CONDUCTORS, ELECTRIC

In terms of their basic electrical attributes, materials at room temperature fall into three categories: insulators, semiconductor, and conductors. Conductors are those materials in which electrons or current can easily flow. They may exist as solids (silver, copper), as liquids (salt water, mercury) or in gaseous form (ionosphere). Simply put, conductors are used to form an electrical connection between two points. For the electrical engineer, conductors are generally employed to deliver power, be it megawatts of hydroelectric power on overhead lines or faint high-speed signals along metallic traces in a microchip. Given the wide variety of applications and the many materials with which to make conductors, choosing the right one can be challenging.

The most common materials encountered in the fabrication of electric conductors are metals, and the most popular metals used are silver, copper, gold, and aluminum. These metals have low electrical resistivity at room temperature and additional properties that make them quite attractive for certain applications. Silver is the best electrical conductor, and it does not tarnish in pure air or pure water. Silver is expensive, so it is rarely used to make electrical cables but is commonly found in solder alloys and electrical connectors or contacts. Copper is almost as good a conductor as silver but is much less expensive. Copper is used in most electrical cables, but it tarnishes easily so that insulators or coatings are used for protection. Gold is also a good electrical conductor and is unaffected by air. It is very expensive and not often used to fabricate cables, though it is often used for plating in order to provide a protective conductive coating on another material. Common electrical applications that make use of gold include connectors and contacts, and at higher frequencies it is used to fabricate microwave interconnects and integrated circuits. Aluminum is substantially more resistive than copper, but it still is considered to be a good electrical conductor. It also has a much smaller volume mass density than the other metals, which means that an electric cable made out of aluminum can be quite light in comparison with copper. The main electrical application of aluminum is in the fabrication of overhead power transmission cables, which should be made as light as possible. A few other metals are used for highly specialized applications. Tin and lead, for example, are poor electrical conductors, but they are commonly found in solder alloys since they have a low melting point. Tin alloys are commonly found in fuses. Tungsten has the highest melting point of all metals, which makes it attractive for high-temperature applications such as the filaments in light bulbs and as heating elements.

This article discusses the electrical performance of conductors. The classical free electron gas model of conductors is presented in the first section. The second section deals with the direct current (dc) performance of conductors. The third section presents an analysis of circular cross-section conductors under alternating current (ac) excitation, and the last section deals with two common transmission line geometries constructed using a pair of conductors.

THE FREE ELECTRON GAS MODEL OF CONDUCTORS

Any material that allows the passage of an appreciable current density may be called a *conductor*. Current density is understood as being a flow of electric charge in motion per unit area of conductor. Conductors are generally fabricated from metals such as copper, aluminum, or gold since these elements, when arranged in a solid, readily give up a valence electron. In a metal conductor, the electron is the elemental charge responsible for the transport of current. The widely accepted classical model for a metal conductor is that of a free electron gas where roughly each atom constituting the conductor donates an electron that is free to move under the application of an electric field. This simple model explains reasonably well the behavior of conductors that is observed experimentally on a macroscopic scale.

The free-electron gas model of a perfect conductor has interesting consequences. First, it implies that a perfect conductor cannot sustain an internal electric field if charges are at rest. Consider a static electric field applied via a voltage source to an isolated conducting block having a net charge of zero, as shown in Fig. 1. The externally applied electric field causes the free electrons in the conductor to move about until they reach a region where the total perceived electric field

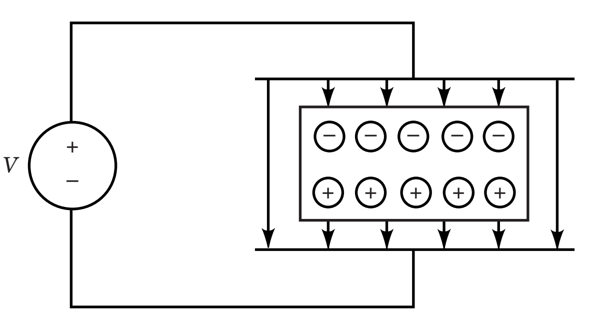

Figure 1. The spatial distribution of electrons in a block of perfectly conductive material when the block is placed in a static electric field. The electrons leave behind ionized atoms such that the total electric field inside the block is zero.

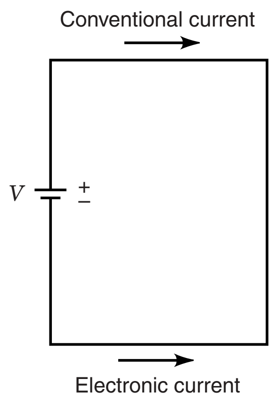

Figure 2. The direction of the electronic and conventional currents in a conductive wire connected to a battery. The conventional current is commonly employed in electrical circuit analysis.

and the force acting upon them is zero. By migrating to the top surface, the free electrons leave behind ionized atoms, which creates diametrically opposed regions of net positive and negative charge concentrations. The positive and negative charge concentrations, ideally, are evenly distributed on the bottom and top surfaces of the conducting block, as shown in Fig. 1, and create an electric field inside the conductor that cancels out exactly with the externally applied electric field. Thus, the net electric field inside the conductor is zero.

A second implication of the free-electron gas model is that an electric field tangential to a perfectly conducting surface must vanish on the surface while an electric field normal to a conducting surface may be nonvanishing. Referring to Fig. 1, the external electric field is tangential to the vertical walls of the block and must vanish on these walls while the field is normal to the top and bottom surfaces of the block and is allowed to exist on these surfaces. Furthermore, in the regions outside the conducting block, the electric field remains undisturbed by the introduction of the block; that is, the field in those regions is the same whether the block is present or not.

Finally, the free-electron gas model implies that an isolated negatively charged conductor will see its net charge migrate toward the exterior surfaces of the conductor where the charge carriers will distribute themselves evenly. Since electrons repel each other, we can imagine that they will tend to move as far away from one another as possible. At equilibrium, the distribution of net electric charge must be such that the sum of all forces perceived by individual electrons is zero. The application of an external electric field to a charged conductor will modify the charge distribution in the same way as in an uncharged conductor.

CONDUCTORS UNDER DC EXCITATION

When a conducting wire is connected to a battery to form a closed loop, a current carried by free electrons in motion flows in the wire. As shown in Fig. 2, the path followed by the negative charge carriers in the wire is out of the negative terminal of the battery, through the wire and into the positive terminal of the battery. The flow of current along this path is commonly referred to as the *electronic current.* For historical reasons, we assign the current direction in the opposite direction, along the path followed by a hypothetical positive charge carrier. This direction is known as the *conventional current flow* and is assigned such that it leaves the positive terminal of a battery or voltage source. The direction of the conventional current is consistent with the direction of an electric field, which is set by the motion of a positive test charge placed within the field.

Macroscopic View of Current Flow in Conductors

In a perfect conductor, the current flowing under the application of a voltage source is infinite. In a real conductor, the current may be quite large but remains finite due to energy losses encountered by the electrons in motion. Furthermore, since the electrons are in motion, an electric field is allowed to exist in the conductor and a voltage drop can be measured across the wire. Ohm's law states that the resistance of a circuit element is given by the ratio of the voltage drop across the element to the current flowing through it in the direction of the drop:

$$R = \frac{V}{I} \tag{1}$$

Resistance is quoted in units of Ω.

The resistance of a conductive element depends on its geometry as well as its material composition. The resistance of a conductor having a length L and invariant cross section A is found experimentally to be proportional to L and inversely proportional to A:

$$R = \rho\frac{L}{A} \tag{2}$$

The constant of proportionality is the resistivity ρ which is material- and temperature-dependent and has units of $\Omega \cdot m$. The conductivity σ is the reciprocal of the resistivity and has units of siemens per meter (S/m). The simple equation given above for the resistance of a conductive element holds as long as the cross section of the element remains invariant along its length. In such a case, the dc current density flowing in the element can be assumed to be constant over its cross section.

The resistivity of a material can be determined experimentally by measuring (1) the voltage drop V_c across a cube of the material 1 m on a side and (2) the current I_c flowing through the cube. The cube is clamped as shown in Fig. 3 in order to ensure a uniform distribution of current density in the cross-section of the material. According to Eqs. (1) and (2), the resistivity of the material comprising the unit cube is obtained directly as the ratio V_c/I_c.

The resistivity of metals generally increases with temperature. For a small range of temperature near 20 °C, ρ can be assumed to vary in a linear fashion according to

$$\rho_T = \rho_0[1 + \alpha(T - T_0)] \tag{3}$$

where ρ_T is the resistivity at the temperature T, ρ_0 is the known resistivity of the material at a standard temperature T_0, and α is the temperature coefficient of resistivity for the particular metal. The temperatures in Eq. (3) are usually in °C and α is usually quoted in °C^{-1}. Table 1 gives values for

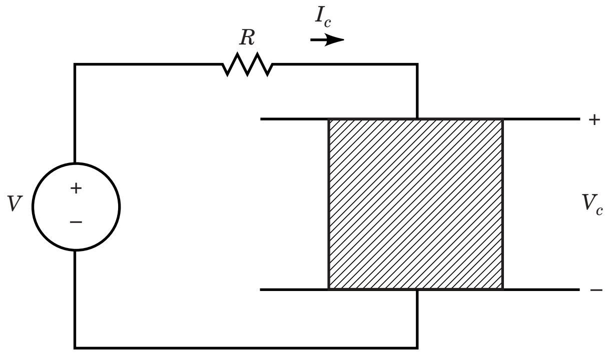

Figure 3. A typical experimental setup used to measure the resistivity of a unit cube of material, shown as the hatched region. The block is clamped and a known dc voltage source V is applied; the current I_c is then measured and the resistivity ρ is deduced via Eqs. (1) and (2). The lumped resistance R models the resistance of the setup without the block.

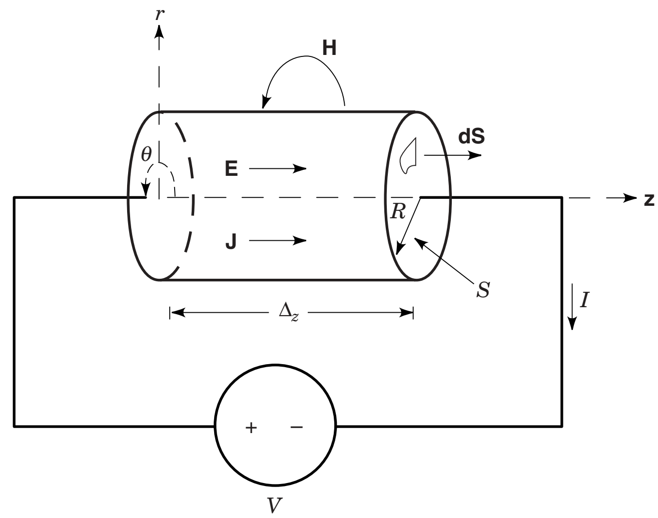

Figure 4. A conductive cylinder of length Δ_z and radius R connected to a voltage source V. The voltage source generates the current density J and the electric field E inside the conductor. The magnetic field H is generated by the current flowing in the cylinder and loops around the conductor, as shown. The cylindrical coordinate system is also given for reference.

the resistivity and for the temperature coefficient of resistivity for the metals most commonly encountered in the fabrication of conductors. Most of the values quoted in this table have been obtained from Ref. 1. To be considered an electric conductor, a material must have a resistivity less than $10^{-5}\,\Omega\cdot m$.

Microscopic View of Current Flow in Conductors

The current density **J** is a vector function that describes the magnitude and direction of the current flow per unit area at a point inside a conductor; its units are A/m^2. The current I is a macroscopic scalar quantity and is obtained from **J** via integration:

$$I = \int\int_S \mathbf{J}\cdot\mathbf{dS} \tag{4}$$

where **dS** is a surface element of the area S through which the current I flows, as shown in Fig. 4. The positive direction of **J** at any point is taken as the direction of a positive test charge placed at that point and is generally in the direction of the local electric field; this direction is consistent with that of the conventional current. If the current density is constant over the surface of integration and perpendicular to it, then the current flowing through is simply $I = JS$.

It can easily be shown that Ohm's law, given by Eq. (1), holds for microscopic quantities (2). The microscopic version of Ohm's law is called Ohm's law at a point, and it relates the electric field to the current density at any point inside a linear, homogeneous, and isotropic conductive material via

$$\rho = \frac{\mathbf{E}}{\mathbf{J}} \tag{5}$$

The units of the electric field intensity **E** are V/m. A conductive material is linear if its resistivity does not depend on **E** or **J**, is homogeneous if its resistivity is the same everywhere, and is isotropic if its resistivity is independent of the orientation of **E**.

In a real conductor, electrons move under the application of an electric field, since a force proportional to **E** acts upon

Table 1. Resistivity ρ and Temperature Coefficient of Resistivity α at 20 °C for the Metals Most Commonly Used for the Fabrication of Conductors[a]

Metal	ρ ($\Omega\cdot$m)	α (°C^{-1})	Range of Validity (°C)
Silver (high purity)	1.586×10^{-8}	0.0061	0–100
Copper (high purity)	1.678×10^{-8}	0.0068	0–500
Gold (high purity)	2.24×10^{-8}	0.0083	0–100
Aluminum (99.996%)	2.6548×10^{-8}	0.00429	
Magnesium	4.45×10^{-8}	0.0165	
Tungsten	5.6×10^{-8}	0.0045	
Zinc	5.916×10^{-8}	0.00419	0–100
Nickel	6.84×10^{-8}	0.0069	0–100
Iron (99.99%)	9.71×10^{-8}	0.00651	
Platinum (99.85%)	10.6×10^{-8}	0.003927	0–100
Tin	12.034×10^{-8}	0.0047	0–100
Lead	20.648×10^{-8}	0.00336	20–40

[a] The temperature range of validity for α is given if it is known.

them. This force does not lead to an infinite velocity since the electrons collide repeatedly with other particles in the material. The collisions cause the electrons to lose energy and to change their direction of motion in a random manner. However, if **E** is constant and the material is linear and homogeneous, the electrons will drift at a constant average velocity in the direction opposite to the electric field. The drift velocity $\mathbf{v_d}$ is proportional to the electric field and is given by

$$\mathbf{v_d} = \mu\mathbf{E} \tag{6}$$

where constant of proportionality μ is defined as the mobility of the electrons in the conductive material; the mobility has units of $m^2/(V \cdot s)$. The drift velocity points along the direction of the electric field which is also in the direction of the conventional current density; $\mathbf{v_d}$ has units of m/s.

The current density at a point in the conductor may be related to $\mathbf{v_d}$ via

$$\mathbf{J} = q\mathbf{v_d} \tag{7}$$

where q is the volume charge density in C/m^3 at the same point in the conductor. From Eqs. (5)–(7), we observe that the conductivity of a material is related to its mobility through

$$\sigma = \frac{1}{\rho} = q\mu \tag{8}$$

The volume charge density q is defined as the number of free electrons per unit volume times the elementary unit of charge:

$$q = Ne \tag{9}$$

where $e = 1.6021892 \times 10^{-19}$ C and N has units of m^{-3}. If we assume that every atom of the conductor makes available one valence electron for conduction, then the number of free electrons per unit volume of conductor is given by

$$N = \frac{N_A D}{W_a} \tag{10}$$

where $N_A = 6.022045 \times 10^{23}$ mol^{-1} is Avogadro's number, D is the metal's volume mass density in kg/m^3, and W_a is the atomic weight of the metal in kg/mol. The conductivity is therefore related to fundamental material quantities via

$$\sigma = \frac{N_A D}{W_a} e\mu \tag{11}$$

Table 2 gives the atomic weight, volume mass density, mobility and volume charge density of electrons for some of the most popular metals used to fabricate conductors. The values quoted in this table have been obtained from a number of references.

CONDUCTORS UNDER AC EXCITATION

Many applications make use of conductors to transmit time varying electrical signals. Most time-varying signals are sinusoidal in form or can be decomposed into a linear combination of sinusoidal signals at different frequencies. The analysis of circuit elements, including conductors, can therefore be made assuming a sinusoidal or ac excitation without any loss of generality.

The behavior of conductors under ac excitation may be significantly different from their dc behavior depending on the frequency of the signal. The frequency-dependent electrical parameters of an isolated conductor are its resistance and its inductive reactance. Both of these parameters generally increase with frequency and cause the ac impedance of a conductor to be larger than its dc resistance.

Current Density and the Skin Effect in a Conductor

The current density is uniformly distributed over the cross-section of a conductor under dc excitation only. Under ac excitation, the current density is nonuniform. In a circular cross-section conductor, the current density is usually greatest around the outside perimeter and decreases toward the center. This effect is referred to as the *skin effect,* and it becomes more pronounced as the frequency of excitation increases. The expression for the ac current density **J** in a conductor is obtained by deriving from Maxwell's equations the governing differential equation for **J** and finding the appropriate solutions (3).

Consider the conducting wire of length ΔZ and radius R shown in Fig. 4, across which a time-harmonic voltage V is applied and through which the current density phasor **J** flows. For a conductor having a finite conductivity σ, a time-varying electric field **E** in the longitudinal direction shown is present. In general, a time-varying electric field induces a time-varying magnetic field and vice versa. The relationship between these fields in our conductor is formulated mathematically as Maxwell's equations which read in the frequency domain and in differential form:

$$\nabla \times \mathbf{H} = \sigma\mathbf{E} + j\omega\epsilon\mathbf{E} \tag{12}$$

$$\nabla \times \mathbf{E} = -j\omega\mu\mathbf{H} \tag{13}$$

$$\nabla \cdot \mathbf{E} = 0 \tag{14}$$

$$\nabla \cdot \mathbf{H} = 0 \tag{15}$$

where $\omega = 2\pi f$ is the angular frequency of excitation in rad/s, σ is the conductivity in S/m of the material comprising the

Table 2. Atomic Weight, Volume Mass Density, Mobility, and Volume Charge Density of Some Metals at 20 °C

Metal	W_a (g/mol)	D (kg/m^3)	μ ($m^2/(V \cdot s)$)	q (C/m^3)
Silver (high purity)	107.86815	10.5×10^3	0.00671	9.39×10^9
Copper (high purity)	63.546	8.92×10^3	0.00440	1.35×10^{10}
Gold (high purity)	196.9665	19.32×10^3	0.00472	9.464×10^9
Aluminum (high purity)	26.98154	2.7×10^3	0.0039	9.7×10^9

conductor, ϵ is its electrical permittivity in F/m, μ is its magnetic permeability in H/m, not to be confused with the mobility of charge carriers, and $\mathbf{H}$ is the magnetic field intensity in A/m. Since our conductor has a net electric charge of zero, Maxwell's equation from Gauss' law for electric fields, stated as Eq. (14), must equal zero.

The differential equation governing the conduction current density in our conductor is derived as follows. Taking the curl of Eq. (13)

$$\nabla \times \nabla \times \mathbf{E} = -j\omega\mu\nabla \times \mathbf{H} \tag{16}$$

and substituting into Eq. (12) yields

$$\frac{-1}{j\omega\mu}\nabla \times \nabla \times \mathbf{E} = \sigma\mathbf{E} + j\omega\epsilon\mathbf{E} \tag{17}$$

Substituting the vector identity

$$\nabla \times \nabla \times \mathbf{E} = \nabla(\nabla \cdot \mathbf{E}) - \nabla^2\mathbf{E} \tag{18}$$

into Eq. (17)

$$\frac{-1}{j\omega\mu}[\nabla(\nabla \cdot \mathbf{E}) - \nabla^2\mathbf{E}] = \sigma\mathbf{E} + j\omega\epsilon\mathbf{E} \tag{19}$$

and making use of Eq. (14) yields

$$\frac{1}{j\omega\mu}\nabla^2\mathbf{E} = \sigma\mathbf{E} + j\omega\epsilon\mathbf{E} \tag{20}$$

The permittivity ϵ of a metal is usually near that of free space: $\epsilon = \epsilon_0 = 8.85418782 \times 10^{-12}$ F/m. It is quite clear from the above and Eq. (12) that the displacement current density $j\omega\epsilon\mathbf{E}$ is negligible in a good conductor compared to the conduction current density $\sigma\mathbf{E}$ since usually $\sigma \gg \omega\epsilon$. Neglecting the displacement current density, substituting Ohm's law at a point, stated as Eq. (5), and rearranging the above yields the differential equation governing the vector current density in our conductor:

$$\nabla^2\mathbf{J} - j\omega\mu\sigma\mathbf{J} = 0 \tag{21}$$

The main current component of $\mathbf{J}$ is directed along z, as shown in Fig. 4, such that the radial and angular components may be neglected without much loss of accuracy; thus $J_r = J_\theta = 0$. Furthermore, since the structure is circular symmetric about the z axis, we may simplify the functional dependence of J_z on θ by setting $\partial J_z/\partial\theta = 0$. Finally, since the structure is invariant along z and ΔZ is very small compared to the wavelength, we may also set $\partial J_z/\partial z = 0$. Applying these simplifications to the above equation and expanding $\nabla^2 J_z$ in cylindrical coordinates yields the scalar ordinary differential equation that governs J_z:

$$\frac{d^2}{dr^2}J_z + \frac{1}{r}\frac{d}{dr}J_z - j\omega\mu\sigma J_z = 0 \tag{22}$$

where r is the radial dimension in m. Multiplying the above by r^2 and introducing the notation

$$T^2 = -j\omega\mu\sigma \tag{23}$$

yields

$$r^2\frac{d^2}{dr^2}J_z + r\frac{d}{dr}J_z + (Tr)^2J_z = 0 \tag{24}$$

which is recognized as Bessel's equation of order zero. The parameter T simplifies to

$$T = \frac{1-j}{\delta} \tag{25}$$

where we have introduced δ, which is known as the skin depth of the material. For a good conductor ($\sigma \gg \omega\epsilon$), the skin depth is given by

$$\delta = \sqrt{\frac{2}{\omega\mu\sigma}} \tag{26}$$

and has units of m. The parameter T is thus seen to have units of m^{-1}.

The solution to Bessel's equation [Eq. (24)] is found in a number of advanced applied mathematics or electromagnetics textbooks (4,5). A general solution is the following linear combination of Bessel functions:

$$J_z(r) = AJ_0(Tr) + BY_0(Tr) \tag{27}$$

where A and B are constants, J_0 is Bessel's function of the first kind of order zero, not to be confused with current density, and Y_0 is Bessel's function of the second kind of order zero. These functions are defined, tabulated, and graphed in a number of mathematical textbooks and handbooks (6). A glance at the graph of Y_0 reveals that this function has a pole at the origin, where $Tr = 0$. Since our current must be finite at that point, we must impose $B = 0$. The remaining constant A can be determined by imposing a boundary condition at $r = R$. According to Eq. (5), the current density at $r = R$ must satisfy $J_z(R) = \sigma E_z(R)$, where $E_z(R)$ is the longitudinal electric field, tangential to the surface of the conductor. Applying this condition and $B = 0$ to Eq. (27) and solving for A yields

$$A = \frac{\sigma E_z(R)}{J_0(TR)} \tag{28}$$

Substituting the above into Eq. (27) yields the expression for the current density in the conductor:

$$J_z(r) = \sigma E_z(R)\frac{J_0(Tr)}{J_0(TR)} \qquad \frac{\mathrm{A}}{\mathrm{m}^2} \tag{29}$$

where $J_0(u)$ is expressed as the infinite sum:

$$J_0(u) = 1 - \frac{u^2}{2^2} + \frac{u^4}{2^2 \cdot 4^2} - \frac{u^6}{2^2 \cdot 4^2 \cdot 6^2} + \frac{u^8}{2^2 \cdot 4^2 \cdot 6^2 \cdot 8^2} - \cdots \tag{30}$$

The arguments of the Bessel functions in Eq. (29) are complex; and based on the above expression, if u is complex, then so is the Bessel function.

The magnitude of the current density in a circular cross-section conductor is often taken as

$$|J_z(r)| = \sigma|E_z(R)|e^{-(R-r)/\delta} \qquad \frac{\mathrm{A}}{\mathrm{m}^2} \tag{31}$$

Table 3. Skin Depth of Some Metals at 20 °C for Three Frequencies

Metal	δ at 60 Hz (m)	δ t 1 MHz (m)	δ at 30 GHz (m)
Silver (high purity)	8.183×10^{-3}	6.338×10^{-5}	3.659×10^{-7}
Copper (high purity)	8.417×10^{-3}	6.520×10^{-5}	3.764×10^{-7}
Gold (high purity)	9.73×10^{-3}	7.53×10^{-5}	4.35×10^{-7}
Aluminum (high purity)	1.0587×10^{-2}	8.2004×10^{-5}	4.7345×10^{-7}

which is the current density in a flat conductive medium due to an infinite plane wave, normally incident at $r = R$. The above is a simple expression that provides some physical insight and is a good approximation to Eq. (29) as long as the ratio R/δ is large. From the above, the skin depth is seen as being the radial distance δ from the outside surface of the conductor where the current density is reduced to $1/e$ or about 36.8% of its maximum value at $r = R$. Hence δ is also known as the depth of penetration. As can be seen from Eq. (26), the skin depth depends on the inverse square root of the conductivity and of the frequency. Table 3 gives δ at a few frequencies for the most popular metals used to fabricate conductors. As can be seen from these data, the depth of current penetration at low frequencies is of the order of a centimeter and is about four orders of magnitude larger than the depth of penetration at millimeter-wave frequencies, which is of the order of half a micrometer. It is also noteworthy that the depth of penetration increases as the conductivity of a material decreases.

Figure 5 shows the variation of the normalized current density with radial position in a copper conductor having a radius $R = 0.5$ mm for a number of frequencies. The solid curves are computed using the exact expression stated as Eq. (29), and the dashed curves are computed via the exponential approximation given by Eq. (31). The skin effect is evident at higher frequencies since the current density decreases dramatically with decreasing r, from its maximum value at the conductor perimeter. We note also that the approximate expression agrees reasonably well with the exact expression as long as R/δ is large, at least greater than 8.

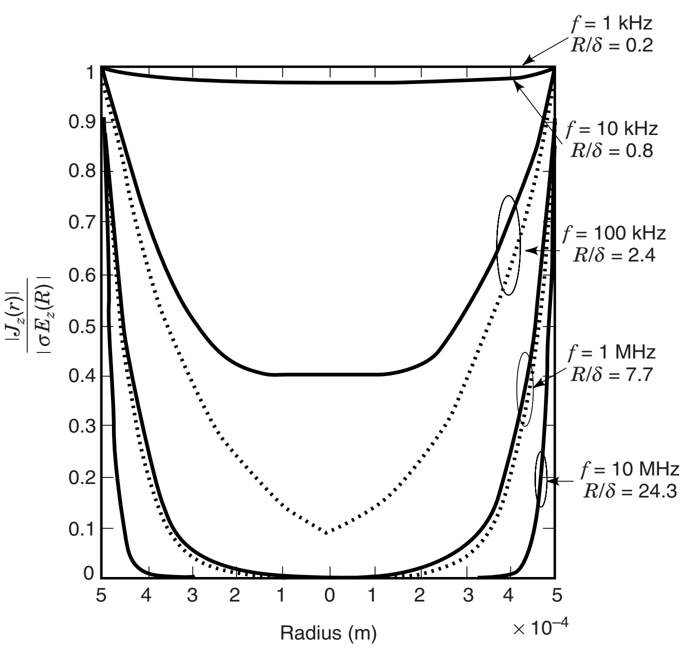

Figure 5. The skin effect. The normalized magnitude of the current density is plotted versus radial position in a copper circular cross-section conductor of radius $R = 0.5$ mm. The solid curves are computed via the exact expression, given by Eq. (29), and the dashed curves are computed using the exponential approximation, given by Eq. (31).

When $\delta \gg R$, the products TR and Tr are very small, and according to Eq. (30) the Bessel functions tend toward unity. In this case the current density may be assumed uniform over the cross section of the conductor, and the skin effect is negligible; this is also shown in Fig. 5. When δ is of the same order of magnitude as the conductor radius, $\delta \approx R$, then the skin effect is non-negligible and the current density must be computed using Eq. (29). When $\delta \ll R$, then again the skin effect is non-negligible and the current density may be computed using Eq. (29) or is well approximated by Eq. (31), as shown in Fig. 5.

Impedance of a Conductor

The impedance of a circuit element is defined as

$$Z = \frac{V}{I} \tag{32}$$

where V and I are phasors. The above is consistent with Ohm's law for dc quantities stated as Eq. (1). For our conducting cylinder shown in Fig. 4, the current I can be obtained either from the current density $\mathbf{J}$ using Eq. (4) or through Maxwell's equation from Ampere's law, given by Eq. (12). Using the latter approach provides a direct route to the expression for the impedance of a conductive cylinder.

Maxwell's equation from Ampere's law, rewritten in the frequency domain and in integral form, reads

$$\oint \mathbf{H} \cdot \mathbf{dl} = \int\int_S (\sigma\mathbf{E} + j\omega\epsilon\mathbf{E}) \cdot \mathbf{dS} \tag{33}$$

Again, neglecting the displacement current density $j\omega\epsilon\mathbf{E}$ compared to the conduction current density $\sigma\mathbf{E}$, substituting Ohm's law at a point given by Eq. (5), and integrating the remainder of the right-hand side over the area S defined in Fig. 4, yields Ampere's law in the familiar form:

$$\oint \mathbf{H} \cdot \mathbf{dl} = I \tag{34}$$

where $\mathbf{H}$ and I are phasors.

The path of integration is chosen to trace out a circle of radius R from the z axis. The line element dl is thus given by the elemental arc length, $Rd\theta$, where θ is the angle measured up from the horizontal plane passing through the page and the center axis of the conductor, as shown in Fig. 4. Since the assigned current is flowing along the z axis, the associated magnetic field coincides exactly with the chosen path of integration. Ampere's law thus becomes

$$\int_0^{2\pi} H_\theta(R)(R d\theta) = I \tag{35}$$

which upon integration yields

$$I = 2\pi R H_\theta(R) \tag{36}$$

where $H_\theta(R)$ is the θ directed component of the magnetic field on the outside surface of the conductor.

Again neglecting the radial and angular components of the current density and electric field, as well as angular dependencies, Eq. (13) is expanded as

$$-\frac{\partial}{\partial r} E_z(r) = -j\omega\mu H_\theta(r) \tag{37}$$

E_z is related to J_z via Ohm's law at a point, given by Eq. (5), and J_z has been obtained in the previous section and is given by Eq. (29). Substituting these relations into the above, evaluating the derivative with respect to r, and isolating for $H_\theta(r)$ yields

$$H_\theta(r) = \frac{E_z(R)}{j\omega\mu} \frac{T J_0'(Tr)}{J_0(TR)} \tag{38}$$

where

$$J_0'(Tr) = \frac{d}{d(Tr)} J_0(Tr) \tag{39}$$

and $J_0'(u)$ is easily obtained by deriving Eq. (30):

$$J_0'(u) = \frac{d}{du} J_0(u) = -\frac{u}{2} + \frac{u^3}{2^2 \cdot 4} - \frac{u^5}{2^2 \cdot 4^2 \cdot 6} + \frac{u^7}{2^2 \cdot 4^2 \cdot 6^2 \cdot 8} - \cdots \tag{40}$$

Evaluating Eq. (38) at R and substituting into Eq. (36) yields the desired expression for the current I:

$$I = 2\pi R \frac{E_z(R)}{j\omega\mu} \frac{T J_0'(TR)}{J_0(TR)} \tag{41}$$

By definition, the voltage V across the conductive cylinder is related to the electric field on the surface of the cylinder according to

$$V = \int \mathbf{E}(R) \cdot \mathbf{dl} \tag{42}$$

Evaluating the above line integral yields

$$V = E_z(R)\Delta z \tag{43}$$

Substituting the above and Eq. (41) into Eq. (32) and dividing through by Δz yields the expression for the impedance per unit length of our circular cross-section conductor:

$$Z_l = \frac{j\omega\mu}{2\pi R} \frac{J_0(TR)}{T J_0'(TR)} \quad \frac{\Omega}{\text{m}} \tag{44}$$

where the subscript l is used to differentiate between per unit length quantities and total quantities.

Low-Frequency Approximation. When $\delta \gg R$, which occurs at low frequencies for a conductor having a small radius, the argument TR in Eq. (44) is small: $|TR| \ll 1$. Neglecting terms that are second order and higher in u, in Eqs. (30) and (40), yields the following approximation:

$$\frac{J_0(TR)}{T J_0'(TR)} = \frac{2}{-T^2 R} \tag{45}$$

Substituting the above into Eq. (44), introducing Eq. (23), and working out the algebra yields

$$Z_l = \frac{1}{\sigma(\pi R^2)} \quad \frac{\Omega}{\text{m}} \tag{46}$$

By comparison with Eq. (2), the above is recognized as being the dc resistance per unit length of our conductive cylinder and may be used to approximate the ac impedance of the conductor as long as $\delta \gg R$.

High-Frequency Approximation. Figure 6 shows that the ratio $J_0(TR)/J_0'(TR)$ tends toward $-j$ as the frequency and the argument TR tend toward infinity. Substituting this approximation into Eq. (44) and working out the algebra yields the high-frequency approximation for the impedance per unit length of our conductive cylinder:

$$Z_l = \frac{1}{\sigma(2\pi R \cdot \delta)}(1 + j) \quad \frac{\Omega}{\text{m}} \tag{47}$$

The above holds as long as $\delta \ll R$, which holds at high frequencies. The above shows that the impedance of a conductor is inductive since the imaginary part is positive. Furthermore, the ac resistance is as though the current density were distributed uniformly over an area of $2\pi R$ times δ, which is the product of the conductor's circumference with the depth of penetration.

Other Conductor Geometries. The geometry of a conductor's cross-section has a direct impact on the level of ac power transmitted through the conductor and the impact is greater at higher frequencies. Square or triangular cross-section wires, for example, have corners where the current density is higher compared to other regions near the edges. These types of wires will thus have greater ac losses compared to a circular cross-section wire of identical area.

Inductive Reactance of an Isolated Conductor

The expression for the internal impedance of a conductor, given by Eq. (44), is rigorous and can be used whenever an accurate value is required. Though accurate, this equation does not account for the material surrounding the conductor. Since a magnetic field is associated with the current flowing

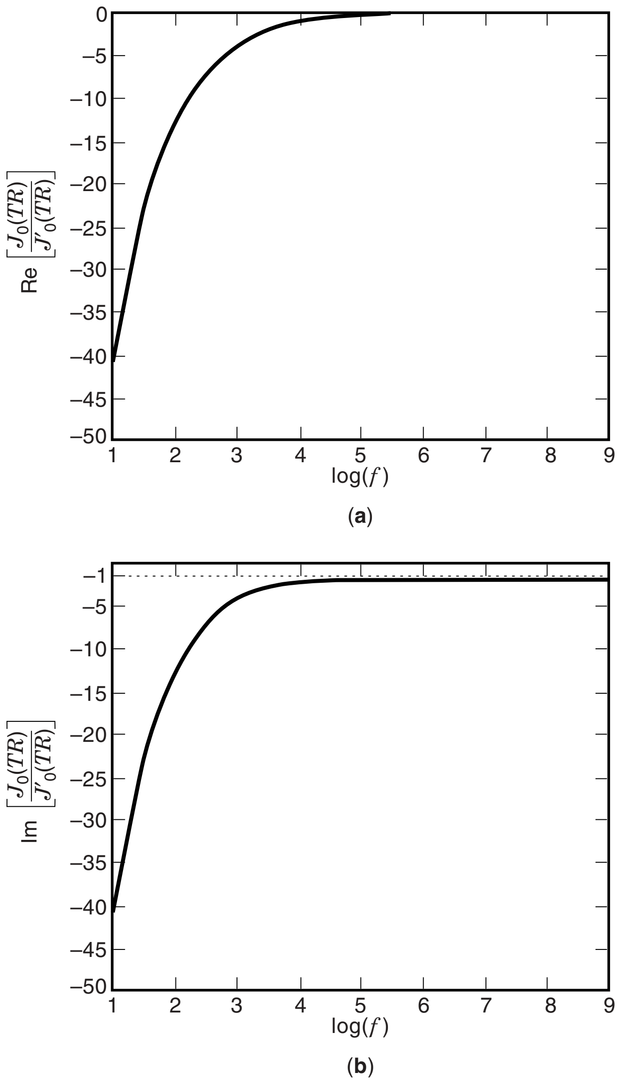

Figure 6. The convergence of $J_0(TR)/J_0'(TR)$ to $-j$ as the frequency tends toward infinity. (a) Real part. (b) Imaginary part.

through a conductor, and this magnetic field permeates the space surrounding the conductor, we expect that the dielectric material filling this space will affect the reactance of the conductor. It is therefore desirable to obtain a simple expresson for the self-inductance per unit length of an isolated conductor as a function of its physical features and those of the medium surrounding it.

The self-inductive reactance of a conductor is perceived by all time-varying signals. For a time harmonic signal, the inductive reactance increases linearly with frequency and adds in series with the ac resistance of the conductor, causing an increase in impedance. The self-inductance of an isolated conductor can be decomposed into internal and external inductances due to the time varying magnetic fields that exist inside and outside of the conductor. The derivation of expressions for these inductances is presented in this section. Our derivation is based on magnetic flux linkage considerations (7,8).

Consider the long length of wire shown in Fig. 7 through which a current phasor I flows. For a conductor having a finite conductivity σ, an electric field $\mathbf{E}$ in the longitudinal direction shown is present. Furthermore, Ampere's law states that the current I has an associated magnetic field $\mathbf{H}$ and magnetic flux density $\mathbf{B} = \mu\mathbf{H}$ in Wb/m^2 looping around the conductor in the direction shown in Fig. 7.

The magnetic flux phasor Φ through the area A is defined as

$$\Phi = \int\int_A \mathbf{B} \cdot \mathbf{dA} \tag{48}$$

If the magnetic flux density $\mathbf{B}$ is uniform over A and normal to it, then the above simplifies to $\Phi = BA$. The magnetic flux has units of Wb.

The magnetic flux linkage phasor Ψ is related to Φ by

$$\Psi = N\Phi \tag{49}$$

where N is the number of times that the flux lines link the conductor carrying the current I. The magnetic flux linkage has units of Wbt for Weber-turns.

According to Faraday's law, a time-domain voltage v with the polarity shown in Fig. 7 will be induced over the length of the wire. This voltage is equal to the time derivative of the total flux linking the conductor:

$$v = \frac{d\psi}{dt} \tag{50}$$

where ψ is the total time domain flux linkage.

Since the total flux linkage is directly proportional to the current i flowing in the conductor, the induced voltage must also be proportional to the time derivative of the current. The constant of proportionality is defined as the inductance L:

$$v = L\frac{di}{dt} \tag{51}$$

Based on the above, it is quite clear that a voltage drop over the length of the conductor is induced only for time-varying currents. Equating the above two equations:

$$\frac{d\psi}{dt} = L\frac{di}{dt} \tag{52}$$

and isolating for L yields

$$L = \frac{d\psi}{dt}\frac{dt}{di} \tag{53}$$

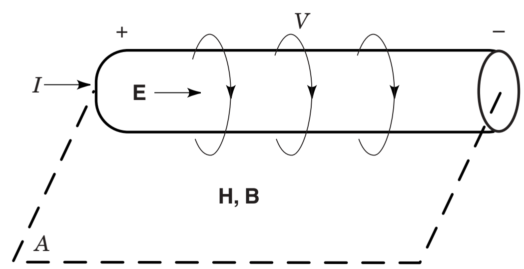

Figure 7. A long length of wire carrying the current I and across which the voltage drop V exists. The electric field E is present inside the conductor and the current I induces the magnetic field H and the magnetic flux density B, as shown.

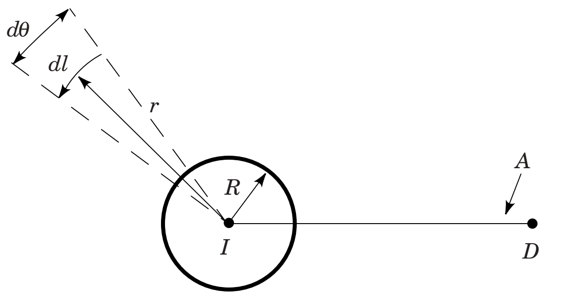

Figure 8. The cross-section of a cylindrical conductor of radius R through which a current I is flowing (out of the page). The associated geometry is used to help compute the external inductance of the wire L_{ext}.

For a sinusoidal time variation, the time derivatives in the above equations are replaced with $j\omega$ and the above becomes, in the phasor domain,

$$L = \frac{\Psi}{I} \tag{54}$$

According to the above, the inductance of a conductor can be computed by finding the ratio of Ψ to I for the geometry of interest.

External Inductance of an Isolated Conductor. The self-inductance due to the magnetic flux linkage permeating the region $R \leq r \leq D$ as defined in Fig. 8 is referred to as the external inductance L_{ext}. The external inductance depends on the radius of the conductor, on the magnetic permeability of the region outside of the conductor, and on the width $D - R$ of the region considered.

Our starting point in deriving an expression for L_{ext} is Ampere's law stated as Eq. (34). As shown in Fig. 8, the path of integration is chosen to trace out a counterclockwise circle at a radial distance of r from the center. The line element dl is given by the elemental arc length $r\, d\theta$, where θ is the angle measured up from the horizontal axis. Since the assigned current is flowing out of the page, the associated magnetic field loops around the conductor in the counterclockwise direction and coincides exactly with the chosen path of integration. Ampere's law is thus written:

$$\int_0^{2\pi} H_\theta(r)(r\, d\theta) = I \tag{55}$$

which upon integration yields the magnitude of the magnetic field at a radial distance r from the center of the conductor:

$$H_\theta(r) = \frac{I}{2\pi r} \tag{56}$$

The above expression holds for the region outside of the conductor $r > R$.

The magnitude of the magnetic flux density associated with the magnetic field at r is

$$B_\theta(r) = \frac{\mu_e I}{2\pi r} \tag{57}$$

where μ_e is the magnetic permeability of the region external to the conductor. Usually this region is filled with air or a dielectric material such that the magnetic permeability is that of free space: $\mu_e = \mu_0 = 4\pi \times 10^{-7}$ H/m.

The magnetic flux per unit length of conductor is obtained by applying Eq. (48), where the area A is the rectangle bounded by a 1 m length of conductor along the z axis, which points out of the page, and the width $D - R$, as shown in Fig. 8. The magnetic flux density is everywhere normal to the area A such that the surface integral simplifies to

$$\Phi = \int_0^1 \int_R^D \frac{\mu_e I}{2\pi r}\, dr\, dz \tag{58}$$

which upon integration yields the expression for the flux per unit length of conductor in the region $R \leq r \leq D$:

$$\Phi_l = \frac{\mu_e I}{2\pi} \ln\left(\frac{D}{R}\right) \tag{59}$$

The flux links the total current I exactly once so that $N = 1$ in Eq. (49) and the magnetic flux linkage per unit length is

$$\Psi_l = \frac{\mu_e I}{2\pi} \ln\left(\frac{D}{R}\right) \tag{60}$$

The external inductance per unit length of conductor due to the flux in the region $R \leq r \leq D$ is obtained by applying Eq. (54) to the above:

$$L_{\text{ext},l} = \frac{\mu_e}{2\pi} \ln\left(\frac{D}{R}\right) \quad \frac{\text{H}}{\text{m}} \tag{61}$$

It is clear from the above that the external inductance depends on the medium surrounding the conductor and the geometry of the configuration. If the relative permeability of the medium is unity, then the external inductance simplifies to

$$L_{\text{ext},l} = 2 \times 10^{-7} \ln\left(\frac{D}{R}\right) \quad \frac{\text{H}}{\text{m}} \tag{62}$$

Internal Inductance of an Isolated Conductor. The time-varying flux linkage inside the isolated conductor shown in Fig. 9 also contributes to the total self-inductance. This component is referred to as the internal inductance, L_{int}.

Again our starting point in deriving an expression for L_{int} is Ampere's law. Recall that Ampere's law relates the magnetic field intensity to the current flowing through an enclosed area. The magnetic field intensity associated with the current $I(r)$ confined within the circle of radius r, as shown in Fig. 9, is given by

$$\oint \mathbf{H} \cdot \mathbf{dl} = I(r) \tag{63}$$

Integrating counterclockwise along this circle yields

$$H_\theta(r) = \frac{I(r)}{2\pi r} \tag{64}$$

Assuming that the skin effect is negligible, we take the current density as being uniform over the cross section of the

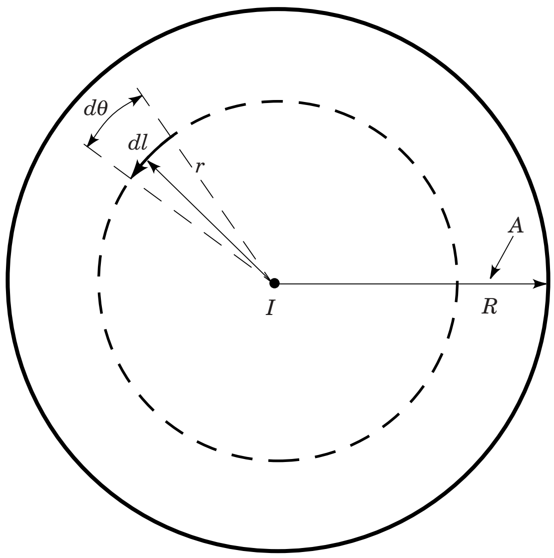

Figure 9. The enlarged cross-sectional view of a cylindrical conductor of radius R through which a current I is flowing (out of the page). The associated geometry is used to help compute the internal inductance of the wire L_{int}.

conductor:

$$J_z = \frac{I}{\pi R^2} \tag{65}$$

where I is the total current flowing through the conductor. The current flowing within the circle of radius r is therefore

$$I(r) = \pi r^2 J_z = \frac{\pi r^2}{\pi R^2} I \tag{66}$$

Substituting the above into Eq. (64) yields the expression for the magnitude of the magnetic field intensity within the conductor:

$$H_\theta(r) = \frac{rI}{2\pi R^2} \tag{67}$$

which holds for the region $r < R$.

The magnitude of the magnetic flux density associated with the magnetic field at r is

$$B_\theta(r) = \frac{\mu_i r I}{2\pi R^2} \tag{68}$$

where μ_i is the magnetic permeability of the conductor.

The magnetic flux density is everywhere normal to the rectangular area A which is bounded by the center axis of the conductor and its radius as shown in Fig. 9. The differential magnetic flux through A at any position r is given by

$$d\Phi = \frac{\mu_i r I}{2\pi R^2}\, dr\, dz \tag{69}$$

The differential flux linkage associated with this differential magnetic flux is from Eq. (49):

$$d\Psi = N(r)\, d\Phi \tag{70}$$

where $N(r)$ is the fraction of current linked by the flux at position r. Clearly, $N(r)$ must be less than or equal to 1 and at position r is given by

$$N(r) = \frac{I(r)}{I} = \frac{\pi r^2}{\pi R^2} \tag{71}$$

The total flux linkage per unit length of conductor is obtained by integrating $d\Psi$ over the area A, which is the rectangle bounded by a 1 m length of conductor and its radius R:

$$\Psi = \int_0^1 \int_0^R \left(\frac{\pi r^2}{\pi R^2}\right)\left(\frac{\mu_i r I}{2\pi R^2}\right) dr\, dz \tag{72}$$

Working out the above integration yields

$$\Psi_l = \frac{\mu_i I}{8\pi} \tag{73}$$

According to Eq. (54), the internal inductance per unit length of conductor is

$$L_{\text{int},l} = \frac{\mu_i}{8\pi} \qquad \frac{\text{H}}{\text{m}} \tag{74}$$

From the above, we note that the internal inductance depends on the magnetic properties of the material. For the most popular metals used to fabricate conductors, the magnetic permeability is near that of free space: $\mu_i = \mu_0$. In such a case, the above simplifies to

$$L_{\text{int},l} = \frac{1}{2} \times 10^{-7} \qquad \frac{\text{H}}{\text{m}} \tag{75}$$

Total Inductance of an Isolated Conductor. The total self-inductance per unit length of our isolated conductor is given by the sum of the external and internal inductances:

$$L_l = L_{\text{ext},l} + L_{\text{int},l} \tag{76}$$

which, upon substitution, yields

$$L_l = \frac{\mu_e}{2\pi} \ln\left(\frac{D}{R}\right) + \frac{\mu_i}{8\pi} \qquad \frac{\text{H}}{\text{m}} \tag{77}$$

It is quite clear from the above that the external component can dominate the total inductance. If the ratio D/R is greater than e and the permeabilities of the conductor and material surrounding it are similar, $\mu_e \approx \mu_i$, then the external inductance is at least four times greater than the internal inductance.

COMMON TRANSMISSION LINE GEOMETRIES

When conductors are arranged in such a way as to carry power efficiently from one point to another, we refer to the resulting structure as a *transmission line.* Generally speaking, transmission lines consist of two or more conductors in parallel and connect a source to a load. The source might be a hydroelectric generator and the load might be a steel factory, in which case the transmission line would carry megawatts of power at a low frequency; however, the source could

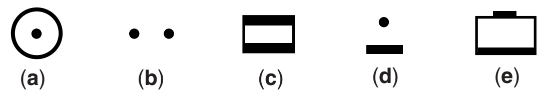

Figure 10. The cross-section of some commonly used transmission lines. (a) Coaxial line; (b) two-wire line; (c) parallel-plate line; (d) wire above ground plane; (e) microstrip line.

also be an antenna and the load a radio receiver, and then the transmission line would carry microwatt power levels at very high frequencies. Several different transmission line structures, primarily defined by their conductor geometries, exist to fulfill a wide variety of power delivery applications. Common types include the coaxial line, two-wire line, parallel-plate line, wire above a ground plane, and microstrip line, as shown in cross-sectional view in Fig. 10.

Here the term *transmission* line will refer to a pair of conductors of constant cross-section and spacing throughout their length, operating in the transverse electromagnetic mode (TEM). Other types of conductor geometries and operating modes exist and are discussed at length in the literature on electromagnetics (5) and power systems analysis (8). In the remainder of this section we will focus on obtaining the equivalent model of a transmission line based on its conductor geometry and material properties. Such a model is useful to determine how the line will behave under transient or sinusoidal steady-state excitation, and it leads to an understanding of the many transmission line effects, including signal delay, attenuation, reflections, standing waves, and pulse dispersion.

Equivalent Model of a Transmission Line

In general a transmission line may be a considerable fraction of the operating wavelength or even several wavelengths long. Hence, unlike ordinary circuit theory where a model consists of lumped elements, the transmission line model contains distributed parameters, in the form of resistance per unit length R_l in Ω/m, inductance per unit length L_l in H/m, capacitance per unit length C_l in F/m, and conductance per unit length G_l in S/m.

Consider an infinitesimal length Δz of a two-wire transmission line, as shown in Fig. 11(a), in which the applied voltage gives rise to the current flow I and associated electric and magnetic fields **E** and **H**. Intuitively we can arrive at the equivalent model given in Fig. 11(b) or the more commonly found equivalent circuit shown in Fig. 11(c). $R_l\Delta z$ represents the conductor losses in the metal, $L_l\Delta z$ and $C_l\Delta z$ account for the magnetic and electric fields, respectively, which exist between the two conductors, and $G_l\Delta z$ represents the losses in the dielectric medium separating the conductors. Such a model may represent any of the two-conductor transmission lines of Fig. 10, as long as we keep Δz much smaller than a wavelength, less than $\lambda/10$. To completely model a longer line would require placing several of the Δz equivalent circuits in cascade.

We now proceed to derive expressions for the elements of the equivalent circuit shown in Fig. 11(c). By so doing, a general methodology will be presented which can be applied to the modeling of an arbitrary transmission line. The parallel-plate and two-wire lines are purposely chosen here since they are common and simple geometries. Other transmission line structures can be treated using the same approach.

Parallel-Plate Transmission Line

Figure 12 depicts a short length Δz of parallel-plate transmission line, consisting of two parallel conducting plates of thickness t and width w, and separated by a homogeneous dielectric material of thickness d. The conductors are characterized by their conductivity σ_c and are assumed to have the permeability and permittivity of free-space μ_0 and ϵ_0; the surrounding lossy isotropic homogeneous dielectric is characterized by σ_d, μ_d, and ϵ_d. Since we are assuming a TEM mode of operation for the line, a quasistatic analysis provides the most straightforward route to values of the equivalent circuit (2,3).

Distributed Resistance. The series resistance of the transmission line structure accounts for ohmic losses in both conductors encountered by the currents flowing in opposite directions in the top and bottom plates. The dc resistance of a

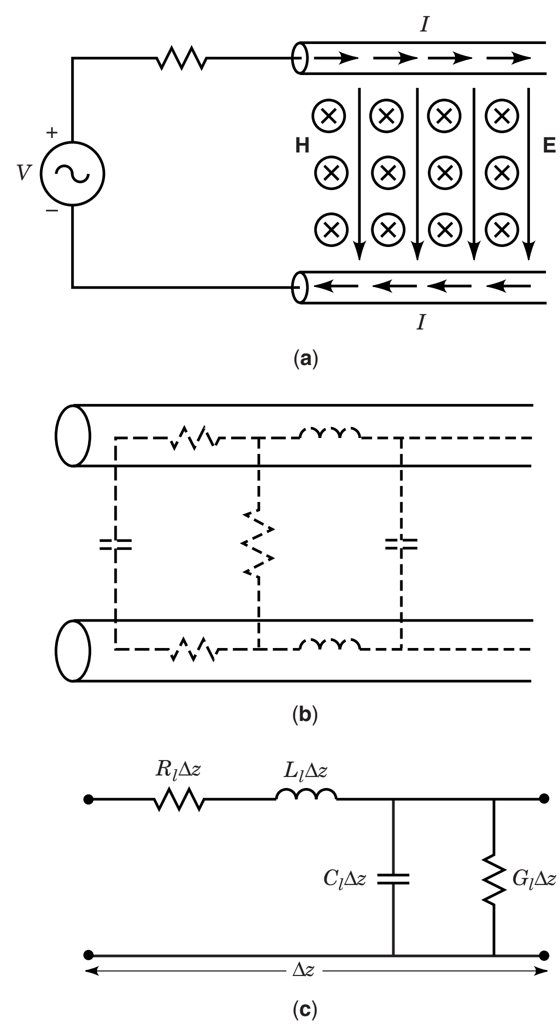

Figure 11. Illustration of how a two-wire transmission line can be modeled as a network of electrical elements. (a) Fields and currents along a two-wire transmission line excited by a generator. (b) Distributed parameter equivalent circuit. (c) Equivalent circuit of an infinitesimal length Δz of transmission line.

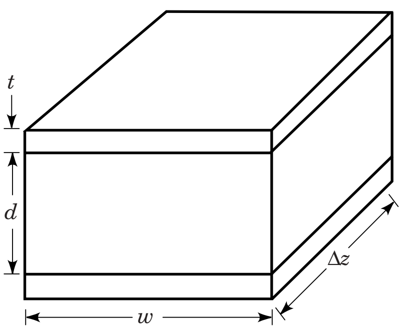

Figure 12. Parallel-plate transmission line of length Δz, width w, and height d. The thickness of the conducting plates is t. A dielectric material fills the space between the plates.

single plate is easily obtained using Eq. (2):

$$R_{\mathrm{dc}} = \frac{\Delta z}{\sigma_c w t} \tag{78}$$

The dc series resistance per unit length for both conductors is therefore

$$R_{\mathrm{dc},l} = \frac{2R_{\mathrm{dc}}}{\Delta z} = \frac{2}{\sigma_c w t} \qquad \frac{\Omega}{\mathrm{m}} \tag{79}$$

Due to the skin effect at high frequencies, the current is confined to a thin layer of thickness δ at the surface of the conductors. As a result, the expression for the ac resistance per unit length is

$$R_{\mathrm{ac},l} = \frac{2}{\sigma_c w \delta} \qquad \frac{\Omega}{\mathrm{m}} \tag{80}$$

where δ is given by Eq. (26). The above holds for $\delta \ll t$.

Distributed Capacitance. The capacitance per unit length models the coupling between lines due to the electric field created by the potential difference V, as shown in Fig. 13. Capacitance in F is defined in general as

$$C = \frac{Q}{V} \tag{81}$$

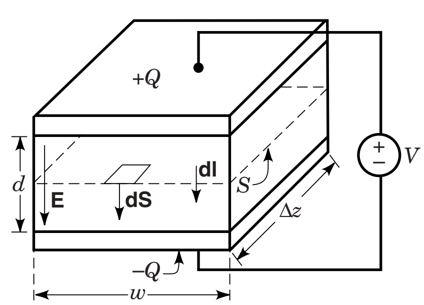

Figure 13. Parallel-plate transmission line connected to the dc voltage source V. The charge $+Q$ has accumulated onto the top plate while the charge $-Q$ has accumulated onto the bottom plate. The electric field E exists in the dielectric region between the plates, as shown.

where Q is the charge in C stored on the plates and V is the potential difference applied to them.

Gauss's law states that the electric flux density integrated over a closed surface is equal to the total charge enclosed. Gauss's law written for the top conductor is

$$Q = \epsilon_d \int\int_S \mathbf{E} \cdot \mathbf{dS} \tag{82}$$

where S and $\mathbf{dS}$ are as defined in Fig. 13. S is taken as the open surface shown instead of a closed surface as required by Gauss's law. This approximation is well-justified in this case, since the electric field is concentrated between the plates, it is everywhere uniform and normal to the conductors, and for small d/w the external and fringing electric fields can be neglected.

The potential difference V is related to the electric field by definition:

$$V = \int \mathbf{E} \cdot \mathbf{dl} \tag{83}$$

where $\mathbf{dl}$ is as shown in Fig. 13.

Substituting the above two relations into the definition of capacitance:

$$C = \frac{\epsilon_d \int\int_S \mathbf{E} \cdot \mathbf{dS}}{\int \mathbf{E} \cdot \mathbf{dl}} \tag{84}$$

and working out the surface and line integrals yields

$$C = \frac{\epsilon_d E w \Delta z}{E d} \tag{85}$$

The capacitance per unit length of line is therefore

$$C_l = \frac{\epsilon_d w}{d} \qquad \frac{\mathrm{F}}{\mathrm{m}} \tag{86}$$

Neglecting the external and fringing fields which extend beyond the edges of the plate limits the validity of the above expression to small d/w.

Distributed External Inductance. The external inductance of the conductors, defined as the ratio of flux linkage to enclosed current, can be obtained by applying Eq. (54). The current I flows into the page in the top conductor and out of the page in the bottom, as shown in Fig. 14. The associated magnetic field $\mathbf{H}$ is concentrated predominantly between the conductors, as illustrated, where the contributions from the top and bottom plates add constructively. Furthermore, for small d/w, $\mathbf{H}$ is oriented along the width of the plates and is essentially uniform between them. The magnetic field is relatively weak above the top and below the bottom plates since the contributions due to these conductors add destructively in those regions.

The magnetic flux between the plates is given by Eq. (48), where the area A is shown in Fig. 14. Since $\mathbf{H}$ is everywhere normal to A and uniform along d and Δz, integration is straightforward and yields

$$\Phi = \mu_d H d \Delta z \tag{87}$$

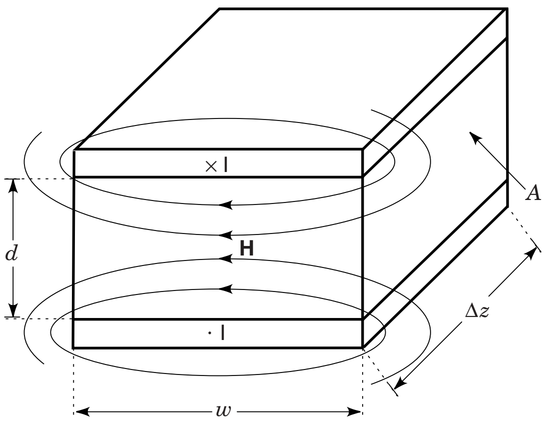

Figure 14. Parallel-plate transmission line with the current I flowing into the page in the top conductor and out of the page in the bottom conductor. The associated magnetic field H is concentrated predominantly between the conductors, as shown.

Since the flux links a conductor exactly once, we write the flux linkage according to Eq. (49) as

$$\Psi = \mu_d \mathrm{H} d \Delta z \quad (88)$$

The relationship between current and magnetic field is given by Ampere's law, stated as Eq. (34). Integrating **H** around the top conductor while neglecting the thickness t of the latter and the magnetic field above the conductor yields

$$I = \mathrm{H}w \quad (89)$$

Substituting the above and Eq. (88) into Eq. (54) and dividing through by ΔZ yields the inductance per unit length of parallel-plate transmission line:

$$L_l = \frac{\mu_d d}{w} \quad \frac{\mathrm{H}}{\mathrm{m}} \quad (90)$$

This result is again most accurate for small d/w.

Distributed Conductance. The conductance due to losses in the dielectric medium separating the two conductors can be derived from

$$G = \frac{I_d}{V} \quad (91)$$

where I_d is the current flowing through the dielectric between the top and bottom plates, and V is the potential difference applied to them, as shown in Fig. 13. Introducing Eq. (4) and the definition for V into the above yields

$$G = \frac{\int\int_S \mathbf{J_d} \cdot \mathbf{dS}}{\int \mathbf{E} \cdot \mathbf{dl}} \quad (92)$$

where $\mathbf{J_d}$ is the current density flowing between the plates. $\mathbf{J_d}$ is normal to S and uniform over w and ΔZ.

Substituting Ohm's law at a point, given by Eq. (5), into the above yields

$$G = \frac{\sigma_d \int\int_S \mathbf{E} \cdot \mathbf{dS}}{\int \mathbf{E} \cdot \mathbf{dl}} \quad (93)$$

which is identical to the expression for capacitance given by Eq. (84) with σ_d and ϵ_d interchanged. Consequently, we can write by analogy with Eq. (86) the conductance per unit length of our parallel plate transmission line:

$$G_l = \frac{\sigma_d w}{d} \quad \frac{\mathrm{S}}{\mathrm{m}} \quad (94)$$

TEM Transmission Line Relationships. By combining the equations for the distributed parameters of our parallel-plate transmission line, we can derive very useful relationships that hold for the TEM mode supported by any TEM transmission line structure. Multiplying Eqs. (86) and (90) yields

$$L_l C_l = \mu_d \epsilon_d \quad (95)$$

while Eq. (94) divided by Eq. (86) leads to

$$\frac{G_l}{C_l} = \frac{\sigma_d}{\epsilon_d} \quad (96)$$

The above are very important results for transmission lines. Indeed, complicated derivations for inductance and conductance, for example, are avoided once the capacitance per unit length of a line is known.

Two-Wire Line

The two-wire structure shown in Fig. 15 is a commonly encountered implementation of a transmission line. The conductors of diameter $2a$ are separated by a center-to-center distance d and surrounded by a homogeneous dielectric characterized by σ_d, μ_d, and ϵ_d. The conductors are again assumed to be characterized by σ_c, μ_0, and ϵ_0. Like the parallel-plate transmission line, the two-wire structure supports a TEM mode; thus a quasistatic analysis again provides the most straightforward route to the expressions for the equivalent circuit.

Distributed Resistance. At dc or very low frequencies the resistance of a single wire is given by Eq. (2), which for this geometry becomes

$$R_{\mathrm{dc}} = \frac{\Delta z}{\sigma_c \pi a^2} \quad (97)$$

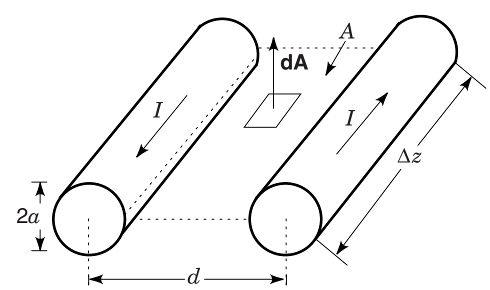

Figure 15. Geometry of a two-wire transmission line of length Δ_z. The conductors of radius a are separated by a center-to-center distance d. A dielectric material fills the entire space surrounding the conductors. The current I is flowing into the page in the right conductor and out of the page in the left conductor.

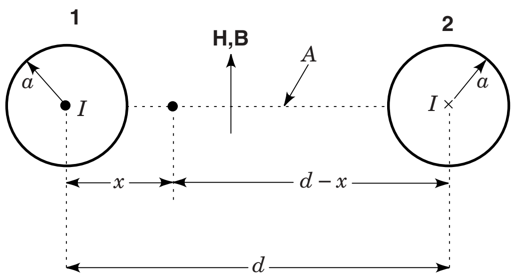

Figure 16. Cross-sectional view of a two-wire transmission line. The cylindrical conductors of radius a, through which a current I is flowing (out of the page in the left conductor, into the page in the right conductor), are separated by a center-to-center distance d. The induced magnetic field H and magnetic flux density B are as shown. The associated geometry is used to help compute the external inductance of the transmission line.

Hence the resistance per unit length due to both wires is given by

$$R_{\mathrm{dc},l} = \frac{2}{\sigma_c \pi a^2} \qquad \frac{\Omega}{\mathrm{m}} \tag{98}$$

At high frequencies, the resistance of a single conductor is given by the real part of Eq. (47) times the length of our lines:

$$R_{\mathrm{ac}} = \frac{\Delta Z}{\sigma_c (2\pi a \cdot \delta)} \tag{99}$$

where δ is the skin depth given by Eq. (26). The ac resistance per unit length due to both conductors is therefore

$$R_{\mathrm{ac},l} = \frac{1}{\sigma_c (\pi a \delta)} \qquad \frac{\Omega}{\mathrm{m}} \tag{100}$$

Distributed External Inductance. The external self-inductance of a short length ΔZ of two-wire line can be computed using Eq. (54), where the flux linkage Ψ is computed through the surface A between the conductors as shown in Fig. 15. The current I is assumed to flow in the conductors as shown so that the associated magnetic fields add constructively between the wires and destructively to the left of the left conductor and to the right of the right conductor. If d is small, we can neglect the flux in these regions compared to the flux between the wires. Also, we suppose that the wires are thin compared to the separation distance $a \ll d$, which allows us to ignore the flux passing through the wires themselves.

Referring to Fig. 16, the magnitude of the magnetic flux density B_1 at a position x due to wire 1 is given by analogy with Eq. (57):

$$B_1 = \frac{\mu_d I}{2\pi x} \tag{101}$$

B_1 is oriented in the direction shown along the center axis between both wires. Since the current flowing in wire 2 is in the opposite direction to the current in wire 1, the magnetic field due to wire 2 at position x is in the same direction as that created by wire 1, with the magnitude of magnetic flux density now being given by

$$B_2 = \frac{\mu_d I}{2\pi (d - x)} \tag{102}$$

The total magnetic flux density in the region between the wires is given by the sum of B_1 and B_2. Adding the magnitudes yields

$$B = \frac{\mu_d I}{2\pi}\left(\frac{1}{x} + \frac{1}{d - x}\right) \tag{103}$$

or

$$B = \frac{\mu_d I}{2\pi} \frac{d}{x(d - x)} \tag{104}$$

The flux through the area A is given by Eq. (48). Since B is everywhere normal to the surface A, the integration is written

$$\Phi = \int_a^{d-a} \int_0^{\Delta Z} \frac{\mu_d I}{2\pi} \frac{d}{x(d - x)}\, dz\, dx \tag{105}$$

Carrying out the above yields

$$\Phi = \frac{\mu_d I \Delta Z}{\pi} \ln\left(\frac{d - a}{a}\right) \tag{106}$$

Since the number of times the flux links the current I in this structure is 1, Eq. (49) states that $\Psi = \Phi$ such that the inductance is given by substituting the above into Eq. (54) and simplifying:

$$L = \frac{\mu_d \Delta Z}{\pi} \ln\left(\frac{d - a}{a}\right) \tag{107}$$

Given that $d \gg a$, the inductance per unit length of two-wire line is

$$L_l = \frac{\mu_d}{\pi} \ln\left(\frac{d}{a}\right) \qquad \frac{\mathrm{H}}{\mathrm{m}} \tag{108}$$

Distributed Capacitance. The distributed capacitance of our two-wire transmission line is easily obtained by substituting Eq. (108) into Eq. (95), which holds in general for TEM transmission lines, and solving for C_l:

$$C_l = \frac{\pi \epsilon_d}{\ln(d/a)} \qquad \frac{\mathrm{F}}{\mathrm{m}} \tag{109}$$

Distributed Conductance. Similarly, the distributed conductance is obtained by substituting Eq. (109) into Eq. (96) and solving for G_l:

$$G_l = \frac{\pi \sigma_d}{\ln(d/a)} \qquad \frac{\mathrm{S}}{\mathrm{m}} \tag{110}$$

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CONDUCTORS, ELECTRIC. See BUSBARS.

CONDUCTORS, OVERHEAD LINE. See OVERHEAD LINE CONDUCTORS.