SKIN EFFECT

INTRODUCTION

The term *skin effect* is used to indicate that electromagnetic waves penetrate only a small distance into a conductor. Equivalently, the skin effect is the phenomenon that time-varying current densities are concentrated in a thin layer near the surface or skin of a conductor. The current densities are concentrated at the side of the conductor exposed to the source of these currents. These currents are often referred to by the term *eddy currents*.

For time-harmonic fields the amplitude of these fields decays exponentially with the distance from the surface. The distance over which the amplitude of the fields, and hence of the current density, decreases by a factor e is called the skin depth δ given by

$$\delta = \sqrt{\frac{2}{\omega\mu\sigma}} \tag{1}$$

Thus, the layer in which the fields are concentrated becomes thinner when the frequency $f = \omega/(2\pi)$ increases, the permeability μ increases, or the conductivity σ increases. In the limit of perfect conductors, fields and current density are concentrated in a vanishingly thin layer at the surface of the conductor.

If a current flows in a conductor with finite conductivity this will cause a voltage drop along the flow of the current. The ratio of the voltage drop to the current is the internal impedance of the conductor. If the current density is concentrated in a thin layer at the surface of the conductor this internal impedance, expressed per unit square, that is, for a unit length and a unit width of the conductor, is called the surface impedance Z_s. For a cylindrical wire with radius r this amounts to an internal impedance $Z_s/(2\pi r)$ per unit length. This surface impedance Z_s is given by

$$Z_s = \frac{1+j}{\sigma\delta} = \frac{1+j}{\sqrt{2}}\sqrt{\frac{\omega\mu}{\sigma}} \tag{2}$$

If Z_s is decomposed in its real and imaginary part as $Z_s = R_s + j\omega L_i$ then R_s is the surface resistance and L_i the internal inductance per unit square. The surface impedance also expresses the ratio between the tangential electric field and tangential magnetic field at the surface of a conductor.

For a conductor several skin depths thick, the electromagnetic fields will not penetrate through the conductor. For a hollow conductor, such as a coaxial cable, this means that the internal electromagnetic field is decoupled from the external field. This is the electromagnetic shielding property of conductors. Since for copper at 60 Hz the skin depth $\delta = 8.5$ mm, the skin effect is not only important at radiofrequencies but also plays an important role for the design of power transmission lines, electrical machines, and electrification of railways. Because of the concentration of the current density at the surface of the wires, the Ohmic losses per unit length and voltage drops per unit length are substantially higher than what would be expected from a uniform distribution of the current density over the cross section of the wire. This concentration also imposes a limitation on the useful diameter of wires for power transmission.

The dissipated power per unit length P_d by a current I in the skin layer is given by

$$P_d = \frac{1}{2}R_s|I|^2 = \frac{1}{2}\sqrt{\frac{\omega\mu}{2\sigma}}|I|^2 \tag{3}$$

It is thus seen that for a given current this dissipation increases with frequency and permeability and decreases with conductivity.

Strictly speaking, the previous expressions for δ, Z_s, and P_d are valid only in good conductors with planar surfaces. However, if the radii of curvature of the conductor surface are large compared to the skin depth δ, these formulas are still good approximations. Table 1 shows the skin depth δ and surface resistance R_s for a number of conductors. Except for iron, the relative permeability μ/μ_0, with μ_0 the free-space permeability, is equal to one for all metals.

When the surface of the conductor is rough, the losses inside the conductor will increase because the surface resistance increases. Porosity of the surface also will increase the surface resistance considerably.

To take the skin effect into account in numerical electromagnetic simulations, two approaches are possible. First, one can replace the boundary condition of a vanishing tangential electric field for a perfect conductor by an impedance boundary condition relating the tangential electric field to the tangential magnetic field through the surface impedance. In most cases this is an approximation, but it can be a very good one. A second approach is to also perform a simulation of the fields inside the conductor. For numerical techniques such as finite-element techniques, finite-difference techniques, or volume integral equation techniques, which discretize the volume of the conductors, a fine discretization is needed in order to accurately model the exponential decay of the fields inside the conductors. For boundary integral equation techniques the second approach does not entail extra complications.

THEORY

Plane Interfaces

To study the skin effect quantitatively consider the structure of Fig. 1, consisting of a conductor with planar interface that occupies the semiinfinite region $z > 0$ and that is characterised by the material parameters ε, μ, and σ. The region $z < 0$ is assumed to be free space with parameters ε_0 and μ_0. Outside the conductor and in the absence of sources, the electromagnetic fields in time-harmonic regime $e^{j\omega t}$ satisfy the Maxwell curl equations

$$\nabla\times \mathbf{E} = -j\omega\mu_0\mathbf{H} \tag{4}$$

$$\nabla\times \mathbf{H} = j\omega\epsilon_0\mathbf{E} \tag{5}$$

Inside the conductor the fields satisfy

$$\nabla\times \mathbf{E} = -j\omega\mu\mathbf{H} \tag{6}$$

$$\nabla\times \mathbf{H} = (j\omega\epsilon + \sigma)\mathbf{E} \approx \sigma\mathbf{E} \tag{7}$$

Table 1. Conductivities, Skin Depths and Surface Resistances for a Number of Conductors (*f* in H_z)

	σ	δ	R_s
Gold	41 MS/m	79 mm/$\sqrt{f}$	310 n$\Omega\sqrt{f}$
Silver	61.7 MS/m	64.1 mm/$\sqrt{f}$	253 n$\Omega\sqrt{f}$
Copper	58 MS/m	66.1 mm/$\sqrt{f}$	261 n$\Omega\sqrt{f}$
Aluminum	37.2 MS/m	82.5 mm/$\sqrt{f}$	326 n$\Omega\sqrt{f}$
Brass	15.7 MS/m	127 mm/$\sqrt{f}$	501 n$\Omega\sqrt{f}$
Iron (μ_r = 120)	10 MS/m	15 mm/$\sqrt{f}$	6.88 $\mu\Omega\sqrt{f}$
Tin	8.69 MS/m	171 mm/$\sqrt{f}$	674 n$\Omega\sqrt{f}$
Mercury	1.04 MS/m	494 mm/$\sqrt{f}$	1.95 $\mu\Omega\sqrt{f}$
Zinc	17.4 MS/m	121 mm/$\sqrt{f}$	476 n$\Omega\sqrt{f}$
Lead	4.8 MS/m	230 mm/$\sqrt{f}$	910 n$\Omega\sqrt{f}$
Platinum	9.66 MS/m	162 mm/$\sqrt{f}$	639 n$\Omega\sqrt{f}$
Human tissue (around 1 GHz)	1.2 S/m	460 m/$\sqrt{f}$	1.81 m$\Omega\sqrt{f}$

In the last equation the displacement current term was neglected compared to the conduction current term, which is allowed for all practical conductors at practical frequencies. For example, in platinum $\omega\varepsilon$ becomes comparable to σ only at 1.5 PHz. Taking the curl of Eqs. (4) and (<xref target="W4949-mdis-0006" style="unformatted">6</xref>), substituting Eqs. (5) and (<xref target="W4949-mdis-0007" style="unformatted">7</xref>) and using the Maxwell divergence equations $\nabla \cdot \mathbf{E} = \nabla \cdot \mathbf{H} = 0$, both in free space and in the conductor, shows that the electric field in free space satisfies the Helmholtz equation

$$\Delta^2\mathbf{E} + k_0^2\mathbf{E} = 0 \tag{8}$$

with $k^2{}_0 = \omega^2\varepsilon_0\mu_0$ and in the conductor the Helmholtz equation

$$\Delta^2\mathbf{E} - j\omega\mu\sigma\,\mathbf{E} = 0 \tag{9}$$

Now illuminate the conductor by a plane electromagnetic wave (see Fig. 1). Assume that this plane wave has *TE* (transverse electric) polarization, that is, according to the coordinate system of Fig. 1 the electric field has only a component in the *y*-direction. This incident electric field is given by

$$\mathbf{E}^{\mathrm{i}} = E_0\mathbf{u}_y e^{-jk_x x}e^{-jk_z z} \tag{10}$$

where $\mathbf{u}_y$ the unit vector in the *y*-direction. Substituting in Eq. (8) and taking into account an angle of incident θ^{i} yields $k_x = k_0 \sin\theta^{\mathrm{i}}$ and $k_y = k_0 \cos\theta^{\mathrm{i}}$. From Eq. (4) the incident magnetic field is then found to be

$$\mathbf{H}^{\mathrm{i}} = \frac{k_x\mathbf{u}_z - k_z\mathbf{u}_x}{\omega\mu_0}\, E_0 e^{-jk_x x}e^{-jk_z z} \tag{11}$$

where $\mathbf{u}_x$ and $\mathbf{u}_z$ unit vectors in *x*- and *z*-direction respectively. The incident plane wave will give rise to a reflected plane wave in free space and a transmitted plane wave in the conductor. All these plane waves will have the same phase variation $e^{-jk_x x}$ in the *x*-direction along the interface and they will all be TE polarized. This means that the reflected plane wave takes the form

$$\mathbf{E}^{\mathrm{r}} = RE_0\mathbf{u}_y e^{-jk_x x}e^{jk_z z} \tag{12}$$

$$\mathbf{H}^{\mathrm{r}} = \frac{k_x\mathbf{u}_z + k_z\mathbf{u}_x}{\omega\mu_0} RE_0 e^{-jk_x x}e^{jk_z z} \tag{13}$$

with R the reflection coefficient, which is still to be determined. The transmitted electric field in the conductor takes the form

$$\mathbf{E}^{\mathrm{t}} = E_0\mathbf{u}_y e^{-jk_x x} f(z) \tag{14}$$

Substituting in Eq. (9) shows that $f(z)$ satisfies

$$\frac{d^2 f(z)}{dz^2} - (j\omega\mu\sigma + k_x^2) f(z) = 0 \tag{15}$$

Because $k_x = k_0 \sin\theta^{\mathrm{i}}$ is of the same order of magnitude as k_0, we neglect the term $k^2{}_x$ in the previous equation on the same grounds for neglecting $\omega\varepsilon$ in Eq. (7). Taking into account that $f(z)$ should remain bounded for $z \to +\infty$ yields $f(z) = Te^{-(1+j)z/\delta}$ with δ the skin depth defined in Eq. (1). From Eqs. (14) and (6) the fields in the conductor are thus found to be

$$\mathbf{E}^{\mathrm{t}} = TE_0\mathbf{u}_y e^{-jk_x x}e^{-(1+j)z/\delta} \tag{16}$$

$$\mathbf{H}^{\mathrm{t}} = \frac{k_x\mathbf{u}_z - (1-j)/\delta\,\mathbf{u}_x}{\omega\mu}\, TE_0 e^{-jk_x x}e^{-(1+j)z/\delta} \tag{17}$$

where T the transmission coefficient. It is seen that the amplitudes of the fields decrease by a factor of e when they propagate over a distance perpendicular to the interface equal to the skin depth δ. The reflection and transmission coefficients follow from imposing the continuity of the tangential fields, that is of E_y

$$1 + R = T \tag{18}$$

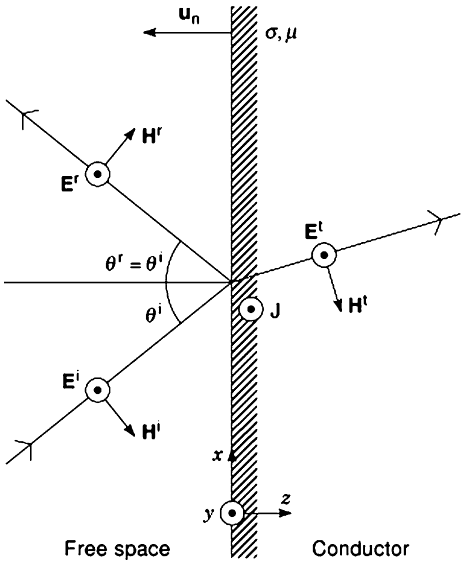

Figure 1. Incident plane electromagnetic TE wave on a plane conductor with conductivity σ and permeability μ. The angle of incidence is θ^i.

and H_x

$$-\frac{k_z}{\omega\mu_0} + R\frac{k_z}{\omega\mu_0} = -\frac{(1-j)T}{\omega\mu\delta} \tag{19}$$

at the interface $z = 0$. The solution of this set of equations is

$$R = \frac{k_z\mu_r\delta + 1 - j}{k_z\mu_r\delta - 1 + j} \tag{20}$$

and

$$T = \frac{2k_z\mu_r\delta}{k_z\mu_r\delta - 1 + j} \tag{21}$$

At the interface $z = 0$ the relation between the tangential electric field $\mathbf{E}_t$ and the tangential magnetic field $\mathbf{H}_t$ can be expressed as

$$\mathbf{E}_t = Z_S\mathbf{u}_n \times \mathbf{H}_t \tag{22}$$

where Z_s the surface impedance defined in Eq. (2) and $\mathbf{u}_n = -\mathbf{u}_z$ for the configuration of Fig. 1. Indeed from Eq. (16) it follows that $\mathbf{E}_t = TE_0\mathbf{u}_y e^{-jk_z z}$ and from Eq. (17) that $\mathbf{H}_t = -1 - j/\omega\mu\delta \;\; TE_0\mathbf{u}_x e^{-jk_x x}$ and hence that Z_s is given by Eq. (2).

The current density $\mathbf{J}$ in the conductor is $\sigma\mathbf{E}$ and from Eq. (16)

$$\mathbf{J} = \sigma TE_0\mathbf{u}_y e^{-jk_x x} e^{-(1+j)z/\delta} \tag{23}$$

The total current per unit length I flowing in the conductor from the integration of Eq. (23) over all $z > 0$ is given by $I = \mathbf{J}_s \cdot \mathbf{u}_y$ with

$$\mathbf{J}_s = \int_0^{+\infty} \sigma TE_0\mathbf{u}_y e^{-jk_x x} e^{-(1+j)z/\delta}\, dz = \frac{\sigma\delta}{1+j}\; TE_0\mathbf{u}_y e^{-jk_x x} \tag{24}$$

where the notation $\mathbf{J}_s$ indicates that this can be viewed as an equivalent surface current density. When this current flows across the conductor in the y-direction over a unit distance it causes a voltage drop V equal to $\mathbf{E}^t(z = 0) \times \mathbf{u}_y = TE_0 e^{-jk_x x}$. The ratio V/I can be seen as the internal impedance per unit square of the conductor and is given by the surface impedance Z_s. The real part of the surface impedance is the surface resistance R_s and the imaginary part the internal reactance ωL_i which are both equal to $1/(\sigma\delta)$. This means that the surface resistance can be seen as the resistance per unit square when all the current is homogeneously distributed over and concentrated in a layer with thickness δ at the surface of the conductor, that is, as the dc resistance of a planar conductor with thickness δ.

The dissipated power P_d in the skin layer, per unit distance in the x-direction, due to ohmic losses is given by

$$P_d = \frac{1}{2}\Re\int_0^{+\infty} \mathbf{J}^* \cdot \mathbf{E}^t\, dz = \frac{\sigma}{2}\int_0^{+\infty} |\mathbf{E}^t|^2\, dz = \frac{\sigma\delta}{4}\,|E_0|^2|T|^2 \tag{25}$$

From Eq. (23) it follows that

$$|I| = \frac{\sigma\delta}{\sqrt{2}}|T||E_0| \tag{26}$$

which allows one to recast Eq. (25) as

$$P_d = \frac{|I|^2}{2\sigma\delta} = \frac{1}{2}R_s|I|^2 \tag{27}$$

This shows that P_d can be seen as the power dissipated per unit length in the surface impedance by the surface current I.

The skin depth δ and the surface impedance Z_s are independent of $k_x = k_0 \sin\theta^i$, that is, independent of the angle of incidence of the plane wave. The previous derivations can also be repeated for a TM (transverse magnetic) polarized plane wave leading to the same conclusions. Only the expressions Eqs. (20) and (<xref target="W4949-mdis-0021" style="unformatted">21</xref>) for R and T will change. This means that the previous analysis remains valid for arbitrary plane waves incident on the conductor surface. The amplitudes of the fields inside the conductor will always decrease by a factor e after having traveled a distance δ given by Eq. (1), and the surface impedance Z_s is always given by Eq. (2) independent of the angle of incidence. Since an arbitrary incident field can always be expressed as a superposition of plane waves, these conclusions remain valid for arbitrary illuminations of the conductor. For more on the plane-wave interaction with conductors, we refer the reader to Stratton (1).

Curved Interfaces

To investigate the effect of curvature on the penetration of electromagnetic fields in a conductor, consider the structure of Fig. 2 consisting of a round wire, with radius r_0, conductivity σ, and permeability μ stretched along the z axis (see also Ref. 2). It is assumed that some time-harmonic z-directed current flows inside the wire that depends only on the radial coordinate r. Since $\mathbf{J} = \sigma\mathbf{E}$, it follows from Eq. (9) that the longitudinal current density J_z satisfies the

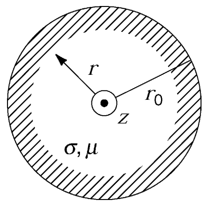

Figure 2. Round wire with radius r_0, conductivity σ and permeability μ stretched along the z-axis.

equation

$$\frac{d^2J_z}{dr^2} + \frac{1}{r}\frac{dJ_z}{dr} - j\omega\mu\sigma J_z = 0 \tag{28}$$

A general solution of this equation is $J_z = AJ_0[(1-j)r/\delta] + BY_0[(1-j)r/\delta]$ with $J_0(x)$ the Bessel function and $Y_0(x)$ the Neumann function of order zero and argument x. Since the current density needs to remain finite at the center of the wire, B should be zero. If J_z, at $r = r_0$, is denoted by $J_{z,0}$ then J_z can be expressed as

$$J_z = J_{z,0}\frac{J_0[(1-j)r/\delta]}{J_0[(1-j)r_0/\delta]} \tag{29}$$

In Fig. 3 $|J_z/J_{z,0}|$ is shown for different values of $a = \delta/r_0$ and compared with

$$\left|\frac{J_z}{J_{z,0}}\right| = \left|e^{-(1+j)(r_0-r)/\delta}\right| = e^{-(r_0-r)/\delta} \tag{30}$$

for a planar conductor with $r_0 - r$ the distance from the surface [see Eq. (23) with a change of coordinates]. One notes that for $r_0 > 7\delta$ there is a good agreement between both results. This means that for $r_0 > 7\delta$ the conductor can be regarded to be planar (i.e., the curvature can be neglected) with respect to the skin effect.

The total current I flowing inside the wire is given by

$$\begin{aligned} I &= \oint_S J_z\,dS = 2\pi\int_0^{r_0} J_{z,0}\frac{J_0[(1-j)r/\delta]}{J_0[(1-j)r_0/\delta]}\,dr \\ &= -\sqrt{2}\pi r_0 J_{z,0}(1+j)\delta\frac{J_0'[(1-j)rj_0/\delta]}{J_0[(1-j)r_0/\delta]} \end{aligned} \tag{31}$$

The voltage drop V per unit length is given by $E_z = J_z/\sigma$ at $r = r_0$ or

$$V = \frac{J_{z,0}}{\sigma} \tag{32}$$

From Eqs. (31) and (32) it follows that the internal impedance per unit length of the wire is given by

$$\begin{aligned} Z_i &= \frac{V}{I} = -\frac{1-j}{2\sqrt{2}\pi r_0\sigma\delta}\frac{J_0[(1-j)r_0/\delta]}{J_0'[(1-j)r_0/\delta]} \\ &= \frac{Z_s}{2\pi r_0}\frac{j}{\sqrt{2}}\frac{J_0[(1-j)r_0/\delta]}{J_0'[(1-j)r_0/\delta]} \end{aligned} \tag{33}$$

Comparing Z_i with $Z_s/(2\pi r_0)$ gives an indication of the radius of curvature above which it is possible to use the surface impedance Z_s for a planar conductor to calculate the internal impedance of a curved conductor. For a 10% error on R_s it is easily determined that r_0/δ should be larger than 5.5 and for the same error on ωL_i, r_0/δ should be larger than 2.2. In (3) the results shown in Fig. 3 are compared to spherically curved surfaces.

Thin Conducting Layer

Instead of a semi-infinite conducting space as shown in Fig. 1, consider a thin conducting layer with thickness d, conductivity σ, and permeability μ. This thin layer between $z = 0$ and $z = d$ is embedded in free space. Assume fields inside this layer that only depend on the z coordinate. Taking an x and y dependence into account will not change the conclusions of this section. Assume also, without loss of generality, that the electric field is oriented along the y axis and the magnetic field along the x axis. If the displacement current in the layer is neglected, the total current I flowing in the conductor per unit length in the x direction is given by $I = \mathbf{J}_s \cdot \mathbf{u}_y$ with

$$\begin{aligned} \mathbf{J}_s &= \mathbf{u}_z \times [\mathbf{H}(z=d) - \mathbf{H}(z=0)] \\ &= [H_x(z=d) - H_x(z=0)]\mathbf{u}_y \end{aligned} \tag{34}$$

Maxwell's equations Eqs. (<xref target="W4949-mdis-0006" style="unformatted">6</xref>) and (<xref target="W4949-mdis-0007" style="unformatted">7</xref>) relate the electric and magnetic fields at $z = d$ and $z = 0$:

$$\begin{aligned} H_x(z=d) &= -\frac{j}{Z_s\sin[(1-j)d/\delta]} \\ &\times \{\cos[\frac{(1-j)d}{\delta}]E_y(z=0) - E_y(z=0)\} \end{aligned} \tag{35}$$

$$\begin{aligned} H_x(z=d) &= -\frac{j}{Z_s\sin[(1-j)d/\delta]} \\ &\times \{E_y(z=0) - \cos[\frac{(1-j)d}{\delta}]E_y(z=0)\} \end{aligned} \tag{36}$$

From Eq. (34) it now follows that

$$I = -\frac{j\{\cos[(1-j)d/\delta] - 1\}}{Z_s\sin[(1-j)d/\delta]}[E_y(z=d) + E_y(z=0)] \tag{37}$$

The voltage drop over a unit distance caused by the current I is with good approximation given by $V \approx E_y(z = d/2) \approx [E_y(z = d) + E_y(z = 0)]$. This approximation is valid as long as the conductor is thin compared to the wavelength in free space. Hence, the internal impedance per unit square of the thin layer is given by

$$Z_i = \frac{V}{I} = \frac{Z_s\sin[(1-j)d/\delta]}{2j\{1-\cos[(1-j)d/\delta]\}} = \frac{Z_s}{2j}\tan\frac{(1-j)d}{2\delta} \tag{38}$$

This means that under the aforementioned restrictions a thin conducting layer can be replaced with an infinitely thin conducting sheet with impedance Z_i.

In the low-frequency limit, that is, in the limit where the thickness d of the layer is small compared to the skin depth δ, one verifies that

$$Z_i = \frac{1}{\sigma d} \tag{39}$$

In this case the current density is homogeneously distributed over the conductor and the internal impedance becomes equal to the DC resistance of a planar conductor with thickness d.

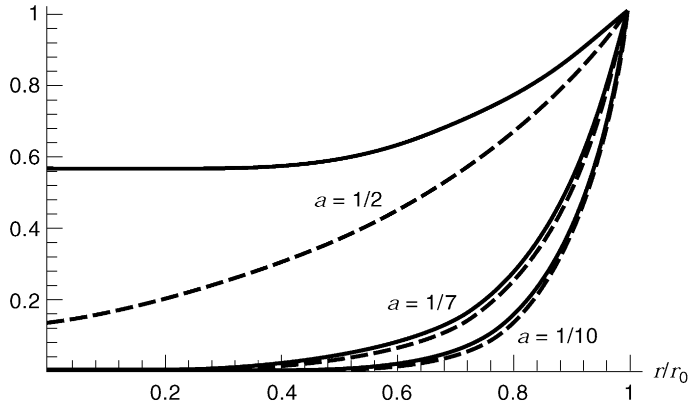

Figure 3. Current density distribution in a conducting wire (continuous lines) compared with the current distribution in a plane conductor (dashed lines) for different values of $a = r_0/\delta$ where r_0 is the radius of the wire and δ the skin depth.

At high frequencies, that is, when the thickness d of the layer exceeds several skin depths δ, one verifies that

$$Z_i = \frac{Z_s}{2} \tag{40}$$

In this case all the current is concentrated in thin sheets at the top and the bottom of the layer. Each of these current sheets yields an internal impedance Z_s. Since both impedances are in parallel, the total internal impedance is $Z_s/2$.

A General Conducting Object

Consider a conducting object with surface S embedded in free space. If the radius of curvature or the thickness of this object is not too small compared to the skin depth δ then, as described in Section 2.2, Equation (22) can be used to express the relation between the tangential electric field $\mathbf{E}_t$ and the tangential magnetic field $\mathbf{H}_t$. If this is not the case, one can, in principle, still replace the object by a boundary condition that relates $\mathbf{E}_t$ and $\mathbf{H}_t$ at its surface. The most general form of this relation is

$$\mathbf{E}_t(\mathbf{r}) = \int_S \bar{\bar{Z}}_{s,tt}(\mathbf{r}|\mathbf{r}') \cdot [\mathbf{u}_n \times \mathbf{H}_t(\mathbf{r}')]dS', \tag{41}$$

where $\bar{\bar{Z}}_{s,tt}(\mathbf{r}|\mathbf{r}')$ is a planar dyadic function and where $\mathbf{u}_n$ is the normal vector to the surface S. For high frequencies $\bar{\bar{Z}}_{s,tt}(\mathbf{r}|\mathbf{r}')$ will become $Z_s\bar{\bar{I}}_{tt}\delta(\mathbf{r}-\mathbf{r}')$ with $\bar{\bar{I}}_{tt}$ the planar unit dyadic. The dyadic function $\bar{\bar{Z}}_{s,tt}(\mathbf{r}|\mathbf{r}')$ takes the shape and the material parameters of the object into account. It can be derived using numerical techniques such as boundary integral equations. For an example we refer to (4). Note that the object can have an inhomogeneous material distribution.

Surface Roughness

Because of the increased surface area, the surface resistance increases when the conductor has a rough surface. For limited surface roughness the relative increase of the surface resistance $\Delta R_s/R_s$ is proportional to the RMS roughness ρ according to the empirical law

$$\frac{\Delta R_s}{R_s} = 0.29\frac{\rho}{\delta} \tag{42}$$

This law is valid up to $\rho/\delta \approx 1.5$, independent of the type of material or frequency. At $\rho/\delta \approx 3$ the relative increase saturates at $\Delta R_s/R_s \approx 0.6$ [see (4)]. Not only surface roughness but also porosity of the surface will substantially increase the surface resistance. The increase in resistivity will result in an increase of the dissipation according to the first part of Eq. (3).

Numerical Simulations

In numerical electromagnetic simulation techniques the finite conductivity is most easily taken into account by imposing the impedance boundary condition Eq. (22) instead of $\mathbf{E}_t = 0$ for perfect conductors. The relation Eq. (22) can also be expressed as

$$\mathbf{E}_t = \mathbf{Z}_s\mathbf{J}_s \tag{43}$$

where $\mathbf{J}_s$ is the current density concentrated at the surface.

Consider the electric field integral equation for perfect conductors of the form

$$0 = \lim_{r\to S}[\mathbf{E}_t^i(\mathbf{r}) + \int_S \bar{\bar{G}}_{tt}(\mathbf{r}|\mathbf{r}') \cdot \mathbf{J}_s(\mathbf{r}')dS'] \tag{44}$$

where $\mathbf{E}^i(\mathbf{r})$ is an incident electric field and $\bar{\bar{G}}_{tt}$ is the electric-electric Green dyadic. This integral equation is a Fredholm equation of the first kind. For a conductor with finite conductivity this integral equation is replaced by

$$\mathbf{Z}_s\mathbf{J}_s = \lim_{r\to S}[\mathbf{E}_t^i(\mathbf{r}) + \int_S \mathring{G}_{tt}(\mathbf{r}|\mathbf{r}') \cdot \mathbf{J}_s(\mathbf{r}')dS'] \tag{45}$$

which is a Fredholm integral equation of the second kind.

Inclusion of the surface impedance in the finite-element method goes along the same lines as for integral equation techniques. Consider the functional for a volume V with surface S and internal sources $\mathbf{J}$

$$\begin{aligned} F(\mathbf{E}) = & \frac{1}{2}\int_V [(\nabla\times\mathbf{E})\cdot(\nabla\times\mathbf{E}) - k^2\mathbf{E}\cdot\mathbf{E} + 2\,j\omega\mu\mathbf{E}\cdot\mathbf{J}]dV \\ & + j\omega\mu\int_S \mathbf{E}\cdot(\mathbf{u}_n\times\mathbf{H})dS \end{aligned} \tag{46}$$

with $\mathbf{u}_n$ the unit normal pointing into V and $k^2 = \omega^2\varepsilon\mu$. The surface impedance is now taken into account by replacing $\mathbf{u}_n \times \mathbf{H}$ in the surface integral term by $Z_s\mathbf{E}$.

The finite-difference time-domain technique is more complicated because of the frequency dependence of Z_s.

Equation (22) has to be expressed in the time domain, which involves a convolution integral

$$\mathbf{e}_t(t) = \mathbf{u}_n \times \int_0^t z_s(t-\tau)\mathbf{h}_t(\tau)d\tau \tag{47}$$

where $z_s(t)$ is the inverse Fourier transform of Z_s. After discretization with respect to time this convolution implies that in principle the magnetic fields of all previous time steps need to be remembered. However, several techniques (5) have been developed to limit the number of field values that have to be stored. In these techniques the surface impedance is approximated by a series of first-order rational functions in ω.

When analyzing the eigenmodes of resonators, an important quantity is the quality factor of the resonances. If the walls of the resonator consist of good conducting material, the quality factor Q can be estimated very well from a calculation of the eigenmodes in a resonator with perfectly conducting walls followed by a perturbation analysis taking into account the wall losses due to the skin effect. If $\mathbf{H}_m$ are the magnetic fields corresponding to a mode in the resonator, then Q is given by

$$Q = \frac{2}{\delta}\frac{\int_V |\mathbf{H}_m|^2 dV}{\int_S |\mathbf{u}_n \times \mathbf{H}_m|^2 dS} \tag{48}$$

with V the volume of the resonator and S its surface (6).

For an eigenmode in a waveguide with conducting walls, the skin effect will give rise to an attenuation of the eigenmodes. Just as the quality factor for a resonator, the attenuation constant α_m of an eigenmode can be estimated from the fields $\mathbf{E}_m$ and $\mathbf{H}_m$ of the eigenmode propagating in a waveguide with perfectly conducting walls. If the waveguide is oriented along the z axis, then α_m is given by

$$\alpha_m = \frac{R_s}{2}\frac{\int_c |\mathbf{u}_n \times \mathbf{H}_m|^2 dc}{\int_S (\mathbf{E}_m \times \mathbf{H}_m^*)\cdot \mathbf{u}_z dS} \tag{49}$$

where S is the crosssection of the waveguide and c is the contour cut out of the crosssection by the perfectly conducting walls. The attenuation of the eigenmode in dB/m is then given by 8.69 α_m.

In numerical simulations a thin conducting layer can be replaced by an infinitely thin sheet with a sheet condition given by

$$\mathbf{E}_t = Z_i \mathbf{u}_n \times \Delta\mathbf{H}_t \tag{50}$$

where Z_i is as given by Eq. (38) and $\Delta\mathbf{H}_t$ is the jump in the tangential magnetic field over the sheet. The tangential electric field $\mathbf{E}_t$ remains continuous over the sheet.

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