VOLTERRA SERIES

Functional expansions are used in every branch of nonlinear system theory: identification and modelling, realization, stability, optimal control, stochastic differential equations and filtering, etc. Almost all the expansions used are of the Volterra type or, in the stochastic case, of the Wiener type. There exist a great number of publications on these expansions. Let us here mention only the early works by Wiener (1), Barrett (2), and George (3) and the two books by Rugh (4) and Schetzen (5).

After recalling the definition of the Volterra series expansion and some of its convergence issues, we will study various methods in order to derive the Volterra kernels and the response to typical inputs. The analysis is then applied to the study of weakly nonlinear circuits in order to derive distortion rates or intermodulation products.

FUNCTIONAL REPRESENTATION OF NONLINEAR SYSTEMS

Volterra Functional Series

For simplicity of presentation, we shall consider time-invariant systems. If a system is linear and time-invariant, then the output $y(t)$ can be expressed as the convolution of the input $u(t)$ with the system unit impulse response $h(t)$:

$$y(t) = \int_{-\infty}^{\infty} h(\tau)u(t-\tau)\,d\tau \tag{1}$$

The system unit impulse response $h(t)$ completely characterizes the linear time-invariant system since, once known, the response to any input can be determined from Eq. (1). A system is said to be causal if the output at any given time does not depend on future values of the input. That is, for any time t_1,

$$y(t_1) = \int_{-\infty}^{0} h(\tau)u(t_1-\tau)\,d\tau = 0$$

This will be so if and only if

$$h(\tau) = 0 \quad \text{for} \quad \tau < 0$$

The extension of Eq. (1) to nonlinear time-invariant systems with memory is the Volterra series

$$y(t) = h_0 + \sum_{n=1}^{\infty} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} h_n(\tau_1, \tau_2, \ldots, \tau_n) u(t-\tau_1)u(t-\tau_2) \cdots u(t-\tau_n)\,d\tau_1\,d\tau_2 \cdots d\tau_n \tag{2}$$

This functional form was first studied by Volterra. Much of his work in this area is summarized in his book (6). The functions $h_n(\tau_1, \tau_2, \ldots, \tau_n)$ are called the Volterra kernels of the system. A nonlinear system which can be represented by a Volterra series is completely characterized by its Volterra kernels. Also, with an argument similar to that of linear systems, it can be shown that the nonlinear system is causal if

$$h_n(\tau_1, \tau_2, \ldots, \tau_n) = 0 \quad \text{for} \quad \tau_j < 0, \quad j = 1, \ldots, n$$

It is well known that without loss of generality, the kernels can be assumed to be symmetric. In fact, any kernel $h_n(\tau_1, \tau_2, \ldots, \tau_n)$ can be replaced by a symmetric one by setting

$$h_n^{\text{sym}}(\tau_1, \tau_2, \ldots, \tau_n) = \frac{1}{n!} \sum_{(\tau_{i_1}, \tau_{i_2}, \ldots, \tau_{i_n}) \in S} h_n(\tau_{i_1}, \tau_{i_2}, \ldots, \tau_{i_n})$$

where S is the set of all permutations of $\tau_1 \ldots, \tau_n$.

The multiple Laplace transform $L[.]$ of the nth-order Volterra kernel $n > 0$ (one-sided in each variable)

$$H_n(s_1, \ldots, s_n) = \int_0^\infty \cdots \int_0^\infty h_n(\tau_1, \ldots, \tau_n) e^{-s_1\tau_1} \cdots e^{-s_n\tau_n}\, d\tau_1\, d\tau_2 \cdots d\tau_n$$

is called the nth-order transfer function. Since $h_n(\tau_1, \ldots, \tau_n)$ is symmetric, so is $H_n(s_1, \ldots, s_n)$.

On the Convergence of Volterra Series

The Volterra series is a nonlinear power series with memory (7). The nonlinearity can be seen by changing the input by a gain factor c so that the new input is $cu(t)$. By using Eq. (2), the new output is

$$y(t) = h_0 + \sum_{n=1}^{\infty} c^n \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} h_n(\tau_1, \tau_2, \ldots, \tau_n) u(t-\tau_1) u(t-\tau_2) \cdots u(t-\tau_n)\, d\tau_1\, d\tau_2 \cdots d\tau_n$$

which is a power series in the amplitude factor c. It is a series with memory since the integrals are convolutions. As a consequence of its power series character, there are some limitations associated with the application of the Volterra series to nonlinear problems. One major limitation is the convergence of this series.

In order to illustrate this let us consider the system of Fig. 1, where the system L is a linear system with the unit impulse response $h(t)$

$$z(t) = \int_{-\infty}^{\infty} h(\tau) u(t-\tau)\, d\tau \tag{3}$$

and the system N is a nonlinear no-memory system with the input-output relation

$$y(t) = N[z(t)] = \frac{z(t)}{1 + z^2(t)}$$

The Taylor series expansion of this expression is

$$y(t) = \sum_{n=0}^{\infty} (-1)^n [z(t)]^{2n+1} \tag{4}$$

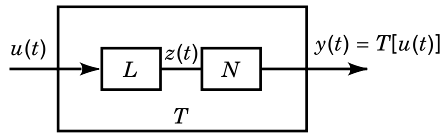

Figure 1. An example of a nonlinear system.

which converges only for $z^2(t) < 1$. The Volterra series representation of the overall system T is now easily derived by substituting Eq. (3) for Eq. (4) to obtain

$$y(t) = \sum_{n=0}^{\infty} (-1)^n \left[\int_{-\infty}^{\infty} h(\tau) u(t-\tau)\, d\tau \right]^{2n+1}$$

in which the Volterra kernels are

$$h_{2n+1}(\tau_1, \ldots, \tau_{2n+1}) = (-1)^n h(\tau_1) h(\tau_2) \ldots h(\tau_{2n+1})$$

and

$$h_{2n}(\tau_1, \ldots, \tau_{2n}) = 0, \quad n \geq 0$$

Since the Taylor series converges only for $z^2(t) < 1$, the above Volterra series will diverge at those times for which $|z(t)| \geq 1$. The Volterra series, thus, is valid only for the class of inputs $u(t)$ for which the amplitude of $z(t)$ is less than one.

Now let N be replaced by the following nonlinear, no-memory system

$$y(t) = E\,\text{sign}[z(t)]$$

Clearly, the system T cannot be represented by a Volterra series. It is, therefore, evident that generally, many types of nonlinear systems, such as those that include saturating elements, cannot be characterized by a Volterra series that converges for all inputs.

Proofs are presented in Volterra (6), Brillant (8), and Blackman (9) which show that under certain conditions, a functional $y(t) = T[x(t)]$ can be approximated to any desired degree of accuracy by a finite series of the form of Eq. (2). Such a functional is called continuous. In particular, it is easy to show that the functional relation between the solution (output) and the forcing function (input) of a nonlinear differential equation with constant coefficients which satisfies the Lipschitz condition is continuous. If $T[x(t)]$ can exactly be represented by a converging infinite series of the form of Eq. (2), it is called analytic or weak. Conditions for convergence are discussed by Volterra and Brillant. Brillant also notes that two special types of systems, for which the functional relation between input and output is analytic, are a linear system and a nonlinear no-memory system with a power series relation between input and output. He then shows that various combinations such as cascading, adding, or multiplying such systems results in an analytic system.

In practice, most of the analog circuits used in communication systems, such as modulators, mixers, amplifiers, harmonic oscillators, etc., are of a weak nature and, therefore, analyzed and designed in the frequency domain. For such weakly nonlinear circuits (having, say, distortion components of 20 dB or more below the fundamental one), the Volterra series technique can be readily used in the frequency domain to obtain results both quantitatively and qualitatively.

Given an input-output map described by a nonlinear control system $\dot{x} = f(x, u)$ and a nonlinear output $y = h(x)$, Lesiak and Krener (10) present a simple means for obtaining a series representation of the output $y(t)$ in terms of the input $u(t)$. When the control enters linearly, $\dot{x} = f(x) + ug(x)$, the method yields the existence of a Volterra series representation. The uniqueness of Volterra series representations is also discussed in (10). This work generalizes Brockett's technique (11), the work of Gilbert (12) and the method described by Bruni, Di Pillo, and Koch (13) for bilinear systems where explicit formulas for the calculation of the kernel functions were given. Later Boyd and Chua (14) show that any time-invariant continuous nonlinear operator can be approximated by a Volterra series.

Properties of the Multiple Laplace Transform

Before going on, let us recall some properties of the multiple Laplace transform (4). In the following list of results, one-sidedness is assumed, and the capital letter notation is used for transforms.

1. The Laplace transform operation is linear

$$L[f(\tau_1,\ldots,\tau_n)+\alpha g(\tau_1,\ldots,\tau_n)] = F(s_1,\ldots,s_n)+\alpha G(s_1,\ldots,s_n),\ \alpha\in\mathbf{R}$$

2. If $f(\tau_1, \ldots, \tau_n)$ can be written as a product of two factors of the form

$$f(\tau_1,\ldots,\tau_n)=h(\tau_1,\ldots,\tau_k)g(\tau_{k+1},\ldots,\tau_n)$$

then

$$F(s_1,\ldots,s_n)=H(s_1,\ldots,s_k)G(s_{k+1},\ldots,s_n)$$

3. If $f(\tau_1, \ldots, \tau_n)$ can be written as a convolution of the form

$$f(\tau_1,\ldots,\tau_n)=\int_0^\infty h(\sigma)g(\tau_1-\sigma,\ldots,\tau_n-\sigma)\,d\sigma$$

then

$$F(s_1,\ldots,s_n)=H(s_1+\cdots+s_n)G(s_1,\ldots,s_n)$$

4. If $f(\tau_1, \ldots, \tau_n)$ can be written as an n-fold convolution of the form

$$f(\tau_1,\ldots,\tau_n)=\int_0^\infty \cdots\int_0^\infty h(\tau_1-\sigma_1,\ldots,\tau_n-\sigma_n)g(\sigma_1,\ldots,\sigma_n)\,d\sigma_1\ldots d\sigma_n$$

then

$$F(s_1,\ldots,s_n)=H(s_1,\ldots,s_n)G(s_1,\ldots,s_n)$$

5. If $c_1, \ldots, c_n$ are nonnegative constants, then

$$L[f(\tau_1-c_1,\ldots,\tau_n-c_n)]=F(s_1,\ldots,s_n)\mathrm{e}^{-s_1c_1\cdots-s_nc_n}$$

6. If $f(\tau_1, \ldots, \tau_n)$ is given by the product

$$f(\tau_1,\ldots,\tau_n)=h(\tau_1,\ldots,\tau_n)g(\tau_1,\ldots,\tau_n)$$

then

$$F(s_1,\ldots,s_n)=\frac{1}{(2\pi i)^n}\int_{\sigma_n-i\infty}^{\sigma_n+i\infty}\cdots\int_{\sigma_1-i\infty}^{\sigma_1+\infty}H(s_1-w_1,\ldots,s_n-w_n)G(w_1,\ldots,w_n)\,dw_1\cdots dw_n$$

Restricting our attention to one-sided input signals, and using the convolution property of the Laplace transform, the input-output relation for a stationary linear system

$$y(t)=\int_{-\infty}^\infty h(\tau)u(t-\tau)\,d\tau=\int_0^t h(\tau)u(t-\tau)\,d\tau$$

can be written in the form

$$Y(s)=H(s)U(s) \tag{5}$$

Therefore, if a system transfer function $H(s)$ is known, and the input signal of interest has a simple Laplace transform $U(s)$, then the utility of this representation for computing the corresponding output signal is clear. Let us now consider a homogeneous system of degree n with one-sided input signals represented by

$$\begin{aligned} y(t)&=\int_{-\infty}^\infty\cdots\int_{-\infty}^\infty h_n(\tau_1,\tau_2,\ldots,\tau_n)u(t_1-\tau_1)u(t_2-\tau_2)\cdots u(t_n-\tau_n)\,d\tau_1\,d\tau_2\cdots d\tau_n\\ &=\int_0^t\cdots\int_0^t h_n(\tau_1,\tau_2,\ldots,\tau_n)u(t-\tau_1)u(t-\tau_2)\cdots u(t-\tau_n)\,d\tau_1\,d\tau_2\cdots d\tau_n \end{aligned} \tag{6}$$

Inspection of the above list of properties of the multivariable Laplace transform yields no direct way to write this in a form similar to Eq. (5). Therefore, an indirect approach is adopted by writing Eq. (6) as a pair of equations

$$\begin{aligned} y_n(t_1,\ldots,t_n)&=\int_0^{t_1}\cdots\int_0^{t_n} h_n(\tau_1,\tau_2,\ldots,\tau_n)u(t-\tau_1)u(t-\tau_2)\cdots u(t-\tau_n)\,d\tau_1\,d\tau_2\cdots d\tau_n\\ y(t)&=y_n(t_1,\ldots,t_n)|_{t_1=\cdots=t_n=t}=y_n(t,\ldots,t) \end{aligned} \tag{7}$$

Now, Eq. (7) yields

$$Y_n(s_1,\ldots,s_n)=H_n(s_1,\ldots,s_n)U(s_1)\cdots U(s_n) \tag{8}$$

where $H_m(s_1, \ldots, s_n) = L[h(t_1, \ldots, t_n)]$ is a (multivariable) transfer function of the homogeneous system. Therefore, given $H_n(s_1, \ldots, s_n)$ and $U(s)$, it is easy to compute $Y_n(s_1, \ldots, s_n)$. However, the inverse Laplace transform must be computed before $y(t)$ can be found, and often this is not easy.

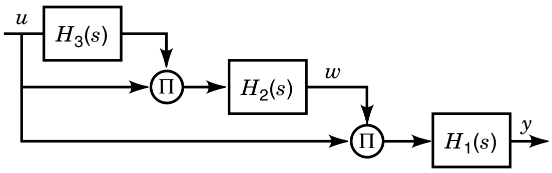

Figure 2. Association of three linear systems.

Example: The overall transfer function of the system shown in Fig. 2 is

$$H(s_1, s_2, s_3) = H_1(s_1 + s_2 + s_3)H_2(s_1 + s_2)H_3(s_1)$$

RECURSIVE COMPUTATION OF THE KERNELS

Several methods have been developed in the literature for determining the kernels or the associated transfer functions based on the classical symbolic method of Brillant (8), George (3), Bedrosian and Rice (15), Bussgang, Ehrman and Graham (16), Chua and Ng (17), and Flake (18). Among them, the method of exponential inputs is particularly used. After recalling this method, we describe a differential geometry approach (10) and an algebraic approach based on generating power series (19) when the system is described by a set of differential equations. We shall see that the algebraic approach has the advantage of being easily implementable on a computer by using algebraic computing software.

Exponential Input Method

Let us consider the Volterra series expansion of a nonlinear system of the form

$$y(t) = \sum_{n=1}^{\infty} \int_0^{t_1} \cdots \int_0^{t_n} h_n(\tau_1, \tau_2, \ldots, \tau_n)u(t-\tau_1)u(t-\tau_2) \cdots u(t-\tau_n)\, d\tau_1\, d\tau_2 \cdots d\tau_n \tag{9}$$

Let the input $u(t)$ be a sum of exponentials

$$u(t) = \mathrm{e}^{s_1 t} + \mathrm{e}^{s_2 t} + \cdots + \mathrm{e}^{s_k t}$$

where $s_1, s_2, \ldots, s_k$ are rationally independent. This means that there are no rational numbers $\alpha_1, \alpha_2, \ldots, \alpha_k$ such that the sum $\alpha_1 s_1 + \alpha_2 s_2 + \cdots \alpha_k s_k$ is rational. Then Eq. (9) becomes

$$y(t) = \sum_{n=1}^{\infty} \left[\sum_{k_1=1}^{k} \cdots \sum_{k_n=1}^{k} H_n(s_{k_1}, \ldots, s_{k_n})\mathrm{e}^{(s_{k_1}+\cdots+s_{k_n})t} \right] \tag{10}$$

If each s_i occurs in $(s_{k_1}, \ldots, s_{k_n})$ m_i times, then there are

$$\frac{n!}{m_1!m_2!\ldots m_k!}$$

identical terms in the expression between brackets. Thus, Eq. (10) can be written in the form

$$y(t) = \sum_{n=1}^{\infty} \sum_{m} \frac{n!}{m_1!m_2!\ldots m_k!} H_n(s_{k_1}, \ldots, s_{k_n})\mathrm{e}^{(s_{k_1}+\cdots+s_{k_n})t} \tag{11}$$

where m under the summation sign indicates that the sum includes all the distinct vectors $(m_1, \ldots, m_k)$ such that $\sum_{i=1}^{k} m_i = n$. Note that if $m_1 = m_2 = \cdots = m_k = 1$, then the amplitude associated with the exponential component $\mathrm{e}^{(s_{k_1}+\cdots+s_{k_n})t}$ is simply $k!H_k(s_1, \ldots, s_k)$. This suggests a recursive procedure for determining all the nonlinear transfer functions from the behavior of a system.

Let us apply the method to the simple nonlinear circuit (16) of Fig. 3 consisting of a capacitor, a linear resistor, and a nonlinear resistor in parallel with the current source $i(t)$.

The nonlinear differential equation relating the current excitation $i(t)$ and the voltage $v(t)$ across the capacitor is given by

$$\dot{v} + k_1 v + k_2 v^2 = i \tag{12}$$

Let $i(t) = e^{st}$. Equating the coefficients of e^{st} on both sides of Eq. (12) after the substitution of (11) for $v(t)$ we get

$$H_1(s) = \frac{1}{s + k_1}$$

In order to determine $H_2(s_1, s_2)$, let us take $i(t) = e^{s_1 t} + e^{s_2 t}$ and identify the coefficient of the term $2!e^{(s_1+s_2)t}$ after the substitution of Eq. (11) for $v(t)$ in both sides of Eq. (12). We obtain $H_2(s_1, s_2)$ in term of $H_1(s)$ as follows

$$H_2(s_1, s_2) = -k_2 H_1(s_1)H_1(s_2)H_1(s_1 + s_2)$$

Similarly, the third-order transfer function is obtained by injecting a sum of three exponentials inputs

$$i(t) = e^{s_1 t} + e^{s_2 t} + e^{s_3 t}$$

It follows

$$H_3(s_1, s_2, s_3) = -\frac{2}{3}[H_2(s_1, s_2)H_1(s_3) + H_2(s_2, s_3)H_1(s_1) + H_2(s_1, s_3)H_1(s_2)]h_1(s_1 + s_2 + s_3)$$

Repeating this process indefinitely gives higher order nonlinear transfer functions in terms of lower-order nonlinear transfer functions.

Differential Geometry Approach

Consider a control system Σ of the general form (10)

$$\dot{x} = f(x, u), \quad x(0) = x^0 \quad y = h(x)$$

where the input takes values in $\mathbf{R}^l$, the state x is an element of $\mathbf{R}^m$, and the output y takes values in $\mathbf{R}^n$. The vector field f

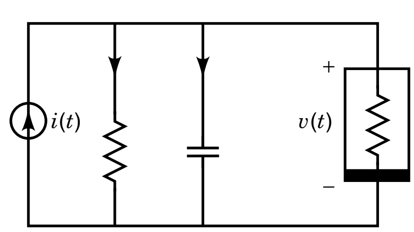

Figure 3. A simple nonlinear circuit.

and the output function h are assumed to possess a sufficient degree of smoothness. The input function u belongs to $L^1([0, T], \mathbf{R}^l)$, the space of absolutely integrable functions on $[0, T]$, or belongs to $L^\infty([0, T], \mathbf{R}^l)$, the space of bounded and measurable functions on $[0, T]$. In either case, the output is a member of the space of continuous functions $C^0([0, T], \mathbf{R}^n)$. Therefore, it is natural to associate with Σ the input-output map

$$\Phi : L^1([0,T], \mathbf{R}^l) \quad \text{or} \quad L^\infty([0,T], \mathbf{R}^l) \to C^0([0,T], \mathbf{R}^n)$$

Definition: Φ has a Volterra series representation if there exists a set of kernels $h_0, h_1, h_2, \ldots$ such that

1. h_0 is defined on $[0, T]$, and h_i, $i = 1, 2, \ldots$, is defined on $\{(t, \tau_1, \ldots, \tau_i) | 0 \le \tau_i \le \cdots \le \tau_1 \le t \le T\}$
2. Each h_i, $i = 0, 1, 2, \ldots$ is continuous on its domain of definition
3. There exists a $\delta > 0$ such that whenever $\|u\| < \delta$

$$\Phi(u)(t) = h_0(t) + \sum_{i=1}^{\infty} \int_0^t \int_0^{\tau_1} \cdots \int_0^{\tau_{i-1}} h_i(t, \tau_1, \ldots, \tau_i) u(\tau_1) \cdots u(\tau_i)\, d\tau_i \cdots d\tau_1$$

The series converges in norm topology on $C^0([0, T], \mathbf{R}^n)$ for all $\|u\| < \delta$.

Theorem: Let f, g be analytic vector fields and h an analytic function. If $\dot{x} = f(x)$, and $x(0) = x^0$ has a solution on $[0, T]$, then Φ has a Volterra series representation, and it is unique.

The proof of this theorem is given by Lesiak and Krener in (10). Let us sketch the idea. For simplicity of notation, the input and output are taken to be scalar valued. Let $\gamma_0(t, \tau, x)$ denote the solution of the differential equation

$$\frac{d}{dt}\gamma_0(t, \tau, x) = f(\gamma_0(t, \tau, x))$$

such that

$$\gamma_0(\tau, \tau, x) = x$$

Given u, let $\gamma_u(t, \tau, x)$ be the solution of the differential equation

$$\frac{d}{dt}\gamma_u(t, \tau, x) = f(\gamma_u(t, \tau, x)) + u(t) g(\gamma_u(t, \tau, x))$$

satisfying

$$\gamma_u(\tau, \tau, x) = x$$

For a fixed t, the curve $\rho(\tau) = \gamma_0(t, \tau, \gamma_u(\tau, 0, x^0))$ satisfies $\rho(0) = \gamma_0(t, 0, x^0)$ and $\rho(t) = \gamma_u(t, 0, x^0)$. Further, for any smooth function h, the fundamental theorem of calculus yields

$$h\big(\gamma_u(t, 0, x^0)\big) = h\big(\gamma_0(t, 0, x^0)\big) + \int_0^t \frac{d}{d\tau} h(\rho(\tau))\, d\tau \qquad (13)$$

Direct calculations yield

$$\frac{d}{d\tau} h(\rho(\tau)) = u(\tau) \left[\frac{\partial h(\gamma_0(t, \tau, x))}{\partial x} g(x) \right]_{x = \gamma_u(\tau, 0, x^0)}$$

Given

$$h_0(t) = h(\gamma_0(t, 0, x^0))$$

and

$$w_1(t, \tau, x) = \left[\frac{\partial h(\gamma_0(t, \tau, x))}{\partial x} g(x) \right]_{x = \gamma_u(\tau, 0, x^0)}$$

Eq. (13) can be reduced to

$$h\big(\gamma_u(t, 0, x^0)\big) = h_0(t) + \int_0^t u(\tau_1) w_1\big(t, \tau_1, \gamma_u(\tau_1, 0, x^0)\big)\, d\tau_1 \quad (14)$$

Replacing $h(.)$ by $w_1(t, \tau_1, .)$ and applying Eq. (13) yields

$$w_1(t\tau_1, \gamma_u(\tau_1, 0, x^0)) = h_1(t, \tau_1) + \int_0^{\tau_1} u(\tau_2) w_2(t, \tau_1, \tau_2, \gamma_u(\tau_2, 0, x^0))\, d\tau_2$$

where

$$h_1(t, \tau_1) = w_1(t, \tau_1, \gamma_0(\tau_1, 0, x^0))$$

and

$$w_2(t, \tau_1, \tau_2, x) = \frac{\partial w_1(t, \tau_1, \gamma_0(t, \tau_1, x))}{\partial x} g(x)$$

Hence, Eq. (14) becomes

$$h(\gamma_u(t, 0, x^0)) = h_0(t) + \int_0^t h_1(t\tau_1) u(\tau_1)\, d\tau_1 + \int_0^t \int_0^{\tau_1} w_2(t, \tau_1, \tau_2, \gamma_0(\tau_2, 0, x^0)) u(\tau_1) u(\tau_2)\, d\tau_2\, d\tau_1$$

After k repetitions of this process, we obtain the output representation

$$h(\gamma_u(t, 0, x^0)) = h_0(t) + \sum_{i=1}^{k} \int_0^t \int_0^{\tau_1} \cdots \int_0^{\tau_{i-1}} h_i(t, \tau_1, \tau_2, \ldots, \tau_i) u(\tau_1) \cdots u(\tau_i)\, d\tau_i \cdots d\tau_1 + \int_0^t \int_0^{\tau_1} \cdots \int_0^{\tau_k} w_{k+1}(t, \tau_1, \ldots, \tau_{k+1}), \gamma_0(\tau_2, 0, x^0)) u(\tau_1) u(\tau_2) \cdots u(\tau_{k+1})\, d\tau_{k+1} \cdots$$

with

$$h_i(t, \tau_1, \tau_2, \ldots, \tau_i) = w_i(t, \tau_1, \ldots, \tau_i, \gamma_0(\tau_1, 0, x^0))$$

and

$$w_i(t, \tau_1, \ldots, \tau_i, x) = \frac{\partial w_{i-1}(t, \tau_1, \ldots, \tau_{i-1}, \gamma_0(\tau_{i-1}, \tau_i, x))}{\partial x} g(x)$$

Continuing indefinitely, we generate the Volterra series representation in Eq. (13).

Algebraic Approach

Fliess' algebraic framework (19) summarized below allows deriving an explicit expression of the Volterra kernel by using an algebraic computing software.

Let us recall some definitions and results from this algebraic approach (20). Let $u_1(t)$, $u_2(t)$, . . ., $u_m(t)$ be some piecewise continuous inputs and $Z = \{z_0, z_1, . . ., z_m\}$ be a finite set called alphabet. We denote by Z^* the set of words generated by Z. The algebraic approach introduced by Fliess may be sketched as follows. Let us consider the letter z_0 as an operator which codes the integration with respect to time and the letter z_i, $i = 1, . . ., m$, as an operator which codes the integration with respect to time after multiplying by the input $u_i(t)$. In this way, any word $w \in Z^*$ gives rise to an iterated integral, denoted by $\mathbf{I}^t\{w\}$, which can be defined recursively as follows:

$$\mathbf{I}^t\{\theta\} = 1$$

$$\mathbf{I}^t\{w\} = \begin{cases} \int_0^t d\tau \mathbf{I}^\tau\{v\} & \text{if } w = z_0 v \\ \int_0^t u_i(\tau)\, d\tau \mathbf{I}^\tau\{v\} & \text{if } w = z_1 v \end{cases}, \quad v \in Z^* \tag{15}$$

Using the previous formalism and an iterative scheme, the solution $y(t)$ of the nonlinear control system

$$\begin{cases} \dot{x}(t) = f(x(t)) + \sum_{i=1}^m u_i(t) g_i(x(t)), \quad x(0) = x_0 \\ y(t) = h(x(t)) \end{cases} \tag{16}$$

may be written (20)

$$y(t) = h(x_0) + \sum_{\nu 0} \sum_{j_0, j_1, \ldots, j_\nu = 0}^{m} L_{f_{j_\nu}} \cdots L_{f_{j_2}} L_{f_{j_1}} h(x_0) \mathbf{I}^t\{z_{j_0} z_{j_1} \cdots z_{j_{nu}}\} \tag{17}$$

with the series converging uniformly for a small t and small $|u_i(\tau)|$, $0 \le \tau \le t$; $i = 1, . . ., m$. This functional expansion is called the Fliess fundamental formula or Fliess expansion of the solution. To this expansion, it can also be associated (20) with an absoluting converging power series for small t and small $|u_i(\tau)|$, $0 \le \tau \le t$; $i = 1, . . ., m$, called the Fliess generating power series or Fliess series denoted by $\mathbf{g}$ of the following form

$$\mathbf{g} = h(x_0) + \sum_{\nu 0} \sum_{j_0, j_1, \cdots, j_\nu = 0}^{m} L_{f_{j_\nu}} \cdots L_{f_{j_2}} L_{f_{j_1}} h(x_0) z_{j_0} z_{j_1} \cdots z_{j_\nu} \tag{18}$$

This algebraic setting allows us to generalize the Heaviside calculus for linear system to the nonlinear domain. This will clearly appear in the next section devoted to the efficient computation of the Volterra series.

Links Between Volterra and Fliess Series

The following result (19) gives the expression of the Volterra kernels of the response of the nonlinear control system [Eq. (16)] in terms of the vector fields and the output function defining the system,

$$y(t) = w_0(t) + \sum_{n=1}^{\infty} \int_0^t \int_0^{\tau_2} \cdots \int_0^{\tau_n} w_n(t, \tau_n, \ldots, \tau_1) u(\tau_n) \cdots u(\tau_1)\, d\tau_n \cdots d\tau_1 \tag{19}$$

where the kernels are analytic functions of the form

$$\begin{aligned} w_0(t) &= \textstyle\sum_{\nu 0} L_f^\nu h(x_0) \frac{t^\nu}{\nu!} = e^{tL_f} h(x_0) \\ w_1(t, \tau_1) &= \textstyle\sum_{\nu_0, \nu_1 0} L_f^{\nu_0} L_g L_f^{\nu_1} h(x_0) \frac{(t-\tau_1)^{\nu_1} \tau_1^{\nu_0}}{\nu_1! \nu_0!} \\ &= e^{\tau_1 L_f} L_g e^{(t-\tau_1) L_f} h(x_0) \\ &\vdots \\ w_n(t, \tau_n, \tau_{n-1}, \ldots, \tau_1) &= \textstyle\sum_{\nu_0, \nu_1, \ldots, \nu_n 0} L_f^{\nu_0} L_g L_f^{\nu_1} \cdots L_g L_f^{\nu_n} h(x_0) \\ &\quad \frac{(t-\tau_n)^{\nu_n} \ldots \tau_1^{\nu_0}}{\nu_n! \ldots \nu_0!} \\ &= e^{\tau_1 L_f} L_g e^{(\tau_2 - \tau_1) L_f} \cdots L_g e^{(t-\tau_n) L_f} h(x_0) \end{aligned} \tag{20}$$

In order to show this, let us use the fundamental formula [Eq. (17)]. The zero order kernel is the free response of the system. Indeed, from Eq. (17) we have

$$w_0(t) = h(x_0) + \sum_{\nu 0} \sum_{j_0, \ldots, j_\nu = 0} L_{f_{j_\nu}} \cdots L_{f_{j_\nu}} L_{f_{j_1}} h(x_0) \int_0^t d\xi_{j_\nu} d\xi_{j_{\nu-1}} \cdots d\xi_{j_0}$$

which can also be written

$$y(t) = \sum_{l 0} L_f^l h(x_0) \frac{t^l}{l!}$$

or using a formal notation,

$$y(t) = e^{tL_f} h(x_0)$$

This formula is nothing else than the classical formula given by Gröbner (21).

For the computation of the first-order kernel, let us consider the terms of Eq. (17) which contain only one contribution of the input u. Therefore,

$$\int_0^t w_1(t, \tau_1) u(\tau_1)\, d\tau_1 = \sum_{\nu_0, \nu_1 0} L_f^{\nu_0} L_g L_f^{\nu_1} h(x_0) \int_0^t \underbrace{d\xi_0 \cdots d\xi_0}_{\nu_1 - \text{times}} d\xi_1 \underbrace{d\xi_0 \cdots d\xi_0}_{\nu_0 - \text{times}}$$

But the iterated integral inside can be proved to be equal to

$$\int_0^t \frac{(t-\tau_1)^{\nu_1} \tau_1^{\nu_0}}{\nu_1! \nu_0!} u(\tau_1)\, d\tau_1$$

So, the first order kernel may be written as

$$w_1(t,\tau_1) = \sum_{\nu_0,\nu_1 0} L_f^{\nu_0} L_g L_f^{\nu_1} h(x_0) \frac{(t-\tau_1)^{\nu_1}\tau_1^{\nu_0}}{\nu_1!\nu_0!} = e^{\tau_1 L_f} L_g e^{(t-\tau_1)L_f} h(x_0)$$

For the computation of the second order kernel, let us regroup the terms of Eq. (17) which contain exactly two contributions of the input u; therefore,

$$\int_0^t \int_0^{\tau_2} w_2(t,\tau_1,\tau_2) u(\tau_1) u(\tau_2)\, d\tau_1\, d\tau_2$$

$$\sum_{\nu_0,\nu_1,\nu_2 0} L_f^{\nu_0} L_g L_f^{\nu_1} L_g L_f^{\nu_2} h(x_0) \int_0^t \int_0^{\tau_2} \underbrace{d\xi_0 \cdots d\xi_0}_{\nu_2-\text{times}} d\xi_1 \underbrace{d\xi_0 \cdots d\xi_0}_{\nu_1-\text{times}} d\xi_1 \underbrace{d\xi_0 \cdots d\xi_0}_{\nu_0-\text{times}}$$

The iterated integral inside this expression can be proved to be equal to

$$\int_0^t \int_0^{\tau_2} \frac{(t-\tau_2)^{\nu_2}(\tau_2-\tau_1)^{\nu_1}\tau_1^{\nu_0}}{\nu_2!\nu_1!\nu_0!} u(\tau_1)u(\tau_2)\, d\tau_1\, d\tau_2$$

Thus, the second-order kernel may be written as

$$w_2(t,\tau_1,\tau_2) = \sum_{\nu_0,\nu_1,\nu_2 0} L_f^{\nu_0} L_g L_f^{\nu_1} L_g L_f^{\nu_2} h(x_0) \frac{(t-\tau_2)^{\nu_2}(\tau_2-\tau_1)^{\nu_1}\tau_1^{\nu_0}}{\nu_2!\nu_1!\nu_0!} = e^{\tau_2 L_f} L_g e^{(\tau_1-\tau_2)L_f} L_g e^{(t-\tau_1)L_f} h(x_0)$$

The higher-order kernels are obtained in the same way.

Using the Campbell-Baker-Hausdorff formula (21)

$$e^{\sigma L_f} L_g e^{-\sigma L_f} h(x_0) = \sum_{i=1}^{\infty} \frac{\sigma^i}{i!} ad_{L_f}^i L_g$$

the expressions for the kernels Eq. (20) may be written

$$\begin{aligned} w_0(t) &= e^{tL_f} h(x_0) \\ w_1(t,\tau_1) &= e^{\tau_1 L_f} L_g e^{(t-\tau_1)L_f} h(x_0) \\ &= \sum_{i=1}^{\infty} \frac{\tau_1^i}{i!} ad_{L_f}^i L_g e^{tL_f} h(x_0) \end{aligned} \tag{21}$$

$$\begin{aligned} w_2(t,\tau_n,\tau_{n-1},\ldots,\tau_1) &= e^{\tau_1 L_f} L_g e(\tau_2-\tau_1) L_f L_g e^{(t-\tau_2)L_f} h(x_0) \\ &= \sum_{i,j=1}^{\infty} \frac{\tau_1^i}{i!}\frac{\tau_2^j}{j!} ad_{L_f}^i L_g ad_{L_f}^j L_g e^{tL_f} h(x_0) \\ &\vdots \end{aligned} \tag{22}$$

These kernel expressions lead to techniques which can, for example, be used in singular optimal control problems (22). This will be sketched in a next section.

However, efficient computation remains an open problem for the moment. Indeed, the computation of the operator

$$L_f^{\nu_0} L_g L_f^{\nu_1} \cdots L_g L_f^{\nu_n} h(x_0) \tag{23}$$

is very heavy in general. Let us consider, for instance, the Duffing equation,

$$\ddot{y} + a\dot{y} + by + cy^3 = u(t)$$

or

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -ax_2 - bx_1 - cx_1^3 + u(t) \end{cases}$$
$$y = x_1$$

Here,

$$L_f = x_2 \frac{\partial}{\partial x_1} - (ax_2 + bx_1 + c_1 x_1^3)\frac{\partial}{\partial x_2}$$

and

$$L_g = \frac{\partial}{\partial x_2}$$

In the following, we will show, through a simple example, how to obtain the algebraic expression of the terms of the Volterra series and how to derive the expression of the Volterra kernels.

Efficient Computation of Volterra Kernels

Let us consider the system (4),

$$\ddot{y}(t) + (\omega^2 + u(t))y(t) = 0, \quad t \ge 0, \quad y(0) = 0, \quad \dot{y}(0) = 1$$

After two integrations, we obtain

$$y(t) + \omega^2 \int_0^t \int_0^{\tau} y(\sigma)\, d\sigma\, d\tau + \int_0^t \int_0^{\tau} u(\sigma) y(\sigma)\, d\sigma\, d\tau - t = 0$$

The associated algebraic equation for $\mathbf{g}$ is

$$(1 + \omega^2 z_0^2)\mathbf{g} + z_0 z_1 \mathbf{g} - z_0 = 0$$

In order to solve this equation, let us use the following iterative scheme

$$\mathbf{g} = \mathbf{g}_0 + \mathbf{g}_1 + \mathbf{g}_2 + \cdots + \mathbf{g}_i + \cdots$$

where $\mathbf{g}_i$ contains all the terms of the solution $\mathbf{g}$ having exactly i occurrences in the variable z_1,

$$\begin{aligned} \mathbf{g}_0 &= (1+\omega^2 z_0^2)^{-1} z_0 \\ \mathbf{g}_1 &= -(1+\omega^2 z_0^2)^{-1} z_0 z_1 \mathbf{g}_0 = -(1+\omega^2 z_0^2)^{-1} z_0 z_1 (1+\omega^2 z_0^2)^{-1} z_0 \\ \mathbf{g}_2 &= -(1+\omega^2 z_0^2)^{-1} z_0 z_1 \mathbf{g}_1 \\ &\quad (1+\omega^2 z_0^2)^{-1} z_0 z_1 (1+\omega^2 z_0^2)^{-1} z_0 z_1 (1+\omega^2 z_0^2)^{-1} z_0 \\ &\vdots \end{aligned}$$

Each g_i, $i = 0, 1, 2, \ldots$ is a (rational) generating power series of analytic causal functionals y_i, $i = 0, 1, 2, \ldots$ which represents the ith order term of the Volterra associated with the solution $y(t)$. Let us now compute $y_i(t)$, $i0$.

First,

$$\mathbf{g}_0 = -\frac{1}{2j\omega}(1+j\omega z_0)^{-1} + \frac{1}{2j\omega}(1-j\omega z_0)^{-1}$$

and

$$y_0(t) = w_0(t) = -\frac{1}{2j\omega}e^{-j\omega t} + \frac{1}{2j\omega}e^{j\omega t} = \frac{1}{\omega}\sin(wt)$$

The power series

$$\mathbf{g}_1 = -(1+\omega^2 z_0^2)^{-1} z_0 z_1 (1+\omega^2 z_0^2)^{-1} z_0$$

after decomposing into partial fractions the term on the right-hand side and on the left-hand side of z_1,

$$\left[\frac{1}{2j\omega}(1+j\omega z_0)^{-1} - \frac{1}{2j\omega}(1-j\omega z_0)^{-1}\right] z_1 \left[-\frac{1}{2j\omega}(1+j\omega z_0)^{-1} + \frac{1}{2j\omega}(1-j\omega z_0)^{-1}\right]$$

or

$$\begin{aligned}\mathbf{g}_1 = \frac{1}{4\omega^2}[&(1+j\omega z_0)^{-1} z_1 (1+j\omega z_0)^{-1} \\ &- (1+j\omega z_0)^{-1} z_1 (1-j\omega z_0)^{-1} \\ &- (1-j\omega z_0)^{-1} z_1 (1+j\omega z_0)^{-1} \\ &+ (1-j\omega z_0)^{-1} z_1 (1-j\omega z_0)^{-1}]\end{aligned}$$

In order to obtain the equivalent expression in the time domain, we need the following result (23).

The rational power series can be written as

$$(1-a_0 z_0)^{-p_0} z_1 (1-a_1 z_0)^{-p_1} z_1 \ldots z_1 (1-a_l z_0)^{-p_l} \tag{24}$$

where $a_0, a_1, \ldots, a_l \in \mathbf{C}$, $p_0, p_1, \ldots, p_l \in \mathbf{N}$, in the symbolic representation of

$$\int_0^t \int_0^{\tau_l} \cdots \int_0^{\tau_2} f_{a_0}^{p_0}(t-\tau_l)\ldots f_{a_{l-1}}^{p_{l-1}}(\tau_2-\tau_1) f_{a_l}^{p_l}(\tau_1) u(\tau_l) \ldots u(\tau_1) d\tau_l \ldots d\tau_1 \tag{25}$$

where $f_a^p(t)$ denotes the exponential polynomial

$$\left(\sum_{j=0}^{p-1} \frac{\binom{j}{p-1}}{j!} a^j t^j\right) e^{at}$$

From the previous example, we can see

$$\begin{aligned}y_1(t) = &\int_0^t \frac{1}{2j\omega} e^{-j\omega(t-\tau)} u(\tau)\left[\frac{-1}{2j\omega}e^{-j\omega\tau} + \frac{1}{2j\omega}e^{j\omega\tau}\right] d\tau \\ &- \int_0^t \frac{1}{2j\omega} e^{j\omega(t-\tau)} u(\tau)\left[\frac{-1}{2j\omega}e^{-j\omega\tau} + \frac{1}{2j\omega}e^{j\omega\tau}\right] d\tau\end{aligned}$$

Therefore,

$$y_1(t) = \int_0^1 w_1(t,\tau) u(\tau)\, d\tau$$

with $w_1(t, \tau) = -1/\omega^2 \sin[\omega(t-\tau)] \sin \omega t$

The higher-order kernel can be computed in the same way after decomposing into partial fractions each rational power series. A recent implementation of this algorithm can be found in (24).

COMPUTATION OF THE RESPONSE TO TYPICAL INPUTS

The next objective is to show how the Volterra series can be used to determine the output of a system subject to various deterministic excitations (steps, slopes, harmonics, etc.). In the linear case, Laplace and Fourier transforms are systematic and powerful tools of operational calculus. A direct generalization of these techniques to the nonlinear domain leads to multidimensional Laplace and Fourier transforms, but the computation based on these transforms is often tedious, even for low-order Volterra kernels, and seems difficult to implement on a computer. An alternative method, presented here, based on noncommutative variables and on the properties of iterated integrals leads to a simple nonlinear generalization of Heaviside symbolic calculus and to an easy implementation on a computer. It is compared with the association of variables introduced by George (3) and which we shall now briefly recall.

Transfer Function Approach: Association of Variables

If the Volterra kernels are known for a system, then the output $y(t)$ for a given input $u(t)$ could be obtained. Let us consider relation Eq. (8), and let us assume that the nth-order Laplace transform of $y_n(t_1, \ldots, t_n)$, denoted $Y_n(s_1, s_2, \ldots, s_n)$, is given. The question is how to derive $y_n(t)$? Obviously, one can perform the nth-order inverse Laplace transform of $Y_n(s_1, s_2, \ldots, s_n)$

$$y_n(t_1,\ldots,t_n) = \frac{1}{(2\pi i)^n}\int_{\sigma_n - i\infty}^{\sigma_n + i\infty} \cdots \int_{\sigma_1 - i\infty}^{\sigma_1 + i\infty} Y_n(s_1, s_2, \ldots, s_n) e^{s_1 t_1 + \cdots + s_n t_n}\, ds_1 \cdots ds_n \tag{26}$$

and set $t_1 = t_2 = \cdots = t_n = t$. However, this computation is often unwieldy. In order to bypass this difficulty, George (3) developed a method whereby the t_i variables can be set equal or associated without leaving the transform domain, leading to a one-dimensional Laplace transform $Y_n(s)$. Indeed, let us consider a two variable transform $Y_2(s_1, s_2)$; setting $t_1 = t_2 = t$ in Eq. (26) yields

$$y_2(t,t) = \frac{1}{(2\pi i)}\int_{\sigma_2 - i\infty}^{\sigma_2 + i\infty}\left[\frac{1}{(2\pi i)}\int_{\sigma_1 - i\infty}^{\sigma_1 + i\infty} Y_2(s_1, s_2) e^{s_1 t}\, ds_1\right] \times e^{s_2 t}\, ds_2$$

Changing the variable of integration s_1 to $s = s_1 + s_2$ gives

$$y_2(t,t) = \frac{1}{(2\pi i)} \int_{\sigma_2 - i\infty}^{\sigma_2 + i\infty} \left[\frac{1}{(2\pi i)} \int_{\sigma_1 - i\infty}^{\sigma_1 + i\infty} Y_2(s - s_2, s_2) e^{s - s_2 t}\, ds \right] \times e^{s_2 t}\, ds_2$$

or by interchanging the order of integration

$$y_2(t,t) = \frac{1}{(2\pi i)} \int_{\sigma_1 - i\infty}^{\sigma_1 + i\infty} \left[\frac{1}{(2\pi i)} \int_{\sigma_2 - i\infty}^{\sigma_2 + i\infty} Y_2(s - s_2, s_2) e^{s - s_2 t}\, ds_2 \right] e^{s_2 t}\, ds$$

Thus, the associated transform $Y_2(s)$ is

$$Y_2(s) = \frac{1}{(2\pi i)} \int_{\sigma_2 - i\infty}^{\sigma_2 + i\infty} Y_2(s - s_2, s_2) e^{s - s_2 t}\, ds_2 \tag{27}$$

Similarly, a transform of any order can be reduced to a first-order transform by successive pairwise associations. For example, let us consider the third-order term

$$\frac{1}{(s_1 + s_2 + s_3 + a)(s_1 + a)(s_2 + a)(s_3 + a)}$$

Associating the variables s_2 and s_3 yields

$$\frac{1}{(s_1 + s_2 + a)(s_1 + a)(s_2 + 2a)}$$

Then, associating s_1 and s_2 yields

$$\frac{1}{(s + a)(s + 3a)}$$

The procedure for computing $Y_n(s)$ from $Y_n(s_1, s_2, \ldots, s_n)$ is called *association of variables.* Although an explicit formula for performing the associating operation in a large class of Laplace transforms has been obtained in the literature (see Rugh (4) and the references herein), this technique has seldom been used. The main reason for this seems to be the tedious manipulations involved and the difficulty in decomposing them onto a computer.

Algebraic Approach

In this part, we show how to compute the response of nonlinear systems to typical inputs. This method, based on the use of the formal representation of the Volterra kernels Eq. (24), is also easily implementable on a computer using formal languages like AXIOM (24). These algebraic tools for the first time enable one to derive exponential polynomial expressions depending explicitly on time for the truncated Volterra series associated with response (19) and, therefore, lead to a finer analysis than pure numerical results.

To continue our use of algebraic tools, let us introduce the Laplace-Borel transform associated with a given analytic function input

$$u(t) = \sum_{n0} a_n \frac{t^n}{n!}$$

Its Laplace–Borel transform is

$$\mathbf{g}_u = \sum_{n0} a_n z_0^n$$

Example:

$$\cos \omega t = \frac{1}{2} e^{j\omega t} + \frac{1}{2} e^{-j\omega t}$$

Its Borel transform is given by

$$\mathbf{g}_u = \frac{1}{2}(1 - j\omega t z_0)^{-1} + \frac{1}{2}(1 + j\omega t z_0)^{-1} = (1 + \omega^2 z_0^2)^{-1}$$

Before seeing the algebraic computation itself in order to compute the first terms of the response to typical inputs, let us introduce a new operation on formal power series, *the shuffle product.*

Given two formal power series,

$$\mathbf{g}_1 = \sum_{w \in Z^*} (\mathbf{g}_1, w) w \quad \text{and} \quad \mathbf{g}_2 = \sum_{w \in Z^*} (\mathbf{g}_2, w) w$$

The shuffle product of two formal power series $\mathbf{g}_1$ and $\mathbf{g}_2$ is given by

$$\mathbf{g}_1 \varpi \mathbf{g}_2 = \sum_{w_1, w_2 \in Z^*} (\mathbf{g}_1, w_1)(\mathbf{g}_2, w_2) w_1 \varpi w_2$$

where the shuffle product of two words is defined as follows:

- $1 \varpi 1 = 1$
- $\forall z \in Z, \quad 1 \varpi z = z \varpi 1 = z$
- $\forall z, z' \in Z, \quad \forall w, w' \in Z^*$
  $zw \varpi z'w' = z[w \varpi z'w'] + z'[zw \varpi w']$

This operation consists in shuffling all the letters of the two words by keeping the order of the letters in the two words. For instance,

$$z_0 z_1 \varpi z_1 z_0 = 2 z_0 z_1^2 z_0 + z_0 z_1 z_0 z_1 + z_1 z_0 z_1 z_0 + z_1 z_0^2 z_1$$

It has been shown that the Laplace-Borel transform of expression Eq. (24), for a given input $u(t)$ with the Laplace-Borel transform $\mathbf{g}_u$, is obtained by substituting from the right each variable z_1 by the operator $z_0[\mathbf{g}_u \varpi \cdot]$.

Therefore, in order to apply this result, we need to know how to compute a shuffle product of algebraic expressions of the form

$$\mathbf{g}_n = (1 + a_0 z_0)^{-1} z_{i_1} (1 + a_1 z_0)^{-1} z_{i_2} \ldots (1 + a_{n-1} z_0)^{-1} z_{i_n} (1 + a_n z_0)^{-1} \tag{28}$$

where $i_1, i_2, \ldots, i_n \in \{0, 1\}$.

This computation is very simple; it amounts to adding some singularities. For instance,

$$(1 + a z_0)^{-1} \varpi (1 + b z_0)^{-1} = (1 + (a + b) z_0)^{-1}$$

Consider two generating power series of the form Eq. (28)

$$\mathbf{g}_p = (1 + a_0 z_0)^{-1} z_{i_1} (1 + a_1 z_0)^{-1} z_{i_2} \cdots (1 + a_{p-1} z_0)^{-1} z_{i_p} (1 + a_p z_0)^{-1}$$

and

$$\mathbf{g}_q = (1 + b_0 z_0)^{-1} z_{j_1} (1 + b_1 z_0)^{-1} z_{j_2} \cdots (1 + b_{q-1} z_0)^{-1} z_{j_q} (1 + b_q z_0)^{-1}$$

where p and $q \in \mathbf{N}$, the indices $i_1, i_2, \ldots, i_p \in \{0, 1\}$, $j_1, j_2, \ldots, j_q \in \{0, 1\}$, and $a_i, b_j \in \mathbf{C}$. The shuffle product of these expressions is given by induction on the length

$$\mathbf{g}_p \varpi \mathbf{g}_q = \mathbf{g}_p \varpi \mathbf{g}_{q-1} z_{j_q} (1 + a_p + b_q) z_0)^{-1} + \mathbf{g}_{p-1} \varpi \mathbf{g}_q z_{i_p} (1 + (a_p + b_q) z_0)^{-1}$$

See (25) for case-study examples and some other rules for computing directly the stationary response to harmonic inputs or the response of a Dirac function, and see (26) for the algebraic computation of the response to white noise inputs. This previous computation of the rational power series $\mathbf{g}$ and of the response to typical entries has been applied to the analysis of nonlinear electronics circuits (27) and to the study of laser semi-conductors (28) and (29).

Application to Nonlinear Circuits

Description of Nonlinear Circuits. Most of the nonlinear electronic circuits encountered can be described in terms of elementary nonlinear components such as nonlinear resistors, capacitors, inductors, and independent sources which are usually represented as shown in Fig. 4, where v and i denote, respectively, the voltage across a branch of the circuit and the current flowing in it; v_x and i_x are respectively, a voltage and a current controlling variable. Representations 1 and 2 correspond, respectively, to impedance and admittance descriptions of the nonlinear element.

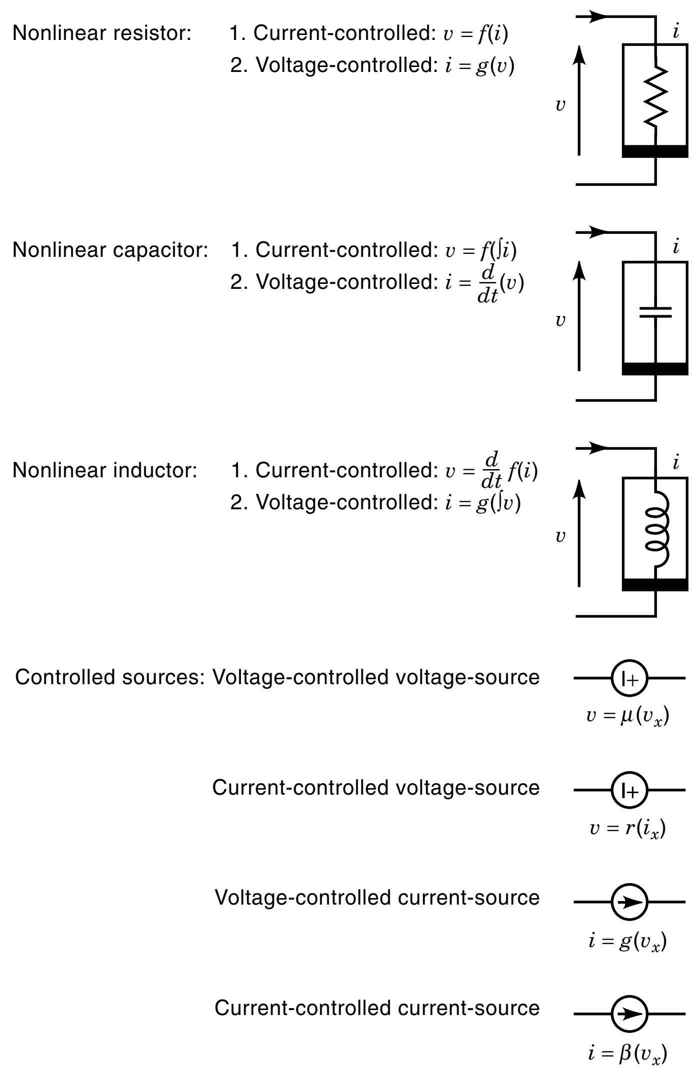

Figure 4. Representation of lumped electronic nonlinear elements.

Note that elements that operate in a monotonic region of their characteristic possess both representations. These components generally operate in a region where their behavior is described by a power series expansion on their quiescent or DC points. These expansions can be expressed in one of the following general forms which correspond to the Taylor expansions of the functions f, g, μ, r, and α:

$$w(t) = \sum_{n \geq 1} a_n z^n(t)$$
$$w(t) = \sum_{n \geq 1} b_n \left[\int_0^t z(\tau)\, d\tau \right]^n \tag{29}$$
$$w(t) = \frac{d}{dt} \left[\sum_{n \geq 1} c_n z^n(t) \right]$$

Depending on the nonlinear element considered and on its representation (impedance or admittance), w and z may represent either a current or a voltage incremental variable. z is called the controlling variable and w the controlled one. Note that even when both representations 1 and 2 exist for an element, it may be preferable to use the one of which the power series expansion Eq. (29) is more rapidly convergent. Separating the summations in Eq. (29) into a linear part plus second- and higher-order terms suggests that each nonlinear element may be seen as a parallel (if w is a current) or a cascade (if w is a voltage) combination of a linear element ($n = 1$) and a strictly nonlinear element ($n \geq 2$). This leads to an equivalent representation of the nonlinear elements given in Fig. 5.

Let us first consider these strictly nonlinear elements as independent sources and modify the circuit by imbedding the linear component of each nonlinear element into the linear circuit. This results in a linear circuit called the modified linear circuit. Using Kirchhoff's current and voltage laws, a standard linear analysis can be carried out.

To avoid dealing with certain types of networks whose functional representation may fail to exist, we shall assume that the networks meet certain requirements.

Consider each nonlinear capacitor (inductor) described by an admittance (impedance) representation and its associated nonlinear independent current (voltage) source. Let i and v denote, respectively, the source current and its branch voltage. Assume that all the other independent current (voltage) sources, inputs and sources associated with the other nonlin-

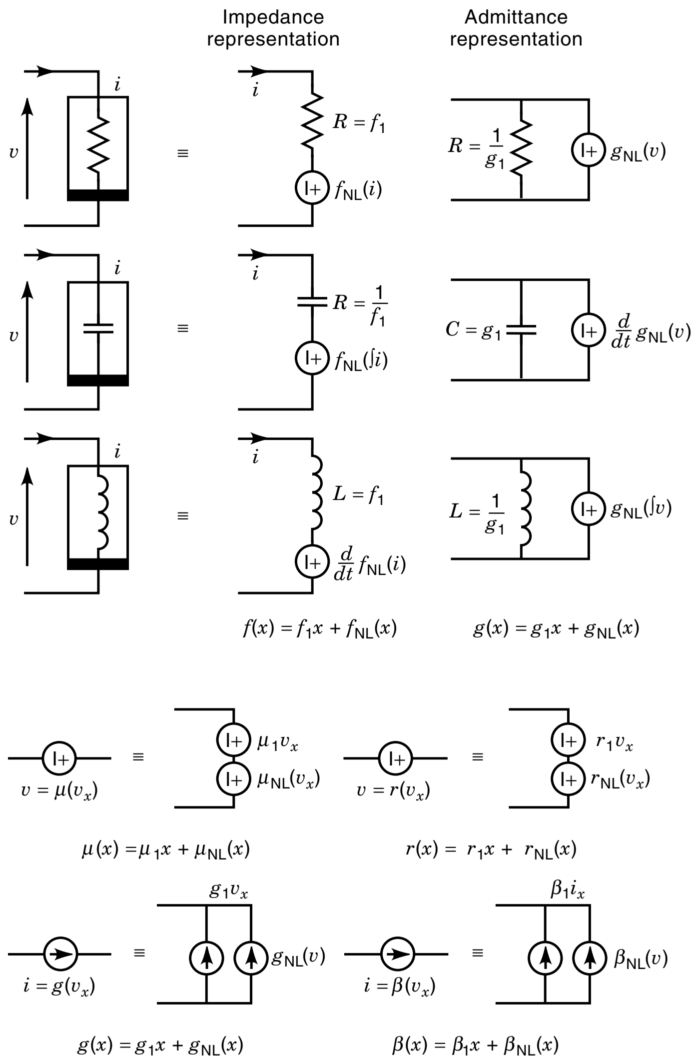

Figure 5. Equivalent representation of the nonlinear elements.

ear elements, are open circuited (short circuited), and that all independent voltage (current) sources, inputs and sources associated with nonlinear elements, are short circuited (open circuited); then, the linear transfer function linking i and v and associated with the resulting linear circuit must be strictly proper. Recall that in linear system theory, a rational function $G(s)$ is said to be strictly proper if $G(\infty) = 0$.

Circuits which do not satisfy H_1 or H_2 depend on an infinite number of higher-order derivatives of some inputs. An example of this situation is provided by considering the circuit of Fig. 6. For the modified linear circuit associated with this cir-

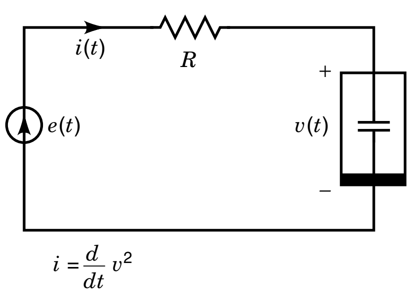

Figure 6. Example of a nonlinear circuit.

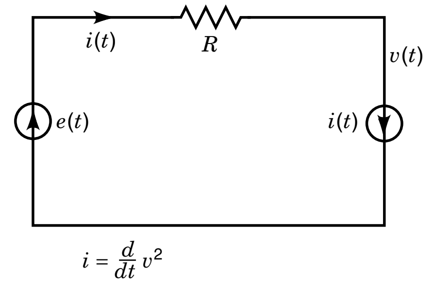

Figure 7. Modified linear circuit.

cuit (shown on Fig. 7), the nonlinear capacitor, which consists only of a strictly nonlinear element, has been replaced by an independent current source. In order to show that the hypothesis H_1 is not verified (the capacitor being described in an admittance form), the independent voltage source is short circuited, and the linear transfer function, linking the current i through the current source and the voltage across it, must be searched for: here $(v/i) = R$, which is obviously not a strictly proper rational function. The nonlinear differential equation describing the behavior of this circuit is

$$e + v\dot{v} = v \tag{30}$$

which can be solved iteratively following the Picard iterative scheme

$$v_0 = e, \quad v_n = e + v_{n-1}\dot{v}_{n-1}, \quad n \geq 1$$

yielding

$$v = e + e\dot{e} + 2e(\dot{e})^2 + e^2\ddot{e} + \cdots \tag{31}$$

Expression Eq. (31) makes explicit the dependency of v on the derivatives of the voltage input e. On the other hand, Eq. (30) can be solved analytically, at least for a constant input voltage e. One finds

$$v + e \log\left(1 - \frac{v}{e}\right) = t$$

which must be considered only for $t \geq 0$. This formula shows that the solution has a nondefined first-order derivative at zero which is a sufficient condition for the nonexistence of a Volterra analytical functional expansion of the solution $v(t)$.

The modified linear circuit must be well-behaved. This means that the modified linear circuit possesses a unique defined solution, and that, in particular, no circuit variable tends to infinity with the input frequency. For example, the nonlinear circuit of Fig. 8 with its linear modified linear associated circuit shown on Fig. 9 does not satisfy H_3.

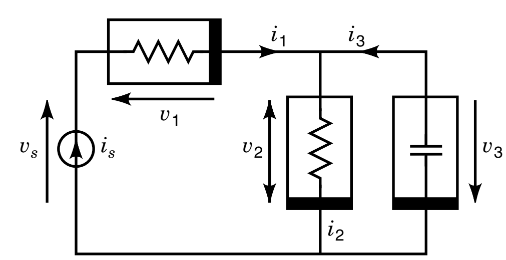

Figure 8. A nonlinear circuit: $v_1 = f(i_1)$; $v_2 = h(i_2)$; $v_3 = r(\int i_3)$.

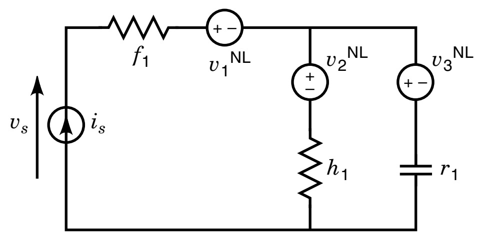

Figure 9. Circuit obtained from that of Figure 8 by imbedding the linear part of the nonlinear elements into the linear circuit.

Note that in practical circuits, H_1, H_2, and H_3 are generally fulfilled.

Descriptive Equations. Any lumped circuit obeys three basic laws: Kirchhoff's voltage law (KVL), Kirchhoff's current law (KCL), and the elements' law (branch characteristics). Let each passive nonlinear element (resistor, capacitor, inductor) be described by its controlling variable: current, if it has an impedance representation, or voltage for the admittance. Let the current sources be described by the voltages across their branches and the voltage sources by the currents flowing in their branches. For a nonlinear circuit containing p branches and n nodes, one may then write $n - 1$ KCL equations and $p - (n - 1)$ KVL equations. Finally, if one keeps in these equations only the descriptive variables using the branch characteristics, one gets p equations linking the p unknown variables. These equations are of three types:

E_1: **Dynamical equations.** These are generally integro-differential equations linking a set, as reduced as possible, of variables of the circuit to be described totally.

E_2: **Output equations.** These are functions connecting variables described by the dynamical equations to the remaining variables.

E_3: **Reduction equations.** These equations are linear; they allow the number of unknowns in the previous set of equations to be reduced. They correspond to

1. KCL at nodes joining only passive elements described in an admittance form or independent current input sources or dependent voltages sources if the current flowing through their branch appears as a controlling variable of another element.
2. KVL for loops containing only passive elements described in an impedance form or independent voltage input sources or dependent current sources if the voltage across their branch appears as a controlling variable of another element.

For example, let us consider again the nonlinear circuit of Fig. 8. This circuit is described by the following set of equations derived as shown previously:

$$E_1 \to h(i_2) + r\left(\int i_3\right) = 0$$
$$E_2 \to v_s = f(i_1) + h(i_2)$$
$$E_3 \to \begin{cases} i_s = i_1 \\ i_1 + i_3 = i_2 \end{cases}$$

Using E_3, E_1 and E_2 can be written:

$$\begin{cases} h(i_2) + r\left(\int (i_2 - i_s)\right) = 0 \\ v_s = f(i_s) + h(i_2) \end{cases} \tag{32}$$

Derivation of the Generating Power Series Associated with Nonlinear Circuits. Using the algebraic approach described earlier, it is not difficult to derive the generating power series associated with the unknown variables of the set of equations obtained from E_3, E_1, and E_2. Instead of showing this in general, let us here illustrate the main ideas through the above example of Fig. 8. Given

$$f(i) = \sum_{n\geq 1} f_n i^n$$
$$h(i) = \sum_{n\geq 1} h_n i^n$$

and

$$r\left(\int i\right) = \sum_{n\geq 1} r_n \left(\int i\right)^n$$

If $\mathbf{g}_2$ and $\mathbf{g}_s$ denote the generating power series associated respectively with i_2 and i_s from Eq. (32), we obtain the following set of algebraic equations

$$\begin{cases} \sum_{n\geq 1} h_n \mathbf{g}_2^{\varpi n} + \sum_{n\geq 1} r_n (x_0 g_2 - x_1)^{\varpi n} = 0 \\ g_s = \sum_{n\geq 1} h_n \mathbf{g}_2^{\varpi n} + \sum_{n\geq 1} f_n \mathbf{g}_s^{\varpi n} \end{cases} \tag{33}$$

where $\mathbf{g}^{\varpi n} = \mathbf{g}\varpi \ . \ . \ . \ \varpi\mathbf{g}$ (n-times). From the algebraic rules defined earlier, we can derive iteratively the expressions for $[\mathbf{g}]_i$, the power series containing exactly i occurrences of the letter x_1 in $\mathbf{g}$.

These computations are easily implementable on a computer using a formal computing software. In the same way, we can systematically derive the response to typical inputs as we previously described. In the last two parts, we use these Volterra series expansions in a time domain in order to derive physical quantities like signal distortions or intermodulation products.

DISTORTION ANALYSIS

In this part, we are interested in the analysis of the response of weakly nonlinear systems driven by harmonic inputs. When the input signal is of the form $\sin(\omega t)$, its response is in general also periodic, but the output signal contains components with a multiple integer of the input pulsation. When the signal input is composed of two harmonics of pulsation ω_1 and ω_2, respectively, then the output signal is a sum of harmonics with pulsation $p\omega_1 + q\omega_2$, where p and q are negative or positive integers.

The study of the harmonic components of the response is of great importance in the study of distortions existing in nonlinear circuits, like the transistors, the amplifiers, the modulators, etc. One can cite, for instance, the works of Bedrosian and Rice (15), Goldman (30), Narayanan (31,32), Bussgang, Ehrman and Graham (16) and Crippa (33). A Volterra series offers an efficient tool for this study because for weakly nonlinear systems, often only first, second, and third terms of the Volterra series are sufficient in order to obtain significant quantitative results.

Harmonic Analysis

Let us consider a stationary nonlinear system described by the Volterra series

$$y(t) = \sum_{n=1}^{\infty} \int_0^{t_1} \cdots \int_0^{t_n} h_n(t-\tau_1, t-\tau_2, \ldots, t-\tau_n) u(\tau_1) u(\tau_2) \ldots u(\tau_n)\, d\tau_1\, d\tau_2 \ldots d\tau_n \quad (34)$$

The output of this system driven by the input is

$$u(t) = |A_1| \cos(\omega_1 t + \phi_1) = \frac{1}{2}(A_1 e^{i\omega_1 t} + A_{-1} e^{-i\omega_1 t})$$

where $A_1 = A_{-1}^* = |A_1| e^{i\phi_1}$ is given by

$$y(t) = y_1(t) + y_2(t) + \cdots + y_n(t) + \cdots$$

with

$$y_n(t) = \frac{1}{2^n} \sum_{j_1=-1}^{+1} \cdots \sum_{j_n=-1}^{+1} A_{j_1} \ldots A_{j_n} H_n(\omega_{j_1}, \ldots, \omega_{j_n}) e^{i(\omega_{j_1} + \cdots + \omega_{j_n})} \quad (35)$$

where $H_n(\omega_{j_1}, \ldots, \omega_{j_n})$ is the multidimensional Laplace transform of $h_n(t_1, t_2, \ldots, t_n)$. The Laplace transform H_n is, like h_n, a symmetric function. This allows regrouping identical terms in expression Eq. (35). In order to do so, let us denote by $m_1(m_{-1})$ the occurrence number of the pulsation $\omega_1(\omega_{-1})$ in $(\omega_{j_1}, \ldots, \omega_{j_n})$. Equation (35) may also be written

$$y_n(t) = \frac{1}{2^n} \sum_{m \in M, m=(m_1, m_{-1})} \frac{n!}{(m_{-1})! m_1!} (A_1^*)^{m_{-1}} (A_1)^{m_1} H_n(\omega_{-1}, \ldots, \omega_{-1}, \omega_1, \ldots, \omega_1) e^{i(m_1 - m_{-1})\omega_1 t} \quad (36)$$

where M represents the set of the couples (m_1, m_{-1}), such that $m_1 + m_{-1} = n$. By regrouping conjugate complexes in Eq. (36), we obtain

$$y_n(t) = \frac{1}{2^{n-1}} \sum_{m \in M, m_1 \geq m_{-1}} \frac{n!}{(m_{-1})! m_1!} |A_1|^{(m_{-1}+m_1)} |H_n(\underbrace{\omega_{-1}, \ldots, \omega_{-1}}_{m_{-1}}, \underbrace{\omega_1, \ldots, \omega_1}_{m_1})| \cos((m_1 - m_{-1})\omega_1 t + (m_1 - m_{-1})\phi_1 + \mathrm{Arg}|H_n(m)|) \quad (37)$$

For a signal input with pulsation ω_1, the response of a weakly nonlinear system is, therefore, periodic, and it is composed of multiple integer terms of this pulsation.

More generally, it can be shown that for a multi-pulsation input signal of the form

$$u(t) = \sum_{k=1}^{K} |A_k| \cos(\omega_k t + \phi_k) = \frac{1}{2} \sum_{k=-K, k \neq 0}^{k=K} A_k e^{i\omega_k t}$$

with $A_k = |A_k| e^{i\phi_k}$, the response of the system is given by

$$y(t) = y_1(t) + y_2(t) + \cdots + y_n(t) + \cdots$$

with

$$y_n(t) = \frac{1}{2^{n-1}} \sum_{m \in M, \omega_m \geq 0} \frac{n!}{(m_{-K})! \ldots m_K!)} |A_1|^{(m_{-1}+m_1)} \ldots |A_1|^{(m_{-K}+m_K)} |H_n(\Omega)| \cos(\omega_m t + \phi_m + \mathrm{Arg}|H_n(\Omega)|) \quad (38)$$

where

$$\Omega = (\underbrace{\omega_K, \ldots, \omega_K}_{m_{-K}}, \ldots \underbrace{\omega_K, \ldots \omega_K}_{m_K})$$

$$\omega_m = (m_K - m_{-K})\phi_K + \cdots + (m_1 - m_{-1})\phi_1$$

and $m = (m_{-K}, \ldots, m_{-1}, m_1, \ldots, m_K)$ is such that $m_{-K} + \cdots + m_{-1} + m_1 + \cdots + m_K = n$. Therefore, in the terms of y_n with $n \geq 1$ we find

- Terms with the same pulsation as the input one
- Terms with a pulsation equal to an integer multiple of one of the input pulsations (harmonic terms)
- Terms resulting in an interference between several input pulsations (intermodulation terms)

Nonlinear Distortions

The analysis of the resulting spectrum is of great importance in numerous electronic applications. In order to characterize nonlinear circuit performances, several nonlinear distortion rates have been introduced in the literature depending on the application considered. The most popular are described in the following sections.

Distortion Rate. Let us consider the signal input

$$u(t) = |E_1| \cos \omega_1 t$$

of a weakly nonlinear system described by Eq. (34). As previously discussed, the first terms of the output are of the form

$$A_0 + |A_1| \cos(\omega_1 t + \phi_1) + |A_2| \cos(2\omega_1 t + \phi_2) + \cdots$$

The amplitude of the various harmonic terms is not really significant. On the other hand, their ratio with respect to the amplitude of the fundamental frequency of the input signal may serve as a distortion measure. The distortion ratio of the k-th harmonic is defined as

$$\frac{|A_1|}{|A_k|}$$

Harmonic Distortion Rate. The value of this rate indicates the global relative importance of the output harmonic level with respect to the fundamental frequency term. It is defined by

$$\frac{|A_1|^2}{|A_2|^2 + |A_3|^2 + \cdots}$$

Gain Distortion. Let us consider again the input signal

$$u(t) = |E_1| \cos \omega_1 t$$

and the fundamental frequency output

$$|A_1| \cos(\omega_1 t + \phi_1)$$

Given a linear system $A_1 = E_1 H_1(i\omega_1)$, where H_1 is the transfer function, the ratio

$$\frac{|A_1|}{|E_1|} = |H_1(i\omega_1)|$$

is a classic definition of the *linear gain.* For a nonlinear system, the contributions to the fundamental frequency of the higher-order Volterra kernel are nonzero in general. For a fixed input frequency, the gain is no longer constant but depends on the amplitude of the input:

$$\frac{|A_1|}{|E_1|} = |H_1(i\omega_1) + \frac{3}{4}|E_1|^2 H_3(-i\omega_1, i\omega_1, i\omega_1) + \cdots|$$

Intermodulation. Let us now consider an input of the following form

$$u(t) = |E_1| \cos \omega_1 t + |E_2| \cos \omega_2 t$$

The 2nd-order intermodulation ratio (IMR2) is defined as the difference (dB) between the level of the output signal at the fundamental frequency and the level of the distortion term at the frequency $\omega_1 + \omega_2$ or $\omega_1 - \omega_2$. The 3rd-order intermodulation ratio (IMR3) is defined in the same way, that is, the difference of the level of the output signal at the fundamental frequency and the level of the distortion term at the frequency $2\omega_1 + \omega_2$ or $\omega_1 + 2\omega_2$ or $2\omega_1 - \omega_2$ or $\omega_1 - 2\omega_2$, and so on. In general, the measure of the intermodulation terms is taken by choosing the input with the same amplitude $|E|$,

$$|E_1| = |E_2| = |E|$$

and neighboring pulsations,

$$\omega_1 \simeq \omega \quad \text{and} \quad \omega_2 \simeq \omega$$

From the previous part it is not difficult to see that, for instance,

$$\text{IMR2}(\omega_1 \pm \omega_2) = \frac{|H_1(\omega)|}{|E||H_2(\omega, \pm\omega)|}$$

and

$$\text{IMR3}(2\omega_1 \pm \omega_2) = \frac{4|H_1(\omega)|}{3|E|^2|H_3(\omega, \omega, \pm\omega)|}$$

Transmodulation. In order to analyze the transmodulation, that is the effect that a modulation is transferred from one signal to another through a weakly nonlinear system, the following input signal is considered

$$u(t) = |E|(1 + m \cos \omega_m t) \cos \omega_1 t + |E| \cos \omega_2 t$$

Technical results on the output signal may be found in the paper by Meyer, Shensa and Eschenbach (34).

Note that the algebraic framework described earlier allows us to easily compute all the previous distortion rates (35).

BIBLIOGRAPHY

1. N. Wiener, *Nonlinear Problems in Random Theory,* New York: Wiley, 1958.
2. J. Barrett, The use of functionals in the analysis of nonlinear physical systems, *J. Electronics Control,* **15**: 567–615, 1963.
3. D. George, Continuous nonlinear systems, *MIT RLE Technical Report No. 355,* 1959.
4. W. J. Rugh, *Nonlinear System Theory,* Baltimore, MD: John Hopkins Univ. Press, 1981.
5. M. Schetzen, *The Volterra and Wiener Theories of Nonlinear Systems,* New York: Wiley, 1980.
6. V. Volterra, *Theory of Functionals* (translated from the Spanish), London: Blackie, 1930 (reprinted by Dover, New York, 1959).
7. M. Schetzen, Nonlinear system modeling based on the Wiener theory, *Proc. IEEE,* **69** (12): 1981.
8. M. Brillant, Theory of the analysis of nonlinear systems, MIT RLE Technical Report No. 345, 1958.
9. J. Blackman, The representation of nonlinear networks, Syracuse University Research Institute, Report No. 81560 for Air Force Cambridge Research Center, Cambridge, Mass,
10. C. Lesiak and A. J. Krener, The existence and uniqueness of Volterra series for nonlinear systems, *IEEE Trans. Autom. Control,* **23**: 1090–1095, 1978.
11. R. W. Brockett, Volterra series and geometric control theory, *Automatica,* **12**: 167–176, 1976. R. W. Brockett and E. G. Gilbert, An addendum to Volterra series and geometric control theory, *Automatica,* **12**: 635, 1976.
12. E. G. Gilbert, Functional expansions for the response of nonlinear differential systems, *IEEE Trans. Autom. Control,* **22**: 909–921, 1977.
13. C. Bruni, G. Di Pillo, and G. Koch, On the mathematical models of bilinear systems, *Ricerche di Automatica,* **2** (1): 1971.
14. S. Boyd and L. O. Chua, Fading memory and the problem of approximating nonlinear operators with Volterra series, *IEEE Circuits Syst.,* **32** (11): 1150–1161, 1985.
15. E. Bedrosian and S. Rice, The output properties of Volterra systems (nonlinear systems with memory) driven by harmonic and Gaussian input, *Proc. IEEE,* **59**: 1688–1707, 1971.
16. J. Bussgang, L. Ehrman, and J. W. Graham, Analysis of nonlinear systems with multiples inputs, *Proc. IEEE,* **62**: 1088–1119, 1974.
17. L. O. Chua and C. Y. Ng, Frequency domain analysis of nonlinear systems: general theory, *Electronics Circuits and Systems,* **3** (4): 165–185, 1979 and Frequency domain analysis of nonlinear systems: formulation of transfer functions, *Electronics Circuits and Systems,* **3** (6); 257–269, 1979.
18. R. H. Flake, Volterra series representation of nonlinear systems, *IEEE Trans. Ind. Appl.,* **81**: 330–335, 1963.
19. M. Fliess, M. Lamnabhi, and F. Lamnabhi-Lagarrigue, Algebraic approach to nonlinear functional expansions, *IEEE Circuits Syst.,* **30**: 550–570, 1983.
20. M. Fliess, Fonctionnelles causales non linéaires et indéterminées non commutatives, *Bull. Soc. Math. France,* **109**: 3–40, 1981.
21. W. Gröbner, Die Lie-Reihen und ihre Anwendungen (2e édition), *VEB Deutscher Verlag der Wissenschaften,* Berlin: 1967.

22. F. Lamnabhi-Lagarrigue and G. Stefani, Singular optimal control problems: On the necessary conditions for optimality, *SIAM J. Contr. Optim.,* **28**: 823–840, 1990.

23. F. Lamnabhi-Lagarrigue and M. Lamnabhi, Détermination algébrique des noyaux de Volterra associés certains systèmes non linéaires, *Ricerche di Automatica,* **10**: 17–26, 1979.

24. A. Martin, Calcul d'approximations de la solution d'un système non linéaire utilisant le logiciel SCRATCHPAD, in G. Jacob and F. Lamnabhi-Lagarrigue (eds.), *Algebraic Computing in Control,* Lect. Note Contr. Inform. Sc., Springer Verlag, 165, 1991.

25. F. Lamnabhi-Lagarrigue, *Analyse des systèmes non linéaires,* Ed. Hermes, 1994.

26. M. Fliess and F. Lamnabhi-Lagarrigue, Application of a new functional expansion to the cubic anharmonic oscillator, *J. Math. Physics,* **23**: 495–502, 1982.

27. S. Baccar, F. Lamnabhi-Lagarrigue, and G. Salembier, Utilisation du calcul formel pour la modélisation et la simulation des circuits électroniques faiblement non linéaires, *Annales des Télécommunications,* **46**: 282–288, 1991.

28. L. Hassine et al., Volterra functional series expansions for semiconductor lasers under modulation, *IEEE J. Quantum Electron.,* **30**: 918–928, 1994.

29. L. Hassine et al., Volterra functional series for noise in semiconductor lasers, *IEEE J. Quantum Electron.,* **30**: 2534–2546, 1994.

30. J. Goldman, A Volterra series description of crosstalk interference in communications systems, *Bell Syst. Tech. J.,* **52**: 649–668, 1973.

31. S. Narayanan, Transistor distortion analysis using Volterra series representation, *Bell Syst. Tech. J.,* **46**: 991–1024, 1967.

32. S. Narayanan, Application of Volterra series to intermodulation distortion analysis of a transistor feedback amplifier, *IEEE Trans. Circuit Theory,* **17**: 518–527, 1970.

33. G. Crippa, Evaluation of distortion and intermodulation in nonlinear transmission systems by means of Volterra series expansion, *Alta Frequenza,* **38**: 332–336, 1969.

34. R. G. Meyer, M. J. Shensa, and R. Eschenbach, Cross-modulation and intermodulation in amplifiers at high frequencies, *IEEE Solid State Circuits,* **7**: 16–23, 1972.

35. M. Fliess and M. Lamnabhi, Application d'une technique nouvelle de calculs de développements fonctionnels à la détermination de distortions non linéaires, *Proc. Huitième Colloque sur le Traitement du Signal et ses Applications,* Nice, 1981.

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VOLT–OHM METERS. See MULTIMETERS.

VOLT RAMP GENERATION. See RAMP GENERATOR.