Instructor’s Solution Manual for ADVANCED CALCULUS

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Contents

NOTE: Users of *Advanced Calculus* should be aware of the web site

`www.math.washington.edu/~folland/Homepage/index.html`

where a list of corrections to the book can be found. In particular, some errors in the exercises and in the answers in the back of the book were discovered in the course of preparing this solution manual. The solutions given here pertain to the *corrected* exercises.

Chapter 1

Setting the Stage

1.1 Euclidean Spaces and Vectors

1. $|\mathbf{x}| = \sqrt{3^2 + (-1)^2 + (-1)^2 + 1^2} = 2\sqrt{3}$, $|\mathbf{y}| = \sqrt{(-2)^2 + 2^2 + 1^2} = 3$, $\mathbf{x}\cdot\mathbf{y} = 3(-2) + (-1)2 + (-1)1 + 1\cdot 0 = -9$, $\theta = \arccos(-9/3\cdot 2\sqrt{3}) = \arccos(-\sqrt{3}\,/2) = 5\pi/6$.

2. $|\mathbf{x}\pm\mathbf{y}|^2 = (\mathbf{x}\pm\mathbf{y})\cdot(\mathbf{x}\pm\mathbf{y}) = |\mathbf{x}|^2 \pm 2\mathbf{x}\cdot\mathbf{y} + |\mathbf{y}|^2$. Taking the plus sign gives (a); adding these identities with the plus and minus signs gives (b).

3. $|\mathbf{x}_1 + \cdots + \mathbf{x}_k|^2 = \sum_{j=1}^k |\mathbf{x}_j|^2 + 2\sum_{1\le i<j\le k} \mathbf{x}_i\cdot\mathbf{x}_j$. The Pythagorean theorem follows immediately.

4. With $f(t) = |\mathbf{a} - t\mathbf{b}|^2$ as in the proof, equality holds precisely when the minimum value of $f(t)$ is 0, that is, when $\mathbf{a} = t\mathbf{b}$ for some $t\in\mathbb{R}$. Thus equality holds in Cauchy's inequality precisely when $\mathbf{a}$ and $\mathbf{b}$ are linearly dependent.

5. The triangle inequality is an equality precisely when $\mathbf{a}\cdot\mathbf{b} = |\mathbf{a}|\,|\mathbf{b}|$, that is, when the angle from $\mathbf{a}$ to $\mathbf{b}$ is 0, or when $\mathbf{a}$ is a positive scalar multiple of $\mathbf{b}$ or vice versa.

6. $|\mathbf{a}| = |(\mathbf{a}-\mathbf{b}) + \mathbf{b}| \le |\mathbf{a}-\mathbf{b}| + |\mathbf{b}|$, so $|\mathbf{a}| - |\mathbf{b}| \le |\mathbf{a}-\mathbf{b}|$. Likewise, $|\mathbf{b}| - |\mathbf{a}| \le |\mathbf{a}-\mathbf{b}|$.

7. (a) If $\mathbf{a}\cdot\mathbf{b} = 0$ then $\mathbf{a}\perp\mathbf{b}$, so $|\mathbf{a}\times\mathbf{b}| = |\mathbf{a}|\,|\mathbf{b}|$; hence if also $\mathbf{a}\times\mathbf{b} = \mathbf{0}$ then $\mathbf{a} = \mathbf{0}$ or $\mathbf{b} = \mathbf{0}$.

(b) If $\mathbf{a}\cdot\mathbf{c} = \mathbf{b}\cdot\mathbf{c}$ and $\mathbf{a}\times\mathbf{c} = \mathbf{b}\times\mathbf{c}$ then $(\mathbf{a}-\mathbf{b})\cdot\mathbf{c} = 0$ and $(\mathbf{a}-\mathbf{b})\times\mathbf{c} = \mathbf{0}$, so by (a), either $\mathbf{a}-\mathbf{b} = \mathbf{0}$ or $\mathbf{c} = \mathbf{0}$; the latter possibility is excluded.

(c) We always have $\mathbf{a}\times\mathbf{a} = \mathbf{0}$. If $\mathbf{a}$ and $\mathbf{b}$ are proportional, then $\mathbf{a}\times\mathbf{b} = \mathbf{0}$ too. If not, then $\mathbf{a}\times\mathbf{b}$ is a nonzero vector perpendicular to $\mathbf{a}$, so $\mathbf{a}\times(\mathbf{a}\times\mathbf{b}) \ne \mathbf{0}$.

8. This follows from the definitions by a simple calculation.

1.2 Subsets of Euclidean Space

1. (a)–(d): See the answers in the back of the text.
(e) $S^{\text{int}} = \varnothing$ and $\partial S = \overline{S} = S \cup \{(y,0) : -1 \le y \le 1\}$.
(f) $S^{\text{int}} = S \setminus \{(0,0)\}$, $\overline{S} = \{(x,y) : x^2 + y^2 \le 1\}$, and ∂S is the union of the unit circle and the line segment $[-1,0]\times\{0\}$.
(g) $S^{\text{int}} = \varnothing$ and $\partial S = \overline{S} = [0,1]\times[0,1]$.

2. If $\mathbf{x} \in S^{\text{int}}$, there is a ball $B = B(r, \mathbf{x})$ contained in S. B is open, so every point of B is an interior point of B and hence of S, so in fact $B \subset S^{\text{int}}$ and $\mathbf{x}$ is an interior point of S^{int}. Thus S^{int} is open by Proposition 1.4a. Next, $\overline{S}$ and ∂S are the complements of $(S^c)^{\text{int}}$ and $S^{\text{int}} \cup (S^c)^{\text{int}}$, respectively, so they are closed by Proposition 1.4b.

3. We use Proposition 1.4a. If $\mathbf{x} \in S_1 \cup S_2$, some ball centered at $\mathbf{x}$ is contained in either S_1 or S_2 and hence in $S_1 \cup S_2$, so $\mathbf{x}$ is an interior point of $S_1 \cup S_2$. If $\mathbf{x} \in S_1 \cap S_2$, there are balls B_1 and B_2 centered at $\mathbf{x}$ and contained in S_1 and S_2, respectively; the smaller of these balls is contained in $S_1 \cap S_2$, so again $\mathbf{x}$ is an interior point of $S_1 \cap S_2$.

4. The complements of $S_1 \cup S_2$ and $S_1 \cap S_2$ are $S_1^c \cap S_2^c$ and $S_1^c \cup S_2^c$, respectively, which are both open by Exercise 3 and Proposition 1.4b.

5. This follows from the remarks preceding Proposition 1.4: $\mathbb{R}^n$ is the disjoint union of S^{int}, ∂S, and $(S^c)^{\text{int}}$, whereas $\overline{S} = S^{\text{int}} \cup \partial S$ and $\overline{S^c} = (S^c)^{\text{int}} \cup \partial S$.

6. One example (in $\mathbb{R}^1$) is $S_j = [0,\ 1 - j^{-1}]$, for which $\bigcup_1^\infty S_j = [0, 1)$.

7. $\mathbb{R}^n$ and $\varnothing$.

8. The sets in Exercise 1a and 1f are both examples.

9. If $|\mathbf{x} - \mathbf{a}| < r$ then $|\mathbf{x}| = |(\mathbf{x} - \mathbf{a}) + \mathbf{a}| \le r + |\mathbf{a}|$. Thus, if $S \subset B(r, \mathbf{a})$ then $S \subset B(r + |\mathbf{a}|,\ \mathbf{0})$.

1.3 Limits and Continuity

1. (a) $f(0, y) = 1$ for $y > 0$ and $f(0, y) = -1$ for $y < 0$.
(b) $f(x, 0) = x^{-3} \to \infty$ as $x \to 0$.
(c) $f(t, t) = 1/8t^4 \to \infty$ as $t \to 0$.

2. (a) Since $|xy| \le \frac{1}{2}(x^2 + y^2)$, we have $|f(x, y)| \le \frac{1}{4}(x^2 + y^2) \to 0$ as $x, y \to 0$.
(b) Since $|3x^4 - y^4| \le 3(x^4 + y^4)$, we have $|f(x, y)| \le 3|x| \to 0$ as $x, y \to 0$.

3. $f(x, y) \to y$ as $x \to 0$, so take $f(0, y) = y$.

4. $f(x, a)$ and $f(a, y)$ are continuous for $a \ne 0$ since f is continuous except at $(0, 0)$. Moreover, $f(x, 0) = f(0, y) = 0$ for all x, y, also continuous.

5. The two formulas for f agree along the curves $y = 0$ and $y = x^2$, $x \ne 0$, so f is continuous except at the origin. It is discontinuous there since $f(0, 0) = 0$ but $f(x, \frac{1}{2}x^2) = \frac{1}{2} \not\to 0$ as $x \to 0$.

6. Since $|f(x)| \le |x|$ for all x, we have $f(x) \to 0 = f(0)$ as $x \to 0$. Suppose $a \ne 0$. If a is irrational, then $f(a) = a \ne 0$, but there are points x arbitrarily close to a with $f(x) = 0$. If a is rational, then $f(a) = 0$, but there are points x arbitrarily close to a with $|f(x)| > \frac{1}{2}|a|$. In both cases f is discontinuous at a.

7. Clearly $|f(x)| \le |x|$ for all x, so f is continuous at 0. If $a \ne 0$ is rational, then $f(a) \ne 0$, but there are points x arbitrarily close to a with $f(x) = 0$; hence f is discontinuous at a. If a is irrational and δ is the distance from a to the nearest rational number with denominator $\le k$, then $|f(x)| < 1/k$ for $|x - a| < \delta$; hence f is continuous at a. (There are only finitely many rational numbers with denominator $\le k$ in any bounded interval.)

8. Given $\mathbf{a} \in \mathbb{R}^n$ and $\epsilon > 0$, let $U = B(\epsilon, \mathbf{f}(\mathbf{a}))$. Then U is open, and hence so is $V = \{\mathbf{x} : \mathbf{f}(\mathbf{x}) \in U\}$. We have $\mathbf{a} \in V$, so there exists $\delta > 0$ such that $B(\delta, \mathbf{a}) \subset V$. But this says that $|\mathbf{f}(\mathbf{x}) - \mathbf{f}(\mathbf{a})| < \epsilon$ whenever $|\mathbf{x} - \mathbf{a}| < \delta$, so $\mathbf{f}$ is continuous at $\mathbf{a}$. One can replace "open" by "closed" in the hypothesis by the reasoning of the second paragraph of the proof of Theorem 1.13.

9. The fact that since $\mathbf{f}$ is a one-to-one correspondence between the points of U and the points of V has the following consequences that we shall use: (i) If $A \subset U$, $\mathbf{f}(U \setminus A) = V \setminus \mathbf{f}(A)$. (ii) If $B \subset V$, $\{\mathbf{x} : \mathbf{f}(\mathbf{x}) \in B\} = \mathbf{f}^{-1}(B)$.

Suppose $\mathbf{b} \in \mathbf{f}(\partial S)$, and let $\epsilon > 0$ be small enough so that $B(\epsilon, \mathbf{b}) \subset V$. Since $\mathbf{f}$ is continuous, $\mathbf{f}^{-1}(B(\epsilon, \mathbf{b}))$ is a neighborhood of $\mathbf{f}^{-1}(\mathbf{b})$ by Theorem 1.13 and the remarks following it. Hence it contains points in S and points not in S, and therefore $B(\epsilon, \mathbf{b})$ contains points in $\mathbf{f}(S)$ and points not in $\mathbf{f}(S)$. It follows that $\mathbf{b} \in \partial(\mathbf{f}(S))$.

Conversely, suppose $\mathbf{b} \in \partial(\mathbf{f}(S))$, and let $\mathbf{a} = \mathbf{f}^{-1}(\mathbf{b})$; let $\epsilon > 0$ be small enough so that $B(\epsilon, \mathbf{a}) \subset U$. Since $\mathbf{f}^{-1}$ is continuous, $\mathbf{f}(B(\epsilon, \mathbf{a})) = (\mathbf{f}^{-1})^{-1}(B(\epsilon, \mathbf{a}))$ is a neighborhood of $\mathbf{b}$ by Theorem 1.13 again. Hence it contains points in $\mathbf{f}(S)$ and points not in $\mathbf{f}(S)$, and so $B(\epsilon, \mathbf{a})$ contains points in S and points not in S. It follows that $\mathbf{a} \in \partial S$ and hence $\mathbf{b} \in \mathbf{f}(\partial S)$.

1.4 Sequences

1. (a) Divide top and bottom by $\sqrt{k}$ to get $x_k = \dfrac{\sqrt{2 + k^{-1}}}{2 + k^{-1/2}} \to \dfrac{\sqrt{2}}{2}$.

(b) $|\sin k/k| \le 1/k \to 0$.

(c) Diverges since x_k is 0, $\frac{1}{2}\sqrt{3}$, and $-\frac{1}{2}\sqrt{3}$ for infinitely many k each.

2. $|x_k - 3| = 19/|k - 5| < \epsilon$ whenever $k > 5 + 19\epsilon^{-1}$.

3. $x_k = 1 \cdot \dfrac{1}{2} \cdot \dfrac{2}{3} \cdots \dfrac{k-1}{k} = \dfrac{1}{k} \to 0$.

4. If $x_k \to a$ and $y_k \to b$, then $(x_k, y_k) \to (a, b)$. By continuity of addition and multiplication (Theorem 1.10) and the sequential characterization of continuity (Theorem 1.15), the result follows.

5. If $\mathbf{f}(\mathbf{x}) \to \mathbf{l}$ as $\mathbf{x} \to \mathbf{a}$, for any $\epsilon > 0$ there exists $\delta > 0$ such that $|\mathbf{f}(\mathbf{x}) - \mathbf{l}| < \epsilon$ whenever $0 < |\mathbf{x} - \mathbf{a}| < \delta$. If $\mathbf{x}_k \to \mathbf{a}$, there exists K such that $|\mathbf{x}_k - \mathbf{a}| < \delta$ whenever $k > K$, and hence $|\mathbf{f}(\mathbf{x}_k) - \mathbf{l}| < \epsilon$. On the other hand, if $\mathbf{f}(\mathbf{x}) \not\to \mathbf{l}$ as $\mathbf{x} \to \mathbf{a}$, there exists $\epsilon > 0$ such that for every $\delta > 0$ there is an $\mathbf{x}$ with $0 < |\mathbf{x} - \mathbf{a}| < \delta$ but $|\mathbf{f}(\mathbf{x}) - \mathbf{l}| > \epsilon$. Let $\mathbf{x}_k$ be such a point for $\delta = 1/k$. Then $\mathbf{x}_k \to \mathbf{a}$ but $\mathbf{f}(\mathbf{x}_k) \not\to \mathbf{l}$.

6. If $\mathbf{x}_k \in S$, $\mathbf{x}_k \neq \mathbf{a}$, and $\mathbf{x}_k \to \mathbf{a}$, then the sequence $\{\mathbf{x}_k\}$ must assume infinitely many distinct values, and for $\epsilon > 0$, all but finitely many of them are in $B(\epsilon, \mathbf{a})$; thus $\mathbf{a}$ is an accumulation point of S. Conversely, if $\mathbf{a}$ is an accumulation point of S, for each positive integer k the ball $B(\mathbf{a}, 1/k)$ contains points of S other than $\mathbf{a}$; let $\mathbf{x}_k$ be one.

7. If $\mathbf{a}$ is an accumulation point of S, then $\mathbf{a} \in \overline{S}$ by Theorem 1.14 and Exercise 6. If $\mathbf{a} \notin S$ and $\mathbf{a}$ is not an accumulation point of S, there is a neighborhood of $\mathbf{a}$ that contains only finitely many points of S. If ϵ is less than the minimum distance from $\mathbf{a}$ to any of these points (which do not coincide with $\mathbf{a}$ since $\mathbf{a} \notin S$), $B(\epsilon, \mathbf{a})$ is a neighborhood of $\mathbf{a}$ that is disjoint from S, and hence $\mathbf{a} \notin \overline{S}$.

1.5 Completeness

1. (a) $S = (-1,\ -1/\sqrt{2}) \cup (1/\sqrt{2},\ 1)$, so the inf and sup are -1 and 1.

(b) The supremum is the 0th element of the sequence; the infimum is the limit of the odd-numbered subsequence.

(c) $S = [\pi/4, \infty)$, so the inf and sup are $\pi/4$ and ∞.

2. One example is $x_k = \sin(k\pi/3)$ (Exercise 1c in §1.4).

3. If $a = 0.a_1a_2a_3\ldots \in (0,1)$, let $x_k = 0.a_1a_2\ldots a_k$, considered as a fraction with denominator 10^k. Then $\{x_k\}$ is a subsequence of the given sequence that converges to a. For $a = 0$, take $x_k = 1/k$; for $a = 1$, take $x_k = (k-1)/k$.

4. (a) If $\lim x_k = l$, then $l = l^2$ and hence $l = 0$ or 1.

(b) The limit is zero if $|a| < 1$, 1 if $a = \pm 1$, and nonexistent (or infinite) if $|a| > 1$.

5. We have $x_1 = \sqrt{2} < 2$. If $x_k < 2$, then $x_{k+1} = \sqrt{2 + x_k} < \sqrt{2+2} = 2$. By induction, $x_k < 2$ for all k. This being the case, $x_{k+1} = \sqrt{2 + x_k} > \sqrt{x_k + x_k} = \sqrt{2x_k} > \sqrt{x_k \cdot x_k} = x_k$. Thus the sequence $\{x_k\}$ is increasing and bounded above by 2, so it converges to a limit l. We have $l = \sqrt{2 + l}$, hence $l^2 = l + 2$, and hence $l = 2$ or $l = -1$. The latter alternative is impossible since $x_k > 0$ for all k.

6. (a) Let x_k be the kth term of the Fibonacci sequence, so $r_k = x_{k+1}/x_k$. Since $x_{k+2} = x_{k+1} + x_k$, we obtain $r_{k+1} = 1 + (1/r_k) = (r_k + 1)/r_k$ by dividing through by x_{k+1}. Replacing k by $k+1$ we get $r_{k+2} = (r_{k+1} + 1)/r_{k+1}$, and substituting in $r_{k+1} = (r_k + 1)/r_k$ gives $r_{k+2} = (2r_k + 1)/(r_k + 1)$.

(b) The function $f(x) = (2x+1)/(x+1) = 2-(x+1)^{-1}$ is an increasing function of x, and $f(\varphi) = \varphi$. Hence, if $r_k < \varphi$ then $r_{k+2} = f(r_k) < f(\varphi) = \varphi$, and if $r_k > \varphi$ then $r_{k+2} = f(r_k) > f(\varphi) = \varphi$. Since $r_1 = 1 < \varphi$ and $r_2 = 2 > \varphi$, it follows that $r_k < \varphi$ for k odd and $r_k > \varphi$ for k even. Next, $r_{k+2} - r_k = f(r_k) - r_k = (1 + r_k - r_k^2)/(r_k + 1)$, which by the hint is positive for k odd and negative for k even.

(c) By (b), $\{r_{2j-1}\}$ is an increasing sequence and $\{r_{2j}\}$ is a decreasing sequence, bounded above and below, respectively, by φ. Their limits l_1 and l_2 both satisfy $f(l_j) = l_j$, and hence both are equal to φ.

7. If $\{\mathbf{x}_{k_j}\}$ converges to $\mathbf{a}$, and $\epsilon > 0$, then $B(\epsilon, \mathbf{a})$ contains $\mathbf{x}_{k_j}$ for all sufficiently large j. Conversely, if every ball about $\mathbf{a}$ contains infinitely many $\mathbf{x}_k$, we can pick $\mathbf{x}_{k_1} \in B(1, \mathbf{a})$, and then for $j = 2, 3, 4, \ldots$, we can pick $k_j > k_{j-1}$ so that $\mathbf{x}_{k_j} \in B(j^{-1}, \mathbf{a})$; then $\mathbf{x}_{k_j} \to \mathbf{a}$.

8. If S is bounded and infinite, let $\{\mathbf{x}_k\}$ be a sequence of distinct points of S. By Theorem 1.19, this set has a convergent subsequence, and by Exercise 6 in §1.4 its limit $\mathbf{a}$ is an accumulation point of S. (At most one $\mathbf{x}_k$ can be equal to $\mathbf{a}$; throw it out if necessary.)

9. If there are infinitely many k for which $x_k > a - \epsilon$, then $\sup\{x_k : k \geq m\} > a - \epsilon$ for all m and hence $\limsup x_k \geq a - \epsilon$. If there are only finitely many k for which $x_k > a + \epsilon$, then $\sup\{x_k : k \geq m\} \leq a + \epsilon$ for m sufficiently large, and hence $\limsup x_k \leq a + \epsilon$. Since ϵ is arbitrary, we have $a \leq \limsup x_k \leq a$ and hence $a = \limsup x_k$.

10. We define a subsequence $\{x_{k_j}\}$ recursively. We take $k_1 = 1$, and for $j > 1$, we choose $k_j > k_{j-1}$ so that $x_{k_j} > \sup\{x_k : k \geq k_{j-1} + 1\} - (1/j)$. Then, with Y_m as in the definition of $\limsup$, we have $Y_{k_{j-1}+1} - (1/j) < x_{k_j} < Y_{k_{j-1}+1}$. It follows that $\lim x_{k_j} = \lim Y_m = \limsup x_k$. Similarly for $\liminf$.

11. If $x_{k_j} \to a$, then for any $\epsilon > 0$ we have $a - \epsilon < x_k < a + \epsilon$ for infinitely many k. It follows that $\limsup x_k \geq a - \epsilon$ and $\liminf x_k \leq a + \epsilon$; since ϵ can be arbitrarily small, the same is true with $\epsilon = 0$.

12. With Y_m and y_m as in the definition of $\limsup$ and $\liminf$, the assertion that $|x_k - a| \leq \epsilon$ for $k \geq K$ is equivalent to the assertion that $Y_m \leq a + \epsilon$ and $y_m \geq a - \epsilon$ for $m \geq K$. If this holds, then $a - \epsilon \leq \liminf x_k \leq \limsup x_k \leq a + \epsilon$ for every ϵ, and hence $\liminf x_k = a = \limsup x_k$. Conversely, if the latter condition holds, then for any $\epsilon > 0$ there exists M such that $a - \epsilon \leq y_m \leq Y_m \leq a + \epsilon$ for $m \geq M$, and so $|x_k - a| \leq \epsilon$ for $k \geq M$; hence $\lim x_k = a$.

1.6 Compactness

1. (a) One example is $S = \mathbb{R}$, $f(x) = e^x$.

(b) One example is $S = \mathbb{R}$, $f(x) = x^2$.

2. (a) One example is $S = (0, 1)$, $f(x) = 1/x$.

(b) S bounded $\Longrightarrow \overline{S}$ compact $\Longrightarrow f(\overline{S})$ compact $\Longrightarrow f(S) \subset f(\overline{S})$ bounded.

3. If S is compact and V is an infinite subset of S, let $\mathbf{x}_1, \mathbf{x}_2, \ldots$ be a sequence of distinct points of V. This sequence has a convergent subsequence whose limit $\mathbf{l}$ lies in S, and $\mathbf{l}$ is an accumulation point of V (Exercise 6, §1.4). Conversely, suppose S is not compact. If S is not closed, there is a sequence $\{\mathbf{x}_k\}$ in S that converges to a point $\mathbf{l} \in S^c$, and if S is not bounded, there is a sequence $\{\mathbf{x}_k\}$ in S with $|\mathbf{x}_k| \to \infty$. In either case, the *set* $\{\mathbf{x}_1, \mathbf{x}_2, \ldots\}$ is an infinite subset of S with no accumulation point in S. (In the first case, the only accumulation point is $\mathbf{l}$; in the second case, there is no accumulation point at all.)

4. If not, there is a sequence $\{\mathbf{x}_k\}$ in S such that $f(\mathbf{x}_k) < 1/k$. Some subsequence $\{\mathbf{x}_{k_j}\}$ has a limit $\mathbf{l} \in S$; but then $f(\mathbf{l}) = \lim f(\mathbf{x}_{k_j}) = 0$, contrary to assumption.

5. By Bolzano-Weierstrass: For $k \geq 1$, pick $\mathbf{x}_k \in S_k$. Then $\mathbf{x}_k \in S_1$ for all k, so some subsequence $\{\mathbf{x}_{k_j}\}$ converges to a point $\mathbf{l} \in S_1$. But since $\mathbf{x}_j \in S_k$ for $j > k$, $\mathbf{l}$ is actually in S_k for all k, i.e., $\mathbf{l} \in \bigcap S_k$. By Heine-Borel: Let B be an open ball containing S_1, and let $U_j = B \setminus S_j$. If the sets U_k covered S_1, there would be a finite subcover; that is, $S_1 \subset \bigcup_1^K U_k = U_K$. But this is false since $S \setminus U_K = S_K \neq \varnothing$. Thus the sets U_k do not cover S, that is, $\bigcap S_k = S \setminus \bigcup U_k \neq \varnothing$.

6. (a) If $\mathbf{x} \in U \cap \overline{V}$, there is a sequence $\{\mathbf{y}_k\}$ in V that converges to $\mathbf{x}$, i.e., $|\mathbf{x} - \mathbf{y}_k| \to 0$; thus $d(U, V) = 0$.

(b) Suppose U is compact, V is closed, but $d(U, V) = 0$. Then there exist $\mathbf{x}_k \in U$, $\mathbf{y}_k \in V$ such that $|\mathbf{x}_k - \mathbf{y}_k| \to 0$. Since U is compact, by passing to a subsequence we may assume that $\mathbf{x}_k \to \mathbf{x} \in U$. But then also $\mathbf{y}_k \to \mathbf{x}$, so $\mathbf{x} \in \overline{V} = V$, contradicting $U \cap V = \varnothing$.

(c) One example is $U = \{(x, y) : y \leq 0\}$, $V = \{(x, y) : y \geq e^x\}$.

1.7 Connectedness

1. (a) The two branches ($x > 0$ and $x < 0$).

(b) One point in the set and the rest of the set.

(c) The intersections with the half-spaces $x > 0$ and $x < 0$.

2. If $\mathbf{a}, \mathbf{b}$ are points in the unit sphere S, the plane through $\mathbf{a}$, $\mathbf{b}$, and the origin (that is, the linear span of $\mathbf{a}$ and $\mathbf{b}$) intersects S in a circle, and either of the two arcs between $\mathbf{a}$ and $\mathbf{b}$ provides a continuous path in S from $\mathbf{a}$ to $\mathbf{b}$. (If $\mathbf{b} = -\mathbf{a}$, any great circle through $\mathbf{a}$ will do.) This argument works in any number of dimensions.

3. If f is neither strictly increasing nor strictly decreasing, one can find points $x, y, z \in I$ such that (i) $x < y < z$, and (ii) either $f(x) \leq f(y)$ and $f(y) \geq f(z)$, or $f(x) \geq f(y)$ and $f(y) \leq f(z)$; we assume the former alternative. If $f(x) = f(y)$ or $f(y) = f(z)$, then f is not one-to-one. Otherwise, the intervals $(f(x), f(y))$ and $(f(z), f(y))$ are nonempty, and one is contained in the other. Assuming f is continuous, the intermediate value theorem implies that $f((x,y)) \supset (f(x), f(y))$ and $f((y,z)) \supset (f(z), f(y))$, so there are points in (x,y) and (y,z) at which f takes the same value, and again f is not one-to-one.

4. Suppose $S_1 \cup S_2$ is disconnected, so $S_1 \cup S_2 = U \cup V$ where neither U nor V intersects the closure of the other one. Then $S_1 = (S_1 \cap U) \cup (S_1 \cap V)$ is a disconnection of S_1 unless either $S_1 \cap V$ or $S_1 \cap U$ is empty, i.e., $S_1 \subset U$ or $S_1 \subset V$. Likewise, we must have $S_2 \subset U$ or $S_2 \subset V$. It cannot be that S_1 and S_2 are both contained in U (resp. V), for then V (resp. U) would be empty; so $S_1 \subset U$ and $S_2 \subset V$ or vice versa. Either alternative contradicts the assumption that $S_1 \cap S_2 \neq \varnothing$.

$S_1 \cap S_2$ is connected when $n = 1$ by Theorem 1.25, but not when $n > 1$. For example, take S_1 to be the unit sphere (Exercise 2) and S_2 to be a line through the origin; the intersection consists of two points.

5. Suppose $S = U \cup V$ where U and V are open and disjoint. If $\mathbf{x} \in U$, there is a ball centered at $\mathbf{x}$ that is contained in U and hence is disjoint from V; hence $\mathbf{x} \notin \overline{V}$. Likewise $V \cap \overline{U} = \varnothing$, so $U \cup V$ is a disconnection of S.

Conversely, suppose S is open and $S = U \cup V$ is a disconnection. If $\mathbf{x} \in U$, there is a ball centered at $\mathbf{x}$ that is contained in S (since S is open) and a ball centered at $\mathbf{x}$ that does not intersect V (since $U \cap \overline{V} = \varnothing$). The smaller of these two is a ball centered at $\mathbf{x}$ that is contained in U. Thus every point of U is an interior point of U, so U is open; likewise, V is open.

6. If $S = U \cup V$ where U and V are closed and disjoint, it is immediate that $U \cup V$ is a disconnection of S. Conversely, if S is closed and $U \cup V$ is a disconnection of S, suppose $\mathbf{a} \in \overline{U}$. Since S is closed, we have $\mathbf{a} \in S$; since $V \cap \overline{U} = \varnothing$, we have $\mathbf{a} \notin V$. Hence $\mathbf{a} \in U$, so U is closed. Likewise, V is closed.

7. If $S = S_1 \cup S_2$ is a disconnection of S, define $f(\mathbf{x}) = 0$ for $\mathbf{x} \in S_1$ and $f(\mathbf{x}) = 1$ for $\mathbf{x} \in S_2$. Each point of S_1 has a neighborhood U that does not intersect S_2, so that f is constant on $S \cap U$; likewise with S_1 and S_2 switched. It follows that f is continuous on S.

Conversely, if f maps S continuously onto $\{0, 1\}$, let $S_1 = f^{-1}(\{0\})$ and $S_2 = f^{-1}(\{1\})$. If $\mathbf{x} \in S \cap \overline{S}_1$, then $f(\mathbf{x}) = 0$ since f is continuous, so $\mathbf{x} \notin S_2$. Thus $\overline{S}_1 \cap S_2 = \varnothing$, and likewise with S_1 and S_2 switched, so $S_1 \cup S_2$ is a disconnection of S.

8. Suppose $\overline{S} = U \cup V$ is a disconnection of $\overline{S}$. Then $(S \cap U) \cup (S \cap V)$ is a disconnection of S unless $S \cap U$ or $S \cap V$ is empty. The latter alternatives are impossible: If $S \cap U = \varnothing$, say, then $S \subset V$; but since U does not intersect the closure of V, we would have $U = U \cap \overline{S} \subset U \cap \overline{V} = \varnothing$, contrary to the definition of disconnection.

9. Pick $\mathbf{x} \in S$. If $g(\mathbf{x}) = 0$ we are done. Otherwise, either $g(\mathbf{x}) > 0$ or $g(\mathbf{x}) < 0$, in which case $g(-\mathbf{x}) < 0$ or $g(-\mathbf{x}) > 0$ respectively; either way, the intermediate value theorem implies that $g(\mathbf{y}) = 0$ for some $\mathbf{y} \in S$.

10. $f(1,3) = -2$ and $f(4,-1) = 5$, so there is a point $(x,y) \in S$ such that $f(x,y) = 0$, i.e., $x = y$.

11. (a) The graph $y = \sin(\pi/x)$, $0 < x \leq 2$, is arcwise connected almost by definition (it's an arc!), and S is its closure. (Check that every point in S^c has a neighborhood that does not intersect S, and that every neighborhood of every point on the vertical line segment $\{0\} \times [-1,1]$ contains points of the graph $y = \sin(\pi/x)$.) So S is connected by Exercise 8.

(b) Suppose $\mathbf{f} : [0,1] \to S$ is continuous and satisfies $\mathbf{f}(0) = (2,0)$ and $\mathbf{f}(1) = (0,1)$. The first component f_1 of $\mathbf{f}$ is continuous, so by the intermediate value theroem, for each k there exists $t_k \in [0,1]$ so that $f_1(t_k) = 1/2k$ and hence $\mathbf{f}(t_k) = (1/2k, 0)$ (= the only point in S with x-coordinate $1/2k$). As t goes from t_k to t_j ($j \neq k$,) f_1 must assume all values between $1/2k$ and $1/2j$, and hence f_2 must assume all values between -1 and 1 (again because there is only one point in S with a given x-coordinate in this range, and the y-coordinates of these points range from -1 to 1). By passing to a subsequence, by Bolzano-Weierstrass we may assume that $t_k \to t_0$. Every neighborhood of t_0 contains points at which f_2 assumes any given value between -1 and 1, so f_2 cannot be continuous at t_0, contrary to assumption.

1.8 Uniform Continuity

1. Given $\epsilon > 0$, if $|\mathbf{x} - \mathbf{y}| < (\epsilon/C)^{1/\lambda}$ then $|\mathbf{f}(\mathbf{x}) - \mathbf{f}(\mathbf{y})| < \epsilon$.

2. (a) $\int_0^b (a+t)^{\lambda-1}\, dt < \int_0^b t^{\lambda-1}\, dt$, so $(a+b)^\lambda - a^\lambda < b^\lambda$.

(b) For any x, y we have $|x|^\lambda \leq (|x-y| + |y|)^\lambda \leq |x-y|^\lambda + |y|^\lambda$, and likewise with x and y switched; hence $\big|\, |x|^\lambda - |y|^\lambda \big| \leq |x-y|^\lambda$.

3. Given $\epsilon > 0$, we can choose $\delta_1, \delta_2 > 0$ so that $|\mathbf{f}(\mathbf{x}) - \mathbf{f}(\mathbf{y})| < \frac{1}{2}\epsilon$ whenever $\mathbf{x}, \mathbf{y} \in S$ and $|\mathbf{x} - \mathbf{y}| < \delta_1$ and $|\mathbf{g}(\mathbf{x}) - \mathbf{g}(\mathbf{y})| < \frac{1}{2}\epsilon$ whenever $\mathbf{x}, \mathbf{y} \in S$ and $|\mathbf{x} - \mathbf{y}| < \delta_2$. Let $\delta = \min(\delta_1, \delta_2)$. Then $|(\mathbf{f}+\mathbf{g})(\mathbf{x}) - (\mathbf{f}+\mathbf{g})(\mathbf{y})| \leq |\mathbf{f}(\mathbf{x}) - \mathbf{f}(\mathbf{y})| + |\mathbf{g}(\mathbf{x}) - \mathbf{g}(\mathbf{y})| < \epsilon$ whenever $\mathbf{x}, \mathbf{y} \in S$ and $|\mathbf{x} - \mathbf{y}| < \delta$.

4. Suppose $\mathbf{f}$ is uniformly continuous and $\{\mathbf{x}_k\}$ is Cauchy. Given $\epsilon > 0$, there exists $\delta > 0$ so that $|\mathbf{f}(\mathbf{x}) - \mathbf{f}(\mathbf{y})| < \epsilon$ whenever $|\mathbf{x} - \mathbf{y}| < \delta$, and there exists K such that $|\mathbf{x}_j - \mathbf{x}_k| < \delta$ whenever $j, k > K$. It follows that $|\mathbf{f}(\mathbf{x}_j) - \mathbf{f}(\mathbf{x}_k)| < \epsilon$ whenever $j, k > K$, so $\{\mathbf{f}(\mathbf{x}_k)\}$ is Cauchy. For the counterexample, take $x_k = 1/k$ and $f(x) = 1/x$ or $f(x) = \cos(\pi/x)$.

5. If $\mathbf{f}(S)$ is unbounded, we can find a sequence $\{\mathbf{x}_k\}$ in S such that $|\mathbf{f}(\mathbf{x}_k)| \to \infty$. If also S is bounded, by passing to a subsequence we may assume that $\{\mathbf{x}_k\}$ converges to some limit (which may not be in S). Then $|\mathbf{x}_j - \mathbf{x}_k|$ will be as small as we please provided j and k are sufficiently large, but for any j we can find $k > j$ such that $|\mathbf{f}(\mathbf{x}_k)| > |\mathbf{f}(\mathbf{x}_j)| + 1$ and hence $|\mathbf{f}(x_j) - \mathbf{f}(\mathbf{x}_k)| \geq 1$. Thus $\mathbf{f}$ is not uniformly continuous on S.

Chapter 2

Differential Calculus

2.1 Differentiability in One Variable

1. Suppose $a, b \in I$ and $a < b$. If $f' > 0$ on (a, b), then $f(a) < f(b)$ by the mean value theorem. Otherwise, let $x_1, \ldots, x_k$ be the points of (a, b) at which f' vanishes, in increasing order. Then $f(b) - f(a) = [f(b) - f(x_k)] + [f(x_k) - f(x_{k-1}] + \cdots + [f(x_1) - f(a)]$. Each of the differences on the right is positive by the mean value theorem.

2. We have $f'(0) = \lim_{x \to 0} f(x)/x = \lim_{x\to 0} x \sin(1/x) = 0$ since $|x \sin(1/x)| \le |x|$. For $x \neq 0$ we have $f'(x) = 2x \sin(1/x) - \cos(1/x)$; the first term on the right approaches 0 as $x \to 0$, but the second term has no limit.

3. $g'(0) = f'(0) + \frac{1}{2} = \frac{1}{2}$, and for $x \neq 0$, $g'(x) = 2x\sin(1/x) - \cos(1/x) + \frac{1}{2}$. In particular, $g'(1/2k\pi) = -\frac{1}{2}$, so every interval about 0 contains small subintervals on which $g' < 0$.

4. $h'(0) = \lim_{x\to 0} h(x)/x = 0$ since $|h(x)| \le x^2$ for all x.

5. For $h > 0$, $[f(a+h) - f(a)]/h = f'(c)$ for some $c \in (a,\, a+h)$. As $h \to 0$, $c \to 0$ also, and so $f'(c) \to L$.

6. These formulas are obtained by applying l'Hôpital's rule 2 or 3 times. The general result is that $\lim_{h\to 0} \Delta_h^n f(a)/h^n = f^{(n)}(a)$, where Δ_h is the operator defined by $\Delta_h f(a) = f(a+h) - f(a)$. (Explicitly, $\Delta_h^n f(a) = \sum_{j=0}^{n} (-1)^{n-j} \binom{n}{j} f(a + jh)$.)

7. $\log[(1 + ax)^{b/x}] = (b/x)\log(1 + ax)$. By l'Hôpital, the latter quantity tends to ab as $x \to 0$.

8. $(\mathbf{f} \cdot \mathbf{g})' = (\sum f_j g_j)' = \sum (f_j' g_j + f_j g_j') = \mathbf{f}' \cdot \mathbf{g} + \mathbf{f} \cdot \mathbf{g}'$. The calculation for cross products is similar.

9. (a) $\lim_{x\to 0} e^{-1/x^2}/x^n = \lim_{y\to\infty} y^{n/2}/e^y = 0$ by Corollary 2.12.

(b) This is the case $n = 1$ of (a).

(c) For $k = 1$, $f'(x) = (-2/x^3)e^{-1/x^2}$. Assume $f^{(k)}(x) = P(1/x)e^{-1/x^2}$; then $f^{(k+1)}(x) = [(-1/x^2)P'(1/x) - (2/x^3)P(1/x)]e^{-1/x^2}$. The first term in brackets is a polynomial of degree $(3k-1) + 2 = 3k + 1$ in $1/x$, and the second term is a polynomial of degree $3k + 3$ in $1/x$.

(d) The case $k = 1$ is (b). Assuming by inductive hypothesis that $f^{(k-1)}(0) = 0$, we have $f^{(k)}(0) = \lim_{x\to 0} f^{(k-1)}(x)/x$. By (c), $f^{(k-1)}(x)/x = Q(1/x)e^{-1/x^2}$ where Q is a polynomial; hence the limit as $x \to 0$ is 0 by (a).

10. h is well defined since the two formulas agree at $x = 1$; it is continuous on $(0, 2)$ by inspection and continuous at 0 and 2 by definition of $f'(0)$ and $f'(1)$. By the intermediate value theorem, for any v between $f'(0)$ and $f'(1)$ there exists $u \in (0, 2)$ such that $h(u) = v$, and by the mean value theorem, $h(u) = f'(c)$ for some $c \in (0, u)$ (if $u \le 1$) or $c \in (u - 1,\ 1)$ (if $u \ge 1$).

2.2 Differentiability in Several Variables

1. (a) See the answer in the back of the text.
(b) $\nabla f(x, y) = (4e^{4x-y^2},\ -2ye^{4x-y^2})$; $\nabla f(1, -2) = (4, 4)$; $\partial_{(3/5,4/5)} f(1, -2) = \frac{3}{5} \cdot 4 + \frac{4}{5} \cdot 4 = \frac{28}{5}$.
(c) $\nabla f(x, y) = (-11y - 14,\ 11x - 12)/(7x + 3y)^2$; $\nabla f(1, -2) = (8, -1)$; $\partial_{(3/5,4/5)} f(1, -2) = \frac{3}{5} \cdot 8 - \frac{4}{5} \cdot 1 = 4$.

2. (a) See the answer in the back of the text.
(b) $df(x, y, z) = (x + z^2)^{-1}\, dx + 3y^2\, dy + 2z(x + z^2)^{-1}\, dz$; $df(1, 1, 0) = dx + 3\, dy$; $f(1.1, 1.2, -0.1) - f(1, 1, 0) \approx 0.1 + 3(0.2) = 0.7$.

3. $dw = \dfrac{2xy^{3/2}z}{z + 1}\, dx + \dfrac{3x^2y^{1/2}z}{2(z + 1)}\, dy + \dfrac{x^2y^{3/2}}{(z + 1)^2}\, dz$, so $dw|_{(5,4,1)} = 40\, dx + \frac{75}{2}\, dy + 50\, dz$.
(a) $0 = 40(.03) + \frac{75}{2}(-.08) + 50\, dz \Longrightarrow dz = [-1.2 + 3]/50 = .036$.
(b) The coefficient of dz is largest.

4. $x\partial_x u + 2y\partial_y u + \partial_z u = xe^{2z} - 2y^{-1}e^{5z} + (2xe^{2z} + 5y^{-1}e^{5z}) = 3xe^{2z} + 3y^{-1}e^{5z} = 3u$.

5. $\partial_x u = -y^2/(xy - y + 2x)^2$ and $\partial_y u = 2x^2/(xy - y + 2x)^2$; the result follows.

6. Since $\partial_i(|\mathbf{x}|^{-1}) = -x_i|\mathbf{x}|^{-3}$, we have $df_j = |\mathbf{x}|^{-1}\, dx_j - \sum_{i=1}^n x_i x_j |\mathbf{x}|^{-3}\, dx_i$ and hence

$$\begin{aligned} \sum_j x_j\, df_j &= \sum_j x_j|\mathbf{x}|^{-1}\, dx_j - \sum_i \sum_j x_i x_j^2 |\mathbf{x}|^{-3}\, dx_i \\ &= \sum_j x_j|\mathbf{x}|^{-1}\, dx_j - \sum_i x_i|\mathbf{x}|^{-1}\, dx_i = 0. \end{aligned}$$

7. (a) $|f(x, y)| \le \frac{1}{2}|x|$, so $f(x, y) \to 0$ as $(x, y) \to (0, 0)$.
(b) With $\mathbf{u} = (\cos\theta, \sin\theta)$, $\partial_{\mathbf{u}} f(0, 0) = \lim_{t\to 0} f(t\cos\theta,\ t\sin\theta)/t = \cos^2\theta \sin\theta$.
(c) Taking $\theta = 0$ or $\theta = \frac{1}{2}\pi$ we see that $\partial_1 u(0, 0) = \partial_2 u(0, 0) = 0$. If f were differentiable it would follow that $\partial_{\mathbf{u}} f(0, 0) = 0$ for all $\mathbf{u}$; but this is false.

8. Assume $n = 2$ for simplicity; the general case follows by an elaboration of the argument as in the proof of Theorem 2.19. Suppose $|\partial_1 f| \le C$ and $|\partial_2 f| \le C$ on S. Given $\mathbf{a} \in S$, let $r > 0$ be small enough so that $B(r, \mathbf{a}) \subset S$. If $|\mathbf{h}| < r$, by the mean value theorem we have

$$\begin{aligned} |f(\mathbf{a} + \mathbf{h}) - f(\mathbf{a})| &\le |f(a_1 + h_1,\ a_2 + h_2) - f(a_1,\ a_2 + h_2)| + |f(a_1,\ a_2 + h_2) - f(a_1, a_2)| \\ &\le C(|h_1| + |h_2|), \end{aligned}$$

which implies the continuity of f at $\mathbf{a}$.

2.3 The Chain Rule

1. See the answers in the back of the text.

2. (a) and (c) See the answer in the back of the text.
(b) $\partial_x w = e^{x-3y} f_1 + 2x f_2/(x^2+1)$, $\partial_y w = -3e^{x-3y} f_1 + 2y^3 f_3/\sqrt{y^4+4}$.

3. (a) $2\partial_x u - 3\partial_y u = (6-6)f'(3x+2y) = 0$.
(b) $x\partial_x u + y\partial_y u - u = x[y + f(yx^{-1}) - yx^{-1} f'(yx^{-1})] + y[x + f'(yx^{-1})] - xy - xf(yx^{-1}) = xy$.
(c) $x\partial_x u + y\partial_y u = xzf_1 + yzf_2 = z\partial_z u$.

4. $\partial u/\partial x_j = x_j f'(r)/r$, so $\sum(\partial u/\partial x_j)^2 = (\sum x_j^2) f'(r)^2/r^2 = f'(r)^2$.

5. Both formulas for the tangent plane at $(a, b, f(a,b))$ amount to $z - f(a,b) = A(x-a) + B(y-b)$ where $A = \partial_x f(a,b)$ and $B = \partial_y f(a,b)$.

6. These are all similar; we just do (d). If $F(x,y,z) = xyz^2 - \log(z-1)$, we have $\partial_x F = yz^2$, $\partial_y F = xz^2$, and $\partial_z F = 2xyz - (z-1)^{-1}$, so $\nabla F(-2,-1,2) = (-4,-8,7)$. Hence the tangent plane is given by $-4(x+2) - 8(y+1) + 7(z-2) = 0$, or $7z = 4x + 8y + 30$.

7. We have $\varphi(x) = F(x, x, \ldots, x)$, so $\varphi'(x) = \partial_1 F + \partial_2 F + \cdots + \partial_n F$, these derivatives being evaluated at $(x, x, \ldots, x)$.

2.4 The Mean Value Theorem

1. (a) The conclusion is that the directional derivative $\partial_{\mathbf{u}} f$ vanishes at some point on the line segment, where $\mathbf{u} = (\mathbf{b} - \mathbf{a})/|\mathbf{b} - \mathbf{a}|$. (Apply Rolle's theorem to the function $\varphi(t) = f(t\mathbf{b} + (1-t)\mathbf{a})$ on the interval $[0,1]$.)

(b) The conclusion is that there is a point $\mathbf{a} \in S$ such that $\nabla f(\mathbf{a}) = \mathbf{0}$. (Suppose $f = c$ on ∂S. If $f \equiv c$ on S then $\nabla f \equiv \mathbf{0}$ on S. Otherwise, either the maximum or the minimum of f on $\overline{S}$ [which exist since S is compact] is achieved at a point $\mathbf{a} \in S$, and then $\nabla f(\mathbf{a}) = \mathbf{0}$.)

2. (a) If S is convex, we can apply Theorem 2.39 to get $f(\mathbf{b}) - f(\mathbf{a}) = \partial_1 f(\mathbf{c})(b_1 - a_1) = 0$.

(b) An example with $n = 2$: let S be the square $(-1,1) \times (-1,1)$ with the segment $[0,1)$ on the y-axis removed. Define f on S by $f(x,y) = 0$ if $y \le 0$, $f(x,y) = -y^2$ if $y > 0$ and $x < 0$, and $f(y) = y^2$ if $y > 0$ and $x > 0$. Then $\partial_x f \equiv 0$ on S, but $f(-x,y) \neq f(x,y)$ for $y > 0$.

2.5 Functional Relations and Implicit Functions: A First Look

1. (a) See the answer in the back of the text.

(b) Differentiation in x gives $4x + (2z + e^{-z})\partial_x z = 0$, or $\partial_x z = -4x/(2z + e^{-z})$. Likewise, differentiation in y gives $\partial_y z = -6y/(2z + e^{-z})$.

2. (a) Elimination of z gives $y^2 + 4y = x^2 - 2x$, so $y = -2 \pm \sqrt{x^2 - 2x + 4}$ and hence $z = 2x - 8 \pm 4\sqrt{x^2 - 2x + 4}$. Therefore $dy/dx = \pm(x-1)/\sqrt{x^2 - 2x + 4}$, and $dz/dx = 2 \pm 4(x-1)/\sqrt{x^2 - 2x + 4}$.

(b) Differentiating the original equations in x gives $z' = 2x - 2yy'$ and $z' = 2 + 4y'$. Thus $2x - 2yy' = 2 + 4y'$, so $y' = (x-1)/(y+2)$ and $z' = 2 + 4(x-1)/(y+2)$.

3. Differentiating the equations gives $5y^4y' + (y'z + yz')e^{yz} + z't^2 + 2zt = 0$ and $2yy' + 4z^3z' = 2t$, which are linear equations in y' and z' to be solved simultaneously.

4. If (x, y) are IVs, $u = x^2+3y^2 \implies u_x = 2x$. If (x, z) are IVs, $u = x^2+3(xz)^2 \implies u_x = 2x+6xz^2$.

5. For $(\partial V/\partial h)|_r$, just use $V = \pi r^2 h$. For $(\partial V/\partial h)|_S$, r is implicitly a function of h and S; the equation $S = 2\pi r(r+h)$ yields $\partial r/\partial h = -r/(2r+h)$, so $\partial V/\partial h = \pi r^2+2\pi r(\partial r/\partial h)h = \pi r^2-2\pi r^2 h/(2r+h)$. For $(\partial V/\partial S)|_r$, V and h are implicitly functions of S and r; differentiating the given equations in S gives $\partial V/\partial S = \pi r^2(\partial h/\partial S)$ and $1 = 2\pi r(\partial h/\partial S)$, whence $\partial V/\partial S = r/2$. For $(\partial S/\partial V)|_r$, S and h are implicitly functions of V and r; differentiating the given equations in V gives $1 = \pi r^2(\partial h/\partial V)$ and $\partial S/\partial V = 2\pi r(\partial h/\partial V)$, whence $\partial S/\partial V = 2/r$.

6. $\partial x/\partial y$ is the derivative when x is considered as a function of y and z; by (2.44) it equals $-\partial_y F/\partial_x F$. Likewise $\partial y/\partial z = -\partial_z F/\partial_y F$ and $\partial z/\partial x = -\partial_x F/\partial_z F$. The product of these quantities is -1.

7. Taking V and T as independent variables means that E and P are determined as functions of V and T, say

$$E = \varphi(V, T), \qquad P = \psi(V, T), \tag{*}$$

and the equation $\partial_V E - T\partial_T P + P = 0$ then becomes $\partial_1\varphi - T\partial_2\psi + P = 0$. Now take P and T as the independent variables. Differentiating (*) with respect to P gives $\partial_P E = (\partial_1\varphi)(\partial_P V)$ and $1 = (\partial_1\psi)(\partial_P V)$, so $\partial_1\varphi = \partial_P E/\partial_P V$ and $\partial_1\psi = 1/\partial_P V$. Differentiating (*) with respect to T gives $0 = (\partial_1\psi)(\partial_T V) + \partial_2\psi$, so $\partial_2\psi = -(\partial_1\psi)(\partial_T V) = -\partial_T V/\partial_P V$. Substituting these into $\partial_1\varphi - T\partial_2\psi + P = 0$ gives $\partial_P E + T\partial_T V + P\partial_P V = 0$.

2.6 Higher-Order Partial Derivatives

1. These are routine exercises in elementary calculus. Half of the calculations were performed in Exercise 1, §2.2.

2. From Example 4, $u_r = (\cos\theta)f_x + (\sin\theta)f_y$ and hence $u_{r\theta} = -(\sin\theta)f_x + (\cos\theta)(\partial f_x/\partial\theta) + (\cos\theta)f_y + (\sin\theta)(\partial f_y/\partial\theta)$; furthermore, $\partial f_x/\partial\theta = -(r\sin\theta)f_{xx} + (r\cos\theta)f_{xy}$ and $\partial f_y/\partial\theta = -(r\sin\theta)f_{xy} + (r\cos\theta)f_{yy}$.

3. (a) $\partial_x w = 2f_1 + (\sin 3y)f_2 + 4x^3 f_3$. Hence $\partial_x^2 w = 2\partial_x f_1 + (\sin 3y)\partial_x f_2 + 4x^3\partial_x f_3 + 12x^2 f_3$. The derivatives $\partial_x f_j$ are just like $\partial_x w$ but with an extra subscript j on the f's: $\partial_x f_1 = 2f_{11} + (\sin 3y)f_{12} + 4x^3 f_{13}$, etc. Now it's just a matter of collecting terms. Similarly, $\partial_x\partial_y w = 2\partial_y f_1 + (\sin 3y)\partial_y f_2 + 3(\cos 3y)f_2 + (3\cos 3y)\partial_y f_2$, etc. (The $3x\cos 3y$ multiplying f_{12} in the answer in the book should be $6x\cos 3y$.)

(b) $\partial_y w = -3e^{x-3y}f_1 + 2y^3(y^4+4)^{-1/2}f_3$. Hence $\partial_x\partial_y w = -3e^{x-3y}(e^{x-3y}f_{11} + 2x(x^2+1)^{-1}f_{12}) - 3e^{x-3y}f_1 + 2y^3(y^4+4)^{-1/2}(e^{x-3y}f_{13} + 2x(x^2+1)^{-1}f_{23})$. Similarly, $\partial_y^2 w = -3e^{x-3y}\partial_y f_1 + 9e^{x-3y}f_1 + 2y^3(y^4+4)^{-1/2}\partial_y f_3 + \partial_y[2y^3(y^4+4)^{-1/2}]f_3$, which works out to be $9e^{2x-6y}f_{11} - 12y^3e^{x-3y}(y^4+4)^{-1/2}f_{13} + 4y^6(y^4+4)^{-1}f_{33} + 9e^{x-3y}f_1 + (2y^6+24y^2)(y^4+4)^{-3/2}f_3$.

4. We have $u_x = F'(x+g(y))$; hence $u_{xy} = F''(x+g(y))g'(y)$ and $u_{xx} = F''(x+g(y))$; also $u_y = F'(x+g(y))g'(y)$. The result follows.

5. $\sum_{j,k} x_j x_k \partial_j\partial_k f(\mathbf{x})$ and $a(a-1)f(\mathbf{x})$ are both equal to $(d^2/dt^2)f(t\mathbf{x})|_{t=1}$.

6. $u_s = 2sf_x + 2tf_y$, so $u_{ss} = 2f_x + 4s^2 f_{xx} + 4t^2 f_{yy}$. Also $u_t = -2tf_x + 2sf_y$, so $u_{tt} = -2f_x + 4t^2 f_{xx} + 4s^2 f_{yy}$. The result follows.

7. $u_{tt} = c^2[f''(x-ct) + g''(x+ct)] = c^2 u_{xx}$.

8. We have $\partial r/\partial x = x/r$, so $F_x = -xr^{-3}g(ct-r) - xr^{-2}g'(ct-r)$, and hence

$$F_{xx} = -\frac{g(ct-r)}{r^3} + \frac{3x^2 g(ct-r)}{r^5} + \frac{x^2 g'(ct-r)}{r^4} - \frac{g'(ct-r)}{r^2} + \frac{2x^2 g'(ct-r)}{r^4} + \frac{x^2 g''(ct-r)}{r^3}.$$

F_{yy} and F_{zz} are the same, with x replaced by y and z. Since $x^2 + y^2 + z^2 = r^2$, adding these gives

$$\begin{aligned} -\frac{3g(ct-r)}{r^3} + \frac{3g(ct-r)}{r^3} + \frac{g'(ct-r)}{r^2} - \frac{3g'(ct-r)}{r^2} + \frac{2g'(ct-r)}{r^2} + \frac{g''(ct-r)}{r} &= \frac{g''(ct-r)}{r} \\ &= c^{-2}F_{tt}. \end{aligned}$$

9. $F_j = x_j r^{-1} f'(r)$, so $F_{jj} = r^{-1}f'(r) - x_j^2 r^{-3} f'(r) + x_j^2 r^{-2} f''(r)$. Adding these up gives $nr^{-1}f'(r) - r^{-1}f'(r) + f''(r)$ since $\sum x_j^2 = r^2$.

10. In one variable, the assertion is that $(fg)^{(k)} = \sum_{j=0}^{k} \binom{k}{j} f^{(j)} g^{(k-j)}$, which is proved just like the binomial theorem. (Induction on k, using the fact that $\binom{k}{j} + \binom{k}{j-1} = \binom{k+1}{j}$.) The n-variable result follows by applying the one-variable result in each variable separately; the facts that $\alpha = \beta + \gamma \iff \alpha_j = \beta_j + \gamma_j$ for all j and $\alpha! = \alpha_1! \cdots \alpha_n!$ make everything turn out right. (This could be phrased as an induction on n.)

11. This follows by applying the one-dimensional binomial theorem in each variable as in the preceding problem.

2.7 Taylor's Theorem

1. (a) $f(x) = x^2[x - (x - \frac{1}{6}x^3 + \cdots)] = \frac{1}{6}x^5 + \cdots$, and $g(x) = [(1 + x + \cdots) - 1][(1 - 2x^2 + \cdots) - 1]^2 = (x + \cdots)(4x^4 + \cdots) = 4x^5 + \cdots$.

(b) $f(x)/g(x) = (\frac{1}{6}x^5 + \cdots)/(4x^5 + \cdots) = (\frac{1}{6} + \cdots)/(4 + \cdots) \to \frac{1}{24}$.

2. (a) $f'(x) = x^{-1}$, $f''(x) = -x^{-2}$, and $f^{(3)}(x) = 2x^{-3}$, so $P_{1,3}(h) = h - \frac{1}{2}h^2 + \frac{1}{3}h^3$; also $|f^{(4)}(x)| = |-6x^{-4}| \le 96$ for $|x-1| \le \frac{1}{2}$, so $C = 96/4! = 4$.

(b) $f'(x) = \frac{1}{2}x^{-1/2}$, $f''(x) = -\frac{1}{4}x^{-3/2}$, and $f^{(3)}(x) = \frac{3}{8}x^{-5/2}$, so $P_{1,3}(h) = 1 + \frac{1}{2}h - \frac{1}{8}h^2 + \frac{1}{16}h^3$; also $|f^{(4)}(x)| = |-\frac{15}{16}x^{-7/2}| \le \frac{15}{16}2^{7/2}$ for $|x-1| \le \frac{1}{2}$, so $C = \frac{15}{16}2^{7/2}/4! = 5 \cdot 2^{-7/2}$.

(c) $f'(x) = -(x+3)^{-2}$, $f''(x) = 2(x+3)^{-3}$, and $f^{(3)}(x) = -6(x+3)^{-4}$, so $P_{1,3}(h) = \frac{1}{4} - \frac{1}{16}h + \frac{1}{64}h^2 - \frac{1}{256}h^3$; also $|f^{(4)}(x)| = |24(x+3)^{-5}| \le 24(3.5)^{-5}$ for $|x-1| \le \frac{1}{2}$, so $C = (3.5)^{-5}$.

3. By Lagrange's formula, $|\sin x - x - \frac{1}{6}x^3| = |R_{0,4}(x)| \le |x|^5/5!$ since $|\sin^{(5)}(x)| = |\sin x| \le 1$, and $(\frac{1}{2}\pi)^5/5! \approx 0.0797$. In general, we have $|R_{0,2m-1}(x)| = |R_{0,2m}(x)| \le |x|^{2m+1}/(2m+1)!$. This is less than 0.01 for $|x| \le \frac{1}{2}\pi$ provided $m \ge 3$; so the 5th order polynomial suffices.

4. For $f(t) = e^{-t}$ the remainder $R_{0,k}(t)$ is bounded by $t^{k+1}/(k+1)!$ for $t \in [0,1]$, so $|\int_0^1 R_{0,k}(x^2)\,dx| \le \int_0^1 x^{2k+2}\,dx/(k+1)! = 1/(k+1)!(2k+3)$. This is less than 0.0005 for $k \ge 6$, so to three decimal places $\int_0^1 e^{-x^2}\,dx = \sum_0^5 (-1)^k \int_0^1 x^{2k}\,dx/k! = \sum_0^5 (-1)^k/k!(2k+1) = 0.747$.

5. (a) $x\sin(x+y) = x[(x+y) - \frac{1}{6}(x+y)^3 + \cdots] = x^2 + xy - \frac{1}{6}x^4 - \frac{1}{2}x^3y - \frac{1}{2}x^2y^2 - \frac{1}{6}xy^3 + \cdots$.
(b) $e^{xy}\cos(x^2+y^2) = [1+xy+\frac{1}{2}(xy)^2+\cdots][1-\frac{1}{2}(x^2+y^2)^2+\cdots] = 1+xy-\frac{1}{2}(x^4+x^2y^2+y^4)+\cdots$.
(c)

$$\begin{aligned}\frac{e^{x-2y}}{1+x^2-y} &= \left[1+(x-2y)+\frac{1}{2}(x-2y)^2+\frac{1}{6}(x-2y)^3+\frac{1}{24}(x-2y)^4+\cdots\right]\times\\ &\quad\times\left[1+(y-x^2)+(y-x^2)^2+(y-x^2)^3+(y-x^2)^4+\cdots\right]\\ &= \Big[1+x-2y+\frac{1}{2}x^2-2xy+2y^2+\frac{1}{6}x^3-x^2y+2xy^2-\frac{4}{3}y^3+\frac{1}{24}x^4-\frac{1}{3}x^3y+x^2y^2\\ &\quad -\frac{4}{3}xy^3+\frac{2}{3}y^4+\cdots\Big]\times\left[1+y-x^2+y^2-2x^2y+y^3+x^4-3x^2y^2+y^4+\cdots\right]\\ &= 1+x-y-\frac{1}{2}x^2-xy+y^2-\frac{5}{6}x^3-\frac{1}{2}x^2y+xy^2-\frac{1}{3}y^3+\frac{13}{24}x^4-\frac{1}{6}x^3y\\ &\quad -\frac{1}{2}x^2y^2-\frac{1}{3}xy^3+\frac{1}{3}y^4+\cdots.\end{aligned}$$

6. Setting $x = 3+h$, $y = 1+k$, we have $f(x,y) = 3+h-\cos\pi k+(3+h)\log(1+k) = 3+h-(1-\frac{1}{2}(\pi k)^2+\cdots)+(3+h)(k-\frac{1}{2}k^2+\frac{1}{3}k^3+\cdots) = 2+h+3k+\frac{1}{2}(\pi^2-3)k^2-\frac{1}{2}hk^2+k^3+\cdots$.

7. With $x = 1+h$, $y = 2+k$, $z = 1+l$, we have $f(x,y,z) = (1+h)^2(2+k)+(1+l) = 3+4k+k+l+2h^2+2hk+h^2k$, with no remainder. (The remainder is also known to vanish since all 4th order derivatives of f vanish.)

8. A $(k-1)$-fold application of l'Hôpital gives

$$\begin{aligned}\lim_{h\to 0}\frac{f(a+h)-P_{a,k}(h)}{h^k} &= \lim_{h\to 0}\frac{f^{(k-1)}(a+h)-f^{(k-1)}(a)-f^{(k)}(a)h}{h}\\ &= \lim_{h\to 0}\frac{f^{(k-1)}(a+h)-f^{(k-1)}(a)}{h}-f^{(k)}(a)=0\end{aligned}$$

by definition of $f^{(k)}(a)$.

9. We have $f(a+h) = f(a)+f^{(k)}(a)h^k/k!+R_{a,k}(h)$, and by Corollary 2.60, for h sufficiently small we have $|R_{a,k}(h)| \le \frac{1}{2}|f^{(k)}(a)h^k|/k!$ Thus for k even, if $f^{(k)}(a) > 0$ we have $f(a+h)-f(a) \ge \frac{1}{2}f^{(k)}(a)h^k/k! > 0$ for small h, and likewise if $f^{(k)}(a) < 0$ we have $f(a+h)-f(a) < 0$ for small h. For k odd, the same reasoning shows that $f(a+h)-f(a)$ changes sign along with h^k at $h = 0$.

10. By (2.70) we have $|R_{\mathbf{a},k}(\mathbf{h})| \le k\sum_{|\alpha|=k}(|\mathbf{h}^\alpha|/\alpha!)\int_0^1(1-t)^{k-1}Ct^\lambda|\mathbf{h}|^\lambda\,dt$. Since each component of $\mathbf{h}$ is less than $|\mathbf{h}|$ in absolute value, we have $|\mathbf{h}^\alpha| \le |\mathbf{h}|^k$ for $|\alpha| = k$. Hence $|R_{\mathbf{a},k}(\mathbf{h})| \le C'|\mathbf{h}|^{k+\lambda}$ where $C' = Ck\sum_{|\alpha|=k}\alpha!^{-1}\int_0^1(1-t)^{k-1}t^\lambda\,dt$.

2.8 Critical Points

1. We employ the notation α, β, γ as in Theorem 2.82.

(a) $f_x = 2x$ and $f_y = 12y(y+2)(y-1)$, so the critical points are $(0,0)$, $(0,-2)$, and $(0,1)$. Also $\alpha = 2$, $\beta = 0$, $\gamma = 36y^2 + 24y - 24$, so these points are respectively a saddle, a minimum, and a minimum.

(b) $f_x = 4x(x^2-1)$ and $f_y = 3(y^2-2)$, so the critical points are $(0, \pm\sqrt{2})$, $(\pm 1, \sqrt{2})$, and $(\pm 1, -\sqrt{2})$. Also $\alpha = 12x^2 - 4$, $\beta = 0$, $\gamma = 6y$, so $(0, \sqrt{2})$ and $(\pm 1, -\sqrt{2})$ are saddles, $(0, -\sqrt{2})$ is a maximum, and $(\pm 1, \sqrt{2})$ are minima.

(c) $f_x = 3x^2 - 2x - y^2$ and $f_y = 2y(1-x)$. If $f_y = 0$ then either $y = 0$ or $x = 1$. In the first case, $f_x = 0 \Longrightarrow x = 0$ or $x = \frac{2}{3}$; in the second case, $f_x = 0 \Longrightarrow y = \pm 1$. So the critical points are $(0,0)$, $(\frac{2}{3}, 0)$, and $(1, \pm 1)$. Also, $\alpha = 6x - 2$, $\beta = -2y$, and $\gamma = 2(1-x)$, so $(1, \pm 1)$ and $(0,0)$ are saddles and $(\frac{2}{3}, 0)$ is a minimum.

(d) $f_x = xy^2(4 - 3x - 2y)$ and $f_y = x^2y(4 - 2x - 3y)$. If either $x = 0$ or $y = 0$ then $f_x = f_y = 0$; otherwise, $f_x = f_y = 0$ only when $3x + 2y = 2x + 3y = 4$, that is, $x = y = \frac{4}{5}$. Thus the critical points are $(\frac{4}{5}, \frac{4}{5})$ and all points on the x and y axes. Note that $f = 0$ on the lines $x = 0$, $y = 0$, and $x + y = 2$; elsewhere, $f < 0$ when $x + y > 2$ and $f < 0$ when $x + y < 2$. Thus the points $(a, 0)$ and $(0, a)$ are local (nonstrict) minima when $a < 2$, local (nonstrict) maxima when $a > 2$, and saddle points when $a = 2$. Also, $f(\frac{4}{5}, \frac{4}{5}) > 0$, and $(\frac{4}{5}, \frac{4}{5})$ is inside the triangle bounded by the lines on which $f = 0$, so it must be a maximum. (One could also check this by Theorem 2.82.)

(e) $f_x = 2x(2 - 2x^2 - y^2)e^{-x^2-y^2}$ and $f_y = 2y(1 - 2x^2 - y^2)e^{-x^2-y^2}$. Thus $f_x = 0 \iff x = 0$ or $2x^2 + y^2 = 2$. In the first case, $f_y = 2y(1-y^2) = 0 \Longrightarrow y = 0$ or $y = \pm 1$. In the second case, $f_y = -2y = 0 \Longrightarrow y = 0$ and hence $2x^2 = 2$, or $x = \pm 1$. Thus the critical points are $(0,0)$, $(0, \pm 1)$, and $(\pm 1, 0)$. $(0,0)$ is obviously the global minimum. A straightforward but tedious application of Theorem 2.82 shows that $(\pm 1, 0)$ are maxima and $(0, \pm 1)$ are saddles. (See also the solution to Exercise 1h below.)

(f) $f_x = -ax^{-2} + y$ and $f_y = -by^{-2} + x$. If $f_x = 0$ then $y = ax^{-2}$; substituting this into $f_y = 0$ gives $x = (a^2/b)^{1/3}$ and $y = (b^2/a)^{1/3}$. At this critical point, $\alpha = 2b/a$, $\beta = 1$, and $\gamma = 2a/b$, so the point is a minimum if $b/a > 0$ and a maximum if $b/a < 0$.

(g) $f_x = 3(x^2 - 1)$, $f_y = -3(y^2 - 3)$, and $f_z = 2z$, so the critical points are those where $x = \pm 1$, $y = \pm\sqrt{3}$, and $z = 0$. The Hessian matrix is diagonal with diagonal entries $6x$, $-6y$, 2. Thus $(1, -\sqrt{3}, 0)$ is a minimum, $(-1, \sqrt{3}, 0)$ is a maximum, and $\pm(1, \sqrt{3})$ are saddles.

(h) With $E = e^{-x^2-y^2-z^2}$ we have $f_x = 2x(3 - 3x^2 - 2y^2 - z^2)E$, $f_y = 2y(2 - 3x^2 - 2y^2 - z^2)E$, and $f_z = 2z(1 - 3x^2 - 2y^2 - z^2)E$. If $f_x = 0$, then either $x = 0$ or $3x^2 + y^2 + z^2 = 3$. In the first case the equations $f_y = f_z = 0$ give $y = 0$ or $2y^2 + z^2 = 2$, and $z = 0$ or $2y^2 + z^2 = 1$; the solutions are $y = z = 0$; $y = 0$, $z = \pm 1$; $y = \pm 1$, $z = 0$. In the second case, the equations $f_y = f_z = 0$ give $y = z = 0$ and hence $x = \pm 1$. So the critical points are $(0,0,0)$, $(\pm 1, 0, 0)$, $(0, \pm 1, 0)$, and $(0, 0, \pm 1)$. One can analyze them without the tedium of computing all the second derivatives as follows. Since $f(\mathbf{x}) > 0$ for $\mathbf{x} \neq \mathbf{0}$, $(0,0,0)$ is obviously the global minimum. Since $f(\mathbf{x}) \to 0$ as $\mathbf{x} \to \infty$, f must have a global maximum; by examining the values at the critical points one sees that the maximum is at $(\pm 1, 0, 0)$. Now consider $f(\mathbf{x})$ for $\mathbf{x}$ near $(0, \pm 1, 0)$. Since te^{-t} has a maximum at $t = 1$, $f(0, y, 0) = 2y^2 e^{-y^2}$ has a maximum at $y = \pm 1$. On the other hand, $f(x, \pm 1, 0) = (3x^2 + 2)e^{-x^2-1} = e^{-1}[(3x^2+2)(1 - x^2 + \cdots)] = e^{-1}(2 + x^2 + \cdots)$ has a local minimum at $x = 0$. Thus $(0, \pm 1, 0)$ is a saddle, and likewise so is $(0, 0, \pm 1)$.

(i) Note that $f = 0$ precisely on the planes $x = 0$, $y = 0$, $z = 0$, and $x + y + z = 4$ that include the faces of the tetrahedron with vertices at $\mathbf{0}$, $4\mathbf{i}$, $4\mathbf{j}$, and $4\mathbf{k}$. Since $\nabla(f_1 f_2) = \mathbf{0}$ on the set where $f_1 = f_2 = 0$, all points on the lines where these planes intersect are degenerate critical points, and none of them are maxima or minima since f changes sign whenever one crosses one of these planes. Since $f_x = yz(4 - 2x - y - z)$, $f_y = xz(4 - x - 2y - z)$, and $f_z = xy(4 - x - y - 2z)$, one sees that the only other critical point is where $2x + y + z = x + 2y + z = x + y + 2z = 4$, i.e., $(1, 1, 1)$. This point is inside the tetrahedron, $f > 0$ there, and $f = 0$ on the faces of the tetrahedron; hence this point is a local maximum.

2. We have $f_{xx} = 2a$, $f_{xy} = b$, and $f_{yy} = 2c$, so the origin is a minimum if $4ac \geq b^2$ and $a, c \geq 0$, a maximum if $4ac \geq b^2$ and $a, c \leq 0$, and a saddle if $4ac < b^2$. (If $4ac = b^2$, then $f(x, y) = \pm(|a|^{1/2}x \pm |c|^{1/2}y)^2$ [any combination of the two signs can occur], so the origin is still an extremum.)

3. The origin is a global minimum for f_1 and a saddle point for f_2. For f_3 it is a "shoulder point" as in the right-hand graph of Figure 2.5 (but upside down).

4. (a) The 2nd-order Taylor polynomial of f at the origin is y^2, so the origin is a degenerate critical point.

(b) For $b \neq 0$ we have $f(at, bt) = b^2t^2 +$ higher order, and $f(at, 0) = 2a^4t^4$; these all have local minima at the origin. However, f is negative in the region between the two parabolas $y = x^2$ and $y = 2x^2$ and positive in the regions inside both or outside both. The origin is on the boundary of all these regions, so f has neither a maximum or a minimum there.

5. The second directional derivative of f in the direction $\mathbf{u}$ at $\mathbf{a}$ is

$$\frac{d^2}{dt^2} f(\mathbf{a} + t\mathbf{u})\bigg|_{t=0} = \frac{d}{dt} \sum u_j \partial_j f(\mathbf{a} + t\mathbf{u})\bigg|_{t=0} = \sum_{j,k} u_j u_k \partial_j \partial_k f(\mathbf{a}) = H(\mathbf{a})\mathbf{u} \cdot \mathbf{u}.$$

2.9 Extreme Value Problems

1. $f_x = 4x + 2$ and $f_y = 2y$, so the only critical point is $(-\frac{1}{2}, 0)$, and $f(-\frac{1}{2}, 0) = -\frac{1}{2}$. On the unit circle, $f(\cos\theta, \sin\theta) = 2\cos^2\theta + \sin^2\theta + 2\cos\theta = (1 + \cos\theta)^2$, whose maximum and minimum are 4 and 0 (at $\theta = 0$ and $\theta = \pi$). So the maximum is 4 and the minimum is $-\frac{1}{2}$.

2. $f_x = 6x$ and $f_y = -4y + 2$, so the only critical point is $(0, \frac{1}{2})$, and $f(0, \frac{1}{2}) = \frac{1}{2}$. On the unit circle, $f(\cos\theta, \sin\theta) = 3 - 5\sin^2\theta + 2\sin\theta$, so $(d/d\theta)f(\cos\theta, \sin\theta) = -10\cos\theta\sin\theta + 2\cos\theta = 2\cos\theta(1 - 5\sin\theta) = 0$ when $\cos\theta = 0$ or $\sin\theta = \frac{1}{5}$. In the first case, $\sin\theta = \pm 1$, so $f = 0$ or -4. In the second case, $f = \frac{16}{5}$. So the minimum is -4 and the maximum is $\frac{16}{5}$.

3. $f_x = 3x^2 - 1$ and $f_y = 2y - 2$, so the critical points are at $(\pm 1/\sqrt{3}, 1)$, which are both outside the triangle. The sides of the triangle are segments in the lines $y = 0$, $y = 2 - 2x$, and $y = 2 + 2x$. We have $f(x, 0) = x^3 - x$, whose critical points are at $x = \pm 1/\sqrt{3}$, and $f(\pm 1\sqrt{3}, 0) = \mp 2/3\sqrt{3}$. Also, $f(x, 2 - 2x) = x^3 + 4x^2 - 5x$, whose only critical point in the interval $0 \leq x \leq 1$ is at $x = (\sqrt{31} - 4)/3$, where $f = (308 - 62\sqrt{31})/27$. Next, $f(x, 2 + 2x) = x^3 + 4x^2 + 3x$, whose only critical point in the interval $-1 \leq x \leq 0$ is at $x = (\sqrt{7} - 4)/3$, where $f = (20 - 14\sqrt{7})/27$. Finally, $f = 0$ at all three vertices. Comparing all these values, we see that the minimum is $(308 - 62\sqrt{31})/27 \approx -1.378$ and the maximum is $2/3\sqrt{3} \approx 0.3849$.

4. $f_x = 6x - 8y + 2$ and $f_y = -8x - 8y + 16$; setting these simultaneously equal to 0 gives $x = y = 1$, and $f(1,1) = 9$. Next one analyzes f on the four sides: (i) $f(0,y) = -4y^2 + 16y$; critical point at $y = 2$, and $f(0,2) = 16$. (ii) $f(x,0) = 3x^2 + 2x$; critical point at $x = -\frac{1}{3}$, not in the range of interest. (iii) $f(4,y) = 56 - 16y - 4y^2$; critical point at $y = -2$, not in the range of interest. (iv) $f(x,3) = 3x^2 - 22x + 12$; critical point at $x = \frac{11}{3}$; $f(\frac{11}{3},3) = -\frac{85}{3}$. Finally one checks the corners: $f(0,0) = 0$, $f(0,3) = 12$, $f(4,0) = 56$, and $f(4,3) = -28$. The minimum is $-\frac{85}{3}$ and the maximum is 56.

5. Clearly $|f(x,y)| \geq |(x,y)|^2$, so f has a minimum on $\mathbb{R}^2$ by Theorem 2.83a. We have $f_x = -2b(A - bx - cy) + 2x$ and $f_y = -2c(A - bx - cy) + 2y$. Setting these equal to 0 simultaneously gives $x = Ab/(1 + b^2 + c^2)$ and $y = Ac/(1 + b^2 + c^2)$. Substituting these values into $f(x,y)$ and simplifying gives $f = A^2/(1 + b^2 + c^2)$.

6. Clearly $f(x,y) > 0$ except when $x = y = 0$, so $f(0,0) = 0$ is the absolute minimum. f has an absolute maximum by Theorem 2.83b, namely $f(0,\pm 1) = 2e^{-1}$. (See Exercise 1e in §2.8 for the analysis of the critical points.)

7. As in Exercise 1e in §2.8, one finds that the critical points are $(\pm 1, 0)$ and $(0, \pm 1)$, and $f(\pm 1, 0) = e^{-1}$ and $f(0, \pm 1) = -2e^{-1}$. Since f assumes both positive and negative values and vanishes at infinity, by Theorem 2.83b it has both an absolute maximum and an absolute minimum, which must occur at critical points; hence the maximum is e^{-1} and the minimum is $-2e^{-1}$.

8. If (x,y) is in the first quadrant but outside the "triangle" bounded by the lines $x = 1/3C$ and $y = 1/4C$ and the hyperbola $xy = C$, then $f(x,y) > C$; hence f has a minimum but no maximum by Theorem 2.83a. The only critical point is at $x = (9/4)^{1/3}$, $y = (16/3)^{1/3}$ (see Exercise 1f in §2.8), and the value of f there is $(12)^{1/3}$; this has to be the minimum.

9. Lagrange's method works easily here: one has to solve $2x = 2\lambda x$, $4y = 2\lambda y$, and $6z = 2\lambda z$ subject to the constraint $x^2 + y^2 + z^2 = 1$. The first equation implies that $\lambda = 1$ or $x = 0$. If $\lambda = 1$, the second and third equations imply that $y = z = 0$ and hence $x = \pm 1$. If $x = 0$, the second equation forces $\lambda = 2$ or $y = 0$. If $\lambda = 2$, then $z = 0$, so $y = \pm 1$; if $y = 0$, then $z = \pm 1$. So the constrained critical points are $\pm(1,0,0)$, $\pm(0,1,0)$, and $\pm(0,0,1)$. Clearly the first pair gives the minimum of 1 and the last pair gives the maximum of 3.

10. With $f(a,b) = \sum(y_j - ax_j - b)^2$, we have $f_a = -2\sum x_j(y_j - ax_j - b) = 2[a\sum x_j^2 + bk\overline{x} - \sum x_j y_j]$ and $f_b = -2\sum(y_j - ax_j - b) = 2k[a\overline{x} + b - \overline{y}]$. Solving this pair of linear equations for a and b yields the asserted result.

11. By Lagrange's method, we solve $1 = -2\lambda a x^{-2} = -2\lambda b y^{-2} = -2\lambda c z^{-2}$ subject to $(a/x) + (b/y) + (c/z) = 1$. With $\mu = -2\lambda$, the first set of equations gives $x = \sqrt{\mu a}$, $y = \sqrt{\mu b}$, $z = \sqrt{\mu c}$, and the constraint equation then gives $\sqrt{\mu} = \sqrt{a} + \sqrt{b} + \sqrt{c}$, and hence $x + y + z = (\sqrt{a} + \sqrt{b} + \sqrt{c})^2$. (This is a minimum because $x + y + z \to \infty$ as $(x,y,z) \to \infty$ on the constraint surface.)

12. We wish to minimize $x + y + z$ subject to the constraint $xyz = V$ (and $x, y, z > 0$). By Lagrange's method, we solve $1 = \lambda yz = \lambda xz = \lambda xy$ subject to the constraint $xyz = V$; the solution is obviously $x = y = z = V^{1/3}$, so $x + y + z = 3V^{1/3}$. There is no maximum; $(x,y,z) = (V, c, c^{-1})$ satisfies $xyz = V$ no matter how large c is.

13. Parametrizing the first line by $\mathbf{f}(s) = (1-s, s, 0)$ and the second one by $\mathbf{g}(t) = (t, t, t)$, we wish to minimize $\varphi(s,t) = |\mathbf{f}(s) - \mathbf{g}(t)|^2 = (1-s-t)^2 + (s-t)^2 + t^2 = 2s^2 + 3t^2 - 2s - 2t + 1$. We have $\varphi_s = 4s - 2$ and $\varphi_t = 6t - 2$. so the critical point is $s = \frac{1}{2}$, $t = \frac{1}{3}$. The point on the first line closest to the second one is $\mathbf{f}(\frac{1}{2}) = (\frac{1}{2}, \frac{1}{2}, 0)$ (and the minimum distance is $\sqrt{1/6}$).

14. We wish to minimize $V = xyz$ subject to the constraint $xy + 2xz + 2yz = A$. Solving the constraint equation for z gives $V = xy(A - xy)/(2x + 2y)$. After a little calculation we find $V_x = 2y^2(A - x^2 - 2xy)/(2x+2y)^2$ and $V_y = 2x^2(A - y^2 - 2xy)/(2x+2y)^2$. Setting these equal to 0 gives $x^2 + 2xy = A = y^2 + 2xy$, so that $x = y = \sqrt{A/3}$; hence $z = \frac{1}{2}\sqrt{A/3}$ and $V = \frac{1}{2}(\sqrt{A/3})^3$.

15. We wish to minimize $x^2 + y^2 + z^2$ subject to the constraints $x + z = 4$ and $3x - y = 6$. By Lagrange's method, we solve $2x = \lambda + 3\mu$, $2y = -\mu$, and $2z = \lambda$ to obtain $2x = 2z - 6y$; solving this simultaneously with the constraint equations gives $x = z = 2$, $y = 0$.

16. (a) Lagrange's method gives the equations $2v(xv - yu) = 2\lambda x$, $-2u(xv - yu) = 2\lambda y$, $-2y(xv - yu) = 2\mu u$, $2x(xv - yu) = 2\mu v$. Eliminating λ and μ (and assuming $xv - yu \neq 0$, which is OK since the maximum is clearly positive) we obtain $v/x = -u/y$ or $x/v = -y/u$ (whichever avoids division by zero); either way, the critical points are those (x, y, u, v) such that (u, v) is proportional to $(-y, x)$. By the constraints, the constant of proportionality is $\pm b/a$, and $(xv - yu)^2 = (b/a)^2(x^2 + y^2)^2 = a^2b^2$.

(b) Using the parametrization, $(xv - yu)^2 = a^2b^2(\cos\theta\sin\varphi - \sin\theta\cos\varphi)^2 = a^2b^2\sin^2(\theta - \varphi)$, whose maximum is obviously a^2b^2.

17. The distance from P_1 to Q is $y_1 \sec\theta_1$, and the distance from Q to P_2 is $-y_2\sec\theta_2$ (remember that $y_2 < 0$). Thus the total travel time from P_1 to P_2 is $(y_1/v_1)\sec\theta_1 - (y_2/v_2)\sec\theta_2$. The constraint is that the x-distances have to add up right: $y_1\tan\theta_1 - y_2\tan\theta_2 = |x_2 - x_1|$. Thus, Lagrange's method gives the equations $(y_1/v_1)\sec\theta_1\tan\theta_1 = \lambda y_1\sec^2\theta_1$ and $(y_2/v_2)\sec\theta_2\tan\theta_2 = \lambda y_2\sec^2\theta_2$, whence $\sin\theta_1/v_1 = \lambda = \sin\theta_2/v_2$.

18. Let P_j be the product of the x_i's with the jth term omitted. Lagrange's method gives the equations $P_j = \lambda$ for all j, from which it follows that the x_j's are all equal. Thus the maximum value of $x_1 \cdots x_n$ occurs at $x_1 = \cdots = x_n = c/n$, and that value is $(c/n)^n$. In other words, $(x_1 \cdots x_n)^{1/n}$ is at most c/n when $x_1 + \cdots + x_n = c$, that is, $(x_1 \cdots x_n)^{1/n} \leq (x_1 + \cdots + x_n)/n$, with equality only when the x_j's are all equal.

19. Let $\mathbf{e}_1, \ldots, \mathbf{e}_n$ be an orthonormal eigenbasis for A, with eigenvalues $\lambda_1, \ldots, \lambda_n$. If $\mathbf{x} = \sum t_j\mathbf{e}_j$, we have $|\mathbf{x}|^2 = \sum t_j^2$ and $A\mathbf{x}\cdot\mathbf{x} = \sum \lambda_j t_j^2$; thus we wish to maximize and minimize $\sum \lambda_j t_j^2$ subject to the constraint $\sum t_j^2 = 1$. By an easy extension of the argument in Exercise 9, Lagrange's method shows that the critical points are the unit eigenvectors of A. (Things are a little messier if the eigenvalues are not all distinct.) If $\mathbf{x}$ is a unit eigenvector with eigenvalue λ_j, then $A\mathbf{x}\cdot\mathbf{x} = \lambda_j$, so the maximum and minimum are the largest and smallest of the λ_j's.

2.10 Vector-Valued Functions and Their Derivatives

1. $D\mathbf{f}(x,y,z) = \begin{pmatrix} yz^2 & xz^2 - 8y & 2xyz \\ 3y^2 & 6xy - z & -y \end{pmatrix}$.

2. $D\mathbf{f}(x,y) = \begin{pmatrix} 1 & 2 \\ 3y & 3x \\ 2x & -6y \end{pmatrix}$.

3. (a) $D\mathbf{f}(u,v) = \begin{pmatrix} 2u & -5 \\ 2ve^{2u} & e^{2u} \\ 2 & -2v/(1+v^2) \end{pmatrix}, \quad D\mathbf{f}(0,0) = \begin{pmatrix} 0 & -5 \\ 0 & 1 \\ 2 & 0 \end{pmatrix}$.

(b) $D(\mathbf{f} \circ \mathbf{g})(1,2) = \begin{pmatrix} 0 & -5 \\ 0 & 1 \\ 2 & 0 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} = \begin{pmatrix} -15 & -20 \\ 3 & 4 \\ 2 & 4 \end{pmatrix}$.

4. (a) $D\mathbf{f}(x,y,z) = \begin{pmatrix} 2 & 2(y-1) & -\cos z \\ 3 & 4e^{2y-5z} & -10e^{2y-5z} \end{pmatrix}$.

(b) $D(\mathbf{g} \circ \mathbf{f})(0,0,0) = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} 2 & -2 & -1 \\ 3 & 4 & -10 \end{pmatrix} = \begin{pmatrix} 8 & 6 & -21 \\ 18 & 10 & -43 \end{pmatrix}$.

5. $\mathbf{f}(\mathbf{a}+\mathbf{h}) - \mathbf{f}(\mathbf{a}) = A\mathbf{h}$ for all $\mathbf{a}$ and $\mathbf{h}$, so $A = D\mathbf{f}(\mathbf{a})$ for all $\mathbf{a}$.

6. This is immediate from the definitions.

7. $h = \sum_k f_k g_k$, so $\partial_j h = \sum_k (\partial_j f_k) g_k + \sum_k (\partial_j g_k) f_k$; this is the jth component of $(D\mathbf{f})^* \mathbf{g} + (D\mathbf{g})^* \mathbf{f}$.

8. By (2.86),

$$\begin{pmatrix} \partial_t w & \partial_s w \end{pmatrix} = \begin{pmatrix} \partial_1 f & \partial_2 f & \partial_3 f & \partial_4 f \end{pmatrix} \begin{pmatrix} \partial_t x & \partial_s x \\ \partial_t y & \partial_s y \\ 1 & 0 \\ 0 & 1 \end{pmatrix},$$

which gives the desired result.

9. (a) The unit sphere S is compact, and φ is continuous on it, so φ achieves a maximum M on S at some point $\mathbf{a} \in S$.

(b) Obviously $|A\mathbf{0}| = 0 = M|\mathbf{0}|$. If $\mathbf{x} \neq \mathbf{0}$, let $r = |\mathbf{x}|$ and $\mathbf{u} = \mathbf{x}/r$. Then $|A\mathbf{x}| = r|A\mathbf{u}| \leq rM = M|\mathbf{x}|$, with equality if $\mathbf{x} = \mathbf{a}$. Thus M is the smallest constant such that $|A\mathbf{x}| \leq M|\mathbf{x}|$ for all $\mathbf{x}$, i.e., $M = \|A\|$.

10. (a) If $\mathbf{x} \in \mathbb{R}^n$, we have $|A\mathbf{x}| \leq \sqrt{m}\max_j |(A\mathbf{x})_j| = \sqrt{m}\max_j |\sum_k A_{jk} x_k| \leq \sqrt{m}\max_j \sum_k |A_{jk}|\,|\mathbf{x}|$, so $\|A\| \leq \sqrt{m}\max_j \sum_k |A_{jk}|$.

(b) If $A_{j1} = 1$ and $A_{jk} = 0$ for $k > 1$, the bound for $\|A\|$ in (a) is $\sqrt{m}$. Also, $A\mathbf{x} = (x_1, x_1, \ldots, x_1)$ for any $\mathbf{x}$; in particular, if $\mathbf{x} = (1, 0, \ldots, 0)$, then $|\mathbf{x}| = 1$ and $|A\mathbf{x}| = |(1, 1, \ldots, 1)| = \sqrt{m}$, so $\|A\| \geq \sqrt{m}$.

Chapter 3

The Implicit Function Theorem and its Applications

3.1 The Implicit Function Theorem

1. With $F(x,y,z) = x^2 - 4x + 2y^2 - yz$ we have $F_x = 2x - 4$, $F_y = 4y - z$, $F_z = -y$, so $\nabla F(2,-1,3) = (0,-7,1)$. Hence the equation $F = 1$ can be solved locally for y or z but not x. Explicitly, $z = (x^2 - 4x + 2y^2 - 1)/y$ and $y = (z - \sqrt{z^2 + 8(1 - x^2 + 4x)})/4$; but $x = 2 \pm \sqrt{5 - 2y^2 + yz}$, and the square root vanishes at $(y,z) = (-1,3)$, so there are two values of x for some nearby values of y and z and none for others.

2. With $F(x,y) = x^2 + 2xy + 3y^2$ we have $F_x = 2x + 2y$ and $F_y = 2x + 6y$, so at least one of F_x and F_y is nonzero when $(x,y) \neq (0,0)$, which is the case when $F(x,y) = c > 0$. If $c = 0$ then $F_x = F_y = 0$ and the set where $F = c$ is a single point, and if $c < 0$ then the set where $F = c$ is empty. (Clearly $F(x,y) = (x+y)^2 + 2y^2 > 0$ when $(x,y) \neq (0,0)$.)

3. With $u = (x^2 + y^2 + 2z^2)^{1/2}$ and $F(x,y,z) = u - \cos z$, we have $F_y = y/u$ and $F_z = 2z/u + \sin z$, so $F_y(0,1,0) = 1$ and $F_z(0,1,0) = 0$. Hence the equation can be solved for y but not z.

4. The x and y intercepts of the graph are at $(1,0)$ and $(0,-e^{1/3})$; the tangent line is vertical at the former point. Writing $y = f(x) = (x - e^{1-x})^{1/3}$, we see that the graph is asymptotic to the curve $y = x^{1/3}$ as $x \to +\infty$ and asymptotic to the curve $y = -e^{(1-x)/3}$ as $x \to -\infty$; hence f maps $\mathbb{R}$ onto $\mathbb{R}$. Also, $(d/dx)(x - e^{1-x}) = 1 + e^{1-x} > 1$, so f is one-to-one (in fact, strictly increasing). Hence $f^{-1} : \mathbb{R} \to \mathbb{R}$ exists.

5. With $G(x,y) = F(F(x,y),y)$, we have $G_y = F_1(F(x,y),y)F_2(x,y) + F_2(F(x,y),y)$, so $G_y(0,0) = F_2(0,0)[F_1(0,0) + 1] \neq 0$ when $F_2(0,0) \neq 0$ and $F_1(0,0) \neq -1$.

6. With $(u,v) = \mathbf{F}(x,y,z) = (xy + 2yz - 3xz,\ xyz + x - y)$, we have $D\mathbf{F} = \begin{pmatrix} y - 3z & x + 2z & 2y - 3x \\ yz + 1 & xz - 1 & xy \end{pmatrix}$. At $(1,1,1)$, then, we have $D\mathbf{F} = \begin{pmatrix} -2 & 3 & -1 \\ 2 & 0 & 1 \end{pmatrix}$; $\partial(u,v)/\partial(x,y) = -6$, $\partial(u,v)/\partial(x,z) = 0$, $\partial(u,v)/\partial(y,z) = 3$. So the equations can be solved for x and y or for y and z.

7. With $(z,w) = \mathbf{F}(x,y,u,v) = (u^3+xv-y,\ v^3+yu-x)$, we have $D\mathbf{F} = \begin{pmatrix} v & -1 & 3u^2 & x \\ -1 & u & y & 3v^2 \end{pmatrix}$. At $(0,1,1,-1)$, then, we have $D\mathbf{F} = \begin{pmatrix} -1 & -1 & 3 & 0 \\ -1 & 1 & 1 & 3 \end{pmatrix}$. The determinants of all the 2×2 submatrices of this matrix are nonzero, so the equations can be solved for any pair of variables.

8. With $\begin{pmatrix} s \\ t \end{pmatrix} = \begin{pmatrix} xy^2+xzu+yv^2 \\ u^3yz+2xv-u^2v^2 \end{pmatrix}$, we have $\dfrac{\partial(s,t)}{\partial(u,v)} = \det\begin{pmatrix} xz & 2yv \\ 3u^2z-2uv^2 & -2u^2v+2x \end{pmatrix}$, which equals -2 when all variables are set equal to 1. Hence the equations can be solved for u and v.

9. With $u = x^2+y^2+z^2$, $v = xy+tz$, and $w = xz+ty+e^t$, we have $\dfrac{\partial(u,v,w)}{\partial(x,y,z)} = \det\begin{pmatrix} 2x & 2y & 2z \\ y & x & t \\ z & t & x \end{pmatrix}$, which equals 8 at $(x,y,z,t) = (-1,-2,1,0)$. So the equations can be solved for x,y,z.

3.2 Curves in the Plane

1. (a) S is a smooth curve (an ellipse); $\nabla F = \mathbf{0}$ only at $(0,0) \notin S$; $y = f(x)$ near any point except $(\pm\sqrt{3},0)$; $x = f(y)$ near any point except $(0,\pm1)$

(b) S is the union of two smooth curves (a hyperbola); $\nabla F = \mathbf{0}$ only at $(0,0) \notin S$; $y = f(x)$ near any point except $(\pm\sqrt{3},0)$; $x = f(y)$ near any point.

(c) S is a smooth curve (one branch of a hyperbola); ∇F never vanishes; $y = f(x)$ except near $(\sqrt{3},0)$; $x = f(y)$ globally.

(d) S is the union of the three lines $x = 0$, $y = 0$, and $x+y = 1$; $\nabla F = \mathbf{0}$ at the three points where these lines intersect; elsewhere, $y = f(x)$ near all points of the lines $y = 0$ and $x+y = 1$ and $x = f(y)$ near all points of the lines $x = 0$ and $x+y = 1$.

(e) S is the union of $\{(0,0)\}$ and the parabola $y = x^2+1$; $\nabla F = \mathbf{0}$ at $(0,0)$ but not on the parabola; $y = f(x)$ on the parabola; $x = f(y)$ near any point of the parabola except $(0,1)$.

(f) S is the parabola $y = x^2$; $\nabla F(0,0) = \mathbf{0}$ but S is still a smooth curve there; $y = f(x)$ globally; $x = f(y)$ near any point except $(0,0)$.

(g) S is the discrete set $\{(0,\ (2n+\frac{1}{2})\pi) : n \in \mathbb{Z}\}$; $\nabla F = \mathbf{0}$ at each point of S.

2. (a) $\nabla(x^p+y^p) = p(x^{p-1},y^{p-1}) \neq (0,0)$ except at $(x,y) = (0,0) \notin S_p$, so S is locally a smooth curve. S_p is also connected; see part (b).

(b) If $p = 1$, S_p is a straight line. If $p = 2$, S_p is the unit circle. If $p > 1$ is odd, S_p is the graph of $y = (1-x^p)^{1/p}$, a curve that is asymptotic to the line $y = -x$ as $x \to \pm\infty$ but has a "bump" in the middle where it goes through the first quadrant between $(0,1)$ and $(1,0)$. If $p > 2$ is even, S_p is a simple closed curve intermediate between the unit circle and the square with vertices at $x = \pm1$, $y = \pm1$.

(c) If p is even, the top and bottom halves of S_p are graphs of $y = \pm(1-x^p)^{1/p}$, a continuous function on $[-1,1]$ that is differentiable except at the endpoints; likewise with x and y switched. If p is odd, S_p is the graph of $y = (1-x^p)^{1/p}$, a continuous function that is differentiable except at $x = 1$; likewise with x and y switched.

3. (a) S is the parabola $x = (y-1)^2-1$.

(b) S is the half-line $y = x+2$, $x \geq -1$; the endpoint $(-1,1)$ is $\mathbf{f}(0)$, and $\mathbf{f}'(0) = 0$.

(c) S is the line $y = x + 2$; $\mathbf{f}'(0) = 0$ but S is still smooth at $\mathbf{f}(0) = (-1, 1)$.

(d) S is the astroid $x^{2/3} + y^{2/3} = 1$, a simple closed curve with cusps at $\pm(1, 0)$ and $\pm(0, 1)$, which are the points $\mathbf{f}(t)$ where $\mathbf{f}'(t) = \mathbf{0}$ ($t = \frac{1}{2}n\pi$, $n \in \mathbb{Z}$).

(e) S is a limaçon; a reasonable sketch can be obtained by drawing a smooth curve from $(2, 0)$ to $(0, \sqrt{3})$ to $(-1, 0)$ to $(0, 0)$ to $(-1, 0)$ to $(0, -\sqrt{3})$ to $(2, 0)$ (corresponding to $t = n\pi/3$ for $n = 0, \ldots, 6$). It is a smooth curve except at $(-1, 0)$, where it has a self-intersection; $\mathbf{f}'(t)$ never vanishes.

4. (a) The left-hand and right-hand derivatives of φ at $s = 0$ are both 0.

(b) Since $\sin^2 t + \cos^2 t = 1$, as t traverses the intervals $[0, \frac{1}{2}\pi]$, $[\frac{1}{2}\pi, \pi]$, $[\pi, \frac{3}{2}\pi]$, and $[\frac{3}{2}\pi, 2\pi]$, $\mathbf{f}(t)$ traces out the line segments from $(1, 0)$ to $(0, 1)$ to $(-1, 0)$ to $(0, -1)$ to $(1, 0)$. Since $(d/dt)\varphi(\cos t) = \pm 2\cos t \sin t$ and $(d/dt)\varphi(\sin t) = \pm 2\sin t \cos t$, $\mathbf{f}'(t) = 0$ when $t = \frac{1}{2}n\pi$ for $n \in \mathbb{Z}$; the points $\mathbf{f}(\frac{1}{2}n\pi)$ are the corners of the square.

5. (a) One sees that $y^2(1 - x) = x^2(1 + x)$ on S by substituting in $x = (t^2 - 1)/(t^2 + 1)$ and $y = t(t^2 - 1)/(t^2 + 1)$ and simplifying. Conversely, if (x, y) satisfies this equation, then $(x, y) = \mathbf{f}(t)$ where $t = y/x$ ($t = \pm 1$ if $(x, y) = (0, 0)$), because $x = (y^2 - x^2)/(y^2 + x^2) = ((y/x)^2 - 1)/((y/x)^2 + 1) = (t^2 - 1)/(t^2 + 1)$ and $y = (y/x)x = tx$.

(b) For $|t|$ very large, $(x, y) \approx (1, t)$, so S is asymptotic to $x = 1$. As t goes from $-\infty$ to -1 to 1 to ∞, $\mathbf{f}(t)$ goes from $(1, -\infty)$ to $(0, 0)$, makes a loop through $(-1, 0)$ and back to $(0, 0)$, and goes from $(0, 0)$ to $(1, \infty)$.

(c) We have $(0, 0) = \mathbf{f}(\pm 1)$. The curves described by $\mathbf{f}(t)$ for t near -1 or 1 are both smooth; they are tangent to the lines with slope ± 1 at the origin. One can see this from the nonparametric representation too: if x is small, then $1 \pm x \approx 1$, so the equation $y^2(1 - x) = x^2(1 + x)$ gives $y^2 \approx x^2$, or $y \approx \pm x$.

6. (a) This is obvious: $F_3 = 0$ if and only if either $F_1 = 0$ or $F_2 = 0$.

(b) $\nabla F_3 = F_1 \nabla F_2 + F_2 \nabla F_1 = \mathbf{0}$ when $F_1 = F_2 = 0$.

3.3 Surfaces and Curves in Space

1. (a) S is the plane $x + y = z$; $\partial_u \mathbf{f}$ and $\partial_v \mathbf{f}$ everywhere independent.

(b) S is the cone $(x/a)^2 + (y/b)^2 = z^2$; $\partial_v \mathbf{f} = \mathbf{0}$ at the origin (the vertex of the cone).

(c) S is the one-sheeted hyperboloid $x^2 + y^2 - z^2 = 1$; $\partial_u \mathbf{f}$ and $\partial_v \mathbf{f}$ everywhere independent.

(d) S is the paraboloid $z = x^2 + y^2$; $\partial_v \mathbf{f} = \mathbf{0}$ when $u = 0$, but the surface is nonsingular.

2. (a) With $\mathbf{f}(u, v) = (e^{u-v},\ u - 3v,\ \frac{1}{2}(u^2 + v^2))$ we have $(1, -2, 1) = \mathbf{f}(1, 1)$, $\partial_u \mathbf{f}(1, 1) = (1, 1, 1)$, and $\partial_v \mathbf{f}(1, 1) = (-1, -3, 1)$. The cross product of the latter vectors is $(4, -2, -2)$, so the tangent plane is $4(x - 1) - 2(y + 2) - 2(z - 1) = 0$ or $2x - y - z = 3$.

(b) With $\mathbf{f}(u, v) = ((u + v)^{-1},\ -u - e^v,\ u^3)$, we have $(1, -2, 1) = \mathbf{f}(1, 0)$, $\partial_u \mathbf{f}(1, 0) = (-1, -1, 3)$, and $\partial_v \mathbf{f}(1, 0) = (-1, -1, 0)$. The cross product of the latter vectors is $(3, -3, 0)$, so the tangent plane is $3(x - 1) - 3(y + 2) = 0$ or $x - y = 3$.

3. (a) Using polar coordinates in the xy-plane, $\mathbf{f}(u, v) = (u\cos v,\ u\sin v,\ f(u))$.

(b) Using polar coordinates in the yz-plane, $\mathbf{f}(u, v) = (u,\ f(v)\cos\theta,\ f(v)\sin\theta)$.

(c) $\mathbf{f}(x,y) = (x, y, -\sqrt{1+2x^2+y^2})$ and $\mathbf{g}(u,v) = (\sinh u\cos(v/\sqrt{2}),\ \sinh u\sin v,\ -\cosh u)$ are two of the possibilities.

(d) $\mathbf{f}(u,v) = (3\cos u,\ v,\ 3\sin u)$.

4. (a) The cross product of the normal vectors $(1,-2,1)$ and $(2,-1,-1)$ is $3(1,1,1)$, and one point in the intersection is $(1,\frac{1}{3},\frac{8}{3})$ (set $x=1$ and solve for y and z), so $\mathbf{f}(t) = (1,\frac{1}{3},\frac{8}{3}) + t(1,1,1)$ works.

(b) The cross product of the normal vectors $(1,2,0)$ and $(0,1,-3)$ is $(-6,3,1)$, and one point in the intersection is $(3,0,-\frac{2}{3})$ (set $y=0$ and solve for x and z), so $\mathbf{f}(t) = (3,0,-\frac{2}{3}) + t(-6,3,1)$ works.

5. (a) Substituting $z = 1-x$ in the equation of the sphere gives $2x^2-2x+y^2 = 0$ or $(x-\frac{1}{2})^2+\frac{1}{2}y^2 = \frac{1}{4}$. Parametrize this ellipse in the usual way: $x = \frac{1}{2}+\frac{1}{2}\cos t$, $y = \frac{1}{\sqrt{2}}\sin t$; then $z = 1-x = \frac{1}{2}-\frac{1}{2}\cos t$.

(b) $(\frac{1}{2},-\frac{1}{\sqrt{2}},\frac{1}{2})$ corresponds to $t = -\frac{1}{2}\pi$; we have $(x,y,z)'(t) = (-\frac{1}{2}\sin t, \frac{1}{\sqrt{2}}\cos t, \frac{1}{2}\sin t)$, which equals $(\frac{1}{2},0,-\frac{1}{2})$ at $t=-\frac{1}{2}\pi$; so $\mathbf{f}(t) = (\frac{1}{2},-\frac{1}{\sqrt{2}},\frac{1}{2}) + t(\frac{1}{2},0,-\frac{1}{2})$ works.

6. Perhaps the best way to nail this is to perform a rotation around the y-axis to make the plane horizontal. Namely, let $c = \sqrt{1+a^2}$ and

$$\begin{pmatrix} u \\ v \end{pmatrix} = \frac{1}{c}\begin{pmatrix} 1 & a \\ -a & 1 \end{pmatrix}\begin{pmatrix} x \\ z \end{pmatrix} = \frac{1}{c}\begin{pmatrix} x+az \\ -ax+z \end{pmatrix}.$$

(The matrix is orthogonal since its columns are orthonormal, so this transformation preserves shapes.) In these coordinates the plane is $v = 1/c$ and the cone is $(au+v)^2 = (u-av)^2 + c^2y^2$ or $c^2y^2 + (1-a^2)u^2 - 4auv = (1-a^2)v^2$. Thus the intersection is the curve $c^2y^2+(1-a^2)u^2-4au/c = (1-a^2)/c^2$ in the plane $v = 1/c$ with coordinates (y,u). This is clearly a circle if $a=0$, an ellipse if $0<|a|<1$, a parabola if $|a| = 1$, and a hyperbola if $|a| > 1$. Parametrizations may be obtained in the form $y = \alpha\sin t$, $u = \beta\cos t+\gamma$ in the elliptic case and $y = \alpha\sinh t$, $u = \beta\cosh t+\gamma$ in the hyperbolic case (for suitable α,β,γ), and $y = t$, $u = \pm\frac{1}{\sqrt{2}}t^2$ in the parabolic case.

7. The statement is as follows, and the proof is as outlined in the text on p. 130. (a) Let F and G be real-valued functions of class C^1 on an open set in $\mathbb{R}^3$, and let $S = \{\mathbf{x} : F(\mathbf{x}) = G(\mathbf{x}) = 0\}$. If $\mathbf{a}\in S$ and $\nabla F(\mathbf{a})$ and $\nabla G(\mathbf{a})$ are linearly independent, there is a neighborhood N of $\mathbf{a}$ in $\mathbb{R}^3$ such that $S\cap N$ is the graph of a C^1 function from some interval in $\mathbb{R}$ into $\mathbb{R}^2$ ($(y,z) = \mathbf{g}(x)$, or similarly with the variables permuted). (b) Let $\mathbf{f} : (a,b)\to\mathbb{R}^3$ be of class C^1. If $\mathbf{f}'(t_0)\neq\mathbf{0}$, there is an open interval I containing t_0 such that the set $\{\mathbf{f}(t) : t\in I\}$ is the graph of a C^1 function as in part (a).

3.4 Transformations and Coordinate Systems

1. (a) x constant: circles centered at the origin. y constant: half-lines starting at the origin. (See answer in back of text for the inverse and Jacobian.)

(b) $\det D\mathbf{f}\equiv 2$. x constant: the lines $u = c$ $(c>0)$. y constant: the curves $u = c/v^2$ $(c>0)$. $\mathbf{f}^{-1}(u,v) = (\sqrt{u},\ v\sqrt{u})$.

(c) $\det D\mathbf{f}\equiv 0$. The range of $\mathbf{f}$ is the parabola $u = \frac{1}{4}v^2$, and each line $x = c$ or $y = c$ maps onto this parabola.

2. (a) $x = -\frac{1}{3}(u-2v)$, $y = -\frac{1}{3}(2u-v)$ (i.e., $\mathbf{f}^{-1} = -\frac{1}{3}\mathbf{f}$).

(b) Substituting the formulas for x and y into the equations for the lines gives $v = -u$, $v = u$, and $5v - 4u = 3$; the image is the triangle bounded by these lines.

(c) $(0,0) = \mathbf{f}(0,0)$, $(-1,2) = \mathbf{f}(\frac{5}{3}, \frac{4}{3})$, and $(2,1) = \mathbf{f}(0,-1)$; the region is the triangle with these vertices.

3. (a) If $\sin x = a$ and $\cos x = b$, we have $(u/a)^2 - (v/b)^2 = 1$. If $\cosh y = \alpha$ and $\sinh y = \beta$, we have $(u/\alpha)^2 + (v/\beta)^2 = 1$.

(b) $\dfrac{\partial(u,v)}{\partial(x,y)} = \det\begin{pmatrix} \cos x\cosh y & \sin x \sinh y \\ -\sin x \sinh y & \cos x \cosh y\end{pmatrix} = \cos^2 x \cosh^2 y + \sin^2 x \sinh^2 y = \cos^2 x(1 + \sinh^2 y) + \sin^2 x \sinh^2 y = \cos^2 x + \sinh^2 y$, which vanishes when $\cos x = \sinh y = 0$, i.e., when $y = 0$ and $x = (n + \frac{1}{2})\pi$; for these values, $u = \pm 1$ and $v = 0$.

(c) The foci of the hyperbola $(u/a)^2 - (v/b)^2 = 1$ are at $(\pm c, 0)$ where $c^2 = a^2 + b^2$; with a, b as in (a) we have $c = 1$. When $|a| \geq |b|$, the foci of the ellipse $(u/\alpha)^2 + (v/\beta)^2 = 1$ are at $(\pm\gamma, 0)$ where $\gamma^2 = \alpha^2 - \beta^2$; with α, β as in (a) we have $\gamma = 1$.

4. (a) The lines $x - y = 0$ and $x - y = 1$ meet the hyperbolas $xy = 1$ and $xy = 4$ in the first quadrant and again in the third quadrant.

(b) $D\mathbf{f} = \begin{pmatrix} 1 & -1 \\ y & x \end{pmatrix}$, $J = x + y$.

(c) The lines $x - y$ = constant are tangent to the hyperbolas xy = constant along the line $y = -x$.

(d) We have $x = y + u$, so $v = (y+u)y$ or $y^2 + uy - v = 0$; hence $y = \frac{1}{2}(-u - \sqrt{u^2 + 4v})$, where the minus sign is necessary to make $y = -3$ when $(u,v) = (5,-6)$, and then $x = y + u = \frac{1}{2}(u - \sqrt{u^2+4v})$. With $w = 1/\sqrt{u^2 + 4v}$, we have $D\mathbf{g} = \frac{1}{2}\begin{pmatrix} 1 - uw & -2w \\ -1 - uw & -2w \end{pmatrix}$.

(e) $D\mathbf{f}(2,-3)D\mathbf{g}(5,-6) = \begin{pmatrix} 1 & -1 \\ -3 & 2 \end{pmatrix}\begin{pmatrix} -2 & -1 \\ -3 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

5. Let $(u,v) = \mathbf{f}(x,y) = (y/x^2, xy)$. Then $D\mathbf{f} = \begin{pmatrix} -2y/x^3 & 1/x^2 \\ y & x \end{pmatrix}$ and $\det D\mathbf{f} = -3y/x^2$. Also $y = ux^2$, so $x = v/y = v/ux^2$, whence $x = (v/u)^{1/3}$ and $y = ux^2 = (uv^2)^{1/3}$.

6. (a) $r = c$: the sphere of radius c about 0. $\varphi = c$: the (half) cone $z = (\cot c)\sqrt{x^2 + y^2}$ (the positive or negative z-axis if $c = 0$ or $c = \pi$, the plane $z = 0$ if $c = \frac{1}{2}\pi$). $\theta = c$: the vertical half-plane corresponding to the ray $\theta = c$ (polar coordinates) in the xy-plane.

(b) $D\mathbf{f} = \begin{pmatrix} \sin\varphi\cos\theta & r\cos\varphi\cos\theta & -r\sin\varphi\sin\theta \\ \sin\varphi\sin\theta & r\cos\varphi\sin\theta & r\sin\varphi\cos\theta \\ \cos\varphi & -r\sin\varphi & 0 \end{pmatrix}$, so one easily computes that $\det D\mathbf{f} = r^2\sin\varphi(\cos^2\varphi + \sin^2\varphi)(\cos^2\theta + \sin^2\theta) = r^2\sin\varphi$.

(c) $r_0 \neq 0$ and $\varphi_0 \neq 0$ or π; in other words, $(x_0, y_0) \neq (0,0)$.

7. With $\mathbf{G}(\mathbf{x},\mathbf{y}) = (\mathbf{x}, \mathbf{F}(\mathbf{x},\mathbf{y}))$, we have $D\mathbf{G} = \begin{pmatrix} I & 0 \\ D_{\mathbf{x}}\mathbf{F} & D_{\mathbf{y}}\mathbf{F} \end{pmatrix}$ and hence $\det D\mathbf{G} = \det D_{\mathbf{y}}\mathbf{F}$. If the latter determinant is nonzero at $(\mathbf{a}, \mathbf{b})$, where $\mathbf{F}(\mathbf{a},\mathbf{b}) = \mathbf{0}$, then $\mathbf{G}$ is locally invertible there. Because of the form of $\mathbf{G}$, $\mathbf{G}^{-1}$ has the form $\mathbf{G}^{-1}(\mathbf{u},\mathbf{v}) = (\mathbf{u}, \mathbf{g}(\mathbf{u},\mathbf{v}))$ where $\mathbf{g} : \mathbb{R}^{n+k} \to \mathbb{R}^k$ is C^1. For $\mathbf{x}$ near $\mathbf{a}$, let $\mathbf{f}(\mathbf{x}) = \mathbf{g}(\mathbf{x}, \mathbf{0})$. Then $\mathbf{G}(\mathbf{x}, \mathbf{f}(\mathbf{x})) = (\mathbf{x}, \mathbf{0})$, that is, $\mathbf{F}(\mathbf{x}, \mathbf{f}(\mathbf{x})) = \mathbf{0}$, so $\mathbf{f}$ solves the implicit function problem. Uniqueness follows from the uniqueness of $\mathbf{G}^{-1}$.

3.5 Functional Dependence

1. (a) $D\mathbf{f} = \begin{pmatrix} 1 & 1 & -1 \\ 1 & -1 & 1 \\ 2x & 2y-2z & -2y+2z \end{pmatrix}$ has rank 2 everywhere (the first two rows are independent; the third is $x+2y$ times the first plus $x+2z$ times the second). $h = \frac{1}{4}(f+g)^2 + \frac{1}{4}(f-g)^2$.

(b) $D\mathbf{f} = \begin{pmatrix} 2x & 2y & 2z \\ 1 & 1 & 1 \\ 0 & 1 & -1 \end{pmatrix}$ is nonsingular except on the plane $y+z = 2x$.

(c) $D\mathbf{f} = \begin{pmatrix} y^{1/2}\cos x & \frac{1}{2}y^{-1/2}\sin x & 0 \\ -2y\cos x \sin x & \cos^2 x - 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ is defined for $y > 0$ and has rank 2 there (the second row is $-2y^{1/2}\sin x$ times the first). $g = -f^2$.

(d) $D\mathbf{f} = \begin{pmatrix} y & x & 1 \\ 2(xy+z)y & 2(xy+z)x & 2(xy+z) \\ -y & -x & -1 \end{pmatrix}$ has rank 1 everywhere (the second and third rows are $2(xy+z)$ and -1 times the first). $h = 2 - f$ and $g = f^2$.

(e) $D\mathbf{f} = \begin{pmatrix} x^{-1} & -y^{-1} & 1 \\ x^{-1} & -y^{-1} & -1 \\ y^{-1} - 2yx^{-2} & 2x^{-1} - xy^{-2} & 0 \end{pmatrix}$ has rank 2 on the set $x, y > 0$ where $\mathbf{f}$ is defined (the first two rows are independent; the third is $(x^2 - 2y^2)/2xy$ times their sum). $h = e^{(f+g)/2} + 2e^{-(f+g)/2}$.

(f) $D\mathbf{f} = \begin{pmatrix} 1 & -1 & 1 \\ 2x & -2y & 0 \\ 1 & 0 & 1 \end{pmatrix}$ is nonsingular except on the plane $x = 0$.

2. (a) THEOREM. *Let $\mathbf{f} = (f, g)$ be a C^1 map from a connected open set $U \subset \mathbb{R}^2$ into $\mathbb{R}^2$. Suppose that $D\mathbf{f}(\mathbf{x})$ has rank 1 at every $\mathbf{x} \in U$. Then every $\mathbf{x}_0 \in U$ has a neighborhood N such that f and g are functionally dependent on N and $\mathbf{f}(N)$ is a smooth curve in $\mathbb{R}^2$.* PROOF: Let $u = f(x, y)$ and $v = g(x, y)$. Since $D\mathbf{f}(\mathbf{x}_0)$ has rank 1, it has at least one nonzero entry, which we may take to be $\partial_x f(\mathbf{x}_0)$. By the implicit function theorem, the equation $u = f(x, y)$ can be solved near $\mathbf{x} = \mathbf{x}_0$ and $u = u_0 = f(\mathbf{x}_0)$ to yield x as a function of u and y. Then $v = g(x, y)$ becomes a function of u and y too. Implicit differentiation of the equations $u = f(x, y)$ and $v = g(x, y)$ with respect to y (taking u as the other independent variable) gives $0 = (\partial_x f)(\partial_y x) + (\partial_y f)$ and $\partial_y v = (\partial_x g)(\partial_y x) + (\partial_y g)$. Solving the first of these for $\partial_y x$ and substituting in the second gives $\partial_y v = (\partial_x f)^{-1} \det D\mathbf{f} = 0$, so v is a (smooth) function of u alone, say $v = \varphi(u)$. Hence $g(x, y) = \varphi(f(x, y))$, and the range of $\mathbf{f}$ near $\mathbf{x}_0$ is the smooth curve $v = \varphi(u)$.

(b) THEOREM. *Let $\mathbf{f} = (f, g, h)$ be a C^1 map from a connected open set $U \subset \mathbb{R}^2$ into $\mathbb{R}^3$. Suppose that $D\mathbf{f}(\mathbf{x})$ has rank 1 at every $\mathbf{x} \in U$. Then every $\mathbf{x}_0$ in U has a neighborhood N such that f, g, and h are functionally dependent on N and $\mathbf{f}(N)$ is a smooth curve in $\mathbb{R}^3$.* PROOF: The proof is similar to part (a), so we just sketch it. Let $u = f(x, y)$, $v = g(x, y)$, $w = h(x, y)$. We may assume $\partial_x f(\mathbf{x}_0) \neq 0$. Then we can solve the equation $u = f(x, y)$ for x as a function of u and y, making v and w into functions of u and y. One calculates that $\partial_y v = (\partial_x f)^{-1}[\partial(f, g)/\partial(x, y)]$ and $\partial_y w = (\partial_x f)^{-1}[\partial(f, h)/\partial(x, y)]$, which both vanish since $D\mathbf{f}$ has rank 1. Thus $v = \varphi(u)$ and $w = \psi(u)$ (which describes a smooth curve), and $g = \varphi(f)$ and $h = \psi(f)$.

Chapter 4

Integral Calculus

4.1 Integration on the Line

1. The sup and inf of f on any nontrivial interval are 1 and 0, respectively. It follows that for any partition P of any interval $[a,b]$ we have $S_P f = b - a$ and $s_P f = 0$, so $\overline{I}_a^b(f) = b - a$ and $\underline{I}_a^b(f) = 0$.

2. If $c \geq 0$, for any interval I we have $\sup_I(cf) = c\sup_I(f)$ and $\inf_I(cf) = c\inf_I(f)$; it follows that $S_P(cf) = cS_P(f)$ and $s_P(cf) = cs_P(f)$ for any partition P of $[a,b]$ and hence that $\int_a^b cf = c\int_a^b f$. If $c < 0$, the orders are reversed ($\sup_I(cf) = c\inf_I(f)$, etc.) but the final result is the same.

3. We use Lemma 4.5. Given $\epsilon > 0$, let P be a partition of $[a,b]$ such that $S_P f - s_P f < \epsilon$. Let P' be the partition obtained by adding the subdivision points c and d to P (if they are not already there). By Lemma 4.3, $S_{P'} f - s_{P'} f < \epsilon$. Let Q be the partition of $[c,d]$ obtained from P' by omitting the subintervals belonging to $[a,c]$ or $[d,b]$; then $S_Q f - s_Q f < \epsilon$. By Lemma 4.5, f is integrable on $[c,d]$.

4. If $f \leq g$ then $s_P f \leq s_P g$ and $S_P f \leq S_P g$ for any partition P of $[a,b]$; it follows that $\int_a^b f \leq \int_a^b g$.

5. Given an interval $I \subset [a,b]$, let M and m be the sup and inf of f on I, and let L and l be the sup and inf of $|f|$ on I. If $f \geq 0$ on I then $L = M$ and $l = m$, and if $f \leq 0$ on I then $L = -m$ and $l = -M$; in either case, $L - l = M - m$. If f changes sign on I, then $L = \max(M, |m|)$ and $l \geq 0$, so $L - l \leq L < M + |m| = M - m$. It follows that for any partition P of $[a,b]$ we have $S_P|f| - s_P|f| \leq S_P f - s_P f$, and it follows from Lemma 4.5 that $|f|$ is integrable. Moreover $\pm f \leq |f|$, so $\pm\int_a^b f \leq \int_a^b |f|$, hence $|\int_a^b f| \leq \int_a^b |f|$.

6. Let $l = \lim x_n$. Given $\epsilon > 0$, the interval I_0 of length $\frac{1}{2}\epsilon$ centered at l contains x_n for n sufficiently large, say $n > N$. For $j = 1, \ldots, N$, let I_j be the interval of length $\epsilon/2N$ centered at x_j. Then every x_j is contained in $\bigcup_0^N I_j$, and the sum of the lengths of the I_j's is ϵ.

7. If $A = f(x_0) > 0$ and f is continuous at x_0, there is an interval $[c,d] \subset [a,b]$ containing x_0 such that $f(x) \geq \frac{1}{2}A$ for $x \in [c,d]$. By Theorem 4.9c, $\int_a^b f = \int_a^c f + \int_c^d f + \int_d^b f \geq \int_c^d f \geq \frac{1}{2}A(d-c) > 0$.

8. (a) If $P = \{x_0, \ldots, x_J\}$, let $P' = \{x_0/c, \ldots, x_J/c\}$; then $P \longleftrightarrow P'$ is a one-to-one correspondence between partitions of $[a,b]$ and partitions of $[a/c, b/c]$. The sup and inf of $f(x)$ on $[x_{j-1}, x_j]$ are equal to the sup and inf of $f(cx)$ on $[x_{j-1}/c, x_j/c]$, and the length of the former interval is c times the length of the latter. It follows that $S_P f = cS_{P'} f(c\,\cdot\,)$ and $s_P f = cs_{P'} f(c\,\cdot\,)$; the result follows.

(b) and (c) are similar; one considers $P' = \{-x_J, \ldots, -x_0\}$ or $P' = \{x_0 + c, \ldots, x_J + c\}$.

9. g and h are uniformly continuous on $[a, b]$, and $K = g([a, b]) \times h([a, b])$ is a compact set in the plane, so f is uniformly continuous on it. Thus, given $\epsilon > 0$, we can choose $\eta > 0$ such that $|f(u, v) - f(u', v')| < \epsilon/2(b - a)$ whenever (u, v) and (u', v') are points in K with $|u - u'| < \eta$ and $|v - v'| < \eta$, and we can choose $\delta_0 > 0$ small enough so that $|g(x) - g(x')| < \eta$ and $|h(x) - h(x')| < \eta$ whenever $x, x' \in [a, b]$ and $|x - x'| < \delta_0$. Next, let $\varphi(x) = f(g(x), h(x))$. By Proposition 4.16 we can choose $\delta > 0$ small enough so that $\delta \le \delta_0$ and $\int_a^b \varphi(x)\,dx - s_P\varphi < \frac{1}{2}\epsilon$ for any partition $P = \{x_j\}_0^J$ of $[a, b]$ with $\max(x_j - x_{j-1}) < \delta$. Since φ is continuous, its infimum on $[x_{j-1}, x_j]$ is achieved at some point ξ_j, so $s_P\varphi = \sum \varphi(\xi_j)(x_j - x_{j-1})$. Then for any $x_j', x_j'' \in [x_{j-1}, x_j]$ we have

$$\left|\int_a^b f(g(x), h(x))\,dx - \sum f(g(x_j'), h(x_j''))(x_j - x_{j-1})\right|$$
$$\le \left|\int_1^b \varphi(x)\,dx - \sum \varphi(\xi_j)(x_j - x_{j-1})\right| + \sum |\varphi(\xi_j) - f(g(x_j'), h(x_j''))|(x_j - x_{j-1})$$
$$\le \frac{\epsilon}{2} + \sum \frac{\epsilon}{2(b-a)}(x_j - x_{j-1}) = \epsilon.$$

4.2 Integration in Higher Dimensions

1. (a) If $Z \subset \bigcup_1^M R_m$ where $\sum A(R_m) < \epsilon$, the same is true of U.

(b) Given $\epsilon > 0$, choose a finite collection of rectangles R_{jm} ($j = 1, \ldots, k$ and $m = 1, \ldots M_j$) such that $Z_j \subset \bigcup_{m=1}^{M_j} R_{jm}$ and $\sum_m A(R_{jm}) < \epsilon/k$ for each j. Then $\bigcup_1^k Z_j \subset \bigcup_{j,m} R_{jm}$ and $\sum_{j,m} A(R_{jm}) < \epsilon$.

2. (a) Given a partition $P = \{x_j\}_0^J$ of $[a, b]$, let m_j and M_j be the inf and sup of f on $[x_{j-1}, x_j]$. Then the graph of f is contained in $\bigcup_1^J R_j$ where $R_j = [x_{j-1}, x_j] \times [m_j, M_j]$, and the sum of the areas of these rectangles is $S_Pf - s_Pf$. This can be made arbitrarily small (Lemma 4.5), so the graph has zero content.

(b) The boundary of S is the union of the graph of f and three line segments, all of which have zero content, so S is measurable. Let $M = \max_{[a,b]} f$. Given a partition $P = \{x_j\}$ of $[a, b]$, let $Q = \{x_j; y_k\}$ be a partition of $[a, b] \times [0, M]$ with the same x_j's, such that each m_j and M_j (as in part (a)) is among the y_k's. Then the upper and lower approximations to the area of S corresponding to this partition are just S_Pf and s_Pf; as these can be made arbitrarily close to $\int_a^b f$, it follows that $\int_a^b f$ is the area of S.

3. Given $\epsilon > 0$, choose a partition P of a rectangle containing S such that $\underline{A}(S) - s_P\chi_S < \frac{1}{2}\epsilon$. Let $R_1, \ldots, R_M$ be the subrectangles of the partition P that are contained in S (so $s_P\chi_S$ is the sum of their areas). For each m, let $\widetilde{R}_m$ be the rectangle with the same center as R_m and side lengths $\sqrt{1-\delta}$ times as big, where $\delta < \epsilon/2\underline{A}(S)$; then $A(\widetilde{R}_m) > (1-\delta)A(R_m)$, and $\widetilde{R}_m \subset S^{\text{int}}$. Let Q be the partition obtained by adding all the x (resp. y) coordinates of the sides of the $\widetilde{R}_m$'s into the collection of x_j's (resp. y_k's) in P. Then the $\widetilde{R}_m$'s are among the subrectangles of the partition Q, so $s_Q\chi_{S^{\text{int}}} \ge \sum A(\widetilde{R}_m) > (1-\delta)s_P\chi_S > (1-(\epsilon/2\underline{A}(S)))(\underline{A}(S) - \frac{1}{2}\epsilon) > \underline{A}(S) - \epsilon$. It follows that $\underline{A}(S^{\text{int}}) \ge \underline{A}(S)$, and the reverse inequality is obvious.

4. This really follows from Exercise 3 and the observation that if R is a rectangle whose interior includes $\overline{S}$, then the outer area of S plus the inner area of $R \setminus S$ equals the area of R, and likewise with S replaced by $\overline{S}$. The inner area of $R \setminus S$ is the same as the inner area of its interior $R^{\text{int}} \setminus \overline{S}$, which is the same as the inner area of $R \setminus \overline{S}$ since the boundary of R has zero content; it follows that $\overline{A}(\overline{S}) = \overline{A}(S)$. Alternatively, one can redo the argument of Exercise 3 by considering a partition of R and shrinking the rectangles that do not meet S slightly to produce rectangles that do not meet $\overline{S}$.

5. Given any partition of a rectangle whose interior contains $\overline{S}$, the subrectangles of the partition fall into three classes: (i) those contained in S^{int}, (ii) those that intersect ∂S, and (iii) those that do not intersect $\overline{S}$. (Any rectangle R that intersects both S^{int} and $(\overline{S})^c$ also intersects ∂S; otherwise $R \cap S^{\text{int}}$ and $R \cap (\overline{S})^c$ would be a disconnection of R.) The sum of the areas of the rectangles in class (i) (resp. class (ii), classes (i) and (ii)) approximates $\underline{A}(S^{\text{int}})$ (resp. $\overline{A}(\partial S)$, $\overline{A}(\overline{S})$). It follows that $\underline{A}(S^{\text{int}}) + \overline{A}(\partial S) = \overline{A}(\overline{S})$, and hence by Exercises 3 and 4 that $\underline{A}(S) + \overline{A}(\partial S) = \overline{A}(S)$.

6. (a) U is a union of open rectangles, so it is open (any point is an interior point).

(b) Let U_n be the approximation to U obtained by stopping at the nth stage, with the rectangles of length 4^{-n}. U_n is the union of 1 rectangle of area $\frac{1}{4}$, two rectangles of area $\frac{1}{16}, \ldots, 2^{n-1}$ rectangles of area 4^{-n}. The sum of these areas is $\frac{1}{2}(1-2^{-n})$, and there is some overlap, so $A(U_n) < \frac{1}{2}$; in fact $A(U_n) \le c < \frac{1}{2}$ with c independent of n. If P is any partition of the unit square, the union of the (closed) subrectangles of P that are contained in U is a compact set, and the open rectangles out of which U is built are an open cover of it, so by Heine-Borel there is a finite subcover; in other words, all these subrectangles are contained in U_n for some n. It follows that $s_P\chi_U \le c < \frac{1}{2}$, so $\underline{A}(U) < \frac{1}{2}$.

(c) U contains every $(x, y) \in R$ such that $0 < y < 1$ and x has a terminating base-2 decimal expansion; the set of all such (x, y) is dense. Hence $\overline{A}(U) = A(R) = 1$ by Exercise 4.

(d) We have $\underline{A}(V) = A(R) - \overline{A}(U) = 1 - 1 = 0$ and $\overline{A}(V) = A(R) - \underline{A}(U) > 1 - \frac{1}{2} = \frac{1}{2}$.

7. Suppose $\varphi(b) = 0$. With $F(x) = \int_a^x f(t)\,dt$, $\int_a^b f(x)\varphi(x)\,dx = F(x)\varphi(x)|_a^b - \int_a^b F(x)\varphi'(x)\,dx = -\int_a^b F(x)\varphi'(x)\,dx$ since $F(a) = \varphi(b) = 0$. Since $\varphi' \ge 0$, we can apply the mean value theorem to get $-\int_a^b F(x)\varphi'(x)\,dx = -F(c)\int_a^b \varphi'(x)\,dx = F(c)\varphi(a) = \varphi(a)\int_a^c f(x)\,dx$, the claimed result for the case $\varphi(b) = 0$. Moreover, for any constant C, $\int_a^b f(x)[\varphi(x) + C]\,dx = \varphi(a)\int_a^c f(x)\,dx + C\int_a^b f(x)\,dx = [\varphi(a) + C]\int_a^c f(x)\,dx + C\int_c^b f(x)\,dx$. Finally, if $\varphi(b) \ne 0$, we apply this result with $\varphi(x)$ replaced by $\varphi(x) - \varphi(b)$ and $C = \varphi(b)$ to get the desired conclusion.

4.3 Multiple Integrals and Iterated Integrals

1. (a) $\iint_S (x + 3y^3)\,dA = \int_{-1}^1 \int_0^{\sqrt{1-x^2}} (x + 3y^3)\,dy\,dx = \int_{-1}^1 [x\sqrt{1-x^2} + \frac{3}{4}(1-x^2)^2]\,dx = [-\frac{1}{3}(1-x^2)^{3/2} + \frac{3}{4}(x - \frac{2}{3}x^3 + \frac{1}{5}x^5)]_{-1}^1 = \frac{4}{5}$.

(b) $\iint_S (x^2 - \sqrt{y})\,dA = \int_0^2 \int_{y^2}^{2y} (x^2 - \sqrt{y})\,dx\,dy = \int_0^2 [\frac{1}{3}x^3 - x\sqrt{y}]_{y^2}^{2y}\,dy = \int_0^2 (\frac{8}{3}y^3 - 2y^{3/2} - \frac{1}{3}y^6 + y^{5/2})\,dy = [\frac{2}{3}y^4 - \frac{4}{5}y^{5/2} - \frac{1}{21}y^7 + \frac{2}{7}y^{7/2}]_0^2 = \frac{32}{35}(5 - \sqrt{2})$.

2. $V = \int_0^{1-x}\int_0^1 6xy(1-x-y)\,dy\,dx = \int_0^1 [3x(1-x)y^2 - 2xy^3]_0^{1-x}\,dx = \int_0^1 x(1-x)^3\,dx = [\frac{1}{2}x^2 - x^3 + \frac{3}{4}x^4 - \frac{1}{5}x^5]_0^1 = \frac{1}{20}$.

3. (a) and (b): See answers in back of text.

(c) The parabolas intersect at $(-3,9)$ and $(1,1)$, and they are given by $x = \pm\sqrt{y}$ and $x = -2 \pm \sqrt{10-y}$. Thus $\iint_S f\,dA = \int_{-3}^{1}\int_{x^2}^{6-4x-x^2} f(x,y)\,dy\,dx = \int_0^1\int_{-\sqrt{y}}^{\sqrt{y}} f(x,y)\,dx\,dy + \int_1^9\int_{-\sqrt{y}}^{-2+\sqrt{10-y}} f(x,y)\,dy\,dx + \int_9^{10}\int_{-2-\sqrt{10-y}}^{-2+\sqrt{10-y}} f(x,y)\,dy\,dx$.

4. (a) and (b): See answers in back of text.
(c) $\int_0^{\ln 2}\int_{e^y}^2 f(x,y)\,dx\,dy$.

5. (a) $\int_1^3\int_1^y ye^{2x}\,dx\,dy = \int_1^3 \frac{1}{2}(ye^{2y} - e^2y)\,dy = [\frac{1}{4}ye^{2y} - \frac{1}{8}e^{2y} - \frac{1}{4}e^2y^2]_1^3 = \frac{5}{8}e^6 - \frac{33}{8}e^2$.
(b) $\int_0^1\int_{\sqrt{x}}^1 \cos(y^3+1)\,dy\,dx = \int_0^1\int_0^{y^2}\cos(y^3+1)\,dy\,dx = \int_0^1 y^2\cos(y^3+1)\,dy = \frac{1}{3}\sin(y^3+1)|_0^1 = \frac{1}{3}(\sin 2 - \sin 1)$.
(c) $\int_1^2\int_{1/x}^1 ye^{xy}\,dy\,dx = \int_{1/2}^1\int_{1/y}^2 ye^{xy}\,dx\,dy = \int_{1/2}^1 (e^{2y} - e)\,dy = [\frac{1}{2}e^{2y} - ey]_{1/2}^1 = \frac{1}{2}e^2 - e$.

6. The region of integration is bounded above by $y = x+1$, below by $y = 2x^2$, and on the left by the y-axis. Reversing the order of integration gives $\int_0^1\int_0^{\sqrt{y/2}} f(y)\,dx\,dy + \int_1^2\int_{y-1}^{\sqrt{y/2}} f(y)\,dx\,dy = \int_0^1 f(y)\sqrt{y/2}\,dy + \int_1^2 f(y)(\sqrt{y/2} - y + 1)\,dy$.

7. Reversing the order of integration gives $h(x) = \int_0^x\int_t^x g(t)\,dy\,dt = \int_0^x (x-t)g(t)\,dt$.

8. See answer in back of text.

9. (a) The region of integration is bounded below by the region in the first quadrant of the xy-plane under the parabola $x = 1 - y^2$ and above by the plane $z = y$.
(b) and (c): See answers in back of text.

10. The volume is $\int_0^a\int_0^{b(1-(x/a))} c(1 - \frac{x}{a} - \frac{y}{b})\,dy\,dx = \int_0^a \frac{1}{2}bc(1-\frac{x}{a})^2\,dx = \frac{1}{6}abc$. The x-moment is $\int_0^a\int_0^{b(1-(x/a))} cx(1-\frac{x}{a}-\frac{y}{b})\,dy\,dx = \int_0^a \frac{1}{2}bcx(1-\frac{x}{a})^2\,dx = \frac{1}{24}a^2bc$, so the x-coordinate of the center of mass is $\frac{1}{4}a$. By symmetry, the y and z coordinates are $\frac{1}{4}b$ and $\frac{1}{4}c$.

11. The mass is $\int_0^2\int_0^2\int_0^2 yz\,dx\,dy\,dz = \int_0^2 dx\int_0^2 y\,dy\int_0^2 z\,dz = 8$. The x-moment is $\int_0^2\int_0^2\int_0^2 xyz\,dx\,dy\,dz = 8$; the y-moment is $\int_0^2\int_0^2\int_0^2 y^2z\,dx\,dy\,dz = \frac{32}{3}$, and likewise the z-moment is $\frac{32}{3}$. So the center of mass is $(1, \frac{4}{3}, \frac{4}{3})$.

12. $x^2 - 3 < x - 1$ when $-1 < x < 2$, so the net charge is $\int_{-1}^2\int_0^2\int_{x^2-3}^{x-1} 2z\,dz\,dy\,dx = 2\int_{-1}^2[(x-1)^2 - (x^2-3)^2]\,dx = 2\int_{-1}^2(-x^4 + 7x^2 - 2x - 8)\,dx = -\frac{126}{5}$.

13. (a) f is not integrable on S because it is unbounded ($f(x,2x) = (2x)^{-2} \to \infty$ as $x \to 0$). However, for fixed y_0, $f(x,y_0)$ is continuous on $[0,1]$ except at the three points 0, y_0, and 1 and is bounded in absolute value by y_0^{-2}; hence it is integrable on $[0,1]$. Likewise $f(x_0,y)$ is integrable on $[0,1]$.
(b) $\int_0^1\int_0^1 f(x,y)\,dx\,dy = \int_0^1[\int_0^y y^{-2}\,dx - \int_y^1 x^{-2}\,dx]\,dy = \int_0^1[y^{-1} - (-1 + y^{-1})]\,dy = 1$, and $\int_0^1\int_0^1 f(x,y)\,dy\,dx = \int_0^1[-\int_0^x x^{-2}\,dy + \int_x^1 y^{-2}\,dy]\,dx = \int_0^1[-x^{-1} + (-1 + x^{-1})]\,dx = -1$.

4.4 Change of Variables for Multiple Integrals

1. $A = \int_0^{2\pi}\int_0^{1+\cos\theta} r\,dr\,d\theta = \int_0^{2\pi}\frac{1}{2}(1+\cos\theta)^2\,d\theta = \frac{3}{2}\pi$.

2. The volume is $\int_{-\pi/2}^{\pi/2}\int_0^1\int_r^1 r\,dz\,dr\,d\theta = \pi\int_0^1 r(1-r)\,dr = \pi[\frac{1}{2}r^2 - \frac{1}{3}r^3]_0^1 = \frac{1}{6}\pi$. The x-moment is $\int_{-\pi/2}^{\pi/2}\int_0^1\int_r^1 r^2\cos\theta\,dz\,dr\,d\theta = [\sin\theta]_{-\pi/2}^{\pi/2}[\frac{1}{3}r^3 - \frac{1}{4}r^4]_0^1 = \frac{1}{6}$. The y-moment vanishes by symmetry. The z-moment is $\int_{-\pi/2}^{\pi/2}\int_0^1\int_r^1 zr\,dz\,dr\,d\theta = \pi\int_0^1 \frac{1}{2}(1-r^2)r\,dr = \pi[\frac{1}{2}r^2 - \frac{1}{4}r^4]_0^1 = \frac{1}{4}\pi$. Thus $\overline{x} = 1/\pi$, $\overline{y} = 0$, and $\overline{z} = \frac{3}{4}$.

3. The top and bottom hemispheres are given by $z = \pm\sqrt{4-r^2}$, so $V = \int_0^{2\pi}\int_0^1 2\sqrt{4-r^2}\,r\,dr\,d\theta = -\frac{4}{3}\pi(4-r^2)^{3/2}|_0^1 = \frac{4}{3}\pi(8 - 3^{3/2})$.

4. The equation of the cylinder is $x^2 + y^2 = 2x$, or $r = 2\cos\theta$ ($-\frac{1}{2}\pi \le \theta \le \frac{1}{2}\pi$). Hence $V = \int_{\pi/2}^{\pi/2}\int_0^{2\cos\theta}(2-r)r\,dr\,d\theta = \int_{-\pi/2}^{\pi/2}(4\cos^2\theta - \frac{8}{3}\cos^3\theta)\,d\theta = [2\theta + \sin 2\theta - \frac{8}{3}(\sin\theta - \frac{1}{3}\sin^3\theta)]_{-\pi/2}^{\pi/2} = 2\pi - \frac{32}{9}$.

5. In cylindrical coordinates with the origin at the center of the base, the mass is $\int_0^{2\pi}\int_0^R\int_0^h czr\,dz\,dr\,d\theta = c(2\pi)(\frac{1}{2}R^2)(\frac{1}{2}h^2) = \frac{1}{2}c\pi R^2h^2$.

6. In cylindrical coordinates, $V = \int_0^{2\pi}\int_0^{\sqrt{3}}(\sqrt{4-r^2} - 1)r\,dr\,d\theta = 2\pi[-\frac{1}{3}(4-r^2)^{3/2} - \frac{1}{2}r^2]_0^{\sqrt{3}} = \frac{5}{3}\pi$. Alternatively, in spherical coordinates, the plane $z = 1$ is given by $r = \sec\varphi$, so $V = \int_0^{2\pi}\int_0^{\pi/3}\int_{\sec\varphi}^2 r^2\sin\varphi\,dr\,d\varphi\,d\theta = \frac{2}{3}\pi\int_0^{\pi/3}(8 - \sec^3\varphi)\sin\varphi\,d\varphi = \frac{2}{3}\pi[-8\cos\varphi - \frac{1}{2}\sec^2\varphi]_0^{\pi/3} = \frac{5}{3}\pi$.

7. $M = \int_0^{2\pi}\int_0^{\pi}\int_0^R c(1-r)r^2\sin\varphi\,dr\,d\varphi\,d\theta = c(2\pi)(2)(\frac{1}{3}R^3 - \frac{1}{4}R^4) = \frac{1}{3}\pi cR^4$.

8. By symmetry the coordinates of the centroid are all equal, so it suffices to calculate $\overline{z}$. The z-moment is $\int_0^{\pi/2}\int_0^{\pi/2}(r\cos\varphi)r^2\sin\varphi\,dr\,d\varphi\,d\theta = (\frac{1}{2}\pi)[\frac{1}{2}\sin^2\varphi]_0^{\pi/2}[\frac{1}{4}r^4]_0^1 = \frac{1}{16}\pi$, and the volume is $\frac{1}{8}(\frac{4}{3}\pi) = \frac{1}{6}\pi$, so $\overline{z} = \frac{3}{8}$.

9. Let $u = x - 3y$, $v = 2x + y$; then $\partial(u,v)/\partial(x,y) = 7$, and $x = \frac{1}{7}(u + 3v)$, $y = \frac{1}{7}(-2u + v)$. The given parallelogram P in the xy-plane becomes the rectangle R in the uv-plane given by $0 \le u \le 10$, $0 \le v \le 15$. So the area of P is $\frac{1}{7}(10)(15)$; the x-moment is $\iint_P x\,dx\,dy = \frac{1}{49}\int_0^{15}\int_0^{10}(u + 3v)\,du\,dv = \frac{3125}{49}$, and the y-moment is $\iint_P y\,dx\,dy = \frac{1}{49}\int_0^{15}\int_0^{10}(-2u + v)\,du\,dv = - = \frac{375}{49}$, so $\overline{x} = \frac{55}{14}$ and $\overline{y} = -\frac{5}{14}$. (Shortcut: The centroids of P and R are their geometric centers, which correspond under the map $(x,y) \to (u,v)$. The center of R is clearly $(5, \frac{15}{2})$, so the center of P is $(\frac{55}{14}, -\frac{5}{14})$.)

10. Let $u = x + y$ and $v = x - y$; then $\partial(u,v)/\partial(x,y) = -2$, so $\iint_S (x+y)^4(x-y)^{-5}\,dA = \frac{1}{2}\int_{-1}^1\int_1^3 u^4v^{-5}\,du\,dv = \frac{1}{2}[\frac{1}{5}u^5]_{-1}^1[-\frac{1}{4}v^{-4}]_1^3 = \frac{4}{81}$.

11. Let $u = x + 2y$, $v = x - 2y + z$, $w = \sqrt{3}\,z$; then $\partial(u,v,w)/\partial(x,y,z) = -4\sqrt{3}$, and the given ellipsoid in xyz-space becomes the unit ball in uvw-space. Thus the volume of the ellipsoid is $(1/4\sqrt{3})(4\pi/3) = \pi/3\sqrt{3}$.

12. We have $x = \sqrt{u/v}$ and $y = \sqrt{uv}$, so $\dfrac{\partial(x,y)}{\partial(u,v)} = \det\dfrac{1}{2}\begin{pmatrix}\sqrt{1/uv} & -\sqrt{u/v^3} \\ \sqrt{v/u} & \sqrt{u/v}\end{pmatrix} = \dfrac{1}{2v}$. Thus the area is $\int_1^4\int_1^4(1/2v)\,du\,dv = 3[\frac{1}{2}\log v]_1^4 = \frac{3}{2}\log 4$, the x-moment is $\int_1^4\int_1^4\sqrt{u/v}\,(1/2v)\,du\,dv = \frac{1}{2}[\frac{2}{3}u^{3/2}]_1^4[-2v^{-1/2}]_1^4 = \frac{7}{3}$, and the y-moment is $\int_1^4\int_1^4\sqrt{uv}\,(1/2v)\,du\,dv = \frac{1}{2}[\frac{2}{3}u^{3/2}]_1^4[2v^{1/2}]_1^4 = \frac{14}{3}$. Thus $\overline{x} = \frac{14}{9\log 4}$ and $\overline{y} = \frac{28}{9\log 4}$.

13. $\det D\mathbf{G} = \det\begin{pmatrix} y & x \\ 2x & -2y\end{pmatrix} = -2(x^2+y^2)$, so $\iint_S (x^2+y^2)\,dA = \frac{1}{2}\iint_S |\det D\mathbf{G}|\,dA = \frac{1}{2}\int_1^3\int_1^4 du\,dv = 3$.

14. We have $\dfrac{\partial(x,y)}{\partial(u,v)} = \det\begin{pmatrix} 1-v & -u \\ v & u\end{pmatrix} = u$. Also, $x+y=u$ and $x/y = v^{-1}-1$, so $u = x+y$ and $v = y/(x+y)$; hence the x and y axes correspond to $v=0$ and $v=1$. Therefore $\iint_S (x+y)^{-1}\,dA = \int_0^1\int_1^4 u^{-1}\cdot u\,du\,dv = 3$.

15. In double polar coordinates, the unit sphere in $\mathbb{R}^4$ is given by $r^2+s^2=R^2$, and the volume element is $rs\,dr\,ds\,d\theta\,d\varphi$. Thus the volume of the ball is $\int_0^{2\pi}\int_0^{2\pi}\int_0^R\int_0^{\sqrt{R^2-s^2}} rs\,dr\,ds\,d\theta\,d\varphi = (2\pi)^2\int_0^R \frac{1}{2}(R^2-s^2)s\,ds = 2\pi^2[\frac{1}{2}R^2s^2-\frac{1}{4}s^4]_0^R = \frac{1}{2}\pi^2R^4$.

4.5 Functions Defined by Integrals

1. (a) f is obviously C^1 as a function of x for each y, and $f(0,y)\equiv 0$. For fixed $x\neq 0$, one studies the behavior of $f(x,y)$ as $y\to 0+$ as in Exercise 9, §2.1. First one verifies (by l'Hôpital) that $f(x,y)/y^k\to 0$ as $y\to 0+$ for every k and (by induction) that $\partial_y^k f(x,y) = P_k(x,y^{-1})e^{-x^2/y}$ for $y>0$ where P_k is a polynomial. By induction again, it follows that for all $k\geq 0$, $\partial_y^k f(x,y)/y\to 0$ as $y\to 0+$, and hence that $\partial_y^{k+1}f(x,0)$ exists and equals zero.

(b) $F(x) = \int_0^1 f(x,y)\,dy = x^3[x^{-2}e^{-x^2/y}]_0^1 = xe^{-x^2}$, so $F'(x) = (1-2x^2)e^{-x^2}$ and $F'(0)=1$. But $\partial_x f(x,y) = (3x^2-2x^4)y^{-2}e^{-x^2/y}$ for $y>0$, so $\partial_x f(0,y)\equiv 0$.

2. (a) $F'(x) = \int_0^1 e^y/(1+xe^y)\,dy = \int_1^e 1/(1+xu)\,du = x^{-1}\log(1+xu)|_1^e = x^{-1}\log((1+ex)/(1+x))$.

(b) $F'(x) = x^{-2}\cos(x^5)\cdot 2x - \int_1^{x^2} y\sin(xy^2)\,dy = 2x^{-1}\cos(x^5) - (2x)^{-1}\int_x^{x^5}\sin u\,du = 2x^{-1}\cos x^5 + (2x)^{-1}(\cos x^5 - \cos x)$.

(c) $F'(x) = (3x)^{-1}e^{3x^2}\cdot 3 + \int_1^{3x} e^{xy}\,dy = x^{-1}e^{3x^2} + [x^{-1}e^{xy}]_{y=1}^{3x} = x^{-1}(2e^{3x^2}-e^x)$.

3. $h'(x) = (x-x)e^{x-x}g(x) + \int_0^x (x-y+1)e^{x-y}g(y)\,dy = h(x) + \int_0^x e^{x-y}g(y)\,dy$, so $h''(x) = h'(x) + e^{x-x}g(x) + \int_0^x e^{x-y}g(y)\,dy = h'(x) + g(x) + [h'(x)-h(x)]$.

4. $h'(x) = \frac{1}{2}\sin 2(x-x)\,g(x) + \int_0^x \cos 2(x-y)\,g(y)\,dy = \int_0^x \cos 2(x-y)\,g(y)\,dy$, and so $h''(x) = \cos 2(x-x)\,g(x) - 2\int_0^x \sin 2(x-y)\,g(y)\,dy = g(x) - 4h(x)$.

5. $F'(x) = f(x,\varphi(x))\varphi'(x) - f(x,\psi(x))\psi'(x) + \int_{\psi(x)}^{\varphi(x)} \partial_x f(x,y)\,dy$.

6. For $n>1$ we have $(f^{[n]})'(x) = [(n-1)!]^{-1}[(x-x)^{n-1}f(x) + \int_0^x (n-1)(x-y)^{n-2}f(y)\,dy] = [(n-2)!]^{-1}\int_0^x (x-y)^{n-2}f(y)\,dy = f^{[n-1]}(x)$. For $n=1$, $f^{[1]}(x) = \int_0^x f(y)\,dy$, so $(f^{[1]})' = f$.

7. (a) $\partial_t u = -\frac{1}{2}t^{-3/2}\int_0^1 e^{-(x-y)^2/4t}f(y)\,dy + t^{-1/2}\int_0^1 e^{-(x-y)^2/4t}[(x-y)^2/4t^2]f(y)\,dy$. On the other hand, $\partial_x u = t^{-1/2}\int_0^1 e^{-(x-y)^2/4t}[-(x-y)/2t]\,dy$, so $\partial_x^2 u = t^{-1/2}\int_0^1 e^{-(x-y)^2/4t}[((x-y)/2t)^2 - (1/2t)]f(y)\,dy = \partial_t u$.

(b) For short, let $w = (x-y)^2+t^2$. Then $\partial_x v = t\int_0^1 (-1)w^{-2}2(x-y)f(y)\,dy$, so

$$\begin{aligned}\partial_x^2 v &= t\int_0^1 2w^{-3}4(x-y)^2 f(y)\,dy + t\int_0^1 (-1)w^{-2}2f(y)\,dy \\ &= 8t\int_0^1 (x-y)^2w^{-3}f(y)\,dy - 2t\int_0^1 w^{-2}f(y)\,dy.\end{aligned}$$

On the other hand, $\partial_t v = \int_0^1 w^{-1} f(y)\,dy + 2t^2 \int_0^1 (-1)w^{-2} f(y)\,dy$, so

$$\begin{aligned}\partial_t^2 v &= t\int_0^1 (-1)w^{-2} 2t\,dy - 4t\int_0^1 w^{-2} f(y)\,dy - 2t^2 \int_0^1 (-2)w^{-3}(2t) f(y)\,dy \\ &= 8t^3 \int_0^1 w^{-3} f(y)\,dy - 6t\int_0^1 w^{-2} f(y)\,dy.\end{aligned}$$

Thus, in $\partial_x^2 v + \partial_t^2 v$, the terms involving w^{-3} add up to $8t\int_0^1 w^{-2} f(y)\,dy$, and the terms involving w^{-2} add up to the negative of this quantity.

8. We show that for each $\mathbf{x} \in T$, $\partial_1 F(\mathbf{x})$ exists and is given by (4.48), and hence is continuous by Corollary 4.53a. Let $\mathbf{h}_j = (h_j, 0, \dots, 0)$ where $h_j \to 0$ and each h_j is small enough so that the points $\mathbf{x} + \mathbf{h}_j$ are all in a ball contained in T. By the mean value theorem, $|f(\mathbf{x} + \mathbf{h}_j,\, \mathbf{y}) - f(\mathbf{x},\, \mathbf{y})|/|h_j| = |\partial_1 f(\mathbf{x} + t\mathbf{h}_j,\, \mathbf{y})| \le C$, and $[f(\mathbf{x} + \mathbf{h}_j,\, \mathbf{y}) - f(\mathbf{x},\, \mathbf{y})]/h_j \to \partial_{x_1} f(\mathbf{x}, \mathbf{y})$ for each $\mathbf{y}$ as $j \to \infty$. Hence, by the bounded convergence theorem, the integrals of these difference quotients, which are $[F(\mathbf{x} + \mathbf{h}_j) - F(\mathbf{x})]/h_j$, converge to $\int\cdots\int_S \partial_{x_1} f(\mathbf{x}, \mathbf{y})\, d^n\mathbf{y}$, which is therefore $\partial_1 F(\mathbf{x})$.

4.6 Improper Integrals

1. (a) Converges by comparison to $x^{-3/2}$.

(b) Diverges by comparison to x^{-1}.

(c) Converges by comparison to e^{-x} (for example), using the fact that $x^2 < e^x$ and $e^{-x^2} < e^{-2x}$ for large x.

(d) The integrand is bounded in absolute value by $(x^2 - x - 2)^{-1}$, which is comparable to x^{-2} for large x; hence converges absolutely.

(e) Diverges by comparision to x^{-1}. ($\tan(0) = 0$ and $\tan'(0) = 1$, so $\tan(u) \approx u$ for small u.)

2. (a) Converges since $x/\sqrt{1 - x^2} \approx 1/\sqrt{2(1 - x)}$ for x near 1.

(b) Diverges by comparision to $(x - \frac{1}{2}\pi)^{-1}$ since $\sin x \approx 1$ and $\cos x = \sin(\frac{1}{2}\pi - x) \approx \frac{1}{2}\pi - x$ for x near $\frac{1}{2}\pi$.

(c) The integrand equals $[(3-x)\sqrt{1 - x}]^{-1}$, which is roughly $1/2\sqrt{1 - x}$ for x near 1; hence converges.

(d) Converges since $1/x^{1/2}(x^2 + x)^{1/3} = 1/x^{5/6}(x + 1) \approx x^{-5/6}$ for x near 0.

(e) For x near 0, $1 - \cos x \approx \frac{1}{2}x^2$ and $\sin^3 2x \approx (2x)^3$, so the integrand is $\approx 1/16x$; hence diverges.

3. (a) The integrand is comparable to $x^{-3/4}$ near 0 and less than e^{-x} near infinity; hence converges.

(b) The integrand is $\approx x^{-1/3}$ near 0 and $\approx (1 - x)^{-2}$ near 1; hence $\int_0^{1/2}$ converges but $\int_{1/2}^1$ diverges.

(c) For x near 0, $e^x - 1 \approx x$, so $\sqrt{x}/(e^x - 1) \approx x^{-1/2}$ and $\int_0^1$ converges. For x large, the integrand is less than $e^{-x/2}$; hence $\int_1^\infty$ converges.

(d) The integrand is $\approx -1/x$ for x near 0; hence $\int_0^{1/2}$ diverges. ($\int_{1/2}^1$, $\int_1^2$ and $\int_2^\infty$ all converge, though.)

(e) $|x^{-1/5} \sin x^{-1}| \le x^{-1/5}$ for x near 0 and $x^{-1/5} \sin x^{-1} \approx x^{-6/5}$ for x near infinity; hence converges.

(f) For $x \ll 0$, $e^x/(e^x + x^2) < e^x$, so $\int_{-\infty}^0$ converges; but for $x \gg 0$, $e^x/(e^x + x^2) \approx 1$, so $\int_0^\infty$ diverges.

4. (a) This follows from Corollary 2.12 ($\lim_{x\to\infty}(\log x/x^{\epsilon/p}) = 0$, so $(\log x)^{-p} > x^{-\epsilon}$ for large x).
(b) $\int_2^\infty x^{-1}(\log x)^{-p}\,dx = \int_{\log 2}^\infty u^{-p}\,du$ converges if and only if $p > 1$.

5. (a) As in the preceding exercise, $(\log\log x)^{-p} > (\log x)^{-\epsilon}$ for x large.
(b) $\int_3^\infty (x\log x)^{-1}(\log\log x)^{-p}\,dx = \int_{\log\log 3}^\infty u^{-p}\,du$ converges if and only if $p > 1$.

6. (a) $\int_0^k f(x)\,dx = 2^{-1} + 2^{-2} + \cdots + 2^{1-k} = 1 - 2^{1-k}$, so since $f \geq 0$, for $k \leq b \leq k+1$ we have $1 - 2^{1-k} \leq \int_0^b f(x)\,dx \leq 1 - 2^{-k}$. It follows that $\int_0^\infty f(x)\,dx = 1$.
(b) One possibility: $f(x) = 2^k$ for $x \in [k,\, k + 2^{-2k}]$ $(k = 1, 2, 3, \ldots)$ and $f(x) = 0$ elsewhere.

7. Let $M = \sup\{\varphi(x) : x \geq a\}$ (which exists since φ is bounded). Given $\epsilon > 0$, there exists x_0 such that $\varphi(x_0) > M - \epsilon$. Since φ is increasing, we have $M - \epsilon < \varphi(x) \leq M$ and hence $|\varphi(x) - M| < \epsilon$ for all $x \geq x_0$. It follows that $\lim_{x\to\infty}\varphi(x) = M$.

8. Granted the hint, we have $\int_1^\infty x^{-1}|\sin x|\,dx \geq c\int_1^\infty x^{-1}\,dx = \infty$. One way to carry out the hint is as follows. On the one hand, $\int_{n\pi}^{(n+1)\pi} x^{-1}|\sin x|\,dx > \int_{n\pi}^{(n+1)\pi}[(n+1)\pi]^{-1}|\sin x|\,dx = 2/(n+1)\pi$. On the other hand, $\int_{n\pi}^{(n+1)\pi} x^{-1}\,dx < \int_{n\pi}^{(n+1)\pi}[n\pi]^{-1}\,dx = 1/n$. Since $2n/(n+1) \geq 1$ for all $n \geq 1$, we can take $c = 1/\pi$.

9. Integrating by parts, $\int_a^b f(x)g(x)\,dx = F(x)g(x)|_a^b - \int_a^b F(x)g'(x)\,dx$. As $b \to \infty$, $F(b)$ remains bounded and $g(b) \to 0$, so $F(b)g(b) \to 0$. On the other hand, since $g' \leq 0$, $|F(x)g'(x)| \leq C|g'(x)| = -Cg'(x)$, and $-\int_a^\infty g'(x)\,dx = g(a) - \lim_{b\to\infty} g(b) = g(a)$. Hence $\int_a^\infty F(x)g'(x)\,dx$ is absolutely convergent.

10. The antiderivative of $1/x(x+2)$ is $\frac{1}{2}\log|x/(x+2)|$, so

$$\begin{aligned}
P.V.\int_{-1}^1 \frac{dx}{x(x+2)} &= \lim_{\epsilon\to 0+}\frac{1}{2}\left(\left[\log\left|\frac{x}{x+2}\right|\right]_{-1}^{-\epsilon} + \left[\log\left|\frac{x}{x+2}\right|\right]_{\epsilon}^{1}\right)\\
&= \lim_{\epsilon\to 0+}\frac{1}{2}\left(\log\frac{\epsilon}{-\epsilon+2} - \log 1 + \log\frac{1}{3} - \log\frac{\epsilon}{\epsilon+2}\right)\\
&= -\frac{1}{2}\log 3 + \lim_{\epsilon\to 0+}\log\frac{\epsilon+2}{-\epsilon+2} = -\frac{1}{2}\log 3.
\end{aligned}$$

11. We have $\varphi(x) = \varphi(0) + \varphi'(0)x + \frac{1}{2}\varphi''(0)x^2 + R(x)$ where $|R(x)| \leq Cx^3$ for $|x| \leq 1$, so

$$P.V.\int_{-1}^1 \frac{\varphi(x)}{x^3}\,dx = \varphi(0)\cdot P.V.\int_{-1}^1\frac{dx}{x^3} + \varphi'(0)\cdot P.V.\int_{-1}^1\frac{dx}{x^2} + \frac{\varphi''(0)}{2}\cdot P.V.\int_{-1}^1\frac{dx}{x} + \int_{-1}^1\frac{R(x)}{x^3}\,dx.$$

The last integral is proper, and the first and third P.V. integrals exist (and are zero), but $P.V.\int_{-1}^1 x^{-2}\,dx = \infty$; hence the original P.V. integral exists if and only if the coefficient $\varphi'(0)$ of this term vanishes.

4.7 Improper Multiple Integrals

1. Spherical coordinates turn $\iiint_{|\mathbf{x}|<1}|\mathbf{x}|^{-p}\,d^3\mathbf{x}$ into $4\pi\int_0^1 r^{-p+2}\,dr$, which converges precisely when $p < 3$. Likewise, the integral over $|\mathbf{x}| > 1$ becomes $4\pi\int_1^\infty r^{-p+2}\,dr$, which converges precisely when $p > 3$.

2. (a) In spherical coordinates, the integral is $4\pi \int_0^\infty r^2\,dr/(1+r^2)$, which diverges since the integrand tends to 1 at infinity.

(b) In polar coordinates, the integral is $\int_0^\infty \int_0^{\pi/2} r\,d\theta\,dr/(1+r^2)^2 = -\frac{1}{4}\pi(1+r^2)^{-1}|_0^\infty = \frac{1}{4}\pi$.

(c) In spherical coordinates, the integral is $\int_0^{2\pi}\int_0^\pi\int_0^1 r\cos^2\varphi\sin\varphi\,dr\,d\varphi\,d\theta = 2\pi[\frac{1}{3}\cos^3\varphi]_0^\pi[\frac{1}{2}r^2]_0^1 = 2\pi/3$.

(d) $\iint_{x>0} xe^{-x^2-y^2}\,dA = \int_0^\infty xe^{-x^2}\,dx\int_{-\infty}^\infty e^{-y^2}\,dy = [-\frac{1}{2}e^{-x^2}]_0^\infty\sqrt{\pi} = \frac{1}{2}\sqrt{\pi}$.

(e) In polar coordinates, the integral is $\int_0^{2\pi}\int_0^1 r^{-1}\cos^2\theta\,dr\,d\theta$; the r-integral diverges.

3. By the extreme value theorem, ρ is bounded, say $|\rho| \le C$. For a given $\mathbf{x}$, let R be big enough so that $\rho(\mathbf{x}-\mathbf{y}) = 0$ for $|\mathbf{y}| \ge R$. Then the integral defining $\varphi(x)$ is dominated by $(C/4\pi)\iiint_{|\mathbf{y}|<R}|\mathbf{y}|^{-1}\,d^3\mathbf{y} = C\int_0^R r\,dr < \infty$.

4. (a) Since the first quadrant of the unit disc is contained in the square S, using polar coordinates we have $\iint_S |f|\,dA \ge \int_0^{\pi/2}\int_0^1 r^{-1}|\cos^2\theta - \sin^2\theta|\,dr\,d\theta$; the r-integral diverges. (This could also be done by a calculation like that in part (b) below, using the fact that $\iint_S |f|\,dA = 2\int_0^1\int_0^x f(x,y)\,dy\,dx$.)

(b) We have $f(x,y) = (x^2+y^2)^{-1} - 2y^2(x^2+y^2)^{-2}$, so for a fixed $y > 0$ the substitution $x = y\tan u$ turns the indefinite integral $\int f(x,y)\,dx$ into $y^{-1}\int(1-2\cos^2 u)\,du = -y^{-1}\int\cos 2u\,du = -y^{-1}\sin u\cos u = -x/(x^2+y^2)$. Hence $\int_0^1 f(x,y)\,dx = -1/(1+y^2)$ and so $\int_0^1\int_0^1 f(x,y)\,dx\,dy = -\frac{1}{4}\pi$. Since $f(x,y) = -f(y,x)$, it follows that $\int_0^1\int_0^1 f(x,y)\,dy\,dx = \frac{1}{4}\pi$.

Chapter 5

Line and Surface Integrals; Vector Analysis

5.1 Arc Length and Line Integrals

1. (a) $\int_0^{2\pi} |\mathbf{g}'(t)|\,dt = \int_0^{2\pi} \sqrt{a^2(\sin^2 t + \cos^2 t) + b^2}\,dt = \int_0^{2\pi} \sqrt{a^2+b^2}\,dt = 2\pi\sqrt{a^2+b^2}$.
(b) $\int_0^2 |\mathbf{g}'(t)|\,dt = \int_0^2 \sqrt{(t^2-1)^2+4t^2}\,dt = \int_0^2 (t^2+1)\,dt = \frac{14}{3}$.
(c) $\int_1^e |\mathbf{g}'(t)|\,dt = \int_1^e \sqrt{t^{-2}+4+4t^2}\,dt = \int_1^e (t^{-1}+2t)\,dt = 1+e^2-1 = e^2$.
(d) $\int_0^2 |\mathbf{g}'(t)|\,dt = \int_0^2 \sqrt{36+36t+36t+36t^2}\,dt = 6\int_0^2 (1+t)\,dt = 24$.

2. (a) With the center at the origin and the major axis on the y axis, the ellipse is described parametrically by $x = b\cos t$, $y = a\sin t$, and the whole length L is 4 times the length in the first quadrant. Hence $L = 4\int_0^{\pi/2} \sqrt{b^2\sin^2 t + a^2\cos^2 t}\,dt = 4\int_0^{\pi/2} \sqrt{a^2-(a^2-b^2)\sin^2 t}\,dt = 4aE(k)$ where $k^2 = 1-(b/a)^2$.
(b) The base of the cylinder is the circle $x^2+(y-1)^2 = 1$ in the xy-plane; the semicircle where both coordinates are positive is given by $x = \cos t$, $y = 1+\sin t$, $-\frac{1}{2}\pi \le t \le \frac{1}{2}\pi$. On the sphere we then have $z = \sqrt{4-x^2-y^2} = \sqrt{2-2\sin t}$. Hence the arc length is

$$L = \int_{-\pi/2}^{\pi/2} \sqrt{\sin^2 t + \cos^2 t + \frac{\cos^2 t}{(2-2\sin t)}}\,dt = \int_{-\pi/2}^{\pi/2} \sqrt{1 + \frac{\cos^2 t}{(2-2\sin t)}}\,dt.$$

Let $s = \frac{1}{2}(\frac{1}{2}\pi - t)$, so $t = \frac{1}{2}\pi - 2s$. Then $\int_{-\pi/2}^{\pi/2} \cdots\,dt = 2\int_0^{\pi/2} \cdots\,ds$; $2-2\sin t = 2-2\cos 2s = 4\sin^2 s$, and $\cos^2 t = \sin^2 2s = 4\sin^2 s\cos^2 s$, so $L = 2\int_0^{\pi/2} \sqrt{1+\cos^2 s}\,ds = 2\int_0^{\pi/2} \sqrt{2-\sin^2 s}\,ds = 2^{3/2}E(2^{-1/2})$.

3. The element of arc length is $ds = \sqrt{1+\sinh^2 x}\,dx = \cosh x\,dx$. Thus the arc length or "mass" of the curve is $\int_{-1}^1 \cosh x\,dx = 2\sinh 1$, and the y-moment is $\int y\,ds = \int_{-1}^1 \cosh^2 x\,dx = \frac{1}{2}\int_{-1}^1 (1+\cosh 2x)\,dx = \frac{1}{2}(2+\sinh 2)$. Thus $\overline{y} = (2+\sinh 2)/4\sinh 1$, and $\overline{x} = 0$ by symmetry.

4. $ds = |\mathbf{g}'(t)|\,dt = \sqrt{4\sin^2 t + 4\cos^2 t + 4t^2}\,dt = 2\sqrt{1+t^2}\,dt$, so $\int_C \sqrt{z}\,ds = \int_0^{2\pi} t\sqrt{1+t^2}\,dt = \frac{1}{3}[(1+4\pi^2)^{3/2}-1]$.

5. (a) Parametrize C by $\mathbf{g}(t) = (t,t,t)$, $0 \le t \le 1$; then $\mathbf{F}(\mathbf{g}(t)) = (t^2,t^2,t^2)$ and $\mathbf{g}'(t) = (1,1,1)$, so $\int_C \mathbf{F}\cdot d\mathbf{x} = \int_0^1 3t^2\,dt = 1$.

(b) Parametrize C by $\mathbf{g}(t) = (t, t^2, t^3)$, $0 \le t \le 1$; then $\mathbf{F}(\mathbf{g}(t)) = (t^5, t^2, t^4)$ and $\mathbf{g}'(t) = (1, 2t, 3t^2)$, so $\int_C \mathbf{F} \cdot d\mathbf{x} = \int_0^1 (t^5 + 2t^3 + 3t^6)\, dt = \frac{1}{6} + \frac{1}{2} + \frac{3}{7} = \frac{23}{21}$.

(c) Parametrize C by $\mathbf{g}(t) = (\sin t, \cos t)$, $0 \le t \le 2\pi$ (remember that C is oriented clockwise!); then $\mathbf{F}(\mathbf{g}(t)) = (\sin t - \cos t, \sin t + \cos t)$ and $\mathbf{g}'(t) = (\cos t, -\sin t)$, so $\int_C \mathbf{F} \cdot d\mathbf{x} = \int_0^{2\pi} (-\sin^2 t - \cos^2 t)\, dt = -2\pi$.

(d) Parametrize the parabolic portion of C by $\mathbf{g}(t) = (t, t^2)$, $-2 \le t \le 2$, and the straight portion by $\mathbf{g}(t) = (-t, 4)$, $-2 \le t \le 2$. Then $\int_C \mathbf{F} \cdot d\mathbf{x} = \int_{-2}^2 (t^4, t^7) \cdot (1, 2t)\, dt + \int_{-2}^2 (4t^2, -16t^3) \cdot (-1, 0)\, dt = \int_{-2}^2 (t^4 + 2t^8 - 4t^2)\, dt = 2(\frac{32}{5} + \frac{1024}{9} - \frac{32}{3}) = \frac{9856}{45}$.

6. (a) $\int_C (xe^{-y}\, dx + \sin \pi x\, dy) = \int_0^1 (xe^{-x^2} + (\sin \pi x) 2x)\, dx = \left[-\frac{1}{2} e^{-x^2} - (2/\pi) x \cos \pi x + (2/\pi^2) \sin \pi x \right]_0^1 = \frac{1}{2}(1 - e^{-1}) + (2/\pi)$.

(b) $\int_C (y\, dx + z\, dy + xy\, dz) = \int_0^{2\pi} (-\sin^2 t + t \cos t + \sin t \cos t)\, dt = \left[\frac{1}{4} \cos 2t - \frac{1}{2} + t \sin t + \cos t + \frac{1}{2} \sin^2 t \right]_0^{2\pi} = -\pi$.

(c) On the segment from (0,0) to (1,0) we have $y = 0$ and $dy = 0$; on the segment from (1,0) to (1,1) we have $x = 1$ and $dx = 0$, and on the segment from (1,1) to (0,0) we have $y = x$ and $dy = dx$. Hence the integral is $\int_0^1 0\, dx + \int_0^1 (-2)\, dy + \int_1^0 (x^2 - 2x)\, dx = -2 + \frac{2}{3} = -\frac{4}{3}$.

7. (a) Parametrize C by $\mathbf{x} = \mathbf{g}(t)$, $a \le t \le b$. Then we have $\left| \int_C \mathbf{F}\, ds \right| = \left| \int_a^b \mathbf{F}(\mathbf{g}(t)) |\mathbf{g}'(t)|\, dt \right| \le \int_a^b |\mathbf{F}(\mathbf{g}(t))|\, |\mathbf{g}'(t)|\, dt = \int_C |\mathbf{F}|\, ds$.

(b) With $\mathbf{g}$ as in part (a), we have $\left| \int_C \mathbf{F} \cdot d\mathbf{x} \right| = \left| \int_a^b (\mathbf{F}(\mathbf{g}(t)) \cdot \mathbf{g}'(t)\, dt \right| \le \int_C |\mathbf{F}(\mathbf{g}(t)) \cdot \mathbf{g}'(t)|\, dt \le \int_C |\mathbf{F}(\mathbf{g}(t))|\, |\mathbf{g}'(t)|\, dt = \int_C |\mathbf{F}|\, ds$, where the next-to-last step uses Cauchy's inequality.

8. As noted in the text, if P is a partition of $[a, b]$ and P' is the partition obtained from P by adding in the point c if it is not already there, then $L_{P'}(C) \ge L_P(C)$, so in computing $L(C) = \sup_P L_P(C)$ it is enough to consider partitions P that contain c. If $P = \{t_0, \ldots, t_K\}$ is such a partition with $c = t_J$, let $P_1 = \{t_0, \ldots, t_J\}$ and $P_2 = \{t_J, \ldots, t_K\}$; then P_1 and P_2 are partitions of $[a, c]$ and $[c, b]$, respectively. Conversely, if P_1 and P_2 are partitions of $[a, c]$ and $[c, b]$, we can concatenate them to obtain the partition P of $[a, b]$. In these cases we clearly have $L_P(C) = L_{P_1}(C_1) + L_{P_2}(C_2)$. It follows that $L_P(C) \le L(C_1) + L(C_2)$, and taking the supremum over all P gives $L(C) \le L(C_1) + L(C_2)$. On the other hand, given $\epsilon > 0$, choose P_1 and P_2 so that $L_{P_j}(C_j) \ge L(C_j) - \epsilon$ for $j = 1, 2$; then $L(C) \ge L_P(C) = L_{P_1}(C_1) + L_{P_2}(C_2) \ge L(C_1) + L(C_2) - 2\epsilon$. Since ϵ is arbitrary, $L(C) \ge L(C_1) + L(C_2)$, and we are done.

9. By the mean value theorem, the displayed expression equals $\sqrt{g'(t_j')^2 + h'(t_j'')^2}\, (t_j - t_{j-1})$ for some points $t_j', t_j'' \in [t_{j-1}, t_j]$. By Exercise 9, §4.1, the sum of these quantities from $j = 1$ to $j = J$, which is $L_P(C)$, can be made as close to $\int_a^b \sqrt{g'(t)^2 + h'(t)^2}\, dt = \int_a^b |\mathbf{g}'(t)|\, dt$ as we wish by taking P sufficiently fine. It follows that this integral equals $L(C)$.

5.2 Green's Theorem

1. (a) Let D be the unit disc; then $\int_C \mathbf{F} \cdot d\mathbf{x} = -\iint_D 2\, dx\, dy = -2\pi$ (the minus sign is there because the circle has the wrong orientation).

(b) $\int_C (y^2\, dx - 2x\, dy) = \int_0^1 \int_0^x (-2 - 2y)\, dy\, dx = \int_0^1 (-2x - x^2)\, dx = -\frac{4}{3}$.

(c) $\int_C[(x^2+10xy+y^2)\,dx+(5x^2+5xy)\,dy] = \int_0^2\int_0^2[(10x+5y)-(10x+2y)]\,dx\,dy = \int_0^2 dx\int_0^2 3y\,dy = 2\cdot 6 = 12$.

(d) $\partial_x(2x^3y\cos y^2) = 6x^2y\cos y^2 = \partial_y(3x^2\sin y^2)$, so the integrand of the double integral in Green's theorem vanishes.

2. Directly, by using the polar angle θ as the parameter for both circles: $\int_0^{2\pi} 32\cos^2\theta\sin^2\theta\,d\theta - \int_0^{2\pi} 2\cos^2\theta\sin^2\theta\,d\theta = \frac{15}{2}\int_0^{2\pi}\sin^2 2\theta\,d\theta = \frac{15}{2}\pi$. By using Green's theorem: $\iint_S(y^2+x^2)\,dx\,dy = \int_0^{2\pi}\int_1^2 r^3\,dr\,d\theta = 2\pi\cdot\frac{15}{4} = \frac{15}{2}\pi$.

3. If S is the region inside C, $\int_C[y^3\,dx + (3x-x^3)\,dy] = \iint_S 3(1-x^2-y^2)\,dx\,dy$. The integrand is positive inside the unit disc and negative outside, so the integral is maximized by taking S to be the unit disc and C to be the unit circle.

4. Take the arch given by $0 \le t \le 2\pi$: the region under it is bounded on the bottom by the segment $[0, 2\pi]$ of the x-axis (where $dy = 0$) and on top by the cycloid (traversed from right to left). Thus the area is $\int_{\partial S} x\,dy = \int_{2\pi}^0 R^2(t-\sin t)\sin t\,dt$ or $-\int_{\partial S} y\,dx = \int_0^{2\pi} R^2(1-\cos t)^2\,dt$; both integrals are equal to $3\pi R^2$.

5. The oriented boundary of S consists of two vertical line segments, a segment of the x-axis, and the curve $y = f(x)$, traversed from right to left. The vertical segments contribute nothing to $-\int_{\partial S} y\,dx$ since $dx = 0$ on them, and the segment of the x-axis contributes nothing since $y = 0$ on it. The integral over the curve is $-\int_b^a f(x)\,dx = \int_a^b f(x)\,dx$.

6. We have $f(\partial g/\partial n) = \mathbf{F}\cdot\mathbf{n}$ where $\mathbf{F} = f\nabla g$, and $\partial_j F_j = \partial_j(f\partial_j g) = f\partial_j^2 g + (\partial_j f)(\partial_j g)$. The result therefore follows from Corollary 5.17.

7. (a) The image of U under the map $\det D\mathbf{G}$ is a connected subset of $\mathbb{R}$, by Theorem 1.26. It does not contain 0, hence must be contained in either $(0,\infty)$ or $(-\infty,0)$; otherwise it would be disconnected.

(b,c) Let $\mathbf{g} = (g,h)$, $\mathbf{u} = (u,v)$, so $x = g(u,v)$, $y = h(u,v)$, and $dy = \partial_u h\,du + \partial_v h\,dv$. Thus $A = \int_{\partial S} x\,dy = \int_{\partial T} g(\partial_u h\,du + \partial_v h\,dv)$, where ∂S is given the positive orientation with respect to S and ∂T is given the orientation induced from the one on ∂S by the change of variable $(x,y)\to(u,v)$. This may or may not be the positive orientation of ∂T with respect to T, so in applying Green's theorem to the integral over ∂T there will be an ambiguity of sign. The result is $A = \pm\iint_T[\partial_u(g\partial_v h) - \partial_v(g\partial_u h)]\,du\,dv = \pm\iint_T(\partial_u g\partial_v h - \partial_v g\partial_u h)\,du\,dv = \pm\iint_T \det D\mathbf{G}\,dA$. The easiest way to determine which sign is right is to observe that the area A is positive; hence the sign must be $+$ if $\det D\mathbf{G} > 0$ and $-$ if $\det D\mathbf{G} < 0$.

5.3 Surface Area and Surface Integrals

1. $A = \iint_{x^2+y^2\le a^2}\sqrt{1+y^2+x^2}\,dx\,dy = \int_0^{2\pi}\int_0^a\sqrt{1+r^2}\,r\,dr\,d\theta = \frac{2}{3}\pi[(1+a^2)^{3/2}-1]$.

2. $A = \iint_{x^2+y^2\le a^2}\sqrt{1+4x^2+4y^2}\,dx\,dy = \int_0^{2\pi}\int_0^a\sqrt{1+4r^2}\,r\,dr\,d\theta = \frac{1}{6}\pi[(1+4a^2)^{3/2}-1]$.

3. With $\mathbf{G}(\varphi,\theta) = ((b+a\cos\varphi)\cos\theta,\ (b+a\cos\varphi)\sin\theta,\ a\sin\varphi)$, we have $\partial_\varphi\mathbf{G} = -a\sin\varphi\sin\theta\mathbf{i} - a\sin\varphi\sin\theta\mathbf{j} + a\cos\varphi\mathbf{k}$ and $\partial_\theta\mathbf{G} = -(b+a\cos\varphi)\sin\theta\mathbf{i} + (b+a\cos\varphi)\cos\theta\mathbf{j}$, so $(\partial_\varphi\mathbf{G})\times(\partial_\theta\mathbf{G}) = -a(b+a\cos\theta)\cos\varphi\cos\theta\mathbf{i} + a(b+a\cos\varphi)\cos\varphi\sin\theta\mathbf{j} - a(b+a\cos\varphi)\sin\varphi\mathbf{k}$ and hence

$|(\partial_\varphi \mathbf{G}) \times (\partial_\theta \mathbf{G})|^2 = a^2(b + a\cos\varphi)^2(\cos^2\varphi\cos^2\theta + \cos^2\varphi\sin^2\theta + \sin^2\varphi) = a^2(b + a\cos\varphi)^2$. Since $0 < a < b$, $b + a\cos\varphi$ is always positive, so finally $A = \int_0^{2\pi}\int_0^{2\pi} a(b + a\cos\varphi)\,d\varphi\,d\theta = 4\pi^2 ab$.

4. The integral can be set up in two ways, as in Example 1 (p. 232). First way: The upper half of the surface is $z = (b/a)\sqrt{a^2 - x^2 - y^2}$, so the area is

$$2\iint_{x^2+y^2\le a^2} \sqrt{1 + \frac{b^2}{a^2}\cdot\frac{x^2+y^2}{a^2-x^2-y^2}}\,dx\,dy = 4\pi\int_0^a \sqrt{\frac{a^4 + (b^2-a^2)r^2}{a^2(a^2-r^2)}}\,r\,dr.$$

The substitution $s = \sqrt{a^2 - r^2}$ simplifies this to $(4\pi/a)\int_0^a \sqrt{b^2a^2 + (a^2-b^2)s^2}\,ds$, and the substitution $u = s/a$ turns this into $4\pi a\int_0^1 \sqrt{b^2 + (a^2-b^2)u^2}\,du$. Second way: Use modified spherical coordinates to parametrize the ellipsoid. With $x = a\sin\varphi\cos\theta$, $y = a\sin\varphi\sin\theta$, and $z = b\cos\varphi$, the formula (5.20) for area yields the integral $2\pi a\int_0^\pi \sin\varphi\sqrt{b^2\sin^2\varphi + a^2\cos^2\varphi}\,d\varphi$, and the substitution $u = \cos\varphi$ turns this into $2\pi a\int_{-1}^1 \sqrt{b^2 + (a^2-b^2)u^2}\,du$. This is the same as the integral obtained in the first way since $\int_{-1}^1 = 2\int_0^1$ for even functions. Finally, one uses a trig substitution ($u = (b\tan t)/\sqrt{a^2-b^2}$ if $a > b$, $u = (b\sin t)/\sqrt{b^2-a^2}$ if $b > a$) or a table of integrals to evaluate the integral as $2\pi a^2 + 2\pi ab^2 C/\sqrt{|a^2-b^2|}$, where $C = \log((a + \sqrt{a^2-b^2})/b)$ if $a > b$ and $C = \arcsin(\sqrt{b^2-a^2}/b)$ if $a < b$. (Both expressions for the area have the limiting value $4\pi a^2$ as $b \to a$.)

5. Clearly $\overline{x} = \overline{y} = 0$ by symmetry. The z-moment, in spherical coordinates, is $\iint_S z\,dA = 2\pi\int_0^{\pi/2}\cos\varphi\sin\varphi\,d\varphi = \pi\sin^2\varphi\big|_0^{\pi/2} = \pi$, and the area is 2π; hence $\overline{z} = \frac{1}{2}$.

6. $\int_0^{2\pi}\int_0^{\pi/3}(4\sin^2\varphi)(4\sin\varphi)\,d\varphi\,d\theta = 32\pi[\frac{1}{3}\cos^3\varphi - \cos\varphi]_0^{\pi/3} = \frac{20}{3}\pi$.

7. By symmetry, the integrals of x^2, y^2, and z^2 over the unit sphere are equal, so the integral of $x^2 + y^2 - 2z^2$ vanishes. The integral is also easily computed in spherical coordinates: it is $2\pi\int_0^\pi(\sin^2\varphi - 2\cos^2\varphi)\sin\varphi\,d\varphi = 2\pi\int_0^\pi(1 - 3\cos^2\varphi)\sin\varphi\,d\varphi = 2\pi\big[\cos\varphi - \cos^3\varphi\big]_0^\pi = 0$.

8. (a) $\mathbf{n}\,dA = (-y\mathbf{i} - x\mathbf{j} + \mathbf{k})\,dy\,dx$, so $\iint_S \mathbf{F}\cdot\mathbf{n}\,dA = \int_0^1\int_0^2(-x^2y^2 - xy)\,dy\,dx = \int_0^1(-\frac{8}{3}x^2 - 2x)\,dx = -\frac{8}{9} - 1 = -\frac{17}{9}$.

(b) Since lines through the center of a sphere are perpendicular to the sphere, the unit normal to the unit sphere S at a point $\mathbf{x} \in S$ is simply the vector $\mathbf{x} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$. Hence $\iint_S \mathbf{F}\cdot\mathbf{n}\,dA = \iint_S x^3\,dA$, which vanishes since x^3 is an odd function. Alternatively, in spherical coordinates one finds $\mathbf{n}\,dA = (\sin\varphi\cos\theta\mathbf{i} + \sin\varphi\sin\theta\mathbf{j} + \cos\varphi\mathbf{k})\sin\varphi\,d\theta\,d\varphi$, so $\iint_S \mathbf{F}\cdot\mathbf{n}\,dA = \int_0^\pi\int_0^{2\pi}\sin^4\varphi\cos^3\theta\,d\theta\,d\varphi = 0$.

(c) The triangle lies in the plane $x + y + z = 2$; taking x, y as parameters, we have $\mathbf{n}\,dA = (\mathbf{i} + \mathbf{j} + \mathbf{k})\,dy\,dx$, so $\iint_S \mathbf{F}\cdot\mathbf{n}\,dA = \int_0^2\int_0^{2-x}(xy + 2 - x - y)\,dy\,dx = \int_0^2(\frac{1}{2}(x-1)(2-x)^2 + (2-x)^2)\,dx = \frac{1}{2}\int_0^2(x^3 - 3x^2 + 4)\,dx = 2$.

(d) The normal is horizontal on the vertical side of the cylinder, so $\mathbf{F}\cdot\mathbf{n} = 0$ there. On the top ($z = b$) we have $z^2 = b^2$, $\mathbf{n} = \mathbf{k}$; on the bottom ($z = a$), we have $z^2 = a^2$, $\mathbf{n} = -\mathbf{k}$. Hence $\iint_S \mathbf{F}\cdot\mathbf{n}\,dA = \pi(b^2 - a^2)$.

(e) $S = S_1 \cap S_2$ where S_1 and S_2 are the portions of the sphere $z = \sqrt{2 - x^2 - y^2}$ and the paraboloid $z = x^2 + y^2$ with $x^2 + y^2 \le 1$, oriented with the normal pointing up and down, respectively. As in (b), the normal at a point $\mathbf{x}$ on the sphere is $\mathbf{x}/\sqrt{2}$, so $\mathbf{F}\cdot\mathbf{n} = |\mathbf{x}|^2/\sqrt{2} = \sqrt{2}$. Also, the element of area in spherical coordinates is $r^2 \sin\varphi\, d\varphi\, d\theta = 2\sin\varphi\, d\varphi\, d\theta$, so $\iint_{S_1} \mathbf{F}\cdot\mathbf{n}\, dA = \sqrt{2}\cdot 2\pi \int_0^{\pi/4} 2\sin\varphi\, d\varphi = \pi(2^{5/2}-4)$. On the paraboloid, with x, y as parameters, we have $\mathbf{n}\, dA = (2x\mathbf{i} + 2y\mathbf{j} - \mathbf{k})$ (remember that the z-component must be negative), so $\iint_{S_2} \mathbf{F}\cdot\mathbf{n}\, dA = \iint_{x^2+y^2\le 1}(2x^2+2y^2-(x^2+y^2))\, dy\, dx = 2\pi\int_0^1 r^3\, dr = \frac{1}{2}\pi$. Hence $\iint_S \mathbf{F}\cdot\mathbf{n}\, dA = \pi(2^{5/2}-\frac{7}{2})$.

5.4 Vector Derivatives

1. These are all simple computations.

2. These are simple computations too. For (c), with $r = |\mathbf{x}|$ and $g(r) = r^a$, one has $\nabla^2 f(\mathbf{x}) = g''(r) + (n-1)r^{-1}g'(r) = a(a-1)r^{a-2} + (n-1)ar^{a-2} = a(a+n-2)r^{a-2}$.

3. The first two formulas are most easily obtained just by writing out the indicated products and taking the indicated derivatives. The last one follows from the first one by using (5.29).

4. (Also **5**, **6**, **7**.) These are straightforward calculations, but the verifications of (5.25), (5.27), and (5.33) require a little masochism. It is less frustrating to start with the complicated expressions (on the right in (5.25) and (5.27), on the left in (5.33)) and work toward the simpler expressions on the other side.

8. $\mathbf{F}\times\mathbf{G}$ is skew-symmetric in $\mathbf{F}$ and $\mathbf{G}$.

9. This follows immediately from (5.29) and (5.30).

5.5 The Divergence Theorem

1. (a) $\operatorname{div}\mathbf{F} = 2x$, and the integral of $2x$ over the unit ball vanishes by symmetry since $2x$ is odd.
(b) $\operatorname{div}\mathbf{F} = 3$, so polar coordinates yield $3\int_0^1\int_0^{2\pi}(\sqrt{2-r^2}-r^2)r\, d\theta\, d\rho = 6\pi[-\frac{1}{3}(2-r^2)^{3/2}-\frac{1}{4}r^4]_0^1 = \pi[2^{5/2}-\frac{7}{2}]$.
(c) $\operatorname{div}\mathbf{F} = 2(x+y+z)$, and the integrals of x, y, and z over the cube are equal by symmetry, so we get $6\int_0^a\int_0^a\int_0^a x\, dx\, dy\, dz = 3a^4$.
(d) $\operatorname{div}\mathbf{F} = a^{-2}+b^{-2}+c^{-2}$, and the volume of the ellipsoid is $\frac{4}{3}\pi abc$ (reduce it to the volume of the unit sphere by the change of variable $u = x/a$, $v = y/b$, $w = z/c$), so the integral is $(a^{-2}+b^{-2}+c^{-2})\frac{4}{3}\pi abc = 4\pi(b^2c^2+a^2c^2+a^2b^2)/3abc$.
(e) $\operatorname{div}\mathbf{F} = 2z$, so the integral is $\iint_W\int_1^2 2z\, dz\, dA = a[z^2]_1^2 = 3A$.

2. Directly: $\mathbf{F}\cdot\mathbf{n} = (x^2+y^2+z^2)^2/a = a^3$ on S (see the remark at the beginning of the exercises), so the integral is a^3 times the area of S, i.e., $4\pi a^5$. By the divergence theorem: $\operatorname{div}\mathbf{F} = 5(x^2+y^2+z^2)$, so the integral is $5\int_0^\pi\int_0^{2\pi}\int_0^a r^2\cdot r^2\sin\varphi\, dr\, d\theta\, d\varphi = 4\pi a^5$.

3. $\operatorname{div}\mathbf{F} = 3$, so $\frac{1}{3}\iint_{\partial R}\mathbf{F}\cdot\mathbf{n}\, dA = \frac{1}{3}\iiint_R \operatorname{div}\mathbf{F}\, dV = \iiint_R dV =$ volume of R.

4. Let $\mathbf{F} = fg\mathbf{i}$; then $\mathbf{F}\cdot\mathbf{n} = fgn_x$ and $\operatorname{div}\mathbf{F} = \partial_x(fg) = f\partial_x g + g\partial_x f$, so the result follows from the divergence theorem.

5. (a) $\iint_{\partial R}(\partial f/\partial n)\, dA = \iint_{\partial R}\nabla f\cdot\mathbf{n}\, dA = \iiint_R \operatorname{div}(\nabla f)\, dV = \iiint_R \nabla^2 f\, dV$.
(b) By (5.28) we have $\operatorname{div}(f\nabla f) = |\nabla f|^2 + f\nabla^2 f$; apply the divergence theorem to $\mathbf{F} = f\nabla f$.

6. (a) $\partial_x g = -x(x^2+y^2+z^2)^{-3/2} = -x/|\mathbf{x}|^3$, and likewise for y and z, so $\nabla g(\mathbf{x}) = -\mathbf{x}/|\mathbf{x}|^3$.

(b) See Exercise 2c in §5.4.

(c) If S is the sphere of radius a about $\mathbf{0}$, on S we have $\mathbf{n} = \mathbf{x}/|\mathbf{x}|$, so $\nabla g \cdot \mathbf{n} = -|\mathbf{x}|^2/|\mathbf{x}|^4 = -a^{-2}$; hence $\iint_S (\partial g/\partial n)\, dA = -a^{-2} \cdot 4\pi a^2 = -4\pi$.

(d) g is not of class C^1 on the region inside the sphere, so the divergence theorem doesn't apply.

(e) Choose ϵ so small that the ball $B_\epsilon = \{\mathbf{x} : |\mathbf{x}| \le \epsilon\}$ is contained in the interior of R. Then $\nabla^2 g = 0$ on $R \setminus B^\epsilon$, so by the divergence theorem and part (c), $0 = \iiint_{R\setminus B_\epsilon} \nabla^2 g\, dV = \iint_{\partial R} (\partial g/\partial n)\, dA - \iint_{\partial B_\epsilon} (\partial g/\partial n)\, dA = \iint_{\partial R} (\partial g/\partial n)\, dA + 4\pi$.

7. (a) Since $\nabla^2 f = \nabla^2 g = 0$ on R, by (5.39) we have $0 = \iint_{\partial R} (f\nabla g - g\nabla f) \cdot n\, dA$. ∂R is the union of the sphere $|\mathbf{x}| = r$ (with the outward normal) and the sphere $|\mathbf{x}| = \epsilon$ (with the inward normal). On $|\mathbf{x}| = r$ we have $g = r^{-1}$ and $\partial g/\partial n = -r^{-2}$ (as in Exercise 6c), so by Exercise 5a, $\iint_{|\mathbf{x}|=r} (f\nabla g - g\nabla f) \cdot \mathbf{n}\, dA = -r^{-2} \iint_{|\mathbf{x}|=r} f\, dA - r^{-1} \iiint_{|\mathbf{x}|\le r} \nabla^2 f\, dV = -r^{-2} \int_{|\mathbf{x}|=r} f\, dA$, which is -4π times the mean value of f on the sphere $|\mathbf{x}| = r$. Likewise, the integral over $|\mathbf{x}| = \epsilon$ (with the inward normal) is 4π times the mean value of f on this sphere. The sum is zero, so the mean values are equal.

(b) Since f is continuous, $f(\mathbf{0})$ is the limit of the mean value of f on the sphere $|\mathbf{x}| = \epsilon$ as $\epsilon \to 0$.

5.6 Some Applications to Physics

1. To evaluate the potential $u(\mathbf{x}) = -\int_{|\mathbf{p}|=r} |\mathbf{p} - \mathbf{x}|^{-1}\, dA$ (we take the density ρ to be 1) at a particular point $\mathbf{x}$, the key to proceeding efficiently is to rotate the coordinates so that $\mathbf{x}$ is on the positive z-axis (or, equivalently, to observe that $u(\mathbf{x})$ is spherically symmetric so that it suffices to take $\mathbf{x}$ on the positive z-axis). For $\mathbf{x} = (0, 0, z)$ and $|\mathbf{p}| = r$ we have $|\mathbf{p} - \mathbf{x}| = \sqrt{p_1^2 + p_2^2 + (p_3 - z)^2} = \sqrt{r^2 - 2zp_3 + z^2}$, so in spherical coodinates we have

$$u(\mathbf{x}) = -\int_0^\pi \int_0^{2\pi} \frac{r^2 \sin\varphi\, d\theta\, d\varphi}{\sqrt{r^2 - 2zr\cos\varphi + z^2}} = -\frac{2\pi r}{z} \sqrt{r^2 - 2zr\cos\varphi + z^2}\,\bigg|_0^\pi$$
$$= -\frac{2\pi}{z}\big(|r+z| - |r-z|\big).$$

For $0 \le z \le r$ this is $-4\pi r$; for $z \ge r$ it is $-4\pi r^2/z = -4\pi r^2/|\mathbf{x}|$. The latter is the potential for a mass $4\pi r^2$ located at the origin. The corresponding field $-\nabla u(\mathbf{x})$ is $\mathbf{0}$ for $|\mathbf{x}| < r$ and $-4\pi r^2 \mathbf{x}/|\mathbf{x}|^3$ for $|\mathbf{x}| > r$.

2. Think of the ball as the union of thin spherical shells of radius r ($0 < r < R$) and thickness dr. For a given $\mathbf{x}$, the shells with $r > |\mathbf{x}|$ contribute nothing to the field, and the shells with $r < |\mathbf{x}|$ contribute $-4\pi r^2 \mathbf{x}/|\mathbf{x}|^3$. Integrating from 0 to $\min(R, |\mathbf{x}|)$ gives the field as $-\frac{4}{3}\pi\mathbf{x}$ for $|\mathbf{x}| < R$ and $-\frac{4}{3}\pi R^3 \mathbf{x}/|\mathbf{x}|^3$ for $|\mathbf{x}| > R$, as claimed. (The potential can also be found by integrating in r; it is $\frac{2}{3}\pi(|\mathbf{x}|^2 - 3R^2)$ for $|\mathbf{x}| < R$ and $-\frac{4}{3}\pi R^3/|\mathbf{x}|$ for $|\mathbf{x}| > R$.)

3. (a) We take $\rho = 1$. The field is

$$\mathbf{E} = \int_{-\infty}^{\infty} \frac{x\mathbf{i} + y\mathbf{i} + (z-p)\mathbf{k}}{(x^2 + y^2 + (z-p)^2)^{3/2}}\, dp = \int_{-\infty}^{\infty} \frac{x\mathbf{i} + y\mathbf{i} + t\mathbf{k}}{(x^2 + y^2 + t^2)^{3/2}}\, dt \qquad (t = z - p).$$

The x and y components of the integrand decay like $|t|^{-3}$ as $|t| \to \infty$, while the z component decays like t^{-2}, so the integral converges. The z-component vanishes since its integrand is odd. Since $\int_{-\infty}^{\infty}(a^2+t^2)^{-3/2}\,dp = \big[t/a^2\sqrt{a^2+t^2}\big]_{-\infty}^{\infty} = 2/a^2$ (via the substitution $t = a\tan\theta$), we obtain $\mathbf{E} = 2(x\mathbf{i}+y\mathbf{j})/(x^2+y^2)$.

(b) $u(\mathbf{x})$ should be $\int_{-\infty}^{\infty}(x^2+y^2+(p-z)^2)^{-1/2}\,dp$, but the integrand decays like $|p|^{-1}$ as $|p| \to \infty$, so the integral diverges.

(c) Since $\log(\sqrt{a^2+t^2}+t)$ is an antiderivative of $1/\sqrt{a^2+t^2}$,

$$u_R(\mathbf{x}) = \int_{z-R}^{z+R} \frac{dt}{\sqrt{x^2+y^2+t^2}} = \log\left[\frac{\sqrt{x^2+y^2+(z+R)^2}+z+R}{\sqrt{x^2+y^2+(z-R)^2}+z-R}\right].$$

Multiplying top and bottom of the fraction by $\sqrt{x^2+y^2+(z-R)^2}-z+R$ turns this into

$$\log\left[\frac{(\sqrt{x^2+y^2+(z+R)^2}+z+R)(\sqrt{x^2+y^2+(z-R)^2}-z+R)}{x^2+y^2}\right]$$
$$= \log\left[\frac{(\sqrt{x^2+y^2+(z+R)^2}+z+R)(\sqrt{x^2+y^2+(z-R)^2}-z+R)}{R^2}\right] + 2\log R - \log(x^2+y^2).$$

As $R \to \infty$, the first term approaches $\log 4$, so subtracting off the $2\log R$ yields $\log 4 - \log(x^2+y^2)$, whose gradient is $-2(x\mathbf{i}+y\mathbf{j})/(x^2+y^2)$. (Of course the $\log 4$ can be discarded too.)

4. The argument is essentially the same as the proof of Theorem 5.46, using the facts that$\nabla \log \mathbf{y} = \mathbf{y}/|\mathbf{y}|^2$ and $\nabla^2 \log|\mathbf{y}| = 0$ for $\mathbf{y} \neq \mathbf{0}$ (Exercise 2d, §5.4). The two-dimensional analogue of Green's formula (5.39) is easily obtained from Exercise 6, §5.2, and it yields the following analogue of (5.47):

$$\nabla^2 u(\mathbf{x}) = \lim_{\epsilon\to 0}\int_{|\mathbf{y}|=\epsilon}\left[(\log|\mathbf{y}|)\nabla\rho(\mathbf{x}+\mathbf{y}) - \frac{\rho(\mathbf{x}+\mathbf{y})\mathbf{y}}{|\mathbf{y}|^2}\right]\cdot\mathbf{n}\,ds,$$

where $\mathbf{n}$ is the unit *inward* normal to the circle, namely, $\mathbf{n} = -\mathbf{y}/\epsilon$. The estimate (5.48) for the first term on the right becomes

$$\left|\int_{|\mathbf{y}|=\epsilon}(\log|\mathbf{y}|)\nabla\rho(\mathbf{x}+\mathbf{y})\cdot\mathbf{n}\,ds\right| \le C|\log\epsilon|2\pi\epsilon \to 0 \text{ as } \epsilon \to 0,$$

and from the second term, since $|\mathbf{y}| = \epsilon$ on the circle, one obtains

$$\nabla^2 u(\mathbf{x}) = \lim_{\epsilon\to 0}\frac{1}{\epsilon}\int_{|\mathbf{y}|=\epsilon}\rho(\mathbf{x}+\mathbf{y})\,ds = 2\pi\rho(\mathbf{x}).$$

5.7 Stokes's Theorem

1. $\operatorname{curl}\mathbf{F}\cdot\mathbf{n}\,dA = (\mathbf{i}-\mathbf{j}+\mathbf{k})\cdot(-\mathbf{j}+\mathbf{k})\,dx\,dy = 2\,dx\,dy$, so the integral is twice the area of the disc $x^2+y^2 \le 1$, i.e., 2π.

2. $\operatorname{curl}\mathbf{F}\cdot\mathbf{n}\,dA = (-\mathbf{j}-\mathbf{k})\cdot(\mathbf{j}+\mathbf{k})\,dx\,dy = -2\,dx\,dy$. The curve C lies over the curve in the xy-plane given by $x^2+y^2+(a-y)^2 = a^2$, or $2x^2/a^2 + 4(y-\frac{1}{2}a)^2/a^2 = 1$, an ellipse with semiaxes $a/\sqrt{2}$ and $a/2$. The integral is -2 times the area of the region inside the ellipse, i.e., $(-2)\pi(a/\sqrt{2})(a/2) = -\pi a^2/\sqrt{2}$.

3. The equation of a nonvertical plane parallel to the x-axis has the form $z = by + c$. Thus $\operatorname{curl}\mathbf{F}\cdot\mathbf{n}\,dA = (-x\mathbf{i} + y\mathbf{j} + 2\mathbf{k})\cdot(-b\mathbf{j} + \mathbf{k})\,dx\,dy = (-by + 2)\,dx\,dy$. The integral of this over the disc $x^2 + y^2 = a^2$ is twice the area of the disc, i.e., $2\pi a^2$ (the integral of by vanishes by symmetry).

4. ∂S is the circle of radius a about the origin in the xy-plane. First method: ∂S is also the boundary of the disc D of radius a in the xy-plane, so $\iint_S \operatorname{curl}\mathbf{F}\cdot\mathbf{n}\,dA = \iint_D \operatorname{curl}\mathbf{F}\cdot\mathbf{k}\,dA = \iint_D xy^3\,dx\,dy = 0$, by symmetry. Second method: parametrize ∂S by $x = a\cos\theta$, $y = a\sin\theta$, $z = 0$. Then $\iint_S \operatorname{curl}\mathbf{F}\cdot\mathbf{n}\,dA = \int_{\partial S}\mathbf{F}\cdot d\mathbf{x} = \int_0^{2\pi}(-\sin^2\theta + \cos^2\theta)\,d\theta = 0$.

5. Like Exercise 4, this can be done two ways. ∂S is the ellipse $(x^2/4) + (y^2/9) = 1$ in the xy-plane. If D is the region inside this ellipse, we have $\iint_S \operatorname{curl}\mathbf{F}\cdot\mathbf{n}\,dA = \iint_D \operatorname{curl}\mathbf{F}\cdot\mathbf{k}\,dA = 0$ since $\operatorname{curl}\mathbf{F}\cdot\mathbf{k} = 0$. Alternatively, parametrize ∂S by $x = 2\cos\theta$, $y = 3\sin\theta$, $z = 0$; then $\iint_S \operatorname{curl}\mathbf{F}\cdot\mathbf{n}\,dA = \int_{\partial S}\mathbf{F}\cdot d\mathbf{x} = \int_0^{2\pi} 10\cos\theta\sin\theta\,d\theta = 0$.

6. (a) This is a simple calculation.

(b) Parametrize C by $x = a\cos\theta$, $y = a\sin\theta$, $z = c$; then $\mathbf{F} = (-(\sin\theta)\mathbf{i} + (\cos\theta)\mathbf{j})/a$ and $d\mathbf{x} = a(-(\sin\theta)\mathbf{i} + (\cos\theta)\mathbf{j})\,d\theta$, so $\int_C \mathbf{F}\cdot d\mathbf{x} = \int_0^{2\pi} d\theta = 2\pi$.

(c) $\mathbf{F}$ is not of class C^1 on any surface bounded by C (such a surface must intersect the z-axis), so Stokes's theorem doesn't apply.

7. Let A_r be the annulus in between the circles C_r and C_1 in the xz-plane, oriented so that $\mathbf{n} = \mathbf{j}$. Then $\iint_{C_r}\mathbf{F}\cdot d\mathbf{x} - \int_{C_1}\mathbf{F}\cdot d\mathbf{x} = \pm\iint_{A_r}\operatorname{curl}\mathbf{F}\cdot\mathbf{n}\,dA$, the sign being $+$ if $r > 1$ and $-$ if $r < 1$. But $\operatorname{curl}\mathbf{F}\cdot\mathbf{n} = 3$, and the area of A_r is $\pi|r^2 - 1|$. It follows that $\int_{C_r}\mathbf{F}\cdot d\mathbf{x} = 5 + 3\pi(r^2 - 1)$.

8. Use (5.26) and (5.30): $\operatorname{curl}(f\nabla g) = f\operatorname{curl}(\nabla g) + \nabla f\times\nabla g = \nabla f\times\nabla g$. Now apply Stokes's theorem.

5.8 Integrating Vector Derivatives

1. (a) $f(x,y) = \int(2xy + x^2)\,dx = x^2y + \frac{1}{3}x^3 + \varphi(y)$; $\partial_y f(x,y) = x^2 + \varphi'(y) = x^2 - y^2$, so $\varphi(y) = -\frac{1}{3}y^3 + C$.

(b) $\partial_y(3y^2 + 5x^4y) = 6y + 5x^4 \neq 5x^4 - 6y = \partial_x(x^5 - 6xy)$.

(c) $f(x,y) = \int(2e^{2x}\sin y - 3y + 5)\,dx = e^{2x}\sin y - 3xy + 5x + \varphi(y)$; $\partial_y f(x,y) = e^{2x}\cos y - 3x + \varphi'(y) = e^{2x}\cos y - 3x$, so $\varphi(y) = C$.

(d) $f(x,y,z) = \int(yz - y\sin xy)\,dx = xyz + \cos xy + \varphi(y,z)$; $\partial_y f(x,y,z) = xz - x\sin xy + \partial_y\varphi(y,z) = xz - x\sin xy + z\cos yz$, so $\varphi(y,z) = \sin yz + \psi(z)$; $\partial_z f(x,y,z) = xy + y\cos yz + \psi'(z) = xy + y\cos yz$, so $\psi(z) = C$.

(e) $\partial_z(y - z) = -1 \neq 1 = \partial_x(x - y)$.

(f) $f(x,y,z) = \int 2xy\,dx = x^2y + \varphi(y,z)$; $\partial_y f(x,y,z) = x^2 + \partial_y\varphi(y,z) = x^2 + \log z$, so $\varphi(y,z) = y\log z + \psi(z)$; $\partial_z f(x,y,z) = (y/z) + \psi'(z) = (y+2)/z$, so $\psi(z) = 2\log z + C$.

(g) $f(x,y,z,w) = \int(xw^2 + yzw)\,dx = \frac{1}{2}x^2w^2 + xyzw + \varphi(y,z,w)$; $\partial_y f(x,y,z) = xzw + \partial_y\varphi(y,z,w) = xzw + yz^2 - 2e^{2y+z}$, so $\varphi(y,z,w) = \frac{1}{2}y^2z^2 - e^{2y+z} + \psi(z,w)$; $\partial_z f(x,y,z,w) = xyw + y^2z - e^{2y+z} + \partial_z\psi(z,w) = xyw + y^2z - e^{2y+z} - w\sin zw$, so $\psi(z,w) = \cos zw + \chi(w)$; $\partial_w f(x,y,z,w) = x^2w + xyz - z\sin zw + \chi'(w) = xyz + x^2w - z\sin zw$, so $\chi(w) = C$.

2. (a) $\operatorname{div}\mathbf{G} = 3x^2 + (1-3x^2) + 0 = 1 \neq 0$.

(b) Take $\mathbf{F} = F_1\mathbf{i} + F_2\mathbf{j}$, so $\operatorname{curl}\mathbf{F} = -\partial_z F_2 + \partial_z F_1 + (\partial_x F_2 - \partial_y F_1)\mathbf{k}$. Then $-\partial_z F_2 = xy + z$, so $F_2 = -xyz - \frac{1}{2}z^2 + \varphi(x,y)$, and $\partial_z F_1 = xz$, so $F_1 = \frac{1}{2}xz^2 + \psi(x,y)$. Hence, $\partial_x F_2 - \partial_y F_1 = -yz + \partial_x\varphi(x,y) - \partial_y\psi(x,y) = -yz - x$. One solution is $\varphi(x,y) = -\frac{1}{2}x^2$ and $\psi(x,y) = 0$.

(c) Proceeding as in (b), we have $-\partial_z F_2 = xe^{-x^2z^2} - 6x$, so $F_2 = -x\int_0^z e^{-x^2t^2}\,dt + 6xz + \varphi(x,y)$, and $\partial_z F_1 = 5y + 2z$, so $F_1 = 5yz + z^2 + \psi(x,y)$. Hence, $\partial_x F_2 - \partial_y F_1 = -\int_0^z e^{-x^2t^2}\,dt + 2x^2\int_0^z t^2 e^{-x^2t^2}\,dt + 6z + \partial_x\varphi(x,y) - 5z - \partial_y\psi(x,y) = z - ze^{-x^2z^2}$. This does not look hopeful, but by integration by parts, $2x^2\int_0^z t^2e^{-x^2t^2}\,dt = -ze^{-x^2z^2} + \int_0^z e^{-x^2t^2}\,dt$, so we can take $\varphi = \psi = 0$.

3. By the procedure outlined before Theorem 5.64 we can find f such that $\nabla^2 f = \operatorname{div}\mathbf{H}$. Then $\operatorname{div}(\mathbf{H} - \nabla f) = \operatorname{div}\mathbf{H} - \nabla^2 f = 0$, so by Theorem 5.63 there exists $\mathbf{G}$ such that $\operatorname{curl}\mathbf{G} = \mathbf{H} - \nabla f$.

4. (a) If $0 < r < s$, let A_{rs} be the annulus between C_r and C_s. Then $\int_{C_r}\mathbf{F}\cdot d\mathbf{x} - \int_{C_s}\mathbf{F}\cdot d\mathbf{x} = \int_{A_{rs}}(\partial_1 F_2 - \partial_2 F_1)\,dA = 0$ by Green's theorem. Thus $\alpha = \int_{C_r}\mathbf{F}\cdot d\mathbf{x}$ is independent of r.

(b) Let $\epsilon > 0$ be the minimum distance from points in the compact set C to the origin, and let $r = \frac{1}{2}\epsilon$. Then the curve C and the circle C_r together bound a region R, so by Green's theorem again, $0 = \int_R(\partial_1 F_2 - \partial_2 F_1)\,dA = \int_C \mathbf{F}\cdot d\mathbf{x} - \int_{C_r}\mathbf{F}\cdot d\mathbf{x} = \int_C\mathbf{F}\cdot d\mathbf{x} - \alpha$.

(c) From Example 1 and part (b), for any closed curve in S (oriented counterclockwise) we have $\int_C \mathbf{F}_0\cdot d\mathbf{x} = 2\pi$ and hence $\int_C(\mathbf{F} - (\alpha/2\pi)\mathbf{F}_0)\cdot d\mathbf{x} = 0$. By Proposition 5.60, $\mathbf{F} - (\alpha/2\pi)\mathbf{F}_0$ is a gradient.

Chapter 6

Infinite Series

6.1 Definitions and Examples

1. (a) This is a geometric series with initial term $2(x+1)$ and ratio $2(x+1)^3$; it converges for $2|x+1|^3 < 1$, i.e., $-1-2^{-1/3} < x < -1+2^{-1/3}$, to the sum $2(x+1)/[1-2(x+1)^3]$.

(b) This is a geometric series with initial term $10x^{-2}$ and ratio $2x^{-2}$; it converges for $2|x|^{-2} < 1$, i.e., $x < -\sqrt{2}$ or $x > \sqrt{2}$, to the sum $10x^{-2}/(1-2x^{-2}) = 10/(x^2-2)$.

(c) This is a geometric series with intial term 1 and ratio $(1-x)/(1+x)$. The ratio is less than 1 in absolute value if and only if x is closer to 1 than to -1, i.e., $x > 0$. The sum is $1/[1-(1-x)/(1+x)] = (1+x)/2x$.

(d) This is a geometric series with intial term $\log x$ and ratio $\log x$; it converges for $|\log x| < 1$, i.e., $e^{-1} < x < e$, to the sum $\log x/(1-\log x)$.

2. (a) The nth term is $\frac{1}{2} + 2^{-n}$, which does not tend to 0; the series diverges.

(b) This is a telescoping series; the sum of the first n terms is $1-(n+1)^{-1}$, so the full sum is 1.

(c) This is a telescoping series; the sum of the first n terms is $\sqrt{n+1}-1$, so the series diverges.

(d) The odd-numbered terms do not tend to zero; the series diverges.

3. If $f(x) = \log(1+x)$, then $f^{(k+1)}(x) = (-1)^k k!/(1+x)^{k+1}$, so Lagrange's formula reads $R_{0,k}(x) = (-1)^k x^{k+1}/(k+1)(1+c)^{k+1}$ where c is between 0 and x. For $x > 0$ we have $1+c > 1$, so $|R_{0,k}(x)| \leq x^{k+1}/(k+1)$, and for $x \leq 1$ this vanishes as $k \to \infty$. (For $-\frac{1}{2} < x < 0$ we still have $|x| < \frac{1}{2} < 1+c$, so $|R_{0,k}(x)| \leq 1/(k+1) \to 0$.) The integral formula (2.56) gives $R_{0,k}(x) = (-1)^k x^{k+1} \int_0^1 (1-t)^n (1+tx)^{-k-1}\, dt$, and the mean value theorem for integrals then gives $R_{0,k}(x) = (-1)^k x^{k+1}(1-\tau)^n(1+\tau x)^{-n-1}$ for some $\tau \in [0,1]$. Now, for $-1 < x < 0$, we have $0 \leq 1-\tau \leq 1+\tau x$ and $1+\tau x > 1+x$, so $|R_{0,k}(x)| = |x|^{n+1}(1+\tau x)^{-1}[(1-\tau)/(1+\tau x)]^n \leq |x|^{n+1}/(1+x)$, which vanishes as $k \to \infty$.

4. (a) Note that if $P \neq 0$ then all a_n are nonzero. Let $P_k = a_1 a_2 \cdots a_k$; then $a_k = P_k/P_{k-1} \to P/P = 1$ as $k \to \infty$.

(b) Assume all a_n are positive. With P_k as in (a), if $\prod_1^\infty a_n = P$, then $\sum_1^k \log a_n = \log P_k \to \log P$. Conversely, if $\sum \log a_n = S$, then $P_k = \exp(\sum_1^k a_n) \to e^S$.

6.2 Series with Nonnegative Terms

1. $a_n \sim n^{-3/2}$; converges by comparison to $\sum n^{-3/2}$.

2. Practically any test you can think of will work on this one! (Ratio test, root test, integral test using Corollary 2.12, comparison to geometric series $\sum 2^{-n}, \ldots$)

3. $a_n \sim n^{-2/3}$; diverges by comparison to $\sum n^{-2/3}$.

4. Converges by ratio test: $a_{n+1}/a_n = (n+2)/(n+1)^2 \to 0$.

5. Diverges by root test: $a_n^{1/n} = (2n+1)^3/(3n+1)^2 \sim \frac{8}{9}n \to \infty$.

6. Converges by ratio test: $a_{n+1}/a_n = (2n+3)^2/3(2n+1)(2n+2) \to \frac{1}{3}$.

7. Diverges by ratio test: $a_{n+1}/a_n = (n+1)/10 \to \infty$.

8. Diverges by comparison to $\sum n^{-1}$, using (2.13).

9. Converges by ratio test: $a_{n+1}/a_n = (2n+3)/(3n+5) \to \frac{2}{3}$.

10. Converges by ratio test: $a_{n+1}/a_n = (n+1)^2/(2n+1)(2n+2) \to \frac{1}{4}$.

11. Diverges by ratio test: $a_{n+1}/a_n = 3[n/(n+1)]^n \to 3/e > 1$. (See Exercise 7 in §2.1, with $x = n^{-1}$. The root test is a little easier if you know Stirling's formula.)

12. Converges by root test: $a_n^{1/n} = [n/(n+1)]^n \to 1/e$. (See Exercise 7 in §2.1, with $x = n^{-1}$.)

13. By l'Hôpital's rule or Taylor's theorem, $n^2[1 - \cos(1/n)] \to \frac{1}{2}$; series converges by comparison to $\sum n^{-2}$.

14. By rationalizing the numerator, $a_n = 1/(\sqrt{n+1} + \sqrt{n})\sqrt{n+2} \sim 1/2n$; series diverges by comparison to $\sum n^{-1}$.

15. $\sin[n/(n^2+3)] \sim n/(n^2+3) \sim 1/n$; series diverges by comparison to $\sum n^{-1}$.

16. Converges by the extended root test (part (a) of Theorem 6.14): $a_n^{1/n} = n^{2/n}[\pi + (-1)^n]/5 \le n^{2/n}(\pi + 1)/5 < .9$ for large n.

17. Converges by Raabe's test: $a_{n+1}/a_n = (2n+1)/(2n+4)$, so $n[1 - (a_{n+1}/a_n)] = 3n/(2n+4) \to \frac{3}{2}$.

18. Diverges by Raabe's test: $a_{n+1}/a_n = (2n+2)/(2n+3)$, so $n[1 - (a_{n+1}/a_n)] = n/(2n+3) \to \frac{1}{2}$.

19. If $\sum a_n$ converges, then $a_n \to 0$, so $a_n \le 1$ for large n. For such n we have $a_n^p < a_n$ for $p > 1$, so $\sum a_n^p$ converges by comparison to $\sum a_n$.

20. Use the integral test: $\int dx/x(\log x)^p = (\log x)^{1-p}/(1-p)$ for $p \ne 1$ and $\int dx/x \log x = \log\log x$, and $(\log x)^{1-p}$ remains bounded as $x \to \infty$ precisely when $p > 1$.

21. Use the integral test: $\int dx/x(\log x)(\log\log x)^p = (\log\log x)^{1-p}/(1-p)$ for $p \ne 1$ and $\int dx/x(\log x)(\log\log x) = \log\log\log x$ by the substitution $u = \log\log x$, so as in the preceding problem, the series converges if and only if $p > 1$.

22. From Theorem 6.7, $\sum_2^{10^{40}-1} 1/n\log n \geq \int_2^{10^{40}} dx/x\log x \geq \sum_3^{10^{40}} 1/n\log n$. The integral is $\log\log x\big|_2^{10^{40}} = \log 40 + \log\log 10 - \log\log 2 \approx 4.889$, and $1/10^{40}\log(10^{40})$ is negligible, so $4.889 < \sum_2^{10^{40}} 1/n\log n < 4.889 + (1/2\log 2) \approx 5.611$. Also from Theorem 6.7, $\sum_{10^{40}}^{\infty} 1/n(\log n)^2 \approx \int_{10^{40}}^{\infty} dx/x(\log x)^2 = -(\log x)^{-1}\big|_{10^{40}}^{\infty} = 1/40\log 10 \approx 0.11$. (The error in adding or removing the initial term in the sum is negligible.)

23. The derivative of $x/(x^2+1)^2$ is $(1-3x^2)/(x^2+1)^3$, which is nonpositive for $x \geq 1/\sqrt{3}$. By Theorem 6.7, $\sum_3^\infty n/(n^2+1)^2 \geq \int_3^\infty x\,dx/(x^2+1)^2 \geq \sum_4^\infty n/(n^2+1)^2$. The integral is $-1/2(x^2+1)\big|_3^\infty = .05$, so $.05 < \sum_3^\infty n/(n^2+1)^2 < .05 + 3/(3^2+1)^2 = .08$. Adding on the first two terms, namely .25 and .08, gives $.38 < \sum_1^\infty n/(n^2+1)^2 < .41$.

24. $c_n = \sum_1^n 1/k - \int_1^n dx/x > 0$ as in Theorem 6.7. Also, $c_{n+1} - c_n = 1/(n+1) - \log(n+1) + \log n = 1/(n+1) - \int_n^{n+1} dx/x < 1/(n+1) - \int_n^{n+1} dx/(n+1) = 0$, so $\{c_n\}$ is decreasing.

25. Let $r = (1+L)/2$. If $L < 1$ then $r < 1$, and $a_n^{1/n} < r$ for all but finitely many n; it follows from Theorem 6.14a that $\sum a_n$ converges. If $L > 1$ then $a_n^{1/n} > 1$ for infinitely many n and so $a_n > 1$ for infinitely many n; it follows that $\sum a_n$ diverges.

6.3 Absolute and Conditional Convergence

1. For (a) and (b), just use the fact that $|\cos n\theta| \leq 1$ and $|\sin n\theta| \leq 1$ for all n to get a comparison with the convergent series $\sum |x|^n$ and $\sum n^{-2}$. For (c), use the ratio test.

2. To get a rearrangement whose sum is $+\infty$, add up the positive terms until the sum exceeds 1; then put in one negative term; then add more positive terms until the sum exceeds 2; then put in another negative term, etc. Since the original series converges, only finitely many negative terms can be less than -1. After they are all used up, once the sum exceeds $K+1$ it never drops below K, so it tends to $+\infty$ as more terms are added.

3. For the series $0 + \frac{1}{2} + 0 - \frac{1}{4} + 0 + \frac{1}{6} + \cdots$, the $(2m-1)$th and $(2m)$th partial sums both coincide with the mth partial sum of $\frac{1}{2}(1 - \frac{1}{2} + \frac{1}{3} - \cdots)$; hence they converge to $\frac{1}{2}\log 2$. When this series is added to $1 - \frac{1}{2} + \frac{1}{3} - \cdots = \log 2$, the odd-numbered (positive) terms of the latter series survive unchanged, and for all $k \geq 0$, the $(2k+2)$th terms of the two series cancel and the $(4k)$th terms add up to give the negative terms $-\frac{1}{2}, -\frac{1}{4}, -\frac{1}{6}, \ldots$. After omitting the resulting zero terms one obtains the stated rearranged series, whose sum is therefore $\frac{3}{2}\log 2$.

4. Let $s_k = \sum_0^k a_n$ and $t_k = \sum_0^k b_n$. Then $t_k = s_k$ if k is odd, and $t_k = s_k - a_{k+1} - a_k$ if k is even. Since $a_k \to 0$, $\lim s_k = \lim t_k$.

5. (a) Suppose $\sum |a_n| < \infty$. After throwing out finitely many terms we may assume $|a_n| \leq \frac{1}{2}$ for all n, in which case $|\log(1+a_n)| < .7$ for all n. By Taylor's theorem, $\log(1+a_n) = a_n + R(a_n)$ where $|R(a_n)| \leq C|a_n|^2 \leq C|a_n|$, so $\sum \log(1+a_n)$ is the sum of the two absolutely convergent series $\sum a_n$ and $\sum R(a_n)$. Conversely, suppose $\sum |b_n| < \infty$ where $b_n = \log(1+a_n)$. We have $a_n = e^{b_n} - 1$, and Taylor's theorem again gives $a_n = b_n + r(b_n)$ where $|r(b_n)| \leq C|b_n|^2 \leq C|b_n|$, so $\sum a_n$ is the sum of the two absolutely convergent series $\sum b_n$ and $\sum r(b_n)$.

(b) $\sum a_n$ is convergent by the alternating series test, but not absolutely convergent since $\sum n^{-1/2} = \infty$. By Taylor's theorem, $\log(1 + a_n) = a_n - \frac{1}{2}a_n^2 + R(a_n)$ where $|R(a_n)| \le C|a_n|^3 = Cn^{-3/2}$. $\sum a_n$ converges as above; $\sum R(a_n)$ is absolutely convergent; but $\sum \frac{1}{2}a_n^2 = \sum 1/2n$ diverges.

6.4 More Convergence Tests

1. By the ratio test, the series converges absolutely for $|x + 2| < 1$, i.e., $-3 < x < -1$, and diverges for $|x + 2| > 1$. At $x = -3$ or $x = -1$ the series becomes $\sum(-1)^n/(n^2 + 1)$ or $\sum 1/(n^2 + 1)$, both of which are absolutely convergent by comparison to $\sum 1/n^2$.

2. By the ratio test or the root test, the series converges absolutely for $|2x - 1| < 1$, i.e., $0 < x < 1$, and diverges for $|2x - 1| > 1$. At $x = 0$ or $x = 1$ the series becomes $\sum(-1)^n n^3$ or $\sum n^3$, both of which diverge since $n^3 \not\to 0$.

3. By the ratio test, the series converges absolutely for all x: $|a_{n+1}/a_n| = x^2/(2n + 3) \to 0$ as $n \to \infty$.

4. By the ratio test, the series converges absolutely for $|x/5| < 1$, i.e., $-5 < x < 5$, and diverges for $|x/5| > 1$. At $x = 5$ the series is $\sum 25n/(n + 1)^2$, which diverges by comparison with $\sum 1/n$. At $x = -5$ it is $\sum 25n(-1)^n/(n + 1)^2$, which converges (conditionally) by the alternating series test.

5. The series converges absolutely for $|x - 4| < 2$, i.e., $2 < x < 6$, and diverges for $|x - 4| > 2$, by the ratio test. (In detail: $|a_{n+1}/a_n| = |x - 4|(2^n - 3)\log(n + 3)/(2^{n+1} - 3)\log(n + 4)$. We have $(2^n - 3)/(2^{n+1} - 3) = (1 - 3 \cdot 2^{-n})/(2 - 3 \cdot 2^{-n}) \to \frac{1}{2}$, and $[\log(n + 3)/\log(n + 4)] - 1 = \log[(n+3)/(n+4)]/\log(n+4) \to 0/\infty = 0$.) At $x = 2$ the series is $\sum 2^n/(2^n - 3)\log(n+3)$, which diverges by comparison to (for example) $\sum 1/n$. At $x = 6$ the series is $\sum(-1)^n 2^n/(2^n - 3)\log(n+3)$, which converges (conditionally) by the alternating series test.

6. By the ratio test or the root test, the series converges absolutely or diverges according as $|(x-1)/(x+1)|$ is < 1 or > 1, i.e., $x > 0$ or $x < 0$. At $x = 0$ the series is $\sum(-1)^n/\sqrt{n}$, which converges (conditionally) by the alternating series test.

7. By the ratio test, the series converges absolutely for $|\frac{1}{2}x - 3| < 1$, i.e., $4 < x < 8$, and diverges for $|\frac{1}{2}x - 3| > 1$. The numerator of the coefficient of $(\frac{1}{2}x - 3)^n$ is clearly bigger than the denominator, so at $x = 4$ or $x = 8$ the terms of the series do not tend to zero and the series diverges.

8. By the ratio test, the series converges absolutely for $|x + 1| < 1$, i.e., $-2 < x < 0$, and diverges for $|x+1| > 1$. At both $x = -2$ and $x = 0$ the series is $\sum(-1)^n/(3n+2)$, which converges (conditionally) by the alternating series test.

9. We have $|a_{n+1}/a_n| = (2n + 3)|x|/(3n + 5) \to \frac{2}{3}|x|$, so by the ratio test, the series converges absolutely for $|x| < \frac{3}{2}$ and diverges for $|x| > \frac{3}{2}$. At $x = \frac{3}{2}$, we have $a_{n+1}/a_n = (6n + 9)/(6n + 10)$, and $n[1 - (6n + 9)/(6n + 10)] = n/(6n + 10) \to \frac{1}{6}$, so the series diverges by Raabe's test. However, by the proof of Raabe's test, the nth term is comparable to $n^{-1/6}$, and the terms decrease since $(6n+9)/(6n+10) < 1$. Hence, at $x = -\frac{3}{2}$, where there is an extra factor of $(-1)^n$, the series converges conditionally by the alternating series test.

10. By Taylor's theorem, $\log[1 + (1/n)] = (1/n) + r_n$ where $|r_n| \le Cn^{-2}$. The series $\sum(-1)^n/n$ converges conditionally, while $\sum(-1)^n r_n$ converges absolutely; hence the original series converges conditionally.

11. Let $a_n = \int_n^{n+1} \log(x+7)\,dx/x$. Then $a_n > \int_n^{n+1} dx/n = 1/n$, so the series is not absolutely convergent. On the other hand, $\log(x+7)/x$ is decreasing for $x > 0$, and hence so is a_n, and $a_n < \log(n+7)/n \to 0$, so $\sum(-1)^n a_n$ converges by the alternating series test.

12. $n^{1/n} \to 1 \neq 0$, so the series diverges.

13. By Taylor's theorem, $n \sin n^{-1} = 1 + r_n$ where $|r_n| \leq Cn^{-2}$, and hence $\log(n \sin n^{-1}) = \log(1 + r_n) = r_n + R_n$ where $|R_n| \leq C'|r_n|^2 \leq C''n^{-4}$. Hence the series converges absolutely by comparison to $\sum n^{-2}$.

14. By Taylor's theorem, $\log(1+n^{-1}) = n^{-1} - \frac{1}{2}n^{-2} + r_n$ where $|r_n| \leq Cn^{-3}$, so $\log((n+1)/n)^n = n\log(1+n^{-1}) = 1 - (2n)^{-1} + nr_n$. Hence,

$$\left(\frac{n+1}{n}\right)^n = \exp\left(1 - \frac{1}{2n} + nr_n\right) = e\exp\left(-\frac{1}{2n} + nr_n\right) = e\left(1 - \frac{1}{2n} + nr_n + R_n\right),$$

where $|R_n| \leq C'[-(2n)^{-1} + nr_n]^2 \leq C''n^{-2}$. Finally,

$$e - \left(\frac{n+1}{n}\right)^n = \frac{e}{2n} - enr_n - eR_n.$$

$\sum(-1)^{n-1}e/2n$ is conditionally convergent, and $\sum(-1)^n e(nr_n + R_n)$ is absolutely convergent by comparison to $\sum n^{-2}$, so the original series is conditionally convergent.

15. The power series $\sum_0^\infty (-1)^n x^{2n}/(2n+1)!$ for $x^{-1}\sin x$ is an alternating series (for any x, since the powers of x are all even), and the absolute value of the ratio of the $(n+1)$th term to the nth is $x^2/(2n+2)(2n+3)$. For $|x| \leq \pi$ this is less than 1 when $n \geq 1$, so the terms decrease after the first one. Hence the error is smaller than the first neglected term, namely $x^8/9! \leq \pi^8/9! \approx .261$.

16. Let $b = \lim b_n$. *First Method:* We have $\sum a_n b_n = \sum a_n(b_n - b) + b\sum a_n$. Since $b_n - b$ decreases to 0 and the partial sums of $\sum a_n$ converge to the full sum, Dirichlet's test gives the convergence of $\sum a_n(b_n - b)$. *Second Method:* Let $A_n = a_0 + \cdots + a_n$ and $b_n' = b_n - b_{n-1}$. Then by Lemma 6.23, $\sum_0^k a_n b_n = A_k b_k + \sum_1^k A_{n-1}b_n'$. We have $A_k b_k \to (\sum a_n)b$, and the series $\sum A_{n-1}b_n'$ is absolutely convergent since $\sum |A_{n-1}b_n'| \leq C\sum |b_n'| = C\sum(-b_n') = C(b_0 - b)$.

17. The convergence of $\sum n^{-p}a_n$ follows from Dirichlet's test. (Take $a_n = n^{-p}$ in Theorem 6.25, then relabel b_n as a_n.) Absolute convergence is guaranteed for $p > 1$, since $a_n \to 0$ and hence $|a_n| \leq C$.

18. The series converges absolutely by comparison to the geometric series $\sum |x|^n$ when $|x| < 1$ (for any θ). When $x = 1$ it converges for $\theta \neq 2k\pi$ by Corollary 6.27. When $x = -1$ it converges for $\theta \neq (2k+1)\pi$ by Corollary 6.27, since $(-1)^n \cos n\theta = \cos n(\theta - \pi)$.

6.5 Double Series; Products of Series

1. (a) $(1-x)^{-1}(1-x)^{-1} = (\sum_0^\infty x^n)(\sum_0^\infty x^m) = \sum_{j=0}^\infty (\sum_{m+n=j} 1)x^j = \sum_{j=0}^\infty (j+1)x^j$.

(b) $(1-x)^{-1}(1-x)^{-2} = (\sum_0^\infty x^n)(\sum_0^\infty (m+1)x^m) = \sum_{j=0}^\infty (\sum_{m+n=j}(m+1))x^j = \sum \frac{1}{2}(j+1)(j+2)x^j$ since $\sum_{m+n=j}(m+1) = 1 + 2 + \cdots + (j+1) = \frac{1}{2}(j+1)(j+2)$.

2. $f(x)f(y) = (\sum_0^\infty x^n/n!)(\sum_0^\infty y^m/m!) = \sum_{j=0}^\infty \sum_{m+n=j} x^n y^m/n!m!$. But by the binomial theorem, $\sum_{m+n=j} x^n y^m/n!m! = \sum_{n=0}^j x^n y^{j-n}/n!(j-n)! = (x+y)^j/j!$, so $f(x)f(y) = \sum_{j=0}^\infty (x+y)^j/j! = f(x+y)$.

3. If $f(x) = (1-4x)^{-1/2}$, then $f^{(n)}(x) = (-\frac{1}{2})(-\frac{3}{2})\cdots(-n+\frac{1}{2})(-4)^n(1-4x)^{-n-(1/2)} = 1\cdot 3\cdots(2n-1)2^n(1-4x)^{-n-(1/2)}$. Moreover, $1\cdot 3\cdots(2n-1) = (2n)!/2\cdot 4\cdots(2n) = (2n)!/2^n n!$. It follows that the coefficient of x^n in the Taylor series is $(2n)!/(n!)^2$, so that series converges for $|x| < \frac{1}{4}$ by the ratio test. The product of this series with itself is the series $\sum c_j x^j$ where $c_j = \sum_{m+n=j}(2n)!(2m)!/(n!m!)^2 = \sum_{n=0}^j (2n)!(2j-2n)!/[n!(j-n)!]^2$. On the other hand, the sum of the latter series is $(1-4x)^{-1}$, whose Taylor series is the geometric series $\sum_{j=0}^\infty (4x)^j$. Equating coefficients of x^j in these two series gives the asserted formula $c_j = 4^j$. (The justification for the last step is contained in Theorem 2.77: $|(1-4x)^{-1} - \sum_0^k c_j x^j| = |x|^{k+1}|\sum_{k+1}^\infty c_j x^{j-k-1}| \le C|x|^{k+1}$ for $|x| \le \frac{1}{8}$, say, so $\sum_0^k c_j x^j$ is the kth Taylor polynomial of $(1-4x)^{-1}$. See also Corollary 7.22.)

4. $\sum_0^\infty (-1)^n (n+1)^{-1/2}$ is convergent by the alternating series test, but not absolutely convergent. The Cauchy product of the series with itself is $\sum_{j=0}^\infty (-1)^j c_j$ where $c_j = \sum_{n=0}^j [(n+1)(j-n+1)]^{-1/2}$. By the hint, $c_j \ge \sum_{n=0}^j (\frac{1}{2}j+1)^{-1} = (j+1)/(\frac{1}{2}j+1) \not\to 0$ as $j \to \infty$; hence $\sum (-1)^j c_j$ diverges.

5. Let $S = \sum_{m,n=0}^\infty a_{mn}$, which we think of as the limit of the square partial sums $s_k^\square$, and for each m, let $S_m = \sum_{n=0}^\infty a_{mn}$. The claim is that $S = \sum_0^\infty S_m$, whether these quantities are finite or not.

Case I: $\sum S_m < \infty$. Clearly $s_M^\square \le \sum_{m=0}^M S_m$, so $S \le \sum S_m$. On the other hand, given $\epsilon > 0$, we can find M such that $\sum_0^M S_m > \sum_0^\infty S_m - \epsilon$. We can then find N such that $\sum_{n=0}^N a_{mn} > S_m - \epsilon(M+1)^{-1}$ for $m = 0, \ldots, M$. Let $K = \max(M, N)$; then $s_K^\square \ge \sum_{m=0}^M \sum_{n=0}^N a_{mn} \ge \sum_0^\infty S_m - 2\epsilon$. It follows that $S \ge \sum S_m$ and hence $S = \sum S_m$.

Case II: $S_m < \infty$ for each m, but $\sum S_m = \infty$. Given $C > 0$, we can find M such that $\sum_0^M S_m > C$. We can then find N such that $\sum_{n=0}^N a_{mn} > S_m - (M+1)^{-1}$ for $m = 0, \ldots, M$. Let $K = \max(M, N)$; then $s_K^\square \ge \sum_{m=0}^M \sum_{n=0}^N a_{mn} \ge C - 1$. It follows that $S = \infty$.

Case III: $S_m = \infty$ for some $m = m_0$. Given $C > 0$, we can find N such that $\sum_0^N a_{m_0 n} > C$. Let $K = \max(m_0, N)$; then $s_K^\square > C$. It follows that $S = \infty$.

6. Since $\sum_{m,n=0}^\infty |a_{mn}| < \infty$, it is approximated by its square partial sums, so given $\epsilon > 0$ we have $\sum_{\max(m,n)>K} |a_{mn}| < \epsilon$ for K sufficiently large. This is a double series (it's obtained from $\sum_{m,n=0}^\infty |a_{mn}|$ by replacing a_{mn} by 0 when $\max(m,n) \le K$), so by Exercise 5, it can be summed as an iterated series first in n, then in m. Summing over only those m such that $m \le K$ gives $\sum_{m=0}^K \sum_{n=K+1}^\infty |a_{mn}| < \epsilon$, that is, the sum of the tail ends of the series $\sum_{n=0}^\infty |a_{mn}|$ over $m = 0, \ldots, K$ is less than ϵ. With notation as in Exercise 5, this implies that $|s_K^\square - \sum_{m=0}^K S_m| < \epsilon$. But also $|S - s_K^\square| \le \sum_{\max(m,n)>K} |a_{mn}| < \epsilon$, so $|S - \sum_{m=0}^K S_m| < 2\epsilon$. It follows that $S = \sum_{m=0}^\infty S_m$.

7. $s_k^\triangle = \sum_{j=1}^k \sum_{m+n=j} (m+n)^{-p} = \sum_{j=1}^k j \cdot j^{-p} = \sum_{j=1}^k j^{1-p}$. This has a finite limit as $k \to \infty$ if and only if $p > 2$.

8. The only nonzero terms in $\sum_{m=0}^\infty a_{mn}$ are $+1$ at $m = n$ and -1 at $m = n+1$, so $\sum_{m=0}^\infty a_{mn} = 0$ for all n. On the other hand, the only nonzero terms in $\sum_{n=0}^\infty a_{mn}$ are $+1$ at $n = m$ and -1 at $n = m-1$, and the latter term is missing when $m = 0$. Hence the sum is 0 for $m > 0$ and 1 for $m = 0$. It follows that the sum of the first iterated series is 0, whereas the sum of the second one is 1.

Chapter 7

Functions Defined by Series and Integrals

7.1 Sequences and Series of Functions

1. (a) $\lim f_k(x) = 0$ if $0 \le x < 1$, $\lim f_k(1) = 1$. We have $|f_k(x) - 0| \le (1-\delta)^k \to 0$ for $x \in [0, 1-\delta]$, so the convergence is uniform there.

(b) $\lim f_k(0) = 0$, $\lim f_k(x) = 1$ if $0 < x \le 1$. We have $|f_k(x) - 1| = 1 - \delta^{1/k} \to 0$ for $x \in [\delta, 1]$, so the convergence is uniform there.

(c) $\lim f_k(x) = 0$ if $x \in [0, \pi] \setminus \{\frac{1}{2}\pi\}$, $\lim f_k(\frac{1}{2}\pi) = 1$. We have $|f_k(x) - 0| \le \sin^k(\frac{1}{2}\pi - \delta) \to 0$ for $x \in [0, \frac{1}{2}\pi - \delta]$ or $x \in [\frac{1}{2}\pi + \delta, 1]$, so the convergence is uniform there.

(d) $|f_k(x)| \le k^{-1}$ for all x, so $f_k \to 0$ uniformly on $\mathbb{R}$.

(e) $\lim f_k(x) = 0$ for all $x \in [0, \infty)$, but the maximum of f_k on this interval is e^{-1} (at $x = k^{-1}$), so the convergence is not uniform. However, $|f_k(x) - 0| \le k\delta e^{-k\delta}$ for $x \ge \delta$ provided $k \ge \delta^{-1}$, and $\lim_{k\to\infty} k\delta e^{-k\delta} = 0$; hence the convergence is uniform on $[\delta, \infty)$.

(f) $\lim f_k(x) = 0$ for all $x \in [0, \infty)$, but the maximum of f_k on this interval is e^{-1} (at $x = k$), so the convergence is not uniform. However, $|f_k(x) - 0| \le b/k$ for $x \le b$, so the convergence is uniform on $[0, b]$ for any b.

(g) $\lim f_k(x) = 0$ for $x \ne 1$ since $f_k(x) < x^k$ for $x < 1$ and $f_k(x) < x^{-k}$ for $x > 1$, and $\lim f_k(1) = \frac{1}{2}$. For any $\delta > 0$ we have $|f_k(x) - 0| \le (1-\delta)^k$ for $x \in [0, 1-\delta]$ and $|f_k(x) - 0| \le (1+\delta)^{-k}$ for $x \in [1+\delta, \infty)$, so the convergence is uniform on these intervals.

2. (a) The series is a geometric series, convergent for $x > 0$ to the sum $1/(1-e^{-x})$. The convergence is absolute and uniform on $[\delta, \infty)$ for any $\delta > 0$, by the M-test with $M_n = e^{-n\delta}$. The sum is continuous on $(0, \infty)$.

(b) The series is absolutely and uniformly convergent on $[-1, 1]$ by the M-test with $M_n = 1/n^2$ ($n > 0$); it diverges elsewhere since the nth term $\not\to 0$. The sum is continuous on $[-1, 1]$.

(c) The series is absolutely and uniformly convergent on $[-2+\delta, 2-\delta]$ for any $\delta > 0$ by the M-test with $M_n = \frac{1}{8}n(1 - \frac{1}{2}\delta)^n$ ($\sum M_n$ converges by the ratio test). It diverges for $|x| \ge 2$ since the nth term $\not\to 0$. The sum is continuous on $(-2, 2)$.

(d) The series is absolutely and uniformly convergent on $\mathbb{R}$ by the M-test with $M_n = 1/n^3$; the sum is everywhere continuous.

(e) The series is absolutely and uniformly convergent on $\mathbb{R}$ by the M-test with $M_n = 1/n^2$; the sum is everywhere continuous.

(f) The series is absoutely and uniformly convergent on $[1+\delta,\infty)$ for any $\delta > 0$ by the M-test with $M_n = n^{-1-\delta}$, and it diverges for $x \le 1$ (Theorem 6.9). The sum is continuous on $(1,\infty)$.

3. Let M be the maximum value of $|g(x)|$ on $[0,1]$. Given $\epsilon > 0$, choose $\delta > 0$ so that $|g(x)| < \epsilon$ for $1-\delta \le x \le 1$. Then if k is large enough so that $(1-\delta)^k < \epsilon/M$ we have $|f_k(x)| \le Mx^k < \epsilon$ for $x \le 1-\delta$ and $|f_k(x)| \le |g(x)| < \epsilon$ for $x > 1-\delta$. That is, $|f_k(x)| < \epsilon$ for all $x \in [0,1]$ when k is sufficiently large, so $f_k \to 0$ uniformly on $[0,1]$.

4. Given $\delta > 0$, let $I_1 = [-1+\delta, 1-\delta]$, and for $k \ge 2$ let $I_k = [k-1+\delta,\ k-\delta]$. For a given k, let $M_n = \max_{x\in I_k} |x^2 - n^2|^{-1}$. Then $M_n < \infty$ for all n, and $M_n/n^{-2} \to 1$ as $n \to \infty$, so $\sum M_n < \infty$. The M-test therefore gives the uniform convergence of the series for $x \in I_k$ or $-x \in I_k$.

5. The series fails to converge absolutely by comparison to $\sum 1/n$. However, $1/(x^2+n)$ decreases to 0 as $n \to \infty$ for each x, so by the alternating series test, the series converges for each x, and the absolute difference between the kth partial sum and the full sum is at most $1/(x^2+k+1) \le 1/(k+1)$. The latter quantity is independent of x and tends to zero as $k \to \infty$, so the convergence is uniform.

6. (a) Since $\sum c_n$ converges we have $|c_n| \le C$, so for $x \in [-a,a]$ $(a < 1)$ we have $|c_n x^n/(1-x^n)| \le Ca^n/(1-a)$. Hence $\sum c_n x^n/(1-x^n)$ converges absolutely and uniformly on $[-a,a]$ by the M-test.

(b) By the hint, $\sum c_n x^n/(1-x^n) = \sum c_n/(1-x^n) - \sum c_n$. The first series on the right converges absolutely and uniformly for $|x| \ge b > 1$ by the M-text, since $|c_n/(1-x^n)| \le C/(b^n-1)$, and $\sum 1/(b^n-1)$ converges by comparison to $\sum 1/b^n$. The second one is independent of x, so it does not affect uniform convergence, but it decides the issue of absolute convergence.

7. If $f_k \to f$ uniformly on S_m, we have $|f_k(x)-f(x)| \le C_k^m$ for $x \in S_m$, where $C_k^m \to 0$ as $k \to \infty$. But then $|f_k(x) - f(x)| \le \max(C_k^1,\ldots,C_k^M)$ for $x \in \bigcup_1^M S_m$, and $\max(C_k^1,\ldots,C_k^M) \to 0$ as $k \to \infty$.

8. By Theorem 7.7, the point is to show that $\{f_k\}$ is uniformly Cauchy on $[a,b]$. But it is uniformly Cauchy on (a,b), so given $\epsilon > 0$ there exists K such that $|f_j(x) - f_k(x)| < \epsilon$ for $x \in (a,b)$ whenever $j,k > K$. Since f_j and f_k are continuous on $[a,b]$, we can take the limit as $x \to a, b$ to conclude that $|f_j(x) - f_k(x)| \le \epsilon$ for $x \in [a,b]$ whenever $j,k > K$, and hence $\{f_k\}$ is uniformly Cauchy on $[a,b]$.

9. Since $f - f_k$ is continuous, the set S_k in the hint is closed and hence compact. Moreover, since f_k increases to f, we have $S_1 \supset S_2 \supset \cdots$. By Exercise 5 in §1.6, if each S_k is nonempty then so is $\bigcap_1^\infty S_k$. But if x is in the latter set we have $f(x) - f_k(x) \ge \epsilon$ for all k, which is false since $\lim f_k(x) = f(x)$. Hence some S_K (and hence every S_k with $k > K$) is empty, i.e., $|f(x) - f_k(x)| < \epsilon$ for all $x \in S$ when $k \ge K$. Thus $f_k \to f$ uniformly on S.

7.2 Integrals and Derivatives of Sequences and Series

1. The series defining f converges absolutely and uniformly on $\mathbb{R}$ by the M-test with $M_n = n^{-2}$. Hence f is continuous, and termwise integration is permissible: $\int_0^{\pi/2} f(x)\,dx = \sum_1^\infty n^{-2} \int_0^{\pi/2} \sin nx\,dx = \sum_1^\infty [-n^{-3}\cos nx]_0^{\pi/2}$. Now, $\cos \frac{1}{2}n\pi$ is 0 when n is odd and $(-1)^{n/2}$ when n is even; hence the nth term of the last series is n^{-3} when n is odd, $2n^{-3}$ when $n = 2, 6, 10, \ldots$, and 0 when $n = 4, 8, 12, \ldots$.

2. The series defining f converges absolutely and uniformly on $[0,\infty)$ by the M-test with $M_n = n^{-2}$. Hence f is continuous there, and $\int_0^1 f(x)\,dx = \sum_1^\infty \int_0^1 (x+n)^{-2}\,dx = \sum_1^\infty [n^{-1} - (n+1)^{-1}]$. This is a telescoping series; the nth partial sum is $1 - (n+1)^{-1}$, so the full sum is 1.

3. (a) Just observe that $\lim_{k\to\infty} \arctan kx$ is $\frac{1}{2}\pi$ if $x > 0$, $-\frac{1}{2}\pi$ if $x < 0$, and 0 if $x = 0$.

(b) We have $f_k'(x) = \arctan kx + kx/(1 + k^2x^2)$. If $x \neq 0$ the second term tends to zero like $1/k$ as $k \to \infty$, and if $x = 0$ it vanishes to begin with. Hence $\lim_{k\to\infty} f_k'(x)$ is $\frac{1}{2}\pi$, 0, or $-\frac{1}{2}\pi$ for $x > 0$, $x = 0$, or $x < 0$ respectively. The convergence cannot be uniform near $x = 0$ because the limit is discontinuous.

4. In each case it is a matter of using the M-test to establish the uniform convergence of the derived series on compact subsets of the interval of convergence. In what follows we write down the derived series and the constants M_n in the M-test.

(a) $-\sum ne^{-nx}$; $M_n = ne^{-\delta n}$ for $x \geq \delta$.

(b) $\sum nx^{n-1}/(n^2 + n + 1)$; $M_n = (1 - \delta)^{n-1}$ for $|x| \leq 1 - \delta$.

(c) $\sum n^2x^{n-1}/2^{n+3}$; $M_n = n^2(1 - \frac{1}{2}\delta)^{n-1}$ for $|x| \leq 2 - \delta$.

(d) $-\sum(\sin nx)/n^2$; $M_n = 1/n^2$.

(e) $\sum(-2x)/(x^2 + n^2)^2$; $M_n = 2K/n^4$ for $|x| \leq K$.

(f) $-\sum n^{-x}\log n$; $M_n = n^{-1-d}\log n$ for $|x| \geq 1 + \delta$.

5. Using the observation that $2x/(x^2 - n^2) = (x - n)^{-1} + (x + n)^{-1}$, we see that the derived series of f is $-\sum_1^\infty[(x - n)^{-2} + (x + n)^{-2}]$. This series converges uniformly on compact subsets of $\mathbb{R} \setminus \{\pm1, \pm2, \ldots\}$ by the same argument as in Exercise 4, §7.1 (basically, a comparison to $\sum n^{-2}$), so the termwise differentiation is justified.

6. (a) For $x \geq \delta > 0$, $0 \leq kf(kx) \leq Ck(kx)^{-1-\epsilon} \leq C\delta^{-1-\epsilon}k^{-\epsilon} \to 0$ as $k \to \infty$.

(b) With $y = kx$, $\int_0^1 kf(kx)\,dx = \int_0^k f(y)\,dy \to \int_0^\infty f(y)\,dy = a$.

(c) Given $\epsilon > 0$, pick $\delta > 0$ so that $|g(x) - g(0)| < \epsilon/3a$ when $0 \leq x \leq \delta$. Then $\int_0^1 f_k(x)g(x)\,dx$ is the sum of $g(0)\int_0^1 f_k(x)\,dx$, $\int_0^\delta f_k(x)[g(x) - g(0)]\,dx$, and $\int_\delta^1 f_k(x)[g(x) - g(0)]\,dx$. By part (b), the first term is within $\epsilon/3$ of $ag(0)$ provided k is suffuciently large. The absolute value of the second term is at most $(\epsilon/3a)\int_0^\delta f_k(x)\,dx \leq \epsilon/3$. Finally, since integrable functions are bounded we have $|g(x) - g(0)| \leq C$, so the third term is no bigger than $C\int_\delta^1 f_k(x)\,dx$, which is less than $\epsilon/3$ for large k by part (a). In short, $|\int_0^1 f_k(x)g(x)\,dx - ag(0)| < \epsilon$ for k sufficiently large, and we are done.

7.3 Power Series

1. (a) $|a_{n+1}x^{n+1}/a_nx^n| = |a_{n+1}/a_n|\,|x| \to L|x|$, so by the ratio test, $\sum a_nx^n$ converges when $L|x| < 1$ and diverges when $L|x| > 1$.

(b) $|a_nx^n|^{1/n} \to L|x|$, so by the root test, $\sum a_nx^n$ converges when $L|x| < 1$ and diverges when $L|x| > 1$.

2. If $|a_n| \leq C$, then $|a_nx^n|^{1/n} \leq C^{1/n}|x|$. If $|x| < 1$ then $C^{1/n}|x| < 1$ for large n, so $\sum a_nx^n$ converges by the root test (Theorem 6.14a).

3. $\sum_0^\infty a_nx^{kn}$ converges when $|x^k| < R$ and diverges when $|x^k| > R$, so the radius of convergence is $R^{1/k}$.

4. Let $1/R = \limsup |a_n|^{1/n}$. If $|x| < R$, let $r = (R + |x|)/2$. Then $|a_n x^n|^{1/n} < |x|/r < 1$ for large n, so $\sum a_n x^n$ converges by the root test (Theorem 6.14a). If $|x| > R$, then $|a_n x^n|^{1/n} > 1$ for infinitely many n, so $a_n x^n \not\to 0$ and $\sum a_n x^n$ diverges.

5. (a) $e^{-t^2} = \sum_0^\infty (-t^2)^n/n!$ for $t \in \mathbb{R}$. By Theorem 7.18, $\int_0^x e^{-t^2}\,dt = \sum_0^\infty (-1)^n x^{2n+1}/n!(2n+1)$ for $x \in \mathbb{R}$.

(b) $\cos t^2 = \sum_0^\infty (-1)^n (t^2)^{2n}/(2n)!$ for $t \in \mathbb{R}$. By Theorem 7.18, $\int_0^x \cos t^2\,dt = \sum_0^\infty (-1)^n x^{4n+1}/(2n)!(4n+1)$ for $x \in \mathbb{R}$.

(c) $t^{-1}\log(1+2t) = \sum_1^\infty (-1)^{n-1} 2^n t^{n-1}/n$ for $-1 < 2t \le 1$ (Exercise 3 in §6.1), with the understanding that $t^{-1}\log(1+2t)|_{t=0} = 2$. By Theorem 7.18, $\int_0^x t^{-1}\log(1+2t)\,dt = \sum_1^\infty (-1)^{n-1}(2x)^n/n^2$ for $|x| < \frac{1}{2}$. However, the integral on the left is continuous for $|x| \le \frac{1}{2}$ (it is improper but convergent at $x = -\frac{1}{2}$), and the series on the right converges absolutely and uniformly for $|2x| \le 1$ by comparison to $\sum n^{-2}$. Thus the equality persists for $|x| = \frac{1}{2}$.

6. Each of the series for the three integrals in question, namely, $\sum_0^\infty (-1)^n/n!(2n+1)$, $\sum_0^\infty (-1)^n/(2n)!(4n+1)$, and $\sum_1^\infty (-1)^{n-1}/n^2$, is an alternating series whose terms decrease monotonically to zero in absolute value, so the full sum lies in between any two successive partial sums. One simply computes the partial sums (with a calculator or otherwise) until the desired accuracy is attained.

(a) $.74672\ldots = \sum_0^5 (-1)^n/n!(2n+1) < \int_0^1 e^{-t^2}\,dt < \sum_0^6 (-1)^n/n!(2n+1) = .74683\ldots$, so the answer to three decimal places is .747.

(b) $.90452\ldots = 1-(1/2!5)+(1/4!9)-(1/6!13) < \int_0^1 \cos t^2\,dt < 1-(1/2!5)+(1/4!9) = .90462\ldots$, so the answer to three decimal places is .905.

(c) This series, alas, converges much more slowly. One has to go to the 123rd partial sum to be sure that the answer to three decimal places is .822 and not .823. I used Maple to find that the 123rd partial sum is $.8224998\ldots$ and the 124th partial sum is $.8224347\ldots$; the full sum is in between. (It is actually $\pi^2/12 = .8224670\ldots$.)

7. If $f(x) = f(-x)$ then $\sum a_n x^n = \sum (-1)^n a_n x^n$, so by Corollary 7.22, $a_n = (-1)^n a_n$ for all n and hence $a_n = 0$ for n odd. Likewise, if $f(x) = -f(-x)$ then $a_n = (-1)^{n+1} a_n$ for all n and hence $a_n = 0$ for n even.

8. (a) The absolute value of the ratio of the $(n+1)$th term to the nth term is $x^2/4(n+1)(n+k+1)$, which vanishes as $n \to \infty$ for all x; hence the series converges for all x.

(b) $\displaystyle \frac{d}{dx}\sum_{n=0}^\infty \frac{(-1)^n x^{2n+2k}}{2^{2n+k} n!(n+k)!} = \sum_{n=0}^\infty \frac{(-1)^n x^{2n+2k-1}}{2^{2n+k-1} n!(n+k-1)!} = x^k J_{k-1}(x).$

(c) $\displaystyle \frac{d}{dx}\sum_{n=0}^\infty \frac{(-1)^n x^{2n}}{2^{2n+k} n!(n+k)!} = \sum_{n=1}^\infty \frac{(-1)^n x^{2n-1}}{2^{2n+k-1}(n-1)!(n+k)!} = \sum_{m=0}^\infty \frac{(-1)^m x^{2m+1}}{2^{2m+k+1} m!(m+k+1)!} = -\frac{J_{k+1}(x)}{x^k}$. (For the second equality, $m = n-1$.)

(d) $\displaystyle x^2u'' + xu' - k^2u = \sum_{n=0}^\infty \frac{(-1)^n}{n!(n+k)!}\big[(2n+k)(2n+k-1) + (2n+k) - k^2\big]\left[\frac{x}{2}\right]^{2n+k} = \sum_{n=1}^\infty \frac{(-1)^n}{(n-1)!(n+k-1)!}\frac{x^{2n+k}}{2^{2n+k-2}} = \sum_{m=0}^\infty \frac{(-1)^{m+1}}{m!(m+k)!}\frac{x^{2m+k+2}}{2^{2m+k}} = -x^2 J_k(x)$. (In the third equality, $m = n-1$.)

9. The series converges for all x by the ratio test. Let $c_n = [2\cdot 3\cdot 5\cdot 6\cdots(3n-1)(3n)]^{-1}$; then $(d^2/dx^2)c_n x^{3n} = c_{n-1}x^{3n-2}$, so $f''(x) = (d^2/dx^2)\sum_0^\infty c_n x^{3n} = x\sum_1^\infty c_{n-1}x^{3(n-1)}$ $= x\sum_0^\infty c_m x^m = xf(x)$.

10. (a) $\sum_1^\infty nx^n/(n+1)! = x(d/dx)\sum_0^\infty x^n/(n+1)! = x(d/dx)\big[x^{-1}\sum_0^\infty x^{n+1}/(n+1)!\big] = x(d/dx)\big[x^{-1}(e^x-1)\big] = e^x - x^{-1}(e^x-1)$. Alternatively, since $n=(n+1)-1$, $\sum_1^\infty nx^n/(n+1)! = \sum_0^\infty x^n/n! - \sum_0^\infty x^n/(n+1)! = e^x - x^{-1}(e^x-1)$.

(b) $\sum_0^\infty(-1)^n x^{2n+1}/(2n+1)\cdot(2n+2)! = \int_0^x\sum_0^\infty(-1)^n t^{2n}\,dt/(2n+2)! = \int_0^x t^{-2}\sum_1^\infty(-1)^{m-1}t^{2m}\,dt/(2m)! = \int_0^x t^{-2}(1-\cos t)\,dt$.

(c) $\sum_0^\infty x^n/(n+1)^2 n! = x^{-1}\sum_0^\infty x^{n+1}/(n+1)^2 n! = x^{-1}\int_0^x\sum_0^\infty t^n\,dt/(n+1)! = x^{-1}\int_0^x t^{-1}(e^t-1)\,dt$ (see part (a)).

(d) $\sum_0^\infty(-1)^n(2n+1)x^{2n}/(2n)! = (d/dx)\sum_0^\infty(-1)^n x^{2n+1}/(2n)! = (d/dx)(x\cos x) = \cos x - x\sin x$.

11. (a) Integrate by parts: $\int_0^x \arctan t\,dt = t\arctan t|_0^x - \int_0^x t\,dt/(t^2+1) = x\arctan x - \frac{1}{2}\log(x^2+1)$.

(b) From Example 2 and Abel's theorem, we have $\arctan x = \sum_0^\infty(-1)^n x^{2n+1}/(2n+1)$ for $x\in[-1,1]$, the series converging uniformly on that interval. (The convergence at ±1 comes from the alternating series test, and Abel's theorem then gives the uniformity on $[-1,1]$ and the validity of the expansion at $x=\pm1$, since $\arctan$ is continuous.) It then follows from Theorem 7.13a that $f(x) = \sum_0^\infty(-1)^n x^{2n+2}/(2n+1)(2n+2)$ for $x\in[-1,1]$.

(c) Setting $x=1$ in (b) and using (a) gives $\frac{1}{4}\pi - \frac{1}{2}\log 2 = f(1) = \sum_0^\infty(-1)^n/(2n+1)(2n+2)$, and the observation that $[(2n+1)(2n+2)]^{-1} = (2n+1)^{-1}-(2n+2)^{-1}$ then yields the desired result.

7.4 The Complex Exponential and Trig Functions

1. (a) $\sinh ix = \frac{1}{2}(e^{ix}-e^{-ix}) = i\sin x$ and $\cosh ix = \frac{1}{2}(e^{ix}+e^{-ix}) = \cos x$ by (7.30) and (7.32). Alternatively, one can examine the power series expansions.

(b) First method: express the sinh and cosh of z and w in terms of $e^{\pm z}$ and $e^{\pm w}$, multiply out the right sides of the asserted formulas, and simplify. Second method: set $z=ix$ and $w=iy$, and use (7.34) and part (a).

(c) $\sinh(x+iy) = \sinh x\cosh iy + \cosh x\sinh iy = \sinh x\cos y + i\cosh x\sin y$, and $\cosh(x+iy) = \cosh x\cosh iy + \sinh x\sinh iy = \cosh x\cos y + i\sinh x\sin y$.

2. If $c=a+ib$ with $a,b\in\mathbb{R}$, $(d/dx)e^{(a+ib)x} = (d/dx)e^{ax}(\cos bx + i\sin bx)] = ae^{ax}(\cos bx+i\sin bx) + be^{-ax}(-\sin bx + i\cos bx) = (a+ib)e^{ax}(\cos bx+i\sin bx) = ce^{cx}$.

3. $\int e^{ax}\cos bx\,dx + i\int e^{ax}\sin bx\,dx = \int e^{(a+ib)x}\,dx$. By Exercise 2, this is

$$\frac{e^{(a+ib)x}}{a+ib} = \frac{e^{ax}(\cos bx + i\sin bx)}{a+ib}\cdot\frac{a-ib}{a-ib} = \frac{e^{ax}(a\cos bx + b\sin bx) + ie^{ax}(a\sin bx - b\cos bx)}{a^2+b^2}.$$

The asserted formulas follow by taking real and imaginary parts. (Of course some constants of integration are being suppressed here.)

7.5 Functions Defined by Improper Integrals

1. For all $x \in I$ we have $\int_c^\infty |f(x,t)|\,dt \le \int_c^\infty g(t)\,dt < \infty$, so the integral converges absolutely. Likewise, $|\int_d^\infty f(x,t)\,dt| \le \int_d^\infty g(t)\,dt \to 0$ as $d \to \infty$, so the convergence is uniform.

2. Let $g(x) = \int_c^\infty \partial_x f(x,t)\,dt$, and pick a point $a \in I$. By Theorem 7.39 (with f replaced by $\partial_x f$), $\int_a^x g(y)\,dy = \int_c^\infty \int_a^x \partial_x f(y,t)\,dy\,dt = \int_c^\infty f(x,t)\,dt - \int_c^\infty f(a,t)\,dt$. The last integral is a constant, so by the fundamental theorem of calculus, $g(x) = (d/dx)\int_c^\infty f(x,t)\,dt$.

3. $\int_0^\infty e^{-xt}\,dt = -x^{-1}e^{-xt}\big|_0^\infty = x^{-1}$. Formally differentiating $\int_0^\infty e^{-xt}\,dt$ n times gives $(-1)^n\int_0^\infty t^n e^{-xt}\,dt$. The latter integral is uniformly convergent on $[\delta,\infty)$ for any $\delta > 0$, by Theorem 7.38 with $g(t) = t^n e^{-\delta t}$, so the differentiation is justified. On the other hand, $(d/dx)^n x^{-1} = (-1)^n n! x^{-n-1}$; the result follows.

4. $\int_0^\infty (t^2+x)^{-1}\,dt = x^{-1/2}\arctan(t/x^{1/2})\big|_0^\infty = \frac{1}{2}\pi x^{-1/2}$. Formally differentiating $\int_0^\infty (t^2+x)^{-1}\,dt$ $n-1$ times gives $(-1)^{n-1}(n-1)!\int_0^\infty (t^2+x)^{-n}\,dt$. The latter integral is uniformly convergent on $[\delta,\infty)$ for any $\delta > 0$ by Theorem 7.38 with $g(t) = (t^2+\delta)^{-n}$, so the differentiation is justified. On the other hand, $(d/dx)^{n-1}x^{-1/2} = (-\frac{1}{2})(-\frac{3}{2})\cdots(-\frac{2n-3}{2})x^{-(2n-1)/2}$. Hence

$$\int_0^\infty \frac{dt}{(t^2+x)^n} = \frac{\pi}{2}\cdot\frac{1\cdot 3\cdots(2n-3)}{2^{n-1}(n-1)!}x^{(1-2n)/2} = \frac{\pi}{2}\cdot\frac{1\cdot 3\cdots(2n-3)}{2\cdot 4\cdots(2n-2)}x^{(1-2n)/2}.$$

5. $\int_0^\infty x^{-1}(e^{-bx}-e^{-ax})\,dx = -\int_0^\infty\int_a^b e^{-xt}\,dt\,dx$. Since $\int_0^\infty e^{-xt}\,dt$ is uniformly convergent on any compact interval in $(0,\infty)$ (see Exercise 3), we can reverse the order of integration to get $-\int_a^b x^{-1}\,dx = \log(a/b)$.

6. $\int_0^\infty x^{-1}(e^{-bx}-e^{-ax})\cos x\,dx = -\int_0^\infty\int_a^b e^{-tx}\cos x\,dt\,dx$. The integral $\int_0^\infty e^{-tx}\cos x\,dx$ is uniformly convergent for $t \in [\delta,\infty)$ for any $\delta > 0$ by Theorem 7.38 with $g(x) = e^{-\delta x}$, so we can reverse the order of integration to get $-\int_a^b\int_0^\infty e^{-tx}\cos x\,dx\,dt = -\int_a^b t\,dt/(t^2+1) = -\frac{1}{2}\log(t^2+1)\big|_a^b$. (For the first equality, use the result of Exercise 3, §7.4, or integrate by parts twice.)

7. $\int_0^\infty e^{-x}x^{-1}(1-\cos ax)\,dx = \int_0^\infty\int_0^a e^{-x}\sin tx\,dt\,dx$. The integral $\int_0^\infty e^{-x}\sin tx\,dx$ is uniformly convergent for $t \in [0,\infty)$ by Theorem 7.38 with $g(x) = e^{-x}$, so we can reverse the order of integration to get $\int_0^a\int_0^\infty e^{-x}\sin tx\,dx\,dt = \int_0^a t\,dt/(t^2+1) = \frac{1}{2}\log(a^2+1)$. (For the first equality, use the result of Exercise 3, §7.4, or integrate by parts twice.)

8. If $x > 0$, set $s = xt$; then $dt/t = ds/s$, so $\int_0^\infty \sin xt\,dt/t = \int_0^\infty \sin s\,ds/s = \frac{1}{2}\pi$. It follows that $\int_0^\infty \sin xt\,dt/t = -\frac{1}{2}\pi$ for $x < 0$ since the sine function is odd, and of course the integrand vanishes if $x = 0$. The convergence cannot be uniform on I if $0 \in I$ by Theorem 7.39, since the resulting function is discontinuous at 0. However, the convergence is uniform for $x \ge \delta$ or $x \le -\delta$ ($\delta > 0$). To see this, use integration by parts as in Example 3, §4.6, or Example 3, §7.5: we have $\int_b^\infty \sin xt\,dt/t = (-\cos xt)/xt\big|_b^\infty - \int_b^\infty \cos xt\,dt/xt^2$, and for $|x| \ge \delta$ this is bounded in absolute value by $\delta^{-1}[b^{-1} + \int_b^\infty dt/t^2] = 2/\delta b$, which vanishes as $b \to \infty$.

9. Let $F(x) = \int_0^\infty \sin^2 xt\,dt/t^2$ for $x \ge 0$. Since $\sin^2 xt \le \min((xt)^2, 1)$, for $0 \le x \le C$ the integrand is less than $\min(C^2, t^{-2})$, whose integral is finite. Hence the convergence is uniform on $[0,C]$ for any C, and so F is continuous on $[0,\infty)$. Next, formally $F'(x) = \int_0^\infty 2\sin xt\cos xt\,dt/t = \int_0^\infty \sin 2xt/t\,dt$. By Exercise 8, the differentiation is justified and $F'(x) = \frac{1}{2}\pi$ for $x > 0$. Hence $F(x) = \frac{1}{2}\pi x + c$, and $c = F(0) = 0$.

10. (a) The integral over $[1,\infty)$ converges by comparison to $\int_1^\infty x^{-2}\,dx$, and the integral over $[0,1]$ is proper since $\lim_{x\to 0}(\cos bx - \cos ax)/x^2 = \frac{1}{2}(a^2 - b^2)$. One way to obtain the uniformity easily is to use the identity $\cos bx - \cos ax = 2\sin\frac{1}{2}(a+b)x \sin\frac{1}{2}(a-b)x$ and the estimate $|\sin u| \le \min(|u|, 1)$ to get $|\cos bx - \cos ax|/x^2 \le \min(\frac{1}{2}|a^2 - b^2|,\, x^{-2})$, which for a, b in a bounded set is less than a fixed integrable function $\min(C, x^{-2})$.

(b) $I(a,b) = -\int_0^\infty \int_a^b x^{-1}\sin tx\,dt\,dx$. By Exercise 8, this is $-\int_a^b \int_0^\infty x^{-1}\sin tx\,dx\,dt = -\int_a^b \frac{1}{2}\pi\,dt = \frac{1}{2}\pi(a-b)$.

(c) Since $I(a,b)$ is continuous in a and b by part (a), the formula $I(a,b) = \frac{1}{2}\pi(a-b)$ persists for $a, b \ge 0$. Also, clearly $I(a,b)$ is even in both a and b, so $I(a,b) = I(|a|,|b|) = \frac{1}{2}\pi(|a| - |b|)$ for any a, b.

11. (a) The differentiated integral is $-\int_0^\infty te^{-t^2}\sin xt\,dt$, which converges absolutely and uniformly on $\mathbb{R}$ by Theorem 7.38 with $g(t) = te^{-t^2}$. Hence, by integration by parts, $F'(x) = -\int_0^\infty te^{-t^2}\sin xt\,dt = \frac{1}{2}e^{-t^2}\sin xt\big|_0^\infty - \frac{1}{2}x\int_0^\infty e^{-t^2}\cos xt\,dt = -\frac{1}{2}xF(x)$.

(b) Solving the differential equation from (a), $F'/F = -\frac{1}{2}x$, so $\log|F(x)| = -\frac{1}{4}x^2 + \log|C|$, hence $F(x) = Ce^{-x^2/4}$, and $C = F(0) = \int_0^\infty e^{-t^2}\,dt = \frac{1}{2}\sqrt{\pi}$ (Proposition 4.66).

12. As in Exercise 11, $G'(x) = \int_0^\infty te^{-t^2}\cos xt\,dt = -\frac{1}{2}e^{-t^2}\cos xt\big|_0^\infty - \frac{1}{2}x\int_0^\infty e^{-t^2}\sin xt\,dt = \frac{1}{2}(1 - xG(x))$. Write this equation as $G'(x) + \frac{1}{2}xG(x) = \frac{1}{2}$ and multiply through by the integrating factor $e^{x^2/4}$ to get $(e^{x^2/4}G)' = \frac{1}{2}e^{x^2/4}$. It follows that $e^{x^2/4}G(x) = \int_0^x e^{t^2/4}\,dt + C$, and $C = G(0) = 0$.

13. Let $g(x,t) = (1 - e^{-xt^2})/t^2$ and $F(x) = \int_0^\infty g(x,t)\,dt$. $g(x,t)$ is continuous on the region $x \ge 0$, $t \ge 0$ if we define $g(x,0) = x$. Hence $\int_0^1 g(x,t)\,dt$ is continuous in x, and so is $\int_1^\infty g(x,t)\,dt$ since the convergence is uniform ($|g(x,t)| \le t^{-2}$). Thus $F(x)$ is continuous for $x \ge 0$. We have $F'(x) = \int_0^\infty e^{-xt^2}\,dt$ for $x > 0$, the differentiation being justified since the latter integral converges uniformly for $x \ge \delta > 0$. Hence, by the substitution $s = t\sqrt{x}$ and Proposition 4.66, $F'(x) = \int_0^\infty e^{-s^2}\,ds/\sqrt{x} = \frac{1}{2}\sqrt{\pi/x}$. Thus $F(x) = \sqrt{\pi x} + C$ for $x > 0$, and $C = \lim_{x\to 0} F(x) = F(0) = 0$.

14. (a) The integrand is at most e^{-t^2} for all x, so the integral converges uniformly to a continuous function of x. Formal differentiation of the integral yields $-2x\int_0^\infty e^{-t^2-(x^2/t^2)}\,dt/t^2$; this integral is still convergent for $x \ne 0$ because the factor e^{-x^2/t^2} kills the factor $1/t^2$ near $t = 0$, and the convergence is uniform for $|x| \ge \delta > 0$. Hence $F'(x) = -2x\int_0^\infty e^{-t^2-(x^2/t^2)}\,dt/t^2$ for $x \ne 0$. If $x > 0$, let $u = x/t$, so $du = -x\,dt/t^2$, and $F'(x) = 2\int_\infty^0 e^{-(x^2/u^2)-u^2}\,du = -2F(x)$. If $x < 0$, the substitution $u = -x/t$ likewise yields $F'(x) = 2F(x)$.

(b) The differential equations for F give $F(x) = C_\pm e^{\mp 2x}$ for $\pm x > 0$. Since F is continuous at 0, we have $C_\pm = F(0) = \int_0^\infty e^{-t^2}\,dt = \frac{1}{2}\sqrt{\pi}$ by Proposition 4.66, so $F(x) = \frac{1}{2}\sqrt{\pi}\,e^{-2|x|}$.

(c) By the substitution $u = \sqrt{p}\,t$, $\int_0^\infty e^{-pt^2-(q/t^2)}\,dt = \int_0^\infty e^{-u^2-(pq/t^2)}\,du/\sqrt{p} = F(\sqrt{pq})/\sqrt{p} = \frac{1}{2}\sqrt{\pi/p}\,e^{-2\sqrt{pq}}$.

15. (a) Formal differentiation of the integral n times yields $(-1)^n\int_0^\infty e^{-sx}x^n f(x)\,dx$. The convergence of the latter integral, for any n, is uniform for $s \ge b + \delta$ ($\delta > 0$) since $|e^{-sx}x^n f(x)| \le ae^{-\delta x}(1+x)^{N+n}$, so the differentiation is justified.

(b) Integrate by parts: $\int_0^\infty e^{-sx}f'(x)\,dx = e^{-sx}f(x)\big|_0^\infty + \int_0^\infty se^{-sx}f(x)\,dx = -f(0) + sL[f](s)$; the assumed estimate on f guarantees that $e^{-sx}f(x) \to 0$ as $x \to \infty$.

7.6 The Gamma Function

1. $\dfrac{2^{2n-1}}{\sqrt{\pi}}\Gamma(n)\Gamma(n+\frac{1}{2}) = 2^{2n-1}1\cdot 2\cdots(n-1)\cdot\frac{1}{2}\cdot\frac{3}{2}\cdots(n-\frac{1}{2}) = 2\cdot 4\cdots(2n-2)\cdot 1\cdot 3\cdots(2n-1) = (2n-1)! = \Gamma(2n)$.

2. Let $u = \log(1/t)$, so $t = e^{-u}$, $dt = -e^{-u}\,du$, and u goes from ∞ to 0 as t goes from 0 to 1. Thus $\int_0^1 [\log(1/t)]^{a-1}t^{b-1}\,dt = \int_0^\infty u^{a-1}e^{-bu}\,du = b^{-a}\Gamma(a)$ by (7.51).

3. (a) By (7.52), $\int_0^\infty x^4 e^{-x^2}\,dx = \frac{1}{2}\Gamma(\frac{5}{2}) = \frac{1}{2}\cdot\frac{3}{2}\cdot\frac{1}{2}\sqrt{\pi} = \frac{3}{8}\sqrt{\pi}$.

(b) By (7.51), $\int_0^\infty e^{-3x}\sqrt{x}\,dx = 3^{-3/2}\Gamma(\frac{3}{2}) = 3^{-3/2}\frac{1}{2}\sqrt{\pi} = \frac{1}{2}\sqrt{\pi/27}$.

(c) Let $u = x^4$, $du = 4x^3\,dx$; then $\int_0^\infty x^9 e^{-x^4}\,dx = \frac{1}{4}\int_0^\infty u^{3/2}e^{-u}\,du = \frac{1}{4}\Gamma(\frac{5}{2}) = \frac{1}{4}\cdot\frac{3}{2}\cdot\frac{1}{2}\sqrt{\pi} = \frac{3}{16}\sqrt{\pi}$.

4. (a) The substitution $u = 1-t$ in (7.53) turns $B(x,y)$ into $B(y,x)$.

(b) $B(x,1) = \int_0^1 t^{x-1}\,dt = x^{-1}t^x\big|_0^1 = x^{-1}$.

(c) $t^x(1-t)^{y-1} + t^{x-1}(1-t)^y = t^{x-1}(1-t)^{y-1}[t+(1-t)] = t^{x-1}(1-t)^{y-1}$; the result follows by integrating from 0 to 1.

(d) The substitution $t = 1/(u+1)$ turns (7.53) into $B(x,y) = \int_0^\infty (1+u)^{-x-y}u^{y-1}\,du$.

5. Let $u = x^b$; then $dx = b^{-1}u^{b^{-1}-1}\,du$, so $\int_0^1 x^a(1-x^b)^c\,dx = b^{-1}\int_0^1 u^{[(a+1)/b]-1}(1-u)^c\,du = b^{-1}B((a+1)/b,\ c+1)$; use Theorem 7.55.

6. We have $\Gamma(2x) = (2x-1)(2x-2)\Gamma(2x-2)$, $\Gamma(x) = (x-1)\Gamma(x-1)$, and $\Gamma(x+\frac{1}{2}) = (x-\frac{1}{2})\Gamma(x-\frac{1}{2})$. Hence, given that $\Gamma(2x) = \pi^{-1/2}2^{2x-1}\Gamma(x)\Gamma(x+\frac{1}{2})$, we have

$$\Gamma(2x-2) = \frac{\pi^{-1/2}2^{2x-1}(x-1)\Gamma(x-1)\cdot(x-\frac{1}{2})\Gamma(x-\frac{1}{2})}{(2x-1)(2x-2)} = \pi^{-1/2}2^{2(x-1)-1}\Gamma(x-1)\Gamma(x-\tfrac{1}{2}),$$

so the duplication formula is valid for $x-1$. Thus, if the formula is valid for $x > -n$ it is also valid for $x > -n-1$, and by induction, it is therefore valid for all x.

7. $\int_0^{\pi/2}\sin^k x\,dx = \frac{1}{2}B(\frac{1}{2}(k+1),\frac{1}{2}) = \frac{1}{2}\Gamma(\frac{1}{2}(k+1))\Gamma(\frac{1}{2})/\Gamma(\frac{1}{2}k+1)$. If k is even, this is

$$\frac{1}{2}\cdot\frac{\frac{1}{2}\cdot\frac{3}{2}\cdot(\frac{1}{2}k-\frac{1}{2})\sqrt{\pi}\cdot\sqrt{\pi}}{1\cdot 2\cdots(\frac{1}{2}k)} = \frac{1\cdot 3\cdots(k-1)}{2\cdot 4\cdots k}\cdot\frac{\pi}{2}.$$

If k is odd, it is

$$\frac{1}{2}\cdot\frac{1\cdot 2\cdots(\frac{1}{2}k-\frac{1}{2})\cdot\sqrt{\pi}}{\frac{1}{2}\cdot\frac{3}{2}\cdots(\frac{1}{2}k)\sqrt{\pi}} = \frac{2\cdot 4\cdots(k-1)}{1\cdot 3\cdots k}.$$

(There is one more factor in the denominator than in the numerator, which absorbs the extra factor of 2. If $k=1$, we simply have $\Gamma(1)\Gamma(\frac{1}{2})/2\Gamma(\frac{3}{2}) = 1$.)

8. Since $\int_0^{\pi/2}\sin^{2n+1}x\,dx < \int_0^{\pi/2}\sin^{2n}x\,dx < \int_0^{\pi/2}\sin^{2n-1}x\,dx$, by Exercise 7 we have

$$\frac{2\cdot 4\cdots(2n)}{1\cdot 3\cdots(2n+1)} < \frac{1\cdot 3\cdots(2n-1)}{2\cdot 4\cdots(2n)}\cdot\frac{\pi}{2} < \frac{2\cdot 4\cdots(2n-2)}{1\cdot 3\cdots(2n-1)}.$$

Multiplying through by $2 \cdot 4 \cdots (2n)/1 \cdot 3 \cdots (2n-1)$ gives $c_n < \frac{1}{2}\pi < (2n+1)c_n/2n$. Since $c_n = c_{n-1}(4n^2)/(4n^2-1) > c_{n-1}$, the sequence $\{c_n\}$ is increasing, and we have just seen that it is bounded above by $\frac{1}{2}\pi$, so it converges to a limit $L \le \frac{1}{2}\pi$. The sequence $\{(2n+1)c_n/2n\}$ also converges to L, but its terms are $> \frac{1}{2}\pi$, so $L \ge \frac{1}{2}\pi$. In short, $L = \frac{1}{2}\pi$.

9. (a) For $\alpha > 0$, $(d/dx)I_{\alpha+1}[f](x) = \Gamma(\alpha+1)^{-1}[(x-x)^\alpha f(x) + \alpha \int_0^x (x-t)^{\alpha-1} f(t)\,dt] = I_\alpha[f](x)$ since $\Gamma(\alpha+1) = \alpha\Gamma(\alpha)$. Actually there is more to be said when $\alpha < 1$, since then the integral defining $I_\alpha[f](x)$ is improper: the integrand blows up at $t = x$. But $(d/dx)\int_0^{x-\epsilon}(x-t)^\alpha f(t)\,dt = \epsilon^\alpha f(x-\epsilon) + \alpha\int_0^{x-\epsilon}(x-t)^{\alpha-1}f(t)\,dt$. For x in a finite interval, say $[0, C]$, we have $|f(x)| \le M < \infty$, so $|\epsilon^\alpha f(x-\epsilon)| \le M\epsilon^\alpha$ and $\left|\alpha\int_0^{x-\epsilon}(x-t)^{\alpha-1}f(t)\,dt - \alpha\int_0^x (x-t)^{\alpha-1}f(t)\,dt\right| \le M\alpha\int_{x-\epsilon}^x (x-t)^{\alpha-1}\,dt = M\epsilon^\alpha$. Hence $(d/dx)\int_0^{x-\epsilon}(x-t)^\alpha f(t)\,dt \to \alpha\int_0^x (x-t)^{\alpha-1}f(t)\,dt$ uniformly for $x \in [0, C]$, so the differentiation in the limit as $\epsilon \to 0$ is justified.

(b) $I_\alpha[I_\beta[f]](x) = [\Gamma(\alpha)\Gamma(\beta)]^{-1}\int_0^x\int_0^t (x-t)^{\alpha-1}(t-s)^{\beta-1}f(s)\,ds\,dt$. Reversing the order of integration (OK even if $\alpha, \beta < 1$ since the integral is absolutely convergent) turns $\int_0^x\int_0^t \cdots ds\,dt$ into $\int_0^x\int_s^x \cdots dt\,ds$. Now, in the inner integral, make the substitution $u = (t-s)/(x-s)$, so that u goes from 0 to 1 when t goes from s to x. Then $t - s = (x-s)u$, $x - t = (x-s)(1-u)$, and $dt = (x-s)\,du$, so

$$I_\alpha[I_\beta[f]](x) = \frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_0^x\int_0^1 u^{\alpha-1}(1-u)^{\beta-1}(x-s)^{\alpha+\beta-1}f(s)\,dt\,ds$$
$$= \frac{B(\alpha,\beta)}{\Gamma(\alpha)\Gamma(\beta)}\int_0^x (x-s)^{\alpha+\beta-1}f(s)\,ds = \frac{1}{\Gamma(\alpha+\beta)}\int_0^x (x-s)^{\alpha+\beta-1}f(s)\,ds = I_{\alpha+\beta}[f](x).$$

10. (a) The series is $[\Gamma(\frac{2}{3})/\Gamma(\frac{1}{3})]\sum_0^\infty \Gamma(n+\frac{4}{3})/\Gamma(n+\frac{5}{3})$. The nth term is asymptotic to $n^{-1/3}$, so the series diverges.

(b) The series is $\frac{1}{4}\Gamma(\frac{5}{4})\sum_0^\infty \Gamma(n+1)/\Gamma(n+\frac{9}{4})$. The nth term is asymptotic to $n^{-5/4}$, so the series converges.

11. The series is $\pi^{-p/2}\sum_0^\infty [\Gamma(n+\frac{1}{2})/\Gamma(n+1)]^p$. The nth term is asymptotic to $n^{-p/2}$, so the series converges if and only if $p > 2$. For Raabe's test, we have

$$n\left[1 - \frac{a_{n+1}}{a_n}\right] = n\left[1 - \left(\frac{2n+1}{2n+2}\right)^p\right] = n\left[1 - \left(1 - \frac{1}{2n+2}\right)^p\right].$$

Since $(1+t)^p \approx 1 + pt$ for t small, this is approximately $np/(2n+2)$ for n large, and the limit is $p/2$. Hence Raabe's test gives convergence for $p > 2$ and divergence for $p < 2$ but is indecisive for $p = 2$.

12. $\Gamma(a+n)/\Gamma(c+n) \approx n^{a-c}$ and $\Gamma(b+n)/n! = \Gamma(b+n)/\Gamma(n+1) \approx n^{b-1}$ for n large, so the nth term is asymptotic to $n^{a+b-c-1}$. Hence the series converges if and only if $a+b-c-1 < -1$, i.e., $a+b < c$.

7.7 Stirling's Formula

1. Since the second derivative of $\log(1+t)$ is $-(1+t)^{-2}$, Lagrange's form of the remainder immediately gives $\log(1+t) = t + R(t)$ where $-\frac{1}{2}t^2 \le R(t) \le 0$ for $t \ge 0$. Thus, $a - (a+n-\frac{1}{2})\log(1 + (a/n)) = -(a-\frac{1}{2})(a/n) - (a+n-\frac{1}{2})R(a/n)$, and for $0 \le a \le 1$ this is, in absolute value, at most $1/2n + (n+\frac{1}{2})/2n^2 = 1/n + 1/4n^2 \le 5/4n$.

2. $(2n)!/(n!)^2 2^{2n} \sim (2n)^{2n+(1/2)} e^{-2n}\sqrt{2\pi}/n^{2n+1}e^{-2n}(2\pi)2^{2n} = 1/\sqrt{\pi n}$, where $\sim$ means that the ratio approaches 1 as $n \to \infty$.

3. Note that $2 \cdot 4 \cdots 2n = 2^n(1 \cdot 2 \cdots n) = 2^n n!$, and $1 \cdot 3 \cdots (2n-1) = (2n)!/2 \cdot 4 \cdots 2n = (2n)!/2^n n!$. Thus the numerator and denominator of Wallis's fraction are $[2^n n!]^2$ and $[(2n)!]^2(2n+1)/[2^n n!]^2$, so the whole fraction is

$$\frac{[2^n n!]^4}{[(2n)!]^2(2n+1)} = \frac{[n!]^4}{n^{4n+2}e^{-4n}} \cdot \frac{(2n)^{4n+1}e^{-4n}}{[(2n)!]^2} \cdot \frac{n}{2(2n+1)},$$

which tends to $L^4 \cdot L^{-2} \cdot \frac{1}{4}$ as $n \to \infty$. Thus $\frac{1}{4}L^2 = \frac{1}{2}\pi$, or $L = \sqrt{2\pi}$.

Chapter 8

Fourier Series

8.1 Periodic Functions and Fourier Series

1. f is odd, so $a_n = 0$ and $b_n = (1/\pi)\int_{-\pi}^{\pi} f(\theta)\sin n\theta\,d\theta = (2/\pi)\int_0^\pi \sin n\theta\,d\theta = -(2/n\pi)\cos n\theta\big|_0^\pi = (2/n\pi)[1-(-1)^n]$. This is 0 if n is even and $4/n\pi$ if n is odd, say $n = 2m-1$; thus the Fourier series is $(4/\pi)\sum_1^\infty[\sin(2m-1)\theta]/(2m-1)$.

2. By the double angle formula, $\sin^2\theta = \frac{1}{2}(1-\cos 2\theta)$, and the thing on the right is a Fourier series!

3. f is even, so $b_n = 0$, $a_0 = (2/\pi)\int_0^\pi \sin\theta\,d\theta = 4/\pi$, and for $n > 0$, $a_n = (1/\pi)\int_{-\pi}^{\pi} f(\theta)\cos n\theta\,d\theta = (2/\pi)\int_0^\pi \sin\theta\cos n\theta\,d\theta = (1/\pi)\int_0^\pi[\sin(n+1)\theta - \sin(n-1)\theta]\,d\theta = (1/\pi)[(n+1)^{-1}-(n-1)^{-1}][1-(-1)^{n-1}] = -2[1-(-1)^{n-1}]/(n^2-1)\pi$. This is 0 if n is odd and $-4/(n^2-1)\pi$ if n is even, say $n = 2m$; thus the Fourier series is $\frac{1}{2}a_0 + \sum_1^\infty a_n\cos n\theta = (2/\pi) - (4/\pi)\sum_1^\infty(\cos 2m\theta)/(4m^2-1)$.

4. f is even, so $b_n = 0$ and $a_n = (2/\pi)\int_0^\pi \theta^2\cos n\theta\,d\theta$. The constant term is $\frac{1}{2}a_0 = (1/\pi)\int_0^\pi \theta^2\,d\theta = \pi^2/3$. For $n > 0$, integration by parts gives $\int\theta^2\cos n\theta\,d\theta = (2/n^2)\theta\cos n\theta + [(\theta^2/n)-(2/n^3)]\sin n\theta$, so $a_n = (2/\pi)(2/n^2)\pi(-1)^n = 4(-1)^n/n^2$.

5. Here it is easier to use the exponential form of the series: $c_n = (1/2\pi)\int_{-\pi}^{\pi} e^{b\theta}e^{-in\theta}\,d\theta = \big[e^{(b-in)\theta}\big]_{-\pi}^{\pi}/2\pi(b-in) = (-1)^n[e^{b\pi}-e^{-b\pi}]/2\pi(b-in) = (-1)^n(\sinh b\pi)/\pi(b-in)$.

6. f is odd, so $a_n = 0$ and $b_n = (2/\pi)\int_0^\pi \theta(\pi-\theta)\sin n\theta\,d\theta$. Integration by parts gives $\int\theta(\pi-\theta)\sin n\theta\,d\theta = (1/n)\theta(\pi-\theta)\cos n\theta + (1/n^2)(\pi-2\theta)\sin n\theta - (2/n^3)\cos n\theta$, so $b_n = (2/\pi)(2/n^3)[1-(-1)^n]$. This is 0 when n is even and $8/\pi n^3$ when n is odd, say $n = 2m-1$; hence the Fourier series is $(8/\pi)\sum_1^\infty(2m-1)^{-3}\sin(2m-1)\theta$.

7. f is even, so $b_n = 0$. The constant term $\frac{1}{2}a_0$ is the mean value of f on $[-\pi,\pi]$, which is 0 by construction of f. For $n > 0$, $a_n = (2/\pi a)\int_0^a\cos n\theta\,d\theta - [2/\pi(\pi-a)]\int_a^\pi\cos n\theta\,d\theta = (2/\pi an)\sin na + [2/\pi(\pi-a)n]\sin na = [2/a(\pi-a)n]\sin na$.

8. f is even, so $b_n = 0$. The constant term $\frac{1}{2}a_0$ is the mean value of f on $[-\pi,\pi]$, namely $1/2\pi$. For $n > 0$, $a_n = (2/\pi a^2)\int_0^a(a-\theta)\cos n\theta\,d\theta = \big[(2/\pi a^2 n)(a-\theta)\sin n\theta - (2/\pi a^2n^2)\cos n\theta\big]_0^a = 2(1-\cos na)/\pi a^2n^2$.

9. *First method:* Suppose $(k-1)P \le a < kP$. Then $\int_a^{a+P} = \int_a^{kP} + \int_{kP}^{a+P}$ (the integrand is $f(x)\,dx$ in all integrals). By periodicity of f, the second integral on the right equals $\int_{(k-1)P}^a$, so adding it to the first integral gives $\int_{(k-1)P}^{kP}$. Another application of periodicity shows that this is equal to $\int_0^P$. *Second method:* Let $g(a) = \int_a^{a+P} f(x)\,dx$. $g(a)$ is a continuous function of a, and except at the finitely many points where f is discontinuous we have $g'(a) = f(a+P) - f(a) = 0$. It follows that g is constant, so $g(a) = g(0)$.

8.2 Convergence of Fourier Series

1. Let f be the sawtooth wave of Example 1.

(a) The function depicted is $\frac{1}{2}f(\pi - 2\theta) = \sum_1^\infty [(-1)^{n+1}/n]\sin(n\pi - 2n\theta) = \sum_1^\infty (1/n)\sin 2n\theta$.

(b) The function depicted is $1 + (1/\pi)f(2\theta - \pi) = 1 + (2/\pi)\sum_1^\infty [(-1)^{n+1}/n]\sin(2n\theta - n\pi) = 1 - \sum_1^\infty (2/\pi n)\sin 2n\theta$.

2. The function $f(\theta)$ here is the function of Exercise 4, §8.1, shifted to the right by $\frac{1}{4}\pi$. Hence $f(\theta) = (\pi^2/3) + 4\sum_1^\infty [(-1)^n/n^2]\cos n(\theta - \frac{1}{4}\pi)$, and $\cos n(\theta - \frac{1}{4}\pi) = \cos\frac{1}{4}n\pi\cos n\theta + \sin\frac{1}{4}n\pi\sin n\theta$.

3. (a) $f(\theta) = \frac{1}{2}(1 + g(\theta))$ where g is the square wave of Exercise 1, §8.1; hence $f(\theta) = \frac{1}{2} + (2/\pi)\sum_1^\infty (2m-1)^{-1}\sin(2m-1)\theta$.

(b) $f(\theta) = \frac{1}{2}(|\sin\theta| + \sin\theta)$ for $|\theta| < \pi$, so by Exercise 3, §8.1, we have $f(\theta) = (1/\pi) - (2/\pi)\sum_1^\infty (\cos 2m\theta)/(4m^2 - 1) + \frac{1}{2}\sin\theta$.

(c) $f(\theta) = [(\pi - a)/2\pi][g(\theta) + (\pi - a)^{-1}] = (1/2\pi) + [(\pi - a)/2\pi]g(\theta)$ where g is the function of Exercise 7, §8.1, so $f(\theta) = (1/2\pi) + (1/\pi)\sum_1^\infty [(\sin na)/na]\cos n\theta$.

(d) $f(\theta) = \frac{1}{2}(e^\theta - e^{-\theta})$ for $|\theta| < \pi$, so by Exercise 5, §8.1, with $b = 1$, $f(\theta) = [(\sinh\pi)/2\pi]\sum_{-\infty}^\infty [(-1)^n/(1-in)][e^{in\theta} - e^{-in\theta}] = [(\sinh\pi)/\pi]\sum_{-\infty}^\infty [(-1)^n/(1-in)]i\sin n\theta$. The sum of the nth and $(-n)$th terms is $(-1)^n[(1-in)^{-1} - (1+in)^{-1}]i\sin n\theta = (-1)^{n-1}[2n/(1+n^2)]\sin n\theta$.

4. (a) Setting $\theta = 0$ gives $(2/\pi) - (4/\pi)\sum_1^\infty 1/(4m^2-1) = 0$ or $\sum_1^\infty 1/(4m^2-1) = \frac{1}{2}$, a result also obtainable from the observation that $(4m^2-1)^{-1} = \frac{1}{2}[(2m-1)^{-1} - (2m+1)^{-1}]$, so that the series telescopes. Setting $\theta = \frac{1}{2}\pi$ gives $1 = (2/\pi) - (4/\pi)\sum(-1)^m/(4m^2-1)$, or $\sum_1^\infty (-1)^{m+1}/(4m^2-1) = (\pi-2)/4$.

(b) Setting $\theta = \pi$ gives $\pi^2 = (\pi^2/3) + 4\sum_1^\infty 1/n^2$ or $\sum_1^\infty 1/n^2 = \pi^2/6$; setting $\theta = 0$ gives $0 = (\pi^2/3) + 4\sum_1^\infty (-1)^n/n^2$ or $\sum_1^\infty (-1)^{n+1}/n^2 = \pi^2/12$.

(c) Setting $\theta = 0$ gives $1 = [(\sinh\pi b)/\pi]\sum_{-\infty}^\infty (-1)^n/(b - in)$. The $n = 0$ term is $1/b$, and for $n > 0$ the sum of the nth and $(-n)$th terms is $2b(-1)^n/(b^2+n^2)$; thus $1 = [(\sinh\pi b)/\pi][(1/b) + 2b\sum_1^\infty (-1)^n/(b^2+n^2)]$, or $\sum_1^\infty (-1)^n/(b^2+n^2) = (\pi b\operatorname{csch}\pi b - 1)/2b^2$. Setting $\theta = \pi$ gives $[(\sinh\pi b)/\pi]\sum_{-\infty}^\infty 1/(b - in) = \frac{1}{2}(e^{\pi b} + e^{-\pi b}) = \cosh\pi b$. (The function represented by the series is discontinuous at $\theta = \pi$, so the sum of the series is the average of the left and right hand limits!) Again the $n = 0$ term is $1/b$, and for $n > 0$ the sum of the nth and $(-n)$th terms is $2b/(b^2+n^2)$, so $(1/b) + \sum_1^\infty 2b/(b^2+n^2) = \pi\coth\pi b$ and hence $\sum_1^\infty 1/(b^2+n^2) = (\pi b\coth\pi b - 1)/2b^2$.

(d) Setting $\theta = \frac{1}{2}\pi$ gives $\frac{1}{4}\pi^2 = (8/\pi)\sum_1^\infty (-1)^{m+1}/(2m-1)^3$ or $\sum_1^\infty (-1)^{m+1}/(2m-1)^3 = \pi^3/32$.

5. Given $\theta \in \mathbb{R}$ and $\epsilon > 0$, choose $\delta > 0$ small enough so that $|f(\theta+\varphi) - f(\theta+)| < \epsilon/3$ when $0 < \varphi < \delta$ and $|f(\theta+\varphi) - f(\theta-)| < \epsilon/3$ when $-\delta < \varphi < 0$. Let $M = \sup_{\theta\in[-\pi,\pi]}|f(\theta)|$. By (8.23), there exists r_0 such that $0 \le P_r(\varphi) \le \epsilon/6\pi M$ for $\varphi \in [-\pi, -\delta] \cup [\delta, \pi]$ when $r_0 < r < 1$.

Assume now that $r_0 < r < 1$. We have $|\int_\delta^\pi f(\theta+\varphi)P_r(\varphi)\,d\varphi| \le M(\epsilon/6\pi M)(\pi-\delta) < \frac{1}{6}\epsilon$, and likewise $|\int_{-\pi}^{-\delta} f(\theta+\varphi)P_r(\varphi)\,d\varphi| < \frac{1}{6}\epsilon$. Moreover, $\int_0^\delta P_r(\varphi)\,d\varphi = \int_0^\pi P_r(\varphi)\,d\varphi - \int_\delta^\pi P_r(\varphi)\,d\varphi = \frac{1}{2} - \int_\delta^\pi P_r(\varphi)\,d\varphi$, and this last integral is between 0 and $\epsilon/6M$. Now, we have

$$\int_0^\delta f(\theta+\varphi)P_r(\varphi)\,d\varphi - \frac{1}{2}f(\theta+) = \int_0^\delta [f(\theta+\varphi)-f(\theta+)]P_r(\varphi)\,d\varphi + f(\theta+)\left[\int_0^\delta P_r(\varphi)\,d\varphi - \frac{1}{2}\right].$$

The first term on the right is at most $(\epsilon/3)\int_0^\delta P_r(\varphi)\,d\varphi = \frac{1}{6}\epsilon$ in absolute value, and the second term is at most $M(\epsilon/6M) = \frac{1}{6}\epsilon$. Similarly, $|\int_{-\delta}^0 f(\theta+\varphi)P_r(\varphi)\,d\varphi - \frac{1}{2}f(\theta-)| < \frac{1}{3}\epsilon$. Adding up these results, abbreviating $f(\theta+\varphi)P_r(\varphi)\,d\varphi$ as $(*)$ we have

$$\left|\int_{-\pi}^\pi (*) - \frac{1}{2}[f(\theta+)+f(\theta-)]\right| \le \left|\int_{-\pi}^{-d}(*)\right| + \left|\int_{-\delta}^0 (*) - \frac{1}{2}f(\theta-)\right| + \left|\int_0^\delta (*) - \frac{1}{2}f(\theta+)\right| + \left|\int_\delta^\pi (*)\right|$$
$$< \frac{\epsilon}{6} + \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{6} = \epsilon$$

for $r_0 < r < 1$, as desired.

6. Since f is uniformly continuous, given $\epsilon > 0$ we can choose $\delta > 0$ small enough so that $|f(\theta+\varphi) - f(\theta)| < \epsilon/3$ for any θ when $|\varphi| < \delta$. The argument of the preceding exercise (slightly simplified, since $f(\theta+) = f(\theta-) = f(\theta)$) shows that $|A_r f(\theta) - f(\theta)| < \epsilon$ for all θ when $r_0 < r < 1$.

8.3 Derivatives, Integrals, and Uniform Convergence

1. $a_n' = \pi^{-1}\int_{-\pi}^\pi f'(\theta)\cos n\theta\,d\theta = \pi^{-1}f(\theta)\cos n\theta\big|_{-\pi}^\pi - \pi^{-1}\int_{-\pi}^\pi f(\theta)(-n\sin n\theta)\,d\theta = 0 + nb_n$, and $b_n' = \pi^{-1}\int_{-\pi}^\pi f'(\theta)\sin n\theta\,d\theta = \pi^{-1}f(\theta)\sin n\theta\big|_{-\pi}^\pi - \pi^{-1}\int_{-\pi}^\pi f(\theta)(n\cos n\theta)\,d\theta = 0 - na^n$.

2. (a) $g(\theta)$ is $-1 + (a+\theta)/(a-\pi)$ for $-\pi \le \theta < -a$, $a^{-1}\theta$ for $|\theta| \le a$, and $1 - (\theta-a)/(\pi-a)$ for $a < \theta \le \pi$. The graph is the broken line joining the points $(-\pi,0)$, $(-a,-1)$, $(a,1)$, and $(\pi,0)$.

(b) Termwise integration of the series in Exercise 7, §8.1 gives $[2/a(\pi-a)]\sum_1^\infty(\sin na\sin n\theta)/n^2$.

3. (a) From Exercise 4, §8.1, we have $3\theta^2 - \pi^2 = 12\sum_1^\infty[(-1)^n/n^2]\cos n\theta$ ($|\theta| \le \pi$), and termwise integration yields $\theta^3 - \pi^2\theta = 12\sum_1^\infty[(-1)^n/n^3]\sin n\theta$. ($\theta^3 - \pi^2\theta$ is odd, so its mean value on $[-\pi,\pi]$ is 0.)

(b) Integration of the result of (a) and multiplication by 4 gives $\theta^4 - 2\pi^2\theta^2 = C_0 + 48\sum_1^\infty[(-1)^{n+1}/n^4]\cos n\theta$, where $C_0 = (1/2\pi)\int_{-\pi}^\pi(\theta^4 - 2\pi^2\theta^2)\,d\theta = (1/\pi)(\frac{1}{5}\pi^5 - \frac{2}{3}\pi^5) = -\frac{7}{15}\pi^4$.

(c) Setting $\theta = \pi$ in (b) gives $-\pi^4 = -\frac{7}{15}\pi^4 - 48\sum_1^\infty(1/n^4)$, or $\sum_1^\infty(1/n^4) = \pi^4/90$.

4. The function $f(\theta) = |\sin\theta|$ is continuous and piecewise smooth, and its derivative is $s(\theta)\cos\theta$ where s is the square wave (Exercise 1, §8.1). So by Corollary 8.27, we have $s(\theta)\cos\theta = (8/\pi)\sum_1^\infty(n\sin 2n\theta)/(4n^2-1)$, and in particular the latter series converges to $\cos\theta$ for $0 < \theta < \pi$. On the other hand, termwise integration of the series for $|\sin\theta|$ from 0 to θ gives, for $0 < \theta < \pi$, $1 - \cos\theta = (2/\pi)\theta - (2/\pi)\sum_1^\infty(\sin 2n\theta)/n(4n^2-1)$, or $\cos\theta = 1 - (2/\pi)\theta + (2/\pi)\sum_1^\infty(\sin 2n\theta)/n(4n^2-1)$. Now, since $4n/(4n^2-1) = (1/n(4n^2-1)) + (1/n)$, the equality of the two series for $\cos\theta$ amounts to the assertion that $\frac{1}{2}\pi - \theta = \sum_1^\infty(\sin 2n\theta)/n$ for $0 < \theta < \pi$, and one verifies this by substituting $\pi - 2\theta$ for θ in Example 1, §8.2.

5. $f(\theta)$ is not continuous (it has jumps at the odd multiples of π), so Theorem 8.26 does not apply. (For those who like distributions: The derivative of $f(\theta)$ is really $f(\theta) - (2\sinh\pi)\delta_\pi(\theta)$, where δ_π is the periodic delta-function with singularities at the odd multiples of π. The Fourier series of δ_π is $(1/2\pi)\sum_{-\infty}^{\infty}(-1)^n e^{in\theta}$, so the correct conclusion is not $c_n = inc_n$ but rather $inc_n = c_n - (-1)^n(\sinh\pi)/\pi$, which is true by Exercise 5, §8.1.)

6. (a) The series $\sum_{n\neq 0} n^{k-(6/5)}/(1+n^6)$ converges if and only if $k \leq 6$, so the given series can be differentiated 6 times.

(b) The series $\sum_0^\infty n^k/2^n$ converges for all k, so the given series can be differentiated any number of times.

(c) The given series converges uniformly (M-test with $M_n = 2^{-n}$), so its sum is continuous, but the differentiated series $-\sum \sin 2^n\theta$ does not converge at most points (the terms do not tend to zero as $n \to \infty$).

8.4 Fourier Series on Intervals

1. (a) The even periodic extension of f is the constant function 1, which is its own Fourier series. The odd periodic extension is the square wave (Exercise 1, §8.1).

(b) The even periodic extension of f is $|\sin\theta|$ (Exercise 3, §8.1). The odd periodic extension is $\sin\theta$, which is its own Fourier series.

(c) The even periodic extension of f is the function of Exercise 4, §8.1. The odd periodic extension is given by $f(\theta) = \theta|\theta|$ for $|\theta| < \pi$, so $f(\theta) = \pi\theta - \theta(\pi - |\theta|)$; the Fourier series of the latter two functions are given by Example 1 and Exercise 6, §8.1.

(d) Let g be the triangle wave of Example 2, §8.1. Then the even periodic extension of f is $\frac{1}{2}g(2\theta)$, and the odd periodic extension is $g(\theta + \frac{1}{2}\pi) - \frac{1}{2}\pi$. The Fourier series of these are, respectively, $(\pi/4) - (2/\pi)\sum_1^\infty(\cos(4m-2)\theta)/(2m-1)^2$ and $(4/\pi)\sum_1^\infty(-1)^{m+1}(\sin(2m-1)\theta)/(2m-1)^2$ (since $\cos(2m-1)(\theta+\frac{1}{2}\pi) = (-1)^m\sin(2m-1)\theta)$.

2. (a) The odd 2-periodic extension of f is $s(\pi x)$ where s is the square wave of Exercise 1, §8.1; its Fourier series is $(4/\pi)\sum_1^\infty(\sin(2m-1)\pi x)/(2m-1)$.

(b) The even 4-periodic extension of f is a square wave, namely, $s(\frac{1}{4}\pi x + \frac{1}{2}\pi)$ where s is the function of Exercise 1, §8.1; since $\sin(2m-1)(\frac{1}{4}\pi x + \frac{1}{2}\pi) = (-1)^{m+1}\cos(2m-1)\frac{1}{4}\pi x = (-1)^{m+1}\cos(\frac{1}{2}m - \frac{1}{4})\pi x$, its Fourier series is $(4/\pi)\sum_1^\infty(-1)^{m+1}(\cos(\frac{1}{2}m-\frac{1}{4})\pi x)/(2m-1)$.

(c) The odd $2l$-periodic extension of f is given by $f(x) = x(l-|x|)$ on $[-l,l]$, that is, $f(x) = (l/\pi)^2 g(\pi x/l)$ where g is the function of Exercise 6, §8.1; its Fourier series is $(8l^2/\pi^3)\sum_1^\infty(\sin(2m-1)\pi x/l)/(2m-1)^3$.

(d) The 1-periodic extension of f is $e^{1/2}g(2\pi x - \pi)$ where g is the function of Exercise 5, §8.1, with $b = 1/2\pi$. Its Fourier series is $[(e^{1/2}\sinh\frac{1}{2})/\pi]\sum_{-\infty}^{\infty}(-1)^n e^{in(2\pi x-\pi)}/((2\pi)^{-1} - in) = (e-1)\sum_{-\infty}^{\infty} e^{2\pi inx}/(1-2\pi in)$.

3. We have $a_n = (1/l)\int_0^{2l} f(x)\cos(n\pi x/2l)\,dx$ and similarly for b_n. Replace $f(x)$ by $f(2l-x)$, then set $x = 2l - u$ and use the facts that $\cos(n\pi(2l-u)/2l) = (-1)^n\cos(n\pi u/2l)$ and $\sin(n\pi(2l-u)/2l) = (-1)^{n+1}\sin(n\pi u/2l)$ to deduce that $a_n = (-1)^n a_n$ and $b_n = (-1)^{n+1}b_n$, whence $a_n = 0$ for n odd and $b_n = 0$ for n even.

4. Extend f as suggested in the hint and expand it in a Fourier sine series on $[0, 2l]$: $f(x) = \sum_1^\infty b_n \sin(n\pi x/2l)$ where $b_n = (1/l)\int_0^{2l} f(x)\sin(n\pi x/2l)\,dx$. Then for n even, $b_n = 0$ by Exercise 3; and for n odd, $f(x)\sin(n\pi x/2l)$ is symmetric about $x = l$ (as observed in the solution of Exercise 3), so its integral over $[0, 2l]$ is twice its integral over $[0, l]$. Setting $n = 2m - 1$ thus yields $f(x) = \sum_1^\infty \beta_m \sin(m - \frac{1}{2})\pi x/l$ where $\beta_m = (2/l)\int_0^l f(x)(\sin(m - \frac{1}{2})\pi x/l)\,dx$.

8.5 Applications to Differential Equations

1. (a) The Fourier cosine series for $f(x) = x$ on $[0, 100]$ is $50 - (400/\pi^2)\sum_1^\infty(\cos(2m-1)\pi x/100)/(2m-1)^2$, so the solution (8.35) of the heat equation is $u(x,t) = 50 - (400/\pi^2)\sum_1^\infty e^{-(.00011)(2m-1)^2\pi^2 t}(\cos(2m-1)\pi x/100)/(2m-1)^2$.

(b) When $t = 60$, the error in discarding the terms after $m = 2$ is

$$\left|\frac{400}{\pi^2}\sum_3^\infty e^{-(.0066)(2m-1)^2\pi^2}\frac{\cos(2m-1)(\pi x/100)}{(2m-1)^2}\right| \le \frac{400}{\pi^2}e^{-(.0066)5^2\pi^2}\sum_3^\infty \frac{1}{(2m-1)^2}$$
$$\approx \frac{400}{\pi^2}e^{-1.628}(.123) \approx .98.$$

To within this error, $u(x, 60) = 50 - (400/\pi^2)[e^{-.0066\pi^2}\cos(\pi x/100) + \frac{1}{9}e^{-(.0066)9\pi^2}\cos(3\pi x/100)] \approx 50 - (37.97)\cos(\pi x/100) - (2.51)\cos(3\pi x/100)$, which is about 10 when $x = 0$, 12 when $x = 10$, and 40 when $x = 40$.

(c) For $t \ge 3600$, $|u(x,t) - 50| \le (400/\pi^2)e^{-(.0011)\pi^2(3600)}\sum_1^\infty 1/(2m-1)^2 = 50e^{-(.396)\pi^2} \approx 1.0037$. Almost good enough, but not quite! A slightly less crude estimate works: $|u(x,t) - 50| \le (400/\pi^2)[e^{-(.396)\pi^2} + e^{-9(.396)\pi^2}\sum_2^\infty 1/(2m-1)^2] = (400/\pi^2)[e^{-(.396)\pi^2} + e^{-(3.564)\pi^2}((\pi^2/8) - 1)] \approx .81$.

2. One follows the separation-of variables procedure as on p. 382 to find solutions of the form $e^{-k\alpha t}(C_1\cos\sqrt{\alpha}\,\theta + C_2\sin\sqrt{\alpha}\,\theta)$. The periodicity condition then forces $\sqrt{\alpha} = n$, so the resulting analog of (8.35) is $u(\theta, t) = \sum_0^\infty e^{-n^2kt}(a_n\cos n\theta + b_n\sin n\theta)$. To satisfy the initial condition one takes $\sum_0^\infty(a_n\cos n\theta + b_n\sin n\theta)$ to be the Fourier series of $f(\theta)$. (The result looks a little neater in exponential form: $u(\theta,t) = \sum_{-\infty}^\infty c_n e^{-n^2kt + in\theta}$ where $f(\theta) = \sum_{-\infty}^\infty c_n e^{in\theta}$.)

3. If $u(x,t) = \sum_1^\infty b_n(t)\sin(n\pi x/l)$ is to satisfy $\partial_t u = k\partial_x^2 u + G$ where $G(x,t) = \sum_1^\infty \beta_n(t)\sin(n\pi x/l)$, we must have $b_n'(t) = -k(n\pi/l)^2 b_n(t) + \beta_n(t)$, assuming that termwise differentiation of the series is justified. To solve this ordinary differential equation, multiply through by the integrating factor $e^{k(n\pi/l)^2 t}$ to obtain $(d/dt)[b_n(t)e^{k(n\pi/l)^2 t}] = e^{k(n\pi/l)^2 t}\beta_n(t)$, whence $b_n(t)e^{k(n\pi/l)^2 t} = b_n(0) + \int_0^t e^{k(n\pi/l)^2 s}\beta_n(s)\,ds$. For this to work, the following conditions are (more than) sufficient: (1) f is of class C^1 on $[0, l]$, and $f(0) = f(l) = 0$. (2) $G(x,t)$ is C^2 as a function of $x \in [0, l]$ for each t, $G(0,t) = G(l,t) = 0$, and $G(x,t)$, $\partial_x G(x,t)$, and $\partial_x^2 G(x,t)$ are jointly continuous as functions of $x \in [0, l]$ and $t \ge 0$. The boundary conditions on f and G guarantee that their odd periodic extensions are still at least C^1, and that of $\partial_x^2 G$ is at least piecewise continuous. It follows that the Fourier sine coefficients of f (namely, $b_n(0)$) are absolutely summable, and those of G (namely, $\beta_n(t)$) are continuous in t and satisfy $|\beta_n(t)| \le Cn^{-2}$ for t in any finite interval $[0, T]$. We then have

$$|b_n(t)| \le e^{-k(n\pi/l)^2 t}\left[|b_n(0)| + Cn^{-2}\int_0^t e^{-k(n\pi/l)^2 s}\,ds\right] \le e^{-k(n\pi/l)^2 t}|b_n(0)| + \frac{C}{k(\pi/l)^2 n^4}.$$

This is enough to guarantee the absolute and uniform convergence of the series defining $u(x,t)$ for $x \in [0,l]$ and $t \in [0,T]$, as well as the absolute and uniform convergence of the series defining $\partial_t u(x,t)$ and $\partial_x^2 u(x,t)$ for $x \in [0,l]$ and $t \in [\epsilon, T]$ ($\epsilon > 0$), so that all formal calculations are justified.

4. (a) The odd periodic extension of the initial displacement $u(x,0)$ is $mg(\pi x/l)$ where g is as in Exercise 2, §8.3, with $a = \pi b/l$, so its Fourier sine series can be read off from the answer to that exercise. The series for $u(x,t)$ can then be read off from (8.37).

(b) When $b = (0.4)l$ we have $2l^2/\pi^2 b(l-b) = 200/24\pi^2 \approx .844$, and $n^{-2}\sin((.4)n\pi) \approx .951$, .147, $-.065$, $-.059$, 0 when $n = 1,2,3,4,5$, so the first five coefficients (up to the overall factor of m) are .803, .124, $-.055$, $-.050$, 0. When $b = (0.1)l$ we have $2l^2/\pi^2 b(l-b) = 200/9\pi^2 \approx 2.252$, and $n^{-2}\sin((.1)n\pi) \approx .309$, .147, .090, .059, .040 when $n = 1,2,3,4,5$, so the first five coefficients are (m times) .696, .331, .203, .133, .090. (Note: The L^2 norm of the initial displacement $u(\cdot,0)$ is $m\sqrt{l/3}$, independent of b, so the total energy of these waves is independent of b and a direct comparision of the coefficients is appropriate.)

5. For $u(x,t) = \varphi(x)\psi(t)$ to be a solution of the modified wave equation, we must have $\varphi(x)\psi''(t) + 2\delta\varphi(x)\psi'(t) = c^2\varphi''(x)\psi(t)$, or $[\psi''(t) + 2\delta\psi(t)]/c^2\psi(t) = \varphi''(x)/\varphi(x) = -\alpha$, a constant. As on page 385, the boundary conditions force $\alpha = (n\pi/l)^2$ and $\varphi(x) = \sin(n\pi x/l)$. Then $\psi''(t) + 2\delta\psi'(t) + (n\pi c/l)^2\psi(t) = 0$. The roots of $\lambda^2 + 2\delta\lambda + (n\pi c/l)^2 = 0$ are $\lambda = -\delta \pm i\omega_n$ where $\omega_n = \sqrt{(n\pi c/l)^2 - \delta^2}$, so $\psi(t) = e^{-\delta t}(b_n \cos\omega_n t + B_n \sin\omega_n t)$. Taking linear combinations of these solutions for $n = 1,2,3,\ldots$ gives the desired result analogous to (8.37). (This is assuming $\delta < \pi c/l$. If not, the solutions for $n \le \delta l/\pi c$ have pure exponential decay with no oscillation.)

6. (a) If u_1 solves the problem for $g_1 = g_2 = 0$ and u_2 solves the problem for $f_1 = f_2 = 0$, then $u = u_1 + u_2$ solves the problem in the general case.

(b) First we note that $\sinh c(L-y) = \sinh cL \cosh cy - \cosh cL \sinh cy$ (Exercise 1b, §7.4), so $\cosh cy = \operatorname{csch} cL \sinh c(L-y) + \coth cL \sinh cy$; hence any linear combination of $\sinh cy$ and $\cosh cy$ is also a linear combination of $\sinh cy$ and $\sinh c(L-y)$ (and vice versa). Now, for $u(x,y) = \varphi(x)\psi(y)$ to satisfy Laplace's equation, we need $\psi''(y)/\psi(y) = -\varphi''(x)/\varphi(x) = \alpha$, and the boundary conditions become $\varphi(0) = \varphi(l) = 0$. As in the text, this forces $\alpha = (n\pi/l)^2$ and $\varphi(x) = \sin(n\pi x/l)$. Hence $\psi''(y) = (n\pi/l)^2\psi(y)$, so (by the preceding remark) $\psi(y) = b_n^1 \sinh(n\pi(L-y)/l) + b_n^2 \sinh(n\pi y/l)$. Taking linear combinations, we arrive at the general solution $u(x,y) = \sum_1^\infty \sin(n\pi x/l)[b_n^1 \sinh(n\pi(L-y)/l) + b_n^2 \sinh(n\pi y/l)]$. We then have $u(x,0) = \sum_1^\infty b_n^1 \sinh(n\pi L/l)\sin(n\pi x/l)$, which must be the Fourier sine series of $f_1(x)$, and $u(x,L) = \sum_1^\infty b_n^2 \sinh(n\pi L/l)\sin(n\pi x/l)$, which must be the Fourier series of $f_2(x)$.

7. (a) For $u(r,\theta) = \varphi(r)\psi(\theta)$ to satisfy the polar Laplace equation, we need $r^2\varphi''(r)\psi(\theta) + r\varphi'(r)\psi(\theta) + \varphi(r)\psi''(\theta) = 0$, or $[r^2\varphi''(r) + r\varphi'(r)]/\varphi(r) = -\psi''(\theta)/\psi(\theta) = \alpha$, a constant. This differential equation for ψ, together with the periodicity requirement, yields $\alpha = n^2$ and $\psi(\theta) = c_n e^{in\theta} + c_{-n}e^{-in\theta}$ ($n = 0,1,2,\ldots$). The general solution of the differential equation for φ is then a linear combination of r^n and r^{-n} (or 1 and $\log r$ if $n = 0$). But r^{-n} and $\log r$ blow up at the origin and must be discarded. Hence we obtain the general solution $u(r,\theta) = \sum_{-\infty}^{\infty} c_n r^{|n|} e^{in\theta}$, and the requirement that $u(1,\theta) = f(\theta)$ means that the c_n's are the Fourier coefficients of f; thus $u(r,\theta) = A_r f(\theta)$.

(b) This follows immediately from part (a), (8.20), (8.22), and the observation that the Poisson kernel P_r is an even function, so that one can replace φ by $-\varphi$ in the integral.

8.6 The Infinite-Dimensional Geometry of Fourier Series

1. Use the identities $\cos nx \cos mx = \frac{1}{2}[\cos(m+n)x + \cos(m-n)x]$ and $\sin nx \sin mx = \frac{1}{2}[\cos(m-n)x - \cos(m+n)x]$, or $\cos nx \cos mx = \frac{1}{4}(e^{inx}+e^{-inx})(e^{imx}+e^{-imx})$, etc., to see that $\int_0^\pi \cos nx \cos mx\,dx$ and $\int_0^\pi \sin nx \sin mx\,dx$ are 0 when $m \neq n$ and $\pi/2$ when $m = n > 0$ (of course $\int_0^\pi \cos^2 nx\,dx = \pi$ when $n = 0$). The norm of $\cos nx$ or $\sin nx$ is $\sqrt{\pi/2}$ for $n > 0$; the norm of $1 = \cos 0x$ is $\sqrt{\pi}$.

2. If $f \in L^2(0,\pi)$, let $\widetilde{f}$ be its odd 2π-periodic extension and let $\sum_1^\infty b_n \sin n\theta$ be the Fourier series of $\widetilde{f}$, whose restriction to $[0,\pi]$ is the Fourier sine series of f. Then $\int_0^\pi |f(\theta) - \sum_1^N b_n \sin n\theta|^2\,d\theta = \frac{1}{2}\int_{-\pi}^\pi |\widetilde{f}(\theta) - \sum_1^N b_n \sin n\theta|^2\,d\theta \to 0$ as $N \to \infty$. Likewise for the cosine series.

3. $\langle f_0, f_1\rangle = \int_0^1 (x+a)\,dx = \frac{1}{2} + a$ and $\langle f_0, f_2\rangle = \int_0^1 (x^2+bx+c)\,dx = \frac{1}{3} + \frac{1}{2}b + c$, so to make $\langle f_0, f_1\rangle = \langle f_0, f_2\rangle = 0$ we need $a = -\frac{1}{2}$ and $c = -\frac{1}{3} - \frac{1}{2}b$. Then $\langle f_1, f_2\rangle = \int_0^1 (x-\frac{1}{2})(x^2+bx-\frac{1}{2}b-\frac{1}{3})\,dx = \int_0^1 [x^3 + (b-\frac{1}{2})x^2 - (b+\frac{1}{3})x + \frac{1}{4}b + \frac{1}{6}]\,dx = \frac{1}{4} + \frac{1}{3}(b-\frac{1}{2}) - \frac{1}{2}(b+\frac{1}{3}) + \frac{1}{4}b + \frac{1}{6} = \frac{1}{12}(b+1)$, so we must have $b = -1$ and hence $c = \frac{1}{6}$.

4. $\int_{-l}^{l} [\varphi_n^+(x)/\sqrt{2}]\overline{[\varphi_m^+(x)/\sqrt{2}]}\,dx = \int_0^l \varphi_n^+(x)\overline{\varphi_m^+(x)}\,dx = \delta_{mn}$ since the integrand is even; likewise, $\int_{-l}^{l} [\varphi_n^-(x)/\sqrt{2}]\overline{[\varphi_m^-(x)/\sqrt{2}]}\,dx = \delta_{mn}$. Finally, $\int_{-l}^{l} \varphi_n^+(x)\overline{\varphi_m^-(x)}\,dx = 0$ since the integrand is odd.

5. $\langle \psi_n, \psi_m\rangle = \int_{(a-d)/c}^{(b-d)/c} \varphi_n(cx+d)\overline{\varphi_m(cx+d)}\,c\,dx = \int_a^b \varphi_n(u)\overline{\varphi_m(u)}\,du = \delta_{mn}$ $(u = cx+d)$.

6. $\langle \psi_n, \psi_m\rangle = \int_0^1 \varphi_n(x^2)\overline{\varphi_m(x^2)}2x\,dx = \int_0^1 \varphi_n(u)\overline{\varphi_m(u)}\,du = \delta_{mn}$ $(u = x^2)$.

7. Suppose, to begin with, that f is continuous on $[a,b]$ except for a jump at $c \in (a,b)$. To simplify notation, we assume $a < 0 = c < b$. For n large enough so that $a < -\frac{1}{n}$ and $\frac{1}{n} < b$, define $f_n(x)$ to be $f(x)$ if $a \le x \le -\frac{1}{n}$ or $\frac{1}{n} \le x \le b$, and $f_n(x) = f(-\frac{1}{n}) + \frac{n}{2}[f(\frac{1}{n}) - f(-\frac{1}{n})](x + \frac{1}{n})$ if $-\frac{1}{n} < x < \frac{1}{n}$. Then f_n is continuous on $[a,b]$. Moreover, if $|f(x)| \le M$ for all $x \in [a,b]$, the same is true of $|f_n(x)|$, since the values of f_n on $[-\frac{1}{n}, \frac{1}{n}]$ lie in between $f(-\frac{1}{n})$ and $f(\frac{1}{n})$; so $|f_n(x) - f(x)| \le 2M$. Hence $\int_a^b |f_n(x) - f(x)|^2\,dx = \int_{-1/n}^{1/n} |f_n(x) - f(x)|^2\,dx \le 8M^2/n \to 0$ as $n \to \infty$. In the general case of discontinuities at $c_1, \dots, c_k \in [a,b]$, one likewise defines $f_n(x)$ to be $f(x)$ except on the intervals $(c_j - \frac{1}{n}, c_j + \frac{1}{n})$, on which f_n interpolates linearly between $f(c_j - \frac{1}{n})$ and $f(c_j + \frac{1}{n})$, and one finds that $\int_a^b |f_n(x) - f(x)|^2\,dx \le 8kM^2/n \to 0$ as $n \to \infty$.

8. We have $|a_n|^2 = |c_n + c_{-n}|^2 = |c_n|^2 + |c_{-n}|^2 + 2\,\mathrm{Re}(c_n\overline{c}_{-n})$ and $|b_n|^2 = |i(c_n - c_{-n})|^2 = |c_n|^2 + |c_{-n}|^2 - 2\,\mathrm{Re}(c_n\overline{c}_{-n})$, so $|a_n|^2 + |b_n|^2 = 2(|c_n|^2 + |c_{-n}|^2)$. (When $n = 0$ we have $a_0 = 2c_0$ and $b_0 = 0$, and this formula becomes $|a_0|^2 = 4|c_0|^2$.) Hence $\frac{1}{2}\pi|a_0|^2 + \pi\sum_1^\infty(|a_n|^2 + |b_n|^2) = 2\pi\sum_{-\infty}^\infty |c_n|^2 = \int_{-\pi}^\pi |f(\theta)|^2\,d\theta$.

9. (a) From Exercise 4, §8.1, $2\pi^4/9 + 16\sum_1^\infty 1/n^4 = (1/\pi)\int_{-\pi}^\pi \theta^4\,d\theta = 2\pi^4/5$, or $\sum_1^\infty 1/n^4 = \pi^4/90$.

(b) From Exercise 6, §8.1, $(8/\pi)^2\sum_1^\infty 1/(2n-1)^6 = (1/\pi)\int_{-\pi}^\pi \theta^2(\pi - |\theta|)^2\,d\theta = (2/\pi)\int_0^\pi (\pi^2\theta^2 - 2\pi\theta^3 + \theta^4)\,d\theta = \pi^4/15$, or $\sum_1^\infty 1/(2n-1)^6 = \pi^6/960$.

(c) From Exercise 3b, §8.3, $2(7\pi^4/15)^2 + (48)^2\sum_1^\infty 1/n^8 = (1/\pi)\int_{-\pi}^\pi (\theta^4 - 2\pi^2\theta^2)^2\,d\theta = (2/\pi)\int_0^\pi (\theta^8 - 4\pi^2\theta^6 + 4\pi^4\theta^4)\,d\theta = 214\pi^8/315$, or $\sum_1^\infty 1/n^8 = \pi^8/9450$.

(d) From Exercise 7, §8.1, for $0 < a < \pi$ we have $[4/a^2(\pi - a)^2] \sum_1^\infty (\sin na)^2/n^2 = (2/\pi)[\int_0^a a^{-2}\,d\theta + \int_a^\pi (\pi - a)^{-2}\,d\theta] = (2/\pi)[a^{-1} + (\pi - a)^{-1}] = 2/a(\pi - a)$, or $\sum_1^\infty (\sin na)^2/n^2 = a(\pi - a)/2$. This formula is still valid when $a = 0$ or $a = \pi$ (both sides vanish then), and the sum is clearly π-periodic as a function of a.

10. First way: $\langle f, f' \rangle = \int_{-\pi}^{\pi} f(\theta)f'(\theta)\,d\theta = \frac{1}{2}\int_{-\pi}^{\pi}(f^2)'(\theta)\,d\theta = \frac{1}{2}f(\theta)^2\big|_{-\pi}^{\pi} = 0$ since $f(-\pi) = f(\pi)$. Second way: If $\{c_n\}$ are the Fourier coefficients of f, then the Fourier coefficients of f' are $\{inc_n\}$, and $c_{-n} = \int_{-\pi}^{\pi} f(\theta)e^{in\theta}\,d\theta = \int_{-\pi}^{\pi} \overline{f(\theta)e^{-in\theta}}\,d\theta = \bar{c}_n$. Hence, by (8.46), $\langle f, f' \rangle = 2\pi \sum_{-\infty}^{\infty} c_n(-in\bar{c}_n) = -2\pi \sum_1^\infty in(|c_n|^2 - |c_{-n}|^2) = 0$.