MULTIRATE FILTERBANKS

As the computation speed in modern chip design increases in an exponential way, our world is turning rapidly from analog to digital mode, such as digital communications, digital audios, and digital televisions and videos. This change has been providing higher quality services than before, making the information highway possible, and revolutionizing our living standards. In this section, we briefly describe concepts of analog-to-digital (A/D) and digital-to-analog (D/A) converters, rate conversion, some applications of multirate filtering, such as transmultiplexing and multiresolution image analysis and coding, and finally the outline of this article.

INTRODUCTION

A/D and D/A Converters and Rate Conversion

In A/D transitions, analog signals, such as audio signals and video signals, from our physical world are first converted to digital signals, then these digital signals are processed using digital signal processing (DSP) techniques, such as filtering, detection, and compression. Finally the processed digital signals are converted back to physical analog signals. This process is illustrated in Fig. 1.

An A/D converter samples an analog signal $x_a(t)$ into a discrete sequence $x(n) = x_a(nT_s)$ with a sampling frequency $f_s = 1/T_s$, where T_s is the sampling period length. It is shown in Fig. 2(a). For the sampled signal $x(n)$ to convey all the information that the original analog signal $x_a(t)$ has, it is necessary and sufficient for the sampling frequency (or sampling rate) f_s not to be below the Nyquist frequency, i.e., twice of the bandwidth (Hz) of the analog signal $x_a(t)$:

$$f_s \geq 2f_0 \tag{1}$$

where f_0 is the bandwidth (Hz) of $x_a(t)$. A sampled signal $x(n)$ may be resampled with a reduced sampling rate $1/M$ shown in Fig. 2(b), where $M\downarrow$ is called downsampling (or decimation) by factor M. By doing so, the new rate for $x(Mn)$ is $1/M$ times of the original rate of $x(n)$.

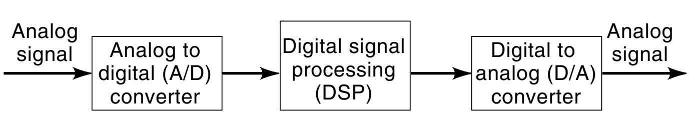


Figure 1. A general signal processing diagram. It is used in current application systems.

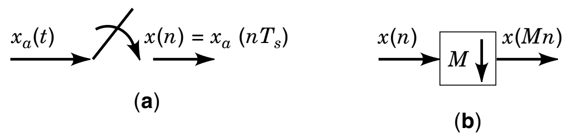


Figure 2. An analog-to-digital (A/D) signal converter converts an analog signal to a digital sequence and is implemented before digital signal processing.

The resampled signal $x(Mn)$ can be expanded to signal $\hat{x}(n)$ with the data rate the same as $x(n)$ by inserting $M - 1$ zeros between each two samples of $x(Mn)$, which is denoted by $\uparrow M$ and called upsampling (or expansion) by factor M shown in Fig. 3(b). A discrete signal $x(n)$ may be converted to an analog signal $\hat{x}_a(t)$ as shown in Fig. 3(a), where f_0 is the bandwidth of the original analog signal $x_a(t)$. This is called a D/A converter. The analog signal $\hat{x}_a(t)$ can be mathematically represented as

$$\hat{x}_a(t) = \sum_n x(n)\frac{\sin \pi(f_st - n)}{\pi(f_st - n)} \tag{2}$$

which is called the Shannon sampling theorem.

A noninteger (fractional) multiple rate change is shown in Fig. 4, where the data rate of $\hat{x}(n)$ is N/M times of the data rate of $x(n)$. For more about rate conversion, *see* Ref. 1.

There are many applications of multirate filtering. We next want to briefly discuss two of them: transmultiplexing and multiresolution image analysis and coding.

Transmultiplexing

Multiuser communications play important roles in current communication systems, where multiple users share a common channel. Such examples include phone (wireline and wireless mobile) networks and satellite communications. In multiuser communication systems, there are three common multiplexing methods: (1) time division multiple access (TDMA), where different users use different time slots; (2) frequency division multiple access (FDMA), where different users use different frequency slots; and (3) code division multiple access (CDMA), where different users use different codes. These multiplexing methods are called transmultiplexing. A general transmultiplexing block diagram with P users is

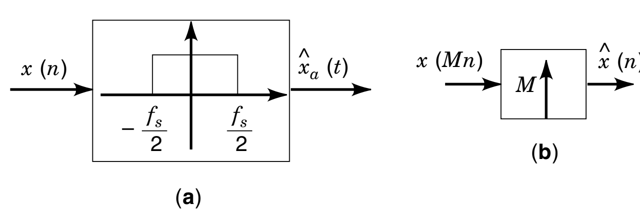


Figure 3. A digital-to-analog (D/A) signal converter, which converts a digital signal to an analog signal and is implemented after digital signal processing.

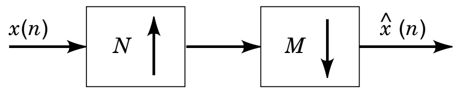


Figure 4. Rate conversion changes the data rates of a signal.

shown in Fig. 5. Each transmitter uses a linear time-invariant (LTI) finite impulse response (FIR) filter $F_p(z)$ and each receiver uses another LTI FIR filter $H_p(z)$.

The FIR filters in Fig. 5 for TDMA, FDMA, and CDMA systems have the following features shown in Fig. 6.

1. In TDMA transmultiplexing, each user occupies one fixed time slot in total M time slots, i.e., $F_p(z) = z^{-p}$ and $H_p(z) = z^p$ for $0 \le p \le P - 1$ [see Fig. 6(a)].
2. In FDMA transmultiplexing, each user occupies one fixed frequency slot in total M frequency slots, i.e., $F_p(z)$ and $H_p(z)$ are both supported mainly in a frequency band with $1/M$ of the total bandwidth [see Fig. 6(b)].
3. In CDMA transmultiplexing, each user uses a pseudo-random sequence to spread the information sequence, i.e., $F_p(z) = \sum_{n=0}^{M-1} f_p(n)z^{-n}$ with a pseudo-random binary sequence $f_p(n)$ for each p, $0 \le p \le P - 1$ [see Fig. 6(c)].

Since synchronization for FDMA systems is not as important as the one for TDMA and CDMA systems, in some applications FDMA is preferred and mixed with other systems, such as in satellite communications. Because FIR filters cannot be ideal bandpass filters, it is not obvious that the receivers in the FDMA system in Fig. 5 are able to recover the original transmitted information sequences, that is, $\hat{x}_p(n) = c_p x_p(n - n_p)$ for integers n_p and nonzero constants c_p. We will see later that this perfect recovery problem is equivalent to the perfect reconstruction of a multirate filterbank. For more about multirate filterbanks and transmultiplexers, see, for example, Refs. 2 and 3. More applications include ADSL, DMT, etc.

Multiresolution Image Analysis and Coding

Multiresolution image analysis is an important technique in digital image processing and has applications in target detection, image browsing, image/video compression, etc. The basic idea is to decompose an image into different resolutions and then process these images with different resolutions according to different needs. The multiresolution decomposition and reconstruction are built upon tree-structured multirate filterbanks as illustrated in Fig. 7(a), which are used along the x-axis and the y-axis separately in two-dimensional image processing. As an example, an original image is shown in Fig. 8 and its three-level multiresolution decomposition is shown in Fig. 9. In multiresolution image compression, such as embedded zero-tree wavelet (EZW) coding (4), the correlations between the images at different resolutions are fully taken into account. It has been shown that this compression method has better performance than the conventional discrete cosine transform (DCT) compression method. For more details about EZW and the comparisons between EZW and DCT approaches, one may consult Refs. 4 and 5. By using the EZW image coding, the original image in Fig. 8 is compressed with compression ratio 64, and its decompressed image is shown in Fig. 10.

In many multiresolution image analyses, such as image/video compression, the perfect reconstruction of the multirate filterbank shown in Fig. 7(a) is needed, which is equivalent to the perfect reconstruction of the 2-channel multirate filterbank shown in Fig. 7(b). Certain tree-structured multirate filterbanks also lead to wavelets (6–13).

Outline

From the two multirate filtering applications, one can see the importance of the perfect reconstruction property of a multirate filterbank. The rest of this article is devoted to developing a systematic theory of building perfect reconstruction (PR) multirate filterbanks. It should be pointed out that the PR property may not be necessary for good performance in some applications. This article is organized as follows. In the second section, we study some necessary properties on some basic building blocks for multirate filterbanks. In the third section, we study perfect reconstruction multirate filterbanks. In the fourth section, we develop factorization and construction for multirate filterbanks with perfect reconstruction. In the fifth section, we study two special kinds of multirate filterbanks: discrete Fourier transform (DFT) filterbanks and cosine modulated multirate filterbanks, which are related to short-time Fourier transforms and Gabor transforms. In the sixth section, we describe some recent developments on multirate filterbanks. In the last section, we summarize this article. There have been extensive studies on multirate filterbanks in the past two decades. Due to the length limitation, this article is by no means a complete survey on this subject but provides some basics on multirate filterbanks. For more comprehensive descriptions and historical

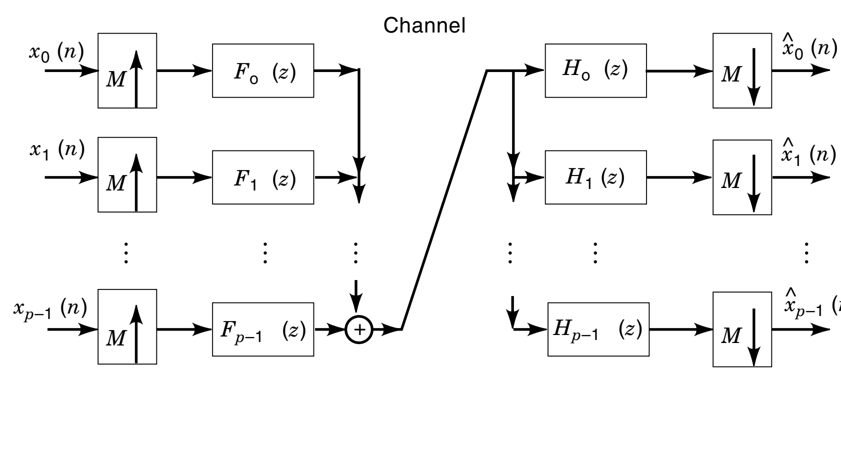


Figure 5. Transmultiplexer which is used in many communication systems including multiuser systems.

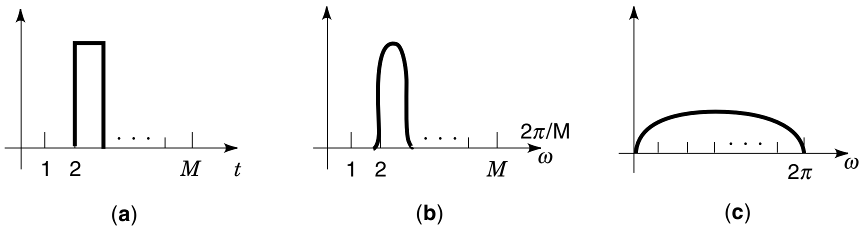


Figure 6. Multi-access characteristics having three common multi-access methods: (a) TDMA; (b) FDMA; and (c) CDMA.

events on multirate filterbanks, we refer the reader to Refs. 1 and 8–13.

NOTATIONS, BASIC BUILDING BLOCKS, AND THEIR PROPERTIES

Notations

Throughout this article, all lowercase letters, $x(n)$ and $y(n)$, denote scalar values and scalar-valued sequences; capital letters, $X(z)$ and $Y(z)$, denote the z-transforms of scalar-valued sequences; bold-faced lowercase letters, $\mathbf{x}(n)$ and $\mathbf{y}(n)$, denote vector sequences; and bold-faced capital letters, $\mathbf{H}(z)$ and $\mathbf{F}(z)$, denote matrices and matrix polynomials. For a sequence $x(n)$, its z-transform is defined by

$$X(z) = \sum_n x(n) z^{-n}$$

and $X(e^{j\omega})$ is its discrete time Fourier transform (DTFT).

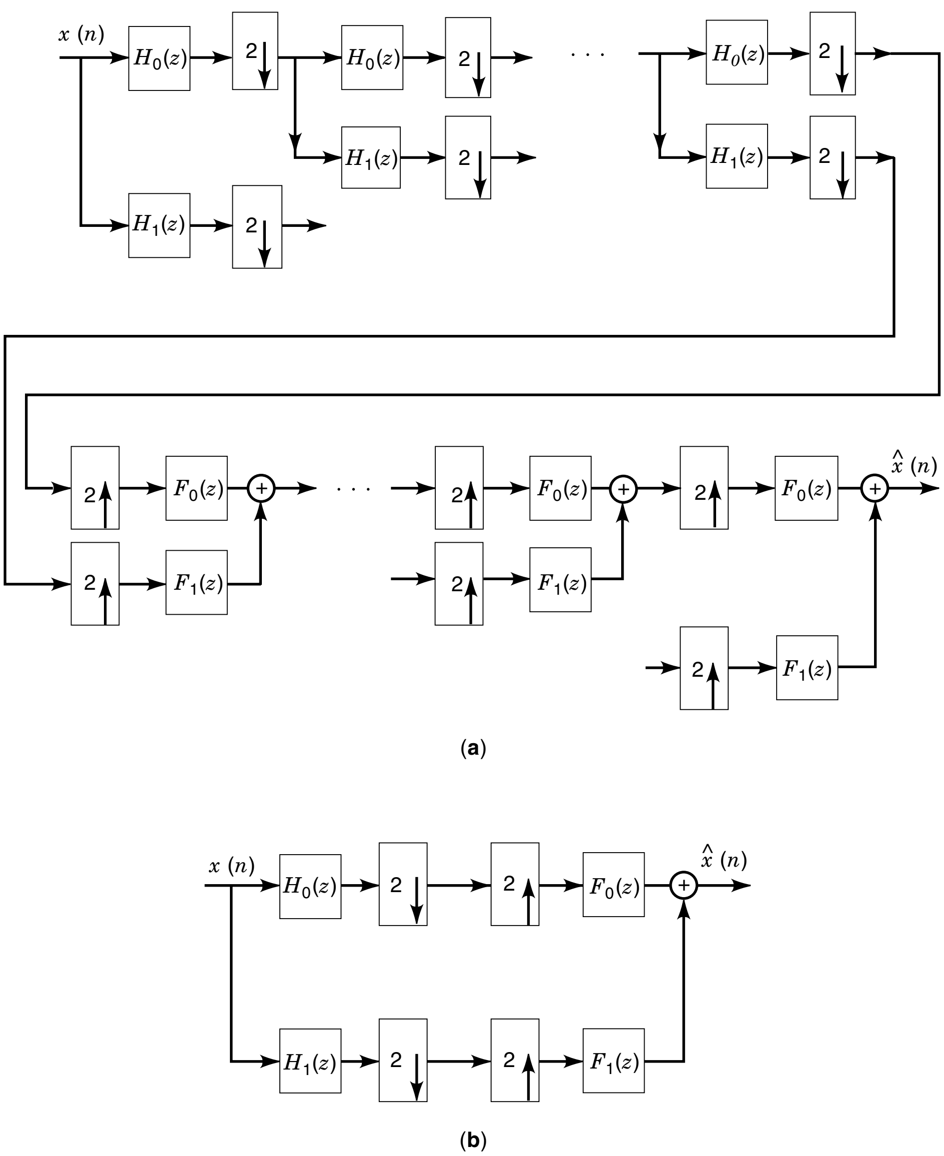


Figure 7. Multiresolution decomposition and reconstruction, where a signal is decomposed into different resolutions and then can be reconstructed from the various resolution decompositions.

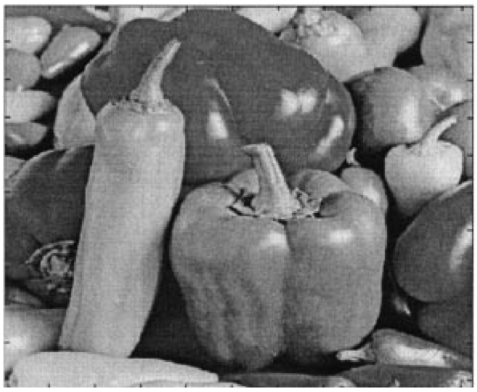


Figure 8. Original image, which is a test image from compression-using wavelets.

A matrix polynomial $\mathbf{H}(z)$ is

$$\mathbf{H}(z) = \sum_n \mathbf{H}_n z^{-n}$$

where $\mathbf{H}_n$ are constant matrices with same size. For a square matrix polynomial $\mathbf{H}(z)$, $\det(\mathbf{H}(z))$ denotes its determinant. For a matrix $\mathbf{A}$, matrices $\mathbf{A}^T$, $\mathbf{A}^\dagger$, and $\mathbf{A}^*$ denote the transpose, transpose conjugate, and conjugate, respectively. For a matrix polynomial $\mathbf{H}(z)$, its tilde operation $\tilde{\mathbf{H}}(z)$ denotes $\mathbf{H}^\dagger(1/z^*)$, that is,

$$\tilde{\mathbf{H}}(z) = \sum_n \mathbf{H}_n^\dagger z^n, \quad \text{if}\, \mathbf{H}(z) = \sum_n \mathbf{H}_n z^{-n} \tag{3}$$

A matrix polynomial $\mathbf{H}(z)$ has *FIR inverse* if and only if there exists a matrix polynomial $\mathbf{F}(z)$ such that $\mathbf{F}(z)\mathbf{H}(z) = \mathbf{I}$, where $\mathbf{I}$ is the identity matrix.

The matrix $\mathbf{W}_M$ denotes the M-point DFT matrix, that is,

$$\mathbf{W}_M = (W_M^{mn})_{0 \le m,n \le M-1}$$

where $W_m \triangleq e^{-j2\pi/M}$.

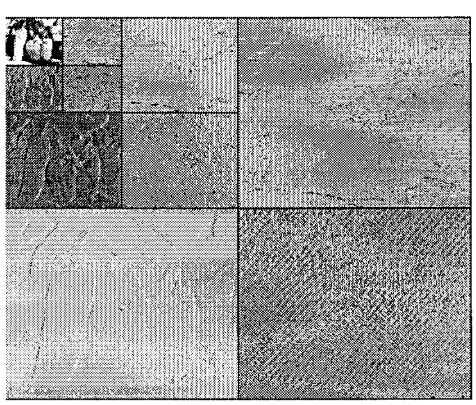


Figure 9. Multiresolution decomposition of the original image, which is an example of a multiresolution decomposition of the test image in Fig. 8, by using two-channel filterbanks (or wavelets).

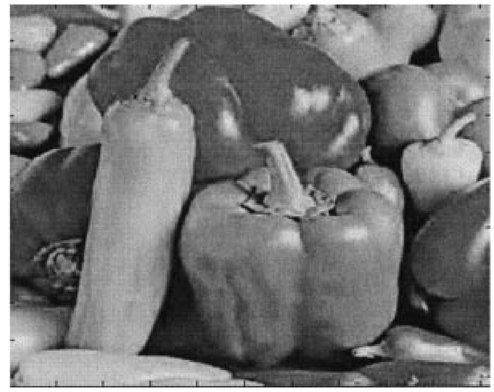


Figure 10. Decompressed image with compression ratio 64, which is obtained by using the wavelet-based compression algorithms developed by Said and Pearlman (5).

Decimator and Expander

An M-fold decimator (or downsampling) and L-fold expander (or upsampling) are depicted in Fig. 11(a) with an example in Fig. 11(b), and Fig. 11(c) with an example in Fig. 11(d), respectively, where

$$y_D(n) = x(Mn)$$

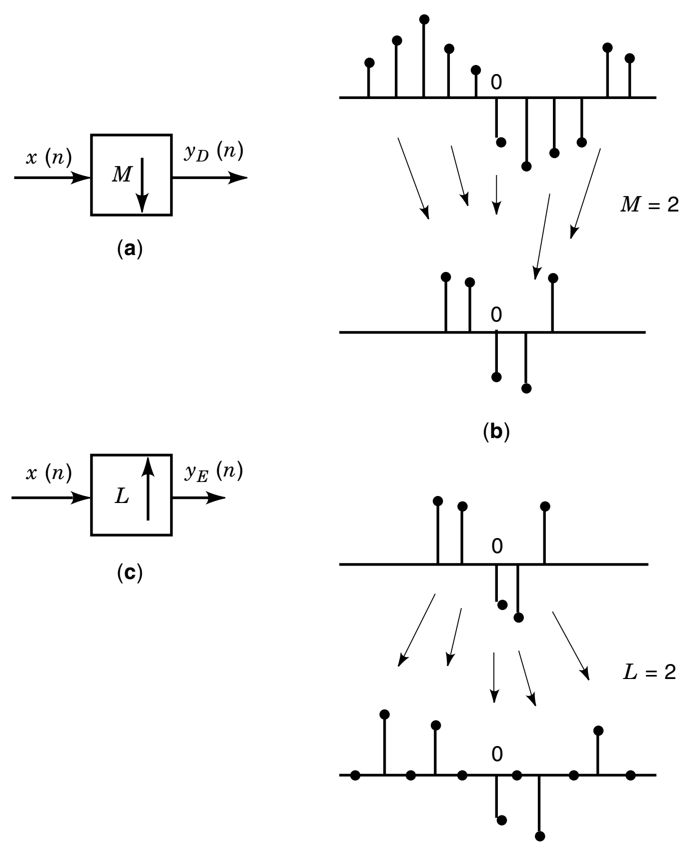


Figure 11. Decimator and expander. In the decimator, a data rate is reduced M times. In the expander, a data rate is increased L times. These two operations play important roles in multirate signal processing.

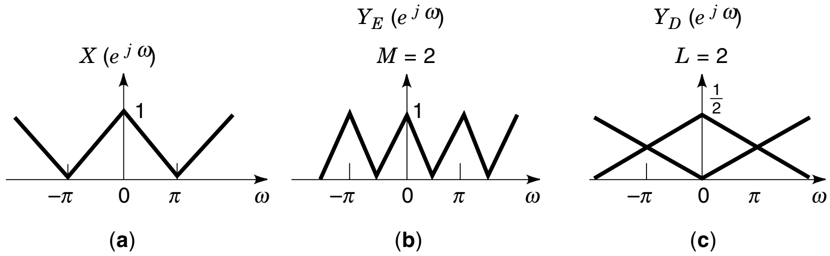


Figure 12. The frequency domain representation examples of the decimator and expander.

and

$$y_E(n) = \begin{cases} x[n/L], & \text{if } n \text{ is a multiple of } L, \\ 0, & \text{otherwise.} \end{cases}$$

In the frequency and z-transform domains (1,9),

$$Y_D(e^{j\omega}) = \sum_n y_D(n)e^{-j\omega n} = \frac{1}{M}\sum_{k=0}^{M-1} X(e^{j(\omega-2\pi k)/M}) \qquad (4)$$

and

$$Y_E(z) = \sum_n y_E(n)z^{-n} = X(z^L), \quad \text{and } Y_E(e^{j\omega}) = X(e^{j\omega L})$$

The graphical meaning for the expander is that the DTFT of the expanded $y_E(n)$ is an L-fold compressed version of the uncompressed $X(e^{j\omega})$ shown in Fig. 12(a,b). The graphical meaning for the decimator is the following [shown in Fig. 12(a,c)]:

1. Stretch $X(e^{j\omega})$ by a factor M to obtain $X(e^{j\omega/M})$;
2. Obtain $M - 1$ copies of this stretched version by shifting it uniformly in successive amounts of 2π;
3. Add all these shifted and stretched versions to the original unshifted and stretched version $X(e^{j\omega/M})$, and divide by M.

The $M - 1$ shifted and stretched versions of $X(e^{j\omega})$ in Eq. (4) are the aliasing created by the down-sampling.

Noble Identities

The following two Noble identities play important roles in the multirate filterbank theory. They tell us when the orders of the decimator/expander and an LTI system can be switched. The two Noble identities are shown in Fig. 13, where $y_1(n) = y_2(n)$ and $y_3(n) = y_4(n)$.

Polyphase Representations

The polyphase representation was first invented by Bellanger et al. (14) and Vary (15) and first recognized by Vaidyanathan and Vetterli in the simplifications of multirate filterbank theory studies. It can be briefly described as follows. For any given integer N, any filter $H(z)$ can be decomposed into

$$H(z) = \sum_{l=0}^{N-1} z^{-l}E_l(z^N) \qquad (5)$$

where

$$E_l(z) = \sum_n h[Nn + l]z^{-n}$$

and $h[n]$ is the impulse response of $H(z)$. The decomposition, Eq. (5), is called the *Type 1 polyphase representation* of $H(z)$. Meanwhile, $H(z)$ can be decomposed into

$$H(z) = \sum_{l=0}^{N-1} z^{-N+1+l}R_l(z^N) \qquad (6)$$

where $R_l(z) = E_{N-1-l}(z)$, which is called the *Type 2 polyphase representation* of $H(z)$. For $l = 0, 1, \ldots, N - 1$, $E_l(z)$ and $R_l(z)$ are called the lth Type 1 and Type 2 *polyphase components* of $H(z)$, respectively. We will see later that the Type 1 polyphase representation is for the analysis bank and the Type 2 polyphase representation is for the synthesis bank in a multirate filterbank.

The main purpose for introducing these polyphase representations is to move the decimator from the right side of an LTI filter to the left side (expander from the left side of an LTI filter to the right side) by using the Noble identities in Fig. 13. In the Noble identities, the power of the variable z in an LTI filter needs to rise, which usually does not hold for an

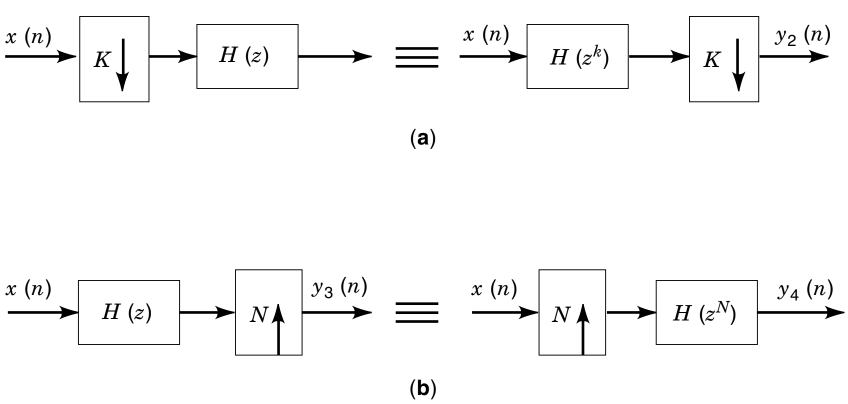


Figure 13. The Noble identities, the rules of switching the order of LTI system and decimator/expander.

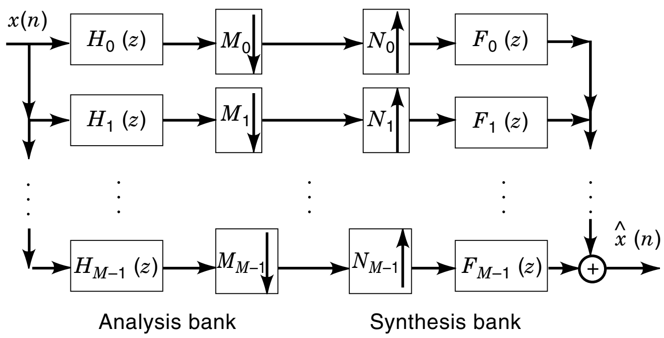


Figure 14. A general M-channel multirate filterbank.

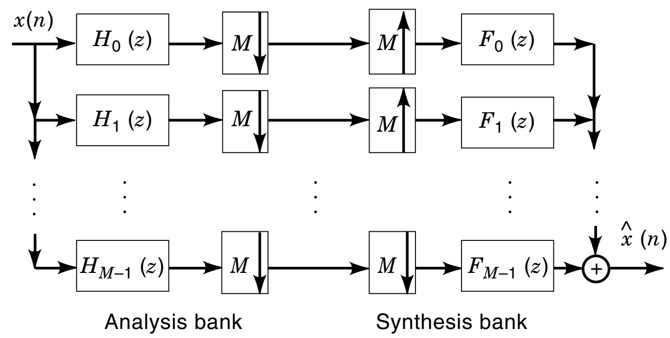


Figure 16. M-channel maximally decimated multirate filterbank.

LTI filter but does hold for the polyphase representations of an LTI filter as shown in Eqs. (5) and (6).

M-CHANNEL MULTIRATE FILTERBANKS

A general M-channel multirate filterbank is depicted in Fig. 14, where the left side is an analysis bank and the right side is a synthesis bank, each of which has M LTI filters. In many applications, such as FDMA, these M LTI filters occupy M different frequency bands as shown in Fig. 15.

There are several cases for an M-channel multirate filterbank in Fig. 14:

1. when $M_0 = M_1 = \cdots = M_{M-1} = N_0 = N_1 = \cdots = N_{M-1} = M$, the filterbank is called maximally decimated;
2. when $M_0 = M_1 = \cdots = M_{M-1} = N_0 = N_1 = \cdots = N_{M-1} < M$, the filterbank is called nonmaximally decimated;
3. when $M_0 = M_1 = \cdots = M_{M-1} = N_0 = N_1 = \cdots = N_{M-1} > M$, the filterbank is called over decimated;
4. when M_k and N_l are not all equal, the filterbank is called nonuniformally decimated.

Although there are increasing discussions on the cases 2–4 lately, such as applications of nonmaximally decimated multirate filterbanks in intersymbol interference (ISI) cancellation (16–21), and studies of nonuniformally decimated multirate filterbanks (22–27), the first case is the most well studied and the most important case. In this section, we focus on the first case, that is, maximally decimated multirate filterbanks.

Maximally Decimated Multirate Filterbanks: Perfect Reconstruction and Aliasing Component Matrix

An M-channel maximally decimated multirate filterbank is shown in Fig. 16. By comparing it with the transmultiplexer in Fig. 5, one can see that, when $P = M$, the transmitter side in Fig. 5 is the same as the synthesis bank in Fig. 16, and the receiver side in Fig. 5 is the same as the analysis bank in Fig. 16. The multiresolution decomposition filterbank in Fig. 7(b) is a 2-channel maximally decimated filterbank. In both applications, it is desired that the output signal $\hat{x}(n)$ is equal to the input $x(n)$ in some sense of the multirate filterbank in Fig. 16, which is the perfect reconstruction property as follows.

A multirate filterbank in Fig. 16 is called perfect reconstruction (PR) if and only if $\hat{x}(n) = cx(n - n_0)$ for a nonzero constant c and an integer n_0.

The question now becomes how to construct a PR multirate filterbank, in other words, what conditions on $H_m(z)$ and $F_m(z)$ are for the PR. Since in many applications, such as transmultiplexing and image analysis and coding, FIR filters are preferred, in what follows we are only interested in FIR filters $H_m(z)$ and $F_m(z)$ in Fig. 16 In this case, the multirate filterbank is called FIR. Examples of 2-channel PR filterbanks were first obtained by Smith and Barnwell (28) and Mintzer (29) independently. Some early studies on multirate filterbanks can also be found in Refs. 29a–c.

In the z-transform domain, the PR property becomes

$$\hat{X}(z) = cz^{-n_0}X(z) \tag{7}$$

In terms of an input signal $X(z)$, by using Eq. (4) the output $\hat{X}(z)$ in Fig. 16 can be formulated as follows:

$$\hat{X}(z) = A_0(z)X(z) + \sum_{l=1}^{M-1} A_l(z)X(zW_M^l) \tag{8}$$

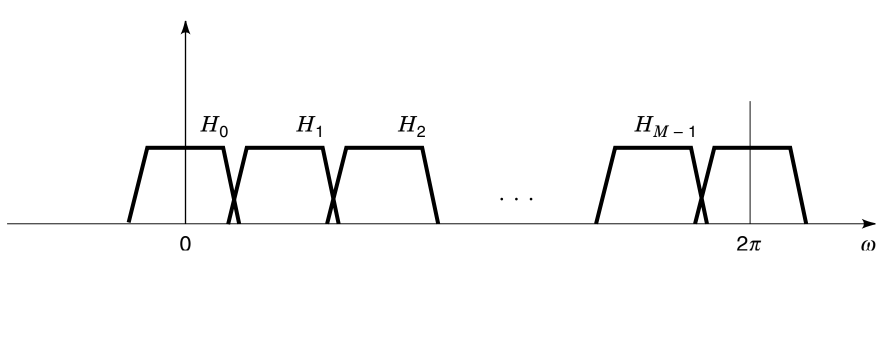


Figure 15. M-channel analysis filter frequency response example.

where

$$A_l(z) = \frac{1}{M}\sum_{k=0}^{M-1} H_k(zW_M^l)F_k(z), \quad 0 \le l \le M-1 \tag{9}$$

Clearly, the second term in the right hand side of Eq. (8) is the aliasing term. For the PR property Eq. (7), we need

$$A_0(z) = cz^{-n_0} \quad \text{and}\, A_l(z) = 0 \quad \text{for}\ 1 \le l \le M-1 \tag{10}$$

We now want to simplify the PR condition Eq. (10). To do so, let

$$\mathbf{t}(z) = [cz^{-n_0}, 0, \ldots, 0]^T$$
$$\mathbf{f}(z) = [F_0(z), F_1(z), \ldots, F_{M-1}(z)]^T$$

and

$$\mathbf{H}(z) = \begin{bmatrix} H_0(z) & \ldots & H_{M-1}(z) \\ \vdots & \vdots & \vdots \\ H_0(zW_M^{M-1}) & \ldots & H_{M-1}(zW_M^{M-1}) \end{bmatrix} \tag{11}$$

Then

$$\mathbf{H}(z)\mathbf{f}(z) = M\mathbf{t}(z) \tag{12}$$

Thus, given an analysis bank, $H_0(z), \ldots, H_{M-1}(z)$, if the matrix polynomial $\mathbf{H}(z)$ has an FIR inverse, then the synthesis bank $\mathbf{f}(z)$ can be solved from the Eq. (12). In other words, the multirate filterbank in Fig. 16 with this synthesis bank is PR. The matrix polynomial $\mathbf{H}(z)$ in Eq. (11) is called the *aliasing component* (AC) matrix. In conclusion, we have the following theorem.

Theorem 1. An FIR multirate filterbank in Fig. 16 is perfect reconstruction if and only if its AC matrix $\mathbf{H}(z)$ has FIR inverse.

One can see that the AC matrix $\mathbf{H}(z)$ is a structured matrix, where its components are not free but related. This limits the study and construction of PR multirate filterbanks. We next want to use the polyphase representations and Noble identities introduced in the second section and convert the AC matrix to the polyphase matrix in which all components are free.

Maximally Decimated Multirate Filterbanks: Perfect Reconstruction and Polyphase Matrix

The analysis in the previous subsection is a direct analysis of the relationship between the input and the output in Fig. 16. We next want to first simplify the block diagram in Fig. 16 by using some properties of building blocks studied in the second section, such as Noble identities and polyphase representations, and then study the PR property for the simplified system. The main idea for the simplification is to switch the orders of decimator/expander and FIR filters, and then convert the multirate filterbank into a multi-input and multi-output (MIMO) system.

For each analysis filter $H_m(z)$ in Fig. 16, let $E_{m,k}(z)$ be its kth Type 1 polyphase component, and for each synthesis filter $F_m(z)$ in Fig. 16, let $R_{l,m}(z)$ be its lth Type 2 polyphase component, for $0 \le m, k, l \le M-1$. Let

$$\mathbf{E}(z) = (E_{m,k}(z))_{0\le m,k\le M-1}, \quad \text{and}\ \mathbf{R}(z) = (R_{l,m}(z))_{0\le l,m\le M-1}$$

which are called the *polyphase matrices* of the analysis bank and the synthesis bank in Fig. 16, respectively. Then, it is not hard to see that

$$\mathbf{h}(z) = \mathbf{E}(z^M)\mathbf{e}(z) \quad \text{and}\ \mathbf{f}(z) = \tilde{\mathbf{e}}(z)\mathbf{R}(z^M), \tag{13}$$

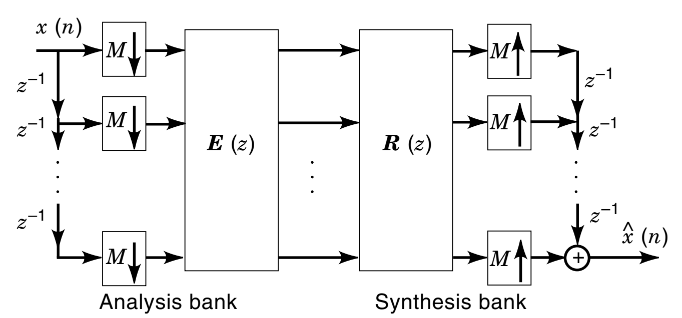
where $\mathbf{h}(z) = (H_0(z), \ldots, H_{M-1}(z))^T$, $\mathbf{e}(z) = (1, z^{-1}, \ldots, z^{-M+1})^T$, $\mathbf{f}(z) = (F_0(z), \ldots, F_{M-1}(z))^T$, and $\tilde{\mathbf{e}}(z)$ is the tilde operation of $\mathbf{e}(z)$. Thus, by using the Noble identities, the multirate filterbank in Fig. 16 is the same as the one shown in Fig. 17(a), which is called the polyphase representation of the multirate filterbank in Fig. 16.

Let $\mathbf{P}(z) = \mathbf{R}(z)\mathbf{E}(z)$, then the multirate filterbank in Fig. 16 is equivalent to the MIMO system in Fig. 17(b) with system transform matrix $\mathbf{P}(z)$. Clearly, the PR property is equivalent to the invertibility of the polyphase matrix $\mathbf{E}(z)$.

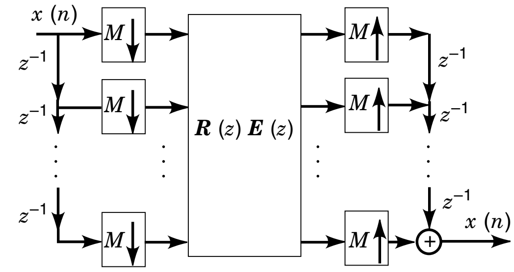
Theorem 2. An FIR multirate filterbank in Fig. 16 is perfect reconstruction if and only if the polyphase matrix $\mathbf{E}(z)$ has an FIR inverse.

Theorem 1 deals with the AC matrix $\mathbf{H}(z)$ while Theorem 2 deals with the polyphase matrix $\mathbf{E}(z)$. From Eq. (13), it is not hard to see the following relationships between these two matrices:

$$\mathbf{H}(z) = \mathbf{W}_M^{\dagger}\mathbf{D}(z)\mathbf{E}^T(z^M) \quad \text{and}\ \mathbf{E}(z^M) = \mathbf{H}^T(z)\mathbf{W}_M\mathbf{D}(z^{-1}) \tag{14}$$



(a)



(b)

Figure 17. Polyphase representation of M-channel maximally decimated multirate filterbank.

where $\mathbf{W}_N$ is the DFT matrix and $\mathbf{D}(z)$ is the diagonal matrix polynomial:

$$\mathbf{D}(z) = \mathrm{diag}(1, z^{-1}, \ldots, z^{-M+1})$$

From Eq. (14), it is clear that the FIR invertibilities of the AC matrix $\mathbf{H}(z)$ and the polyphase matrix $\mathbf{E}(z)$ are equivalent. Unlike matrix $\mathbf{H}(z)$, matrix $\mathbf{E}(z)$ does not have any relationship between its components, which leads to the systematic construction and factorization discussed later.

In some applications, such as the cross-talk cancellation in transmultiplexers in Fig. 5, PR may not be necessary as long as the aliasing (cross-talk) is cancelled in a multirate filterbank, i.e., the second term at the right hand side of Eq. (8) is zero. A necessary and sufficient condition on such filterbanks was obtained by Vaidyanathan and Mitra (30), which is stated as follows. A square matrix polynomial $\mathbf{P}(z)$ is called *pseudo-circulant* if and only if it has the following form:

$$\mathbf{P}(z) = \begin{bmatrix} P_0(z) & P_{M-1}(z) & \cdots & P_1(z) \\ z^{-1}P_1(z) & P_0(z) & \cdots & z^{-1}P_2(z) \\ \vdots & \vdots & \vdots & \vdots \\ z^{-1}P_{M-2}(z) & z^{-1}P_{M-3}(z) & \cdots & P_{M-1}(z) \\ z^{-1}P_{M-1}(z) & z^{-1}P_{M-2}(z) & \cdots & P_0(z) \end{bmatrix}$$

Notice that when no z^{-1} appeared on the lower triangular components in $\mathbf{P}(z)$, it would be circulant.

Theorem 3. An M-channel multirate filterbank in Fig. 16 is aliasing free if and only if the polyphase matrix $\mathbf{P}(z) = \mathbf{R}(z)\mathbf{E}(z)$ is pseudo-circulant. Under this condition, the filterbank output and the input are related by $\hat{X}(z) = A_0(z)X(z)$ as in Eq. (8), where

$$A_0(z) = z^{-M+1}(P_0(z^M) + z^{-1}P_1(z^M) + \cdots + z^{-M+1}P_{M-1}(z^M))$$

PERFECT RECONSTRUCTION FIR MULTIRATE FILTERBANK FACTORIZATION AND CONSTRUCTION

From the studies in the previous section, PR FIR M-channel multirate filterbanks are converted to $M \times M$ matrix polynomials. In this section, we focus on $M \times M$ matrix polynomials $\mathbf{E}(z) = (E_{k,l}(z))_{0 \le k,l \le M-1}$. Without loss of generality, in what follows we only consider FIR and causal matrix polynomials, i.e.,

$$\mathbf{E}(z) = \sum_{n=0}^{L} E_n z^{-n}, L \text{ is a nonnegative integer}$$

As studied in the third section, when the polyphase matrices of analysis banks have FIR inverses, the corresponding synthesis banks can be obtained by using the inverses for the PR multirate filterbanks, i.e., PR FIR multirate filterbanks are constructed. In this section, we first study general $\mathbf{E}(z)$ with FIR inverses and then study paraunitary matrix polynomials $\mathbf{E}(z)$ that are corresponding to paraunitary multirate filterbanks.

Factorization of FIR Polyphase Matrices with FIR Inverses

The goal of this subsection is to characterize all FIR causal $M \times M$ matrix polynomials with FIR inverses. Since the determinant of the FIR inverse of an $M \times M$ matrix polynomial is the inverse of its determinant, we have the following lemma.

Lemma 1. An FIR matrix polynomial has FIR inverse if and only if its determinant is cz^{-n_0} for a nonzero constant c and an integer n_0.

Let $\mathbf{H}(z)$ be an FIR causal matrix polynomial with an FIR inverse. If $\det(\mathbf{H}(z)) = cz^{-\rho}$, then ρ is called its *McMillan degree*, the minimal number of delay elements to implement the MIMO system (9). A matrix polynomial $\mathbf{H}(z)$ is called *unimodular* if and only if its McMillan degree is 0, that is, its determinant is a nonzero constant. To introduce the complete factorization of FIR matrix polynomials with FIR inverses, let us first introduce three types of elementary row (column) operations:

Type 1. Interchange two rows (or columns).

Type 2. Multiply a row (or column) with a nonzero constant c.

Type 3. Add a polynomial multiple of a row (or column) to another row (or column).

The corresponding matrices of these elementary operations are called elementary matrices, which have the following forms.

Let $\mathbf{e}_i$ be the M dimensional vector with its ith entry 1 and other entries 0 for $i = 1, 2, \ldots, M$, i.e.,

$$\mathbf{e}_i = (0 \quad \cdots \quad 0 \quad \underset{i}{1} \quad 0 \quad \cdots \quad 0)^T$$

A Type 1 elementary matrix $\mathbf{A}$ can be written as

$$\mathbf{A} = \mathbf{I} + (\mathbf{e}_i - \mathbf{e}_j)(\mathbf{e}_j - e_i)^{\dagger},$$

for certain $i \ne j$ and $1 \le i, j \le M$.

A Type 2 elementary matrix $\mathbf{A}$ can be written as

$$\mathbf{A} = \mathbf{I} + c\mathbf{e}_i\mathbf{e}_i^{\dagger}$$

for certain $c \ne -1$ and a certain i, $1 \le i \le M$.

A Type 3 elementary matrix $\mathbf{U}(z)$ can be written as

$$\mathbf{U}(z) = \mathbf{I} + \alpha(z)\mathbf{e}_i\mathbf{e}_j^{\dagger}$$

where $\alpha(z)$ is a polynomial of z^{-1} and $i \ne j$ with $1 \le i, j \le M$.

With these three elementary operations/matrices, any $M \times N$ matrix polynomial can be diagonalized and the resulted decomposition is called the *Smith–McMillan decomposition,* which is stated as follows. An $M \times N$ matrix polynomial has the following Smith–McMillan decomposition; *see,* for example, Ref. (9):

$$\mathbf{H}(z) = \mathbf{W}(z)\begin{bmatrix} \gamma_0(z) & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & \gamma_1(z) & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & \gamma_p(z) & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \end{bmatrix}\mathbf{U}(z) \tag{15}$$

where $\mathbf{W}(z)$ and $\mathbf{U}(z)$ are products of some elementary matrix polynomials with sizes $M \times M$ and $N \times N$, respectively, $\gamma_i(z)$ are polynomials of z^{-1}, $\gamma_i(z)$ divides $\gamma_{i+1}(z)$, for $i = 0, 1, \ldots, p - 1$, i.e.,

$$\gamma_i(z) | \gamma_{i+1}(z), i = 0, 1, \ldots, p-1$$

and

$$\gamma_i(z) = \frac{\Delta_{i+1}(z)}{\Delta_i(z)}$$

where $\Delta_0(z) = 1$, $\Delta_i(z)$ for $i > 0$ is the greatest common divisor of all the $i \times i$ minors of $\mathbf{H}(z)$.

When $\mathbf{H}(z)$ is a square causal matrix polynomial with an FIR inverse, the diagonal matrix in the Smith–McMillan decomposition has the form of $\mathrm{diag}(c_1 z^{-n_1}, \ldots, c_M z^{-n_M})$ with $n_m \geq 0$ and $c_m \neq 0$ for $1 \leq m \leq M$, and this is a complete characterization of all square causal matrix polynomials with FIR inverses. The factorization in Eq. (15) is, however, not convenient to be incorporated in the optimal design studied later. We next want to introduce another factorization. For more details, see Refs. 31 and 32.

We define three kinds of basic matrices.

Class I.
$\mathscr{O} \stackrel{\Delta}{=} \{\mathbf{V}(z)\colon \mathbf{V}(z) = \mathbf{I} - \mathbf{v}\mathbf{v}^\dagger + z^{-1}\mathbf{v}\mathbf{v}^\dagger$ where $\mathbf{v}$ is an $M \times 1$ constant vector with unit norm$\}$.
Let $\mathbf{V}(z) = \mathbf{I} - \mathbf{v}\mathbf{v}^\dagger + z^{-1}\mathbf{v}\mathbf{v}^\dagger \in \mathscr{O}$. Then, its inverse $\mathbf{V}^{-1}(z) = \mathbf{I} - \mathbf{v}\mathbf{v}^\dagger + z\mathbf{v}\mathbf{v}^\dagger$.

Class II.
$\mathscr{U} \stackrel{\Delta}{=} \{\mathbf{U}(z)\colon \mathbf{U}(z) = \mathbf{I} + \alpha z^{-m}\mathbf{e}_i\mathbf{e}_j^\dagger$ where α is a constant, m is a nonnegative integer, and $i \neq j$ with $1 \leq i, j \leq M\}$.
Let $\mathbf{U}(z) = \mathbf{I} + \alpha z^{-m}\mathbf{e}_i\mathbf{e}_j^\dagger \in \mathscr{U}$. Then, its inverse $\mathbf{U}^{-1}(z) = \mathbf{I} - \alpha z^{-m}\mathbf{e}_i\mathbf{e}_j^\dagger$.

Class III.
$\mathscr{A} \stackrel{\Delta}{=} \{\mathbf{A}\colon \mathbf{A} = \mathbf{I} + (\mathbf{e}_i - \mathbf{e}_j)(\mathbf{e}_j - \mathbf{e}_i)^\dagger$ for certain $i \neq j$ and $1 \leq i, j \leq M$ or $\mathbf{A} = \mathbf{I} + c\mathbf{e}_i\mathbf{e}_i^\dagger$ for certain $c \neq -1$ and a certain i, $1 \leq i \leq M\}$.
Let $\mathbf{A} = \mathbf{I} + (\mathbf{e}_i - \mathbf{e}_j)(\mathbf{e}_j - \mathbf{e}_i)^\dagger \in \mathscr{A}$. Then, its inverse $\mathbf{A}^{-1} = \mathbf{A}$. Let $\mathbf{A} = \mathbf{I} + c\mathbf{e}_i\mathbf{e}_i^\dagger \in \mathscr{A}$. Then, its inverse $\mathbf{A}^{-1} = \mathbf{I} - c/(c+1)\mathbf{e}_i\mathbf{e}_i^\dagger$.

With these three cases of matrices, we have the following complete factorization.

Theorem 4. A causal FIR $M \times M$ matrix polynomial $\mathbf{H}(z)$ has an FIR inverse if and only if $\mathbf{H}(z)$ has the following form

$$\mathbf{H}(z) = \mathbf{V}_\rho(z)\cdots\mathbf{V}_1(z)\mathbf{A}_\sigma\mathbf{U}_\sigma(z)\cdots\mathbf{A}_1\mathbf{U}_1(z) \tag{16}$$

where ρ is the McMillan degree of $\mathbf{H}(z)$, σ is a certain nonnegative integer, $\mathbf{V}_i(z) \in \mathscr{O}$ for $i = 1, 2, \ldots, \rho$, $\mathbf{A}_i \in \mathscr{A}$ and $\mathbf{U}_i(z) \in \mathscr{U}$ for $i = 1, 2, \ldots, \sigma$.

Factorization of Paraunitary FIR Matrix Polynomials

In this subsection, we introduce paraunitary matrix polynomials and corresponding multirate filterbanks, which are special FIR multirate filterbanks with FIR inverses.

An $M \times N$ matrix polynomial $\mathbf{H}(z)$ is called *paraunitary* if and only if

$$\tilde{\mathbf{H}}(z)\mathbf{H}(z) = d\mathbf{I}_N, \quad \text{for all complex values } z$$

where d is a positive constant and $\mathbf{I}_N$ is the $N \times N$ identity matrix. When we restrict the complex value z on the unit circle, i.e., in the Fourier transform domain, it becomes the concept of lossless matrices. An $M \times N$ matrix polynomial $\mathbf{H}(z)$ is called *lossless* if and only if

$$\mathbf{H}^\dagger(e^{j\omega})\mathbf{H}(e^{j\omega}) = d\mathbf{I}_N, \quad \text{for all real values } \omega$$

where d is a positive constant. When $\mathbf{H}(z)$ is FIR, lossless is equivalent to paraunitary. When $\mathbf{H}(z)$ exists for all $z = e^{j\omega}$ but not all complex values z, lossless is not equivalent to paraunitary, while paraunitary always implies lossless. Since we are interested in FIR $\mathbf{H}(z)$, we only consider paraunitary matrix polynomials in this subsection. An example of paraunitary matrix polynomial is:

$$\mathbf{H}(z) = \begin{bmatrix} z^{-1}+1 & z^{-1}-1 \\ z^{-1}-1 & z^{-1}+1 \end{bmatrix}$$

In this case

$$\tilde{\mathbf{H}}(z) = \begin{bmatrix} z+1 & z-1 \\ z-1 & z+1 \end{bmatrix}$$

and $\tilde{\mathbf{H}}(z)\mathbf{H}(z) = 4\mathbf{I}_2$. A paraunitary multirate filterbank is shown in Fig. 18.

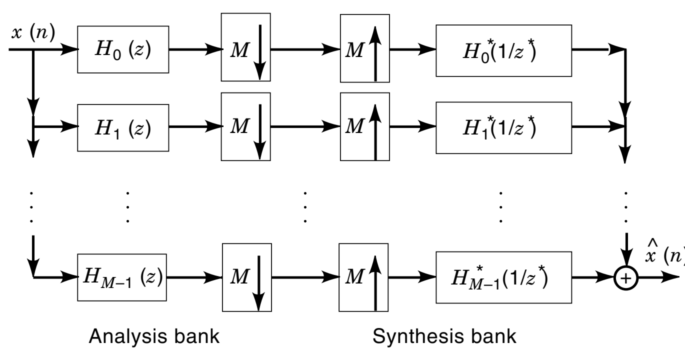


Figure 18. Paraunitary M-channel maximally decimated multirate filterbank.

Similar to orthogonal transformations (matrices), the advantages of paraunitary multirate filterbanks include that they preserve signal energies in the decompositions (or transformations) and the synthesis banks (or inverse transformations) are simply the tilde operations of the analysis banks. In this sense, paraunitary multirate filterbanks are generalizations of orthogonal transformations, such as DFT, by adding delay variables (or memory) into the transformations. In the following, we want to present a complete characterization of all paraunitary matrix polynomials obtained by Vaidyanathan (9,33–35).

Theorem 5. An $M \times M$ causal FIR matrix polynomial $\mathbf{H}(z)$ is paraunitary if and only if it can be factorized as

$$\mathbf{H}(z) = d\mathbf{V}_\rho(z)\cdots\mathbf{V}_1(z)\mathbf{H}_0 \tag{17}$$

where d is a positive constant, ρ is the McMillan degree of $\mathbf{H}(z)$, $\mathbf{H}_0$ is an $M \times M$ unitary constant matrix, $\mathbf{V}_i(z) \in \mathscr{O}$ for $i = 1, 2, \ldots, \rho$, and if $\rho = 0$ then $\mathbf{H}(z) = d\mathbf{H}_0$.

For a nonsquare paraunitary matrix polynomial, the following similar factorization holds (9,36–38).

Theorem 6. An $M \times N$ causal FIR matrix polynomial $\mathbf{H}(z)$ is paraunitary if and only if it can be factorized as

$$\mathbf{H}(z) = d\mathbf{V}_\rho(z)\cdots\mathbf{V}_1(z)\mathbf{H}_0$$

where d is a positive constant, $\mathbf{H}_0$ is an $M \times N$ unitary constant matrix, $\mathbf{V}_i(z) \in \mathscr{O}$ for $i = 1, 2, \ldots, \rho$, and if $\rho = 0$ then $\mathbf{H}(z) = d\mathbf{H}_0$.

For 2-channel paraunitary matrix polynomials, the above factorization is simplified as the following lattice representation (9).

Corollary 1. A 2×2 causal FIR matrix polynomial $\mathbf{H}(z)$ is paraunitary if and only if it can be factorized as

$$\mathbf{H}(z) = d\mathbf{R}_\rho\Lambda(z)\cdots\mathbf{R}_1\Lambda(z)\mathbf{R}_0\begin{bmatrix}1 & 0\\ 0 & \pm 1\end{bmatrix}$$

where d is a positive constant, ρ is the McMillan degree of $\mathbf{H}(z)$, and

$$\mathbf{R}_i = \begin{bmatrix}\cos\theta_i & \sin\theta_i\\ -\sin\theta_i & \cos\theta_i\end{bmatrix}, \quad \text{and } \Lambda(z) = \begin{bmatrix}1 & 0\\ 0 & z^{-1}\end{bmatrix}$$

and θ_i is an angle for $i = 0, 1, 2, \ldots, \rho$.

A lattice realization of a 2-channel paraunitary analysis bank is shown in Fig. 19, where $\alpha = \sqrt{d}$, $c_i = \cos\theta_i$ and $s_i = \sin\theta_i$ for $i = 0, 1, \ldots, \rho$, where ρ delays are needed.

Table 1. Optimized Analysis Filter Impulse Responses of 3-Channel Paraunitary Filterbank

n	$h_0(n)$	$h_1(n)$	$h_2(n)$
0	−0.0429753	−0.0927704	0.0429888
1	0.0000139	0.0000008	−0.0000139
2	0.1489104	0.0087654	−0.1489217
3	0.2971954	0.0000226	0.2972354
4	0.3537539	0.1864025	−0.3537496
5	0.2672266	−0.0000020	0.2672007
6	0.0870758	−0.3543303	−0.0870508
7	−0.0521155	−0.0000363	−0.0520909
8	−0.0875973	0.3564594	0.0875756
9	−0.0427096	−0.0000049	−0.0427067
10	0.0474530	−0.1931082	−0.0474452
11	0.0429618	0.0000230	0.0429677
12	0.0	0.0	0.0
13	−0.0232765	−0.0000026	−0.0232749
14	0.0000022	0.0	0.0000022

Perfect Reconstruction Multirate Filterbank Design

After the complete characterizations of PR multirate filterbanks, the next important issue is the design of a desired PR multirate filterbank. In this subsection, we want to briefly describe a method for the design. The goal here is to design an M-channel PR multirate filterbank such that all analysis filters have good filter properties, i.e., with good passband and stopband attenuation properties. What one can do is to use the factorizations Eqs. (16) and (17) to parameterize these analysis filters and then formulate the minimization problem for the parameters:

$$\min \sum_{m=0}^{M-1} \int_{m\text{th stopband}} |H_m(e^{j\omega})|^2 d\omega \tag{18}$$

The following is a design example obtained from Ref. 9. Consider a 3-channel causal FIR paraunitary filterbank, that is, $M = 3$ in Fig. 17 and Eq. (17). The maximal length of all analysis filters $H_m(z)$ equals 15. By implementing this minimization, the optimized 3 analysis filter impulse responses are listed in Table 1 and their frequency responses are shown in Fig. 20, where the coefficients in Table 1 are from Ref. 9. Since all filter coefficients are real, their Fourier spectra are symmetric about the origin. For more details on the design issues, see Refs. 9, 12, and 12a.

DFT AND COSINE MODULATED FILTERBANKS

In the previous section, we have studied general M-channel maximally decimated multirate filterbanks, where the Fourier spectra of M analysis filters may not necessarily have the same shape. In many applications, such as FDMA communi-

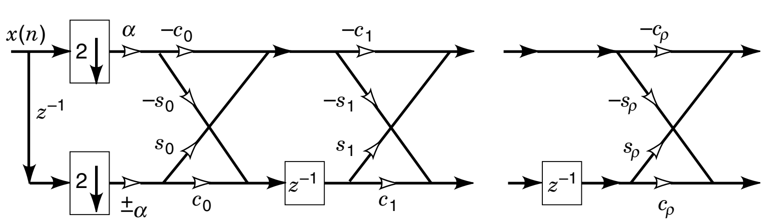


Figure 19. Lattice realization of 2-channel paraunitary analysis bank.

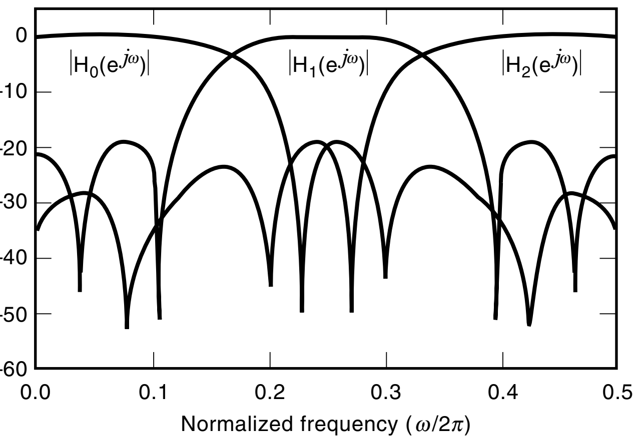


Figure 20. Frequency responses of 3 analysis filters. (Illustration courtesy of Prentice-Hall, Englewood Cliffs, NJ © 1993 by Prentice-Hall.)

cation systems, it is however quite often that all M analysis filters are derived from a single prototype filter and therefore have the same shape of their Fourier spectra. The advantage of such systems is the implementation simplicity. In this section, we introduce two kinds of such filterbanks. One is the discrete Fourier transform (DFT) filterbank, where analysis filters are single-sided shifts of a prototype filter in the frequency domain (or exponential modulation). The other is the cosine modulated filterbank, where analysis filters are double-sided shifts of a prototype filter.

DFT Filterbanks

DFT filterbanks form a class of the simplest multirate filterbanks, where all analysis filters are shifted from a single prototype filter in the frequency domain. The question then becomes when DFT filterbanks are PR. To study this question, let us formulate the analysis filters. Let

$$P(z) = \sum_{n=0}^{L-1} p(n) z^{-n}$$

be an FIR filter with length L, which is usually a good lowpass filter. The M analysis filters are

$$H_m(z) = P(zW_M^m) = \sum_{n=0}^{L-1} p(n) W_M^{-mn} z^{-n}, \quad 0 \le m \le M-1 \quad (19)$$

where their Fourier spectra are illustrated in Fig. 15. Notice that M analysis filter coefficients $h_m(n) = W_M^{-mn}p(n)$ are no longer real even when the prototype filter coefficients $p(n)$ are real.

Let $P_l(z)$, $0 \le l \le M-1$, be the Type 1 polyphase components of the prototype filter $P(z)$ with total M components. Then it is not hard to see that the polyphase matrix of the analysis bank $H_m(z)$, $0 \le m \le M-1$, in Eq. (19) is

$$\mathbf{E}(z) = \mathbf{W}_M^{\dagger} \operatorname{diag}(P_0(z), P_1(z), \ldots, P_{M-1}(z))$$

which is shown in Fig. 21 with $\mathbf{P}(z) = \operatorname{diag}(P_0(z), P_1(z), \ldots, P_{M-1}(z))$.

By the study in the third section, it is clear that the DFT filterbanks are PR if and only if

$$P_m(z) = c_m z^{-n_m}, \quad c_m \ne 0, 0 \le m \le M-1, \\ \text{for some integer } n_m$$

or

$$P(z) = \sum_{m=0}^{M-1} c_m z^{-Mn_m - l}, \quad c_m \ne 0 \quad (20)$$

The paraunitariness of this DFT filterbank forces $|c_m| = c \ne 0$ for all $0 \le m \le M-1$, which is basically equivalent to the DFT. When $P_m(z) = 1/\sqrt{M}$ for $0 \le m \le M-1$, the DFT filterbank is precisely reduced to the DFT as shown in Fig. 22, which is the reason for the name of the DFT filterbanks.

One can see that the condition on the prototype filter $P(z)$ in Eq. (20) for the PR property is very restrictive and usually limits their applications. There are three ways to get around this condition. The first one, which is also the most intuitive one, is to design $P(z)$ with excellent lowpass property. Then, the DFT filterbank is almost PR because the whole frequency band is almost divided with a wall-cut manner by M analysis filters. The second way is to use nonmaximally decimated DFT filterbanks, that is, the decimation factor is less than the number of channels (or users), which corresponds to oversampled short-time Fourier transforms or discrete Gabor transforms. For more details, *see,* for example, Refs. 1, 39–42, 42a. The third way is to use double-sided shifts instead of the single-sided shifts as in Eq. (20), which leads to cosine modulated filterbanks as we shall see in the next subsection.

Cosine Modulated Filterbanks

The DFT filterbanks in the previous subsection 5.1 have two disadvantages. One is that analysis filter coefficients are complex-valued and the other is that the PR condition is too restrictive. We now want to use double-sided shifts or cosine modulations to construct M analysis filters with real coefficients and better filter properties in PR multirate banks.

Let $P(z)$ be a prototype filter with length L as before. Let

$$U_m(z) = P(zW_{2M}^{m+0.5}) \quad \text{and} \quad V_m(z) = P(zW_{2M}^{-(m+0.5)}), \\ 0 \le m \le M-1$$

and for $0 \le m \le M-1$,

$$\begin{aligned} H_m(z) &= a_m U_m(z) + a_m^* V_m(z) \\ &= \sum_{n=0}^{L-1} 2\,\mathrm{real}(a_m W_{2M}^{-(m+0.5)n}) p(n) z^{-n} \end{aligned}$$

This tells us that the analysis filter coefficients are all real. Furthermore, let

$$a_m = W_{2M}^{(m+0.5)(L-1)/2 + (-1)^m \pi/4}$$

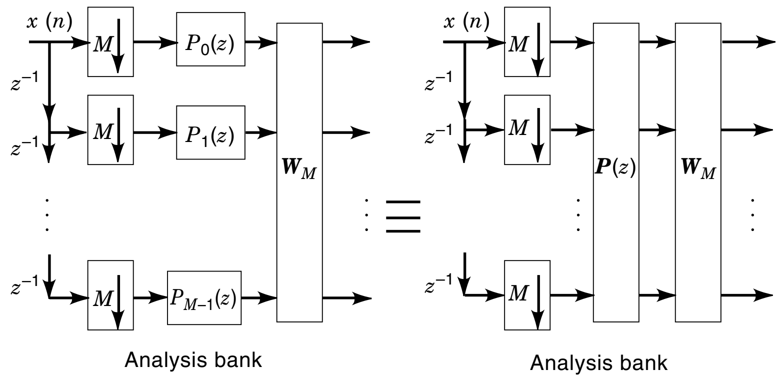


Figure 21. DFT analysis bank, a special multirate filterbank.

Then, the analysis filter coefficients are

$$h_m(n) = 2\cos\left[\frac{\pi}{M}(m+0.5)\left(n - \frac{L-1}{2}\right) + (-1)^l\frac{\pi}{4}\right]p(n) \tag{21}$$

The corresponding synthesis filter coefficients are

$$f_m(n) = 2\cos\left[\frac{\pi}{M}(m+0.5)\left(n - \frac{L-1}{2}\right) - (-1)^l\frac{\pi}{4}\right]p(n) \tag{22}$$

From these filters, one can see why they are called cosine modulated filterbanks. We now present a necessary and sufficient condition for the PR property (9,10,12,43).

Theorem 7. Let $P_l(z)$, $0 < l \leq 2M - 1$, be the Type 1 polyphase components of a prototype filter $P(z)$ of length L with $2M$ total components. If $L = 2KM + 1$ for some positive integer K and M analysis filters and synthesis filters are defined in Eqs. (21) and (22), respectively, then the cosine modulated filterbank is paraunitary if and only if

$$\tilde{P}_m(z)P_m(z) + \tilde{P}_{m+M}(z)P_{m+M}(z) = \alpha, \quad 0 \leq m \leq M-1 \tag{23}$$

for a constant $\alpha > 0$.

This theorem suggests the following method to construct paraunitary cosine modulated filterbanks. Define

$$\mathbf{Q}_m(z) = \begin{bmatrix} P_m(z) \\ P_{m+M}(z) \end{bmatrix}, \quad m = 0, 1, 2, \ldots, M-1$$

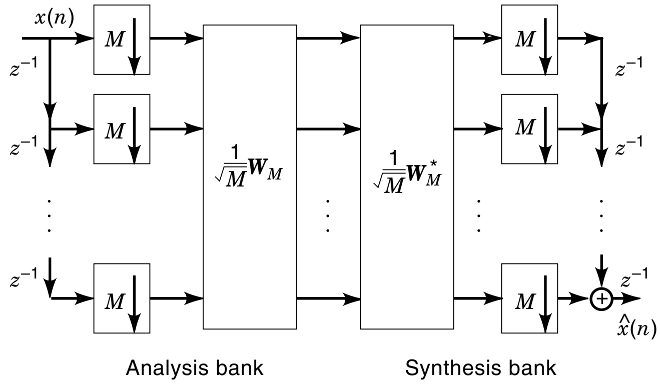


Figure 22. DFT (analysis bank) and IDFT (synthesis bank).

then condition Eq. (23) is equivalent to

$$\tilde{\mathbf{Q}}_m(z)\mathbf{Q}_m(z) = \alpha, m = 0, 1, 2, \ldots, M-1$$

i.e., all $\mathbf{Q}_m(z)$ are 2×1 paraunitary matrix polynomials. They have been completely characterized in the previous section and the factorization in Theorem 6 can be used to construct optimal cosine modulated filterbanks similar to those studied in the same section. The following example is from Ref. 12. Figure 23(a) shows the frequency response of an optimized prototype filter of length 129, and Figure 23(b) shows 8 frequency responses for the 8-channel PR cosine modulated filterbank with the prototype filter in Fig. 23(a). One can see that their stopband attenuations are about 80dB.

Several updated research results on cosine modulated filterbanks are:

1. Notice that the previous paraunitary cosine modulated filterbanks require the prototype filters of length $2KM$. This requirement is relaxed in Ref. 44.
2. Notice that in the cosine modulated filterbanks, the analysis filters may not be linear phase although the prototype filter is. All linear phase analysis and synthesis filters in cosine modulated filterbanks are obtained in Ref. 45 by using two prototype filters simultaneously.
3. Two dimensional cosine modulated filterbanks are studied and constructed in Refs. 46–48.

SOME ADDITIONAL AND RECENT RESEARCH TOPICS

In this section, we want to briefly mention several other topics on multirate filterbank theory and applications.

Linear-Phase Perfect Reconstruction Filterbanks

In some applications, such as image/video processing, it is desired that analysis and synthesis filters have no phase distortion. This has motivated the study of linear-phase FIR PR multirate filterbank factorization and construction. Linear-phase multirate filterbanks were first studied independently by Nguyen and Vaidyanathan (49,50), and Vetterli and Le Gall (51). For more details, see, for example Refs. 7, 9, 11, 12, 45, 49–54.

Lapped Orthogonal Transforms

Blocked discrete cosine transform (DCT) plays an important role in the current image/video coding standards, such as

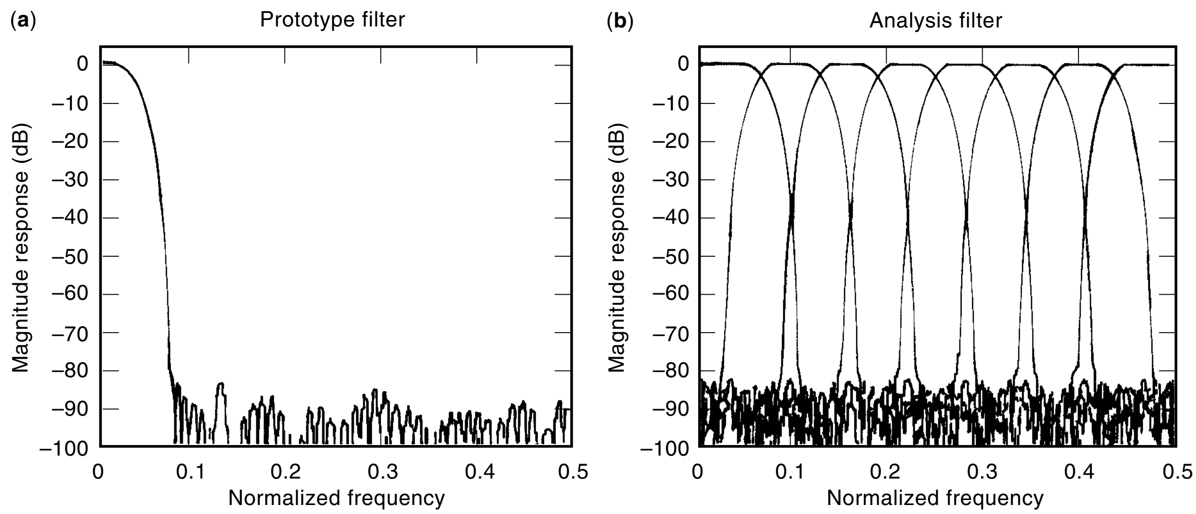


Figure 23. Cosine modulated filterbank prototype and analysis filters (12).

JPEG, MPEG1 and MPEG2, H.261 and H.263 in blocked DCT transform coding, one first decomposes an image on a rectangular region into 8 × 8 blocks, and then implement 8 × 8 DCT on each block. Then, one quantizes the blocked DCT coefficients to get compression gain. Due to the truncation of an image in the block decomposition, the blocking effects degrade the compression performance at low bit rates. The blocking effects basically come from a hard truncation between blocks, which causes discontinuities of sinusoidal bases at the boundaries of these blocks. Based on this observation, overlaps between adjacent blocks are used, such that a smooth truncation between blocks is achieved while no more coefficients in the transform domain are used and moreover the orthogonality is still maintained. These overlapped blocked DCT are called lapped orthogonal transforms (LOT) (55–58). LOT can be thought of as special cases of cosine modulated filterbanks, (9,12,54), and order 1 PR multirate filterbanks, (9,31,32). Time-varying filterbanks and LOT are studied in Refs. 59 and 60. LOT over finite fields are studied in Ref. 61, where a complete factorization is given. In the continuous-time domain, LOT is called Malvar wavelets or local sinusoidal bases. For more details, see Refs. 62 and 63. For multidimensional Malvar wavelets, see Refs. 64 and 65.

Multidimensional Multirate Filterbanks

Multirate filterbanks have been used in multidimensional signal processing, such as image/video processing. One way to use multirate filterbanks is to use one-dimensional filterbanks at each dimension of signals. Another way to use multirate filterbanks is to use multidimensional filterbanks directly to multidimensional signals, which has the potential to take advantage of multidimensional signals themselves, such as nonhorizontal or nonvertical lines in image analysis. This motivates the study of nonseparable multidimensional multirate filterbanks. It is known that all one-dimensional paraunitary filterbanks can be decomposed into degree-one building blocks, which are very useful in the construction of optimal paraunitary filterbanks and also their implementations. This property, however, does not hold for multidimensional paraunitary filterbanks. For more details, *see* Refs. 66 and 80. For more properties on multidimensional basic building blocks, such as decimator and expander, delay chain systems, and constructions, *see* for example, Refs. 67–85, 85a. For multidimensional cosine modulated filterbanks, *see* Refs. 46–48. For multidimensional LOT, *see* Refs. 64, 65, 86, 86a.

Optimal Multirate Filterbanks in Quantization

One of the most important applications of multirate filter banks is data compression, such as speech, image, and video compression. In this application, an analysis filter bank is used to decompose a signal and then a quantization scheme is used to compress the signal. In the reconstruction, a dequantization scheme and a corresponding synthesis filter bank are used. This implies that the decomposition and reconstruction of a filter bank and the quantization should be considered together to achieve the maximal compression gain. Along this direction, several research works can be found (87–93).

Multirate Filterbanks over Finite Fields

What we have studied on multirate filterbanks are all over the complex field, i.e., all signals and filter coefficients are complex values. These values can certainly be in any finite field. Due to the incompleteness of a finite field, it is impossible to factorize all FIR PR filterbanks over finite fields. A counter example was given in Ref. 94. Although this is the case, a completely study on LOT (order one multirate filterbanks) over finite fields was obtained in Ref. 61. It is found that nonmaximally decimated multirate filterbanks over finite fields are the same as convolutional codes for the error control purpose (31,32,61,95). For filterbanks over finite fields, see also Refs. 96, 97.

Nonuniformally Decimated Multirate Filterbanks

In the previous sections, we have studied uniformally decimated multirate filterbanks. There have been also some research works on nonuniformally decimated multirate filterbanks (22–27). In particular, tree-structured multirate filterbanks that may lead to wavelets are nonuniformally decimated multirate filterbanks (6–13).

Nonmaximally Decimated Multirate Filterbanks

Nonmaximally decimated multirate filterbanks have been recently applied as precoders in communication systems, such as intersymbol interference (ISI) cancellation (16–21,98,99) and image quantization and coding (100,101). They are applied in the prefilter design for discrete multiwavelet transforms (98,99).

Time-Varying, Vector Filterbanks and Multirate Filterbanks with Block Sampling

Because of the nonstationarity of signals, such as speech, audio, and images, time-varying filter banks have been studied to exploit the nonstationarity, where filter banks change dynamically with the nature of a signal. An important problem in time-varying filter bank theory is how to construct time-varying filter banks with PR property from local PR filter banks. For this problem, there have been several discussions (60,102–114).

Nonlinear Multirate Filterbanks

Conventional filter banks are all linear transformations. It is well-known that all linear transformations may smear edges that are important in image/video processing. Since nonlinear transformations, such as median filtering, may preserve edges better than linear transformations do, nonlinear filter banks have been recently studied (115–118).

SUMMARY

In this article, we introduced some fundamentals on multirate filterbank theory and applications. Beginning with some application backgrounds, such as transmultiplexer including TDMA, FDMA, and CDMA communications, and multiresolution image/video analysis and coding, we briefly mentioned the importance of the construction of PR multirate filterbanks. For systematic construction of PR multirate filterbanks, in particular paraunitary multirate filterbanks, some basic building blocks, such as decimator and expander, the Noble identities, and polyphase representations, and their properties were briefly described. These properties were used to convert a maximally decimated multirate filterbank into an MIMO system that has a system transfer matrix polynomial with free variables (in mathematics, a free-module). This conversion makes the systematic study easier and the complete lattice factorization possible. After the study of general maximally decimated multirate filterbanks, we then briefly introduced the DFT and cosine modulated filterbanks, which are often used in practical applications. Finally, we briefly mentioned some additional and recent research topics in multirate filterbank theory and applications. As we mentioned in the introduction, this article covers only some basics of multirate filterbanks.

BIBLIOGRAPHY

1. R. E. Crochiere and L. R. Rabiner, *Multirate Digital Signal Processing,* Englewood Cliffs, NJ: Prentice-Hall, 1983.
2. M. Vetterli, Perfect transmultiplexers, *Proc. IEEE Int. Conf. Acoust. Speech Signal Process.,* pp. 2567–2570, Tokyo, Japan, 1986.
3. R. D. Koilpillai, T. Q. Nguyen, and P. P. Vaidyanathan, Some results in the theory of cross-talk free transmultiplexers, *IEEE Trans. Signal Process.,* **SP-39**: 2174–2183, 1991.
4. J. M. Shapiro, Embedded image coding using zerotree of wavelet coefficients, *IEEE Trans. Signal Process.,* **SP-41**: 3445–3462, 1993.
5. A. Said and W. A. Pearlman, A new and efficient image codec based on set partitioning in hierarchical trees, *IEEE Trans. Circuits Syst. Video Tech.,* June 1996.
6. O. Rioul and M. Vetterli, Wavelets and signal processing, *IEEE Signal Process. Magazine,* **8**: 14–38, 1991.
7. M. Vetterli and C. Herley, Wavelets and filter banks: Theory and design, *IEEE Trans. Signal Process.,* **SP-40**: 2207–2232, 1992.
8. A. Akansu and R. Haddad, *Multiresolution Signal Decomposition: Transforms, Subbands, and Wavelets,* Boston: Academic Press, 1992.
9. P. P. Vaidyanathan, *Multirate Systems and Filter Banks,* Englewood Cliffs, NJ: Prentice Hall, 1993.
10. N. J. Fliege, *Multirate Digital Signal Processing,* New York: Wiley, 1994.
11. M. Vetterli and J. Kovačević, *Wavelets and Subband Coding,* Englewood Cliffs, NJ: Prentice-Hall, 1995.
12. G. Strang and T. Q. Nguyen, *Wavelets and Filter Banks,* Wellesley-Cambridge Press, 1996.

12a. P. P. Vaidyanathan et al., Improved technique for design of perfect reconstruction FIR QMF banks with lossless polyphase matrices, *IEEE Trans. Acoust. Speech Signal Process.,* **ASSP-37**: 1042–1056, 1989.

13. A. N. Akansu and M. J. T. Smith, *Subband and Wavelet Transforms: Design and Applications,* Boston: Kluwer, 1996.
14. M. Bellanger, G. Bonnerot, and M. Coudreuse, Digital filtering by polyphase network: Application to sample rate alteration and filter banks, *IEEE Trans. Acoust. Speech Signal Process.,* **ASSP-24**: 109–114, 1976.
15. P. Vary, On the design of digital filter banks based on a modified principle of polyphase, *AEU,* **33**: 293–300, 1979.
16. X.-G. Xia, New precoding for intersymbol interference cancellation using nonmaximally decimated multirate filterbanks with ideal FIR equalizers, preprint, 1995, *IEEE Trans. Signal Process.,* **45**, 2431–2441, 1997.
17. H. Liu and X.-G. Xia, Precoding techniques for undersampled multi-receiver communication systems, Tech. Report #97-3-1, Dept. Elect. Eng., Univ. of Delaware, 1997.
18. X.-G. Xia and H. Liu, Polynomial ambiguity resistant precoders: Theory and applications in ISI/multipath cancelation, Tech. Report #97-5-1, Dept. Elect. Eng., Univ. of Delaware, 1997.
19. X.-G. Xia and G. Zhou, On optimal ambiguity resistant precoders in ISI/multipath cancellation, Tech. Report #97-5-2, Dept. Elect. Eng., Univ. of Delaware, 1997.
20. X.-G. Xia, Modulated coding and least square decoding via coded modulation and Viterbi decoding, Tech. Report #97-6-2, Dept. Elect. Eng., Univ. of Delaware, 1997.
21. G. B. Giannakis, Filterbanks for blind channel identification and equalization, *IEEE Signal Process. Lett.,* **4**: 184–187, 1997.

22. P.-Q. Hoang and P. P. Vaidyanathan, Nonuniform multirate filter banks: Theory and design, *Proc. IEEE Int. Symp. Circuits Syst.,* pp. 371–374, Portland, Oregon, 1989.
23. K. Nayebi, T. P. Barnwell III, and M. J. T. Smith, Nonuniform filter banks: A reconstruction and design theory, *IEEE Trans. Signal Process.,* **SP-41**, 1993.
24. J. Li, T. Q. Nguyen, and S. Tantaratana, A simple design method for nonuniform multirate filter banks, *Proc. Asilomar Conf. Signals Systems Computers,* pp. 1015, Pacific Grove, CA, 1994.
25. J. Princen, The design of nonuniform modulated filterbanks, *IEEE Trans. Signal Process.,* **SP-43**: 2550, 1995.
26. J.-J. Lee and B. G. Lee, A design of nonuniform cosine modulated filter banks, *IEEE Trans. Circuits Syst. II Analog Digit. Signal Process.,* **42**: 732, 1995.
27. T. Chen, Nonuniform multirate filter banks: analysis and design with an H(∞) performance measure, *IEEE Trans. Signal Process.,* **SP-45**: 572, 1997.
28. M. J. T. Smith and T. P. Barnwell III, Exact reconstruction techniques for tree-structured subband coders, *IEEE Trans. Acoust. Speech Signal Process.,* **ASSP-34**: 434–441, 1986.
28a. M. J. T. Smith and T. P. Barnwell III, A new filter-bank theory for time-frequency representation, *IEEE Trans. Acoust. Speech Signal Process.,* **ASSP-35**: 314–327, 1987.
29. F. Mintzer, Filters for distortion-free two-band multirate filter banks, *IEEE Trans. Acoust. Speech Signal Process.,* **ASSP-33**: 626–630, 1985.
29a. H. J. Nussbaumer, Pseudo QMF filter bank, *IBM Tech. Disclosure Bulletin,* **24**: 3081–3087, 1981.
29b. P. L. Chu, Quadrature mirror filter design for an arbitrary number of equal bandwidth channels, *IEEE Trans. Acoust. Speech Signal Process.,* **ASSP-33**: 203–218, 1985.
29c. R. V. Cox, The design of uniformly and nonuniformly spaced pseudo QMF, *IEEE Trans. Acoust. Speech Signal Process.,* **ASSP-34**: 1090–1096, 1986.
30. P. P. Vaidyanathan and S. K. Mitra, Polyphase networks, block digital filtering, LPTV systems, and alias-free QMF banks: A unified approach based on pseudocirculants, *IEEE Trans. Acoust. Speech Signal Process.,* **ASSP-36**: 381–391, 1988.
31. P. P. Vaidyanathan and T. Chen, Role of anticausal inverses in multirate filter-banks—Part I: System-theoretic fundamentals, *IEEE Trans. Signal Process.,* **SP-43**: 1090–1102, 1995.
32. P. P. Vaidyanathan and T. Chen, Role of anticausal inverses in multirate filter-banks—Part II: The FIR case, factorizations, and biorthogonal lapped transforms, *IEEE Trans. Signal Process.,* **SP-43**: 1103–1115, 1995.
33. P. P. Vaidyanathan, Theory and design of M-channel maximally decimated quadrature mirror filters with arbitrary M, having perfect reconstruction property, *IEEE Trans. Acoust. Speech Signal Process.,* **ASSP: 35**: 476–492, 1987.
34. P. P. Vaidyanathan, Quadrature mirror filter banks, M-band extensions and perfect-reconstruction techniques, *IEEE ASSP Magazine,* **4**: 4–20, 1987.
35. P. P. Vaidyanathan, Multirate digital filters, filter banks, polyphase networks, and applications: A tutorial, *Proc. IEEE,* **78**: 56–93, 1990.
36. Z. Doğanata, P. P. Vaidyanathan, and T. Q. Nguyen, General synthesis procedures for FIR lossless transfer matrices, for perfect-reconstruction multirate filter bank applications, *IEEE Trans. Acoust. Speech Signal Process.,* **ASSP-36**: 1561–1574, 1988.
37. X.-G. Xia and B. W. Suter, FIR paraunitary filter banks given several analysis filters: Factorization and construction, *IEEE Trans. Signal Process.,* **SP-44**: 720–723, 1996.
38. H.-J. Park, *A Computational Theory of Laurent Polynomial Rings and Multidimensional FIR Systems,* Ph.D. dissertation, Dept. of Mathematics, Univ. of Berkeley, 1995.
39. M. R. Portnoff, Time-frequency representation of digital signals and systems based on short-time Fourier analysis, *IEEE Trans. Acoust. Speech Signal Process.,* **ASSP-28**: 55–69, 1980.
40. M. J. Bastiaans, Gabor's expansion of a signal into Gaussian elementary signals, *Proc. IEEE,* **68**: 594–598, 1980.
41. J. Wexler and S. Raz, Discrete Gabor expansions, *Signal Process.,* **21**: 207–220, 1990.
42. S. Qian and D. Chen, Discrete Gabor transform, *IEEE Trans. Signal Process.,* **SP-41**: 2429–2438, 1993.
42a. S. Qian and D. Chen, *Joint Time-Frequency Analysis,* Prentice-Hall, New Jersey, 1996.
43. R. D. Koilpillai and P. P. Vaidyanathan, Cosine-modulated FIR filter banks satisfying perfect reconstruction, *IEEE Trans. Signal Process.,* **SP-40**: 770–783, 1992.
44. T. Q. Nguyen and R. D. Koilpillai, The theory and design of arbitrary-length cosine-modulated filter banks and wavelets, satisfying perfect reconstruction, *IEEE Trans. Signal Process.,* **SP-44**: 473–483, 1996.
45. Y.-P. Lin and P. P. Vaidyanathan, Linear phase cosine modulated maximally decimated filter banks with perfect reconstruction, *IEEE Trans. Signal Process.,* **SP-43**: 2525, 1995.
46. Y.-P. Lin and P. P. Vaidyanathan, A review of two-dimensional nonseparable filter bank design techniques, *Multidim. Syst. Signal Process.,* Special Issue on Multidimensional Filter Banks, 1995.
47. Y.-P. Lin and P. P. Vaidyanathan, On the study of four-parallelogram filter banks, *IEEE Trans. Signal Process.,* **SP-44**: 2707, 1996.
48. Y.-P. Lin and P. P. Vaidyanathan, Theory and design of two-parallelogram filter banks, *IEEE Trans. Signal Process.,* **SP-44**: 2688, 1996.
49. T. Q. Nguyen and P. P. Vaidyanathan, Two-channel perfect reconstruction FIR QMF structures which yield linear phase FIR analysis and synthesis filters, *IEEE Trans. Acoust. Speech Signal Process.,* **ASSP-37**: 676–690, 1989.
50. T. Q. Nguyen and P. P. Vaidyanathan, Structures for M-channel perfect reconstruction FIR QMF banks which yield linear phase analysis filters, *IEEE Trans. Acoust. Speech Signal Process.,* **ASSP-38**: 433–446, 1990.
51. M. Vetterli and D. Le Gall, Perfect reconstruction FIR filter banks: Some properties and factorizations, *IEEE Trans. Acoust. Speech Signal Process.,* **ASSP-37**: 1057–1071, 1989.
52. S.-M. Phoong, C. W. Kim, P. P. Vaidyanathan, and R. Ansari, A new class of two-channel biorthogonal filter banks and wavelet bases, *IEEE Trans. Signal Process.,* **SP-43**: 649, 1995.
53. L. Chen, T. Q. Nguyen, and K.-P. Chan, Symmetric extension methods for M-channel linear-phase perfect-reconstruction filter banks, *IEEE Trans. Signal Process.,* **SP-43**: 2505, 1995.
54. R. L. de Queiroz, T. Q. Nguyen, and K. R. Rao, The GenLot: Generalized linear-phase lapped orthogonal transform, *IEEE Trans. Signal Process.,* **SP-44**: 497–507, 1996.
55. H. S. Malvar and D. H. Staelin, The LOT: Transform coding without blocking effects, *IEEE Trans. Acoust. Speech Signal Process.,* **ASSP-37**: 553–559, 1989.
56. H. S. Malvar, Lapped transforms for efficient transform/subband coding, *IEEE Trans. Acoust. Speech Signal Process.,* **ASSP-38**: 969–978, 1990.
57. H. S. Malvar, *Signal Processing with Lapped Transforms,* Boston: Artech House, 1992.
58. A. N. Akansu and F. E. Wadas, On lapped orthogonal transforms, *IEEE Trans. Signal Process.,* **40**: 439–443, 1992.

59. R. L. de Queiroz and K. R. Rao, Time-varying lapped transforms and wavelet packets, *IEEE Trans. Signal Process.,* **SP-41**: 3293–3305, 1993.

60. X.-G. Xia and B. W. Suter, Construction of perfect reconstruction time-varying FIR multirate filter banks with overlaps, *Proc. IEEE Int. Symp. Time-Freq. Time-Scale Anal.,* Philadelphia, Oct., 1994.

61. S.-M. Phoong and P. P. Vaidyanathan, Paraunitary filter banks over finite fields, *IEEE Trans. Signal Process.,* **SP-45**: 1443–1457, 1997.

62. R. Coifman and Y. Meyer, Remarques sur l'analyse de Fourier à fenêtre, *C.R. Acad. Sci. Paris,* **312**, Serie I: 259–261, 1991.

63. P. Auscher, G. Weiss, and M. V. Wickerhauser, Local sine and cosine basis of Coifman and Meyer and the construction of smooth wavelets, in C. K. Chui (ed.), *Wavelets: A Tutorial in Theory and Applications,* Boston: Academic Press, 1992.

64. X.-G. Xia and B. W. Suter, A family of two dimensional nonseparable Malvar wavelets, *Appl. Comp. Harm. Anal.,* **2**: 243–256, 1995.

65. X.-G. Xia and B. W. Suter, Construction of Malvar wavelets on hexagons, *Appl. Comp. Harm. Anal.,* **3**: 65–71, 1996.

66. S. Venkataraman and B. C. Levy, State space representations of 2-D FIR lossless transfer matrices, *IEEE Trans. Circuits Syst. II, Analog Digit. Signal Process.,* **41**: 117–132, 1994.

67. M. Vetterli, Multidimensional sub-band coding: Some theory and algorithms, *Signal Processing,* **6**: 97–112, 1984.

68. D. E. Dudgeon and R. M. Mersereau, *Multidimensional Digital Signal Processing,* Englewood Cliffs, NJ: Prentice-Hall, 1984.

69. J. W. Woods, *Subband Image Coding,* Boston: Kluwer Academic Publishers, 1991.

70. V. C. Liu and P. P. Vaidyanathan, On the factorization of a subclass of 2-D digital FIR lossless matrices for 2-D QMF bank applications, *IEEE Trans. Circuits Syst.,* **CAS-37**: 852–854, 1990.

71. G. Karlsson and M. Vetterli, Theory of two-dimensional multirate filter banks, *IEEE Trans. Acoust. Speech Signal Process.,* **ASSP-38**: 925–937, 1990.

72. E. Viscito and J. P. Allebach, The analysis and design of multidimensional FIR perfect reconstruction filter banks for arbitrary sampling lattices, *IEEE Trans. Circuits and Syst.,* **CAS-38**: 29–41, 1991.

73. P. P. Vaidyanathan, Fundamentals of multidimensional multirate digital signal processing, *Sadhana,* **15**: 157–176, 1990.

74. J. Kovačević and M. Vetterli, The commutativity of up/downsampling in two dimensions, *IEEE Trans. Inf. Theory,* **IT-37**: 695–698, 1991.

75. J. Kovačević and M. Vetterli, Nonseparable multidimensional perfect reconstruction filter banks and wavelet bases for $\mathbf{R}^n$, *IEEE Trans. Inf. Theory,* **IT-38**: Feb. 1992.

76. R. H. Bamberger and M. J. T. Smith, A filter bank for the directional decomposition of images: theory and design, *IEEE Trans. Signal Process.,* **SP-40**: 882–893, 1992.

77. A. A. C. M. Kalker, Commutativity of up/down sampling, *Electron. Lett.,* **28**: 567–569, 1992.

78. T. Chen and P. P. Vaidyanathan, The role of integer matrices in multidimensional multirate systems, *IEEE Trans. Signal Process.,* **SP-41**: 1035–1047, 1993.

79. T. Chen and P. P. Vaidyanathan, Recent developments in multidimensional multirate systems, *IEEE Trans. Circuits Syst. Video Technol.,* **3**: 115–137, 1993.

80. S. Basu and C. Chiang, A complete parameterization of 2-D nonseparable orthogonal wavelets, *Proc. IEEE Signal Process. Symp. Time-Freq. Time-Scale Anal.,* Victoria BC, Canada, 1992.

81. S. Venkataraman and B. C. Levy, A comparison of design methods for 2-D FIR orthogonal perfect reconstruction filter banks, *IEEE Trans. Circuits Syst. II, Analog Digit. Signal Process.,* **42**: 525, 1995.

82. M. Ikehara, K. Sawaaki, H. Isobe, A. Yamashita, and H. Kuroda, Design of 2-D perfect reconstruction filter banks for arbitrary sampling lattices—Design of 2-D N-th-band digital filters, *Electron. and Commun. Japan. Part 3, Fundamental Electron. Sci.,* **76**: 85, 1993.

83. B. L. Evans, R. H. Bamberger, and J. H. McClellan, Rules for multidimensional multirate structures, *IEEE Trans. Signal Process.,* **SP-42**: 762–771, 1994.

84. R. A. Gopinath and C. S. Burrus, On upsampling, downsampling, and rational sampling rate filter banks, *IEEE Trans. Signal Process.,* **SP-42**: 812–824, 1994.

85. X.-G. Xia, B. W. Suter, and M. E. Oxley, On necessary and sufficient conditions for perfect reconstruction multidimensional delay chain systems, *IEEE Trans. Signal Process.,* **SP-43**: 1515–1519, 1995.

85a. T. A. C. M. Kalker and I. A. Shah, A group theoretic approach to multidimensional filter banks: theory and applications, *IEEE Trans. Signal Process.,* **SP-44**:1392, 1996.

86. X.-G. Xia and B. W. Suter, A systematic construction method for spatial-varying FIR filter banks with perfect reconstruction, *Proc. 1994 IEEE Int. Conf. Image Process.,* pp. 830–833, Austin, Texas, 1994.

86a. J. Kovačević, Local cosine bases in two dimensions, *Proc. Int. Conf. Acoust. Speech Signal Process.,* pp. 2125–2128, Detroit, 1995.

87. R. A. Haddad and K. Park, Modeling, analysis, and optimum design of quantized M-band filter banks, *IEEE Trans. Signal Process.,* **SP-43**: 2540, 1995.

88. A. Benyassine and A. N. Akansu, Subspectral Modeling in Filter Banks, *IEEE Trans. Signal Process.,* **SP-43**: 3050, 1995.

89. S. Ohno and H. Sakai, Optimization of filter banks using cyclostationary spectral analysis, *IEEE Trans. Signal Process.,* **SP-44**: 2718, 1996.

90. S. Wada, H. Yagi, and T. Saito, A flexible design of nonuniformly oversampled multirate filter banks, *Proc. IEEE Int. Symp. Circuits Syst.,* 1996.

91. B. Xuan and R. H. Bamberger, Complete FIR principal component filter banks, *Proc. IEEE Int. Symp. Circuits Syst.,* 1996.

92. A. Delopoulos and S. Kollias, Optimal filter banks for signal reconstruction from noisy subband components, *IEEE Trans. Signal Process.,* **SP-44**: 212, 1996.

93. P. P. Vaidyanathan, Theory of optimal orthonormal filter banks, *Proc. Int. Conf. Acoust. Speech Signal Process.,* pp. 1487–1490, Atlanta, Georgia, 1996.

94. P. P. Vaidyanathan, Unitary and paraunitary systems in finite fields, *Proc. IEEE ISCAS,* New Orleans, LA., pp. 1189–1192, 1990.

95. X.-G. Xia, Applications of multirate filterbanks in error correction codes: Partial response channels, *Proc. of SPIE: Wavelet Applications III,* **2762**: 478–489, Orlando, 1996.

96. M. Vetterli, Filter banks allowing for perfect reconstruction, *Signal Process.,* **10**: 219–244, 1986.

97. M. D. Swanson and A. H. Twefik, A binary wavelet decomposition of binary images, *IEEE Trans. Image Process.,* **5**: 1637–1650, 1996.

98. X.-G. Xia, J. S. Geronimo, D. P. Hardin, and B. W. Suter, Design of prefilters for discrete multiwavelet transforms, *IEEE Trans. Signal Process.,* **SP-44**: 25–35, 1996.

99. X.-G. Xia, A new prefilter design for discrete multiwavelet transforms, Tech. Report #96-11-2, Dept. of Elect. Eng., Univ. of Delaware, 1996.

100. Z. Cvetkovic and M. Vetterli, Oversampled Filter Banks, *IEEE Trans. Signal Process.,* to appear.

101. Z. Cvetkovic and M. Vetterli, Robustness of overcomplete expansions, in J. Zeevi and R. Coifman (eds.), *Signal and Image Representation in Combined Spaces,* Boston: Academic Press, 1996.

102. K. Nayebi, T. P. Barnwell, and M. J. T. Smith, Analysis-synthesis systems with time-varying filter bank structures, *Proc. IEEE Int. Conf. Acoust. Speech Signal Process.,* pp. 617–620, San Francisco, 1992.

103. I. Sodagar, K. Nayebi, T. P. Barnwell III, and M. J. T. Smith, Time-varying analysis-synthesis systems based on filter banks and post filtering, *IEEE Trans. Signal Process.,* **SP-43**: 2512, 1995.

104. C. Herley and M. Vetterli, Orthogonal time-varying filter banks and wavelet packets, *IEEE Trans. Signal Process.,* **SP-42**: 2650, 1994.

105. C. Herley, Boundary filters for finite-length signals and time-varying filter banks, *IEEE Trans. Circuits Syst. 2, Analog Digit. Signal Process.,* **42**: 102, 1995.

106. T. Chen and P. P. Vaidyanathan, Time reversed inversion for time-varying filter banks, with applications in subband coding of images, *Proc. 27th Annual Asilomar Conf. Signals, Syst. Computers,* Pacific Grove, CA, 1993.

107. R. L. de Queiroz and K. R. Rao, Structure for time-varying paraunitary filter banks with perfect reconstruction, *Electronic Letters,* **29** (2): 217, 1993.

108. R. A. Gopinath and C. S. Burrus, Factorization approach to unitary time-varying filter bank trees and wavelets, *IEEE Trans. Signal Process.,* **SP-43**: 666–680, 1995.

109. S.-M. Phoong and P. P. Vaidyanathan, Time-varying filters and filter banks: Some basic principles, *IEEE Trans. Signal Process.,* **SP-44**: 2971, 1996.

110. M. R. K. Khansari and A. Leon-Garcia, Subband decomposition of signals with generalized sampling, *IEEE Trans. Signal Process.,* **SP-41**: 3365–3376, 1993.

111. W. Li and Y.-Q. Zhang, A proposal for very low bit rate video coding standard using vector subband coding (VSC), ISO/IEC JTC1/SC29/WG11, MPEG-94-209, July 1994.

112. V. M. Gadre and R. K. Patney, Vector multirate filtering and matrix filter banks, *Proc. IEEE Int. Symp. Circuits Syst.,* pp. 1360–1363, 1994.

113. X.-G. Xia and B. M. Suter, Multirate filter banks with block sampling, *IEEE Trans. Signal Process.,* **SP-44**: 484–496, 1996.

114. X.-G. Xia and B. M. Suter, Vector-valued wavelets and vector filter banks, *IEEE Trans. Signal Process.,* **SP-44**: 508–518, 1996.

115. D. W. Redmill and D. R. Bull, Nonlinear perfect reconstruction critically decimated filter banks, *Electron. Lett.,* **32** (4): 310, 1996.

116. F. Hampson and J. Pesquet, A nonlinear subband decomposition with perfect reconstruction, *Proc. IEEE Int. Conf. Acoust. Speech Signal Process.,* pp. 1523–1527, Atlanta, GA, 1996.

117. R. L. de Queiroz and D. Florencio, A pyramidal coder using a non-linear filter bank, *Proc. IEEE Int. Conf. Acoust. Speech Signal Process.,* pp. 1570–1573, Atlanta, GA, 1996.

118. D. Florencio and R. Schafer, Perfect reconstruction non-linear filter banks, *Proc. IEEE Int. Conf. Acoust. Speech Signal Process.,* pp. 1814–1817, Atlanta, GA, 1996.

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MULTISENSOR DATA FUSION. See DATA FUSION.

MULTISPECTRAL IMAGE ANALYSIS. See IMAGE CLASSIFICATION.

MULTISPECTRAL PATTERN RECOGNITION. See IMAGE CLASSIFICATION.

MULTISTAGE DECISION MAKING. See DYNAMIC PROGRAMMING.

MULTITARGET TRACKING. See SONAR TRACKING.

MULTITHREADED ARCHITECTURES. See DATA-FLOW AND MULTITHREADED ARCHITECTURES.

MULTIUSER SYSTEMS. See TIME-SHARING SYSTEMS.

MULTIVARIABLE CONTROL SYSTEMS. See H_∞ CONTROL.