

Robust control can be generally defined as the control, by means of fixed compensators, of uncertain plants (i.e., of systems with uncertain dynamics and unknown disturbance signals). Robustness in a control system refers to its ability to cope with uncertainties in a satisfactory manner, maintaining its stability and performance as desired. Uncertainty in signals and systems is inevitable, reflecting both the complexity of the physical world and the limitation in human understanding. Uncertain signals typically arise as a result of the randomness and unpredictability of environmental effects and are of an unmeasurable and unpredictable nature. Uncertain system dynamics, on the other hand, can be attributed to changes in the actual system and to modeling errors, be they accidental or deliberate. Generally, uncertain dynamics may come from the following sources:

1. Imperfect or incomplete knowledge of physical processes. This represents the information unattainable because of one's limited knowledge or inadequate measurements. It can be particularly acute for complex systems and processes (e.g., those found in biomedical engineering).
2. Parameter variations. Every physical system will undergo a change in parameter values under different operating conditions. Aging itself can be a factor.
3. Neglected high-frequency dynamics, time delays, nonlinearities, and the like. It may occur as a result of a sheer lack of knowledge, or the difficulty to model these characteristics. It may also occur because of the desire for models of low complexity.

While robustness is a concept of universal significance, robustness analysis for control systems is the study of whether a system, however designed, can meet specified stability and performance goals in the presence of uncertainty within a prescribed range.

Uncertain plants can be modeled in various ways. In particular, models can be stochastic or purely deterministic. In robust control, uncertain systems are typically modeled deterministically, as bounded sets of system models. A property is then said to hold robustly if it holds for every model in the set. The simplest case is that of *unstructured* uncertainty: the model set consists of *all* systems in a certain neighborhood (e.g., all transfer functions lying within a certain "distance" of a distinguished "nominal" system). Such a description is particularly appropriate to account for unmodeled dynamics. One rather typical example is the modeling of flexible structures. It is well known that, in general, a flexible structure cannot be accurately represented by a finite-dimensional system. For control design purposes, however, we desire, and most often are compelled to find, an approximate finite-dimensional model with a relatively low order. In doing so, a common practice is to include in the nominal model a small number of dominant modes in the low-frequency range and to treat the high-frequency modes as modeling uncertainty. Evidently, this description is also appropriate for modeling errors resulting from model reduction, or from any frequency response truncation. Moreover, it can be used to cover, albeit in a conservative way, parameter variations. The latter amounts to drawing up a frequency response envelope to describe the range of parameter variation in frequency domain.

ROBUST CONTROL ANALYSIS

Robustness is a property inherently sought after in engineering systems. The concept is directly linked to such issues as design viability and system reliability. In broad terms, robustness can be regarded as the capability to withstand unknown, unexpected, and often hostile conditions that can adversely affect a system's behavior. A system must be sufficiently robust in order to function properly under undesirable circumstances, conducting its task as designed. As engineering systems are becoming more and more complex and are required to operate in increasingly more uncertain environments, robustness has become increasingly more crucial.

Finally, in robustness analysis, it is common to introduce a *fictitious* uncertainty to represent a performance objective.

A more accurate account of parametric uncertainty calls for model sets within which individual models are uniquely characterized by the value(s) of one or more parameter(s) (e.g., transfer function coefficients). Typically each such parameter takes values in a known range. Such models account for the fact that parameters in physical devices are bound to vary with time, environment, and operating conditions. Ackermann's car steering problem (1) offers a good illustrative example. In his study of a four-wheel steering vehicle, he found that the vehicle mass and the adhesion between tires and road surface are significant uncertain parameters. This is easily understandable. The vehicle mass certainly varies with load, and in a more subtle sense, it varies with fuel consumption. The adhesion changes with road condition, wearing of tires, and weather condition. We can think of other uncertain parameters by considering the human-vehicle system as a whole or by considering a whole batch of vehicles as a family of systems. In these scenarios, differences among individual drivers and vehicles can all constitute significant uncertain factors. Yet a more striking example is an aircraft, whose aerodynamic coefficients vary in large magnitudes due to changes in altitude, maneuvering, and weather.

There is an inherent trade-off between fidelity and simplicity in modeling uncertain systems. In a sense, unstructured and parametric uncertainties may be considered the two extremes. While an unstructured perturbation furnishes a simple characterization and is useful for simplifying robustness analysis, it contains little information and may often be too conservative. On the other hand, uncertain parameters often yield a more natural and accurate model, but such elaborate descriptions tend to complicate analysis. The process of robustness analysis, therefore, calls for a judicious balance between uncertainty description and complexity of analysis. In its full generality, however, the description of an uncertain system should take into account both parametric variations and unmodeled dynamics. Uncertainty descriptions of this kind are called *structured*.

To be sure, robustness is not entirely a new concept in control system analysis and design. In retrospect, the need for a control system to tolerate unmodeled dynamics and parameter variations is precisely what motivated feedback control, and it has been a primary goal in control system design since its birth. This is well recognized in classical control design and, at least implicitly, is embedded in classical loop shaping methods. Concepts such as gain and phase margins may well be regarded as elementary robustness measures. However, it was not until the late 1970s that the term began to appear routinely in the control literature, when the need for robustness was reexamined and was gaining increasing recognition. Robust control as a research direction soon thrived and became a defining theme. After two decades of intense activity, it has evolved into a broad research area rich in theory and potential applications. The progress has been rapid and vast, leading to the development of a variety of key concepts and techniques, among which notably are the \mathcal{H}_∞/μ theory, the Kharitonov/polynomial approach, and analyses based on state-space formulations and the Lyapunov theory.

The structured singular value (2), also known as μ , was introduced in the early 1980s as a very general framework for studying structured uncertainty in linear time-invariant

models. It defines an exact measure of robust stability and performance in the frequency domain. The method is a natural progression of earlier work on using *singular values* to extend the concept of stability margin to multivariable systems, an idea that was heavily influenced by operator theoretic results such as the Small Gain Theorem. The main impetus for the development of the structured singular value theory, evidently, has been the recognition that unstructured uncertainty is too crude a model, often leading to excessive conservatism.

From a computational point of view, the success of μ hinges critically upon whether and how it can be computed both accurately and efficiently. Unfortunately, this is known to be difficult. Recent studies have shown that computation of μ generally amounts to solving a so-called NP-hard decision problem, which, in the language of computing theory, is one that suffers from an exponential growth in its computational complexity. Although this by no means implies that every μ problem is computationally difficult, it nevertheless points to the unfortunate conclusion that the computation of μ in general poses an insurmountable difficulty in the worst case. In retrospect, it thus comes as no surprise that the major progress in computing μ has been made by some forms of approximation, specifically, readily computable bounds. While no definitive conclusion has been drawn concerning the gap between μ and such bounds, it is reassuring that the bounds are often reasonably tight and that they have other interpretations of engineering significance.

The "Kharitonov" theory is another robustness analysis approach, developed in parallel with μ , which deals almost entirely with robust stability issues under parametric uncertainty; only in rare cases is unstructured uncertainty also taken into consideration. The research in this area has a natural heritage from such a classical stability test as the Routh-Hurwitz criterion and was mainly triggered by a landmark result published by V. L. Kharitonov (3) in 1978, later referred to as Kharitonov's theorem. Kharitonov considered the question of "stability" of parameterized families of polynomials. Here polynomials are thought of as characteristic polynomials of systems described by rational transfer functions and thus are "stable" if their zeros all lie in the open left-half of the complex plane (continuous time) or in the open unit disk (discrete time). In the continuous-time case, Kharitonov showed that, for an uncertain "interval" polynomial whose coefficients each vary independently in a given interval, stability of the entire family can be assessed by testing merely four simply constructed extreme polynomials. From an aesthetic point of view, Kharitonov's theorem possesses remarkable elegance, reducing an otherwise seemingly impossible task to a simple problem. From an engineering perspective, however, the theorem is likely to find only limited utility because very rarely would an uncertain system yield a family of characteristic polynomials of the form required in the problem description. Thus, Kharitonov's work triggered a flurry of activities in the search for more realistic results, and soon came various generalizations. Two notable features stand out from the robust stability conditions available in this category. First, they are stated either in terms of a finite number of polynomials or as graphical tests requiring a frequency sweep. Second, the main success to date pertains to uncertainty descriptions in which polynomial coefficients depend linearly on uncertain parameters.

Robust stability analysis in state-space formulation often comes under the names of stability radius, interval matrix, and stability bound problems. The key issue here is to determine the largest uncertainty size under which stability is preserved. Unlike in the aforementioned two approaches, parametric uncertainty in state-space representation is defined in terms of perturbations to system matrices, which can be either unstructured or structured, but most often are only allowed to be real. In a natural way, the robust stability problem translates to one of how the perturbations may affect the eigenvalues of the system matrix, the solution of which can draw upon rich theories from linear algebra and Lyapunov analysis. Thus, unsurprisingly, most of the robustness conditions have this flavor. Alternatively, the problem can also be recast as one of μ analysis or one of polynomial stability. In the former case, we need to compute μ with respect to solely real uncertainties, for which the μ computation schemes are known to be conservative. In the latter case, the coefficients in the resultant polynomial will depend on the unknown parameters in a multilinear or multinomial fashion. The Khari- tonov approach cannot provide a satisfactory answer for such polynomial families either. The problem thus appears to be a very difficult one and is known to have been solved only for a number of isolated cases. Most notably, recent progress has made it possible to compute the stability radius in an efficient manner when the matrix perturbation is unstructured. For structured uncertainty, unfortunately, the problem is completely open; recent studies showed too that it is in general NP-hard. Accordingly, the majority of results in the latter class are sufficient conditions for robust stability.

In conclusion, robustness is a key concept, vital for engineering design in general and for control system design in particular. Robust control has matured into a field rich in theory and potential applications. By focusing on these three selected areas, this chapter is limited only to robustness analysis of linear time-invariant systems, that is, control of linear time-invariant plants by linear time-invariant controllers, which itself is condensed from a vast collection of results and techniques. Nevertheless, we should note that the concept and theory of robust control goes far beyond the boundary of linear time-invariant systems and, in fact, has been quickly branching to the domains of nonlinear control, adaptive control, and the like. As a whole, it has been, and will continue to be, a driving force behind the evolution of control theory.

THE STRUCTURED SINGULAR VALUE

Uncertain Systems

Throughout this article, we consider model sets \mathcal{P} with the following property: \mathcal{P} can be represented by a block diagram with some of the blocks being fully known systems, and others being “elementary uncertain blocks.” The latter are elementary sets, namely, unit balls in “simple” vector spaces. For example, some uncertainty blocks might be the real interval $[-1, 1]$ and others might be the unit ball of \mathcal{H}_∞ , the set of transfer function matrices (linear time-invariant systems) that are causal and bounded-input bounded-output (BIBO) stable; the size or “ \mathcal{H}_∞ -norm” of $\Delta \in \mathcal{H}_\infty$ is defined to be the supremum of its largest singular value over the imaginary axis (continuous time) or unit disk (discrete time).

Thus each plant in the model set corresponds to the selection of one element in each of the uncertain blocks. The *nominal plant* corresponds to the choice “0” for all elementary uncertain blocks. Note that the assumption that all the uncertainty balls have unity radius is made at no cost: any size information can be embedded into known blocks (e.g., connected in cascade with the uncertain blocks). It turns out that, for “uncertain” block diagrams of this type, the transfer function (or transfer function matrix) between any two nodes (or tuples of nodes) in the block diagram is given by a *linear fractional transformation* (LFT). Similarly, when such an uncertain plant is connected with a feedback compensator, the transfer function between any two nodes will also be an LFT. LFTs have attractive mathematical properties that can be used to advantage at the modeling, analysis, and synthesis stages. More on LFTs can be found, e.g., in Ref. 7.

Robust Stability

It should be intuitively clear that block diagrams of the type just described can always be “redrawn” in the form of an M - Δ loop as depicted in Fig. 1 (external input and outputs have been left out). Here M corresponds to the nominal system, which is comprised of closed-loop transfer functions as elements and has an input channel and an output channel for each elementary uncertain block, and Δ is a block-diagonal matrix whose diagonal blocks are the elementary uncertain blocks. For its generality, the M - Δ loop paradigm has found wide acceptance in robust control (see, for example, Refs. 2 and 4–8).

Throughout most of this article, we will assume that the nominal system, or equivalently M , is linear and time-invariant, as are all instances of the uncertainty blocks, equivalently, of Δ . We will also assume that M and all instances of Δ are in \mathcal{H}_∞ . In this case, an immediate payoff of the LFT uncertainty description and the ensuing representation of the system via the M - Δ loop is the following strong form of the Small Gain Theorem, a necessary and sufficient condition for well-posedness and stability of the M - Δ loop, in the case where Δ consists of a single uncertainty block, ranging over the unit ball in \mathcal{H}_∞ .

Small Gain Theorem. Let $M \in \mathcal{H}_\infty$. Then the M - Δ loop is well-posed and BIBO stable for all $\Delta \in \mathcal{H}_\infty$ with $\|\Delta\|_\infty \leq 1$ if and only if $\|M\|_\infty < 1$.

As alluded to earlier, the \mathcal{H}_∞ norm of a causal, stable, continuous-time transfer function matrix M is defined as

$$\|M\|_\infty = \sup_{\omega \in \mathbb{R}} \bar{\sigma}(M(j\omega))$$

where $\bar{\sigma}$ denotes the largest singular value.

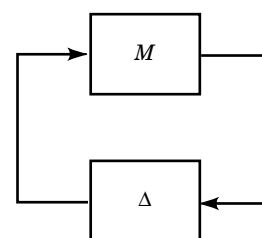


Figure 1. M - Δ loop.

As an example consider a model set of the “output multiplicative uncertainty” type. Specifically, let

$$P = (I + w\Delta)P_0$$

where P_0 is the transfer matrix of a linear, time-invariant nominal model, w is a scalar “weighting” transfer function, and Δ ranges over the unit ball in \mathcal{H}_∞ . The weight w is introduced to account for the fact that the amount of uncertainty is usually frequency-dependent; in particular, system dynamics are often poorly known at high frequencies. Suppose that a feedback controller K has been tentatively selected to stabilize the nominal system P_0 . (We use the negative feedback convention, i.e., the loop transfer function is $-KP_0$.) Isolating Δ from the nominal closed-loop system, we obtain an M - Δ loop with

$$M = -wKP_0(I + KP_0)^{-1}$$

Since K stabilizes P_0 , M is stable.

As a note of interest, we expect a keen connection between the Small Gain Theorem and the classical Nyquist criterion. Indeed, this can be best observed by examining single-input/single-output systems. In such case P and K are scalar, and thus so is M . Since both M and Δ are stable, Nyquist’s criterion implies that the M - Δ loop is stable whenever the Nyquist plot of $M\Delta$ does not encircle the critical point $-1 + j0$. Clearly, this will be the case for every Δ satisfying $|\Delta(j\omega)| \leq 1$ for all ω if and only if $|M(j\omega)| < 1$ holds at all frequencies (including ∞).

The Small Gain Theorem suggests that one way to obtain a robustly stable system, or more generally to obtain a robust design, is to make sure that the \mathcal{H}_∞ norm of a certain system transfer function is small enough. This has triggered an entire field of research known as \mathcal{H}_∞ design, which is discussed elsewhere in this encyclopedia. The focus of the present article is the case when Δ is block-diagonal (i.e., when the uncertainty model consists of several blocks or, in other words, when the uncertainty is structured). Typically, two types of uncertainty blocks are considered in the literature: (i) the set of real, constant scalar multiples of the identity, with the scalar having magnitude no larger than one, and (ii) the set of causal and BIBO stable (\mathcal{H}_∞) transfer function matrices, with \mathcal{H}_∞ -norm no larger than one. The latter corresponds to unmodeled dynamics. The former, on the other hand, is used to represent parametric uncertainty, particularly when a same uncertain parameter affects more than one coefficients in a transfer function. For example, concurrent variation as a function of temperature (e.g., dilation) of multiple quantities in a mechanical system can result in such a block. This description is more general than the simpler “scalar nonrepeated blocks.”

Examples with structured uncertainty arise with plants modeled as being affected by uncertainty at more than one physical location, e.g.,

$$P = (I + w_1\Delta_1)P_0(I + w_2\Delta_2)$$

where both input and output uncertainty are accounted for. Another instance arises in the context of the robust performance problem, discussed in a later section. For an example including both parametric uncertainty and unmodeled dy-

namics, consider now the model set $(1 + w_1\Delta^c)P_a$ with P_a explicitly given as

$$P_a(s) = \frac{1}{s - a}$$

where a can take any value in the interval $[-0.9, 1.1]$. We may write $a = 1 + w_2\Delta^r$ with $w_2 = 0.1$ and $|\Delta^r| \leq 1$, and P_a can be represented as a loop with $P_0(s) = 1/(s - 1)$ in the forward path and $-w_2\Delta^r$ in the feedback path (again using the negative feedback convention). Let K be a feedback controller (still with the negative feedback convention) that stabilizes P_0 . By “extracting” the uncertainty blocks Δ^c and Δ^r , we obtain an M - Δ loop with

$$M = \begin{bmatrix} w_2P_0(I + P_0K)^{-1} & -w_2KP_0(I + P_0K)^{-1} \\ w_1P_0(I + P_0K)^{-1} & -w_1KP_0(I + P_0K)^{-1} \end{bmatrix}$$

$$\Delta = \begin{bmatrix} \Delta^r & 0 \\ 0 & \Delta^c \end{bmatrix}$$

where Δ^r is a real number ranging over $[-1, 1]$ and Δ^c is a scalar transfer function ranging over the unit ball in \mathcal{H}_∞ .

Clearly, the condition $\|M\|_\infty < 1$ in the Small Gain Theorem remains sufficient for robust stability when Δ is restricted to be block-diagonal. However, it is in general no longer necessary. A refinement of the Small Gain Theorem for the structured uncertainty case was proposed by Doyle and Safonov in the early 1980s (2,4). We adopt here the framework introduced by Doyle, that of the structured singular value also known as μ . The Small μ Theorem states that, if the uncertainty is restricted to be block-diagonal, then the correct refinement is essentially (see Ref. 9 for a precise statement) to replace $\|M\|_\infty$ with $\|M\|_\mu$, where for a continuous-time transfer function matrix M ,

$$\|M\|_\mu = \sup_{\omega \in \mathbb{R}_e} \mu(M(j\omega))$$

and $\mu(\cdot)$ denotes the structured singular value of its matrix argument with respect to the block-structure under consideration. The set $\mathbb{R}_e = \mathbb{R} \cup \{\infty\}$ is the extended real line; if no parametric uncertainty is present, however, $\mu(\cdot)$ is continuous and \mathbb{R}_e can be replaced by \mathbb{R} . Similarly, for a discrete-time system,

$$\|M\|_\mu = \sup_{\theta \in [0, 2\pi)} \mu(M(e^{j\theta}))$$

But what specifically is μ ? This is discussed next.

The Structured Singular Value

Let us denote by Γ the set of values taken by $\Delta(j\omega)$ [or $\Delta(e^{j\theta})$] as Δ ranges over the set of block diagonal transfer function matrices of interest, with the “unit ball” restriction lifted, namely let

$$\Gamma = \{\text{diag}(\Gamma^r, \Gamma^c, \Gamma^c) : \Gamma^r \in \Gamma_r, \Gamma^c \in \Gamma_c, \Gamma^c \in \Gamma_c\}$$

with

$$\begin{aligned}\Gamma_r &:= \{\text{diag}(\gamma_1^r I_{k_1}, \dots, \gamma_{m_r}^r I_{k_{m_r}}) : \gamma_i^r \in \mathbb{R}\} \\ \Gamma_c &:= \{\text{diag}(\gamma_1^c I_{k_{m_r+1}}, \dots, \gamma_{m_c}^c I_{k_{m_r+m_c}}) : \gamma_i^c \in \mathbb{C}\} \\ \Gamma_C &:= \{\text{diag}(\Gamma_1^C, \dots, \Gamma_{m_c}^C) : \Gamma_i^C \in \mathbb{C}^{k_{m_r+m_c+i} \times k_{m_r+m_c+i}}\}\end{aligned}$$

The first and third block types, often referred to as repeated real and full complex blocks, correspond to values of parametric and dynamic uncertainty, respectively. The second block type, known as repeated complex, often arises in analyzing multidimensional (10) and time-delay systems (11), and is also used sometimes when an LFT state-space representation of transfer functions is sought (5). It is worth noting that while Γ as just defined is usually adequate for representing uncertainties frequently encountered, it can be extended further to accommodate more general situations. For example, full real blocks (i.e., unknown real matrices) may be added whenever desired.

The *structured singular value* $\mu(M)$ of a matrix M with respect to the block structure Γ is defined to be 0 if there is no $\Gamma \in \Gamma$ such that $\det(I - \Gamma M) = 0$, and

$$\mu(M) = \left(\min_{\Gamma \in \Gamma} \{\bar{\sigma}(\Gamma) : \det(I - \Gamma M) = 0\} \right)^{-1}$$

otherwise. It can be checked that for structures simply consisting of one full complex block as in the Small Gain Theorem, $\mu(M(j\omega))$ becomes the largest singular value of $M(j\omega)$, and $\|M\|_\mu$ is thus equal to $\|M\|_\infty$.

Given a matrix M and a block structure Γ , computing $\mu(M)$ is generally not an easy task. Indeed, this computation is known to be NP-hard, even when Γ is simplified to a structure containing only full complex blocks. Thus estimates of $\mu(M)$, e.g., upper and lower bounds on $\mu(M)$, are often used instead. These, as well as other properties of μ , are discussed next.

Let \mathcal{U} be the set of unitary matrices in Γ and \mathcal{D} be the set of nonsingular matrices that commute with every $\Gamma \in \Gamma$. The latter consist of block-diagonal matrices with scalar multiples of the identity in correspondence with full complex blocks (Γ_c), and with arbitrary blocks in correspondence with those constrained to be scalar multiples of the identity (Γ_r, Γ_C). Then the following result holds:

$$\max_{U \in \mathcal{U}} \rho_R(MU) \leq \mu(M) \leq \inf_{D \in \mathcal{D}} \bar{\sigma}(DMD^{-1}) \quad (1)$$

Here ρ_R is the largest absolute value of a real eigenvalue of its matrix argument. Inequalities [see Eq. (1)] are of special interest in the case of purely complex uncertainty structures. In that case, (i) the lower bound is equal to $\mu(M)$ and ρ_R can be replaced by the spectral radius ρ , and (ii) the upper bound is equal to $\mu(M)$ whenever $m_c + 2m_r$ is no greater than 3, and extensive numerical experimentation suggests that it is never (or at least “seldom”) much larger. Moreover, the upper bound can be computed efficiently by solving a convex optimization problem, in fact, a linear matrix inequality (LMI) problem. LMIs define a special class of convex optimization problems and are discussed elsewhere in this encyclopedia.

For uncertainty structures where real (scalar multiple of the identity) blocks are present, inequalities [Eq. (1)] can be

tightened to the following (12):

$$\max_{Q \in \mathcal{Q}} \rho_R(MQ) \leq \mu(M) \leq \inf_{D \in \mathcal{D}^+, G \in \mathcal{G}} \{\gamma \geq 0 : MDM^H + GM^H - MG - \gamma^2 D < 0\} \quad (2)$$

Here the superscript H indicates complex conjugate transpose, \mathcal{Q} is the subset of Γ consisting of matrices whose complex blocks are unitary, \mathcal{D}^+ is the subset of \mathcal{D} consisting of Hermitian positive definite matrices, \mathcal{G} is the subset of \mathcal{D} consisting of skew-Hermitian matrices (i.e., $G^H = -G$) with zero blocks in correspondence with repeated real blocks in Γ , and the $<$ sign indicates that the matrix expression is constrained to be negative definite. The lower bound in condition (2) is always equal to $\mu(M)$. The upper bound is never greater than that in condition (1) (it reduces to it when $G = 0$ is imposed) and, as was the case for condition (1), can be computed by solving an LMI problem. See, for example, Section 8.12 in Ref. 8.

For the class of problems where the matrix M has rank one, Young (13) showed that the right-hand side in inequalities [Eq. (2)] is equal to $\mu(M)$. In that case, Chen et al. (14) obtained an explicit formula for $\mu(M)$. Let $M = ba^H$, where a and b are column vectors. Let also a and b be partitioned into subvectors a_i and b_i compatibly with Γ . For $i = 1, \dots, m_r + m_c$, let $\phi_i = a_i^H b_i$. Moreover, define

$$\gamma = \sum_{i=m_r+1}^{m_r+m_c} |\phi_i| + \sum_{i=m_r+m_c+1}^{m_r+m_c+m_c} \|a_i\|_2 \|b_i\|_2$$

Then,

$$\mu(M) = \inf_{x \in \mathbb{R}} \left(\sum_{i=1}^{m_r} |\text{Re}(\phi_i) + x \text{Im}(\phi_i)| + \gamma \sqrt{1+x^2} \right) \quad (3)$$

Furthermore, if we assume with no loss of generality that for some $l \leq m_r$, $\text{Im}(\phi_i) \neq 0$ for $1 \leq i \leq l$, and $\text{Im}(\phi_i) = 0$ for $i > l$, and that

$$-\frac{\text{Re}(\phi_1)}{\text{Im}(\phi_1)} \leq \dots \leq -\frac{\text{Re}(\phi_l)}{\text{Im}(\phi_l)}$$

then the infimum is achieved at one of the following points:

$$\begin{aligned}x_k &= -\frac{\text{Re}(\phi_k)}{\text{Im}(\phi_k)}, \quad k = 1, \dots, l \\ x'_0 &= \pm \frac{\sum_{i=1}^l |\text{Im}(\phi_i)|}{\sqrt{\gamma^2 - \left(\sum_{i=1}^l |\text{Im}(\phi_i)|\right)^2}} \\ x'_k &= \pm \frac{\sum_{i=1}^k |\text{Im}(\phi_i)| - \sum_{i=k+1}^l |\text{Im}(\phi_i)|}{\sqrt{\gamma^2 - \left(\sum_{i=1}^k |\text{Im}(\phi_i)| - \sum_{i=k+1}^l |\text{Im}(\phi_i)|\right)^2}}\end{aligned}$$

Finally, the infimum cannot be achieved at x'_0 unless $x'_0 \in (-\infty, x_1]$, and for $k = 1, \dots, l$, it cannot be achieved at x'_k unless $x'_k \in [x_k, x_{k+1}]$.

The rank-one case just alluded to is one of the rare instances for which one can obtain an explicit expression for $\mu(M)$. This expression not only simplifies the computation of the upper bound in condition (2) but also was found useful in studying robust stability of uncertain polynomials. Indeed, as

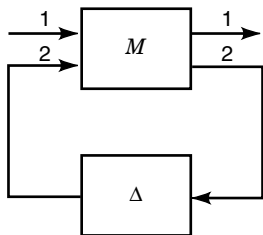


Figure 2. Robust performance setup.

will be discovered shortly, an important class of stability problems for uncertain polynomials can be formulated in terms of a rank-one μ problem. Consequently, the result furnishes a unifying tool for the stability problems and a link between μ analysis and the Kharitonov approach to robustness analysis.

Robust Performance

A key reason for the popularity of the μ framework is that it encompasses not only the robust stability problem but also the following robust performance problem: determine whether, for all plants in the given model set, the energy (integral of the square of the magnitude) in a specified error output signal remains below a specified threshold whenever the disturbance input's energy is less than a specified value.

Consider the block diagram of Fig. 2 where, as compared to Fig. 1, external (disturbance) input and (error) output are made explicit. Given a block diagram such as the one of Fig. 2, the input-output transfer function in the continuous-time case is given by the *linear-fractional transformation*

$$F_\ell(M(s), \Delta(s)) = M_{11}(s) + M_{12}(s)\Delta(s)(I - M_{22}(s)\Delta(s))^{-1}M_{21}(s)$$

where $M_{ij}(s)$ is the transfer function from input j to output i of $M(s)$, $i, j = 1, 2$, when the feedback connection through $\Delta(s)$ is removed. [Thus $M_{22}(s)$ is the transfer function matrix formerly denoted $M(s)$.]

The issue at hand is to determine, under the assumption that $M(s) \in \mathcal{H}_\infty$, whether robust performance holds, that is, whether it is the case that, for all $\Delta(s)$ in our unit uncertainty ball,

$$\|F_\ell(M(s), \Delta(s))\|_\infty < 1 \quad (4)$$

This is readily handled by noting that, in view of the Small Gain Theorem, for any fixed $\Delta(s)$ such that the system is stable, condition (4) is equivalent to the stability of the augmented system depicted in Fig. 3 for all *fictitious* uncertainty

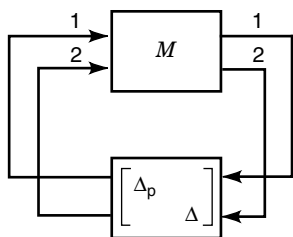


Figure 3. Fictitious uncertainty for robust performance.

blocks $\Delta_p(s)$ such that $\|\Delta_p\|_\infty \leq 1$. It thus follows that robust performance holds if and only if

$$\sup_{\omega \in \mathbb{R}_e} \mu(M(j\omega)) < 1$$

where μ now denotes the structured singular value corresponding to the “augmented” block structure $\text{diag}(\mathbb{C}^{k_p \times k_p}, \Gamma)$ (i.e., the block-structure Γ corresponding to the actual uncertainty, augmented with a full complex block).

For an example of a typical robust performance problem, consider an uncertain plant described by the multiplicative uncertainty model set

$$P = (I + w_1 \Delta)P_0$$

with a fixed feedback controller K . It is desired to determine whether $\|w_2 S\|_\infty < 1$ for all possible Δ in a possibly structured unit uncertainty ball, where S is the sensitivity function (i.e., using the negative feedback convention, $S = (I + PK)^{-1}$). Here w_1 and w_2 are stable transfer functions introduced for frequency-weighting purposes. For simplicity, w_1 and w_2 are assumed to be scalars. Using the transformation just outlined, we obtain

$$M = \begin{bmatrix} w_2(I + P_0 K)^{-1} & w_2(I + P_0 K)^{-1}P_0 \\ -w_1(I + KP_0)^{-1}K & -w_1(I + KP_0)^{-1}KP_0 \end{bmatrix} \quad (5)$$

In the single-input/single-output case, M has rank one. In the present case $m_r = m_c = 0$, so the right-hand side of Eq. (3) is simply γ and the right-hand side of the expression defining γ reduces to its second term. Thus

$$\begin{aligned} \mu(M(j\omega)) &= |w_2(j\omega)(1 + P_0(j\omega)K(j\omega))^{-1}| \\ &\quad + |w_1(j\omega)(1 + K(j\omega)P_0(j\omega))^{-1}K(j\omega)P_0(j\omega)| \end{aligned}$$

and the condition for robust performance can be expressed as

$$|w_2(j\omega)| + |w_1(j\omega)K(j\omega)P_0(j\omega)| < |1 + P_0(j\omega)K(j\omega)| \quad \forall \omega \in \mathbb{R}_e$$

Extensions

The structured singular value may be further generalized in many directions, depending on uncertainty descriptions and characterizations. Some of these generalizations are summarized next.

- *Nondiagonal uncertainty structure.* The uncertainty structure need not be diagonal. It can contain unknown, independently bounded blocks in every entry. Doyle (2) and Kouvaritakis and Latchman (15) showed that the analysis may be converted into one based on the standard μ , but this may lead to a substantial increase in computational effort. Chen et al. (16,17) proposed a computational scheme that renders the computation growth insignificant.
- *Uncertainty with phase information.* Tits et al. (18) adopted a notion of μ with phase, in which not only are uncertainties known to be bounded by given quantities, but also their phases are known to vary in given ranges. The formulation gives a more detailed uncertainty de-

scription, and it requires extensions of the concept of phase and of μ .

- *ℓ_1 -norm bounded uncertainty.* Khammash and Pearson (19,20) studied structured uncertainties bounded in ℓ_1 norm, which is another active research area in robust control, concerning peak-to-peak system response. They showed that robust stability can be assessed by computing the spectral radius of a positive matrix constructed from the impulse response of the nominal system.
- *Time-varying uncertainty.* Shamma (21) and Megretsky (22) examined the robust stability problem with respect to structured time-varying uncertainties. They showed that if the uncertainty is allowed to vary arbitrarily over time, robust stability holds if and only if for some $D \in \mathcal{D}$, $\|DM(s)D\|_\infty < 1$. It is readily checked that the left-hand side (known as scaled \mathcal{H}_∞ -norm), is similar to the right-hand side in condition (1), except that here the same D must be used at all frequencies. Subsequently, Poolla and Tikku (23) showed that, if the time variation of the uncertainty is arbitrarily slow, then robust stability holds if and only if the right-hand side in condition (1) is less than 1 at all frequencies.

Finally, while μ may be custom made and seems to be an all-encompassing paradigm when extended appropriately, it cannot be applied to models in which the uncertainty block Δ is allowed to be unstable. An effective robustness measure for the latter situation is furnished by the *gap metric*, a concept discussed elsewhere in this encyclopedia.

THE KHARITONOV APPROACH

The Kharitonov approach, named after Russian mathematician V. L. Kharitonov whose celebrated 1978 theorem is often considered to be the cornerstone of the field, is largely concerned with the issue of determining zero locations for a family of polynomials whose coefficients vary in a bounded set. Hence, by nature, it can be best presented in a framework different from that of the M - Δ loop, or μ , namely, directly as a polynomial stability problem. This issue, however, is connected to the M - Δ loop paradigm in an intimate fashion. To see this, simply consider a model set comprising proper real rational functions whose coefficients take values in certain bounded intervals. To determine robust stability of such a plant together with any compensator will then amount to checking whether the set of all resultant closed-loop characteristic polynomials have zeros in the “stability region.” For continuous-time systems, our main focus, the stability region of interest, is the open left half of the complex plane. Other regions of interest include the open unit disk, a shifted left half plane, and a sector; these regions can be imposed to study stability of discrete-time systems or to enforce pole placement constraints. A polynomial is generally said to be Hurwitz stable, or is referred to as a Hurwitz polynomial, if its zeros lie in the open left half plane.

A general description for a set of polynomials of interest is

$$p(s, q) = \sum_{k=0}^n a_k(q) s^k, \quad (6)$$

where q is an unknown vector that may or may not represent physical parameters. When q varies over a bounded set $Q \subset \mathbb{R}^m$, a family of polynomials are generated:

$$\mathcal{P} = \{p(s, q) : q \in Q\} \quad (7)$$

The problem of concern is to determine if the polynomial family \mathcal{P} is robustly Hurwitz stable, by which we mean that every member in \mathcal{P} is Hurwitz stable. We shall assume that the coefficients $a_k(q)$ are continuous functions of q . Furthermore, we assume that $a_n(q) > 0$ for all $q \in Q$ (i.e., all polynomials in \mathcal{P} have the same degree). For control system analysis, it is typical to restrict the polynomial family \mathcal{P} to the following classes, arranged by order of increased complexity.

1. \mathcal{P}_a : the coefficients $a_k(q)$ are affine functions of q . For example,

$$p(s, q) = s^2 + (q_1 + 2q_2 + 3)s + (4q_1 + 5q_2 + 6)$$

2. \mathcal{P}_m : the coefficients $a_k(q)$ are multiaffine functions of q . For example,

$$p(s, q) = s^3 + (2q_1q_2 + 2q_1q_3 + q_3 + 1)s^2 + (4q_2q_3 + 5)s + (q_1q_2q_3 + 1)$$

3. \mathcal{P}_p : the coefficients $a_k(q)$ are multivariate polynomials in q . For example,

$$p(s, q) = s^3 + (2q_1^2q_2 + 2q_1q_3^2 + q_1q_3 + 1)s^2 + (4q_2q_3 + 5)s + (q_1^2q_2^2q_3^2 + 1)$$

It should be rather evident that $\mathcal{P}_a \subset \mathcal{P}_m \subset \mathcal{P}_p$ and hence that the complexity in analysis increases in that same order. At present, the only available methods for tackling \mathcal{P}_m and \mathcal{P}_p are largely ad hoc, via either local optimization or graphical approaches, and they are either conservative or computationally formidable. In particular, when Q is an ℓ_∞ ball [i.e., a hyperrectangle (“box”) parallel to the coordinate axes], the problem of testing the stability of \mathcal{P}_m is known to be NP-hard.

The class \mathcal{P}_a , as the sole tractable case, merits a particularly thorough study. A polynomial family \mathcal{P} in this class consists of all polynomials of the form

$$p(s, q) = p(s, q^0) + \sum_{k=0}^m (q_k - q_k^0) p_k(s) \quad (8)$$

Here q^0 belongs to Q and may be regarded as the “nominal” value of uncertain parameter vector q , and the $p_k(s)$ ’s are fixed polynomials. Evidently, one can assume with no loss of generality that $p(s, q^0)$ is Hurwitz stable, which is necessary for \mathcal{P} to be robustly Hurwitz stable as q varies over Q .

Let $p \in [1, \infty]$, and let $\|\cdot\|_p$ be the standard ℓ_p Hölder norm defined on the Euclidean space \mathbb{R}^m . That is,

$$\|q\|_p = \begin{cases} \left(\sum_{i=1}^m |q_i|^p \right)^{1/p}, & 1 \leq p < \infty \\ \max_{1 \leq i \leq m} |q_i|, & p = \infty \end{cases}$$

Then a common description for Q adopts the notion of unit ℓ_p balls centered at q^0 , defined as

$$Q = \{q: \|q - q^0\|_p \leq 1\} \quad (9)$$

When the coefficient $a_k(q)$ depends on q_k alone [i.e., when $p_k(s)$ is a constant multiple of s^k], the ℓ_p ball description gives rise to a class of most studied polynomial families. Such families can be expressed as

$$\mathcal{P} = \left\{ \sum_{k=0}^n \gamma_k q_k s^k : \|q - q^0\|_p \leq 1 \right\} \quad (10)$$

for some (possibly different) q^0 and some fixed scalars γ_k . In particular, for $p = \infty, 1, 2$, such \mathcal{P} is referred to as an interval polynomial, a diamond polynomial, and a spherical polynomial, respectively.

Arguably, the entire success of the Kharitonov approach may be best summarized as a triumph over the polynomial family in Eq. (8) with the ℓ_p norm characterization in Eq. (9), but not beyond, of which the most shining example is Kharitonov's celebrated theorem.

Interval Polynomials

An interval polynomial can be equivalently expressed as

$$\mathcal{P} = \left\{ \sum_{k=0}^n q_k s^k : \underline{q}_k \leq q_k \leq \bar{q}_k \right\} \quad (11)$$

where the \underline{q}_k s and the \bar{q}_k s are fixed. Kharitonov's original treatment of the interval polynomial problem is of an algebraic nature. He constructed four extremal members of \mathcal{P} ,

$$\begin{aligned} K_1(s) &= \underline{q}_0 + \underline{q}_1 s + \bar{q}_2 s^2 + \bar{q}_3 s^3 + \underline{q}_4 s^4 + \underline{q}_5 s^5 + \dots \\ K_2(s) &= \underline{q}_0 + \bar{q}_1 s + \bar{q}_2 s^2 + \underline{q}_3 s^3 + \underline{q}_4 s^4 + \bar{q}_5 s^5 + \dots \\ K_3(s) &= \bar{q}_0 + \bar{q}_1 s + \underline{q}_2 s^2 + \underline{q}_3 s^3 + \bar{q}_4 s^4 + \bar{q}_5 s^5 + \dots \\ K_4(s) &= \bar{q}_0 + \underline{q}_1 s + \underline{q}_2 s^2 + \bar{q}_3 s^3 + \bar{q}_4 s^4 + \underline{q}_5 s^5 + \dots \end{aligned}$$

later dubbed Kharitonov polynomials. In a remarkable fashion, Kharitonov showed that the stability of the entire interval polynomial family can be ascertained by testing merely these four.

Kharitonov's Theorem. The interval polynomial is Hurwitz stable if and only if $K_1(s)$, $K_2(s)$, $K_3(s)$, and $K_4(s)$ are all Hurwitz stable.

Subsequent development showed that for polynomials of degree 5, 4, and 3, the test can be further simplified, requiring checking only 3, 2, and 1 of the four Kharitonov polynomials, respectively.

Dasgupta (24) gave a geometrical interpretation to Kharitonov's Theorem in frequency domain, which sheds light on why such a puzzling result would hold. The interpretation makes use of the concept of *value set*. The idea is to examine the image of the polynomial family when s lies on the boundary of the stability region, namely

$$p(j\omega, Q) = \{p(j\omega, q) : q \in Q\}$$

Because $p(s, q^0)$ is stable and by assumption $a_n(q)$ never vanishes on Q , we can conclude from the continuity properties of the zeros of polynomials that if some member in \mathcal{P} is not Hurwitz stable, then some polynomial in \mathcal{P} must have a zero on the imaginary axis. Thus the entire family \mathcal{P} is Hurwitz stable if and only if

$$0 \notin p(j\omega, Q), \quad \forall \omega$$

that is, the value set never contains the origin. This is a rather straightforward fact known as early as in 1929 and is generally referred to in the literature as the *zero exclusion principle*. Note, in particular, that for a polynomial $p(s, q)$ in the family described by Eq. (11), for any $q \in Q$, the real and imaginary parts of $p(j\omega, q)$ are respectively given by

$$\operatorname{Re} p(j\omega, q) = q_0 - q_2 \omega^2 + q_4 \omega^4 - \dots$$

and

$$\operatorname{Im} p(j\omega, q) = j\omega(q_1 - q_3 \omega^2 + q_5 \omega^4 - \dots)$$

Thus the real part (resp. imaginary part) of $p(j\omega, q)$ depends only on the parameters with even (resp. odd) subscript. For an interval polynomial, therefore, it becomes clear that

$$\begin{aligned} \operatorname{Re} K_1(j\omega) = \operatorname{Re} K_2(j\omega) &\leq \operatorname{Re} p(j\omega, q) \leq \operatorname{Re} K_3(j\omega) = \operatorname{Re} K_4(j\omega) \\ \operatorname{Im} K_1(j\omega) = \operatorname{Im} K_4(j\omega) &\leq \operatorname{Im} p(j\omega, q) \leq \operatorname{Im} K_3(j\omega) = \operatorname{Im} K_2(j\omega) \end{aligned}$$

Because the four Kharitonov polynomials are themselves in \mathcal{P} , it follows that, at each ω , the value set is a rectangle in the complex plane with vertices $K_i(j\omega)$, as depicted in Fig. 4.

As ω increases, the rectangle evolves in a continuous fashion. Its dimensions vary, but its edges remain parallel to the axes, and the relative positions of the $K_i(j\omega)$ s do not change. Now, a polynomial $p(s)$ of degree n with positive coefficients is Hurwitz stable if and only if the phase of $p(j\omega)$ monotonically increases from 0 to $n\pi/2$ as ω goes from 0 to ∞ . This is best seen by noting that $p(s)$ can always be factorized as $p(s) = a_n \Pi(s - s_i)$ in terms of its zeros s_i , and by considering the phase of each factor separately. When this is understood, Kharitonov's Theorem becomes self-evident. Indeed, if the K_i s are Hurwitz stable, then the phase of each $K_i(j\omega)$ increases monotonically from 0 to $n\pi/2$ and the entire Kharitonov rectangle rotates around the origin by a total angle of $n\pi/2$ without ever touching it. Stability of \mathcal{P} then follows from the zero exclusion principle.

Kharitonov's Theorem opened an era epitomized by the search for so-called vertex results in robust stability analysis, as manifested by the construction of the four Kharitonov polynomials at the vertices of the hyperrectangle Q . This theme

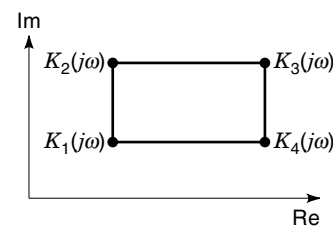


Figure 4. Kharitonov rectangle.

persisted in much of the subsequent work. Although Kharitonov's Theorem is without question a milestone in control theory, and perhaps a measured intellectual triumph in general, we should note that the interval polynomial family is nevertheless a very special instance of an affine uncertainty class, and hence the theorem has a rather limited scope in application. The quest thus continues.

Edge Theorem

In light of Kharitonov's Theorem, it is tempting to contemplate the possibility that a similar vertex result would hold for the general affine polynomial family with Q a hyperrectangle. This, unfortunately, turns out to be false; a counterexample can be readily constructed to demonstrate the opposite. What can be said about a polynomial family \mathcal{P} in class \mathcal{P}_a ? Bartlett et al. (25) provided an answer. In what is now known as the Edge Theorem, they took the bounding set Q to be a convex polytope in \mathbb{R}^m . Let $q^i, i = 1, \dots, l$, be the vertices of Q . Then, it is well known that Q can be represented as a convex hull of the vertices. That is,

$$Q = \text{conv}\{q^1, \dots, q^l\} = \left\{ \sum_{i=1}^l \lambda_i q^i : \sum_{i=1}^l \lambda_i = 1, \lambda_i \geq 0 \right\}$$

Because $a_k(q)$ is an affine function of q , it follows that

$$\mathcal{P} = \text{conv}\{p(s, q^1), \dots, p(s, q^l)\}$$

This implies that the value set $p(j\omega, Q)$ is a polygon, generated by the vertex polynomials $p(s, q^i)$. It should be rather clear that the interval polynomial family is generated by a polytope—a hyperrectangle—and so is the diamond polynomial family.

Edge Theorem. The affine polynomial family (polytope of polynomials) \mathcal{P}_a is Hurwitz stable if and only if for each edge point q of Q , $p(s, q)$ is Hurwitz stable.

Let q^i and q^j be two vertices of Q connected by an edge. When λ varies from 0 to 1, the polynomial

$$p_{ij}(s, \lambda) = \lambda p(s, q^i) + (1 - \lambda)p(s, q^j)$$

defines a line segment (of polynomials) connecting $p(s, q^i)$ and $p(s, q^j)$. The theorem shows that—quoting directly from Ref. 25—it suffices to check the edges. Because an edge polynomial involves only one parameter, its stability can be readily tested, by resorting to either a graphical test based upon a root locus, or the numerical solution of a generalized eigenvalue problem (see, for example, Chapter 4 in Ref. 26).

The heuristics behind the Edge Theorem are simple. Because for large values of ω , the value set does not contain the origin (this is easy to see), the polynomial family will be Hurwitz stable if and only if there is no frequency ω at which the origin belongs to the boundary of the (polygonal) value set. It should be intuitively clear that every point on the boundary of the value set must be the image of a point on an edge of the polytope of polynomials. Thus, if for some ω the origin does belong to this boundary, the corresponding edge polynomial must be unstable. Note that it is wasteful to check the entire set of all polynomial “segments” joining two vertices of

the polytope because some of these segments are not edges. In some cases, however, it is not an easy task to check which such segments are edges and which are not.

The Edge Theorem generalizes Kharitonov's Theorem in two important aspects. First, it is applicable to general affine polynomial families, with Q an arbitrary polytope. Secondly, the stability region may be any open, simply connected set (27,28), although only stated here in terms of the open left half plane. In contrast, Kharitonov's Theorem addresses only Hurwitz stability. On the other hand, as the number of parameters increases, the computation effort required in the Edge Theorem can be enormous. For example, when Q is a box in \mathbb{R}^m , with 2^m vertices, the number of edges is $m2^{m-1}$, a very large number even for moderate values of m . Finally, to a lesser extent, the Edge Theorem is not applicable to situations where the boundary of Q is curved, of which the spherical polynomial family is an example.

Graphical Tests

The complexity of the Edge Theorem provides a direct motivation to search for computationally more tractable stability criteria, and graphical conditions become a natural avenue to explore. Not only are graphical criteria time-honored tools in classical stability analysis, but the zero exclusion principle, with all its simplicity and transparency, also prompts a deeper investigation of such tools. We can generally feel that an important asset of the zero exclusion principle is its generality, both in terms of uncertain polynomial families and in terms of stability regions.

Barmish (29) studied the issue systematically; earlier conditions had appeared sporadically on stability of perturbed polynomials in isolated cases. Barmish's approach stems from a geometrical argument: a convex polygon in the complex plane does not intersect the origin as long as it can be separated from it by a straight line or, equivalently, as long as the vertices can be separated, as a whole, from the origin by such a line. This observation led to his construction of a testing function, which is to be evaluated along the boundary of the stability region. After this is accomplished, we can determine stability by plotting the testing function. Barmish's test is certainly one step forward compared to a pure brute-force computation; however, it remains somewhat ad hoc and is computationally overwhelming. Because it requires evaluations at all vertices, it does not clear the hurdle of exponential growth in complexity.

On a lesser scale, Tsypkin and Polyak (30) obtained a graphical test for a simpler problem. They examined the polynomial family in Eq. (10). Let $p \in [1, \infty]$ be given. Furthermore, for $r \in [1, \infty]$ such that $(1/p) + (1/r) = 1$, define

$$X_r(\omega) = (\gamma_0^r + (\gamma_2\omega^2)^r + (\gamma_4\omega^4)^r + \dots)^{1/r}$$

$$Y_r(\omega) = (\gamma_1^r + (\gamma_3\omega^2)^r + (\gamma_5\omega^4)^r + \dots)^{1/r}$$

$$R(\omega) = \gamma_0 q_0^0 - \gamma_2 q_2^0 \omega^2 + \gamma_4 q_4^0 \omega^4 + \dots$$

$$I(\omega) = \gamma_1 q_1^0 - \gamma_3 q_3^0 \omega^2 + \gamma_5 q_5^0 \omega^4 + \dots$$

Note that $R(\omega) = \text{Re}(p(j\omega, q^0))$ and $\omega I(\omega) = \text{Im}(p(j\omega, q^0))$.

Tsypkin-Polyak Criterion. The polynomial family \mathcal{P} of Eq. (10) is Hurwitz stable if and only if $p(s, q^0)$ is Hurwitz stable,

$|q_n^0| > 1$, $|q_0^0| > 1$, and

$$\left(\frac{|R(\omega)|}{X_r(\omega)}\right)^p + \left(\frac{|I(\omega)|}{Y_r(\omega)}\right)^p > 1, \quad \forall \omega \in (0, \infty) \quad (12)$$

This condition was independently obtained by Hinrichsen and Pritchard (31).

Up to this point, we could assert that the stability problem for affine polynomial families remains largely unresolved. However, as yet another observation, we find that at each ω , the zero exclusion condition defines two linear constraints in terms of perturbed coefficients, imposed on the real and imaginary parts of $p(j\omega, q)$, respectively. These constraints, together with a convex bounding set \mathcal{Q} , define in turn a convex feasibility condition; when the parameters vary independently, it reduces further to a linear program. This is a simple but conceptually appealing observation. It led to a reformulation via linear programming, due to Saridereli and Kern (32) and to Tesi and Vicino (33), which can be solved readily for each ω and then plotted graphically. Qiu and Davison (34) went further to demonstrate that for very general bounding sets it suffices to solve an optimization problem with one variable only, and the problem can be solved explicitly in special cases. Finally, Chen et al. (14,35) recognized that the problem can be reformulated as a special rank-one μ problem for each ω , and showed that stability can be ascertained by evaluating an explicit formula. These results led to the final resolution of the affine polynomial family.

μ and the Kharitonov Approach

Indeed, there is an inherent linkage between μ analysis and Kharitonov approach, whenever the latter applies, in that both approaches yield necessary and sufficient conditions for problems of the same nature. However, for a rather long time a clear link seemed elusive. The main cause, it seems, lay in how to reconcile an optimization-based formulation such as μ , and explicit results from the Kharitonov approach. Can one, for example, derive Kharitonov's theorem from μ , or vice versa?

The explicit formula of rank-one μ given earlier lends an answer. Specifically, for a general affine polynomial family \mathcal{P}_a [(Eq. (8)] with \mathcal{Q} the unit ℓ_∞ ball, robust stability can be checked by computing μ , with a rank-one matrix $M(s)$ constructed as

$$M(s) = \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix} \begin{bmatrix} p_1(s) & \dots & p_m(s) \\ p(s, q^0) & \dots & p(s, q^0) \end{bmatrix}$$

To see this, observe that $p(s, q)$ in Eq. (8) is the characteristic polynomial of the M - Δ loop of Fig. 1 with $\Delta = -\text{diag}(q_1 - q_1^0, \dots, q_m - q_m^0)$. Indeed,

$$\det(I - \Delta M(s)) = 1 + \sum_{k=0}^m \frac{p_k(s)}{p(s, q^0)} (q_k - q_k^0) = \frac{p(s, q)}{p(s, q^0)}$$

Thus, stability of $p(s, q)$ for all $q \in \mathcal{Q}$ is equivalent to stability of the M - Δ loop for all diagonal matrices Δ with real (parametric) entries lying in $[-1, 1]$. The condition for this is that $\|M\|_\mu < 1$ where the structured singular value is computed

with respect to the block structure $\Gamma = \{\text{diag}(\delta_1, \dots, \delta_m) : \delta_i \in \mathbb{R}\}$. In light of the formula for rank-one μ , an explicit condition can then be stated. Such a result clearly applies to general stability regions, and it furnishes a frequency sweeping condition for robust stability. Note that we may interpret this result alternatively based upon the zero exclusion principle. Indeed, under the condition that \mathcal{Q} is the unit ℓ_∞ ball centered at q^0 , all the polynomials in \mathcal{P} will have zeros in a specified region if and only if the zeros of $p(s, q^0)$ are in that region and $\mu(M(s)) < 1$ for all s on the boundary of the region. This follows because, according to the zero exclusion principle, it is both necessary and sufficient that

$$\min \left\{ \|q - q^0\|_\infty : p(s, q^0) + \sum_{k=0}^m (q_k - q_k^0) p_k(s) = 0 \right\} > 1$$

in order for the polynomial family $p(s, q)$ in Eq. (8) to have no zero on or exterior to the boundary of the region, for all possible $q \in \mathcal{Q}$.

More generally, it is possible to extend the definition of μ by means of more general norms and to use this extended μ to study the robust stability of an affine family \mathcal{P} with a more general bounding set \mathcal{Q} . Such a generalization also leads to a similar expression when the M matrix in question has rank one (14,35). In particular, when the stability region is restricted to the open left half plane, and \mathcal{Q} is the unit ℓ_p ball centered at q^0 with $a_k(q) = \gamma_k q_k$, the expression for the generalized rank-one μ , denoted as $\mu_p(\cdot)$ for purpose of distinction, is found to be

$$\mu_p(M(j\omega)) = \begin{cases} Y_r(\omega)/|I(\omega)| & \text{if } R(\omega) = 0 \\ X_r(\omega)/|R(\omega)| & \text{if } \omega I(\omega) = 0 \\ \frac{X_r(\omega)Y_r(\omega)}{(X_r^p(\omega)|I(\omega)|^p + Y_r^p(\omega)|R(\omega)|^p)^{1/p}} & \text{otherwise} \end{cases}$$

which leads to a similar condition for robust Hurwitz stability. This condition is slightly more general than, but essentially replicates, the graphical criterion by Tsympkin and Polyak. Note that for $p = \infty$, the polynomial family becomes an interval polynomial, and the stability condition reduces to checking whether $p(s, q^0)$ is Hurwitz stable, $|q_0| > 1$, and

$$\min \left\{ \frac{X_1(\omega)}{|R(\omega)|}, \frac{Y_1(\omega)}{|I(\omega)|} \right\} < 1, \quad \forall \omega \in (0, \infty)$$

A little thought reveals that the latter is equivalent to determining whether one of the four conditions $\text{Re}(K_1(j\omega)) > 0$, $\text{Re}(K_3(j\omega)) < 0$, $\text{Im}(K_1(j\omega)) > 0$, and $\text{Im}(K_3(j\omega)) < 0$ holds. Clearly, this is further equivalent to the requirement that the rectangular value set in Fig. 4 never contains the origin.

Extensions

There is an immense body of literature devoted to polynomial stability problems. Various extensions to Kharitonov's Theorem have been obtained. They generally fall into the categories of vertex results and frequency-sweeping conditions, consisting of delicate studies and intricate technical details. We summarize some of the highlights next. A recent and comprehensive account can be found in the books by Barmish (26) and by Bhattacharyya et al. (36).

Vertex and Edge Results. Much of the work in this direction continues the thread in Kharitonov's Theorem, focusing on simple uncertainty descriptions and leading to stability tests based on vertex and/or edge polynomials. Some notable examples follow.

- *Complex Interval Polynomials.* The polynomial family has complex coefficients whose real and imaginary parts are allowed to vary independently in given intervals. Eight vertex polynomials need to be tested to ascertain stability.
- *Diamond Polynomials.* This polynomial family is described in Eq. (10), with $p = 1$. Eight vertex polynomials are required as well.
- *Stabilization of Interval Plants via First-Order Compensators.* The numerator and denominator of the plant transfer function are interval polynomials, and it is to be stabilized by a first-order compensator in closed loop. It suffices to stabilize 16 vertex plants, constructed based upon the vertex numerator and denominator polynomials.
- *Generalized Kharitonov Theorem.* It concerns linear combination of interval polynomials and requires checking certain polynomial segments in addition to vertices.

Other stability conditions based on vertex polynomials are also available. As the complexity of uncertainty structure increases slightly, they usually require testing more (e.g., 32 or 64) vertex polynomials. A clear insight concerning uncertainty structure and the required number of vertices, however, remains unavailable.

Performance Issues. The entire Kharitonov theory is largely successful for determining stability of uncertain polynomials. However, a number of results are also available regarding properties of transfer functions, which have implications toward performance issues. Some examples follow.

- *\mathcal{H}_∞ Norm of Interval Transfer Functions.* When the numerator and denominator of a transfer function are both interval polynomials, the \mathcal{H}_∞ norm of the transfer function can be computed over 16 vertex transfer functions, provided that the four Kharitonov polynomials associated with the denominator are stable.
- *Peak Magnitudes of Closed-Loop Transfer Functions.* The peak \mathcal{H}_∞ norm of closed-loop transfer functions can be computed over the edges of the plant family, when it is an interval plant.
- *Nyquist and Bode Envelopes.* The Nyquist and Bode plots of open or closed-loop transfer functions associated with an interval plant lie in envelopes determined by plots generated by vertex and edge plants.

Other Extensions. Additional extensions may be found in the following categories.

- *Schur Stability.* The Kharitonov theory has been extended with varying degrees of success to other stability regions, such as the unit circle. These results are useful for studying stability of discrete-time systems and for addressing other performance issues.

- *Unstructured Uncertainty.* Unmodeled dynamics may be included along with parametric uncertainties. They may be accommodated either in the rank-one μ formula or by small gain-type conditions involving vertex and edge plants.
- *Nonlinear Systems.* In a system consisting of an interval plant and a static, sector bounded nonlinear component, stability conditions similar to the Popov and circle criteria have been obtained, which also require the testing of vertex and edge plants.
- *Multilinear Uncertainty Structure.* The entire success in the Kharitonov approach relies on the key assumption that polynomial coefficients depend linearly on uncertain parameters, and the utility of all the results in this area is thus measured by how much the uncertainty can deviate from this description. Little success has been achieved in this endeavor. A fundamental barrier, as implicated by the zero exclusion principle, is that the stability problem is one of optimization subject to nonlinear, nonconvex constraints.

STATE-SPACE APPROACH

Dynamical systems are often described by state-space equations. Accordingly, it is common to model system uncertainty as perturbations to system matrices. An uncertain continuous-time system in this spirit may be described by

$$\dot{x}(t) = (A + B\Delta C)x(t) \quad (13)$$

Here A , B , and C are known matrices of appropriate dimensions. The matrix A is assumed to be stable. The system uncertainty is represented by a set Δ of allowed values for the *real* matrix Δ , which may be unstructured or structured. Typical perturbation classes considered in the literature are as follows, arranged in increasing order of generality. In all cases, $\gamma > 0$ is given.

- *Unstructured Perturbation.* The set Δ consist of all real matrices with spectral norm less than a given number:

$$\Delta_U = \{\Delta \text{ real} : \bar{\sigma}(\Delta) \leq \gamma\}$$

- *Element-by-Element Perturbation.* Each element in Δ varies in a given interval. Let $r_{ij} \geq 0$ be given. The set Δ is defined as

$$\Delta_E = \left\{ \Delta \text{ real} : \Delta = \begin{bmatrix} r_{11}\delta_{11} & \cdots & r_{1m}\delta_{1m} \\ \vdots & \vdots & \vdots \\ r_{n1}\delta_{n1} & \cdots & r_{nm}\delta_{nm} \end{bmatrix}, \right. \\ \left. \|\Delta\| = \max_{i,j} \{|\delta_{ij}| : r_{ij} > 0\} \leq \gamma \right\}$$

- *Linear Combination.* The allowable set of perturbations is described by

$$\Delta_L = \left\{ \Delta \text{ real} : \Delta = \sum_{i=1}^k \delta_i E_i, \|\Delta\| = \max_i |\delta_i| \leq \gamma \right\}$$

where the E_i s are given.

Evidently, an uncertain discrete-time system can be described in exactly the same manner.

The problem of interest is to determine the size of the perturbation matrix, measured by a norm of choice, so that the system remains stable. This issue naturally translates into one concerning how the eigenvalues of a stable matrix A would vary when it is perturbed by Δ . More specifically, would the eigenvalues cross the stability boundary? And if they do, what is the minimal γ such that at least one of the eigenvalues leaves the stability region? These questions may be addressed by examining the characteristic polynomial

$$\Phi(s, \Delta) = \det(sI - A - B\Delta C)$$

or equivalently, the characteristic equation

$$\det(I - C(sI - A)^{-1}B\Delta) = 0$$

Thus, it becomes clear at the outset that the problem may be tackled in principle by using a polynomial approach. Similarly, it can also be analyzed as a μ problem. The latter can be easily seen with respect to Δ_E and Δ_L , by rearranging the elements of these sets into diagonal matrices and by defining the M matrix appropriately. For Δ_U , we may simply adopt a full real block structure and define μ accordingly. It should be pointed out, nevertheless, that both μ and the polynomial approach will lead to complications in the present context. On the one hand, the computation of μ with respect to a real Δ is generally very difficult, and approximation by its upper bound can be very conservative. On the other hand, the characteristic polynomial $\Phi(s, \Delta)$ will generally exhibit a multilinear or multinomial dependence of its coefficients on Δ , for which the Kharitonov theory is ill-equipped; indeed, it is not difficult to see that the coefficients of $\Phi(s, \Delta)$ are multilinear in δ_{ij} if $\Delta \in \Delta_E$, and are multinomial functions of δ_k if $\Delta \in \Delta_L$. In summary, both approaches are ineffective and conservative.

By far this uncertainty description poses the most difficult challenge in robust stability analysis, and the state-space approach is the least developed. Results are scarce, and only in rare cases are they nonconservative.

Stability Radius

A notion frequently encountered in studying the state-space uncertainty description is that of stability radius. This notion is closely related to μ , but it is less developed. Let \mathbb{D} be a stability region of concern, and $\partial\mathbb{D}$ be its boundary. Furthermore, denote by $\sigma(A) \subset \mathbb{D}$ the spectrum of A . Then for any norm $\|\cdot\|$ of interest, the stability radius associated with the triple (A, B, C) is defined by

$$r(A, B, C) = \inf\{\|\Delta\| : \Delta \in \mathbf{\Delta}, \sigma(A + B\Delta C) \cap \partial\mathbb{D} \neq \emptyset\}$$

In other words, it defines the minimal perturbation size leading to instability, or the “distance” of A to the set of unstable matrices. By definition, it thus follows directly that the matrix family $\{A + B\Delta C : \Delta \in \mathbf{\Delta}, \|\Delta\| \leq \gamma\}$ has all eigenvalues in \mathbb{D} whenever $r(A, B, C) > \gamma$. Moreover, in view of the preceding discussion, we may regard the stability radius as the reciprocal of the maximum of a certain μ , with respect to an appropriate block structure and a matrix M . For further distinction, the stability radius is said to be unstructured if Δ is unstructured and structured otherwise.

There essentially exists no result for the structured stability radius other than those already known for μ . For the unstructured stability radius, much of the early work was devoted to derivation of bounds. One representative example is

$$r(A, B, C) \geq \frac{1}{\|C(sI - A)^{-1}B\|_\infty} \quad (14)$$

This, of course, is a rather straightforward consequence of the Small Gain Theorem. Recently, however, Qiu et al. (37) obtained the following exact, readily computable formula.

Real Stability Radius. Let $G(s) = C(sI - A)^{-1}B$, and $\sigma_2(\cdot)$ be the second largest singular value. Then,

$$r(A, B, C)^{-1} = \sup_{s \in \partial\mathbb{D}} \inf_{\gamma \in (0, 1]} \sigma_2 \left(\begin{bmatrix} \operatorname{Re}[G(s)] & -\gamma \operatorname{Im}[G(s)] \\ \frac{1}{\gamma} \operatorname{Im}[G(s)] & \operatorname{Re}[G(s)] \end{bmatrix} \right)$$

The significance of this result lies in that for any $s \in \partial\mathbb{D}$, the function $\sigma_2(\cdot)$ is unimodal in γ over $(0, 1)$, and hence its infimum can be computed effectively. Furthermore, when \mathbb{D} is the open left half plane or the open unit disk, that is, when Hurwitz or Schur stability is of concern, Sreedhar et al. (38) developed a fast-converging algorithm for the maximization with respect to s . Consequently, from a computational standpoint, the unstructured stability radius problem can be considered largely resolved.

Interval Matrices

An interval matrix is a family of real matrices in which all elements are known only within certain closed intervals. In precise terms, the interval matrix $A_I = [\underline{A}, \bar{A}]$ is the set of matrices defined by

$$A_I = \{A : \underline{a}_{ij} \leq a_{ij} \leq \bar{a}_{ij}\}$$

that is, each a_{ij} of A is confined elementwise to lie within an interval determined by \underline{a}_{ij} and \bar{a}_{ij} , the corresponding elements of \underline{A} and \bar{A} , respectively. An interval matrix A_I is said to be stable if every $A \in A_I$ is stable. Evidently, interval matrix and set Δ_E share the same uncertainty description.

Interval matrices are direct matrix analogues of interval polynomials, and hence there has been a lingering temptation for extension of Kharitonov’s Theorem to the former. Unfortunately, neither vertex nor edge results exist for interval matrices. In fact, more recent studies showed that in order to determine stability of an interval matrix, we must solve an NP-hard decision problem. This in a way explains why only sufficient stability conditions are available.

One approach of attack is to analyze eigenvalue distribution. Heinen (39) and Argoun (40) examined the problem on the basis of Gershgorin’s Theorem, and their developments culminated in a subsequent work of Chen (41), leading to a number of simple, albeit conservative, stability conditions. As a representative example of these results, consider an interval matrix A_I such that $\bar{a}_{ii} < 0$. Let W be constructed as

$$W = [w_{ij}], \quad w_{ij} = \begin{cases} 0 & i = j \\ \frac{\max\{|\underline{a}_{ij}|, |\bar{a}_{ij}|\}}{|\bar{a}_{ii}|} & i \neq j \end{cases}$$

Then a sufficient condition for Hurwitz stability of A_I is found by Chen (41) to be

$$\rho(W) < 1$$

A useful feature of this result, and more generally of conditions obtained by using Gershgorin's Theorem, is that it lends a ready characterization via the so-called M-matrices. The latter aspect makes it possible to unify a number of sufficient stability conditions in different forms.

Alternatively, Yedavalli and others studied interval matrices from a Lyapunov analysis standpoint. This is collectively inspected next.

Lyapunov Analysis

The Lyapunov theory, as anticipated, is widely employed in robust stability analysis pertaining to state-space formulation, yielding various results concerning stability radius and interval matrices. One common thread in this approach is to find a *single* quadratic Lyapunov function applicable to the entire family of the perturbed matrices; the technique is often referred to in the literature as *quadratic stability*. Another lies in the simplicity of the stability conditions.

Let us begin with the unstructured uncertainty set Δ_U . By constructing the usual Lyapunov function

$$V(x) = \frac{1}{2}x^T P x$$

we find that the entire family of matrices $\{A + B\Delta C: \Delta \in \Delta_U\}$ is Hurwitz stable if there exists a positive definite matrix $P > 0$ such that

$$PA + A^T P + \gamma^2 B B^T + C^T C < 0 \quad (15)$$

This, of course, does not come as a surprise. According to the well-known Bounded Real Lemma (42), it is equivalent to

$$\|C(sI - A)^{-1}B\|_\infty < 1/\gamma$$

The latter condition clearly coincides with condition (14).

More results are available for the special case when $B = C = I$. For the structured uncertainty set Δ_E , Yedavalli (43) gave the sufficient condition

$$\bar{\sigma} \left(\frac{|P|R + R^T|P|}{2} \right) < 1/\gamma \quad (16)$$

for the Hurwitz stability of the matrix family $\{A + \Delta: \Delta \in \Delta_E\}$. Here $P > 0$ is the unique solution to the Lyapunov equation

$$PA + A^T P = -2I \quad (17)$$

$|P|$ denotes the modulus matrix of P (i.e., each entry of $|P|$ is the absolute value of the corresponding entry of P) and R is given by

$$R = \begin{bmatrix} r_{11} & \cdots & r_{1n} \\ \vdots & \vdots & \vdots \\ r_{n1} & \cdots & r_{nn} \end{bmatrix}$$

Furthermore, Zhou and Khargonekar (42) observed that the uncertainty description Δ_E can be regarded as a special case of Δ_L , for which they provided the stronger Hurwitz stability condition

$$\bar{\sigma} \left(\frac{1}{2} \sum_{i=1}^k (PE_i + E_i^T P) \right) < 1/\gamma \quad (18)$$

where, again, $P > 0$ is the unique solution to Eq. (17). Subsequent developments led to further extensions for problems with even more detailed uncertainty descriptions. For example, the δ_i s may be allowed to vary in asymmetric intervals. Moreover, because rather obviously any interval matrix can be represented alternatively in the form of $\{A + \Delta: \Delta \in \Delta_E\}$, these conditions can be applied to determine the Hurwitz stability of an interval matrix as well.

Yet another issue clearly of interest is whether it is possible to derive vertex versions of these sufficient conditions. Boyd and Yang (44) examined stability problems for matrix polytopes. Specifically, they postulated the uncertainty description

$$A = \text{conv}\{A_1, \dots, A_k\}$$

A sufficient condition for A to be Hurwitz stable can be easily found to be the existence of a $P > 0$ such that

$$PA_i + A_i^T P < 0, \quad i = 1, \dots, k \quad (19)$$

Similarly, for the uncertainty set Δ_L , a vertex condition can be obtained as

$$\bar{\sigma} \left(\frac{1}{2} \sum_{i=1}^k \epsilon_i (PE_i + E_i^T P) \right) < 1/\gamma \quad (20)$$

for all combinations of the ϵ_i in $\{-1, +1\}$. It should be rather evident that this condition improves upon inequality (18). Both conditions (19) and (20) may be regarded as vertex results in the matrix perturbation case, and both can be posed and solved as LMI problems.

CONCLUSION

Summary

For the past two decades, modeling uncertainty and robustness has resurfaced as a dominating theme in control theory and application and is now held unanimously by theoreticians and practitioners as the most important concern in control system design. For both its intrinsic appeal and practical significance, robust control as a whole attracted considerable interest and underwent a period of immense development, bringing control theory to a new height. Many important issues have been addressed. Many remain unresolved. The ultimate puzzle, it now appears, lies in the fundamental conflict between problem complexity and computational tractability.

Of the three main research areas surveyed in this article, the structured singular value provides the most general formulation for uncertainty modeling and is the most systematically developed tool in robustness analysis. The major issues

in μ analysis are clearly generality of uncertainty description, conservatism of analysis, and ease of computation. The main success achieved with this approach, unquestionably, lies in the progress in computing μ . While it cannot be computed exactly in general, various computational schemes have been developed for computing it approximately, and commercial software programs are now available. This paves the way for its application to a series of engineering design problems, ranging from disk drive control to flight control. Successful applications to other potential areas, including robot manipulators, flexible structures, magnetic bearings, and chemical processes, have also been reported in laboratory experiments.

The Kharitonov approach, unlike μ analysis, was more restrictive in scope in its early phase of development. However, it has undergone a “bottom-up” growth pattern as the uncertainty descriptions become progressively more general and sophisticated. Overall, the Kharitonov and state-space methods may be broadly classified as a parametric approach toward robustness analysis, originating from interval polynomials and culminating at state-space uncertainty descriptions. The main appeal of this approach, it appears, lies in its quest for analytical solutions, more appealing than mere computation-based tools. The main success in the entire parametric approach, which remains the state-of-the-art today, is the resolution of the affine uncertain polynomial family case, for which necessary and sufficient stability conditions are available, in terms of both edge tests and graphical conditions. On the other hand, the multilinear/multinomial polynomial family and the state-space uncertainty description are the weakest link, for which only sufficient stability conditions are available with unknown conservatism, and more systematic, efficient, computation-based approximate tests are called for. At present, only a few applications of the Kharitonov theory are reported in the literature, including Ackermann’s car steering problem and an automotive engine control problem investigated by Abate et al. (45) (see also Chapter 3 in Ref. 26). It should be rather evident that the fundamental bottleneck in all robustness analysis methods, be it μ analysis or Kharitonov approach, lies in computational complexity, and the ultimate challenge is in the conquest over the “curse of dimensionality.” No matter whether this can be achieved or not, we should be consciously aware that the dilemma is the natural cause of problem generality and hence complexity and results from the search of optimal solutions. In engineering system design, we should therefore reconcile and seek a judicious trade-off between these conflicting requirements.

To Probe Further

In light of the difficulties encountered in robustness analysis with respect to structured and/or parametric uncertainties, a number of researchers recently examined complexity issues from a computational standpoint, drawing upon concepts and techniques from computing science and operation research. The main discoveries are in the following areas.

- μ with Real Uncertainties. Braatz et al. (46) showed that the computation of real μ is NP-hard.
- μ with Complex Uncertainties. Toker and Ozbay (47) proved that the computation of complex μ is also NP-hard.

- μ with Real and Complex Uncertainties. Braatz et al. (46) and Toker and Ozbay (47) both showed that the computation of μ is NP-hard.
- The μ Bounds. Toker (48) and Fu (49) showed that the problem of finding an accurate bound for μ is NP-hard.
- Interval Matrix. Coxson and DeMarco (50) showed that stability of interval matrices amounts to an NP-hard problem.

These results indicate that a worst-case instance exists in each class of the problems, for which it is rather unlikely that computational complexity can be bounded via a polynomial function of the problem dimension. It thus comes as no surprise that the problems are difficult, and indeed are intractable in general.

From a technical standpoint, the computational difficulty in question may be best seen as an outcome of nonlinear, non-convex optimization problems. Although only explored systematically in recent years, complexity issues have been under contemplation for a long time and have led to alternative, computationally tractable approximations and formulations. One notable remedy is to resort to formulations based upon LMIs, and problems in this class include those that can be described via *integral quadratic constraints* (IQC). Both LMIs and IQCs offer in essence an energy-based perspective toward system analysis, and they draw heavily upon concepts in classical passivity and dissipativity theory, leading to readily computable, albeit only sufficient, robust stability conditions. For a comprehensive treatment of control-relevant LMI and convex programming problems, see Ref. 51, or the relevant chapter in this encyclopedia. Megretsky and Rantzer (52) provided a detailed account of the IQC technique.

The computational complexity results just discussed are strongly linked to the worst-case nature of the robustness problems; that is, the requirement of robustness must be met for all possible instances. Is so stringent a requirement truly necessary? This question prompted a reexamination of robustness issues, and it led to a recent venture departing almost entirely from the worst-case formulation. A number of researchers argued that worst-case scenarios hardly occur in practice, that a worst-case analysis is not only overly demanding but also too pessimistic, and that, after all, worst-case analysis problems are often intractable. The argument thus motivated the description of uncertainty via probabilistic measures, and accordingly probabilistic approaches to robustness analysis. In this new thinking, the deterministic uncertainty description is discarded altogether and is replaced by a probability description characterizing the likelihood that the uncertainty may lie in a bounded set. The robustness condition then amounts to determining the probability under which the system may become unstable. Recent studies (53–56) show that a variety of problems, which are NP-hard in the deterministic setting, become readily solvable computationally when formulated probabilistically. The area, however, is entirely open and is not without obstacles of its own.

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ROBUSTNESS ANALYSIS. See ROBUST CONTROL ANALYSIS.

ROBUST SIGNAL PROCESSING. See NONLINEAR SYSTEMS.

ROLLBACK RECOVERY. See PROGRAM DIAGNOSTICS.

ROLLING METALS. See METALS INDUSTRY.

ROTATION SPEED. See TACHOMETERS.