MAGNETIC FLUX

The magnetic flux is related to the number of magnetic lines of force crossing a given area. It is therefore analogous to the flux of a flowing quantity. The magnetic flux is defined as the integral of the product of an elemental area and the magnetic induction perpendicular to it. If the magnetic induction is constant over a given area, then the flux is the product of the magnetic induction and the area. The magnetic induction is referred to as the magnetic flux density, since it is the magnetic flux per unit area.

Since the flux represents the number of lines of force and the lines of force are fixed in a infinitely conducting medium, the flux is conserved in an ideal conducting medium. Finite high-conductivity media such as copper and high-temperature plasma conserve (enclosed) flux when an external field change occurs or the cross section is changed over a time short compared to the time scale in which the lines of force (and the magnetic field) can diffuse across the medium.

The magnetic flux is related to the energy stored in the magnetic field and represents the capability of a primary magnet to induce voltage in a coupled secondary circuit over a time duration. When a magnet current is changed the flux (magnetic induction) changes, and by Lenz's law (discussed later) it induces a voltage in a secondary circuit. The magnitude of the voltage depends upon the rate of change of flux. The duration over which this voltage can be maintained is thus proportional to the flux:

$$\Phi = \int (\boldsymbol{B} \cdot d\boldsymbol{S})$$

where the integral of the magnetic flux density B is over the area of interest S. Since the surface area vector is normal to the surface, the integral gives the flux that intersects the surface. If B is constant over S,

$$\Phi = BS$$

Since the flux density is related to the magnetic field intensity H by the relation $B = \mu_r \mu_0 H$, where μ_r is the relative permeability of the medium and μ_0 is the permeability of vacuum,

$$\Phi = \mu_0 \int (\mu_r \boldsymbol{H} \cdot d\boldsymbol{S})$$

The MKS unit for magnetic flux is Tesla = m^2 or Webers or volt-seconds.

Magnetic Flux and Vector Potential

The vector potential is a quantity closely related to flux and is defined as

$$\boldsymbol{B} = \operatorname{curl} \boldsymbol{A}$$

The flux is then given by

$$\Phi = \int (\operatorname{curl} \boldsymbol{A} \cdot d\boldsymbol{S})$$

Applying Stoke's theorem,

$$\Phi = \int (\boldsymbol{A} \cdot d\boldsymbol{l})$$

where the integral is over the closed loop enclosing the surface area of interest. Therefore the flux is the line integral of the vector potential around the perimeter of the area of interest. For example, for a coil which has only an azimuthal component A_θ, the flux enclosed by a circle of radius r is $2\pi r A_\theta$. The equation is commonly used to create flux plots (lines of force) by using constant-vector-potential lines and to calculate the flux enclosed by a given area, since many electromagnetic and magnetostatic problems are solved by solving for the vector potential.

Magnetic Flux and Inductance

The inductance L of an electrical circuit depicts the ability of the circuit to oppose a change of current in its own circuit or a mutually (magnetically) coupled circuit. More fundamentally it is the ability of a circuit to oppose the change in magnetic flux enclosed by the current circuit. The flux enclosed by a circuit is proportional to the current in the circuit or the circuit that is mutually coupled to it, and the proportionality constant is the self-inductance L or the mutual inductance M, or

$$\Phi = LI \qquad \text{or} \qquad \Phi' = MI'$$

where I is the current in the circuit whose flux is of interest and I' is the current in the mutually coupled circuit. The mutual inductance of two circuits with self-inductance L_i and L_j is given by

$$M_{ij} = K(L_i L_j)^{1/2}$$

where K is the coupling coefficient. The mutual inductance may be positive or negative.

For a coil with N turns, the total inductance is given by

$$L_{\text{total}} = \sum_{i=1}^{N} \sum_{j=1}^{N} (L_i + M_{ij})$$

If the turns are identical and fully coupled ($K = 1$) to each other,

$$L_{\text{total}} = \sum_{i=1}^{N} [L_i + (N-1)L_i] = N^2 L_i = N^2 \Phi / I$$

where Φ is the flux due to one turn when a current I passes through it. Since the flux Φ_N induced by N turns is $N\phi$,

$$L_{\text{total}} = N\Phi_N / I$$

Magnetic Energy Density

The volume permeated by a magnetic field stores energy and therefore any device which generates a magnetic field also stores energy. The energy density associated with the magnetic field of a region is given by

$$u = (\boldsymbol{B} \cdot \boldsymbol{H})/8 = B^2/(8\mu_r\mu_0)$$

so that the energy stored in a flux tube cylinder with a cross sectional area A (perpendicular to $\boldsymbol{B}$) and length l is

$$E_f = \Phi^2 l/(8\mu_r\mu_0 A)$$

Lenz's Law and Flux Conservation

As stated previously, an electrical circuit such as a loop of wire or a metallic cylinder opposes a change in the flux enclosed by itself. A voltage is induced in the circuit in a direction such that the voltage can drive a current that opposes the change in the flux. Therefore Lenz's law states that the induced voltage is given by the time rate of change of the flux, that is,

$$V = -d\Phi/dt \tag{1}$$

The voltage drives a current I given by

$$V = L\,dI/dt + IR \tag{2}$$

where L and R are the inductance and resistance of the circuit. Equating Eqs. (1) and (2) and integrating over time,

$$LI + \Phi = \text{const} - \int (IR\,dt)$$

Now, the left-hand side is the total flux (sum of the initial flux and the induced flux). Therefore the flux is conserved if the electrical circuit has zero resistance, that is, for an ideal electrical circuit such as a loop of conductor or a cylinder with zero resistance, the flux enclosed by it does not change when the flux density (magnetic field) or the area enclosed by the circuit is changed (Fig. 1). Analogously, the flux enclosed by a circuit is conserved if the current or the inductance of the circuit is changed. However, in nonideal conductors with nonzero resistance the current induced by the changing flux would decay with a time constant of L/R and the flux would change with the same time constant.

In mutually coupled circuits Eq. (2) is modified to include voltage induced by the mutually inductance. Therefore for two circuits p and q,

$$V_p = L_p\,dI_p/dt + M_{pq}\,dI_q/dt + I_p R_p$$

and the same flux conservation concept would apply if the resistance is zero, that is,

$$L_p I_p + M_{pq} I_q + \Phi = \text{const when } R_p = 0$$

The currents induced for the conservation of flux are called diamagnetic (1) or eddy currents and in resistive conductors, such currents cause losses in the conductors when the field (flux) is changed.

It can be easily shown that a diamagnetic material placed inside a coil reduces the inductance of the coil (the total magnetic flux in the coil is reduced), while a paramagnetic and ferromagnetic material placed inside a coil increases its inductance.

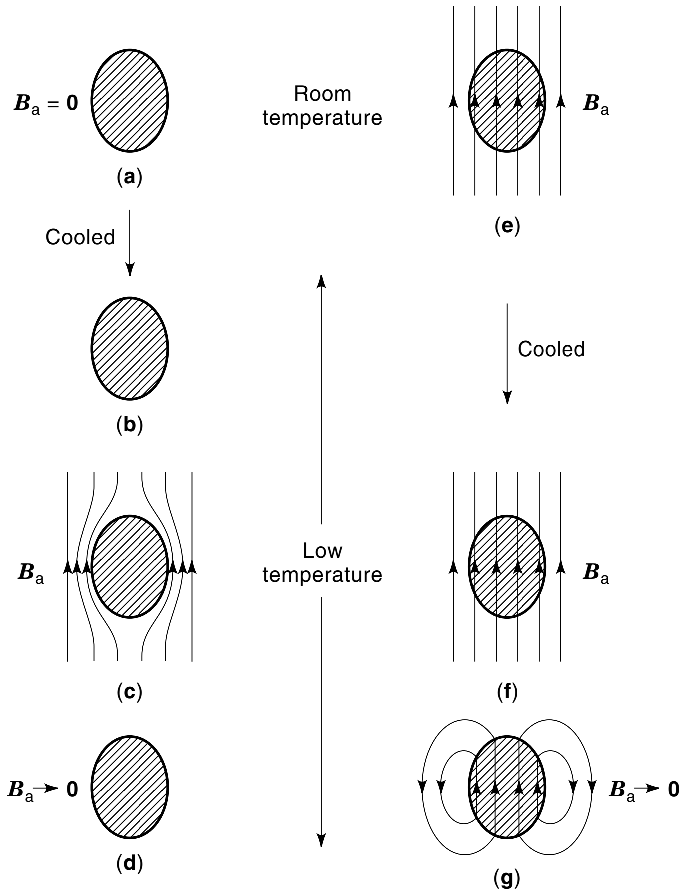

Figure 1. Magnetic behavior of a "perfect" conductor. (a) and (b) Specimen becomes resistanceless in absence of field. (c) Magnetic field applied to resistanceless specimen. (d) Magnetic field removed. (e) and (f) Specimen becomes resistanceless in applied magnetic field. (g) Applied magnetic field removed (subscript 'a' refers to applied field) (Ref. 4).

Poynting Flux

When the magnetic field is not constant in time, by Maxwells Law

$$\mathrm{curl}\,\boldsymbol{E} = -d\boldsymbol{B}/dt$$

Therefore, an electric field $\boldsymbol{E}$ is always associated with a time varying magnetic field. The medium therefore stores both electric and magnetic energy and this energy is, in general, time dependent and in addition, as the changing fields penetrate the volume, there may be energy dissipation in volume V due to resistive currents driven by the electric field at a rate given by

$$dE_{\mathrm{dis}}/dt = \mathrm{Int}[\boldsymbol{J}\cdot\boldsymbol{E}]\,dV$$

where $\boldsymbol{J}$ is the current density in the volume. Maxwells Law can be written as

$$dE_{\mathrm{dis}}/dt = \int[\boldsymbol{E}\cdot\mathrm{curl}\,\boldsymbol{H} - \boldsymbol{E}\cdot dD/dt]\,dV$$

where $\boldsymbol{D}$ is the electric displacement vector. Using vector identity and assuming linear properties, this can be written as

$$dE_{\mathrm{dis}}/dt = \int[\boldsymbol{J}\cdot\boldsymbol{E}]\,dV = -\int[du/dt + \mathrm{div}(\boldsymbol{E}\times\boldsymbol{H})]$$

where now the energy density of the region,

$$u = (\boldsymbol{E}\cdot\boldsymbol{D} + \boldsymbol{B}\cdot\boldsymbol{H})/2$$

In differential form this leads to the energy conservation equation

$$du/dt + \mathrm{div}(\boldsymbol{E}\times\boldsymbol{H}) + \boldsymbol{J}\cdot\boldsymbol{E} = 0$$

The first term represents the rate of change of the energy density, the term $\boldsymbol{S} = (\boldsymbol{E}\times\boldsymbol{H})$ represents energy flow in or out of the volume and the last term represents the energy dissipation. $\boldsymbol{S}$ is called the Poynting vector and is particularly relevant to electromagnetic fields and waves.

Flux Penetration and Diffusion

As stated in the previous sections, if a magnetic field is applied to (or changed on) the exterior of a material (the flux enclosed by the area of the material is changed), the material gets an induced voltage that drives diamagnetic currents opposing the change in flux. If the material has a finite resistance, the currents will then decay and the flux will penetrate into the material. The flux will penetrate diffusively much like the diffusion of heat over time. The following relations illustrate this phenomenon.

Maxwells law gives

$$\mathrm{curl}\,\boldsymbol{E} = -\partial\boldsymbol{B}/\partial t$$
$$\mathrm{curl}\,\boldsymbol{H} = \boldsymbol{J} + \frac{\partial\boldsymbol{D}}{\partial t}$$

and Ohm's law gives

$$\boldsymbol{E} = \rho\boldsymbol{J}$$

where $\boldsymbol{E}$ and $\boldsymbol{J}$ are the induced electric field and (eddy) current density, respectively, and ρ is the resistivity of the material.

Therefore,

$$\mathrm{curl}\,(\rho/\mu)\,\mathrm{curl}\,\boldsymbol{B} = -\frac{\partial\boldsymbol{B}}{\partial t}$$

Since div $\boldsymbol{B} = \boldsymbol{0}$ and for uniform resistivity ρ,

$$(\rho/\mu)\nabla^2\boldsymbol{B} = \partial\boldsymbol{B}/\partial t$$

which is the equation for the diffusion of the flux into the material and $\rho/\mu_0 = D_{\mathrm{m}}$ is the magnetic diffusion coefficient. The flux and the field diffuse into the material thickness of l in a time given by l^2/D_{m}.

If the conducting material is in motion such as a moving plasma, an additional induced electric (convective) field $\boldsymbol{v}\times\boldsymbol{B}$ is present. The net time rate of change of the magnetic field is given by

$$d\boldsymbol{B}/dt = (\rho/\mu)\nabla^2\boldsymbol{B} + \mathrm{curl}(\boldsymbol{v}\times\boldsymbol{B})$$

and in analogy with viscous flow, a "magnetic" Reynolds number can be defined as

$$R_{\mathrm{M}} = Lv/D_{\mathrm{m}}$$

where L is the characteristic dimension of the flow. The magnetic Reynold's number can vary from a value much less than 1 for laboratory devices to values on the order of 100 for fusion plasmas, while for geophysical or astronomical conditions, R_{m} can be as high as 10^6 to 10^{10}. Therefore the flux can be diffused by the flow as it penetrates the conducting material. This convective flow can be a mechanism for converting one type of flux into another (see the section entitled "Flux Conversion").

FLUX LINE AND FLUX TUBE

The flux lines are directed lines of force (LOF) and lie in the direction that a north (mono-) pole would point to when placed in the magnetic field. The LOF is defined by the equation

$$dx/B_x = dy/B_y = dz/B_z$$

where B_x, B_y, and B_z are components of the flux density in the directions x, y, and z. The equation may be integrated to give surfaces of the type (2)

$$f(x, y, z) = a$$
$$g(x, y, z) = b$$

the intersection of which gives a specific line of force. In this case, the local unit vector of the LOF is given by

$$\boldsymbol{k} = \text{grad}\,\boldsymbol{f}\,\text{grad}\boldsymbol{g}/(|\text{grad}\,\boldsymbol{f}\,\text{grad}\,\boldsymbol{g}|^2)^{1/2}$$

which involves components of tensorial products. A tube of force is a collection or a group of lines of force. Since div $\boldsymbol{B} = 0$, the flux in a tube is conserved as the lines of force diverge and converge. If a tube branches into a number of tubes, then the sum of fluxes remains the same.

Since the flux in a tube of force is conserved, the cross section of the tube of force traversing through materials of different permeability would be inversely proportional to the permeability (Fig. 2); however, continuity equations require this variation in cross section to be gradual.

A useful concept is the specific volume of a magnetic tube of force given by

$$U = dV/d\Phi$$

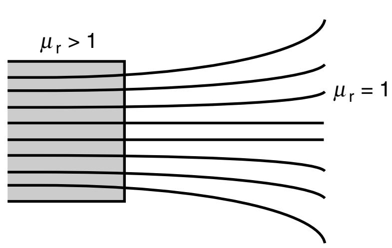

Figure 2. Spreading of field lines (LOF) in low permeability region for the same flux.

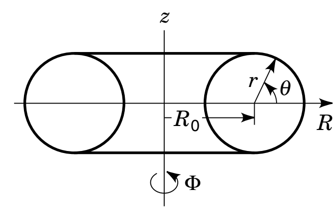

Figure 3. Toroidal geometry, R_0 is the major radius, Φ the azimuthal angle, γ the minor radius, and θ the poloidal angle.

where now V is the volume of the tube and Φ is the flux enclosed by it.

Since the flux in a tube is conserved,

$$U = \int\!\!\int (dS\,\mathrm{dl})/(\mathrm{B\,dS})]$$

or

$$U = \int (dl/B)$$

where dS and dl are the cross sectional area and length of a volume element, B is the flux density, and the integral is over the whole tube.

FLUX AND FIELDS IN A TOROIDAL GEOMETRY

The toroidal geometry (Fig. 3) has applications especially to plasma-confinement devices, and the topology of the field and the constituent magnetic flux are of specific interest in such devices and astrophysics. In a toroidal geometry, a pure toroidal field (field lines going around the major circumference) or a pure poloidal field (field lines going around the minor circumference) would give closed field lines. In most plasmas of interest, both fields would be present and the toroidal and poloidal fluxes are also called longitudinal and azimuthal fluxes. The toroidal or azimuthal flux χ is the flux enclosed by the surface ϕ = const, where ϕ is the azimuthal angle around the major axis of the toroid. The poloidal or the longitudinal flux Φ is the flux enclosed by the surface θ = const (Fig. 4). If

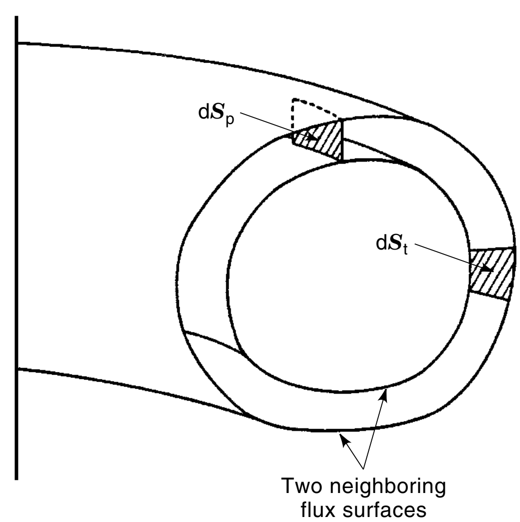

Figure 4. Illustration of the poloidal and toroidal surface elements $d\boldsymbol{S}$p and $d\boldsymbol{S}_r$.

both fields are present, the lines of force go around helical paths around the torus. In general, a line of force starting at a certain poloidal angle will arrive at a different poloidal angle after one traverse or more around the major circumference. The rotational transform is defined as the change in angle averaged over a large number of transits around the major circumference:

$$\iota = \lim \sum_{\kappa=1}^{n} \iota_\kappa / n, \qquad n \to \infty$$

For a toroid with a toroidal current of I_ϕ and a uniform toroidal field of B_0, the rotational transform at the minor radius r is given by

$$\begin{aligned} \iota &= B_\theta(r) 2\pi R / r B_0 \\ &= 2\pi d\chi / d\Phi \end{aligned}$$

where $B_\theta(r) = \mu_0 I_\phi / 2\pi r$ is the poloidal field at minor radius r and R is the major radius. The quantity $q = 2\pi/\iota$ is known as the factor of safety in fusion-device terminology.

The surface on which the helical lines that close on themselves after a number of transits is called a rational surface.

MAGNETOMOTIVE FORCE AND RELUCTANCE

These terms are defined analogousy to electrical circuits. A magnetic circuit consists of flux threading the circuit, analogous to current. The flux is "driven" by the magnetomotive force (mmf) E_M, and the flux Φ is limited by the reluctance, so that reluctance is analogous to resistance in an electrical circuit. In most applications the mmf can be defined as ampere-turns, that is, the product of the instantaneous current and the number of turns. The reluctance of a circuit element is given by

$$\mathscr{R} = E_M / \Phi$$

The reluctance of an element is related to the characteristics of the element by

$$\mathscr{R} = L / \mu_r \mu_0 A$$

where μ_r and μ_0 are the relative permeability of the circuit element material and the permeability of vacuum, respectively, and L and A are the length and cross-sectional area of the circuit element. Therefore materials with high permeability such as iron have low reluctance, and vacuum or air has high reluctance. The concept of reluctance can be used in magnetic circuits analogous to electric circuits. If the mmf is analogous to the emf (applied voltage), then flux is analogous to resistive current, and the reluctance is analogous to electrical resistance (with permeability being equivalent to electrical conductivity). For example, the flux generated by a coil with ampere turns NI and threading two adjacent (in series) volumes with reluctances R_1 and R_2 is given (in one dimensional approximation) by

$$\Phi = NI / (R_1 + R_2)$$

The magnetization in a material is given by

$$M = B - \mu_0 H = (\mu_r - 1)\mu_0 H$$

Thus low-reluctance materials also have high magnetization. For $\mu_r \gg 1$,

$$M \sim H(\mathrm{L}/\mathscr{R}A)$$

Ferromagnetic Materials and Shielding

Ferromagnetic materials (e.g., iron) have high permeability and therefore low reluctance. Therefore in magnetic devices, where the flux is to be linked effectively between two electrical circuits, e.g., transformers and motors, a ferromagnetic path is usually employed. Conversely, to shield regions from magnetic fields, a low reluctance magnetic path may be provided for the field so that the field lines prefer to pass through the ferromagnetic region rather than the region that has to be shielded. Such a shielding iron may cover the source or cover the region to be shielded.

DEMAGNETIZATION FACTOR

While the flux inside a perfect conductor is conserved, it must be remembered that the magnetic field intensity H is not. In fact, the field intensity inside the diamagnetic material can be shown to increase by a factor depending upon the geometry of the material, for the same conditions of field excitation, for example, magnetic current. (For paramagnetic and ferromagnetic materials, the field intensity decreases by some factor.) The demagnetization factor is 2 for a sphere and is 1.5 for a cylindrical cross section. This fact can be explained as follows. (For an alternate description, see Ref. 3 on the analogous characteristic of depolarizing factor.)

Consider a long solenoidal magnet (4) that produces a nearly uniform external field intensity H_e in the direction x at the center of the solenoid. Now if a sphere of diamagnetic material is placed at the center and the solenoidal field is established, the diamagnetic material will exclude this flux from inside and it can be shown (solution to the Laplace's equation) that the field lines will be as shown in Fig. 5. While the field intensity H_e' outside the sphere is unaffected far away from the sphere, the field intensity H_i is zero just out-

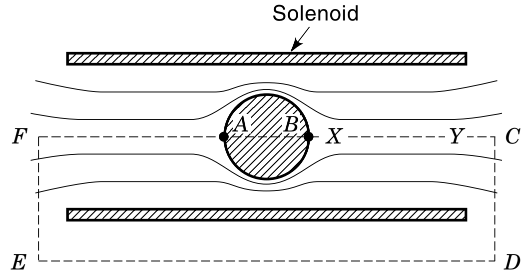

Figure 5. A diamagnetic sphere in a solenoid. The field strength at a point close to the sphere, such as X, is less than it would be if the sphere were absent, while the field strength at a point far away, such as Y, is essentially unchanged. The line integral of H around the broken line is independent of whether the sphere is present or not, so the field strength inside the sphere must exceed the applied field H_a (Ref. 4).

side of the sphere along the diameter parallel to the field direction, and near that region the external field intensity H'_e will be less than the value in the absence of the sphere. Therefore the field intensity H'_e outside the sphere will be less than or equal to the field intensity H_e in the absence of the sphere. Now, the $\int \boldsymbol{H} \cdot d\boldsymbol{l}$ along the closed path $ABCDEFA$ gives (by Ampere's law) the total ampere turns in the solenoid, which was held constant when the sphere was placed. Therefore

$$\int (\boldsymbol{H}_i \cdot d\boldsymbol{l})_{AB} + \int (\boldsymbol{H}_e \cdot d\boldsymbol{l})_{BC} + \int (\boldsymbol{H}_e \cdot d\boldsymbol{l})_{CDEF} + \int (\boldsymbol{H}_e \cdot d\boldsymbol{l})_{FA}$$
$$= \int (\boldsymbol{H}'_i \cdot d\boldsymbol{l})_{AB} + \int (\boldsymbol{H}'_e \cdot d\boldsymbol{l})_{BC} + \int (\boldsymbol{H}'_e \cdot d\boldsymbol{l})_{CDEF} + \int (\boldsymbol{H}'_e \cdot d\boldsymbol{l})_{FA}$$

Since, H'_e is less than or equal to H_e along BC and FA and the integral over $CDEF$ is unaffected (too far away), $\int (\boldsymbol{H}'_i \cdot d\boldsymbol{l})_{AB}$ is larger than $\int (\boldsymbol{H}_i \cdot d\boldsymbol{l})_{AB}$ to preserve the sum of the integrals, which essentially means that H'_e is larger than H_e along CD, that is, inside the sphere. This effect was first noted for paramagnetic and ferromagnetic materials for which the relative permeability is greater than 1, and therefore the field intensity inside the sphere would be less than the field in the absence of the sphere. Since this effect was first noted in such materials, the effect is considered to be a demagnetization and can be stated as

$$H_i = H_a - nI_m$$

where H_i is the field intensity inside the object with magnetization I_m and H_a is the applied field intensity. The quantity n is called the demagnetization factor and depends on the geometry of the object. For spheres $n = \frac{1}{3}$ and for cylinders $n = \frac{1}{2}$. Clearly if the applied field is perpendicular to a thin cylindrical wire, because of the volume average, $n = 0$. It must be remembered that for paramagnetic and ferromagnetic materials, I_m is positive and the field intensity inside the material decreases, while in diamagnetic materials, I_m is negative and the field intensity inside the material increases. This demagnetization factor has important consequences for nonlinear magnetization and critical characteristics of materials such as iron and superconductors.

MAGNETIC HELICITY

The topology of magnetic surfaces and the complexity of the structure of the magnetic field can be described by a quantity known as magnetic helicity, which is defined as

$$H = \int (\boldsymbol{B} \cdot \boldsymbol{A}\, dV)$$

where the integral is over the volume of interest. The magnetic helicity describes the linking of field lines and tubes of force. Considering the two linked tubes in Fig. 6, the helicity can be written as

$$H = \int\int [d\boldsymbol{S} \cdot d\boldsymbol{l})(\boldsymbol{B} \cdot \boldsymbol{A})]$$

where S is the cross section of tube 1 and l is the length of tube 1. Since B is approximately normal to S, this can be written as

$$\mathrm{H} = \oint (\boldsymbol{A} \cdot d\boldsymbol{l}) \int (\boldsymbol{B} \cdot d\boldsymbol{S})$$

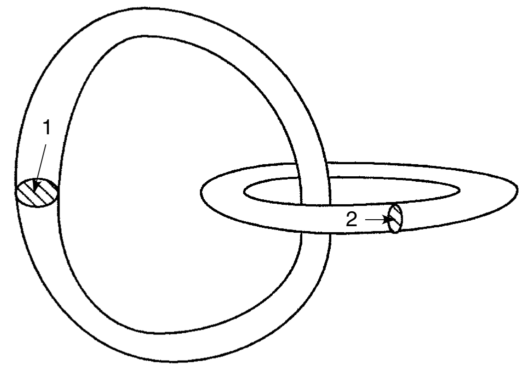

Figure 6. Linkage of flux tubes, tubes with cross sections 1 and 2 are threaded by flux Φ_1 and Φ_2 respectively.

The surface integral is the flux Φ_1 enclosed by tube 1 while the line integral is the flux Φ_2 enclosed by tube 2. The linked system helicity is then equal to the sum of the two helicities equal to $2\Phi_1\Phi_2$. If the tubes are linked N times the helicity will be equal to $2N\Phi_1\Phi_2$. In many systems of interest, the helicity is conserved as the magnetic configuration evolves.

Flux Conversion

If the field configuration is confined in a closed, perfectly conducting and nonpermeable surface (the normal component of $\boldsymbol{B}$ and velocity of any conducting medium $\boldsymbol{v}$ are zero), then helicity is conserved. This means that in such a configuration, although individual fluxes of different tubes (components of flux density), for example, toroidal and poloidal fluxes, are not conserved independently, the product is conserved. The symmetry then permits conversion of one type of flux into another. Such flux conversions are observed in plasma devices and in geomagnetic phenomena (5–7). The presence of turbulent structures and coherent magnetic field fluctuations may provide a mechanism for the conversion of flux and geomagnetic phenomena (8).

A simple generation of flux conversion is illustrated by using the diffusion time for flux lines in a good conductor. Consider a magnetic field applied externally to a pair of conducting materials [Fig. 7(a)]. After a certain time, depending on the conductivity of the material, the flux will diffuse in the two conductors [Fig. 7(b)]. Now, if one of the conductors is moved fast compared to the diffusion time, the flux lines will be bent and appear as shown in Fig. 7(c) until the lines can redistribute themselves inside the conductor. It is clear that in this process, a portion of the magnetic field that was pre-

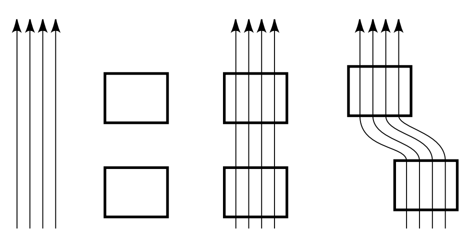

Figure 7. Conversion of vertical field to horizontal field. (a) Field before penetrating two blocks of conductor. (b) Fields after penetration. (c) Fields after the lower conductor is moved—a horizontal component is created in the gap between the two conductors.

viously in the vertical direction has been converted into a horizontal field.

Dynamo Action and Geomagnetism

The fact that convective motion of conducting fluids can generate magnetic fields has been invoked in explaining spontaneous flux generation from seed magnetic flux and is considered to be the source of the dynamo action in the earth's core, which produces magnetic fields. In perfectly conducting fluids, the lines of force are frozen (see the section on plasma equilibrium and Ref. 9). While the earth's core is conducting, any generated magnetic field must have short decay time due to the finite resistivity of the earth's melted core. Therefore a continuous dynamo action is necessary to maintain this field. Such a dynamo action is caused by the correlation between velocity and field perturbations in the turbulent motion of the core (10,11). Two effects, the α effect and the ω effect, are invoked to explain the dynamo action (12).

The α effect is a direct result of the Faraday effect. Consider Ohm's law,

$$\boldsymbol{J} = \sigma \boldsymbol{E} + \sigma(\boldsymbol{v} \times \boldsymbol{B})$$

where the second term is due to the induced Faraday emf. If a turbulent system is present, so that $\boldsymbol{v} = \boldsymbol{v}_0 + \boldsymbol{v}'$ and $\boldsymbol{B} = \boldsymbol{B}_0 + \boldsymbol{B}'$, the average induced electric field $\boldsymbol{E} = \boldsymbol{v}_0 \times \boldsymbol{B}_0 + \boldsymbol{v}' \times \boldsymbol{B}'$, since the averages of $\boldsymbol{v}'$ and $\boldsymbol{B}'$ are zero in turbulent perturbations. Therefore an additional emf $\boldsymbol{E}' = \boldsymbol{v}' \times \boldsymbol{B}'$ associated with the correlated velocity and magnetic fields occurs. In specific systems, this electric field can be written as $\boldsymbol{E}' = \alpha \boldsymbol{B}_0$, where α is a tensor in general. This electric field then has a component of current which maintains the dynamo action. Consider Fig. 8, where the turbulent velocity of the fluid can be resolved into an axial component $\boldsymbol{v}_2$ and a rotational component $\boldsymbol{v}_1$. If the initial magnetic field is in the x direction, the $\boldsymbol{v}_1$ component will produce an electric field $\boldsymbol{v}_1\boldsymbol{B}_0$ and a current $\boldsymbol{J}_1$ perpendicular to the y axis. This current will then produce a magnetic field $\boldsymbol{B}'$ in the y direction. An electric field $\boldsymbol{E}' = \boldsymbol{v}_2\boldsymbol{B}'$ will be produced in the x direction (parallel to the original magnetic field), as stated previously.

The ω effect is caused by convective effects illustrated in the preceding section. In the illustration shown in Fig. 9, a radial or poloidal field is convected by a toroidal flow. When a toroidal flow is impressed upon a poloidal field, the velocity field shears the magnetic field and produces a toroidal magnetic field such that the direction at the top and the bottom are opposite, preserving helicity (13). The dynamo is again due to the correlation between the turbulent velocity field and the turbulent magnetic field.

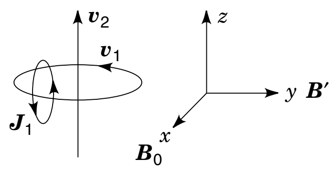

Figure 8. Illustration of the α effect. Consider a right-handed helical velocity field depicted by ($\boldsymbol{v}_1$ and $\boldsymbol{v}_2$ in the presence of a field $\boldsymbol{B}_0$ aligned along the x axis. This will produce current loops such as $\boldsymbol{J}_1$, lying in the x-z plane. Associated with the current loop $\boldsymbol{J}_1$, is a field $\boldsymbol{B}'$ aligned parallel to the y axis. This new field $\boldsymbol{B}'$ interacts with $\boldsymbol{v}_2$ to produce an electric field parallel to the x axis.

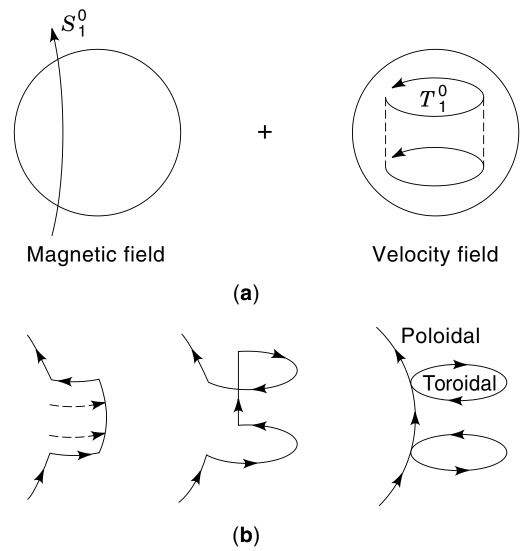

Figure 9. Production of a toroidal magnetic field in the core. (a) An initial poloidal magnetic field passing through the Earth's core is shown on the left, and an initial cylindrical shear velocity field, T_1^0, is shown on the right. (b) The interaction between the velocity and the magnetic field in (a) is shown at three successive times moving from left to right. The velocity field is only shown on the left by dotted lines. After one complete circuit two new toroidal magnetic field loops of opposite sign (T_2^0) have been produced. After Ref. 9.

ELECTRICAL MACHINES

Transformers

Transformers essentially use Lenz's law. In transformers a "primary" coil is supplied with a time varying current and a "secondary" coil mutually coupled to the primary coil receives an induced voltage that can then be used to drive a current into another circuit (Fig. 10). This then permits isolating the secondary circuit electrically from the primary circuit while enabling the indirect use of the source that powers the primary circuit. In addition, the transformer permits the "stepping up or down" of the voltage, that is, the secondary voltage can be larger or smaller than the primary voltage by the ratio of the number of turns in the primary and secondary coils. In a transformer the secondary coil is made to link nearly all the flux due to the primary coil by placing the primary and secondary coils around iron, which provides a closed low-reluctance path for the magnetic flux. The changing current in the primary coil causes a change in the flux and the secondary coil receives an induced voltage that opposes this changing flux.

$$V_\mathrm{p} = N_\mathrm{p}\, d\Phi/dt$$
$$V_\mathrm{s} = -N_\mathrm{s}\, d\Phi/dt$$

The negative sign indicates that the secondary coil opposes the change in flux caused by the primary coil

$$V_\mathrm{s}/V_\mathrm{p} = -N_\mathrm{s}/N_\mathrm{p}$$

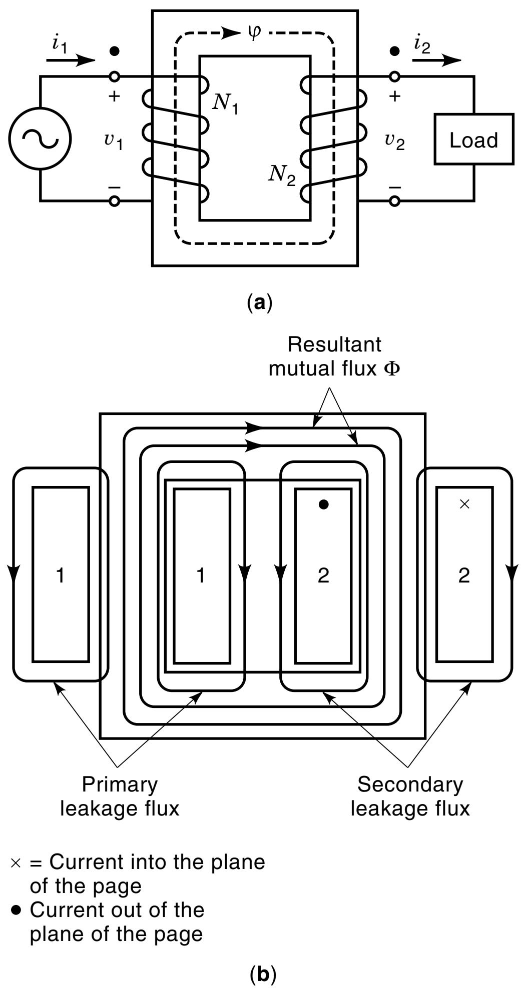

Figure 10. (a) Ideal transformer and load. (b) Component fluxes.

Since the same flux is linked and the flux path is the same (the reluctance and the flux are equal in the primary and secondary circuits),

$$N_sI_s = N_pI_p \; I_s/I_p = N_p/N_s$$

or

$$V_sI_s = V_pI_p$$

However in a nonideal transformer (14), part of the voltage applied to the primary coil is expended in generating the flux in the core and part is expended for compensating for eddy currents in the coil and iron and losses in iron due to hysterisis. The flux generated in the core by the primary current links the secondary current as a mutual flux and the remaining current leaks out into the air (which is outside the iron core and there is no linkage with the secondary current) as leakage flux. Similarly the flux due to current in the secondary coil (under load conditions) also has two parts: mutual and leakage flux. As is evident from the terminology, the mutual flux of the primary and secondary coils are equal and the leakage flux is dependent on the core size and permeability—the larger the area and permeability, the smaller the fraction of leakage flux.

The mutual flux is subject to saturation effects in iron. The reduced permeability of the iron at high excitation currents (flux densities) causes a smaller increase in mutual flux for an increase in the current, and the induced voltage exhibits saturation. This increases the high-harmonic components in the secondary voltage. Since the leakage flux is in the air, it is proportional to the current.

The total primary and secondary flux can be written as

$$\Phi_{tp} = N_p\Phi_{lp} + N_p\Phi_m$$
$$\Phi_{ts} = N_s\Phi_{ls} + N_s\Phi_m$$

where subscript 1 refers to the leakage flux and m refers to the mutual flux. Similarly the total voltage is also the sum of that induced by (or used in creating) the leakage flux and that induced by the mutual flux. The leakage flux

$$N_p\Phi_{lp} = N_p^2I_p/\mathscr{R}_p$$
$$N_s\Phi_{ls} = N_s^2I_s/\mathscr{R}_s$$

where $\mathscr{R}_p$ and $\mathscr{R}_s$ are the reluctances of the primary and secondary leakage paths, respectively. The corresponding voltages associated with the leakage paths can be defined as

$$V_{lp} = L_{lp}\,dI_p/dt$$
$$V_{ls} = -L_{ls}\,dI_s/dt$$

where $L_{lp} = N_p^2/\mathscr{R}_p$ and $L_{ls} = N_s^2/\mathscr{R}_s$ are the leakage inductances. Taking into account the resistance of the coils, r_p and r_s, the total voltage is then given by

$$V_p = E_p + L_{lp}\,dI_p/dt + I_pr_p$$
$$V_s = E_s - L_{ls}\,dI_s/dt - I_sr_s$$

where E_p is the voltage inducing the mutual flux in the primary and E_s is the voltage induced by the mutual flux. In well-designed transformers, the leakage and resistive terms are usually negligible.

Since

$$V_p = N_p d\Phi_{tp}/dt$$
$$\Phi_{tp} = (1/N_p)\int(V_p\,dt)$$

For a sinusoidal voltage with a frequency $f = \omega/2\pi$, $V_p = V_0 \sin(\omega t + \alpha)$

$$\Phi_{tp} = (V_0/\omega N_p)\cos(\omega t + \alpha) + \Phi_c$$

where Φ_c is a transient flux that decays after switching on due to eddy currents and hysteresis losses. Therefore the flux induced in the transformer is inversely proportional to the frequency of the applied voltage and lags in phase angle by $\pi/2$.

An approximate equivalent circuit of the transformer can be constructed in a single circuit taking into account the mutual coupling, where the circuit consists of primary inductance and resistance, the mutual coupling inductance and magnetization, the leakage flux, and the secondary impedance (inductance and resistance referred to the primary). Other nonideal effects such as saturation of the iron core, ac losses in the core, and eddy currents can be taken into account in such a circuit (15).

Dc Electric Generators

In a generator (Fig. 11 from Ref. 14), a coil of conductors on the armature (rotor) moves across the north and south magnetic poles (stator). If the coil has N_c turns and the poles generate a flux Φ, the coil will link a flux Φ under the north pole, then zero flux between poles, and a flux $-\Phi$ under the south pole. Therefore the voltage induced in the coil is

$$V_c = \Delta\Phi/\Delta t$$

where $\Delta\Phi = 2N_c\Phi$ is the change in the flux seen by the coil and Δt is the time over which the flux change occurs. If the coil is rotating at a rate of n rotations per second and there are p poles in the stator, then $\Delta t = 1/np$, so that

$$V_c = 2N_c pn\Phi$$

If C coils are connected in series and a are connected in parallel, the generated voltage is

$$V_g = 2CN_c pn\Phi/a = K_a\Phi\omega$$

where ω is the angular frequency of rotation and $K_a = CN_c/\pi ap$ is known as the armature constant. As shown later, the voltage induced is alternating, and dc generators require so called commutators to change the brush polarity alternately to generate dc voltage.

Generalized ac Machines

Induced Voltage in an ac Generator. In ac motors and generators, a number of multipole excitation coils are placed in a stationary high permeability core and a set of secondary coils are placed in a rotary core. The secondary and the primary excitation coils are placed around a common axis and have a small gap. (A simple example of a two-pole ac machine is shown in Fig. 12.) At an arbitrary angle θ between the rotor

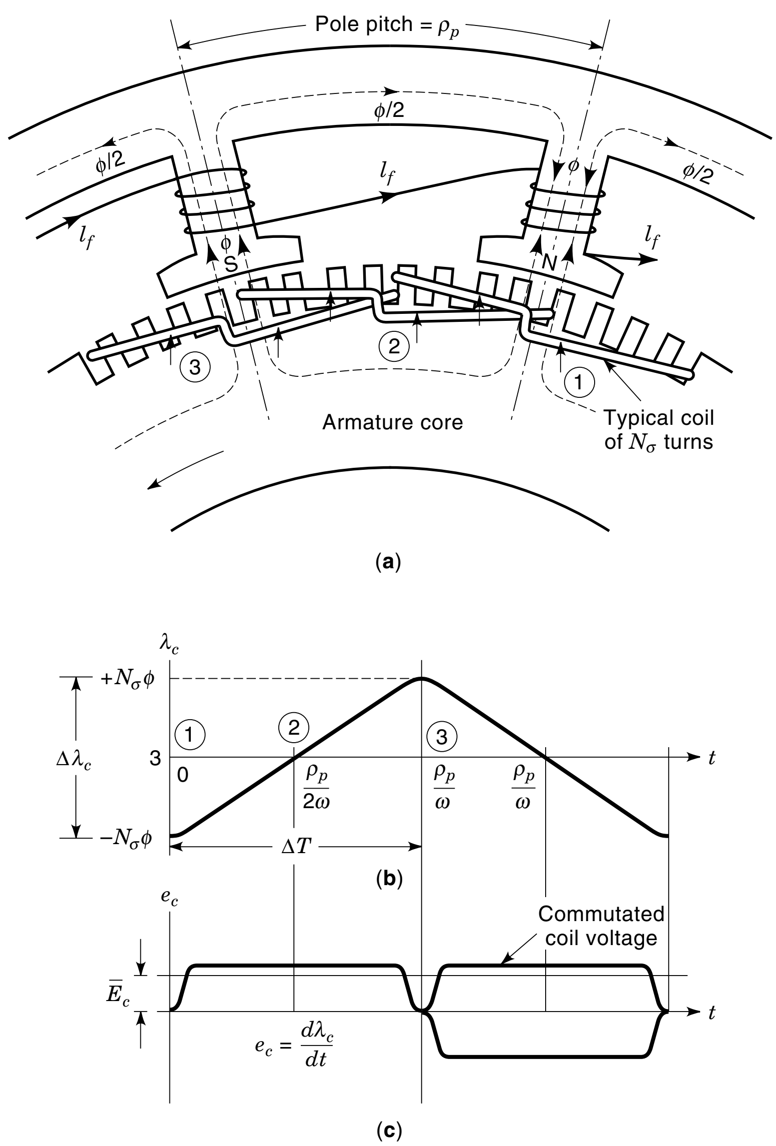

Figure 11. Schematic of a generator.

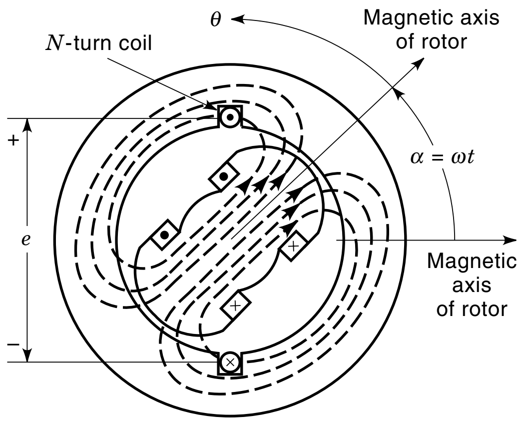

Figure 12. Elementary two-pole ac machine with stator coil of N turns.

and the stator with N turns, the flux linked by the stator is

$$N\Phi = N\int_{-\pi/2}^{\pi/2} [B_{\max}(\cos\theta) lr\, d\theta] = 2NB_{\max} lr$$

where l is the length of the rotor (normal to the figure) and r is the radius of the stator at the gap. For p poles, the flux is $(2/p)(2Nb_{\max}lr)$. If the rotor spins with an angular velocity ω, the flux links changes with time as

$$\Phi' = N\Phi\cos(\omega t)$$

The voltage induced due to the time variation of the flux is given by

$$e = -d\Phi'/dt = \omega N\Phi\,\sin(\omega t) - N\cos(\omega t)\,d\Phi/dt$$

If the flux produced by the coils is independent of time, the second term is zero, but it is clear that the generated voltage is alternating.

Rotating Magnetic Field. In three-phase ac machines (where three legs of the ac supply each have a phase difference of 120°), three sets of stator coils are connected to the three phases (Fig. 13), so that the currents in the coils are given by

$$I_a = I_{\max}\cos(\omega t)$$
$$I_b = I_{\max}\cos(\omega t - \pi/3)$$
$$I_c = I_{\max}\cos(\omega t - 2\pi/3)$$

where $\omega = 2\pi f$ and f is the frequency of the ac supply. In such a case, the total instantaneous force on the armature at an arbitrary angle θ due to the three coils is proportional to the flux linked, which, in turn, is proportional to the current and is given by

$$\begin{aligned}\Phi(\theta, t) &= \Phi_{\max}\cos\theta\cos(\omega t) + \Phi_{\max}\cos\theta\cos(\omega t - \pi/3)\\ &\quad + \Phi_{\max}\cos\theta\cos(\omega t - 2\pi/3)\\ &= 1.5\Phi_{\max}\cos(\theta - \omega t)\end{aligned}$$

which represents a traveling wave of flux (also an mmf or force in a motor, or induced emf in a generator). If at $t = 0$

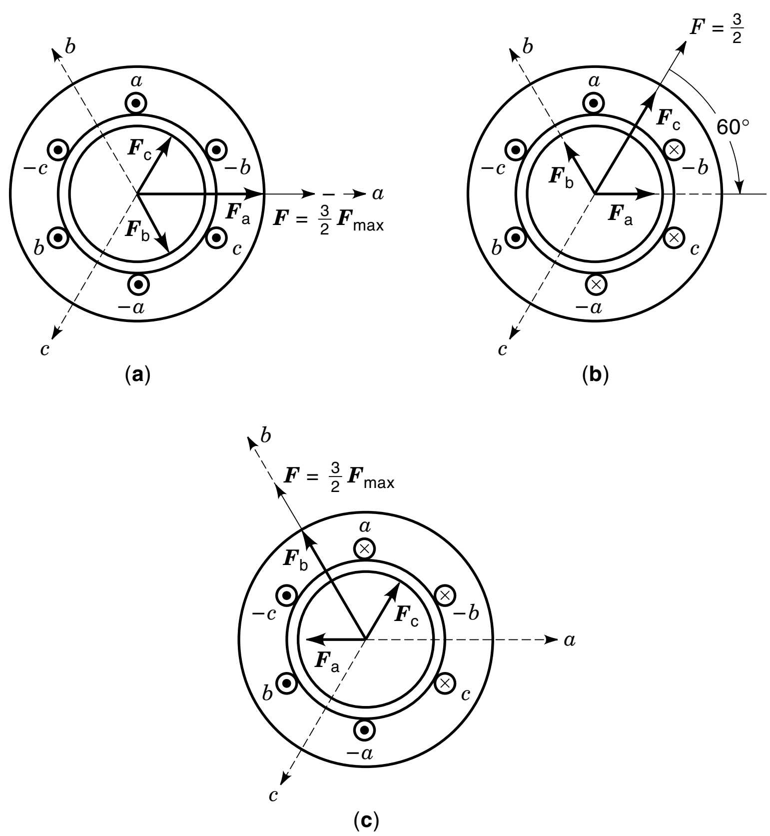

Figure 13. Production of a rotating magnetic field by means of three currents.

the peak of the flux was at $\theta = \theta_0$, then in a time t_0 the peak moves to $\theta = \theta_0 - \omega t$ and therefore the field appears to rotate in time. Figure 13 shows this rotation, where F is the force (proportional to the flux linked) experienced by the armature.

If the armature also rotates but with an angular velocity ω_a, the linkage is given by

$$\Phi(t) = 1.5\Phi_{\max}\cos(\omega_a - \omega)t$$

so that when $\omega_a = \omega$, the linked flux appears to be a constant and the motor or generator is *synchronous*.

This description can also be represented by a coupled-circuit description (14) using the stator and rotor inductance and flux linkages and resistance of the coils. The circuit description then leads to a set of two differential equations with time derivatives of current. Solutions of these equations give the instantaneous values of current and magnetic energy in the machine. The derivative of this energy with respect to the mechanical angle gives the torque produced.

Sinusoidally Wound Stators. The windings are arranged in such a fashion that the number of turns in the excitation and primary coils is a sinusoidal function of the angle, that is

$$N_i = N_{i0}\sin(p\phi) = N_{i0}\sin(p\pi x/L)$$

where i refers to the excitation or the secondary coil, N_0 is the maximum number of turns, p is the number of pairs of poles, x is the position along the circumference, and L is total circumferential length. [If the coils are not arranged in a sinusoidal fashion and are as shown in Fig. 12, then the fundamental component is given by $N_i = (2/\pi)N_{i0}\sin(p\phi/2)$.]

Now if the field in the gap is H_g, the integral of the field around a closed loop enclosing a coil (see Fig. 14) has two legs of the loop in the iron core that contribute negligibly if the permeability is very high and has two legs that cross the gap. Because the field direction remains along the integration direction, these add and the integral

$$\int(\boldsymbol{H}\cdot d\boldsymbol{x}) = 2H_g g$$

where g is the gap. But, by Ampere's law, the integral is also equal to the total current (ampere turns enclosed), IN_0 $\sin(p\pi x/L)$. Therefore

$$H_g = (IN_0/2g)\sin(p\pi x/L)$$

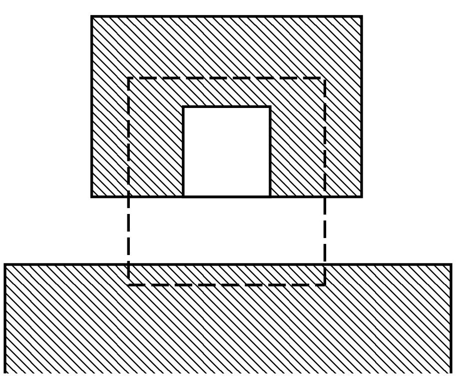

Figure 14. Flux in the gap between the stator and the rotor. Most of the reflectance is in the gap since the stator and the rotor have high permeability iron path.

Now the flux coupled to the coil,

$$\Phi = l\int(\boldsymbol{B}\cdot d\boldsymbol{x}) = (\mu_0 lIN_0/2g)\int[\sin(p\pi x/L)\,dx]$$

where l is the length of the coil (or the area under consideration). Over one length of the pole, p/L, the integral gives

$$\Phi_0 = (\mu_0 lIN_0L/2gp\pi)IN_0G_g$$

where $G_g = (\mu_0 lL/2gp\pi)$ is the gap permeance per pole (or is the inverse of the reluctance of the gap per pole).

Now, if the current is alternating and the rotor is in motion, at any instance the flux coupled to the coil is given by

$$\begin{aligned}\Phi &= I_0N_0G_g\sin(\omega t + f)\\ &= IN_0G_g\sin(\omega t)\cos\phi + IN_0G_g\cos(\omega t)\sin\phi\\ &= \Phi_x\sin(\omega t) + \Phi_y\cos(\omega t)\end{aligned}$$

where $\phi = p\pi x/L$ represents the angle or the spatial phase of the rotor at time t.

If we use the designations j and J to distinguish between and separately account for the rotational time dependence and the angular position

$$\sin\phi = (e^{J\phi} - e^{-J\phi})/2J$$
$$\sin(\omega t) = (e^{j\omega t} - e^{-j\omega t})/2j$$

The above expression can be rewritten as

$$\Phi = \Phi_{td} + \Phi_{top} = (J\Phi_y - j\Phi_x)e^{j\omega t} + (J\Phi_y + j\Phi_x)e^{-j\omega t}$$

which represents two counterrotating components of flux, one direct and another opposite, that is, two components of flux with a phase difference of π. The two components are illustrated in Fig. 15.

The preceding description of the flux is useful in the design of devices such as sine-cosine transformers (SCT), remote and point control systems, tachogenerators, and servomotors.

CHARGED-PARTICLE MOTION IN A MAGNETIC FIELD

A charged particle is deflected from its original path by a magnetic field if it has a velocity component perpendicular to the magnetic field (that is, charged particles with velocity in the direction of the magnetic field do not experience a force). The particle moves in a direction perpendicular both to the initial velocity and the magnetic field. Since the motion is perpendicular to the magnetic field, no work is done by the magnetic field and the particle energy does not change. It can be seen then that the particle exercises circular motion around the field direction (flux lines), and if the particle has a parallel velocity (which remains unaffected by the field), the particle executes spiral motion. The radius of the circular motion is called the Larmor or gyro radius and the rotational frequency is called the Larmor or gyro frequency.

It can be shown that if the field varies slowly in space and in time, the flux enclosed by the charged particle is constant.

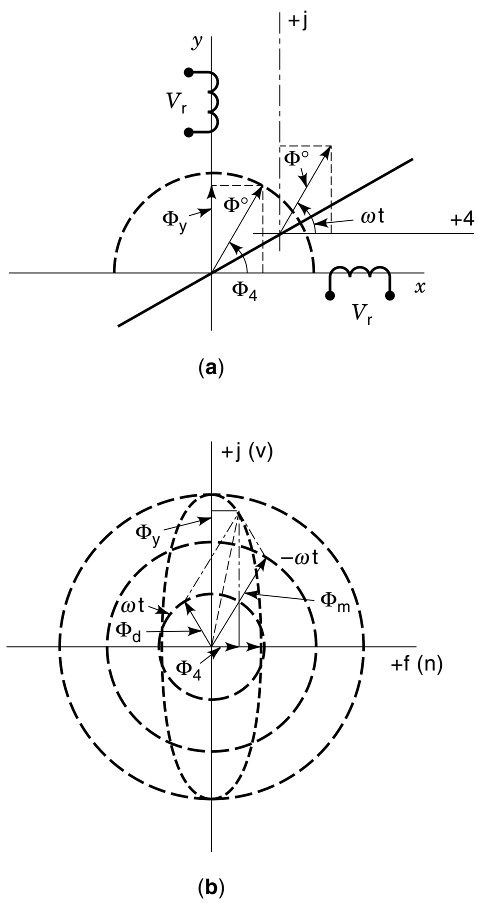

Figure 15. (a) Vector diagrams to illustrate spatial flux vector and (b) time vector diagram to illustrate geometrical meaning of the symmetrical component decomposition.

This conservation of flux is true in an "adiabatic" sense and leads to other adiabatic constants of motion, which enable the development of magnetic traps for plasmas and particle beams as well as particle accelerators and particle detectors.

The equation of motion of the charged particle in an electric and a magnetic field is given by

$$d\boldsymbol{p}/dt = q(\boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B})$$

where $\boldsymbol{p} = \gamma m\boldsymbol{v}$ and m, $\boldsymbol{v}$, γ, and q are the particle momentum, mass, velocity, relativistic factor and charge, respectively, and $\boldsymbol{E}$ and $\boldsymbol{B}$ are the electric and magnetic fields. The equation can be written in component form for $\boldsymbol{E} = 0$ as (for simplicity shown only for the magnetic field in the z direction, i.e., $\boldsymbol{B} = B\boldsymbol{e}_z$).

$$dv_x/dt = (qB/\gamma m)v_y$$
$$dv_y/dt = -(qB/\gamma m)v_x$$
$$dv_z/dt = 0$$

which are the equations for circular or spiral orbits with the gyro frequency $\Omega_g = qB/\gamma m$. Solving the equations for displacements, one gets

$$x = (v_p/\Omega_g)\sin(\Omega_g t)$$
$$y = \pm(v_p/\Omega_g)\cos(\Omega_g t)$$

where the + or − sign corresponds to the positive or negative charge (which may be dropped if the gyrofrequency includes the sign of the charge) and $v_p = (v_x^2 + v_y^2)^{1/2}$. Therefore $r_g = v_p/\Omega_g$ is the Larmor radius. (For a full relativistic treatment of the charged particle motion, see Ref. 16.)

Motion in a Time-Varying Field

If a particle is performing gyro orbits in a time-varying magnetic field, the energy is not a constant, since there is an associated electric field given by

$$\text{curl}\,\boldsymbol{B} = -\partial\boldsymbol{E}/\partial t$$

The energy gain is given by (2)

$$\Delta U_p = \int(q\,d\boldsymbol{r}\cdot\boldsymbol{E})$$

where the integral is around the orbit and $\boldsymbol{r}$ is the displacement along the path of the orbit. For this approximately closed path, the Stoke's theorem gives

$$\Delta U_p = \int(q\,d\boldsymbol{S}\cdot\text{curl}\,\boldsymbol{E})$$
$$\int(q\,d\boldsymbol{S}\cdot\partial\boldsymbol{B}/\partial t)$$
$$\sim q\pi r_g^2\,dB/dt$$

For time scales much larger than the time period of the gyro motion,

$$dU_p/dt \sim \Omega_g \Delta U_p/2\pi = (q\Omega_g r_g^2/2)\,dB/dt$$

which gives

$$(1/U_p)\,dU_p/dt = (1/B)\,dB/dt$$

or

$$d\mu/dt = d(U_p/B)/dt = 0$$

where μ is known as the magnetic moment of the particle, $\mu = q\Omega_g r_g^2/2$.

Substituting for Ω_g, $\mu = (q^2/2\pi m)(\pi r_g^2 B) = (q^2/2\pi m)\Phi_\gamma$, where Φ_γ is the flux enclosed by the circular orbit. Since $d\mu/dt = 0$, $d\Phi_\gamma/dt = 0$, the flux enclosed by the particle orbit is conserved if the rate of change of the magnetic field is adiabatic, that is, the change occurs over a period much larger than the gyro time period.

Motion in an Inhomogeneous Magnetic Field

The flux enclosed by a particle orbit also remains constant if the spatial variation of the magnetic field is adiabatic, that is, the scale length of variation is much larger than the gyro radius of the particle orbit. This can be shown simply by the fact that the situation is essentially same as for slow time variation of the field.

The magnetic field variation experienced by the charged particle as it moves in an inhomogeneous magnetic field with a velocity v is given by

$$dB/dr = v_r dB/dt$$

where v_r is the component of the velocity v in the direction r. Again, as shown before, in such a case, the magnetic moment is conserved and therefore the flux enclosed by the particle orbit is conserved.

Other adiabatic invariants such as the bounce invariant in trapped orbits and the line integral of the canonical angular momentum in a periodic motion (17) are also the result of flux conservation.

PLASMA EQUILIBRIUM AND FLUX SURFACES

The most common devices for nuclear fusion and plasma applications employ a toroidal geometry, where the plasma carries toroidal and poloidal currents (see the section entitled "Flux and Fields in Toroidal Geometry") and are confined by toroidal and poloidal fields. In such cases, the equilibrium is obtained as a balance between the Lorentz body force, which is generated by the interaction of the plasma current with the magnetic field, and the pressure force due to gradients in pressure. Such a confinement scheme is used in the Z, theta, and screw pinches, tokamaks, spheromaks, stellarators, and compact toroids. In many of these applications the primary configuration of the plasma is axisymmetric (except for, e.g., helical devices), that is, the variation of the current, magnetic field, pressure, and plasma properties are small and only appear as perturbations. Plasmas in such toroidal geometries attain equilibria (position and shape of the plasma, conditions of magnetic field and plasma current profiles, etc.) based on the solution to the Grad–Shafranov equation. It can be shown that the poloidal flux [see Fig. 16(b)] is constant on specific surfaces. While it is obvious that in the absence of pressure, the surfaces of constant flux are concentric, they are not so when the plasma pressure is finite. Since the outermost flux surface is usually fixed by a flux-conserving boundary or by an external vertical field, this means that the center of the plasma is shifted from the minor axis of the toroid by the so-called Shafranov shift. An equilibrium pressure limit (the so-called equilibrium β limit, where β is the ratio of the plasma pressure to the pressure due to the magnetic field) is obtained when the shift exceeds the radius.

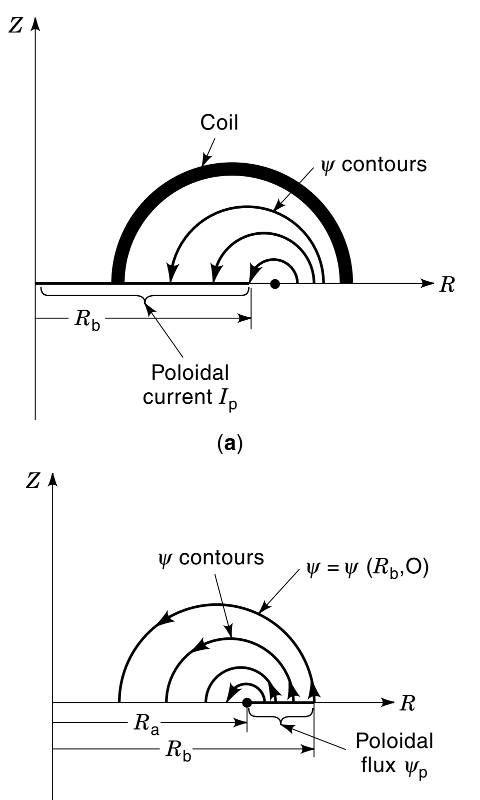

Figure 16. (a) Disk-shaped surface through which the total (plasma plus coil) poloidal current I_p flows. (b) Washer-shaped surface through which the poloidal flux ψ_p passes.

For the geometry shown in Fig. 3, the primary coordinates are the major radius R, the azimuthal angle ϕ around the major axis, and the vertical coordinate z. Additional coordinates are the minor radius r and the poloidal angle θ. We limit ourselves to axisymmetric equilibria so that $\partial/\partial\phi = 0$.

Maxwell's equations are

$$\operatorname{div} \boldsymbol{B} = 0 \tag{3a}$$

$$\operatorname{curl} \boldsymbol{B} = \mu_0 \boldsymbol{J} \tag{3b}$$

and the plasma force balance equation is

$$\boldsymbol{J} \times \boldsymbol{B} = \operatorname{grad} \mathrm{p} \tag{3c}$$

where $\boldsymbol{B}$ is the magnetic field, $\boldsymbol{J}$ the current density and p the pressure. Expanding the first equation,

$$(1/R)(\partial/\partial R)(RB_{\mathrm{R}}) + (1/R)\partial B_\phi/\partial\phi + \partial B_Z/\partial Z = 0$$

where the second term is zero due to axisymmetry. If we define a flux function ψ, such that

$$B_Z = (1/R)\partial\psi/\partial R$$
$$B_R = -(1/R)\partial\psi/\partial Z$$

then

$$\boldsymbol{B} = \boldsymbol{B}_\phi + \boldsymbol{B}_{\mathrm{p}} = B_\phi \boldsymbol{e}_\phi + (1/R)\operatorname{grad}\psi \times \boldsymbol{e}_\phi \tag{4}$$

The poloidal flux

$$\Phi_{\mathrm{P}} = \int (\boldsymbol{B}_{\mathrm{p}} \cdot \mathrm{d}\boldsymbol{S}) = \int_{R_0}^{R} [2\pi R(1/R)(\partial\psi/\partial R)\, dR] = 2\pi\psi$$

so that the flux function is essentially equal to the poloidal flux except for a constant of 2π.

Now taking a scalar product of Eq. (3c) and (3b),

$$\boldsymbol{B} \cdot \operatorname{grad} \mathrm{p} = 0$$
$$(B_\phi/R)(\partial p/\partial\phi) + (1/R)\operatorname{grad}\psi \times \boldsymbol{e}_\phi \cdot \operatorname{grad} p = 0$$

The first term is zero by axisymmetry; therefore

$$\operatorname{grad}\psi \times \operatorname{grad} p \cdot \boldsymbol{e}_\phi = 0$$

which shows that the pressure is constant if ψ is constant or $p = p(\psi)$, that is, the flux surfaces are constant-pressure surfaces. This is an important result that says the solution of flux surfaces gives the plasma equilibrium.

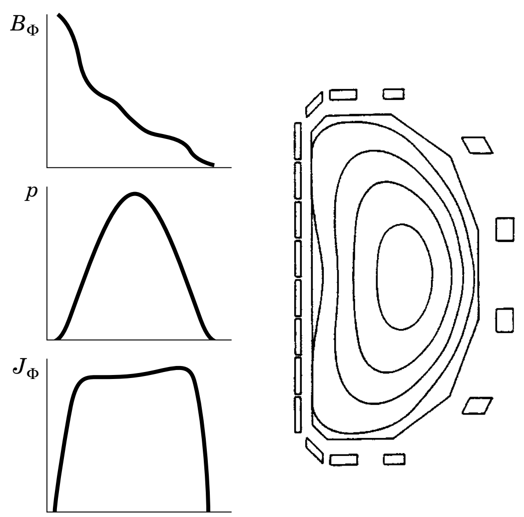

Figure 17. Numerically computed equilibrium of the noncircular, high-β tokamak DIII-D located at GA Technologies. Shown are flux surface plots and midplane profiles. Courtesy J. Helton, GA Technologies.

Substituting the expression for B as in Eq. (4) in Maxwell's Eq. (3b) and using the axisymmetric condition,

$$\mu_0 \boldsymbol{J} = \mathrm{grad}(RB_\phi) \times \boldsymbol{e}_\phi / R - (1/R)[R(\partial/\partial R)(1/R)(\partial\psi/\partial R) + \partial^2\psi/\partial Z^2)\boldsymbol{e}_\phi$$

The total current density can be divided into poloidal and toroidal components

$$\begin{aligned} \mu_0 \boldsymbol{J} &= \mu_0 \boldsymbol{J}_\mathrm{p} + \mu_0 \boldsymbol{J}_\phi \\ \mu_0 \boldsymbol{J}_\mathrm{p} &= \mathrm{grad}(R\boldsymbol{B}_\phi) \times \boldsymbol{e}_\phi R \\ \mu_0 \boldsymbol{J}_\phi &= \Delta^* \psi / R \end{aligned} \tag{5}$$

where

$$\Delta^*\psi = R(\partial/\partial R)(1/R)(\partial\psi/\partial R) + \partial^2\psi/\partial Z^2$$

The quantity RB_ϕ is designated $F(\psi)$, which can be shown to be proportional to the total poloidal plasma current enclosed by the flux surface, $\psi(R, 0) = \text{const}$,

$$\begin{aligned} I_\mathrm{p} &= \int (\boldsymbol{J}_\mathrm{p} \cdot d\boldsymbol{S}) = \int dR \int \{R\, df[\mathrm{grad}(R\boldsymbol{B}_\phi) \times \boldsymbol{e}_\phi]z\} \\ &= 2\pi \int (dR \partial F/\partial R) = 2\pi F(\psi) \end{aligned}$$

Now taking a scalar product of Eq. (3c) with grad ψ

$$\mathrm{grad}\,\psi \cdot (\boldsymbol{J} \times \boldsymbol{B} - \mathrm{grad}\, p) = 0$$

which after using Eqs. (4) and (5) gives

$$\Delta^*\psi = -\mu_0 R^2 dp/d\psi - F\, dF/d\psi \tag{6}$$

where the property grad $p = dp/d\psi \cdot$ grad ψ is used.

Equation (6) is known as the Grad–Shafranov equation. With appropriate boundary conditions, the equation can be solved to obtain plasma position and equilibrium. The solution is obtained as the solution to the shapes and locations of different flux surfaces (surfaces of constant ψ), and since each flux surface has an associated pressure, the flux surfaces define the plasma shape and location. Figure 17 shows an example of the equilibrium for the Doublet IIID tokamak.

Additional discussions on the applications and solutions to the Grad–Shafranov equation can be found in Refs. 4 and 18.

SUPERCONDUCTORS AND MAGNETIC FLUX

Superconducting Properties

Superconductors are materials that have special properties below the so-called critical temperature, critical field, and critical current density. When such materials are superconducting, they have zero resistivity and in addition they exhibit the Meissner effect (19,20). Earlier it was shown that Maxwell's equations lead to the fact that perfect conductors with zero resistivity exclude flux when the magnetic field (flux) is increased from zero to some finite value. Such perfect conductors maintain the initial flux, and diamagnetic currents cancel any change in the flux. However, in 1933, Meissner and Ochsenfeld observed that superconductors that are in the Meissner regime (e.g., lead) exclude all flux whether it was initially present or not (see Fig. 18). This is a significant characteristic of superconductors that distinguishes them from perfect conductors. These so-called type I superconductors receive induced surface currents, called Meissner currents, in the presence of a magnetic field which cancel all the

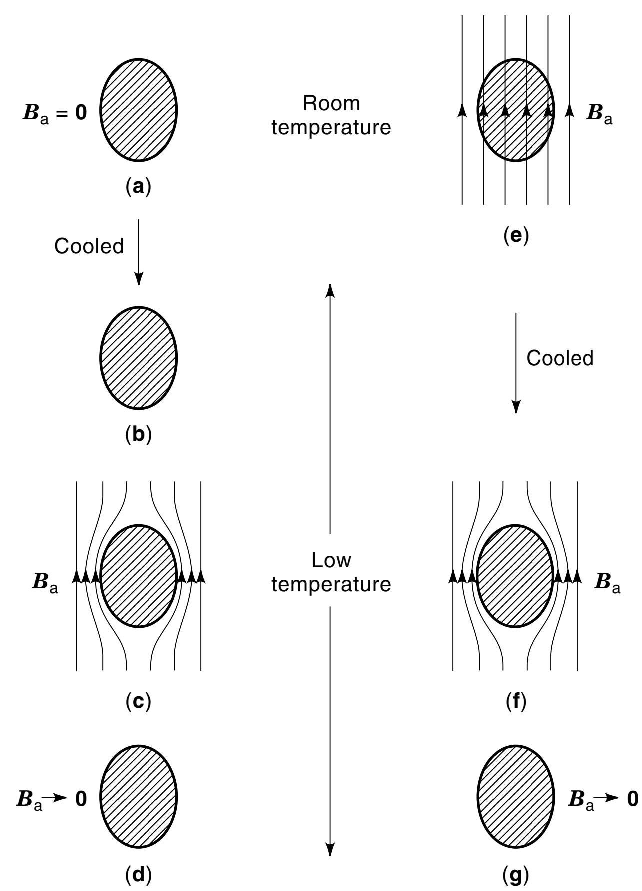

Figure 18. Magnetic behaviour of a superconductor. (a) and (b) Specimen becomes resistanceless in absence of magnetic field. (c) Magnetic field applied to superconducting specimen. (d) Magnetic field removed. (e) and (f) Specimen becomes superconducting in applied magnetic field. (g) Applied magnetic field removed. B_a is the applied magnetic flux density (Ref. 4).

flux inside the superconductor volume, independent of whether the initial flux (flux prior to the material becoming superconductor) was zero or finite. Another way of stating this is that superconductors are not just diamagnetic materials but have a relative permeability μ_r that is equal to zero, so that the magnetization is equal and opposite to the applied magnetic induction, that is, $M = -\mu_0 H$.

The superconducting property arises from the fact that below the critical temperature and field, the free energy for such materials is lower in the superconducting state compared to the normal (nonsuperconducting) state. This is due to the formation of Cooper pairs of superconducting electrons, which, on average, do not lose any energy by collisions with the lattice ions. The density of superconducting electrons and the free-energy gap at $T = 0$ and $H = 0$ are properties of the material. (See Ref. 21 for detailed references on energy-gap measurements.) As the temperature is raised, the free-energy gap is reduced, and above the critical temperature the superconducting state has an unfavorable free energy, and therefore the material would be normal. As the magnetic field is increased, the total energy, which includes the energy due to magnetization, is increased until again at the critical field, the normal state with zero magnetization is favored and the material would be normal.

While the description of the field being excluded from the volume of the superconductor is reasonable, in reality the external field penetrates to a small depth, the so-called London penetration length $\lambda_L = (m_e/\mu_0 n_s e^2)^{1/2}$, where e and m_e are the electron charge and mass and n_s is the density of superconducting electrons (20,22).

However, it must be noted that there is a class of alloy superconductors, known as type II superconductors, which are commonly used in electrical and magnetic applications, the flux (field) is allowed to penetrate into the superconductor above the thermodynamic critical field, while preserving the superconducting (zero resistance) property.

Intermediate State

As was noted in the section entitled "Demagnetization Factor," the field intensity in a diamagnetic material is higher than the applied field intensity. Since ideal superconductors exhibiting the Meissner effect have $I = -H_i$ ($M = -B_a$)

$$H_i = H_a/(1-n)$$

Now, since the superconductor would become normal at $H_i = H_c$ where H_c is the critical field intensity, this means that the applied field is less than the critical field. This is a paradoxical situation, since this means that as the material would become normal at $H_a < H_c$, which in turn would make $I = 0$ and we would have the material in a normal state for $H_a < H_c$. This is resolved by the realization of the fact that normal and superconducting regions coexist inside the material for $H_a > (1 - n)H_c$ (Fig. 19). The cross-sectional area of normal material is such that the average magnetization is such as to satisfy the boundary condition $H_i = H_a/(1 - n)$ for $H_i < H_c$ and $H_i = H_a$ for $H_a = H_c$. This condition is obtained if

$$H_i = H_a/[1 + n(f-1)]$$

where f is the fraction of normal cross section. This state is known as the intermediate state in superconducting materi-

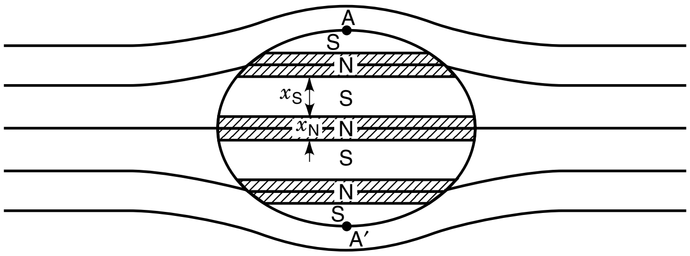

Figure 19. Ellipsoid split into normal and superconducting laminae in a magnetic field.

als and is analogous to an equilibrium of solid and liquid phases of matter near transition conditions. For actual observations on the intermediate state see Refs. 23 and 24.

Type II Superconductors

In a concept proposed by Pippard in 1953, the density of superelectrons cannot change abruptly and changes gradually only over a distance called the coherence length, that is, there cannot be a sharp boundary between normal and superconducting regions. The coherence length is a property of the material and if impurities are present, it is considerably reduced (by an order of 10 or more) to the geometric mean of the pure coherence length and the electron mean free path.

If the coherence length is shorter than the penetration length (see the section entitled "Superconducting Properties"), the formation of coexisting normal and superconducting zones is favored, since then the total free energy of the material is reduced because the surface energy of the boundaries between the normal and superconducting zones is negative for short coherence length (Fig. 20). For the Ginsburg–Landau constant $\kappa = \lambda/\xi > 0.71$, where λ is the penetration length and ξ is the coherence length, the material favors a mixed state of normal and superconducting regions over a fully normal state for applied fields greater than the thermodynamic critical field. Intrinsic superconductors, such as niobium, have $\kappa > 0.71$ (0.78 to 0.9) even without impurities, but alloys such as niobium–titanium have even higher values of κ.

Therefore in type II superconductors, once the applied field exceeds the thermodynamic (first) critical field, small zones of normal state are formed and the excess field lines are localized along these cores of normal zones, which have circulating currents on their surfaces that preserve the superconducting state of the regions outside the cores. Since the surface energy is negative, the formation of the smallest and maximum number of "flux" cores is favored to maximize the total surface area of such cores. Since the coherence length is small in such materials, there can be many fluxons that require sharp transition zones. The flux core is therefore of such a size as to give the minimum flux, that is, the flux of a so called "fluxon" or flux core, $\Phi_0 = 2.07 \times 10^{-15}$ W (25). The material acquires a lattice of fluxons as shown in Fig. 21. The number of fluxons depends on the amount of flux that needs to pass through the material. Because such fluxons are maintained by circular currents around the flux cores, the fluxons are also called vortices).

Under such conditions, the superconductor does not become normal until the fluxons with the transition regions fill up the area of cross section and the new critical field called

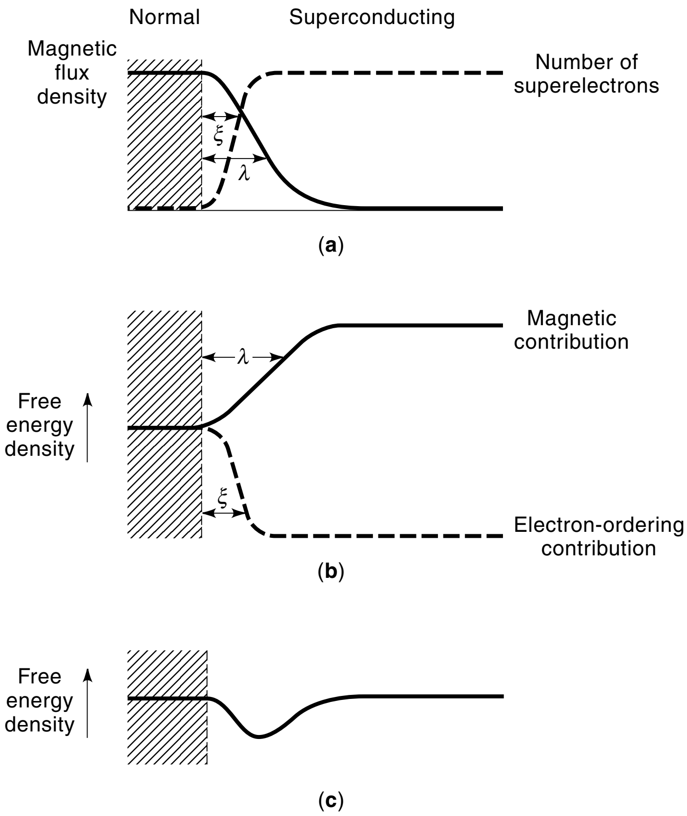

Figure 20. Negative surface energy; coherence range less than penetration depth. (Compare this with Fig. 6.9.) (a) Penetration depth and coherence range. (b) Contributions to free energy. (c) Total free energy.

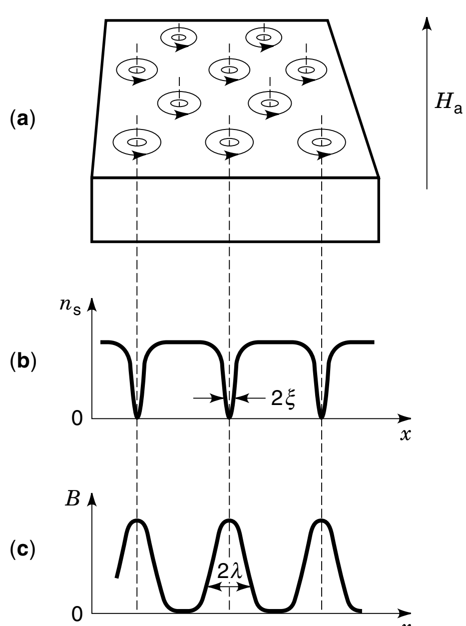

Figure 21. Mixed state in applied magnetic field of strength just greater than H_{c1}. (a) Lattice of cores and associated vortices. (b) Variation with position of concentration of superelectrons. (c) Variation of flux density.

H_{c2} is given by

$$H_{c2} \sim 1.41\kappa H_c$$

where H_c is the thermodynamic critical field at which the magnetic energy is equal to the difference between the free energy in the normal state and the superconducting state.

Flux Flow in Type II Superconductors. While the foregoing is true for a superconductor with no transport current (e.g., current from an external circuit), the amount of current the superconductor can carry in a magnetic field or the critical current requires additional considerations. When a superconducting fluxon lattice is also carrying current, the fluxons experience a Lorentz body force per unit volume of the conductor equal to the vector cross product of the current density and the magnetic field threading the fluxon (in most cases the applied field) (26–28). These forces would move the fluxons perpendicular to both the current density and the applied field, for example, in a wire with a transverse magnetic field, the fluxons would move radially perpendicular to the field. But these vortices or fluxons are pinned by imperfections in the lattice. Such imperfections in the lattice may be created due to working of the metal or impurities in the metal. Therefore, the superconducting state will be maintained as long as the pinning force per unit volume is larger than the Lorentz force.

As the Lorentz force approaches and exceeds the pinning force, the flux cores start moving and there will be some viscous resistance to such motion. Such a resistance would require work to be performed and energy to be supplied. This power requirement would then manifest itself as an electrical resistance and an associated voltage drop. The critical pinning force is not a constant and increases from zero with field and then reduces again to drop to zero at the upper critical field. Figure 22 (29) shows the increase of the flux flow resistivity with increasing field (30). This is called flux flow, and in this regime the flux cores move with a velocity relative to the electrons carrying the transport current. Since the cores are carried forward (in the direction of the transport current)

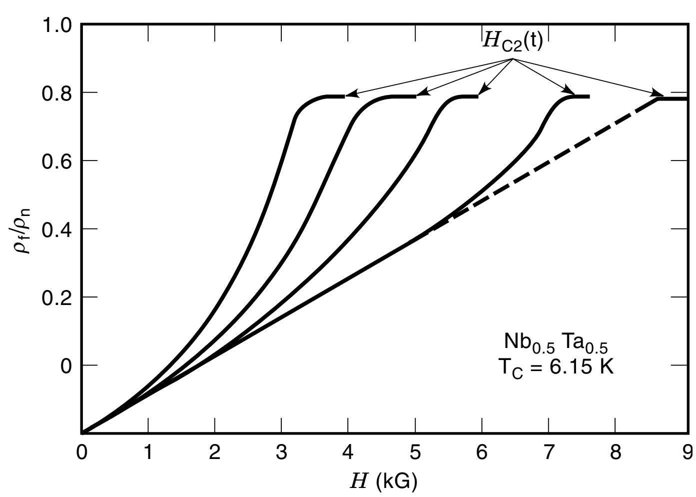

Figure 22. Variation of the flux-flow resistivity of a NbTa specimen with field at various constant reduced temperatures. (right to left—0, 30%, 50%, 60%, and 70% of the critical temperatures)

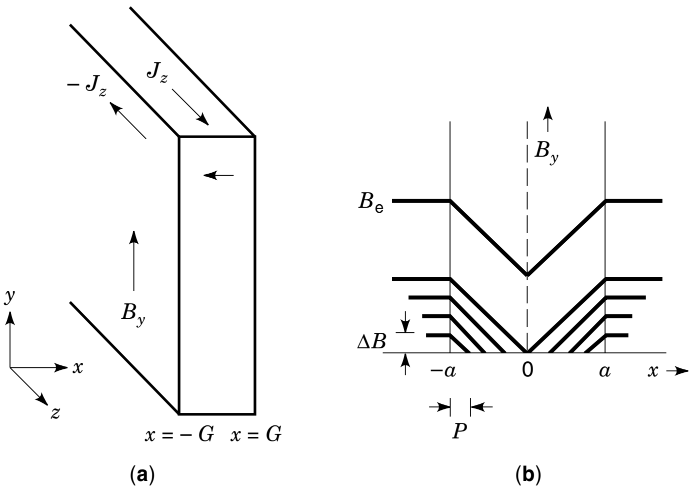

Figure 23. (a) Screening currents induced to flow in a slab by a magnetic field parallel to the slab surface; (b) Magnetic field pattern across the slab showing the reduction of internal field by screening currents.

as well, the net motion of the cores is at an angle to the transport current, but in most cases the angle is close to 90°. The electric field induced by the motion is given by

$$E = n_f \Phi_0 v_f$$

where n_f is the number of fluxons and v_f is the velocity of the fluxons (the voltage is given by the product of the electric field and length along the current direction).

Superconducting applications require zero or infinitesimally small resistance and therefore the regime of flux flow is required to be as close to the critical current as possible. Therefore, most applications require a high n value, which is given by

$$E \sim (I/I_c)^n$$

where I is the transport current and I_c is the critical current.

Flux Penetration and Flux Jump in a Type II Superconductor. According to the critical state model, when a field is applied in the exterior of the superconductor, screening currents would be induced to exclude the field. The cross section (proportional thickness in the slab shown in Fig. 23) of the current flow is equal to the total current divided by the critical current of the specimen. The current density in the material is always equal to the critical current density. As the field is increased, so is the thickness of the current sheet and after current flows throughout the cross section of the material, the field is fully penetrated and then increases inside the material as the field is raised.

Unlike in normal conducting materials, the penetration of the field in the superconductor does not reverse when the field change direction is reversed. If a field has penetrated well inside the material during one direction of change (say the field is increased) and if the field change direction is reversed (say the field is decreased), the field inside the superconductor initially decreases on the edge of the superconductor while the field inside the superconductor remains unaffected. As the field change is continued further this reduction continues into the thickness. Figure 24 shows this phenomenon schematically. This behavior of the diamagnetism causes the superconductor magnetization to be hysteretic (31). The magnetization hysteresis for a typical superconductor is shown in Fig. 25.

Now the critical state can be unstable because if there is perturbation in the form of a temperature increase, the critical current density is reduced, which then causes the fields in the superconductor to redistribute requiring motion of flux in the superconductor. This flux motion generates heat, resulting in the further increase of temperature. The larger the sensitivity of the critical current density to the temperature the larger the heat produced. The smaller the specific heat of the material, the larger the temperature increase for a given heat generated. Therefore, if conditions are unfavorable, the superconducting material will run away in temperature, resulting in the material quenching and the flux jumping inside the material. The condition beyond which such a flux jump would occur is given by

$$\mu_0 J_c^2 a^2 / [3\rho_m C(T_c - T_0)] < 1$$

where ρ_m, T_c, J_c, C, and a are the density, critical temperature, critical current density, specific heat, and thickness of the superconductor, and T_0 is the bath (initial) temperature (32). The result of the flux jump is that filaments of a superconductor cannot be larger than a certain size and when a superconductor is made of a large number of superconducting filaments of wire, these must be twisted to cancel the diamagnetic currents over short distances (33).

Superconductor Performance under ac Conditions

When superconductors are operated under ac conditions (applied magnetic field and currents), the superconductor response is significantly different from that of a good conductor. There are two reasons for this difference: (a) the magnetization of the superconductor is hysteretic, (b) there are saturation effects due to the criticality with respect to magnetic field and superconducting current capability. In addition, since usual superconductors are composites of multiple superconducting filaments as well as normal stabilizing conductors like copper, there are coupling effects due to mutual induc-

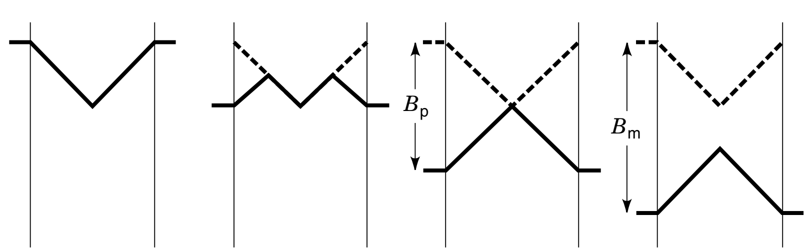

Figure 24. (a) Field pattern within a superconducting slab subjected to large field change; (b) as the field is reduced; (c) when the field change penetrates to center of slab; (d) when the field reaches a minimum value before rising again.

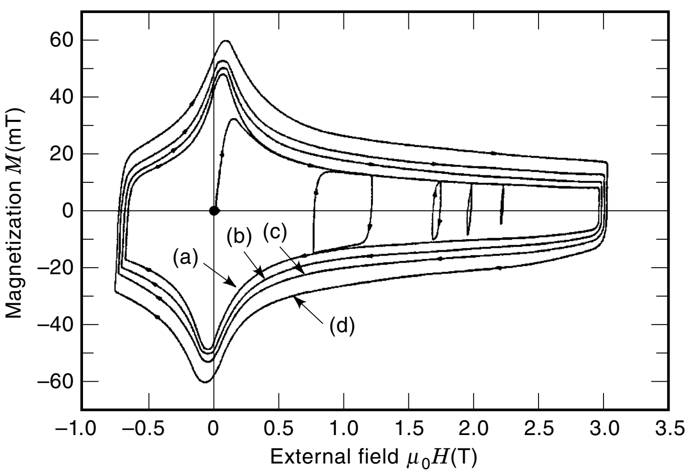

Figure 25. Magnetization of a 361-filament NbTi/copper composite with a twist pitch of 25.4 mm, measured at (a) 0.0075 $T \cdot s^{-1}$, (b) 0.0375 $T \cdot s^{-1}$, (c) 0.075 $T \cdot s^{-1}$, and (d) 0.15 $T \cdot s^{-1}$.

tance and cross conductance. The subject is wide, and details are strongly dependent upon the details of the superconductors and applications (34,35).

Under alternating magnetic field conditions, diamagnetic currents induce reverse magnetization, and since this magnetization is hysteritic, in a full cycle, an energy loss per unit volume of $\int [\boldsymbol{M} \cdot \boldsymbol{B}]\, d\boldsymbol{B}$ is incurred. This loss is similar to the hysteresis loss in ferromagnetic materials.

With an alternating magnetic field transverse to the superconductors, diamagnetic saddle currents flow similar to the normal conductors, but these currents are not limited by conductivity but rather by the critical current. If the superconductor is a monofilament, large currents would mostly flow in the skin of the superconductor, and the superconductor would be unstable. In order to limit the current, the superconductor is made of several filaments, twisted together and separated by a conducting matrix, for example, of copper. The induced voltage is then limited by the twist length, and the currents are distributed over all the filaments. However, for cryogenic stability and stability against flux jump, the matrix has to be made of high conductivity material, and this leads to the fact that the filaments couple somewhat through the matrix. Thus currents flowing along the filaments cross over through the matrix, and such currents induce ac losses in the superconductors. Such ac losses have to be limited to prevent heating of the conductor and subsequent quenching (transition to normal state).

Another issue under alternating field and current conditions is that the filaments or a cable of superconductors are not exactly identical due to differences in superconducting characteristics, twist pitches and end effects, and therefore currents may not be shared equally. A strong nonuniform current distribution can result from very small differences. The superconductor performance under those conditions would be significantly poorer than the sum of the individual filaments or superconductors.

MEASUREMENT OF FLUX

It is usually only necessary to measure magnetic fields, magnetization, etc. However, in special cases where the measurement of flux is desired in a static field region, a good way is to place a coil and move it transverse to the field direction so that the coupled flux changes. The time integral of induced voltage then gives the change in flux over the amplitude of motion. In instances where the flux is changed over time, a stationary coil enclosing the flux can be used. It is also appropriate to use flux density probes such as "Hall" probes over the area of interest and integrate the flux density over the area.

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RAGHAVAN JAYAKUMAR
Lawrence Livermore National Laboratory

MAGNETIC HYSTERESIS. See MAGNETIC NOISE, BARKHAUSEN EFFECT.