# Student Solutions Manual

for use with

# Complex Variables and Applications

***Seventh Edition***

Selected Solutions to Exercises in Chapters 1-7

by

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# Chapter 1

## SECTION 2

1. *(a)* $(\sqrt{2}-i) - i(1-\sqrt{2}i) = \sqrt{2} - i - i - \sqrt{2} = -2i$;

   *(b)* $(2,-3)(-2,1) = (-4+3, 6+2) = (-1,8)$;

   *(c)* $(3,1)(3,-1)\left(\frac{1}{5},\frac{1}{10}\right) = (10,0)\left(\frac{1}{5},\frac{1}{10}\right) = (2,1)$.

2. *(a)* $\operatorname{Re}(iz) = \operatorname{Re}[i(x+iy)] = \operatorname{Re}(-y+ix) = -y = -\operatorname{Im} z$;

   *(b)* $\operatorname{Im}(iz) = \operatorname{Im}[i(x+iy)] = \operatorname{Im}(-y+ix) = x = \operatorname{Re} z$.

3. $$(1+z)^2 = (1+z)(1+z) = (1+z)\cdot 1 + (1+z)z = 1\cdot(1+z) + z(1+z)$$
   $$= 1 + z + z + z^2 = 1 + 2z + z^2.$$

4. If $z = 1 \pm i$, then $z^2 - 2z + 2 = (1 \pm i)^2 - 2(1 \pm i) + 2 = \pm 2i - 2 \mp 2i + 2 = 0$.

5. To prove that multiplication is commutative, write

$$\begin{aligned} z_1 z_2 &= (x_1, y_1)(x_2, y_2) = (x_1 x_2 - y_1 y_2, y_1 x_2 + x_1 y_2) \\ &= (x_2 x_1 - y_2 y_1, y_2 x_1 + x_2 y_1) = (x_2, y_2)(x_1, y_1) = z_2 z_1. \end{aligned}$$

6. *(a)* To verify the associative law for addition, write

$$\begin{aligned} (z_1 + z_2) + z_3 &= [(x_1, y_1) + (x_2, y_2)] + (x_3, y_3) = (x_1 + x_2, y_1 + y_2) + (x_3, y_3) \\ &= ((x_1 + x_2) + x_3, (y_1 + y_2) + y_3) = (x_1 + (x_2 + x_3), y_1 + (y_2 + y_3)) \\ &= (x_1, y_1) + (x_2 + x_3, y_2 + y_3) = (x_1, y_1) + [(x_2, y_2) + (x_3, y_3)] \\ &= z_1 + (z_2 + z_3). \end{aligned}$$

(b) To verify the distributive law, write

$$
\begin{aligned}
z(z_1+z_2) &= (x,y)[(x_1,y_1)+(x_2,y_2)] = (x,y)(x_1+x_2, y_1+y_2)\\
&= (xx_1+xx_2-yy_1-yy_2,\ yx_1+yx_2+xy_1+xy_2)\\
&= (xx_1-yy_1+xx_2-yy_2,\ yx_1+xy_1+yx_2+xy_2)\\
&= (xx_1-yy_1,\ yx_1+xy_1)+(xx_2-yy_2,\ yx_2+xy_2)\\
&= (x,y)(x_1,y_1)+(x,y)(x_2,y_2) = zz_1+zz_2.
\end{aligned}
$$

10. The problem here is to solve the equation $z^2+z+1=0$ for $z=(x,y)$ by writing

$$(x,y)(x,y)+(x,y)+(1,0)=(0,0).$$

Since

$$(x^2-y^2+x+1,\ 2xy+y)=(0,0),$$

it follows that

$$x^2-y^2+x+1=0 \quad \text{and} \quad 2xy+y=0.$$

By writing the second of these equations as $(2x+1)y=0$, we see that either $2x+1=0$ or $y=0$. If $y=0$, the first equation becomes $x^2+x+1=0$, which has no real roots (according to the quadratic formula). Hence $2x+1=0$, or $x=-1/2$. In that case, the first equation reveals that $y^2=3/4$, or $y=\pm\sqrt{3}/2$. Thus

$$z=(x,y)=\left(-\frac{1}{2}, \pm\frac{\sqrt{3}}{2}\right).$$

SECTION 3

1. (a) $\dfrac{1+2i}{3-4i}+\dfrac{2-i}{5i}=\dfrac{(1+2i)(3+4i)}{(3-4i)(3+4i)}+\dfrac{(2-i)(-5i)}{(5i)(-5i)}=\dfrac{-5+10i}{25}+\dfrac{-5-10i}{25}=-\dfrac{2}{5};$

(b) $\dfrac{5i}{(1-i)(2-i)(3-i)}=\dfrac{5i}{(1-3i)(3-i)}=\dfrac{5i}{-10i}=-\dfrac{1}{2};$

(c) $(1-i)^4=[(1-i)(1-i)]^2=(-2i)^2=-4.$

2. (a) $(-1)z=-z$ since $z+(-1)z=z[1+(-1)]=z\cdot 0=0;$

(b) $\dfrac{1}{1/z}=\dfrac{1}{z^{-1}}\cdot\dfrac{z}{z}=\dfrac{z}{1}=z \quad (z\neq 0).$

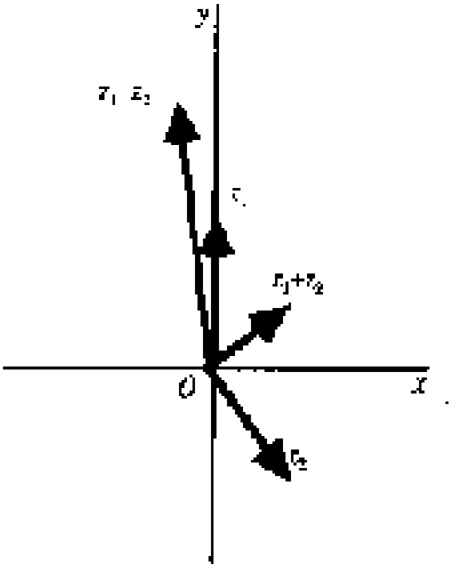
5. $(z_1z_2)(z_3z_4) = z_1[z_2(z_3z_4)] = z_1[(z_2z_3)z_4] = z_1[(z_3z_2)z_4] = z_1[z_3(z_2z_4)] = (z_1z_3)(z_2z_4).$

6. $$\frac{z_1z_2}{z_3z_4} = z_1z_2\left(\frac{1}{z_3z_4}\right) = z_1z_2\left(\frac{1}{z_3}\right)\left(\frac{1}{z_4}\right) = z_1\left(\frac{1}{z_3}\right)z_2\left(\frac{1}{z_4}\right) = \left(\frac{z_1}{z_3}\right)\left(\frac{z_2}{z_4}\right) \qquad (z_3 \neq 0, z_4 \neq 0).$$

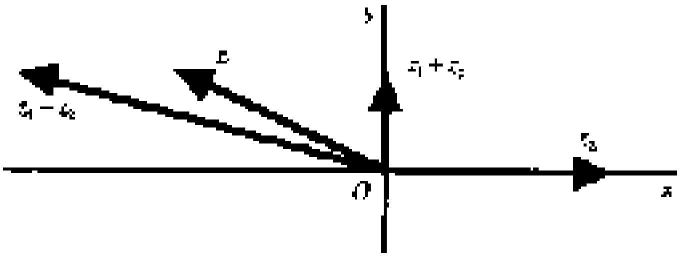
7. $$\frac{z_1z}{z_2z} = \left(\frac{z_1}{z_2}\right)\left(\frac{z}{z}\right) = \left(\frac{z_1}{z_2}\right)z\left(\frac{1}{z}\right) = \left(\frac{z_1}{z_2}\right)(zz^{-1}) = \left(\frac{z_1}{z_2}\right)\cdot 1 = \frac{z_1}{z_2} \qquad (z_2 \neq 0, z \neq 0).$$

## SECTION 4

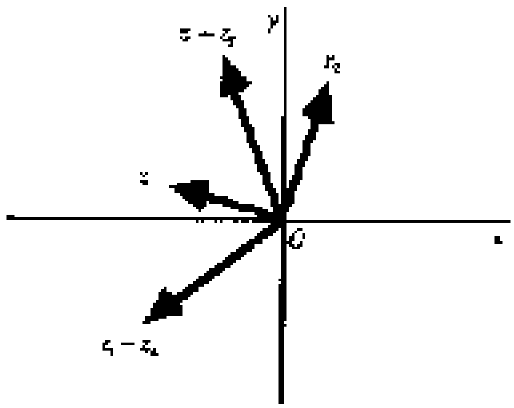
1. (a) $z_1 = 2i, \quad z_2 = \frac{2}{3} - i$



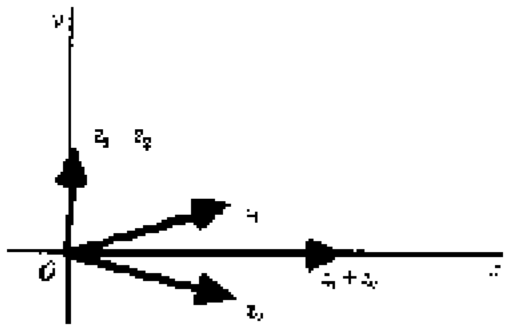

(b) $z_1 = (-\sqrt{3}, 1), \quad z_2 = (\sqrt{3}, 0)$



(c) $z_1 = (-3, 1)$, $z_2 = (1, 4)$




(d) $z_1 = x_1 + iy_1$, $z_2 = x_1 - iy_1$




**2.** Inequalities (3), Sec. 4, are

$$\operatorname{Re} z \le |\operatorname{Re} z| \le |z| \quad \text{and} \quad \operatorname{Im} z \le |\operatorname{Im} z| \le |z|.$$

These are obvious if we write them as

$$x \le |x| \le \sqrt{x^2 + y^2} \quad \text{and} \quad y \le |y| \le \sqrt{x^2 + y^2}.$$

**3.** In order to verify the inequality $\sqrt{2}\,|z| \ge |\operatorname{Re} z| + |\operatorname{Im} z|$, we rewrite it in the following ways:

$$\sqrt{2}\sqrt{x^2 + y^2} \ge |x| + |y|,$$

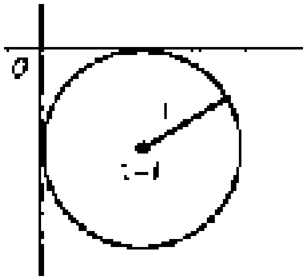
$$2(x^2 + y^2) \ge |x|^2 + 2|x||y| + |y|^2,$$

$$|x|^2 - 2|x||y| + |y|^2 \ge 0,$$

$$(|x| - |y|)^2 \ge 0.$$

This last form of the inequality to be verified is obviously true since the left-hand side is a perfect square.

4. *(a)* Rewrite $|z-1+i|=1$ as $|z-(1-i)|=1$. This is the circle centered at $1-i$ with radius 1. It is shown below.




5. *(a)* Write $|z-4i|+|z+4i|=10$ as $|z-4i|+|z-(-4i)|=10$ to see that this is the locus of all points $z$ such that the sum of the distances from $z$ to $4i$ and $-4i$ is a constant. Such a curve is an ellipse with foci $\pm 4i$.

   *(b)* Write $|z-1|=|z+i|$ as $|z-1|=|z-(-i)|$ to see that this is the locus of all points $z$ such that the distance from $z$ to 1 is always the same as the distance to $-i$. The curve is, then, the perpendicular bisector of the line segment from 1 to $-i$.

SECTION 3

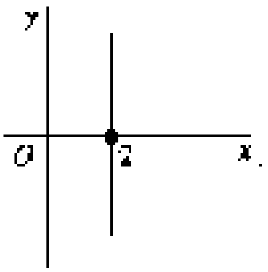
1. *(a)* $\overline{\bar{z}+3i} = z+\overline{3i} = z-3i$;

   *(b)* $\overline{iz} = \bar{i}\bar{z} = -i\bar{z}$;

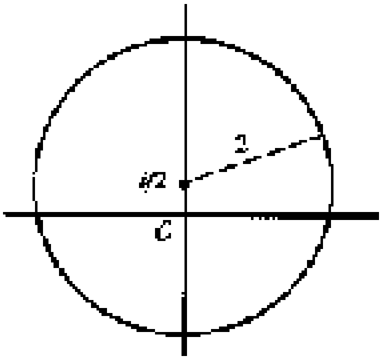
   *(c)* $\overline{(2+i)^2} = \left(\overline{2+i}\right)^2 = (2-i)^2 = 4-4i+i^2 = 4-4i-1 = 3-4i$;

   *(d)* $|(2\bar{z}+5)(\sqrt{2}-i)| = |2\bar{z}+5||\sqrt{2}-i| = |\overline{2z+5}|\sqrt{2+1} = \sqrt{3}\,|2z+5|$.

2. *(a)* Rewrite $\operatorname{Re}(\bar{z}-i)=2$ as $\operatorname{Re}[x+i(-y-1)]=2$, or $x=2$. This is the vertical line through the point $z=2$, shown below.



(b) Rewrite $|2z - i| = 4$ as $2\left|z - \frac{i}{2}\right| = 4$, or $\left|z - \frac{i}{2}\right| = 2$. This is the circle centered at $\frac{i}{2}$ with radius 2, shown below.




3. Write $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$. Then

$$\overline{z_1 - z_2} = \overline{(x_1 + iy_1) - (x_2 + iy_2)} = \overline{(x_1 - x_2) + i(y_1 - y_2)}$$
$$= (x_1 - x_2) - i(y_1 - y_2) = (x_1 - iy_1) - (x_2 - iy_2) = \bar{z}_1 - \bar{z}_2$$

and

$$\overline{z_1 z_2} = \overline{(x_1 + iy_1)(x_2 + iy_2)} = \overline{(x_1x_2 - y_1y_2) + i(y_1x_2 + x_1y_2)}$$
$$= (x_1x_2 - y_1y_2) - i(y_1x_2 + x_1y_2) = (x_1 - iy_1)(x_2 - iy_2) = \bar{z}_1\bar{z}_2.$$

4. (a) $\overline{z_1z_2z_3} = \overline{(z_1z_2)z_3} = \overline{z_1z_2}\,\bar{z}_3 = (\bar{z}_1\bar{z}_2)\bar{z}_3 = \bar{z}_1\bar{z}_2\bar{z}_3$;

(b) $\overline{z^4} = \overline{z^2z^2} = \overline{z^2}\,\overline{z^2} = \overline{zz}\,\overline{zz} = (\bar{z}\bar{z})(\bar{z}\bar{z}) = \bar{z}\bar{z}\bar{z}\bar{z} = \bar{z}^4$.

6. (a) $\overline{\left(\frac{z_1}{z_2z_3}\right)} = \frac{\bar{z}_1}{\overline{z_2z_3}} = \frac{\bar{z}_1}{\bar{z}_2\bar{z}_3}$;

(b) $\left|\frac{z_1}{z_2z_3}\right| = \frac{|z_1|}{|z_2z_3|} = \frac{|z_1|}{|z_2||z_3|}$.

8. In this problem, we shall use the inequalities (see Sec. 4)

$$|\text{Re}\,z| \le |z| \quad \text{and} \quad |z_1 + z_2 + z_3| \le |z_1| + |z_2| + |z_3|.$$

Specifically, when $|z| \le 1$,

$$|\text{Re}(2 + \bar{z} + z^3)| \le |2 + \bar{z} + z^3| \le 2 + |\bar{z}| + |z^3| = 2 + |z| + |z|^3 \le 2 + 1 + 1 = 4.$$

**10.** First write $z^4 - 4z^2 + 3 = (z^2 - 1)(z^2 - 3)$. Then observe that when $|z| = 2$,

$$|z^2 - 1| \ge \left||z^2| - |1|\right| = \left||z|^2 - 1\right| = |4 - 1| = 3$$

and

$$|z^2 - 3| \ge \left||z^2| - |3|\right| = \left||z|^2 - 3\right| = |4 - 3| = 1.$$

Thus, when $|z| = 2$,

$$|z^4 - 4z^2 + 3| = |z^2 - 1||z^2 - 3| \ge 3 \cdot 1 = 3.$$

Consequently, when $z$ lies on the circle $|z| = 2$,

$$\left|\frac{1}{z^4 - 4z^2 + 3}\right| = \frac{1}{|z^4 - 4z^2 + 3|} \le \frac{1}{3}.$$

**11.** *(a)* Prove that $z$ is real $\Leftrightarrow$ $\bar{z} = z$.

($\Leftarrow$) Suppose that $\bar{z} = z$, so that $x - iy = x + iy$. This means that $i2y = 0$, or $y = 0$. Thus $z = x + i0 = x$, or $z$ is real.

($\Rightarrow$) Suppose that $z$ is real, so that $z = x + i0$. Then $\bar{z} = x - i0 = x + i0 = z$.

*(b)* Prove that $z$ is either real or pure imaginary $\Leftrightarrow$ $\bar{z}^2 = z^2$.

($\Leftarrow$) Suppose that $\bar{z}^2 = z^2$. Then $(x - iy)^2 = (x + iy)^2$, or $i4xy = 0$. But this can be only if either $x = 0$ or $y = 0$, or possibly $x = y = 0$. Thus $z$ is either real or pure imaginary.

($\Rightarrow$) Suppose that $z$ is either real or pure imaginary. If $z$ is real, so that $z = x$, then $\bar{z}^2 = x^2 = z^2$. If $z$ is pure imaginary, so that $z = iy$, then $\bar{z}^2 = (-iy)^2 = (iy)^2 = z^2$.

**12.** *(a)* We shall use mathematical induction to show that

$$\overline{z_1 + z_2 + \cdots + z_n} = \bar{z}_1 + \bar{z}_2 + \cdots + \bar{z}_n \qquad (n = 2, 3, \ldots).$$

This is known when $n = 2$ (Sec. 5). Assuming now that it is true when $n = m$, we may write

$$\begin{aligned}\overline{z_1 + z_2 + \cdots + z_m + z_{m+1}} &= \overline{(z_1 + z_2 + \cdots + z_m) + z_{m+1}} \\ &= \overline{(z_1 + z_2 + \cdots + z_m)} + \bar{z}_{m+1} \\ &= (\bar{z}_1 + \bar{z}_2 + \cdots + \bar{z}_m) + \bar{z}_{m+1} \\ &= \bar{z}_1 + \bar{z}_2 + \cdots + \bar{z}_m + \bar{z}_{m+1}.\end{aligned}$$

*(b)* In the same way, we can show that

$$\overline{z_1 z_2 \cdots z_n} = \bar{z}_1 \bar{z}_2 \cdots \bar{z}_n \qquad (n = 2, 3, \ldots).$$

This is true when $n = 2$ (Sec. 5). Assuming that it is true when $n = m$, we write

$$\begin{aligned}\overline{z_1 z_2 \cdots z_m z_{m+1}} &= \overline{(z_1 z_2 \cdots z_m) z_{m+1}} = \overline{(z_1 z_2 \cdots z_m)}\, \bar{z}_{m+1} \\ &= (\bar{z}_1 \bar{z}_2 \cdots \bar{z}_m)\bar{z}_{m+1} = \bar{z}_1 \bar{z}_2 \cdots \bar{z}_m \bar{z}_{m+1}.\end{aligned}$$

14. The identities (Sec. 5) $z\bar{z} = |z|^2$ and $\operatorname{Re} z = \dfrac{z + \bar{z}}{2}$ enable us to write $|z - z_0| = R$ as

$$(z - z_0)(\bar{z} - \bar{z}_0) = R^2,$$

$$z\bar{z} - (z\bar{z}_0 + \overline{z\bar{z}_0}) + z_0\bar{z}_0 = R^2,$$

$$|z|^2 - 2\operatorname{Re}(z\bar{z}_0) + |z_0|^2 = R^2.$$

15. Since $x = \dfrac{z + \bar{z}}{2}$ and $y = \dfrac{z - \bar{z}}{2i}$, the hyperbola $x^2 - y^2 = 1$ can be written in the following ways:

$$\left(\frac{z + \bar{z}}{2}\right)^2 - \left(\frac{z - \bar{z}}{2i}\right)^2 = 1,$$

$$\frac{z^2 + 2z\bar{z} + \bar{z}^2}{4} + \frac{z^2 - 2z\bar{z} + \bar{z}^2}{4} = 1,$$

$$\frac{2z^2 + 2\bar{z}^2}{4} = 1,$$

$$z^2 + \bar{z}^2 = 2.$$

SECTION 7

1. *(a)* Since

$$\arg\left(\frac{i}{-2 - 2i}\right) = \arg i - \arg(-2 - 2i),$$

one value of $\arg\left(\dfrac{i}{-2 - 2i}\right)$ is $\dfrac{\pi}{2} - \left(-\dfrac{3\pi}{4}\right)$, or $\dfrac{5\pi}{4}$. Consequently, the principal value is

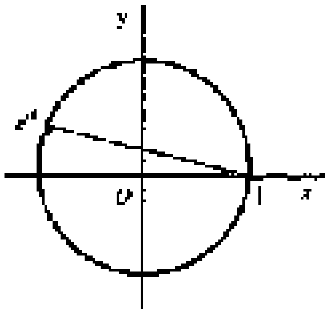
$$\frac{5\pi}{4} - 2\pi, \text{ or } -\frac{3\pi}{4}.$$

*(b)* Since

$$\arg(\sqrt{3}-i)^6 = 6\arg(\sqrt{3}-i),$$

one value of $\arg(\sqrt{3}-i)^6$ is $6\left(-\frac{\pi}{6}\right)$, or $-\pi$. So the principal value is $-\pi+2\pi$, or $\pi$.

4. The solution $\theta=\pi$ of the equation $|e^{i\theta}-1|=2$ in the interval $0\le\theta<2\pi$ is geometrically evident if we recall that $e^{i\theta}$ lies on the circle $|z|=1$ and that $|e^{i\theta}-1|$ is the distance between the points $e^{i\theta}$ and 1. See the figure below.




5. We know from de Moivre's formula that

$$(\cos\theta+i\sin\theta)^3=\cos 3\theta+i\sin 3\theta,$$

or

$$\cos^3\theta+3\cos^2\theta(i\sin\theta)+3\cos\theta(i\sin\theta)^2+(i\sin\theta)^3=\cos 3\theta+i\sin 3\theta.$$

That is,

$$(\cos^3\theta-3\cos\theta\sin^2\theta)+i(3\cos^2\theta\sin\theta-\sin^3\theta)=\cos 3\theta+i\sin 3\theta.$$

By equating real parts and then imaginary parts here, we arrive at the desired trigonometric identities:

(*a*) $\cos 3\theta=\cos^3\theta-3\cos\theta\sin^2\theta$; (*b*) $\sin 3\theta=3\cos^2\theta\sin\theta-\sin^3\theta$.

6. Here $z=re^{i\theta}$ is any nonzero complex number and $n$ a negative integer $(n=-1,-2,\ldots)$. Also, $m=-n=1,2,\ldots$. By writing

$$(z^m)^{-1}=(r^m e^{im\theta})^{-1}=\frac{1}{r^m}e^{i(-m\theta)}$$

and

$$(z^{-1})^m=\left[\frac{1}{r}e^{i(-\theta)}\right]^m=\left(\frac{1}{r}\right)^m e^{i(-m\theta)}=\frac{1}{r^m}e^{i(-m\theta)},$$

we see that $(z^m)^{-1}=(z^{-1})^m$. Thus the definition $z^n=(z^{-1})^m$ can also be written as $z^n=(z^m)^{-1}$.

9. First of all, given two nonzero complex numbers $z_1$ and $z_2$, suppose that there are complex numbers $c_1$ and $c_2$ such that $z_1 = c_1c_2$ and $z_2 = c_1\overline{c_2}$. Since

$$|z_1| = |c_1||c_2| \quad \text{and} \quad |z_2| = |c_1||\overline{c_2}| = |c_1||c_2|,$$

it follows that $|z_1| = |z_2|$.

Suppose, on the other hand, that we know only that $|z_1| = |z_2|$. We may write

$$z_1 = r_1\exp(i\theta_1) \quad \text{and} \quad z_2 = r_1\exp(i\theta_2).$$

If we introduce the numbers

$$c_1 = r_1\exp\left(i\frac{\theta_1+\theta_2}{2}\right) \quad \text{and} \quad c_2 = \exp\left(i\frac{\theta_1-\theta_2}{2}\right),$$

we find that

$$c_1c_2 = r_1\exp\left(i\frac{\theta_1+\theta_2}{2}\right)\exp\left(i\frac{\theta_1-\theta_2}{2}\right) = r_1\exp(i\theta_1) = z_1$$

and

$$c_1\overline{c_2} = r_1\exp\left(i\frac{\theta_1+\theta_2}{2}\right)\exp\left(-i\frac{\theta_1-\theta_2}{2}\right) = r_1\exp i\theta_2 = z_2.$$

That is,

$$z_1 = c_1c_2 \quad \text{and} \quad z_2 = c_1\overline{c_2}.$$

10. If $S = 1 + z + z^2 + \cdots + z^n$, then

$$S - zS = (1 + z + z^2 + \cdots + z^n) - (z + z^2 + z^3 + \cdots + z^{n+1}) = 1 - z^{n+1}.$$

Hence $S = \dfrac{1 - z^{n+1}}{1 - z}$, provided $z \neq 1$. That is,

$$1 + z + z^2 + \cdots + z^n = \frac{1 - z^{n+1}}{1 - z} \qquad (z \neq 1).$$

Putting $z = e^{i\theta}$ $(0 < \theta < 2\pi)$ in this identity, we have

$$1 + e^{i\theta} + e^{i2\theta} + \cdots + e^{in\theta} = \frac{1 - e^{i(n+1)\theta}}{1 - e^{i\theta}}.$$

Now the real part of the left-hand side here is evidently

$$1+\cos\theta+\cos 2\theta+\cdots+\cos n\theta;$$

and, to find the real part of the right-hand side, we write that side in the form

$$\frac{1-\exp[i(n+1)\theta]}{1-\exp(i\theta)}\cdot\frac{\exp\left(-i\frac{\theta}{2}\right)}{\exp\left(-i\frac{\theta}{2}\right)}=\frac{\exp\left(-i\frac{\theta}{2}\right)-\exp\left[i\frac{(2n+1)\theta}{2}\right]}{\exp\left(-i\frac{\theta}{2}\right)-\exp\left(i\frac{\theta}{2}\right)},$$

which becomes

$$\frac{\cos\frac{\theta}{2}-i\sin\frac{\theta}{2}-\cos\frac{(2n+1)\theta}{2}-i\sin\frac{(2n+1)\theta}{2}}{-2i\sin\frac{\theta}{2}}\cdot\frac{i}{i},$$

or

$$\frac{\left[\sin\frac{\theta}{2}+\sin\frac{(2n+1)\theta}{2}\right]-i\left[\cos\frac{\theta}{2}-\cos\frac{(2n+1)\theta}{2}\right]}{2\sin\frac{\theta}{2}}.$$

The real part of this is clearly

$$\frac{1}{2}+\frac{\sin\frac{(2n+1)\theta}{2}}{2\sin\frac{\theta}{2}},$$

and we arrive at *Lagrange's trigonometric identity*:

$$1+\cos\theta+\cos 2\theta+\cdots+\cos n\theta=\frac{1}{2}+\frac{\sin\frac{(2n+1)\theta}{2}}{2\sin\frac{\theta}{2}}\qquad(0<\theta<2\pi).$$

## SECTION 9

1. *(a)* Since $2i = 2\exp\left[i\left(\frac{\pi}{2}+2k\pi\right)\right]$ $(k = 0,\pm1,\pm2,\ldots)$, the desired roots are

$$(2i)^{1/2} = \sqrt{2}\exp\left[i\left(\frac{\pi}{4}+k\pi\right)\right] \qquad (k = 0,1).$$

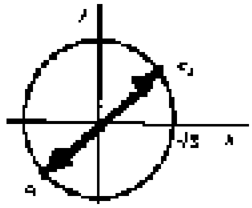
That is,

$$c_0 = \sqrt{2}e^{i\pi/4} = \sqrt{2}\left(\cos\frac{\pi}{4}+i\sin\frac{\pi}{4}\right) = \sqrt{2}\left(\frac{1}{\sqrt{2}}+\frac{i}{\sqrt{2}}\right) = 1+i$$

and

$$c_1 = (\sqrt{2}e^{i\pi/4})e^{i\pi} = -c_0 = -(1+i),$$

$c_0$ being the principal root. These are sketched below.



*(b)* Observe that $1-\sqrt{3}i = 2\exp\left[i\left(-\frac{\pi}{3}+2k\pi\right)\right]$ $(k = 0,\pm1,\pm2,\ldots)$. Hence

$$(1-\sqrt{3}i)^{1/2} = \sqrt{2}\exp\left[i\left(-\frac{\pi}{6}+k\pi\right)\right] \qquad (k = 0,1).$$

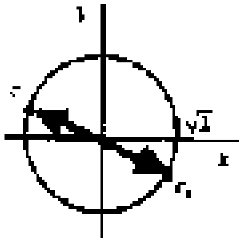
The principal root is

$$c_0 = \sqrt{2}e^{-i\pi/6} = \sqrt{2}\left(\cos\frac{\pi}{6}-i\sin\frac{\pi}{6}\right) = \sqrt{2}\left(\frac{\sqrt{3}}{2}-\frac{i}{2}\right) = \frac{\sqrt{3}-i}{\sqrt{2}},$$

and the other root is

$$c_1 = (\sqrt{2}e^{-i\pi/6})e^{i\pi} = -c_0 = -\frac{\sqrt{3}-i}{\sqrt{2}}.$$

These roots are shown below.



2. (a) Since $-16 = 16\exp[i(\pi - 2k\pi)]$ $(k = 0, \pm 1, \pm 2, \ldots)$, the needed roots are

$$(-16)^{1/4} = 2\exp\left[i\left(\frac{\pi}{4} + \frac{k\pi}{2}\right)\right] \qquad (k = 0,1,2,3).$$

The principal root is

$$c_0 = 2e^{i\pi/4} = 2\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right) = 2\left(\frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}\right) = \sqrt{2}(1+i)$$

The other three roots are

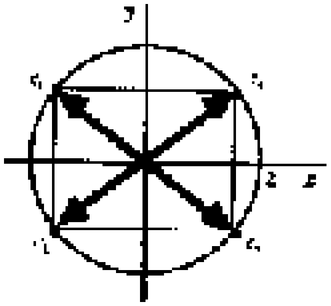
$$c_1 = (2e^{i\pi/4})e^{i\pi/2} = c_0 i = \sqrt{2}(1+i)i = -\sqrt{2}(1-i),$$

$$c_2 = (2e^{i\pi/4})e^{i\pi} = -c_0 = -\sqrt{2}(1+i),$$

and

$$c_3 = (2e^{i\pi/4})e^{i3\pi/2} = c_0(-i) = \sqrt{2}(1+i)(-i) = \sqrt{2}(1-i).$$

The four roots are shown below.




(b) First write $-8 - 8\sqrt{3}i = 16\exp\left[i\left(-\frac{2\pi}{3} + 2k\pi\right)\right]$ $(k = 0, \pm 1, \pm 2, \ldots)$. Then

$$(-8 - 8\sqrt{3}i)^{1/4} = 2\exp\left[i\left(-\frac{\pi}{6} + \frac{k\pi}{2}\right)\right] \qquad (k = 0,1,2,3).$$

The principal root is

$$c_0 = 2e^{-i\pi/6} = 2\left(\cos\frac{\pi}{6} - i\sin\frac{\pi}{6}\right) = 2\left(\frac{\sqrt{3}}{2} - \frac{i}{2}\right) = \sqrt{3} - i.$$

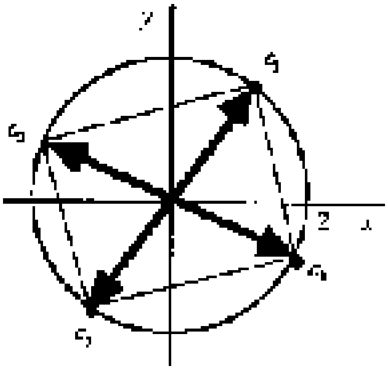
The others are

$$c_1 = (2e^{-i\pi/6})e^{i\pi/2} = c_0 i = 1+\sqrt{3}i,$$

$$c_2 = (2e^{-i\pi/6})e^{i\pi} = -c_0 = -(\sqrt{3}-i),$$

$$c_3 = (2e^{-i\pi/6})e^{i3\pi/2} = c_0(-i) = -(1+\sqrt{3}i).$$

These roots are all shown below.




3. (a) By writing $-1 = 1\exp[i(\pi + 2k\pi)]$ $(k = 0, \pm 1, \pm 2, \ldots)$, we see that

$$(-1)^{1/3} = \exp\left[i\left(\frac{\pi}{3} + \frac{2k\pi}{3}\right)\right] \qquad (k = 0,1,2).$$

The principal root is

$$c_0 = e^{i\pi/3} = \cos\frac{\pi}{3} + i\sin\frac{\pi}{3} = \frac{1+\sqrt{3}i}{2}.$$

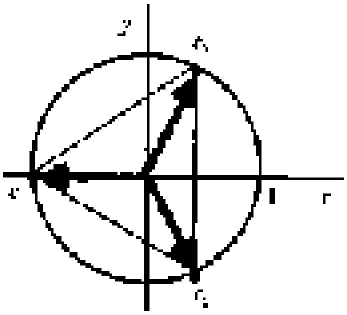
The other two roots are

$$c_1 = e^{i\pi} = -1$$

and

$$c_2 = e^{i5\pi/3} = e^{i2\pi}e^{-i\pi/3} = \cos\frac{\pi}{3} - i\sin\frac{\pi}{3} = \frac{1-\sqrt{3}i}{2}.$$

All three roots are shown below.



(*b*) Since $8 = 8\exp[i(0+2k\pi)]$ $(k = 0, \pm1, \pm2, \dots)$, the desired roots of 8 are

$$8^{1/6} = \sqrt{2}\exp\left(i\frac{k\pi}{3}\right) \qquad (k = 0,1,2,3,4,5),$$

the principal one being

$$c_0 = \sqrt{2}.$$

The others are

$$c_1 = \sqrt{2}e^{i\pi/3} = \sqrt{2}\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right) = \sqrt{2}\left(\frac{1}{2} + \frac{\sqrt{3}}{2}i\right) = \frac{1+\sqrt{3}i}{\sqrt{2}},$$

$$c_2 = (\sqrt{2}e^{-i\pi/3})e^{i\pi} = \sqrt{2}\left(\cos\frac{\pi}{3} - i\sin\frac{\pi}{3}\right)(-1) = -\sqrt{2}\left(\frac{1}{2} - \frac{\sqrt{3}}{2}i\right) = -\frac{1-\sqrt{3}i}{\sqrt{2}},$$

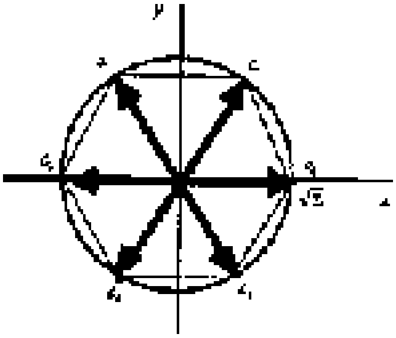
$$c_3 = \sqrt{2}e^{i\pi} = -\sqrt{2},$$

$$c_4 = (\sqrt{2}e^{i\pi/3})e^{i\pi} = -c_1 = -\frac{1+\sqrt{3}i}{\sqrt{2}},$$

and

$$c_5 = (\sqrt{2}e^{i2\pi/3})e^{i\pi} = -c_2 = \frac{1-\sqrt{3}i}{\sqrt{2}}.$$

All six roots are shown below.




4. The three cube roots of the number $z_0 = -4\sqrt{2} + 4\sqrt{2}i = 8\exp\left(i\frac{3\pi}{4}\right)$ are evidently

$$(z_0)^{1/3} = 2\exp\left[i\left(\frac{\pi}{4} + \frac{2k\pi}{3}\right)\right] \qquad (k = 0,1,2).$$

In particular,

$$c_0 = 2\exp\left(i\frac{\pi}{4}\right) = \sqrt{2}(1+i).$$

With the aid of the number $\omega_3 = \frac{-1+\sqrt{3}i}{2}$, we obtain the other two roots:

$$c_1 = c_0\omega_3 = \sqrt{2}(1+i)\left(\frac{-1+\sqrt{3}i}{2}\right) = \frac{-(\sqrt{3}+1)+(\sqrt{3}-1)i}{\sqrt{2}},$$

$$c_2 = c_0\omega_3^2 = (c_0\omega_3)\omega_3 = \left[\frac{-(\sqrt{3}+1)+(\sqrt{3}-1)i}{\sqrt{2}}\right]\left(\frac{-1+\sqrt{3}i}{2}\right) = \frac{(\sqrt{3}-1)-(\sqrt{3}+1)i}{\sqrt{2}}.$$

**5.** *(a)* Let $a$ denote any fixed real number. In order to find the two square roots of $a+i$ in exponential form, we write

$$A = |a+i| = \sqrt{a^2+1} \quad \text{and} \quad \alpha = \text{Arg}(a+i).$$

Since

$$a+i = A\exp[i(\alpha+2k\pi)] \qquad (k = 0,\pm 1,\pm 2,\ldots),$$

we see that

$$(a+i)^{1/2} = \sqrt{A}\exp\left[i\left(\frac{\alpha}{2}+k\pi\right)\right] \qquad (k = 0,1).$$

That is, the desired square roots are

$$\sqrt{A}e^{i\alpha/2} \quad \text{and} \quad \sqrt{A}e^{i\alpha/2}e^{i\pi} = -\sqrt{A}e^{i\alpha/2}.$$

*(b)* Since $a+i$ lies above the real axis, we know that $0<\alpha<\pi$. Thus $0<\frac{\alpha}{2}<\frac{\pi}{2}$, and this tells us that $\cos\left(\frac{\alpha}{2}\right)>0$ and $\sin\left(\frac{\alpha}{2}\right)>0$. Since $\cos\alpha = \frac{a}{A}$, it follows that

$$\cos\frac{\alpha}{2} = \sqrt{\frac{1+\cos\alpha}{2}} = \frac{1}{\sqrt{2}}\sqrt{1+\frac{a}{A}} = \frac{\sqrt{A+a}}{\sqrt{2}\sqrt{A}}$$

and

$$\sin\frac{\alpha}{2} = \sqrt{\frac{1-\cos\alpha}{2}} = \frac{1}{\sqrt{2}}\sqrt{1-\frac{a}{A}} = \frac{\sqrt{A-a}}{\sqrt{2}\sqrt{A}}.$$

Consequently,

$$\pm\sqrt{A}e^{i\alpha/2} = \pm\sqrt{A}\left(\cos\frac{\alpha}{2}+i\sin\frac{\alpha}{2}\right) = \pm\sqrt{A}\left(\frac{\sqrt{A+a}}{\sqrt{2}\sqrt{A}}+i\frac{\sqrt{A-a}}{\sqrt{2}\sqrt{A}}\right)$$

$$= \pm\frac{1}{\sqrt{2}}(\sqrt{A+a}+i\sqrt{A-a}).$$

**6.** The four roots of the equation $z^4 + 4 = 0$ are the four fourth roots of the number $-4$. To find those roots, we write $-4 = 4\exp[i(\pi + 2k\pi)]$ $(k = 0, \pm 1, \pm 2, \ldots)$. Then

$$(-4)^{1/4} = \sqrt{2}\exp\left[i\left(\frac{\pi}{4} + \frac{k\pi}{2}\right)\right] = \sqrt{2}e^{i\pi/4}e^{ik\pi/2} \qquad (k = 0,1,2,3).$$

To be specific,

$$c_0 = \sqrt{2}e^{i\pi/4} = \sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right) = \sqrt{2}\left(\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}\right) = 1 + i,$$

$$c_1 = c_0 e^{i\pi/2} = (1+i)i = -1 + i,$$

$$c_2 = c_0 e^{i\pi} = (1+i)(-1) = -1 - i,$$

$$c_3 = c_0 e^{i3\pi/2} = (1+i)(-i) = 1 - i.$$

This enables us to write

$$\begin{aligned}
z^4 + 4 &= (z - c_0)(z - c_1)(z - c_2)(z - c_3) \\
&= [(z - c_1)(z - c_2)]\,[(z - c_0)(z - c_3)] \\
&= [(z+1) - i][(z+1) + i]\cdot[(z-1) - i][(z-1) + i] \\
&= \left[(z+1)^2 + 1\right]\cdot\left[(z-1)^2 + 1\right] \\
&= \left(z^2 + 2z + 2\right)\left(z^2 - 2z + 2\right).
\end{aligned}$$

**7.** Let $c$ be any $n$th root of unity other than unity itself. With the aid of the identity (see Exercise 10, Sec. 7),

$$1 + z + z^2 + \cdots + z^{n-1} = \frac{1 - z^n}{1 - z} \qquad (z \neq 1),$$

we find that

$$1 + c + c^2 + \cdots + c^{n-1} = \frac{1 - c^n}{1 - c} = \frac{1 - 1}{1 - c} = 0.$$

**9.** Observe first that

$$(z^{1/m})^{-1} = \left[\sqrt[m]{r}\exp\frac{i(\theta + 2k\pi)}{m}\right]^{-1} = \frac{1}{\sqrt[m]{r}}\exp\frac{i(-\theta - 2k\pi)}{m} = \frac{1}{\sqrt[m]{r}}\exp\frac{i(-\theta)}{m}\exp\frac{i(-2k\pi)}{m}$$

and

$$(z^{-1})^{1/m} = \sqrt[m]{\frac{1}{r}}\exp\frac{i(-\theta+2k\pi)}{m} = \frac{1}{\sqrt[m]{r}}\exp\frac{i(-\theta)}{m}\exp\frac{i(2k\pi)}{m},$$

where $k = 0,1,2,\ldots,m-1$. Since the set

$$\exp\frac{i(-2k\pi)}{m} \qquad (k = 0,1,2,\ldots,m-1)$$

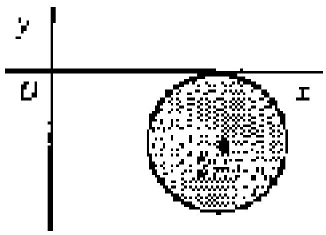
is the same as the set

$$\exp\frac{i(2k\pi)}{m} \qquad (k = 0,1,2,\ldots,m-1),$$

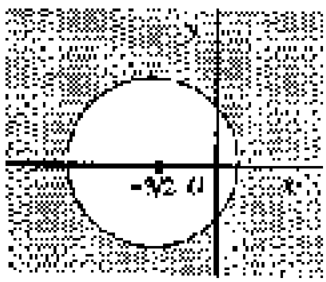
but in reverse order, we find that $(z^{1/m})^{-1} = (z^{-1})^{1/m}$.

## SECTION 10

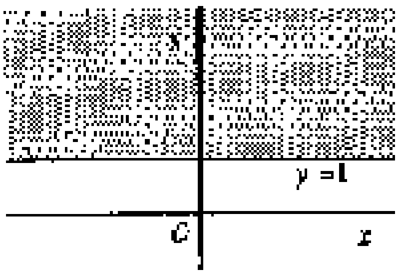
1. *(a)* Write $|z-2+i|\le 1$ as $|z-(2-i)|\le 1$ to see that this is the set of points inside and on the circle centered at the point $2-i$ with radius 1. It is *not* a domain.



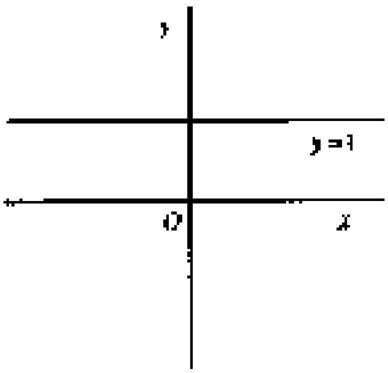

*(b)* Write $|2z+3|>4$ as $\left|z-\left(-\frac{3}{2}\right)\right|>2$ to see that the set in question consists of all points exterior to the circle with center at $-3/2$ and radius 2. It is a domain.



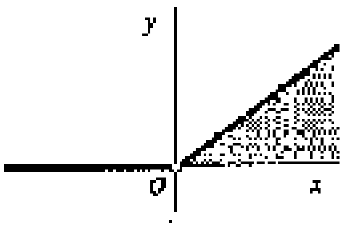
(c) Write $\operatorname{Im} z > 1$ as $y > 1$ to see that this is the half plane consisting of all points lying above the horizontal line $y = 1$. It is a domain.



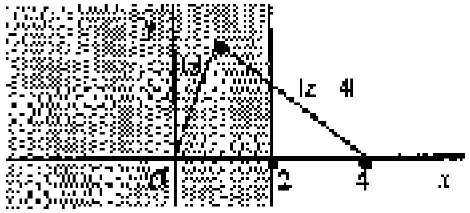

(d) The set $\operatorname{Im} z = 1$ is simply the horizontal line $y = 1$. It is *not* a domain.



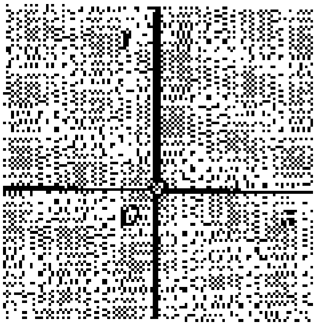

(e) The set $0 \le \arg z \le \frac{\pi}{4}$ $(z \ne 0)$ is indicated below. It is *not* a domain.



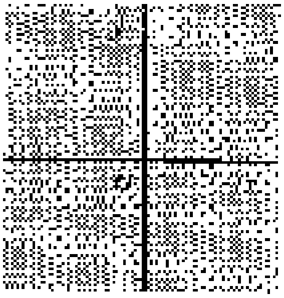

(f) The set $|z-4| \ge |z|$ can be written in the form $(x-4)^2 + y^2 \ge x^2 + y^2$, which reduces to $x \le 2$. This set, which is indicated below, is *not* a domain. The set is also geometrically evident since it consists of all points $z$ such that the distance between $z$ and 4 is greater than or equal to the distance between $z$ and the origin.



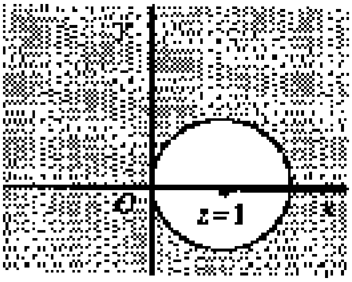
4. *(a)* The closure of the set $-\pi < \arg z < \pi$ $(z \neq 0)$ is the entire plane.



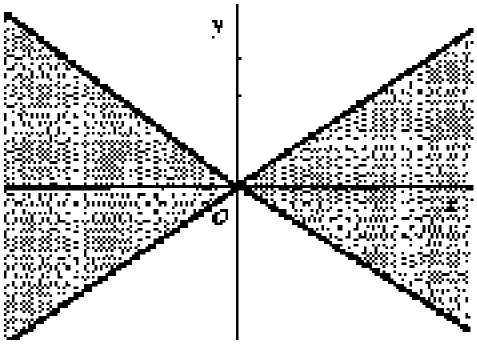

*(b)* We first write the set $|\text{Re}\, z| < |z|$ as $|x| < \sqrt{x^2 + y^2}$, or $x^2 < x^2 + y^2$. But this last inequality is the same as $y^2 > 0$, or $|y| > 0$. Hence the closure of the set $|\text{Re}\, z| < |z|$ is the entire plane.



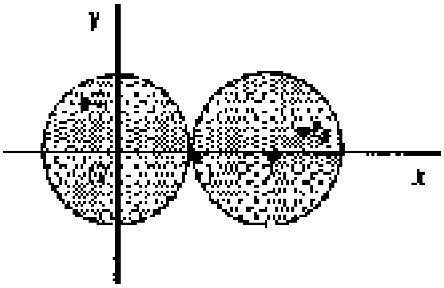
*(c)* Since $\dfrac{1}{z} = \dfrac{\bar{z}}{z\bar{z}} = \dfrac{\bar{z}}{|z|^2} = \dfrac{x - iy}{x^2 + y^2}$, the set $\text{Re}\left(\dfrac{1}{z}\right) \leq \dfrac{1}{2}$ can be written as $\dfrac{x}{x^2 + y^2} \leq \dfrac{1}{2}$, or $(x^2 - 2x) + y^2 \geq 0$. Finally, by completing the square, we arrive at the inequality $(x-1)^2 + y^2 \geq 1^2$, which describes the circle, together with its exterior, that is centered at $z = 1$ with radius 1. The closure of this set is itself.



(d) Since $z^2 = (x+iy)^2 = x^2 - y^2 + i2xy$, the set $\operatorname{Re}(z^2) > 0$ can be written as $y^2 < x^2$, or $|y| < |x|$. The closure of this set consists of the lines $y = \pm x$ together with the shaded region shown below.




5. The set $S$ consists of all points $z$ such that $|z| < 1$ or $|z - 2| < 1$, as shown below.




Since every polygonal line joining $z_1$ and $z_2$ must contain at least one point that is not in $S$, it is clear that $S$ is not connected.

8. We are given that a set $S$ contains each of its accumulation points. The problem here is to show that $S$ must be closed. We do this by contradiction. We let $z_0$ be a boundary point of $S$ and suppose that it is not a point in $S$. The fact that $z_0$ is a boundary point means that every neighborhood of $z_0$ contains at least one point in $S$; and, since $z_0$ is not in $S$, we see that every *deleted* neighborhood of $S$ must contain at least one point in $S$. Thus $z_0$ is an accumulation point of $S$, and it follows that $z_0$ is a point in $S$. But this contradicts the fact that $z_0$ is not in $S$. We may conclude, then, that each boundary point $z_0$ must be in $S$. That is, $S$ is closed.

# Chapter 2

SECTION 11

1. *(a)* The function $f(z)=\dfrac{1}{z^2+1}$ is defined everywhere in the finite plane except at the points $z=\pm i$, where $z^2+1=0$.

   *(b)* The function $f(z)=\operatorname{Arg}\left(\dfrac{1}{z}\right)$ is defined throughout the entire finite plane except for the point $z=0$.

   *(c)* The function $f(z)=\dfrac{z}{z+\overline{z}}$ is defined everywhere in the finite plane except for the imaginary axis. This is because the equation $z+\overline{z}=0$ is the same as $x=0$.

   *(d)* The function $f(z)=\dfrac{1}{1-|z|^2}$ is defined everywhere in the finite plane except on the circle $|z|=1$, where $1-|z|^2=0$.

3. Using $x=\dfrac{z+\overline{z}}{2}$ and $y=\dfrac{z-\overline{z}}{2i}$, write

$$\begin{aligned}
f(z)&=x^2-y^2-2y+i(2x-2xy)\\
&=\frac{(z+\overline{z})^2}{4}+\frac{(z-\overline{z})^2}{4}+i(z-\overline{z})+i(z+\overline{z})-\frac{(z+\overline{z})(z-\overline{z})}{2}\\
&=\frac{z^2}{2}+\frac{\overline{z}^2}{2}+2iz-\frac{z^2}{2}+\frac{\overline{z}^2}{2}=\overline{z}^2+2iz.
\end{aligned}$$

SECTION 17

5. Consider the function

$$f(z)=\left(\frac{z}{\overline{z}}\right)^2=\left(\frac{x+iy}{x-iy}\right)^2 \qquad (z\neq 0),$$

where $z=x+iy$. Observe that if $z=(x,0)$, then

$$f(z)=\left(\frac{x+i0}{x-i0}\right)^2=1;$$

and if $z=(0,y)$,

$$f(z)=\left(\frac{0+iy}{0-iy}\right)^2=1$$

But if $z = (x, x)$,

$$f(z) = \left(\frac{x+ix}{x-ix}\right)^2 = \left(\frac{1+i}{1-i}\right)^2 = -1.$$

This shows that $f(z)$ has value 1 at all nonzero points on the real and imaginary axes but value $-1$ at all nonzero points on the line $y = x$. Thus the limit of $f(z)$ as $z$ tends to 0 cannot exist.

10. *(a)* To show that $\lim_{z\to\infty} \frac{4z^2}{(z-1)^2} = 4$, we use statement (2), Sec. 16, and write

$$\lim_{z\to 0} \frac{4\left(\frac{1}{z}\right)^2}{\left(\frac{1}{z}-1\right)^2} = \lim_{z\to 0} \frac{4}{(1-z)^2} = 4.$$

*(b)* To establish the limit $\lim_{z\to 1} \frac{1}{(z-1)^3} = \infty$, we refer to statement (1), Sec. 16, and write

$$\lim_{z\to 1} \frac{1}{1/(z-1)^3} = \lim_{z\to 1} (z-1)^3 = 0.$$

*(c)* To verify that $\lim_{z\to\infty} \frac{z^2+1}{z-1} = \infty$, we apply statement (3), Sec. 16, and write

$$\lim_{z\to 0} \frac{\frac{1}{z}-1}{\left(\frac{1}{z}\right)^2+1} = \lim_{z\to 0} \frac{z-z^2}{1+z^2} = 0.$$

11. In this problem, we consider the function

$$T(z) = \frac{az+b}{cz+d} \qquad (ad-bc \neq 0).$$

*(a)* Suppose that $c = 0$. Statement (3), Sec. 16, tells us that $\lim_{z\to\infty} T(z) = \infty$ since

$$\lim_{z\to 0} \frac{1}{T(1/z)} = \lim_{z\to 0} \frac{c+dz}{a+bz} = \frac{c}{a} = 0.$$

*(b)* **Suppose that** $c \neq 0$. Statement (2), Sec. 16, reveals that $\lim_{z\to\infty} T(z) = \frac{a}{c}$ since

$$\lim_{z\to 0} T\left(\frac{1}{z}\right) = \lim_{z\to 0} \frac{a+bz}{c+dz} = \frac{a}{c}.$$

Also, we know from statement (1), Sec. 16, that $\lim_{z\to -d/c} T(z) = \infty$ since

$$\lim_{z\to -d/c} \frac{1}{T(z)} = \lim_{z\to -d/c} \frac{cz+d}{az+b} = 0.$$

SECTION 19

1. *(a)* If $f(z) = 3z^2 - 2z + 4$, then

$$f'(z) = \frac{d}{dz}(3z^2 - 2z + 4) = 3\frac{d}{dz}z^2 - 2\frac{d}{dz}z + \frac{d}{dz}4 = 3(2z) - 2(1) + 0 = 6z - 2.$$

*(b)* If $f(z) = (1-4z^2)^3$, then

$$f'(z) = 3(1-4z^2)^2\frac{d}{dz}(1-4z^2) = 3(1-4z^2)^2(-8z) = -24z(1-4z^2)^2.$$

*(c)* If $f(z) = \frac{z-1}{2z+1}$ $\left(z \neq -\frac{1}{2}\right)$, then

$$f'(z) = \frac{(2z+1)\frac{d}{dz}(z-1) - (z-1)\frac{d}{dz}(2z+1)}{(2z+1)^2} = \frac{(2z+1)(1) - (z-1)2}{(2z+1)^2} = \frac{3}{(2z+1)^2}.$$

*(d)* If $f(z) = \frac{(1+z^2)^4}{z^2}$ $(z \neq 0)$, then

$$f'(z) = \frac{z^2\frac{d}{dz}(1+z^2)^4 - (1+z^2)^4\frac{d}{dz}z^2}{(z^2)^2} = \frac{z^2 4(1+z^2)^3(2z) - (1+z^2)^4 2z}{(z^2)^2}$$

$$= \frac{2z(1+z^2)^3[4z^2 - (1+z^2)]}{z^4} = \frac{2(1+z^2)^3(3z^2-1)}{z^3}.$$

3. If $f(z) = 1/z$ $(z \neq 0)$, then

$$\Delta w = f(z+\Delta z) - f(z) = \frac{1}{z+\Delta z} - \frac{1}{z} = \frac{-\Delta z}{(z+\Delta z)z}.$$

Hence

$$f'(z) = \lim_{\Delta z \to 0} \frac{\Delta w}{\Delta z} = \lim_{\Delta z \to 0} \frac{-1}{(z+\Delta z)z} = -\frac{1}{z^2}.$$

4. We are given that $f(z_0) = g(z_0) = 0$ and that $f'(z_0)$ and $g'(z_0)$ exist, where $g'(z_0) \neq 0$. According to the definition of derivative,

$$f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} = \lim_{z \to z_0} \frac{f(z)}{z - z_0}.$$

Similarly,

$$g'(z_0) = \lim_{z \to z_0} \frac{g(z) - g(z_0)}{z - z_0} = \lim_{z \to z_0} \frac{g(z)}{z - z_0}.$$

Thus

$$\lim_{z \to z_0} \frac{f(z)}{g(z)} = \lim_{z \to z_0} \frac{f(z)/(z - z_0)}{g(z)/(z - z_0)} = \frac{\lim\limits_{z \to z_0} f(z)/(z - z_0)}{\lim\limits_{z \to z_0} g(z)/(z - z_0)} = \frac{f'(z_0)}{g'(z_0)}.$$

## SECTION 22

1. (a) $f(z) = \bar{z} = x - iy$. So $u = x$, $v = -y$.
Inasmuch as $u_x = v_y \Rightarrow 1 = -1$, the Cauchy-Riemann equations are not satisfied anywhere.

(b) $f(z) = z - \bar{z} = (x + iy) - (x - iy) = 0 + i2y$. So $u = 0$, $v = 2y$.
Since $u_x = v_y \Rightarrow 0 = 2$, the Cauchy-Riemann equations are not satisfied anywhere.

(c) $f(z) = 2x + ixy^2$. Here $u = 2x$, $v = xy^2$.
$u_x = v_y \Rightarrow 2 = 2xy \Rightarrow xy = 1$.
$u_y = -v_x \Rightarrow 0 = -y^2 \Rightarrow y = 0$.
Substituting $y = 0$ into $xy = 1$, we have $0 = 1$. Thus the Cauchy-Riemann equations do not hold anywhere.

(d) $f(z) = e^x e^{-iy} = e^x(\cos y - i \sin y) = e^x \cos y - ie^x \sin y$. So $u = e^x \cos y$, $v = -e^x \sin y$.
$u_x = v_y \Rightarrow e^x \cos y = -e^x \cos y \Rightarrow 2e^x \cos y = 0 \Rightarrow \cos y = 0$. Thus

$$y = \frac{\pi}{2} + n\pi \qquad (n = 0, \pm 1, \pm 2, \ldots).$$

$u_y = -v_x \Rightarrow -e^x \sin y = e^x \sin y \Rightarrow 2e^x \sin y = 0 \Rightarrow \sin y = 0$. Hence

$$y = n\pi \qquad (n = 0, \pm 1, \pm 2, \ldots).$$

Since these are two different sets of values of $y$, the Cauchy-Riemann equations cannot be satisfied anywhere.

3. (a) $f(z)=\dfrac{1}{z}=\dfrac{1}{z}\cdot\dfrac{\bar{z}}{\bar{z}}=\dfrac{\bar{z}}{|z|^2}=\dfrac{x}{x^2+y^2}+i\dfrac{-y}{x^2+y^2}$. So

$$u=\frac{x}{x^2+y^2}\quad\text{and}\quad v=\frac{-y}{x^2+y^2}.$$

Since

$$u_x=\frac{y^2-x^2}{(x^2+y^2)^2}=v_y\quad\text{and}\quad u_y=\frac{-2xy}{(x^2+y^2)^2}=-v_x\qquad (x^2+y^2\neq 0),$$

$f'(z)$ exists when $z\neq 0$. Moreover, when $z\neq 0$,

$$f'(z)=u_x+iv_x=\frac{y^2-x^2}{(x^2+y^2)^2}+i\frac{2xy}{(x^2+y^2)^2}=-\frac{x^2-i2xy-y^2}{(x^2+y^2)^2}$$

$$=-\frac{(x-iy)^2}{(x^2+y^2)^2}=-\frac{(\bar{z})^2}{(z\bar{z})^2}=-\frac{(\bar{z})^2}{(z)^2(\bar{z})^2}=-\frac{1}{z^2}.$$

(b) $f(z)=x^2+iy^2$. Hence $u=x^2$ and $v=y^2$. Now

$$u_x=v_y\Rightarrow 2x=2y\Rightarrow y=x\quad\text{and}\quad u_y=-v_x\Rightarrow 0=0.$$

So $f'(z)$ exists only when $y=x$, and we find that

$$f'(x+ix)=u_x(x,x)+iv_x(x,x)=2x+i0=2x.$$

(c) $f(z)=z\operatorname{Im} z=(x+iy)y=xy+iy^2$. Here $u=xy$ and $v=y^2$. We observe that

$$u_x=v_y\Rightarrow y=2y\Rightarrow y=0\quad\text{and}\quad u_y=-v_x\Rightarrow x=0.$$

Hence $f'(z)$ exists only when $z=0$. In fact,

$$f'(0)=u_x(0,0)+iv_x(0,0)=0+i0=0.$$

4. (a) $f(z)=\dfrac{1}{z^4}=\underbrace{\left(\dfrac{1}{r^4}\cos 4\theta\right)}_{u}+i\underbrace{\left(-\dfrac{1}{r^4}\sin 4\theta\right)}_{v}\quad (z\neq 0)$. Since

$$ru_r=-\frac{4}{r^4}\cos 4\theta=v_\theta\quad\text{and}\quad u_\theta=-\frac{4}{r^4}\sin 4\theta=-rv_r,$$

$f$ is analytic in its domain of definition. Furthermore,

$$f'(z) = e^{-i\theta}(u_r + iv_r) = e^{-i\theta}\left(-\frac{4}{r^5}\cos 4\theta + i\frac{4}{r^5}\sin 4\theta\right)$$

$$= -\frac{4}{r^5}e^{-i\theta}(\cos 4\theta - i\sin 4\theta) = -\frac{4}{r^5}e^{-i\theta}e^{-i4\theta}$$

$$= -\frac{4}{r^5 e^{i5\theta}} = -\frac{4}{(re^{i\theta})^5} = -\frac{4}{z^5}.$$

(b) $f(z) = \sqrt{r}e^{i\theta/2} = \underbrace{\sqrt{r}\cos\frac{\theta}{2}}_{u} + i\underbrace{\sqrt{r}\sin\frac{\theta}{2}}_{v} \quad (r > 0, \alpha < \theta < \alpha + 2\pi)$. Since

$$ru_r = \frac{\sqrt{r}}{2}\cos\frac{\theta}{2} = v_\theta \quad \text{and} \quad u_\theta = -\frac{\sqrt{r}}{2}\sin\frac{\theta}{2} = -rv_r,$$

$f$ is analytic in its domain of definition. Moreover,

$$f'(z) = e^{-i\theta}(u_r + iv_r) = e^{-i\theta}\left(\frac{1}{2\sqrt{r}}\cos\frac{\theta}{2} + i\frac{1}{2\sqrt{r}}\sin\frac{\theta}{2}\right)$$

$$= \frac{1}{2\sqrt{r}}e^{-i\theta}\left(\cos\frac{\theta}{2} + i\sin\frac{\theta}{2}\right) = \frac{1}{2\sqrt{r}}e^{-i\theta}e^{i\theta/2}$$

$$= \frac{1}{2\sqrt{r}e^{i\theta/2}} = \frac{1}{2f(z)}.$$

(c) $f(z) = \underbrace{e^{-\theta}\cos(\ln r)}_{u} + i\underbrace{e^{-\theta}\sin(\ln r)}_{v} \quad (r > 0, 0 < \theta < 2\pi)$. Since

$$ru_r = -e^{-\theta}\sin(\ln r) = v_\theta \quad \text{and} \quad u_\theta = -e^{-\theta}\cos(\ln r) = -rv_r,$$

$f$ is analytic in its domain of definition. Also,

$$f'(z) = e^{-i\theta}(u_r + iv_r) = e^{-i\theta}\left[-\frac{e^{-\theta}\sin(\ln r)}{r} + i\frac{e^{-\theta}\cos(\ln r)}{r}\right]$$

$$= \frac{i}{re^{i\theta}}\left[e^{-\theta}\cos(\ln r) + ie^{-\theta}\sin(\ln r)\right] = i\frac{f(z)}{z}.$$

5. When $f(z) = x^3 + i(1-y)^3$, we have $u = x^3$ and $v = (1-y)^3$. Observe that

$$u_x = v_y \Leftrightarrow 3x^2 = -3(1-y)^2 \Leftrightarrow x^2 + (1-y)^2 = 0 \quad \text{and} \quad u_y = -v_x \Leftrightarrow 0 = 0.$$

Evidently, then, the Cauchy-Riemann equations are satisfied only when $x = 0$ and $y = 1$. That is, they hold only when $z = i$. Hence the expression

$$f'(z) = u_x + iv_x = 3x^2 + i0 = 3x^2$$

is valid only when $z = i$, in which case we see that $f'(i) = 0$.

6. Here $u$ and $v$ denote the real and imaginary components of the function $f$ defined by means of the equations

$$f(z) = \begin{cases} \dfrac{\bar{z}^2}{z} & \text{when } z \neq 0, \\ 0 & \text{when } z = 0. \end{cases}$$

Now

$$f(z) = \underbrace{\frac{x^3 - 3xy^2}{x^2 + y^2}}_{u} + i\underbrace{\frac{y^3 - 3x^2y}{x^2 + y^2}}_{v}$$

when $z \neq 0$, and the following calculations show that

$$u_x(0,0) = v_y(0,0) \quad \text{and} \quad u_y(0,0) = -v_x(0,0):$$

$$u_x(0,0) = \lim_{\Delta x \to 0} \frac{u(0 + \Delta x, 0) - u(0,0)}{\Delta x} = \lim_{\Delta x \to 0} \frac{\Delta x}{\Delta x} = 1,$$

$$u_y(0,0) = \lim_{\Delta y \to 0} \frac{u(0, 0 + \Delta y) - u(0,0)}{\Delta y} = \lim_{\Delta y \to 0} \frac{0}{\Delta y} = 0,$$

$$v_x(0,0) = \lim_{\Delta x \to 0} \frac{v(0 + \Delta x, 0) - v(0,0)}{\Delta x} = \lim_{\Delta x \to 0} \frac{0}{\Delta x} = 0,$$

$$v_y(0,0) = \lim_{\Delta y \to 0} \frac{v(0, 0 + \Delta y) - v(0,0)}{\Delta y} = \lim_{\Delta y \to 0} \frac{\Delta y}{\Delta y} = 1.$$

7. Equations (2), Sec. 22, are

$$u_x \cos\theta + u_y \sin\theta = u_r,$$
$$-u_x r\sin\theta + u_y r\cos\theta = u_\theta.$$

Solving these simultaneous linear equations for $u_x$ and $u_y$, we find that

$$u_x = u_r \cos\theta - u_\theta \frac{\sin\theta}{r} \quad \text{and} \quad u_y = u_r \sin\theta + u_\theta \frac{\cos\theta}{r}.$$

Likewise,

$$v_x = v_r \cos\theta - v_\theta \frac{\sin\theta}{r} \quad \text{and} \quad v_y = v_r \sin\theta + v_\theta \frac{\cos\theta}{r}.$$

Assume now that the Cauchy-Riemann equations in polar form,

$$r u_r = v_\theta, \quad u_\theta = -r v_r,$$

are satisfied at $z_0$. It follows that

$$u_x = u_r \cos\theta - u_\theta \frac{\sin\theta}{r} = v_\theta \frac{\cos\theta}{r} + v_r \sin\theta = v_r \sin\theta + v_\theta \frac{\cos\theta}{r} = v_y;$$

$$u_y = u_r \sin\theta + u_\theta \frac{\cos\theta}{r} = v_\theta \frac{\sin\theta}{r} - v_r \cos\theta = -\left(v_r \cos\theta - v_\theta \frac{\sin\theta}{r}\right) = -v_x.$$

9\. (a) Write $f(z) = u(r,\theta) + iv(r,\theta)$. Then recall the polar form

$$r u_r = v_\theta, \quad u_\theta = -r v_r$$

of the Cauchy-Riemann equations, which enables us to rewrite the expression (Sec. 22)

$$f'(z_0) = e^{-i\theta}(u_r + i v_r)$$

for the derivative of $f$ at a point $z_0 = (r_0, \theta_0)$ in the following way:

$$f'(z_0) = e^{-i\theta}\left(\frac{1}{r} v_\theta - \frac{i}{r} u_\theta\right) = \frac{-i}{re^{i\theta}}(u_\theta + i v_\theta) = \frac{-i}{z_0}(u_\theta + i v_\theta).$$

*(b)* Consider now the function

$$f(z) = \frac{1}{z} = \frac{1}{re^{i\theta}} = \frac{1}{r}e^{-i\theta} = \frac{1}{r}(\cos\theta - i\sin\theta) = \frac{\cos\theta}{r} - i\frac{\sin\theta}{r}.$$

With

$$u(r,\theta) = \frac{\cos\theta}{r} \quad \text{and} \quad v(r,\theta) = -\frac{\sin\theta}{r},$$

the final expression for $f'(z_0)$ in part *(a)* tells us that

$$f'(z) = \frac{-i}{z}\left(-\frac{\sin\theta}{r} - i\frac{\cos\theta}{r}\right) = -\frac{1}{z}\left(\frac{\cos\theta - i\sin\theta}{r}\right)$$

$$= -\frac{1}{z}\left(\frac{e^{-i\theta}}{r}\right) = -\frac{1}{z}\left(\frac{1}{re^{i\theta}}\right) = -\frac{1}{z^2}$$

when $z \neq 0$.

**10.** *(a)* We consider a function $F(x,y)$, where

$$x = \frac{z+\bar{z}}{2} \quad \text{and} \quad y = \frac{z-\bar{z}}{2i}.$$

Formal application of the chain rule for multivariable functions yields

$$\frac{\partial F}{\partial \bar{z}} = \frac{\partial F}{\partial x}\frac{\partial x}{\partial \bar{z}} + \frac{\partial F}{\partial y}\frac{\partial y}{\partial \bar{z}} = \frac{\partial F}{\partial x}\left(\frac{1}{2}\right) + \frac{\partial F}{\partial y}\left(-\frac{1}{2i}\right) = \frac{1}{2}\left(\frac{\partial F}{\partial x} + i\frac{\partial F}{\partial y}\right).$$

*(b)* Now define the operator

$$\frac{\partial}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right),$$

suggested by part *(a)*, and formally apply it to a function $f(z) = u(x,y) + iv(x,y)$:

$$\frac{\partial f}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}\right) = \frac{1}{2}\frac{\partial f}{\partial x} + \frac{i}{2}\frac{\partial f}{\partial y}$$

$$= \frac{1}{2}(u_x + iv_x) + \frac{i}{2}(u_y + iv_y) = \frac{1}{2}\left[(u_x - v_y) + i(v_x + u_y)\right].$$

If the Cauchy-Riemann equations $u_x = v_y$, $u_y = -v_x$ are satisfied, this tells us that $\partial f/\partial \bar{z} = 0$.

SECTION 24

1. (a) $f(z) = \underbrace{3x + y}_{u} + i\underbrace{(3y - x)}_{v}$ is entire since

$$u_x = 3 = v_y \quad \text{and} \quad u_y = 1 = -v_x.$$

(b) $f(z) = \underbrace{\sin x \cosh y}_{u} - i\underbrace{\cos x \sinh y}_{v}$ is entire since

$$u_x = \cos x \cosh y = v_y \quad \text{and} \quad u_y = \sin x \sinh y = -v_x.$$

(c) $f(z) = e^{-y}\sin x - ie^{-y}\cos x = \underbrace{e^{-y}\sin x}_{u} + i\underbrace{(-e^{-y}\cos x)}_{v}$ is entire since

$$u_x = e^{-y}\cos x = v_y \quad \text{and} \quad u_y = -e^{-y}\sin x = -v_x.$$

(d) $f(z) = (z^2 - 2)e^{-x}e^{-iy}$ is entire since it is the product of the entire functions

$$g(z) = z^2 - 2 \quad \text{and} \quad h(z) = e^{-x}e^{-iy} = e^{-x}(\cos y - i\sin y) = \underbrace{e^{-x}\cos y}_{u} + i\underbrace{(-e^{-x}\sin y)}_{v}.$$

The function $g$ is entire since it is a polynomial, and $h$ is entire since

$$u_x = -e^{-x}\cos y = v_y \quad \text{and} \quad u_y = -e^{-x}\sin y = -v_x.$$

2. (a) $f(z) = \underbrace{xy}_{u} + i\underbrace{y}_{v}$ is nowhere analytic since

$$u_x = v_y \Rightarrow y = 1 \quad \text{and} \quad u_y = -v_x \Rightarrow x = 0,$$

which means that the Cauchy-Riemann equations hold only at the point $z = (0,1) = i$.

(c) $f(z) = e^y e^{ix} = e^y(\cos x + i\sin x) = \underbrace{e^y\cos x}_{u} + i\underbrace{e^y \sin x}_{v}$ is nowhere analytic since

$$u_x = v_y \Rightarrow -e^y\sin x = e^y \sin x \Rightarrow 2e^y \sin x = 0 \Rightarrow \sin x = 0$$

and

$$u_y = -v_x \Rightarrow e^y \cos x = -e^y\cos x \Rightarrow 2e^y\cos x = 0 \Rightarrow \cos x = 0.$$

More precisely, the roots of the equation $\sin x = 0$ are $n\pi$ $(n = 0, \pm 1, \pm 2, \ldots)$, and $\cos n\pi = (-1)^n \neq 0$. Consequently, the Cauchy-Riemann equations are not satisfied anywhere.

7. (a) Suppose that a function $f(z) = u(x,y) + iv(x,y)$ is analytic and real-valued in a domain $D$. Since $f(z)$ is real-valued, it has the form $f(z) = u(x,y) + i0$. The Cauchy-Riemann equations $u_x = v_y, u_y = -v_x$ thus become $u_x = 0, u_y = 0$, and this means that $u(x,y) = a$, where $a$ is a (real) constant. (See the proof of the theorem in Sec. 23.) Evidently, then, $f(z) = a$. That is, $f$ is constant in $D$.

(b) Suppose that a function $f$ is analytic in a domain D and that its modulus $|f(z)|$ is constant there. Write $|f(z)| = c$, where $c$ is a (real) constant. If $c = 0$, we see that $f(z) = 0$ throughout $D$. If, on the other hand, $c \neq 0$, write $f(z)\overline{f(z)} = c^2$, or

$$\overline{f(z)} = \frac{c^2}{f(z)}.$$

Since $f(z)$ is analytic and never zero in $D$, the conjugate $\overline{f(z)}$ must be analytic in $D$. Example 3 in Sec. 24 then tells us that $f(z)$ must be constant in $D$.

## SECTION 25

1. (a) It is straightforward to show that $u_{xx} + u_{yy} = 0$ when $u(x,y) = 2x(1-y)$. To find a harmonic conjugate $v(x,y)$, we start with $u_x(x,y) = 2 - 2y$. Now

$$u_x = v_y \Rightarrow v_y = 2 - 2y \Rightarrow v(x,y) = 2y - y^2 + \phi(x).$$

Then

$$u_y = -v_x \Rightarrow -2x = -\phi'(x) \Rightarrow \phi'(x) = 2x \Rightarrow \phi(x) = x^2 + c.$$

Consequently,

$$v(x,y) = 2y - y^2 + (x^2 + c) = x^2 - y^2 + 2y + c.$$

(b) It is straightforward to show that $u_{xx} + u_{yy} = 0$ when $u(x,y) = 2x - x^3 + 3xy^2$. To find a harmonic conjugate $v(x,y)$, we start with $u_x(x,y) = 2 - 3x^2 + 3y^2$. Now

$$u_x = v_y \Rightarrow v_y = 2 - 3x^2 + 3y^2 \Rightarrow v(x,y) = 2y - 3x^2 y + y^3 + \phi(x).$$

Then

$$u_y = -v_x \Rightarrow 6xy = 6xy - \phi'(x) \Rightarrow \phi'(x) = 0 \Rightarrow \phi(x) = c.$$

Consequently,

$$v(x,y) = 2y - 3x^2 y + y^3 + c.$$

(c) It is straightforward to show that $u_{xx} + u_{yy} = 0$ when $u(x,y) = \sinh x \sin y$. To find a harmonic conjugate $v(x,y)$, we start with $u_x(x,y) = \cosh x \sin y$. Now

$$u_x = v_y \Rightarrow v_y = \cosh x \sin y \Rightarrow v(x,y) = -\cosh x \cos y + \phi(x).$$

Then

$$u_y = -v_x \Rightarrow \sinh x \cos y = \sinh x \cos y - \phi'(x) \Rightarrow \phi'(x) = 0 \Rightarrow \phi(x) = c.$$

Consequently,

$$v(x,y) = -\cosh x \cos y + c.$$

(*d*) It is straightforward to show that $u_{xx} + u_{yy} = 0$ when $u(x,y) = \dfrac{y}{x^2+y^2}$. To find a harmonic conjugate $v(x,y)$, we start with $u_x(x,y) = -\dfrac{2xy}{(x^2+y^2)^2}$. Now

$$u_x = v_y \Rightarrow v_y = -\frac{2xy}{(x^2+y^2)^2} \Rightarrow v(x,y) = \frac{x}{x^2+y^2} + \phi(x)$$

Then

$$u_y = -v_x \Rightarrow \frac{x^2-y^2}{(x^2+y^2)^2} = \frac{x^2-y^2}{(x^2+y^2)^2} - \phi'(x) \Rightarrow \phi'(x) = 0 \Rightarrow \phi(x) = c$$

Consequently,

$$v(x,y) = \frac{x}{x^2+y^2} + c.$$

2. Suppose that $v$ and $V$ are harmonic conjugates of $u$ in a domain $D$. This means that

$$u_x = v_y, \quad u_y = -v_x \quad \text{and} \quad u_x = V_y, \quad u_y = -V_x.$$

If $w = v - V$, then,

$$w_x = v_x - V_x = -u_y + u_y = 0 \quad \text{and} \quad w_y = v_y - V_y = u_x - u_x = 0.$$

Hence $w(x,y) = c$, where $c$ is a (real) constant (compare the proof of the theorem in Sec. 23). That is, $v(x,y) - V(x,y) = c$.

3. Suppose that $u$ and $v$ are harmonic conjugates of each other in a domain $D$. Then

$$u_x = v_y, \quad u_y = -v_x \quad \text{and} \quad v_x = u_y, \quad v_y = -u_x.$$

It follows readily from these equations that

$$u_x = 0, \quad u_y = 0 \quad \text{and} \quad v_x = 0, \quad v_y = 0.$$

Consequently, $u(x,y)$ and $v(x,y)$ must be constant throughout $D$ (compare the proof of the theorem in Sec. 23).

5. The Cauchy-Riemann equations in polar coordinates are

$$ru_r = v_\theta \quad \text{and} \quad u_\theta = -rv_r.$$

Now

$$ru_r = v_\theta \Rightarrow ru_{rr} + u_r = v_{\theta r}$$

and

$$u_\theta = -r v_r \Rightarrow u_{\theta\theta} = -r v_{r\theta}.$$

Thus

$$r^2 u_{rr} + r u_r + u_{\theta\theta} = r v_{\theta r} - r v_{r\theta};$$

and, since $v_{\theta r} = v_{r\theta}$, we have

$$r^2 u_{rr} + r u_r + u_{\theta\theta} = 0,$$

which is the polar form of Laplace's equation. To show that $v$ satisfies the same equation, we observe that

$$u_\theta = -r v_r \Rightarrow v_r = -\frac{1}{r}u_\theta \Rightarrow v_{rr} = \frac{1}{r^2}u_\theta - \frac{1}{r}u_{\theta r}$$

and

$$r u_r = v_\theta \Rightarrow v_{\theta\theta} = r u_{r\theta}.$$

Since $u_{\theta r} = u_{r\theta}$, then

$$r^2 v_{rr} + r v_r + v_{\theta\theta} = u_\theta - r u_{\theta r} - u_\theta + r u_{r\theta} = 0.$$

6. If $u(r,\theta) = \ln r$, then

$$r^2 u_{rr} + r u_r + u_{\theta\theta} = r^2\left(-\frac{1}{r^2}\right) + r\left(\frac{1}{r}\right) + 0 = 0.$$

This tells us that the function $u = \ln r$ is harmonic in the domain $r > 0, 0 < \theta < 2\pi$. Now it follows from the Cauchy-Riemann equation $r u_r = v_\theta$ and the derivative $u_r = \frac{1}{r}$ that $v_\theta = 1$; thus $v(r,\theta) = \theta + \phi(r)$, where $\phi(r)$ is at present an arbitrary differentiable function of $r$. The other Cauchy-Riemann equation $u_\theta = -r v_r$ then becomes $0 = -r\phi'(r)$. That is, $\phi'(r) = 0$; and we see that $\phi(r) = c$, where $c$ is an arbitrary (real) constant. Hence $v(r,\theta) = \theta + c$ is a harmonic conjugate of $u(r,\theta) = \ln r$.

# Chapter 3

SECTION 28

1. (a) $\exp(2\pm 3\pi i) = e^2\exp(\pm 3\pi i) = -e^2$, since $\exp(\pm 3\pi i) = -1$.

(b) $$\exp\frac{2+\pi i}{4} = \left(\exp\frac{1}{2}\right)\left(\exp\frac{\pi i}{4}\right) = \sqrt{e}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$$
$$= \sqrt{e}\left(\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}\right) = \sqrt{\frac{e}{2}}(1+i).$$

(c) $\exp(z+\pi i) = (\exp z)(\exp \pi i) = -\exp z$, since $\exp \pi i = -1$.

3. First write
$$\exp(\bar{z}) = \exp(x - iy) = e^x e^{-iy} = e^x\cos y - ie^x \sin y,$$
where $z = x + iy$. This tells us that $\exp(\bar{z}) = u(x,y) + iv(x,y)$, where
$$u(x,y) = e^x\cos y \quad \text{and} \quad v(x,y) = -e^x\sin y.$$
Suppose that the Cauchy-Riemann equations $u_x = v_y$ and $u_y = -v_x$ are satisfied at some point $z = x + iy$. It is easy to see that, for the functions $u$ and $v$ here, these equations become $\cos y = 0$ and $\sin y = 0$. But there is no value of $y$ satisfying this pair of equations. We may conclude that, since the Cauchy-Riemann equations fail to be satisfied anywhere, the function $\exp(\bar{z})$ is not analytic anywhere.

4. The function $\exp(z^2)$ is entire since it is a composition of the entire functions $z^2$ and $\exp z$; and the chain rule for derivatives tells us that
$$\frac{d}{dz}\exp(z^2) = \exp(z^2)\frac{d}{dz}z^2 = 2z\exp(z^2).$$
Alternatively, one can show that $\exp(z^2)$ is entire by writing
$$\exp(z^2) = \exp\left[(x+iy)^2\right] = \exp(x^2-y^2)\exp(i2xy)$$
$$= \underbrace{\exp(x^2-y^2)\cos(2xy)}_{u} + i\underbrace{\exp(x^2-y^2)\sin(2xy)}_{v}$$
and using the Cauchy-Riemann equations. To be specific,
$$u_x = 2x\exp(x^2-y^2)\cos(2xy) - 2y\exp(x^2-y^2)\sin(2xy) = v_y$$
and
$$u_y = -2y\exp(x^2-y^2)\cos(2xy) - 2x\exp(x^2-y^2)\sin(2xy) = -v_x.$$

Furthermore,

$$\frac{d}{dz}\exp\left(z^2\right) = u_x + iv_x = 2(x+iy)\left[\exp\left(x^2-y^2\right)\cos(2xy) + i\exp\left(x^2-y^2\right)\sin(2xy)\right]$$
$$= 2z\exp\left(z^2\right)$$

5. We first write

$$|\exp(2z+i)| = |\exp[2x+i(2y+1)]| = e^{2x}$$

and

$$\left|\exp\left(iz^2\right)\right| = \left|\exp\left[-2xy+i\left(x^2-y^2\right)\right]\right| = e^{-2xy}.$$

Then, since

$$\left|\exp(2z+i)+\exp\left(iz^2\right)\right| \le |\exp(2z+i)| + \left|\exp\left(iz^2\right)\right|,$$

it follows that

$$\left|\exp(2z+i)+\exp\left(iz^2\right)\right| \le e^{2x}+e^{-2xy}.$$

6. First write

$$\left|\exp\left(z^2\right)\right| = \left|\exp\left[(x+iy)^2\right]\right| = \left|\exp\left(x^2-y^2+i2xy\right)\right| = \exp\left(x^2-y^2\right)$$

and

$$\exp\left(|z|^2\right) = \exp\left(x^2+y^2\right).$$

Since $x^2-y^2 \le x^2+y^2$, it is clear that $\exp\left(x^2-y^2\right) \le \exp\left(x^2+y^2\right)$. Hence it follows from the above that

$$\left|\exp\left(z^2\right)\right| \le \exp\left(|z|^2\right).$$

7. To prove that $|\exp(-2z)| < 1 \Leftrightarrow \operatorname{Re} z > 0$, write

$$|\exp(-2z)| = |\exp(-2x - i2y)| = \exp(-2x).$$

It is then clear that the statement to be proved is the same as $\exp(-2x) < 1 \Leftrightarrow x > 0$, which is obvious from the graph of the exponential function in calculus.

8. *(a)* Write $e^z = -2$ as $e^x e^{iy} = 2e^{i\pi}$. This tells us that

$$e^x = 2 \quad \text{and} \quad y = \pi + 2n\pi \qquad (n = 0, \pm 1, \pm 2, \ldots).$$

That is,

$$x = \ln 2 \quad \text{and} \quad y = (2n+1)\pi \qquad (n = 0, \pm 1, \pm 2, \ldots).$$

Hence

$$z = \ln 2 + (2n+1)\pi i \qquad (n = 0, \pm 1, \pm 2, \ldots).$$

*(b)* Write $e^z = 1 + \sqrt{3}\,i$ as $e^x e^{iy} = 2e^{i\pi/3}$, from which we see that

$$e^x = 2 \quad \text{and} \quad y = \frac{\pi}{3} + 2n\pi \qquad (n = 0, \pm 1, \pm 2, \ldots).$$

That is,

$$x = \ln 2 \quad \text{and} \quad y = \left(2n + \frac{1}{3}\right)\pi \qquad (n = 0, \pm 1, \pm 2, \ldots).$$

Consequently,

$$z = \ln 2 + \left(2n + \frac{1}{3}\right)\pi i \qquad (n = 0, \pm 1, \pm 2, \ldots).$$

*(c)* Write $\exp(2z - 1) = 1$ as $e^{2x-1} e^{i2y} = 1e^{i0}$ and note how it follows that

$$e^{2x-1} = 1 \quad \text{and} \quad 2y = 0 + 2n\pi \qquad (n = 0, \pm 1, \pm 2, \ldots).$$

Evidently, then,

$$x = \frac{1}{2} \quad \text{and} \quad y = n\pi \qquad (n = 0, \pm 1, \pm 2, \ldots);$$

and this means that

$$z = \frac{1}{2} + n\pi i \qquad (n = 0, \pm 1, \pm 2, \ldots).$$

9. This problem is actually to find all roots of the equation

$$\overline{\exp(iz)} = \exp(i\bar{z}).$$

To do this, set $z = x + iy$ and rewrite the equation as

$$e^{-y}e^{-ix} = e^{y}e^{ix}.$$

Now, according to the statement in italics at the beginning of Sec. 8 in the text,

$$e^{-y} = e^{y} \quad \text{and} \quad -x = x + 2n\pi,$$

where $n$ may have any one of the values $n = 0, \pm 1, \pm 2, \ldots$. Thus

$$y = 0 \quad \text{and} \quad x = n\pi \qquad (n = 0, \pm 1, \pm 2, \ldots).$$

The roots of the original equation are, therefore,

$$z = n\pi \qquad (n = 0, \pm 1, \pm 2, \ldots).$$

**10.** *(a)* Suppose that $e^z$ is real. Since $e^z = e^x \cos y + ie^x \sin y$, this means that $e^x \sin y = 0$. Moreover, since $e^x$ is never zero, $\sin y = 0$. Consequently, $y = n\pi \; (n = 0, \pm 1, \pm 2, \ldots)$; that is, $\operatorname{Im} z = n\pi \; (n = 0, \pm 1, \pm 2, \ldots)$.

*(b)* On the other hand, suppose that $e^z$ is pure imaginary. It follows that $\cos y = 0$, or that $y = \dfrac{\pi}{2} + n\pi \; (n = 0, \pm 1, \pm 2, \ldots)$. That is, $\operatorname{Im} z = \dfrac{\pi}{2} + n\pi \; (n = 0, \pm 1, \pm 2, \ldots)$.

**12.** We start by writing

$$\frac{1}{z} = \frac{\bar{z}}{z\bar{z}} = \frac{\bar{z}}{|z|^2} = \frac{x - iy}{x^2 + y^2} = \frac{x}{x^2 + y^2} + i\frac{-y}{x^2 + y^2}.$$

Because $\operatorname{Re}(e^z) = e^x \cos y$, it follows that

$$\operatorname{Re}(e^{1/z}) = \exp\left(\frac{x}{x^2 + y^2}\right)\cos\left(\frac{-y}{x^2 + y^2}\right) = \exp\left(\frac{x}{x^2 + y^2}\right)\cos\left(\frac{y}{x^2 + y^2}\right).$$

Since $e^{1/z}$ is analytic in every domain that does not contain the origin, Theorem 1 in Sec. 25 ensures that $\operatorname{Re}(e^{1/z})$ is harmonic in such a domain.

**13.** If $f(z) = u(x, y) + iv(x, y)$ is analytic in some domain $D$, then

$$e^{f(z)} = e^{u(x,y)} \cos v(x, y) + ie^{u(x,y)} \sin v(x, y).$$

Since $e^{f(z)}$ is a composition of functions that are analytic in $D$, it follows from Theorem 1 in Sec. 25 that its component functions

$$U(x, y) = e^{u(x,y)} \cos v(x, y), \quad V(x, y) = e^{u(x,y)} \sin v(x, y)$$

are harmonic in $D$. Moreover, by Theorem 2 in Sec. 25, $V(x,y)$ is a harmonic conjugate of $U(x,y)$.

14. The problem here is to establish the identity

$$(\exp z)^n = \exp(nz) \qquad (n = 0, \pm 1, \pm 2, \ldots).$$

(a) To show that it is true when $n = 0,1,2,\ldots$, we use mathematical induction. It is obviously true when $n = 0$. Suppose that it is true when $n = m$, where $m$ is any nonnegative integer. Then

$$(\exp z)^{m+1} = (\exp z)^m (\exp z) = \exp(mz)\exp z = \exp(mz + z) = \exp[(m+1)z].$$

(b) Suppose now that $n$ is a negative integer $(n = -1, -2, \ldots)$, and write $m = -n = 1,2,\ldots$. In view of part (a),

$$(\exp z)^n = \left(\frac{1}{\exp z}\right)^m = \frac{1}{(\exp z)^m} = \frac{1}{\exp(mz)} = \frac{1}{\exp(-nz)} = \exp(nz).$$

SECTION 30

1. (a) $\operatorname{Log}(-ei) = \ln|-ei| + i\operatorname{Arg}(-ei) = \ln e - \dfrac{\pi}{2}i = 1 - \dfrac{\pi}{2}i.$

(b) $\operatorname{Log}(1-i) = \ln|1-i| + i\operatorname{Arg}(1-i) = \ln\sqrt{2} - \dfrac{\pi}{4}i = \dfrac{1}{2}\ln 2 - \dfrac{\pi}{4}i.$

2. (a) $\log e = \ln e + i(0 + 2n\pi) = 1 + 2n\pi i \quad (n = 0, \pm 1, \pm 2, \ldots).$

(b) $\log i = \ln 1 + i\left(\dfrac{\pi}{2} + 2n\pi\right) = \left(2n + \dfrac{1}{2}\right)\pi i \quad (n = 0, \pm 1, \pm 2, \ldots).$

(c) $\log\left(-1 + \sqrt{3}i\right) = \ln 2 + i\left(\dfrac{2\pi}{3} + 2n\pi\right) = \ln 2 + 2\left(n + \dfrac{1}{3}\right)\pi i \quad (n = 0, \pm 1, \pm 2, \ldots).$

3. (a) Observe that

$$\operatorname{Log}(1+i)^2 = \operatorname{Log}(2i) = \ln 2 + \frac{\pi}{2}i$$

and

$$2\operatorname{Log}(1+i) = 2\left(\ln\sqrt{2} + i\frac{\pi}{4}\right) = \ln 2 + \frac{\pi}{2}i.$$

Thus

$$\operatorname{Log}(1+i)^2 = 2\operatorname{Log}(1+i).$$

*(b)* On the other hand,

$$\text{Log}(-1+i)^2 = \text{Log}(-2i) = \ln 2 - \frac{\pi}{2}i$$

and

$$2\,\text{Log}(-1+i) = 2\left(\ln\sqrt{2} + i\frac{3\pi}{4}\right) = \ln 2 + \frac{3\pi}{2}i.$$

Hence

$$\text{Log}(-1+i)^2 \neq 2\,\text{Log}(-1+i).$$

4. *(a)* Consider the branch

$$\log z = \ln r + i\theta \qquad \left(r > 0, \frac{\pi}{4} < \theta < \frac{9\pi}{4}\right).$$

Since

$$\log(i^2) = \log(-1) = \ln 1 + i\pi = \pi i \quad \text{and} \quad 2\log i = 2\left(\ln 1 + i\frac{\pi}{2}\right) = \pi i,$$

we find that $\log(i^2) = 2\log i$ when this branch of $\log z$ is taken.

*(b)* Now consider the branch

$$\log z = \ln r + i\theta \qquad \left(r > 0, \frac{3\pi}{4} < \theta < \frac{11\pi}{4}\right).$$

Here

$$\log(i^2) = \log(-1) = \ln 1 + i\pi = \pi i \quad \text{and} \quad 2\log i = 2\left(\ln 1 + i\frac{5\pi}{2}\right) = 5\pi i.$$

Hence, for this particular branch, $\log(i^2) \neq 2\log i$.

5. *(a)* The two values of $i^{1/2}$ are $e^{i\pi/4}$ and $e^{i5\pi/4}$. Observe that

$$\log(e^{i\pi/4}) = \ln 1 + i\left(\frac{\pi}{4} + 2n\pi\right) = \left(2n + \frac{1}{4}\right)\pi i \qquad (n = 0, \pm 1, \pm 2, \ldots)$$

and

$$\log(e^{i5\pi/4}) = \ln 1 + i\left(\frac{5\pi}{4} + 2n\pi\right) = \left[(2n+1) + \frac{1}{4}\right]\pi i \qquad (n = 0, \pm 1, \pm 2, \ldots).$$

Combining these two sets of values, we find that

$$\log(i^{1/2}) = \left(n + \frac{1}{4}\right)\pi i \qquad (n = 0, \pm 1, \pm 2, \ldots).$$

On the other hand,

$$\frac{1}{2}\log i = \frac{1}{2}\left[\ln 1 + i\left(\frac{\pi}{2}+2n\pi\right)\right] = \left(n+\frac{1}{4}\right)\pi i \qquad (n = 0, \pm 1, \pm 2, \ldots).$$

Thus the set of values of $\log(i^{1/2})$ is the same as the set of values of $\frac{1}{2}\log i$, and we may write

$$\log(i^{1/2}) = \frac{1}{2}\log i.$$

*(b)* Note that

$$\log(i^2) = \log(-1) = \ln 1 + (\pi + 2n\pi)i = (2n+1)\pi i \qquad (n = 0, \pm 1, \pm 2, \ldots)$$

but that

$$2\log i = 2\left[\ln 1 + i\left(\frac{\pi}{2}+2n\pi\right)\right] = (4n+1)\pi i \qquad (n = 0, \pm 1, \pm 2, \ldots).$$

Evidently, then, the set of values of $\log(i^2)$ is *not* the same as the set of values of $2\log i$. That is,

$$\log(i^2) \neq 2\log i.$$

7. To solve the equation $\log z = i\pi/2$, write $\exp(\log z) = \exp(i\pi/2)$, or $z = e^{i\pi/2} = i$.

10. Since $\ln(x^2+y^2)$ is the real component of any (analytic) branch of $2\log z$, it is harmonic in every domain that does not contain the origin. This can be verified directly by writing $u(x,y) = \ln(x^2+y^2)$ and showing that $u_{xx}(x,y) + u_{yy}(x,y) = 0$.

## SECTION 31

1. Suppose that $\operatorname{Re} z_1 > 0$ and $\operatorname{Re} z_2 > 0$. Then

$$z_1 = r_1 \exp i\Theta_1 \quad \text{and} \quad z_2 = r_2 \exp i\Theta_2,$$

where

$$-\frac{\pi}{2} < \Theta_1 < \frac{\pi}{2} \quad \text{and} \quad -\frac{\pi}{2} < \Theta_2 < \frac{\pi}{2}.$$

The fact that $-\pi < \Theta_1 + \Theta_2 < \pi$ enables us to write

$$\operatorname{Log}(z_1 z_2) = \operatorname{Log}[(r_1 r_2)\exp i(\Theta_1 + \Theta_2)] = \ln(r_1 r_2) + i(\Theta_1 + \Theta_2)$$

$$= (\ln r_1 + i\Theta_1) + (\ln r_2 + i\Theta_2) = \operatorname{Log}(r_1 \exp i\Theta_1) + \operatorname{Log}(r_2 \exp i\Theta_2)$$

$$= \operatorname{Log} z_1 + \operatorname{Log} z_2.$$

3. We are asked to show in two different ways that

$$\log\left(\frac{z_1}{z_2}\right) = \log z_1 - \log z_2 \qquad (z_1 \neq 0, z_2 \neq 0).$$

(a) One way is to refer to the relation $\arg\left(\frac{z_1}{z_2}\right) = \arg z_1 - \arg z_2$ in Sec. 7 and write

$$\log\left(\frac{z_1}{z_2}\right) = \ln\left|\frac{z_1}{z_2}\right| + i\arg\left(\frac{z_1}{z_2}\right) = (\ln|z_1| + i\arg z_1) - (\ln|z_2| + i\arg z_2) = \log z_1 - \log z_2.$$

(b) Another way is to first show that $\log\left(\frac{1}{z}\right) = -\log z \ (z \neq 0)$. To do this, we write $z = re^{i\theta}$ and then

$$\log\left(\frac{1}{z}\right) = \log\left(\frac{1}{r}e^{-i\theta}\right) = \ln\left(\frac{1}{r}\right) + i(-\theta + 2n\pi) = -[\ln r + i(\theta - 2n\pi)] = -\log z,$$

where $n = 0, \pm 1, \pm 2, \ldots$. This enables us to use the relation

$$\log(z_1 z_2) = \log z_1 + \log z_2$$

and write

$$\log\left(\frac{z_1}{z_2}\right) = \log\left(z_1 \frac{1}{z_2}\right) = \log z_1 + \log\left(\frac{1}{z_2}\right) = \log z_1 - \log z_2.$$

**5.** The problem here is to verify that

$$z^{1/n} = \exp\left(\frac{1}{n}\log z\right) \qquad (n = -1, -2, \ldots),$$

given that it is valid when $n = 1, 2, \ldots$. To do this, we put $m = -n$, where $n$ is a negative integer. Then, since $m$ is a positive integer, we may use the relations $z^{-1} = 1/z$ and $1/e^z = e^{-z}$ to write

$$z^{1/n} = (z^{1/m})^{-1} = \left[\exp\left(\frac{1}{m}\log z\right)\right]^{-1}$$
$$= 1\Big/\left[\exp\left(\frac{1}{m}\log z\right)\right] = \exp\left(-\frac{1}{m}\log z\right) = \exp\left(\frac{1}{n}\log z\right).$$

SECTION 32

**1.** In each part below, $n = 0, \pm 1, \pm 2, \ldots$.

*(a)* $$(1+i)^i = \exp[i\log(1+i)] = \exp\left\{i\left[\ln\sqrt{2} + i\left(\frac{\pi}{4} + 2n\pi\right)\right]\right\}$$
$$= \exp\left[\frac{i}{2}\ln 2 - \left(\frac{\pi}{4} + 2n\pi\right)\right] = \exp\left(-\frac{\pi}{4} - 2n\pi\right)\exp\left(\frac{i}{2}\ln 2\right).$$

Since $n$ takes on all integral values, the term $-2n\pi$ here can be replaced by $+2n\pi$. Thus

$$(1+i)^i = \exp\left(-\frac{\pi}{4} + 2n\pi\right)\exp\left(\frac{i}{2}\ln 2\right).$$

*(b)* $$(-1)^{1/\pi} = \exp\left[\frac{1}{\pi}\log(-1)\right] = \exp\left\{\frac{1}{\pi}[\ln 1 + i(\pi + 2n\pi)]\right\} = \exp[(2n+1)i]$$

**2.** *(a)* $$\text{P.V. } i^i = \exp(i\,\text{Log}\, i) = \exp\left[i\left(\ln 1 + i\frac{\pi}{2}\right)\right] = \exp\left(-\frac{\pi}{2}\right).$$

*(b)* $$\text{P.V. } \left[\frac{e}{2}(-1-\sqrt{3}i)\right]^{3\pi i} = \exp\left\{3\pi i\,\text{Log}\left[\frac{e}{2}(-1-\sqrt{3}i)\right]\right\} = \exp\left[3\pi i\left(\ln e - i\frac{2\pi}{3}\right)\right]$$
$$= \exp(2\pi^2)\exp(i3\pi) = -\exp(2\pi^2).$$

(c) P.V. $(1-i)^{4i} = \exp[4i\,\mathrm{Log}(1-i)] = \exp\left[4i\left(\ln\sqrt{2} - i\frac{\pi}{4}\right)\right] = e^{\pi}e^{i4\ln\sqrt{2}}$

$$= e^{\pi}[\cos(4\ln\sqrt{2}) + i\sin(4\ln\sqrt{2})] = e^{\pi}[\cos(2\ln 2) + i\sin(2\ln 2)].$$

3. Since $-1+\sqrt{3}i = 2e^{i2\pi/3}$, we may write

$$(-1+\sqrt{3}i)^{3/2} = \exp\left[\frac{3}{2}\log(-1+\sqrt{3}i)\right] = \exp\left\{\frac{3}{2}\left[\ln 2 + i\left(\frac{2\pi}{3} + 2n\pi\right)\right]\right\}$$

$$= \exp[\ln(2^{3/2}) + (3n+1)\pi i] = 2\sqrt{2}\exp[(3n+1)\pi i],$$

where $n = 0, \pm 1, \pm 2, \ldots$. Observe that if $n$ is even, then $3n+1$ is odd; and so $\exp[(3n+1)\pi i] = -1$. On the other hand, if $n$ is odd, $3n+1$ is even; and this means that $\exp[(3n+1)\pi i] = 1$. So only two distinct values of $(-1+\sqrt{3}i)^{3/2}$ arise. Specifically,

$$(-1+\sqrt{3}i)^{3/2} = \pm 2\sqrt{2}.$$

5. We consider here any nonzero complex number $z_0$ in the exponential form $z_0 = r_0\exp i\Theta_0$, where $-\pi < \Theta_0 \le \pi$. According to Sec. 8, the principal value of $z_0^{1/n}$ is $\sqrt[n]{r_0}\exp\left(i\frac{\Theta_0}{n}\right)$; and, according to Sec. 32, that value is

$$\exp\left(\frac{1}{n}\mathrm{Log}\,z_0\right) = \exp\left[\frac{1}{n}(\ln r_0 + i\Theta_0)\right] = \exp(\ln\sqrt[n]{r_0})\exp\left(i\frac{\Theta_0}{n}\right) = \sqrt[n]{r_0}\exp\left(i\frac{\Theta_0}{n}\right).$$

These two expressions are evidently the same.

7. Observe that when $c = a + bi$ is any fixed complex number, where $c \ne 0, \pm 1, \pm 2, \ldots$, the power $i^c$ can be written as

$$i^c = \exp(c\log i) = \exp\left\{(a+bi)\left[\ln 1 + i\left(\frac{\pi}{2} + 2n\pi\right)\right]\right\}$$

$$= \exp\left[-b\left(\frac{\pi}{2} + 2n\pi\right) + ia\left(\frac{\pi}{2} + 2n\pi\right)\right] \qquad (n = 0, \pm 1, \pm 2, \ldots).$$

Thus

$$|i^c| = \exp\left[-b\left(\frac{\pi}{2} + 2n\pi\right)\right] \qquad (n = 0, \pm 1, \pm 2, \ldots),$$

and it is clear that $|i^c|$ is multiple-valued unless $b = 0$, or $c$ is real. Note that the restriction $c \ne 0, \pm 1, \pm 2, \ldots$ ensures that $i^c$ is multiple-valued even when $b = 0$.

SECTION 33

1. The desired derivatives can be found by writing

$$\frac{d}{dz}\sin z = \frac{d}{dz}\left(\frac{e^{iz}-e^{-iz}}{2i}\right) = \frac{1}{2i}\left(\frac{d}{dz}e^{iz} - \frac{d}{dz}e^{-iz}\right)$$
$$= \frac{1}{2i}(ie^{iz}+ie^{-iz}) = \frac{e^{iz}+e^{-iz}}{2} = \cos z$$

and

$$\frac{d}{dz}\cos z = \frac{d}{dz}\left(\frac{e^{iz}+e^{-iz}}{2}\right) = \frac{1}{2}\left(\frac{d}{dz}e^{iz} + \frac{d}{dz}e^{-iz}\right)$$
$$= \frac{1}{2}(ie^{iz}-ie^{-iz})\cdot\frac{i}{i} = -\frac{e^{iz}-e^{-iz}}{2i} = -\sin z.$$

2. From the expressions

$$\sin z = \frac{e^{iz}-e^{-iz}}{2i} \quad \text{and} \quad \cos z = \frac{e^{iz}+e^{-iz}}{2},$$

we see that

$$\cos z + i\sin z = \frac{e^{iz}+e^{-iz}}{2} + \frac{e^{iz}-e^{-iz}}{2} = e^{iz}.$$

3. Equation (4), Sec. 33 is

$$2\sin z_1 \cos z_2 = \sin(z_1+z_2) + \sin(z_1-z_2).$$

Interchanging $z_1$ and $z_2$ here and using the fact that $\sin z$ is an odd function, we have

$$2\cos z_1 \sin z_2 = \sin(z_1+z_2) - \sin(z_1-z_2).$$

Addition of corresponding sides of these two equations now yields

$$2(\sin z_1 \cos z_2 + \cos z_1 \sin z_2) = 2\sin(z_1+z_2),$$

or

$$\sin(z_1+z_2) = \sin z_1 \cos z_2 + \cos z_1 \sin z_2.$$

4. Differentiating each side of equation (5), Sec. 33, with respect to $z_1$, we have

$$\cos(z_1+z_2) = \cos z_1 \cos z_2 - \sin z_1 \sin z_2.$$

**7.** *(a)* From the identity $\sin^2 z + \cos^2 z = 1$, we have

$$\frac{\sin^2 z}{\cos^2 z} + \frac{\cos^2 z}{\cos^2 z} = \frac{1}{\cos^2 z}, \quad \text{or} \quad 1 + \tan^2 z = \sec^2 z.$$

*(b)* Also,

$$\frac{\sin^2 z}{\sin^2 z} + \frac{\cos^2 z}{\sin^2 z} = \frac{1}{\sin^2 z}, \quad \text{or} \quad 1 + \cot^2 z = \csc^2 z.$$

**9.** From the expression

$$\sin z = \sin x \cosh y + i \cos x \sinh y,$$

we find that

$$\begin{aligned}|\sin z|^2 &= \sin^2 x \cosh^2 y + \cos^2 x \sinh^2 y \\ &= \sin^2 x(1 + \sinh^2 y) + (1 - \sin^2 x)\sinh^2 y \\ &= \sin^2 x + \sinh^2 y.\end{aligned}$$

The expression

$$\cos z = \cos x \cosh y - i \sin x \sinh y,$$

on the other hand, tells us that

$$\begin{aligned}|\cos z|^2 &= \cos^2 x \cosh^2 y + \sin^2 x \sinh^2 y \\ &= \cos^2 x(1 + \sinh^2 y) + (1 - \cos^2 x)\sinh^2 y \\ &= \cos^2 x + \sinh^2 y\end{aligned}$$

**10.** Since $\sinh^2 y$ is never negative, it follows from Exercise 9 that

*(a)* $$|\sin z|^2 \geq \sin^2 x, \quad \text{or} \quad |\sin z| \geq |\sin x|$$

and that

*(b)* $$|\cos z|^2 \geq \cos^2 x, \quad \text{or} \quad |\cos z| \geq |\cos x|.$$

**11.** In this problem we shall use the identities

$$|\sin z|^2 = \sin^2 x + \sinh^2 y, \quad |\cos z|^2 = \cos^2 x + \sinh^2 y.$$

*(a)* Observe that

$$\sinh^2 y = |\sin z|^2 - \sin^2 x \le |\sin z|^2$$

and

$$|\sin z|^2 = \sin^2 x + (\cosh^2 y - 1) = \cosh^2 y - (1 - \sin^2 x)$$
$$= \cosh^2 y - \cos^2 x \le \cosh^2 y.$$

Thus

$$\sinh^2 y \le |\sin z|^2 \le \cosh^2 y, \quad \text{or} \quad |\sinh y| \le |\sin z| \le \cosh y.$$

*(b)* On the other hand,

$$\sinh^2 y = |\cos z|^2 - \cos^2 x \le |\cos z|^2$$

and

$$|\cos z|^2 = \cos^2 x + (\cosh^2 y - 1) = \cosh^2 y - (1 - \cos^2 x)$$
$$= \cosh^2 y - \sin^2 x \le \cosh^2 y.$$

Hence

$$\sinh^2 y \le |\cos z|^2 \le \cosh^2 y, \quad \text{or} \quad |\sinh y| \le |\cos z| \le \cosh y.$$

13. By writing $f(z) = \sin\bar{z} = \sin(x - iy) = \sin x \cosh y - i\cos x \sinh y$, we have

$$f(z) = u(x, y) + iv(x, y),$$

where

$$u(x, y) = \sin x \cosh y \quad \text{and} \quad v(x, y) = -\cos x \sinh y.$$

If the Cauchy-Riemann equations $u_x = v_y$, $u_y = -v_x$ are to hold, it is easy to see that

$$\cos x \cosh y = 0 \quad \text{and} \quad \sin x \sinh y = 0.$$

Since $\cosh y$ is never zero, it follows from the first of these equations that $\cos x = 0$; that is, $x = \frac{\pi}{2} + n\pi$ $(n = 0, \pm 1, \pm 2, \ldots)$. Furthermore, since $\sin x$ is nonzero for each of these values of $x$, the second equation tells us that $\sinh y = 0$, or $y = 0$. Thus the Cauchy-Riemann equations hold only at the points

$$z = \frac{\pi}{2} + n\pi \qquad (n = 0, \pm 1, \pm 2, \ldots).$$

Evidently, then, there is no neighborhood of any point throughout which $f$ is analytic, and we may conclude that $\sin\bar{z}$ is not analytic anywhere.

The function $f(z) = \cos\bar{z} = \cos(x - iy) = \cos x \cosh y + i\sin x \sinh y$ can be written as

$$f(z) = u(x, y) + iv(x, y),$$

where

$$u(x, y) = \cos x \cosh y \quad \text{and} \quad v(x, y) = \sin x \sinh y.$$

If the Cauchy-Riemann equations $u_x = v_y$, $u_y = -v_x$ hold, then

$$\sin x \cosh y = 0 \quad \text{and} \quad \cos x \sinh y = 0.$$

The first of these equations tells us that $\sin x = 0$, or $x = n\pi$ $(n = 0, \pm 1, \pm 2, \ldots)$. Since $\cos n\pi \neq 0$, it follows that $\sinh y = 0$, or $y = 0$. Consequently, the Cauchy-Riemann equations hold only when

$$z = n\pi \qquad (n = 0, \pm 1, \pm 2, \ldots).$$

So there is no neighborhood throughout which $f$ is analytic, and this means that $\cos \bar{z}$ is nowhere analytic.

**16.** *(a)* Use expression (12), Sec. 33, to write

$$\overline{\cos(iz)} = \overline{\cos(-y + ix)} = \cos y \cosh x - i \sin y \sinh x$$

and

$$\cos(i\bar{z}) = \cos(y - ix) = \cos y \cosh x - i \sin y \sinh x.$$

This shows that $\overline{\cos(iz)} = \cos(i\bar{z})$ for all $z$.

*(b)* Use expression (11), Sec. 33, to write

$$\overline{\sin(iz)} = \overline{\sin(-y + ix)} = -\sin y \cosh x - i \cos y \sinh x$$

and

$$\sin(i\bar{z}) = \sin(y + ix) = \sin y \cosh x + i \cos y \sinh x.$$

Evidently, then, the equation $\overline{\sin(iz)} = \sin(i\bar{z})$ is equivalent to the pair of equations

$$\sin y \cosh x = 0, \quad \cos y \sinh x = 0.$$

Since $\cosh x$ is never zero, the first of these equations tells us that $\sin y = 0$. Consequently, $y = n\pi$ $(n = 0, \pm 1, \pm 2, \ldots)$. Since $\cos n\pi = (-1)^n \neq 0$, the second equation tells us that $\sinh x = 0$, or that $x = 0$. So we may conclude that $\overline{\sin(iz)} = \sin(i\bar{z})$ if and only if $z = 0 + in\pi = n\pi i$ $(n = 0, \pm 1, \pm 2, \ldots)$.

**17.** Rewriting the equation $\sin z = \cosh 4$ as $\sin x \cosh y + i \cos x \sinh y = \cosh 4$, we see that we need to solve the pair of equations

$$\sin x \cosh y = \cosh 4, \quad \cos x \sinh y = 0$$

for $x$ and $y$. If $y=0$, the first equation becomes $\sin x=\cosh 4$, which cannot be satisfied by any $x$ since $\sin x \le 1$ and $\cosh 4>1$. So $y \neq 0$, and the second equation requires that $\cos x=0$. Thus

$$x=\frac{\pi}{2}+n\pi \qquad (n=0,\pm 1,\pm 2,\ldots).$$

Since

$$\sin\left(\frac{\pi}{2}+n\pi\right)=(-1)^n,$$

the first equation then becomes $(-1)^n\cosh y=\cosh 4$, which cannot hold when $n$ is odd. If $n$ is even, it follows that $y=\pm 4$. Finally, then, the roots of $\sin z=\cosh 4$ are

$$z=\left(\frac{\pi}{2}+2n\pi\right)\pm 4i \qquad (n=0,\pm 1,\pm 2,\ldots)$$

**18.** The problem here is to find all roots of the equation $\cos z=2$. We start by writing that equation as $\cos x\cosh y-i\sin x\sinh y=2$. Thus we need to solve the pair of equations

$$\cos x\cosh y=2, \qquad \sin x\sinh y=0$$

for $x$ and $y$. We note that $y\neq 0$ since $\cos x=2$ if $y=0$, and that is impossible. So the second in the pair of equations to be solved tells us that $\sin x=0$, or that $x=n\pi$ $(n=0,\pm 1,\pm 2,\ldots)$. The first equation then tells us that $(-1)^n\cosh y=2$; and, since $\cosh y$ is always positive, $n$ must be even. That is, $x=2n\pi$ $(n=0,\pm 1,\pm 2,\ldots)$. But this means that $\cosh y=2$, or $y=\cosh^{-1}2$. Consequently, the roots of the given equation are

$$z=2n\pi+i\cosh^{-1}2 \qquad (n=0,\pm 1,\pm 2,\ldots).$$

To express $\cosh^{-1}2$, which has two values, in a different way, we begin with $y=\cosh^{-1}2$, or $\cosh y=2$. This tells us that $e^y+e^{-y}=4$; and, rewriting this as

$$(e^y)^2-4(e^y)+1=0,$$

we may apply the quadratic formula to obtain $e^y=2\pm\sqrt{3}$, or $y=\ln(2\pm\sqrt{3})$. Finally, with the observation that

$$\ln(2-\sqrt{3})=\ln\left[\frac{(2-\sqrt{3})(2+\sqrt{3})}{2+\sqrt{3}}\right]=\ln\left(\frac{1}{2+\sqrt{3}}\right)=-\ln(2+\sqrt{3}),$$

we arrive at this alternative form of the roots:

$$z=2n\pi\pm i\ln(2+\sqrt{3}) \qquad (n=0,\pm 1,\pm 2,\ldots).$$

SECTION 34

1. To find the derivatives of sinh z and cosh z, we write

$$\frac{d}{dz}\sinh z = \frac{d}{dz}\left(\frac{e^z - e^{-z}}{2}\right) = \frac{1}{2}\frac{d}{dz}(e^z - e^{-z}) = \frac{e^z + e^{-z}}{2} = \cosh z$$

and

$$\frac{d}{dz}\cosh z = \frac{d}{dz}\left(\frac{e^z + e^{-z}}{2}\right) = \frac{1}{2}\frac{d}{dz}(e^z + e^{-z}) = \frac{e^z - e^{-z}}{2} = \sinh z.$$

3. Identity (7), Sec. 33, is $\sin^2 z + \cos^2 z = 1$. Replacing $z$ by $iz$ here and using the identities

$$\sin(iz) = i\sinh z \quad \text{and} \quad \cos(iz) = \cosh z,$$

we find that $i^2 \sinh^2 z + \cosh^2 z = 1$, or

$$\cosh^2 z - \sinh^2 z = 1.$$

Identity (6), Sec. 33, is $\cos(z_1 + z_2) = \cos z_1 \cos z_2 - \sin z_1 \sin z_2$. Replacing $z_1$ by $iz_1$ and $z_2$ by $iz_2$ here, we have $\cos[i(z_1 + z_2)] = \cos(iz_1)\cos(iz_2) - \sin(iz_1)\sin(iz_2)$. The same identities that were used just above then lead to

$$\cosh(z_1 + z_2) = \cosh z_1 \cosh z_2 + \sinh z_1 \sinh z_2.$$

6. We wish to show that

$$|\sinh x| \le |\cosh z| \le \cosh x$$

in two different ways.

(a) Identity (12), Sec. 34, is $|\cosh z|^2 = \sinh^2 x + \cos^2 y$. Thus $|\cosh z|^2 - \sinh^2 x \ge 0$; and this tells us that $\sinh^2 x \le |\cosh z|^2$, or $|\sinh x| \le |\cosh z|$. On the other hand, since $|\cosh z|^2 = (\cosh^2 x - 1) + \cos^2 y = \cosh^2 x - (1 - \cos^2 y) = \cosh^2 x - \sin^2 y$, we know that $|\cosh z|^2 - \cosh^2 x \le 0$. Consequently, $|\cosh z|^2 \le \cosh^2 x$, or $|\cosh z| \le \cosh x$.

(b) Exercise 11(*b*), Sec. 33, tells us that $|\sinh y| \le |\cos z| \le \cosh y$. Replacing $z$ by $iz$ here and recalling that $\cos iz = \cosh z$ and $iz = -y + ix$, we obtain the desired inequalities.

7. (a) Observe that

$$\sinh(z + \pi i) = \frac{e^{z+\pi i} - e^{-(z+\pi i)}}{2} = \frac{e^z e^{\pi i} - e^{-z} e^{-\pi i}}{2} = \frac{-e^z + e^{-z}}{2} = -\frac{e^z - e^{-z}}{2} = -\sinh z.$$

*(b)* Also,

$$\cosh(z+\pi i) = \frac{e^{z+\pi i} + e^{-(z+\pi i)}}{2} = \frac{e^z e^{\pi i} + e^{-z} e^{-\pi i}}{2} = \frac{-e^z - e^{-z}}{2} = -\frac{e^z + e^{-z}}{2} = -\cosh z.$$

*(c)* From parts *(a)* and *(b)*, we find that

$$\tanh(z+\pi i) = \frac{\sinh(z+\pi i)}{\cosh(z+\pi i)} = \frac{-\sinh z}{-\cosh z} = \frac{\sinh z}{\cosh z} = \tanh z.$$

9. The zeros of the hyperbolic tangent function

$$\tanh z = \frac{\sinh z}{\cosh z}$$

are the same as the zeros of $\sinh z$, which are $z = n\pi i$ $(n = 0, \pm 1, \pm 2, \ldots)$. The singularities of $\tanh z$ are the zeros of $\cosh z$, or $z = \left(\frac{\pi}{2} + n\pi\right)i$ $(n = 0, \pm 1, \pm 2, \ldots)$.

15. *(a)* Observe that, since $\sinh z = i$ can be written as $\sinh x \cos y + i\cosh x \sin y = i$, we need to solve the pair of equations

$$\sinh x \cos y = 0, \quad \cosh x \sin y = 1.$$

If $x = 0$, the second of these equations becomes $\sin y = 1$; and so $y = \frac{\pi}{2} + 2n\pi$ $(n = 0, \pm 1, \pm 2, \ldots)$. Hence

$$z = \left(2n + \frac{1}{2}\right)\pi i \qquad (n = 0, \pm 1, \pm 2, \ldots).$$

If $x \neq 0$, the first equation requires that $\cos y = 0$, or $y = \frac{\pi}{2} + n\pi$ $(n = 0, \pm 1, \pm 2, \ldots)$. The second then becomes $(-1)^n \cosh x = 1$. But there is no nonzero value of $x$ satisfying this equation, and we have no additional roots of $\sinh z = i$.

*(b)* Rewriting $\cosh z = \frac{1}{2}$ as $\cosh x \cos y + i \sinh x \sin y = \frac{1}{2}$, we see that $x$ and $y$ must satisfy the pair of equations

$$\cosh x \cos y = \frac{1}{2}, \quad \sinh x \sin y = 0$$

If $x = 0$, the second equation is satisfied and the first equation becomes $\cos y = \frac{1}{2}$. Thus $y = \cos^{-1}\frac{1}{2} = \pm\frac{\pi}{3} + 2n\pi$ $(n = 0, \pm 1, \pm 2, \ldots)$, and this means that

$$z = \left(2n \pm \frac{1}{3}\right)\pi i \qquad (n = 0, \pm 1, \pm 2, \ldots).$$

If $x \neq 0$, the second equation tells us that $y = n\pi$ $(n = 0, \pm 1, \pm 2, \ldots)$. The first then becomes $(-1)^n \cosh x = \frac{1}{2}$. But this equation in $x$ has no solution since $\cosh x \geq 1$ for all $x$. Thus no additional roots of $\cosh z = \frac{1}{2}$ are obtained.

16. Let us rewrite $\cosh z = -2$ as $\cosh x \cos y + i \sinh x \sin y = -2$. The problem is evidently to solve the pair of equations

$$\cosh x \cos y = -2, \quad \sinh x \sin y = 0.$$

If $x = 0$, the second equation is satisfied and the first reduces to $\cos y = -2$. Since there is no $y$ satisfying this equation, no roots of $\cosh z = -2$ arise.
If $x \neq 0$, we find from the second equation that $\sin y = 0$, or $y = n\pi$ $(n = 0, \pm 1, \pm 2, \ldots)$. Since $\cos n\pi = (-1)^n$, it follows from the first equation that $(-1)^n \cosh x = -2$. But this equation can hold only when $n$ is odd, in which case $x = \cosh^{-1} 2$. Consequently,

$$z = \cosh^{-1} 2 + (2n+1)\pi i \qquad (n = 0, \pm 1, \pm 2, \ldots).$$

Recalling from the solution of Exercise 18, Sec 35, that $\cosh^{-1} 2 = \pm\ln(2+\sqrt{3})$, we note that these roots can also be written as

$$z = \pm\ln(2+\sqrt{3}) + (2n+1)\pi i \qquad (n = 0, \pm 1, \pm 2, \ldots).$$

# Chapter 4

SECTION 37

**2.** (a) $$\int_1^2 \left(\frac{1}{t}-i\right)^2 dt = \int_1^2 \left(\frac{1}{t^2}-1\right)dt - 2i\int_1^2 \frac{dt}{t} = -\frac{1}{2} - 2i\ln 2 = -\frac{1}{2} - i\ln 4;$$

(b) $$\int_0^{\pi/6} e^{i2t}\,dt = \left[\frac{e^{i2t}}{2i}\right]_0^{\pi/6} = \frac{1}{2i}\left[\cos\frac{\pi}{3} + i\sin\frac{\pi}{3} - 1\right] = \frac{\sqrt{3}}{4}+\frac{i}{4};$$

(c) Since $|e^{-zt}| = e^{-xt}$, we find that

$$\int_0^\infty e^{-zt}\,dt = \lim_{b\to\infty}\int_0^b e^{-zt}\,dt = \lim_{b\to\infty}\left[\frac{e^{-zt}}{-z}\right]_{t=0}^{t=b} = \frac{1}{z}\lim_{b\to\infty}(1-e^{-bz}) = \frac{1}{z} \quad \text{when } \operatorname{Re} z > 0.$$

**3.** The problem here is to verify that

$$\int_0^{2\pi} e^{im\theta}e^{-in\theta}\,d\theta = \begin{cases} 0 & \text{when } m \neq n, \\ 2\pi & \text{when } m = n. \end{cases}$$

To do this, we write

$$I = \int_0^{2\pi} e^{im\theta}e^{-in\theta}\,d\theta = \int_0^{2\pi} e^{i(m-n)\theta}\,d\theta$$

and observe that when $m \neq n$,

$$I = \left[\frac{e^{i(m-n)\theta}}{i(m-n)}\right]_0^{2\pi} = \frac{1}{i(m-n)} - \frac{1}{i(m-n)} = 0.$$

When $m = n$, $I$ becomes

$$I = \int_0^{2\pi} d\theta = 2\pi;$$

and the verification is complete.

**4.** First of all,

$$\int_0^\pi e^{(1+i)x}\,dx = \int_0^\pi e^x \cos x\,dx + i\int_0^\pi e^x \sin x\,dx.$$

But also,

$$\int_0^\pi e^{(1+i)x}\,dx = \left[\frac{e^{(1+i)x}}{1+i}\right]_0^\pi = \frac{e^\pi e^{i\pi}-1}{1+i} = \frac{-e^\pi-1}{1+i}\cdot\frac{1-i}{1-i} = -\frac{1+e^\pi}{2}+i\,\frac{1+e^\pi}{2}.$$

Equating the real parts and then the imaginary parts of these two expressions, we find that

$$\int_0^\pi e^x \cos x\, dx = -\frac{1-e^\pi}{2} \quad \text{and} \quad \int_0^\pi e^x \sin x\, dx = \frac{1+e^\pi}{2}.$$

5. Consider the function $w(t) = e^{it}$ and observe that

$$\int_0^{2\pi} w(t)dt = \int_0^{2\pi} e^{it}dt = \left[\frac{e^{it}}{i}\right]_0^{2\pi} = \frac{1}{i} - \frac{1}{i} = 0.$$

Since $|w(c)(2\pi - 0)| = |e^{ic}|2\pi = 2\pi$ for every real number $c$, it is clear that there is no number $c$ in the interval $0 < c < 2\pi$ such that

$$\int_0^{2\pi} w(t)dt = w(c)(2\pi - 0).$$

6. (a) Suppose that $w(t)$ is even. It is straightforward to show that $u(t)$ and $v(t)$ must be even. Thus

$$\int_{-a}^{a} w(t)dt = \int_{-a}^{a} u(t)dt + i\int_{-a}^{a} v(t)dt = 2\int_0^a u(t)dt + 2i\int_0^a v(t)dt$$

$$= 2\left[\int_0^a u(t)dt + i\int_0^a v(t)dt\right] = 2\int_0^a w(t)dt.$$

(b) Suppose, on the other hand, that $w(t)$ is odd. It follows that $u(t)$ and $v(t)$ are odd, and so

$$\int_{-a}^{a} w(t)dt = \int_{-a}^{a} u(t)dt + i\int_{-a}^{a} v(t)dt = 0 + i0 = 0.$$

7. Consider the functions

$$P_n(x) = \frac{1}{\pi}\int_0^\pi \left(x + i\sqrt{1-x^2}\cos\theta\right)^n d\theta \qquad (n = 0,1,2,\ldots),$$

where $-1 \le x \le 1$. Since

$$\left|x + i\sqrt{1-x^2}\cos\theta\right| = \sqrt{x^2 + (1-x^2)\cos^2\theta} \le \sqrt{x^2 + (1-x^2)} = 1,$$

it follows that

$$|P_n(x)| \le \frac{1}{\pi}\int_0^\pi \left|x + i\sqrt{1-x^2}\cos\theta\right|^n d\theta \le \frac{1}{\pi}\int_0^\pi d\theta = 1.$$

## SECTION 38

**1.** *(a)* Start by writing

$$I=\int_{-b}^{-a} w(-t)\,dt=\int_{-b}^{-a} u(-t)\,dt+i\int_{-b}^{-a} v(-t)\,dt.$$

The substitution $\tau=-t$ in each of these two integrals on the right then yields

$$I=-\int_{b}^{a} u(\tau)\,d\tau-i\int_{b}^{a} v(\tau)\,d\tau=\int_{a}^{b} u(\tau)\,d\tau+i\int_{a}^{b} v(\tau)\,d\tau=\int_{a}^{b} w(\tau)\,d\tau.$$

That is,

$$\int_{-b}^{-a} w(-t)\,dt=\int_{a}^{b} w(\tau)\,d\tau.$$

*(b)* Start with

$$I=\int_{a}^{b} w(t)\,dt=\int_{a}^{b} u(t)\,dt+i\int_{a}^{b} v(t)\,dt$$

and then make the substitution $t=\phi(\tau)$ in each of the integrals on the right. The result is

$$I=\int_{\alpha}^{\beta} u[\phi(\tau)]\phi'(\tau)\,d\tau+i\int_{\alpha}^{\beta} v[\phi(\tau)]\phi'(\tau)\,d\tau=\int_{\alpha}^{\beta} w[\phi(\tau)]\phi'(\tau)\,d\tau.$$

That is,

$$\int_{a}^{b} w(t)\,dt=\int_{\alpha}^{\beta} w[\phi(\tau)]\phi'(\tau)\,d\tau.$$

**3.** The slope of the line through the points $(\alpha,a)$ and $(\beta,b)$ in the $\tau t$ plane is

$$m=\frac{b-a}{\beta-\alpha}.$$

So the equation of that line is

$$t-a=\frac{b-a}{\beta-\alpha}(\tau-\alpha).$$

Solving this equation for $t$, one can rewrite it as

$$t = \frac{b-a}{\beta-\alpha}\tau + \frac{a\beta - b\alpha}{\beta-\alpha}.$$

Since $t = \phi(\tau)$, then,

$$\phi(\tau) = \frac{b-a}{\beta-\alpha}\tau - \frac{a\beta - b\alpha}{\beta-\alpha}.$$

4. If $Z(\tau) = z[\phi(\tau)]$, where $z(t) = x(t) + iy(t)$ and $t = \phi(\tau)$, then

$$Z(\tau) = x[\phi(\tau)] + iy[\phi(\tau)].$$

Hence

$$Z'(\tau) = \frac{d}{d\tau}x[\phi(\tau)] + i\frac{d}{d\tau}y[\phi(\tau)] = x'[\phi(\tau)]\phi'(\tau) + iy'[\phi(\tau)]\phi'(\tau)$$

$$= \{x'[\phi(\tau)] + iy'[\phi(\tau)]\}\phi'(\tau) = z'[\phi(\tau)]\phi'(\tau).$$

5. If $w(t) = f[z(t)]$ and $f(z) = u(x,y) + iv(x,y)$, $z(t) = x(t) + iy(t)$, we have

$$w(t) = u[x(t),y(t)] + iv[x(t),y(t)].$$

The chain rule tells us that

$$\frac{du}{dt} = u_x x' + u_y y' \quad \text{and} \quad \frac{dv}{dt} = v_x x' + v_y y',$$

and so

$$w'(t) = (u_x x' + u_y y') + i(v_x x' + v_y y').$$

In view of the Cauchy-Riemann equations $u_x = v_y$ and $u_y = -v_x$, then,

$$w'(t) = (u_x x' - v_x y') + i(v_x x' + u_x y') = (u_x + iv_x)(x' + iy')$$

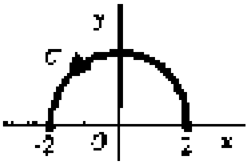
That is,

$$w'(t) = \{u_x[x(t),y(t)] + iv_x[x(t),y(t)]\}[x'(t) + iy'(t)] = f'[z(t)]z'(t)$$

when $t = t_0$.

SECTION 40

1. (a) Let $C$ be the semicircle $z = 2e^{i\theta}$ $(0 \le \theta \le \pi)$, shown below.

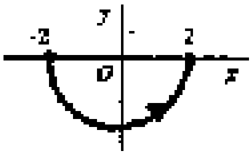



Then

$$\int_C \frac{z+2}{z}\,dz = \int_C\left(1+\frac{2}{z}\right)dz = \int_0^\pi\left(1+\frac{2}{2e^{i\theta}}\right)2ie^{i\theta}\,d\theta = 2i\int_0^\pi (e^{i\theta}+1)\,d\theta$$

$$= 2i\left[\frac{e^{i\theta}}{i}+\theta\right]_0^\pi = 2i(i+\pi+i) = -4+2\pi i.$$

(b) Now let $C$ be the semicircle $z = 2e^{i\theta}$ $(\pi \le \theta \le 2\pi)$ just below.




This is the same as part (a), except for the limits of integration. Thus

$$\int_C \frac{z+2}{z}\,dz = 2i\left[\frac{e^{i\theta}}{i}+\theta\right]_\pi^{2\pi} = 2i(-i+2\pi-i-\pi) = 4+2\pi i.$$

(c) Finally, let $C$ denote the entire circle $z = 2e^{i\theta}$ $(0 \le \theta \le 2\pi)$. In this case,

$$\int_C \frac{z+2}{z}\,dz = 4\pi i,$$

the value here being the sum of the values of the integrals in parts (a) and (b).

2. (a) The arc is $C: z = 1+e^{i\theta}$ $(\pi \le \theta \le 2\pi)$. Then

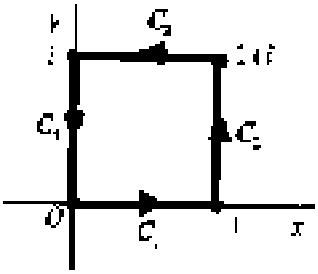
$$\int_C (z-1)\,dz = \int_\pi^{2\pi}(1+e^{i\theta}-1)ie^{i\theta}\,d\theta = i\int_\pi^{2\pi} e^{i2\theta}\,d\theta = i\left[\frac{e^{i2\theta}}{2i}\right]_\pi^{2\pi}$$

$$= \frac{1}{2}\left(e^{i4\pi}-e^{i2\pi}\right) = \frac{1}{2}(1-1) = 0.$$

(b) Here $C: z = x\ (0 \le x \le 2)$. Then

$$\int_C (z-1)\,dz = \int_0^2 (x-1)\,dx = \left[\frac{x^2}{2} - x\right]_0^2 = 0.$$

3. **In this problem, the path $C$ is the sum of the paths $C_1$, $C_2$, $C_3$, and $C_4$ that are shown below.**




The function to be integrated around the closed path $C$ is $f(z) = \pi e^{\pi \bar{z}}$. We observe that $C = C_1 + C_2 + C_3 + C_4$ and find the values of the integrals along the individual legs of the square $C$.

(i) Since $C_1$ is $z = x\ (0 \le x \le 1)$,

$$\int_{C_1} \pi e^{\pi \bar{z}}\,dz = \pi \int_0^1 e^{\pi x}\,dx = e^{\pi} - 1.$$

(ii) Since $C_2$ is $z = 1 + iy\ (0 \le y \le 1)$,

$$\int_{C_2} \pi e^{\pi \bar{z}}\,dz = \pi \int_0^1 e^{\pi(1-iy)}\,i\,dy = e^{\pi}\pi i \int_0^1 e^{-i\pi y}\,dy = 2e^{\pi}.$$

(iii) Since $C_3$ is $z = (1-x) + i\ (0 \le x \le 1)$,

$$\int_{C_3} \pi e^{\pi \bar{z}}\,dz = \pi \int_0^1 e^{\pi[(1-x)-i]}(-1)\,dx = \pi e^{\pi} \int_0^1 e^{-\pi x}\,dx = e^{\pi} - 1.$$

(iv) Since $C_4$ is $z = i(1-y)\ (0 \le y \le 1)$,

$$\int_{C_4} \pi e^{\pi \bar{z}}\,dz = \pi \int_0^1 e^{-\pi(1-y)i}(-i)\,dy = \pi i \int_0^1 e^{i\pi y}\,dy = -2.$$

**Finally**, then, since

$$\int_C \pi e^{\pi \bar{z}}\,dz = \int_{C_1} \pi e^{\pi \bar{z}}\,dz + \int_{C_2} \pi e^{\pi \bar{z}}\,dz + \int_{C_3} \pi e^{\pi \bar{z}}\,dz + \int_{C_4} \pi e^{\pi \bar{z}}\,dz,$$

we find that

$$\int_C \pi e^{\pi \bar{z}}\,dz = 4(e^{\pi} - 1).$$

**4.** The path $C$ is the sum of the paths

$$C_1: z = x + ix^3 \ (-1 \le x \le 0) \quad \text{and} \quad C_2: z = x + ix^3 \ (0 \le x \le 1).$$

Using

$$f(z) = 1 \text{ on } C_1 \quad \text{and} \quad f(z) = 4y = 4x^3 \text{ on } C_2,$$

we have

$$\int_C f(z)dz = \int_{C_1} f(z)dz + \int_{C_2} f(z)dz = \int_{-1}^{0} 1(1+i3x^2)dx + \int_0^1 4x^3(1+i3x^2)dx$$

$$= \int_{-1}^{0} dx + 3i\int_{-1}^{0} x^2 dx + 4\int_0^1 x^3 dx + 12i\int_0^1 x^5 dx$$

$$= [x]_{-1}^{0} + i[x^3]_{-1}^{0} + [x^4]_0^1 + 2i[x^6]_0^1 = 1 + i + 1 + 2i = 2 + 3i.$$

**5.** The contour C has some parametric representation $z = z(t)$ $(a \le t \le b)$, where $z(a) = z_1$ and $z(b) = z_2$. Then

$$\int_C dz = \int_a^b z'(t)dt = [z(t)]_a^b = z(b) - z(a) = z_2 - z_1.$$

**6.** To integrate the branch

$$z^{-1+i} = e^{(-1+i)\log z} \qquad (|z| > 0,\ 0 < \arg z < 2\pi)$$

around the circle $C: z = e^{i\theta}$ $(0 \le \theta \le 2\pi)$, write

$$\int_C z^{-1+i} dz = \int_C e^{(-1+i)\log z} dz = \int_0^{2\pi} e^{(-1+i)(\ln 1 + i\theta)} ie^{i\theta} d\theta = i\int_0^{2\pi} e^{-i\theta} e^{-\theta} e^{i\theta} d\theta = i\int_0^{2\pi} e^{-\theta} d\theta = i(1 - e^{-2\pi}).$$

**7.** Let $C$ be the positively oriented circle $|z| = 1$, with parametric representation $z = e^{i\theta}$ $(0 \le \theta \le 2\pi)$, and let $m$ and $n$ be integers. Then

$$\int_C z^m \bar{z}^n dz = \int_0^{2\pi} (e^{i\theta})^m (e^{-i\theta})^n ie^{i\theta} d\theta = i\int_0^{2\pi} e^{i(m+1)\theta} e^{-in\theta} d\theta.$$

But we know from Exercise 3, Sec. 37, that

$$\int_0^{2\pi} e^{im\theta} e^{-in\theta} d\theta = \begin{cases} 0 & \text{when } m \ne n, \\ 2\pi & \text{when } m = n. \end{cases}$$

Consequently,

$$\int_C z^m \bar{z}^n dz = \begin{cases} 0 & \text{when} \quad m+1 \neq n, \\ 2\pi i & \text{when} \quad m+1 = n. \end{cases}$$

8. Note that $C$ is the right-hand half of the circle $x^2 + y^2 = 4$. So, on $C$, $x = \sqrt{4-y^2}$. This suggests the parametric representation $C: z = \sqrt{4-y^2} + iy \ (-2 \le y \le 2)$, to be used here. With that representation, we have

$$\int_C \bar{z}\,dz = \int_{-2}^{2} \left(\sqrt{4-y^2} - iy\right)\left(\frac{-y}{\sqrt{4-y^2}} + i\right) dy$$

$$= \int_{-2}^{2} (-y - y)\,dy + i\int_{-2}^{2} \left(\frac{y^2}{\sqrt{4-y^2}} + \sqrt{4-y^2}\right) dy$$

$$= i\int_{-2}^{2} \frac{y^2 + 4 - y^2}{\sqrt{4-y^2}}\,dy = 4i\int_{-2}^{2} \frac{dy}{\sqrt{4-y^2}} = 4i\left[\sin^{-1}\left(\frac{y}{2}\right)\right]_{-2}^{2}$$

$$= 4i\left[\sin^{-1}(1) - \sin^{-1}(-1)\right] = 4i\left[\frac{\pi}{2} - \left(-\frac{\pi}{2}\right)\right] = 4\pi i.$$

10. Let $C_0$ be the circle $z = z_0 + Re^{i\theta} \ (-\pi \le \theta \le \pi)$.

(a) $$\int_{C_0} \frac{dz}{z - z_0} = \int_{-\pi}^{\pi} \frac{1}{Re^{i\theta}} Rie^{i\theta} d\theta = i\int_{-\pi}^{\pi} d\theta = 2\pi i.$$

(b) When $n = \pm 1, \pm 2, \ldots$,

$$\int_{C_0} (z - z_0)^{n-1} dz = \int_{-\pi}^{\pi} \left(Re^{i\theta}\right)^{n-1} Rie^{i\theta} d\theta = iR^n \int_{-\pi}^{\pi} e^{in\theta} d\theta$$

$$= \frac{R^n}{n}\left(e^{in\pi} - e^{-in\pi}\right) = i\frac{2R^n}{n}\sin n\pi = 0.$$

11. In this case, where $a$ is any real number other than zero, the same steps as in Exercise 10(*b*), with $a$ instead of $n$, yield the result

$$\int_{C_0} (z - z_0)^{a-1} dz = i\frac{2R^a}{a}\sin(a\pi).$$

**12.** (*a*) The function $f(z)$ is continuous on a smooth arc $C$, which has a parametric representation $z = z(t)$ $(a \le t \le b)$. Exercise 1(*b*), Sec. 38, enables us to write

$$\int_a^b f[z(t)]z'(t)\,dt = \int_\alpha^\beta f[Z(\tau)]z'[\phi(\tau)]\phi'(\tau)\,d\tau,$$

where

$$Z(\tau) = z[\phi(\tau)] \qquad (\alpha \le \tau \le \beta).$$

But expression (14), Sec. 38, tells us that

$$z'[\phi(\tau)]\phi'(\tau) = Z'(\tau);$$

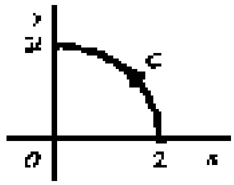
and so

$$\int_a^b f[z(t)]z'(t)\,dt = \int_\alpha^\beta f[Z(\tau)]Z'(\tau)\,d\tau.$$

(*b*) Suppose that $C$ is any contour and that $f(z)$ is piecewise continuous on $C$. Since $C$ can be broken up into a finite chain of smooth arcs on which $f(z)$ is continuous, the identity obtained in part (*a*) remains valid.

SECTION 41

1. Let $C$ be the arc of the circle $|z| = 2$ shown below.




Without evaluating the integral, let us find an upper bound for $\left|\int_C \frac{dz}{z^2-1}\right|$. To do this, we note that if $z$ is a point on $C$,

$$|z^2-1| \ge \left||z^2|-1\right| = \left||z|^2-1\right| = |4-1| = 3.$$

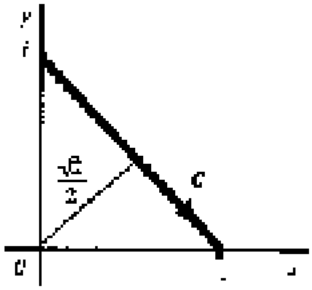
Thus

$$\left|\frac{1}{z^2-1}\right| = \frac{1}{|z^2-1|} \le \frac{1}{3}.$$

Also, the length of $C$ is $\frac{1}{4}(4\pi) = \pi$. So, taking $M = \frac{1}{3}$ and $L = \pi$, we find that

$$\left|\int_C \frac{dz}{z^2-1}\right| \le ML = \frac{\pi}{3}.$$

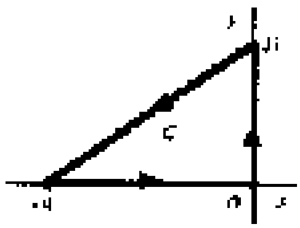
2. The path $C$ is as shown in the figure below. The midpoint of $C$ is clearly the closest point on $C$ to the origin. The distance of that midpoint from the origin is clearly $\frac{\sqrt{2}}{2}$, the length of $C$ being $\sqrt{2}$.




Hence if $z$ is any point on $C$, $|z| \ge \frac{\sqrt{2}}{2}$. This means that, for such a point, $\left|\frac{1}{z^4}\right| = \frac{1}{|z|^4} \le 4$. Consequently, by taking $M = 4$ and $L = \sqrt{2}$, we have

$$\left|\int_C \frac{dz}{z^4}\right| \le ML = 4\sqrt{2}.$$

3. The contour $C$ is the closed triangular path shown below.




To find an upper bound for $\left|\int_C (e^z - \bar{z})\,dz\right|$, we let $z$ be a point on $C$ and observe that

$$|e^z - \bar{z}| \le |e^z| + |\bar{z}| = e^x + \sqrt{x^2 + y^2}.$$

But $e^x \le 1$ since $x \le 0$, and the distance $\sqrt{x^2+y^2}$ of the point $z$ from the origin is always less than or equal to 4. Thus $|e^z - \bar{z}| \le 5$ when $z$ is on $C$. The length of $C$ is evidently 12. Hence, by writing $M = 5$ and $L = 12$, we have

$$\left|\int_C (e^z - \bar{z})\,dz\right| \le ML = 60.$$

4. Note that if $|z| = R$ $(R > 2)$, then

$$|2z^2 - 1| \le 2|z|^2 + 1 = 2R^2 + 1$$

and

$$|z^4 + 5z^2 + 4| = |z^2 + 1||z^2 + 4| \ge \left||z|^2 - 1\right|\left||z|^2 - 4\right| = (R^2 - 1)(R^2 - 4).$$

Thus

$$\left|\frac{2z^2 - 1}{z^4 + 5z^2 + 4}\right| = \frac{|2z^2 - 1|}{|z^4 + 5z^2 + 4|} \le \frac{2R^2 + 1}{(R^2 - 1)(R^2 - 4)}$$

when $|z| = R$ $(R > 2)$. Since the length of $C_R$ is $\pi R$, then,

$$\left|\int_{C_R} \frac{2z^2 - 1}{z^4 + 5z^2 + 4}\,dz\right| \le \frac{\pi R(2R^2 + 1)}{(R^2 - 1)(R^2 - 4)} = \frac{\dfrac{\pi}{R}\left(2 + \dfrac{1}{R^2}\right)}{\left(1 - \dfrac{1}{R^2}\right)\left(1 - \dfrac{4}{R^2}\right)};$$

and it is clear that the value of the integral tends to zero as $R$ tends to infinity.

5. Here $C_R$ is the positively oriented circle $|z| = R$ $(R > 1)$. If $z$ is a point on $C_R$, then

$$\left|\frac{\operatorname{Log} z}{z^2}\right| = \frac{|\ln R + i\Theta|}{R^2} \le \frac{\ln R + |\Theta|}{R^2} \le \frac{\pi + \ln R}{R^2},$$

since $-\pi < \Theta \le \pi$. The length of $C_R$ is, of course, $2\pi R$. Consequently, by taking

$$M = \frac{\pi + \ln R}{R^2} \quad \text{and} \quad L = 2\pi R,$$

we see that

$$\left|\int_{C_R} \frac{\text{Log}\, z}{z^2}\, dz\right| \le ML - 2\pi\left(\frac{\pi - \ln R}{R}\right).$$

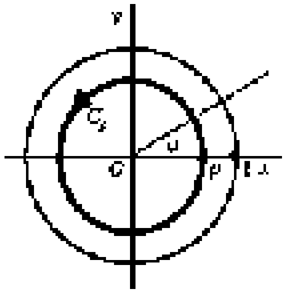
Since

$$\lim_{R\to\infty} \frac{\pi + \ln R}{R} = \lim_{R\to\infty} \frac{1/R}{1} = 0,$$

it follows that

$$\lim_{R\to\infty} \int_{C_R} \frac{\text{Log}\, z}{z^2}\, dz = 0.$$

6. Let $C_\rho$ be the positively oriented circle $|z| = \rho$ $(0 < \rho < 1)$, shown in the figure below, and suppose that $f(z)$ is analytic in the disk $|z| \le 1$.




We let $z^{-1/2}$ represent any particular branch

$$z^{-1/2} = \exp\left(-\frac{1}{2}\log z\right) = \exp\left[-\frac{1}{2}(\ln r + i\theta)\right] = \frac{1}{\sqrt{r}}\exp\left(-i\frac{\theta}{2}\right) \qquad (r > 0, \alpha < \theta < \alpha + 2\pi)$$

of the power function here; and we note that, since $f(z)$ is continuous on the *closed bounded* disk $|z| \le 1$, there is a nonnegative constant $M$ such that $|f(z)| \le M$ for each point $z$ in that disk. We are asked to find an upper bound for $\left|\int_{C_\rho} z^{-1/2} f(z)\, dz\right|$. To do this, we observe that if $z$ is a point on $C_\rho$,

$$\left|z^{-1/2} f(z)\right| = \left|z^{-1/2}\right| |f(z)| \le \frac{M}{\sqrt{\rho}}.$$

Since the length of the path $C_\rho$ is $2\pi\rho$, we may conclude that

$$\left|\int_{C_\rho} z^{-1/2} f(z)\, dz\right| \le \frac{M}{\sqrt{\rho}}\, 2\pi\rho = 2\pi M\sqrt{\rho}.$$

Note that, inasmuch as $M$ is independent of $\rho$, it follows that

$$\lim_{\rho\to 0} \int_{C_\rho} z^{-1/2} f(z)\, dz = 0.$$

SECTION 43

1. The function $z^n$ $(n = 0,1,2,\ldots)$ has the antiderivative $z^{n+1}/(n+1)$ everywhere in the finite plane. Consequently, for any contour $C$ from a point $z_1$ to a point $z_2$,

$$\int_C z^n dz = \int_{z_1}^{z_2} z^n dz = \frac{z^{n+1}}{n+1}\Bigg]_{z_1}^{z_2} = \frac{z_2^{n+1}}{n+1} - \frac{z_1^{n+1}}{n+1} = \frac{1}{n+1}\left(z_2^{n+1} - z_1^{n+1}\right).$$

2. (a) $$\int_i^{i/2} e^{\pi z} dz = \frac{e^{\pi z}}{\pi}\Bigg]_i^{i/2} = \frac{e^{i\pi/2} - e^{i\pi}}{\pi} = \frac{i+1}{\pi} = \frac{1+i}{\pi}.$$

(b) $$\int_0^{\pi+2i} \cos\left(\frac{z}{2}\right) dz = 2\sin\left(\frac{z}{2}\right)\Bigg]_0^{\pi+2i} = 2\sin\left(\frac{\pi}{2}+i\right) = 2\,\frac{e^{i\left(\frac{\pi}{2}+i\right)} - e^{-i\left(\frac{\pi}{2}+i\right)}}{2i} = -i\left(e^{i\pi/2}e^{-1} - e^{-i\pi/2}e\right)$$

$$= -i\left(\frac{i}{e} + ie\right) = \frac{1}{e} + e = e + \frac{1}{e}.$$

(c) $$\int_1^3 (z-2)^3 dz = \frac{(z-2)^4}{4}\Bigg]_1^3 = \frac{1}{4} - \frac{1}{4} = 0.$$

3. Note the function $(z-z_0)^{n-1}$ $(n = \pm1, \pm2, \ldots)$ always has an antiderivative in any domain that does not contain the point $z = z_0$. So, by the theorem in Sec. 42,

$$\int_{C_0} (z - z_0)^{n-1} dz = 0$$

for any closed contour $C_0$ that does not pass through $z_0$.

5. Let $C$ denote any contour from $z = -1$ to $z = 1$ that, except for its end points, lies above the real axis. This exercise asks us to evaluate the integral

$$I = \int_{-1}^{1} z^i dz,$$

where $z^i$ denotes the principal branch

$$z^i = \exp(i\,\mathrm{Log}\,z) \qquad (|z| > 0, -\pi < \mathrm{Arg}\,z < \pi).$$

An antiderivative of this branch *cannot* be used since the branch is not even defined at $z = -1$. But the integrand can be replaced by the branch

$$z^i = \exp(i \log z) \qquad \left(|z| > 0, -\frac{\pi}{2} < \arg z < \frac{3\pi}{2}\right)$$

since it agrees with the integrand along $C$. Using an antiderivative of this new branch, we can now write:

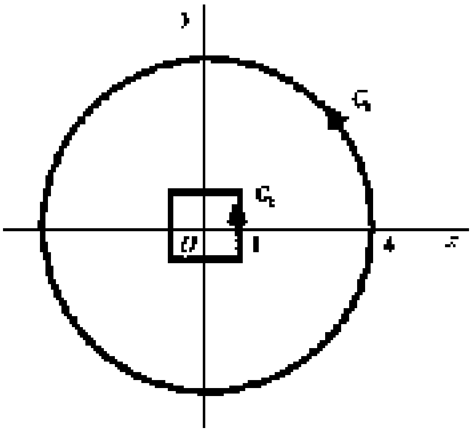
$$I = \frac{z^{i+1}}{i+1}\Bigg]_{-1}^{1} = \frac{1}{i+1}\left[(1)^{i+1} - (-1)^{i+1}\right] = \frac{1}{i+1}\left[e^{(i+1)\log 1} - e^{(i+1)\log(-1)}\right]$$

$$= \frac{1}{i+1}\left[e^{(i+1)(\ln 1 + i0)} - e^{(i+1)(\ln 1 + i\pi)}\right] = \frac{1}{i+1}\left(1 - e^{-\pi}e^{i\pi}\right) = \frac{1+e^{-\pi}}{1+i} \cdot \frac{1-i}{1-i}$$

$$= \frac{1+e^{-\pi}}{2}(1-i).$$

SECTION 46

2. The contours $C_1$ and $C_2$ are as shown in the figure below.




In each of the cases below, the singularities of the integrand lie outside $C_1$ or inside $C_2$; and so the integrand is analytic on the contours and between them. Consequently,

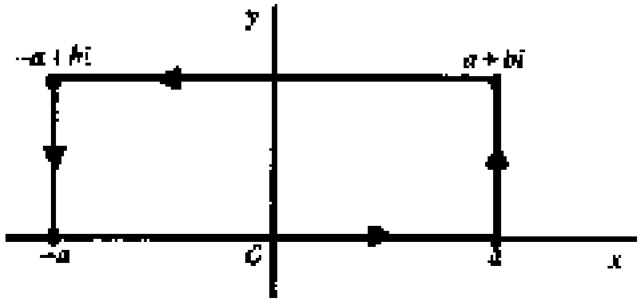
$$\int_{C_1} f(z)\,dz = \int_{C_2} f(z)\,dz.$$

(a) When $f(z)=\dfrac{1}{3z^2+1}$, the singularities are the points $z=\pm\dfrac{1}{\sqrt{3}}i$.

(b) When $f(z)=\dfrac{z+2}{\sin(z/2)}$, the singularities are at $z=2n\pi\ (n=0,\pm1,\pm2,\ldots)$.

(c) When $f(z)=\dfrac{z}{1-e^z}$, the singularities are at $z=2n\pi i\ (n=0,\pm1,\pm2,\ldots)$.

4. (a) In order to derive the integration formula in question, we integrate the function $e^{-z^2}$ around the closed rectangular path shown below.




Since the lower horizontal leg is represented by $z=x\ (-a\le x\le a)$, the integral of $e^{-z^2}$ along that leg is

$$\int_{-a}^{a} e^{-x^2}\,dx = 2\int_0^a e^{-x^2}\,dx.$$

Since the opposite direction of the upper horizontal leg has parametric representation $z=x+bi\ (-a\le x\le a)$, the integral of $e^{-z^2}$ along the upper leg is

$$-\int_{-a}^{a} e^{-(x+bi)^2}\,dx = -e^{b^2}\int_{-a}^{a} e^{-x^2}e^{-i2bx}\,dx = -e^{b^2}\int_{-a}^{a} e^{-x^2}\cos 2bx\,dx + ie^{b^2}\int_{-a}^{a} e^{-x^2}\sin 2bx\,dx,$$

or simply

$$-2e^{b^2}\int_0^a e^{-x^2}\cos 2bx\,dx.$$

Since the right-hand vertical leg is represented by $z=a+iy\ (0\le y\le b)$, the integral of $e^{-z^2}$ along it is

$$\int_0^b e^{-(a+iy)^2}\,i\,dy = ie^{-a^2}\int_0^b e^{y^2}e^{-i2ay}\,dy.$$

Finally, since the opposite direction of the left hand vertical leg has the representation $z = -a + iy$ $(0 \le y \le b)$, the integral of $e^{-z^2}$ along that vertical leg is

$$-\int_0^b e^{-(-a+iy)^2}\,i\,dy = -ie^{-a^2}\int_0^b e^{y^2}e^{i2ay}\,dy.$$

According to the Cauchy-Goursat theorem, then,

$$2\int_0^a e^{-x^2}dx - 2e^{b^2}\int_0^a e^{-x^2}\cos 2bx\,dx + ie^{-a^2}\int_0^b e^{y^2}e^{-i2ay}dy - ie^{-a^2}\int_0^b e^{y^2}e^{i2ay}dy = 0;$$

and this reduces to

$$\int_0^a e^{-x^2}\cos 2bx\,dx = e^{-b^2}\int_0^a e^{-x^2}dx + e^{-(a^2+b^2)}\int_0^b e^{y^2}\sin 2ay\,dy.$$

(b) We now let $a \to \infty$ in the final equation in part (a), keeping in mind the known integration formula

$$\int_0^\infty e^{-x^2}dx = \frac{\sqrt{\pi}}{2}$$

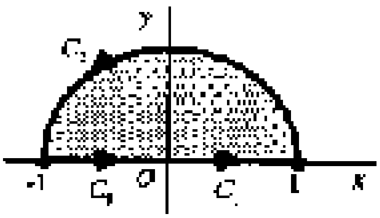
and the fact that

$$\left| e^{-(a^2+b^2)}\int_0^b e^{y^2}\sin 2ay\,dy \right| \le e^{-(a^2+b^2)}\int_0^b e^{y^2}\,dy \to 0 \text{ as } a \to \infty.$$

The result is

$$\int_0^\infty e^{-x^2}\cos 2bx\,dx = \frac{\sqrt{\pi}}{2}e^{-b^2} \qquad (b > 0).$$

6. We let $C$ denote the entire boundary of the semicircular region appearing below. It is made up of the leg $C_1$ from the origin to the point $z = 1$, the semicircular arc $C_2$ that is shown, and the leg $C_3$ from $z = -1$ to the origin. Thus $C = C_1 + C_2 + C_3$.



We also let $f(z)$ be a continuous function that is defined on this closed semicircular region by writing $f(0)=0$ and using the branch

$$f(z)=\sqrt{r}\,e^{i\theta/2}\qquad\left(r>0,-\frac{\pi}{2}<\theta<\frac{3\pi}{2}\right)$$

of the multiple-valued function $z^{1/2}$. The problem here is to evaluate the integral of $f(z)$ around $C$ by evaluating the integrals along the individual paths $C_1$, $C_2$, and $C_3$ and then adding the results. In each case, we write a parametric representation for the path (or a related one) and then use it to evaluate the integral along the particular path.

*(i)* $C_1$: $z=re^{i0}$ $(0\le r\le 1)$. Then

$$\int_{C_1} f(z)dz=\int_0^1\sqrt{r}\cdot 1dr=\left[\frac{2}{3}r^{3/2}\right]_0^1=\frac{2}{3}.$$

*(ii)* $C_2$: $z=1\cdot e^{i\theta}$ $(0\le\theta\le\pi)$. Then

$$\int_{C_2} f(z)dz=\int_0^\pi e^{i\theta/2}\cdot ie^{i\theta}d\theta=i\int_0^\pi e^{i3\theta/2}d\theta=i\left[\frac{2}{3i}e^{i3\theta/2}\right]_0^\pi=\frac{2}{3}(-i-1)=-\frac{2}{3}(1+i).$$

*(iii)* $-C_3$: $z=re^{i\pi}$ $(0\le r\le 1)$. Then

$$\int_{C_3} f(z)dz=-\int_{-C_3} f(z)dz=-\int_0^1\sqrt{r}\,e^{i\pi/2}(-1)dr=i\int_0^1\sqrt{r}dr=i\left[\frac{2}{3}r^{3/2}\right]_0^1=\frac{2}{3}i.$$

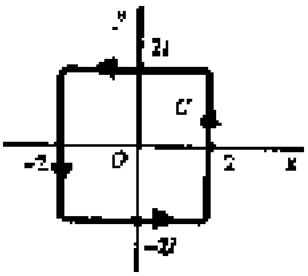
The desired result is

$$\int_C f(z)dz=\int_{C_1} f(z)dz+\int_{C_2} f(z)dz+\int_{C_3} f(z)dz=\frac{2}{3}-\frac{2}{3}(1+i)+\frac{2}{3}i=0.$$

The Cauchy-Goursat theorem does not apply since $f(z)$ is not analytic at the origin, or even defined on the negative imaginary axis.

SECTION 48

1. In this problem, we let $C$ denote the square contour shown in the figure below.



(a) $\int_C \frac{e^{-z}\,dz}{z-(\pi i/2)} = 2\pi i\left[e^{-z}\right]_{z=\pi i/2} = 2\pi i(-i) = 2\pi.$

(b) $\int_C \frac{\cos z}{z(z^2+8)}dz = \int_C \frac{(\cos z)/(z^2+8)}{z-0}dz = 2\pi i\left[\frac{\cos z}{z^2+8}\right]_{z=0} = 2\pi i\left(\frac{1}{8}\right) = \frac{\pi i}{4}.$

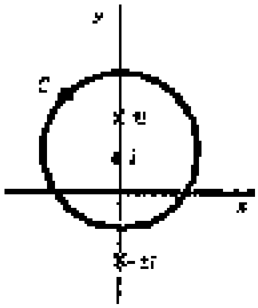
(c) $\int_C \frac{z\,dz}{2z+1} = \int_C \frac{z/2}{z-(-1/2)}dz = 2\pi i\left[\frac{z}{2}\right]_{z=-1/2} = 2\pi i\left(-\frac{1}{4}\right) = -\frac{\pi i}{2}.$

(d) $\int_C \frac{\cosh z}{z^4}dz = \int_C \frac{\cosh z}{(z-0)^{3+1}}dz = \frac{2\pi i}{3!}\left[\frac{d^3}{dz^3}\cosh z\right]_{z=0} = \frac{\pi i}{3}(0) = 0.$

(e) $\int_C \frac{\tan(z/2)}{(z-x_0)^2}dz = \int_C \frac{\tan(z/2)}{(z-x_0)^{1+1}}dz = \frac{2\pi i}{1!}\left[\frac{d}{dz}\tan\left(\frac{z}{2}\right)\right]_{z=x_0}$

$$= 2\pi i\left(\frac{1}{2}\sec^2\frac{x_0}{2}\right) = i\pi\sec^2\left(\frac{x_0}{2}\right) \text{ when } -2 < x_0 < 2.$$

**2.** Let $C$ denote the positively oriented circle $|z-i| = 2$, shown below.




(a) The Cauchy integral formula enables us to write

$$\int_C \frac{dz}{z^2+4} = \int_C \frac{dz}{(z-2i)(z+2i)} = \int_C \frac{1/(z+2i)}{z-2i}dz = 2\pi i\left(\frac{1}{z+2i}\right)_{z=2i} = 2\pi i\left(\frac{1}{4i}\right) = \frac{\pi}{2}.$$

(b) Applying the extended form of the Cauchy integral formula, we have

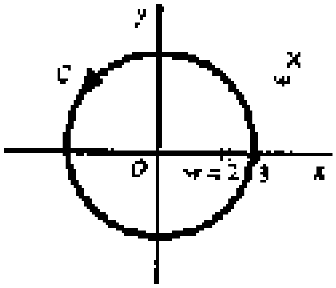
$$\int_C \frac{dz}{(z^2+4)^2} = \int_C \frac{dz}{(z-2i)^2(z+2i)^2} = \int_C \frac{1/(z+2i)^2}{(z-2i)^{1+1}}dz = \frac{2\pi i}{1!}\left[\frac{d}{dz}\frac{1}{(z+2i)^2}\right]_{z=2i}$$

$$= 2\pi i\left[\frac{-2}{(z+2i)^3}\right]_{z=2i} = \frac{-4\pi i}{(4i)^3} = \frac{-4\pi i}{-(16)(4)i} = \frac{\pi}{16}.$$

3. Let $C$ be the positively oriented circle $|z| = 3$, and consider the function

$$g(w) = \int_C \frac{2z^2 - z - 2}{z - w}\,dz \qquad (|w| \neq 3).$$

We wish to find $g(w)$ when $w = 2$ and when $|w| > 3$ (see the figure below).




We observe that

$$g(2) = \int_C \frac{2z^2 - z - 2}{z - 2}\,dz = 2\pi i\left[2z^2 - z - 2\right]_{z=2} = 2\pi i(4) = 8\pi i.$$

On the other hand, when $|w| > 3$, the Cauchy-Goursat theorem tells us that $g(w) = 0$.

5. Suppose that a function $f$ is analytic inside and on a simple closed contour $C$ and that $z_0$ is not on $C$. If $z_0$ is inside $C$, then

$$\int_C \frac{f'(z)\,dz}{z - z_0} = 2\pi i f'(z_0) \quad \text{and} \quad \int_C \frac{f(z)\,dz}{(z - z_0)^2} = \int_C \frac{f(z)\,dz}{(z - z_0)^{1+1}} = \frac{2\pi i}{1!} f'(z_0).$$

Thus

$$\int_C \frac{f'(z)\,dz}{z - z_0} = \int_C \frac{f(z)\,dz}{(z - z_0)^2}.$$

The Cauchy-Goursat theorem tells us that this last equation is also valid when $z_0$ is exterior to $C$, each side of the equation being 0.

7. Let $C$ be the unit circle $z = e^{i\theta}$ $(-\pi \le \theta \le \pi)$, and let $a$ denote any real constant. The Cauchy integral formula reveals that

$$\int_C \frac{e^{az}}{z}\,dz = \int_C \frac{e^{az}}{z - 0}\,dz = 2\pi i\left[e^{az}\right]_{z=0} = 2\pi i.$$

On the other hand, the stated parametric representation for $C$ gives us

$$\int_C \frac{e^{az}}{z}dz = \int_{-\pi}^{\pi} \frac{\exp(ae^{i\theta})}{e^{i\theta}} ie^{i\theta}d\theta = i\int_{-\pi}^{\pi} \exp[a(\cos\theta + i\sin\theta)]d\theta$$

$$= i\int_{-\pi}^{\pi} e^{a\cos\theta}e^{ia\sin\theta}d\theta = i\int_{-\pi}^{\pi} e^{a\cos\theta}[\cos(a\sin\theta) + i\sin(a\sin\theta)]d\theta$$

$$= -\int_{-\pi}^{\pi} e^{a\cos\theta}\sin(a\sin\theta)d\theta - i\int_{-\pi}^{\pi} e^{a\cos\theta}\cos(a\sin\theta)d\theta.$$

Equating these two different expressions for the integral $\int_C \frac{e^{az}}{z}dz$, we have

$$-\int_{-\pi}^{\pi} e^{a\cos\theta}\sin(a\sin\theta)d\theta + i\int_{-\pi}^{\pi} e^{a\cos\theta}\cos(a\sin\theta)d\theta = 2\pi i.$$

Then, by equating the imaginary parts on each side of this last equation, we see that

$$\int_{-\pi}^{\pi} e^{a\cos\theta}\cos(a\sin\theta)d\theta = 2\pi;$$

and, since the integrand here is even,

$$\int_{0}^{\pi} e^{a\cos\theta}\cos(a\sin\theta)d\theta = \pi.$$

8. (*a*) The binomial formula enables us to write

$$P_n(z) = \frac{1}{n!2^n}\frac{d^n}{dz^n}(z^2-1)^n = \frac{1}{n!2^n}\frac{d^n}{dz^n}\sum_{k=0}^{n}\binom{n}{k}z^{2n-2k}(-1)^k.$$

We note that the highest power of $z$ appearing under the derivative is $z^{2n}$, and differentiating it $n$ times brings it down to $z^n$. So $P_n(z)$ is a polynomial of degree $n$.

(*b*) We let $C$ denote any positively oriented simple closed contour surrounding a fixed point $z$. The Cauchy integral formula for derivatives tells us that

$$\frac{d^n}{dz^n}(z^2-1)^n = \frac{n!}{2\pi i}\int_C \frac{(s^2-1)^n}{(s-z)^{n+1}}ds \qquad (n = 0,1,2,\ldots).$$

Hence the polynomials $P_n(z)$ in part (*a*) can be written

$$P_n(z) = \frac{1}{2^{n+1}\pi i}\int_C \frac{(s^2-1)^n}{(s-z)^{n+1}}ds \qquad (n = 0,1,2,\ldots).$$

*(c)* Note that

$$\frac{(s^2-1)^n}{(s-1)^{n+1}} = \frac{(s-1)^n(s+1)^n}{(s-1)^{n+1}} = \frac{(s+1)^n}{s-1}.$$

Referring to the final result in part *(b)*, then, we have

$$P_n(1) = \frac{1}{2^{n+1}\pi i}\int_C \frac{(s^2-1)^n}{(s-1)^{n+1}}\,ds = \frac{1}{2^n}\cdot\frac{1}{2\pi i}\int_C \frac{(s+1)^n}{s-1}\,ds = \frac{1}{2^n}2^n = 1 \qquad (n=0,1,2,\ldots).$$

Also, since

$$\frac{(s^2-1)^n}{(s+1)^{n+1}} = \frac{(s-1)^n(s+1)^n}{(s+1)^{n+1}} = \frac{(s-1)^n}{s+1},$$

we have

$$P_n(-1) = \frac{1}{2^{n+1}\pi i}\int_C \frac{(s^2-1)^n}{(s+1)^{n+1}}\,ds = \frac{1}{2^n}\cdot\frac{1}{2\pi i}\int_C \frac{(s-1)^n}{s+1}\,ds = \frac{1}{2^n}(-2)^n = (-1)^n \quad (n=0,1,2,\ldots).$$

9. We are asked to show that

$$f''(z) = \frac{1}{\pi i}\int_C \frac{f(s)\,ds}{(s-z)^3}.$$

*(a)* In view of the expression for $f'(z)$ in the lemma,

$$\begin{aligned}\frac{f'(z+\Delta z)-f'(z)}{\Delta z} &= \frac{1}{2\pi i}\int_C\left[\frac{1}{(s-z-\Delta z)^2}-\frac{1}{(s-z)^2}\right]\frac{f(s)\,ds}{\Delta z}\\ &= \frac{1}{2\pi i}\int_C \frac{2(s-z)-\Delta z}{(s-z-\Delta z)^2(s-z)^2}f(s)\,ds.\end{aligned}$$

Then

$$\begin{aligned}\frac{f'(z+\Delta z)-f'(z)}{\Delta z} - \frac{1}{\pi i}\int_C\frac{f(s)\,ds}{(s-z)^3} &= \frac{1}{2\pi i}\int_C\left[\frac{2(s-z)-\Delta z}{(s-z-\Delta z)^2(s-z)^2}-\frac{2}{(s-z)^3}\right]f(s)\,ds\\ &= \frac{1}{2\pi i}\int_C\frac{3(s-z)\Delta z-2(\Delta z)^2}{(s-z-\Delta z)^2(s-z)^3}f(s)\,ds.\end{aligned}$$

(*b*) We must show that

$$\left|\int_C \frac{3(s-z)\Delta z - 2(\Delta z)^2}{(s-z-\Delta z)^2(s-z)^3} f(s)\,ds\right| \le \frac{(3D|\Delta z| + 2|\Delta z|^2)M}{(d-|\Delta z|)^2 d^3} L.$$

Now $D$, $d$, $M$, and $L$ are as in the statement of the exercise in the text. The triangle inequality tells us that

$$|3(s-z)\Delta z - 2(\Delta z)^2| \le 3|s-z|\cdot|\Delta z| + 2|\Delta z|^2 \le 3D|\Delta z| + 2|\Delta z|^2.$$

Also, we know from the verification of the expression for $f'(z)$ in the lemma that $|s-z-\Delta z| \ge d-|\Delta z| > 0$; and this means that

$$|(s-z-\Delta z)^2(s-z)^3| \ge (d-|\Delta z|)^2 d^3 > 0.$$

This gives the desired inequality.

(*c*) If we let $\Delta z$ tend to 0 in the inequality obtained in part (*b*) we find that

$$\lim_{\Delta z \to 0} \frac{1}{2\pi i}\int_C \frac{3(s-z)\Delta z - 2(\Delta z)^2}{(s-z-\Delta z)^2(s-z)^3} f(s)\,ds = 0.$$

This, together with the result in part (*a*), yields the desired expression for $f''(z)$.

# Chapter 5

SECTION 52

1. We are asked to show in two ways that the sequence

$$z_n = -2 + i\frac{(-1)^n}{n^2} \qquad (n = 1,2,\ldots)$$

converges to $-2$. One way is to note that the two sequences

$$x_n = -2 \quad \text{and} \quad y_n = \frac{(-1)^n}{n^2} \qquad (n = 1,2,\ldots)$$

of real numbers converge to $-2$ and 0, respectively, and then to apply the theorem in Sec. 51. Another way is to observe that $|z_n - (-2)| = \frac{1}{n^2}$. Thus for each $\varepsilon > 0$,

$$|z_n - (-2)| < \varepsilon \quad \text{whenever} \quad n > n_0,$$

where $n_0$ is any positive integer such that $n_0 \geq \frac{1}{\sqrt{\varepsilon}}$.

2. Observe that if $z_n = -2 + i\frac{(-1)^n}{n^2}$ $(n = 1,2,\ldots)$, then

$$r_n = |z_n| = \sqrt{4 + \frac{1}{n^4}} \to 2.$$

But, since

$$\Theta_{2n} = \operatorname{Arg} z_{2n} \to \pi \quad \text{and} \quad \Theta_{2n-1} = \operatorname{Arg} z_{2n-1} \to -\pi \qquad (n = 1,2,\ldots),$$

the sequence $\Theta_n$ $(n = 1,2,\ldots)$ does not converge.

3. Suppose that $\lim_{n\to\infty} z_n = z$. That is, for each $\varepsilon > 0$, there is a positive integer $n_0$ such that $|z_n - z| < \varepsilon$ whenever $n > n_0$. In view of the inequality (see Sec. 4)

$$|z_n - z| \geq ||z_n| - |z||,$$

it follows that $||z_n| - |z|| < \varepsilon$ whenever $n > n_0$. That is, $\lim_{n\to\infty} |z_n| = |z|$.

**4.** The summation formula found in the example in Sec. 52 can be written

$$\sum_{n=1}^{\infty} z^n = \frac{z}{1-z} \quad \text{when} \quad |z|<1.$$

If we put $z = re^{i\theta}$, where $0 < r < 1$, the left-hand side becomes

$$\sum_{n=1}^{\infty} (re^{i\theta})^n = \sum_{n=1}^{\infty} r^n e^{in\theta} = \sum_{n=1}^{\infty} r^n \cos n\theta + i\sum_{n=1}^{\infty} r^n \sin n\theta,$$

and the right-hand side takes the form

$$\frac{re^{i\theta}}{1-re^{i\theta}} \cdot \frac{1-re^{-i\theta}}{1-re^{-i\theta}} = \frac{re^{i\theta} - r^2}{1 - r(e^{i\theta} + e^{-i\theta}) + r^2} = \frac{r\cos\theta - r^2 + ir\sin\theta}{1-2r\cos\theta + r^2}.$$

Thus

$$\sum_{n=1}^{\infty} r^n \cos n\theta + i\sum_{n=1}^{\infty} r^n \sin n\theta = \frac{r\cos\theta - r^2}{1-2r\cos\theta + r^2} + i\frac{r\sin\theta}{1-2r\cos\theta + r^2}.$$

Equating the real parts on each side here and then the imaginary parts, we arrive at the summation formulas

$$\sum_{n=1}^{\infty} r^n \cos n\theta = \frac{r\cos\theta - r^2}{1-2r\cos\theta + r^2} \quad \text{and} \quad \sum_{n=1}^{\infty} r^n \sin n\theta = \frac{r\sin\theta}{1-2r\cos\theta + r^2},$$

where $0 < r < 1$. These formulas clearly hold when $r = 0$ too.

**6.** Suppose that $\sum_{n=1}^{\infty} z_n = S$. To show that $\sum_{n=1}^{\infty} \overline{z_n} = \overline{S}$, we write $z_n = x_n + iy_n$, $S = X + iY$ and appeal to the theorem in Sec. 52. First of all, we note that

$$\sum_{n=1}^{\infty} x_n = X \quad \text{and} \quad \sum_{n=1}^{\infty} y_n = Y.$$

Then, since $\sum_{n=1}^{\infty} (-y_n) = -Y$, it follows that

$$\sum_{n=1}^{\infty} \overline{z_n} = \sum_{n=1}^{\infty} (x_n - iy_n) = \sum_{n=1}^{\infty} [x_n + i(-y_n)] = X - iY = \overline{S}$$

**8.** **Suppose that** $\sum_{n=1}^{\infty} z_n = S$ and $\sum_{n=1}^{\infty} w_n = T$. In order to use the theorem in Sec. 52, we write

$$z_n = x_n + iy_n, \quad S = X + iY \quad \text{and} \quad w_n = u_n + iv_n, \quad T = U + iV.$$

Now

$$\sum_{n=1}^{\infty} x_n = X, \quad \sum_{n=1}^{\infty} y_n = Y \quad \text{and} \quad \sum_{n=1}^{\infty} u_n = U, \quad \sum_{n=1}^{\infty} v_n = V.$$

Since

$$\sum_{n=1}^{\infty} (x_n + u_n) = X + U \quad \text{and} \quad \sum_{n=1}^{\infty} (y_n + v_n) = Y + V,$$

it follows that

$$\sum_{n=1}^{\infty} [(x_n + u_n) + i(y_n + v_n)] = X + U + i(Y + V).$$

That is,

$$\sum_{n=1}^{\infty} [(x_n + iy_n) + (u_n + iv_n)] = X + iY + (U + iV),$$

or

$$\sum_{n=1}^{\infty} (z_n + w_n) = S + T.$$

SECTION 54

**1.** Replace $z$ by $z^2$ in the known series

$$\cosh z = \sum_{n=0}^{\infty} \frac{z^{2n}}{(2n)!} \qquad (|z| < \infty)$$

to get

$$\cosh(z^2) = \sum_{n=0}^{\infty} \frac{z^{4n}}{(2n)!} \qquad (|z| < \infty).$$

Then, multiplying through this last equation by $z$, we have the desired result:

$$z\cosh(z^2) = \sum_{n=0}^{\infty} \frac{z^{4n+1}}{(2n)!} \qquad (|z| < \infty).$$

**2.** *(b)* Replacing $z$ by $z-1$ in the known expansion

$$e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!} \qquad (|z| < \infty),$$

we have

$$e^{z-1} = \sum_{n=0}^{\infty} \frac{(z-1)^n}{n!} \qquad (|z| < \infty).$$

So

$$e^z = e^{z-1}e = e\sum_{n=0}^{\infty} \frac{(z-1)^n}{n!} \qquad (|z| < \infty).$$

**3.** We want to find the Maclaurin series for the function

$$f(z) = \frac{z}{z^4+9} = \frac{z}{9} \cdot \frac{1}{1+(z^4/9)}.$$

To do this, we first replace $z$ by $-(z^4/9)$ in the known expansion

$$\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n \qquad (|z| < 1),$$

as well as its condition of validity, to get

$$\frac{1}{1+(z^4/9)} = \sum_{n=0}^{\infty} \frac{(-1)^n}{3^{2n}} z^{4n} \qquad (|z| < \sqrt{3}).$$

Then, if we multiply through this last equation by $\frac{z}{9}$, we have the desired expansion:

$$f(z) = \sum_{n=0}^{\infty} \frac{(-1)^n}{3^{2n+2}} z^{4n+1} \qquad (|z| < \sqrt{3}).$$

**6.** Replacing $z$ by $z^2$ in the representation

$$\sin z = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n+1}}{(2n+1)!} \qquad (|z| < \infty),$$

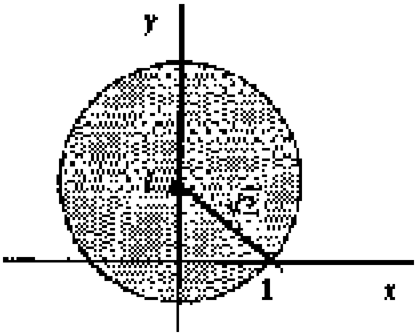
we have

$$\sin(z^2) = \sum_{n=0}^{\infty} (-1)^n \frac{z^{4n+2}}{(2n+1)!} \qquad (|z| < \infty).$$

Since the coefficient of $z^n$ in the Maclaurin series for a function $f(z)$ is $f^{(n)}(0)/n!$, this shows that

$$f^{(4n)}(0) = 0 \quad \text{and} \quad f^{(4n+1)}(0) = 0 \qquad (n = 0, 1, 2, \ldots).$$

7. The function $\dfrac{1}{1-z}$ has a singularity at $z = 1$. So the Taylor series about $z = i$ is valid when $|z - i| < \sqrt{2}$, as indicated in the figure below.




To find the series, we start by writing

$$\frac{1}{1-z} = \frac{1}{(1-i)-(z-i)} = \frac{1}{1-i} \cdot \frac{1}{1-(z-i)/(1-i)}.$$

This suggests that we replace $z$ by $(z-i)/(1-i)$ in the known expansion

$$\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n \qquad (|z| < 1)$$

and then multiply through by $\dfrac{1}{1-i}$. The desired Taylor series is then obtained:

$$\frac{1}{1-z} = \sum_{n=0}^{\infty} \frac{(z-i)^n}{(1-i)^{n+1}} \qquad (|z-i| < \sqrt{2}).$$

9. The identity $\sinh(z - \pi i) = -\sinh z$ and the periodicity of $\sinh z$, with period $2\pi i$, tell us that

$$\sinh z = -\sinh(z + \pi i) = -\sinh(z - \pi i).$$

So, if we replace $z$ by $z - \pi i$ in the known representation

$$\sinh z = \sum_{n=0}^{\infty} \frac{z^{2n+1}}{(2n+1)!} \qquad (|z| < \infty)$$

and then multiply through by $-1$, we find that

$$\sinh z = -\sum_{n=0}^{\infty} \frac{(z-\pi i)^{2n+1}}{(2n+1)!} \qquad (|z-\pi i|<\infty).$$

13. Suppose that $0<|z|<4$. Then $0<|z/4|<1$, and we can use the known expansion

$$\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n \qquad (|z|<1).$$

To be specific, when $0<|z|<4$,

$$\frac{1}{4z-z^2} = \frac{1}{4z}\cdot\frac{1}{1-\frac{z}{4}} = \frac{1}{4z}\sum_{n=0}^{\infty}\left(\frac{z}{4}\right)^n = \sum_{n=0}^{\infty}\frac{z^{n-1}}{4^{n+1}} = \frac{1}{4z}+\sum_{n=1}^{\infty}\frac{z^{n-1}}{4^{n+1}} = \frac{1}{4z}+\sum_{n=0}^{\infty}\frac{z^n}{4^{n+2}}.$$

SECTION 56

1. We may use the expansion

$$\sin z = \sum_{n=0}^{\infty}(-1)^n \frac{z^{2n+1}}{(2n+1)!} \qquad (|z|<\infty)$$

to see that when $0<|z|<\infty$,

$$z^2 \sin\left(\frac{1}{z^2}\right) = \sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}\cdot\frac{1}{z^{4n}} = 1+\sum_{n=1}^{\infty}\frac{(-1)^n}{(2n+1)!}\cdot\frac{1}{z^{4n}}.$$

3. Suppose that $1<|z|<\infty$ and recall the Maclaurin series representation

$$\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n \qquad (|z|<1).$$

This enables us to write

$$\frac{1}{1+z} = \frac{1}{z}\cdot\frac{1}{1+\frac{1}{z}} = \frac{1}{z}\sum_{n=0}^{\infty}\left(-\frac{1}{z}\right)^n = \sum_{n=0}^{\infty}\frac{(-1)^n}{z^{n+1}} \qquad (1<|z|<\infty).$$

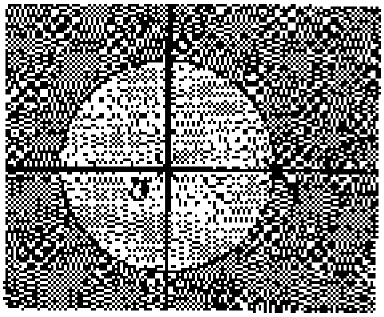
Replacing $n$ by $n-1$ in this last series and then noting that

$$(-1)^{n-1} = (-1)^{n-1}(-1)^2 = (-1)^{n+1},$$

we arrive at the desired expansion:

$$\frac{1}{1+z} = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{z^n} \qquad (1<|z|<\infty).$$

4. The singularities of the function $f(z)=\dfrac{1}{z^2(1-z)}$ are at the points $z=0$ and $z=1$. Hence there are Laurent series in powers of $z$ for the domains $0<|z|<1$ and $1<|z|<\infty$ (see the figure below).



To find the series when $0<|z|<1$, recall that $\dfrac{1}{1-z}=\sum_{n=0}^{\infty} z^n$ $(|z|<1)$ and write

$$f(z)=\frac{1}{z^2}\cdot\frac{1}{1-z}=\frac{1}{z^2}\sum_{n=0}^{\infty} z^n=\sum_{n=0}^{\infty} z^{n-2}=\frac{1}{z^2}+\frac{1}{z}+\sum_{n=2}^{\infty} z^{n-2}=\sum_{n=0}^{\infty} z^n+\frac{1}{z}+\frac{1}{z^2}.$$

As for the domain $1<|z|<\infty$, note that $|1/z|<1$ and write

$$f(z)=-\frac{1}{z^3}\cdot\frac{1}{1-(1/z)}=-\frac{1}{z^3}\sum_{n=0}^{\infty}\left(\frac{1}{z}\right)^n=-\sum_{n=0}^{\infty}\frac{1}{z^{n+3}}=-\sum_{n=3}^{\infty}\frac{1}{z^n}.$$

5. *(a)* The Maclaurin series for the function $\dfrac{z+1}{z-1}$ is valid when $|z|<1$. To find it, we recall the Maclaurin series representation

$$\frac{1}{1-z}=\sum_{n=0}^{\infty} z^n \qquad (|z|<1)$$

for $\dfrac{1}{1-z}$ and write

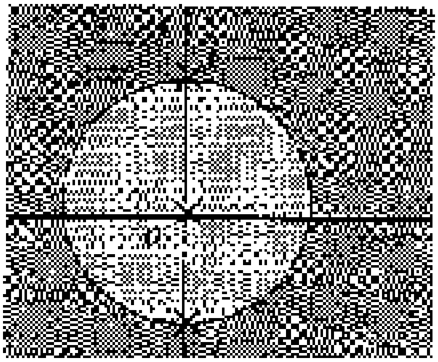
$$\begin{aligned}\frac{z+1}{z-1} &= (z+1)\frac{1}{1-z}=(-z-1)\sum_{n=0}^{\infty} z^n=-\sum_{n=0}^{\infty} z^{n+1}-\sum_{n=0}^{\infty} z^n\\ &=-\sum_{n=1}^{\infty} z^n-\sum_{n=0}^{\infty} z^n=-1-2\sum_{n=1}^{\infty} z^n \qquad (|z|<1).\end{aligned}$$

(b) To find the Laurent series for the same function when $1<|z|<\infty$, we recall the Maclaurin series for $\dfrac{1}{1-z}$ that was used in part (a). Since $\left|\dfrac{1}{z}\right|<1$ here, we may write

$$\frac{z+1}{z-1}=\frac{1+\frac{1}{z}}{1-\frac{1}{z}}=\left(1+\frac{1}{z}\right)\frac{1}{1-\frac{1}{z}}=\left(1+\frac{1}{z}\right)\sum_{n=0}^{\infty}\left(\frac{1}{z}\right)^n=\sum_{n=0}^{\infty}\frac{1}{z^n}+\sum_{n=0}^{\infty}\frac{1}{z^{n+1}}$$

$$=\sum_{n=0}^{\infty}\frac{1}{z^n}+\sum_{n=1}^{\infty}\frac{1}{z^n}=1+2\sum_{n=1}^{\infty}\frac{1}{z^n}\qquad(1<|z|<\infty).$$

7. The function $f(z)=\dfrac{1}{z(1+z^2)}$ has isolated singularities at $z=0$ and $z=\pm i$, as indicated in the figure below. Hence there is a Laurent series representation for the domain $0<|z|<1$ and also one for the domain $1<|z|<\infty$, which is exterior to the circle $|z|=1$.



To find each of these Laurent series, we recall the Maclaurin series representation

$$\frac{1}{1-z}=\sum_{n=0}^{\infty}z^n\qquad(|z|<1).$$

For the domain $0<|z|<1$, we have

$$f(z)=\frac{1}{z}\cdot\frac{1}{1+z^2}=\frac{1}{z}\sum_{n=0}^{\infty}(-z^2)^n=\sum_{n=0}^{\infty}(-1)^n z^{2n-1}=\frac{1}{z}+\sum_{n=1}^{\infty}(-1)^n z^{2n-1}=\sum_{n=0}^{\infty}(-1)^{n+1}z^{2n+1}+\frac{1}{z}.$$

On the other hand, when $1<|z|<\infty$,

$$f(z)=\frac{1}{z^3}\cdot\frac{1}{1+\frac{1}{z^2}}=\frac{1}{z^3}\sum_{n=0}^{\infty}\left(-\frac{1}{z^2}\right)^n=\sum_{n=0}^{\infty}\frac{(-1)^n}{z^{2n+3}}=\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{z^{2n+1}}.$$

In this second expansion, we have used the fact that $(-1)^{n-1}=(-1)^{n-1}(-1)^2=(-1)^{n+1}$.

9. (a) Let $a$ denote a real number, where $-1 < a < 1$. Recalling that

$$\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n \qquad (|z| < 1)$$

enables us to write

$$\frac{a}{z-a} = \frac{a}{z} \cdot \frac{1}{1-(a/z)} = \sum_{n=0}^{\infty} \frac{a^{n+1}}{z^{n+1}},$$

or

$$\frac{a}{z-a} = \sum_{n=1}^{\infty} \frac{a^n}{z^n} \qquad (|a| < |z| < \infty).$$

(b) Putting $z = e^{i\theta}$ on each side of the final result in part (a), we have

$$\frac{a}{e^{i\theta} - a} = \sum_{n=1}^{\infty} a^n e^{-in\theta}.$$

But

$$\frac{a}{e^{i\theta} - a} = \frac{a}{(\cos\theta - a) + i\sin\theta} \cdot \frac{(\cos\theta - a) - i\sin\theta}{(\cos\theta - a) - i\sin\theta} = \frac{a\cos\theta - a^2 - ia\sin\theta}{1 - 2a\cos\theta + a^2}$$

and

$$\sum_{n=1}^{\infty} a^n e^{-in\theta} = \sum_{n=1}^{\infty} a^n \cos n\theta - i\sum_{n=1}^{\infty} a^n \sin n\theta.$$

Consequently,

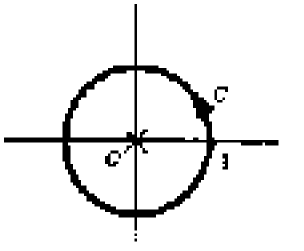
$$\sum_{n=1}^{\infty} a^n \cos n\theta = \frac{a\cos\theta - a^2}{1 - 2a\cos\theta + a^2} \quad \text{and} \quad \sum_{n=1}^{\infty} a^n \sin n\theta = \frac{a\sin\theta}{1 - 2a\cos\theta + a^2}$$

when $-1 < a < 1$.

10. (a) Let $z$ be any fixed complex number and $C$ the unit circle $w = e^{i\phi}$ $(-\pi \le \phi \le \pi)$ in the $w$ plane. The function

$$f(w) = \exp\left[\frac{z}{2}\left(w - \frac{1}{w}\right)\right]$$

has the one singularity $w = 0$ in the $w$ plane. That singularity is, of course, interior to $C$, as shown in the figure below.




Now the function $f(w)$ has a Laurent series representation in the domain $0<|w|<\infty$. According to expression (5), Sec. 55, then,

$$\exp\left[\frac{z}{2}\left(w-\frac{1}{w}\right)\right]=\sum_{n=-\infty}^{\infty} J_n(z)w^n \qquad (0<|w|<\infty),$$

where the coefficients $J_n(z)$ are

$$J_n(z)=\frac{1}{2\pi i}\int_C \frac{\exp\left[\frac{z}{2}\left(w-\frac{1}{w}\right)\right]}{w^{n+1}}dw \qquad (n=0,\pm1,\pm2,\ldots).$$

Using the parametric representation $w=e^{i\phi}$ $(-\pi\le\phi\le\pi)$ for $C$, let us rewrite this expression for $J_n(z)$ as follows:

$$J_n(z)=\frac{1}{2\pi i}\int_{-\pi}^{\pi}\frac{\exp\left[\frac{z}{2}\left(e^{i\phi}-e^{-i\phi}\right)\right]}{e^{i(n+1)\phi}}ie^{i\phi}d\phi=\frac{1}{2\pi}\int_{-\pi}^{\pi}\exp[iz\sin\phi]e^{-in\phi}d\phi.$$

That is,

$$J_n(z)=\frac{1}{2\pi}\int_{-\pi}^{\pi}\exp[-i(n\phi-z\sin\phi)]d\phi \qquad (n=0,\pm1,\pm2,\ldots).$$

*(b)* The last expression for $J_n(z)$ in part *(a)* can be written as

$$J_n(z)=\frac{1}{2\pi}\int_{-\pi}^{\pi}[\cos(n\phi-z\sin\phi)-i\sin(n\phi-z\sin\phi)]d\phi$$

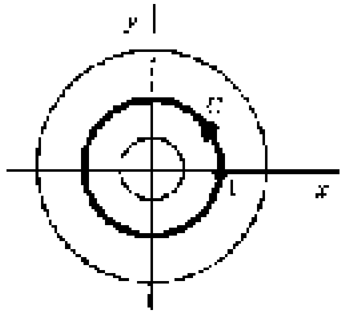
$$=\frac{1}{2\pi}\int_{-\pi}^{\pi}\cos(n\phi-z\sin\phi)d\phi-\frac{i}{2\pi}\int_{-\pi}^{\pi}\sin(n\phi-z\sin\phi)d\phi$$

$$=\frac{1}{2\pi}2\int_{0}^{\pi}\cos(n\phi-z\sin\phi)d\phi-\frac{i}{2\pi}0 \qquad (n=0,\pm1,\pm2,\ldots)$$

That is,

$$J_n(z) = \frac{1}{\pi}\int_0^{\pi} \cos(n\phi - z\sin\phi)\,d\phi \qquad (n = 0, \pm 1, \pm 2, \ldots).$$

11. (a) The function $f(z)$ is analytic in some annular domain centered at the origin; and the unit circle $C: z = e^{i\phi}\ (-\pi \le \phi \le \pi)$ is contained in that domain, as shown below.




For each point $z$ in the annular domain, there is a Laurent series representation

$$f(z) = \sum_{n=0}^{\infty} a_n z^n + \sum_{n=1}^{\infty} \frac{b_n}{z^n},$$

where

$$a_n = \frac{1}{2\pi i}\int_C \frac{f(z)\,dz}{z^{n+1}} = \frac{1}{2\pi i}\int_{-\pi}^{\pi} \frac{f(e^{i\phi})}{e^{i\phi(n+1)}}\, ie^{i\phi}\,d\phi = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(e^{i\phi})e^{-in\phi}\,d\phi \qquad (n = 0, 1, 2, \ldots)$$

and

$$b_n = \frac{1}{2\pi i}\int_C \frac{f(z)\,dz}{z^{-n+1}} = \frac{1}{2\pi i}\int_{-\pi}^{\pi} \frac{f(e^{i\phi})}{e^{i\phi(-n+1)}}\, ie^{i\phi}\,d\phi = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(e^{i\phi})e^{in\phi}\,d\phi \qquad (n = 1, 2, \ldots).$$

Substituting these values of $a_n$ and $b_n$ into the series, we then have

$$f(z) = \sum_{n=0}^{\infty} \frac{1}{2\pi}\int_{-\pi}^{\pi} f(e^{i\phi})e^{-in\phi}\,d\phi\, z^n + \sum_{n=1}^{\infty} \frac{1}{2\pi}\int_{-\pi}^{\pi} f(e^{i\phi})e^{in\phi}\,d\phi\,\frac{1}{z^n},$$

or

$$f(z) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(e^{i\phi})\,d\phi + \frac{1}{2\pi}\sum_{n=1}^{\infty}\int_{-\pi}^{\pi} f(e^{i\phi})\left[\left(\frac{z}{e^{i\phi}}\right)^n + \left(\frac{e^{i\phi}}{z}\right)^n\right]d\phi.$$

*(b)* Put $z = e^{i\theta}$ in the final result in part *(a)* to get

$$f(e^{i\theta}) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(e^{i\phi})\,d\phi + \frac{1}{2\pi}\sum_{n=1}^{\infty}\int_{-\pi}^{\pi} f(e^{i\phi})\left[e^{in(\theta-\phi)} + e^{-in(\theta-\phi)}\right]d\phi,$$

or

$$f(e^{i\theta}) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(e^{i\phi})\,d\phi + \frac{1}{\pi}\sum_{n=1}^{\infty}\int_{-\pi}^{\pi} f(e^{i\phi})\cos[n(\theta-\phi)]\,d\phi.$$

If $u(\theta) = \operatorname{Re} f(e^{i\theta})$, then, equating the real parts on each side of this last equation yields

$$u(\theta) = \frac{1}{2\pi}\int_{-\pi}^{\pi} u(\phi)\,d\phi + \frac{1}{\pi}\sum_{n=1}^{\infty}\int_{-\pi}^{\pi} u(\phi)\cos[n(\theta-\phi)]\,d\phi.$$

SECTION 60

1. Differentiating each side of the representation

$$\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n \qquad (|z|<1),$$

we find that

$$\frac{1}{(1-z)^2} = \frac{d}{dz}\sum_{n=0}^{\infty} z^n = \sum_{n=0}^{\infty}\frac{d}{dz}z^n = \sum_{n=1}^{\infty} nz^{n-1} = \sum_{n=0}^{\infty}(n+1)z^n \qquad (|z|<1).$$

Another differentiation gives

$$\frac{2}{(1-z)^3} = \frac{d}{dz}\sum_{n=0}^{\infty}(n+1)z^n = \sum_{n=0}^{\infty}(n+1)\frac{d}{dz}z^n = \sum_{n=1}^{\infty} n(n+1)z^{n-1} = \sum_{n=0}^{\infty}(n+1)(n+2)z^n \qquad (|z|<1).$$

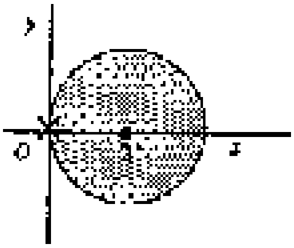
2. Replace $z$ by $1/(1-z)$ on each side of the Maclaurin series representation (Exercise 1)

$$\frac{1}{(1-z)^2} = \sum_{n=0}^{\infty}(n+1)z^n \qquad (|z|<1),$$

as well as in its condition of validity. This yields the Laurent series representation

$$\frac{1}{z^2} = \sum_{n=2}^{\infty}\frac{(-1)^n(n-1)}{(z-1)^n} \qquad (1<|z-1|<\infty).$$

3. Since the function $f(z) = 1/z$ has a singular point at $z = 0$, its Taylor series about $z_0 = 2$ is valid in the open disk $|z-2| < 2$, as indicated in the figure below.




To find that series, write

$$\frac{1}{z} = \frac{1}{2+(z-2)} = \frac{1}{2} \cdot \frac{1}{1+(z-2)/2}$$

to see that it can be obtained by replacing $z$ by $-(z-2)/2$ in the known expansion

$$\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n \qquad (|z| < 1).$$

Specifically,

$$\frac{1}{z} = \frac{1}{2}\sum_{n=0}^{\infty}\left[-\frac{(z-2)}{2}\right]^n \qquad (|z-2| < 2),$$

or

$$\frac{1}{z} = \sum_{n=0}^{\infty}\frac{(-1)^n}{2^{n+1}}(z-2)^n \qquad (|z-2| < 2).$$

Differentiating this series term by term, we have

$$-\frac{1}{z^2} = \sum_{n=1}^{\infty}\frac{(-1)^n}{2^{n+1}}n(z-2)^{n-1} = \sum_{n=0}^{\infty}\frac{(-1)^{n+1}}{2^{n+2}}(n+1)(z-2)^n \qquad (|z-2| < 2).$$

Thus

$$\frac{1}{z^2} = \frac{1}{4}\sum_{n=0}^{\infty}(-1)^n(n+1)\left(\frac{z-2}{2}\right)^n \qquad (|z-2| < 2).$$

4. Consider the function defined by the equations

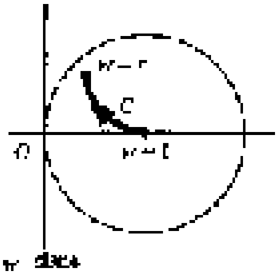
$$f(z) = \begin{cases} \dfrac{e^z - 1}{z} & \text{when } z \neq 0, \\ 1 & \text{when } z = 0. \end{cases}$$

When $z \neq 0$, $f(z)$ has the power series representation

$$f(z) = \frac{1}{z}\left[\left(1 + \frac{z}{1!} + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots\right) - 1\right] = 1 + \frac{z}{2!} + \frac{z^2}{3!} + \cdots.$$

Since this representation clearly holds when $z = 0$ too, it is actually valid for all $z$. Hence $f$ is entire.

**6.** Let $C$ be a contour lying in the open disk $|w - 1| < 1$ in the $w$ plane that extends from the point $w = 1$ to a point $w = z$, as shown in the figure below.




According to Theorem 1 in Sec. 59, we can integrate the Taylor series representation

$$\frac{1}{w} = \sum_{n=0}^{\infty}(-1)^n (w-1)^n \qquad (|w - 1| < 1)$$

term by term along the contour $C$. Thus

$$\int_C \frac{dw}{w} = \int_C \sum_{n=0}^{\infty}(-1)^n (w-1)^n\, dw = \sum_{n=0}^{\infty}(-1)^n \int_C (w-1)^n\, dw.$$

But

$$\int_C \frac{dw}{w} = \int_1^z \frac{dw}{w} = [\operatorname{Log} w]_1^z = \operatorname{Log} z - \operatorname{Log} 1 = \operatorname{Log} z$$

and

$$\int_C (w-1)^n = \int_1^z (w-1)^n\, dw = \left[\frac{(w-1)^{n+1}}{n+1}\right]_1^z = \frac{(z-1)^{n+1}}{n+1}.$$

Hence

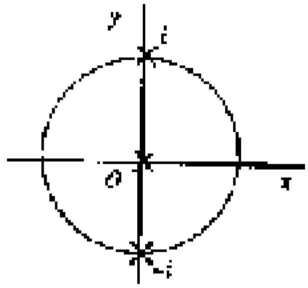
$$\operatorname{Log} z = \sum_{n=0}^{\infty}\frac{(-1)^n}{n+1}(z-1)^{n+1} = \sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}(z-1)^n \qquad (|z - 1| < 1);$$

and, since $(-1)^{n-1} = (-1)^{n-1}(-1)^2 = (-1)^{n+1}$, this result becomes

$$\operatorname{Log} z = \sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}(z-1)^n \qquad (|z - 1| < 1).$$

1. The singularities of the function $f(z)=\dfrac{e^z}{z(z^2+1)}$ are at $z=0,\pm i$. The problem here is to find the Laurent series for $f$ that is valid in the punctured disk $0<|z|<1$, shown below.




We begin by recalling the Maclaurin series representations

$$e^z=1+\frac{z}{1!}+\frac{z^2}{2!}+\frac{z^3}{3!}+\cdots \qquad (|z|<\infty)$$

and

$$\frac{1}{1-z}=1+z+z^2+z^3+\cdots \qquad (|z|<1),$$

which enable us to write

$$e^z=1+z+\frac{1}{2}z^2+\frac{1}{6}z^3+\cdots \qquad (|z|<\infty)$$

and

$$\frac{1}{z^2+1}=1-z^2+z^4-z^6+\cdots \qquad (|z|<1).$$

Multiplying these last two series term by term, we have the Maclaurin series representation

$$\begin{aligned}\frac{e^z}{z^2+1}&=1+z+\frac{1}{2}z^2+\frac{1}{6}z^3+\cdots\\ &\qquad\quad -z^2\quad -z^3\quad -\cdots\\ &\qquad\qquad\qquad\qquad z^4+\cdots\\ &\qquad\qquad\qquad\qquad\quad\vdots\\ &=1+z-\frac{1}{2}z^2-\frac{5}{6}z^3+\cdots,\end{aligned}$$

which is valid when $|z|<1$. The desired Laurent series is then obtained by multiplying each side of the above representation by $\frac{1}{z}$:

$$\frac{e^z}{z(z^2+1)}=\frac{1}{z}+1-\frac{1}{2}z-\frac{5}{6}z^2+\cdots \qquad (0<|z|<1).$$

4. We know the Laurent series representation

$$\frac{1}{z^2 \sinh z} = \frac{1}{z^3} - \frac{1}{6}\cdot\frac{1}{z} + \frac{7}{360}z + \cdots \qquad (0 < |z| < \pi)$$

from Example 2, Sec. 61. Expression (3), Sec. 55, for the coefficients $b_n$ in a Laurent series tells us that the coefficient $b_1$ of $\frac{1}{z}$ in this series can be written

$$b_1 = \frac{1}{2\pi i}\int_C \frac{dz}{z^2 \sinh z},$$

where $C$ is the circle $|z| = 1$, taken counterclockwise. Since $b_1 = -\frac{1}{6}$, then,

$$\int_C \frac{dz}{z^2 \sinh z} = 2\pi i\left(-\frac{1}{6}\right) = -\frac{\pi i}{3}.$$

6. The problem here is to use mathematical induction to verify the differentiation formula

$$[f(z)g(z)]^{(n)} = \sum_{k=0}^{n}\binom{n}{k} f^{(k)}(z)g^{(n-k)}(z) \qquad (n = 1, 2, \ldots).$$

The formula is clearly true when $n = 1$ since in that case it becomes

$$[f(z)g(z)]' = f(z)g'(z) + f'(z)g(z).$$

We now assume that the formula is true when $n = m$ and show how, as a consequence, it is true when $n = m + 1$. We start by writing

$$[f(z)g(z)]^{(m+1)} = \{[f(z)g(z)]'\}^{(m)} = [f(z)g'(z) + f'(z)g(z)]^{(m)}$$

$$= [f(z)g'(z)]^{(m)} + [f'(z)g(z)]^{(m)}$$

$$= \sum_{k=0}^{m}\binom{m}{k} f^{(k)}(z)g^{(m-k+1)}(z) + \sum_{k=0}^{m}\binom{m}{k} f^{(k+1)}(z)g^{(m-k)}(z)$$

$$= \sum_{k=0}^{m}\binom{m}{k} f^{(k)}(z)g^{(m-k+1)}(z) + \sum_{k=1}^{m+1}\binom{m}{k-1} f^{(k)}(z)g^{(m+1-k)}(z)$$

$$= f(z)g^{(m+1)}(z) + \sum_{k=1}^{m}\left[\binom{m}{k} + \binom{m}{k-1}\right] f^{(k)}(z)g^{(m+1-k)}(z) + f^{(m+1)}(z)g(z).$$

But

$$\binom{m}{k}+\binom{m}{k-1}=\frac{m!}{k!(m-k)!}+\frac{m!}{(k-1)!(m-k+1)!}=\frac{(m+1)!}{k!(m+1-k)!}=\binom{m+1}{k},$$

and so

$$[f(z)g(z)]^{(m+1)}=f(z)g^{(m+1)}(z)+\sum_{k=1}^{m}\binom{m+1}{k}f^{(k)}(z)g^{(m+1-k)}(z)+f^{(m+1)}(z)g(z),$$

or

$$[f(z)g(z)]^{(m+1)}=\sum_{k=0}^{m+1}\binom{m+1}{k}f^{(k)}(z)g^{(m+1-k)}(z).$$

The desired verification is now complete.

7. We are given that $f(z)$ is an entire function represented by a series of the form

$$f(z)=z+a_2z^2+a_3z^3+\cdots \qquad (|z|<\infty).$$

(a) Write $g(z)=f[f(z)]$ and observe that

$$f[f(z)]=g(0)+\frac{g'(0)}{1!}z+\frac{g''(0)}{2!}z^2+\frac{g'''(0)}{3!}z^3+\cdots \qquad (|z|<\infty).$$

It is straightforward to show that

$$g'(z)=f'[f(z)]f'(z),$$

$$g''(z)=f''[f(z)][f'(z)]^2+f'[f(z)]f''(z),$$

and

$$g'''(z)=f'''[f(z)][f'(z)]^3+2f'(z)f''(z)f''[f(z)]+f''[f(z)]f'(z)f''(z)+f'[f(z)]f'''(z).$$

Thus

$$g(0)=0,\quad g'(0)=1,\quad g''(0)=4a_2,\quad \text{and}\quad g'''(0)=12(a_2^2+a_3),$$

and so

$$f[f(z)]=z+2a_2z^2+2(a_2^2+a_3)z^3+\cdots \qquad (|z|<\infty).$$

(b) Proceeding formally, we have

$$f[f(z)] = f(z) + a_2[f(z)]^2 + a_3[f(z)]^3 + \cdots$$

$$= (z + a_2z^2 + a_3z^3 + \cdots) + a_2(z + a_2z^2 + a_3z^3 + \cdots)^2 + a_3(z + a_2z^2 + a_3z^3 + \cdots)^3 + \cdots$$

$$= (z + a_2z^2 + a_3z^3 + \cdots) + (a_2z^2 + 2a_2^2z^3 + \cdots) + (a_3z^3 + \cdots)$$

$$= z + 2a_2z^2 + 2(a_2^2 + a_3)z^3 + \cdots.$$

(c) Since

$$\sin z = z - \frac{z^3}{3!} + \cdots = z + 0z^2 + \left(-\frac{1}{6}\right)z^3 + \cdots \qquad (|z| < \infty),$$

the result in part (a), with $a_2 = 0$ and $a_3 = -\frac{1}{6}$, tells us that

$$\sin(\sin z) = z - \frac{1}{3}z^3 + \cdots \qquad (|z| < \infty).$$

8. We need to find the first four nonzero coefficients in the Maclaurin series representation

$$\frac{1}{\cosh z} = \sum_{n=0}^{\infty} \frac{E_n}{n!} z^n \qquad \left(|z| < \frac{\pi}{2}\right).$$

This representation is valid in the stated disk since the zeros of $\cosh z$ are the numbers $z = \left(\frac{\pi}{2} + n\pi\right)i$ $(n = 0, \pm 1, \pm 2, \ldots)$, the ones nearest to the origin being $z = \pm\frac{\pi}{2}i$. The series contains only even powers of $z$ since $\cosh z$ is an even function, that is, $E_{2n+1} = 0$ $(n = 0, 1, 2, \ldots)$. To find the series, we divide the series

$$\cosh z = 1 + \frac{z^2}{2!} + \frac{z^4}{4!} + \frac{z^6}{6!} + \cdots = 1 + \frac{1}{2}z^2 + \frac{1}{24}z^4 + \frac{1}{720}z^6 + \cdots \qquad (|z| < \infty)$$

into 1. The result is

$$\frac{1}{\cosh z} = 1 - \frac{1}{2}z^2 + \frac{5}{24}z^4 - \frac{61}{720}z^6 + \cdots \qquad \left(|z| < \frac{\pi}{2}\right),$$

or

$$\frac{1}{\cosh z} = 1 - \frac{1}{2!}z^2 + \frac{5}{4!}z^4 - \frac{61}{6!}z^6 + \cdots \qquad \left(|z| < \frac{\pi}{2}\right).$$

Since

$$\frac{1}{\cosh z} = E_0 + \frac{E_2}{2!}z^2 + \frac{E_4}{4!}z^4 + \frac{E_6}{6!}z^6 + \cdots \qquad \left(|z| < \frac{\pi}{2}\right),$$

this tells us that

$$E_0 = 1, \quad E_2 = -1, \quad E_4 = 5, \quad \text{and} \quad E_6 = -61.$$

# Chapter 6

## SECTION 64

1. *(a)* Let us write

$$\frac{1}{z+z^2}=\frac{1}{z}\cdot\frac{1}{1+z}=\frac{1}{z}(1-z+z^2-z^3+\cdots)=\frac{1}{z}-1+z-z^2+\cdots \qquad (0<|z|<1).$$

The residue at $z=0$, which is the coefficient of $\frac{1}{z}$, is clearly 1.

*(b)* We may use the expansion

$$\cos z=1-\frac{z^2}{2!}+\frac{z^4}{4!}-\frac{z^6}{6!}+\cdots \qquad (|z|<\infty)$$

to write

$$z\cos\left(\frac{1}{z}\right)=z\left(1-\frac{1}{2!}\cdot\frac{1}{z^2}+\frac{1}{4!}\cdot\frac{1}{z^4}-\frac{1}{6!}\cdot\frac{1}{z^6}+\cdots\right)=z-\frac{1}{2}\cdot\frac{1}{z}+\frac{1}{4!}\cdot\frac{1}{z^3}-\frac{1}{6!}\cdot\frac{1}{z^5}+\cdots$$

$$(0<|z|<\infty).$$

The residue at $z=0$, or coefficient of $\frac{1}{z}$, is now seen to be $-\frac{1}{2}$.

*(c)* Observe that

$$\frac{z-\sin z}{z}=\frac{1}{z}(z-\sin z)=\frac{1}{z}\left[z-\left(z-\frac{z^3}{3!}+\frac{z^5}{5!}-\cdots\right)\right]=\frac{z^2}{3!}-\frac{z^4}{5!}+\cdots \qquad (0<|z|<\infty).$$

Since the coefficient of $\frac{1}{z}$ in this Laurent series is 0, the residue at $z=0$ is 0.

*(d)* Write

$$\frac{\cot z}{z^4}=\frac{1}{z^4}\cdot\frac{\cos z}{\sin z}$$

and recall that

$$\cos z=1-\frac{z^2}{2!}+\frac{z^4}{4!}-\cdots=1-\frac{z^2}{2}+\frac{z^4}{24}-\cdots \qquad (|z|<\infty)$$

and

$$\sin z=z-\frac{z^3}{3!}+\frac{z^5}{5!}-\cdots=z-\frac{z^3}{6}+\frac{z^5}{120}-\cdots \qquad (|z|<\infty).$$

Dividing the series for $\sin z$ into the one for $\cos z$, we find that

$$\frac{\cos z}{\sin z} = \frac{1}{z} - \frac{z}{3} - \frac{z^3}{45} + \cdots \qquad (0 < |z| < \pi).$$

Thus

$$\frac{\cot z}{z^4} = \frac{1}{z^4}\left(\frac{1}{z} - \frac{z}{3} - \frac{z^3}{45} + \cdots\right) = \frac{1}{z^5} - \frac{1}{3}\cdot\frac{1}{z^3} - \frac{1}{45}\cdot\frac{1}{z} + \cdots \qquad (0 < |z| < \pi).$$

Note that the condition of validity for this series is due to the fact that $\sin z = 0$ when $z = n\pi$ $(n = 0, \pm 1, \pm 2, \ldots)$. It is now evident that $\frac{\cot z}{z^4}$ has residue $-\frac{1}{45}$ at $z = 0$.

(e) Recall that

$$\sinh z = z + \frac{z^3}{3!} + \frac{z^5}{5!} + \cdots \qquad (|z| < \infty)$$

and

$$\frac{1}{1-z^2} = 1 + z^2 + z^4 + \cdots \qquad (|z| < 1).$$

There is a Laurent series for the function

$$\frac{\sinh z}{z^4(1-z^2)} = \frac{1}{z^4}\cdot(\sinh z)\left(\frac{1}{1-z^2}\right)$$

that is valid for $0 < |z| < 1$. To find it, we first multiply the Maclaurin series for $\sinh z$ and $\frac{1}{1-z^2}$:

$$\begin{aligned}(\sinh z)\left(\frac{1}{1-z^2}\right) &= \left(z + \frac{1}{6}z^3 + \frac{1}{120}z^5 + \cdots\right)(1 + z^2 + z^4 + \cdots) \\ &= z + \frac{1}{6}z^3 + \frac{1}{120}z^5 + \cdots \\ &\qquad\quad z^3 + \frac{1}{6}z^5 + \cdots \\ &\qquad\qquad\qquad z^5 + \cdots \\ &= z + \frac{7}{6}z^3 + \cdots \qquad (0 < |z| < 1).\end{aligned}$$

We then see that

$$\frac{\sinh z}{z^4(1-z^2)} = \frac{1}{z^3} + \frac{7}{6}\cdot\frac{1}{z} + \cdots \qquad (0<|z|<1).$$

This shows that the residue of $\dfrac{\sinh z}{z^4(1-z^2)}$ at $z=0$ is $\dfrac{7}{6}$.

**2.** In each part, $C$ denotes the positively oriented circle $|z|=3$.

*(a)* To evaluate $\int_C \dfrac{\exp(-z)}{z^2}\,dz$, we need the residue of the integrand at $z=0$. From the Laurent series

$$\frac{\exp(-z)}{z^2} = \frac{1}{z^2}\left(1-\frac{z}{1!}+\frac{z^2}{2!}-\frac{z^3}{3!}+\cdots\right) = \frac{1}{z^2}-\frac{1}{1!}\cdot\frac{1}{z}+\frac{1}{2!}-\frac{z}{3!}+\cdots \qquad (0<|z|<\infty),$$

we see that the required residue is $-1$. Thus

$$\int_C \frac{\exp(-z)}{z^2}\,dz = 2\pi i(-1) = -2\pi i.$$

*(c)* Likewise, to evaluate the integral $\int_C z^2\exp\left(\dfrac{1}{z}\right)dz$, we must find the residue of the integrand at $z=0$. The Laurent series

$$z^2\exp\left(\frac{1}{z}\right) = z^2\left(1+\frac{1}{1!}\cdot\frac{1}{z}+\frac{1}{2!}\cdot\frac{1}{z^2}+\frac{1}{3!}\cdot\frac{1}{z^3}+\frac{1}{4!}\cdot\frac{1}{z^4}+\cdots\right)$$

$$= z^2+\frac{z}{1!}+\frac{1}{2!}+\frac{1}{3!}\cdot\frac{1}{z}+\frac{1}{4!}\cdot\frac{1}{z^2}+\cdots,$$

which is valid for $0<|z|<\infty$, tells us that the needed residue is $\dfrac{1}{6}$. Hence

$$\int_C z^2\exp\left(\frac{1}{z}\right)dz = 2\pi i\left(\frac{1}{6}\right) = \frac{\pi i}{3}.$$

*(d)* As for the integral $\int_C \frac{z+1}{z^2-2z}\,dz$, we need the two residues of

$$\frac{z+1}{z^2-2z}=\frac{z+1}{z(z-2)},$$

one at $z=0$ and one at $z=2$. The residue at $z=0$ can be found by writing

$$\frac{z+1}{z(z-2)}=\left(\frac{z+1}{z}\right)\left(\frac{1}{z-2}\right)=\left(-\frac{1}{2}\right)\left(1+\frac{1}{z}\right)\cdot\frac{1}{1-(z/2)}$$

$$=\left(-\frac{1}{2}-\frac{1}{2}\cdot\frac{1}{z}\right)\left(1+\frac{z}{2}+\frac{z^2}{2^2}+\cdots\right),$$

which is valid when $0<|z|<2$, and observing that the coefficient of $\frac{1}{z}$ in this last product is $-\frac{1}{2}$. To obtain the residue at $z=2$, we write

$$\frac{z+1}{z(z-2)}=\frac{(z-2)+3}{z-2}\cdot\frac{1}{2+(z-2)}=\frac{1}{2}\left(1+\frac{3}{z-2}\right)\cdot\frac{1}{1+(z-2)/2}$$

$$=\frac{1}{2}\left(1+\frac{3}{z-2}\right)\left[1-\frac{z-2}{2}+\frac{(z-2)^2}{2^2}-\cdots\right],$$

which is valid when $0<|z-2|<2$, and note that the coefficient of $\frac{1}{z-2}$ in this product is $\frac{3}{2}$. Finally, then, by the residue theorem,

$$\int_C \frac{z+1}{z^2-2z}\,dz=2\pi i\left(-\frac{1}{2}+\frac{3}{2}\right)=2\pi i.$$

**3.** In each part of this problem, $C$ is the positively oriented circle $|z|=2$.

*(a)* If $f(z)=\frac{z^5}{1-z^3}$, then

$$\frac{1}{z^2}f\left(\frac{1}{z}\right)=\frac{1}{z^7}\cdot\frac{1}{z^3-1}\cdot z^3\cdot\ldots=-\frac{1}{z^4}\cdot\frac{1}{1-z^3}=-\frac{1}{z^4}(1+z^3+z^6+\cdots)=-\frac{1}{z^4}-\frac{1}{z}-z^2-\cdots$$

when $0<|z|<1$. This tells us that

$$\int_C f(z)\,dz=2\pi i\operatorname*{Res}_{z=0}\frac{1}{z^2}f\left(\frac{1}{z}\right)=2\pi i(-1)=-2\pi i.$$

*(b)* When $f(z)=\dfrac{1}{1+z^2}$, we have

$$\frac{1}{z^2}f\left(\frac{1}{z}\right)=\frac{1}{1+z^2}=\frac{1}{1-(-z^2)}=1-z^2+z^4-\cdots \qquad (0<|z|<1).$$

Thus

$$\int_C f(z)\,dz=2\pi i\operatorname*{Res}_{z=0}\frac{1}{z^2}f\left(\frac{1}{z}\right)=2\pi i(0)=0.$$

*(c)* If $f(z)=\dfrac{1}{z}$, it follows that $\dfrac{1}{z^2}f\left(\dfrac{1}{z}\right)=\dfrac{1}{z}$. Evidently, then,

$$\int_C f(z)\,dz=2\pi i\operatorname*{Res}_{z=0}\frac{1}{z^2}f\left(\frac{1}{z}\right)=2\pi i(1)=2\pi i.$$

**4.** Let $C$ denote the circle $|z|=1$, taken counterclockwise.

*(a)* The Maclaurin series $e^z=\displaystyle\sum_{n=0}^{\infty}\frac{z^n}{n!}$ $(|z|<\infty)$ enables us to write

$$\int_C \exp\left(z+\frac{1}{z}\right)dz=\int_C e^z e^{1/z}\,dz=\int_C e^{1/z}\sum_{n=0}^{\infty}\frac{z^n}{n!}\,dz=\sum_{n=0}^{\infty}\frac{1}{n!}\int_C z^n\exp\left(\frac{1}{z}\right)dz.$$

*(b)* Referring to the Maclaurin series for $e^z$ once again, let us write

$$z^n\exp\left(\frac{1}{z}\right)=z^n\sum_{k=0}^{\infty}\frac{1}{k!}\cdot\frac{1}{z^k}=\sum_{k=0}^{\infty}\frac{1}{k!}z^{n-k} \qquad (n=0,1,2,\ldots).$$

Now the $\dfrac{1}{z}$ in this series occurs when $n-k=-1$, or $k=n+1$. So, by the residue theorem,

$$\int_C z^n\exp\left(\frac{1}{z}\right)dz=2\pi i\frac{1}{(n+1)!} \qquad (n=0,1,2,\ldots).$$

The final result in part *(a)* thus reduces to

$$\int_C \exp\left(z+\frac{1}{z}\right)dz=2\pi i\sum_{n=0}^{\infty}\frac{1}{n!(n+1)!}.$$

5. We are given two polynomials

$$P(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_n z^n \qquad (a_n \neq 0)$$

and

$$Q(z) = b_0 + b_1 z + b_2 z^2 + \cdots + b_m z^m \qquad (b_m \neq 0),$$

where $m \geq n + 2$.

It is straightforward to show that

$$\frac{1}{z^2} \cdot \frac{P(1/z)}{Q(1/z)} = \frac{a_0 z^{m-2} + a_1 z^{m-3} + a_2 z^{m-4} + \cdots + a_n z^{m-n-2}}{b_0 z^m + b_1 z^{m-1} + b_2 z^{m-2} + \cdots + b_m} \qquad (z \neq 0).$$

Observe that the numerator here is, in fact, a polynomial since $m - n - 2 \geq 0$. Also, since $b_m \neq 0$, the quotient of these polynomials is represented by a series of the form $d_0 + d_1 z + d_2 z^2 + \cdots$. That is,

$$\frac{1}{z^2} \cdot \frac{P(1/z)}{Q(1/z)} = d_0 + d_1 z + d_2 z^2 + \cdots \qquad (0 < |z| < R_0);$$

and we see that $\dfrac{1}{z^2} \cdot \dfrac{P(1/z)}{Q(1/z)}$ has residue 0 at $z = 0$.

Suppose now that all of the zeros of $Q(z)$ lie inside a simple closed contour $C$, and assume that $C$ is positively oriented. Since $P(z)/Q(z)$ is analytic everywhere in the finite plane except at the zeros of $Q(z)$, it follows from the theorem in Sec. 64 and the residue just obtained that

$$\int_C \frac{P(z)}{Q(z)}\, dz = 2\pi i \operatorname*{Res}_{z=0}\left[\frac{1}{z^2} \cdot \frac{P(1/z)}{Q(1/z)}\right] = 2\pi i \cdot 0 = 0.$$

If $C$ is negatively oriented, this result is still true since then

$$\int_C \frac{P(z)}{Q(z)}\, dz = -\int_{-C} \frac{P(z)}{Q(z)}\, dz = 0.$$

SECTION 65

1. *(a)* From the expansion

$$e^z = 1 + \frac{z}{1!} + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots \qquad (|z| < \infty),$$

we see that

$$z \exp\left(\frac{1}{z}\right) = z + 1 + \frac{1}{2!} \cdot \frac{1}{z} + \frac{1}{3!} \cdot \frac{1}{z^2} + \cdots \qquad (0 < |z| < \infty).$$

The principal part of $z\exp\left(\frac{1}{z}\right)$ at the isolated singular point $z=0$ is, then,

$$\frac{1}{2!}\cdot\frac{1}{z}+\frac{1}{3!}\cdot\frac{1}{z^2}+\cdots;$$

and $z=0$ is an essential singular point of that function.

*(b)* The isolated singular point of $\frac{z^2}{1+z}$ is at $z=-1$. Since the principal part at $z=-1$ involves powers of $z+1$, we begin by observing that

$$z^2=(z+1)^2-2z-1=(z+1)^2-2(z+1)+1.$$

This enables us to write

$$\frac{z^2}{1+z}=\frac{(z+1)^2-2(z+1)+1}{z+1}=(z+1)-2+\frac{1}{z+1}.$$

Since the principal part is $\frac{1}{z+1}$, the point $z=-1$ is a (simple) pole.

*(c)* The point $z=0$ is the isolated singular point of $\frac{\sin z}{z}$, and we can write

$$\frac{\sin z}{z}=\frac{1}{z}\left(z-\frac{z^3}{3!}+\frac{z^5}{5!}-\cdots\right)=1-\frac{z^2}{3!}+\frac{z^4}{5!}-\cdots \qquad (0<|z|<\infty).$$

The principal part here is evidently 0, and so $z=0$ is a removable singular point of the function $\frac{\sin z}{z}$.

*(d)* The isolated singular point of $\frac{\cos z}{z}$ is $z=0$. Since

$$\frac{\cos z}{z}=\frac{1}{z}\left(1-\frac{z^2}{2!}+\frac{z^4}{4!}-\cdots\right)=\frac{1}{z}-\frac{z}{2!}+\frac{z^3}{4!}-\cdots \qquad (0<|z|<\infty),$$

the principal part is $\frac{1}{z}$. This means that $z=0$ is a (simple) pole of $\frac{\cos z}{z}$.

*(e)* Upon writing $\frac{1}{(2-z)^3}=\frac{-1}{(z-2)^3}$, we find that the principal part of $\frac{1}{(2-z)^3}$ at its isolated singular point $z=2$ is simply the function itself. That point is evidently a pole (of order 3).

**2.** *(a)* The singular point is $z = 0$. Since

$$\frac{1-\cosh z}{z^3} = \frac{1}{z^3}\left[1 - \left(1 + \frac{z^2}{2!} + \frac{z^4}{4!} + \frac{z^6}{6!} + \cdots\right)\right] = -\frac{1}{2!}\cdot\frac{1}{z} - \frac{z}{4!} - \frac{z^3}{6!} - \cdots$$

when $0 < |z| < \infty$, we have $m = 1$ and $B = -\frac{1}{2!} = -\frac{1}{2}$.

*(b)* Here the singular point is also $z = 0$. Since

$$\frac{1-\exp(2z)}{z^4} = \frac{1}{z^4}\left[1 - \left(1 + \frac{2z}{1!} + \frac{2^2z^2}{2!} + \frac{2^3z^3}{3!} + \frac{2^4z^4}{4!} + \frac{2^5z^5}{5!} + \cdots\right)\right]$$

$$= -\frac{2}{1!}\cdot\frac{1}{z^3} - \frac{2^2}{2!}\cdot\frac{1}{z^2} - \frac{2^3}{3!}\cdot\frac{1}{z} - \frac{2^4}{4!} - \frac{2^5}{5!}z - \cdots$$

when $0 < |z| < \infty$, we have $m = 3$ and $B = -\frac{2^3}{3!} = -\frac{4}{3}$.

*(c)* The singular point of $\frac{\exp(2z)}{(z-1)^2}$ is $z = 1$. The Taylor series

$$\exp(2z) = e^{2(z-1)}e^2 = e^2\left[1 + \frac{2(z-1)}{1!} + \frac{2^2(z-1)^2}{2!} + \frac{2^3(z-1)^3}{3!} + \cdots\right] \qquad (|z| < \infty)$$

enables us to write the Laurent series

$$\frac{\exp(2z)}{(z-1)^2} = e^2\left[\frac{1}{(z-1)^2} + \frac{2}{1!}\cdot\frac{1}{z-1} + \frac{2^2}{2!} + \frac{2^3}{3!}(z-1) + \cdots\right] \qquad (0 < |z-1| < \infty).$$

Thus $m = 2$ and $B = e^2\frac{2}{1!} = 2e^2$.

**3.** Since $f$ is analytic at $z_0$, it has a Taylor series representation

$$f(z) = f(z_0) + \frac{f'(z_0)}{1!}(z-z_0) + \frac{f''(z_0)}{2!}(z-z_0)^2 + \cdots \qquad (|z-z_0| < R_0).$$

Let $g$ be defined by means of the equation

$$g(z) = \frac{f(z)}{z-z_0}.$$

(*a*) Suppose that $f(z_0) \neq 0$. Then

$$g(z) = \frac{1}{z - z_0}\left[f(z_0) + \frac{f'(z_0)}{1!}(z - z_0) + \frac{f''(z_0)}{2!}(z - z_0)^2 + \cdots\right]$$

$$= \frac{f(z_0)}{z - z_0} + \frac{f'(z_0)}{1!} + \frac{f''(z_0)}{2!}(z - z_0) + \cdots \qquad (0 < |z - z_0| < R_0).$$

This shows that $g$ has a simple pole at $z_0$, with residue $f(z_0)$.

(*b*) Suppose, on the other hand, that $f(z_0) = 0$. Then

$$g(z) = \frac{1}{z - z_0}\left[\frac{f'(z_0)}{1!}(z - z_0) + \frac{f''(z_0)}{2!}(z - z_0)^2 + \cdots\right]$$

$$= \frac{f'(z_0)}{1!} + \frac{f''(z_0)}{2!}(z - z_0) + \cdots \qquad (0 < |z - z_0| < R_0).$$

Since the principal part of $g$ at $z_0$ is just 0, the point $z = 0$ is a removable singular point of $g$.

4. Write the function

$$f(z) = \frac{8a^3 z^2}{(z^2 + a^2)^3} \qquad (a > 0)$$

as

$$f(z) = \frac{\phi(z)}{(z - ai)^3} \quad \text{where} \quad \phi(z) = \frac{8a^3 z^2}{(z + ai)^3}.$$

Since the only singularity of $\phi(z)$ is at $z = -ai$, $\phi(z)$ has a Taylor series representation

$$\phi(z) = \phi(ai) + \frac{\phi'(ai)}{1!}(z - ai) + \frac{\phi''(ai)}{2!}(z - ai)^2 + \cdots \qquad (|z - ai| < 2a)$$

about $z = ai$. Thus

$$f(z) = \frac{1}{(z - ai)^3}\left[\phi(ai) + \frac{\phi'(ai)}{1!}(z - ai) + \frac{\phi''(ai)}{2!}(z - ai)^2 + \cdots\right] \qquad (0 < |z - ai| < 2a).$$

Now straightforward differentiation reveals that

$$\phi'(z) = \frac{16a^4 iz - 8a^3 z^2}{(z + ai)^4} \quad \text{and} \quad \phi''(z) = \frac{16a^3(z^2 - 4aiz - a^2)}{(z + ai)^5}.$$

Consequently,

$$\phi(ai) = -a^2 i, \quad \phi'(ai) = -\frac{a}{2}, \quad \text{and} \quad \phi''(ai) = -i.$$

This enables us to write

$$f(z) = \frac{1}{(z-ai)^3}\left[-a^2 i - \frac{a}{2}(z-ai) - \frac{i}{2}(z-ai)^2 + \cdots\right] \qquad (0 < |z-ai| < 2a).$$

The principal part of $f$ at the point $z = ai$ is, then,

$$-\frac{i/2}{z-ai} - \frac{a/2}{(z-ai)^2} - \frac{a^2 i}{(z-ai)^3}.$$

SECTION 67

1. *(a)* The function $f(z) = \dfrac{z^2+2}{z-1}$ has an isolated singular point at $z = 1$. Writing $f(z) = \dfrac{\phi(z)}{z-1}$, where $\phi(z) = z^2 + 2$, and observing that $\phi(z)$ is analytic and nonzero at $z = 1$, we see that $z = 1$ is a pole of order $m = 1$ and that the residue there is $B = \phi(1) = 3$.

*(b)* If we write

$$f(z) = \left(\frac{z}{2z+1}\right)^3 = \frac{\phi(z)}{\left[z - \left(-\frac{1}{2}\right)\right]^3}, \quad \text{where} \quad \phi(z) = \frac{z^3}{8},$$

we see that $z = -\dfrac{1}{2}$ is a singular point of $f$. Since $\phi(z)$ is analytic and nonzero at that point, $f$ has a pole of order $m = 3$ there. The residue is

$$B = \frac{\phi''(-1/2)}{2!} = -\frac{3}{16}.$$

*(c)* The function

$$\frac{\exp z}{z^2 + \pi^2} = \frac{\exp z}{(z-\pi i)(z+\pi i)}$$

has poles of order $m = 1$ at the two points $z = \pm\pi i$. The residue at $z = \pi i$ is

$$B_1 = \frac{\exp \pi i}{2\pi i} = \frac{-1}{2\pi i} = \frac{i}{2\pi},$$

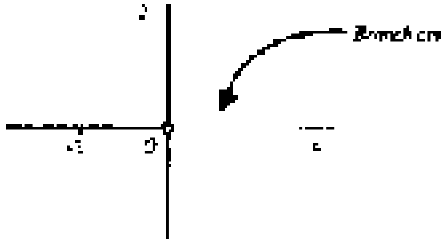
and the one at $z = -\pi i$ is

$$B_2 = \frac{\exp(-\pi i)}{-2\pi i} = \frac{-1}{-2\pi i} = -\frac{i}{2\pi}.$$

**2.** *(a)* Write the function $f(z)=\dfrac{z^{1/4}}{z+1}$ $(|z|>0, 0<\arg z<2\pi)$ as

$$f(z)=\frac{\phi(z)}{z+1}, \quad \text{where} \quad \phi(z)=z^{1/4}=e^{\frac{1}{4}\log z} \; (|z|>0, 0<\arg z<2\pi).$$

The function $\phi(z)$ is analytic throughout its domain of definition, indicated in the figure below




Also,

$$\phi(-1)=(-1)^{1/4}=e^{\frac{1}{4}\log(-1)}=e^{\frac{1}{4}(\ln 1+i\pi)}=e^{i\pi/4}=\cos\frac{\pi}{4}+i\sin\frac{\pi}{4}=\frac{1+i}{\sqrt{2}}\neq 0.$$

This shows that the function $f$ has a pole of order $m=1$ at $z=-1$, the residue there being

$$B=\phi(-1)=\frac{1+i}{\sqrt{2}}.$$

*(b)* Write the function $f(z)=\dfrac{\operatorname{Log} z}{(z^2+1)^2}$ as

$$f(z)=\frac{\phi(z)}{(z-i)^2} \quad \text{where} \quad \phi(z)=\frac{\operatorname{Log} z}{(z+i)^2}.$$

From this, it is clear that $f(z)$ has a pole of order $m=2$ at $z=i$. Straightforward differentiation then reveals that

$$\operatorname*{Res}_{z=i}\frac{\operatorname{Log} z}{(z^2+1)^2}=\phi'(i)=\frac{\pi-2i}{8}.$$

(c) Write the function

$$f(z)=\frac{z^{1/2}}{(z^2+1)^2} \qquad (|z|>0, 0<\arg z<2\pi)$$

as

$$f(z)=\frac{\phi(z)}{(z-i)^2} \quad \text{where} \quad \phi(z)=\frac{z^{1/2}}{(z+i)^2}.$$

Since

$$\phi'(z)=\frac{(z+i)z^{-1/2}-4z^{1/2}}{2(z+i)^3}$$

and

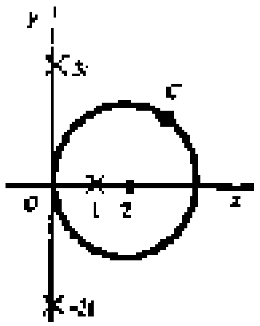
$$i^{-1/2}=e^{-i\pi/4}=\frac{1}{\sqrt{2}}-\frac{i}{\sqrt{2}}, \qquad i^{1/2}=e^{i\pi/4}=\frac{1}{\sqrt{2}}+\frac{i}{\sqrt{2}},$$

$$\operatorname*{Res}_{z=i}\frac{z^{1/2}}{(z^2+1)^2}=\phi'(i)=\frac{1-i}{8\sqrt{2}}.$$

3. (a) We wish to evaluate the integral

$$\int_C \frac{3z^3+2}{(z-1)(z^2+9)}dz,$$

where $C$ is the circle $|z-2|=2$, taken in the counterclockwise direction. That circle and the singularities $z=1, \pm 3i$ of the integrand are shown in the figure just below.



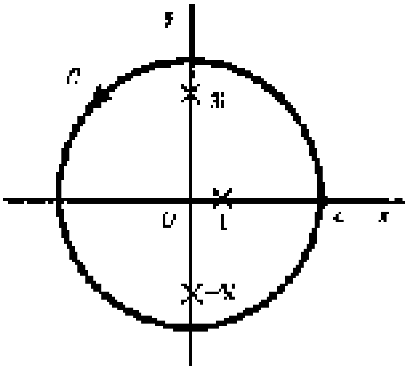

Observe that the point $z=1$, which is the only singularity inside $C$, is a simple pole of the integrand and that

$$\operatorname*{Res}_{z=1}\frac{3z^3+2}{(z-1)(z^2+9)}=\frac{3z^3+2}{z^2+9}\bigg]_{z=1}=\frac{1}{2}.$$

According to the residue theorem, then,

$$\int_C \frac{3z^3+2}{(z-1)(z^2+9)}dz=2\pi i\left(\frac{1}{2}\right)=\pi i.$$

*(b)* Let us redo part *(a)* when $C$ is changed to be the positively oriented circle $|z| = 4$, shown in the figure below.




In this case, all three singularities $z = 1, \pm 3i$ of the integrand are interior to $C$. We already know from part *(a)* that

$$\operatorname*{Res}_{z=1} \frac{3z^3 + 2}{(z-1)(z^2 - 9)} = \frac{1}{2}.$$

It is, moreover, straightforward to show that

$$\operatorname*{Res}_{z=3i} \frac{3z^3 + 2}{(z-1)(z^2 + 9)} = \left. \frac{3z^3 + 2}{(z-1)(z+3i)} \right]_{z=3i} = \frac{15 + 49i}{12}$$

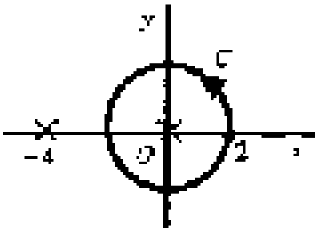
and

$$\operatorname*{Res}_{z=-3i} \frac{3z^3 - 2}{(z-1)(z^2 + 9)} = \left. \frac{3z^3 + 2}{(z-1)(z-3i)} \right]_{z=-3i} = \frac{15 - 49i}{12}.$$

The residue theorem now tells us that

$$\int_C \frac{3z^3 + 2}{(z-1)(z^2+9)} dz = 2\pi i \left( \frac{1}{2} + \frac{15 + 49i}{12} + \frac{15 - 49i}{12} \right) = 6\pi i.$$

4. *(a)* Let $C$ denote the positively oriented circle $|z| = 2$, and note that the integrand of the integral $\int_C \frac{dz}{z^3(z+4)}$ has singularities at $z = 0$ and $z = -4$. (See the figure below.)



To find the residue of the integrand at $z=0$, we recall the expansion

$$\frac{1}{1-z}=\sum_{n=0}^{\infty} z^n \qquad (|z|<1)$$

and write

$$\frac{1}{z^3(z+4)}=\frac{1}{4z^3}\left[\frac{1}{1+(z/4)}\right]=\frac{1}{4z^3}\sum_{n=0}^{\infty}\left(-\frac{z}{4}\right)^n=\sum_{n=0}^{\infty}\frac{(-1)^n}{4^{n+1}}z^{n-3} \qquad (0<|z|<4).$$

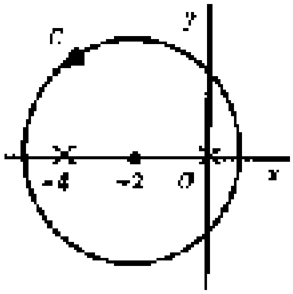
Now the coefficient of $\frac{1}{z}$ here occurs when $n=2$, and we see that

$$\operatorname*{Res}_{z=0}\frac{1}{z^3(z+4)}=\frac{1}{64}.$$

Consequently,

$$\int_C \frac{dz}{z^3(z+4)}=2\pi i\left(\frac{1}{64}\right)=\frac{\pi i}{32}.$$

*(b)* Let us replace the path $C$ in part *(a)* by the positively oriented circle $|z+2|=3$, centered at $-2$ and with radius 3. It is shown below.




We already know from part *(a)* that

$$\operatorname*{Res}_{z=0}\frac{1}{z^3(z+4)}=\frac{1}{64}.$$

To find the residue at $-4$, we write

$$\frac{1}{z^3(z+4)}=\frac{\phi(z)}{z-(-4)}, \quad \text{where} \quad \phi(z)=\frac{1}{z^3}.$$

This tells us that $z=-4$ is a simple pole of the integrand and that the residue there is $\phi(-4)=-1/64$. Consequently,

$$\int_C \frac{dz}{z^3(z+4)}=2\pi i\left(\frac{1}{64}-\frac{1}{64}\right)=0$$

5. Let us evaluate the integral $\int_C \frac{\cosh \pi z \, dz}{z(z^2+1)}$, where $C$ is the positively oriented circle $|z|=2$. All three isolated singularities $z = 0, \pm i$ of the integrand are interior to $C$. The desired residues are

$$\operatorname*{Res}_{z=0} \frac{\cosh \pi z}{z(z^2+1)} = \frac{\cosh \pi z}{z^2+1}\bigg]_{z=0} = 1,$$

$$\operatorname*{Res}_{z=i} \frac{\cosh \pi z}{z(z^2+1)} = \frac{\cosh \pi z}{z(z+i)}\bigg]_{z=i} = \frac{1}{2},$$

and

$$\operatorname*{Res}_{z=-i} \frac{\cosh \pi z}{z(z^2+1)} = \frac{\cosh \pi z}{z(z-i)}\bigg]_{z=-i} = \frac{1}{2}.$$

Consequently,

$$\int_C \frac{\cosh \pi z \, dz}{z(z^2+1)} = 2\pi i\left(1 + \frac{1}{2} + \frac{1}{2}\right) = 4\pi i.$$

6. In each part of this problem, $C$ denotes the positively oriented circle $|z|=3$.

*(a)* It is straightforward to show that

$$\text{if } f(z) = \frac{(3z+2)^2}{z(z-1)(2z+5)}, \quad \text{then} \quad \frac{1}{z^2} f\left(\frac{1}{z}\right) = \frac{(3+2z)^2}{z(1-z)(2+5z)}.$$

Thus function $\frac{1}{z^2} f\left(\frac{1}{z}\right)$ has a simple pole at $z = 0$, and

$$\int_C \frac{(3z+2)^2}{z(z-1)(2z+5)} \, dz = 2\pi i \operatorname*{Res}_{z=0}\left[\frac{1}{z^2} f\left(\frac{1}{z}\right)\right] = 2\pi i\left(\frac{9}{2}\right) = 9\pi i.$$

*(b)* Likewise,

$$\text{if } f(z) = \frac{z^3(1-3z)}{(1+z)(1+2z^4)}, \quad \text{then} \quad \frac{1}{z^2} f\left(\frac{1}{z}\right) = \frac{z-3}{z(z+1)(z^4+2)}.$$

The function $\frac{1}{z^2} f\left(\frac{1}{z}\right)$ has a simple pole at $z = 0$, and we find here that

$$\int_C \frac{z^3(1-3z)}{(1+z)(1+2z^4)} \, dz = 2\pi i \operatorname*{Res}_{z=0}\left[\frac{1}{z^2} f\left(\frac{1}{z}\right)\right] = 2\pi i\left(-\frac{3}{2}\right) = -3\pi i.$$

*(c)* Finally,

$$\text{if } f(z) = \frac{z^3 e^{1/z}}{1+z^3}, \quad \text{then} \quad \frac{1}{z^2} f\left(\frac{1}{z}\right) = \frac{e^z}{z^2(1+z^3)}.$$

The point $z = 0$ is a pole of order 2 of $\dfrac{1}{z^2} f\left(\dfrac{1}{z}\right)$. The residue is $\phi'(0)$, where

$$\phi(z) = \frac{e^z}{1+z^3}.$$

Since

$$\phi'(z) = \frac{(1+z^3)e^z - e^z 3z^2}{(1+z^3)^2},$$

the value of $\phi'(0)$ is 1. So

$$\int_C \frac{z^3 e^{1/z}}{1+z^3}\,dz = 2\pi i \operatorname*{Res}_{z=0}\left[\frac{1}{z^2} f\left(\frac{1}{z}\right)\right] = 2\pi i(1) = 2\pi i.$$

SECTION 69

1. *(a)* Write

$$\csc z = \frac{1}{\sin z} = \frac{p(z)}{q(z)}, \quad \text{where} \quad p(z) = 1 \text{ and } q(z) = \sin z.$$

Since

$$p(0) = 1 \neq 0, \quad q(0) = \sin 0 = 0, \quad \text{and} \quad q'(0) = \cos 0 = 1 \neq 0,$$

$z = 0$ must be a simple pole of $\csc z$, with residue

$$\frac{p(0)}{q'(0)} = \frac{1}{1} = 1.$$

*(b)* From Exercise 2, Sec. 61, we know that

$$\csc z = \frac{1}{z} + \frac{1}{3!}z + \left[\frac{1}{(3!)^2} - \frac{1}{5!}\right]z^3 + \cdots \qquad (0 < |z| < \pi).$$

Since the coefficient of $\dfrac{1}{z}$ here is 1, it follows that $z = 0$ is a simple pole of $\csc z$, the residue being 1.

2. *(a)* Write

$$\frac{z-\sinh z}{z^2\sinh z}=\frac{p(z)}{q(z)}, \quad \text{where} \quad p(z)=z-\sinh z \text{ and } q(z)=z^2\sinh z.$$

Since

$$p(\pi i)=\pi i\neq 0, \quad q(\pi i)=0, \quad \text{and} \quad q'(\pi i)=\pi^2\neq 0,$$

it follows that

$$\operatorname*{Res}_{z=\pi i}\frac{z-\sinh z}{z^2\sinh z}=\frac{p(\pi i)}{q'(\pi i)}=\frac{\pi i}{\pi^2}=\frac{i}{\pi}.$$

*(b)* Write

$$\frac{\exp(zt)}{\sinh z}=\frac{p(z)}{q(z)}, \quad \text{where} \quad p(z)=\exp(zt) \text{ and } q(z)=\sinh z.$$

It is easy to see that

$$\operatorname*{Res}_{z=\pi i}\frac{\exp(zt)}{\sinh z}=\frac{p(\pi i)}{q'(\pi i)}=-\exp(i\pi t) \quad \text{and} \quad \operatorname*{Res}_{z=-\pi i}\frac{\exp(zt)}{\sinh z}=\frac{p(-\pi i)}{q'(-\pi i)}=-\exp(-i\pi t).$$

Evidently, then,

$$\operatorname*{Res}_{z=\pi i}\frac{\exp(zt)}{\sinh z}+\operatorname*{Res}_{z=-\pi i}\frac{\exp(zt)}{\sinh z}=-2\,\frac{\exp(i\pi t)+\exp(-i\pi t)}{2}=-2\cos\pi t.$$

3. *(a)* Write

$$f(z)=\frac{p(z)}{q(z)}, \quad \text{where} \quad p(z)=z \text{ and } q(z)=\cos z.$$

Observe that

$$q\left(\frac{\pi}{2}+n\pi\right)=0 \qquad (n=0,\pm1,\pm2,\ldots).$$

Also, for the stated values of $n$,

$$p\left(\frac{\pi}{2}+n\pi\right)=\frac{\pi}{2}+n\pi\neq 0 \quad \text{and} \quad q'\left(\frac{\pi}{2}+n\pi\right)=-\sin\left(\frac{\pi}{2}+n\pi\right)=(-1)^{n+1}\neq 0.$$

So the function $f(z)=\dfrac{z}{\cos z}$ has poles of order $m=1$ at each of the points

$$z_n=\frac{\pi}{2}+n\pi \qquad (n=0,\pm1,\pm2,\ldots).$$

The corresponding residues are

$$B=\frac{p(z_n)}{q'(z_n)}=(-1)^{n+1}z_n.$$

*(b)* Write

$$\tanh z=\frac{p(z)}{q(z)}, \quad \text{where} \quad p(z)=\sinh z \text{ and } q(z)=\cosh z.$$

Both $p$ and $q$ are entire, and the zeros of $q$ are (Sec. 34)

$$z=\left(\frac{\pi}{2}+n\pi\right)i \qquad (n=0,\pm1,\pm2,\ldots).$$

In addition to the fact that $q\left(\left(\frac{\pi}{2}-n\pi\right)i\right)=0$, we see that

$$p\left(\left(\frac{\pi}{2}-n\pi\right)i\right)=\sinh\left(\frac{\pi}{2}i+n\pi i\right)=i\cos n\pi=i(-1)^n\neq0$$

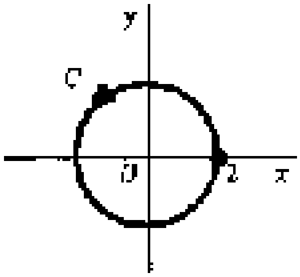
and

$$q'\left(\left(\frac{\pi}{2}+n\pi\right)i\right)=\sinh\left(\frac{\pi}{2}i+n\pi i\right)=i(-1)^n\neq0.$$

So the points $z=\left(\frac{\pi}{2}+n\pi\right)i$ $(n=0,\pm1,\pm2,\ldots)$ are poles of order $m=1$ of $\tanh z$, the residue in each case being

$$B=\frac{p\left(\left(\frac{\pi}{2}+n\pi\right)i\right)}{q'\left(\left(\frac{\pi}{2}+n\pi\right)i\right)}=\frac{i(-1)^n}{i(-1)^n}=1.$$

**4.** Let $C$ be the positively oriented circle $|z| = 2$, shown just below.




*(a)* To evaluate the integral $\int_C \tan z\,dz$, we write the integrand as

$$\tan z = \frac{p(z)}{q(z)}, \quad \text{where} \quad p(z) = \sin z \text{ and } q(z) = \cos z,$$

and recall that the zeros of $\cos z$ are $z = \frac{\pi}{2} + n\pi$ $(n = 0, \pm 1, \pm 2, \ldots)$. Only two of those zeros, namely $z = \pm\pi/2$, are interior to $C$, and they are the isolated singularities of $\tan z$ interior to $C$. Observe that

$$\operatorname*{Res}_{z=\pi/2} \tan z = \frac{p(\pi/2)}{q'(\pi/2)} = -1 \quad \text{and} \quad \operatorname*{Res}_{z=-\pi/2} \tan z = \frac{p(-\pi/2)}{q'(-\pi/2)} = -1.$$

Hence

$$\int_C \tan z\,dz = 2\pi i(-1 - 1) = -4\pi i.$$

*(b)* The problem here is to evaluate the integral $\int_C \frac{dz}{\sinh 2z}$. To do this, we write the integrand as

$$\frac{1}{\sinh 2z} = \frac{p(z)}{q(z)}, \quad \text{where} \quad p(z) = 1 \text{ and } q(z) = \sinh 2z.$$

Now $\sinh 2z = 0$ when $2z = n\pi i$ $(n = 0, \pm 1, \pm 2, \ldots)$, or when

$$z = \frac{n\pi i}{2} \qquad (n = 0, \pm 1, \pm 2, \ldots).$$

Three of those zeros of $\sinh 2z$, namely $0$ and $\pm\frac{\pi i}{2}$, are inside $C$ and are the isolated singularities of the integrand that need to be considered here. It is straightforward to show that

$$\operatorname*{Res}_{z=0} \frac{1}{\sinh 2z} = \frac{p(0)}{q'(0)} = \frac{1}{2\cosh 0} = \frac{1}{2},$$

$$\operatorname*{Res}_{z=\pi i/2} \frac{1}{\sinh 2z} = \frac{p(\pi i/2)}{q'(\pi i/2)} = \frac{1}{2\cosh(\pi i)} = \frac{1}{2\cos\pi} = -\frac{1}{2},$$

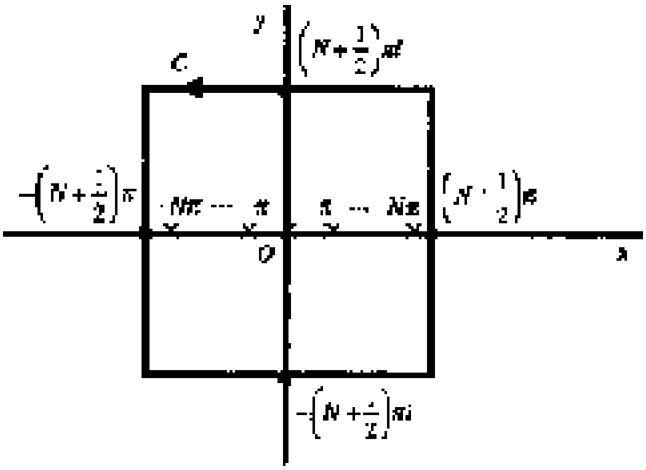
and

$$\operatorname*{Res}_{z=-\pi i/2} \frac{1}{\sinh 2z} = \frac{p(-\pi i/2)}{q'(-\pi i/2)} = \frac{1}{2\cosh(-\pi i)} = \frac{1}{2\cos(-\pi)} = -\frac{1}{2}.$$

Thus

$$\int_C \frac{dz}{\sinh 2z} = 2\pi i\left(\frac{1}{2} - \frac{1}{2} - \frac{1}{2}\right) = -\pi i.$$

5. The simple closed contour $C_N$ is as shown in the figure below.




Within $C_N$, the function $\dfrac{1}{z^2 \sin z}$ has isolated singularities at

$$z = 0 \quad \text{and} \quad z = \pm n\pi \ \ (n = 1, 2, \ldots, N).$$

To find the residue at $z = 0$, we recall the Laurent series for $\csc z$ that was found in Exercise 2, Sec. 61, and write

$$\frac{1}{z^2 \sin z} = \frac{1}{z^2}\csc z = \frac{1}{z^2}\left\{\frac{1}{z} + \frac{1}{3!}z + \left[\frac{1}{(3!)^2} - \frac{1}{5!}\right]z^3 + \cdots\right\}$$

$$= \frac{1}{z^3} + \frac{1}{6}\cdot\frac{1}{z} + \left[\frac{1}{(3!)^2} - \frac{1}{5!}\right]z + \cdots \qquad (0 < |z| < \pi).$$

This tells us that $\dfrac{1}{z^2 \sin z}$ has a pole of order 3 at $z = 0$ and that

$$\operatorname*{Res}_{z=0} \frac{1}{z^2 \sin z} = \frac{1}{6}.$$

As for the points $z = \pm n\pi$ $(n = 1, 2, \ldots, N)$, write

$$\frac{1}{z^2 \sin z} = \frac{p(z)}{q(z)}, \quad \text{where} \quad p(z) = 1 \text{ and } q(z) = z^2 \sin z.$$

Since

$$p(\pm n\pi) = 1 \neq 0, \quad q(\pm n\pi) = 0, \quad \text{and} \quad q'(\pm n\pi) = n^2\pi^2 \cos n\pi = (-1)^n n^2 \pi^2 \neq 0,$$

it follows that

$$\operatorname*{Res}_{z=\pm n\pi} \frac{1}{z^2 \sin z} = \frac{1}{(-1)^n n^2 \pi^2} \cdot \frac{(-1)^n}{(-1)^n} = \frac{(-1)^n}{n^2 \pi^2}.$$

So, by the residue theorem,

$$\int_{C_N} \frac{dz}{z^2 \sin z} dz = 2\pi i \left[ \frac{1}{6} + 2 \sum_{n=1}^{N} \frac{(-1)^n}{n^2 \pi^2} \right].$$

Rewriting this equation in the form

$$\sum_{n=1}^{N} \frac{(-1)^{n+1}}{n^2} = \frac{\pi^2}{12} - \frac{\pi}{4i} \int_{C_N} \frac{dz}{z^2 \sin z}$$

and recalling from Exercise 7, Sec. 41, that the value of the integral here tends to zero as $N$ tends to infinity, we arrive at the desired summation formula:

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2} = \frac{\pi^2}{12}.$$

6. The path $C$ here is the positively oriented boundary of the rectangle with vertices at the points $\pm 2$ and $\pm 2 + i$. The problem is to evaluate the integral

$$\int_C \frac{dz}{(z^2 - 1)^2 + 3}$$

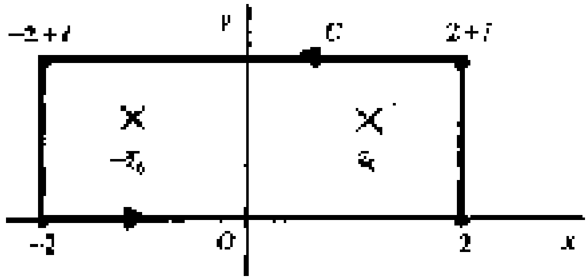
The isolated singularities of the integrand are the zeros of the polynomial

$$q(z) = (z^2 - 1)^2 + 3.$$

Setting this polynomial equal to zero and solving for $z^2$, we find that any zero $z$ of $q(z)$ has the property $z^2 = 1 \pm \sqrt{3}i$. It is straightforward to find the two square roots of $1+\sqrt{3}i$ and also the two square roots of $1-\sqrt{3}i$. These are the four zeros of $q(z)$. Only two of those zeros,

$$z_0 = \sqrt{2}e^{i\pi/6} = \frac{\sqrt{3}+i}{\sqrt{2}} \quad \text{and} \quad -\bar{z}_0 = -\sqrt{2}e^{-i\pi/6} = \frac{-\sqrt{3}+i}{\sqrt{2}},$$

lie inside $C$. They are shown in the figure below.




To find the residues at $z_0$ and $-\bar{z}_0$, we write the integrand of the integral to be evaluated as

$$\frac{1}{(z^2-1)^2+3} = \frac{p(z)}{q(z)}, \quad \text{where} \quad p(z) = 1 \text{ and } q(z) = (z^2-1)^2 + 3.$$

This polynomial $q(z)$ is, of course, the same $q(z)$ as above; hence $q(z_0) = 0$. Note, too, that $p$ and $q$ are analytic at $z_0$ and that $p(z_0) \neq 0$. Finally, it is straightforward to show that $q'(z) = 4z(z^2-1)$ and hence that

$$q'(z_0) = 4z_0(z_0^2 - 1) = -2\sqrt{6} + 6\sqrt{2}i \neq 0.$$

We may conclude, then, that $z_0$ is a simple pole of the integrand, with residue

$$\frac{p(z_0)}{q'(z_0)} = \frac{1}{-2\sqrt{6}+6\sqrt{2}i}.$$

Similar results are to be found at the singular point $-\bar{z}_0$. To be specific, it is easy to see that

$$q'(-\bar{z}_0) = -q'(\bar{z}_0) = -\overline{q'(z_0)} = 2\sqrt{6} + 6\sqrt{2}i \neq 0,$$

the residue of the integrand at $-\bar{z}_0$ being

$$\frac{p(-\bar{z}_0)}{q'(-\bar{z}_0)} = \frac{1}{2\sqrt{6}+6\sqrt{2}i}.$$

Finally, by the residue theorem,

$$\int_C \frac{dz}{(z^2-1)^2+3} = 2\pi i\left(\frac{1}{-2\sqrt{6}+6\sqrt{2}i}+\frac{1}{2\sqrt{6}+6\sqrt{2}i}\right)=\frac{\pi}{2\sqrt{2}}.$$

7. We are given that $f(z)=1/[q(z)]^2$, where $q$ is analytic at $z_0$, $q(z_0)=0$, and $q'(z_0)\neq 0$. These conditions on $q$ tell us that $q$ has a zero of order $m=1$ at $z_0$. Hence $q(z)=(z-z_0)g(z)$, where $g$ is a function that is analytic and nonzero at $z_0$; and this enables us to write

$$f(z)=\frac{\phi(z)}{(z-z_0)^2},\quad \text{where}\quad \phi(z)=\frac{1}{[g(z)]^2}.$$

So $f$ has a pole of order 2 at $z_0$, and

$$\operatorname*{Res}_{z=z_0} f(z)=\phi'(z_0)=-\frac{2g'(z_0)}{[g(z_0)]^3}.$$

But, since $q(z)=(z-z_0)g(z)$, we know that

$$q'(z)=(z-z_0)g'(z)+g(z)\quad\text{and}\quad q''(z)=(z-z_0)g''(z)+2g'(z).$$

Then, by setting $z=z_0$ in these last two equations, we find that

$$q'(z_0)=g(z_0)\quad\text{and}\quad q''(z_0)=2g'(z_0).$$

Consequently, our expression for the residue of $f$ at $z_0$ can be put in the desired form:

$$\operatorname*{Res}_{z=z_0} f(z)=-\frac{q''(z_0)}{[q'(z_0)]^3}.$$

8. *(a)* To find the residue of the function $\csc^2 z$ at $z=0$, we write

$$\csc^2 z=\frac{1}{[q(z)]^2},\quad\text{where}\quad q(z)=\sin z.$$

Since $q$ is entire, $q(0)=0$, and $q'(0)=1\neq 0$, the result in Exercise 7 tells us that

$$\operatorname*{Res}_{z=0}\csc^2 z=-\frac{q''(0)}{[q'(0)]^3}=0.$$

(*b*) The residue of the function $\dfrac{1}{(z+z^2)^2}$ at $z=0$ can be obtained by writing

$$\frac{1}{(z+z^2)^2}=\frac{1}{[q(z)]^2}, \quad \text{where} \quad q(z)=z+z^2.$$

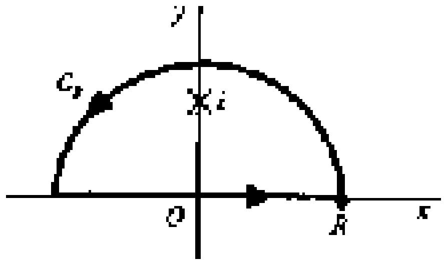
Inasmuch as $q$ is entire, $q(0)=0$, and $q'(0)=1\neq 0$, we know from Exercise 7 that

$$\operatorname*{Res}_{z=0}\frac{1}{(z+z^2)^2}=-\frac{q''(0)}{[q'(0)]^3}=-2.$$

# Chapter 7

SECTION 72

1. To evaluate the integral $\int_0^\infty \frac{dx}{x^2+1}$, we integrate the function $f(z)=\frac{1}{z^2+1}$ around the simple closed contour shown below, where $R>1$.




We see that

$$\int_{-R}^{R}\frac{dx}{x^2+1}+\int_{C_R}\frac{dz}{z^2+1}=2\pi i B,$$

where

$$B=\operatorname*{Res}_{z=i}\frac{1}{z^2+1}=\operatorname*{Res}_{z=i}\frac{1}{(z-i)(z+i)}=\frac{1}{z+i}\bigg]_{z=i}=\frac{1}{2i}.$$

Thus

$$\int_{-R}^{R}\frac{dx}{x^2+1}=\pi-\int_{C_R}\frac{dz}{z^2+1}.$$

Now if $z$ is a point on $C_R$,

$$|z^2+1|\geq ||z|^2-1|=R^2-1;$$

and so

$$\left|\int_{C_R}\frac{dz}{z^2+1}\right|\leq\frac{\pi R}{R^2-1}=\frac{\frac{\pi}{R}}{1-\frac{1}{R^2}}\to 0 \quad \text{as} \quad R\to\infty.$$

Finally, then

$$\int_{-\infty}^{\infty}\frac{dx}{x^2+1}=\pi, \quad \text{or} \quad \int_0^\infty\frac{dx}{x^2+1}=\frac{\pi}{2}.$$

**2.** The integral $\int_0^\infty \frac{dx}{(x^2+1)^2}$ can be evaluated using the function $f(z)=\frac{1}{(z^2+1)^2}$ and the same simple closed contour as in Exercise 1. Here

$$\int_{-R}^{R}\frac{dx}{(x^2+1)^2}+\int_{C_R}\frac{dz}{(z^2+1)^2}=2\pi iB,$$

where $B=\operatorname*{Res}_{z=i}\frac{1}{(z^2+1)^2}$. Since

$$\frac{1}{(z^2+1)^2}=\frac{\phi(z)}{(z-i)^2},\quad \text{where}\quad \phi(z)=\frac{1}{(z+i)^2},$$

we readily find that $B=\phi'(i)=\frac{1}{4i}$, and so

$$\int_{-R}^{R}\frac{dx}{(x^2+1)^2}=\frac{\pi}{2}-\int_{C_R}\frac{dz}{(z^2+1)^2}.$$

If $z$ is a point on $C_R$, we know from Exercise 1 that

$$|z^2+1|\ge R^2-1;$$

thus

$$\left|\int_{C_R}\frac{dz}{(z^2+1)^2}\right|\le\frac{\pi R}{(R^2-1)^2}=\frac{\dfrac{\pi}{R^3}}{\left(1-\dfrac{1}{R^2}\right)^2}\to 0\quad\text{as}\quad R\to\infty.$$

The desired result is, then,

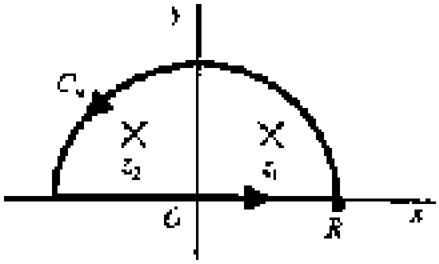
$$\int_{-\infty}^{\infty}\frac{dx}{(x^2+1)^2}=\frac{\pi}{2},\quad\text{or}\quad\int_0^\infty\frac{dx}{(x^2+1)^2}=\frac{\pi}{4}.$$

3. We begin the evaluation of $\int_0^\infty \frac{dx}{x^4+1}$ by finding the zeros of the polynomial $z^4+1$, which are the fourth roots of $-1$, and noting that two of them are below the real axis. In fact, if we consider the simple closed contour shown below, where $R>1$, that contour encloses only the two roots

$$z_0=e^{i\pi/4}=\frac{1}{\sqrt{2}}+\frac{i}{\sqrt{2}}$$

and

$$z_1=e^{i3\pi/4}=e^{i\pi/4}e^{i\pi/2}=\left(\frac{1}{\sqrt{2}}+\frac{i}{\sqrt{2}}\right)i=-\frac{1}{\sqrt{2}}+\frac{i}{\sqrt{2}}.$$




Now

$$\int_{-R}^{R}\frac{dx}{x^4+1}+\int_{C_R}\frac{dz}{z^4+1}=2\pi i(B_1+B_2),$$

where

$$B_1=\operatorname*{Res}_{z=z_1}\frac{1}{z^4+1}\quad\text{and}\quad B_2=\operatorname*{Res}_{z=z_2}\frac{1}{z^4+1}.$$

The method of Theorem 2 in Sec. 69 tells us that $z_1$ and $z_2$ are simple poles of $\dfrac{1}{z^4+1}$ and that

$$B_1=\frac{1}{4z_1^3}\cdot\frac{z_1}{z_1}=-\frac{z_1}{4}\quad\text{and}\quad B_2=\frac{1}{4z_2^3}\cdot\frac{z_2}{z_2}=-\frac{z_2}{4},$$

since $z_1^4=-1$ and $z_2^4=-1$. Furthermore,

$$B_1+B_2=-\frac{1}{4}(z_1+z_2)=-\frac{1}{4}\left[\left(\frac{1}{\sqrt{2}}+\frac{i}{\sqrt{2}}\right)+\left(-\frac{1}{\sqrt{2}}+\frac{i}{\sqrt{2}}\right)\right]=-\frac{i}{2\sqrt{2}}.$$

Hence

$$\int_{-R}^{R}\frac{dx}{x^4+1}=\frac{\pi}{\sqrt{2}}-\int_{C_R}\frac{dz}{z^4+1}.$$

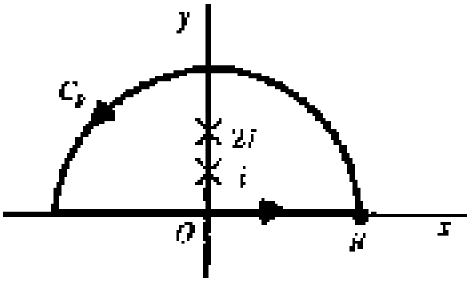
Since

$$\left|\int_{C_R}\frac{dz}{z^4+1}\right|\le\frac{\pi R}{R^4-1}\to 0\text{ as }R\to\infty,$$

we have

$$\int_{-\infty}^{\infty}\frac{dx}{x^4+1}=\frac{\pi}{\sqrt{2}},\quad\text{or}\quad\int_{0}^{\infty}\frac{dx}{x^4+1}=\frac{\pi}{2\sqrt{2}}.$$

4. We wish to evaluate the integral $\int_0^\infty \frac{x^2\,dx}{(x^2+1)(x^2+4)}$. We use the simple closed contour shown below, where $R > 2$.




We must find the residues of the function $f(z) = \frac{z^2}{(z^2+1)(z^2+4)}$ at its simple poles $z = i$ and $z = 2i$. They are

$$B_1 = \operatorname*{Res}_{z=i} f(z) = \frac{z^2}{(z+i)(z^2+4)}\bigg]_{z=i} = -\frac{1}{6i}$$

and

$$B_2 = \operatorname*{Res}_{z=2i} f(z) = \frac{z^2}{(z^2+1)(z+2i)}\bigg]_{z=2i} = \frac{1}{3i}.$$

Thus

$$\int_{-R}^{R} \frac{x^2\,dx}{(x^2+1)(x^2+4)} + \int_{C_R} \frac{z^2\,dz}{(z^2+1)(z^2+4)} = 2\pi i(B_1 + B_2),$$

or

$$\int_{-R}^{R} \frac{x^2\,dx}{(x^2+1)(x^2+4)} = \frac{\pi}{3} - \int_{C_R} \frac{z^2\,dz}{(z^2+1)(z^2+4)}.$$

If $z$ is a point on $C_R$, then

$$|z^2+1| \ge ||z|^2-1| = R^2-1 \quad \text{and} \quad |z^2+4| \ge ||z|^2-4| = R^2-4.$$

Consequently,

$$\left|\int_{C_R} \frac{z^2\,dz}{(z^2+1)(z^2+4)}\right| \le \frac{\pi R^3}{(R^2-1)(R^2-4)} = \frac{\dfrac{\pi}{R}}{\left(1-\dfrac{1}{R^2}\right)\left(1-\dfrac{4}{R^2}\right)} \to 0 \text{ as } R \to \infty;$$

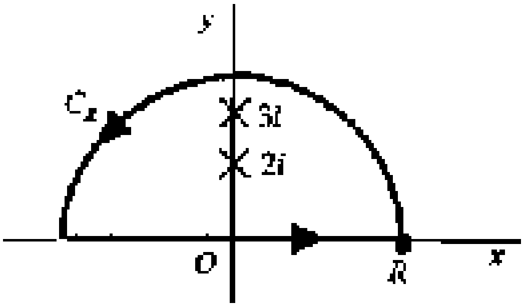
and we may conclude that

$$\int_{-\infty}^{\infty} \frac{x^2\,dx}{(x^2+1)(x^2+4)} = \frac{\pi}{3}, \quad \text{or} \quad \int_0^{\infty} \frac{x^2\,dx}{(x^2+1)(x^2+4)} = \frac{\pi}{6}.$$

5. The integral $\int_0^\infty \frac{x^2\,dx}{(x^2+9)(x^2+4)^2}$ can be evaluated with the aid of the function

$$f(z)=\frac{z^2}{(z^2+9)(z^2+4)^2}$$

and the simple closed contour shown below, where $R>3$.




We start by writing

$$\int_{-R}^{R}\frac{x^2\,dx}{(x^2+9)(x^2+4)^2}+\int_{C_R}\frac{z^2\,dz}{(z^2+9)(z^2+4)^2}=2\pi i(B_1+B_2),$$

where

$$B_1=\operatorname*{Res}_{z=3i}\frac{z^2}{(z^2+9)(z^2+4)^2}\quad\text{and}\quad B_2=\operatorname*{Res}_{z=2i}\frac{z^2}{(z^2+9)(z^2+4)^2}.$$

Now

$$B_1=\left.\frac{z^2}{(z+3i)(z^2+4)^2}\right]_{z=3i}=-\frac{3}{50i}.$$

To find $B_2$, we write

$$\frac{z^2}{(z^2+9)(z^2+4)^2}=\frac{\phi(z)}{(z-2i)^2},\quad\text{where}\quad \phi(z)=\frac{z^2}{(z^2+9)(z+2i)^2}.$$

Then

$$B_2=\phi'(2i)=\frac{13}{200i}.$$

This tells us that

$$\int_{-R}^{R}\frac{x^2\,dx}{(x^2+9)(x^2+4)^2}=\frac{\pi}{100}-\int_{C_R}\frac{z^2\,dz}{(z^2+9)(z^2+4)^2}.$$

Finally, since

$$\left|\int_{C_R}\frac{z^2\,dz}{(z^2+9)(z^2+4)^2}\right|\le\frac{\pi R^3}{(R^2-9)(R^2-4)^2}\to 0\ \text{as}\ R\to\infty,$$

we find that

$$\int_{-\infty}^{\infty}\frac{x^2\,dx}{(x^2+9)(x^2+4)^2}=\frac{\pi}{100},\quad\text{or}\quad \int_0^\infty\frac{x^2\,dx}{(x^2+9)(x^2+4)^2}=\frac{\pi}{200}.$$

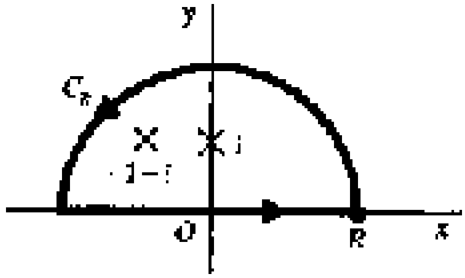
7. In order to show that

$$\text{P.V.}\int_{-\infty}^{\infty}\frac{x\,dx}{(x^2+1)(x^2+2x+2)}=-\frac{\pi}{5},$$

we introduce the function

$$f(z)=\frac{z}{(z^2+1)(z^2+2z+2)}$$

and the simple closed contour shown below.




Observe that the singularities of $f(z)$ are at $i$, $z_0=-1+i$ and their conjugates $-i$, $\overline{z}_0=-1-i$ in the lower half plane. Also, if $R>\sqrt{2}$, we see that

$$\int_{-R}^{R}f(x)\,dx+\int_{C_R}f(z)\,dz=2\pi i(B_0+B_1),$$

where

$$B_0=\operatorname*{Res}_{z=z_0}f(z)=\left.\frac{z}{(z^2+1)(z-\overline{z}_0)}\right]_{z=z_0}=-\frac{1}{10}-\frac{3}{10}i$$

and

$$B_1=\operatorname*{Res}_{z=i}f(z)=\left.\frac{z}{(z+i)(z^2+2z+2)}\right]_{z=i}=\frac{1}{10}-\frac{1}{5}i.$$

Evidently, then,

$$\int_{-R}^{R}\frac{x\,dx}{(x^2+1)(x^2+2x+2)}=-\frac{\pi}{5}-\int_{C_R}\frac{z\,dz}{(z^2+1)(z^2+2z+2)}.$$

Since

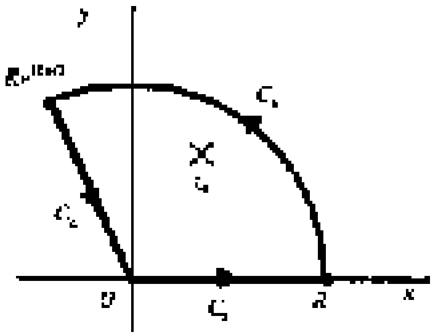
$$\left|\int_{C_R}\frac{z\,dz}{(z^2+1)(z^2+2z+2)}\right|=\left|\int_{C_R}\frac{z\,dz}{(z^2+1)(z-z_0)(z-\overline{z}_0)}\right|\le\frac{\pi R^2}{(R^2-1)(R-\sqrt{2})^2}\to 0$$

as $R\to\infty$, this means that

$$\lim_{R\to\infty}\int_{-R}^{R}\frac{x\,dx}{(x^2+1)(x^2+2x+2)}=-\frac{\pi}{5}.$$

This is the desired result.

8. The problem here is to establish the integration formula $\int_0^\infty \frac{dx}{x^3+1} = \frac{2\pi}{3\sqrt{3}}$ using the simple closed contour shown below, where $R > 1$.




There is only one singularity of the function $f(z) = \frac{1}{z^3+1}$, namely $z_0 = e^{i\pi/3}$, that is interior to the closed contour when $R > 1$. According to the residue theorem,

$$\int_{C_1} \frac{dz}{z^3+1} + \int_{C_R} \frac{dz}{z^3+1} + \int_{C_2} \frac{dz}{z^3+1} = 2\pi i \operatorname*{Res}_{z=z_0} \frac{1}{z^3+1},$$

where the legs of the closed contour are as indicated in the figure. Since $C_1$ has parametric representation $z = r$ $(0 \le r \le R)$,

$$\int_{C_1} \frac{dz}{z^3+1} = \int_0^R \frac{dr}{r^3+1};$$

and, since $-C_2$ can be represented by $z = re^{i2\pi/3}$ $(0 \le r \le R)$,

$$\int_{C_2} \frac{dz}{z^3+1} = -\int_{-C_2} \frac{dz}{z^3+1} = -\int_0^R \frac{e^{i2\pi/3}\,dr}{(re^{i2\pi/3})^3+1} = -e^{i2\pi/3}\int_0^R \frac{dr}{r^3+1}.$$

Furthermore,

$$\operatorname*{Res}_{z=z_0} \frac{1}{z^3+1} = \frac{1}{3z_0^2} = \frac{1}{3e^{i2\pi/3}}.$$

Consequently,

$$(1-e^{i2\pi/3})\int_0^R \frac{dr}{r^3+1} = \frac{2\pi i}{3e^{i2\pi/3}} - \int_{C_R} \frac{dz}{z^3+1}.$$

But

$$\left|\int_{C_R} \frac{dz}{z^3+1}\right| \le \frac{1}{R^3-1}\cdot\frac{2\pi R}{3} \to 0 \text{ as } R \to \infty.$$

This gives us the desired result, with the variable of integration $r$ instead of $x$:

$$\int_0^\infty \frac{dr}{r^3+1} = \frac{2\pi i}{3(e^{i2\pi/3} - e^{i4\pi/3}\cdot e^{-i\pi/3})} = \frac{2\pi i}{3(e^{i\pi/3} - e^{-i\pi/3})} = \frac{\pi}{3\sin(2\pi/3)} = \frac{2\pi}{3\sqrt{3}}.$$

**9.** Let $m$ and $n$ be integers, where $0 \le m < n$. The problem here is to derive the integration formula

$$\int_0^\infty \frac{x^{2m}}{x^{2n}+1}\,dx = \frac{\pi}{2n}\csc\left(\frac{2m+1}{2n}\pi\right).$$

*(a)* The zeros of the polynomial $z^{2n}+1$ occur when $z^{2n}=-1$. Since

$$(-1)^{1/(2n)} = \exp\left[i\frac{(2k+1)\pi}{2n}\right] \qquad (k=0,1,2,\ldots,2n-1),$$

it is clear that the zeros of $z^{2n}+1$ in the upper half plane are

$$c_k = \exp\left[i\frac{(2k+1)\pi}{2n}\right] \qquad (k=0,1,2,\ldots,n-1)$$

and that there are none on the real axis.

*(b)* With the aid of Theorem 2 in Sec. 69, we find that

$$\operatorname*{Res}_{z=c_k}\frac{z^{2m}}{z^{2n}+1} = \frac{c_k^{2m}}{2nc_k^{2n-1}} = \frac{1}{2n}c_k^{2(m-n)+1} \qquad (k=0,1,2,\ldots,n-1).$$

Putting $\alpha = \dfrac{2m+1}{2n}\pi$, we can write

$$c_k^{2(m-n)+1} = \exp\left[i\frac{(2k+1)\pi(2m-2n+1)}{2n}\right]$$

$$= \exp\left[i\frac{(2k+1)(2m+1)\pi}{2n}\right]\exp[-i(2k+1)\pi] = -e^{i(2k+1)\alpha}.$$

Thus

$$\operatorname*{Res}_{z=c_k}\frac{z^{2m}}{z^{2n}+1} = -\frac{1}{2n}e^{i(2k+1)\alpha} \qquad (k=0,1,2,\ldots,n-1).$$

In view of the identity (see Exercise 10, Sec. 7)

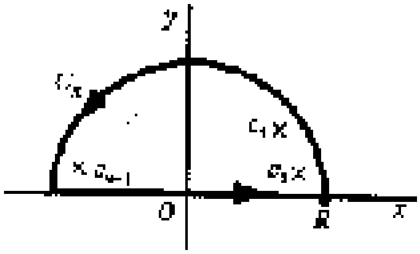
$$\sum_{k=0}^{n-1} z^k = \frac{1-z^n}{1-z} \qquad (z \ne 1),$$

then,

$$2\pi i\sum_{k=0}^{n-1}\operatorname*{Res}_{z=c_k}\frac{z^{2m}}{z^{2n}+1}=-\frac{\pi i}{n}e^{i\alpha}\sum_{k=0}^{n-1}(e^{i2\alpha})^k=-\frac{\pi i}{n}e^{i\alpha}\frac{1-e^{i2n\alpha}}{1-e^{i2\alpha}}\cdot\frac{e^{-i\alpha}}{e^{-i\alpha}}=-\frac{\pi i}{n}\cdot\frac{e^{i2n\alpha}-1}{e^{i\alpha}-e^{-i\alpha}}$$

$$=-\frac{\pi i}{n}\cdot\frac{e^{i(2m+1)\pi}-1}{e^{i\alpha}-e^{-i\alpha}}=\frac{\pi}{n}\cdot\frac{2i}{e^{i\alpha}-e^{-i\alpha}}=\frac{\pi}{n\sin\alpha}.$$

(c) Consider the path shown below, where $R>1$.




The residue theorem tells us that

$$\int_{-R}^{R}\frac{x^{2m}}{x^{2n}+1}dx+\int_{C_R}\frac{z^{2m}}{z^{2n}+1}dz=2\pi i\sum_{k=0}^{n-1}\operatorname*{Res}_{z=c_k}\frac{z^{2m}}{z^{2n}+1},$$

or

$$\int_{-R}^{R}\frac{x^{2m}}{x^{2n}+1}dx=\frac{\pi}{n\sin\alpha}-\int_{C_R}\frac{z^{2m}}{z^{2n}+1}dz.$$

Observe that if $z$ is a point on $C_R$, then

$$|z^{2m}|=R^{2m}\quad\text{and}\quad|z^{2n}+1|\ge R^{2n}-1.$$

Consequently,

$$\left|\int_{C_R}\frac{z^{2m}}{z^{2n}+1}dz\right|\le\frac{R^{2m}}{R^{2n}-1}\pi R\cdot\frac{R^{-2n}}{R^{-2n}}=\pi\frac{\dfrac{1}{R^{2(n-m)-1}}}{1-\dfrac{1}{R^{2n}}}\to 0;$$

and the desired integration formula follows.

10. The problem here is to evaluate the integral

$$\int_0^\infty\frac{dx}{[(x^2-a)^2+1]^2},$$

where $a$ is any real number. We do this by following the steps below.

*(a)* Let us first find the four zeros of the polynomial

$$q(z) = (z^2 - a)^2 + 1.$$

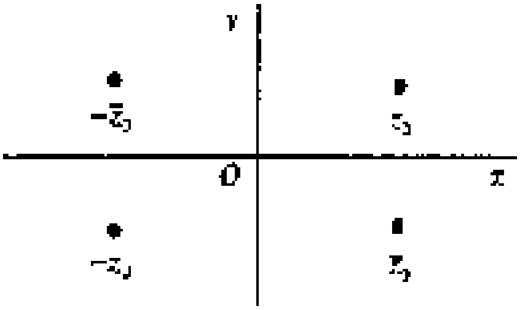
Solving the equation $q(z) = 0$ for $z^2$, we obtain $z^2 = a \pm i$. Thus two of the zeros are the square roots of $a + i$, and the other two are the square roots of $a - i$. By Exercise 5, Sec. 9, the two square roots of $a + i$ are the numbers

$$z_0 = \frac{1}{\sqrt{2}}\left(\sqrt{A + a} + i\sqrt{A - a}\right) \quad \text{and} \quad -z_0,$$

where $A = \sqrt{a^2 + 1}$. Since $(\pm\bar{z}_0)^2 = \overline{z_0^2} = \overline{a + i} = a - i$, the two square roots of $a - i$ are evidently

$$\bar{z}_0 \quad \text{and} \quad -\bar{z}_0.$$

The four zeros of $q(z)$ just obtained are located in the plane in the figure below, which tells us that $z_0$ and $-\bar{z}_0$ lie above the real axis and that the other two zeros lie below it.




*(b)* Let $q(z)$ denote the polynomial in part *(a)*; and define the function

$$f(z) = \frac{1}{[q(z)]^2},$$

which becomes the integrand in the integral to be evaluated when $z = x$. The method developed in Exercise 7, Sec. 69, reveals that $z_0$ is a pole of order 2 of $f$. To be specific, we note that $q$ is entire and recall from part *(a)* that $q(z_0) = 0$. Furthermore, $q'(z) = 4z(z^2 - a)$ and $z_0^2 = a + i$, as pointed out above in part *(a)*. Consequently, $q'(z_0) = 4z_0(z_0^2 - a) = 4iz_0 \neq 0$. The exercise just mentioned, together with the relations $z_0^2 = a + i$ and $1 + a^2 = A^2$, also enables us to write the residue $B_1$ of $f$ at $z_0$:

$$B_1 = -\frac{q''(z_0)}{[q'(z_0)]^3} = -\frac{12z_0^2 - 4a}{(4iz_0)^3} = \frac{3z_0^2 - a}{16iz_0^2 z_0} = \frac{3(a+i) - a}{16i(a+i)z_0} \cdot \frac{a - i}{a - i} = \frac{a - i(2a^2 + 3)}{16A^2 z_0}.$$

As for the point $-\bar{z}_0$, we observe that

$$q'(-\bar{z}) = -\overline{q'(z)} \quad \text{and} \quad q''(-\bar{z}) = \overline{q''(z)}.$$

Since $q(-\bar{z}_0)=0$ and $q'(-\bar{z}_0) = -\overline{q'(z_0)} = 4i\bar{z}_0 \neq 0$, the point $-\bar{z}_0$ is also a pole of order 2 of $f$. Moreover, if $B_2$ denotes the residue there,

$$B_2 = -\frac{q''(-\bar{z}_0)}{[q'(-\bar{z}_0)]^3} = \frac{\overline{q''(z_0)}}{\overline{[q'(z_0)]^3}} = \overline{\left\{\frac{q''(z_0)}{[q'(z_0)]^3}\right\}} = -\overline{B_1}.$$

Thus

$$B_1 + B_2 = B_1 - \overline{B_1} = 2i \operatorname{Im} B_1 = \frac{1}{8A^2 i}\operatorname{Im}\left[\frac{-a + i(2a^2+3)}{z_0}\right].$$

(c) We now integrate $f(z)$ around the simple closed path in the figure below, where $R > |z_0|$ and $C_R$ denotes the semicircular portion of the path. The residue theorem tells us that

$$\int_{-R}^{R} f(x)\,dx + \int_{C_R} f(z)\,dz = 2\pi i(B_1 + B_2),$$

or

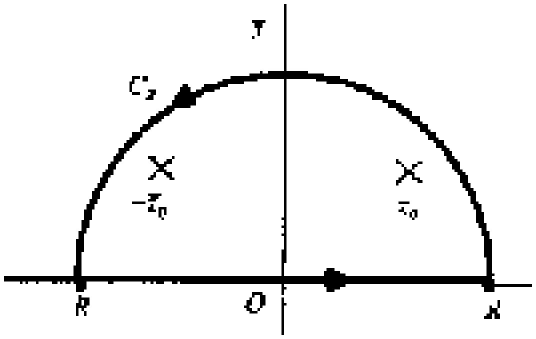
$$\int_{-R}^{R} \frac{dx}{[(x^2-a)^2+1]^2} = \frac{\pi}{4A^2}\operatorname{Im}\left[\frac{-a + i(2a^2+3)}{z_0}\right] - \int_{C_R} \frac{dz}{[q(z)]^2}.$$

In order to show that

$$\lim_{R\to\infty} \int_{C_R} \frac{dz}{[q(z)]^2} = 0,$$

we start with the observation that the polynomial $q(z)$ can be factored into the form

$$q(z) = (z - z_0)(z + z_0)(z - \bar{z}_0)(z + \bar{z}_0).$$




Recall now that $R > |z_0|$. If $z$ is a point on $C_R$, so that $|z| = R$, then

$$|z - z_0| \geq ||z| - |z_0|| = R - |z_0| \quad \text{and} \quad |z \pm \bar{z}_0| \geq ||z| - |\bar{z}_0|| = R - |z_0|.$$

This enables us to see that $|q(z)| \geq (R-|z_0|)^4$ when $z$ is on $C_R$. Thus

$$\left|\frac{1}{[q(z)]^2}\right| \leq \frac{1}{(R-|z_0|)^8}$$

for such points, and we arrive at the inequality

$$\left|\int_{C_R} \frac{1}{[q(z)]^2}\,dz\right| \leq \frac{\pi R}{(R-|z_0|)^8} = \frac{\dfrac{\pi}{R^7}}{\left(1-\dfrac{|z_0|}{R}\right)^8},$$

which tells us that the value of this integral does, indeed, tend to 0 as $R$ tends to $\infty$. Consequently,

$$\text{P.V.}\int_{-\infty}^{\infty} \frac{dx}{[(x^2-a)^2+1]^2} = \frac{\pi}{4A^2}\,\text{Im}\left[\frac{-a+i(2a^2+3)}{z_0}\right].$$

But the integrand here is even, and

$$\text{Im}\left[\frac{-a+i(2a^2+3)}{z_0}\right] = \text{Im}\left[\sqrt{2}\,\frac{-a+i(2a^2+3)}{\sqrt{A+a}+i\sqrt{A-a}}\cdot\frac{\sqrt{A+a}-i\sqrt{A-a}}{\sqrt{A+a}-i\sqrt{A-a}}\right].$$

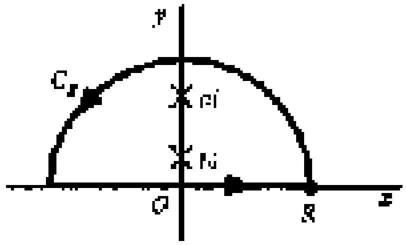
So, the desired result is

$$\int_0^{\infty} \frac{dx}{[(x^2-a)^2+1]^2} = \frac{\pi}{8\sqrt{2}A^3}\left[(2a^2+3)\sqrt{A+a}+a\sqrt{A-a}\right],$$

where $A = \sqrt{a^2+1}$.

## SECTION 74

1. The problem here is to evaluate the integral $\displaystyle\int_{-\infty}^{\infty} \frac{\cos x\,dx}{(x^2+a^2)(x^2+b^2)}$, where $a > b > 0$. To do this, we introduce the function $f(z) = \dfrac{1}{(z^2+a^2)(z^2+b^2)}$, whose singularities $ai$ and $bi$ lie inside the simple closed contour shown below, where $R > a$. The other singularities are, of course, in the lower half plane.



According to the residue theorem,

$$\int_{-R}^{R} \frac{e^{ix}\,dx}{(x^2+a^2)(x^2+b^2)} + \int_{C_R} f(z)e^{iz}\,dz = 2\pi i(B_1+B_2),$$

where

$$B_1 = \operatorname*{Res}_{z=ai}[f(z)e^{iz}] = \frac{e^{iz}}{(z+ai)(z^2+b^2)}\Bigg]_{z=ai} = \frac{e^{-a}}{2a(b^2-a^2)i}$$

and

$$B_2 = \operatorname*{Res}_{z=bi}[f(z)e^{iz}] = \frac{e^{iz}}{(z^2+a^2)(z+bi)}\Bigg]_{z=bi} = \frac{e^{-b}}{2b(a^2-b^2)i}.$$

That is,

$$\int_{-R}^{R} \frac{e^{ix}\,dx}{(x^2+a^2)(x^2+b^2)} = \frac{\pi}{a^2-b^2}\left(\frac{e^{-b}}{b}-\frac{e^{-a}}{a}\right) - \int_{C_R} f(z)e^{iz}\,dz,$$

or

$$\int_{-R}^{R} \frac{\cos x\,dx}{(x^2+a^2)(x^2+b^2)} = \frac{\pi}{a^2-b^2}\left(\frac{e^{-b}}{b}-\frac{e^{-a}}{a}\right) - \operatorname{Re}\int_{C_R} f(z)e^{iz}\,dz.$$

Now, if $z$ is a point on $C_R$,

$$|f(z)| \le M_R \quad \text{where} \quad M_R = \frac{1}{(R^2-a^2)(R^2-b^2)}$$

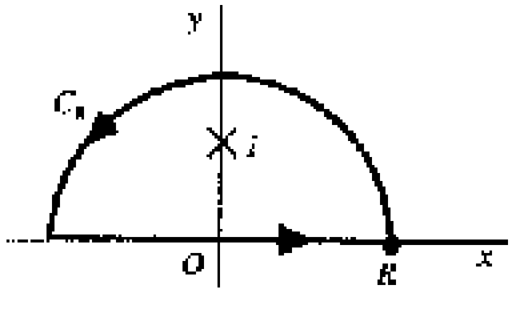
and $|e^{iz}| = e^{-y} \le 1$. Hence

$$\left|\operatorname{Re}\int_{C_R} f(z)e^{iz}\,dz\right| \le \left|\int_{C_R} f(z)e^{iz}\,dz\right| \le M_R\pi R = \frac{\pi R}{(R^2-a^2)(R^2-b^2)} \to 0 \text{ as } R\to\infty.$$

So it follows that

$$\int_{-\infty}^{\infty} \frac{\cos x\,dx}{(x^2+a^2)(x^2+b^2)} = \frac{\pi}{a^2-b^2}\left(\frac{e^{-b}}{b}-\frac{e^{-a}}{a}\right) \qquad (a>b>0).$$

**2.** This problem is to evaluate the integral $\displaystyle\int_0^\infty \frac{\cos ax}{x^2+1}\,dx$, where $a \ge 0$. The function $f(z) = \dfrac{1}{z^2+1}$ has the singularities $\pm i$, and so we may integrate around the simple closed contour shown below, where $R > 1$.



We start with

$$\int_{-R}^{R}\frac{e^{iax}}{x^2+1}dx+\int_{C_R} f(z)e^{iaz}dz = 2\pi i B,$$

where

$$B=\operatorname*{Res}_{z=i}\left[f(z)e^{iaz}\right] = \frac{e^{iaz}}{z+i}\bigg|_{z=i} = \frac{e^{-a}}{2i}.$$

Hence

$$\int_{-R}^{R}\frac{e^{iax}}{x^2+1}dx = \pi e^{-a} - \int_{C_R} f(z)e^{iaz}dz,$$

or

$$\int_{-R}^{R}\frac{\cos ax}{x^2+1}dx = \pi e^{-a} - \operatorname{Re}\int_{C_R} f(z)e^{iaz}dz.$$

Since

$$|f(z)|\le M_R \quad \text{where} \quad M_R = \frac{1}{R^2-1},$$

we know that

$$\left|\operatorname{Re}\int_{C_R} f(z)e^{iaz}dz\right| \le \left|\int_{C_R} f(z)e^{iaz}dz\right| \le \frac{\pi R}{R^2-1};$$

and so

$$\int_{-\infty}^{\infty}\frac{\cos ax}{x^2-1}dx = \pi e^{-a}.$$

That is,

$$\int_{0}^{\infty}\frac{\cos ax}{x^2+1}dx = \frac{\pi}{2}e^{-a} \qquad (a>0)$$

4. To evaluate the integral $\int_0^\infty \frac{x\sin 2x}{x^2+3}dx$, we first introduce the function

$$f(z)=\frac{z}{z^2+3}=\frac{z}{(z-z_1)(z-\bar{z}_1)},$$

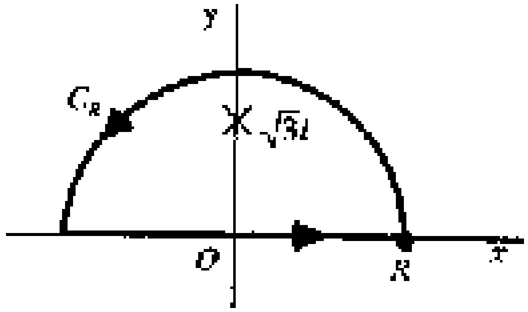
where $z_1=\sqrt{3}i$. The point $z_1$ lies above the $x$ axis, and $\bar{z}_1$ lies below it. If we write

$$f(z)e^{i2z} = \frac{\phi(z)}{z-z_1} \quad \text{where} \quad \phi(z)=\frac{z\exp(i2z)}{z-\bar{z}_1},$$

we see that $z_1$ is a simple pole of the function $f(z)e^{i2z}$ and that the corresponding residue is

$$B_1 = \phi(z_1) = \frac{\sqrt{3}i\exp(-2\sqrt{3})}{2\sqrt{3}i} = \frac{\exp(-2\sqrt{3})}{2}.$$

Now consider the simple closed contour shown in the figure below, where $R > \sqrt{3}$.




Integrating $f(z)e^{i2z}$ around the closed contour, we have

$$\int_{-R}^{R} \frac{xe^{i2x}}{x^2+3}\,dx = 2\pi i B_1 - \int_{C_R} f(z)e^{i2z}\,dz.$$

Thus

$$\int_{-R}^{R} \frac{x\sin 2x}{x^2+3}\,dx = \mathrm{Im}(2\pi i B_1) - \mathrm{Im}\int_{C_R} f(z)e^{i2z}\,dz.$$

Now, when $z$ is a point on $C_R$,

$$|f(z)| \le M_R, \quad \text{where} \quad M_R = \frac{R}{R^2-3} \to 0 \text{ as } R \to \infty;$$

and so, by limit (1), Sec. 74,

$$\lim_{R\to\infty} \int_{C_R} f(z)e^{i2z}\,dz = 0.$$

Consequently, since

$$\left|\mathrm{Im}\int_{C_R} f(z)e^{i2z}\,dz\right| \le \left|\int_{C_R} f(z)e^{i2z}\,dz\right|,$$

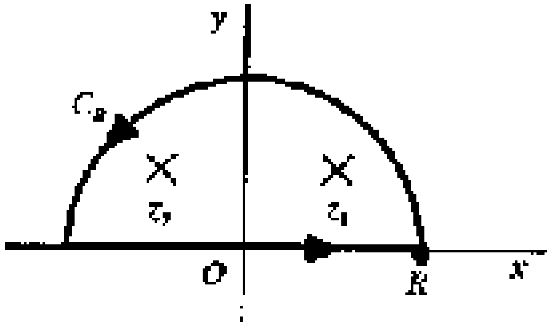
we arrive at the result

$$\int_{-\infty}^{\infty} \frac{x\sin 2x}{x^2+3}\,dx = \pi\exp(-2\sqrt{3}), \quad \text{or} \quad \int_{0}^{\infty} \frac{x\sin 2x}{x^2+3}\,dx = \frac{\pi}{2}\exp(-2\sqrt{3}).$$

**6.** The integral to be evaluated is $\int_{-\infty}^{\infty} \frac{x^3 \sin ax}{x^4+4}\,dx$, where $a > 0$. We define the function $f(z) = \frac{z^3}{z^4+4}$; and, by computing the fourth roots of $-4$, we find that the singularities

$$z_1 = \sqrt{2}e^{i\pi/4} = 1+i \quad \text{and} \quad z_2 = \sqrt{2}e^{i3\pi/4} = \sqrt{2}e^{i\pi/4}e^{i\pi/2} = (1+i)i = -1+i$$

both lie inside the simple closed contour shown below, where $R > \sqrt{2}$. The other two singularities lie below the real axis.




The residue theorem and the method of Theorem 2 in Sec. 69 for finding residues at simple poles tell us that

$$\int_{-R}^{R} \frac{x^3 e^{iax}}{x^4+4}\,dx + \int_{C_R} f(z)e^{iaz}\,dz = 2\pi i(B_1 + B_2),$$

where

$$B_1 = \operatorname*{Res}_{z=z_1} \frac{z^3 e^{iaz}}{z^4+4} = \frac{z_1^3 e^{iaz_1}}{4z_1^3} = \frac{e^{iaz_1}}{4} = \frac{e^{ia(1+i)}}{4} = \frac{e^{-a}e^{ia}}{4}$$

and

$$B_2 = \operatorname*{Res}_{z=z_2} \frac{z^3 e^{iaz}}{z^4+4} = \frac{z_2^3 e^{iaz_2}}{4z_2^3} = \frac{e^{iaz_2}}{4} = \frac{e^{ia(-1+i)}}{4} = \frac{e^{-a}e^{-ia}}{4}.$$

Since

$$2\pi i(B_1 + B_2) = \pi i e^{-a}\left(\frac{e^{ia} + e^{-ia}}{2}\right) = i\pi e^{-a}\cos a,$$

we are now able to write

$$\int_{-R}^{R} \frac{x^3 \sin ax}{x^4+4}\,dx = \pi e^{-a}\cos a - \operatorname{Im}\int_{C_R} f(z)e^{iaz}\,dz.$$

Furthermore, if $z$ is a point on $C_R$, then

$$|f(z)| \le M_R \quad \text{where} \quad M_R = \frac{R^3}{R^4 - 4} \to 0 \text{ as } R \to \infty;$$

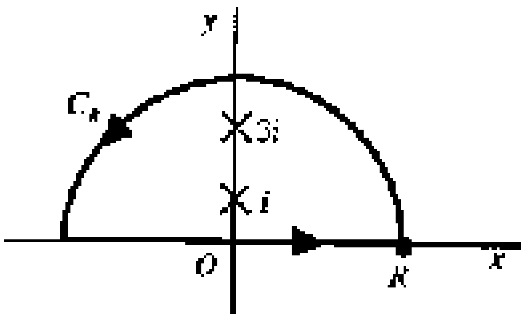
and this means that

$$\left| \operatorname{Im} \int_{C_R} f(z)e^{iaz}dz \right| \le \left| \int_{C_R} f(z)e^{iaz}dz \right| \to 0 \text{ as } R \to \infty,$$

according to limit (1), Sec. 74. Finally, then,

$$\int_{-\infty}^{\infty} \frac{x^3 \sin ax}{x^4 + 4} dx = \pi e^{-a} \cos a \qquad (a > 0).$$

8. In order to evaluate the integral $\int_0^{\infty} \frac{x^3 \sin x \, dx}{(x^2+1)(x^2+9)}$, we introduce here the function $f(z) = \frac{z^3}{(z^2+1)(z^2+9)}$. Its singularities in the upper half plane are $i$ and $3i$, and we consider the simple closed contour shown below, where $R > 3$.




Since

$$\operatorname*{Res}_{z=i}\left[f(z)e^{iz}\right] = \left.\frac{z^3 e^{iz}}{(z+i)(z^2+9)}\right]_{z=i} = -\frac{1}{16e}$$

and

$$\operatorname*{Res}_{z=3i}\left[f(z)e^{iz}\right] = \left.\frac{z^3 e^{iz}}{(z^2+1)(z+3i)}\right]_{z=3i} = \frac{9}{16e^3},$$

the residue theorem tells us that

$$\int_{-R}^{R} \frac{x^3 e^{ix}\, dx}{(x^2+1)(x^2+9)} + \int_{C_R} f(z)e^{iz}dz = 2\pi i\left(-\frac{1}{16e} + \frac{9}{16e^3}\right),$$

or

$$\int_{-R}^{R} \frac{x^3 \sin x \, dx}{(x^2+1)(x^2+9)} = \frac{\pi}{8e}\left(\frac{9}{e^2} - 1\right) - \operatorname{Im} \int_{C_R} f(z)e^{iz}dz.$$

Now if $z$ is a point on $C_R$, then

$$|f(z)| \le M_R \quad \text{where} \quad M_R = \frac{R}{(R^2-1)(R^2-9)} \to 0 \text{ as } R \to \infty.$$

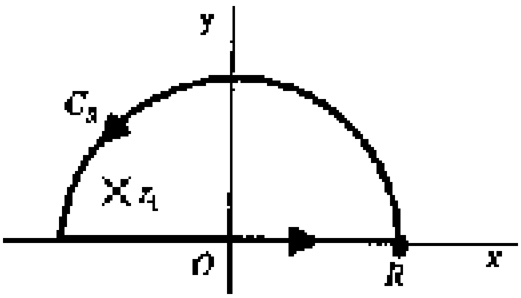
So, in view of limit (1), Sec. 74,

$$\left|\mathrm{Im}\int_{C_R} f(z)e^{iz}dz\right| \le \left|\int_{C_R} f(z)e^{iz}dz\right| \to 0 \text{ as } R \to \infty;$$

and this means that

$$\int_{-\infty}^{\infty} \frac{x^3 \sin x\,dx}{(x^2+1)(x^2+9)} = \frac{\pi}{8e}\left(\frac{9}{e^2}-1\right), \quad \text{or} \quad \int_{0}^{\infty} \frac{x^3 \sin x\,dx}{(x^2+1)(x^2+9)} = \frac{\pi}{16e}\left(\frac{9}{e^2}-1\right).$$

9. The Cauchy principal value of the integral $\int_{-\infty}^{\infty} \frac{\sin x\,dx}{x^2+4x+5}$ can be found with the aid of the function $f(z) = \frac{1}{z^2+4z+5}$ and the simple closed contour shown below, where $R > \sqrt{5}$. Using the quadratic formula to solve the equation $z^2+4z+5=0$, we find that $f$ has singularities at the points $z_1 = -2+i$ and $\bar{z}_1 = -2-i$. Thus $f(z) = \frac{1}{(z-z_1)(z-\bar{z}_1)}$, where $z_1$ is interior to the closed contour and $\bar{z}_1$ is below the real axis.




The residue theorem tells us that

$$\int_{-R}^{R} \frac{e^{ix}\,dx}{x^2+4x+5} + \int_{C_R} f(z)e^{iz}dz = 2\pi i B,$$

where

$$B = \operatorname*{Res}_{z=z_1}\left[\frac{e^{iz}}{(z-z_1)(z-\bar{z}_1)}\right] = \frac{e^{iz_1}}{(z_1-\bar{z}_1)};$$

and so

$$\int_{-R}^{R} \frac{\sin x\,dx}{x^2+4x+5} = \mathrm{Im}\left[\frac{2\pi i e^{iz_1}}{(z_1-\bar{z}_1)}\right] - \mathrm{Im}\int_{C_R} f(z)e^{iz}dz.$$

or

$$\int_{-R}^{R} \frac{\sin x\,dx}{x^2+4x+5} = -\frac{\pi}{e}\sin 2 - \operatorname{Im}\int_{C_R} f(z)e^{iz}dz.$$

Now, if $z$ is a point on $C_R$, then $|e^{iz}| = e^{-y} \le 1$ and

$$|f(z)| \le M_R \quad \text{where} \quad M_R = \frac{1}{(R-|z_1|)(R-|\bar{z}_1|)} = \frac{1}{(R-\sqrt{5})^2}.$$

Hence

$$\left|\operatorname{Im}\int_{C_R} f(z)e^{iz}dz\right| \le \left|\int_{C_R} f(z)e^{iz}dz\right| \le M_R\pi R = \frac{\pi R}{(R-\sqrt{5})^2} \to 0 \text{ as } R\to\infty,$$

and we may conclude that

$$\text{P.V.}\int_{-\infty}^{\infty} \frac{\sin x\,dx}{x^2+4x+5} = -\frac{\pi}{e}\sin 2.$$

**10.** To find the Cauchy principal value of the improper integral $\int_{-\infty}^{\infty} \frac{(x+1)\cos x}{x^2+4x+5}dx$, we shall use the function $f(z) = \frac{z+1}{z^2+4z+5} = \frac{z+1}{(z-z_1)(z-\bar{z}_1)}$, where $z_1 = -2+i$, and $\bar{z}_1 = -2-i$, and the same simple closed contour as in Exercise 9. In this case,

$$\int_{-R}^{R} \frac{(x+1)e^{ix}dx}{x^2+4x+5} + \int_{C_R} f(z)e^{iz}dz = 2\pi i B,$$

where

$$B = \operatorname*{Res}_{z=z_1}\left[\frac{(z+1)e^{iz}}{(z-z_1)(z-\bar{z}_1)}\right] = \frac{(z_1+1)e^{iz_1}}{(z_1-\bar{z}_1)} = \frac{(-1+i)e^{-2i}}{2ei}.$$

Thus

$$\int_{-R}^{R} \frac{(x+1)\cos x}{x^2+4x+5}dx = \operatorname{Re}(2\pi i B) - \int_{C_R} f(z)e^{iz},$$

or

$$\int_{-R}^{R} \frac{(x+1)\cos x}{x^2+4x+5}dx = \frac{\pi}{e}(\sin 2 - \cos 2) - \int_{C_R} f(z)e^{iz}dz.$$

Finally, we observe that if $z$ is a point on $C_R$, then

$$|f(z)| \le M_R \quad \text{where} \quad M_R = \frac{R+1}{(R-|z_1|)(R-|\bar{z}_1|)} = \frac{R+1}{(R-\sqrt{5})^2} \to 0 \text{ as } R\to\infty.$$

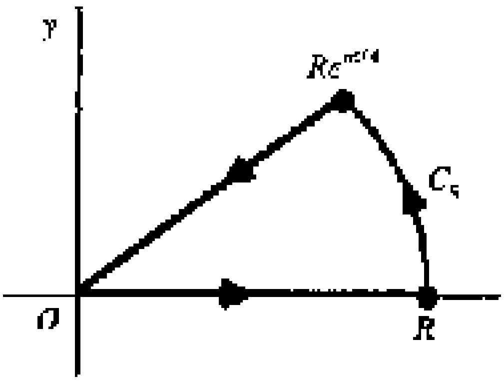
Limit (1), Sec. 74, then tells us that

$$\left|\operatorname{Re}\int_{C_R} f(z)e^{iz}dz\right| \le \left|\int_{C_R} f(z)e^{iz}dz\right| \to 0 \text{ as } R\to\infty,$$

and so

$$\text{P.V.}\int_{-\infty}^{\infty} \frac{(x+1)\cos x}{x^2+4x+5}\,dx = \frac{\pi}{e}(\sin 2-\cos 2).$$

12. (a) Since the function $f(z)=\exp(iz^2)$ is entire, the Cauchy-Goursat theorem tells us that its integral around the positively oriented boundary of the sector $0\le r\le R,\ 0\le\theta\le\pi/4$ has value zero. The closed path is shown below.




A parametric representation of the horizontal line segment from the origin to the point $R$ is $z=x\ (0\le x\le R)$, and a representation for the segment from the origin to the point $Re^{i\pi/4}$ is $z=re^{i\pi/4}\ (0\le r\le R)$. Thus

$$\int_0^R e^{ix^2}dx+\int_{C_R} e^{iz^2}dz-e^{i\pi/4}\int_0^R e^{-r^2}dr=0,$$

or

$$\int_0^R e^{ix^2}dx=e^{i\pi/4}\int_0^R e^{-r^2}dr-\int_{C_R} e^{iz^2}dz.$$

By equating real parts and then imaginary parts on each side of this last equation, we see that

$$\int_0^R \cos(x^2)\,dx=\frac{1}{\sqrt{2}}\int_0^R e^{-r^2}dr-\operatorname{Re}\int_{C_R} e^{iz^2}dz$$

and

$$\int_0^R \sin(x^2)\,dx=\frac{1}{\sqrt{2}}\int_0^R e^{-r^2}dr-\operatorname{Im}\int_{C_R} e^{iz^2}dz.$$

(*b*) A parametric representation for the arc $C_R$ is $z = Re^{i\theta}$ $(0 \le \theta \le \pi/4)$. Hence

$$\int_{C_R} e^{iz^2} dz = \int_0^{\pi/4} e^{iR^2 e^{i2\theta}} Rie^{i\theta} d\theta = iR \int_0^{\pi/4} e^{iR^2 \cos 2\theta} e^{-R^2 \sin 2\theta} e^{i\theta} d\theta.$$

Since $\left|e^{iR^2 \cos 2\theta}\right| = 1$ and $\left|e^{i\theta}\right| = 1$, it follows that

$$\left|\int_{C_R} e^{iz^2} dz\right| \le R \int_0^{\pi/4} e^{-R^2 \sin 2\theta} d\theta.$$

Then, by making the substitution $\phi = 2\theta$ in this last integral and referring to the form (3), Sec. 74, of Jordan's inequality, we find that

$$\left|\int_{C_R} e^{iz^2} dz\right| \le \frac{R}{2} \int_0^{\pi/2} e^{-R^2 \sin \phi} d\phi < \frac{R}{2} \cdot \frac{\pi}{2R^2} = \frac{\pi}{4R} \to 0 \text{ as } R \to \infty.$$

(*c*) In view of the result in part (*b*) and the integration formula

$$\int_0^\infty e^{-x^2} dx = \frac{\sqrt{\pi}}{2},$$

it follows from the last two equations in part (*a*) that

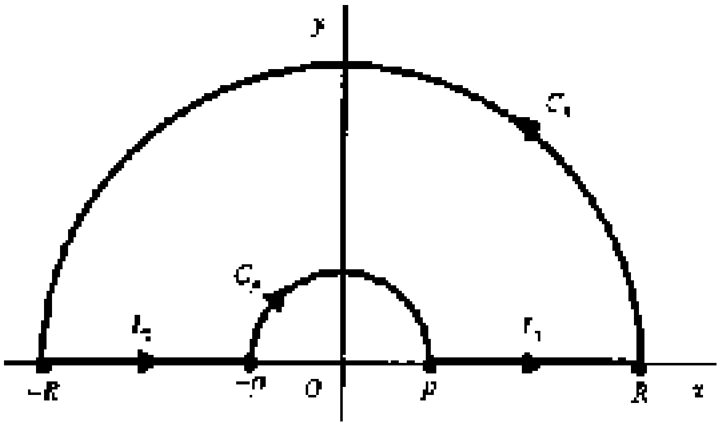
$$\int_0^\infty \cos(x^2) dx = \frac{1}{2}\sqrt{\frac{\pi}{2}} \quad \text{and} \quad \int_0^\infty \sin(x^2) dx = \frac{1}{2}\sqrt{\frac{\pi}{2}}.$$

SECTION 77

**1.** The main problem here is to derive the integration formula

$$\int_0^\infty \frac{\cos(ax) - \cos(bx)}{x^2} dx = \frac{\pi}{2}(b - a) \qquad (a \ge 0, b \ge 0),$$

using the indented contour shown below.



Applying the Cauchy-Goursat theorem to the function

$$f(z) = \frac{e^{iaz} - e^{ibz}}{z^2},$$

we have

$$\int_{L_1} f(z)\,dz + \int_{C_R} f(z)\,dz + \int_{L_2} f(z)\,dz + \int_{C_\rho} f(z)\,dz = 0,$$

or

$$\int_{L_1} f(z)\,dz - \int_{-L_2} f(z)\,dz = -\int_{C_\rho} f(z)\,dz - \int_{C_R} f(z)\,dz.$$

Since $L_1$ and $-L_2$ have parametric representations

$$L_1 : z = re^{i0} = r \ (\rho \le r \le R) \quad \text{and} \quad -L_2 : z = re^{i\pi} = -r \ (\rho \le r \le R),$$

we can see that

$$\int_{L_1} f(z)\,dz + \int_{L_2} f(z)\,dz = \int_{L_1} f(z)\,dz - \int_{-L_2} f(z)\,dz = \int_{\rho}^{R} \frac{e^{iar} - e^{ibr}}{r^2}\,dr + \int_{\rho}^{R} \frac{e^{-iar} - e^{-ibr}}{r^2}\,dr$$

$$= \int_{\rho}^{R} \frac{(e^{iar} + e^{-iar}) - (e^{ibr} + e^{-ibr})}{r^2}\,dr = 2\int_{\rho}^{R} \frac{\cos(ar) - \cos(br)}{r^2}\,dr.$$

Thus

$$2\int_{\rho}^{R} \frac{\cos(ar) - \cos(br)}{r^2}\,dr = -\int_{C_\rho} f(z)\,dz - \int_{C_R} f(z)\,dz.$$

In order to find the limit of the first integral on the right here as $\rho \to 0$, we write

$$f(z) = \frac{1}{z^2}\left[\left(1 + \frac{iaz}{1!} + \frac{(iaz)^2}{2!} + \frac{(iaz)^3}{3!} + \cdots\right) - \left(1 - \frac{ibz}{1!} + \frac{(ibz)^2}{2!} - \frac{(ibz)^3}{3!} + \cdots\right)\right]$$

$$= \frac{i(a-b)}{z} + \cdots \quad (0 < |z| < \infty).$$

From this we see that $z = 0$ is a simple pole of $f(z)$, with residue $B_0 = i(a-b)$. Thus

$$\lim_{\rho \to 0} \int_{C_\rho} f(z)\,dz = -B_0\pi i = -i(a-b)\pi i = \pi(a-b).$$

As for the limit of the value of the second integral as $R \to \infty$, we note that if $z$ is a point on $C_R$, then

$$f(z) \le \frac{|e^{iaz}| + |e^{ibz}|}{|z|^2} = \frac{e^{-ay} + e^{-by}}{R^2} \le \frac{1+1}{R^2} = \frac{2}{R^2}.$$

Consequently,

$$\left| \int_{C_R} f(z)\,dz \right| \le \frac{2}{R^2}\,\pi R = \frac{2\pi}{R} \to 0 \text{ as } R \to \infty.$$

It is now clear that letting $\rho \to 0$ and $R \to \infty$ yields

$$2\int_0^\infty \frac{\cos(ar) - \cos(br)}{r^2}\,dr = \pi(b-a).$$

This is the desired integration formula, with the variable of integration $r$ instead of $x$. Observe that when $a = 0$ and $b = 2$, that result becomes

$$\int_0^\infty \frac{1 - \cos(2x)}{x^2}\,dx = \pi.$$

But $\cos(2x) = 1 - 2\sin^2 x$, and we arrive at

$$\int_0^\infty \frac{\sin^2 x}{x^2}\,dx = \frac{\pi}{2}.$$

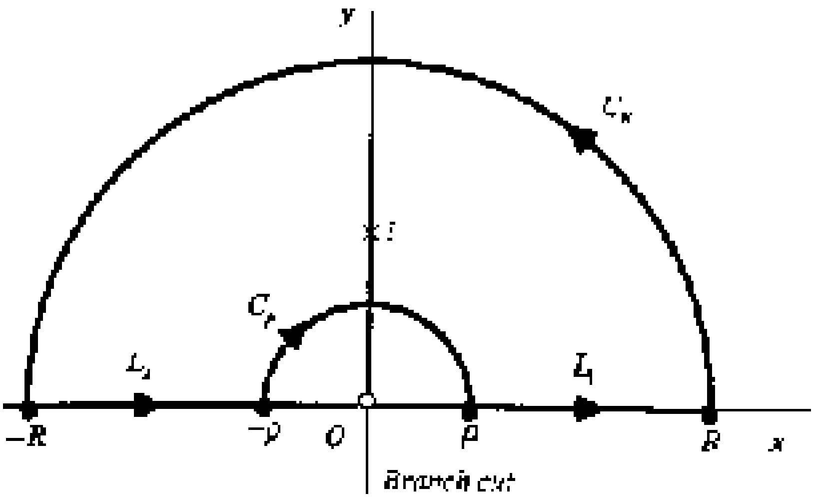
2. Let us derive the integration formula

$$\int_0^\infty \frac{x^a}{(x^2+1)^2}\,dx = \frac{(1-a)\pi}{4\cos(a\pi/2)} \qquad (-1 < a < 3),$$

where $x^a = \exp(a \ln x)$ when $x > 0$. We shall integrate the function

$$f(z) = \frac{z^a}{(z^2+1)^2} = \frac{\exp(a \log z)}{(z^2+1)^2} \qquad \left( |z| > 0, -\frac{\pi}{2} < \arg z < \frac{3\pi}{2} \right),$$

whose branch cut is the origin and the negative imaginary axis, around the simple closed path shown below.




By Cauchy's residue theorem,

$$\int_{L_1} f(z)\,dz + \int_{C_R} f(z)\,dz + \int_{L_2} f(z)\,dz + \int_{C_\rho} f(z)\,dz = 2\pi i \operatorname*{Res}_{z=i} f(z).$$

That is,

$$\int_{L_1} f(z)\,dz + \int_{L_2} f(z)\,dz = 2\pi i \operatorname*{Res}_{z=i} f(z) - \int_{C_\rho} f(z)\,dz - \int_{C_R} f(z)\,dz.$$

Since

$$L_1: z = re^{i0} = r \ (\rho \le r \le R) \quad \text{and} \quad -L_2: z = re^{i\pi} = -r \ (\rho \le r \le R),$$

the left-hand side of this last equation can be written

$$\int_{L_1} f(z)\,dz - \int_{-L_2} f(z)\,dz = \int_\rho^R \frac{e^{a(\ln r + i0)}}{(r^2+1)^2}\,dr - \int_\rho^R \frac{e^{a(\ln r + i\pi)}}{(r^2+1)^2} e^{i\pi}\,dr$$

$$= \int_\rho^R \frac{r^a}{(r^2+1)^2}\,dr + e^{ia\pi}\int_\rho^R \frac{r^a}{(r^2+1)^2}\,dr = (1+e^{ia\pi})\int_\rho^R \frac{r^a}{(r^2+1)^2}\,dr.$$

Also,

$$\operatorname*{Res}_{z=i} f(z) = \phi'(i) \quad \text{where} \quad \phi(z) = \frac{z^a}{(z+i)^2},$$

the point $z = i$ being a pole of order 2 of the function $f(z)$. Straightforward differentiation reveals that

$$\phi'(z) = e^{(a-1)\log z}\left[\frac{a(z+i) - 2z}{(z+i)^3}\right],$$

and from this it follows that

$$\operatorname*{Res}_{z=i} f(z) = -ie^{ia\pi/2}\left(\frac{1-a}{4}\right).$$

We now have

$$(1+e^{ia\pi})\int_\rho^R \frac{r^a}{(r^2+1)^2}\,dr = \frac{\pi(1-a)}{2}e^{ia\pi/2} - \int_{C_\rho} f(z)\,dz - \int_{C_R} f(z)\,dz.$$

Once we show that

$$\lim_{\rho\to 0}\int_{C_\rho} f(z)\,dz = 0 \quad \text{and} \quad \lim_{R\to\infty}\int_{C_R} f(z)\,dz = 0,$$

we arrive at the desired result:

$$\int_0^\infty \frac{r^a}{(r^2+1)^2}\,dr = \frac{\pi(1-a)}{2}\cdot\frac{e^{ia\pi/2}}{1+e^{ia\pi}}\cdot\frac{e^{-ia\pi/2}}{e^{-ia\pi/2}} = \frac{\pi(1-a)}{4}\cdot\frac{2}{e^{ia\pi/2}+e^{-ia\pi/2}} = \frac{(1-a)\pi}{4\cos(a\pi/2)}.$$

The first of the above limits is shown by writing

$$\left|\int_{C_\rho} f(z)\,dz\right| \le \frac{\rho^a}{(1-\rho^2)^2}\pi\rho = \frac{\pi\rho^{a+1}}{(1-\rho^2)^2}$$

and noting that the last term tends to 0 as $\rho \to 0$ since $a+1>0$. As for the second limit,

$$\left|\int_{C_R} f(z)\,dz\right| \le \frac{R^a}{(R^2-1)^2}\pi R = \frac{\pi R^{a+1}}{(R^2-1)^2}\cdot\frac{\dfrac{1}{R^4}}{\dfrac{1}{R^4}} = \frac{\pi\dfrac{1}{R^{3-a}}}{\left(1-\dfrac{1}{R^2}\right)^2};$$

and the last term here tends to 0 as $R\to\infty$ since $3-a>0$.

3. The problem here is to derive the integration formulas

$$I_1 = \int_0^\infty \frac{\sqrt[3]{x}\ln x}{x^2+1}\,dx = \frac{\pi^2}{6} \quad \text{and} \quad I_2 = \int_0^\infty \frac{\sqrt[3]{x}}{x^2+1}\,dx = \frac{\pi}{\sqrt{3}}$$

by integrating the function

$$f(z) = \frac{z^{1/3}\log z}{z^2+1} = \frac{e^{(1/3)\log z}\log z}{z^2+1} \qquad \left(|z|>0, -\frac{\pi}{2}<\arg z<\frac{3\pi}{2}\right).$$

around the contour shown in Exercise 2. As was the case in that exercise,

$$\int_{L_1} f(z)\,dz + \int_{L_2} f(z)\,dz = 2\pi i \operatorname*{Res}_{z=i} f(z) - \int_{C_\rho} f(z)\,dz - \int_{C_R} f(z)\,dz.$$

Since

$$f(z) = \frac{\phi(z)}{z-i} \quad \text{where} \quad \phi(z) = \frac{e^{(1/3)\log z}\log z}{z+i},$$

the point $z = i$ is a simple pole of $f(z)$, with residue

$$\operatorname*{Res}_{z=i} f(z) = \phi(i) = \frac{\pi}{4} e^{i\pi/6}.$$

The parametric representations

$$L_1: z = re^{i0} = r \ (\rho \le r \le R) \quad \text{and} \quad -L_2: z = re^{i\pi} = -r \ (\rho \le r \le R)$$

can be used to write

$$\int_{L_1} f(z)\,dz = \int_\rho^R \frac{\sqrt[3]{r}\ln r}{r^2+1}\,dr \quad \text{and} \quad \int_{L_2} f(z)\,dz = e^{i\pi/3}\int_\rho^R \frac{\sqrt[3]{r}\ln r + i\pi\sqrt[3]{r}}{r^2+1}\,dr.$$

Thus

$$\int_\rho^R \frac{\sqrt[3]{r}\ln r}{r^2+1}\,dr + e^{i\pi/3}\int_\rho^R \frac{\sqrt[3]{r}\ln r + i\pi\sqrt[3]{r}}{r^2+1}\,dr = \frac{\pi^2}{2} i e^{i\pi/6} - \int_{C_\rho} f(z)\,dz - \int_{C_R} f(z)\,dz.$$

By equating real parts on each side of this equation, we have

$$\int_\rho^R \frac{\sqrt[3]{r}\ln r}{r^2+1}\,dr + \cos(\pi/3)\int_\rho^R \frac{\sqrt[3]{r}\ln r}{r^2+1}\,dr - \pi\sin(\pi/3)\int_\rho^R \frac{\sqrt[3]{r}}{r^2+1}\,dr = -\frac{\pi^2}{2}\sin(\pi/6)$$
$$- \operatorname{Re}\int_{C_\rho} f(z)\,dz - \operatorname{Re}\int_{C_R} f(z)\,dz;$$

and equating imaginary parts yields

$$\sin(\pi/3)\int_\rho^R \frac{\sqrt[3]{r}\ln r}{r^2+1}\,dr + \pi\cos(\pi/3)\int_\rho^R \frac{\sqrt[3]{r}}{r^2+1}\,dr = \frac{\pi^2}{2}\cos(\pi/6)$$
$$- \operatorname{Im}\int_{C_\rho} f(z)\,dz - \operatorname{Im}\int_{C_R} f(z)\,dz.$$

Now $\sin(\pi/3) = \dfrac{\sqrt{3}}{2}$, $\cos(\pi/3) = \dfrac{1}{2}$, $\sin(\pi/6) = \dfrac{1}{2}$, $\cos(\pi/6) = \dfrac{\sqrt{3}}{2}$ and it is routine to show that

$$\lim_{\rho\to 0}\int_{C_\rho} f(z)\,dz = 0 \quad \text{and} \quad \lim_{R\to\infty}\int_{C_R} f(z)\,dz = 0.$$

Thus

$$\frac{3}{2}\int_0^\infty \frac{\sqrt[3]{r}\ln r}{r^2+1}\,dr - \frac{\pi\sqrt{3}}{2}\int_0^\infty \frac{\sqrt[3]{r}}{r^2+1}\,dr = -\frac{\pi^2}{4},$$

$$\frac{\sqrt{3}}{2}\int_0^\infty \frac{\sqrt[3]{r}\ln r}{r^2+1}\,dr + \frac{\pi}{2}\int_0^\infty \frac{\sqrt[3]{r}}{r^2+1}\,dr = \frac{\pi^2\sqrt{3}}{4}.$$

That is,

$$\frac{3}{2}I_1 - \frac{\pi\sqrt{3}}{2}I_2 = -\frac{\pi^2}{4},$$

$$\frac{\sqrt{3}}{2}I_1 + \frac{\pi}{2}I_2 = \frac{\pi^2\sqrt{3}}{4}.$$

Solving these simultaneous equations for $I_1$ and $I_2$, we arrive at the desired integration formulas.

4. Let us use the function

$$f(z) = \frac{(\log z)^2}{z^2+1} \qquad \left(|z| > 0, -\frac{\pi}{2} < \arg z < \frac{3\pi}{2}\right)$$

and the contour in Exercise 2 to show that

$$\int_0^\infty \frac{(\ln x)^2}{x^2+1}\,dx = \frac{\pi^3}{8} \quad \text{and} \quad \int_0^\infty \frac{\ln x}{x^2+1}\,dx = 0.$$

Integrating $f(z)$ around the closed path shown in Exercise 2, we have

$$\int_{L_1} f(z)\,dz + \int_{L_2} f(z)\,dz = 2\pi i \operatorname*{Res}_{z=i} f(z) - \int_{C_\rho} f(z)\,dz - \int_{C_R} f(z)\,dz.$$

Since

$$f(z) = \frac{\phi(z)}{z-i} \quad \text{where} \quad \phi(z) = \frac{(\log z)^2}{z+i},$$

the point $z = i$ is a simple pole of $f(z)$ and the residue is

$$\operatorname*{Res}_{z=i} f(z) = \phi(i) = \frac{(\log i)^2}{2i} = \frac{(\ln 1 + i\pi/2)^2}{2i} = -\frac{\pi^2}{8i}.$$

Also, the parametric representations

$$L_1: z = re^{i0} = r \ (\rho \le r \le R) \quad \text{and} \quad -L_2: z = re^{i\pi} = -r \ (\rho \le r \le R)$$

enable us to write

$$\int_{L_1} f(z)dz = \int_\rho^R \frac{(\ln r)^2}{r^2+1}dr \quad \text{and} \quad \int_{L_2} f(z)dz = \int_\rho^R \frac{(\ln r + i\pi)^2}{r^2+1}dr.$$

Since

$$\int_{L_1} f(z)dz + \int_{L_2} f(z)dz = 2\int_\rho^R \frac{(\ln r)^2}{r^2+1}dr - \pi^2 \int_\rho^R \frac{dr}{r^2+1} + 2\pi i \int_\rho^R \frac{\ln r}{r^2+1}dr,$$

then,

$$2\int_\rho^R \frac{(\ln r)^2}{r^2+1}dr - \pi^2 \int_\rho^R \frac{dr}{r^2+1} + 2\pi i \int_\rho^R \frac{\ln r}{r^2+1}dr = -\frac{\pi^3}{4} - \int_{C_\rho} f(z)dz - \int_{C_R} f(z)dz.$$

Equating real parts on each side of this equation, we have

$$2\int_\rho^R \frac{(\ln r)^2}{r^2+1}dr - \pi^2 \int_\rho^R \frac{dr}{r^2+1} = -\frac{\pi^3}{4} - \mathrm{Re}\int_{C_\rho} f(z)dz - \mathrm{Re}\int_{C_R} f(z)dz;$$

and equating imaginary parts yields

$$2\pi \int_\rho^R \frac{\ln r}{r^2+1}dr = \mathrm{Im}\int_{C_\rho} f(z)dz - \mathrm{Im}\int_{C_R} f(z)dz.$$

It is straightforward to show that

$$\lim_{\rho\to 0}\int_{C_\rho} f(z)dz = 0 \quad \text{and} \quad \lim_{R\to\infty}\int_{C_R} f(z)dz = 0.$$

Hence

$$2\int_0^\infty \frac{(\ln r)^2}{r^2+1}dr - \pi^2 \int_0^\infty \frac{dr}{r^2+1} = -\frac{\pi^3}{4}$$

and

$$2\pi \int_0^\infty \frac{\ln r}{r^2+1}dr = 0.$$

Finally, inasmuch as (see Exercise 1, Sec. 72),

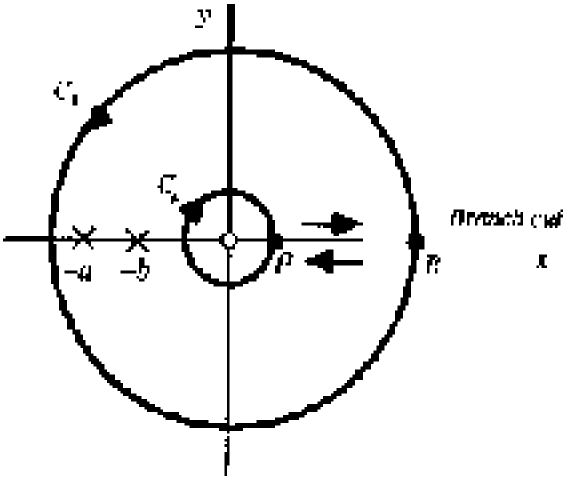
$$\int_0^\infty \frac{dr}{r^2+1} = \frac{\pi}{2},$$

we arrive at the desired integration formulas.

5. Here we evaluate the integral $\int_0^\infty \frac{\sqrt[3]{x}}{(x+a)(x+b)}dx$, where $a > b > 0$. We consider the function

$$f(z) = \frac{z^{1/3}}{(z+a)(z+b)} = \frac{\exp\left(\frac{1}{3}\log z\right)}{(z+a)(z+b)} \qquad (|z|>0,\ 0<\arg z<2\pi)$$

and the simple closed contour shown below, which is similar to the one used in Sec. 77. The numbers $\rho$ and $R$ are small and large enough, respectively, so that the points $z=-a$ and $z=-b$ are between the circles.




A parametric representation for the upper edge of the branch cut from $\rho$ to $R$ is $z = re^{i0}$ $(\rho \le r \le R)$, and so the value of the integral of $f$ along that edge is

$$\int_\rho^R \frac{\exp\left[\frac{1}{3}(\ln r + i0)\right]}{(r+a)(r+b)}dr = \int_\rho^R \frac{\sqrt[3]{r}}{(r+a)(r+b)}dr.$$

A representation for the lower edge from $\rho$ to $R$ is $z = re^{i2\pi}$ $(\rho \le r \le R)$. Hence the value of the integral of $f$ along that edge from $R$ to $\rho$ is

$$\int_R^\rho \frac{\exp\left[\frac{1}{3}(\ln r + i2\pi)\right]}{(r+a)(r+b)}dr = -e^{i2\pi/3}\int_\rho^R \frac{\sqrt[3]{r}}{(r+a)(r+b)}dr.$$

According to the residue theorem, then,

$$\int_\rho^R \frac{\sqrt[3]{r}}{(r+a)(r+b)}dr + \int_{C_R} f(z)dz - e^{i2\pi/3}\int_\rho^R \frac{\sqrt[3]{r}}{(r+a)(r+b)}dr + \int_{C_\rho} f(z)dz = 2\pi i(B_1 + B_2).$$

where

$$R_1 = \operatorname*{Res}_{z=-a} f(z) = \frac{\exp\left[\frac{1}{3}\log(-a)\right]}{-a+b} = -\frac{\exp\left[\frac{1}{3}(\ln a + i\pi)\right]}{a-b} = -\frac{e^{i\pi/3}\sqrt[3]{a}}{a-b}$$

and

$$R_2 = \operatorname*{Res}_{z=-b} f(z) = \frac{\exp\left[\frac{1}{3}\log(-b)\right]}{-b+a} = \frac{\exp\left[\frac{1}{3}(\ln b + i\pi)\right]}{-b+a} = \frac{e^{i\pi/3}\sqrt[3]{b}}{a-b}.$$

Consequently,

$$\left(1-e^{i2\pi/3}\right)\int_\rho^R \frac{\sqrt[3]{r}}{(r+a)(r+b)}\,dr = -\frac{2\pi i e^{i\pi/3}(\sqrt[3]{a}-\sqrt[3]{b})}{a-b} - \int_{C_\rho} f(z)\,dz - \int_{C_R} f(z)\,dz.$$

Now

$$\left|\int_{C_\rho} f(z)\,dz\right| \le \frac{\sqrt[3]{\rho}}{(a-\rho)(b-\rho)}2\pi\rho = \frac{2\pi\sqrt[3]{\rho}\,\rho}{(a-\rho)(b-\rho)} \to 0 \text{ as } \rho \to 0$$

and

$$\left|\int_{C_R} f(z)\,dz\right| \le \frac{\sqrt[3]{R}}{(R-a)(R-b)}2\pi R = \frac{2\pi R^2}{(R-a)(R-b)}\cdot\frac{1}{\sqrt[3]{R^2}} \to 0 \text{ as } R \to \infty.$$

Hence

$$\int_0^\infty \frac{\sqrt[3]{r}}{(r+a)(r+b)}\,dr = -\frac{2\pi i e^{i\pi/3}(\sqrt[3]{a}-\sqrt[3]{b})}{\left(1-e^{i2\pi/3}\right)(a-b)}\cdot\frac{e^{-i\pi/3}}{e^{-i\pi/3}} = \frac{2\pi i(\sqrt[3]{a}-\sqrt[3]{b})}{\left(e^{i\pi/3}-e^{-i\pi/3}\right)(a-b)}$$

$$= \frac{\pi(\sqrt[3]{a}-\sqrt[3]{b})}{\sin(\pi/3)(a-b)} = \frac{\pi(\sqrt[3]{a}-\sqrt[3]{b})}{\frac{\sqrt{3}}{2}(a-b)} = \frac{2\pi}{\sqrt{3}}\cdot\frac{\sqrt[3]{a}-\sqrt[3]{b}}{a-b}.$$

Replacing the variable of integration $r$ here by $x$, we have the desired result:

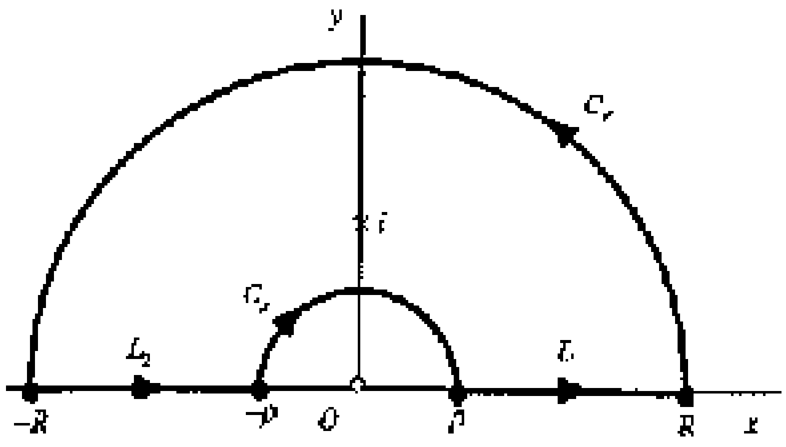
$$\int_0^\infty \frac{\sqrt[3]{x}}{(x+a)(x+b)}\,dx = \frac{2\pi}{\sqrt{3}}\cdot\frac{\sqrt[3]{a}-\sqrt[3]{b}}{a-b} \qquad (a > b > 0).$$

**6.** *(a)* Let us first use the branch

$$f(z)=\frac{z^{-1/2}}{z^2+1}=\frac{\exp\left(-\frac{1}{2}\log z\right)}{z^2+1}\qquad\left(|z|>0,\ -\frac{\pi}{2}<\arg z<\frac{3\pi}{2}\right)$$

and the indented path shown below to evaluate the improper integral

$$\int_0^\infty \frac{dx}{\sqrt{x}\,(x^2+1)}.$$




Cauchy's residue theorem tells us that

$$\int_{L_1} f(z)dz+\int_{C_R} f(z)dz+\int_{L_2} f(z)dz+\int_{C_\rho} f(z)dz=2\pi i\operatorname*{Res}_{z=i} f(z),$$

or

$$\int_{L_1} f(z)dz+\int_{L_2} f(z)dz=2\pi i\operatorname*{Res}_{z=i} f(z)-\int_{C_\rho} f(z)dz-\int_{C_R} f(z)dz.$$

Since

$$L_1: z=re^{i0}=r\ (\rho\le r\le R)\quad\text{and}\qquad L_2: z=re^{i\pi}=-r\ (\rho\le r\le R),$$

we may write

$$\int_{L_1} f(z)dz+\int_{L_2} f(z)dz=\int_\rho^R\frac{dr}{\sqrt{r}\,(r^2+1)}-i\int_\rho^R\frac{dr}{\sqrt{r}\,(r^2+1)}=(1-i)\int_\rho^R\frac{dr}{\sqrt{r}\,(r^2+1)}.$$

Thus

$$(1-i)\int_\rho^R\frac{dr}{\sqrt{r}\,(r^2+1)}=2\pi i\operatorname*{Res}_{z=i} f(z)-\int_{C_\rho} f(z)dz-\int_{C_R} f(z)dz.$$

Now the point $z = i$ is evidently a simple pole of $f(z)$, with residue

$$\operatorname*{Res}_{z=i} f(z) = \left.\frac{z^{-1/2}}{z+i}\right]_{z=i} = \frac{\exp\left[-\frac{1}{2}\log i\right]}{2i} = \frac{\exp\left[-\frac{1}{2}\left(\ln 1 + i\frac{\pi}{2}\right)\right]}{2i} = \frac{e^{-i\pi/4}}{2i} = \frac{1}{2i}\left(\frac{1-i}{\sqrt{2}}\right).$$

Furthermore,

$$\left|\int_{C_\rho} f(z)dz\right| \le \frac{\pi\rho}{\sqrt{\rho}(1-\rho^2)} = \frac{\pi\sqrt{\rho}}{1-\rho^2} \to 0 \text{ as } \rho \to 0$$

and

$$\left|\int_{C_R} f(z)dz\right| \le \frac{\pi\sqrt{R}}{(R^2-1)} = \frac{\pi}{\sqrt{R}\left(R - \frac{1}{R}\right)} \to 0 \text{ as } R \to \infty.$$

Finally, then, we have

$$(1-i)\int_0^\infty \frac{dr}{\sqrt{r}(r^2+1)} = \frac{\pi(1-i)}{\sqrt{2}},$$

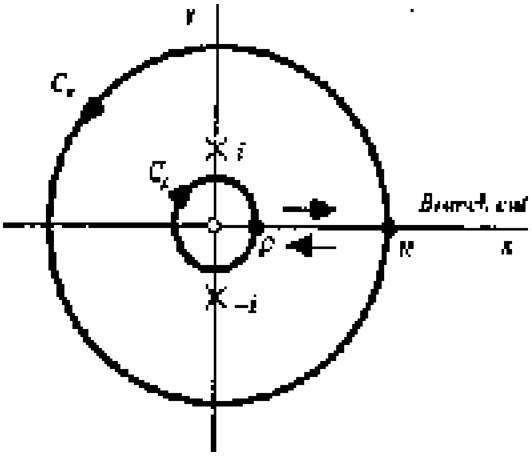
which is the same as

$$\int_0^\infty \frac{dx}{\sqrt{x}(x^2+1)} = \frac{\pi}{\sqrt{2}}.$$

(b) To evaluate the improper integral $\displaystyle\int_0^\infty \frac{dx}{\sqrt{x}(x^2+1)}$, we now use the branch

$$f(z) = \frac{z^{-1/2}}{z^2+1} = \frac{\exp\left(-\frac{1}{2}\log z\right)}{z^2+1} \qquad (|z| > 0,\ 0 < \arg z < 2\pi)$$

and the simple closed contour shown in the figure below, which is similar to Fig. 99 in Sec. 77. We stipulate that $\rho < 1$ and $R > 1$, so that the singularities $z = \pm i$ are between $C_\rho$ and $C_R$.



Since a parametric representation for the upper edge of the branch cut from $\rho$ to $R$ is $z = re^{i0}$ $(\rho \le r \le R)$, the value of the integral of $f$ along that edge is

$$\int_\rho^R \frac{\exp\left[-\frac{1}{2}(\ln r + i0)\right]}{r^2+1}\,dr = \int_\rho^R \frac{1}{\sqrt{r}(r^2+1)}\,dr.$$

A representation for the lower edge from $\rho$ to is $R$ is $z = re^{i2\pi}$ $(\rho \le r \le R)$, and so the value of the integral of $f$ along that edge from $R$ to $\rho$ is

$$-\int_\rho^R \frac{\exp\left[-\frac{1}{2}(\ln r + i2\pi)\right]}{r^2+1}\,dr = -e^{-i\pi}\int_\rho^R \frac{1}{\sqrt{r}(r^2+1)}\,dr = \int_\rho^R \frac{1}{\sqrt{r}(r^2+1)}\,dr.$$

Hence, by the residue theorem,

$$\int_\rho^R \frac{1}{\sqrt{r}(r^2+1)}\,dr + \int_{C_R} f(z)\,dz + \int_\rho^R \frac{1}{\sqrt{r}(r^2+1)}\,dr + \int_{C_\rho} f(z)\,dz = 2\pi i(B_1 + B_2),$$

where

$$B_1 = \operatorname*{Res}_{z=i} f(z) = \frac{z^{-1/2}}{z+i}\bigg]_{z=i} = \frac{\exp\left[-\frac{1}{2}\log i\right]}{2i} = \frac{\exp\left[-\frac{1}{2}\left(\ln 1 + i\frac{\pi}{2}\right)\right]}{2i} = \frac{e^{-i\pi/4}}{2i}$$

and

$$B_2 = \operatorname*{Res}_{z=-i} f(z) = \frac{z^{-1/2}}{z-i}\bigg]_{z=-i} = \frac{\exp\left[-\frac{1}{2}\log(-i)\right]}{-2i} = -\frac{\exp\left[-\frac{1}{2}\left(\ln 1 + i\frac{3\pi}{2}\right)\right]}{2i} = -\frac{e^{-i3\pi/4}}{2i}.$$

That is,

$$2\int_\rho^R \frac{1}{\sqrt{r}(r^2+1)}\,dr = \pi(e^{-i\pi/4} - e^{-i3\pi/4}) - \int_{C_\rho} f(z)\,dz - \int_{C_R} f(z)\,dz$$

Since

$$\left|\int_{C_\rho} f(z)\,dz\right| \le \frac{2\pi\rho}{\sqrt{\rho}(1-\rho^2)} = \frac{2\pi\sqrt{\rho}}{1-\rho^2} \to 0 \text{ as } \rho \to 0$$

and

$$\left|\int_{C_R} f(z)\,dz\right| \le \frac{2\pi R}{\sqrt{R}(R^2-1)} = \frac{2\pi}{\sqrt{R}\left(R - \frac{1}{R}\right)} \to 0 \text{ as } R \to \infty.$$

we now find that

$$\int_0^\infty \frac{1}{\sqrt{r}(r^2+1)}\,dr = \pi\frac{e^{-i\pi/4} - e^{-i3\pi/4}}{2} = \pi\frac{e^{-i\pi/4} + e^{-i3\pi/4}e^{i\pi}}{2}$$

$$= \pi\frac{e^{i\pi/4} + e^{-i\pi/4}}{2} = \pi\cos\left(\frac{\pi}{4}\right) = \frac{\pi}{\sqrt{2}}.$$

When $x$, instead of $r$, is used as the variable of integration here, we have the desired result:

$$\int_0^\infty \frac{dx}{\sqrt{x}(x^2+1)} = \frac{\pi}{\sqrt{2}}.$$

SECTION 78

1. Write

$$\int_0^{2\pi} \frac{d\theta}{5+4\sin\theta} = \int_C \frac{1}{5+4\left(\dfrac{z-z^{-1}}{2i}\right)}\cdot\frac{dz}{iz} = \int_C \frac{dz}{2z^2+5iz-2},$$

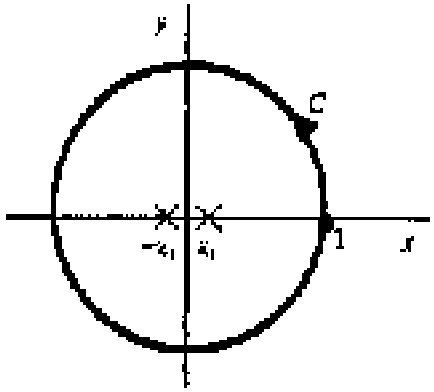
where $C$ is the positively oriented unit circle $|z|=1$. The quadratic formula tells us that the singular points of the integrand on the far right here are $z=-i/2$ and $z=-2i$. The point $z=-i/2$ is a simple pole interior to $C$; and the point $z=-2i$ is exterior to $C$. Thus

$$\int_0^{2\pi} \frac{d\theta}{5+4\sin\theta} = 2\pi i \operatorname*{Res}_{z=-i/2}\left[\frac{1}{2z^2+5iz-2}\right] = 2\pi i\left[\frac{1}{4z+5i}\right]_{z=-i/2} = 2\pi i\left(\frac{1}{3i}\right) = \frac{2\pi}{3}.$$

2. To evaluate the definite integral in question, write

$$\int_{-\pi}^{\pi} \frac{d\theta}{1+\sin^2\theta} = \int_C \frac{1}{1+\left(\dfrac{z-z^{-1}}{2i}\right)^2}\cdot\frac{dz}{iz} = \int_C \frac{4iz\,dz}{z^4-6z^2+1},$$

where $C$ is the positively oriented unit circle $|z|=1$. This circle is shown below.



Solving the equation $(z^2)^2 - 6(z^2) + 1 = 0$ for $z^2$ with the aid of the quadratic formula, we find that the zeros of the polynomial $z^4 - 6z^2 + 1$ are the numbers $z$ such that $z^2 = 3 \pm 2\sqrt{2}$. Those zeros are, then, $z = \pm\sqrt{3+2\sqrt{2}}$ and $z = \pm\sqrt{3-2\sqrt{2}}$. The first two of these zeros are exterior to the circle, and the second two are inside of it. So the singularities of the integrand in our contour integral are

$$z_1 = \sqrt{3-2\sqrt{2}} \quad \text{and} \quad z_2 = -z_1,$$

indicated in the figure. This means that

$$\int_{-\pi}^{\pi} \frac{d\theta}{1+\sin^2\theta} = 2\pi i(B_1 + B_2),$$

where

$$B_1 = \operatorname*{Res}_{z=z_1} \frac{4iz}{z^4 - 6z^2 + 1} = \frac{4iz_1}{4z_1^3 - 12z_1} = \frac{i}{z_1^2 - 3} = \frac{i}{(3-2\sqrt{2})-3} = -\frac{i}{2\sqrt{2}}$$

and

$$B_2 = \operatorname*{Res}_{z=-z_1} \frac{4iz}{z^4 - 6z^2 + 1} = \frac{-4iz_1}{-4z_1^3 + 12z_1} = \frac{i}{z_1^2 - 3} = -\frac{i}{2\sqrt{2}}.$$

Since

$$2\pi i(B_1 + B_2) = 2\pi i\left(-\frac{i}{\sqrt{2}}\right) = \frac{2\pi}{\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}} = \sqrt{2}\pi,$$

the desired result is

$$\int_{-\pi}^{\pi} \frac{d\theta}{1+\sin^2\theta} = \sqrt{2}\pi.$$

7. Let $C$ be the positively oriented unit circle $|z| = 1$. In view of the binomial formula (Sec. 3)

$$\int_0^{\pi} \sin^{2n}\theta\, d\theta = \frac{1}{2}\int_{-\pi}^{\pi} \sin^{2n}\theta\, d\theta = \frac{1}{2}\int_C \left(\frac{z - z^{-1}}{2i}\right)^{2n} \frac{dz}{iz} = \frac{1}{2^{2n+1}(-1)^n i}\int_C \frac{(z-z^{-1})^{2n}}{z}\, dz$$

$$= \frac{1}{2^{2n+1}(-1)^n i}\int_C \sum_{k=0}^{2n} \binom{2n}{k} z^{2n-k}(-z^{-1})^k z^{-1}\, dz$$

$$= \frac{1}{2^{2n+1}(-1)^n i}\sum_{k=0}^{2n} \binom{2n}{k}(-1)^k \int_C z^{2n-2k-1}\, dz.$$

Now each of these last integrals has value zero except when $k = n$:

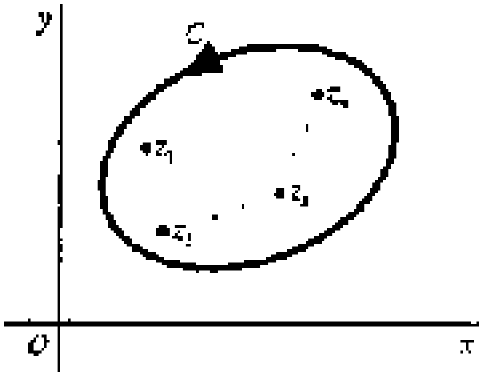
$$\int_C z^{-1}\,dz = 2\pi i.$$

Consequently,

$$\int_0^{\pi} \sin^{2n}\theta\,d\theta = \frac{1}{2^{2n+1}(-1)^n i}\cdot\frac{(2n)!(-1)^n 2\pi i}{(n!)^2} = \frac{(2n)!}{2^{2n}(n!)^2}\pi.$$

SECTION 80

5. We are given a function $f$ that is analytic inside and on a positively oriented simple closed contour $C$, and we assume that $f$ has no zeros on $C$. Also, $f$ has $n$ zeros $z_k$ $(k = 1, 2, \ldots, n)$ inside $C$, where each $z_k$ is of multiplicity $m_k$. (See the figure below.)




The object here is to show that

$$\int_C \frac{zf'(z)}{f(z)}\,dz = 2\pi i\sum_{k=1}^{n} m_k z_k.$$

To do this, we consider the $k$th zero and start with the fact that

$$f(z) = (z - z_k)^{m_k} g(z),$$

where $g(z)$ is analytic and nonzero at $z_k$. From this, it is straightforward to show that

$$\frac{zf'(z)}{f(z)} = \frac{m_k z}{z - z_k} + \frac{zg'(z)}{g(z)} = \frac{m_k(z - z_k) + m_k z_k}{z - z_k} + \frac{zg'(z)}{g(z)} = m_k + \frac{zg'(z)}{g(z)} + \frac{m_k z_k}{z - z_k}.$$

Since the term $\dfrac{zg'(z)}{g(z)}$ here has a Taylor series representation at $z_k$, it follows that $\dfrac{zf'(z)}{f(z)}$ has a simple pole at $z_k$ and that

$$\operatorname*{Res}_{z=z_k} \frac{zf'(z)}{f(z)} = m_k z_k.$$

An application of the residue theorem now yields the desired result.

6\. *(a)* To determine the number of zeros of the polynomial $z^6 - 5z^4 + z^3 - 2z$ inside the circle $|z| = 1$, we write

$$f(z) = -5z^4 \quad \text{and} \quad g(z) = z^6 + z^3 - 2z.$$

We then observe that when $z$ is on the circle,

$$|f(z)| = 5 \quad \text{and} \quad |g(z)| \le |z|^6 + |z|^3 + 2|z| = 4.$$

Since $|f(z)| > |g(z)|$ on the circle and since $f(z)$ has 4 zeros, counting multiplicities, inside it, the theorem in Sec. 80 tells is that the sum

$$f(z) + g(z) = z^6 - 5z^4 + z^3 - 2z$$

also has four zeros, counting multiplicities, inside the circle.

*(b)* Let us write the polynomial $2z^4 - 2z^3 + 2z^2 - 2z + 9$ as the sum $f(z) + g(z)$, where

$$f(z) = 9 \quad \text{and} \quad g(z) = 2z^4 - 2z^3 + 2z^2 - 2z.$$

Observe that when $z$ is on the circle $|z| = 1$,

$$|f(z)| = 9 \quad \text{and} \quad |g(z)| \le 2|z|^4 + 2|z|^3 + 2|z|^2 + 2|z| = 8.$$

Since $|f(z)| > |g(z)|$ on the circle and since $f(z)$ has no zeros inside it, the sum $f(z) + g(z) = 2z^4 - 2z^3 + 2z^2 - 2z + 9$ has no zeros there either.

7\. Let $C$ denote the circle $|z| = 2$.

*(a)* The polynomial $z^4 + 3z^3 + 6$ can be written as the sum of the polynomials

$$f(z) = 3z^3 \quad \text{and} \quad g(z) = z^4 + 6.$$

On $C$,

$$|f(z)| = 3|z|^3 = 24 \quad \text{and} \quad |g(z)| = |z^4 + 6| \le |z|^4 + 6 = 22.$$

Since $|f(z)| > |g(z)|$ on $C$ and $f(z)$ has 3 zeros, counting multiplicities, inside $C$, it follows that the original polynomial has 3 zeros, counting multiplicities, inside $C$.

*(b)* The polynomial $z^4 - 2z^3 + 9z^2 + z - 1$ can be written as the sum of the polynomials

$$f(z) = 9z^2 \quad \text{and} \quad g(z) = z^4 - 2z^3 + z - 1.$$

On $C$,

$$|f(z)| = 9|z|^2 = 36 \quad \text{and} \quad |g(z)| = |z^4 - 2z^3 + z - 1| \le |z|^4 + 2|z|^3 + |z| + 1 = 35.$$

Since $|f(z)|>|g(z)|$ on $C$ and $f(z)$ has 2 zeros, counting multiplicities, inside $C$, it follows that the original polynomial has 2 zeros, counting multiplicities, inside $C$.

(c) The polynomial $z^5+3z^3+z^2+1$ can be written as the sum of the polynomials

$$f(z)=z^5 \quad \text{and} \quad g(z)=3z^3+z^2+1.$$

On $C$,

$$|f(z)|=|z|^5=32 \quad \text{and} \quad |g(z)|=|3z^3+z^2+1|\le 3|z|^3+|z|^2+1=29.$$

Since $|f(z)|>|g(z)|$ on $C$ and $f(z)$ has 5 zeros, counting multiplicities, inside $C$, it follows that the original polynomial has 5 zeros, counting multiplicities, inside $C$.

10. The problem here is to give an alternative proof of the fact that any polynomial

$$P(z)=a_0+a_1z+\cdots+a_{n-1}z^{n-1}+a_nz^n \qquad (a_n\neq 0),$$

where $n\ge 1$, has precisely $n$ zeros, counting multiplicities. Without loss of generality, we may take $a_n=1$ since

$$P(z)=a_n\left(\frac{a_0}{a_n}+\frac{a_1}{a_n}z+\cdots+\frac{a_{n-1}}{a_n}z^{n-1}+z^n\right).$$

Let

$$f(z)=z^n \quad \text{and} \quad g(z)=a_0+a_1z+\cdots+a_{n-1}z^{n-1}.$$

Then let $R$ be so large that

$$R>1+|a_0|+|a_1|+\cdots+|a_{n-1}|.$$

If $z$ is a point on the circle $C: |z|=R$, we find that

$$\begin{aligned}|g(z)|&\le|a_0|+|a_1||z|+\cdots+|a_{n-1}||z|^{n-1}=|a_0|+|a_1|R+\cdots+|a_{n-1}|R^{n-1}\\&<|a_0|R^{n-1}+|a_1|R^{n-1}+\cdots+|a_{n-1}|R^{n-1}=(|a_0|+|a_1|+\cdots+|a_{n-1}|)R^{n-1}\\&<RR^{n-1}=R^n=|z|^n=|f(z)|.\end{aligned}$$

Since $f(z)$ has precisely $n$ zeros, counting multiplicities, inside $C$ and since $R$ can be made arbitrarily large, the desired result follows.

**1.** The singularities of the function

$$F(s)=\frac{2s^3}{s^4-4}$$

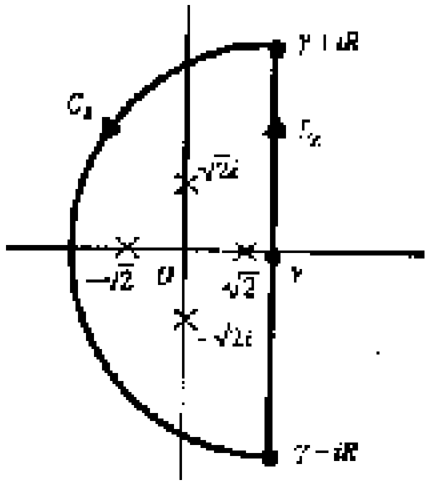
are the fourth roots of 4. They are readily found to be

$$s=\sqrt{2}\,e^{ik\pi/2} \qquad (k=0,1,2,3),$$

or

$$\sqrt{2},\quad \sqrt{2}i,\quad -\sqrt{2},\quad \text{and}\quad -\sqrt{2}i.$$

See the figure below, where $\gamma>\sqrt{2}$ and $R>\sqrt{2}+\gamma$.




The function

$$e^{st}F(s)=\frac{2s^3e^{st}}{s^4-4}$$

has simple poles at the points

$$s_0=\sqrt{2},\quad s_1=\sqrt{2}i,\quad s_2=-\sqrt{2},\quad \text{and}\quad s_3=-\sqrt{2}i;$$

and

$$\sum_{n=0}^{3}\operatorname*{Res}_{s=s_n}\left[e^{st}F(s)\right]=\sum_{n=0}^{3}\operatorname*{Res}_{s=s_n}\frac{2s^3e^{st}}{s^4-4}=\sum_{n=0}^{3}\frac{2s_n^3e^{s_nt}}{4s_n^3}=\sum_{n=0}^{3}\frac{1}{2}e^{s_nt}$$

$$=\frac{1}{2}e^{\sqrt{2}t}+\frac{1}{2}e^{i\sqrt{2}t}+\frac{1}{2}e^{-\sqrt{2}t}+\frac{1}{2}e^{-i\sqrt{2}t}$$

$$=\frac{e^{\sqrt{2}t}+e^{-\sqrt{2}t}}{2}+\frac{e^{i\sqrt{2}t}+e^{-i\sqrt{2}t}}{2}$$

$$=\cosh\sqrt{2}t+\cos\sqrt{2}t.$$

Suppose now that $s$ is a point on $C_R$, and observe that

$$|s| = |\gamma + Re^{i\theta}| \leq \gamma + R = R + \gamma \quad \text{and} \quad |s| = |\gamma + Re^{i\theta}| \geq |\gamma - R| = R - \gamma > \sqrt{2}.$$

It follows that

$$|2s^3| = 2|s|^3 \leq 2(R+\gamma)^3$$

and

$$|s^4 - 4| \geq ||s|^4 - 4| \geq (R-\gamma)^4 - 4 > 0.$$

Consequently,

$$|F(s)| \leq \frac{2(R+\gamma)^3}{(R-\gamma)^4 - 4} \to 0 \text{ as } R \to \infty.$$

This ensures that

$$f(t) = \cosh\sqrt{2}t + \cos\sqrt{2}t.$$

2. The polynomials in the denominator of

$$F(s) = \frac{2s-2}{(s+1)(s^2+2s+5)}$$

have zeros at $s = -1$ and $s = -1 \pm 2i$. Let us, then, write

$$e^{st}F(s) = \frac{e^{st}(2s-2)}{(s+1)(s-s_1)(s-\bar{s}_1)},$$

where $s_1 = -1+2i$. The points $-1$, $s_1$, and $\bar{s}_1$ are evidently simple poles of $e^{st}F(s)$ with the following residues:

$$B_1 = \operatorname*{Res}_{s=-1}\left[e^{st}F(s)\right] = \left.\frac{e^{st}(2s-2)}{(s-s_1)(s-\bar{s}_1)}\right]_{s=-1} = -e^{-t},$$

$$B_2 = \operatorname*{Res}_{s=s_1}\left[e^{st}F(s)\right] = \frac{e^{s_1 t}(2s_1-2)}{(s_1+1)(s_1-\bar{s}_1)} = \left(\frac{1}{2} - \frac{i}{2}\right)e^{-t}e^{i2t},$$

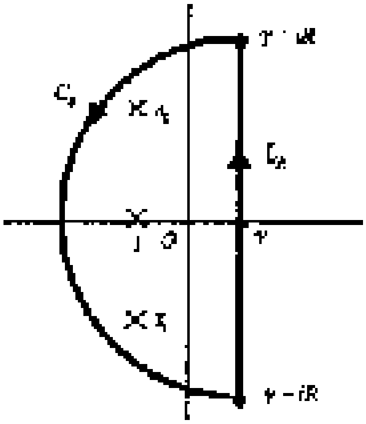
$$B_3 = \operatorname*{Res}_{s=\bar{s}_1}\left[e^{st}F(s)\right] = \frac{e^{\bar{s}_1 t}(2\bar{s}_1-2)}{(\bar{s}_1+1)(\bar{s}_1-s_1)} = \overline{\left[\frac{e^{s_1 t}(2s_1-2)}{(s_1+1)(s_1-\bar{s}_1)}\right]} = \bar{B}_2 = \left(\frac{1}{2} + \frac{i}{2}\right)e^{-t}e^{-i2t}.$$

It is easy to see that

$$R_1+R_2+R_3=-e^{-t}+\left(\frac{1}{2}-\frac{i}{2}\right)e^{-t}e^{i2t}+\left(\frac{1}{2}+\frac{i}{2}\right)e^{-t}e^{-i2t}$$

$$=-e^{-t}+e^{-t}\left(\frac{e^{i2t}-e^{-i2t}}{2i}+\frac{e^{i2t}+e^{-i2t}}{2}\right)=e^{-t}(\sin 2t+\cos 2t-1).$$

Now let $s$ be any point on the semicircle shown below, where $\gamma>0$ and $R>\sqrt{5}+\gamma$.




Since

$$|s|=|\gamma+Re^{i\theta}|\le\gamma+R=R+\gamma\quad\text{and}\quad|s|=|\gamma+Re^{i\theta}|\ge|\gamma-R|=R-\gamma>\sqrt{5},$$

we find that

$$|2s-2|\le 2|s|+2\le 2(R+\gamma)+2,$$

$$|s+1|\ge||s|-1|\ge(R-\gamma)-1>0,$$

and

$$|s^2-2s+5|=|s-s_1||s-\bar{s}_1|\ge(|s|-|s_1|)^2\ge\left[(R-\gamma)^2-\sqrt{5}\right]^2>0.$$

Thus

$$|F(s)|=\frac{|2s-2|}{|s+1||s^2+2s+5|}\le\frac{2(R+\gamma)+2}{\left[(R-\gamma)-1\right]\left[(R-\gamma)^2-\sqrt{5}\right]^2}\to 0\text{ as }R\to\infty,$$

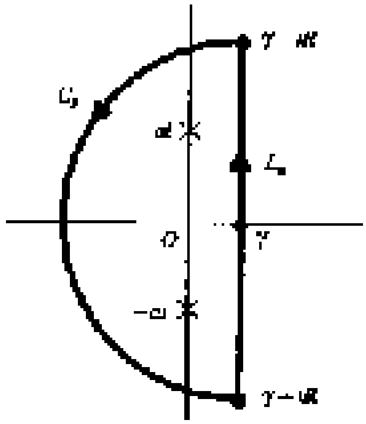
and we may conclude that

$$f(t)=e^{-t}(\sin 2t+\cos 2t-1).$$

4. The function

$$F(s) = \frac{s^2 - a^2}{(s^2 + a^2)^2} \qquad (a > 0)$$

has singularities at $s = \pm ai$. So we consider the simple closed contour shown below, where $\gamma > 0$ and $R > a + \gamma$.




Upon writing

$$F(s) = \frac{\phi(s)}{(s - ai)^2} \quad \text{where} \quad \phi(s) = \frac{s^2 - a^2}{(s + ai)^2},$$

we see that $\phi(s)$ is analytic and nonzero at $s_1 = ai$. Hence $s_1$ is a pole of order $m = 2$ of $F(s)$. Furthermore, $\overline{F(s)} = F(\bar{s})$ at points where $F(s)$ is analytic. Consequently, $\bar{s}_1$ is also a pole of order 2 of $F(s)$; and we know from expression (2), Sec. 82, that

$$\operatorname*{Res}_{s=s_1}\left[e^{st}F(s)\right] + \operatorname*{Res}_{s=\bar{s}_1}\left[e^{st}F(s)\right] = 2\operatorname{Re}\left[e^{iat}(b_1 + b_2 t)\right],$$

where $b_1$ and $b_2$ are the coefficients in the principal part

$$\frac{b_1}{s - ai} + \frac{b_2}{(s - ai)^2}$$

of $F(s)$ at $ai$. These coefficients are readily found with the aid of the first two terms in the Taylor series for $\phi(s)$ about $s_1 = ai$:

$$F(s) = \frac{1}{(s - ai)^2}\,\phi(s) = \frac{1}{(s - ai)^2}\left[\phi(ai) + \frac{\phi'(ai)}{1!}(s - ai) + \cdots\right]$$

$$= \frac{\phi(ai)}{(s-ai)^2} + \frac{\phi'(ai)}{s-ai} + \cdots \qquad (0 < |s-ai| < 2a).$$

It is straightforward to show that $\phi(ai) = 1/2$ and $\phi'(ai) = 0$, and we find that $b_1 = 0$ and $b_2 = 1/2$. Hence

$$\operatorname*{Res}_{s=s_1}\left[e^{st}F(s)\right] + \operatorname*{Res}_{s=\bar{s}_1}\left[e^{st}F(s)\right] = 2\operatorname{Re}\left[e^{iat}\left(\frac{1}{2}t\right)\right] = t\cos at.$$

We can, then, conclude that

$$f(t) = t\cos at \qquad (a > 0),$$

provided that $F(s)$ satisfies the desired boundedness condition. As for that condition, when $s$ is a point on $C_R$,

$$|s| = |\gamma + Re^{i\theta}| \le \gamma + R = R + \gamma \quad \text{and} \quad |s| = |\gamma + Re^{i\theta}| \ge |\gamma - R| = R - \gamma > a;$$

and this means that

$$|s^2 - a^2| \le |s|^2 + a^2 \le (R+\gamma)^2 + a^2 \quad \text{and} \quad |s^2 + a^2| \ge ||s|^2 - a^2| \ge (R-\gamma)^2 - a^2 > 0.$$

Hence

$$|F(s)| \le \frac{(R+\gamma)^2 + a^2}{[(R-\gamma)^2 - a^2]^2} \to 0 \text{ as } R \to \infty.$$

6. We are given

$$F(s) = \frac{\sinh(xs)}{s^2 \cosh s} \qquad (0 < x < 1),$$

which has isolated singularities at the points

$$s_0 = 0, \quad s_n = \frac{(2n-1)\pi}{2}i, \quad \text{and} \quad \bar{s}_n = -\frac{(2n-1)\pi}{2}i \qquad (n = 1, 2, \ldots).$$

This function has the property $\overline{F(s)} = F(\bar{s})$, and so

$$f(t) = \operatorname*{Res}_{s=s_0}\left[e^{st}F(s)\right] + \sum_{n=1}^{\infty}\left\{\operatorname*{Res}_{s=s_n}\left[e^{st}F(s)\right] + \operatorname*{Res}_{s=\bar{s}_n}\left[e^{st}F(s)\right]\right\}.$$

To find the residue at $s_0 = 0$, we write

$$\frac{\sinh(xs)}{s^2\cosh s} = \frac{xs + (xs)^3/3! + \cdots}{s^2(1 + s^2/2! + \cdots)} = \frac{x + x^3 s^2/6 + \cdots}{s + s^3/2 + \cdots} \qquad \left(0 < |s| < \frac{\pi}{2}\right).$$

Division of series then reveals that $s_0$ is a simple pole of $F(s)$, with residue $x$; and, according to expression (5), Sec. 82,

$$\operatorname*{Res}_{s=s_0}\left[e^{st}F(s)\right] = \operatorname*{Res}_{s=s_0} F(s) = x.$$

As for the residues of $F(s)$ at the singular points $s_n$ $(n=1,2,\ldots)$, we write

$$F(s)=\frac{p(s)}{q(s)} \quad \text{where} \quad p(s)=\sinh(xs) \text{ and } q(s)=s^2\cosh s.$$

We note that

$$p(s_n)= i\sin\frac{(2n-1)\pi x}{2}\neq 0 \quad \text{and} \quad q(s_n)=0;$$

furthermore, since

$$q'(s)=2s\cosh s + s^2\sinh s,$$

we find that

$$q'(s_n) = -\frac{(2n-1)^2\pi^2}{4} i\sin\frac{(2n-1)\pi}{2} = -i\frac{(2n-1)^2\pi^2}{4}\sin\left(n\pi-\frac{\pi}{2}\right)$$

$$= -i\frac{(2n-1)^2\pi^2}{4}\left(\sin n\pi\cos\frac{\pi}{2}-\cos n\pi\sin\frac{\pi}{2}\right) = \frac{(2n-1)^2\pi^2}{4}(-1)^n i\neq 0.$$

In view of Theorem 2 in Sec. 69, then, $s_n$ is a simple pole of $F(s)$, and

$$\operatorname*{Res}_{s=s_n}F(s)=\frac{p(s_n)}{q'(s_n)} = \frac{4}{\pi^2}\cdot\frac{(-1)^n}{(2n-1)^2}\sin\frac{(2n-1)\pi x}{2}.$$

Expression (4), Sec. 82, now gives us

$$\operatorname*{Res}_{s=s_n}\left[e^{st}F(s)\right] + \operatorname*{Res}_{s=\bar{s}_n}\left[e^{st}F(s)\right] = 2\operatorname{Re}\left\{\frac{4}{\pi^2}\cdot\frac{(-1)^n}{(2n-1)^2}\sin\frac{(2n-1)\pi x}{2}\exp\left[i\frac{(2n-1)\pi t}{2}\right]\right\}$$

$$=\frac{8}{\pi^2}\cdot\frac{(-1)^n}{(2n-1)^2}\sin\frac{(2n-1)\pi x}{2}\cos\frac{(2n-1)\pi t}{2}.$$

Summing all of the above residues, we arrive at the final result:

$$f(t) = x + \frac{8}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^n}{(2n-1)^2}\sin\frac{(2n-1)\pi x}{2}\cos\frac{(2n-1)\pi t}{2}.$$

7. The function

$$F(s) = \frac{1}{s\cosh(s^{1/2})},$$

where it is agreed that the branch cut of $s^{1/2}$ does not lie along the negative real axis, has isolated singularities at $s_0 = 0$ and when $\cosh(s^{1/2}) = 0$, or at the points $s_n = -\frac{(2n-1)^2\pi^2}{4}$ $(n = 1, 2, \ldots)$. The point $s_0$ is a simple pole of $F(s)$, as is seen by writing

$$\frac{1}{s\cosh(s^{1/2})} = \frac{1}{s\left[1 + (s^{1/2})^2/2! + (s^{1/2})^4/4! + \cdots\right]} = \frac{1}{s + s^2/2 + s^3/24 + \cdots}$$

and dividing this last denominator into 1. In fact, the residue is found to be 1; and expression (3), Sec. 82, tells us that

$$\operatorname*{Res}_{s=s_0}\left[e^{st}F(s)\right] = \operatorname*{Res}_{s=s_0} F(s) = 1.$$

As for the other singularities, we write

$$F(s) = \frac{p(s)}{q(s)} \quad \text{where} \quad p(s) = 1 \text{ and } q(s) = s\cosh(s^{1/2}).$$

Now

$$p(s_n) = 1 \neq 0 \quad \text{and} \quad q(s_n) = 0;$$

also, since

$$q'(s) = \frac{1}{2}s^{1/2}\sinh(s^{1/2}) + \cosh(s^{1/2}),$$

it is straightforward to show that

$$q'(s_n) = -\frac{(2n-1)\pi}{4}\sin\left(n\pi - \frac{\pi}{2}\right) = \frac{(2n-1)\pi}{4}(-1)^n \neq 0.$$

So each point $s_n$ is a simple pole of $F(s)$, and

$$\operatorname*{Res}_{s=s_n} F(s) = \frac{p(s_n)}{q'(s_n)} = \frac{4}{\pi}\cdot\frac{(-1)^n}{2n-1}.$$

Consequently, according to expression (3), Sec. 82,

$$\operatorname*{Res}_{s=s_n}\left[e^{st}F(s)\right] = e^{s_n t}\operatorname*{Res}_{s=s_n} F(s) = \frac{4}{\pi}\cdot\frac{(-1)^n}{2n-1}\exp\left[-\frac{(2n-1)^2\pi^2 t}{4}\right] \qquad (n=1,2,\ldots).$$

Finally, then,

$$f(t) = \operatorname*{Res}_{s=s_0}\left[e^{st}F(s)\right] + \sum_{n=1}^{\infty}\operatorname*{Res}_{s=s_n}\left[e^{st}F(s)\right],$$

or

$$f(t) = 1 + \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{2n-1}\exp\left[-\frac{(2n-1)^2\pi^2 t}{4}\right].$$

8. Here we are given the function

$$F(s) = \frac{\coth(\pi s/2)}{s^2+1} = \frac{\cosh(\pi s/2)}{(s^2+1)\sinh(\pi s/2)},$$

which has the property $\overline{F(s)} = F(\bar{s})$. We consider first the singularities at $s = \pm i$. Upon writing

$$F(s) = \frac{\phi(s)}{s-i} \quad \text{where} \quad \phi(s) = \frac{\cosh(\pi s/2)}{(s+i)\sinh(\pi s/2)},$$

we find that, since $\phi(i) = 0$, the point $i$ is a removable singularity of $F(s)$ [see Exercise 3(b), Sec. 65]; and the same is true of the point $-i$. At each of these points, it follows that the residue of $e^{st}F(s)$ is 0. The other singularities occur when $\pi s/2 = n\pi i$ $(n = 0,\pm1,\pm2,\ldots)$, or at the points $s = 2ni$ $(n = 0,\pm1,\pm2,\ldots)$. To find the residues, we write

$$F(s) = \frac{p(s)}{q(s)} \quad \text{where} \quad p(s) = \cosh\left(\frac{\pi s}{2}\right) \text{ and } q(s) = (s^2+1)\sinh\left(\frac{\pi s}{2}\right).$$

and note that

$$p(2ni) = \cosh(n\pi i) = \cos(n\pi) = (-1)^n \neq 0 \quad \text{and} \quad q(2ni) = 0.$$

Furthermore, since

$$q'(s) = (s^2+1)\frac{\pi}{2}\cosh\left(\frac{\pi s}{2}\right) + 2s\sinh\left(\frac{\pi s}{2}\right),$$

we have

$$q'(2ni) = (-4n^2+1)\frac{\pi}{2}\cosh(n\pi i) = (-4n^2+1)\frac{\pi}{2}\cos(n\pi) = -\frac{\pi(4n^2-1)}{2}(-1)^n \neq 0$$

Thus

$$\operatorname*{Res}_{s=2ni} F(s) = \frac{p(2ni)}{q'(2ni)} = -\frac{2}{\pi}\cdot\frac{1}{4n^2-1} \qquad (n = 0, \pm 1, \pm 2, \ldots).$$

Expressions (3) and (4) in Sec. 82 now tell us that

$$\operatorname*{Res}_{s=0}\left[e^{st}F(s)\right] = \operatorname*{Res}_{s=0} F(s) = \frac{2}{\pi}$$

and

$$\operatorname*{Res}_{s=2ni}\left[e^{st}F(s)\right] + \operatorname*{Res}_{s=-2ni}\left[e^{st}F(s)\right] = 2\operatorname{Re}\left[e^{i2nt}\left(-\frac{2}{\pi}\cdot\frac{1}{4n^2-1}\right)\right] = -\frac{4}{\pi}\cdot\frac{\cos 2nt}{4n^2-1} \qquad (n = 1, 2, \ldots).$$

The desired function of $t$ is, then,

$$f(t) = \frac{2}{\pi} - \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{\cos 2nt}{4n^2-1}.$$

9. The function

$$F(s) = \frac{\sinh(xs^{1/2})}{s^2\sinh(s^{1/2})} \qquad (0 < x < 1),$$

where it is agreed that the branch cut of $s^{1/2}$ does not lie along the negative real axis, has isolated singularities at $s = 0$ and when $\sinh(s^{1/2}) = 0$, or at the points $s = -n^2\pi^2$ $(n = 1, 2, \ldots)$. The point $s = 0$ is a pole of order 2 of $F(s)$, as is seen by first writing

$$\frac{\sinh(xs^{1/2})}{s^2\sinh(s^{1/2})} = \frac{xs^{1/2} + (xs^{1/2})^3/3! + (xs^{1/2})^5/5! + \cdots}{s^2\left[s^{1/2} + (s^{1/2})^3/3! + (s^{1/2})^5/5! + \cdots\right]} = \frac{x + x^3s/6 + x^5s^2/120 + \cdots}{s^2 + s^3/6 + s^4/120 + \cdots}$$

and dividing the series in the denominator into the series in the numerator. The result is

$$\frac{\sinh(xs^{1/2})}{s^2\sinh(s^{1/2})} = x\frac{1}{s^2} + \frac{1}{6}(x^3 - x)\frac{1}{s} + \cdots \qquad (0 < |s| < \pi^2).$$

In view of expression (1), Sec. 82, then,

$$\operatorname*{Res}_{s=0}\left[e^{st}F(s)\right] = \frac{1}{6}(x^3 - x) + xt = \frac{1}{6}x(x^2 - 1) + xt.$$

As for the singularities $s = -n^2\pi^2$ $(n = 1, 2, \ldots)$, we write

$$F(s) = \frac{p(s)}{q(s)} \quad \text{where} \quad p(s) = \sinh(xs^{1/2}) \text{ and } q(s) = s^2\sinh(s^{1/2}).$$

Observe that $p(-n^2\pi^2) \neq 0$ and $q(-n^2\pi^2) = 0$. Also, since

$$q'(s) = 2s\sinh(s^{1/2}) + \frac{1}{2}s s^{1/2}\cosh(s^{1/2}),$$

it is easy to see that $q'(-n^2\pi^2) \neq 0$. So the points $s = -n^2\pi^2$ $(n = 1, 2, \ldots)$, are simple poles of $F(s)$, and

$$\operatorname*{Res}_{s=-n^2\pi^2} F(s) = \frac{p(s)}{q'(s)}\bigg]_{s=-n^2\pi^2} = \frac{2\sinh(xs^{1/2})}{ss^{1/2}\cosh(s^{1/2})}\bigg]_{s=-n^2\pi^2} = \frac{2}{\pi^3}\cdot\frac{(-1)^{n+1}}{n^3}\sin n\pi x \qquad (n = 1, 2, \ldots).$$

Thus, in view of expression (3), Sec. 82,

$$\operatorname*{Res}_{s=-n^2\pi^2}\left[e^{st}F(s)\right] = \frac{2}{\pi^3}\cdot\frac{(-1)^{n+1}}{n^3}e^{-n^2\pi^2 t}\sin n\pi x \qquad (n = 1, 2, \ldots).$$

Finally, since

$$f(t) = \operatorname*{Res}_{s=0}\left[e^{st}F(s)\right] + \sum_{n=1}^{\infty}\operatorname*{Res}_{s=-n^2\pi^2}\left[e^{st}F(s)\right],$$

we arrive at the expression

$$f(t) = \frac{1}{6}x(x^2 - 1) + xt + \frac{2}{\pi^3}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^3}e^{-n^2\pi^2 t}\sin n\pi x.$$

10. The function

$$F(s) = \frac{1}{s^2} - \frac{1}{s\sinh s}$$

has isolated singularities at the points

$$s_0 = 0 \quad \text{and} \quad s_n = n\pi i,\ \overline{s}_n = -n\pi i \quad (n = 1, 2, \ldots).$$

Now

$$s\sinh s = s\left(s+\frac{1}{6}s^3+\cdots\right)=s^2+\frac{1}{6}s^4+\cdots \qquad (0<|s|<\infty),$$

and division of this series into 1 reveals that

$$F(s)=\frac{1}{s^2}-\left(\frac{1}{s^2}+\frac{1}{6}-\cdots\right)=-\frac{1}{6}+\cdots \qquad (0<|s|<\pi).$$

This shows that $F(s)$ has a removable singularity at $s_0$. Evidently, then, $e^{st}F(s)$ must also have a removable singularity there; and so

$$\operatorname*{Res}_{s=s_0}\left[e^{st}F(s)\right]=0.$$

To find the residue of $F(s)$ at $s_n=n\pi i$ $(n=1,2,\ldots)$, we write

$$F(s)=\frac{p(s)}{q(s)} \quad \text{where} \quad p(s)=\sinh s-s \;\text{ and }\; q(s)=s^2\sinh s$$

and observe that

$$p(n\pi i)=-n\pi i\neq 0, \quad q(n\pi i)=0, \quad \text{and} \quad q'(n\pi i)=n^2\pi^2(-1)^{n+1}\neq 0.$$

Consequently, $F(s)$ has a simple pole at $s_n$, and

$$\operatorname*{Res}_{s=s_n}F(s)=\frac{p(n\pi i)}{q'(n\pi i)}=\frac{-n\pi i}{n^2\pi^2(-1)^{n+1}}=\frac{(-1)^n}{n\pi}i \quad (n=1,2,\ldots).$$

Since $F(\overline{s})=\overline{F(s)}$, the points $\overline{s}_n$ are also simple poles of $F(s)$; and we may write

$$\operatorname*{Res}_{s=s_n}\left[e^{st}F(s)\right]+\operatorname*{Res}_{s=\overline{s}_n}\left[e^{st}F(s)\right]=2\operatorname{Re}\left[\frac{(-1)^n}{n\pi}ie^{in\pi t}\right]=2\operatorname{Re}\left[\frac{(-1)^n}{n\pi}(i\cos n\pi t-\sin n\pi t)\right]$$

$$=2\frac{(-1)^{n+1}}{n\pi}\sin n\pi t.$$

Hence the desired result is

$$f(t)=\operatorname*{Res}_{s=s_0}\left[e^{st}F(s)\right]+\sum_{n=1}^{\infty}\left\{\operatorname*{Res}_{s=s_n}\left[e^{st}F(s)\right]+\operatorname*{Res}_{s=\overline{s}_n}\left[e^{st}F(s)\right]\right\},$$

or

$$f(t)=\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\sin n\pi t.$$

11. We consider here the function

$$F(s)=\frac{\sinh(xs)}{s(s^2+\omega^2)\cosh s} \qquad (0<x<1),$$

where $\omega>0$ and $\omega\neq\omega_n=\frac{(2n-1)\pi}{2}$ $(n=1,2,\ldots)$. The singularities of $F(s)$ are at

$$s=0,\quad s=\pm\omega i,\quad \text{and}\quad s=\pm\omega_n i\ \ (n=1,2,\ldots).$$

Because the first term in the Maclaurin series for $\sinh(xs)$ is $xs$, it is easy to see that $s=0$ is a removable singularity of $e^{st}F(s)$ and that

$$\operatorname*{Res}_{s=0}\left[e^{st}F(s)\right]=0.$$

To find the residue of $F(s)$ at $s=\omega i$, we write

$$F(s)=\frac{\phi(s)}{s-\omega i}\quad \text{where}\quad \phi(s)=\frac{\sinh(xs)}{s(s+\omega i)\cosh s},$$

from which it follows that $s=\omega i$ is simple pole and

$$\operatorname*{Res}_{s=\omega i}F(s)=\phi(\omega i)=\frac{\sinh(x\omega i)}{\omega i 2\omega i\cosh(\omega i)}=\frac{i\sin\omega x}{-2\omega^2\cos\omega}.$$

Since $\overline{F(s)}=F(\bar{s})$, then,

$$\operatorname*{Res}_{s=\omega i}\left[e^{st}F(s)\right]+\operatorname*{Res}_{s=-\omega i}\left[e^{st}F(s)\right]=2\operatorname{Re}\left[\frac{i\sin\omega x}{-2\omega^2\cos\omega}ie^{i\omega t}\right]=2\frac{\sin\omega x}{2\omega^2\cos\omega}\sin\omega t=\frac{\sin\omega x\sin\omega t}{\omega^2\cos\omega}.$$

As for the residues at $s=\omega_n i$ $(n=1,2,\ldots)$, we put $F(s)$ in the form

$$F(s)=\frac{p(s)}{q(s)}\quad \text{where}\quad p(s)=\sinh(xs)\ \ \text{and}\ \ q(s)=(s^3+\omega^2 s)\cosh s.$$

Now $p(\omega_n i)=\sinh(x\omega_n i)=i\sin\omega_n x\neq 0$ and $q(\omega_n i)=0$. Also, since

$$q'(s)=(s^3+\omega^2 s)\sinh s+(3s^2+\omega^2)\cosh s,$$

we find that

$$q'(\omega_n i)=(-\omega_n^3 i+\omega^2\omega_n i)\sinh(\omega_n i)=-\omega_n(\omega^2-\omega_n^2)\sin\omega_n\neq 0$$

Hence we have a simple pole at $s=\omega_n i$, with residue

$$\operatorname*{Res}_{s=\omega_n i}F(s)=\frac{p(\omega_n i)}{q'(\omega_n i)}=\frac{i\sin\omega_n x}{-\omega_n(\omega^2-\omega_n^2)\sin\omega_n}.$$

Consequently,

$$\operatorname*{Res}_{s=\omega_n i}\left[e^{st}F(s)\right]+\operatorname*{Res}_{s=-\omega_n i}\left[e^{st}F(s)\right]=2\operatorname{Re}\left[\frac{i\sin\omega_n x}{-\omega_n(\omega^2-\omega_n^2)\sin\omega_n}\cdot e^{i\omega_n t}\right]=2\frac{\sin\omega_n x\sin\omega_n t}{\omega_n(\omega^2-\omega_n^2)\sin\omega_n}.$$

But $\sin\omega_n=\sin\left(n\pi-\dfrac{\pi}{2}\right)=(-1)^{n+1}$, and this means that

$$\operatorname*{Res}_{s=\omega_n i}\left[e^{st}F(s)\right]+\operatorname*{Res}_{s=-\omega_n i}\left[e^{st}F(s)\right]=2\frac{(-1)^{n+1}}{\omega_n}\cdot\frac{\sin\omega_n x\sin\omega_n t}{\omega^2-\omega_n^2}.$$

Finally,

$$f(t)=\operatorname*{Res}_{s=0}\left[e^{st}F(s)\right]+\left\{\operatorname*{Res}_{s=\omega i}\left[e^{st}F(s)\right]+\operatorname*{Res}_{s=-\omega i}\left[e^{st}F(s)\right]\right\}+\sum_{n=1}^{\infty}\left\{\operatorname*{Res}_{s=\omega_n i}\left[e^{st}F(s)\right]+\operatorname*{Res}_{s=-\omega_n i}\left[e^{st}F(s)\right]\right\}.$$

That is,

$$f(t)=\frac{\sin\omega x\sin\omega t}{\omega^2\cos\omega}+2\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{\omega_n}\cdot\frac{\sin\omega_n x\sin\omega_n t}{\omega^2-\omega_n^2}.$$