CHAPTER 10
MECHANICAL IMPEDANCE

Elmer L. Hixson

INTRODUCTION

The *mechanical impedance* at a given point in a vibratory system is the ratio of the sinusoidal force applied to the system at that point to the velocity at the same point. For example, mechanical impedance is discussed in Chap. 6 as it relates to dynamic absorbers and auxiliary mass dampers. In the following sections of this chapter, the mechanical impedance of basic elements that make up vibratory systems is presented. This is followed by a discussion of combinations of these elements. Then, various mechanical circuit theorems are described. Such theorems can be used as an aid in the modeling of mechanical circuits and in determining the response of vibratory systems; they are the mechanical equivalents of well-known theorems employed in the analysis of electric circuits. The measurement of mechanical impedance and some applications are also given.

MECHANICAL IMPEDANCE OF VIBRATORY SYSTEMS

The *mechanical impedance* Z of a system is the ratio of a sinusoidal driving force F acting on the system to the resulting velocity v of the system. Its *mechanical mobility* $\mathfrak{M}$ is the reciprocal of the mechanical impedance.

Consider a sinusoidal driving F that has a magnitude F_0 and an angular frequency ω:

$$F = F_0\, e^{j\omega t} \tag{10.1}$$

The application of this force to a linear mechanical system results in a velocity v:

$$v = v_0 e^{j(\omega t + \phi)} \tag{10.2}$$

where v_0 is the magnitude of the velocity and ϕ is the phase angle between F and v.

Then by definition, the mechanical impedance of the system Z (at the point of application of the force) is given by

$$Z = F/v \tag{10.3}$$

BASIC MECHANICAL ELEMENTS

The idealized mechanical systems considered in this chapter are considered to be represented by combinations of basic mechanical elements assembled to form linear mechanical systems. These basic elements are *mechanical resistances* (*dampers*), *springs,* and *masses.* In general, the characteristics of real masses, springs, and mechanical resistance elements differ from those of ideal elements in two respects:

1. A spring may have a nonlinear force-deflection characteristic; a mass may suffer plastic deformation with motion; and the force presented by a resistance may not be exactly proportional to velocity.
2. All materials have some mass; thus, a perfect spring or resistance cannot be made. Some compliance or spring effect is inherent in all elements. Energy can be dissipated in a system in several ways: friction, acoustic radiation, hysteresis, etc. Such a loss can be represented as a resistive component of the element impedance.

Mechanical Resistance (Damper). A mechanical resistance is a device in which the relative velocity between the end points is proportional to the force applied to the end points. Such a device can be represented by the dashpot of Fig. 10.1*a,* in which the force resisting the extension (or compression) of the dashpot is the result of viscous friction. An ideal resistance is assumed to be made of massless, infinitely rigid elements. The velocity of point *A*, v_1, with respect to the velocity at point *B*, v_2, is

$$v = (v_1 - v_2) = \frac{F_a}{c} \tag{10.4}$$

where *c* is a constant of proportionality called the *mechanical resistance* or *damping constant.* For there to be a relative velocity *v* as a result of force at *A*, there must be an equal reaction force at *B*. Thus, the transmitted force F_b is equal to F_a. The velocities v_1 and v_2 are measured with respect to the stationary reference *G;* their difference is the relative velocity *v* between the end points of the resistance.

With the sinusoidal force of Eq. (10.1) applied to point *A* with point *B* attached to a fixed (immovable) point, the velocity v_1 is obtained from Eq. (10.4):

$$v_1 = \frac{F_0 e^{j\omega t}}{c} = v_0 e^{j\omega t} \tag{10.5}$$

Because *c* is a real number, the force and velocity are said to be "in phase."

The mechanical impedance of the resistance is obtained by substituting from Eqs. (10.1) and (10.5) in Eq. (10.3):

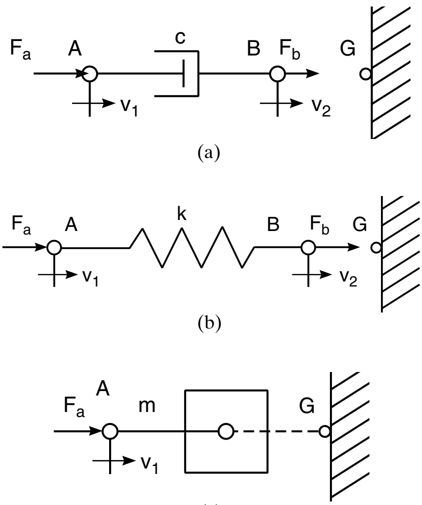

FIGURE 10.1 Schematic representations of basic mechanical elements. (*a*) An ideal mechanical resistance. (*b*) An ideal spring. (*c*) An ideal mass.

$$Z_c = \frac{F}{v} = c \tag{10.6}$$

The mechanical impedance of a resistance is the value of its damping constant c.

Spring. A linear spring is a device for which the relative displacement between its end points is proportional to the force applied. It is illustrated in Fig. 10.1*b* and can be represented mathematically as follows:

$$x_1 - x_2 = \frac{F_a}{k} \tag{10.7}$$

where x_1, x_2 are displacements relative to the reference point G and k is the *spring stiffness.* The stiffness k can be expressed alternately in terms of a *compliance* $C = 1/k$. The spring transmits the applied force, so that $F_b = F_a$.

With the force of Eq. (10.1) applied to point A and with point B fixed, the displacement of point A is given by Eq. (10.7):

$$x_1 = \frac{F_0 e^{j\omega t}}{k} = x_0 e^{j\omega t}$$

The displacement is thus sinusoidal and in phase with the force. The relative velocity of the end connections is required for impedance calculations and is given by the differentiation of x with respect to time:

$$\dot{x} = v = \frac{j\omega F_0 e^{j\omega t}}{k} = \frac{\omega}{k} F_0 e^{j(\omega t + 90°)} \tag{10.8}$$

Substituting Eqs. (10.1) and (10.8) in Eq. (10.3), the impedance of the spring is

$$Z_k = -\frac{jk}{\omega} \tag{10.9}$$

Mass. In the ideal mass illustrated in Figs. 2.2 and 10.1*c,* the acceleration $\ddot{x}$ of the rigid body is proportional to the applied force F:

$$\ddot{x}_1 = \frac{F_a}{m} \tag{10.10}$$

where m is the mass of the body. By Eq. (10.10), the force F_a is required to give the mass the acceleration $\ddot{x}_1$, and the force F_b is transmitted to the reference G. When a sinusoidal force is applied, Eq. (10.10) becomes

$$\ddot{x}_1 = \frac{F_0 e^{j\omega t}}{m} \tag{10.11}$$

The acceleration is sinusoidal and in phase with the applied force.

Integrating Eq. (10.11) to find velocity,

$$\dot{x} = v = \frac{F_0 e^{j\omega t}}{j\omega m}$$

The mechanical impedance of the mass is the ratio of F to v, so that

$$Z_m = \frac{F_0 e^{j\omega t}}{F_0 e^{j\omega t}/j\omega m} = j\omega m \tag{10.12}$$

Thus, the impedance of a mass is an imaginary quantity that depends on the magnitude of the mass and on the frequency.

COMBINATIONS OF MECHANICAL ELEMENTS

In analyzing the properties of mechanical systems, it is often advantageous to combine groups of basic mechanical elements into single impedances. Methods for calculating the impedances of such combined elements are described in this section. An extensive coverage of mechanical impedance theory and a table of combined elements is given in Ref. 1.

Parallel Elements. Consider the combination of elements shown in Fig. 10.2, a spring and a mechanical resistance. They are said to be in parallel since the same force is applied to both, and both are constrained to have the same relative velocities between their connections. The force F_c required to give the resistance the velocity v is found from Eqs. (10.3) and (10.6).

$$F_c = vZ_c = vc$$

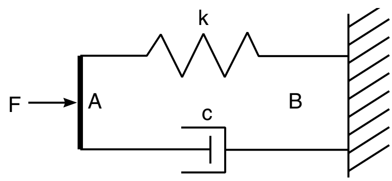

FIGURE 10.2 Schematic representation of a parallel spring-resistance combination.

The force required to give the spring this same velocity is, from Eqs. (10.8) and (10.9),

$$F_k = vZ_k = \frac{vk}{j\omega}$$

The total force F is

$$F = F_c + F_k$$

Since $Z = F/v$,

$$Z = c - j\,\frac{k}{\omega}$$

Thus, the total mechanical impedance is the sum of the impedances of the two elements.

By extending this concept to any number of parallel elements, the driving force F equals the sum of the resisting forces:

$$F = \sum_{i=1}^{n} vZ_i = v\sum_{i=1}^{n} Z_i \qquad \text{and} \qquad Z_p = \sum_{i=1}^{n} Z_i \tag{10.13}$$

where Z_p is the total mechanical impedance of the parallel combination of the individual elements Z_i.

Since mobility is the reciprocal of impedance, when the properties of the parallel elements are expressed as mobilities, the total mobility of the combination follows from Eq. (10.13):

$$\frac{1}{\mathfrak{M}_p} = \sum_{i=1}^{n} \frac{1}{\mathfrak{M}_i} \tag{10.14}$$

Series Elements. In Fig. 10.3 a spring and damper are connected so that the applied force passes through both elements to the inertial reference. Then the velocity v is the sum of v_k and v_c. This is a series combination of elements. The method for determining the mechanical impedance of the combination follows.

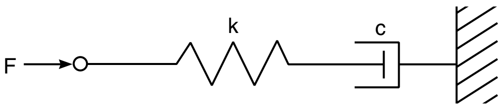

FIGURE 10.3 Schematic representation of a series combination of a spring and a damper.

Consider the more general case of three arbitrary impedances shown in Fig. 10.4. Determine the impedance presented by the end of a number of series-connected elements. Elements Z_1 and Z_2 must have no mass, since a mass always has one end connected to a stationary inertial reference. However, the impedance Z_3 may be a mass. The relative velocities between the end connections of each element are indicated by v_a, v_b, and v_c; the velocities of the connections with respect to the stationary reference point G are indicated by v_1, v_2, and v_3:

$$v_3 = v_c \qquad v_2 = v_3 + (v_2 - v_3) = v_c + v_b$$

$$v_1 = v_2 + (v_1 - v_2) = v_a + v_b + v_c$$

The impedance at point 1 is F/v_1, and the force F is transmitted to all three elements. The relative velocities are

$$v_a = \frac{F}{Z_1} \qquad v_b = \frac{F}{Z_2} \qquad v_c = \frac{F}{Z_3}$$

Thus, the total impedance is defined by

$$\frac{1}{Z} = \frac{F/Z_1 + F/Z_2 + F/Z_3}{F} = \frac{1}{Z_1} + \frac{1}{Z_2} + \frac{1}{Z_3}$$

Extending this principle to any number of massless series elements,

$$\frac{1}{Z_s} = \sum_{i=1}^{n} \frac{1}{Z_i} \tag{10.15}$$

where Z_s is the total mechanical impedance of the elements Z_i connected in series.

Since mobility is the reciprocal of impedance, the total mobility of series connected elements (expressed as mobilities) is

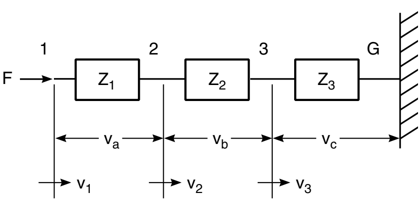

FIGURE 10.4 Generalized three-element system of series-connected mechanical impedances.

$$\mathfrak{M}_s = \sum_{i=1}^{n} \mathfrak{M}_i \tag{10.16}$$

Using Eqs. (10.15) and (10.16), the mobility and impedance for Fig. 10.3 become:

$$\mathfrak{M} = 1/c + j\omega/k \qquad \text{and} \qquad Z = (ck/j\omega)/(c + k/j\omega)$$

MECHANICAL CIRCUIT THEOREMS

The following theorems are the mechanical analogs of theorems widely used in analyzing electric circuits. They are statements of basic principles (or combinations of them) that apply to elements of mechanical systems. In all but Kirchhoff's laws, these theorems apply only to systems composed of linear, bilateral elements. A *linear element* is one in which the magnitudes of the basic elements (c, k, and m) are constant, regardless of the amplitude of motion of the system; a *bilateral element* is one in which forces are transmitted equally well in either direction through its connections.

KIRCHHOFF'S LAWS

1. *The sum of all the forces acting at a point (common connection of several elements) is zero:*

$$\sum_{i}^{n} F_i = 0 \qquad \text{(at a point)} \tag{10.17}$$

This follows directly from the considerations leading to Eq. (10.13).

2. *The sum of the relative velocities across the connections of series mechanical elements taken around a closed loop is zero:*

$$\sum_{i}^{n} v_i = 0 \qquad \text{(around a closed loop)} \tag{10.18}$$

This follows from the considerations leading to Eq. (10.14).

Kirchhoff's laws apply to any system, even when the elements are not linear or bilateral.

Example 10.1. Find the velocity of all the connection points and the forces acting on the elements of the system shown in Fig. 10.5. The system contains two velocity generators v_1 and v_6. Their magnitudes are known, their frequencies are the same, and they are 180° out-of-phase.

A. Using Eq. (10.17), write a force equation for each connection point except *a* and *e*.
At point *b*: $F_1 - F_2 - F_3 = 0$. In terms of velocities and impedances:

$$(v_1 - v_2)Z_1 - (v_2 - v_3)Z_2 - (v_2 - v_4)Z_4 = 0 \tag{a}$$

At point *c*, the two series elements have the same force acting: $F_2 - F_2 = 0$. In terms of velocities and impedances:

$$(v_2 - v_3)Z_2 - (v_3 - v_4)Z_3 = 0 \tag{b}$$

At point *d*: $F_2 + F_3 - F_4 - F_5 = 0$. In terms of velocities and impedances:

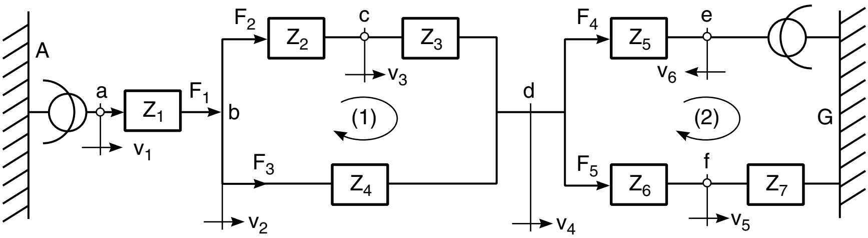

FIGURE 10.5 System of mechanical elements and vibration sources analyzed in Example 10.1 to find the velocity of each connection and the force acting on each element.

$$(v_3 - v_4)Z_3 + (v_2 - v_4)Z_4 - (v_4 + v_6)Z_5 - (v_4 - v_5)Z_6 = 0 \qquad (c)$$

Note that v_6 is (+) because of the 180° phase relation to v_1.
At point *f:* $F_5 - F_5 = 0$. In terms of velocities and impedances:

$$(v_4 - v_5)Z_6 - v_5 Z_7 = 0 \qquad (d)$$

Since v_1 and v_6 are known, the four unknown velocities v_2, v_3, v_4, and v_5 may be determined by solving the four simultaneous equations above. After the velocities are obtained, the forces may be determined from the following:

$$F_1 = (v_1 - v_2)Z_1 \qquad F_2 = (v_2 - v_3)Z_2 = (v_3 - v_4)Z_3$$

$$F_3 = (v_2 - v_4)Z_4 \qquad F_4 = (v_4 + v_6)Z_5$$

$$F_5 = (v_4 - v_5)Z_6 = v_5 Z_7$$

B. The method of *node forces.* Equations (*a*) through (*d*) above can be rewritten as follows:

$$v_1 Z_1 = (Z_1 + Z_2 + Z_3)v_2 - Z_2 v_3 - Z_4 v_4 \qquad (a')$$

$$0 = -Z_2 v_2 + (Z_2 + Z_3)v_3 - Z_3 v_4 \qquad (b')$$

$$0 = -Z_4 v_2 - Z_3 v_3 + (Z_3 + Z_4 + Z_5 + Z_6)v_4 - Z_6 v_5 \qquad (c')$$

$$-v_6 Z_5 = -Z_6 v_4 + (Z_6 + Z_7)v_5 \qquad (d')$$

These equations can be written by inspection of the schematic diagram by the following rule: *At each point with a common velocity (force node), equate the force generators to the sum of the impedances attached to the node multiplied by the velocity of the node, minus the impedances multiplied by the velocities of their other connection points.*

When the equations are written so that the unknown velocities form columns, the equations are in the proper form for a determinant solution for any of the unknowns. Note that the determinant of the *Z*'s is symmetrical about the main diagonal. This condition always exists and provides a check for the correctness of the equations.

C. Using Eq. (10.18), write a velocity equation in terms of force and mobility around enough closed loops to include each element at least once. In Fig. 10.5, note that

$$F_3 = F_1 - F_2 \qquad \text{and} \qquad F_5 = F_1 - F_4$$

Around loop (1):

$$F_2(\mathfrak{M}_2 + \mathfrak{M}_3) - (F_1 - F_2)\mathfrak{M}_4 = 0 \tag{e}$$

The minus sign preceding the second term results from going across the element 4 in a direction opposite to the assumed force acting on it.

Around loop (2):

$$F_4\mathfrak{M}_5 - v_6 - (F_1 - F_4)(\mathfrak{M}_6 + \mathfrak{M}_7) = 0 \tag{f}$$

A summation of velocities from A to G along the upper path forms the following closed loop:

$$v_1 + F_1\mathfrak{M}_1 + F_2(\mathfrak{M}_2 + \mathfrak{M}_3) + F_4\mathfrak{M}_5 - v_6 = 0 \tag{g}$$

Equations (e), (f), and (g) then may be solved for the unknown forces F_1, F_2, and F_4. The other forces are $F_3 = F_1 - F_2$ and $F_5 = F_1 - F_4$. The velocities are:

$$v_2 = v_1 - F_1\mathfrak{M}_1 \qquad v_3 = v_2 - F_2\mathfrak{M}_2 \qquad v_4 = v_2 - F_3\mathfrak{M}_4 \qquad v_5 = F_5\mathfrak{M}_7$$

When a system includes more than one source of vibration energy, a Kirchhoff's law analysis with impedance methods can be made only if all the sources are operating at the same frequency. This is the case because sinusoidal forces and velocities can add as phasors only when their frequencies are identical. However, they may differ in magnitude and phase. Kirchhoff's laws still hold for instantaneous values and can be used to write the differential equations of motion for any system.

RECIPROCITY THEOREM

If a force generator operating at a particular frequency at some point (1) *in a system of linear bilateral elements produces a velocity at another point* (2), *the generator can be removed from* (1) *and placed at* (2); *then the former velocity at* (2) *will exist at* (1), *provided the impedances at all points in the system are unchanged.* This theorem also can be stated in terms of a vibration generator that produces a certain velocity at its point of attachment (1), regardless of force required, and the force resulting on some element at (2).

Reciprocity is an important characteristic of linear bilateral elements. It indicates that a system of such elements can transmit energy equally well in both directions. It further simplifies the calculation on two-way energy transmission systems since the characteristics need be calculated for only one direction.

SUPERPOSITION THEOREM

If a mechanical system of linear bilateral elements includes more than one vibration source, the force or velocity response at a point in the system can be determined by adding the response to each source, taken one at a time (the other sources supplying no energy but replaced by their internal impedances).

The internal impedance of a vibrational generator is that impedance presented at its connection point when the generator is supplying no energy. This theorem finds useful application in systems having several sources. A very important application arises when the applied force is nonsinusoidal but can be represented by a Fourier series. Each term in the series can be considered a separate sinusoidal generator. The response at any point in the system can be calculated for each generator by using the impedance values at that frequency. Each response term becomes a term in the Fourier series representation of the total response function. The over-all response as a function of time then can be synthesized from the series.

Figure 10.6 illustrates an application of superposition. The velocities v_c' and v_c'' can be determined by the methods of Example 10.1. Then the velocity v_c is the sum of v_c' and v_c''.

THÉVENIN'S EQUIVALENT SYSTEM

If a mechanical system of linear bilateral elements contains vibration sources and produces an output to a load at some point at any particular frequency, the whole system can be represented at that frequency by a single constant-force generator F_c in parallel with a single impedance Z_i connected to the load. Thévenin's equivalent-system representation for a physical system may be determined by the following experimental procedure: Denote by F_c the force which is transmitted by the attachment point of the system to an infinitely rigid fixed point; this is called the *clamped force.* When the load connection is disconnected and perfectly free to move, a free velocity v_f is measured. Then the parallel impedance Z_i is F_c/v_f. The impedance Z_i also can be determined by measuring the internal impedance of the system when no source is supplying motional energy.

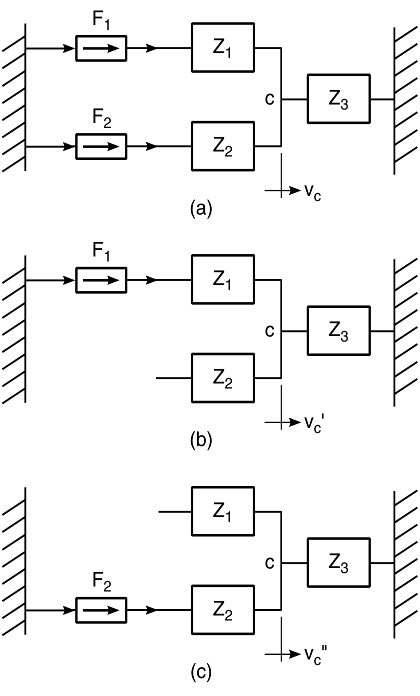

FIGURE 10.6 System of mechanical elements including two force generators used to illustrate the principle of superposition.

If the values of all the system elements in terms of ideal elements are known, F_c and Z_i may be determined analytically. A great advantage is derived from this representation in that attention is focused on the characteristics of a system at its output point and not on the details of the elements of the system. This allows an easy prediction of the response when different loads are attached to the output connection. After a final load condition has been determined, the system may be analyzed in detail for strength considerations.

NORTON'S EQUIVALENT SYSTEM

A mechanical system of linear bilateral elements having vibration sources and an output connection may be represented at any particular frequency by a single constant-velocity generator v_f in series with an internal impedance Z_i.

This is the series system counterpart of Thévenin's equivalent system where v_f is the free velocity and Z_i is the impedance as defined above. The same advantages in analysis exist as with Thévenin's parallel representation. The most advantageous one depends upon the type of structure to be analyzed. In the experimental determination of an equivalent system, it is usually easier to measure the free velocity than the clamped force on large heavy structures, while the converse is true for light structures. In any case, one representation is easily derived from the other. When v_f and Z_i are determined, $F_c = v_f Z_i$.

MECHANICAL 2-PORTS

Consider the "black box" shown in Fig. 10.7. It may have many elements between terminals (ports) (1) and (2). The forces and velocities at the ports can be determined by the use of 2-port equations in terms of impedances and mobilities. The impedance parameter equations are

$$F_1 = Z_{11}v_1 + Z_{12}v_2 \qquad \text{and} \qquad F_2 = Z_{21}v_1 + Z_{22}v_2$$

The Z parameters can be determined by measurements or from a known circuit model. These parameters are defined as follows:

1. For $v_2 = 0$ (port 2 clamped), $Z_{11} = F_1/v_1$ and $Z_{21} = F_2/v_1$.
2. For $v_1 = 0$ (port 1 clamped), $Z_{12} = F_1/v_2$ and $Z_{22} = F_2/v_2$

The mobility parameter equations for this situation are as follows:

$$v_1 = \mathfrak{M}_{11}F_1 + \mathfrak{M}_{12}F_2 \qquad \text{and} \qquad v_2 = \mathfrak{M}_{12}F_1 + \mathfrak{M}_{22}F_2$$

These $\mathfrak{M}$ parameters can be determined by measurement or from a model. The definitions are as follows:

1. For $F_2 = 0$ (port 2 free), $\mathfrak{M}_{11} = v_1/F_1$ and $\mathfrak{M}_{12} = v_2/F_1$.
2. For $F_1 = 0$ (port 1 free), $\mathfrak{M}_{21} = v_1/F_2$ and $\mathfrak{M}_{22} = v_2/F_2$.

Note that for large, massive structures, it may be difficult to clamp the ports to measure the impedance parameters. In this case, the mobility parameters requiring free conditions may be more appropriate. Likewise, for very light structures, the impedance parameters may be more appropriate. In any case, one set of parameters can be determined from the other by matrix inversion.

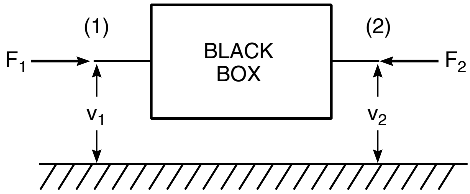

FIGURE 10.7 "Black box" representation of a mechanical system.

MECHANICAL IMPEDANCE MEASUREMENTS AND APPLICATIONS

Measurements

Transducers (Chap. 12), instrumentation (Chap. 13), and spectrum analyzers (Chap. 14) are essential subjects related to impedance measurements. Some special considerations are given here. The measurement of mechanical impedance involves the application of a sinusoidal force and the measurement of the complex ratio of force to the resulting velocity. Many combinations of transducers are capable of performing these measurements. However, the most effective method is to use an impedance transducer such as that shown in Fig. 10.8. These devices are available from suppliers of vibration-measuring sensors. As shown in Fig. 10.8, the force supplied by the vibration exciter passes through a force sensor to the unknown Z_x, and the motion is measured by an accelerometer whose output is integrated to obtain velocity. The accelerometer measures the true motion, but the force sensor measures the force required to move the accelerometer and its mounting structure, as well as the force to Z_x. This extra mass is usually called the "mass below the force gage." The impedance is then as follows:

$$Z_x = j\omega[K_f/K_a](e_f/e_a) - j\omega m_o$$

where e_f and e_a are the force gage and accelerometer phasor potentials, K_f in volts/N is the force gage sensitivity, K_a in volts/m/sec^2 is the accelerometer sensitivity, and m_o is the mass below the force gage. The ratio K_f/K_a and m_o can be determined by a calibration as follows:

1. With no attachment, $Z_x = 0$. Then $m_o = [K_f/K_a]\ (e_f/e_a)_0$.
2. Attach a known mass, M. Then $M + m_o = [K_f/K_a]\ (e_f/e_a)_1$, $m_o = M/\{[(e_f/e_a)_0/(e_f/e_a)_1] - 1\}$.
3. Thus $[K_f/K_a] = m_o/(e_f/e_a)_0$.

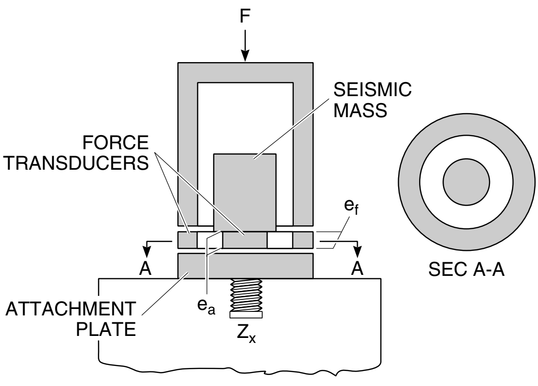

FIGURE 10.8 Device for the measurement of mechanical impedance in which force and acceleration are measured.

With the aid of a two-channel analyzer (see Chap. 14) or appropriate signal processing software (see Chap. 22), forces such as sine-sweeps, broad bandwidth random noise, or impacts can be used for these measurements. The Fourier transform of the force and acceleration potentials will provide correct sinusoidal terms. The impact method can be implemented with a hammer equipped with a force gage and accelerometer, as detailed in Chap. 21.

APPLICATIONS

The impedance concept is widely used in the study of mechanical systems.[2–4,6] Three practical applications are presented here.

Application 1. Assume one wishes to determine the free motion at a point on a structure that would be altered by the attachment of a sensor such as an accelerometer. The procedure is illustrated in Fig. 10.9, and involves the following steps.

1. Turn off the source causing the vibration v_f.
2. Measure the internal impedance Z_0 at a point A over the expected frequency range.
3. Attach the measuring device whose known impedance is Z_m and measure v_m.
4. Draw the Norton equivalent circuit at point A with Z_m attached. Note that Z_0 is attached to the reference since it may be masslike.
5. Calculate the free velocity from

$$v_f = v_m Z_m / (Z_0 + Z_m)$$

Application 2. Assume one wishes to choose a vibration isolator between a vibrating machine and a flexible structure. The criteria are to reduce the ratio of the velocity of the structure to the free velocity of the machine below some desired value, or to reduce the ratio of the force transmitted to the structure to the clapped force of the machine below some desired value. The procedure is as follows:

1. Model the system as shown in Fig. 10.10, where F_{cm} is the clamped force and Z_m is the impedance at the attachment point. The structural impedance at the attachment point is Z_{st} and "Z" is a set of Z parameters of the isolator that satisfy

$$F_1 = Z_{11}v_1 + Z_{12}v_2 \qquad \text{and} \qquad F_2 = Z_{21}v_1 + Z_{22}v_2$$

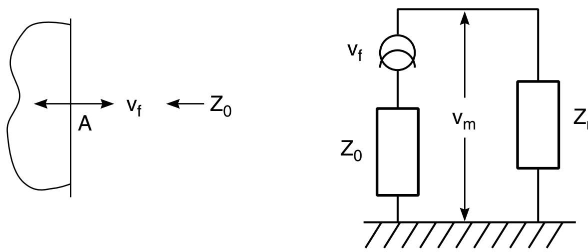

FIGURE 10.9 Measurement of free motion.

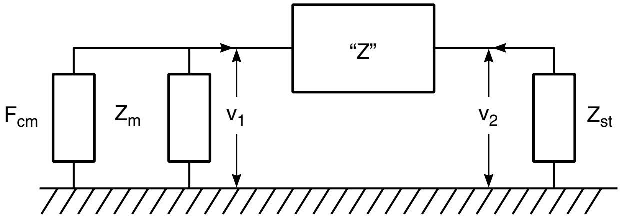

FIGURE 10.10 Vibration isolation application.

2. Add the source and structure to obtain

$$F_1 = F_{cm} - Z_m v_1 \qquad \text{and} \qquad F_2 = -Z_{st} v_2$$

The system equations then become

$$F_{cm} = (Z_{11} + Z_m)v_1 + Z_{12}v_2 \qquad \text{and} \qquad 0 = Z_{21}v_1 + (Z_{22} + Z_{st})v_1$$

3. Solve for the force to the structure $F_{st} = F_2$ from

$$F_{st}/F_{cm} = Z_{12}Z_{st}/[(Z_{11} + Z_m)(Z_{22} + Z_{st}) - Z_{12}Z_{21}]$$

This result follows from $v_{st} = F_{st}/Z_{st}$ and $v_{fm} = F_{cm}/Z_m$.

4. The ratio of the velocity of the structure to the free velocity of the machine is then given by

$$v_{st}/v_{fm} = Z_{21}Z_m/[(Z_{11} + Z_m)(Z_{22} + Z_{st}) - Z_{12}Z_{21}]$$

Typical vibration isolators can be modeled as shown in Fig. 10.11, where the Z parameters are given by

$$Z_{11} = c + j\omega m_1 + k/j\omega; \quad Z_{22} = c + j\omega m_2 + k/j\omega; \quad Z_{12} = Z_{21} = c + k/j\omega$$

The values of c, k, m_1, and m_2 should be available from the manufacturer, or they can be measured. Using the measured values of Z_m and Z_{st}, the transmissibilities of the force and velocity can be computed from the expression above, and plots of these functions versus frequency can be compared to the desired criteria.

Application 3. Assume one wishes to isolate a piece of equipment from a vibrating structure. The procedure is essentially the same as detailed in Application 2.

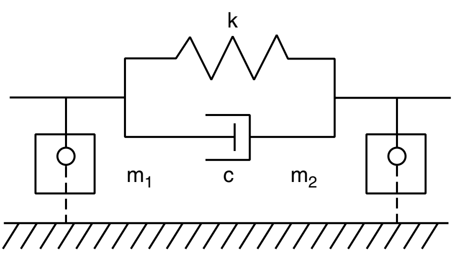

FIGURE 10.11 Vibration isolator model.

Specifically, measure the clamped force F_{st}, or the free velocity v_{st}, of the structure. Then in Fig. 10.10, replace the F_{cm} and Z_m with F_{st} and Z_{st}, and replace Z_{st} with Z_m. Proceed to write the system 2-port equations and solve for the force or velocity transmissibility.

REFERENCES

1. Hixson, E. L.: "Mechanical Impedance and Mobility," chap. 10, in C. M. Harris and C. E. Crede (eds.), "Shock and Vibration Handbook," 1st ed., McGraw-Hill Book Company, Inc., New York, 1961.
2. Morse, P. M.: "Vibration and Sound," p. 29, McGraw-Hill Book Company, Inc., New York, 1948.
3. Crafton, P. A.: "Shock and Vibration in Linear Systems," p. 99, Harper & Sons, Inc., New York, 1961.
4. Snowdon, J. C., "Vibration and Shock in Damped Mechanical Systems," p. 105, John Wiley & Sons, Inc., 1968.
5. Plunkett, R.: ASME Paper 53-A-45, *J. Appl. Mechanics,* **76**:250 (1954).
6. Carlson, U.: *J. Acous. Soc. Am.,* **97**(2):1345 (1995).