STABILITY THEORY, NONLINEAR

Stability analysis plays a central role in systems engineering. There are two general notions of stability that arise in the study of dynamical systems: input-output stability and stability of a particular response or a set of responses. In the first notion, the system is viewed as a map from the space of input signals to the space of output signals. It is said to be stable if an input that is well behaved in some sense (e.g., signals with finite amplitude or energy) will always produce an output that is well behaved in the same sense. In the second notion, the input to the system is fixed so that the response of the system over time is determined solely by the initial state of the system at the initial time. Such a response is said to be stable if other responses starting at nearby initial states stay nearby; otherwise it is unstable. It is said to be asymptotically stable if all responses starting at nearby initial states not only stay nearby, but also approach this particular response as time becomes large. This article deals with the latter notion of stability. An asymptotically stable response is the basis of a steady-state behavior whereby other responses asymptotically approach the steady-state response. A steady-state response can be as simple as a constant (time-invariant) response, or it can be a periodic one. These, as well as other more complicated steady-state responses, are described in the first section.

In our study of dynamical systems, we model the system by a finite number of coupled first-order ordinary differential equations

$$\begin{aligned} \dot{x}_1 &= f_1(t, x_1, \ldots, x_n) \\ \dot{x}_2 &= f_2(t, x_1, \ldots, x_n) \\ \vdots\ &\quad \vdots \\ \dot{x}_n &= f_n(t, x_1, \ldots, x_n) \end{aligned}$$

where $\dot{x}_i$ denotes the derivative of x_i with respect to the time variable t. We call the variables $x_1, x_2, \ldots, x_n$ the state variables. They represent the memory that the dynamical system has of its past. They are usually chosen as physical variables that represent the energy-storing elements. For example, in an *RLC* electrical circuit, the state variables could be voltages across capacitors and currents through inductors, while in a spring-mass-damper mechanical system, the state variables could be positions and velocities of moving masses. We usually use vector notation to write the above equations in a compact form. Define

$$x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, \quad f(t, x) = \begin{bmatrix} f_1(t, x) \\ f_2(t, x) \\ \vdots \\ f_n(t, x) \end{bmatrix}$$

and rewrite the n first-order differential equations as an n-dimensional first-order vector differential equation

$$\dot{x} = f(t, x)$$

which we call the state equation, and x is referred to as the *state*. The response of the system due to initial state x_0 at initial time t_0 is the solution of the differential equation $\dot{x} = f(t, x)$ subject to the initial condition $x(t_0) = x_0$. This solution is unique provided the function f is locally of Lipschitz character in the domain of interest D; that is, every point in D has a neighborhood D_0 and a nonnegative constant L such that the Lipschitz condition

$$\|f(t, x) - f(t, y)\| \leq L\|x - y\|$$

is satisfied for all x and y in D_0. $\|x\|$ is a measure of the length of the vector x in the n-dimensional state space (the space of n-dimensional real vectors), and is defined by $\|x\|^2 = \sum_i^n x_i^2$. The locus of the solution $x(t)$ in the state space is usually referred to as a *trajectory*. A special case of the state equation arises when the function f does not depend explicitly on t; that is, $\dot{x} = f(x)$, in which case the system is said to be *autonomous;* otherwise it is said to be *nonautonomous*.

In the first section, we introduce the most common forms of steady-state responses, namely, equilibrium points (constant solutions), limit cycles (periodic solutions), tori, and strange attractors (chaos). The most effective stability analysis tools are available for the case of equilibrium points, which we treat in the second section. We make the notions of stability, instability, and asymptotic stability, introduced earlier, precise and present Lyapunov's method and illustrate it by examples. We treat the special case of linear systems and show how stability of an equilibrium point of a nonlinear system can be studied by linearizing the system about this point. We use the center-manifold theorem to treat the critical case when linearization fails and end the section with an extension of Lyapunov's method to nonautonomous systems. An important issue in the analysis of dynamical systems is the effect of changes in the system's parameters on its behavior. Smooth changes in the system's behavior are usually studied via sensitivity analysis tools, but when the change in the parameters results in a change in the qualitative behavior of the system, like the disappearance of an equilibrium point or a limit cycle, it is studied via bifurcation theory, which we introduce in the final section.

STEADY-STATE BEHAVIOR

The steady-state behavior of a system is described by the asymptotic nature of solutions as time becomes large. For nonlinear systems, this will depend on the system as well as on the initial conditions provided. The possible types of steady-state behavior are more varied than one might think; they include the well-known constant time behavior (asymptotically stable equilibria) and periodic time behavior (asymptotically stable limit cycles), as well as more complicated behaviors, such as multiperiodic behavior (asymptotically stable tori) and chaos (strange attractors). We begin with some definitions that are general enough to capture this range of possibilities, and then provide some examples.

The steady-state behavior of a system takes place on a subset of the state space called an *attractor.* The key ingredients for defining an attractor $\mathscr{A}$ are the following: (1) If a solution is started in $\mathscr{A}$, it never leaves $\mathscr{A}$. That is, $\mathscr{A}$ is an *invariant set,* defined by saying that for each $x(0) = x_0 \in \mathscr{A}$, $x(t) \in \mathscr{A}$ $\forall\, t$. (2) Solutions started sufficiently close to $\mathscr{A}$ will approach $\mathscr{A}$ as $t \to \infty$. That is, $\mathscr{A}$ is *locally attractive.* (3) A feature of an attractor $\mathscr{A}$ is that it contains a solution that comes arbitrarily close to every point in $\mathscr{A}$ at some time. This implies that $\mathscr{A}$ is minimal in the sense that there are no subsets of $\mathscr{A}$ that satisfy conditions (1) and (2).

The *domain of attraction* (or region, or basin, of attraction) for an attractor $\mathscr{A}$ is defined to be the set of initial conditions in the state space that are asymptotic to $\mathscr{A}$ as $t \to \infty$. It can (at least formally) be constructed by considering a neighborhood $U_{\mathscr{A}}$ of $\mathscr{A}$ that is used in proving its asymptotic stability, and taking $\bigcup_{t\le 0}\{x(t)\}\ \forall\, x(0) \in U_{\mathscr{A}}$. This simply starts solutions in a neighborhood in which one knows they will approach $\mathscr{A}$ in forward time and runs time backward. In this way all solutions that will approach $\mathscr{A}$ are collected. Except in simple problems it is impossible to determine the domain of attraction, although parts of it can often be estimated using Lyapunov methods, as described in the next section. It is important to realize that a nonlinear system may possess multiple attractors of various types. The domains of attraction for different attractors must, of course, be distinct. Typically, different domains of attraction are separated by solutions that are asymptotic to saddle-type invariant sets (that is, invariant sets that are generally unstable but that have some stable directions).

The most well-known type of attractor is an asymptotically stable equilibrium point of an autonomous system, $\dot{x} = f(x)$. An equilibrium point for this system is a point $\bar{x}$ such that $f(\bar{x}) = 0$, representing a constant, time-invariant, steady-state response. Obviously the solution for the system initiated at such a point is simply $x(t) = \bar{x}$, and $\bar{x}$ is invariant and minimal. The matter of its stability is considered in detail in the following section.

The next simplest type of attractor is represented by a stable closed trajectory in the state space, called a limit cycle, γ. A solution $x(t)$ that lies on γ is necessarily periodic, since by starting from a point on γ, the solution takes time T to traverse the trajectory and return to the starting point, after which the motion continually repeats. Thus, $x(t + T) = x(t)$ for all points on γ, where T is the period of the limit cycle. The frequency content of a limit cycle is composed of a fundamental harmonic plus multiples of the fundamental. Limit cycles can arise in autonomous or nonautonomous systems, examples of which follow. A simple autonomous system that possesses a limit cycle is the following:

$$\dot{r} = r(1 - r^2), \quad \dot{\theta} = \omega \tag{1}$$

which is expressed in polar coordinate form. This two-dimensional system has an invariant set $\{(r, \theta): r = 1, \theta \in [0, 2\pi)\}$, a circle, which attracts all solutions except for the unstable equilibrium at $r = 0$, and on which solutions wind around with constant speed leading to a period $T = 2\pi/\omega$. Furthermore, since all points on the circle are visited by every solution started on it, it satisfies all the conditions for an attractor. Such a closed trajectory can, of course, exist in higher-order systems as well. Another example of an asymptotically stable periodic attractor is offered by the simple linear equation

$$\dot{x} = -\alpha x + \beta \cos(\omega t) \tag{2}$$

which has a known steady-state solution of the form $x_{ss}(t) = A\cos(\omega t - \nu)$, which is obviously asymptotically stable for $\alpha > 0$ since the transient decays to zero. In order to view this steady state as a limit cycle, one considers the *extended state space,* constructed by supplementing the preceding equation with the trivial equation $\dot{\theta} = \omega$ and replacing ωt by θ in Eq. (2). (This renders the system autonomous.) The extended state space, (x, θ), is a cylinder, shown in Fig. 1, on which the

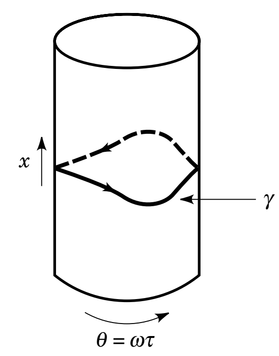

Figure 1. Limit cycle in a cylindrical state space.

steady-state solution is a closed trajectory γ with period $T = 2\pi/\omega$. While both of these examples have circular limit cycles around which solutions move at constant speed, this, of course, is not generally the case.

A steady-state response that is composed of multiple, noncommensurate frequencies corresponds to an asymptotically stable torus in the state space. Roughly speaking, each angular coordinate on the torus has an associated frequency. (Note that a limit cycle can be viewed as a one-dimensional torus.) A simple example of an asymptotically stable torus is given by a state model that is a simple generalization of that used for the first limit-cycle example,

$$\dot{r} = r(1 - r^2), \quad \dot{\theta}_1 = \omega_1, \quad \dot{\theta}_2 = \omega_2 \tag{3}$$

Tori can exist in Euclidean spaces of dimension three and higher. It is easy to show that the $r = 0$ solution is unstable and the solution with $r = 1$, $\theta_1 = \omega_1 t + \theta_{10}$, $\theta_2 = \omega_2 t + \theta_{20}$ is asymptotically stable. Here ω_1 and ω_2 represent the frequencies of the steady-state response. Note that if ω_1/ω_2 is rational, then every solution on the torus is periodic and closed, representing a one-parameter family of periodic responses. In contrast, when ω_1/ω_2 is irrational, the torus will be covered by a single solution (a dense solution) for any initial condition. Thus, the torus satisfies condition (3) for an attractor only in the incommensurable case. Also, note that in more general examples, the various rotation speeds are not constant on the torus, and the tori can be highly distorted. The response of a torus is generally composed of a set of discrete frequencies that include ω_1 and ω_2, as well as various linear combinations of them that result from nonlinear interactions. Also, one can encounter tori with more than two frequencies.

A chaotic steady-state response corresponds to a complicated set in the state space known as a *strange attractor.* While chaos is observed in many simulations and experiments, it is virtually impossible to prove the existence of a strange attractor for a given system model. The essence of these difficulties lies in the fact that there exist extremely complicated invariant sets in such systems, and it is not possible to prove that an asymptotically stable periodic solution does not exist nearby. However, these subtle issues fall outside the main topic of this article. The response of a strange attractor has a broadband frequency content, which is rather unexpected for a deterministic system.

Note that the complexity of these attractors is related to their dimensionality. The simplest, an equilibrium, has dimension 0, the limit cycle has dimension 1, and a torus with N frequencies has dimension N. It is interesting to note that a chaotic attractor, if it exists, will have a noninteger, or fractal, dimension due to the rich structure of the invariant manifold on which it exists. The difficulties associated with determining the stability of various types of invariant sets is similarly related to their dimensionality. For equilibria many analysis techniques exist, as described in detail in the following. Techniques also exist for limit cycles, tori, and chaos, but can rarely be applied without computational tools.

STABILITY OF EQUILIBRIUM POINTS

We consider the autonomous system $\dot{x} = f(x)$, where the components of the n-dimensional vector $f(x)$ are locally Lipschitz functions of x, defined for all x in a domain $D \subset \mathbb{R}^n$ that contains the origin $x = 0$. Suppose the origin is an equilibrium point of $\dot{x} = f(x)$; that is, $f(0) = 0$. Our goal is to characterize and study the stability of the origin. There is no loss of generality in taking the equilibrium point at the origin, for any equilibrium point $\overline{x} \neq 0$ can be shifted to the origin via the change of variables $y = x - \overline{x}$.

The equilibrium point $x = 0$ of $\dot{x} = f(x)$ is *stable,* if for each $\epsilon > 0$, there is $\delta = \delta(\epsilon) > 0$ such that $\|x(0)\| < \delta$ implies that $\|x(t)\| < \epsilon$, for all $t \geq 0$. It is said to be *asymptotically stable* if it is stable and δ can be chosen such that $\|x(0)\| < \delta$ implies that $x(t)$ approaches the origin as t tends to infinity. When the origin is asymptotically stable, the *domain of attraction* is defined as the set of all points x such that the solution of $\dot{x} = f(x)$ that starts from x at time $t = 0$ approaches the origin as t tends to ∞. When the domain of attraction is the whole space $\mathbb{R}^n$, we say that the origin is *globally asymptotically stable.*

Lyapunov's Method

In 1892, Lyapunov introduced a method to determine the stability of equilibrium points without solving the state equation. Let $V(x)$ be a continuously differentiable scalar function defined in D. A function $V(x)$ is said to be *positive definite* if $V(0) = 0$ and $V(x) > 0$ for every $x \neq 0$. It is said to be *positive semidefinite* if $V(x) \geq 0$ for all x. A function $V(x)$ is said to be *negative definite* or *negative semidefinite* if $-V(x)$ is positive definite or positive semidefinite, respectively. The derivative of V along the trajectories of $\dot{x} = f(x)$ is given by $\dot{V}(x) = \sum_{i=1}^{n} (\partial V/\partial x_i)\dot{x}_i = (\partial V/\partial x)f(x)$, where $\partial V/\partial x$ is a row vector whose ith component is $\partial V/\partial x_i$.

Lyapunov's stability theorem states that *the origin is stable if there is a continuously differentiable positive definite function $V(x)$ so that $\dot{V}(x)$ is negative semidefinite, and it is asymptotically stable if $\dot{V}(x)$ is negative definite.* A function $V(x)$ satisfying the conditions for stability is called a *Lyapunov function.* The surface $V(x) = c$, for some $c > 0$, is called a *Lyapunov surface* or a level surface. Using Lyapunov surfaces, Fig. 2 makes the theorem intuitively clear. It shows Lyapunov surfaces for decreasing constants $c_3 > c_2 > c_1 > 0$. The condition $\dot{V} \leq 0$ implies that $V(x(t))$ decreases along the trajectory $x(t)$. Therefore, when a trajectory crosses a Lyapunov surface $V(x) = c$, it moves inside the set $\{V(x) \leq c\}$ and can never come out again. When $\dot{V} < 0$, the trajectory moves from one Lyapunov surface to an inner Lyapunov surface with a smaller c. As c decreases, the Lyapunov surface $V(x) = c$ shrinks to the origin, showing that the trajectory approaches the origin as time progresses. If we only know that $\dot{V} \leq 0$, we cannot be

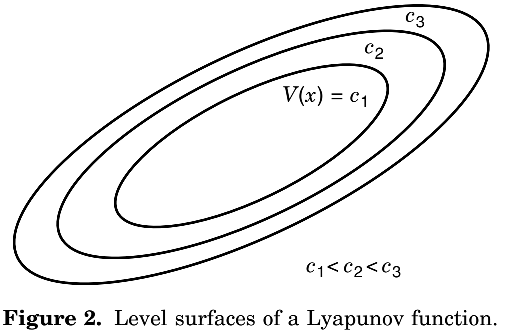

Figure 2. Level surfaces of a Lyapunov function.

sure that the trajectory will approach the origin, but we can conclude that the origin is stable since the trajectory can be contained inside any ϵ neighborhood of the origin by requiring the initial state $x(0)$ to lie inside a Lyapunov surface contained in that neighborhood.

When $\dot{V}(x)$ is only negative semidefinite, we may still be able to conclude asymptotic stability of the origin if we can show that no solution can stay identically in the set $\{\dot{V}(x) = 0\}$, other than the trivial solution $x(t) \equiv 0$. Under this condition, $V(x(t))$ must decrease toward 0, and consequently $x(t) \to 0$ as $t \to \infty$. This extension of the basic theorem is usually referred to as *the invariance principle.*

Lyapunov functions can be used to estimate the domain of attraction of an asymptotically stable origin, that is, to find sets contained in the domain of attraction. If there is a Lyapunov function that satisfies the conditions of asymptotic stability over a domain D and if $\{V(x) < c\}$ is bounded and contained in D, then every trajectory starting in $\{V(x) < c\}$ remains in $\{V(x) < c\}$ and approaches the origin as $t \to \infty$. Thus, $\{V(x) < c\}$ is an estimate of the domain of attraction. If the Lyapunov function $V(x)$ is radially unbounded that is, $\|x\| \to \infty$ implies that $V(x) \to \infty$, then any point $x \in \mathbb{R}^n$ can be included in the bounded set $\{V(x) < c\}$. Therefore, *the origin is globally asymptotically stable if there is a continuously differentiable, radially unbounded function $V(x)$ such that for all $x \in \mathbb{R}^n$, $V(x)$ is positive definite and $\{\dot{V}(x)$ is either negative definite or negative semidefinite but no solution can stay identically in the set $\{\dot{V}(x) = 0\}$ other than the trivial solution $x(t) \equiv 0$.*

Lyapunov's method is a very powerful tool for studying the stability of equilibrium points. However, there are two drawbacks to the method of which the reader should be aware. First, there is no systematic method for finding a Lyapunov function for a given system. In some cases, there are natural Lyapunov function candidates like *energy functions* in electrical or mechanical systems (see Example 2). In other cases, it is basically a matter of trial and error. Second, the conditions of the method are only sufficient; they are not necessary. Failure of a Lyapunov function candidate to satisfy the conditions for stability or asymptotic stability does not mean that the origin is not stable or asymptotically stable.

Example 1. Consider the second-order system

$$\dot{x}_1 = -x_1 + x_1^2 x_2, \qquad \dot{x}_2 = x_1 - x_2$$

The system has three equilibrium points at (0,0), (1,1), and $(-1,-1)$. We want to study the stability of the origin (0,0). We take the quadratic function

$$V(x) = ax_1^2 + 2bx_1x_2 + dx_2^2 = x^{\mathrm{T}}Px$$

where

$$P = \begin{bmatrix} a & b \\ b & d \end{bmatrix}$$

as a Lyapunov-function candidate. For $V(x)$ to be positive definite, we must have $a > 0$, $d > 0$, and $ad - b^2 > 0$. The derivative of $V(x)$ along the trajectories of the system is given by

$$\begin{aligned}\dot{V}(x) &= 2(ax_1 + bx_2)\dot{x}_1 + 2(bx_1 + dx_2)\dot{x}_2 \\ &= 2(ax_1 + bx_2)(-x_1 + x_1^2x_2) + 2(bx_1 + dx_2)(x_1 - x_2)\end{aligned}$$

Choosing $b = 0$ yields

$$\dot{V}(x) = -x^{\mathrm{T}}Qx + 2ax_1^3x_2$$

where

$$Q = 2\begin{bmatrix} a & -0.5d \\ -0.5d & d \end{bmatrix}$$

The matrix Q is positive definite when $ad - d^2/4 > 0$. Choose $a = d = 1$. Near the origin, the quadratic term $-x^{\mathrm{T}}Qx$ dominates the fourth-order term $2x_1^3x_2$. Thus, $\dot{V}(x)$ is negative definite and the origin is asymptotically stable. Notice that the origin is not globally asymptotically stable since there are other equilibrium points. We can use $V(x)$ to estimate the domain of attraction of the origin. The function $V(x)$ is positive definite for all x. We need to determine a domain D about the origin where $\dot{V}(x)$ is negative definite and a set $\{V(x) < c\} \subset D$, which is bounded. We are interested in the largest set $\{V(x) < c\}$ that we can determine, that is, the largest value for the constant c. Using the inequalities $x^{\mathrm{T}}Qx \geq \lambda_{\min}(Q)\|x\|^2 = \|x\|^2$ and $2x_1^3x_2 \leq x_1^2|2x_1x_2| \leq \|x\|^4$, we see that $\dot{V}(x) \leq -\|x\|^2 + \|x\|^4$. Hence $\dot{V}(x)$ is negative definite in the domain $\{\|x\| < 1\}$. We would like to choose a positive constant c such that $\{V(x) < c\}$ is a subset of this domain. Since $x^{\mathrm{T}}Px \geq \lambda_{\min}(P)\|x\|^2 = \|x\|^2$, we can choose $c = 1$. Thus, the set $\{V(x) < 1\}$ is an estimate of the domain of attraction.

Example 2. A simple pendulum moving in a vertical plane can be modeled by the second-order differential equation

$$ml\ddot{\theta} = -mg\sin\theta - kl\dot{\theta}$$

where l is the length of the rod, m is the mass of the bob, θ is the angle subtended by the rod and the vertical line through the pivot point, g is the acceleration due to gravity, and k is a coefficient of friction. Taking $x_1 = \theta$ and $x_2 = \dot{\theta}$ as the state variables, we obtain the state equation

$$\dot{x}_1 = x_2, \qquad \dot{x}_2 = -a\sin x_1 - bx_2$$

where $a = g/l > 0$ and $b = k/m \geq 0$. The case $b = 0$ is an idealized frictionless pendulum. To find the equilibrium points, we set $\dot{x}_1 = \dot{x}_2 = 0$ and solve for x_1 and x_2. The first equation gives $x_2 = 0$ and the second one gives $\sin x_1 = 0$. Thus, the equilibrium points are located at $(n\pi, 0)$, for $n = 0, \pm 1, \pm 2, \ldots$. From the physical description of the pendulum it is clear that the pendulum has only two equilibrium positions corresponding to the equilibrium points (0,0) and $(\pi,0)$. Other equilibrium points are repetitions of these two positions that correspond to the number of full swings the pendulum would make before it rests at one of the two equilibrium positions. Let us use Lyapunov's method to study the stability of the equilibrium point at the origin. As a Lyapunov-function candidate, we use the energy of the pendulum, which is defined as the sum of its potential and kinetic energies, namely,

$$V(x) = \int_0^{x_1} a \sin y \, dy + \tfrac{1}{2} x_2^2 = a(1 - \cos x_1) + \tfrac{1}{2} x_2^2$$

The reference of the potential energy is chosen such that $V(0) = 0$. The function $V(x)$ is positive definite over the domain $-2\pi < x_1 < 2\pi$. The derivative of $V(x)$ along the trajectories of the system is given by

$$\dot{V}(x) = a\dot{x}_1 \sin x_1 + x_2 \dot{x}_2 = -bx_2^2$$

When friction is neglected ($b = 0$), $\dot{V}(x) = 0$ and we can conclude that the origin is stable. Moreover, $V(x)$ is constant during the motion of the system. Since $V(x) = c$ forms a closed contour around $x = 0$ for small $c > 0$, we see that the trajectory will be confined to one such contour and will not approach the origin. Hence the origin is not asymptotically stable. On the other hand, in the case with friction ($b > 0$), $\dot{V}(x) = -bx_2^2 \leq 0$ is negative semidefinite and we can conclude that the origin is stable. Notice that $\dot{V}(x)$ is only negative semidefinite and not negative definite because $\dot{V}(x) = 0$ for $x_2 = 0$ irrespective of the value of x_1. Therefore, we cannot conclude asymptotic stability using Lyapunov's stability theorem. Here comes the role of the invariance principle. Consider the set $\{\dot{V}(x) = 0\} = \{x_2 = 0\}$. Suppose that a solution of the state equation stays identically in this set. Then

$$x_2(t) \equiv 0 \Rightarrow \dot{x}_2(t) \equiv 0 \Rightarrow \sin x_1(t) \equiv 0$$

Hence, on the segment $-\pi < x_1 < \pi$ of the $x_2 = 0$ line, the system can maintain the $\dot{V}(x) = 0$ condition only at the origin $x = 0$. Noting that the solution is confined to a set $\{V(x) \leq c\}$ and, for sufficiently small c, $\{V(x) \leq c\} \subset \{-\pi < x_1 < \pi\}$, we conclude that no solution can stay identically in the set $\{V(x) \leq c\} \cap \{x_2 = 0\}$ other than the trivial solution $x(t) \equiv 0$. Hence, the origin is asymptotically stable. We can also estimate the domain of attraction by the set $\{V(x) \leq c\}$ where $c < \min_{\{x_1 = \pm\pi\}} V(x) = 2a$ is chosen such $V(x) = c$ is a closed contour contained in the strip $\{-\pi < x_1 < \pi\}$.

Example 3. Consider the system

$$\dot{x}_1 = x_2, \qquad \dot{x}_2 = -g_1(x_1) - g_2(x_2)$$

where $g_1(\cdot)$ and $g_2(\cdot)$ are locally Lipschitz functions and satisfy $g_i(0) = 0$, $yg_i(y) > 0 \ \forall \ y \neq 0$, $i = 1, 2$, and $\int_0^y g_1(z) \, dz \to \infty$, as $|y| \to \infty$. The system has an isolated equilibrium point at the origin. The equation of this system can be viewed as a generalized pendulum equation with $g_2(x_2)$ as the friction term. Therefore, a Lyapunov-function candidate may be taken as the energylike function $V(x) = \int_0^{x_1} g_1(y) \, dy + \frac{1}{2}x_2^2$, which is positive definite in $\mathbb{R}^2$ and radially unbounded. The derivative of $V(x)$ along the trajectories of the system is given by

$$\dot{V}(x) = g_1(x_1)x_2 + x_2[-g_1(x_1) - g_2(x_2)] = -x_2 g_2(x_2) \leq 0$$

Thus, $\dot{V}(x)$ is negative semidefinite. Note that $\dot{V}(x) = 0$ implies $x_2 g_2(x_2) = 0$, which implies $x_2 = 0$. The only solution that can stay identically in the set $\{x \in \mathbb{R}^2 | x_2 = 0\}$ is the trivial solution $x(t) \equiv 0$. Thus, by the invariance principle, the origin is globally asymptotically stable.

Linear Systems

The linear time-invariant system $\dot{x} = Ax$ has an equilibrium point at the origin. The equilibrium point is isolated if and only if $\det(A) \neq 0$. Stability properties of the origin can be characterized by the locations of the eigenvalues of the matrix A. Recall from linear system theory that the solution of $\dot{x} = Ax$ for a given initial state $x(0)$ is given by $x(t) = \exp(At)x(0)$ and that for any matrix A there is a nonsingular matrix P (possibly complex) that transforms A into its Jordan form; that is,

$$P^{-1}AP = J = \text{block diag}[J_1, J_2, \ldots, J_r]$$

where J_i is the Jordan block associated with the eigenvalue λ_i of A. Therefore,

$$\exp(At) = P\exp(Jt)P^{-1} = \sum_{i=1}^{r} \sum_{k=1}^{m_i} t^{k-1} \exp(\lambda_i t) R_{ik} \tag{4}$$

where m_i is the order of the Jordan block associated with the eigenvalue λ_i. If one of the eigenvalues of A is in the open right-half complex plane, the corresponding exponential term $\exp(\lambda_i t)$ in Eq. (4) will grow unbounded as $t \to \infty$. Therefore, to guarantee stability we must restrict the eigenvalues to be in the closed left-half complex plane. But those eigenvalues on the imaginary axis (if any) could give rise to unbounded terms if the order of the associated Jordan block is higher than 1, due to the term t^{k-1} in Eq. (4). Therefore, we must restrict eigenvalues on the imaginary axis to have Jordan blocks of order 1. For asymptotic stability of the origin, $\exp(At)$ must approach 0 as $t \to \infty$. From Eq. (4), this is the case if and only if $\text{Re}(\lambda_i) < 0$, $\forall \ i$.

When all eigenvalues of A satisfy $\text{Re}(\lambda_i) < 0$, A is called a *Hurwitz matrix.* The origin of $\dot{x} = Ax$ is asymptotically stable if and only if A is a Hurwitz matrix. The asymptotic stability of the origin can be also investigated using Lyapunov's method. Consider a quadratic Lyapunov-function candidate $V(x) = x^T P x$ where P is a real symmetric positive-definite matrix. The derivative of V along the trajectories of $\dot{x} = Ax$ is given by

$$\dot{V}(x) = x^T P \dot{x} + \dot{x}^T P x = x^T(PA + A^T P)x = -x^T Q x$$

where Q is a symmetric matrix defined by

$$PA + A^T P = -Q \tag{5}$$

If Q is positive definite, we can conclude that the origin is asymptotically stable; that is, A is a Hurwitz matrix. Suppose we start by choosing Q as a real symmetric positive definite matrix, and then solve Eq. (5) for P. If Eq. (5) has a positive definite solution, then again we can conclude that the origin is asymptotically stable. Equation (5) is called the *Lyapunov equation.* It turns out that A *is a Hurwitz matrix if and only if for any given positive definite symmetric matrix* Q *there exists a positive definite symmetric matrix* P *that satisfies the Lyapunov equation* (5). *Moreover, if* A *is a Hurwitz matrix, then* P *is the unique solution of* Eq. (5).

Equation (5) is a linear algebraic equation that can be solved by rearranging it in the form $Mx = y$, where x and y are defined by stacking the elements of P and Q in vectors.

Almost all commercial software programs for control systems include commands for solving the Lyapunov equation.

Linearization

Consider the nonlinear system $\dot{x} = f(x)$ and suppose that $f(x)$ is continuously differentiable for all $x \in D \subset \mathbb{R}^n$. The Jacobian matrix $\partial f/\partial x$ is an $n \times n$ matrix whose (i, j) element is $\partial f_i/\partial x_j$. Let A be the Jacobian matrix evaluated at the origin $x = 0$. By applying the mean-value theorem to each component of f, it can be shown that $f(x) = Ax + g(x)$, where $\|g(x)\|/\|x\| \to 0$ as $\|x\| \to 0$. Suppose A is a Hurwitz matrix and let P be the solution of the Lyapunov equation (5) for some positive definite Q. Taking $V(x) = x^T Px$, it can be shown that $\dot{V}(x)$ is negative definite in the neighborhood of the origin. Hence, *the origin is asymptotically stable if all the eigenvalues of A have negative real parts.* Using some advanced results of Lyapunov stability, it can be also shown that *the origin is unstable if one (or more) of the eigenvalues of A has a positive real part.* This provides us with a simple procedure for determining stability of the origin of a nonlinear system by calculating the eigenvalues of its linearization about the origin. Note, however, that linearization fails when $\mathrm{Re}(\lambda_i) \leq 0$ for all i, with $\mathrm{Re}(\lambda_i) = 0$ for some i.

Example 4. The pendulum equation has two equilibrium points at (0,0) and (π,0). Let us investigate stability of each point using linearization. The Jacobian matrix is given by

$$\frac{\partial f}{\partial x} = \begin{bmatrix} \dfrac{\partial f_1}{\partial x_1} & \dfrac{\partial f_1}{\partial x_2} \\ \dfrac{\partial f_2}{\partial x_1} & \dfrac{\partial f_2}{\partial x_2} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -a\cos x_1 & -b \end{bmatrix}$$

To determine stability of the origin we evaluate the Jacobian at $x = 0$, to obtain

$$A = \left.\frac{\partial f}{\partial x}\right|_{x=0} = \begin{bmatrix} 0 & 1 \\ -a & -b \end{bmatrix}$$

The eigenvalues of A are $\lambda_{1,2} = -b/2 \pm \frac{1}{2}\sqrt{b^2 - 4a}$. For all positive values of a and b, the eigenvalues satisfy $\mathrm{Re}(\lambda_i) < 0$. Hence, the equilibrium point at the origin is asymptotically stable. In the absence of friction ($b = 0$), both eigenvalues are on the imaginary axis. In this case we cannot determine the stability of the origin through linearization. We have seen before that in this case the origin is a stable equilibrium point, as determined by an energy Lyapunov function. To determine the stability of the equilibrium point at (π,0), we evaluate the Jacobian at this point. This is equivalent to performing a change of variables $z_1 = x_1 - \pi$, $z_2 = x_2$ to shift the equilibrium to the origin, and then evaluating the Jacobian $\partial f/\partial z$ at $z = 0$.

$$\tilde{A} = \left.\frac{\partial f}{\partial x}\right|_{x_1=\pi,\, x_2=0} = \begin{bmatrix} 0 & 1 \\ a & -b \end{bmatrix}$$

The eigenvalues of $\tilde{A}$ are $\lambda_{1,2} = -b/2 \pm \frac{1}{2}\sqrt{b^2 + 4a}$. For all $a > 0$ and $b \geq 0$, there is one eigenvalue in the open right-half plane. Hence, the equilibrium point at (π,0) is unstable.

The Center-Manifold Theorem

When the Jacobian of an equilibrium point has one or more eigenvalues with zero real parts and all other eigenvalues with negative real parts, the stability of the equilibrium cannot be ascertained from linearization. In these cases, the local nonlinear nature of the system will dictate the stability of the equilibrium, and the center-manifold theorem allows one to determine precisely the nature of the nonlinear terms that determine the stability. The main idea behind this technique is that the critical behavior occurs in a low-dimensional invariant manifold of the state space, one with dimension equal to the number of eigenvalues with zero real parts. The stability in the other dimensions is dominated by the exponential behavior associated with the eigenvalues that have negative real parts, but the nonlinear coupling between the marginal and asymptotically stable modes can play a critical role in determining stability. The center-manifold theorem makes these ideas precise.

We begin with the following motivating example:

$$\dot{y} = zy - y^3, \quad \dot{z} = -z + ay^2 \tag{6}$$

Here, from the linearization point of view, the z dynamics are asymptotically stable and y is neutral. Based on this, one might be tempted to make the assumption that $z \to 0$, and therefore y is governed by $\dot{y} = -y^3$, and thus the origin is asymptotically stable. However, as is shown in an example below, this is incorrect; the stability of the origin is dictated by the sign of $(a - 1)$. The problem with the naive assumption made previously is that z approaches something small, but nonzero, and the correction, which stems from the nonlinear coupling terms and is captured by the center manifold, is crucial for determining stability.

The development of the center manifold technique begins with the autonomous system $\dot{x} = f(x)$, which has an equilibrium at $x = 0$. The Jacobian A is defined as before and the state equation is written as $\dot{x} = Ax + g(x)$, where $g(x) = f(x) - Ax$ contains terms that are essentially nonlinear about the origin. In order to split the dynamics into linearly asymptotically stable and neutral parts, the linear part of the equation is put into real Jordan form via a matrix P, as follows:

$$J = P^{-1}AP = \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix} \tag{7}$$

where all eigenvalues of A_1 have zero real parts and all eigenvalues of A_2 have negative real parts. The coordinate transformation

$$P\begin{bmatrix} y \\ z \end{bmatrix} = x$$

then puts the state equation into the split form

$$\dot{y} = A_1 y + g_1(y, z), \quad \dot{z} = A_2 z + g_2(y, z) \tag{8}$$

The z component of the dynamics of this system is dominated by the relatively fast linear system $\dot{z} = A_2 z$, whereas the y dynamics are slower than any exponential order. The key to the center-manifold technique is to capture the small, but crucial, coupling effects correctly in the nonlinear terms.

A *center manifold* for this system is simply a smooth invariant manifold of the form $z = h(y)$ with $h(0) = 0$ and $(\partial h/\partial y)(0) = 0$. Under some smoothness conditions on the nonlinear terms in Eq. (8) (that are inherited from the original equation), a local center manifold exists, although it is not in general unique. The power of the center manifold is that it can be used to reduce the dimensionality of the problem, as follows. By restricting the system dynamics to the center manifold one obtains $\dot{y} = A_1 y + g_1(y, h(y))$, referred to as the *reduced system.* The *center manifold theorem states that if the origin of the reduced system is asymptotically stable (unstable), then the origin of the original system $\dot{x} = f(x)$ is likewise asymptotically stable (unstable).*

The construction of the center manifold can be carried out as follows. We take the time derivative of $z = h(y)$ to obtain $\dot{z} = [\partial h(y)/\partial y]\dot{y}$. Equation (8) is used to substitute in for $\dot{z}$ and $\dot{y}$, and z is replaced everywhere by $h(y)$. This leads to $A_2 h(y) + g_2(y, h(y)) = [\partial h(y)/\partial y][A_1 y + g_1(y, h(y))]$. This equation for $h(y)$, which must satisfy the conditions $h(0) = 0$ and $\partial h/\partial y(0) = 0$, is generally impossible to solve. However, since only local information is needed for stability considerations, an approximation for $h(y)$ can be obtained by assuming a series expansion for $h(y)$, substituting it into the equation, and matching coefficients, as demonstrated in the forthcoming Example 5. Once the expansion form for $h(y)$ is determined and the expanded version of the reduced equation is in hand, various techniques can be employed for determining the stability of the reduced system. In general, this task is made much easier due to the lower dimensionality of the reduced system.

Example 5. Consider the system given in Eq. (6). Here $A_1 = 0$, $A_2 = -1$, $g_1(y,z) = yz - y^3$, and $g_2(y,z) = ay^2$. The center manifold is assumed to be of the form $h(y) = c_1 y^2 + c_2 y^3 + \cdots$. This is substituted into the equation for $h(y)$ and expanded in powers of y, and the coefficients of y^2, y^3, etc., are gathered and solved. This leads to the result that $c_1 = a$ and $c_2 = 0$. Therefore, $h(y) = ay^2 + O(|y|^4)$. [We use the notation $f(y) = O(|y|^p)$ when $|f(y)| \le k|y|^p$ for sufficiently small $|y|$.] The reduced system is given by taking the equation for $\dot{y}$ and simply replacing z by the expansion for $h(y)$, resulting in $\dot{y} = (a - 1)y^3 + O(|y|^5)$. Thus, the conclusion is reached that for $a - 1 < 0$, $x = 0$ is an asymptotically stable equilibrium point for system Eq. (6), while for $a - 1 > 0$ the origin is unstable. For $a - 1 = 0$ no conclusions regarding stability can be drawn without considering higher-order expansions.

Nonautonomous Systems

Suppose the origin $x = 0$ is an equilibrium point of the nonautonomous system $\dot{x} = f(t, x)$; that is, $f(t, 0) = 0$ for all $t \ge 0$. For nonautonomous systems we allow the Lyapunov-function candidate V to depend on t. Let $V(t, x)$ be a continuously differentiable function defined for all $t \ge 0$ and all $x \in D$. The derivative of V along the trajectories of $\dot{x} = f(t, x)$ is given by $\dot{V}(t, x) = \partial V/\partial t + (\partial V/\partial x) f(t, x)$. If there are positive-definite functions $W_1(x)$, $W_2(x)$, and $W_3(x)$ such that

$$W_1(x) \le V(t,x) \le W_2(x), \qquad \dot{V}(t,x) \le -W_3(x) \tag{9}$$

for all $t \ge 0$ and all $x \in D$, then the origin is uniformly asymptotically stable, where "uniformly" indicates that the ϵ–δ definition of stability and the convergence of $x(t)$ to zero are independent of the initial time t_0. Such uniformity annotation is not needed with autonomous systems since the solution of an autonomous state equation starting at time t_0 depends only on the difference $t - t_0$, which is not the case for nonautonomous systems. If inequalities in Eq. (9) hold globally and $W_1(x)$ is radially unbounded, then the origin is globally uniformly asymptotically stable.

Example 6. Consider the nonautonomous system

$$\dot{x}_1 = -x_1 - g(t)x_2, \qquad \dot{x}_2 = x_1 - x_2$$

where $g(t)$ is continuously differentiable and satisfies $0 \le g(t) \le k$ and $\dot{g} \le g(t)$ for all $t \ge 0$. The system has an equilibrium point at the origin. Consider a Lyapunov-function candidate $V(t, x) = x_1^2 + [1 + g(t)]x_2^2$. The function V satisfies the inequalities

$$x_1^2 + x_2^2 \le V(t,x) \le x_1^2 + (1+k)x_2^2$$

The derivative of V along the trajectories of the system is given by

$$\dot{V} = -2x_1^2 + 2x_1x_2 - [2 + 2g(t) - \dot{g}(t)]x_2^2$$

Using the bound on $\dot{g}(t)$, we have $2 + 2g(t) - \dot{g}(t) \ge 2 + 2g(t) - g(t) \ge 2$. Therefore

$$\dot{V} \le -2x_1^2 + 2x_1x_2 - 2x_2^2 = -x^{\mathrm{T}} \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} x = -x^{\mathrm{T}} Q x$$

The matrix Q is positive definite. Hence the origin is uniformly asymptotically stable. Since all inequalities are satisfied globally and $x_1^2 + x_2^2$ is radially unbounded, the origin is globally uniformly asymptotically stable.

BIFURCATION THEORY

The term *bifurcation,* strictly speaking, refers to the splitting of a whole into two parts. While this is relevant to its meaning in dynamical systems, it has taken on a much broader definition. The general bifurcation problem deals with qualitative changes in system behavior as parameters are varied in a quasistatic manner. The simplest example is the case when a parameter is varied in such a manner that the real part of an eigenvalue of the Jacobian at an equilibrium point changes sign, corresponding to the change in stability of the equilibrium. There are two generic ways in which such transitions can occur. The first is a real eigenvalue passing through zero; the second is a complex conjugate pair of eigenvalues passing through the imaginary axis. While linearization and the center-manifold theory allow one to determine the stability of the equilibrium point, the larger question looms as to what changes take place *near* the equilibrium through such a transition. Questions such as these are the basic motivation behind bifurcation theory.

Consider the system $\dot{x} = f(x, \lambda)$, where λ represents a system parameter. The technical definition of a bifurcation is as follows: *A bifurcation is said to occur at $\lambda = \lambda_0$ if the state space for $\lambda < \lambda_0$ is not topologically equivalent to that for $\lambda > \lambda_0$. It is said that λ_0 is the bifurcation value of the parameter.* The reason for emphasizing topological equivalence in the state space is that one is interested in transitions that cause *qualitative* changes in the structure of the system response. For example, a smooth increase in the amplitude of a limit cycle is not a bifurcation, but the disappearance of a limit cycle is.

A very important feature of bifurcation theory is that it allows for one to build a catalog of the generic qualitative changes systems may undergo as parameters are varied. For some simple, but widely encountered cases, this classification is complete. However, there are many important issues that remain unresolved. At the end of this section we will further address these classifications. The first convenient classification of bifurcations is to separate them into *local* and *global.* Local bifurcations describe changes in a neighborhood of an equilibrium, whereas global bifurcations involve nonlocal changes. (It is interesting to note that the study of local bifurcations in which two parameters are varied requires knowledge of global bifurcations.) We focus our attention on the analysis of local bifurcations and demonstrate some global bifurcations at the end of the discussion.

Center-manifold theory is a powerful tool for analyzing these local bifurcations. It, along with *normal-form theory* (described briefly later), allows one to reduce local bifurcations to their simplest possible forms, facilitating the classification mentioned above. To fix the ideas, consider the original system $\dot{x} = f(x, \lambda)$ augmented by the trivial dynamics of the parameter $\dot{\lambda} = 0$. Clearly the λ dynamics are linearly neutral and have a zero eigenvalue (the case of multiple parameters goes through in the same manner). The center-manifold theory described earlier is redeveloped by carrying along the equation $\dot{\lambda} = 0$. The result is that the slow dynamics now contain y and λ and the center manifold is of the form $z = h(y, \lambda)$. The procedure is carried out as before, with the key observation that λ is considered as a state variable, that is, terms such as λy are taken to be nonlinear. [The equation for $h(y, \lambda)$ simplifies to that of the original case since $\dot{\lambda} = 0$, with λ carried along in a straightforward manner.] This results in a reduced system describing the y dynamics that depends on the parameter, along with $\dot{\lambda} = 0$. The equations are valid for a range of λ values near the bifurcation value, and therefore this process allows for the *unfolding* of the behavior of the system about the bifurcation.

Example 7. Consider the following parameterized version of the system given in Eq. (6):

$$\dot{\lambda} = 0, \qquad \dot{y} = \lambda y + zy - y^3, \qquad \dot{z} = -z + ay^2 \tag{10}$$

Here $A_1 = 0$ (the 2×2 zero matrix), $A_2 = -1$, $g_1(\lambda, y, z) = (0, \lambda y + yz - y^3)^T$, and $g_2(\lambda, y, z) = ay^2$. Note that λ is treated as a state variable. The center manifold is assumed to be of the form $h(\lambda, y) = b_1 y^2 + b_2 \lambda y + b_3 \lambda^2 + \cdots$. This is substituted into the equation for $h(\lambda, y)$ and expanded in powers of y and λ and the coefficients of y^2, λy, λ^2, etc., are gathered and solved. This leads to the result that $b_1 = a$, $b_2 = 0$, and $b_3 = 0$. Therefore $h(\lambda, y) = ay^2 + \cdots$, and the reduced system is given by taking the equation for $\dot{y}$ and simply replacing z by the expansion for $h(\lambda, y)$, resulting in $\dot{y} = \lambda y + (a - 1)y^3 + \cdots$, which is valid in some neighborhood of $(\lambda, y) = (0,0)$.

Clearly, for $\lambda \neq 0$ the stability of the origin of the original system is dictated by the sign of λ; this is known from linearization. The center-manifold results confirm this, but offer much more. It is seen that the approximate reduced equation has equilibria at $y = 0$ and $\pm\sqrt{\lambda/(1 - a)}$. It is not difficult to show that the nonzero equilibria have the following properties: for $a > 1$ they exist for $\lambda < 0$ and are unstable when they exist, while for $a < 1$ they exist for $\lambda > 0$ and are asymptotically stable when they exist. Thus, for $a > 1$ there exist two unstable equilibria near the asymptotically stable origin for $\lambda < 0$, and they collapse onto the origin and disappear at $\lambda = 0$, leaving the origin unstable for $\lambda > 0$. This *subcritical pitchfork bifurcation* is shown in Fig. 3. The case for $a < 1$, a *supercritical pitchfork bifurcation,* is shown in the same figure, from which the reader can infer the dynamic behavior as λ is varied through zero.

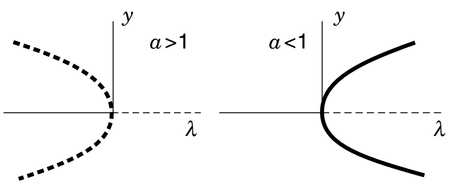

Figure 3. Subcritical pitchfork bifurcation (left); supercritical pitchfork bifurcation (right).

This example points out some important features of local bifurcations. First, bifurcations that involve n eigenvalues with zero real parts require analysis of an nth-order dynamical system, obtained by reduction to the parametrized center manifold. Second, one can easily extend the ideas to include more than one parameter. The selection of the proper parameters to uncover all critical behavior near a bifurcation is a subtle matter that involves techniques beyond the scope of this article. Third, the analysis of this example is very straightforward, as there is only one nonlinear term in the approximate reduced equation. In more complicated problems several nonlinear terms may occur, rendering the stability or bifurcation analysis much more difficult. In such cases one can use *normal-form theory* to simplify the problem. This technique, which shares many similarities with feedback linearization, involves a systematic sequence of coordinate changes that remove as many nonlinear terms as is possible for the given system. This technique, when the bifurcation parameters are included, allows one to systematically reduce entire classes of problems into generic forms. This produces the classification scheme described above. Here we offer the results of this classification for the simplest bifurcations. As an eigenvalue of an equilibrium passes through zero, there are three generic things that can occur. The most general is the *saddle-node* bifurcation, in which a pair of equilibria merge and annihilate one another. If all other eigenvalues of the equilibrium of interest are stable, then the bifurcation diagram as the parameter is varied is as shown in Fig. 4. If

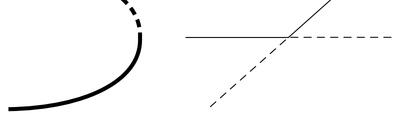

Figure 4. Saddle-node bifurcation (left); transcritical bifurcation (right).

the equilibrium persists through the instability, a *transcritical* bifurcation, shown in Fig. 4, generally occurs. When a special symmetry in the problem exists, the *pitchfork* bifurcation encountered in the preceding example can occur. These cover the generic bifurcations involving a single eigenvalue going through zero. When a complex conjugate pair of eigenvalues passes through the imaginary axis, something more interesting happens. First, the center-manifold reduction for this problem leads to a dynamical system consisting of two equations. When expressed in polar coordinates these are $\dot{r} = \lambda r + ar^3 + \cdots$ and $\dot{\theta} = \omega + br^2 + \cdots$. The radial variable, representing the amplitude of the oscillation, undergoes a pitchfork bifurcation (although $r < 0$ is meaningless) that is super- ($a < 0$) or sub- ($a > 0$) critical as λ is increased through zero. The angular variable simply describes rotation at a nominal frequency of ω with a small amplitude-dependent shift arising from nonlinear effects. This is the *Hopf bifurcation.* The result of this bifurcation is shown in Fig. 5, and it results in the birth of stable limit cycles as an asymptotically stable equilibrium goes unstable (for $a < 0$) or in the merging of an asymptotically stable equilibrium with an unstable limit cycle (for $a > 0$). This completes the list of bifurcations that generically occur as a single parameter is varied and an equilibrium point changes stability.

Center-manifold and normal-form methods also exist for analyzing the bifurcations that occur when limit cycles change stability. These problems can be handled by defining and using a *Poincaré map* near the limit cycle and studying the generic bifurcations of fixed points of maps. It is found that limit cycles can undergo saddle-node, transcritical, and pitchfork bifurcations that are essentially the same as shown in Figs. 3 and 4. The analogy to the Hopf bifurcation, the *Neimark-Sacker* bifurcation, involves a limit cycle changing stability and merging with a two-dimensional torus around it (in a super- or subcritical manner). However, this situation is complicated by the fact that resonances can occur between the two frequencies of the torus, and this can lead to secondary nonlinear phenomena. In addition, limit cycles have an additional bifurcation, the *period-doubling* bifurcation, in which a limit cycle changes stability and merges with another limit cycle that has a period twice that of the original. Again, this can be super- or subcritical in nature.

Before leaving the topic of local bifurcations it should be noted that if several parameters are being simultaneously varied, one can encounter situations in which several eigenvalues simultaneously have zero real parts. Similarly, one may find that some critical terms in a given normal form change sign as parameters are varied (for example, the parameter a in the Hopf bifurcation). The bifurcation analysis of such problems is extremely rich and often exceedingly complicated, but some classifications along these lines have been carried out for two and three parameter systems.

Global bifurcations result in qualitative changes in the state space that cannot be described as local to any equilibrium. Perhaps the simplest interesting example of a global bifurcation is the saddle connection shown in Fig. 6. Here, as a parameter is varied in the two-dimensional state space, a stable limit cycle moves toward and eventually merges with a saddle point, forming a saddle loop. As the parameter is pushed beyond this point, the limit cycle disappears. Note that this occurs without any changes to the stability types of the saddle point or the limit cycle. Global bifurcations involving saddle connections in systems of dimension three and higher are the source of chaos in dynamical systems.

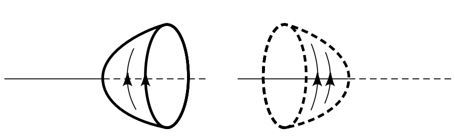

Figure 5. Hopf bifurcation.

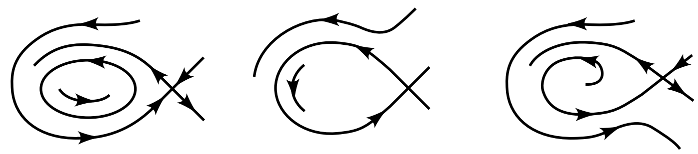

Figure 6. Saddle connection global bifurcation.

FURTHER READING

For further reading on the Lyapunov stability and its applications, we refer the reader to Refs. 1 and 2. As for the steady-state behavior of dynamical systems, chaos, and bifurcations, we recommend Refs. 3–5.

BIBLIOGRAPHY

1. H. K. Khalil, *Nonlinear Systems,* 2nd ed., Upper Saddle River, NJ: Prentice Hall, 1996.
2. M. Vidyasagar, *Nonlinear Systems Analysis,* 2nd ed., Englewood Cliffs, NJ: Prentice Hall, 1993.
3. S. H. Strogatz, *Nonlinear Dynamics and Chaos,* Reading, MA: Addison-Wesley, 1994.
4. J. Guckenheimer and P. Holmes, *Nonlinear Oscillations, Dynamical Systems, and Bifurcations of Vector Fields,* 2nd ed., New York: Springer Verlag, 1986.
5. D. K. Arrowsmith and C. M. Place, *An Introduction to Dynamical Systems,* Cambridge, UK: Cambridge University Press, 1990.

HASSAN K. KHALIL
STEVEN W. SHAW
Michigan State University

STABILIZATION. See BILINEAR SYSTEMS.