# FREQUENCY-DOMAIN CIRCUIT ANALYSIS

When a designer is examining the behavior of an electrical circuit, the first thing to do, as in the case of any other physical system, is to write down a suitable set of equations describing the existing links between the involved physical variables, which are, in this case, voltages and currents. Generally, these equations arise from Kirchhoff's voltage and current laws and from the branch relations which describe circuit components.

If there are *dynamic* components (i.e., described by branch relations where time derivatives of voltages and/or currents are present), then the circuit is governed by a system of mixed algebraic and integrodifferential equations. The solution of such a system may be carried out using time as the independent variable and evaluating the wanted voltages and/or currents by numerical or analytical integration techniques. In this case the analysis is said to be performed in the *time domain*.

A completely different approach may be followed for linear time-invariant circuits. By using suitable transformations—that is, Laplace transform or Fourier transform—the original integrodifferential equations, which are linear with constant coefficients, are reduced to algebraic equations, where the original time functions are substituted by complex-valued functions of a complex variable $s = \sigma + j\omega$ in the case of Laplace transform, or simply $j\omega$ in the case of Fourier transform. Then the system of linear algebraic equations is solved for the new transformed functions. Finally, an inverse transformation recovers the requested voltages or currents as functions of time. When following this procedure, the analysis is said to be performed in the frequency domain or, alternatively, in the $s$-domain when an explicit reference to the Laplace transform is preferred.

The frequency-domain approach offers several advantages over the direct solution of time-domain circuit equations:

1. The solution is reduced to algebra and is greatly simplified by the extensive use of tables.
2. The conditions of energy storage elements within the circuit at the time when the input signal is applied (i.e., the initial conditions) become part of circuit equations and hence are automatically accounted for.
3. There is no need to evaluate initial conditions at $t = 0+$, as in the case of time-domain analysis, when a jump discontinuity occurs at $t = 0$. Only their values immediately before the beginning of the transient (i.e., at $t = 0-$) are required.
4. The sinusoidal steady-state behavior of a linear circuit may be easily analyzed by resorting to network functions defined in terms of Laplace transforms.
5. Frequency-domain analysis provides a deeper insight in the behavior of linear circuits. For example, it is possible to effectively compute sensitivities, to give stability conditions, to establish necessary and sufficient conditions for the realization of one-port and multiport networks, and so on.
6. Many design techniques are based on the description of the circuit behavior in the frequency domain, via network functions. The knowledge of these functions is of paramount importance in designing feedback amplifier, one-ports, multiports, active and passive filters, equalizers, oscillators, and so on.
7. Frequency-domain formulation allows an effective description of transmission lines, which would otherwise require partial differential equation in the time domain.
8. Frequency-domain formulation allows model simplification techniques not possible in the time domain.

In what follows, we first describe the different methods developed to write circuit equations in the frequency-domain. Then we introduce the concept of network function and its fundamental properties. As a third step, the use of two-sided Laplace and Fourier transforms in circuit analysis is illustrated. Finally, a Section on the applications of frequency-domain analysis with some examples completes the manuscript.

## FREQUENCY-DOMAIN CIRCUIT EQUATIONS

This section starts introducing the one-sided Laplace transform and a number of its properties, relevant to circuit analysis. A complete discussion of this topic may be found in Refs. 1 and 2. Successively, we will illustrate the use of Laplace transform to write frequency-domain circuit equations for lumped, linear, time-invariant circuits. The case of distributed circuits will be considered at the end of the paper.

### The One-Sided LaplaceTransform

Given a function of time $f(t)$ defined for all $t \geq 0$, its one-sided Laplace transform is defined as

$$F(s) = \mathcal{L}[f(t)] = \int_{0-}^{\infty} f(t)e^{-st}\,dt \tag{1}$$

where $s = \sigma + j\omega$ is a complex variable, called the *complex frequency.* Note that in Eq. (1) the lower limit of integration equals $0-$, to include functions which have a discontinuity or an impulse at $t = 0$.

Equation (1) establishes a correspondence between the time function $f(t)$ defined for all $t \geq 0$ and its Laplace transform $F(s)$. This fact ensures that the solution coming from the frequency-domain approach is identical to the one obtainable by operating in the time domain.

The integral which defines the Laplace transformation exists under mild conditions on the function $f(t)$, generally met for the signals used in engineering. A sufficient condition is that $f(t)$ is exponentially bounded—that is, $|f(t)| \leq Me^{ct}$—for some constant $M > 0$ and some constant $c$, for all $t > 0$ (or only for $t$ greater than some $t_0$).

The greatest lower bound $\sigma_+$ of the values of $\sigma$ for which the Laplace integral (1) exists is called *abscissa of convergence.* It can be shown that the Laplace integral defines an analytic function within the half-plane of convergence $\sigma > \sigma_+$.

Extensive tables of transform pairs have been built (see Refs. 3 and 4) by using the definition given in Eq. (1). Examples of Laplace transforms of elementary functions are reported in Table 1. Some of the main properties of one-sided Laplace transform, relevant in circuit analysis, are shown in Table 2. For proofs and a complete collection of these properties, see Refs. 1 and 2.

The inverse transform, which gives $f(t)$ from its Laplace transform $F(s)$, is denoted by $\mathcal{L}^{-1}[F(s)]$ and is expressed by

$$f(t) = 0, \qquad t < 0$$
$$f(t) = \mathcal{L}^{-1}[F(s)] = \frac{1}{2\pi j}\int_{\sigma_1 - j\infty}^{\sigma_1 + j\infty} F(s)e^{st}\,ds, \quad t > 0, \quad \sigma_1 > \sigma_+ \tag{2}$$

In many practical situations, the inverse transform may be simply obtained by resorting to a partial-fraction expansion of the original function. Then, the inverse transforms of the elementary fractions are found by inspection, resorting to tables of Laplace transforms.

### Tableau Equations

We start by considering the time-domain tableau equations for a lumped, linear time-invariant circuit (5, 6). The (nodal) tableau equations are

$$\boldsymbol{A}\boldsymbol{i} = 0 \tag{3}$$

$$\boldsymbol{v} - \boldsymbol{A}^{\mathrm{T}}\boldsymbol{e} = 0 \tag{4}$$

$$(\boldsymbol{M}_0\mathrm{D} + \boldsymbol{M}_1)\boldsymbol{v} + (\boldsymbol{N}_0\mathrm{D} + \boldsymbol{N}_1)\boldsymbol{i} = \boldsymbol{u}_s(t) \tag{5}$$

where $i$ is the vector of branch currents, $v$ is the vector of branch voltages, and $e$ is the vector of nodal voltages with respect to a datum node. Equations (3) and (4) are the matrix formulation of Kirchhoff's current and voltage laws: $A$ is the $(n-1) \times b$ reduced incidence matrix, $n$ being the number of nodes and $b$ the number of branches; the superscript T denotes transposition. The branch equations are given (in matrix form) by Eq. (5): $M_0, M_1, N_0, N_1$ are $b \times b$ matrices with real constant elements; D is the differentiation operator $d/dt$ and $u_s(t)$ is the vector of independent sources.

The application of Laplace transform to both sides of Eqs. 3–5, taking into account the properties shown in Table 2, gives the (nodal) tableau equations in the frequency domain

$$\boldsymbol{A}\boldsymbol{I}(s) = 0 \tag{6}$$

$$\boldsymbol{V}(s) - \boldsymbol{A}^{\mathrm{T}}\boldsymbol{E}(s) = 0 \tag{7}$$

$$(\boldsymbol{M}_0 s + \boldsymbol{M}_1)\boldsymbol{V}(s) + (\boldsymbol{N}_0 s + \boldsymbol{N}_1)\boldsymbol{I}(s) = \boldsymbol{U}_s(s) + \boldsymbol{U}_0 \tag{8}$$

where $I(s)$, $V(s)$, and $E(s)$, are vectors of transformed branch currents, branch voltages, and node-to-datum voltages, respectively. $U_s(s)$ is the vector of transformed independent sources and $U_0 = M_0 v(0) + N_0 i(0)$ is the vector of initial conditions. In matrix form we have

$$\boldsymbol{T}(s)\boldsymbol{W}(s) \equiv \begin{bmatrix} 0 & 0 & \boldsymbol{A} \\ -\boldsymbol{A}^{\mathrm{T}} & 1 & 0 \\ 0 & \boldsymbol{M}_0 s + \boldsymbol{M}_1 & \boldsymbol{N}_0 s + \boldsymbol{N}_1 \end{bmatrix} \begin{bmatrix} \boldsymbol{E}(s) \\ \boldsymbol{V}(s) \\ \boldsymbol{I}(s) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \boldsymbol{U}_s(s) + \boldsymbol{U}_0 \end{bmatrix} \tag{9}$$

where matrix $T$ is constituted by elements which are either real constants or real polynomials of degree 1 and $W$ is the vector of output variables.

Other forms of the tableau equations are possible if reference is made to fundamental cut-sets or fundamental loops of the connected digraph associated to the considered circuit. These forms of the tableau equations are not considered here, and the interested reader is referred to Ref. 6, pp. 715–717.

As a concluding remark, let us consider tableau equations 6–8 in more detail. Equations (6) and (7) may be obtained from Eqs. (3) and (4) by a simple substitution of time functions with their Laplace transforms: They may be viewed as Kirchhoff's current and voltage laws in the frequency domain. Equation (8) contains the frequency-domain branch equations. They are listed in the fourth column of Table 3, for the components shown in Fig. 1(a), while in the third column the corresponding time-domain equations are shown for comparison.

**Impedance and Admittance.** Consider a two-terminal element, with zero initial conditions. Let us choose *associated reference directions* for input current and terminal voltage, (e.g., see Fig. 1(a)) and denote their transforms by $I(s)$ and $V(s)$, respectively. The ratio

$$Z(s) = \frac{V(s)}{I(s)} \tag{10}$$

is called the *impedance* of the two-terminal element. The reciprocal of the impedance

$$Y(s) = \frac{1}{Z(s)} = \frac{I(s)}{V(s)} \tag{11}$$

is referred to as the *admittance* of the two-terminal element. In the frequency domain, impedance and admittance play the same role as resistance and conductance in Ohm's laws, respectively. Table 4 gives impedances and admittances of resistors, capacitors, and inductors.

**Writing Circuit Equations by Inspection.** Taking into account Eqs. (10) and (11), one can easily verify that the circuits shown in Fig. 1(b) are governed by the algebraic frequency-domain equations reported in the fourth column of Table 3. These circuits are called the equivalent circuits in the frequency domain for resistors, capacitors, inductors, and coupled inductors.

As a consequence, the frequency-domain equations of any circuit may be written directly, avoiding the preliminary step of writing time-domain equations. In fact, it is sufficient to apply the same analysis methods used for resistive circuits (see Ref. 6, Chapter 5; and Ref. 7, Chapter 4) to a frequency-domain equivalent circuit, hereafter called the *transformed circuit,* obtained by replacing each

Table 1. Elementary One-Sided Laplace Trasform Pairs

| Time Function $f(t)$ | Type | Laplace Trasform $F(s) = £[f(t)]$ |
|---|---|---|
| $\delta(t)$ | Impulse function | $1$ |
| $u(t)$ | Unit step function | $\frac{1}{s}$ |
| $t$ | Ramp | $\frac{1}{s^2}$ |
| $e^{-at}$ | Exponential | $\frac{1}{s+a}$ |
| $e^{-at}\, sin\, \omega_0 t$ | Damped sine | $\frac{\omega_0}{(s+a)^2+\omega_0^2}$ |
| $e^{-at}\, cos\, \omega_0 t$ | Damped cosine | $\frac{s+a}{(s+a)^2+\omega_0^2}$ |

Table 2. Main Properties of One-Sided Laplace Transform

| Property | Transform Pair |
|---|---|
| Linearity | $\mathcal{L}[a_1\ f_1(t) + a_2\ f_2(t)] = a_1 F_1(s) + a_2 F_2(s)$ |
| Time differentiation | $\mathcal{L}\left[\frac{df}{dt}\right] = sF(s) - f(0^-)$ |
| Time integration | $\mathcal{L}\left[\int_{0^-}^{t} f(\tau)d\tau\right] = \frac{1}{s}F(s)$ |
| Time shift | $\mathcal{L}[f(t-t_0)u(t-t_0)] = e^{-t_0 s}F(s),\ t_0 > 0$ |
| Frequency shift | $\mathcal{L}[e^{s_0 t}\ f(t)] = F(s-s_0)$ |
| Initial-value theorem | $f(0^+) = lim_{t\to 0^+}\ f(t) = lim_{s\to\infty} sF(s)$ |
| Final-value theorem | $\lim_{t\to\infty} f(t) = \lim_{s\to 0} sF(s)$, $sF(s)$ regular on the $j\omega$ axis and in the right half-plane |

Table 3. Branch Equations in the Frequency Domain

| Component | Parameter(s) | Time Domain | Frequency Domain[a] |
|---|---|---|---|
| Resistor | $R$ | $v(t) = Ri(t)$ | $V(s) = RI(s)$ |
| Capacitor | $C$ | $i(t) = C\,\frac{dv}{dt}$ | $I(s) = sCV(s) - CV_0$ or: $V(s) = \frac{1}{sC}I(s) + \frac{V_0}{s}$ |
| Inductor | $L$ | $v(t) = L\,\frac{di}{dt}$ | $V(s) = sLI(s) - LI_0$ or: $I(s) = \frac{1}{sL}V(s) + \frac{I_0}{s}$ |
| Coupled inductors | $M, L_1, L_2$ | $v_1(t) = L_1\frac{di_1}{dt} + M\frac{di_2}{dt}$ <br> $v_2(t) = M\frac{di_1}{dt} + L_2\frac{di_2}{dt}$ | $V_1(s) = sL_1I_1(s) + sMI_2(s) - L_1I_{10} - MI_{20}$ <br> $V_2(s) = sMI_1(s) + sL_2I_2(s) - MI_{10} - L_2I_{20}$ <br> or: <br> $V_1(s) = sL_1\left(I_1(s) - \frac{I_{10}}{s}\right) + sM\left(I_2(s) - \frac{I_{20}}{s}\right)$ <br> $V_2(s) = sM\left(I_1(s) - \frac{I_{10}}{s}\right) + sL_2\left(I_2(s) - \frac{I_{20}}{s}\right)$ |

[a]Note that $V_0, I_0, I_{10}$, and $I_{20}$ are initial values computed at $t = 0^-$.

**Table 4.** Impedance $Z(s)$ and Admittance $Y(s)$ of Resistors, Capacitors, and Inductors

| Component | Parameter | $Z(s)$ | $Y(s)$ |
|---|---|---|---|
| Resistor | $R$ | $R$ | $G-1/R$ |
| Capacitor | $C$ | $\frac{1}{sC}$ | $sC$ |
| Inductor | $L$ | $sL$ | $\frac{1}{sL}$ |

element with its impedance or admittance and adding the appropriate sources to take into account initial conditions.

Furthermore, for this transformed circuit all general theorems, valid for linear time-invariant resistive circuits, (superposition, Thevenin's, Norton's, Tellegen's, etc.), still hold.

**Example** The above results are used to write tableau equations for the very simple circuit of Fig. 2(a). The transformed circuit is shown in Fig. 2(b), where $v_3(0-) = V_0$ and $i_4(0-) = I_0$ are initial values of capacitor voltage and inductor current, respectively. The frequency-domain tableau equations, written in matrix form and partitioned accord-

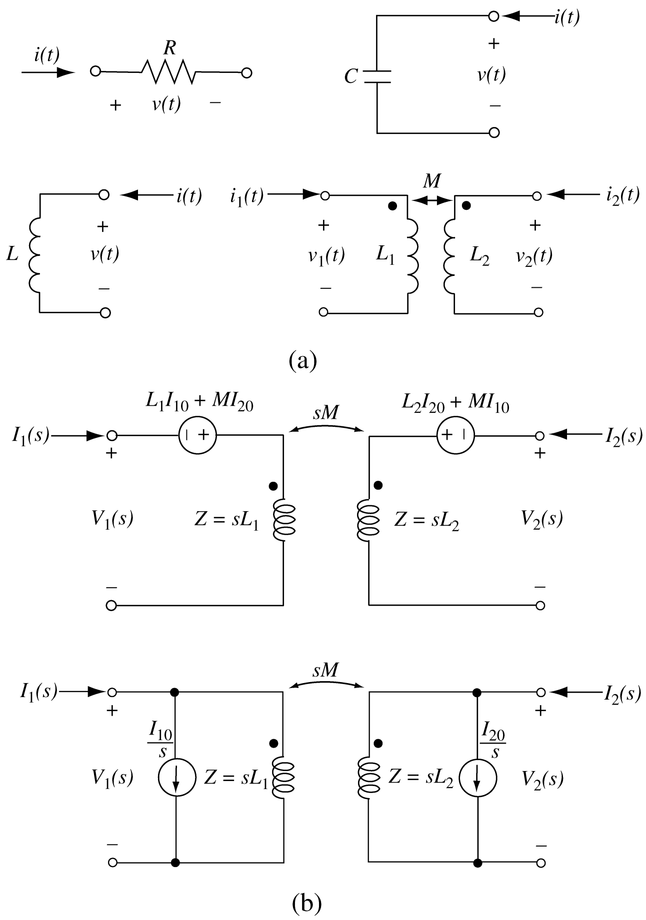


**Figure 1.** (a) Symbols and associated reference directions for voltages and currents of resistors, capacitors, inductors and coupled inductors; (b) Frequency-domain equivalent circuits of resistors, capacitors, inductors and coupled inductors. For capacitors and inductors both the series and the parallel frequency-domain equivalent circuits are shown. Initial values are denoted by $V_0 = v(0-)$, $I_0 = i(0-)$, $I_{10} = i_1(0-)$ and $I_{20} = i_2(0-)$.

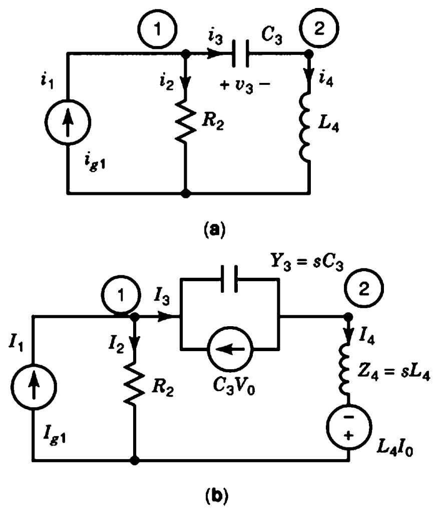


**Figure 2.** (a) A simple *RLC* circuit; at $t = 0-$ the capacitor has an initial voltage $v_3(0-) = V_0$ and the inductor has an initial current $i_4(0-) = I_0$. Associated reference directions are assumed. (b) The frequency-domain transformed circuit.

ing to Eq. (9), are

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & -R_2 & 0 & 0 \\ 0 & 0 & 0 & 0 & -sC_3 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & -sL_4 \end{bmatrix} \times \begin{bmatrix} E_1 \\ E_2 \\ V_1 \\ V_2 \\ V_3 \\ V_4 \\ I_1 \\ I_2 \\ I_3 \\ I_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ I_{g1} \\ 0 \\ -C_3V_0 \\ -L_4I_0 \end{bmatrix} \quad (12)$$

### Loop Equations

The set of loop equations may be written directly in the frequency domain by substituting each circuit element with the appropriate series equivalent circuit of Fig. 1(b) and then applying the standard technique used for resistive circuits. Note that for nonzero initial conditions, supplemental voltage sources of the form $LI_0$ and/or $V_0/s$ are added to the original circuit.

The final set of loop equations has the form

$$\boldsymbol{Z}_l(s)\boldsymbol{I}_l(s) = \boldsymbol{V}_s(s) + \boldsymbol{V}_0(s) \quad (13)$$

where $Z_l(s)$ is called the *loop impedance matrix,* $V_s(s)$ is the vector of independent voltage sources, and $V_0(s)$ is the voltage vector due to initial conditions.

The restriction imposed in loop analysis is that all elements must be current-controlled. If this is not the case, one may resort to circuit transformations or to the modified loop analysis (8).

Loop equations may be written by inspection. If the circuit is formed only by resistors, capacitors, and inductors, the rules are particularly simple: The matrix $Z_l$ is symmetric, the $k$th diagonal element $z_{kk}$ is the sum of all impedances in loop $k$, and any off-diagonal element $z_{jk}$ is the sum of all impedances common to loops $j$ and $k$ if the reference directions for the two loops are the same, the negative of the sum otherwise.

Finally, for a circuit with a planar graph, meshes may be used instead of fundamental loops. In this case, mesh equations are obtained in terms of fictitious circulating currents, generally referred to as mesh currents.

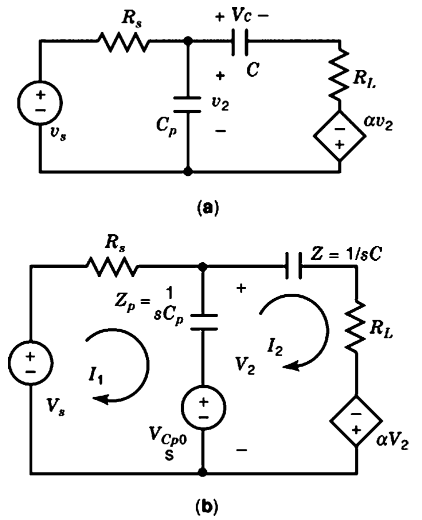


**Figure 3.** (a) Loop (mesh) analysis; at $t = 0-$ capacitors $C_p$ and $C$ have initial voltages $v_2(0-) = V_{Cp0}$ and $v_c(0-) = 0$ V, respectively. (b) The frequency-domain transformed circuit.

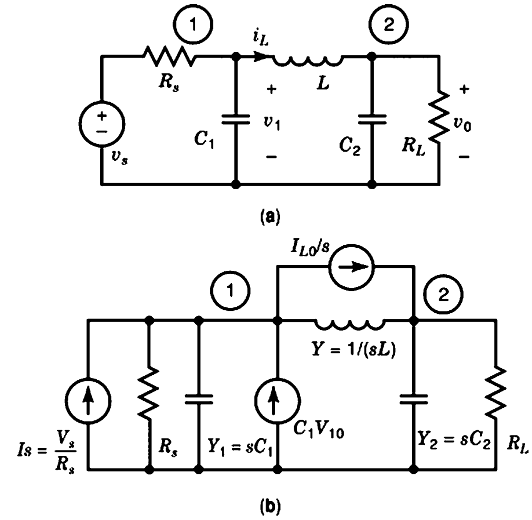


**Figure 4.** (a) Nodal analysis; capacitor $C_1$ has an initial voltage $v_1(0-) = V_{10}$. The initial value of the current $i_L$ through the inductor $L$ is $I_{L0}$. (b) The frequency-domain transformed circuit.

**Example** Consider the (planar) circuit of Fig. 3(a). The initial values of voltages across capacitors $C_p$ and $C$ are $v_2(0-) = V_{Cp0}$ and $v_C(0-) = 0$ $V$, respectively. In this case, meshes can be chosen as fundamental loops. Assuming for the mesh currents $I_1$ and $I_2$ the reference directions shown in the transformed circuit of Fig. 3(b), we write the following frequency-domain mesh equations:

$$\begin{bmatrix} 1/(sC_p) + R_s & -1/(sC_p) \\ -(1+\alpha)/(sC_p) & R_L + 1/(sC) + (1+\alpha)/(sC_p) \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \end{bmatrix} = \begin{bmatrix} V_s(s) \\ 0 \end{bmatrix} + \begin{bmatrix} -V_0/s \\ (1+\alpha)V_0/s \end{bmatrix} \tag{14}$$

### Nodal Equations

Nodal equations may be written directly in the frequency domain by substituting each circuit element with the appropriate parallel equivalent circuit of Fig. 1(b) and then applying the standard technique used for resistive circuits. Note that for nonzero initial conditions, supplemental current sources of the form $I_0/s$ and/or $CV_0$ are added to the original circuit.

The final form of nodal equations is

$$\boldsymbol{Y}_n(s)\boldsymbol{E}(s) = \boldsymbol{I}_s(s) + \boldsymbol{I}_0(s) \tag{15}$$

where $Y_n(s)$ is called the *nodal admittance matrix,* $I_s(s)$ is the vector of independent current sources, and $I_0(s)$ is the current vector due to initial conditions.

Nodal equations may be written by inspection. If the circuit is formed only by resistors, capacitors, and inductors, the rules are particularly simple: The matrix $Y_n$ is symmetric, the $k$th diagonal element $y_{kk}$ is the sum of all admittances connected to node $k$, and any off-diagonal element $y_{jk}$ is the negative of the sum of all admittances connecting node $j$ and node $k$.

Nodal analysis may be directly applied only to circuits containing voltage-controlled elements. If there are elements which are not voltage-controlled, circuit transformations should be performed. Finally, nodal analysis may be effectively extended to circuits with ideal voltage sources and/or ideal operational amplifier (9, Section 4.5).

**Example** Consider the circuit of Fig. 4(a). $V_{10}$ and $I_{L0}$ are initial values of capacitor voltage $v_1$ and inductor current $i_L$, respectively. The initial voltage across $C_2$ is zero. To effectively write nodal equations, the voltage source is first transformed into a current source by a Thevenin–Norton transformation. The final circuit, used to write nodal equations in the frequency domain, is shown in Fig. 4(b). The following equations are obtained by inspection:

$$\begin{bmatrix} G_s + sC_1 + 1/(sL) & -1/(sL) \\ -1/(sL) & G_L + sC_2 + 1/(sL) \end{bmatrix} \begin{bmatrix} E_1 \\ E_2 \end{bmatrix} = \begin{bmatrix} V_s(s)G_s \\ 0 \end{bmatrix} + \begin{bmatrix} C_1V_{10} - I_{L0}/s \\ I_{L0}/s \end{bmatrix} \tag{16}$$

### Modified Nodal Equations

Modified nodal analysis (*MNA*) has been introduced to cope with the problems associated with nodal analysis (6, 10). MNA follows the logical steps of nodal analysis. When an element is found which is *not* voltage-controlled, its current is added to the set of unknowns. To balance the number of unknowns and the number of equations, the element equation is added to the set of equations.

In matrix form we have

$$\boldsymbol{P}(s)\begin{bmatrix} \boldsymbol{E}(s) \\ \boldsymbol{I}(s) \end{bmatrix} = \boldsymbol{U}_s(s) + \boldsymbol{U}_0 \tag{17}$$

where $E(s)$ is the vector of nodal voltages, $I(s)$ is the vector of unknown currents, $U_s(s)$ is the vector of transformed independent sources, and $U_0$ is the vector due to initial conditions.

The elements of the coefficient matrix $P(s)$ depend solely on the circuit topology, on complex frequency $s$, and on element values. They are either real constants or real polynomials of degree 1. If inductor currents are not added to the unknowns, then also monomials of degree $-1$ are present. In such a case, $U_0$ may contain terms of the form $I_0/s$.

MNA has many attractive features. The coefficient matrix and the right-hand side vector of Eq. (17) can be assembled by inspection through "stamps" describing the contribution of each circuit element (10). MNA can be applied to any circuit, including those containing transmission lines, without the need of circuit transformations and keeping the number of equations to a minimum. Furthermore, if any current is required as output variable, it may be directly inserted in vector $I$ of Eq. (17), adding the corresponding branch relation to the set of MNA equations.

**Example** MNA is illustrated by resorting to the circuit of Fig. 5(a), where the operational amplifier is supposed ideal. The frequency-domain transformed circuit is shown in Fig. 5(b), where $V_{20}$ and $V_{30}$ are initial values of capacitor voltages $v_2$ and $v_3$, respectively. The modified nodal equations are

$$\begin{bmatrix} G_{11} & -G_{11} & 0 & 0 & 0 & 1 & 0 \\ -G_{11} & G_{11}+G_{12}+s(C_2+C_3) & -sC_3 & -sC_2 & 0 & 0 & 0 \\ 0 & -sC_3 & G_4+sC_3 & -G_4 & 0 & 0 & 0 \\ 0 & -sC_2 & -G_4 & G_4+G_6+sC_2 & -G_6 & 0 & 1 \\ 0 & 0 & 0 & -G_6 & G_5+G_6 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} E_1 \\ E_2 \\ E_3 \\ E_4 \\ E_5 \\ I_1 \\ I_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ V_i(s) \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ C_3V_{30} - C_2V_{20} \\ -C_3V_{30} \\ C_2V_{20} \\ 0 \\ 0 \\ 0 \end{bmatrix} \tag{18}$$

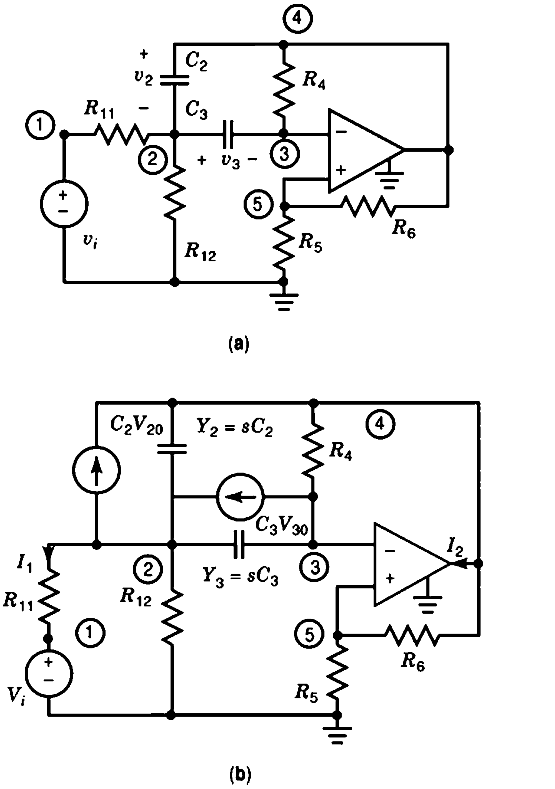


**Figure 5.** (a) A simple $RC$-active circuit; at $t = 0-$ the capacitors $C_2$ and $C_3$ have initial voltages $v_2(0-) = V_{20}$ and $v_3(0-) = V_{30}$, respectively. (b) The frequency-domain transformed circuit.

## STATE EQUATIONS

State equations are best suited for time-domain analysis, even if sometimes it may be useful to solve these equations in the frequency domain. Also in this case, all known methods used to write time-domain state equations (inspection, equivalent sources, or network graph theory; see Ref. 11, Chapter 6) may be applied to the transformed circuit obtained by substituting inductors and capacitors with the series and parallel circuits of Fig. 1(b), respectively. In spite of that, state equations are generally written first in the time domain and then transformed to the frequency domain, where they take the following form:

$$s\boldsymbol{X}(s) = \boldsymbol{A}\boldsymbol{X}(s) + \boldsymbol{B}\boldsymbol{U}(s) + \boldsymbol{X}_0 \tag{19}$$

In the above equation, $X(s)$, $X_0$, and $U(s)$ are the vectors of transformed state variables, of initial conditions, and of transformed independent sources, respectively. $A$ and $B$ are matrices of appropriate dimensions (note that $A$ is not the incidence matrix).

Equation (19) may be rearranged to yield

$$[s\mathbf{1} - \boldsymbol{A}]\boldsymbol{X}(s) = \boldsymbol{B}\boldsymbol{U}(s) + \boldsymbol{X}_0 \tag{20}$$

where $\mathbf{1}$ is the identity matrix.

### Natural Frequencies

Let us consider a (linear time-invariant) circuit described by its tableau equations [Eq. (9)]. This system of linear algebraic equations admits a unique solution if and only if matrix $T(s)$ is invertible; that is, $\det[T(s)]$ is not identically zero. Furthermore, since all elements of $T(s)$) are polynomials of degree either 0 or 1 in $s$, $\det[T(s)]$ is a polynomial in $s$ too, with real coefficients, called the *characteristic polynomial* of the circuit. The roots $\lambda_i$ of

$$\det[\boldsymbol{T}(s)] = 0 \tag{21}$$

are called the *natural frequencies* of the circuit. They are either real or occur in complex-conjugate pairs.

Note that when working in the time domain, $T(s)$ becomes $T(\mathrm{D})$, where D is the differentiation operator $d/dt$, and hence the above definition of natural frequencies coincides with that given in the time domain (Ref. 12, Chapter 12).

If all independent sources are set to zero (i.e., considering the circuit in the zero-input state), then, applying Kramer's rule, a generic output variable $W_k(s)$ is given by

$$W_k(s) = \frac{\sum_{i=1}^{b} \Delta_{ik} U_{i0}}{\det[\boldsymbol{T}(s)]} \tag{22}$$

where $\Delta_{ik}$ is the cofactor of the element $(b + n - 1 + i)$, $k$ of $T(s)$ and $U_{i0}$ denotes the $i$th source due to initial conditions.

Expanding the right-hand side of Eq. (22) in partial fractions and then taking the inverse Laplace transform of both sides, it turns out that, in general, any zero-input response is a linear combination of exponentials $e^{\lambda_i t}$(or polynomials in $t$ times $e^{\lambda i t}$), called *modes,* where each $\lambda_i$ is a natural frequency of the circuit. For some particular response, some modes may be absent.

If all the natural frequencies of the circuit have negative real part, then the zero-input response, for any initial state, goes to zero exponentially with increasing time. In this case, the circuit is referred to as *strictly* (or *exponentially*) stable.

Natural frequencies are an intrinsic feature of the circuit: They depend only on the circuit and not on the method used to analyze it. In fact, it can be shown that the nonzero natural frequencies of any linear time-invariant circuit are identical to the nonzero roots of the determinantal polynomial of the matrix of any system of equations describing the circuit, when framed as a set of linear algebraic equations of the form $R(s)W(s) = F(s)$ (see Ref. 13, Chapter 14). The total number of natural frequencies is not greater than the number of energy-storing elements present in the circuit and it defines the order of complexity of the circuit (see Ref. 14, Chapter 8). Zero natural frequencies are of limited physical significance, and in some cases their number may be determined by inspection (see Ref. 12, Chapter 11).

In the case of tableau analysis and MNA, the determinantal polynomials of matrices $T(s)$ and $P(s)$ differ from each other only for a multiplicative constant, and hence they give the same set of natural frequencies, including those at $s = 0$ (if any), with the same multiplicities (see Ref. 6, Section 10.4.2).

In the case of state equations, let us consider them in the form given byEq. (20). The natural frequencies are the roots of the determinantal polynomial

$$\det[s\mathbf{1} - \boldsymbol{A}] = 0 \tag{23}$$

and hence they coincide with the eigenvalues of matrix $A$. As in the case of tableau analysis and MNA, Eq. (23) supplies all natural frequencies, including zero natural frequencies, with their own multiplicities.

## NETWORK FUNCTIONS

Given a lumped, linear time-invariant network in the zero state—that is, with zero initial voltages across all capacitors and zero initial current through all inductors—a *network function* $H(s)$ is defined as the ratio of the Laplace transform $W(s)$ of an *output variable* $w(t)$ to the Laplace transform $Q(s)$ of an input variable $q(t)$:

$$H(s) = W(s)/Q(s) \tag{24}$$

If the input is an unit impulse, then $Q(s) = 1$ and hence $H(s)$ turns out to be the Laplace transform of the impulse response of the circuit.

If input and output variables are defined at the same terminal pair, the network function is either an impedance or an admittance, as defined in Eqs. (10) and (11), respectively. If input and output are measured at two different terminal pairs, the network function is referred to as a *transfer function.* A transfer function may have the dimension of impedance or admittance or be dimensionless.

In the case of multiple-input multiple-output circuits, a matrix transfer function $H(s)$ may be defined, whose $k$, $i$ element is the ratio of the Laplace transform of the $k$th output variable to the Laplace transform of the $i$th input variable, when all others input variables are set to zero.

Network functions may be efficiently obtained by resorting to network symbolic analysis programs. In general, these programs are also able to perform mixed numerical–symbolic analysis (15, 16) and to evaluate sensitivities (e.g., see Ref. 17, Chapter 10); and Ref. 18, Chapter 8).

### Fundamental Properties

Consider, for simplicity, a single-input single-output lumped, linear circuit. Any network function may be determined starting from any set of circuit equations described in the previous section. Referring again to tableau equations [Eq. (9)] and using Kramer's rule, we have

$$H_{ki}(s) = \frac{W_k(s)}{Q_i(s)} = \frac{\Delta_{ik}}{\det[\boldsymbol{T}(s)]} \tag{25}$$

where $W_k(s)$ denotes the $k$th output variable and $\Delta_{ik}$ is the cofactor of the element $i$, $k$ of $T(s)$, where $i$ is the position of the independent source $Q_i(s)$ into the right-hand side vector of Eq. (9).

From Eq. (25), it follows that any network function of a lumped, linear circuit is a real rational function of $s$—that is, the ratio of two polynomials with real coefficients. Its finite poles are natural frequencies of the circuit, according to Eq. (21); due to a possible pole-zero cancellation, some natural frequencies may not appear as poles of the network function (see Ref. 6, p. 612).

The transform $W_k(s)$ of the output variable is given by

$$W_k(s) = H_{ki}(s)Q_i(s) = \frac{\Delta_{ik}}{\det[\boldsymbol{T}(s)]}Q_i(s) \tag{26}$$

Transient response is computed by finding a partial-fraction expansion of the right-hand side of Eq. (26) and then taking the inverse Laplace transform of each term of the expansion. If the network function has no poles inside the right half-plane and if its $j\omega$-axis poles (if any) are simple, then the (zero-state) impulse response of the circuit remains bounded and the network function is said to be *stable.* If there are no poles on the $j\omega$-axis, then the impulse response decays with time, and moreover, any (zero-state) response remains bounded for any bounded input. In this case the network function is said to be *strictly stable* (see, for example, Ref. 19, Chapter 9).

When the network functions are impedances or admittances of *RCLM* networks—that is, networks composed of a finite number of resistors, capacitors, inductors, and coupled inductors—they must be positive real functions; other constraints are added for two-element-kind (i.e., $RC$, $RL$, and $LC$) networks (20).

### The Sinusoidal Steady State

The large use of network functions in circuit analysis in only partially due to their utility in evaluating complicated transients. One of the reasons for their acknowledged importance is related to their capability to describe the sinusoidal steady state of stable networks according to the following theorem, usually referred to as the fundamental theorem of sinusoidal steady state (19, Section 9.4; 6, Section 10.5).

**Theorem** Consider any linear time-invariant circuit, driven by sinusoidal independent sources, all at the same frequency $\omega$. If the circuit is strictly stable (i.e., all the natural frequencies have negative real part), then, for any set of initial conditions, all voltages and currents tend, as time goes to infinity, to a unique sinusoidal steady-state at the same frequency $\omega$.

In the case of a single-input single-output circuit described by Eq. (24), magnitude $W_m$ and phase $\phi_w$ of the output waveform are given by (see Ref. 6, Section 10.5)

$$W_m = |H(j\omega)|Q_m \tag{27}$$

$$\phi_w = \arg[H(j\omega)] + \phi_q \tag{28}$$

where $Q_m$ and $\phi_q$ are magnitude and phase of the input signal, respectively.

In terms of phasors (see Ref. 6, Chapter 9) we have

$$\hat{W} = H_\omega \hat{Q} \tag{29}$$

where $\hat{W} = |\hat{W}|ej\phi_w$ and $\hat{Q} = |\hat{Q}|ej\phi_q$ are the phasors of output and input waveforms, respectively, and $H_\omega$ is the transfer function defined in terms of the sinusoidal steady state. A comparison of Eq. (29) with Eqs. (27) and (28) shows that any phasor transfer function, defined in the sinusoidal steady state, may be obtained simply by setting $s = j\omega$ in the corresponding transfer function $H(s)$ defined in terms of Laplace transforms.

$H(j\omega)$ is a complex valued function of $\omega$ and may be written as

$$H(j\omega) = \mathscr{R}(\omega) + j\mathscr{I}(\omega) = |H(j\omega)|e^{j\phi_h(\omega)} \tag{30}$$

The parts of $H(j\omega)$—that is, real part $\mathscr{R}(\omega)$, imaginary part $\mathscr{I}(\omega)$, magnitude $|H(j\omega)|$ (or gain $20 \log|H(j\omega)|$), and phase $\phi_h(\omega)$—are the quantities involved in the steady-state response to sinusoidal excitations. Generally speaking, the overall information contained in any pair of these parts [i.e., magnitude (or gain) and phase, or real and imaginary parts], when considered as a function of frequency $f = \omega/(2\pi)$, is referred to as *frequency response* of the circuit.

It is easily verified that real part $\mathscr{R}(\omega)$ and magnitude-squared function $|H(j\omega)|^2$ are even functions of $\omega$, whereas imaginary part $\mathscr{I}(\omega)$ and $\tan \phi_h(\omega)$ are odd functions of $\omega$.

Transfer functions play an important role in engineering, since they can be easily and accurately measured, resorting to stable sinusoidal oscillators and to precise measurement equipments—for example, to network analyzers. On the other hand, they are the starting point for designing networks with a prescribed frequency behavior, as in the case of electrical filters and equalizers.

When poles and zeros of network functions are known, gain and phase versus frequency curves can be easily plotted, resorting to the so-called Bode plots (Ref. 21, Section 8.2). On the contrary, if curves of magnitude or phase, or real or imaginary parts, versus frequency are given, methods have been developed to build realizable network functions (20).

It is important to remember that real and imaginary parts of any stable network function are related to each other and, hence, constraints on them cannot be assigned arbitrarily. In fact, for a network function with no poles in the right half-plane and on the $j\omega$ axis (infinity included), they satisfy the following equations (see, for example, Ref. 18, Chapter 7):

$$\mathscr{R}(\omega) = \mathscr{R}(\infty) + \frac{1}{\pi}\int_{-\infty}^{+\infty} \frac{\mathscr{I}(x)}{\omega - x}\,dx \tag{31}$$

$$\mathscr{I}(\omega) = -\frac{1}{\pi}\int_{-\infty}^{+\infty} \frac{\mathscr{R}(x)}{\omega - x}\,dx \tag{32}$$

where $\mathscr{R}(\infty)$ is the value of the network function at infinity. The above equations state that if the imaginary part is specified over all frequencies, then the real part is determined to within an additive constant and that, if the real part is specified, then the imaginary part is completely determined. Similar results hold also for gain and phase of a network function, provided that it has no zeros in the right half-plane—that is, it is a minimum-phase function (18).

## TWO-SIDED LAPLACE AND FOURIER TRANSFORMS IN CIRCUIT ANALYSIS

In addition to the one-sided Laplace transform, other frequency representations of time functions are possible and yield circuit equations closely related to the ones introduced in the previous sections. In this section, we discuss the use of two of such representations, the two-sided Laplace Transform and the Fourier Transform. The two-sided Laplace transform has a marginal role in circuit applications, however it is considered here because it completes the theoretical framework of frequency-domain analysis. The Fourier transform, instead, adds significant possibilities to the frequency-domain analysis of circuits and, therefore, is discussed for both its theoretical and practical importance. In the following, we briefly review the definition and the properties of the two transforms, along with the frequency-domain circuit equations which arise when such transforms are used.

### Two-Sided Laplace Transform

The two-sided Laplace transform of the function $f(t)$ is

$$\bar{F}(s) = \int_{-\infty}^{+\infty} f(t)e^{-st}\,dt \tag{33}$$

where $s = \sigma + j\omega$, and lowercase and overlined uppercase letters are used for transform pairs (22).

The following decomposition is useful to compute the two-sided Laplace transform:

$$\begin{aligned} \bar{F}(s) &= F_{-}(s) + F(s) \\ F_{-}(s) &= \int_{-\infty}^{0-} f(t)e^{-st}\,dt = \int_{0+}^{+\infty} f(-t)e^{st}\,dt \end{aligned} \tag{34}$$

where $F(s)$ is the one-sided Laplace transform of $f(t)$ and $F_{-}(s)$ is the one-sided Laplace transform of $f(-t)$ computed for $t \in [0+, \infty]$ and argument $-s$. The two-sided Laplace transform exists for any $s$ such that both $F_{-}(s)$ and $F(s)$ exist. If $F_{-}(s)$ has abscissa of convergence $\sigma_{-}$ (i.e., it exists for $\sigma < \sigma_{-}$), $F(s)$ has abscissa of convergence $\sigma_{+}$ (i.e., it exists for $\sigma > \sigma_{+}$), and $\sigma_{+} < \sigma_{-}$, then $\bar{F}(s)$ exists and is an analytic function of s in the strip of convergence $\sigma_{+} < \sigma < \sigma_{-}$. When $\sigma_{+} = \sigma_{-}$, $\bar{F}(s)$ can exist as a distribution, whereas when $\sigma_{+} > \sigma_{-}$, $\bar{F}(s)$ does not exist.

The inversion of the two-sided Laplace transform can be obtained by the line integral

$$f(t) = \frac{1}{2\pi j}\int_{\sigma_0 - j\infty}^{\sigma_0 + j\infty} \bar{F}(s)e^{st}\,ds \tag{35}$$

where the integration line is in the strip of convergence, or by the $F_{-}(s)$ and $F(s)$ decomposition and tables of one-sided Laplace transform pairs. In the latter case, $F_{-}(s)$ and $F(s)$ are identified by their poles, since the poles of $F_{-}(s)$ are on the right of $\sigma_{-}$ and those of $F(s)$ are on the left of $\sigma_{+}$.

### Fourier Transform

The Fourier transform of the function $f(t)$ is

$$\mathscr{F}(\omega) = \int_{-\infty}^{+\infty} f(t)e^{-j\omega t}\,dt \tag{36}$$

where lowercase and script uppercase letters are used for transform pairs (23). The inverse transformation is

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \mathscr{F}(\omega)e^{j\omega t}\,d\omega \tag{37}$$

The term spectrum of $f(t)$ is also used to indicate $\mathscr{F}(\omega)$, or, less often, the magnitude of $\mathscr{F}(\omega)$.

A sufficient condition for the existence of $\mathscr{F}(\omega)$ requires that $f(t)$ has bounded variations (i.e., finite variations for finite time increments) and is absolutely integrable. For time functions with nonvanishing asymptotic values, the Fourier transform may exist as a distribution (23).

### Properties of Two-Sided Laplace Transform and Fourier Transform

Table 5 summarizes two-sided Laplace transform and Fourier transform pairs of common use, with emphasis on two-sided time functions, as algebraic, rational and harmonic signals (22, 23). Table 6 lists the main properties of the two transformation methods.

Most properties of Table 6 are identical to the corresponding properties of Table 2 for the one-sided Laplace transform, or follow from these by simply replacing $s$ with $j\omega$. A major difference in the properties of the two-sided Laplace transform and of the Fourier transform concerns the derivation formula, where the $f(0-)$ term is dropped. Owing to the latter point, these transforms are not suited for the inclusion of the initial conditions in the solution of circuit equations.

Other properties, involving the parts of $\mathscr{F}(\omega)$, are stated for network functions in the section entitled "The Sinusoidal Steady State."

### Relations Between Transforms

The different transforms of a waveform can be obtained one from the other via direct relations. Such relations ease the shift between frequency representations, allowing one to exploit their specific properties.

The two Laplace transforms coincide for one-sided functions ($f(t) = 0$ for $t < 0$).

The Fourier transform $\mathscr{F}(\omega)$ exists when $\bar{F}(s)$ exists on the $j\omega$ axis and is given by $\mathscr{F}(\omega) = \bar{F}(j\omega)$ [see Eqs. (33) and (36)]. Graphically, each part of $\mathscr{F}(\omega)$ is the profile of the corresponding part of $\bar{F}(s)$ along the $j\omega$ axis. If the $j\omega$ axis does not belong to the strip of convergence of $\bar{F}(s)$, $\mathscr{F}(\omega)$ does not exist, whereas if the $j\omega$ axis is a boundary of the strip, $\mathscr{F}(\omega)$ exists as a distribution (e.g., see Ref. 23, Section 9.2).

Finally, $\bar{F}(s)$ can be interpreted as the Fourier transform of the function $f(t)\exp(-\sigma t)$, which is $\mathscr{F}(\omega + \sigma/j) = \mathscr{F}(s/j)$ (see Table 6, frequency shift) for every $\sigma$ where $f(t)\exp(-\sigma t)$ is Fourier-transformable.

### Circuit Equations and Network Functions

Frequency-domain circuit equations based on the two-sided Laplace transform and on the Fourier transform arise by applying such transformations to the time-domain circuit equations. The equations obtained in this way differ from those based on the one-sided Laplace transform only in the branch relations of dynamic elements. The transformation of linear time-invariant dynamic branch relations via two-sided Laplace transform and Fourier transform replaces the time-derivative operator with $s$ and $j\omega$, respectively, and adds no initial contribution.

For the two-sided Laplace transform, furthermore, the time-integral operator is simply replaced by $1/s$, and, hence, the equivalent circuits of basic elements follow from those of Fig. 1(b) by simply dropping the initial condition sources. Accordingly, the one-sided Laplace equations written in the section entitled "Frequency-domain Circuit Equations" become two-sided Laplace equations by setting initial conditions to zero and representing variables by two-sided Laplace transform (i.e., $F(s) \to \bar{F}(s)$).

For the Fourier transform, instead, some additional care is required. In order to avoid frequency-domain equations containing $\delta(\omega)$ terms, it is expedient to avoid the Fourier transformation of time-domain equations involving the time-integral operator (e.g., see row 3 of Table 6). Every one-sided Laplace equation written in the section entitled

Table 5. Elementary two-Sided Laplace Transform and Fourier Transform Pairs

| $w(t)$ | $\overline{W}(s)$, s.c.[a] | $\mathcal{W}(\omega)$ |
|---|---|---|
| 1 | $2\pi\delta(s), \sigma = 0$ | $2\pi\delta(\omega)$ |
| $t$ | $-2\pi\delta'(s), \sigma = 0$ | $-2\pi\delta'(\omega)$ |
| $t^2$ | $2\pi\delta''(s), \sigma = 0$ | $2\pi\delta''(\omega)$ |
| $\delta(t)$ | $1, -\infty < \sigma < \infty$ | 1 |
| $exp(s_0 t)$ | $2\pi\delta(s - s_0), \sigma - \sigma_0$ | $2\pi\delta(\omega + js_0)$ |
| $cos(\omega_0 t)$ | $\pi[\delta(s - j\omega_0) + \delta(s + j\omega_0)], \sigma = 0$ | $\pi[\delta(\omega - \omega_0) + \delta(\omega + \omega_0)]$ |
| $sign(t)$ | $2/s, \sigma = 0$ | $2/j\omega$ |
| $\sum_{n=-\infty}^{\infty} \delta(t - nT)$ | | $\omega_T \sum_{n=-\infty}^{\infty} \delta(\omega - n\omega_T)$ |

[a]Note that s.c. indecates the strip of convergence of $\overline{W}(s)$; $\delta'(\cdot)$ and $\delta''(\cdot)$ are the first and second derivative of the delta function, respectively; $s_0 = \sigma_0 + j\omega_0$ is a complex constant; and $\omega_T = 2\pi/T$.

Table 6. Main Properties of Two-Sided Laplace Transform and Fourier Transform[a]

| Property | $\omega(t)$ | $\overline{W}(s)$ | $\mathcal{W}(\omega)$ |
|---|---|---|---|
| Linearity | $a_1 f_1(t) + a_2 f_2(t)$ | $a_1\overline{F}_1(s) + a_2\overline{F}_2(s)$ | $a_1\mathscr{F}_1(\omega) + a_2\mathscr{F}_2(\omega)$ |
| Time differentiation | $\frac{d f(t)}{dt}$ | $s\overline{F}(s)$ | $j\omega\mathscr{F}(\omega)$ |
| Time integration | $\int_{-\infty}^{t} f(t')dt'$ | $\frac{1}{s}\overline{F}(s)$ | $\frac{1}{j\omega}\mathscr{F}(\omega) + \pi\mathscr{F}(0)\delta(\omega)$ |
| Time shift | $f(t - t_0)$ | $\overline{F}(s)exp(-st_0)$ | $\mathscr{F}(\omega)exp(-\omega t_0)$ |
| Frequency shift | $f(t)exp(s_0 t)$ | $\overline{F}(s - s_0)$ | $\mathscr{F}(\omega + js_0)$ |
| Convolution | $f(t) * g(t)$ | $\overline{F}(s)\overline{G}(s)$ | $\mathscr{F}(\omega)\mathcal{G}(\omega)$ |
| Moment theorem | $\int_{-\infty}^{\infty} t^n f(t)dt$ | $(-1)^n \frac{d^n\overline{F}(s)}{ds^n}$ | $j^n \frac{d^n\mathscr{F}(\omega)}{d\omega^n}$ |

[a]Note that * and $s_0$ indicate convolution and a complex constant, respectively.

"Frequency-domain Circuit Equations" which does not contain the factor 1/$s$ can be turned into an equation based on the Fourier transform by replacing $s$ with $j\omega$, setting initial conditions to zero and representing variables by a Fourier transform. As an example, the tableau equations in the Fourier domain are

$$\boldsymbol{T}(j\omega)\begin{bmatrix} \boldsymbol{\mathcal{E}}(\omega) \\ \boldsymbol{\mathcal{V}}(\omega) \\ \boldsymbol{\mathcal{I}}(\omega) \end{bmatrix} = \begin{bmatrix} \boldsymbol{0} \\ \boldsymbol{0} \\ \boldsymbol{\mathcal{U}}_s(\omega) \end{bmatrix} \tag{38}$$

Similarly in the section "Network Functions," we define network functions in terms of two-sided Laplace and Fourier transforms by

$$\overline{H}(s) = \frac{\overline{W}(s)}{\overline{Q}(s)}, \qquad \mathscr{H}(\omega) = \frac{\mathcal{W}(\omega)}{\mathcal{Q}(\omega)} \tag{39}$$

The network functions $H$, $\bar{H}$ and $\mathscr{H}$ are the different transforms of the same impulse response $h(t)$. Since $h(t)$ is supposed to be a causal (i.e., one-sided) function, $\bar{H}(s) = H(s)$ and, provided $H(s)$ is strictly stable (i.e., the $j\omega$ axis belongs to its strip of convergence), $\mathscr{H}(\omega) = \bar{H}(j\omega) = H(j\omega)$ (see the section entitled "Relations Between Transforms"). The latter relation highlights also that $\mathscr{H}(\omega)$ yields the steady-state response to a harmonic input signal (see the section entitled "The Sinusoidal Steady State"). In the following, network functions are indicated only by $H(s)$ and $H(j\omega)$.

### Additional Fourier Transform Properties

Additional properties of the Fourier transform relevant to circuit analysis are briefly reviewed.

**Physical Meaning of the Fourier Representation.** The Fourier representation describes signals in terms of harmonic components and the behavior of linear time-invariant systems in terms of transformation of harmonic components. This interpretation arises from Eqs. (37) and (39):

$$w(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \mathcal{W}(\omega)e^{j\omega t}\,d\omega = \frac{1}{2\pi}\int_{-\infty}^{+\infty} H(j\omega)\mathcal{Q}(\omega)e^{j\omega t}\,d\omega \tag{40}$$

where $q(t)$ and $w(t)$ are the input and output signals of a circuit with network function $H(s)$, respectively, and $\mathcal{W}$ and $\mathcal{Q}$ are their Fourier transforms. In the above equation, $[1/2\pi\mathcal{W}(\omega)\,d\omega]\,e^{j\omega t}$ are the complex harmonic signals composing $w(t)$, each of which comes from the corresponding component of the input signal $[1/2\pi\mathcal{Q}(\omega)\,d\omega]\,e^{j\omega t}$ modified by $H(j\omega)$. Such an equation formalizes the operation of linear frequency selective circuits and is the basis for the physical interpretation of frequency responses.

**Energy and Power Spectra.** The Fourier transform allows also a frequency representation of the energy content of signals (24). The energy of $q(t)$ can be expressed by the sum of the energies of its harmonic components

$$\mathcal{E} \stackrel{\text{def}}{=} \int_{-\infty}^{+\infty} q(t)q^*(t)\,dt = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \mathcal{Q}(\omega)\mathcal{Q}^*(\omega)\,d\omega \tag{41}$$

where * denotes complex conjugation. The above equation stems from the orthogonality of harmonic components and is known as the Parseval's formula for finite energy signals. This relation leads to the definition of the energy spectrum $G_q(\omega)$ $\mathcal{Q}(\omega)\mathcal{Q}^*(\omega)/2\pi$ (i.e., the energy density of $q(t)$ within

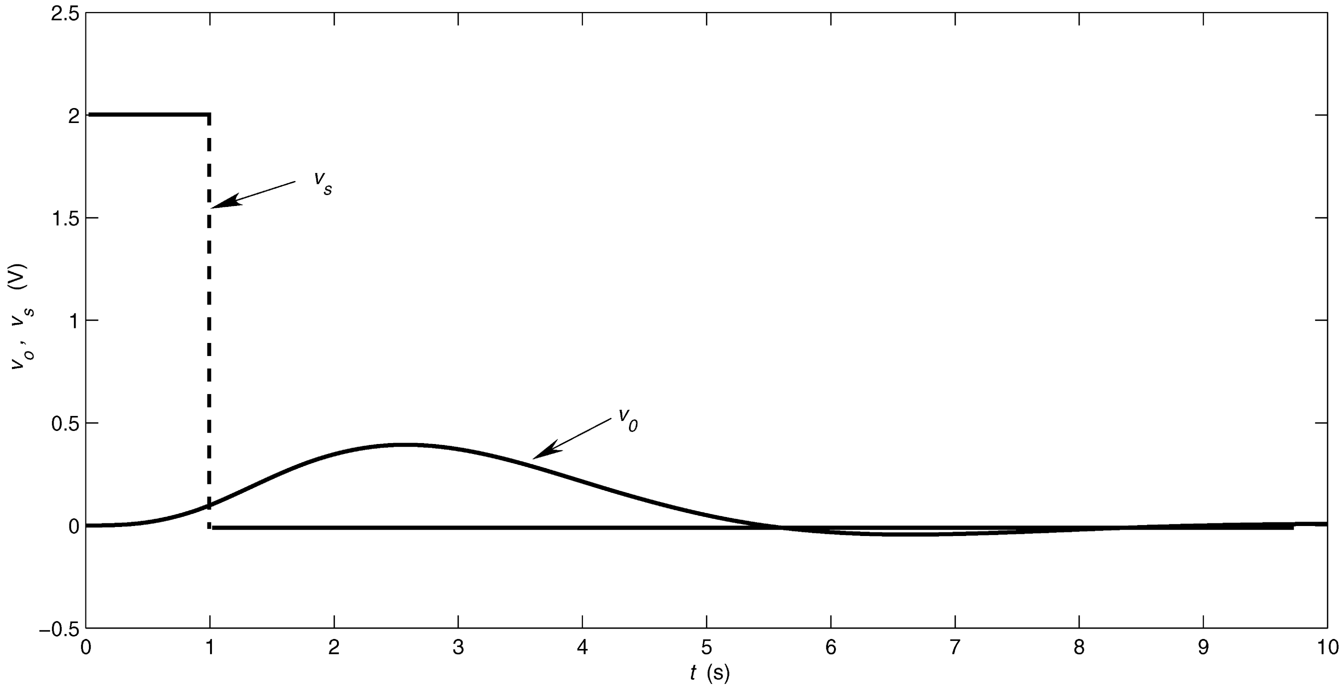


**Figure 6.** Behavior of $v_o$ and $v_s$ versus time.

$[\omega,\omega + d\omega]$) and of the autocorrelation function $R_q(\tau)$ as the inverse Fourier transform of $G_q(\omega)$.

The autocorrelation and energy spectrum concepts can be extended to signals with finite average power and to stationary stochastic processes, thereby providing a frequency representation also for these important class of signals (24). In this case, the energy spectrum is renamed power spectrum and the frequency-domain analysis results extend to these signals by the transfer relation

$$G_w(\omega) = |H(j\omega)|^2 G_q(\omega) \tag{42}$$

where $G_q(\omega)$ and $G_w(\omega)$ are the power spectra of the input and output related by the network function $H(j\omega)$, respectively.

## APPLICATIONS OF FREQUENCY-DOMAIN ANALYSIS

The procedure for the frequency-domain analysis of circuits is almost independent of the transformation method used. This latter decides only which waveforms can be represented, how they are represented, and which elements of the circuit behavior are highlighted. Apparently, Laplace transform seems able to handle a more general class of functions and, therefore, seems preferable. This point, however, is controversial (e.g., see Ref. 23) and the transformation method, instead, should be chosen according to applications.

Roughly speaking, network functions in the $s$ domain offer zero-pole portraits of the circuit behavior and provide the most reliable information on system dynamics and stability. Furthermore, the one-sided Laplace transform takes into account the initial conditions of energy-storing elements and is the preferred transformation method for the frequency-domain solution of transient problems.

When network functions are computed on the $j\omega$ axis (i.e., the Fourier representation is used), they offer a frequency-by-frequency portrait of the circuit behavior, describing how the harmonic components of the input signals are changed. This is useful both in obtaining and in specifying the circuit frequency responses, helps the physical interpretation of frequency-domain results, and allows the frequency characterization of circuits by measurements. For its properties, the Fourier approach is common in problems involving steady-state analysis, signal propagation, and stochastic signals.

In this section we show examples of frequency-domain circuit analysis, which illustrate typical applications and remark the features of the different transformation methods.

### Transient and Frequency Responses

In order to illustrate the evaluation of transient and frequency responses, we consider the circuit of Fig. 4(a). For such a circuit, we compute the zero-state response of voltage $v_o(t)$ across resistor $R_L$ to a rectangular input pulse of duration $T$ and height $E_0$; that is, $v_s(t) = E_0[u(t) - u(t-T)]$. Besides, we compute also the frequency response of $v_o$ to the input $v_s$. The components have the following (normalized) values: $R_s = R_L = 1\ \Omega$, $C_1 = C_2 = 1$ F, $L = 2$ H, $E_0 = 2$ V, and $T = 1$ s.

To solve this problem, we first compute the voltage transfer function $H(s) = V_o(s)/V_s(s)$, by using Eq. (16) and setting to zero the sources related to initial conditions.

$$H(s) = \frac{V_o(s)}{V_s(s)} = \frac{\Delta_{12}}{\Delta} = \frac{0.5}{s^3 + 2s^2 + 2s + 1} \tag{43}$$

where $\Delta$ is the determinant of the nodal admittance matrix $Y_n$ and $\Delta_{12}$ is the cofactor of element 1, 2 of $Y_n$.

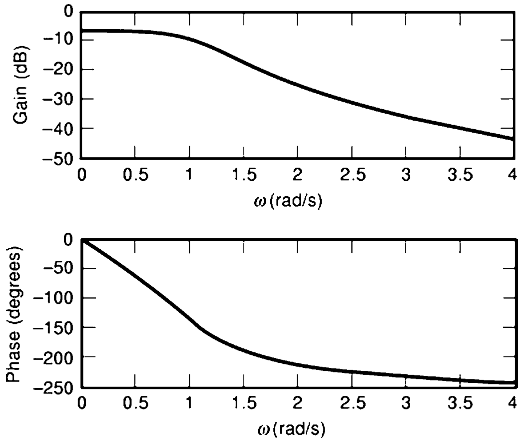


**Figure 7.** Frequency response for the circuit of Fig. 4.

From Tables 1 and 2, the Laplace transform of the input waveform turns out to be $V_s(s) = E_0(1 - e^{-Ts})/s = 2(1 - e^{-s})/s$. As a consequence, the transform of output voltage is

$$V_o(s) = H(s)V_s(s) = \frac{1}{s(s^3 + 2s^2 + 2s + 1)}(1 - e^{-s}) \tag{44}$$

By expanding in partial fractions, we obtain

$$V_o(s) = \left[\frac{1}{s} - \frac{1}{s+1} - \frac{1}{(s+0.5)^2 + (\sqrt{3}/2)^2}\right](1 - e^{-s}) \tag{45}$$

Taking the inverse Laplace transform of each term of the above equation and resorting again to Tables 1 and 2, we have

$$\begin{aligned} v_o(t) = {} & \left[1 - e^{-t} - \frac{2}{\sqrt{3}}e^{-0.5t}\sin\frac{\sqrt{3}}{2}t\right]u(t) \\ & - \left[1 - e^{-t+1} - \frac{2}{\sqrt{3}}e^{-0.5t+0.5}\sin\left(\frac{\sqrt{3}}{2}(t-1)\right)\right]u(t-1) \end{aligned} \tag{46}$$

where $v_o$ is expressed in volts and $t$ in seconds. Fig. 6 shows the behavior of $v_o$ and $v_s$ versus time.

To obtain the frequency response, we set $s = j\omega$ in Eq. (43):

$$H(j\omega) = \frac{0.5}{(1 - 2\omega^2) + j\omega(2 - \omega^2)} \tag{47}$$

The parts of $H(j\omega)$ are easily computed from the above equation. Figure 7 shows the curves of gain $g_H = 20\log|H(j\omega)|$ and phase $\phi_H = \arg[H(j\omega)]$ for $0 \le \omega \le 4$ rad/s.

## Asymptotic Responses of Circuits

The response of (strictly) stable circuits can be intuitively divided into a *transient part* and an *asymptotic part* (see Ref. 25 for a complete discussion). The two parts can be visualized and computed by a partial-fraction expansion of the circuit response. Fractions from the poles of the circuit transfer function $H(s)$ represent decreasing time functions and form the transient part, whereas the other terms form the asymptotic part. The asymptotic part can be also computed directly by the two-sided Laplace transform (e.g., see Ref. 26, Chapter 9).

The evaluation of the asymptotic part is particularly important for periodic input signals. In this case, the asymptotic response is usually named *steady-state response.* Here, we illustrate two typical approaches to the evaluation of such a response. For this, we write a periodic input signal of period $T$ as

$$q(t) = \sum_{n=-\infty}^{\infty} q_T(t - nT) = q_T(t) * \sum_{n=-\infty}^{\infty} \delta(t - nT) \tag{48}$$

where $q_T(t)$ is the *cycle function,* which coincides with $q(t)$ in $[0, T]$ and is null elsewhere, and * denotes convolution.

**One-Sided Approach.** The steady-state response $w_a(t)$ is a periodic function of period $T$. The one-sided approach consists in applying the periodic input from $t = 0$ and leads to the cycle function $w_{aT}(t)$ of $w_a(t)$.

With this approach, the transform $W(s)$ of the complete response $w(t)$ is given by

$$W(s) = H(s)Q(s) = H(s)\frac{Q_T(s)}{1 - e^{-sT}} = W_t(s) + W_a(s) \tag{49}$$

where $W_t$ and $W_a = W_{aT}(s)/(1 - e^{-sT})$ collect the poles of $H(s)$ and $1/(1 - \exp(-sT))$, respectively ($Q_T(s)$ is analytic everywhere but at infinity). In Eq. (49), $W_t(s)$ can be explicitly computed from the partial fraction terms of $W(s)$ involving the poles of $H(s)$. Once $W_t(s)$ is computed, $W_{aT}(s)$ is obtained as

$$W_{aT}(s) = H(s)Q_T(s) - W_t(s)(1 - e^{-sT}) \tag{50}$$

The interested reader is referred to Ref. 27 for a detailed discussion and illustrative examples.

**Fourier Approach.** In this approach $w_a(t)$ is obtained as a Fourier series via $H(j\omega)$ and the Fourier transform of the periodic input.

The Fourier transform of the periodic input is a *line spectrum* composed of equispaced ideal pulses (see Eq. (48) and Table 5) and $\mathcal{W}_a(\omega)$ is composed of the same input lines modified by $H(j\omega)$

$$\begin{aligned} \mathcal{Q}(\omega) &= \mathcal{Q}_T(\omega)\sum_{n=-\infty}^{\infty}\omega_T\delta(\omega - n\omega_T), \qquad \omega_T = 2\pi/T \\ \mathcal{W}_a(\omega) &= H(j\omega)\mathcal{Q}(\omega) = H(j\omega)\mathcal{Q}_T(\omega)\sum_{n=-\infty}^{\infty}\omega_T\delta(\omega - n\omega_T) \end{aligned} \tag{51}$$

From above, the Fourier series of $w_a(t)$ follows

$$w_a(t) = \sum_{n=-\infty}^{\infty}\frac{1}{T}H(jn\omega_T)\mathcal{Q}(n\omega_T)e^{jn\omega_T t} \tag{52}$$

Though the computation of $w_a(t)$ via its Fourier series is simple, it is practically useful only when the number of significant harmonic component is small. On the other hand, this representation can be exploited also for the asymptotic responses of weakly nonlinear circuits driven by periodic sources. For such an analysis, each nonlinear circuit element is characterized by generalized network functions describing how the element combines its input spectral lines into output ones (28).

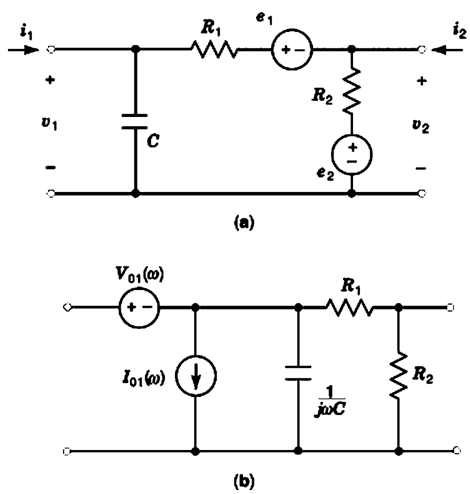


**Figure 8.** Evaluation of equivalent noise sources for a two-port element with noisy resistors: (a) Original problem; (b) transformed circuit with equivalent sources $\mathcal{V}_{01}(\omega)$ and $\mathcal{I}_{01}(\omega)$.

## Noise Sources

In this section we illustrate the frequency-domain analysis of circuits containing stochastic sources characterized by their power spectra (24, Chapter 10).

We consider the very simple example of Fig. 8(a). Noise sources $e_1(t)$ and $e_2(t)$ model the thermal noise generated by the two resistors. A typical problem in noisy two-port elements is the evaluation of the source terms of their chain matrix constitutive relations—that is, the equivalent sources $v_{01}$ and $i_{01}$ shown in Fig. 8(b).

In order to obtain the equivalent sources for this example, we replace $e_1(t)$ and $e_2(t)$ with deterministic signals with spectra $\mathcal{E}_1(\omega)$ and $\mathcal{E}_2(\omega)$ and generate the transformed circuit of the problem. $\mathcal{V}_{01}(\omega)$ and $\mathcal{I}_{01}(\omega)$ can be computed as the voltage and current at port one, ensuring null voltage and current at port two. This analysis yields the following transfer relations:

$$\begin{aligned} \mathcal{V}_{01}(\omega) &= \mathcal{E}_1(\omega) + (R_1/R_2)\mathcal{E}_2(\omega) \\ \mathcal{I}_{01}(\omega) &= j\omega C\mathcal{E}_1(\omega) + [j\omega C(R_1/R_2) - 1/R_2]\mathcal{E}_2(\omega) \end{aligned} \tag{53}$$

When noise sources are described by their power spectra $G_{e1}(\omega)$ and $G_{e2}(\omega)$, the power spectra of the equivalent sources are obtained by using the statistical independence of $e_1$ and $e_2$„ so that Eq. (42) yields

$$\begin{aligned} G_{v01}(\omega) &= G_{e1}(\omega) + (R_1/R_2)^2 G_{e2}(\omega) \\ G_{i01}(\omega) &= (\omega C)^2 G_{e1}(\omega) + [(\omega C(R_1/R_2))^2 + 1/R_2^2] G_{e2}(\omega) \end{aligned} \tag{54}$$

## Frequency-Domain Analysis of Large Circuits

The evaluation of network functions and of frequency responses amounts to the symbolic or numerical solution of frequency-domain circuit equations. For large networks, with hundreds or thousands of dynamic elements, both tasks can be prohibitively expensive. In these cases, approximate frequency solutions are sought, which reproduce the exact solution in a limited frequency range. The approximate solutions are defined by a reduced number of poles, which approximate some of the poles of the exact solution.

The simplest approach to generate reduced order approximations of network functions relies on Padé approximants (30). A rational approximation $\hat{H}(s)$ of order $p$ to $H(s)$,

$$\hat{H}(s) = \sum_{j=1}^{p} \frac{\hat{k}_j}{s - \hat{s}_j} \tag{55}$$

can be sought by expanding the Maclaurin series of $\hat{H}(s)$ and $H(s)$, up to order $n = 2p - 1$

$$\sum_{n=0}^{2p-1} s^n \left[ -\sum_{j=1}^{p} \frac{\hat{k}_j}{\hat{s}_j^{n+1}} \right] = \sum_{n=0}^{2p-1} s^n \left[ \frac{H^{(n)}(0)}{n!} \right] \tag{56}$$

The above equation requires

$$-\sum_{j=1}^{p} \frac{\hat{k}_j}{\hat{s}_j^{n+1}} = \frac{H^{(n)}(0)}{n!}, \qquad n = 0, 1, \ldots, 2p - 1 \tag{57}$$

which yield the unknown parameters $\hat{k}_j$ and $\hat{s}_j$ as functions of the coefficients $H^{(n)}(0) = [d^n H(s)/ds^n]_{s=0}$. Such coefficients are shortly named moments of $H(s)$ (see Table 6 moment theorem), and are much easier to compute than $H(s)$ itself. In this basic version, $\hat{H}(s)$ approximates $H(s)$ for small $s$ values (*i.e.*, in the low-frequency range), however, different series expansion and coefficient identities have been devised to cope with different frequency ranges (the interested reader may refer to (29) for a detailed discussion).

The moments of network functions can be easily obtained by MNA, however their evaluation is affected by numerical error and the generation of rational approximations from them is an ill conditioned problem. As a result, rational approximations with more than a few tens of poles can be hardly obtained via explicit moment calculation. Recently, several methods have been developed to generate accurate high order rational approximation via Krylov subspace projections and implicit moment calculation. These methods operate by projecting the vector of the nodal voltages and supplemental currents of the network on the Krylov subspace generated by the moments of the network function matrix of the problem. Let the zero-state frequency-domain MNA equations of a network be written as

$$\begin{cases} \mathbf{X} = s\mathbf{A}\mathbf{X} + \mathbf{R}\mathbf{U} \\ \mathbf{Y} = \mathbf{L}^T\mathbf{X} \end{cases} \tag{58}$$

where $\mathbf{U}$ is the vector of sources, $\mathbf{X}$ is the vector collecting the $N$ nodal voltages and supplemental currents of the network, and $\mathbf{Y}$ is the output vector. The Krylov subspace of order $p$ of this problem is

$$Kr(\mathbf{A},\ \mathbf{R},\ p) = \text{span}\{\mathbf{R},\ \mathbf{A}\mathbf{R}, \ldots,\ \mathbf{A}^{p-1}\mathbf{R}\} \tag{59}$$

and the reduced order problem is defined by the reduced unknown vector $\mathbf{X}_p$, $\mathbf{X} = \mathbf{V}_p\mathbf{X}_p$, with $p \ll N$ and $\mathbf{V}_p$ the matrix collecting the elements of an orthonormal basis of $Kr(\mathbf{A}, \mathbf{R}, p)$. Such a basis is obtained without computing the moments, *e.g.*, via the Arnoldi's algorithm. This reduction approach allows to handle huge networks with thousands of dynamic elements, obtaining accurate wideband approximations, possibly preserving the stability and passivity properties of the original network (*e.g.*, see (31)).

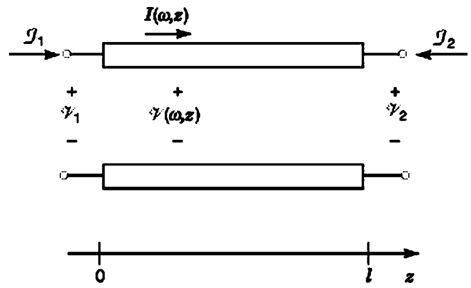


**Figure 9.** A two-conductor transmission line and the quantities relevant to its analysis.

### Transmission Line Frequency-Domain Analysis

Transmission lines (*TL*s) are important circuit elements, because of their many applications (e.g., the modeling of electrically long interconnects). As linear time-invariant elements, TLs are effectively treated by the frequency-domain approach. Both one-sided Laplace transform (transient problems) and Fourier transform (steady-state problems and physical interpretation) are used for TL analysis.

The Fourier transform of voltages and currents along a two-conductors uniform TL supporting a *quasi*-TEM field are related by the *telegrapher equations* (32):

$$\begin{aligned} \frac{d\mathcal{V}(\omega,z)}{dz} &= -\tilde{Z}(j\omega)\mathcal{I}(\omega,z) \\ \frac{d\mathcal{I}(\omega,z)}{dz} &= -\tilde{Y}(j\omega)\mathcal{V}(\omega,z) \end{aligned} \tag{60}$$

In the above equations, $z$ is the longitudinal coordinate (see Fig. 9), and $\tilde{Z}(j\omega)$ and $\tilde{y}(j\omega)$ are the TL per unit length impedance and admittance, respectively (31).

The solution of Eqs. (60) is

$$\begin{aligned} \mathcal{V}(\omega,z) &= \mathcal{V}^{+}(\omega)e^{-Kz} + \mathcal{V}^{-}(\omega)e^{Kz} \\ \mathcal{I}(\omega,z) &= (\mathcal{V}^{+}(\omega)e^{-Kz} + \mathcal{V}^{-}(\omega)e^{Kz})Y(j\omega) \end{aligned} \tag{61}$$

where $\mathcal{V}^{+}(\omega)$ and $\mathcal{V}^{-}(\omega)$ are arbitrary functions and $K$ and $Y$ are the TL complex propagation constant and characteristic admittance, respectively:

$$\begin{aligned} K(j\omega) &= \alpha(\omega) + j\beta(\omega) = \sqrt{\tilde{Z}(j\omega)\tilde{Y}(j\omega)} \\ Y(j\omega) &= \sqrt{\tilde{Y}(j\omega)/\tilde{Z}(j\omega)} \end{aligned} \tag{62}$$

The waveform described by $\mathcal{V}^{+}(\omega)\exp(-Kz)$ has harmonic components $\mathcal{V}^{+}(\omega)\exp[-\alpha(\omega)z - j\beta(\omega)z + j\omega t]\ d\omega/2\pi$. Each of such components is a harmonic function $\mathcal{V}^{+}(\omega)\exp(j\omega t)$ traveling toward increasing $z$ with phase velocity $v_\phi = \omega/\beta(\omega)$ and attenuating according to $\exp(-\alpha(\omega)z)$. It describes a transverse electromagnetic field concentrated on the line crossection and propagating along $+z$ as a plane wave.

The frequency-domain analysis of a circuit containing the TL can be carried out by relating voltages and currents at the line ends through the TL solution [Eq. (61)]. The relations are (see Fig. 9)

$$\begin{aligned} \mathcal{V}_1(\omega) &= \mathcal{V}^{+} + \mathcal{V}^{-} \\ \mathcal{V}_2(\omega) &= \mathcal{V}^{+}\exp(-\Theta) + \mathcal{V}^{-}\exp(+\Theta) \\ \mathcal{I}_1(\omega) &= Y(\mathcal{V}^{+} - \mathcal{V}^{-}) \\ \mathcal{I}_2(\omega) &= -Y\mathcal{V}^{+}\exp(-\Theta) + Y\mathcal{V}^{-}\exp(+\Theta) \end{aligned} \tag{63}$$

with $\Theta = K(j\omega)\updownarrow$, where $\updownarrow$ is the line length. The arbitrary functions $\mathcal{V}^{+}(\omega)$ and $\mathcal{V}^{-}(\omega)$ are determined by the above relations and the constitutive relations of the circuit connected at the TL ends. Alternatively, the TL can be characterized as a two-port circuit by a set of network parameters. The network parameters can be obtained by expressing two of the TL end variables as a function of the other two through Eqs. (63). As an example, the chain matrix of the TL is obtained by computing $\mathcal{V}^{+}(\omega)$ and $\mathcal{V}^{-}(\omega)$) via the second and fourth equations of Eqs. (63) and by using the first and third equations of Eqs. (63) to obtain $\mathcal{V}_1(\omega)$ and $\mathcal{I}_1(\omega)$. The chain relations are

$$\mathcal{V}_1(\omega) = \cosh(\Theta)\mathcal{V}_2(\omega) + [\sinh(\Theta)/Y][-\mathcal{I}_2(\omega)] \tag{64}$$

$$\mathcal{I}_1(\omega) = Y\sinh(\Theta)\mathcal{V}_2(\omega) + \cosh(\Theta)[-\mathcal{I}_2(\omega)] \tag{65}$$

Chain parameters can be interpreted as transfer functions in the $s$ domain by replacing $j\omega$ with $s$ in $\Theta = K(j\omega)\updownarrow$ and $Y(j\omega)$. The network functions of distributed circuits, however, are not rational functions of $s$, because they have infinitely many poles. For small $\Theta(s)$ values (i.e., electrically short lines), the lumped parameter formulation can be recovered by approximating the TL transfer functions with rational functions.

Multiconductor TLs can be treated similarly, by replacing scalar relations for voltages and currents along the line with vector relations for voltages and currents on the different conductors (31). The formulation becomes considerably complicated, yet it maintains the same properties of the two-conductor case.

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