CHAPTER 7
VIBRATION OF SYSTEMS HAVING DISTRIBUTED MASS AND ELASTICITY

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INTRODUCTION

Preceding chapters consider the vibration of lumped parameter systems; i.e., systems that are idealized as rigid masses joined by massless springs and dampers. Many engineering problems are solved by analyses based on ideal models of an actual system, giving answers that are useful though approximate. In general, more accurate results are obtained by increasing the number of masses, springs, and dampers; i.e., by increasing the number of degrees-of-freedom. As the number of degrees-of-freedom is increased without limit, the concept of the system with distributed mass and elasticity is formed. This chapter discusses the free and forced vibration of such systems. Types of systems include rods vibrating in torsional modes and in tension-compression modes, and beams and plates vibrating in flexural modes. Particular attention is given to the calculation of the natural frequencies of such systems for further use in other analyses. Numerous charts and tables are included to define in readily available form the natural frequencies of systems commonly encountered in engineering practice.

FREE VIBRATION

Degrees-of-Freedom. Systems for which the mass and elastic parts are lumped are characterized by a finite number of degrees-of-freedom. In physical systems, all elastic members have mass, and all masses have some elasticity; thus, all real systems have distributed parameters. In making an analysis, it is often assumed that real systems have their parameters lumped. For example, in the analysis of a system consisting of a mass and a spring, it is commonly assumed that the mass of the spring is negligible so that its only effect is to exert a force between the mass and the support to which the spring is attached, and that the mass is perfectly rigid so that it does not deform and exert any elastic force. The effect of the mass of the spring on the motion of the system may be considered in an approximate way, while still maintaining the assumption of one degree-of-freedom, by assuming that the spring moves so that the deflection of each of its elements can be described by a single parameter. A commonly used assumption is that the deflection of each section of the spring is proportional to its distance from the support, so that if the deflection of the mass is given, the deflection of any part of the spring is defined. For the exact solution of the problem, even though the mass is considered to be perfectly rigid, it is necessary to consider that the deformation of the spring can occur in any manner consistent with the requirements of physical continuity.

Systems with distributed parameters are characterized by having an infinite number of degrees-of-freedom. For example, if an initially straight beam deflects laterally, it may be necessary to give the deflection of each section along the beam in order to define completely the configuration. For vibrating systems, the coordinates usually are defined in such a way that the deflections of the various parts of the system from the equilibrium position are given.

Natural Frequencies and Normal Modes of Vibration. The number of natural frequencies of vibration of any system is equal to the number of degrees-of-freedom; thus, any system having distributed parameters has an infinite number of natural frequencies. At a given time, such a system usually vibrates with appreciable amplitude at only a limited number of frequencies, often at only one. With each natural frequency is associated a shape, called the normal or natural mode, which is assumed by the system during free vibration at the frequency. For example, when a uniform beam with simply supported or hinged ends vibrates laterally at its lowest or fundamental natural frequency, it assumes the shape of a half sine wave; this is a normal mode of vibration. When vibrating in this manner, the beam behaves as a system with a single degree-of-freedom, since its configuration at any time can be defined by giving the deflection of the center of the beam. When any linear system, i.e., one in which the elastic restoring force is proportional to the deflection, executes free vibration in a single natural mode, each element of the system except those at the supports and nodes executes simple harmonic motion about its equilibrium position. All possible free vibration of any linear system is made up of superposed vibrations in the normal modes at the corresponding natural frequencies. The total motion at any point of the system is the sum of the motions resulting from the vibration in the respective modes.

There are always nodal points, lines, or surfaces, i.e., points which do not move, in each of the normal modes of vibration of any system. For the fundamental mode, which corresponds to the lowest natural frequency, the supported or fixed points of the system usually are the only nodal points; for other modes, there are additional nodes. In the modes of vibration corresponding to the higher natural frequencies of some systems, the nodes often assume complicated patterns. In certain problems involving forced vibrations, it may be necessary to know what the nodal patterns are, since a particular mode usually will not be excited by a force acting at a nodal point. Nodal lines are shown in some of the tables.

Methods of Solution. The complete solution of the problem of free vibration of any system would require the determination of all the natural frequencies and of the mode shape associated with each. In practice, it often is necessary to know only a few of the natural frequencies, and sometimes only one. Usually the lowest frequencies are the most important. The exact mode shape is of secondary importance in many problems. This is fortunate, since some procedures for finding natural frequencies involve assuming a mode shape from which an approximation to the natural frequency can be found.

Classical Method. The fundamental method of solving any vibration problem is to set up one or more equations of motion by the application of Newton's second law of motion. For a system having a finite number of degrees-of-freedom, this procedure gives one or more ordinary differential equations. For systems having distributed parameters partial differential equations are obtained. Exact solutions of the equations are possible for only a relatively few configurations. For most problems other means of solution must be employed.

Rayleigh's and Ritz's Methods. For many elastic bodies, Rayleigh's method is useful in finding an approximation to the fundamental natural frequency. While it is possible to use the method to estimate some of the higher natural frequencies, the accuracy often is poor; thus, the method is most useful for finding the fundamental frequency. When any elastic system without damping vibrates in its fundamental normal mode, each part of the system executes simple harmonic motion about its equilibrium position. For example, in lateral vibration of a beam the motion can be expressed as $y = X(x) \sin \omega_n t$ where X is a function only of the distance along the length of the beam. For lateral vibration of a plate, the motion can be expressed as $w = W(x,y) \sin \omega_n t$ where x and y are the coordinates in the plane of the plate. The equations show that when the deflection from equilibrium is a maximum, all parts of the body are motionless. At that time all the energy associated with the vibration is in the form of elastic strain energy. When the body is passing through its equilibrium position, none of the vibrational energy is in the form of strain energy so that all of it is in the form of kinetic energy. For conservation of energy, the strain energy in the position of maximum deflection must equal the kinetic energy when passing through the equilibrium position. Rayleigh's method of finding the natural frequency is to compute these maximum energies, equate them, and solve for the frequency. When the kinetic-energy term is evaluated, the frequency always appears as a factor. Formulas for finding the strain and kinetic energies of rods, beams, and plates are given in Table 7.1.

If the deflection of the body during vibration is known exactly, Rayleigh's method gives the true natural frequency. Usually the exact deflection is not known, since its determination involves the solution of the vibration problem by the classical method. If the classical solution is available, the natural frequency is included in it, and nothing is gained by applying Rayleigh's method. In many problems for which the classical solution is not available, a good approximation to the deflection can be assumed on the basis of physical reasoning. If the strain and kinetic energies are computed using such an assumed shape, an approximate value for the natural frequency is found. The correctness of the approximate frequency depends on how well the assumed shape approximates the true shape.

In selecting a function to represent the shape of a beam or a plate, it is desirable to satisfy as many of the boundary conditions as possible. For a beam or plate supported at a boundary, the assumed function must be zero at that boundary; if the boundary is built in, the first derivative of the function must be zero. For a free boundary, if the conditions associated with bending moment and shear can be satisfied, better accuracy usually results. It can be shown[2] that the frequency that is found by using any shape except the correct shape always is higher than the actual frequency. Therefore, if more than one calculation is made, using different assumed shapes, the lowest computed frequency is closest to the actual frequency of the system.

In many problems for which a classical solution would be possible, the work involved is excessive. Often a satisfactory answer to such a problem can be obtained

TABLE 7.1 Strain and Kinetic Energies of Uniform Rods, Beams, and Plates

Member	Strain energy V	Kinetic energy T: General	Kinetic energy T: Maximum*
Rod in tension or compression	$\frac{SE}{2}\int_0^l \left(\frac{\partial u}{\partial x}\right)^2 dx$	$\frac{S\gamma}{2g}\int_0^l \left(\frac{\partial u}{\partial t}\right)^2 dx$	$\frac{S\gamma\omega_n^2}{2g}\int_0^l V^2\, dx$
Rod in torsion	$\frac{GI_p}{2}\int_0^l \left(\frac{\partial \phi}{\partial x}\right)^2 dx$	$\frac{I_p\gamma}{2g}\int_0^l \left(\frac{\partial \phi}{\partial t}\right)^2 dx$	$\frac{I_p\gamma\omega_n^2}{2g}\int_0^l \Phi^2\, dx$
Beam in bending	$\frac{EI}{2}\int_0^l \left(\frac{\partial^2 y}{\partial x^2}\right)^2 dx$	$\frac{S\gamma}{2g}\int_0^l \left(\frac{\partial y}{\partial t}\right)^2 dx$	$\frac{S\gamma\omega_n^2}{2g}\int_0^l Y^2\, dx$
Rectangular plate in bending[1]	$\frac{D}{2}\int_S\int \left\{\left(\frac{d^2w}{dx^2}+\frac{d^2w}{dy^2}\right)^2 - 2(1-\mu)\left[\frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2} - \left(\frac{\partial^2 w}{\partial x\,\partial y}\right)^2\right]\right\} dx\, dy$	$\frac{\gamma h}{2g}\int_S\int \left(\frac{\partial w}{\partial t}\right)^2 dx\, dy$	$\frac{\gamma h\omega_n^2}{2g}\int_S\int W^2\, dx\, dy$
Circular plate (deflection symmetrical about center)[1]	$\pi D\int_0^a \left\{\left(\frac{\partial^2 w}{\partial r^2}+\frac{1}{r}\frac{\partial w}{\partial r}\right)^2 - 2(1-\mu)\frac{\partial^2 w}{\partial r^2}\frac{1}{r}\frac{\partial w}{\partial r}\right\} r\, dr$	$\frac{\pi\gamma h}{g}\int_0^a \left(\frac{\partial w}{\partial t}\right)^2 r\, dr$	$\frac{\pi\gamma h\omega_n^2}{g}\int_0^a W^2 r\, dr$

u = longitudinal deflection of cross section of rod
ϕ = angle of twist of cross section of rod
y = lateral deflection of beam
w = lateral deflection of plate
Capitals denote values at extreme deflection for simple harmonic motion.
l = length of rod or beam
a = radius of circular plate
h = thickness of beam or plate

S = area of cross section
I_p = polar moment of inertia
I = moment of inertia of beam
γ = weight density
E = modulus of elasticity
G = modulus of rigidity
μ = Poisson's ratio
$D = Eh^3/12(1-\mu^2)$

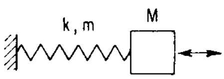
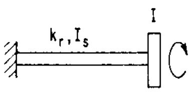
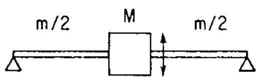
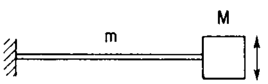
* This is the maximum kinetic energy in simple harmonic motion.

by the application of Rayleigh's method. In this chapter several examples are worked using both the classical method and Rayleigh's method. In all, Rayleigh's method gives a good approximation to the correct result with relatively little work. Many other examples of solutions to problems by Rayleigh's method are in the literature.[3–5]

Ritz's method is a refinement of Rayleigh's method. A better approximation of the fundamental natural frequency can be obtained by its use, and approximations of higher natural frequencies can be found. In using Ritz's method, the deflections which are assumed in computing the energies are expressed as functions with one or more undetermined parameters; these parameters are adjusted to make the computed frequency a minimum. Ritz's method has been used extensively for the determination of the natural frequencies of plates of various shapes and is discussed in the section on the lateral vibrations of plates.

Lumped Parameters. A procedure that is useful in many problems for finding approximations to both the natural frequencies and the mode shapes is to reduce the system with distributed parameters to one having a finite number of degrees-of-freedom. This is done by lumping the parameters for each small region into an equivalent mass and elastic element. Several formalized procedures for doing this and for analyzing the resulting systems are described in Chap. 28. If a system consists of a rigid mass supported by a single flexible member whose mass is not negligible, the elastic part of the system sometimes can be treated as an equivalent spring; i.e., some of its mass is lumped with the rigid mass. Formulas for several systems of this kind are given in Table 7.2.

TABLE 7.2 Approximate Formulas for Natural Frequencies of Systems Having Both Concentrated and Distributed Mass

TYPE OF SYSTEM	NATURAL FREQUENCY $\omega_n = 2\pi f_n$	STIFFNESS
 SPRING WITH MASS ATTACHED	$\sqrt{\frac{k}{M + m/3}}$	$k = \frac{Gd^4}{8nD^3}$ D = COIL DIA d = WIRE DIA n = NUMBER OF TURNS
 CIRCULAR ROD, WITH DISC ATTACHED, IN TORSION	$\sqrt{\frac{k_r}{I + I_s/3}}$	$k_r = \frac{G\pi D^4}{32\,l}$ D = ROD DIAMETER l = ROD LENGTH
 UNIFORM SIMPLY SUPPORTED BEAM WITH MASS IN CENTER	$\sqrt{\frac{k}{M + m/2}}$	$k = \frac{48EI}{l^3}$ l = BEAM LENGTH I = MOMENT OF INERTIA
 UNIFORM CANTILEVER BEAM WITH MASS ON END	$\sqrt{\frac{k}{M + 0.23m}}$	$k = \frac{3EI}{l^3}$ l = BEAM LENGTH I = MOMENT OF INERTIA

Orthogonality. It is shown in Chap. 2 that the normal modes of vibration of a system having a finite number of degrees-of-freedom are orthogonal to each other. For a system of masses and springs having n degrees-of-freedom, if the coordinate system is selected in such a way that X_1 represents the amplitude of motion of the first mass, X_2 that of the second mass, etc., the orthogonality relations are expressed by $(n - 1)$ equations as follows:

$$m_1 X_1^a X_1^b + m_2 X_2^a X_2^b + \cdots = \sum_{i=1}^{n} m_i X_i^a X_i^b = 0 \qquad [a \neq b]$$

where X_1^a represents the amplitude of the first mass when vibrating only in the ath mode, X_1^b the amplitude of the first mass when vibrating only in the bth mode, etc.

For a body such as a uniform beam whose parameters are distributed only lengthwise, i.e., in the X direction, the orthogonality between two normal modes is expressed by

$$\int_0^l \rho\phi_a(x)\phi_b(x)\,dx = 0 \qquad [a \neq b] \tag{7.1}$$

where $\phi_a(x)$ represents the deflection in the ath normal mode, $\phi_b(x)$ the deflection in the bth normal mode, and ρ the density.

For a system, such as a uniform plate, in which the parameters are distributed in two dimensions, the orthogonality condition is

$$\int_A\int \rho\phi_a(x,y)\phi_b(x,y)\,dx\,dy = 0 \qquad [a \neq b] \tag{7.2}$$

LONGITUDINAL AND TORSIONAL VIBRATIONS OF UNIFORM CIRCULAR RODS

Equations of Motion. A circular rod having a uniform cross section can execute longitudinal, torsional, or lateral vibrations, either individually or in any combination. The equations of motion for longitudinal and torsional vibrations are similar in form, and the solutions are discussed together. The lateral vibration of a beam having a uniform cross section is considered separately.

In analyzing the longitudinal vibration of a rod, only the motion of the rod in the longitudinal direction is considered. There is some lateral motion because longitudinal stresses induce lateral strains; however, if the rod is fairly long compared to its diameter, this motion has a minor effect.

Consider a uniform circular rod, Fig. 7.1*A*. The element of length dx, which is formed by passing two parallel planes A–A and B–B normal to the axis of the rod, is shown in Fig. 7.1*B*. When the rod executes only longitudinal vibration, the force acting on the face A–A is F, and that on face B–B is $F + (\partial F/\partial x)\,dx$. The net force acting to the right must equal the product of the mass of the element $(\gamma/g)S\,dx$ and its acceleration $\partial^2u/\partial t^2$, where γ is the weight density, S the area of the cross section, and u the longitudinal displacement of the element during the vibration:

$$\left(F + \frac{\partial F}{\partial x}\,dx\right) - F = \frac{\partial F}{\partial x}\,dx = \left(\frac{\gamma}{g}\right) S\,dx\,\frac{\partial^2 u}{\partial t^2} \qquad \text{or} \qquad \frac{\partial F}{\partial x} = \frac{\gamma S}{g}\,\frac{\partial^2 u}{\partial t^2} \tag{7.3}$$

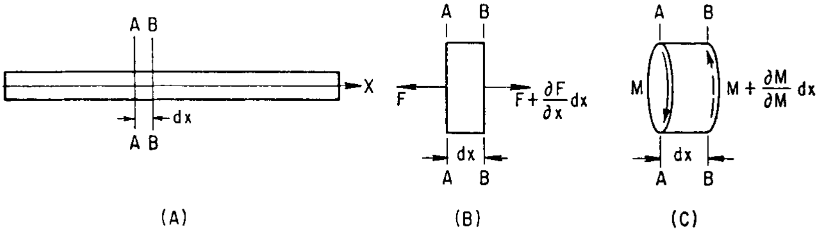


FIGURE 7.1 (*A*) Rod executing longitudinal or torsional vibration. (*B*) Forces acting on element during longitudinal vibration. (*C*) Moments acting on element during torsional vibration.

This equation is solved by expressing the force F in terms of the displacement. The elastic strain at any section is $\partial u/\partial x$, and the stress is $E\,\partial u/\partial x$. The force F is the product of the stress and the area, or $F = ES\,\partial u/\partial x$, and $\partial F/\partial x = ES\,\partial^2 u/\partial x^2$. Equation (7.3) becomes $Eu'' = \gamma/g\ddot{u}$, where $u'' = \partial^2 u/\partial x^2$ and $\ddot{u} = \partial^2 u/\partial t^2$. Substituting $a^2 = Eg/\gamma$,

$$a^2 u'' = \ddot{u} \tag{7.4}$$

The equation governing the torsional vibration of the circular rod is derived by equating the net torque acting on the element, Fig. 7.1*C*, to the product of the moment of inertia J and the angular acceleration $\ddot{\phi}$, ϕ being the angular displacement of the section. The torque on the section A–A is M and that on section B–B is $M + (\partial M/\partial x)\,dx$. By an analysis similar to that for the longitudinal vibration, letting $b^2 = Gg/\gamma$,

$$b^2 \phi'' = \ddot{\phi} \tag{7.5}$$

Solution of Equations of Motion. Since Eqs. (7.4) and (7.5) are of the same form, the solutions are the same except for the meaning of a and b. The solution of Eq. (7.5) is of the form $\phi = X(x)T(t)$ in which X is a function of x only and T is a function of t only. Substituting this in Eq. (7.5) gives $b^2 X''T = X\ddot{T}$. By separating the variables,[6]

$$\begin{aligned} T &= A \cos(\omega_n t + \theta) \\ X &= C \sin \frac{\omega_n x}{b} + D \cos \frac{\omega_n x}{b} \end{aligned} \tag{7.6}$$

The natural frequency ω_n can have infinitely many values, so that the complete solution of Eq. (7.5) is, combining the constants,

$$\phi = \sum_{n=1}^{n=\infty} \left(C_n \sin \frac{\omega_n x}{b} + D_n \cos \frac{\omega_n x}{b} \right) \cos(\omega_n t + \theta_n) \tag{7.7}$$

The constants C_n and D_n are determined by the end conditions of the rod and by the initial conditions of the vibration. For a built-in or clamped end of a rod in torsion, $\phi = 0$ and $X = 0$ because the angular deflection must be zero. The torque at any section of the shaft is given by $M = (GI_p)\phi'$, where GI_p is the torsional rigidity of the shaft; thus, for a free end, $\phi' = 0$ and $X' = 0$. For the longitudinal vibration of a rod, the boundary conditions are essentially the same; i.e., for a built-in end the displacement is zero ($u = 0$) and for a free end the stress is zero ($u' = 0$).

EXAMPLE 7.1. The natural frequencies of the torsional vibration of a circular steel rod of 2-in. diameter and 24-in. length, having the left end built in and the right end free, are to be determined.

SOLUTION. The built-in end at the left gives the condition $X = 0$ at $x = 0$ so that $D = 0$ in Eq. (7.6). The free end at the right gives the condition $X' = 0$ at $x = l$. For each mode of vibration, Eq. (7.6) is $\cos \omega_n l/b = 0$ from which $\omega_n l/b = \pi/2, 3\pi/2, 5\pi/2, \ldots$. Since $b^2 = Gg/\gamma$, the natural frequencies for the torsional vibration are

$$\omega_n = \frac{\pi}{2l}\sqrt{\frac{Gg}{\gamma}}, \frac{3\pi}{2l}\sqrt{\frac{Gg}{\gamma}}, \frac{5\pi}{2l}\sqrt{\frac{Gg}{\gamma}}, \ldots \qquad \text{rad/sec}$$

For steel, $G = 11.5 \times 10^6$ lb/in.^{2} and $\gamma = 0.28$ lb/in.^{3} The fundamental natural frequency is

$$\omega_n = \frac{\pi}{2(24)}\sqrt{\frac{(11.5 \times 10^6)(386)}{0.28}} = 8240 \text{ rad/sec} = 1311 \text{ Hz}$$

The remaining frequencies are 3, 5, 7, etc., times ω_n.

Since Eq. (7.4), which governs longitudinal vibration of the bar, is of the same form as Eq. (7.5), which governs torsional vibration, the solution for longitudinal vibration is the same as Eq. (7.7) with u substituted for ϕ and $a = \sqrt{Eg/\gamma}$ substituted for b. The natural frequencies of a uniform rod having one end built in and one end free are obtained by substituting a for b in the frequency equations found above in Example 7.1:

$$\omega_n = \frac{\pi}{2l}\sqrt{\frac{Eg}{\gamma}}, \frac{3\pi}{2l}\sqrt{\frac{Eg}{\gamma}}, \frac{5\pi}{2l}\sqrt{\frac{Eg}{\gamma}}, \ldots$$

The frequencies of the longitudinal vibration are independent of the lateral dimensions of the bar, so that these results apply to uniform noncircular bars. Equation (7.5) for torsional vibration is valid only for circular cross sections.

Torsional Vibrations of Circular Rods with Discs Attached. An important type of system is that in which a rod which may twist has mounted on it one or more rigid discs or members that can be considered as the equivalents of discs. Many systems can be approximated by such configurations. If the moment of inertia of the rod is small compared to the moments of inertia of the discs, the mass of the rod may be neglected and the system considered to have a finite number of degrees-of-freedom. Then the methods described in Chaps. 2 and 38 are applicable. Even if the moment of inertia of the rod is not negligible, it usually may be lumped with the moment of inertia of the disc. For a shaft having a single disc attached, the formula in Table 7.2 gives a close approximation to the true frequency.

The exact solution of the problem requires that the effect of the distributed mass of the rod be considered. Usually it can be assumed that the discs are rigid enough that their elasticity can be neglected; only such systems are considered. Equation (7.5) and its solution, Eq. (7.7), apply to the shaft where the constants are determined by the end conditions. If there are more than two discs, the section of shaft between each pair of discs must be considered separately; there are two constants for each section. The constants are determined from the following conditions:

1. For a disc at an end of the shaft, the torque of the shaft at the disc is equal to the product of the moment of inertia of the disc and its angular acceleration.
2. Where a disc is between two sections of shaft, the angular deflection at the end of each section adjoining the disc is the same; the difference between the torques in the two sections is equal to the product of the moment of inertia of the disc and its angular acceleration.

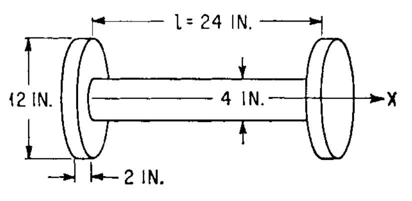


FIGURE 7.2 Rod with disc attached at each end.

EXAMPLE 7.2. The fundamental frequency of vibration of the system shown in Fig. 7.2 is to be calculated and the result compared with the frequency obtained by considering that each half of the system is a simple shaft-disc system with the end of the shaft fixed. The system consists of a steel shaft 24 in. long and 4 in. in diameter having attached to it at each end a rigid steel disc 12 in. in diameter and 2 in. thick. For the approximation, add one-third of the moment of inertia of half the shaft to that of the disc (Table 7.2). (Because of symmetry, the center of the shaft is a nodal point; i.e., it does not move. Thus, each half of the system can be considered as a rod-disc system.)

EXACT SOLUTION. The boundary conditions are: at $x = 0$, $M = GI_p\phi' = I_1\ddot{\phi}$; at $x = l$, $M = GI\phi' = -I_2\ddot{\phi}$, where I_1 and I_2 are the moments of inertia of the discs. The signs are opposite for the two boundary conditions because, if the shaft is twisted in a certain direction, it will tend to accelerate the disc at the left end in one direction and the disc at the right end in the other. In the present example, $I_1 = I_2$; however, the solution is carried out in general terms.

Using Eq. (7.7), the following is obtained for each value of n:

$$\phi' = \frac{\omega_n}{b}\left(C\cos\frac{\omega_n x}{b} - D\sin\frac{\omega_n x}{b}\right)\cos(\omega_n t + \theta)$$

$$\ddot{\phi} = \omega_n^2\left(C\sin\frac{\omega_n x}{b} + D\cos\frac{\omega_n x}{b}\right)[-\cos(\omega_n t + \theta)]$$

The boundary conditions give the following:

$$GI_p\frac{\omega_n}{b}C = -\omega_n^2 DI_1 \qquad \text{or} \qquad C = -\frac{b\omega_n I_1}{GI_p}D$$

$$\frac{\omega_n}{b}GI_p\left(C\cos\frac{\omega_n l}{b} - D\sin\frac{\omega_n l}{b}\right) = \omega_n^2 I_2\left(C\sin\frac{\omega_n l}{b} + D\cos\frac{\omega_n l}{b}\right)$$

These two equations can be combined to give

$$-\frac{\omega_n}{b}GI_p\left(\frac{b\omega_n I_1}{GI_p}\cos\frac{\omega_n l}{b} + \sin\frac{\omega_n l}{b}\right) = \omega_n^2 I_2\left(-\frac{b\omega_n I_1}{GI_p}\sin\frac{\omega_n l}{b} + \cos\frac{\omega_n l}{b}\right)$$

The preceding equation can be reduced to

$$\tan\alpha_n = \frac{(c+d)\alpha_n}{cd\alpha_n^2 - 1} \tag{7.8}$$

where $\alpha_n = (\omega_n l)/b$, $c = I_1/I_s$, $d = I_2/I_s$, and I_s is the polar moment of inertia of the shaft as a rigid body. There is a value for X in Eq. (7.6) corresponding to each root of Eq. (7.8) so that Eq. (7.7) becomes

$$\theta = \sum_{n=1}^{n=\infty} A_n\left(\cos\frac{\omega_n x}{b} - c\alpha_n\sin\frac{\omega_n x}{b}\right)\cos(\omega_n t + \theta_n)$$

For a circular disc or shaft, $I = ½mr^2$ where m is the total mass; thus $c = d = (D^4/d^4)(h/l) = 6.75$. Equation (7.8) becomes $(45.56\alpha_n^2 - 1)\tan\alpha_n = 13.5\alpha_n$, the lowest root of which is $\alpha_n = 0.538$. The natural frequency is $\omega_n = 0.538\sqrt{Gg/\gamma l^2}$ rad/sec.

APPROXIMATE SOLUTION. From Table 7.2, the approximate formula is

$$\omega_n = \left(\frac{k_r}{I + I_s/3}\right)^{1/2} \qquad \text{where } k_r = \frac{\pi d^4}{32}\frac{G}{l}$$

For the present problem where the center of the shaft is a node, the values of moment of inertia I_s and torsional spring constant for half the shaft must be used:

$$½I_s = \frac{\pi d^4}{32}\frac{\gamma}{g}\frac{l}{2} \qquad \text{and} \qquad k_r = 2\left[\frac{\pi d^4}{32}\frac{G}{l}\right]$$

From the previous solution:

$$I_1 = 6.75 I_s \qquad I_1 + \frac{1}{2}\left(\frac{I_s}{3}\right) = \frac{\pi d^4}{32}\,\frac{\gamma}{g}\,\frac{l}{2}\,[2(6.75) + 0.333]$$

Substituting these values into the frequency equation and simplifying gives

$$\omega_n = 0.538\sqrt{\frac{Gg}{\gamma l^2}}$$

In this example, the approximate solution is correct to at least three significant figures. For larger values of I_s/I, poorer accuracy can be expected.

For steel, $G = 11.5 \times 10^6$ lb/in.^{2} and $\gamma = 0.28$ lb/in.^{3}; thus

$$\omega_n = 0.538\sqrt{\frac{(11.5 \times 10^6)(386)}{(0.28)(24)^2}} = 0.538 \times 5245 = 2822 \text{ rad/sec} = 449 \text{ Hz}$$

Longitudinal Vibration of a Rod with Mass Attached. The natural frequencies of the longitudinal vibration of a uniform rod having rigid masses attached to it can be solved in a manner similar to that used for a rod in torsion with discs attached. Equation (7.4) applies to this system; its solution is the same as Eq. (7.7) with a substituted for b. For each value of n,

$$u = \left(C_n \sin \frac{\omega_n x}{a} + D_n \cos \frac{\omega_n x}{a}\right) \cos(\omega_n t + \theta)$$

In Fig. 7.3, the rod of length l is fixed at $x = 0$ and has a mass m_2 attached at $x = l$. The boundary conditions are: at $x = 0$, $u = 0$ and at $x = l$, $SEu' = -m_2\ddot{u}$. The latter expresses the condition that the force in the bar equals the product of the mass and its acceleration at the end with the mass attached. The sign is negative because the force is tensile or positive when the acceleration of the mass is negative. From the first boundary condition, $D_n = 0$. The second boundary condition gives

$$\frac{\omega_n SE}{a} C_n \cos \frac{\omega_n l}{a} = m_2 \omega_n^2 C_n \sin \frac{\omega_n l}{a}$$

from which

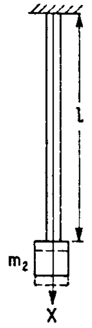


FIGURE 7.3 Rod, with mass attached to end, executing longitudinal vibration.

$$\frac{SEl}{m_2 a^2} = \frac{\omega_n l}{a} \tan \frac{\omega_n l}{a}$$

Since $a^2 = Eg/\gamma$, this can be written

$$\frac{m_1}{m_2} = \frac{\omega_n l}{a} \tan \frac{\omega_n l}{a}$$

where m_1 is the mass of the rod. This equation can be applied to a simple mass-spring system by using the relation that the constant k of a spring is equivalent to SE/l for the rod, so that $l/a = (m_1/k)^{1/2}$, where m_1 is the mass of the spring:

$$\frac{m_1}{m_2} = \omega_n \sqrt{\frac{m_1}{k}} \tan \omega_n \sqrt{\frac{m_1}{k}} \qquad (7.9)$$

Rayleigh's Method. An accurate approximation to the fundamental natural frequency of this system can be found by using Rayleigh's method. The motion of the mass can be expressed as $u_m = u_0 \sin \omega t$. If it is assumed that the deflection u at each section of the rod is proportional to its distance from the fixed end, $u = u_0(x/l) \sin \omega_n t$. Using this relation in the appropriate equation from Table 7.1, the strain energy V of the rod at maximum deflection is

$$V = \frac{SE}{2}\int_0^l \left(\frac{\partial u}{\partial x}\right)^2 dx = \frac{SE}{2}\int_0^l \left(\frac{u_0}{l}\right)^2 dx = \frac{SEu_0^2}{2l}$$

The maximum kinetic energy T of the rod is

$$T = \frac{S\gamma}{2g}\int_0^l V_{\max}^2\, dx = \frac{S\gamma}{2g}\int_0^l \left(\omega_n u_0 \frac{x}{l}\right)^2 dx = \frac{S\gamma}{2g}\,\omega_n^2 u_0^2\,\frac{l}{3}$$

The maximum kinetic energy of the mass is $T_m = m_2\omega_n^2 u_0^2/2$. Equating the total maximum kinetic energy $T + T_m$ to the maximum strain energy V gives

$$\omega_n = \left(\frac{SE}{l(m_2 + m_1/3)}\right)^{1/2}$$

where $m_1 = S\gamma l/g$ is the mass of the rod. Letting $SE/l = k$,

$$\omega_n = \sqrt{\frac{k}{M + m/3}} \tag{7.10}$$

This formula is included in Table 7.2. The other formulas in that table are also based on analyses by the Rayleigh method.

EXAMPLE 7.3. The natural frequency of a simple mass-spring system for which the weight of the spring is equal to the weight of the mass is to be calculated and compared to the result obtained by using Eq. (7.10).

SOLUTION. For $m_1/m_2 = l$, the lowest root of Eq. (7.9) is $\omega_n\sqrt{m/k} = 0.860$. When $m_2 = m_1$,

$$\omega_n = 0.860\sqrt{\frac{k}{m_2}}$$

Using the approximate equation,

$$\omega_n = \sqrt{\frac{k}{m_2(1 + \frac{1}{3})}} = 0.866\sqrt{\frac{k}{m_2}}$$

LATERAL VIBRATION OF STRAIGHT BEAMS

Natural Frequencies from Nomograph. For many practical purposes the natural frequencies of uniform beams of steel, aluminum, and magnesium can be determined with sufficient accuracy by the use of the nomograph, Fig. 7.4. This nomograph applies to many conditions of support and several types of load. Figure 7.4*A* indicates the procedure for using the nomograph.

Classical Solution. In the derivation of the necessary equation, use is made of the relation

$$EI\frac{d^2y}{dx^2} = M \tag{7.11}$$

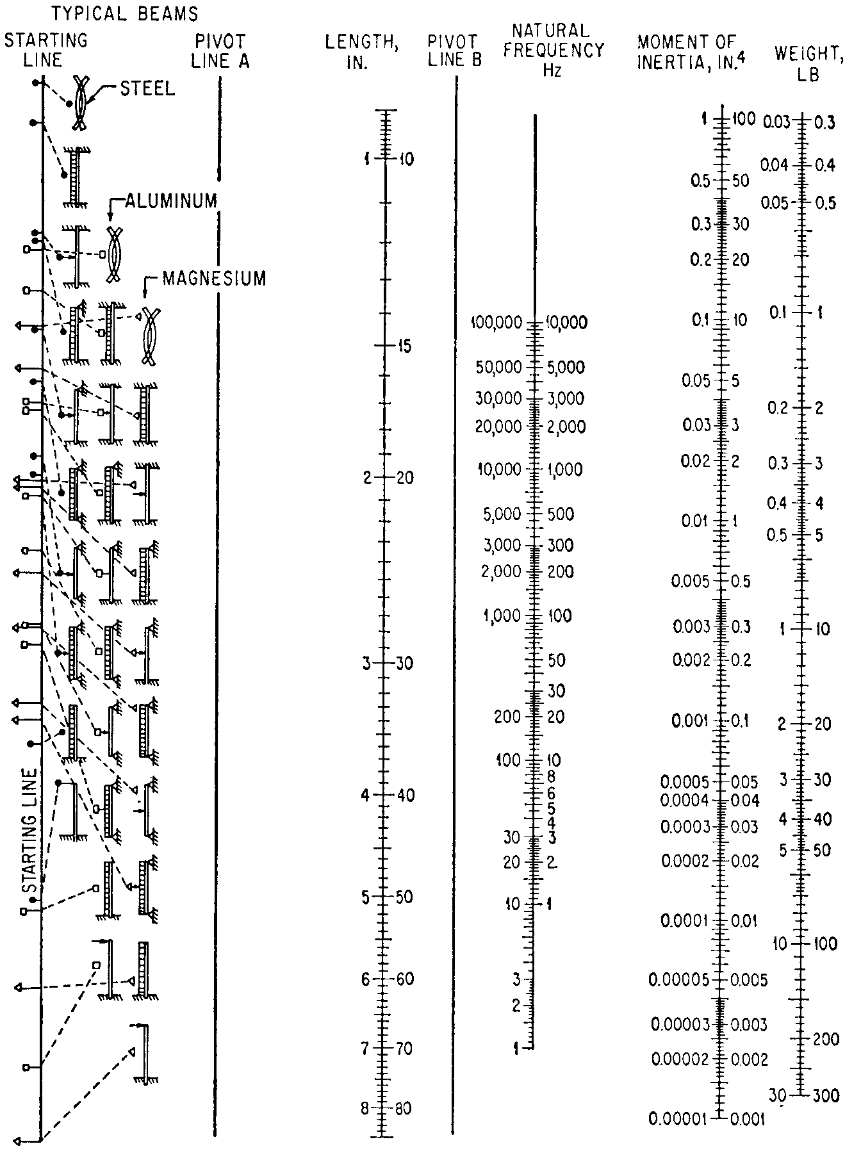


FIGURE 7.4 Nomograph for determining fundamental natural frequencies of beams. From the point on the starting line which corresponds to the loading and support conditions for the beam, a straight line is drawn to the proper point on the length line. (If the length appears on the left side of this line, subsequent readings on all lines are made to the left; and if the length appears to the right, subsequent readings are made to the right.) From the intersection of this line with pivot line A, a straight line is drawn to the moment of inertia line; from the intersection of this line with pivot line B, a straight line is drawn to the weight line. (For concentrated loads, the weight is that of the load; for uniformly distributed loads, the weight is the total load on the beam, including the weight of the beam.) The natural frequency is read where the last line crosses the natural frequency line. (*J. J. Kerley.*[7])

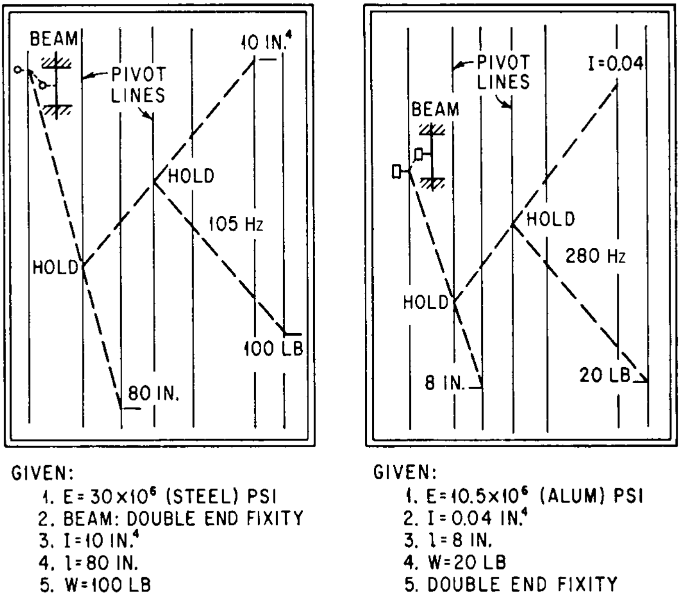


FIGURE 7.4A Example of use of Fig. 7.4. The natural frequency of the steel beam is 105 Hz and that of the aluminum beam is 280 Hz. (*J. J. Kerley.*[7])

This equation relates the curvature of the beam to the bending moment at each section of the beam. This equation is based upon the assumptions that the material is homogeneous, isotropic, and obeys Hooke's law and that the beam is straight and of uniform cross section. The equation is valid for small deflections only and for beams that are long compared to cross-sectional dimensions since the effects of shear deflection are neglected. The effects of shear deflection and rotation of the cross sections are considered later.

The equation of motion for lateral vibration of the beam shown in Fig. 7.5*A* is found by considering the forces acting on the element, Fig. 7.5*B*, which is formed by passing two parallel planes *A–A* and *B–B* through the beam normal to the longitudinal axis. The vertical elastic shear force acting on section *A–A* is *V*, and that on section *B–B* is $V + (\partial V/\partial x)\, dx$. Shear forces acting as shown are considered to be positive. The total vertical elastic shear force at each section of the beam is composed of two parts: that caused by the static load including the weight of the beam

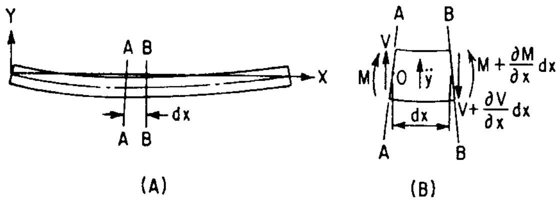


FIGURE 7.5 (*A*) Beam executing lateral vibration. (*B*) Element of beam showing shear forces and bending moments.

and that caused by the vibration. The part of the shear force caused by the static load exactly balances the load, so that these forces need not be considered in deriving the equation for the vibration if all deflections are measured from the position of equilibrium of the beam under the static load. The sum of the remaining vertical forces acting on the element must equal the product of the mass of the element $S\gamma/g\,dx$ and the acceleration $\partial^2 y/\partial t^2$ in the lateral direction: $V + (\partial V/\partial x)\,dx - V = (\partial V/\partial x)\,dx = -(S\gamma/g)(\partial^2 y/\partial t^2)\,dx$, or

$$\frac{\partial V}{\partial x} = -\frac{\gamma S}{g}\frac{\partial^2 y}{\partial t^2} \tag{7.12}$$

If moments are taken about point 0 of the element in Fig. 7.5*B*, $V\,dx = (\partial M/\partial x)\,dx$ and $V = \partial M/\partial x$. Other terms contain differentials of higher order and can be neglected. Substituting this in Eq. (7.12) gives $-\partial^2 M/\partial x^2 = (S\gamma/g)(\partial^2 y/\partial t^2)$. Substituting Eq. (7.11) gives

$$-\frac{\partial^2}{\partial x^2}\left(EI\,\frac{\partial^2 y}{\partial x^2}\right) = \frac{\gamma S}{g}\frac{\partial^2 y}{\partial t^2} \tag{7.13}$$

Equation (7.13) is the basic equation for the lateral vibration of beams. The solution of this equation, if EI is constant, is of the form $y = X(x)\,[\cos(\omega_n t + \theta)]$, in which X is a function of x only. Substituting

$$\kappa^4 = \frac{\omega_n^{\,2}\gamma S}{EIg} \tag{7.14}$$

and dividing Eq. (7.13) by $\cos(\omega_n t + \theta)$:

$$\frac{d^4 X}{dx^4} = \kappa^4 X \tag{7.15}$$

where X is any function whose fourth derivative is equal to a constant multiplied by the function itself. The following functions satisfy the required conditions and represent the solution of the equation:

$$X = A_1 \sin \kappa x + A_2 \cos \kappa x + A_3 \sinh \kappa x + A_4 \cosh \kappa x$$

The solution can also be expressed in terms of exponential functions, but the trigonometric and hyperbolic functions usually are more convenient to use.

For beams having various support conditions, the constants A_1, A_2, A_3, and A_4 are found from the end conditions. In finding the solutions, it is convenient to write the equation in the following form in which two of the constants are zero for each of the usual boundary conditions:

$$X = A\,(\cos \kappa x + \cosh \kappa x) + B(\cos \kappa x - \cosh \kappa x) + C(\sin \kappa x + \sinh \kappa x) + D(\sin \kappa x - \sinh \kappa x) \tag{7.16}$$

In applying the end conditions, the following relations are used where primes indicate successive derivatives with respect to x:

The deflection is proportional to X and is zero at any rigid support.

The slope is proportional to X' and is zero at any built-in end.

The moment is proportional to X'' and is zero at any free or hinged end.

The shear is proportional to X''' and is zero at any free end.

The required derivatives are:

$$X' = \kappa[A(-\sin\kappa x + \sinh\kappa x) + B(-\sin\kappa x - \sinh\kappa x) + C(\cos\kappa x + \cosh\kappa x) + D(\cos\kappa x - \cosh\kappa x)]$$

$$X'' = \kappa^2[A(-\cos\kappa x + \cosh\kappa x) + B(-\cos\kappa x - \cosh\kappa x) + C(-\sin\kappa x + \sinh\kappa x) + D(-\sin\kappa x - \sinh\kappa x)]$$

$$X''' = \kappa^3[A(\sin\kappa x + \sinh\kappa x) + B(\sin\kappa x - \sinh\kappa x) + C(-\cos\kappa x + \cosh\kappa x) + D(-\cos\kappa x - \cosh\kappa x)]$$

For the usual end conditions, two of the constants are zero, and there remain two equations containing two constants. These can be combined to give an equation which contains only the frequency as an unknown. Using the frequency, one of the unknown constants can be found in terms of the other. There always is one undetermined constant, which can be evaluated only if the amplitude of the vibration is known.

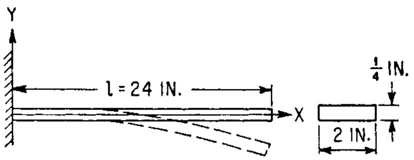


FIGURE 7.6 First mode of vibration of beam with left end clamped and right end free.

EXAMPLE 7.4. The natural frequencies and modes of vibration of the rectangular steel beam shown in Fig. 7.6 are to be determined and the fundamental frequency compared with that obtained from Fig. 7.4. The beam is 24 in. long, 2 in. wide, and ¼ in. thick, with the left end built in and the right end free.

SOLUTION. The boundary conditions are: at $x = 0$, $X = 0$, and $X' = 0$; at $x = l$, $X'' = 0$, and $X''' = 0$. The first condition requires that $A = 0$ since the other constants are multiplied by zero at $x = 0$. The second condition requires that $C = 0$. From the third and fourth conditions, the following equations are obtained:

$$0 = B(-\cos\kappa l - \cosh\kappa l) + D(-\sin\kappa l - \sinh\kappa l)$$

$$0 = B(\sin\kappa l - \sinh\kappa l) + D(-\cos\kappa l - \cosh\kappa l)$$

Solving each of these for the ratio D/B and equating, or making use of the mathematical condition that for a solution the determinant of the two equations must vanish, the following equation results:

$$\frac{D}{B} = -\frac{\cos\kappa l + \cosh\kappa l}{\sin\kappa l + \sinh\kappa l} = \frac{\sin\kappa l - \sinh\kappa l}{\cos\kappa l + \cosh\kappa l} \tag{7.17}$$

Equation (7.17) reduces to $\cos\kappa l \cosh\kappa l = -1$. The values of κl which satisfy this equation can be found by consulting tables of hyperbolic and trigonometric functions. The first five are: $\kappa_1 l = 1.875$, $\kappa_2 l = 4.694$, $\kappa_3 l = 7.855$, $\kappa_4 l = 10.996$, and $\kappa_5 l = 14.137$. The corresponding frequencies of vibration are found by substituting the length of the beam to find each κ and then solving Eq. (7.14) for ω_n:

$$\omega_n = \kappa_n^2 \sqrt{\frac{EIg}{\gamma S}}$$

For the rectangular section, $I = bh^3/12 = 1/384$ in.^{4} and $S = bh = 0.5$ in.^{2} For steel, $E = 30 \times 10^6$ lb/in.^{2} and $\gamma = 0.28$ lb/in.^{3} Using these values,

$$\omega_1 = \frac{(1.875)^2}{(24)^2} \sqrt{\frac{(30 \times 10^6)(386)}{(0.28)(384)(0.5)}} = 89.6 \text{ rad/sec} = 14.26 \text{ Hz}$$

The remaining frequencies can be found by using the other values of κ. Using Fig. 7.4, the fundamental frequency is found to be about 12 Hz.

To find the mode shapes, the ratio D/B is found by substituting the appropriate values of κl in Eq. (7.17). For the first mode:

$$\cosh 1.875 = 3.33710 \qquad \sinh 1.875 = 3.18373$$

$$\cos 1.875 = -0.29953 \qquad \sin 1.875 = 0.95409$$

Therefore, $D/B = -0.73410$. The equation for the first mode of vibration becomes

$$y = B_1[(\cos \kappa x - \cosh \kappa x) - 0.73410 (\sin \kappa x - \sinh \kappa x)] \cos (\omega_1 t + \theta_1)$$

in which B_1 is determined by the amplitude of vibration in the first mode. A similar equation can be obtained for each of the modes of vibration; all possible free vibration of the beam can be expressed by taking the sum of these equations.

Frequencies and Shapes of Beams. Table 7.3 gives the information necessary for finding the natural frequencies and normal modes of vibration of uniform beams having various boundary conditions. The various constants in the table were determined by computations similar to those used in Example 7.4. The table includes (1) diagrams showing the modal shapes including node locations, (2) the boundary conditions, (3) the frequency equation that results from using the boundary conditions in Eq. (7.16), (4) the constants that become zero in Eq. (7.16), (5) the values of κl from which the natural frequencies can be computed by using Eq. (7.14), and (6) the ratio of the nonzero constants in Eq. (7.16). By the use of the constants in this table, the equation of motion for any normal mode can be written. There always is a constant which is determined by the amplitude of vibration.

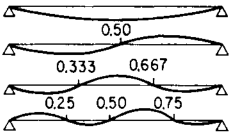
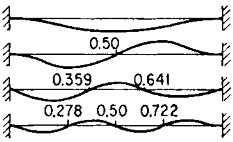
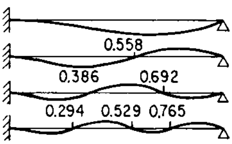
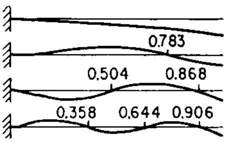
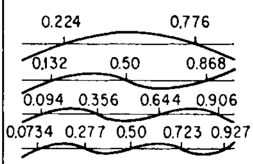
Values of characteristic functions representing the deflections of beams, at 50 equal intervals, for the first five modes of vibration have been tabulated.[8] Functions are given for beams having various boundary conditions, and the first three derivatives of the functions are also tabulated.

Rayleigh's Method. This method is useful for finding approximate values of the fundamental natural frequencies of beams. In applying Rayleigh's method, a suitable function is assumed for the deflection, and the maximum strain and kinetic energies are calculated, using the equations in Table 7.1. These energies are equated and solved for the frequency. The function used to represent the shape must satisfy the boundary conditions associated with deflection and slope at the supports. Best accuracy is obtained if other boundary conditions are also satisfied. The equation for the static deflection of the beam under a uniform load is a suitable function, although a simpler function often gives satisfactory results with less numerical work.

EXAMPLE 7.5. The fundamental natural frequency of the cantilever beam in Example 7.4 is to be calculated using Rayleigh's method.

SOLUTION. The assumed deflection $Y = (a/3l^4)[x^4 - 4x^3l + 6x^2l^2]$ is the static deflection of a cantilever beam under uniform load and having the deflection $Y = a$ at $x = l$. This deflection satisfies the conditions that the deflection Y and the slope Y' be zero at $x = 0$. Also, at $x = l$, Y'' which is proportional to the moment and Y''' which is proportional to the shear are zero. The second derivative of the function is $Y'' = (4a/l^4)[x^2 - 2xl + l^2]$. Using this in the expression from Table 7.1, the maximum strain energy is

TABLE 7.3 Natural Frequencies and Normal Modes of Uniform Beams

SUPPORTS	MODE n	(A) SHAPE AND NODES (NUMBERS GIVE LOCATION OF NODES IN FRACTION OF LENGTH FROM LEFT END)	(B) BOUNDARY CONDITIONS EQ (7.16)	(C) FREQUENCY EQUATION	(D) CONSTANTS EQ (7.16)	(E) kl EQ (7.14) $\omega_n = k^2\sqrt{\frac{EIg}{A\gamma}}$	(F) R RATIO OF NON-ZERO CONSTANTS COLUMN (D)
HINGED-HINGED	1		$x=0\begin{cases}X=0\\X''=0\end{cases}$	SIN $kl=0$	$A=0$	3.1416	1.0000
	2				$B=0$	6.283	1.0000
	3				$\frac{C}{D}=1$	9.425	1.0000
	4		$x=l\begin{cases}X=0\\X''=0\end{cases}$			12.566	1.0000
	n>4					$\approx n\pi$	1.0000
CLAMPED-CLAMPED	1		$x=0\begin{cases}X=0\\X'=0\end{cases}$	(COS kl) (COSH kl) $=1$	$A=0$	4.730	-0.9825
	2				$C=0$	7.853	-1.0008
	3				$\frac{D}{B}=R$	10.996	-1.0000-
	4		$x=l\begin{cases}X=0\\X'=0\end{cases}$			14.137	-1.0000+
	n>4					$\approx\frac{(2n+1)\pi}{2}$	-1.0000-
CLAMPED-HINGED	1		$x=0\begin{cases}X=0\\X'=0\end{cases}$	TAN $kl=$ TANH kl	$A=0$	3.927	-1.0008
	2				$C=0$	7.069	-1.0000+
	3				$\frac{D}{B}=R$	10.210	-1.0000
	4		$x=l\begin{cases}X=0\\X''=0\end{cases}$			13.352	-1.0000
	n>4					$\approx\frac{(4n+1)\pi}{4}$	-1.0000
CLAMPED-FREE	1		$x=0\begin{cases}X=0\\X'=0\end{cases}$	(COS kl) (COSH kl) $=-1$	$A=0$	1.875	-0.7341
	2				$C=0$	4.694	-1.0185
	3				$\frac{D}{B}=R$	7.855	-0.9992
	4		$x=l\begin{cases}X''=0\\X'''=0\end{cases}$			10.996	-1.0000+
	n>4					$\approx\frac{(2n-1)\pi}{2}$	-1.0000-
FREE-FREE	1		$x=0\begin{cases}X''=0\\X'''=0\end{cases}$	(COS kl) (COSH kl) $=1$	$B=0$	0 (REPRESENTS TRANSLATION)	
	2				$D=0$	4.730	-0.9825
	3				$\frac{C}{A}=R$	7.853	-1.0008
	4		$x=l\begin{cases}X''=0\\X'''=0\end{cases}$			10.996	-1.0000-
	5					14.137	-1.0000+
	n>5					$\approx\frac{(2n-1)\pi}{2}$	-1.0000-

$$V = \frac{EI}{2} \int_0^l \left(\frac{d^2Y}{dx^2} \right)^2 dx = \frac{8}{5} \frac{EIa^2}{l^3}$$

The maximum kinetic energy is

$$T = \frac{\omega_n^2 \gamma S}{2g} \int_0^l Y^2 \, dx = \frac{52}{405} \frac{\omega_n^2 \gamma S l a^2}{g}$$

Equating the two energies and solving for the frequency,

$$\omega_n = \sqrt{\frac{162}{13} \times \frac{EIg}{\gamma S l^4}} = \frac{3.530}{l^2} \sqrt{\frac{EIg}{\gamma S}}$$

The exact frequency as found in Example 7.4 is $(3.516/l^2)\sqrt{EIg/\gamma S}$; thus, Rayleigh's method gives good accuracy in this example.

If the deflection is assumed to be $Y = a[1 - \cos(\pi x/2l)]$, the calculated frequency is $(3.66/l^2)\sqrt{EIg/\gamma S}$. This is less accurate, but the calculations are considerably shorter. With this function, the same boundary conditions at $x = 0$ are satisfied; however, at $x = l$, $Y'' = 0$, but Y''' does not equal zero. Thus, the condition of zero shear at the free end is not satisfied. The trigonometric function would not be expected to give as good accuracy as the static deflection relation used in the example, although for most practical purposes the result would be satisfactory.

Effects of Rotary Motion and Shearing Force. In the preceding analysis of the lateral vibration of beams it has been assumed that each element of the beam moves only in the lateral direction. If each plane section that is initially normal to the axis of the beam remains plane and normal to the axis, as assumed in simple beam theory, then each section rotates slightly in addition to its lateral motion when the beam deflects.[9] When a beam vibrates, there must be forces to cause this rotation, and for a complete analysis these forces must be considered. The effect of this rotation is small except when the curvature of the beam is large relative to its thickness; this is true either for a beam that is short relative to its thickness or for a long beam vibrating in a higher mode so that the nodal points are close together.

Another factor that affects the lateral vibration of a beam is the lateral shear force. In Eq. (7.11) only the deflection associated with the bending stress in the beam is included. In any beam except one subject only to pure bending, a deflection due to the shear stress in the beam occurs. The exact solution of the beam vibration problem requires that this deflection be considered. The analysis of beam vibration including both the effects of rotation of the cross section and the shear deflection is called the *Timoshenko beam theory.* The following equation governs such vibration:[10]

$$a^2 \frac{\partial^4 y}{\partial x^4} + \frac{\partial^2 y}{\partial t^2} - \rho^2 \left(1 + \frac{E}{\kappa G}\right) \frac{\partial^4 y}{\partial x^2 \partial t^2} + \rho^2 \frac{\gamma}{g\kappa G} \frac{\partial^4 y}{\partial t^4} = 0 \tag{7.18}$$

where $a^2 = EIg/S\gamma$, E = modulus of elasticity, G = modulus of rigidity, and $\rho = \sqrt{I/S}$, the radius of gyration; $\kappa = F_s/GS\beta$, F_s being the total lateral shear force at any section and β the angle which a cross section makes with the axis of the beam because of shear deformation. Under the assumptions made in the usual elementary beam theory, κ is $\frac{2}{3}$ for a beam with a rectangular cross section and $\frac{3}{4}$ for a circular beam. More refined analysis shows[11] that, for the present purposes, $\kappa = \frac{5}{6}$ and $\frac{9}{10}$ are more accurate values for rectangular and circular cross sections, respectively. Using a solution of the form $y = C \sin(n\pi x/l) \cos \omega_n t$, which satisfies the necessary end conditions, the following frequency equation is obtained for beams with both ends simply supported:

$$a^2 \frac{n^4\pi^4}{l^4} - \omega_n^2 - \omega_n^2 \frac{n^2\pi^2\rho^2}{l^2} - \omega_n^2 \frac{n^2\pi^2\rho^2}{l^2} \frac{E}{\kappa G} + \frac{\rho^2\gamma}{g\kappa G} \omega_n^4 = 0 \qquad (7.18a)$$

If it is assumed that $nr/l << 1$, Eq. (7.18a) reduces to

$$\omega_n = \frac{a\pi^2}{(l/n)^2}\left[1 - \frac{\pi^2 n^2}{2}\left(\frac{\rho}{l}\right)^2\left(1 + \frac{E}{\kappa G}\right)\right] \qquad (7.18b)$$

When $nr/l < 0.08$, the approximate equation gives less than 5 percent error in the frequency.[11]

Values of the ratio of ω_n to the natural frequency uncorrected for the effects of rotation and shear have been plotted,[11] using Eq. (7.18a) for three values of $E/\kappa G$, and are shown in Fig. 7.7.

For a cantilever beam the frequency equation is quite complicated. For $E/\kappa G$ = 3.20, corresponding approximately to the value for rectangular steel or aluminum beams, the curves in Fig. 7.8 show the effects of rotation and shear on the natural frequencies of the first six modes of vibration.

EXAMPLE 7.6. The first two natural frequencies of a rectangular steel beam 40 in. long, 2 in. wide, and 6 in. thick, having simply supported ends, are to be computed with and without including the effects of rotation of the cross sections and shear deflection.

SOLUTION. For steel $E = 30 \times 10^6$ lb/in.^{2}, $G = 11.5 \times 10^6$ lb/in.^{2}, and for a rectangular cross section $\kappa = 5/6$; thus $E/\kappa G = 3.13$. For a rectangular beam $\rho = h/12$ where

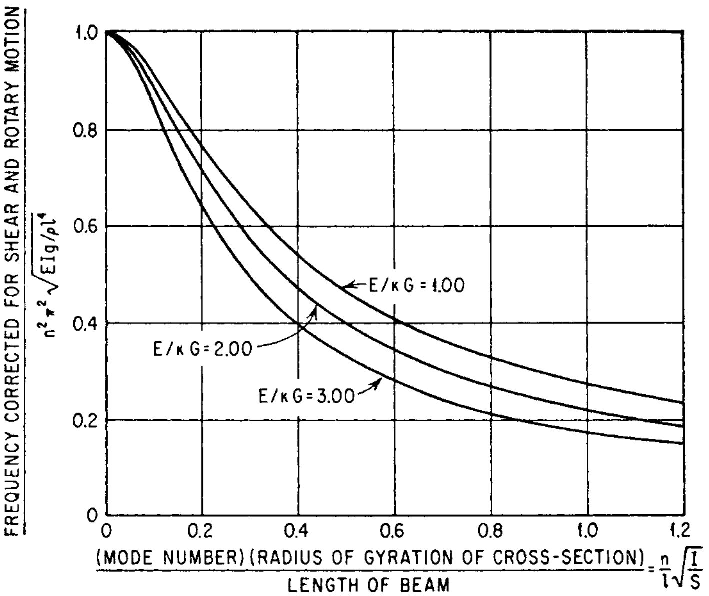


FIGURE 7.7 Influence of shear force and rotary motion on natural frequencies of simply supported beams. The curves relate the corrected frequency to that given by Eq. (7.14). (*J. G. Sutherland and L. E. Goodman.*[11])

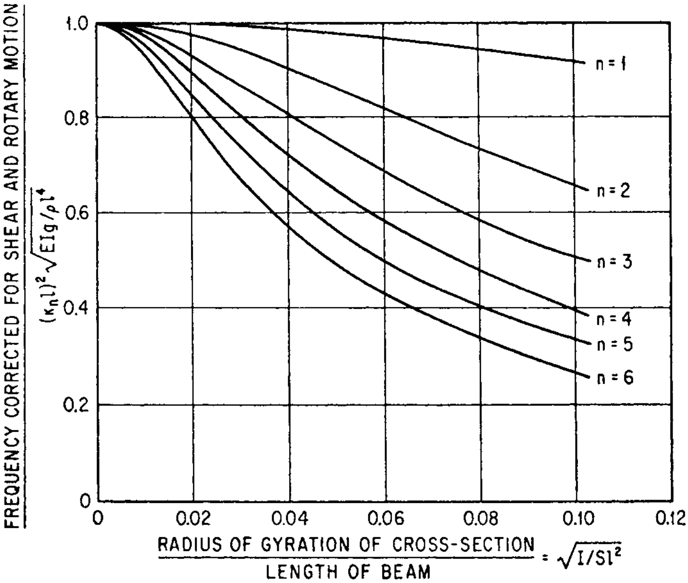


FIGURE 7.8 Influence of shear force and rotary motion on natural frequencies of uniform cantilever beams ($E/\kappa G = 3.20$). The curves relate the corrected frequency to that given by Eq. (7.14). (*J. G. Sutherland and L. E. Goodman.*[11])

h is the thickness; thus $\rho/l = 6/(40\sqrt{12}) = 0.0433$. The approximate frequency equation, Eq. (7.18*b*), becomes

$$\omega_n = \frac{a\pi^2}{(l/n)^2}\left[1 - \frac{\pi^2}{2}(0.0433n)^2(1 + 3.13)\right]$$

$$= \frac{a\pi^2}{(l/n)^2}(1 - 0.038n^2)$$

Letting $\omega_0 = a\pi^2/(l/n)^2$ be the uncorrected frequency obtained by neglecting the effect of n in Eq. (7.18*b*):

For $n = 1$: $$\frac{\omega_n}{\omega_0} = 1 - 0.038 = 0.962$$

For $n = 2$: $$\frac{\omega_n}{\omega_0} = 1 - 0.152 = 0.848$$

Comparing these results with Fig. 7.7, using the curve for $E/\kappa G = 3.00$, the calculated frequency for the first mode agrees with the curve as closely as the curve can be read. For the second mode, the curve gives $\omega_n/\omega_0 = 0.91$; therefore the approximate equation for the second mode is not very accurate. The uncorrected frequencies are, since $I/S = \rho^2 = h^2/12$,

For $n = 1$: $$\omega_0 = \frac{\pi^2}{l^2}\sqrt{\frac{EIg}{S\gamma}} = \frac{\pi^2}{(40)^2}\sqrt{\frac{(30\times 10^6)(36)386}{(12)(0.28)}} = 2170 \text{ rad/sec} = 345 \text{ Hz}$$

For $n = 2$: $\omega_0 = 345 \times 4 = 1380$ Hz

The frequencies corrected for rotation and shear, using the value from Fig. 7.7 for correction of the second mode, are:

For $n = 1$: $f_n = 345 \times 0.962 = 332$ Hz

For $n = 2$: $f_n = 1380 \times 0.91 = 1256$ Hz

Effect of Axial Loads. When an axial tensile or compressive load acts on a beam, the natural frequencies are different from those for the same beam without such load. The natural frequencies for a beam with hinged ends, as determined by an energy analysis, assuming that the axial force F remains constant, are[12]

$$\omega_n = \frac{\pi^2 n^2}{l^2}\sqrt{\frac{EIg}{S\gamma}}\sqrt{1 \pm \frac{\alpha^2}{n^2}} = \omega_0\sqrt{1 \pm \frac{\alpha^2}{n^2}}$$

where $\alpha^2 = Fl^2/EI\pi^2$, n is the mode number, ω_0 is the natural frequency of the beam with no axial force applied, and the other symbols are defined in Table 7.1. The plus sign is for a tensile force and the minus sign for a compressive force.

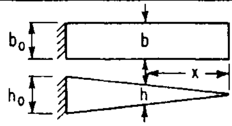
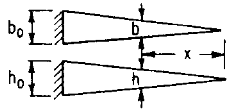
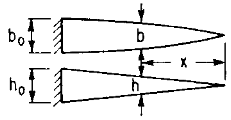
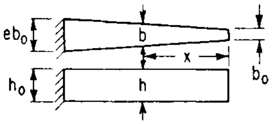
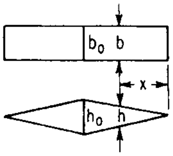
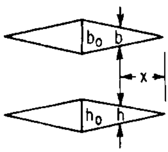
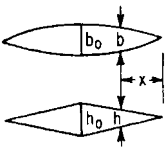
For a cantilever beam with a constant axial force F applied at the free end, the natural frequency is found by an energy analysis[13] to be $[1 + \frac{5}{14}(Fl^2/EI)]^{1/2}$ times the natural frequency of the beam without the force applied. If a uniform axial force is applied along the beam, the effect is the same as if about seven-twentieths of the total force were applied at the free end of the beam.

If the amplitude of vibration is large, an axial force may be induced in the beam by the supports. For example, if both ends of a beam are hinged but the supports are rigid enough so that they cannot move axially, a tensile force is induced as the beam deflects. The force is not proportional to the deflection; therefore, the vibration is of the type characteristic of nonlinear systems in which the natural frequency depends on the amplitude of vibration. The natural frequency of a beam having immovable hinged ends is given in the following table where the axial force is zero at zero deflection of the beam[14] and where x_0 is the amplitude of vibration, I the moment of inertia, and S the area of the cross section; ω_0 is the natural frequency of the unrestrained bar.

$\frac{x_0}{\sqrt{I/S}}$	0	0.1	0.2	0.4	0.6	0.8
$\frac{\omega_n}{\omega_0}$	1	1.0008	1.0038	1.015	1.038	1.058
$\frac{x_0}{\sqrt{I/S}}$	1.0	1.5	2	3	4	5
$\frac{\omega_n}{\omega_0}$	1.089	1.190	1.316	1.626	1.976	2.35

Beams Having Variable Cross Sections. The natural frequencies for beams of several shapes having cross sections that can be expressed as functions of the distance along the beam have been calculated.[15] The results are shown in Table 7.4. In the analysis, Eq. (7.13) was used, with EI considered to be variable.

TABLE 7.4 Natural Frequencies of Variable-Section Steel Beams (*J. N. Macduff and R. P. Felgar.*[16,17])

BEAM STRUCTURE	$(f_n l^2/\rho)/10^4$				
	$\frac{b}{b_0}$	$\frac{h}{h_0}$	n = 1	n = 2	n = 3
	1	$\frac{x}{l}$	17.09	48.89	96.57
	$\frac{x}{l}$	$\frac{x}{l}$	26.08	68.08	123.64
	$\left(\frac{x}{l}\right)^{\frac{1}{2}}$	$\frac{x}{l}$	22.30	58.18	109.90
	$e^{x/l}$	1	15.23	77.78	206.07
	1	$\frac{x}{l}$	21.21* 35.05†	56.97	
	$\frac{x}{l}$	$\frac{x}{l}$	32.73* 49.50†	76.57	
	$\left(\frac{x}{l}\right)^{\frac{1}{2}}$	$\frac{x}{l}$	25.66* 42.02†	66.06	

* SYMMETRIC † ANTISYMMETRIC

f_n = natural frequency, Hz
$\rho = \sqrt{I/S}$ = radius of gyration, in.
h = depth of beam, in.
l = beam length, in.
n = mode number
b = width of beam, in.

For materials other than steel: $f_n = f_{ns}\sqrt{\frac{E\gamma_s}{E_s\gamma}}$

E = modulus of elasticity, lb/in.^{2}
γ = density, lb/in.^{3}
Terms with subscripts refer to steel
Terms without subscripts refer to other material

Rayleigh's or Ritz's method can be used to find approximate values for the frequencies of such beams. The frequency equation becomes, using the equations in Table 7.1, and letting $Y(x)$ be the assumed deflection,

$$\omega_n^2 = \frac{Eg}{\gamma} \frac{\int_0^l I\,(d^2Y/dx^2)^2\,dx}{\int_0^l SY^2\,dx}$$

where $I = I(x)$ is the moment of inertia of the cross section and $S = S(x)$ is the area of the cross section. Examples of the calculations are in the literature.[18] If the values of $I(x)$ and $S(x)$ cannot be defined analytically, the beam may be divided into two or more sections, for each of which I and S can be approximated by an equation. The strain and kinetic energies of each section may be computed separately, using an appropriate function for the deflection, and the total energies for the beam found by adding the values for the individual sections.

Continuous Beams on Multiple Supports. In finding the natural frequencies of a beam on multiple supports, the section between each pair of supports is considered as a separate beam with its origin at the left support of the section. Equation (7.16) applies to each section. Since the deflection is zero at the origin of each section, $A = 0$ and the equation reduces to

$$X = B(\cos \kappa x - \cosh \kappa x) + C(\sin \kappa x + \sinh \kappa x) + D(\sin \kappa x - \sinh \kappa x)$$

There is one such equation for each section, and the necessary end conditions are as follows:

1. At each end of the beam the usual boundary conditions are applicable, depending on the type of support.
2. At each intermediate support the deflection is zero. Since the beam is continuous, the slope and the moment just to the left and to the right of the support are the same.

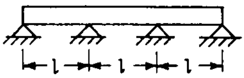
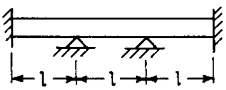
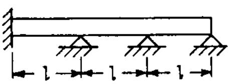
General equations can be developed for finding the frequency for any number of spans.[19,20] Table 7.5 gives constants for finding the natural frequencies of uniform continuous beams on uniformly spaced supports for several combinations of end supports.

Beams with Partly Clamped Ends. For a beam in which the slope at each end is proportional to the moment, the following empirical equation gives the natural frequency:[21]

$$f_n = f_0 \left[n + \frac{1}{2} \left(\frac{\beta_L}{5n + \beta_L} \right) \right] \left[n + \frac{1}{2} \left(\frac{\beta_R}{5n + \beta_R} \right) \right]$$

where f_0 is the frequency of the same beam with simply supported ends and n is the mode number. The parameters $\beta_L = k_L l/EI$ and $\beta_R = k_R l/EI$ are coefficients in which k_L and k_R are stiffnesses of the supports as given by $k_L = M_L/\theta_L$, where M_L is the moment and θ_L the angle at the left end, and $k_R = M_R/\theta_R$, where M_R is the moment and θ_R the angle at the right end. The error is less than 2 percent except for bars having one end completely or nearly clamped ($\beta > 10$) and the other end completely or nearly hinged ($\beta < 0.9$).

TABLE 7.5 Natural Frequencies of Continuous Uniform Steel* Beams (*J. N. Macduff and R. P. Felgar.*[16,17])

Beam structure	$(f_n l^2/\rho)/10^4$					
	N	$n = 1$	$n = 2$	$n = 3$	$n = 4$	$n = 5$
	Extreme Ends Simply Supported					
	1	31.73	126.94	285.61	507.76	793.37
	2	31.73	49.59	126.94	160.66	285.61
	3	31.73	40.52	59.56	126.94	143.98
	4	31.73	37.02	49.59	63.99	126.94
	5	31.73	34.99	44.19	55.29	66.72
	6	31.73	34.32	40.52	49.59	59.56
	7	31.73	33.67	38.40	45.70	53.63
	8	31.73	33.02	37.02	42.70	49.59
	9	31.73	33.02	35.66	40.52	46.46
	10	31.73	33.02	34.99	39.10	44.19
	11	31.73	32.37	34.32	37.70	41.97
	12	31.73	32.37	34.32	37.02	40.52
	Extreme Ends Clamped					
	1	72.36	198.34	388.75	642.63	959.98
	2	49.59	72.36	160.66	198.34	335.20
	3	40.52	59.56	72.36	143.98	178.25
	4	37.02	49.59	63.99	72.36	137.30
	5	34.99	44.19	55.29	66.72	72.36
	6	34.32	40.52	49.59	59.56	67.65
	7	33.67	38.40	45.70	53.63	62.20
	8	33.02	37.02	42.70	49.59	56.98
	9	33.02	35.66	40.52	46.46	52.81
	10	33.02	34.99	39.10	44.19	49.59
	11	32.37	34.32	37.70	41.97	47.23
	12	32.37	34.32	37.02	40.52	44.94
	Extreme Ends Clamped-Supported					
	1	49.59	160.66	335.2	573.21	874.69
	2	37.02	63.99	137.30	185.85	301.05
	3	34.32	49.59	67.65	132.07	160.66
	4	33.02	42.70	56.98	69.51	129.49
	5	33.02	39.10	49.59	61.31	70.45
	6	32.37	37.02	44.94	54.46	63.99
	7	32.37	35.66	41.97	49.59	57.84
	8	32.37	34.99	39.81	45.70	53.63
	9	31.73	34.32	38.40	43.44	49.59
	10	31.73	33.67	37.02	41.24	46.46
	11	31.73	33.67	36.33	39.81	44.19
	12	31.73	33.02	35.66	39.10	42.70

* For materials other than steel, use equation at bottom of Table 7.4.
f_n = natural frequency, Hz; n = mode number
$\rho = \sqrt{I/S}$ = radius of gyration, in.; N = number of spans
l = span length, in.

LATERAL VIBRATION OF BEAMS WITH MASSES ATTACHED

The use of Fig. 7.4 is a convenient method of estimating the natural frequencies of beams with added loads.

Exact Solution. If the masses attached to the beam are considered to be rigid so that they exert no elastic forces, and if it is assumed that the attachment is such that the bending of the beam is not restrained, Eqs. (7.13) and (7.16) apply. The section of the beam between each two masses, and between each support and the adjacent mass, must be considered individually. The constants in Eq. (7.16) are different for each section. There are $4N$ constants, N being the number of sections into which the beam is divided. Each support supplies two boundary conditions. Additional conditions are provided by:

1. The deflection at the location of each mass is the same for both sections adjacent to the mass.
2. The slope at each mass is the same for each section adjacent thereto.
3. The change in the lateral elastic shear force in the beam, at the location of each mass, is equal to the product of the mass and its acceleration $\ddot{y}$.
4. The change of moment in the beam, at each mass, is equal to the product of the moment of inertia of the mass and its angular acceleration $(\partial^2/\partial t^2)(\partial y/\partial x)$.

Setting up the necessary equations is not difficult, but their solution is a lengthy process for all but the simplest configurations. Even the solution of the problem of a beam with hinged ends supporting a mass with negligible moment of inertia located anywhere except at the center of the beam is fairly long. If the mass is at the center of the beam, the solution is relatively simple because of symmetry and is illustrated to show how the result compares with that obtained by Rayleigh's method.

Rayleigh's Method. Rayleigh's method offers a practical method of obtaining a fairly accurate solution of the problem, even when more than one mass is added. In carrying out the solution, the kinetic energy of the masses is added to that of the beam. The strain and kinetic energies of a uniform beam are given in Table 7.1. The kinetic energy of the ith mass is $(m_i/2)\omega_n^2 Y^2(x_i)$, where $Y(x_i)$ is the value of the amplitude at the location of mass. Equating the maximum strain energy to the total maximum kinetic energy of the beam and masses, the frequency equation becomes

$$\omega_n^2 = \frac{EI \int_0^l (Y'')^2\,dx}{\dfrac{\gamma S}{g} \int_0^l Y^2\,dx + \sum_{i=1}^{n} m_i Y^2(x_i)} \tag{7.19}$$

where $Y(x)$ is the maximum deflection. If $Y(x)$ were known exactly, this equation would give the correct frequency; however, since Y is not known, a shape must be assumed. This may be either the mode shape of the unloaded beam or a polynomial that satisfies the necessary boundary conditions, such as the equation for the static deflection under a load.

Beam as Spring. A method for obtaining the natural frequency of a beam with a single mass mounted on it is to consider the beam to act as a spring, the stiffness of which is found by using simple beam theory. The equation $\omega_n = \sqrt{k/m}$ is used. Best accuracy is obtained by considering m to be made up of the attached mass plus some portion of the mass of the beam. The fraction of the beam mass to be used depends on the type of beam. The equations for simply supported and cantilevered beams with masses attached are given in Table 7.2.

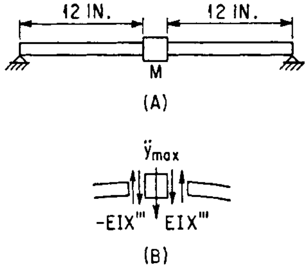


FIGURE 7.9 (*A*) Beam having simply supported ends with mass attached at center. (*B*) Forces exerted on mass, at extreme deflection, by shear stresses in beam.

EXAMPLE 7.7. The fundamental natural frequencies of a beam with hinged ends 24 in. long, 2 in. wide, and ¼ in. thick having a mass m attached at the center (Fig. 7.9) are to be calculated by each of the three methods, and the results compared for ratios of mass to beam mass of 1, 5, and 25. The result is to be compared with the frequency from Fig. 7.4.

EXACT SOLUTION. Because of symmetry, only the section of the beam to the left of the mass has to be considered in carrying out the exact solution. The boundary conditions for the left end are: at $x = 0$, $X = 0$, and $X'' = 0$. The shear force just to the left of the mass is negative at maximum deflection (Fig. 7.9*B*) and is $F_s = -EIX'''$; to the right of the mass, because of symmetry, the shear force has the same magnitude with opposite sign. The difference between the shear forces on the two sides of the mass must equal the product of the mass and its acceleration. For the condition of maximum deflection,

$$2EIX''' = m\ddot{y}_{\max} \tag{7.20}$$

where X''' and $\ddot{y}_{\max}$ must be evaluated at $x = l/2$. Because of symmetry the slope at the center is zero. Using the solution $y = X \cos \omega_n t$ and $\ddot{y}_{\max} = -\omega_n^2 X$, Eq. (7.20) becomes

$$2EIX''' = -m\omega_n^2 X \tag{7.21}$$

The first boundary condition makes $A = 0$ in Eq. (7.16) and the second condition makes $B = 0$. For simplicity, the part of the equation that remains is written

$$X = C \sin \kappa x + D \sinh \kappa x \tag{7.22}$$

Using this in Eq. (7.20) gives

$$2EI\left(-C\kappa^3 \cos \frac{\kappa l}{2} + D\kappa^3 \cosh \frac{\kappa l}{2}\right) = -m\omega_n^2 \left(C \sin \frac{\kappa l}{2} + D \sinh \frac{\kappa l}{2}\right) \tag{7.23}$$

The slope at the center is zero. Differentiating Eq. (7.22) and substituting $x = l/2$,

$$\kappa\left(C \cos \frac{\kappa l}{2} + D \cosh \frac{\kappa l}{2}\right) = 0 \tag{7.24}$$

Solving Eqs. (7.23) and (7.24) for the ratio C/D and equating, the following frequency equation is obtained:

$$2\,\frac{m_b}{m} = \frac{\kappa l}{2}\left(\tan \frac{\kappa l}{2} - \tanh \frac{\kappa l}{2}\right)$$

where $m_b = \gamma Sl/g$ is the total mass of the beam. The lowest roots for the specified ratios m/m_b are as follows:

m/m_b	1	5	25
$\kappa l/2$	1.1916	0.8599	0.5857

The corresponding natural frequencies are found from Eq. (7.14) and are tabulated, with the results obtained by the other methods, at the end of the example.

Solution by Rayleigh's Method. For the solution by Rayleigh's method it is assumed that $Y = B \sin(\pi x/l)$. This is the fundamental mode for the unloaded beam (Table 7.3). The terms in Eq. (7.19) become

$$\int_0^l (Y'')^2\, dx = B^2 \left(\frac{\pi}{l}\right)^4 \int_0^l \sin^2 \frac{\pi x}{l}\, dx = B^2 \frac{l}{2} \left(\frac{\pi}{l}\right)^4$$

$$\int_0^l Y^2\, dx = B^2 \int_0^l \sin^2 \frac{\pi x}{l}\, dx = B^2 \frac{l}{2}$$

$$Y^2(x_1) = B^2$$

Substituting these terms, Eq. (7.19) becomes

$$\omega_n = \sqrt{\frac{EIB^2(l/2)(\pi/l)^4}{(S\gamma B^2 l/2g) + mB^2}} = \frac{\pi^2}{\sqrt{1 + 2m/m_b}} \sqrt{\frac{EIg}{S\gamma l^4}}$$

The frequencies for the specified values of m/m_b are tabulated at the end of the example. Note that if $m = 0$, the frequency is exactly correct, as can be seen from Table 7.3. This is to be expected since, if no mass is added, the assumed shape is the true shape.

Lumped Parameter Solution. Using the appropriate equation from Table 7.2, the natural frequency is

$$\omega_n = \sqrt{\frac{48EI}{l^3(m + 0.5m_b)}}$$

Since $m_b = \gamma Sl/g$, this becomes

$$\omega_n = \sqrt{\frac{48}{(m/m_b) + 0.5}} \sqrt{\frac{EIg}{S\gamma l^4}}$$

Comparison of Results. The results for each method can be expressed as a coefficient α multiplied by $\sqrt{EIg/S\gamma l^4}$. The values of α for the specified values by m/m_b for the three methods of solution are:

m/m_b	1	5	25
Exact	5.680	2.957	1.372
Rayleigh	5.698	2.976	1.382
Spring	5.657	2.954	1.372

The results obtained by all the methods agree closely. For large values of m/m_b the third method gives very accurate results.

Numerical Calculations. For steel, $E = 30 \times 10^6$ lb/in.^{2}, $\gamma = 0.28$ lb/in.^{3}; for a rectangular beam, $I = bh^3/12 = 1/384$ in.^{4} and $S = bh = ½$ in.^{2}. The fundamental frequency using the value of α for the exact solution when $m/m_b = 1$ is

$$\omega_1 = \frac{\alpha}{l^2} \sqrt{\frac{EIg}{S\gamma}} = \frac{5.680}{576} \sqrt{\frac{(30 \times 10^6)(386)}{(0.5)(384)(0.28)}} = 145 \text{ rad/sec} = 23 \text{ Hz}$$

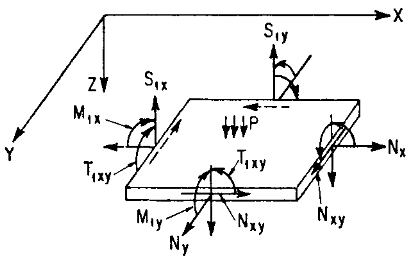


FIGURE 7.10 Element of plate showing bending moments, normal forces, and shear forces.

Other frequencies can be found by using the other values of α. Nearly the same result is obtained by using Fig. 7.4, if half the mass of the beam is added to the additional mass.

LATERAL VIBRATION OF PLATES

General Theory of Bending of Rectangular Plates. For small deflections of an initially flat plate of uniform thickness (Fig. 7.10) made of homogeneous isotropic material and subjected to normal and shear forces in the plane of the plate, the following equation relates the lateral deflection w to the lateral loading:[22]

$$D\nabla^4 w = D\left(\frac{\partial^4 w}{\partial x^4} + 2\,\frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4}\right) = P + N_x\,\frac{\partial^2 w}{\partial x^2} + 2N_{xy}\,\frac{\partial^2 w}{\partial x\,\partial y} + N_y\,\frac{\partial^2 w}{\partial y^2} \tag{7.25}$$

where $D = Eh^3/12(1 - \mu^2)$ is the plate stiffness, h being the plate thickness and μ Poisson's ratio. The parameter P is the loading intensity, N_x the normal loading in the X direction per unit of length, N_y the normal loading in the Y direction, and N_{xy} the shear load parallel to the plate surface in the X and Y directions.

The bending moments and shearing forces are related to the deflection w by the following equations:[23]

$$M_{1x} = -D\left(\frac{\partial^2 w}{\partial x^2} + \mu\,\frac{\partial^2 w}{\partial y^2}\right) \qquad M_{1y} = -D\left(\frac{\partial^2 w}{\partial y^2} + \mu\,\frac{\partial^2 w}{\partial x^2}\right)$$

$$T_{1xy} = D(1-\mu)\,\frac{\partial^2 w}{\partial x\,\partial y} \tag{7.26}$$

$$S_{1x} = -D\left(\frac{\partial^3 w}{\partial x^3} + \frac{\partial^3 w}{\partial x\,\partial y^2}\right) \qquad S_{1y} = -D\left(\frac{\partial^3 w}{\partial y^3} + \frac{\partial^3 w}{\partial x^2\,\partial y}\right)$$

As shown in Fig. 7.10, M_{1x} and M_{1y} are the bending moments per unit of length on the faces normal to the X and Y directions, respectively, T_{1xy} is the twisting or warping moment on these faces, and S_{1x}, S_{1y} are the shearing forces per unit of length normal to the plate surface.

The boundary conditions that must be satisfied by an edge parallel to the X axis, for example, are as follows:
Built-in edge:

$$w = 0 \qquad \frac{\partial w}{\partial y} = 0$$

Simply supported edge:

$$w = 0 \qquad M_{1y} = -D\left(\frac{\partial^2 w}{\partial y^2} + \mu\,\frac{\partial^2 w}{\partial x^2}\right) = 0$$

Free edge:

$$M_{1y} = -D\left(\frac{\partial^2 w}{\partial y^2} + \mu \frac{\partial^2 w}{\partial x^2}\right) = 0 \qquad T_{1xy} = 0 \qquad S_{1y} = 0$$

which together give

$$\frac{\partial}{\partial y}\left[\frac{\partial^2 w}{\partial y^2} + (2 - \mu)\frac{\partial^2 w}{\partial x^2}\right] = 0$$

Similar equations can be written for other edges. The strains caused by the bending of the plate are

$$\epsilon_x = -z\frac{\partial^2 w}{\partial x^2} \qquad \epsilon_y = -z\frac{\partial^2 w}{\partial y^2} \qquad \gamma_{xy} = 2z\frac{\partial^2 w}{\partial x\,\partial y} \tag{7.27}$$

where z is the distance from the center plane of the plate.

Hooke's law may be expressed by the following equations:

$$\begin{aligned}
\epsilon_x &= \frac{1}{E}(\sigma_x - \mu\sigma_y) & \sigma_x &= \frac{E}{1-\mu^2}(\epsilon_x + \mu\epsilon_y) \\
\epsilon_y &= \frac{1}{E}(\sigma_y - \mu\sigma_x) & \sigma_y &= \frac{E}{1-\mu^2}(\epsilon_y + \mu\epsilon_x) \\
\gamma_{xy} &= \frac{\tau_{xy}}{G} & \tau_{xy} &= G\gamma_{xy}
\end{aligned} \tag{7.28}$$

Substituting the expressions giving the strains in terms of the deflections, the following equations are obtained for the bending stresses in terms of the lateral deflection:

$$\begin{aligned}
\sigma_x &= -\frac{Ez}{1-\mu^2}\left(\frac{\partial^2 w}{\partial x^2} + \mu\frac{\partial^2 w}{\partial y^2}\right) = \frac{12M_{1x}}{h^3}z \\
\sigma_y &= -\frac{Ez}{1-\mu^2}\left(\frac{\partial^2 w}{\partial y^2} + \mu\frac{\partial^2 w}{\partial x^2}\right) = \frac{12M_{1y}}{h^3}z \\
\tau_{xy} &= 2G\frac{\partial^2 w}{\partial x\,\partial y}z = \frac{12T_{1xy}}{h^3}z
\end{aligned} \tag{7.29}$$

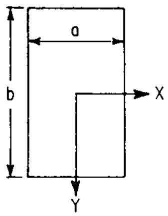
Table 7.6 gives values of maximum deflection and bending moment at several points in plates which have various shapes and conditions of support and which are subjected to uniform lateral pressure. The results are all based on the assumption that the deflections are small and that there are no loads in the plane of the plate. The bending stresses are found by the use of Eqs. (7.29). Bending moments and deflections for many other types of load are in the literature.[22]

The stresses caused by loads in the plane of the plate are found by assuming that the stress is uniform through the plate thickness. The total stress at any point in the plate is the sum of the stresses caused by bending and by the loading in the plane of the plate.

For plates in which the lateral deflection is large compared to the plate thickness but small compared to the other dimensions, Eq. (7.25) is valid. However, additional equations must be introduced because the forces N_x, N_y, and N_{xy} depend not only on the initial loading of the plate but also upon the stretching of the plate due to the

TABLE 7.6 Maximum Deflection and Bending Moments in Uniformly Loaded Plates under Static Conditions

RECTANGULAR PLATES



$\alpha = w_{MAX}/(Pa^4/Eh^3)$

$\beta = M_{1x}/Pa^2$

$\gamma = M_{1y}/Pa^2$

w = LATERAL DEFLECTION

P = UNIFORM PRESSURE

h = PLATE THICKNESS

E = MODULUS OF ELASTICITY

μ = POISSON'S RATIO

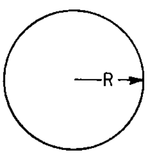
SIMPLY SUPPORTED EDGES ($\mu = 0.3$)[24]

b/a	1	1.2	1.4	1.6	1.8	2.0	3.0	∞
$(\alpha)_{x=0,\ y=0}$	0.044	0.062	0.077	0.091	0.102	0.111	0.134	0.142
$(\beta)_{x=0,\ y=0}$	0.048	0.063	0.075	0.086	0.095	0.102	0.119	0.125
$(\gamma)_{x=0,\ y=0}$	0.048	0.050	0.051	0.049	0.048	0.046	0.040	0.038

BUILT-IN EDGES ($\mu = 0.3$)[25]

b/a	1	1.2	1.4	1.6	1.8	2.0
$(\alpha)_{x=0,\ y=0}$	0.014	0.019	0.023	0.025	0.027	0.028
$(\beta)_{x=a/2,\ y=0}$	-0.051	-0.064	-0.073	-0.078	-0.081	-0.083
$(\gamma)_{x=0,\ y=b/2}$	-0.051	-0.055	-0.057	-0.057	-0.057	-0.057
$(\beta)_{x=0,\ y=0}$		0.030	0.035	0.038	0.040	
$(\gamma)_{x=0,\ y=0}$		0.023	0.021	0.019	0.017	

CIRCULAR PLATES[26]



M_r = RADIAL MOMENT

M_t = TANGENTIAL MOMENT

$$D = \frac{Eh^3}{12(1-\mu^2)}$$

	SIMPLY SUPPORTED EDGES	BUILT-IN EDGES	
	CENTER	CENTER	EDGE
$w/(PR^4/D)$	$\frac{5+\mu}{64(1+\mu)}$	$\frac{1}{64}$	0
M_r/PR^2	$\frac{3+\mu}{16}$	$\frac{1+\mu}{16}$	$-\frac{1}{8}$
M_t/PR^2	$\frac{3+\mu}{16}$	$\frac{1+\mu}{16}$	$-\frac{\mu}{8}$

bending. The equations of equilibrium for the X and Y directions in the plane of the plate are

$$\frac{\partial N_x}{\partial x} + \frac{\partial N_{xy}}{\partial y} = 0 \qquad \frac{\partial N_{xy}}{\partial x} + \frac{\partial N_y}{\partial y} = 0 \tag{7.30}$$

It can be shown[27] that the strain components are given by

$$\epsilon_x = \frac{\partial u}{\partial x} + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2 \qquad \epsilon_y = \frac{\partial v}{\partial y} + \frac{1}{2}\left(\frac{\partial w}{\partial y}\right)^2$$
$$\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} + \frac{\partial w}{\partial x}\frac{\partial w}{\partial y} \tag{7.31}$$

where u is the displacement in the X direction and v is the displacement in the Y direction. By differentiating and combining these expressions, the following relation is obtained:

$$\frac{\partial^2 \epsilon_x}{\partial y^2} + \frac{\partial^2 \epsilon_y}{\partial x^2} - \frac{\partial^2 \gamma_{xy}}{\partial x\,\partial y} = \left(\frac{\partial^2 w}{\partial x\,\partial y}\right)^2 - \frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2} \tag{7.32}$$

If it is assumed that the stresses caused by the forces in the plane of the plate are uniformly distributed through the thickness, Hooke's law, Eqs. (7.28), can be expressed:

$$\epsilon_x = \frac{1}{hE}(N_x - \mu N_y) \qquad \epsilon_y = \frac{1}{hE}(N_y - \mu N_x) \qquad \gamma_{xy} = \frac{1}{hG}N_{xy} \tag{7.33}$$

The equilibrium equations are satisfied by a stress function ϕ which is defined as follows:

$$N_x = h\frac{\partial^2 \phi}{\partial y^2} \qquad N_y = h\frac{\partial^2 \phi}{\partial x^2} \qquad N_{xy} = -h\frac{\partial^2 \phi}{\partial x\,\partial y} \tag{7.34}$$

If these are substituted into Eqs. (7.33) and the resulting expressions substituted into Eq. (7.32), the following equation is obtained:

$$\frac{\partial^4 \phi}{\partial x^4} + 2\frac{\partial^4 \phi}{\partial x^2\,\partial y^2} + \frac{\partial^4 \phi}{\partial y^4} = E\left[\left(\frac{\partial^2 w}{\partial x\,\partial y}\right)^2 - \frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2}\right] \tag{7.35}$$

A second equation is obtained by substituting Eqs. (7.34) in Eq. (7.25):

$$D\nabla^4 w = P + h\left(\frac{\partial^2 \phi}{\partial y^2}\frac{\partial^2 w}{\partial x^2} - 2\frac{\partial^2 \phi}{\partial x\,\partial y}\frac{\partial^2 w}{\partial x\,\partial y} + \frac{\partial^2 \phi}{\partial x^2}\frac{\partial^2 w}{\partial y^2}\right) \tag{7.36}$$

Equations (7.35) and (7.36), with the boundary conditions, determine ϕ and w, from which the stresses can be computed. General solutions to this set of equations are not known, but some approximate solutions can be found in the literature.[28]

Free Lateral Vibrations of Rectangular Plates. In Eq. (7.25), the terms on the left are equal to the sum of the rates of change of the forces per unit of length in the X and Y directions where such forces are exerted by shear stresses caused by bending normal to the plane of the plate. For a rectangular element with dimensions dx and dy, the net force exerted normal to the plane of the plate by these stresses is $D\nabla^4 w\, dx\, dy$. The last three terms on the right-hand side of Eq. (7.25) give the net force normal to the plane of the plate, per unit of length, which is caused by the forces acting in the plane of the plate. The net force caused by these forces on an element with dimensions dx and dy is $(N_x\, \partial^2 w/\partial x^2 + 2N_{xy}\, \partial^2 w/\partial x\, \partial y + N_y\, \partial^2 w/\partial y^2)\, dx\, dy$. As in the corresponding beam problem, the forces in a vibrating plate consist of two parts: (1) that which balances the static load P including the weight of the plate and (2) that which is induced by the vibration. The first part is always in equilibrium with the load and together with the load can be omitted from the equation of motion if the deflection is taken from the position of static equilibrium. The force exerted normal to the plane of the plate by the bending stresses must equal the sum of the force exerted normal to the plate by the loads acting in the plane of the plate; i.e., the product of the mass of the element $(\gamma h/g)\, dx\, dy$ and its acceleration $\ddot{w}$. The term involving the acceleration of the element is negative, because when the bending force is positive the acceleration is in the negative direction. The equation of motion is

$$D\nabla^4 w = -\frac{\gamma}{g} h\ddot{w} + \left(N_x \frac{\partial^2 w}{\partial x^2} + 2N_{xy} \frac{\partial^2 w}{\partial x\, \partial y} + N_y \frac{\partial^2 w}{\partial y^2}\right) \tag{7.37}$$

This equation is valid only if the magnitudes of the forces in the plane of the plate are constant during the vibration. For many problems these forces are negligible and the term in parentheses can be omitted.

When a system vibrates in a natural mode, all parts execute simple harmonic motion about the equilibrium position; therefore, the solution of Eq. (7.37) can be written as $w = AW(x,y) \cos (w_n t + \theta)$ in which W is a function of x and y only. Substituting this in Eq. (7.37) and dividing through by $A \cos (w_n t + \theta)$ gives

$$D\nabla^4 W = \frac{\gamma h \omega_n^2}{g} W + \left(N_x \frac{\partial^2 W}{\partial x^2} + 2N_{xy} \frac{\partial^2 W}{\partial x\, \partial y} + N_y \frac{\partial^2 W}{\partial y^2}\right) \tag{7.38}$$

The function W must satisfy Eq. (7.38) as well as the necessary boundary conditions.

The solution of the problem of the lateral vibration of a rectangular plate with all edges simply supported is relatively simple; in general, other combinations of edge conditions require the use of other methods of solution. These are discussed later.

EXAMPLE 7.8. The natural frequencies and normal modes of small vibration of a rectangular plate of length a, width b, and thickness h are to be calculated. All edges are hinged and subjected to unchanging normal forces N_x and N_y.

SOLUTION. The following equation, in which m and n may be any integers, satisfies the necessary boundary conditions:

$$W = A \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} \tag{7.39}$$

Substituting the necessary derivatives into Eq. (7.38),

$$D\left[\left(\frac{m}{a}\right)^4 + 2\left(\frac{m}{a}\right)^2\left(\frac{n}{b}\right)^2 + \left(\frac{n}{b}\right)^4\right]\pi^4 \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b}$$
$$= \frac{\gamma h \omega_n^2}{g} \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} - \pi^2\left[N_x\left(\frac{m}{a}\right)^2 + N_y\left(\frac{n}{b}\right)^2\right] \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b}$$

Solving for ω_n^2,

$$\omega_n^2 = \frac{g}{\gamma h}\left\{\pi^4 D\left[\left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2\right]^2 + \pi^2\left[N_x\left(\frac{m}{a}\right)^2 + N_y\left(\frac{n}{b}\right)^2\right]\right\} \tag{7.40}$$

By using integral values of m and n, the various frequencies are obtained from Eq. (7.40) and the corresponding normal modes from Eq. (7.39). For each mode, m and

n represent the number of half sine waves in the X and Y directions, respectively. In each mode there are $m - 1$ evenly spaced nodal lines parallel to the Y axis, and $n - 1$ parallel to the X axis.

Rayleigh's and Ritz's Methods. The modes of vibration of a rectangular plate with all edges simply supported are such that the deflection of each section of the plate parallel to an edge is of the same form as the deflection of a beam with both ends simply supported. In general, this does not hold true for other combinations of edge conditions. For example, the vibration of a rectangular plate with all edges built in does not occur in such a way that each section parallel to an edge has the same shape as does a beam with both ends built in. A function that is made up using the mode shapes of beams with built-in ends obviously satisfies the conditions of zero deflection and slope at all edges, but it cannot be made to satisfy Eq. (7.38).

The mode shapes of beams give logical functions with which to formulate shapes for determining the natural frequencies, for plates having various edge conditions, by the Rayleigh or Ritz methods. By using a single mode function in Rayleigh's method an approximate frequency can be determined. This can be improved by using more than one of the modal shapes and using Ritz's method as discussed below.

The strain energy of bending and the kinetic energy for plates are given in Table 7.1. Finding the maximum values of the energies, equating them, and solving for ω_n^2 gives the following frequency equation:

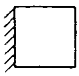
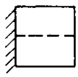
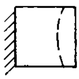
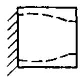
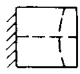
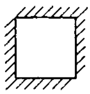
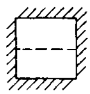
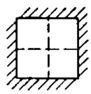
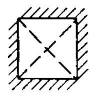
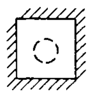
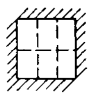
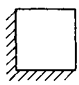
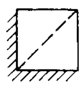
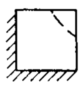
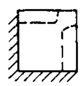
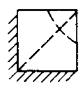
$$\omega_n^2 = \frac{V_{max}}{\dfrac{\gamma h}{2g} \int_A \int W^2 \, dx \, dy} \tag{7.41}$$

where V is the strain energy.

In applying the Rayleigh method, a function W is assumed that satisfies the necessary boundary conditions of the plate. An example of the calculations is given in the section on circular plates. If the shape assumed is exactly the correct one, Eq. (7.41) gives the exact frequency. In general, the correct shape is not known and a frequency greater than the natural frequency is obtained. The Ritz method involves assuming W to be of the form $W = a_1 W_1(x,y) + a_2 W_2(x,y) + \cdots$ in which $W_1, W_2, \ldots$ all satisfy the boundary conditions, and $a_1, a_2, \ldots$ are adjusted to give a minimum frequency. Reference 29 is an extensive compilation, with references to sources, of calculated and experimental results for plates of many shapes. Some examples are cited in the following sections.

Square, Rectangular, and Skew Rectangular Plates. Tables of the functions necessary for the determination of the natural frequencies of rectangular plates by the use of the Ritz method are available,[30] these having been derived by using the modal shapes of beams having end conditions corresponding to the edge conditions of the plates. Information is included from which the complete shapes of the vibrational modes can be determined. Frequencies and nodal patterns for several modes of vibration of square plates having three sets of boundary conditions are shown in Table 7.7. By the use of functions which represent the natural modes of beams, the frequencies and nodal patterns for rectangular and skew cantilever plates have been determined[31] and are shown in Table 7.8. Comparison of calculated frequencies with experimentally determined values shows good agreement. Natural frequencies of rectangular plates having other boundary conditions are given in Table 7.9.

TABLE 7.7 Natural Frequencies and Nodal Lines of Square Plates with Various Edge Conditions (*After D. Young.*[29])

	1ST MODE	2ND MODE	3RD MODE	4TH MODE	5TH MODE	6TH MODE
$\omega_n / \sqrt{Dg/\gamma h a^4}$	3.494	8.547	21.44	27.46	31.17	
NODAL LINES						
$\omega_n / \sqrt{Dg/\gamma h a^4}$	35.99	73.41	108.27	131.64	132.25	165.15
NODAL LINES						
$\omega_n / \sqrt{Dg/\gamma h a^4}$	6.958	24.08	26.80	48.05	63.14	
NODAL LINES						

$\omega_n = 2\pi f_n$ h = PLATE THICKNESS

$D = Eh^3/12(1-\mu^2)$ a = PLATE LENGTH

γ = WEIGHT DENSITY

Triangular and Trapezoidal Plates. Nodal patterns and natural frequencies for triangular plates have been determined[33] by the use of functions derived from the mode shapes of beams, and are shown in Table 7.10. Certain of these have been compared with experimental values and the agreement is excellent. Natural frequencies and nodal patterns have been determined experimentally for six modes of vibration of a number of cantilevered triangular plates[34] and for the first six modes of cantilevered trapezoidal plates derived by trimming the tips of triangular plates parallel to the clamped edge.[35] These triangular and trapezoidal shapes approximate the shapes of various delta wings for aircraft and of fins for missiles.

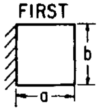
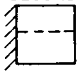
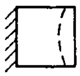
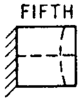
Circular Plates. The solution of the problem of small lateral vibration of circular plates is obtained by transforming Eq. (7.38) to polar coordinates and finding the solution that satisfies the necessary boundary conditions of the resulting equation. Omitting the terms involving forces in the plane of the plate,[36]

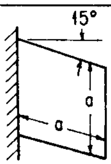
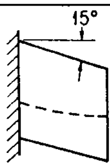
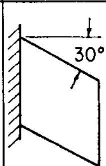
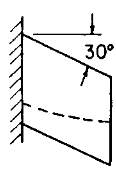
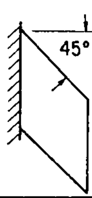
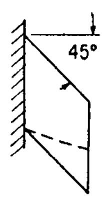
$$\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r}\frac{\partial^2}{\partial \theta^2}\right)\left(\frac{\partial^2 W}{\partial r^2} + \frac{1}{r}\frac{\partial W}{\partial r} + \frac{1}{r}\frac{\partial^2 W}{\partial \theta^2}\right) = \kappa^4 W \tag{7.42}$$

where

$$\kappa^4 = \frac{\gamma h \omega_n^2}{gD}$$

TABLE 7.8 Natural Frequencies and Nodal Lines of Cantilevered Rectangular and Skew Rectangular Plates ($\mu = 0.3$)* (*M. V. Barton.*[30])

MODE \ a/b	1/2	1	2	5
FIRST 	3.508	3.494	3.472	3.450
SECOND 	5.372	8.547	14.93	34.73
THIRD 	21.96	21.44	21.61	21.52
FOURTH 	10.26	27.46	94.49	563.9
FIFTH 	24.85	31.17	48.71	105.9

MODE	FIRST	SECOND	FIRST	SECOND	FIRST	SECOND
$\omega_n/\sqrt{Dg/\gamma h a^4}$	3.601	8.872	3.961	10.190	4.824	13.75
NODAL LINES						

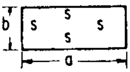
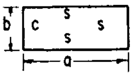
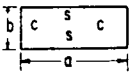
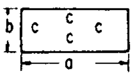
* For terminology, see Table 7.7.

The solution of Eq. (7.42) is[36]

$$W = A \cos (n\theta - \beta)[J_n(\kappa r) + \lambda J_n(i\kappa r)] \tag{7.43}$$

where J_n is a Bessel function of the first kind. When $\cos (n\theta - \beta) = 0$, a mode having a nodal system of n diameters, symmetrically distributed, is obtained. The term in

TABLE 7.9 Natural Frequencies of Rectangular Plates (*R. F. S. Hearman.*[32])

	b/a	1.0	1.5	2.0	2.5	3.0	∞
	$\omega_n/\sqrt{Dg/\gamma h a^4}$	19.74	14.26	12.34	11.45	10.97	9.87
	b/a	1.0	1.5	2.0	2.5	3.0	∞
	$\omega_n/\sqrt{Dg/\gamma h a^4}$	23.65	18.90	17.33	16.63	16.26	15.43
	a/b	1.0	1.5	2.0	2.5	3.0	∞
	$\omega_n/\sqrt{Dg/\gamma h b^4}$	23.65	15.57	12.92	11.75	11.14	9.87
	b/a	1.0	1.5	2.0	2.5	3.0	∞
	$\omega_n/\sqrt{Dg/\gamma h a^4}$	28.95	25.05	23.82	23.27	22.99	22.37
	a/b	1.0	1.5	2.0	2.5	3.0	∞
	$\omega_n/\sqrt{Dg/\gamma h b^4}$	28.95	17.37	13.69	12.13	11.36	9.87
	b/a	1.0	1.5	2.0	2.5	3.0	∞
	$\omega_n/\sqrt{Dg/\gamma h a^4}$	35.98	27.00	24.57	23.77	23.19	22.37

s DENOTES SIMPLY SUPPORTED EDGE

c DENOTES BUILT-IN OR CLAMPED EDGE

a = LENGTH OF PLATE

b = WIDTH OF PLATE

FOR OTHER TERMINOLOGY SEE TABLE 7.7

brackets represents modes having concentric nodal circles. The values of κ and λ are determined by the boundary conditions, which are, for radially symmetrical vibration:

Simply supported edge:

$$W = 0 \qquad M_{1r} = D\left(\frac{d^2W}{dr^2} + \frac{\mu}{a}\frac{dW}{dr}\right) = 0$$

Fixed edge:

$$W = 0 \qquad \frac{dW}{dr} = 0$$

TABLE 7.10 Natural Frequencies and Nodal Lines of Triangular Plates (*B. W. Anderson.*[33])

MODE \ k	2	4	8	14
FIRST (a/k, a)	7.194	7.122	7.080	7.068
SECOND	30.803	30.718	30.654	30.638
THIRD	61.131	90.105	157.70	265.98
FOURTH	148.8	259.4	493.4	853.6
k	2	4	7	
FIRST	5.887	6.617	6.897	
SECOND (a/k, a)	25.40	28.80	30.28	

a = LENGTH OF TRIANGLE

k = RATIO OF LENGTH TO WIDTH OF TRIANGLE

FOR OTHER TERMINOLOGY SEE TABLE 7.7

Free edge:

$$M_{1r} = D\left(\frac{d^2W}{dr^2} + \frac{\mu}{a}\frac{dW}{dr}\right) = 0 \qquad \frac{d}{dr}\left(\frac{d^2W}{dr^2} + \frac{1}{r}\frac{dW}{dr}\right) = 0$$

EXAMPLE 7.9. The steel diaphragm of a radio earphone has an unsupported diameter of 2.0 in. and is 0.008 in. thick. Assuming that the edge is fixed, the lowest three frequencies for the free vibration in which only nodal circles occur are to be calculated, using the exact method and the Rayleigh and Ritz methods.

EXACT SOLUTION. In this example $n = 0$, which makes $\cos(n\theta - \beta) = 1$; thus, Eq. (7.43) becomes

$$W = A[J_0(\kappa r) + \lambda I_0(\kappa r)]$$

where $J_0(i\kappa r) = I_0(\kappa r)$ and I_0 is a modified Bessel function of the first kind.

At the boundary where $r = a$,

$$\frac{\partial W}{\partial r} = A\kappa[-J_1(\kappa a) + \lambda I_1(\kappa a)] = 0 \qquad -J_1(\kappa a) + \lambda I_1(\kappa a) = 0$$

The deflection at $r = a$ is also zero:

$$J_0(\kappa a) + \lambda I_0(\kappa a) = 0$$

The frequency equation becomes

$$\lambda = \frac{J_1(\kappa a)}{I_1(\kappa a)} = -\frac{J_0(\kappa a)}{I_0(\kappa a)}$$

The first three roots of the frequency equation are: $\kappa a = 3.196, 6.306, 9.44$. The corresponding natural frequencies are, from Eq. (7.42),

$$\omega_n = \frac{10.21}{a^2}\sqrt{\frac{Dg}{\gamma h}} \qquad \frac{39.77}{a^2}\sqrt{\frac{Dg}{\gamma h}} \qquad \frac{88.9}{a^2}\sqrt{\frac{Dg}{\gamma h}}$$

For steel, $E = 30 \times 10^6$ lb/in.^{2}, $\gamma = 0.28$ lb/in.^{3}, and $\mu = 0.28$. Hence

$$D = \frac{Eh^3}{12(1-\mu^2)} = \frac{30 \times 10^6(0.008)^3}{12(1-0.078)} = 1.38 \text{ lb-in.}$$

Thus, the lowest natural frequency is

$$\omega_1 = 10.21\sqrt{\frac{(1.38)(386)}{(0.28)(0.008)}} = 4960 \text{ rad/sec} = 790 \text{ Hz}$$

The second frequency is 3070 Hz, and the third is 6880 Hz.

SOLUTION BY RAYLEIGH'S METHOD. The equations for strain and kinetic energies are given in Table 7.1. The strain energy for a plate with clamped edges becomes

$$V = \pi D \int_0^a \left(\frac{\partial^2 W}{\partial r^2} + \frac{1}{r}\frac{\partial W}{\partial r}\right)^2 r\,dr$$

The maximum kinetic energy is

$$T = \frac{\omega_n^2 \pi \gamma h}{g}\int_0^a W^2 r\,dr$$

An expression of the form $W = a_1[1-(r/a)^2]^2$, which satisfies the conditions of zero deflection and slope at the boundary, is used. The first two derivatives are $\partial W/\partial r = a_1(-4r/a^2 + 4r^3/a^4)$ and $\partial^2 W/\partial r^2 = a_1(-4/a^2 + 12r^2/a^4)$. Using these values in the equations for strain and kinetic energy, $V = 32\pi D a_1^2/3a^2$ and $T = \omega_n^2 \pi \gamma h a^2 a_1^2/10g$. Equating these values and solving for the frequency,

$$\omega_n = \sqrt{\frac{320\,Dg}{3a^4\,\gamma h}} = \frac{10.33}{a^2}\sqrt{\frac{Dg}{\gamma h}}$$

This is somewhat higher than the exact frequency.

SOLUTION BY RITZ'S METHOD. Using an expression for the deflection of the form

$$W = a_1[1-(r/a)^2]^2 + a_2[1-(r/a)^2]^3$$

and applying the Ritz method, the following values are obtained for the first two frequencies:

$$\omega_1 = \frac{10.21}{a^2}\sqrt{\frac{Dg}{\gamma h}} \qquad \omega_2 = \frac{43.04}{a^2}\sqrt{\frac{Dg}{\gamma h}}$$

The details of the calculations giving this result are in the literature.[37] The first frequency agrees with the exact answer to four significant figures, while the second frequency is somewhat high. A closer approximation to the second frequency and approximations of the higher frequencies could be obtained by using additional terms in the deflection equation.

The frequencies of modes having n nodal diameters are:[37]

$n = 1$: $$\omega_1 = \frac{21.22}{a^2}\sqrt{\frac{Dg}{\gamma h}}$$

$n = 2$: $$\omega_2 = \frac{34.84}{a^2}\sqrt{\frac{Dg}{\gamma h}}$$

For a plate with its center fixed and edge free, and having m nodal circles, the frequencies are:[38]

m	0	1	2	3
$\omega_n a^2 \Big/ \sqrt{\dfrac{Dg}{\gamma h}}$	3.75	20.91	60.68	119.7

Stretching of Middle Plane. In the usual analysis of plates, it is assumed that the deflection of the plate is so small that there is no stretching of the middle plane. If such stretching occurs, it affects the natural frequency. Whether it occurs depends on the conditions of support of the plate, the amplitude of vibration, and possibly other conditions. In a plate with its edges built in, a relatively small deflection causes a significant stretching. The effect of stretching is not proportional to the deflection; thus, the elastic restoring force is not a linear function of deflection. The natural frequency is not independent of amplitude but becomes higher with increasing amplitudes. If a plate is subjected to a pressure on one side, so that the vibration occurs about a deflected position, the effect of stretching may be appreciable. The effect of stretching in rectangular plates with immovable hinged supports has been discussed.[39] The effect of the amplitude on the natural frequency is shown in Fig. 7.11; the effect on the total stress in the plate is shown in Fig. 7.12. The natural frequency increases rapidly as the amplitude of vibration increases.

Rotational Motion and Shearing Forces. In the foregoing analysis, only the motion of each element of the plate in the direction normal to the plane of the plate is considered. There is also rotation of each element, and there is a deflection associated with the lateral shearing forces in the plate. The effects of these factors becomes significant if the curvature of the plate is large relative to its thickness, i.e., for a plate in which the thickness is large compared to the lateral dimensions or when the plate is vibrating in a mode for which the nodal lines are close together. These effects have been analyzed for rectangular plates[40] and for circular plates.[41]

Complete Circular Rings. Equations have been derived[42,43] for the natural frequencies of complete circular rings for which the radius is large compared to the thickness of the ring in the radial direction. Such rings can execute several types of free vibration, which are shown in Table 7.11 with the formulas for the natural frequencies.

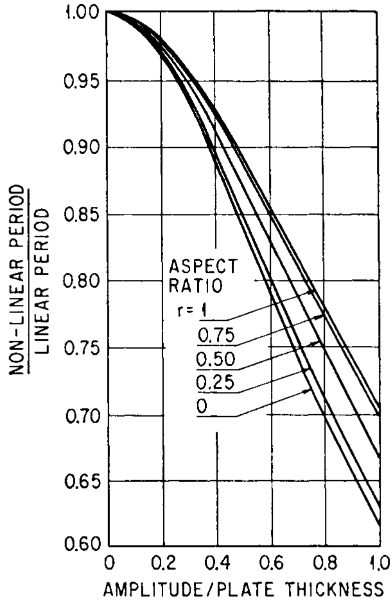


FIGURE 7.11 Influence of amplitude on period of vibration of uniform rectangular plates with immovable hinged edges. The aspect ratio *r* is the ratio of width to length of the plate. (*H. Chu and G. Herrmann.*[39])

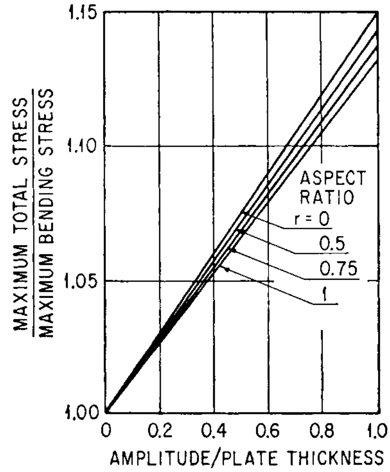


FIGURE 7.12 Influence of amplitude on maximum total stress in rectangular plates with immovable hinged edges. The aspect ratio *r* is the ratio of width to length of the plate. (*H. Chu and G. Herrmann.*[39])

TRANSFER MATRIX METHOD

In some assemblies which consist of various types of elements, e.g., beam segments, the solution for each element may be known. The transfer matrix method[44,45] is a procedure by means of which the solution for such elements can be combined to yield a frequency equation for the assembly. The associated mode shapes can then be determined. The method is an extension to distributed systems of the Holzer method, described in Chap. 38, in which torsional problems are solved by dividing an assembly into lumped masses and elastic elements, and of the Myklestad method,[46] in which a similar procedure is applied to beam problems. The method has been used[47] to find the natural frequencies and mode shapes of the internals of a nuclear reactor by modeling the various elements of the system as beam segments.

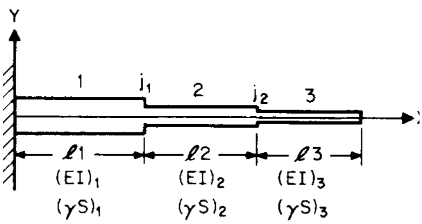
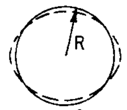
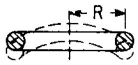
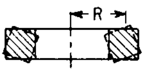
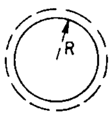


FIGURE 7.13 Cantilever beam made up of three segments having different section properties.

The method will be illustrated by setting up the frequency equation for a cantilever beam, Fig. 7.13, composed of three segments, each of which has uni-

TABLE 7.11 Natural Frequencies of Complete Circular Rings Whose Thickness in Radial Direction Is Small Compared to Radius

TYPE OF VIBRATION	SHAPE OF LOWEST MODE	RECTANGULAR CROSS SECTION	CIRCULAR CROSS SECTION
		ω_n	ω_n
FLEXURAL IN PLANE OF RING WITH n COMPLETE WAVE-LENGTH IN CIRCUM-FERENCE	 n = 2	$\sqrt{\frac{Eg}{\gamma}\frac{I}{AR^4}\frac{n^2(n^2-1)^2}{n^2+1}}$ n ANY INTEGER > 1	$\sqrt{\frac{E\pi r^4}{4mR^4}\frac{n^2(n^2-1)^2}{n^2+1}}$ n ANY INTEGER > 1
FLEXURAL NORMAL TO PLANE OF RING	 n = 2		$\sqrt{\frac{E\pi r^4}{4mR^4}\frac{n^2(n^2-1)^2}{n^2+1+\mu}}$ n ANY INTEGER > 1
TORSIONAL		FIRST MODE $\sqrt{\frac{Eg}{\gamma R^2}\frac{I_x}{I_p}}$	$\sqrt{\frac{G\pi r^2}{mR^2}(n^2+1+\mu)}$ n=0, OR ANY INTEGER
EXTENSIONAL		$\sqrt{\frac{Eg}{\gamma R^2}}$	$\sqrt{\frac{E\pi r^2}{mR^2}(1+n^2)}$ n=0, OR ANY INTEGER

E = MODULUS OF ELASTICITY
G = MODULUS OF RIGIDITY
γ = WEIGHT DENSITY
n : DEFINED FOR EACH TYPE OF VIBRATION
R = RADIUS OF RING
μ = POISSON'S RATIO

PROPERTIES OF CROSS SECTIONS
I = MOMENT OF INERTIA WITH RESPECT TO AXIS OF SECTION
I_x = MOMENT OF INERTIA WITH RESPECT TO RADIAL LINE
I_p = POLAR MOMENT OF INERTIA
A = AREA
r = RADIUS
m = MASS PER UNIT OF LENGTH

form section properties. Only the effects of bending will be considered, but the method can be extended to include other effects, such as shear deformation and rotary motion of the cross section.[45] Application to other geometries is described in Ref. 45.

Depending on the type of element being considered, the values of appropriate parameters must be expressed at certain sections of the piece in terms of their values at other sections. In the beam problem, the deflection and its first three derivatives must be used.

Transfer Matrices. Two types of transfer matrix are used. One, which for the beam problem is called the **R** matrix (after Lord Rayleigh[44]), yields the values of the parameters at the right end of a uniform segment of the beam in terms of their values at the left end of the segment. The other type of transfer matrix is the point matrix, which yields the values of the parameters just to the right of a joint between segments in terms of their values just to the left of the joint.

As can be seen by looking at the successive derivatives, the coefficients in Eq. (7.16) are equal to the following, where the subscript 0 indicates the value of the indicated parameter at the left end of the beam:

$$A = \frac{X_0}{2} \qquad C = \frac{X_0'}{2\kappa} \qquad B = \frac{-X_0''}{2\kappa^2} \qquad D = \frac{-X_0'''}{2\kappa^3}$$

Using the following notation, X and its derivatives at the right end of a beam segment can be expressed, by the matrix equation, in terms of the values at the left end of the segment. The subscript n refers to the number of the segment being considered, the subscript l to the left end of the segment and the subscript r to the right end.

$$C_{0n} = \frac{\cos \kappa_n l_n + \cosh \kappa_n l_n}{2}$$

$$S_{1n} = \frac{\sin \kappa_n l_n + \sinh \kappa_n l_n}{2\kappa_n}$$

$$C_{2n} = \frac{-(\cos \kappa_n l_n - \cosh \kappa_n l_n)}{2\kappa_n^2}$$

$$S_{3n} = \frac{-(\sin \kappa_n l_n - \sinh \kappa_n l_n)}{2\kappa_n^3}$$

where κ_n takes the value shown in Eq. (7.14) with the appropriate values of the parameters for the segment and l_n is the length of the segment.

$$\begin{bmatrix} X \\ X' \\ X'' \\ X''' \end{bmatrix}_{rn} = \begin{bmatrix} C_{0n} & S_{1n} & C_{2n} & S_{3n} \\ \kappa_n^4 S_{3n} & C_{0n} & S_{1n} & C_{2n} \\ \kappa_n^4 C_{2n} & \kappa_n^4 S_{3n} & C_{0n} & S_{1n} \\ \kappa_n^4 S_{1n} & \kappa_n^4 C_{2n} & \kappa_n^4 S_{3n} & C_{0n} \end{bmatrix} \begin{bmatrix} X \\ X' \\ X'' \\ X''' \end{bmatrix}_{ln}$$

or $\mathbf{x}_{rn} = \mathbf{R}_n \mathbf{X}_{ln}$, where the boldface capital letter denotes a square matrix and the boldface lowercase letters denote column matrices. Matrix operations are discussed in Chap. 28.

At a section where two segments of a beam are joined, the deflection, the slope, the bending moment, and the shear must be the same on the two sides of the joint. Since $M = EI \cdot X''$ and $V = EI \cdot X'''$, the point transfer matrix for such a joint is as follows, where the subscript jn refers to the joint to the right of the nth segment of the beam:

$$\begin{bmatrix} X \\ X' \\ X'' \\ X''' \end{bmatrix}_{rjn} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & (EI)_l/(EI)_r & 0 \\ 0 & 0 & 0 & (EI)_l/(EI)_r \end{bmatrix} \begin{bmatrix} X \\ X' \\ X'' \\ X''' \end{bmatrix}_{ljn}$$

or $\mathbf{x}_{rjn} = \mathbf{J}_n \mathbf{x}_{ljn}$.

The Frequency Equation. For the cantilever beam shown in Fig. 7.13, the coefficients relating the values of X and its derivatives at the right end of the beam to their values at the left end are found by successively multiplying the appropriate **R** and **J** matrices, as follows:

$$\mathbf{x}_{r3} = \mathbf{R}_3 \mathbf{J}_2 \mathbf{R}_2 \mathbf{J}_1 \mathbf{R}_1 \mathbf{x}_{l1}$$

Carrying out the multiplication of the square **R** and **J** matrices and calling the resulting matrix **P** yields

$$\begin{bmatrix} X \\ X' \\ X'' \\ X''' \end{bmatrix}_{r3} = \begin{bmatrix} P_{11} & P_{12} & P_{13} & P_{14} \\ P_{21} & P_{22} & P_{23} & P_{24} \\ P_{31} & P_{32} & P_{33} & P_{34} \\ P_{41} & P_{42} & P_{43} & P_{44} \end{bmatrix} \begin{bmatrix} X \\ X' \\ X'' \\ X''' \end{bmatrix}_{l1}$$

The boundary conditions at the fixed left end of the cantilever beam are $X = X' = 0$. Using these and performing the multiplication of **P** by $\mathbf{x}_{l1}$ yields the following:

$$\begin{aligned} X_{r3} &= P_{13}X_{l1}'' + P_{14}X_{l1}''' \\ X_{r3}' &= P_{23}X_{l1}'' + P_{24}X_{l1}''' \\ X_{r3}'' &= P_{33}X_{l1}'' + P_{34}X_{l1}''' \\ X_{r3}''' &= P_{43}X_{l1}'' + P_{44}X_{l1}''' \end{aligned} \tag{7.44}$$

The boundary conditions for the free right end of the beam are $X'' = X''' = 0$. Using these in the last two equations results in two simultaneous homogeneous equations, so that the following determinant, which is the frequency equation, results:

$$\begin{vmatrix} P_{33} & P_{34} \\ P_{43} & P_{44} \end{vmatrix} = 0$$

It can be seen that for a beam consisting of only one segment, this determinant yields a result which is equivalent to Eq. (7.17).

While in theory it would be possible to multiply the successive **R** and **J** matrices and obtain the **P** matrix in literal form, so that the transcendental frequency equation could be written, the process, in all but the simplest problems, would be long and time-consuming. A more practicable procedure is to perform the necessary multiplications with numbers, using a digital computer, and finding the roots by trial and error.

Mode Shapes. Either of the last two equations of Eq. (7.44) may be used to find the ratio X_{l1}''/X_{l1}'''. These are used in Eq. (7.16), with $\kappa = \kappa_1$ to find the shape of the first segment. By the use of the **R** and **J** matrices the values of the coefficients in Eq. (7.16) are found for each of the other segments.

With intermediate rigid supports or pinned connections, numerical difficulties occur in the solution of the frequency equation. These difficulties are eliminated by the use of delta matrices, the elements of which are combinations of the elements of the **R** matrix. These delta matrices, for various cases, are tabulated in Refs. 44 and 45.

In Ref. 47 transfer matrices are developed and used for structures which consist, in part, of beams that are parallel to each other.

FORCED VIBRATION

CLASSICAL SOLUTION

The classical method of analyzing the forced vibration that results when an elastic system is subjected to a fluctuating load is to set up the equation of motion by the application of Newton's second law. During the vibration, each element of the system is subjected to elastic forces corresponding to those experienced during free vibration; in addition, some of the elements are subjected to the disturbing force. The equation which governs the forced vibration of a system can be obtained by adding the disturbing force to the equation for free vibration. For example, in Eq. (7.13) for the free vibration of a uniform beam, the term on the left is due to the elastic forces in the beam. If a force $F(x,t)$ is applied to the beam, the equation of motion is obtained by adding this force to Eq. (7.13), which becomes, after rearranging terms,

$$EI \frac{\partial^4 y}{\partial x^4} + \frac{\gamma S}{g} \frac{\partial^2 y}{\partial t^2} = F(x,t)$$

where EI is a constant. The solution of this equation gives the motion that results from the force F. For example, consider the motion of a beam with hinged ends subjected to a sinusoidally varying force acting at its center. The solution is obtained by representing the concentrated force at the center by its Fourier series:

$$EIy'''' + \frac{\gamma S}{g}\ddot{y} = \frac{2F}{l}\left[\sin\frac{\pi x}{l} - \sin\frac{3\pi x}{l} + \sin\frac{5\pi x}{l} \cdots\right]\sin\omega t$$

$$= \frac{2F}{l}\sum_{n=1}^{n=\infty}\left(\sin\frac{n\pi}{2}\sin\frac{n\pi x}{l}\right)\sin\omega t \tag{7.45}$$

where $\sin(n\pi/2)$, which appears in each term of the series, makes the nth term positive, negative, or zero. The solution of Eq. (7.45) is

$$y = \sum_{n=1}^{n=\infty}\left[A_n \sin\frac{n\pi x}{l}\sin\omega_n t + B_n \sin\frac{n\pi x}{l}\cos\omega_n t + \sin\frac{n\pi}{2}\frac{2Fg/S\gamma l}{(n\pi/l)^4(EIg/S\gamma) - \omega^2}\sin\frac{n\pi x}{l}\sin\omega t\right] \tag{7.46}$$

The first two terms of Eq. (7.46) are the values of y which make the left side of Eq. (7.45) equal to zero. They are obtained in exactly the same way as in the solution of the free-vibration problem and represent the free vibration of the beam. The constants are determined by the initial conditions; in any real beam, damping causes the free vibration to die out. The third term of Eq. (7.46) is the value of y which makes the left-hand side of Eq. (7.45) equal the right-hand side; this can be verified by substitution. The third term represents the forced vibration. From Table 7.3, $\kappa_n l = n\pi$ for a beam with hinged ends; then from Eq. (7.14), $\omega_n^2 = n^4\pi^4 EIg/S\gamma l^4$. The term representing the forced vibration in Eq. (7.46) can be written, after rearranging terms,

$$y = \frac{2Fg}{S\gamma l}\sum_{n=1}^{n=\infty}\frac{\sin(n\pi/2)}{\omega_n^2[1 - (\omega/\omega_n)^2]}\sin\frac{n\pi x}{l}\sin\omega t \tag{7.47}$$

From Table 7.3 and Eq. (7.16), it is evident that this deflection curve has the same shape as the nth normal mode of vibration of the beam since, for free vibration of a beam with hinged ends, $X_n = 2C\sin\kappa x = \sin(n\pi x/l)$.

The equation for the deflection of a beam under a distributed static load $F(x)$ can be obtained by replacing $-(\gamma S/g)\ddot{y}$ with F in Eq. (7.12); then Eq. (7.13) becomes

$$y_s'''' = \frac{F(x)}{EI} \tag{7.48}$$

where EI is a constant. For a static loading $F(x) = 2F/l \sin n\pi/2 \sin n\pi x/l$ corresponding to the nth term of the Fourier series in Eq. (7.45), Eq. (7.48) becomes $y_{sn}'''' = 2F/EIl \sin n\pi/2 \sin n\pi x/l$. The solution of this equation is

$$y_{sn} = \frac{2F}{EIl}\left(\frac{l}{n\pi}\right)^4 \sin\frac{n\pi}{2} \sin\frac{n\pi x}{l}$$

Using the relation $\omega_n^2 = n^4\pi^4 EIg/S\gamma l^4$, this can be written

$$y_{sn} = \frac{2Fg}{\omega_n^2 S\gamma l} \sin\frac{n\pi x}{l} \sin\frac{n\pi}{2}$$

Thus, the nth term of Eq. (7.47) can be written

$$y_n = y_{sn}\frac{1}{1-(\omega/\omega_n)^2}\sin\omega t$$

Thus, the amplitude of the forced vibration is equal to the static deflection under the Fourier component of the load multiplied by the "amplification factor" $1/[1 - (\omega/\omega_n)^2]$. This is the same as the relation that exists, for a system having a single degree-of-freedom, between the static deflection under a load F and the amplitude under a fluctuating load $F \sin \omega t$. Therefore, insofar as each mode alone is concerned, the beam behaves as a system having a single degree-of-freedom. If the beam is subjected to a force fluctuating at a single frequency, the amplification factor is small except when the frequency of the forcing force is near the natural frequency of a mode. For all even values of n, $\sin(n\pi/2) = 0$; thus, the even-numbered modes are not excited by a force acting at the center, which is a node for those modes. The distribution of the static load that causes the same pattern of deflection as the beam assumes during each mode of vibration has the same form as the deflection of the beam. This result applies to other beams since a comparison of Eqs. (7.15) and (7.48) shows that if a static load $F = (\omega_n^2\gamma S/g)y$ is applied to any beam, it will cause the same deflection as occurs during the free vibration in the nth mode.

The results for the simply supported beam are typical of those which are obtained for all systems having distributed mass and elasticity. Vibration of such a system at resonance is excited by a force which fluctuates at the natural frequency of a mode, since nearly any such force has a component of the shape necessary to excite the vibration. Even if the force acts at a nodal point of the mode, vibration may be excited because of coupling between the modes.

METHOD OF VIRTUAL WORK

Another method of analyzing forced vibration is by the use of the theorem of virtual work and D'Alembert's principle. The theorem of virtual work states that when any elastic body is in equilibrium, the total work done by all external forces during any virtual displacement equals the increase in the elastic energy stored in the body. A virtual displacement is an arbitrary small displacement that is compatible with the geometry of the body and which satisfies the boundary conditions.

In applying the principle of work to forced vibration of elastic bodies, the problem is made into one of equilibrium by the application of D'Alembert's principle. This permits a problem in dynamics to be considered as one of statics by adding to the equation of static equilibrium an "inertia force" which, for each part of the body, is equal to the product of the mass and the acceleration. Using this principle, the theorem of virtual work can be expressed in the following equation:

$$\Delta V = \Delta(F_I + F_E) \tag{7.49}$$

in which V is the elastic strain energy in the body, F_I is the inertia force, F_E is the external disturbing force, and Δ indicates the change of the quantity when the body undergoes a virtual displacement. The various quantities can be found separately.

For example, consider the motion of a uniform beam having hinged ends with a sinusoidally varying force acting at the center, and compare the result with the solution obtained by the classical method. All possible motions of any beam can be represented by a series of the form

$$y = q_1X_1 + q_2X_2 + q_3X_3 + \cdots = \sum_{n=1}^{n=\infty} q_nX_n \tag{7.50}$$

in which the X's are functions representing displacements in the normal modes of vibration and the q's are coefficients which are functions of time. The determination of the values of q_n is the problem to be solved. For a beam having hinged ends, Eq. (7.50) becomes

$$y = \sum_{n=1}^{n=\infty} q_n \sin \frac{n\pi x}{l} \tag{7.51}$$

This is evident by using the values of $\kappa_n l$ from Table 7.3 in Eq. (7.16). A virtual displacement, being any arbitrary small displacement, can be assumed to be

$$\Delta y = \Delta q_m X_m = \Delta q_m \sin \frac{m\pi x}{l}$$

The elastic strain energy of bending of the beam is

$$V = \frac{EI}{2}\int_0^l \left(\frac{\partial^2 y}{\partial x^2}\right)^2 dx = \frac{EI}{2}\sum_{n=1}^{n=\infty} q_n^2 \int_0^l \left[\frac{\partial^2}{\partial x^2}\left(\sin\frac{n\pi x}{l}\right)\right]^2 dx$$

$$= \frac{EI}{2}\sum_{n=1}^{n=\infty} q_n^2 \left(\frac{n\pi}{l}\right)^4 \int_0^l \left(\sin\frac{n\pi x}{l}\right)^2 dx = \frac{EI}{2}\sum_{n=1}^{n=\infty} q_n^2 \left(\frac{n\pi}{l}\right)^4 \frac{l}{2}$$

For the virtual displacement, the change of elastic energy is

$$\Delta V = \frac{\partial V}{\partial q_m}\Delta q_m = \frac{EI}{2l^3}(n\pi)^4 q_m \Delta q_m = \frac{EI}{2l^3}(\kappa_n l)^4 q_m \Delta q_m$$

The value of the inertia force at each section is

$$F_I = -\frac{\gamma S}{g}\ddot{y} = -\frac{\gamma S}{g}\sum_{n=1}^{n=\infty}\frac{d^2q_n}{dt^2}\sin\frac{n\pi x}{l}$$

The work done by this force during the virtual displacement Δy is

$$\Delta F_I = F_I\,\Delta y = -\frac{\gamma S}{g}\sum_{n=1}^{n=\infty}\frac{d^2q_n}{dt^2}\Delta q_m \int_0^l \sin\frac{n\pi x}{l}\sin\frac{m\pi x}{l}\,dx$$

$$= -\frac{\gamma S l}{2g}\frac{d^2q_m}{dt^2}\Delta q_m$$

The orthogonality relation of Eq. (7.1) is used here, making the integral vanish when $n = m$. For a disturbing force F_E, the work done during the virtual displacement is

$$\Delta F_E = F_E \, \Delta y = F(X_m)_{x=c} \, \Delta q_m$$

in which $(X_m)_{x=c}$ is the value of X_m at the point of application of the load. Substituting the terms into Eq. (7.49),

$$\frac{\gamma S l}{2g} \ddot{q}_m + \frac{EI}{2l^3} (\kappa_m l)^4 q_m = F(X_m)_{x=c}$$

Rearranging terms and letting $EI/S\gamma = a^2$,

$$\ddot{q}_m + \kappa_m{}^4 a^2 q_m = \frac{2g}{\gamma S l} F(X_m)_{x=c} \tag{7.52}$$

If F_E is a force which varies sinusoidally with time at any point $x = c$,

$$F\,(X_m)_{x=c} = \bar{F} \sin \frac{m\pi c}{l} \sin \omega t$$

and Eq. (7.52) becomes

$$\ddot{q}_m + \kappa_m{}^4 a^2 q_m = \frac{2g\bar{F}}{\gamma S l} \sin \frac{m\pi c}{l} \sin \omega t$$

The solution of this equation is

$$q_m = A_m \sin \kappa_m{}^2 at + B_m \cos \kappa_m{}^2 at + \frac{2\bar{F}g}{\gamma S l} \frac{\sin (m\pi c/l)}{\kappa_m{}^4 a^2 - \omega^2} \sin \omega t$$

Since $\kappa_m{}^2 a = \omega_m$,

$$q_m = A_m \sin \omega_m t + B_m \cos \omega_m t + \frac{2\bar{F}g}{\gamma S l} \frac{\sin (m\pi c/l)}{\omega_m{}^2 - \omega^2} \sin \omega t$$

when the force acts at the center $c/l = ½$. Substituting the corresponding values of q in Eq. (7.51), the solution is identical to Eq. (7.46), which was obtained by the classical method.

VIBRATION RESULTING FROM MOTION OF SUPPORT

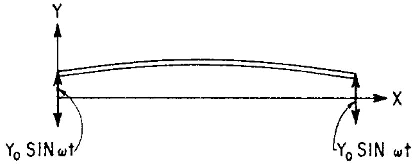


FIGURE 7.14 Simply supported beam undergoing sinusoidal motion induced by sinusoidal motion of the supports.

When the supports of an elastic body are vibrated by some external force, forced vibration may be induced in the body.[48] For example, consider the motion that results in a uniform beam, Fig. 7.14, when the supports are moved through a sinusoidally varying displacement $(y)_{x=0,l} = Y_0 \sin \omega t$. Although Eq. (7.13) was developed for the free vibration of beams, it is applicable to the present problem because there is no force acting on any section of the beam except the elastic force associated with the bending of the beam. If a solution of the form $y = X(x) \sin \omega t$ is assumed and substituted into Eq. (7.13):

$$X'''' = \frac{\omega^2 \gamma S}{EIg} X \tag{7.53}$$

This equation is the same as Eq. (7.15) except that the natural frequency ω_n^2 is replaced by the forcing frequency ω^2. The solution of Eq. (7.53) is the same except that κ is replaced by $\kappa' = (\omega^2\gamma S/EIg)^{1/4}$:

$$X = A_1 \sin \kappa' x + A_2 \cos \kappa' x + A_3 \sinh \kappa' x + A_4 \cosh \kappa' x \tag{7.54}$$

The solution of the problem is completed by finding the constants, which are determined by the boundary conditions. Certain boundary conditions are associated with the supports of the beam and are the same as occur in the solution of the problem of free vibration. Additional conditions are supplied by the displacement through which the supports are forced. For example, if the supports of a beam having hinged ends are moved sinusoidally, the boundary conditions are: at $x = 0$ and $x = l$, $X'' = 0$, since the moment exerted by a hinged end is zero, and $X = Y_0$, since the amplitude of vibration is prescribed at each end. By the use of these boundary conditions, Eq. (7.54) becomes

$$X = \frac{Y_0}{2} \left[\tan \frac{\kappa' l}{2} \sin \kappa' x + \cos \kappa' x - \tanh \frac{\kappa' l}{2} \sinh \kappa' x + \cosh \kappa' x \right] \tag{7.55}$$

The motion is defined by $y = X \sin \omega t$. For all values of κ', each of the coefficients except the first in Eq. (7.55) is finite. The tangent term becomes infinite if $\kappa' l = n\pi$, for odd values of n. The condition for the amplitude to become infinite is $\omega = \omega_n$ because $\kappa'/\kappa = \omega^2/\omega_n^2$ and, for natural vibration of a beam with hinged ends, $\kappa_n l = n\pi$. Thus, if the supports of an elastic body are vibrated at a frequency close to a natural frequency of the system, vibration at resonance occurs.

DAMPING

The effect of damping on forced vibration can be discussed only qualitatively. Damping usually decreases the amplitude of vibration, as it does in systems having a single degree-of-freedom. In some systems, it may cause coupling between modes, so that motion in a mode of vibration that normally would not be excited by a certain disturbing force may be induced.

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