the vertically forced simple pendulum. In this figure, $m = g = \ell = I = 1$.

where $\Omega_1^2 = -1/I\,[mg\ell - m\ell\omega^2\beta] > 0$ and $\Omega_2^2 = 1/I\,[mg\ell + m\ell\omega^2\beta] > 0$. Stability of the fixed point may be determined from the eigenvalues λ_1, λ_2 of $\Phi(0, 1/\omega)$, which are the roots of the characteristic polynomial

$$\lambda^2 - \left[\left(\frac{\Omega_2}{\Omega_1} - \frac{\Omega_1}{\Omega_2}\right)\sin\frac{\Omega_1}{2\omega}\sinh\frac{\Omega_2}{2\omega} + 2\cos\frac{\Omega_1}{2\omega}\cosh\frac{\Omega_2}{2\omega}\right]\lambda + 1 = 0$$

and the usual stability condition is that both eigenvalues must lie within the unit disk in the complex plane. Given the form of the characteristic polynomial, an equivalent condition for stability is trace $\Phi(0, 1/\omega) \le 2$, which in this case may be written

$$2\cos\frac{\Omega_1}{2\omega}\cosh\frac{\Omega_2}{2\omega} + \left(\frac{\Omega_2}{\Omega_1} - \frac{\Omega_1}{\Omega_2}\right)\sin\frac{\Omega_1}{2\omega}\sinh\frac{\Omega_2}{2\omega} < 2$$

The boundary between regions of stability and instability may be approximated by expanding the trigonometric functions in Taylor series around zero and solving a truncated inequality for ω in terms of β. Note beforehand that in doing so, it has been implicitly assumed that $\Omega_1/2\omega \sim 0$ and $\Omega_2/2\omega$ which implies ω is large. For the present example, the stability regions displayed in Fig. 5 have been obtained by numerically calculating λ_1 and λ_2 over the indicated ranges of β and ω.

Remark 7. Note that in the absence of damping, $\Phi(0, 1/\omega)$ is an area-preserving map. This fact implies that λ_1 and λ_2 are constrained to either lie on the real axis such that $\lambda_2 = 1/\lambda_1$, or lie on the unit disk in the complex plane. In this case, stability results are fairly weak (lack of asymptotic stability), but are typical of Hamiltonian, and more generally, conservative systems. In the presence of dissipation, the eigenvalues may occur as conjugate pairs inside the unit disk, implying asymptotic stability.

Remark 8. Note that results obtained through Floquet theory are strictly local results. In the case where the controlled system is nonlinear, proofs of stability *and instability* are strictly for the equilibrium or periodic orbit, and no information is given about the asymptotic behavior of solutions nearby. This remark is not surprising for neutrally stable equilibria, but it is in fact sometimes true for unstable equilibria as well. As an example, the reader is referred to the parametrically excited pendulum example later in this article, where in Fig. 8(b) it is seen that while the origin is unstable, orbits which pass arbitrarily close to the origin are in fact bounded by KAM (for Kolmogorov, Arnold, and Maser) tori and subharmonic resonance bands. Understanding when proofs of instability obtained by Floquet theory imply unbounded solutions for such systems is a current topic of research.

PERFORMANCE IMPROVEMENTS AND TRANSIENT BEHAVIOR

Sometimes stability in a system is not in question, but certain performance specifications are the design constraints. This section begins by revisiting the particle accelerator. Then a method to analyze the transient behavior of a system is given and followed by an example.

Example 6. Suppression of betatron oscillations in cyclic accelerators. Consider the cyclic accelerator subject to oscillations of focusing and defocusing sectors of the magnetic lens given in Eq. (1). The work of (1) describes an experimental method of betatron oscillations by alternating gradient focusing. To make the focusing improve, it is desirable that both the $\omega^2(1 - n)$ and the $\omega^2 n$ terms increase simultaneously, with $0 < n < 1$. Clearly, though, if n is viewed as the control variable, then it is not possible to both increase and decrease n and $1 - n$ simultaneously. Instead, the technique of focusing and defocusing a lens is introduced, which, according to (2) is modelled

$$\begin{aligned}\frac{d^2x}{d\theta^2} + \omega^2[1 - (n + K(\Omega,\theta))]x = 0\\ \frac{d^2z}{d\theta^2} + \omega^2[n + K(\Omega,\theta)]z = 0\end{aligned} \qquad (26)$$

where K is a PAZ function with frequency Ω; for example, $K(\Omega, \theta) = \alpha(\Omega)\sin \Omega\theta$.

Suppose that the oscillatory control input is given by $K(\Omega, \theta) = \beta\Omega \sin(\Omega\theta)$, that is, $\alpha(\Omega) = \omega\Omega$, where β is a constant. Define $x_1 = x$, $x_2 = \dot{x}$, $x_3 = z$ and $x_4 = \dot{z}$. Then the state space representation of Eq. (26) becomes

$$\begin{aligned}\frac{dx_1}{d\theta} &= x_2\\ \frac{dx_2}{d\theta} &= -\omega^2(1-n)x_1 + \beta\omega^2\Omega\sin(\Omega\theta)x_1\\ \frac{dx_3}{d\theta} &= x_4\\ \frac{dx_4}{d\theta} &= \omega^2 n x_3 - \beta\omega^2\Omega\sin(\Omega\theta)x_3\end{aligned} \qquad (27)$$

which is precisely the form of Eq. (7). Using the previously discussed techniques, the equation corresponding to Eq. (11) becomes

$$\begin{aligned}\frac{d^2y}{d\theta^2} + \omega^2(1 - n + 0.5\beta^2)y = 0\\ \frac{d^2\zeta}{d\theta^2} + \omega^2(n + 0.5\beta^2)\zeta = 0\end{aligned} \qquad (28)$$

It is seen that the net effect was to increase both the $\omega^2(1 - n)$ and the $\omega^2 n$ terms, which as a result, improved system performance by suppressing the betatron oscillations in both the x and z directions (this simplified model did not take into account the interaction of oscillations along x and z). Hence, performance has been improved via the introduction of oscillatory open-loop control, provided that Ω is sufficiently large. This example demonstrates that the benefits of introducing oscillations can, at times, help system performance even when stabilization is not an issue.

Analysis Method For Transient Behavior

From the theory presented above, it is possible to approximate the transient behavior of Eqs. (7) and (9), using the techniques found in Refs. (38) and (25). Assume that Eq. (11) has an asymptotically stable equilibrium point, z_s, in the vicinity of the equilibrium point of Eq. (2), x_s, with $u = constant = \lambda_0$.

- By the method of averaging, for sufficiently large ω, the solutions of Eqs. (8) and (11) remain arbitrarily close to each other provided they have the same initial conditions. That is, $y(t) \approx z(t)$ for all time.
- The solution to Eq. (7) is given by $x(t) = h(\omega t, z(t))$, where $z(t)$ is the solution to Eq. (8). This transformation is a homeomorphism.

Therefore, the quantities in Eq. (5) can be selected so that $h(t, c)$ is the solution to the generating equation and P is as defined in Eq. (11). Now define $w = h(\omega t, y(t))$ where $y(t)$ is the solution to $\dot{y}(t) = P(y(t))$ in Eq. (11). For sufficiently large ω,

$$x(t) \approx h(\omega t, y(t)) = w(t)$$

It is possible to analyze the transient behavior of Eq. (7) in two different manners:

1. Analyze the transient behavior of $y(t)$ and then examine the behavior of $x(t)$ through the direct relation $x(t) \approx w(t)$. This technique predicts many of the fast oscillatory parts of the trajectory x.
2. Analyze a moving average of x(t), given by $\overline{x}(t)$ where

$$\overline{x}(t) \equiv H(z(t)); \quad H(c) \equiv \frac{1}{T}\int_0^T h(\lambda, c)\,d\lambda \qquad (29)$$

   and T is the period of $h(t, \cdot)$. This is done by approximating $\overline{x}(t) \approx H(y(t))$ and once again, analyzing the transient behavior of y (making this technique more typical of Eq. (4)). Since the fast dynamics in h are averaged out, this technique introduces some error. On the other hand, since H does not explicitly depend on t, the analysis becomes simpler.

In either of the two methods, controlling $y(t)$ in Eq. (11) governs how to control $x(t)$.

Example 7. Transient behavior of the vertically oscillating pendulum. Referring back to Example 3 and using the same notation, it is possible to now approximate the trajectories of

θ, the angular position, and $\dot{\theta}$, the pendulum's angular velocity. The averaged equations of the system are

$$\dot{y}_1 = y_2$$
$$\dot{y}_2 = -\frac{1}{2}\left(\frac{\eta m\ell}{I}\right)^2 \cos y_1 \sin y_1 - \frac{mg\ell}{I}\sin y_1 - \frac{b}{I}y_2$$

The transformations utilized are

$$x_1 = z_1$$
$$x_2 = -\frac{\eta m\ell}{I}\cos\omega t \sin z_1 + z_2$$

Therefore, it is possible to obtain estimates on the transient behavior of the vertically oscillating pendulum using one of the two following methods:

Method 1. The estimate on θ is given by $\theta \approx y_1$. The estimate on $\dot{\theta}$ is given by

$$\dot{\theta} \approx -\frac{\eta m\ell}{I}\cos\omega t \sin y_1 + y_2$$

Method 2. The estimate on $\bar{\theta}$ is given by $\bar{\theta} \approx y_1$. The estimate on $\bar{\dot{\theta}}$ is given by

$$\bar{\dot{\theta}} \approx H(y)$$

where

$$H(y) \equiv \frac{1}{2\pi}\int_0^{2\pi}\left[-\frac{\eta m\ell}{I}\cos s \sin y_1 + y_2\right]ds$$
$$= y_2$$

Note that the approximation obtained by Method 2 is merely the averaged system. The approximation obtained by Method 1, as well as the transient solution obtained from the original equation of motion, are compared to the averaged solution in Fig. 6. It is clear from the figure that the averaged system accurately captures the averaged behavior of the system as it stabilizes. The phase plot in the lower left of the figure shows some discrepancies between the trajectory obtained by Method 1 with the trajectory of the original system, but in the lower right figure it is seen that the Method 1 approximation of $\dot{\theta}$ is actually quite close to the velocity of the original system. As Fig. 6 shows, Method 1 is more accurate than Method 2. However, Method 1 utilizes a time-varying output equation, making it a more complicated technique.

RESONANCE PHENOMENA IN PERIODICALLY FORCED SYSTEMS

Resonances arise as the natural result of subjecting any system to periodic excitation. The resonances produced vary

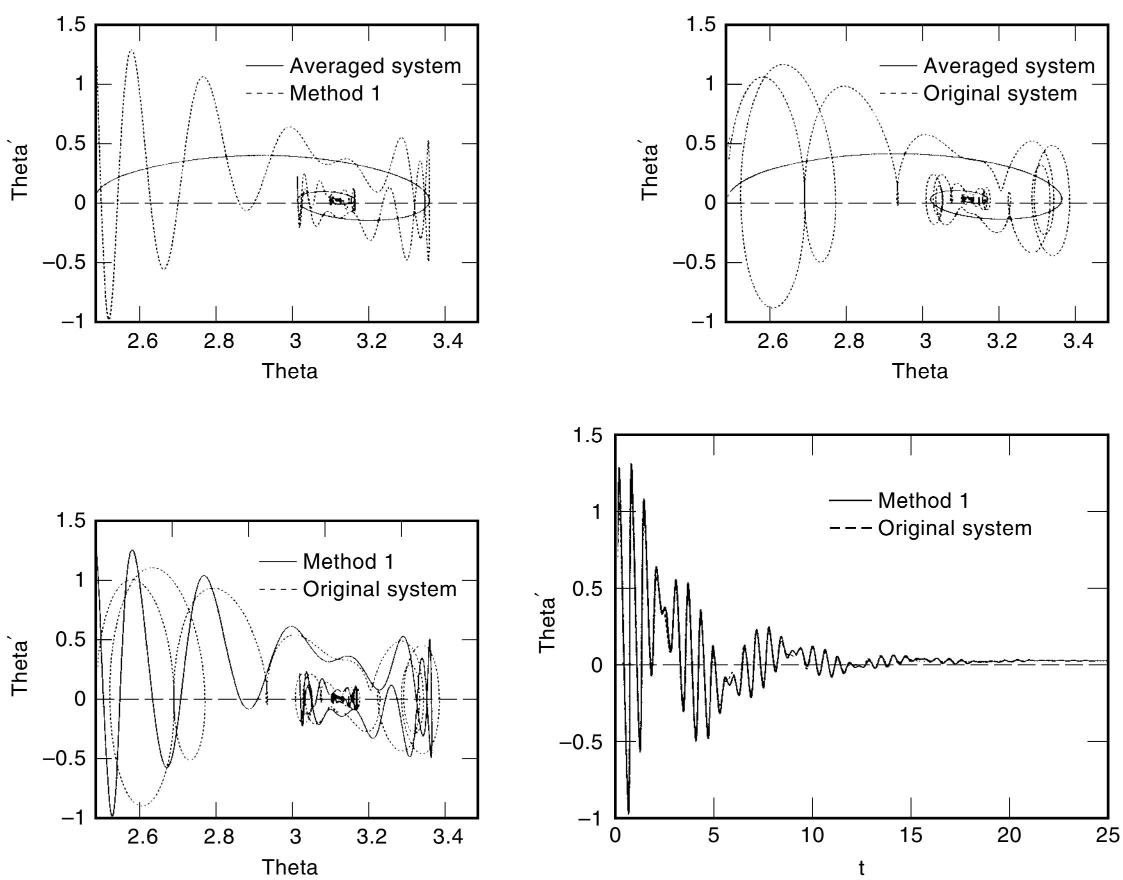

Figure 6. A comparison of the behaviors of the averaged/Method 2 system, the Method 1 system, and the original system. In these plots, $m = g = \ell = I = 1$, $b = 0.5$, $\beta = 0.2$, and $\omega = 10$.

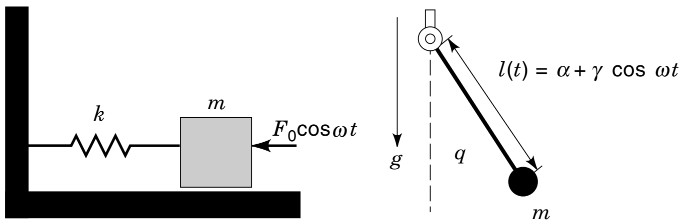

Figure 7. The periodically forced spring–mass system (left) and the parametrically excited pendulum (right).

based on the type of system being excited; for example the simple resonance produced by exciting the spring-mass system illustrated in the left frame of Fig. 7 is different from the parametric resonance exhibited by the parametrically excited pendulum illustrated on the right. The difference in the resonant behavior exhibited by the two systems is that the forced mass-spring system has a resonance at a single frequency, where the parametrically excited pendulum has resonances at subharmonics of the natural frequency, similar to those of Mathieu's equation. As a nonlinear system, the parametrically excited pendulum also has resonances which form as the result of the breaking and tangling separatrix solutions, as visualized with the period-1 return map or Poincaré map.

In designing open-loop oscillatory controls for the type of stabilization described in this article, a primary objective should be to choose the forcing frequencies such that undesirable resonances are avoided. The main obstacle to prevent the control theorist from ensuring this using the methods presented in this article is that (i) averaging techniques do not generally capture phenomena of periods different than the period over which the system is averaged, and (ii) Floquet theory gives only local stability information for a single equilibrium or periodic orbit. Experience and the application of more powerful tools in the analysis of such systems has shown that these effects are often avoided by choosing the forcing frequency to be sufficiently large (10). This observation is very consistent with the "ϵ sufficiently small" conditions imposed by averaging theory, although the arguments used by averaging theory are for purely analytical reasons. The literature concerning the global dynamics of nonlinear systems excited by periodic inputs is vast, fairly technical, and tends to lie completely outside the boundaries of conventional control theory. In this section, only broad concepts and observations are presented in the form of an example, and the reader is referred to such texts as (36,51,52) for extensive overviews of the field.

Example 8. Qualitative features of parametrically excited pendulum dynamics. To illustrate the kind of responses which might arise in subjecting a nonlinear system to periodic excitation, consider the parametrically excited pendulum (PEP) with no damping, which is often described by the differential equation

$$\ddot{q} + (1 + \gamma \sin \omega t) \sin q = 0 \tag{30}$$

Remark 9. Note that after a rescaling of time $\tau = \omega t$, Eq. (30) takes the form

$$q'' + \left(\frac{1}{\omega^2} + \frac{\gamma}{\omega^2} \sin \tau\right) \sin q = 0$$

which, if $\alpha = 1/\omega^2$ and $\beta = \gamma/\omega^2$, is recognizable as a nonlinear version of Mathieu's equation

$$q'' + (\alpha + \beta \sin t) q = 0$$

Note also that by letting $\gamma = \omega^2 \beta$, Eq. (30) is merely the equation of the motion of the vertically forced pendulum discussed in Examples 3, 4, and 5. In this example, however, it is assumed that $\gamma < 1$.

The natural tool for visualizing the global dynamics of single-degree-of-freedom periodically forced system is the Poincaré map, where Poincaré sections are taken at the end of every forcing period. Features of the phase portrait typically have equivalent features in the Poincaré map. For example, fixed points and periodic orbits in the phase portrait are associated with periodic points of the Poincaré map. The Poincaré map also preserves hyperbolic structures of the phase portrait; for example, normally hyperbolic periodic points of the Poincaré map reflect the normally hyperbolic structure of the corresponding equilibriums and periodic orbits of the phase portrait. Poincaré maps clearly show the bands of stochasticity, indicating the presence of chaotic behavior and subharmonic resonances, which are periodic orbits periodic of some rational multiple of the forcing period.

The results of simple simulations of the PEP are shown in Figs. 8(a–f). Using averaging techniques described previously, it can be shown that the averaged system is merely the unperturbed ($\gamma = 0$) system, the phase portrait of which is shown in Fig. 8(a). In general, in implementing an open-loop oscillatory control law, an objective is to choose the forcing parameters so that the Poincaré map closely resembles the averaged phase portrait. In the present examples, there are three ways to systematically adjust γ and ω: (i) fix γ and let ω vary, (ii) fix ω and let γ vary, or (iii) let $\gamma = \gamma(\omega)$ or $\omega = \omega(\gamma)$ and adjust the independent variable. Physically, it is often most reasonable to fix the forcing amplitude and control the frequency, hence for the current example attention is restricted to this case.

The five other plots in Fig. 8 reflect the changes which take place in the Poincaré map when γ is fixed at 0.5 and ω increases from 1 to 9. Passing from the upper right frame [(b), $\omega = 1$)] to the frames in the middle row [(c, d), $\omega = 3, 5$] to the lower frames [(e, f), $\omega = 7, 9$], the general trend is clearly that subharmonic resonances and separatrix splitting is suppressed as ω increases. In Fig. 8(b), the origin is unstable, and this instability would be predicted by a Floquet analysis of the origin. In Figs. 8(c)–(f), the origin is stable, as indicated by the presence of KAM tori. What varies with the excitation is the minimum radius at which subharmonic resonance bands exist. Progressing from Figs. 8(b) to (f), observe that the inner resonance bands are being pushed out towards the separatrix as frequency increases. As a consequence, the region in which regular quasiperiodic flow dominates increases.

In addition to pushing subharmonic resonance bands out, increasing frequency also has the effect of reducing the area of lobes formed by the intersection of stable and unstable manifolds of the two periodic points. The significance of this observation is that as frequency increases, the set of initial conditions which are transported out of the region between the separatrices of the averaged system decreases. This is an important observation, because the averaged phase portrait

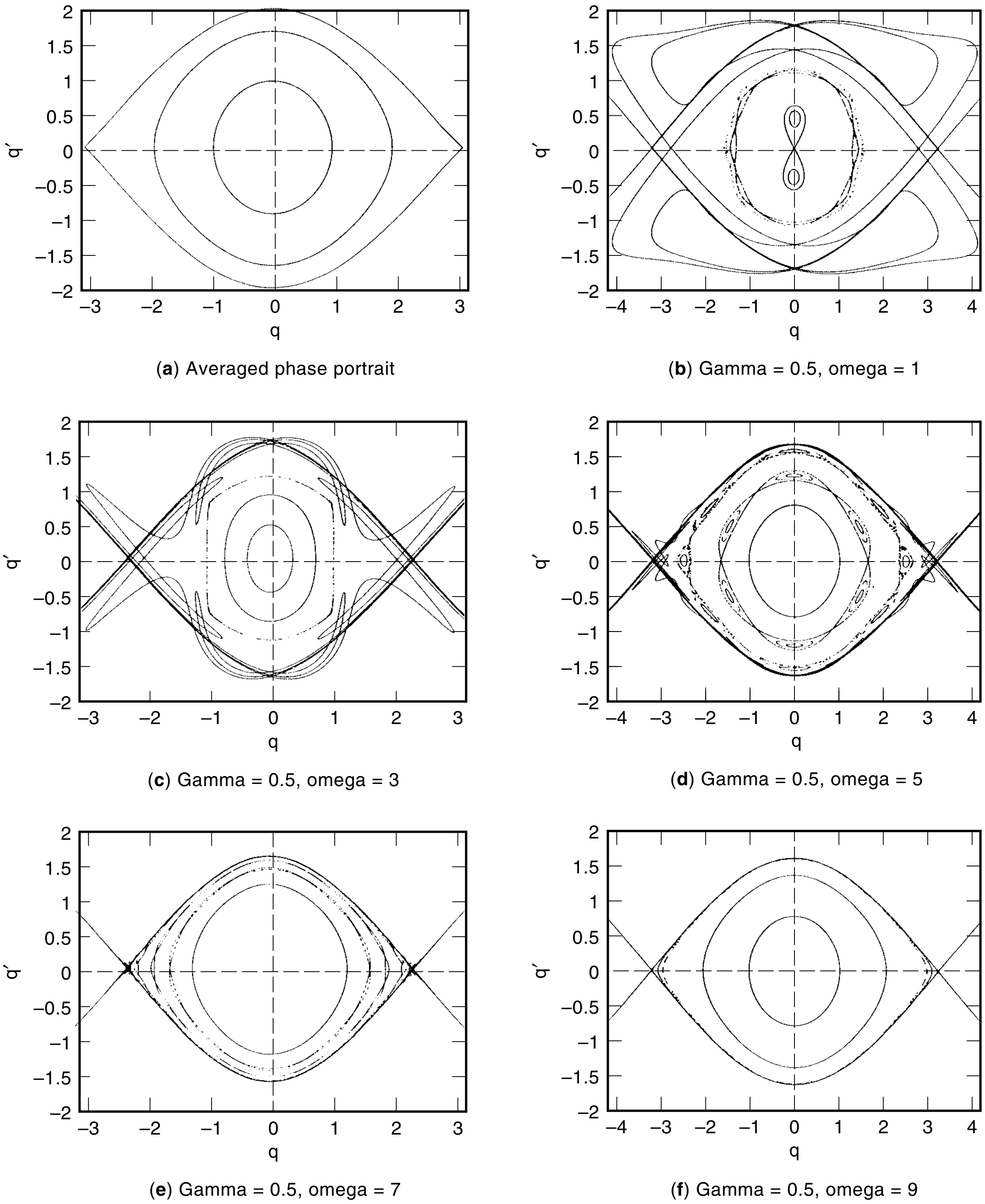

Figure 8. Poincaré maps showing separatrix splitting and resonance bands for the parametrically excited pendulum. The phase portrait for the unperturbed simple pendulum is shown in the upper left frame, with Poincaré maps of the forced system to the right and underneath. Forcing parameters used in the Poincaré maps are indicated in each plot. Note that the Poincaré maps of the forced system more closely resemble the averaged phase portrait as the forcing frequency becomes large.

provides no information whatsoever on the existence of a set of initial conditions which lies inside the separatrices but are transported out. In (f), it is unlikely that any subharmonic resonances exist, and the stable and unstable manifolds of the periodic points have closed to form a barrier to transport as indicated by the averaged phase portrait in (a).

Remark 10. The example gives somewhat anecdotal evidence that choosing a sufficiently high forcing frequency tends to suppress the negative features of periodic excitation. This has also been found to be the case in the cart and pendulum problem described in Examples 3, 4, and 5 [see (10)], the vertically forced rotating chain (33), and in an entire class of periodically forced single-degree-of-freedom systems (33). Current work revolves around extending this understanding to other classes of single-degree-of-freedom systems and multi-degree-of-freedom systems.

Remark 11. It has also been observed that dissipation tends to have beneficial effects beyond guaranteeing the asymptotic stability of certain equilibriums. Dissipation generally has the effect of breaking phase space separatrices and imposing hyperbolicity on systems with elliptic structures. As a result, elliptic structures, such as KAM tori and resonance bands, are destroyed and initial conditions which, in the absence of dissipation, belong to a KAM torus or resonance limit on a fixed point or periodic orbit. In addition, with sufficient dissipation, intersecting stable and unstable manifolds completely separate, giving rise to distinct basins of attraction. As with frequency, the extent to which dissipation helps eliminate undesirable nonlinear effects is largely dependent on its magnitude. In (10), it was seen for the cart and pendulum problem of Example 4 that there exists a minimum damping coefficient such that for all values less than this minimum value, manifold intersections and resonances persist. Recent work has suggested that the same issue arises in multi-degree-of-freedom systems.

Remark 12. Unfortunately, there is no known generally applicable rule-of-thumb for deciding what constitutes a sufficiently large forcing frequency. Experiments, simulation, and preliminary analysis suggest that for many systems, a rule of thumb might be for the forcing frequency to be an order of magnitude larger than the largest natural frequency of the controlled system. This rule of thumb presents a problem for many-degree-of-freedom systems like rotating chains or infinite dimensional systems like strings and beams, where natural frequencies tend to be very large. In addition, there exist counterexamples where the averaged phase portrait and Poincaré map resemble each other for small forcing frequencies, but possess completely different features at high frequencies. These topics represent the current focus of much of the research in this field.

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APPENDIX 1. CLASSICAL AVERAGING THEORY

The classical method of averaging was originally developed for periodic systems of the form

$$\dot{x} = \epsilon f(t, x) \tag{1.1}$$

where $x \in \mathbb{R}^n$, $f: \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$, $f(t + T, \cdot) = f(t, \cdot)$ and $0 < \epsilon \ll 1$. For simplicity, assume that f has continuous second partial derivatives in its second argument. Classical averaging theory addresses the relationship between the original, time-varying Eq. (1.1) and the autonomous averaged system given by

$$\dot{y} = \epsilon f_0(y) \tag{1.2}$$

where $f_0: \mathbb{R}^n \to \mathbb{R}^n$ is defined

$$f_0(y) \equiv \frac{1}{T}\int_0^T f(s, y)\, ds$$

and where y is treated as a constant in the integration. For sufficiently small ϵ, solutions of Eq. (1.2) provide good approximations to solutions of Eq. (1.1). Since there are many mathematical tools that can be used to analyze and control the time-invariant system Eq. (1.2), the problem of determining the behavior of time-varying periodic system Eq. (1.1) has been greatly simplified.

Specifically, if it is assumed that $x(t_0) = y(t_0)$ then for sufficiently small ϵ, the following statements hold:

- On any finite time interval, assuming the same initial conditions at time t_0, the solutions to Eqs. (1.1) and (1.2) remain close to each other. As ϵ becomes smaller, then the approximation becomes better and tends to zero in the limit.
- If the solution to Eq. (1.2) approaches a uniformly asymptotically stable equilibrium point, then, under additional mild assumptions, the solutions to Eqs. (1.2) and (1.1) remain close to each other on infinite time intervals. As ϵ becomes smaller, then the approximation becomes better and tends to zero in the limit.
- If Eq. (1.2) has a uniformly asymptotically stable equilibrium point, then Eq. (1.1) has an asymptotically stable periodic solution in the vicinity of this equilibrium point.

For a detailed discussion on the theoretical framework of averaging, the reader is referred to (36,47,53,54).

APPENDIX 2. FLOQUET THEORY

Floquet theory is concerned with local stability for systems of the form

$$\dot{x} = A(t)x \tag{2.1}$$

where $x \in \mathbb{R}^n$, $A(t): \mathbb{R}^{n\times n} \times \mathbb{R} \to \mathbb{R}^{n\times n}$, and $A(t + T) = A(t)$. Such systems arise as linearizations of Eq. (2) around an equilibrium or periodic orbit. As described in many standard texts on differential equations, the fundamental result upon which Floquet theory is built is that the fundamental matrix solution of Eq. (2.1) can be written as the product of a periodic matrix $P(t)$ and a constant exponential growth or decay; that is,

$$\Phi(t) = P(t)e^{Bt}$$

where $\Phi(t)$ is a fundamental matrix associated with Eq. (2.1), $P(t)$ is a $n \times n$ matrix periodic in t of period T, and e^{Bt} is the matrix exponential of a constant $n \times n$ matrix B. By the periodicity of $A(t)$, if $\Phi(t)$ is a fundamental matrix, so is $\Phi(t + T)$

and $\Phi(t + T)$ is linearly dependent on $\Phi(t)$. Then there exists a constant $n \times n$ matrix M such that

$$\Phi(t+T) = \Phi(t)M$$

where $M = e^{BT}$ and B is a constant $n \times n$ matrix. Without loss of generality, let $t = 0$ and $\Phi(0) = I$ where I is the $n \times n$ identity matrix. Then

$$\Phi(T) = IM = M$$

and therefore M represents the rate of growth or decay of the solution. Stable solutions decay or at least remain bounded. Hence, the condition for an equilibrium to be stable is that all the eigenvalues λ_i of M satisfy $|\lambda_i| \leq 1$. M itself is called the monodromy matrix, and eigenvalues λ_i of M are called Floquet multipliers. Complex numbers ξ_i such that $\lambda_i = e^{\xi_i T}$ are called Floquet exponents.

Floquet theory is a very classical method, and there is a vast literature describing the basic theory and applications. Some of the classic texts on the topic include Hale (47), Magnus and Winkler (55), Stoker (56), and Yakubovich and Starzhinskii (57). Floquet theory is most famous for its application in the study of Hill's equation and Mathieu's equation, which, because of their second order structure, bear special relevance in the stability of periodically forced mechanical systems. Hill's equation is the topic of (55), and various classic applications to (electrical) engineering are given in (56). References (47) and (57) give comprehensive summaries of the fundamental theory as well as examples.

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