FUZZY LOGIC

In this article, a brief introduction is given to fuzzy systems. The materials in this article can be used as basic knowledge on fuzzy set theory and fuzzy logic in support of other articles on the subject in this publication. However, many introductions similar to this one have been published over the years in other reports, articles, and books on fuzzy logic and control.

Fuzzy sets are those with *unsharp* boundaries.These sets are generally in better agreement with the human mind that works with shades of gray, rather than with just black or white. Fuzzy sets are typically able to represent linguistic terms, for example, *warm, hot, high, low.* Today, in Japan, Europe, the United States, and many other parts of the world, fuzzy logic and its applications are widely accepted and applied. In many consumer products like washing machines and cameras, fuzzy controllers are used in order to obtain higher machine intelligence quotient (IQ) and user-friendly products. A few interesting applications can be mentioned: control of subway systems, image stabilization of video cameras, and autonomous control of helicopters. Although industries in the United States and Europe hesitated in accepting fuzzy logic at first, they have become more enthusiastic about applying this technology in recent years.

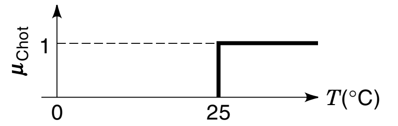

Figure 1. The characteristic function μ_{Chot} for a crisp set.

FUZZY SETS VERSUS CRISP SETS

In the classical set theory, a set is denoted as a so-called *crisp set* and can be described by its characteristic function as follows:

$$\mu_C: U \rightarrow \{0, 1\} \tag{1}$$

In Eq. (1), U is called the *universe of discourse,* that is, a collection of elements that can be continuous or discrete. In a crisp set, each element of the universe of discourse either belongs to the crisp set ($\mu_C = 1$) or does not belong to the crisp set ($\mu_C = 0$).

Consider a characteristic function μ_{Chot} representing the crisp set *hot,* a set with all "hot" temperatures. Figure 1 graphically describes this crisp set, considering temperatures higher than 25°C as hot. (Note that for all the temperatures T, we have $T \in U$.)

The definition of a fuzzy set, proposed by Zadeh (1), is given by the characteristic function

$$\mu_F: U \Rightarrow [0, 1] \tag{2}$$

In this case, the elements of the universe of discourse can belong to the fuzzy set with any value between 0 and 1. This value is called the *degree of membership.* If an element has a value close to 1, the degree of membership, or "truth" value is high. The characteristic function of a fuzzy set is called the *membership function,* for it gives the degree of membership for each element of the universe of discourse. If now the characteristic function of μ_{Fhot} is considered, one can express the human opinion, for example, that 24°C is still fairly hot, and that 26°C is hot, but not as hot as 30°C and higher. This results in a gradual transition from membership (completely true) to nonmembership (not true at all). Figure 2 shows the membership function μ_{Fhot} of the fuzzy set F_{hot}.

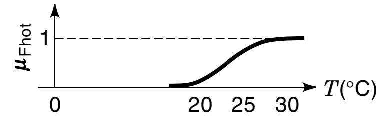

Figure 2. The membership function μ_{Fhot} for a fuzzy set.

J. Webster (ed.), *Wiley Encyclopedia of Electrical and Electronics Engineering.* Copyright © 1999 John Wiley & Sons, Inc.

In this figure, the membership function has a gradual transition. However, every individual can construct a different transition according to his/her own opinion. Membership functions can have many possible shapes, depending on the subjectivity of the issues involved. In practice, the transitions may be linear to simplify the computations.

Example 1. Suppose someone wants to describe the class of cars having the property of being expensive by considering cars such as BMW, Buick, Cadillac, Ferrari, Fiat, Lada, Mercedes, Nissan, Peugot, and Rolls Royce. Describe a fuzzy set "expensive cars."

Some cars, like Ferrari or Rolls Royce, definitely belong to the class "expensive," while other cars, like Fiat or Lada, do not belong to it. But there is a third group of cars, which are not really expensive, but which are also not cheap. Using fuzzy sets, the fuzzy set of "expensive cars" is, for example,

$$\{(Ferrari, 1), (Rolls\,Royce, 1), (Mercedes, 0.9), (BMW, 0.8), (Cadillac, 0.8), (Nissan, 0.7), (Buick, 0.6), (Peugot, 0.5), (Fiat, 0.2), (Lada, 0.1)\}$$

Example 2. Suppose one wants to define the set of natural numbers "close to 5." Find a fuzzy set representation.

This can be expressed in the discrete case by the fuzzy set:

$$\underset{\sim}{5} = (3, 0.2) + (4, 0.5) + (5, 1) + (6, 0.5) + (7, 0.2)$$

The underscore ∼ under number 5 designates fuzziness. The sign "+" represents membership of new elements in the fuzzy set "close to 5" and not a summation operator. The membership function in the continuous case of the fuzzy set of real numbers "close to 5" is, for example,

$$\mu_{\underset{\sim}{5}} = \frac{1}{1 + (x-5)^2} \tag{3}$$

and the fuzzy set $\underset{\sim}{5}$ contains, for example, the elements (5, 1), (6, 0.5). In general, we denote any discrete fuzzy set by

$$\underset{\sim}{A} = \sum_{x_i \in X} \mu_A(x_i)/x_i \tag{4}$$

or for the continuous case:

$$\underset{\sim}{A} = \int_{x \in X} \mu_A(x_i)/x_i \tag{5}$$

Note that the Σ and $\int$ signs do not denote the mathematical sum or integral.

THE SHAPE OF FUZZY SETS

The membership function of a fuzzy set can have different shapes. This depends on its definition. Membership functions with piecewise straight lines with a platform are called Π-functions (e.g., trapezoidal function) (see Fig. 3).

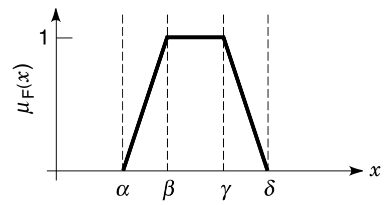

Figure 3. Example of a Π-function.

Other common shapes and forms are shown in Fig. 4, below.

Definition 1. The function Π: $X \to [0, 1]$ is defined by four parameters (α, β, γ, δ):

$$\Pi(x; \alpha, \beta, \gamma, \delta) = \begin{cases} 0 & x < \alpha \\ (x-\alpha)/(\beta-\alpha) & \alpha \le x \le \beta \\ 1 & \beta \le x \le \gamma \\ 1-(x-\gamma)/(\delta-\gamma) & \gamma \le x \le \delta \\ 0 & x > \delta \end{cases} \tag{6}$$

Further, we have a decreasing membership function with straight lines, the L-function; an increasing membership function with straight lines, the Γ-function; a triangular function with straight lines, the Λ-function; and a membership function with the membership function value 1 for only one value and the rest zero, the singleton. They are all special cases of the Π-function. This is shown in Eqs. (7–10). Suppose that the underlying domain is [−6, 6]; then the following equations hold:

$$\Gamma(x; \alpha, \beta) = \Pi(x; \alpha, \beta, 6, 6) \tag{7}$$

$$L(x; \gamma, \delta) = \Pi(x; -6, -6, \gamma, \delta) \tag{8}$$

$$\Lambda(x; \alpha, \beta, \delta) = \Pi(x; \alpha, \beta, \beta, \delta) \tag{9}$$

$$\text{singleton}\,(x; \alpha) = \Pi(x; \alpha, \alpha, \alpha, \alpha) \tag{10}$$

Hence, most standard shapes are special cases of the Π function.

FUZZY SETS OPERATIONS

As in the traditional crisp sets, logical operations, for example, union, intersection, and complement, can be applied to fuzzy sets (1). Some of the more common operations are discussed in this section.

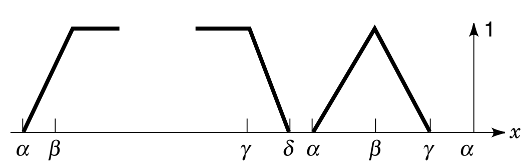

Figure 4. Examples of the Γ, L, Λ, and singleton.

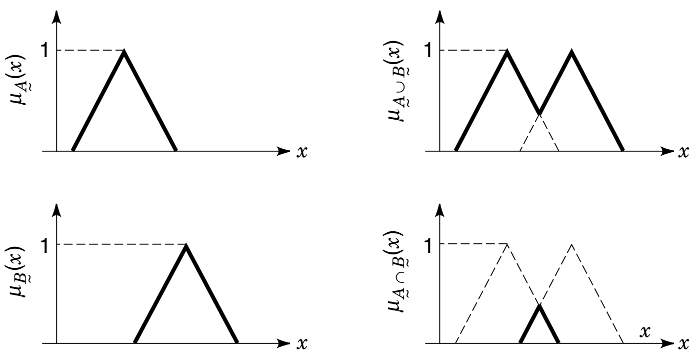

Figure 5. The fuzzy set operations union and intersection.

Union

The union operation (and the intersection operation as well) can be defined in many different ways. Here, the definition that is used in most cases is discussed. The union of two fuzzy sets A and B with the membership functions $\mu_A(x)$ and $\mu_B(x)$ is a fuzzy set $\underset{\sim}{C}$, written as $\underset{\sim}{C} = \underset{\sim}{A} \cup \underset{\sim}{B}$, whose membership function is related to those of A and B as follows:

$$\forall x \in U: \mu_C(x) = \max[\mu_{\underset{\sim}{A}}(x), \mu_{\underset{\sim}{B}}(x)] \qquad (11)$$

where U is the universe of discourse. The operator in this equation is referred to as the max-operator.

Intersection

According to the min-operator, the intersection of two fuzzy sets A and B with the membership functions $\mu_{\underset{\sim}{A}}(x)$ and $\mu_{\underset{\sim}{B}}(x)$, respectively, is a fuzzy set $\underset{\sim}{C}$, written as $\underset{\sim}{C} = \underset{\sim}{A} \cup \underset{\sim}{B}$, whose membership function is related to those of A and B as follows:

$$\forall x \in U: \mu_{\underset{\sim}{C}}(x) = \min[\mu_{\underset{\sim}{A}}(x), \mu_{\underset{\sim}{B}}(x)] \qquad (12)$$

Both the intersection and the union operation are explained by Fig. 5. Min and max operators are special cases of more general operators called t-norm and t-conorm (s-norm), respectively.

Complement

The complement of a fuzzy set $\underset{\sim}{A}$ is denoted $\bar{\underset{\sim}{A}}$ as with a membership function defined as (see also Fig. 6):

$$\forall x \in U: \mu_{\bar{\underset{\sim}{A}}}(x) = 1 - \mu_{\underset{\sim}{A}}(x) \qquad (13)$$

Most of the properties that hold for classical sets (e.g., commutativity, associativity, and idempotence) hold also for fuzzy sets, manipulated by the specific operations in Eqs. (11–13), except for two properties:

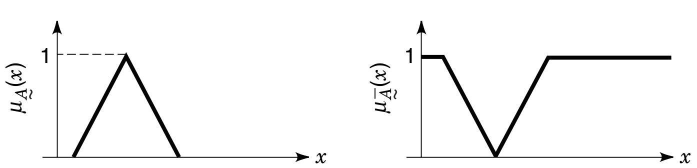

Figure 6. Fuzzy set and its complement.

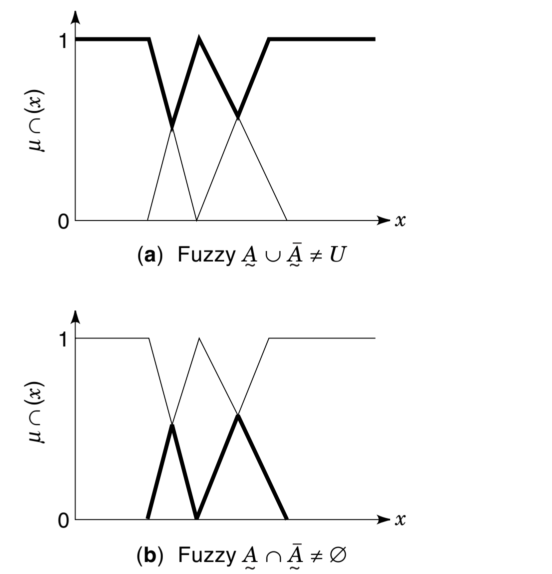

Figure 7. Excluded middle laws for fuzzy sets. (a) Fuzzy $\underset{\sim}{A} \cup \bar{\underset{\sim}{A}} \neq U$. (b) Fuzzy $\underset{\sim}{A} \cap \bar{\underset{\sim}{A}} \neq \emptyset$.

1. *Law of contradiction* ($\underset{\sim}{A} \cap \bar{\underset{\sim}{A}} \neq \varnothing$). One can easily notice that the intersection of a fuzzy set and its complement results in a fuzzy set with membership values up to $\frac{1}{2}$ in our case and thus does not equal the empty set (see the equation below and Fig. 7).

$$\forall x \in U: \mu_{\underset{\sim}{A}\cap\bar{\underset{\sim}{A}}}(x) = \min\big[\mu_{\underset{\sim}{A}}(x), \big(1 - \mu_{\underset{\sim}{A}}(x)\big)\big] \leq \frac{1}{2} \neq \varnothing \qquad (14)$$

2. *Law of excluded middle.* The union of a fuzzy set and its complement does not give the universe of discourse (see Fig. 7).

$$\forall x \in U: \mu_{\underset{\sim}{A}\cap\bar{\underset{\sim}{A}}}(x) = \max\big[\mu_{\underset{\sim}{A}}(x), \big(1 - \mu_{\underset{\sim}{A}}(x)\big)\big] \leq \frac{1}{2} \neq U \qquad (15)$$

Fuzzification and α-Cut Sets

It is the crisp domain in which we perform all computations with today's computers. The conversion from crisp to fuzzy and fuzzy to crisp sets can be done by the following means.

Definition 2. The process of assigning a set of fuzzy linquistic labels to a physical variable within a range [−U, U] is termed as fuzzification of that variable.

As an example, the temperatures ranging from −30° to +100°C can be partitioned into seven segments, leading to seven linguistic labels: very cold, cold, zero, moderate, warm, hot, very hot, as shown in Fig. 8.

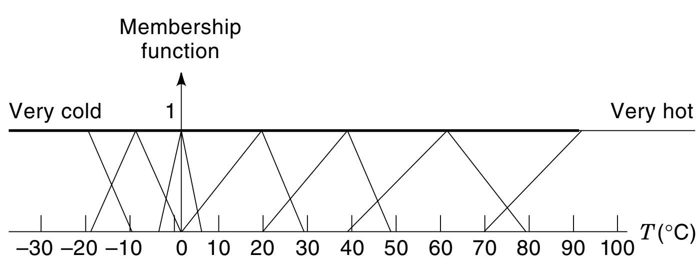

Figure 8. Fuzzification of physical variable temperature.

Definition 3. Given a fuzzy set A, the α-cut (or λ-cut) set of A is defined by

$$A_\alpha = \{x | \mu_{\underset{\sim}{A}}(x) \geq \alpha\} \tag{16}$$

Note that by virtue of the condition on $\mu_A(x)$ in Eq. (16), that is, a common property, the set A_α in Eq. (16) is now a crisp set. In fact, any fuzzy set can be converted to an infinite number of cut sets.

Example 3. Consider a fuzzy set

$$A = \left\{ \frac{1}{x_1} + \frac{0.9}{x_2} + \frac{0.8}{x_3} + \frac{0.75}{x_4} + \frac{0.5}{x_5} + \frac{0.2}{x_6} + \frac{0.15}{x_7} + \frac{0.1}{x_8} + \frac{0.05}{x_9} + \frac{0}{x_{10}} \right\}$$

It is desired to find the number of α-cut sets for different values of α.

The fuzzy set $\underset{\sim}{A}$ is shown in Fig. 9. The α-cut sets A_1, $A_{0.8}$, $A_{0.5}$, $A_{0.1}$, A_{0+} and A_0 are defined by

$$\begin{aligned} A_1 &= \{x_1\}, A_{0.8} = \{x_1, x_2, x_3\} \\ A_{0.5} &= \{x_1, x_2, \ldots, x_5\} \\ A_{0.1} &= \{x_1, x_2, \ldots, x_8\} \\ A_{0+} &= \{x_1, x_2, \ldots, x_9\} \text{ and } A_0 = U \end{aligned}$$

Note that by definition, the 0-cut set A_0 is the universe of discourse. Figure 10 shows these α-cut sets.

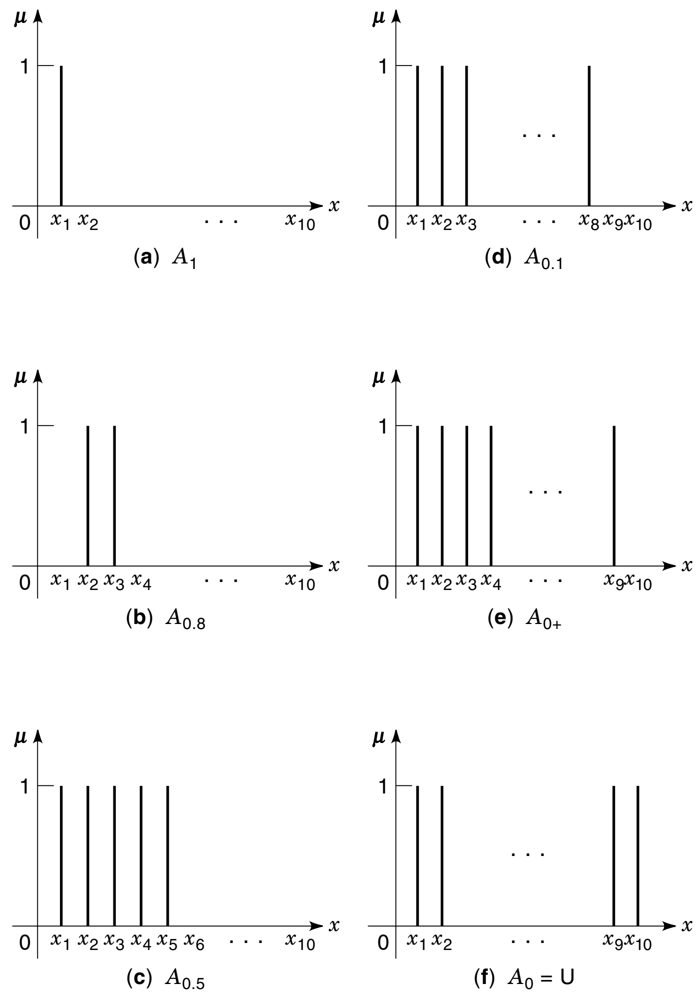

Figure 10. Schematic of 5 α-cut sets for fuzzy set of Example 3.

Extension Principle

In fuzzy sets, just as in crisp sets, one needs to find a means to extend the domain of a function; that is, given a fuzzy set A and a function $f(\cdot)$, then what is the value of function $f(A)$? This notion is called the extension principle (2–4).

Let the function f be defined by

$$f : U \rightarrow V \tag{17}$$

where U and V are domain and range sets, respectively. Define a fuzzy set A $\subset U$ as

$$\underset{\sim}{A} = \left\{ \frac{\mu_1}{u_1} + \frac{\mu_2}{u_2} + \cdots + \frac{\mu_n}{u_n} \right\} \tag{18}$$

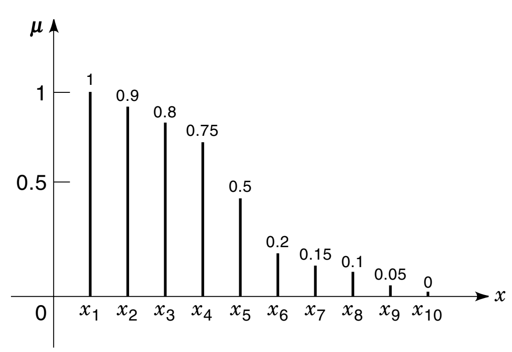

Figure 9. Fuzzy set $\underset{\sim}{A}$ of Example 3.

Then, the extension principle asserts that the function f is a fuzzy set, as well, which is defined, in the simplest case, by:

$$\underset{\sim}{B} = f(\underset{\sim}{A}) = \frac{\mu_1}{f(u_1)} + \frac{\mu_2}{f(u_2)} + \cdots + \frac{\mu_n}{f(u_n)} \tag{19}$$

In other words, the resulting fuzzy set has the same membership values corresponding to the functions of the elements u_i, $i = 1, 2, \ldots, n$. The following examples illustrate the use of the extension principle, which are also an illustration of fuzzy arithmatic.

Example 4. Given two universes of discourse $U_1 = U_2 = \{1, 2, \ldots, 10\}$ and two fuzzy sets (numbers) defined by

$$\text{“Approximately 2”} = \underset{\sim}{2} = \frac{0.5}{1} + \frac{1}{2} + \frac{0.8}{3} \text{ and}$$
$$\text{“Almost 5”} = \underset{\sim}{5} = \frac{0.6}{3} + \frac{0.8}{4} + \frac{1}{5}$$

It is desired to find “approximately 10,” that is, $\underset{\sim}{10} = \underset{\sim}{2} \times \underset{\sim}{5}$.

The function $f = u_1 \times u_2: \rightarrow v$ represents the arithmetic product of these two fuzzy numbers which is given by

$$
\begin{aligned}
\underset{\sim}{10} &= \underset{\sim}{2} \times \underset{\sim}{5} \\
&= \left(\frac{0.5}{1} + \frac{1}{2} + \frac{0.8}{3}\right) \times \left(\frac{0.6}{3} + \frac{0.8}{4} + \frac{1}{5}\right) = \frac{\min(0.5, 0.6)}{3} \\
&+ \frac{\min(0.5, 0.8)}{4} + \frac{\min(0.5, 1)}{5} + \frac{\min(1, 0.6)}{6} + \frac{\min(1, 0.8)}{8} \\
&+ \frac{\min(1, 1)}{10} + \frac{\min(0.8, 0.6)}{9} + \frac{\min(0.8, 0.8)}{12} + \frac{\min(0.8, 1)}{15} \\
&= \frac{0.5}{3} + \frac{0.5}{4} + \frac{0.5}{5} + \frac{0.6}{6} + \frac{0.8}{8} + \frac{0.6}{9} + \frac{1}{10} + \frac{0.8}{12} + \frac{0.8}{15}
\end{aligned}
$$

Here, intersection properties of fuzzy sets have been used. The above resulting fuzzy number has its prototype, that is, value 10 with a membership value 1 and the other 8 pairs are spread around the point (1, 10).

The complexity of the extension principle would increase when more than one member of $u_1 \times u_2$ is mapped to only one member of v; one would take the maximum membership grades of these members in the fuzzy set A. The following example illustrates this case.

Example 5. Consider the two fuzzy numbers,

$$
\begin{aligned}
\underset{\sim}{2} &= \text{“approximately 2”} = \frac{0.5}{1} + \frac{1}{2} + \frac{0.5}{3} \\
\underset{\sim}{4} &= \text{“approximately 4”} = \frac{0.8}{2} + \frac{0.9}{3} + \frac{1}{4}
\end{aligned}
$$

It is desired to find $\underset{\sim}{8}$.

The product $\underset{\sim}{2} \times \underset{\sim}{4}$ would be given by the following expression:

$$
\begin{aligned}
\underset{\sim}{2} \times \underset{\sim}{4} &= \frac{\min(0.5, 0.8)}{2} + \frac{\min(0.5, 0.9)}{3} \\
&+ \frac{\max\{\min(0.5, 1), \min(1, 0.8)\}}{4} \\
&+ \frac{\max\{\min(1, 0.9), \min(0.5, 0.8)\}}{6} + \frac{\min(1, 1)}{8} \\
&+ \frac{\min(0.5, 0.9)}{9} + \frac{\min(0.5, 1)}{12} = \frac{0.5}{2} + \frac{0.5}{3} + \frac{0.8}{4} \\
&+ \frac{0.9}{6} + \frac{1}{8} + \frac{0.5}{9} + \frac{0.5}{12}
\end{aligned}
$$

Note here that due to multipoint mapping case, max–min composition has been used.

Fuzzy Relations

Consider the Cartesian product of two universes U and V, defined by

$$U \times V = \{(u, v) | u \in U, v \in V\} \tag{20}$$

which combines elements of U and V in a set of ordered pairs. As an example, if $U = \{1, 2\}$ and $V = \{a, b, c\}$, then $U \times V = \{(1, a), (1, b), (1, c), (2, a), (2, b), (2, c)\}$. The above product is said to be a crisp relation which can be expressed by either a

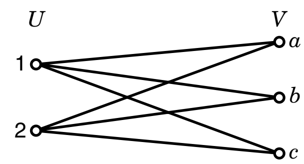

Figure 11. A crisp Sagittal diagram.

matrix expression

$$R_c = U \times V = \begin{matrix} & a & b & c \\ 1 & 1 & 1 & 1 \\ 2 & 1 & 1 & 1 \end{matrix} \tag{21}$$

or in a so-called *Sagittal* diagram (see Fig. 11) (see Ref. 3, Chap. 2). In classical set relations, one can perform operations on crisp relations using max-min composition, similar to those in Example 5.

The fuzzy relations, similarly, map elements of one universe, say U, to elements of another universe V through Cartesian product, but the strength of the relationship is measured by the grade of a membership function (4). In other words, a fuzzy relation R is a mapping:

$$\underset{\sim}{R}: U \times V \rightarrow [0, 1] \tag{22}$$

The following example illustrates this relationship; that is,

$$\mu_{\underset{\sim}{R}}(u, v) = \mu_{\underset{\sim}{A} \times \underset{\sim}{B}}(u, v) = \min\left(\mu_{\underset{\sim}{A}}(u), \mu_{\underset{\sim}{B}}(v)\right) \tag{23}$$

Example 6. Consider two fuzzy sets $\underset{\sim}{A}_1 = 0.2/x_1 + 0.9/x_2$ and $\underset{\sim}{A}_2$ $0.3/y_1 + 0.5/y_2 + 1/y_3$. Determine the fuzzy relation among these sets.

The fuzzy relation $\underset{\sim}{R}$, using Eq. (23), is

$$
\begin{aligned}
\underset{\sim}{R} = \underset{\sim}{A}_1 \times \underset{\sim}{A}_2 &= \begin{bmatrix} 0.2 \\ 0.9 \end{bmatrix} \times [0.3 \quad 0.5 \quad 1] \\
&= \begin{bmatrix} \min(0.2, 0.3) & \min(0.2, 0.5) & \min(0.2, 1) \\ \min(0.9, 0.3) & \min(0.9, 0.5) & \min(0.9, 1) \end{bmatrix} \\
&= \begin{bmatrix} 0.2 & 0.2 & 0.2 \\ 0.3 & 0.5 & 0.9 \end{bmatrix}
\end{aligned}
$$

In crisp or fuzzy relations, the composition of two relations, using the max-min rule, is given below. Given two fuzzy relations $\underset{\sim}{R}(u, v)$ and $\underset{\sim}{S}(v, w)$, then the composition of these relations is

$$\underset{\sim}{T} = \underset{\sim}{R} \circ \underset{\sim}{S} \; with \; \mu_T(u, w) = \max_{v \in V} \left\{\min\left(\mu_R(u, v), \mu_S(v, w)\right)\right\}$$

or using the max-product rule, the characteristic function is given by

$$\mu_{\underset{\sim}{T}}(u, w) = \max_{v \in V}\{\mu_{\underset{\sim}{R}}(u, v) \cdot \mu_{\underset{\sim}{S}}(v, w)\}$$

The same compositional rules hold for crisp relations. In general, $\underset{\sim}{R} \circ \underset{\sim}{S} \neq \underset{\sim}{S} \circ \underset{\sim}{R}$. The following example illustrates this point.

Example 7. Consider two fuzzy relations

$$\underset{\sim}{R} = \begin{matrix} & y_1 & y_2 \\ x_1 & 0.6 & 0.8 \\ x_2 & 0.7 & 0.9 \end{matrix} \quad \text{and } \underset{\sim}{S} = \begin{matrix} & z_1 & z_2 \\ y_1 & 0.3 & 0.1 \\ y_2 & 0.2 & 0.8 \end{matrix}$$

It is desired to evaluate $\underset{\sim}{R} \circ \underset{\sim}{S}$ and $\underset{\sim}{S} \circ \underset{\sim}{R}$.

Using the max-min composition, we have

$$\underset{\sim}{T}_1 = \underset{\sim}{R} \circ \underset{\sim}{S} = \begin{bmatrix} 0.3 & 0.8 \\ 0.3 & 0.8 \end{bmatrix}$$

where, for example, the (1, 1) element is obtained by max{min(0.6, 0.3), min(0.8, 0.2)} = 0.3.

The max-min composition of results in $\underset{\sim}{S} \circ \underset{\sim}{R}$

$$\underset{\sim}{S} \circ \underset{\sim}{R} = \begin{bmatrix} 0.3 & 0.3 \\ 0.7 & 0.8 \end{bmatrix} \neq R \circ S$$

which is expected.

Using the max-product rule, we have

$$T_2 = \underset{\sim}{R} \circ \underset{\sim}{S} = \begin{bmatrix} 0.18 & 0.64 \\ 0.21 & 0.72 \end{bmatrix}$$

where, for example, the term (2, 2) is obtained by max{(0.7) (0.1), (0.9) (0.8)} = 0.72.

The max-product composition $\underset{\sim}{S} \circ \underset{\sim}{R}$ results in

$$\underset{\sim}{S} \circ \underset{\sim}{R} = \begin{bmatrix} 0.3 & 0.3 \\ 0.7 & 0.8 \end{bmatrix} \neq R \circ S$$

which is, once again, expected.

FUZZY LOGIC AND APPROXIMATE REASONING

In the final section of this article, an introduction to fuzzy logic and approximate reasoning is given. Parts of this section are based on the work of Jamshidi (2) and Ross (4).

Predicate Logic

Let a predicate logic proposition P be a linguistic statement contained within a universe of propositions which are either completely true or false.

The truth value of the proposition, P can be assigned a binary truth value, called $T(P)$, just as an element in a universe is assigned a binary quantity to measure its membership in a particular set. For binary (Boolean) predicate logic, $T(P)$ is assigned a value of 1 (true) or 0 (false). If U is the universe of all propositions, then T is a mapping of these propositions to the binary quantities (0, 1), or

$$T: U \to \{0, 1\}$$

Now, let P and Q be two simple propositions on the same universe of discourse that can be combined using the following five logical connectives:

1. disjunction ($\vee$)
2. conjunction ($\wedge$)
3. negation ($-$)
4. implication ($\rightarrow$)
5. equality ($\leftrightarrow$ *or* $\equiv$)

to form logical expressions involving two simple propositions. These connectives can be used to form new propositions from simple propositions.

Now, define sets A and B from universe X where these sets might represent linguistic ideas or thoughts. Then, a propositional calculus will exist for the case where proposition P measures the truth of the statement that an element, x, from the universe X is contained in set A, and the truth of the statement that this element, x, is contained in set B, or more conventionally let

P: truth that $x \in A$

Q: truth that $x \in B$, where truth is measured in terms of the truth value; that is,

If $x \in A$, $T(P) = 1$; otherwise, $T(P) = 0$. If $x \in B$, $T(Q) = 1$; otherwise, $T(Q) = 0$, or using the characteristic function to represent truth (1) and false (0),

$$\chi_A(x) = \begin{cases} 1, x \in A \\ 0, x \notin A \end{cases}$$

The above five logical connectives can be used to create compound propositions, where a compound proposition is defined as a logical proposition formed by logically connecting two or more simple propositions. Just as one is interested in the truth of a simple proposition, predicate logic also involves the assessment of the truth of compound propositions. For two simple proposition cases, the resulting compound propositions are defined below in terms of their binary truth values,

$$P: x \in A, \overline{P}: x \notin A$$

$$P \vee Q \Rightarrow x \in A \text{ or } B$$

$$\text{Hence, } T(P \vee Q) = \max(T(P), T(Q))$$

$$P \wedge Q \Rightarrow x \in A \text{ and } B$$

$$\text{Hence, } T(P \wedge Q) = \min(T(P), T(Q))$$

$$\text{If } T(P) = 1, \text{ then } T(\overline{P}) = 0; \textit{If } T(P) = 0, \text{ then } T(\overline{P}) = 1$$

$$P \leftrightarrow Q \Rightarrow x \in A, B$$

$$\text{Hence, } T(P \leftrightarrow Q) = T(P) = T(Q)$$

The logical connective implication presented here is also known as the classical implication to distinguish it from an alternative form due to Lukasiewicz, a Polish mathematician in the 1930s, who was first credited with exploring logics other than Aristotelian (classical or binary logic) logic. This classical form of the implication operation requires some explanation.

For a proposition P defined on set A and a proposition Q defined on set B, the implication "P implies Q" is equivalent to taking the union of elements in the complement of set A with the elements in the set B. That is, the logical implication

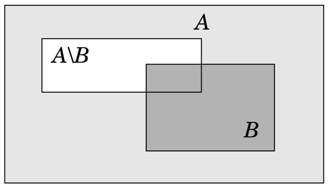

Figure 12. Venn diagram for implication $P \to Q$.

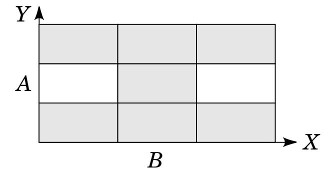

Figure 13. The Cartesian space for the implication IF A, THEN B.

is analogous to the set-theoretic form,

$$P \to Q \equiv \overline{A} \cup B \text{ is true either ``not in } A\text{'' or ``in } B\text{''}$$
$$\text{So that } (P \to Q) \leftrightarrow (\overline{P} \vee Q)$$
$$T(P \to Q) = T(\overline{P} \vee Q) = \max\big(T(\overline{P}), T(Q)\big)$$

This is linguistically equivalent to the statement, "P implies Q is true" when either "not A" or "B" is true. Graphically, this implication and the analogous set operation is represented by the Venn diagram in Fig. 12. As noted, the region represented by the difference $A \backslash B$ is the set region where the implication "P implies Q" is false (the implication "fails"). The shaded region in Fig. 12 represents the collection of elements in the universe where the implication is true; that is, the shaded area is the set

$$\begin{aligned} &\overline{A \setminus B} = \overline{A} \cup B = (\overline{A \cap \overline{B}}) \\ &\text{If } x \text{ is in } A \text{ and } x \text{ is not in } B \text{ then} \\ &A \to B \text{ fails } A \setminus B \text{(difference)} \end{aligned} \tag{24}$$

Now, with two propositions (P and Q) each being able to take on one of two truth values (true or false, 1 or 0), there will be a total of $2^2 = 4$ propositional situations. These situations are illustrated in Table 1, along with the appropriate truth values, for the propositions P and Q and the various logical connectives between them.

Suppose the implication operation involves two different universes of discourse; P is a proposition described by set A, which is defined on universe X, and Q is a proposition described by set B, which is defined on universe Y. Then, the implication "P implies Q" can be represented in set-theoretic terms by the relation R, where R is defined by

$$\begin{aligned} &R = (A \times B) \cup (\overline{A} \times Y) \equiv \text{ IF } A, \text{ THEN } B \\ &\text{If } x \in A \text{ where } x \in X, A \subset X \\ &\text{then } y \in B \text{ where } y \in Y, B \subset Y \end{aligned} \tag{25}$$

This implication is also equivalent to the linguistic rule form: IF A, THEN B. The graphic shown below in Fig. 13 represents the Cartesian space of the product $X \times Y$, showing typical sets A and B, and superposed on this space is the set-theoretic equivalent of the implication. That is,

$$P \to Q \Rightarrow \text{ IF } x \in A, \text{ then } y \in B, \text{ or } P \to Q \equiv \overline{A} \cup B$$

The shaded regions of the compound Venn diagram in Fig. 12 represent the truth domain of the implication, IF A, THEN B (P implies Q). In the problem section, the case of IF A, THEN B, ELSE C is considered.

Table 1. Truth Table

P	Q	$\overline{P}$	$P \vee Q$	$P \wedge Q$	$P \to Q$	$P \leftrightarrow Q$
T(1)	T(1)	F(0)	T(1)	T(1)	T(1)	T(1)
T(1)	F(0)	F(0)	T(1)	F(0)	F(0)	F(0)
F(0)	T(1)	T(1)	T(1)	F(0)	T(1)	F(0)
F(0)	F(0)	T(1)	F(0)	F(0)	T(1)	T(1)

Tautologies

In predicate logic, it is useful to consider compound propositions that are always true, irrespective of the truth values of the individual simple propositions. Classical logical compound propositions with this property are called tautologies. Tautologies are useful for deductive reasoning and for making deductive inferences. So if a compound proposition can be expressed in the form of a tautology, the truth value of that compound proposition is known to be true. Inference schemes in expert systems often employ tautologies. The reason for this is that tautologies are logical formulas that are true on logical grounds alone (4).

One of these, known as the *modus ponens* deduction, is a very common inference scheme used in forward chaining rule-based expert systems. It is an operation whose task is to find the truth value of a consequent in a production rule, given the truth value of the antecedent in the rule. A modus ponens deduction concludes that given two propositions, a and a-implies-b, both of which are true, then the truth of the simple proposition b is automatically inferred. Another useful tautology is the *modus tollens* inference, which is used in backward-chaining expert systems. In modus tollens, an implication between two propositions is combined with a second proposition, and both are used to imply a third proposition. Some common tautologies are listed below.

$$\overline{B} \cup B \leftrightarrow X \qquad \big(A \wedge (A \to B)\big) \to B \text{ (Modus Ponens)} \tag{26}$$

$$\begin{aligned} &A \cup X;\ \overline{A} \cup X \leftrightarrow X\big(\overline{B} \wedge (A \to B)\big) \to \overline{A} \text{ (Modus Tollens)} \\ &A \leftrightarrow B \end{aligned} \tag{27}$$

Contradictions

Compound propositions that are always false, regardless of the truth value of the individual simple propositions comprising the compound proposition, are called contradictions. Some simple contradictions are listed below.

$$\begin{aligned} &\overline{B} \cap B \\ &A \cap \phi;\ \overline{A} \cap \phi \end{aligned} \tag{28}$$

Deductive Inferences

The modus ponens deduction is used as a tool for inferencing in rule-based systems. A typical IF-THEN rule is use to determine whether an antecedent (cause or action) infers a consequent (effect or action). Suppose we have a rule of the form,

$$\text{IF } A,\ \text{THEN } B$$

This rule could be translated into a relation using the Cartesian product of sets A and B; that is,

$$R = A \times B$$

Now, suppose a new antecedent, say A' is known. Can we use the modus ponens deduction to infer a new consequent, say B', resulting from the new antecedent? That is, in rule form

$$\text{IF } A' : \text{THEN } B'?$$

The answer, of course, is yes, through the use of the composition relation. Since "A implies B" is defined on the Cartesian space $X \times Y$, B' can be found through the following set-theoretic formulation,

$$B' = A' \circ R = A' \circ \big((A \times B) \cup (\overline{A} \times Y)\big) \tag{29}$$

A modus ponens deduction can also be used for the compound rule,

$$\text{IF } A,\ \text{THEN } B,\ \text{ELSE } C$$

using the relation defined as

$$R = (A \times B) \cup (\overline{A} \times C) \tag{30}$$

and hence.

Example 8. Let two universes of discourse described by $X = \{1, 2, 3, 4, 5, 6\}$ and $Y = \{1, 2, 3, 4\}$, and define the crisp set $A = \{3, 4\}$ on X and $B = \{2, 3\}$ on Y. Determine the deductive inference IF A, THEN B.

The deductive inference yields the following characteristic function in matrix form, following the relation,

$$R = (A \times B) \cup (\overline{A} \times Y) = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{array}\!\!\begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \end{bmatrix} \end{array}$$

Fuzzy Logic

The extension of the above discussions to fuzzy deductive inference is straightforward. The fuzzy proposition $\underset{\sim}{P}$ has a value on the closed interval [0, 1]. The truth value of a proposition $\underset{\sim}{P}$ is given by

$$T(P) = \mu_A(x) \text{ where } 0 \le \mu_A \le 1$$

Thus, the degree of truth for $\underset{\sim}{P}$: $x \in A$ is the membership grade of x in $\underset{\sim}{A}$. The logical connectives of negation, disjunction, conjunction, and implication are similarly defined for fuzzy logic, for example, disjunction

$$\underset{\sim}{P} \vee \underset{\sim}{Q} \Rightarrow x \text{ is } \underset{\sim}{A} \text{ or } \underset{\sim}{B}$$
$$T(\underset{\sim}{P} \vee \underset{\sim}{Q}) = \max\big(T(\underset{\sim}{P}), T(\underset{\sim}{Q})\big)$$

the implication is given by

$$\underset{\sim}{P} \to \underset{\sim}{Q} \Rightarrow x \text{ is } \underset{\sim}{A} \text{ THEN } x \text{ is } \underset{\sim}{B}$$

or

$$T(\underset{\sim}{P} \to \underset{\sim}{Q}) = T(\underset{\sim}{\overline{P}} \vee \underset{\sim}{Q}) = \max\big(T(\underset{\sim}{\overline{P}}), T(\underset{\sim}{Q})\big)$$

Thus, a fuzzy logic implication would result in a fuzzy rule

$$\underset{\sim}{P} \to \underset{\sim}{Q} \Rightarrow \text{ If } x \text{ is } \underset{\sim}{A} \text{ THEN } y \text{ is } \underset{\sim}{B}$$

and is eqivalent to the following fuzzy relation

$$\underset{\sim}{R} = (\underset{\sim}{A} \times \underset{\sim}{B}) \cup (\underset{\sim}{\overline{A}} \times Y) \tag{31}$$

with a grade membership function,

$$\mu_{\underset{\sim}{R}}(x, y) = \max\big\{\big(\mu_{\underset{\sim}{A}}(x) \wedge \mu_{\underset{\sim}{B}}(y)\big), \big(1 - \mu_{\underset{\sim}{A}}(x)\big)\big\}$$

Example 9. Consider two universes of discourse $X = \{1, 2, 3, 4\}$ and $Y = \{1, 2, 3, 4, 5, 6\}$. Let two fuzzy sets A and B be given by

$$\underset{\sim}{A} = \frac{0.8}{2} + \frac{1}{3} + \frac{0.3}{4}$$
$$\underset{\sim}{B} = \frac{0.4}{2} + \frac{1}{3} + \frac{0.6}{4} + \frac{0.2}{5}$$

It is desired to find a fuzzy relation $\underset{\sim}{R}$ corresponding to IF $\underset{\sim}{A}$, THEN $\underset{\sim}{B}$.

Using the relation in Eq. (31) would give

$$\underset{\sim}{A} \times \underset{\sim}{B} = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array}\!\!\begin{array}{c} \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0.4 & 0.8 & 0.6 & 0.2 & 0 \\ 0 & 0.4 & 1 & 0.6 & 0.2 & 0 \\ 0 & 0.3 & 0.3 & 0.3 & 0.2 & 0 \end{bmatrix} \end{array}$$

$$\underset{\sim}{\overline{A}} \times Y = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array}\!\!\begin{array}{c} \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 0.2 & 0.2 & 0.2 & 0.2 & 0.2 & 0.2 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0.7 & 0.7 & 0.7 & 0.7 & 0.7 & 0.7 \end{bmatrix} \end{array}$$

and, hence, $\underset{\sim}{R} = \max\{\underset{\sim}{A} \times \underset{\sim}{B}, \underset{\sim}{\bar{A}} \times Y\}$

$$\underset{\sim}{R} = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array}\!\!\begin{array}{c} \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 0.2 & 0.4 & 0.8 & 0.6 & 0.2 & 0.2 \\ 0 & 0.4 & 1 & 0.6 & 0.2 & 0 \\ 0.7 & 0.7 & 0.7 & 0.7 & 0.7 & 0.7 \end{bmatrix} \end{array} \tag{32}$$

Approximate Reasoning

The primary goal of fuzzy systems is to formulate a theoretical foundation for reasoning about imprecise propositions, which is termed approximate reasoning in fuzzy logic technological systems.

Let us have a rule-based format to represent fuzzy information. These rules are expressed in conventional antecedent-consequent form, such as

Rule 1: IF x is $\underset{\sim}{A}$ THEN y is $\underset{\sim}{B}$ where $\underset{\sim}{A}$ and $\underset{\sim}{B}$ represent fuzzy propositions (sets).

Now let us introduce a new antecedent, say $\underset{\sim}{A}'$, and consider the following rule:

Rule 2: IF x is $\underset{\sim}{A}'$, THEN y is $\underset{\sim}{B}'$.

From information derived from Rule 1, is it possible to derive the consequent Rule 2, $\underset{\sim}{B}'$? The answer is yes, and the procedure is a fuzzy composition. The consequent $\underset{\sim}{B}'$ can be found from the composition operation

$$\underset{\sim}{B}' = \underset{\sim}{A}' \circ \underset{\sim}{R}' \tag{33}$$

Example 10. Reconsider the fuzzy system of Example 9. Let a new fuzzy set $\underset{\sim}{A}'$ be given by $\underset{\sim}{A}' = (0.5/1) + (1/2) + (0.2/3)$. It is desired to find an approximate reason (consequent) for the rule IF $\underset{\sim}{A}'$ THEN $\underset{\sim}{B}'$.

The relations of Eqs. (32) and (33) are used to determine $\underset{\sim}{B}'$.

$$\underset{\sim}{B}' = \underset{\sim}{A}' \circ \underset{\sim}{R}' = [0.5 \quad 0.5 \quad 0.8 \quad 0.6 \quad 0.5 \quad 0.5]$$

or

$$\underset{\sim}{B}' = \frac{0.5}{1} + \frac{0.5}{2} + \frac{0.8}{3} + \frac{0.6}{4} + \frac{0.5}{5} + \frac{0.5}{6}$$

Note the inverse relation between fuzzy antecedents and fuzzy consequences arising from the composition operation. More exactly, if we have a fuzzy relation $\underset{\sim}{R}: \underset{\sim}{A} \to \underset{\sim}{B}$, then will the value of the composition $\underset{\sim}{A} \circ \underset{\sim}{R} = \underset{\sim}{B}$? The answer is no, and one should not expect an inverse to exist for fuzzy composition. This is not, however, the case in crisp logic, that is, where all these latter sets and relations are crisp. The following example illustrates the nonexistence of the inverse.

Example 11. Let us reconsider the fuzzy system of Examples 9 and 10. Let $\underset{\sim}{A}' = A$, and evaluate $\underset{\sim}{B}'$.

We have

$$\underset{\sim}{B}' = \underset{\sim}{A}' \circ \underset{\sim}{R} = \underset{\sim}{A} \circ \underset{\sim}{R} = \frac{0.3}{1} + \frac{0.4}{2} + \frac{0.8}{3} + \frac{0.6}{4} + \frac{0.3}{5} + \frac{0.3}{6} \neq \underset{\sim}{B}$$

which yields a new consequent, since the inverse is not guaranteed. The reason for this situation is the fact that fuzzy inference is not precise, but approximate. The inference, in this situation, represents approximate linguistic characteristics of the relation between two universes of discourse.

BIBLIOGRAPHY

1. L. A. Zadeh, Fuzzy sets, *Information and Control,* **8**: 338–353, 1965.
2. M. Jamshidi, *Large Scale Systems: Modeling, Control and Fuzzy Logic,* Englewood Cliffs, NJ: Prentice-Hall, 1996.
3. M. Jamshidi, N. Vadiee, and T. J. Ross (eds.), *Fuzzy Logic and Control: Software and Hardware Applications.* Englewood Cliffs, NJ: Prentice-Hall, 1993.
4. T. J. Ross, *Fuzzy Logic and Engineering Application,* New York: McGraw-Hill, 1995.

MOHAMMAD JAMSHIDI
The University of New Mexico