NETWORK EQUATIONS

In an electrical network or an electrical circuit, the operation of every single element is determined at each moment by the interaction between the element itself and the rest of the network. In other words, we can say it results from the action due to different requirements: (1) that the element behaves in a manner compatible with its specific nature; and (2) that such behavior is in turn compatible with the behavior of all other elements present in the network. The constitutive relations describe the operation of the single elements and Kirchhoff's laws regulate the interaction. The equations that derive from them are the *network equations* and are the subject of this article.

Within the limits of the circuit model approximation, the operating conditions of the single elements are identified by the voltages and currents at their terminals; these are the unknowns of the circuit equations. They are linked by the constitutive relations, which are expressions of the nature of the elements and, as a whole, describe their operations. The constitutive relations do not depend on the way in which the elements are connected to one another. For the sake of simplicity, we will refer here to a network of one-ports only, that is, elements with two terminals. Its extension to the case where there are also n-terminal elements does not involve any conceptual difficulty and will be considered later.

Voltages and currents are algebraic variables, and so it is necessary to choose a reference direction in advance for every single one-port. In this context the concepts of voltage and current are introduced axiomatically; thus, it may be difficult to understand fully the need for the choice of a reference direction for them. However, if one remembers that the current is nothing but the intensity of the electric charge flow crossing the one-port, one readily understands why it is necessary to indicate the reference direction in advance in order to assess the intensity of the flow charge indicated by the symbol $i(t)$. Similarly, for the voltage it is sufficient to recall that it is the work done by the electric field to bring a unit charge from one terminal to another. Therefore, it is necessary here to indicate also the starting and ending terminals in advance to evaluate the work indicated by the symbol $v(t)$. These concepts, which are the basis of the circuit theory, would merit a more profound discussion. However, due to space limitations we cannot develop them in this article. We therefore refer the reader to LINEAR NETWORK ELEMENTS and TIME-DOMAIN NETWORK ANALYSIS. There the problems implicit in the previous simple definitions, and their connections with the general theory of the electromagnetic fields, are dealt with in greater depth.

Returning to the circuit model, if we represent the one-port as a closed box with two terminals, as shown in Fig. 1a, we can indicate the chosen reference directions with an arrow placed on one of the two terminals for the current and with the signs + and − placed near the terminals for the voltage. Suppose that a current reference direction is chosen from the two available. Then the voltage reference direction can be chosen with either the + sign or the − sign at the terminal where the current arrow enters the element. The former choice is called the *normal* convention, which we will always adopt here. In the general case, one thinks, for example, of the p–n junction diode; the four possible alternatives give rise to different expressions of the constitutive equations.

One-ports can be classified as dynamic and nondynamic. For nondynamic one-ports, which we will call resistive one-ports, the relations between the voltage and the current are of the "algebraic" type, that is, the value assumed by the voltage at any time depends only on the value of the current at that time and vice versa. Resistors, diodes, voltage and current sources are examples of resistive one-ports. By contrast, the operations of dynamic one-ports are described by means of differential or integral-type equations. Thus the value of the voltage or of the current at any time depends also on their past histories. Capacitors and inductors are examples of dynamic one-ports. For now and for the sake of simplicity, we will refer only to resistive one-ports.

The voltage and the current of a resistive one-port, therefore, identify its operating point. This expression derives from the fact that, as the one-port constitutive relation is of an algebraic type, $f(v, i) = 0$, and thus is graphically representable in the plane (v, i), a given voltage and a given current identify a point on the characteristic curve $f(v, i) = 0$. If one considers, for example, the linear resistor, the characteristic $v = Ri$ is representable in the plane (v, i) by a straight line passing through the origin. The points on this straight line represent the possible operating conditions that the one-port in question can allow. What the effective operating point of the one-port is at a given time is determined by the operating conditions of the remaining part of the network into which it is inserted. The laws governing this interaction are the two Kirchhoff laws, which we will briefly recall after the introduction of some simple definitions.

Let us call *node* the connecting point of at least two terminals of distinct elements. Between two nodes effectively connected to one-ports we will say that the network has *branches*—one branch for each one-port. The set of branches

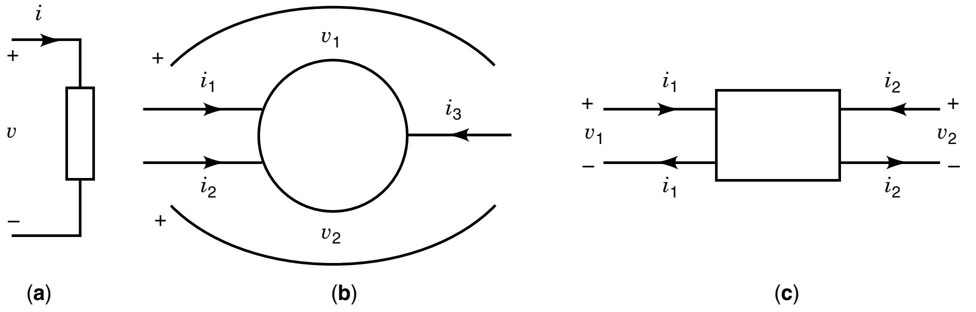

Figure 1. Graphic representation for (a) two-terminal elements; (b) three-terminal elements or three-poles; (c) four-terminal elements operating as two-ports. The normal convention is used.

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and nodes of the network constitute the *graph* of the circuit. Finally, we will call *loop* any closed path starting from any node, traversing different branches and nodes of the network itself and ending at the same node, where precisely two branches are incident with each node. The graph of the network and its topologic properties are considered in TIME-DOMAIN NETWORK ANALYSIS. With this premise one can state Kirchhoff's two laws.

Kirchhoff's Current Law (node law). The algebraic sum of all the currents at any node is equal to zero.

Kirchhoff's Voltage Law (loop law). The algebraic sum of all the voltages along any loop is equal to zero.

In this context, and because of the way in which they have been introduced, Kirchhoff's laws appear as postulates or experimental laws. In effect they can be derived, in a more general model of the electromagnetic field, from Maxwell's equations and are strictly exact in the stationary regime, that is, when voltages and currents do not vary with time. However, they are only approximate, but generally closely so, in a dynamic regime (see LINEAR NETWORK ELEMENTS).

The equations obtained by applying Kirchhoff's laws do not depend on the nature of the single elements, but only on the way in which they are connected. For greater clarity, let us assume that the network consists of b branches and n nodes. Kirchhoff's current law allows us to write n equations for the currents, while Kirchhoff's voltage law allows us to write an imprecise number m of equations for the voltages, where m is in fact the number of possible loops present in the network. To analyze a circuit it is not necessary to consider all these equations. In reality, it is easy to show (see TIME-DOMAIN NETWORK ANALYSIS) that only $(n - 1)$ equations for the nodes and $[b - (n - 1)]$ equations for the loops are in effect independent of each other. Every other additional equation would result in a linear combination of them and would not provide any additional information. Thus Kirchhoff's laws allow independent b equations to be written in the $2b$ unknowns $i_1, i_2, \ldots, i_b$ and $v_1, v_2, \ldots, v_b$.

It is important to emphasize that Kirchhoff's equations are linear, algebraic and homogeneous; hence the interaction that they describe is linear and instantaneous, and time derivatives and integrals do not appear in them. Every dependence on the past history of the circuit, every nonlinearity, as every "source" term, can only derive from the constitutive relations of the network elements. This is a point of extreme importance that is at the origin of many important properties of the circuit equations.

If we now join the b equations that express the constitutive relations and the independent b equations derived by applying Kirchhoff's laws, we obtain $2b$ equations in $2b$ unknowns. Naturally, such a system of equations will still be algebraic and linear only if the equations introduced from the constitutive relations are such. If this is so, the solution of the circuit equations does not present any particular problems, provided we ignore those deriving from the dimension of the resulting system of linear algebraic equations and thus from the number of branches of the network. Ultimately, it is only necessary to invert the coefficient matrix of the equation system that, because of the way in which it has formed, is certainly not singular in significant cases.

Naturally, it is not generally convenient to deal directly with the system of $2b$ equations in $2b$ unknowns. An immediate reduction of the equation system can be obtained by eliminating the currents or the voltages from the Kirchhoff b equations, making use of the constitutive relations. In this way, one obtains a system of b equations in b unknowns—voltages or currents.

A more significant reduction can be obtained by exploiting the fact that the sets of Kirchhoff equations for voltages and that for currents are independent of each other. Thus, it is possible to introduce new unknowns that by definition make it possible to satisfy one of the two sets of equations. This way of dealing with the problem leads to the introduction of the method of *node potentials,* where the $n - 1$ unknowns are the node potentials, and to the method of *loop currents,* where the $b - (n - 1)$ unknowns are, in fact, the loop currents (see TIME-DOMAIN NETWORK ANALYSIS, where these methods are dealt with in detail). Naturally, one can resort to the method of the node potentials or loop currents even when the one-ports are neither resistive nor linear, because they are based on a particular technique of reformulating the node and loop equations, which, we recall, are not dependent on the nature of the elements present in the network.

Where the one-ports of the network (even if they are still linear) are not all resistive, the overall system of circuit equations becomes of the algebraic-differential type. This is because the constitutive relations of the dynamic elements contain differential equations. For such networks, circuit equations alone are not sufficient to determine the solution of the circuit starting from a given time t_0. This is due to the fact that the behavior of a dynamic one-port for $t < t_0$ depends also on the state of the one-port at $t = t_0$: The state contains all the information of the one-port past history necessary to determine its future behavior. For example, the state variable of linear capacitors is the voltage, while that of linear inductors is the current. As a result the mathematical model of a circuit generally consists of the circuit equations and the initial state of all the dynamic elements. This aspect will be dealt with in detail later. However, even here, searching for the unknown functions $i_1(t), i_2(t), \ldots, i_b(t)$ and $v_1(t), v_2(t), \ldots, v_b(t)$ for $t_0 < t < +\infty$ does not present particular problems for linear circuits, as it falls within the well consolidated field of solving a system of linear algebraic-differential equations.

The whole system of circuit equations can be reduced to a system of linear ordinary differential equations of the first order, where the state variables of the circuit are the only unknowns; these are the state equations of the circuit. The possibility of effecting such a reduction has an important significance: The first-order time derivative of each state variable and all the other circuit variables at any time t depend only on the values that all the state variables and the sources of the circuit have at the same time t. By means of the state equations it is possible to study many of the properties of linear circuits without having necessarily to solve them.

The study of circuit equations is considerably complicated when the circuit contains nonlinear elements (for example, diodes, nonlinear inductor, nonlinear capacitors). This is because it is no longer possible to apply the superposition property, a property that is the basis of the whole analysis of linear circuits (see NETWORK THEOREMS).

In the case of nonlinear circuits with only resistive elements, the circuit equations are still algebraic, but a part of them is nonlinear. The properties of these equations can be very different from those of linear circuits. For example, it is possible to have several solutions that are compatible with the same sources. In general, the problem of their solution is very complicated and there are still many unsolved questions.

The analysis is further complicated (and considerably so) if the nonlinear circuit contains dynamic elements. Because many results can be found in the literature on nonlinear ordinary differential equations having the normal form $d\boldsymbol{x}/dt = \boldsymbol{H}(\boldsymbol{x}; t)$, where $\boldsymbol{x}$ is a vector whose components are the state variables of the circuit and $\boldsymbol{H}$ is a single-valued vectorial function, much attention has been given to determining when is it possible to reduce the equations of nonlinear dynamic circuits to a system of differential equations in normal form for the state variables.

In general, for nonlinear circuits it is not always possible to reduce the system of circuit equations to a system of state equations in normal form. It can happen that the first derivative of some state variables depends on the state variables of the circuit through multivalued functions. As we will then see, if this happens, it means that the circuit equations and the initial state alone are not sufficient by themselves to determine the circuit evolution. That is, the circuit model under consideration is "incomplete"; essential phenomena for an adequate representation of the "physical" circuit are absent in the circuit model because of the approximations introduced in the modeling phase.

CONSTITUTIVE AND KIRCHHOFF EQUATIONS

Every circuit element is described by a mathematical model that approximates the behavior of a physical device. It may happen that, depending on the application, the same physical device can be represented by different models. Generally, complex circuit elements are obtained by interconnecting several basic elements. First, we will consider the general features of constitutive relations of basic circuit elements to show how they affect the network equation structures. Later, we will deal with Kirchhoff's equations, which describe the interaction between single elements.

The first basic classification divides one-ports, and, more generally, circuit elements with several terminals, into resistive and dynamic.

Resistive One-Ports

A resistive one-port is a two-terminal circuit element described by a constitutive relation of the type

$$f_s(v, i) = 0 \tag{1}$$

Equation (1) defines an *instant* type relation between the current i and the voltage v, that is, the value of the voltage at any time t depends only on the value that the current assumes at that time and vice versa.

The linear resistor is a particular type of resistive one-port where

$$v = Ri \tag{2}$$

and the resistance R is a constant. If the relation between the voltage and the current is not linear, we say that the one-port is nonlinear. Examples of nonlinear resistive one-ports are diodes and varistors. For example, the exponential model of the junction diodes is described by the equation $i = I_s[\exp(v/nV_T) - 1]$ where I, V_T and n are characteristic parameters and a varistor model is described by $v = \alpha i^{\beta}$, with α and β constant. Also, ideal voltage and current sources are resistive one-ports, respectively, imposing the voltage and current.

Generally, it is not always possible to express the constitutive relation of a nonlinear resistive one-port by combinations of elementary functions. This difficulty can be overcome by observing that Eq. (1) can be represented graphically in the (v, i) plane. The curve thus obtained is the *characteristic curve* of the one-port. The points on such a curve represent the possible operating conditions of the one-port. A graphic representation allows the effective operating point to be easily determined for simple circuits of the type shown in Fig. 2(a). Figure 2(b) shows the characteristic curves of two one-ports. In this case the only operating point compatible with the two characteristics and Kirchhoff's laws (which in this simple case are reduced to $i_1 = i_2$ and $v_1 = v_2$) is given by the intersection of the two curves. It is to be noted that the normal convention has not been adopted for the second one-port. Naturally, there can be cases, which we will examine later, where the characteristic curves do not meet—or meet at more than one point. There are one-ports that have characteristic curves that are variable in the time; they are called time-varying one-ports. The function f_s for these one-ports depends also on time as a parameter. When the characteristic of the one-ports does not vary in time, they are said to be time-invariant. The resistor with a time-independent resistance and diodes are examples of time-invariant one-ports, while the switches are examples of time-variant one-ports.

A voltage (current) value is called admissible voltage (current) if there is at least one current (voltage) value such that Eq. (1) is satisfied. It can happen that not all the current and voltage values are admissible for a static one-port. It is sufficient to consider the characteristic of an ideal voltage or current source, or of the ideal diode. This is possible since Eq. (1) is only the representation of a physical device in the context of the circuit model that we are adopting. The presence of elements with characteristics of this type can give rise to problems of incongruency of the circuit model. For example, for the ideal diode, positive voltages and negative currents are inadmissible (appropriate references having been chosen).

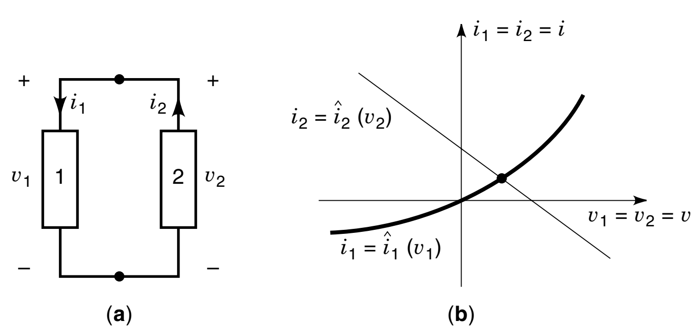

Figure 2. (a) Simple resistive circuit used to illustrate the graphic solution method; (b) the two one-port characteristic curves are superimposed on the same (v,i) plane.

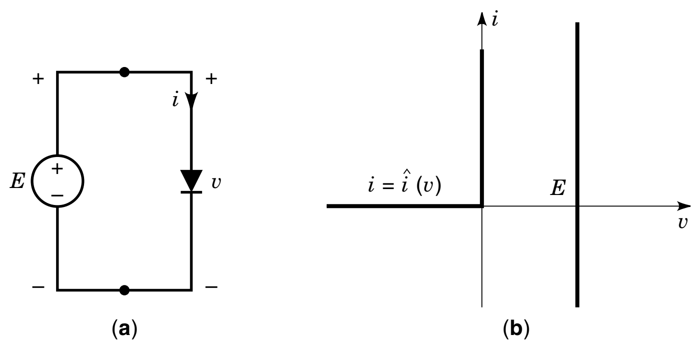

Figure 3. Example of an unsolvable resistive circuit for $E > 0$ [$i = \hat{\imath}(v)$ is the characteristic of an ideal diode].

Consequently, a circuit consisting of an ideal diode connected to an ideal voltage source may not admit solution as in the case shown in Fig. 3, where the two characteristic curves do not intersect each other. That is, the circuit model in question does not adequately describe the physical circuit it represents. To obtain a circuit model that admits one and only one solution, it would be sufficient to add a resistor with an arbitrarily small resistance in series to the voltage source.

If for any admissible voltage value there exists one current value that verifies Eq. (1), then this can be rewritten as

$$i = \hat{\imath}(v) \tag{3}$$

where $\hat{\imath}(\cdot)$ is a single-valued function defined for the admissible voltage values. These one-ports are called *voltage controlled.* If, however, for any admissible current value there is one voltage value that verifies Eq. (1), it can be rewritten as

$$v = \hat{v}(i) \tag{4}$$

where $\hat{v}(\cdot)$ is a single-valued function defined for the admissible current values. These one-ports are called *current controlled.* Naturally, if the one-ports are time-varying, then the functions $\hat{\imath}$ and $\hat{v}$ depend on the time.

There are static one-ports that are voltage and current controlled at the same time; for example, linear resistors (with $R \neq 0$ and $R \neq \infty$), junction diodes, zener diodes and varistors. In these cases the characteristic curves are strictly increasing, Eq. (1) can be rewritten indifferently as either Eq. (3) or Eq. (4) and the function g is the inverse of the function r and vice versa. One-ports that are only voltage or only current controlled have characteristic curves that are not strictly increasing. Therefore, it is clear that in the case where a one-port is voltage controlled only, the function g is not wholly invertible with respect to the voltage; in the case where a one-port is current controlled only, the function r is not wholly invertible with respect to the current.

The ideal current source, the open circuit and the tunnel diode, are examples of voltage-controlled one-ports. The ideal voltage source, the short-circuit and the thyristor with disconnected gate are examples of current-controlled one-ports. The ideal switch is current controlled when it is *on* and voltage controlled when it is *off.* A circuit consisting of a voltage-controlled one-port (such as the tunnel diode) connected to a current source will admit more than one solution; see Fig. 4. This is a further example of an inadequate circuit model. In this case to obtain a circuit that admits one and only one solution, it is sufficient to add a capacitor with an arbitrary small capacitance in parallel to the tunnel diode.

Dynamic One-Ports

The voltage and the current in dynamic one-ports are related through differential equations. Thus, the value of the voltage or of the current also depends at every instant on their time history. Basic dynamic one-ports are the capacitor and the inductor.

The constitutive relation of the capacitor is

$$f_c(v, q) = 0 \tag{5}$$

where q is the capacitor charge related to the current of the capacitor by the differential equation

$$i = \frac{dq}{dt} \tag{6}$$

or the integral equation

$$q(t) = \int_{t_0}^{t} i(\tau)\,d\tau + q(t_0), \qquad t \geq t_0 \tag{7}$$

Equation (5) is an instantaneous relation between the charge and the voltage of the capacitor, that is, the charge at any time t depends only on the voltage value at the same time t and vice versa. By contrast, the relation between the voltage and the current, because of Eq. (6) or (7), is not instantaneous; the charge, and hence the voltage, at any time $t > t_0$ depends on the charge value at $t = t_0$ and on the entire time history of the current on the interval $[t_0, t]$. The value of the charge $q(t_0)$ summarizes the whole electric history of the capacitor for $t < t_0$. Therefore, capacitors have memory. In general, Eq. (5) can be represented graphically in the (v, q) plane. This defines the characteristic curve of the capacitor; for this reason Eq. (5) is called the *characteristic equation* of the capacitor.

For linear capacitors we obtain

$$q = Cv \tag{8}$$

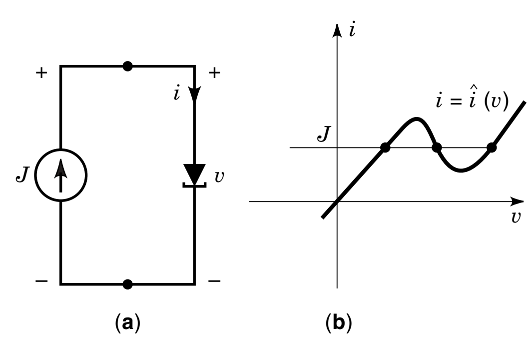

Figure 4. Example of a resistive circuit with three solutions [$i = \hat{\imath}(v)$ is the characteristic of a tunnel diode].

with C constant, and thus

$$i = C\frac{dv}{dt} \tag{9}$$

or, equivalently,

$$v(t) = \frac{1}{C}\int_{t_0}^{t} i(\tau)\,d\tau + v(t_0) \qquad t \geq t_0 \tag{10}$$

From Eq. (10), it is more evident that $v(t)$ depends on the time history of the current in the time interval (t_0, t) and on the voltage value at $t = t_0$. A capacitor having a constitutive relation that is nonlinear is called nonlinear. As an example of a nonlinear capacitor, we can consider the model of the varactor diode described by the equation $q = -3/2\ C_0V_0(1 - v/V_0)^{2/3}$ for $v < V$, where C_0 and V_0 are two constants.

The other fundamental dynamic one-port, which, with respect to the capacitor has a dual operation, is the inductor, defined by the constitutive relation

$$f_i(i, \phi) = 0 \tag{11}$$

where ϕ is the inductor flux, connected to the voltage v by the differential equation

$$v = \frac{d\phi}{dt} \tag{12}$$

or, equivalently, by the integral equation

$$\phi(t) = \int_{t_0}^{t} v(\tau)\,d\tau + \phi(t_0) \qquad t \geq t_0 \tag{13}$$

Equation (9) is an instantaneous relation between the flux and current of the inductor, that is the inductor flux at any time t depends only on the current at that time t and vice versa, while the relation between the current and the voltage is not of an instantaneous type: the flux, and hence the current, at any time $t > t_0$ depends on the flux value at $t = t_0$ and on the entire time history of the voltage on the interval $[t_0, t]$. For the inductor the value of the flux $\phi(t_0)$ summarizes the whole electric history for $t < t_0$. Therefore the inductors have memory like the capacitors. In general, Eq. (9) can be represented graphically in the (i, ϕ) plane and it defines the characteristic curve of the inductor; it is called the *characteristic equation* of the inductor.

For linear inductors we have

$$\phi = Li \tag{14}$$

with L constant, and thus

$$v = L\frac{di}{dt} \tag{15}$$

or, equivalently,

$$i(t) = \frac{1}{L}\int_{t_0}^{t} v(\tau)\,d\tau + i(t_0) \qquad t \geq t_0 \tag{16}$$

From Eq. (16), it is evident that $i(t)$ depends on the time history of the voltage in the interval (t_0, t) and on the current value at $t = t_0$. An inductor whose characteristic is nonlinear is called nonlinear. For example, the behavior of an inductor realized on a ferromagnetic core can be described by the nonlinear constitutive equation $i = a\phi + b\phi^3$ if the hysteresis phenomena are negligible, and a Josephson junction can be represented by the equation $i = I_0 \sin(\phi/\Phi_0)$; a, b, I_0, and Φ_0 are characteristic constants.

It is possible to extend all the concepts introduced for static one-ports to the characteristic curves of capacitors and inductors (time-variant capacitors, time-variant inductors, voltage and charge admissible values, charge and voltage-controlled capacitors, current and flux admissible values, flux and current-controlled inductor. For example, the varactor diode model given in the foregoing describes a voltage-controlled capacitor with $v < V_0$ admissible voltages. For more details on linear and nonlinear one-ports we refer the reader to LINEAR NETWORK ELEMENTS and to NONLINEAR NETWORK ELEMENTS.

Circuit Elements With More Than Two Terminals

An element with $M + 1$ terminals is characterized by $M + 1$ currents $i_1, i_2, \ldots$ and $(M + 1)M/2$ voltages $v_{12}, v_{13}, \ldots$; the current i_k is associated with the kth terminal and the voltage v_{kh} is associated with the two terminals k and h. The reference directions of the currents are those entering the element, and for the voltages those going from the terminal k to the terminal h. An example is shown in Fig. 1(b), where $(M + 1) = 3$, and in Fig. 1(c), where $(M + 1) = 4$.

In agreement with Kirchhoff's law for currents, at any time the following equation has to be verified:

$$\sum_{h=1}^{M+1} i_h = 0 \tag{17}$$

Therefore, only M currents are independent. Similarly, in agreement with Kirchhoff's law for voltages, it has to be

$$v_{kh} = v_{kp} - v_{ph} \tag{18}$$

and, thus, as for currents, only M voltages are independent. To identify a set of independent currents and voltages one may choose a reference terminal, for example, the terminal labeled "$M + 1$," and consider the currents of the first M terminals $i_1, i_2, \ldots, i_M$ and the voltages $v_1, v_2, \ldots, v_M$ between the first M terminals and the reference terminal (in Fig. 1(b) an element with three terminals is considered, $M = 2$; the terminal labeled "3" is chosen as reference terminal). The variables thus obtained are all independent of each other and any other electrical variable of the element under consideration can be expressed as a linear combination of them by means of Eqs. (17) and (18). They are called descriptive variables of the element. In general, the constitutive relations link the descriptive currents and the descriptive voltages of the single elements of a circuit. This method is one among many other possible methods for identifying a set of independent currents and voltages of a multi-terminal element (see MULTIPOLE AND MULTIPORT ANALYSIS).

The one-port is an element with two terminals ($M + 1 = 2$) and is characterized by a single descriptive current and a single descriptive voltage [Fig. 1(a)]. The three-pole is an element with three terminals ($M + 1 = 3$) and is characterized by two descriptive currents and two descriptive voltages [Fig. 1(b)]. The four-pole is an element with four terminals, and so on. Examples of three-poles are transistors and three-phase voltage sources, while among four-poles there are voltage- and current-controlled sources, ideal transformers, operational amplifiers, gyrators, and mutual inductances.

Typological analysis undertaken for the constitutive relations of one-ports can be extended to constitutive relations of elements with $M + 1$ terminals without any problem of principle. Consider, for example, a resistive three-pole and let i_1, i_2 and v_1, v_2 be, respectively, the descriptive currents and voltages [Fig. 1(b)]. The constitutive relation is of the type

$$\begin{aligned} f_1(v_1, v_2, i_1, i_2) = 0 \\ f_2(v_1, v_2, i_1, i_2) = 0 \end{aligned} \tag{19}$$

where f_1 and f_2 are two functions, generally nonlinear. For resistive linear three-poles the relations between the descriptive variables are linear. In general, as Eq. (19) may not be described by means of a finite combination of elementary functions, it might be useful to represent them graphically, considering some of the descriptive variables as parameters. An appropriate choice of reference terminals can simplify the representation of the characteristic curves considerably. As with the one-port, the possibility of making Eq. (19) explicit with reference to two descriptive variables depends on the control variables of the element.

For dynamic n-terminal elements, however, only the relations between fluxes and currents or charges and voltages are of instantaneous kind. For example, the generic descriptive voltage v_k of a multi-terminal element of inductive type (e.g., coupled inductors), is equal to the time derivative of the flux ϕ_k. Besides, ϕ_k is related to all the descriptive currents of the element through an algebraic relation.

The operation of an element with more than two terminals may be conditioned by the topology of the circuit into which it is inserted. The simplest and at the same time the most significant example is that in which an element with $2M$ terminals is connected to M distinct circuits, each of which is representable as a one-port [in Fig. 1(c) an example with $M = 2$ is considered]. In this case the current that enters a given terminal is equal to the current that exits from another terminal. Each terminal couple having this property is called a port of the element, and the element in this operating state is called an M-port. Each port of an M-port is characterized by the current circulating in one of the two terminals and by the voltage between the two terminals (for every port we can adopt the normal convention). It is clear that in this operating condition the constitutive relations must be taken as those that relate the currents and voltages of the single ports of the element. We should remember that there are also elements with $2M$ terminals that can operate only as M-ports. The ideal transformer and the ideal operational amplifier are two examples of elements with four terminals that can operate only as two ports. For more details on linear and nonlinear multi-terminal elements we refer the reader to LINEAR NETWORK ELEMENTS and to NONLINEAR NETWORK ELEMENTS.

Kirchhoff's Equations

In the beginning of this article Kirchhoff's laws were recalled in reference to a circuit consisting of only one-ports. Their extension to a circuit with more than two terminals is straightforward if one is referring to the circuit graph. Such a circuit graph can be formed by associating N nodes and ($N - 1$) branches with elements with N terminals, for as many descriptive variables as there are (a reference node having previously been chosen for each element), and to each M-port $2M$ nodes and M branches, for as many ports as extant. Then, let us consider a circuit with b branches n nodes and let v_k, i_k, $k = 1, \ldots, b$ be the unknowns of the circuit. As we have mentioned earlier, only ($n - 1$) equations at the nodes and ($b - n + 1$) equations at the loops are linearly independent. To simplify the discussion we are implicitly assuming that the circuit graph is connected. Where this is not so, what has been said holds true for every single connected part of the whole graph.

To determine ($n - 1$) linear-independent equations at the nodes one need only consider any ($n - 1$) nodes of the circuit, but determining ($b - n + 1$) linear-independent equations at the loops is not as easy as it is for the nodes. In general, ($b - n + 1$) linearly independent equations at the loops can be determined by applying Kirchhoff's law for voltages at a set of fundamental loops of the circuit.

The concept of the fundamental loop, which we will recall briefly here, is linked to the tree and the co-tree of a circuit. We recall that a tree is a subset of branches that pass through all the nodes of the circuit without forming loops. Even if a circuit has different trees, each tree consists always of ($n - 1$) branches; the remaining ($b - n + 1$) branches of the graph constitute the co-tree. Therefore, each branch of the co-tree belongs to a loop consisting of itself and branches of the corresponding tree. Such a loop is called a fundamental loop. Thus for every choice of the tree there exist ($b - n + 1$) fundamental loops. It is evident that equations obtained by applying Kirchhoff's loop law to ($b - n + 1$) fundamental loops are linearly independent because each of them exclusively contains the voltage of the corresponding co-tree branch.

Thus Kirchhoff's laws allow independent b equations to be written in $2b$ unknown voltages and currents. These equations can be written in compact form using the vector notation. To this end we define the column vectors

$$\begin{aligned} \boldsymbol{i} = [i_1, \ldots, i_b]^{\mathrm{T}} \\ \boldsymbol{v} = [v_1, \ldots, v_b]^{\mathrm{T}} \end{aligned} \tag{20}$$

representing, respectively, the circuit currents and voltages. Then the linearly independent Kirchhoff b equations can be rewritten using two matrices A and B whose elements are 0, +1, or −1

$$\boldsymbol{Ai} = \boldsymbol{0} \tag{21}$$

$$\boldsymbol{Bv} = \boldsymbol{0} \tag{22}$$

The $(n - 1) \times b$ matrix A is a reduced incidence matrix of the circuit and the $(b - n + 1) \times b$ matrix B is a fundamental loop matrix (for more details see TIME-DOMAIN NETWORK ANALYSIS). The system of Eqs. (21) and (22) is a maximal independent set of Kirchhoff equations. The ($n - 1$) independent equations for the currents can also be obtained with a fundamental cut set of the circuit; in this case A is a fundamental cut set matrix.

THE SYSTEM OF CIRCUIT EQUATIONS

As we have seen, the "dynamics" of a circuit is described by a maximal set of Kirchhoff independent equations and by the constitutive relations of the single circuit elements, which by their very nature are independent of each other. The Kirchhoff equations are linear, algebraic and homogeneous and thus the interaction they describe is always instantaneous and linear. By contrast, the constitutive relations of the circuit elements can radically modify the nature of the overall system of equations by transforming it into algebraic-differential and nonlinear. From this viewpoint the most interesting classifications are resistive elements and dynamic elements, linear elements, and nonlinear elements.

Let us, for the moment, consider a circuit consisting of linear and time-invariant capacitors and inductors and resistive one-ports that can, instead, be nonlinear and time variant. Later, we will refer to a circuit consisting of n_c capacitors and n_i inductors, as well as n_s resistive one-ports. The number of circuit branches will then be $b = n_c + n_i + n_s$. For the sake of simplicity, we will number the circuit branches in the following order: the capacitor branches are numbered from 1 to n_c; those corresponding to the inductors from $n_c + 1$ to $n_c + n_i$; and, finally, those corresponding to the resistive one-ports from $n_c + n_i + 1$ to $n_c + n_i + n_s = b$. For this type of circuit the system of circuit equations is made up of the b Kirchhoff Eqs. (21) and (22), n_c first-order differential equations of the type of Eq. (9), n_i first-order differential equations of the type of Eq. (15), and n_s algebraic equations of the type of Eq. (1).

The presence of linear and time-invariant dynamic one-ports in the network introduces first-order differential equations, and, therefore, it is necessary to know the initial values of the variables that appear in them under the derivative operations. In fact, from the integral form Eq. (10) of the capacitor constitutive relation, it is clearly seen that, in order to know the voltage value at any given time t, it is not enough to know the current in the interval (t_0, t); one also needs to know the voltage or the charge at $t = t_0$. The initial condition $v(t_0)$ summarizes the effects of the entire past history of the capacitor (from $t = -\infty$ to $t = t_0$) on the present value of $v(t)$ for $t > t_0$. From the physical viewpoint this is due to the fact that capacitor voltage determines the energy value E_C stored in the one-port at every instant throughout the relation (see NETWORK THEOREMS).

$$E_c(t) = \frac{1}{2}Cv^2(t) = \frac{1}{2C}q^2(t) \tag{23}$$

Similarly, for the inductor, it is necessary to know the current value or the flux at the initial instant. We will call the capacitor voltages and the inductor currents *state variables* of the circuit under examination insofar as they describe the initial state. Naturally, the same role can be played by the capacitor charge and the inductor flux, respectively.

In the final analysis the state of the circuit at the generic instant t_1 summarizes the whole electric history of the circuit. No matter how the circuit has been brought to its state at the instant t_1, its subsequent behavior will depend only on the state value at $t = t_1$ and on the independent sources (see TIME-DOMAIN NETWORK ANALYSIS). The set of functions $\{v_k(t), i_k(t); k = 1, . . ., b\}$ defined in the interval $t_0 \le t < \infty$, is called the circuit solution in the interval $t_0 \le t < \infty$ if it is a solution of the circuit equations and it is compatible with the initial conditions for the state variables.

Resistive Circuits

Even if circuits without inductors and capacitors are a particular case, their study is of fundamental importance in circuit theory. In fact, as we will see, in the study of dynamic circuits one often resorts to auxiliary circuits consisting of resistive elements related to the dynamic circuits being studied. When an electric network is without dynamic elements, that is, $n_c = n_i = 0$, the circuit is said to be *resistive* and the circuit equations are algebraic and generally nonlinear. Even if the sources present are time variant, the solution at each instant has no memory of its operating point at preceding instants. For these circuits the time appears as a parameter in the equations. Therefore the relations between the overall circuit variables and the voltages and currents imposed by the independent sources are instant type relations. If the nonlinearities of the one-ports present allow it, the values of the $2b$ unknowns can be determined univocally by solving the relative system of algebraic equations instant by instant. In the case of linear one-ports—apart from pathological cases, which we will refer to shortly—this is always possible. In the presence of nonlinearities, the circuit may have no solutions, one solution or several solutions.

The pathological situations in which a resistive network, even linear, may not admit solutions, are those where there is incongruency or dependence between the constitutive relations of some circuit elements and the Kirchhoff equations regulating the interaction. In general, bearing in mind what has been said in the introduction to this article about the interaction between each single element and the rest of the network, we may say that cases of incompatibility between Kirchhoff's equations and the constitutive relations may occur when there are elements for which there are inadmissible voltages and/or currents. The case of two ideal sources of voltage $e_1(t)$ and $e_2(t)$ connected in parallel is emblematic. It is evident that no solution is possible when there is incongruency [i.e., $e_1(t) \neq e_2(t)$], while the number of solutions is infinite, at least so far as the currents of the sources are concerned, when there is dependence [i.e., $e_1(t) = e_2(t)$]. Naturally, the incongruency or the dependence is entirely in the model used to represent the real circuit; a more realistic model that includes the "internal" resistance of the sources would resolve every problem of incongruency or dependence. In general, the existence and the uniqueness of the solution of a linear resistive one-port network are guaranteed if there are no loops consisting of voltage sources only, no cut-sets consisting of current sources only, and the resistances of the circuit resistors are strictly positive. If controlled sources, ideal transformers, gyrators, nullators, and norators are also present in the circuit, further pathological situations of a different nature may be present. There can also be cases where solutions do not exist because of the presence of nonlinear elements. A typical case is that of an ideal current or voltage source that feeds an ideal diode, as we have seen in the foregoing. In this case the incongruency disappears if more realistic models of source or diode are also adopted. Hereafter we will assume that such situations are absent. Finally, it should be said that even where the system of circuit equations admits more than one solution, such behavior can be attributed to a weakness of the model. In reality, it is not possible to determine which of the solutions is the one for the real circuit if other factors, able to provide a single solution, are not forthcoming. For nonlinear resistive networks the existence and uniqueness of the solution are certainly guaranteed if, besides the topological hypotheses on sources made for the linear case, all the current and voltage values are admissible for the nonlinear resistors, and their characteristic curves are strictly increasing. If the first hypothesis is not satisfied, there can be no solution; if the second one is not satisfied, the solution may not be unique. However, the question of existence and uniqueness of the solution for nonlinear resistive circuit requires further discussion which we cannot undertake here. See Ref. 1 where the problem is considered in detail.

To solve nonlinear resistive circuit equations it is generally necessary to resort to approximate methods. Only for linear circuits is it possible to determine the solution by analytical methods, as, for example, Gauss's method. However, many properties of the solution of nonlinear resistive circuits may be determined without necessarily having to resolve the circuit equations.

Dynamic Circuits and Global State Equations

Returning to circuits with dynamic one-ports, we observe that, in general, the system of $2b$ circuit equations of the network is of the algebraic-differential type

$$\begin{cases} C\dfrac{d\boldsymbol{v}_c}{dt} = \boldsymbol{i}_c \\ L\dfrac{d\boldsymbol{i}_i}{dt} = \boldsymbol{v}_i \end{cases} \tag{24}$$

$$\mathbf{0} = \boldsymbol{F}(\boldsymbol{v}, \boldsymbol{i}; t) \tag{25}$$

where $\boldsymbol{v}_c = (v_1, \ldots, v_{n_c})^{\mathrm{T}}$, $\boldsymbol{i}_i = (i_{n_c+1}, \ldots, i_{n_c+n_i})^{\mathrm{T}}$ are the vectors that represent, respectively, the capacitor voltages and the inductor currents, that is, the state variables of the circuit, $C = \mathrm{diag}(C_1, \ldots, C_{n_c})$, $L = \mathrm{diag}(L_{n_c+1}, \ldots, L_{n_c+n_i})$ are two diagonal matrices, respectively, representative of the capacitances and inductances of the circuit, $\boldsymbol{i}_c = (i_1, \ldots, i_{n_c})^{\mathrm{T}}$, $\boldsymbol{v}_i = (v_{n_c+1}, \ldots, v_{n_c+n_i})^{\mathrm{T}}$ are the vectors that represent, respectively, the capacitor currents and the inductor voltages, and $\boldsymbol{v}$, $\boldsymbol{i}$ are representative of all the network voltages and currents. The system of algebraic equations (25) consists of the b Kirchhoff equations (21) and (22) and $b - (n_c + n_i)$ characteristic equations of the resistive one-ports, that is, $b - (n_c + n_i)$ equations of the type of Eq. (1).

The system of $2b$ Eqs. (24) and (25) can be reduced to the canonical form wherein only the state variables appear as unknowns. That this is possible is evident from the following considerations. If we assign the state variables at a given time, that is, $n_c + n_i$ circuit variables, the overall system of Eqs. (24) and (25) can be interpreted as a system of $2b$ equations, which still has $2b$ unknowns, where now the derivatives of the state variables have assumed the role of unknowns instead of the state variables themselves. Such a system can be resolved to furnish the values of the derivatives of the state variables at that given time. In other words, it is possible to express the state variable derivatives as function of the state variables themselves, and this constitutes the canonical form to which we referred. Operatively, this result can be obtained as follows: the system of algebraic Eqs. (25) is resolved by expressing the $2b - (n_c + n_i)$ nonstate variables of the circuit as function of the $n_c + n_i$ state variables. Then the expressions of the capacitor currents $\boldsymbol{i}_c$ and the inductor voltages $\boldsymbol{v}_i$ so obtained are substituted in Eqs. (24).

To clarify the matter it is useful to refer to a concrete example of the type shown in Fig. 5. All the voltages and currents have been ordered in accordance with the convention we have previously adopted. The equations that describe the dynamics of the circuit are

$$\begin{cases} C\dfrac{dv_1}{dt} = i_1 \\ L\dfrac{di_2}{dt} = v_2 \end{cases} \tag{26}$$

$$\begin{cases} 0 = i_1 + i_2 - i_3 \\ 0 = i_3 + i_4 \\ 0 = v_1 - v_2 \\ 0 = v_2 + v_3 - v_4 \\ 0 = v_3 - R_3 i_3 \\ 0 = v_4 - e(t) \end{cases} \tag{27}$$

The system of circuit Eqs. (26) and (27) consists of 8 equations in 8 unknowns; Eq. (26) expresses, respectively, the constitutive relations of the capacitor and the inductor. The first four equations of Eq. (27) constitute the maximal set of linearly independent Kirchhoff equations and the remaining two equations are the constitutive relations of the resistive one-ports present in the circuit—the resistor and the voltage source.

To reduce the differential algebraic Eqs. (26) and (27) into canonical form, one need only determine the expression of the capacitor current i_1 and the inductor voltage v_2, from Eq. (27), as function of the sole state variables v_1, i_2 and the voltage source $e(t)$. To this end it is sufficient to consider the voltage v_1 and the current i_2 as assigned, and interpret Eq. (27) as a system of 6 equations in the 6 unknowns i_1, v_2, i_3, v_3, i_4, v_4. The solution of this system is

$$\begin{aligned} i_1(t) &= \frac{e(t) - v_1(t)}{R} - i_2(t) \\ v_2(t) &= v_1(t) \end{aligned} \tag{28}$$

and

$$\begin{aligned} i_3(t) &= -i_4(t) \\ v_3(t) &= e(t) - v_1(t) \\ i_4(t) &= \frac{v_1(t) - e(t)}{R} \\ v_4(t) &= e(t) \end{aligned} \tag{29}$$

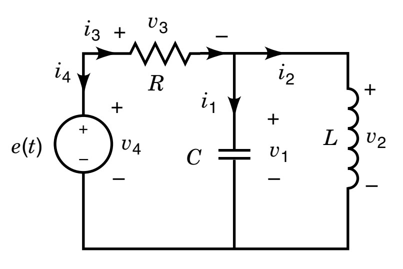

Figure 5. Simple dynamic circuit used to illustrate the determination of the state equations in normal form.

The result obtained is very significant; the "nonstate" variables can be expressed at each instant as function of the sole state variables and the voltage source. The result further justifies the name of state variables given to v_1 and i_2; their knowledge at a given time in fact implies the knowledge of all the other circuit variables at the same time and thus univocally determines the "state" of the circuit. Substituting Eq. (28) in the differential Eqs. (26), the equations for the state variables are obtained

$$\frac{dv_1}{dt} = -\frac{v_1}{RC} - \frac{i_2}{C} + \frac{e(t)}{RC}$$
$$\frac{di_2}{dt} = \frac{v_1}{L} \tag{30}$$

This is a system of two first-order ordinary differential equations in *normal form,* that is, in general, of the type

$$\frac{dx_1}{dt} = H_1(x_1, \ldots, x_N; t)$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots \tag{31}$$
$$\frac{dx_N}{dt} = H_N(x_1, \ldots, x_N; t)$$

where x_k is the generic state variable, the value of which is known at instant $t = t_0$, and $H_1, \ldots, H_N$ are single-valued functions defined for every $x_1, \ldots, x_N$, where $N = n_c + n_i$. The differential equations that regulate the dynamics of the circuit state variables, written in the normal form of Eq. (31), are called *global state equations* of the circuit. Global state Eqs. (30) are linear and with constant coefficients because the circuit under examination consists of linear and time invariant one-ports.

Naturally, the presence of a nonlinear one-port of particular nature can hinder the reduction of the circuit equation to a system of normal form state equations. Let us again consider the circuit in Fig. 5 where, however, we have substituted the linear resistor with a nonlinear one. If the nonlinear resistor is both voltage and current controlled (for example, a junction diode) or only voltage controlled [for example, a tunnel diode; see Fig. 4(b)], its constitutive relation is of type $i_3 = \hat{\imath}(v_3)$, where the function $\hat{\imath}(v_3)$ is a single-valued function. Here, too, it is possible to express the circuit variables as functions of the state variables by means of a univocal relation of instantaneous type, resolving the algebraic part of the circuit in respect to the nonstate variables. In this case the global state equations are nonlinear and are

$$\frac{dv_1}{dt} = \frac{\hat{\imath}[e(t) - v_1(t)]}{C} - \frac{i_2}{C}$$
$$\frac{di_2}{dt} = \frac{v_2}{L} \tag{32}$$

If, instead, the nonlinear resistor is controllable only in the current (for example, a thyristor with disconnected gate) the constitutive relation is of the type $v_3 = \hat{v}(i_3)$, where the function $\hat{v}(i_3)$ is not globally invertible, that is, it is not invertible

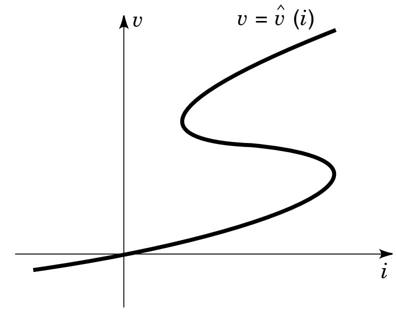

Figure 6. Characteristic curve of a current-controlled one-port.

for every value of i_3 (see Fig. 6). In this case the capacitor current i_1 can be expressed as a function of the state variables only in the implicit form through the nonlinear equation

$$\hat{v}[i_1(t) + i_2(t)] + v_1(t) - e(t) = 0 \tag{33}$$

Thus the state equations are

$$\hat{v}\left[C\frac{dv_1}{dt} + i_2(t)\right] = -v_1(t) + e(t)$$
$$\frac{di_2}{dt} = \frac{v_2}{L} \tag{34}$$

As the function $\hat{v}(i_3)$ is not globally invertible, the first equation of the system in Eq. (34) cannot be rewritten in normal form. This is a direct consequence of the fact that in this circuit even though all the nonstate circuit variables are linked to the state by means of the instantaneous relations imposed by the algebraic part of the circuit equations, the state does not determine them univocally. From Eq. (34) it is evident that the capacitor current can be expressed in general as function of the state variable and the voltage source only through multivalued functions. In this case, starting from the assigned initial conditions, the solution of the state equations cannot be unique because the time derivative of the capacitor voltage is a multivalued function of the state variables and the voltage source. From the physical point of view this is another very interesting example of an incomplete model. The incongruency can be resolved by adding an inductor with an arbitrary small inductance in series with the nonlinear resistor. In this way the current i_3 also becomes a state variable.

It is clear, then, that to reduce the circuit equations to a system of global state equations, one needs to be able to express the nonstate circuit variables, and in particular the capacitor currents and the inductor voltages, as functions of the state and source variables by means of single-valued functions. This is the same as resolving a resistive circuit obtained from the actual circuit by substituting each capacitor with a voltage source whose voltage is equal to that of the capacitor and each inductor with a current source whose current is equal to that of the inductor. We call this circuit *associated resistive circuit;* interested readers may refer to Ref. 1. Thus it is evident that the necessary and sufficient condition to express the nonstate circuit variables of a dynamic circuit as functions of the state variables by means of single-valued functions is that the associated resistive circuit admits one and only one solution for every admissible state value. In this way the possibility of reducing the circuit equations to a system of global state equations is brought back to the study of the existence and uniqueness of the solution for a resistive circuit.

If the dynamic circuit is linear the associated resistive circuit also is linear. In this case, in agreement with what has been said about linear resistive circuits, it is possible to reduce the circuit equations to a system of global state equations if: (1) the dynamic circuit does not have loops consisting of only capacitors and independent voltage sources and cutsets consisting of only inductors and independent current sources; and (2) all the circuit resistances are strictly positive. Naturally the presence of controlled sources (or elements that can be returned to the controlled sources) of norators or nullators in the associated resistive circuit implies that the very difficulties that we mentioned previously regarding the existence and the uniqueness of solutions are reflected in the possibility of writing the state equations in normal form. Even if the nonlinear case does not lend itself to a systematic general treatment, the preceding criterion continues to be valid if the characteristics of the nonlinear resistors are defined for all voltage and current values and are strictly increasing. By contrast, if the dynamic circuit also contains resistors with nonmonotone characteristics, nonlinear multi-terminal elements as transistors, and operational amplifiers, then the associated resistive circuit can have more than one solution, as we have demonstrated in the example; thus, a global state equation system may not exist.

Reference 2 proposes algorithms for the formulation of circuit state equations relevant to linear circuits that can easily be implemented by a computer, based on the solution of the associated resistive circuits. Again Ref. 2 indicates a criterion that allows the verification of the existence or nonexistence of global state equations $d\boldsymbol{x}/dt = \boldsymbol{H}(\boldsymbol{x}; t)$ for a nonlinear dynamic circuit and proposes an algorithm for the numerical evaluation of the vector-valued function $\boldsymbol{H}$.

Once the global state equations are obtained it is necessary first to ascertain the existence and uniqueness of the solution for fixed initial conditions. The problems connected with the existence and uniqueness of the solution of the global state equations are also dealt with in detail in NONLINEAR DYNAMIC PHENOMENA IN CIRCUITS. Here, we limit ourselves to recalling that in the case of linear state equations the solution exists and is unique regardless of the initial conditions for the state. The existence and uniqueness of the solution in the nonlinear case are ensured if the characteristics of the nonlinear oneports are regular and every resistive element absorbs electrical power as soon as the current or the voltage exceeds a certain value; interested readers may refer to Refs. 1–3.

The solution of the state equations can be determined analytically only in the case of linear and time-invariant circuits. In this case the circuit state equations are of the type

$$\frac{d\boldsymbol{x}}{dt} = A\boldsymbol{x} + \boldsymbol{d}(t) \tag{35}$$

where A is the so-called dynamic matrix and $\boldsymbol{d}(t)$ is a known term dependent on the sources. The properties and the solution techniques of linear time-invariant dynamic circuits are also illustrated in TRANSIENT ANALYSIS. Here, we will limit ourselves to recalling that two different terms can be distinguished for each circuit variable because of the linearity; one that goes to zero if all the initial conditions for the state variables are annulled, which is called *free evolution,* and one that is annulled if the sources are all turned off, which is called *forced evolution.* Each of these can be represented through the $n_c + n_i$ natural modes of the circuits $e^{\lambda_i t}$, where λ_i $i = 1, \ldots, n_c + n_i$ are the natural, generally complex, frequencies of the circuit (we have implicitly assumed that the natural frequencies are all distinct); the natural circuit frequencies are the eigenvalues of the dynamic matrix A. For "dissipative" circuits the free evolution tends to zero when $t \to \infty$, while the forced one will depend on the waveforms of the independent sources of the network. For example, when all the circuit sources are constant, the forced evolution term for every variable tends towards a constant function for $t \to \infty$. Where all the sources are sinusoidal and isofrequential, the forced evolution term of each variable tends towards a sinusoidal waveform with the same frequency as the sources for $t \to \infty$. By contrast, for time-variant and/or nonlinear circuits the solution cannot be determined in closed form. In these cases one has to use approximate solution methods, such as perturbative techniques (see NETWORK ANALYSIS USING LINEARIZATION) and more generally numerical methods (see CIRCUIT ANALYSIS COMPUTING and PERIODIC NONLINEAR CIRCUITS). The general properties of the solutions of circuit state equations, even nonlinear, can be determined without having to resolve them. In the last thirty years many analysis techniques have been developed for predicting the qualitative behavior of the solutions of state equations of a circuit merely starting from their structures and properties (see NONLINEAR DYNAMIC PHENOMENA IN CIRCUITS, QUALITATIVE ANALYSIS OF DYNAMIC CIRCUITS, and Refs. 1–3).

Dc Operating Points

Let us now consider time-invariant dynamic circuits with only stationary sources. These circuits are called *autonomous circuits.* An autonomous circuit generally admits stationary solutions. These solutions are called *dc operating points* of the circuit. The fundamental characteristic of these solutions is that the current in the capacitors and the voltage across the inductors are zero at each instant. The dc operating points of an autonomous circuit can be determined in various ways. It is possible to determine the stationary solutions of the state equations first, and then determine the remaining variables through their instantaneous relations with the state variables. The stationary solutions of the state equations are obtained by making all the system derivatives in Eqs. (31) zero, and thus they are the solutions of the algebraic equation system

$$\begin{aligned} &\boldsymbol{H}_1(x_1, \ldots, x_N) = \boldsymbol{0} \\ &\ldots\ldots\ldots\ldots\ldots\ldots \\ &\boldsymbol{H}_N(x_1, \ldots, x_N) = \boldsymbol{0} \end{aligned} \tag{36}$$

The dc operating points of an autonomous circuit are also the solutions of the resistive circuit, obtained by substituting an open circuit for each capacitor and a short-circuit for each inductor that can be considered as current and voltage sources turned off, respectively.

In general, except for the cases of model incongruency, which we have already discussed for resistive circuits, an autonomous circuit will admit one or more dc operating points.

If the resistors of the autonomous circuit have strictly increasing characteristics and topological hypotheses on sources are verified, then there is only one dc operating point. By contrast, there can be more than one dc operating point if the circuit also contains resistors that are only voltage or current controlled. The possibility of having more than one dc operating point in a dynamic circuit does not mean, in fact, that the circuit model is ill-posed. Which of the dc operating points is effectively reached depends on the stability of the corresponding stationary solution of the state equations and on the initial value of the state (see QUALITATIVE ANALYSIS OF DYNAMIC CIRCUITS).

The determination of dc operating points assumes particular significance when besides stationary sources the network also contains variable sources with small amplitudes. In such cases nonlinear characteristic curves of the network can be approximated with straight lines passing through the dc operating points of the elements and tangent to the characteristic curves. In this way it is possible to determine the circuit solution by superimposing the solution of two distinct problems, the first nonlinear but static, and the second, dynamic but linear.

GEOMETRIC DESCRIPTION OF THE EVOLUTION OF A DYNAMIC CIRCUITS

The structure of circuit Eqs. (24) and (25) clearly shows that dynamic and resistive one-ports play two different roles in the mechanism determining the circuit time evolution. In particular, the constitutive relations of resistive one-ports play a role similar to that played by the Kirchhoff equations. In analogy with mechanics, algebraic Eqs. (25) can be considered as *holonomic constraints,* generally time variant, on the voltages and currents of the circuit, while differential Eqs. (24) recall the motion equations (see Refs. 1 and 4). To understand this parallel better we will use a geometric approach. The voltages and currents of the circuit elements identify a point of coordinates $(v_1, \ldots, v_b, i_1, \ldots, i_b)$ in a space of $2b$ dimensions. Because the voltages and currents must be compatible with the holonomic constraints given by Eqs. (25), consisting of the Kirchhoff equations and the constitutive relations of the resistive elements, the point $(v_1, \ldots, v_b, i_1, \ldots, i_b)$ is forced to move on a surface of the space $\boldsymbol{R}^{2b}$. Let us call a point $P = (v_1, \ldots, v_b, i_1, \ldots, i_b)$, compatible with the holonomic constraints (25), the *circuit operating point,* and in analogy with mechanics, let us call the set of all circuit operating points the *configuration space* of the circuit, which we will indicate with Σ. Normally, if there is no dependence or contradiction between the Kirchhoff equations and the constitutive relations of the resistive one-ports, the dimension of the configuration space is $N = n_c + n_i$. In such cases the circuit solution is represented by the motion of the operating point $P(t)$ on the surface Σ; the motion laws are given by the system of differential Eqs. (24) which describe the operation of the dynamic one-ports. Returning to the analogy of mechanics once again and to understand these definitions better, one may imagine a body sliding along an inclined plane under the action of gravity. In this case the configuration space is a plane of the three-dimensional space.

It is evident that the trajectory of the operating point in the configuration space is univocally determined if, for every value of the state circuit variables $\boldsymbol{v}_c$ and $\boldsymbol{i}_i$, there is one and only one operating point P compatible with it. In other words, the projection of the circuit configuration space Σ onto the state space of the circuit determines a one-to-one correspondence. This means that the resistive circuit associated with the dynamic circuit admits one and only one solution, and thus it is possible to express the nonstate variables as functions of state variables by means of a single-valued function. If the projection of the circuit configuration space onto the state space does not determine a one-to-one correspondence, there may be more than one trajectory of the circuit operating point compatible with the initial conditions for the state; thus the circuit model is *ill-posed.* As an example let us consider the circuit illustrated in Fig. 7. For the sake of simplicity, this consists of a sole dynamic element. The resistor N is described by the constitutive relation $f(v_4, i_4) = 0$. The circuit equations are then

$$L\frac{di_1}{dt} = v_1 \tag{37}$$

$$\begin{cases} 0 = i_1 - i_3 \\ 0 = i_2 + i_3 \\ 0 = i_1 - i_4 \\ 0 = v_1 - v_2 + v_3 + v_4 \\ 0 = v_2 - E \\ 0 = v_3 - R_3 i_3 \\ 0 = f(v_4, i_4) \end{cases} \tag{38}$$

An operating point $P = (v_1, v_2, v_3, v_4, i_1, i_2, i_3, i_4)$ of this circuit is a point of the eight-dimensional space $\boldsymbol{R}^8$, compatible with the holonomic constraints determined by the algebraic equations (38). The set of Eqs. (38) consists of 7 independent and compatible equations, and thus has infinite solutions. The solutions of these equations form a surface of dimension $N = 8 - 7 = 1$, that is, a curve of the space $\boldsymbol{R}^8$. This is the configuration space of the circuit under examination.

The only way to visualize this configuration space is to project it onto two-dimensional planes. The projection of Σ onto planes with two currents or two voltages as coordinates is a straight line, as is evident from the first 4 equations of system [Eq. (38)]. If the projection is made, for example, onto the plane with coordinates (i_1, i_3), the straight line is the bisector of the first and third quadrants, while if the projection is made onto the plane (v_1, v_3), the straight line does not generally pass through the origin. If Σ is projected onto the

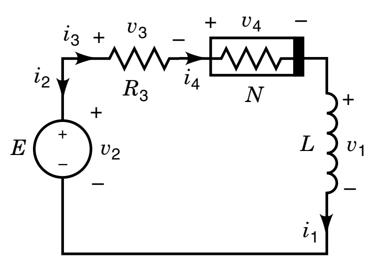

Figure 7. Circuit used to highlight the problems arising in the writing of the state equations due to nonlinear elements.

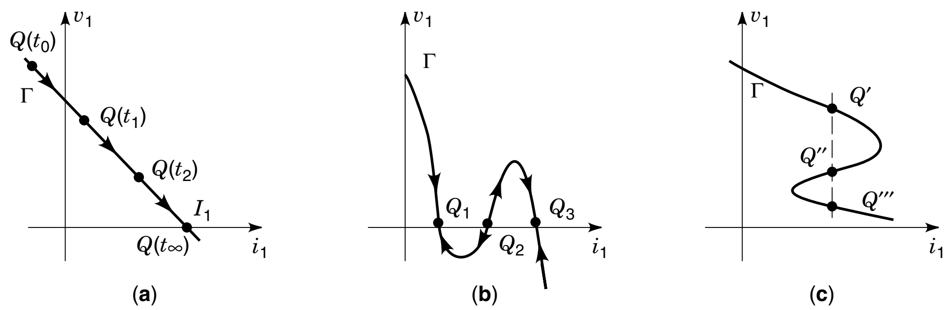

Figure 8. Projections of the configuration space Σ of the circuit shown in Fig. 7 onto the (v_1,i_1) plane for different types of resistor $\boldsymbol{N}$.

planes (v_2, i_2), (v_3, i_3) and (v_4, i_4), we obtain the characteristic curves of the circuit resistive one-ports. In particular, the projection onto the plane (v_2, i_2) is a straight line, which moves parallel to the i_2 axis. The projection of Σ onto the plane (v_1, i_1), which we will denote with Γ, is the most interesting one, since it involves all the constitutive relations of the resistive elements. Combining the equations of system [Eq. (38)] we obtain the equation for the curve Γ

$$f(-v_1 + E - R_3 i_1, i_1) = 0 \tag{39}$$

Let us consider first the case where resistor N is linear, that is, $f(v_4, i_4) = v_4 - R_4 i_4 = 0$. In this case Eq. (39) becomes

$$v_1 + i_1(R_3 + R_4) - E = 0 \tag{40}$$

and thus Γ is a straight line with a slope constant in time [Fig. 8(a) reports the case wherein $E > 0$]. The projection of the operating point P onto the plane (v_1, i_1), marked Q in Fig. 8(a), moves along the straight line Γ with the law of motion specified by the differential Eq. (37). In this case at every value of the state variable i_1 corresponds a single point Q and thus a single operating point P and vice versa. Therefore, once the initial state $i_1(t_0)$ is assigned, the initial point $Q(t_0)$ is univocally determined, and thus the initial operating point $P(t_0)$ of the circuit. The projection of point $Q(t_0)$ onto the v_1 axis gives the inductor voltage at that time, which, by means of Eq. (37), determines the increase per unit of time of the current i_1, and thus the elementary displacement of the point Q, and hence of the circuit operating point P. Proceeding, then, the trajectory of the operating point for $t > t_0$ can be determined. In the case we are examining, the circuit has a single dc operating point. This is represented by the intersection of the configuration space with the hyperplane $v_1 = 0$. The projection of the dc operating point onto the plane (v_1, i_1) is represented by $Q(t_\infty)$. The operating point, irrespective of its initial position, tends asymptotically towards the dc operating point for $t \to +\infty$.

If the resistor is nonlinear and both voltage and current controlled (for example, a junction diode) or only current controlled (for example, a thyristor with disconnected gate; see Fig. 6), its constitutive relation is of the type $v_4 = \hat{v}(i_4)$, where $\hat{v}(i_4)$ is a single-valued function. Then Eq. (39) becomes

$$v_1 + R_3 i_1 + \hat{v}(i_1) - E = 0 \tag{41}$$

and a possible curve Γ is shown in Fig. 8(b). In this case too, the projection of curve Γ onto the i_1 axis determines a one-to-one correspondence between the circuit state and the operating point. Therefore, once assigned, the initial state univocally determines the motion of the circuit operating point. For the case described by a curve Γ of the type illustrated in Fig. 8(b), three dc operating points are possible, Q_1, Q_2 and Q_3. One can easily see that the point Q_2 is unstable, whereas Q_1 and Q_3 are stable. The operating point reaches equilibrium points Q_1 or Q_3 according to its initial position as indicated by the arrows in the figure.

Finally, let us consider the case where the resistor is only voltage controlled, $i_4 = \hat{\imath}(v_4)$ [for example, a tunnel diode; in Fig. 4(b) the characteristic of a tunnel diode is represented], where $\hat{\imath}(v_4)$ is a single-valued function but not globally invertible. The equation of curve Γ is given by

$$i_1 - \hat{\imath}(-v_1 - R_3 i_1 + E) = 0 \tag{42}$$

and a possible curve is illustrated in Fig. 8(c). In this case, as the function $\hat{\imath}(v_4)$ is not globally invertible, the projection of curve Γ onto the i_1 axis does not determine a one-to-one correspondence, that is, more than one circuit operating point can correspond to the same state value [see Fig. 8(c)]. Hence, the motion of the operating point in the configuration space, starting from an assigned initial condition, might be undetermined. This is another example of an ill-posed dynamic circuit model. To obtain a well-posed model it is sufficient to add an arbitrary small capacitance in parallel to the voltage-controlled resistor. In this way the voltage across the nonlinear resistor also "becomes" a state variable.

For the dynamic circuit reported in Fig. 5, which we examined in the previous section, the configuration space Γ determined by Eqs. (27), has dimensions $N = 8 - 6 = 2$ and it is the hyperplane defined by Eqs. (28). The motion of the circuit operating point on Γ is described by the differential Eqs. (26).

FINAL CONSIDERATIONS

So far, we have considered dynamic circuits consisting of linear and time-invariant capacitors and inductors and generally time-variant and nonlinear resistive one-ports. The analysis made can readily be extended to time-variant linear and/or nonlinear dynamic one-ports. In such cases, generally, one has to consider the capacitor charges and the inductor fluxes as circuit state variables and thus the unknowns of the problem become $(n_c + n_i) + 2b$; n_c charges, which we denote with the vector $\boldsymbol{q} = (q_1, \ldots, q_{n_c})^T$, n_i fluxes, which we denote with the vector $\boldsymbol{\phi} = (\phi_{n_c+1}, \ldots, \phi_{n_c+n_i})^T$, and $b = (n_c + n_i + n_s)$ currents and voltages. The circuit always consists of n_c capacitors, n_i inductors, and n_s resistive one-ports. The circuit equations also become $(n_c + n_i) + 2b$; the b linearly independent Kirchhoff Eqs. (21) and (22); the n_c characteristic capacitor equations type [Eq. (5)], which express the voltages of the single capacitors as functions of the respective charges; the n_i characteristic inductor equations type [Eq. (11)], which express the single inductor currents as functions of the respective fluxes; the n_s characteristic resistive one-ports type [Eqs. (1) and (9)], the n_c first-order differential equations type [Eq. (6)], which link the single capacitor currents to the respective charges; and the n_i first-order differential equations type [Eq. (12)] which link the single inductor voltages to the respective fluxes. Therefore, the system of circuit equations in these cases is of the following type:

$$\frac{d\boldsymbol{q}}{dt} = \boldsymbol{i}_c \qquad \frac{d\boldsymbol{\phi}}{dt} = \boldsymbol{v}_i \tag{43}$$

$$\boldsymbol{0} = \boldsymbol{F}(\boldsymbol{q}, \boldsymbol{\phi}, \boldsymbol{v}, \boldsymbol{i}; t) \tag{44}$$

Now algebraic equation system Eq. (44), besides the b Kirchhoff equations and the n_s characteristic resistive one-port equations, also includes the $(n_c + n_i)$ characteristic equations of circuit dynamic one-ports. In this case too the circuit equation system can be reduced to the canonic form wherein only the circuit state variables $\boldsymbol{q}$ and $\boldsymbol{\phi}$ appear as unknowns. By means of algebraic equations (44) it is possible to express the capacitor currents $\boldsymbol{i}_c$ and the inductor voltages $\boldsymbol{v}_i$ as functions of the sole state variables and sources

$$\boldsymbol{i}_c = \boldsymbol{H}_c(\boldsymbol{q}, \phi, \boldsymbol{i}, \boldsymbol{v}; t) \qquad \boldsymbol{v}_i = \boldsymbol{H}_i(\boldsymbol{q}, \phi, \boldsymbol{i}, \boldsymbol{v}; t) \tag{45}$$

In fact if one considers the charge $\boldsymbol{q}$ and the fluxes $\boldsymbol{\phi}$ as assigned, it is possible to interpret system [Eq. (44)] as a system of $2b$ equations in the $2b$ unknowns $\boldsymbol{i}$ and $\boldsymbol{v}$. If the solution of this system exists and is unique, then $\boldsymbol{H}_c$ and $\boldsymbol{H}_i$ are single-valued functions defined for every value of the state variables $\boldsymbol{q}$ and $\boldsymbol{\phi}$. When this is so, the system of state equations

$$\frac{d\boldsymbol{q}}{dt} = \boldsymbol{H}_c(\boldsymbol{q}, \boldsymbol{\phi}, \boldsymbol{i}, \boldsymbol{v}; t) \qquad \frac{d\phi}{dt} = \boldsymbol{H}_i(\boldsymbol{q}, \boldsymbol{\phi}, \boldsymbol{i}, \boldsymbol{v}; t) \tag{46}$$

is in normal form.

Only if the capacitors are charge controlled and the inductors flux controlled does the state circuit at a given time determine the capacitor voltages $\boldsymbol{v}_c$ and the inductor currents $\boldsymbol{i}_i$ univocally at that time, and thus it is possible to define an associated resistive circuit as for circuits consisting of linear and time-invariant dynamic one-ports. Otherwise it is evident that it will never be possible to build a system of global state equations. If all the capacitors are at least charge controlled and the inductors at least flux controlled, and the resistive circuit associated with the dynamic circuit admits one and only one solution, then it is possible to reduce the systems of circuit Eqs. (43) and (44) to a system of global state Eqs. (46).

Similar results can naturally be extended to a circuit containing elements with more than two terminals. In such cases the algebraic equation system in Eq. (44) will also contain the characteristic equations of resistive and dynamic elements with several terminals (i.e., controlled sources, gyrators, ideal transformers, operational amplifiers, mutual inductances, and transistors), and the associated resistive circuit will contains static n-poles and multiports as well as simple one-ports. The more general approach to the problem of the existence of a system of normal form differential equations for circuit state variables is therefore that of bringing the problem back to the study of the existence and uniqueness of the solution of the resistive circuit associated with the dynamic circuit under examination. In this way it is possible to use everything that is known regarding nonlinear resistive circuits (see, e.g., Ref. 1 and CIRCUIT STABILITY OF DC OPERATING POINTS). There are many reports in the literature that deal with the problem of constructing global state equations for certain classes of circuits without explicitly referring to the associated resistive circuits. The results of these works are summarized clearly and fully in Willson's review (3) of 1973.

However, it should be mentioned here that to solve a circuit it is not necessary to determine the state equations in normal form previously, but it is possible to resolve the algebraic-differential equations of the circuit directly by using, for example, difference methods (see, for example, Ref. 2). Nonetheless the problem of the existence of normal form state equations remains a fundamental question. If the circuit does not admit a system of global state equations the circuit model would be incomplete and thus could have more than one solution. In such cases numerical methods would still produce a set of numbers, which, of course, would be meaningless.

As we tried to show, and as is also described in more detail in other articles (*see,* for example, CHAOTIC CIRCUIT BEHAVIOR and TRANSMISSION USING CHAOTIC SYSTEMS), the circuit equations regulate the dynamics of a very rich and complex model, which is common to other different dynamic systems. This richness and complexity can be verified experimentally in a very simple way because electric circuits are generally very easy to build. For these reasons the research in this field has developed remarkably in recent years, not only for the intrinsic practical interest but also at a speculative level.

BIBLIOGRAPHY

1. M. Hasler and J. Neirynck, *Nonlinear Circuits,* Norwood, MA: Artech House, 1986.
2. L. O. Chua and P. M. Lin, *Computer-Aided Analysis of Electronic Circuits,* Englewood Cliffs, NJ: Prentice-Hall, 1975.
3. A. N. Willson, Some aspects of the theory of nonlinear networks, *Proc. IEEE,* **61**: 1092–1113, 1973.
4. C. A. Desoer and F. F. Wu, Trajectories of nonlinear RLC networks: a geometric approach, *IEEE Trans. Circuit Theory,* **CT-19**: 562–571, 1972.

Reading List

L. O. Chua, C. A. Desoer, and E. S. Kuh, *Linear and Nonlinear Circuits,* New York: McGraw-Hill, 1987.

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