CHAPTER 8
TRANSIENT RESPONSE TO STEP AND PULSE FUNCTIONS*

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INTRODUCTION

In analyses involving shock and transient vibration, it is essential in most instances to begin with the time-history of a quantity that describes a motion, usually displacement, velocity, or acceleration. The method of reducing the time-history depends upon the purpose for which the reduced data will be used. When the purpose is to compare shock motions, to design equipment to withstand shock, or to formulate a laboratory test as means to simulate an environmental condition, the *response spectrum* is found to be a useful concept. This concept in data reduction is discussed in Chap. 23, and its application to environmental conditions is discussed in Chap. 24.

This chapter deals briefly with methods of analysis for obtaining the response spectrum from the time-history, and includes in graphical form certain significant spectra for various regular step- and pulse-type excitations. The usual concept of the response spectrum is based upon the single degree-of-freedom system, usually considered linear and undamped, although useful information sometimes can be obtained by introducing nonlinearity or damping. The single degree-of-freedom system is considered to be subjected to the shock or transient vibration, and its response determined.

The *response spectrum* is a graphical presentation of a selected quantity in the response taken with reference to a quantity in the excitation. It is plotted as a function of a dimensionless parameter that includes the natural period of the responding system and a significant period of the excitation. The excitation may be defined in terms of various physical quantities, and the response spectrum likewise may depict various characteristics of the response.

* Chapter 8 is based on Chaps. 3 and 4 of "Engineering Vibrations," by L. S. Jacobsen and R. S. Ayre, McGraw-Hill Book Company, Inc., 1958.

LINEAR, UNDAMPED, SINGLE DEGREE-OF-FREEDOM SYSTEMS

DIFFERENTIAL EQUATION OF MOTION

It is assumed that the system is linear and undamped. The excitation, which is a known function of time alone, may be a force function $F(t)$ acting directly on the mass of the system (Fig. 8.1*A*) or it may be a ground motion, i.e., foundation or base motion, acting on the spring anchorage. The ground motion may be expressed as a ground displacement function $u(t)$ (Fig. 8.1*B*). In many cases, however, it is more useful to express it as a ground acceleration function $\ddot{u}(t)$ (Fig. 8.1*C*).

The differential equation of motion, written in terms of each of the types of excitation, is given in Eqs. (8.1*a*), (8.1*b*), and (8.1*c*).

$$m\ddot{x} = -kx + F(t) \quad \text{or} \quad \frac{m\ddot{x}}{k} + x = \frac{F(t)}{k} \tag{8.1a}$$

$$m\ddot{x} = -k[x - u(t)] \quad \text{or} \quad \frac{m\ddot{x}}{k} + x = u(t) \tag{8.1b}$$

$$m[\ddot{\delta}_x + \ddot{u}(t)] = -k\delta_x \quad \text{or} \quad \frac{m\ddot{\delta}_x}{k} + \delta_x = -\frac{m\ddot{u}(t)}{k} \tag{8.1c}$$

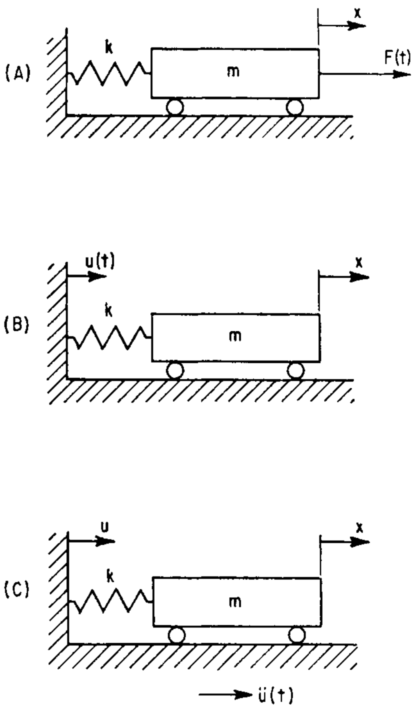


FIGURE 8.1 Simple oscillator acted upon by known excitation functions of time: (*A*) force $F(t)$, (*B*) ground displacement $u(t)$, (*C*) ground acceleration $\ddot{u}(t)$.

where x is the displacement (absolute displacement) of the mass relative to a *fixed reference* and δ_x is the displacement relative to a *moving anchorage* or ground. These displacements are related to the ground displacement by $x = u + \delta_x$. Similarly, the accelerations are related by $\ddot{x} = \ddot{u} + \ddot{\delta}_x$.

Furthermore, if Eq. (8.1*b*) is differentiated twice with respect to time, a differential equation is obtained in which ground acceleration $\ddot{u}(t)$ is the excitation and the absolute acceleration $\ddot{x}$ of the mass m is the variable. The equation is

$$\frac{m}{k}\frac{d^2\ddot{x}}{dt^2} + \ddot{x} = \ddot{u}(t) \tag{8.1d}$$

If Eq. (8.1*d*) is treated as a second-order equation in $\ddot{x}$ as the dependent variable, it is of the same general form as Eqs. (8.1*a*), (8.1*b*), and (8.1*c*).

Occasionally, the excitation is known in terms of ground velocity $\dot{u}(t)$. Differentiating Eq. (8.1*b*) once with respect to time, the following second-order equation in $\dot{x}$ is obtained:

$$\frac{m}{k}\frac{d^2\dot{x}}{dt^2} + \dot{x} = \dot{u}(t) \tag{8.1e}$$

The analogy represented by Eqs. (8.1*b*), (8.1*d*), and (8.1*e*) may be extended further since it is generally possible to differentiate Eq. (8.1*b*) any number of times *n*:

$$\frac{m}{k}\frac{d^2}{dt^2}\left(\frac{d^n x}{dt^n}\right)+\left(\frac{d^n x}{dt^n}\right)=\left(\frac{d^n u}{dt^n}\right)(t) \tag{8.1f}$$

This is of the same general form as the preceding equations if it is considered to be a second-order equation in $(d^n x/dt^n)$ as the response variable, with $(d^n u/dt^n)(t)$, a known function of time, as the excitation.

ALTERNATE FORMS OF THE EXCITATION AND OF THE RESPONSE

The foregoing equations are alike, mathematically, and a solution in terms of one of them may be applied to any of the others by making simple substitutions. Therefore, the equations may be expressed in the single general form:

$$\frac{m}{k}\ddot{\nu}+\nu=\xi(t) \tag{8.2}$$

where ν and ξ are the *response* and the *excitation,* respectively, at time *t.*

A general notation (ν and ξ) is desirable in the presentation of response functions and response spectra for general use. However, in the discussion of examples of solution, it sometimes is preferable to use more specific notations. Both types of notation are used in this chapter. For ready reference, the alternate forms of the excitation and the response are given in Table 8.1 where $\omega_n^2 = k/m$.

TABLE 8.1 Alternate Forms of Excitation and Response in Eq. (8.2)

Excitation $\xi(t)$		Response ν	
Force	$\frac{F(t)}{k}$	Absolute displacement	x
Ground displacement	$u(t)$	Absolute displacement	x
Ground acceleration	$\frac{-\ddot{u}(t)}{\omega_n^2}$	Relative displacement	δ_x
Ground acceleration	$\ddot{u}(t)$	Absolute acceleration	$\ddot{x}$
Ground velocity	$\dot{u}(t)$	Absolute velocity	$\dot{x}$
*n*th derivative of ground displacement	$\frac{d^n u}{dt^n}(t)$	*n*th derivative of absolute displacement	$\frac{d^n x}{dt^n}$

METHODS OF SOLUTION OF THE DIFFERENTIAL EQUATION

A brief review of four methods of solution is given in the following sections.

Classical Solution. The complete solution of the linear differential equation of motion consists of the sum of the *particular integral* x_1 and the *complementary function* x_2, that is, $x = x_1 + x_2$. Since the differential equation is of second order, two constants of integration are involved. They appear in the complementary function and are evaluated from a knowledge of the initial conditions.

Example 8.1: Versed-sine Force Pulse. In this case the differential equation of motion, applicable for the duration of the pulse, is

$$\frac{m\ddot{x}}{k} + x = \frac{F_p}{k}\frac{1}{2}\left(1 - \cos\frac{2\pi t}{\tau}\right) \qquad [0 \le t \le \tau] \tag{8.3a}$$

where, in terms of the general notation, the excitation function $\xi(t)$ is

$$\xi(t) \equiv \frac{F(t)}{k} = \frac{F_p}{k}\frac{1}{2}\left(1 - \cos\frac{2\pi t}{\tau}\right)$$

and the response ν is displacement x. The maximum value of the pulse excitation force is F_p.

The particular integral (particular solution) for Eq. (8.3*a*) is of the form

$$x_1 = M + N\cos\frac{2\pi t}{\tau} \tag{8.3b}$$

By substitution of the particular solution into the differential equation, the required values of the coefficients M and N are found.

The complementary function is

$$x_2 = A\cos\omega_n t + B\sin\omega_n t \tag{8.3c}$$

where A and B are the constants of integration. Combining x_2 and the explicit form of x_1 gives the complete solution:

$$x = x_1 + x_2 = \frac{F_p/2k}{1 - \tau^2/T^2}\left(1 - \frac{\tau^2}{T^2} + \frac{\tau^2}{T^2}\cos\frac{2\pi t}{\tau}\right) + A\cos\omega_n t + B\sin\omega_n t \tag{8.3d}$$

If it is assumed that the system is initially at rest, $x = 0$ and $\dot{x} = 0$ at $t = 0$, and the constants of integration are

$$A = -\frac{F_p/2k}{1 - \tau^2/T^2} \qquad \text{and} \qquad B = 0 \tag{8.3e}$$

The complete solution takes the following form:

$$\nu \equiv x = \frac{F_p/2k}{1 - \tau^2/T^2}\left(1 - \frac{\tau^2}{T^2} + \frac{\tau^2}{T^2}\cos\frac{2\pi t}{\tau} - \cos\omega_n t\right) \tag{8.3f}$$

If other starting conditions had been assumed, A and B would have been different from the values given by Eqs. (8.3*e*). It may be shown that if the starting conditions are general, namely, $x = x_0$ and $\dot{x} = \dot{x}_0$ at $t = 0$, it is necessary to superimpose on the complete solution already found, Eq. (8.3*f*), only the following additional terms:

$$x_0\cos\omega_n t + \frac{\dot{x}_0}{\omega_n}\sin\omega_n t \tag{8.3g}$$

For values of time equal to or greater than τ, the differential equation is

$$m\ddot{x} + kx = 0 \qquad [\tau \le t] \tag{8.4a}$$

and the complete solution is given by the complementary function alone. However, the constants of integration must be redetermined from the known conditions of the system at time $t = \tau$. The solution is

$$\nu \equiv x = \frac{F_p}{k} \frac{\sin(\pi\tau/T)}{1 - \tau^2/T^2} \sin \omega_n \left(t - \frac{\tau}{2}\right) \qquad [\tau \le t] \tag{8.4b}$$

The additional terms given by expressions (8.3*g*) may be superimposed on this solution if the conditions at time $t = 0$ are general.

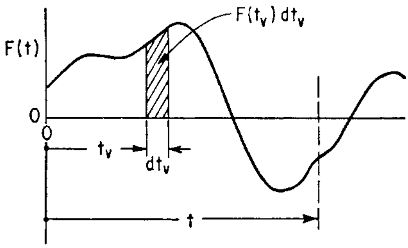


FIGURE 8.2 General excitation and the elemental impulse.

Duhamel's Integral. The use of Duhamel's integral (convolution integral or superposition integral) is a well-known approach to the solution of transient vibration problems in linear systems. Its development[7] is based on the *superposition* of the responses of the system to a sequence of impulses.

A general excitation function is shown in Fig. 8.2, where $F(t)$ is a known force function of time, the variable of integration is t_v between the limits of integration 0 and t, and the elemental impulse is $F(t_v)\,dt_v$. It may be shown that the complete solution of the differential equation is

$$x = \left(x_0 - \frac{1}{m\omega_n}\int_0^t F(t_v)\sin\omega_n t_v\,dt_v\right)\cos\omega_n t + \left(\frac{\dot{x}_0}{\omega_n} + \frac{1}{m\omega_n}\int_0^t F(t_v)\cos\omega_n t_v\,dt_v\right)\sin\omega_n t \tag{8.5}$$

where x_0 and $\dot{x}_0$ are the initial conditions of the system at zero time.

Example 8.2: Half-cycle Sine, Ground Displacement Pulse. Consider the following excitation:

$$\xi(t) \equiv u(t) = \begin{cases} u_p \sin \dfrac{\pi t}{\tau} & [0 \le t \le \tau] \\ 0 & [\tau \le t] \end{cases}$$

The maximum value of the excitation displacement is u_p. Assume that the system is initially at rest, so that $x_0 = \dot{x}_0 = 0$. Expressing the excitation function in terms of the variable of integration t_v, Eq. (8.5) may be rewritten for this particular case in the following form:

$$x = \frac{ku_p}{m\omega_n}\left(-\cos\omega_n t\int_0^t \sin\frac{\pi t_v}{\tau}\sin\omega_n t_v\,dt_v + \sin\omega_n t\int_0^t \sin\frac{\pi t_v}{\tau}\cos\omega_n t_v\,dt_v\right) \tag{8.6a}$$

Equation (8.6*a*) may be reduced, by evaluation of the integrals, to

$$\nu \equiv x = \frac{u_p}{1 - T^2/4\tau^2}\left(\sin\frac{\pi t}{\tau} - \frac{T}{2\tau}\sin\omega_n t\right) \qquad [0 \le t \le \tau] \tag{8.6b}$$

where $T = 2\pi/\omega_n$ is the natural period of the responding system.

For the second era of time, where $\tau \le t$, it is convenient to choose a new time variable $t' = t - \tau$. Noting that $u(t) = 0$ for $\tau \le t$, and that for continuity in the system response the initial conditions for the second era must equal the closing conditions for the first era, it is found from Eq. (8.5) that the response for the second era is

$$x = x_\tau \cos \omega_n t' + \frac{\dot{x}_\tau}{\omega_n} \sin \omega_n t' \tag{8.7a}$$

where x_τ and $\dot{x}_\tau$ are the displacement and velocity of the system at time $t = \tau$ and hence at $t' = 0$. Equation (8.7*a*) may be rewritten in the following form:

$$\nu \equiv x = u_p \frac{(T/\tau) \cos (\pi\tau/T)}{(T^2/4\tau^2) - 1} \sin \omega_n \left(t - \frac{\tau}{2}\right) \qquad [\tau \leq t] \tag{8.7b}$$

Phase-Plane Graphical Method. Several numerical and graphical methods,[18,23] all related in general but differing considerably in the details of procedure, are available for the solution of linear transient vibration problems. Of these methods, the phase-plane graphical method is one of the most useful. The procedure is basically very simple, it gives a clear physical picture of the response of the system, and it may be applied readily to some classes of *nonlinear* systems.[3,5,6,8,13,15,21,22]

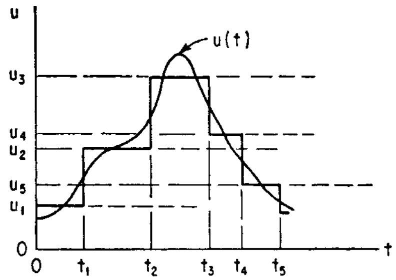


FIGURE 8.3 General excitation approximated by a sequence of finite rectangular steps.

In Fig. 8.3 a general excitation in terms of ground displacement is represented, approximately, by a sequence of finite steps. The *i*th step has the total height u_i, where u_i is constant for the duration of the step. The differential equation of motion and its complete solution, applying for the duration of the step, are

$$\frac{m\ddot{x}}{k} + x = u_i \qquad [t_{i-1} \leq t \leq t_i] \tag{8.8a}$$

$$x - u_i = (x_{i-1} - u_i) \cos \omega_n (t - t_{i-1}) + \frac{\dot{x}_{i-1}}{\omega_n} \sin \omega_n (t - t_{i-1}) \tag{8.8b}$$

where x_{i-1} and $\dot{x}_{i-1}$ are the displacement and velocity of the system at time t_{i-1}; consequently, they are the initial conditions for the *i*th step. The system velocity (divided by ω_n) during the *i*th step is

$$\frac{\dot{x}}{\omega_n} = -(x_{i-1} - u_i) \sin \omega_n (t - t_{i-1}) + \frac{\dot{x}_{i-1}}{\omega_n} \cos \omega_n (t - t_{i-1}) \tag{8.8c}$$

Squaring Eqs. (8.8*b*) and (8.8*c*) and adding them,

$$\left(\frac{\dot{x}}{\omega_n}\right)^2 + (x - u_i)^2 = \left(\frac{\dot{x}_{i-1}}{\omega_n}\right)^2 + (x_{i-1} - u_i)^2 \tag{8.8d}$$

This is the equation of a circle in a rectangular system of coordinates $\dot{x}/\omega_n$, x. The center is at $0, u_i$; and the radius is

$$R_i = \left[\left(\frac{\dot{x}_{i-1}}{\omega_n}\right)^2 + (x_{i-1} - u_i)^2\right]^{1/2} \tag{8.8e}$$

The solution for Eq. (8.8*a*) for the *i*th step may be shown, as in Fig. 8.4, to be the arc of the circle of radius R_i and center $0, u_i$, subtended by the angle $\omega_n(t_i - t_{i-1})$ and starting at the point $\dot{x}_{i-1}/\omega_n, x_{i-1}$. Time is positive in the counterclockwise direction.

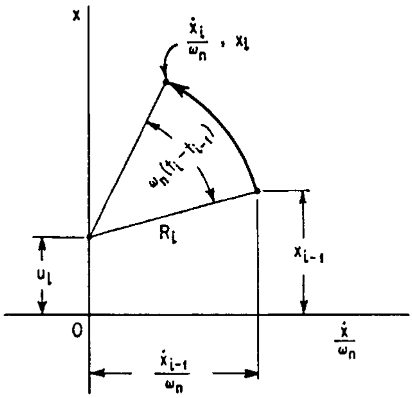


FIGURE 8.4 Graphical representation in the phase-plane of the solution for the ith step.

Example 8.3: Application to a General Pulse Excitation. Figure 8.5 shows an application of the method for the general excitation $u(t)$ represented by seven steps in the time-displacement plane. Upon choice of the step heights u_i and durations $(t_i - t_{i-1})$, the arc-center locations can be projected onto the X axis in the phase-plane and the arc angles $\omega_n(t_i - t_{i-1})$ can be computed. The graphical construction of the sequence of circular arcs, the *phase trajectory,* is then carried out, using the system conditions at zero time (in this example, 0,0) as the starting point.

Projection of the system displacements from the phase-plane into the time-displacement plane at once determines the time-displacement response curve. The time-velocity response can also be determined by projection as shown. The velocities and displacements at particular instants of time can be found directly from the phase trajectory coordinates without the necessity for drawing the time-response curves. Furthermore, the times of occurrence and the magnitudes of all the maxima also can be obtained directly from the phase trajectory.

Good accuracy is obtainable by using reasonable care in the graphical construction and in the choice of the steps representing the excitation. Usually, the time intervals should not be longer than about one-fourth the natural period of the system.[22]

The Laplace Transformation. The Laplace transformation provides a powerful tool for the solution of linear differential equations. The following discussion of the technique of its application is limited to the differential equation of the type applying to the undamped linear oscillator. Application to the linear oscillator with viscous damping is illustrated in a later part of this chapter.

Definitions. The Laplace transform $F(s)$ of a known function $f(t)$, where $t > 0$, is defined by

$$F(s) = \int_0^\infty e^{-st} f(t)\, dt \tag{8.9a}$$

where s is a complex variable. The transformation is abbreviated as

$$F(s) = \mathcal{L}[f(t)] \tag{8.9b}$$

The limitations on the function $f(t)$ are not discussed here. For the conditions for existence of $\mathcal{L}[f(t)]$, for complete accounts of the technique of application, and for extensive tables of function-transform pairs, the references should be consulted.[16,17]

General Steps in Solution of the Differential Equation. In the solution of a differential equation by Laplace transformation, the first step is to transform the differential equation, in the variable t, into an algebraic equation in the complex variable s. Then, the algebraic equation is solved, and the solution of the differential equation is determined by an *inverse* transformation of the solution of the algebraic equation. The process of inverse Laplace transformation is symbolized by

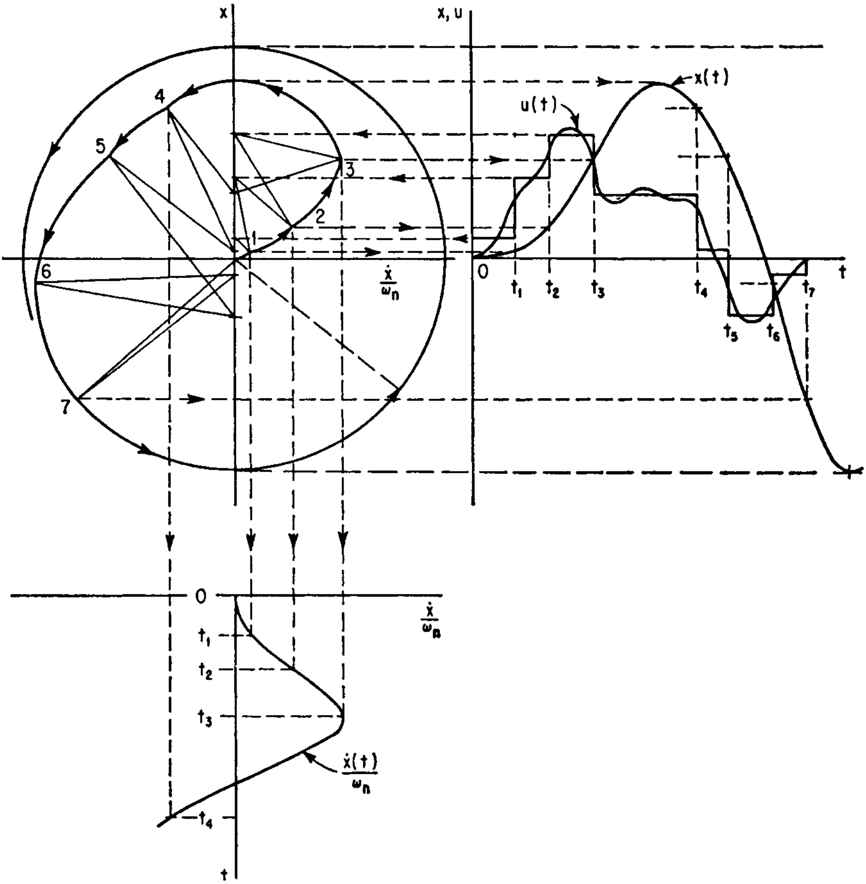


FIGURE 8.5 Example of phase-plane graphical solution.[2]

$$\mathcal{L}^{-1}[F(s)] = f(t) \tag{8.10}$$

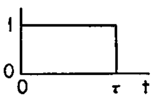
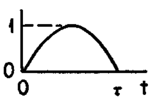
Tables of Function-Transform Pairs. The processes symbolized by Eqs. (8.9*b*) and (8.10) are facilitated by the use of tables of function-transform pairs. Table 8.2 is a brief example. Transforms for general operations, such as differentiation, are included as well as transforms of explicit functions.

In general, the transforms of the explicit functions can be obtained by carrying out the integration indicated by the definition of the Laplace transformation. For example:

For $f(t) = 1$:

$$F(s) = \int_0^\infty e^{-st}dt = -\frac{1}{s}\, e^{-st}\Big]_0^\infty = \frac{1}{s}$$

TABLE 8.2 Pairs of Functions $f(t)$ and Laplace Transforms $F(s)$

	$f(t)$	$F(s)$
	Operation Transforms	
1	Definition, $f(t)$	$F(s) = \int_0^\infty e^{-st} f(t)\, dt$
2	First derivative, $f'(t)$	$sF(s) - f(0)$
3	nth derivative, $f^{(n)}(t)$	$s^n F(s) - s^{n-1} f(0) - s^{n-2} f'(0) - \cdots - s f^{(n-2)}(0) - f^{(n-1)}(0)$ †
4	Superposition, $C_1 f_1(t) + C_2 f_2(t) + \cdots + C_n f_n(t)$	$C_1 F_1(s) + C_2 F_2(s) + \cdots + C_n F_n(s)$
5	Shifting in s plane, $e^{at} f(t)$	$\int_0^\infty e^{-st} e^{at} f(t)\, dt = \int_0^\infty e^{-(s-a)t} f(t)\, dt = F(s-a)$
6	Shifting in t plane $\begin{cases} f(t-b) \text{ when } t > b, \\ 0 \text{ when } t < b \end{cases}$	$e^{-bs} F(s)$
	Function Transforms	
7	1	$\frac{1}{s}$
8	$\frac{t^{n-1}}{(n-1)!}$	$\frac{1}{s^n}$, for $n = 1, 2, \ldots$
9	e^{-at}	$\frac{1}{s+a}$
10	$\frac{1}{a}(1 - e^{-at})$	$\frac{1}{s(s+a)}$
11	te^{-at}	$\frac{1}{(s+a)^2}$
12	$\frac{1}{a}\sin at$	$\frac{1}{s^2 + a^2}$
13	$\frac{1}{a^2}(1 - \cos at)$	$\frac{1}{s(s^2 + a^2)}$
14	$\frac{1}{a^3}(at - \sin at)$	$\frac{1}{s^2(s^2 + a^2)}$
15	$\frac{1}{(b-a)}(e^{-at} - e^{-bt})$	$\frac{1}{(s+a)(s+b)}$
16	$\frac{1}{ab} + \frac{be^{-at} - ae^{-bt}}{ab(a-b)}$	$\frac{1}{s(s+a)(s+b)}$
17	$\frac{a \sin bt - b \sin at}{ab(a^2 - b^2)}$	$\frac{1}{(s^2 + a^2)(s^2 + b^2)}$
18	$e^{-at}(1 - at)$	$\frac{s}{(s+a)^2}$
19	$\cos at$	$\frac{s}{s^2 + a^2}$
20	Rectangular pulse 	$\frac{1 - e^{-s\tau}}{s}$
21	Sine pulse 	$\frac{\pi/\tau}{s^2 + \pi^2/\tau^2}(1 + e^{-s\tau})$

† $f(t)$ and its derivatives through $f^{(n-1)}(t)$ must be continuous.

Transformation of the Differential Equation. The differential equation for the undamped linear oscillator is given in general form by

$$\frac{1}{\omega^2_n}\ddot{\nu} + \nu = \xi(t) \tag{8.11}$$

Applying the operational transforms (items 1 and 3, Table 8.2), Eq. (8.11) is transformed to

$$\frac{1}{\omega^2_n} s^2 F_r(s) - \frac{1}{\omega^2_n} sf(0) - \frac{1}{\omega^2_n} f'(0) + F_r(s) = F_e(s) \tag{8.12a}$$

where $F_r(s)$ = the transform of the unknown response $\nu(t)$, sometimes called the *response transform*

$s^2F_r(s) - sf(0) - f'(0)$ = the transform of the second derivative of $\nu(t)$

$f(0)$ and $f'(0)$ = the known *initial values* of ν and $\dot{\nu}$, i.e., ν_0 and $\dot{\nu}_0$

$F_e(s)$ = the transform of the known excitation function $\xi(t)$, written $F_e(s) = \mathfrak{L}[\xi(t)]$, sometimes called the *driving transform*

It should be noted that the initial conditions of the system are explicit in Eq. (8.12*a*).

The Subsidiary Equation. Solving Eq. (8.12*a*) for $F_r(s)$,

$$F_r(s) = \frac{sf(0) + f'(0) + \omega_n^{\ 2}F_e(s)}{s^2 + \omega_n^{\ 2}} \tag{8.12b}$$

This is known as the *subsidiary equation* of the differential equation. The first two terms of the transform derive from the initial conditions of the system, and the third term derives from the excitation.

Inverse Transformation. In order to determine the response function $\nu(t)$, which is the solution of the differential equation, an inverse transformation is performed on the subsidiary equation. The entire operation, applied explicitly to the solution of Eq. (8.11), may be abbreviated as follows:

$$\nu(t) = \mathfrak{L}^{-1}[F_r(s)] = \mathfrak{L}^{-1}\left[\frac{s\nu_0 + \dot{\nu}_0 + \omega_n^{\ 2}\mathfrak{L}[\xi(t)]}{s^2 + \omega_n^{\ 2}}\right] \tag{8.13}$$

Example 8.4: Rectangular Step Excitation. In this case $\xi(t) = \xi_c$ for $0 \le t$ (Fig. 8.6*A*). The Laplace transform $F_e(s)$ of the excitation is, from item 7 of Table 8.2,

$$\mathfrak{L}[\xi_c] = \xi_c\mathfrak{L}[1] = \xi_c\frac{1}{s}$$

Assume that the starting conditions are general, that is, $\nu = \nu_0$ and $\dot{\nu} = \dot{\nu}_0$ at $t = 0$. Substituting the transform and the starting conditions into Eq. (8.13), the following is obtained:

$$\nu(t) = \mathfrak{L}^{-1}\left[\frac{s\nu_0 + \dot{\nu}_0 + \omega^2_n\xi_c(1/s)}{s^2 + \omega^2_n}\right] \tag{8.14a}$$

The foregoing may be rewritten as three separate inverse transforms:

$$\nu(t) = \nu_0\mathfrak{L}^{-1}\left[\frac{s}{s^2 + \omega_n^{\ 2}}\right] + \dot{\nu}_0\mathfrak{L}^{-1}\left[\frac{1}{s^2 + \omega_n^{\ 2}}\right] + \xi_c\omega_n^{\ 2}\mathfrak{L}^{-1}\left[\frac{1}{s(s^2 + \omega_n^{\ 2})}\right] \tag{8.14b}$$

The inverse transforms in Eq. (8.14*b*) are evaluated by use of items 19, 12 and 13, respectively, in Table 8.2. Thus, the time-response function is given explicitly by

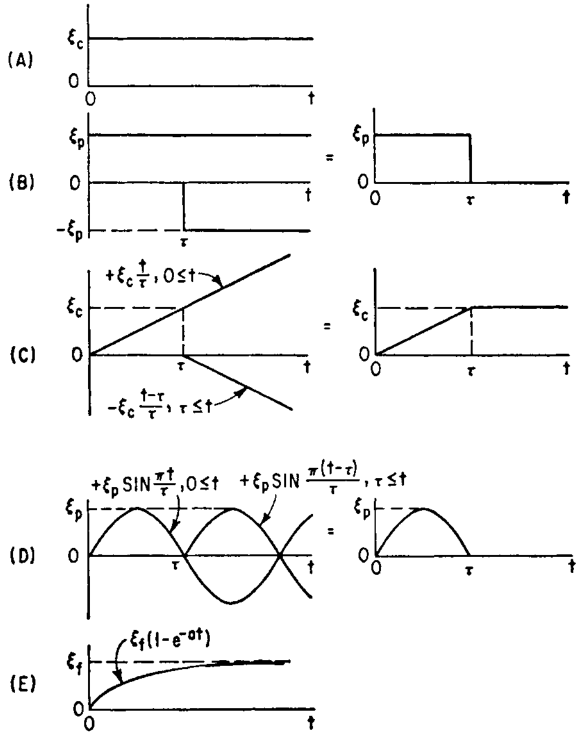


FIGURE 8.6 Excitation functions in examples of use of the Laplace transform: (*A*) rectangular step, (*B*) rectangular pulse, (*C*) step with constant-slope front, (*D*) sine pulse, and (*E*) step with exponential asymptotic rise.

$$\nu(t) = \nu_0 \cos \omega_n t + \frac{\dot{\nu}_0}{\omega_n} \sin \omega_n t + \xi_c(1 - \cos \omega_n t) \tag{8.14c}$$

The first two terms are the same as the starting condition response terms given by expressions (8.20*a*). The third term agrees with the response function shown by Eq. (8.22), derived for the case of a start from rest.

Example 8.5: Rectangular Pulse Excitation. The excitation function, Fig. 8.6*B*, is given by

$$\xi(t) = \begin{cases} \xi_p & \text{for } 0 \le t \le \tau \\ 0 & \text{for } \tau \le t \end{cases}$$

For simplicity, assume a start from rest, i.e., $\nu_0 = 0$ and $\dot{\nu}_0 = 0$ when $t = 0$.

During the first time interval, $0 \le t \le \tau$, the response function is of the same form as Eq. (8.14*c*) except that, with the assumed start from rest, the first two terms are zero.

During the second interval, $\tau \le t$, the transform of the excitation is obtained by applying the delayed-function transform (item 6, Table 8.2) and the transform for the rectangular step function (item 7) with the following result:

$$F_e(s) = \mathfrak{L}[\xi(t)] = \xi_p\left(\frac{1}{s} - \frac{e^{-s\tau}}{s}\right)$$

This is the transform of an excitation consisting of a rectangular step of height $-\xi_p$ starting at time $t = \tau$, superimposed on the rectangular step of height $+\xi_p$ starting at time $t = 0$.

Substituting for $\mathfrak{L}[\xi(t)]$ in Eq. (8.13),

$$\nu(t) = \xi_p\omega_n^2\left\{\mathfrak{L}^{-1}\left[\frac{1}{s(s^2+\omega_n^2)}\right] - \mathfrak{L}^{-1}\left[\frac{e^{-s\tau}}{s(s^2+\omega_n^2)}\right]\right\} \tag{8.15a}$$

The first inverse transform in Eq. (8.15*a*) is the same as the third one in Eq. (8.14*b*) and is evaluated by use of item 13 in Table 8.2. However, the second inverse transform requires the use of items 6 and 13. The function-transform pair given by item 6 indicates that when $t < b$ the inverse transform in question is zero, and when $t > b$ the inverse transform is evaluated by replacing t by $t - b$ (in this particular case, by $t - \tau$). The result is as follows:

$$\begin{aligned}\nu(t) &= \xi_p\omega_n^2\left\{\frac{1}{\omega_n^2}(1-\cos\omega_n t) - \frac{1}{\omega_n^2}[1-\cos\omega_n(t-\tau)]\right\}\\ &= 2\xi_p \sin\frac{\pi\tau}{T}\sin\omega_n\left(t-\frac{\tau}{2}\right) \qquad [\tau \le t]\end{aligned} \tag{8.15b}$$

Theorem on the Transform of Functions Shifted in the Original (t) Plane. In Example 8.5, use is made of the theorem on the transform of functions shifted in the original plane. The theorem (item 6 in Table 8.2) is known variously as the second shifting theorem, the theorem on the transform of delayed functions, and the time-displacement theorem. In determining the transform of the excitation, the theorem provides for shifting, i.e., displacing the excitation or a component of the excitation in the positive direction along the time axis. This suggests the term *delayed function.* Examples of the shifting of component parts of the excitation appear in Fig. 8.6*B*, 8.6*C*, and 8.6*D*. Use of the theorem also is necessary in determining, by means of inverse transformation, the response following the delay in the excitation. Further illustration of the use of the theorem is shown by the next two examples.

Example 8.6: Step Function with Constant-slope Front. The excitation function (Fig. 8.6*C*) is expressed as follows:

$$\xi(t) = \begin{cases}\xi_c\dfrac{t}{\tau} & [0 \le t \le \tau]\\ \xi_c & [\tau \le t]\end{cases}$$

Assume that $\nu_0 = 0$ and $\dot{\nu}_0 = 0$.

The driving transforms for the first and second time intervals are

$$\mathfrak{L}[\xi(t)] = \begin{cases}\xi_c\dfrac{1}{\tau}\dfrac{1}{s^2} & [0 \le t \le \tau]\\ \xi_c\dfrac{1}{\tau}\left(\dfrac{1}{s^2} - \dfrac{e^{-s\tau}}{s^2}\right) & [\tau \le t]\end{cases}$$

The transform for the second interval is the transform of a *negative* constant slope excitation, $-\xi_c(t-\tau)/\tau$, starting at $t = \tau$, superimposed on the transform for the positive constant slope excitation, $+\xi_c t/\tau$, starting at $t = 0$.

Substituting the transforms and starting conditions into Eq. (8.13), the responses for the two time eras, in terms of the transformations, are

$$\nu(t) = \begin{cases} \xi_c \dfrac{\omega_n^2}{\tau} \mathcal{L}^{-1}\left[\dfrac{1}{s^2(s^2+\omega_n^2)}\right] & [0 \le t \le \tau] \\ \xi_c \dfrac{\omega_n^2}{\tau} \left\{\mathcal{L}^{-1}\left[\dfrac{1}{s^2(s^2+\omega_n^2)}\right] - \mathcal{L}^{-1}\left[\dfrac{e^{-s\tau}}{s^2(s^2+\omega_n^2)}\right]\right\} & [\tau \le t] \end{cases} \tag{8.16a}$$

Evaluation of the inverse transforms by reference to Table 8.2 [item 14 for the first of Eqs. (8.16*a*), items 6 and 14 for the second] leads to the following:

$$\nu(t) = \begin{cases} \xi_c \dfrac{\omega_n^2}{\tau} \dfrac{1}{\omega_n^3} (\omega_n t - \sin \omega_n t) & [0 \le t \le \tau] \\ \xi_c \dfrac{\omega_n^2}{\tau} \left\{\dfrac{1}{\omega_n^3} (\omega_n t - \sin \omega_n t) - \dfrac{1}{\omega_n^3} [\omega_n(t-\tau) - \sin \omega_n(t-\tau)]\right\} & [\tau \le t] \end{cases}$$

Simplifying,

$$\nu(t) = \begin{cases} \xi_c \dfrac{1}{\omega_n \tau} (\omega_n t - \sin \omega_n t) & [0 \le t \le \tau] \\ \xi_c \left[1 + \dfrac{2}{\omega_n \tau} \sin \dfrac{\omega_n \tau}{2} \cos \omega_n \left(t - \dfrac{\tau}{2}\right)\right] & [\tau \le t] \end{cases} \tag{8.16b}$$

Example 8.7: Half-cycle Sine Pulse. The excitation function (Fig. 8.6*D*) is

$$\xi(t) = \begin{cases} \xi_p \sin \dfrac{\pi t}{\tau} & [0 \le t \le \tau] \\ 0 & [\tau \le t] \end{cases}$$

Let the system start from rest. The driving transforms are

$$\mathcal{L}[\xi(t)] = \begin{cases} \xi_p \dfrac{\pi}{\tau} \dfrac{1}{s^2 + \pi^2/\tau^2} & [0 \le t \le \tau] \\ \xi_p \dfrac{\pi}{\tau} \left(\dfrac{1}{s^2 + \pi^2/\tau^2} + \dfrac{e^{-s\tau}}{s^2 + \pi^2/\tau^2}\right) & [\tau \le t] \end{cases}$$

The driving transform for the second interval is the transform of a sine wave of *positive* amplitude ξ_p and frequency π/τ starting at time $t = \tau$, superimposed on the transform of a sine wave of the same amplitude and frequency starting at time $t = 0$.

By substitution of the driving transforms and the starting conditions into Eq. (8.13), the following are found:

$$\nu(t) = \begin{cases} \xi_p \dfrac{\pi}{\tau} \omega_n^2 \mathcal{L}^{-1}\left[\dfrac{1}{s^2 + \pi^2/\tau^2} \cdot \dfrac{1}{s^2 + \omega_n^2}\right] & [0 \le t \le \tau] \\ \xi_p \dfrac{\pi}{\tau} \omega_n^2 \left\{\mathcal{L}^{-1}\left[\dfrac{1}{s^2 + \pi^2/\tau^2} \cdot \dfrac{1}{s^2 + \omega_n^2}\right] + \mathcal{L}^{-1}\left[\dfrac{e^{-s\tau}}{s^2 + \pi^2/\tau^2} \cdot \dfrac{1}{s^2 + \omega_n^2}\right]\right\} & [\tau \le t] \end{cases} \tag{8.17a}$$

Determining the inverse transforms from Table 8.2 [item 17 for the first of Eqs. (8.17*a*), items 6 and 17 for the second]:

$$\nu(t) = \begin{cases} \xi_p \dfrac{\pi}{\tau} \; \omega_n^2 \dfrac{\omega_n \sin(\pi t/\tau) - (\pi/\tau)\sin \omega_n t}{(\pi\omega_n/\tau)\,(\omega_n^2 - \pi^2/\tau^2)} & [0 \le t \le \tau] \\ \xi_p \dfrac{\pi}{\tau} \; \omega_n^2 \left[\dfrac{\omega_n \sin(\pi t/\tau) - (\pi/\tau)\sin \omega_n t}{(\pi\omega_n/\tau)\,(\omega_n^2 - \pi^2/\tau^2)} \right. \\ \qquad \left. + \dfrac{\omega_n \sin[\pi(t-\tau)/\tau] - (\pi/\tau)\sin \omega_n (t-\tau)}{(\pi\omega_n/\tau)\,(\omega_n^2 - \pi^2/\tau^2)} \right] & [\tau \le t] \end{cases}$$

Simplifying,

$$\nu(t) = \begin{cases} \xi_p \dfrac{1}{1 - T^2/4\tau^2} \left(\sin \dfrac{\pi t}{\tau} - \dfrac{T}{2\tau} \sin \omega_n t \right) & [0 \le t \le \tau] \\ \xi_p \dfrac{(T/\tau)\cos(\pi\tau/T)}{(T^2/4\tau^2) - 1} \sin \omega_n \left(t - \dfrac{\tau}{2} \right) & [\tau \le t] \end{cases} \tag{8.17b}$$

where $T = 2\pi/\omega_n$ is the natural period of the responding system. Equations (8.17*b*) are equivalent to Eqs. (8.6*b*) and (8.7*b*) derived previously by the use of Duhamel's integral.

Example 8.8: Exponential Asymptotic Step. The excitation function (Fig. 8.6*E*) is

$$\xi(t) = \xi_f (1 - e^{-at}) \qquad [0 \le t]$$

Assume that the system starts from rest. The driving transform is

$$\mathfrak{L}[\xi(t)] = \xi_f \left(\frac{1}{s} - \frac{1}{s+a} \right) = \xi_f a \frac{1}{s(s+a)}$$

It is found by Eq. (8.13) that

$$\nu(t) = \xi_f a \omega_n^2 \mathfrak{L}^{-1} \left[\frac{1}{s(s+a)(s^2 + \omega_n^2)} \right] \qquad [0 \le t] \tag{8.18a}$$

It frequently happens that the inverse transform is not readily found in an available table of transforms. Using the above case as an example, the function of s in Eq. (8.18*a*) is first *expanded in partial fractions;* then the inverse transforms are sought, thus:

$$\frac{1}{s(s+a)(s^2 + \omega_n^2)} = \frac{\kappa_1}{s} + \frac{\kappa_2}{s+a} + \frac{\kappa_3}{s + j\omega_n} + \frac{\kappa_4}{s - j\omega_n} \tag{8.18b}$$

where $j = \sqrt{-1}$

$$\kappa_1 = \left[\frac{1}{(s+a)(s+j\omega_n)\,(s-j\omega_n)} \right]_{s=0} = \frac{1}{a\omega_n^2}$$

$$\kappa_2 = \left[\frac{1}{s(s+j\omega_n)\,(s-j\omega_n)} \right]_{s=-a} = \frac{1}{-a(a^2 + \omega_n^2)}$$

$$\kappa_3 = \left[\frac{1}{s(s+a)\,(s-j\omega_n)} \right]_{s=-j\omega_n} = \frac{1}{-2\omega_n^2(a - j\omega_n)}$$

$$\kappa_4 = \left[\frac{1}{s(s+a)\,(s+j\omega_n)}\right]_{s\,=\,+j\omega_n} = \frac{1}{-2\omega_n^2(a+j\omega_n)}$$

Consequently, Eq. (8.18*a*) may be rewritten in the following expanded form:

$$\nu(t) = \xi_f\left\{\mathcal{L}^{-1}\left[\frac{1}{s}\right] - \frac{\omega_n^2}{a^2+\omega_n^2}\,\mathcal{L}^{-1}\left[\frac{1}{s+a}\right] - \frac{a}{2(a-j\omega_n)}\cdot \mathcal{L}^{-1}\left[\frac{1}{s+j\omega_n}\right] - \frac{a}{2(a+j\omega_n)}\,\mathcal{L}^{-1}\left[\frac{1}{s-j\omega_n}\right]\right\} \tag{8.18c}$$

The inverse transforms may now be found readily (items 7 and 9, Table 8.2):

$$\nu(t) = \xi_f\left[1 - \frac{\omega_n^2}{a^2+\omega_n^2}\,e^{-at} - \frac{a}{2(a-j\omega_n)}\,e^{-j\omega_n t} - \frac{a}{2(a+j\omega_n)}\,e^{j\omega_n t}\right]$$

Rewriting,

$$\nu(t) = \xi_f\left[1 - \frac{\omega_n^2 e^{-at} + a^2 \tfrac{1}{2}(e^{j\omega_n t} + e^{-j\omega_n t}) - aj\omega_n \tfrac{1}{2}(e^{j\omega_n t} - e^{-j\omega_n t})}{a^2+\omega_n^2}\right]$$

Making use of the relations, $\cos z = (\tfrac{1}{2})(e^{jz} + e^{-jz})$ and $\sin z = -j(\tfrac{1}{2})\,(e^{jz} - e^{-jz})$, the equation for $\nu(t)$ may be expressed as follows:

$$\nu(t) = \xi_f\left[1 - \frac{(a/\omega_n)[\sin \omega_n t + (a/\omega_n)\cos \omega_n t] + e^{-at}}{1 + a^2/\omega_n^2}\right] \tag{8.18d}$$

Partial Fraction Expansion of F(s). The partial fraction expansion of $F_r(s)$, illustrated for a particular case in Eq. (8.18*b*), is a necessary part of the technique of solution. In general $F_r(s)$, expressed by the subsidiary equation (8.12*b*) and involved in the inverse transformation, Eqs. (8.10) and (8.13), is a quotient of two polynomials in *s*, thus

$$F_r(s) = \frac{A(s)}{B(s)} \tag{8.19}$$

The purpose of the expansion of $F_r(s)$ is to divide it into simple parts, the inverse transforms of which may be determined readily. The general procedure of the expansion is to factor $B(s)$ and then to rewrite $F_r(s)$ in partial fractions.[16,17]

INITIAL CONDITIONS OF THE SYSTEM

In all the solutions for response presented in this chapter, unless otherwise stated, it is assumed that the initial conditions (ν_0 and $\dot{\nu}_0$) of the system are both zero. Other starting conditions may be accounted for merely by superimposing on the time-response functions given the additional terms

$$\nu_0 \cos \omega_n t + \frac{\dot{\nu}_0}{\omega_n}\,\sin \omega_n t \tag{8.20a}$$

These terms are the complete solution of the homogeneous differential equation, $m\ddot{\nu}/k + \nu = 0$. They represent the free vibration resulting from the initial conditions.

The two terms in Eq. (8.20*a*) may be expressed by either one of the following combined forms:

$$\sqrt{\nu_0^2 + \left(\frac{\dot{\nu}_0}{\omega_n}\right)^2} \sin (\omega_n t + \theta_1) \qquad \text{where } \tan \theta_1 = \frac{\nu_0 \omega_n}{\dot{\nu}_0} \tag{8.20b}$$

$$\sqrt{\nu_0^2 + \left(\frac{\dot{\nu}_0}{\omega_n}\right)^2} \cos (\omega_n t - \theta_2) \qquad \text{where } \tan \theta_2 = \frac{\dot{\nu}_0}{\nu_0 \omega_n} \tag{8.20c}$$

where $\sqrt{\nu_0^2 + \left(\frac{\dot{\nu}_0}{\omega_n}\right)^2}$ is the resultant amplitude and θ_1 or θ_2 is the phase angle of the *initial-condition free vibration.*

PRINCIPLE OF SUPERPOSITION

When the system is linear, the *principle of superposition* may be employed. Any number of component excitation functions may be superimposed to obtain a prescribed total excitation function, and the corresponding component response functions may be superimposed to arrive at the total response function. However, the superposition must be carried out on a time basis and with complete regard for algebraic sign. The superposition of maximum component responses, disregarding time, may lead to completely erroneous results. For example, the response functions given by Eqs. (8.31) to (8.34) are defined completely with regard to time and algebraic sign, and may be superimposed for any combination of the excitation functions from which they have been derived.

COMPILATION OF RESPONSE FUNCTIONS AND RESPONSE SPECTRA; SINGLE DEGREE-OF-FREEDOM, LINEAR, UNDAMPED SYSTEMS

STEP-TYPE EXCITATION FUNCTIONS

Constant-Force Excitation (Simple Step in Force). The excitation is a constant force applied to the mass at zero time, $\xi(t) \equiv F(t)/k = F_c/k$. Substituting this excitation for $F(t)/k$ in Eq. (8.1*a*) and solving for the absolute displacement x,

$$x = \frac{F_c}{k}(1 - \cos \omega_n t) \tag{8.21a}$$

Constant-Displacement Excitation (Simple Step in Displacement). The excitation is a constant displacement of the ground which occurs at zero time, $\xi(t) \equiv u(t) = u_c$. Substituting for $u(t)$ in Eq. (8.1*b*) and solving for the absolute displacement x,

$$x = u_c(1 - \cos \omega_n t) \tag{8.21b}$$

Constant-Acceleration Excitation (Simple Step in Acceleration). The excitation is an instantaneous change in the ground acceleration at zero time, from zero to a constant value $\ddot{u}(t) = \ddot{u}_c$. The excitation is thus

$$\xi(t) \equiv -m\ddot{u}_c/k = -\ddot{u}_c/\omega_n^2$$

Substituting in Eq. (8.1c) and solving for the *relative* displacement δ_x,

$$\delta_x = \frac{-\ddot{u}_c}{\omega_n}\,(1 - \cos \omega_n t) \tag{8.21c}$$

When the excitation is defined by a function of acceleration $\ddot{u}(t)$, it is often convenient to express the response in terms of the absolute acceleration $\ddot{x}$ of the system. The force acting on the mass in Fig. 8.1C is $-k\,\delta_x$; the acceleration $\ddot{x}$ is thus $-k\,\delta_x/m$ or $-\delta_x\omega_n^2$. Substituting $\delta_x = -\ddot{x}/\omega_n^2$ in Eq. (8.21c),

$$\ddot{x} = \ddot{u}_c(1 - \cos \omega_n t) \tag{8.21d}$$

The same result is obtained by letting $\xi(t) \equiv \ddot{u}(t) = \ddot{u}_c$ in Eq. (8.1d) and solving for $\ddot{x}$. Equation (8.21d) is similar to Eq. (8.21b) with acceleration instead of displacement on both sides of the equation. This analogy generally applies in step- and pulse-type excitations.

The absolute displacement of the mass can be obtained by integrating Eq. (8.21d) twice with respect to time, taking as initial conditions $x = \dot{x} = 0$ when $t = 0$,

$$x = \frac{\ddot{u}_c}{\omega_n^2}\left[\frac{\omega_n^2 t^2}{2} - (1 - \cos \omega_n t)\right] \tag{8.21e}$$

Equation (8.21e) also may be obtained from the relation $x = u + \delta_x$, noting that in this case $u(t) = \ddot{u}_c t^2/2$.

Constant-Velocity Excitation (Simple Step in Velocity). This excitation, when expressed in terms of ground or spring anchorage motion, is equivalent to prescribing, at zero time, an instantaneous change in the ground velocity from zero to a constant value $\dot{u}_c$. The excitation is $\xi(t) \equiv u(t) = \dot{u}_c t$, and the solution for the differential equation of Eq. (8.1b) is

$$x = \frac{\dot{u}_c}{\omega_n}\,(\omega_n t - \sin \omega_n t) \tag{8.21f}$$

For the velocity of the mass,

$$\dot{x} = \dot{u}_c(1 - \cos \omega_n t) \tag{8.21g}$$

The result of Eq. (8.21g) could have been obtained directly by letting $\xi(t) \equiv \dot{u}(t) = \dot{u}_c$ in Eq. (8.1e) and solving for the *velocity* response $\dot{x}$.

General Step Excitation. A comparison of Eqs. (8.21a), (8.21b), (8.21c), (8.21d), and (8.21g) with Table 8.1 reveals that the response ν and the excitation ξ are related in a common manner. This may be expressed as follows:

$$\nu = \xi_c(1 - \cos \omega_n t) \tag{8.22}$$

where ξ_c indicates a constant value of the excitation. The excitation and response of the system are shown in Fig. 8.7.

Absolute Displacement Response to Velocity-Step and Acceleration-Step Excitations. The absolute displacement responses to the velocity-step and the acceleration-step excitations are given by Eqs. (8.21f) and (8.21e) and are shown in Figs. 8.8 and 8.9, respectively. The comparative effects of displacement-step, velocity-step, and acceleration-step excitations, in terms of *absolute displacement* response, may be seen by comparing Figs. 8.7 to 8.9.

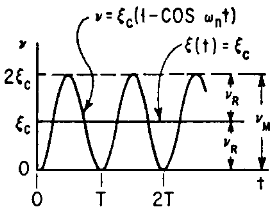


FIGURE 8.7 Time response to a simple step excitation (general notation).

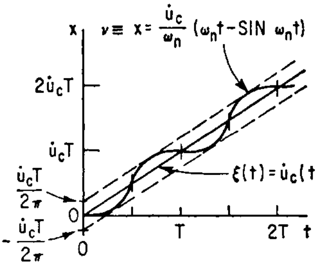


FIGURE 8.8 Time-displacement response to a constant-velocity excitation (simple step in velocity).

In the case of the velocity-step excitation, the *velocity* of the system is always positive, except at $t = 0, T, 2T, \ldots$, when it is zero. Similarly, an acceleration-step excitation results in system *acceleration* that is always positive, except at $t = 0, T, 2T, \ldots$, when it is zero. The natural period of the responding system is $T = 2\pi/\omega_n$.

Response Maxima. In the response of a system to step or pulse excitation, the maximum value of the response often is of considerable physical significance. Several kinds of maxima are important. One of these is the *residual response amplitude,* which is the amplitude of the free vibration about the final position of the excitation as a base. This is designated ν_R, and for the response given by Eq. (8.22):

$$\nu_R = \pm\xi_c \tag{8.22a}$$

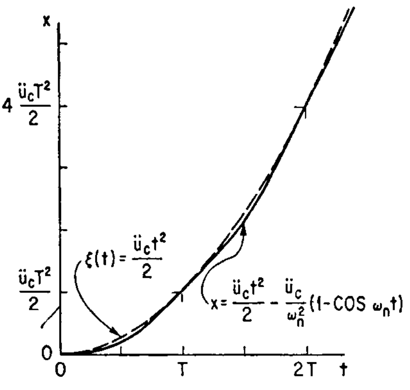


FIGURE 8.9 Time-displacement response to a constant-acceleration excitation (simple step in acceleration).

Another maximum is the *maximax response,* which is the greatest of the maxima of ν attained at *any time* during the response. In general, it is of the same sign as the excitation. For the response given by Eq. (8.22), the maximax response ν_M is

$$\nu_M = 2\xi_c \tag{8.22b}$$

Asymptotic Step. In the exponential function $\xi(t) = \xi_f(1 - e^{-at})$, the maximum value ξ_f of the excitation is approached asymptotically. This excitation may be defined alternatively by $\xi(t) = (F_f/k)(1 - e^{-at})$; $u_f(1 - e^{-at})$; $(-\ddot{u}_f/\omega_n^2)(1 - e^{-at})$; etc. (see Table 8.1). Substituting the excitation $\xi(t) = \xi_f(1 - e^{-at})$ in Eq. (8.2), the response ν is

$$\nu = \xi_f\left[1 - \frac{(a/\omega_n)\,[\sin \omega_n t + (a/\omega_n)\cos \omega_n t] + e^{-at}}{1 + a^2/\omega_n^2}\right] \tag{8.23a}$$

The excitation and the response of the system are shown in Fig. 8.10. For large values of the exponent *at,* the motion is nearly simple harmonic. The residual ampli-

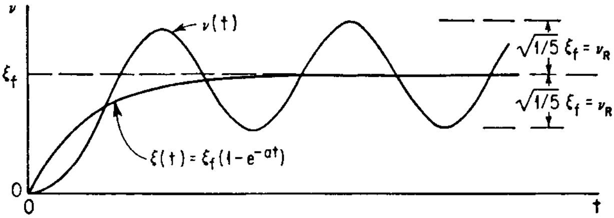


FIGURE 8.10 Time response to an exponentially asymptotic step for the particular case $\omega_n/a = 2$.

tude, relative to the final position of equilibrium, approaches the following value asymptotically.

$$\nu_R \rightarrow \xi_f \frac{1}{\sqrt{1 + \omega_n^2/a^2}} \tag{8.23b}$$

The maximax response $\nu_M = \nu_R + \xi_f$ is plotted against ω_n/a to give the response spectrum in Fig. 8.11.

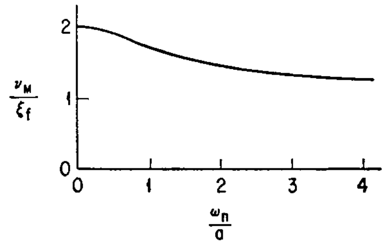


FIGURE 8.11 Spectrum for maximax response resulting from exponentially asymptotic step excitation.

Step-type Functions Having Finite Rise Time. Many step-type excitation functions rise to the constant maximum value ξ_c of the excitation in a finite length of time τ, called the *rise time.* Three such functions and their first three time derivatives are shown in Fig. 8.12. The step having a *cycloidal* front is the only one of the three that does not include an infinite third derivative; i.e., if the step is a ground displacement, it does not have an infinite rate of change of ground acceleration (infinite "jerk").

The excitation functions and the expressions for maximax response are given by the following equations:

Constant-slope front:

$$\xi(t) = \begin{cases} \xi_c \dfrac{t}{\tau} & [0 \le t \le \tau] \\ \xi_c & [\tau \le t] \end{cases} \tag{8.24a}$$

$$\frac{\nu_M}{\xi_c} = 1 + \left| \frac{T}{\pi\tau} \sin \frac{\pi\tau}{T} \right| \tag{8.24b}$$

Versed-sine front:

$$\xi(t) = \begin{cases} \dfrac{\xi_c}{2}\left(1 - \cos \dfrac{\pi t}{\tau}\right) & [0 \le t \le \tau] \\ \xi_c & [\tau \le t] \end{cases} \tag{8.25a}$$

$$\frac{\nu_M}{\xi_c} = 1 + \left| \frac{1}{(4\tau^2/T^2) - 1} \cos \frac{\pi\tau}{T} \right| \tag{8.25b}$$

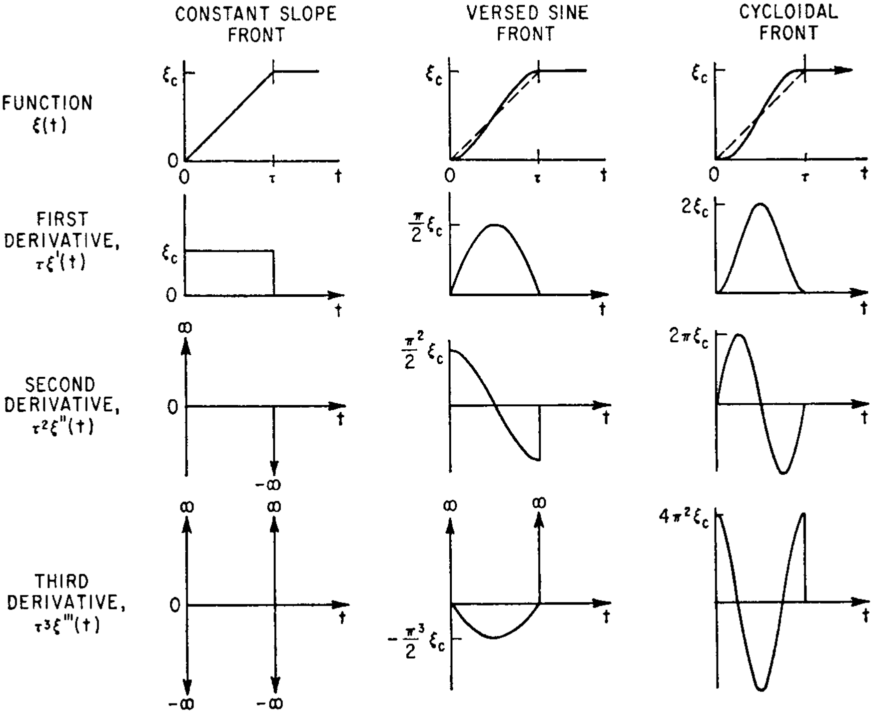


FIGURE 8.12 Three step-type excitation functions and their first three time derivatives. (*Jacobsen and Ayre.*[22])

Cycloidal front:

$$\xi(t) = \begin{cases} \dfrac{\xi_c}{2\pi}\left(\dfrac{2\pi t}{\tau} - \sin\dfrac{2\pi t}{\tau}\right) & [0 \le t \le \tau] \\ \xi_c & [\tau \le t] \end{cases} \tag{8.26a}$$

$$\frac{\nu_M}{\xi_c} = 1 + \left|\frac{T}{\pi\tau(1 - \tau^2/T^2)}\sin\frac{\pi\tau}{T}\right| \tag{8.26b}$$

where $T = 2\pi/\omega_n$ is the natural period of the responding system.

In the case of step-type excitations, the maximax response occurs after the excitation has reached its constant maximum value ξ_c and is related to the residual response amplitude by

$$\nu_M = \nu_R + \xi_c \tag{8.27}$$

Figure 8.13 shows the spectra of maximax response versus step rise time τ expressed relative to the natural period T of the responding system. In Fig. 8.13*A* the comparison is based on *equal rise times,* and in Fig. 8.13*B* it relates to *equal maximum slopes of the step fronts.* The residual response amplitude has values of zero

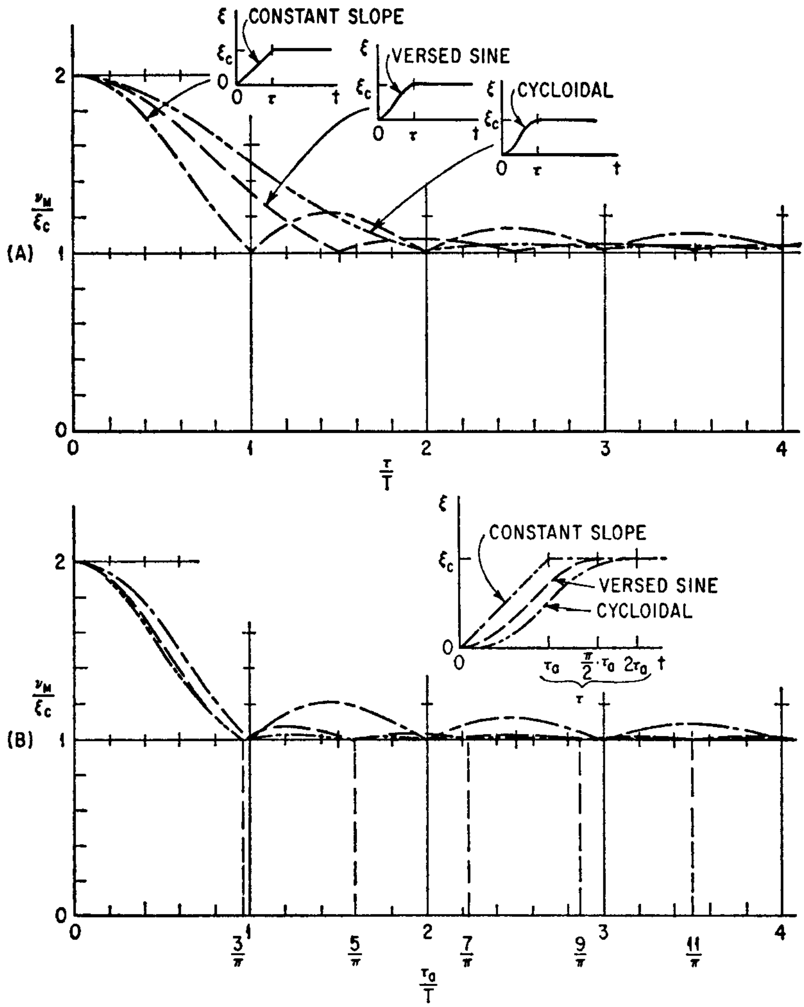


FIGURE 8.13 Spectra of maximax response resulting from the step excitation functions of Fig. 8.12. (*A*) For step functions having equal rise time τ. (*B*) For step functions having equal maximum slope ξ_c/τ_a. (*Jacobsen and Ayre.*[22])

($\nu_M/\xi_c = 1$) in all three cases; for example, the step excitation having a constant-slope front results in zero residual amplitude at $\tau/T = 1, 2, 3, \ldots$.

A Family of Exponential Step Functions Having Finite Rise Time. The inset diagram in Fig. 8.14 shows and Eqs. (8.28*a*) define a family of step functions having fronts which rise exponentially to the constant maximum ξ_c in the rise time τ. Two limiting cases of vertically fronted steps are included in the family: When $a \to -\infty$, the vertical front occurs at $t = 0$; when $a \to +\infty$, the vertical front occurs at $t = \tau$. An

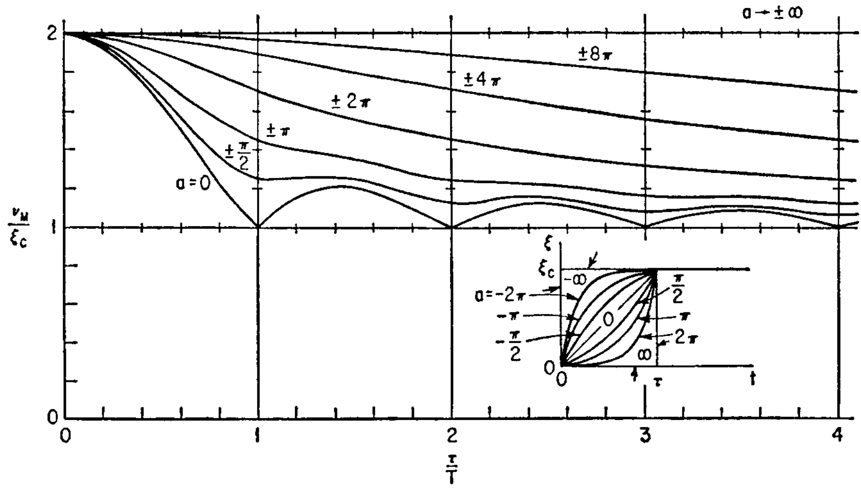


FIGURE 8.14 Spectra of maximax response for a family of step functions having exponential fronts, including the vertical fronts $a \to \pm\infty$, and the constant-slope front $a = 0$, as special cases. (*Jacobsen and Ayre.*[22])

intermediate case has a constant-slope front ($a = 0$). The maximax responses are given by Eq. (8.28*b*) and by the response spectra in Fig. 8.14. The values of the maximax response are independent of the sign of the parameter *a*.

$$\xi(t) = \begin{cases} \xi_c \dfrac{1 - e^{at/\tau}}{1 - e^a} & [0 \le t \le \tau] \\ \xi_c & [\tau \le t] \end{cases} \tag{8.28a}$$

$$\frac{\nu_M}{\xi_c} = 1 + \left| \frac{a}{1 - e^a} \left[\frac{1 - 2e^a \cos(2\pi\tau/T) + e^{2a}}{a^2 + 4\pi^2\tau^2/T^2} \right]^{1/2} \right| \tag{8.28b}$$

where T is the natural period of the responding system.

There are zeroes of residual response amplitude ($\nu_M/\xi_c = 1$) at finite values of τ/T only for the constant-slope front ($a = 0$). Each of the step functions represented in Fig. 8.13 results in zeroes of residual response amplitude, and each function has antisymmetry with respect to the half-rise time $\tau/2$. This is of interest in the selection of cam and control-function shapes, where one of the criteria of choice may be *minimum residual amplitude of vibration* of the driven system.

PULSE-TYPE EXCITATION FUNCTIONS

The Simple Impulse. If the duration τ of the pulse is short relative to the natural period T of the system, the response of the system may be determined by equating the impulse J, i.e., the force-time integral, to the momentum $m\dot{x}_J$:

$$J = \int_0^\tau F(t)\, dt = m\dot{x}_J \tag{8.29a}$$

Thus, it is found that the impulsive velocity $\dot{x}_J$ is equal to J/m. Consequently, the velocity-time response is given by $\dot{x} = \dot{x}_J \cos \omega_n t = (J/m) \cos \omega_n t$. The displacement-time response is obtained by integration, assuming a start from rest,

$$x = x_J \sin \omega_n t$$

where

$$x_J = \frac{J}{m\omega_n} = \omega_n \int_0^\tau \frac{F(t)\, dt}{k} \tag{8.29b}$$

The impulse concept, used for determining the response to a short-duration force pulse, may be generalized in terms of ν and ξ by referring to Table 8.1. The *generalized impulsive response* is

$$\nu = \nu_J \sin \omega_n t \tag{8.30a}$$

where the amplitude is

$$\nu_J = \omega_n \int_0^\tau \xi(t)\, dt \tag{8.30b}$$

The impulsive response amplitude ν_J and the generalized impulse $k \int_0^\tau \xi(t)\, dt$ are used in comparing the effects of various pulse shapes when the pulse durations are short.

Symmetrical Pulses. In the following discussion a comparison is made of the responses caused by single symmetrical pulses of rectangular, half-cycle sine, versed-sine, and triangular shapes. The excitation functions and the time-response equations are given by Eqs. (8.31) to (8.34). Note that the residual response amplitude factors are set in brackets and are identified by the time interval $\tau \le t$.

Rectangular:

$$\left.\begin{aligned} \xi(t) &= \xi_p \\ \nu &= \xi_p(1 - \cos \omega_n t) \end{aligned}\right\} \quad [0 \le t \le \tau] \tag{8.31a}$$

$$\left.\begin{aligned} \xi(t) &= 0 \\ \nu &= \xi_p \left[2 \sin \frac{\pi\tau}{T} \right] \sin \omega_n \left(t - \frac{\tau}{2} \right) \end{aligned}\right\} \quad [\tau \le t] \tag{8.31b}$$

Half-cycle sine:

$$\left.\begin{aligned} \xi(t) &= \xi_p \sin \frac{\pi t}{\tau} \\ \nu &= \frac{\xi_p}{1 - T^2/4\tau^2} \left(\sin \frac{\pi t}{\tau} - \frac{T}{2\tau} \sin \omega_n t \right) \end{aligned}\right\} \quad [0 \le t \le \tau] \tag{8.32a}$$

$$\left.\begin{aligned} \xi(t) &= 0 \\ \nu &= \xi_p \left[\frac{(T/\tau) \cos (\pi\tau/T)}{(T^2/4\tau^2) - 1} \right] \sin \omega_n \left(t - \frac{\tau}{2} \right) \end{aligned}\right\} \quad [\tau \le t] \tag{8.32b}$$

Versed-sine:

$$\left.\begin{aligned} \xi(t) &= \frac{\xi_p}{2}\left(1 - \cos\frac{2\pi t}{\tau}\right) \\ \nu &= \frac{\xi_p/2}{1 - \tau^2/T^2}\left(1 - \frac{\tau^2}{T^2} + \frac{\tau^2}{T^2}\cos\frac{2\pi t}{\tau} - \cos\omega_n t\right) \end{aligned}\right\} \quad [0 \le t \le \tau] \qquad (8.33a)$$

$$\left.\begin{aligned} \xi(t) &= 0 \\ \nu &= \xi_p\left[\frac{\sin\pi\tau/T}{1 - \tau^2/T^2}\right]\sin\omega_n\left(t - \frac{\tau}{2}\right) \end{aligned}\right\} \quad [\tau \le t] \qquad (8.33b)$$

Triangular:

$$\left.\begin{aligned} \xi(t) &= 2\xi_p\frac{t}{\tau} \\ \nu &= 2\xi_p\left(\frac{t}{\tau} - \frac{T}{\tau}\frac{\sin\omega_n t}{2\pi}\right) \end{aligned}\right\} \quad \left[0 \le t \le \frac{\tau}{2}\right] \qquad (8.34a)$$

$$\left.\begin{aligned} \xi(t) &= 2\xi_p\left(1 - \frac{t}{\tau}\right) \\ \nu &= 2\xi_p\left(1 - \frac{t}{\tau} - \frac{T}{\tau}\frac{\sin\omega_n t}{2\pi} + \frac{T}{\tau}\frac{\sin\omega_n(t - \tau/2)}{\pi}\right) \end{aligned}\right\} \quad \left[\frac{\tau}{2} \le t \le \tau\right] \qquad (8.34b)$$

$$\left.\begin{aligned} \xi(t) &= 0 \\ \nu &= \xi_p\left[2\frac{\sin^2(\pi\tau/2T)}{\pi\tau/2T}\right]\sin\omega_n(t - \tau/2) \end{aligned}\right\} \quad [\tau \le t] \qquad (8.34c)$$

where T is the natural period of the responding system.

Equal Maximum Height of Pulse as Basis of Comparison. Examples of time response, for six different values of τ/T, are shown separately for the rectangular, half-cycle sine, and versed-sine pulses in Fig. 8.15, and for the triangular pulse in Fig. 8.22*B.* The basis of comparison is equal maximum height of excitation pulse ξ_p.

Residual Response Amplitude and Maximax Response. The spectra of maximax response ν_M and residual response amplitude ν_R are given in Fig. 8.16 by (*A*) for the rectangular pulse, by (*B*) for the sine pulse, and by (*C*) for the versed-sine pulse. The maximax response may occur either within the duration of the pulse or after the pulse function has dropped to zero. In the latter case the maximax response is equal to the residual response amplitude. In general, the maximax response is given by the residual response amplitude only in the case of short-duration pulses; for example, see the case $\tau/T = ¼$ in Fig. 8.15 where T is the natural period of the responding system. The response spectra for the triangular pulse appear in Fig. 8.24.

Maximax Relative Displacement When the Excitation Is Ground Displacement. When the excitation $\xi(t)$ is given as *ground displacement* $u(t)$, the response ν is the absolute displacement x of the mass (Table 8.1). It is of practical importance in the investigation of the maximax *distortion* or *stress* in the elastic element to know the maximax value of the relative displacement. In this case the relative displacement is a *derived quantity* obtained by taking the difference between the response and the excitation, that is, $x - u$ or, in terms of the general notation, $\nu - \xi$.

If the excitation is given as ground acceleration, the response is determined directly as relative displacement and is designated δ_x (Table 8.1). To avoid confusion, relative displacement determined as a *derived quantity,* as described in the first case

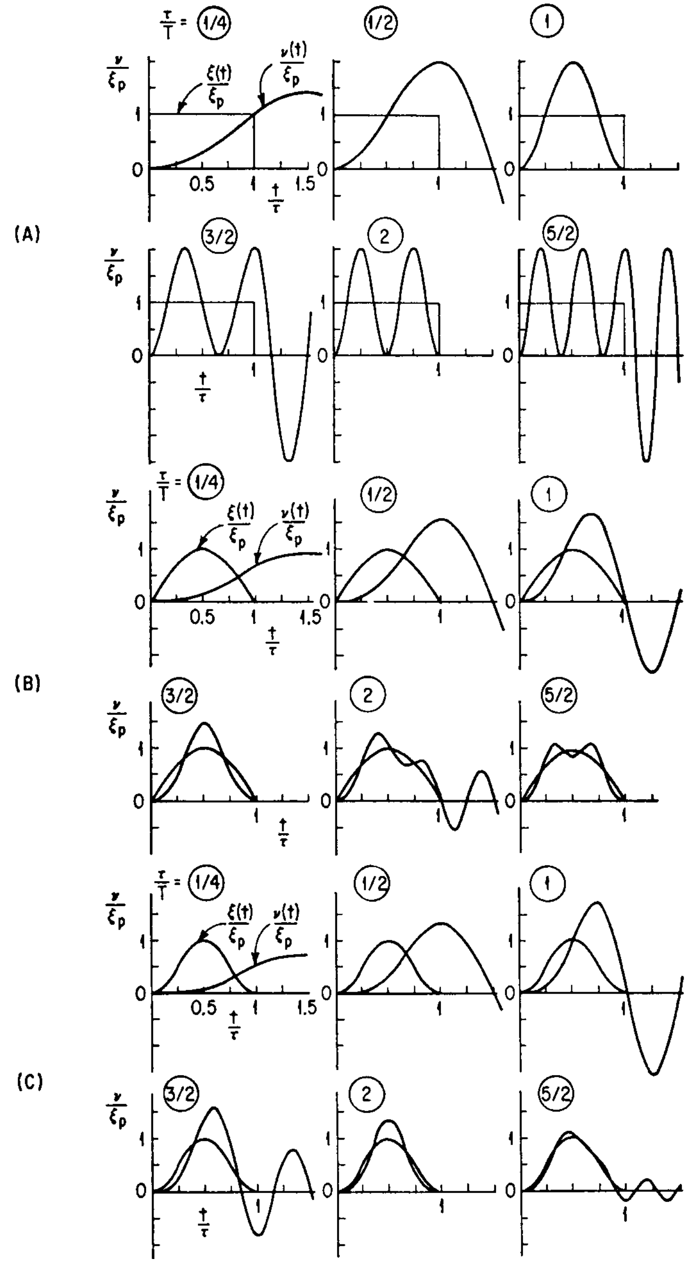


FIGURE 8.15 Time response curves resulting from single pulses of (*A*) rectangular, (*B*) half-cycle sine, and (*C*) versed-sine shapes.[19]

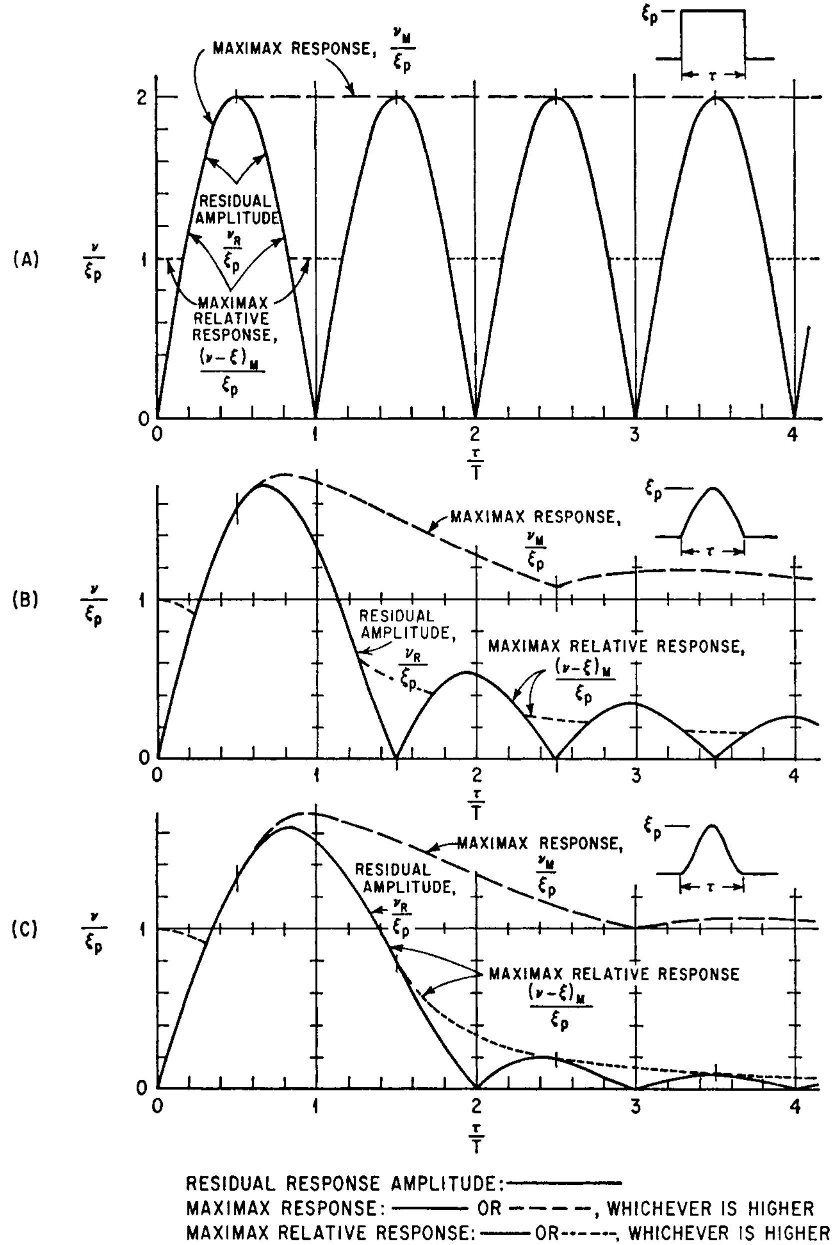


FIGURE 8.16 Spectra of maximax response, residual response amplitude, and maximax relative response resulting from single pulses of (*A*) rectangular, (*B*) half-cycle sine, and (*C*) versed-sine shapes.[19] The spectra are shown on another basis in Fig. 8.18.

above, is designated by $x - u$; relative displacement determined directly as the *response variable* (second case above) is designated by δ_x. The distinction is made readily in the general notation by use of the symbols $\nu - \xi$ and ν, respectively, for relative response and for response. The maximax values are designated $(\nu - \xi)_M$ and ν_M, respectively.

The maximax relative response may occur *either* within the duration of the pulse or during the residual vibration era ($\tau \le t$). In the latter case the maximax relative response is equal to the residual response amplitude. This explains the discontinuities which occur in the spectra of maximax relative response shown in Fig. 8.16 and elsewhere.

The meaning of the relative response $\nu - \xi$ may be clarified further by a study of the time-response and time-excitation curves shown in Fig. 8.15.

Equal Area of Pulse as Basis of Comparison. In the preceding section on the comparison of responses resulting from pulse excitation, the pulses are assumed of equal maximum height. Under some conditions, particularly if the pulse duration is short relative to the natural period of the system, it may be more useful to make the comparison on the basis of equal pulse area; i.e., equal impulse (equal time integral).

The *areas* for the pulses of maximum height ξ_p and duration τ are as follows: rectangle, $\xi_p\tau$; half-cycle sine, $(2/\pi)\xi_p\tau$; versed-sine $(\frac{1}{2})\xi_p\tau$; triangle, $(\frac{1}{2})\xi_p\tau$. Using the area of the *triangular pulse* as the basis of comparison, and requiring that the areas of the other pulses be equal to it, it is found that the pulse *heights,* in terms of the height ξ_{p0} of the *reference triangular pulse,* must be as follows: rectangle, $(\frac{1}{2})\xi_{p0}$; half-cycle sine, $(\pi/4)\xi_{p0}$; versed-sine, ξ_{p0}.

Figure 8.17 shows the time responses, for four values of τ/T, redrawn on the basis of *equal pulse area* as the criterion for comparison. Note that the response reference is the constant ξ_{p0}, which is the height of the triangular pulse. To show a direct comparison, the response curves for the various pulses are superimposed on each other. For the shortest duration shown, $\tau/T = \frac{1}{4}$, the response curves are nearly alike. Note that the responses to two different rectangular pulses are shown, one of duration τ and height $\xi_{p0}/2$, the other of duration $\tau/2$ and height ξ_{p0}, both of area $\xi_{p0}\tau/2$.

The response spectra, plotted on the basis of equal pulse area, appear in Fig. 8.18. The residual response spectra are shown altogether in (A), the maximax response spectra in (B), and the spectra of maximax relative response in (C).

Since the pulse area is $\xi_{p0}\tau/2$, the generalized impulse is $k\xi_{p0}\tau/2$, and the amplitude of vibration of the system computed on the basis of the generalized impulse theory, Eq. (8.30b), is given by

$$\nu_J = \omega_n \xi_{p0}\frac{\tau}{2} = \pi\frac{\tau}{T}\ \xi_{p0} \tag{8.35}$$

A comparison of this straight-line function with the response spectra in Fig. 8.18B shows that *for values of τ/T less than one-fourth the shape of the symmetrical pulse is of little concern.*

Family of Exponential, Symmetrical Pulses. A continuous variation in shape of pulse may be investigated by means of the family of pulses represented by Eqs. (8.36a) and shown in the inset diagram in Fig. 8.19A:

$$\xi(t) = \begin{cases} \xi_p \dfrac{1 - e^{2at}/\tau}{1 - e^a} & \left[0 \le t \le \dfrac{\tau}{2}\right] \\ \xi_p \dfrac{1 - e^{2a(1 - t/\tau)}}{1 - e^a} & \left[\dfrac{\tau}{2} \le t \le \tau\right] \\ 0 & [\tau \le t] \end{cases} \tag{8.36a}$$

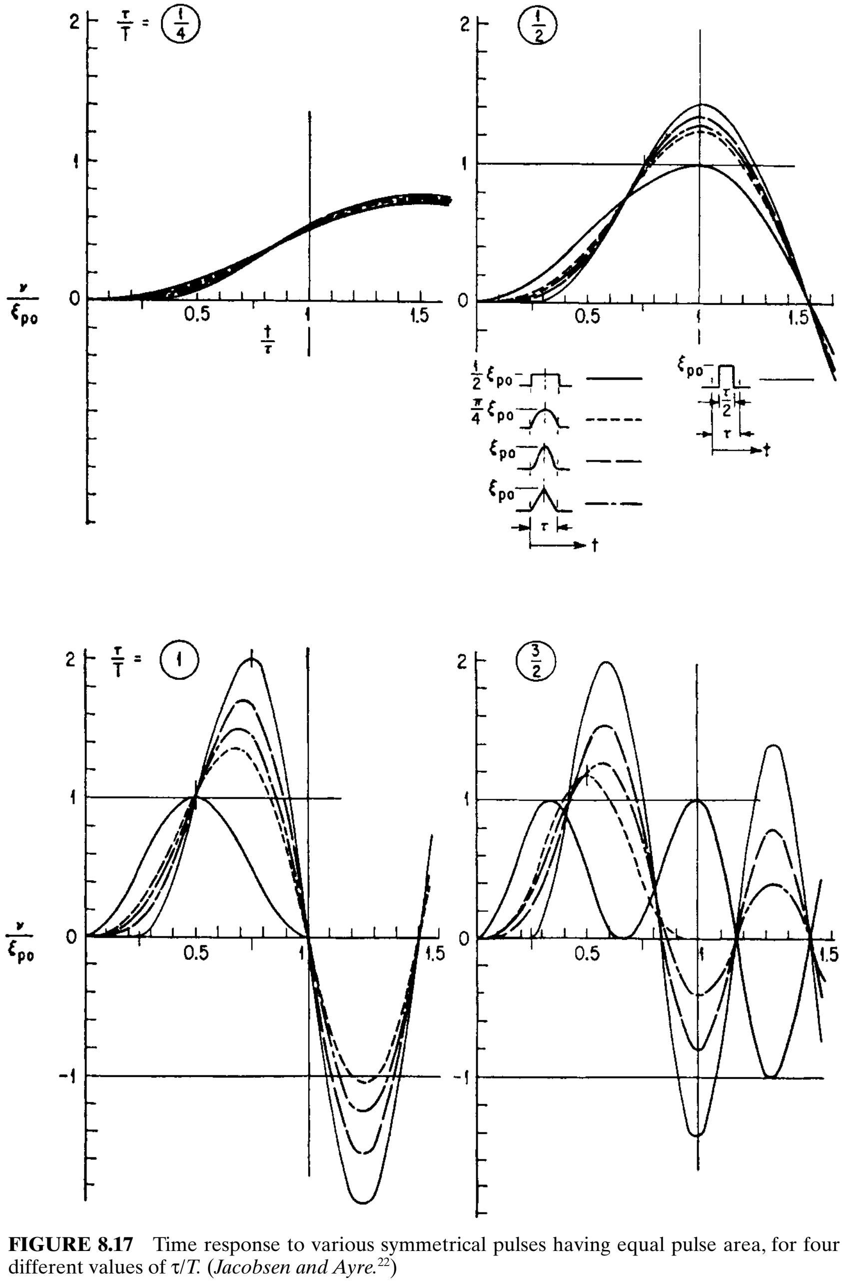


FIGURE 8.17 Time response to various symmetrical pulses having equal pulse area, for four different values of τ/T. (*Jacobsen and Ayre.*[22])

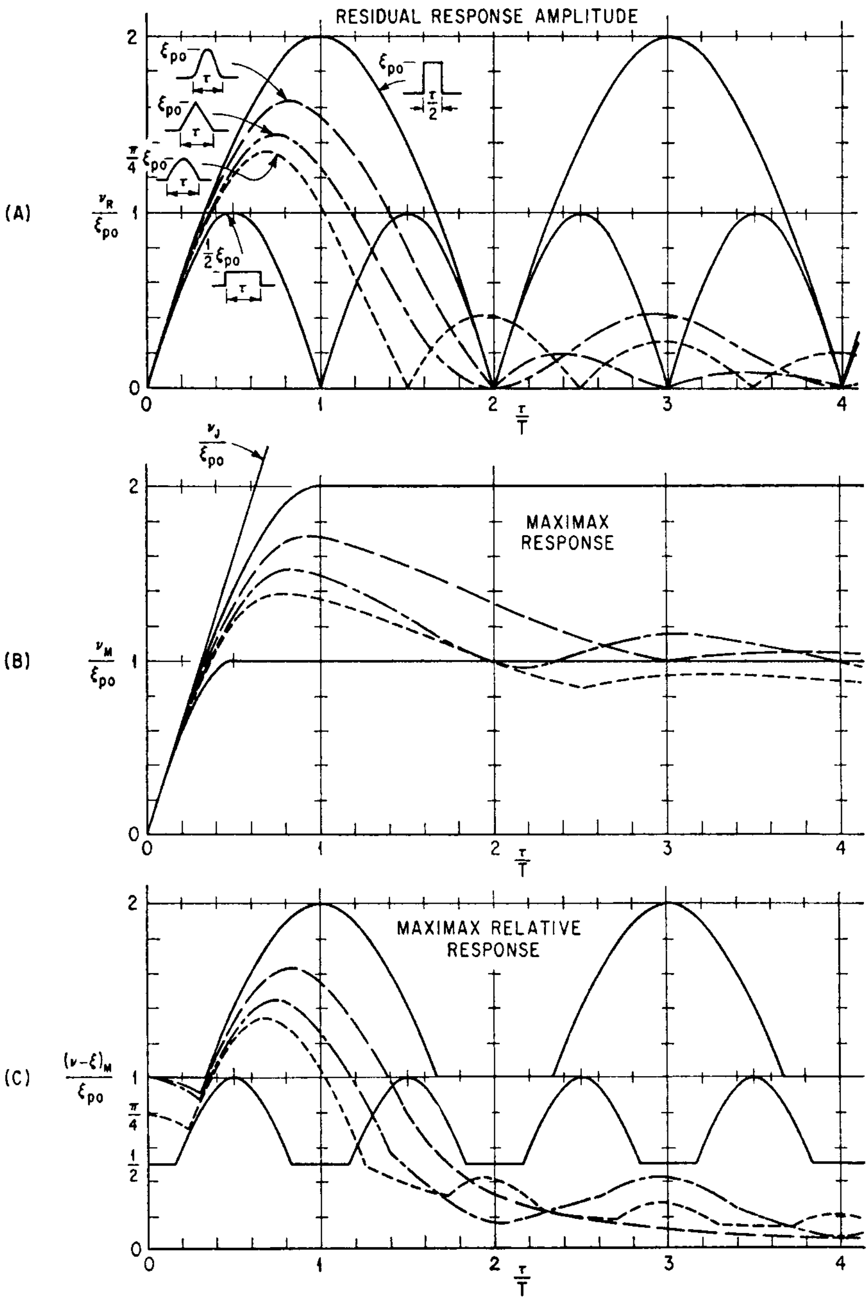
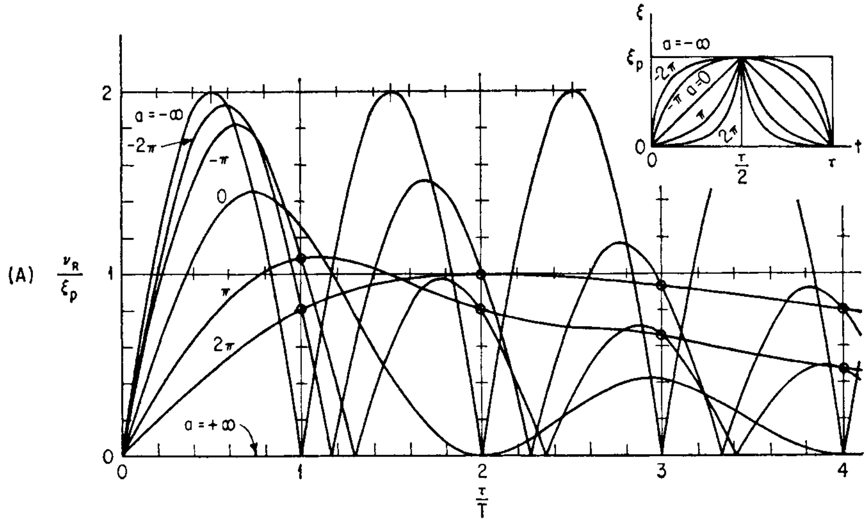


FIGURE 8.18 Response spectra for various symmetrical pulses having equal pulse area: (*A*) residual response amplitude, (*B*) maximax response, and (*C*) maximax relative response. (*Jacobsen and Ayre.*[22])



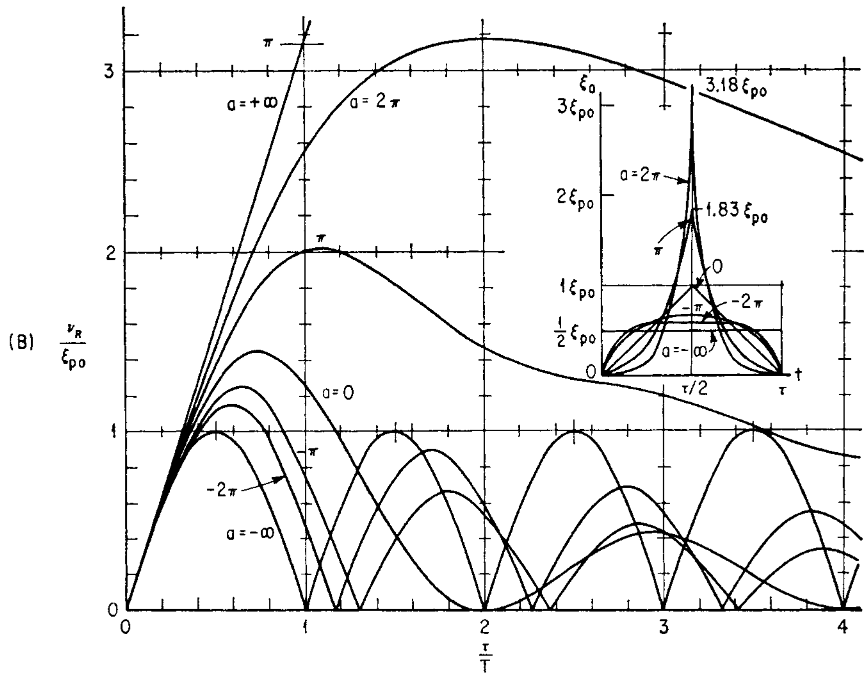


FIGURE 8.19 Spectra for residual response amplitude for a family of exponential, symmetrical pulses: (*A*) pulses having equal height; (*B*) pulses having equal area. (*Jacobsen and Ayre.*[22])

The family includes the following special cases:

$a \to -\infty$: rectangle of height ξ_p and duration τ
$a = 0$: triangle of height ξ_p and duration τ
$a \to +\infty$: spike of height ξ_p and having zero area

The residual response amplitude of vibration of the system is

$$\frac{\nu_R}{\xi_p} = \frac{2aT}{\pi\tau}\left(\frac{e^a - \cos(\pi\tau/T) - (aT/\pi\tau)\sin(\pi\tau/T)}{(1-e^a)(1+a^2T^2/\pi^2\tau^2)}\right) \tag{8.36b}$$

where T is the natural period of the responding system. Figure 8.19*A* shows the spectra for residual response amplitude for seven values of the parameter a, compared on the basis of *equal pulse height.* The zero-area spike ($a \to +\infty$) results in zero response.

The area of the general pulse of height ξ_p is

$$A_p = \xi_p \frac{\tau}{a}\left(\frac{1-e^a+a}{1-e^a}\right) \tag{8.36c}$$

If a comparison is to be drawn on the basis of *equal pulse area* using the area $\xi_{p0}\tau/2$ of the triangular pulse as the reference, the height ξ_{pa} of the general pulse is

$$\xi_{pa} = \xi_{p0}\frac{a}{2}\left(\frac{1-e^a}{1-e^a+a}\right) \tag{8.36d}$$

The residual response amplitude spectra, based on the equal-pulse-area criterion, are shown in Fig. 8.19*B.* The case $a \to +\infty$ is equivalent to a generalized impulse of value $k\xi_{p0}\tau/2$ and results in the straight-line spectrum given by Eq. (8.35).

Symmetrical Pulses Having a Rest Period of Constant Height. In the inset diagrams of Fig. 8.20 each pulse consists of a rise, a central rest period or "dwell" having constant height, and a decay. The expressions for the pulse *rise* functions may be obtained from Eqs. (8.24*a*), (8.25*a*), and (8.26*a*) by substituting $\tau/2$ for τ. The pulse *decay* functions are available from symmetry.

If the *rest* period is long enough for the maximax displacement of the system to be reached during the duration τ_r of the pulse rest, the maximax may be obtained from Eqs. (8.24*b*), (8.25*b*), and (8.26*b*) and, consequently, from Fig. 8.13. The substitution of $\tau/2$ for τ is necessary.

Equations (8.37) to (8.39) give the residual response amplitudes. The spectra computed from these equations are shown in Fig. 8.20.

Constant-slope rise and decay:

$$\frac{\nu_R}{\xi_p} = \frac{2T}{\pi\tau}\left[1 - \cos\frac{\pi\tau}{T} + \frac{1}{2}\cos\frac{2\pi\tau_r}{T} - \cos\frac{\pi(\tau+2\tau_r)}{T} + \frac{1}{2}\cos\frac{2\pi(\tau+\tau_r)}{T}\right]^{1/2} \tag{8.37}$$

Versed-sine rise and decay:

$$\frac{\nu_R}{\xi_p} = \frac{1}{1-\tau^2/T^2}\left[1 + \cos\frac{\pi\tau}{T} - \frac{1}{2}\cos\frac{2\pi\tau_r}{T} - \cos\frac{\pi(\tau+2\tau_r)}{T} - \frac{1}{2}\cos\frac{2\pi(\tau+\tau_r)}{T}\right]^{1/2} \tag{8.38}$$

Cycloidal rise and decay:

$$\frac{\nu_R}{\xi_p} = \frac{2T/\pi\tau}{1-\tau^2/4T^2}\left[\cos\frac{\pi\tau_r}{T} - \cos\frac{\pi(\tau+\tau_r)}{T}\right] \tag{8.39}$$

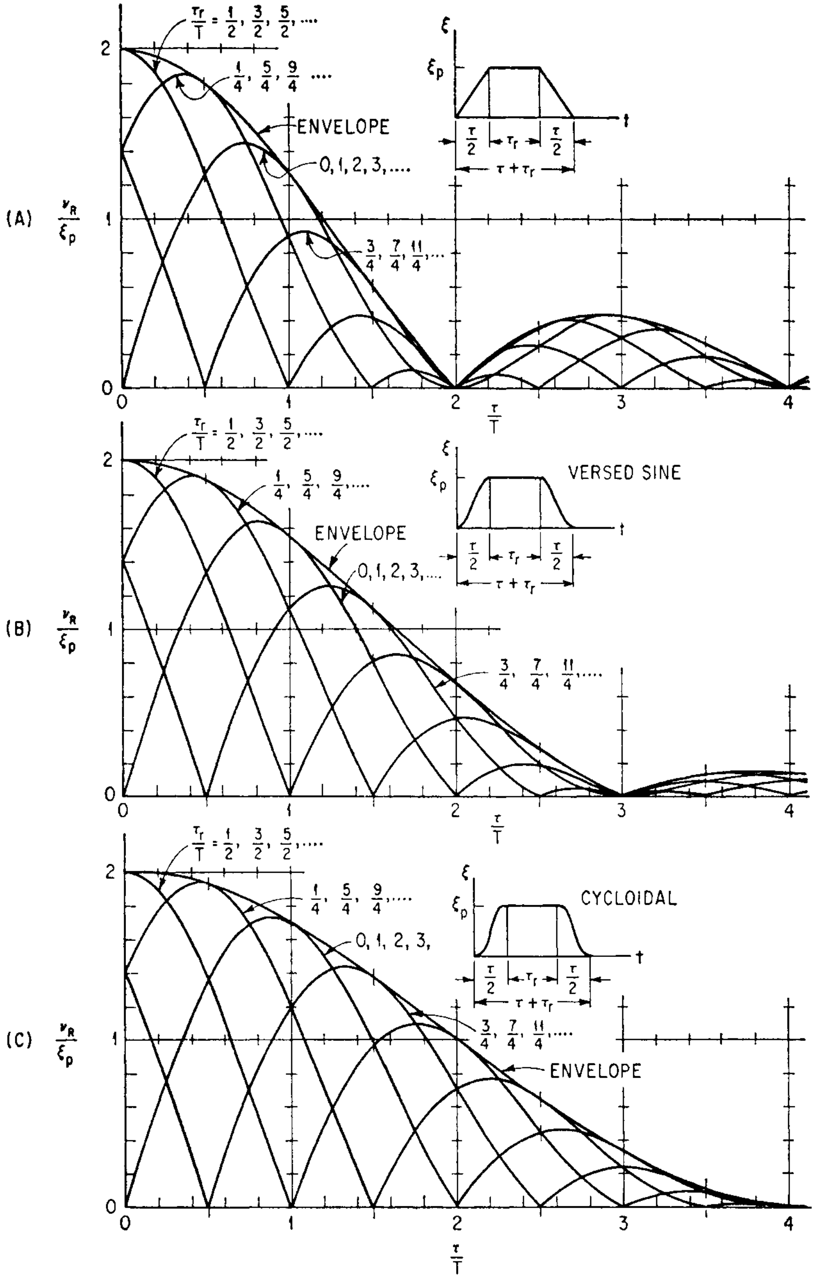


FIGURE 8.20 Residual response amplitude spectra for three families of symmetrical pulses having a central rest period of constant height and of duration τ_r. Note that the abscissa is τ/T, where τ is the sum of the rise time and the decay time. (*A*) Constant-slope rise and decay. (*B*) Versed-sine rise and decay. (*C*) Cycloidal rise and decay. (*Jacobsen and Ayre.*[22])

Note that τ in the abscissa is the sum of the rise time and the decay time and is *not* the total duration of the pulse. Attached to each spectrum is a set of values of τ_r/T where T is the natural period of the responding system.

When $\tau_r/T = 1, 2, 3, \ldots$, the residual response amplitude is equal to that for the case $\tau_r = 0$, and the spectrum starts at the origin. If $\tau_r/T = \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \ldots$, the spectrum has the maximum value 2.00 at $\tau/T = 0$. The *envelopes* of the spectra are of the same forms as the *residual-response-amplitude* spectra for the related *step functions;* see the spectra for $[(\nu_M/\xi_c) - 1]$ in Fig. 8.13*A*. In certain cases, for example, at $\tau/T = 2, 4, 6, \ldots$, in Fig. 8.20*A*, $\nu_R/\xi_p = 0$ for all values of τ_r/T.

Unsymmetrical Pulses. Pulses having only slight asymmetry may often be represented adequately by symmetrical forms. However, if there is considerable asymmetry, resulting in appreciable steepening of either the rise or the decay, it is necessary to introduce a parameter which defines the *skewing* of the pulse.

The ratio of the rise time to the pulse period is called the *skewing constant,* $\sigma = t_1/\tau$. There are three special cases:

$\sigma = 0$: The pulse has an *instantaneous* (vertical) *rise,* followed by a decay having the duration τ. This case may be used as an elementary representation of a *blast pulse.*

$\sigma = \frac{1}{2}$: The pulse may be *symmetrical.*

$\sigma = 1$: The pulse has an *instantaneous decay,* preceded by a rise having the duration τ.

Triangular Pulse Family. The effect of asymmetry in pulse shape is shown readily by means of the family of triangular pulses (Fig. 8.21). Equations (8.40) give the excitation and the time response.

Rise era: $0 \leq t \leq t_1$

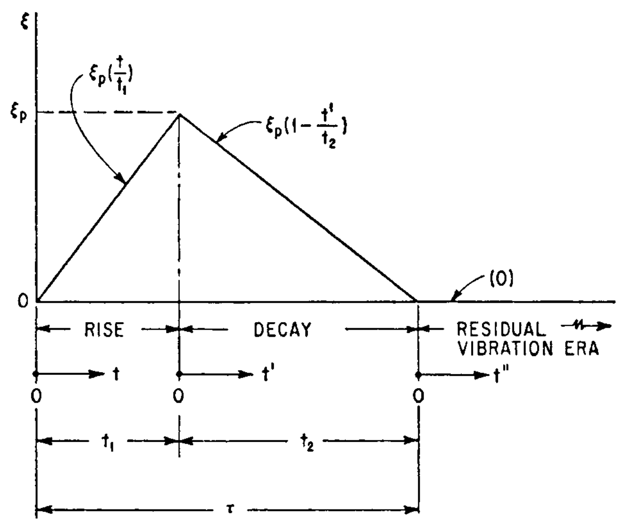


FIGURE 8.21 General triangular pulse.

$$\xi(t) = \xi_p \frac{t}{t_1}$$
$$\nu = \xi_p \left(\frac{t}{t_1} - \frac{T}{2\pi t_1} \sin \omega_n t \right) \tag{8.40a}$$

Decay era: $0 \le t' \le t_2$, where $t' = t - t_1$

$$\xi(t) = \xi_p \left(1 - \frac{t'}{t_2} \right)$$
$$\nu = \xi_p \left[1 - \frac{t'}{t_2} + \frac{T}{2\pi t_2} \left(1 + 4 \frac{t_2}{t_1} \frac{\tau}{t_1} \sin^2 \frac{\pi t_1}{T} \right)^{1/2} \sin (\omega_n t' + \theta') \right] \tag{8.40b}$$

where

$$\tan \theta' = \frac{\sin (2\pi t_1/T)}{\cos (2\pi t_1/T) - \tau/t_2}$$

Residual-vibration era: $0 \le t''$, where $t'' = t - \tau = t - t_1 - t_2$

$$\xi(t) = 0$$
$$\nu = \xi_p \frac{1}{\pi} \left[\frac{T}{t_1} \frac{T}{t_2} \left(\frac{\tau}{t_1} \sin^2 \frac{\pi t_1}{T} + \frac{\tau}{t_2} \sin^2 \frac{\pi t_2}{T} - \sin^2 \frac{\pi \tau}{T} \right) \right]^{1/2} \sin (\omega_n t'' + \theta_R) \tag{8.40c}$$

where

$$\tan \theta_R = \frac{(\tau/t_2) \sin (2\pi t_2/T) - \sin (2\pi\tau/T)}{(\tau/t_2) \cos (2\pi t_2/T) - \cos (2\pi\tau/T) - t_1/t_2}$$

For the special cases $\sigma = 0$, ½, and 1, the time responses for six values of τ/T are shown in Fig. 8.22, where T is the natural period of the responding system. Some of the curves are superposed in Fig. 8.23 for easier comparison. The response spectra appear in Fig. 8.24. The straight-line spectrum ν_J/ξ_p for the amplitude of response based on the impulse theory also is shown in Fig. 8.24*A*. In the two cases of extreme skewing, $\sigma = 0$ and $\sigma = 1$, the residual amplitudes are *equal* and are given by Eq. (8.41*a*). For the symmetrical case, $\sigma = ½$, ν_R is given by Eq. (8.41*b*).

$$\sigma = 0 \text{ and } 1: \quad \frac{\nu_R}{\xi_p} = \left[1 - \frac{T}{\pi\tau} \sin \frac{2\pi\tau}{T} + \left(\frac{T}{\pi\tau} \right)^2 \sin^2 \frac{\pi\tau}{T} \right]^{1/2} \tag{8.41a}$$

$$\sigma = ½: \quad \frac{\nu_R}{\xi_p} = 2 \frac{\sin^2 (\pi\tau/2T)}{\pi\tau/2T} \tag{8.41b}$$

The residual response amplitudes for other cases of skewness may be determined from the amplitude term in Eqs. (8.40*c*); they are shown by the response spectra in Fig. 8.25. The residual response amplitudes resulting from single pulses that are mirror images of each other in time are equal. In general, the phase angles for the residual vibrations are unequal.

Note that in the cases $\sigma = 0$ and $\sigma = 1$ for *vertical rise* and *vertical decay,* respectively, there are *no zeroes of residual amplitude,* except for the trivial case, $\tau/T = 0$.

The family of triangular pulses is particularly advantageous for investigating the effect of varying the skewness, because both criteria of comparison, equal pulse height and equal pulse area, are satisfied simultaneously.

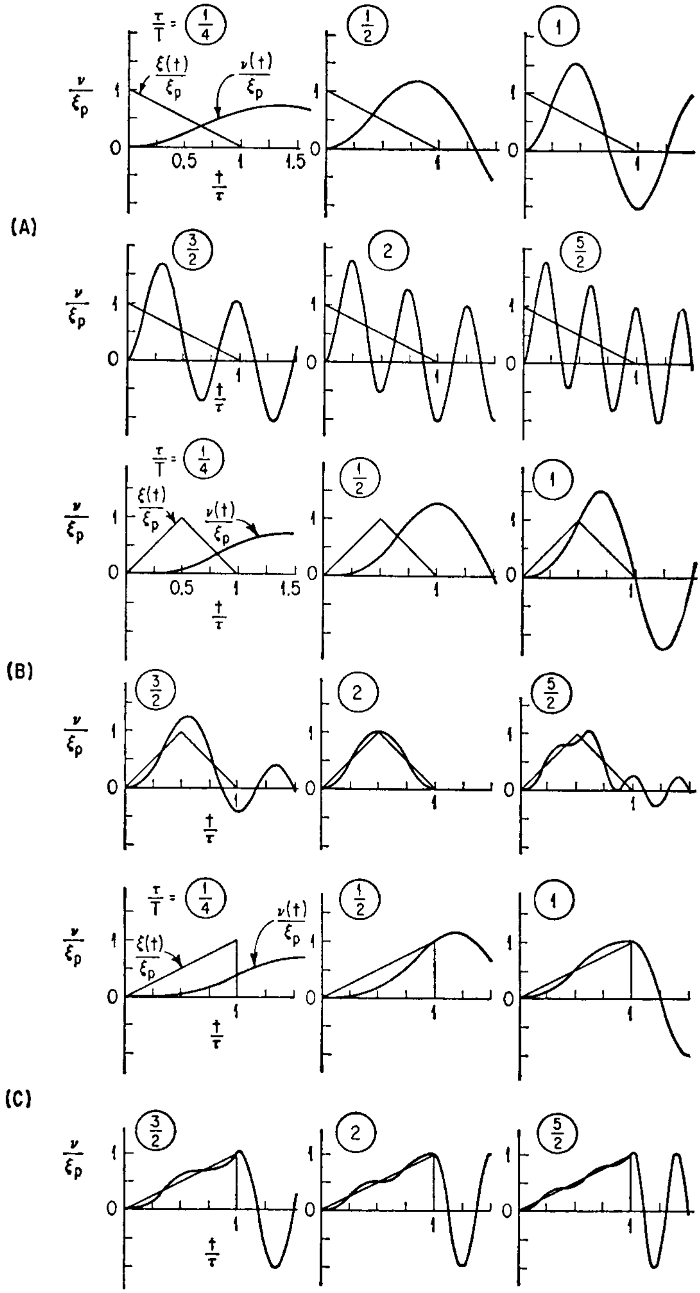


FIGURE 8.22 Time response curves resulting from single pulses of three different triangular shapes: (*A*) vertical rise (elementary blast pulse), (*B*) symmetrical, and (*C*) vertical decay.[19]

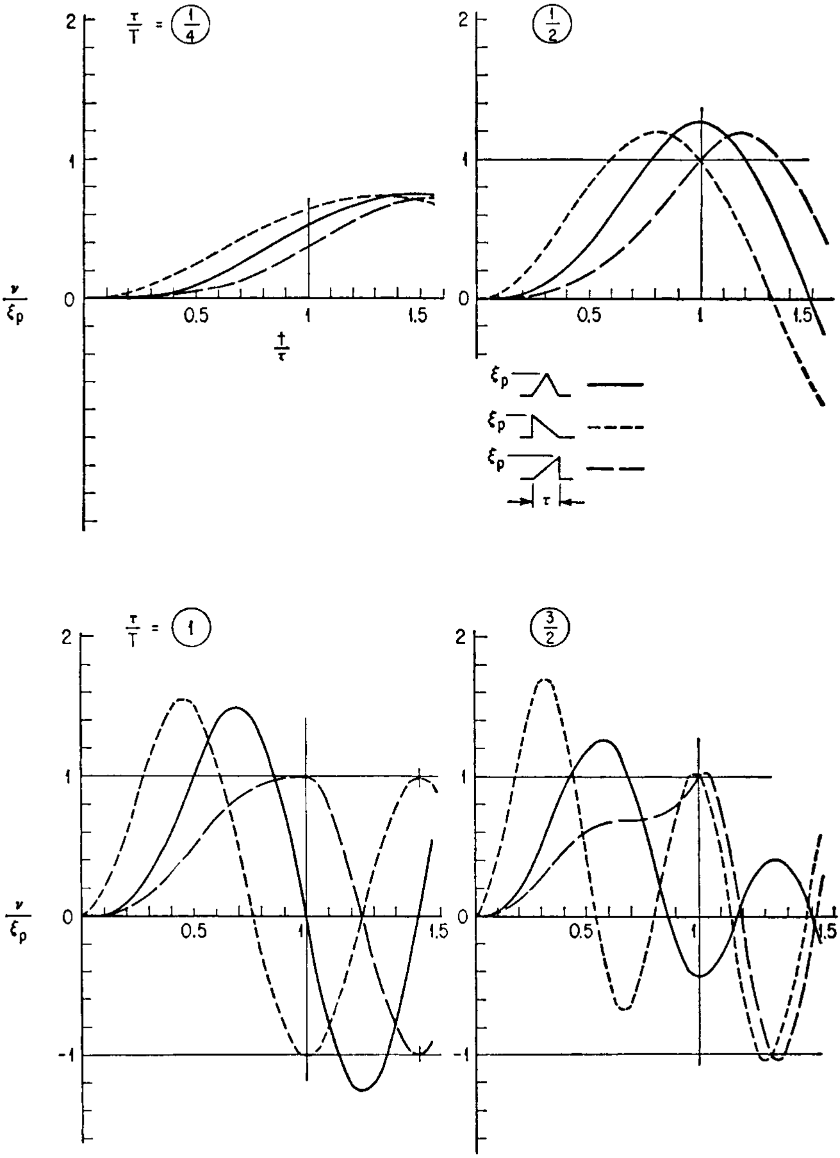


FIGURE 8.23 Time response curves of Fig. 8.22 superposed, for four values of τ/*T*. (*Jacobsen and Ayre.*[22])

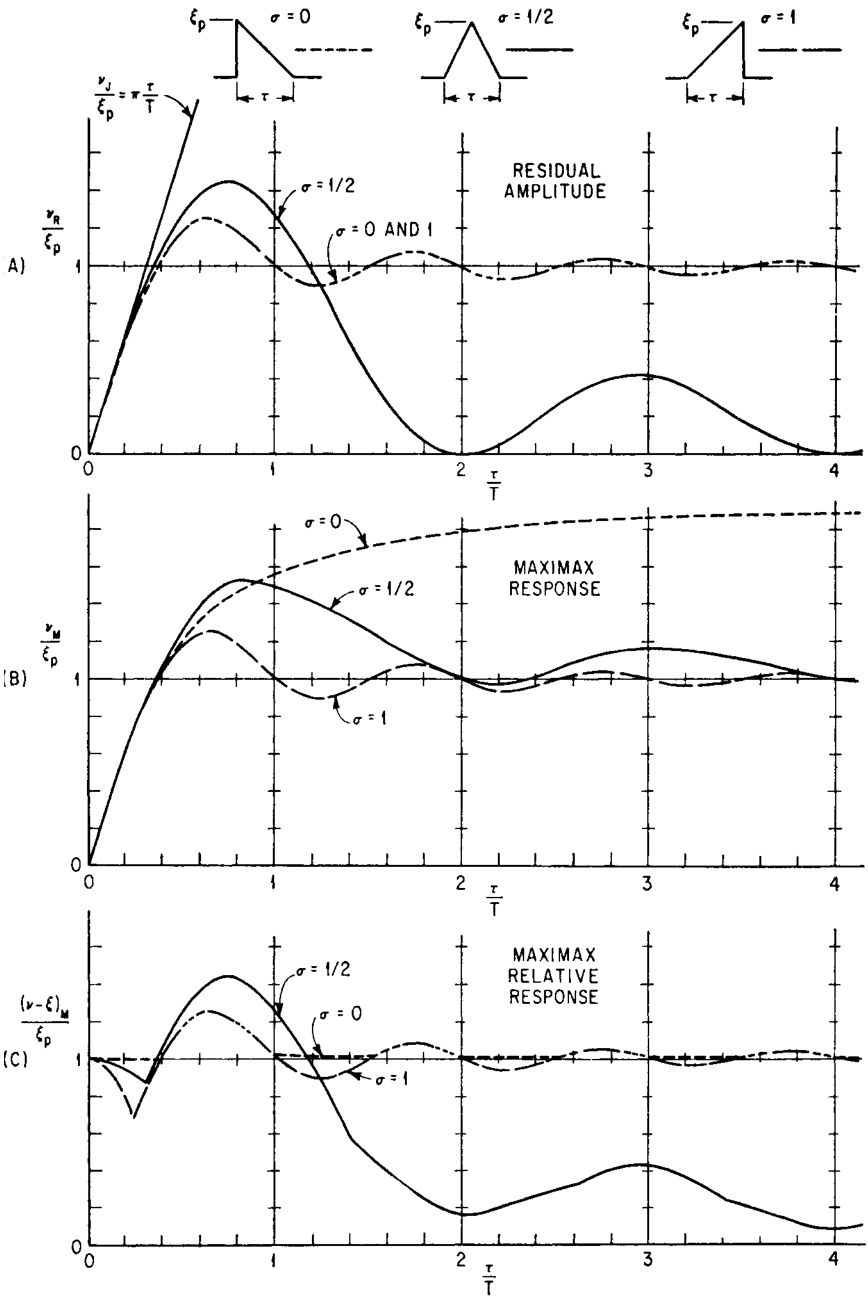


FIGURE 8.24 Response spectra for three types of triangular pulse: (*A*) Residual response amplitude. (*B*) Maximax response. (*C*) Maximax relative response. (*Jacobsen and Ayre.*[22])

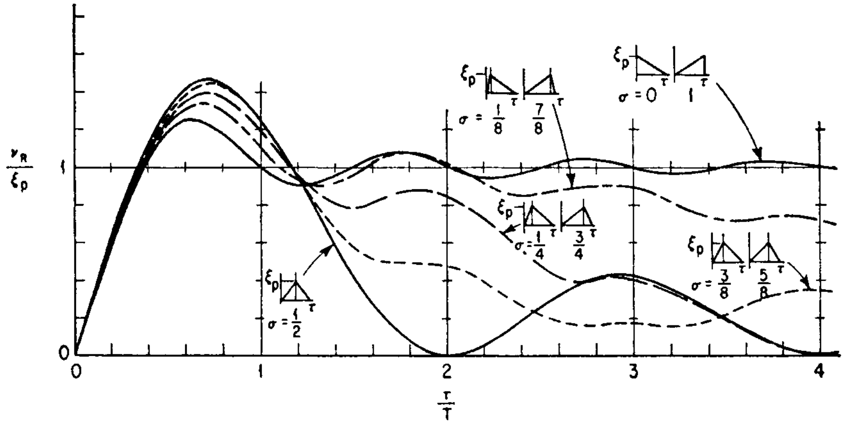


FIGURE 8.25 Spectra for residual response amplitude for a family of triangular pulses of varying skewness. (*Jacobsen and Ayre.*[22])

Various Pulses Having Vertical Rise or Vertical Decay. Figure 8.26 shows the spectra of residual response amplitude plotted on the basis of *equal pulse area.* The rectangular pulse is included for comparison. The expressions for residual response amplitude for the rectangular and the triangular pulses are given by Eqs. (8.31*b*) and (8.41*a*), and for the quarter-cycle sine and the half-cycle versed-sine pulses by Eqs. (8.42) and (8.43).

Quarter-cycle "sine":

$$\xi(t) = \xi_p \begin{cases} \sin \dfrac{\pi t}{2\tau} & \text{for vertical decay} \\ \text{or} & \\ \cos \dfrac{\pi t}{2\tau} & \text{for vertical rise} \end{cases} \qquad [0 \le t \le \tau]$$

$$\xi(t) = 0 \qquad [\tau \le t]$$

$$\frac{\nu_R}{\xi_p} = \frac{4\tau/T}{(16\tau^2/T^2) - 1}\left(1 + \frac{16\tau^2}{T^2} - \frac{8\tau}{T}\sin\frac{2\pi\tau}{T}\right)^{1/2} \tag{8.42}$$

Half-cycle "versed-sine":

$$\xi(t) = \xi_p \begin{cases} \dfrac{1}{2}\left(1 - \cos\dfrac{\pi t}{\tau}\right) & \text{for vertical decay} \\ \text{or} & \\ \dfrac{1}{2}\left(1 + \cos\dfrac{\pi t}{\tau}\right) & \text{for vertical rise} \end{cases} \qquad [0 \le t \le \tau]$$

$$\xi(t) = 0 \qquad [\tau \le t]$$

$$\frac{\nu_R}{\xi_p} = \frac{1/2}{(4\tau^2/T^2) - 1}\left[1 + \left(1 - \frac{8\tau^2}{T^2}\right)^2 - 2\left(1 - \frac{8\tau^2}{T^2}\right)\cdot\cos\frac{2\pi\tau}{T}\right]^{1/2} \tag{8.43}$$

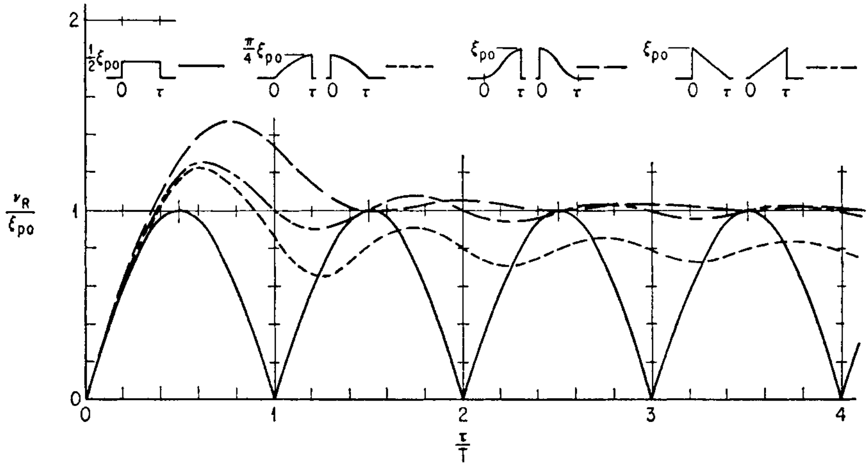


FIGURE 8.26 Spectra for residual response amplitude for various unsymmetrical pulses having either vertical rise or vertical decay. Comparison on the basis of equal pulse area.[19]

where T is the natural period of the responding system.

Note again that the residual response amplitudes, caused by single pulses that are mirror images in time, are equal. Furthermore, it is seen that the unsymmetrical pulses, having either vertical rise or vertical decay, result in no zeroes of residual response amplitude, except in the trivial case $\tau/T = 0$.

Exponential Pulses of Finite Duration, Having Vertical Rise or Vertical Decay. Families of exponential pulses having either a vertical rise or a vertical decay, as shown in the inset diagrams in Fig. 8.27, can be formed by Eqs. (8.44*a*) and (8.44*b*).

Vertical rise with exponential decay:

$$\xi(t) = \begin{cases} \xi_p \left(\dfrac{1 - e^{a(1 - t/\tau)}}{1 - e^a} \right) & [0 \le t \le \tau] \\ 0 & [\tau \le t] \end{cases} \tag{8.44a}$$

Exponential rise with vertical decay:

$$\xi(t) = \begin{cases} \xi_p \left(\dfrac{1 - e^{at/\tau}}{1 - e^a} \right) & [0 \le t \le \tau] \\ 0 & [\tau \le t] \end{cases} \tag{8.44b}$$

Residual response amplitude for *either form of pulse:*

$$\frac{\nu_R}{\xi_p} = \frac{a}{1 - e^a} \left\{ \frac{[(2\pi\tau/T)(1 - e^a)/a + \sin(2\pi\tau/T)]^2 + [1 - \cos(2\pi\tau/T)]^2}{a^2 + 4\pi^2\tau^2/T^2} \right\}^{1/2} \tag{8.44c}$$

When $a = 0$, the pulses are triangular with vertical rise or vertical decay. If $a \to +\infty$ or $-\infty$, the pulses approach the shape of a zero-area spike or of a rectangle, respectively. The spectra for residual response amplitude, plotted on the basis of *equal pulse height,* are shown in Fig. 8.27.

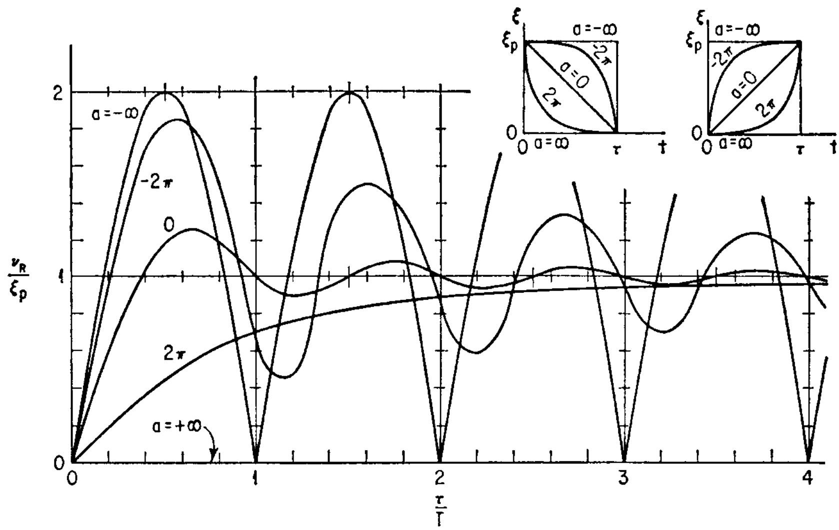


FIGURE 8.27 Spectra for residual response amplitude for unsymmetrical exponential pulses having either vertical rise or vertical decay. Comparison on the basis of equal pulse height.[19]

Figure 8.28 shows the spectra of residual response amplitude in greater detail for the range in which the parameter a is limited to positive values. This group of pulses is of interest in studying the effects of a simple form of blast pulse, in which the peak height and the duration are constant but the rate of decay is varied.

The areas of the pulses of equal height ξ_p, and the heights of the pulses of equal area $\xi_{p0}\tau/2$ are the same as for the symmetrical exponential pulses [see Eqs. (8.36*c*) and (8.36*d*)]. If the spectra in Fig. 8.28 are redrawn, using *equal pulse area* as the criterion for comparison, they appear as in Fig. 8.29. The limiting pulse case $a \to +\infty$ represents a generalized impulse of value $k\xi_{p0}\tau/2$. The asymptotic values of the spectra are equal to the peak heights of the equal area pulses and are given by

$$\frac{\nu_R}{\xi_{p0}} \to \frac{a(1-e^a)}{2\,(1-e^a+a)} \qquad \text{as } \frac{\tau}{T} \to \infty \tag{8.44d}$$

Exponential Pulses of Infinite Duration. Five different cases are included as follows:

1. The excitation function, consisting of a vertical rise followed by an exponential decay, is

$$\xi(t) = \xi_p e^{-at} \qquad [0 \le t] \tag{8.45a}$$

It is shown in Fig. 8.30. The response time equation for the system is

$$\nu = \xi_p \frac{(a/\omega_n)\sin\omega_n t - \cos\omega_n t + e^{-at}}{1 + a^2/\omega_n^{\,2}} \tag{8.45b}$$

and the asymptotic value of the residual amplitude is given by

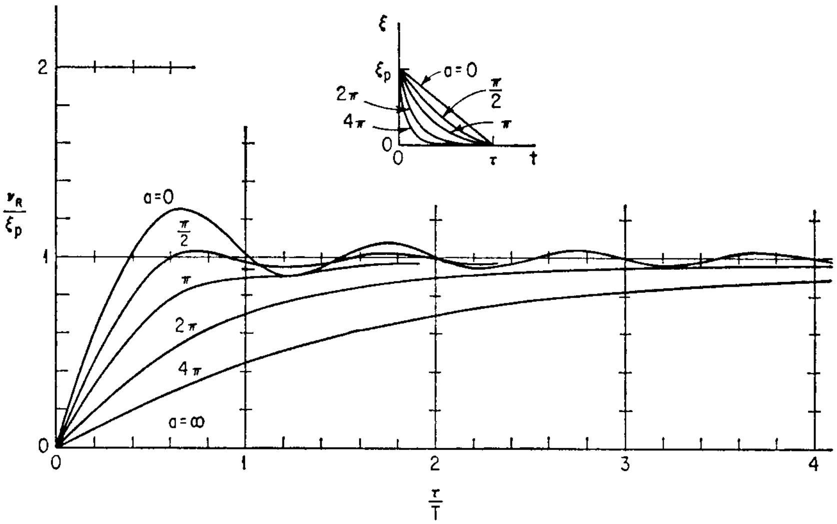


FIGURE 8.28 Spectra for residual response amplitude for a family of simple blast pulses, the same family shown in Fig. 8.27 but limited to positive values of the exponential decay parameter *a*. Comparison on the basis of equal pulse height. (These spectra also apply to mirror-image pulses having vertical decay.) (*Jacobsen and Ayre.*[22])

$$\frac{\nu_R}{\xi_p} \rightarrow \frac{1}{\sqrt{1 + a^2/\omega_n^2}} \tag{8.45c}$$

The maximax response is the first maximum of ν. The time response, for the particular case $\omega_n/a = 2$, and the response spectra are shown in Figs. 8.31 and 8.32, respectively.

2. The *difference of two exponential functions,* of the type of Eq. (8.45*a*), results in the pulse given by Eq. (8.46*a*):

$$\xi(t) = \xi_0(e^{-bt} - e^{-at})$$

$$a > b \qquad [0 \leq t] \tag{8.46a}$$

The shape of the pulse is shown in Fig. 8.33. Note that ξ_0 is the ordinate of each of the exponential functions at $t = 0$; it is *not* the pulse maximum. The asymptotic residual response amplitude is

$$\nu_R \rightarrow \xi_0 \frac{(b/\omega_n) - (a/\omega_n)}{[(1 + a^2/\omega_n^2)(1 + b^2/\omega_n^2)]^{1/2}} \tag{8.46b}$$

3. The *product of the exponential function e^{-at} by time* results in the excitation given by Eq. (8.47*a*) and shown in Fig. 8.34.

$$\xi(t) = C_0 t e^{-at} \tag{8.47a}$$

where C_0 is a constant. The peak height of the pulse ξ_p is equal to C_0/ae, and occurs at the time $t_1 = 1/a$. Equations (8.47*b*) and (8.47*c*) give the time response and the asymptotic residual response amplitude:

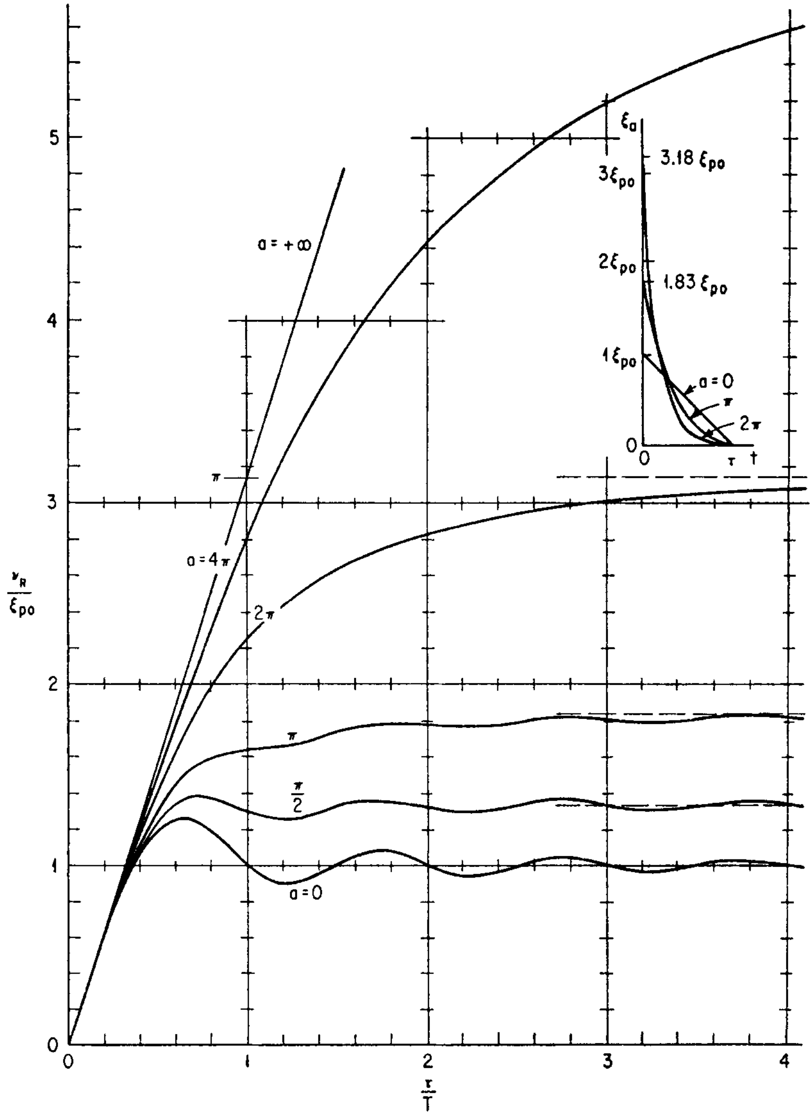


FIGURE 8.29 Spectra for residual response amplitude for the family of simple blast pulses shown in Fig. 8.28, compared on the basis of equal pulse area. (*Jacobsen and Ayre.*[22])

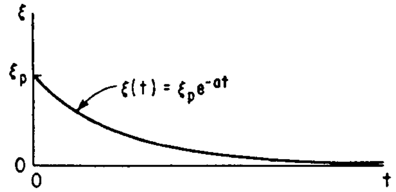


FIGURE 8.30 Pulse consisting of vertical rise followed by exponential decay of infinite duration.

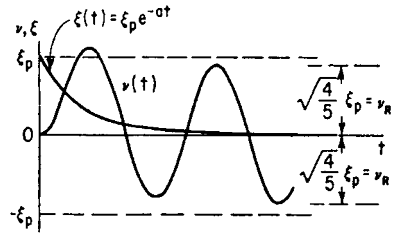


FIGURE 8.31 Time response to the pulse having a vertical rise and an exponential decay of infinite duration (Fig. 8.30), for the particular case $\omega_n/a = 2$.

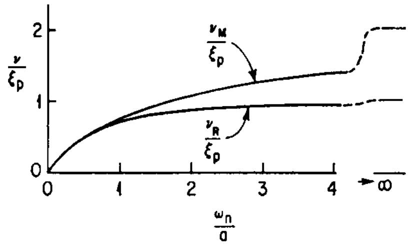


FIGURE 8.32 Spectra for maximax response and for asymptotic residual response amplitude, for the pulse shown in Fig. 8.30.

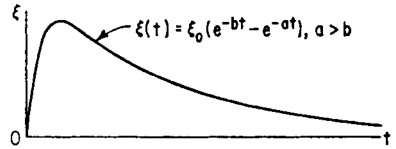


FIGURE 8.33 Pulse formed by taking the difference of two exponentially decaying functions.

$$\nu = \xi_p \frac{ae/\omega_n}{(1 + a^2/\omega_n^2)^2} \left\{ \left[\frac{2a}{\omega_n} + \left(1 + \frac{a^2}{\omega_n^2}\right) \omega_n t \right] e^{-at} - \frac{2a}{\omega_n} \cos \omega_n t - \left(1 - \frac{a^2}{\omega_n^2}\right) \sin \omega_n t \right\} \tag{8.47b}$$

$$\frac{\nu_R}{\xi_p} \to \frac{e}{(a/\omega_n) + (\omega_n/a)} \tag{8.47c}$$

The *maximum* value of ν_R occurs in the case $a/\omega_n = 1$, and is given by

$$(\nu_R)_{\max} = \xi_p e/2 = 1.36\xi_p$$

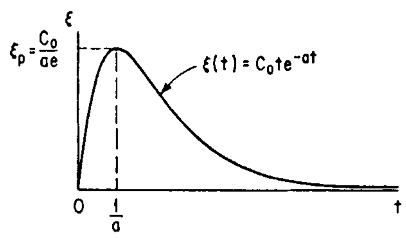


FIGURE 8.34 Pulse formed by taking the product of an exponentially decaying function by time.

Both of the excitation functions described by Eqs. (8.46*a*) and (8.47*a*) include finite times of rise to the pulse peak. These rise times are dependent on the exponential decay constants.

4. The rise time may be made independent of the decay by inserting a separate rise function before the decay function, as in Fig. 8.35, where a *straight-line rise precedes the exponential decay.* The response-time equations are as follows:

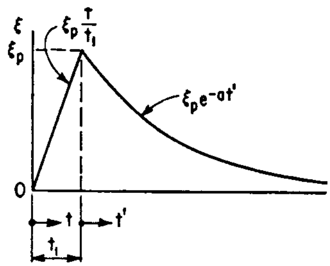


FIGURE 8.35 Pulse formed by a straight-line rise followed by an exponential decay asymptotic to the time axis.

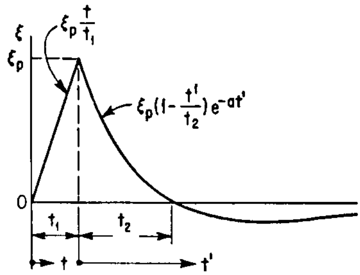


FIGURE 8.36 Pulse formed by a straight-line rise followed by a continuous exponential decay through positive and negative phases. (*Frankland.*[14])

Pulse rise era:

$$\nu = \xi_p \frac{\omega_n t - \sin \omega_n t}{\omega_n t_1} \qquad [0 \le t \le t_1] \tag{8.48a}$$

Pulse decay era:

$$\nu = \xi_p \left[\frac{e^{-at'}}{1 + a^2/\omega_n^2} + \left(\frac{a^2/\omega_n^2}{1 + a^2/\omega_n^2} - \frac{\sin \omega_n t_1}{\omega_n t_1} \right) \cos \omega_n t' + \left(\frac{a/\omega_n}{1 + a^2/\omega_n^2} + \frac{1 - \cos \omega_n t_1}{\omega_n t_1} \right) \sin \omega_n t' \right] \tag{8.48b}$$

where $t' = t - t_1$ and $0 \le t'$.

5. Another form of pulse, which is a more complete representation of a blast pulse since it includes the possibility of a negative phase of pressure,[14] is shown in Fig. 8.36. It consists of a straight-line rise, followed by an exponential decay through the positive phase, into the negative phase, finally becoming asymptotic to the time axis. The rise time is t_1 and the duration of the positive phase is $t_1 + t_2$.

Unsymmetrical Exponential Pulses with Central Peak. An interesting family of unsymmetrical pulses may be formed by using Eqs. (8.36*a*) and changing the sign of the exponent of *e* in both the numerator and the denominator of the second of the equations. The resulting family consists of pulses whose maxima occur at the mid-period time and which satisfy simultaneously both criteria for comparison (equal pulse height and equal pulse area).

Figure 8.37 shows the spectra of residual response amplitude and, in the inset diagrams, the pulse shapes. The limiting cases are the symmetrical triangle of duration τ and height ξ_p, and the rectangles of duration $\tau/2$ and height ξ_p. All pulses in the family have the area $\xi_p\tau/2$. Zeroes of residual response amplitude occur for all values of *a,* at even integer values of τ/T. The residual response amplitude is

$$\frac{\nu_R}{\xi_p} = \frac{aT/\pi\tau}{1 - \cosh a} \cdot \left[\frac{\cosh 2a - \cosh a - (1 - \cosh a) \cos (2\pi\tau/T) + (1 - \cosh 2a) \cos (\pi\tau/T)}{1 + a^2T^2/\pi^2\tau^2} \right]^{1/2} \tag{8.49}$$

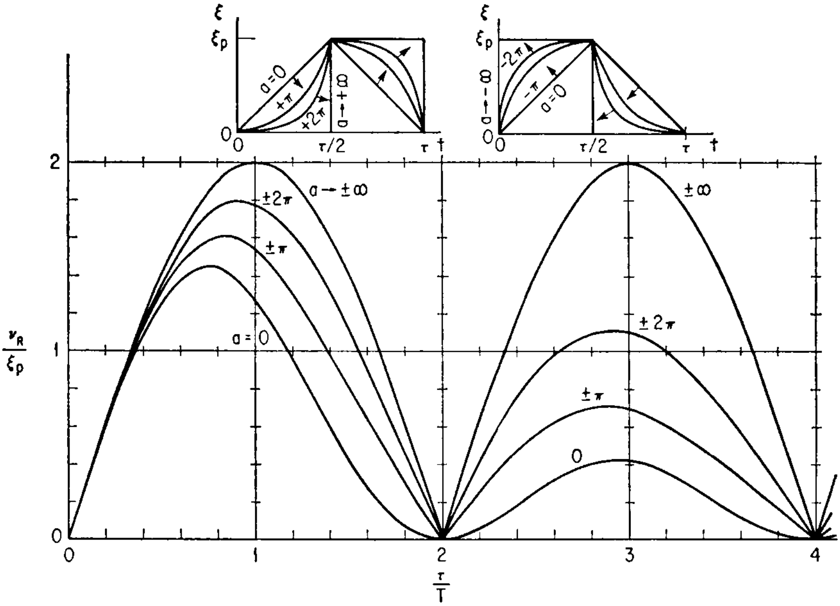


FIGURE 8.37 Spectra of residual response amplitude for a family of unsymmetrical exponential pulses of equal area and equal maximum height, having the pulse peak at the mid-period time. (*Jacobsen and Ayre.*[22])

Pulses which are mirror images of each other in time result in equal residual amplitudes.

Skewed Versed-sine Pulse. By taking the product of a decaying exponential and the versed-sine function, a family of pulses with varying skewness is obtained.[13,22] The family is described by the following equation:

$$\xi(t) = \begin{cases} \xi_p \dfrac{e^{2\pi(\sigma - t/\tau)} \cot \pi\sigma}{1 - \cos 2\pi\sigma} (1 - \cos 2\pi t/\tau) & [0 \le t \le \tau] \\ 0 & [\tau \le t] \end{cases} \tag{8.50}$$

These pulses are of particular interest when the excitation is a ground displacement function because they have continuity in both velocity and displacement; thus, they do not involve theoretically infinite accelerations of the ground. When the skewing constant σ equals one-half, the pulse is the symmetrical versed sine. When $\sigma \to 0$, the front of the pulse approaches a straight line with infinite slope, and the pulse area approaches zero.

The spectra of residual response amplitude and of maximax relative response, plotted on the basis of equal pulse height, are shown in Fig. 8.38 for several values of σ. The residual response amplitude spectra are reasonably good approximations to the spectra of maximax relative response except at the lower values of τ/T.

Figure 8.39 compares the residual response amplitude spectra on the basis of equal pulse area. The required pulse heights, for a constant pulse area of $\xi_{p0}\tau/2$, are shown in the inset diagram. On this basis, the pulse for $\sigma \to 0$ represents a generalized impulse of value $k\xi_{p0}\tau/2$.

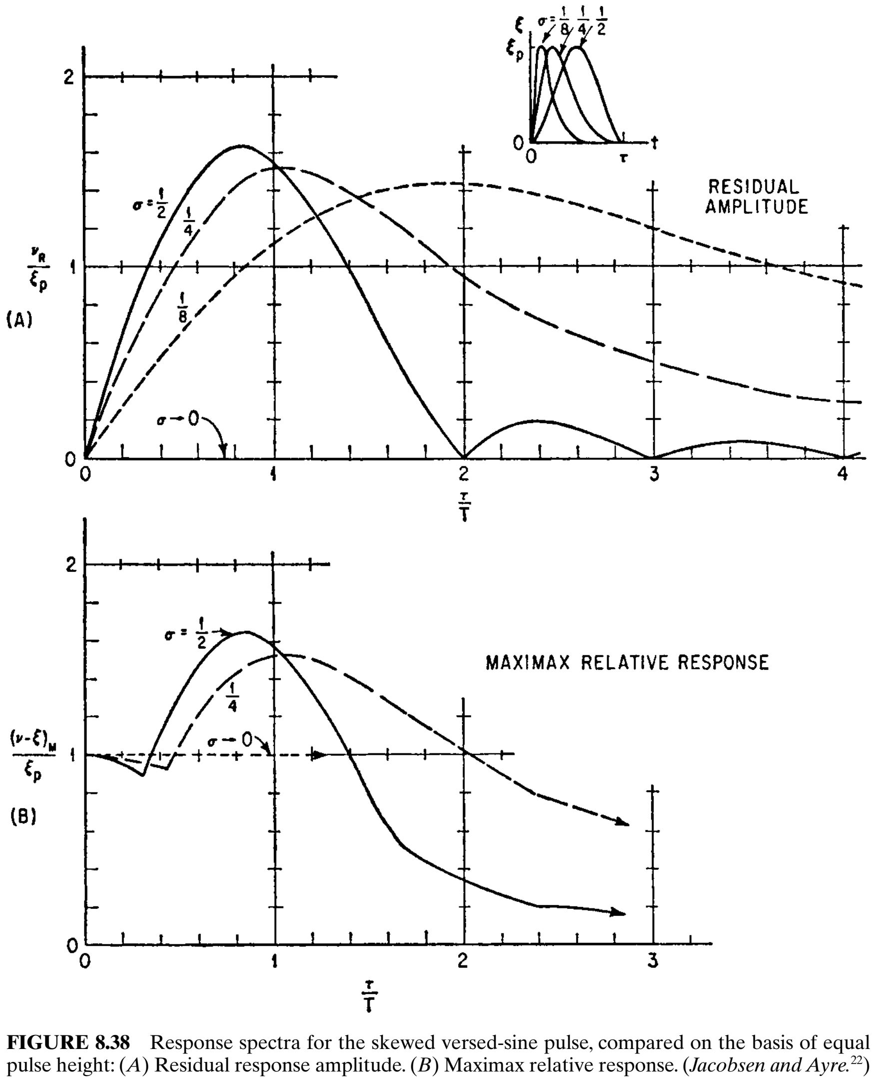


FIGURE 8.38 Response spectra for the skewed versed-sine pulse, compared on the basis of equal pulse height: (*A*) Residual response amplitude. (*B*) Maximax relative response. (*Jacobsen and Ayre.*[22])

Full-cycle Pulses (Force-Time Integral = 0). The residual response amplitude spectra for three groups of *full-cycle pulses* are shown as follows: in Fig. 8.40 for the rectangular, the sinusoidal, and the symmetrical triangular pulses; in Fig. 8.41 for three types of pulse involving sine and cosine functions; and in Fig. 8.42 for three forms of triangular pulse. The pulse shapes are shown in the inset diagrams. Expressions for the residual response amplitudes are given in Eqs. (8.51) to (8.53).

Full-cycle rectangular pulse:

$$\frac{\nu_R}{\xi_p} = 2 \sin \frac{\pi\tau}{T} \left[2 \sin \frac{\pi\tau}{T} \right] \tag{8.51}$$

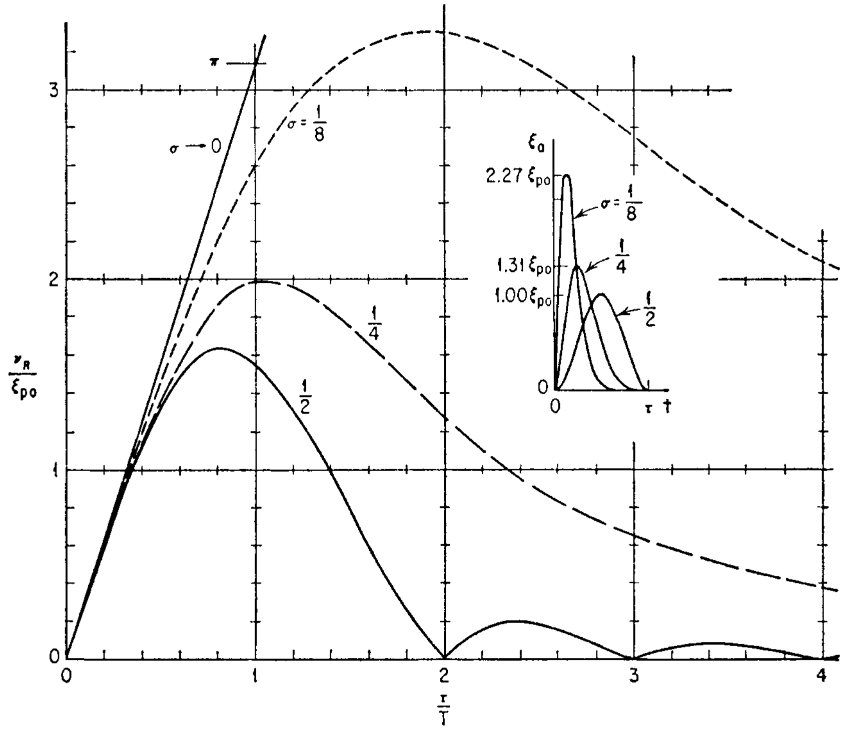


FIGURE 8.39 Spectra of residual response amplitude for the skewed versed-sine pulse, compared on the basis of equal pulse area.[19]

Full-cycle "sinusoidal" pulses:

Symmetrical half cycles

$$\frac{\nu_R}{\xi_p} = 2 \sin \frac{\pi\tau}{T} \left[\frac{T/\tau}{(T^2/4\tau^2) - 1} \cos \frac{\pi\tau}{T} \right] \tag{8.52a}$$

Vertical front and vertical ending

$$\frac{\nu_R}{\xi_p} = \frac{2}{1 - T^2/16\tau^2} \cos \frac{2\pi\tau}{T} \tag{8.52b}$$

Vertical jump at mid-cycle

$$\frac{\nu_R}{\xi_p} = \frac{2}{1 - T^2/16\tau^2} \left(1 - \frac{T}{4\tau} \sin \frac{2\pi\tau}{T} \right) \tag{8.52c}$$

Full-cycle triangular pulses:

Symmetrical half cycles

$$\frac{\nu_R}{\xi_p} = 2 \sin \frac{\pi\tau}{T} \left[\frac{4T}{\pi\tau} \sin^2 \frac{\pi\tau}{2T} \right] \tag{8.53a}$$

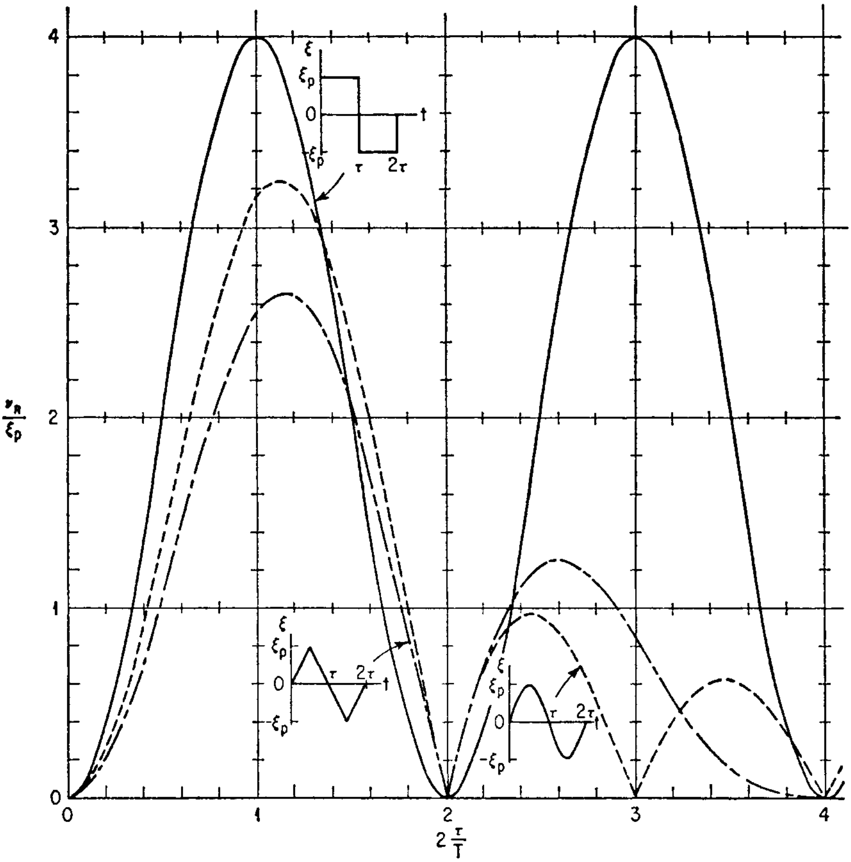


FIGURE 8.40 Spectra of residual response amplitude for three types of full-cycle pulses. Each half cycle is symmetrical.[19]

Vertical front and vertical ending

$$\frac{\nu_R}{\xi_p} = 2\left(\frac{T}{2\pi\tau}\sin\frac{2\pi\tau}{T} - \cos\frac{2\pi\tau}{T}\right) \tag{8.53b}$$

Vertical jump at mid-cycle

$$\frac{\nu_R}{\xi_p} = 2\left(1 - \frac{T}{2\pi\tau}\sin\frac{2\pi\tau}{T}\right) \tag{8.53c}$$

In the case of full-cycle pulses having symmetrical half cycles, note that the residual response amplitude equals the residual response amplitude of the symmetrical one-half-cycle pulse of the same shape, multiplied by the dimensionless residual response amplitude function $2\sin(\pi\tau/T)$ for the single rectangular pulse. Compare the bracketed functions in Eqs. (8.51), (8.52*a*), and (8.53*a*) with the bracketed functions in Eqs. (8.31*b*), (8.32*b*), and (8.34*c*), respectively.

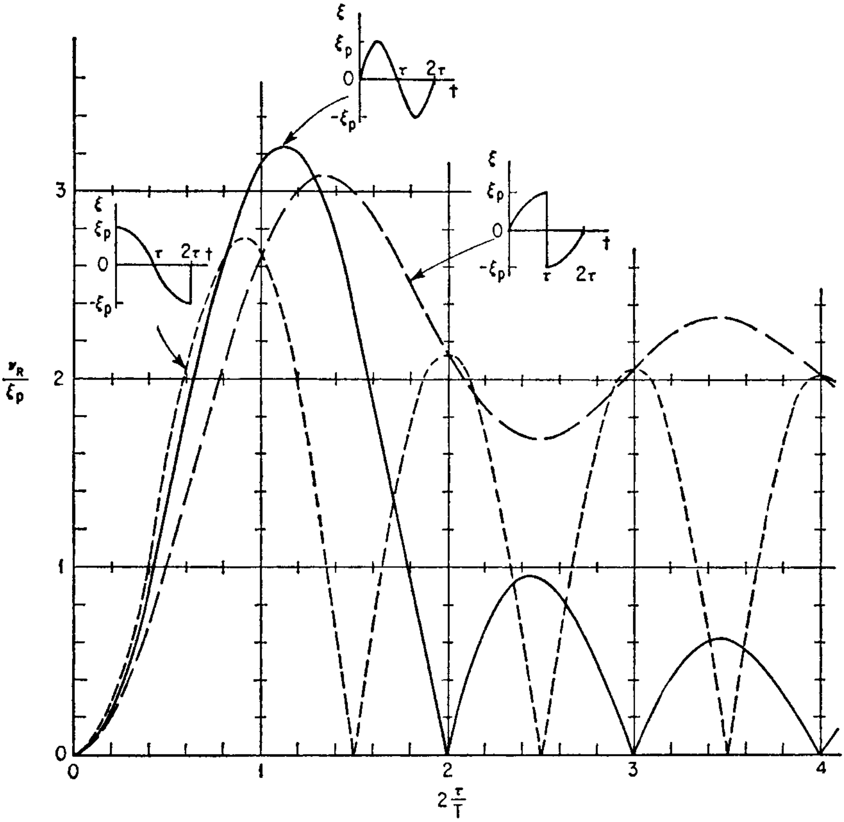


FIGURE 8.41 Spectra of residual response amplitude for three types of full-cycle "sinusoidal" pulses.[19]

SUMMARY OF TRANSIENT RESPONSE SPECTRA FOR THE SINGLE DEGREE-OF-FREEDOM, LINEAR, UNDAMPED SYSTEM

Initial Conditions. The following conclusions are based on the assumption that the system is initially at rest.

Step-type Excitations

1. The maximax response ν_M occurs *after* the step has risen (monotonically) to full value ($\tau \leq t$, where τ is the step rise time). It is equal to the residual response amplitude plus the constant step height ($\nu_M = \nu_R + \xi_c$).
2. The extreme values of the ratio of maximax response to step height ν_M/ξ_c are 1 and 2. When the ratio of step rise time to system natural period τ/T approaches zero, the step approaches the simple rectangular step in shape and ν_M/ξ_c approaches the upper extreme of 2. If τ/T approaches infinity, the step loses the character of a dynamic excitation; consequently, the inertia forces of the system approach zero and ν_M/ξ_c approaches the lower extreme of 1.

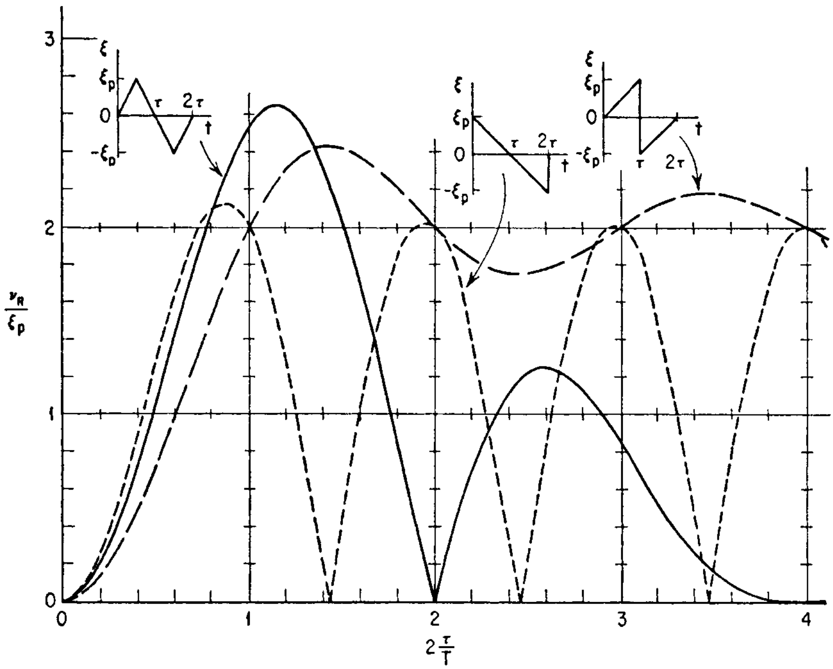


FIGURE 8.42 Spectra of residual response amplitude for three types of full-cycle triangular pulses.[19]

3. For some particular shapes of step rise, ν_M/ξ_c is equal to 1 at certain finite values of τ/T. For example, for the step having a constant-slope rise, $\nu_M/\xi_c = 1$ when $\tau/T = 1, 2, 3, \ldots$ The lowest values of $\tau/T = (\tau/T)_{min}$, for which $\nu_M/\xi_c = 1$, are, for three shapes of step rise: constant-slope, 1.0; versed-sine, 1.5; cycloidal, 2.0. The lowest possible value of $(\tau/T)_{min}$ is 1.

4. In the case of step-type excitations, when $\nu_M/\xi_c = 1$ the residual response amplitude ν_R is zero. Sometimes it is of practical importance in the design of cams and dynamic control functions to achieve the smallest possible residual response.

Single-Pulse Excitations

1. When the ratio τ/T of pulse duration to system natural period is less than ½, the time shapes of certain types of equal area pulses are of secondary significance in determining the maxima of system response [maximax response ν_M, maximax relative response $(\nu - \xi)_M$, and residual response amplitude ν_R]. If τ/T is less than ¼, the pulse shape is of little consequence in almost all cases and the system response can be determined to a fair approximation by use of the simple impulse theory. If τ/T is larger than ½, the pulse shape may be of great significance.

2. The maximum value of maximax response for a given shape of pulse, $(\nu_M)_{max}$, usually occurs at a value of the period ratio τ/T between ½ and 1. The maximum value of the ratio of maximax response to the reference excitation, $(\nu_M)_{max}/\xi_p$, is usually between 1.5 and 1.8.

3. If the pulse has a *vertical rise,* ν_M is the first maximum occurring, and $(\nu_M)_{max}$ is an asymptotic value approaching $2\xi_p$ as τ/T approaches infinity. In the special case of the rectangular pulse, $(\nu_M)_{max}$ is equal to $2\xi_p$ and occurs at values of τ/T equal to or greater than ½.

4. If the pulse has a *vertical decay,* $(\nu_M)_{max}$ is equal to the maximum value $(\nu_R)_{max}$ of the residual response amplitude.

5. The maximum value $(\nu_R)_{max}$ of the residual response amplitude, for a given shape of pulse, often is a reasonably good approximation to $(\nu_M)_{max}$, except if the pulse has a steep rise followed by a decay. A few examples are shown in Table 8.3. Furthermore, if $(\nu_M)_{max}$ and $(\nu_R)_{max}$ for a given pulse shape are approximately equal in magnitude, they occur at values of τ/T not greatly different from each other.

6. Pulse shapes that are mirror images of each other in time result in equal values of residual response amplitude.

7. The residual response amplitude ν_R generally has zero values for certain finite values of τ/T. However, if the pulse has either a vertical rise or a vertical decay, but not both, there are no zero values except the trivial one at $\tau/T = 0$. In the case of the rectangular pulse, $\nu_R = 0$ when $\tau/T = 1, 2, 3, \ldots$. For several shapes of pulse the values of $(\tau/T)_{min}$ (lowest values of τ/T for which $\nu_R = 0$) are as follows: rectangular, 1.0; sine, 1.5; versed-sine, 2.0; symmetrical triangle, 2.0. The lowest possible value of $(\tau/T)_{min}$ is 1.

8. In the formulation of pulse as well as of step-type excitations, it may be of practical consequence for the residual response to be as small as possible; hence, attention is devoted to the case, $\nu_R = 0$.

TABLE 8.3 Comparison of Greatest Values of Maximax Response and Residual Response Amplitude

Pulse shape	$(\nu_M)_{max}/(\nu_R)_{max}$
Symmetrical:	
Rectangular	1.00
Sine	1.04
Versed sine	1.05
Triangular	1.06
Vertical-decay pulses	1.00
Vertical-rise pulses:	
Rectangular	1.00
Triangular	1.60
Asymptotic exponential decay	2.00

SINGLE DEGREE-OF-FREEDOM LINEAR SYSTEM WITH DAMPING

The calculation of the effects of damping on transient response may be laborious. If the investigation is an extensive one, use should be made of an analog computer.

DAMPING FORCES PROPORTIONAL TO VELOCITY (VISCOUS DAMPING)

In the case of steady forced vibration, even very small values of the viscous damping coefficient have great effect in limiting the system response at or near resonance. If the excitation is of the single step- or pulse-type, however, the effect of damping on the maximax response may be of relatively less importance, unless the system is highly damped.

For example, in a system under steady sinusoidal excitation at resonance, a tenfold increase in the fraction of critical damping c/c_c from 0.01 to 0.1 results in a theoretical tenfold decrease in the magnification factor from 50 to 5. In the case of the same system, initially at rest and acted upon by a half-cycle sine pulse of "resonant duration" $\tau = T/2$, the same increase in the damping coefficient results in a decrease in the maximax response of only about 9 percent.

Half-cycle Sine Pulse Excitation. Figure 8.43 shows the spectra of maximax response for a viscously damped system excited by a half-cycle sine pulse.[12] The system is initially at rest. The results apply to the cases indicated by the following differential equations of motion:

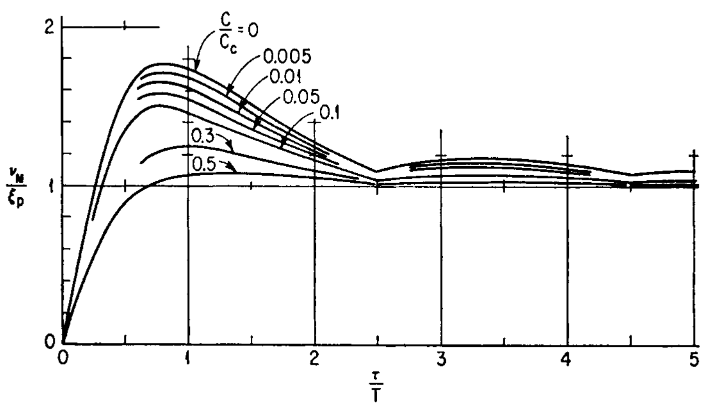


FIGURE 8.43 Spectra of maximax response for a viscously damped single degree-of-freedom system acted upon by a half-cycle sine pulse. (*R. D. Mindlin, F. W. Stubner, and H. L. Cooper.*[23])

$$\frac{m\ddot{x}}{k} + \frac{c\dot{x}}{k} + x = \frac{F_p}{k} \sin \frac{\pi t}{\tau} \tag{8.54a}$$

$$\frac{m\ddot{x}}{k} + \frac{c\dot{x}}{k} + x = u_p \sin \frac{\pi t}{\tau} \tag{8.54b}$$

$$\frac{m\ddot{\delta}_x}{k} + \frac{c\dot{\delta}_x}{k} + \delta_x = \frac{-m\ddot{u}_p}{k} \sin \frac{\pi t}{\tau} \tag{8.54c}$$

and in general

$$\frac{m\ddot{\nu}}{k} + \frac{c\dot{\nu}}{k} + \nu = \xi_p \sin \frac{\pi t}{\tau} \tag{8.54d}$$

where $0 \le t \le \tau$.

For values of t greater than τ, the excitation is zero. The distinctions among these cases may be determined by referring to Table 8.1. The fraction of critical damping c/c_c in Fig. 8.43 is the ratio of the damping coefficient c to the critical damping coefficient $c_c = \sqrt{2mk}$. The damping coefficient must be defined in terms of the velocity $(\dot{x}, \dot{\delta}_x, \dot{\nu})$ appropriate to each case. For $c/c_c = 0$, the response spectrum is the same as the spectrum for maximax response shown for the undamped system in Fig. 8.16*B*.

Other Forms of Excitation; Methods. Qualitative estimates of the effects of viscous damping in the case of other forms of step or pulse excitation may be made by the use of Fig. 8.43 and of the appropriate spectrum for the undamped response to the excitation in question.

Quantitative calculations may be effected by extending the methods described for the undamped system. If the excitation is of general form, given either numerically or graphically, the *phase-plane-delta*[21,22] method described in a later section of this chapter may be used to advantage. Of the analytical methods, the *Laplace transformation* is probably the most useful. A brief discussion of its application to the viscously damped system follows.

Laplace Transformation. The differential equation to be solved is

$$\frac{m\ddot{\nu}}{k} + \frac{c\dot{\nu}}{k} + \nu = \xi(t) \tag{8.55a}$$

Rewriting Eq. (8.55*a*),

$$\frac{\ddot{\nu}}{\omega_n^2} + \frac{2\zeta\dot{\nu}}{\omega_n} + \nu = \xi(t) \tag{8.55b}$$

where $\zeta = c/c_c$ and $\omega_n^2 = k/m$.

Applying the operation transforms of Table 8.2 to Eq. (8.55*b*), the following algebraic equation is obtained:

$$\frac{1}{\omega_n^2}[s^2F_r(s) - sf(0) - f'(0)] + \frac{2\zeta}{\omega_n}[sF_r(s) - f(0)] + F_r(s) = F_e(s) \tag{8.56a}$$

The *subsidiary* equation is

$$F_r(s) = \frac{(s + 2\zeta\omega_n)f(0) + f'(0) + \omega_n^2F_e(s)}{s^2 + 2\zeta\omega_n s + \omega_n^2} \tag{8.56b}$$

where the initial conditions $f(0)$ and $f'(0)$ are to be expressed as ν_0 and $\dot{\nu}_0$, respectively.

By performing an *inverse* transformation of Eq. (8.56*b*), the response is determined in the following operational form:

$$\nu(t) = \mathfrak{L}^{-1}[F_r(s)]$$

$$= \mathfrak{L}^{-1}\left[\frac{(s + 2\zeta\omega_n)\nu_0 + \dot{\nu}_0 + \omega_n^2F_e(s)}{s^2 + 2\zeta\omega_n s + \omega_n^2}\right] \tag{8.57}$$

Example 8.9: Rectangular Step Excitation. Assume that the damping is less than critical ($\zeta < 1$), that the system starts from rest ($\nu_0 = \dot{\nu}_0 = 0$), and that the system is acted upon by the rectangular step excitation: $\xi(t) = \xi_c$ for $0 \le t$. The transform of the excitation is given by

$$F_e(s) = \mathfrak{L}[\xi(t)] = \mathfrak{L}[\xi_c] = \xi_c\frac{1}{s}$$

Substituting for ν_0, $\dot{\nu}_0$ and $F_e(s)$ in Eq. (8.57), the following equation is obtained:

$$\nu(t) = \mathfrak{L}^{-1}[F_r(s)] = \xi_c \omega_n^2 \mathfrak{L}^{-1}\left[\frac{1}{s(s^2 + 2\zeta\omega_n s + \omega_n^2)}\right] \tag{8.58a}$$

Rewriting,

$$\nu(t) = \xi_c \omega_n^2 \mathfrak{L}^{-1}\left[\frac{1}{s[s + \omega_n(\zeta - j\sqrt{1-\zeta^2})]\,[s + \omega_n(\zeta + j\sqrt{1-\zeta^2})]}\right] \tag{8.58b}$$

where $j = \sqrt{-1}$.

To determine the inverse transform $\mathfrak{L}^{-1}[F_r(s)]$, it may be necessary to expand $F_r(s)$ in partial fractions as explained previously. However, in this particular example the transform pair is available in Table 8.2 (see item 16). Thus, it is found readily that $\nu(t)$ is given by the following:

$$\nu(t) = \xi_c \omega_n^2\left[\frac{1}{ab} + \frac{be^{-at} - ae^{-bt}}{ab(a-b)}\right] \tag{8.59a}$$

where $a = \omega_n(\zeta - j\sqrt{1-\zeta^2})$ and $b = \omega_n(\zeta + j\sqrt{1-\zeta^2})$. By using the relations, $\cos z = (½)(e^{jz} + e^{-jz})$ and $\sin z = -(½)j(e^{jz} - e^{-jz})$, Eq. (8.59*a*) may be expressed in terms of *cosine* and *sine* functions:

$$\nu(t) = \xi_c\left[1 - e^{-\zeta\omega_n t}\left(\cos\omega_d t + \frac{\zeta}{\sqrt{1-\zeta^2}}\sin\omega_d t\right)\right] \qquad [\zeta < 1] \tag{8.59b}$$

where the *damped* natural frequency $\omega_d = \omega_n\sqrt{1-\zeta^2}$.

If the damping is negligible, $\zeta \to 0$ and Eq. (8.59*b*) reduces to the form of Eq. (8.22) previously derived for the case of zero damping:

$$\nu(t) = \xi_c(1 - \cos\omega_n t) \qquad [\zeta = 0] \tag{8.22}$$

CONSTANT (COULOMB) DAMPING FORCES; PHASE-PLANE METHOD

The phase-plane method is particularly well suited to the solving of transient response problems involving Coulomb damping forces.[21, 22] The problem is truly a stepwise linear one, provided the usual assumptions regarding Coulomb friction are valid. For example, the differential equation of motion for the case of ground displacement excitation is

$$m\ddot{x} \pm F_f + kx = ku(t) \tag{8.60a}$$

where F_f is the Coulomb friction force. In Eq. (8.60*b*) the friction force has been moved to the right side of the equation and the equation has been divided by the spring constant k:

$$\frac{m\ddot{x}}{k} + x = u(t) \mp \frac{F_f}{k} \tag{8.60b}$$

The effect of friction can be taken into account readily in the construction of the phase trajectory by modifying the ordinates of the stepwise excitation by amounts equal to $\mp F_f/k$. The quantity F_f/k is the Coulomb friction "displacement," and is equal to one-fourth the decay in amplitude in each cycle of a *free* vibration under the influence of Coulomb friction. The algebraic sign of the friction term changes when the velocity changes sign. When the friction term is placed on the right-hand side of the differential equation, it must have a *negative* sign when the velocity is positive.

Example 8.10: Free Vibration. Figure 8.44 shows an example of free vibration with the initial conditions $x = x_0$ and $\dot{x} = 0$. The locations of the arc centers of the phase trajectory alternate each half cycle from $+F_f/k$ to $-F_f/k$.

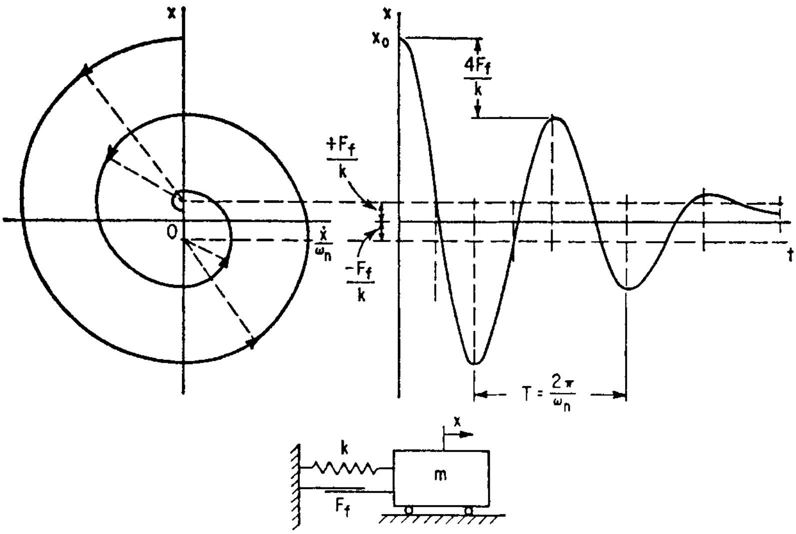


FIGURE 8.44 Example of phase-plane solution of free vibration with Coulomb friction[2]; the natural frequency is $\omega_n = \sqrt{k/m}$.

Example 8.11: General Transient Excitation. A general stepwise excitation $u(t)$ and the response x of a system under the influence of a friction force F_f are shown in Fig. 8.45. The case of zero friction is also shown. The initial conditions are $x = 0$, $\dot{x} = 0$. The arc centers are located at ordinates of $u(t) \mp F_f/k$. During the third step in the excitation, the velocity of the system changes sign from positive to negative (at $t = t_2'$); consequently, the friction displacement must also change sign, but from negative to positive.

SINGLE DEGREE-OF-FREEDOM NONLINEAR SYSTEMS

PHASE-PLANE-DELTA METHOD

The transient response of damped linear systems and of nonlinear systems of considerable complexity can be determined by the *phase-plane-delta* method.[21, 22] Assume that the differential equation of motion of the system is

$$m\ddot{x} = G(x,\dot{x},t) \tag{8.61a}$$

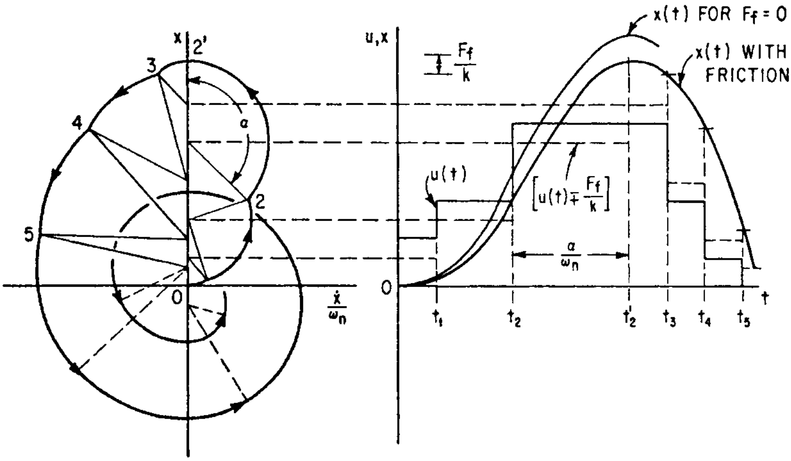


FIGURE 8.45 Example of phase-plane solution for a general transient excitation with Coulomb friction in the system.[2]

where $G(x,\dot{x},t)$ is a general function of x, $\dot{x}$, and t to any powers. The coefficient of $\ddot{x}$ is constant, either inherently or by a suitable division.

In Eq. (8.61*a*) the general function may be replaced by another general function minus a linear, constant-coefficient, restoring force term:

$$G(x,\dot{x},t) = g(x,\dot{x},t) - kx$$

By moving the linear term kx to the left side of the differential equation, dividing through by m, and letting $k/m = \omega_n^2$, the following equation is obtained:

$$\ddot{x} + \omega_n^2 x = \omega_n^2 \delta \tag{8.61b}$$

where the *operative displacement* δ is given by

$$\delta = \frac{1}{k} g(x,\dot{x},t) \tag{8.61c}$$

The separation of the kx term from the G function does not require that the kx term exist physically. Such a term can be separated by first adding to the G function the *fictitious* terms, $+kx - kx$.

With the differential equation of motion in the δ form, Eq. (8.61*b*), the response problem can now be solved readily by stepwise linearization. The left side of the equation represents a simple, undamped, linear oscillator. Implicit in the δ function on the right side of the equation are the nonlinear restoration terms, the linear or nonlinear dissipation terms, and the excitation function.

If the δ function is held constant at a value δ for an interval of time Δt, the response of the linear oscillator in the phase-plane is an arc of a circle, with its center on the X axis at δ and subtended by an angle equal to $\omega_n\,\Delta t$. The graphical construction may be similar in general appearance to the examples already shown for linear systems in Figs. 8.5, 8.44, and 8.45. Since in the general case the δ function involves the dependent variables, it is necessary to estimate, before constructing each step, appropriate average values of the system displacement and/or velocity to be expected during the step. In some cases, more than one trial may be required before suitable accuracy is obtained.

Many examples of solution for various types of systems are available in the literature.[3,5,6,8,13,15,20–22]

MULTIPLE DEGREE-OF-FREEDOM, LINEAR, UNDAMPED SYSTEMS

Some of the transient response analyses, presented for the single degree-of-freedom system, are in complete enough form that they can be employed in determining the responses of linear, undamped, multiple degree-of-freedom systems. This can be done by the use of *normal* (*principal*) *coordinates.* A system of normal coordinates is a system of generalized coordinates chosen in such a way that vibration in each normal mode involves only one coordinate, a normal coordinate. The differential equations of motion, when written in normal coordinates, are all independent of each other. Each differential equation is related to a particular normal mode and involves only one coordinate. The differential equations are of the same general form as the differential equation of motion for the single degree-of-freedom system. The response of the system in terms of the physical coordinates, for example, displacement or stress at various locations in the system, is determined by superposition of the normal coordinate responses.

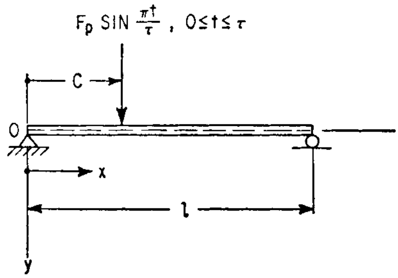


FIGURE 8.46 Simply supported beam loaded by a concentrated force sine pulse of half-cycle duration.

Example 8.12: Sine Force Pulse Acting on a Simple Beam. Consider the flexural vibration of a prismatic bar with simply supported ends, Fig. 8.46. A sine-pulse concentrated force F_p sin $(\pi t/\tau)$ is applied to the beam at a distance c from the left end (origin of coordinates). Assume that the beam is initially at rest. The displacement response of the beam, during the time of action of the pulse, is given by the following series:

$$y = \frac{2F_p l^3}{\pi^4 EI} \sum_{i=1}^{i=\infty} \frac{1}{i^4} \sin \frac{i\pi c}{l} \sin \frac{i\pi x}{l} \left[\frac{1}{1 - T_i^2/4\tau^2} \left(\sin \frac{\pi t}{\tau} - \frac{T_i}{2\tau} \sin \omega_i t \right) \right] \qquad [0 \le t \le \tau] \tag{8.62a}$$

where $$i = 1, 2, 3, \ldots; T_i = \frac{2\pi}{\omega_i} = \frac{2l^2}{i^2\pi} \sqrt{\frac{A\gamma}{EIg}} = \frac{T_1}{i^2}, \text{sec}$$

A comparison of Eqs. (8.62*a*) and (8.32*a*) shows that the time function [sin $(\pi t/\tau)$ – $(T_i/2\tau)$ sin $\omega_i t$] for the *i*th term in the beam-response series is of exactly the same form as the time function [sin $(\pi t/\tau)$ – $(T/2\tau)$ sin $\omega_n t$] in the response of the single degree-of-freedom system. Furthermore, the magnification factors $1/(1 - T_i^2/4\tau^2)$ and $1/(1 - T^2/4\tau^2)$ in the two equations have identical forms.

Following the end of the pulse, beginning at $t = \tau$, the vibration of the beam is expressed by

$$y = \frac{2F_p l^3}{\pi^4 EI} \sum_{i=1}^{i=\infty} \frac{1}{i^4} \sin\frac{i\pi c}{l} \sin\frac{i\pi x}{l} \left[\frac{(T_i/\tau)\cos(\pi\tau/T_i)}{(T_i^2/4\tau^2) - 1} \sin\omega_i\left(t - \frac{\tau}{2}\right) \right] \qquad [\tau \le t] \tag{8.62b}$$

A comparison of Eqs. (8.62*b*) and (8.32*b*) leads to the same conclusion as found above for the time era $0 \le t \le \tau$.

Excitation and Displacement at Mid-span. As a specific case, consider the displacement at mid-span when the excitation is applied at mid-span ($c = x = l/2$). The even-numbered terms of the series now are all zero and the series take the following forms:

$$y_{l/2} = \frac{2F_p l^3}{\pi^4 EI} \sum_{i=1,3,5,\ldots}^{\infty} \frac{1}{i^4} \left[\frac{1}{1 - T_i^2/4\tau^2} \left(\sin\frac{\pi t}{\tau} - \frac{T_i}{2\tau} \sin\omega_i t \right) \right] \qquad [0 \le t \le \tau] \tag{8.63a}$$

$$y_{l/2} = \frac{2F_p l^3}{\pi^4 EI} \sum_{i=1,3,5,\ldots}^{\infty} \frac{1}{i^4} \left[\frac{(T_i/\tau)\cos(\pi\tau/T_i)}{(T_i^2/4\tau^2) - 1} \sin\omega_i\left(t - \frac{\tau}{2}\right) \right] \qquad [\tau \le t] \tag{8.63b}$$

Assume, for example, that the pulse period τ equals two-tenths of the fundamental natural period of the beam ($\tau/T_1 = 0.2$). It is found from Fig. 8.16*B*, by using an abscissa value of 0.2, that the maximax response in the *fundamental* mode ($i = 1$) occurs in the residual vibration era ($\tau \le t$). The value of the corresponding ordinate is 0.75. Consequently, the maximax response for $i = 1$ is 0.75 ($2F_p l^3/\pi^4 EI$).

In order to determine the maximax for the *third* mode ($i = 3$), an abscissa value of $\tau/T_i = i^2\tau/T_1 = 3^2 \times 0.2 = 1.8$, is used. It is found that the maximax is greater than the residual amplitude and consequently that it occurs during the time era $0 \le t \le \tau$. The value of the corresponding ordinate is 1.36; however, this must be multiplied by $1/3^4$, as indicated by the series. The maximax for $i = 3$ is thus 0.017 ($2F_p l^3/\pi^4 EI$).

The maximax for $i = 5$ also occurs in the time era $0 \le t \le \tau$ and the ordinate may be estimated to be about 1.1. Multiplying by $1/5^4$, it is found that the maximax for $i = 5$ is approximately 0.002 ($2F_p l^3/\pi^4 EI$), a negligible quantity when compared with the maximax value for $i = 1$.

To find the maximax total response to a reasonable approximation, it is necessary to sum on a time basis several terms of the series. In the particular example above, the maximax total response occurs in the residual vibration era and a reasonably accurate value can be obtained by considering only the first term ($i = 1$) in the series, Eq. (8.63*b*).

GENERAL INVESTIGATION OF TRANSIENTS

An extensive (and efficient) investigation of transient response in multiple degree-of-freedom systems requires the use of an automatic computer. In some of the simpler cases, however, it is feasible to employ numerical or graphical methods. For example, the phase-plane method may be applied to multiple degree-of-freedom linear systems[1, 2] through the use of normal coordinates. This involves independent phase-planes having the coordinates q_i and $\dot{q}_i/\omega_i$, where q_i is the *i*th normal coordinate.

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