Appendix A —————————————————————

Useful solutions to standard problems

Introduction and synopsis

Modelling is a key part of design. In the early stage, approximate modelling establishes whether the concept will work at all, and identifies the combination of material properties which maximize performance. At the embodiment stage, more accurate modelling brackets values for the forces, the displacements, the velocities, the heat fluxes and the dimensions of the components. And in the final stage, modelling gives precise values for stresses, strains and failure probability in key components; power, speed, efficiency and so forth.

Many components with simple geometries and loads have been modelled already. Many more complex components can be modelled approximately by idealizing them as one of these. There is no need to reinvent the beam or the column or the pressure vessel; their behaviour under all common types of loading has already been analysed. The important thing is to know that the results exist and where to find them.

This appendix summarizes the results of modelling a number of standard problems. Their usefulness cannot be overstated. Many problems of conceptual design can be treated, with adequate precision, by patching together the solutions given here; and even the detailed analysis of non-critical components can often be tackled in the same way. Even when this approximate approach is not sufficiently accurate, the insight it gives is valuable.

The appendix contains 15 double page sections which list, with a short commentary, results for constitutive equations; for the loading of beams, columns and torsion bars; for contact stresses, cracks and other stress concentrations; for pressure vessels, vibrating beams and plates; and for the flow of heat and matter. They are drawn from numerous sources, listed under Further reading in Section A.16.

A.1 Constitutive equations for mechanical response

The behaviour of a component when it is loaded depends on the *mechanism* by which it deforms. A beam loaded in bending may deflect elastically; it may yield plastically; it may deform by creep; and it may fracture in a brittle or in a ductile way. The equation which describes the material response is known as a *constitutive equation*. Each mechanism is characterized by a different constitutive equation. The constitutive equation contains one or more than one *material property*: Young's modulus, E, and Poisson's ratio, v, are the material properties which enter the constitutive equation for linear-elastic deformation; the yield strength, σ_y, is the material property which enters the constitutive equation for plastic flow; creep constants, $\dot{\varepsilon}_0$, σ_0 and n enter the equation for creep; the fracture toughness, K_{IC}, enters that for brittle fracture.

The common constitutive equations for mechanical deformation are listed on the facing page. In each case the equation for uniaxial loading by a tensile stress σ is given first; below it is the equation for multiaxial loading by principal stresses σ_1, σ_2 and σ_3, always chosen so that σ_1 is the most tensile and σ_3 the most compressive (or least tensile) stress. They are the basic equations which determine mechanical response.

Constitutive equations for mechanical response

Elastic deformation

Uniaxial	$\varepsilon_1 = \dfrac{\sigma_1}{E}$
General	$\varepsilon_1 = \dfrac{\sigma_1}{E} - \dfrac{\nu}{E}(\sigma_2 + \sigma_3)$

Plastic deformation

Uniaxial	$\sigma_1 \geq \sigma_y$
General $(\sigma_1 > \sigma_2 > \sigma_3)$	$\sigma_1 - \sigma_3 = \sigma_y$ (Tresca) $\sigma_e \geq \sigma_y$ (Von Mises) with $\sigma_e^2 = \dfrac{1}{2}\left[(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2\right]$

Creep deformation

Uniaxial	$\dot{\varepsilon}_1 = \dot{\varepsilon}_0 \left(\dfrac{\sigma}{\sigma_0}\right)^n$
General	$\dot{\varepsilon}_1 = \dot{\varepsilon}_0 \left(\dfrac{\sigma^{n-1}}{\sigma_0^n}\right)\left(\sigma_1 - \dfrac{1}{2}(\sigma_2 + \sigma_3)\right)$

Fracture

Uniaxial	$\sigma_1 = \dfrac{CK_{Ic}}{\sqrt{\pi a}}$
General	$\sigma_1 = \dfrac{CK_{Ic}}{\sqrt{\pi a}}(\sigma_1 > \sigma_2 > \sigma_3)$

Material properties

E, ν Elastic constants	K_{Ic} Fracture toughness
σ_y Plastic yield strength	a Crack length
$\dot{\varepsilon}_0, \sigma_0, n$ Creep constants	$C \approx 1$ tension $C \approx 15$ compression

A.2 Moments of sections

A beam of uniform section, loaded in simple tension by a force F, carries a stress $\sigma = F/A$ where A is the area of the section. Its response is calculated from the appropriate constitutive equation. Here the important characteristic of the section is its area, A. For other modes of loading, higher moments of the area are involved. Those for various common sections are given on the facing page. They are defined as follows.

The second moment I measures the resistance of the section to bending about a horizontal axis (shown as a broken line). It is

$$I = \int_{\text{section}} y^2\, b(y)\, dy$$

where y is measured vertically and $b(y)$ is the width of the section at y. The moment K measures the resistance of the section to twisting. It is equal to the polar moment J for circular sections, where

$$J = \int_{\text{section}} 2\pi r^3\, dr$$

where r is measured radially from the centre of the circular section. For non-circular sections K is less than J.

The section modulus $Z = I/y_m$ (where y_m is the normal distance from the neutral axis of bending to the outer surface of the beam) measures the surface stress generated by a given bending moment, M:

$$\sigma = \frac{M\, y_m}{I} = \frac{M}{Z}$$

Finally, the moment H, defined by

$$H = \int_{\text{section}} y\, b(y)\, dy$$

measures the resistance of the beam to fully-plastic bending. The fully plastic moment for a beam in bending is

$$M_p = H\sigma_y$$

Thin or slender shapes may buckle before they yield or fracture. It is this which sets a practical limit to the thinness of tube walls and webs.

Moments of sections

Section Shape	$A(m^2)$	$I_{xx}(m^4)$	$K(m^4)$	$Z(m^3)$	$Q(m^3)$
	πr^2	$\dfrac{\pi}{4}r^4$	$\dfrac{\pi}{2}r^4$	$\dfrac{\pi}{4}r^3$	$\dfrac{\pi}{2}r^3$
	b^2	$\dfrac{b^4}{12}$	$0.14b^4$	$\dfrac{b^3}{6}$	$0.21b^3$
	πab	$\dfrac{\pi}{4}a^3b$	$\dfrac{\pi a^3 b^3}{(a^2+b^2)}$	$\dfrac{\pi}{4}a^2b$	$\dfrac{\pi a^2 b}{2}$ $(a<b)$
	bh	$\dfrac{bh^3}{12}$	$\dfrac{b^3 h}{3}\left(1-0.58\dfrac{b}{h}\right)$ $(h>b)$	$\dfrac{bh^2}{6}$	$\dfrac{b^2 h^2}{3h+1.8b}$ $(h>b)$
	$\dfrac{\sqrt{3}}{4}a^2$	$\dfrac{a^4}{32\sqrt{3}}$	$\dfrac{a^4\sqrt{3}}{80}$	$\dfrac{a^3}{32}$	$\dfrac{a^3}{20}$
	$\pi(r_o^2-r_i^2)$ $\approx 2\pi rt$	$\dfrac{\pi}{4}(r_o^4-r_i^4)$ $\approx \pi r^3 t$	$\dfrac{\pi}{2}(r_o^4-r_i^4)$ $\approx 2\pi r^3 t$	$\dfrac{\pi}{4r_o}(r_o^4-r_i^4)$ $\approx \pi r^2 t$	$\dfrac{\pi}{2r_o}(r_o^4-r_i^4)$ $\approx 2\pi r^2 t$
	$4bt$	$\dfrac{2}{3}b^3 t$	$b^3 t\left(1-\dfrac{t}{b}\right)^4$	$\dfrac{4}{3}b^2 t$	$2b^2 t\left(1-\dfrac{t}{b}\right)^2$
	$\pi(a+b)t$	$\dfrac{\pi}{4}a^3 t\left(1+\dfrac{3b}{a}\right)$	$\dfrac{4\pi(ab)^{5/2}t}{(a^2+b^2)}$	$\dfrac{\pi a^2 t}{4}\left(1+\dfrac{3b}{a}\right)$	$2\pi t(a^3 b)^{1/2}$ $(b>a)$
	$b(h_o-h_i)$ $\approx 2bt$	$\dfrac{b}{12}(h_o^3-h_i^3)$ $\approx \dfrac{1}{2}bth_o^2$	—	$\dfrac{b}{6h_o}(h_o^3-h_i^3)$ $\approx bth_o$	—
	$2t(h+b)$	$\dfrac{1}{6}h^3 t\left(1+\dfrac{3b}{h}\right)$	$\approx\dfrac{2tb^2h^2}{h+b}$ ⊏⊐ □ $\dfrac{2}{3}bt^3\left(1+\dfrac{4h}{b}\right)$	$\dfrac{h^2 t}{3}\left(1+\dfrac{3b}{h}\right)$	$2tbh$ ⊏⊐ □ $\dfrac{2}{3}bt^2\left(1+\dfrac{4h}{b}\right)$
	$2t(h+b)$	$\dfrac{t}{6}(h^3+4bt^2)$	$\dfrac{t^3}{3}(8b+h)$ ⊢ ⊢ $\dfrac{2}{3}ht^3\left(1+\dfrac{4b}{h}\right)$	$\dfrac{t}{3h}(h^3+4bt^2)$	$\dfrac{t^2}{3}(8b+h)$ ⊢ ⊢ $\dfrac{2}{3}ht^2\left(1+\dfrac{4b}{h}\right)$
	$t\lambda\left(1+\dfrac{\pi^2 d^2}{4\lambda^2}\right)$	$\dfrac{t\lambda d^2}{8}$	—	$\dfrac{t\lambda d}{4}$	—
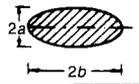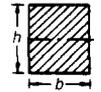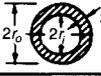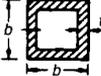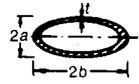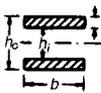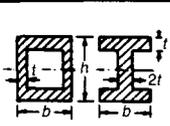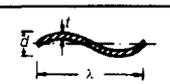
A.3 Elastic bending of beams

When a beam is loaded by a force F or moments M, the initially straight axis is deformed into a curve. If the beam is uniform in section and properties, long in relation to its depth and nowhere stressed beyond the elastic limit, the deflection δ, and the angle of rotation, θ, can be calculated using elastic beam theory (see Further reading in Section A.16). The basic differential equation describing the curvature of the beam at a point x along its length is

$$EI\frac{\mathrm{d}y^2}{\mathrm{d}x^2} = M$$

where y is the lateral deflection, and M is the bending moment at the point x on the beam. E is Young's modulus and I is the second moment of area (Section A.2). When M is constant this becomes

$$\frac{M}{I} = E\left(\frac{1}{R} - \frac{1}{R_0}\right)$$

where R_0 is the radius of curvature before applying the moment and R the radius after it is applied. Deflections δ and rotations θ are found by integrating these equations along the beam. Equations for the deflection, δ, and end slope, θ, of beams, for various common modes of loading are shown on the facing page.

The stiffness of the beam is defined by

$$S = \frac{F}{\delta} = \frac{C_1 EI}{\ell^3}$$

It depends on Young's modulus, E, for the material of the beam, on its length, ℓ, and on the second moment of its section, I. The end-slope of the beam, θ, is given by

$$\theta = \frac{F\ell^2}{C_2 EI}$$

Values of C_1 and C_2 are listed opposite.

Elastic bending of beams
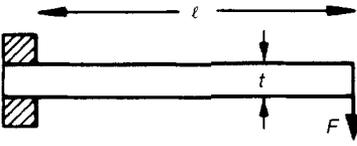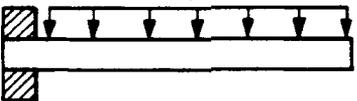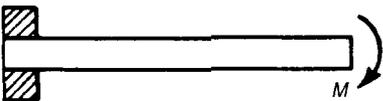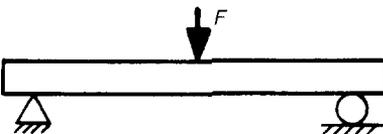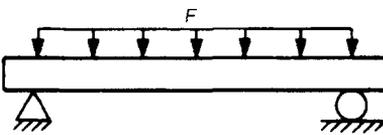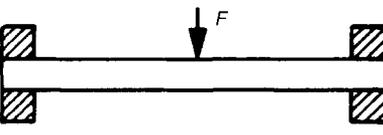
	C_1	C_2
	3	2
	8	6
	2	1
	48	16
	$\dfrac{384}{5}$	24
	192	–
	384	–
	6	–
	–	4
	–	3
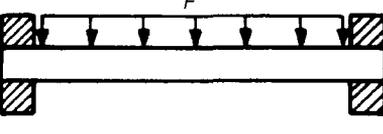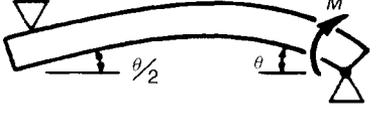
$$\delta = \frac{F\ell^3}{C_1 EI} = \frac{M\ell^2}{C_1 EI}$$

$$\theta = \frac{F\ell^2}{C_2 EI} = \frac{M\ell}{C_2 EI}$$

$E =$ Young's modulus (N/m²)

$\delta =$ deflection (m)

$F =$ force (N)

$M =$ moment (Nm)

$\ell =$ length (m)

$b =$ width (m)

$t =$ depth (m)

$\theta =$ end slope (–)

$I =$ see Table 2 (m⁴)

$y =$ distance from N.A. (m)

$R =$ radius of curvature (m)

$$\frac{\sigma}{y} = \frac{M}{I} = \frac{E}{R}$$

A.4 Failure of beams and panels

The longitudinal (or 'fibre') stress σ at a point y from the neutral axis of a uniform beam loaded elastically in bending by a moment M is

$$\frac{\sigma}{y} = \frac{M}{I} = E\left(\frac{1}{R} - \frac{1}{R_0}\right)$$

where I is the second moment of area (Section A.2), E is Young's modulus, R_0 is the radius of curvature before applying the moment and R is the radius after it is applied. The tensile stress in the outer fibre of such a beam is

$$\sigma = \frac{M y_m}{I} = \frac{M}{Z}$$

where y_m is the perpendicular distance from the neutral axis to the outer surface of the beam. If this stress reaches the yield strength σ_y of the material of the beam, small zones of plasticity appear at the surface (top diagram, facing page). The beam is no longer elastic, and, in this sense, has failed. If, instead, the maximum fibre stress reaches the brittle fracture strength, σ_f (the 'modulus of rupture', often shortened to MOR) of the material of the beam, a crack nucleates at the surface and propagates inwards (second diagram); in this case, the beam has certainly failed. A third criterion for failure is often important: that the plastic zones penetrate through the section of the beam, linking to form a plastic hinge (third diagram).

The failure moments and failure loads, for each of these three types of failure, and for each of several geometries of loading, are given on the diagram. The formulae labelled 'Onset' refer to the first two failure modes; those labelled 'Full plasticity' refer to the third. Two new functions of section shape are involved. Onset of failure involves the quantity Z; full plasticity involves the quantity H. Both are listed in the table of Section A.2, and defined in the text which accompanies it.

Failure of beams and panels
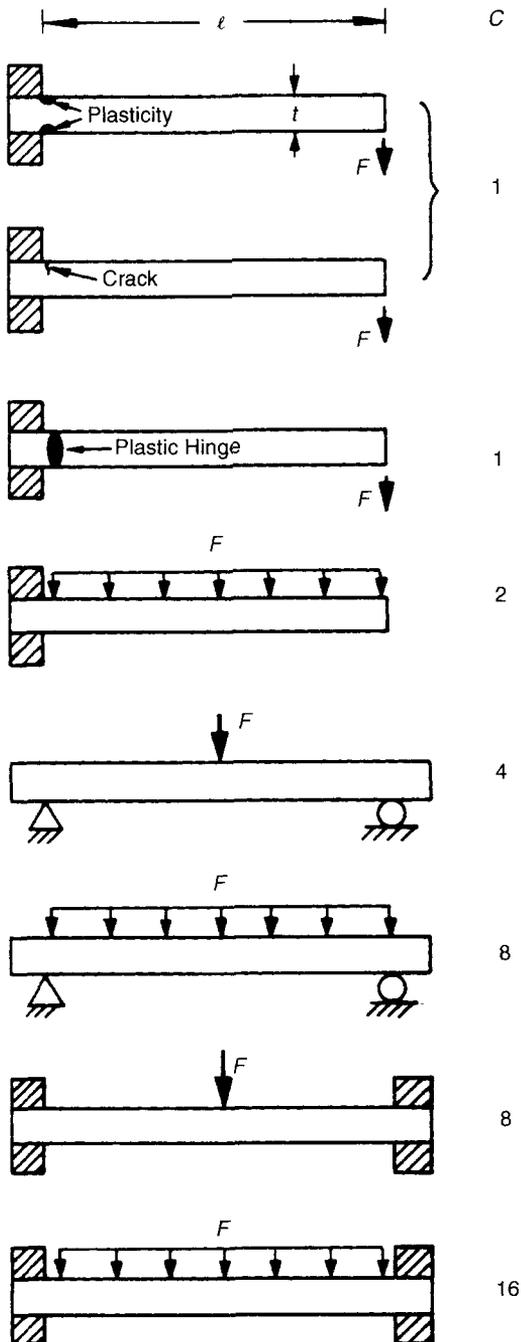
$$M_f = \left(\frac{I}{y_m}\right)\sigma^* \text{ (Onset)}$$

$$M_f = H\sigma_y \text{ (Full plasticity)}$$

$$F_f = C\left(\frac{I}{y_m}\right)\frac{\sigma^*}{\ell} \text{ (Onset)}$$

$$F_f = \frac{CH\sigma_y}{\ell} \text{ (Full plasticity)}$$

M_f = failure moment (Nm)

F_f = force at failure (N)

ℓ = length (m)

t = depth (m)

b = width (m)

I = see Table 2 (m^4)

$\dfrac{I}{y_m}$ = see Table 2 (m^3)

H = see Table 2 (m^3)

σ_y = yield strength (N/m^2)

σ_f = modulus of rupture (N/m^2)

σ^* = σ_y (plastic material)

= σ_f (brittle material)

A.5 Buckling of columns and plates

If sufficiently slender, an elastic column, loaded in compression, fails by elastic buckling at a critical load, F_{crit}. This load is determined by the end constraints, of which four extreme cases are illustrated on the facing page: an end may be constrained in a position and direction; it may be free to rotate but not translate (or 'sway'); it may sway without rotation; and it may both sway and rotate. Pairs of these constraints applied to the ends of column lead to the five cases shown opposite. Each is characterized by a value of the constant n which is equal to the number of half-wavelengths of the buckled shape.

The addition of the bending moment M reduces the buckling load by the amount shown in the second box. A negative value of F_{crit} means that a tensile force is necessary to prevent buckling.

An elastic foundation is one that exerts a lateral restoring pressure, p, proportional to the deflection ($p = ky$ where k is the foundation stiffness per unit depth and y the local lateral deflection). Its effect is to increase F_{crit}, by the amount shown in the third box.

A thin-walled elastic tube will buckle inwards under an external pressure p', given in the last box. Here I refers to the second moment of area of a section of the tube wall cut parallel to the tube axis.

Buckling of columns and plates
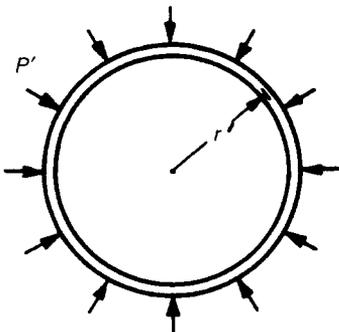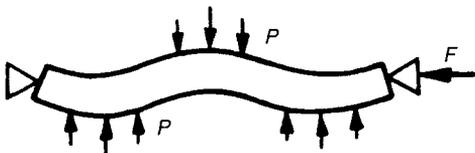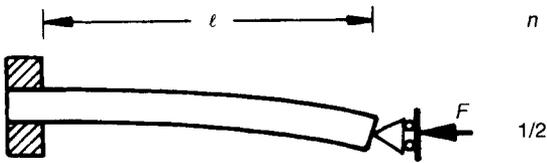
$$F_{CRIT} = \frac{n^2\pi^2 EI}{\ell^2}$$

or

$$\frac{F_{CRIT}}{A} = \frac{n^2\pi^2 E}{(\ell/r)^2}$$

n

1/2

1

1

3/2

2

F = force (N)

M = moment (Nm)

E = Young's modulus (N/m^2)

ℓ = length (m)

A = section area (m^2)

I = see Table 2 (m^4)

r = gyration rad. $\left(\dfrac{I}{A}\right)^{1/2}$ (m)

k = foundation stiffness (N/m^2)

n = half-wavelengths in buckled shape

p' = pressure (N/m^2)

$$F_{CRIT} = \frac{\pi^2 EI}{\ell^2} - \frac{M^2}{4EI}$$

$$F_{CRIT} = \frac{n^2\pi^2 EI}{\ell^2} + \frac{k\ell^2}{n^2}$$

$$p'_{CRIT} = \frac{3EI}{(r')^3}$$

A.6 Torsion of shafts

A torque, T, applied to the ends of an isotropic bar of uniform section, and acting in the plane normal to the axis of the bar, produces an angle of twist θ. The twist is related to the torque by the first equation on the facing page, in which G is the shear modulus. For round bars and tubes of circular section, the factor K is equal to J, the polar moment of inertia of the section, defined in Section A.2. For any other section shape K is less than J. Values of K are given in Section A.2.

If the bar ceases to deform elastically, it is said to have failed. This will happen if the maximum surface stress exceeds either the yield strength σ_y of the material or the stress at which it fractures. For circular sections, the shear stress at any point a distance r from the axis of rotation is

$$\tau = \frac{Tr}{K} = \frac{G\theta r}{\ell}$$

The maximum shear stress, τ_{max}, and the maximum tensile stress, σ_{max}, are at the surface and have the values

$$\tau_{max} = \sigma_{max} = \frac{Td_0}{2K} = \frac{G\theta d_0}{2\ell}$$

If τ_{max} exceeds $\sigma_y/2$ (using a Tresca yield criterion), or if σ_{max} exceeds the MOR, the bar fails, as shown on the figure. The maximum surface stress for the solid ellipsoidal, square, rectangular and triangular sections is at the points on the surface closest to the centroid of the section (the mid-points of the longer sides). It can be estimated approximately by inscribing the largest circle which can be contained within the section and calculating the surface stress for a circular bar of that diameter. More complex section-shapes require special consideration, and, if thin, may additionally fail by buckling.

Helical springs are a special case of torsional deformation. The extension of a helical spring of n turns of radius R, under a force F, and the failure force F_{crit}, are given on the facing page.

Torsion of shafts
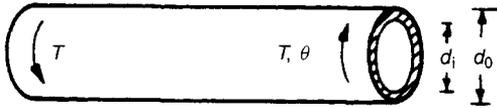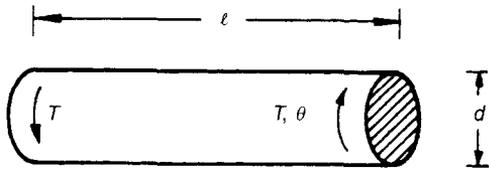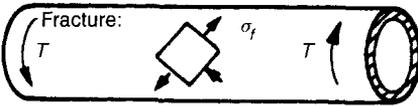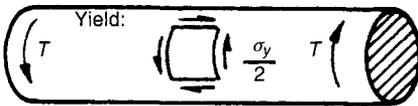
Elastic deflection

$$\theta = \frac{\ell T}{KG}$$

Failure

$$T_f = \frac{K\sigma_y}{d_0} \text{ (Onset of yield)}$$

$$T_f = \frac{2K\sigma_f}{d_0} \text{ (Brittle fracture)}$$

ETC

Yield:

Fracture:

T = torque (Nm)

θ = angle of twist

G = shear modulus (N/m^2)

ℓ = length (m)

d = diameter (m)

K = see Table 1 (m^4)

σ_y = yield strength (N/m^2)

σ_f = modulus of rupture (N/m^2)

Spring deflection and failure

$$u = \frac{64FR^3n}{Gd^4}$$

$$F_f = \frac{\pi}{32}\frac{d^3\sigma_y}{R}$$

F = force (N)

u = deflection (m)

R = coil radius (m)

n = number of turns
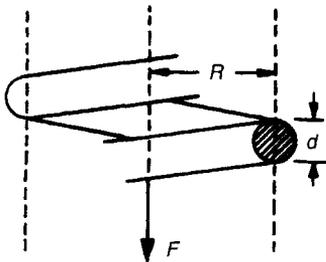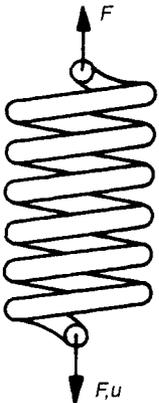
A.7 Static and spinning discs

A thin disc deflects when a pressure difference Δp is applied across its two surfaces. The deflection causes stresses to appear in the disc. The first box on the facing page gives deflection and maximum stress (important in predicting failure) when the edges of the disc are simply supported. The second gives the same quantities when the edges are clamped. The results for a thin horizontal disc deflecting under its own weight are found by replacing Dp by the mass-per-unit-area, $\rho g t$, of the disc (here ρ is the density of the material of the disc and g is the acceleration due to gravity). Thick discs are more complicated; for those, see Further reading.

Spinning discs, rings and cylinders store kinetic energy. Centrifugal forces generate stresses in the disc. The two boxes list the kinetic energy and the maximum stress σ_{max} in discs and rings rotating at an angular velocity ω (radians/sec). The maximum rotation rate and energy are limited by the burst-strength of the disc. They are found by equating the maximum stress in the disc to the strength of the material.

Static and spinning discs
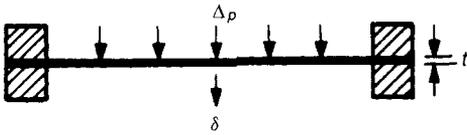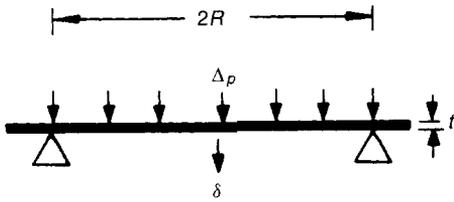
Simple

$$\delta = \frac{3}{4}(1 - \nu^2)\frac{\Delta p R^4}{E t^3}$$

$$\sigma_{max} = \frac{3}{8}(3 + \nu)\frac{\Delta p R^2}{t^2}$$

Clamped

$$\delta = \frac{3}{16}(1 - \nu^2)\frac{\Delta p R^4}{E t^3}$$

$$\sigma_{max} = \frac{3}{8}(1 + \nu)\frac{\Delta p R^2}{t^2}$$

δ = deflection (m)

E = Young's modulus (N/m)

Δp = pressure diff. (N/m)

ν = Poisson's ratio

Disc

$$u = \frac{\pi}{4}\rho t \omega^2 R^4$$

$$\sigma_{max} = \frac{1}{8}(3 + \nu)\rho \omega^2 R^2$$

Ring

$$u = \pi \rho t \omega^2 R^3 x$$

$$\sigma_{max} = \rho R^2 \omega^2$$

u = energy (J)

ω = angular vel. (rad/s)

ρ = density kg/m^3
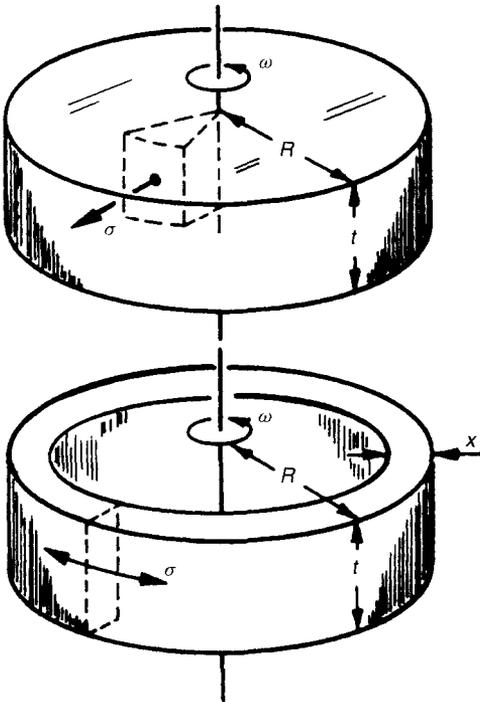
A.8 Contact stresses

When surfaces are placed in contact they touch at one or a few discrete points. If the surfaces are loaded, the contacts flatten elastically and the contact areas grow until failure of some sort occurs: failure by crushing (caused by the compressive stress, σ_c), tensile fracture (caused by the tensile stress, σ_t) or yielding (caused by the shear stress σ_s). The boxes on the facing page summarize the important results for the radius, a, of the contact zone, the centre-to-centre displacement u and the peak values of σ_c, σ_t and σ_s.

The first box shows results for a sphere on a flat, when both have the same moduli and Poisson's ratio has the value 1/3. Results for the more general problem (the 'Hertzian Indentation' problem) are shown in the second box: two elastic spheres (radii R_1 and R_2, moduli and Poisson's ratios E_1, ν_1 and E_2, ν_2) are pressed together by a force F.

If the shear stress σ_s exceeds the shear yield strength $\sigma_y/2$, a plastic zone appears beneath the centre of the contact at a depth of about $a/2$ and spreads to form the fully-plastic field shown in the two lower figures. When this state is reached, the contact pressure is approximately 3 times the yield stress, as shown in the bottom box.

Contact stresses
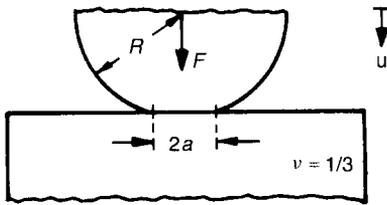
$$a = 0.7 \left(\frac{FR}{E} \right)^{1/3}$$

$$u = 1.0 \left(\frac{F^2}{E^2 R} \right)^{1/3}$$

$$\left. \vphantom{\frac{FR}{E}} \right\} \quad \nu = \frac{1}{3}$$
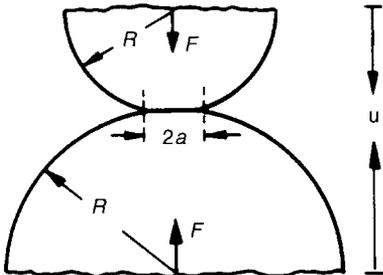
$$a = \left(\frac{3}{4} \frac{F}{E^*} \frac{R_1 R_2}{(R_1 + R_2)} \right)^{1/3}$$

$$u = \left(\frac{9}{16} \frac{F^2}{(E^*)^2} \frac{(R_1 + R_2)}{R_1 R_2} \right)^{1/3}$$
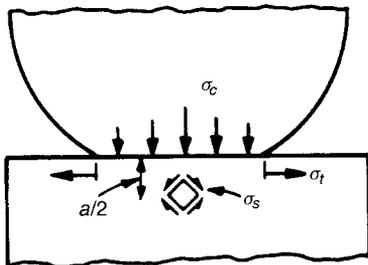
$$(\sigma_c)_{\max} = \frac{3F}{2\pi a^2}$$

$$(\sigma_s)_{\max} = \frac{F}{2\pi a^2}$$

$$(\sigma_t)_{\max} = \frac{F}{6\pi a^2}$$
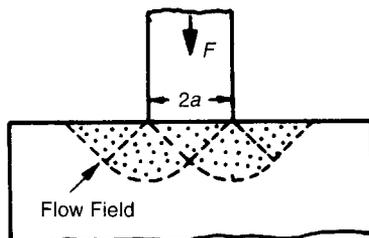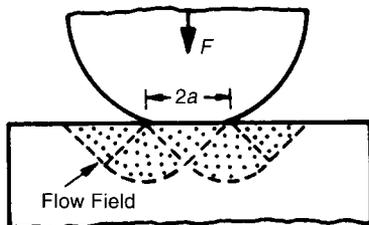
$R_1 R_2$	radii of spheres (m)
$E_1 E_2$	modulii of spheres (N/m^2)
$\nu_1 \nu_2$	Poisson's ratios
F	load (N)
a	radius of contact (m)
u	displacement (m)
σ	stresses (N/m^2)
σ_y	yield stress (N/m^2)
E^*	$\left(\dfrac{1 - \nu_1^2}{E_1} + \dfrac{1 - \nu_2^2}{E_2} \right)^{-1}$

$$\frac{F}{\pi a^2} = 3\sigma_y$$

A.9 Estimates for stress concentrations

Stresses and strains are concentrated at holes, slots or changes of section in elastic bodies. Plastic flow, fracture and fatigue cracking start at these places. The local stresses at the stress concentrations can be computed numerically, but this is often unnecessary. Instead, they can be estimated using the equation shown on the facing page.

The stress concentration caused by a change in section dies away at distances of the order of the characteristic dimension of the section-change (defined more fully below), an example of St Venant's principle at work. This means that the maximum local stresses in a structure can be found by determining the nominal stress distribution, neglecting local discontinuities (such as holes or grooves), and then multiplying the nominal stress by a stress concentration factor. Elastic stress concentration factors are given approximately by the equation. In it, σ_{nom} is defined as the load divided by the minimum cross-section of the part, r is the minimum radius of curvature of the stress-concentrating groove or hole, and c is the characteristic dimension: either the half-thickness of the remaining ligament, the half-length of a contained crack, the length of an edge-crack or the height of a shoulder, whichever is *least*. The drawings show examples of each such situation. The factor α is roughly 2 for tension, but is nearer 1/2 for torsion and bending. Though inexact, the equation is an adequate working approximation for many design problems.

The maximum stress is limited by plastic flow or fracture. When plastic flow starts, the strain concentration grows rapidly while the stress concentration remains constant. The strain concentration becomes the more important quantity, and may not die out rapidly with distance (St Venant's principle no longer applies).

Estimates for stress concentrations
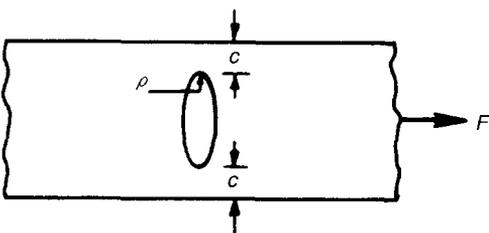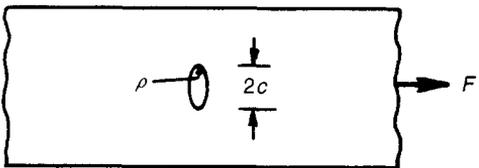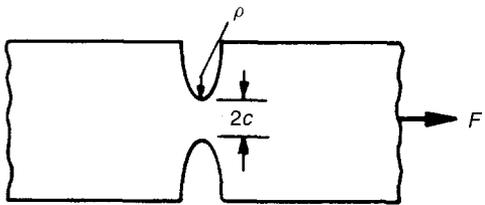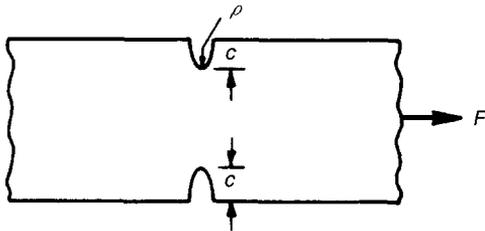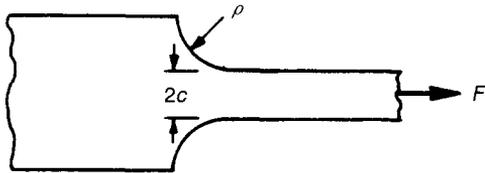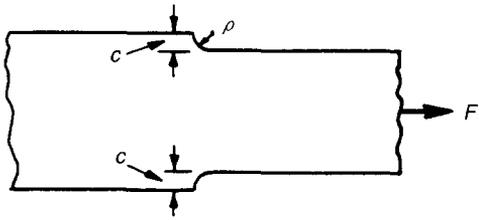
$$\frac{\sigma_{max}}{\sigma_{nom}} = 1 + \alpha \left(\frac{c}{\rho}\right)^{1/2}$$

F = force (N)

A_{min} = minimum section (m^2)

$\sigma_{nom} = F/A_{min}$ (N/m^2)

ρ = radius of curvature (m)

c = characteristic length (m)

$\alpha \approx 0.5$ (tension)

$\alpha \approx 2.0$ (torsion)

A.10 Sharp cracks

Sharp cracks (that is, stress concentrations with a tip radius of curvature of atomic dimensions) concentrate stress in an elastic body more acutely than rounded stress concentrations do. To a first approximation, the local stress falls off as $1/r^{1/2}$ with radial distance r from the crack tip. A tensile stress σ, applied normal to the plane of a crack of length $2a$ contained in an infinite plate (as in the top figure on the facing page) gives rise to a local stress field σ_ℓ which is tensile in the plane containing the crack and given by

$$\sigma_\ell = \frac{C\sigma\sqrt{\pi a}}{\sqrt{2\pi r}}$$

where r is measured from the crack tip in the plane $\theta = 0$, and C is a constant. The mode 1 *stress intensity factor* K_I, is defined as

$$K_I = C\sigma\sqrt{\pi a}$$

Values of the constant C for various modes of loading are given on the figure. (The stress σ for point loads and moments is given by the equations at the bottom.) The crack propagates when $K_I > K_{IC}$, the *fracture toughness*.

When the crack length is very small compared with all specimen dimensions and compared with the distance over which the applied stress varies, C is equal to 1 for a contained crack and 1.1 for an edge crack. As the crack extends in a uniformly loaded component, it interacts with the free surfaces, giving the correction factors shown opposite. If, in addition, the stress field is non-uniform (as it is in an elastically bent beam), C differs from 1; two examples are given on the figure. The factors, C, given here, are approximate only, good when the crack is short but not when the crack tips are very close to the boundaries of the sample. They are adequate for most design calculations. More accurate approximations, and other less common loading geometries can be found in the references listed in Further reading.

Sharp cracks

$$C$$

1.0 $(a \ll w)$

$$\left(\frac{2w}{\pi a} \tan \frac{\pi a}{2w}\right)^{1/2}$$

1.1 $(a \ll w)$

$$\frac{\left(\frac{2w}{\pi a} \tan \frac{\pi a}{2w}\right)^{1/2}}{1 - a/w}$$

1.1 $(a \ll w)$

$$\frac{1.1(1 - 0.2\, a/w)}{(1 - a/w)^{3/2}}$$

$$\frac{2}{\pi}\left(\frac{a+c}{a-c}\right)^{1/2}$$

$$\frac{1.1\left(1 - \frac{3}{2}\frac{a}{t}\right)}{(1 - a/t)^{3/2}}$$

$$\frac{1.1\left(1 - \frac{3}{2}\frac{a}{t}\right)}{(1 - a/t)^{3/2}}$$
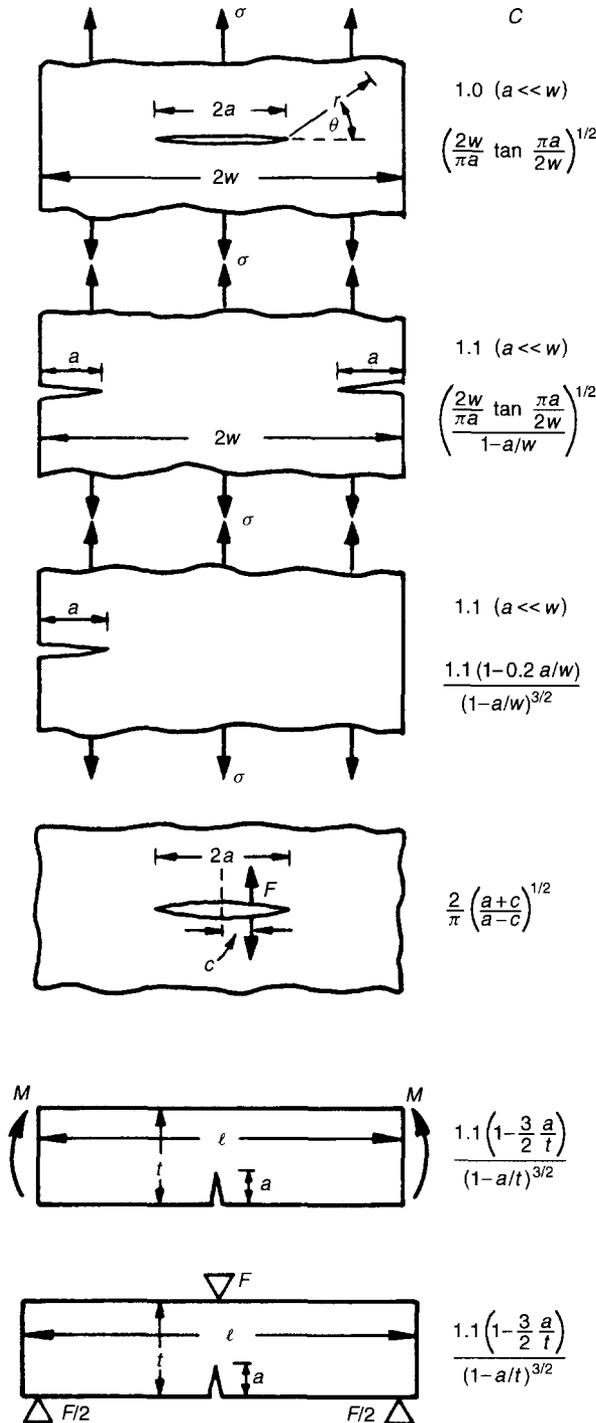
$$\boxed{\begin{array}{c} K_1 = C\sigma\sqrt{\pi a} \\ \text{failure when} \\ K_1 \geq K_{IC} \end{array}}$$

K_1 = stress intensity $(\text{N/m}^{3/2})$

σ = remote stress (N/m^2)

F = load (N)

M = moment (Nm)

a = crack half-length

   = surface crack length (m)

w = half-width (centre) (m)

   = width (edge crack) (m)

b = sample depth (m)

t = beam thickness (m)

point load on crack face:

$$\sigma = \frac{F}{2ab}$$

moment on beam:

$$\sigma = \frac{6M}{bt}$$

3-point bending:

$$\sigma = \frac{3F\ell}{2bt}$$

A.11 Pressure vessels

Thin-walled pressure vessels are treated as membranes. The approximation is reasonable when $t < b/4$. The stresses in the wall are given on the facing page; they do not vary significantly with radial distance, r. Those in the plane tangent to the skin, σ_θ and σ_z for the cylinder and σ_θ and σ_ϕ for the sphere, are just equal to the internal pressure amplified by the ratio b/t or $b/2t$, depending on geometry. The radial stress σ_r is equal to the mean of the internal and external stress, $p/2$ in this case. The equations describe the stresses when an external pressure p_e is superimposed if p is replaced by $(p - p_e)$.

In thick-walled vessels, the stresses vary with radial distance r from the inner to the outer surfaces, and are greatest at the inner surface. The equations can be adapted for the case of both internal and external pressures by noting that when the internal and external pressures are equal, the state of stress in the wall is

$$\sigma_\theta = \sigma_r = -p \quad \text{(cylinder)}$$

or
$$\sigma_\theta = \sigma_\phi = \sigma_r = -p \quad \text{(sphere)}$$

allowing the term involving the external pressure to be evaluated. It is not valid to just replace p by $(p - p_e)$.

Pressure vessels fail by yielding when the Von Mises equivalent stress first exceed the yield strength, σ_y. They fail by fracture if the largest tensile stress exceeds the fracture stress σ_f, where

$$\sigma_f = \frac{CK_{IC}}{\sqrt{\pi a}}$$

and K_{IC} is the fracture toughness, a the half-crack length and C a constant given in Section A.10.

Pressure vessels
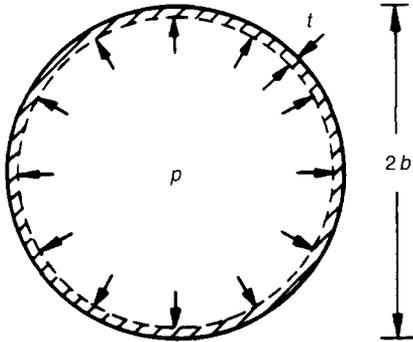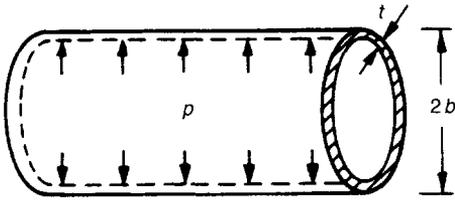
Thin Walled
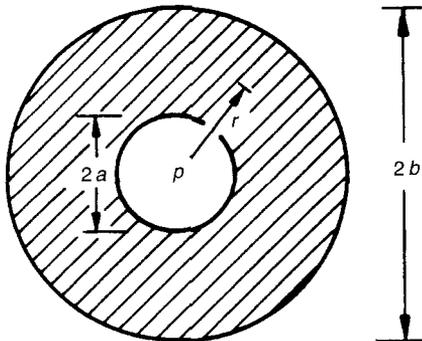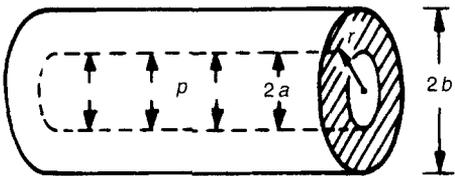
Cylinder

$$\sigma_\theta = \frac{pb}{t}$$

$$\sigma_r = -p/2$$

$$\sigma_z = \frac{pb}{2t} \quad \text{(closed ends)}$$

Sphere

$$\sigma_\theta = \sigma_\phi = \frac{pb}{2t}$$

$$\sigma_r = -p/2$$

$p =$ pressure (N/m^2)

$t =$ wall thickness (m)

$a =$ inner radius (m)

$b =$ outer radius (m)

$r =$ radial coordinate (m)

Thick Walled

$$\sigma_\theta = \frac{pa^2}{r^2} \left(\frac{b^2 - r^2}{b^2 - a^2} \right)$$

$$\sigma_r = -\frac{pa^2}{r^2} \left(\frac{b^2 + r^2}{b^2 - a^2} \right)$$

$$\sigma_\theta = \sigma_\phi = \frac{pa^3}{2r^3} \left(\frac{b^3 + 2r^3}{b^3 - a^3} \right)$$

$$\sigma_r = -\frac{pa^3}{r^3} \left(\frac{b^3 - r^3}{b^3 - a^3} \right)$$

A.12 Vibrating beams, tubes and discs

Any undamped system vibrating at one of its natural frequencies can be reduced to the simple problem of a mass m attached to a spring of stiffness K. The lowest natural frequency of such a system is

$$f = \frac{1}{2\pi}\sqrt{\frac{K}{m}}$$

Specific cases require specific values for m and K. They can often be estimated with sufficient accuracy to be useful in approximate modelling. Higher natural frequencies are simple multiples of the lowest.

The first box on the facing page gives the lowest natural frequencies of the flexural modes of uniform beams with various end-constraints. As an example, the first can be estimated by assuming that the effective mass of the beam is one quarter of its real mass, so that

$$m = \frac{m_0\ell}{4}$$

where m_0 is the mass per unit length of the beam and that K is the bending stiffness (given by F/δ from Section A.3); the estimate differs from the exact value by 2%. Vibrations of a tube have a similar form, using I and m_0 for the tube. Circumferential vibrations can be found approximately by 'unwrapping' the tube and treating it as a vibrating plate, simply supported at two of its four edges.

The second box gives the lowest natural frequencies for flat circular discs with simply-supported and clamped edges. Discs with doubly-curved faces are stiffer and have higher natural frequencies.

Vibrating beams, tubes and discs

Beams, tubes

C_1

3.52

$$F_1 = \frac{C_1}{2\pi}\sqrt{\frac{EI}{m_0\ell^4}}$$

9.87
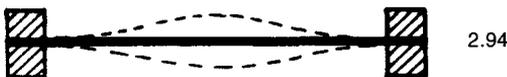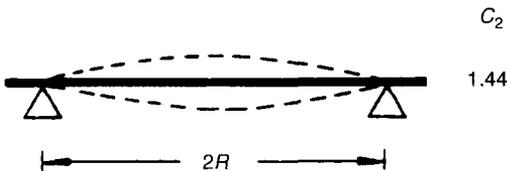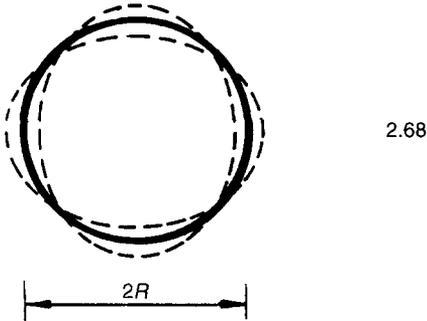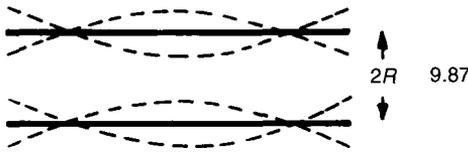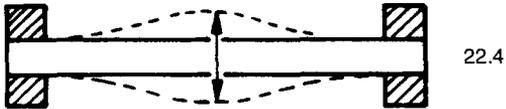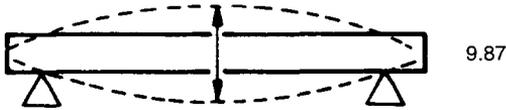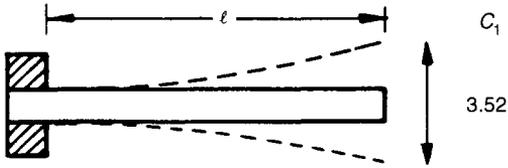
f = natural frequency (s^{-1})

$m_0 = \rho A$ = mass/length (kg/m)

22.4

ρ = density (kg/m^3)

A = section area (m^2)

I = see Table A1

$$\left\{ \begin{array}{r} \text{with } A = 2\pi R \\ I = \pi R^3 t \end{array} \right.$$

$2R$ 9.87

$$\left\{ \text{with } A = \frac{\ell t^3}{12} \right.$$

Discs

$$f_1 = \frac{C_2}{2\pi}\sqrt{\frac{Et^3}{m_1 R^4(1 - \nu^2)}}$$

2.68

$m_1 = \rho t$ = mass/area (kg/m^2)

t = thickness (m)

R = radius (m)

ν = Poisson's ratio

$2R$

C_2

1.44

$2R$

2.94

A.13 Creep and creep fracture

At temperatures above $1/3\ T_m$ (where T_m is the absolute melting point), materials creep when loaded. It is convenient to characterize the creep of a material by its behaviour under a tensile stress σ, at a temperature T_m. Under these conditions the steady-state tensile strain rate $\dot{\varepsilon}$ is often found to vary as a power of the stress and exponentially with temperature:

$$\dot{\varepsilon}_{SS} = A \left(\frac{\sigma}{\sigma_0} \right)^n \exp - \frac{Q}{RT}$$

where Q is an activation energy, A is a kinetic constant and R is the gas constant. At constant temperature this becomes

$$\dot{\varepsilon}_{SS} = \dot{\varepsilon}_0 \left(\frac{\sigma}{\sigma_0} \right)^n$$

where $\dot{\varepsilon}_0 (s^{-1})$, $\sigma_0 (N/m^2)$ and n are creep constants.

The behaviour of creeping components is summarized on the facing page which give the deflection rate of a beam, the displacement rate of an indenter and the change in relative density of cylindrical and spherical pressure vessels in terms of the tensile creep constants.

Prolonged creep causes the accumulation of creep damage which ultimately leads, after a time t_f, to fracture. To a useful approximation

$$t_f \dot{\varepsilon}_{SS} = C$$

where C is a constant characteristic of the material. Creep-ductile material have values of C between 0.1 and 0.5; creep-brittle materials have values of C as low as 0.01.

Creep and creep fracture

$$\dot{\varepsilon} = \dot{\varepsilon}_0 \left(\frac{\sigma}{\sigma_0} \right)^n$$
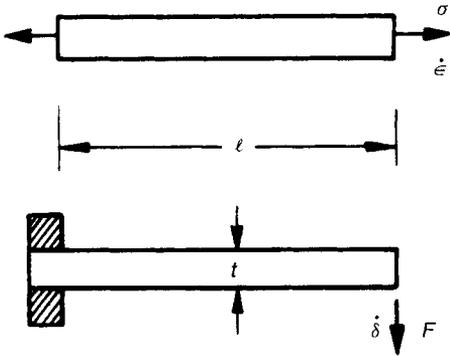
$$\dot{\delta} = \frac{2\dot{\varepsilon}_0}{n+2} \left(\frac{(2n+1)}{n\sigma_0} \frac{E}{bt} \right)^n \left(\frac{\ell}{t} \right)^{n+1} \ell$$

$$\dot{u} = C_1 \dot{\varepsilon}_0 \sqrt{A} \left(\frac{C_2 F}{\sigma_0 A} \right)^n$$
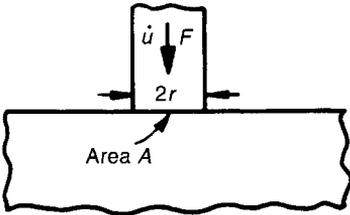
$$\dot{\rho} = 2\dot{\varepsilon}_0 \frac{\rho(1-\rho)}{\left(1 - (1-\rho)^{1/n} \right)^n}$$
$$\times \left(\frac{2}{n} \frac{(p_e - p_i)}{\sigma_0} \right)^n$$
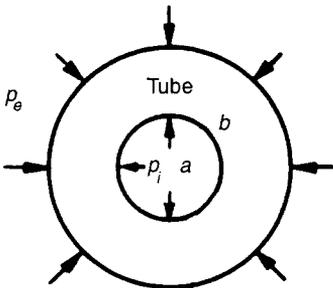
$$\dot{\rho} = \frac{3}{2} \dot{\varepsilon}_0 \frac{\rho(1-\rho)}{\left(1 - (1-\rho)^{1/n} \right)^n}$$
$$\times \left(\frac{3}{2n} \frac{(p_e - p_i)}{\sigma_0} \right)^n$$
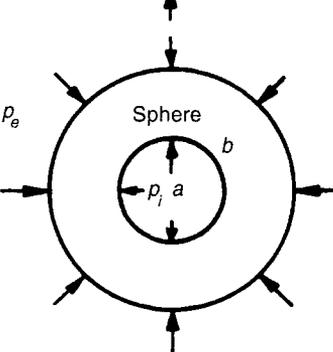
$\sigma =$ stress N/m^2

$F =$ force (N)

$\dot{\delta}, \dot{u} =$ displacement rates (m/s)

$n, \dot{\varepsilon}_0, \sigma_0 =$ creep constants

$\ell, b, t =$ beam dimensions (m)

$a, b =$ radii of pressure vessels (m)

$\rho =$ relative density, $\dfrac{b^3 - a^3}{b^3}$

$C_1, C_2 =$ constants

A.14 Flow of heat and matter

Heat flow can be limited by conduction, convection or radiation. The constitutive equations for each are listed on the facing page. The first equation is Fourier's first law, describing steady-state heat flow; it contains the thermal conductivity, λ. The second is Fourier's second law, which treats transient heat-flow problems; it contains the thermal diffusivity, a, defined by

$$a = \frac{\lambda}{\rho C}$$

where r is the density and C the specific heat at constant pressure. Solutions to these two differential equations are given in Section A.15.

The third equation describes convective heat transfer. It, rather than conduction, limits heat flow when the Biot number

$$B_i = \frac{hs}{\lambda} < 1$$

where h is the heat-transfer coefficient and s is a characteristic dimension of the sample. When, instead, $B_i > 1$, heat flow is limited by conduction. The final equation is the Stefan-Boltzmann law for radiative heat transfer. The emissivity, ε, is unity for black bodies; less for all other surfaces.

Diffusion of matter follows a pair of differential equations with the same form as Fourier's two laws, and with similar solutions. They are commonly written

$$J = -D\nabla C = -D\frac{dC}{dx} \quad \text{(steady state)}$$

and

$$\frac{\partial C}{\partial t} = D\nabla C^2 = D\frac{\partial^2 C}{\partial x^2} \quad \text{(time-dependent flow)}$$

where J is the flux, C is the concentration, x is the distance and t is time. Solutions are given in the next section.

Flow of heat and matter
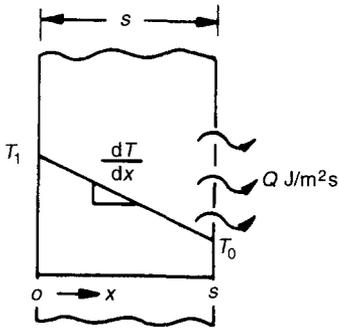
$$Q = -\lambda \nabla T = -\lambda \frac{dT}{dx}$$

Q = heat flux (J/m²s)

T = temperature (K)

x = distance (m)

λ = thermal conductivity (W/mK)
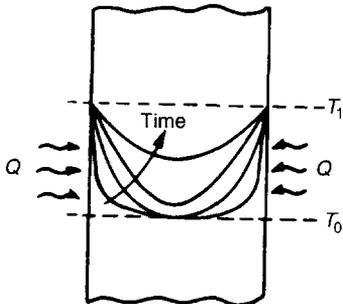
$$\frac{\partial T}{\partial t} = a\nabla^2 T = a\frac{\partial^2 T}{\partial x^2}$$

t = time (s)

ρ = density (kg/m³)

C = specific heat (J/m³K)

a = thermal diffusivity, $\dfrac{\lambda}{\rho c}$ (m²/s)
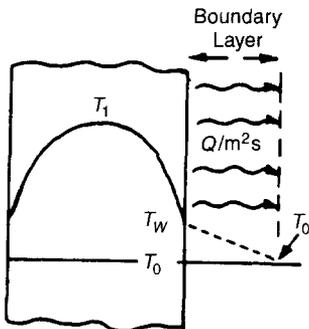
$$Q = h(T_w - T_0)$$

T_w = surface temperature (K)

T = fluid temperature (K)

h = heat transfer coeff. (W/m²K)

    = 5–50 W/m²K in air

    = 1000–5000 W/m²K in water
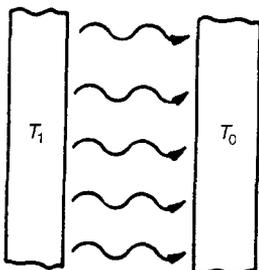
$$Q = \varepsilon\sigma(T_1^4 - T_0^4)$$

ε = emissivity (1 for black body)

σ = Stefan constant

    = 5.67×10^{-8} W/m²K⁴

A.15 Solutions for diffusion equations

Solutions exist for the diffusion equations for a number of standard geometries. They are worth knowing because many real problems can be approximated by one of these.

At steady-state the temperature or concentration profile does not change with time. This is expressed by equations in the box within the first box at the top of the facing page. Solutions for these are given below for uniaxial flow, radial flow in a cylinder and radial flow in a sphere. The solutions are fitted to individual cases by matching the constants A and B to the boundary conditions. Solutions for matter flow are found by replacing temperature, T, by concentration, C, and conductivity, λ, by diffusion coefficient, D.

The box within the second large box summarizes the governing equations for time-dependent flow, assuming that the diffusivity (a or D) is not a function of position. Solutions for the temperature or concentration profiles, $T(x, t)$ or $C(x, t)$, are given below. The first equation gives the 'thin-film' solution: a thin slab at temperature T_1, or concentration C_1 is sandwiched between two semi-infinite blocks at T_0 or C_0, at $t = 0$, and flow allowed. The second result is for two semi-infinite blocks, initially at T_1 and T_0, (or C_1 or C_0) brought together at $t = 0$. The last is for a T or C profile which is sinusoidal, of amplitude A at $t = 0$.

Note that all transient problems end up with a characteristic time constant t^* with

$$t^* = \frac{x^2}{\beta a} \quad \text{or} \quad \frac{x^2}{\beta D}$$

where x is a dimension of the specimen; or a characteristic length x^* with

$$x^* = \sqrt{\beta a t} \quad \text{or} \quad \sqrt{\beta D t}$$

where t is the timescale of observation, with $1 < \beta < 4$, depending on geometry.

Solutions for diffusion equations

Steady state

1-D

2-D

3-D
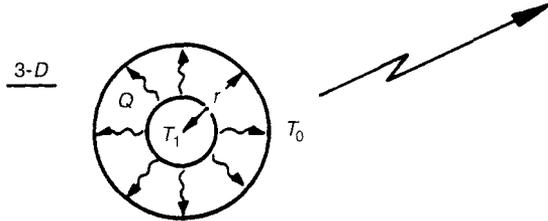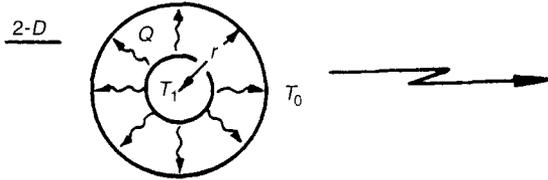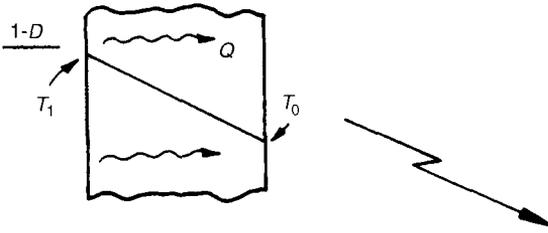
$$\lambda \nabla^2 T = 0$$

$$D \nabla^2 C = 0$$

$$\lambda T(x) = Ax + B(1 - D)$$

$$\lambda T(r) = A \ell n r + B(2 - D)$$

$$\lambda T(r) = A/r + B(3 - D) \text{ etc.}$$

$A, B =$ constants of integration

$x =$ linear distance (m)

$r =$ radial distance (m)

Transient
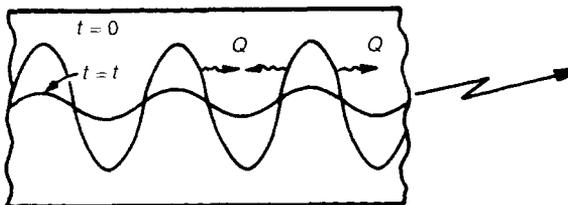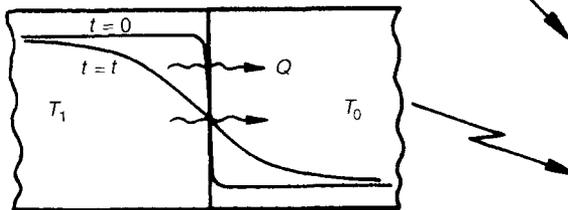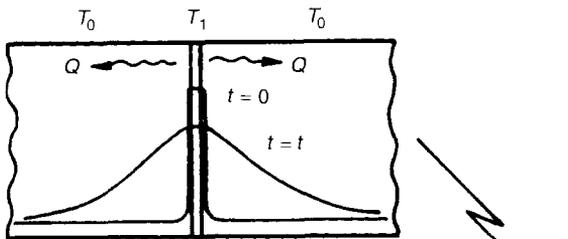
$$\frac{\partial T}{\partial t} = a \nabla^2 T$$

$$\frac{\partial C}{\partial t} = D \nabla^2 C$$

$$T(x, t) = \frac{A}{\sqrt{at}} \exp - \left(\frac{x^2}{4at} \right) + B$$

$$T(x, t) = A \left(1 + erf \left(\frac{x}{2\sqrt{at}} \right) \right) + B$$

$$T(x, t) = A \sin \lambda x, \exp -\lambda^2 at \text{ etc.}$$

A.16 Further reading

Constitutive laws

Cottrell, A.H. *Mechanical Properties of Matter*, Wiley NY (1964).
Gere, J.M. and Timoshenko, S.P. *Mechanics of Materials*, 2nd SI edition, Wadsworth International, California (1985).

Moments of area

Young, W.C. *Roark's Formulas for Stress and Strain*, 6th edition, McGraw-Hill (1989).

Beams, shafts, columns and shells

Calladine, C.R. *Theory of Shell Structures*, Cambridge University Press, Cambridge (1983).
Gere, J.M. and Timoshenko, S.P. *Mechanics of Materials*, 2nd edition, Wadsworth International, California USA (1985).
Timoshenko, S.P. and Goodier, J.N. *Theory of Elasticity*, 3rd edition, McGraw-Hill, (1970).
Timoshenko, S.P. and Gere, J.M. *Theory of Elastic Stability*, 2nd edition, McGraw-Hill (1961).
Young, W.C. *Roark's Formulas for Stress and Strain*, 6th edition, McGraw-Hill (1989).

Contact stresses and stress concentration

Timoshenko, S.P. and Goodier, J.N. *Theory of Elasticity*, 3rd edition, McGraw-Hill, (1970).
Hill, R. *Plasticity*, Oxford University Press, Oxford (1950).
Johnson, K.L. *Contact Mechanics*, Oxford University Press, Oxford (1985).

Sharp cracks

Hertzberg, R.W. *Deformation and Fracture of Engineering Materials*, 3rd edition, Wiley, New York, 1989.
Tada, H., Paris, P.C. and Irwin, G.R. *The Stress Analysis of Cracks Handbook*, 2nd edition, Paris Productions and Del Research Group, Missouri.

Pressure vessels

Timoshenko, S.P. and Goodier, J.N. *Theory of Elasticity*, 3rd edition, McGraw-Hill, (1970).
Hill, R. *Plasticity*, Oxford University Press, Oxford (1950).
Young, W.C. *Roark's Formulas for Stress and Strain*, 6th edition, McGraw-Hill (1989).

Vibration

Young, W.C. *Roark's Formulas for Stress and Strain*, 6th edition, McGraw-Hill (1989).

Creep

Finnie, I. and Heller, W.R. *Creep of Engineering Materials*, McGraw-Hill, New York. (1976).

Heat and matter flow

Hollman, J.P. *Heat Transfer*, 5th edition, McGraw-Hill, New York (1981).
Carslaw, H.S. and Jaeger, J.C. *Conduction of Heat in Solids*, 2nd edition, Oxford University Press, Oxford (1959).
Shewmon, P.G. *Diffusion in Solids*, 2nd edition, TMS Warrendale, PA (1989).