NETWORK THEOREMS

In this article, we consider electrical networks from the Kirchhoff point of view. The descriptive equations are decomposed into the constitutive relations for the network elements and the Kirchhoff equations for the connection element (see Fig. 1). In the following, these networks are called Kirchhoff networks.

This decomposition was already used implicitly in the nineteenth century by the founders of network theory. It was Belevitch (1) who clarified this concept and used port currents and voltages of the connection element for describing Kirchhoff networks. An advantage of this decomposition is that the connection element can be described by linear equations in linear as well as nonlinear networks. If dynamic network elements that are characterized by differential and/or integral equations are included, we obtain a mixture of differential equations and algebraic or transcendent equations that describe the network. Equations of this type are called differential–algebraic equations (DAE) [see, e.g., Chua, Desoer, and Kuh (2); Hasler and Neirynck (3); Mathis (4); or Vlach and Singhal (5) for further details]. We adopt the point of view of mathematical dynamical systems [see, e.g., Arrowsmith and Place (6)] where DAEs, their set of solutions (the flow), and the right-hand side (the vector field) are, under certain restrictions, simply different representations of the same abstract subject.

It is sufficient to consider subsets of $\mathbb{R}^n$ as solution manifolds in the case of resistive networks, and subsets of suitable function spaces as solution manifolds in the case of more general dynamic circuits. Manifold theory is not really needed at this stage. However, we emphasize the term *manifold* in order to recall that the solution sets of network equations are not mere points-sets: In the vicinity of any solution point there is a continuum of solutions that locally behaves as a Euclidean space, and there exists a smooth transition between any two discrete points within the solution set. In the following, network theory is considered as a mathematical theory consisting of definitions, theorems, and corollaries although a consistent presentation of network theory is rather rare [see Slepian (7) and Reibiger (8)]. Roughly speaking, network theorems may be classified as follows:

1. Theorems that consider properties of an individual network
2. Theorems that consider interrelations of at least two networks.

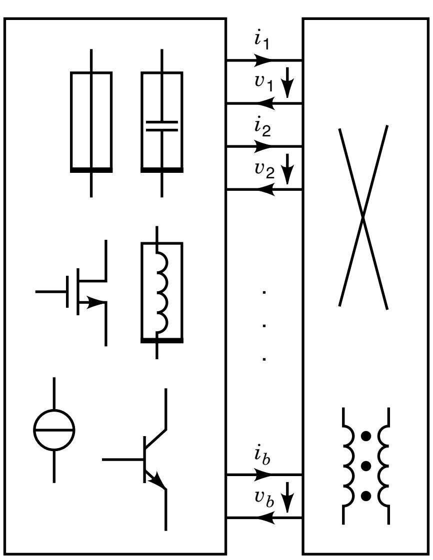

Figure 1. Decomposition of a Kirchhoff network into circuits elements and the connection element (wires and ideal transformers).

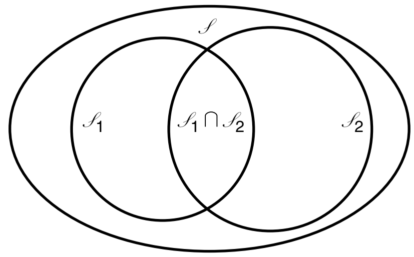

Figure 2. The sets of solutions $\mathscr{S}_1$ and $\mathscr{S}_2$ the descriptive equations of two networks (solution manifolds) and its intersection embedded into a solution space $\mathscr{S}$.

We will concentrate mainly on theorems of the second class. Based on Kirchhoff's point of view, two networks can be different with respect to the connection element (network topology), the kinds of network elements, and/or the associated network parameters. Interestingly, many of the network theorems and certain properties of networks can be discussed in a unified manner using this classification. This was first pointed out by Ghenzi (9), but in a rather restricted manner (e.g., the duality theory presented in the following is based on Ghenzi's ideas). In addition, the abstract network theory of Reibiger has been very useful.

In the next subsection, useful superposition theorems for arbitrary linear networks are discussed. These theorems may be applied when the solution manifolds differ only in the value of parameters characterizing the independent voltage and/or current sources. The following section is devoted to networks that include different types of network elements but have the same connection element. A well-known theorem was first published by Weyl (10) and later by Tellegen (11) and Ghenzi [see Mathis (4)] that analyses the energy or power flow into the connection element. We discuss the relationship of at least two networks and their solution manifolds that differ partly and/or entirely with respect to the types of the network elements and their parameters, as well as the connection element where only certain network characteristics (impedances or admittances) are fixed. In a more abstract set-theoretical framework, this can be illustrated by Fig. 2, where the intersection of the solution manifolds $\mathscr{S}_1$ and $\mathscr{S}_2$ of two networks that are embedded in a solution space $\mathscr{S}$ is nonzero.

Although this presentation can be used to give an idea of what is behind theorems of this kind, a rather concrete formulation needs to present a mapping between two networks. We will illustrate this point by formulating known network theorems like substitution theorems, interreciprocity, and duality as special cases rather than separate theorems.

SUPERPOSITION THEOREMS

Although superposition theorems consider crucial properties to be a single linear network, these theorems use at least two networks to test it. It is well known that Helmholtz (12) derived a superposition theorem in 1853. Helmholtz's result was extended by other researchers and was included by Maxwell in his monumental treatise (13). But it seems that Hausrath (14) was the first who derived these theorems under very general assumptions and studied their conditions of validity as well as their many applications. Hausrath formulated the principle of superposition in the following manner:

> If a [linear time-invariant] network includes a number of arbitrarily distributed current and voltage sources, then each source results in currents and voltages associated to the network elements as if the other sources are eliminated and the current or the voltage of some network element can be calculated by summing up the currents and voltages of each source, respectively.

He added a mathematical interpretation: the resulting current is a linear function of the potential distribution, or, in reverse order, the resulting potential distribution is a linear function of the currents.

Hausrath proved this statement by means of Maxwell's equations, where inductors and currents can be included in the networks. He emphasized "the superposition equations in the case of ac currents differ from the dc case in that their coefficients will be complex." A more general reasoning of superposition theorems can be given in the following manner:

If we assume that a network can be described by currents and voltages and the network equations have the following form

$$\mathbf{L}_1 \boldsymbol{x} = \mathbf{L}_2 \boldsymbol{y} \tag{1}$$

where $\mathbf{L}_1$ and $\mathbf{L}_2$ are linear operators, $\boldsymbol{x}$ is a vector of voltages and currents of the network elements, and $\mathbf{y}$ is a vector including independent current and voltage sources, in a linear manner, then the principle of superposition can be proved very easily. In the case of linear resistive networks, the operators are matrices and the result follows from linear algebra. On the other hand, the superposition principle is also valid in the case of differential operators with time-dependent coefficients, that is, for linear time-variant networks. Assuming that the source vector $\boldsymbol{y}$ is determined by s currents and voltage sources, it can be decomposed into

$$\boldsymbol{y} = \boldsymbol{y}_1 + \cdots + \boldsymbol{y}_s \tag{2}$$

If $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_s$ are the (unique) solutions of $\mathbf{L}_1\boldsymbol{x}_i = \mathbf{L}_2\boldsymbol{y}_i$ ($i = 1, \ldots, s$), the complete solution $\mathbf{x}$ is calculated by

$$\boldsymbol{x} = \boldsymbol{x}_1 + \cdots + \boldsymbol{x}_s \tag{3}$$

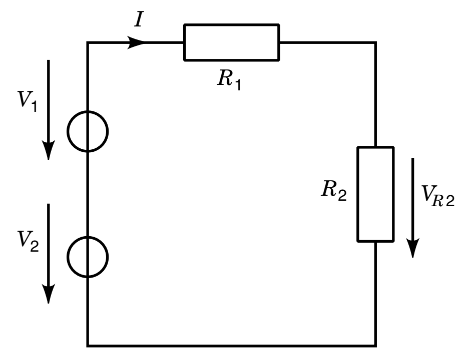

Figure 3. Elementary circuit with two independent voltage sources.

The idea of a superposition theorem can be illustrated by using the simple network in Fig. 3 which contains two independent voltage sources.

Example. For determination of the voltage V_{R_2} in this network, the best way is to replace both sources by one source, $V = V_1 + V_2$, and to calculate V_{R_2} by means of the well-known rule of voltage dividers. On the other hand, the idea of superposition theorems suggests constructing two networks by setting all values of the voltage source (with the exception of one) to zero. Using the rule of voltage dividers, the voltages $V_{R_2}^{(1)}$ and $V_{R_2}^{(2)}$ are calculated and as a result of the superposition theorem we have $V_{R_2} = V_{R_2}^{(1)} + V_{R_2}^{(2)}$.

The approach in the last example can be generalized very easily. We associate a number of "test" networks (corresponding to the number of independent sources) with the original network by setting all sources except one to zero (i.e., current sources $I_0 = 0$ and voltage sources $V_0 = 0$) and calculate the desired network variable. The value of this variable in the original network can be determined by a superposition theorem.

Although superposition theorems can be very useful in network analysis, it should be kept in mind that they are generally not valid when considering power quantities (e.g., average power) as description variables. Commonly, they will not be true even in linear time-invariant resistive networks. We demonstrate this statement by means of the next example, omitting the rather simple calculations.

Example. In Fig. 3 a linear resistive network that includes two dc voltage sources V_1 and V_2 is shown. If the power of the oneports is defined by $P = V \times I$, the current I' and the voltage V'_{R2} are calculated under the condition that V_2 is zero. The power of R_2 is $P' = V'_{R2} \times I'$. In the same manner P'' is calculated under the condition that V_1 is zero; we obtain $P'' = V''_{R2} \times I''$. But the superposition $P' + P''$ does not correspond to the correct power P if both sources are included. In this case, we have

$$P = (V'_{R2} + V''_{R2}) \times (I' + I'') \tag{4}$$

$$= (V'_{R2} \times I' + V''_{R2} \times I'') + (V'_{R2} \times I'' + V'_{R2} \times I'') \tag{5}$$

$$=: (P' + P'') + P_{IA} \tag{6}$$

where P_{IA} is the interaction term.

In contrast to this, it can be shown that the superposition principle of (average) power quantities of reciprocal networks with current and voltage sources can be valid if we add the powers of current sources and voltage sources separately. Martens and Lê (15) proved the following theorem:

Let P be the power dissipated in a linear network in which there are only resistors, current sources, and voltage sources. Let $P_E(P_J)$ be the power dissipated when all the current (voltage) sources are open circuited (short circuited). Then $P = P_E + P_J$.

A first theorem of this kind was given by Guillemin (16), pp. 127–128) and more general cases for networks including capacitors and inductors can be found in the paper of Martens. A simple example is shown in Fig. 4. It can be shown that the superposition theorem in linear time-invariant networks is generally valid if the frequencies of the two sources are different [see, e.g., Desoer and Kuh, Chap. 7.1 (17)].

ENERGY AND POWER

Definitions

Although other network variables are possible (see e.g., Mathis (4, chap. 6) we restrict this discussion to currents and voltages. Furthermore, we consider only Kirchhoff networks that include the standard set of network elements (R, L, C, independent and controlled sources as well as the Kirchhoff connection element). In this case, the descriptive equations are formulated as

$$\mathbf{Ai} = \mathbf{0}, \mathbf{Bv} = \mathbf{0} \tag{7}$$

$$\mathbf{f}(\mathbf{i}, \mathbf{v}, t) = \mathbf{0}, \mathbf{M}(\mathbf{x})\frac{dx}{dt} = \mathbf{g}(\mathbf{x}) \tag{8}$$

where $\mathbf{x} = (\mathbf{i}, \mathbf{v})$. The pair of matrices ($\mathbf{A}$, $\mathbf{B}$) that describes the Kirchhoff connection element is exact in the following sense [Ghenzi (9), Mathis (4)]:

$$\mathbf{AB}^T = \mathbf{0}$$
$$\mathrm{Rk}(\mathbf{A}) + \mathrm{Rk}(\mathbf{B}^T) = b$$

where b is the number of ports of the connection element.

In many applications, the solution manifold of these network equations with respect to the currents and voltages needs to be calculated. But sometimes the power quantities are of interest because a thermodynamic interpretation (e.g., by means of Joule's theorem) is available. In general, products $v_k \times i_k$ of currents i_k and voltages v_k are instantaneous powerlike quantities with respect to physical dimension, but a thermodynamic interpretation exists only if both quantities are associated to the same port (or two-terminal element). In the latter case, the integrals of these powerlike quantities also have an energetic interpretation.

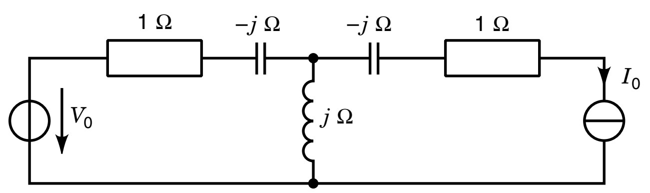

Figure 4. Nambiar's example of a circuit where superposition of power is valid (both generators have the same frequency and phase).

WEYL–TELLEGEN THEOREM

We consider two networks $\mathcal{N}$ and $\tilde{\mathcal{N}}$ that have the same connection element and where the port quantities are denoted by v_k, i_k and $\tilde{v}_k$, $\tilde{i}_k$, respectively. These quantities satisfy the following Kirchhoff equations where the port currents and voltages are described by the vectors $\boldsymbol{i}$, $\boldsymbol{v}$, $\tilde{\boldsymbol{i}}$, $\tilde{\boldsymbol{v}}$:

$$\mathbf{A}\boldsymbol{i} = \mathbf{0}, \mathbf{B}\boldsymbol{v} = \mathbf{0} \quad \text{and} \quad \mathbf{A}\tilde{\boldsymbol{i}} = \mathbf{0}, \mathbf{B}\tilde{\boldsymbol{v}} = \mathbf{0} \tag{9}$$

It is easy to prove that the following relations hold:

$$(\boldsymbol{i}, \tilde{\boldsymbol{v}}) = 0, (\tilde{\boldsymbol{i}}, \boldsymbol{v}) = 0 \tag{10}$$

The proofs are a direct consequence of the exactness of $\mathbf{A}$ and $\mathbf{B}$ (see Mathis (4), appendix A, 1.17). Weyl (10) [see also Cauer's monograph (18)] presented a proof of the following relations in 1923,

$$(\boldsymbol{i}, \boldsymbol{v}) = 0 \tag{11}$$

but there is no difference from a mathematical point of view to the above relations since both networks have the same connection element. Tellegen (11) reinvented Weyl's result and illustrated it with several examples and applications. A detailed discussion of the history of these theorems (as well as many further applications) can be found in Penfield, Spence, Duinker (19). It should be mentioned that the theorem of Weyl–Tellegen is very useful in sensitivity analysis (see also the section "Interreciprocity" in this article).

EQUIVALENCE OF *n*-PORTS

Imagine two different multiports (black boxes) with the same number n of externally accessible ports and assume they are equivalent in the sense that they cannot be distinguished from each other by any measurements of electrical parameters at the ports. Though they behave externally in an identical manner, their internal circuitry may be completely different; not even the numbers of internal nodes and meshes need be the same. This concept of external equivalence of multiports is the central theme in classical network synthesis since the early work of Foster and Cauer (18). As opposed to the analysis viewpoint that still pervades circuit theory textbooks, equivalence theorems like those of the Norton–Thévenin type may be attacked in a very clear and straightforward manner from the synthesis point of view, that is, one starts with a mathematical description of the external behavior and then asks for different internal realizations.

When terminating a multiport with any fixed-load network and collecting the n currents and n voltages measured simultaneously at the n ports in n-vectors $\boldsymbol{i}$ and $\boldsymbol{v}$, respectively, one obtains an *admissible pair* ($\mathbf{i}$, $\mathbf{v}$). The set of all admissible ($\mathbf{i}$, $\mathbf{v}$)-pairs of an n-port $\mathcal{N}$ is called the driving point characteristic or the graph $G(\mathcal{N})$ of $\mathcal{N}$. Assume $\mathcal{N}$ to be linear and time invariant and all signals to be real valued (complex signals

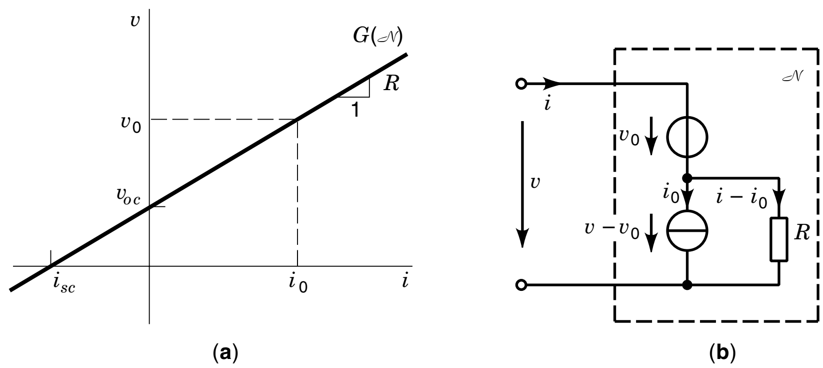

Figure 5. (a) (i, v)-characteristic or graph $G(\mathcal{N})$ of a linear resistive affine oneport $\mathcal{N}$, and (b) its elementary circuit model.

may be used as well in the *spot-frequency sense*—otherwise, see "Complex Variables" below). Equivalently, $G(\mathcal{N})$ is a fixed linear manifold, that is, the linear affine subspace

$$G(\mathcal{N}) = \begin{bmatrix} \mathbf{i}_0 \\ \mathbf{v}_0 \end{bmatrix} + \text{span}\begin{bmatrix} \mathbf{I} \\ \mathbf{V} \end{bmatrix} \subset IR^n \times IR^n \qquad (12)$$

where span[·] denotes the space spanned by the columns of a matrix. Except for pathological degenerations due to physically meaningless interconnections of nullors (cf. Ref. 20), $\mathbf{I}$ and $\mathbf{V}$ are square matrices such that the columns of the $2n \times n$ matrix

$$\begin{bmatrix} \mathbf{I} \\ \mathbf{V} \end{bmatrix}$$

form a basis for the *direction space*

$$\text{span}\begin{bmatrix} \mathbf{I} \\ \mathbf{V} \end{bmatrix}$$

This latter term is suggested by a geometric interpretation of Eq. (12): $G(\mathcal{N})$ is generated by a parallel shift of the direction space from the origin $(\mathbf{0}, \mathbf{0})$ to any reference point $(\mathbf{i}_0, \mathbf{v}_0) \in G(\mathcal{N})$.

In case of a real oneport, the graph $G(\mathcal{N})$ is simply a straight line in $IR \times IR$ as shown in Fig. 5(a). Note that the intercept points $(i_{sc}, 0)$ and $(0, v_{oc})$ are measured as admissible pairs when the oneport is connected to an external short circuit ($v \equiv 0$) or open circuit ($i \equiv 0$), respectively. Moreover, admissible pairs outside the section between these intercept points may occur exclusively in the presence of external sources.

When the slope R of the graph $G(\mathcal{N})$ in Fig. 5(a) is nonzero and finite, we may write down the equation of a straight line in point-slope form

$$(v - v_0) = R(i - i_0), \quad (i_0, v_0) \in G(\mathcal{N}) \qquad (13)$$

where the reference point (i_0, v_0) may be any admissible pair. Clearly, Eq. (13) is Ohm's law for not necessarily passive $\mathcal{N}$, and it has the obvious and most elementary circuit realization shown in Fig. 5(b). From each of the three representations for $\mathcal{N}$ [Eq. (13), Fig. 5(a), or (b)] the following facts are readily seen:

- The slope of $G(\mathcal{N})$ equals the internal resistance R of the circuit realization. According to Eq. (13), R is determined by two admissible pairs (i_i, v_i), $i = 0, 1$, as

$$R = \frac{v_1 - v_0}{i_1 - i_0} \qquad (14)$$

- If $\mathcal{N}$ is not a "dead" circuit (free of independent sources), one may choose the admissible pairs $(i_0, v_0) = (i_{sc}, 0)$ (external short circuit) and $(i_1, v_1) = (0, v_{oc})$ (external open circuit) to determine in the well-known method

$$R = -\frac{v_{oc}}{i_{sc}} \qquad (15)$$

  as the ratio of open-circuit voltage and short-circuit current.
- When the $\mathcal{N}$ does not contain any independent sources, the equivalent circuit of $\mathcal{N}$ shrinks to a single resistor R and $G(\mathcal{N})$ is a proper linear subspace [a line passing through (0, 0)]. Hence, one may choose $(i_0, v_0) = (0, 0)$ and determine $R = v_1/i_1$ by the usual form of Ohm's law.
- For $R = 0$, the graph $G(\mathcal{N})$ is a horizontal line and $\mathcal{N}$ reduces to an independent voltage source with $v_0 = v_{oc}$ or to a short circuit in the case $v_0 = 0$.
- For $R = \infty$, the graph $G(\mathcal{N})$ is a vertical line and $\mathcal{N}$ reduces to an independent current source with $i_0 = i_{sc}$ or to an open circuit in case $i_0 = 0$.

Thévenin–Norton Theorem

The Thévenin–Norton equivalent circuits as shown in Fig. 6 are the earliest and most elementary equivalence results for

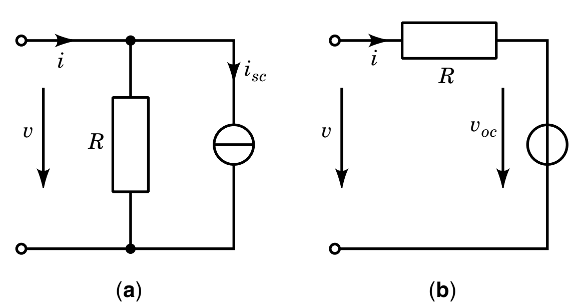

Figure 6. (a) Norton and (b) Thévenin equivalent circuits for a linear resistive affine oneport.

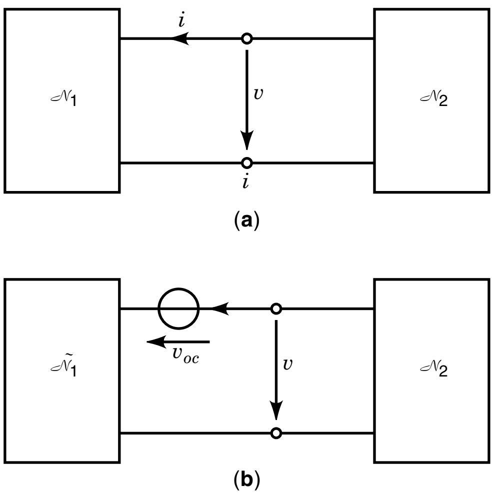

Figure 7. (a) Decomposition of a network into a purely linear-resistive affine oneport $\mathcal{N}_1$ and a more general (possibly nonlinear, time-varying and dynamic) remainder $\mathcal{N}_2$. (b) Replacing $\mathcal{N}_1$ by a voltage source and a oneport $\tilde{\mathcal{N}}_1$ without independent sources.

oneport networks. The main value of these equivalent circuits is that they may replace *any* generic linear resistive oneport. No matter how complicated its internal structure is, and insofar as its external behavior is of concern, it is equivalent to the most simple circuits in Fig. 6. The main applications of Thévenin–Norton equivalents are in modeling noisy circuits (21) and in conventional circuit analysis (17). In the latter case, one may split a complicated network into two parts, $\mathcal{N}_1$ and $\mathcal{N}_2$, as shown in Fig. 7. When collecting all nonlinear, time-varying, and dynamic components in $\mathcal{N}_2$ such that $\mathcal{N}_1$ embraces exclusively linear-resistive parts of the network (and no components whose individual electrical quantities are of any interest for the solution), $\mathcal{N}_1$ may be replaced by a single voltage source in series with a oneport $\tilde{\mathcal{N}}_1$ that contains no independent sources (see Fig. 7), and which, in turn, may be replaced by a single resistor. In the theory of noisy networks, equivalent sources serve to replace a noisy resistor or a complicated noisy oneport network by a noise (current or voltage) source and a noiseless resistor.

Before discussing Thévenin–Norton equivalents in more detail, a warning seems appropriate. One should bear in mind that these equivalences are strictly limited to the data at the accessible (external) ports; for example, the currents through the internal resistors in the equivalent Thévenin and Norton circuits in Fig. 6 are not the same. Consequently, internal power consumption of the "equivalent" sources is different!

Linear Resistive Oneports. Define a linear oneport $\mathcal{N}$ to be *generic* or *nondegenerate* when its graph $G(\mathcal{N})$ is a one-dimensional linear manifold (cf. Fig. 5(a); this condition excludes nullors) and it is neither an independent voltage, a current source, nor a short or open circuit.

Theorem (Thévenin–Norton). Any generic, linear, time-invariant, resistive oneport $\mathcal{N}$ may be replaced by the equivalent realizations shown in Fig. 6.

Proof. The graph $G(\mathcal{N})$ being an affine line as shown in Fig. 5(a), the Thévenin and Norton equivalent sources trivially follow from Fig. 5(b) when choosing as a reference point (i_0, v_0), the admissible pairs $(0, v_{oc})$ and $(i_{sc}, 0)$, respectively.

Note that this proof (besides the usual assumptions of nondegeneracy) invokes exclusively the linearity property of the oneport. No recourse is made to other theorems (such as the substitution theorem) or to external circuitry. A conceptually similar proof, based on an abstract network model and a rigorous exploitation of the consequences of linearity, has been given in Ref. (22).

Depending on the applications at hand, one may determine the parameters R, v_{oc} and i_{sc} of the Thévenin and Norton equivalents by external *black box* measurements or by analysis of a given internal circuit diagram. Since $G(\mathcal{N})$ is a straight line, it is sufficient to measure two admissible pairs (i_i, v_i), $i = 1, 2$ as discussed above [see Eqs. (13)–(15)]. Clearly, the easiest way is to measure v_{oc} and i_{sc} directly and to determine R from Eq. (15). However, when the oneport might be damaged by open- or short-circuit measurements, one must use Eq. (14) and calculate v_{oc} and i_{sc} from Eq. (13).

In case of a given circuit diagram for $\mathcal{N}$, the equivalent circuit parameters must be calculated by means of network analysis. This is usually done by two independent analyses. Define $\tilde{\mathcal{N}}$ to be a copy of $\mathcal{N}$ with all (say s) independent sources "off," that is, setting $i_{0i} = v_{0i} = 0$, $i = 1, \cdots, s$, and hence open-circuiting current sources and short-circuiting voltage sources. Then, one analysis serves for computing R as the input resistance of $\tilde{\mathcal{N}}$ and another one for computing v_{oc} or i_{sc} (with all sources "on"). Many instructive examples including networks with controlled sources may be found in circuit theory text books (2).

When $\mathcal{N}$ is a quite complex circuit, one may proceed as follows in order to avoid two complete circuit analyses: (1) Pull all s independent sources out of N as new ports and compute a matrix representation for the resulting $(1 + s)$-port; (2) connect the independent sources to the s source ports and determine i_{sc} or v_{oc} by ordinary matrix calculus as well as R (with all s sources "off"). A similar procedure has been proposed in Ref. 23 based on the concept of interreciprocity (see also "Interreciprocity").

Linear Resistive Multiports. There are natural generalizations of Thévenin–Norton equivalent circuits to nondegenerate n-ports $\mathcal{N}$ that is, when $G(\mathcal{N})$ in Eq. (12) has dimension n. Clearly, Eqs. (12)–(13) and Fig. 5 hold mutatis mutandis, that is, all currents $\mathbf{i}$ and voltages $\mathbf{v}$ are n-vector valued. The resistor in circuit in Fig. 5(b) symbolizes an n-port with $n \times n$ resistance matrix $\mathbf{R}$, where each port is loaded separately by a parallel-connected current and a series-connected voltage source. The question of existence of $\mathbf{R}$ is not crucial, since one may choose any other basis

$$\begin{bmatrix} \mathbf{I} \\ \mathbf{V} \end{bmatrix}$$

for the direction space

$$\text{span} \begin{bmatrix} \mathbf{I} \\ \mathbf{R} \end{bmatrix} = \text{span} \begin{bmatrix} \mathbf{I} \\ \mathbf{V} \end{bmatrix}$$

where $\mathbf{1}$ denotes the $n \times n$ unit-matrix. Clearly, since

$$\begin{bmatrix} \mathbf{1} \\ \mathbf{R} \end{bmatrix} = \begin{bmatrix} \mathbf{I} \\ \mathbf{V} \end{bmatrix} \mathbf{I}^{-1}$$

existence of $\mathbf{R} = \mathbf{V}\mathbf{I}^{-1}$ requires $\det\mathbf{I} \neq \mathbf{0}$, that is, the n-port has to be current controlled.

As to the choice of a reference point $(\mathbf{i}_0, \mathbf{v}_0) \in IR^n \times IR^n$ along $G(\mathcal{N})$, there is no further restriction beyond $\det\mathbf{I} \neq 0$ in choosing $(\mathbf{0}, \mathbf{v}_0)$. Equivalently, when the n-port is voltage controlled ($\det\mathbf{V} \neq 0$), one may reduce $(\mathbf{i}_0, \mathbf{v}_0)$ to $(\mathbf{i}_0, \mathbf{0})$. This way, the number of independent sources is reduced by n and one ends up with the n-port analogs for the Thévenin and Norton equivalent circuits in Fig. 6 [see Fig. 8(a) for a twoport Thévenin circuit].

Unlike oneport equivalents, n-ports have more than the standard solutions $(\mathbf{0}, \mathbf{v}_0)$ and $(\mathbf{i}_0, \mathbf{0})$ for choosing a reference point in order to turn n source currents or voltages to zero. In general, there are $\binom{2n}{n}$ different choices, each resulting in a different equivalent circuit. Figure 8 shows two nonstandard equivalent circuits from the six possible choices in the case $n = 2$. These may be described by the equations

$$\begin{bmatrix} v_1 - v_{01} \\ i_2 - i_{02} \end{bmatrix} = [\mathbf{H}] \begin{bmatrix} i_1 \\ v_2 \end{bmatrix}, \quad \begin{bmatrix} v_1 - v_{01} \\ i_1 - i_{01} \end{bmatrix} = [\mathbf{A}] \begin{bmatrix} v_2 \\ -i_2 \end{bmatrix} \tag{16}$$

where $\mathbf{H}$ and $\mathbf{A}$ denote the hybrid and chain matrices of the twoport.

In order to determine the parameters of an affine n-port, $n + 1$ admissible pairs $(\mathbf{i}, \mathbf{v})$ are necessary: one reference point $(\mathbf{i}_0, \mathbf{v}_0)$ and n basis vectors for the direction space span

$$\begin{bmatrix} \mathbf{I} \\ \mathbf{V} \end{bmatrix}$$

This can be done by network analysis quite in the same way as for oneports. Measurements, however, may be tricky. In case of a hybrid equivalent circuit [Fig. 8(b) and left part of Eq. (16)] one gets the reference data $(0, i_{02}, v_{01}, 0)$ for the independent sources in the familiar way from simultaneous open- and short-circuit measurements according to $i_1 = v_2 = 0$. It is instructive to verify that a chain-type circuit [Fig. 8(a) and right part of Eq. (16)] would require a rather exotic measurement setup in order to determine for instance $v_{01} = v_1|_{i_2=v_2=0}$.

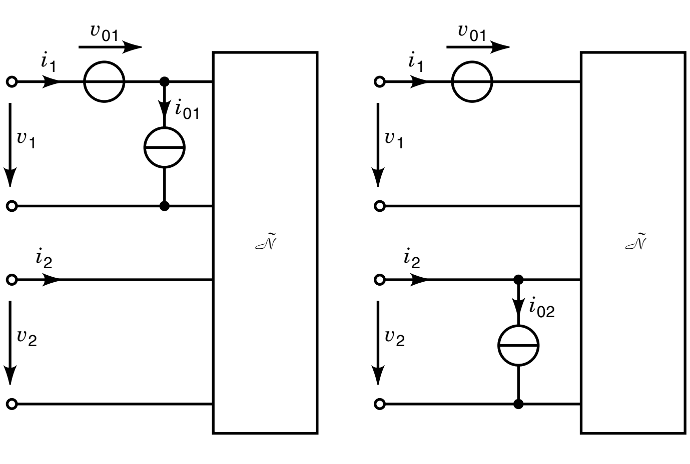

Figure 8. Equivalent circuits for a linear affine twoport deduced from (a) a chain matrix and (b) a hybrid matrix representation. $\tilde{\mathcal{N}}$ contains no independent sources.

Complex Variables. There is no limitation in using Laplace transform and in applying the Thévenin–Norton theorem on a complex variables basis for circuits which contain capacitors and inductors. Because of initial values there may be additional sources that have to be treated as independent ones. As a consequence, the characteristics of the equivalent current and voltage sources may depend on the initial values of the internal inductors and capacitors. Moreover, when using complex quantities, one implicitly works in sinusoidal steady state when the complex variable s is replaced by jω. Therefore, referring to a circuit decomposition as shown in Fig. 7, not only $\mathcal{N}_1$ but also the external circuitry in $\mathcal{N}_2$ must be linear and time invariant.

Darlington Theorem

Darlington's theorem certainly is the most penetrating frequency domain result of classical network theory. It is not usually presented outside specialized books on network synthesis [(1,18); see also (24 and 25)], since its derivation requires spectral factorization and, therefore, quite a lot of analytic function theory. However, in the present context, it is essential to get an idea of what it is. In many cases, a spot-frequency version of the theorem may be sufficient.

The most elementary version of this theorem is for lumped oneports that are passive and therefore have a positive real rational input impedance $Z(s)$ (i.e., the real part of $Z(s)$ has no zeros in the right-half plane and does not vanish identically on the real frequency axis $s = j\omega$).

Theorem (Darlington, 1939). Any positive real function $Z(s)$ may be realized as the input impedance of a lossless twoport terminated in a positive resistor R.

Clearly, one may choose $R = 1$ by using an added ideal transformer in the lossless twoport. Darlington's theorem has become generalized for multiports [see, e.g., Newcomb (26) and Belevitch (1); for a correct form of the multiport-cascade Darlington-realization of rational matrices, however, it is still indicated to consult Ref. 27]. Extensions of the Darlington theory to nonpassive devices have been demonstrated by Ball and Helton (28). Furthermore, affine multiports $\mathcal{N}$ containing independent sources may be treated as well, since Darlington representation deals with the input impedance $\mathbf{Z}$ of $\tilde{\mathcal{N}}$ (with all independent sources "off"). For that reason only the direction space of the external behavior in terms of the "slope" or "angle operator" $\mathbf{Z}$ is affected. Hence, a large class of linear lumped multiports has a Darlington-type representation like that in Fig. 9(b). In order to figure out the precise limitations, it is useful to think of the equivalence of the Thévenin and Darlington representations shown in Fig. 9 as a special case of a lossless cascade transformation $\mathcal{F}\{\cdot\}$ between two Thévenin circuits $(\tilde{\mathbf{V}}_0, \tilde{\mathbf{Z}})$ and $(\mathbf{V}_0, \mathbf{Z}) = \mathcal{F}\{(\tilde{\mathbf{V}}_0, \tilde{\mathbf{Z}})\}$, where, in case of the Darlington circuit, we have $\tilde{\mathbf{Z}} = \mathbf{R} = \text{diag}(R_i)$. Adhering to the familiar impedance coordinates and defining $\text{Re}\{\mathbf{Z}\} :=$

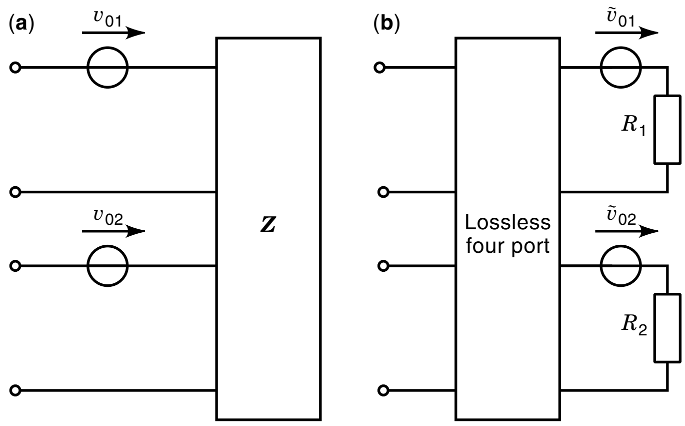

Figure 9. (a) Thévenin equivalent circuit for an affine passive two-port with impedance matrix Z and (b) its Darlington equivalent two-port with real positive resistors R_1 and R_2.

$\frac{1}{2}(\mathbf{Z} + \mathbf{Z}^*)$, where $\mathbf{Z}^*$ denotes the Hermitian conjugate matrix, it is not difficult to show that

$$\mathbf{V}_0 = \mathbf{T}\tilde{\mathbf{V}}_0, \quad \mathrm{Re}\{\mathbf{Z}(\mathrm{j}\omega)\} = \mathbf{T}\,\mathrm{Re}\{\tilde{\mathbf{Z}}(\mathrm{j}\omega)\}\mathbf{T}^* \tag{17}$$

Clearly, $\mathbf{T} = \mathbf{T}(\mathrm{j}\omega)$ but at a fixed frequency it can be *any* nonsingular constant matrix by proper choice of the lossless transformation. It becomes clear that the question about what can be done by means of lossless cascade transformations amounts to a study of the congruence classes of $\mathrm{Re}\{\mathbf{Z}(\mathrm{j}\omega)\}$ or, equivalently, $\mathrm{Re}\{\tilde{\mathbf{Z}}(\mathrm{j}\omega)\}$. In case of Darlington equivalents with all resistors R_i normalized to ± 1, $\mathrm{Re}\{\mathbf{Z}(\mathrm{j}\omega)\} = \mathrm{diag}(R_i)$ becomes a signature matrix $\mathbf{J} = \mathrm{diag}(\pm 1)$; as a consequence, such circuits exist only if the factorization problem

$$\mathrm{Re}\{\mathbf{Z}(\mathrm{j}\omega)\} = \mathbf{T}(\mathrm{j}\omega)\mathbf{J}\mathbf{T}^*(\mathrm{j}\omega), \quad \mathbf{J} = \mathrm{diag}(\pm 1) \tag{18}$$

has a solution. When fixing a particular frequency, the problem boils down to the congruence diagonalization of *constant* indefinite Hermitian matrices; hence, standard linear algebra software packages may be used to determine *spot-frequency* equivalent circuits as they are widely used in noise *analysis* (29). In network *synthesis* and modeling of stationary stochastic processes (30), one is interested in circuits that are *continuously* valid for all frequencies, that is, one looks for a rational matrix $\mathbf{T}(s)$ that in addition to Eq. (17) fulfills certain analyticity requirements. In case of a passive impedance $\mathbf{Z}(s)$, all resistors R_i are positive; hence we have $\mathbf{J} = \mathbf{1}$. Solutions of $\mathrm{Re}\{\mathbf{Z}(\mathrm{j}\omega)\} = \mathbf{T}(\mathrm{j}\omega)\mathbf{T}^*(\mathrm{j}\omega)$ may be found by classical spectral or Wiener-Hopf factorization.

In case of an indefinite matrix $\mathbf{J}$ it should be intuitively clear that Eq. (17) does not have a solution when the inertia of $\mathrm{Re}\{\mathbf{Z}(\mathrm{j}\omega)\}$ varies with $\mathrm{j}\omega$: One cannot imagine a lossless $2n$-port that transforms positive resistors into negative ones. In fact, it is known in mathematics that this condition is necessary and sufficient to solve Hermitian factorization problems of this kind (28,31); as a result, continuous-frequency Darlington equivalents exist only in case of *constant inertia* of the Hermitian matrix $\mathrm{Re}\{\mathbf{Z}(\mathrm{j}\omega)\}$. In this case, Eq. (17) can be solved by $\mathbf{J}$-spectral factorization; for this reason, this extension of the classical theorem is also called a $\mathbf{J}$-Darlington theorem (28).

Beyond the celebrated low-sensitivity properties of lossless cascade realization for selective filters, the main value of Darlington-type models lies in the invariance properties of the lossless $2n$-port. Lossless transformations do not alter the characteristics of stationary power or energy flow into the embedded n-port and, consequently, leave its most fundamental physical properties invariant. Whereas Thévenin–Norton circuits are based exclusively on the linear properties of a device, Darlington models additionally reflect the quadratic constraints imposed by conservation of power or energy. This fact makes them natural canonical models *par excellence* for linear physical systems as well as for spectral models in a thermodynamic interpretation when only second-order properties are of interest.

Though computation of Darlington circuits may be cumbersome (especially for continuous models), they offer much information and place the fundamental characteristics directly in evidence (e.g., passivity of $\mathcal{N}$ is obvious when all resistors $R_i > 0$). Moreover, they provide for a clear partitioning of the model into pure resistors and dynamical elements, thus limiting frequency dependence and complex numbers to the lossless $2n$-port. In fact, Darlington representation may be viewed as real diagonalization of a complex impedance matrix $\mathbf{Z}$ by means of a linear fractional map $\mathscr{F} : \mathbf{Z} \to \mathbf{R}$.

When emphasis is on analysis or modeling of power flow in linear systems, it is conceptually more appealing to work with scattering parameters instead of voltages, currents, and impedances. The inherent normalization to external loads was limited originally to resistors. In order to extend applicability of scattering analysis to complex as well as to multiport loads, the theory of *complex normalization* of scattering matrices has been developed (32–34). Rather late it became evident that this formalism amounts to replacing the complex load multiports by their Darlington equivalents and to performing normalization with respect to the decoupled real resistors R_i (35).

EQUIVALENT AND PARTIALLY EQUIVALENT NETWORKS

Foundations

According to Belevitch (1) electrical networks are characterized by network elements and their connection element, where branch currents and voltages are used to describe a network mathematically.

The interaction with other networks and/or the observation of certain network variables requires the introduction of oneports (pairs of terminals) in the networks. From a systematic point of view, it is suitable to replace the oneports by norators. By definition, branch currents and voltages of these "singular" network elements [see Carlin (20)] are arbitrary, and therefore each nonlinear and linear resistive characteristic (including current and voltage sources as well as "open circuits" and "short circuits") can be represented. In this manner, twoports can be represented as network elements with "private" network variables if a mathematical representation of such a network is given by both the Kirchhoff equations and the constitutive relations of all network elements (including the norators). In this section, networks with a certain set of b norators are called n-port networks.

There are several levels at which one can compare two networks and their descriptions. The strongest condition seems to be that the network topology and structure of the Kirchhoff matrices, respectively, as well as the constitutive relations of the network elements be the same. Obviously these conditions lead to the same manifold of solutions of the descriptive equations. If dynamic networks are considered, then such a condition may be valid only for a certain time point or, in the case of linear time-invariant networks, only for a specified frequency.

Whereas this strong comparability is rather trivial and useless, it is easy to define weaker kinds. In 1929, based on ideas of Cauer (36), Ghenzi (9) defined that two networks are homological if parts of their network topologies and the solutions of the associated branch variables have the same behavior.

In network theory, many results on homological networks are known. The famous theorem of Helmholtz (12) published in 1853 (often called as theorem of Thévenin) was the first example of this kind. This was generalized by Mayer (37) and Norton [see Brittain (38) for some historical remarks] to networks with controlled sources in 1926. Another remarkable network theorem was derived by Kennelly (39), who proved the Y–Δ equivalence. In both cases, the homology of two networks was considered with respect to the network part that includes one and three norators, respectively. In the case of Helmholtz's theorem, the quotient of the norator current and voltage has to be the same, whereas the Y–Δ equivalence is based on the condition that the norator currents have to be equal. Cauer (36) was the first to define a general concept of equivalence where the fixed part of the network topology consists only of the norators that replace the oneports for interaction with network surroundings. He pointed out that the theory of equivalent networks is the central subject of network synthesis, where the main goal is to design different networks of equivalent behavior with respect to the ports. In the next subsections we will present some prominent results of network theory that can be understood in a unified manner using the framework of homological networks. For example, the duality of networks is studied in the literature apart from the subjects discussed above. But as Ghenzi emphasized, duality of networks is only a special case in the homology of networks.

Substitution Theorem

There are many interesting applications of the homology of networks in network theory. A very general theorem is the so-called substitution theorem that allows us "to replace any particular branch of a network by a suitable chosen independent source without changing any branch current or any branch voltage. In many instances, the substitute network is easier to solve than the original one." [See Desoer and Kuh (17).] We can find several versions of this idea in the monograph of Chua, Desoer, and Kuh (2) where nonlinear resistive as well as dynamic networks are considered. A very general substitution theorem was published by Haase and Reibiger (40), but we restrict ourselves in this subsection to describing special cases of this general theorem.

This statement can be demonstrated in a simple manner if we consider nonlinear resistive networks. For this case Chua, Desoer, and Kuh (2) presented the following theorem together with a proof:

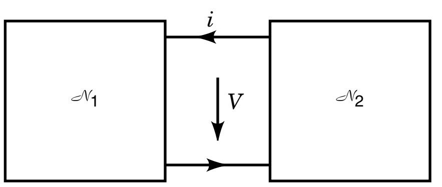

Figure 10. Decomposition of a resistive network into oneport networks $\mathcal{N}_1$ and $\mathcal{N}_2$. The electrical behavior at the common port is completely determined by $i(t)$ and $u(t)$.

Theorem. Let a resistive network $\mathcal{N}$ including time-variant sources be decomposed into two oneport subnetworks $\mathcal{N}_1$ and $\mathcal{N}_2$ that are connected at their ports (see Fig. 10); this port is characterized by a current i and a voltage u. If $\mathcal{N}$ has a unique solution $i(t)$ (for all t), then $\mathcal{N}_2$ may be substituted by a voltage source $u(t)$ without affecting the branch voltages and the branch currents inside $\mathcal{N}_1$, provided the substituted circuit $\mathcal{N}_S$ has a unique solution (for all t).

Remark. If a voltage source is used for replacing $\mathcal{N}_2$, an analogous theorem holds.

Example. Let $\mathcal{N}_1$ in Fig. 10 be a network consisting of a nonlinear resistor ($\mathcal{N}_1$) connected with an independent voltage source in series with a linear resistor ($\mathcal{N}_2$). If a unique intersection of the nonlinear characteristic $\mathcal{N}_1$ and the "load" characteristic $\mathcal{N}_2$ is considered, the intersection can be determined by means of a horizontal and a vertical characteristic (voltage or current source) if these lines do not intersect the nonlinear characteristic a second time. This case is illustrated in Fig. 11.

Remark. If $\mathcal{N}_1$ is a linear resistive network with sources, it can be characterized at its port by means of another line. With the exception of singular cases the intersection can be determined with a current or voltage source, too.

Miller Theorem. In many monographs about circuit design, the so-called Miller theorem and its applications are discussed [e.g., Millman and Grabel (41)]. Probably the first detailed presentation of Miller's result is included in the Radiation Laboratory Series [see Vol. 19 "Waveforms" written by Chance et al. (42)]. After some extensions of this theorem

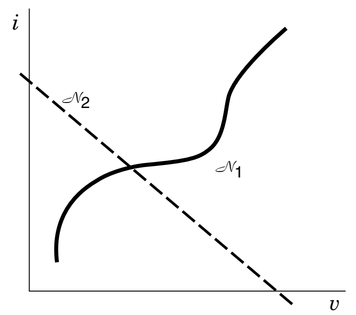

Figure 11. Uniqueness of intersection for a decomposition according to Fig. 10 when $\mathcal{N}_1$ is a nonlinear resistor and $\mathcal{N}_2$ is a linear affine oneport (an independent voltage source in series with a linear resistor).

were published, a more complete version was presented recently by Rathore (43). Based on the connections of twoports using parallel (P) or/and series (S) connections, Rathore showed that the different versions of Miller's theorem can be derived in a unique manner. The four corresponding connections of twoports are denoted by PP, PS, SP, SS if P or S is the type of connection of the input and output ports of both twoports. In each case, one of the twoports includes a controlled source that depends on the kind of connection, whereas the other twoport is arbitrary. By means of an equivalent network of the latter twoport that includes controlled voltage or current sources and the corresponding gain (PP or SS) or transfer impedance or admittance (SP or PS), we obtain the Miller equivalent network. Note that the first twoport can be fully characterized by its voltage or current gain. If we have PP connection, the first twoport consists only of a single impedance Z and the gain of the second twoport is $\mu = U_2/U_1$. This will be represented in the Miller equivalent network by means of input and output impedances with the following values:

$$Z_1 = Z\frac{1}{1-\mu} \quad \text{and} \quad Z_2 = Z\frac{\mu}{\mu-1} \tag{19}$$

This example (see Fig. 12) is the one presented by Miller to illustrate his theorem. Note that at least one of these impedances can be negative if $\mu \neq 1$. Therefore, the Miller theorem results in an equivalent network with a negative resistor and cannot be realized with real devices. On the other hand, it is very suitable in circuit design, for example, to study the frequency behavior of this network.

Wye–Delta and Star–Mesh Transformation. A theorem that considers two networks with equivalent three-pole-subnetworks of special topologies is well known as the star–delta transformation, since the network topology of one of the subnetworks looks like a star and the other subnetwork looks like a delta. This type of equivalence was given for the first time by Kennelly (39) in 1899 and studied intensively with respect to the theory of electrical transmission lines by Herzog and Feldmann (44) in 1903. Later on it became a standard subject in elementary textbooks of network theory [e.g., Guillemin (45), Desoer and Kuh (17)]. In Guillemin's book (see p. 131) we find a nice statement for the calculation of the delta conductances or resistances *"the product of the adjacent two, divided by the sum of all three."* For example, the wye resistors of delta–wye transformation are calculated by

$$\tilde{R}_i = \frac{R_j R_k}{R_1 + R_2 + R_3} \tag{20}$$

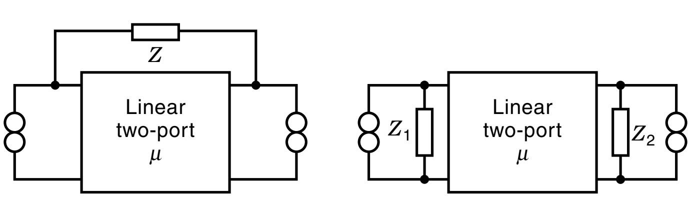

Figure 12. The decomposition of the impedance Z by the Miller theorem into Z_1 and Z_2.

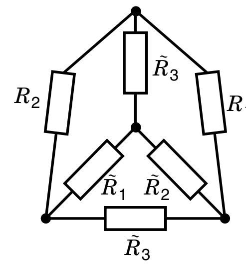

Figure 13. A simple replacement of a delta-type resistor network (without tilde) into a wye-type resistor network (with tilde).

where (i, j, k) is a permutation of (1, 2, 3), the R's are the delta resistors and the $\tilde{R}$'s are the wye resistors. The delta–wye transformation is illustrated in Fig. 13. Furthermore, Guillemin pointed out that with the wye–delta transformation, a node of the network is eliminated (that is a node potential) and with a delta–wye transformation an elementary mesh is eliminated (that is a mesh current). Repeated applications of this procedure lead to a simplified network. Obviously, this statement is closely related to the Gauss algorithm for solving linear algebraic equations (LU factorization).

It should be emphasized that not every linear resistive three-pole-network can be represented by a subnetwork with one of these topologies if it is assumed that all resistive values are positive. Therefore, this theorem is only applicable to such networks that include three-pole subnetworks with a wye or a delta topology. It should be mentioned that a generalization of the wye–delta transformation to networks with sources can be found in a paper of Herzog (46) [see also Chang and Chu (47)].

In their famous paper from 1964, Brayton and Moser (48) presented a derivation of this theorem in a geometrical framework using the so-called Legrendre transformation. They proved that it has no analog for nonlinear networks although some exceptions under certain restrictions are known (see also Chua (49). This result is related to a more recent result published by Boyd and Chua (50). These authors showed that even a simple cascade of two linear blocks (where a nonlinear block is embedded the input–output behavior) is not preserved by any commutation operation. That is to say that the classes of equivalent nonlinear networks essentially consist of one network only.

The more generalized star-mesh transformations for n-pole-subnetworks were published in 1924 with a star or mesh topology [see Küpfmüller (51) and Rosen (52)] but in these cases no bijective relationships between these two subnetworks exist if there are more than three poles [further literature and remarks are included in Bedrosian (53)]. However, it is pointed out in these papers that these results can be applied in certain areas of network analysis.

RECIPROCITY

Reciprocity is an essential property of electrical networks, although its physical idea has been applied previously to other physical systems. Rayleigh considered a reciprocity relation in his famous monograph *Theory of Sound* (54). In all cases, reciprocity is related to an interchange between cause and response of a system or network. In order to define as well as test this property of a single network, we associate a family of networks to it. If the network contains n norators (that cannot combine with nullators into a nullor) the networks are called n-ports, where a port is characterized by the voltage and the current of the corresponding norator. One of these networks can be constructed in the following manner (restricted at first to the classical case of linear time-invariant n-ports):

- All norators, with exception of two, are replaced by short circuits.
- Network 1: Both of the remaining two norators I and II are replaced by an independent voltage source where $v_I^1 \neq 0$ and $v_{II}^1 = 0$.
- Network 2: Both of the remaining two norators are replaced by an independent voltage source where $v_{II}^2 = 0$ and $v_{II}^2 \neq 0$.

This n-port is "reciprocal with respect to these two ports" (norators) if the following conclusion is satisfied: $v_I^1 = v_{II}^2 \Rightarrow i_{II}^1 = i_I^2$ where the currents are the corresponding port (norator) currents. This n-port is "reciprocal" if it is reciprocal with respect to all pairs of ports (norators). Obviously we have an operational definition that can be used to construct a measurement process in order to test this property. The idea is illustrated with a two-port network.

Unfortunately, it is not possible to use voltage sources for the test of reciprocity in all cases of linear time-invariant n-ports. But a more general approach uses independent current sources and open circuits, or a mixture of current and voltage sources [for further details see Balabanian, Bickart, Seshu (55), Chap. 9].

Although the basic definition of reciprocity uses a family of networks, it is more suitable to formulate criteria that are related to properties of a certain network. For this purpose, the Weyl–Tellegen theorem can be used since it is formulated for two different networks with identical connection elements. For simplicity, we will consider only the case of a pair of ports (norators). If currents and voltages of these ports are denoted by index 1 and 2, respectively, we have by the properties of a Kirchhoff network

$$(\tilde{v}_1 i_1 - v_1 \tilde{i}_1) + (\tilde{v}_2 i_2 - v_2 \tilde{i}_2) + \sum_{k=3}^{b} (\tilde{v}_k i_k - v_k \tilde{i}_k) = 0 \qquad (21)$$

where b is the number of branches of the network (including the two ports or norators). Using the special properties of the two associated networks and reciprocity with respect to these ports, the following condition for the twoport network arises

$$\sum_{k=3}^{b} (\tilde{v}_k i_k - v_k \tilde{i}_k) \qquad (22)$$

where the sum encompasses all nonport branches of the network. In contrast to the external definition, this relation is based on internal quantities only. Therefore an internal characterization of reciprocity is given. In the case of linear time-invariant n-ports, a similar relation can be derived from the Weyl–Tellegen theorem

$$-\sum_{\text{ports}} (-\tilde{v}_k i_k + v_k \tilde{i}_k) = \sum_{\text{internal branches}} (\tilde{v}_k i_k - v_k \tilde{i}_k) \qquad (23)$$

where the reciprocity of the n-port network results in the vanishing of the left side (external condition) or the right side (internal condition). In the case of linear reciprocal n-ports, reciprocity can be characterized by means of certain invariant properties of the n-port matrices. For example, the impedance and the admittance matrices have to be symmetric. For further details, see Balabanian, Bickart, and Seshu (55). These invariant properties are formulated in a different manner if different types of excitations (different n-port matrices) are considered. Fortunately, a geometric characterization is available to unify these different formulations. In the case of linear time-invariant n-ports, the external behavior is characterized by a totally isotropic linear space. A more general characterization of n-ports, including the nonlinear n-ports based on paper of Brayton and Moser (48), was generalized by Brayton (56) and Chua, Matsumoto, and Ichiraku (57); see also the monograph of Mathis (4). The main idea behind this approach is that the reciprocity of an n-port is characterized by a 2-form (in the sense of Cartan)

$$\sum_{\text{ports}} di_k \wedge dv_k \qquad (24)$$

that vanishes on the set of all admissible currents and voltages. If this 2-form is represented in a suitable coordinate system, one of the classical characterizations of reciprocity can be derived. In this sense, we speak of a geometrical formulation of the internal representation of reciprocity.

It is well known that linear reciprocal n-ports can be analyzed in a simplified manner using the symmetry of the n-port matrices. In this case, only half of the nondiagonal coefficients have to be calculated. An elegant formulation of the dynamic state space equations of linear or nonlinear *RLC* networks can be derived if the network is reciprocal and complete (see, e.g., Weiss and Mathis (58) for recent results). Based on these conditions, Brayton and Moser (48) proved that a scalar function $P(\mathbf{v}_C, \mathbf{i}_L)$ exists (where ($\mathbf{v}_C$ and $\mathbf{i}_L$ are the voltages of the capacitors and currents of the inductors, respectively) that can be used to formulate the dynamic equations

$$\mathbf{C}(\mathbf{v}_C)\,\frac{d\mathbf{v}_C}{dt} = \frac{\partial P}{\partial \mathbf{v}_C} \qquad (25)$$

$$\mathbf{L}(\mathbf{i}_L)\,\frac{d\mathbf{i}_L}{dt} = -\frac{\partial P}{\partial \mathbf{i}_L} \qquad (26)$$

For the proof of the existence of P, Brayton and Moser used the 2-form as an integrability condition. These authors applied their *mixed potential function* P to derive stability conditions for this class of nonlinear networks without solving the dynamic equations. Furthermore, it should be mentioned that reciprocity is a sensitive assumption for a thermodynamic interpretation of electrical networks and other physical systems [see, e.g., Weiss and Mathis (58) and Stratonovich (59)].

INTERRECIPROCITY

Unfortunately, many interesting networks are not included in the class of reciprocal networks. Therefore Bordewijk (60) introduced a new property in 1956 that extends the reciprocity in some sense. In order to define the reciprocity of a network including norators (ports), a family of networks was generated where the norators are replaced by different excitations (independent sources) as well as open and short circuits. It is assumed that the connection element and the other network elements are not changed.

Bordewijk assumed the external condition of reciprocity

$$\sum_{\text{ports}} (-\tilde{v}_k i_k + v_k \tilde{i}_k) = 0 \tag{27}$$

for two n-port networks with the same connection element but possibly different network elements. Therefore, we say that two n-port networks with the same connection element are interreciprocal if this condition for all admissible port currents and voltages is satisfied.

Using the internal condition of interreciprocity, it is easily shown that the admittance and impedance matrices of linear and interreciprocal n-port networks $\mathcal{N}$ and $\tilde{\mathcal{N}}$ that are related by

$$\tilde{\mathbf{Y}} = \mathbf{Y}^T, \quad \tilde{\mathbf{Z}} = \mathbf{Z}^T \tag{28}$$

It was already known to Bordewijk that, in general, there is more than one way of associating one network with another in such a manner that the two networks are interreciprocal, even in the linear case (61). Therefore, he introduced the transposition operation to linear n-port networks, where the separate network elements are replaced by network elements that are interreciprocal with regards to the aforesaid network elements. A complete table of network elements and their unique interreciprocal network elements can be found in [Balabanian, Bickart, Seshu (55), p. 376].

Bordewijk proved that two n-port networks that arise from one another by transposition are interreciprocal. Note that it is not generally true that two interreciprocal n-port networks arise one from the other by transposition. Obviously, transposition is a self-inverse operation. Therefore this approach is closely related to duality of networks (see the next section).

It is easy to conclude that an n-port network composed of reciprocal network elements is invariant for transposition. Therefore, reciprocal n-port networks form an invariant set in the set of all linear n-port networks. With respect to this prominent property, a pair of n-port networks that are generated by transposition are said to be adjoint of each other.

The paper of Bordewijk contained many applications of interreciprocity. In particular, he studied the analysis of amplifiers with pairwise interreciprocal network models. Furthermore, he considered an extended noise theory of linear and interreciprocal networks. Another prominent application of adjoint networks is the sensitivity analysis of networks with respect to network parameters. Although a straightforward method is available to calculate small absolute or relative variations of currents and voltages with respect to variations of certain network parameters, the analysis results of the adjoint network are of some advantage if more than one sensitivity has to be calculated. In contrast to the method of first-order approximation, where each parameter has to be considered separately, only one analysis of the adjoint network is necessary. For further details, see Hasler and Neirynck [(3), pp. 236ff]. A more detailed comparison is contained in Vallese (62). The relationship between the concepts of adjoint networks and transposed networks was discussed by Bordewijk (61).

DUALITY

A very interesting relationship between two different networks and their solution manifolds, respectively, is discussed by means of the so-called duality theory of electrical networks. Introduced by Russel (63), the first results were published by Matthies and Strecker (64). Since these results were based on considerations of the corresponding network graph, this duality concept was restricted to networks with a planar network graph. Cauer (36) reformulated these early results in the framework of mathematical graph theory and presented conditions that an arbitrary graph is planar. More detailed information can be found, for instance, in Weinberg (65). An illustration of this approach to dual networks is shown in the following example.

Example. Obviously the *RLC* network in Fig. 14 has a planar network graph. Therefore it is possible to map this network on the surface of a ball where it decomposes the surface of the ball into three areas. Now we associate a node to each of these three areas and connect each of these nodes by a branch that crosses a branch of the original network. As a result, the skeleton of another graph is constructed that can be interpreted as a network graph. This is called the dual network graph. As the next problem, we have to determine the kind of network element in each branch. A table of network elements and its corresponding dual elements can be found in Hasler and Neirynck [(3), p. 218]. In the most simple cases, an Ohmian resistor has to be replaced by an admittance $\tilde{G} = R/R_0^2$, where R_0 is the duality constant. Using these correspondences, the dual network can be constructed (see Fig. 15).

In order to clarify this duality operation and to extend it to more general networks (where planarity of the network

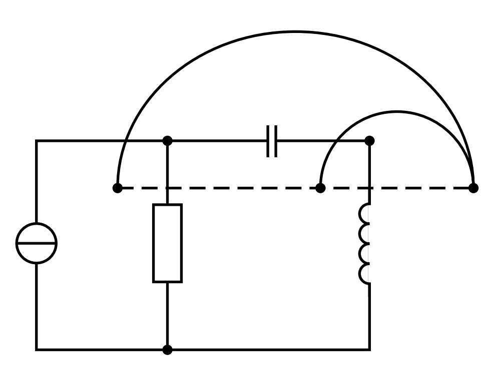

Figure 14. Illustrating the construction of the dual graph of a simple *RLC* network.

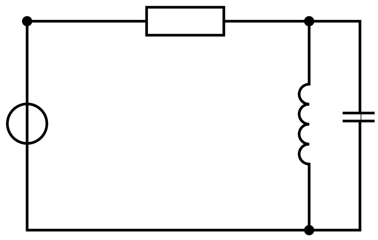

Figure 15. The corresponding dual *RLC* network (see Fig. 14).

graph is not needed), Ghenzi's axiomatic representation of network theory was applied by Mathis and Marten (66). Ghenzi reformulated the description of the connection element by means of an exact pair of incidence matrices (**A**, **B**) mentioned above. The dual network is then defined by interchanging these two matrices ($\mathbf{B}^T$, $\mathbf{A}^T$) and by replacing each network element by its dual (where the above mentioned table is used). In those cases where the connection element with (**A**, **B**) can be represented by a planar graph, a planar graph can be constructed for the dual connection element with ($\mathbf{B}^T$, $\mathbf{A}^T$). In this manner, the classical duality theory is reformulated. Unfortunately, this statement is invalid if (**A**, **B**) represents a nonplanar graph. Using a theorem of Belevitch (1) Mathis and Marten (66) showed that for each pair of real and exact matrices an ideal transformer *b*-port can be constructed. Therefore, the dual network can also be represented by the usual network elements ideal transformers in each case. The relationships of duality in the sets of resistive networks can be illustrated by means of Fig. 16. Essentially, duality is in a property of the connection element of a network. Therefore, the consideration of resistive networks with ideal transformers (Rü networks) and without ideal transformers is no substantial restriction. It is known that a dual planar network can be constructed for each planar resistive network. In the case of nonplanar networks, ideal transformers are needed for the construction of the dual network. An example can be found in Ref. 66. It should be mentioned that an alternative representation of this duality theory was described by Reibiger using so-called bondgraphs (67).

Since the solution manifolds of dual networks are closely related by transformations of currents and voltages, the solutions of one network can be used to represent the solutions of the other. As a simple illustration of this approach, we mention the celebrated duality relation between the parallel and series *RLC* circuits.

BARTLETT THEOREM AND OTHER SYMMETRIES

In many areas of physics, geometric symmetries of a system can be used to simplify the analysis. In linear network theory, symmetries of this kind were also applied in a successful manner. Probably the first example was the application of symmetries to polyphase networks; Fortescue (68) published some results in 1918. In 1931, Bartlett (69) published his now well-known result on reflection symmetric networks. This was very useful in filter theory and has been generalized to electronic circuits. In order to illustrate Bartlett's theorem, consider a network with two independent voltage sources that can be decomposed with respect to a mirror plane into two parts containing the same network elements. This is shown in Fig. 17 using identical *n*-ports $\mathcal{N}$. Obviously, each block contains a voltage source that was extracted. Both parts of the network are connected by a number of wires. If the network is excited with symmetric voltages $v_1 = v_2$, the currents in the connections are zero and open circuits occur. Therefore, both parts can be analyzed independently. On the other hand, if the voltages are antisymmetric $v_1 = -v_2$, the voltages between the connections are zero and there are short circuits at both ports. Again, these parts can be analyzed in a separate manner. By means of this approach, difference amplifiers can be analyzed.

Unfortunately, the occurrence of three-dimensional geometric symmetries in a linear network is rather an exception. Furthermore, the results of Fortescue and Bartlett are proven by the superposition theorem, and for this reason, a direct generalization to nonlinear networks is impossible. However, some interesting symmetry results for nonlinear networks were published. The papers of Chua and Vandewalle (70) and Vandewalle and Chua (71) contain many references and results where the framework of permutation groups is applied to generalize many ad hoc techniques that are well known in circuit design for special circuits. They show that for a certain choice of the reference nodes a symmetric network has a symmetric solution, provided the network has a unique solution. Furthermore, the authors present a reduction technique for nonlinear symmetric networks that generalizes Bartlett's method for linear networks and unifies various algebraic and graphical reduction methods. Interesting applications of these results are the analysis of networks with complementary

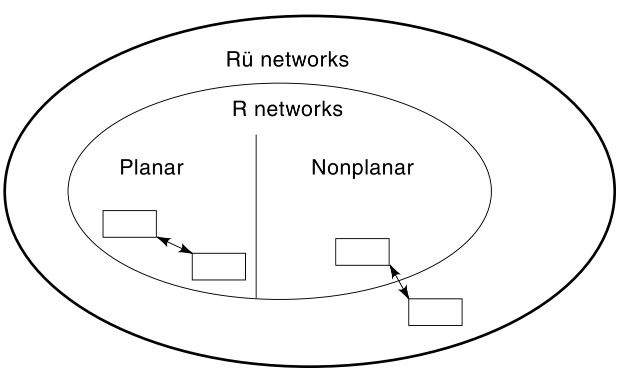

Figure 16. Decomposition of the set of linear resistor networks with ideal transformers with respect to duality classes.

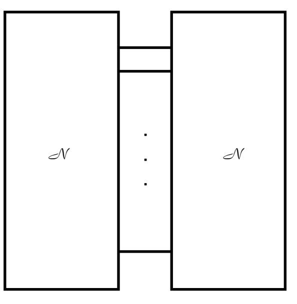

Figure 17. A decomposition of an arbitrary network into symmetric blocks.

symmetric network elements (e.g., *npn* and *pnp* bipolar transistors). An example is the push-pull transistor amplifier.

A fundamental symmetry of network equations as well as other physical descriptive equations is the balance of physical dimensions in addition to the numerical values of an equation. It is known that in the dimensional theory multiparameter Lie groups can be helpful. An overview of dimensional theory with its applications is given by Mathis (72). The method of normalized linear networks that is useful in filter design is a simple example of dimensional theory. In general, normalized equations include a number of dimensionless constants that can be determined by dimensional theory in a systematic manner. Mathis showed that different normalized representations of nonlinear network equations are very useful if the singular perturbation theory is applied (72). A certain representation determines the initial solution for the perturbation series.

BIBLIOGRAPHY

1. V. Belevitch, *Classical Network Theory,* San Francisco: Holden-Day, 1968.
2. L. O. Chua, C. A. Desoer, and E. S. Kuh, *Linear and Nonlinear Circuits,* Singapore: McGraw-Hill, 1987.
3. M. Hasler and J. Neirynck, *Nonlinear Networks,* Norwood, CT: Artech House, 1986.
4. W. Mathis, *Nichtlineare Netzwerke,* Berlin: Springer-Verlag, 1987.
5. J. Vlach and K. Singhal, *Computer Methods for Circuit Analysis and Design,* New York: Van Nostrand Reinhold, 1983.
6. D. K. Arrowsmith and C. M. Place, *An Introduction to Dynamical Systems,* Cambridge: Cambridge University Press, 1991.
7. P. Slepian, *Mathematical foundations of network analysis,* Berlin: Springer-Verlag, 1968.
8. A. Reibiger, On the terminal behavior of networks. *Proc. ECCTD'85,* Prag, 1985, pp. 224–228. See also an extended version: Über das Klemmenverhalten von Netzwerken. *Wiss. Zeitschr. TU Dresden* **35**: 165–173, 1986.
9. A. G. Ghenzi, Studien über die algebraische Theorie der elektrischen Netzwerke, Dissertation, ETH Zürich, 1953.
10. H. Weyl, Repartición de corriente en una red conductora. (Distribution of current in a conduction network) *Rev. Mat. Hispano-Americana,* **5**: 153–164, 1923. (Trans. J. Friedman, George Washington University Logistics Research Project, 1951.)
11. B. D. H. Tellegen, A general network theorem, with applications. *Philips Res. Rep.,* **7**: 259–269, 1952.
12. H. Helmholtz, Ueber einige Gesetze der Vertheilung elektrischer Ströme in körperlichen Leitern mit Anwendung auf die thierisch-elektrischen Versuche. *Pogg. Ann.,* **89**: 211–233, 353–377, 1853.
13. J. C. Maxwell, *A Treatise on Electricity and Magnetism,* 2 vols., New York: Dover, 1954.
14. H. Hausrath, Die Untersuchung elektrischer Systeme auf Grundlage der Superpositionsprinzipien, Berlin: Verlagsbuchhandlung Julius Springer, 1907.
15. G. O. Martens and H. H. Lê, On the superposition of power in linear time-invariant networks, *Proc. IEEE,* **59**: 1721–1722, 1971.
16. E. A. Guillemin, *Theory of Linear Physical Systems,* New York: Wiley, 1963.
17. C. A. Desoer and E. S. Kuh, *Basic Circuit Theory,* New York: McGraw-Hill, 1969.
18. W. Cauer, Synthesis of Linear Communication Networks, vols. 1, 2. New York: McGraw-Hill, 1958.
19. P. Penfield, R. Spence, and S. Duinker, *Tellegen's Theorem and Electrical Networks,* Cambridge: MIT Press, 1970.
20. H. J. Carlin, Singular network elements. *IEEE Trans. Circuit Theory,* **11**: 67–72, 1964.
21. H. A. Haus and R. B. Adler, *Circuit Theory of Linear Noisy Networks,* New York: Wiley, 1959.
22. A. Reibiger, Geometrical proof of the Thévenin–Norton theorem, *Proc. ECCTD'97,* Budapest, 1997, pp. 24–28.
23. S. W. Director and D. A. Wayne, Computational efficiency in the determination of Thévenin and Norton equivalents, *IEEE Trans. Circuit Theory,* **19**: 96–98, 1972.
24. Ph. Delsarte, Y. Genin, and Y. Kamp, On the role of the Nevanlinna-Pick interpolation problem in circuit and system theory, *Int. J. Circuit Theory Appl.,* **9**: 89–96, 1981.
25. P. Dewilde et al., The role of losslessness in theory and applications. *Int. J. Electron. Commun., (AEÜ)* special issue, **5/6**: 241–382, 1995.
26. R. W. Newcomb, *Linear Multiport Synthesis,* New York: McGraw-Hill, 1966.
27. P. Dewilde, Cascade scattering matrix synthesis, Ph.D. thesis, Stanford Univ. Tech. Rept. No. 6560–21, June 1970.
28. J. Ball and J. W. Helton, Lie groups over the field of rational functions, signed spectral factorization, signed interpolation, and amplifier design, *J. Operator Theory,* **8**: 19–64, 1982.
29. H. A. Haus and R. B. Adler, Canonical form of linear noisy networks, *IRE Trans. Circuit Theory,* **5**: 161–167, 1958.
30. P. Dewilde, A. Vieira, and T. Kailath, On a generalized Szegö-Levinson realization algorithm for optimal linear predictors based on a network synthesis approach, *IEEE Trans. Circuits Syst.,* **25**: 663–675, 1978.
31. A. M. Nikolaichuk and I. M. Spitkovskii, Factorization of Hermitian matrix-functions and their application to boundary value problems, *Ukrainian Math. J.,* **27**: 629–639, 1975.
32. D. C. Youla, On scattering matrices normalized to complex port numbers, *Proc. IRE,* **49**: 1221, 1961.
33. D. C. Youla, An extension to the concept of scattering matrix, *IEEE Trans. Circuit Theory,* **11**: 310–311, 1964.
34. R. A. Rohrer, The scattering matrix: Normalized to complex *n*-port load networks, *IEEE Trans. Circuit Theory,* **12**: 223–230, 1965.
35. R. Pauli, Darlington's theorem and complex normalization, *Int. J. Circuit Theory Appl.,* **17**: 429–446, 1989.
36. W. Cauer, Topologische dualitätssätze und reziprozitätstheoreme der schaltungstheorie, *Zeitschr. Angew. Math. Mech.,* **14**: 349–350, 1934.
37. H. F. Mayer, Über das ersatzschema der verstärkerröhre, *Telegr.u.Fernsprechtech.,* **15**: 325–327, 1926.
38. J. E. Brittain, Thévenin theorem, *IEEE Spectrum,* **27** (3): 42, March 1990.
39. A. E. Kennelly, The equivalence of triangles and three-pointed stars conducting networks, *Electrical World,* **34**: 413, 1899.
40. J. Haase and A. Reibiger, Verallgemeinerung und Anwendung des Substitutions-theorems der Netzwerktheorie, *Wiss. Zeitschr. TU Dresden,* **34**: 125–129, 1985.
41. J. Millman and A. Grabel, *Microelectronics,* New York: McGraw-Hill, 1987.
42. B. Chance et al., *Waveforms,* New York: McGraw-Hill, 1949.

43. T. S. Rathore, Generalized Miller theorem and its applications, *IEEE Trans. Educ.,* **32**: 386–390, 1989 with corrections *IEEE Trans. Educ.,* **33**: 224, 1990.
44. Herzog and Feldmann, Die Berechnung elektrischer Leitungsnetze in Theorie und Praxis. Berlin: 1903.
45. E. A. Guillemin, *Introductory Circuit Theory,* New York: Wiley, 1953.
46. W. Herzog, Stern-Dreieck-Umwandlung mit Quellen, *Bull. SEV,* **61**: 1097–1106, 1970.
47. S. Chang and Y.-I. Chu, Active Δ-to-Y transformation, *Proc. IEEE,* **59**: 326–328, 1971.
48. R. K. Brayton and J. K. Moser, A theory of nonlinear networks I and II, *Quart. Appl. Math.,* **12**: 1–33, 81–104, 1964.
49. L. O. Chua, Δ-Y and Y-Δ transformation for nonlinear networks, *Proc. IEEE,* **59**: 417–419, 1971.
50. S. Boyd and L. O. Chua, Uniqueness of a basic nonlinear structure, *IEEE Trans.,* **CAS-30**: 648–651, 1983.
51. K. Küpfmüller, Über einen Umwandlungssatz zur Theorie der linearen Netze, *Wiss. Veröffentl. Siemens-Konzern,* **3**: 130–134, 1923–1924.
52. A. Rosen, A new network theorem, *J. IEE,* **62**: 916–918, 1924.
53. S. D. Bedrosian, Converse of the Star-Mesh Transformation, *IRE Trans. Circuit Theory,* 491–493, 1961.
54. J. W. S. Lord Rayleigh, *Theory of Sound,* vol. 1, London: MacMillan, 1924.
55. N. Balabanian, T. A. Bickert, and S. Seshu, *Electrical Network Theory,* New York: Wiley, 1969.
56. R. K. Brayton, Nonlinear reciprocal networks, in *Mathematical Aspects of Electrical Network Analysis,* Providence, RI: Amer. Math. Soc., 1971.
57. L. O. Chua, T. Matsumoto, and S. Ichiraku, Geometric properties of resistive nonlinear n-ports: Transversality, structural stability, reciprocity, and antireciprocity, *IEEE Trans. Circuits Syst.,* **27**: 577–603, 1980.
58. L. Weiss and W. Mathis, A thermodynamical approach to noise in nonlinear networks, *Int. J. Circuit Theory Appl.,* **26** (2): 147–165, 1998.
59. R. L. Stratonovich, *Linear and nonlinear fluctuation dissipation theorems,* Berlin: Springer-Verlag, 1992.
60. J. L. Bordewijk, Inter-reciprocity applied to electrical networks, *Appl. Sci. Res.,* **B, 6**: 1–74, 1956/57.
61. J. L. Bordewijk, Comments on "Automated network design and the interreciprocity concept," *IEEE Trans. Circuit Theory,* **18**: 179, 1971.
62. L. M. Vallese, Incremental versus adjoint network models for network sensitivity analysis, *IEEE Trans. Circuits Syst.,* **21**: 46–49, 1974.
63. A. Russel, *Alternating Currents,* Chap. 27, Cambridge: Cambridge University Press, 1904.
64. K. Matthies and F. Strecker, Über reziprozität bei wechselstromkreisen, *Arch. f. Elektrotechn.,* **14**: 1–15, 1924.
65. L. Weinberg, *Network Analysis and Synthesis,* New York: McGraw-Hill, 1962.
66. W. Mathis and W. Marten, On the structure of networks and duality theory. *Proc. 31st Midwest Symp. Circuits Syst.,* St. Louis, USA, 1988.
67. A. Reibiger, Über die Darstellung von Minty-Netzwerken durch Bondgraphen. TU Dresden, TU Informationen 09-02-90, 1990.
68. J. C. L. Fortescue, Method of symmetrical coordinates applied to the solution of polyphase networks, *Trans. AIEE,* **37**: 1027–1040, 1918.
69. A. C. Barlett, *The Theory of Electrical Artificial Lines and Filters,* New York: Wiley, 1931.
70. L. O. Chua and J. Vandewalle, A unified theory of symmetry for nonlinear multiports and multiterminal resistors, *Int. J. Circuit Theory Appl.,* **7**: 337–371, 1979.
71. J. Vandewalle and L. O. Chua, A unified theory of symmetry for nonlinear resistive networks, *J. Franklin Inst.,* **308**: 533–577, 1979.
72. W. Mathis, Equivalence and transformation, in W. K. Chen (ed.), *The Circuits and Filters Handbook,* Boca Raton, FL: CRC Press and IEEE Press, 1995.

Wolfgang Mathis
University of Magdeburg
Rainer Pauli
Technical University of Munich