J. Webster (ed.), *Wiley Encyclopedia of Electrical and Electronics Engineering*

QUALITATIVE ANALYSIS OF DYNAMIC CIRCUITS

Explicit solutions of the voltages and currents in a nonlinear dynamic circuit are generally not available. In many cases, however, a qualitative description of the circuit behavior is possible and useful. Instead of solving the circuit equations exactly, we ask questions such as: Is the circuit stable? Does the circuit oscillate? Do the solutions remain bounded? Although circuit simulators, such as SPICE, give accurate solutions to the circuit equations, generally they cannot answer these questions because they give us only the solution for a specific set of initial conditions. To know whether the circuit is stable for all initial conditions, a new run needs to be performed for each set of initial conditions, which is impractical. Furthermore, a qualitative analysis of a circuit also tells us about its behavior when the circuit parameters are changed, which is difficult to do with a circuit simulator.

Many techniques of the mathematics of dynamic systems theory are useful for qualitatively analyzing dynamic (and resistive) circuits. A very useful technique for analyzing complicated dynamical systems is to study a quantity derived from the system, such as energy, and use it to draw conclusions regarding the system. Most of the methods of analysis in this article are of this nature. We assume that the reader is familiar with basic circuit theory (1). We assume that the voltages or currents in external sources in a dynamic circuit are bounded functions of time.

Abstractly speaking, a dynamic circuit is a network whose behavior changes over time. The following definition, however, is sufficient to classify a large enough class of useful circuits, especially under the lumped circuit approximation, where we do not consider wave propagative phenomena. A dynamic circuit is a network composed of n-terminal (possibly nonlinear) resistors, capacitors, and inductors. A resistor is a *resistive* element whereas a capacitor or an inductor are called *dynamic* elements. For one-port or two-terminal elements, their notation along with reference directions are shown in Fig. 1. For simplicity in this article, we assume that all elements are two-terminal elements. A nonlinear resistor is described by a relationship between the voltage across it and the current through it. Similarly, a nonlinear capacitor is described by a relationship between the charge on the capacitor q and the voltage across the capacitor v, and a nonlinear inductor is described by a relationship between the flux ϕ and the current i. More formally, nonlinear resistors, capacitors and inductors are defined by the relationships

$$R(v, i) = 0, \qquad C(q, v) = 0, \qquad L(\phi, i) = 0 \tag{1}$$

respectively. These equations are the *constituency relationships* of the elements. If an element is time-varying, then we need to write the relationships as $R(v, i, t) = 0$, $C(q, v, t) = 0$, and $L(\phi, i, t) = 0$. A resistor is voltage-controlled if the current i is a function of the voltage across the resistor, that is, the constituency relationship $R(i, v) = 0$ is expressed as $i = \hat{\imath}(v)$. A resistor is current-controlled if the constituency relationship is written $v = \hat{v}(i)$. A capacitor is voltage-controlled if the charge is a function of the voltage across the capacitor, that is, $C(q, v) = 0$ is rewritten $q = \hat{q}(v)$. Similarly, a capacitor is charge-controlled if we write the constituency relationship $v = \hat{v}(q)$. An inductor is flux-controlled if it is described by a function $i = \hat{\imath}(\phi)$ and current-controlled if it is described by $\phi = \hat{\phi}(i)$.

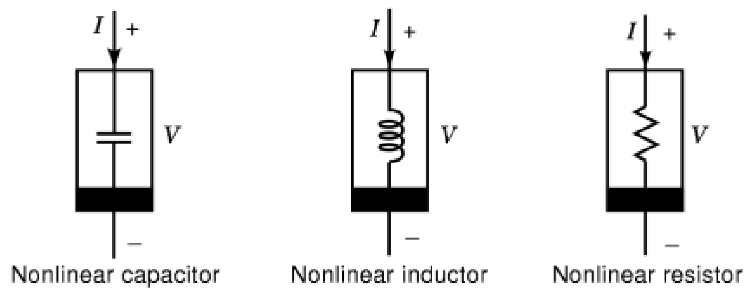

Fig. 1. Two-terminal nonlinear elements along with their reference directions.

Passivity

Because many of the results in this article rely on energy-like quantities, the concept of passivity is important. Passivity implies that the nonlinear element dissipates energy (or at least does not supply energy), which is the case for elements without a power source.

Definition 1. *A nonlinear resistor with a constituency relationship* $i = g(v)$ *is passive if* $vg(v) \geq 0$ *for all* v. *It is strictly passive if* $vg(v) > 0$ *for all* $v \neq 0$. *It is eventually passive if there exists a* $k > 0$ *such that* $vg(v) \geq 0$ *for* $|v| > k$. *It is eventually strictly passive if* $vg(v) > 0$ *for* $|v| > k$.

If the constituency relationship is given by $v = r(i)$, the previous definition still holds when v is replaced by i. Passive resistors are such that the power vi fed into the resistor is always nonnegative, that is, the constituency relationship lies in the first or third quadrant of the v-i plane.

Definition 2. *A nonlinear element (resistor, inductor, or capacitor) with a constituency relationship* $y = f(x)$ *is strongly locally passive if there exists* $a, b > 0$ *such that the following holds for all* x, x':

$$a\|x - x'\|^2 \leq (x - x')[f(x) - f(x')] \leq b\|x - x'\|^2 \qquad (2)$$

It is eventually strongly locally passive if Eq. (2) holds for all $|x| > k$, $|x'| > k$ *for some* $k > 0$.

A strictly locally passive nonlinear element has a constituency relationship that is strictly increasing and is not necessarily passive.

Stationary or Equilibrium Points

The state of a capacitor and an inductor is the charge and the flux, respectively. The state of a dynamic circuit consists of the charges and fluxes of the capacitors and inductors, respectively. A dynamic circuit is *autonomous* (or time-independent) if it does not contain any time-varying circuit elements or sources. A *stationary* or *equilibrium point* of a dynamic circuit is defined as the state of the circuit such that the state does not change with time. This is also called the *dc-operating point*. Because the change of charge and flux with respect to time is the current through the capacitor and the voltage across the inductor, respectively, an equilibrium point is the state where all the branch currents of the capacitors and branch voltages of the inductors are zero. Thus, at the equilibrium point, we can replace a capacitor by an open circuit, and an inductor by a short circuit, and reduce the dynamic circuit to a resistive circuit. Thus finding equilibrium points of a dynamic circuit consists of two steps. First the operating points of the corresponding resistive circuit are found. These operating points, which are expressed as branch currents and branch voltages, are used to find the equilibrium points of the dynamic circuit as follows: for each capacitor C_i, the branch voltage v_i at the operating point is used to find all the q_i's which satisfy $C_i(q_i, v_i) = 0$. Similarly this is done for inductors to find the ϕ_i's. Each operating point of the resistive circuit can correspond to many equilibrium points of the dynamic circuit.

State Equations and Uniqueness and Existence of Solutions

The equations governing a dynamic circuit consists of Kirchhoff's current law ($\mathbf{A}\mathrm{i} = 0$), Kirchhoff's voltage law ($\mathbf{B}\mathrm{v} = 0$), the constituency relationships of the elements, and the equations $i = dq/dt$, $v = d\phi/dt$. This forms a set of constrained differential equations. Suppose that we can eliminate the variables v and i, so that we are left with the equations:

$$\begin{aligned}\frac{dq_c}{dt} &= \mathbf{f}_c(q_c, \phi_l, t)\\ \frac{d\phi_l}{dt} &= \mathbf{f}_l(q_c, \phi_l, t)\end{aligned} \tag{3}$$

where q_c is the vector corresponding to the charges on the capacitors and ϕ_l is the vector of fluxes of the inductors. These are the *state equations* of the dynamic circuit.

If all the capacitors are charge-controlled and have differentiable and invertible constituency relationships $v = \hat{v}(q)$ and all the inductors are flux-controlled and have differentiable and invertible constituency relationships $i = \hat{\imath}(\phi)$, then we can express the state equations in terms of capacitor voltages v_c and inductor currents i_l. Let us write $v_c = g_c(q_c)$ and $i_l = g_l(\phi_l)$. The hypothesis implies that ∇g_c, $g^{-1}{}_c$, ∇g_l, and $g^{-1}{}_l$ exist.

$$\begin{aligned}\frac{dv_c}{dt} &= \nabla g_c(q_c)\cdot\frac{dq_c}{dt} = \nabla g_c(x)|_{x=g_c^{-1}(v_c)}\cdot \mathbf{f}_c[g_c^{-1}(v_c), g_l^{-1}(i_l), t]\\ &= \mathbf{h}_c(v_c, i_l, t)\\ \frac{di_l}{dt} &= \nabla g_l(\phi_l)\cdot\frac{d\phi_l}{dt} = \nabla g_l(x)|_{x=g_l^{-1}(i_l)}\cdot \mathbf{f}_l[g_c^{-1}(v_c), g_l^{-1}(i_l), t]\\ &= \mathbf{h}_l(v_c, i_l, t)\end{aligned}$$

This form is more desirable in practice because voltages and currents are more easily measured than charge or flux.

Even if the state equations of the system exist, there is still no guarantee that solutions to the equations exist, let alone that they are unique for each initial conditions. A mild requirement guaranteeing the existence and uniqueness of solutions for all time t of ordinary differential equations of the form $\dot{x} = f(x, t)$ is that f is continuous and uniformly Lipschitz continuous with respect to x (2).

In this article, unless otherwise stated, we assume that we can always write the state equations [Eq. (3)] of a dynamic circuit and that solutions to $\dot{x} = f(x, t)$ exist and are unique for all time and that f is continuous.

Simple First- and Second-Order Nonlinear Dynamic Circuits

For a dynamic circuit with only two-terminal dynamic elements is called the *order* of the system, because, in general, the resulting state equations (when they can be written) consist of this many first-order equations

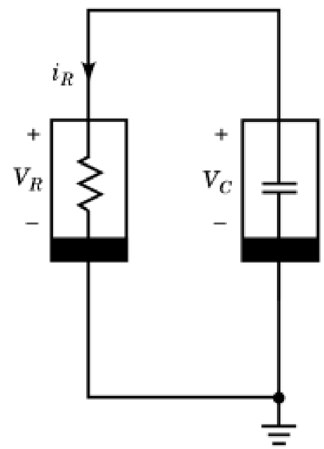

Fig. 2. First-order autonomous dynamic circuit consisting of a resistor in parallel with a capacitor.

(except for cases such as where there are loops of capacitors, in which case there are less equations than the number of dynamic elements). The simplest types of dynamic circuits are the first-order and second-order circuits. In this section we study some simple first-order and second-order dynamic circuits in some detail and illustrate how the qualitative behavior of these systems is derived.

One Nonlinear Resistor and One Nonlinear Capacitor. Consider the first-order circuit consisting of a nonlinear capacitor coupled with a nonlinear resistor (Fig. 2). If the resistor is voltage-controlled with driving-point characteristic $i = g(v)$ and the capacitor is charge controlled with driving-point characteristic $v = c(q)$, then the state equation of the circuit is written

$$\frac{dq}{dt} = -g[c(q)] \tag{4}$$

What is the possible behavior of this system? The following theorem shows that nothing very interesting can happen in this case:

Theorem 1. Suppose that c and g are continuous functions. Any solution $q(t)$ of Eq. (4) is monotonic with no inflection points (isolated points with slope 0). Thus, $q(t)$ converges to an equilibrium point, or $q(t)$ becomes unbounded.

This can be proved as follows. Existence and uniqueness of solutions imply that, if $dq/dt = 0$ at some time t_0, $dq/dt = 0$ for all time t. Because g and c are continuous and $q(t)$ is continuous, $\dot{q} = dq/dt = -g[c(q)]$ is a continuous function of time. Suppose that $\dot{q} > 0$ at some time t_0 and that $\dot{q} < 0$ at some time t_1. Then, by the mean value theorem (18), $\dot{q} = 0$ at some time t_2, which contradicts the above. So $\dot{q}$ always has the same sign, and q is either constant or a strictly monotonic function. If $q(t)$ is bounded, then it converges to some number $q*$. Because $q(t)$ is monotonic, there exists a sequence $t_n \to \infty$ such that $-g\{c[q(t)]\} = \dot{q} \to 0$, as $n \to \infty$. Therefore $-g[c(q*)] = 0$ which implies that $q*$ is an equilibrium point.

One Resistor, One Capacitor, and One Periodic Input. Now assume that a periodic voltage source $u(t)$ is connected in series with a nonlinear capacitor and resistor as shown in Fig. 3. We assume that $u(t)$ is a continuous function of t. Again assuming that the capacitor is charge-controlled and that the resistor is voltage

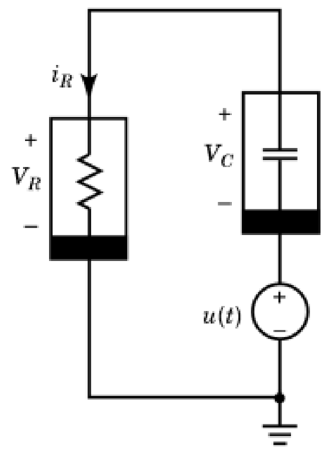

Fig. 3. First-order nonautonomous dynamic circuit obtained from Fig. 2 by adding a periodic voltage source in series with the capacitor.

controlled, we get the state equation:

$$\frac{dq}{dt} = -g[c(q) - u(t)]$$

For the initial condition $q(t_0) = q_0$, we denote the corresponding solution as $q(t, t_0, q_0)$. Let T be the period of $u(t)$. Then, by uniqueness of solutions, $q(t + nT, t_0, q_0)$ for n an integer depends only on $q(t, t_0, q_0)$. This implies that a map P_{t_0} exists, which maps q_0 to $q(t_0 + T, t_0, q_0)$. Note that P_{t_0} depends on t_0. The map P_{t_0} is called a *Poincaré* map. By continuity of solutions of ordinary differential equations, P_{t_0} is continuous because $u(t)$ is continuous. P_{t_0} is also invertible by uniqueness of solutions. This implies that P_{t_0} is a strictly monotonic map. Note that uniqueness of solutions also implies that $q(t_0 + 2T, t_0, q_0) = P_{t_0}[P_{t_0}(q_0)]$ etc., so that $q(t_0 + nT, t_0, q_0)$ is just the nth iterate of the map P_{t_0} evaluated at q_0. Because P_{t_0} is monotonic, its iterates either diverge or converge toward an equilibrium point. This implies that $q(t, t_0, q_0)$ as a function of t either diverges to infinity or converges toward a periodic waveform with period T.

One Resistor, One Capacitor, and One Inductor. Consider a second-order circuit consisting of a capacitor, an inductor, and a resistor connected in parallel (Fig. 4). Assuming that the resistor is voltage-controlled, the inductor is flux-controlled, and the capacitor is charge-controlled with constituency relationships $i = f(v)$, $i = g(\phi)$, $v = h(q)$, respectively, then the state equations are given by

$$\begin{aligned} \frac{dq}{dt} &= -i_R + i_L = -f[h(q)] + q(\phi) \\ \frac{d\phi}{dt} &= -v_C = -h(q) \end{aligned} \tag{5}$$

The Poincaré–Bendixson theorem (2) states that second order autonomous circuits cannot exhibit behavior more complicated than periodic solutions.

Theorem 2. If the trajectory of an autonomous second-order system is bounded and does not approach an equilibrium point, then it must approach a periodic solution.

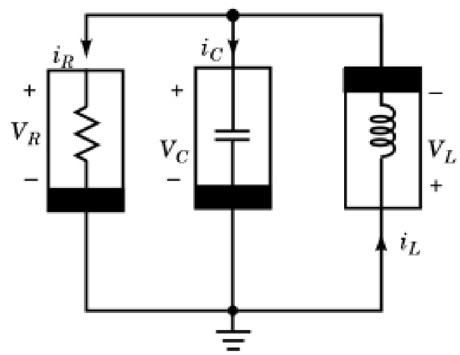

Fig. 4. Second-order autonomous dynamic circuit consisting of a resistor, a capacitor, and an inductor in parallel.

In particular, in second-order autonomous systems with one unstable equilibrium point, almost all bounded trajectories must approach a periodic solution.

If the characteristics of the nonlinear elements are of a certain form, the Levinson–Smith theorem (2) allows us to conclude that there exists a periodic solution which is stable in the sense that nearby solutions converge toward it.

Theorem 3. If the following conditions are satisfied:

(1) the inductor is linear and strictly passive;
(2) the capacitor is strictly passive such that h is odd and differentiable;
(3) the function $F(x) = f(h(x))$ is odd and differentiable and there exists $a > 0$ such that $F(x) < 0$ on $0 < x < a$, $F(x) > 0$ on $x > a$, and $F'(x) > 0$ on $x > a$:
(4) $\int^{x}{}_{0}\, h(s)ds \to \infty$, as $|x| \to \infty$; and
(5) $F(x) \to \infty$, as $x \to \infty$;

then Eq. (5) has a nonconstant periodic solution which is stable.

This nonconstant periodic solution is unique if, in addition, the following conditions are satisfied (3):

(1) $F(x)$ is zero only at $x = 0$, $x = a$ and $x = -a$; and
(2) $F(x) \to \infty$ monotonically, as $x \to \infty$ for $x > a$.

By using index theory (19), it can be proved that any periodic solution encircles at least one stationary point.

When the circuit is autonomous but higher than second-order (e.g., Chua's circuit) or the circuit is second-order but nonautonomous, that is, driven by an external input, the behavior can be very complex. In fact, circuits of this type can be chaotic and can exhibit complicated bifurcation phenomena (4,5,6). See the article (Circuits exhibiting chaotic behavior) for an introduction to bifurcation and chaos in nonlinear circuits and systems.

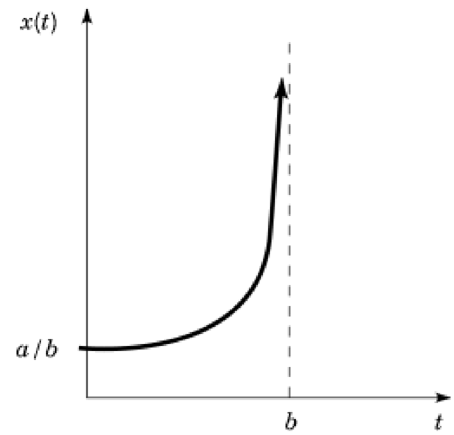

Fig. 5. The function $x(t) = a/(b - t)$ goes to infinity in finite time.

No Finite Escape-Time Criteria

General ordinary differential equations can have solutions which diverge to infinity in finite time, such as $x(t) = a/(b - t)$ (Fig. 5). In this case the solutions do not exist for all time and this is a nonphysical situation. The following criterion guarantees that dynamic circuits do not have such solutions (7).

Theorem 4. A dynamic circuit does not have solutions which escape to infinity in finite time if the following conditions are satisfied:

(1) There are no loops nor cut sets consisting only of capacitors and/or inductors.
(2) All capacitors and inductors are eventually strongly locally passive.
(3) All resistors are eventually passive.

Eventually Uniformly Boundedness of Trajectories

Even if the solutions exist for all time, we do not want the voltages and currents in practical circuits to diverge to infinity. Therefore, we want the trajectories of the system to be bounded.

Definition 3. *A dynamic circuit is eventually uniformly bounded if there exists a bounded set K such that, for each initial condition $x(t_0)$, there exists a time T for which the state x(t) remains in K for all $t \geq T$.*

The following result states that dynamic circuits which consist of independent sources and passive elements eventually have uniformly bounded solutions (7).

Theorem 5. Suppose that the constituency relationships of the elements are differentiable functions. A dynamic circuit is eventually uniformly bounded if the following conditions are satisfied:

(1) There is no loop in the circuit formed exclusively by capacitors, inductors, and/or voltages sources.
(2) There is no cut set in the circuit formed exclusively by capacitors, inductors, and/or current sources.
(3) All resistors (not including sources) are eventually strongly locally passive.
(4) All capacitors and inductors are eventually strongly locally passive.

Stability of Equilibrium Points

Consider an autonomous dynamic circuit. Let us rewrite the state equations [Eq. (3)] in a general form as:

$$\dot{x} = f(x) \tag{6}$$

In terms of Eq. (6), an equilibrium point is a state $x*$ such that $\dot{x} = 0$, that is, $f(x*) = 0$.

The property of uniqueness and existence of solutions allows us to show that, if the state of the circuit is at an equilibrium point, then it remains there for all time. Because of physical noise, we are also interested in behavior near the equilibrium point. Compare this with a ball balanced on the tip of a needle. This is an equilibrium state, and theoretically the ball remains balanced for ever. But, because of small noise, the ball invariably leaves the equilibrium state. We call such equilibrium states unstable. On the other hand, a ball lying at the bottom of a bowl is at a stable equilibrium state because a ball lying near the bottom will move toward the bottom of the bowl. More precisely, an equilibrium point is (Lyapunov) asymptotically *stable* if all initial conditions nearby do not leave a neighborhood of the equilibrium and converge toward the equilibrium point as time goes on. If initial conditions nearby do not stay within a neighborhood of the equilibrium point, the equilibrium point is called *unstable.*

A consequence of the Hartman–Grobman linearization theorem (8) is that it allows us to deduce the stability of an equilibrium point by looking at the eigenvalues of the linearization.

Theorem 6. Consider Eq. (6). Let x_e be an equilibrium point, that is, $f(x_e) = 0$. Suppose that $Df(x_e)$, the Jacobian matrix of f at x_e, does not have purely imaginary eigenvalues. If all the eigenvalues have negative real parts, then x_e is asymptotically stable, and otherwise it is unstable.

Lyapunov Functions and Lyapunov's Direct Method

As alluded to before, many of the techniques for studying the qualitative behavior of dynamic circuits are based on studying a simple quantity related to the circuit. One class of such techniques is called *Lyapunov's methods.* In these methods, a *Lyapunov function* is constructed that maps the state of the system x into a simple (usually scalar) quantity $V(x)$. We then observe how $V(x)$ evolves with time and draw conclusions about the system. For general nonlinear systems, there are no systematic ways for constructing Lyapunov functions. In some cases, an energy-like quantity is used for the Lyapunov function.

Definition 4. *Let a and b be strictly increasing continuous functions such that $a(0) = b(0) = 0$. A function $V : \mathbb{R}^n \to \mathbb{R}$ is a Lyapunov function if $b(|x|) \le V(x) \le a(|x|)$ for all x.*

The basic Lyapunov asymptotic stability theorem is as follows (20):

Theorem 7. Consider the system $\dot{x} = f(x)$. If V is a differentiable Lyapunov function such that $\nabla V \cdot f(x) \le -c(|x|)$ for some strictly increasing function c with $c(0) = 0$, then the origin is globally asymptotically stable, that is, all trajectories converge toward the origin.

The construction of V to show stability in this way is called *Lyapunov's direct method.* A local version of this theorem is also true when "for all x" in Definition 4 is replaced by "in a neighborhood of 0."

An extension of this theorem, called LaSalle's invariant principle (9), is stated (for autonomous systems) as follows:

Theorem 8. Consider the system $\dot{x} = f(x)$. Let $V : \mathbb{R}^n \to \mathbb{R}$ be a differentiable function such that $V(x) \geq C$ for all x and some constant C and

$$\frac{dV}{dt} \triangleq \nabla V \cdot f(x) \leq -W(x) \leq 0$$

for all $t > 0$ and some continuous W. Define

$$E = \{x : W(x) = 0\}$$

Then each solution $x(t)$ of $\dot{x} = f(x)$ approaches $E \cup \{\infty\}$. Although V does not necessarily satisfy the conditions in Definition 4, we still call V a Lyapunov function when used in the context of Theorem 8.

Content, Co-Content, Energy, and Co-Energy of Two-Terminal Elements

Consider a dynamic circuit consisting of two-terminal elements. We can define energy-like quantities for each element. For resistive elements, we define content and co-content whereas for dynamic elements, we define energy and co-energy.

Content and Co-content. A resistive two-terminal element's driving-point characteristic is a relationship between v and i. For a current-controlled resistor, the *content* is defined as

$$G(I) = \int_0^I \hat{v}(i)\,di$$

where $\hat{v}(i)$ is the voltage-vs-current characteristic of the nonlinear resistor. Similarly, the co-content of a voltage-controlled resistor is defined as

$$\overline{G}(V) = \int_0^V \hat{\imath}(v)\,dv$$

This is illustrated in Fig. 6. The co-content is the area below the curve, and the content is the area above the curve within the rectangle. For resistors, which are both voltage- and current-controlled, it is easy to see from Fig. 6 that, for $V = \hat{v}(I)$, $G(I) + \bar{G}(V) = VI$ is the power dissipated by the resistor.

Energy and Co-energy. We assume that the charge on the capacitors and the flux in the inductors are zero at time $t = 0$. For a two-terminal, charge-controlled capacitor with the device characteristic $v = v(q)$, the capacitive *energy* (10) is defined as:

$$U(Q) = \int_0^Q v(q)\,dq$$

The capacitive energy is derived as the work done by the capacitor:

$$U(Q) = \int_0^t v(t)i(t)\,dt = \int_0^t v\frac{dq}{dt}dt = \int_{q(0)}^{q(t)} v\,dq = \int_0^Q v\,dq$$

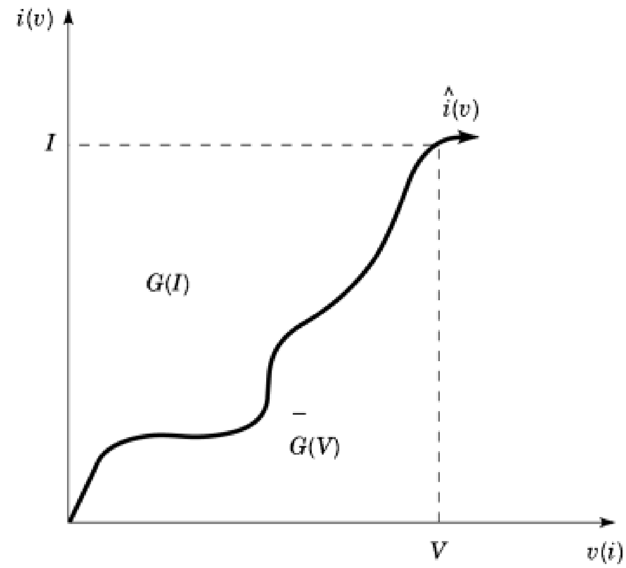

Fig. 6. The co-content of a resistor is the area under the curve bounded by the coordinate axes and the dashed vertical line. The content of a resistor is the area above the curve bounded by the coordinate axes and the dashed horizontal line.

For a two-terminal, voltage-controlled capacitor with the device characteristic $q = q(v)$, the capacitor *co-energy* is defined as

$$\overline{U}(V) = \int_0^V q(v)\,dv$$

A graphical interpretation of the energy and the co-energy is shown in Fig. 7 for the case when the capacitor is both voltage-controlled and charge-controlled. The co-energy is the area below the curve, and the energy is the area above the curve bounded by the rectangle. It is clear that, for $V = \hat{v}(Q)$, $U(Q) + \bar{U}(V) = QV$.

Dually, the energy and co-energy of an inductor are defined as

$$T(\Phi) = \int_0^\Phi i(\phi)\,d\phi$$

and

$$\overline{T}(I) = \int_0^I \phi(i)\,di$$

respectively, and, for $I = \hat{\imath}(\Phi)$, $T(\Phi) + \bar{T}(I) = \Phi I$.

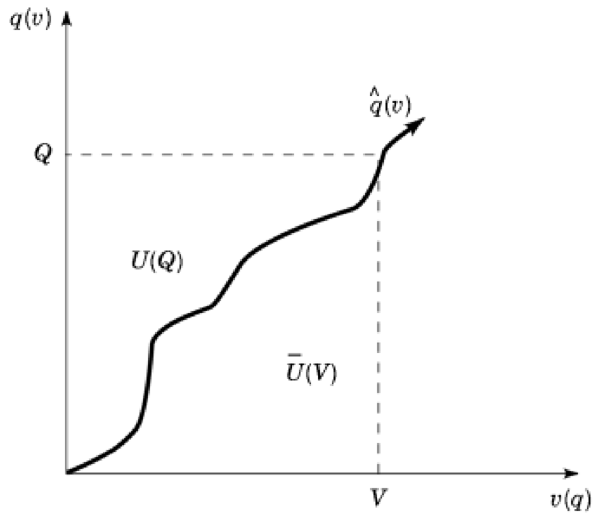

Fig. 7. The co-energy of a capacitor is the area under the curve bounded by the coordinate axes and the dashed vertical line. The energy of a capacitor is the area above the curve bounded by the coordinate axes and the dashed horizontal line.

Using Content and Energy to Derive the Qualitative Behavior of Dynamic Circuits

Conservation of Energy. Consider a dynamic circuit where all the capacitors are charge-controlled and all inductors are flux-controlled. Let $D_j = v_j i_j$ be the power dissipated by the nonlinear resistor R_j. Let U_k and T_l be the energies of capacitor C_k and inductor T_l, respectively. Then the law of conservation of energy (expressed as a particular form of Tellegen's theorem: $\Sigma\, v_n i_n = 0$) implies that

$$\sum_j D_j + \sum_k \frac{dT_k}{dt} + \sum_l \frac{dU_l}{dt} = 0$$

at each instant.

Completely Stable Behavior. **Definition 5**. *A dynamic circuit is convergent (or completely stable) if all of its trajectories converge toward an equilibrium point.*

Theorem 9. Consider a dynamic circuit with only charge-controlled capacitors and resistors. Assume that the characteristics $v = \hat{v}(q)$ of the capacitors are differentiable and have positive slope everywhere. Assume also that the resistors are either independent voltage sources or voltage-controlled, eventually strictly passive resistors with differentiable constituency relationships. If the circuit has a finite number of equilibrium points, then the system is convergent.

The proof is based on LaSalle's invariant principle. The resistors are combined into a resistive n-port whose constituency relationship is the gradient of a potential function which we choose as the Lyapunov function. The passivity is used to guarantee eventually bounded solutions. The local passivity of the capacitor guarantees the negativity of the derivative of the Lyapunov function with respect to the trajectories. Here we give the simple proof from (11). The Lyapunov function equals the sum of the co-contents of the resistors: $V = \Sigma \int_0^{v_k} \hat{i}_k(v)dv$ where the summation is over all resistors. Because of the eventual strict passivity of the resistors, a C can be chosen such that $V \geq C$. By Theorem 5, the solution of the circuit is eventually bounded. The derivative of V with respect to t is given by

$$\frac{dV}{dt} = \sum_{\text{resistors}} \frac{dv_k}{dt} i_k = - \sum_{\text{capacitors}} \frac{dv_k}{dt} i_k$$
$$= - \sum_{\text{capacitors}} \frac{dv_k}{dq} i_k^2 \leq 0$$

where the second equality is from Tellegen's theorem. By applying LaSalle's theorem (Theorem 8), the conclusion follows.

The following theorems provide criteria for a circuit to have a unique equilibrium point and be convergent (7):

Theorem 10. An autonomous dynamic circuit is convergent and has a unique equilibrium point if the following conditions are satisfied:

(1) There is no loop in the circuit formed exclusively by capacitors, inductors, and/or voltages sources.
(2) There is no cut set in the circuit formed exclusively by capacitors, inductors, and/or current sources.
(3) All resistors (not including sources) are strongly locally passive.
(4) All capacitors and inductors are strongly locally passive.

Theorem 11. An autonomous dynamic circuit is convergent and has a unique equilibrium point if the following conditions are satisfied:

(1) There is no loop in the circuit formed exclusively by capacitors, inductors, and/or voltages sources.
(2) There is no cut set in the circuit formed exclusively by capacitors, inductors, and/or current sources.
(3) Every loop containing a voltage source also contains a capacitor.
(4) Every cut set containing a current source also contains an inductor.
(5) All resistors (not including sources) are strictly passive.
(6) All capacitors and inductors are eventually strongly locally passive.

Equivalent Resistors, Capacitors, and Inductors. Even though the elements are nonlinear, the content, co-content, energy, and co-energy enjoy the following linear superposable property. Consider a one-port N consisting only of arbitrarily connected, nonlinear, charge-controlled capacitors and its equivalent capacitor, an example of which is shown in Fig. 8. The energy of the equivalent capacitor is the sum of the energies of the capacitors in N. Similarly, this result is also true for the co-energy of a one-port of voltage-controlled capacitors. By duality, this result is also true for a one-port consisting only of inductors.

We give the proof here for the energy of a one-port N composed only of capacitors. The sum of the energies of the capacitors is given by

$$U_{sum} = \sum \int_0^{q_b} v_b(q_b)\, dq_b = \sum \int_0^t v_b(\tau) i_b(\tau)\, d\tau$$
$$= \int_0^t \sum v_b(\tau) i_b(\tau)\, d\tau$$

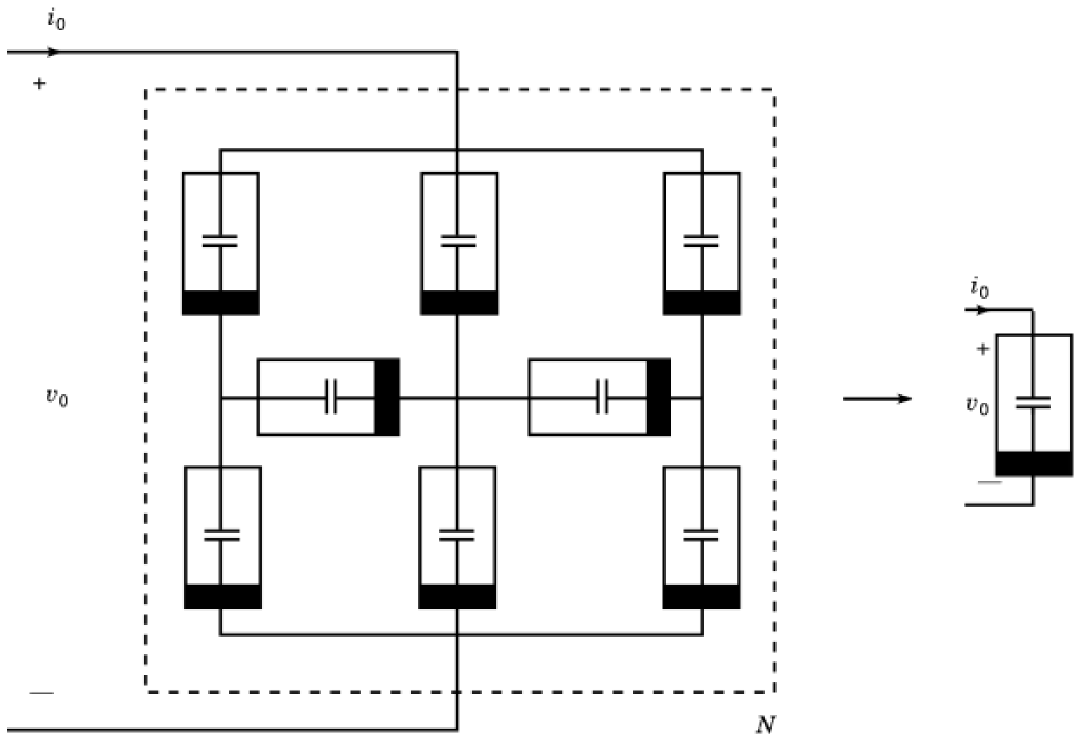

Fig. 8. A one-port N consisting of arbitrarily connected capacitors and its representation as an equivalent capacitor.

where the summation is over all branches b in the one-port N. By Tellegen's theorem, $\Sigma\, v_b(\tau)i_b(\tau) = v_0(\tau)i_0(\tau)$, and thus

$$U_{sum} = \int_0^t v_0(\tau)i_0(\tau)\,d\tau$$

which is the energy of the equivalent capacitor.

For a one-port consisting only of resistors, this theorem is true for the content and co-content, provided that all of the resistors are passive (12).

We give the proof here for the content. Consider a nonlinear one-port as in Fig. 8, except that all the capacitors are replaced by passive resistors. We wish to show that the sum of the contents of the resistors in the one-port is equal to the content of the equivalent resistor. The derivative of the sum of the contents of the resistors with respect to i_0 is given by

$$\frac{\partial \sum_b G_b}{\partial i_0} = \sum_b \frac{\partial G_b}{\partial i_0} = \sum_b \frac{\partial G_b}{\partial i_b}\frac{\partial i_b}{\partial i_0} = \sum_b v_b \frac{\partial i_b}{\partial i_0}$$

where the summation is over all capacitors in the one-port. By Tellegen's theorem,

$$\sum_b v_b \frac{\partial i_b}{\partial i_0} = v_0 \frac{\partial i_0}{\partial i_0} = v_0$$

Therefore $\partial \Sigma_b G_b / \partial i_0 = v_0$ and

$$\sum_b G_b = \int_0^{i_0} v_0 di_0 + c$$

Suppose that $i_0 = 0$. Passivity of the resistors implies that $i_b = 0$ for all b and consequently $G_b = 0$ for all b. Putting this into the above equation, we get $c = 0$ and therefore $\Sigma_b G_b = \int_0^{i_0} v_0 di_0$ which is the content of the equivalent resistor.

Stationary Principles. The total content $\mathscr{G}$ of a circuit consisting only of current-controlled resistors is the sum of the contents of each resistor. The total co-content $\bar{\mathscr{G}}$ of a circuit consisting only of voltage-controlled resistors is the sum of the co-contents of each resistor. Because the branch currents are uniquely determined by a link (or co-tree) branch vector, the total content $\mathscr{G}$ is expressed as a function of the link branch vector i_l. Similarly, the total co-content $\bar{\mathscr{G}}$ is a function of the tree branch voltage vector v_t. Millar's theorem on stationary content (12) states that a set of link branch currents is a solution of the circuit if and only if it is a stationary point of the total content $\mathscr{G}$, that is, $\partial \mathscr{G} / \partial i_l = 0$ if and only if i_l is the link branch current vector corresponding to a solution of the circuit equations. Similarly, a tree branch voltage vector is a stationary point of the total co-content if and only if it is a solution of the circuit.

Here we give the proof presented in (13). Let i be the branch current vector. By Kirchhoff's Current Law, $i = B^T i_l$, where i_l is the link branch current vector. Let v be the corresponding voltage vector across the resistors, v and i satisfy the constituency relationships of the resistors:

$$\frac{\partial \mathscr{G}}{\partial i_l} = \frac{\partial \mathscr{G}}{\partial i} \frac{\partial i}{\partial i_l} = v^T B^T = (Bv)^T$$

which is zero if and only if v satisfies Kirchhoff voltage law, that is, i and v are solutions of the circuit.

These results are also valid for inductor-only and capacitor-only circuits. The total energy of a dynamic circuit consisting only of charge-controlled capacitors is defined as the sum of the energies of the capacitors. Similar definitions exist for the total co-energy. Then, in a circuit of charge-controlled capacitors, the link branch current vector corresponding to a solution of the circuit is a stationary point of the total energy at every instant. Similar for a circuit of voltage-controlled capacitors, a tree branch voltage vector is at a stationary point of the total co-energy. In a circuit of flux-controlled inductors, the tree branch voltage vector is at a stationary point of the total energy, whereas, in a circuit of current-controlled inductors, the link branch current vector is at a stationary point of the total co-energy.

Unique Asymptotic Behavior

In many applications, we want the circuit to behave roughly the same way, no matter what the initial charges and fluxes are on the capacitors and inductors, respectively. This is difficult to achieve because the initial conditions vary widely, so we are content if the circuit behaves roughly the same way after a long enough time regardless of the initial conditions, that is, the transient behavior dies down.

In linear systems, we can separate the system response into two parts: the transient or zero-input response due to the initial states and the zero-state response due to the input. The transient response goes to zero if the system is stable, leaving us with only the zero-state response, which we call the steady-state solution, and this gives us the desired property.

In nonlinear circuits and systems, this separation does not exist any longer, but the convergence toward a steady-state solution regardless of initial condition can still be defined as follows (14):

Definition 6. *Assume that the state equations of the system exist and are written as*

$$\dot{x} = f(x, t) \tag{7}$$

If all solutions are bounded and any two solutions of Eq. (7) converge toward each other asymptotically, that is if x(t) and y(t) are solutions of Eq. (7), we get

$$\lim_{t\to\infty} \|x(t) - y(t)\| = 0$$

then we say the system has a unique steady-state solution.

The following theorem gives criteria for a dynamic circuit to have a unique steady-state solution (7,15):

Theorem 12. A dynamic circuit has a unique steady-state solution if the following conditions are satisfied:

(1) There is no loop in the circuit formed exclusively by capacitors, inductors, and/or voltage sources.
(2) There is no cut set in the circuit formed exclusively by capacitors, inductors, and/or current sources.
(3) All capacitors and inductors are weakly nonlinear and passive.
(4) All resistors (not including sources) are strongly locally passive.

There is a tradeoff between the local passivity of the resistors and the nonlinearity of the capacitors and inductors. The more locally passive the resistors are, the more nonlinear the capacitors and inductors can be, and the theorem still remains valid.

Qualitative Behavior of Dynamic Circuits Driven by Periodic Inputs

We have mentioned earlier that even simple, second-order, nonautonomous circuits driven by periodic input can exhibit complicated chaotic behavior. In these cases the state trajectory can contain frequency components which are not integral combinations of the driving frequencies. In this section we present some conditions (7) guaranteeing that solutions of circuits driven by periodic inputs generate only frequency components at integral combinations of the driving frequencies (harmonics).

Theorem 13. Consider a dynamic circuit driven by periodic inputs. The circuit has a unique steady-state waveform with frequency components only at integral combinations of the driving frequencies if the following conditions are satisfied:

(1) There is no loop in the circuit formed exclusively by capacitors, inductors, and/or voltages sources.
(2) There is no cut set in the circuit formed exclusively by capacitors, inductors, and/or current sources.
(3) All capacitors and inductors are linear and passive.
(4) All resistors (not including sources) are strongly locally passive.

(5) The external driving sources are defined by differentiable periodic functions.

Theorem 14. Consider a dynamic circuit driven by periodic inputs. The conclusion of the previous theorem holds if the following conditions are satisfied:

(1) There is no loop in the circuit formed exclusively by capacitors, inductors, and/or voltages sources.
(2) There is no cut set in the circuit formed exclusively by capacitors, inductors, and/or current sources.
(3) The circuit either has no capacitors or no inductors.
(4) All resistors (not including sources) are linear and passive.
(5) The external driving sources are defined by differentiable periodic functions.

Theorem 15. Consider a dynamic circuit driven by periodic inputs. The conclusion of the previous theorem holds if the following conditions are satisfied:

(1) There is no loop in the circuit formed exclusively by capacitors, inductors, and/or voltages sources.
(2) There is no cut set in the circuit formed exclusively by capacitors, inductors, and/or current sources.
(3) All capacitors and inductors are strongly locally passive and have twice-differentiable (C^2) constituency relationships.
(4) All resistors (not including sources) are strongly locally passive.
(5) The external driving sources are combinations of dc sources and sufficiently small periodic sources.

Manley–Rowe Equations

A characteristic feature of nonlinear circuits is the ability to generate beat frequencies $\omega_{m,n} = m\omega_1 \pm n\omega_2$ given two sinusoidal input signals of radial frequencies ω_1 and ω_2. For resistive networks this can be seen as follows. Given an input signal $\sin\,\omega_1 t + \sin\,\omega_2 t$, the output is written as $y(t) = f(\sin\,\omega_1 t + \sin\,\omega_2 t)$. Expressing f as a Taylor expansion around zero, we obtain

$$
\begin{aligned}
y(t) = f(0) &+ \frac{f'(0)}{2}(\sin\omega_1 t + \sin\omega_2 t) \\
&+ \frac{f''(0)}{6}(\sin\omega_1 t + \sin\omega_2 t)^2 + \cdots
\end{aligned}
$$

which gives terms of the form $\alpha_{m,n}\,\sin[(m\omega_1 + n\omega_2)t]$.

The Manley–Rowe equations describe the relative power in the various beat frequency components. In particular, consider the following nonlinear circuit driven by two sinusoidal voltage sources $v_1(t) = E_1\,\sin(\omega_1 t)$ and $v_2(t) = E_2\,\sin(\omega_2 t)$ with *incommensurable* frequencies ω_1 and ω_2 as shown in Fig. 9. Two frequencies ω_1 and ω_2 are *incommensurable* if their ratio ω_1/ω_2 is an irrational number, that is, there do not exist integers n and m such that $n\omega_1 + m\omega_2 = 0$. The ω_1 filter indicates an ideal filter, which is a short circuit for sinusoids with frequency ω_1 and an open circuit for all other frequencies. We assume that the capacitor is charge-controlled. Let $P_{m,n}$ denote the average power corresponding to the frequency $\omega_{m,n}$ which is flowing into the nonlinear

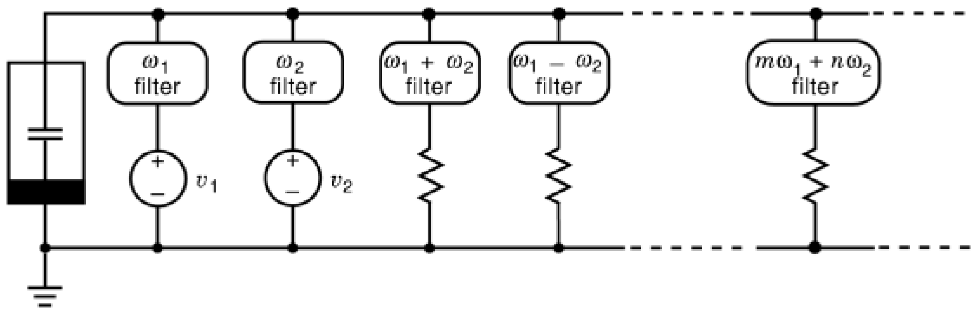

Fig. 9. A circuit driven by two periodic voltage sources illustrating the Manley–Rowe equations.

capacitor. The Manley–Rowe equations are expressed as (16):

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{mP_{m,n}}{\omega_{m,n}} = 0$$

and

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{nP_{m,n}}{\omega_{m,n}} = 0$$

The same result is valid when the capacitor in Fig. 9 is replaced by a flux-controlled inductor.

Duality

Many of the results in this chapter have dual counterparts. A theorem about *RC* circuits is also valid for *RL* circuits, etc., because of the dual properties of current and voltage. If we look at the circuit equations ($\mathbf{A}i = 0$, $\mathbf{B}v = 0$, constituency relationships, $i = dq/dt$, $v = d\phi/dt$) and interchange v with i and interchange q with ϕ, then we obtain another set of constrained differential equations. When do these correspond to the circuit equations of another circuit (the dual circuit)? The equations $i = dq/dt$, $v = d\phi/dt$ are interchanged after the variable interchange. For each constituency relationship, we get the constituency relationship of a dual element. For a resistor with constituency relationship $f(v, i) = 0$, the dual element is a resistor with constituency relationship $f(i, v) = 0$. For a capacitor with constituency relationship $g(q, v) = 0$, the dual element is an inductor with constituency relationship $g(\phi, i) = 0$. For an inductor with constituency relationship $h(\phi, i) = 0$, the dual element is a capacitor with constituency relationship $h(q, v) = 0$. Finally Kirchhoff's Laws becomes $\mathbf{A}v = 0$, $\mathbf{B}i = 0$. Are these the Kirchhoff Laws of another circuit? The answer is affirmative if and only if the underlying graph of the circuit is planar, that is, it can be drawn on the two-dimensional plane so that branches intersect only at the nodes.

For a circuit with a connected planar underlying graph, the dual circuit is found by the following algorithm (17):

(1) Draw the underlying graph in planar form with n nodes and b branches. It partitions the plane into $(b - n + 2)$ connected regions by Euler's formula. The dual graph has $(b - n + 2)$ nodes and b branches. Draw exactly one node of the dual graph lying in each region.

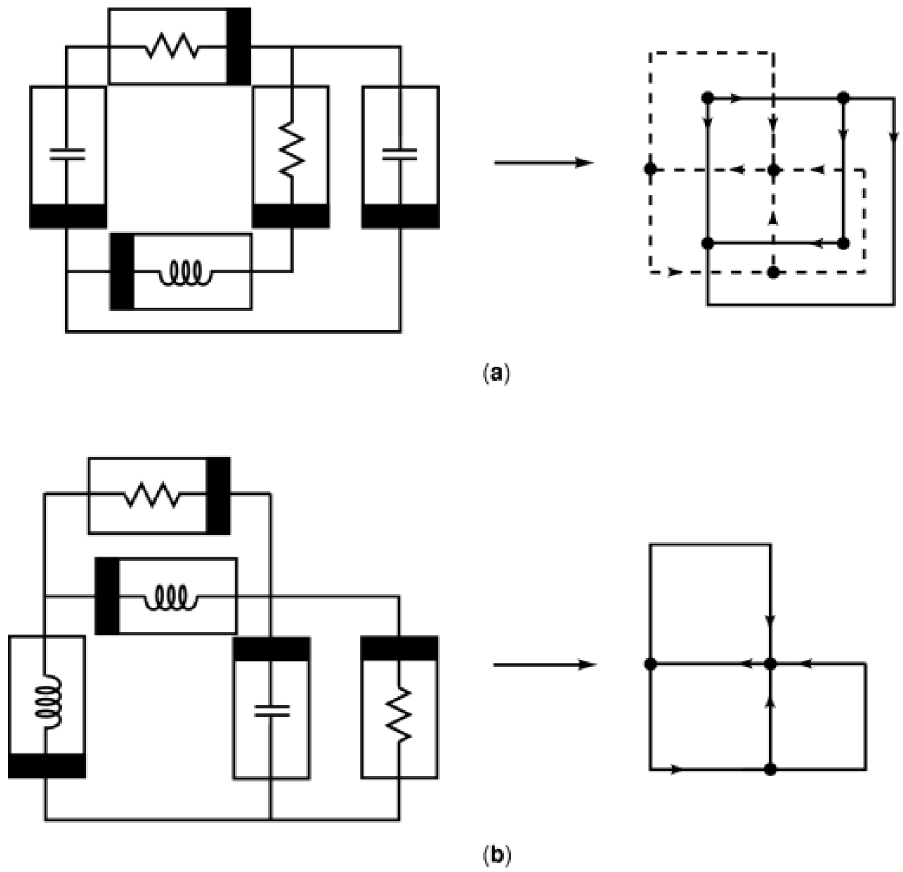

Fig. 10. (a) A nonlinear circuit and its associated graph. The dual graph is shown with dashed branches. (b) The dual circuit of (a) and its associated graph.

(2) Each branch of the graph lies exactly between two regions. Draw a branch of the dual graph connecting the nodes of the dual graph in these two regions.

(3) The direction of the branch in the dual graph is found as follows. For a branch in the graph, find the region adjacent to the branch which is bounded. If the branch encircles this region clockwise, the corresponding branch of the dual graph points toward the region. Otherwise, it points away from this region.

(4) A branch of the dual graph corresponds to each branch of the original graph. The dual circuit has the topology of the dual graph, and the element of each branch is the dual element of the original circuit at the corresponding branch.

For example, in Fig. 10(a), we show a circuit and its corresponding graph. The dual graph is shown with dashed lines. In Fig. 10(b), we show the dual circuit along with its corresponding graph. From the previous discussion, a circuit and its dual circuit have the same qualitative dynamics, because they share the same state equations except for a renaming of the variables.

Note that the dual of the dual circuit is the original circuit, but with all the nonlinear elements turned upside down.

Thus for a circuit with a planar underlying graph, the operation of turning all the elements upside down results in a circuit with the same qualitative dynamics as the original circuit. In fact the underlying graph needs not be planar.

Theorem 16. By turning all the elements of a circuit upside down, we obtain a circuit with the same state equations.

This can be shown as follows. Turning all the elements upside down reverses the directions of all the branch currents and branch voltages but does not change the constituency relationships nor the equations $i = dq/dt$ and $v = d\phi/dt$. If Kirchhoff's laws are $\mathbf{A}i = 0$, $\mathbf{B}v = 0$ for the original circuit, they are now $-\mathbf{A}i = 0$ and $-\mathbf{B}v = 0$ for the new circuit. So the same Kirchhoff laws for the original circuit also hold for the new circuit. Note that the underlying graph does not need to be planar for this theorem to be true.

Many of the results in this article can be extended to general circuits with multi-terminal devices. In some of the results, e.g., the results on completely stable behavior, an additional technical requirement of reciprocity is imposed on the circuit elements. Unfortunately, many of today's circuits contain nonreciprocal circuit elements such as transistors (17,18,19).

BIBLIOGRAPHY

1. L. O. Chua C. A. Desoer E. S. Kuh *Linear and Nonlinear Circuits*, New York: McGraw–Hill, 1987.
2. R. K. Miller A. N. Michel *Ordinary Differential Equations*, New York: Academic Press, 1982.
3. D. W. Jordan P. Smith *Nonlinear Ordinary Differential Equations*, Oxford Applied Mathematics And Computing Science Series, Oxford, 1977, Chap. 11, 333–338.
4. K. Murali M. Lakshmanan L. O. Chua The simplest dissipative nonautonomous chaotic circuit, *IEEE Trans. Circuits Syst. I, Fundam. Theory Appl.*, **41**: 462–463, 1994.
5. C. W. Wu G. Q. Zhong L. O. Chua Synchronizing nonautonomous chaotic systems without phase-locking, *J. Circuits, Syst., Comput.*, **6** (3): 227–241, 1996.
6. L. O. Chua *et al.* A universal circuit for studying and generating chaos, parts I–II, *IEEE Trans. Circuits Syst. I, Fundam. Theory Appl.*, **40**: 732–761, 1993. Special Issue on Chaos in Electronic Circuits, Part A.
7. L. O. Chua Dynamic nonlinear networks: State-of-the-art, *IEEE Trans. Circuits Syst.*, **27**: 1059–1087, 1980.
8. P. Hartman *Ordinary Differential Equations.* New York: Wiley, 1964.
9. J. P. LaSalle An invariance principle in the theory of stability, in *Differential Equations and Dynamical Systems* (J. K. Hale and J. P. LaSalle, eds.), New York: Academic Press, 1967, pp. 277–286.
10. C. Cherry Some general theorems for non-linear systems possessing reactance, *Philos. Mag.*, **42** (333): 1161–1177, 1951.
11. M. Hasler Qualitative analysis, in *The Circuits and Filters Handbook* (W.-K. Chen, ed.), Boca Raton: CRC Press, 1995, Chap. 31, pp. 914–934.
12. W. Millar Some general theorems for non-linear systems possessing resistance, *Philos. Mag.*, **42** (333): 1150–1160, 1951.
13. L. O. Chua Stationary principles and potential functions for nonlinear networks, *J. Franklin Inst.*, **296** (2): 91–114, 1973.
14. L. O. Chua D. N. Green A qualitative analysis of the behavior of dynamic nonlinear networks: Steady-state solutions of nonautonomous networks, *IEEE Trans. Circuits Syst.*, **23**: 530–550, 1976.
15. M. J. Hasler P. Verburgh On the uniqueness of the steady state for nonlinear circuits with time-dependent sources, *IEEE Trans. Circuits Syst.*, **31**: 702–713, 1984.
16. J. M. Manley H. E. Rowe Some general properties of nonlinear elements—part I. general energy relations, *Proc. IRE*, **44**: 904–913, July 1956.
17. L. O. Chua *Introduction to Nonlinear Network Theory*, New York: McGraw–Hill, 1964.
18. W. Rudin *Principles of Mathematical Analysis, 3rd Ed.*, New York: McGraw–Hill, 1990.
19. J. Guckenheimer P. Holmes *Nonlinear Oscillations, Dynamical Systems, and Bifurcations of Vector Fields*, New York: Springer-Verlag, 1983.
20. M. Vidyasagar *Nonlinear Systems Analysis*, Englewood Cliffs, NJ: Prentice-Hall, 1978.

READING LIST

Reference 17 is an excellent classic text on nonlinear circuit theory.

M. P. Kennedy L. O. Chua Circuit Theoretic Solutions for Neural Networks—an Old Approach to A New Problem, *Proc. IEEE ICNN*, San Diego, CA, June 1987, II-169–II-176. This paper uses the co-content function of a nonlinear network to study the stability of the Hopfield neural network

Reference 7 is a good source of more advanced results in the qualitative analysis of nonlinear circuits.

L. O. Chua Nonlinear Circuits, *IEEE Trans. Circuits Syst.*, **31** (1): 69–87, 1984. Contains extensive bibliography.

CHAI WAH WU
IBM T. J. Watson Research Center