CHAPTER 21

NAVIGATIONAL MATHEMATICS

GEOMETRY

2100. Definition

Geometry deals with the properties, relations, and measurement of lines, surfaces, solids, and angles. **Plane geometry** deals with plane figures, and **solid geometry** deals with three–dimensional figures.

A **point**, considered mathematically, is a place having position but no extent. It has no length, breadth, or thickness. A point in motion produces a **line**, which has length, but neither breadth nor thickness. A **straight or right line** is the shortest distance between two points in space. A line in motion in any direction except along itself produces a **surface**, which has length and breadth, but not thickness. A **plane surface** or **plane** is a surface without curvature. A straight line connecting any two of its points lies wholly within the plane. A plane surface in motion in any direction except within its plane produces a **solid**, which has length, breadth, and thickness. **Parallel** lines or surfaces are those which are everywhere equidistant. **Perpendicular** lines or surfaces are those which meet at right or 90° angles. A perpendicular may be called a **normal**, particularly when it is perpendicular to the tangent to a curved line or surface at the point of tangency. All points equidistant from the ends of a straight line are on the perpendicular bisector of that line. The shortest distance from a point to a line is the length of the perpendicular between them.

2101. Angles

An **angle** is formed by two straight lines which meet at a point. It is measured by the arc of a circle intercepted between the two lines forming the angle, the center of the circle being at the point of intersection. In Figure 2101, the angle formed by lines *AB* and *BC*, may be designated "angle *B*," "angle *ABC*," or "angle *CBA*"; or by Greek letter as "angle α." The three letter designation is preferred if there is more than one angle at the point. When three letters are used, the middle one should always be that at the **vertex** of the angle.

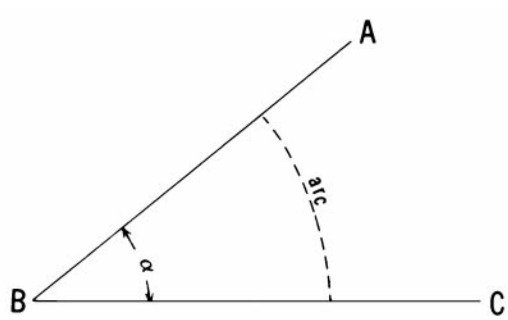

Figure 2101. An angle.

An **acute angle** is one less than a right angle (90°).

A **right angle** is one whose sides are perpendicular (90°).

An **obtuse angle** is one greater than a right angle (90°) but less than 180°.

A **straight angle** is one whose sides form a continuous straight line (180°).

A **reflex angle** is one greater than a straight angle (180°) but less than a circle (360°). Any two lines meeting at a point form two angles, one less than a straight angle of 180° (unless exactly a straight angle) and the other greater than a straight angle.

An **oblique angle** is any angle not a multiple of 90°.

Two angles whose sum is a right angle (90°) are **complementary angles**, and either is the **complement** of the other.

Two angles whose sum is a straight angle (180°) are **supplementary angles**, and either is the **supplement** of the other.

Two angles whose sum is a circle (360°) are **explementary angles**, and either is the **explement** of the other. The two angles formed when any two lines terminate at a common point are explementary.

If the sides of one angle are perpendicular to those of another, the two angles are either equal or supplementary. Also, if the sides of one angle are parallel to those of another, the two angles are either equal or supplementary.

When two straight lines intersect, forming four angles, the two opposite angles, called **vertical angles**, are equal. Angles which have the same vertex and lie on opposite sides of a common side are **adjacent angles**. Adjacent angles formed by intersecting lines are supplementary, since each pair of adjacent angles forms a straight angle.

A **transversal** is a line that intersects two or more other lines. If two or more parallel lines are cut by a transversal, groups of adjacent and vertical angles are formed,

A **dihedral angle** is the angle between two intersecting planes.

2102. Triangles

A **plane triangle** is a closed figure formed by three straight lines, called **sides**, which meet at three points called **vertices**. The vertices are labeled with capital letters and the sides with lowercase letters, as shown in Figure 2102a.

An **equilateral triangle** is one with its three sides equal in length. It must also be **equiangular**, with its three angles equal.

An **isosceles triangle** is one with two equal sides, called **legs**. The angles opposite the legs are equal. A line which **bisects** (divides into two equal parts) the unequal angle of an isosceles triangle is the perpendicular bisector of the opposite side, and divides the triangle into two equal right triangles.

A **scalene triangle** is one with no two sides equal. In such a triangle, no two angles are equal.

An **acute triangle** is one with three acute angles.

A **right triangle** is one having a right angle. The side opposite the right angle is called the **hypotenuse**. The other two sides may be called **legs**. A plane triangle can have only one right angle.

An **obtuse triangle** is one with an obtuse angle. A plane triangle can have only one obtuse angle.

An **oblique triangle** is one which does not contain a right angle.

The **altitude** of a triangle is a line or the distance from any vertex perpendicular to the opposite side.

A **median** of a triangle is a line from any vertex to the center of the opposite side. The three medians of a triangle meet at a point called the **centroid** of the triangle. This point divides each median into two parts, that part between the centroid and the vertex being twice as long as the other part.

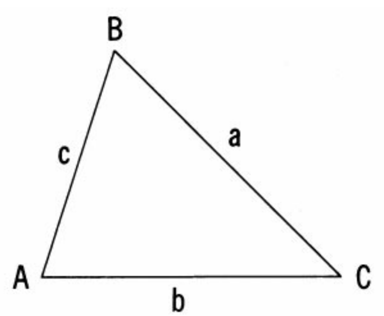

Figure 2102a. A triangle.

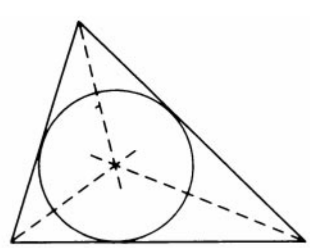

Figure 2102b. A circle inscribed in a triangle.

Lines bisecting the three *angles* of a triangle meet at a point which is equidistant from the three sides, which is the center of the **inscribed circle**, as shown in Figure 2102b. This point is of particular interest to navigators because it is the point theoretically taken as the fix when three lines of position of equal weight and having only random errors do not meet at a common point. In practical navigation, the point is found visually, not by construction, and other factors often influence the chosen fix position.

The perpendicular bisectors of the three *sides* of a triangle meet at a point which is equidistant from the three vertices, which is the center of the **circumscribed circle**, the circle through the three vertices and the smallest circle which can be drawn enclosing the triangle. The center of a circumscribed circle is *within* an acute triangle, *on the hypotenuse* of a right triangle, and *outside* an obtuse triangle.

A line connecting the mid–points of two sides of a triangle is always parallel to the third side and half as long. Also, a line parallel to one side of a triangle and intersecting the other two sides divides these sides proportionally. This principle can be used to divide a line into any number of equal or proportional parts.

The sum of the angles of a plane triangle is always 180°. Therefore, the sum of the acute angles of a right triangle is 90°, and the angles are complementary. If one side of a triangle is extended, the **exterior angle** thus formed is supplementary to the adjacent **interior angle** and is therefore equal to the sum of the two non adjacent angles. If two angles of one triangle are equal to two angles of another triangle, the third angles are also equal, and the triangles are **similar**. If the area of one triangle is equal to the area of another, the triangles are **equal**. Triangles having equal bases and altitudes also have equal areas. Two figures are **congruent** if one can be placed over the other to make an exact fit. Congruent figures are both similar and equal. If any side of one triangle is equal to any side of a similar triangle, the triangles are congruent. For example, if two right triangles have equal sides, they are congruent; if two right triangles have two corresponding sides equal, they are congruent. Triangles are congruent only if the sides and angles are equal.

The sum of two sides of a plane triangle is always greater than the third side; their difference is always less than the third side.

The area of a triangle is equal to 1/2 of the area of the polygon formed from its base and height. This can be stated algebraically as:

$$\text{Area of plane triangle } A = \frac{bh}{2}$$

The square of the hypotenuse of a right triangle is equal to the sum of the squares of the other two sides, or $a^2 + b^2$

$= c^2$. Therefore the length of the hypotenuse of plane right triangle can be found by the formula :

$$c = \sqrt{a^2 + b^2}$$

2103. Circles

A **circle** is a plane, closed curve, all points of which are equidistant from a point within, called the **center**.

The distance around a circle is called the **circumference**. Technically the length of this line is the **perimeter**, although the term "circumference" is often used. An arc is part of a circumference. A major arc is more than a semicircle (180°), a **minor** are is less than a semicircle (180°). A **semi–circle** is half a circle (180°), a **quadrant** is a quarter of a circle (90°), a **quintant** is a fifth of a circle (72°), a **sextant** is a sixth of a circle (60°), an **octant** is an eighth of a circle (45°). Some of these names have been applied to instruments used by navigators for measuring altitudes of celestial bodies because of the part of a circle used for the length of the arc of the instrument.

Concentric circles have a common center. A **radius** (plural **radii**) or **semidiameter** is a straight line connecting the center of a circle with any point on its circumference.

A **diameter** of a circle is a straight line passing through its center and terminating at opposite sides of the circumference. It divides a circle into two equal parts. The ratio of the length of the circumference of any circle to the length of its diameter is 3.14159+, or π (the Greek letter pi), a relationship that has many useful applications.

A **sector** is that part of a circle bounded by two radii and an arc. The angle formed by two radii is called a **central angle**. Any pair of radii divides a circle into sectors, one less than a semicircle (180°) and the other greater than a semicircle (unless the two radii form a diameter).

A **chord** is a straight line connecting any two points on the circumference of a circle. Chords equidistant from the center of a circle are equal in length.

A **segment** is the part of a circle bounded by a chord and the intercepted arc. A chord divides a circle into two segments, one less than a semicircle (180°), and the other greater than a semicircle (unless the chord is a diameter). A diameter perpendicular to a chord bisects it, its arc, and its segments. Either pair of vertical angles formed by intersecting chords has a combined number of degrees equal to the sum of the number of degrees in the two arcs intercepted by the two angles.

An **inscribed angle** is one whose vertex is on the circumference of a circle and whose sides are chords. It has half as many degrees as the arc it intercepts. Hence, an angle inscribed in a semicircle is a right angle if its sides terminate at the ends of the diameter forming the semicircle.

A **secant** of a circle is a line intersecting the circle, or a chord extended beyond the circumference.

A **tangent** to a circle is a straight line, in the plane of the circle, which has only one point in common with the circumference. A tangent is perpendicular to the radius at the **point of tangency**. Two tangents from a common point to opposite sides of a circle are equal in length, and a line from the point to the center of the circle bisects the angle formed by the two tangents. An angle formed outside a circle by the intersection of two tangents, a tangent and a secant, or two secants has *half* as many degrees as the *difference* between the two intercepted arcs. An angle formed by a tangent and a chord, with the apex at the point of tangency, has half as many degrees as the arc it intercepts. A **common tangent** is one tangent to more than one circle. Two circles are tangent to each other if they touch at one point only. If of different sizes, the smaller circle may be either inside or outside the larger one.

Parallel lines intersecting a circle intercept equal arcs.

If A = area; r = radius; d = diameter; C = circumference; s = linear length of an arc; a = angular length of an arc, or the angle it subtends at the center of a circle, in degrees; b = angular length of an arc, or the angle it subtends at the center of a circle, in radians:

$$\text{Area of circle } A = \pi r^2 = \frac{\pi d^2}{4}$$

$$\text{Circumference of a circle } C = 2\pi r = \pi d = 2\pi \text{ rad}$$

$$\text{Area of sector} = \frac{\pi r^2 a}{360} = \frac{r^2 b}{2} = \frac{rs}{2}$$

$$\text{Area of segment} = \frac{r^2(b - \sin a)}{2}$$

2104. Spheres

A **sphere** is a solid bounded by a surface every point of which is equidistant from a point within called the **center**. It may also be formed by rotating a circle about any diameter.

A **radius** or **semidiameter** of a sphere is a straight line connecting its center with any point on its surface. A **diameter** of a sphere is a straight line through its center and terminated at both ends by the surface of the sphere.

The intersection of a plane and the surface of a sphere is a circle, a **great circle** if the plane passes through the center of the sphere, and a **small circle** if it does not. The shorter arc of the great circle between two points on the surface of a sphere is the shortest distance, on the surface of the sphere, between the points. Every great circle of a sphere bisects every other great circle of that sphere. The **poles** of a circle on a sphere are the extremities of the sphere's diameter which is perpendicular to the plane of the circle. All points on the circumference of the circle are equidistant from either of its poles. In the ease of a great circle, *both* poles are 90° from any point on the circumference of the circle. Any great circle may be considered a **primary**, particularly when it serves as the origin of measurement of a coordinate. The great circles through its poles are called **secondary**. Secondaries are perpendicular to their primary.

A **spherical triangle** is the figure formed on the surface of a sphere by the intersection of three great circles. The lengths of the sides of a spherical triangle are measured in degrees, minutes, and seconds, as the angular lengths of the arcs forming them. The sum of the three sides is always less than 360°. The sum of the three angles is always *more* than 180° and *less* than 540°.

A **lune** is the part of the surface of a sphere bounded by halves of two great circles.

2105. Coordinates

Coordinates are magnitudes used to define a position. Many different types of coordinates are used. Important navigational ones are described below.

If a position is known to be on a given *line*, only one magnitude (coordinate) is needed to identify the position if an origin is stated or understood.

If a position is known to be on a given *surface*, two magnitudes (coordinates) are needed to define the position.

If nothing is known regarding a position other than that it exists in space, three magnitudes (coordinates) are needed to define its position.

Each coordinate requires an origin, either stated or implied. If a position is known to be on a given plane, it might be defined by means of its distance from each of two intersecting lines, called **axes**. These are called **rectangular coordinates**. In Figure 2105, OY is called the **ordinate**, and OX is called the **abscissa**. Point O is the **origin**, and lines OX and OY the axes (called the X and Y axes, respectively). Point A is at position x,y. If the axes are not perpendicular but the lines x and y are drawn parallel to the axes, **oblique coordinates** result. Either type are called **Cartesian coordinates**. A three–dimensional system of Cartesian coordinates, with X Y, and Z axes, is called **space coordinates**.

Another system of plane coordinates in common usage consists of the *direction* and *distance* from the origin (called the **pole**). A line extending in the direction indicated is called a **radius vector**. Direction and distance from a fixed point constitute **polar coordinates**, sometimes called the rho–theta (the Greek ρ, to indicate distance, and the Greek θ, to indicate direction) system. An example of its use is the radar scope.

Spherical coordinates are used to define a position on the surface of a sphere or spheroid by indicating angular distance from a primary great circle and a reference secondary great circle. Examples used in navigation are latitude and longitude, altitude and azimuth, and declination and hour angle.

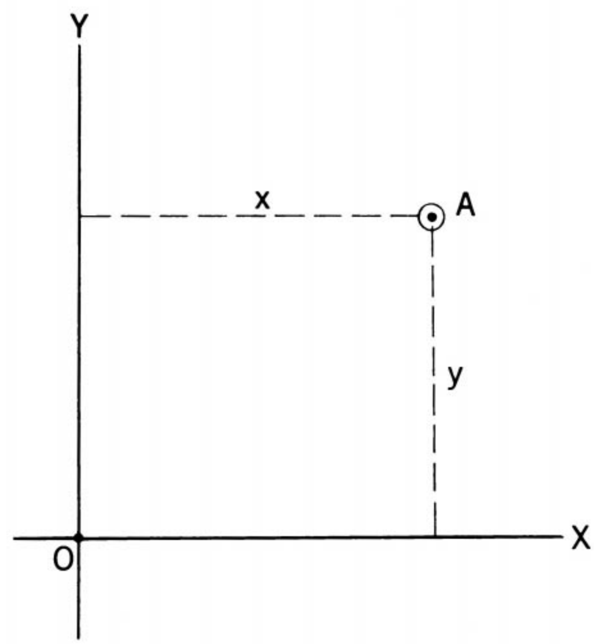

Figure 2105. Rectangular coordinates.

TRIGONOMETRY

2106. Definitions

Trigonometry deals with the relations among the angles and sides of triangles. **Plane trigonometry** deals with plane triangles, those on a plane surface. Spherical trigonometry deals with spherical triangles, which are drawn on the surface of a sphere. In navigation, the common methods of celestial sight reduction use spherical triangles on the surface of the earth. For most navigational purposes, the earth is assumed to be a sphere, though it is somewhat flattened.

2107. Angular Measure

A circle may be divided into 360 **degrees** (°), which is the **angular length** of its circumference. Each degree may be divided into 60 **minutes** ('), and each minute into 60 seconds ("). The angular length of an arc is usually expressed in these units. By this system a right angle or quadrant has 90° and a straight angle or semicircle 180°. In marine navigation, altitudes, latitudes, and longitudes are usually expressed in degrees, minutes, and tenths (27°14.4'). Azimuths are usually expressed in degrees and tenths (164.7°). The system of degrees, minutes, and seconds indicated above is the **sexagesimal system**. In the **centesimal system** used chiefly in France, the circle is divided into 400 **centesimal degrees** (sometimes called **grades**) each of which is divided into 100 centesimal minutes of 100 centesimal seconds each.

A **radian** is the angle subtended at the center of a circle by an arc having a linear length equal to the radius of the circle. A circle (360°) = 2π radians, a semicircle (180°) = π radians, a right angle (90°) = $\pi/2$ radians. The length of the arc of a circle is equal to the radius multiplied by the angle subtended in radians.

2108. Trigonometric Functions

Trigonometric functions are the various proportions or ratios of the sides of a plane right triangle, defined in relation to one of the acute angles. In Figure 2108a, let θ be any acute angle. From any point R on line OA, draw a line perpendicular to OB at F. From any other point R' on OA, draw a line perpendicular to OB at F'. Then triangles OFR and OF'R' are similar right triangles because all their corresponding *angles* are equal. Since in any pair of similar triangles the ratio of any two sides of one triangle is equal to the ratio of the corresponding two sides of the other triangle,

$$(RF,OF) = \frac{R'F'}{OF'} = \frac{RF}{OR} = \frac{R'F'}{OR'} \quad \text{and} \quad \frac{OF}{OR} = \frac{OF'}{OR'}.$$

No matter where the point R is located on OA, the ratio between the lengths of any two sides in the triangle OFR has a constant value. Hence, for any value of the acute angle θ, there is a fixed set of values for the ratios of the various sides of the triangle. These ratios are defined as follows:

$$\text{sine } \theta = \sin\theta = \frac{\text{side opposite}}{\text{hypotenuse}}$$

$$\text{cosine } \theta = \cos\theta = \frac{\text{side adjacent}}{\text{hypotenuse}}$$

$$\text{tangent } \theta = \tan\theta = \frac{\text{side opposite}}{\text{side adjacent}}$$

$$\text{cosecant } \theta = \csc\theta = \frac{\text{hypotenuse}}{\text{side opposite}}$$

$$\text{secant } \theta = \sec\theta = \frac{\text{hypotenuse}}{\text{side adjacent}}$$

$$\text{cotangent } \theta = \cot\theta = \frac{\text{side adjacent}}{\text{side opposite}}$$

Of these six principal functions, the second three are the reciprocals of the first three; therefore

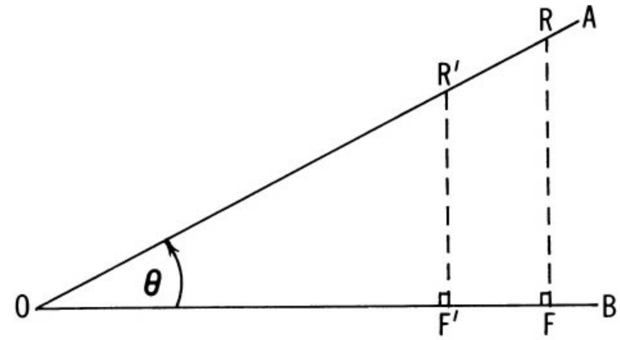

Figure 2108a. Similar right triangles.

$$\sin\theta = \frac{1}{\csc\theta} \qquad \csc\theta = \frac{1}{\sin\theta}$$

$$\cos\theta = \frac{1}{\sec\theta} \qquad \sec\theta = \frac{1}{\cos\theta}$$

$$\tan\theta = \frac{1}{\cot\theta} \qquad \cot\theta = \frac{1}{\tan\theta}$$

In Figure 2108b, A, B, and C are the angles of a plane right triangle, with the right angle at C. The sides are labeled a, b, c, opposite angles A, B, and C respectively. The six principal trigonometric functions of angle B are:

$$\sin B = \frac{b}{c} = \cos A = \cos(90° - B)$$

$$\cos B = \frac{a}{c} = \sin A = \sin(90° - B)$$

$$\tan B = \frac{b}{a} = \cot A = \cot(90° - B)$$

$$\cot B = \frac{a}{b} = \tan A = \tan(90° - B)$$

$$\sec B = \frac{c}{a} = \csc A = \csc(90° - B)$$

$$\csc B = \frac{c}{b} = \sec A = \sec(90° - B)$$

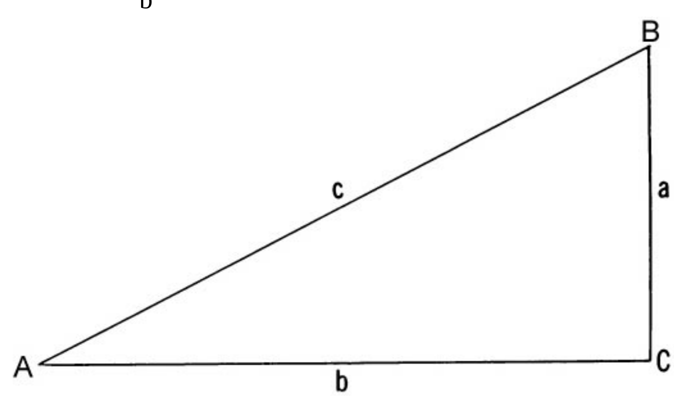

Figure 2108b. A right triangle.

Since A and B are *complementary*, these relations show that the sine of an angle is the cosine of its complement, the tangent of an angle is the cotangent of its complement, and the secant of an angle is the cosecant of its complement. Thus, the co–function of an angle is the function of its complement.

$$\sin(90° - A) = \cos A$$
$$\cos(90° - A) = \sin A$$
$$\tan(90° - A) = \cot A$$
$$\csc(90° - A) = \sec A$$
$$\sec(90° - A) = \csc A$$
$$\cot(90° - A) = \tan A$$

The numerical value of a trigonometric function is sometimes called the **natural function** to distinguish it from the logarithm of the function, called the **logarithmic function**. Numerical values of the six principal functions are given at 1' intervals in the table of natural trigonometric functions. Logarithms are given at the same intervals in another table.

Since the relationships of 30°, 60°, and 45° right triangles are as shown in Figure 2108c, certain values of the basic functions can be stated exactly, as shown in Table 2108.

2109. Functions In Various Quadrants

To make the definitions of the trigonometric functions more general to include those angles greater than 90°, the

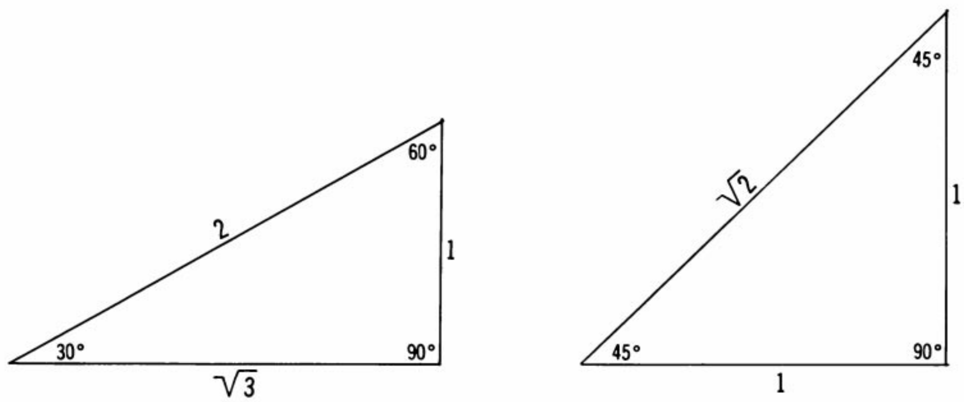

Figure 2108c. Numerical relationship of sides of 30°, 60°, and 45° triangles.

Function	*30°*	*45°*	*60°*
sine	$\frac{1}{2}$	$\frac{1}{\sqrt{2}} = \frac{1}{2}\sqrt{2}$	$\frac{\sqrt{3}}{2} = \frac{1}{2}\sqrt{3}$
cosine	$\frac{\sqrt{3}}{2} = \frac{1}{2}\sqrt{3}$	$\frac{1}{\sqrt{2}} = \frac{1}{2}\sqrt{2}$	$\frac{1}{2}$
tangent	$\frac{1}{\sqrt{3}} = \frac{1}{3}\sqrt{3}$	$\frac{1}{1} = 1$	$\frac{\sqrt{3}}{1} = \sqrt{3}$
cotangent	$\frac{\sqrt{3}}{1} = \sqrt{3}$	$\frac{1}{1} = 1$	$\frac{1}{\sqrt{3}} = \frac{1}{3}\sqrt{3}$
secant	$\frac{2}{\sqrt{3}} = \frac{2}{3}\sqrt{3}$	$\frac{\sqrt{2}}{1} = \sqrt{2}$	$\frac{2}{1} = 2$
cosecant	$\frac{2}{1} = 2$	$\frac{\sqrt{2}}{1} = \sqrt{2}$	$\frac{2}{\sqrt{3}} = \frac{2}{3}\sqrt{3}$

Table 2108. Values of various trigonometric functions for angles 30°, 45°, and 60°.

functions are defined in terms of the rectangular Cartesian coordinates of point R of Figure 2108a, due regard being given to the sign of the function. In Figure 2109a, OR is assumed to be a *unit* radius. By convention the sign of OR is always positive. This radius is imagined to rotate in a counterclockwise direction through 360° from the horizontal position at 0°, the positive direction along the X axis. Ninety degrees (90°) is the positive direction along the Y axis. The angle between the original position of the radius and its position at any time increases from 0° to 90° in the first quadrant (I), 90° to 180° in the second quadrant (II), 180° to 270° in the third quadrant (III), and 270° to 360° in the fourth quadrant (IV).

The numerical value of the sine of an angle is equal to the projection of the unit radius on the Y–axis. According to the definition given in section 2108, the sine of angle in the first quadrant of Figure 2109a is $\frac{+y}{+OR}$. If the radius OR is equal to one, $\sin\theta = +y$. Since +y is equal to the projection of the unit radius OR on the Y axis, the sine function of an angle in the first quadrant defined in terms of rectangular Cartesian coordinates does not contradict the definition in section 2108. In Figure 2109a,

$$\sin\theta = +y$$

$$\sin(180°-\theta) = +y = \sin\theta$$

$$\sin(180°+\theta) = -y = -\sin\theta$$

$$\sin(360°-\theta) = -y = \sin(-\theta) = -\sin\theta$$

The numerical value of the cosine of an angle is equal to the projection of the unit radius on the X axis. In Figure 2109a,

$$\cos\theta = +x$$

$$\cos(180°-\theta) = -x = -\cos\theta$$

$$\cos(180°+\theta) = -x = -\cos\theta$$

$$\cos(360°-\theta) = +x = \cos(-\theta) = \cos\theta\ .$$

The numerical value of the tangent of an angle is equal to the ratio of the projections of the unit radius on the Y and X axes. In Figure 2109a

$$\tan\theta = \frac{+y}{+x}$$

$$(180°-\theta) = \frac{+y}{-x} = -\tan\theta$$

$$\tan(180°+\theta) = \frac{-y}{-x} = \tan\theta$$

$$\tan(360°-\theta) = \frac{-y}{+x} = \tan(-\theta) = -\tan\theta\ .$$

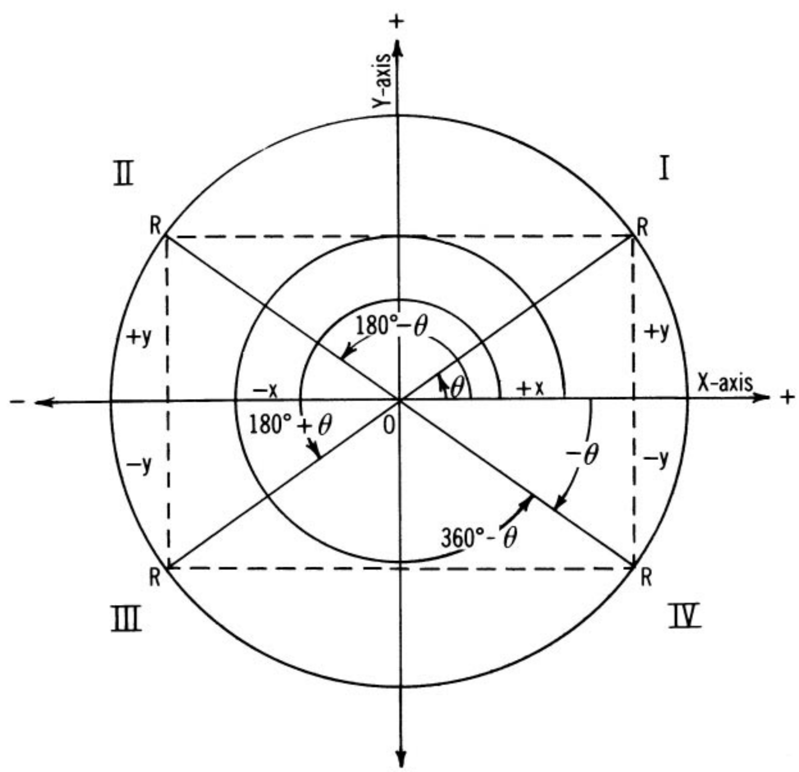

Figure 2109a. The functions in various quadrants, mathematical convention.

The cosecant, secant, and cotangent functions of angles in the various quadrants are similarly determined.

$$\csc\theta = \frac{1}{+y}$$

$$\csc(180° - \theta) = \frac{1}{+y} = \csc\theta$$

$$\csc(180° + \theta) = \frac{1}{-y} = -\csc\theta$$

$$\csc(360° - \theta) = \frac{1}{-y} = \csc(-\theta) = -\csc\theta$$

$$\sec\theta = \frac{1}{+x}$$

$$\sec(180° - \theta) = \frac{1}{-x} = -\sec\theta$$

$$\sec(180° + \theta) = \frac{1}{-x} = -\sec\theta$$

$$\sec(360° - \theta) = \frac{1}{+x} = \sec(-\theta) = \sec\theta$$

$$\cot\theta = \frac{+x}{+y}$$

$$\cot(180° - \theta) = \frac{-x}{+y} = -\cot\theta$$

$$\cot(180° + \theta) = \frac{-x}{-y} = \cot\theta$$

$$\cot(360° - \theta) = \frac{+x}{-y} = \cot(-\theta) = -\cot\theta.$$

The signs of the functions in the four different quadrants are shown below:

	I	II	III	IV
sine and cosecant	+	+	-	-
cosine and secant	+	-	-	+
tangent and cotangent	+	-	+	-

The numerical values vary by quadrant as shown above:

	I	II	III	IV
sin	0 to +1	+1 to 0	0 to –1	–1 to 0
csc	+∞ to +1	+1 to 0	–∞ to –1	–1 to –∞
cos	+1 to 0	0 to –1	–1 to 0	0 to +1
sec	+1 to +∞	–∞ to –1	–1 to –∞	+∞ to +1
tan	0 to +∞	–∞ to 0	0 to +∞	–∞ to 0
cot	+∞ to 0	–∞ to 0	+∞ to 0	0 to –∞

These relationships are shown graphically in Figure 2109b.

2110. Trigonometric Identities

A **trigonometric identity** is an equality involving trigonometric functions of θ which is true for all values of θ, except those values for which one of the functions is not defined or for which a denominator in the equality is equal to zero. The **fundamental identities** are those identities from which other identities can be derived.

$$\sin\theta = \frac{1}{\csc\theta} \qquad \csc\theta = \frac{1}{\sin\theta}$$

$$\cos\theta = \frac{1}{\sec\theta} \qquad \sec\theta = \frac{1}{\cos\theta}$$

$$\tan\theta = \frac{1}{\cot\theta} \qquad \cot\theta = \frac{1}{\tan\theta}$$

$$\tan\theta = \frac{\sin\theta}{\cos\theta} \qquad \cot\theta = \frac{\cos\theta}{\sin\theta}$$

$$\sin^2\theta + \cos^2\theta = 1 \qquad \tan^2\theta + 1 = \sec^2\theta$$

$$\sin(90° - \theta) = \cos\theta \qquad \csc(90° - \theta) = \sec\theta$$
$$\cos(90° - \theta) = \sin\theta \qquad \sec(90° - \theta) = \csc\theta$$
$$\tan(90° - \theta) = \cot\theta \qquad \cot(90° - \theta) = \tan\theta$$

$$\sin(-\theta) = -\sin\theta \qquad \csc(-\theta) = -\csc\theta$$
$$\cos(-\theta) = \cos\theta \qquad \sec(-\theta) = \sec\theta$$
$$\tan(-\theta) = -\tan\theta \qquad \cot(-\theta) = -\cot\theta$$

$$\sin(90 + \theta) = \cos\theta \qquad \csc(90 + \theta) = \sin\theta$$
$$\cos(90 + \theta) = -\sin\theta \qquad \sec(90 + \theta) = -\csc\theta$$
$$\tan(90 + \theta) = -\cot\theta \qquad \cot(90 + \theta) = -\tan\theta$$

$$\sin(180° - \theta) = \sin\theta \qquad \csc(180° - \theta) = \csc\theta$$
$$\cos(180° - \theta) = -\cos\theta \qquad \sec(180° - \theta) = -\sec\theta$$
$$\tan(180° - \theta) = -\tan\theta \qquad \cot(180° - \theta) = -\cot\theta$$

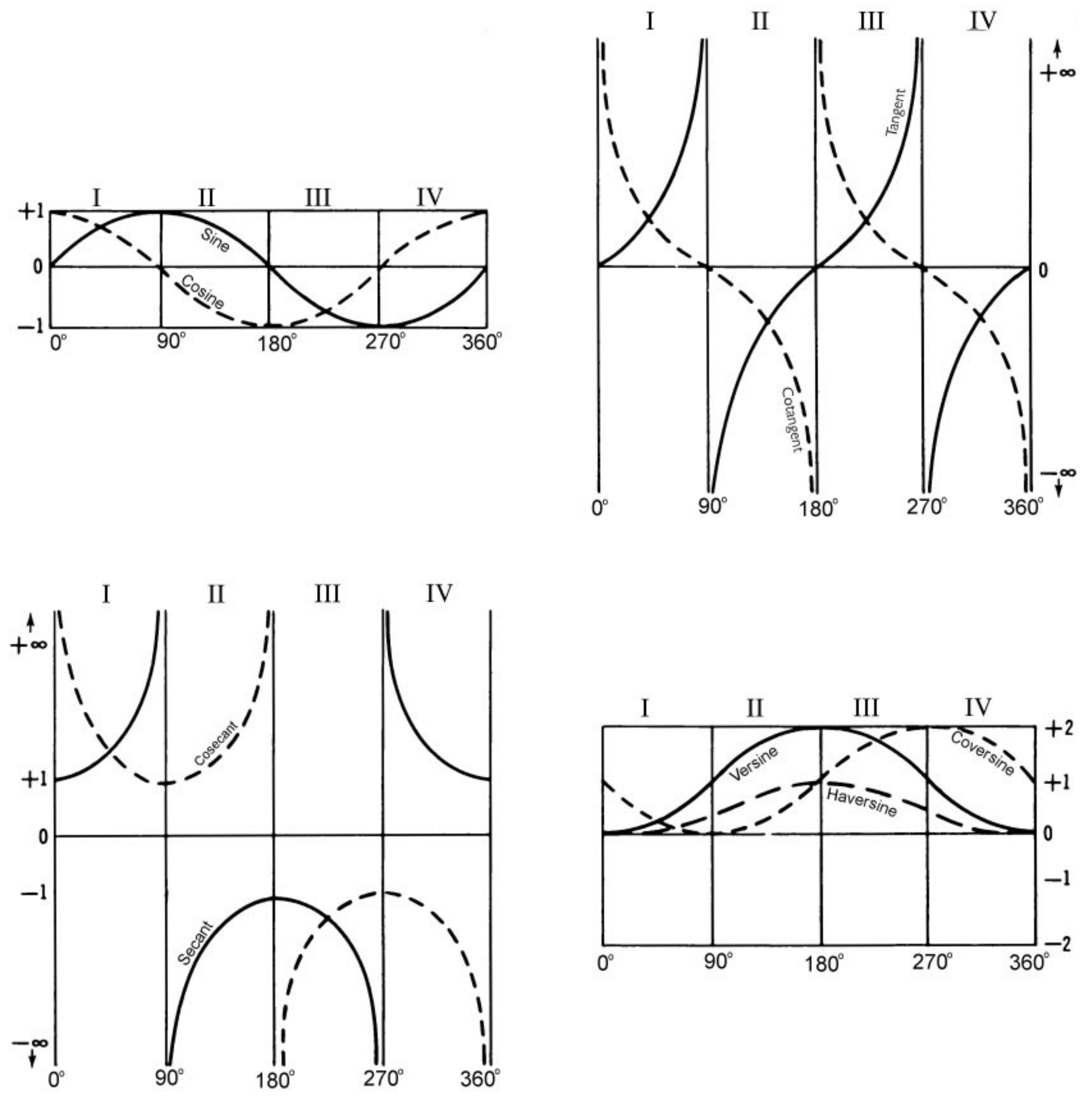

Figure 2109b. Graphic representation of values of trionometric functions in various quadrants.

$\sin(180°+\theta) = -\sin\theta$ $\quad\csc(180°+\theta) = -\csc\theta$

$\cos(180°+\theta) = \cos\theta$ $\quad\sec(180°+\theta) = \sec\theta$

$\tan(180°+\theta) = \tan\theta$ $\quad\cot(180°+\theta) = \cot\theta$

$\sin(360°-\theta) = -\sin\theta$ $\quad\csc(360°-\theta) = -\csc\theta$

$\cos(360°-\theta) = \cos\theta$ $\quad\sec(360°-\theta) = \sec\theta$

$\tan(360°-\theta) = -\tan\theta$ $\quad\cot(360°-\theta) = -\cot\theta$

2111. Inverse Trigonometric Functions

An angle having a given trigonometric function may be indicated in any of several ways. Thus, sin y = x, y = arc sin x, and y = $\sin^{-1}$ x have the same meaning. The superior "–1" is not an exponent in this case. In each case, y is "the angle whose sine is x." In this case, y is the **inverse sine** of x. Similar relationships hold for all trigonometric functions.

SOLUTION OF TRIANGLES

A triangle is composed of six parts: three angles and three sides. The angles may be designated A, B, and C; and the sides opposite these angles as *a*, *b*, and *c*, respectively. In general, when any three parts are known, the other three parts can be found, unless the known parts are the three angles.

2112. Right Plane Triangles

In a right plane triangle it is only necessary to substitute numerical values in the appropriate formulas representing the basic trigonometric functions and solve. Thus, if *a* and *b* are known,

$$\tan A = \frac{a}{b}$$
$$B = 90° - A$$
$$c = a \csc A$$

Similarly, if *c* and *B* are given,

$$A = 90° - B$$
$$a = c \sin A$$
$$b = c \cos A$$

2113. Oblique Plane Triangles

When solving an oblique plane triangle, it is often desirable to draw a rough sketch of the triangle approximately to scale, as shown in Figure 2113. The following laws are helpful in solving such triangles:

Law of sines: $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$

Law of cosines: $a^2 = b^2 + c^2 - 2bc \cos A.$

The unknown parts of oblique plane triangles can be computed by the formulas in Table 2113, among others. By reassignment of letters to sides and angles, these formulas can be used to solve for all unknown parts of oblique plane triangles.

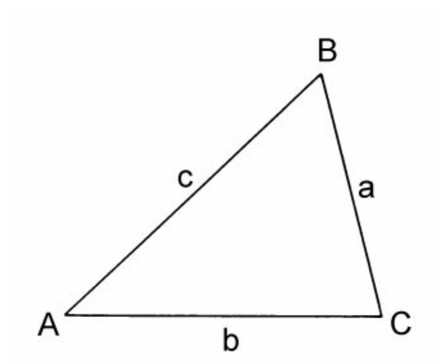

Figure 2113. A plane oblique triangle.

Known	*To find*	*Formula*	*Comments*
a, *b*, *c*	A	$\cos A = \frac{c^2 + b^2 - a^2}{2bc}$	Cosine law
a, *b*, A	B	$\sin B = \frac{b \sin A}{a}$	Sine law. Two solutions if *b*>*a*
	C	$C = 180° - (A + B)$	A + B + C = 180°
	c	$c = \frac{a \sin C}{\sin A}$	Sine law
a, *b*, C	A	$\tan A = \frac{a \sin C}{b - a \cos C}$	
	B	$B = 180° - (A + C)$	A + B + C = 180°
	c	$c = \frac{a \sin C}{\sin A}$	Sine law
a, A, B	*b*	$b = \frac{a \sin B}{\sin A}$	Sine law
	C	$C = 180° - (A + B)$	A + B + C = 180°
	c	$c = \frac{a \sin C}{\sin A}$	Sine law

Table 2113. Formulas for solving oblique plane triangles.

SPHERICAL TRIGONOMETRY

2114. Napier's Rules

Right spherical triangles can be solved with the aid of **Napier's Rules of Circular Parts**. If the right angle is omitted, the triangle has five parts: two angles and three sides, as shown in Figure 2114a. Since the right angle is already known, the triangle can be solved if any two other parts are known. If the two sides forming the right angle, and the *complements* of the other three parts are used, these elements (called "parts" in the rules) can be arranged in five sectors of a circle in the same order in which they occur in the triangle, as shown in Figure 2114b. Considering any part as the middle part, the two parts nearest it in the diagram are considered the adjacent parts, and the two farthest from it the opposite parts.

Napier's Rules state: The sine of a middle part equals the product of (1) the tangents of the adjacent parts or (2) the cosines of the opposite parts.

In the use of these rules, the co–function of a complement can be given as the function of the element. Thus, the cosine of co–A is the same as the sine of A. From these rules the following formulas can be derived:

$$\sin a = \tan b \cot B = \sin c \sin A$$

$$\sin b = \tan a \cot A = \sin c \sin B$$

$$\cos c = \cot A \cot B = \cos a \cos b$$

$$\cos A = \tan b \cot c = \cos a \sin B$$

$$\cos B = \tan a \cot c = \cos b \sin A$$

The following rules apply:

1. An oblique angle and the side opposite are in the same quadrant.

2. Side c (the hypotenuse) is less then 90° when a and b are in the same quadrant, and more than 90° when a and b are in different quadrants.

If the known parts are an angle and its opposite side, two solutions are possible.

A **quadrantal spherical triangle** is one having one side of 90°. A **biquadrantal spherical triangle** has two sides of 90°. A **triquadrantal spherical triangle** has three sides of 90°. A biquadrantal spherical triangle is isosceles and has two right angles opposite the 90° sides. A triquadrantal spherical triangle is equilateral, has three right angles, and bounds an octant (one–eighth) of the surface of the sphere. A quadrantal spherical triangle can be solved by Napier's rules provided any two elements in addition to the 90° side are known. The 90° side is omitted and the other parts are arranged in order in a five–sectored circle, using the complements of the three parts farthest from the 90° side. In the case of a quadrantal triangle, rule 1 above is used, and rule 2 restated: angle C (the angle opposite the side of 90°) is *more* than 90° when A and B are in the same quadrant, and *less* than 90° when A and B are in different quadrants. If the rule requires an angle of more than 90° and the solution produces an angle of less than 90°, subtract the solved angle from 180°.

2115. Oblique Spherical Triangles

An oblique spherical triangle can be solved by dropping a perpendicular from one of the apexes to the opposite side, subtended if necessary, to form two right spherical triangles. It can also be solved by the following formulas in Table 2115, reassigning the letters as necessary.

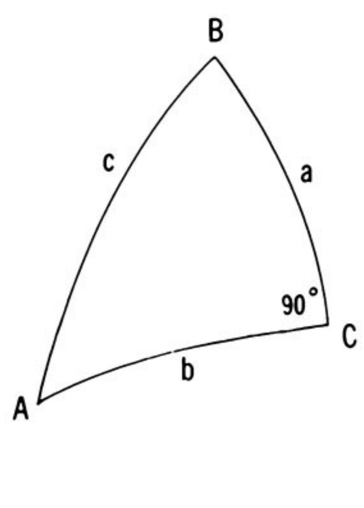

Figure 2114a. Parts of a right spherical triangle as used in Napier's rules.

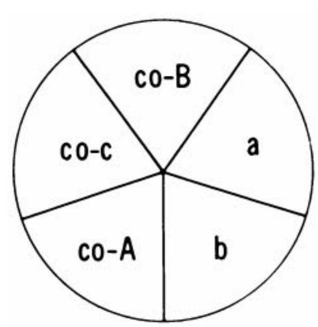

Figure 2114b. Diagram for Napier's Rules of Circular Parts.

Known	*To find*	*Formula*	*Comments*
a, b, C	A	$\tan A = \dfrac{\sin D \tan C}{\sin (b - D)}$	$\tan D = \tan a \cos C$
	B	$\sin B = \dfrac{\sin C \sin b}{\sin c}$	
c, A, B	C	$\cos C = \sin A \sin B \cos c - \cos A \cos B$	
	a	$\tan a = \dfrac{\tan c \sin E}{\sin (B + E)}$	$\tan E = \tan A \cos c$
	b	$\tan b = \dfrac{\tan c \sin F}{\sin (A + F)}$	$\tan F = \tan B \cos c$
a, b, A	c	$\sin(c + G) = \dfrac{\cos a \sin G}{\cos b}$	$\cot G = \cos A \tan b$ Two solutions
	B	$\sin B = \dfrac{\sin A \sin b}{\sin a}$	Two solutions
	C	$\sin(C + H) = \sin H \tan b \cot a$	$\tan H = \tan A \cos b$ Two solutions
a, A, B	C	$\sin(C - K) = \dfrac{\cos A \sin K}{\cos B}$	$\cot K = \tan B \cos a$ Two solutions
	b	$\sin b = \dfrac{\sin a \sin B}{\sin A}$	Two solutions
	c	$\sin(c - M) = \cot A \tan B \sin M$	$\tan M = \cos B \tan a$ Two solutions

Table 2115. Formulas for solving oblique spherical triangles.