ROBUST CONTROL

The role of feedback in automatic control systems is to exploit evolving real-time measurements to obtain more precise control over system behavior. A robust control system is one that performs within specified tolerances despite uncertain variations in the controlled plant within given bounds. Robust control theory is that branch of mathematical system theory concerned with the design and analysis of robust feedback control systems.

The block diagram in Fig. 1 represents a typical feedback control system consisting of an "uncertain plant" and a "controller." The plant might be an aircraft or missile, a robotic lunar surface explorer, a chemical process, an automobile engine, or a nuclear power plant. It might be something as small as the magnetic read-head positioning system in a computer disk drive or as large as the global economy. It could be almost any complex dynamic system. The controller, on the other hand, is typically a computer or a microprocessor, though it could be simply the thermostat in a home heating control system or the mechanical linkage in the eighteenth-century flyball governor invented by James Watt for controlling steam engines. From a control theorist's perspective the uncertain plant and the controller are simply mathematical relationships, for example, differential equations. Robust control theory is focused on the quantitative analysis of the consequences of plant uncertainty. The need for robust control arises when a control system design must be based on an inexact mathematical model of the true physical plant.

ORIGINS OF ROBUST CONTROL THEORY

Since Watt's invention of the flyball governor in 1788, designers of feedback control systems have implicitly sought robust designs. Robustness considerations are implicit in the classical feedback design methods based on root locus and frequency responses. But prior to 1960, the issue of robustness had not been generally recognized by mathematical control theorists, and the term robustness itself did not appear in the literature of mathematical control theory before 1975. Apparently the first clear mathematical formulation of a robust feedback control problem was produced by I. Horowitz (6) in the early 1960s. Horowitz studied simple linear time-invariant feedback control systems with a single feedback loop for plant models with several uncertain parameters. He correctly recognized that plant uncertainty is a dominant factor in determining what can and cannot be achieved with feedback control, at least for simple single-loop feedback systems, and he showed how to design uncertainty-tolerant controllers using classical root-locus and frequency-response methods.

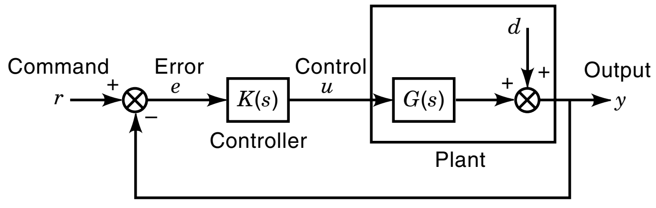


Figure 1. Robust control concerns designing feedback controllers so that the output $y(t)$ will precisely track command signals r even when the plant is uncertain.

Unfortunately, the significance of Horowitz's contribution was not immediately noticed by control researchers. A gap between mathematical control theory and control engineering practice emerged in the 1960s and early 1970s, as the mathematical theory of feedback control increasingly decoupled from engineering practice. Time-tested root-locus and frequency-response techniques were regarded by researchers as ad hoc and simplistic. They sought to pose feedback design as a mathematical optimization. But, explicit mathematical representations of robustness issues were not incorporated in their mathematical representations of the problems of feedback control. In those early days, mathematical control research generally presumed that available mathematical models were "sufficiently accurate." The profound significance of this omission was not immediately evident.

In the 1960s, mathematical proofs of "optimality" were widely regarded as sufficient evidence that the emergent, but as yet untried, modern mathematical theories would produce superior results. But in the early 1970s, researchers began to be jolted as early attempts to apply mathematical optimization theories to the design of complex 'multivariable' feedback controllers for military aircraft and submarines resulted in some surprising failures. In one unpublished design study carried out at Systems Control Inc. under the supervision of D. Kleinman with MIT's Michael Athans as consultant, a linear quadratic Gaussian (LQG) controller for a Trident submarine caused the vessel to unexpectedly surface in nonlinear simulations involving moderately rough seas. In another example, a 1977 journal paper describing the disappointing results of an LQG control design study for the F-8C Crusader aircraft concluded euphemistically with the observation that "The study has pinpointed certain theoretical weaknesses . . . as well as the need for using common sense pragmatic techniques to modify the design based on 'pure' theory." A lack of attention to robustness was quickly identified as the dominant factor in these failures.

In 1976, the modern field of robust control theory was born. The term robustness was introduced into the control theory lexicon in papers by E. J. Davison and by M. Safonov and M. Athans. Mathematical control theorists in Michael Athans' MIT laboratory began to refocus their attention on methods for analyzing and optimizing the robustness of feedback control systems. Researchers sought to salvage as much as possible of the mathematical theory developed in the 1960s. They worked to link linear optimal control theory to classical root-locus/pole-placement methods. In 1978 researcher Gunter Stein summarized the mood of the time, explaining that his goal was "to make modern control theoretical methods work." Researchers were beginning to reexamine classical frequency-response methods and to take a closer look at Horowitz's work on uncertainty-tolerant single-loop feedback design. They were seeking multivariable generalizations of Horowitz's ideas that would mesh well with the sophisticated mathematics of modern optimal control theory.

Though attempts at direct generalizations of Horowitz's ideas to more complex 'multivariable feedback' systems with several feedback control loops proved unfruitful, mathematical control researchers quickly discovered that multivariable robustness analysis methods could be developed based on the so-called *small-gain theorem* of nonlinear stability theory.

Along with the Lyapunov, Zames–Sandberg and other nonlinear stability theories, this now became the core of the emergent mathematical theory of robust control. Prior to 1976, the Lyapunov and Zames–Sandberg theories were generally regarded as esoteric tools for nonlinear stability analysis. Despite some suggestive early remarks by stability theory pioneers G. Zames and V. Popov, by 1975, the robustness implications of nonlinear stability theories had yet to be developed. Today, many still regard the mid 1976 change of focus of mathematical control theory from optimal control to robustness as a revolutionary paradigm shift, though, unlike many scientific revolutions, the shift occurred with remarkably little controversy.

In 1977 Safonov showed that the general problem of robust stability analysis is equivalent to computing a topological separation of graphs of feedback operators. Lyapunov, conic sector, positivity small-gain and other nonlinear stability theories emerge as special cases. Other researchers, such as J. C. Doyle, G. Stein and N. R. Sandell exploited H. H. Rosenbrock's multivariable version of the Nyquist stability criterion to develop a simplified, and hence more widely accessible, 'linear' explanation of multivariable robustness criteria, singular-value (σ-plots), and structured singular-value (μ-plots) frequency-response plots.

By 1981, the concept of singular-value, loop shaping and the closely related concept of mixed sensitivity became central to understanding robust control. For a multivariable control system with loop transfer function matrix $L(s)$, design specifications were posed in terms of the "sensitivity matrix" $S(s) = [I + L(s)]^{-1}$ and the complementary sensitivity matrix $T(s) = L(s)[I + L(s)]^{-1}$. From these definitions, it follows that there is the fundamental constraint

$$S(s) + T(s) = I \tag{1}$$

whence there is a fundamental tradeoff between sensitivity $S(s)$ and complementary sensitivity $T(s)$.

In the mixed-sensitivity framework, the concept of multiplicative uncertainty Δ_M became central. Given a "true" plant $G(s)$ and a "nominal" plant model G_0, a multiplicative uncertainty Δ_M is defined by

$$G(s) = [I + \Delta_M(s)]G_0(s)$$

When knowledge of plant uncertainty is limited to bounds on the size of the multiplicative uncertainty matrix Δ_M, good control performance results when the size of $S(s)$ is smaller than unity over a specified control bandwidth, say $\omega < \omega_B$, with $s = j\omega$. The singular values of a matrix are a particularly useful measure of its "size."

To this day in the majority practical robust control designs, performance is specified in terms of S and T. Singular values of $S(j\omega)$ are constrained to be small compared to unity over that desired control bandwidth ($\omega < \omega_B$). And, singular values of $T(j\omega)$ are constrained to be smaller than unity at higher frequencies (beyond ω_B) at which multiplicative uncertainties are expected to be large because of the effects of small unmodeled time delays, phase lags, and other sorts of "parasitic dynamics."

Optimal H_2, H_∞, and μ-synthesis are very flexible robust control theories introduced in the 1980s. They are currently the principal methods employed for designing robust, multivariable, feedback control systems. Though each of these methods has the flexibility to optimize robustness in a very general setting, their most common use is in conjunction with weighted, mixed-sensitivity, performance criteria where they are used to find a controller $K(s)$ for a plant $G(s)$ so as to minimize the "size" of the following transfer function matrix:

$$\begin{bmatrix} W_1(s)S(s) \\ W_2(s)K(s)S(s) \\ W_3(s)T(s) \end{bmatrix}$$

Here W_1, W_2, W_3 are frequency-dependent weights specified by the designer, $L(s) = G(s)K(s)$ is the loop transfer function, $K(s)$ is the controller, and $G(s)$ is the plant to be controlled.

SOURCES OF UNCERTAINTY

Uncertainty is always present to some degree in mathematical models of physical systems. It arises from ignorance of physical details, from approximations, or from unpredictable external effects, such as noise or weather changes.

Models derived from idealized physical laws and theoretical principles are uncertain. This includes, for instance, Newton's force law, the second law of thermodynamics, Kirchhoff's current law, Boyle's ideal gas law, Maxwell's electromagnetic equations, and the law of supply and demand from economic theory. Though some laws are very good predictors of behavior, all such laws are idealizations with fundamental limits of accuracy. Moreover, applications of such laws and principles usually entail assumptions that further degrade the model's accuracy of prediction.

Approximations produce modeling uncertainty. Linearized models are uncertain, and neglected "nonlinear distortion" becomes uncertainty. Engineers designing stability augmentation systems for aircraft, robot manipulator arms, or similar mechanical systems employ a "rigid-body approximation" neglecting flexible bending modes, thereby introducing uncertainty into the mathematical models of these systems. Deliberate truncation of modes to simplify a model is called "model reduction." Uncertainty arises even when all flexible modes are included in models, because the numerical values in the equations for all modes are seldom known with precision.

Mathematical models derived from experimental data are also uncertain. Knowledge of the past is uncertain and predictions of the future are always uncertain. The approximate curve-fitting processes implicit in most algorithms for system identification mean that experimentally identified models usually can not exactly reproduce even the past data. And even when they do, forecasts of future behavior derived from such models are still never guaranteed to be accurate.

STABILITY AND ROBUSTNESS

It turns out that the effects of even very small model uncertainties are amplified many times by a feedback system. Thus, it is vital that control engineers quantify the tolerance of their designs to such uncertainties. Because modeling uncertainty is usually a dominant factor in determining how precisely it is possible to control a plant's behavior, an engineer wishing to optimize control performance must quantify the intrinsic tradeoff between uncertainty tolerance and control precision, that is the intrinsic tradeoff between robustness and performance.

Examples of extreme sensitivity to small effects abound in nature. For example, a ball balanced on the top of a hill is unstable. Left undisturbed, it would, in principle, remain forever at rest on the mountain peak. Yet, a disturbance smaller than the flapping of a butterfly wing causes the ball to plummet. Such extreme sensitivity to small disturbances is called *instability.* Unstable systems are never robust.

On the other hand, a ball at rest in the cusp of a valley is stable and is not dislodged even by a major earthquake. Further, so long as gravity remains an attractive force, then balls at rest in valleys remain stably at rest. Not only is the ball stable, but it remains stable for any large nonnegative variation in the gravitational constant. A ball in a valley is said to possess the property of *stability robustness.*

But even the property of stability itself can be extremely sensitive to small effects. If one had a knob which could reduce the Earth's gravitational attraction until gravitational attraction became negative, then balls in valleys would no longer have stability robustness. As the pointer on the knob approached zero, the property of stability itself would be extremely sensitive to the position of the knob. With the knob set a little bit above zero, a ball in a valley would remain stably at rest. But with the pointer on the knob ever so slightly below zero, attraction would become repulsion, and stability robustness would no longer hold. The ball would be launched into space in response to the slightest change in the knob's setting. The property of stability for a ball in a valley is thus not robust to negative variations in the gravitational constant.

Of course, although there is little prospect that gravity will reverse itself, there are many situations encountered by engineers in which stability is not robust to variations in physical parameters.

Robustness in Classical Single-Loop Control

Although the term *robustness* was not generally used in control theory until the 1970s, robustness has always concerned feedback control system designers. For example, consider the flyball governor system invented by James Watt in 1788 to regulate the rotational velocity of a steam engine. A sketch of Watt's invention is shown in Fig. 2. The flyball governor device is coupled to the main shaft of a steam engine via a pulley. It maintains a set value for the rotational velocity of the shaft by opening the steam valve when velocity decreases and closing it when it increases.

A simplified linear model for a steam engine feedback system incorporating a flyball governor is given by the following differential equations:

$$\text{Steam engine: } \frac{d^2}{dt^2}y(t) + 2\frac{d}{dt}y(t) + y(t) = K_s u(t)$$
$$\text{Flyball governor: } \frac{d}{dt}u(t) + u(t) = K_g(r(t) - y(t))$$

Here $y(t)$ denotes the angular velocity of the steam engine's output shaft at time t, $u(t)$ indicates that the steam valve position at time t, and $r(t)$ denotes an externally specified desired shaft velocity. The constants K_g and K_s are constants

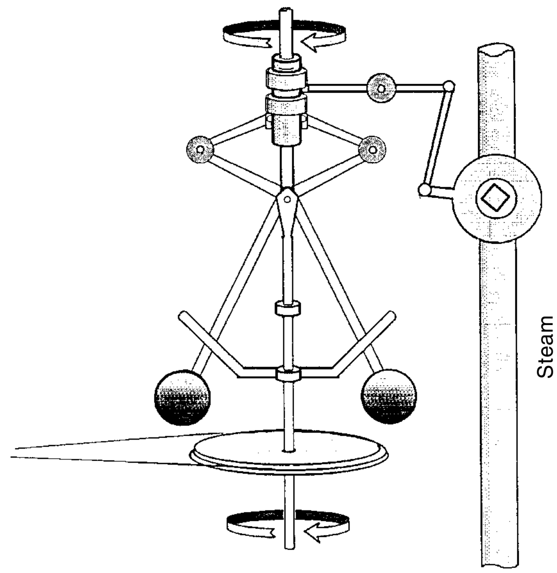


Figure 2. An early example of a feedback control system is the Watt flyball governor. The device enhances the robustness of steam plants against engine load variation and steam pressure changes.

corresponding, respectively, to the position of the fulcrum in the mechanical linkage between the flyball governor and the steam valve and the physical properties of the steam valve. The equilibrium solution of these equations (obtained by setting all the d/dt and d^2/dt^2 terms to zero) is $y(t) = K/(1 + K)r$ where $K = K_g K_s$. Hence, if K is much larger than one, then $y(t)$ is very nearly equal to its desired value r, assuming that equilibrium is achieved.

A more detailed analysis of the differential equations that model the flyball governor shows that they have a time-varying solution of the form

$$y(t) = \frac{K}{1+K}r + c_1 e^{\lambda t} + c_2 e^{\sigma t}\sin(\omega t + \phi)$$

where $\lambda = -1 - K^{1/3}$, $\sigma = -1 + 0.5K^{1/3}$, $\omega = 0.5\sqrt{3}\,K^{1/3}$, and the three numbers c_1, c_2, ϕ are determined by the initial conditions $(d/dt)y(0)$, $y(0)$, and $u(0)$. When $-1 \le K < 8$, the time-varying terms $c_1 e^{\lambda t}$ and $c_2 e^{\sigma t}\sin(\omega t + \phi)$ both decay in magnitude to zero as time t increases toward infinity, so that equilibrium is approached asymptotically. However, if $K > 8$, then $\sigma > 0$, and the $e^{\sigma t}$ term becomes increasingly large, predicting that the steam engine's velocity $y(t)$ alternately increases and decreases with ever growing amplitude. For example, if the steam valve has gain constant $K_s = 1$, then the mathematics predicts that the flyball governor system is unstable for governor gain $K_g > 8$. A governor gain of $K_g = 7$ would be fine. But, if the true value of the engine gain K_s were uncertain and known to within only $\pm 33\%$, then one might have $K_s = 1.33$ in which case the model predicts that the flyball governor system is unstable for governor gain $K_g > 8/1.33 = 6$. Whence, a value of governor gain K_g offers insufficient robustness to accommodate a $\pm 33\%$ variation in steam engine gain about a nominal value of $K_s = 1$.

MULTIVARIABLE ROBUSTNESS

In the 1960s and 1970s as increasingly complex control systems with multiple feedback loops were deployed, engineers typically evaluated the robustness of these systems against variations *one-loop-at-at-a-time.* That is, they examined the effects of varying each feedback while holding all the other gains fixed at their nominal values. This proved unwise, because a system may tolerate large, even infinite, variations in individual feedback gains, yet may be destabilized by even very small simultaneous variations in several feedback gains. The term *multivariable robustness* refers to the tolerance of feedback systems to such simultaneous variations.

As early as 1971, H. H. Rosenbrock and M. Athans urged greater attention to the issue of simultaneous variations in several feedback gains. And Rosenbrock even developed crude methods for designing robust controllers for such systems, provided that the loops were approximately decoupled, a condition that Rosenbrock called diagonal dominance. But, for many, the telling example that made the case for multivariable robustness theory was the following very simple example produced by J. C. Doyle in 1978.

Multivariable Robustness

Consider a two-loop multivariable feedback system with nominal loop transfer function

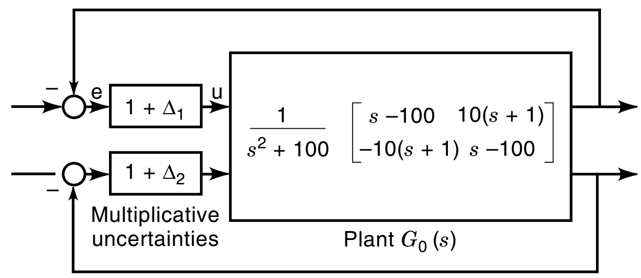
$$G_0(s) = \frac{1}{s^2+100}\begin{bmatrix} s-100 & 10(s+1) \\ -10(s+1) & s-100 \end{bmatrix}$$

If one closes either one of the two feedback loops around this system, the open-loop transfer function in the other loop is simply given by

$$g(s) = \frac{1}{s}$$

whence the system remains stable if the gain in the other loop assumes any nonnegative value. Moreover, this implies that gain margin is infinite and the phase margin is $\pm 90°$ in each loop, when evaluated one-loop-at-a-time. The system seems to be extraordinarily robust when examined one-loop-at-a-time.

However, this is not the case when gains in both loops are varied simultaneously, as the stability diagram in Fig. 3 indicates.



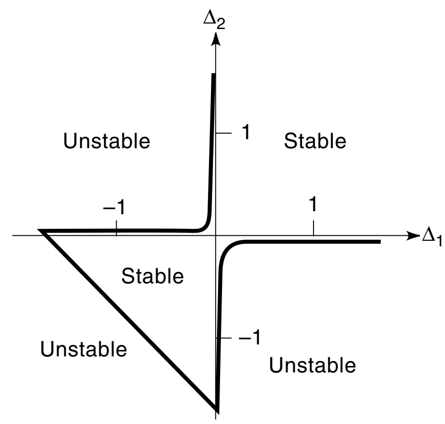


Figure 3. Doyle's 1978 two-loop feedback example shows that one-loop-at-a-time stability robustness analysis is unwise. In this case, the feedback system remains stable for large variations in either of the two gain perturbations Δ_1 and Δ_2 when the other is fixed to be zero, but is unstable for small simultaneous variations.

Uncertainty

An important aspect of robust control theory is quantifying the difference between two transfer function matrices, say $G(s)$ and $G_0(s)$. Typically, $G_0(s)$ is a nominal mathematical model, and $G(s)$ is the corresponding "true system" and the difference is called "uncertainty." The uncertainty may be viewed as additive or multiplicative, or it may be regarded in terms of more exotic linear fractional transformation (LFT) or normalized coprime matrix fraction description (MFD) transformations. Useful quantitative measures of size are singular values, H_∞ norms, and the gap metric.

Additive and Multiplicative Uncertainty. An *additive uncertainty* is a matrix, say $\Delta_A(s)$, which produces the true system $G(s)$ when added the nominal model G_0, that is

$$G(s) = G_0(s) + \Delta_A(s)$$

A *multiplicative uncertainty* is a matrix $\Delta_M(s)$ such that the true and nominal are obtained by multiplying the model $G_0(s)$ by $[I + \Delta_M(s)]$. Because matrix multiplication is not generally commutative, the "right" multiplicative uncertainty [say $\Delta_{Mr}(s)$] and "left" multiplicative uncertainty [say $\Delta_{Ml}(s)$] are generally different, that is,

$$G(s) = G_0(s)[I + \Delta_{Mr}(s)] = [I + \Delta_{Ml}(s)]G(s)$$

does not imply that $\Delta_{Mr}(s)$ equals $\Delta_{Ml}(s)$. Feedback control systems with additive and multiplicative uncertainties are depicted in Fig. 4.

LFT Uncertainty. LFT is an acronym standing for linear fractional transformation. In the case of LFT uncertainty, the "true plant" is presumed to be related to a perturbational matrix Δ via a relationship of the form

$$G(s) = P_{22} + P_{21}\Delta(I - P_{22}\Delta)^{-1}P_{12}. \tag{2}$$

where

$$P(s) = \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix} \tag{3}$$

is a given transfer function matrix. The LFT transformation Eq. (2) is also sometimes called the Redheffer *star product*

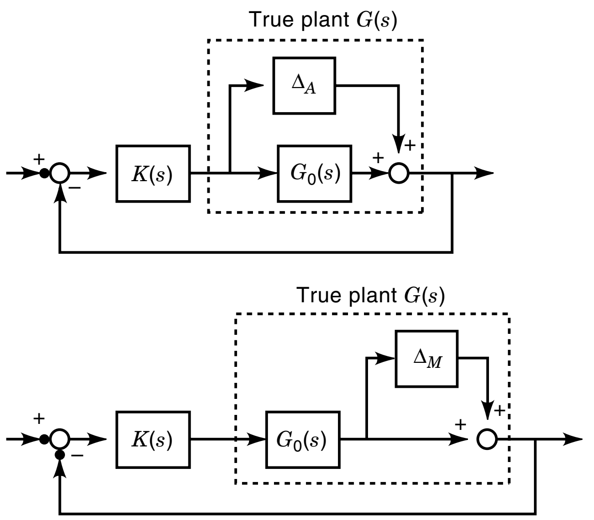


Figure 4. Inaccuracies in a plant model $G(s)$ can be represented as additive uncertainty Δ_A or multiplicative uncertainty Δ_M.

and denoted

$$G(s) \triangleq \Delta \star P$$

MFD Uncertainty and the Gap Metric. Every real transfer function matrix $G(s)$ can be decomposed into a ratio

$$G(s) = N(s)D^{-1}(s)$$

where $N(s)$ and $D(s)$ are both stable transfer function matrices. Such a decomposition is called a stable right matrix fraction description (MFD) of $G(s)$.

Stable MFD's are not unique. For example, another stable MFD is given by

$$\begin{bmatrix} \tilde{N}(s) \\ \tilde{D}(s) \end{bmatrix} = \begin{bmatrix} N(s) \\ D(s) \end{bmatrix} Q(s) \quad (4)$$

where $Q(s)$ is any stable invertible transfer function matrix. A stable right MFD $N(s)$, $D(s)$ of $G(s)$ is said to be *coprime* if every other stable MFD of $G(s)$ can be written in the form of Eq. (4) for some stable $Q(s)$. A right MFD is said to be *normalized* if the matrix

$$U(s) = \begin{bmatrix} N(s) \\ D(s) \end{bmatrix}$$

has the all-pass property $U'(-s)U(s) = I$.

MFDs play a role in some more exotic quantitative measures of uncertainty used in robust control theory. An example is the gap metric $\delta(G, G_0)$ which, roughly speaking, is the sine of the angle between the graphs of $G(s)$ and $G_0(s)$. Formally, the gap is computed via solution of a certain H_∞ optimization problem, namely

$$\delta(G(s), G_0(s)) = \min_{Q(s)\ \text{stable}} \left\| \begin{bmatrix} N(s) \\ D(s) \end{bmatrix} - \begin{bmatrix} N_0(s) \\ D_0(s) \end{bmatrix} Q(s) \right\|_\infty$$

where $[N_0(s), D_0(s)]$ and $[N(s), D(s)]$ are stable normalized coprime left MFDs of $G_0(s)$ and $G(s)$, respectively. The gap metric has attracted much interest from theorists because it has a rich mathematical structure with many subtle symmetries and geometric interpretations.

Singular Values and H_∞ Norms. Given a true plant $G(s)$ and a nominal plant $G_0(s)$, it is useful to have a quantitative measure of the "size" of the difference between the two. In robust control theory, singular values and H_∞ norms provide the desired measures of size.

An $n \times m$ matrix, say A, is a mapping of m-vectors x into n-vectors Ax. Likewise, a transfer function matrix, say $A(s)$, is mapping of signal vectors $x(s)$ into signal vectors $A(s)x(s)$. Singular values and H_∞-norms indicate the "size" or, more precisely, the gain of such mappings.

Every complex $n \times m$ matrix A with rank r has a singular-value decomposition (SVD)

$$A = U\Sigma V^*$$

where U and V are square unitary matrices

$$\Sigma = \begin{bmatrix} \tilde{\Sigma} & 0 \\ 0 & 0 \end{bmatrix}$$

and $\tilde{\Sigma}$ is an $r \times r$ diagonal matrix

$$\tilde{\Sigma} = \begin{bmatrix} \sigma_1 & 0 & 0 & \cdots & 0 \\ 0 & \sigma_2 & 0 & \cdots & 0 \\ \vdots & & \ddots & & \vdots \\ 0 & 0 & 0 & \cdots & \sigma_r \end{bmatrix}$$

with $\sigma_1 \geq \sigma_2 \geq \ldots \geq \sigma_r > 0$. The positive numbers $\sigma_1, \sigma_2, \ldots, \sigma_r$ are called the *singular values* of A. The orthogonal columns $\{u_1, \ldots, u_n\}$. $\{v_1, \ldots, v_m\}$ of U and V are called the *singular vectors.* The SVD of A is sometimes also written in terms of the singular vectors as

$$A = \sum_{i=1}^{r} \sigma_i u_i v_i^*$$

The largest singular value of A is also a measure of the vector gain of the matrix A, that is

$$\sigma_1(A) = \max_x \frac{\|Ax\|}{\|x\|}$$

where $\|x\|$ denotes the Euclidean norm

$$x \triangleq \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}$$

The H_∞ norm of a stable transfer function matrix, say $P(s)$, is defined in terms of the largest singular value of its frequency response

$$\|P\|_\infty \triangleq \sup_\omega \sigma_1[P(j\omega)]$$

For an unstable transfer function $P(s)$, the H_∞-norm by definition is infinite, that is, whenever $P(s)$ has unstable poles $\|P\|_\infty \triangleq \infty$.

The following are other useful properties of singular values:

1. $\sigma_1(A + B) \leq \sigma_1(A) + \sigma_1(B)$.
2. $\sigma_1(AB) \leq \sigma_1(A)\sigma_1(B)$.
3. $\sigma_1(A) = 1/\sigma_n(A^{-1})$ for A an $n \times n$ invertible matrix.
4. $\text{Trace}(A^*A) = \sum_{i=1}^{r} \sigma_i^2(A)$.
5. $\sigma_1(A_{ij}) \leq \sigma_1(A)$ for any submatrix A_{ij} of A.
6. $$\max\{\sigma_1(A), \sigma_1(B)\} \leq \sigma_1\left(\begin{bmatrix} A \\ B \end{bmatrix}\right) \leq \sqrt{2}\max\{\sigma_1(A), \sigma_1(B)\} \tag{5}$$

Canonical Robust Control Problem

The canonical robust control problem concerns systems of the general form depicted in Fig. 5. The nth fictitious uncertainty is not an actual uncertain gain. This fictitious uncertainty of size $\|\Delta_n\|_\infty \leq 1$ is wrapped as a purely imaginary loop from a disturbance input d to regulated error output signal e to represent the performance specification that the gain of the transfer function matrix from d to e should be less than one, that is,

$$\sigma_1[T_{ed}(j\omega)] < 1 \quad \text{for all } \omega$$

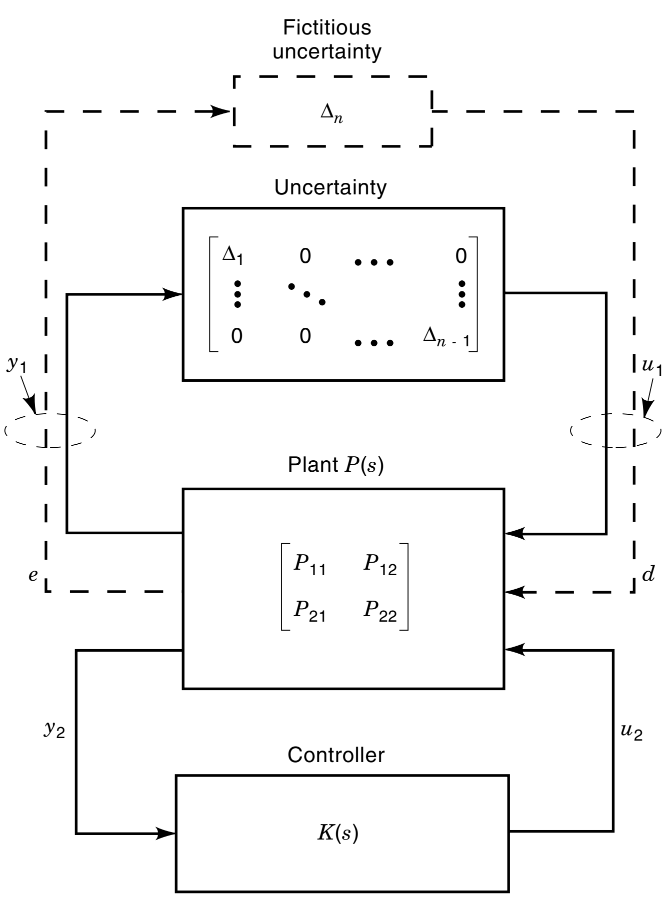


Figure 5. The canonical robust control problem represents a wide spectrum of uncertain control design problems. It accommodates robust performance requirements via a fictitious uncertainty Δ_n.

According to an interpretation of the small-gain theorem known as the performance robustness theorem, robustness of stability for all n uncertainties $\|\Delta_i\|_\infty \leq 1$ ($i = 1, \ldots, n$) is equivalent to performance robustness in the absence of the fictitious uncertainty Δ_n.

The canonical robust control problem is described mathematically as follows. Given a multi-input, multi-output (MIMO) plant transfer function matrix $P(s)$, find a stabilizing controller $F(s)$ so that the closed-loop transfer function matrix remains stable for all diagonal uncertainty matrices

$$\Delta = \begin{bmatrix} \Delta_1 & 0 & 0 & \cdots & 0 \\ 0 & \Delta_2 & 0 & \cdots & 0 \\ \vdots & & \ddots & & \vdots \\ 0 & 0 & 0 & \cdots & \Delta_n \end{bmatrix}$$

with $\|\Delta\|_\infty \leq 1$ and, perhaps, subject to an additional constraint $\Delta \in \mathsf{D}$ where D is a specified subset of the set of diagonal matrices with H_∞-norm less than one.

Closely related to the canonical robust control problem are the concept of the "multivariable stability margin" and the "structured singular value." Consider the system in Fig. 5. Denote by $T_{y_1u_1}(s)$ the closed-loop transfer function from u_1 to y_1 with the uncertainty Δ removed. Loosely, $T_{y_1u_1}(s)$ is the closed-loop transfer function that the diagonal uncertainty Δ sees as it looks back into the rest of the system. Given a stable transfer function matrix $T_{y_1u_1}(s)$, the "multivariable stability margin" $K_m(T_{y_1u_1})$ associated with $T_{y_1u_1}(s)$, by definition, is the "size" of the smallest destabilizing $\Delta \in \mathsf{D}$. The "structured singular value" $\mu[T_{y_1u_1}(s)]$, by definition, is the reciprocal of the multivariable stability margin. More formally, for any stable transfer function $T_{y_1u_1}(j\omega)$,

$$\begin{aligned} \frac{1}{\mu[T_{y_1u_1}(j\omega)]} &\triangleq K_m[T_{y_1u_1}(j\omega)] \\ &\triangleq \inf_{\Delta \in \mathsf{D}} \{\sigma_1(\Delta) \mid \det[I - T_{y_1u_1}(j\omega)\Delta] = 0\} \end{aligned}$$

Sigma Plots and Mu Plots

A well-known stability result called the *small-gain theorem* says that, if the loop gain of a feedback system is less than one, then it is stable. For linear time-invariant systems, the H_∞ norm of the system transfer function is the relevant measure of gain. Hence, for the canonical robust control problem, a sufficient condition for closed-loop stability is that $\|\Delta\|_\infty\|T_{y_1u_1}\|_\infty < 1$. Now, the uncertainty Δ has, by hypothesis, $\|\Delta\|_\infty \leq 1$. Hence, by the small-gain theorem, a sufficient condition closed-loop stability is that $\|T_{y_1u_1}\|_\infty < 1$ or, equivalently,

$$\sigma_1[T_{y_1u_1}(j\omega)] < 1 \text{ for all } \omega$$

The problem of choosing the feedback $K(s)$ so that the foregoing H_∞ inequality holds is known as the H_∞ *control problem.*

To gain a sense of the tolerance of uncertainty at each frequency, control engineers plot singular value Bode plots, such as in Fig. 6. The number $1/\sigma_1[T_{y_1u_1}(j\omega)]$ is interpreted as a lower bound on the size $K_m(j\omega)$ of the smallest destabilizing uncertainty $\Delta(j\omega)$. Hence, singular values provide a convenient upper bound on the structured singular value, namely,

$$\mu[T_{y_1u_1}(j\omega)] \leq \sigma_1[T_{y_1u_1}(j\omega)] \tag{6}$$

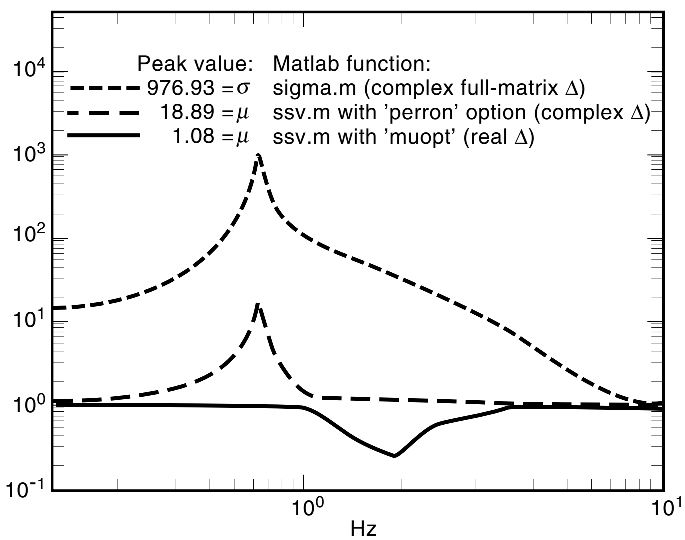


Figure 6. Singular value Bode plots are widely used to evaluate robustness, but other less-conservative upper-bounds on $\mu(T_{y_1u_1})$ are better when the uncertainty Δ is structured. Shown here are Bode plots of several upper-bounds computed using the Matlab Robust Control Toolbox functions for a particular $T_{y_1u_1}$.

The bound is conservative in the sense that generally the inequality is strict. In the special case in which the uncertainty Δ is *unstructured,* however, then $\mu[T_{y_1u_1}(j\omega)] = \sigma_1]T_{y_1u_1}(j\omega)]$. An uncertain matrix Δ is said to be unstructured if it is a full (not a diagonal) matrix about which nothing else is known (so the set D is the set of all complex matrices).

From the definition of μ, it follows that a necessary and sufficient condition for the linear time-invariant in Fig. 5 to be stable is given by

$$\sigma_1[\Delta(j\omega)]\mu[T_{y_1u_1}(j\omega)] < 1 \text{ for all } \omega$$

If one 'defines' σ_1 and μ as the gains of Δ and $T_{y_1u_1}$, respectively, then this constitutes a sort of nonconservative version of the loop-gain-less-than-one small gain. It is known informally as the *small μ theorem.* This interpretation of μ as a nonconservative replacement for the singular value in applications of the small-gain theorem involving diagonally structured Δs is in fact the reason why μ is called the *structured singular value.*

Conservativeness, Scaling, and Multipliers. As noted previously, the singular value $\sigma_1(T)$ of a matrix T is generally only a conservative upper bound on the desired quantity $\mu(T)$. And in general this bound can be arbitrarily poor. For example, if

$$T = \begin{bmatrix} 0 & 1000 \\ 0 & 0 \end{bmatrix}$$

then one may readily compute from the definitions that $\sigma_1(T) = 1000$ and $\mu(T) = 0$. So, the ratio of σ_1/μ can be very large. It can even be infinite.

Diagonal scaling is one technique for reducing the conservativeness of the bound on μ provided by singular values. In particular, it can be shown that, for any diagonal scaling matrix

$$D = \begin{bmatrix} d_1 & 0 & 0 & \cdots & 0 \\ 0 & d_2 & 0 & \cdots & 0 \\ \vdots & & \ddots & & \vdots \\ 0 & 0 & 0 & \cdots & d_n \end{bmatrix}$$

that $\mu(T) \le \sigma_1(DTD^{-1})$, whence,

$$\mu(T) \le \overline{\mu} \triangleq \inf_{D \text{ diagonal}} \sigma_1(DTD^{-1})$$

Usually, the upper bound $\overline{\mu}$ is a fairly tight bound on the true value of μ.

Computation of μ via LMIs. In practice, the foregoing optimization over D is usually reduced to an equivalent optimization problem involving a linear matrix inequality (LMI), namely

$$\max_X \alpha$$
subject to
$$X - \alpha T^* X T \ge 0$$

The greatest value of α for which this optimization is feasible yields $\overline{\mu}(T) = \sqrt{\alpha}$, and the optimal diagonal scaling $D = X^{1/2}$. Solutions to LMI optimization problems are computed via standard algorithms. An LMI-based "*D*, *G*-scaling" technique extends the LMI concept to further tighten the upper bound on μ when some of the uncertainties Δ_i are known to be real constants. For example, the Matlab Robust Control Toolbox function `muopt.m` computes $\overline{\mu}$ this way.

Kharitonov Theorem

A system is stable if and only if all its poles have negative real parts. The *characteristic polynomial* associated with a transfer function $T_{y_1u_1}(s)$ is any polynomial whose roots are its poles. One very simple but useful test for robustness allows evaluating the stability of a system whose characteristic polynomial has several uncertain real coefficients. Consider the polynomial $p(s) = a_n s^n + a_{n-1}s^{n-1} + \cdots + a_1 + a_0$. Further, suppose that each of the coefficients a_i $(i = 1, . . ., n)$ is a real number which varies in the range $\underline{a}_i \le a_i \le \overline{a}_i$. The Kharitonov theorem (6) gives necessary and sufficient conditions for all of the roots of the polynomial $p(s)$ to be robustly stable (i.e., have negative real parts) for all values of the coefficients in the specified ranges. The Kharitonov theorem says that robust stability is assured if, and only if, four specific polynomials are stable. The four Kharitonov polynomials are obtained by setting all of the even and all of the odd coefficients to their minimal values and maximal values, respectively. Specifically, the four Kharitonov polynomials are

$$p_1(s) = \sum_{i \text{ odd}} \underline{a}_i s^i + \sum_{i \text{ even}} \underline{a}_i s^i$$
$$p_2(s) = \sum_{i \text{ odd}} \underline{a}_i s^i + \sum_{i \text{ even}} \overline{a}_i s^i$$
$$p_3(s) = \sum_{i \text{ odd}} \overline{a}_i s^i + \sum_{i \text{ even}} \underline{a}_i s^i$$
$$p_4(s) = \sum_{i \text{ odd}} \overline{a}_i s^i + \sum_{i \text{ even}} \overline{a}_i s^i$$

For example, if $p(s)$ is a degree three polynomial with each coefficient a_i ranging between 3 and 5, then

$$p_1(s) = 3s^3 + 3s^2 + 3s + 3$$
$$p_2(s) = 3s^3 + 5s^2 + 3s + 5$$
$$p_3(s) = 5s^3 + 3s^2 + 5s + 3$$
$$p_4(s) = 5s^3 + 5s^2 + 5s + 5$$

Robust Controller Synthesis. From the small μ theorem, it follows that the canonical robust control problems are solved if the controller $K(s)$ stabilizes $P(s)$ and if the resultant closed-loop system $T_{y_1u_1}(s)$ satisfies the condition $\mu[T_{y_1u_1}(j\omega)] < 1$ for all ω. Formally, this leads to the μ-synthesis optimization

$$\mu_{\text{opt}} \triangleq \min_{K(s)\text{ stabilizing}} \sup_{\omega} \mu[T_{y_1u_1}(j\omega)] \tag{7}$$

The previous μ-synthesis problem Eq. (7) constitutes a formal mathematical formulation of the canonical robust control problem. The canonical robust problem is solved if and only if $\mu_{\text{opt}} < 1$.

H_∞ Synthesis

Computing μ, a nonconvex optimization, is generally difficult. The solution to the μ-synthesis problem Eq. (7) is generally even harder. Therefore, in practice, engineers often choose to minimize a conservative singular-value upper bound on μ; see Eq. (6). This leads to the problem

$$\sigma_{\text{opt}} \triangleq \min_{K(s)\text{stabilizing}} \sup_{\omega} \sigma_1[T_{y_1u_1}(j\omega)] \tag{8}$$

or, equivalently,

$$\sigma_{\text{opt}} \triangleq \min_{K(s)\text{ stabilizing}} \|T_{y_1u_1}\|_\infty. \tag{9}$$

This is the H_∞ optimal control problem. It is closely related to the standard H_∞ control problem which is to find, if it exists, a controller $K(s)$ for the system in Fig. 5 such that

$$\|T_{y_1u_1}\|_\infty < 1 \tag{10}$$

Optimal H_∞ controllers possess the property that their closed-loop singular-value frequency-response is completely flat, that is

$$\sigma_{\text{opt}} = \|T_{y_1u_1}\|_\infty = \sigma_1[T_{y_1u_1}(j\omega)] \text{ for all } \omega$$

This flatness of optimal H_∞ controllers is called the *all-pass* property, analogous to the flat response of all-pass filters that arise in circuit theory and signal processing.

H_∞ Control Riccati Equations. Suppose that the plant $P(s)$ in Fig. 5 is given in state-space form as

$$\begin{bmatrix} \dot{x} \\ y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} A & B_1 & B_2 \\ C_1 & D_{11} & D_{12} \\ C_2 & D_{21} & D_{22} \end{bmatrix} \begin{bmatrix} x \\ u_1 \\ u_2 \end{bmatrix}$$

Further suppose, for simplicity, that the following conditions hold:

$$\begin{matrix} D_{11} = 0 & D_{12}^* D_{12} = I & D_{12}^* C_1 = 0 \\ D_{22} = 0 & D_{21} D_{21}^* = I & B_1 D_{21}^* = 0 \end{matrix} \tag{11}$$

The foregoing conditions are not very restrictive. Indeed, by a suitable choice variables, these simplifying conditions can be made to hold in nearly all cases [see (21)].

The formula for an H_∞ controller $K(s)$ that solves the standard H_∞ control problem is most commonly expressed in terms of stabilizing solutions P and Q of the following two algebraic Riccati equations:

$$0 = PA + A'P - P(B_2B_2' - B_1B_1')P + C_1'C_1$$

and

$$0 = AQ + QA' - Q(C_2'C_2 - C_1'C_1)P + B_1B_1'$$

or, equivalently, in terms of the stable eigenspaces of two Hamiltonian matrices

$$H_P \triangleq \begin{bmatrix} A & B_2B_2' - B_1B_1' \\ C_1C_1' & A' \end{bmatrix} \tag{12}$$

$$H_Q \triangleq \begin{bmatrix} A' & C_2'C_2 - C_1'C_1 \\ B_1'B_1 & A \end{bmatrix} \tag{13}$$

Provided that there exists a controller $K(s)$ that solves the standard H_∞ problem, one such $K(s)$ is given in terms of solutions to P, Q of these two Riccati equations as

$$u_2 = -F(I - QP)^{-1}\hat{x}$$
$$\dot{\hat{x}} = (A + QC_1^TC_1)\hat{x} + B_2u_2 + H(y_2 - C_2\hat{x})$$

where $F = B_2'P$ and $H = QC_2'$.

H_∞ Existence Conditions. The foregoing formula gives a solution to the standard H_∞ control problem Eq. (10), provided that a solution exists. In general, no solution may exist, so it is important to have precise and numerically testable mathematical existence conditions. The following are the necessary and sufficient conditions for the existence of a solution:

1. Rank(D_{12}) = dim(u_2) and rank(D_{21}) = dim(y_2).
2. $\sigma_1((D_{12}^\perp)'D_{11}) \leq 1$ and $\sigma_1(D_{11}(D_{21}')^\perp) \leq 1$ where $(D)^\perp$ denotes an orthogonal matrix whose column span is null(D').
3. Neither of the Hamiltonians Eqs. (12–13) has any purely imaginary eigenvalues; that is, $\text{Imag}[\lambda_i(H_P)] \neq 0$ $\forall i$ and $\text{Imag}[\lambda_i(H_Q)] \neq 0$ $\forall i$. (This is necessary and sufficient for the two Riccati equations to have 'stabilizing' solutions that is, solutions such that $A + (B_1B_1' - B_2B_2')P$ and $A + Q(C_1'C_1 - C_2'C_2)$ have only eigenvalues with negative real parts.)
4. Both $A - B_2F$ and $A - HC_2$ are stable (i.e., both have all eigenvalues with strictly negative real parts). (Note: In theory, this condition holds if and only if the Riccati solutions P and Q are both positive-semidefinite matrices, but in practice semidefiniteness cannot be checked reliably with a finite precision computation.)

5. All of the eigenvalues of PQ have moduli less than one:

$$|\lambda_i(PQ)| < 1 \quad \text{for all } i.$$

Implicit in condition 4 above is the requirement that the plant be both stabilizable and detectable, that is, the pair (A, B_2) must be stabilizable and the pair (C_2, A) must be detectable. If this is not so, then condition 4 cannot hold.

γ-Iteration. The solution to the H_∞ optimal control problem Eq. (9) is computed via successive solutions to the standard H_∞ problem Eq. (10) via a technique called *γ-iteration.* The basic idea is to scale the plant matrix $P(s)$ in Eq. (3) by multiplying its first row by a nonnegative scalar γ and then to test the H_∞ existence conditions for various values of γ. The greatest value of γ, for which all five of the H_∞ existence conditions hold, yields the minimal value of the H_∞ cost σ_{opt} as $\sigma_{\text{opt}} = 1/\gamma$. The H_∞ optimal controller is the solution to the corresponding standard H_∞ problem.

μ-Synthesis

The problem of μ-synthesis involves computing a controller $K(s)$ that solves optimization Eq. (7). No such algorithm exists, but the combination of H_∞ optimal control together with diagonal scaling is employed to produce good suboptimal solutions. The algorithm is called the *DK*-iteration, and it proceeds as follows:

Input: The "plant" $P(s)$ shown in Fig. 5.

Initialize: Set $D(s) = I$, $K(s) = 0$, $T_{y_1u_1}(s) = P_{11}(s)$.

Step 1. Replace the plant by the diagonally scaled plant

$$P(s) \leftarrow \begin{bmatrix} D & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix} \begin{bmatrix} D & 0 \\ 0 & I \end{bmatrix}^{-1} \tag{14}$$

Step 2. Compute the solution $K(s)$ to the H_∞ optimization problem Eq. (9) for the diagonally scaled plant Eq. (14) using the γ-iterative technique, and set

$$T_{y_1u_1} = P_{11} + P_{12}K(I - P_{11}K)^{-2}P_{21}$$

Step 3. At each frequency ω, compute the diagonal scaling $D(j\omega)$ that minimizes $\sigma_1(DT_{y_1u_1}D^{-1})$.

Step 4. Using a curve-fitting method, find a polynomial transfer function $D(s)$ that approximates the frequency response $D(j\omega)$ computed in Step 3.

Step 5. Go to Step 1.

There are no theoretical guarantees that this algorithm converges to a joint minimum in $K(s)$ and $D(s)$. In fact, it may not even find a local minimum. But, at least it improves the controller, which is enough for it to be useful in some engineering applications. Refinements on this algorithm have recently appeared in which D, G-iteration handles real uncertainties with less conservatism. Also, a technique for bypassing the curve fitting in Step 4 is available. The optimal $D(s)$ is a prespecified order computed via an LMI-related technique [see (22)].

Mixed Sensitivity and Loop Shaping

There is one H_∞ robust control problem of considerable practical significance, the weighted mixed-sensitivity problem in which

$$T_{y_1u_1} = \begin{bmatrix} W_1(s)S(s) \\ W_2(s)K(s)S(s) \\ W_3(s)T(s) \end{bmatrix}$$

Figure 7 shows how a conventional control system is augmented with "weights" W_1, W_2, W_3 to produce an augmented plant $P(s)$ such that the closed-loop $T_{y_1u_1}$ has the indicated form, namely,

$$P(s) = \left[\begin{array}{c|c} W_1 & -W_1G \\ 0 & W_2 \\ 0 & W_3G \\ \hline I & -G \end{array}\right]$$

Let us suppose (as is often the case in applications of the theory) that the control-signal weight W_2 is absent, so that $T_{y_1u_1}$ reduces to

$$T_{y_1u_1} = \begin{bmatrix} W_1(s)S(s) \\ W_3(s)T(s) \end{bmatrix}$$

The all-pass property of H_∞ optimal controllers ensures that

$$\sigma_{\text{opt}} = \|T_{y_1u_1}\|_\infty = \sigma_1[Ty_1u_1(j\omega)] \quad \text{for all } \omega.$$

From this, combined with the singular-value property Eq. (5) and the fundamental constraint $S(s) + T(s) = I$, it follows that, to within a factor of $\sqrt{2}$,

$$\sigma_{\text{opt}} \approx \max\{\sigma_1[W_1(j\omega)S(j\omega)], \sigma_1[W_3(j\omega)T(j\omega)]\}.$$

Whence, it follows from the properties of singular values that, at frequencies where $\sigma_1[S(j\omega)] < 1$,

$$\sigma_i[L(j\omega)] \ge \sigma_n[L(j\omega)] \approx \frac{1}{\sigma_1[S(j\omega)]} \approx \frac{\|W_1(j\omega)\|}{\sigma_{\text{opt}}}$$

and, at other frequencies where $\sigma_1(T(j\omega)) < 1$,

$$\sigma_i[L(j\omega)] \le \sigma_1[L(j\omega)] \approx \sigma_1[T(j\omega)] \approx \frac{\sigma_{\text{opt}}}{\|W_3(j\omega)\|}$$

where $L(s) = G(s)K(s)$ is the loop transfer function matrix of the control system in Fig. 7. The situation is depicted in Fig. 8. The key feature to notice is that, inside the control loop bandwidth, the singular values $\sigma_i[L(j\omega)]$ of the loop transfer function are bounded below by $W_1(j\omega)/\sigma_{\text{opt}} > 1$ and, outside the control bandwidth, the singular values are bounded above by $\sigma_{\text{opt}}/W_3(j\omega) < 1$. All of the loop transfer function singular values cross over in the intermediate frequency range, as depicted in Fig. 7.

The implication is that the shapes of the Bode magnitude plots of $W_1(j\omega)$ and $1/W_3(j\omega)$ specify with considerable precision the actual shapes of the optimal singular-value plots of

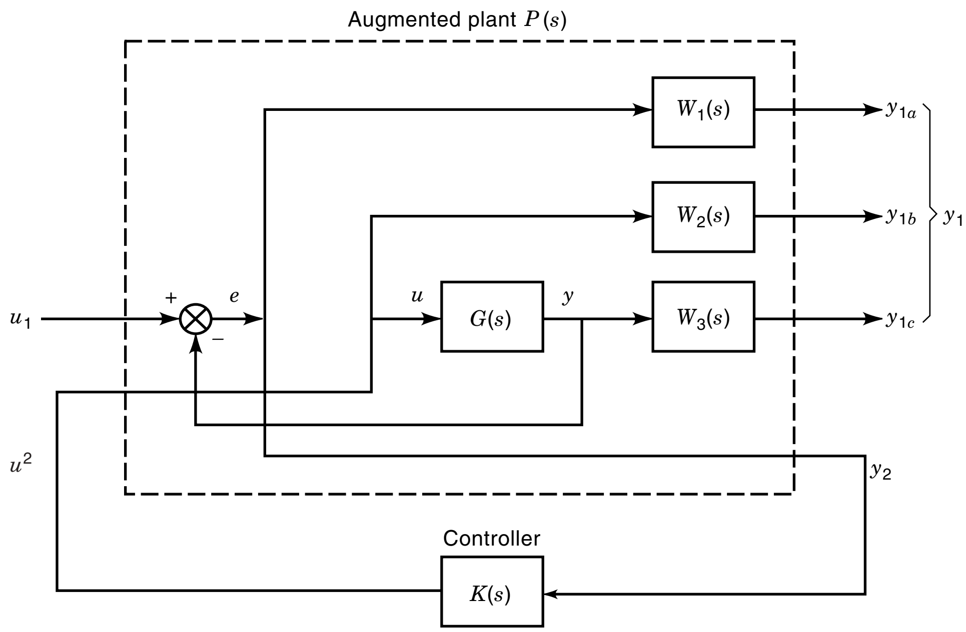


Figure 7. In mixed-sensitivity H_∞ robust control design, the plant model is augmented with weighting filters $W_1(s)$, $W_2(s)$ and $W_3(s)$ that determine the shape of the resulting closed-loop sensitivity and complementary sensitivity Bode plots.

the optimal loop-transfer function $L(j\omega) \triangleq G(j\omega)K(j\omega)$. Thus, the weights $W_1(s)$ and $W_3(s)$ are high-precision "knobs" for shaping the loop-transfer function's singular values. The H_∞ theory automatically ensures that the controller $K(s)$ is stabilizing. Because of this, the weighted mixed-sensitivity H_∞ loop shaping is a highly popular and useful method for multiloop control design.

FURTHER READING

History. For pre-1900 control history, including an account of the invention of the Watt flyball governor, see Ref. 1. For an accurate account of the early developments in robust control theory, see Ref. 3 and the monograph Ref. 2. Reprints of key journal papers are provided in the anthology (5). Reference 4 is an extensive bibliography on robust control.

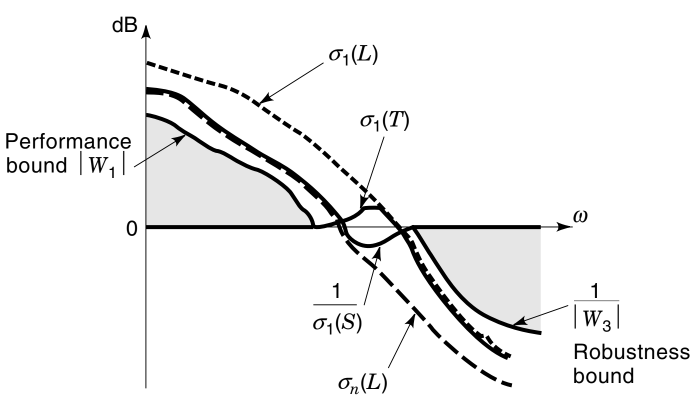


Figure 8. The mixed-sensitivity weights $W_1(s)$ and $W_3(s)$ provide engineers with the means to directly specify the desired shape of the compensated loop transfer function $L(s)$. The Bode plot of $L(s)$ is sandwiched between the plot of $W_1(s)$ and the plot of $1/W_3(s)$. Loop-shaping is achieved by manipulating the weights $W_1(s)$ and $W_3(s)$.

Textbooks. Reference 6 describes the early 1960s work of Horowitz on uncertainty tolerant design for single-loop feedback control systems. References 7 and 8 provide a good discussion of the most useful techniques of robust control theory along with numerous design examples. More theoretical treatments are in the texts (9), (11), and (12). Adaptive robust control theory is treated in Ref. 10. A more specialized focus on robustness analysis methods related to the Kharitonov theory is in (13).

Software. Software for robustness analysis, H_∞ control design, μ-synthesis, and related topics is provided by the Matlab toolboxes (14), (15), and (16). The user's guides accompanying these software products all contain extensive tutorial texts covering the theory, its use, and numerous design examples.

Advanced Topics. The robustness implications of the gap metric is examined in Ref. 19. Recently LMI-based methods and have become increasingly important in robust control theory, leading to significant reductions in conservatism and extensions to difficult multi-objective problems, simultaneous stabilization, and gain scheduling for slowly varying plants; see Refs. 17 and 18. A key issue in applying robust control theory is the question of how evolving experimental data are incorporated to identify uncertainty sizes and to adaptively enhance robustness. The latter is the focus of unfalsified control theory; see Ref. 20.

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ROBUST CONTROL ANALYSIS

Robustness is a property inherently sought after in engineering systems. The concept is directly linked to such issues as design viability and system reliability. In broad terms, robustness can be regarded as the capability to withstand unknown, unexpected, and often hostile conditions that can adversely affect a system's behavior. A system must be sufficiently robust in order to function properly under undesirable circumstances, conducting its task as designed. As engineering systems are becoming more and more complex and are required to operate in increasingly more uncertain environments, robustness has become increasingly more crucial.

Robust control can be generally defined as the control, by means of fixed compensators, of uncertain plants (i.e., of systems with uncertain dynamics and unknown disturbance signals). Robustness in a control system refers to its ability to cope with uncertainties in a satisfactory manner, maintaining its stability and performance as desired. Uncertainty in signals and systems is inevitable, reflecting both the complexity of the physical world and the limitation in human understanding. Uncertain signals typically arise as a result of the randomness and unpredictability of environmental effects and are of an unmeasurable and unpredictable nature. Uncertain system dynamics, on the other hand, can be attributed to changes in the actual system and to modeling errors, be they accidental or deliberate. Generally, uncertain dynamics may come from the following sources:

1. Imperfect or incomplete knowledge of physical processes. This represents the information unattainable because of one's limited knowledge or inadequate measurements. It can be particularly acute for complex systems and processes (e.g., those found in biomedical engineering).
2. Parameter variations. Every physical system will undergo a change in parameter values under different operating conditions. Aging itself can be a factor.
3. Neglected high-frequency dynamics, time delays, nonlinearities, and the like. It may occur as a result of a sheer lack of knowledge, or the difficulty to model these characteristics. It may also occur because of the desire for models of low complexity.

While robustness is a concept of universal significance, robustness analysis for control systems is the study of whether a system, however designed, can meet specified stability and performance goals in the presence of uncertainty within a prescribed range.

Uncertain plants can be modeled in various ways. In particular, models can be stochastic or purely deterministic. In robust control, uncertain systems are typically modeled deterministically, as bounded sets of system models. A property is then said to hold robustly if it holds for every model in the set. The simplest case is that of *unstructured* uncertainty: the model set consists of *all* systems in a certain neighborhood (e.g., all transfer functions lying within a certain "distance" of a distinguished "nominal" system). Such a description is particularly appropriate to account for unmodeled dynamics. One rather typical example is the modeling of flexible structures. It is well known that, in general, a flexible structure cannot be accurately represented by a finite-dimensional system. For control design purposes, however, we desire, and most often are compelled to find, an approximate finite-dimensional model with a relatively low order. In doing so, a common practice is to include in the nominal model a small number of dominant modes in the low-frequency range and to treat the high-frequency modes as modeling uncertainty. Evidently, this description is also appropriate for modeling errors resulting from model reduction, or from any frequency response truncation. Moreover, it can be used to cover, albeit in a conservative way, parameter variations. The latter amounts to drawing up a frequency response envelope to describe the range of parameter variation in frequency domain.

Finally, in robustness analysis, it is common to introduce a *fictitious* uncertainty to represent a performance objective.

A more accurate account of parametric uncertainty calls for model sets within which individual models are uniquely characterized by the value(s) of one or more parameter(s) (e.g., transfer function coefficients). Typically each such parameter takes values in a known range. Such models account for the fact that parameters in physical devices are bound to vary with time, environment, and operating conditions. Ackermann's car steering problem (1) offers a good illustrative example. In his study of a four-wheel steering vehicle, he found that the vehicle mass and the adhesion between tires and road surface are significant uncertain parameters. This is easily understandable. The vehicle mass certainly varies with load, and in a more subtle sense, it varies with fuel consumption. The adhesion changes with road condition, wearing of tires, and weather condition. We can think of other uncertain parameters by considering the human-vehicle system as a whole or by considering a whole batch of vehicles as a family of systems. In these scenarios, differences among individual drivers and vehicles can all constitute significant uncertain factors. Yet a more striking example is an aircraft, whose aerodynamic coefficients vary in large magnitudes due to changes in altitude, maneuvering, and weather.

There is an inherent trade-off between fidelity and simplicity in modeling uncertain systems. In a sense, unstructured and parametric uncertainties may be considered the two extremes. While an unstructured perturbation furnishes a simple characterization and is useful for simplifying robustness analysis, it contains little information and may often be too conservative. On the other hand, uncertain parameters often yield a more natural and accurate model, but such elaborate descriptions tend to complicate analysis. The process of robustness analysis, therefore, calls for a judicious balance between uncertainty description and complexity of analysis. In its full generality, however, the description of an uncertain system should take into account both parametric variations and unmodeled dynamics. Uncertainty descriptions of this kind are called *structured.*

To be sure, robustness is not entirely a new concept in control system analysis and design. In retrospect, the need for a control system to tolerate unmodeled dynamics and parameter variations is precisely what motivated feedback control, and it has been a primary goal in control system design since its birth. This is well recognized in classical control design and, at least implicitly, is embedded in classical loop shaping methods. Concepts such as gain and phase margins may well be regarded as elementary robustness measures. However, it was not until the late 1970s that the term began to appear routinely in the control literature, when the need for robustness was reexamined and was gaining increasing recognition. Robust control as a research direction soon thrived and became a defining theme. After two decades of intense activity, it has evolved into a broad research area rich in theory and potential applications. The progress has been rapid and vast, leading to the development of a variety of key concepts and techniques, among which notably are the $\mathcal{H}_\infty/\mu$ theory, the Kharitonov/polynomial approach, and analyses based on state-space formulations and the Lyapunov theory.

The structured singular value (2), also known as μ, was introduced in the early 1980s as a very general framework for studying structured uncertainty in linear time-invariant models. It defines an exact measure of robust stability and performance in the frequency domain. The method is a natural progression of earlier work on using *singular values* to extend the concept of stability margin to multivariable systems, an idea that was heavily influenced by operator theoretic results such as the Small Gain Theorem. The main impetus for the development of the structured singular value theory, evidently, has been the recognition that unstructured uncertainty is too crude a model, often leading to excessive conservatism.

From a computational point of view, the success of μ hinges critically upon whether and how it can be computed both accurately and efficiently. Unfortunately, this is known to be difficult. Recent studies have shown that computation of μ generally amounts to solving a so-called NP-hard decision problem, which, in the language of computing theory, is one that suffers from an exponential growth in its computational complexity. Although this by no means implies that every μ problem is computationally difficult, it nevertheless points to the unfortunate conclusion that the computation of μ in general poses an insurmountable difficulty in the worst case. In retrospect, it thus comes as no surprise that the major progress in computing μ has been made by some forms of approximation, specifically, readily computable bounds. While no definitive conclusion has been drawn concerning the gap between μ and such bounds, it is reassuring that the bounds are often reasonably tight and that they have other interpretations of engineering significance.

The "Kharitonov" theory is another robustness analysis approach, developed in parallel with μ, which deals almost entirely with robust stability issues under parametric uncertainty; only in rare cases is unstructured uncertainty also taken into consideration. The research in this area has a natural heritage from such a classical stability test as the Routh-Hurwitz criterion and was mainly triggered by a landmark result published by V. L. Kharitonov (3) in 1978, later referred to as Kharitonov's theorem. Kharitonov considered the question of "stability" of parameterized families of polynomials. Here polynomials are thought of as characteristic polynomials of systems described by rational transfer functions and thus are "stable" if their zeros all lie in the open left-half of the complex plane (continuous time) or in the open unit disk (discrete time). In the continuous-time case, Kharitonov showed that, for an uncertain "interval" polynomial whose coefficients each vary independently in a given interval, stability of the entire family can be assessed by testing merely four simply constructed extreme polynomials. From an aesthetic point of view, Kharitonov's theorem possesses remarkable elegance, reducing an otherwise seemingly impossible task to a simple problem. From an engineering perspective, however, the theorem is likely to find only limited utility because very rarely would an uncertain system yield a family of characteristic polynomials of the form required in the problem description. Thus, Kharitonov's work triggered a flurry of activities in the search for more realistic results, and soon came various generalizations. Two notable features stand out from the robust stability conditions available in this category. First, they are stated either in terms of a finite number of polynomials or as graphical tests requiring a frequency sweep. Second, the main success to date pertains to uncertainty descriptions in which polynomial coefficients depend linearly on uncertain parameters.

Robust stability analysis in state-space formulation often comes under the names of stability radius, interval matrix, and stability bound problems. The key issue here is to determine the largest uncertainty size under which stability is preserved. Unlike in the aforementioned two approaches, parametric uncertainty in state-space representation is defined in terms of perturbations to system matrices, which can be either unstructured or structured, but most often are only allowed to be real. In a natural way, the robust stability problem translates to one of how the perturbations may affect the eigenvalues of the system matrix, the solution of which can draw upon rich theories from linear algebra and Lyapunov analysis. Thus, unsurprisingly, most of the robustness conditions have this flavor. Alternatively, the problem can also be recast as one of μ analysis or one of polynomial stability. In the former case, we need to compute μ with respect to solely real uncertainties, for which the μ computation schemes are known to be conservative. In the latter case, the coefficients in the resultant polynomial will depend on the unknown parameters in a multilinear or multinomial fashion. The Kharitonov approach cannot provide a satisfactory answer for such polynomial families either. The problem thus appears to be a very difficult one and is known to have been solved only for a number of isolated cases. Most notably, recent progress has made it possible to compute the stability radius in an efficient manner when the matrix perturbation is unstructured. For structured uncertainty, unfortunately, the problem is completely open; recent studies showed too that it is in general NP-hard. Accordingly, the majority of results in the latter class are sufficient conditions for robust stability.

In conclusion, robustness is a key concept, vital for engineering design in general and for control system design in particular. Robust control has matured into a field rich in theory and potential applications. By focusing on these three selected areas, this chapter is limited only to robustness analysis of linear time-invariant systems, that is, control of linear time-invariant plants by linear time-invariant controllers, which itself is condensed from a vast collection of results and techniques. Nevertheless, we should note that the concept and theory of robust control goes far beyond the boundary of linear time-invariant systems and, in fact, has been quickly branching to the domains of nonlinear control, adaptive control, and the like. As a whole, it has been, and will continue to be, a driving force behind the evolution of control theory.

THE STRUCTURED SINGULAR VALUE

Uncertain Systems

Throughout this article, we consider model sets $\mathscr{P}$ with the following property: $\mathscr{P}$ can be represented by a block diagram with some of the blocks being fully known systems, and others being "elementary uncertain blocks." The latter are elementary sets, namely, unit balls in "simple" vector spaces. For example, some uncertainty blocks might be the real interval $[-1, 1]$ and others might be the unit ball of $\mathscr{H}_\infty$, the set of transfer function matrices (linear time-invariant systems) that are causal and bounded-input bounded-output (BIBO) stable; the size or "$\mathscr{H}_\infty$-norm" of $\Delta \in \mathscr{H}_\infty$ is defined to be the supremum of its largest singular value over the imaginary axis (continuous time) or unit disk (discrete time).

Thus each plant in the model set corresponds to the selection of one element in each of the uncertain blocks. The *nominal plant* corresponds to the choice "0" for all elementary uncertain blocks. Note that the assumption that all the uncertainty balls have unity radius is made at no cost: any size information can be embedded into known blocks (e.g., connected in cascade with the uncertain blocks). It turns out that, for "uncertain" block diagrams of this type, the transfer function (or transfer function matrix) between any two nodes (or tuples of nodes) in the block diagram is given by a *linear fractional transformation* (LFT). Similarly, when such an uncertain plant is connected with a feedback compensator, the transfer function between any two nodes will also be an LFT. LFTs have attractive mathematical properties that can be used to advantage at the modeling, analysis, and synthesis stages. More on LFTs can be found, e.g., in Ref. 7.

Robust Stability

It should be intuitively clear that block diagrams of the type just described can always be "redrawn" in the form of an M–Δ loop as depicted in Fig. 1 (external input and outputs have been left out). Here M corresponds to the nominal system, which is comprised of closed-loop transfer functions as elements and has an input channel and an output channel for each elementary uncertain block, and Δ is a block-diagonal matrix whose diagonal blocks are the elementary uncertain blocks. For its generality, the M–Δ loop paradigm has found wide acceptance in robust control (see, for example, Refs. 2 and 4–8).

Throughout most of this article, we will assume that the nominal system, or equivalently M, is linear and time-invariant, as are all instances of the uncertainty blocks, equivalently, of Δ. We will also assume that M and all instances of Δ are in $\mathscr{H}_\infty$. In this case, an immediate payoff of the LFT uncertainty description and the ensuing representation of the system via the M–Δ loop is the following strong form of the Small Gain Theorem, a necessary and sufficient condition for well-posedness and stability of the M–Δ loop, in the case where Δ consists of a single uncertainty block, ranging over the unit ball in $\mathscr{H}_\infty$.

Small Gain Theorem. Let $M \in \mathscr{H}_\infty$. Then the M–Δ loop is well-posed and BIBO stable for all $\Delta \in \mathscr{H}_\infty$ with $\|\Delta\|_\infty \leq 1$ if and only if $\|M\|_\infty < 1$.

As alluded to earlier, the $\mathscr{H}_\infty$ norm of a causal, stable, continuous-time transfer function matrix M is defined as

$$\|M\|_\infty = \sup_{\omega \in \mathbb{R}} \overline{\sigma}(M(j\omega))$$

where $\overline{\sigma}$ denotes the largest singular value.

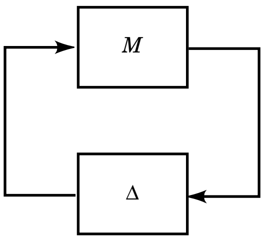

Figure 1. M–Δ loop.

As an example consider a model set of the "output multiplicative uncertainty" type. Specifically, let

$$P = (I + w\Delta)P_0$$

where P_0 is the transfer matrix of a linear, time-invariant nominal model, w is a scalar "weighting" transfer function, and Δ ranges over the unit ball in $\mathscr{H}_\infty$. The weight w is introduced to account for the fact that the amount of uncertainty is usually frequency-dependent; in particular, system dynamics are often poorly known at high frequencies. Suppose that a feedback controller K has been tentatively selected to stabilize the nominal system P_0. (We use the negative feedback convention, i.e., the loop transfer function is $-KP_0$.) Isolating Δ from the nominal closed-loop system, we obtain an M–Δ loop with

$$M = -wKP_0(I + KP_0)^{-1}$$

Since K stabilizes P_0, M is stable.

As a note of interest, we expect a keen connection between the Small Gain Theorem and the classical Nyquist criterion. Indeed, this can be best observed by examining single-input/single-output systems. In such case P and K are scalar, and thus so is M. Since both M and Δ are stable, Nyquist's criterion implies that the M—Δ loop is stable whenever the Nyquist plot of $M\Delta$ does not encircle the critical point $-1 + j0$. Clearly, this will be the case for every Δ satisfying $|\Delta(jw)| \le 1$ for all w if and only if $|M(jw)| < 1$ holds at all frequencies (including ∞).

The Small Gain Theorem suggests that one way to obtain a robustly stable system, or more generally to obtain a robust design, is to make sure that the $\mathscr{H}_\infty$ norm of a certain system transfer function is small enough. This has triggered an entire field of research known as $\mathscr{H}_\infty$ design, which is discussed elsewhere in this encyclopedia. The focus of the present article is the case when Δ is block-diagonal (i.e., when the uncertainty model consists of several blocks or, in other words, when the uncertainty is structured). Typically, two types of uncertainty blocks are considered in the literature: (i) the set of real, constant scalar multiples of the identity, with the scalar having magnitude no larger than one, and (ii) the set of causal and BIBO stable ($\mathscr{H}_\infty$) transfer function matrices, with $\mathscr{H}_\infty$-norm no larger than one. The latter corresponds to unmodeled dynamics. The former, on the other hand, is used to represent parametric uncertainty, particularly when a same uncertain parameter affects more than one coefficients in a transfer function. For example, concurrent variation as a function of temperature (e.g., dilation) of multiple quantities in a mechanical system can result in such a block. This description is more general than the simpler "scalar nonrepeated blocks."

Examples with structured uncertainty arise with plants modeled as being affected by uncertainty at more than one physical location, e.g.,

$$P = (I + w_1\Delta_1)P_0(I + w_2\Delta_2)$$

where both input and output uncertainty are accounted for. Another instance arises in the context of the robust performance problem, discussed in a later section. For an example including both parametric uncertainty and unmodeled dynamics, consider now the model set $(1 + w_1\Delta^c)P_a$ with P_a explicitly given as

$$P_a(s) = \frac{1}{s - a}$$

where a can take any value in the interval $[-0.9,1.1]$. We may write $a = 1 + w_2\Delta^r$ with $w_2 = 0.1$ and $|\Delta^r| \le 1$, and P_a can be represented as a loop with $P_0(s) = 1/(s - 1)$ in the forward path and $-w_2\Delta^r$ in the feedback path (again using the negative feedback convention). Let K be a feedback controller (still with the negative feedback convention) that stabilizes P_0. By "extracting" the uncertainty blocks Δ^c and Δ^r, we obtain an M—Δ loop with

$$M = \begin{bmatrix} w_2P_0(I + P_0K)^{-1} & -w_2KP_0(I + P_0K)^{-1} \\ w_1P_0(I + P_0K)^{-1} & -w_1KP_0(I + P_0K)^{-1} \end{bmatrix}$$

$$\Delta = \begin{bmatrix} \Delta^r & 0 \\ 0 & \Delta^c \end{bmatrix}$$

where Δ^r is a real number ranging over $[-1,1]$ and Δ^c is a scalar transfer function ranging over the unit ball in $\mathscr{H}_\infty$.

Clearly, the condition $\|M\|_\infty < 1$ in the Small Gain Theorem remains sufficient for robust stability when Δ is restricted to be block-diagonal. However, it is in general no longer necessary. A refinement of the Small Gain Theorem for the structured uncertainty case was proposed by Doyle and Safonov in the early 1980s (2,4). We adopt here the framework introduced by Doyle, that of the structured singular value also known as μ. The Small μ Theorem states that, if the uncertainty is restricted to be block-diagonal, then the correct refinement is essentially (see Ref. 9 for a precise statement) to replace $\|M\|_\infty$ with $\|M\|_\mu$, where for a continuous-time transfer function matrix M,

$$\|M\|_\mu = \sup_{\omega \in \mathbb{R}_e} \mu(M(\mathrm{j}\omega))$$

and $\mu(\cdot)$ denotes the structured singular value of its matrix argument with respect to the block-structure under consideration. The set $\mathbb{R}_e = \mathbb{R} \cup \{\infty\}$ is the extended real line; if no parametric uncertainty is present, however, $\mu(\cdot)$ is continuous and $\mathbb{R}_e$ can be replaced by $\mathbb{R}$. Similarly, for a discrete-time system,

$$\|M\|_\mu = \sup_{\theta \in [0,2\pi)} \mu(M(e^{j\theta}))$$

But what specifically is μ? This is discussed next.

The Structured Singular Value

Let us denote by $\boldsymbol{\Gamma}$ the set of values taken by $\Delta(j\omega)$ [or $\Delta(e^{j\theta})$] as Δ ranges over the set of block diagonal transfer function matrices of interest, with the "unit ball" restriction lifted, namely let

$$\boldsymbol{\Gamma} = \{\mathrm{diag}(\Gamma^r, \Gamma^c, \Gamma^C) : \Gamma^r \in \boldsymbol{\Gamma}_r, \Gamma^c \in \boldsymbol{\Gamma}_c, \Gamma^C \in \boldsymbol{\Gamma}_C\}$$

with

$$\mathbf{\Gamma}_{\rm r} := \{\mathrm{diag}(\gamma_1^{\rm r} I_{k_1}, \ldots, \gamma_{m_{\rm r}}^{\rm r} I_{k_{m_{\rm r}}}) : \gamma_i^{\rm r} \in \mathbb{R}\}$$
$$\mathbf{\Gamma}_{\rm c} := \{\mathrm{diag}(\gamma_1^{\rm c} I_{k_{m_{\rm r}+1}}, \ldots, \gamma_{m_{\rm c}}^{\rm c} I_{k_{m_{\rm r}+m_{\rm c}}}) : \gamma_i^{\rm c} \in \mathbb{C}\}$$
$$\mathbf{\Gamma}_{\rm C} := \{\mathrm{diag}(\Gamma_1^{\rm C}, \ldots, \Gamma_{m_{\rm C}}^{\rm C}) : \Gamma_i^{\rm c} \in \mathbb{C}^{k_{m_{\rm r}+m_{\rm c}+i} \times k_{m_{\rm r}+m_{\rm c}+i}}\}$$

The first and third block types, often referred to as repeated real and full complex blocks, correspond to values of parametric and dynamic uncertainty, respectively. The second block type, known as repeated complex, often arises in analyzing multidimensional (10) and time-delay systems (11), and is also used sometimes when an LFT state-space representation of transfer functions is sought (5). It is worth noting that while $\mathbf{\Gamma}$ as just defined is usually adequate for representing uncertainties frequently encountered, it can be extended further to accommodate more general situations. For example, full real blocks (i.e., unknown real matrices) may be added whenever desired.

The *structured singular value* $\mu(M)$ of a matrix M with respect to the block structure $\mathbf{\Gamma}$ is defined to be 0 if there is no $\Gamma \in \mathbf{\Gamma}$ such that $\det(I - \Gamma M) = 0$, and

$$\mu(M) = \left(\min_{\Gamma\in\mathbf{\Gamma}}\{\overline{\sigma}(\Gamma) : \det(I - \Gamma M) = 0\}\right)^{-1}$$

otherwise. It can be checked that for structures simply consisting of one full complex block as in the Small Gain Theorem, $\mu(M(\mathrm{j}\omega))$ becomes the largest singular value of $M(\mathrm{j}\omega)$, and $\|M\|_\mu$ is thus equal to $\|M\|_\infty$.

Given a matrix M and a block structure $\mathbf{\Gamma}$, computing $\mu(M)$ is generally not an easy task. Indeed, this computation is known to be NP-hard, even when $\mathbf{\Gamma}$ is simplified to a structure containing only full complex blocks. Thus estimates of $\mu(M)$, e.g., upper and lower bounds on $\mu(M)$, are often used instead. These, as well as other properties of μ, are discussed next.

Let $\mathscr{U}$ be the set of unitary matrices in $\mathbf{\Gamma}$ and $\mathscr{D}$ be the set of nonsingular matrices that commute with every $\Gamma \in \mathbf{\Gamma}$. The latter consist of block-diagonal matrices with scalar multiples of the identity in correspondence with full complex blocks ($\mathbf{\Gamma}_{\rm C}$), and with arbitrary blocks in correspondence with those constrained to be scalar multiples of the identity ($\mathbf{\Gamma}_{\rm r}$, $\mathbf{\Gamma}_{\rm c}$). Then the following result holds:

$$\max_{U\in\mathscr{U}} \rho_R(MU) \le \mu(M) \le \inf_{D\in\mathscr{D}} \overline{\sigma}(DMD^{-1}) \tag{1}$$

Here ρ_R is the largest absolute value of a real eigenvalue of its matrix argument. Inequalities [see Eq. (1)] are of special interest in the case of purely complex uncertainty structures. In that case, (i) the lower bound is equal to $\mu(M)$ and ρ_R can be replaced by the spectral radius ρ, and (ii) the upper bound is equal to $\mu(M)$ whenever $m_{\rm C} + 2m_{\rm c}$ is no greater than 3, and extensive numerical experimentation suggests that it is never (or at least "seldom") much larger. Moreover, the upper bound can be computed efficiently by solving a convex optimization problem, in fact, a linear matrix inequality (LMI) problem. LMIs define a special class of convex optimization problems and are discussed elsewhere in this encyclopedia.

For uncertainty structures where real (scalar multiple of the identity) blocks are present, inequalities [Eq. (1)] can be tightened to the following (12):

$$\max_{Q\in\mathscr{Q}} \rho_R(MQ) \le \mu(M) \le \inf_{D\in\mathscr{D}^+, G\in\mathscr{G}} \{\gamma \ge 0 : MDM^{\rm H} + GM^{\rm H} - MG - \gamma^2 D < 0\} \tag{2}$$

Here the superscript H indicates complex conjugate transpose, $\mathscr{Q}$ is the subset of $\mathbf{\Gamma}$ consisting of matrices whose complex blocks are unitary, $\mathscr{D}^+$ is the subset of $\mathscr{D}$ consisting of Hermitian positive definite matrices, $\mathscr{G}$ is the subset of $\mathscr{D}$ consisting of skew-Hermitian matrices (i.e., $G^{\rm H} = -G$) with zero blocks in correspondence with repeated real blocks in $\mathbf{\Gamma}$, and the $<$ sign indicates that the matrix expression is constrained to be negative definite. The lower bound in condition (2) is always equal to $\mu(M)$. The upper bound is never greater than that in condition (1) (it reduces to it when $G = 0$ is imposed) and, as was the case for condition (1), can be computed by solving an LMI problem. See, for example, Section 8.12 in Ref. 8.

For the class of problems where the matrix M has rank one, Young (13) showed that the right-hand side in inequalities [Eq. (2)] is equal to $\mu(M)$. In that case, Chen et al. (14) obtained an explicit formula for $\mu(M)$. Let $M = ba^{\rm H}$, where a and b are column vectors. Let also a and b be partitioned into subvectors a_i and b_i compatibly with $\mathbf{\Gamma}$. For $i = 1, \ldots, m_{\rm r} + m_{\rm c}$, let $\phi_i = a_i^{\rm H} b_i$. Moreover, define

$$\gamma = \sum_{i=m_{\rm r}+1}^{m_{\rm r}+m_{\rm c}} |\phi_i| + \sum_{i=m_{\rm r}+m_{\rm c}+1}^{m_{\rm r}+m_{\rm c}+m_{\rm C}} \|a_i\|_2 \|b_i\|_2$$

Then,

$$\mu(M) = \inf_{x\in\mathbb{R}} \left(\sum_{i=1}^{m_{\rm r}} |\mathrm{Re}(\phi_i) + x\,\mathrm{Im}(\phi_i)| + \gamma\sqrt{1+x^2}\right) \tag{3}$$

Furthermore, if we assume with no loss of generality that for some $l \le m_{\rm r}$, $\mathrm{Im}(\phi_i) \ne 0$ for $1 \le i \le l$, and $\mathrm{Im}(\phi_i) = 0$ for $i > l$, and that

$$-\frac{\mathrm{Re}(\phi_1)}{\mathrm{Im}(\phi_1)} \le \cdots \le -\frac{\mathrm{Re}(\phi_l)}{\mathrm{Im}(\phi_l)}$$

then the infimum is achieved at one of the following points:

$$x_k = -\frac{\mathrm{Re}(\phi_k)}{\mathrm{Im}(\phi_k)}, \quad k = 1, \ldots, l$$
$$x'_0 = \pm\frac{\sum_{i=1}^{l} |\mathrm{Im}(\phi_i)|}{\sqrt{\gamma^2 - \left(\sum_{i=1}^{l} |\mathrm{Im}(\phi_i)|\right)^2}}$$
$$x'_k = \pm\frac{\sum_{i=1}^{k} |\mathrm{Im}(\phi_i)| - \sum_{i=k+1}^{l} |\mathrm{Im}(\phi_i)|}{\sqrt{\gamma^2 - \left(\sum_{i=1}^{k} |\mathrm{Im}(\phi_i)| - \sum_{i=k+1}^{l} |\mathrm{Im}(\phi_i)|\right)^2}}$$

Finally, the infimum cannot be achieved at x'_0 unless $x'_0 \in (-\infty, x_1]$, and for $k = 1, \ldots, l$, it cannot be achieved at x'_k unless $x'_k \in [x_k, x_{k+1}]$.

The rank-one case just alluded to is one of the rare instances for which one can obtain an explicit expression for $\mu(M)$. This expression not only simplifies the computation of the upper bound in condition (2) but also was found useful in studying robust stability of uncertain polynomials. Indeed, as

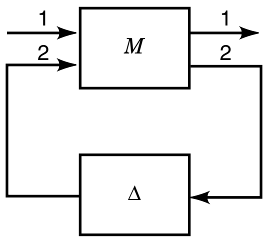

Figure 2. Robust performance setup.

will be discovered shortly, an important class of stability problems for uncertain polynomials can be formulated in terms of a rank-one μ problem. Consequently, the result furnishes a unifying tool for the stability problems and a link between μ analysis and the Kharitonov approach to robustness analysis.

Robust Performance

A key reason for the popularity of the μ framework is that it encompasses not only the robust stability problem but also the following robust performance problem: determine whether, for all plants in the given model set, the energy (integral of the square of the magnitude) in a specified error output signal remains below a specified threshold whenever the disturbance input's energy is less than a specified value.

Consider the block diagram of Fig. 2 where, as compared to Fig. 1, external (disturbance) input and (error) output are made explicit. Given a block diagram such as the one of Fig. 2, the input-output transfer function in the continuous-time case is given by the *linear-fractional transformation*

$$F_\ell(M(s),\Delta(s)) = M_{11}(s) + M_{12}(s)\Delta(s)(I - M_{22}(s)\Delta(s))^{-1}M_{21}(s)$$

where $M_{ij}(s)$ is the transfer function from input j to output i of $M(s)$, $i,j = 1,2$, when the feedback connection through $\Delta(s)$ is removed. [Thus $M_{22}(s)$ is the transfer function matrix formerly denoted $M(s)$.]

The issue at hand is to determine, under the assumption that $M(s) \in \mathscr{H}_\infty$, whether robust performance holds, that is, whether it is the case that, for all $\Delta(s)$ in our unit uncertainty ball,

$$\|F_\ell(M(s),\Delta(s))\|_\infty < 1 \tag{4}$$

This is readily handled by noting that, in view of the Small Gain Theorem, for any fixed $\Delta(s)$ such that the system is stable, condition (4) is equivalent to the stability of the augmented system depicted in Fig. 3 for all *fictitious* uncertainty blocks $\Delta_p(s)$ such that $\|\Delta_p\|_\infty \le 1$. It thus follows that robust performance holds if and only if

$$\sup_{\omega \in \mathbb{R}_e} \mu(M(\mathrm{j}\omega)) < 1$$

where μ now denotes the structured singular value corresponding to the "augmented" block structure diag($\mathbb{C}^{k_p \times k_p}$, $\mathbf{\Gamma}$) (i.e., the block-structure $\mathbf{\Gamma}$ corresponding to the actual uncertainty, augmented with a full complex block).

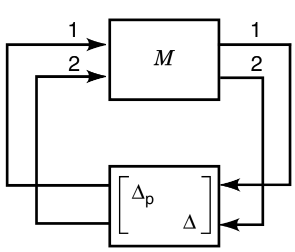

Figure 3. Fictitious uncertainty for robust performance.

For an example of a typical robust performance problem, consider an uncertain plant described by the multiplicative uncertainty model set

$$P = (I + w_1\Delta)P_0$$

with a fixed feedback controller K. It is desired to determine whether $\|w_2S\|_\infty < 1$ for all possible Δ in a possibly structured unit uncertainty ball, where S is the sensitivity function (i.e., using the negative feedback convention, $S = (I + PK)^{-1}$). Here w_1 and w_2 are stable transfer functions introduced for frequency-weighting purposes. For simplicity, w_1 and w_2 are assumed to be scalars. Using the transformation just outlined, we obtain

$$M = \begin{bmatrix} w_2(I+P_0K)^{-1} & w_2(I+P_0K)^{-1}P_0 \\ -w_1(I+KP_0)^{-1}K & -w_1(I+KP_0)^{-1}KP_0 \end{bmatrix} \tag{5}$$

In the single-input/single-output case, M has rank one. In the present case $m_r = m_c = 0$, so the right-hand side of Eq. (3) is simply γ and the right-hand side of the expression defining γ reduces to its second term. Thus

$$\mu(M(j\omega)) = |w_2(j\omega)(1 + P_0(j\omega)K(j\omega))^{-1}| + |w_1(j\omega)(1 + K(j\omega)P_0(j\omega))^{-1}K(j\omega)P_0(j\omega)|$$

and the condition for robust performance can be expressed as

$$|w_2(j\omega)| + |w_1(j\omega)K(j\omega)P_0(j\omega)| < |1 + P_0(j\omega)K(j\omega)| \; \forall \omega \in \mathbb{R}_e$$

Extensions

The structured singular value may be further generalized in many directions, depending on uncertainty descriptions and characterizations. Some of these generalizations are summarized next.

- *Nondiagonal uncertainty structure.* The uncertainty structure need not be diagonal. It can contain unknown, independently bounded blocks in every entry. Doyle (2) and Kouvaritakis and Latchman (15) showed that the analysis may be converted into one based on the standard μ, but this may lead to a substantial increase in computational effort. Chen et al. (16,17) proposed a computational scheme that renders the computation growth insignificant.
- *Uncertainty with phase information.* Tits et al. (18) adopted a notion of μ with phase, in which not only are uncertainties known to be bounded by given quantities, but also their phases are known to vary in given ranges. The formulation gives a more detailed uncertainty de-

scription, and it requires extensions of the concept of phase and of μ.

- *ℓ_1-norm bounded uncertainty.* Khammash and Pearson (19,20) studied structured uncertainties bounded in ℓ_1 norm, which is another active research area in robust control, concerning peak-to-peak system response. They showed that robust stability can be assessed by computing the spectral radius of a positive matrix constructed from the impulse response of the nominal system.
- *Time-varying uncertainty.* Shamma (21) and Megretsky (22) examined the robust stability problem with respect to structured time-varying uncertainties. They showed that if the uncertainty is allowed to vary arbitrarily over time, robust stability holds if and only if for some $D \in \mathscr{D}$, $\|DM(s)D\|_\infty < 1$. It is readily checked that the left-hand side (known as scaled $\mathscr{H}_\infty$-norm), is similar to the right-hand side in condition (1), except that here the same D must be used at all frequencies. Subsequently, Poolla and Tikku (23) showed that, if the time variation of the uncertainty is arbitrarily slow, then robust stability holds if and only if the right-hand side in condition (1) is less than 1 at all frequencies.

Finally, while μ may be custom made and seems to be an all-encompassing paradigm when extended appropriately, it cannot be applied to models in which the uncertainty block Δ is allowed to be unstable. An effective robustness measure for the latter situation is furnished by the *gap metric,* a concept discussed elsewhere in this encyclopedia.

THE KHARITONOV APPROACH

The Kharitonov approach, named after Russian mathematician V. L. Kharitonov whose celebrated 1978 theorem is often considered to be the cornerstone of the field, is largely concerned with the issue of determining zero locations for a family of polynomials whose coefficients vary in a bounded set. Hence, by nature, it can be best presented in a framework different from that of the M–Δ loop, or μ, namely, directly as a polynomial stability problem. This issue, however, is connected to the M–Δ loop paradigm in an intimate fashion. To see this, simply consider a model set comprising proper real rational functions whose coefficients take values in certain bounded intervals. To determine robust stability of such a plant together with any compensator will then amount to checking whether the set of all resultant closed-loop characteristic polynomials have zeros in the "stability region." For continuous-time systems, our main focus, the stability region of interest, is the open left half of the complex plane. Other regions of interest include the open unit disk, a shifted left half plane, and a sector; these regions can be imposed to study stability of discrete-time systems or to enforce pole placement constraints. A polynomial is generally said to be Hurwitz stable, or is referred to as a Hurwitz polynomial, if its zeros lie in the open left half plane.

A general description for a set of polynomials of interest is

$$p(s,q) = \sum_{k=0}^{n} a_k(q) s^k, \tag{6}$$

where q is an unknown vector that may or may not represent physical parameters. When q varies over a bounded set $Q \subset \mathbb{R}^m$, a family of polynomials are generated:

$$\mathscr{P} = \{ p(s,q) : q \in Q \} \tag{7}$$

The problem of concern is to determine if the polynomial family $\mathscr{P}$ is robustly Hurwitz stable, by which we mean that every member in $\mathscr{P}$ is Hurwitz stable. We shall assume that the coefficients $a_k(q)$ are continuous functions of q. Furthemore, we assume that $a_n(q) > 0$ for all $q \in Q$ (i.e., all polynomials in $\mathscr{P}$ have the same degree). For control system analysis, it is typical to restrict the polynomial family $\mathscr{P}$ to the following classes, arranged by order of increased complexity.

1. $\mathscr{P}_{\mathrm{a}}$: the coefficients $a_k(q)$ are affine functions of q. For example,

$$p(s,q) = s^2 + (q_1 + 2q_2 + 3)s + (4q_1 + 5q_2 + 6)$$

2. $\mathscr{P}_{\mathrm{m}}$: the coefficients $a_k(q)$ are multiaffine functions of q. For example,

$$p(s,q) = s^3 + (2q_1q_2 + 2q_1q_3 + q_3 + 1)s^2 + (4q_2q_3 + 5)s + (q_1q_2q_3 + 1)$$

3. $\mathscr{P}_{\mathrm{p}}$: the coefficients $a_k(q)$ are multivariate polynomials in q. For example,

$$p(s,q) = s^3 + (2q_1^2q_2 + 2q_1q_3^2 + q_1q_3 + 1)s^2 + (4q_2q_3 + 5)s + (q_1^2q_2^2q_3^2 + 1)$$

It should be rather evident that $\mathscr{P}_{\mathrm{a}} \subset \mathscr{P}_{\mathrm{m}} \subset \mathscr{P}_{\mathrm{p}}$ and hence that the complexity in analysis increases in that same order. At present, the only available methods for tackling $\mathscr{P}_{\mathrm{m}}$ and $\mathscr{P}_{\mathrm{p}}$ are largely ad hoc, via either local optimization or graphical approaches, and they are either conservative or computationally formidable. In particular, when Q is an ℓ_∞ ball [i.e., a hyperrectangle ("box") parallel to the coordinate axes], the problem of testing the stability of $\mathscr{P}_{\mathrm{m}}$ is known to be NP-hard.

The class $\mathscr{P}_{\mathrm{a}}$, as the sole tractable case, merits a particularly thorough study. A polynomial family $\mathscr{P}$ in this class consists of all polynomials of the form

$$p(s,q) = p(s,q^0) + \sum_{k=0}^{m} (q_k - q_k^0) p_k(s) \tag{8}$$

Here q^0 belongs to Q and may be regarded as the "nominal" value of uncertain parameter vector q, and the $p_k(s)$'s are fixed polynomials. Evidently, one can assume with no loss of generality that $p(s, q^0)$ is Hurwitz stable, which is necessary for $\mathscr{P}$ to be robustly Hurwitz stable as q varies over Q.

Let $p \in [1, \infty]$, and let $\|\cdot\|_p$ be the standard ℓ_p Hölder norm defined on the Euclidean space $\mathbb{R}^m$. That is,

$$\|q\|_p = \begin{cases} \left(\sum_{i=1}^{m} |q_i|^p \right)^{1/p}, & 1 \le p < \infty \\ \max_{1 \le i \le m} |q_i|, & p = \infty \end{cases}$$

Then a common description for Q adopts the notion of unit ℓ_p balls centered at q^0, defined as

$$Q = \{q : \|q - q^0\|_p \le 1\} \tag{9}$$

When the coefficient $a_k(q)$ depends on q_k alone [i.e., when $p_k(s)$ is a constant multiple of s^k], the ℓ_p ball description gives rise to a class of most studied polynomial families. Such families can be expressed as

$$\mathscr{P} = \left\{ \sum_{k=0}^{n} \gamma_k q_k s^k : \|q - q^0\|_p \le 1 \right\} \tag{10}$$

for some (possibly different) q^0 and some fixed scalars γ_k. In particular, for $p = \infty$, 1, 2, such $\mathscr{P}$ is referred to as an interval polynomial, a diamond polynomial, and a spherical polynomial, respectively.

Arguably, the entire success of the Kharitonov approach may be best summarized as a triumph over the polynomial family in Eq. (8) with the ℓ_p norm characterization in Eq. (9), but not beyond, of which the most shining example is Kharitonov's celebrated theorem.

Interval Polynomials

An interval polynomial can be equivalently expressed as

$$\mathscr{P} = \left\{ \sum_{k=0}^{n} q_k s^k : \underline{q}_k \le q_k \le \overline{q}_k \right\} \tag{11}$$

where the $\underline{q}_k$s and the $\overline{q}_k$s are fixed. Kharitonov's original treatment of the interval polynomial problem is of an algebraic nature. He constructed four extremal members of $\mathscr{P}$,

$$\begin{aligned}
K_1(s) &= \underline{q}_0 + \underline{q}_1 s + \overline{q}_2 s^2 + \overline{q}_3 s^3 + \underline{q}_4 s^4 + \underline{q}_5 s^5 + \cdots \\
K_2(s) &= \underline{q}_0 + \overline{q}_1 s + \overline{q}_2 s^2 + \underline{q}_3 s^3 + \underline{q}_4 s^4 + \overline{q}_5 s^5 + \cdots \\
K_3(s) &= \overline{q}_0 + \overline{q}_1 s + \underline{q}_2 s^2 + \underline{q}_3 s^3 + \overline{q}_4 s^4 + \overline{q}_5 s^5 + \cdots \\
K_4(s) &= \overline{q}_0 + \underline{q}_1 s + \underline{q}_2 s^2 + \overline{q}_3 s^3 + \overline{q}_4 s^4 + \underline{q}_5 s^5 + \cdots
\end{aligned}$$

later dubbed Kharitonov polynomials. In a remarkable fashion, Kharitonov showed that the stability of the entire interval polynomial family can be ascertained by testing merely these four.

Kharitonov's Theorem. The interval polynomial is Hurwitz stable if and only if $K_1(s)$, $K_2(s)$, $K_3(s)$, and $K_4(s)$ are all Hurwitz stable.

Subsequent development showed that for polynomials of degree 5, 4, and 3, the test can be further simplified, requiring checking only 3, 2, and 1 of the four Kharitonov polynomials, respectively.

Dasgupta (24) gave a geometrical interpretation to Kharitonov's Theorem in frequency domain, which sheds light on why such a puzzling result would hold. The interpretation makes use of the concept of *value set.* The idea is to examine the image of the polynomial family when s lies on the boundary of the stability region, namely

$$p(j\omega, Q) = \{p(j\omega, q) : q \in Q\}$$

Because $p(s, q^0)$ is stable and by assumption $a_n(q)$ never vanishes on Q, we can conclude from the continuity properties of the zeros of polynomials that if some member in $\mathscr{P}$ is not Hurwitz stable, then some polynomial in $\mathscr{P}$ must have a zero on the imaginary axis. Thus the entire family $\mathscr{P}$ is Hurwitz stable if and only if

$$0 \notin p(j\omega, Q), \quad \forall \omega$$

that is, the value set never contains the origin. This is a rather straightforward fact known as early as in 1929 and is generally referred to in the literature as the *zero exclusion principle.* Note, in particular, that for a polynomial $p(s, q)$ in the family described by Eq. (11), for any $q \in Q$, the real and imaginary parts of $p(j\omega, q)$ are respectively given by

$$\operatorname{Re} p(j\omega, q) = q_0 - q_2\omega^2 + q_4\omega^4 - \cdots$$

and

$$\operatorname{Im} p(j\omega, q) = j\omega(q_1 - q_3\omega^2 + q_5\omega^4 - \cdots)$$

Thus the real part (resp. imaginary part) of $p(j\omega, q)$ depends only on the parameters with even (resp. odd) subscript. For an interval polynomial, therefore, it becomes clear that

$$\begin{aligned}
\operatorname{Re} K_1(j\omega) = \operatorname{Re} K_2(j\omega) \le \operatorname{Re} p(j\omega, q) \le \operatorname{Re} K_3(j\omega) = \operatorname{Re} K_4(j\omega) \\
\operatorname{Im} K_1(j\omega) = \operatorname{Im} K_4(j\omega) \le \operatorname{Im} p(j\omega, q) \le \operatorname{Im} K_3(j\omega) = \operatorname{Im} K_2(j\omega)
\end{aligned}$$

Because the four Kharitonov polynomials are themselves in $\mathscr{P}$, it follows that, at each ω, the value set is a rectangle in the complex plane with vertices $K_i(j\omega)$, as depicted in Fig. 4.

As ω increases, the rectangle evolves in a continuous fashion. Its dimensions vary, but its edges remain parallel to the axes, and the relative positions of the $K_i(j\omega)$s do not change. Now, a polynomial $p(s)$ of degree n with positive coefficients is Hurwitz stable if and only if the phase of $p(j\omega)$ monotonically increases from 0 to $n\pi/2$ as ω goes from 0 to ∞. This is best seen by noting that $p(s)$ can always be factorized as $p(s) = a_n\Pi(s - s_i)$ in terms of its zeros s_i, and by considering the phase of each factor separately. When this is understood, Kharitonov's Theorem becomes self-evident. Indeed, if the K_is are Hurwitz stable, then the phase of each $K_i(j\omega)$ increases monotonically from 0 to $n\pi/2$ and the entire Kharitonov rectangle rotates around the origin by a total angle of $n\pi/2$ without ever touching it. Stability of $\mathscr{P}$ then follows from the zero exclusion principle.

Kharitonov's Theorem opened an era epitomized by the search for so-called vertex results in robust stability analysis, as manifested by the construction of the four Kharitonov polynomials at the vertices of the hyperrectangle Q. This theme

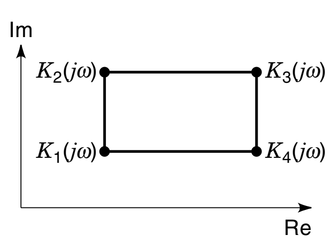

Figure 4. Kharitonov rectangle.

persisted in much of the subsequent work. Although Kharitonov's Theorem is without question a milestone in control theory, and perhaps a measured intellectual triumph in general, we should note that the interval polynomial family is nevertheless a very special instance of an affine uncertainty class, and hence the theorem has a rather limited scope in application. The quest thus continues.

Edge Theorem

In light of Kharitonov's Theorem, it is tempting to contemplate the possibility that a similar vertex result would hold for the general affine polynomial family with Q a hyperrectangle. This, unfortunately, turns out to be false; a counterexample can be readily constructed to demonstrate the opposite. What can be said about a polynomial family $\mathscr{P}$ in class $\mathscr{P}_a$? Bartlett et al. (25) provided an answer. In what is now known as the Edge Theorem, they took the bounding set Q to be a convex polytope in $\mathbb{R}^m$. Let q^i, $i = 1, \ldots, l$, be the vertices of Q. Then, it is well known that Q can be represented as a convex hull of the vertices. That is,

$$Q = \text{conv}\{q^1, \ldots, q^l\} = \left\{\sum_{i=1}^{l} \lambda_i q^i : \sum_{i=0}^{l} \lambda_i = 1, \lambda_i \geq 0\right\}$$

Because $a_k(q)$ is an affine function of q, it follows that

$$\mathscr{P} = \text{conv}\{p(s, q^1), \ldots, p(s, q^l)\}$$

This implies that the value set $p(j\omega, Q)$ is a polygon, generated by the vertex polynomials $p(s, q^i)$. It should be rather clear that the interval polynomial family is generated by a polytope—a hyperrectangle—and so is the diamond polynomial family.

Edge Theorem. The affine polynomial family (polytope of polynomials) $\mathscr{P}_a$ is Hurwitz stable if and only if for each edge point q of Q, $p(s, q)$ is Hurwitz stable.

Let q^i and q^j be two vertices of Q connected by an edge. When λ varies from 0 to 1, the polynomial

$$p_{ij}(s, \lambda) = \lambda p(s, q^i) + (1 - \lambda) p(s, q^j)$$

defines a line segment (of polynomials) connecting $p(s, q^i)$ and $p(s, q^j)$. The theorem shows that—quoting directly from Ref. 25—*it suffices to check the edges.* Because an edge polynomial involves only one parameter, its stability can be readily tested, by resorting to either a graphical test based upon a root locus, or the numerical solution of a generalized eigenvalue problem (see, for example, Chapter 4 in Ref. 26).

The heuristics behind the Edge Theorem are simple. Because for large values of ω, the value set does not contain the origin (this is easy to see), the polynomial family will be Hurwitz stable if and only if there is no frequency ω at which the origin belongs to the *boundary* of the (polygonal) value set. It should be intuitively clear that every point on the boundary of the value set must be the image of a point on an *edge* of the polytope of polynomials. Thus, if for some ω the origin does belong to this boundary, the corresponding edge polynomial must be unstable. Note that it is wasteful to check the entire set of all polynomial "segments" joining two vertices of the polytope because some of these segments are not edges. In some cases, however, it is not an easy task to check which such segments are edges and which are not.

The Edge Theorem generalizes Kharitonov's Theorem in two important aspects. First, it is applicable to general affine polynomial families, with Q an arbitrary polytope. Secondly, the stability region may be any open, simply connected set (27,28), although only stated here in terms of the open left half plane. In contrast, Kharitonov's Theorem addresses only Hurwitz stability. On the other hand, as the number of parameters increases, the computation effort required in the Edge Theorem can be enormous. For example, when Q is a box in $\mathbb{R}^m$, with 2^m vertices, the number of edges is $m2^{m-1}$, a very large number even for moderate values of m. Finally, to a lesser extent, the Edge Theorem is not applicable to situations where the boundary of Q is curved, of which the spherical polynomial family is an example.

Graphical Tests

The complexity of the Edge Theorem provides a direct motivation to search for computationally more tractable stability criteria, and graphical conditions become a natural avenue to explore. Not only are graphical criteria time-honored tools in classical stability analysis, but the zero exclusion principle, with all its simplicity and transparency, also prompts a deeper investigation of such tools. We can generally feel that an important asset of the zero exclusion principle is its generality, both in terms of uncertain polynomial families and in terms of stability regions.

Barmish (29) studied the issue systematically; earlier conditions had appeared sporadically on stability of perturbed polynomials in isolated cases. Barmish's approach stems from a geometrical argument: a convex polygon in the complex plane does not intersect the origin as long as it can be separated from it by a straight line or, equivalently, as long as the vertices can be separated, as a whole, from the origin by such a line. This observation led to his construction of a testing function, which is to be evaluated along the boundary of the stability region. After this is accomplished, we can determine stability by plotting the testing function. Barmish's test is certainly one step forward compared to a pure brute-force computation; however, it remains somewhat ad hoc and is computationally overwhelming. Because it requires evaluations at all vertices, it does not clear the hurdle of exponential growth in complexity.

On a lesser scale, Tsypkin and Polyak (30) obtained a graphical test for a simpler problem. They examined the polynomial family in Eq. (10). Let $p \in [1, \infty]$ be given. Furthermore, for $r \in [1, \infty]$ such that $(1/p) + (1/r) = 1$, define

$$X_r(\omega) = (\gamma_0^r + (\gamma_2\omega^2)^r + (\gamma_4\omega^4)^r + \cdots)^{1/r}$$
$$Y_r(\omega) = (\gamma_1^r + (\gamma_3\omega^2)^r + (\gamma_5\omega^4)^r + \cdots)^{1/r}$$
$$R(\omega) = \gamma_0 q_0^0 - \gamma_2 q_2^0 \omega^2 + \gamma_4 q_4^0 \omega^4 + \cdots$$
$$I(\omega) = \gamma_1 q_1^0 - \gamma_3 q_3^0 \omega^2 + \gamma_5 q_5^0 \omega^4 + \cdots$$

Note that $R(\omega) = \text{Re}(p(j\omega, q^0))$ and $\omega I(\omega) = \text{Im}(p(j\omega, q^0))$.

Tsypkin-Polyak Criterion. The polynomial family $\mathscr{P}$ of Eq. (10) is Hurwitz stable if and only if $p(s, q^0)$ is Hurwitz stable,

$|q_n^0| > 1$, $|q_0^0| > 1$, and

$$\left(\frac{|R(\omega)|}{X_r(\omega)}\right)^p + \left(\frac{|I(\omega)|}{Y_r(\omega)}\right)^p > 1, \quad \forall \omega \in (0, \infty) \tag{12}$$

This condition was independently obtained by Hinrichsen and Pritchard (31).

Up to this point, we could assert that the stability problem for affine polynomial families remains largely unresolved. However, as yet another observation, we find that at each ω, the zero exclusion condition defines two linear constraints in terms of perturbed coefficients, imposed on the real and imaginary parts of $p(j\omega, q)$, respectively. These constraints, together with a convex bounding set Q, define in turn a convex feasibility condition; when the parameters vary independently, it reduces further to a linear program. This is a simple but conceptually appealing observation. It led to a reformulation via linear programming, due to Saridereli and Kern (32) and to Tesi and Vicino (33), which can be solved readily for each ω and then plotted graphically. Qiu and Davison (34) went further to demonstrate that for very general bounding sets it suffices to solve an optimization problem with one variable only, and the problem can be solved explicitly in special cases. Finally, Chen et al. (14,35) recognized that the problem can be reformulated as a special rank-one μ problem for each ω, and showed that stability can be ascertained by evaluating an explicit formula. These results led to the final resolution of the affine polynomial family.

μ and the Kharitonov Approach

Indeed, there is an inherent linkage between μ analysis and Kharitonov approach, whenever the latter applies, in that both approaches yield necessary and sufficient conditions for problems of the same nature. However, for a rather long time a clear link seemed elusive. The main cause, it seems, lay in how to reconcile an optimization-based formulation such as μ, and explicit results from the Kharitonov approach. Can one, for example, derive Kharitonov's theorem from μ, or vice versa?

The explicit formula of rank-one μ given earlier lends an answer. Specifically, for a general affine polynomial family $\mathscr{P}_a$ [(Eq. (8)] with Q the unit ℓ_∞ ball, robust stability can be checked by computing μ, with a rank-one matrix $M(s)$ constructed as

$$M(s) = \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix} \begin{bmatrix} \dfrac{p_1(s)}{p(s,q^0)} & \cdots & \dfrac{p_m(s)}{p(s,q^0)} \end{bmatrix}$$

To see this, observe that $p(s, q)$ in Eq. (8) is the characteristic polynomial of the M–Δ loop of Fig. 1 with $\Delta = -\text{diag}(q_1 - q_1^0, \ldots, q_m - q_m^0)$. Indeed,

$$\det(I - \Delta M(s)) = 1 + \sum_{k=0}^{m} \frac{p_k(s)}{p(s,q^0)}(q_k - q_k^0) = \frac{p(s,q)}{p(s,q^0)}$$

Thus, stability of $p(s, q)$ for all $q \in Q$ is equivalent to stability of the M–Δ loop for all diagonal matrices Δ with real (parametric) entries lying in $[-1, 1]$. The condition for this is that $\|M\|_\mu < 1$ where the structured singular value is computed with respect to the block structure $\boldsymbol{\Gamma} = \{\text{diag}(\delta_1, \ldots, \delta_m)\colon \delta_i \in \mathbb{R}\}$. In light of the formula for rank-one μ, an explicit condition can then be stated. Such a result clearly applies to general stability regions, and it furnishes a frequency sweeping condition for robust stability. Note that we may interpret this result alternatively based upon the zero exclusion principle. Indeed, under the condition that Q is the unit ℓ_∞ ball centered at q^0, all the polynomials in $\mathscr{P}$ will have zeros in a specified region if and only if the zeros of $p(s, q^0)$ are in that region and $\mu(M(s)) < 1$ for all s on the boundary of the region. This follows because, according to the zero exclusion principle, it is both necessary and sufficient that

$$\min\left\{\|q - q^0\|_\infty : p(s,q^0) + \sum_{k=0}^{m}(q_k - q_k^0)p_k(s) = 0\right\} > 1$$

in order for the polynomial family $p(s, q)$ in Eq. (8) to have no zero on or exterior to the boundary of the region, for all possible $q \in Q$.

More generally, it is possible to extend the definition of μ by means of more general norms and to use this extended μ to study the robust stability of an affine family $\mathscr{P}$ with a more general bounding set Q. Such a generalization also leads to a similar expression when the M matrix in question has rank one (14,35). In particular, when the stability region is restricted to the open left half plane, and Q is the unit ℓ_p ball centered at q^0 with $a_k(q) = \gamma_k q_k$, the expression for the generalized rank-one μ, denoted as $\mu_p(\cdot)$ for purpose of distinction, is found to be

$$\mu_p(M(j\omega)) = \begin{cases} Y_r(\omega)/|I(\omega)| & \text{if } R(\omega) = 0 \\ X_r(\omega)/|R(\omega)| & \text{if } \omega I(\omega) = 0 \\ \dfrac{X_r(\omega)Y_r(\omega)}{(X_r^p(\omega)|I(\omega)|^p + Y_r^p(\omega)|R(\omega)|^p)^{1/p}} & \text{otherwise} \end{cases}$$

which leads to a similar condition for robust Hurwitz stability. This condition is slightly more general than, but essentially replicates, the graphical criterion by Tsypkin and Polyak. Note that for $p = \infty$, the polynomial family becomes an interval polynomial, and the stability condition reduces to checking whether $p(s, q^0)$ is Hurwitz stable, $|q_0| > 1$, and

$$\min\left\{\frac{X_1(\omega)}{|R(\omega)|}, \frac{Y_1(\omega)}{|I(\omega)|}\right\} < 1, \qquad \forall \omega \in (0, \infty)$$

A little thought reveals that the latter is equivalent to determining whether one of the four conditions $\text{Re}(K_1(j\omega)) > 0$, $\text{Re}(K_3(j\omega)) < 0$, $\text{Im}(K_1(j\omega)) > 0$, and $\text{Im}(K_3(j\omega)) < 0$ holds. Clearly, this is further equivalent to the requirement that the rectangular value set in Fig. 4 never contains the origin.

Extensions

There is an immense body of literature devoted to polynomial stability problems. Various extensions to Kharitonov's Theorem have been obtained. They generally fall into the categories of vertex results and frequency-sweeping conditions, consisting of delicate studies and intricate technical details. We summarize some of the highlights next. A recent and comprehensive account can be found in the books by Barmish (26) and by Bhattacharyya et al. (36).

Vertex and Edge Results. Much of the work in this direction continues the thread in Kharitonov's Theorem, focusing on simple uncertainty descriptions and leading to stability tests based on vertex and/or edge polynomials. Some notable examples follow.

- *Complex Interval Polynomials.* The polynomial family has complex coefficients whose real and imaginary parts are allowed to vary independently in given intervals. Eight vertex polynomials need to be tested to ascertain stability.
- *Diamond Polynomials.* This polynomial family is described in Eq. (10), with $p = 1$. Eight vertex polynomials are required as well.
- *Stabilization of Interval Plants via First-Order Compensators.* The numerator and denominator of the plant transfer function are interval polynomials, and it is to be stabilized by a first-order compensator in closed loop. It suffices to stabilize 16 vertex plants, constructed based upon the vertex numerator and denominator polynomials.
- *Generalized Kharitonov Theorem.* It concerns linear combination of interval polynomials and requires checking certain polynomial segments in addition to vertices.

Other stability conditions based on vertex polynomials are also available. As the complexity of uncertainty structure increases slightly, they usually require testing more (e.g., 32 or 64) vertex polynomials. A clear insight concerning uncertainty structure and the required number of vertices, however, remains unavailable.

Performance Issues. The entire Kharitonov theory is largely successful for determining stability of uncertain polynomials. However, a number of results are also available regarding properties of transfer functions, which have implications toward performance issues. Some examples follow.

- *$\mathcal{H}_\infty$ Norm of Interval Transfer Functions.* When the numerator and denominator of a transfer function are both interval polynomials, the $\mathcal{H}_\infty$ norm of the transfer function can be computed over 16 vertex transfer functions, provided that the four Kharitonov polynomials associated with the denominator are stable.
- *Peak Magnitudes of Closed-Loop Transfer Functions.* The peak $\mathcal{H}_\infty$ norm of closed-loop transfer functions can be computed over the edges of the plant family, when it is an interval plant.
- *Nyquist and Bode Envelopes.* The Nyquist and Bode plots of open or closed-loop transfer functions associated with an interval plant lie in envelopes determined by plots generated by vertex and edge plants.

Other Extensions. Additional extensions may be found in the following categories.

- *Schur Stability.* The Kharitonov theory has been extended with varying degrees of success to other stability regions, such as the unit circle. These results are useful for studying stability of discrete-time systems and for addressing other performance issues.
- *Unstructured Uncertainty.* Unmodeled dynamics may be included along with parametric uncertainties. They may be accommodated either in the rank-one μ formula or by small gain-type conditions involving vertex and edge plants.
- *Nonlinear Systems.* In a system consisting of an interval plant and a static, sector bounded nonlinear component, stability conditions similar to the Popov and circle criteria have been obtained, which also require the testing of vertex and edge plants.
- *Multilinear Uncertainty Structure.* The entire success in the Kharitonov approach relies on the key assumption that polynomial coefficients depend linearly on uncertain parameters, and the utility of all the results in this area is thus measured by how much the uncertainty can deviate from this description. Little success has been achieved in this endeavor. A fundamental barrier, as implicated by the zero exclusion principle, is that the stability problem is one of optimization subject to nonlinear, nonconvex constraints.

STATE-SPACE APPROACH

Dynamical systems are often described by state-space equations. Accordingly, it is common to model system uncertainty as perturbations to system matrices. An uncertain continuous-time system in this spirit may be described by

$$\dot{x}(t) = (A + B\Delta C)x(t) \tag{13}$$

Here A, B, and C are known matrices of appropriate dimensions. The matrix A is assumed to be stable. The system uncertainty is represented by a set $\mathbf{\Delta}$ of allowed values for the *real* matrix Δ, which may be unstructured or structured. Typical perturbation classes considered in the literature are as follows, arranged in increasing order of generality. In all cases, $\gamma > 0$ is given.

- *Unstructured Perturbation.* The set $\mathbf{\Delta}$ consist of all real matrices with spectral norm less than a given number:

$$\mathbf{\Delta}_{\mathrm{U}} = \{\Delta \text{ real} : \overline{\sigma}(\Delta) \le \gamma\}$$

- *Element-by-Element Perturbation.* Each element in Δ varies in a given interval. Let $r_{ij} \ge 0$ be given. The set $\mathbf{\Delta}$ is defined as

$$\mathbf{\Delta}_{\mathrm{E}} = \left\{ \Delta \text{ real} : \Delta = \begin{bmatrix} r_{11}\delta_{11} & \cdots & r_{1m}\delta_{1m} \\ \vdots & \vdots & \vdots \\ r_{n1}\delta_{n1} & \cdots & r_{nm}\delta_{nm} \end{bmatrix}, \right.$$
$$\left. \|\Delta\| = \max_{i,j}\{|\delta_{ij}| : r_{ij} > 0\} \le \gamma \right\}$$

- *Linear Combination.* The allowable set of perturbations is described by

$$\mathbf{\Delta}_{\mathrm{L}} = \left\{ \Delta \text{ real} : \Delta = \sum_{i=1}^{k} \delta_i E_i,\ \|\Delta\| = \max_i |\delta_i| \le \gamma \right\}$$

where the E_is are given.

Evidently, an uncertain discrete-time system can be described in exactly the same manner.

The problem of interest is to determine the size of the perturbation matrix, measured by a norm of choice, so that the system remains stable. This issue naturally translates into one concerning how the eigenvalues of a stable matrix A would vary when it is perturbed by Δ. More specifically, would the eigenvalues cross the stability boundary? And if they do, what is the minimal γ such that at least one of the eigenvalues leaves the stability region? These questions may be addressed by examining the characteristic polynomial

$$\Phi(s, \Delta) = \det(sI - A - B\Delta C)$$

or equivalently, the characteristic equation

$$\det(I - C(sI - A)^{-1}B\Delta) = 0$$

Thus, it becomes clear at the outset that the problem may be tackled in principle by using a polynomial approach. Similarly, it can also be analyzed as a μ problem. The latter can be easily seen with respect to $\mathbf{\Delta}_E$ and $\mathbf{\Delta}_L$, by rearranging the elements of these sets into diagonal matrices and by defining the M matrix appropriately. For $\mathbf{\Delta}_U$, we may simply adopt a full real block structure and define μ accordingly. It should be pointed out, nevertheless, that both μ and the polynomial approach will lead to complications in the present context. On the one hand, the computation of μ with respect to a real Δ is generally very difficult, and approximation by its upper bound can be very conservative. On the other hand, the characteristic polynomial $\Phi(s, \Delta)$ will generally exhibit a multilinear or multinomial dependence of its coefficients on Δ, for which the Kharitonov theory is ill-equipped; indeed, it is not difficult to see that the coefficients of $\Phi(s, \Delta)$ are multilinear in δ_{ij} if $\Delta \in \mathbf{\Delta}_E$, and are multinomial functions of δ_k if $\Delta \in \mathbf{\Delta}_L$. In summary, both approaches are ineffective and conservative.

By far this uncertainty description poses the most difficult challenge in robust stability analysis, and the state-space approach is the least developed. Results are scarce, and only in rare cases are they nonconservative.

Stability Radius

A notion frequently encountered in studying the state-space uncertainty description is that of stability radius. This notion is closely related to μ, but it is less developed. Let $\mathbb{D}$ be a stability region of concern, and $\partial\mathbb{D}$ be its boundary. Furthermore, denote by $\sigma(A) \subset \mathbb{D}$ the spectrum of A. Then for any norm $\|\cdot\|$ of interest, the stability radius associated with the triple (A, B, C) is defined by

$$r(A, B, C) = \inf\{\|\Delta\| : \Delta \in \mathbf{\Delta}, \sigma(A + B\Delta C) \cap \partial\mathbb{D} \neq \emptyset\}$$

In other words, it defines the minimal perturbation size leading to instability, or the "distance" of A to the set of unstable matrices. By definition, it thus follows directly that the matrix family $\{A + B\Delta C: \Delta \in \mathbf{\Delta}, \|\Delta\| \leq \gamma\}$ has all eigenvalues in $\mathbb{D}$ whenever $r(A, B, C) > \gamma$. Moreover, in view of the preceding discussion, we may regard the stability radius as the reciprocal of the maximum of a certain μ, with respect to an appropriate block structure and a matrix M. For further distinction, the stability radius is said to be unstructured if Δ is unstructured and structured otherwise.

There essentially exists no result for the structured stability radius other than those already known for μ. For the unstructured stability radius, much of the early work was devoted to derivation of bounds. One representative example is

$$r(A, B, C) \geq \frac{1}{\|C(sI - A)^{-1}B\|_\infty} \tag{14}$$

This, of course, is a rather straightforward consequence of the Small Gain Theorem. Recently, however, Qiu et al. (37) obtained the following exact, readily computable formula.

Real Stability Radius. Let $G(s) = C(sI - A)^{-1}B$, and $\sigma_2(\cdot)$ be the second largest singular value. Then,

$$r(A, B, C)^{-1} = \sup_{s \in \partial D} \inf_{\gamma \in (0,1]} \sigma_2 \left(\begin{bmatrix} \mathrm{Re}[G(s)] & -\gamma \mathrm{Im}[G(s)] \\ \frac{1}{\gamma}\mathrm{Im}[G(s)] & \mathrm{Re}[G(s)] \end{bmatrix} \right)$$

The significance of this result lies in that for any $s \in \partial\mathbb{D}$, the function $\sigma_2(\cdot)$ is unimodal in γ over (0, 1), and hence its infimum can be computed effectively. Furthermore, when $\mathbb{D}$ is the open left half plane or the open unit disk, that is, when Hurwitz or Schur stability is of concern, Sreedhar et al. (38) developed a fast-converging algorithm for the maximization with respect to s. Consequently, from a computational standpoint, the unstructured stability radius problem can be considered largely resolved.

Interval Matrices

An interval matrix is a family of real matrices in which all elements are known only within certain closed intervals. In precise terms, the interval matrix $A_I = [\underline{A}, \overline{A}]$ is the set of matrices defined by

$$A_I = \{A : \underline{a}_{ij} \leq a_{ij} \leq \overline{a}_{ij}\}$$

that is, each a_{ij} of A is confined elementwise to lie within an interval determined by $\underline{a}_{ij}$ and $\overline{a}_{ij}$, the corresponding elements of $\underline{A}$ and $\overline{A}$, respectively. An interval matrix A_I is said to be stable if every $A \in A_I$ is stable. Evidently, interval matrix and set $\mathbf{\Delta}_E$ share the same uncertainty description.

Interval matrices are direct matrix analogues of interval polynomials, and hence there has been a lingering temptation for extension of Kharitonov's Theorem to the former. Unfortunately, neither vertex nor edge results exist for interval matrices. In fact, more recent studies showed that in order to determine stability of an interval matrix, we must solve an NP-hard decision problem. This in a way explains why only sufficient stability conditions are available.

One approach of attack is to analyze eigenvalue distribution. Heinen (39) and Argoun (40) examined the problem on the basis of Gershgorin's Theorem, and their developments culminated in a subsequent work of Chen (41), leading to a number of simple, albeit conservative, stability conditions. As a representative example of these results, consider an interval matrix A_I such that $\overline{a}_{ii} < 0$. Let W be constructed as

$$W = [w_{ij}], \qquad w_{ij} = \begin{cases} 0 & i = j \\ \dfrac{\max\{|\underline{a}_{ij}|, |\overline{a}_{ij}|\}}{|\overline{a}_{ii}|} & i \neq j \end{cases}$$

Then a sufficient condition for Hurwitz stability of A_I is found by Chen (41) to be

$$\rho(W) < 1$$

A useful feature of this result, and more generally of conditions obtained by using Gershgorin's Theorem, is that it lends a ready characterization via the so-called M-matrices. The latter aspect makes it possible to unify a number of sufficient stability conditions in different forms.

Alternatively, Yedavalli and others studied interval matrices from a Lyapunov analysis standpoint. This is collectively inspected next.

Lyapunov Analysis

The Lyapunov theory, as anticipated, is widely employed in robust stability analysis pertaining to state-space formulation, yielding various results concerning stability radius and interval matrices. One common thread in this approach is to find a *single* quadratic Lyapunov function applicable to the entire family of the perturbed matrices; the technique is often referred to in the literature as *quadratic stability*. Another lies in the simplicity of the stability conditions.

Let us begin with the unstructured uncertainty set $\mathbf{\Delta}_{\mathrm{U}}$. By constructing the usual Lyapunov function

$$V(x) = \tfrac{1}{2}x^{\mathrm{T}}Px$$

we find that the entire family of matrices $\{A + B\Delta C: \Delta \in \mathbf{\Delta}_{\mathrm{U}}\}$ is Hurwitz stable if there exists a positive definite matrix $P > 0$ such that

$$PA + A^{\mathrm{T}}P + \gamma^2 BB^{\mathrm{T}} + C^{\mathrm{T}}C < 0 \tag{15}$$

This, of course, does not come as a surprise. According to the well-known Bounded Real Lemma (42), it is equivalent to

$$\|C(sI - A)^{-1}B\|_\infty < 1/\gamma$$

The latter condition clearly coincides with condition (14).

More results are available for the special case when $B = C = I$. For the structured uncertainty set $\mathbf{\Delta}_{\mathrm{E}}$, Yedavalli (43) gave the sufficient condition

$$\overline{\sigma}\left(\frac{|P|R + R^{\mathrm{T}}|P|}{2}\right) < 1/\gamma \tag{16}$$

for the Hurwitz stability of the matrix family $\{A + \Delta: \Delta \in \mathbf{\Delta}_{\mathrm{E}}\}$. Here $P > 0$ is the unique solution to the Lyapunov equation

$$PA + A^{\mathrm{T}}P = -2I \tag{17}$$

$|P|$ denotes the modulus matrix of P (i.e., each entry of $|P|$ is the absolute value of the corresponding entry of P) and R is given by

$$R = \begin{bmatrix} r_{11} & \cdots & r_{1n} \\ \vdots & \vdots & \vdots \\ r_{n1} & \cdots & r_{nn} \end{bmatrix}$$

Furthermore, Zhou and Khargonekar (42) observed that the uncertainty description $\mathbf{\Delta}_{\mathrm{E}}$ can be regarded as a special case of $\mathbf{\Delta}_{\mathrm{L}}$, for which they provided the stronger Hurwitz stability condition

$$\overline{\sigma}\left(\frac{1}{2}\sum_{i=1}^{k}(|PE_i + E_i^{\mathrm{T}}P|)\right) < 1/\gamma \tag{18}$$

where, again, $P > 0$ is the unique solution to Eq. (17). Subsequent developments led to further extensions for problems with even more detailed uncertainty descriptions. For example, the δ_ks may be allowed to vary in asymmetric intervals. Moreover, because rather obviously any interval matrix can be represented alternatively in the form of $\{A + \Delta: \Delta \in \Delta_{\mathrm{E}}\}$, these conditions can be applied to determine the Hurwitz stability of an interval matrix as well.

Yet another issue clearly of interest is whether it is possible to derive vertex versions of these sufficient conditions. Boyd and Yang (44) examined stability problems for matrix polytopes. Specifically, they postulated the uncertainty description

$$A = \mathrm{conv}\{A_1, \ldots, A_k\}$$

A sufficient condition for A to be Hurwitz stable can be easily found to be the existence of a $P > 0$ such that

$$PA_i + A_i^{\mathrm{T}}P < 0, \qquad i = 1, \ldots, k \tag{19}$$

Similarly, for the uncertainty set $\mathbf{\Delta}_{\mathrm{L}}$, a vertex condition can be obtained as

$$\overline{\sigma}\left(\frac{1}{2}\sum_{i=1}^{k}\epsilon_i(PE_i + E_i^{\mathrm{T}}P)\right) < 1/\gamma \tag{20}$$

for all combinations of the ϵ_i in $\{-1, +1\}$. It should be rather evident that this condition improves upon inequality (18). Both conditions (19) and (20) may be regarded as vertex results in the matrix perturbation case, and both can be posed and solved as LMI problems.

CONCLUSION

Summary

For the past two decades, modeling uncertainty and robustness has resurfaced as a dominating theme in control theory and application and is now held unanimously by theoreticians and practitioners as the most important concern in control system design. For both its intrinsic appeal and practical significance, robust control as a whole attracted considerable interest and underwent a period of immense development, bringing control theory to a new height. Many important issues have been addressed. Many remain unresolved. The ultimate puzzle, it now appears, lies in the fundamental conflict between problem complexity and computational tractability.

Of the three main research areas surveyed in this article, the structured singular value provides the most general formulation for uncertainty modeling and is the most systematically developed tool in robustness analysis. The major issues in μ analysis are clearly generality of uncertainty description, conservatism of analysis, and ease of computation. The main success achieved with this approach, unquestionably, lies in the progress in computing μ. While it cannot be computed exactly in general, various computational schemes have been developed for computing it approximately, and commercial software programs are now available. This paves the way for its application to a series of engineering design problems, ranging from disk drive control to flight control. Successful applications to other potential areas, including robot manipulators, flexible structures, magnetic bearings, and chemical processes, have also been reported in laboratory experiments.

The Kharitonov approach, unlike μ analysis, was more restrictive in scope in its early phase of development. However, it has undergone a "bottom-up" growth pattern as the uncertainty descriptions become progressively more general and sophisticated. Overall, the Kharitonov and state-space methods may be broadly classified as a parametric approach toward robustness analysis, originating from interval polynomials and culminating at state-space uncertainty descriptions. The main appeal of this approach, it appears, lies in its quest for analytical solutions, more appealing than mere computation-based tools. The main success in the entire parametric approach, which remains the state-of-the-art today, is the resolution of the affine uncertain polynomial family case, for which necessary and sufficient stability conditions are available, in terms of both edge tests and graphical conditions. On the other hand, the multilinear/multinomial polynomial family and the state-space uncertainty description are the weakest link, for which only sufficient stability conditions are available with unknown conservatism, and more systematic, efficient, computation-based approximate tests are called for. At present, only a few applications of the Kharitonov theory are reported in the literature, including Ackermann's car steering problem and an automotive engine control problem investigated by Abate et al. (45) (see also Chapter 3 in Ref. 26). It should be rather evident that the fundamental bottleneck in all robustness analysis methods, be it μ analysis or Kharitonov approach, lies in computational complexity, and the ultimate challenge is in the conquest over the "curse of dimensionality." No matter whether this can be achieved or not, we should be consciously aware that the dilemma is the natural cause of problem generality and hence complexity and results from the search of optimal solutions. In engineering system design, we should therefore reconcile and seek a judicious trade-off between these conflicting requirements.

To Probe Further

In light of the difficulties encountered in robustness analysis with respect to structured and/or parametric uncertainties, a number of researchers recently examined complexity issues from a computational standpoint, drawing upon concepts and techniques from computing science and operation research. The main discoveries are in the following areas.

- *μ with Real Uncertainties.* Braatz et al. (46) showed that the computation of real μ is NP-hard.
- *μ with Complex Uncertainties.* Toker and Ozbay (47) proved that the computation of complex μ is also NP-hard.
- *μ with Real and Complex Uncertainties.* Braatz et al. (46) and Toker and Ozbay (47) both showed that the computation of μ is NP-hard.
- *The μ Bounds.* Toker (48) and Fu (49) showed that the problem of finding an accurate bound for μ is NP-hard.
- *Interval Matrix.* Coxson and DeMarco (50) showed that stability of interval matrices amounts to an NP-hard problem.

These results indicate that a worst-case instance exists in each class of the problems, for which it is rather unlikely that computational complexity can be bounded via a polynomial function of the problem dimension. It thus comes as no surprise that the problems are difficult, and indeed are intractable in general.

From a technical standpoint, the computational difficulty in question may be best seen as an outcome of nonlinear, nonconvex optimization problems. Although only explored systematically in recent years, complexity issues have been under contemplation for a long time and have led to alternative, computationally tractable approximations and formulations. One notable remedy is to resort to formulations based upon LMIs, and problems in this class include those that can be described via *integral quadratic constraints* (IQC). Both LMIs and IQCs offer in essence an energy-based perspective toward system analysis, and they draw heavily upon concepts in classical passivity and dissipativity theory, leading to readily computable, albeit only sufficient, robust stability conditions. For a comprehensive treatment of control-relevant LMI and convex programming problems, see Ref. 51, or the relevant chapter in this encyclopedia. Megretsky and Rantzer (52) provided a detailed account of the IQC technique.

The computational complexity results just discussed are strongly linked to the worst-case nature of the robustness problems; that is, the requirement of robustness must be met for all possible instances. Is so stringent a requirement truly necessary? This question prompted a reexamination of robustness issues, and it led to a recent venture departing almost entirely from the worst-case formulation. A number of researchers argued that worst-case scenarios hardly occur in practice, that a worst-case analysis is not only overly demanding but also too pessimistic, and that, after all, worst-case analysis problems are often intractable. The argument thus motivated the description of uncertainty via probabilistic measures, and accordingly probabilistic approaches to robustness analysis. In this new thinking, the deterministic uncertainty description is discarded altogether and is replaced by a probability description characterizing the likelihood that the uncertainty may lie in a bounded set. The robustness condition then amounts to determining the probability under which the system may become unstable. Recent studies (53–56) show that a variety of problems, which are NP-hard in the deterministic setting, become readily solvable computationally when formulated probabilistically. The area, however, is entirely open and is not without obstacles of its own.

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ROBUSTNESS ANALYSIS. See ROBUST CONTROL ANALYSIS.
ROBUST SIGNAL PROCESSING. See NONLINEAR SYSTEMS.
ROLLBACK RECOVERY. See PROGRAM DIAGNOSTICS.
ROLLING METALS. See METALS INDUSTRY.
ROTATION SPEED. See TACHOMETERS.