
CHAPTER 6

DYNAMIC VIBRATION ABSORBERS AND AUXILIARY MASS DAMPERS

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INTRODUCTION

Auxiliary masses are frequently attached to vibrating systems by springs and damping devices to assist in controlling the amplitude of vibration of the system. Depending upon the application, these auxiliary mass systems fall into two distinct classes.

1. If the primary system is excited by a force or displacement that has a constant frequency, or in some cases by an exciting force that is a constant multiple of a rotational speed, then it is possible to modify the vibration pattern and to reduce its amplitude significantly by the use of an auxiliary mass on a spring tuned to the frequency of the excitation. When the auxiliary mass system has as little damping as possible, it is called a *dynamic absorber*.
2. If it is impossible to incorporate damping into a structure that vibrates excessively, it may be possible to provide the damping in an auxiliary system attached to the structure. When used in this manner, the auxiliary mass system is one form of a damper. (Other forms may be incorporated as an integral part of the system.) The names *damped absorber* or *auxiliary mass damper* are given to this type of system.

It is sometimes useful to analyze the auxiliary mass system in terms of its electrical analog.

FORMS OF DYNAMIC ABSORBERS AND AUXILIARY MASS DAMPERS

In its simplest form, as applied to a single degree-of-freedom system, the character of the auxiliary mass system is the same as that of the primary system. Thus a tor-

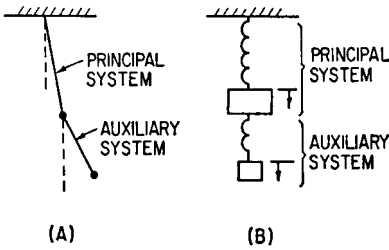


FIGURE 6.1 Dynamic vibration absorbers in pendulum form (A) and linear form (B).

For example, consider the two degree-of-freedom system shown in Fig. 6.5A consisting of two masses m_1 and m_2 on a rigid, massless bar. A dynamic absorber of the type shown in Fig. 6.5B is effective for the vertical translational motion; however, if the auxiliary masses are on cantilever beams mounted on the rigid bar, as shown in Fig. 6.5C, the absorber can be made effective for both vertical translational motion and rotational motion about an axis normal to the page.

WAYS OF EXPRESSING THE EFFECTS OF AUXILIARY MASS SYSTEMS

Suppose a linear auxiliary mass system, consisting of one or more masses, springs, and dampers, is attached to a vibrating primary system. The reaction back on the primary system is proportional to the amplitude of motion at the point of attachment. It is a function of the frequency of excitation and of the masses, spring stiffnesses, and damping constants of the auxiliary mass system. If there is no damping in the auxiliary mass system, the reaction forces are either in phase or 180° out of phase with the displacement and the acceleration at the point of attachment. However, where there is damping in the auxiliary system, the reaction has a component that is 90° out of phase with the acceleration and the displacement.

Since the reaction is proportional to the amplitude of motion, it is possible to express the properties of the auxiliary mass system in terms of the motion at the point of attachment. This can be done in three ways: (1) the ratio of the reaction force to the displacement at the point of attachment, (2) the ratio of the reaction force to the velocity at the

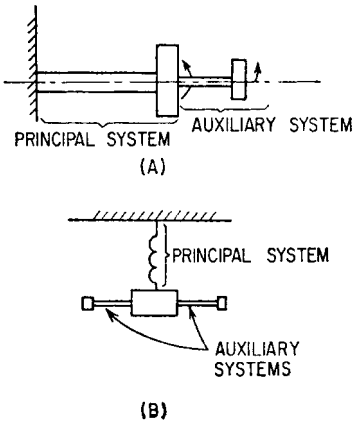


FIGURE 6.2 Typical dynamic vibration absorbers. The principal and auxiliary systems vibrate in torsion in the arrangement at (A); the auxiliary system is in the form of masses and beams at (B).

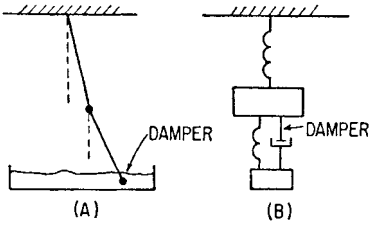


FIGURE 6.3 Damped auxiliary mass systems corresponding to the undamped vibration absorbers shown in Fig. 6.1.

impedance Z of the auxiliary system. The third ratio corresponds to a mass and is designated *equivalent mass* m_{eq} . The equivalent mass of a system is $-1/\omega^2$ that of the equivalent spring k_{eq} of the system.

point of attachment, or (3) the ratio of the reaction force to the acceleration at the point of attachment. The first ratio can be considered equivalent to a spring whose stiffness changes with frequency. The second ratio can be considered equivalent to a damper; at any frequency it is equal in magnitude to the force-displacement ratio divided by the angular frequency. The phase angle between the force and the velocity is 90° from the phase angle between the force and the displacement. This force-velocity ratio is called the *mechanical*

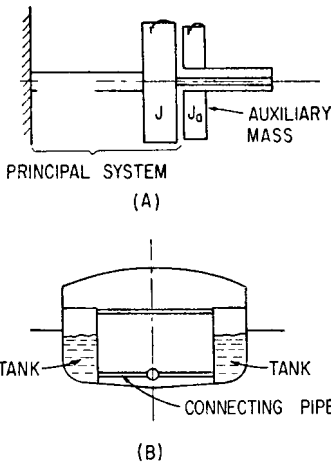


FIGURE 6.4 Typical damped auxiliary mass systems. In the torsional system at (A), damping is provided by relative motion of the flywheels J, J_a . In the antirroll tanks for ships shown at (B), water flows from one tank to the other and damping is provided by a constriction in the connecting pipe.

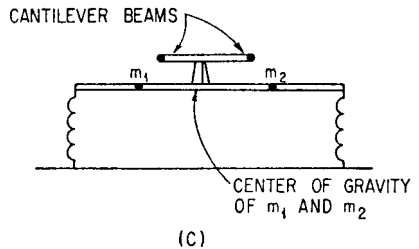
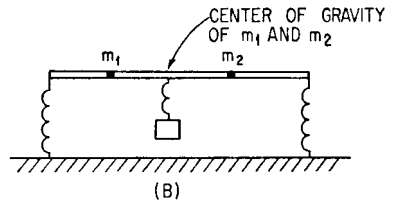
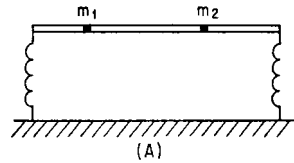


FIGURE 6.5 Application of a dynamic absorber to reduce the vibration of the spring-mounted bar at (A) in both vertical translational and rotational modes. The linear mass-spring system at (B) is effective for only translational motion, whereas the cantilever beams at (C) are effective for rotational as well as translational motion.

Because of the phase relations between the force and the displacement, velocity, and acceleration at the point of connection, it is customary to represent the ratios as complex quantities. Thus $Z = k_{eq}/j\omega = j\omega m_{eq}$. Most dynamic analyses of mechanical systems are made on purely reactive systems, i.e., systems having masses and stiffnesses only, and no damping. The effects of auxiliary mass systems are most easily understood if the effect of the auxiliary system is represented as a reactive subsystem. For this reason, and because the hypothetical addition of a mass to a system is often more easily comprehended than the addition of a spring, the effects of auxiliary mass systems are treated in terms of the equivalent masses in this chapter, i.e., in terms of the ratio of the force exerted by the auxiliary system upon the primary system to the acceleration at the point of attachment of the auxiliary system.

THE INFLUENCE OF A SIMPLE AUXILIARY MASS SYSTEM UPON A VIBRATING SYSTEM

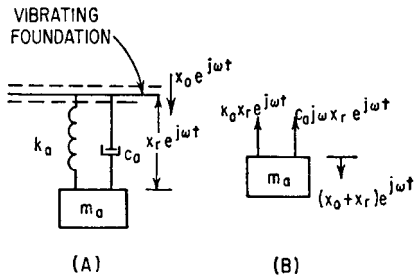


FIGURE 6.6 Auxiliary mass damper. The arrangement of the damper is shown at (A), and the forces acting on the mass are indicated at (B).

The magnitude of the equivalent mass of a simple auxiliary mass system, consisting of a mass m_a , spring k_a , and viscous damper c_a , can be determined readily by evaluating the forces exerted by such a system upon a foundation vibrating at a frequency $f = \omega/2\pi$. The system with its assumed constants and displacements is shown in Fig. 6.6A. The spring and damping forces acting on m are shown in Fig. 6.6B, and the equation of motion is

$$(-k_a x_r - c_a j\omega x_r) e^{j\omega t} = -m_a (x_0 + x_r) \omega^2 e^{j\omega t}$$

Solving for x_r ,

$$x_r = \frac{m_a \omega^2 x_0}{-m_a \omega^2 + j c_a \omega + k_a} \tag{6.1}$$

The force acting on the foundation is

$$F e^{j\omega t} = (k_a + j c_a \omega) x_r e^{j\omega t}$$

Eliminating x_r from the preceding equations,

$$F = \frac{(k_a + j c_a \omega) m_a \omega^2}{-m_a \omega^2 + j c_a \omega + k_a} x_0 \tag{6.2}$$

Since the force exerted by an equivalent mass m_{eq} rigidly attached to the moving foundation is $F = m_{eq} \omega^2 x_0$:

$$m_{eq} = \frac{k_a + j c_a \omega}{k_a + j c_a \omega - m_a \omega^2} m_a \tag{6.3}$$

Equation (6.3) can be written in terms of nondimensional quantities:

$$m_{\text{eq}} = \frac{1 + 2\zeta\beta_a j}{(1 - \beta_a^2) + 2\zeta\beta_a j} m_a \quad (6.4)$$

where $\beta_a = \frac{\omega}{\omega_a}$, a tuning parameter

$\omega_a^2 = \frac{k_a}{m_a}$, the natural frequency of the auxiliary system

$\zeta = \frac{c_a}{c_{ca}}$, a damping parameter

$c_{ca} = 2\sqrt{k_a m_a}$, critical damping of the auxiliary system

Equation (6.4) can be divided into the following real and imaginary components:

$$m_{\text{eq}} = \frac{(1 - \beta_a^2) + (2\zeta\beta_a)^2}{(1 - \beta_a^2)^2 + (2\zeta\beta_a)^2} m_a - \frac{2\zeta\beta_a^3}{(1 - \beta_a^2)^2 + (2\zeta\beta_a)^2} j m_a \quad (6.5)$$

The real and imaginary parts of m_{eq} are shown in Fig. 6.7A and B, respectively. If there is no damping, $\zeta = 0$ and

$$m_{\text{eq}} = \frac{1}{1 - \beta_a^2} m_a \quad (6.6)$$

If $\beta_a = 1$ in Eq. (6.6), m_{eq} becomes infinite and a finite force produces no displacement. Thus, the auxiliary mass enforces a point of no motion (i.e., a node) at its point of attachment.

This concept can be applied to reduce the amplitude of the forced vibration of a single degree-of-freedom system by attaching a damped absorber.^{1,2} A sketch of the system with a damped auxiliary mass system is shown in Fig. 6.8A. In the equivalent system shown in Fig. 6.8B, there is no force acting on the mass m but instead the support is given a motion $ue^{j\omega t}$. The equations for the system of Fig. 6.8B are similar to those for the system of Fig. 6.8A with the value ku substituted for F . The amplitude of forced vibration of a single degree-of-freedom system, Eq. (2.24), is

$$x_0 = \frac{F/k}{1 - m\omega^2/k}$$

The effect of the auxiliary mass system is to increase the mass m of the primary system by the equivalent mass of the auxiliary system as given by Eq. (6.4):

$$x_0 = \frac{F/k}{1 - \frac{\omega^2}{k} \left[m + m_a \frac{(1 + 2\zeta\beta_a j)}{(1 - \beta_a^2) + 2\zeta\beta_a j} \right]}$$

Substituting $\mu = m_a/m$, the mass ratio, $\delta_{\text{st}} = F/k$, the static deflection of the spring of the primary system, and $\beta = \sqrt{m\omega^2/k}$, the ratio of the forcing frequency to the natural frequency of the primary system, and writing in dimensionless form,

$$\frac{x_0}{\delta_{\text{st}}} = \frac{(1 - \beta_a^2) + 2\zeta\beta_a j}{(1 - \beta_a^2) + 2\zeta\beta_a j - \beta^2[(1 - \beta_a^2) + 2\zeta\beta_a j + \mu(1 + 2\zeta\beta_a j)]}$$

The amplitude of motion of the primary mass, without regard to phase, is

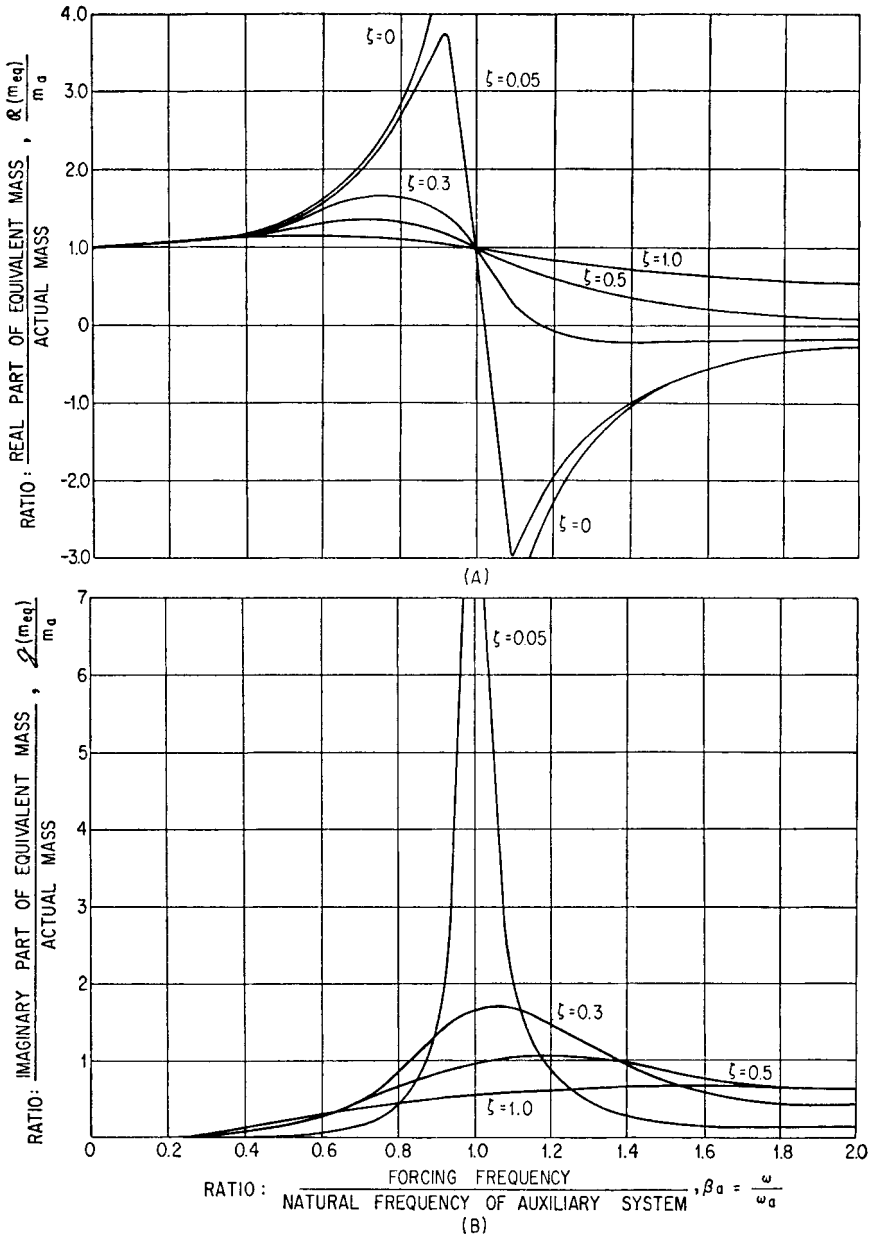


FIGURE 6.7 Equivalent mass m_{eq} of the auxiliary-mass system shown in Fig. 6.6. The real part of the equivalent mass is shown at (A) and the imaginary part at (B).

$$\frac{x_0}{\delta_{st}} = \left\{ \frac{(1 - \beta_a^2)^2 + (2\zeta\beta_a)^2}{[(1 - \beta_a^2)(1 - \beta^2) - \beta^2\mu]^2 + (2\zeta\beta_a)^2[1 - \beta^2 - \beta^2\mu]^2} \right\}^{1/2} \tag{6.7}$$

If $\zeta = 0$ (no damping), then

$$\frac{x_0}{\delta_{st}} = \frac{1 - \beta_a^2}{(1 - \beta_a^2)(1 - \beta^2) - \beta^2\mu} \tag{6.8}$$

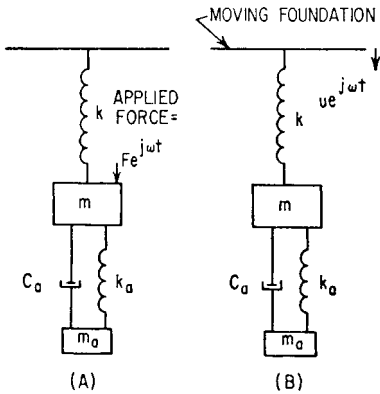


FIGURE 6.8 Schematic diagram of auxiliary mass m_a coupled by a spring k_a and viscous damper c_a to a primary system k, m . The primary system is excited by the force F at (A), or alternatively by the foundation motion u at (B).

the amplitude of motion of the auxiliary mass is u_0 , is equal and of opposite sign to the exciting force F . Hence

$$F = m_a \omega^2 u_0 \tag{6.9}$$

Since the frequency is known, the mass and amplitude of motion necessary to neutralize a given excitation force are determined by Eq. (6.9). The spring stiffness in the auxiliary system is determined by the requirement that the auxiliary system be tuned to the frequency of the exciting force:

$$k_a = m_a \omega^2 \tag{6.10}$$

Although the concept of tuning a dynamic absorber appears simple, practical considerations make it difficult to tune any system exactly. When the auxiliary mass is small relative to the mass of the primary system, its effectiveness depends upon accurate tuning. If the tuning is incorrect, the addition of the auxiliary mass may bring the composite system (primary and auxiliary systems) into resonance with the exciting force.

Consider the natural frequencies of the composite system. The natural frequency of the primary system is $\omega_0 = \sqrt{k/m}$. With this relation, Eq. (6.8) in which the damping is zero ($\zeta = 0$) becomes

$$\frac{x_0}{\delta_{st}} = \frac{1 - \omega^2/\omega_a^2}{(1 - \omega^2/\omega_a^2)(1 - \omega^2/\omega_0^2) - (\omega^2/\omega_0^2)\mu}$$

If $\beta_a = 1, x_0 = 0$; that is, the vibration of the primary system is eliminated entirely when the auxiliary system is undamped and is tuned to the forcing frequency.

THE DYNAMIC ABSORBER

If the auxiliary mass system has no damping and is tuned to the forcing frequency, it acts as a dynamic absorber and enforces a node at its point of attachment. The auxiliary mass must be sufficiently large so that it will not have an excessive amplitude.³ For a dynamic absorber attached to the primary system at the point where the excitation is introduced, the required mass of the auxiliary body is easily determined. Since the primary mass is motionless, the force exerted by the absorber, when

At resonance the denominator is zero and ω is designated ω_n :

$$(\omega_n^2 - \omega_a^2)(\omega_n^2 - \omega_0^2) - \omega_n^2 \omega_a^2 \mu = 0 \tag{6.11}$$

The natural frequencies are found from the roots ω_n^2 of Eq. (6.11):

$$\omega_n^2 = \frac{\omega_a^2(1 + \mu) + \omega_0^2}{2} \pm \sqrt{\left[\frac{\omega_a^2(1 + \mu) - \omega_0^2}{2} \right]^2 + \omega_a^2 \omega_0^2 \mu} \tag{6.12}$$

This last relation may be represented by Mohr's circle, Fig. 6.9.

Since the absorber is nominally tuned to the frequency of the excitation, the root ω_{n2}^2 that is closer to the forcing frequency is of interest. The ratio ω_{n2}/ω_a is a measure of the sensitivity of the tuning required to avoid resonance. This is given as a function of μ for various ratios of ω_0/ω_a in Fig. 6.10. Dynamic absorbers are most generally used when the primary system without the absorber is nearly in resonance with the excitation. If the natural frequency of the primary system is less than the forcing frequency, it is preferable to tune the dynamic absorber to a frequency slightly lower than the forcing frequency to avoid the resonance that lies above the natural frequency of the primary system. Likewise if the natural frequency of the primary system is above the forcing frequency, it is well to tune the damper to a frequency slightly greater than the forcing frequency. Figure 6.10 shows that the tuning for a primary system with high natural frequency is more sensitive than that for a primary system with low natural frequency. Mohr's circle of Fig. 6.9 provides a useful graphical representation.

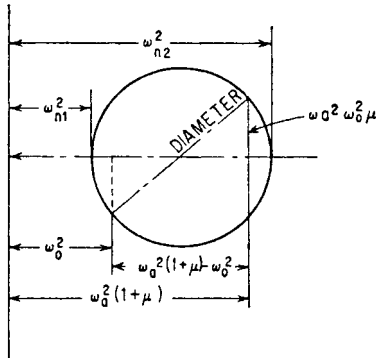


FIGURE 6.9 Representation of the natural frequencies ω_n of the composite system by Mohr's circle. The circle is constructed on the diameter located by the natural frequencies ω_0 , ω_a of the primary and auxiliary systems, respectively. The natural frequencies of the composite system are indicated by the intercept of the circle with the horizontal axis.

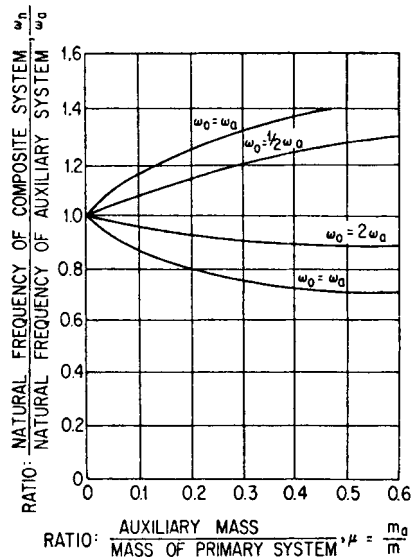


FIGURE 6.10 Curves showing effect of mass ratio m_a/m on the natural frequencies ω_n of the composite system, for several ratios of the natural frequency ω_a of the auxiliary system to the natural frequency ω_0 of the primary system.

Where the natural frequency of the composite system is nearly equal to the tuned frequency of the absorber, the amplitude of motion of the primary mass at resonance is much smaller than that of the absorber. Consequently, the motion of the primary mass does not become large even at resonance; but the motion of the absorber, unless limited by damping, may become so large that failure occurs.

The use of the dynamic absorber is not restricted to single degree-of-freedom systems or to locations in simple systems where the exciting forces act. However,

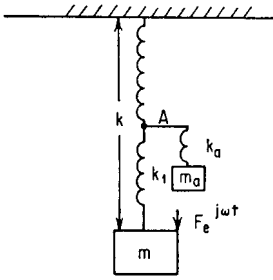


FIGURE 6.11 Dynamic absorber attached to the spring of the primary system. The analysis shows that this is not as effective as if it were attached to the rigid body on which the force acts.

dynamic absorbers are most effective if located where the excitation force acts. For example, consider a dynamic absorber that is attached to the spring in the simple system shown in Fig. 6.11. When the absorber is tuned so that $\sqrt{k_a/m_a} = \omega$, the equivalent mass is infinite at its point of attachment and enforces a node at point A. If the stiffness of the spring between A and the mass m is k_1 , then the force F' exerted by the absorber to enforce the node is equal to that exerted by a system composed of the mass m and the spring k_1 attached to a fixed foundation at A and acted upon by the force $F e^{j\omega t}$. The force F' is

$$F' = \frac{F}{1 - (m\omega^2/k_1)}$$

Thus the amplitude of motion of the auxiliary mass is

$$u_0 = \frac{F}{1 - (m\omega^2/k_1)} \times \frac{1}{m_a\omega^2} \tag{6.13}$$

The amplitude of motion of the primary mass is

$$x = \frac{F}{k_1} \left(1 - \frac{m\omega^2}{k_1} \right)^{-1} \tag{6.14}$$

Hence, an absorber attached to the spring is not as effective as one attached to the body where the force is acting. It is possible for the primary system to come into resonance about the new node at A.

AUXILIARY MASS DAMPERS

In general, the dynamic absorber is effective only for a system that is subjected to a constant frequency excitation. In the special case of a pendulum absorber (discussed later in this chapter), it is effective for an excitation that is a constant multiple of a rotating shaft speed. When excited at frequencies other than the frequency to which it is tuned, the absorber acts as an attached mass of positive value at frequencies below the tuned frequency and of negative value at frequencies above the tuned frequency. It introduces an additional degree-of-freedom and an additional natural frequency into the primary system.

In a multiple degree-of-freedom system, the introduction of an auxiliary mass system tends to lower those original natural frequencies of the primary system that are below the tuned frequency of the auxiliary system. This is because the auxiliary mass system adds a positive equivalent mass at frequencies below the tuned frequency. The original natural frequencies of the primary system that are higher than the tuned frequency of the auxiliary system are raised by adding the auxiliary mass system, because the equivalent mass of the auxiliary system is negative. A new natural mode of vibration corresponding to the vibration of the auxiliary mass system against the primary system is injected between the displaced initial natural frequencies of the primary system. Because the equivalent mass of the auxiliary mass system is large only at frequencies near the tuned frequency, those frequencies of the primary system that are closest to the tuned frequency are most strongly influenced by the auxiliary mass system. The addition of damping in the auxiliary mass system can be effective in reducing the amplitudes of motion of the primary system at the natural frequencies. For this reason auxiliary mass dampers are used quite commonly to reduce over-all vibration stresses and amplitudes.

Studies of the effects of a damped auxiliary mass system upon the amplitude of motion of an undamped, single degree-of-freedom system¹⁻⁵ have been applied to a multimass system.^{6,7} In analyzing dampers utilizing auxiliary masses, it is desirable to consider a composite system in which the characteristics of both the primary and auxiliary systems are fixed. This composite system is excited by a harmonic force of varying frequency. It is desirable to express the tuned frequency of the auxiliary mass system in terms of the natural frequency of the primary system rather than the ratio β_a of the excitation frequency ω to the tuned frequency ω_a of the auxiliary system. Defining a new ratio α ,

$$\alpha = \frac{\omega_a}{\omega_0} = \frac{\beta}{\beta_a}$$

Then Eq. (6.7) becomes

$$\frac{x_0}{\delta_{st}} = \left\{ \frac{(\alpha^2 - \beta^2)^2 + (2\zeta\alpha\beta)^2}{[(\alpha^2 - \beta^2)(1 - \beta^2) - \alpha^2\beta^2\mu]^2 + (2\zeta\alpha\beta)^2(1 - \beta^2 - \beta^2\mu)^2} \right\}^{1/2} \quad (6.15)$$

This equation is plotted in Fig. 6.12. Note that all curves pass through two points *A*, *B* on the graph, independent of the damping parameter ζ . These points are known as *fixed points*. Their locations are independent of the value of ζ if the ratio of the coefficient of ζ^2 to the term independent of ζ is the same in both numerator and denominator of Eq. (6.15):

$$\frac{(2\alpha\beta)^2}{(\alpha^2 - \beta^2)^2} = \frac{2\alpha\beta(1 - \beta^2 - \beta^2\mu)^2}{[(\alpha^2 - \beta^2)(1 - \beta^2) - \alpha^2\beta^2\mu]^2} \quad (6.16)$$

This equation is satisfied if

$$\begin{aligned} (2\alpha\beta)^2 &= 0 \\ \frac{1}{\alpha^2 - \beta^2} + \frac{(1 - \beta^2 - \beta^2\mu)}{(\alpha^2 - \beta^2)(1 - \beta^2) - \alpha^2\beta^2\mu} &= 0 \\ \frac{1}{\alpha^2 - \beta^2} - \frac{(1 - \beta^2 - \beta^2\mu)}{(\alpha^2 - \beta^2)(1 - \beta^2) - \alpha^2\beta^2\mu} &= 0 \end{aligned}$$

The first two solutions are trivial. The third yields the equation

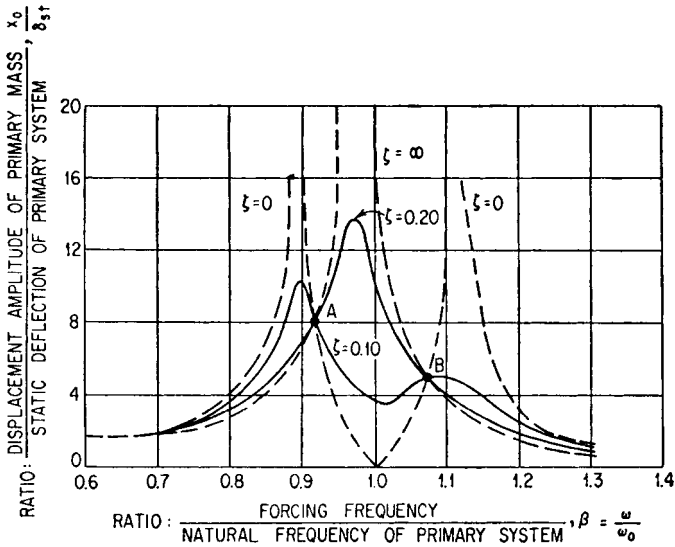


FIGURE 6.12 Curves for auxiliary mass damper showing amplitude of vibration of mass of primary system, as given by Eq. (6.15), as a function of the ratio of forcing frequency ω to natural frequency of primary system $\omega = \sqrt{k/m}$. The mass ratio $m_a/m = 0.05$, and the natural frequency ω_a of the auxiliary mass system is equal to the natural frequency ω_0 of the primary system. Curves are included for several values of damping in the auxiliary system.

$$\beta^4 \left(1 + \frac{\mu}{2} \right) - \beta^2 (1 + \alpha^2 + \alpha^2 \mu) + \alpha^2 = 0 \tag{6.17}$$

The solution of this equation gives two values of β , designated β_c , one corresponding to each fixed point.

The amplitude of motion at each fixed point may be found by substituting each value of β_c given by Eq. (6.17) into Eq. (6.15). Since the amplitude is independent of ζ , the value that gives the simplest calculation (namely, $\zeta = \infty$) can be used for the calculation:

$$\frac{x_0}{\delta_{st}} \Big|_c = \left[\frac{1}{(1 - \beta_c^2 - \beta_c^2 \mu)^2} \right]^{1/2} \tag{6.18}$$

For the auxiliary mass damper to be most effective in limiting the value of x_0/δ_{st} over a full range of excitation frequencies, it is necessary to select the spring and damping constants of the system as given by the parameters α and ζ , respectively, so that the amplitude x_0 of the primary mass is a minimum. First consider the influence of the ratio α . As α is varied, the values of β_c computed from Eq. (6.17) are substituted in Eq. (6.18) to obtain values of x_0/δ_{st} for the fixed points A and B. The optimum value of α is that for which the amplitude x_0 at A is equal to that at B.

Let the two roots of Eq. (6.17) be β_1^2 and β_2^2 , where β_1^2 is less than 1 and β_2^2 is greater than 1. When x_0/δ_{st} has the same value for both β_1 and β_2 in Eq. (6.18),

$$\beta_1^2 + \beta_2^2 = \frac{2}{1 + \mu}$$

In an equation having unity for the coefficient of its highest power, the sum of the roots is equal to the coefficient of the second term with its sign changed:

$$\beta_1^2 + \beta_2^2 = \frac{1 + \alpha^2 + \alpha^2\mu}{1 + \mu/2}$$

From the two preceding equations, the optimum tuning (i.e., that required to give the same amplitude of motion at both fixed points) is obtained:

$$\alpha_{opt} = \frac{1}{1 + \mu} \tag{6.19}$$

where α is defined by the equation preceding Eq. (6.15).

If the effect of the damping is considered, it is possible to choose a value of the damping parameter ζ that will make the fixed points nearly the points of greatest amplitude of the motion. Consider Fig. 6.13, which represents the curves defining the motion of a single degree-of-freedom system to which an ideally tuned damped vibration absorber is attached (Fig. 6.8). The solid curves (1) represent the response of a system fitted with an undamped absorber. Curve 2 represents infinite damping of the auxiliary system. Curves 3 have horizontal tangents at the fixed points *A* and *B*, respectively. Since it is difficult to determine the required damping from maxima at the fixed points, the assumption is made that an optimum damping gives the same value of x_0/δ_{st} at a convenient point between *A* and *B* as at these fixed points. First find the values of β at *A* and *B*. This is done by solving Eq. (6.17) with the values of α as determined by Eq. (6.19) substituted:

$$\beta^4 - \frac{2\beta^2}{1 + \mu} + \frac{2}{(2 + \mu)(1 + \mu)^2} = 0$$

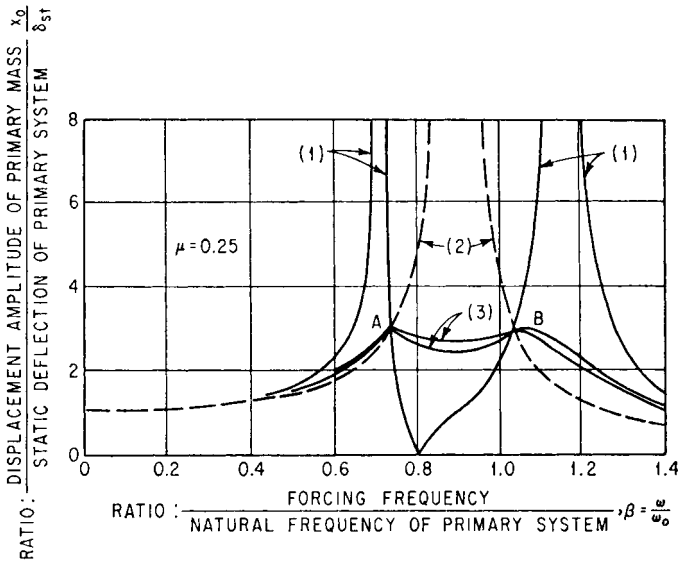


FIGURE 6.13 Curves similar to Fig. 6.12 but with optimum tuning. Curves 1 apply to an undamped absorber, curve 2 represents infinite damping in the auxiliary system, and curves 3 have horizontal tangents at the fixed points *A* and *B*.

Solving for β to obtain the abscissas at the fixed points,

$$\beta^2 = \frac{1}{1 + \mu} \sqrt{\left(1 \pm \frac{\mu}{2 + \mu}\right)} \tag{6.20}$$

A convenient value for β lying between the two fixed points A and B is defined by

$$\beta_l^2 = \frac{1}{1 + \mu} \tag{6.21}$$

The frequency corresponding to this frequency ratio β_l is the natural frequency of the composite system when the damping is infinite; it is called the locked frequency.⁷ The value of x_0/δ_{st} at the fixed points is found by substituting Eq. (6.20) into Eq. (6.18):

$$\frac{x_0}{\delta_{st}} \text{ at fixed point} = \sqrt{1 + \frac{2}{\mu}} \tag{6.22}$$

An approximate value for the maximum damping is obtained by solving for the value of ζ in Eq. (6.15) that gives a value of $x_0/\delta_{st} = \sqrt{1 + 2/\mu}$ when β_l^2 (the locked frequency) is given by Eq. (6.21) and α has the optimum value given by Eq. (6.19). This gives the following value for the optimum damping parameter:

$$\zeta_{opt} = \sqrt{\frac{\mu}{2(1 + \mu)}} \tag{6.23}$$

It is possible to find the value of ζ^2 that makes the fixed point A a maximum on the x_0/δ_{st} vs. β plot, Fig. 6.13, and also to find the value of ζ^2 that makes the point B a maximum. The average of the two values so obtained indicates optimum damping:⁴

$$\zeta_{opt} = \sqrt{\frac{3\mu}{8(1 + \mu)^3}} \tag{6.24}$$

Optimum Damping for an Auxiliary Mass Absorber Connected to the Primary System with Damping Only. In general, the most effective damping is obtained where the auxiliary mass damping system includes a spring in its connection to the primary system. However, such a design requires a calculation of the optimum stiffness of the spring. Sometimes it is more expedient to add an oversize mass, coupled only by damping to the primary system, than it is to compute the optimum system. However, if use is made of such a simplified damper by taking it from a list of standard dampers and applying it with a minimum of calculations, the stock dampers should be as efficient as the application will permit.

In computing the optimum damping characteristic for an auxiliary mass absorber, attached to a single degree-of-freedom system by damping only, from the relations that have been developed, note in Eq. (6.4) that $\zeta = \infty$ and $\beta_a = \infty$ when $k = 0$. Then $\alpha = \beta/\beta_a = 0$. However, the product $\zeta\alpha = \zeta\beta/\beta_a$ is finite; thus, substituting $\alpha = 0$ but retaining the product $\zeta\alpha$ in Eq. (6.15),

$$\frac{x_0}{\delta_{st}} = \sqrt{\frac{\beta^2 + 4(\zeta\alpha)^2}{\beta^2(1 - \beta^2)^2 + 4(\zeta\alpha)^2[1 - \beta^2(1 + \mu)]^2}} \tag{6.25}$$

The value of x_0/δ_{st} is independent of $\zeta\alpha$ where the ratio of the coefficient of $\zeta\alpha$ to the term independent of $\zeta\alpha$ in the numerator is the same as the corresponding ratio in the denominator:

$$\frac{4}{\beta^2} = \frac{4[1 - \beta^2(1 + \mu)]^2}{\beta^2(1 - \beta^2)^2}$$

The solution of this equation for β gives the fixed points

$$\beta^2 = 0 \quad \text{and} \quad \beta^2 = \frac{2}{2 + \mu} \tag{6.26}$$

The amplitude of motion of the primary mass where $\beta^2 = 2/(2 + \mu)$ is

$$\frac{x_0}{\delta_{st}} = \frac{2 + \mu}{\mu} \tag{6.27}$$

Curves showing the motion of the mass of a primary system fitted with an auxiliary mass system connected by damping only are given in Fig. 6.14. The optimum damping is that which makes the maximum amplitude occur at the fixed point *B*. By finding the value of $\zeta\alpha$ that makes the slope of x_0/δ_{st} versus β equal to zero at $\beta^2 = 2/(2 + \mu)$, the optimum damping is defined by

$$(\zeta\alpha)_{opt} = \sqrt{\frac{1}{2(2 + \mu)(1 + \mu)}} \tag{6.28}$$

The values for the amplitude of vibration of the primary mass, the relative amplitude between the primary and auxiliary masses, and the optimum damping constants are given in Figs. 6.15 to 6.17 as functions of the mass ratio $\mu = m_0/m$.

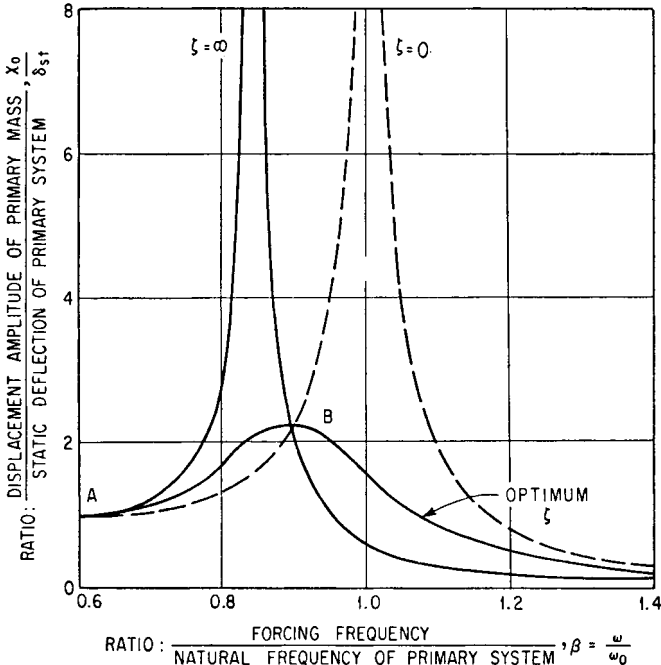


FIGURE 6.14 Curves similar to Fig. 6.12 for system having auxiliary mass coupled by damping only. Several values of damping are included.

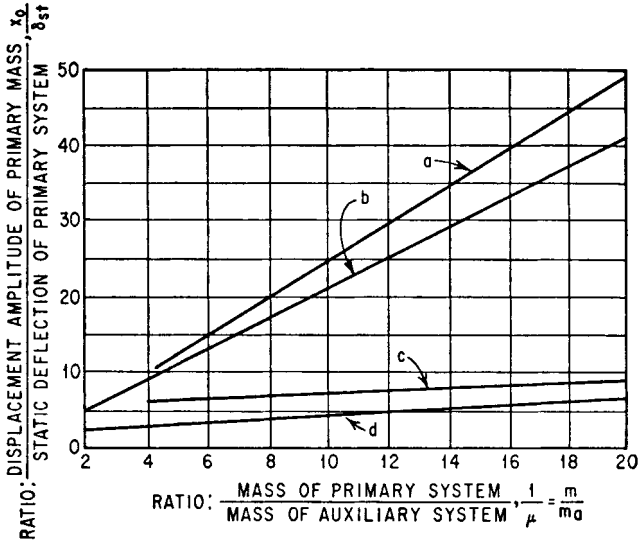


FIGURE 6.15 Displacement amplitude of the primary mass as a function of the size of the auxiliary mass: (a) auxiliary system coupled only by Coulomb friction ($\alpha = 0$) with optimum damping; (b) auxiliary system coupled only by viscous damping ($\alpha = 0$) of optimum value; (c) auxiliary system coupled by spring and damper tuned to frequency of primary system ($\alpha = 1$) with optimum damping; (d) auxiliary system coupled by spring and damper with optimum tuning [$\alpha = 1/(1 + \mu)$] and optimum damping.

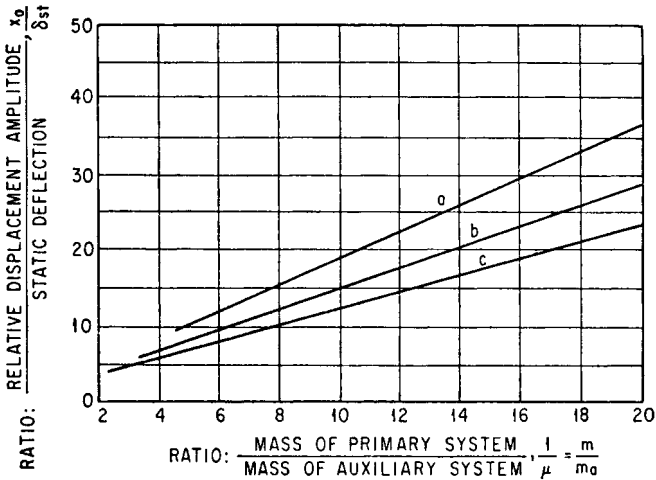


FIGURE 6.16 Relative displacement amplitude between the primary mass and the auxiliary mass as a function of the size of the auxiliary mass: (a) auxiliary system coupled by spring and damper with optimum tuning [$\alpha = 1/(1 + \mu)$] and optimum damping; (b) auxiliary system coupled only by viscous damping ($\alpha = 0$) of optimum value; (c) auxiliary system coupled by spring and damper tuned to frequency of primary system ($\alpha = 1$) with optimum damping.

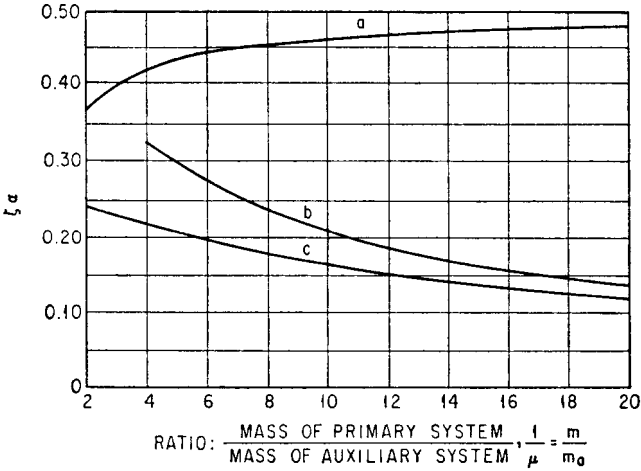


FIGURE 6.17 Curves showing damping required in auxiliary mass systems to minimize vibration amplitude of primary system: (a) auxiliary mass coupled by viscous damping only ($\alpha = 0$); (b) auxiliary system coupled by spring and damper tuned to frequency of primary system ($\alpha = 1$); (c) auxiliary system coupled by spring and damper with optimum tuning [$\alpha = 1/(1 + \mu)$]. The ordinate of the curves is $\zeta\alpha$, where ζ is the fraction of critical damping in the auxiliary system [Eq. (6.4)] and α is the tuning parameter [Eq. (6.15)].

The Use of Auxiliary Mass Absorbers for Vibration Energy Dissipation.

When a complicated mass-spring system is analyzed for possible vibration troubles, it is customary to compute the natural frequencies of the several modes of vibration of the system. The vibration amplitudes and stresses are estimated by making an energy balance between the energy input from the various exciting forces and the energy dissipated in the system and external reactions. From this point of view, it is desirable to know how much energy is dissipated in auxiliary mass systems and what value the damping constant should have in an auxiliary mass system of limited size to give maximum energy absorption. This is not the best criterion for determining the optimum damping because it neglects the effects of damping upon the mode shapes and the frequencies of the system, but it is generally adequate when compared with the other uncertainties of the calculations. Methods of designing dampers for torsional systems are given in Chap. 38.

Optimum Viscous Damping to Give Large Energy Absorption in an Auxiliary Mass Absorber.⁸ Suppose the amplitude of motion of the primary system is unaffected by the auxiliary mass system which is attached to it. All energy absorption occurs in the damping element of the auxiliary mass system and is obtained by integrating the differential work done in the damper over a vibration cycle. The force exerted by damping is $c\dot{x}_r$ (the subscripts *a* are dropped), where x_r is the relative motion and the increment of work is $c\dot{x}_r dx_r = c\dot{x}_r^2 dt$. If $x_r = x_{r0} \cos \omega t$, the work done over a cycle is

$$V = \oint c\omega^2 x_{r0}^2 \sin^2 \omega t dt = \pi c x_{r0}^2 \omega \tag{6.29}$$

For a damper attached to a support moving in harmonic motion of amplitude x_0 , the relative motion x_r is given by Eq. (6.1). The amplitude of relative motion is

$$x_{r0} = \frac{m\omega^2 x_0}{\sqrt{(k - m\omega^2)^2 + c^2\omega^2}} = \frac{\beta_a^2 x_0}{\sqrt{(1 - \beta_a^2)^2 + (2\zeta\beta_a)^2}}$$

Substituting the above value of x_{r0} in Eq. (6.29) and integrating,

$$V = \frac{\pi c \omega x_0^2 m^2 \omega^4}{(k - m\omega^2)^2 + c^2 \omega^2} = \frac{\pi x_0^2 m \omega^2 (2\zeta\beta_a)\beta_a^2}{(1 - \beta_a^2)^2 + (2\zeta\beta_a)^2} \tag{6.30}$$

Equation (6.30) can be used to find the tuning and the damping that gives the maximum energy dissipation when the amplitude of the forcing motion remains constant. Placing $\partial V/\partial\beta_a = 0$, the optimum value of β_a for given values of ζ is found from

$$(\beta_a)_{opt}^2 = (2\zeta^2 - 1) \pm 2\sqrt{1 - \zeta^2 + \zeta^4} \tag{6.31}$$

Placing $\partial V/\partial\zeta = 0$, the optimum value of ζ for a given value of β_a is

$$\zeta_{opt} = \frac{1 - \beta_a^2}{2\beta_a} \tag{6.32}$$

Where $k = 0$, the optimum damping is determined most conveniently by setting $\partial V/\partial c = 0$, using the dimensional form of Eq. (6.30), and determining c for maximum energy absorption:

$$c_{opt} = m\omega \tag{6.33}$$

Auxiliary Mass Damper Using Coulomb Friction Damping.⁹ Dampers relying on Coulomb friction (i.e., friction whose force is constant) have been widely used. A damper relying on dry friction and connected to its primary system with a spring is too complicated to be analyzed or to be adjusted by experiment. For this reason, a damper with Coulomb friction has been used with only friction damping connecting the seismic mass (usually in a torsional application) to the primary system.^{1,2,9} Because the motion is irregular, it is necessary to use energy methods of analysis. The analysis given here applies to the case of linear vibration. By analogy, the application to torsional or other vibration can be made easily (see Table 2.1 for analogous parameters).

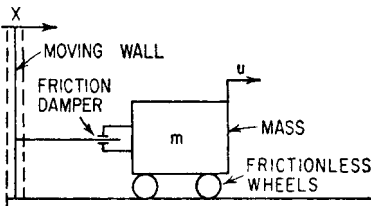


FIGURE 6.18 Schematic diagram of auxiliary mass absorber with Coulomb friction damping.

Consider the system shown in Fig. 6.18. It consists of a mass resting on wheels that provide no resistance to motion and are connected through a friction damper to a wall that is moving sinusoidally. The friction damper consists of two friction facings that are held on opposite sides of a plate by a spring that can be adjusted to give a desired clamping force. The maximum force that can be transmitted through each interface of the damper is the product of the normal force and the coefficient of friction; the maximum total force for the damper is the summation over the number of interfaces.

Consider the velocity diagrams shown in Fig. 6.19A, B, and C. In these diagrams the velocity of the moving wall, $\dot{x} = x_0\omega \sin \omega t$, is shown by curve 1; the velocity \dot{u} of the mass is shown by curve 2. The force exerted by the damper when slipping occurs is F_s . When $F_s \geq m\ddot{u}$, the mass moves sinusoidally with the wall. When $F_s < m\ddot{u}$, slip-

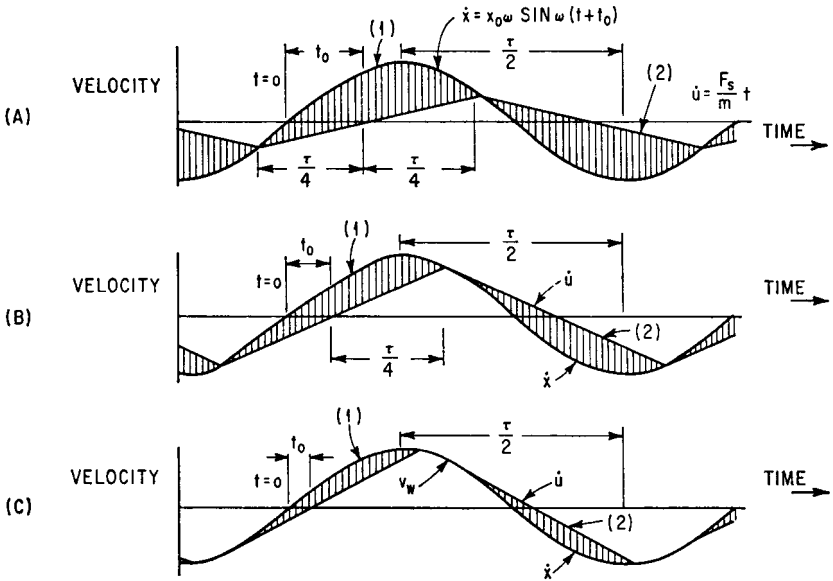


FIGURE 6.19 Velocity-time diagrams for motion of wall (curve 1) and mass (curve 2) of Fig. 6.18. The conditions for a small damping force are shown at (A), for an intermediate damping force at (B), and for a large damping force at (C). The relative velocity between the wall and the mass is indicated by vertical shading.

ping occurs in the damper and the mass is accelerated at a constant rate. Since a constant acceleration produces a uniform change in velocity, the velocity of the mass when the damper is slipping is shown by straight lines. The relative velocity between the wall and the mass is shown by the vertical shading.

Figure 6.19A applies to a damper with a low friction force. The damper slips continuously. In Fig. 6.19B the velocities resulting from a larger friction force are shown. Slipping disappears for certain portions of the cycle. Where the wall and the mass have the same velocity, their accelerations also are equal. Slipping occurs when the force transmitted by the damper is not large enough to keep the mass accelerating with the wall. Since at the breakaway point the accelerations of the wall and mass are equal, their velocity-time curves have the same slope; i.e., the curves are tangent at this point. In Fig. 6.19C, the damping force is so large that the mass follows the wall for a considerable portion of the cycle and slips only where its acceleration becomes greater than the value of F_s/m . A slight increase in the clamping force or in the coefficient of friction locks the mass to the wall; then there is no relative motion and no damping.

Because of the nature of the damping force, the damping provided by the friction damper can be computed most practically in terms of energy. If the friction force exerted through the damper is F_s , the energy dissipated by the damper is the product of the friction force and the total relative motion between the mass and the moving wall. The time reference is taken at the moment when the auxiliary mass m has a zero velocity and is being accelerated to a positive velocity, Fig. 6.19A. Let the period of the vibratory motion of the wall be $\tau = 2\pi/\omega$, where ω is the angular frequency of the wall motion. By symmetry, the points of no slippage in the damper occur at times

$-\tau/4, \tau/4,$ and $3\tau/4$. Let the time when the velocity of the wall is zero be $-t_0$; then the velocity of the wall \dot{x} is

$$\dot{x} = +x_0\omega \sin \omega(t + t_0)$$

The velocity \dot{u} of the mass for $-\tau/4 < t < \tau/4$ is

$$\dot{u} = \dot{u}t = \frac{F_s}{m}t$$

The velocities of the wall and the mass are equal at time $t = \tau/4$:

$$x_0\omega \sin \omega \left(\frac{\tau}{4} + t_0 \right) = \frac{F_s}{m} \frac{\tau}{4}$$

Since $\omega\tau/4 = \pi/2$, $\sin \omega(\tau/4 + t_0) = \cos \omega t_0$. Therefore

$$\cos \omega t_0 = \frac{F_s}{m} \frac{\pi}{2x_0\omega^2}$$

The relative velocity between the moving wall and the mass is $\dot{x} - \dot{u}$, and the total relative motion is the integral of the relative velocity over a cycle. Note that the area between the two curves for the second half of the cycle is the same as for the first. Hence, the work V per cycle is

$$V = 2 \int_{-\tau/4}^{\tau/4} F_s(\dot{x} - \dot{u}) dt = 4F_s x_0 \sqrt{1 - \left(\frac{F_s \pi}{2m x_0 \omega^2} \right)^2} \tag{6.34}$$

Optimum damping occurs when the work per cycle is a maximum. It can be determined by setting the derivative of V with respect to F_s in Eq. (6.34) equal to zero and solving for F_s :

$$(F_s)_{\text{opt}} = \frac{\sqrt{2}}{\pi} m\omega^2 x_0 \tag{6.35}$$

Energy absorption per cycle with optimum damping is, from Eq. (6.34),

$$V_{\text{opt}} = \frac{4}{\pi} m\omega^2 x_0^2 \tag{6.36}$$

A comparison of the effectiveness of the Coulomb friction damper with other types is given in Fig. 6.15.

EFFECT OF NONLINEARITY IN THE SPRING OF AN AUXILIARY MASS DAMPER

It is possible to extend the range of frequency over which a dynamic absorber is effective by using a nonlinear spring.¹⁰⁻¹² When a nonlinear spring is used, the natural frequency of the absorber is a function of the amplitude of vibration; it increases or decreases, depending upon whether the spring stiffness increases or decreases with deflection. Figure 6.20A shows a typical response curve for a system with increasing spring stiffness; Fig. 6.20B illustrates types of systems having increasing spring stiffness and shows typical force-deflection curves. Figure 6.21A shows a typical response

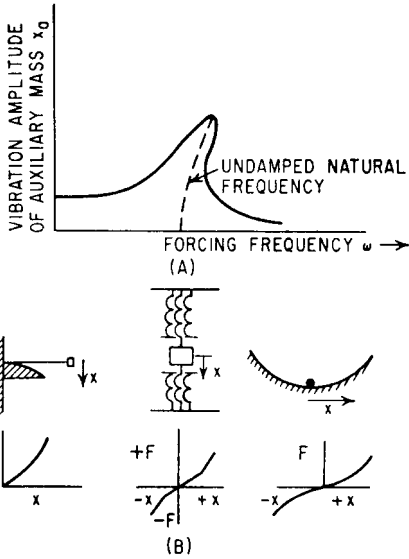


FIGURE 6.20 Auxiliary mass damper with nonlinear spring having stiffness that increases as deflection increases. The response to forced vibration and the natural frequency are shown at (A). Several arrangements of nonlinear systems with the corresponding force-deflection curves are shown at (B).

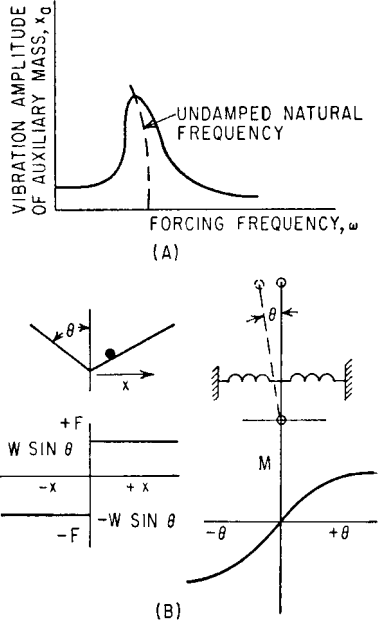


FIGURE 6.21 Auxiliary mass damper with nonlinear spring having stiffness that decreases as deflection increases. The response to forced vibration and the natural frequency are shown at (A). Two arrangements of nonlinear systems with the corresponding force-deflection curves are shown at (B).

curve for a system of decreasing spring stiffness; Fig. 6.21B illustrates types of systems having decreasing stiffnesses and shows typical force-deflection curves.

To compute the equivalent mass at a given frequency when a nonlinear spring is used, it is necessary to use a trial-and-error procedure. By the methods given in Chap. 4, compute the natural frequency of the auxiliary mass system, assuming the point of attachment fixed, as a function of the amplitude of motion of the auxiliary mass. This will result in a curve similar to the dotted curves in Figs. 6.20A and 6.21A. At the given frequency, compute β_a in Eq. (6.4) in terms of the tuned frequency of the absorber at zero amplitude. (The tuned frequency will change with amplitude because the spring constant changes.) With this value of β_a compute the equivalent mass from Eq. (6.6). With this mass in the system, compute the amplitude of motion x_0 of the primary mass to which the auxiliary system is attached [Eq. (6.7)] and the amplitude of the relative motion $x_{r0} = v^2(1 - v^2)x_0$. Using this value of x_{r0} , ascertain the corresponding value of resonance frequency of the system from the computed curve, and compute the new value of β_a . Repeat the process until the value of β_a remains unchanged upon repeated calculation.

A dynamic absorber having a nonlinear characteristic can be used to introduce nonlinearity into a resonant system. This can be useful in the case where a machine passes through a resonance rapidly as the speed is increased but slowly as the speed is decreased. In bringing this machine up to speed, there is a natural frequency that comes into strong resonance, giving a critical speed. A strongly nonlinear dynamic absorber tuned at low amplitudes to the optimum frequency for the damped absorber

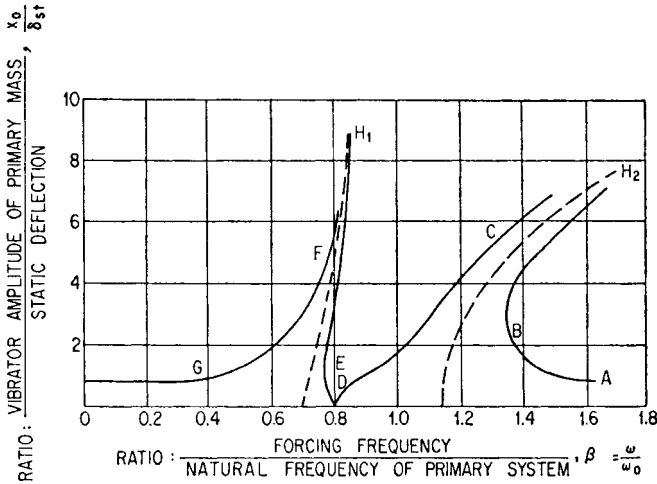


FIGURE 6.22 Motion of the primary mass, as a function of forcing frequency, in a system having a nonlinear dynamic absorber whose natural frequency increases with amplitude. The mass of the absorber is 0.25 times the mass of the primary system ($\mu = 0.25$).

can be used to reduce the effects of the critical speed. Two resonant peaks will be introduced, as shown on curve 1 of Fig. 6.13. By making the dynamic absorber nonlinear, so that the stiffness becomes greater as the amplitude of vibration is increased, the peaks are bent over to provide the response curve shown in Fig. 6.22. In starting, the machine is accelerated through the two critical speeds so fast that a resonance is unable to build up. In coasting to a stop, there would be ample time for significant amplitudes to build up if the nonlinearity did not exist. Because of the nonlinearity, the amplitude of vibration as a function of speed (since β is proportional to speed) follows the path A, B, C, D, E, F, G and never reaches the extreme amplitudes H_1 and H_2 .

MULTIMASS ABSORBERS

In general, only one mass is used in a dynamic absorber. However, it is possible to provide a dynamic absorber that is effective for two or more frequencies by attaching an auxiliary mass system that resonates at the frequencies that are objectionable. The principle that would make such a dynamic absorber effective is utilized in the design of the elastic system of a ship's propulsion plant driven by independent high-pressure and low-pressure turbines. By making the frequencies of the two branches about the reduction gear identical, the gear becomes a node for one of the resonant modes. Then it is impossible to excite the mode of vibration where one turbine branch vibrates against the other as a result of excitation transmitted by the propeller shaft to that node.

DISTRIBUTED MASS ABSORBERS

It is possible to use distributed masses as vibration dampers. Consider an undamped rod of distributed mass and elasticity attached to a foundation that vibrates the rod

axially, as shown in Fig. 6.23. The differential equation for the motion of this rod is derived in Chap. 7. The values of the constants are set by the boundary conditions:

$$\begin{aligned} \text{Stress} &= E \frac{\partial u}{\partial x} = 0 \quad \text{where } x = l \\ u &= u_0 \cos \omega t \quad \text{where } x = 0 \end{aligned} \tag{6.37}$$

The solution of the equation of motion is

$$\begin{aligned} u &= u_0 \cos \omega t \left(\cos \sqrt{\frac{\gamma \omega^2}{Eg}} x + \tan \sqrt{\frac{\gamma \omega^2}{Eg}} l \sin \sqrt{\frac{\gamma \omega^2}{Eg}} x \right) \\ &= u_0 \cos \omega t \frac{\cos \sqrt{\gamma \omega^2 / Eg} (1-x)}{\cos \sqrt{\gamma \omega^2 / Eg} l} \end{aligned} \tag{6.38}$$

where E is the modulus of elasticity and γ is the weight density of the material. When $x = 0$, the force F on the foundation is

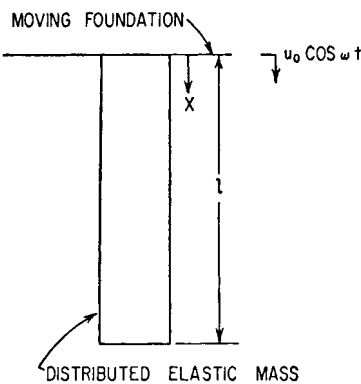
$$F = SE \frac{\partial u}{\partial x} \Big|_0 = SEu_0 \sqrt{\frac{\gamma \omega^2}{Eg}} \left(\tan \sqrt{\frac{\gamma \omega^2}{Eg}} l \right) \tag{6.39}$$

where S is the cross-sectional area of the bar. It is apparent that as the argument of the tangent has successive values of $\pi/2, 3\pi/2, 5\pi/2, \dots$, the force exerted on the foundation becomes infinite. The distributed mass acts as a dynamic absorber enforcing a node at its point of attachment. By tuning the mass so that

$$\sqrt{\frac{\gamma \omega^2}{Eg}} l = \frac{n\pi}{2} \quad \text{or} \quad l = \frac{n\pi}{2\omega} \sqrt{\frac{Eg}{\gamma}} \tag{6.40}$$

the distributed mass acts as a dynamic absorber for not only the fundamental frequency $\omega/2\pi$ but also for the third, fifth, seventh, . . . harmonics of the fundamental.

The above solution neglects damping. It is possible to consider the effect of damping by including a damping term in the differential equation. The stress in an element is assumed to be the sum of a deformation stress and a stress related to the velocity of strain:



$$\sigma = E\epsilon + \mu \frac{d\epsilon}{dt} \tag{6.41}$$

where $\epsilon = \partial u / \partial x$ is the strain. Then the differential equation becomes

$$E \frac{\partial^2 u}{\partial x^2} + \mu \frac{\partial^3 u}{\partial x^2 \partial t} = \frac{\gamma}{g} \frac{\partial^2 u}{\partial t^2} \tag{6.42}$$

Since the absorber is excited by a foundation moving with a frequency $f = \omega/2\pi$, u may be expressed as $\Re u_1 e^{j\omega t}$ and the partial differential equation can be written as the ordinary linear differential equation

$$E \frac{d^2 u_1}{dx^2} + j\omega\mu \frac{d^2 u_1}{dx^2} + \frac{\gamma \omega^2 u_1}{g} = 0$$

This equation may be written

FIGURE 6.23 Elastic body with distributed mass used as auxiliary mass damper.

$$\left(1 + \frac{\mu\omega j}{E}\right) \frac{d^2u_1}{dx^2} + \frac{\gamma\omega^2}{Eg} u_1 = 0 \tag{6.43}$$

Since Eq. (6.43) is a second-order linear differential equation, the solution may be written

$$u = A_1e^{\beta_1x} + A_2e^{\beta_2x} \tag{6.44}$$

where β_1 and β_2 are the two roots of the equation

$$\beta^2 = \frac{-(\gamma/Eg)\omega^2}{1 + (j\mu\omega/E)} \tag{6.45}$$

For small values of μ , by a binomial expansion of the denominator,

$$\pm\beta = \frac{1}{2} \sqrt{\frac{\gamma}{Eg} \frac{\mu\omega^2}{E}} + j\sqrt{\frac{\gamma}{Eg}} \omega \tag{6.46}$$

where μ is defined by Eq. (6.41).

The boundary conditions to be met by the damper are:

At $x = 0$: $u = u_0$ therefore, $A_1 + A_2 = u_0$ (6.47)

At $x = l$: $\sigma = (E + j\omega\mu) \frac{\partial u}{\partial x} = 0$ therefore, $A_1e^{\beta l} - A_2e^{-\beta l} = 0$

Solving Eqs. (6.47) for A_1 and A_2 and substituting the result in Eq. (6.44),

$$u = u_0 \frac{\cosh \beta(l-x)}{\cosh \beta l} \tag{6.48}$$

The force exerted on the foundation by the damper is

$$F_{(x=0)} = S\sigma_{(x=0)} = -Su_0(E + j\omega\mu) \beta \tanh \beta l \tag{6.49}$$

where S is the cross-section area of the bar. When the complex value of β as given in Eq. (6.46) is substituted in Eq. (6.49), the following value for the dynamic force exerted on the foundation is obtained:

$$\begin{aligned} \frac{F_{(x=0)}}{SE\sqrt{\frac{\gamma}{Eg}}\omega u_0} = & \frac{\left(1 + \frac{\mu^2\omega^2}{2E^2}\right) \sin 2\left(\sqrt{\frac{\gamma}{Eg}}\omega l\right) + \frac{\mu\omega}{2E} \sinh\left(\frac{\mu\omega}{E} \sqrt{\frac{\gamma}{Eg}}\omega l\right)}{\cos 2\left(\sqrt{\frac{\gamma}{Eg}}\omega l\right) + \cosh\left(\frac{\mu\omega}{E} \sqrt{\frac{\gamma}{Eg}}\omega l\right)} \\ & + j \frac{\frac{\mu\omega}{2E} \sin 2\left(\sqrt{\frac{\gamma}{Eg}}\omega l\right) - \left(1 + \frac{\mu^2\omega^2}{2E^2}\right) \sinh\left(\frac{\mu\omega}{E} \sqrt{\frac{\gamma}{Eg}}\omega l\right)}{\cos 2\left(\sqrt{\frac{\gamma}{Eg}}\omega l\right) + \cosh\left(\frac{\mu\omega}{E} \sqrt{\frac{\gamma}{Eg}}\omega l\right)} \tag{6.50} \end{aligned}$$

A plot of the real and imaginary values of $F_{(x=0)}/SE\sqrt{\frac{\gamma}{Eg}}\omega u_0$ is given in Fig. 6.24 for zero damping and for a damping coefficient $\mu\omega/E = 0.1$ as a function of a tuning

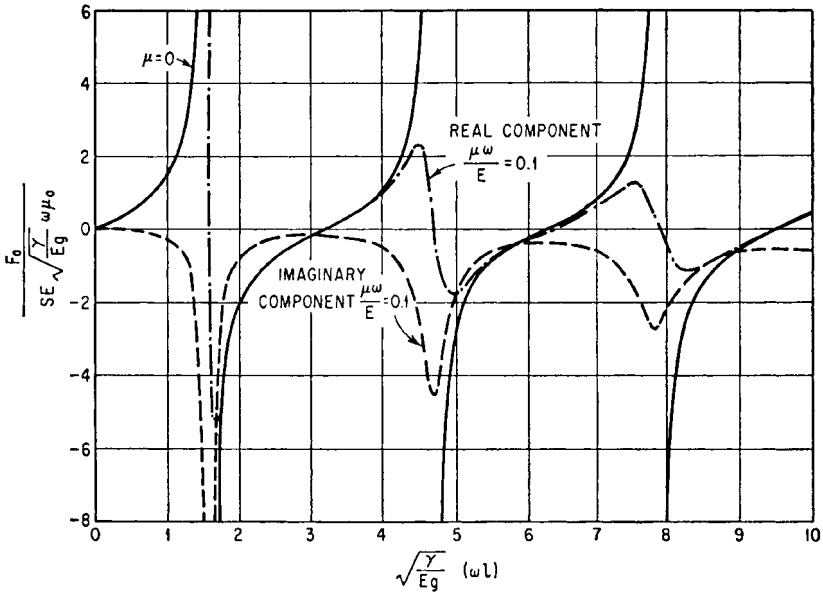


FIGURE 6.24 Real and imaginary components of the force applied to a vibrating body by the distributed mass damper shown in Fig. 6.23. These relations are given mathematically by Eq. (6.50), and the terms are defined in connection with Eq. (6.38). The curves are for a value of the damping coefficient $\mu\omega/E = 0.1$, where μ is defined by Eq. (6.41).

parameter $\sqrt{\gamma/Eg(\omega)}$. Damping decreases the effectiveness of the distributed mass damper substantially, particularly for the higher modes.

Use of a distributed mass as a vibration absorber is practical only at very high frequency; otherwise, too long a length is required.

PRACTICAL APPLICATIONS OF AUXILIARY MASS DAMPERS AND ABSORBERS TO SINGLE DEGREE-OF-FREEDOM SYSTEMS

THE DYNAMIC ABSORBER

The dynamic absorber, because of its tuning, can be used to eliminate vibration only where the frequency of the vibration is constant. Many pieces of equipment to which it is applied are operated by alternating current. So that it can be used for time keeping, the frequency of this alternating current is held remarkably constant. For this reason, most applications of dynamic absorbers are made to mechanisms that operate in synchronism from an ac power supply.

An application of a dynamic absorber to the pedestal of an ac generator having considerable vibration is shown in Fig. 6.25, where the relative sizes of absorber and pedestal are shown approximately to scale. In this case, the application is made to a complicated structure and the mass of the absorber is much less than that of the primary system; however, since the frequency of the excitation is constant, the dynamic

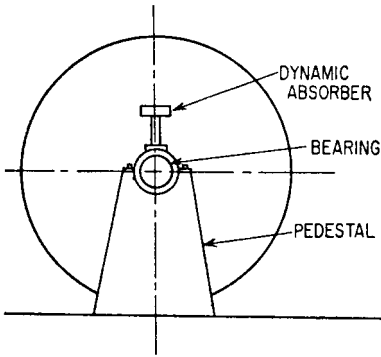


FIGURE 6.25 Application of a dynamic absorber to the bearing pedestal of an ac generator.

cutter blade and its driving mechanism is actuated by the magnetic field at a frequency of 120 Hz, as a result of the 60-Hz ac power supply. The forces and torques required to move the blade are balanced by reactions on the housing, causing it to vibrate. The dynamic absorber tuned to a frequency of 120 Hz enforces a node at the location of its mass. Since this is approximately the center-of-gravity of the assembly of the cutter and its driving mechanism, the absorber effectively neutralizes the unbalanced force. The moment caused by the rotation of the moving parts is still unbalanced. A second very small dynamic absorber placed in the handle of the clipper could enforce a node at the handle and substantially eliminate all vibration.

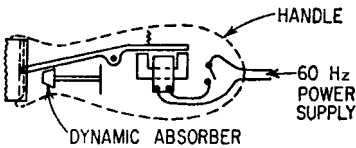


FIGURE 6.26 Application of a dynamic absorber to a hair clipper.

$\omega = 240\pi$), must equal the unbalanced force generated by the clipper mechanism. The torque generated by the two absorbers must balance the torque of the mechanism. Since the value of ω^2 is known, the values of m_1x_1 and m_2x_2 can be determined. Weights that fit into the available space with adequate room to move are chosen, and a spring is designed of such stiffness that the natural frequency is 120 Hz.

Because of the desirable balancing properties of the simple dynamic absorber and the constancy of frequency of ac power, it might be expected that devices operating at a frequency of 120 Hz would be used more widely. However, their application is limited because the frequency of vibration is too high to allow large amplitudes of motion.

REDUCTION OF ROLL OF SHIPS BY AUXILIARY TANKS

An interesting application of auxiliary mass absorbers is found in the auxiliary tanks used to reduce the rolling of ships,^{1,13} as shown in Fig. 6.27. When a ship is heeled, the

absorber reduces the vibration. When the mass ratio is small, it is important that the absorber be accurately tuned and that the damping be small. In this case, the excitation was the unbalance in the turbine rotor which was elastically connected to the pedestal through the flexibility of the shaft. If the absorber were ideally effective, there would be no forces at the frequency of the shaft speed; therefore, there would be no displacements from the pedestal where the force is neutralized through the remainder of the structure.

The dynamic absorber has been applied to the electric clipper shown in Fig. 6.26. The structure consisting of the cutter blade and its driving mechanism is actuated by the magnetic field at a frequency of 120 Hz, as a result of the 60-Hz ac power supply. The forces and torques required to move the blade are balanced by reactions on the housing, causing it to vibrate. The dynamic absorber tuned to a frequency of 120 Hz enforces a node at the location of its mass. Since this is approximately the center-of-gravity of the assembly of the cutter and its driving mechanism, the absorber effectively neutralizes the unbalanced force. The moment caused by the rotation of the moving parts is still unbalanced. A second very small dynamic absorber placed in the handle of the clipper could enforce a node at the handle and substantially eliminate all vibration. The design of these absorbers is simple after the unbalanced forces and torques generated by the cutter mechanism are computed. The sum of the inertia forces generated by the two absorbers, $m_1x_1\omega^2 + m_2x_2\omega^2$ (where m_1 and x_1 are the mass and amplitude of motion of the first absorber, m_2 and x_2 are the corresponding values for the second absorber, and

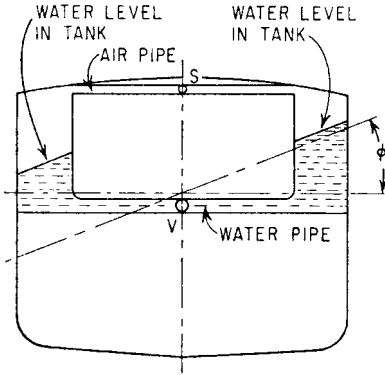


FIGURE 6.27 Cross section of ship equipped with antiroll tanks. The flow of water from one tank to the other tends to counteract rolling of the ship.

tanks. The damping is controlled by restricting the flow of water, either with a valve S in the line that allows air to flow between the tanks (Fig. 6.27) or with a valve V in the water line. Since the tanks occupy valuable space, the mass ratio of the water in the tanks to the ship is small. Fortunately, the excitation from waves generally is not large relative to the restoring moments, and roll becomes objectionable only because the normal damping of a ship in rolling motion is not very large. The use of antirolling tanks in the German luxury liners *Bremen* and *Europa* reduced the maximum roll from 15 to 5°.

REDUCTION OF ROLL OF SHIPS BY GYROSCOPES

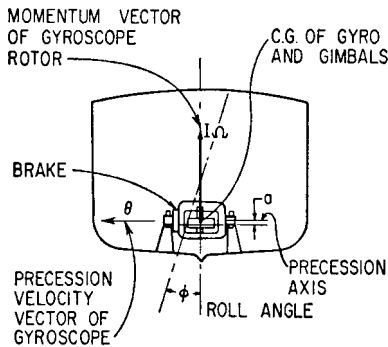


FIGURE 6.28 Application of a gyroscope to a ship to reduce roll.

restoring moment $k_r\phi$ acting on it is proportional to the angle of heel (or roll). This restoring moment acts to return the ship (and the water that moves with it) to its equilibrium position. If I_s represents the polar moment of inertia of the ship and its entrained water, the differential equation for the rolling motion of the ship is

$$I_s\ddot{\phi} + k_r\phi = M_s \tag{6.51}$$

where M_s represents the rolling moments exerted on the ship, usually by waves.

To reduce rolling of the ship, auxiliary wing tanks connected by pipes are used. The water flowing from one tank to another has a natural frequency that is determined by the length and cross-sectional area of the tube connecting the

A large gyroscope may be used to reduce roll in ships, as shown in Fig. 6.28.^{1,14} In response to the velocity of roll of a ship, the gyroscope precesses in the plane of symmetry of the ship. By braking this precession, energy can be dissipated and the roll reduced. The torque exerted by the gyroscope is proportional to the rate of change of the angular momentum about an axis perpendicular to the torque. Letting I represent the polar moment of inertia of the gyroscope about its spin axis and $\dot{\theta}$ the angular velocity of precession of the gyroscope, then the equation of motion of the ship is

$$I_s\ddot{\phi} + k_r\phi + I\Omega\dot{\theta} = M_s \tag{6.52}$$

Assume that the gyroscope has (1) a moment of inertia about the precession axis of I_g , (2) a weight of W , and (3) that its center-of-gravity is below the gimbal axis (as

it must be for the gyro to come to equilibrium in a working position) a distance a , as shown in Fig. 6.28. Then the equation of motion of the gyroscope is

$$I_g \ddot{\theta} + W_a \theta + c \dot{\theta} - I \Omega \dot{\phi} = 0 \quad (6.53)$$

where Ω is the spin velocity of the gyroscope. From Eq. (6.53), for a roll frequency of ω , the angle of precession of the gyroscope is

$$\theta = \frac{j I \Omega \omega \phi}{-I_g \omega^2 + W_a - j c \omega} \quad (6.54)$$

The torque exerted on the ship is

$$I \Omega \dot{\theta} = \frac{-(I \Omega)^2 \omega^2 \phi}{-I_g \omega^2 + W_a + j c \omega} \quad (6.55)$$

The equivalent moment of inertia of the gyroscope system in its reaction on the ship is

$$\frac{I \Omega^2}{-I_g \omega^2 + W_a + c j \omega} \quad (6.56)$$

By analogy with the steps of Eqs. (6.2) through (6.7), it follows that

$$\frac{\phi}{\phi_{st}} = \sqrt{\frac{(1 - \beta_g^2)^2 + (2\zeta\beta_g)^2}{[(1 - \beta_g^2)(1 - \beta^2) - \beta^2\mu]^2 + (2\zeta\beta_g)^2(1 - \beta^2)^2}} \quad (6.57)$$

where the parameters are defined in terms of ship and gyro constants as follows:

$$\beta_g = \frac{\omega}{\sqrt{W_a/I_g}} \quad \beta = \frac{\omega}{\sqrt{k_r/I_s}} \quad \zeta = \frac{c}{2\sqrt{W_a I_g}} \quad \mu = \frac{(I \Omega)^2}{W_a I_s} \quad \psi_{st} = \frac{M_s}{k_r}$$

Because $I \Omega$ can be made large by using a large gyro rotor and spinning it at a high speed, and W_a can be made small by choice of a design, the value of μ can be made quite large even though I_s is large. In one experimental ship, $\mu = 20$ was obtained. Even with this large value of μ , the precession angle of the gyroscope would become very large for optimum damping. Therefore it is necessary to use much more damping than optimum. Gyro stabilizers were used on the Italian ship *Conte di Savoia*; they are sometimes installed on yachts.

Both antirolling tanks and gyro stabilizers are more effective if they are active rather than passive. Activated dampers are considered below.

AUXILIARY MASS DAMPERS APPLIED TO ROTATING MACHINERY

An important industrial use of auxiliary mass systems is to neutralize the unbalance of centrifugal machinery. A common application is the balance ring in the spin dryer of home washing machines. The operation of such a balancer is dependent upon the basket of the washer rotating at a speed greater than the natural frequency of its support. The balance ring is attached to the washing machine basket concentric with its axis of rotation, as shown in Fig. 6.29.

Consider the washing machine basket shown in Fig. 6.29. When its center-of-gravity does not coincide with its axis of rotation and it is rotating at a speed lower

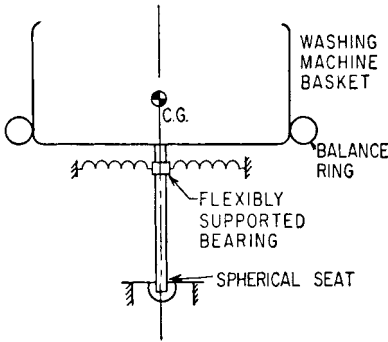


FIGURE 6.29 Schematic diagram showing location of balance ring on basket of a spin dryer.

of-gravity. The relative positions of the bearing center and the center-of-gravity are shown in Fig. 6.30A and B.

Since the balance ring is circular with a smooth inner surface, any weights or fluids contained in the ring can be acted upon only by forces directed radially. When the ring is rotated about a vertical axis, the weights or fluids will move within the ring in such a manner as to be concentrated on the side farthest from the axis of rotation. If this concentration occurs below the natural frequency (Fig. 6.30A), the weights tend to move further from the axis and the resultant center-of-gravity is displaced so as to give a greater eccentricity. The points A and G rotate about the axis O at the frequency ω . The initial eccentricity of the center-of-gravity of the washer basket and its load from the axis of rotation is represented by e , and p is the elastic displacement of this center of rotation due to the centrifugal force. Where the off-center rotating weight is W , the unbalanced force is $(W/g)(\rho + e)\omega^2$ [where $\rho = e/(1 - \beta^2)$ and $\beta^2 = \omega^2/\omega_n^2 < 1$] and acts in the direction from A to G.

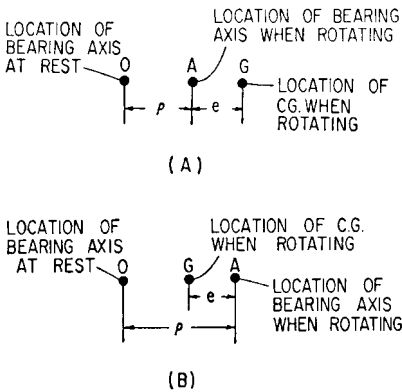


FIGURE 6.30 Diagram in plane normal to axis of rotation of spin dryer in Fig. 6.29. Relative positions of axes when rotating speed is less than natural frequency are shown at (A); corresponding diagram for rotation speed greater than natural frequency is shown at (B).

frequency ω . The unbalanced force is $(W/g)(\rho + e)\omega^2$ [where $\rho = e/(1 - \beta^2)$ and $\beta^2 = \omega^2/\omega_n^2 > 1$]. This gives a negative force that acts in a direction from G to A. Thus the eccentricity is brought toward zero and the rotor is automatically balanced. Because it is necessary to pass through the critical speed in bringing the rotor up to speed and in stopping it, it is desirable to heavily damp the balancing elements, either fluid or weights.

In practical applications, the balancing elements can take several forms. The earliest form consisted of two or more spheres or cylinders free to move in a race con-

than its critical speed (corresponding to the natural frequency in rocking motion about the spherical seat), the centrifugal force tends to pull the rotational axis in the direction of the unbalance. This effect increases with an increase in rotational speed until the critical speed is reached. At this speed the amplitude would become infinite if it were not for the damping in the system. Above the critical speed, the phase position of the axis of rotation relative to the center-of-gravity shifts so that the basket tends to rotate about its center-of-gravity with the flexibly supported bearing moving in a circle about an axis through the center-

of-gravity. The relative positions of the bearing center and the center-of-gravity are shown in Fig. 6.30A and B. Since the balance ring is circular with a smooth inner surface, any weights or fluids contained in the ring can be acted upon only by forces directed radially. When the ring is rotated about a vertical axis, the weights or fluids will move within the ring in such a manner as to be concentrated on the side farthest from the axis of rotation. If this concentration occurs below the natural frequency (Fig. 6.30A), the weights tend to move further from the axis and the resultant center-of-gravity is displaced so as to give a greater eccentricity. The points A and G rotate about the axis O at the frequency ω . The initial eccentricity of the center-of-gravity of the washer basket and its load from the axis of rotation is represented by e , and p is the elastic displacement of this center of rotation due to the centrifugal force. Where the off-center rotating weight is W , the unbalanced force is $(W/g)(\rho + e)\omega^2$ [where $\rho = e/(1 - \beta^2)$ and $\beta^2 = \omega^2/\omega_n^2 < 1$] and acts in the direction from A to G.

If the displacement of the weights or fluids in the balance ring occurs above the natural frequency, the center-of-gravity tends to move closer to the dynamic location of the axis. The action in this case is shown in Fig. 6.30B. Then the points A and G rotate about O at the frequency ω . The unbalanced force is $(W/g)(\rho + e)\omega^2$ [where $\rho = e/(1 - \beta^2)$ and $\beta^2 = \omega^2/\omega_n^2 > 1$]. This gives a negative force that acts in a direction from G to A. Thus the eccentricity is brought toward zero and the rotor is automatically balanced. Because it is necessary to pass through the critical speed in bringing the rotor up to speed and in stopping it, it is desirable to heavily damp the balancing elements, either fluid or weights.

In practical applications, the balancing elements can take several forms. The earliest form consisted of two or more spheres or cylinders free to move in a race con-

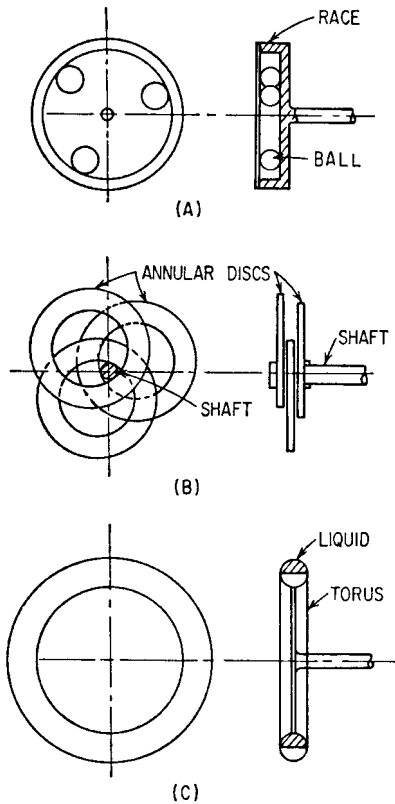


FIGURE 6.31 Examples of balancing means for rotating machinery: (A) spheres (or cylinders) in a race; (B) annular discs rotating on shaft; (C) damping fluid in torus.

6 in. (15.2 cm) in diameter], can neutralize 250 lb-in. (290 kg-cm); and half of a 2-in. (5-cm) diameter torus filled with fluid of density 0.2 lb/in^3 (5.5 gram/cm^3) can neutralize 609 lb-in. (700 kg-cm). Only the fluid-filled torus would be initially balanced.

AUXILIARY MASS DAMPERS APPLIED TO TORSIONAL VIBRATION

Dampers and absorbers are used widely for the control of torsional vibration of internal-combustion engines. The most common absorber is the viscous-damped, untuned auxiliary mass unit shown in Fig. 6.32. The device is comprised of a cylindrical housing carrying an inertia mass that is free to rotate. There is a preset clearance between the housing and the inertia mass that is filled with a silicone oil of proper viscosity. Silicone oil is used because of its high viscosity index; i.e., its viscosity changes relatively little with temperature. With the inertia mass and the damping medium contained, the housing is seal-welded to provide a leakproof and simple

centric with the axis of the rotor, as shown in Fig. 6.31A. A later modification consists of three annular discs that rotate about an enlarged shaft concentric with the axis, as indicated in Fig. 6.31B. These are contained in a sealed compartment with oil for lubrication and damping. A fluid type of damper is shown in Fig. 6.31C, the fluid usually being a high-density viscous material. With proper damping, mercury would be excellent, but it is too expensive. Therefore a more viscous, high-density halogenated fluid is used.

The balancers must be of sufficient weight and operate at such a radius that the product of their weight and the maximum eccentricity they can attain is equivalent to the unbalanced moment of the load. This requirement makes the use of the spheres or cylinders difficult because they cannot be made large; it makes the annular plates large because they are limited in the amount of eccentricity that can be obtained.

In a cylindrical volume 24 in. (61 cm) in diameter and 2 in. (5 cm) thick, seven spheres 2 in. (5 cm) in diameter can neutralize 98.6 lb-in. (114 kg-cm) of unbalance; three cylinders 4 in. (10 cm) in diameter by 2 in. (5 cm) thick can neutralize 255 lb-in. (295 kg-cm); three annular discs, each $\frac{3}{8}$ in. (1.6 cm) thick with an outside diameter of 19.55 in. (50 cm) and an inside diameter of 10.45 in. (26.5 cm) [the optimum for a center post

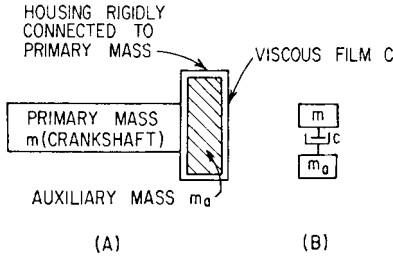


FIGURE 6.32 Untuned auxiliary mass damper with viscous damping. The application to a torsional system is shown at (A), and the linear analog at (B).

absorber. However, the silicone oil has poor boundary lubricating properties and if decomposed by a local hot spot (such as might be caused by a reduced clearance at some particular spot), the decomposed damping fluid is abrasive.

Because of the simplicity of this untuned damper, it is commonly used in preference to the more effective tuned damper. However, it is possible to use the same construction methods for a tuned damper, as shown in Fig. 6.33. It is also possible to mount the standard damper with the housing for the unsprung inertia mass attached to the main

mass by a spring, as shown in Fig. 6.34. If the viscosity of the oil and the dimensions of the masses and the clearance spaces are known, the damping effects of the dampers shown in Figs. 6.32 and 6.34 can be computed directly in terms of the equations previously developed. The damper in Fig. 6.34 can be analyzed by treating the spring and housing as additional elements in the main system and the untuned mass as a viscous damped auxiliary mass. If the inertia of the housing is negligible, the inertia mass is effectively connected to the main mass through a spring and a dashpot in series. The two elements in series can be represented by a complex spring constant equal to

$$\frac{1}{(1/jc\omega) + (1/k)} = \frac{kcj\omega}{k + cj\omega}$$

Where there is no damping in parallel with the spring, Eq. (6.3) becomes

$$m_{eq} = km/(k - m\omega^2)$$

Substituting the complex value of the spring constant, the effective mass is

$$m_{eq} = \frac{ckj\omega}{k + cj\omega} \left[\frac{m}{-m\omega^2 + ckj\omega/(k + cj\omega)} \right] \tag{6.58}$$

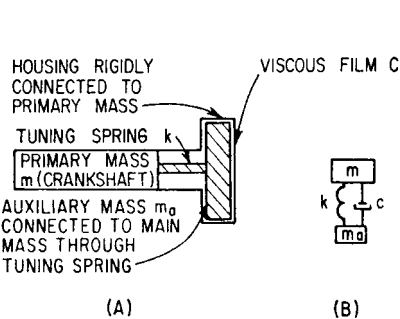


FIGURE 6.33 Tuned auxiliary mass damper with viscous damping. The application to a torsional system is shown at (A), and the linear analog at (B).

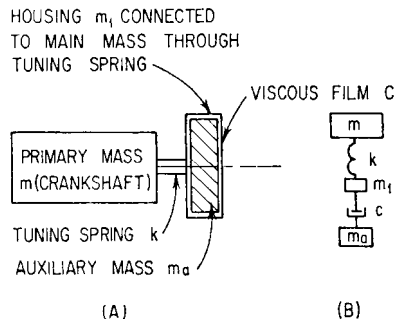


FIGURE 6.34 Auxiliary mass damper with viscous damping and spring-mounted housing. The application to a torsional system is shown at (A), and the linear analog at (B).

In terms of the nondimensional parameters defined in Eq. (6.4):

$$m_{eq} = \frac{(2\zeta\beta_a)^2(1 - \beta_a^2)}{\beta_a^4 - (2\zeta\beta_a)^2(1 - \beta_a^2)} m + \frac{-2\zeta\beta_a^3 m}{\beta_a^4 - (2\zeta\beta_a)^2(1 - \beta_a^2)} j \quad (6.59)$$

Before the advent of silicone oil with its chemical stability and relatively constant viscosity over service temperature conditions, the damper most commonly used for absorbing torsional vibration energy was the dry friction or Lanchester damper shown in Fig. 6.35. The damping is determined by the spring tension and the coefficient of friction at the sliding interfaces. Its optimum value is determined by the equation for a torsional system analogous to Eq. (6.35) for a linear system:

$$(T_s)_{opt} = \frac{\sqrt{2}}{\pi} I\omega^2\theta_0 \quad (6.60)$$

where T_s is the slipping torque, I is the moment of inertia of the flywheels, and θ_0 is the amplitude of angular motion of the primary system. The dry-friction-based Lanchester damper requires frequent adjustment, as the braking material wears, to maintain a constant braking force.

It is possible to use torque-transmitting couplings that can absorb vibration energy, as the spring elements for tuned dampers. The Bibby coupling (Fig. 6.36) is used in this manner. Since the stiffness of this coupling is nonlinear, the optimum tuning of such an absorber is secured for only one amplitude of motion.

A discussion of dampers and of their application to engine systems is given in Chap. 38.

DYNAMIC ABSORBERS TUNED TO ORDERS OF VIBRATION RATHER THAN CONSTANT FREQUENCIES

In the torsional vibration of rotating machinery, it is generally found that exciting torques and forces occur at the same frequency as the rotational speed or at multiples of this frequency. The ratio of the frequency of vibration to the rotational speed is called the *order of the vibration* q . Thus a power plant driving a four-bladed propeller may have a torsional vibration whose frequency is 4 times the rotational speed of the drive shaft; sometimes it may have a second torsional vibration whose frequency is 8 times the rotational speed. These are called the fourth-order and eighth-order torsional vibrations.

If a dynamic absorber in the form of a pendulum acting in a centrifugal field is used, then its natural frequency increases linearly with speed. Therefore it can be used to neutralize an order of vibration.¹⁵⁻¹⁹

Consider a pendulum of length l and of mass m attached at a distance R from the center of a rotating shaft, as shown in Fig. 6.37. Since the pendulum is excited by torsional vibration in the shaft, let the radius R be rotating at a constant speed Ω with a

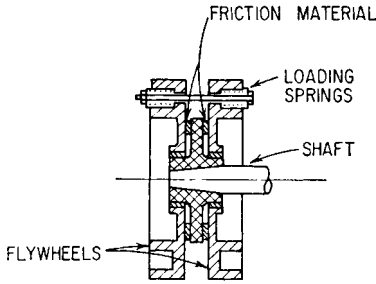


FIGURE 6.35 Schematic cross section through Lanchester damper.

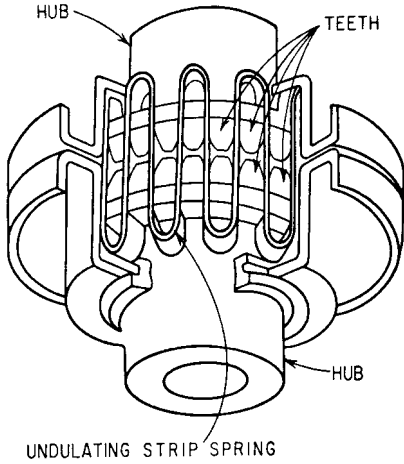


FIGURE 6.36 Coupling used as elastic and damping element in auxiliary mass damper for torsional vibration. The torque is transmitted by an undulating strip of thin steel interposed between the teeth on opposite hubs. The stiffness of the strip is nonlinear, increasing as torque increases. Oil pumped between the strip and teeth dissipates energy.

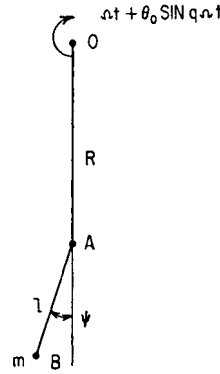


FIGURE 6.37 Schematic diagram of pendulum absorber.

superposed vibration $\theta = \theta_0 \cos q\Omega t$, where q represents the order of the vibration. Then the angle of R with respect to any desired reference is $\Omega t + \theta_0 \cos q\Omega t$. The angle of the pendulum with respect to the radius R is defined as $\psi = \psi_0 \cos q\Omega t$, as shown by Fig. 6.37.

The acceleration acting on the mass m at position B is most easily ascertained by considering the change in velocity during a short increment of time Δt . The components of velocity of the mass m at time t are shown graphically in Fig. 6.38A; at time $t + \Delta t$, the corresponding velocities are shown in Fig. 6.38B. The change in velocity during the time interval Δt is shown in Fig. 6.38C. Since the acceleration is the change in velocity per unit of time, the accelerations along and perpendicular to l are:

Acceleration along l :

$$\frac{-l(\Omega + \dot{\theta} + \dot{\psi}^2) \Delta t - R(\Omega + \dot{\theta})^2 \Delta t \cos \psi + R\ddot{\theta} \Delta t \sin \psi}{\Delta t} \tag{6.61}$$

Acceleration perpendicular to l :

$$\frac{l(\ddot{\theta} + \ddot{\psi}) \Delta t + R(\Omega + \dot{\theta})^2 \Delta t \sin \psi + R\ddot{\theta} \Delta t \cos \psi}{\Delta t} \tag{6.62}$$

Only the force $-F$ directed along the pendulum, acts on the mass m . Therefore the equations of motion are

$$\begin{aligned} -F &= -ml(\Omega + \dot{\theta} + \dot{\psi})^2 - mR(\Omega + \dot{\theta})^2 \cos \psi + R\ddot{\theta} \sin \psi \\ 0 &= ml(\ddot{\theta} + \ddot{\psi}) + mR(\Omega + \dot{\theta})^2 \sin \psi + mR\ddot{\theta} \cos \psi \end{aligned} \tag{6.63}$$

Assuming that ψ and θ are small, Eqs. (6.63) simplify to

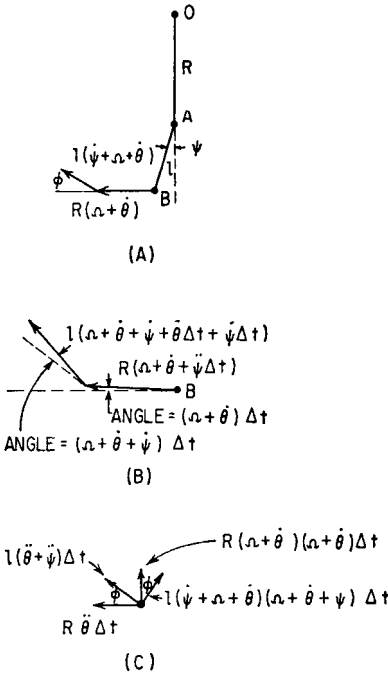


FIGURE 6.38 Velocity vectors for the pendulum absorber: (A) velocities at time t ; (B) velocities at time $t + \Delta t$; (C) change in velocities during time increment Δt .

$$I_{eq} = \frac{mR(R+l)^2}{R-q^2l} = \frac{m(R+l)^2}{1-q^2l/R} \tag{6.67}$$

When

$$\frac{R}{l} = q^2 \tag{6.68}$$

the equivalent inertia is infinite and the pendulum acts as a dynamic absorber by enforcing a node at its point of attachment.

Where the pendulum is damped, the equivalent moment of inertia is given by an equation analogous to Eqs. (6.4) and (6.5):

$$\begin{aligned} I_{eq} &= \frac{1 + 2\zeta v j}{(1 - v^2) + 2\zeta v j} m(R+l)^2 \\ &= m(R+l)^2 \left[\frac{1 - v^2 + (2\zeta v)^2}{(1 - v^2)^2 + (2\zeta v)^2} - \frac{2\zeta v^3 j}{(1 - v^2)^2 + (2\zeta v)^2} \right] \end{aligned} \tag{6.69}$$

where $v^2 = q^2 l/R$ and $\zeta = (c/2m\Omega)\sqrt{l/R}$.

When the pendulum is attached to a single degree-of-freedom system as is shown in Fig. 6.39, the amplitude of motion θ_0 of the flywheel of inertia I is given, by analogy to Eq. (6.7), as

$$\begin{aligned} Ft &= m(R+l)\Omega^2 \\ I(\ddot{\theta} + \ddot{\psi}) + R\Omega^2\psi + R\ddot{\theta} &= 0 \end{aligned} \tag{6.64}$$

The second of Eqs. (6.64) upon substitution of $\theta = \theta_0 \cos q\Omega t$ and $\psi = \psi_0 \cos q\Omega t$ yields

$$\frac{\psi_0}{\theta_0} = \frac{(q\Omega)^2(l+R)}{-(q\Omega)^2l + \Omega^2R} = \frac{q^2(l+R)}{R-q^2l} \tag{6.65}$$

The torque M exerted at point 0 by the force F is

$$M = RF \sin \psi = RF\psi \quad \text{when } \psi \text{ is small}$$

From Eqs. (6.64) and (6.65), when ψ is small,

$$M = \frac{mq^2R(R+l)^2\Omega^2}{R-q^2l} \tag{6.66}$$

If a flywheel having a moment of inertia I is accelerated by a shaft having an amplitude of angular vibratory motion θ_0 and a frequency $q\Omega$, the torque amplitude exerted on the shaft is $I(q\Omega)^2\theta_0$. Therefore, the equivalent moment of inertia I_{eq} of the pendulum is

$$\frac{\theta_a}{\theta_{st}} = \sqrt{\frac{(1 - v^2)^2 + (2\zeta v)^2}{[(1 - v^2)(1 - \beta_p^2) - \beta_p^2 \mu]^2 + (2\zeta v)^2 [1 - \beta_p^2 - \beta_p^2 \mu_p]^2}} \quad (6.70)$$

where

$$2\zeta v = \frac{cql}{mR}$$

$$\mu_p = \frac{m(R + l)^2}{I}$$

$$\beta_p = \frac{q}{k_r I}$$

$$\theta_{st} = \frac{m_0}{k_r}$$

The pendulum tends to detune when the amplitude of motion of the pendulum is large, thereby introducing harmonics of the torque that it neutralizes.¹⁷ Suppose the shaft rotates at a constant speed Ω , i.e., $\theta_0 = 0$, and consider the torque exerted on the shaft as m moves through a large amplitude ψ_0 about its equilibrium position. Equations (6.63) become

$$\begin{aligned} F &= ml(\Omega + \dot{\psi})^2 + mR\Omega^2 \cos \psi \\ l\ddot{\psi} + R\Omega^2 \sin \psi &= 0 \end{aligned} \quad (6.71)$$

A solution for the second of Eqs. (6.71) is

$$\psi = \sqrt{\frac{2\Omega^2 R}{l}} \sqrt{\cos \psi - \cos \psi_0} \quad (6.72)$$

The solution of Eq. (6.72) involves elliptic integrals and is given approximately by

$$\psi = \psi_0 \sin \omega t$$

where

$$\omega = \sqrt{\frac{R}{l}} \frac{\pi/2}{F(\psi_0/2, \pi/2)} \Omega$$

and $F(\psi_0/2, \pi/2)$ is an elliptic function of the first kind whose value may be obtained from tables.

Since $\omega/\Omega = q$ (the order of the disturbance), the tuning of the damper will be changed for large angles and becomes

$$q^2 = \frac{R}{l} \left(\frac{\pi/2}{F(\psi_0/2, \pi/2)} \right)^2 \quad (6.73)$$

The value of $q^2 l/R = v^2$ used in Eqs. (6.69) and (6.70) is given in Fig. 6.40 as a function of the amplitude of the pendulum.

Since the force exerted by the mass m is directed along the rod connecting it to the pivot A (Fig. 6.37), the reactive torque on the shaft is

$$\begin{aligned} M &= FR \sin \psi \\ &= mR^2 \Omega^2 \left[\frac{l}{R} \left(1 + \frac{\ddot{\psi}}{\Omega} \right)^2 \sin \psi + \sin \psi \cos \psi \right] \\ &= mR^2 \Omega^2 (A_1 \sin q\Omega t + A_2 \sin 2q\Omega t + A_3 \sin 3q\Omega t + \dots) \end{aligned} \quad (6.74)$$

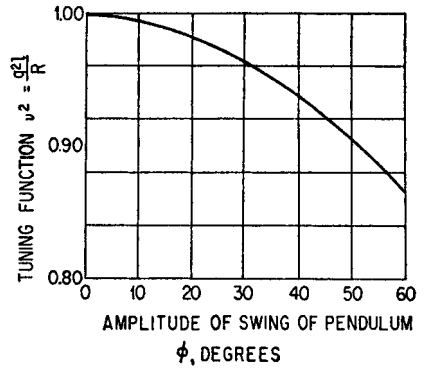
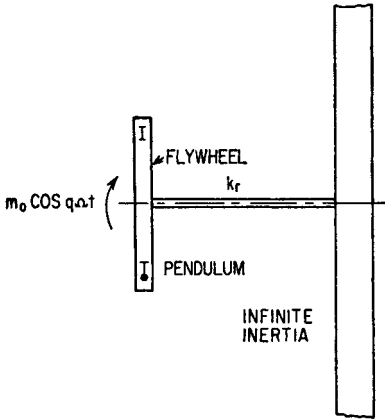


FIGURE 6.39 Application of pendulum absorber to a rotational single degree-of-freedom system.

FIGURE 6.40 Tuning function for a pendulum absorber used in Eqs. (6.69) and (6.70).

The values of the fundamental torque corresponding to the tuned frequency and to the second and third harmonics of this tuned frequency are given in Fig. 6.41 as a function of the angle of swing of the pendulum, for a typical installation. In this case, the pendulum is tuned to the $4\frac{1}{2}$ order of vibration. (The $4\frac{1}{2}$ order of vibration is one whose frequency is $4\frac{1}{2}$ times the rotational frequency and 9 times the fundamental frequency. The latter is called the half order and occurs at half of the rotational frequency. This is common in four-cycle engines.)

Two types of pendulum absorber are used. The one most commonly used is shown in Fig. 6.42. The counterweight, which also is used to balance rotating forces in the engine, is suspended from a hub carried by the crankshaft by pins that act through holes with clearance, Fig. 6.42A. By suspending the pendulum from two pins, the pendulum when oscillating does not rotate but rather moves as shown in Fig. 6.42B. Since it is not subjected to angular acceleration, it may be treated as a particle located at its center-of-gravity. Referring to Fig. 6.42A and B, the expression for acceleration [Eqs. (6.61) and (6.62)] and the equations of motion [Eqs. (6.63)] apply if

$$R = H_1 + H_2$$

$$l = \frac{D_c + D_p}{2} - D_b \tag{6.75}$$

- where H_1 = distance from center of rotation to center of holes in crank hub
- H_2 = distance from center of holes in pendulum to center-of-gravity of pendulum
- D_c = diameter of hole in crank hub
- D_p = diameter of hole in pendulum
- D_b = diameter of pin

In practice, difficulty arises from the wear of the holes and the pin. Moreover, the motion on the pins generally is small and the loads due to centrifugal forces are large so that fretting is a problem. Because the radius of motion of the pendulum is short,

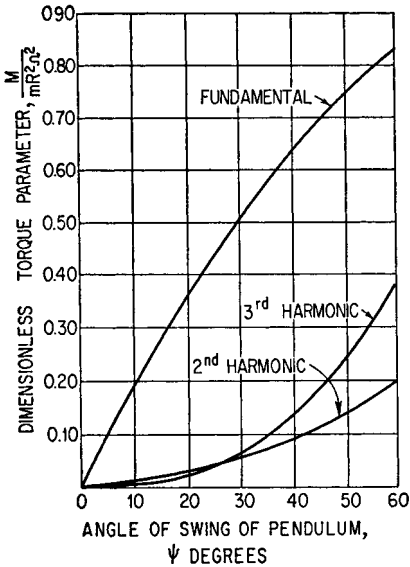


FIGURE 6.41 Harmonic components of torque generated by a pendulum absorber as a function of its angle of swing. The torque is expressed by the parameters used in Eq. (6.74).

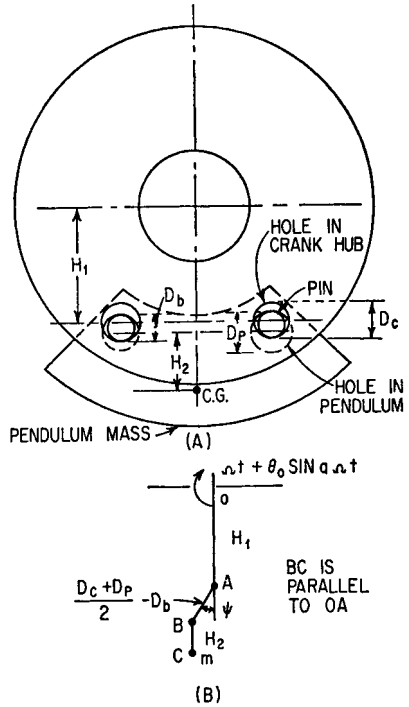


FIGURE 6.42 Bifilar type of pendulum absorber. The mechanical arrangement is shown at (A), and a schematic diagram at (B).

only a small amount of wear can be tolerated. Hardened pins and bushings are used to reduce the wear.

The pendulum is most easily designed if it is recognized that the inertia torques generated by the pendulum must neutralize the forcing torques. Thus

$$m\omega^2 l \psi_0 R = M \tag{6.76}$$

The radii l and R are set by the design of the crank and the order of vibration to be neutralized. The original motion ψ_0 is generally limited to a small angle, approximately 20° . It is probable that the most stringent condition is at the lowest operating speed, although the absorber may be required only to avoid difficulty at some particular critical speed. Knowing the excitation M , it is possible to compute the required mass of the pendulum weight.

A second type of pendulum absorber is a cylinder that rolls in a hole in a counterweight, as shown in Fig. 6.43. In this type, the radius of the pendulum corresponds to the difference in the radii of the hole and of the cylinder. It is found, by observing tests and checking the tuning of actual systems using cylindrical pendulums, that the weight rotates with a uniform angular velocity. Therefore the tuning is independent of the moments of inertia of the cylinder. It is common to allow a

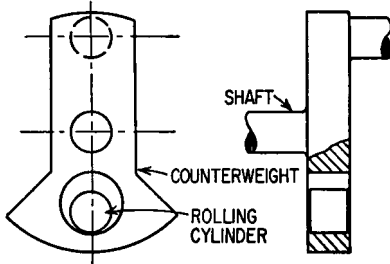


FIGURE 6.43 Roller type of pendulum absorber.

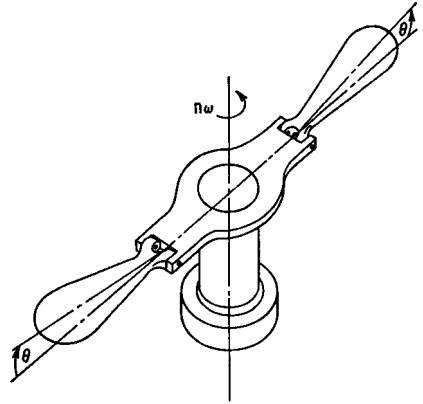


FIGURE 6.44 Application of pendulum absorbers to counteract linear vibration.

larger amplitude of motion with the absorber of Fig. 6.43 than with the absorber of Fig. 6.40.

Applications of pendulum absorbers to torsional-vibration problems are given in Chap. 38.

PENDULUM ABSORBER FOR LINEAR VIBRATION

The principle of the pendulum absorber can be applied to linear vibration as well as to torsional vibration. To neutralize linear vibration, pendulums are rotated about an axis parallel to the direction of vibration, as shown in Fig. 6.44. This can be accomplished with an absorber mounted on the moving body. Two or more pendulums are used so that centrifugal forces are balanced. Free rotational movement of each pendulum in the plane of the axis allows the axial forces to be neutralized. The pendulum assembly must rotate about the axis at some submultiple of the frequency of vibration. The size of the absorber is determined by the condition that the components of the inertia forces of the weights in the axial direction $[\Sigma m\omega^2 r\theta]$ must balance the exciting forces. This device can be applied where the vibration is generated by the action of rotating members but the magnitude of the vibratory forces is uncertain. A discussion of this absorber, including the influence of moments of inertia and damping of the pendulum, together with some applications to the elimination of vibration in special locations on a ship, is given in Ref. 20.

APPLICATIONS OF DAMPERS TO MULTIPLE DEGREE-OF-FREEDOM SYSTEMS

Auxiliary mass dampers as applied to systems of several degrees-of-freedom can be represented most effectively by equivalent masses or moments of inertia, as determined by Eq. (6.5) or Eq. (6.6). The choice of proper damping constants is more dif-

difficult. For the case of torsional vibration, the practical problems of designing dampers and selecting the proper damping are considered in Chap. 38.

There are many applications of dampers to vibrating structures that illustrate the use of different types of auxiliary mass damper. One such application has been to ships.²¹ These absorbers had low damping and were designed to be filled with water so that they could be tuned to the objectionable frequencies. In one case, the absorber was located near the propeller (the source of excitation) and when properly tuned was found to be effective in reducing the resonant vibration of the ship. In another case, the absorber was located on an upper deck but was not as effective. It enforced a node at its point of attachment but, because of the flexibility between the upper deck and the bottom of the ship, there was appreciable motion in the vicinity of the propeller and vibratory energy was fed to the ship's structure. To operate properly, the absorbers must be closely tuned and the propeller speed closely maintained. Because the natural frequencies of the ship vary with the types of loading, it is not sufficient to install a fixed frequency absorber that is effective at only one natural frequency of the hull, corresponding to a particular loading condition.

An auxiliary mass absorber has been applied to the reduction of vibration in a heavy building that vibrated at a low frequency under the excitation of a number of looms.²² The frequency of the looms was substantially constant. However, the magnitude of the excitation was variable as the looms came into and out of phase. The dynamic absorber, consisting of a heavy weight hung as a pendulum, was tuned to the frequency of excitation. Because the frequency was low and the forces large, the absorber was quite large. However, it was effective in reducing the amplitude of vibration in the building and was relatively simple to construct.

ACTIVATED VIBRATION ABSORBERS

The cost and space that can be allotted to ship antirolling devices are limited. Therefore it is desirable to activate the absorbers so that their full capacity is used for small amplitudes as well as large. Activated dampers can be made to deliver as large restoring forces for small amplitudes of motion of the primary body as they would be required to deliver if the motions were large. For example, the gyrostabilizer that is used in the ship is precessed by a motor through its full effective range, in the case of small angles as well as large. Thus, it introduces a restoring torque that is much larger than would be introduced by the normal damped precession.¹⁴ In the same manner the water in antiroll tanks is always pumped to the tank where it will introduce the maximum torque to counteract the roll. By pumping, much larger quantities of water can be transferred and larger damping moments obtained than can be obtained by controlled gravity flows.

Devices for damping the roll are desirable for ships. It has been common practice to install bilge keels (long fins which extend into the water) in steel ships. Some ships are now fitted with activated, retractable hydrofoils located at the bilge at the maximum beam. Both these devices are effective only when the ship is in motion and add to the resistance of the ship.

Activated vibration absorbers are essentially servomechanisms designed to maintain some desired steady state. Steam and gas turbine speed governors, wicket gate controls for frequency regulation in water turbines, and temperature control equipment can be considered as special forms of activated vibration absorbers.²³

THE USE OF AUXILIARY MASS DEVICES TO REDUCE TRANSIENT AND SELF-EXCITED VIBRATIONS

Where the vibration is self-excited or caused by repeated impact, it is necessary to have sufficient damping to prevent a serious build-up of vibration amplitude. This damping, which need not always be large, may be provided by a loosely coupled auxiliary mass. A simple application of this type is the ring fitted to the interior of a gear, as shown in Fig. 6.45. By fitting this ring with the proper small clearance so that relative motion occurs between it and the gear, it is possible to obtain enough energy dissipation to damp the high-frequency, low-energy vibration that causes the gear to ring. The rubbery coatings applied to large, thin-metal panels such as automobile doors to give them a solid rather than a "tinny" sound depend for their effectiveness on a proper balance of mass, elasticity, and damping (see Chap. 37).

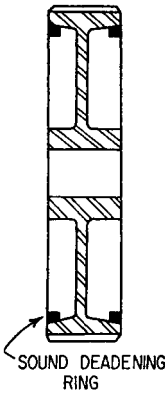


FIGURE 6.45 Application of auxiliary mass damper to deaden noise in gear.

Another application where auxiliary mass dampers are useful is in the prevention of fatigue failures in turbines. At the high-pressure end of an impulse turbine, steam or hot gas is admitted through only a few nozzles. Consequently, as the blade passes the nozzle it is given an impulse by the steam and set into vibration at its natural frequency. It is a characteristic of alloy steels that they have very little internal damping at high operating temperature. For this reason the free vibration persists with only a slightly diminished amplitude until the blade again is subjected to the steam impulse. Some of these second impulses will be out of phase with the motion of the blade and will reduce its amplitude; however, successive impulses may increase the amplitude on subsequent

passes until failure occurs. Damping can be increased by placing a number of loose wires in a cylindrical hole cut in the blade in a radial direction. The damping of a number of these wires has been computed in terms of the geometry of the application (number of wires, density of wires, size of the hole, radius of the blade, rotational speed, etc.) and the amplitude of vibration.²⁴ These computations show reasonable agreement with experimental results.

An auxiliary mass has been used to damp the cutting tool chatter set up in a boring bar.²⁵ Because of the characteristics of the metal-cutting process or of some coupling between motions of the tool parallel and perpendicular to the work face, it is sometimes found that a self-excited vibration is initiated at the natural frequency of the cutter system. Since the self-excitation energy is low, the vibration usually is initiated only if the damping is small. Chatter of the tool is most common in long, poorly supported tools, such as boring bars (see Chap. 40). To eliminate this chatter, a loose auxiliary mass is incorporated in the boring bar, as shown in Fig. 6.46. This may be air-damped or fluid-damped. Since the excitation is at the natural frequency of the tool, the damping should be such that the tool vibrates with a minimum ampli-

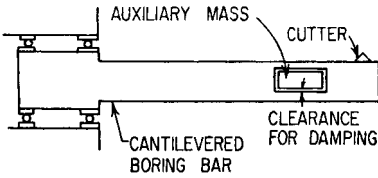


FIGURE 6.46 Application of auxiliary mass damper to reduce chatter in boring bar.

tion in amplitude can be obtained by a shift in natural frequency than by damping. However, such a shift cannot be attained because the frequency of the excitation always coincides with the natural frequency of the complete system. Instead, a better technique is to determine the damping that gives the maximum decrement of the free vibration.

Let the boring bar and damper be represented by a single degree-of-freedom system with a damper mass coupled to the main mass by viscous damping, as shown in Fig. 6.47A. The forces acting on the masses are shown in Fig. 6.47B. The equations of motion are

$$\begin{aligned}
 -kx_1 - c\dot{x}_1 + c\dot{x}_2 &= m_1\ddot{x}_1 \\
 c\dot{x}_1 - c\dot{x}_2 &= m_2\ddot{x}_2
 \end{aligned}
 \tag{6.78}$$

Substituting $x = Ae^{st}$, the resulting frequency equation is

$$s^3 + \frac{c(m_1 + m_2)}{m_1 m_2} s^2 + \frac{k}{m_1} s + \frac{kc}{m_1 m_2} = 0
 \tag{6.79}$$

Where chatter occurs, this equation has three roots, one real and two complex. The complex roots correspond to decaying free vibrations. Let the roots be as follows:

$$\alpha_1, \alpha_2 + j\beta, \alpha_2 - j\beta$$

The value of β determines the frequency of the free vibration, and the value of α_2 determines the decrement (rate of decrease of amplitude) of the free vibration. The decrement α_2 is of primary interest; it is most easily found from the conditions that when the coefficient of s^3 is unity, (1) the sum of the roots is equal to the negative of the coefficient of s^2 , (2) the sum of the products of the roots taken two at a time is the negative of the coefficient of s , and (3) the product of the roots is the negative of the constant term. The equations thus obtained are

$$\alpha_1 + 2\alpha_2 = -\frac{c(1 + \mu)}{\mu m_1}
 \tag{6.80}$$

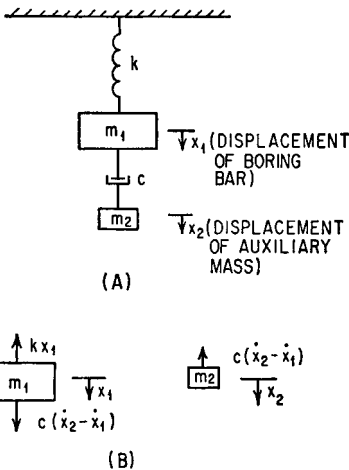


FIGURE 6.47 Schematic diagram of damper shown in Fig. 6.46. The arrangement is shown at (A), and the forces acting on the boring bar and auxiliary mass are shown at (B).

$$2\alpha_1\alpha_2 + \alpha_2^2 + \beta^2 = -\omega_n^2 \tag{6.81}$$

$$\alpha_1(\alpha_2^2 + \beta^2) = -\omega_n^2 \frac{c}{m_1\mu} \tag{6.82}$$

where $\omega_n^2 = k/m_1$ and $\mu = m_2/m_1$. It is not practical to find the optimum damping by solving these equations for α_2 and then setting the derivative of α_2 with respect to c equal to zero. However, it is possible to find the optimum damping by the following process. Eliminate $(\alpha_2^2 + \beta^2)$ between Eqs. (6.81) and (6.82) to obtain

$$2\alpha_1^2\alpha_2 = \omega_n^2 \left(\frac{c}{\mu m_1} - \alpha_1 \right) \tag{6.83}$$

Substituting the value of α_1 from Eq. (6.80) in Eq. (6.83),

$$2\alpha_2 \left[2\alpha_2 + \frac{c(1+\mu)}{\mu m_1} \right]^2 = \frac{c\omega_n^2}{\mu m_1} + \omega_n^2 \left[2\alpha_2 + \frac{c(1+\mu)}{\mu m_1} \right] \tag{6.84}$$

To find the damping that gives the maximum decrement, differentiate with respect to c and set $d\alpha_2/dc = 0$:

$$2\alpha_2 \left[2\alpha_2 + \frac{c(1+\mu)}{\mu m_1} \right] = \frac{1}{2}\omega_n^2 \frac{2+\mu}{1+\mu} \tag{6.85}$$

Solving Eqs. (6.84) and (6.85) simultaneously,

$$c_{opt} = \frac{\mu^2 m_1 \omega_n}{2(1+\mu)^{3/2}} \tag{6.86}$$

$$(\alpha_2)_{opt} = -\frac{(2+\mu)\omega_n}{4(1+\mu)^{1/2}} \tag{6.87}$$

These values may be obtained by proper choice of clearance between the auxiliary mass and the hole in which it is located. Air damping is preferable to oil because it requires less clearance. Therefore the plug is not immobilized by the centrifugal forces that, with the rotating boring bar, become larger as the clearance is increased.

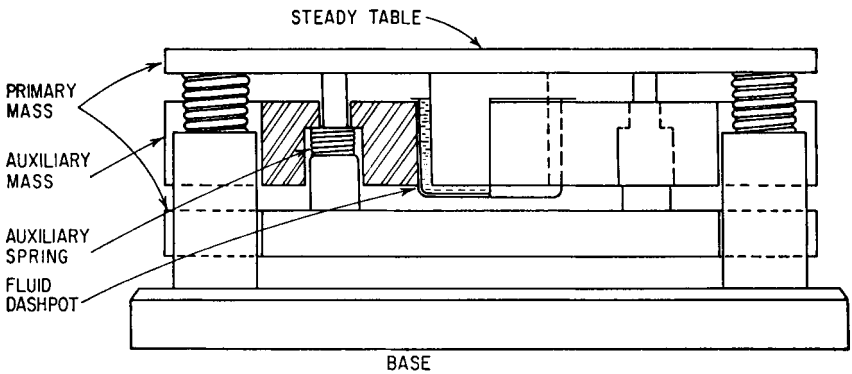


FIGURE 6.48 Application of auxiliary mass to spring-mounted table to reduce vibration of table. (Macinante.²⁶)

In precision measurements, it is necessary to isolate the instruments from effects of shock and vibration in the earth and to damp any oscillations that might be generated in the measuring instruments. A heavy spring-mounted table fitted with a heavy auxiliary mass that is attached to the table by a spring and submerged in an oil bath (Fig. 6.48) has proved to be effective.²⁶ In this example the table has a top surface of 13½ in. (34 cm) by 13½ in. (34 cm) and a height of 6 in. (15 cm). Each auxiliary mass weighs about 70 lb (32 kg). The springs for both the primary table and the auxiliary system are designed to give a natural frequency between 2 and 4 Hz in both the horizontal and vertical directions. By trying different fluids in the bath, suitable damping may be obtained experimentally.

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