CHAPTER 23
CONCEPTS IN SHOCK DATA ANALYSIS

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INTRODUCTION

This chapter discusses the interpretation of shock measurements and the reduction of data to a form adapted to further engineering use. Methods of data reduction also are discussed. A shock measurement is a trace giving the value of a shock parameter versus time over the duration of the shock, referred to hereafter as a *time-history.* The shock parameter may define a motion (such as displacement, velocity, or acceleration) or a load (such as force, pressure, stress, or torque). It is assumed that any corrections that should be applied to eliminate distortions resulting from the instrumentation have been made. The trace may be a pulse or transient. Concepts in vibration data analysis are discussed in Chap. 22.

Examples of sources of shock to which this discussion applies are earthquakes (see Chap. 24), free-fall impacts, collisions, explosions, gunfire, projectile impacts, high-speed fluid entry, aircraft landing and braking loads, and spacecraft launch and staging loads.

BASIC CONSIDERATIONS

Often, a shock measurement in the form of a time-history of a motion or loading parameter is not useful directly for engineering purposes. Reduction to a different form is then necessary, the type of data reduction employed depending upon the ultimate use of the data.

Comparison of Measured Results with Theoretical Prediction. The correlation of experimentally determined and theoretically predicted results by comparison of records of time-histories is difficult. Generally, it is impractical in theoretical analyses to give consideration to all the effects which may influence the experimentally obtained results. For example, the measured shock often includes the vibrational response of the structure to which the shock-measuring device is attached. Such vibration obscures the determination of the shock input for which an applicable theory is being tested; thus, data reduction is useful in minimizing or eliminating the irrelevancies of the measured data to permit ready comparison of theory with corresponding aspects of the experiment. It often is impossible to make such comparisons on the basis of original time-histories.

Calculation of Structural Response. In the design of equipment to withstand shock, the required strength of the equipment is indicated by its response to the shock. The response may be measured in terms of the deflection of a member of the equipment relative to another member or by the magnitude of the dynamic loads imposed upon the equipment. The structural response can be calculated from the time-history by known means; however, certain techniques of data reduction result in descriptions of the shock that are related directly to structural response.

As a design procedure it is convenient to represent the equipment by an appropriate model that is better adapted to analysis (see Chap. 41). A typical model is shown in Fig. 23.1; it consists of a secondary structure supported by a primary structure. Each structure is represented as a lumped-parameter single degree-of-freedom system with the secondary mass m much smaller than the primary mass M so that the response of the primary mass is unaffected by the response of the secondary mass. The response of the primary mass to an input shock motion is the input shock motion to the secondary structure. Depending upon the ultimate objective of the design work, certain characteristics of the response of the model must be known:

1. If design of the secondary structure is to be effected, it is necessary to know the time-history of the motion of the primary structure. Such motion constitutes the excitation for the secondary structure.

2. In the design of the primary structure, it is necessary to know the deflection of such structure as a result of the shock, either the time-history or the maximum value.

By selection of suitable data reduction methods, response information useful in the design of the equipment is obtained from the original time-history.

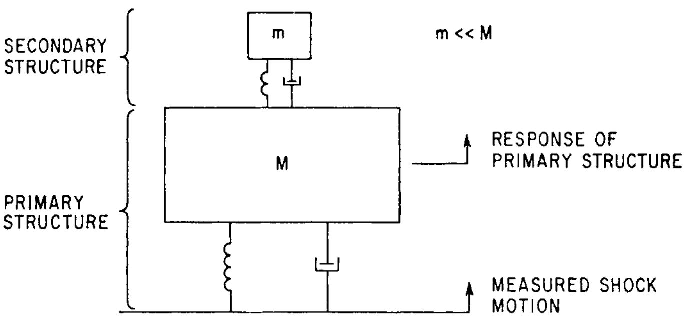


FIGURE 23.1 Commonly used structural model consisting of a primary and a secondary structure.

Laboratory Simulation of Measured Shock. Because of the difficulty of using analytical methods in the design of equipment to withstand shock, it is common practice to prove the design of equipments by laboratory tests that simulate the anticipated actual shock conditions. Unless the shock can be defined by one of a few simple functions, it is not feasible to reproduce in the laboratory the complete time-history of the actual shock experienced in service. Instead, the objective is to synthesize a shock having the characteristics and severity considered significant in causing damage to equipment. Then, the data reduction method is selected so that it extracts from the original time-history the parameters that are useful in specifying an appropriate laboratory shock test. Shock testing machines are discussed in Chap. 26.

EXAMPLES OF SHOCK MOTIONS

Five examples of shock motions are illustrated in Fig. 23.2 to show typical characteristics and to aid in the comparison of the various techniques of data reduction. The acceleration impulse and the acceleration step are the classical limiting cases of shock motions. The half-sine pulse of acceleration, the decaying sinusoidal acceleration, and the complex oscillatory-type motion typify shock motions encountered frequently in practice.

In selecting data reduction methods to be used in a particular circumstance, the applicable physical conditions must be considered. The original record, usually a time-history, may indicate any of several physical parameters; e.g., acceleration, force, velocity, or pressure. Data reduction methods discussed in subsequent sections of this chapter are applicable to a time-history of any parameter. For purposes of illustration in the following examples, the primary time-history is that of acceleration; time-histories of velocity and displacement are derived therefrom by integration. These examples are included to show characteristic features of typical shock motions and to demonstrate data reduction methods.

ACCELERATION IMPULSE OR STEP VELOCITY

The *delta function* $\boldsymbol{\delta}(t)$ is defined mathematically as a function consisting of an infinite ordinate (acceleration) occurring in a vanishingly small interval of abscissa (time) at time $t = 0$ such that the area under the curve is unity. An acceleration time-history of this form is shown diagrammatically in Fig. 23.2*A*. If the velocity and displacement are zero at time $t = 0$, the corresponding velocity time-history is the velocity step and the corresponding displacement time-history is a line of constant slope, as shown in the figure. The mathematical expressions describing these time histories are

$$\ddot{u}(t) = \dot{u}_0\boldsymbol{\delta}(t) \tag{23.1}$$

where $\boldsymbol{\delta}(t) = 0$ when $t \neq 0$, $\boldsymbol{\delta}(t) = \infty$ when $t = 0$, and $\int_{-\infty}^{\infty} \boldsymbol{\delta}(t)\, dt = 1$. The acceleration can be expressed alternatively as

$$\ddot{u}(t) = \lim_{\epsilon \to 0} \dot{u}_0/\epsilon \qquad [0 < t < \epsilon] \tag{23.2}$$

where $\ddot{u}(t) = 0$ when $t < 0$ and $t > \epsilon$. The corresponding expressions for velocity and displacement for the initial conditions $u = \dot{u} = 0$ when $t < 0$ are

$$\dot{u}(t) = \dot{u}_0 \qquad [t > 0] \tag{23.3}$$

$$u(t) = \dot{u}_0 t \qquad [t > 0] \tag{23.4}$$

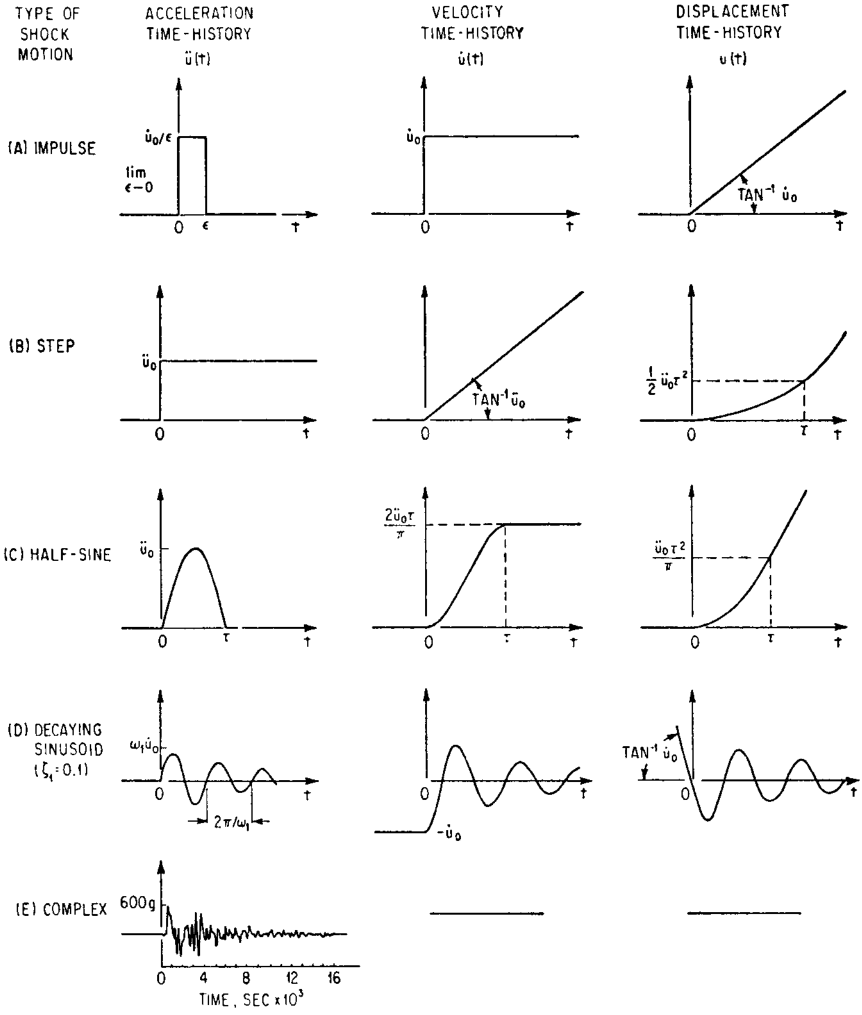


FIGURE 23.2 Five examples of shock motions.

ACCELERATION STEP

The *unit step function* $\mathbf{1}(t)$ is defined mathematically as a function which has a value of zero at time less than zero ($t < 0$) and a value of unity at time greater than zero ($t > 0$). The mathematical expression describing the acceleration step is

$$\ddot{u}(t) = \ddot{u}_0 \mathbf{1}(t) \tag{23.5}$$

where $\mathbf{1}(t) = 1$ for $t > 0$ and $\mathbf{1}(t) = 0$ for $t < 0$. An acceleration time-history of the unit step function is shown in Fig. 23.2*B;* the corresponding velocity and displacement time-histories are also shown for the initial conditions $u = \dot{u} = 0$ when $t = 0$.

$$\dot{u}(t) = \ddot{u}_0 t \qquad [t > 0] \tag{23.6}$$

$$u(t) = \tfrac{1}{2}\ddot{u}_0 t^2 \qquad [t > 0] \tag{23.7}$$

The unit step function is the time integral of the delta function:

$$\mathbf{1}(t) = \int_{-\infty}^{t} \boldsymbol{\delta}(t)\, dt \qquad [t > 0] \tag{23.8}$$

HALF-SINE ACCELERATION

A half-sine pulse of acceleration of duration τ is shown in Fig. 23.2*C;* the corresponding velocity and displacement time-histories also are shown, for the initial conditions $u = \dot{u} = 0$ when $t = 0$. The applicable mathematical expressions are

$$\begin{aligned} \ddot{u}(t) &= \ddot{u}_0 \sin\left(\frac{\pi t}{\tau}\right) && [0 < t < \tau] \\ \ddot{u}(t) &= 0 && \text{when } t < 0 \quad \text{and } t > \tau \end{aligned} \tag{23.9}$$

$$\begin{aligned} \dot{u}(t) &= \frac{\ddot{u}_0 \tau}{\pi}\left(1 - \cos\frac{\pi t}{\tau}\right) && [0 < t < \tau] \\ \dot{u}(t) &= \frac{2\ddot{u}_0 \tau}{\pi} && [t > \tau] \end{aligned} \tag{23.10}$$

$$\begin{aligned} u(t) &= \frac{\ddot{u}_0 \tau^2}{\pi^2}\left(\frac{\pi t}{\tau} - \sin\frac{\pi t}{\tau}\right) && [0 < t < \tau] \\ u(t) &= \frac{\ddot{u}_0 \tau^2}{\pi}\left(\frac{2t}{\tau} - 1\right) && [t > \tau] \end{aligned} \tag{23.11}$$

This example is typical of a class of shock motions in the form of acceleration pulses not having infinite slopes.

DECAYING SINUSOIDAL ACCELERATION

A decaying sinusoidal trace of acceleration is shown in Fig. 23.2*D;* the corresponding time-histories of velocity and displacement also are shown for the initial conditions $\dot{u} = -\dot{u}_0$ and $u = 0$ when $t = 0$. The applicable mathematical expression is

$$\ddot{u}(t) = \frac{\dot{u}_0 \omega_1}{\sqrt{1 - \zeta_1^2}} e^{-\zeta_1 \omega_1 t} \sin\left(\sqrt{1 - \zeta_1^2}\,\omega_1 t + \sin^{-1}\left(2\zeta_1\sqrt{1 - \zeta_1^2}\right)\right) \qquad [t > 0] \tag{23.12}$$

where ω_1 is the frequency of the vibration and ζ_1 is the fraction of critical damping corresponding to the decrement of the decay. Corresponding expressions for velocity and displacement are

$$\dot{u}(t) = \frac{\dot{u}_0}{\sqrt{1 - \zeta_1^2}} e^{-\zeta_1 \omega_1 t} \cos\left(\sqrt{1 - \zeta_1^2}\, \omega_1 t + \sin^{-1} \zeta_1\right) \qquad [t > 0] \qquad (23.13)$$

where $\dot{u}(t) = -\dot{u}_0$ when $t < 0$.

$$u(t) = -\frac{\dot{u}_0}{\omega_1 \sqrt{1 - \zeta_1^2}} e^{-\zeta_1 \omega_1 t} \sin\left(\sqrt{1 - \zeta_1^2}\, \omega_1 t\right) \qquad [t > 0] \qquad (23.14)$$

where $u(t) = -\dot{u}_0 t$ when $t < 0$.

COMPLEX SHOCK MOTION

The trace shown in Fig. 23.2*E* is an acceleration time-history representing typical field data. It cannot be defined by an analytic function. Consequently, the corresponding velocity and displacement time-histories can be obtained only by numerical, graphical, or analog integration of the acceleration time-history.

CONCEPTS OF DATA REDUCTION

Consideration of the engineering uses of shock measurements indicates two basically different methods for describing a shock: (1) a description of the shock in terms of its inherent properties, in the time domain or in the frequency domain; and (2) a description of the shock in terms of the effect on structures when the shock acts as the excitation. The latter is designated reduction to the response domain. The following sections discuss concepts of data reduction to the frequency and response domains.

Whenever practical, the original time-history should be retained even though the information included therein is reduced to another form. The purpose of data reduction is to make the data more useful for some particular application. The reduced data usually have a more limited range of applicability than the original time-history. These limitations must be borne in mind if the data are to be applied intelligently.

DATA REDUCTION TO THE FREQUENCY DOMAIN

Any nonperiodic function can be represented as the superposition of sinusoidal components, each with its characteristic amplitude and phase.[1] This superposition is the Fourier spectrum, as defined in Eq. (23.55). It is analogous to the Fourier components of a periodic function (Chap. 22). The Fourier components of a periodic function occur at discrete frequencies, and the composite function is obtained by superposition of components. By contrast, the classical Fourier spectrum for a nonperiodic function is a continuous function of frequency, and the composite function is achieved by integration. The following sections discuss the application of the continuous Fourier spectrum to describe the shock motions illustrated in Fig. 23.2. A discrete realization of the Fourier spectrum is given by Eq. (22.26).

Acceleration Impulse. Using the definition of the acceleration pulse given by Eq. (23.2) and substituting this for $f(t)$ in Eq. (23.55),

$$\mathbf{F}(\omega) = \lim_{\epsilon \to 0} \int_0^{\epsilon} \frac{\dot{u}_0}{\epsilon} e^{-j\omega t}\, dt \qquad (23.15)$$

Carrying out the integration,

$$\mathbf{F}(\omega) = \lim_{\epsilon \to 0} \frac{\dot{u}_0(1 - e^{-j\omega\epsilon})}{j\omega\epsilon} = \dot{u}_0 \tag{23.16}$$

The corresponding amplitude and phase spectra are

$$F(\omega) = \dot{u}_0; \qquad \theta(\omega) = 0 \tag{23.17}$$

These spectra are shown in Fig. 23.3*A*. The magnitude of the Fourier amplitude spectrum is a constant, independent of frequency, equal to the area under the acceleration-time curve.

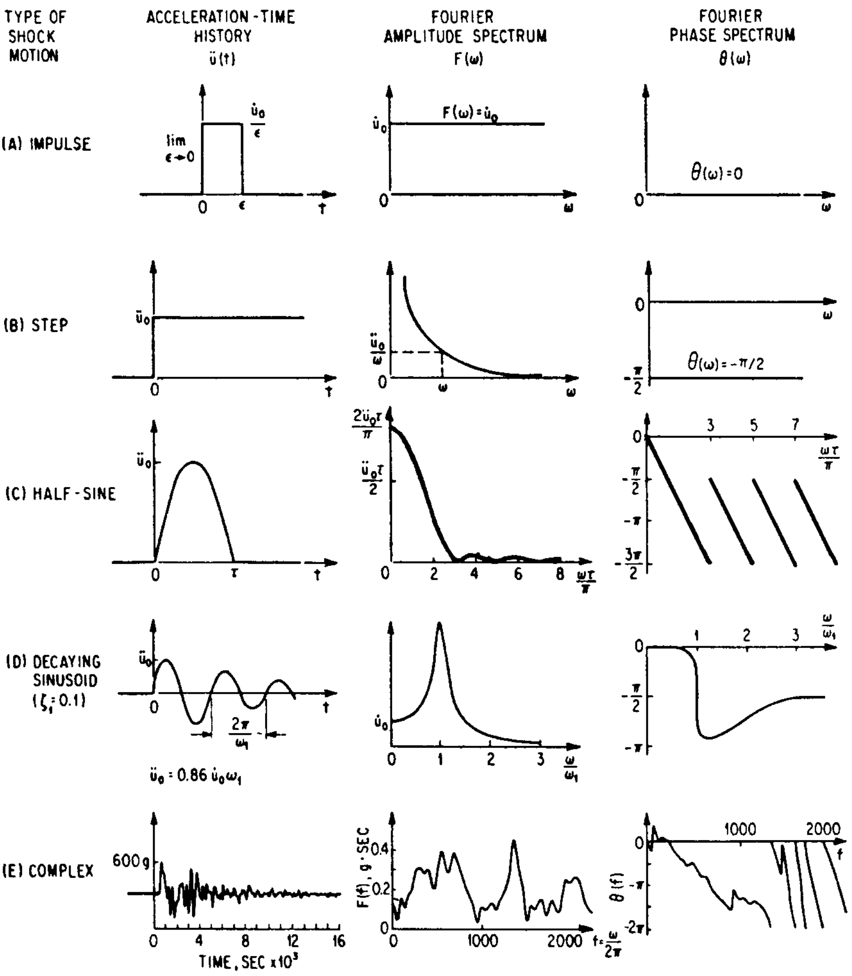


FIGURE 23.3 Fourier amplitude and phase spectra for the shock motions in Fig. 23.2.

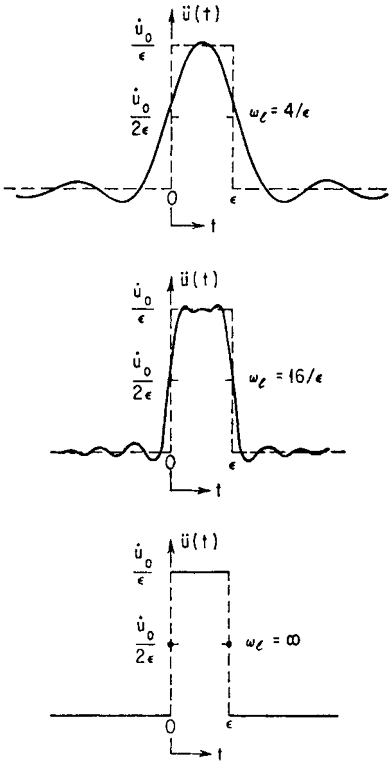


FIGURE 23.4 Time-histories which result from the superposition of the Fourier components of a rectangular pulse for several different upper limits of frequency ω_l of the components.

The physical significance of the spectra in Fig. 23.3*A* is shown in Fig. 23.4, where the rectangular acceleration pulse of magnitude $\dot{u}_0/\epsilon$ and duration $t = \epsilon$ is shown as approximated by superposed sinusoidal components for several different upper limits of frequency for the components. With the frequency limit $\omega_l = 4/\epsilon$, the pulse has a noticeably rounded contour formed by the superposition of all components whose frequencies are less than ω_l. These components tend to add in the time interval $0 < t < \epsilon$ and, though existing for all time from $-\infty$ to $+\infty$, cancel each other outside this interval, so that $\ddot{u}$ approaches zero. When $\omega_l = 16/\epsilon$, the pulse is more nearly rectangular and $\ddot{u}$ approaches zero more rapidly for time $t < 0$ and $t > \epsilon$. When $\omega_l = \infty$, the superposition of sinusoidal components gives $\ddot{u} = \dot{u}_0/\epsilon$ for the time interval of the pulse, and $\ddot{u} = \dot{u}_0/2\epsilon$ at $t = 0$ and $t = \epsilon$. The components cancel completely for all other times. As $\epsilon \to 0$ and $\omega_l \to \infty$, the infinitely large number of superimposed frequency components gives $\ddot{u} = \infty$ at $t = 0$. The same general result is obtained when the Fourier components of other forms of $\ddot{u}(t)$ are superimposed.

Acceleration Step. The Fourier spectrum of the acceleration step does not exist in the strict sense since the integrand of Eq. (23.55) does not tend to zero as $\omega \to \infty$. Using a convergence factor, the Fourier transform is found by substituting $\ddot{u}(t)$ for $f(t)$ in Eq. (23.55):

$$\mathbf{F}(\omega - ja) = \int_0^\infty \ddot{u}_0 e^{-j(\omega - ja)t}\, dt = \frac{\ddot{u}_0}{j(\omega - ja)} \tag{23.18}$$

Taking the limit as $a \to 0$,

$$\mathbf{F}(\omega) = \frac{\ddot{u}_0}{j\omega} \tag{23.19}$$

The amplitude and phase spectra are

$$F(\omega) = \frac{\ddot{u}_0}{\omega}; \qquad \theta(\omega) = -\frac{\pi}{2} \tag{23.20}$$

These spectra are shown in Fig. 23.3*B;* the amplitude spectrum decreases as frequency increases, whereas the phase is a constant independent of frequency. Note that the spectrum of Eq. (23.19) is $1/j\omega$ times the spectrum for the impulse given by Eq. (23.16).

Half-sine Acceleration. Substitution of the half-sine acceleration time-history, Eq. (23.9), into Eq. (23.57) gives

$$\mathbf{F}(\omega) = \int_0^\tau \ddot{u}_0 \sin \frac{\pi t}{\tau} e^{-j\omega t}\, dt \tag{23.21}$$

Performing the indicated integration gives

$$\begin{aligned} \mathbf{F}(\omega) &= \frac{\ddot{u}_0 \tau/\pi}{1 - (\omega\tau/\pi)^2}(1 + e^{-j\omega\tau}) \qquad [\omega \neq \pi/\tau] \\ \mathbf{F}(\omega) &= -\frac{j\ddot{u}_0\tau}{2} \qquad [\omega = \pi/\tau] \end{aligned} \tag{23.22}$$

Applying Eqs. (23.63) and (23.64) to find expressions for the spectra of amplitude and phase,

$$\begin{aligned} F(\omega) &= \frac{2\ddot{u}_0\tau}{\pi} \left| \frac{\cos(\omega\tau/2)}{1 - (\omega\tau/\pi)^2} \right| \qquad [\omega \neq \pi/\tau] \\ F(\omega) &= \frac{\ddot{u}_0\tau}{2} \qquad [\omega = \pi/\tau] \end{aligned} \tag{23.23}$$

$$\theta(\omega) = -\frac{\omega\tau}{2} + n\pi \tag{23.24}$$

where n is the smallest integer that prevents $|\theta(\omega)|$ from exceeding $3\pi/2$. The Fourier spectra of the half-sine pulse of acceleration are plotted in Fig. 23.3*C*.

Decaying Sinusoidal Acceleration. The application of Eq. (23.57) to the decaying sinusoidal acceleration defined by Eq. (23.12) gives the following expression for the Fourier spectrum:

$$\mathbf{F}(\omega) = \dot{u}_0 \frac{1 + j2\zeta_1\omega/\omega_1}{(1 - \omega^2/\omega_1^2) + j2\zeta_1\omega/\omega_1} \tag{23.25}$$

This can be converted to a spectrum of absolute values by applying Eq. (23.63):

$$F(\omega) = \dot{u}_0 \sqrt{\frac{1 + (2\zeta_1\omega/\omega_1)^2}{(1 - \omega^2/\omega_1^2)^2 + (2\zeta_1\omega/\omega_1)^2}} \tag{23.26}$$

A spectrum of phase angle is obtained from Eq. (23.64):

$$\theta(\omega) = -\tan^{-1} \frac{2\zeta_1(\omega/\omega_1)^3}{(1 - \omega^2/\omega_1^2) + (2\zeta_1\omega/\omega_1)^2} \tag{23.27}$$

These spectra are shown in Fig. 23.3*D* for a value of $\zeta = 0.1$. The peak in the amplitude spectrum near the frequency ω_1 indicates a strong concentration of Fourier components near the frequency of occurrence of the oscillations in the shock motion.

Complex Shock. The complex shock motion shown in Fig. 23.3*E* is the result of actual measurements; hence, its functional form is unknown. Its Fourier spectrum must be computed numerically. The Fourier spectrum shown in Fig. 23.3*E* was evaluated digitally using 100 time increments of 0.00015 sec duration. The peaks in the amplitude spectrum indicate concentrations of sinusoidal components near the frequencies of various oscillations in the shock motion. The portion of the phase spectrum at the high frequencies creates an appearance of discontinuity. If the phase angle were not returned to zero each time it passes through –360°, as a convenience in plotting, the curve would be continuous.

Application of the Fourier Spectrum. The Fourier spectrum description of a shock is useful in linear analysis when the properties of a structure on which the shock acts are defined as a function of frequency. Such properties are designated by the general term *frequency response function;* in shock and vibration technology, commonly used frequency response functions are mechanical impedance, mobility, and transmissibility. Such functions are often inappropriately called "transfer functions." This terminology should be reserved for functions of the Laplace variable (see Chap. 21).

When a shock acts on a structure, the structure responds in a manner that is essentially oscillatory. The frequencies that appear predominantly in the response are (1) the preponderant frequencies of the shock and (2) the natural frequencies of the structure. The Fourier spectrum of the response $\mathbf{R}(\omega)$ is the product of the Fourier spectrum of the shock $\mathbf{F}(\omega)$ and an appropriate frequency response function for the structure, as given by Eq. (21.27). For example, if $\mathbf{F}(\omega)$ and $\mathbf{R}(\omega)$ are Fourier spectra of acceleration, the frequency response function is the transmissibility of the structure, i.e., the ratio of acceleration at the responding station to the acceleration at the driving station, as a function of frequency. However, if $\mathbf{R}(\omega)$ is a Fourier spectrum of velocity and $\mathbf{F}(\omega)$ is a Fourier spectrum of force, the frequency response function is mobility as a function of frequency.

The Fourier spectrum also finds application in evaluating the effect of a load upon a shock source. A source of shock generally consists of a means of shock excitation and a resilient structure through which the excitation is transmitted to a load. Consequently, the character of the shock delivered by the resilient structure of the shock source is influenced by the nature of the load being driven. The characteristics of the source and load may be defined in terms of mechanical impedance or mobility (see Chap. 10). If the shock motion at the source output is measured with no load and expressed in terms of its Fourier spectrum, the effect of the load upon this shock motion can be determined by Eq. (41.1). The resultant motion with the load attached is described by its Fourier spectrum.

The frequency response function of a structure may be determined by applying a transient force to the structure and noting the response. This is analogous to the more commonly used method of applying a sinusoidally varying force whose frequency can be varied over a wide range and noting the sinusoidally varying motion at the frequency of the force application. In some circumstances, it may be more convenient to apply a transient. From the measured time-histories of the force and the response, the corresponding Fourier spectra can be calculated. The frequency response function is the quotient of the Fourier spectrum of the force divided by the Fourier spectrum of the response (see Chap. 21).

DATA REDUCTION TO THE RESPONSE DOMAIN

A structure or physical system has a characteristic response to a particular shock applied as an excitation to the structure. The magnitudes of the response peaks can be used to define certain effects of the shock by considering systematically the properties of the system and relating the peak responses to such properties. This is in contrast to the Fourier spectrum description of a shock in the following respects:

1. Whereas the Fourier spectrum defines the shock in terms of the amplitudes and phase relations of its frequency components, the response spectrum describes only the effect of the shock upon a structure in terms of peak responses. This effect is of considerable significance in the design of equipments and in the specification of laboratory tests.

2. The time-history of a shock cannot be determined from the knowledge of the peak responses of a system excited by the shock; i.e., the calculation of peak responses is an irreversible operation. This contrasts with the Fourier spectrum, where the Fourier spectrum can be determined from the time-history, and vice versa.

By limiting consideration to the response of a linear, viscously damped single degree-of-freedom structure with lumped parameters (hereafter referred to as a simple structure and illustrated in Fig. 23.5), there are only two structural parameters upon which the response depends: (1) the undamped natural frequency and (2) the fraction of critical damping. With only two parameters involved, it is feasible to obtain from the shock measurement a systematic presentation of the peak responses of many simple structures. This process is termed *data reduction to the response domain.* This type of reduced data applies directly to a system that responds in a single degree-of-freedom; it is useful to some extent by normal-mode superposition to evaluate the response of a linear system that responds in more than one degree-of-freedom. The conditions of a particular application determine the magnitude of errors resulting from superposition.[1–4]

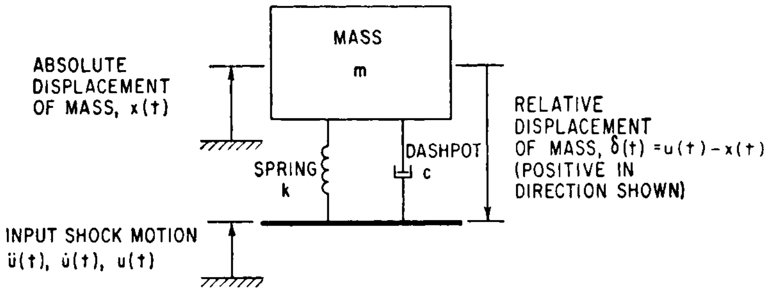


FIGURE 23.5 Representation of a simple structure used to accomplish the data reduction of a shock motion to the response domain.

Shock Response Spectrum. The response of a system to a shock can be expressed as the time-history of a parameter that describes the motion of the system. For a simple system, the magnitudes of the response peaks can be summarized as a function of the natural frequency or natural period of the responding system, at various values of the fraction of critical damping. This type of presentation is termed a *shock response spectrum*, or simply a *response spectrum* or a *shock spectrum*. In the shock response spectrum, or more specifically the two-dimensional shock response spectrum, only the maximum value of the response found in a single time-history is plotted. The three-dimensional shock response spectrum conceptually takes the form of a surface and shows the distribution of response peaks throughout the time-history. The two-dimensional spectrum is more common and is discussed in considerable detail in the immediately following section. The three-dimensional spectrum is discussed in less detail in a later section.

Parameters for the Shock Response Spectrum. The peak response of the simple structure may be defined, as a function of natural frequency, in terms of any one of several parameters that describe its motion. The parameters often are related to each other by the characteristics of the structure. However, inasmuch as one of the advantages of the shock response spectrum method of data reduction and presentation is convenience of application to physical situations, it is advantageous to give careful consideration in advance to the particular parameter that is best adapted to the attainment of particular objectives. Referring to the simple structure shown in Fig. 23.5, the following significant parameters may be determined directly from measurements on the structure:

1. Absolute displacement $x(t)$ of mass m. This indicates the displacement of the responding structure with reference to an inertial reference plane, i.e., coordinate axes fixed in space.
2. Relative displacement $\delta(t)$ of mass m. This indicates the displacement of the responding structure relative to its support, a quantity useful for evaluating the distortions and strains within the responding structure.
3. Absolute velocity $\dot{x}(t)$ of mass m. This quantity is useful for determining the kinetic energy of the structure.
4. Relative velocity $\dot{\delta}(t)$ of mass m. This quantity is useful for determining the stresses generated within the responding structure due to viscous damping and the maximum energy dissipated by the responding structure.
5. Absolute acceleration $\ddot{x}(t)$ of mass m. This quantity is useful for determining the stresses generated within the responding structure due to the combined elastic and damping reactions of the structure.

The *equivalent static acceleration* is that steadily applied acceleration, expressed as a multiple of the acceleration of gravity, which distorts the structure to the maximum distortion resulting from the action of the shock.[5] For the simple structure of Fig. 23.5, the relative displacement response δ indicates the distortion under the shock condition. The corresponding distortion under static conditions, in a $1g$ gravitational field, is

$$\delta_{st} = \frac{mg}{k} = \frac{g}{\omega_n^2} \tag{23.28}$$

By analogy, the maximum distortion under the shock condition is

$$\delta_{\max} = \frac{A_{eq}g}{\omega_n^2} \tag{23.29}$$

where A_{eq} is the equivalent static acceleration in units of gravitational acceleration. From Eq. (23.29),

$$A_{eq} = \frac{\delta_{\max}\omega_n^2}{g} \tag{23.30}$$

The maximum relative displacement $\delta_{\max}$ and the equivalent static acceleration A_{eq} are directly proportional.

If the shock is a loading parameter, such as force, pressure, or torque, as a function of time, the corresponding equivalent static parameter is an equivalent static force, pressure, or torque, respectively. Since the supporting structure is assumed to be motionless when a shock loading acts, the relative response motions and absolute response motions become identical.

The differential equation of motion for the system shown in Fig. 23.5 is

$$-\ddot{x}(t) + 2\zeta\omega_n\dot{\delta}(t) + \omega_n^2\delta(t) = 0 \tag{23.31}$$

where ω_n is the undamped natural frequency and ζ is the fraction of critical damping. When $\zeta = 0$, $\ddot{x}_{max} = A_{eq}g$; this follows directly from the relation of Eq. (23.29). When $\zeta \neq 0$, the acceleration $\ddot{x}$ experienced by the mass m results from forces transmitted by the spring k and the damper c. Thus, in a damped system, the maximum acceleration of mass m is not exactly equal to the equivalent static acceleration. However, in most mechanical structures, the damping is relatively small; therefore, the equivalent static acceleration and the maximum absolute acceleration often are interchangeable with negligible error.

Referring to the model in Fig. 23.1, suppose the equivalent static acceleration A_{eq} and the maximum absolute acceleration $\ddot{x}_{max}$ are known for the primary structure. Then A_{eq} is useful directly for calculating the maximum relative displacement response of the primary structure. When the natural frequency of the secondary structure is much higher than the natural frequency of the primary structure, the maximum acceleration $\ddot{x}_{max}$ of M is useful for calculating the maximum relative displacement of m with respect to M. The secondary structure then responds in a "static manner" to the acceleration of the mass M; i.e., the maximum acceleration of m is approximately equal to that of M. Consequently, both A_{eq} and $\ddot{x}_{max}$ can be used for design purposes to calculate equivalent static loads on structures or equipment.

If the damping in the responding structure is large ($\zeta > 0.2$), the values of A_{eq} and $\ddot{x}_{max}$ are significantly different. Because the maximum distortion of primary structures often is the type of information required and the equivalent static acceleration is an expression of this response in terms of an equivalent static loading, the following discussion is limited to shock response spectra in terms of A_{eq}.

The response of a simple structure with small damping to oscillatory-type shock excitation often is substantially sinusoidal at the natural frequency of the structure, i.e., the envelope of the oscillatory response varies in a relatively slow manner, as depicted in Fig. 23.6. The maximum relative displacement δ_{max}, the maximum relative velocity $\dot{\delta}_{max}$, and the maximum absolute acceleration $\ddot{x}_{max}$ are related approximately as follows:

$$\dot{\delta}_{max} = \omega_n\delta_{max}; \qquad \ddot{x}_{max} = \omega_n\dot{\delta}_{max}; \qquad \ddot{x}_{max} = \omega_n^2\delta_{max} \tag{23.32}$$

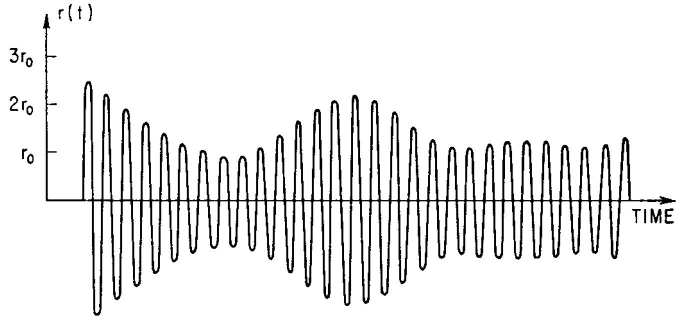


FIGURE 23.6 Examples of an oscillatory response time-history $r(t)$ for which the envelope of the response varies in a relatively slow manner.

where the sign may be neglected since the positive and negative maxima are approximately equal. When applicable, these relations may be used to convert from a spectrum expressed in one parameter to a spectrum expressed in another parameter.

For idealized shock motions which often are approximated in practice, it is desirable to use a dimensionless ratio for the ordinate of the shock response spectrum. Some of the more common dimensionless ratios are

$$\frac{gA_{\text{eq}}}{\ddot{u}_{\max}} = \frac{\omega_n^2\delta_{\max}}{\ddot{u}_{\max}}; \qquad \frac{\ddot{x}_{\max}}{\ddot{u}_{\max}}; \qquad \frac{\omega_n\delta_{\max}}{\Delta\dot{u}}; \qquad \frac{\dot{\delta}_{\max}}{\Delta\dot{u}}; \qquad \frac{\delta_{\max}}{u_{\max}}$$

where $\ddot{u}_{\max}$ and $u_{\max}$ are the maximum acceleration and displacement, respectively, of the shock motion and $\Delta\dot{u}$ is the velocity change of the shock motion (equal to the area under the acceleration time-history). Sometimes these ratios are referred to as *shock amplification factors.*

Calculation of Shock Response Spectrum. The relative displacement response of a simple structure (Fig. 23.5) resulting from a shock defined by the acceleration $\ddot{u}(t)$ of the support is given by the *Duhamel integral*[6]

$$\delta(t) = \frac{1}{\omega_d} \int_0^t \ddot{u}(t_v)e^{-\zeta\omega_n(t - t_v)} \sin \omega_d(t - t_v)\, dt_v \tag{23.33}$$

where $\omega_n = (k/m)^{1/2}$ is the undamped natural frequency, $\zeta = c/2m\omega_n$ is the fraction of critical damping, and $\omega_d = \omega_n(1 - \zeta^2)^{1/2}$ is the damped natural frequency. The excitation $\ddot{u}(t_v)$ is defined as a function of the variable of integration t_v, and the response $\delta(t)$ is a function of time t. The relative displacement δ and relative velocity $\dot{\delta}$ are considered to be zero when $t = 0$. The equivalent static acceleration, defined by Eq. (23.30), as a function of ω_n and ζ is

$$A_{\text{eq}}(\omega_n,\zeta) = \frac{\omega_n^2}{g}\, \delta_{\max}(\omega_n,\zeta) \tag{23.34}$$

If a shock loading such as the input force $F(t)$ rather than an input motion acts on the simple structure, the response is

$$\delta(t) = \frac{1}{m\omega_d} \int_0^t F(t_v)e^{-\zeta\omega_n(t - t_v)} \sin \omega_d(t - t_v)\, dt_v \tag{23.35}$$

and an equivalent static force is given by

$$F_{\text{eq}}(\omega_n,\zeta) = k\delta_{\max}(\omega_n,\zeta) = m\omega_n^2\delta_{\max}(\omega_n,\zeta) \tag{23.36}$$

The equivalent static force is related to equivalent static acceleration by

$$F_{\text{eq}}(\omega_n,\zeta) = mA_{\text{eq}}(\omega_n,\zeta) \tag{23.37}$$

It is often of interest to determine the maximum relative displacement of the simple structure in Fig. 23.5 in both a positive and a negative direction. If $\ddot{u}(t)$ is positive as shown, positive values of $\ddot{x}(t)$ represent upward acceleration of the mass m. Initially, the spring is compressed and the positive direction of $\delta(t)$ is taken to be positive as shown. Conversely, negative values of $\delta(t)$ represent extension of spring k from its original position. It is possible that the ultimate use of the reduced data would require that both extension and compression of spring k be determined. Correspondingly, a positive and a negative sign may be associated with an equivalent static acceleration A_{eq} of the support, so that A_{eq}^+ is an upward acceleration producing a positive deflection δ and A_{eq}^{-} is a downward acceleration producing a negative deflection δ.

For some purposes it is desirable to distinguish between the maximum response which occurs during the time in which the measured shock acts and the maximum response which occurs during the free vibration existing after the shock has terminated. The shock spectrum based on the former is called a *primary shock response spectrum* and that based on the latter is called a *residual shock response spectrum.* For instance, the response $\delta(t)$ to the half-sine pulse in Fig. 23.2*C* occurring during the period $(t < \tau)$ is the primary response and the response $\delta(t)$ occurring during the period $(t > \tau)$ is the residual response. Reference is made to primary and residual shock response spectra in the next section on *Examples of Shock Response Spectra* and in the section on *Relationship between Shock Response Spectrum and Fourier Spectrum.*

Examples of Shock Response Spectra. In this section the shock response spectra are presented for the five acceleration time-histories in Fig. 23.2. These spectra, shown in Fig. 23.7, are expressed in terms of equivalent static acceleration for the undamped responding structure, for $\zeta = 0.1, 0.5$, and other selected fractions of critical damping. Both the maximum positive and the maximum negative responses are indicated. In addition, a number of relative displacement response time-histories $\delta(t)$ are plotted to show the nature of the responses. A large number of shock response spectra, based on various response parameters, are given in Chap. 8.

ACCELERATION IMPULSE: The application of Eq. (23.33) to the acceleration impulse shown in Fig. 23.2*A* and defined by Eq. (23.1) yields

$$\delta(t) = \frac{\dot{u}_0}{\omega_d} e^{-\zeta\omega_n t} \sin \omega_d t \qquad [\zeta < 1] \tag{23.38}$$

This response is plotted in Fig. 23.7*A* for $\zeta = 0, 0.1$, and 0.5. The response peaks are reached at the times $t = (\cos^{-1}\zeta)/\omega_d$, $\cos^{-1}\zeta$ increasing by π for each succeeding peak. The values of the response at the peaks are

$$\delta_{\max}^{(i)}(\omega_n,\zeta) = \frac{\dot{u}_0}{\omega_n} \exp\left(-\frac{\zeta}{\sqrt{1-\zeta^2}}\,[\cos^{-1}\zeta + (i-1)\pi]\right) \qquad [0 < \cos^{-1}\zeta \le \pi/2] \tag{23.39}$$

where i is the number of the peak ($i = 1$ for the first positive peak, $i = 2$ for the first negative peak, etc.).

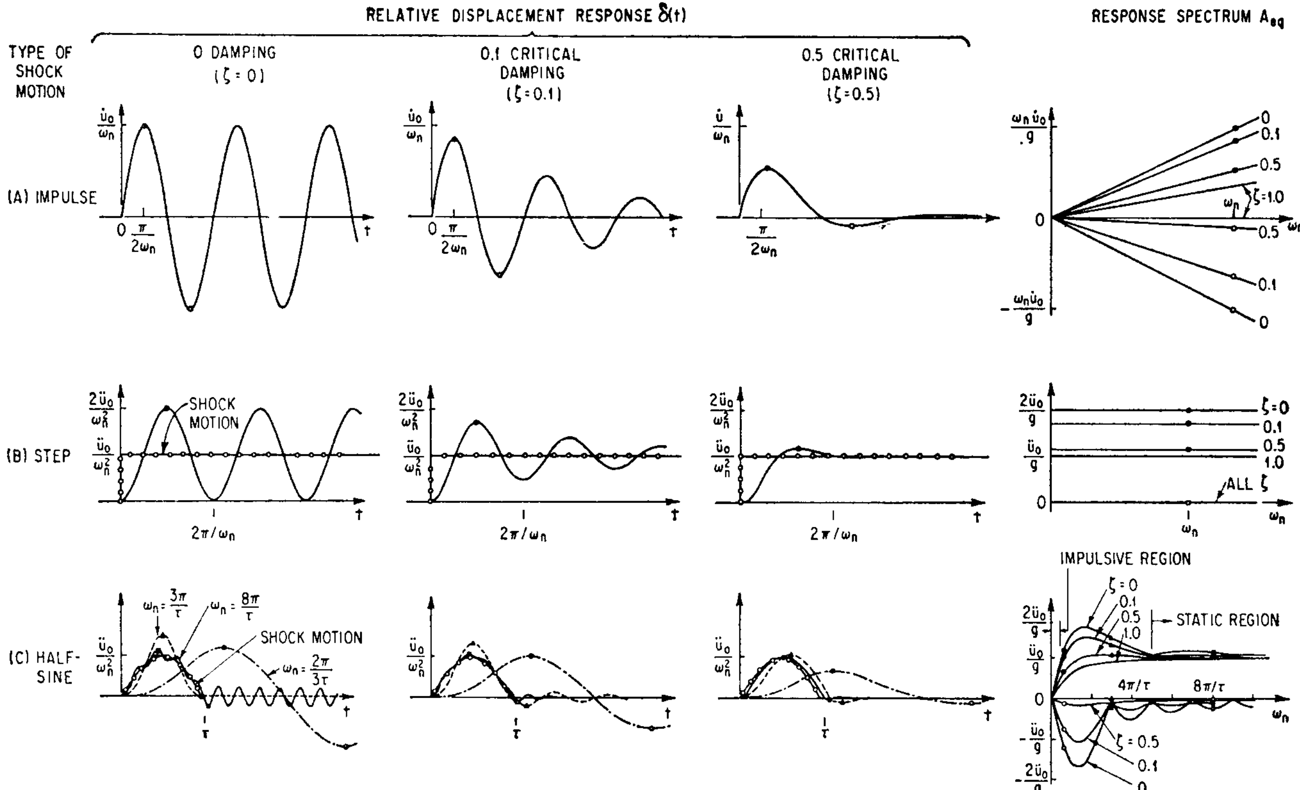
The largest positive response occurs at the first peak, i.e., when $i = 1$, and is shown by the solid dots in Fig. 23.7*A*. Hence, the equivalent static acceleration in the positive direction is obtained by substitution of Eq. (23.39) into Eq. (23.34) with $i = 1$:

$$A_{eq}^{+}(\omega_n,\zeta) = \frac{\omega_n \dot{u}_0}{g} \exp\left(-\frac{\zeta}{\sqrt{1-\zeta^2}}\cos^{-1}\zeta\right) \tag{23.40}$$

The equivalent static acceleration in the negative direction is calculated from the maximum relative deflection at the second peak, i.e., when $i = 2$, and is shown by the hollow dots in Fig. 23.7*A:*

$$A_{eq}^{-}(\omega_n,\zeta) = \frac{\omega_n \dot{u}_0}{g} \exp\left(-\frac{\zeta}{\sqrt{1-\zeta^2}}(\cos^{-1}\zeta + \pi)\right) \tag{23.41}$$

The resulting shock spectrum is shown in Fig. 23.7*A* with curves for $\zeta = 0, 0.1, 0.5$, and 1.0. At any value of damping, a shock response spectrum is a straight line pass-


RELATIVE DISPLACEMENT RESPONSE δ(t)
RESPONSE SPECTRUM A_{eq}
TYPE OF SHOCK MOTION
0 DAMPING (ζ = 0)
0.1 CRITICAL DAMPING (ζ = 0.1)
0.5 CRITICAL DAMPING (ζ = 0.5)
(A) IMPULSE
(B) STEP
(C) HALF-SINE
SHOCK MOTION
$\omega_n = \frac{3\pi}{\tau}$
$\omega_n = \frac{8\pi}{\tau}$
$\omega_n = \frac{2\pi}{3\tau}$
$\frac{\pi}{2\omega_n}$
$2\pi/\omega_n$
ALL ζ
ζ = 1.0
IMPULSIVE REGION
STATIC REGION
$4\pi/\tau$
$8\pi/\tau$
ζ = 0.5
0.1
0

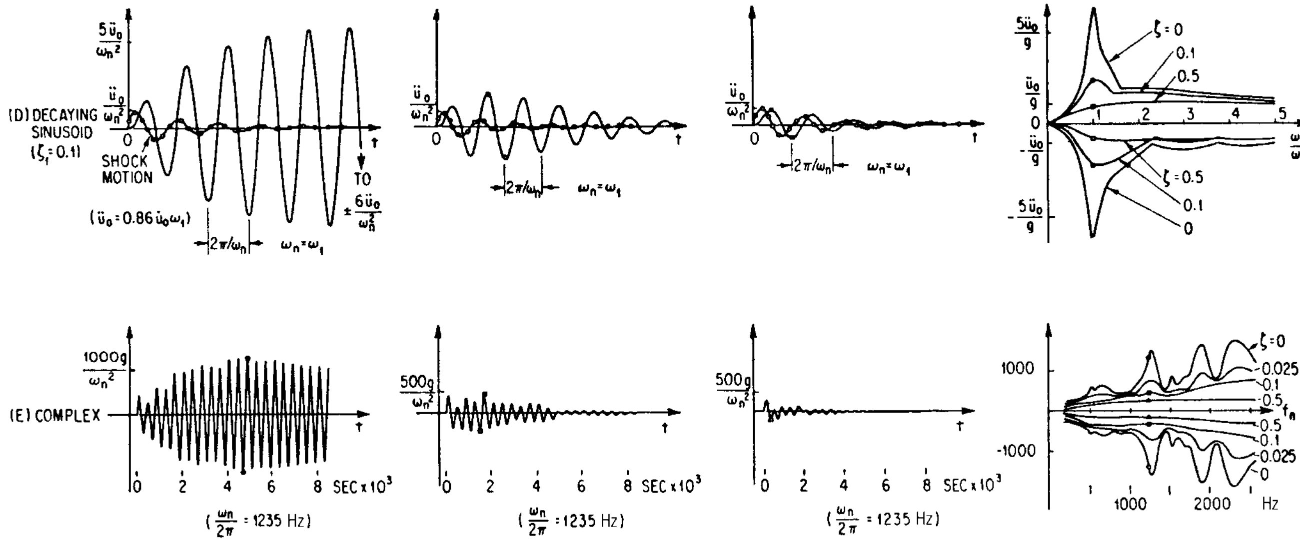


FIGURE 23.7 Time-histories of response to shock motions defined in Fig. 23.2 and corresponding shock response spectra.

ing through the origin. The peak distortion of the structure δ_{max} is inversely proportional to frequency. Thus, the relative displacement of the mass increases as the natural frequency decreases, whereas the equivalent static acceleration has an opposite trend.

ACCELERATION STEP: The response of a simple structure to the acceleration step in Fig. 23.2*B* is found by substituting from Eq. (23.5) in Eq. (23.33) and integrating:

$$\delta(t) = \frac{\ddot{u}_0}{\omega_n^2}\left[1 - \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}} \cos\,(\omega_d t - \sin^{-1}\zeta)\right] \qquad [\zeta < 1] \tag{23.42}$$

The responses $\delta(t)$ are shown in Fig. 23.7*B* for $\zeta = 0, 0.1$, and 0.5. The response overshoots the value $\ddot{u}_0/\omega_n^2$ and then oscillates about this value as a mean with diminishing amplitude as energy is dissipated by damping. An overshoot to $2\ddot{u}_0/\omega_n^2$ occurs for zero damping. A response $\delta = \ddot{u}_0/\omega_n^2$ would result from a steady application of the acceleration $\ddot{u}_0$.

The response maxima and minima occur at the times $t = i\pi/\omega_d$, $i = 0$ providing the first minimum and $i = 1$ the first maximum. The maximum values of the relative displacement response are

$$\delta_{max}(\omega_n,\zeta) = \frac{\ddot{u}_0}{\omega_n^2}\left[1 + \exp\left(-\frac{\zeta i\pi}{\sqrt{1-\zeta^2}}\right)\right] \qquad [i \text{ odd}] \tag{23.43}$$

The largest positive response occurs at the first maximum, i.e., where $i = 1$, and is shown by the solid symbols in Fig. 23.7*B*. The equivalent static acceleration in the positive direction is obtained by substitution of Eq. (23.43) into Eq. (23.34) with $i = 1$:

$$A_{eq}^{+}(\omega_n,\zeta) = \frac{\ddot{u}_0}{g}\left[1 + \exp\left(-\frac{\zeta\pi}{\sqrt{1-\zeta^2}}\right)\right] \tag{23.44a}$$

The greatest negative response is zero; it occurs at $t = 0$, independent of the value of damping, as shown by open symbols in Fig. 23.7*B*. Thus, the equivalent static acceleration in the negative direction is

$$A_{eq}^{-}(\omega_n,\zeta) = 0 \tag{23.44b}$$

Since the equivalent static acceleration is independent of natural frequency, the shock response spectrum curves shown in Fig. 23.7*B* are horizontal lines. The symbols shown on the shock response spectra correspond to the responses shown.

The equivalent static acceleration for an undamped simple structure is twice the value of the acceleration step $\ddot{u}_0/g$. As the damping increases, the overshoot in response decreases; there is no overshoot when the structure is critically damped.

HALF-SINE ACCELERATION: The expressions for the response of the damped simple structure to the half-sine acceleration of Eq. (23.9) are too involved to have general usefulness. For an undamped system, the response $\delta(t)$ is

$$\begin{aligned} \delta(t) &= \frac{\ddot{u}_0}{\omega_n^2}\left(\frac{(\omega_n\tau/\pi)}{1-(\omega_n\tau/\pi)^2}\right)[\sin\,\omega_n t - (\omega_n\tau/\pi)\,\sin\,(\pi t/\tau)] \qquad [0 < t \le \tau] \\ \delta(t) &= \frac{\ddot{u}_0}{\omega_n^2}\left(\frac{(\omega_n\tau/\pi)}{1-(\omega_n\tau/\pi)^2}\right) 2\cos\left(\frac{\omega_n\tau}{2}\right)\sin\left[\omega_n\left(t-\frac{\tau}{2}\right)\right] \qquad [t > \tau] \end{aligned} \tag{23.45}$$

For zero damping the residual response is sinusoidal with constant amplitude. The first maximum in the response of a simple structure with natural frequency less than π/τ occurs during the residual response; i.e., after $t = \tau$. As a result, the magnitude of each succeeding response peak is the same as that of the first maximum. Thus the positive and negative shock response spectrum curves are equal for $\omega_n \leq \pi/\tau$. The dot-dash curve in Fig. 23.7*C* is an example of the response at a natural frequency of $2\pi/3\tau$. The peak positive response is indicated by a solid circle, the peak negative response by an open circle. The positive and negative shock response spectrum values derived from this response are shown on the undamped ($\zeta = 0$) shock response spectrum curves at the right-hand side of Fig. 23.7*C*, using the same symbols.

At natural frequencies below $\pi/2\tau$, the shock response spectra for an undamped system are very nearly linear with a slope $\pm 2\dot{u}_0\tau/\pi g$. In this low-frequency region the response is essentially impulsive; i.e., the maximum response is approximately the same as that due to an ideal acceleration impulse (Fig. 23.7*A*) having a velocity change $\dot{u}_0$ equal to the area under the half-sine acceleration time-history.

The response at the natural frequency $3\pi/\tau$ is the dotted curve in Fig. 23.7*C*. The displacement and velocity response are both zero at the end of the pulse, and hence no residual response occurs. The solid and open triangles indicate the peak positive and negative response, the latter being zero. The corresponding points appear on the undamped shock response spectrum curves. As shown by the negative undamped shock response spectrum curve, the residual spectrum goes to zero for all odd multiples of π/τ above $3\pi/\tau$.

As the natural frequency increases above $3\pi/\tau$, the response attains the character of relatively low amplitude oscillations occurring with the half-sine pulse shape as a mean. An example of this type of response is shown by the solid curve for $\omega_n = 8\pi/\tau$. The largest positive response is slightly higher than $\ddot{u}_0/\omega_n^2$, and the residual response occurs at a relatively low level. The solid and open square symbols indicate the largest positive and negative response.

As the natural frequency becomes extremely high, the response follows the half-sine shape very closely. In the limit, the natural frequency becomes infinite and the response approaches the half-sine wave shown in Fig. 23.7*C*. For natural frequencies greater than $5\pi/\tau$, the response tends to follow the input and the largest response is within 20 percent of the response due to a static application of the peak input acceleration. This portion of the shock response spectrum is sometimes referred to as the "static region" (see *Limiting Values of Shock Response Spectrum* below).

The equivalent static acceleration without damping for the positive direction is

$$
\begin{aligned}
A^{+}_{\text{eq}}(\omega_n,0) &= \frac{\ddot{u}_0}{g}\left(\frac{2(\omega_n\tau/\pi)}{1-(\omega_n\tau/\pi)^2}\right)\cos\left(\frac{\omega_n\tau}{2}\right) && \left[\omega_n \leq \frac{\pi}{\tau}\right] \\
A^{+}_{\text{eq}}(\omega_n,0) &= \frac{\ddot{u}_0}{g}\left(\frac{(\omega_n\tau/\pi)}{(\omega_n\tau/\pi)-1}\right)\sin\left(\frac{2i\pi}{(\omega_n\tau/\pi)+1}\right) && \left[\omega_n > \frac{\pi}{\tau}\right]
\end{aligned}
\tag{23.46}
$$

where i is the positive integer which maximizes the value of the sine term while the argument remains less than π. In the negative direction the peak response always occurs during the residual response; thus, it is given by the absolute value of the first of the expressions in Eq. (23.46):

$$
A^{-}_{\text{eq}}(\omega_n,0) = \frac{\ddot{u}_0}{g}\left(\frac{2(\omega_n\tau/\pi)}{1-(\omega_n\tau/\pi)^2}\right)\cos\left(\frac{\omega_n\tau}{2}\right) \tag{23.47}
$$

Shock response spectra for damped systems are commonly found by use of a digital computer. Spectra for $\zeta = 0.1$ and 0.5 are shown in Fig. 23.7*C*.

The response of a damped structure whose natural frequency is less than $\pi/2\tau$ is essentially impulsive; i.e., the shock response spectra in this frequency region are substantially identical to the spectra for the acceleration impulse in Fig. 23.7*A*.

Except near the zeros in the negative spectrum for an undamped system, damping reduces the peak response. For the positive spectra, the effect is small in the static region since the response tends to follow the input for all values of damping. The greatest effect of damping is seen in the negative spectra because it affects the decay of response oscillations at the natural frequency of the structure.

DECAYING SINUSOIDAL ACCELERATION: Although analytical expressions for the response of a simple structure to the decaying sinusoidal acceleration shown in Fig. 23.2*D* are available, calculation of spectra is impractical without use of a computer. Figure 23.7*D* shows spectra for several values of damping in the decaying sinusoidal acceleration. In the low-frequency region ($\omega_n < 0.2\omega_1$), the response is essentially impulsive. The area under the acceleration time-history of the decaying sinusoid is $\dot{u}_0$; hence, the response of a very low-frequency structure is similar to the response to an acceleration impulse of magnitude $\dot{u}_0$.

When the natural frequency of the responding system approximates the frequency ω_1 of the oscillations in the decaying sinusoid, a resonant type of build-up tends to occur in the response oscillations. The region in the neighborhood of $\omega_1 = \omega_n$ may be termed a quasi-resonant region of the shock response spectrum. Responses for $\zeta = 0, 0.1$, and 0.5 and $\omega_n = \omega_1$ are shown in Fig. 23.7*D*. In the absence of damping in the responding system, the rate of build-up diminishes with time and the amplitude of the response oscillations levels off as the input acceleration decays to very small values. Small damping in the responding system, e.g., $\zeta = 0.1$, reduces the initial rate of build-up and causes the response to decay to zero after a maximum is reached. When damping is as large as $\zeta = 0.5$, no build-up occurs.

COMPLEX SHOCK: The shock spectra for the complex shock of Fig. 23.2*E* are shown in Fig. 23.7*E*. Time-histories of the response of a system with a natural frequency of 1,250 Hz also are shown. The ordinate of the spectrum plot is equivalent static acceleration, and the abscissa is the natural frequency in hertz. Three pronounced peaks appear in the spectra for zero damping, at approximately 1,250 Hz, 1,900 Hz, and 2,350 Hz. Such peaks indicate a concentration of frequency content in the shock, similar to the spectra for the decaying sinusoid in Fig. 23.7*D*. Other peaks in the shock spectra for an undamped system indicate less significant oscillatory behavior in the shock. The two lower frequencies at which the pronounced peaks occur correlate with the peaks in the Fourier spectrum of the same shock, as shown in Fig. 23.3*E*. The highest frequency at which a pronounced peak occurs is above the range for which the Fourier spectrum was calculated.

Because of response limitations of the analysis, the shock spectra do not extend below 200 Hz. Since the duration of the complex shock of Fig. 23.2*E* is about 0.016 sec, an impulsive-type response occurs only for natural frequencies well below 200 Hz. As a result, no impulsive region appears in the shock response spectra. There is no static region of the spectra shown because calculations were not extended to a sufficiently high frequency.

In general, the equivalent static acceleration A_{eq} is reduced by additional damping in the responding structure system except in the region of valleys in the shock spectra, where damping may increase the magnitude of the spectrum. Positive and negative spectra tend to be approximately equal in magnitude at any value of damping; thus, the spectra for a complex oscillatory type of shock may be based on peak response independent of sign to a good approximation.

Limiting Values of Shock Response Spectrum. The response data provided by the shock response spectrum sometimes can be abstracted to simplified parameters that are useful for certain purposes. In general, this cannot be done without definite information on the ultimate use of the reduced data, particularly the natural frequencies of the structures upon which the shock acts. Two important cases are discussed in the following sections.

IMPULSE OR VELOCITY CHANGE: The duration of a shock sometimes is much smaller than the natural period of a structure upon which it acts. Then the entire response of the structure is essentially a function of the area under the time-history of the shock, described in terms of acceleration or a loading parameter such as force, pressure, or torque. Consequently, the shock has an effect which is equivalent to that produced by an impulse of infinitesimally short duration, i.e., an ideal impulse.

The shock response spectrum of an ideal impulse is shown in Fig. 23.7*A*. All equivalent static acceleration curves are straight lines passing through the origin. The portion of the spectrum exhibiting such straight-line characteristics is termed the *impulsive region*. The shock response spectrum of the half-sine acceleration pulse has an impulsive region when ω_n is less than approximately $\pi/2\tau$, as shown in Fig. 23.7*C*. If the area under a time-history of acceleration or shock loading is not zero or infinite, an impulsive region exists in the shock response spectrum. The extent of the region on the natural frequency axis depends on the shape and duration of the shock.

The portions adjacent to the origin of the positive shock response spectra of an undamped system for several single pulses of acceleration are shown in Fig. 23.8. To illustrate the impulsive nature, each spectrum is normalized with respect to the peak impulsive response $\omega_n \, \Delta\dot{u}/g$, where $\Delta\dot{u}$ is the area under the corresponding acceleration time-history. Hence, the spectra indicate an impulsive response where the ordinate is approximately 1. The response to a single pulse of acceleration is impulsive within a tolerance of 10 percent if $\omega_n < 0.25\pi/\tau$; i.e., $f_n < 0.4\tau^{-1}$, where f_n is the natural frequency of the responding structure in hertz and τ is the pulse duration in seconds. This result also applies when the responding system is damped. Thus, it is possible to reduce the description of a shock pulse to a designated velocity change when the natural frequency of the responding structure is less than a specified value. The magnitude of the velocity change is the area under the acceleration pulse:

$$\Delta\dot{u} = \int_0^\tau \ddot{u}(t)\, dt \qquad (23.48)$$

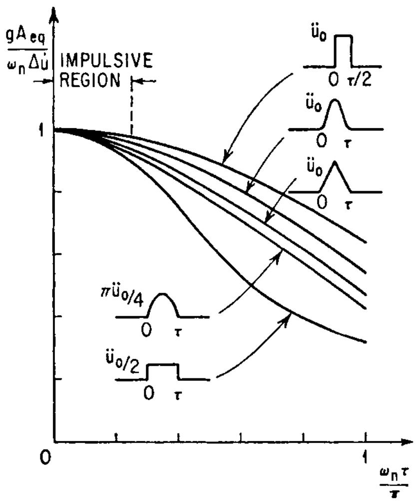


FIGURE 23.8 Portions adjacent to the origin of the positive spectra of an undamped system for several single pulses of acceleration.

PEAK ACCELERATION OR LOADING: The natural frequency of a structure responding to a shock sometimes is sufficiently high that the response oscillations of the structure at its natural frequency have a relatively small amplitude. Examples of such responses are shown in Fig. 23.7*C* for $\omega_n = 8\pi/\tau$ and $\zeta = 0, 0.1, 0.5$. As a result, the maximum response of the structure is approximately equal to the maximum acceleration of the shock and is termed *equivalent static response*. The magnitude of the spectra in such a static region is determined principally by the peak value of the shock acceleration or loading. Portions of the positive spectra of an undamped system in the region of high natural frequencies are shown in Fig. 23.9 for a number of acceleration pulses. Each spectrum is normalized with respect to the maximum acceleration of the pulse. If the ordinate is approximately 1, the shock response spectrum curves behave approximately in a static manner.

The limit of the static region in terms of the natural frequency of the structure is more a function of the slope of the acceleration time-history than of the duration of the pulse. Hence, the horizontal axis of the shock response spectra in Fig. 23.9 is given in terms of the ratio of the rise time τ_r to the maximum value of the pulse. As shown in Fig. 23.9, the peak response to a single pulse of acceleration is approximately equal to the maximum acceleration of the pulse, within a tolerance of 20 percent, if $\omega_n > 2.5\pi/\tau_r$; i.e., $f_n > 1.25\tau_r^{-1}$, where f_n is the natural frequency of the responding structure in hertz and τ_r is the rise time to the peak value in seconds. The tolerance of 20 percent applies to an undamped system; for a damped system, the tolerance is lower, as indicated in Fig. 23.7*C*.

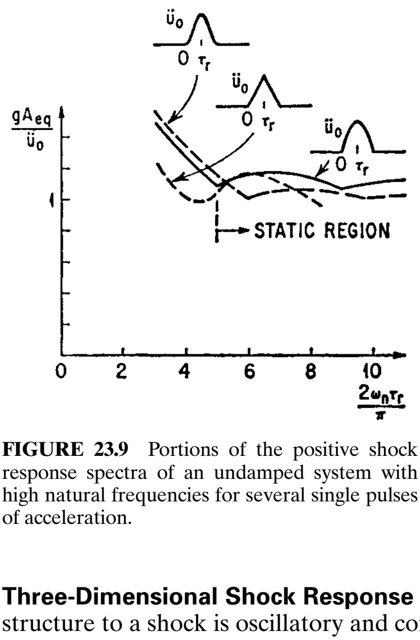


FIGURE 23.9 Portions of the positive shock response spectra of an undamped system with high natural frequencies for several single pulses of acceleration.

The concept of the static region also can be applied to complex shocks. Suppose the shock is oscillatory, as shown in Fig. 23.2*E*. If the response to such a shock is to be nearly static, the response to each of the succession of pulses that make up the shock must be nearly static. This is most significant for pulses of large magnitude because they determine the ordinate of the spectrum in the static region. Therefore, the shock response spectrum for a complex shock in the static region is based upon the pulses of greatest magnitude and shortest rise time.

Three-Dimensional Shock Response Spectrum.[7] In general, the response of a structure to a shock is oscillatory and continues for an appreciable number of oscillations. At each oscillation, the response has an interim maximum value that differs, in general, from the preceding or following maximum value. For example, a typical time-history of response of a simple system of given natural frequency is shown in Fig. 23.6; the characteristics of the response may be summarized by the block diagram of Fig. 23.10. The abscissa of Fig. 23.10 is the peak response at the respective cycles of the oscillation, and the ordinate is the number of cycles at which the peak response exceeds the indicated value. Thus, the time-history of Fig. 23.6 has 29 cycles of oscillation at which the peak response of the oscillation exceeds $0.6r_0$, but only 2 cycles at which the peak response exceeds $2.0r_0$.

In accordance with the concept of the shock response spectrum, the natural frequency of the responding system is modified by discrete increments and the response determined at each increment. This leads to a number of time-histories of response corresponding to Fig. 23.6, one for each natural frequency, and a similar number of block diagrams corresponding to Fig. 23.10. This group of block diagrams can be assembled to form a surface that shows pictorially the characteristics of the shock in terms of the response of a simple system. The axes of the surface are peak response,

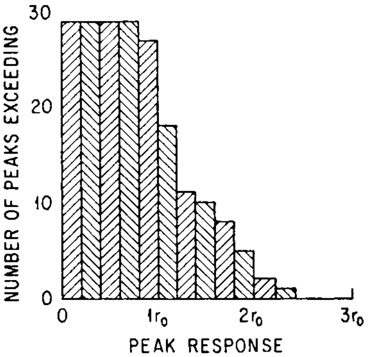


FIGURE 23.10 Bar chart for the response of a system to a shock excitation.

natural frequency of the responding system, and number of response cycles exceeding a given peak value. The block diagram of Fig. 23.10 is arranged on this set of axes at *A,* as shown in Fig. 23.11, at the appropriate position along the natural frequency axis. Other corresponding block diagrams are shown at *B.* The three-dimensional shock response spectrum is conceptually the surface faired through the ends of the bars; the intercept of this surface with the planes of the block diagrams is indicated at *C* and that with the maximum response–natural frequency plane at *D.* Surfaces are obtainable for both positive and negative values of the response, and a separate surface is obtained for each fraction of critical damping in the responding system.

The two-dimensional shock response spectrum is a special case of the three-dimensional surface. The former is a plot of the maximum response as a function of the natural frequency of the responding system; hence, it is a projection on the plane of the response and natural frequency axes of the maximum height of the surface. However, the height of the surface never exceeds that at one response cycle. Thus, the two-dimensional shock response spectrum is the intercept of the surface with a plane normal to the "number of peaks exceeding" axis at the origin.

The response surface is a useful concept and illustrates a physical condition. However, it is not well adapted to quantitative analysis because the distances from

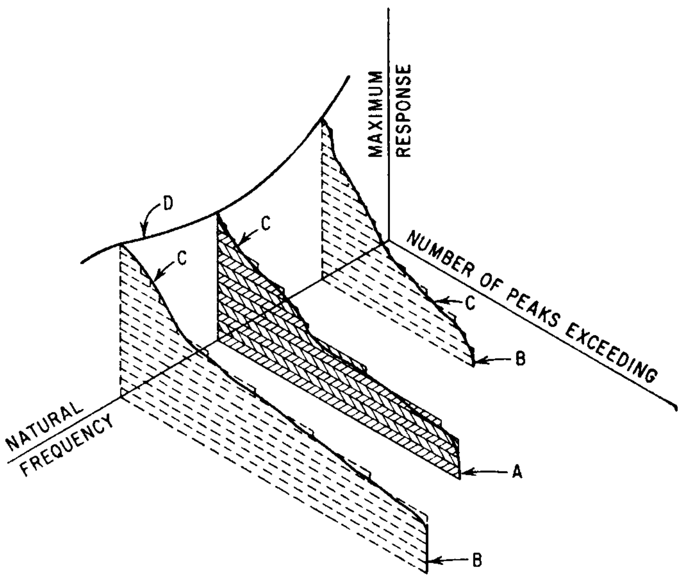


FIGURE 23.11 Example of a three-dimensional shock response spectrum.

the surface to the coordinate planes cannot be determined readily. A group of bar charts, each corresponding to Fig. 23.10, is more useful for quantitative purposes. The differences in lengths of the bars are discrete increments; this corresponds to the data reduction method in which the axis of response magnitudes is divided into discrete increments for purposes of counting the number of peaks exceeding each magnitude. In concept, the width of the increment may be considered to approach zero and the line faired through the ends of the bars represents the smooth intercept with the surface.

Relationship between Shock Response Spectrum and Fourier Spectrum. Although the shock response spectrum and the Fourier spectrum are fundamentally different, there is a partial correlation between them. A direct relationship exists between a running Fourier spectrum, to be defined subsequently, and the response of an undamped simple structure. A consequence is a simple relationship between the Fourier spectrum of absolute values and the peak residual response of an undamped simple structure.

For the case of zero damping, Eq. (23.33) provides the relative displacement response

$$\delta(\omega_n,t) = \frac{1}{\omega_n} \int_0^t \ddot{u}(t_v) \sin \omega_n(t - t_v)\, dt_v \tag{23.49}$$

A form better suited to our needs here is

$$\delta(\omega_n,t) = \frac{1}{\omega_n}\, \mathcal{I}\left[e^{j\omega_n t} \int_0^t \ddot{u}(t_v)\, e^{-j\omega_n t_v}\, dt_v\right] \tag{23.50}$$

The integral above is seen to be the Fourier spectrum of the portion of $\ddot{u}(t)$ which lies in the time interval from zero to t, evaluated at the natural frequency ω_n. Such a time-dependent spectrum can be termed a "running Fourier spectrum" and denoted by $\mathbf{F}(\omega,t)$:

$$\mathbf{F}(\omega,t) = \int_0^t \ddot{u}(t_v) e^{-j\omega t_v}\, dt_v \tag{23.51}$$

It is assumed that the excitation vanishes for $t < 0$. The integral in Eq. (23.50) can be replaced by $\mathbf{F}(\omega_n,t)$; and after taking the imaginary part

$$\delta(\omega_n,t) = \frac{1}{\omega_n}\, F(\omega_n,t) \sin\,[\omega_n t + \theta(\omega_n,t)] \tag{23.52}$$

where $F(\omega_n,t)$ and $\theta(\omega_n,t)$ are the magnitude and phase of the running Fourier spectrum, corresponding to the definitions in Eqs. (23.63) and (23.64). Equation (23.52) provides the previously mentioned direct relationship between undamped structural response and the running Fourier spectrum.

When the running time t exceeds τ, the duration of $\ddot{u}(t)$, the running Fourier spectrum becomes the usual spectrum as given by Eq. (23.57), with τ used in place of the infinite upper limit of the integral. Consequently, Eq. (23.52) yields the sinusoidal residual relative displacement for $t > \tau$:

$$\delta_r(\omega_n,t) = \frac{1}{\omega_n}\, F(\omega_n) \sin\,[\omega_n t + \theta(\omega_n)] \tag{23.53}$$

The amplitude of this residual deflection and the corresponding equivalent static acceleration are

$$
\begin{aligned}
(\delta_r)_{\max} &= \frac{1}{\omega_n} F(\omega_n) \\
(A_{\mathrm{eq}})_r &= \frac{\omega_n^2 (\delta_r)_{\max}}{g} = \frac{\omega_n}{g} F(\omega_n)
\end{aligned}
\tag{23.54}
$$

This result is clearly evident for the Fourier spectrum and undamped shock response spectrum of the acceleration impulse. The Fourier spectrum is the horizontal line (independent of frequency) shown in Fig. 23.3*A* and the shock response spectrum is the inclined straight line (increasing linearly with frequency) shown in Fig. 23.7*A*. Since the impulse exists only at $t = 0$, the entire response is residual. The undamped shock spectra in the impulsive region of the half-sine pulse and the decaying sinusoidal acceleration, Fig. 23.7*C* and *D*, respectively, also are related to the Fourier spectra of these shocks, Fig. 23.3*C* and *D*, in a similar manner. This results from the fact that the maximum response occurs in the residual motion for systems with small natural frequencies. Another example is the entire negative shock response spectrum with no damping for the half-sine pulse in Fig. 23.7*C*, whose values are ω_n/g times the values of the Fourier spectrum in Fig. 23.3*C*.

METHODS OF DATA REDUCTION

Even though preceding sections of this chapter include several analytic functions as examples of typical shocks, data reduction in general is applied to measurements of shock that are not definable by analytic functions. The following sections outline data reduction methods that are adapted for use with any general type of function, obtained in digital form in practice. Standard forms for presenting the analysis results are given in Ref. 8.

FOURIER SPECTRUM

The Fourier spectrum is computed using the discrete Fourier transform (DFT) defined in Eq. (14.6). The DFT is commonly computed using a fast Fourier transform (FFT) algorithm, as discussed in Chap. 14 (see Ref. 9 for details on FFT computations). Fourier spectra can be computed as a function of either radial frequency ω in radians/sec or cyclical frequency f in Hz, that is,

$$
F_1(f) = \int_{-\infty}^{\infty} x(t) e^{-j2\pi f t} dt \qquad \text{or} \qquad F_2(\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt \tag{23.55}
$$

where the two functions are related by $F_2(\omega) = 2\pi F_1(f)$.

SHOCK RESPONSE SPECTRUM

The shock response spectrum can be computed by the following techniques: (*a*) direct numerical or recursive integration of the Duhamel integral in Eq. (23.33), or (*b*) convolution or recursive filtering procedures. One of the most widely used programs for computing the shock response spectrum is the "ramp invariant method" detailed in Ref. 10. Any of these computational procedures can be modified to count the number of response maxima above various discrete increments of maximum response to obtain the results depicted in Fig. 23.11.

Reed Gage. The shock spectrum may be measured directly by a mechanical instrument that responds to shock in a manner analogous to the data reduction techniques used to obtain shock spectra from time-histories. The instrument includes a number of flexible mechanical systems that are considered to respond as single degree-of-freedom systems; each system has a different natural frequency, and means are provided to indicate the maximum deflection of each system as a result of the shock. The instrument often is referred to as a *reed gage* because the flexible mechanical systems are small cantilever beams carrying end masses; these have the appearance of reeds.[11]

The response parameter indicated by the reed gage is maximum deflection of the reeds relative to the base of the instrument; generally, this deflection is converted to equivalent static acceleration by applying the relation of Eq. (23.30). The reed gage offers a convenience in the indication of a useful quantity immediately and in the elimination of auxiliary electronic equipment. Also, it has important limitations: (1) the information is limited to the determination of a shock response spectrum; (2) the deflection of a reed is inversely proportional to its natural frequency squared, thereby requiring high equivalent static accelerations to achieve readable records at high natural frequencies; (3) the means to indicate maximum deflection of the reeds (styli inscribing on a target surface) tend to introduce an undefined degree of damping; and (4) size and weight limitations on the reed gage for a particular application often limit the number of reeds which can be used and the lowest natural frequency for a reed. In spite of these limitations, the instrument sees continued use and has provided significant shock response spectra where more elaborate instruments have failed.

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