SENSITIVITY ANALYSIS

IMPORTANCE OF THE SENSITIVITY CONCEPT

In physics and engineering, we usually mean by *sensitivity* of a system a measure for determining the amount of change an outcome undergoes if the relevant *parameters* fixing the outcome are somewhat modified (1–5). In some situations, for example in a measurement or in a decision-making situation, such an outcome may be an individual result. In others, an outcome may be one or several functions of some further independent variable(s), especially a time or frequency response. Obviously, such a general definition of sensitivity is somewhat vague and, in order to make it quantitatively useful, needs more precise specification.

If we are dealing with only small modifications, appropriate sensitivity measures can usually be based on differential expressions, that is, on *first-order derivatives* of the desired outcome with respect to the component parameters. In other situations, the encountered changes may be so large that a first-order approach may be insufficient. One then must either consider also higher-order derivatives or determine the resulting modified behavior by direct precise computation. The latter approach can be simplified drastically if the functions under consideration have an appropriate simple structure, as is the case for system functions of linear circuits and systems. However, the number of instances in which influences of order higher than the first have to be taken into account is probably much smaller than what is sometimes believed. Indeed, if the expected parameter changes are such that higher-order effects must be taken into account, one is frequently dealing with what may be termed poor engineering design.

Interest in sensitivity aspects may arise in quite different contexts. For *analog circuits,* the most immediate concern results from the errors due to *manufacturing inaccuracies, temperature changes,* and *aging.* The resulting requirements become particularly severe for applications for which stringent criteria have to be met. One such type of application are filters since these may have to present dramatically different behavior in passbands and stopbands. Another type are high-performance amplifiers. Some attention to such circuits will therefore be given later. However, it may be mentioned now already that in order to alleviate the problem, it has been found, in both those cases, to be very advantageous to have recourse to *passivity* and its limiting form *losslessness.* It seems indeed that despite the enormous importance of active devices there exists a universal law of sound engineering practice: Any apparatus having to satisfy very stringent requirements should preferably be built either in purely passive fashion or, especially if that choice is inherently excluded (amplifiers, combustion engines, power stations), in such a way that the most critical performance aspects are determined by passive devices or subsystems.

This does not imply at all, however, that passivity and losslessness are the only criteria one should aim for in the case of critical applications. As an example, the stopband sensitivity of filters in bridge-type configuration increases dramatically with increasing stopband requirements. Hence, even very slight manufacturing inaccuracies, temperature changes, and aging may inadmissibly perturb the behavior of such filters, also in the case of passive (lossless) implementations.

The values of, for example, *parasitic elements* can also be considered as relevant parameters in the sense used at the beginning of this section. Hence, sensitivity analysis encompasses the analysis of the influence that parasitic elements have upon the behavior of a circuit.

In *digital circuits,* the arguments listed so far for emphasizing the importance of sensitivity lose their meaning. Indeed, under the usually permitted assumption of fully reliable digital operation, manufacturing inaccuracies, temperature changes, and aging have no effect. There is, however, a different aspect due to which sensitivity is of direct relevance: the limited number of bits available in the registers for storing the parameters that fix the circuit behavior. Hence, low sensitivity is also of interest for digital systems. Nevertheless, bridge-type configurations are now admissible even for critical filtering purposes. There is no limit to the achievable stopband attenuation provided the filter coefficients have been determined with sufficient accuracy and the relevant registers are long enough.

One of the reasons for the present dominance of digital circuits has itself also partly to do with sensitivity: The ever smaller features of highly integrated circuits cannot be controlled with such precision that accurate analog operation could be ensured. In digital circuits, however, the details of the analog operations, that is, the transitions from one state to another, are irrelevant as long as these transitions occur sufficiently fast.

Sensitivity may also be of relevance in a more *indirect* fashion. Indeed, various types of imperfections such as noise and nonlinear distortion can frequently be interpreted as being caused by parameter fluctuations. If this is the case, a reduction of sensitivity with respect to such a parameter change will also imply a corresponding reduction of the disturbance caused by the imperfection. In this sense, there exists a highly beneficial relationship between sensitivity on the one hand and noise and nonlinear distortion on the other. This holds true for analog as well as for digital circuits (6–8).

At this point, one question immediately comes to mind: While properties such as passivity and losslessness have no natural meaning for digital circuits, is it nevertheless possible to carry them over to the digital domain and thus to profit also there from the sensitivity benefits potentially available from such properties? The answer to this is affirmative, and corresponding structures are known as wave digital filters (8). [These filters, due to their passivity, offer, however, a much broader resistance against disturbing imperfections, a property also referred to as robustness (9), than what we are discussing in the present context.]

So far, we have taken it for granted that a low sensitivity is desired, but the opposite situation also has some interest. This is the case in particular in *measuring* and *sensing* equipment, where one does indeed want to obtain large deviations of the output for small changes of the device to be measured or of the phenomenon to be detected. Hence, circuits such as bridges, which are to be avoided for critical analog filtering, are precisely preferred arrangements, for example, for measuring purposes.

Finally, sensitivity expressions are also of great interest in *numerical work* (10). In particular, optimization problems can usually be solved only by iterative procedures involving individual linear steps, which in turn rely in many cases on the use of sensitivity-type expressions. These should again be reasonably large in order to accelerate convergence. Related to optimization is the problem of computer-based *circuit adjustment.* Corresponding strategies also rely on knowledge of the relevant sensitivities.

The present text is not an attempt to give a comprehensive summary of the subject but rather a brief exposition of those sensitivity aspects that, according to the experience of the author, have proved to be of particular importance in the area of circuit design and operation (with prime emphasis on linear circuits). For other aspects and many further details, the reader may wish to consult specialized books such as Refs. (1–5), or relevant chapters in books with wider scope (11–14).

SENSITIVITY DEFINITIONS

Let F be a real or complex quantity of interest. It is a function of a certain number of parameters, say γ_1 to γ_n; these are frequently real, for example, if they represent component values, but they can also be complex. In general, F also depends on one or several further physical quantities, say on time t or on the frequency ω or the complex frequency $s = \sigma + j\omega$, but such dependencies will be made explicit only if strictly required. Hence, we can write $F = F(\gamma)$, $\gamma = (\gamma_1, \ldots, \gamma_n)^{\mathrm{T}}$. If one of the γ_ν is changed by a small amount $\Delta\gamma_\nu$, F is changed in first approximation by

$$\Delta F = \frac{\partial F}{\partial \gamma_\nu} \Delta \gamma_\nu$$

An appropriate measure for sensitivity, more precisely for what is called *absolute sensitivity* of F with respect to γ_μ, is therefore

$$S(F:\gamma_\nu) = \frac{\partial F}{\partial \gamma_\nu} \tag{1}$$

Frequently, one is more interested in relative changes of F with respect to relative changes of γ_ν, that is, in the *relative sensitivity*

$$S(\ln F:\ln \gamma_\nu) = \frac{\partial \ln F}{\partial \ln \gamma_\nu} = \frac{\gamma_\nu}{F} \frac{\partial F}{\partial \gamma_\nu} \tag{2}$$

sometimes also in one of the *semirelative sensitivities*

$$S(\ln F:\gamma_\nu) = \frac{\partial \ln F}{\partial \gamma_\nu} = \frac{1}{F} \frac{\partial F}{\partial \gamma_\nu}, S(F:\ln \gamma_\nu) = \frac{\partial F}{\partial \ln \gamma_\nu} = \gamma_\nu \frac{\partial F}{\partial \gamma_\nu} \tag{3}$$

which all are closely related to the original expression. The relative sensitivity [Eq. (2)] is frequently the preferred quantity. However, it loses its meaning if $F = 0$ and/or $\gamma_\nu = 0$. Corresponding remarks hold for the semirelative sensitivities.

One can also represent in a compact fashion the complete *sensitivity vectors.* Thus

$$\begin{aligned} S(F;\gamma) &= DF,\, S(\ln F;\gamma) = \frac{1}{F} DF \\ S(\ln F;\ln\gamma) &= \frac{\partial \ln F}{\partial \ln \gamma} = \frac{1}{F}(\gamma_1 \frac{\partial F}{\partial \gamma_1}, \ldots, \gamma_n \frac{\partial F}{\partial \gamma_n})^T \end{aligned} \tag{4}$$

where

$$\begin{aligned} &\boldsymbol{D} = (D_1, \ldots, D_n)^{\mathrm{T}}, D_\nu = \frac{\partial}{\partial \gamma_\nu}, \\ &\nu = 1, \ldots, \mathrm{n}, \ln \boldsymbol{\gamma} = (\ln \gamma_1, \ldots, \ln \gamma_n)^{\mathrm{T}} \end{aligned} \tag{5}$$

If instead of a single F one is interested in a vector $\boldsymbol{F} = (F_1, \ldots, F_m)^{\mathrm{T}}$, one has, for example,

$$S(\boldsymbol{F}:\boldsymbol{\gamma}) = \boldsymbol{D}\boldsymbol{F}^{\mathrm{T}} = \left(\frac{\partial \boldsymbol{F}}{\partial \boldsymbol{\gamma}}\right)^{\mathrm{T}} \tag{6}$$

where $\partial\boldsymbol{F}/\partial\gamma$ is the Jacobian matrix of $\boldsymbol{F}$ with respect to γ.

If F is a system function (impedance, admittance, transmittance, reflectance, etc.) of a linear constant (i.e., time independent) circuit and if γ_ν refers to a one-port constituent (resistance, inductance, capacitance, impedance, admittance, etc.), F is a bilinear function of γ_ν of the form (see below)

$$F = \frac{F_{11} + \gamma_\nu F_{12}}{F_{21} + \gamma_\nu F_{22}} \tag{7}$$

where F_{11}, F_{12}, F_{21}, and F_{22} are independent of γ_ν. The same holds true if γ_ν is the multiplicative parameter characterizing a controlled source. However, if γ_ν is a mutual inductance, a turns ratio of an ideal transformer, or the gyration constant of a gyrator, we have

$$F = \frac{F_{11} + \gamma_\nu F_{12} + \gamma_\nu^2 F_{13}}{F_{21} + \gamma_\nu F_{22} + \gamma_\nu^2 F_{23}} \tag{8}$$

where $F_{11}, F_{12}, F_{13}, F_{21}, F_{22}$, and F_{23} are independent of γ_ν. The case of Eq. (7) is the most frequent, and the simplicity of such an expression can be of considerable help not only if first-order sensitivities as given by Eqs. (1) to (6) are of interest, but in particular if arbitrary large changes in γ_ν are to be taken into account.

The bilinearity of Eq. (7) is itself a consequence of the linearity of the steady-state equations of the circuit. As an example, assume that $F = V_0/E$, where V_0 is a response voltage and E a source voltage; that $\gamma_\nu = Z$, where Z is some impedance in the circuit; and that V and I are the voltage across, and the current through, Z. We may replace Z by a voltage source whose voltage V is controlled by the current according to $V = ZI$. Applying superposition we may write $V_0 = AE + BV$, $I = ace + DV$, where A, B, C, and D are independent of the voltages and currents. Eliminating V and I, one obtains indeed Eq. (7), with $F_{11} = A$, $F_{12} = BC - AD$, $F_{21} = I$, $F_{22} = -D$. In a similar way, Eq. (8) can be shown to hold, observing that in the case of a mutual inductance, an ideal transformer, or a gyrator, one has to make use of two auxiliary controlled sources.

If F is in fact a transfer function to be evaluated at real frequencies (thus for $s = j\omega$), one may be more interested in the loss α and the phase β defined by

$$\alpha(\omega, \boldsymbol{\gamma}) + j\beta(\omega, \boldsymbol{\gamma}) = -\ln F(j\omega, \boldsymbol{\gamma}) \tag{9}$$

$$\alpha(\omega, \boldsymbol{\gamma}) = -\ln |F(j\omega, \boldsymbol{\gamma})|, \quad \beta(\omega, \boldsymbol{\gamma}) = -\operatorname{arc} F(j\omega, \boldsymbol{\gamma}) \tag{10}$$

Hence, we have, for example,

$$S(\alpha: \gamma_\nu) = -S(\ln |F|: \gamma_\nu), S(\alpha: \ln \gamma_\nu) = -S(\ln |F|: \ln \gamma_\nu) \tag{11}$$

In the passband of a good filter, we have $\alpha \approx 0$; hence $S(\ln \alpha; \gamma_\nu)$ and $S(\ln \alpha; \ln \gamma_\nu)$ would there have no meaning.

Usually, one has $|\Delta\gamma_\nu/\gamma_\nu| \le \epsilon$ where ϵ is independent of ν. The change of ΔF is thus in first-order approximation bounded according to

$$|\Delta F| \le \epsilon \sum_{\nu=1}^{n} \left|\frac{\partial F}{\partial \ln \gamma_\nu}\right|, \quad \left|\frac{\Delta F}{F}\right| \le \epsilon \sum_{\nu=1}^{n} \left|\frac{\partial \ln F}{\partial \ln \gamma_\nu}\right| \tag{12}$$

This shows that the sums in the expressions just presented can be interpreted as *worst-case sensitivities.* Such concepts can be extended to apply, for example, to the worst case over a frequency range of interest.

Worst-case sensitivities are often too pessimistic for practical applications. A better way of proceeding is then offered by statistical considerations. For this, let γ_0 be the nominal value of $\gamma = (\gamma_1, \ldots, \gamma_n)^{\mathrm{T}}$, thus $\gamma_{\nu 0}$ the nominal value of γ_ν, and $F_0 = F(\gamma_0)$, the relative changes being $(\gamma_\nu - \gamma_{\nu 0})/\gamma_{\nu 0}$ and $(F - F_0)/F_0$. If we can restrict ourselves to first-order terms, we have

$$F - F_0 = \sum_{\nu=1}^{n} \frac{\gamma_\nu - \gamma_{\nu 0}}{\gamma_{\nu 0}} \cdot \frac{\partial F}{\partial \ln \gamma_\nu}|_{\gamma=\gamma_0} \tag{13}$$

Let then the γ_ν be real random variables, with $\mathring{\gamma}_\nu = \mathrm{E}\{\gamma_\nu\}$ and $\mathring{F} = \mathrm{E}\{F\}$ the expected values of γ_ν and F. Ideally, $\mathring{\gamma}_\nu = \gamma_{\nu 0}$ for $\nu = 1$ to n, in which case also $\mathring{F} = F_0$, but this ideal situation is frequently not achieved, especially if effects due to temperature changes and/or aging have to be taken into account. Since Eq. (13) is linear in F and γ_ν we can also write

$$F - \mathring{F} = \sum_{\nu=1}^{n} \frac{\gamma_\nu - \mathring{\gamma}_\nu}{\gamma_{\nu 0}} \cdot \frac{\partial F}{\partial \ln \gamma_\nu}|_{\gamma=\gamma_0} \tag{14}$$

Finally, we assume that the γ_ν are uncorrelated and such that the $\gamma_\nu/\gamma_{\nu 0}$ all have the same variance σ_γ. Defining the variances σ_F and σ_{F/F_0}, also in the case of complex F, by

$$\sigma_F^2 = E\{|F - \mathring{F}|^2\} \text{ and } \sigma_{F/F_0}^2 = E\{|F - \mathring{F}|^2/F_0^2\}$$

we obtain from (13) and (14),

$$\sigma_F^2 = \sigma_\gamma^2 \sum_{\nu=1}^{n} |\frac{\partial F}{\partial \ln \gamma_\nu}|^2_{\gamma=\gamma_0}, \quad \sigma_{F/F_0}^2 = \sigma_\gamma^2 \sum_{\nu=1}^{n} |\frac{\partial \ln F}{\partial \ln \gamma_\nu}|^2_{\gamma=\gamma_0}$$

Hence, the sums in these two expressions can be interpreted as *statistical sensitivities*, and it may again be appropriate to determine the worst case over the frequency range of interest.

Although worst-case and statistical sensitivities may be useful design tools, they frequently do not offer a sufficiently precise picture. In particular, in the case of filters,

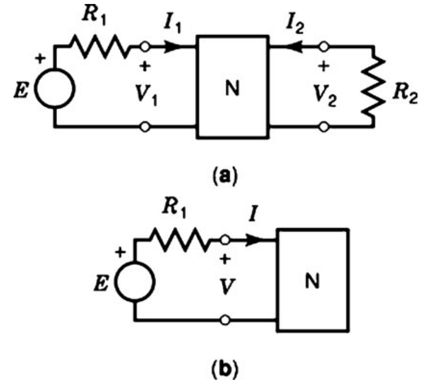

Figure 1. (a) A two-port N driven at port 1 by a source of voltage E and internal resistance R_1 and terminated at port 2 by a load resistance R_2. (b) A one-port N driven by a source of voltage E and internal resistance R_1.

what is really prescribed is a so-called tolerance plot of, for example, the loss $\alpha(\omega)$ with respect to the value $\alpha(\omega_0)$ at some reference frequency ω_0 (see the section entitled "The Passband Sensitivity Theorem").

SENSITIVITY IN PASSIVE, ESPECIALLY LOSSLESS CIRCUITS

Linear Constant Two-Ports at Steady State

In Fig. 1(a) is shown a *two-port* N under canonic operating conditions, that is, inserted between a resistive source (source voltage E, source resistance R_1) and a resistive load R_2. We assume N to be linear and constant (time independent) and consider its steady state at complex frequency $s = \sigma + j\omega$, the voltages E, V_1, and V_2 being assumed to be complex rms values. For N operated from left to right, as shown, the quantities of primary interest are the *transmittance* S_{21}, the *reflectance* S_{11}, the *characteristic function* Ψ, and the *effective* (transducer) *loss* α and *phase* β. In addition to α, the *insertion loss* α_i is frequently used; it is related in a simple way to α, to which it is equal for $R_1 = R_2$, but is less convenient for our purpose. The first three of these quantities are defined by

$$S_{21} = 2\sqrt{\frac{R_1}{R_2}}\frac{V_2}{E} \tag{15a}$$

$$S_{11} = \frac{2V_1 - E}{E} \tag{15b}$$

$$\Psi = \frac{S_{11}}{S_{21}} \tag{15c}$$

and are thus functions of s, while α and β are defined only for $\sigma = 0$, that is, according to

$$S_{21}(j\omega) = e^{-\alpha(\omega) - j\beta(\omega)}, \alpha(\omega) = -\ln |S_{21}(j\omega)| \tag{16}$$

$$e^{2\alpha(\omega)} = \frac{1}{|S_{21}(j\omega)|^2} = 1 + |\Psi(j\omega)|^2 \tag{17}$$

the second equality in Eq. (19) holding (and Ψ thus being fully meaningful) only if N is lossless. For any two-port N,

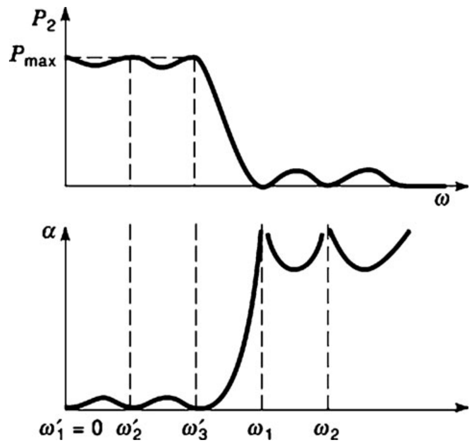

Figure 2. Typical plots of the output power P_2 and the effective loss α versus ω for an optimally designed low-pass filter.

however, we have

$$|S_{21}(j\omega)|^2 = \frac{P_2}{P_{\max}}, P_2 = \frac{|V_2|^2}{R_2}, P_{\max} = \frac{|E|^2}{4R_1} \tag{18}$$

$$|S_{21}(j\omega)|^2 + |S_{11}(j\omega)|^2 = 1 - \frac{P_d}{P_{\max}}, P_d = P_1 - P_2 \tag{19}$$

$$\alpha_i(\omega) = \alpha(\omega) - \ln\frac{R_1 + R_2}{2\sqrt{R_1 R_2}} \le \alpha(\omega) \tag{20}$$

P_1 being the power transmitted to N via port 1, P_2 the power delivered to the load, P_d the power dissipated in N, and P_{max} the maximum power available from the resistive source, thus with $P_1 \le P_{max}$ (the term *power* denoting, in the present context, everywhere *active power* at real frequency, i.e., for $\sigma = 0$). If N is passive, we have $P_d \ge 0$ and thus

$$P_2 \le P_{\max}, \alpha(\omega) \ge 0 \quad \text{for all}\, \omega$$

and furthermore

$$|S_{21}(s)| \le 1, |S_{11}(s)| \le 1 \quad \text{for} \quad \sigma = \text{Re}\, s \ge 0$$

Classical filters are not only passive, but, ideally, even lossless and thus satisfy $P_d = 0$. Hence, if at some ω we have $S_{11}(j\omega) = 0$, we also have there $|S_{21}(j\omega)| = 1$ and $\alpha(\omega) = 0$, and vice versa. Typical plots of P_2 and α versus ω are shown in Fig. 2 for an optimally designed low-pass filter of fifth order (critical regions being enlarged to make them more visible).

The Passband Sensitivity Theorem

We now consider the dependence of the quantities just described not only on ω or s but also on the parameter vector $\gamma = (\gamma_1, \ldots, \gamma_n)^T$, that is, we write $\alpha(\omega, \gamma)$ instead of $\alpha(\omega)$, etc. The components of the circuit of Fig. 1(a) are the elements of N as well as the resistances R_1 and R_2. We assume N to be passive, that is, for any γ_ν referring to a resistance, an inductance, or a capacitance we have $\gamma_\nu > 0$. Strictly speaking, no such restriction exists in the case of an ideal transformer or a gyrator, but in order to simplify our writing we may assume any corresponding γ_ν to be chosen equal to $\pm$ the element value so that we still have $\gamma_\nu > 0$. Coupled inductances may be assumed to be replaced by positive inductances and ideal transformers. Altogether, we may thus assume $\gamma > \mathbf{0}$ (which definitely implies γ to be real). More specifically, let $\gamma_0 = (\gamma_{10}, \ldots, \gamma_{n0})^T > \mathbf{0}$ be the vector of the nominal values of the γ_ν, that is the one for which ideal plots such as those of Fig. 2 are achieved.

Let now ω'_0 be any frequency for which $\alpha(\omega'_0, \gamma_0) = 0$, which in the lossless case is indeed achievable by proper design. Referring to Fig. 2, we may thus choose $\omega'_0 = 0, \omega'_2$, or ω'_3. Let us then change the parameter vector from γ_0 to any other value $\gamma > \mathbf{0}$, in which case N definitely remains passive, ensuring $\alpha(\omega'_0, \gamma) \ge 0$, that is, $\alpha(\omega'_0, \gamma) \ge \alpha(\omega'_0, \gamma_0) = 0$. Hence, the loss $\alpha(\omega'_0, \gamma)$, evaluated at the fixed ω'_0 but considered as a function of γ, has a minimum at $\gamma = \gamma_0$. But since $S_{21}(j\omega'_0, \gamma_0) \ne 0$ and $\ne \infty$, $\alpha(\omega'_0, \gamma)$ defined by

$$\alpha(\omega'_0, \boldsymbol{\gamma}) = -\tfrac{1}{2}\ln[S_{21}(j\omega'_0, \boldsymbol{\gamma})S_{21}(-j\omega'_0, \boldsymbol{\gamma})]$$

is differentiable with respect to all the γ_ν at $\gamma = \gamma_0$. Hence,

$$\frac{\partial\alpha(\omega'_0, \boldsymbol{\gamma})}{\partial\boldsymbol{\gamma}} = \mathbf{0}$$

that is,

$$\left.\frac{\partial\alpha(\omega'_0, \boldsymbol{\gamma})}{\partial\gamma_\nu}\right|_{\boldsymbol{\gamma}=\boldsymbol{\gamma}_0} = 0 \quad \text{for} \quad \nu = 1, \ldots, n \tag{21}$$

Therefore, at any frequency where α is zero the sensitivities of α with respect to any γ_ν vanish for $\gamma = \gamma_0$. This allows us to conclude that in practice all sensitivities are small whenever α is small, thus throughout the passband of the filter.

This is the content of the standard version of what we are here calling the *passband sensitivity theorem.* This content is far reaching and astonishing. A ladder-type low-pass filter capable of giving the performance of Fig. 2 comprises seven reactive elements, to which must be added the terminating resistances. Hence, $n = 9$, that is, the number of sensitivity values that vanish is equal to $3 \times 9 = 27$, which is far more than the available number of degrees of freedom. But these should in the first place be used to obtain a good filtering property. The result of Eq. (21) expresses that the latter property is the only thing we have to look for, and the excellent passband sensitivity is then automatically obtained, without any further expense.

In view of Eq. (20), the sensitivity of α_i with respect to any parameter other than R_1 and R_2 vanishes at the same time as that of α. However, if properly interpreted, the excellent sensitivity behavior of α with respect to R_1 and R_2 also carries over to α_i. Indeed, the difference $\alpha_i - \alpha$ is independent of ω, that is, the plot of $\alpha_i(\omega)$ can always be obtained by simply parallel shifting that of $\alpha(\omega)$, even after R_1 and/or R_2 have been changed. However, what counts in practice is almost always the actual distortion exhibited by the plot, that is, the deviation of $\alpha(\omega)$ with respect to the loss $\alpha(\omega_0)$ at some reference frequency ω_0, or equivalently, the deviation of $\alpha_i(\omega)$ with respect to $\alpha_i(\omega_0)$, and we have $\alpha(\omega) - \alpha(\omega_0) = \alpha_i(\omega) - \alpha_i(\omega_0)$. The same holds, obviously, for any other loss α' defined in such a way that $\alpha'(\omega) - \alpha(\omega)$ is frequency independent, in particular, for $\alpha'(\omega) = \ln|E/V_2|$. In other words, if $\alpha(\omega)$ remains within a given tolerance plot, the same will be true for any such $\alpha'(\omega)$.

The passband sensitivity theorem has apparently been discovered several times independently, although only after filters had been in use for several decades. The present author, for example, discovered it in 1956 after the application of Darlington's so-called *predistortion method* to filters built with inductors of rather mediocre quality factors had led to disappointing sensitivity problems. As required by this method, $\alpha(\omega, \gamma_0)$ [and, equivalently, $|S_{11}(j\omega, \gamma_0)|$] had indeed been raised to a passband level substantially above zero so that the argument that had led to Eq. (28) was completely destroyed. This mechanism had been briefly explained in a tutorial paper published in Dutch in 1960 (15). A short independent exposition by Orchard appeared in 1966 (16, 17), but experts at the Siemens Communications Laboratories had also discovered the theorem. It is being referred to in the literature as Orchard's theorem, the Fettweis–Orchard theorem (11, 13) or the Orchard–Fettweis theorem (18).

The theorem holds for a large variety of filter types, in particular if losslessness can be involved in an appropriate fashion. It is thus immediately applicable to *classical* LC and *microwave filters,* to *crystal filters,* and to *mechanical filters* (19). The transmittance of a periodically switched linear lossless filter is not only a function of s but also a periodic function of t. The coefficients of the corresponding Fourier expansion are the conversion functions, of which usually only one is of primary interest. Although the corresponding effective loss cannot be made strictly equal to zero, it can be made very small by proper design, so that the theorem is essentially applicable (20). To filters of a type for which the concepts of passivity and losslessness have no inherent relevance it is applicable if they are derived in an appropriate way from a *reference* (prototype) *filter* of classical lossless type. This includes not only *active filters,* for which a simple solution consists in realizing inductors by means of capacitively terminated gyrators that in turn are implemented by means of active devices, but also *switched-capacitor* and *switched-current filters* as well as *digital filters* (21–29). The most important solution for the latter are the so-called *wave digital filters* (*WDF*s) (8).

For WDFs, one can make use of a *generalized passband sensitivity theorem* that extends its significance even to parameter changes that are well beyond those for which a first-order theory is sufficient and that would thus be inadmissible in analog filtering (see the first section). In the case of such larger parameter changes, the loss α at any of the frequencies such as 0, ω'_2, and ω'_3 in Fig. 2 can indeed, due to passivity, only move upwards. Hence, the resulting distortion will be substantially less than the one that would be observed if upward movements occurred at some of those frequencies and downward movements at others. For this reason it has been possible to design *ladder* WDFs that have amazingly simple coefficients but still satisfy quite severe requirements (8).

For the phase β, a strict theorem like the passband sensitivity theorem does not hold. Nevertheless, for a minimum-phase circuit (i.e., a circuit whose phase shift is not larger than needed for achieving the given loss behavior), which is the usual situation, β is strictly related to α and we then have, due to the Bayard-Bode relation (6),

$$\boldsymbol{D}\beta(\omega, \boldsymbol{\gamma}) = \frac{\omega}{\pi} \int_{-\infty}^{\infty} \frac{\boldsymbol{D}\alpha(\omega', \boldsymbol{\gamma})}{\omega'^2 - \omega^2}\, d\omega' \tag{22}$$

Hence, any procedure that reduces the sensitivities $\boldsymbol{D}\alpha(\omega, \gamma)$ tends to reduce at the same time the sensitivities $\boldsymbol{D}\beta(\omega, \gamma)$.

Stopband Sensitivity and Tuning

The most important filter sections by means of which classical filters can be composed are of ladder or lattice type. If high selectivity is desired, an input signal arriving at stopband frequencies must be attenuated by many orders of magnitude. This cannot be achieved in a single ladder section but is, at least in principle, feasible in a single lattice section (or one of its equivalents). Let X_1 and X_2 be the two lattice reactances. In the stopband we have $X_1 \approx X_2$ and, as can be shown, $\alpha \geq \ln|2X_0/\Delta X|$, the bound being almost tight, and $X^2{}_0 = X_1X_2$, $\Delta X = X_1 - X_2$. Hence, for any increase of α by 20 dB the ratio $|\Delta X/X_0|$ must be decreased by a factor of 10, at all relevant frequencies. This imposes unrealistic accuracy requirements if large values of α are needed, but even for moderate values the accuracy requirements can only be met by using highly stable components such as quartz crystals. Alternatively, the best way to circumvent the problem is to use ladder instead of lattice structures, that is, configurations that can be composed by chain-connecting simple series and shunt branches.

Such branches produce transmission zeros (attenuation poles) and thus poles of the function Ψ [cf. Eqs. (15c) and (17)] by parallel or series resonances, to which we refer hereafter as *pole resonances* (main resonances). The location of the transmission zeros can easily be determined by measuring the filter output in an arrangement according to Fig. 1(a), and it is thus possible to bring the location of these zeros very close to their desired position by *tuning.* This makes the denominator of Ψ (assumed to be written as a monic polynomial) to becoming very close to its ideal expression. The numerator zeros usually are all located in the passband and are the frequencies at which $\alpha(\omega)$ is zero. If these zeros also have to become more accurate, tuning of some further resonances, referred to hereafter as *auxiliary resonances,* is needed, as is in particular the case for narrow band-pass filters. Such further tuning, however, cannot usually be based directly on the zeros of Ψ, but if the auxiliary resonances are properly selected, the zeros of Ψ will be moved at least close to their ideal locations. The mechanism behind this can be easiest understood by examining a simple related example.

Consider indeed two series resonant circuits with inductances L_1 and L_2 and resonant frequencies ω_1 and ω_2, the nominal values being $L_{10}, L_{20}, \omega_{10}$, and ω_{20}, respectively. We write $L_1 = L_{10} + \Delta L_1$ and $L_2 = L_{20} + \Delta L_2$ and assume that by tuning we have obtained $\omega_1 = \omega_{10}$ and $\omega_2 = \omega_{20}$. If these two circuits are connected in parallel, a parallel resonant frequency ω_3 will be created, with nominal value ω_{30}. Neglecting higher-order terms and assuming ω_{10} to be close to ω_{20}, one can show that

$$\left|\frac{\omega_3-\omega_{30}}{\omega_{30}}\right| \le \frac{1}{4}\left(\left|\frac{\Delta L_1}{L_{10}}\right|+\left|\frac{\Delta L_2}{L_{20}}\right|\right)\left|\frac{\omega_{10}-\omega_{20}}{\omega_{30}}\right| \quad (23)$$

Hence, as a result of tuning the series resonances (which involves two independent operations), the change in the (dependent) parallel resonance is far smaller than what might be expected from ΔL_1 and ΔL_2.

Due to a similar mechanism one can ensure, by properly adjusting a sufficient number of independent auxiliary resonances, that all zeros of Ψ become quite close to their nominal values. After this, even in the case of narrow-band filters, Ψ differs from its nominal shape only by the very small remaining inaccuracies of its zeros and the small change of its constant factor. The influence of these errors is further reduced by the validity of the passband sensitivity theorem. For selecting suitable auxiliary resonances, taking into account the unavoidable presence of parasitic elements (see also the following section), the original circuit should be reduced to circuits of simpler type (each one of them possibly involving more than one branch of the original circuit), but this should be done by applying appropriate short circuits and not by creating interruptions, since these might wrongly affect the location of the parasitic capacitances. In this selection process, one can advantageously make use of *approximate transformations* of the type explained in Ref. 30. On the other hand, in simpler cases, especially for low-pass filters, it may be sufficient to tune only the pole resonances and thus ignore auxiliary resonances. In all cases, resonant frequencies can be maintained close to their nominal values throughout the required temperature range, for example, by using components with mutually compensating temperature coefficients or, if appropriate conditions are met, by implementing critical parts by means of high-quality crystals.

For a ladder structure, the order in which transmission zeros are best implemented is also related to sensitivity and tuning, a mistuning of such a zero being more the critical the closer its nominal value is to a cutoff frequency. On the other hand, taking into account the discussion about Eq. (23), tuning is very helpful near the tuned frequency but much less helpful further away from it. Hence, since resonances in circuit parts close to an access port are strongly damped by the terminating resistance and thus less critical, the following rule emerges for *ordering* the *pole resonances* inside the structure: If no tuning is used, transmission zeros close to the cutoff frequency should be implemented by branches close to the access ports of N [Fig. 1(a)], but the opposite holds if the pole resonances but not the auxiliary resonances are tuned. If pole and auxiliary resonances are tuned, the ordering of the branches implementing the transmission zeros should be quite irrelevant from the sensitivity point of view.

In some important digital filter structures there exists for each transmission zero a one-to-one correspondence between its location and an associated multiplier. *Numerical tuning* may then be applied in the following sense: The transmission zeros are selected to occur at frequencies for which the associated multiplier coefficients are as simple as possible without unduly deteriorating the available stopband performance, these frequencies then being kept fixed during the final optimization of the transfer function.

Sensitivity to Reactive and Resistive Parasitic Elements

Parasitic elements may be interpreted as small changes of parameters, the nominal values of which are zero. Care must therefore be exercised when trying to apply the passband sensitivity theorem since this theorem had been proved under the assumption that the changes $\Delta\gamma_\nu$, whether greater or less than 0, would not affect the validity of $\gamma > \mathbf{0}$. At strictly real frequencies, however, the formal losslessness (i.e., the losslessness at steady state) of a reactive element is not affected by the sign of its component value. Hence, the reasoning that had led to Eq. (21) remains valid, that is, well-designed lossless filters inherently have strongly reduced passband sensitivity to parasitic capacitances and inductances. This argument clearly breaks down in the case of parasitic resistances, that is, if parasitic resistances are introduced into N in Fig. 1(a), we must expect noticeable changes that depend on first-order derivatives.

Let us assume, therefore, as is quite justified in practice, that on the one hand all inductors have the same quality factor $Q_L = 1/\delta_L$ and, on the other, all capacitors have the same quality factor $Q_C = 1/\delta_C$. If $\delta_L = \delta_C = \delta$ the influence upon $S_{21}(j\omega)$ can be shown to be equivalent to replacing ω by $\omega(1 - j\delta)$. This raises α to $\alpha + \Delta\alpha$ and β to $\beta + \Delta\beta$. In first approximation, we then have

$$\Delta\alpha(\omega) = \delta\tau(\omega) \quad (24a)$$

$$\Delta\beta = -\delta\frac{d\alpha(\omega)}{d\omega} \quad (24b)$$

$$\tau(\omega) = \frac{d\beta(\omega)}{d\omega} \quad (24c)$$

where τ is the so-called group delay and $\Delta\beta$ is negligible in the passband of a well-designed filter. If $\delta_L \neq \delta_C$, as is more realistic, we can separate the effect into two parts, at least in the case of narrow-band filters. One of these parts depends on $\delta_L - \delta_C$ and amounts to inserting at each of the ports either a small series reactance or a small shunt susceptance, which can be taken care of by retuning the terminal branches. The remaining effect is then exactly as described above, but with $\delta = (\delta_L + \delta_C)/2$.

Assume next that we are having both dissipation due to lossy components (nonvanishing δ_L and δ_C) and deviations $\Delta\gamma_\nu$ in the parameters γ_ν. Since we are examining first-order effects, both influences can be considered to be independent. In particular, the excellent sensitivity behavior resulting from the passband sensitivity theorem will be fully retained.

This result is in a sense a special case of a more general principle. An effective loss $\alpha > 0$, implying $P_2 < P_{\max}$ [cf. Eqs. (16) and ((18)], can indeed result from two different mechanisms, either from reflection, that is, by $P_1 < P_{\max}$ and $P_d = 0$, or from dissipation, that is, by $P_d > 0$ [Eq. (19)]. In the first case, the assumptions that had led to Eq. (21) are violated. In the second case, it usually so happens that a basic loss $\alpha_0(\omega)$ [such as, for example, the one given by $\delta\tau(\omega)$] is unavoidable, and we thus have, in some neighborhood of γ_0, $\alpha(\omega, \gamma) \ge \alpha_0(\omega)$, with $\alpha(\omega, \gamma_0)$ reaching the bound $\alpha_0(\omega)$ at some frequencies. We are then again led to Eq. (21). The predistortion method (see the section entitled "The Passband Sensitivity Theorem") aims at equalizing the distortion that $\alpha(\omega)$ suffers from the presence of losses in ideally lossless components [see Eq. (24)]; it does this by creating reflection, which explains the poor sensitivity performance obtained by this method. A better way of equalizing $\alpha(\omega)$ is therefore to rely on introducing further dissipation in appropriately chosen locations.

SENSITIVITY ASPECTS IN OTHER TYPES OF CIRCUITS AND IN COMPUTATION

Sensitivity Reduction in Continuous-Time Active Circuits

A classical way of reducing sensitivity in *amplifiers* is to use *feedback*. We consider only the simplest case, in which a forward transfer function μ and a backward transfer function β yield an overall transfer function

$$H = \frac{\mu}{1-\mu\beta} \approx -\frac{1}{\beta}, |\mu| \gg |\mu\beta| \gg 1 \gg |\beta| \tag{25}$$

Clearly, the realization of μ requires the use of active devices, but for β, which essentially determines the value of H, only passive components are needed. Hence, although the gain of pure active devices is quite inaccurate, the value of H can be implemented with the far greater accuracy available from passive components. This is confirmed by means of the sensitivity of H with respect to μ, that is, by

$$\frac{d \ln H}{d \ln \mu} = \frac{1}{1-\mu\beta} \tag{26}$$

Expressions such as Eqs. (25) and (26) assume that there is no interaction between the functions μ and β. In practice, there is at least some interaction, and the expressions can then be made more precise by the use of the concept of *return difference* (6).

The problem of sensitivity is of prime importance also in *active filters.* Many design approaches therefore attempt to model an active filter after some suitable lossless filter of classical type. Others, however, use the desired transfer function H as point of departure and attempt to implement it as directly as possible. We can write

$$H = \frac{a}{b}, b(s) = s^n + B_{n-1}s^{n-1} + \cdots + B_1 s + B_0 = \prod_{\nu=1}^{n}(s - s_\nu)$$

a and b being real polynomials of degree m and $n \geq m$, respectively, with b monic. A direct implementation on the basis of the coefficients B_ν, $\nu = 0, \ldots, n-1$, is very critical because in a good filter the zeros s_ν, thus the *poles* of $H(s)$, are *clustered* (although distinct) near the cutoff frequency(ies). We thus have, as can be shown, for the sensitivities of these zeros with respect to the coefficients

$$\frac{\partial s_\nu}{\partial B_\mu} = \frac{-s_\nu^\mu}{\prod_{i=1, i\neq\nu}^{n}(s_\nu - s_i)}, \mu = 0, \ldots, n-1 \tag{27}$$

Hence, these sensitivities can, in practice, become very high, especially for larger values of n. Consequently, for $n \geq 3$, H should be implemented by *cascading* first- and second-order sections. This conclusion is a simple example of a much wider observation, that is, that proper *parametrization* of functions such as polynomials and rational functions is of decisive importance for keeping the relevant sensitivities as low as possible.

It should be mentioned in this context that the critical property expressed by Eq. (27) is also a major cause for the numerical computation of classical lossless filters by the insertion loss method to be ill-conditioned, and this despite the excellent sensitivity behavior that filters designed this way do offer. This is a reason why in many places this superior design method had not been adopted in practice until after electronic computers had become more easily accessible.

Sensitivity Reduction in Analog Discrete-Time and in Digital Filters

In the case of *discrete-time circuits,* one can express all relevant functions as rational functions in $z = e^{sT}$ or, equivalently, in $\psi = (z-1)/(z+1) = \tanh(sT/2)$, s being the actual complex frequency, as before, and the sampling rate being $F = 1/T$. The parameter ψ largely plays the role of a (normalized) *equivalent complex frequency.* For $\sigma = 0$, we have $\psi = j\varphi$, $\varphi = \tan(\omega T/2)$. All sensitivity aspects discussed earlier in the section entitled "Sensitivity Reduction in Continuous-Time Active Circuits" carry over to the present situation, especially if we adopt ψ as the equivalent of the former s. This applies in particular if filters are designed by modeling them after classical lossless structures and also if cascading is used as a solution to overcoming the problem explained subsequently to Eq. (27).

A few specific aspects should be mentioned, however. In *switched-capacitor filters* (11,14,20-24), the critical frequencies (i.e., the zeros and poles of the relevant functions) are determined by capacitance ratios rather than by absolute capacitance values, thus strongly alleviating the problem of the acceptable tolerances for the capacitance values themselves. A somewhat similar situation holds true for *switched-current filters* (27), for which transistor aspect ratios are the relevant quantities. In *digital filters* (26, 29), for which actual physical circuits indeed play only an indirect role, that is, that of a means to implement algorithms, an added difficulty is the need for ensuring computability. However, WDFs (8) offer a full solution to transposing reference filters of a classical type, such as ladder and lattice filters, into the algorithmic domain.

Lattice WDFs (i.e., WDFs derived from reference filters in classical lattice configuration) often offer very attractive solutions. Although they exhibit a relatively high stopband sensitivity, for which allowance can be made by correspondingly increasing the coefficient word length, their passband sensitivity is even better than that of *ladder* WDFs (i.e., WDFs derived from reference filters in classical ladder configuration). The reason for this is due to the fact that the best possible filter performances are obtained for characteristic functions Ψ [see Eq. (15)c] that are either even or odd. If Ψ is odd, the two-port N [Fig. 1(a)] is symmetric and can then be implemented not only in ladder but also in lattice configuration. If in a ladder configuration one of the element values is changed, even slightly, N usually invariably loses its symmetry. This unavoidably destroys the oddness of Ψ and thus forces all or at least some of the zeros of $\Psi(s)$ to move away from the $j\omega$ axis, which in turn forces the corresponding minima of $\alpha(\omega)$ to move from zero to a value greater than 0. A lattice structure, however, is inherently symmetric, that is, it is symmetric for any parameter value, so that small changes of the parameters will usually allow the minima of $\alpha(\omega)$ to simply move along the ω axis.

Use of Sensitivities in Numerical Procedures

Optimization strategies constitute an important class of numerical procedures used for circuits. The need for optimization arises not only during the design stage but also, for example, during automated tuning and automated adjustment. A typical task then to be solved is to search for that real parameter vector γ for which, for example, the actual time response $f(t, \gamma)$ or, in the case of linear circuits, the actual frequency response $F(j\omega, \gamma)$ is as close as possible to a desired response $f_0(t)$ or $F_0(j\omega)$, respectively. The error to be minimized usually depends on $|f(t, \gamma) - f_0(t)|$ or $|F(j\omega, \gamma) - F_0(j\omega)|$, respectively, and the actual optimization procedure may aim at finding the least-square or the minimax (minimum of the maximum deviation) error (possibly after including some weighting function). Although a wide class of other criteria may be of interest, the least-square criterion has the advantage of analytic simplicity and the minimax criterion that of being of particular engineering relevance.

Except for a few special cases, optimization problems of this sort cannot be solved by analytic procedures. Hence, iterative numerical strategies are needed. Many of these strategies involve the use of the sensitivities of the actual response function with respect to the γ_ν and thus the use of either $\boldsymbol{D}f$ or $\boldsymbol{D}F$ [cf. Eq. (5)], and possibly generalizations thereof (e.g., if f or F is actually to be replaced by a corresponding vector). As an example, if we are interested in an operation lasting from an initial time t_0 to a final time t_1 and if ϵ is the corresponding mean square error, requiring γ to be real but allowing f and f_0 to be complex functions, the gradient $\boldsymbol{D}\epsilon$ is given by

$$\boldsymbol{D}\epsilon = Re \int_{t_0}^{t_1} 2[f^*(t, \boldsymbol{\gamma}) - f_0^*(t)]\boldsymbol{D}f(t, \boldsymbol{\gamma})\, dt \tag{28}$$

where the asterisk designates complex conjugation.

So far we have assumed that an ideal reference function such as $f_0(t)$ is uniquely defined. This is not strictly the case in practice, especially if the design should be based on a criterion of minimax type. This is taken into account in *design centering,* that is, in approaches that aim to keep the tolerances to be imposed on the components as small as possible in order to guarantee that the performance remains within allowed limits. Alternatively, design centering aims for *yield optimization.* Strategies for achieving such goals can be quite involved and can again benefit from using sensitivities.

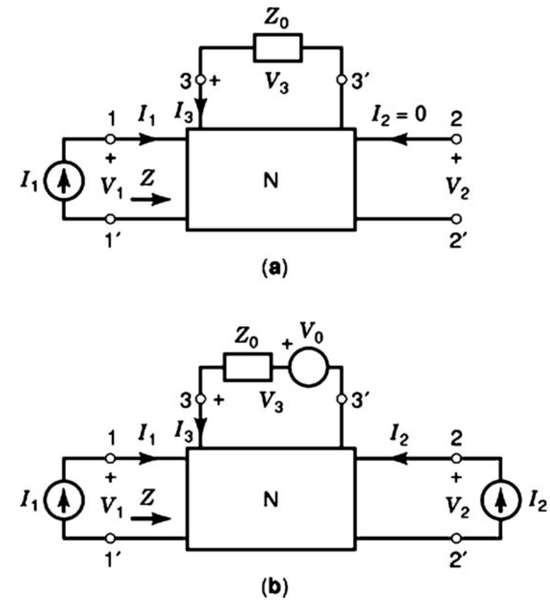

Figure 3. (a) A three-port N terminated at port 3 by Z_0 and open-circuited at port 2, as needed for defining the basic transfer function $H_{21} = V_2/I_1$ as well as the transfer function $H_{31} = I_3/I_1$. (b) Same three-port N in an arrangement obtained from (a) by inserting a voltage source V_0 and a current source I_2. This augmented arrangement reduces to (a) for $V_0 = I_2 = 0$ but allows us to define the transfer functions $H_{23} = V_2/V_0|_{I_1=I_2=0}$ and $H_{32} = I_3/I_2|_{I_1=V_0=0}$.

SOME GENERAL SENSITIVITY PROPERTIES

Sensitivity Calculations in the Frequency Domain

We first consider a *linear constant circuit* operating in steady state. We are interested in the sensitivity of some transfer function H_{21} with respect to some impedance parameter Z_0. One finds that $\partial H_{21}/\partial Z_0$ is equal to $\pm H_{31}H_{23}$, where H_{31} is some transfer function from the input to the location of Z_0, and H_{23} some other transfer function from this location to the output. We show this by means of the example illustrated in Fig. 3. In Fig. 3(a) the circuit is drawn as a three-port N fed by a current source at port 1 and terminated by the impedance Z_0 at port 3, while the open-circuited port 2 is provided in order to give access to the desired output voltage V_2.

The second and third impedance equations of the three-port N can be written

$$V_2 = Z_{21}I_1 + Z_{22}I_2 + Z_{23}I_3, V_3 = Z_{31}I_1 + Z_{32}I_2 + Z_{33}I_3$$

Referring to Fig. 3(a), we thus obtain, since $I_2 = 0$,

$$H_{21} := \frac{V_2}{I_1} = Z_{21} - \frac{Z_{23}Z_{31}}{Z_{33} + Z_0}, \quad H_{31} := \frac{I_3}{I_1} = -\frac{Z_{31}}{Z_{33} + Z_0}$$

and, referring to Fig. 3(b)

$$H_{23} := \left.\frac{V_2}{V_0}\right|_{I_1=I_2=0} = \frac{Z_{23}}{Z_{33} + Z_0}$$

$$H_{32} := \left.\frac{I_3}{I_2}\right|_{I_1=V_0=0} = -\frac{Z_{32}}{Z_{33} + Z_0}$$

This shows that

$$\frac{\partial H_{21}}{\partial Z_0} = -H_{31}H_{23} \tag{29}$$

which can also be obtained from Fig. 3(b) for $I_2 = 0$ by assuming V_0 to be the voltage caused by a small increase ΔZ_0 of Z_0, thus by writing

$$V_0 = -I_3\Delta Z_0 = -H_{31}I_1\Delta Z_0 \quad \text{and} \quad \Delta H_{21} = \Delta V_2/I_1 = H_{23}V_0/I_1$$

Equation (29) is immediately rendered even more practical if N is reciprocal, since then $H_{23} = -H_{32}$. In order to determine functions like H_{31} and H_{32} for all parameters such as Z_0 it then is indeed sufficient to analyze the circuit once by exciting it at port 1 and once at port 2. This advantage can be extended to nonreciprocal circuits by replacing N for the second analysis by its interreciprocal (adjoint), that is, by a three-port for which the reciprocal elements are unchanged while the nonreciprocal elements are replaced by those of opposite type (a gyrator with gyration constant R, e.g., being replaced by one with $-R$). Furthermore, the analysis of the interreciprocal circuit can essentially be reduced to that of the original circuit if the matrix $\boldsymbol{M}$ to be inverted in the process of analysis is first factored in the form $\boldsymbol{M} = \boldsymbol{LU}$, $\boldsymbol{L}$ and $\boldsymbol{U}$ being lower and upper triangular matrices, respectively; indeed, the corresponding matrix for the interreciprocal is $\boldsymbol{M}^{\mathrm{T}} = \boldsymbol{U}^{\mathrm{T}}\boldsymbol{L}^{\mathrm{T}}$ so that its corresponding factorization is immediately deduced. Finally, all this can in full generality be derived in a very systematic way by making use of Tellegen's theorem, more precisely, by a generalized version of the difference form of that theorem (3). On the other hand, an interesting special case occurs if the voltage of actual interest is V_1 rather than V_2 so that H_{21} is equal to the input impedance Z at port 1. We may then replace subscript 2 by 1 and obtain from Eq. (40) in the reciprocal case, using also $Y = 1/Z$, $Y_0 = 1/Z_0$,

$$\frac{\partial Z}{\partial Z_0} = H_{31}^2 = \left(\frac{I_3}{I_1}\right)^2, \frac{\partial Y}{\partial Y_0} = \left(\frac{V_3}{V_1}\right)^2 \tag{30}$$

a result also known as the Vratsanos-Cohn theorem.

Similar results also hold true for *linear constant discrete-time circuits,* the role of Z_0 in Fig. 3 being then assumed by any multiplier coefficient (29).

Sensitivity Calculations in the Time Domain

The generalized version of the difference form of Tellegen's theorem and the adjoint concept are even more useful for the time-domain sensitivity analysis (31), as will be apparent from the brief, simplified outline given hereafter (but nowhere restricted to linear circuits). Let thus $\boldsymbol{v}$ and $\boldsymbol{i}$ be the vectors of all branch voltages and all branch currents, respectively, in the circuit under consideration, say, the *original* circuit. The entries of the matrices $\boldsymbol{DvT}$ and $\boldsymbol{Di}^{\mathrm{T}}$ form the totality of the sensitivities of all voltages and currents with respect to all γ_ν, which we may assume to be real. All vectors $D_\nu\boldsymbol{v}$ and $D_\nu\boldsymbol{i}$ satisfy Kirchhoff's voltage and current laws at the same time as $\boldsymbol{v}$ and $\boldsymbol{i}$, respectively.

Consider then a second circuit, the *adjoint* circuit, with the same topology as the original one, and let $\boldsymbol{v}'$ and $\boldsymbol{i}'$ be its vectors of branch voltages and branch currents. These also satisfy Kirchhoff's laws, say at any time instant t', where t' may be different from t. According to Tellegen's theorem,

$$(\boldsymbol{Di}^{\mathrm{T}})\boldsymbol{v}' - (\boldsymbol{Dv}^{\mathrm{T}})\boldsymbol{i}' = \boldsymbol{0} \tag{31}$$

Let us examine the individual contributions to Eq. (31). Suppose first that some branch β is formed by a resistance described by some (possibly nonlinear) equation that involves at least one (and usually at most a few) of the γ_ν and that we may write as $f(v_\beta, i_\beta, \gamma) = 0$. The quantities v_β and i_β usually depend on *all* γ_ν (a fact sometimes overlooked in the literature). By differentiating $f(v_\beta(t,\gamma), i_\beta(t,\gamma), \gamma)$ with respect to all the γ_ν we can write

$$f_v\boldsymbol{D}v_\beta + f_i\boldsymbol{D}i_\beta + \boldsymbol{D}f = \boldsymbol{0}$$

where

$$f_v = \frac{\partial f}{\partial v_\beta}, f_i = \frac{\partial f}{\partial i_\beta}$$

and where $\boldsymbol{D}\boldsymbol{f}_f$ concerns exclusively the partial derivatives with respect to the γ_ν that appear explicitly (via γ) in $f(v_\beta, i_\beta, \gamma)$. Consequently, provided v'_β and i'_β are related in such a way that

$$i'_\beta f_i + v'_\beta f_v = 0 \tag{32}$$

the vector

$$v'_\beta\boldsymbol{D}i_\beta - i'_\beta\boldsymbol{D}v_\beta \tag{33}$$

which is indeed the contribution of branch β to the left-hand side of Eq. (31), reduces to the simple vector $(i'_\beta/f_v)\boldsymbol{D}f$. This vector (most of whose components are zero) is independent of any of the aforementioned sensitivities. Clearly, Eq. (32) defines a resistance in the adjoint circuit. In the linear constant case we have $f(v_\beta, i_\beta, \gamma) = v_\beta - Ri_\beta$ and thus $f_v = 1$, $f_i = -R$; hence, Eq. (32) then defines a resistance of exactly the same type.

For an algebraically defined element involving two branches, the situation is similar and, in the linear case, again reduces to the known results. If an appropriate range of algebraic two-port elements is available, the only other elements needed for generating the remaining elements of usual interest are, for example, linear capacitances. Assume thus that some branch β is described by $i_\beta = C\dot{v}_\beta$, where C is a constant and the dot on top of the letter designates time derivation, d/dt. Consequently, provided v'_β and i'_β are related by $i'_\beta = -C\dot{v}'_\beta$, that is

$$i'_\beta(t') = -C\frac{d}{dt}v'_\beta(t') \tag{34}$$

the contribution of Eq. (33) becomes equal to the vector

$$C\frac{d}{dt}(v'_\beta\boldsymbol{D}v_\beta) + v'_\beta\dot{v}_\beta\boldsymbol{D}C \tag{35}$$

(where one of the components of $\boldsymbol{DC}$ is equal to 1 and all others are equal to zero).

In order for Eq. (34) to make sense, let t_0 again be the initial time and t_1 the final time of interest [cf. Eq. (28)]

and define $t' = t_1 - t$. We then have

$$i'_\beta(t') = C\frac{d}{dt'}v'_\beta(t')$$

which is the same type of equation as for the capacitance in the original circuit, but the adjoint circuit is assumed to run in opposite time direction, its initial time $t' = 0$ corresponding to $t = t_1$ and its final time $t' = t_1 - t_0$ corresponding to $t = t_0$.

In Eq. (35) the second term is independent of the sensitivities, as desired, but the contribution due to the first term can be made to vanish if we actually have to carry out a time integration between limits as in Eq. (28). In this case, the derivative *d/dt* in Eq. (35) yields

$$(v'_\beta \boldsymbol{D} v_\beta)(t = t_1) - [v'_\beta \boldsymbol{D} v_\beta](t = t_0) \tag{36}$$

However, the initial value $v_\beta(t_0)$ is imposed independently of γ, that is, $\boldsymbol{D}v_\beta(t_0) = 0$. On the other hand, we may impose the initial conditions of the adjoint circuit to be zero, that is, $v'_\beta(0) = 0$. Hence Eq. (36) is then indeed equal to zero.

We still have to consider the access to the circuit. Assume thus that branch 1 consists of an independent voltage source in the original circuit and of a simple short circuit in the adjoint circuit. We then have $\boldsymbol{D}v_1 = \boldsymbol{0}$ and $v'_1 = 0$, so that Eq. (33) vanishes for $\beta = 1$. Similarly, assume that branch 2 is simply open-circuited in the original circuit and that it consists of some current source in the adjoint circuit, so that Eq. (33) reduces to $-i'_2\boldsymbol{D}v_2$ for $\beta = 2$.

These results can, for example, be used in the following way. Assume that the behavior of the original circuit has been calculated for a first choice of γ. Values such as f_i and f_v in Eq. (32), etc., are then known so that the equations governing the elements of the adjoint network are also known, and the same is true for vectors such as $\boldsymbol{D}f$ mentioned subsequently to Eq. (33). The initial conditions of the adjoint circuit being zero, the only freedom left is the choice of $i'_2(t')$. To see how this can be fixed, consider the determination of $\boldsymbol{D}\epsilon$ in an optimization problem such as the one that had led to Eq. (28). We assume that the present ν_2 is actually the function to be optimized; it is therefore identical to the function designated f in Eq. (28), the desired behavior of ν_2 thus being f_0. Hence, we can identify $-i'_2(t')\boldsymbol{D}v_2$ with the integrand in Eq. (28) by choosing

$$i'_2(t') = 2[f_0^*(t_1 - t') - v_2^*(t_1 - t', \boldsymbol{\gamma})]$$

If we then integrate Eq. (31) from t_0 to t_1 and take the real part, the contribution of Eq. (33) for $\beta = 2$ becomes equal to $\boldsymbol{D}\epsilon$, while, according to what we have seen, all other contributions are known. Hence, $\boldsymbol{D}\epsilon$ can be determined from the resulting equation. Extensions to multiple inputs and outputs, to time-varying original circuits, and to inclusion of weighting functions are quite immediate. Note that the above analysis does not exclude $\boldsymbol{v}$ and $\boldsymbol{i}$ to be complex, although in the case of nonlinear circuits one is usually only dealing with real quantities.

Sensitivity Invariants

The steady-state behavior of a linear circuit is entirely determined by quantities that either have the dimension of a resistance or are dimensionless. Let F again be any function of interest (impedance, admittance, transfer function, loss, phase), let $\boldsymbol{L}$, $\boldsymbol{C}$, $\boldsymbol{R}$, and $\boldsymbol{n}$ be the vectors of inductive, capacitive, resistive, and dimensionless parameters fixing the circuit behavior, and let a and b be arbitrary auxiliary parameters. Since F is homogeneous in any set of independent dimensions, we can write

$$\begin{aligned} F\left(a^2\boldsymbol{L}, b^2\boldsymbol{C}, \frac{a}{b}\boldsymbol{R}, \boldsymbol{n}, j\omega\right) &= F\left(\frac{a}{b}ab\boldsymbol{L}, \frac{b}{a}ab\boldsymbol{C}, \frac{a}{b}\boldsymbol{R}, \boldsymbol{n}, j\omega\right) \\ &= \left(\frac{a}{b}\right)^m F(\boldsymbol{L}, \boldsymbol{C}, \boldsymbol{R}, \boldsymbol{n}, jab\omega) \end{aligned} \tag{37}$$

where $m = +1, -1$, or 0. Differentiating Eq. (37) with respect to a and b and then setting $a = b = 1$, we obtain

$$2\sum_\nu \frac{\partial F}{\partial \ln L_\nu} + \sum_\nu \frac{\partial F}{\partial \ln R_\nu} = mF + \omega\frac{\partial F}{\partial \omega} \tag{38}$$

$$2\sum_\nu \frac{\partial F}{\partial \ln C_\nu} - \sum_\nu \frac{\partial F}{\partial \ln R_\nu} = -mF + \omega\frac{\partial F}{\partial \omega} \tag{39}$$

The left-hand sides in Eqs. (38) and (39) comprise simple combinations of individual sensitivities, but the right-hand sides depend only on m and on the behavior of F in terms of ω. A given function F, however, can be realized by several or even infinitely many distinct circuits (32, 33), and for each one of these the individual sensitivities will usually be different. Yet, the overall values of the left-hand sides in Eqs. (38) and (39) are independent of the specific realization; they are therefore known as *sensitivity invariants.* Obviously, they can easily be applied in various specific situations. As an example, we have

$$\sum_\nu \frac{\partial F}{\partial \ln L_\nu} + \sum_\nu \frac{\partial F}{\partial \ln C_\nu} = \omega\frac{\partial F}{\partial \omega} \tag{40}$$

and therefore, if F is a transmittance and thus $\ln F = -\alpha - j\beta$, we can derive

$$\sum_\nu L_\nu\frac{\partial \alpha}{\partial L_\nu} + \sum_\nu C_\nu\frac{\partial \alpha}{\partial C_\nu} = \omega\frac{\partial \alpha}{\partial \omega} \tag{41a}$$

$$\sum_\nu L_\nu\frac{\partial \beta}{\partial L_\nu} + \sum_\nu C_\nu\frac{\partial \beta}{\partial C_\nu} = \omega\tau \tag{41b}$$

where τ is again the group delay [see Eq. (24)c].

Other interesting sensitivity expressions can be obtained for a two-port N operated as in Fig. 1(a). Using known expressions of S_{11} and S_{21} [see Eq. (15)] in terms of, for example, the impedance matrix of N, one finds in view of Eq. (16),

$$2\frac{\partial \alpha(\omega, \boldsymbol{\gamma})}{\partial \ln R_1} = -\mathrm{Re}\, S_{11}(j\omega, \boldsymbol{\gamma}),\; 2\frac{\partial \beta(\omega, \boldsymbol{\gamma})}{\partial \ln R_1} = -\mathrm{Im}\, S_{11}(j\omega, \boldsymbol{\gamma}) \tag{42}$$

$$2\frac{\partial \alpha(\omega, \boldsymbol{\gamma})}{\partial \ln R_2} = -\mathrm{Re}\, S_{22}(j\omega, \boldsymbol{\gamma}),\; 2\frac{\partial \beta(\omega, \boldsymbol{\gamma})}{\partial \ln R_2} = -\mathrm{Im}\, S_{22}(j\omega, \boldsymbol{\gamma}) \tag{43}$$

where Eq. (43) is obtained by defining S_{22} analogously to S_{11}. These results can, for example, usefully be combined with Eqs. (38) and (39), especially if N does not comprise any further resistive parameter. They also hold true for α

and β defined by $\alpha + j\beta = -\ln S_{12}$, S_{12} referring to the direction of transmission opposite to that in Fig. 1(a).

Unfortunately, Eqs. (38) and (39) do not involve sums of magnitudes of sensitivities. Hence, they do not in general offer upper bounds, as would be desirable, but only lower bounds, for example

$$2\sum_{\nu}\left|\frac{\partial \ln F}{\partial \ln L_{\nu}}\right| + \sum_{\nu}\left|\frac{\partial \ln F}{\partial \ln R_{\nu}}\right| \geq \left|m + \omega\frac{\partial \ln F}{\partial \omega}\right|$$
$$2\sum_{\nu}\left|\frac{\partial \ln F}{\partial \ln C_{\nu}}\right| + \sum_{\nu}\left|\frac{\partial \ln F}{\partial \ln R_{\nu}}\right| \geq \left|m - \omega\frac{\partial \ln F}{\partial \omega}\right|$$

In some important special cases, however, some much more useful conclusions can be drawn. Thus, let F be the input impedance $Z(j\omega) = jX(\omega)$ of a lossless circuit. Let us modify this circuit by adding to some inductance L_ν a series resistance R'_ν. This amounts to replacing L_ν by $L_\nu(1 - j\delta_\nu)$, with $\delta_\nu = R'_\nu/\omega L_\nu$, and thus, in a first approximation, to adding to jX a resistive term $\delta_\nu \partial X/\partial \ln L_\nu$. Since due to passivity this term must be nonnegative, we conclude, $\omega \geq 0$, assuming that for any inductance we have $\partial X/\partial L_\nu \geq 0$. Similarly $\partial X/\partial C_\nu \geq 0$ for any capacitance C_ν. Consequently, since due to Eq. (40),

$$\sum_{\nu}\frac{\partial X}{\partial \ln L_{\nu}} + \sum_{\nu}\frac{\partial X}{\partial \ln C_{\nu}} = \omega\frac{\partial X}{\partial \omega} \tag{44}$$

all terms on the left-hand side of this expression are nonnegative so that none of them can grow excessively.

A similar conclusion can be drawn if F is a transmittance. The appearance of the quantity δ_ν just given will indeed add to $\alpha + j\beta$ a first-order correction $(-j\partial\alpha/\partial \ln L_\nu + \partial\beta/\partial \ln L_\nu)\delta_\nu$. If $\alpha = 0$, passivity now requires $\partial\beta/\partial L_\nu \geq 0$, and similarly $\partial\beta/\partial C_\nu \geq 0$, that is, the argument used for Eq. (44) is also valid for Eq. (41)b. This holds true at least for frequencies at which $\alpha(\omega)$ vanishes and thus, in practice, in the entire passband of good filters, confirming the observations we had made with respect to Eq. (22). In fact, it holds true even more generally as will be seen in the next section.

Sensitivity and Energy

Many sensitivity quantities are related to power and energy expressions (34–36). A few important examples for this will be discussed hereafter. We first consider the impedance $Z(j\omega) = jX(\omega)$ of a circuit N composed of inductors, capacitors, ideal transformers, and gyrators. Assume it to be fed by a source of voltage E and internal resistance R_1 [Fig. 1(b)], the internal branches of N to be numbered $\nu = 2, \ldots, n$, and the branch voltages V_ν and currents I_ν to be oriented as usual, while $V_1 = V$, $I_1 = I$. The generalized version of Tellegen's theorem allows us to write

$$M_1 = \sum_{\mu=2}^{n} M_\mu, M_\mu = V_\mu^* I_\mu + V'_\mu I_\mu^*, \mu = 1, \ldots, n \tag{45}$$

the prime indicating, as also everywhere hereafter, a partial derivative with respect to some parameter γ_ν occurring inside of N. If γ_ν is an *inductance* or a *capacitance* characterizing some specific branch ν, the summation Σ in Eq. (62) reduces simply to $j2\omega W_\nu/\gamma_\nu$, where

$$W_\nu = \tfrac{1}{2}L_\nu|I_\nu|^2 \quad \text{for } \gamma_\nu = L_\nu,\; W_\nu = \tfrac{1}{2}C_\nu|V_\nu|^2 \quad \text{for } \gamma_\nu = C_\nu$$

W_ν being thus the average energy stored in the corresponding element. We have

$$V_1 = jXI_1, E = V_1 + R_1 I_1, P_{\max} = \frac{|E|^2}{4R_1}, \rho = \frac{jX - R_1}{jX + R_1} = e^{-j\beta_r}$$

$P_{\max}$ being the maximum power available from the source, ρ the reflectance, and β_r the corresponding phase. One finds $M_1 = j|I_1|^2\ \partial X/\partial\gamma_\nu$, altogether thus

$$2P_{\max}\frac{\partial \beta_r}{\partial \ln \gamma_\nu} = |I_1|^2\frac{\partial X}{\partial \ln \gamma_\nu} = 2\omega W_\nu \tag{46}$$

where the first equality can be verified by using the definition of ρ. The result of Eq. (46) confirms in particular that $\partial X/\partial\gamma_\nu \geq 0$ for $\omega \geq 0$, while we obtain from Eq. (44)

$$|I_1|^2\ \frac{\partial X}{\partial \omega} = 2W, \quad W = \sum W_\nu$$

the sum being extended over all branches that consist either of an inductor or a capacitor and W being thus the total average energy stored in N.

Next, we apply the same analysis to the two-port arrangement of Fig. 1(a) in which N is composed of the same types of elements as before. The first equation, Eq. (45), has to be replaced by

$$M_1 + M_2 = \sum_{\mu=3}^{n} M_\mu \tag{47}$$

the internal branches of N being thus numbered $\mu = 3, \ldots, n$. For the M_μ in the right-hand side of Eq. (64) everything remains as before. Using Eq. (15), one finds for branches ν consisting of an inductor or a capacitor

$$\frac{M_1 + M_2}{2P_{\max 1}} = -S_{11}^* S'_{11} = j\frac{\omega W_{\nu 1}}{\gamma_\nu P_{\max 1}}$$

where the second equality holds in view of Eq. (64) and where we have added a subscript 1 to $P_{\max}$ and W_ν in order to make clear that the source is applied at port 1. After moving E into the terminating branch at port 2, an expression similar to Eq. (64) can be written, with $W_{\nu 2}$ and $P_{\max 2} = |E|^2/4R_2$ taking the role of $W_{\nu 1}$ and $P_{\max 1}$. Adding the two expressions and taking the imaginary part, we obtain altogether

$$\gamma_\nu(|S_{11}|^2\beta'_{11} + |S_{22}|^2\beta'_{22} + |S_{21}|^2\beta'_{21} + |S_{12}|^2\beta'_{12}) = \omega(\frac{W_{\nu 1}}{P_{\max 1}} + \frac{W_{\nu 2}}{P_{\max 2}})$$

where we have made use of $S^*_{11}S'_{11} = |S_{11}|^2\ (\ln S_{11})'$ and $\beta_{11} = -\text{Im} \ln S_{11}$, etc. But for a lossless two-port the scattering parameters satisfy

$$|S_{11}| = |S_{22}|, |S_{21}| = |S_{12}|, |S_{11}|^2 + |S_{21}|^2 = 1,$$
$$S_{11}S_{12}^* = -S_{21}S_{22}^* \tag{50}$$

and therefore also $\beta'_{11} + \beta'_{22} = \beta'_{21} + \beta'_{12}$. Hence, Eq. (69) simplifies to

$$\frac{\partial(\beta_{11} + \beta_{22})}{\partial \ln \gamma_\nu} = \frac{\partial(\beta_{21} + \beta_{12})}{\partial \ln \gamma_\nu} = \omega(\frac{W_{\nu 1}}{P_{\max 1}} + \frac{W_{\nu 2}}{P_{\max 2}})$$

In particular, if N is reciprocal, thus if $S_{21} = S_{12}$ and hence $\beta_{21} = \beta_{12} = \beta$

$$2\frac{\partial \beta}{\partial \ln \gamma_\nu} = \omega\left(\frac{W_{\nu 1}}{P_{\max 1}} + \frac{W_{\nu 2}}{P_{\max 2}}\right),$$

This shows that $\partial\beta/\partial\gamma_\nu \geq 0$ and also, together with Eq. (41b), that

$$2\tau = \frac{W_{01}}{P_{\max 1}} + \frac{W_{02}}{P_{\max 2}}$$

where W_{01} and W_{02} are the total average energies stored in N when E is located in the terminating branch of port 1 and 2, respectively.

If N is not only reciprocal ($S_{12} = S_{21}$) but, more specifically, symmetric ($S_{22} = S_{11}$) or antimetric ($S_{22} = -S_{11}$), in which case $\beta'_{11} = \beta'$, we can proceed directly from Eq. (48) instead of going via Eq. (49). This yields

$$\frac{\partial \beta}{\partial \ln \gamma_\nu} = \frac{\omega W_{\nu 1}}{P_{\max 1}} = \frac{\omega W_{\nu 2}}{P_{\max 2}}, \quad \tau = \frac{W_{01}}{P_{\max 1}} = \frac{W_{02}}{P_{\max 2}}$$

Consider still the real part of the second equality in Eq. (48) [or, equivalently, take the derivative with respect to γ_ν of the third equation, Eq. (50)]. Using

$$\alpha = -\ln |S_{21}| = -\ln |S_{12}|$$

we obtain

$$\mathrm{Re}\, S^*_{11} S'_{11} - |S_{21}|^2 \alpha' = 0$$

that is,

$$\frac{\partial \alpha}{\partial \gamma_\nu} = \mathrm{Re}\left(\frac{S^*_{11}}{1 - |S_{11}|^2}\frac{\partial S_{11}}{\partial \gamma_\nu}\right),$$

The last expression allows us to compute the sensitivities of α in terms of those of S_{11}, in fact for any γ_ν referring to an element inside of N. Since $\alpha = 0$ implies $S_{11} = 0$, the passband sensitivity theorem is immediately confirmed.

The sensitivities S'_{11} can be determined using relations such as Eq. (29). To illustrate this, let us replace the resistive source in Fig. 1(a) by an equivalent current source with parallel resistance R_1. We must, of course, be aware that the meanings of N, V_1, I_1, V_2, and I_2, in Fig. 1(a) differ from those in Fig. 3(a). Furthermore, since the output quantity of interest is now V_1 (cf. Eq. 15b) we may, when addressing Fig. 3(a), assume the access provided by the terminals 2,2′ to be in fact an access to 1,1′. Altogether we may thus assume in Fig. 3(a) to have $I_1 = E/R_1$, $V_1 = V_2$, $I_3 = I_\nu$, and $V_3 = -V_\nu$. In particular, the transfer functions H_{21} and H_{31} defined in the section entitled "Sensitivity Calculations in the Frequency Domain" are now given by $H_{21} = R_1V_1/E$ and $H_{31} = R_1I_\nu/E$, while either $Z_0 = j\omega L_\nu$ or $Y_0 = 1/Z_0 = j\omega C_\nu$. A similar relation holds true for H_{23}, which is also needed in Eq. (29) . On the other hand, if we restrict ourselves to the reciprocal case we may immediately use Eq. (30) with $Z = H_{21}$. On the other hand, in view of Eq. (15)b, we have $S'_{11} = 2H'_{21}/R_1 = 2Z'/R_1$. Finally, one finds for branches consisting of an inductor or a capacitor

$$\gamma_\nu \frac{\partial S_{11}}{\partial \gamma_\nu} = \pm j\omega \frac{\hat{W}_\nu}{\hat{P}_1}, \quad \hat{W}_\nu = \frac{L_\nu I_\nu^2}{2} \quad \text{or} \quad \hat{W}_\nu = \frac{C_\nu V_\nu^2}{2}, \quad \hat{P}_1 = \frac{E^2}{4R_1}$$

the upper sign holding for $\gamma_\nu = L_\nu$ and the lower one for $\gamma_\nu = C_\nu$. The quantities $\hat{W}_\nu$ and $\hat{P}_1$ are, in general, complex, but their magnitudes have the energy and power interpretations encountered before.

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ALFRED FETTWEIS
Ruhr-Universität Bochum ,
Bochum, Germany