MAGNETIC CIRCUITS

MAGNETIC QUANTITIES

Magnetic fields are one way of describing the force exerted by one current-carrying conductor on another. Thus one current is said to produce a magnetic field that exerts a force on the other current, and, in this way, the need to work directly in terms of action at a distance is avoided. The magnetic field produced at distance $\boldsymbol{r}$ from a short current element i of length $\delta\boldsymbol{l}$ is given by the Biot–Savart law (1)

$$\boldsymbol{H} = \frac{i\delta\boldsymbol{l} \times \boldsymbol{a}_r}{4\pi r^2} \tag{1}$$

where $\boldsymbol{a}_r$ is the unit vector in the direction of $\boldsymbol{r}$. Thus the total value of the magnetic field at a given point in space can be determined by integrating Eq. (1) over the complete current-carrying circuit. These currents do not have to be contained only in metallic conductors but can also flow in conductive liquids and gases (plasmas). In addition, the so-called magnetic materials, such as mild steel, only become magnetized by virtue of microscopic currents in the atomic structure of the material created by the orbital motion and spin of the electrons.

Between 1820 and 1822, Ampère, later described by Maxwell as the "Newton of electricity," performed a series of experiments that led him to propose an equivalence between the current i in a small coil and the magnetic strength of the poles of a short bar magnet, in the form

$$\mu_0 i\delta S = Q^* d \tag{2}$$

where d is the length of the magnet and δS is the planar area of the coil. The convenient concept of a magnetic pole Q^* had been suggested about 70 years earlier by Michell, who discovered that the force interaction between magnets could be explained on the basis of an inverse square law between poles, which, in modern SI units, becomes

$$F = \frac{Q_1^* Q_2^*}{4\pi\mu_0 r^2} \tag{3}$$

Equation (3) is precisely analogous to Coulomb's law between two point electric charges Q_1 and Q_2 separated by distance r and leads to the magnetic field produced by a single pole Q_1^*:

$$H = \frac{Q_1^*}{4\pi\mu_0 r^2} \tag{4}$$

Thus we are in good company if we model magnetized materials in terms of distributions of surface magnetic polarity q^* and volume polarity ρ^* within the bulk of the sample. Just as for dielectrics in which the dipole moment per unit volume is defined as $\boldsymbol{P}$, the polarization vector, so in magnetic materials we have the magnetization vector $\boldsymbol{M}$. In this model it can be shown that div $\boldsymbol{M} = -\rho^*$ within the material and $M_n = q^*$ at the surface, where M_n is the component of $\boldsymbol{M}$ normal to, and just inside, the surface.

Now the divergence of the magnetic field is given by div $\boldsymbol{H} = \rho^*/\mu_0$, bearing in mind that the magnetic polarity is associated with the magnetization of the material and that there are no free poles. Thus, replacing ρ^* by $-$div $\boldsymbol{M}$, we have

$$\text{div}(\mu_0\boldsymbol{H} + \boldsymbol{M}) = \text{div}\boldsymbol{B} = 0 \tag{5}$$

where

$$\boldsymbol{B} = \mu_0\boldsymbol{H} + \boldsymbol{M} \tag{6}$$

is the magnetic flux density whose divergence is always zero.

In a linear isotropic material, $\boldsymbol{M}$ is both parallel to and proportional to $\boldsymbol{H}$, so that

$$\boldsymbol{M} = \mu_0\chi_m\boldsymbol{H} \tag{7}$$

where χ_m is a constant known as the magnetic susceptibility. Equation (6) thus becomes

$$\boldsymbol{B} = \mu_0(1 + \chi_m)\boldsymbol{H} = \mu_0\mu_r\boldsymbol{H} = \mu\boldsymbol{H} \tag{8}$$

where the relative permeability $\mu_r = 1 + \chi_m$. The assumption that μ_r is a constant is a severe approximation for nearly all magnetic materials, but it does facilitate mathematical analysis. However, approximate methods can be devised whereby μ_r is taken as some simple function of H.

NONLINEARITY AND SATURATION

In a ferromagnetic material, the magnetic susceptibility not only is a large positive number but also varies strongly with magnetic field strength. These materials are made up of domains that are already magnetized in the sense that each domain contains aligned magnetic dipoles formed by electron spin. When the sample of material is in an unmagnetized

state, the domains are so aligned that their net magnetization cancels. For example, in an iron crystal, there are six directions of "easy" magnetization: both directions along three orthogonal axes. If we assume, for the purpose of the argument only, that all domains are the same size, then in each of the easy directions there will be as many domains pointing in one direction as in the other. If a small magnetic field is now applied to the crystal nearly parallel to one of the easy directions, the domains with polarity in the same direction (i.e., assisting the applied field external to themselves) grow at the expense of adjacent parallel domains with opposite polarity. The mechanism for this is a small movement of the domain walls, and the process is virtually loss-free if it takes place slowly.

Consider next a larger sample made up of many crystals or crystal fragments, each containing many domains, in the form of a ring on which there is a uniformly wound toroidal magnetizing winding producing a uniform applied field H tangential to the internal axis of the ring cross section. This configuration gets over the problem of possible differences between the applied and the local field because there is no surface magnetic polarity present. Thus, with a simple search coil to measure the corresponding flux density B, a curve of B against H, starting with an unmagnetized ring, can be drawn as shown in Fig. 1. The previously described process with a low value of H applied, takes us to point P_1. If this field is removed, the operating point drops back to the origin because the process is reversible. To reach the higher point P_2, the domains already aligned with the field direction continue to grow, other opposing domains completely reverse, and those at other angles reverse polarity if this brings them more nearly into line with the applied field. All this activity involves extensive domain wall movement and the expenditure of energy. As a result, when the field is reduced to zero, the flux density only drops to the point represented by Q_2, and if the applied field is then cycled between equal negative and positive maximum values, the complete loop P_2P_2' is repeatedly traversed. This is known as a *hysteresis loop,* derived from the Greek word meaning to lag, and it can be shown that the expenditure of energy per cycle is proportional to the area of the loop. Alternatively, rising up the initial magnetization curve (also called the normal magnetization curve) to the higher point P_3 takes the material into magnetic saturation whereby almost complete alignment of the domains in the field direction has taken place. For a full discussion of the processes involved see Bozorth (2).

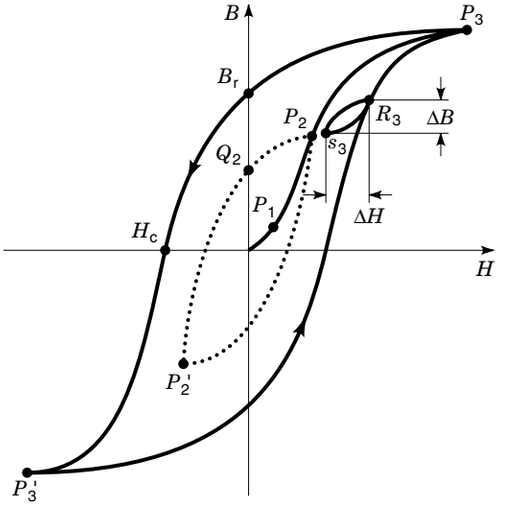

Figure 1. Hysteresis loops for a ferromagnetic material.

Symmetrical hysteresis loops, known as major loops, can be formed from any points such as P_2 or P_3. The value of flux density at which $H = 0$ on any loop is known as the *remanent density* B_r and the value of H at which $B = 0$ is the *coercive force* H_c. The so-called permanent magnetic materials operate in the second quadrant of Fig. 1 and require a very high coercive force, or coercivity, to prevent demagnetization during operation.

Hysteresis loops do not need to be symmetrical. An important example occurs if we are on a major loop such as that represented by P_3P_3' in Fig. 1 but stop at point R_3 and begin to reduce H again but only by a small amount to point S_3. By causing H to oscillate by ΔH produces the minor hysteresis loop shown. The mean slope of this loop (i.e., $\Delta B/\Delta H$) is known as the *incremental permeability* μ_i. Minor loops can also be generated from points on the initial magnetization curve.

Hysteresis is often ignored, being a relatively small effect in a wide range of ferromagnetic materials (the so-called soft materials) whose manufacturing process has been deliberately devised to minimize hysteresis loss, providing very reduced values for both remenence and coercivity. Magnetic saturation, on the other hand, is a different matter and cannot usually be ignored without incurring significant error. Consequently, the magnetic field in soft magnetic materials can be well represented by using the initial magnetization curve and Eq. (8) while for permanent magnetic material simulation the general Eq. (6) must be considered.

There are three ways of handling the representation of the magnetization curve $P_1P_2P_3$ shown in Fig. 1. If a numerical solution of the field is being performed, using the finite-difference or finite-element method for example, the curve can be stored as a set of values of H and the corresponding values of B, so that any required value of B (or H) can be found by some form of interpolation. Most finite-element software has the facility for handling this process automatically, with an option for the user to specify a particular curve that is not already present in the software. The alternative is an algebraic function expressing B in terms of H. A simple equation that is recommended for materials with a relatively smooth knee to the curve (i.e., transition into saturation) is the modified Frolich equation:

$$B = \mu_0 H + \frac{H}{a + b|H|} \tag{9}$$

which provides a neat expression for permeability

$$\mu = \mu_0 + (a + b|H|)^{-1} \tag{10}$$

Another formulation is the rational fraction expression

$$B = \mu_0 H + \frac{a_0 + a_1|H| + a_2|H|^2}{1 + b_1|H| + b_2|H|^2} H \tag{11}$$

and a more complex expression which gives an excellent fit for silicon steel laminations with a sharp knee in the B/H curve is

$$B = \mu_0 H + S_1 \tan^{-1}(a_1 H) + S_2 \tan^{-1}(a_2 H) \tag{12}$$

where, as a guide, typical values of the constants are $S_1 = 0.987$, $S_2 = 0.323$, $a_1 = 0.0308$, and $a_2 = 0.000110$.

Finally, we have the possibility of a quasilinear approximation where a well-defined magnetic circuit is divided into a number of discrete sections, in each of which μ is taken as a constant (defined as the ratio B/H at a given point on the magnetization curve so that the technique is both algebraic and graphical) but adjusted to satisfy conditions that are imposed on the complete circuit. This approach also implies that B and H are uniform in each section of the circuit and so effectively treats a magnetic circuit as a one-dimensional system entirely analogous to an electric circuit (see the section entitled "Duality Between Magnetic and Electric Circuits"). This explains the initial reference to a well-defined circuit because it is not usually possible, using this method, to model leakage flux (see the section entitled "Boundary Conditions") or rapidly changing cross sections of the circuit components.

DUALITY BETWEEN MAGNETIC AND ELECTRIC CIRCUITS

Because there are no free magnetic poles, the net magnetic flux out of any close surface is always zero. In this respect, magnetic flux resembles a steady electric current in a conductor. Indeed, there can be no net outflow of current from any closed surface. This similarity gives rise to the idea known as the *magnetic circuit.* It is based on the analogy (sometimes called a *duality*) between the circulation of magnetic flux in a closed path and the circulation of electric current in a closed circuit. This analogy yields some very simple and reasonably accurate calculations of inherently complicated magnetic field problems. The procedure is illustrated in Fig. 2.

A coil may be used to magnetize an iron core. The iron forms a closed path, and the flux will circulate around this magnetic circuit as indicated. Assuming no magnetic saturation, the leakage of flux into the surrounding air will be small, and thus the total flux is practically the same over all cross sections of the core. This resembles an electric circuit where current is the same over all cross sections of the wire. The mmf of the coil, as given by its ampere-turns $I \times n$, where I is the supply current and n is the number of turns of the coil, could be considered the analogue of the emf of the battery in the electric circuit. Flux in the magnetic circuit is then the analogue of current in the electric circuit. Ampere's equation (47) (see the section entitled "Maxwell's Equation") can be applied in a much simpler form with integration replaced by a summation of terms with constant H in each segment of the closed magnetic path

$$\sum_i H_{li} l_i = In \tag{13}$$

which resembles going round the loop in the electric circuit summing the contributions to the emf.

The analogue of resistance is called *reluctance.* For any part of the magnetic circuit, the reluctance is defined in terms of the magnetic potential and flux

$$R_\mu = \frac{V_1 - V_2}{\Phi} = \frac{1}{\Phi} \int_1^2 H_l \, dl \tag{14}$$

and for the complete circuit

$$\text{reluctance} = \frac{\text{mmf}}{\text{flux}} = \frac{In}{\Phi} \tag{15}$$

The units of reluctance are amperes per weber. The reciprocal of reluctance is called *permeance.*

By analogy with Ohm's law in electric circuits

$$\text{emf} = IR \tag{16}$$

we can thus write

$$\text{mmf} = \Phi R_\mu \tag{17}$$

where for simple parallel-sided segments of iron core, like those in Fig. 2, we have

$$R_\mu = \frac{l}{\mu S} \tag{18}$$

where l is the length, S is the cross section, and μ the permeability. Equation (18) closely resembles expression for the resistance of parallel-sided conductors, if μ is substituted for σ (conductivity).

Table 1 summarizes the established duality between magnetic and electric circuits.

The magnetic circuit analogy is often used in approximate calculations of the magnetizing current which is required to set up a given magnetic flux in the iron circuit of an electrical device. We shall explain the procedure with reference to the

Table 1.

Electric Circuit	Magnetic Circuit
emf (V)	mmf (A)
current I (A)	flux Φ (Wb)
current density (A/m^2)	flux density B (T)
	magnetic field strength H (A/m)
electric potential difference (V)	magnetic potential difference (A)
resistance R (Ω)	reluctance R_μ (A/Wb)
electric conductance (S)	permeance (Wb/A)
electric conductivity σ (S/m)	magnetic permeability μ(H/m)

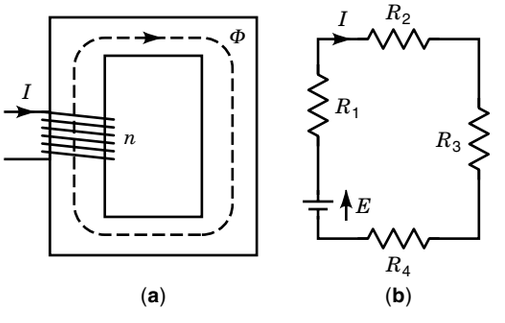

Figure 2. (a) Magnetic circuit, (b) electric circuit analog.

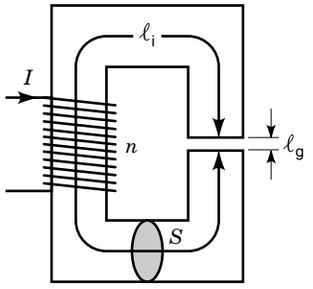

Figure 3. A magnetic circuit with an air gap.

very simple case illustrated in Fig. 3. The presence of a small air gap is quite common and is essential in some devices so that there may be relative motion between different parts.

From Eq. (13) we have

$$\sum H\,dl = H_i l_i + H_g l_g = In \tag{19}$$

so that

$$nI = \frac{Bl_i}{\mu_r \mu_0} + \frac{Bl_g}{\mu_0} = \frac{Bl_g}{\mu_0}\left(1 + \frac{l_i}{\mu_r l_g}\right) \tag{20}$$

Given actual dimensions, the permeability of iron, and the required flux density in the air gap, the ampere-turns and thus magnetizing current may be easily calculated. Suppose $\mu_r = 3000$ and $l_i = 30l_g$. Then

$$nI = \frac{Bl_g}{\mu_0}(1 + 0.01) \tag{21}$$

which shows that only 1% of the mmf is absorbed by the iron path, although it is so much longer; 99% of the mmf is used to force the flux across the air gap.

The principle of the calculation is very easy, but in practice it often becomes modified and refined. First, the analogy between magnetic and electric circuits is not complete because magnetic flux and electric current are very different entities. Electric current is formed by a flow of electrons, whereas nothing flows when there is a magnetic flux. Thus, energy is required to set up a magnetic field, but none is required to maintain it; otherwise, permanent magnets could not exist. Another imperfection of the analogy arises from the fact that the ratio of the conductivities of a typical conductor and insulator is immensely greater than the ratio of the permeabilities of iron and air. Thus air or free space becomes a legitimate, though not a very good, "conductor" of flux (usually in the form of air gaps like the one in our example); at the same time, the possibility of leakage flux must be kept very much in mind. And last, but not least, Ohm's law for electric circuits tells us that resistance is independent of current, whereas the reluctance in magnetic circuits is not independent of flux because the permeability varies with Φ. Magnetic saturation may come into consideration and significantly change the proportions suggested by Eq. (21).

MAXWELL'S EQUATIONS

James Clerk Maxwell (3) was the first investigator to use the very compact form of partial differential equations to describe electromagnetic phenomena. For general time-varying fields, using today's notation (1,4,5), Maxwell's equations may be written as

$$\operatorname{curl} \boldsymbol{H} = \boldsymbol{J} + \frac{\partial \boldsymbol{D}}{\partial t} \tag{22}$$

$$\operatorname{curl} \boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t} \tag{23}$$

$$\operatorname{div} \boldsymbol{D} = \rho \tag{24}$$

$$\operatorname{div} \boldsymbol{B} = 0 \tag{25}$$

where $\boldsymbol{H}$ = magnetic field intensity (amperes per meter), $\boldsymbol{E}$ = electric field intensity (volts per meter), $\boldsymbol{B}$ = magnetic flux density (webers per square meter, or tesla), $\boldsymbol{D}$ = electric flux density (coulombs per square meter), $\boldsymbol{J}$ = electric current density (amperes per square meter), and ρ = electric charge density (coulombs per cubic meter).

The current density $\boldsymbol{J}$ and the charge density ρ are the sources of the field and are related through the equation of continuity, which can be written as

$$\operatorname{div} \boldsymbol{J} = -\frac{\partial \rho}{\partial t} \tag{26}$$

and thus specifies the conservation of charge. These equations are supplemented by the *constitutive equations* describing macroscopic properties of the medium

$$\boldsymbol{D} = \epsilon \boldsymbol{E} \tag{27}$$

$$\boldsymbol{B} = \mu \boldsymbol{H} \tag{28}$$

$$\boldsymbol{J} = \sigma \boldsymbol{E} \tag{29}$$

where the constitutive parameters ϵ, μ, and σ denote, respectively, the permittivity (farads per meter), permeability (henrys per meter), and conductivity (siemens per meter) of the medium. For isotropic media, these parameters are scalars, but for anisotropic materials, they become tensors. Moreover, for nonhomogeneous materials, they are functions of position.

For static fields, where the field quantities do not vary with time, Eqs. (22), (23), and (26) reduce to

$$\operatorname{curl} \boldsymbol{H} = \boldsymbol{J} \tag{30}$$

$$\operatorname{curl} \boldsymbol{E} = 0 \tag{31}$$

$$\operatorname{div} \boldsymbol{J} = 0 \tag{32}$$

which shows that there is no interaction between electric and magnetic fields, and thus we can have separately an electrostatic case or a magnetostatic case.

An important special condition arises also when all field quantities vary sinusoidally with time as $e^{j\omega t}$, where ω is an angular frequency. Using phasor notation, the Maxwell's equations involving time derivatives can now be written as

$$\operatorname{curl} \underline{\boldsymbol{H}} = \underline{\boldsymbol{J}} + j\omega \underline{\boldsymbol{D}} \tag{33}$$

$$\operatorname{curl} \underline{\boldsymbol{E}} = -j\omega \underline{\boldsymbol{B}} \tag{34}$$

$$\operatorname{div} \underline{\boldsymbol{J}} = -j\omega\rho \tag{35}$$

The three vector operators grad, div, and curl are very helpful in all of these compact notations. The definitions of these operators follow:

$$\text{grad}\Phi = \frac{\partial \Phi}{\partial n}\hat{\boldsymbol{n}} \tag{36}$$

which is thus the total or maximum slope of the scalar Φ, where the unit vector points in the "uphill" direction normal to the equipotential surface. Second,

$$\text{div}\boldsymbol{F} = \lim_{v \to 0} \frac{1}{v} \oint \boldsymbol{F} \cdot d\boldsymbol{s} \tag{37}$$

describes the net outflow of vector $\boldsymbol{F}$ per unit volume, for a small volume, and finally

$$\text{curl}\,\boldsymbol{F} = \lim_{s \to 0} \frac{1}{s} \oint \boldsymbol{F} \cdot d\boldsymbol{l} \tag{38}$$

is the circulation per unit area, for a small area. In a *rectangular* coordinate system (x, y, z), it is convenient to introduce a *nabla* (sometimes called a *del*) operator, defined as

$$\nabla = \hat{\boldsymbol{\imath}}\frac{\partial}{\partial x} + \hat{\boldsymbol{\jmath}}\frac{\partial}{\partial y} + \hat{\boldsymbol{k}}\frac{\partial}{\partial z} \tag{39}$$

and the three operations are reduced to

$$\text{grad}\,\Phi = \nabla\Phi \tag{40}$$

$$\text{div}\,\boldsymbol{F} = \nabla \cdot \boldsymbol{F} \tag{41}$$

$$\text{curl}\,\boldsymbol{F} = \nabla \times \boldsymbol{F} \tag{42}$$

which also give alternative notation. In other coordinate systems, definitions of Eqs. (36)–(38) must be obtained directly, but the nabla notation is still often used, although it will no longer imply operation of Eq. (39).

Another form of the equations may be found by applying some fundamental integral relationships of vectors. The two most important are Gauss's or the divergence theorem

$$\iiint \nabla \cdot \boldsymbol{F}\, dv = \oint \boldsymbol{F} \cdot d\boldsymbol{s} \tag{43}$$

and Stoke's or the circulation theorem

$$\iint \nabla \times \boldsymbol{F} \cdot d\boldsymbol{s} = \oint \boldsymbol{F} \cdot d\boldsymbol{l} \tag{44}$$

Hence, for example, for a steady current flow, the integral equation equivalent to Eq. (32) will read

$$\oint_S \boldsymbol{J} \cdot d\boldsymbol{s} = 0 \tag{45}$$

Equations (32) and (45) contain the same information and are vector equivalents of the first Kirchhoff's current law.

In magnetostatics, there is no net polarity as all magnets consist of dipoles so that

$$\oint_S \boldsymbol{B} \cdot d\boldsymbol{s} = 0 \tag{46}$$

which in differential form is expressed by Eq. (25). Moreover, application of Stoke's theorem to Eq. (30) yields

$$\oint \boldsymbol{H} \cdot d\boldsymbol{l} = \iint_S \boldsymbol{J} \cdot d\boldsymbol{s} = I \tag{47}$$

which is known as Ampère's equation and shows that the magnetic field of a current is nonconservative.

Finally, Faraday's law, Eq. (23), can also be written as

$$\oint \boldsymbol{E} \cdot d\boldsymbol{l} = -\frac{d\Phi}{dt} \tag{48}$$

where the magnetic flux is given by

$$\Phi = \iint \boldsymbol{B} \cdot d\boldsymbol{s} \tag{49}$$

POTENTIALS

One way of finding field distributions is to solve Maxwell's equations directly as a system of first-order differential equations in terms of the appropriate field quantities. In most, although not all, important cases, however, it will be more economical to convert the first-order equations into second-order differential equations involving scalar or vector potentials.

In magnetostatics, we often use the vector potential $\boldsymbol{A}$ defined in terms of the magnetic flux density $\boldsymbol{B}$ as

$$\boldsymbol{B} = \nabla \times \boldsymbol{A} \tag{50}$$

With the help of Eqs. (28) and (30), we find

$$\nabla \times \left(\frac{1}{\mu}\nabla \times \boldsymbol{A}\right) = \boldsymbol{J} \tag{51}$$

In conductors, it is likely that $\mu = \mu_0$, so that

$$\nabla \times \nabla \times \boldsymbol{A} = \nabla\nabla \cdot \boldsymbol{A} - \nabla^2\boldsymbol{A} = \mu_0\boldsymbol{J} \tag{52}$$

For convenience we are allowed to put

$$\nabla \cdot \boldsymbol{A} = 0 \tag{53}$$

because there is a free choice in selecting the divergence sources of $\boldsymbol{A}$, since only the curl sources have been defined in Eq. (50). (According to Helmholtz theorem, a vector field is uniquely prescribed if both its curl and divergence are specified.) The condition imposed on the divergence of the vector potential is called a *gauge condition,* and the particular choice of Eq. (53) is known as the *Coulomb gauge.* Hence, if we choose zero divergence,

$$\nabla^2\boldsymbol{A} = -\mu_0\boldsymbol{J} \tag{54}$$

On the other hand, if the magnetostatic field is not required within the regions containing conduction current, Eq. (30) simplifies to

$$\nabla \times \boldsymbol{H} = 0 \tag{55}$$

and we have an alternative formulation available in terms of $\boldsymbol{H}$. Thus, we can write

$$\boldsymbol{H} = -\nabla\varphi_{\mathrm{m}} \tag{56}$$

where φ_{m} is a magnetic scalar potential. This leads to the following equation

$$\nabla \cdot (\mu \nabla\varphi_{\mathrm{m}}) = 0 \tag{57}$$

which when expanded yields

$$\mu\nabla^2\varphi_{\mathrm{m}} + \nabla\mu \cdot \nabla\varphi_{\mathrm{m}} = 0 \tag{58}$$

The formulation in terms of the magnetic scalar potential has some advantages as it reduces the number of unknown functions that must be solved from three to one. In two-dimensional cases this does not matter because the vector potential typically has only one component (in the direction of current flow), but in three-dimensional fields savings in computing effort may be considerable. Either formulation is complicated by the fact that permeability is not only a function of position but depends also on field strength. Moreover, many practical magnetic materials are anisotropic, and so the permeability has to be treated as a tensor.

Generally, in time varying fields, the electric and magnetic fields are coupled, as demonstrated by Eqs. (22) and (23). Both sources ρ and $\boldsymbol{J}$ are present and are linked through the equation of continuity (26). Similarly the electric scalar potential φ and the magnetic vector potential $\boldsymbol{A}$ can be combined using the *Lorentz gauge*

$$\nabla \cdot \boldsymbol{A} + \frac{1}{c^2}\frac{\partial\varphi}{\partial t} = 0 \tag{59}$$

where c is the velocity of light. The Lorentz gauge simplifies the relationship between the potentials and the sources, which may be shown to be

$$\nabla^2\boldsymbol{A} - \frac{1}{c^2}\frac{\partial^2\boldsymbol{A}}{\partial t^2} = -\mu\boldsymbol{J} \tag{60}$$

and

$$\nabla^2\varphi - \frac{1}{c^2}\frac{\partial^2\varphi}{\partial t^2} = -\frac{\rho}{\epsilon} \tag{61}$$

where μ and ϵ have been taken as constants. Thus $\boldsymbol{A}$ depends on $\boldsymbol{J}$ only, whereas φ depends on ρ only. The electric and magnetic fields are obtained from the potentials by the relationships

$$\boldsymbol{E} = -\frac{\partial\boldsymbol{A}}{\partial t} - \nabla\varphi \tag{62}$$

and

$$\boldsymbol{H} = \frac{1}{\mu}\nabla \times \boldsymbol{A} \tag{63}$$

DIFFUSION EQUATION AND EDDY CURRENTS

The term $\partial D/\partial t$ in Eq. (22), known as *displacement current,* was added by Maxwell to what is otherwise Ampere's law in differential form, in order to make the field equations consistent by satisfying Eq. (26). However, it can be shown that displacement current is negligible in good conductors even at very high frequencies. Writing $\boldsymbol{J}$ as $\sigma\boldsymbol{E}$ according to Ohm's law and taking the curl of Ampère's equation, we have

$$\mathrm{curl}\ \mathrm{curl}\boldsymbol{H} = \sigma\ \mathrm{curl}\boldsymbol{E} = -\sigma\mu\frac{\partial\boldsymbol{H}}{\partial t} \tag{64}$$

But

$$\mathrm{curl}\ \mathrm{curl}\boldsymbol{H} = \mathrm{grad}\ \mathrm{div}\boldsymbol{H} - \nabla^2\boldsymbol{H} \tag{65}$$

and so using Eqs. (25) and (28) (assuming constant μ), we have

$$\nabla^2\boldsymbol{H} = \sigma\mu\frac{\partial\boldsymbol{H}}{\partial t} \tag{66}$$

Equation (66) is known as a vector diffusion equation. If vector $\boldsymbol{H}$ has only one component, H_z say in the Cartesian system, then we have the scalar diffusion equation

$$\nabla^2 H_z = \sigma\mu\frac{\partial H_z}{\partial t} \tag{67}$$

where ∇^2 now represents the scalar Laplacian operator $\partial^2/\partial x^2 + \partial^2/\partial y^2 + \partial^2/\partial z^2$.

For sources and fields that vary sinusoidally with time, Eq. (67) can be written in the complex Helmholtz form as

$$\nabla^2\underline{H}_z = j\omega\sigma\mu\underline{H}_z \tag{68}$$

where

$$H_z = \mathrm{Re}\{\underline{H}_z e^{j\omega t}\} \tag{69}$$

As a simple example of the solution of Eq. (68), consider a flat slab of conducting material with its surface on the plane $y = 0$ and filling the positive half space $y \geq 0$. If a uniform field, $H_{\mathrm{s}} \cos \omega t$ is applied parallel to the surface, then H_z inside the slab will be a function of the coordinate y only. Thus, Eq. (68) becomes

$$\frac{d^2\underline{H}_z}{dy^2} = j\omega\sigma\mu\underline{H}_z = \alpha^2\underline{H}_z \tag{70}$$

where

$$\alpha = \sqrt{j\omega\sigma\mu} = \frac{1+j}{\delta} \quad \text{and} \quad \delta = \sqrt{\frac{2}{\omega\sigma\mu}} \tag{71}$$

The general form of the solution of Eq. (70) is

$$\underline{H}_z = K_1 e^{\alpha y} + K_2 e^{-\alpha y} \tag{72}$$

but to satisfy $\underline{H}_z = H_{\mathrm{s}}$ at $y = 0$ and $\underline{H}_z \to 0$ as $y \to \infty$, $K_1 = 0$ and $K_1 = H_s$. Thus

$$\underline{H}_z = H_{\mathrm{s}} e^{-\alpha y} = H_{\mathrm{s}} e^{-y/\delta} e^{-jy/\delta} \tag{73}$$

so that

$$H_z = \mathrm{Re}\{H_s e^{-y/\delta} e^{j(\omega t - y/\delta)}\} = H_s e^{y/\delta} \cos(\omega t - y/\delta) \tag{74}$$

The magnitude of the field decays exponentially with depth, and the instant at which the peak value occurs at depth y lags the peak value on the surface by y/δ (radians). This solution is identical in form to the temperature inside a large flat sheet of material subjected to uniform cyclic heating (of sinusoidal form) on its surface. The name diffusion equation derives from this analogy with heat flow.

The attenuation of the magnetic field with depth is due to the currents induced in the conductor by the applied field. These currents flow in the x direction parallel to the surface with density given by

$$\underline{J}_x = \frac{d\underline{H}_z}{dy} = -\alpha H_s e^{-\alpha y} = -J_s e^{-\alpha y} \tag{75}$$

and the resulting eddy-current loss per unit surface area of the slab is

$$P_e = \frac{1}{2\sigma} \int_0^\infty |\underline{J}_x|^2 \, dy = \frac{H_s^2}{\sigma \delta^2} \int_0^\infty e^{-2y/\delta} \, dy = \frac{H_s^2}{2\sigma\delta} \tag{76}$$

Using Ampère's law, $H_s = I$, the peak current per unit width, the loss can be written in the form

$$P_e = \left(\frac{I}{\sqrt{2}}\right)^2 \frac{1}{\sigma\delta} = I_{\mathrm{rms}}^2 R' \tag{77}$$

where $R' = (\sigma\delta)^{-1}$ is the resistance of unit length of a layer of thickness δ and unit width. For this reason, the parameter δ is known as the eddy-current *skin depth* because the correct value of loss is obtained if the current is assumed to flow with uniform density in a layer of depth δ.

The relative smallness of the actual exponential penetration is particularly important. If we regard anything less than 5% of the surface density as negligible, then the effective penetration is given by 3δ. For copper at 50 Hz, δ is typically about 9 mm so that the penetration, on this criterion, is 27 mm. But δ is inversely proportional to the square root of frequency so that, at 500 kHz, the effective penetration has reduced to 0.27 mm. The same sort of reduction occurs even at power frequencies if the material has a high relative permeability. Thus, for a typical ferromagnetic material at 50 Hz, δ is approximately 1 mm. The result is that this simple one-dimensional approach can be used to model, at least approximately, the surface of conductors of any shape if the major dimensions of the conductor are much greater than δ because locally the surface effect is almost one-dimensional. We only require to know the distribution of the surface field H_s. In numerical finite-difference or finite-element solutions, this can be particularly important because the alternative will be the need to discretize the grid or mesh sufficiently finely to follow the skin phenomenon correctly.

The simpler numerical technique is to render the conducting region impermeable to flux by setting $\mu_r = 0$ and therefore excluding this region from the computation by specifying $H_n = 0$ on its surface. A practical example, although most often solved analytically, is the wall of a waveguide.

In situations where the skin effect is rather less pronounced, but the penetration nevertheless is still small, the conducting surface can be modeled as a surface impedance. This is the equivalent of a local linear one-dimensional solution because the surface impedance is defined as

$$Z_s = \frac{E_s}{H_s} \tag{78}$$

where E_s is the peak value of the surface electric field. From Eqs. (29) and (75),

$$E_s = \frac{J_s}{\sigma} = \frac{\alpha H_s}{\sigma} = (1+j)\frac{H_s}{\sigma\delta} \tag{79}$$

Thus

$$Z_s = \frac{1+j}{\sigma\delta} \tag{80}$$

This situation is one in which the eddy currents are limited by their own field. The currents are said to be *inductance limited.* There is an opposite extreme when the currents are *resistance limited* because of the lack of space. This is precisely what is required in a transformer lamination, for example. The full linear solution for the eddy-current loss per unit surface area of a plate or lamination of thickness $2b$ is (6)

$$P_e = \frac{H_s^2}{\sigma\delta} \frac{\sinh\gamma - \sin\gamma}{\cosh\gamma + \cos\gamma} \tag{81}$$

where $\gamma = 2b/\delta$. It is clear that as γ tends to infinity, the loss tends to $H_s^2/\sigma\delta$ (i.e. twice the loss per unit surface area of a massive slab) because we have two surface effects. However, when $\gamma \ll 1$, the loss tends to

$$P_e' = \frac{H_s^2}{\sigma\delta} \frac{\gamma^3}{6} \tag{82}$$

which is the result that would have been obtained by ignoring the reaction field of the eddy currents altogether. The condition for this resistance-limited regime is that the thickness of the conductor is less than δ, and it can be shown that this condition is also necessary for the lamination to carry magnetic flux without significant reduction. This is why the typical iron lamination thickness for 50 Hz operation is 0.5 mm.

Eddy Currents in a Long Rectangular Bar

Consider now a long rectangular bar of cross section $2a \times 2b$, as shown in Fig. 4, immersed in an alternating magnetic field $H_s \cos \omega t$ parallel to its axis.

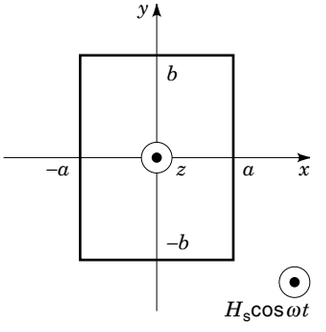

Figure 4. Long rectangular bar in which eddy currents will be induced by the axial applied field.

The Helmholtz equation [Eq. (68)] now becomes

$$\frac{\partial^2 \underline{H}_z}{\partial x^2} + \frac{\partial^2 \underline{H}_z}{\partial y^2} = \alpha^2 \underline{H}_z \tag{83}$$

Using the technique of separation of variables, it can be shown that one solution of Eq. (83) is

$$\underline{H}_z = \sum_{m=0}^{\infty} \{K_m \sin p_m x + L_m \cos p_m x\}\{M_m \sinh \gamma_m y + N_m \cosh \gamma_m y\} \tag{84}$$

where

$$\gamma_m = \sqrt{\alpha^2 + p_m^2}$$

and K_m, L_m, etc., are coefficients to be determined from the boundary conditions of the particular problem. In the same way the separation constant p_m must be determined. An alternative solution can be set up by switching the trigonometric and hyperbolic functions between x and y and using a different separation constant q.

The problem of Fig. 4 has even symmetry since $\underline{H}_z = H_s$ when $x = \pm a$ or $y = \pm b$. Thus, the required solution demands even functions only in both x and y, so that we have

$$\begin{aligned} \underline{H}_z = &\sum_{m=0}^{\infty} L_m \cos p_m x \cosh \sqrt{\alpha^2 + p_m^2}\, y \\ &+ \sum_{n=0}^{\infty} Q_n \cos q_n y \cosh \sqrt{\alpha^2 + q_n^2}\, x \end{aligned} \tag{85}$$

In order to allow the first series in $\cos p_m x$ to match the boundary value of H_s at $y = \pm b$, the second series in Eq. (85) must be zero when $y = b$, i.e., $\cos q_n b = 0$ or

$$q_n = \frac{n\pi}{2b}$$

where $n = 1, 3, 5$, etc. In fact, the first series can satisfy the boundary condition using only one term (i.e., if $p_0 = 0$) so that the coefficient L_0 is given by

$$L_0 = \frac{H_s}{\cosh \alpha b}$$

and $L_m = 0$ for $m = 1, 2, 3$, etc. Thus Eq. (85) becomes

$$\underline{H}_z = H_s \frac{\cosh \alpha y}{\cosh \alpha b} + \sum_{n=1,3,5,\ldots}^{\infty} Q_n \cos \frac{n\pi y}{2b} \cosh \sqrt{\alpha^2 + \frac{n^2\pi^2}{4b^2}}\, x \tag{86}$$

When $x = a$, $\underline{H}_z = H_s$, and the coefficients Q_n can be found by Fourier analysis:

$$Q_n = \frac{\dfrac{4\alpha^2}{n\pi} \sin \dfrac{n\pi}{2}}{\left(\alpha^2 + \dfrac{n^2\pi^2}{4b^2}\right) \cosh \sqrt{\alpha^2 + \dfrac{n^2\pi^2}{4b^2}}\, a} \frac{H_s}{} \tag{87}$$

Equations (86) and (87) completely define the solution. Comparing Eq. (86) with the result for a plate (6), it will be found that the series provides the end effects of the rectangular cross section.

BOUNDARY CONDITIONS

The term *boundary condition* has two connotations. First, a region of space in which the magnetic field is required will be bounded by a surface on which information must be specified in order to convey to the interior region information about any field sources outside. Second, when two subregions having different magnetic properties meet (e.g., iron and air), it is necessary to be able to join the solutions on either side of this interface (i.e., to specify *interface conditions*).

In pursuit of the latter, we first consider a Gaussian surface in the shape of a coin that just squeezes a small piece of the interface δs between its two circular faces. Applying Gauss' theorem to the surface of this coinlike volume we have, in terms of the magnetic flux density $\boldsymbol{B}$

$$\oint \boldsymbol{B} \cdot d\boldsymbol{s} = \boldsymbol{B}_2 \cdot \delta \boldsymbol{s} - \boldsymbol{B}_1 \cdot \delta \boldsymbol{s} = 0 \tag{88}$$

where the contribution of the ribbon edge of the coin is negligible, and the positive direction of the magnetic field is taken to be from region 1 to region 2. Thus,

$$B_{n2} = B_{n1} \tag{89}$$

where the suffix n denotes the component of $\boldsymbol{B}$ normal to the interface.

The second condition is found by applying Ampère's law to a small rectangular path that just squeezes the interface between its two longer sides δl so that

$$\boldsymbol{H}_2 \cdot \delta \boldsymbol{l} - \boldsymbol{H}_1 \cdot \delta \boldsymbol{l} = 0 \tag{90}$$

Even though we can state that the tangential component of $\boldsymbol{H}$ is usually continuous across the boundary (i.e., $H_{t2} = H_{t1}$), the tangential component is actually a two-dimensional vector. However, if there is a surface current density $\boldsymbol{K}$ (Am^{-1}) on the boundary, Eq. (90) can be usefully generalized as

$$(\boldsymbol{H}_2 - \boldsymbol{H}_1) \times \boldsymbol{n} = \boldsymbol{K} \tag{91}$$

where, because $\boldsymbol{K}$ and the discontinuity in $\boldsymbol{H}_t$ must be orthogonal, the vector product with the unit vector $\boldsymbol{n}$ gives the direction of $\boldsymbol{K}$.

Considering again the magnetic interface, if a flux line in region 1 enters the surface at an angle θ_1 to the normal and exits into region 2 at an angle θ_2, then Eqs. (89) and (90) can be written

$$B_2 \cos \theta_2 = B_1 \cos \theta_1 \tag{92}$$

$$H_2 \sin \theta_2 = H_1 \sin \theta_1 \tag{93}$$

Dividing Eq. (93) by Eq. (92), and using Eq. (8), gives

$$\frac{\tan \theta_2}{\tan \theta_1} = \frac{\mu_0 \mu_{r2}}{\mu_0 \mu_{r1}} = \frac{\mu_{r2}}{\mu_{r1}} \tag{94}$$

Suppose now that region 2 is very highly permeable and region 1 is air. The right-hand side of Eq. (94) tends to infinity,

and it therefore appears that θ_1 must be very close to 0° (i.e., the flux enters or leaves a very highly permeable surface at right angles). This is indeed so under most conditions, but there are circumstances where θ_1 can be considerably greater than zero even with very high values of μ_{r2}. For example, if a cylindrical tube of constant and high permeability is placed in a uniform field transverse to its axis, the field in the circular cavity is parallel to the applied field and so cannot leave and enter the interior surface everywhere at right angles. The reason is the dominant tangential field within the tube wall, which may cause θ_2 to be very close to 90° over a large part of the surface. If $\mu_{r2} = 500$ and $\theta_2 = 89.7°$, for example, Eq. (94) yields $\theta_1 = 20.9°$ (i.e., considerably greater than would normally be expected on the air side of a highly permeable surface).

We have already seen that a magnetic polarity density q^* can be used to represent the effect of a magnetic surface, and, by Gauss's theorem,

$$q^* = 2\mu_0 H'_{\mathrm{n}} \tag{95}$$

where H'_{n} is the normal component of the magnetic field due to q^*. If the normal component of the applied field due to all other sources is H_{n}, Eqs. (8) and (89) yield

$$\mu_0\mu_{r2}(H_{\mathrm{n}} - H'_{\mathrm{n}}) = \mu_0\mu_{r1}(H_{\mathrm{n}} + H'_{\mathrm{n}})$$

or

$$H'_{\mathrm{n}} = \frac{\mu_{r2} - \mu_{r1}}{\mu_{r2} + \mu_{r1}} H_{\mathrm{n}} \tag{96}$$

from which q^* follows in terms of H_{n}. Region 1 is assumed here to be the region of lower permeability, usually air, and it may be more convenient to express q^* in terms of the total normal component $H_{\mathrm{n1}} = H_{\mathrm{n}} + H'_{\mathrm{n}}$. Thus from Eqs. (95) and (96)

$$q^* = (\mu_{r2} - \mu_{r1})\frac{\mu_0}{\mu_{r2}} H_{\mathrm{n1}} \tag{97}$$

It is important to take care with the use of Eq. (97) because it is an alternative way of representing the magnetized surface without the direct use of permeability to specify the magnetic property of a region of space, and so the solution must be obtained in terms of $\boldsymbol{H}$ or its associated scalar magnetic potential.

External Boundary Conditions

Some problems have open boundaries that extend to infinity in a mathematical sense. Analytically this is not usually difficult, but to deal with such problems numerically may require special techniques. Quite often a problem will have a natural physical boundary on which one of the components of the magnetic field may be zero or can be assumed to be a well -defined function of position. We note that, from Eq. (95), a surface layer of magnetic polarity can cause a discontinuity in the normal component of $\boldsymbol{H}$, and, from Eq. (91), a surface current can set up a discontinuity in the tangential component of $\boldsymbol{H}$.

It is thus possible to convey information about magnetic field sources external to the region of interest via specification of either $\boldsymbol{H}_t$ or $\boldsymbol{H}_n$ over parts or the whole of the selected boundary surface. However specification of $\boldsymbol{H}_n$ is not sufficient without imposing additionally the satisfaction of Ampère's law for the volume concerned. The three-dimensional generalization of Ampère's law is (7)

$$\iint_S (\boldsymbol{n} \times \boldsymbol{H})\, dS = \iiint_v \operatorname{curl} \boldsymbol{H}\, dv = \iiint_v \boldsymbol{J}\, dv \tag{98}$$

In a two-dimensional problem with the current flow perpendicular to the plane of the region, Eq. (98) reduces to

$$\oint H_t\, dl = \iint J\, dS \tag{99}$$

In terms of the scalar magnetic potential φ_{m}, H_{t} corresponds to specification of φ_{m}. This potential specified boundary condition is known as a Dirichlet condition. Alternatively, H_{n} leads to $\partial\varphi_{\mathrm{m}}/\partial n$, the Neumann condition.

In terms of the magnetic vector potential $\boldsymbol{A}$, $\boldsymbol{n} \times \operatorname{curl} \boldsymbol{A}$ or $\boldsymbol{n} \times \boldsymbol{A}$ respectively are required, which reduce, in the two-dimensional case where the only component of $\boldsymbol{A}$ (A_{p} say) is perpendicular to the two-dimensional plane, to the Dirichlet specification A_{p} (for H_{n}) and the Neumann specification $\partial A_{\mathrm{p}}/\partial n$ (for H_{t}).

TUBES AND SLICES

The method of *tubes and slices* is based on a dual-energy formulation. Foundations of this approach may be traced back to Maxwell, who describes a variational method applied to the calculation of resistance of conductors of varying cross section in his famous treatise on electricity and magnetism (3). The method relies on subdivision of the conductor into *slices* formed by equipotential surfaces and *tubes* separated by very thin insulating sheets. The two calculations yield lower and upper bounds of the resistance, respectively. The approach is applicable to other types of vector fields. For example, in magnetostatics, the equilibrium conditions can be described by two variational principles

$$\langle(\nabla \times \boldsymbol{H} - \boldsymbol{J}'), \delta\boldsymbol{A}\rangle = 0 \tag{100}$$

and

$$\langle(\nabla \times \boldsymbol{A} - \boldsymbol{B}), \delta\boldsymbol{H}\rangle = 0 \tag{101}$$

where $\boldsymbol{J}'$ is the assigned current density. The first variational principle implies that $\boldsymbol{B} = \nabla \times \boldsymbol{A}$, so that $\nabla \cdot \boldsymbol{B} = 0$ and thus there are no divergence sources for the magnetic field. However, the expression $\nabla \times \boldsymbol{H} - \boldsymbol{J}'$ allows a small variation in $\nabla \times \boldsymbol{H}$ from its correct value, so that the variation allows a small additional distribution of curl sources. The product of this small fictitious current multiplied by the small variation of $\boldsymbol{A}$ gives an energy variation of the second order of small quantities which can be put to zero.

The second variational principle assumes that $\nabla \times \boldsymbol{H} = \boldsymbol{J}'$, so that the curl sources of the magnetic field are correct. However, the expression $\nabla \times \boldsymbol{A} - \boldsymbol{B}$ allows a small variation in the divergence sources. The product of this small polarity distribution multiplied by the small variation of $\boldsymbol{H}$ gives an

energy variation of the second order of small quantities which can be put to zero. The field energy can be expressed either in terms of the field vectors $\boldsymbol{H}$ and $\boldsymbol{B}$ by

$$U = \frac{1}{2}\langle \boldsymbol{B}, \boldsymbol{H} \rangle \tag{102}$$

or in terms of the interaction of the current sources with the vector potential $\boldsymbol{A}$ by

$$U = \frac{1}{2}\langle \boldsymbol{J}', \boldsymbol{A} \rangle + \frac{1}{2}[\boldsymbol{I}', \boldsymbol{A}] \tag{103}$$

where $\boldsymbol{I}'$ is the assigned line density of current on the surface, and the brackets [ ] represent integration over the closed boundary surface. $\boldsymbol{I}'$ is related to $\boldsymbol{J}'$ in order to make the total current in the system zero. This isolates the system and gives it a unique energy.

The first variational principle is applied to the energy in terms of $\boldsymbol{A}$ and $\boldsymbol{B} = \nabla \times \boldsymbol{A}$ by writing

$$\delta U(\boldsymbol{A}) = \delta\left(\langle \boldsymbol{J}', \boldsymbol{A} \rangle + [\boldsymbol{I}', \boldsymbol{A}] - \frac{1}{2}\left\langle \boldsymbol{B}, \frac{\boldsymbol{B}}{\mu} \right\rangle\right) = 0 \tag{104}$$

The second variation is therefore negative

$$\delta^2 U(\boldsymbol{A}) \leq 0 \tag{105}$$

The second variational principle is applied to the energy in terms of $\boldsymbol{H}$ by writing

$$\delta U(\boldsymbol{H}) = \delta\left(\frac{1}{2}\langle \boldsymbol{H}, \mu \boldsymbol{H} \rangle\right) = 0 \tag{106}$$

Hence

$$\delta^2 U(\boldsymbol{H}) \geq 0 \tag{107}$$

For simplicity, μ has been assumed constant and this gives the factor $\frac{1}{2}$. However, the method is applicable to permeabilities that are single-valued functions of the field strength. The second variations show the possibility of obtaining both upper and lower bounds for the energy. The first variational principle treats the field as a collection of tubes and the second one as a collection of slices.

In many electrical devices, the magnetic circuit is designed in such a way that very little mmf is absorbed in the iron core and attention is focused on the shape and dimensions of the air gap. The core provides a path for transporting the magnetic flux from the place it is produced (current in the coil) to the place where it can be used (air gap), which may be well away from the winding. An unsaturated iron surface may be assumed to have a constant scalar magnetic potential and thus forms one edge of a slice. The whole air-gap region may be enclosed using a pair of slices (iron surfaces) and at the same time the flux distribution may be described in terms of tubes. Those tubes terminate on iron surfaces. For convenience, we shall work in terms of permeance (which is a reciprocal of reluctance). Permeance in the magnetic circuit is the analogue of conductance in the electric circuit. A parallel-sided tube of flux, of length δl and cross section δS, may be seen to have permeance given by

$$\lambda = \mu \frac{\delta S}{\delta l} \tag{108}$$

If there are m such pieces along a tube, these permeances are in series and are given by

$$\frac{1}{\lambda} = \sum_1^m \frac{\delta l}{\mu \delta S} \tag{109}$$

If there are n tubes in parallel the total permeance is

$$\lambda = \sum_1^n \frac{1}{\sum_1^m \dfrac{\delta l}{\mu \delta S}} \tag{110}$$

If instead we work with the slices, we have for n slices in parallel

$$\lambda = \sum_1^n \frac{\mu \delta S}{\delta l} \tag{111}$$

and for m slices in series

$$\lambda = \frac{1}{\sum_1^m \dfrac{1}{\sum_1^n \dfrac{\mu \delta S}{\delta l}}} \tag{112}$$

The subdivision into tubes produces the lower bound, whereas slices give the upper bound of permeance. Furthermore, for problems outside a current region, the calculation of inductance is reduced to a calculation of permeance. In Eqs. (109)–(112) we may substitute L for λ. The method of tubes and slices is very simple in use and does not require the solution of large systems of equations. It can also be applied to calculation of resistance or capacitance. More details may be found in Refs. 1 and 4.

FINITE DIFFERENCES

The finite difference method (FD), although overshadowed nowadays by the more versatile finite element method introduced in the next section, continues to play an important role in numerical analysis (1,4,6). One of the significant advantages of the finite difference method is the simplicity of its formulation, in both a mathematical and a numerical sense. A basic FD scheme could almost be set up intuitively and a simple computer program for a particular solution could be written even by an inexperienced programmer in a matter of minutes.

Consider Laplace's equation in two dimensions in some arbitrarily shaped region with boundary conditions specified. We will overlay the problem with a mesh of lines "parallel" to the coordinate system used (a rectangular grid is convenient, although not essential), and we will seek an approximate solution at mesh points defined by intersections of the lines. With reference to Fig. 5, we consider the mesh point i, j and

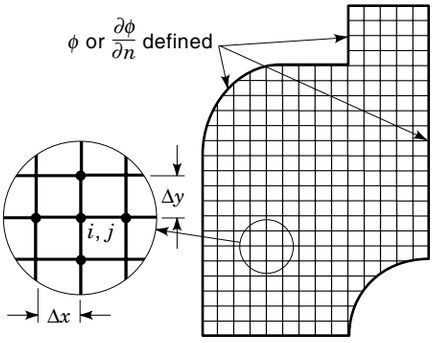

Figure 5. A finite difference mesh.

its immediate neighbors, and by using Taylor's series we can write

$$\varphi_{i+1,j} = \varphi_{i,j} + \Delta x \frac{\partial \varphi}{\partial x}\bigg|_{i,j} + \frac{(\Delta x)^2}{2!}\frac{\partial^2 \varphi}{\partial x^2}\bigg|_{i,j} + \frac{(\Delta x)^3}{3!}\frac{\partial^3 \varphi}{\partial x^3}\bigg|_{i,j} + \cdots \tag{113}$$

and

$$\varphi_{i-1,j} = \varphi_{i,j} - \Delta x \frac{\partial \varphi}{\partial x}\bigg|_{i,j} + \frac{(\Delta x)^2}{2!}\frac{\partial^2 \varphi}{\partial x^2}\bigg|_{i,j} - \frac{(\Delta x)^3}{3!}\frac{\partial^3 \varphi}{\partial x^3}\bigg|_{i,j} + \cdots \tag{114}$$

Adding the last two equations yields

$$\varphi_{i+1,j} + \varphi_{i-1,j} = 2\varphi_{i,j} + (\Delta x)^2 \frac{\partial^2 \varphi}{\partial x^2}\bigg|_{i,j} + O\{(\Delta x)^4\} \tag{115}$$

where $O\{(\Delta x)^4\}$ represents terms containing fourth- and higher-order powers of Δx. Neglecting these terms, we have

$$\frac{\partial^2 \varphi}{\partial x^2}\bigg|_{i,j} = \frac{\varphi_{i+1,j} - 2\varphi_{i,j} + \varphi_{i-1,j}}{(\Delta x)^2} \tag{116}$$

Similarly, under the same assumptions,

$$\frac{\partial^2 \varphi}{\partial y^2}\bigg|_{i,j} = \frac{\varphi_{i,j+1} - 2\varphi_{i,j} + \varphi_{i,j-1}}{(\Delta y)^2} \tag{117}$$

If these finite difference expressions are now substituted into Laplace's equation, the following local approximation is obtained

$$\frac{1}{(\Delta x)^2}(\varphi_{i+1,j} - 2\varphi_{i,j} + \varphi_{i-1,j}) + \frac{1}{(\Delta y)^2}(\varphi_{i,j+1} - 2\varphi_{i,j} + \varphi_{i,j-1}) = 0 \tag{118}$$

If for convenience we choose a square mesh, so that $\Delta x = \Delta y$,

$$\varphi_{i-1,j} + \varphi_{i+1,j} + \varphi_{i,j-1} + \varphi_{i,j+1} - 4\varphi_{i,j} = 0 \tag{119}$$

or

$$\varphi_{i,j} = \frac{1}{4}\{\varphi_{i-1,j} + \varphi_{i+1,j} + \varphi_{i,j-1} + \varphi_{i,j+1}\} \tag{120}$$

which is a well-known five-point scheme for Laplace's equation. The graphical representation is shown in Fig. 6. It is interesting to notice that in this scheme the value at any node is taken as the average of the four values of its immediate neighbors. In this sense the scheme could be considered intuitive, but we have demonstrated that the Laplace's equation is in fact satisfied, subject to errors introduced by neglecting higher-order terms in the Taylor's series. These errors are due to the finite mesh size and are called *discretization* or *truncation errors*.

Equation (120) suggests a possible simple scheme for obtaining the solution by "scanning" all nodes iteratively. Boundary nodes must have appropriate boundary conditions assigned, and then the information about outside sources of the field, as provided by boundary conditions, will gradually spread to the interior of the region through successive application of Eq. (120).

The finite difference method can readily be applied to the other differential equations met in electromagnetism. For example, the diffusion equation, in one space dimension, is of the form

$$\frac{\partial^2 \varphi}{\partial x^2} = \alpha \frac{\partial \varphi}{\partial t} \tag{121}$$

The time derivative can be obtained with the aid of Taylor's series:

$$\frac{\partial \varphi}{\partial t} = \frac{\varphi_{i,k+1} - \varphi_{i,k}}{\Delta t} \tag{122}$$

where Δt is the time interval between successive values of φ_i appearing at the space node i, and the suffix k denotes the time variable. It may be helpful to think of the solution as marching forward in time, each step progressing Δt, generating the electromagnetic transient as it goes.

Hence, it follows that

$$\frac{\varphi_{i+1,k} - 2\varphi_{i,k} + \varphi_{i-1,k}}{(\Delta x)^2} = \alpha \frac{\varphi_{i,k+1} - \varphi_{i,k}}{\Delta t} \tag{123}$$

where i denotes a space variable and Δx is the distance between adjacent nodes. Examination of Eq. (123) reveals that we can obtain one point ahead in time from the values of the previous time row. Let us assume that the boundary conditions are such that $\varphi(x, t)$ is given for all t at $x = 0$ and $x = L$, and that the initial condition $\varphi(x, 0)$ is specified for all x. A suitable computation scheme is shown in Fig. 7.

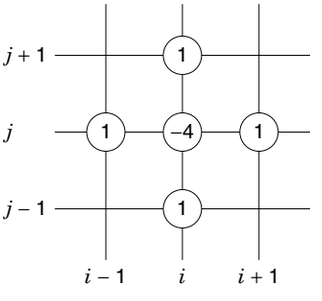

Figure 6. A five-point computation scheme.

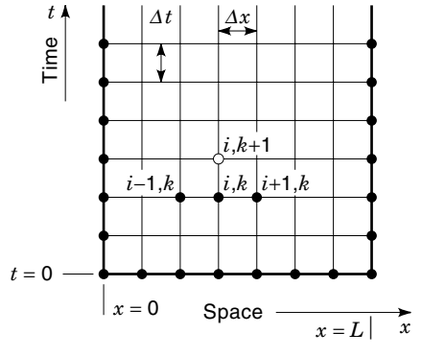

Figure 7. FD scheme for diffusion equation.

The solution moves forward in time, row by row. This is known as an explicit finite difference scheme. Other formulations are also possible, where the solution at each time row is not calculated explicitly, but more than one unknown value is related to several known values in one equation. A very popular implicit scheme is the Crank–Nicolson method (6). Implicit schemes are often solved by iteration. Note also that special equations are required for nodes positioned next to the boundaries, even in the explicit scheme. Finally, any scheme must be compatible with the differential equation and must be stable. Compatibility ensures that the numerical solution converges to the solution of the original equation. Stability may be lost if the various errors accumulate without limit during the computation. Stability conditions can be derived for some equations. Thus for Eq. (123), stability is obtained only if

$$\frac{\Delta t}{\alpha(\Delta x)^2} \leq \frac{1}{2} \tag{124}$$

Compatibility and stability may not always be easy to ensure, and they are important issues. Some implicit schemes are unconditionally stable. A more detailed discussion of these issues may be found in Refs. 4 and 6.

A major difficulty with the finite difference method is caused by the fixed topology (both order and arrangement) of the discretization scheme. It becomes very difficult to match highly irregular boundaries with an appropriate mesh or grid. At the same time, material interfaces, symmetry conditions, and nonlinear characteristics all require special treatment. Another difficulty arises when higher-order terms from Taylor's series are to be introduced to improve the accuracy. Although several special algorithms have been developed, they do not offer as much versatility as the finite element method.

FINITE ELEMENTS

Consider Laplace's equation in two dimensions for a magnetostatic system

$$\nabla^2 V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = 0 \tag{125}$$

where the magnetic field $\boldsymbol{H}$ is given by

$$\boldsymbol{H} = -\text{grad}\, V = -\nabla V = -\left\{ \frac{\partial V}{\partial x}\,\hat{i} + \frac{\partial V}{\partial y}\,\hat{j} \right\} \tag{126}$$

We can apply a variational principle to the defining equation by stating that the principle of equilibrium requires that the potential distribution must be such as to minimize the stored field energy. In our case, this energy can be expressed as

$$\begin{aligned} W &= \int_\Omega \frac{1}{2}\boldsymbol{H}\cdot\boldsymbol{B}\, d\Omega = \frac{1}{2}\int_\Omega \mu H^2\, d\Omega \\ &= \frac{1}{2}\int_\Omega \mu \left(\left[\frac{\partial V}{\partial x}\right]^2 + \left[\frac{\partial V}{\partial y}\right]^2 \right) d\Omega = \frac{1}{2}\int_\Omega \mu |\nabla V|^2\, d\Omega \end{aligned} \tag{127}$$

where integration is carried out over the two-dimensional problem region and is thus taken per unit length. This minimum-energy principle is mathematically equivalent to our original differential equation in the sense that a potential distribution that satisfies Laplace's equation will also minimize the energy, and vice versa.

An alternative formulation that avoids using energy functionals is also possible. It is based on the so-called Galerkin procedure and the method of weighted residuals. The Galerkin procedure is generally easier to apply and leads to a wider class of applications. However, its mathematical formulation is more advanced, and it will not be pursued here. For Laplace's equation, both formulations give identical results.

Consider a single element, and assume that the potential distribution within the element is adequately approximated by the expression

$$V = a + bx + cy + dxy + ex^2 + fy^2 + \cdots \tag{128}$$

We choose as many terms in Eq. (128) as there are "nodes" in the element. Figure 8 shows some examples.

For a rectangle, we choose

$$V = a + bx + cy + dxy \tag{129}$$

For the first-order triangle, we choose

$$V = a + bx + cy \tag{130}$$

In the last case the representation is said to be complete because Eq. (130) contains all the terms necessary for a linear variation in two dimensions. We shall not pursue the higher-order elements, but it is easily seen that finite elements, unlike finite differences, offer a very natural extension to higher-order modeling.

For the three vertices (nodes) of the triangle of Fig. 8(a) the potential assumes the following values:

$$\begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix} = \begin{bmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} \tag{131}$$

where (x_1, y_1), (x_2, y_2), and (x_3, y_3) are the coordinates of the vertices and the determinant of the coefficient matrix in Eq. (131) may be recognized as equal to twice the area of the triangle (A). Rearranging Eq. (131) gives

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{bmatrix}^{-1} \begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix} \tag{132}$$

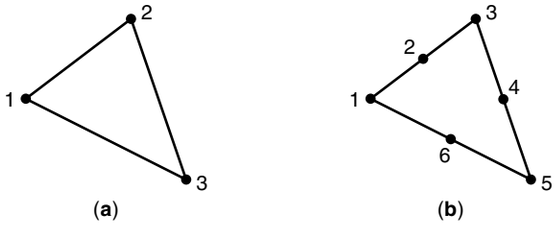

Figure 8. Finite elements; (a) first-order triangle, (b) second-order triangle, (c) rectangle.

and substituting back to Eq. (130) yields

$$V = [1 \quad x \quad y] \begin{bmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{bmatrix}^{-1} \begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix} \tag{133}$$

The last equation may be written as

$$V = \sum_{i=1}^{3} V_i \alpha_i(x, y) \tag{134}$$

where

$$\alpha_1 = \frac{1}{2A}\{(x_2 y_3 - x_3 y_2) + (y_2 - y_3)x + (x_3 - x_2)y\} \tag{135}$$

$$\alpha_2 = \frac{1}{2A}\{(x_3 y_1 - x_1 y_3) + (y_3 - y_1)x + (x_1 - x_3)y\} \tag{136}$$

$$\alpha_3 = \frac{1}{2A}\{(x_1 y_2 - x_2 y_1) + (y_1 - y_2)x + (x_2 - x_1)y\} \tag{137}$$

At the vertices,

$$\begin{aligned} \alpha_i(x_j, y_j) &= 0 \quad i \neq j \\ &= 1 \quad i = j \end{aligned} \tag{138}$$

that is, each function vanishes at all vertices but one, where it assumes the value of one.

We can now associate energy with each element, and remembering that in the two-dimensional field this energy will be taken per unit length, we write

$$W^{(e)} = \frac{1}{2}\int_e \mu |\nabla V|^2 \, dS \tag{139}$$

where integration is performed over the element area. The potential gradient within the element is found from Eq. (134) as

$$\nabla V = \sum_{i=1}^{3} V_i \nabla \alpha_i(x, y) \tag{140}$$

so that the element energy becomes

$$W^{(e)} = \frac{1}{2}\mu \sum_{i=1}^{3}\sum_{j=1}^{3} V_i \left(\int_e \nabla \alpha_i \cdot \nabla \alpha_j \, dS\right) V_j \tag{141}$$

Equation (141) may be written in the following compact form

$$W^{(e)} = \frac{1}{2}\mu [V]^T [N]^{(e)} [V] \tag{142}$$

where $[V]$ is the vector of vertex values of potential, the superscript T denotes transposition, and the 3×3 square element matrix $[N]^{(e)}$ is defined by

$$N_{i,j}^{(e)} = \int_e \nabla \alpha_i \cdot \nabla \alpha_j \, dS \tag{143}$$

For any given triangle, the matrix $[N]$ is readily evaluated with a typical expression of the form

$$\begin{aligned} N_{11}^{(e)} &= \frac{1}{4A}\{(y_2 - y_3)^2 + (x_3 - x_2)^2\} \\ N_{12}^{(e)} &= \frac{1}{4A}\{(y_2 - y_3)(y_3 - y_1) + (x_3 - x_2)(x_1 - x_3)\} \\ N_{13}^{(e)} &= \text{etc.} \end{aligned} \tag{144}$$

and other entries can be obtained by cyclic permutation of suffices.

This completes the specification for an arbitrary element in the finite-element mesh. The total energy associated with the entire region will be found as the sum of individual element energies

$$W = \sum W^{(e)} \tag{145}$$

for all elements. When assembling such elements, we immediately notice that some nodes will be shared between more than one element, as shown in Fig. 9, and thus the topology of the actual mesh will directly affect the way in which the global matrix is formulated. In other words, the global node numbering must be related to the local numbering, and the global matrix must reflect the way in which individual elements are linked to global nodes.

In order to minimize the total energy expression, Eq. (145) must be differentiated with respect to a typical value of V_k

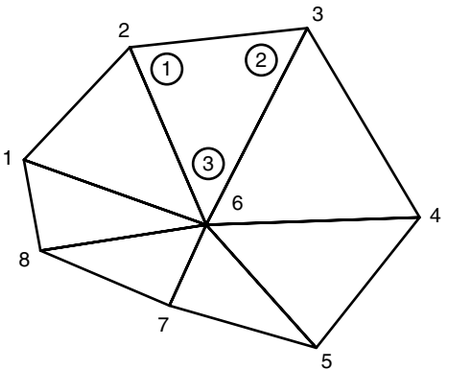

Figure 9. Global and local node numbering.

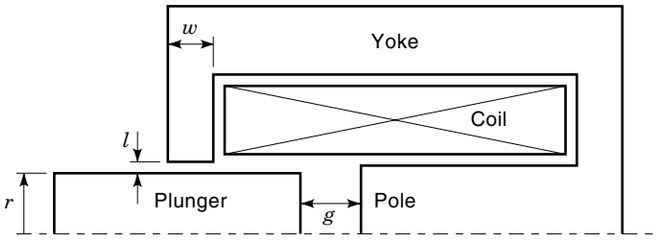

Figure 10. A simple linear magnetic actuator.

and equated to zero. Thus

$$\frac{\partial W}{\partial V_k} = 0 \tag{146}$$

where the index k refers to node numbers in the global numbering scheme. In a typical boundary-value problem, some boundary segments will have specified potential values. Thus a subset of the node potentials contained in the vector $[V]$ will assume exactly those prescribed values. To simplify the discussion, we will assume that we have numbered the nodes that are free to vary as first, thus leaving all nodes with prescribed potential as last. This is not strictly necessary but makes explanation easier. It allows us to rewrite Eq. (146) with the matrices in partitioned form

$$\frac{\partial W}{\partial V_k} = \frac{\partial}{\partial [V_\mathrm{f}]_k}[[V_\mathrm{f}]^T [V_\mathrm{p}]^T] \begin{bmatrix} [N_\mathrm{ff}] & [N_\mathrm{fp}] \\ [N_\mathrm{pf}] & [N_\mathrm{pp}] \end{bmatrix} \begin{bmatrix} [V_\mathrm{f}] \\ [V_\mathrm{p}] \end{bmatrix} = 0 \tag{147}$$

where the subscripts f and p refer to nodes with free and prescribed potentials, respectively. Note that the prescribed potentials cannot vary, and thus differentiation with respect to them is not possible. Hence differentiation with respect to the free potentials results in the following matrix equation:

$$[[N_\mathrm{ff}][N_\mathrm{fp}]] \begin{bmatrix} [V_\mathrm{f}] \\ [V_\mathrm{p}] \end{bmatrix} = 0 \tag{148}$$

and leads to a system of algebraic equations of the form $Ax = b$, namely

$$[N_\mathrm{ff}][V_\mathrm{f}] = -[N_\mathrm{fp}][V_\mathrm{p}] \tag{149}$$

which has a formal solution

$$[V_\mathrm{f}] = -[N_\mathrm{ff}]^{-1}[N_\mathrm{fp}][V_\mathrm{p}] \tag{150}$$

In practice, the number of unknown potentials may be very large, even tens or hundreds of thousands, and special methods for solving large systems of equations are used. Sparse matrices and compact storage schemes are usually employed in combination with iterative or conjugate gradient techniques (4,5).

The finite-element method overcomes the main difficulties of the finite-difference technique, in particular accurate matching of irregular boundary shapes and higher-order approximations, and offers more flexibility.

LINEAR MAGNETIC ACTUATOR

A simple form of linear actuator is shown in Fig. 10, where a cylindrical plunger is attracted toward the fixed circular pole face when the coil is excited.

If the magnetic circuit is highly permeable and leakage and fringing flux are ignored, it can be shown using Maxwell stresses (4) that the axial force on the plunger is given by

$$F = \frac{B_\mathrm{g}^2 \pi r^2}{2\mu_0} \tag{151}$$

The flux density B_g passing normally between plunger and pole surfaces can be simply found as a function of the displacement (air gap) g if the iron is assumed to be infinitely permeable (i.e., the iron path has zero reluctance). Thus, Eq. (13) gives

$$\frac{gB_\mathrm{g}}{\mu_0} + \frac{lB_\mathrm{c}}{\mu_0} = Ni \tag{152}$$

where l is the clearance gap or thickness of the nonmagnetic guide sleeve that surrounds the plunger (not shown in Fig. 10). Similarly, continuity of flux requires

$$2\pi r w B_\mathrm{c} = \pi r^2 B_\mathrm{g}$$

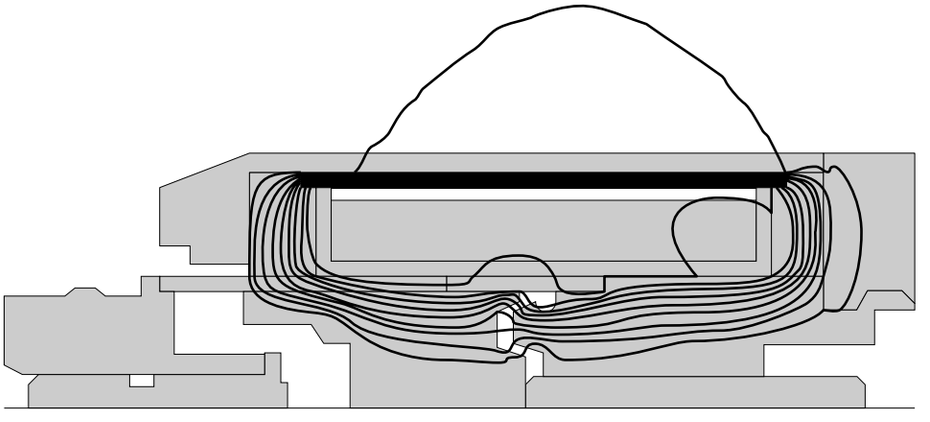

Figure 11. Flux plot in a typical linear actuator obtained using finite element modeling.

or

$$B_c = B_g \frac{r}{2w} \tag{153}$$

Solving Eqs. (152) and (153) for B_g gives

$$B_g = \frac{\mu_0 N i}{\left(g + \dfrac{lr}{2w}\right)} \tag{154}$$

so that the force in Eq. (151) is now fully defined. If g dominates the term in parenthesis in Eq. (154), the force/displacement curve is a function of $1/g^2$.

Magnetic saturation will affect the real situation, especially for small g, and also fringing flux, so that a full solution must be obtained using finite elements. Also the plunger and pole profiles are not usually flat surfaces, and an example of a realistic solution is shown in Fig. 11.

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MAGNETIC CONFINEMENT.

See FUSION PLASMAS.

MAGNETIC EPITAXIAL LAYERS