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Current Trends in Arithmetical Algebraic Geometry

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INTRODUCTION

This volume is the record of a 1985 week-long Joint Summer Research Conference on Algebraic Geometry, held in Arcata, California. The conference, organized by Michael Artin, Barry Mazur, and myself, focused on three current developments in our field:

1. The work of J-M. Fontaine and W. Messing linking p-adic étale cohomology with crystalline cohomology.
2. Diophantine conjectures motivated by hyperbolic geometry.
3. Arakelov theory and its generalization to higher dimensions.

A majority of the lectures given at the conference fell clearly into one of these categories: the lectures of Messing concerned the first topic, those of S. Kobayashi, S. Lang and P. Vojta the second, and those of H. Gillet, C. Soulé and L. Szpiro the third. Other talks at the conference, such as those of G. Anderson, R. Livné and J. Silverman, cannot be so neatly classified.

The manuscripts assembled for this volume include six speakers' contributions, plus four additional papers which do not correspond to talks at the conference:

1. A. Beilinson's article on Height Pairings, frequently cited during the conference, which has been unpublished until now.
2. An updated version of notes by P. Deligne and D. Husemöller on Drinfel'd modules which were written several years ago but never published.
3. A new manuscript of Deligne, "Le déterminant de la cohomologie," which grew out of a letter of Deligne to Quillen that was discussed by Gillet in his lectures at the conference.
4. J-P. Serre's letter to J-F. Mestre (July, 1985) concerning mod p representations of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ which arise from modular forms. The "epsilon" of this letter is the extra information needed to realize G. Frey's idea that Fermat's "Last Theorem" follows from the conjecture that every elliptic curve over $\mathbb{Q}$ is a factor of some Jacobian $J_0(N)$. I am currently preparing a manuscript which should prove a slightly weakened version of "epsilon," thus in fact establishing the link which was hoped for by Frey. For precise conjectures which

predict that all irreducible mod p representations of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ with odd determinant arise from mod p modular forms, see Serre's article, *Sur les representations modulaires de degrê 2 de* $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$, Duke Math. J. 54, 179-230 (1987).

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Height pairing between algebraic cycles

A. Beilinson

Introduction

Let X be a smooth projective variety over $\mathbb{Q}$; assume that its L-functions $L(H^j(X), s)$ satisfy the standard analytic continuation conjectures. Let $CH^i(X)$ be the group of codimension i cycles on X modulo rational equivalence, and $CH^i(X)^o \subset CH^i(X)$ be the subgroup of cycles homologous to zero on $X(\mathbb{C})$; in particular $CH^1(X)^o = Pic^o(X)(\mathbb{Q})$. The conjecture of Birch and Swinnerton-Dyer claims that at $s = 1$ the function $L(H^1(X), s)$ has zero of order $rk\ CH^1(X)^o$ with the leading coefficient equal to the determinant of Néron-Tate canonical height pairing multiplied by the period matrix determinant up to some rational multiple (we do not need its exact value in what follows). As for the other L-functions, Swinnerton-Dyer conjectured [20] that the function $L(H^{2i-1}(X), s)$ has at $s = i$ (=the middle of the critical strip) the zero of order $rk\ CH^i(X)^o$. The aim of this note is to define the canonical height pairing between $CH^i(X)^o$ and $CH^{\dim X+1-i}(X)^o$ that coincides with the Néron-Tate one for $i = 1$ and whose determinant multiplied by the period matrix determinant should conjecturally be equal up to a rational multiple (of the nature I cannot imagine) to the leading coefficient of $L(H^{2i-1}(X), s)$ at $s = i$ *). This pairing should also occur in Riemann-Roch type theorems à la Arakelov-Faltings (see [15]).

*) In fact our height pairing is defined on a certain subgroup of $CH^\cdot(X)^o$; under the very plausible (=of rank of evidence far higher then B-SwD) local conjectures this subgroups should coincide with the whole $CH^\cdot(X)^o$.

The paper goes as follows. To motivate the basic construction, we begin with the simpler geometric case: here our base field is a field $k(C)$ of rational functions on a smooth projective curve C. Then the height pairing $\langle , \rangle$ comes from the global Poincaré duality on ℓ-adic cohomology. We may compute $\langle , \rangle$ in terms of local data round the points of C : if a_1, a_2 are cycles with disjoint supports that are homologous to zero on $X \otimes \overline{k(C)}$ and $v \in C$ is a closed point then the local link index $\langle a_1, a_2 \rangle_v$ is defined, and we have

$$(*) \qquad \langle a_1, a_2 \rangle = \sum_{v \in C} \langle a_1, a_2 \rangle_v$$

In the arithmetic situation, when the base field is a number field, the global construction fails due to the lack of appropriate cohomology theory. But we may still define the local indexes $\langle , \rangle_v$ numbered by the places of the base field, and then use $(*)$ as the definition of $\langle , \rangle$. These indexes are defined using ℓ-adic cohomology for non-archimedean v and using the absolute Hodge-Deligne cohomology (see [2], [3], [19]) for archimedean ones; in case of pairing between divisors and zero cycles they are just Neron's quasifunctions ([17], [13],[22])

We may also consider the intersection pairings. In the geometric case this is just the usual intersection pairing between the cycles of complementary dimensions on the regular scheme X_C proper over C. In the arithmetic case the role of X_C plays the A-variety $X = (X_{\mathbb{Z}}, \omega)$: where $X_{\mathbb{Z}}$ is a regular scheme projective over Spec $\mathbb{Z}$ and ω is a Kahler (1,1)-form on $X_{\mathbb{Z}} \otimes \mathbb{R}$ (see [15]). We define the corresponding Chow groups $CH^{\cdot}(X)$ and the $\mathbb{R}$-valued intersection pairing between $CH^i(X)$ and $CH^{\dim X - i}(X)$. This construction was independently found by H. Gillet and Ch. Soulé [10].

The final § contains some conjectures and motivic speculations about algebraic cycles, heights, L-functions and absolute cohomology groups.

The different construction of height pairing was proposed by S. Bloch [6]; I hope that our pairings coincide.

I would like to thank S. Bloch, P. Deligne, Yu. Manin, V. Schechtman and Ch. Soulé for stimulating ideas and interest.

§1. Height pairing in geometric case (global construction)

In this § k will be an algebraically closed field, and ℓ will be a prime different from char k.

1.0. First recall some basic facts about the intersections. Let Y be a smooth projective scheme over k of dimension $N+1$. The intersection of cycles defines the ring structure on Chow group $CH^{\bullet}(Y)$. This, together with an obvious trace map $CH^{N+1}(Y) \to \mathbb{Q}$, determines the intersection pairing $(,) : CH^{i}(Y) \otimes CH^{N+1-i}(Y) \to \mathbb{Q}$. We also have étale cohomology ring and the trace $H^{2(N+1)}(Y, \mathbb{Q}_{\ell}(N+1)) \to \mathbb{Q}_{\ell}$. The class map $cl : CH^{\bullet}(Y) \to H^{2\bullet}(Y, \mathbb{Q}_{\ell}(\bullet))$ is compatible with these structures, so we may compute $(,)$ using ℓ-adic cohomology classes. In particular, $(,)$ factors through $\overline{CH}^{\bullet}(Y) := \mathrm{Im}(CH^{\bullet}(Y) \to H^{2\bullet}(Y, \mathbb{Q}_{\ell}(\bullet)))$. One hopes that the following standard conjectures hold

- the obvious map $\overline{CH}^{\bullet}(Y) \otimes \mathbb{Q}_{\ell} \to H^{2\bullet}(Y, \mathbb{Q}_{\ell}(\bullet))$ should be injective, so $\overline{CH}^{\bullet}(Y) \otimes \mathbb{Q}$ should be finite-dimensional $\mathbb{Q}$-vector spaces (this is obviously true in char 0)
- $(,)$ should be non-degenerate on $\overline{CH}^{\bullet}(Y) \otimes \mathbb{Q}$
- $\overline{CH}^{\bullet} \otimes \mathbb{Q}$ and $(,)$ should satisfy hard Lefschetz and Hodge-index theorems.
- Finally if k is an algebraic closure of a finite field, then one should have $CH^{\bullet} \otimes \mathbb{Q} = \overline{CH}^{\bullet} \otimes \mathbb{Q}$. More precisely, for an Y_o/F_q, $Y_o \otimes k = Y$, the group $CH^{\bullet}(Y_o) \otimes \mathbb{Q}_{\ell}$ should coincide with invariants of Frobenius in $H^{2\bullet}(Y, \mathbb{Q}_{\ell}(\bullet))$ (Tate's conjecture).

1.1. The height pairing arises in a slightly different situation. Fix a smooth projective irreducible curve C over k; let $\eta \in C$ be its generic point, and $\bar{\eta}/\eta$ be some geometric generic point. Now let X be a smooth projective N-dimensional η-scheme, and $X_{\bar{\eta}} := X \times_{\eta} \bar{\eta}$ be its geometric fiber. Choose some projective scheme $X_C \xrightarrow{\pi} C$ over C with the generic fiber X; for an open $U \subset C$ put $X_U = \pi^{-1}(U)$. Now take $j : U \hookrightarrow C$ s.t. U is affine and $\pi|_U$ is smooth. Put $H^{\bullet}_{!*}(X, \mathbb{Q}_{\ell}(*)) := \mathrm{Im}(H^{\bullet}_{c}(X_U, \mathbb{Q}_{\ell}(*)) \to H^{\bullet}(X_U, \mathbb{Q}_{\ell}(*)))$; the Poincaré duality on X_U induces the perfect pairing between $H^{\bullet}_{!*}(X, \mathbb{Q}_{\ell}(*))$ and $H^{2N+2-\bullet}_{!*}(X, \mathbb{Q}_{\ell}(N+1-*))$. We'll see in a moment that this groups and pairing depend only on X itself (and not on the choice of particular model X_C/C). To do this consider the smooth sheaf $R^{\bullet-1}\pi|_{U*}(\mathbb{Q}_\ell)$ on U and its middle extension $\mathcal{F}^\bullet :=$ $j_* R^{\bullet-1}\pi|_{U*}(\mathbb{Q}_\ell)$ on C. The Poincare duality along the fibers of π defines the perfect pairing $\langle,\rangle : \mathcal{F}^\bullet \otimes \mathcal{F}^{2N+2-\bullet} \longrightarrow \mathbb{Q}_\ell(-N)$ in $D^b(C,\mathbb{Q}_\ell)$ and so the same-noted perfect duality $\langle,\rangle : H^1(C, \mathcal{F}^\bullet) \otimes H^1(C, \mathcal{F}^{2N+2-\bullet}) \longrightarrow \mathbb{Q}_\ell(-N-1)$. Clearly both $\mathcal{F}^\bullet$ and $\langle,\rangle$ depend only on X (and not on X_C).

<u>Lemma 1.1.1.</u> We have $H^\bullet_{!*}(X, \mathbb{Q}_\ell(*)) = H^1(C, \mathcal{F}^\bullet(*))$ and $\langle,\rangle$ coincides with the duality on $H_{!*}$ induced by the pairing between $H_c(X_U)$ and $H(X_U)$.

<u>Proof</u>. I'll prove only the first statement since the second is immediate. Since U is affine, the Leray spectral sequence for π degenerates and reduces to two-step filtrations on $H^\bullet_c(X_U)$ and $H^\bullet(X_U)$ with factors $Gr_{-2}(H^\bullet_c) = H^2_c(U, R^{\bullet-2}\pi_*(\mathbb{Q}_\ell))$, $Gr_{-1}(H^\bullet_c) = H^1_c(U, R^{\bullet-1}\pi_*(\mathbb{Q}_\ell))$ and $Gr_{-1}(H^\bullet) = H^1(U, R^{\bullet-1}\pi_*(\mathbb{Q}_\ell))$, $Gr_0(H^\bullet) = H^0(U, R\pi^\bullet_*(\mathbb{Q}_\ell))$. So $H^\bullet_{!*}(X, \mathbb{Q}_\ell) = \mathrm{Im}\,(H^1_c(U, R^{\bullet-1}\pi_*(\mathbb{Q}_\ell)) \longrightarrow H^1(U, R^{\bullet-1}\pi_*(\mathbb{Q}_\ell))) = H^1(C,\mathcal{F}^\bullet)$, q.e.d. ■

For a moment put $H^\bullet(X, \mathbb{Q}_\ell) := \varinjlim H^\bullet(X_U, \mathbb{Q}_\ell) = \varinjlim H^\bullet(U, R\pi_*(\mathbb{Q}_\ell))$, $H^\bullet(X, \mathbb{Q}_\ell)^\circ := \mathrm{Ker}(H^\bullet(X, \mathbb{Q}_\ell) \longrightarrow H^\bullet(X_{\bar\eta}, \mathbb{Q}_\ell)) = \varinjlim H^1(U,R^{\bullet-1}\pi_*(\mathbb{Q}_\ell))$. Clearly $H^\bullet_{!*}(X) \subset H^\bullet(X)^\circ \subset H^\bullet(X)$ and this groups depend on X only. Now consider the cycles on X; we have the class map $cl : CH^\bullet(X) \longrightarrow H^{2\bullet}(X, \mathbb{Q}_\ell(\cdot))$, put $CH^\bullet(X)^\circ := cl^{-1}(H^{2\bullet}(X, \mathbb{Q}_\ell(\cdot))^\circ)$ to be the subgroup of cycles whose intersection with generic geometric fiber is homologous to zero.

<u>Key-lemma 1.1.2.</u> One has $cl(CH^\bullet(X)^\circ) \subset H^{2\bullet}_{!*}(X, \mathbb{Q}_\ell(\cdot))$.

<u>Proof</u>. The problem is local round the points of C (and exists only at points of bad reduction). So let η_v be the generic point of a henselisation of C at some closed point, $\bar\eta_v$ be a separable closure of η_v, and $X_{\eta_v} = X \times \eta_v$, $X_{\bar\eta_v} = X \times \bar\eta_v$. We have to show that whenever $a \in CH^\bullet(X_{\eta_v})$ is a cycle s.t. $cl\ a \in H^{2\bullet}(X_{\eta_v}, \mathbb{Q}_\ell(\cdot))^\circ :=$ $\mathrm{Ker}(H^{2\bullet}(X_{\eta_v}, \mathbb{Q}_\ell(\cdot)) \longrightarrow H^2(X_{\bar\eta_v}, \mathbb{Q}_\ell(\cdot)))$ then $cl\ a$ is zero. But $H^{2\bullet}(X_{\eta_v}, \mathbb{Q}_\ell(\cdot))^\circ = H^{2\bullet-1}(X_{\bar\eta_v}, \mathbb{Q}_\ell(\cdot-1))_{\mathrm{Gal}\,\bar\eta_v/\eta_v}$. According to [9] th.1.8.4. the weights on this group are ≥ 1. Since the class of an algebraic cycle has weight zero, we are done. ■

Now, by lemma, we may define the height pairing $\langle , \rangle : CH^{i}(X)^{o} \times CH^{N+1-i}(X)^{o} \longrightarrow \mathbb{Q}_{\ell}$ by means of class map followed by the duality between $H^{\bullet}_{!*}$.

One may conjecture that this pairing is $\mathbb{Q}$-valued and independent of ℓ ; of course this is obviously true in char 0, and also true in any char if $i = 1$ (for the discussion of this see §2). The variant of standard conjectures claims that $\overline{CH^{\bullet}(X)^{o}} := \mathrm{Im}\,(CH^{\bullet}(X)^{o} \otimes \mathbb{Q} \longrightarrow H^{2\bullet}_{!*}(X, \mathbb{Q}_{\ell}(\cdot)))$ should be finite-dimensional $\mathbb{Q}$-vector space, that this groups satisfy hard Lefschetz, the form $\langle , \rangle$ should be non-degenerate on them and satisfy Hodge-index theorem. If k = algebraic closure of finite field, then one should have $CH^{\bullet}(X)^{o} \otimes \mathbb{Q} = \overline{CH^{\bullet}(X)^{o}}$.

Problem. Consider the case $\dim \eta/k > 1$.

1.2. The both types of pairings - the intersection and the height one - are related as follows. Suppose that we have $\pi : X_{C} \longrightarrow C$ s.t. X_{C} is regular projective with generic fiber X. Define $CH^{\bullet}(X_{C})^{o} := \mathrm{Ker}(CH^{\bullet}(X_{C}) \rightarrow H^{0}(C, R^{2\bullet}\pi_{*}(\mathbb{Q}_{\ell}(\cdot)))$ to be the subgroup of cycles whose intersection with any geometric fiber of π is homologous to zero. The restriction map $CH^{\bullet}(X_{C}) \rightarrow CH^{\bullet}(X)$, $a_{C} \mapsto a$ maps $CH^{\bullet}(X_{C})^{o}$ into $CH^{\bullet}(X)^{o}$ and we have an obvious

Lemma 1.2.1. For any $a_{1C} \in CH^{\bullet}(X_{C})^{o}$, $a_{2C} \in CH^{N+1-\bullet}(X_{C})^{o}$ one has $(a_{1C}, a_{2C}) = \langle a_{1}, a_{2} \rangle$ ■

One may suppose that the image of $CH\ (X\ )^{o}$ under the restriction map coincides with $CH^{\bullet}(X)^{o}$; see §2 for details.

§2. Local indexes over non-archimedean places

In this § we'll define the local pairing between cycles, and will show how to decompose the global pairings of the previous § into the sum of local ones.

In what follows C_{v} will be any strictly henselian trait with the generic point η_{v}, the special point s_{v}, and an algebraic generic geometric point $\bar{\eta}_{v}$; X_{v} will be a smooth projective scheme over η_{v} of dimension N, and $X_{\bar{v}} := X_{v} \times_{\eta_{v}} \bar{\eta}_{v}$ will be its geometric fiber; ℓ will be a prime $\neq$ char s.

2.0. Let me start with the local intersection pairing. Let $X_{C_{v}}$ be a regular projective scheme over C_{v} with the generic fiber X_{v} and the special one X_{s}. If $a_{C_{v}}$ is any cycle on $X_{C_{v}}$ of codimension d then one has its classes $\widetilde{cl}\,(a_{C_{v}})$ in cohomology groups with supports:

the universal one $\widetilde{cl}_M(a_{C_v}) \in H^{2d}_{M\ \mathrm{supp}a_{C_v}}(X_{C_v}, \mathbb{Q}(d)) = H^{2d}_M(X_{C_v}, X_{C_v} \setminus \mathrm{supp}a_{C_v}, \mathbb{Q}(d))$ and the ℓ-adic one $\widetilde{cl}_{\mathbb{Q}_\ell}(a_{C_v}) \in H^{2d}_{\mathrm{supp}a_{C_v}}(X_{C_v}, \mathbb{Q}_\ell(d))$; clearly $cl_M \longmapsto cl_{\mathbb{Q}_\ell}$ under canonical map (see [18], [3]).

Now let $a_{1C_v} \in \mathcal{Z}^{d_1}(X_{C_v})$, $a_{2C_v} \in \mathcal{Z}^{d_2}(X_{C_v})$ be two cycles on X_{C_v} of supports Y_{1C_v}, Y_{2C_v} respectively such that $d_1 + d_2 = N + 1$ and $Y_{1v} \cap Y_{2v} := Y_{1C_v} \cap Y_{2C_v} \cap X_v = \emptyset$. Define the intersection index $(a_{1C_v}, a_{2C_v})_v \in \mathbb{Q}$ to be the image of $\widetilde{cl}_M(a_{1C_v}) \cup \widetilde{cl}_M(a_{2C_v}) \in H^{2N+2}_{M\ Y_1\cap Y_2}(X_{C_v}, \mathbb{Q}(N+1))$ by $H^{2N+2}_{M\ X_s}(X_{C_v}, \mathbb{Q}(N+1)) \xrightarrow{Tr} H^1_{M\ s_v}(C_v, \mathbb{Q}(1)) = \mathbb{Q}$.

We may replace here H_M by ℓ-adic cohomology; since $H_M \longrightarrow H_{\mathbb{Q}_\ell}$ commutes with any canonical map, we'll get the same answer.

If we are in a global geometric situation 1.2, then for any closed point s_v of C we may consider the henselisation C_v of C at s_v and thus get our local situation. If a_{1C}, a_{2C} are cycles on X_C s.t. a_1, a_2 have disjoint supports on X, then for any s_v we get local intersection index $(a_{1C}, a_{2C})_v$ and clearly one has

Lemma 2.0.1. $(a_{1C}, a_{2C}) = \sum (a_{1C}, a_{2C})_v$ ∎

2.1. Now let us consider the local components for height pairing. Let a_1, a_2, $a_i \in \mathcal{Z}^{d_i}(X_v)$, be a cycles on X_v; put $Y_i = \mathrm{supp}\, a_i$ $U_i = X_v - Y_i$. Suppose that $d_1 + d_2 = N+1$, $Y_1 \cap Y_2 = \emptyset$ and both $cl(a_i) \in H^{2d_i}(X_v, \mathbb{Q}_\ell(d_i))$ are zero. In this situation one has link index $\langle a_1, a_2 \rangle_v \in \mathbb{Q}_\ell$. The intuitive picture for $\langle , \rangle_v$ is following: from the homotopy point of view η_v is a circle round s_v and X_v is $2N+1$-dimensional topological manifold fibered over this circle, a_i are $2d_i$-codimensional cycles on it that doesn't intersect and homologous to zero; the link index $\langle a_1, a_2 \rangle_v$ is the intersection number of a_1 and the chain that bounds a_2. Here are the number of exact definitions of $\langle a_1, a_2 \rangle_v$; the proof of their equivalence is left to the reader. In what follows $\alpha_i \in H^{2d_i-1}(U_i, \mathbb{Q}_\ell(d_i))$ are classes that bound a_i : this means that $\alpha_i \longmapsto \widetilde{cl}(a_i)$ under the boundary map $H^{2d_i-1}(U_i) \longrightarrow H^{2d_i}_{Y_i}(X_v)$.

Lemma-definition 2.1.1. The following definitions of $\langle a_1, a_2\rangle_v$ are equivalent.

a) $\langle a_1, a_2\rangle_v$ is the image of $\alpha_1 \cup \widetilde{cl}(a_2)$ by $H^{2N+1}_{Y_1}(U_1, \mathbb{Q}(N+1)) \longrightarrow H^{2N+1}(X_v, \mathbb{Q}_\ell(N+1)) \xrightarrow{\pi_*} H^1(\mathfrak{p}_v, \mathbb{Q}_\ell(1)) = \mathbb{Q}_\ell$

b) $\langle a_1, a_2\rangle_v$ is the image of $\alpha_1 \cup \alpha_2$ by $H^{2N}(U_1 \cap U_2, \mathbb{Q}_\ell(N+1)) \longrightarrow H^{2N+1}(X_v, \mathbb{Q}_\ell(N+1)) \xrightarrow{\pi_*} \mathbb{Q}_\ell$; here the first arrow comes from the Myer-Vietoris exact sequence for the covering $U_1 \cup U_2 = X_v$.

c) Choose a projective $\pi : X_{C_v} \longrightarrow C_v$ with the generic fiber X_v, $\beta_1 \in H^{2d_1}_{\pi^{-1}(s_v)\cup Y_1}(X_{C_v}, \mathbb{Q}_\ell(d_1))$ and $\beta_2 \in H^{2d_2-2N}_{\pi^{-1}(s_v)\cup Y_2}(X_{C_v}, R\pi^!\mathbb{Q}_\ell(d_2-N))$ s.t. the restriction of β_i on X_v coincide with $\widetilde{cl}(a_i)$ and the image of β_1 in $H^{2d_1}(X_{C_v}, \mathbb{Q}_\ell(d_1))$ is zero (e.g. you may take β_1 to be the boundary of α_1 in X_{C_v}). Then $\langle a_1, a_2\rangle_v$ is the image of $\beta_1 \cup \beta_2$ by $H^2_{\pi^{-1}(s_v)}(X_{C_v}, R\pi^!\mathbb{Q}_\ell(1)) \longrightarrow H^1_{s_v}(C_v, \mathbb{Q}_\ell(1)) = \mathbb{Q}_\ell$ ■

This way we get link pairing between the cycles with disjoint supports; clearly 2.1.1. b) shows that this pairing is bilinear and symmetric. Now we are going to show that it behaves well under the action of correspondences. To do this first let us see that the above definition may be easily generalised to the case of arbitrary many cycles. Namely let $a_1, \ldots, a_n$ be cycles of codimensions d_i on X_v s.t. $\sum d_i = N+1$ and $\bigcap Y_i = \emptyset$ (here $Y_i = \operatorname{supp} a_i$, $U_i = X_v \setminus Y_i$) assume that at least one of them is homologous to zero in X. Choose a non-empty subset $S \subset \{1,\ldots,n\}$ s.t. for any $j \in S$ the cycle a_j is homologous to zero, and for $a_j \in S$ some $\alpha_j \in H^{2d_j-1}(U_j, \mathbb{Q}_\ell(d_j))$ that bounds a_j. Define $\langle a_1,\ldots,a_n\rangle_v$ to be $\operatorname{Tr}[\partial(\cup_{j\in S}\alpha_j) \cup (\cup_{j\notin S} a_j)]$; here $\partial : H^\cdot(\bigcap_{j\in S} U_j) \longrightarrow H^{\cdot+\#S-1}(\bigcup_{j\in S} U_j)$ is the differential in the spectral sequence of the covering $\{U_j\}_{j\in S}$ of $\bigcup_{j\in S} U_j$, and

We have the following easy generalisation of 2.1.1:

Lemma 2.1.2. If at least two of a_i are homologous to zero, then $\langle a_1,\ldots,a_n\rangle_v$ depends on $a_1,\ldots,a_n$ only (and not on the choice of S and α_j) ■

Clearly in this case the pairing is also bilinear and symmetric.

Now we may look at correspondences. Consider two schemes X_{1v}, X_{2v} of dimensions N_1, N_2, a cycles a_i of codimensions d_i on

X_{iv} and a cycle b of codimension d_3 on $X_{1v} \times X_{2v}$. Assume that both a_i are homologous to zero, $d_1 + d_2 + d_3 = N_1 + N_2 + 1$ and $p_1^{-1}(\mathrm{supp}\ a_1) \cap \mathrm{supp}\ b \cap p_2^{-1}(\mathrm{supp}\ a_2) = \emptyset$ (here p_i are projections $X_{1v} \times X_{2v} \longrightarrow X_{iv}$). Then 2.1.1 implies

Lemma 2.1.3. $\langle p_1^*(a_1),\ b,\ p_2^*(a_2)\rangle_v = \langle b(a_1),\ a_2\rangle_v = \langle a_1,\ b(a_2)\rangle_v$ ∎

Here $b(a_1)$ is the image of a_1 under the action of correspondence b. (Note that if $p_1^*(a_1)$ and b doesn't intersect properly, then the cycle $b(a_1)$ is not defined in unique way, but it has correctly defined class in the cohomology group with support in $p_2(\mathrm{supp}\ b \cap p_1^{-1}(\mathrm{supp}\ a_1))$ that suffice for our purposes.)

The lemma shows for example that the computation of $\langle a_1,\ a_2\rangle_v$ in case when one of a_i is algebraically equivalent to zero, may be reduced to the computation of link index between a zero cycles on a curve.

Our pairing behaves in a usual way under the change of the base field. Namely let $\eta_{v'}/\eta_v$ be a degree n extension, X_v is a scheme over η_v and $Y_{v'}$ is one over $\eta_{v'}$. Put $X_{v'} = X_v \times_{\eta_v} \eta_{v'}$ and Y_v be $Y_{v'}$ considered as a scheme over η_v. We have the obvious arrows $\mathcal{Z}^{\cdot}(X_v) \hookrightarrow \mathcal{Z}^{\cdot}(X_{v'})$, $\mathcal{Z}^{\cdot}(Y_{v'}) = \mathcal{Z}^{\cdot}(Y_v)$ and the following holds (recall that k is separably closed):

Lemma 2.1.4. Let a_i be cycles on X_v and b_i the ones on $Y_{v'}$. Then $\langle a_1,\ a_2\rangle_v = 1/n\ \langle a_1,\ a_2\rangle_{v'}$, $\langle b_1,\ b_2\rangle_{v'} = \langle b_1,\ b_2\rangle_v$ ∎

In particular by means of first formula we may define the height pairing between cycles on $X_{\bar{\eta}_v}$; this pairing is clearly Galois-invariant.

Let us relate local link pairings with the global height pairing of §1. Assume that we are in global geometric situation of 1.1. As in 2.0 to each closed point on C corresponds the local picture over corresponding local field η_v. Let a_1, a_2 be cycles on X of codimensions d_i s.t. $d_1 + d_2 = N+1$ and $\mathrm{supp} a_1 \cap \mathrm{supp} a_2 = \emptyset$. If a_i belongs to $CH^{d_i}(X)^{\circ}$ then 1.12.shows that a_i is homologous to zero on any X_v. So the link index $\langle a_1, a_2\rangle_v$ is defined, and it is easy to see that the definition 2.1.1.c implies the formula (*) from the introduction:

Lemma 2.1.5. We have $\langle a_1,\ a_2\rangle = \sum \langle a_1,\ a_2\rangle_v$ (sum over all closed points of C). ∎

Finally let me compare the local pairings from 2.0 and 2.1. Suppose that we are in a situation 2.0. Put $CH^{\cdot}(X_{C_v})^{\circ} := \mathrm{Ker}(CH^{\cdot}(X_{C_v}) \longrightarrow H^{2\cdot}(X_{C_v},\ \mathbb{Q}_\ell(\cdot)))$. Let a_{1C_v}, a_{2C_v} be cycles on X_{C_v} as in 2.0; assume that both a_i are homologous to zero on X_v (here $a_i :=$

$a_{1C_v} \cap X_v$). Then 2.1.1c implies.

Lemma 2.1.6. If one of a_{iC_v} also belongs to $CH^{\bullet}(X_{C_v})^{o}$, then $(a_{1C_v}, a_{2C_v})_v = \langle a_1, a_2 \rangle_v$. ■

This is local analog of 1.2.1. In particular in this situation $\langle a_1, a_2 \rangle_v \in \mathbb{Q}$ and doesn't depends on the choice of $\ell \neq$ char k.

2.2. Here are some conjectures about local pairings. Put $CH^{\bullet}(X_v)^{o} := \mathrm{Ker}\,(CH^{\bullet}(X_v) \longrightarrow H^{2\bullet}(X_{\bar{\eta}_v}, \mathbb{Q}_\ell(\cdot)))$

Conjecture 2.2.1. One has $CH^{i}(X_v)^{o} = \mathrm{Ker}\,(CH^{i}(X_v) \to H^{2i}(X_v, \mathbb{Q}_\ell(i)))$ i.e. any cycle whose intersection with a generic geometric fiber is homologous to zero is homologous to zero on X_v ■

Lemma 2.2.2. This conjecture is true in following cases:

a. Good reduction case.

b. For cycles algebraically equivalent to zero (in particular the case $i = 1$ and $i = N$).

c. Geometric case.

Proof. a is obvious; c was proved in 1.1.2; b may be reduced using correspondences to the case of zero cycles on curve, where it follows, say, from 2.2.6b. ■

Remark. The conjecture would follow if one knows the information on weights on $H^{\bullet}(X_{\bar{\eta}_v}, \mathbb{Q}_\ell)$ similar to those one has in geometric case. If η_v is a p-adic local field, then the thing we need is the usual conjecture on poles of local L-multiples.

Conjecture 2.2.3. The local link pairing is Q-valued and independent of $\ell \neq$ char k.

Lemma 2.2.4. This conjecture is true:

a. In good reduction case.

b. In case char $s = 0$.

c. When one of the cycles is algebraically equivalent to zero (in particular for the pairing between divisors and zero cycles).

Proof. b is obvious; a and c follow from 2.2.6, for c use correspondences to reduce to 2.2.6b. ■

Suppose that we are in a situation 2.0. Clearly the restriction arrow $CH^{\cdot}(X_{C_v}) \to CH^{\cdot}(X_v)$ maps $CH^{\cdot}(X_{C_v})^{o}$ into $CH^{\cdot}(X_v)^{o}$.

Conjecture 2.2.5. $CH\,(X_{C}\,)^{o} \twoheadrightarrow CH\,(X_v)^{o}$. ■

Clearly 2.2.5 implies both 2.2.1 and 2.2.3.

Lemma 2.2.6. This conjecture is true

a. In good reduction case.

b. If X_v is a curve.

Proof. a is obvious; b follows from the well-known fact that the intersection matrix between components of the special fiber is almost negative-definite. ■

The property, 2.2.6b together with 2.1.6, shows that in case of zero cycles on a curve our link index coincides with Néron's local symbol; using the correspondences one may see that the same is true for the pairing between divisors and zero cycles in arbitrary dimensions (or you may directly verify that $\langle,\rangle_v$ satisfies the conditions of Néron's theorem [12].) For the review of heights on curves together with the study of important examples see [7].

We wish to use the formula 2.1.5. for varieties over number field as the definition of left-hand side, just as Neron did in the case of divisors and zero cycles. The only thing that remains to be defined is the link index for archimedean local fields.

§3. Local indexes over $\mathbb{C}$ and $\mathbb{R}$

In this § our base field will be $\mathbb{C}$ or $\mathbb{R}$

3.0. We have the following dictionary (see [2] for notations and details)

Non-archimedean case (η_v is the spectrum of a p-adic local field)	Archimedean case (η_v = Spec $\mathbb{C}$ or Spec $\mathbb{R}$)
Gal $\bar{\eta}_v/\eta_v$-modules	R-mixed Hodge structures
Étale cohomology groups $H^{\cdot}(X_{\bar{v}}, \mathbb{Q}_\ell(*))$ of the geometric fiber with the Galois action	Ordinary cohomology groups $H^{\cdot}(X_v(\mathbb{C}), \mathbb{R}(*))$ with Deligne's mixed Hodge structure
Étale cohomology groups $H^{\cdot}(X_v, \mathbb{Q}_\ell(*))$	Absolute Hodge-Deligne cohomology groups $H^{\cdot}_{\mathcal{H}}(X_v, \mathbb{R}(*))$
Canonical arrow $H^{\cdot}(X_v, \mathbb{Q}_\ell(*)) \longrightarrow H^{\cdot}(X_{\bar{v}}, \mathbb{Q}_\ell(*))$	Canonical arrow $H^{\cdot}_{\mathcal{H}}(X_v, \mathbb{R}(*)) \longrightarrow H^{\cdot}_{\mathcal{B}}(X_v, \mathbb{R}(*)) \subset H^{\cdot}(X_v(\mathbb{C}), \mathbb{R}(*))$

Note that the canonical arrow $H^{2i}(X_v, \mathbb{R}(i)) \longrightarrow H^{2i}(X_v, \mathbb{R}(i))$ is injective; so whenever we have an algebraic cycle homologous to zero as topological cycle it has zero class in $H_{\mathcal{H}}$. This shows that the analog of conjecture 2.2.1 is obviously true in archimedean case (this is one of the sides of the fact that smooth proper varieties have "good reduction" at archimedean places from the Hodge-theoretic point of view). We may translate the definition 2.1.1 a or b using the above dictionary to define the link index in our situation. Namely, this way for any two cycles a_1, a_2 on X_v of codimensions d_1, d_2 s.t. $d_1 + d_2 = N+1$, the supports of a_i doesn't intersect and both a_i are homologous to zero, we get the link index $\langle a_1, a_2\rangle_v \in \mathbb{R}$. The lemmas 2.1.1 ($a \Leftrightarrow b$), 2.1.2. - 2.1.4. remain true, together with their proofs, in our situation.

3.1. Now let me sketch the definition of the analog of intersection index à la Arakelov; since this will not be needed in the main body of the paper I'll omit the details. The similar construction was found by H. Gillet and Ch. Soulé [10]. The role of the model X_{C_v} of X_v over the ring of integers of p-adic field plays now (in our archimedean situation) the Kähler metrics ω on X_v (see [15]). Let us define the $\mathbb{R}$-vector space $\mathcal{Z}^i(X_v, \omega)$ - an analog of the group $\mathcal{Z}^i(X_{C_v})$. In what follows for $c \in H^{\bullet}(X_v, \mathbb{C})$ we denote by $\tilde{c}$ its ω-harmonic representative, and for a cycle $z \in \mathcal{Z}^{\bullet}(X_v) \otimes \mathbb{R}$ let δ_z be its δ-current. Say that a current α is i-Green current if it is $\mathbb{R}(i-1)$-valued of type (i-1, i-1) and for certain (unique) $\alpha_f \in \mathcal{Z}^i(X_v) \otimes \mathbb{R}$ one has $\partial\bar\partial\alpha = \delta_{\alpha_f} - \widetilde{cl\,\alpha_f}$; say that α is regular if it is of C^∞-class off the support of α_f. A (regular) Green current α is trivial if $\alpha = \partial\nu \pm \bar\partial\bar\nu$ for certain (C^∞-class) current ν. Put $\mathcal{Z}^i(X_v, \omega)$ to be the factorspace of i-Green currents modulo trivial ones (you may take all Green currents or regular ones only - this doesn't changes $\mathcal{Z}^i(X_v, \omega)$). For $\alpha \in \mathcal{Z}^i(X_v, \omega)$ let $\alpha_\infty \in H^{i-1,i-1}(X_v, \mathbb{R}(i-1)) (:= H^{i-1,i-1}(X_v) \cap H^{2i-2}_{\mathcal{B}}(X_v, \mathbb{R}(i-1))$ be the (only) class such that for any ω-harmonic 2(N-i+1)-form ν one has $\int_{X_v} \alpha \wedge \nu = \int_{X_v} \alpha_\infty \cdot \nu$. One may see that the arrow $\mathcal{Z}^i(X_v, \omega) \longrightarrow H^{i-1,i-1}(X_v, \mathbb{R}(i-1)) \oplus \mathcal{Z}^i(X_v) \otimes \mathbb{R}$, $\alpha \longmapsto (\alpha_\infty, \alpha_f)$, is isomorphism. This direct sum decomposition of $\mathcal{Z}^i(X_v, \omega)$ is analogous to the decomposition of $\mathcal{Z}^i(X_C)$ into the sum of the group of cycles supported in the special fiber and the group of cycles that intersect the special fiber properly.

Here is an important example of Green current; we'll need it in n° 4.1. For a scheme S put $C^i(S) = \bigoplus_{\substack{\eta \in S \\ \mathrm{Codim}\,\eta = i}} \mathcal{O}^*(\eta)$: an element $\varphi \in C^i(S)$

is a finite number of invertible meromorphic functions $\varphi_{\mathfrak{p}}$ defined at the generic points of codimension i subschemes $\mathfrak{p}$ (this is $E^{-i-1,i}$-term of Quillen's spectral sequence for $K'_{\bullet}(S)$); we have the exact sequence $C^{i-1}(S) \xrightarrow{\operatorname{div}} \mathcal{Z}^{i}(S) \longrightarrow CH^{i}(S) \longrightarrow 0$. Now in our situation for $\varphi \in C^{i-1}(X_v)$ we have Green current $\overline{\operatorname{div}}(\varphi)$ defined by $\int \overline{\operatorname{div}}(\varphi) \wedge \nu = \sum_{\mathfrak{p}} \int_{\mathfrak{p}} \log|\varphi_{\mathfrak{p}}| \cdot \nu|_{\mathfrak{p}}$; clearly $(\overline{\operatorname{div}}\,\varphi)_f = \operatorname{div}\varphi$. This way we get an arrow $\overline{\operatorname{div}} : C^{i-1}(X_v) \longrightarrow \mathcal{Z}^{i}(X_v, \omega)$.

Now let me define the intersection index. Let $a_1 \in \mathcal{Z}^{d_1}(X_v, \omega)$, $a_2 \in \mathcal{Z}^{d_2}(X_v, \omega)$ be elements such that $d_1 + d_2 = N+1$ and the supports of a_{if} are disjoint. Put $(a_1, a_2)_v = \int_{X_v} (\alpha_1 \wedge \delta_{a_{2f}} + \delta_{a_{1f}} \wedge \overline{a_{2\infty}}) \in \mathbb{R}$ (here α_1 is a regular Green current that represents a_1). One may see that $(a_1, a_2)_v$ so defined is independent of the choice of α_1, that $(,)_v$ is symmetric, and in case when both a_{if} are homologous to zero one has $(a_1, a_2)_v = \langle a_{1f}, a_{2f}\rangle_v$. In $N = 1$ case this construction coincides with the original Arakelov one [1] (see also [13]).

§4. Height pairing over number fields

Let us return to the global situation. In this § our base field K will be a finite extension of $\mathbb{Q}$; let $\bar{K}$ be an algebraic closure of K, $\eta = \operatorname{Spec} K$, $\bar{\eta} = \operatorname{Spec}\bar{K}$. For any place v of K denote by K_v the corresponding local field, and by η_v the spectrum of K_v^{nr}: maximal non-ramified extension of K_v. Define the real number $r(v)$ to be log (number of elements in the residese field of K_v) for non-archimedean v, and $r(v) = 1$ for $K_v = \mathbb{R}$, $r(v) = 2$ for $K_v = \mathbb{C}$.

4.0. Let $X = X_K$ be a N-dimensional smooth projective variety over K; put $CH^i(X)^{\circ} := \operatorname{Ker}(CH^i(X) \longrightarrow H^{2i}(X_{\bar{\eta}}, \mathbb{Q}_{\ell}(i)))$, $\overline{CH}^i(X) := CH^i(X)/CH^i(X)^{\circ} \subset H^{2i}(X_{\bar{\eta}}, \mathbb{Q}_{\ell}(i))$. We'll assume that for any non-archimedean v the η_v-scheme $X_v = X \times \eta_v$ satisfies the conjectures 2.2.1 and 2.2.3.

<u>Remark 4.0.1</u>. a. If you do not wish to assume this, then change the notations and put $CH^i(X)^{\circ} := \operatorname{Im}(CH^i(X_{\mathbb{Z}})^{\circ} \longrightarrow CH^i(X))$; here $X_{\mathbb{Z}}$ is certain regular model of X over the ring of integers of K (we

suppose that it exists) and $CH^i(X_{\mathbb{Z}})^o \subset CH^i(X_{\mathbb{Z}})$ is a subgroup of cycles whose intersection with the fiber of $X_{\mathbb{Z}}$ over any prime is homologous to zero. It is easy to see that $CH^i(X)^o$ so defined is independent of choice of particular model $X_{\mathbb{Z}}$. According to conj. 2.2.5. this $CH^i(X)^o$ should coincide with the above one.

b. The same thing happens with the definition of groups $H^i_M(X, \mathbb{Q}(j))_{\mathbb{Z}}$, $i < 2j$ (see [2]); namely we may define them in two ways. The first one: assume the existence of $X_{\mathbb{Z}}$ and take the definition [2] (8.3). The second one: put $H^i_M(X, \mathbb{Q}(j))_{\mathbb{Z}} := \bigcap_{p\ \text{prime}} \mathrm{Ker}(H^i_M(X, \mathbb{Q}(j)) \longrightarrow H^1(\mathrm{Gal}\ \bar{\mathbb{Q}}_p/\mathbb{Q}_p^{nr}, H^{i-1}(X_{\bar{\mathbb{Q}}_p}, \mathbb{Q}_\ell(j)))$; here $\ell \neq p$ and the arrow under the bracket is canonical map $H^i_M(X, \mathbb{Q}(j)) \longrightarrow \mathrm{Ker}(H^i(X_{\mathbb{Q}_p^{nr}}, \mathbb{Q}_\ell(j)) \longrightarrow H^i(X_{\bar{\mathbb{Q}}_p}, \mathbb{Q}_\ell(j))) = H^1(\mathrm{Gal}\ \bar{\mathbb{Q}}_p/\mathbb{Q}_p^{nr}, H^{i-1}(X_{\bar{\mathbb{Q}}_p}, \mathbb{Q}_\ell(j)))$. According to [2] conj. 8.3.3 these two definitions should give the same $H_M(X, \mathbb{Q}(j))_{\mathbb{Z}}$. ■

Let a_1, a_2 be a cycles on X of codimensions d_1, d_2 s.t. $d_1 + d_2 = N+1$, both $cl(a_i) \in CH^{d_i}(X)^o$ and supports of a_i doesn't intersect. For any place v we get the corresponding objects on X_v. This way we get the rational numbers $\langle a_1, a_2\rangle_v$ for non-archimedean v (according to 2.2.1. and 2.2.3, or, if you prefer 4.0.1, according to 2.1.6) and real numbers $\langle a_1, a_2\rangle_v$ for archimedean v. Note, that all but finitely many $\langle a_1, a_2\rangle_v$ are zero. Put $\langle a_1, a_2\rangle := \sum_v r(v)\langle a_1, a_2\rangle_v \in \mathbb{R}$.

Lemma-definition 4.0.2. There exists unique bilinear pairing $\langle,\rangle : CH^{\cdot}(X)^o \times CH^{N+1-\cdot}(X)^o \longrightarrow \mathbb{R}$ s.t. $\langle cl\ a_1, cl\ a_2\rangle = \langle a_1, a_2\rangle$ if supports of a_i doesn't intersect.

Proof. By moving lemma any pair of classes in $CH^{\cdot}$, $CH^{N+1-\cdot}$ may be represented by non-intersecting cycles, so it remains to prove that $\langle a_1, a_2\rangle$ depends only on rational-equivalence class of a_i (the bilinearity is obvious). This reduces to the proof that $\langle a_1, a_2\rangle = 0$ whenever a_1 is rationally equivalent to zero. To see this choose a correspondence b between P^1 and X s.t. $b(0) = a_1$, $b(\infty) = 0$. If the supports of cycles $p_1^*((0) - (\infty))$, b, $p_2^*(a_2)$ intersect by an empty set (here $p_i : P^1 \times X \longrightarrow P^1, X$ are projections), we may apply 2.1.3. to see that $\langle a_1, a_2\rangle = \langle (0) - (\infty), {}^t b(a_2)\rangle$ and we are done by standard arguments with product formula. If not (this means that $\mathrm{supp} a_2 \cap Z \neq \emptyset$, where $Z := (\mathrm{supp}\ b \cap p_1^{-1}(0)) \cup (\mathrm{supp}\ b \cap p_1^{-1}(\infty)) \supset \mathrm{supp}\ a_1$ is a d_1-codimensional subset in X), then you can move a_2 a little (in the same rational equivalence class) to get a_2' s.t. $\mathrm{supp}\ a_2' \cap Z = \emptyset$. Then, by above, $\langle a_1, a_2'\rangle = 0$ and by the same arguments $\langle a_1, a_2\rangle = \langle a_1, a_2'\rangle$, q.e.d. ∎

This is the desired height-pairing. Here are its first properties, that follow directly from the corresponding local properties plus moving lemma.

4.0.3. Let Y be another variety, and $b \in CH^d(X \times Y)$ be a correspondence. Then for $a_1 \in CH^{i_1}(X)^o$, $a_2 \in CH^{i_2}(Y)^o$, $i_1 + i_2 + d = \dim X + \dim Y + 1$, one has $\langle b(a_1), a_2\rangle = \langle a_1, {}^t b(a_2)\rangle$.

4.0.4. The height pairing between divisors and zero cycles coincides with the one of Néron [17], [13].

4.0.5. The pairing $\langle , \rangle$ doesn't depend on the choice of the base field: if you'll consider X as a scheme over $\mathbb{Q}$ this will give the same pairing between cycles.

4.0.6. Let K'/K be an extension of degree n. Then $CH(X_K)^o \otimes \mathbb{Q} \subset CH(X_{K'})^o \otimes \mathbb{Q}$ and for $a_i \in CH(X_K)^o$ one has $\langle a_1, a_2\rangle_K = 1/n\langle a_1, a_2\rangle_{K'}$ By means of this formula you may define the height pairing on $CH(X_{\bar{K}})^o$; this pairing is Galois-invariant.

The following lemma is a particular case of the conjecture 5.6.

Lemma 4.0.7. Let $a \in CH^i(X)^o$ be a cycle such that a is algebraically equivalent to zero and, under some point $K \hookrightarrow \mathbb{C}$, the image of a under the Abel-Jacobi map $CH^i(X_{\mathbb{C}}) \to \mathcal{J}^i(X_{\mathbb{C}})$ is zero (here $\mathcal{J}^i$ is i-th Griffiths Jacobian). Then a lies in the kernel of height pairing.

Proof. Consider a correspondence b between a curve C and X such that $a = b(a_c)$ for certain $a_c \in CH^1(C)^o = J(C)(K)$. Let $A \subset J(C)$ be the abelian subvariety generated by ${}^t b(CH^{N+1-i}(X)^o)$. According to 4.0.3 to prove our lemma it suffice to show that a_c lies in an orthogonal complement to A (under the usual self-duality on $J(C)$); clearly we may consider the situation over $\mathbb{C}$. But there we have the arrow ${}^t b : J^{N+1-i}(X_{\mathbb{C}}) \to J(C_{\mathbb{C}})$ between intermediate Jacobians that commutes with Abel-Jacobi map, and so ${}^t bJ^{N+1-i}(X_{\mathbb{C}}) \supset A_{\mathbb{C}}$. The condition of lemma claims that a_c lies in the orthogonal complement to ${}^t bJ^{N+1-i}(X_{\mathbb{C}})$ and so to A. We are done. ∎

Remark 4.0.8. In fact the height pairing between cycles algebraically equivalent to zero may be canonically reduced to the Neron-Tate pairing. Namely for fixed $K \hookrightarrow \mathbb{C}$ consider the Abel-Jacobi images of cycles on $X_{\mathbb{C}}$ algebraically equivalent to zero. They form an abelian subvariety $A^i_{\mathbb{C}}$ in $J^i(X_{\mathbb{C}})$; it is easy to see that A^i is defined over K. The duality between $J^i(X_{\mathbb{C}})$ and $J^{N+1-i}(X_{\mathbb{C}})$ defines the pairing between A^i and A^{N+1-i}. This pairing gives rise to the Neron-Tate pairing between $A^i(K)$ and $A^{N+1-i}(K)$ that may be identified with the height pairing by means of Abel-Jacobi map.

4.1. In the rest of the § I'll sketch a construction of the Arakelov-type global intersection pairing on A-varieties. The similar construction was found by H. Gillet and Ch. Soulé; I refer to their paper [10] for details. Let (X, ω) be an A-variety [15], i.e. a regular scheme X projective over Spec $\mathbb{Z}$ together with Kähler metrics ω on $X_{\mathbb{R}}$. Let $\mathcal{Z}^i(X_{\mathbb{Z}}, \omega) \subset \mathcal{Z}^i(X_{\mathbb{R}}, \omega) \oplus \mathcal{Z}^i(X_{\mathbb{Z}})$ be the subgroup of elements (a, b) such that $a_f = b_{\mathbb{R}}$. We have the natural map $\overline{\mathrm{div}} : C^{i-1}(X_{\mathbb{Z}}) \longrightarrow \mathcal{Z}^i(X_{\mathbb{Z}}, \omega)$; denote its cokernel by $CH^i(X_{\mathbb{Z}}, \omega)$. There is an obvious exact sequence $C^{i-1}(X_{\mathbb{Z}})^{\circ} \xrightarrow{\overline{\mathrm{div}}^{\circ}} H^{i-1,i-1}(X_{\mathbb{R}}, \mathbb{R}(i-1)) \longrightarrow CH^i(X_{\mathbb{Z}}, \omega) \xrightarrow{\pi} CH^i(X_{\mathbb{Z}}) \longrightarrow 0$; here $C^{i-1}(X_{\mathbb{Z}})^{\circ} := \mathrm{Ker}(\mathrm{div}: C^{i-1}(X_{\mathbb{Z}}) \longrightarrow \mathcal{Z}^i(X_{\mathbb{Z}}))$. Put $CH^i(X_{\mathbb{Z}}, \omega)_{\infty} := \mathrm{Ker}\,\pi = \mathrm{Coker}\,\overline{\mathrm{div}}^{\circ}$. Note that the map $\overline{\mathrm{div}}^{\circ}$ coincides with the composition $C^{i-1}(X_{\mathbb{Z}})^{\circ} \longrightarrow H^{2i-1}_{M}(X_{\mathbb{Z}}, \mathbb{Q}(i)) \xrightarrow{r} H^{2i-1}_{\mathcal{H}}(X_{\mathbb{R}}, \mathbb{R}(i)) = H^{i-1,i-1}(X_{\mathbb{R}}, \mathbb{R}(i-1))$, where the first arrow comes from Quillen's spectral sequence and r is regulator map (see [2] n 6). According to [2] conj. 8.4.1.b $\overline{\mathrm{div}}^{\circ}(C^{i-1}(X_{\mathbb{Z}})^{\circ}) + cl_B(CH^{i-1}(X_{\mathbb{Z}}))$ is a $\mathbb{Z}$-lattice in $H^{i-1,i-1}(X_{\mathbb{R}}, \mathbb{R}(i-1))$, so $CH^i(X_{\mathbb{Z}}, \omega)_{\infty}$ should be equal to the product of $(\mathcal{I}m\,\overline{\mathrm{div}}^{\circ}) \otimes \mathbb{R}/\mathcal{I}m\,\overline{\mathrm{div}}^{\circ}$ (= maximal compact subgroup) and $\mathbb{R}$-vector space $\overline{CH}^{i-1}(X_{\mathbb{Q}}) \otimes \mathbb{R}$.

Now let $a_1 \in Z^{d_1}(X_{\mathbb{Z}}, \omega)$; $a_2 \in Z^{d_2}(X_{\mathbb{Z}}, \omega)$ be two cycles such that $d_1 + d_2 = N+1$ and supports of a_{if} doesn't intersect on $X_{\mathbb{Q}}$ (or $X_{\mathbb{R}}$). Put $(a_1, a_2) := (a_1, a_2)_{\infty} + \sum_p (\log p)(a_1, a_2)_p$ (see 2.0, 3.1.). One may see that this formula defines the pairing $(\,,) : CH^{d_1}(X_{\mathbb{Z}}, \;) \otimes CH^{d_2}(X_{\mathbb{Z}}, \;) \longrightarrow \mathbb{R}$. It has the following properties: ① $(\,,)$ is symmetric ② the image of $\overline{\mathrm{div}}^{\circ}(C^{i-1}(X_{\mathbb{Z}})^{\circ})\cdot\mathbb{R}$ in $CH^i(X_{\mathbb{Z}}, \omega)_{\infty}$ lies in the kernel of $(\,,)$ ③ if both $\pi(a_i) \in CH^{d_i}(X_{\mathbb{Z}})^{\circ}$ (see 4.0.1.), then $(a_1, a_2) = \langle \pi(a_1), \pi(a_2) \rangle$ ④ Let $a_1 \in CH^d(X_{\mathbb{Z}}, \omega)_{\infty}$ be the image of $\tilde{a}_1 \in H^{i-1,i-1}(X_{\mathbb{R}}, \mathbb{R}(i-1))$; then $(a_1, a_2) = \int_{a_{2f\mathbb{R}}} \tilde{a}_1$. (Note that 2 and 4 imply that for $i \leqslant \frac{N+1}{2}$ the

$\mathbb{R}$-subspaces in $H^{i-1,i-1}$ generated by Im $\overline{div}^{o}$ and $\overline{CH}^{i-1}(X_{\mathbb{Q}})$ doesn't intersect; this fits into above conjecture on $CH^{i}(X_{\mathbb{Z}}, \omega)_{\infty})$.

In the next § the conjectures will be formulated for the pairing <,>; but some of them have sense for (,) also. To find such a variants of them is an exercise for the reader.

§5. Some conjectures and problems

Let X be a smooth projective N-dimensional scheme over $\mathbb{Q}$. Assume for X the local conjectures 2.2.1, 2.2.3, and also assume the standard conjectures about analytic continuation and functional equation for $L(H^{\bullet}(X), s)$

Conjecture 5.0 (Swinnerton-Dyer). The groups $CH^{\bullet}(X)^{o}$ are finitely generated and rk $CH^{i}(X)^{o}$ = order of zero of $L(H^{2i-1}(X), s)$ at $s = i$ ∎

The only fact I know supporting this conjecture off the B-Sw-D case (i.e. $i = 1$ case) is Bloch's calculation for the Jacobian of a quartic curve, see [4].

Lemma 5.1. (modulo the conjecture 5.0) The canonical arrow $CH^{N}(X)^{\circ} \twoheadrightarrow \mathrm{Alb}(X)(\mathbb{Q})$ is isomorphism up to torsion.

Proof. It is clear (e.g. by norm map considerations) that the arrow is epimorphic up to torsion so it suffice to see that the ranks are the same. According to 5.0. we have rk $CH^{N}(X)^{\circ} = \mathrm{ord}_{s=N} L(H^{2N-1}(X), s) = \mathrm{ord}_{s=1} L(H^{1}(X), s)$ = rk $CH^{1}(X)^{\circ}$ = rk $\mathrm{Pic}^{\circ}(X)(\mathbb{Q})$ = rk $\mathrm{Alb}(X)(\mathbb{Q})$, since Pic° is isogeneous to Alb and $H^{2N-1}(X)(N-1) \simeq H^{1}(X)$ ∎

In view of Roitman's theorem on torsion zero cycles (see e.g. [7] lecture 5), the statement of 5.1 is equivalent to the following.

Conjecture 5.2. For any smooth projective $\tilde{X}$ over $\bar{\mathbb{Q}}$ the Chow group of zero cycles of degree zero on $\tilde{X}$ coincides with $\mathrm{Alb}(\tilde{X})(\bar{\mathbb{Q}})$ ∎

Note that according to Mumford's theorem ([16], [7] lecture 1) the situation radically changes if there are parameters in the base field (see the discussion below), so it would be very interesting to try to handle 5.2. in any non-trivial particular case. For example, whether every zero-cycle of degree zero on K3 over $\bar{\mathbb{Q}}$ is rationally equivalent to zero?

Conjecture 5.3. a. (hard Lefschetz). Let $\ell \in CH^{1}(X)$ be the class of a hyperplane section. Then for $i \leqslant (N+1)/2$ the arrow $\ell^{N+1-2i} : CH^{i}(X)^{\circ} \otimes \mathbb{Q} \longrightarrow CH^{N+1-i}(X)^{\circ} \otimes \mathbb{Q}$ is isomorphism

b. Let y be a smooth N-dimensional affine scheme over $\mathbb{Q}$, and, as in the proper case, put $CH^{i}(y)^{o} := \mathrm{Ker}(CH^{i}(y) \to H_{B}^{2i}(y \otimes \mathbb{C}, \mathbb{Z}(i)))$. Then one should have $CH^{i}(y)^{o} = 0$ for $i > \frac{\dim y + 1}{2}$. ∎

For $a_1, a_2 \in \mathbb{R}$ say that $a_1 \sim a_2$ if they differ by non-zero rational multiple. Consider the height pairing $\langle,\rangle$ between $\mathbb{R}$-vector spaces $CH^i(X)^\circ \otimes \mathbb{R}$ and $CH^{N+1-i}(X)^\circ \otimes \mathbb{R}$. Denote by $\det\langle,\rangle_i$ its determinant in some rational baseses of $CH^i(X)^\circ \otimes \mathbb{Q}$, $CH^{N+1-i}(X)^\circ \otimes \mathbb{Q}$; clearly $\det\langle,\rangle_i$ is correctly defined equivalence class of real numbers (we assume rk CH (X) is finite, see 5.0) and $\det\langle,\rangle_i \sim \det\langle,\rangle_{N+1-i}$.

Conjecture 5.4.*) a) The height pairing is non-degenerate.

b) $\det\langle,\rangle_i$, multiplied by the determinant of the period matrix for $H^{2i-1}(X)$ (see [21]), is equivalent to the leading coefficient in Taylor serie expansion of $L(H^{2i-1}(X), s)$ at $s = i$. ∎

Conjecture 5.5. (Hodge-index). Assume 5.3 and consider the primitive-cycle decomposition. Then the form $\langle\cdot, \ell^{N+1-2i}\cdot\rangle$ is definite of sign $(-1)^i$ on the primitive i-cycles $(i \leqslant \frac{N+1}{2})$. ∎

For $i = 1$ 5.4a and 5.5 is the well-known positively property of Neron-Tate height, and 5.4b is the weak form of Birch-Swinnerton-Dyer conjecture.

Note that 5.4a together with 4.0.7 imply the following generalisation of 5.1.

Lemma 5.6.**) (modulo 5.4). The Abel-Jacobi map $CH^i(X)^\circ \longrightarrow \mathcal{J}^i(X_{\mathbb{C}})$ is injective up to torsion on the subgroup of cycles algebraically equivalent to zero. ∎

It would be natural to suppose that Abel-Jacobi is injective on the whole $CH^i(X)^\circ \otimes \mathbb{Q}$ but I have no definite reason for this.

Now I am going to formulate the motivic versions of 5.0 and 5.4. To do this I need to assume.

Conjecture 5.7. The usual intersection pairing $CH^i(X)^\circ \times CH^j(X)^\circ \longrightarrow CH^{i+j}(X)^\circ$ is zero up to torsion. ∎

This is a particular case of the [2], conj. 8.5.1. Note that its geometric analog follows directly from Leray spectral sequence considerations (since H^j of the affine curve are zero for $j > 2$). Modulo 5.4 the conjecture 5.7 is equivalent to

Conjecture 5.8. For any three cycles $a_i \in CH^{d_i}(X)^\circ$, $\sum d_i = N + 1$, one has $\langle a_1, a_2, a_3\rangle = 0$. ∎

Clearly 5.7. implies that a correspondence homologous to zero acts in a trivial way on the CH ()°-groups. This shows that we may

*) This conjecture has been formulated independently by S.Bloch [6].
**) This conjecture has been formulated independently by S.Bloch[5](1.4).

define $CH^{\cdot}()^{\circ}$ groups for Grothendieck's motives. More precisely, consider the category M of motives over $\mathbb{Q}$ with coefficients in $\bar{\mathbb{Q}}$ for the theory of algebraic correspondences modulo homological equivalence. For any i we have $\bar{\mathbb{Q}}$-linear functor $CH^{i}()^{\circ}$ on M with values in finite dimensional $\bar{\mathbb{Q}}$-vector spaces, such that $CH^{i}(\mathcal{M}(X))^{\circ} = CH^{i}(X)^{\circ} \otimes \bar{\mathbb{Q}}$ where $\mathcal{M}(X)$ is motive of a smooth projective variety X (see [2] for details). Note that now (due to 5.7) the conjecture 5.3. is implied by the standard conjectures on algebraic cycles. We get also the $\bar{\mathbb{Q}} \otimes \mathbb{R}$-valued height pairing between $CH^{\cdot}()^{\circ}$-groups of a motive M and its dual; according to 5.4. this pairing is non-degenerate. Denote by $\det_i\langle,\rangle(M)$ its determinant in the $\bar{\mathbb{Q}}$-basises of $CH^{\cdot}()^{\circ}$; this is an element of $(\bar{\mathbb{Q}} \otimes \mathbb{R})^*$ correctly defined up to $\bar{\mathbb{Q}}$-multiple. Also for any motive M we have $\bar{\mathbb{Q}} \otimes \mathbb{Q}_{\ell}[\mathrm{Gal}]$-modules $H^{\cdot}(M, \mathbb{Q}_{\ell})$ and $\bar{\mathbb{Q}} \otimes \mathbb{C}$-valued L-functions $L(H^{\cdot}(M), s)$. According to Gross's conjecture , for $i \in \mathbb{Z}$ all the $\mathbb{C}$-valued components of $L(H^{\cdot}(M), s)$ (these components are numbered by embeddings $\bar{\mathbb{Q}} \hookrightarrow \mathbb{C}$) should have the same order of zero at i. Denote it by $\mathrm{ord}_i L(H\ (M), s)$; so we have $L(H^{\cdot}(M), s-i) = \ell(H^{\cdot}(M), i)\cdot(s-i)^{\mathrm{ord}_i L(H\ (M), s)} + O(s-i)$ for some $\ell(H^{\cdot}(M), i) \in (\bar{\mathbb{Q}} \otimes \mathbb{R})^*$. The following conjecture generalises 5.0 and 5.4:

<u>Conjecture 5.9</u>. We have $\mathrm{ord}_i L(H^{2i-1}(M), s) = \dim_{\bar{\mathbb{Q}}} CH^{i}(M)^{\circ}$ and $\ell(H^{2i-1}(M), i) = \det_i\langle,\rangle(M)$, multiplied by the determinant of the period matrix for $H^{2i-1}(M)$ (see[21]) (up to a multiple from $\mathbb{Q}^*$). ■

See the beautiful computations of Gross-Zagier [11] in favor of this.

Recall that according to standard conjectures on algebraic cycles the category M is abelian semisimple. This semisimplicity implies that any additive functor on M, e.g. $CH^{\cdot}()^{\circ}$, $H^{\cdot}(\ , \mathbb{Q}_{\ell})$, is exact. On any motive M we have the colevel filtration $N.$ defined by $N_j(M)$ = the sum of components of M that occure in some $\mathcal{M}(Y)(*)$, where Y is projective smooth scheme of dimension $\leq j$. It is easy to see that in case $M = \mathcal{M}(X)$ the corresponding filtrations on $CH^{\cdot}(\mathcal{M}(X))^{\circ} = CH^{\cdot}(X)^{\circ} \otimes \bar{\mathbb{Q}}$, $H^{\cdot}(X, \mathbb{Q}_{\ell})$ are $N_j CH^{\cdot}(X)^{\circ} = \{cl\ a$: there exists Y, $\mathrm{supp}\, a \subset Y \subset X$ s.t. $\mathrm{codim}\, Y \geq \cdot - j$ and a is homologous to zero in $Y\}$ (in particular N_1 = cycles algebraically equivalent to zero), $N_j H^{\cdot}(X, \mathbb{Q}_{\ell}) = \{\alpha$: there exists $Y \subset X$ s.t. $\mathrm{codim}\, Y \geq \frac{\cdot - j}{2}$ and $\alpha|_{X-Y} = 0\}$. Now 5.9. claims that the rank of $Gr^N_j(CH^{i}(X)^{\circ})$ equals to the order of zero of $L(Gr^N_j H^{2i-1}(X), s)$ at $s = i$. This sharpened form of Swinnerton-Dyer conjecture was first found by Bloch [5] (1.3). Note that this conjecture for $j = 1$ also implies 5.6. (see [5] (1.4)).

5.10. In the rest of the paper I will try to discuss some general conjectural framework for motives, regulators and the like.

A. Motivic sheaves. One hopes that for any scheme S and an appropriate commutative ring A (say $A = \mathbb{Z}, \mathbb{Z}/n, \mathbb{Q}, \bar{\mathbb{Q}}$) there exists certain abelian A-category $M(S, A)$ of (mixed) motivic (perverse) A-sheaves over S together with corresponding derived category $DM(S,A)$. This categories should resemble very much the categories of mixed ℓ-adic sheaves;

- there should be inner $\otimes$ and Hom
- there should be Tate sheaves $A_{\mathcal{M}}(i)$; the Tate twist $M \mapsto M(i) = M \otimes A_{\mathcal{M}}(i)$ is automorphism of $M(S,A)$
- for any morphism $S_1 \to S_2$ of finite type there should be the corresponding functors $f_*, f^*, f_!, f^!$ between $DM(S_i)$
- for $A \supset \mathbb{Q}$ there should be canonical weight filtration $W_\cdot$ on the objects of $M(S,A)$ such that any morphism is strictly compatible with $W_\cdot$ and any $Gr^W_\cdot(M)$ is semisimple.

All this things should behave the same way as in mixed ℓ-adic situation. One should also have realisation functors r on $M(S)$ and $DM(S)$ with values in mixed ℓ-adic sheaves, or Hodge sheaves (if S is an $\mathbb{R}$-scheme, see n° E below)... If $A \supset \mathbb{Q}$ this realisation functors should be exact, faithful on $M(S)$, and should commute with all the above structures. Note that such r induces the morphism between corresponding (absolute) cohomology groups: say $r : H^\cdot(S, M(*)) :=$
$\mathrm{Ext}^i(\mathbb{Q}_{\mathcal{M}}(0), M(*)) \to \mathrm{Ext}^i(\mathbb{Q}_\ell, r_{\mathbb{Q}_\ell}(M)(*)) = H^i(S, r_{\mathbb{Q}_\ell}(M)(*))$.

If $S = \mathrm{Spec}\, k$, k is a field, and $A \supset \mathbb{Q}$, then the category of semisimple objects in $M(S,A)$ should be equivalent to the category of Grothendieck's A-motives over k constructed by means of algebraic correspondences modulo homological equivalence.

B. Relations with K-theory and relative cycles. The cohomology groups $H^\cdot(S, A_{\mathcal{M}}(*))$ form "Poincaré" duality theory with supports" in the sense of Bloch-Ogus. The corresponding canonical map $H^\cdot_{\mathcal{M}}(S,\mathbb{Q}(*)) \to H^\cdot(S,\mathbb{Q}_{\mathcal{M}}(*))$ should be isomorphism if S is regular (recall that $H^\cdot_{\mathcal{M}}(S,\mathbb{Q}(*))$ is a part of $K_\cdot(S) \otimes \mathbb{Q}$ and the canonical map comes from the Chern character). All the canonical maps $H^\cdot_{\mathcal{M}} \to H^\cdot_?$ where $H^\cdot_?$ is ℓ-adic, Hodge-Deligne or some other cohomology, should coincide with $H^\cdot_{\mathcal{M}}(S,\mathbb{Q}(*)) \xrightarrow{\sim} H^\cdot(S,\mathbb{Q}_{\mathcal{M}}(*)) \xrightarrow{r_?} H^\cdot(S,\mathbb{Q}_?(*)) = H^\cdot_?(S,\mathbb{Q}(*))$.

More precisely, for S regular there should exist the Atiyah-Hirzebruch type spectral sequence converging to $K_\cdot(S)$ with second term

$H^{\cdot}(S, \mathbb{Z}_{\mathcal{M}}(*))$; it should degenerate up to factorials at E_2 (one likes to have the same picture for arbitrary singular S, but this possibly forces to replace M(S) by more clever category of properties I can't imagine). Note that for a smooth varieties over a field S. Landsburg [12] has found a spectral sequence of such a type using relative cycles machine; hopefully it should coincide with the conjectural spectral sequence above. V. Schectman (report on Landsburg's work, February 1983) and S. Bloch [23] independently have shown that Landsburg's E_2 form Poincaré duality theory with supports and Bloch [23] has shown that Landsburg's spectral sequence degenerates at E_2 up to torsion. Note that for $i<0$ one should have $H^i(S, \mathbb{Z}_{\mathcal{M}}(*)) = 0$ (as Ext^i in a t-category); the corresponding property for Landsburg's E_2 remains unproven (it was conjectured independently by Ch. Soulé and the author [3], [18]).

C. <u>Application to algebraic cycles</u>. Let us see what B gives being applied to cycles (compare with [7], lecture 1). For regular X one should have $H^{2i}(X, \mathbb{Z}_{\mathcal{M}}(i)) = CH^i(X)$. Let S be regular and $\pi: X_S \longrightarrow S$ be a smooth projective map. We should have the Leray spectral sequence $E_2^{p,q} = H^p(S, R^q\pi_* \ (i)) \Rightarrow H^{p+q}(X_S, \mathbb{Z}_{\mathcal{M}}(i))$ that degenerates up to torsion. So on any $H^{\cdot}(X_S, \mathbb{Z}_{\mathcal{M}}(i))$ we have canonical filtration $\mathcal{F}^{\cdot}$ s.t. $Gr^p_{\mathcal{F}^{\cdot}}H^n(X_S, \mathbb{Q}_{\mathcal{M}}(i)) = H^p(S, R^{n-p}\pi_*\mathbb{Q}_{\mathcal{M}}(i))$. (Note that $CH^{\cdot}()^{\circ}$ in the main body of the paper is $\mathcal{F}^{\circ}CH^{\cdot}()$). In particular $CH^i(X_S)$ is controlled by relative motivic cohomology of X with numbers $\leqslant 2i$.

The group $\mathcal{F}^2CH^i(X_S)$ is something like "nonrepresentable" part of $CH^i(X_S)$: if S is of finite type over $\mathbb{Z}$ or $\mathbb{Q}$, then any element of $\mathcal{F}^2$ is zero at any $\bar{\mathbb{Q}}$-point of S. More precisely, assuming S affine of finite type over $\mathbb{Z}$ we may describe the filtration $\mathcal{F}^{\cdot}$ as follows. Just as in ℓ-adic situation for $M \in M(S)$ one should have $H^j(S,M) = 0$ for $j > \dim S$. Moreover, for any $j \leqslant \dim S$ there exists certain $T \subset S$, $\dim T = j$, such that $H^j(S,M) \hookrightarrow H^j(T, M|_T)$. This shows that

$$\mathcal{F}^jCH^{\cdot}(X_S) = \bigcap_{\substack{T\subset S\\ \dim T = j-1}} \mathrm{Ker}(CH^{\cdot}(X_S) \longrightarrow CH^{\cdot}(X_T)).$$

Now assume that S is a spectrum of a field. For any smooth projective Y_S of dimension $i-1$ one has $CH^i(Y_S) = 0$, so in particular any $H^{2i-\ell}(S, R^{\ell}\pi_{Y*}\mathbb{Q}_{\mathcal{M}}(*))$ is zero. For a motive M over S define the level filtration $N^{\cdot}$ on M by formula $N^jM := \bigcap_{\dim Y=j} \mathrm{Ker}(M \to R^{\cdot}\pi_{Y_S*}\mathbb{Q})$. The above shows that $H^{2i-\ell}(S, R^{\ell}\pi_{X*}\mathbb{Q}_{\mathcal{M}}(i)) = H^{2i-\ell}(S, (N^{i-1}R^{\ell}\pi_{X*}\mathbb{Q}_{\mathcal{M}})(i))$.

In particular this groups are non-zero for $\ell = 2i,\dots,i$ only.

D. <u>The structure of $H^{\cdot}(\ ,A_{\mathcal{M}}(*))$</u>. Consider the complexes $R\Gamma(\ ,A_{\mathcal{M}}(i)) = R\,\mathrm{Hom}(A_{\mathcal{M}}(0),\ A_{\mathcal{M}}(i))$ Zariski locally. These are the complexes $A_{\mathcal{M}}(i)_{Zar}$ in the derived category of sheaves on the big Zariski topology together with canonical commutative multiplication $A_{\mathcal{M}}(i)_{Zar} \overset{L}{\otimes} A_{\mathcal{M}}(j)_{Zar} \longrightarrow A_{\mathcal{M}}(i+j)_{Zar}$. We have $H^{\cdot}(S,A_{\mathcal{M}}(i)) = H^{\cdot}(S_{Zar},A_{\mathcal{M}}(i)_{Zar})$. The following properties of $A_{\mathcal{M}}(i)$ should hold:

(i) $A_{\mathcal{M}}(i)_{Zar} = A \overset{L}{\otimes} \mathbb{Z}_{\mathcal{M}}(i)_{Zar}$; $A_{\mathcal{M}}(i)_{Zar}$ are zero for $i < 0$

(ii) $\mathbb{Z}_{\mathcal{M}}(1)_{Zar} = \mathcal{O}^{*}[-1]$; for any $i > 0$ the complex $\mathbb{Z}_{\mathcal{M}}(i)_{Zar}$ is acyclic off the degrees $1,\dots,i$

(iii) If k is a field then $H^{i}(\mathrm{Spec}\ k,\ \mathbb{Z}_{\mathcal{M}}(i)_{Zar}) = K_{i}^{Milnor}(k)$; the isomorphism induced by the product $\mathbb{Z}_{\mathcal{M}}(1)_{Zar}^{\otimes i} \longrightarrow \mathbb{Z}_{\mathcal{M}}(i)_{Zar}$. So the sheaf $H^{i}(\mathbb{Z}_{\mathcal{M}}(i)_{Zar})$ coincide on regular S with the sheaf of Milnor's K-functors (and the whole $\mathbb{Z}_{\mathcal{M}}(i)_{Zar}$ is something like non-abelian left-derived functor of K^{Milnor})

(iv) If k is a field, then there exists a canonical differential bigraded algebra $\tilde{\mathbb{Q}}^{\cdot}_{\mathcal{M}}(*)$ that represents $\mathbb{Q}_{\mathcal{M}}(*)_{Zar}$ ($\tilde{\mathbb{Q}}^{\cdot}_{\mathcal{M}}(*)$ is the Sullivan's minimal model of $\mathbb{Q}_{\mathcal{M}}(*)_{Zar}$) such that $\tilde{\mathbb{Q}}^{\cdot}_{\mathcal{M}}(*) = \Lambda^{\cdot}(\bigoplus_{i\geq 1} \tilde{\mathbb{Q}}^{1}(i))$. So $\tilde{\mathbb{Q}}^{\cdot}_{\mathcal{M}}(*)$ is standard chain algebra of the graded Lie coalgebra $\bigoplus_{i\geq 1} \tilde{\mathbb{Q}}^{1}(i)$. This conjecture is due to Schechtman. It is closely related to the conjecture that we may compute $H^{\cdot}(\mathrm{Spec}\ k,\ \mathbb{Q}_{\mathcal{M}}(i))$ as Ext's not in the whole category $M(\mathrm{Spec}\ k)$, but in the far smaller category $M_{Tate}(\mathrm{Spec}\ k)$ of objects with all irreducible constituents isomorphic to some Tate's modules.

(v) (Étale descent). We may consider the analogous complexes $A_{\mathcal{M}}(i)_{\acute{e}t}$ on the big étale topology; if $\pi : Zar \rightarrow \acute{E}t$ is a canonical morphism, then $A_{\mathcal{M}}(i)_{\acute{e}t} = \pi^{*}A_{\mathcal{M}}(i)_{Zar}$. Note, that one should have usual étale descent for $A_{\mathcal{M}}(i)_{Zar}$ only in the trivial case $A \supset \mathbb{Q}$. In general one hopes that $A_{\mathcal{M}}(i)_{Zar} = \tau_{\leq i}R\pi_{*}A_{\mathcal{M}}(i)_{\acute{e}t}$ and also $H^{i+1}R\pi_{*}A_{\mathcal{M}}(i)_{\acute{e}t} = 0$ (this "Hilbert theorem 90" property was found independently by S. Lichtenbaum [14]; his paper also contains some discussion on this subject).

(vi) (Finite coefficients). Let ℓ be prime to characteristics of our schemes. The theorem of Suslin on K-functor of an algebraically closed fields dictates that one should have $(\mathbb{Z}/\ell^n)_{\mathcal{M}}(i)_{ét} = \mathbb{Z}/\ell^n(i)_{ét}$. So, by (v), one should have $(\mathbb{Z}/\ell^n)_{\mathcal{M}}(i)_{Zar} = \tau_{\leq i} R\pi_* \mathbb{Z}/\ell^n(i)$. In particular, if k is a field, then $H^j(\mathrm{Spec}\ k, (\mathbb{Z}/\ell^n)_{\mathcal{M}}(i))$ is zero for $j > i$, and coincides with $H^j(\mathrm{Gal}\ \bar{k}/k, \mathbb{Z}/\ell^n(i))$ for $j \leq i$. The n°B implies that we should have the corresponding spectral sequence that converges to $K_{\cdot}(\mathrm{Spec}\ k, \mathbb{Z}/\ell^n)$. R. Thomason noticed (letter from 10-5-1984) that such a spectral sequence degenerates at E_2 if $\mu_{\ell^n} \subset k$, and so the usual Atiyah-Hirzebruch spectral sequence for $K/\ell^n{}^{top}$ also should degenerate - this thing is unknown yet. Anyway, it is easy show that $H^{\cdot}_{fine}(S, \mathbb{Z}/\ell^n(i)) := H^{\cdot}(S_{Zar}, R\pi_* \mathbb{Z}/\ell^n(i))$ form the Poincaré duality theory with supports, and so we have the characteristic classes with values in this groups; note that $H^{2i}_{fine}(S, \mathbb{Z}/\ell^n(i)) = CH^i(S)/\ell^n$. It would be of interest to construct explicitely the category of "fine étale $\mathbb{Z}/\ell^n$-sheaves" or "fine $\mathbb{Z}/\ell^n$-Galois modules" such that Ext's between $\mathbb{Z}/\ell^n(i)$ give $H^{\cdot}_{fine}$.

E. Hodge sheaves. For a scheme S over $\mathbb{C}$ or $\mathbb{R}$ one should have a category $\mathcal{H}_S$ of Hodge sheaves, analogous to the category of mixed $\mathbb{Q}_\ell$ perverse sheaves. At a moment one definitely knows what lisse Hodge sheaves on smooth varieties are (the pure ones are Griffiths's polarisable variations of Hodge structures, and in the mixed case we have the definition of Steenbrink and Zucker); the attempts to define the whole $\mathcal{H}_S$ as well as to define the functors $f_*, ..$ etc. are quite painful. This contrasts much with the clear étale cohomology situation, where all the constructions are quite simple and natural. Thus the Galois action on étale cohomology groups comes from the Galois symmetries of étale site. It would be marvellous if one may do Hodge theory in a parallel way. Namely, whether there exists the "$\mathbb{R}$-site" $\mathrm{Top}_{\mathbb{R}}(S)$ *) of S together with the action of the group $G_{\mathcal{H}}$ (see [8]) **) such that the $\mathbb{R}$-Hodge sheaves on S would be the $G_{\mathcal{H}}$-sheaves on $\mathrm{Top}_{\mathbb{R}}(S)$? Thus the cohomology of $\mathrm{Top}_{\mathbb{R}}(S)$ with values in a constant sheaf $\mathbb{R}$ should coincide with the usual cohomology of S with real coefficients, and the natural action of $G_{\mathcal{H}}$ on them should give

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*) $\mathrm{Top}_{\mathbb{R}}(S)$ should have somewhat a probabilistic nature.

**) $G_{\mathcal{H}}$ is a real proalgebraic group such that the category of real representations of $G_{\mathcal{H}}$ coincides with the category of $\mathbb{R}$-Hodge structures.

the usual Hodge structure. Passing back from $Top_{\mathbb{R}}$ to the usual topology (on the level of cohomology groups this means restoring of the integral structure) breaks the Hodge symmetry.

F. Absolute cohomology with compact supports. Let now S be a scheme over $\mathbb{Z}$. For any $M \in D\mathcal{M}(S)$ there should exist "absolute cohomology groups with compact supports" $H^{\bullet}_{c}(S, M)$ with the following properties

(i) For any $f : S \to S'$ one should have $H^{\bullet}_{c}(S, M) = H^{\bullet}_{c}(S', f_!(M))$.

(ii) Let $S = \operatorname{Spec} \mathbb{Z}$. For $M \in \mathcal{M}(S, \mathbb{Z})$ one should have an exact sequence $\cdots \to H^1_c(S, M) \to H^1(S, M) \xrightarrow{r} H^1_{\mathcal{H}}(M_{\mathcal{H}} \otimes \mathbb{R}) \to H^2_c(S, M)$. where r is a regulator map. So for M of weight ≥ 0 all $H^{\bullet}_c(S, M)$ are zero; for M of weight -1 the only non-zero H_c may be the discret group $H^1_c(S, M)$; for M of weight ≤ -2 the only non-zero H_c is $H^2_c(S, M)$, this group is compact for $M \neq \mathbb{Z}_{\mathcal{M}}(1)$ and $H^2_c(S, \mathbb{Z}_{\mathcal{M}}(1)) = \mathbb{R}$. The Euler-Poincaré volumes of this H^2_c should be related with the values of L-functions, and the height pairing should coincide with $H^1_c(S, M) \otimes H^1_c(S, M^{\circ}) \longrightarrow H^2_c(S, \mathbb{Z}_{\mathcal{M}}(1))$.

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SURVEY OF DRINFEL'D MODULES

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INTRODUCTION

In Deligne [1971], two-dimensional ℓ-adic representations of $Gal(\bar{\mathbf{Q}}/\mathbf{Q})$ are attached to "new" holomorphic modular forms of weight at least two on the Poincaré upper-half plane, or equivalently, to certain automorphic representations of the adèle group $GL_2(\mathbb{A}_{\mathbf{Q}})$. The correspondence preserves L-functions. This theory depends on the properties of moduli varieties of elliptic curves with given level structure. These varieties have a canonical structure over $\mathbf{Q}$ and the ℓ-adic representations are realized in the ℓ-adic H^1 of certain sheaves.

Drinfel'd [1973] transports the theory to the function field case by introducing the concept of elliptic module, which we call a Drinfel'd module, to replace elliptic curves.

Fixed notations throughout the article

- C - absolutely irreducible projective and smooth curve over $\mathbb{F}_q$.
- ∞ - a closed point of C .
- F - the function field $\mathbb{F}_q(C)$ of C over $\mathbb{F}_q$.
- A - the ring $H^0(C-\infty,\mathcal{O}_C)$ of functions regular on $C-\infty$.
- F_∞ - the completion of F at ∞ with valuation ring $\mathcal{O}_\infty$.
- C_∞ - the completion of the algebraic closure of F_∞ .
- $\mathbb{F}_q$ - the finite field of q elements.

Relative to the function field F over $\mathbb{F}_q$, we will define Drinfel'd modules of rank r in the first chapter. Briefly, these are $A = H^0(C-\infty,\mathcal{O}_C)$ module structures on the additive group in characteristic p given by polynomials in Frobenius whose degree is a certain multiple of the rank r . The term elliptic module, which is Drinfel'd's original term, is used for Drinfel'd modules of rank 2 for these are objects which correspond closely to elliptic curves. In fact, we have the following dictionary:

elliptic curve	elliptic module (rank 2 Drinfel'd module)
$\mathbf{Q}$	$\mathbb{F}_q(C) = F$
infinite place	fixed place
$\mathbf{Z}$	$A = H^0(C - \infty, \mathcal{O}_C)$
scheme	scheme over A
n division point	I division point for I an ideal of A
n level structure	I level structure
moduli space	moduli space
lattice in $\mathbf{C}$	discrete A-modules

Most of the above dictionary is explained in chapter 1. Elliptic curves over the complex numbers $\mathbf{C}$ can be interpreted as classes of certain lattices in $\mathbf{C}$. In chapter 2 we describe Drinfel'd modules over C_∞ in terms of discrete A-modules in C_∞ defined over A. Drinfel'd modules defined over $\bar{F}_\infty$, the algebraic closure of F_∞, can be also described by lattices in $\bar{F}_\infty$, and the lattices with Galois invariance properties correspond to Drinfel'd modules over intermediate fields between F_∞ and $\bar{F}_\infty$.

For elliptic curves, indeed for more generally polarized abelian varieties, and for Drinfel'd modules there are moduli problems for families of these objects and moduli spaces. From the point of view of Shimura varieties, basic information about such moduli problems is collected in triples (F,G,h') consisting of:

- a reductive group G over a global field F,
- a conjugacy class of maps $h'\colon G_m \longrightarrow G$.

The above dictionary extends further with the following examples of triples (F,G,h') for the corresponding moduli problems over F:

polarized abelian varieties (rank g) $\left(\mathbf{Q}, CSp_{2g}, \begin{pmatrix} \lambda I_g & 0 \\ 0 & I_g \end{pmatrix}\right)$

elliptic curves $\left(\mathbf{Q}, GL_2, \begin{pmatrix} \lambda & 0 \\ 0 & 1 \end{pmatrix}\right)$

Drinfel'd modules (rank r) $\left(F, GL_r, \begin{pmatrix} \lambda I_1 & 0 \\ 0 & I_{r-1} \end{pmatrix}\right)$

elliptic module $\left(F, GL_2, \begin{pmatrix} \lambda & 0 \\ 0 & 1 \end{pmatrix}\right)$

A critical step in the theory over $\mathbf{Q}$ was the calculation of the cohomology of the complex analytic variety of $\mathbf{C}$-valued points on the moduli scheme and the comparison with the ℓ-adic cohomology. In the function field case Drinfel'd puts a rigid analytic structure on the C_∞-valued points of the moduli scheme, calculates a rigid analytic H^1 for certain simple sheaves, and compares this H^1 with the ℓ-adic H^1. This rigid analytic space is the quotient of a C_∞-analogue of the Poincaré upper-half plane by a discrete group Γ, and the cohomology is in the middle of a short exact sequence with subgroup $H^1(\Gamma)$ and quotient the space of coclosed 1-cochains on the tree of $PGL(2,C_\infty)$ invariant under Γ. This in turn is interpreted in terms of representations. Finally, the various representations in H^1 are sorted out with a congruence formula which is analogous to the Eichler-Shimura congruence formula. These considerations are carried out in chapters 3 and 4.

In chapter 5 the global results on automorphic forms for $GL(2,\mathbb{A}_F)$ are applied to a local result conjectured by Langlands. For a local field K of equal characteristic p, there is a natural bijection between irreducible admissible representations of $GL(2,K)$ and two-dimensional representations of the Weil group $W(\bar{K}/K)$. The global theory of automorphic forms given by elliptic modules applies only to representations $\pi = \otimes_{v\in C} \pi_v$ where π_∞ is the special representation.

In the form of private letters Drinfel'd has a new theory of "shtuka" which handles all automorphic forms on $GL(1)$ and $GL(2)$ over a function field. Even so it seems that elliptic modules are still worth consideration because the theory for $r = 1$ and $r = 2$ may extend easier for general r than the corresponding theory of "shtuka" which depends on the Selberg trace formula. Another method to relate automorphic forms to ℓ-adic representations is in Drinfel'd [1983]. Recently, elliptic modules were used to describe modular forms in the function field case, see D. Goss [1977].

This article grew out of lectures by P. Deligne at the I.H.E.S. in the Winter-Spring of 1975 and a lecture in Bonn during the period of Karneval, 1975.

CHAPTER 1. ALGEBRAIC THEORY OF DRINFEL'D MODULES

In terms of the notations in the introduction, a Drinfel'd module is an A-module structure on the additive group G_a in characteristic p. There is a theory of division points and isogenies of Drinfel'd modules which parallels closely the corresponding theory for abelian varieties. Drinfel'd modules have a characteristic v, which is a valuation of $F = \mathbb{F}_q(C)$ over $\mathbb{F}_q$, and they exhibit a singular and sometimes supersingular behavior at v when $v \neq \infty$. Again, we are reminded of elliptic curves in characteristic p.

Drinfel'd formulated and solved a moduli problem for Drinfel'd modules with prescribed level structure. The constructions take place entirely with affine schemes making the existence of the moduli elementary compared with the existence question of the moduli scheme for elliptic curves with level structure. We introduce Drinfel'd's modification of the notion of level structure which allows for a clearer analysis of level structures at singular and supersingular points for the cases of both Drinfel'd modules and elliptic curves. See additional remarks 1.

§1. ENDOMORPHISMS OF THE ADDITIVE GROUP

The additive group (functor) G_a over a ring R is represented by the polynomial ring $R[X]$ in one variable with structure morphism $\alpha: R[X] \longrightarrow R[X] \otimes_R R[X]$ given by $\alpha(X) = X \otimes 1 + 1 \otimes X$. A morphism $\phi: G_a \longrightarrow G_a$ of the underlying schemes over R is given by a polynomial $\phi(X) \in R[X]$ where $\phi^*(a(X)) = a(\phi(X))$ for $a(X) \in R[X]$, and this morphism ϕ preserves the group scheme structure if and only if $\phi(X)$ is <u>additive</u>, i.e. $\phi(X+Y) = \phi(X) + \phi(Y)$. Composition $\phi\psi$ of morphisms ϕ and ψ is represented by substitution $\phi(\psi(X))$, and the sum of two morphisms $G_a \longrightarrow G_a$ is represented by the sum of two polynomials.

For example, $\phi(X) = aX^{p^i}$ is additive where p is a prime number and $pa = 0$ in R since the binomial coefficient $\binom{p^i}{m}$ is divisible by p for $0 < m < p^i$. For a field k of characteristic 0 the only additive polynomials are of the form cX, $c \in k$, but for a field k of characteristic $p > 0$ the additive polynomials are easily seen to be of the form

$$\phi(X) = \sum_{0 \leqslant i} a_i X^{p^i} = a_0X + a_1X^p + a_2X^{p^2} + \cdots + a_nX^{p^n} .$$

If $\psi(X)$ is a second additive polynomial, then

$$\psi(\phi(X)) = \psi(a_0X) + \psi(a_1X^p) + \cdots + \psi(a_nX^{p^n})$$

is also additive.

(1.1) DEFINITION. Let k be a field of characteristic $p > 0$. The twisted polynomial ring $k\{\tau\}$ is $k \otimes_{\mathbf{Z}} \mathbf{Z}[\tau]$ with the twisted tensor product algebra structure satisfying the commutation rule

$$(a \otimes \tau^i)(b \otimes \tau^j) = a(b)^{p^i} \otimes \tau^{i+j} .$$

We denote $a \otimes \tau^i$ simply by $a\tau^i$ in $k\{\tau\}$, and then the commutation rule becomes $\tau a = a^p\tau$.

Let k denote a field of characteristic $p > 0$ in the remainder of this section.

(1.2) PROPOSITION. The function θ which assigns to an additive polynomial $\phi(X) = a_0X + a_1X^p + \cdots + a_nX^{p^n}$ the element $\tilde{\phi}(X) = a_0 + a_1\tau + \cdots + a_n\tau^n \in k\{\tau\}$ is an isomorphism $\theta\colon \mathrm{End}_k(G_a) \longrightarrow k\{\tau\}$ of rings.

The proof follows from the observation that the relation $(aX^{p^i})^{p^j} = a^{p^j}X^{p^{i+j}}$ becomes $\tau^j a\tau^i = a^{p^j}\tau^{i+j}$ under θ . Note that the multiplicative structure on $\mathrm{End}_k(G_a)$ is substitution of additive polynomials and on $k\{\tau\}$ the twisted polynomial multiplication.

We have two degree functions $\deg\colon \mathrm{End}_k(G_a) \longrightarrow \mathbf{Z}$ and $d\colon k\{\tau\} \longrightarrow \mathbf{Z}$ defined by the relations $\deg(a_0X + a_1X^p + \cdots + a_nX^{p^n}) = p^n$ and $d(a_0 + a_1\tau + \cdots + a_n\tau^n) = n$ where $a_n \neq 0$. The following relations hold for $\phi,\psi \in \mathrm{End}_k(G_a)$ and $a,b \in k\{\tau\}$

$$\deg(\phi\psi) = \deg(\phi)\deg(\psi) \quad , \quad \deg(\phi+\psi) \leqslant \max(\deg(\phi),\deg(\psi))$$

and

$$d(ab) = d(a) + d(b) \quad , \quad d(a+b) \leqslant \max(d(a),d(b)) .$$

We have height functions $\mathrm{ht}\colon \mathrm{End}_k(G_a) \longrightarrow \mathbf{Z}$ and $\mathrm{ht}\colon k\{\tau\} \longrightarrow \mathbf{Z}$ defined by the relations $\mathrm{ht}(a_hX^{p^h} + \cdots + a_nX^{p^n}) = h$ and $\mathrm{ht}(a_h\tau^h + \cdots + a_n\tau^n) = h$ where $a_h \neq 0$. Clearly, $\mathrm{ht}(\phi) = \mathrm{ht}(\theta(\phi))$. The following relations hold $\mathrm{ht}(\phi\psi) = \mathrm{ht}(\phi) + \mathrm{ht}(\psi)$ and $\mathrm{ht}(ab) = \mathrm{ht}(a) + \mathrm{ht}(b)$ for $\phi,\psi \in \mathrm{End}_k(G_a)$ and $a,b \in k\{\tau\}$.

We have substitutions $\partial_0 : \mathrm{End}_k(G_a) \longrightarrow k$ and $\partial : k\{\tau\} \longrightarrow k$ defined by the relations

$$\partial_0\left(\sum a_i X^{p^i}\right) = a_0 \qquad \text{and} \qquad \partial\left(\sum a_i \tau^i\right) = a_0$$

where ∂_0 is the derivative at the origin and ∂ the value at the origin. Clearly $\partial(\theta(\phi)) = \partial_0(\phi)$ for $\phi \in \mathrm{End}_k(G_a)$.

Finally, the following properties of additive polynomials seem to have been known for some time.

(1.3) PROPOSITION. Let $H \subset G_a(k)$ be a finite subgroup, and form the polynomial $P_H(X) = \prod_{h \in H}(X-h)$. Then $P_H(X)$ is an additive polynomial, so $P_H \in \mathrm{End}_k(G_a)$ with $\deg(P_H) = \mathrm{Card}(H)$, and we are able to recover $H = \ker(P_H)(k)$, the set of all $x \in k$ with $P_H(x) = 0$.

PROOF. To show that P_H is additive, consider $Q_Y(X) = P_H(X+Y) - P_H(Y)$ in $k(Y)[X]$. Since H is a subgroup, $Q_Y = 0$ on H and further $\deg(Q_Y) = \deg(P_H) = \mathrm{Card}(H)$. Thus $P_H(X)$ and $Q_Y(X)$ are monic polynomials of the same degree equal to $\mathrm{Card}(H)$ and each equal to zero on $H \subset k \subset k(Y)$. It follows that $P_H(X) = Q_Y(X) = P_H(X+Y) - P_H(Y)$, i.e. P_H is additive, and the other statements hold which proves the proposition.

(1.4) REMARK. As a kind of converse of (1.3) observe that for an additive polynomial $f: G_a \longrightarrow G_a$ over k, the set H of $x \in k$ with $f(x) = 0$ is a subgroup of $G_a(k)$ with $\mathrm{Card}(H)$ dividing $\deg(f)$. If k is algebraically closed, then $\deg(f) = p^{ht(f)} \cdot \mathrm{Card}(H)$, and that the group morphism $f: G_a(k) \longrightarrow G_a(k)$ is surjective for $f \neq 0$. Since $ht(f) = 0$ if and only if $\partial_0(f) \neq 0$, we see that $\partial_0(f) \neq 0$ implies that $\deg(f) = \mathrm{Card}(\ker(f))$ over an algebraically closed k.

§2. DEFINITION OF DRINFEL'D MODULE OVER A FIELD

We return to the basic notations of the introduction, in particular $A = H^0(C - \infty, \mathcal{O}_C)$, and let k denote a field of characteristic p. Before the definition we make some remarks which point out the natural limitations of the definition.

(2.1) REMARK. Since $\mathrm{End}_k(G_a)$ isomorphic to $k\{\tau\}$ is an integral domain, and since A is a Dedekind domain, any ring morphism $A \longrightarrow \mathrm{End}_k(G_a)$ is either injective or has image contained in the constants $k \subset k\{\tau\}$. The second case being relatively trivial means that we will be interested in morphisms which are injective (or faithful).

To the rational point ∞ , we have associated an absolute value $|x|_\infty$ on the function field F with $\mathbb{F}_{q_\infty}$ as residue class field with q_∞ a power of q and the absolute value normalized such that $|a|_\infty = \mathrm{Card}(A/a)$ for $a \in A \subset F$.

(2.2) REMARK. For an injective $\phi: A \longrightarrow \mathrm{End}_k(G_a)$ the composite $A \xrightarrow{\phi} \mathrm{End}_k(G_a) \xrightarrow{\deg} \mathbb{Z}$ denoted $\|a\| = \deg(\phi(a))$ satisfies $\|ab\| = \|a\|\|b\|$ and $\|a+b\| \leqslant \max(\|a\|, \|b\|)$ from properties of deg . Since $\|a\| \geqslant 1$ for $a \neq 0$ and for some a , $\|a\| > 1$, the relation $\|a/b\| = \|a\|/\|b\|$ for $a/b \in F$ defines an extension of $\|a\|$ to F as an absolute value. Since $\|a\| \geqslant 1$ for all $a \in A$, $a \neq 0$, the absolute value $\|x\|$ on F must be equivalent to $|x|_\infty$, i.e. $\|x\| = |x|_\infty^r$ for some $r > 0$ and all $x \in F$.

In fact, $r > 0$ is a natural number, but we defer the proof of this until the next section, see (3.3), and introduce the following basic concept which Drinfel'd called an elliptic module.

(2.3) DEFINITION. For a natural number $r > 0$, a Drinfel'd module over a field k of rank r for the pointed curve (C,∞) over $\mathbb{F}_q$ is a morphism of rings

$$\phi: A \longrightarrow \mathrm{End}_k(G_a)$$

such that G is isomorphic to G_a and $\|a\| = \deg(\phi(a)) = |a|_\infty^r = (\mathrm{Card}(A/a))^r$ for all nonzero $a \in A$.

Let Alg_k denote the category of commutative k-algebras R , and let Mod_A denote the category of A-modules. To a Drinfel'd module ϕ , we associate a functor

$$E: \mathrm{Alg}_k \longrightarrow \mathrm{Mod}_A$$

which assigns to a k-algebra R its underlying additive group $G_a(R)$ together with the A-module structure defined by requiring $\phi(a)(R)$ to be scalar multiplication by a on $G_a(R)$. Observe that the functor E determines ϕ , and frequently we refer to E as the Drinfel'd module instead of ϕ . We will denote $\phi(a)$ by just ϕ_a occasionally.

(2.4) DEFINITION. Let $\phi: A \longrightarrow \mathrm{End}_k(G)$ and $\phi': A \longrightarrow \mathrm{End}_k(G')$ be two Drinfel'd modules. A morphism $u: \phi \longrightarrow \phi'$ is a morphism $u: G \longrightarrow G'$ such that $\phi'_a u = u\phi_a$ for all $a \in A$. A nonzero morphism is called an isogeny.

Equivalently, u is a morphism of functors $E \to E'$ associated to ϕ and ϕ'. Thus Drinfel'd modules over k form a category with the composition of morphisms being evident.

(2.5) DEFINITION. Let $\phi: A \to \mathrm{End}_k(G)$ be a Drinfel'd module and form $\partial_0\phi: A \to k$ where $\partial_0: \mathrm{End}_k(G) \to k$ is the value of the derivative at the origin. The Drinfel'd module ϕ has characteristic ∞ provided $\partial_0\phi: A \to k$ is injective and has characteristic v, a valuation of F over $\mathbb{F}_q$ different from ∞, provided $A \cap m_v = \ker(\partial_0\phi)$.

Since $\ker(\partial_0\phi)$ is either zero or a maximal ideal of A, every Drinfel'd module has characteristic ∞ or some "finite" $v \neq \infty$.

(2.6) REMARK. If $u: \phi \to \phi'$ is a nonzero morphism (isogeny) between two Drinfel'd modules, then ϕ and ϕ' have the same rank and the same characteristic.

(2.7) REMARK. The definition (2.3) of a Drinfel'd module can be formulated in terms of a $(A,\mathrm{End}_k(G_a))$-bimodule N which is free of rank 1 over $\mathrm{End}_k(G_a)$ and satisfying the condition $\|a\| = (\mathrm{Card}(A/a))^r$ for $a \in A$, $a \neq 0$. The choice of a basis element for N identifies N with $\mathrm{End}_k(G_a)$.

§3. DIVISION POINTS

As with abelian varieties, division points of a Drinfel'd module play a basic role in structure and moduli problems. We continue with the notation $E: \mathrm{Alg}_k \to \mathrm{Mod}_A$ equal to the functor associated to a ring morphism $\phi: A \to \mathrm{End}_k(G_a)$.

(3.1) DEFINITION. For $a \in A$ (usually $a \neq 0$) the subfunctor $E_a \subset E$ of a-division points is $\ker(\phi_a)$. For an ideal $I \subset A$ the subfunctor $E_I \subset E$ of I-division points is $\bigcap_{a\in I} E_a$.

If $a_1,\dots,a_r$ generate an ideal I in A, then $E_I = E_{a_1} \cap \cdots \cap E_{a_r}$. So in particular for a principal ideal (b) we have $E_{(b)} = E_b$. Since $E_a = E$ if and only if $a = 0$, the same holds for ideal $E_I = E$ if and only if $I = 0$, and essentially we consider E_I only for nonzero ideals I.

More explicitly, for any k-algebra R, the A-submodule $E_I(R)$ consists of all $x \in E(R)$ such that $ax = 0$ for any $a \in I$. This shows that we can view E_I as a functor defined $E_I: \mathrm{Alg}_k \to \mathrm{Mod}_{A/I}$.

Again as with abelian varieties, $E_I(\bar{k})$ will be a free A/I-module for I prime to the characteristic of E . For the case characteristic ∞ this is no restriction on I and for characteristic v corresponding to the maximal ideal P_v in A it means $I \subset P_v$. In order to prove this assertion we will use the following lemma on torsion modules over a discrete valuation ring V with local uniformizing parameter π . For a V-module L let $\ell(L)$ denote its length and $\pi_L: L \to L$ the action of the scalar π on L .

(3.2) LEMMA. Let L be a V/π^{2m}-module.

(a) We have $2\ell(\ker(\pi_L^m)) \geq \ell(L)$.

(b) For nonzero L , the equality holds in (a) if and only if L is a free V/π^{2m}-module, and in this case $\ker(\pi_L^m)$ is a free V/π^m-module.

PROOF. We can decompose L as the sum of modules N isomorphic to V/π^i with $0 < i \leq 2m$. For $N = V/\pi^i$ $(0 < i \leq 2m)$ observe that

$$\ker(\pi_L^m) = \begin{cases} V/\pi^i & \text{of length } i \text{ for } i \leq m \\ \pi^{i-m}(V/\pi^i) & \text{of length } m \text{ for } m \leq i . \end{cases}$$

In both cases $2\ell(\ker(\pi_N^m)) = 2\cdot\inf(i,m) \geq i = \ell(N)$ so that (a) holds. Also N is free, i.e. $i = 2m$ if and only if $2\ell(\ker(\pi_N^m)) = \ell(N)$, and $\pi^m(V/\pi^{2m})$ is isomorphic to V/π^m as V/π^m-modules which proves (b).

In the following basic structure theorem for $E_I(\bar{k})$ over the algebraic closure $\bar{k}$ of k , we also settle the question left at the end of (2.2). Note that the above definitions apply to any morphism $\phi: A \to \mathrm{End}_k(G_a)$ of rings.

(3.3) THEOREM. If $\phi: A \to \mathrm{End}_k(G_a)$ is a monomorphism of rings, then ϕ is a Drinfel'd module of rank $r > 0$ for an integer r . Moreover, for an ideal I relatively prime to the characteristic of ϕ , the A/I-module $E_I(\bar{k})$ is free of rank r .

PROOF. By (2.2) there exists a real number $r > 0$ with $\deg \phi_a = \mathrm{Card}(A/a)^r$ for all $a \in A$, $a \neq 0$. If $\partial_0\phi_a = 0$, then by (1.4) $\deg \phi_a = \mathrm{Card}\, E_a(\bar{k})$ and also

$$\mathrm{Card}\, E_{a^2}(\bar{k}) = \deg(\phi_{a^2}) = \deg(\phi_a)^2 = \mathrm{Card}\, E_a(\bar{k})^2 .$$

For each irreducible element π of A prime to the characteristic of ϕ , which is a local uniformizing parameter of $A_{\mathfrak{p}} = V$, we apply lemma (3.2) to the

V/π^{2m}-module $E_{\pi^{2m}}(\bar{k})$ to prove that $E_{\pi^m}(\bar{k})$ is a free A/π^m-module of rank d where

$$\mathrm{Card}(E_{\pi^m}(\bar{k})) = \deg(\phi_{\pi^m}) = \mathrm{Card}(A/\pi^m)^r .$$

From this we deduce that r is an integer and so ϕ is a Drinfel'd module of rank r.

Next, for any nonzero $a \in A$ the primary components of $E_a(\bar{k})$ are of the form $E_{\pi^m}(\bar{k})$, so free A/π^m-module of rank r, from which we deduce that $E_a(\bar{k})$ is a free A/a-module of rank r. Finally, if I is an ideal of A prime to the characteristic of ϕ, then there exists another ideal J in A with $A = I + J$ and $IJ = (a)$, a principal ideal prime to the characteristic. Then $A/a = A/I \oplus A/J$, and $E_a = E_I \oplus E_J$ as functors. Since $E_a(\bar{k})$ is free of rank r over A/a, it follows that $E_I(\bar{k})$ is free of rank r over A/I, and this proves the theorem.

(3.4) REMARK. It remains only to consider for a Drinfel'd module E of rank r and characteristic $v \neq \infty$, the A/π^n-modules $E_{\pi^n}(\bar{k})$ where $v(\pi) = 1$. In this case $\partial_0\phi_{\pi^i} = 0$ for all i, and by the discussion in (1.4), the endomorphism ϕ_π has a height h where $\deg(\phi_\pi) = p^h\cdot\mathrm{Card}\,E_\pi(\bar{k})$, or more generally

$$\deg(\phi_{\pi^n}) = (\deg(\phi_\pi))^n = (p^h\cdot\mathrm{Card}\,E_\pi(\bar{k}))^n = p^{nh}\cdot\mathrm{Card}\,E_{\pi^n}(\bar{k}) .$$

Hence again by applying lemma (3.2), we see that $E_{\pi^n}(\bar{k})$ is a free module of rank $r - h < r$ over A/π^n. As finite group schemes over k, we have a splitting when k is separable

$$E_{\pi^i} = E^0_{\pi^i} \times E^{et}_{\pi^i}$$

where $E^{et}_{\pi^i}$ is étale of rank $r - h$ and $E^0_{\pi^i}$ is infinitesimal of rank h.

(3.5) DEFINITION. The height of a Drinfel'd module E of characteristic $v \neq \infty$ is the height of ϕ_π where π is an irreducible with $v(\pi) = 1$.

Since (π) is uniquely determined by v, the height is well defined.

(3.6) REMARK. Let E be a Drinfel'd module of rank r and characteristic v. Let h be the height when $v = \infty$. For an irreducible π of A, let A_π be A localized at π. Then $\varinjlim_n E_{\pi^n}(\bar{k})$ is a divisible A_π-module which is isomorphic to

$$\begin{cases} (F/A_\pi)^r & \text{for } \pi \text{ prime to the characteristic,} \\ (F/A_\pi)^{r-h} & \text{for } v(\pi) = 1 \ . \end{cases}$$

§4. ISOGENIES

Recall that for a Drinfel'd A-module (ϕ, E) over k and any field extension k^* of k, the group $E(k^*)$ has a given A-module structure. For two Drinfel'd A-modules over k and $u \in k\{\tau\}$ with $u \neq 0$, the additive $u: \phi \longrightarrow \phi'$ is an isogeny if and only if the additive $u: E(\bar{k}) \longrightarrow E'(\bar{k})$ is A-linear. This follows from the fact that two polynomials are equal if and only if their associated polynomial maps $\bar{k} \longrightarrow \bar{k}$ are equal. The kernel of $u: E(k^*) \longrightarrow E'(k^*)$ is the A-module $\ker(u)(k^*)$ where $\ker(u)$ is the scheme kernel of $u: \phi \longrightarrow \phi'$.

(4.1) Separable isogenies. An isogeny $u: \phi \longrightarrow \phi'$ is separable when $u(X) = b_0X + b_1X^p + \cdots + b_sX^{p^s}$ where $b_0 \neq 0$. Then u is determined by $\ker(u)(\bar{k})$ a finite A-submodule of $E(\bar{k})$ up to constant factor. For a finite A-submodule $H \subset E(\bar{k})$, we form $P_H(X) = \prod_{h \in H}(X - h)$, and there is a unique Drinfel'd A-module ϕ' such that $\phi'_a P_H = P_H \phi_a$ for all $a \in A$.

If $a \cdot \ker(u)(\bar{k}) = 0$ for $a \neq 0$, then we can form the isogeny v with kernel $u(\ker(\phi_a)(\bar{k}))$ and $vu = \phi_a$. Note that such a nonzero $a \in A$ with $a \cdot \ker(u)(\bar{k}) = 0$ always exists for $u \neq 0$ since it is a finite A-module. When u is separable, we can always choose v to be separable too. By the same construction, if $u: \phi \longrightarrow \phi'$ and $w: \phi \longrightarrow \phi''$ are separable isogenies such that $w(\ker(u)(\bar{k})) = 0$, then there is a separable isogeny $v: \phi' \longrightarrow \phi''$ with $w = vu$ where $\ker(v)(\bar{k}) = u(\ker(w)(\bar{k}))$.

(4.2) Purely inseparable isogenies. These are of the form $\tau^i \in k\{\tau\}$ or $X \longmapsto X^{p^i}$. Since $\phi_a\tau^i = \tau^i\phi_a$ in $k\{\tau\}$ is equivalent to all coefficients of ϕ_a being in $\mathbb{F}_{p^i}$, it follows that a purely inseparable isogeny exists only when the characteristic v of ϕ is unequal to ∞, and in this case p^i is a power of q_v where q_v is the cardinality of the residue class field of v. The finite group scheme $\ker(\tau^i) = \mathrm{Spec}(k[t]/(t^{p^i}))$. This case corresponds to a purely local kernel.

(4.3) REMARKS. Let H be a finite subgroup scheme of G_a, and let ϕ be a Drinfel'd module structure on G_a. Then H is the kernel of some isogeny $u: \phi \longrightarrow \phi'$ if and only if H is stable under the action of A and

$$H^{\mathrm{loc}} = \begin{cases} 0 & \text{if the characteristic} = \infty \\ \operatorname{Spec}(k[t]/(t^{q^h})) & \text{if the characteristic} = v \text{ and} \\ & q = q_v = \operatorname{Card}(F(v)) \ . \end{cases}$$

For any isogeny $u: \phi \longrightarrow \phi'$ there exists an isogeny $v: \phi' \longrightarrow \phi$ and $a \in A$ with $\phi_a = vu$.

Let $\delta(u)$ denote the degree of the additive polynomial $u \in \operatorname{End}_k(E)$. The function $\delta: \operatorname{End}_k(E) \longrightarrow \mathbf{Z}$ also prolongs under extension of scalars to

$$\begin{array}{ccccc} \operatorname{End}_k(E) & \longrightarrow & \operatorname{End}_k(E) \otimes_A F & \longrightarrow & \operatorname{End}_k(E) \otimes_A F \\ \downarrow{\scriptstyle\delta} & & \downarrow{\scriptstyle\delta} & & \downarrow{\scriptstyle\delta} \\ \mathbf{Z} & \longrightarrow & \mathbf{Q} & \longrightarrow & \mathbf{Q} \end{array} \ .$$

(4.4) REMARKS. We study the A-algebra $\operatorname{End}_k(E)$ with δ using the following properties of $\delta: \operatorname{End}_k(E) \otimes_A F_\infty \longrightarrow \mathbf{Q}$:

(1) $\delta(u) \geqslant 0$ and $\delta(u) = 0$ if and only if $u = 0$.

(2) $\delta(au) = \|a\|\delta(u)$ for $a \in F$.

(3) $\delta(u+v) \leqslant \max(\delta(u), \delta(v))$.

(4) $\delta(vu) = \delta(v)\delta(u)$.

Moreover, $\operatorname{End}(E) \hookrightarrow \operatorname{End}_k(E) \otimes_A F_\infty$ will be seen to be a discrete A-module in this normed vector space over F_∞ , see (4.9)(2). The main step will be to show that End(E) is a finitely generated A-module. For this we use the following two lemmas.

(4.5) LEMMA. Let $A^n \subset X \subset F_\infty^n$ where X is a discrete A-module. Then X is finitely generated.

PROOF. For all i observe that $m_\infty^i + A \subset F_\infty$ has finite index, e.g. $k[1/t] + t^i k[[t]] \subset k((t))$ has index $\operatorname{Card}(k)^i$. There exists an i with $X \cap (m_\infty^i)^n = 0$ since X is discrete, and thus X embeds in $(F_\infty/m_\infty^i)^n$ and X/A^n embeds in $(F_\infty/m_\infty^i + A)^n$ which is finite. Since $(X : A^n)$ is finite and A^n is finitely generated, it follows that X is a finitely generated A-module.

(4.6) PROPOSITION. For a finite dimensional subspace V over F of $\operatorname{End}(E) \otimes_A F$, it follows that $V \cap \operatorname{End}(E)$ is a finitely generated A-module which is projective.

PROOF. For $V_\infty = F_\infty \otimes_F V$, it follows that $X = \mathrm{End}(E) \cap V = \mathrm{End}(E) \cap V_\infty$, and we can assume that S generates V and so V_∞ or replace V by a smaller subspace. Let $x_1, \ldots, x_n$ be a basis of V_∞ with $x_i \in X$. Then with this basis $A^n \subset X \subset F_\infty^n \simeq V_\infty$, and we are reduced to the previous lemma since $\mathrm{End}(E)$ is a discrete subspace of $\mathrm{End}(E) \otimes_A F_\infty$. To see that $X = V \cap \mathrm{End}(E)$ is projective, we have only to remark that it is flat since it is torsion free over a Dedekind domain and finitely generated flat modules are projective over a Noetherian ring.

(4.7) COROLLARY. The A-module End(E) is projective.

PROOF. Let $W = \coprod_i W_i$ where $W = \mathrm{End}(E) \otimes_A F$ and $\dim_F W_i$ is finite. Then $X_i = \mathrm{End}(E) \cap W_i$ is projective by (4.6) and the restrictions of the projections $W \to W_i$ to $f_i\colon \mathrm{End}(E) \to X_i$ define a morphism $f\colon \mathrm{End}(E) \to \coprod_i X_i$ onto a projective module with $\ker(f) = 0$. Hence End(E) is projective.

For estimates on the rank we use the following lemma.

(4.8) LEMMA. Let $a \in A$ be prime to the characteristic of E. Then $\mathrm{End}(E) \otimes A/a \to \mathrm{End}(E_a)$ is injective.

PROOF. If $w \in \mathrm{End}(E)$ and $w(\ker(\phi_a)) = 0$, then as in (4.1), it follows that $w = v\phi_a$, and $w|E_a = 0$ implies $w \in \mathrm{End}(E)a$.

Now we summarize all the basic results in the following theorem and remarks.

(4.9) THEOREM. Let E be a Drinfel'd module over an algebraically closed field k of rank r. The

(1) End(E) is a projective A-module of rank $\leq r^2$, and
(2) $\mathrm{End}_k(E) \otimes_A F_\infty$ is a field in which End(E) embeds as a discrete A-module of this normed space over F_∞.

PROOF. (1) The fact that End(E) is projective is contained in (4.7) and the injectivity of $\mathrm{End}(E) \otimes A/a \to \mathrm{End}(E_a)$ coming from (4.8) bounds the rank by r^2 since E_a is an A/a-module of rank r for a prime to the characteristic.

(2) The existence of δ on $\mathrm{End}_k(E) \otimes_A F$ proves that it is a field, and since F_∞/F is a separable extension $\mathrm{End}_k(E) \otimes_A F_\infty$ is also a field.

The subspace where $\delta = 0$ on $\mathrm{End}_k(E) \otimes_A F_\infty$ is zero since $\dim_{F_\infty} \mathrm{End}(E) \otimes_A F_\infty = \mathrm{rank}(\mathrm{End}_k(E))$.

(4.10) REMARK. With the notations of the previous theorem, the ring $\mathrm{End}_k(E)$ is commutative for E of characteristic ∞ and further its rank $\leqslant r$.

(4.11) REMARK. For a place v of F we denote by $D_v(E) = \varinjlim_n E_{(v)^n}$ and then the Tate module would be by definition

$$T_v(E) = \mathrm{Hom}(F_v/A_v, D_v(E)) \quad .$$

Then $\mathrm{End}(E) \otimes_A A_v \hookrightarrow \mathrm{End}\, T_v(E) = \mathrm{End}\, D_v(E)$, and the cokernel is without torsion as an A_v-module.

§5. DRINFEL'D MODULES OVER A BASE SCHEME

Recall that locally free sheaf of rank 1 and invertible sheaf are the same notions, for which we use also the term line bundle.

(5.1) DEFINITION. Let S be a scheme in characteristic p . A Drinfel'd module over S of rank r is an invertible sheaf L and a morphism or rings $\phi: A \longrightarrow \mathrm{End}(L)$ such that locally over open sets where L is trivial $\phi_a(x) = \sum_{i=0}^m a_i x^{p^i}$ where $p^m = \|a\|^r$ and a_m is a unit. We say that ϕ_a is strictly of degree p^m .

In order to analyze the condition that a_m is a unit, we use the following lemma.

(5.2) LEMMA. Let R be a ring with $p = 0$ and $\phi, \psi, f \in R\{\tau\}$ where $f\phi = \psi f$, $\phi = \sum_{i=0}^N a_i \tau^i$ with a_N a unit, $\psi = \sum_{i=0}^N b_i \tau^i$ with b_N a unit, and $f = \sum_{i=0}^M c_i \tau^i$. Then the leading coefficient c_M is either a unit or zero, and so f is either zero or strictly of degree $\leqslant M$.

PROOF. Comparing the coefficients of τ^{M+N} in the relation $\psi f = f\phi$, we deduce $b_N c_M^N = c_M a_N^M$ or $c_M(b_N c_M^{N-1} - a_N^M) = 0$. Since b_N is a unit, this can be written

$$c_M\left(c_M^{N-1} - a_N^M b_N^{-1}\right) = 0 \quad .$$

Thus either $c_M = 0$ or $c_M =$ a unit with $c_M^{N-1} = a_N^M/b_N$.

The next lemma is a version of the Weierstrass preparation theorem.

(5.3) LEMMA. Let R be a ring with $p = 0$. If $\phi = \sum_{i=0}^{N} a_i\tau^i$ where a_M is a unit and $a_{M+1},\ldots,a_N$ are nilpotent, then after a change of coordinates with $\alpha(\tau) = 1 + \sum_{i=1}^{N} c_i\tau^i$ such that

$$(\alpha^{-1}\phi\alpha)(\tau) = \sum_{i=0}^{M} a_i'\tau^i = \phi'(\tau)$$

with a_M' a unit.

The coefficients $c_N, c_{N-1}, \ldots, c_{M+1}$ are chosen by decreasing induction.

(5.4) REMARKS on the definition (5.1). An alternative form of the definition of a Drinfel'd module over a scheme S in characteristic p is to give E/S a group scheme locally isomorphic to G_a and a morphism of rings $\phi: A \longrightarrow \mathrm{End}_S(E)$ with degree $\phi_a = \|a\|_\infty^r$ locally over S .

Using lemma (5.3), we can put an $\mathcal{O}_S$-linear structure on E such that the action of some ϕ_a for $\|a\|_\infty > 1$ is locally given by a polynomial expression with highest coefficient a unit and of degree $\|a\|_\infty^r$. Using lemma (5.2) and the relation $\phi_a\phi_b = \phi_{ab} = \phi_b\phi_a$, it follows that all ϕ_b are given locally by polynomials of strict degree $\|b\|_\infty^r$. Questions of $\mathcal{O}_S$-linearity need only be checked over closed subschemes $\mathrm{Spec}(R) \hookrightarrow S$ where R is a local Artin ring.

(5.5) <u>Analogue of characteristic</u>. The function $a \longmapsto \partial_0\phi_a$ defines a morphism $A \longrightarrow \mathcal{O}_S$ of rings and hence a morphism of schemes $S \longrightarrow \mathrm{Spec}(A)$. For a closed point $s \in S$, the Drinfel'd module L_s over the field F(s) has characteristic $\theta(s) \in \mathrm{Spec}(A)$.

§6. LEVEL STRUCTURE AND THE MODULI SPACE

Let I be an ideal in A , let V(I) denote the set of prime ideals of A containing I , and let E/S be a Drinfel'd module of rank r over S with characteristic morphism $\theta: S \longrightarrow \mathrm{Spec}(A)$. Then form the finite flat group scheme $E_I/S = \bigcap_{a \in I} \ker_S(E \xrightarrow{a} E)$ of rank equal to $\mathrm{Card}(A/I)^r$. The scheme E_I/S is étale outside the characteristics which divide I , i.e.

over $\theta^{-1}(\mathrm{Spec}(V) - V(I)) \subset S$.

In the most elementary sense an I-level structure on a Drinfel'd module E should be an isomorphism $\alpha: (I^{-1}/A)^r \to E_I$ over S , and in fact, this definition works very well away from characteristics dividing I . In order to deal smoothly with characteristics dividing I, we are led to the following definition of Drinfel'd which has also an analogue for elliptic curves with level structures.

(6.1) DEFINITION. An I-level structure on a Drinfel'd module E/S of rank r is an A-linear morphism

$$\alpha: (I^{-1}/A)^r \longrightarrow E_I$$

such that for all i in I^{-1}/A the corresponding sections $\alpha(i)$ of E_I have the property that as divisors on E

$$\sum_{i \in (I^{-1}/A)^r} (\alpha(i)) = (E_I) .$$

Locally E is isomorphic to G_a , and in this case E_I is defined as the kernel of a polynomial map $P: G_a \to G_a$ where

$$P(X) = \prod_{i \in (I^{-1}/A)^r} (X - \alpha(i)) .$$

Let F_I^r denote the contravariant functor from schemes to sets which assigns to a scheme S the set of isomorphism classes of Drinfel'd modules over S of rank r with I-level structure.

(6.2) THEOREM. Let I be an ideal in A with $\mathrm{Card}\, V(I) > 1$. The functor F_I^r is representable by an affine scheme of finite type over A .

PROOF. For $x \in V(I)$ it suffices to show that the functor restricted to the category of schemes over $\mathrm{Spec}(A) - \{x\}$ is representable. Then for a scheme S over $\mathrm{Spec}(A) - \{x\}$, and a Drinfel'd module E over S with an I-level structure, the choice of nonzero elements for the I-level structure gives a trivialization of E . Over these local pieces E is given by coordinates of ϕ_a for each $a \in A$ and elements $\alpha(i)$ for $i \in (I^{-1}/A)^r$ subject to the relations:

(a) $\phi_a \phi_b = \phi_b \phi_a$.

(b) The leading coefficient of ϕ_a is invertible.

(c) $\phi_a(\alpha(i)) = 0$.

(d) $P(X) = \prod(X - \alpha(i))$.

All of these relations are affine in nature and can be represented by an affine scheme.

The book of Katz-Mazur [1985] carries the construction of moduli spaces of elliptic curves using Drinfel'd's definition of level structure.

CHAPTER 2. ANALYTIC THEORY OF THE AFFINE MODULES OF DRINFEL'D

The group of points of an elliptic curve over the complex numbers is of the form $\mathbf{C}/L$ where L is a lattice over $\mathbf{Z}$ (so of rank 2) in $\mathbf{C}$. There is a similar assertion for Drinfel'd modules of rank r. The group of C_∞-valued points is of the form C_∞/L where L is an A-lattice (discrete A-module) of rank r in C_∞.

The description of Drinfel'd modules in terms of lattices gives a calculation of the moduli space of Drinfel'd modules over C_∞ in terms of a quotient $GL(r,A)\backslash \mathrm{Mon}_{F_\infty}(F_\infty^r, C_\infty)$. This is similar to the quotient $GL(2,\mathbf{Z})\backslash \mathrm{Mon}_{\mathbb{R}}(\mathbb{R}^2,\mathbf{C})$ which classifies elliptic curves plus a differential form. In the last section an adèlic description of the points of this moduli space is given.

§1. EXPONENTIAL FUNCTION ASSOCIATED TO A LATTICE

Let X be a subset of a complete nonarchimedean value field K such that $0 \in X$ and $B(a,r) \cap X$ is finite for any ball $B(a,r) \subset K$ around a of radius $r > 0$. Then the Euler product

$$e_X(t) = t \prod_{\gamma \in X - \{0\}} (1 - \frac{t}{\gamma})$$

defines an entire function $e_X\colon K \longrightarrow K$ with zeros on X, because the hypothesis implies $\lim_{\gamma \in X - \{0\}} t/\gamma = 0$ from which we deduce that $e_X(t)$ converges uniformly on any ball.

By changing f and t by scalar factors, we can assume that the power series expansion of an entire function has the form $f(t) = \sum_{0 \leqslant n} a_n t^n$ where $|a_i| \leqslant 1$, $|a_i| < 1$ for $i < r$, and $|a_r| = 1$; when f is nonconstant, also $r > 0$. By solving congruences modulo powers of the maximal ideal of F, we can factor

$$f(t) = (t^r + c_1 t^{r-1} + \cdots + c_r)(b_0 + b_1 t + \cdots) .$$

If, in addition, K is algebraically closed, then it follows that every nonconstant entire function has at least one zero. Thus the Euler product $e_X(t)$

is the unique entire function, up to a constant factor, with one simple zero at each point of X and no other zeros.

(1.1) DEFINITION. A subgroup L of the additive group of a complete value field K is called a lattice provided the intersection $L \cap B(a,r)$ is finite for any ball $B(a,r)$.

When K is of characteristic p , then $pL = 0$ and the torsion group L is a limit of finite subgroups H , i.e.

$$L = \varinjlim_{H \text{ finite} \subset L} H \quad .$$

If the above Euler expansion $e_X(t) = t \prod_{\gamma \in X - \{0\}} (1 - \frac{t}{\gamma})$ is additive, i.e. $e_X(x+y) = e_X(x) + e_X(y)$ for any $x,y \in F$, then X is clearly a lattice and $e_X(t)$ is its exponential function.

(1.2) PROPOSITION. Let L be a lattice in a complete value field K of characteristic p . Then $e_L(x+y) = e_L(x) + e_L(y)$ for any $x,y \in K$.

PROOF. Since $L = \varinjlim_{H \text{ finite} \subset L} H$, it follows that uniformly on any ball

$$e_L(t) = \varinjlim_{H \text{ finite} \subset L} e_H(t) \quad .$$

Each $e_H(t)$ is additive since it equals $c \cdot \prod_{h \in H} (t - h)$ which is additive by 1(1.3). Thus $e_L(x+y) = e_L(x) + e_L(y)$ since it is a limit of functions $e_H(t)$ satisfying this property.

(1.3) REMARK. If L is a lattice in a complete value field K , and if $\lambda \in K^{\bullet}$, then λL is a lattice also and the exponential function $e_L(t)$ of L determines the exponential function of λL by $e_{\lambda L}(t) = \lambda e_L(\lambda^{-1} t)$. The lattice λL is called the dilation of L by $\lambda \in K^{\bullet}$.

Two lattices L and L' are in the same dilation class provided $L' = \lambda L$ for $\lambda \in K^{\bullet}$. We always normalize the exponential function so that

$$e_L(t) = t + (\text{higher order terms}) \quad .$$

§2. CHARACTERIZATION OF DRINFEL'D MODULES OVER C_∞

Lattices $\Gamma \subset C_\infty$ such that $a\Gamma \subset \Gamma$ for $a \in A \subset C_\infty$ with the induced scalar action by $a \in A$ are also A-modules, which we call A-lattices. A dilation $\lambda\Gamma$ of an A-lattice Γ is again an A-lattice for $\lambda \in C_\infty^{\bullet}$, and we can speak of dilation classes of A-lattices.

Given an A-lattice Γ in C_∞, we form the ring morphism $\phi^\Gamma: A \to \mathrm{End}_{C_\infty}(G_a)$ where ϕ_a^Γ for $a \in A$ is defined by the following commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \Gamma & \longrightarrow & C_\infty & \xrightarrow{e_\Gamma} & C_\infty & \longrightarrow & 0 \\ & & \downarrow a & & \downarrow a & & \downarrow \phi_a^\Gamma & & \\ 0 & \longrightarrow & \Gamma & \longrightarrow & C_\infty & \xrightarrow{e_\Gamma} & C_\infty & \longrightarrow & 0 \end{array} .$$

For $\lambda\Gamma$, the relation $\phi_a^{\lambda\Gamma}(e_{\lambda\Gamma}(t)) = e_{\lambda\Gamma}(at)$ becomes in view of (1.3) simply $\phi_a^{\lambda\Gamma}(\lambda e_\Gamma(\lambda^{-1}t)) = \lambda e_\Gamma(a\lambda^{-1}t)$ or $\lambda^{-1}\phi_a^{\lambda\Gamma}(\lambda e_\Gamma(t)) = e_\Gamma(at)$. This implies that $\phi_a^\Gamma = \phi_a^{\lambda\Gamma}$ for $\lambda \in C_\infty$.

(2.1) THEOREM. The function, which assigns to each A-lattice Γ, the ring morphism $\phi^\Gamma: A \to \mathrm{End}_{C_\infty}(G_a)$ defined above, is a bijection from the set of A-lattice dilation classes in C_∞ determined by Γ which are projective of rank r onto the set of isomorphism classes of affine A-modules ϕ_a over C_∞ of rank r with $\partial\phi_a = a$.

PROOF. First we calculate $\ker(\phi_a^\Gamma)$. This is isomorphic to $\ker(\phi_a^\Gamma e_\Gamma)/\Gamma = \ker(e_\Gamma a)/\Gamma$, and $\ker(e_\Gamma a)/\Gamma$ is isomorphic to $a^{-1}\ker(e_\Gamma)/\Gamma = a^{-1}\Gamma/\Gamma$. Hence $\ker(\phi_a^\Gamma)$ is isomorphic to $(A/a)^r$ since Γ is projective of rank r, and moreover $\deg(\phi_a^\Gamma) = \mathrm{Card}(A/a)^r$. Thus $\Gamma \to \phi^\Gamma$ is a well defined function from dilation classes of A-lattices projective of rank r to isomorphism classes of affine A-modules over C_∞ of rank r. The property $\partial\phi_a = a$ follows from $e_\Gamma(t) = t + (\text{higher terms})$, and $\phi_a^{\lambda\Gamma} = \lambda\phi_a^\Gamma\lambda^{-1}$ from the calculation preceding the statement of the theorem.

To show $\Gamma \mapsto \phi^\Gamma$ is a bijection, we consider an affine module $\phi: A \to \mathrm{End}_{C_\infty}(G_a)$ of rank r and construct an additive entire function $e: C_\infty \to C_\infty$ such that $\phi_a(e(X)) = e(aX)$ and $e(t) = t + (\text{higher terms})$. Then the corresponding A-lattice Γ is the set of $x \in C_\infty$ with $e(x) = 0$.

The first step is to show that there is a unique formal solution $e(\tau) \in k\{\{\tau\}\}$ such that $\phi_a(e(x)) = e(ax)$ for a given $a \in A$ with $\phi_a(\tau) = a_0 + \cdots + a_s\tau^s$ and $a = a_0 \neq 0,1$. The condition $\phi_a(e(x)) = e(ax)$ is equivalent to $\phi_a(e(a^{-1}x)) = e(x) = \sum_{0 \leqslant i} e_i x^{p^i}$. Equating coefficients of τ, we derive from the relation

$$\Big(\sum_i a_i\tau^i\Big)\Big(\sum_j e_j(\tau a^{-1})^j\Big) = \sum e_n\tau^n$$

the formula $e_n = \sum_{i+j=n} a_i e_j^{p^i} a^{-p^n}$, and hence the inductive definition for the coefficients e_n starting with $e_0 = 1$

$$e_n(1 - a^{1-p^n}) = \sum_{i=1}^{n} a_i e_{n-i}^{p^i} a^{-p^n} .$$

This shows both the existence and uniqueness of $e(\tau)$.

The second step is to show that for this $e(\tau)$, the relation $\phi_b(e(x)) = e(bx)$ holds for any $b \in A$. Consider

$$(\phi_b e b^{-1})(x) = (\phi_b(\phi_a e a^{-1})b^{-1})(x) = (\phi_a(\phi_b e b^{-1})a^{-1})(x) .$$

From the expression it follows that e and $\phi_b e b^{-1}$ both satisfy the relation $\phi_a e a^{-1} = e$, and from the uniqueness assertion in step one, it follows that $e_b e = eb$ for any $b \in A$.

Next we have to show that $e(x) = \sum_{0 \leqslant n} e_n x^{p^n}$ is an entire function. Assume in the inductive definition of $e(x)$ that $|a| > 1$ and write the above recurrence formula for e_n as

$$e_n(a^{p^n} - a) = \sum_{i=1}^{s} a_i e_{n-i}^{p^n} \quad \text{for } n \geqslant s .$$

Then $|a| \cdot |e_n|^{p-n} \leqslant \max_{1 \leqslant i \leqslant s} \{|a_i|^{p^{-n}} r_{n-i}\}$ where $r_j = |e_j|^{p^{-j}}$. For θ such that $1/|a| < \theta < 1$ and any $n \geqslant$ fixed n_0 , each term $|a_i|^{p^{-n}} \leqslant 1+\varepsilon$, and it follows that

$$r_n \leqslant \theta \cdot \max_{1 \leqslant i \leqslant s} r_{n-i} .$$

Since $\sum_{0 \leqslant n} r_n \leqslant \frac{1}{1-\theta} \max\{r_0,\ldots,r_s\}$, we see that $r_n = |e_n|^{p^{-n}} \longrightarrow 0$, which proves that $e(x)$ is an entire function.

Finally, we calculate ϕ_a for this $e(x)$ where Γ equals the subset of $x \in C_\infty$ with $e(x) = 0$. We have

$$\phi_a(x) = a \cdot x \prod_{\gamma \in a^{-1}\Gamma/\Gamma - \{0\}} \left(1 - \frac{x}{e(\gamma)}\right)$$

since $\phi_a(x) = 0$ on $a^{-1}\Gamma/\Gamma$. Thus the degree of ϕ_a equals $\mathrm{Card}(a^{-1}\Gamma/\Gamma) = \mathrm{Card}(\Gamma/a) = (\mathrm{Card}(A/a))^r$. This proves the theorem.

For our future considerations I-level structures will play a basic role where I is a nonzero ideal in A . This leads to the following definition.

(2.2) DEFINITION. Let Y be a projective A-module of rank r. For a nonzero ideal $I \subset A$ an I-level structure is an isomorphism $\alpha: Y/IY \to (A/I)^r$ of A-modules (or free A/I-modules).

An isomorphism $Y/IY \xrightarrow{u} (A/I)^r$ is equivalent to an isomorphism $I^{-1}Y/Y \xrightarrow{v} (I^{-1}/A)^r$ since $v = I^{-1} \otimes_A u$. In the case $Y = \Gamma$ an A-lattice of rank r and the affine A-module $E = \phi^\Gamma: A \to \mathrm{End}_{C_\infty}(G_a)$ corresponding to Γ by (2.1), we calculated $\ker(\phi_a^\Gamma) = a^{-1}\Gamma/\Gamma$ and thus the subfunctor $E_I = \bigcap_{a \in I} \ker(\phi_a) \subset E$ has the form

$$E_I(C_\infty) = \bigcap_{a \in I} a^{-1}\Gamma/\Gamma = I^{-1}\Gamma/\Gamma .$$

Thus we immediately deduce the following assertion.

(2.3) PROPOSITION. Using $E_I(C_\infty) = I^{-1}\Gamma/\Gamma$, we have that an I-level structure on the affine A-module $E = \phi^\Gamma$ is the same as on I-level structure on the projective A-module Γ.

§3. DISCRETE MODULES IN A VECTOR SPACE OVER A LOCAL FIELD

The following preliminaries are needed in the next section in order to parametrize A-lattices in C_∞ as homogeneous spaces.

(3.1) NOTATIONS. Let A be a discrete subring of a local field K with field of fractions $F \subset K$. We assume that K/A is a compact abelian group.

(3.2) EXAMPLES. (1) $A = \mathbb{Z} \subset F = \mathbb{Q} \subset K = \mathbb{R}$.

(2) $A = \mathbb{Z} + fR \subset K = \mathbb{Q}(\sqrt{-d})$ for $d > 0$, $d \subset \mathbb{Z}$ square free and $F \subset K = \mathbb{C}$.

(3) $A = \mathbb{F}_q[t] \subset F = \mathbb{F}_q(t) = \mathbb{F}_q(t^{-1}) \subset K = \mathbb{F}_q((t^{-1}))$.

(4) $A = \mathbb{F}_q[C-\infty] \subset F = \mathbb{F}_q(C-\infty) = \mathbb{F}_q(C) \subset K = F_\infty$ where C is an affine curve and F_∞ is the completion of F at ∞.

Of course (3) is a special case of (4), and (4) is the case of interest in this part.

For any finite dimensional vector space V over K, the topology is well defined and given by a norm. A subgroup H in V is <u>discrete</u> provided there exists a neighborhood N' of 0 in V with $N' \cap H = 0$. If $N + N \subset N'$, then $N + x \cap N + y = \emptyset$ for $x, y \in H$ if and only if $x = y$.

For an A-module H, the rank of H, denoted $rk_A H$, is $\dim_K(K \otimes_A H)$.

(3.3) PROPOSITION. Let H be a discrete A-module contained in a finite dimensional K-vector space V. Then

$$rk_A(H) \leqslant \dim_K(V) \quad .$$

PROOF. Let $x_1,\ldots,x_n$ denote a set of elements H forming a basis of the vector space $W = K \cdot H \subset V$. Then

$$L = Ax_1 \oplus \cdots \oplus Ax_n \subset H \subset W \subset V \quad .$$

Since H is discrete in V, there exists a neighborhood N' of $0 \in V$ with $H \cap N' = 0$ and a neighborhood N with $N + N \subset N'$. Now $N + L$ is a neighborhood of 0 in V/L intersecting H/L only at 0. Thus H/L is a discrete subgroup of V/L and of the compact abelian group

$$W/L = Kx_1 \oplus \cdots \oplus Kx_n / Ax_1 \oplus \cdots \oplus Ax_n \quad .$$

Thus H/L is finite and $n = \dim_F(F \otimes_A L) = \dim_F(F \otimes_A H) = \dim_K W$ which in turn implies $rk_A(H) = \dim_F(F \otimes_A H) \leqslant \dim_K V$. This proves the proposition.

(3.4) PROPOSITION. Let H be a projective A-submodule contained in a finite dimensional K-vector space V. Then H is discrete in V if and only if $\theta\colon K \otimes_A H \longrightarrow V$ is injective.

PROOF. Assume H is discrete in V. Then $H \subset \mathrm{im}(\theta) \subset V$ is discrete and so $rk_A(H) \leqslant \dim_K(\mathrm{im}(\theta))$ by (3.3). On the other hand, for any H, we have

$$rk_A(H) = \dim_K(K \otimes_A H) \geqslant \dim_K(\mathrm{im}(\theta)) \quad .$$

Hence $\dim_K(K \otimes_A H) = \dim_K(\mathrm{im}(\theta))$ and so θ is injective.

Conversely, assume that θ is injective. There exists H' with $H \oplus H'$ free, and a free module $A^n \subset K^n$ is a discrete A-submodule. Hence H is discrete in $K \otimes_A H$, and since θ is injective, H is discrete in V. This proves the proposition.

(3.5) REMARK. Let C be the completion of the algebraic closure of K. Then for every projective A-module P with $rk_A(P) \leqslant [C : K]$ there exists an A-monomorphism

$$P \longrightarrow C$$

such that P is discrete in C, i.e. by (3.4), the induced $K \otimes_A P \to C$ is injective.

In example (1), $2 = [C:K]$ and free modules P of rank ≤ 2 over $\mathbf{Z}$ can be embedded as discrete subgroups of $\mathbf{C} = C$. In examples (3) and (4), $[C_\infty : F_\infty] = \infty$ and all projective modules embed as A-lattices of C_∞.

§4. MODULI SPACES AS HOMOGENEOUS SPACES (LOCAL THEORY)

Now we return to the notations of the article where $F = \mathbb{F}_q(C)$, $A = \mathbb{F}_q[C-\infty]$, F_∞ completion of F at ∞, and C_∞ the completion of the algebraic closure $\bar{F}_\infty$ of F_∞. Let K denote F_∞ a local field as referred to in §3 where $A \subset K$ is discrete and K/A compact. We denote points in projective space $\mathbb{P}^{r-1}(C_\infty)$ by their homogeneous coordinates $y_1 : \cdots : y_r$. We begin by parametrizing F-vector subspaces or equivalently free A-lattices in C_∞.

(4.1) PROPOSITION. The function $f\colon C_\infty^* \backslash (\mathrm{Hom}_{F_\infty}(F_\infty^r, C_\infty) - \{0\}) \to \mathbb{P}^{r-1}(C_\infty)$ given by $f(u) = u(e_1) : \cdots : u(e_n)$ where $e_i = (0,\ldots,1,\ldots,0)$ is a bijection which restricts to a bijection

$$C_\infty^* \backslash \mathrm{Mon}_{F_\infty}(F_\infty^r, C_\infty) \longrightarrow \mathbb{P}_{r-1}(C_\infty) - \cup(F_\infty\text{-rational hyperplanes}) .$$

PROOF. A nonzero element u of $\mathrm{Hom}_{F_\infty}(F_\infty^r, C_\infty)$ is determined by its values $(u(e_1),\ldots,u(e_r))$ on the canonical basis elements $e_i \in F_\infty^r$. A dilation of u yields a dilation of the r-tuple $(u(e_1),\ldots,u(e_r)) = (x_1,\ldots,x_r)$. Next, $(a_1,\ldots,a_r) \in \ker(u) \subset F_\infty^r$ if and only if $\sum_{i=1}^d a_i x_i = 0$, i.e. for $(a_1,\ldots,a_r) \neq 0$, $(x_1,\ldots,x_r)$ is on the F_∞-rational hyperplane with equation

$$a_1x_1 + \cdots + a_rx_r = 0 .$$

This proves the proposition.

For an A-module Y projective of rank r, we have embeddings and isomorphism $Y \to F_\infty \otimes_A Y \simeq F_\infty^r$ and $GL_A(Y) \to GL_F(F_\infty \otimes_A Y) \simeq GL(r, F_\infty)$ coming from the tensor product. By taking the image of Y or $F_\infty \otimes_A Y$, and using (3.4), the following horizontal arrows are well defined functions.

(4.2) PROPOSITION. Let Y be a projective module of rank r over A. The following diagram is commutative and the horizontal arrows are bijections

$$\begin{array}{ccccccc}
\mathrm{Mon}_{F_\infty}\backslash(F_\infty\otimes_A Y, C_\infty)/GL_A(Y) & & \longrightarrow & & \left\{\begin{array}{l}A\text{-sublattices of } C_\infty\\ \text{isomorphic to } Y\end{array}\right\} & & \\
\downarrow & \searrow & & & \downarrow & \searrow & \\
 & & C_\infty^*\backslash\mathrm{Mon}_F(F\otimes_A Y, C_\infty)/GL_A(Y) & \longrightarrow & & & \left\{\begin{array}{l}A\text{-sublattices of } C_\infty\\ \text{isomorphic to } Y\\ \text{modulo dilations in } C_\infty^*\end{array}\right\} \\
\mathrm{Mon}_{F_\infty}\backslash(F_\infty\otimes_A Y, C_\infty)/GL(r,F) & & \longrightarrow & & \left\{\begin{array}{l}r\text{-dimensional}\\ F\text{-vector subspaces}\\ \text{of } C_\infty\end{array}\right\} & & \downarrow \\
 & \searrow & \downarrow & & & \searrow & \\
 & & C_\infty^*\backslash\mathrm{Mon}_{F_\infty}(F_\infty\otimes_A Y, C_\infty)/GL(r,F) & \longrightarrow & & & \left\{\begin{array}{l}r\text{-dimensional } F\text{-vector}\\ \text{subspaces of } C_\infty\\ \text{modulo dilations in } C_\infty^*\end{array}\right\}.
\end{array}$$

Now putting (4.1) and (4.2) together along with (2.1), we obtain the following assertion. Let P_A^r denote the set of isomorphism classes of projective A-modules of rank r.

(4.3) PROPOSITION. Using the previous notations, we have the following bijections

$$\begin{array}{ccc}
(\mathbb{P}_{r-1}(C_\infty) - \cup(F_\infty\text{-rational hyperplanes}))/GL_A(Y) & & \\
\uparrow & & \\
C_\infty^*\backslash\mathrm{Mon}_{F_\infty}(F_\infty\otimes_A Y, C_\infty)/GL_A(Y) & \longrightarrow & \left\{\begin{array}{l}A\text{-sublattices of } C_\infty\\ \text{isomorphic to } Y\\ \text{modulo dilations in } C_\infty^*\end{array}\right\}
\end{array},$$

and taking disjoint unions

$$\begin{array}{ccc}
\coprod_{Y\in P_A^r}(\mathbb{P}_{r-1}(C_\infty) - \cup(F_\infty\text{-rational hyperplanes}))/GL_A(Y) & & \\
\uparrow & & \\
\coprod_{Y\in P_A^r} C_\infty^*\backslash\mathrm{Mon}_{F_\infty}(F_\infty\otimes_A Y, C_\infty)/GL_A(Y) & \longrightarrow & \left\{\begin{array}{l}A\text{-sublattices of } C_\infty \text{ of rank } r\\ \text{modulo dilations in } C_\infty^*\end{array}\right\} \\
 & & \downarrow \\
 & & \left\{\begin{array}{l}\text{isomorphism classes of affine}\\ A\text{-modules } \phi_a \text{ over } C_\infty \text{ of}\\ \text{rank } r \text{ with } \partial\phi_a = a\end{array}\right\}.
\end{array}$$

(4.4) REMARK. In the case $r = 1$, $P_0(C_\infty)$ - (F_∞-rational hyperplanes) is a point, and (4.3) is a statement about bijections between sets with one point. We will be interested particularly in the case $r = 2$ in latter parts. If $a_1x_1 + a_2x_2 = 0$ where $a_1, a_2 \in F_\infty$, then $x_1 : x_2 = -a_2 : a_1 \in \mathbb{P}_1(F_\infty)$. Thus we have

$$\mathbb{P}_1(C_\infty) - (F_\infty\text{-rational lines}) = \mathbb{P}_1(C_\infty) - \mathbb{P}_1(F_\infty) \quad .$$

The next step is to modify (4.3) to include the case of A-lattices with I-level structure. For a projective A-module Y of rank r the projection $Y \to Y/IY$ includes a group morphism $GL_A(Y) \to GL_A(Y/IY)$ which has kernel $GL_A(Y,I)$. In the following two statements we have a restatement of (4.2) and a modification taking into account I-level structures. Fix an I-level structure

$$\alpha_0 \colon Y/IY \longrightarrow (A/I)^r \quad .$$

(1) Each monomorphism $u\colon F_\infty \otimes_A Y \to C_\infty$ determines a lattice $u(Y) \subset C_\infty$ and an I-level structure $\alpha_0((u|Y)^{-1}) \bmod I) = \alpha$ on $u(Y)$ where $u|Y\colon Y \to u(Y) \subset C_\infty$. Moreover, all A-sublattices of C_∞ , isomorphic to Y together with an I-level structure, come by this construction.

(2) Two monomorphisms $u, u'\colon F_\infty \otimes_A Y \to C_\infty$ determine the same lattice in C_∞ if and only if there exists $h \in GL_A(Y)$ with $u' = uh$. They determine the same lattice in C_∞ with I-level structure if and only if there exists $h \in GL_A(Y,I)$ with $u' = uh$.

This leads to the following modification of the previous proposition.

(4.5) THEOREM. Let I be a nonzero ideal in A , and use the previous notations. We have the following bijections:

$$\coprod_{Y\in P^r_A}(\mathbb{P}_{r-1}(C_\infty) - \cup(F_\infty\text{-rational hyperplanes}))/GL_A(Y,I)$$

$$\Big\uparrow \quad \alpha: I^{-1}Y/Y \to (I^{-1}/A)^r \text{ modulo Aut of } Y$$

$$\coprod_{Y\in P^r_A} C^*_\infty\backslash \mathrm{Mon}_{F_\infty}(F_\infty\otimes_A Y, C_\infty)/GL_A(Y,I) \xrightarrow{\sim} \left\{\begin{matrix}\text{A-sublattices of } C_\infty \\ \text{of rank } r \text{ with an I-level} \\ \text{structure modulo dilations} \\ \text{in } C^*_\infty\end{matrix}\right\}$$

$$\Big\downarrow \quad \alpha: I^{-1}Y/Y \to (I^{-1}/A)^r \text{ modulo Aut of } Y$$

$$\left\{\begin{matrix}\text{isomorphism classes of affine A-modules } \phi_a \\ \text{over } C_\infty \text{ with I-level structure and of} \\ \text{rank} = r \text{ with } \partial\phi_a = a\end{matrix}\right\}$$

or more briefly the bijection

$$\boxed{\coprod_{(\alpha,Y)\in P^r_A(I)} \Omega^r(C_\infty)/GL_A(Y,I) \xrightarrow{\sim} M^r_I(C_\infty)} \ .$$

Here $\Omega^r(C_\infty) = \mathbb{P}_{r-1} = \mathbb{P}_{r-1}(C_\infty) - \cup(F_\infty\text{-rational hyperplanes})$, also M^r_I is the moduli functor for affine A-modules rank r with I-level structure, and $P^r_A(I)$ is the set of ordered pairs (α,Y) with $Y\in P^r_A$ and $\alpha: I^{-1}Y/Y \to (I^{-1}/A)^r$ is an isomorphism.

§5. MODULI SPACES AS HOMOGENEOUS SPACES (ADÈLIC THEORY)

The basic bijection describing $M^r_I(C_\infty)$ in (4.4) involves only the local field at ∞. In this section we will describe $M^r_I(C_\infty)$ using all the primes of F together with $M^r_I(C_\infty)$. This adèlic description is closely related to the adèlic description of vector bundles on a curve which we recall now.

(5.1) REMARKS ON VECTOR BUNDLES. Over any ring space $X = (X,\mathcal{O}_X)$ a vector bundle E of rank n is a (locally free) sheaf on X locally isomorphic to $\mathcal{O}^n_X$. If X is a nonempty open set of the curve C over $\mathbb{F}_q$, for example $X = C - \{\infty\}$, then a vector bundle of rank n can be described as a family $(V,L_v)_{v\in X}$ where X is an n-dimensional vector space over $F = \mathbb{F}_q(C)$ and each L_v is a free module of rank n contained in V over the local ring $\mathcal{O}_{(v)}$ at v. We assume that there exists a basis $x_1,\dots,x_n$

of V over F and a finite set $S \subset X$ with

$$L_v = \mathcal{O}_{(v)} x_1 + \cdots + \mathcal{O}_{(v)} x_n$$

for all $v \in X - S$. The basis $x_1, \ldots, x_n$ is called a trivialization of V on $X - S$. If $y_1, \ldots, y_n$ is any other basis of V, then

$$y_j = \sum_i a_{i,j} x_i \qquad \text{and} \qquad x_i = \sum_j b_{j,i} y_j$$

where $(a_{ij}), (b_{ij}) \in GL(n, \mathcal{O}_{(v)})$ for all but a finite number of v. Hence every basis of V is a trivialization of V over some $X - T$ where $T \subset X$ is finite.

To an $\mathcal{O}_X$-sheaf E, locally isomorphic to $\mathcal{O}_X^n$, we assign $V = E_\eta$ where η is the general point of C. The $\mathcal{O}_{(v)}$-submodule $L_v \subset V$ is defined by trivializing $E|(X-S)$ on an open set $X-S$ around v with $x_1, \ldots, x_n \in V$ and requiring for all $w \in X - S$ with S finite that

$$L_w = \mathcal{O}_{(w)} x_1 + \cdots + \mathcal{O}_{(w)} x_n .$$

These lattices which come up in the above description of a vector bundle can be identified with certain homogeneous spaces. Let F_v denote the completion of F at $v \in X$ with valuation ring $\mathcal{O}_v$. Then using matrix transform between basis of a lattice in a vector space F^n or F_v^n, we have the following commutative diagram of bijections:

$$\begin{array}{ccc} GL(n,F)/GL(n,\mathcal{O}_{(v)}) & \xrightarrow{\sim} & \{\mathcal{O}_{(v)}\text{-lattices in } F^n\} \\ \downarrow \wr & & \downarrow \wr \\ GL(n,F_v)/GL(n,\mathcal{O}_v) & \xrightarrow{\sim} & \{\mathcal{O}_v\text{-lattices in } F_v^n\} \end{array} .$$

(5.2) <u>Homogeneous space description of trivialized vector bundles</u>

A vector bundle E with a trivialization is described by (F^n, L_v) for $v \in X$ where $L_v = \mathcal{O}_{(v)}^n \subset F^n$ for all $v \in X - S$ where S is finite. Thus the vector bundle trivialized on $X - S$ is determined by a finite set of lattices $L_v \subset F^n$ for $v \in S$. This leads to the following bijections:

$$\begin{array}{ccc}
\prod_{v\in S} GL(n,F_v)/GL(n,\mathcal{O}_v) & \xrightarrow{\ \sim\ } & \left\{\begin{array}{l}\text{trivialized over } X-S\\ \text{n-dimensional vector bundles on } X\\ \text{all up to isomorphism}\end{array}\right\} \\
\downarrow & & \\
Vec_S(X) = \prod_{v\in S} GL(n,F_v) \times \prod_{v\in X-S} GL(n,\mathcal{O}_v)\Big/\prod_{v\in X} GL(n,\mathcal{O}_v) & &
\end{array}$$

where the vertical arrow is a homeomorphism of locally compact spaces.

At points of S we can consider a Δ-level structure on the vector bundle E relative to a positive divisor Δ supported on S. This is just an isomorphism of the sheaf $E/E(-\Delta) = E \otimes_{\mathcal{O}} (\mathcal{O}/\mathcal{O}(-\Delta)) \rightarrow (\mathcal{O}/\mathcal{O}(-\Delta))^n$. In terms of the description of E with the data (F^n, L_v) this is an isomorphism $L_v/\pi^{s(v)}L_v \rightarrow (\mathcal{O}_{(v)}/\pi^{s(v)}\mathcal{O}_{(v)})^n$ where $s(v) = \mathrm{ord}_{(v)}(\Delta)$ and π_v is a local uniformizing parameter of $\mathcal{O}_{(v)}$ and hence $\mathcal{O}_v$. If we denote by

$$GL(n,\mathcal{O}_v,\Delta_v) = \ker(GL(n,\mathcal{O}_v) \rightarrow GL(n,\mathcal{O}_v/\pi_v^{s(v)}\mathcal{O}_v))$$

and $K(n,\Delta) = \prod_{v\in X} GL(n,\mathcal{O}_v,\Delta_v)$, then the previous diagram becomes modified as follows for $\mathrm{supp}(\Delta) \subset S$:

$$\begin{array}{ccc}
\prod_{v\ S} GL(n,F_v)/GL(n,\mathcal{O}_v,\Delta_v) & \xrightarrow{\ \sim\ } & \left\{\begin{array}{l}\text{trivialized over } X-S\\ \text{n-dimensional vector bundles}\\ \text{on } X \text{ with } \Delta\text{-level structure}\\ \text{all up to isomorphism}\end{array}\right\} \\
\uparrow\wr & & \\
Vec_S(X;\Delta) = \prod_{v\in S} GL(n,F_v) \times \prod_{v\in X-S} GL(n,\mathcal{O}_v)\Big/K(n,\Delta) & & .
\end{array}$$

Now we consider two special cases for X namely $X = C$ and $X = C - \{\infty\}$ and remove the condition that the vector bundle is trivialized.

(5.3) Vector bundles on C.

The product in the numerator of $Vec_S(X)$ or $Vec_S(X;\Delta)$ is one of the terms in the inductive limit which defines the adèle group $GL(n,\mathbb{A}_F)$ where

$$GL(n,\mathbb{A}_F) = \varinjlim_S \prod_{v\in S} GL(n,F_v) \times \prod_{v\in C-S} GL(n,\mathcal{O}_v) .$$

Hence considering vector bundles with Δ-level structure and trivialization over some open set $C-S$ of C where $\mathrm{Supp}(\Delta) \subset S$, we obtain the bijection

$$GL(n,\mathbb{A}_F)/K(n,\Delta) \longrightarrow \left\{\begin{array}{l}\text{vector bundles of dimension } n\\ \text{with a } \Delta\text{-level structure and}\\ \text{trivialization off the } \mathrm{Supp}(\Delta)\\ \text{all up to isomorphism}\end{array}\right\} .$$

Since the various trivializations are all related by the action of $GL(n,F)$ on basis, we have the following quotient of the previous bijection

$$\boxed{GL(n,F)\backslash GL(n,\mathbb{A}_F)/K(n,\Delta) \longrightarrow \left\{\begin{array}{l}\text{isomorphism classes of}\\ n\text{-dimensional vector bundles}\\ \text{on } C \text{ with } \Delta\text{-level structure}\end{array}\right\}} .$$

(5.4) Vector bundles on $C-\{\infty\}$.

The product in the numerator of $\mathrm{Vec}_S(X)$ or $\mathrm{Vec}_S(X;\Delta)$ for $X = C - \{\infty\}$ is one of the terms in the $\varinjlim$ which defines the finite adèle group $GL(n,\mathbb{A}_F^f)$ where $\mathbb{A}_F^f = \hat{A}\otimes_A F$ and $\hat{A} = \varprojlim_J A/J$, the limit being taken over ideals $J \subset A$. A divisor Δ can be described as an ideal $I \subset A$, and the group $K(n,\Delta)$ is given by

$$K(n,\Delta) = \ker\left(GL(n,\hat{A}) \to GL(n,\hat{A}/\hat{A}I)\right)$$

which we also denote by $GL(n,\hat{A},I)$ to avoid confusion with the previous case discussed in (5.3). As before we obtain in the injective limit the bijection

$$GL(n,\mathbb{A}_F^f)/GL(n,\hat{A},I) \xrightarrow{\sim} \left\{\begin{array}{l}\text{vector bundles of dimension } n \text{ on}\\ C-\{\infty\} \text{ with a } \Delta\text{-level structure}\\ \text{and trivialization off the } \mathrm{Supp}(\Delta)\\ \text{all up to isomorphism}\end{array}\right\} .$$

On the affine $C - \{\infty\}$ with coordinate ring $A = H^0(C-\{\infty\},\mathcal{O})$ a vector bundle of dimension n is just a projective module Y of rank n over A.

Now we combine the previous bijection with

$$C_\infty^*\backslash \mathrm{Mon}_{F_\infty}(F_\infty^r, C_\infty) = \Omega^r(C_\infty) = \mathbb{P}_{r-1}(F_\infty) - \cup\{F_\infty\text{-rational hyperplanes}\} .$$

We have a bijection

$$(GL(r,\mathbb{A}_F^f)/GL(r,\hat{A},I)) \times \Omega^r(C_\infty)$$
$$\downarrow\wr$$
$$\left\{\begin{array}{l}\text{vector bundles } Y \text{ on } C-\infty \text{ of}\\ \text{rank } r \text{ with a trivialization}\\ \text{and a } \Delta\text{-level structure all up}\\ \text{to isomorphism}\end{array}\right\} \times \left\{\begin{array}{l}\text{monomorphisms } F_\infty^r \to C_\infty\\ \text{over } F_\infty \text{ up to dilations in}\\ C_\infty^*\end{array}\right\} .$$

The trivialization of Y is a basis of $Y \otimes_A F$ over F which defines an embedding $Y \longrightarrow F_\infty^r$ as a discrete A-submodule. When this embedding is composed with a monomorphism, $F_\infty^r \longrightarrow C_\infty$ yields an A-sublattice of C_∞ of rank r with an I-level structure modulo dilations in C_∞^*. Hence by (4.5) we have a map

$$(GL(r,\mathbf{A}_F^f)/GL(r,\hat{A},I)) \times \Omega^r(C_\infty) \longrightarrow M_I^r(C_\infty)$$

where the fibres correspond to the various bases of $Y \otimes_A F$ over F. Hence factoring out by the action of GL(r,F) yields the adèlic description of $M_I^r(C_\infty)$.

(5.6) THEOREM. The above map induces a bijection

$$GL(r,F)\backslash GL(r,\mathbf{A}_F^f) \times \Omega^r(C_\infty)/GL(r,\hat{A},I) \xrightarrow{\sim} M_I^r(C_\infty) \quad .$$

(5.7) REMARK. The quotient space descriptions in (4.5) and (5.6) generalize to any open compact subgroup $H \subset GL(r,A)$ to give the C_∞-valued points $M_H^r(C_\infty)$ of a moduli scheme M_H:

$$M_H^r(C_\infty) = GL(r,F)\backslash GL(r,\mathbf{A}_F^f) \times \Omega^r(C_\infty)/H \qquad \text{adèlic version}$$

$$= \coprod_{xH \in GL(r,\mathbf{A}_F^f)/H} \Omega^r(C_\infty)/(xHx^{-1} \cap GL(r,F)) \qquad \text{local version} \quad .$$

The moduli scheme M_H^r arises from H-level structures on Drinfel'd modules.

CHAPTER 3. THE RIGID ANALYTIC MODULI SPACE

Now our aim is to describe the rigid analytic structure on the C_∞-valued points of the moduli space. These points were parametrized by a quotient of $\Omega^r(C_\infty)$ by a discrete group in 2(4.5). The rigid analytic structure on the moduli space is a quotient of the rigid analytic structure on $\Omega^r(C_\infty)$.

In order to define the rigid analytic structure on $\Omega^r(C_\infty)$, we make use of a natural mapping

$$\lambda: \Omega^r(C_\infty) \longrightarrow I(F_\infty^r)_{\mathbf{Q}} \subset I(F_\infty^r)_{\mathbb{R}}$$

onto the rational points $I(F_\infty^r)_{\mathbf{Q}}$ of the geometric realization $I(F_\infty^r)_{\mathbb{R}}$ of the building $I(F_\infty^r)$ of the group $PLG(r)$ over the local field F_∞. The admissible open sets of the rigid analytic structure are inverse images by λ^{-1} of certain open neighborhoods of skeltons in $I(F_\infty^r)_{\mathbb{R}}$.

In this discussion we indicate how both $\Omega^r(C_\infty)$ and the building $I(F_\infty^r)$ are p-adic analogues of real symmetric spaces.

§1. NORMS ON VECTOR SPACES OVER A LOCAL FIELD

(1.1) NOTATIONS. Let K be a local field with valuation ring R, maximal ideal $p = R\pi$, and $q = \mathrm{Card}(k)$ where $k = R/R\pi$. We normalize $|a|$ on K as $|a| = q^{-\mathrm{ord}(a)}$ so that $|\pi| = q^{-1}$ and $|K^\times| = q^{\mathbf{Z}}$. We consider finite dimensional vector spaces over K, and thus the closed unit ball and unit sphere are compact. If V is of dimension m over K, then a lattice is any R-submodule M free of rank m.

(1.2) DEFINITION. A norm on a vector space V over K is a function $\alpha: V \to \mathbb{R}$ satisfying:

(a) $\alpha(x) \geqslant 0$ and $\alpha(x) = 0$ if and only if $x = 0$.

(b) $\alpha(ax) = |a|\alpha(x)$ for $a \in K$ and $x \in V$.

(c) $\alpha(x+y) \leqslant \sup(\alpha(x),\alpha(y))$ for $x,y \in V$.

A norm is called integral provided $\alpha(V) = |K| = \{0\} \cup q^{\mathbf{Z}}$ and rational provided $\alpha(V) \subset \mathbf{Q}$.

If $\alpha(x) < \alpha(y)$, then axiom (c) can be strengthened to the equality $\alpha(x+y) = \alpha(y) = \sup(\alpha(x),\alpha(y))$.

If α is a norm and $t > 0$, then the dilation $t\alpha$ is a norm, and we denote by $N(V)$ the set of dilation classes of norms on V .

(1.3) EXAMPLE (I). For a basis $x_0,\dots,x_m$ of V and real numbers $r_0 > 0 ,\dots, r_m > 0$ the function

$$\alpha(a_0x_0 + \cdots + a_mx_m) = \sup(r_0|a_0| ,\dots, r_m|a_m|)$$

is a norm on V . In fact, every norm can be described by this formula. This is proved by induction on m by considering a nonzero linear form $\lambda: V \to F$ and choosing $x_0 \neq 0$ in V such that $x \mapsto |\lambda(x)|/\alpha(x)$ defined on the compact projective space $\mathbb{P}(V)$ takes its maximum at x_0 . Then $V = Fx_0 \oplus \ker(\lambda)$ for which on $\ker(\lambda)$ the inductive hypothesis applies. See Goldman and Iwahori for the details.

(1.4) EXAMPLE (II). The norm α_M associated to a lattice M in V is given by the formula $\alpha_M(x) = \inf\{1/|a| : ax \in M\}$.

Observe that M is the unit α_M-ball in V of all $x \in V$ with $\alpha_M(x) \leq 1$, and for $x \neq 0$ we have $\alpha_M(x) = q^{-m}$ where $p^m = \{a \in F: ax \in M\}$. For two lattices L and M the inclusion $L \subset M$ holds if and only if $\alpha_L(x) \geq \alpha_M(x)$ for all $x \in V$.

For nonzero $c \in F$ we have $\alpha_{cM}(x) = (1/|c|)\alpha_M(x)$ for all $x \in V$, and in particular, $\alpha_{\pi M}(x) = q\alpha_M(x)$. Finally, if $x_0,\dots,x_m$ is a basis of a lattice M over R , then

$$\alpha_M(a_0x_0 + \cdots + a_mx_m) = \sup\{|a_0| ,\dots, |a_m|\} .$$

Thus this example is a special case of (1.4).

Going back to $\alpha(a_0x_0 + \cdots + a_mx_m) = \sup(r_0|a_0| ,\dots, r_m|a_m|)$, we see that replacing x_i by cx_i for $c \neq 0$ replaces r_i by $r_i/|c|$. In particular, x_i replaced by $\pi^s x_i$ leads to r_i replaced by $q^s r_i$. Hence, by rescaling the basis vectors, we can always require the constants r_i to be in an interval of the form $[r,qr)$ or $(r,qr]$ for some given $r > 0$. So when $\alpha(V) = \{0\} \cup q^{\mathbf{Z}}$, we can choose the x_i with each $r_i = 1$, and this gives the next proposition.

(1.5) PROPOSITION. The following are equivalent for $t > 0$ and a norm α on a vector space V :

(1) $\alpha = t \cdot \alpha_M$ for some lattice M ,

(2) $\alpha(V) = |F| \cdot t$, and

(3) $\alpha(a_0x_0 + \cdots + a_mx_m) = t \cdot \sup(|a_0|, \ldots, |a_m|)$ for some basis $x_0, \ldots, x_m$ of V .

Now we wish to study to what extent the representation

$$\alpha(a_0x_0 + \cdots + a_mx_m) = \sup(r_0|a_0|, \ldots, r_m|a_m|)$$

for a norm α is unique by renormalizing and reordering so that

$$q \geqslant r_0 \geqslant r_1 \geqslant \cdots \geqslant r_{m-1} \geqslant r_m \geqslant 1 .$$

Then the basis elements are unique up to multiplication by $\pi^{\pm 1}$ and cyclic permutation. The requirement that $r_m > 1$ removes this ambiquity, and further after a dilation of α we can assume that $r_0 = q$. The set of values $\alpha(V) = \{0\} \cup q^{\mathbf{Z}}r_0 \cup \cdots \cup q^{\mathbf{Z}}r_m$, and thus the numbers r_i are uniquely determined by α . They make up the set $\alpha(V) \cap (1,q]$ when we require $r_m > 1$. With these notations, consider the lattices L_i for $i = m, m-1, \ldots, 0$ where

$$L_i = \begin{cases} Rx_0 + \cdots + Rx_i + R\pi^{-1}x_{i+1} + \cdots + R\pi^{-1}x_m , & \text{for } r_i > r_{i+1} \\ L_{i+1} , & \text{for } r_i = r_{i+1} , \end{cases}$$

and $L_m = Rx_0 + \cdots + Rx_m$.

(1.6) LEMMA. The lattice L_i is the open ball $B(0,qr_i)$.

PROOF. For $x = \sum a_jx_j \in V$ clearly $x \in L_i$ if and only if $|a_j| \leqslant 1$ if $j \leqslant i$ and $|a_j| \leqslant q$ if $j > i$, or equivalently, $r_j|a_j| \leqslant r_j$ for $j \leqslant i$ and $r_j|a_j| \leqslant qr_j$ for $j > i$. Now assume that $r_i > r_{i+1}$, and we see that both inequalities are equivalent to $r_j|a_j| < r_iq$. Hence, $x \in L_i$ if and only if $\alpha(x) < qr_i$. If $r_i = r_{i+1}$, then $L_i = L_{i+1}$, and so the result holds by induction from m to 0 .

Finally, we obtain the following structure theorem.

(1.7) THEOREM. Let α be a norm on a vector space V with $\alpha(V) \cap (1,q] = S$, and for $r \in S$, let $L(r) = B(0,qr)$ an open ball. Then $\dim V \geqslant \mathrm{Card}(S)$ and $\alpha = \sup_{r\in S}(r\alpha_{L(r)})$.

PROOF. As in (1.3), we can choose a basis $x_0,\dots,x_m$ of V with

$$\alpha(a_0,x_0 + \cdots + a_m x_m) = \sup(r_0|a_0|,\dots,r_m|a_m|)$$

and $q \geqslant r_0 \geqslant r_1 \geqslant \cdots \geqslant r_m \geqslant 1$. The r_i exhaust the set S with possible repetitions and thus $\dim V \geqslant \mathrm{Card}(S)$.

By (1.6) the lattice $L_i = L(r_i)$ for $r_i > r_{i+1}$ is given by $L_i = Rx_0 + \cdots + Rx_i + R\pi^{-1}x_{i+1} + \cdots + R\pi^{-1}x_m$. The norm $\alpha_i = \alpha_{L(r_i)}$ is given by

$$\alpha_i(a_0x_0 + \cdots + a_m x_m) = \sup(|a_0|,\dots,|a_i|, q^{-1}|a_{i+1}|,\dots,q^{-1}|a_m|) ,$$

and since $rq^{-1}|a_{i+1}| \leqslant r_{i+1}|a_{i+1}|$, it follows that

$$\alpha(x) = \sup(r_0\alpha_0(x),\dots,r_m\alpha_m(x)) = \sup_{r\in S}(r\cdot\alpha_{L(r)}(x)) .$$

This proves the theorem.

§2. THE BUILDING FOR PGL(V) OVER A LOCAL FIELD

We continue with the notations (1.1) in this section. The dilation (or homothety) class $\{L\}$ of a lattice L is the set of all lattices λL where $\lambda \in F^{x}$ in the vector space V. Observe that $\{L\}$ is the set of all $\pi^{i}L$ for i in the integers.

Let $PGL(V)$ denote $GL(V)/(\text{scalars})$ for a vector space V.

(2.1) DEFINITION. Let V be a vector space over the local field K. The building $I(V)$ for the group $PGL(V)$ over K is the simplicial complex whose vertices are dilation classes $\{L\}$ of lattices L in V, and whose simplexes $\{v_0,\dots,v_n\}$ are sets of vertices where after reordering $v_i = \{L_i\}$ with $L_0 > L_1 > \cdots > L_n > \pi L_0$.

Observe that the ordering of the vertices $v_0,\dots,v_n$ such that representatives $L_i \in v_i$ can be chosen with $L_0 > \cdots > L_n > \pi L_0$ is unique up to the action of the cyclic group on $n+1$ elements inside the symmetric group.

The simplicial complex $I(V)$ has dimension equal to $\dim V - 1$, and each simplex is contained in a top dimensional simplex (one whose dimension equals

$\dim I(V)$). For $\dim V = 2$, the building $I(V)$ is one dimensional, a graph. In fact, $I(V)$ is simply connected and hence a tree. Each vertex $v = \{L\}$ is contained in $(m+1)$ 1-simplexes $\{v,v'\}$ corresponding to the $m+1$ lattices L' with $L > L' > \pi L$.

This building $I(V)$ is a special case of the buildings (immeubles) which have been associated to general semisimple groups over a local field by Bruhat and Tits. These simplicial complexes are contractible, and the vertices are in natural bijective correspondence with the cosets of $PGL(V)/$(maximal compact subgroup K). Here K is the image of $GL(L)$ in $PGL(V)$ for a lattice L of V. This description together with other features suggest that $I(V)$ is an analogue of the symmetric space for real Lie groups.

Now recall some generalities on geometric realizations as applied to $I(V)$. The geometric realization $I(V)_{\mathbb{R}}$ is the subset of $(t_v) \in \prod_{v \in I(V)} [0,1]$ such that $\{v: t_v \neq 0\}$ is a simplex of $I(V)$ and $\sum_v t_v = 1$. For each simplex $\sigma = \{v_0,\dots,v_n\}$ of $I(V)$ its geometric realization $|\sigma| \subset I(V)_{\mathbb{R}}$ is the subset of $(t_v) \in I(V)_{\mathbb{R}}$ with $t_v = 0$ for $v \notin \sigma$. As a subset of $\mathbb{R}^{n+1} \supset |\sigma|$ is compact, we give $I(V)$ the inductive (weak) topology where M is closed in $I(V)_{\mathbb{R}}$ if and only if $M \cap |\sigma|$ is closed in $|\sigma|$ for each simplex σ of $I(V)$. We will make use of the dense subset $I(V)_{\mathbb{Q}} \subset I(V)_{\mathbb{R}}$ consisting of (t_v) with each $t_v \in \mathbb{Q}$. The set $I(V)_{\mathbb{Z}} \subset I(V)_{\mathbb{Q}}$ of (t_v) with each $t_v \in \mathbb{Z}$ can be identified with the set of vertices of $I(V)$.

The function which links the considerations of this section with those of the previous section is $\theta: I(V)_{\mathbb{R}} \to N(V)$ from the geometric realization of the building to the dilation classes of norms on V as follows: Let $\sigma = \{v_0,\dots,v_n\}$ be a simplex and $t = (t_v) \in |\sigma|$ be a point. We can choose an ordering $\sigma = (v_0,\dots,v_n)$ such that $t_n > 0$ and lattices $L_i \in v_i$ with $L_0 > L_1 > \cdots > L_n > \pi L_0$. Then $\theta(t) = \alpha$ where $\alpha = \sup\left(q^{t_i+\cdots+t_n}\cdot\alpha_{L_i}\right)$.

(2.2) THEOREM. The function $\theta: I(V)_{\mathbb{R}} \to N(V)$ is a well-defined bijection which carries the vertices $I(V)_{\mathbb{Z}}$ onto the set of classes containing integral norms and $I(V)_{\mathbb{Q}}$ onto the set of classes containing rational norms.

PROOF. The function θ is a bijection by (1.10), the structure theorem for norms. In that theorem we proved that each norm α has a unique representation $\alpha = \sup(r_0\alpha_{L_0},\dots,r_m\alpha_{L_m})$ where $q = r_0 \geq r_1 \geq \cdots \geq r_m > 1$ up to dilation. Let $r_i = q^{t_i+\cdots+t_m}$ or $t_i = \log_q(r_i/r_{i+1})$. Then $(t_0,\dots,t_m)$ determines the unique point in $|(\{L_0,\dots,L_m\})|$ which maps to α under θ. The remaining statements are clear from the formulas relating the t_i's and

and r_i's . This proves the theorem.

§3. METRIC ON THE BUILDING

We continue with the notations of (1.1) in this section.

(3.1) DEFINITION. Let V be a vector space over the local field K. For two norms α,β on V, we define the distance $\rho(\alpha,\beta)$ between α and β by the following equivalent formulas:

$$\begin{aligned}\rho(\alpha,\beta) &= \log_q\Big(\sup_{x\in V,\, x\neq 0} \alpha(x)/\beta(x)\Big) + \log_q\Big(\sup_{x\in V,\, x\neq 0} \beta(x)/\alpha(x)\Big)\\ &= \log_q\Big(\sup_{x\in V,\, x\neq 0} \alpha(x)/\beta(x)\Big) - \log_q\Big(\inf_{x\in V,\, x\neq 0} \alpha(x)/\beta(x)\Big)\end{aligned}$$

(3.2) REMARK. From the first formula for $\rho(\alpha,\beta)$ we see that $\rho(\alpha,\beta) = \rho(\beta,\alpha)$, and from the second form it follows that $\rho(\alpha,\beta) \geqslant 0$, and $\rho(\alpha,\beta) = 0$ if and only if $\beta = t\cdot\alpha$ for some $t > 0$. Moreover, for $t_1 > 0$ and $t_2 > 0$ we have $\rho(t_1\alpha,t_2\beta) = \rho(\alpha,\beta)$, and thus

$$\rho(\{\alpha\},\{\beta\}) = \rho(\alpha,\beta)$$

is well defined on dilation classes of norms on V. Finally, it is easy to check the triangle inequality

$$\rho(\alpha,\gamma) \leqslant \rho(\alpha,\beta) + \rho(\beta,\gamma)$$

using the relations of the form

$$\sup_{x\in V,\, x\neq 0} \alpha(x)/\beta(x) = \sup_{x\in V,\, x\neq 0} \alpha(x)/\beta(x)\cdot \sup_{x\in V,\, x\neq 0} \beta(x)/\gamma(x) .$$

Hence ρ is a metric on the space $N(V)$ of dilation classes of norms on V.

Note from the definition of ρ that if $t_1\alpha \leqslant \beta \leqslant t_2\alpha$ on V, then $\rho(\alpha,\beta) \leqslant \log_q(t_1/t_2)$.

(3.3) EXAMPLE. Let M be a lattice in V. Then the integral norm α_M is defined as in (1.4), and

$$M = \{x \in V\colon \alpha_M(x) \leqslant 1\} \quad\text{and}\quad \pi M = \{x \in V\colon \alpha_M(x) < 1\} .$$

Thus $M - \pi M$ is the unit α_M-sphere of all x with $\alpha_M(x) = 1$ in V. Moreover, since every $x \in V$, $x \neq 0$, is proportional to some x' with

$\alpha_M(x') = 1$, we deduce that

$$\rho(\alpha,\alpha_M) = \log_q\Big(\sup_{x\in M-\pi M}\alpha(x)\Big) - \log_q\Big(\inf_{x\in M-\pi M}\alpha(x)\Big) .$$

For two lattices M and N in V , there exists a natural number r with $M \supset \pi^r N$ and $N \supset \pi^r M$ so that $\alpha_M \leqslant q^r\alpha_N$ and $\alpha_N \leqslant q^r\alpha_M$. We obtain $\rho(\alpha_M,\alpha_N) \leqslant 2r$. For a more precise calculation, we reorganize the hypothesis in the next proposition.

(3.4) PROPOSITION. Let $M \supset N \supset \pi^r M$ be lattices in V where $\pi M \not\supset N \not\supset \pi^{r-1}M$. Then $r = \rho(\alpha_M,\alpha_N)$.

PROOF. From $\alpha_M \leqslant \alpha_N \leqslant q^r\alpha_M$ we deduce that $\rho(\alpha_M,\alpha_N) \leqslant r$. Since $\pi M \not\supset N$, we have $x \in (M-\pi M) \cap N$ and therefore $\alpha_N(x) \leqslant 1$. Thus $-\log_q(\inf_{x\in M-\pi M}\alpha_N(x)) \geqslant 0$. Since $\pi^{-r+1}N \not\supset M$ and $\pi^{-r+1}N \supset \pi M$, there exists $x \in M-\pi M$ with $x \in \pi^{-r+1}N$ and so $\alpha_N(x) \geqslant q^r$. Thus $\log_q(\sup_{x\in M-\pi M}\alpha_N(x)) \geqslant r$. By the example (3.3) it follows that $\rho(\alpha_N,\alpha_M) \geqslant r$ and hence $\rho(\alpha_N,\alpha_M) = r$.

(3.5) COROLLARY. A set of lattices $M_0,\ldots,M_r$, or equivalently, a set of integral norms $\alpha_0,\ldots,\alpha_r$ (for example $\alpha_i = \alpha_{M_i}$) , determine a simplex in the building $I(V)_{\mathbf{Z}}$ or $N(V)$ if and only if $\rho(\alpha_i,\alpha_j) = 1$ for $i \neq j$.

PROOF. Arrange, after reordering and dilation, the lattices as follows: $M_0 \supset M_1 \supset \cdots \supset M_{r-1} \supset M_r \supset \pi^i M_0$ with $M_r \not\supset \pi^{i-1}M_0$. Then $1 = \rho(\alpha_{M_0},\alpha_{M_r}) \geqslant i$ by (3.4), so $i = 1$, and the classes $\{M_i\}$ form a simplex. Conversely, $\rho(\alpha_i,\alpha_j) = 1$ for $i \neq j$ when the classes determine a simplex again by (3.4).

§4. THE MAPPING FROM THE p-ADIC SYMMETRIC SPACE TO THE BUILDING

Now we return to the basic situation of the function field $F = \mathbb{F}_q(C)$ of the smooth curve $C/\mathbb{F}_q$, the local field F_∞ at ∞ on the curve, and C_∞ the completion of the algebraic closure of F_∞ .

The simple critical observation is the following: For $z = (z_1,\ldots,z_r) \in C_\infty^r$ the function on F_∞^r

$$a = (a_j) \longmapsto a_z(a) = |z_1a_1 + \cdots + z_ra_r|$$

is a norm on the F_∞-vector space F_∞^r provided $|z_1a_1 + \cdots + z_ra_r| = 0$ implies that $a = (a_1,\ldots,a_r) = 0$. This is the case exactly for

$$z \in C_\infty^r - \{\text{all } F_\infty\text{-rational hyperplanes}\} \quad .$$

Further, for $c \in C_\infty$ and $z \in C_\infty^r$ the relation $\alpha_{cz} = |c|\alpha_z$, which is a dilation of norms, holds and this leads to the following definition.

(4.1) DEFINITION. The building map λ defined on the p-adic symmetric space $\Omega^r(C_\infty)$ to the building $N(F_\infty^r) = I(F_\infty^r)_{\mathbb{R}}$ of the group $PGL(r,F_\infty)$ is given by

$$z = (z_j) \in \Omega^r \longmapsto \lambda(z) = \text{dilation class of } \alpha_z \quad .$$

For a representative r-tuple $(z_1,\ldots,z_r) = z$ of $z \in \Omega^r$, we represent $\lambda(z)$ as the norm α_z, i.e.

$$\lambda(z)(a_1,\ldots,a_r) = |z_1a_1 + \cdots + z_ra_r| \quad .$$

Note, since $|C_\infty^\times| = q^{\mathbb{Q}}$, in fact $\lambda\colon \Omega^r(C_\infty) \longrightarrow I(F_\infty^r)_{\mathbf{Q}}$.

(4.2) PROPOSITION. The building map $\lambda\colon \Omega^r(C_\infty) \to I(F_\infty^r)_{\mathbf{Q}}$ is $GL(r,F_\infty)$-equivariant for right actions. In particular, it is also $GL(r,F)$-equivariant.

PROOF. For $s \in GL(r,F_\infty)$ we view the matrix as acting on the left and a^ts the action on the right. The norm $\lambda(z)$ acted on by s on the right is $\lambda(z)s$, or for $a \in F_\infty^r$, it is $\lambda(z)(s(a)) = |\langle z|sa\rangle| = |\langle z^ts|a\rangle| = \lambda(z^ta)(a)$. Thus $\lambda(z)s = \lambda(z^ts)$ which proves the proposition.

For a subgroup $\Gamma \subset GL(r,F_\infty)$ we have a quotient building mapping

$$\lambda_\Gamma\colon \Omega^r(C_\infty)/\Gamma \longrightarrow I(F_\infty^r)_{\mathbf{Q}}/\Gamma \subset I(F_\infty^r)_{\mathbb{R}}/\Gamma \quad .$$

In the case Γ is a certain subgroup of $GL(r,A)$ this is a mapping of the corresponding moduli space associated with Γ to a quotient of the building by the discrete group Γ.

(4.3) REMARKS. The building map λ is useful for several purposes. First, the sets λ^{-1} (ball around a vertex or simplex) can be used to describe the rigid analytic structure on the p-adic symmetric space. In the special case $r = 2$ so that $I(F_\infty^2)$ is a tree T, we will describe a topological model for $\Omega^2(C_\infty) = C_\infty - F_\infty$ and the map $\lambda\colon C_\infty - F_\infty \longrightarrow T$. This is used to calculate the cohomology of $\Omega^2(C_\infty)/\Gamma$. There are some coverings of T/Γ which induce back to admissible covers of $\Omega^2(C_\infty)/\Gamma$, and these give rise to a spectral sequence which could be thought of as the Leray spectral sequence of the map $\lambda_\Gamma\colon \Omega^2(C_\infty)/\Gamma \longrightarrow T/\Gamma$. In this case $H^*(T/\Gamma)$ is just the cohomology of the group Γ.

In order to illustrate further what is involved in defining the rigid analytic structure, we must calculate $\rho(\lambda(z),\alpha_\Lambda)$ where Λ is the standard lattice $\mathcal{O}_\infty^r \subset F_\infty^r$. Since any other lattice is Λ up to the action of $GL(r,F_\infty)$ and since both ρ and λ are $GL(r,F_\infty)$-invariant, this calculation leads to $\rho(\lambda(z),\alpha_L)$ for any vertex α_L.

By (3.3) we have

$$\rho(\lambda(z),\alpha_\Lambda) = \log_q\Big(\sup_{a\in S(\Lambda)} |a_1z_1+\cdots+a_rz_r|\Big) - \log_q\Big(\inf_{a\in S(\Lambda)} |a_1z_1+\cdots+a_rz_r|\Big)$$

where $S(\Lambda) = \Lambda - \pi\Lambda$, the set of all $(a_1,\ldots,a_r) \in F_\infty^r$ with all $|a_i| \leqslant 1$ and at least one $|a_i| = 1$.

Thus we see that $\rho(\lambda(z),\alpha_\Lambda) \leqslant s$ if and only if for all $a,b \in S(\Lambda)$

$$\frac{1}{q^s} \leqslant \frac{|a_1z_1+\cdots+a_rz_r|}{|b_1z_1+\cdots+b_rz_r|} \leqslant q^r .$$

In the case $|z_1| > |z_2| > \cdots > |z_r|$ with all ratios $|z_i|/|z_j| \notin q^{\mathbf{Z}}$ for $i = j$, we have

$$\rho(\lambda(z),\alpha_\Lambda) = \log_q(|z_1|/|z_r|)$$

by an easy straightforward calculation. When some of the ratios $|z_i|/|z_j| \in q^{\mathbf{Z}}$ the calculation of $\rho(\lambda(z),\alpha_\Lambda)$ is more complicated, and in fact, it is at the basis for the structure of $\Omega^r(C_\infty)$ as a rigid analytic space.

§5. FILTRATION OF THE 1-DIMENSIONAL p-ADIC SYMMETRIC SPACE

In this section we study the case $r = 2$ in detail where $\Omega^2(C_\infty) = \mathbb{P}_1(C_\infty) - \mathbb{P}_1(F_\infty) = C_\infty - F_\infty$. We use the notation Ω for $C_\infty - F_\infty = \Omega^2(C_\infty)$ where $u \in \Omega$ is $u \in C_\infty$ and $T = I(F_\infty^2)$ is the building which in this case is a tree (a contractible 1-dimensional simplicial complex). The building map $\lambda\colon \Omega \longrightarrow T$ is given by

$$\lambda(u)(a,b) = |a + bu| \quad ,$$

and we wish to study

$$\rho(\lambda(u),\alpha_\Lambda) = \log_q \frac{\sup_{(a,b)\in S(\Lambda)}|a+bu|}{\inf_{(a,b)\in S(\Lambda)}|a+bu|} = \rho(\lambda(1/u),\alpha_\Lambda)$$

in terms of congruence properties of u relative to elements of F_∞. This is formulated by using the following notion.

(5.1) DEFINITION. For $u \in C_\infty$ we define the irrational absolute value of u to be $|u|_{ir} = \inf_{a\in F_\infty}|u-a|$.

Observe that this is just the distance from u to $F_\infty \subset C_\infty$. The following properties are easily deduced from the definition.

(5.2) PROPOSITION. The irrational norm satisfies the following:

(1) For $u \in C_\infty$, $|u|_{ir} = 0$ if and only if $u \in F_\infty$.

(2) For $u \in C_\infty$ and $c \in F_\infty$ we have $|cu|_{ir} = |c|\cdot|u|_{ir}$.

(3) For $|u| \notin q^{\mathbb{Z}}$ and $u \in C_\infty$ it follows that $|u|_{ir} = |u|$.

(4) For $|u| = 1$ with residue class $\bar{u} \in \overline{\mathbb{F}}_q$ the irrational norm $|u|_{ir} = |u| = 1$ if and only if $\bar{u} \in \overline{\mathbb{F}}_q - \mathbb{F}_q$.

From the previous section we see for $|u| \notin q^{\mathbb{Z}}$ that $\rho(\lambda(u),\alpha_\Lambda) = \log_q(\max(|u|,1/|u|))$ which for $|u| < 1$ becomes simply $\rho(\lambda(u),\alpha_\Lambda) = -\log_q|u|$.

(5.3) PROPOSITION. The distance from $\lambda(u)$ to the standard vertex α_Λ in the tree T is given by

$$\rho(\lambda(u),\alpha_\Lambda) = \begin{cases} -\log_q|u|_{ir} & \text{for } |u| \leq 1 \\ -\log_q|1/u|_{ir} & \text{for } |u| \geq 1 \ . \end{cases}$$

PROOF. For the case $|u| \leq 1$, we have $\sup_{(a,b)\in S(\Lambda)}|a+bu| = 1$ since $|a+ub| \leq \max(|a|,|b|\cdot|u|) \leq 1$ and $|1+0\cdot u| = 1$. Further, $\inf_{(a,b)\in S(\Lambda)}|a+bu| = \inf_{|a|\leq 1\ ,\ |b|=1}|a+bu| = \inf_{|e|\leq 1}|e+u| = |u|_{ir}$ since $|u| \leq 1$. Hence, we have

$$\rho(\lambda(u),\alpha_\Lambda) = \log_q\ (1/|u|_{ir}) = -\log_q|u|_{ir} \quad .$$

For the case $|u| \geqslant 1$ observe that $\rho(\lambda(u),\alpha_\Lambda) = \rho(\lambda(1/u),\alpha_\Lambda) = -\log_q|1/u|$ by the first part. This proves the proposition.

Now we are in a position to describe the inverse image in Ω of closed balls around the standard lattice vertex $* = \alpha_\Lambda$. First, some notation for residue class reduction $r\colon \mathcal{O}_{C_\infty} \longrightarrow \overline{\mathbb{F}}_q$ or $r\colon \mathbb{P}_1(C_\infty) \longrightarrow \mathbb{P}_1(\overline{\mathbb{F}}_q)$. We choose a cross section $s\colon \mathbb{P}_1(\overline{\mathbb{F}}_q) \longrightarrow \mathbb{P}_1(C_\infty)$ with $s(0) = 0$ and $s(\infty) = \infty$ such that rs is the identity on $\mathbb{P}_1(\overline{\mathbb{F}}_q)$.

(1) Clearly $u \in \lambda^{-1}(*)$ from (2.2) if and only if $|u|_{ir} = 1$. This gives the following relations for $\lambda^{-1}(*) \subset \mathcal{O}^\times_{C_\infty}$,

$$\begin{aligned}
\lambda^{-1}(*) &= r^{-1}(\overline{\mathbb{F}}_q^\times - \mathbb{F}_q^\times) \\
&= \{u \in C_\infty\colon |u| = 1\} - \bigcup_{\xi \in \mathbb{F}_q^\times} \{u \in C : |u - s(\xi)| < 1\} \\
&= \mathcal{O}_{C_\infty} - \bigcup_{\xi \in \mathbb{F}_q} B(s(\xi),1) \\
&= \mathbb{P}_1(C_\infty) - \bigcup_{\eta \in \mathbb{P}_1(\mathbb{F}_q)} B(s(\eta),c)
\end{aligned}$$

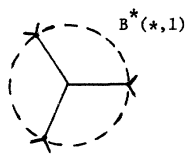
where $B(v,c)$ is the open ball of radius c around v in C_∞ for $v \neq \infty$ and $B(\infty,c) = \{\infty\} \cup \{u \in C\colon |u| > 1/c\} \subset \mathbb{P}_1(C_\infty)$.

(2) For the closed ball $B^*(*,c)$ with $0 \leqslant c < 1$ observe that $B^*(*,c) \subset$ open star of the vertex $* = \alpha_\Lambda$ in the tree T. In this case

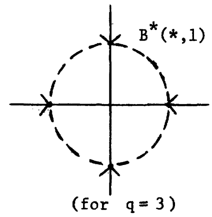
$$\lambda^{-1}(B^*(*,c)) = \mathbb{P}_1(C_\infty) - \bigcup_{\eta \in \mathbb{P}_1(\mathbb{F}_q)} B(s(\eta),q^{-c}) ,$$

and again the inverse image is $\mathbb{P}_1(C_\infty)$ minus $(q+1)$ balls, but this time of slightly smaller radius $q^{-c} \leqslant 1$. Now we see the relation between these balls and the edges of the tree coming out from the vertex $* = \alpha_\Lambda$. All the points of $B(s(\eta),1) - B(s(\eta),q^{-c})$ project to the edge corresponding to $\eta \in \mathbb{P}_1(C_\infty)$.

(3) As the radius c of the closed ball $B^*(*,c)$ approached 1 in the tree, we come to $(q+1)$ new vertices each with q new edges coming out as c increases in $1 < c < 2$. In the building we have the pictures



(for $q=2$)



(for $q=3$)

In terms of $\lambda^{-1}(B^*(*,c))$ this means that as c increases to 1 and through $[1,2)$, that $B(s(\eta),q^{-c})$ will decrease in size and at the moment $c = 1$, it will split into q balls of radius q^c for $c \in [1,2)$ each parametrized by $\mathbb{F}_q$ around the points $s(\eta) + s(\xi_1)$ where $(\eta,\xi_1) \in \mathbb{P}_1(\mathbb{F}_q) \times \mathbb{F}_q$ so that for $c \in [1,2)$

$$\lambda^{-1}(B^*(*,c)) = \bigcup_{(\eta,\xi_1) \in \mathbb{P}_1(\mathbb{F}_q) \times \mathbb{F}_q} B(s(\eta) + s(\xi_1)\pi, q^{-c})$$

Note for $c < 1$ that $B(s(\eta),q^{-c}) = B(s(\eta) + s(\xi_1)\pi, q^{-c})$.

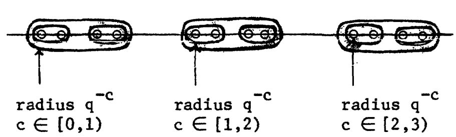
The general result follows by the same considerations as above and proved using $\rho(\lambda(u),*) = -\log_q|u_{ir}|$ for $|u| \leq 1$ and the relation between $|u|$ and $|u|_{ir}$.

(5.4) THEOREM. With the above notations associated with $\lambda\colon \Omega^2(C_\infty) \to T$ we have, for $c \in [m,m+1)$ where m is an integer > 0, the following

$$\lambda^{-1}(B(*,c)) = \mathbb{P}_1(C_\infty) - \bigcup_{\xi \in \mathbb{P}_1(\mathbb{F}_q) \times \mathbb{F}_q^m} B(s(\xi_0) + \cdots + s(\xi_m)\pi^m, q^{-c})$$

where $\xi = (\xi_0,\xi_1,\ldots,\xi_m)$ with $\xi_0 \in \mathbb{P}_1(\mathbb{F}_q)$ and $\xi_i \in \mathbb{F}_q$ for $1 \leq i \leq m$.

Thus for $m \leq c < m+1$ it is the projective line minus $(q+1)q^m$ balls of radius q^{-c}. We have the following picture of $\mathbb{P}_1(C_\infty)$ - {these balls} for $q = 2$:



(5.5) REMARK. Each $\lambda^{-1}(B(*,c))$ has the natural structure of a rigid analytic space. In this way the increasing union of the $\lambda^{-1}(B(*,c))$, which is $\mathbb{P}_1(C_\infty) - \mathbb{P}_1(F_\infty)$ has the structure of a rigid analytic space.

(5.6) REMARK. We have the following intuitive picture of the "topological" space $\mathbb{P}_1(C_\infty) - \mathbb{P}_1(F_\infty)$. For each vertex of the tree T we take a copy of

$$\mathbb{P}_1(C_\infty) - \{q+1 \text{ open discs indexed by the edges with that vertex}\}$$

and for each edge we take a copy of an annulus

$$\mathbb{P}_1(C_\infty) - \{2 \text{ open discs}\} \quad .$$

Now for each edge we glue the two boundary circles of the associated annulus onto the two boundary circles in the spaces

$$\mathbb{P}_1(C_\infty) - \{q + 1 \text{ open discs}\}$$

associated to the vertices of the edge respectively. The result is similar to the boundary of a tubular neighborhood of the tree T embedded in Euclidean space.

CHAPTER 4. COHOMOLOGY OF THE MODULI SPACE

The aim of this chapter is to calculate the étale cohomology of the moduli spaces M_I^2. This is done in two steps. First, we describe the rigid analytic cohomology of $\Omega^2(C_\infty)$ in terms of coclosed cochains on the building, which in this case is a tree. This consists in determining the rigid cohomology of $\mathbb{P}_1$ minus a finite union of discs and then using a patching argument over the edges of the tree with a compatibility condition at each vertex. The étale cohomology of the moduli space, which by comparison is isomorphic to the rigid analytic cohomology $H^1(\Omega^2(C_\infty)/\Gamma)$ of the analytic moduli space, is the middle term of a short exact sequence

$$0 \longrightarrow H^1(\Gamma) \longrightarrow H^1(\Omega^2(C_\infty)/\Gamma) \longrightarrow H^2(\Omega^2(C_\infty))^\Gamma \longrightarrow 0$$

where Γ is the discrete subgroup of $GL(2,F)$ corresponding to the I-level structure. Finally, the action of the inertia subgroup of $Gal(F_{\infty,s}/F_\infty)$, provides an isomorphism of $H^2(\Omega^2(C_\infty))^\Gamma$ onto $H^1(\Gamma)$.

§1. GENERALITIES ON THE COHOMOLOGY OF RIGID ANALYTIC SPACES

For a complete nonarchmedian field K with separable algebraic closure K_s we make the following definitions of $H^i = H^i_{rigid}$ for a rigid analytic space X over K and $i = 0,1$. We do this using the étale cohomology groups for the coefficients $\mathbf{Z}/n$ and μ_n where n^{-1} is in K.

(1.1) DEFINITION OF H^0. In both cases of coefficients $H^0(X,\mathbf{Z}/n) = H^0_{et}(X_s,\mathbf{Z}/n)$ and $H^0(X,\mu_n) = H^0_{et}(X_s,\mu_n)$ where $X_s = X \otimes_K K_s$.

Recall that $H^1_{et}(X,\mu_n)$ can be described as pairs (L,ϕ), up to an evident isomorphism, of an invertible sheaf L on X and an isomorphism $\phi\colon \mathcal{O}_X \xrightarrow{\sim} L^{n\otimes}$ with the group structure given by tensor product. Further $H^1_{et}(X_s,\mu_n) = \varinjlim_L H^1_{et}(X \otimes L,\mu_n)$ for $K \subset L \subset K_s$ and $[L:K]$ finite. These definitions have meaning for rigid analytic spaces.

(1.2) DEFINITION OF H^1. The group $H^1(X,\mu_n)$ is the group under tensor product of pairs (L,ϕ), up to isomorphism, of an invertible sheaf L on X and an isomorphism $\phi: \mathcal{O}_X \xrightarrow{\sim} L^{n\otimes}$. Moreover, $H^1(X_s,\mu_n) = \varinjlim_{K'} H^1(X\otimes K',\mu_n)$ where $K \subset K' \subset K_s$ and $[K':K]$ is finite.

Both H^0 and H^1 are clearly functors under rigid morphisms.

This cohomology has the following properties which we state without any complete proofs.

(1.3) PROPOSITION. If $X = W_{an}$ for W a projective variety or an affine curve over K, then $H^i(W_s) = H^i(X_s)$ for $i = 0,1$, and coefficients μ_n and $\mathbf{Z}/n$.

This follows from the GAGA-type theorems of Kiehl. The projective comparison theorem implies the affine curve comparison theorem since a covering of $0 < |z| < r$ of order n prime to p extends to a ramified covering of $|z| < r$.

(1.4) PROPOSITION. If $f: X \to Y$ is a finite étale morphism of rigid analytic spaces with Galois group C of order prime to n, then $H^1(Y,\mu_n) \to H^1(X,\mu_n)^G$ is an isomorphism.

This is easy from the definitions.

(1.5) PROPOSITION (Kummer sequence). We have an exact sequence over a rigid analytic space:

$$0 \longrightarrow H^0(\mu_n) \longrightarrow H^0(\mathcal{O}_X^*) \xrightarrow{n} H^0(\mathcal{O}_X^*) \longrightarrow H^1(\mu_n) \longrightarrow H^1(\mathcal{O}_X^*) \xrightarrow{n} H^1(\mathcal{O}_X^*) .$$

(1.6) PROPOSITION. Let $\{X_i\}_{i\in I}$ be an admissible open covering of a rigid analytic space X with nerve of dimension $\leqslant 1$, then there is an exact sequence

$$0 \longrightarrow H^0(X,\mu_n) \longrightarrow \prod_i H^0(X_i,\mu_n) \longrightarrow \prod_{i\neq j} H^0(X_i\cap X_j,\mu_n) \longrightarrow$$
$$\longrightarrow H^1(X,\mu_n) \longrightarrow \prod_i H^1(X_i,\mu_n) \longrightarrow \prod_{i\neq j} H^1(X_i\cap X_j,\mu_n) .$$

§2. COHOMOLOGY OF $\Omega^2(C_\infty)$

Using the generalities of the previous section and some specific information, we are able to make the following calculation.

(2.1) PROPOSITION. Let $D_0,\dots,D_m$ be $m+1$ pairwise disjoint open discs in $\mathbb{P}_1(C_\infty)$ of radius in $|C_\infty|$. Then for the rigid analytic space $X = \mathbb{P}_1(C_\infty) - (D_0 \cup \cdots \cup D_m)$ we have for n prime to p :

(a) $H^0(X_s,\mathbf{Z}/n) = \mathbf{Z}/n$

(b) $H^1(X_s,\mu_n) = (\mathbf{Z}/n)^m$

for $X_s = X \otimes_{F_\infty} F_{\infty,s}$.

PROOF. (a) This is a question of connectedness of discs in $\mathbb{P}_1(C_\infty)$ over any finite extension of F_∞ . Since $X \cup D_1 \cup \cdots \cup D_m = \mathbb{P}_1(C_\infty) - D_0$ is a disc and hence connected, and since the spherical boundaries of D_i are connected, it follows that X is connected.

(b) For this, first observe that the rational fuctions without poles on X are dense in $H^0(\mathcal{O}_X)$. Next, if f is a rational function without zeros or poles on X , then there exists $c \in F_\infty$ with $\sup_{x\in X}|cf(x)-1| < 1$, and hence, $cf = g^n$ for some $g \in H^0(\mathcal{O}_X^*)$. For S_i equal to the boundary of D_i , we have an isomorphism

$$H^1(X,\mu_n)/\mathrm{im}(F_\infty^*/(F_\infty^*)^n) \longrightarrow \bigoplus_{i=1}^{m} H^1(S_i,\mu_n)/\mathrm{im}(F_\infty^*/(F_\infty^*)^n)$$

since $H^1(\mathcal{O}_X^*) = 0$ in the Kummer sequence where every divisor on X is principal. Now S_i is a special case of the more general X , and $H^1(S_i,\mu_n)/\mathrm{im}(F_\infty^*/(F_\infty^*)^n) \simeq \mathbf{Z}/n$. This reduces to the assertion that z^j is not an <u>nth</u> power for $n \nmid j$ which follows by writing $z^j = f^n$, $\|f\| = 1$, and reducing modulo the maximal ideal. Hence we have the short exact sequence

$$0 \longrightarrow F_\infty^*/(F_\infty^*)^n \longrightarrow H^1(X,\mu_n) \longrightarrow (\mathbf{Z}/n)^m \longrightarrow 0 \quad .$$

Now pass to the separable algebraic closure of F_∞ through finite extensions to see that $H^1(X_s,\mu_n) = (\mathbf{Z}/n)^m$. This proves the proposition.

Now we can describe the cohomology of $\Omega = \Omega^2(C_\infty)$ using the following notion.

(2.2) DEFINITION. Let M be an abelian group, and let B be a graph (1-dimensional simplicial complex) with B_e the set of oriented edges. The group of 1-cochains $C^1(B,M)$ is the subgroup of $c \in \prod_{e \in B_e} M$ such that $c(-e) = -c(e)$, and the group of coclosed (harmonic) 1-cochains $\underline{H}^1(B,M)$ is the subgroup of $c \in C^1(B,M)$ such that $\sum_{e \in e(b)} c(e) = 0$ where the sum is over $e(b)$ the set of edges ending at b, and this holds for each vertex b of B.

The group of 1-chains $C_1(B,M)$ is the quotient group of $\coprod_{e \in B_e} M$ divided by the subgroup generated by $e - (-e)$, and the group of coclosed 1-chains $\underline{H}_1(B,M)$ is the quotient of $C_1(B,M)$ by the subgroup generated by $\sum_{e \in e(b)} c(e)$ for each vertex b of B.

(2.3) PROPOSITION. We have isomorphisms

$$H^0(\Omega^2(C_\infty)_s, \mathbf{Z}/n) \xrightarrow{\sim} \mathbf{Z}/n \quad \text{and} \quad H^1(\Omega^2(C_\infty)_s, \mu_n) \xrightarrow{\sim} \underline{H}^1(T, \mathbf{Z}/n)$$

which are compatible with the action of $GL(2,F_\infty)$ and $Gal(F_{\infty,s}/F_\infty)$. Here $\lambda\colon \Omega^2(C_\infty) \to T$ onto the tree where $GL(2,F_\infty)$ acts, and $Gal(F_{\infty,s}/F_\infty)$ acts trivially which induces an action on $\underline{H}^1(T,\mathbf{Z}/n)$.

PROOF. We write $X = \bigcup_{i \in I} X_i$ where I is the set of vertices and edges of T. If i is a vertex v, then X_i is $\lambda^{-1}(B^*(v,1/3))$, and if i is an edge e, then X_i is $\lambda^{-1}(e^*)$ where $e^* = e - \bigcup_v B(v,1/4)$ with the union taken over all vertices of T. The nerve of this covering has vertices I and is $sk(T)$ the first barycentric subdivision of T. Let $I_0 \subset I$ be the subset of $i \in I$ corresponding to the edges, choose isomorphisms $H^1(X_i,\mu_n) \to \mathbf{Z}/n$ by (2.1), and choose orientations for each $i \in I_0$.

Then the composite

$$H^1(X,\mu_n) \longrightarrow \prod_{i \in I} H^1(X_i,\mu_n) \longrightarrow \prod_{i \in I_0} H^1(X_i,\mu_n) \longrightarrow \prod_{i \in I_0} \mathbf{Z}/n$$

is seen, with (1.6), to be an isomorphism $H^1(X,\mu_n) \to \underline{H}^1(T,\mathbf{Z}/n)$. It is injective by (1.6), and the image set corresponds exactly to those functions, extended to be alternating on all ordered edges, satisfying the condition of being coclosed. This proves the proposition.

§3. A SIMPLE TOPOLOGICAL MODEL

To illustrate further the patching technique used in the previous proposition (2.3) and the ideas which go into the calculation of the cohomology of $\Omega^2(C_\infty)/\Gamma$, we consider maps of surfaces onto a graph.

(3.1) DEFINITION. A map $f\colon X \to B$ from a surface (real oriented 2-manifold) onto a graph is called regular provided f is proper and

(1) f^{-1}(open 1-simplex) = S^2 - (2 disjoint closed discs) ,

(2) f^{-1}(open star of a vertex) = S^2 - ($\nu(b)$ disjoint closed discs)

where $\nu(b)$ is the number of 1-simplexes of B incident to b .

Observe that the projection f of the boundary X of a tubular neighborhood N of $B \subset \mathbb{R}^3$ is an example of such a regular map $f\colon X \to B$. The boundary ∂X of X is the disjoint union of $f^{-1}(b)$ where $\nu(b) = 1$.

Now a cross section $s\colon B \to X$ of $f\colon X \to B$ always exists and leads to two split short exact sequences

$$0 \longrightarrow \ker H_1(f) \longrightarrow H_1(X) \xrightarrow{H_1(f)} H_1(B) \longrightarrow 0$$

$$0 \longrightarrow H^1(B) \xrightarrow{H^1(f)} H^1(X) \longrightarrow \operatorname{coker} H^1(f) \longrightarrow 0 \quad .$$

The terms $\ker H_1(f) = E^2_{0,1} = H_0(B,\tilde{H}_1(F))$ and $\operatorname{coker} H^1(f) = E_2^{0,1} = H^0(B,\tilde{H}^1(F))$ are part of the spectral sequence of the map $f\colon X \to B$ and $\tilde{H}_1(F)$ and $\tilde{H}^1(F)$ are systems of coefficients on B whose structure is clear from axioms (1) and (2) .

(3.2) REMARK. Following the argument of the proof of (2.3), we have short exact sequences using the groups $\underline{H}_1(B,\mathbf{Z})$ and $\underline{H}^1(B,\mathbf{Z})$ introduced in (2.2)

$$0 \longrightarrow \underline{H}_1(B,\mathbf{Z}) \longrightarrow H_1(X) \xrightarrow{H_1(f)} H_1(B) \longrightarrow 0$$

and

$$0 \longrightarrow H^1(B) \xrightarrow{H^1(f)} H^1(X) \longrightarrow \underline{H}^1(B,\mathbf{Z}) \longrightarrow 0 \quad .$$

For each edge $e \in B_e$ of B , let $X_e = f^{-1}$(open e) , and observe that

$$\coprod_{e\in B_e} H_1(X_e) \longrightarrow H_1(X) \xrightarrow{H_1(f)} H_1(B) \longrightarrow 0$$

and

$$0 \longrightarrow H^1(B) \xrightarrow{H^1(f)} H^1(X) \longrightarrow \prod_{e\in B_e} H^1(X_e)$$

are exact sequences. We choose generators of $H_1(X_e)$ and $H^1(X_e)$ such that the generator for $H_1(X_e)$ and for $H_1(X_{-e})$ are negatives of each other in $H_1(X)$.

In the case of the previous section B was a tree so that $H_1(B) = 0$ and $H^1(B) = 0$, and thus $\underline{H}_1(B,\mathbf{Z}) \xrightarrow{\sim} H_1(X)$ and $H^1(X) \xrightarrow{\sim} \underline{H}^1(B,\mathbf{Z})$ are isomorphisms.

(3.3) REMARK. For a regular map $f\colon X \longrightarrow B$ of a closed surface onto a finite graph we have $\operatorname{rk} H_1(X) = 2\cdot\operatorname{rk} H_1(B)$ and the homology group $H_1(B)$ is isomorphic to $\ker H_1(f) \xrightarrow{\sim} \underline{H}_1(B,\mathbf{Z})$. The same statement holds in cohomology.

This assertion follows from the fact that the symplectic homology pairing $x\cdot y$ on $H_1(X)$ is nonsingular, but $x\cdot y$ restricts to zero on either of the direct summands $\ker H_1(f)$ or $H_1(B) = \operatorname{im} H_1(s)$ for a section s of f . These isotropic submodules are then maximal isotropic and thus of the same rank.

(3.4) REMARK. Let $f\colon X \longrightarrow B$ be a regular map of the surface X onto a graph B which has a finite subgraph B_0 such that $B - B_0$ is the disjoint union of m half lines $L_1,\ldots,L_m$. Then each $f^{-1}(L_i)$ is a topological punctured disc, and X is a closed surface with m points deleted. We define the cuspidal cohomology $H^1_!$ of B and X by

$$H^1_!(B) = \ker(H^1(B) \longrightarrow H^1(B - B_0))$$

and

$$H^1_!(X) = \ker(H^1(X) \longrightarrow H^1(X - f^{-1}(B_0))) \quad .$$

The cohomology exact sequence in (3.2) becomes

$$0 \longrightarrow H^1_!(B) \longrightarrow H^1_!(X) \longrightarrow \underline{H}^1_!(B,\mathbf{Z}) \longrightarrow 0$$

where $\underline{H}^1_!(B,\mathbf{Z})$ is the subgroup of $\underline{H}^1(B,\mathbf{Z})$ consisting of $c = (c_e)_{e\in B_e}$ with $c_e = 0$ for e in $B - B_0$. Again $\operatorname{rk} H^1_!(B) = \operatorname{rk}\underline{H}^1_!(B,\mathbf{Z})$, and $H^1_!(B)$ and $\underline{H}^1_!(B,\mathbf{Z})$ are isomorphic.

§4. COHOMOLOGY OF THE MODULI SPACE WITH FIXED LEVEL STRUCTURE

With the map $\Omega^2(C_\infty) \to T$ onto the tree, we were able to analyze the cohomology of $\Omega^2(C_\infty)$ in (2.3). The simple topological model is contained in (3.2). Now we consider a subgroup Γ of $GL(2,A)$ and study the cohomology of $\Gamma\backslash\Omega^2(C_\infty)$ using $\Gamma\backslash\Omega^2(C_\infty) \to \Gamma\backslash T$, the mod Γ building map.

The Leray spectral sequence of this map is the covering space spectral sequence whose terms in lowest degree yield the exact sequence with coefficients in $\mathbb{Z}/n$ or μ_n

$$0 \longrightarrow H^1(\Gamma\backslash T) \longrightarrow H^1(\Gamma\backslash\Omega^2(C_\infty)) \longrightarrow H^1(\Omega^2(C_\infty))^\Gamma \longrightarrow H^2(\Gamma\backslash T) \quad .$$

The H^1 groups for the graph $\Gamma\backslash T$ and the rigid analytic space $\Gamma\backslash\Omega^2(C_\infty)$ are defined by coverings and the building map induces coverings.

Under the hypothesis that Γ acts on T with stabilizer subgroups of points only p-groups (p prime to n), the groups $H^*(\Gamma\backslash T) = H^*(\Gamma)$ are just the cohomology groups of Γ with coefficients in μ_n or $\mathbb{Z}/n$. Further, since T is a tree, we have $H^2(\Gamma) = 0$. We obtain a short exact sequence whose last term $H^1(\Omega^2(C_\infty))^\Gamma$ is isomorphic, by (2.3), to $\underline{H}^1(\Gamma\backslash T)^\Gamma$ the group of Γ-invariant coclosed cochains on T with values in μ_n^{-1} or $\mathbb{Z}/n$. This gives the following result.

(4.1) PROPOSITION. For a congruence subgroup $\Gamma \subset GL(2,A)$ of the ideal $I \subset A$, which acts on the building T with p-groups for stabilizer subgroups, the mod Γ building map yields the cohomology exact sequence

$$0 \longrightarrow H^1(\Gamma) \longrightarrow H^1(\Gamma\backslash\Omega^2(C_\infty)) \longrightarrow \underline{H}^1(T)^\Gamma \otimes \mu_n^{-1} \longrightarrow 0$$

or with coefficients in $\mathbb{Z}/n$

$$0 \longrightarrow H^1(\Gamma\backslash T,\mathbb{Z}/n) \longrightarrow H^1(M_I^2 \otimes F_{\infty,s},\mathbb{Z}/n) \longrightarrow \underline{H}^1(T,\mathbb{Z}/n)^\Gamma \otimes \mu_n^{-1} \longrightarrow 0 \quad .$$

In order to obtain a useful cohomology, we have to look at a compactification of $\Gamma\backslash\Omega^2(C_\infty)$ or $M_I^2(C_\infty)$, or equivalently a neighborhood of infinity. As in the previous section the cuspidal cohomology $H_!^1(M_I^2 \otimes F_{\infty,s},\mathbb{Z}/n)$ is the subgroup of H^1 consisting of classes equal to zero on some neighborhood of infinity, or equivalently, of classes with image in $\underline{H}_!^1(T,\mathbb{Z}/n)_!^\Gamma \otimes \mu_n^{-1}$, i.e. coclosed Γ-invariant cochains with support compact modulo Γ.

(4.2) REMARK. Let $I_0 \subset Gal(F_{\infty,s}/F_\infty)$ be the inertia subgroup. In the short exact sequence of (4.1), I_0 acts trivially on the subgroup and quotient group (the associated graded group). Then $(\sigma,x) \mapsto \sigma x - x$ defines a map

$I_0 \times H^1(M_I^2 \otimes F_{\infty,s}, \mathbf{Z}/n) \to H^1(\Gamma\backslash T, \mathbf{Z}/n)$ which is bilinear and $\mathrm{Gal}(F_{\infty,s}/F_\infty)$-equivariant. Identifying μ_n with I_0/I_0^n as $\mathrm{Gal}(F_{\infty,s}/F_\infty)$-modules, we factor this map through

$$\mu_n \otimes \underline{H}^1(T, \mathbf{Z}/n)^\Gamma \otimes \mu_n^{-1} \longrightarrow H^1(\Gamma\backslash T, \mathbf{Z}/n) \quad .$$

(4.3) PROPOSITION. Under the assumptions of (4.1) the short exact sequence of (4.1) restricts to

$$0 \longrightarrow H^1(\Gamma\backslash T, \mathbf{Z}/n) \longrightarrow H^1_!(M_I^2 \otimes F_{\infty,s}, \mathbf{Z}/n) \longrightarrow \underline{H}^1_!(T, \mathbf{Z}/n)^\Gamma \otimes \mu_n^{-1} \longrightarrow 0 \quad .$$

In the limit over $n = \ell^i$ the map of (4.2) induces

$$\underline{H}^1(T, \mathbf{Z}_\ell)^\Gamma_! \longrightarrow H^1(\Gamma\backslash T, \mathbf{Z}_\ell) \quad .$$

This map is an isomorphism after tensoring with $\mathbf{Q}_\ell$.

This proposition is the analogue of (3.4). Finally, we incorporate the isomorphism into the exact sequence to obtain a description of $H^1_!$ of the level I-moduli space as follows with the next definition.

(4.4) DEFINITION. The special representation $\mathrm{sp}_{\mathrm{Gal}}$ is the two-dimensional representation of $\mathrm{Gal}(F_{\infty,s}/F_\infty)$ through $\hat{\mathbf{Z}} \ltimes \mathbf{Z}_\ell(1)$ generated by $(\phi,0)$ and $(0,u)$ with $\phi u \phi^{-1} = u^{q_\infty}$. This representation is given by

$$(\phi,0) \longmapsto \begin{pmatrix} 1 & 0 \\ 0 & q_\infty^{-1} \end{pmatrix} \qquad \text{and} \qquad (0,u) \longmapsto \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad .$$

This two-dimensional representation has an invariant one-dimensional subspace, and it allows us to give another interpretation of (4.3).

(4.5) THEOREM. Under the assumptions of (4.3) we have an isomorphism of $\mathrm{Gal}(F_{\infty,s}/F_\infty)$-modules

$$H^1_!(M_I^2 \otimes F_{\infty,s}, \mathbf{Q}_\ell) \simeq \underline{H}^1(T, \mathbf{Q}_\ell)^\Gamma_! \otimes \mathrm{sp}_{\mathrm{Gal}} \quad .$$

In the next chapter, we intepret this isomorphism in terms of automorphic forms on the adèle group.

The proofs of the related results on abelian varieties can be found in [SGA 7, p. 20, exposé I].

CHAPTER 5. APPLICATIONS TO RELATIONS BETWEEN AUTOMORPHIC FORMS AND GALOIS REPRESENTATIONS

In this chapter we calculate the limit cohomology $\underline{H} = \varinjlim_H H^1(M_H^2, \bar{Q}_\ell)$ as a representation of $GL(2,\mathbb{A}_F) \times Gal(\bar{F}/F)$. This representation decomposes into a direct sum of $\pi \otimes \sigma(\pi)$ where $\pi = \pi' \otimes sp_{GL(2,F_\infty)}$ and $\sigma(\pi)$ is a two-dimensional Galois representation with $\sigma(\pi)|D_\infty = sp_{Gal}$.

This defines a certain map $\pi \mapsto \sigma(\pi)$ which is given in §2 and studied further in §3 with the congruence formula for the Frobenius action on the two-dimensional Galois representation. In the last section we sketch the proof of the local Langlands' conjecture in characteristic p for GL(2). This is a proof of a local result using a global theorem.

§1. COCLOSED 1-COCHAINS AND THE SPECIAL REPRESENTATION

For any $X \in \mathbb{P}_1(F_\infty)$ we denote the quotient linear map by $\gamma_X: F_\infty^2 \to F_\infty^2/X$, and observe that if L is a lattice in F_∞^2, then $\gamma_X(L)$ is a lattice in F_∞^2/X. For an ordered 1-simplex $\vec{a} = (L_0 > L_1)$ in $I(F_\infty^2) = T$, the opposite simplex $-\vec{a}$ is represented by $(L_1 > \pi L_0)$, and $\pi\gamma_X(L_0) = \gamma_X(\pi L_0) \subset \gamma_X(L_1) \subset \gamma_X(L_0)$ are lattices in the 1-dimensional space F_∞^2/X so that either $\gamma_X(\pi L_0) = \gamma_X(L_1)$ or $\gamma_X(L_1) = \gamma_X(L_0)$.

(1.1) NOTATIONS. For an ordered 1-simplex $\vec{a}$ of the tree $T = I(F_\infty^2)$, let $P(\vec{a})$ denote the subset of $X \in \mathbb{P}_1(F_\infty)$ with $\gamma_X(L_1) = \gamma_X(L_0)$ where $\vec{a} = (L_0 > L_1)$.

From the above remark we see that $\mathbb{P}_1(F_\infty) = P(\vec{a}) \cup P(-\vec{a})$ is a partition of the projective line. If $\vec{a}_0, \ldots, \vec{a}_s$ are all the ordered 1-simplexes issuing from a vertex $\{L_0\}$, then $\mathbb{P}_1(F_\infty) = P(\vec{a}_0) \sqcup \cdots \sqcup P(\vec{a}_s)$ is also a partition of the projective line. Thus $P(-\vec{a}_0) = P(\vec{a}_1) \sqcup \cdots \sqcup P(\vec{a}_s)$ is a partition of any $P(\vec{a})$ which leads to the assertion that the $P(\vec{a})$ generate the Boolean algebra of open compact subsets of $\mathbb{P}_1(F_\infty)$.

(1.2) REMARK. The end or boundary points of the tree $T = I(F_\infty^2)$ are given by half infinite simplicial paths. Fixing a vertex L_0 of T, we assign to each point $X \in \mathbb{P}_1(F_\infty)$ a sequence of lattices

$L_0 > L_1 > \cdots > L_m > L_{m+1} > \cdots$ with $L_m > L_{m+1} > \pi L_m$ and $L_m \not\supset \pi^{m-1}L_0$ satisfying $\gamma_X(L_i) = \gamma_X(L_{i+1})$ for $i \geq 0$. All half-infinite simplicial paths from L_0 arise this way and each one determines an end of T.

(1.3) NOTATIONS. For a group M we map the coclosed cochains of T into the M-valued measures on $\mathbb{P}_1(F_\infty)$ by $c \mapsto \mu_c$ where $\mu_c(P(\vec{a})) = c(\vec{a})$. The map is defined $\underline{H}^1(T,M) \to \text{Meas}(\mathbb{P}_1(F_\infty),M)$.

For the ordered 1-simplexes $\vec{a}_0,\ldots,\vec{a}_s$ issuing from a vertex, we have

$$\mu_c(P(-\vec{a}_0)) = \mu_c(P(\vec{a}_1)) + \cdots + \mu_c(P(\vec{a}_s))$$

from the coclosed condition $c(-\vec{a}_0) = c(\vec{a}_1) + \cdots + c(\vec{a}_s)$. This relation is sufficient to show that μ_c is a finitely additive set function on the family of compact open subsets of $\mathbb{P}_1(F_\infty)$. From the partition $\mathbb{P}_1(F_\infty) = P(\vec{a}) \sqcup P(-\vec{a})$ we obtain $\mu_c(\mathbb{P}_1(F_\infty)) = 0$ so that μ_c has total mass equal to zero.

(1.4) PROPOSITION. The function $c \mapsto \mu_c$ is an isomorphism $\underline{H}^1(T,M) \to \text{Meas}(\mathbb{P}_1(F_\infty),M)_0$ of the M-valued coclosed cochains of $\dot{T}$ onto the M-valued measures of total mass zero.

PROOF. The inverse of $c \mapsto \mu_c$ is given by $\mu \mapsto c_\mu$ where $c_\mu(\vec{a}) = \mu(P(\vec{a}))$. The coclosed condition for c_μ follows from the finite additivity and total mass zero by the relations made explicit in (1.3). This proves the proposition.

Now the measures on $\mathbb{P}_1(F_\infty)$ are linear functionals on the space C^∞ of locally constant functions on $\mathbb{P}_1(F_\infty)$. This space $C^\infty(\mathbb{P}_1(F_\infty))$ is an important representation space under translation by $GL(2,F_\infty)$ on $\mathbb{P}_1(F_\infty)$. This representation is related to the special representation of the group $GL(2,F_\infty)$. It is the key link between cohomology as described by coclosed cochains and representation theory.

(1.5) DEFINITION. Let D be a ring of scalars. Then the special representation sp (or sp(D)) of $GL(2,F_\infty)$ (with values in D) is defined on the module $V_{sp} = C_0^\infty(\mathbb{P}_1(F_\infty),D)/D$ where D also denotes the subspace of constant functions and the action of $GL(2,F_\infty)$ is given by translation of functions $(sf)(x) = f(s^{-1}x)$ for $f \in C_0^\infty(\mathbb{P}_1(F_\infty),D)$, $x \in \mathbb{P}_1(F_\infty)$, and $s \in GL(2,F_\infty)$.

(1.6) PROPOSITION. The function $c \mapsto \mu_c$ of (1.5) is an isomorphism $\underline{H}^1(T,D) \to \mathrm{Hom}_D(V_{sp},D)$, the algebraic dual.

PROOF. This is immediate from (1.5) and the fact that $\mathrm{Meas}(\mathbb{P}_1(F_\infty),D)$ is the algebraic dual of $C^\infty(\mathbb{P}_1(F_\infty),D)$.

Now we use the group action on V_{sp} to obtain still another version of the previous two propositions.

(1.7) PROPOSITION. The function $\psi(f,s) \mapsto \psi(f,e) = \phi(f)$ defines an isomorphism

$$\mathrm{Hom}_{GL(2,F_\infty)}(V_{sp},C^\infty(GL(2,F_\infty))) \longrightarrow \mathrm{Hom}_D(V_{sp},D)$$

with inverse $\phi(f) \mapsto \psi(f,s) = \phi(sf)$.

PROOF. The $GL(2,F_\infty)$-morphism condition on $\psi(f,s)$ for $s,t \in GL(2,F_\infty)$ takes the form $\psi(tf,s) = \psi(f,st)$. Setting $s = 1$, we obtain $\phi(tf) = \psi(f,t)$ and the two maps are inverse to each other.

We use the notation $L(GL(2,F_\infty))$ for $C^\infty(GL(2,F_\infty))$. Let Γ be a subgroup of $GL(2,F_\infty)$. It acts on the tree T , the space $\mathbb{P}_1(F_\infty)$, the representation V_{sp} and its dual V'_{sp} , and also on $GL(2,F_\infty)$. Hence the isomorphisms of (1.4), (1.6) and (1.7) restrict to isomorphisms

$$\underline{H}^1(T,D)^\Gamma \xrightarrow{\sim} \mathrm{Hom}_D(V_{sp},D)^\Gamma \xleftarrow{\sim} \mathrm{Hom}_{GL(2,F_\infty)}(V_{sp},L(GL(2,F_\infty)/\Gamma)) \quad .$$

Now we assume that Γ is a subgroup of $GL(2,A)$ of finite index. For each parabolic P over the global field F of $GL(2)$ with unipotent radical U , we form $f_P(x) = \int_{u \in U/\Gamma \cap U} f(xu)du$. Note that U is conjugate to $\left\{\begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix}\right\}$ by an element of $GL(2,F)$.

(1.8) DEFINITION. A function f on $GL(2,F_\infty)/\Gamma$ is cuspidal provided $f_P(x) = 0$ for all parabolic P of $GL(2)$ over F . Let $L_0(GL(2,F_\infty)/\Gamma)$ denote the subspace of cuspidal $f \in L(GL(2,F_\infty)/\Gamma)$.

For $f \in L(GL(2,F_\infty)/\Gamma)$ which is the image of an element of V_{sp} by a homomorphism in $\mathrm{Hom}_{GL(2,F_\infty)}(V_{sp},L(GL(2,F_\infty)/\Gamma))$, the function f is cuspidal if it has compact support modulo Γ and the center of $GL(2)$. This gives the next proposition.

(1.9) PROPOSITION. The above isomorphism

$$\underline{H}^1(T,D)^\Gamma \xrightarrow{\sim} \mathrm{Hom}_{GL(2,F_\infty)}(V_{sp},L(GL(2,F_\infty)/\Gamma))$$

restricts to an isomorphism

$$\underline{H}^1(T,D)^\Gamma_! \xrightarrow{\sim} \mathrm{Hom}_{GL(2,F_\infty)}(V_{sp},L_0(GL(2,F_\infty)/\Gamma)) \quad ,$$

where recall $\underline{H}^1(T,D)^\Gamma_!$ denotes the coclosed cochains which are Γ-invariant and have compact support modulo Γ .

§2. LIMIT COHOMOLOGY AND AUTOMORPHIC FORMS

Recall in 2(5.7) we have an adelic and a local description of the C_∞-valued points $M^r_H(C_\infty)$ of the moduli scheme M^r_H for open compact subgroups $H \subset GL(r,\hat{A})$. Our aim in this section is to relate $\bar{\mathbb{Q}}_\ell$-valued automorphic forms to the limit cohomology $\varinjlim_H H^1_!(M^2_H(C_\infty),\bar{\mathbb{Q}}_\ell)$ using the isomorphism in (1.9) and the cohomology calculation (4.5). For this we make use of the special representations sp_{Gal} , see Ch. 4(4.5), and $sp_{GL(2)}$, see (1.5), with values in $\bar{\mathbb{Q}}_\ell$, the algebraic closure of the ℓ-adic numbers.

(2.1) DEFINITION. The space $L_0(GL(2,F)\backslash GL(2,\mathbb{A}_F))$ of cuspidal automorphic forms with values in $\bar{\mathbb{Q}}_\ell$ consists of functions $f: GL(2,F)\backslash GL(2,\mathbb{A}_F) \longrightarrow \bar{\mathbb{Q}}_\ell$ such that

(a) f is invariant by an open compact subgroup,

(b) the $GL(2,F_\infty)$-transforms of f generate a finite direct sum of irreducible representations, and

(c) f is cuspidal, i.e. $\int_{U(F)\backslash U(\mathbb{A}_F)} f(ux)du = 0$ for all x where U consists of matrices

$$\begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \quad .$$

For a function f satisfying (a) and (b) in (2.1), we have that f is in L_0 if and only if it has support compact modulo the center of $GL(2,F)$.

Now we are prepared to relate cohomology and automorphic forms. In (1.9) for a group D of scalars, we studied an isomorphism

$$\theta: \underline{H}^1(T,D) \longrightarrow \mathrm{Hom}_{GL(2,F_\infty)}(V_{sp},C^\infty(GL(2,F_\infty),D))$$

which preserved the action of $GL(2,F_\infty)$ and restricted to certain submodules as an isomorphism. We apply this now to the case $D = C^{\infty}(GL(2,\mathbf{A}_F^f),\bar{\mathbf{Q}}_\ell)$ and obtain an isomorphism

$$\theta: \underline{H}^1(T,C^{\infty}(GL(2,\mathbf{A}_F^f),\bar{\mathbf{Q}}_\ell)) \xrightarrow{\sim} \mathrm{Hom}_{GL(2,F_\infty)}(V_{sp},C^{\infty}(GL(2,\mathbf{A}_F^f) \times GL(2,F_\infty),\bar{\mathbf{Q}}_\ell)) \quad .$$

Using the two coset space descriptions of M_H^r , see 2(5.7), and the cohomology calculation 4(4.5), we have a restriction of this θ to θ_H where θ_H is canonical on the associated graded group and defined

$$\theta_H: H_!^1(M_H,\bar{Q}_\ell) \xrightarrow{\sim} \mathrm{Hom}_{GL(2,F_\infty)}(V_{sp},L_0(GL(2,F)\backslash GL(2,\mathbf{A}_F^f) \times GL(2,F_\infty)/H)) \otimes sp_{Ga}$$

as (Centralizer (H) in $GL(2,\mathbf{A}_F)$) $\times$ $Gal(F_{\infty,s}/F_\infty)$ -representations. For this we use the calculation

$$H_!^1(M_H,\bar{Q}_\ell) = \coprod_{xH \in GL(2,\mathbf{A}_F^f)/H} \left\{\begin{matrix}\text{coclosed cochains of } T \\ \text{invariants by } xHx^{-1} \cap GL(2,F) \\ \text{with compact support}\end{matrix}\right\} \otimes sp_{Gal}$$

$$= \left\{\begin{matrix}\text{coclosed cochains on } Gal(2,\mathbf{A}^f) \times T \\ \text{invariant by } H \text{ and } GL(2,F) \text{ with} \\ \text{compact support}\end{matrix}\right\} \otimes sp_{Gal}$$

which maps by the restriction θ_H of θ . Observe that invariance by H implies that the function is locally constant on $GL(2,\mathbf{A}_F^f)$.

Now we assemble all the isomorphisms θ_H together with the transfer morphisms $M_H \to M_{xHx^{-1}}$ to define an isomorphism in the limit. This limit isomorphism is one of the main results of the theory.

(2.2) THEOREM. With the above notations the limit of the θ_H defines an isomorphism of $GL(2,\mathbf{A}_F) \times Gal(F_{\infty,s}/F_\infty)$ -representations

$$\boxed{\underline{H} = \varinjlim_H H_!^1(M_H,\bar{\mathbf{Q}}_\ell) \to \mathrm{Hom}_{GL(2,F_\infty)}(V_{sp},L_0(GL(2,F)\ GL(2,\mathbf{A}_F))) \otimes sp_{Gal}} \quad .$$

A basic result in the theory of automorphic forms for GL(2) , see [J-L, prop. 11.1.1], is that the representation of $GL(2,\mathbf{A}_F)$ on L_0 decomposes with multiplicity one

$$L_0(GL(2,F)\backslash GL(2,A_F)) = \coprod_{\pi \in \Pi} \pi$$

where Π is a set of irreducible admissible representations of the adèlic group $GL(2,\mathbb{A}_F)$. Now each $\pi \in \Pi$ is of the form $\pi = \otimes_v \pi_v$ where π_v is an irreducible admissible representation with a $GL(2,\mathcal{O}_v)$-invariant vector for almost all v . From the theorem we have the next corollary.

(2.3) COROLLARY. As $GL(2,\mathbb{A}_F^f)$, $Gal(F_{\infty,s}/F_\infty)$ modules, we have an isomorphism

$$\underline{H} \xrightarrow{\sim} \bigoplus_{\pi\in\Pi,\pi_\infty=sp} \left[\left(\bigotimes_{v\neq\infty} \pi_v\right) \otimes sp_{Gal}\right]$$

and as $GL(2,\mathbb{A}_F^f)$, $Gal(F_s/F)$ modules we have a mapping $\pi \mapsto \sigma(\pi)$ of two-dimensional $Gal(F_s/F)$-modules such that $\sigma(\pi)|Gal(F_{\infty,s}/F_\infty) = sp_{Gal}$ and an isomorphism

$$\underline{H} \xrightarrow{\sim} \bigoplus_{\pi\in\Pi,\pi_\infty=sp} \left[\left(\bigotimes_{v\neq\infty} \pi_v\right) \otimes \sigma(\pi)\right] \quad .$$

The mapping $\pi \mapsto \sigma(\pi)$ is a form of the reciprocity mapping of class field theory between automorphic representations equal to the special representation at ∞ and Galois representations of dimension two equal to the special representation at ∞ .

§3. PROPERTIES OF THE CORRESPONDENCE $\pi \mapsto \sigma(\pi)$

Let Π_∞ denote the set of representations π in Π with $\pi_\infty = sp_{GL(2)}$, and let Σ denote the set of compatible families of ℓ-adic representations σ of $Gal(\bar{F}/F)$ which are two dimensional and irreducible for all $\ell \neq p$, and let Σ_∞ denote the $\sigma \in \Sigma$ with $\sigma_\infty = sp_{Gal}$.

From (2.2) and (2.3) we have a function still denoted $\pi \mapsto \sigma(\pi)$ defined $\Pi_\infty \to \Sigma_\infty$. For an irreducible representation π of $GL(2,\mathbb{A}_F)$, let ω_π denote the scalar action defined by π restricted to the center of $GL(2,\mathbb{A}_F)$. Also we use the reciprocity map from abelian class field theory

$$Gal(\bar{F}/F) \longrightarrow GL(1,F)\backslash GL(1,\mathbb{A}_F)$$

so that for a 1-dimensional character χ of the ideal class group we have a character χ of $Gal(\bar{F}/F)$ by composing with the reciprocity map.

(3.1) PROPOSITION. For $\pi \mapsto \sigma(\pi)$ we have

(1) $\sigma(\pi \otimes \chi) = \sigma(\pi) \otimes \chi^{-1}$ for χ a 1-dimensional character.

(2) $\det \sigma(\pi) = \omega_{\pi}^{-1}(-1)$ for the central character ω_{π}.

PROOF. (1) Observe that $\pi \otimes \sigma(\pi) = (\pi \otimes \chi) \otimes (\sigma(\pi) \otimes \chi^{-1})$ is a subrepresentation of $\underline{H}$ from which the first assertion follows.

(2) This relation is rather involved to work out completely. For this one uses the cup product in cohomology and the alternating bilinear form that it defines on $\underline{H}$. In a group GL(2) define $s^{\vee} = s/\det$ where this is the contragredient relative to the alternating form so that

$$sx \wedge [\det(s)]^{-1} sy = s \wedge y \quad .$$

For π a representation on V, $s \mapsto \pi(s^{\vee})$ is isomorphic to the dual representation, see [Deligne 1973, pp. 102-3] for references. For a form ψ with $\psi(\pi(s)x, \pi(s^{\vee})y) = \psi(x,y)$ we have

$$\psi(\pi(s)x, \pi(s)y) = \omega_{\pi}(s)\psi(x,y) \quad ,$$

and ψ is unique up to a constant factor. One has $\psi(x,y) = \omega_{\pi}(-1)\psi(y,x)$. These considerations are coupled with properties of the cup product form to yield the stated relation.

Now we are in a position to study the relation between π_v and $\sigma(\pi)_v | D_v$ for all v such that $GL(2, \mathcal{O}_v)$ leaves a 1-dimensional space of the representation space of π_v invariant. Then $\pi_v = \mathrm{Ind}_B^G(\chi_1, \chi_2)$ (unitary), and it is classified by a quadratic polynomial $T^2 - a_v T + b_v$ with roots $\chi_1(\pi_v)$ and $\chi_2(\pi_v)$, called the Hecke polynomial. Such a place v is called unramified.

Further, if $\sigma(\pi)_v$ is unramified, then it is characterized by the characteristic polynomial of Frobenius Fr_v. This is also a quadratic polynomial $T^2 - a'_v T + b'_v$. The basic result, which we sketch, is the following theorem.

(3.2) THEOREM. For a place v such that π_v and $\sigma(\pi)_v$ are unramified, the Hecke polynomial of π_v twisted by $|\det(s)|^{1/2}$ equals the characteristic polynomial of Frobenius.

PROOF. By an Eichler-Shimura type congruence formula

$$(Fr_v)^2 - a_v(Fr_v) + b_v = 0 \quad .$$

This congruence formula is proved, as in the case of elliptic curves, by considering Drinfel'd modules mod v and the action of Frobenius.

Next, from (3.1) and the considerations related to alternating forms, we deduce that $b_v = b'_v$ since $b'_v = \det \sigma(\pi)$. Hence for roots α and β of the Hecke polynomial, we see that α,β or α,α or β,β are the characteristic roots of Fr_v. The condition on the constant terms of the polynomials implies that the roots of the characteristic polynomial of Fr_v are α and β.

§4. LOCAL LANGLANDS' CONJECTURE IN CHARACTERISTIC p

Now we sketch the steps in the proof of the local Langlands' conjecture in characteristic ρ for GL(2) which was outlined in a letter from Deligne to Drinfel'd dated 1/21/1975.

(4.1) The map $\rho \mapsto \pi(\rho)$. Let ρ be a two-dimensional representation of $W(\bar{K}/K)$ and the local Weil group of the local field K in characteristic ρ. We can take $K = F_v$ for a function field F. Since the Artin conjecture is true for Artin L-functions in the function field case, see A. Weil [1948], we can apply [J-L, proposition 12.6, p. 408] to obtain $\pi(\rho)$ from a global automorphic representation.

(4.2) Injectivity of $\rho \mapsto \pi(\rho)$. This is corollary (1.7) of proposition (1.6) in Gelbart and Jacquet, "A Relation Between Automorphic Representations of GL(2) and GL(3)" [1978]. It is an argument on $GL(2) \times GL(2)$ which uses corollary 19.16 of [J].

(4.3) Surjectivity of $\rho \mapsto \pi(\rho)$. As with the injectivity, this depends on the local result that $\varepsilon(\pi \otimes \chi)$ depends only on ω_π for χ very ramified, see [J-L, proposition 3.8(iii), p. 116] and $\varepsilon(V \otimes \chi)$ depends only on $\det V$ for χ very ramified, see [Deligne, 1973, proof of 4.16, p. 546]. Now consider global π and $\sigma(\pi) = \sigma$ given by (3.2). Here π and σ have the same global L and ε factors and the same local L and ε factors at all, $v \notin S$, where S is finite. We wish to prove that L_v and ε_v are equal for all v. The global functional equations and product formulas have the form

$$\prod_v L_v(\pi \otimes \chi) = \prod_v \varepsilon_v(\pi \otimes \chi) \prod_v L_v(\check{\pi} \otimes \chi^{-1} \otimes \omega_1)$$

and

$$\prod_v L_v(\sigma \otimes \chi) = \prod_v \varepsilon_v(\sigma \otimes \chi) \prod_v L_v(\check{\sigma} \otimes \chi^{-1} \otimes \omega_1) .$$

Now divide the one expression by the other using the equality for $v \notin S$ to obtain

$$\prod_{v \in S} \varepsilon'_v(\pi \otimes \chi) = \prod_{v \in S} \varepsilon'_v(\sigma \otimes \chi)$$

where $\varepsilon'(\tau) = \varepsilon(\tau)(L(\tau^v \times \omega_1)/L(\tau))$. Now choose a global χ which equals 1 at a fixed $v_0 \in S$ and which is very ramified at $w \in S - \{v_0\}$. Then $\varepsilon'_w = \varepsilon_w$ depends only on the central character and determinant. Hence both sides are equal at $w \in S - \{v_0\}$, which implies $\varepsilon'_{v_0}(\pi) = \varepsilon'_{v_0}(\sigma)$. From this we recover $\varepsilon_v(\pi) = \varepsilon_v(\sigma(\pi))$ and $L_v(\pi) = L_v(\sigma(\pi))$ for all v .

For a finite set $S = \{v_0, \ldots, v_m\}$ of places and local discrete series representations π_i at v_i , one can prove by looking at a quaternion algebra ramified at each place of S that there is a global representation π with $\pi_{v_0} = \pi_0$ and with π_{v_i} and π_i differing by an unramified twisting for $i = 1, \ldots, m$. Using (3.2), we form $\sigma = \sigma(\pi)$ and define $\rho = \sigma_{v_0}$. Then $\rho \mapsto \pi_0 = \pi(\rho)$ has the desired property giving surjectivity.

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Le déterminant de la cohomologie

par P. Deligne

Le présent article a pour origine une lettre à D. Quillen, datée du 20 juin 1985. Dans cette lettre, j'exploitais la philosophie d'Arakelov pour "calculer" la torsion analytique d'un fibré vectoriel métrisé sur une surface de Riemann (avec métrique) en terme d'intégrales de quantités élémentaires - tout au moins dans le cas plus facile des fibrés virtuels de dimension 0. Une description de l'énoncé obtenu est donnée au paragraphe 1.

En essayant de comprendre la démonstration, ainsi que G. Faltings [5], H. Gillet et C. Soulé [7], je me suis aperçu que cette lettre établit des bribes d'un programme peut-être plus intéressant que le théorème, dont il éclaire la démonstration. Ce programme est exposé au paragraphe 2. Pour $f : X \to Y$ un morphisme projectif d'intersection complète relative, la formule de Riemann-Roch donne des informations sur l'image directe $f_* : K_0(X) \to K_0(Y)$. En particulier, elle donne une formule pour $c^1 Rf_* E = c^1 \det Rf_* E$, pour E un fibré vectoriel sur X. Un fibré en droite est connu quand il est connu (à isomorphisme unique près) localement. Si on peut "préciser" la formule de Riemann-Roch pour $c^1 \det Rf_* E$ en un isomorphisme canonique entre $\det Rf_* E$ (ou une puissance tensorielle fixe de $\det Rf_* E$) et un autre fibré en droite, le problème de construire cet isomorphisme est local. On acquiert le droit de se localiser sur Y au prix de travailler avec des objets à isomorphisme unique près plutôt qu'à isomorphisme près. Les résultats plus précis obtenus coûtent aussi une débauche de diagrammes commutatifs et des cauchemars de signes. Ils ne s'appliquent qu'à la première classe de Chern. Pour interpréter de même Riemann-Roch pour $ch^n\, Rf_*$, il faudrait sans doute se battre avec des n-champs. Je recule devant cette idée mais j'espère que la philosophie de Grothendieck [9] et le cas $n = 1$ peuvent être un guide heuristique utile. De même: en K-théorie, le langage des espaces de lacets infinis est plus efficace que celui des n-catégories de Picard commutatives qu'envisage Grothendieck, mais une variante catégorique des deux premiers étages de la tour de Postnikov de BQA (§4) peut aider à comprendre ce qui se passe.

Le paragraphe 2 expose ces idées, et un analogue à la Arakelov, avec métriques.

Dans l'analogie entre "métrique hermitienne sur un fibré vectoriel" et "structure entière", je tiens, au contraire de S. Arakelov [1][2] et G. Faltings [5], à pouvoir travailler avec des structures hermitiennes quelconques. Le paragraphe 3 fait de la propagande pour ce point de vue.

Le paragraphe 4 étudie la catégorie V(A) des objets virtuels d'une catégorie exacte A . Cette catégorie est un avatar des deux premiers étages de la tour de Postnikov de BQA . Son introduction nous permettra de cacher de nombreux signes. Au paragraphe 5, on en introduit un analogue à la Arakelov, pour les fibrés métrisés sur un espace analytique complexe X . Cette catégorie incarne une théorie de classes de Chern secondaires à la Bott-Chern [4] et fournit des applications R_p du K_1 de la catégorie des fibrés holomorphes sur X dans l'espace des (p-1 , p-1) formes d'd"-fermées modulo formes d' ou d" exactes.

Enfin, les paragraphes 6 à 11 donnent la preuve du théorème décrit au paragraphe 1, selon les lignes indiquées au paragraphe 2, avec quelques disgressions.

Exposer l'heuristique des paragraphes 2 et 3 est mon excuse pour présenter des résultats incomplets, avec des démonstrations qui ne sont parfois que des esquisses et une généralité pas toujours optimale.

Le présent texte doit beaucoup à l'influence de nombreux mathématiciens. Je suis particulièrement conscient de celles de A. Grothendieck et D. Quillen (§4, d'anciennes conversations à Bures), D. Quillen [13], C. Soulé (§5) et S. Arakelov (§3). Je prie ceux que j'oublie de m'en excuser. Dans la lettre à D. Quillen, je n'utilisais que D. Quillen [13]. La démonstration donnée ici est simplifiée et rendue plus naturelle par l'usage des résultats plus forts de J. Bismuth et D. Freed [3] and D. Freed [6] (1.22).

1. Description du théorème principal.

1.1 Rappels: déterminants ([10]).

Pour V un espace vectoriel, on pose

$$\det V := \overset{\dim V}{\Lambda} V .$$

Pour V* un espace vectoriel $\mathbf{Z}/(2)$-gradué, on pose

$$\det(V^*) := (\det V^+) \otimes (\det V^-)^\vee .$$

On applique la même règle aux espaces vectoriels $\mathbf{Z}$-gradués, en posant $V^+ = \sum V^{2i}$, $V^- = \sum V^{2i+1}$. Il nous sera commode de noter $L^{\otimes(-1)}$ le dual d'un espace vectoriel de rang un. Avec cette règle,

$$\det V^* = \bigotimes_i (\det V^i)^{(-1)^i} .$$

Pour éviter les problèmes de signes, il est préférable de définir det V comme l'espace vectoriel gradué $\overset{\dim V}{\Lambda} V$ placé en degré dim V , et de définir l'isomorphisme de symétrie

$$L_1 \otimes L_2 \xrightarrow{\sim} L_2 \otimes L_1$$

par la règle de Koszul (l'isomorphisme naïf fois $(-1)^{d_1 d_2}$, pour L_i en degré d_i) . L'isomorphisme

$$\det(V) \otimes \det(W) \xrightarrow{\sim} \det(V \oplus W) :$$

$$(e_1 \wedge \ldots \wedge e_m) \otimes (f_1 \wedge \ldots \wedge f_m) \mapsto e_1 \wedge \ldots \wedge e_n \wedge f_1 \wedge \ldots \wedge f_m$$

donne alors lieu à un diagramme commutatif

$$\begin{array}{ccccc} V \oplus W & & \det(V \oplus W) & \xrightarrow{\sim} & \det(V) \otimes \det(W) \\ \downarrow \wr & & \downarrow \wr & & \downarrow \wr \\ W \oplus V & & \det(W \oplus V) & \xrightarrow{\sim} & \det(W) \otimes \det(V) \end{array} .$$

Notons 1 l'espace vectoriel gradué de rang 1 et degré 0 trivial. On définit L^{-1} comme un espace vectoriel gradué de rang 1 muni d'un isomorphisme $L \otimes L^{-1} \simeq 1$. Ceci définit L^{-1} à isomorphisme unique près. L'isomorphisme $L^{-1} \otimes L \simeq L \otimes L^{-1} \simeq 1$ fait de L l'inverse $(L^{-1})^{-1}$ de L^{-1} . Pour plus de détails, cf 4.1. Pour un complexe K , la définition de l'isomorphisme

$$\det K \simeq \det H^*K$$

et ses propriétés ne posent avec ces règles aucun cauchemar de signes.

1.2 Rappels: torsion analytique ([14]).

Soient X une surface de Riemann compacte, munie d'une métrique hermitienne g , et V un fibré vectoriel holomorphe sur X , muni d'une structure hermitienne h . Notons Ω^1_X le fibré en droites de sections locales holomorphes les 1-formes holomorphes et $\Omega^{0,q}(V)$ le faisceau des $(0,q)$-formes C^∞ à valeurs dans V . Nous normaliserons comme suit la structure préhilbertienne sur $\Gamma(X , \Omega^{0,q}(V))$ définie par g et h . (a) On regarde g comme la donnée d'une structure hermitienne sur Ω^1_X . (b) Le produit scalaire est l'intégrale d'une densité sur X . (c) Localement, si α est une section C^∞ de longueur un de Ω^1_X et que v_i est une section de $\Omega^{00}(V)$ (resp $\beta_i = \gamma_i v_i$ une section de $\Omega^{01}(V)$) , cette densité est

pour $\langle v_1 , v_2 \rangle$: $\frac{-1}{2\pi i} \langle v_1 , v_2 \rangle_h \, \alpha \wedge \bar{\alpha}$,

pour $\langle \beta_1 , \beta_2 \rangle$: $\frac{-1}{2\pi i} (v_1 , v_2)_h \, \bar{\gamma}_2 \wedge \gamma_1$.

Rappelons que pour $z = x + iy$, on a $\frac{-1}{2\pi i} dz \wedge d\bar{z} = \frac{1}{\pi} dx \wedge dy$.

Les groupes de cohomologie de V sont ceux du complexe $\bar{\partial}$ de V :

$$\bar{\partial} : \Gamma(X , \Omega^{00}(V)) \longrightarrow \Gamma(X , \Omega^{01}(V)) .$$

Les structures préhilbertiennes des $\Gamma(X , \Omega^{0q}(V))$ induisent des structures hermitiennes sur $H^0(X,V) \xrightarrow{\sim} \mathrm{Ker}(\bar{\partial})$ et $H^1(X,V) \xrightarrow{\sim} \mathrm{coKer}(\bar{\partial}) \xleftarrow{\sim} \mathrm{Im}(\bar{\partial})^\perp = \mathrm{Ker}(\bar{\partial}^*)$. De là, une structure hermitienne $\langle\ ,\ \rangle_0$ sur $\det H^*(X,V)$. Soient λ_i les valeurs propres $\neq 0$ de $\bar{\partial}^* \bar{\partial}$, comptées avec leur multiplicité, et $\det'(\bar{\partial}^*\bar{\partial}) := \Pi\lambda_i := \exp(-\zeta'(0))$, où $\zeta(s)$ est le prolongement analytique de $\sum \lambda_i^{-s}$. La <u>torsion analytique</u> de V est la structure hermitienne

$$\langle\ ,\ \rangle_{t.a.} := \det'(\bar{\partial}^*\bar{\partial})^{-1} \langle\ ,\ \rangle_0$$

sur $\det H^*(X,V)$. On veut la calculer.

Un facteur π a été introduit dans la définition de la structure préhilbertienne de $\Gamma(X , \Omega^{0q}(V))$ pour assurer la validité du lemme suivant.

<u>Lemme</u> 1.3 La torsion analytique est compatible à la dualité de Serre.

Soit $V' := \underline{\mathrm{Hom}}(V , \Omega^1_X)$. La dualité de Serre est une dualité entre $H^q(X,V)$ et $H^{1-q}(X,V')$. Elle induit un isomorphisme

(1.3.1) $\det H^*(X,V) \simeq \det H^*(X,V')$

qu'on veut prouver être une isométrie.

Le morphisme

$$H^q(X,V) \otimes H^{1-q}(X,V') \longrightarrow \mathbb{C}$$

est le composé du cup-produit et du morphisme trace $H^q(X,V) \otimes H^{1-q}(X,V') \to H^1(X\,,\, V \otimes V') \to H^1(X\,,\,\Omega^1_X) \xrightarrow{\text{Tr}} \mathbb{C}$. Si $H^*(X\,,\,\Omega^1_X)$ est calculé par le complexe $\bar\partial$, Tr est $\frac{1}{2\pi i}$ fois l'intégration sur X (à un signe dépendant des conventions près, sans importance pour 1.3). La dualité de Serre est donc induite par la dualité $\pm \frac{1}{2\pi i} \int_X \alpha \wedge \beta$ entre les complexes $\bar\partial$ de V et V' et 1.3 en résulte.

1.4 <u>Calculer det $H^*(X,L)$</u> .

La règle suivante permet d'attacher à deux fibrés en droite L et M sur X un espace vectoriel de rang un $\langle L\,,\,M\rangle$. (a) Deux sections méromorphes à diviseurs disjoints ℓ et m de L et M définissent un élément non nul $\langle \ell\,,\,m\rangle$ de $\langle L\,,\,M\rangle$. (b) Quand on change ℓ ou m , on a

(1.4.1) $$\langle f\ell\,,\,m\rangle = f(\text{div}\, m)\,\langle \ell\,,\,m\rangle$$
$$\langle \ell\,,\,fm\rangle = f(\text{div}\, \ell)\,\langle \ell\,,\,m\rangle\,.$$

Si $v_P(\ell)$ est la valuation de ℓ en P (i.e l'ordre du zéro, ou -l'ordre du pôle), on a $\text{div}\,\ell := \sum_P v_P(\ell)\,.\,P$; on a posé

$$f(\textstyle\sum n_P\,.\,P) := \prod f(P)^{n_P}\,.$$

Que les règles (a) (b) soient cohérentes équivaut au théorème de A. Weil que pour deux fonctions méromorphes à diviseurs disjoints, on a $f(\text{div}\, g) = g(\text{div}\, f)$.

Pour L un fibré en droite sur X , posons

$$\det H^*(X\,,\,L - O) := \det H^*(X,L) \otimes (\det H^*(X,O))^{-1}\,.$$

De façon purement algébrique, on définit un isomorphisme

(1.4.2) $$(\det H^*(X\,,\,L - O))^{\otimes 2} \simeq \langle L\,, L \otimes (\Omega^1_X)^\vee\rangle\,.$$

Ces constructions sont valables sur un corps de base quelconque, voire sur une base quelconque.

1.5 <u>Calculer la torsion analytique</u>.

Si les fibrés en droites L et M sur X sont munis de métriques hermitiennes, on définit comme suit une structure hermitienne sur $\langle L\,,\,M\rangle$. Notons $\|\ \|$ une longueur carrée et posons $f[\sum n_i P_i] = \sum n_i f[P_i]$. Pour des sections méromorphes ℓ et m comme en 1.4, on définit

(1.5.1) $$\log \|\langle \ell , m \rangle\| = \int \frac{1}{2\pi i} d'd'' \log \|\ell\| \cdot \log \|m\| + \log \|\ell\| \, [\mathrm{div}\, m] + \log \|m\| \, [\mathrm{div}\, \ell] ,$$

où $d'd''$ est pris au sens des distributions.

Supposons X et L munis de métriques hermitiennes g et h comme en 0.1. On munit $\mathcal{O}$ de la structure hermitienne triviale $(\langle 1 , 1\rangle = 1)$. On dispose d'une structure hermitienne "torsion analytique" sur $\det H^*(X , L - \mathcal{O})^{\otimes 2}$, déduite de celles de $\det H^*(X,L)$ et $\det H^*(X,\mathcal{O})$.

Notre résultat principal est que (1.4.2) est une isométrie, le membre de gauche étant muni de la structure hermitienne torsion analytique et celui de droite de la structure (1.5.1). Rappelons qu'une métrique sur X est vue comme une structure hermitienne sur Ω^1_X .

Concrètement, ceci signifie qu'une construction purement algébrique permet d'attacher à des sections méromorphes à diviseurs disjoints ℓ et m de L et $L \otimes (\Omega^1)^{-1}$ un élément non nul $\langle \ell , m\rangle$ de $(\det H^*(X , L - \mathcal{O}))^{\otimes 2}$, avec une variance (1.4.1), et que la longueur carrée de cet élément, pour la métrique torsion analytique, est donnée par (1.5.1).

1.6 Pour prouver 1.5, nous utilisons le calcul par D. Quillen [13] et D. Freed [6] de dérivées logarithmiques secondes de la torsion analytique en fonction de paramètres. Quand L est une déformation de $\mathcal{O}$, i.e. est de degré 0, un argument direct, ne requéront pas la normalisation 1.2, est possible.

Pour traiter le cas des fibrés de degré $\neq 0$, il nous faut prouver un résultat analogue à 1.4, 1.5, plus compliqué, pour les fibrés vectoriels de dimension quelconque. Un fibré vectoriel a deux invariants discrets: sa dimension et son degré. Ces invariants sont additifs par somme directe. Traiter le cas des fibrés vectoriels permet de prouver 1.5 à une constante $A(X,g)^{\deg L}$ près, où $A(X,g)$ ne dépend que de la surface de Riemann X et de sa métrique g . La compatibilité à la dualité de Serre donne $A(X,g) = 1$.

2. Programme.

2.1 Incarner une intégrale de classes de Chern.

Soit $f: X \to S$ un morphisme de schémas, propre, plat et purement d'une dimension relative N. Soient I un ensemble fini d'indices, C_i^p $(p > 0 , i \in I)$ des indéterminées, avec C_i^p de poids p, P un polynôme isobare de poids $N+1$ en les C_i^p à coefficients entiers, V_i des fibrés vectoriels sur X, de classes de Chern $c_i^p := c^p(V_i)$ et intégrons sur les fibrés de f le polynôme $P(c_i^p)$ en les classes de Chern des V_i pour obtenir sur S

$$c := \int_{X/S} P(c^p(V_i)) . \tag{2.1.1}$$

La définition (2.1.1) a un sens dans divers contextes. En cohomologie ℓ-adique (ℓ inversible sur S), on a

$$c \in H^2(S , \mathbf{Z}_\ell(1)) .$$

Pour S séparé de type fini sur $\mathbb{C}$, et en théorie de Hodge mixte, c est une classe de cohomologie entière purement de type $(1,1)$ [mise en garde: je n'en ai pas de démonstration avec la généralité dite; que $P(c^p(V_i))$ est purement de type $(N+1 , N+1)$ résulte de (P. Deligne, théorie de Hodge III, Publ. Math. IHES 44 (1974) p 5-78, 9.1.1 et 9.1.2); reste à voir que le morphisme de Gysin $\int_{X/S}$ est un morphisme de structures de Hodge mixtes de $H^{2n+k}(X , \mathbf{Z}(n))$ dans $H^k(S,\mathbf{Z})$; pour X et S lisses, la dualité de Poincaré ramène les propriétés de $\int_{X/S}$ à celles, connues, de l'image inverse].

Pour X et S quasi-projectifs et lisses sur un corps algébriquement clos, on peut travailler dans les anneaux de Chow. Dans ce cas, c est une classe d'équivalence linéaire de diviseurs.

Dans tous les cas, la classe c est candidate à être la première classe de Chern d'un fibré en droites sur S. Quand on peut travailler dans les anneaux de Chow, c est déjà la classe d'isomorphie d'un fibré en droite.

Problème 2.1.2. Dans la situation 2.1, construire "fonctoriellement" un faisceau inversible $I_{X/S}P\ (V_i , i \in I)$ sur S, ayant c comme première classe de Chern.

A cause des difficultés de signes auxquelles donne lieu ce problème, je préférerais y supposer I totalement ordonné. Parmi les fonctorialités voulues, on a :

(2.1.3) Soit $(\text{Vect is})_X$ la catégorie des fibrés vectoriels sur X et des isomorphismes de fibrés vectoriels. Alors, $I_{X/S}\ P$ est un foncteur de la catégorie $(\text{Vect is})_X^I$ dans la catégorie $\underline{\underline{\text{Pic}}}$ (S) des faisceaux inversibles sur S et de leurs isomorphismes. Il est compatible aux changements de base $S' \to S$.

(2.1.4) Soient $j \in I$, P_j le polynôme de poids N

$$P_j := \sum_{p\geq 1} (\text{rg}\ V_j - p + 1)\ c_j^{p-1}\ \partial_{c_j^p}\ P$$

(localement constant sur X) et n_j l'entier (localement constant sur S)

$$n_j := \int_{X/S} P_j(c^p(V_i)) .$$

Pour L_i des faisceaux inversibles sur S , on a

$$P(c^p(V_i \otimes f^*L_i)) = P(c^p(V_i)) + \sum P_j(c^p(V_i)) \ . \ f^*c^1(L_j)$$

modulo le carré de l'idéal engendré par les $f^*(c^1(L_j))$, d'où

$$\int_{X/S} P(c^p(V_i \otimes f^*L_i)) = \int_{X/S} P(c^p(V_i) + \sum n_j\ c^1(L_j) .$$

Pour λ une section sur S de $\mathcal{O}_S^*$, ceci amène à demander que l'automorphisme de multiplication par λ sur V_j agisse sur $I_{X/S}\ P(V_i)$ par λ^{n_j} .

(2.1.5) Additivité en P . Pour $P = P' + P''$, on veut un isomorphisme canonique $I_{X/S}\ P \simeq I_{X/S}\ P' \otimes I_{X/S}\ P''$.

(2.1.6) Additivité par suites exactes. Pour l'énoncer, il sera commode de poser $c_i^0 = 1$. Pour $a \in I$, soit I_1 déduit de I en dédoublant a en a' et a" . Soit P_1 le polynôme en des c_i^p $(i \in I_1)$ obtenu en remplaçant dans P chaque c_a^p par $\sum_{r+s=p} c_{a'}^r\, c_{a''}^s$ $(r\ ,\ s \geq 0)$. Soit enfin une suite exacte

$$0 \to V'_a \to V_a \to V''_a \to 0$$

et posons $V_{a'} := V'_a$, $V_{a''} := V''_a$. On veut un isomorphisme canonique entre $I_{X/S}P(V_i\ ,\ i \in I)$ et $I_{X/S}P_1(V_i\ ,\ i \in I_1)$.

Dans les cas qu'on sait traiter, la définition de cet isomorphisme pose des problèmes de signes. Ils disparaissent dans la version affaiblie suivante, relative à un seul des foncteurs $I_{X/S}\ P$.

(2.1.7) Invariance par semi-simplification. Soient des suites exactes $0 \to V'_i \to V_i \to V''_i \to 0$ et $sV_i := V'_i \oplus V''_i$. On veut un isomorphisme canonique

$$I_{X/S}\ P(V_i\ ,\ i \in I) \simeq I_{X/S}(sV_i\ ,\ i \in I) .$$

Plus précisément, on définira au paragraphe 4 la catégorie des fibrés virtuels; chaque fibré V définit un fibré virtuel $[V]$, chaque suite exacte courte $V' \to V \to V''$ définit un isomorphisme $[V] \simeq [V' \oplus V'']$ et on veut que le foncteur $I_{X/S}P$ se factorise par un foncteur $i_{X/S}$ P défini sur la catégorie des fibrés virtuels:

$$I_{X/S}\, P\, (V_i) = i_{X/S}\, P\, ([V_i])$$

(2.1.8) Identités entre classes de Chern.

L'exigence "additivité par suites exactes" combine l'invariance par semi-simplification et une formule pour $I_{X/S}$ P appliqué à des V_i dont l'un est une somme directe. Cette formule relève la formule donnant les classes de Chern d'une somme directe. De même, les formules pour les classes de Chern d'un produit tensoriel, d'un dual ou d'une puissance extérieure devraient se relever en isomorphismes canoniques. Pour qu'un tel formalisme soit utile, il faudrait aussi déterminer les diagrammes commutatifs auxquels ces isomorphismes donnent lieu.

2.2 Incarner la composante de degré 2 de Riemann-Roch

Soient $f\colon X \to S$ un morphisme propre et lisse, purement d'une dimension relative N , V un fibré vectoriel sur X et $T_{X/S}$ le fibré tangent relatif. La formule de Riemann-Roch s'énonce

(2.2.1) $$\mathrm{ch}\, Rf_*\, V = \int_{X/S} \mathrm{ch}\,(V)\,.\,\mathrm{Td}\,(T_{X/S})\;.$$

La composante de degré deux au membre de gauche est la première classe de Chern du faisceau inversible $\det Rf_*\, V$ sur S . Soit RR_{N+1} la composante de degré $2(N+1)$ de $\mathrm{ch}(V)\,.\,\mathrm{Td}(T_{X/S})$. C'est un polynôme universel de poids $(N+1)$ en les classes de Chern de V et de $T_{X/S}$. Il est à coefficients rationnels. Si le problème 0.1.2 était résolu, on pourrait poser le

Problème 2.2.2 Trouver un entier M (tel que $M.RR_{N+1}$ soit un polynôme à coefficients entier en les classes de Chern de V et $T_{X/S}$) et construire un isomorphisme canonique de fibrés en droites sur S

(2.2.2.1) $$(\det Rf_*\, V)^{\otimes M} \simeq I_{X/S}(M.RR_{N+1})(V\,,\,T_{X/S})\;.$$

Un tel isomorphisme préciserait la composante de degré deux de (2.2.1).

Pour $V = V' \oplus V''$, l'additivité (2.1.6) doit fournir pour le second membre $II(V)$ de (2.2.2.1) que

$$II(V) \simeq II(V') \otimes II(V'')\;.$$

On veut que

(2.2.3) cette additivité est compatible à celle du premier membre de (2.2.2.1).

Il n'est pas nécessaire d'attendre la solution de (2.1.2) pour se poser des cas particuliers de 2.2.2. Par ailleurs, pour V de forme particulière, on peut donner un sens à la question 2.2.2 pour M un entier tel que $M.RR_{N+1}$ ne soit pas à coefficients entiers. Voici des exemples

2.2.3. Soient V_i $(1 \leq i \leq N+2)$ des fibrés virtuels de dimension 0 : $V_i = V'_i - V''_i$ avec V'_i et V''_i de même dimension. On a $ch^0(V_i) = 0$ et

$$ch(\otimes V_i) = \Pi ch(V_i)$$

est un polynôme en les classes de Chern des V'_i et des V''_i de poids $\geq N+2$. Comme polynôme de poids $N+1$ en les classes de Chern des V'_i, V''_i et $T_{X/S}$, la composante de degré $2(N+1)$ de $ch(\otimes V_i) . Td(T_{X/S})$ est identiquement nulle. Ceci amène à désirer une trivialisation canonique de

$$\det Rf_* \otimes V_i .$$

En d'autres termes, et sans mentionner de fibré virtuel: pour $P \subset [1, N+2]$, soit $\mathcal{D}_P$ le fibré en droites

$$\mathcal{D}_P := \det Rf_* (\bigotimes_{i \in P} V'_i \otimes \bigotimes_{i \notin P} V''_i)$$

On veut un isomorphisme canonique entre le produit tensoriel des $\mathcal{D}_p$ pour $\#P$ pair et le produit tensoriel des $\mathcal{D}_p$ pour $\#P$ impair.

Dans le même esprit, si $I = I_0 \amalg I_1 \amalg I_2$ est un ensemble fini d'indices, que les $V_i = V'_i - V''_i$ sont des fibrés vectoriels virtuels sur X $(i \in I)$, que $\dim V'_i = \dim V''_i$ pour $i \in I_1$ ou I_2 et qu'un isomorphisme $\alpha_i : \det V'_i \xrightarrow{\sim} \det V''_i$ est donné pour $i \in I_2$, on désire une trivialisation canonique de $\det Rf_* (\otimes V_i)$ dès que $\#I_1 + 2. \#I_2 > N+1$. En effet, on a $ch^1(V_i) = c^1(V_i) = 0$ pour $i \in I_2$.

Prendre garde que la définition de $\det Rf_* V$ pose des problèmes de signes; les compatibilités désirées pour les isomorphismes canoniques conjecturés ci-dessus en donnent lieu à de pires.

2.3 Métriser.

Si V est un espace vectoriel sur $\mathbb{Q}_p$, la façon la plus simple de se donner une norme sur V est de se donner un $\mathbb{Z}_p$-module libre $V_0 \subset V$, tel que le morphisme

$\mathbb{Q}_p \otimes_{\mathbb{Z}_p} V_0 \to V$ soit un isomorphisme. On pose

$$\| v \| = \inf \{ \|\lambda\| \mid \lambda^{-1} v \in V_0 \} \quad .$$

En termes géométriques, V est un fibré vectoriel sur Spec $(\mathbb{Q}_p)$ et V_0 un prolongement de ce fibré sur le schéma Spec$(\mathbb{Z}_p)$.

D'après Arakelov, si V est un espace vectoriel complexe, il y a lieu par analogie de regarder une structure hermitienne sur V comme un prolongement de V -fibré vectoriel sur Spec($\mathbb{C}$)- à un objet mythique qu'il ne coûte rien de nommer Spec$(\mathbb{C})^-$. Si S est un schéma lisse sur $\mathbb{C}$, une métrique kahlérienne sur S est à regarder comme un prolongement S^- de S sur "Spec$(\mathbb{C})^-$" . Si V est un fibré vectoriel sur S , une structure hermitienne sur V est à regarder comme un prolongement de V de S à S^- . J'omet des grains de sels.

Appliquons cette philosophie à 2.1 et 2.2.

2.4 Soit f: $X \to S$ un morphisme propre et lisse entre schémas lisses sur $\mathbb{C}$, purement d'une dimension relative N . Soient I et P comme en 2.1. Si le problème 2.1.2 est résolu, on dispose d'un foncteur $I_{X/S}$ P attachant à une famille de fibrés vectoriels V_i sur X un fibré en droite $I_{X/S}\, P\, (V_i\ ,\ i \in I)$ sur S .

Problème 2.4.1 Si les V_i sont munis de métriques hermitiennes g_i , définir une métrique hermitienne sur le fibré en droites $I_{X/S}\, P\, (V_i\ ,\ i \in I)$.

Soit V un fibré vectoriel holomorphe. Pour g une structure hermitienne sur V , les formes de Chern $\underline{c}^p(V,g)$ sont des (p,p)-formes formées de classes de cohomologie les classes de Chern $c^p(V)$. Les exemples qui peuvent être traités conduisent à exiger:

Exigence 2.4.2 La métrique 2.4.1 vérifie

$$\underline{c}^1\ I_{X/S}\ P\ (V_i\ ,\ i \in I) = \int_{X/S} P\ (\underline{c}^p(V_i\ ,\ g_i)) \ .$$

On notera que X et S n'ont pas été supposés munis de métriques. Pour définir une métrique sur $I_{X/S}\, P\, (V_i\ ,\ i \in I)$, seule doit importer la donnée de métriques sur les V_i .

Remarque 2.4.3 On exige bien sûr que la métrique 2.4.1 soit de formation compatible à tout changement de base $S' \to S$. Prenant pour S' un point, on voit que le essentiel est celui d'une base réduite à un point. Ecrivons I_X ou simplement

I pour $I_{X/\mathrm{Spec}(\mathbb{C})}$. Pour X propre et lisse de dimension N et les V_i des fibrés vectoriels sur X , I P(V_i , i $\in$ I) est un espace vectoriel complexe de rang un, canoniquement attaché aux V_i . Il s'agit d'attacher à des structures hermitiennes sur les V_i une structure hermitienne sur cet espace vectoriel. On se restreint à ce cas $S = \mathrm{Spec}(\mathbb{C})$ dans la fin de 2.4.

Remarque 2.4.4 Prenons S réduit à un point et fixons X et les V_i . L'exigence (2.4.2) impose comment la métrique sur I P(V_i , i $\in$ I) change quand on change la structure hermitienne sur les V_i . Supposons en effet que des structures hermitiennes $g_i(t)$ sur les V_i dépendent d'un paramètre $t \in \mathbb{R}$. Fixons e $\neq 0$ dans I P(V_i , i $\in$ I) et soit h(t) la longueur carrée $\|e\|$ de e pour la métrique définie par les structures hermitiennes $g_i(t)$ sur les V_i . Soit A la droite affine standard, de coordonnée z , et soit sur A le fibré trivial $X_A := X \times A$ de fibre X . Soit V_{i_A} le fibré sur X_A image inverse de V_i . On munit V_{i_A} de la métrique qui au-dessus de $z \in A$ est $g_i(z\bar{z})$. Appliquant (2.4.2) à cette famille, on obtient une formule pour $\partial\bar{\partial} \log h(z\bar{z})$. On a

$$\partial\bar{\partial} \log h(z\bar{z}) \mid_{z=0} = \partial_t \log h \mid_{t=0}$$

et (2.4.2) détermine donc la dérivée de log h en t = 0 . Translatant en t , on obtient la dérivée de log h en tout point. Intégrant, on détermine le rapport h(1)/h(0) . On vérifie que le nombre obtenu ne dépend pas des chemins $g_i(t)$ reliant les $g_i(0)$ aux $g_i(1)$.

Remarque 2.4.5 Soit $a \in I$ et supposons que V_a soit une extension:

$$0 \to V'_a \to V_a \to V''_a \to 0 .$$

Une structure hermitienne g sur V_a en détermine alors sur V'_a : la structure induite g' , et sur V''_a : la structure quotient g" . Il n'y a pas lieu de considérer (V'_a , g'_a) , (V_a , g_a) , (V''_a , g''_a) comme une suite exacte "sur X^- au-dessus de $\mathrm{Spec}(\mathbb{C})^-$" au sens d'Arakelov. Modèle: pour S^- un schéma sur $\mathrm{Spec}(\mathbb{Z}_p)$, de fibre générale S sur $\mathbb{Q}_p$, une suite de fibrés vectoriels sur S^- $0 \to V' \to V \to V'' \to 0$ peut être exacte sur S sans l'être sur S^- , même si elle l'est aux points génériques de la fibre spéciale.

Si les V_i sont munis de métriques, et que $sV_i := V_i$ pour $i \neq a$, $sV_a := V'_a \oplus V''_a$, il n'y a pas lieu d'espérer que l'isomorphisme (2.1.7) entre I P (V_i , $i \in I$) et I P (sV_i , $i \in I$) soit une isométrie, pour sV_a muni de la métrique $g'_a \oplus g''_a$.

Les exigences (2.1.3) à (2.1.6), (2.4.2) déterminent uniquement le rapport entre les métriques de $IP(V_i, i \in I)$ et de $IP(sV_i, i \in I)$. Fixons en effet $a \in I$ et considérons la famille algébrique d'extensions de V''_a par V'_a déduite de V_a par image inverse par $\lambda . : V''_a \to V''_a$ $(\lambda \in \mathbb{C})$:

$$\begin{array}{ccccccccc} 0 & \to & V'_a & \to & V_a & \to & V''_a & \to & 0 \\ & & \| & & \uparrow\varphi[\lambda] & & \uparrow\lambda & & \\ 0 & \to & V'_a & \to & V_a[\lambda] & \to & V''_a & \to & 0 . \end{array}$$

Par 2.1.7, $IP(V_i, i \in I) \simeq IP(sV_i, i \in I)$ ne change pas pour V_a remplacé par $V_a[\lambda]$. Pour $\lambda \neq 0$, $\varphi[\lambda]$ est un isomorphisme de $V_a[\lambda]$ avec V_a. L'automorphisme correspondant de $IP(V_i, i \in I)$ se calcule par 2.1.6, 2.1.4. Si $g_a[\lambda]$ est une métrique sur $V_a[\lambda]$, 2.4.4 permet alors de calculer comment varie avec λ la métrique correspondante de $IP(V_i, i \in I)$, tant que $\lambda \neq 0$. On récupère le cas $\lambda = 0$ par passage à la limite et, pour $\lambda = 0$, $V_a[\lambda]$ est $V'_a \oplus V''_a$.

2.5 Soient $f: X \to S$ un morphisme propre et lisse entre schémas lisses sur $\mathbb{C}$, purement d'une dimension relative N, V un fibré vectoriel sur X et $T_{X/S}$ le fibré tangent relatif. On suppose V et $T_{X/S}$ munis de structures hermitiennes. Il y a peut-être lieu de supposer que la structure hermitienne de $T_{X/S}$ fournit une structure kählérienne sur chaque fibre $X_s := f^{-1}(s)$ de f. La réponse à (2.5.2) ci-dessous dictera la bonne hypothèse.

Sous ces hypothèses, on dispose ([14]) d'une métrique (la "torsion analytique") sur le fibré vectoriel $\det Rf_* V$. Elle est de formation compatible à tout changement de base. On la suppose correctement normalisée, cf 1.2, 1.3.

Si les problèmes 2.1, 2.2, 2.4 sont supposés résolus, on peut poser le

Problème 2.5.1 L'isomorphisme 2.2.2 est-il une isométrie?

On notera que pour ce problème, l'exposant M de 2.2.2 est sans importance. Remplacer M par $2.M$ libère 2.5.1 des questions de signes qui hantent 2.1 et 2.2.

Notons $\underline{c}^i(V)$ et $\underline{c}^i(T_{X/S})$ les formes de Chern de V et de $T_{X/S}$ définies par les métriques données. Avec les notations de (2.2.2), une réponse positive à (2.5.1) implique, par 2.4.2, une réponse positive à la

Question 2.5.2 A-t-on

$$\underline{c}^1 \det Rf_* V = \int_{X/S} RR_{N+1}(\underline{c}^*(V), \underline{c}^*(T_{X/S})) \ ?$$

Si j'ai bien compris, 2.5.2 a été prouvé par J.M. Bismut et D.S. Freed lorsque $N = 1$ ([3], [6]), i.e. pour X une famille de courbes paramétrées par S . Ils ont obtenus des résultats analogues en toute dimension (et dans des situations non holomorphes), mais j'ignore si leurs résultats couvrent 2.5.2.

Inversément, si l'égalité 2.5.2 vaut universellement (pour N fixé), l'argument 2.4.4 montre que la validité de 2.5.1 est indépendante des métriques sur V et $T_{X/S}$: le logarithme du rapport de similitude de l'isomorphisme 2.2.2 est une fonction F sur S , indépendante des métriques de V et $T_{X/S}$, de formation compatible à tout changement de base. Toujours sous l'hypothèse 2.5.2, on a

— $d'd'' F = 0$ (de sorte que F est constante si S est compacte connexe, ou si S , supposée algébrique, est connexe et qu'on peut prouver F bornée).

— Si V est une extension : $0 \to V' \to V \to V'' \to 0$, F pour V est la somme de F pour V' et V'' (cf 2.4.5).

Une compatibilité à la dualité de Serre assurera que pour X connexe de dimension N et $V' := \underline{Hom}(V , \Omega^N_X)$ le dual de Serre de V

$$F(X,V) = (-1)^{N+1} F(X,V') .$$

Dans le cas des courbes $(N = 1)$, ces propriétés suffisent à assurer que la fonction F est de la forme

$$F(X,V) = rg(V) . G(X) ,$$

donc nulle sur les fibrés virtuels de rang 0 , comme annoncé en 1.5.

2.6 Soulé me dit que 2.5.1 est trop optimiste. Il espère seulement l'existence d'une classe caractéristique correction $(T_{X/S})$ de $T_{X/S}$ telle que le rapport des métriques soit localement constant sur S , donné par

$$\exp(\textstyle\int_{X/S} \text{correction } (T_{X/S}) . \operatorname{ch}(V)) .$$

3. Analogies.

3.1 La théorie d'Arakelov est basée sur l'analogie entre
(a) $\mathbb{C}$, muni de sa valeur absolue $\|z\| = z\bar{z}$, et
(b) un corps valué complet non archimédien non discret K . On note A l'anneau de valuation de K (c'est à dire l'ensemble des $x \in K$ tels que $\|x\| \leq 1$) . On suppose la valuation v discrète, d'ensemble de valeurs $\mathbb{Z}$, on note t une uniformisante ($v(t) = 1$) et on définit le nombre q par $\|x\| = q^{-v(x)}$. L'analogie fait correspondre à $v(x)$ la quantité $\log \|z\|$. Parfois, il faut supposer le corps résiduel fini, de nombre d'éléments q . On veut remplir la case "?" du tableau

Schéma sur $\mathbb{C}$, éventuellement muni de données auxiliaires (fibré vectoriel, ...) .	Schéma sur K , éventuellement muni des mêmes données auxiliaires.
?	Schéma sur Spec(V) , prolongeant le précédent.

Le cas particulier $K = \mathbb{C}[[t]]$ suggère une troisième colonne: famille analytique de schémas, paramétrée par un petit disque épointé D* , et extension de cette famille au-dessus du disque D .

On explique ci-dessous, sous la forme d'un texte sur deux colonnes, comment parfois remplir la case "?". Dans la première (resp seconde) colonne, le corps de base est $\mathbb{C}$ (resp K).

3.2 Soit V un espace vectoriel de dimension finie.

On suppose V muni d'une structure hermitienne. On pose $\|v\| = \langle v,v\rangle$.	On suppose V muni d'une A-structure (un A-module libre $V_A \subset V$ avec $V_A \otimes_A K \xrightarrow{\sim} V$) . On pose $\|v\| = \inf \{\|\lambda\| \mid v \in \lambda V_A\}$.

On a $\|\lambda v\| = \|\lambda\| . \|v\|$.

Il y a parfois lieu de regarder une norme ultramétrique $\|v\|$ quelconque comme l'analogue d'une structure hermitienne. Dans une base convenable et pour des réels $a_i > 0$ convenables, une telle norme s'écrit

$$\|v\| = \sup(a_i \|v_i\|) .$$

Pour qu'elle corresponde à une A-structure, il faut et il suffit qu'elle soit à valeurs dans $\|K\| \subset \mathbb{R}$. Pour K' une extension galoisienne finie de K ,la restriction à V est une bijection des normes ultramétriques sur $V' := V \otimes_K K'$, invariantes par $\mathrm{Gal}(K'/K)$, avec les normes ultramétriques sur V . Ceci permet d'interpréter une norme ultramétrique sur V avec $\log \|v\| \in \mathbb{Q} . \log q$ pour tout $v \in V$ comme la donnée d'une A'-structure $\mathrm{Gal}(K'/K)$-invariante sur V' , pour K' assez grand.

3.3 Soit X un schéma propre et lisse sur $\mathbb{C}$ (resp K) .

La donnée d'une structure hermitienne sur le fibré tangent de X fait de X une variété riemannienne et permet de définir une distance $d(x,y)$.

La donnée d'un schéma propre X_A sur $\mathrm{Spec}(A)$ de fibre générale X définit une distance d sur $X(K)$: si x et y n'ont pas même réduction modulo (t) , $d(x,y) = 1$. Si x et y ont même réduction, on prend leur distance dans une carte affine $Y \subset \mathbb{A}^n$ les contenants.

A droite, une A-structure en fournit une après toute extension finie K' de K , d'où une métrique sur $X(\overline{K})$. Pour X une courbe et X_A normal, l'important n'est pas cette métrique, mais le nombre d'intersection de deux sections (fractionnaire, cf Mumford, Publ. Math. IHES 9 (1961) p 5-22, II(b) p 17).

Une A-structure normale X_A en définit une après toute extension finie des scalaires (normaliser $X_A \otimes_A A'$) et pour $s_1 , s_2 \in X(K') = X_{A'}(A')$, on pose $(s_1 , s_2)_K = 1/[K':K] . (s_1 , s_2)$. Ce nombre est indépendant de K' (assez grand pour que les s_i soient dans $X(K')$) et définit $(s_1 , s_2)_K$ pour $s_i \in X(\overline{K})$. On a $(s_1 , s_2)_K \geq 0$, avec $(s_1 , s_2)_K > 0$ si et seulement si les s_i ont même réduction.

3.4 Soit E un fibré vectoriel sur X .

Une structure hermitienne sur E définit, pour chaque $x \in X$, une structure hermitienne sur la fibre E_x de E en x .

Soit X_A propre sur $\mathrm{Spec}(A)$ prolongeant X et E_A un fibré vectoriel sur X_A prolongeant E . Soit $x \in X(K)$. C'est l'image de $x_A \in X_A(A) \xrightarrow{\sim} X(K)$ et $x_A^* E_A$ est une A-structure sur E_x .

De même pour $x \in X(K')$, K' une extension finie de K . Si le prolongement X'_A domine X_A et que E'_A est l'image inverse de E_A , (X_A , E_A) et (X'_A , E'_A) définissent les mêmes structures entières sur les E_x $(x \ni X(K)$, ou $X(K')$, ou $X(\overline{K}))$. Soit I l'ensemble ordonné filtrant des prolongements propres de X sur $\mathrm{Spec}(A)$ et $\mathrm{Ent}(E,i)$ l'ensemble des prolongements de E sur i (pris à un isomorphisme qui est l'identité sur E près) . On définit une A-structure sur E comme un élément de la limite inductive $\mathrm{Ent}(E)$ des $\mathrm{Ent}(E, i)$. C'est l'analogue d'une structure hermitienne C^∞ .

Soit I^* l'ensemble des prolongements propres normaux. Disons qu'un ouvert U de X_A dans I^* est gros s'il contient X et les points génériques de la fibre spéciale. Si X'_A et X_A sont dans I^* et que X'_A domine X_A , il existe un gros ouvert $U_A \subset X_A$ tel que $\varphi: X'_A \to X_A$ induise un isomorphisme de $\varphi^{-1}(U_A)$ avec U_A . Soit $\mathrm{Ent}^*(E,i)$ l'ensemble des germes de prolongement de E sur un gros ouvert de $i \in I^*$ (germe selon le filtre des gros ouverts). Si X'_A domine X_A et que U'_A est un gros ouvert de X'_A , il existe un gros ouvert U_A de X_A avec $U'_A \supset \varphi^{-1}(U_A) \xrightarrow{\sim} U_A$ et la restriction définit $\mathrm{Ent}^*(E , X'_A) \to \mathrm{Ent}^*(E , X_A)$. Certaines métriques hermitiennes singulières ont pour analogue des éléments de la limite projective $\mathrm{Ent}^*(E)$ des $\mathrm{Ent}^*(E,i)$. Pour X connexe, soit $X(1)$ l'ensemble des anneaux de valuation discrètes B du corps des fonctions rationnelles sur X qui sont anneau local d'un X_A propre normal en un point générique de la fibre spéciale. Un élément de $\mathrm{Ent}^*(E)$ équivaut à la donnée, pour chaque B dans $X(1)$, d'une B-structure sur la fibre générique E_η de E/X .

<u>Exemple</u> 3.4.1 Soit D un diviseur sur X .

Sur $\mathcal{O}(D)$, on définit une structure hermitienne singulière par $\|1\| = 1$	Prolongeons $\mathcal{O}(D)$ de X à $X \cup (X_A - D^-)$ en recollant $\mathcal{O}(D)$ sur X à $\mathcal{O}$ sur $X_A - D^-$ (ils sont égaux à $\mathcal{O}$ sur l'intersection $X - D$) . Ces prolongements définissent un élément de $\mathrm{Ent}^*(\mathcal{O}(D))$.

Un élément E de $\mathrm{Ent}^*(E)$ sera dit régulier sur un ouvert U de X_A contenant X s'il existe un prolongement E_U de E sur U telle que, pour tout $\varphi: X'_A \to X_A$ dominant X_A , E et φ^*E coïncident sur l'intersection de $\varphi^{-1}U$ avec un gros ouvert. L'élément ci-dessus de $\mathrm{Ent}^*(\mathcal{O}(D))$ est régulier sur $X_A - D^-$.

3.5 Dans la fin de ce paragraphe, X est une courbe propre et lisse. Soit L un faisceau inversible sur X .

Supposons L muni d'une structure hermitienne h . La forme de Chern $F_{L,h}$, ou F_h , ou simplement F est localement

$$F := \frac{1}{2\pi i} d'd'' \log \|s\|$$

pour s une section locale holomorphe inversible.

Soit L_A un prolongement de L à X_A , propre et plat sur Spec(A) , prolongeant X , supposé normal. Soit $\underline{\underline{C}}$ l'ensemble de composantes irréductibles de la fibre spéciale réduite et, pour $D \in \underline{\underline{C}}$, soit n(D) la multiplicité de D dans la fibre spéciale. Soit F_L , ou simplement F , la fonction $n(D) \deg_D (L|D)$ sur $\underline{\underline{C}}$.

On a

$$\deg L = \int_X F .$$

$$\deg L = \sum_{D \in \underline{\underline{C}}} F(D) .$$

Pour simplifier, on supposera dorénavant X_A régulier.

Se donner une structure hermitienne sur le fibré en droites trivial $\mathcal{O}$ revient à se donner la fonction réelle sur X

$$L := -\log \|1\| .$$

Les prolongements de $\mathcal{O}$ à X_A s'identifient, par $E \longmapsto \mathcal{O}(E)$, aux diviseurs concentrés sur la fibre spéciale. Pour $E = \sum m(D)D$, notons L_E la fonction sur $\underline{\underline{C}}$: $L_E(D) = m(D)/n(D)$. La fonction constante L = 1 correspond au diviseur "fibre spéciale".

Par analogie, sur $\mathbb{C}$, on appellera encore diviseur E concentré sur la fibre spéciale une métrique L_E sur $\mathcal{O}$, ou la fonction L_E correspondante, et multiple de la fibre une métrique constante sur $\mathcal{O}$.

3.6 Soient E et F deux diviseurs concentrés sur la fibre spéciale. Leur nombre d'intersection (E,F) :=

$$\frac{-1}{2\pi i} \int_X \log \|1\|_E . d'd'' \log \|1\|_F$$

$$= \int_X L_E F_F$$

Intégrand par partie, on obtient (E,F) =

$$\sum (-1)^{i+j} \lg_A H^i(X_A , \mathrm{Tor}^j(\mathcal{O}_E , \mathcal{O}_F))$$

$$= \chi(\mathcal{O}_E \otimes \mathcal{O}_F) = \chi(\mathcal{O}_E) - \chi(\mathcal{O}_E \otimes \mathcal{O}(-F))$$

$$= \sum L_E(D)\, F_{\mathcal{O}(F)}(D)$$

Soit < un ordre total sur $\underline{\underline{C}}$ et, pour $D' \neq D''$ dans $\underline{\underline{C}}$, soit $c(D',D'') := (n(D')D' , n(D'')D'') \geq 0$. Utilisant que $F_{\mathcal{O}(F)}$ est nul pour un diviseur F multiple de la fibre, on obtient (E,F) =

$\frac{1}{2\pi i}\int d'\log \|1\|_E\,(d'\log\|1\|_F)^- \qquad -\sum_{i<j} c(i,j)\,(L_E(i)-L_E(j))(L_F(i)-L_F(j))$.

De là résulte que la forme (,) est symétrique semi-définie négative et que, si X est connexe, son noyau est réduit aux multiples de la fibre spéciale.

3.7 Un <u>diviseur compactifié</u> E est la donnée d'un diviseur $E_{\mathbb{C}}$ sur X et d'une structure hermitienne sur $\mathcal{O}(E_{\mathbb{C}})$. Soit encore $L_E := -\log \|1\|$. Deux diviseurs compactifiés E et F sont <u>disjoints</u> si L_E et L_F sont à supports disjoints.

La construction $E \mapsto$(diviseur E_K sur X , prolongement $\mathcal{O}(E)$ de $\mathcal{O}(E_K)$) identifie un diviseur sur X_A à un diviseur E_K sur X muni d'un prolongement à X_A du faisceau inversible $\mathcal{O}(E_K)$ sur X .

Si $E_{\mathbb{C}} = \sum n_i P_i$, la fonction réelle L_E est C^∞ hors des P_i , avec une singularité en $n_i \log z\bar{z}$ en chaque P_i . Si d'd" est pris au sens des distributions, la forme de Chern F_E de $\mathcal{O}(E_{\mathbb{C}})$ est

$$F = \frac{-1}{2\pi i}\, d'd''L_E + \sum n_i\, \delta_{P_i} .$$

Si E et F sont deux diviseurs compactifiés avec $E_{\mathbb{C}}$ et $F_{\mathbb{C}}$ disjoints, on définit leur <u>nombre d'intersection</u> par

$$(E,F) := \frac{-1}{2\pi i}\int d'd''L_E \cdot L_F + L_E\,[F_{\mathbb{C}}] + L_F\,[E_{\mathbb{C}}]$$
$$= \int F_{\mathcal{O}(E)} \cdot L_F + L_E\,[F_{\mathbb{C}}]$$

(notation [ ] de 1.5.1). Ce produit d'intersection est caractérisé par sa symétrie, son additivité, le fait qu'il est donné par $\int F_E L_F$ pour F à support dans la fibre spéciale et que $(E,F) = 0$ pour E et F disjoints.

Analogue: Si les diviseurs E et F sur X_A ne se coupent que sur la fibre spéciale, le nombre d'intersection (E,F) est encore défini. Si F est à support dans la fibre spéciale, on a

$$(E,F) = \sum F_{\mathcal{O}(E)}(C) \cdot L_F(C) .$$

4. Objets virtuels.

Dans ce paragraphe nous montrons comment attacher à chaque catégorie exacte A([12] p91) la catégorie de ses objets virtuels $V(A)$. Cette catégorie est un avatar, à la Grothendieck [9], des deux premiers étages de la tour de Postnikov de l'espace de lacets infinis de groupes d'homotopie les $K_i(A)$ de Quillen.

4.1 Rappelons qu'une catégorie de Picard est une catégorie P , non vide, dont toute flèche est un isomorphisme, munie d'un foncteur $+ : P \times P \to P$ et d'une contrainte d'associativité pour + (Saavedra [15] I 1.1., Mac Lane [11]) et telle que pour chaque objet P les foncteurs $X \mapsto P + X$ et $X \mapsto X + P$ soient des autoéquivalences de P . Ces axiomes impliquent l'existence d'un objet unité 0 . (Saavedra [15] I 1.3), unique à isomorphisme unique près, et que chaque objet X admet un opposé: un objet -X muni d'un isomorphisme $X + (-X) \xrightarrow{\sim} 0$. Un tel objet est unique à isomorphisme unique près.

Une catégorie de Picard commutative est une catégorie de Picard munie d'une contrainte de commutativité compatible à la contrainte d'associativité (Saavedra [15], I 1.2, Mac Lane [11]). Dans une telle catégorie, on sait définir la "somme" d'une famille finie $(X_i)_{i \in I}$ d'objets, avec les propriétés habituelles. Mise en garde: pour $X = Y$, l'isomorphisme de symétrie $X + Y \to Y + X$ n'est en général pas l'identité. Ce n'est pas inquiétant: on a la même chose pour la somme directe d'espaces vectoriels.

Le foncteur $X \mapsto -X$ est muni de la donnée de compatibilité à + suivante: l'isomorphisme $(X + Y) + ((-X) + (-Y)) = (X + (-X)) + (Y + (-Y)) = 0 + 0 = 0$ fait de $(-X) + (-Y)$ un opposé de $X + Y$. Cette donnée est compatible à l'associativité et à la commutativité. Le foncteur $X \mapsto -X$ est involutif: par symétrie de + , l'isomorphisme $X + (-X) = 0$ définit un isomorphisme $(-X) + X = 0$ qui fait de X l'opposé de -X : $X \xrightarrow{\sim} -(-X)$.

Mise en garde: le diagramme

(4.1.1)
$$\begin{array}{ccc} ((-X) + X) + (-X) & = & (-X) + (X + (-X)) \\ \| & & \| \\ 0 + (-X) = (-X) & = & (-X) + 0 \end{array}$$

n'est en général pas commutatif. A cause de (4.1.1), il faut être prudent quand on écrit un isomorphisme par = et qu'on soumet ce = aux règles du calcul algébrique.

Posons $A - B := A + (-B)$. D'un isomorphisme φ_1 entre A et $B + X$, noté $A = B + X$, on déduit par addition de $(-X)$ aux deux membres un isomorphisme $A - X = B + X - X = B$. Ajoutant X , on trouve un isomorphisme $\varphi_2 : A = B + X$ qui diffère en général de l'original. Par la suite, nous attacherons en principe à un isomorphisme $A - X = B$ l'isomorphisme $A = B + X$ qui lui donne naissance par addition de $(-X)$.

Soit $\mathcal{P}$ une catégorie de Picard commutative. Nous noterons $\pi_0(\mathcal{P})$ (resp $\pi_1(\mathcal{P})$) le groupe commutatif des classes d'isomorphie d'objets de $\mathcal{P}$ (resp le groupe commutatif des automorphismes de 0) . Le foncteur $Y \mapsto X + Y$, évalué en $Y = 0$, identifie $\pi_1(\mathcal{P})$ au groupe des automorphismes d'un quelconque objet X de $\mathcal{P}$. En particulier, chaque objet X définit $\varepsilon(X) \in \pi_1(\mathcal{P})$: l'automorphisme de symétrie de $X + X$. On vérifie que $\varepsilon : \pi_0(\mathcal{P}) \to \pi_1(\mathcal{P})$ est additif. Le défaut de commutativité du diagramme (4.1.1) ci-dessus est $\varepsilon(X)$ et les isomorphismes φ_1 et φ_2 ci-dessus diffèrent par $\varepsilon(X)$. Du point de vue de [9], $\mathcal{P}$ correspond à un espace de lacets infini n'ayant que deux groupes d'homotopie consécutifs non nuls et ε est l'action du générateur du groupe d'homotopie stable $\pi_{n+1}(S^n) = \mathbf{Z}/(2)$.

<u>Exemple</u>. La catégorie $\mathcal{P}$ des fibrés en droites gradués sur un schéma S , avec la contrainte de commutativité donnée par la règle de Koszul. Les objets de $\mathcal{P}$ s'identifient aux paires $(a,\mathcal{L})$, a un entier localement constant sur S (i.e. $a \in \Gamma(S,\mathbf{Z})$) , $\mathcal{L}$ un faisceau inversible. On a $(a,\mathcal{L}) + (b,\mathcal{M}) = (a + b , \mathcal{L} \otimes \mathcal{M})$ et le morphisme de symétrie est $\ell m \mapsto (-1)^{ab} m\ell$. On a $\pi_0 = \Gamma(S,\mathbf{Z}) \times \mathrm{Pic}(S)$, $\pi_1 = \Gamma(S,\mathcal{O}^*)$ et $\varepsilon((a,\mathcal{L})) = (-1)^a$. L'objet 0 est $(0,\mathcal{O})$. L'opposé de A dans $\mathcal{P}$ se notera A^{-1} . Pour $A = (a,\mathcal{L})$, on l'identifiera à $(-a,\mathcal{L}^\vee)$ par la donnée de l'isomorphisme évident

$$(-a,\mathcal{L}^\vee) + (a,\mathcal{L}) = (0 \ , \ \mathcal{L}^\vee \otimes \mathcal{L}) = (0,\mathcal{O}) \ ,$$

i.e. par $(-1)^a$ fois la donnée de l'isomorphisme évident

$$(a,\mathcal{L}) \otimes (-a,\mathcal{L}^\vee) \xrightarrow{\sim} (0,\mathcal{O}) \ .$$

4.2 Soit $\mathcal{A}$ une catégorie exacte. Pour définir les groupes $K_i\mathcal{A}$, Quillen [12] construit une catégorie $Q\mathcal{A}$ de mêmes objets que $\mathcal{A}$. En terme de la réalisation géométrique $BQ\mathcal{A}$ de $Q\mathcal{A}$ ([12] §1) , munie du point base 0 définit par un objet zéro 0 de $\mathcal{A}$, la <u>catégorie</u> $V(\mathcal{A})$ <u>des objets virtuels de</u> $\mathcal{A}$ admet la description suivante:

(a) un objet de $V(A)$ est un lacet de point base 0 de BQA ;
(b) une flèche de γ_1 à γ_2 est une classe d'homotopie d'homomotopies de γ_1 à γ_2 (avec respect du point base).

La composition des lacets fait de $V(A)$ une catégorie de Picard. La somme directe dans A induit une opération $\oplus$ dans QA et $\oplus : BQA \times BQA \to BQA$ fait de BQA un H-espace associatif et commutatif. De là, une contrainte de commutativité pour $+$: notant 0 le lacet trivial, on a un isomorphisme dans $V(A)$

(4.2.1) $\gamma_1 \oplus \gamma_2 = (\gamma_1 + 0) \oplus (0 + \gamma_2) = (\gamma_1 \oplus 0) + (0 \oplus \gamma_2) = \gamma_1 + \gamma_2$

et la commutativité de $\oplus$ fournit la contrainte

(4.2.2) $\gamma_1 + \gamma_2 = \gamma_1 \oplus \gamma_2 = \gamma_2 \oplus \gamma_1 = \gamma_2 + \gamma_1$.

Elle fait de $V(A)$ une catégorie de Picard commutative, avec pour $i = 0 , 1$ $\pi_i(V(A)) = K_i(A)$.

4.3 Voici une description plus algébrique de $V(A)$. Soit (A , is) la catégorie d'objets ceux de A , et de flèches les isomorphismes de A . Pour P une catégorie de Picard, considérons les foncteurs $[\ ]: (A , \text{is}) \to P$, munis de données (a) (b) ci-dessous soumises aux axiomes (c) (d).

(a) (donnée d'additivité) la donnée pour toute suite exacte courte $\Sigma : A' \to A \to A''$ ($A' \to A$ monomorphisme admissible, $A \to A''$ épimorphisme admissible) d'un isomorphisme $\{\Sigma\} : [A] \to [A'] + [A'']$, fonctoriel pour les isomorphismes de suites exactes.

(b) pour les objets zéro de A , un isomorphisme $[0] \to 0$.

(c) Soit $\varphi: A \to B$ un isomorphisme, et Σ la suite exacte $0 \to A \to B$ (resp $A \to B \to 0$) . Alors, $[\varphi]$ (resp $[\varphi]^{-1}$) est le composé

$$[A] \xrightarrow[\{\Sigma\}]{} [0] + [B] \xrightarrow[(b)]{} [B]$$

$$(\text{resp}\quad [B] \xrightarrow[\{\Sigma\}]{} [A] + [0] \xrightarrow[(b)]{} [A]\ ) .$$

(d) (associativité) Pour C muni d'une filtration admissible à trois crans: $C \supset B \supset A \supset 0$, le diagramme d'isomorphismes (a)

$$\begin{array}{ccc} [C] & \longrightarrow & [A] + [C/A] \\ \downarrow & & \downarrow \\ [B] + [C/B] & \longrightarrow & [A] + [B/A] + [C/B] \end{array}$$

est commutatif.

On notera que (c) détermine uniquement la donnée (b). Il nous sera souvent commode d'omettre (b) et de modifier (c) en

(c') (compatibilité aux zéros) il existe une donnée (b) vérifiant (c).

Si la catégorie de Picard P est commutative, nous dirons qu'une donnée d'additivité (a) est commutative, ou compatible à la commutativité si pour $A = A' \oplus A''$, les suites exactes courtes $\Sigma : A' \to A \to A''$ et $\Sigma' : A'' \to A \to A'$ donnent lieu à un triangle commutatif

(4.3.1)
$$\begin{array}{ccc} [A'] + [A''] & \longrightarrow & [A''] + [A'] \\ & {\scriptstyle \{\Sigma\}} \searrow \quad \swarrow {\scriptstyle \{\Sigma'\}} & \\ & [A' \oplus A''] & \end{array} .$$

Pour chaque catégorie de Picard P, les foncteurs $[\ ] : (A,\mathrm{is}) \to P$ munis d'une donnée d'additivité $\{\}$ vérifiant (c') (d) forment une catégorie. Un argument standard montre qu'il existe un système $(P, [\ ], \{\ \})$ universel, noté $(P^{\mathrm{un}}, [\ ], \{\ \})$, tel que pour chaque P, cette catégorie de foncteurs $[\ ]$ soit équivalente à celle des foncteurs additifs $P^{\mathrm{un}} \to P$. Nous allons esquisser la vérification de ce que la catégorie P^{un} universelle est $V(A)$. La donnée de commutativité de $V(A)$ sera caracterisée par la compatiblité de $[\ ]$ à la commutativité.

4.4 Esquisse. Appelons groupoïde de Picard indexé par un ensemble I la donnée de

(a) pour $x,y \in I$, une catégorie $P_{y,x}$ dont toute flèche soit un isomorphisme;

(b) Pour $x,y,z \in I$, une composition $\circ : P_{z,y} \times P_{y,x} \to P_{z,x}$ telle que tout P dans $P_{z,y}$ (resp $P_{y,x}$) le foncteur $X \longmapsto P \circ X$ (resp $X \longmapsto X \circ P$) soit une équivalence;

(c) une donnée d'associativité pour $\circ$ (vérifiant l'axiome du pentagone).

Soit Q une catégorie. Considérons les groupoïdes de Picard P indexés par $\mathrm{Ob}(Q)$, munis des données suivantes: (e) pour toute flèche $\varphi : x \to y$, $[\varphi]$ dans $P_{y,x}$ est donné; (f) pour toute paire de flèches composables $\Sigma = (\varphi_1, \varphi_2)$, un isomorphisme $\{\Sigma\} : [\varphi_2 \circ \varphi_1] \simeq [\varphi_2] \circ [\varphi_1]$ est donné; on exige qe pour trois flèches composables, le diagramme

$$[\varphi_3 \circ \varphi_2 \circ \varphi_1] \simeq [\varphi_3 \circ \varphi_2] \circ [\varphi_1]$$

$$[\varphi_3] \circ [\varphi_2 \circ \varphi_1] \simeq [\varphi_3] \circ [\varphi_2] \circ [\varphi_1]$$

soit commutatif (compatibilité entre associativités).

Pour P un groupoïde de Picard sur $\mathrm{Ob}(Q)$, les données (e) vérifiant (f) forment une catégorie. Un argument standard assure l'existence de $(P^{un}, [\ ], \{\ \})$ universel, i.e. tel que pour tout P la catégorie des foncteurs T compatibles à 0 de P^u dans P soit équivalente par $T \longmapsto T[\ ]$, à celle des données (e) vérifiant (f). Notons Π ce groupoïde de Picard universel.

La description simpliciale de π_1 et π_2 montre que pour tout objet q de Q, $\Pi_{q,q}$ est la catégorie de Picard des lacets de point base q dans BQ, et des classes d'homotopie d'homotopies entre lacets.

Supposons donné un objet q de Q et, pour chaque objet x, une flèche α_x de q vers x. Pour tout système $(P, [\ ], \{\ \})$, les foncteurs $[\alpha_y]^{-1} \circ u \circ [\alpha_x]$ identifient les $P_{y,x}$ à $P_{q,q}$, et il revient au même (équivalence de 2-catégories) de se donner un système $(P, [\ ], \{\ \})$ ou un système du type suivant. Une catégorie de Picard P, munie de: (i) pour tout $\varphi\colon x \to y$, un objet $[\varphi]$ de P ; (ii) pour toute paire $\Sigma = (\varphi_1, \varphi_2)$ de morphismes composables, un isomorphisme $\{\Sigma\} : [\varphi_2 \circ \varphi_1] \xrightarrow{\sim} [\varphi_2] \circ [\varphi_1]$. Cette donnée est supposée compatible à l'associativité. Enfin (iii) des isomorphismes $[\alpha_x] \sim 0$.

Soient A une catégorie exacte et $Q = Q(A)$. Pour tout monomorphisme admissible $i : A \to B$ (resp épimorphisme admissible $j : B \to A$) notons comme dans [12] $\iota_!$ (resp $j^!$) la flèche correspondante, dans $Q(A)$, de A dans B. Pour φ un isomorphisme, on écrit simplement $\varphi := \varphi_! = (\varphi^{-1})^!$. Fixons un objet zéro 0 et notons z_A et z^A les morphismes $A \to 0$ et $0 \to A$. Il reste à vérifier que pour $Q = Q(A)$ muni de $q := 0$ et des $\alpha_x := z^x_!$, une donnée $(P$, (i) à (iii)) comme ci-dessus équivaut à une donnée $(P$, (a) à (d)) comme en 4.3. Rigoureusement, "vérifier" signifie "construire une équivalence de 2-catégories".

4.5 Soit $(P$, (i) à (iii)). Pour tout objet A de A, posons $[A] = [z^!_A]$. Pour tout isomorphisme $\varphi\colon A \to B$, on a $\varphi z^A_! = z^B_!$ et $\varphi z^!_A = z^!_B$. La première égalité fournit par (ii) un isomorphisme $[\varphi][z^A_!] = [z^B_!]$ et par (iii) un isomorphisme $[\varphi] \sim 0$. La deuxième égalité fournit alors par (ii) un isomorphisme encore noté φ de $[A]$ avec $[B]$. On vérifie que cette construction définit un foncteur de (A,is) dans P.

Soit $u : A \to B$ dans $Q(\mathcal{A})$, i.e. un isomorphisme de A avec un sous-quotient B_2/B_1 de B . Pour i l'inclusion de B_1 dans B , le diagramme de $Q(\mathcal{A})$

$$\begin{array}{ccc} A & \xrightarrow{\ u\ } & B \\ {\scriptstyle z_!^A}\uparrow & & \uparrow{\scriptstyle i_!} \\ 0 & \xrightarrow{\ z_{B_1}^!\ } & B_1 \end{array}$$

est commutatif, et $i_! \, z_!^{B_1} = z_!^B$. Ce diagramme fournit donc dans $BQ\mathcal{A}$ une homotopie du lacet $(z_!^B)^{-1} u \, z_!^A$ avec le lacet $(z_!^{B_1})^{-1} z_B^!$, i.e un isomorphisme $[u] \simeq [B_1]$. Si on remplace (B_1 , i) par (B_1' , i') représentant le même sous-objet de B , et que $\varphi: B_1 \to B_1'$ est l'unique isomorphisme tel que $i'\varphi = i$, on vérifie que le diagramme

$$\begin{array}{ccc} & [u] & \\ \swarrow\!\!\sim & & \sim\!\!\searrow \\ [B_1] & \xrightarrow{\ \varphi\ } & [B_1'] \end{array}$$

est commutatif. La construction $u \longmapsto [u]$ est ainsi déterminée à isomorphisme unique près par le foncteur $[\ ] : (\mathcal{A},is) \to \mathcal{P}$. Pour $u = z_!^B : 0 \to B$, on a $B_1 = 0$ et les isomorphismes (iii) proviennent d'un isomorphisme $[0] \sim 0$.

Soit dans $\mathcal{A}$ une suite exacte courte admissible Σ :

$$A \overset{i}{\hookrightarrow} B \overset{j}{\twoheadrightarrow} C .$$

Elle définit dans $Q\mathcal{A}$ un diagramme commutatif

$$\begin{array}{ccc} C & \xrightarrow{\ j^!\ } & B \\ {\scriptstyle z_!^C}\uparrow & & \uparrow{\scriptstyle i_!} \\ 0 & \xrightarrow{\ z_A^!\ } & A \end{array}$$

et $i_! \, z_!^A = z_!^B$ et $j^! \, z_C^! = z_B^!$. De là une homotopie dans $BQ\mathcal{A}$ entre les lacets $(z_!^B)^{-1} \, z_B^!$ et $(z_!^A)^{-1} z_A^! \, (z_!^C)^{-1} z_C^!$, i.e. un isomorphisme $\{\Sigma\} : [B] \sim [A] + [C]$. On vérifie que les conditions (c') (d) de 4.3 sont vérifiées par $\mathcal{P}$, le foncteur $A \longmapsto [A]$ et les isomorphismes $\{\Sigma\}$.

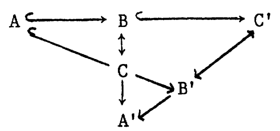
Réciproquement, soit $(\mathcal{P}, [\ ] , \{\ \})$ comme en 4.3. Pour $u: A \to B$ un morphisme dans $Q(\mathcal{A})$, i.e un isomorphisme de A avec un sous-quotient B_2/B_1 de B , on pose $[u] := [B_1]$. On dispose d'un isomorphisme $[z_!^A] = [0] = 0$.

Soit un morphisme composé

$$A \xrightarrow{\ u\ } B \xrightarrow{\ v\ } C \ :$$

u (resp v) est un isomorphisme de A (resp B) avec un sous-quotient B_2/B_1 (resp C_3/C_0) de B (resp C) . Soit C_i l'image inverse dans $C_3 \subset C$ de $B_i \subset B \sim C_3/C_0$, de sorte que vu est l'isomorphisme $A \xrightarrow{\sim} B_2/B_1 \xrightarrow{\sim} C_2/C_1$. On a $[u] = [B_1]$, $[v] = [C_0]$, $[vu] = [C_1]$ et la suite exacte courte $0 \to C_0 \to C_1 \to B_1 \to 0$ fournit un isomorphisme $[vu] = [C_1] = [C_0][B_1] = [v][u]$. Les conditions de (i) à (iii) sont vérifiées, et cette construction est inverse de la précédente.

4.6 Variante. On peut encore définir $V(A)$ par le problème universel suivant, variante de celui considéré en 4.3. On veut une catégorie de Picard munie de (a') la donnée pour tout objet A de $\mathcal{A}$ d'un objet $[A]$ de P ; (b') la donnée pour chaque suite exacte courte admissible Σ : $A' \hookrightarrow A \longrightarrow A''$ d'un isomorphisme $\{\Sigma\}$: $[A] \to [A'] + [A'']$, ces données satisfaisant à l'axiome: (c') pour tout diagramme commutatif



avec (A , B , C') , (B , C , A') , (A , C , B') (et donc (C' , B' , A')) des suites exactes courtes admissibles, le diagramme

$$\begin{array}{ccc} [C] & \longrightarrow & [A] + [B'] \\ \downarrow & & \downarrow \\ [B] + [A'] & \longrightarrow & [A] + [C'] + [A'] \end{array}$$

est commutatif.

C'est apparemment plus simple que 4.3, mais il est périlleux de ne pas imposer a priori la fonctorialité pour les isomorphismes. Avec cette définition, il ne serait pas évident d'emblée que des catégories équivalentes donnent lieu à des catégories d'objets virtuels équivalentes. On ne pourrait plus appliquer [ ] à un objet défini à un isomorphisme unique près. C'est pourquoi dans (c') apparaît C' , non B/A ,... . On vérifie toutefois que (a') - (c') équivaut à (a) - (d) :

pour Z un objet zéro, on reconstitue (b) : $[Z] \to 0$ en considérant la suite exacte courte $Z \to Z \to Z$ et pour φ: $A \to B$ un isomorphisme, on reconstitue $[\varphi]$ en considérant la suite exacte $Z \to A \to B$. On vérifie (a) - (d) .

4.7 Pour X muni d'une filtration admissible finie F , un usage itéré de 4.3 (a) fournit un isomorphisme dans $V(A)$: $[X] \xrightarrow{\sim} \sum[\mathrm{Gr}_F^i(X)]$. Pour X muni d'une bifiltration admissible, le diagramme

$$\begin{array}{ccccc}
\sum[\mathrm{Gr}_G^j(X)] & \xleftarrow{\sim} & [X] & \xrightarrow{\sim} & \sum[\mathrm{Gr}_F^i(X)] \\
\downarrow & & & & \downarrow \\
\sum[\mathrm{Gr}_F^i\,\mathrm{Gr}_G^j(X)] & & = & & \sum[\mathrm{Gr}_G^j\,\mathrm{Gr}_F^i(X)]
\end{array}$$

est commutatif. On le déduit par récurrence sur la longueur des filtrations du lemme suivant

Lemme 4.8 Soit un diagramme commutatif à lignes et colonnes des suites exactes courtes admissibles

$$\begin{array}{ccccc}
X_1^1 & \hookrightarrow & X^1 & \twoheadrightarrow & X_2^1 \\
\downarrow & & \downarrow & & \downarrow \\
X_1 & \hookrightarrow & X & \twoheadrightarrow & X_2 \\
\downarrow & & \downarrow & & \downarrow \\
X_1^2 & \to & X^2 & \twoheadrightarrow & X_2^2
\end{array}$$

Le diagramme

$$\begin{array}{ccccc}
\sum[X^j] & \text{———} & [X] & \text{———} & \sum[X_i] \\
| & & & & | \\
\sum[X_i^j] & & = & & \sum[X_i^j]
\end{array}$$

est commutatif.

Preuve: Si $X_1^1 = X_2^2 = 0$, on a $X = X_2^1 \oplus X_1^2$ et le lemme se réduit à (4.3.1). Dans le cas général, il s'agit de comparer les isomorphismes de $[X]$ avec la somme des $[X_i^j]$ définis par les filtrations

$$X_1^1 \subset X^1 \subset X^1 + X_1 \subset X \qquad \text{et}$$
$$X_1^1 \subset X_1 \subset X^1 + X_1 \subset X ,$$

de quotients successifs les X_i^j . Ces filtrations commençant toutes deux par $X_1^1 = X^1 \cap X_1$ et finissant par $X^1 + X_1$, on se ramène au même problème pour $X^1 + X_1/X^1 \cap X_1$, i.e. au cas où $X_1^1 = X_2^2 = 0$.

4.9 Pour A dans A, soit u le morphisme diagonal de A dans $A \oplus A$, et v: $(a,b) \longmapsto a-b$ de $A \oplus A$ dans A . La suite $0 \to A \xrightarrow{u} A \oplus A \xrightarrow{v} A \to 0$ est exacte admissible et l'automorphisme de symétrie de $A \oplus A$ induit l'identité (resp -1) sur le sous-objet (resp le quotient) A de $A \oplus A$. On en déduit que $\varepsilon(A)$ est défini par l'automorphisme de [A] induit par -1 .

4.10 La catégorie dérivée $D^b(A)$ est le quotient de la catégorie triangulée des complexes bornés à homotopie près $K^b(A)$ par la sous-catégorie épaisse des complexes acycliques. Si P est une catégorie de Picard commutative, munie de $[\ ] : (A,is) \to P$ vérifiant 4.3 (a) à (d) et (4.3.1), les arguments de [10] fournissent une extension encore notée [ ] de $D^b(A)$ dans P . Le cas universel fournit

$$[\ ] : (D^b(A)\ ,\ is) \longrightarrow V(A)$$

4.11 Un foncteur exact $T: A \to B$ induit $T: QA \to QB$. La définition topologique de $V(A)$ (4.2) fournit donc $T: V(A) \to V(B)$. En termes catégoriques, $[\ ] \circ T : (A,is) \to V(B)$ est muni de données 4.3 (a) à (d), donc défini un morphisme de catégories de Picard $V(A) \to V(B)$. Ce morphisme induit les morphismes définis par $T: K_i(A) = \pi_i(V(A)) \to K_i(B) = \pi_i(V(B))$ pour $i = 0, 1$. En particulier, si T induit des isomorphismes $K_i(A) \to K_i(B)$ pour $i = 0, 1$, T induit une équivalence $V(A) \to V(B)$.

Une catégorie exacte et sa duale ont même catégorie Q , donc mêmes catégories d'objets virtuels. En termes catégoriques: un foncteur exact contravariant $T: A \to B$ définit encore $T: (A,is) \to V(B)$ muni de donnée (4.3 (c) à (f), d'où encore $T: V(A) \to V(B)$. Pour $u: X \to Y$ un isomorphisme, $Tu: [TX] \to [TY]$ est $[Tu^{-1}]$.

Un foncteur biexact $\otimes: A \times B \to C$ définit

$$V(A) \times V(B) \to V(C) ,$$

distributif par rapport à + . On prendra garde à la trappe suivante.

(a) Soit $\otimes: P_1 \times P_2 \to P$ un foncteur distributif par rapport à + entre catégories de Picard commutative. Par définition, celà signifie que, à X ou Y fixe,

$X \otimes Y$ est un morphisme de catégories de Picard commutatives, et que tout diagramme

$$\begin{array}{ccc} \sum X_i \otimes \sum Y_j & \text{———} & \sum (X_i \otimes \sum Y_j) \\ | & & | \\ \sum (\sum X_i \otimes Y_j) & \text{———} & \sum_{i,j} X_i \otimes Y_j \end{array}$$

est commutatif.

Si $-Y$ est l'opposé de Y, l'isomorphisme
$$X \otimes Y + X \otimes -Y = X \otimes (Y + -Y) = X \otimes 0 = 0$$
fait de $X \otimes -Y$ l'opposé de $X \otimes Y$: $X \otimes (-Y) \xrightarrow{\sim} -(X \otimes Y)$. De même, $(-X) \otimes Y \xrightarrow{\sim} -(X \otimes Y)$.

(b) Le diagramme

$$\begin{array}{ccc} (-X) \otimes (-Y) & \xrightarrow{\sim} & -((-X) \otimes Y) \\ \downarrow\wr & & \downarrow\wr \\ -(X \otimes (-Y)) & \xrightarrow{\sim} & --(X \otimes Y) = X \otimes Y \end{array}$$

n'est en général pas commutatif. Son défaut de commutativité est $\varepsilon(X \otimes Y)$. La vérification est laissée au lecteur.

4.12 Comme illustré par 4.11, les fonctorialités connues pour les K_i fournissent en général -avec la même construction- une fonctorialité pour les catégories d'objets virtuels.

Pour S un schéma, soit Vect(S) la catégorie exacte des fibrés vectoriels sur S . Pour S noethérien, soit Coh(S) la catégorie abélienne des faisceaux cohérents sur S . On pose $\underline{K}(S) = V(\text{Vect}(S))$ et $\underline{K}'(S) = V(\text{Coh}(S))$. Le résultat analogue pour les K_i ([12] 7.1) montre que $\underline{K}(S) \to \underline{K}'(S)$ est une équivalence pour S régulier (et séparé). De même, si $f\colon X \to S$ est un morphisme lisse, ou simplement à fibres régulières, et que X est quasi-projectif, la catégorie des faisceaux cohérents sur X plats sur S (ou simplement de Tor-dimension finie sur S) donne lieu aux mêmes groupes K , et aux mêmes objets virtuels, que Vect(S) .

Un morphisme de schémas $f\colon X \to S$ définit $f^*\colon \underline{K}(S) \to \underline{K}(X)$. Si f est propre et plat et X quasi-projectif, il définit aussi $f_*\colon \underline{K}(X) \to \underline{K}(S)$. Si Vect'(X) est la catégorie exacte des fibrés vectoriels sur X avec $R^i f_* V = 0$

pour $i > 0$, il résulte en effet du dual de [12] §4 cor.3 que $K_i(\mathrm{Vect}'(S)) \xrightarrow{\sim} K_i(\mathrm{Vect}(S))$. Le foncteur f_* induit un foncteur exact de $\mathrm{Vect}'(X)$ dans $\mathrm{Vect}(S)$ et on définit f_* comme le composé

$$\underline{K}(X) = V(\mathrm{Vect}(X)) \xleftarrow{\sim} V(\mathrm{Vect}'(X)) \xrightarrow{f_*} V(\mathrm{Vect}(S)) = \underline{K}(S)$$

(cf [12] 7.2.7).

4.13 Exemple Soit X un schéma. Le rang $\mathrm{rg}(V)$ d'un fibré vectoriel V sur X est un entier localement constant sur X , i.e. une section de $\Gamma(X,\mathbf{Z})$. Le déterminant $\det(V)$ de V est le fibré en droites gradué puissance extérieure maximale de V , en degré $\mathrm{rg}(V)$, i.e, avec les notations de 4.1 (exemple), $\det(V) := (\mathrm{rg}(V)\ ,\ \overset{\mathrm{rg} V}{\bigwedge} V)$. Pour chaque suite exacte courte

$$V' \hookrightarrow V \twoheadrightarrow V'' \quad ,$$

on dispose d'un isomorphisme

$$\det(V') \otimes \det(V'') \xrightarrow{\sim} \det(V) \ :$$

localement, pour (e'_i) $(1 \leq i \leq n)$ une base de V' et (e''_i) $(1 \leq i \leq m)$ une base de V'' , avec e''_i relevé en $\tilde{e}''_i$ dans V'' ,

$$(e'_1 \wedge \ldots \wedge e'_n) \otimes (e''_1 \wedge \ldots \wedge e''_m) \longmapsto e'_1 \wedge \ldots \wedge e'_n \wedge \tilde{e}''_1 \wedge \ldots \wedge \tilde{e}''_m \ .$$

Les conditions de 4.3 sont vérifiées. La propriété universelle de $\underline{K}(X) = \mathrm{Vect}(X))$ assure donc une factorisation de det par un foncteur, encore noté det , de $\underline{K}(X)$ dans la catégorie de Picard P des fibrés en droites gradués. Si $V = V' \oplus V''$, les suites exactes $V' \hookrightarrow V \twoheadrightarrow V''$ et $V'' \hookrightarrow V \twoheadrightarrow V'$ donnent lieu à un triangle commutatif

$$\begin{array}{ccc} \det(V') \otimes \det(V'') & \xrightarrow{\text{Koszul}} & \det(V'') \otimes \det(V') \\ & \searrow \quad \swarrow & \\ & \det(V) & \end{array}$$

Le foncteur det est donc compatible à la commutativité. C'est ce qui justifie de définir la contrainte de commutativité de P par la règle de Koszul.

Si X est le spectre d'un anneau local A , on a $K_0(A) = \mathbf{Z}$ et $K_1(A) = A^*$. Il en résulte que dans ce cas $\det \underline{K}(X) \to P$ est une équivalence.

Si X est régulier, le foncteur $\underline{K}(X) \to \underline{K}'(X)$ est une équivalence (4.12). De là, une extension du foncteur det de $\mathrm{Vect}(X)$ à $\mathrm{Coh}(X)$:

$$\mathrm{Coh}(X) \longrightarrow \underline{K}'(X) \xleftarrow{\sim} \underline{K}(X) \longrightarrow P \quad .$$

Plus généralement, sur X quelconque, on dispose ([10]) d'une extension de det à la sous-catégorie $D^b_{ctf}(X)$ de la catégorie dérivée $D(X,\mathcal{O})$: complexes bornés, de tor dimension finie et d'∞-présentation finie (= sur X noethérien: à cohomologie cohérente).

4.14 La catégorie de Picard P des fibrés en droites gradués est un quotient de la catégorie de Picard des fibrés virtuels, et les foncteurs $\otimes$ et dual passent au quotient. Pour le dual, on définit le foncteur dual par

$$\mathrm{dual}\ (n,L) = (n,L^\vee) \ ,$$

avec la compatibilité évidente à la somme

$$\mathrm{dual}\ (A + B) \simeq \mathrm{dual}\ (A) + \mathrm{dual}\ (B) \ .$$

Pour V un fibré vectoriel, l'isomorphisme

$$\det(V^\vee) \simeq \mathrm{dual}\ (\det(V))$$

est défini par la condition que des bases duales locales $e_1, \dots, e_n$ et $e'_1, \dots, e'_n$ de V et $V^\vee$ définissent des bases duales $e_1 \wedge \dots \wedge e_n$ et $e'_1 \wedge \dots \wedge e'_n$ de $\overset{n}{\wedge} V$ et $\overset{n}{\wedge} V^\vee$. Pour toute suite exacte $V' \hookrightarrow V \twoheadrightarrow V''$ de suite exacte transposée $V''^\vee \longrightarrow V^\vee \longrightarrow V'^\vee$, le diagramme

$$\begin{array}{ccccc} \det(V^\vee) & \text{——} & \det(V''^\vee) \otimes \det(V'^\vee) & \text{——} & \det(V'^\vee) \otimes \det(V''^\vee) \\ | & & & & | \\ \mathrm{dual}\ \det(V) & - & \mathrm{dual}\ (\det V' \otimes \det V'') & - & \mathrm{dual}\ \det(V') \otimes \mathrm{dual}\ \det(V'') \end{array}$$

(où en première ligne apparaît Koszul) est commutatif.

Pour le produit tensoriel, on pose

$$(a,L)\ \underline{\otimes}\ (b,M) = (ab\ , L^{\otimes b} \otimes M^{\otimes a}) \ ,$$

on définit $\det(V)\ \underline{\otimes}\ \det(W) \xrightarrow{\sim} \det(V \otimes W)$ par

$$(e_1 \wedge \dots e_a)^{\otimes b} \otimes (f_1 \wedge \dots \wedge f_b)^{\otimes a} \longmapsto \wedge \text{ des } e_i \otimes f_j \text{ en ordre lexicographique.}$$

Il faut alors définir

$$A\ \underline{\otimes}\ (B + C) \simeq A\ \underline{\otimes}\ B + A\ \underline{\otimes}\ C$$

par l'isomorphisme évident,

$$(A + B)\ \underline{\otimes}\ C \simeq A\ \underline{\otimes}\ C + B\ \underline{\otimes}\ C$$

avec un signe $(-1)^N$, $N = ab\,\frac{c(c-1)}{2}$, et

$$A \underline{\otimes} B \xrightarrow{\sim} B \underline{\otimes} A$$

avec un signe $(-1)^N$, $N = \frac{a(a-1)}{2}\,\frac{b(b-1)}{2}$.

Les foncteurs $\overset{i}{\wedge}$ passent aussi au quotient.

Pour K dans la catégorie dérivée, de tor-dimension finie et de dual $K^\vee$, on a encore

$$\det(K^\vee) = \text{dual}\ \det(K) .$$

5. Fibrés métriques virtuels.

5.1 Soit X un schéma lisse de type fini sur $\mathbb{C}$. Pour abréger, on appellera métrique sur un fibré vectoriel V sur X une structure hermitienne sur V . On se propose de définir la catégorie $\underline{KM}(X)$ des fibrés métriques virtuels sur X . Trois définitions, avant d'énoncer ses propriétés essentielles. On notera $\underline{K}(X)$ la catégorie des objets virtuels de la catégorie exacte des fibrés vectoriels sur X . On notera $M(X)$ la somme sur p des groupes

$$M^p(X) := \frac{\{(p-1\ ,\ p-1) - \text{forme réelle sur } X\}}{\{\text{celles dans } \operatorname{Im}(d') + \operatorname{Im}(d'')\}} \quad .$$

La fibre d'un morphisme de catégories de Picard commutatives $f\colon \underline{A} \to \underline{B}$ est la catégorie des objets de $\underline{A}$ munis d'un isomorphisme $f(a) \sim 0$. Si f est fidèle, les objets de cette catégorie fibre n'ont pas d'automorphismes non triviaux et on appelle encore fibre le groupe des classes d'isomorphie d'objets de la catégorie fibre.

(i) On veut disposer d'un foncteur d'oubli

$$\omega : \underline{KM}(X) \to \underline{K}(X)$$

(un morphisme de catégories de Picard commutatives) fidèle, surjectif sur les classes d'isomorphie d'objets et de fibre $M(X)$.

La donnée de $\underline{KM}(X)$ muni d'un tel foncteur d'oubli équivaut à celle d'un morphisme μ de catégorie de Picard commutative de $\underline{K}(X)$ dans la catégorie de Picard commutative des $M(X)$-torseurs. Voici comment. Etant donnés $\underline{KM}(X)$ et ω , on attache à x dans $\underline{K}(X)$ l'ensemble $\mu(x)$ des classes d'isomorphie d'objets v de $\underline{KM}(X)$, munis d'un isomorphisme $\omega(v) \sim x$. C'est un $M(X)$-torseur. Réciproquement, si μ est donné, on définit un objet de $\underline{KM}(X)$ comme étant une paire (x,g) d'un objet x de $\underline{K}(X)$ et d'une métrique virtuelle $g \in \mu(x)$ sur x .

Vu la définition de $\underline{K}(X)$ comme solution d'un problème universel, il suffit, pour construire μ , d'attacher à chaque fibré vectoriel V sur X un $M(X)$-torseur $\mu(V)$, et de construire pour μ des isomorphismes 4.3 (a) obéissant au formalisme de 4.3.

(ii) Un fibré métrique (V,g) doit définir un fibré métrique virtuel $[V,g]$ avec $\omega([V,g]) = [V]$. En d'autre termes, une métrique g sur V doit définir une métrique virtuelle $[g]$ sur $[V]$. Si V est une somme directe: $V = V' + V''$, on a $[V] = [V'] + [V'']$, $\mu([V]) = \mu([V']) + \mu([V''])$ et la métrique virtuelle définie par g , somme directe orthogonale de g' et g'' , doit être la somme de $[g']$ et $[g'']$.

(iii) Un fibré métrique virtuel v doit définir une forme de Chern $ch(v)$, avec les propriétés suivantes.

(a) $$ch([V,g]) = ch(V,g) ,$$

avec au second membre la forme définie par [4] représentant la classe de cohomologie $ch(V)$ (caractère de Chern).

(b) $$ch(v_1 + v_2) = ch(v_1) + ch(v_2) .$$

(c) Pour v muni de $\omega(v) \sim 0$, défini par $\alpha \in M(X)$,

$$ch(v) = \frac{1}{2\pi i} d'd''\alpha .$$

5.2 Une théorie 5.1 équivaut à une théorie secondaire à la Chern Bott [4]. Expliquons ce qui est requis.

Supposons donnée une théorie 5.1. Si g et h sont deux métriques sur un fibré V , par (II), elles définissent $[g]$ et $[h]$ dans $\mu([V])$. Définissons $\delta(g ; h) \in M(X)$ par

(5.2.1) $$[g] = [h] + \delta(g ; h) \quad \text{dans } \mu([V]) .$$

Si V est extension de V'' par V' et que g , g' , g'' sont des métriques sur V , V' et V'' , définissons $\delta(g ; g' , g'') \in M(X)$ par

(5.2.2) $$[g] = [g'] + [g''] + \delta(g ; g' , g'') \quad \text{dans } \mu([V]) .$$

On a

(5.2.3) $$\delta(g_1 ; g_2) + \delta(g_2 ; g_3) = \delta(g_1 ; g_3)$$

(5.2.4) $$\delta(g_1 ; g_1' , g_1'') - \delta(g_2 ; g_2' , g_2'') =$$
$$\delta(g_1 ; g_2) - \delta(g_1' ; g_2') - \delta(g_1'' , g_2'')$$

(5.2.5) Soient $V = V_0 \supset V_1 \supset V_2 \supset V_3 = \{0\}$ et $g(i,j)$ une métrique sur V_i/V_j $(i < j)$. On a

$$\delta(g(0,3) ; g(0,2) , g(2,3)) + \delta(g(0,2) ; g(0,1) , g(1,2))$$
$$= \delta(g(0,3) ; g(0,1) , g(1,3)) + \delta(g(1,3) ; g(1,2) , g(2,3))$$

(les deux membres valant $[g(0,3)] - [g(0,1)] - [g(1,2)] - [g(2,3)]$)

(5.2.6) pour $V = V' \oplus V''$,

$$\delta(g' \oplus g'' \; ; \; g' \; , \; g'') = 0$$

(5.2.7) $ch(V,g) = ch(V,h) + \frac{1}{2\pi i} d'd'' \; \delta(g,h)$

(5.2.8) $ch(V,g) = ch(V',g') + ch(V'',g'') + \frac{1}{2\pi i} d'd'' \; \delta(g \; ; \; g',g'')$.

Réciproquement, des fonctions δ vérifiant (5.2.3) à (5.2.8) fournissent une théorie 5.1. Pour V un fibré vectoriel, on définit $\mu(V)$ comme étant le $M(X)$-torseur, unique à isomorphisme unique près, muni de

$$\{\text{métriques sur } V\} \xrightarrow{[\;]} \mu(V)$$

vérifiant (5.2.1). L'existance est garantie par (5.2.3).

Pour V extension de V'' par V' , on définit

$$\mu(V') + \mu(V'') \xrightarrow{\sim} \mu(V)$$

comme étant l'unique isomorphisme pour lequel (5.2.2) est vrai. L'existence est garantie par (5.2.4). Que ces isomorphismes obéissent au formalisme 4.2 résulte de (5.2.5) et (5.2.6) pour $V' = 0$ ou $V'' = 0$. Le foncteur μ fournit donc

$$\mu : \underline{K}(X) \longrightarrow (\text{torseurs sous } M(X)) ,$$

compatible à la commutativité grâce à (5.2.6). Comme expliqué en 5.1, la donnée de μ équivaut à celle de $\underline{KM}(X)$, munie de ω .

Le groupe $KM_0(X)$ des classes d'isomorphie d'objets dans $\underline{KM}(X)$ est engendré par les (V,γ) , V fibré vectoriel et $\gamma \in \mu(V)$, avec les relations suivantes: (a) $(V,\gamma) - (V,\gamma')$ ne dépend que de $\gamma - \gamma'$ dans $M(X)$; (b) pour V extension de V'' par V' , si $\delta(g \; ; \; g',g'') = 0$, alors $(V,[g]) = (V',[g']) + (V'',[g''])$. Les conditions (5.2.7) (5.2.8) équivalent donc à l'existence de

$$ch: \; KM_0(X) \longrightarrow \text{formes}$$

vérifiant 5.1 (iii).

5.3 Supposons données des fonctions δ comme en 5.2, donc une théorie 5.1. Le facteur direct $M^1(X)$ de $M(X)$ est le groupe additif des fonctions réelles sur X et, par passage au quotient par $\bigoplus_{p>1} M^p(X)$, le foncteur μ fournit un foncteur $\overline{\mu}$ de $\underline{K}(X)$ dans les $M^1(X)$-torseurs. Pour tout x dans $\underline{K}(X)$, soit

$\nu(x)$ l'ensemble des métriques sur le fibré en droite $\det(x)$. On en fait un $M^1(X)$-torseur en définissant la métrique $h + a$ $(a \in M^1(X))$ par

$$\|s\|_{h+a} = \|s\|_h \cdot \exp(a) \quad ,$$

où $\| \ \|$ désigne la longueur carrée. Les conditions suivantes (5.3.1) (5.3.2) sur δ assurent l'existence d'un unique isomorphisme $\bar{\mu} \xrightarrow{\sim} \nu$ (un isomorphisme de morphismes de catégories de Picard) qui, pour g une métrique sur un fibré V , envoie $[g]^- \in \bar{\mu}([V])$ sur la métrique de $\det(V) = \det[V]$ déduite de g .

(5.3.1) $$\| \ \|_{\det g} / \| \ \|_{\det h} = \exp(\delta^1(g,h))$$

(5.3.2) Si V est extension de V'' par V' et que la métrique g sur V induit g' sur V' et a pour quotient g'' sur V'' , on a

$$\delta^1(g\ ;\ g',g'') = 0 \ .$$

Pour V un fibré vectoriel, (5.3.1) assure l'existence d'un (unique) isomorphisme de $M(X)$-torseurs $\bar{\mu}([V]) \to \nu([V])$ qui à $[g]$ associe la métrique de $\det(V)$ définie par g . La condition (5.3.2) assure que les données 4.2 coïncident pour $\bar{\mu}$ et ν .

Dit avec moins de précision: si (5.3.1) (5.3.2) sont vérifiés, une métrique virtuelle sur x définit une métrique sur $\det(x)$. La première classe de Chern est respectée.

Remarque 5.4 Soient $KM_0(X)$ le groupe des classes d'isomorphie d'objets de $\underline{KM}(X)$ et $KM_1(X)$ le groupe des automorphismes d'un quelconque objet. Que le foncteur d'oubli: $\underline{KM}(X) \to \underline{K}(X)$ soit fidèle, surjectif sur les classes d'isomorphie d'objets et de fibre $M(X)$ fournit une suite exacte

(5.4.1) $0 \to KM_1(X) \to K_1(X) \xrightarrow{\partial} M(X) \to KM_0(X) \to K_0(X) \to 0$. La définition de la flèche ∂ est qu'un automorphisme $u \in K_1(X)$ d'un objet se transforme une métrique virtuelle γ en $\gamma + \partial u$. L'existence de ch sur $KM_0(X)$ assure que la flèche ∂ est à valeurs dans le noyau de $d'd''$.

Cette suite exacte suggère que $\underline{KM}(X)$ est une version tronquée d'une catégorie $\underline{\widetilde{KM}}(X)$ s'envoyant dans $\underline{K}(X)$ par un foncteur d'oubli non fidèle.

Soit A^p le complexe de composantes les $A_k^p := \{(2p - k - 2)$-formes réelles sur X , modulo celles dans $F^p \oplus \overline{F^p}\}$, avec pour différentielles d , et soit $M_k(X)$ la somme des $M_k^p(X) := H_k(A^p)$. Il devrait exister des groupes $\widetilde{KM}_i(X)$

donnant lieu à une suite exacte longue

(5.4.2) $\to M_i(X) \to \widetilde{KM}_i(X) \to K_i(X) \to$,

et $\widetilde{KM}(X)$ aurait pour groupe de classes d'isomorphie d'objets $KM_o(X) = \widetilde{KM}_o(X)$, pour groupe des automorphismes de 0 l'extension $\widetilde{KM}_1(X)$ de $KM_1(X)$.

5.5 Une théorie de classes de Chern secondaires vérifiant (5.2.3) à (5.2.8) et (5.3.1) (5.3.2) est contenue, pour l'essentiel, dans [4]. Une variante de l'approche de [4] m'a été signalée par C. Soulé.

Expliquons comment définir $\delta(g\ ;\ h)$ vérifiant (5.2.3) .

Rappelons la définition de $ch(V,g)$. Soit $\nabla = \nabla' + \nabla''$, $\nabla' : V \to \Omega^{1,0}(V)$, $\nabla'' : V \to \Omega^{01}(V)$ la connection métrique de V ($\nabla g = 0$, $\nabla'' = \partial''$) . Elle se prolonge en $\nabla : \Omega^{**}(V) \to \Omega^{**}(V)$ vérifiant $\nabla(\alpha x) = d\alpha.x \pm \alpha\nabla x$ ($\pm = (-1)^{\deg\alpha}$) et ∇^2 est un endomorphisme Ω^{**}-linéaire de $\Omega^{**}(V)$, la multiplication par une 2-forme K de type $(1,1)$ à valeurs dans $End(V)$. On a

$$ch(V,g) = Tr\ exp\ (\frac{-1}{2\pi i}\ K)\ .$$

Supposons que g dépende de $t \in T$. Ci-dessous, T sera une variété différentiable, mais il serait plus naturel de se placer dans le cas universel où T est l'espace des structures hermitiennes sur V . La donnée de $g(t)$ est celle d'une structure hermitienne sur l'image inverse pr_1^*V de V sur $X \times T$. Le complexe de de Rham de $X \times T$ est trigradué: nombre de dz , de $d\bar{z}$ et de dt , et la différentielle extérieure se décompose en $d = d' + d'' + d_T$. Les $ch^p(V,g(t))$ dans $\Omega^{p,p}(X)$ fournissent une forme de type $(p,p,0)$, notée $ch^p(V,g)$, sur $X \times T$.

Nous allons construire (cf [4]) une forme réelle A^p , de type $(p-1\ ,\ p-1\ ,\ 1)$, telle que

$$d_T\ ch^p(V,g) = \frac{1}{2\pi i}\ d'd''\ A^p$$

et une forme B^p , de type $(p-2\ ,\ p-1\ ,\ 2)$ telle que

$$d_T\ A^p = d'B^p + \text{complexe conjugué}\ .$$

Ces formes sont de formation compatible à tout changement d'espace de paramètres $T_1 \to T$. Pour ch un chemin de $t_0 \in T$ à $t_1 \in T$, correspondant à des métriques $g(0)$ et $g(1)$ sur V , on pose

(5.5.1) $$\delta^p(g(1)\ ,\ g(0)) := \int_{ch} A^p\ .$$

Si ch^1 et ch^2 sont deux chemins de $g(0)$ à $g(1)$ et que $ch^1 - ch^0$ est le bord d'un cycle H , on a

$$\int_{ch^1} A^p - \int_{ch^2} A^p = d' \int_H B^p + \text{complexe conjugué}$$

L'espace de toutes les formes hermitiennes sur V étant contractile, (5.5.1) définit sans ambiguité $\delta^p(g(1)\ ,\ g(0)) \in M^p(X)$.

5.6 Pour chaque valeur de t , on dispose de la connection métrique ∇ de V . Les composantes ∇' et ∇'' de ∇ se prolongent en ∇' et ∇'' : $\Omega^{***}(pr_1^*V) \to \Omega^{***}(pr_1^*V)$, de type $(1,0,0)$ et $(0,1,0)$, vérifiant $\nabla'(\alpha x) = d'\alpha.x \pm \alpha\nabla' x$ et $\nabla''(\alpha x) = d''\alpha.x \pm \alpha\nabla'' x$ $(\pm = (-1)^{\deg\alpha})$. On dispose aussi d'une connection d_1 "dans la direction T" : $pr_1^*V \to \Omega^{001}(pr_1^*V)$ exprimant que pr_1^*V est une image inverse: $d_1\ pr_1^*(v) = 0$. Soit d_2 la connection "dans le sens de T" caractérisée par

$$d_T(v,w) = (d_1 v,w) + (v,d_2 w) .$$

La métrique identifie pr_1^*V à pr_1^* (antidual de V) , et d_2 correspond au d_1 de l'antidual. Les d_i se prolongent en d_i : $\Omega^{***}(V) \to \Omega^{***}(V)$ avec $d_i(\alpha x) = d_T\alpha.x \pm \alpha.d_i x$. Soit $L := d_2 - d_1$. C'est un endomorphisme (au sens $\mathbf{Z}/2$-gradué) du Ω^{***}-module $\Omega^{***}(V)$, défini par une $(0,0,1)$-forme à valeurs dans End(V) .

On a, les [ ] étant pris au sens $\mathbf{Z}/2$-gradué

$$\nabla'^2 = \nabla''^2 = d_1^2 = d_2^2 = [\nabla''\ ,\ d_1] = [\nabla'\ ,\ d_2] = 0$$

Soit L_0 l'algèbre de Lie (au sens $\mathbf{Z}/2$-gradué) librement engendrée par des éléments impairs ∇' , ∇'' , d_1 , d_2 soumis aux relations précédentes: $[\nabla'\ ,\ \nabla'] = [\nabla''\ ,\ \nabla''] = [d_1\ ,\ d_1] = [d_2\ ,\ d_2] = [\nabla''\ ,\ d_1] = [\nabla'\ ,\ d_2] = 0$. Elle est trigraduée, les générateurs étant de degrés respectifs (1,0,0) (0,1,0) et (0,0,1) pour d_1 et d_2 . Elle admet pour quotient l'algèbre de Lie ($\mathbf{Z}/2$-graduée) commutative de base d' , d'' , d_T (avec $\nabla' \mapsto d'$, $\nabla'' \mapsto d''$, $d_i \mapsto d_T$) . Soit

$$L = \mathrm{Ker}\ (L_0 \longrightarrow \langle d'\ ,\ d''\ ,\ d_T\rangle) .$$

Pour toute situation (X , V , T , g sur X × T) comme ci-dessus, L_0 agit sur $\Omega^{***}(V)$ et L agit par endomorphismes Ω^{***}-linéaires (au sens $\mathbf{Z}/2$-gradué): on dispose d'un morphisme de Lie

$$L \longrightarrow \Omega^{***}(\mathrm{End}(V)) .$$

Soit P une forme invariante de degré k sur $g\ell_n$, où n est le rang de V . Soit ℓ_P la forme linéaire sur $\mathrm{Sym}^k\, g\ell_n$ telle que $P(x) = \ell_P(x^k)$. Etant invariante, ℓ_P définit une section du dual de $\mathrm{Sym}^k\, \underline{\mathrm{End}}(V)$. Les puissances symétriques étant prises au sens $\mathbf{Z}/2$-gradué, on en déduit

$$c(P,\ ) :\ \mathrm{Sym}^k(L) \longrightarrow \Omega^{***}(\mathrm{Sym}^k\, \mathrm{End} V) \longrightarrow \Omega^{***} .$$

Par invariance, $c(P,\ )$ est nul sur $[L\ ,\ \mathrm{Sym}^k(L)]$. L'action de L_0 sur L induit une action de L_0 sur $\mathrm{Sym}^k(L)/[L\ ,\ \mathrm{Sym}^k L\ ]$. Cette action se factorise par $L_0/L = \langle d'\ ,\ d''\ ,\ d_T\rangle$. Le morphisme

$$c_P :\ \mathrm{Sym}^k(L)/[L\ ,\ \mathrm{Sym}^k L\ ] \longrightarrow \Omega^{***}$$

commute à d' , d'' et d_T . C'est un morphisme de tricomplexes. Les formes que nous considérons sont dans l'image de c_P et les calculs essentiels se feront dans $\mathrm{Sym}^k(L)/[L\ ,\ \mathrm{Sym}^k L\ ]$, indépendamment de P .

Soit $\nabla = \nabla' + \nabla''$. Soit dans L

$$K = \nabla^2 := \tfrac{1}{2}[\nabla,\nabla] = [\nabla',\nabla''] ,$$

de type (1,1,0) . On a $[\nabla,\nabla^2] = 0$ (Bianchi) et donc $[\nabla',K] = [\nabla'',K] = 0$. Soit

$$L := d_2 - d_1 ,$$

dans L , de type (0,0,1) . Soit $d_0 = \tfrac{1}{2}(d_1 + d_2)$. On a

$$[d_0,L] = \tfrac{1}{2}[d_2 + d_1\ ,\ d_2 - d_1] = \tfrac{1}{2}([d_2\ ,\ d_2] - [d_1\ ,\ d_1]) = 0$$

$$[d_0,K] = -[[\nabla',\nabla'']\ ,\ d_0] = -[\nabla'\ ,\ [\nabla'',d_0]] - [\nabla''\, [\nabla'\ ,\ d_0]]$$
$$= \tfrac{1}{2}(-[\nabla'\ ,\ [\nabla'',L]] + [\nabla''\ ,\ [\nabla',L]])$$

puisque $d_0 = d_1 + \tfrac{1}{2}L = d_2 - \tfrac{1}{2}L$.

Dans la somme des $\mathrm{Sym}^k(L)/[L\ ,\ \mathrm{Sym}^k L\ ]$, on a donc

$$d_T \exp K = [d_0\ ,\ \exp K] = \exp K.\, [d_0,K]$$
$$= \tfrac{1}{2}(-\mathrm{ad}\nabla'\ \mathrm{ad}\nabla'' + \mathrm{ad}\nabla''\ \mathrm{ad}\nabla')\ (\exp K.L)$$
$$= -d'd''\ (\exp K.L)$$

$$d_T(\exp K.L) = [d_0\ ,\ \exp K.L] = \exp K.[d_0,K].L$$
$$= \exp K.\ \tfrac{1}{2}[-\mathrm{ad}\nabla'\ \mathrm{ad}\nabla'' + \mathrm{ad}\nabla''\ \mathrm{ad}\nabla'](L).L$$
$$= \tfrac{1}{2}\ \{-\mathrm{ad}\nabla'(\exp K.[\nabla'',L].L) + \exp K.[\nabla'',L][\nabla',L]$$

$$+\mathrm{ad}\nabla''(\exp K.[\nabla',L].L) - \exp K.[\nabla',L][\nabla'',L]$$

$$= \tfrac{1}{2}\,\{-d'(\exp K.[\nabla'',L].L) + d''(\exp K.[\nabla',L].L)\}$$

Pour P_k la forme invariante de degré k $\mathrm{Tr}(x^k)$, on a

$$c(P_k\ ,\ \exp K) := c(P_k\ ,\ \frac{K^k}{k!}) = (-2\pi i)^k\ \mathrm{ch}^k(V,g)\ ,$$

d'où

$$d_T\ \mathrm{ch}^k(V,g) = \frac{1}{2\pi i}\ d'd''\ c(P_k\ ,\ \exp(\frac{-1}{2\pi i}\,K).L) \quad \text{et}$$

$$d_T(c(P_k\ ,\ \exp(\frac{-1}{2\pi i}\,K).L) = \tfrac{1}{2}\,\frac{1}{2\pi i}\ d'(c(P_k\ ,\ \exp(\frac{-1}{2\pi i}\,K).[\nabla'',L].L)$$

$$+ \text{ complexe conjugué.}$$

Ceci fournit les formes A^p et B^p de 5.5.

<u>Remarque</u> 5.7 La construction 5.6 ne dépend pas des propriétés particulières de $\mathrm{ch}(V,g)$. Pour toute forme invariante P , si $c_P(V,g)$ est la forme $P(\frac{-1}{2\pi i}\,K)$ correspondante, elle fournit $\delta_P(g,h)$ vérifiant

$$c_P(V,g) - c_P(V,h) = \frac{1}{2\pi i}\,d'd''\ \delta_P(g,h)\ .$$

On a $\delta_{P+Q} = \delta_P + \delta_Q$ et

(5.7.1) $\quad \delta_{PQ}(g,h) = c_P(V,h)\ \delta_Q(g,h) + \delta_P(g,h)\ c_Q(V,g)$.

Noter que dans $M(X)$, on a

(5.7.2) $\quad \delta_P\ .\ \frac{1}{2\pi i}\ d'd''\delta_Q = \frac{1}{2\pi i}\,d'd''\delta_P\ .\ \delta_Q$;

ceci assure la symétrie en P et Q du membre de droite.

Pour la forme de Chern

$$\mathrm{ch}(V,g) = \mathrm{Tr}\ \exp\,(\frac{-1}{2\pi i}\,K)\ ,$$

$\delta(g,h)$ est donné par intégration de

$$A = \mathrm{Tr}(\exp(\frac{-1}{2\pi i}\,K).L)$$

où exp et le produit sont pris au sens de la composition dans <u>End</u>(V) .

<u>Remarque</u> 5.9 Pour la somme de deux fibrés, on a

$$\mathrm{ch}(V_1 \oplus V_2\ ,\ g_1 \oplus g_2) = \mathrm{ch}(V_1\ ,\ g_1) + \mathrm{ch}(V_2\ ,\ g_2) \quad \text{et}$$

(5.9.1) $\quad \delta(g_1 \oplus g_2\ ,\ h_1 \oplus h_2) = \delta(g_1\ ,\ h_1) + \delta(g_2\ ,\ h_2)$.

Pour le produit tensoriel, on a

$$(5.9.2)\qquad \begin{aligned} ch(V_1 \otimes V_2 , g_1 \otimes g_2) &= ch(V_1 , g_1) . ch(V_2 , g_2) \quad \text{et} \\ \delta(g_1 \otimes g_2 , h_1 \otimes h_2) &= ch(V_1 , h_1)\, \delta(g_2 , h_2) \\ &\quad + \delta(g_1 , h_1)\, ch(V_2 , g_2) \end{aligned}$$

(cf 5.7.2 pour la symétrie en V_1 et V_2) .

Sur le fibré en droite trivial $\mathcal{O}$, soit 1 (resp u.1) la métrique pour laquelle $\|1\|$ vaut 1 (resp la constante u) . On a

$$(5.9.3)\qquad \delta(u.1,1) = \log u .$$

Tensorisant (V,g) avec $\mathcal{O}$ muni de ces métriques, on déduit de (5.9.2) (5.9.3) que pour une constante u ,

$$(5.9.4)\qquad \delta(ug,g) = ch(V,g) \log(u) .$$

Noter que $ch(V,ug) = ch(V,g)$, que, comme il se doit, $d'd''\delta(ug,g) = 0$ mais que $\delta(ug,g)$ est en général non nul.

<u>Remarque</u> 5.10 On vérifie facilement que dans l'algèbre de Lie L , les éléments de T-degré 0 sont réduits aux multiples de K . Dans le tricomplexe

$$\mathrm{Sym}^k(L)/[L , \mathrm{Sym}^k L] ,$$

seul K^k est donc de T-degré 0 . On peut montrer que pour $\ell > 0$, la partie de T-degré ℓ est d' et d"-acyclique. Pour $\ell = 1$, il en résulte que la formule donnée pour $\delta(g,h)$ est essentiellement unique.

5.11 Expliquons comment définir $\delta(g ; g',g'')$ (cf 5.2). Si on impose (5.2.4), il suffit de traiter le cas où la métrique g sur V induit la métrique g' sur V' et a pour quotient g" sur V" . Pour g' et g" ainsi choisis, posons

$$\delta(g) := \delta(g ; g',g'') .$$

La condition (5.2.4) se réduit à

$$\delta(g_1) - \delta(g_2) = \delta(g_1 ; g_2) - \delta(g_1' ; g_2') - \delta(g_1'' ; g_2'')$$

et (5.2.6) à ce que pour une somme directe orthogonale, $\delta(g) = 0$.

Soient $\lambda \in \mathbb{C}$, V_λ le pull-back de l'extension V par λ: $V'' \to V''$

$$\begin{array}{ccccccccc} 0 & \longrightarrow & V' & \longrightarrow & V & \longrightarrow & V'' & \longrightarrow & 0 \\ & & \| & & \downarrow{\scriptstyle \sigma(\lambda)} & & \downarrow{\scriptstyle \lambda} & & \\ 0 & \longrightarrow & V' & \longrightarrow & V & \longrightarrow & V'' & \longrightarrow & 0 \end{array}$$

et $\tilde{V}''_\lambda$ l'image inverse par $\sigma(\lambda)$ de l'orthogonal $\tilde{V}''$ de V' dans V. Notons g_λ la métrique sur V_λ, induisant g' et g'' sur V' et V'' et pour laquelle $\tilde{V}''_\lambda$ est l'orthogonal de V_λ. Pour $\lambda = 0$, on a $(V_0, g_0) = (V', g') \oplus (V'', g'')$. Ceci amène à postuler que $\delta(g_\lambda) \to 0$ pour $\lambda \to 0$. Pour $\lambda \neq 0$, $\sigma(\lambda)$ est un isomorphisme. Soit $g(\lambda)$ la métrique $\sigma(\lambda)(g_\lambda)$ sur V. On a $\delta(g_\lambda) = \delta(g(\lambda))$. La formule (5.11.1) pour $h = g(\lambda)$, donc $h' = g'$ et $h'' = \|\lambda\|^{-1} g''$, impose

$$\begin{aligned}\delta(g) &= \lim_{\lambda\to 0} \ \delta(g, g(\lambda)) - \delta(g'', \|\lambda\|^{-1} g'') \\ &= \lim_{\lambda\to 0} \ \delta(g, g(\lambda)) - \log \|\lambda\| \ ch(V'', g'')\end{aligned}$$

(appliquer (5.9.4)). Montrons qu'une telle limite existe.

Les $g(\lambda)$ forment une famille paramétrée par $\mathbb{C}^*$ de métriques sur V. En terme de la décomposition, comme fibré C^∞, de V en $V' \oplus \tilde{V}'' \xrightarrow{\sim} V' \oplus V''$, on a $g(\lambda) = g' \oplus \|\lambda\|^{-1} g''$. Notant pr'' la projection sur $\tilde{V}''$, on calcule que la 1-forme L à valeurs dans $\mathrm{End}\, pr_1^* V$, sur $X \times \mathbb{C}^*$, est

$$L = -d \log \|\lambda\| \ . \ pr'' \ .$$

Calculons la courbure K_λ de (V_λ, g_λ). En terme de la décomposition, comme fibré C^∞, de V_λ en $V' \oplus \tilde{V}''_\lambda \xrightarrow{\sim} V' \oplus V''$, on a $g_\lambda = g' \oplus g''$, et la connection $\nabla_\lambda = \nabla'_\lambda + \nabla''_\lambda$ est

$$\nabla''_\lambda = \begin{pmatrix} \partial'' & \lambda.\alpha \\ 0 & \partial'' \end{pmatrix} \qquad \nabla'_\lambda = \begin{pmatrix} \nabla' & 0 \\ -\bar{\lambda}\alpha^* & \nabla' \end{pmatrix}$$

où α dans Ω^{01} <u>Hom</u>(V'', V') détermine la classe d'extension de V. De là, notant K' et K'' les courbures de (V', g') et (V'', g'') et ∇ la connection de <u>Hom</u>(V'', V'),

$$K_\lambda = \begin{pmatrix} K' - \|\lambda\|\alpha\alpha^* & \lambda\nabla'(\alpha) \\ -\bar{\lambda}\nabla''(\alpha^*) & K'' - \|\lambda\|\alpha^*\alpha \end{pmatrix}$$

Pour $\lambda \neq 0$, la courbure $K(\lambda)$ de (V_λ, g_λ) est l'image par $\sigma(\lambda)$ de K_λ et $\delta(g(\lambda), g)$ se déduit donc par intégration de 1 à λ de

$$\begin{aligned}&\mathrm{Tr}(-\exp(\tfrac{-1}{2\pi i} K(\lambda)) \ . \ d \log \|\lambda\| . pr'') \\ &= \mathrm{Tr}(-\exp(\tfrac{-1}{2\pi i} K_\lambda) . \ pr'') \ . d \log \|\lambda\|\end{aligned}$$

La quantité à intégrer est le produit par $d \log \|\lambda\|$ d'un polynôme en $\|\lambda\|$ dont le terme constant est la trace de $-\exp(\frac{-1}{2\pi i} K'')$. A cause de ce terme constant, l'intégrale diverge pour $\lambda \to 0$, avec une divergence en $-\log \|\lambda\| \cdot ch(V",g")$. La limite qui doit définir $\delta(g)$ existe donc, et

$$(5.11.3)\qquad \delta(g) = \int_1^0 (Tr(\exp(\frac{-1}{2\pi i} K_\lambda)pr") - Tr(\exp(\frac{-1}{2\pi i} K"))) \cdot d \log \|\lambda\| .$$

Plus précisément, nous définissons $\delta(g)$ par (5.11.3). L'espace $M(X)$ (un quotient) n'étant pas nécessairement séparé, la définition par passage à la limite n'a en effet qu'une valeur heuristique.

5.12 Pour vérifier les propriétés (5.2), il est plus commode d'utiliser une autre construction des fonctions δ qui m'a été signalée par C. Soulé. L'idée est la suivante. Pour tout schéma lisse X , X et son produit $X \times A^1$ avec la droite affine ont mêmes K_i . Ce fait, pour $i = 0,1$, implique que le foncteur

$$pr_1^* : \underline{K}(X) \longrightarrow \underline{K}(X \times A^1)$$

est une équivalence. Les foncteurs s^*: $\underline{K}(X \times A^1) \to \underline{K}(X)$ pour s une section de $X \times A^1/X$ sont donc tous naturellement isomorphes. En particulier, pour V un fibré vectoriel sur $X \times A^1$, les $[s^*V]$ sont naturellement isomorphes. Soient i_0 et i_1 les sections $X \to X \times \{0\}$ et $X \to X \times \{1\}$, et $V_0 = i_0^*V$, $V_1 = i_1^*V$.

Supposons provisoirement qu'un formalisme 5.1 est donné. Si g_i est une métrique sur V_i , les $[g_i]$ sont des métriques virtuelles sur $[V_0] \simeq [V_1]$, et $[g_0] - [g_1] \in M(X)$ est défini. C. Soulé propose que, lorsque V est prolongé en un fibré encore noté V sur $X \times \mathbb{P}^1$, $[g_0] - [g_1]$ est donné par la formule suivante. On choisit une métrique g sur V sur $X \times \mathbb{P}^1$, induisant g_0 et g_1 - d'où une forme de Chern $ch(V,g)$ sur $X \times \mathbb{P}^1$ induisant $ch(V_0,g_0)$ et $ch(V,g_1)$ sur $X \times \{0\}$ et $X \times \{1\}$ - et

$$(5.12.1)\qquad [g_0] - [g_1] = -\int_{X \times \mathbb{P}^1/X} ch(V,g). \log(\|z\|/\|z-1\|) .$$

Notons $D(g_0 , g_1)$ le membre de droite. L'identité sur $\mathbb{P}^1$

$$\frac{-1}{2\pi i} d'd" \log(\|z\|/\|z-1\|) = \delta(0) - \delta(1)$$

assure que

$$(5.12.2)\qquad ch(V_0 , g_0) - ch(V_1 , g_1) = \frac{1}{2\pi i} d'd" D(g_0 , g_1) .$$

Soit W un fibré sur X et prenons $V = pr_1^* W$. On a $V_0 = V_1 = W$ et V définit l'isomorphisme identique de $[V_0] = [W]$ avec $[V_1] = [W]$. Si g_0 et g_1 sont deux métriques sur W , et que la métrique g sur pr_1^*W sur $X \times \mathbb{P}^1$

induit g_0 sur $X \times \{0\}$ et g_1 sur $X \times \{1\}$, ce qui précède amène C. Soulé à prendre comme fonction $\delta(g_0 \,;\, g_1)$ de 5.2 la fonction

(5.12.3) $$D(g_0 \,;\, g_1) := -\int_{X \times \mathbb{P}^1/X} ch(V,g) \,.\, \log(\|z\|/\|z-1\|) \,.$$

Soit W un fibré sur X extension de W'' par W' . Les W_λ (comme en 5.9) sont les restrictions aux $X \times \{\lambda\}$ d'un fibré V sur $X \times \mathbb{A}^1$. On a $V_0 = W' \oplus W''$, $V_1 = W$ et V définit l'isomorphisme naturel

$$[W] \xrightarrow{\sim} [W' \oplus W''] = [W'] + [W''] \,.$$

En dehors de la section 0 , les σ_λ de 5.9 fournissent un isomorphisme σ de V avec $pr_1^* W$. Nous noterons encore V le prolongement de V à $X \times \mathbb{P}^1$ auquel σ se prolonge. Si $s\colon\ \mathcal{O}(-X\times\{0\}) \to \mathcal{O}$ est l'inclusion naturelle, V est défini par le diagramme cartésien

$$\begin{array}{ccccccccc}
0 & \longrightarrow & pr^*W' & \longrightarrow & pr^*W & \longrightarrow & pr^*W'' & \longrightarrow & 0 \\
 & & \| & & \uparrow & & \uparrow s & & \\
0 & \longrightarrow & pr^*W' & \longrightarrow & V & \longrightarrow & pr^*W'' \otimes \mathcal{O}(-X\times\{0\}) & \longrightarrow & 0
\end{array}$$

Si une métrique g sur V induit des métriques g_1 sur $V_1 = W$ et $g_0' \oplus g_0''$ sur $V_0 = W' \oplus W''$, ceci amène C. Soulé à prendre comme fonction $\delta(g_1 \,;\, g_0' \,,\, g_0'')$ de (5.2) la fonction

(5.12.4) $$D(g_1 \,;\, g_0' \,,\, g_0'') := \int_{X \times \mathbb{P}^1/X} ch(V,g) \,.\, \log(\|z\|/\|z-1\|) \,.$$

D'après (5.12.1), les exigences (5.2.7) et (5.2.8) sont remplies.

Les définitions de $D(g_0 \,,\, g_1)$ (5.12.1) et (5.12.3) (5.12.4) dépendent d'une métrique g interpolant des métriques données au-dessus de $X \times \{0\}$ et $X \times \{1\}$. Si on change g , on modifie $ch(V,g)$ par un $d'd''\alpha$, où α a une restriction nulle à $X \times \{0\}$ et $X \times \{1\}$ (utiliser 5.5, ou (5.12.1), avec X remplacé par $X \times \mathbb{P}^1$). Dans l'identité

$$\begin{aligned}\int d'd''\alpha .\, \log(\|z\|/\|z-1\|) = {} & d'\int d''\alpha .\, \log(\|z\|/\|z-1\|) \\ & + d''\int \alpha .\, d'\log(\|z\|/\|z-1\|) \\ & + \int \alpha .\, d'd''\log(\|z\|/\|z-1\|)\end{aligned}$$

(intégrales selon les fibrés de $X \times \mathbb{P}^1/X$, prises au sens des distributions), le troisième terme est nul. Les seconds membres de (5.12.1) (5.12.3) (5.12.4) sont donc définis sans ambiguité dans $M(X)$.

5.13 Comparons les constructions 5.6 et (5.12.3). Soit V un fibré vectoriel sur X et g une métrique sur son image inverse sur $X \times \mathbb{P}^1$. Décomposons la forme $c_P(pr_1^* V , g)$, pour P un polynôme invariant de degré p, selon le type $(0,0)$, $(1,0)$, $(0,1)$, $(1,1)$ dans la direction $\mathbb{P}^1$:

$$c_P(pr_1^* V , g) = c + A + B + D .$$

La forme c_P étant fermée de type (p,p), on a

(5.13.1) $$d'_{\mathbb{P}^1} c + d'_X A = d''_{\mathbb{P}^1} c + d''_X B = 0 , \quad d'_X B = d''_X A = 0$$

(5.13.2) $$d'_{\mathbb{P}^1} B + d'_X D = d''_{\mathbb{P}^1} A + d''_X D = 0 .$$

La construction 5.6, elle, fournit une $(p-1 , p-1 , 1)$-forme E avec

(5.13.3) $$d_{\mathbb{P}^1} c = \frac{1}{2\pi i} d'_X d''_X E ,$$

et des $(p-1 , p-2 , 2)$ et $(p-2 , p-1 , 2)$ formes F_1 et F_2 avec

(5.13.4) $$d_{\mathbb{P}^1} E = d''F_1 + d'F_2 .$$

Décomposons E en $E' + E''$ selon le type dans la direction de $\mathbb{P}^1$. Les formes $-d''_X E'$, $d'_X E''$, $-d''_{\mathbb{P}} E' + d''_X F_1 = d'_{\mathbb{P}} E'' - d'_X F_2$, multipliées par $\frac{1}{2\pi i}$, vérifient les identités (5.13.1) (5.13.2) vérifiées par A, B, D. Ceci suggère

(5.13.5) $$2\pi i\, A = -d''_X E' \qquad 2\pi i\, B = d'_X E''$$
$$2\pi i\, D = d'_{\mathbb{P}} E'' - d'_X F_2 = -d''E' + d''_X F_1 .$$

On vérifie que tel est bien le cas. Après un changement de variables, (5.13.3) donne pour les métriques g_0 et g_∞ portées par V sur $X \times \{0\}$ et $X \times \{\infty\}$

$$D(g_0 ; g_\infty) = -\int_{X \times \mathbb{P}^1/X} D.\log \|z\|$$

Modulo $\mathrm{Im}(d')$ et $\mathrm{Im}(d'')$, ou exactement si la métrique au-dessus de $X \times \{z\}$ ne dépend que de $\|z\|$, auquel cas $F_1 = F_2 = 0$, l'intégrale au second membre vaut $\frac{1}{2\pi i}$ fois

$$-\int_{X \times \mathbb{P}^1/X} \tfrac{1}{2}(d'_{\mathbb{P}^1} E'' - d''_{\mathbb{P}^1} E') \log \|z\| = -\int \tfrac{1}{2} E' \frac{d\bar z}{\bar z} - E'' \frac{dz}{z}$$
$$= -\int \tfrac{1}{2} E \left(\frac{d\bar z}{\bar z} - \frac{dz}{z} \right) = \int i\, E\, d\theta$$

et $D(g_0 ; g_\infty)$ est la moyenne des $\delta(g_0 , g_\infty)$ calculés pour les rayons allant de 0 à ∞.

5.14 Vérifions les identités 5.2 pour D. Pour (5.2.3), on munit $pr_1^* V$ sur

$X \times \mathbb{P}^1$ d'une métrique g induisant g_i sur $X \times \{i\}$ $(i = 1,2,3)$; on a

$$D(g_i ; g_j) = -\int ch . \log(z-i/z-j)$$

et on utilise que $\log(ab) = \log a + \log b$.

Pour (5.2.4), on considère sur $X \times \mathbb{P}^1$ le fibré (5.11, 5.12) qui interpole entre $V' \oplus V''$ (sur $X \times \{0\}$) et V (sur $X \times \{1\}$) , et son image inverse sur $X \times \mathbb{P}^1 \times \mathbb{P}^1$. On prend une métrique g qui interpole en $(0,0)$ et $(1,0) \in \mathbb{P}^1$, $g_1' \oplus g_1''$ et g_1 , et en $(0,1)$ et $(1,1)$, $g_2' \oplus g_2''$ et g_2 . Soit ch la forme de Chern. Il s'agit de comparer

$$\int_{X \times \mathbb{P}^1 \times \{0\}/X} ch.\log(\|z\|/\mathbb{R} - 1\|) - \int_{X \times \mathbb{P}^1 \times \{1\}/X} ch.\log(\|z\|/\|z - 1\|$$

$$= \int_{X \times \mathbb{P}^1 \times \mathbb{P}^1/X} ch.\log(\|z\|/\|z - 1\|) . \frac{-1}{2\pi i} d'd'' \log(\|t\|/\|t - 1\|)$$

et

$$\int_{X \times \{0\} \times \mathbb{P}^1/X} ch.\log(\|t\|/\|t - 1\|) - \int_{X \times \{1\} \times \mathbb{P}^1/X} ch.\log(\|t\|/\|t - z\|)$$

$$= \int_{X \times \mathbb{P}^1 \times \mathbb{P}^1/X} ch.\log(\|t\|/\|t - 1\|) . \frac{-1}{2\pi i} d'd'' \log(\|z\|/\|z - 1\|) .$$

Modulo $\mathrm{Im}(d')$ et $\mathrm{Im}(d'')$, on passe de l'un à l'autre par une intégration par partie.

Prouvons (5.2.5). Soit W le sous-faisceau suivant de l'image inverse de V sur $X \times \mathbb{P}^1 \times \mathbb{P}^1$. Si z , t sont les coordonnées de $\mathbb{P}^1 \times \mathbb{P}^1$, on ne touche pas à V en dehors de $z = 0$ ou $t = 0$. Là, on ne prend que les sections d'image dans $pr_1^*(V/V_2)$ divisible par z et d'image dans $pr^*(V/V_1)$ divisible par z et t . Sur $\mathbb{P}^1 \times \{1\}$ (resp $\{1\} \times \mathbb{P}^1$), on obtient le fibré interpolant entre V et $V/V_2 \oplus V_2$ (resp $V/V_1 \oplus V_1$) . Sur $\mathbb{P}^1 \times \{0\}$, on interpole entre $V/V_1 \oplus V_1$ et $V/V_1 \oplus V_1/V_2 \oplus V_2$. Sur $\{0\} \times \mathbb{P}^1$, on interpole entre $V/V_2 \oplus V_2$ et $V/V_1 \oplus V_1/V_2 \oplus V_2$. Prenant une métrique g interpolant les métriques données, et procédant comme dans la preuve de (5.2.4), on obtient (5.2.5).

Prouvons (5.2.6). La définition de D demande de considérer sur $X \times \mathbb{P}^1$ le fibré $pr_1^*V' \oplus pr_2^*V''(-X \times \{0\})$, isomorphe à $pr_1^*V' \oplus pr_2^*V''(-X \times \{\infty\})$, et une métrique g qui est $g' \oplus g''$ sur $X \times \{0\}$ et $X \times \{1\}$. On peut prendre g invariant par $Z \longmapsto 1 - Z$ et la nullité de D résulte alors de l'antiinvariance de $\log(\|z\|/\|1 - z\|)$.

Enfin, 5.2.7 et 5.2.8 résultent de (5.12.2). L'argument heuristique de 5.11 devrait permettre de comparer (5.11.3) et (5.12.4).

5.15 Esquissons la vérification de (5.12.1), la base heuristique des formules avec lesquelles nous avons travaillé. Soient i_0 et i_1 les inclusions de X dans $X \times \mathbb{P}^1$ par $x \longmapsto (x,0)$ et $x \longmapsto (x,1)$. L'isomorphisme considéré en 5.12 entre les foncteurs i_0^* et i_1^* : $\underline{K}(X \times \mathbb{P}^1) \to \underline{K}(X)$ est caractérisé par la propriété de donner lieu à l'isomorphisme évident si appliqué à pr_1^*V ou à $pr_1^*V \otimes \mathcal{O}(-X \times \{\infty\})$. Il définit une trivialisation du $M(X)$-torseur $\mu(i_0^* x) - \mu(i_1^* x)$ pour tout x dans $\underline{K}(X \times \mathbb{P}^1)$. Cette trivialisation est caractérisée par sa compatiblité à $+$, et que c'est l'évidente pour un pr_1^*V ou un $pr_1^*V \otimes \mathcal{O}(-X \times \{\infty\})$. La trivialisation correspondante de $\mu(i_0^*V) - \mu(i_1^*V)$, pour V un fibré vectoriel sur $X \times \mathbb{P}^1$, est caractérisée par les mêmes propriétés de normalisation, et une compatibilité aux suites exactes courtes $0 \to V' \to V \to V'' \to 0$. Si g , g' et g'' sont des métriques sur V , V' et V'' , cette compatibilité s'écrit, pour la formule proposée (5.12.1),

$$D(g_0 , g_1) - D(g_0' , g_1') - D(g_0'' , g_1'') = D(g_0 ; g_0' , g_0'') - D(g_1 ; g_1' , g_1'') ,$$

et se prouve par un argument semblable à celui prouvant 5.2.4 (5.14). La normalisation se déduit, comme 5.2.6, de l'antisymétrie de $\log(\|z\|/\|z - 1\|)$ par $z \longmapsto 1 - z$.

6. Intégrale d'un produit de premières classes de Chern.

6.1 Soit $f\colon X \to S$ un morphisme de schémas propre, plat et purement de dimension relative N. Le programme 2.1, pour P le polynôme $c_1^1 \cdot \cdots \cdot c_{N+1}^1$ demande la construction d'un foncteur (pour les isomorphismes) $I_{X/S}\, P(V_1, \ldots, V_{N+1})$ attachant à $N+1$ fibrés vectoriels sur X un fibré en droites sur S. Dans tous les contextes usuels, la première classe de Chern d'un fibré V ne dépend que de $\det(V)$. Ceci amène à se concentrer sur le cas où les V_i sont des fibrés en droites. Si on sait traiter ce cas, on posera

(6.1.1) $$I_{X/S}\, P(V_1, \ldots, V_{N+1}) := I_{X/S}\, P(\det(V_1), \ldots, \det(V_{N+1}))$$

Pour notre théorème principal, on a besoin du cas où $N = 1$ et où X est une famille de courbes lisses paramétrées par S. Dans ce cas, le foncteur requis a été construit dans SGA4 (XVIII 1.3). Comme dans SGA4, nous le noterons $\langle L, M\rangle$. Pour S spectre d'un corps k, on peut en donner la description directe suivante. On définit $\langle L, M\rangle$ comme l'espace vectoriel de rang un sur k, engendré par des symboles $\langle \ell, m\rangle$ pour ℓ et m des sections rationnelles de L et M, à diviseurs disjoints, avec les relations

(6.1.2) $$\langle \ell, fm\rangle = f(\mathrm{div}(\ell))\,\langle \ell, m\rangle$$
$$\langle f\ell, m\rangle = f(\mathrm{div}(m))\,\langle \ell, m\rangle .$$

Que la construction ne s'effondre pas se ramène au théorème de Weil que, pour f et g deux fonctions rationnelles sur X à diviseurs disjoints, on a ([16] III §4)

$$f(\mathrm{div}(g)) = g(\mathrm{div}(f)) .$$

La construction de SGA4 est plus compliquée, plus instructive et reprouve le théorème de Weil.

Sur une base quelconque, la même construction marche, pour autant qu'on se localise pour la topologie étale: le faisceau des sections locales de $\langle L, M\rangle$ est le faisceau de $\mathcal{O}$-modules engendré par des sections locales $\langle \ell, m\rangle$ avec les relations (6.1.2), avec $f(\mathrm{div}(\ell))$ interprété comme une norme: pour D un diviseur de Cartier relatif, on pose $f(D) := N_{D/S}(f)$. On a $f(D_1 + D_2) = f(D_1).f(D_2)$. On écrit $\mathrm{div}(\ell) = D_1 - D_2$ et on pose $f(\mathrm{div}(\ell)) = f(D_1).f(D_2)^{-1}$.

Pour $L = \mathcal{O}(D)$, et 1 la section canonique de $\mathcal{O}(D)$, on a $\langle 1, fm\rangle =$

$N_{D/S}(f) \cdot \langle 1 , m\rangle$. De là, un isomorphisme

(6.1.3) $$\langle \mathcal{O}(D) , M\rangle = N_{D/S} M$$

avec $\langle 1 , m\rangle \mapsto N_{D/S}(m)$. Pour la définition de $N_{D/S}M$, voir 7.1.

Au paragraphe 8, on indiquera, avec des esquisses de démonstrations, comment traiter l'intégrale d'un produit de premières classes de Chern dans un cas plus général.

6.2 La définition 6.1 rend claire la définition d'isomorphismes de bimultiplicativité:

$$\langle L_1 \otimes L_2 , M\rangle \simeq \langle L_1 , M\rangle \otimes \langle L_2 , M\rangle$$
$$\langle L , M_1 \otimes M_2\rangle \simeq \langle L , M_1\rangle \otimes \langle L , M_2\rangle ,$$

et de symétrie

$$\langle L , M\rangle \xrightarrow{\sim} \langle M , L\rangle .$$

Pour $L = M$, l'isomorphisme de symétrie (un automorphisme de $\langle L , L\rangle$) est la multiplication par $(-1)^{\deg L}$ (SGA4 XVIII 1.3.16.6) .

6.3 Sur $\mathbb{C}$, si L et M sont munis de structures hermitiennes, on définit une structure hermitienne sur $\langle L , M\rangle$ par

(6.3.1) $$\log\|\langle \ell,m\rangle\| = \frac{1}{2\pi i}\int d'd'' \log\|\ell\|.\log\|m\| + \log(\|m\|)\,[\mathrm{div}(\ell)] + \log(\|\ell\|)\,[\mathrm{div}\,(m)] .$$

où d'd" est pris au sens des distributions et où [ ] est comme en 1.5. Regroupant les deux premiers termes, on obtient la formule équivalente

(6.3.2) $$\log\|\langle \ell,m\rangle\| = \int F_L .\log\|m\| + \log(\|\ell\|)\,[\mathrm{div}(m)]$$

qui cache la symétrie mais rend claire que le second membre vérifie l'identité

$$\log\|\langle f\ell,m\rangle\| = \log\|\langle \ell,m\rangle\| + \log\|f(\mathrm{div}(m))\|$$

requise par (6.1.2) pour que (6.3.1) soit légitime.

6.4 Justifions cette définition par l'analogie du §3. Dans le cadre du §3, 2e colonne, soient L et M deux faisceaux inversibles sur X_A et ℓ , m des sections rationnelles sur X_K , à diviseurs disjoints. Elles définissent une base $\langle \ell , m\rangle$ de $\langle L , M\rangle \otimes_A K$. Soient D et E les diviseurs de ℓ et m , vus comme sections rationnelles de L et M sur X_A: on a sur X_A des isomorphismes de faisceaux munis de sections rationnelles $(L , \ell) \simeq (\mathcal{O}(D) , 1)$ et $(M , m) \simeq (\mathcal{O}(E) , 1)$. Avec ces notations, la valuation de $\langle \ell , m\rangle$ est le nombre d'intersection (D,E) . Pour le prouver, on se ramène à supposer div(ℓ) fini sur A et on utilise que $\langle L , M\rangle = N_{\mathrm{div}(\ell)/\mathrm{Spec}\, A}(M)$.

Dans 4.3, ℓ et m définissent de même des diviseurs compactifiés (3.7) et $-\log\|\langle \ell, m\rangle\|$ est leur nombre d'intersection.

6.5 Soient φ et ψ deux fonctions C^∞ avec $\varphi + \psi = 1$. On suppose φ (resp ψ) nul au voisinage de div(ℓ) (resp div(m)) . Une intégration par partie permet de récrire (4.3) sous la forme

(6.5.1) $$\log\|\langle \ell , m\rangle\| = \int F_L .\log\|m\| . \psi + \int F_M .\log\|\ell\| . \varphi + \frac{1}{2\pi i}\int d'' \log\|\ell\| . \log\|m\| . d'\varphi + \frac{1}{2\pi i}\int d' \log\|m\| . \log\|\ell\| . d''\varphi .$$

Cette formule m'a été indiquée par O. Gabber. Elle rend claire que la définition de la métrique de $\langle L , M\rangle$ garde un sens pour L et M munis de métriques singulières, pour autant que leurs singularités soient disjointes. On définit une <u>métrique singulière</u> comme la donnée, pour toute section locale inversible, d'une fonction généralisée (distribution) $\log\|s\|$, avec $\log\|fs\| = \log\|s\| + \log\|f\|$. On ne tient pas à ce que $\|s\|$ ait un sens.

La formule 6.3 peut encore s'exprimer comme suit. La section ℓ de L , de diviseur D , fournit un isomorphisme de L avec $\mathcal{O}$(D) , transformant ℓ en 1 . Avec sa métrique, L est isomorphe au produit tensoriel de $\mathcal{O}$(D) , avec la métrique singulière pour laquelle $\|1\| = 1$ (cf 1.3.4) et de $\mathcal{O}$, avec la métrique singulière pour laquelle $\|1\| = \|\ell\|$. La métrique de $\langle L , M\rangle$ est caractérisé par sa compatibilité à la bimultiplicativité, à la symétrie, et par les règles suivantes:

(i) Si $\mathcal{O}$(D) est muni de la métrique singulière pour laquelle $\|1\| = 1$, l'isomorphisme (6.1.3)

$$\langle \mathcal{O}(D) , M\rangle = N_D(M)$$

est compatible aux métriques.

(ii) On a $\langle \mathcal{O} , \mathcal{O}\rangle = \mathbb{C}$ et la métrique de $\langle(\mathcal{O},g) , (\mathcal{O},h)\rangle$ est celle pour laquelle $-\log\|1\|$ est le nombre d'intersections des "diviseurs concentrés sur la fibre spéciale"(3.5, 3.6) définis par g et h .

6.6 La définition 6.3 s'étend trivialement au cas où S est un schéma lisse de type fini sur $\mathbb{C}$, plutôt que $\mathrm{Spec}(\mathbb{C})$. Dans ce cadre, le résultat suivant est crucial

Proposition. Soient $\underline{c}^1(L)$ et $\underline{c}^1(M)$ les formes de Chern des fibrés en droites métrisés L et M . La forme de Chern du fibré en droite métrisé $\langle L , M\rangle$ est donnée par

$$(6.6.1) \qquad \underline{c}_1(\langle L , M\rangle) = \int_{X/S} c_1(L)\wedge c_1(M) .$$

Rappelons que la première forme de Chern est donnée par

$$\underline{c}_1(L) = \frac{1}{2\pi i} \; d'd''\log\|\ell\|$$

pour ℓ une section locale inversible de L (et $\|\ell\| = \langle \ell,\ell\rangle)$.

On notera que (6.6.1) a un sens pour L et M munis de métriques singulières de lieux singuliers disjoints. Par additivité des deux membres de (6.6.1), il suffit (cf 6.5) de traiter les deux cas suivants: (a) L est $\mathcal{O}(D)$, pour D un diviseur étale sur S , et sa métrique est la métrique singulière pour laquelle $\|1\| = 1$; (b) L et M sont isomorphes à $\mathcal{O}$. Dans le cas (a), $\underline{c}_1(L)$ est le courant défini par le cycle D : pour toute forme β sur X , $\int_{X/S} \underline{c}_1(L)\wedge \beta$ est la trace de D à S de la restriction de β à D . Ceci est compatible au fait que $\langle L , M\rangle$ est, avec sa métrique, la norme de D à S de la restriction de M à D . Dans le cas (b): $L = \mathcal{O}$, avec $\|1\| = g$ et $M = \mathcal{O}$, avec $\|1\| = h$, le fibré $\langle L , M\rangle$ est trivial, avec

$$\log \|1\| = \frac{1}{2\pi i}\int_{X/S} d'd''\log g . \log h \quad ,$$

d'où

$$\frac{1}{2\pi i} \; d'd'' \log\|1\| = \frac{1}{(2\pi i)^2}\int_{X/S} d'd''\log g \;\; d'd''\log h$$
$$= \int_{X/S} \underline{c}_1(L)\wedge \underline{c}_1(M) .$$

6.7 Arakelov et Faltings fixent sur la courbe X une densité positive μ de masse totale 1 et appellent <u>admissible</u> une métrique sur un fibré en droites dont la forme de courbure est proportionnelle à μ . A un facteur près, un fibré en droites a une et une seule métrique admissible. Pour D un diviseur, il existe donc sur $\mathcal{O}(D)$ une et une seule métrique admissible tel que

$$\int \log \|1\| . \; \mu = 0 .$$

Si $\mathcal{O}(D)$ est muni de cette métrique, et L d'une métrique admissible, l'isomorphisme (6.1.3)

$$\langle \mathcal{O}(D)\ ,\ L \rangle \simeq N_D(L)$$

est une isométrie: pour ℓ une section rationnelle de L inversible sur D , $\langle 1,\ell \rangle$ correspond à $N_D(\ell)$,

$$\log\|\langle 1,\ell\rangle\| = \int F_L \ .\ \log\ \|1\| \quad + \quad \log\|\ell\|\ [\mathrm{div}(1)]$$

et l'intégrale au second membre est nulle par hypothèse, tandis que le second terme est $\log\|\ell\|[D]$.

7. $\langle L , M\rangle$ et cohomologie.

7.1 Normes. Pour $g: S' \to S$ fini et plat, le morphisme norme $N_{S'/S}: g_* \mathcal{O}^*_{S'} \to \mathcal{O}^*_S$ induit un foncteur norme de la catégorie de Picard $\underline{\mathrm{Pic}}(S')$ des faisceaux inversibles sur S' dans $\underline{\mathrm{Pic}}(S)$. Pour L un faisceau inversible sur S', de faisceau de sections inversibles L^*, sa norme $N_{S'/S}(L)$ est un faisceau inversible N sur S muni d'un morphisme de faisceaux $N_{S'/S}: g_* L^* \to N^*$ vérifiant $N_{S'/S}(u\ell) = N_{S'/S}(u) N_{S'/S}(\ell)$. Le couple $(N , N_{S'/S})$ est unique à isomorphisme unique près.

Pour F_0 et F_1 des fibrés vectoriels sur S' de même rang, on dispose d'un isomorphisme canonique

(7.1.1) $$\det(g_* F_0 - g_* F_1) = N_{S'/S} \det(F_0 - F_1) \quad \text{, i.e}$$
$$\det g_* F_0 \otimes (\det g_* F_1)^{-1} = N_{S'/S} (\det F_0 \otimes (\det F_1)^{-1})$$

compatible à la localisation sur S, et caractérisé par le fait que pour tout isomorphisme $u: F_1 \to F_0$, les trivialisations des deux membres déduites de u se correspondent. De tels isomorphismes u existent localement sur S et que (1.1) ne dépende pas du u choisi exprime que pour v un automorphisme de F_1, on a

$$\det(v , g_* F_1) = N_{S'/S} \det(v , F_1) .$$

Plus généralement, pour $g: X \to S$ quelconque et E sur X, S-plat, à support S' fini sur S, on définit

$$N_{E/S}: g_* \mathcal{O}^*_{S'} \to \mathcal{O}^*_S : u \longmapsto \det(u , g_* E) ,$$

un foncteur norme correspondant et un isomorphisme

(7.1.2) $$\det(g_* E\otimes F_0 - g_* E\otimes F_1) = N_{E/S} \det(F_0 - F_1) .$$

Montrons que (7.1.1) provient d'un isomorphisme fonctoriel en v, fibré virtuel de rang partout nul sur S'

(7.1.3) $$\det g_* v = N_{S'/S} \det v .$$

Définissons un foncteur Ngr des faisceaux inversibles gradués (a , L) sur S' dans les faisceaux inversibles gradués sur S par

$$\mathrm{Ngr}(a , L) := (\mathrm{Tr}_{S'/S}(a) , \det g_* \mathcal{O}^a \otimes N_{S'/S} L) .$$

La donnée de compatibilité à + évidente sur ce foncteur est compatible à la commutativité et à l'associativité. Pour E un fibré vectoriel sur S', directement ou par (7.1.1), on a un isomorphisme

$$\det g_* E \simeq \mathrm{Ngr} \det E$$

et cet isomorphisme est compatible à l'additivité en E des deux membres. Il se prolonge donc en un isomorphisme fonctoriel en v virtuel:

$$\det g_* v \simeq \mathrm{Ngr} \det(v) \tag{7.1.4}$$

qui, pour v partout de rang 0 donne (7.1.3).

De même, (7.1.2) provient de

$$\det g_*(E \otimes v) = N_{E/S} \det(v) \tag{7.1.5}$$

pour v de rang 0 .

Supposons g plat de dimension relative 1 . Si E est localement le conoyau d'un morphisme u: $\mathcal{O}^n \to \mathcal{O}^n$ avec det(u) fibre à fibre non diviseur de zéro, det u est l'équation d'un diviseur de Cartier relatif div(E) ne dépendant que de E : le fibré en droites det E est trivialisé en dehors de S' , la section trivialisante d se prolonge à X et div(E) est défini par $d = 0$. On a $N_{E/S} = N_{\mathrm{div}(E)/S}$ pour le morphisme norme, donc pour le foncteur norme.

Soient g: $X \to S$ propre et plat purement de dimension relative un et D un diviseur de Cartier relatif. Pour tout faisceau inversible M sur X , on a

$$\langle \mathcal{O}(D) , M\rangle \xrightarrow{\sim} N_{D/S}(M) \quad : \quad \langle 1 , m\rangle \longmapsto N_{D/S}(m) \quad .$$

En d'autres termes, pour tout faisceau inversible L et toute section ℓ de L , fibre à fibre non diviseur de zéro, on a

$$\langle L , M\rangle \xrightarrow{\sim} N_{\mathrm{div}(\ell)/S}(M) \quad : \quad \langle \ell , m\rangle \longmapsto N_{\mathrm{div}(\ell)/S}(m) \quad .$$

<u>Construction</u> 7.2 Soient f: $X \to S$, propre, plat, purement de dimension relative 1, E_0 et E_1 deux fibrés vectoriels sur X , partout du même rang, et F_0 et F_1 de même. On a

$$\langle \det(E_0 - E_1) , \det(F_0 - F_1)\rangle = \det f_*((E_0 - E_1) \otimes (F_0 - F_1)) \tag{7.2.1}$$

Le langage des fibrés virtuels nous sera utile pour ne pas noyer 7.2 dans un océan de signes.

Au membre de gauche, $E_0 - E_1$ est le fibré virtuel somme de $[E_0]$ et de l'opposé de $[E_1]$. Le fibré en droites gradué $\det(E_0 - E_1)$ est $\det(E_0) \otimes \det(E_1)^{-1}$. Il est de degré 0 car E_0 et E_1 sont de même rang et on l'identifie au fibré en droites sous-jacent. Soit $u: E_1 \to E_0$ et notons E_* le complexe $E_1 \to E_0$ (degrés -1 et 0). On a $\det(E_*) = \det(E_0 - E_1)$. Si u est un monomorphisme, son conoyau E est de Tor-dimension finie; la projection de E_0 sur E est un quasi-isomorphisme de E_* avec le complexe réduit à E en degré 0, d'où un isomorphisme

(7.2.2) $$\det(E) = \det(E_*) = \det(E_0 - E_1) = \det(E_0)\det(E_1)^{-1} .$$

Il est déduit par addition de $\det(E_1)^{-1}$ aux deux membres de l'isomorphisme $\det(E_0) = \det(E_1)\det(E)$ défini par la suite exacte courte $E_1 \to E_0 \to E$. Sur l'ouvert où u est inversible, les deux membres de (7.2.2) sont trivialisés, et les trivialisations se correspondent: à gauche, $E = 0$ et $\det(E)$ est donc trivial. A droite, u induit $\det(u)$: $\det(E_1) \xrightarrow{\sim} \det(E_0)$. On note encore $\det(u)$ la trivialisation correspondante de $\det(E_0 - E_1)$.

Au membre de droite, $(E_0 - E_1) \otimes (F_0 - F_1)$ est un produit tensoriel d'objets virtuels. Quels que soient $u: E_1 \to E_0$ et $v: F_1 \to F_0$, c'est le fibré virtuel défini par le complexe $E_* \otimes F_*$ (noter que c'est la "difficulté" 4.11 (b) qui rend ceci possible). S'il existe u et v qui soient des monomorphismes, avec $\det(u)$ et $\det(v)$ fibre à fibre non diviseurs de zéro et non simultanément nuls, de conoyaux E et F, u et v définissent un isomorphisme d'objets virtuels

(7.2.3) $$(E_0 - E_1) \otimes (F_0 - F_1) = [E_* \otimes F_*] = [E \otimes F] = 0 ,$$

et cette trivialisation définit une trivialisation du membre de droite de (7.2.1). Notons $\alpha(u,v)$ l'isomorphisme (7.2.1) par laquelle cette trivialisation correspond à la trivialisation $\langle \det u , \det v \rangle$ du membre de gauche.

Supposons seulement disposer de $u: E_1 \to E_0$ avec $\det(u)$ fibre à fibre non diviseur de zéro. On a par 7.1 un isomorphisme

$$\det f_*((E_0 - E_1) \otimes (F_0 - F_1)) = \det f_*(E \otimes (F_0 - F_1)) = N_{\mathrm{div}\, E/S} \det(F_0 - F_1)$$

et $\det u$ identifie $\langle \det(E_0 - E_1)$, à $N_{\mathrm{div}(E)/S}(\quad)$. Ceci identifie les deux membres de (7.2.1) à $N_{\mathrm{div}(E)/S} \det(F_0 - F_1)$, d'où un isomorphisme $\alpha(u)$ entre les deux membres. On vérifie aussitôt que si $\alpha(u,v)$ est défini, il coïncide avec $\alpha(u)$.

Aux deux membres de (7.2.1), les E_i et les F_i jouent des rôles symétriques. Par symétrie, v: $F_1 \to F_0$ définit donc $\alpha(v)$ avec $\alpha(u,v) = \alpha(v)$ quand $\alpha(u,v)$ est défini.

Etant donnés trois fibrés E_0 , E_1 et E_2 , on a

(7.2.4) $\quad (E_0 - E_2) = (E_0 - E_1) + (E_1 - E_2)$

et une additivité pour les deux membres de (7.2.1). Si on dispose de u': $E_1 \to E_0$ et u": $E_2 \to E_1$ de composé u , l'isomorphisme (7.2.4) coïncide avec celui défini par la suite exacte courte

$$\mathrm{coker}(u'') \longrightarrow \mathrm{coker}(u) \longrightarrow \mathrm{coker}(u') \ .$$

On en déduit que $\alpha(u)$, $\alpha(u')$, $\alpha(u'')$ sont compatibles à l'additivité des deux membres de (7.2.1). Si on dispose seulement de u et u" , soit u' le morphisme rationnel uu''^{-1} et définissons $\alpha(u')$ comme étant l'isomorphisme entre les deux membres de (7.2.1) tels que $\alpha(u)$, $\alpha(u')$, $\alpha(u'')$ soient compatibles à l'additivité. Pour f: $E_2' \to E_2$ sans diviseur de zéro, $\alpha(u')$ est le même, calculé avec u et u" ou uf et u"f . Ceci permet de définir $\alpha(u')$ pour u': $E_1 \to E_0$ un isomorphisme sur un ouvert dense fibre à fibre et contenant fibre à fibre les points non de Cohen Macaulay: localement sur S (pour la topologie étale), il existe E_2 sur X et u": $E_2 \to E_1$ tel que u'u" soit un morphisme, et deux tels sont coiffés par un même troisième; prendre $E_2 = E_1 \otimes L^{\otimes -n}$ avec L ample. Que ce soit possible localement sur S suffit, car la définition d'une flèche (7.2.1) est un problème local sur S .

Symétriquement, un morphisme rationnel v: $F_1 \to F_0$ analogue à u': $E_1 \to E_0$ définit un isomorphisme (7.2.1) $\alpha(v)$. Montrons que, quels que soient u: $E_1 \to E_0$ et v: $F_1 \to F_0$ rationnels (isomorphisme sur un ouvert fibre à fibre dense et contenant les points non de Cohen-Macaulay, on a $\alpha(u) = \alpha(v)$. Les compatibilités à l'additivité et à la localisation sur S ramènent à supposer u et v des morphismes avec div(u) et div(v) disjoints. Dans ce cas, $\alpha(u) = \alpha(u,v) = \alpha(v)$. Il en résulte que $\alpha(u)$ (resp $\alpha(v)$) est indépendant de u (resp v): c'est l'isomorphisme 7.2.1

<u>Variante</u> 7.3 L'isomorphisme (7.2.1) provient d'un isomorphisme biadditif, fonctoriel en des fibrés virtuels de rang 0

(7.3.1) $\quad \langle \det u \ , \ \det v \rangle = \det f_* (u \otimes v)$.

Il s'agit de montrer que pour tout isomorphisme de fibrés virtuels $(F_0 - F_1) \xrightarrow{\sim} (F'_0 - F'_1)$, les isomorphismes (7.2.1) donnent lieu à un diagramme commutatif

$$\begin{array}{ccc} \langle \det(E_0 - E_1) , \det(F_0 - F_1)\rangle & \text{——} & \det f_*((E_0 - E_1) \otimes (F_0 - F_1)) \\ | & & | \\ \langle \det(E_0 - E_1) , \det(F'_0 - F'_1)\rangle & \text{——} & \det f_*((E_0 - E_1) \otimes (F'_0 - F'_1)) \end{array}$$

(et de même, symétriquement, pour E remplaçant F) .
On se ramène à supposer qu'il existe u: $E_1 \to E_0$ non diviseur de zéro et au même problème pour l'isomorphisme

$$\det f_*(E \otimes (F_0 - F_1)) \text{ —— } N_{E/S} \det(F_0 - F_1) \quad ,$$

traité en (7.1.4) (7.1.5).

7.4 Pour u et v virtuels de rang 0 , det(u ⊗ v) est trivialisé: localement, u et v sont isomorphes à 0 et la trivialisation est définie aussi bien par un isomorphisme $u \xrightarrow{\sim} 0$ que par un isomorphisme $v \to 0$ (pour un énoncé plus général, voir 4.14) . Pour f: $X \to S$ comme en 7.2 et u , v , w de rang 0 , on a donc un isomorphisme

(7.4.1) $\det f_*(u \otimes v \otimes w) = \langle \det u , \det(v \otimes w)\rangle = 1$.

Cette définition fait jouer un rôle spécial à u , mais l'isomorphisme obtenu est symétrique en u , v et w : on se ramène à supposer u , v et w définis par $E_0 - E_1$, $F_0 - F_1$, $G_0 - G_1$ et qu'il existe a: $E_1 \to E_0$, b: $F_1 \to F_0$, c: $G_1 \to G_0$ de déterminants non diviseurs de zéro à diviseurs deux à deux disjoints. On trouve que 7.4.1 est déduit de l'isomorphisme

$$\text{coker (a)} \otimes \text{coker (b)} \otimes \text{coker (c)} = 0$$

où E , F , G jouent des rôles symétriques.

<u>Construction</u> 7.5 Soit f: $X \to S$ une courbe sur S , propre, plate et d'intersection complète relative. Le fibré en droites sous-jacent au fibré en droites gradué $\det Rf_*(L - O) := \det Rf_*L \otimes \det Rf_*O^{-1}$ donne lieu à un isomorphisme

(7.5.1) $(\det Rf_*(L - O))^{\otimes 2} = \langle L , L \otimes \omega^{-1}\rangle$.

Pour tout fibré vectoriel F sur X , de dual de Serre $\underline{\mathrm{Hom}}(F , \omega)$, la dualité de Serre

$$R\underline{\mathrm{Hom}}(Rf_*F , O) [-1] = Rf_* \underline{\mathrm{Hom}}(F , \omega)$$

donne lieu à un isomorphisme entre les fibrés en droites sous-jacents aux fibrés en droites gradués $\det Rf_*F$ et $\det Rf_* \underline{Hom}(F , \omega)$, (situés en degrés différant par un entier pair). De là, un isomorphisme

$$\det Rf_*((L - O) \otimes (\omega L^{-1} - O) = \det Rf_* \omega . (\det Rf_* L)^{-1} . (\det Rf_* \omega L^{-1})^{-1} . \det Rf_* O$$
$$= (\det Rf_* (L - O))^{\otimes(-2)}$$

et on applique 7.2.1.

7.6 Les deux membres M de 7.5.1 donnent lieu à des isomorphismes canoniques

(7.6.1) $M(L \otimes M) = M(L) \otimes M(M) \otimes \langle L , M \rangle^{\otimes 2}$

et (à gauche, dualité de Serre)

(7.6.2) $M(L^{-1} \otimes \omega) \simeq M(L)$.

Nous utiliserons que les isomorphismes(7.6.1) (resp (7.6.2)) des deux membres se correspondent par (7.5.1). L'usage essentiel, en 7.11, ne le requiert qu'au signe près.

8. Intégrale d'un produit de premières classes de Chern. Cas général (esquisse).

8.1 Avec les notations de 6.1, et comme expliqué en 6.1, il s'agit d'attacher à des faisceaux inversibles L_i $(0 \leq i \leq N)$ sur X un faisceau inversible $\langle L_0, \ldots, L_N\rangle$ sur S. Nous supposerons X projectif sur S. Pour S réduit à un point, on peut définir $\langle L_0, \ldots, L_N\rangle$ comme l'espace vectoriel engendré par des symboles $\langle \ell_0, \ldots, \ell_N\rangle$, pour les ℓ_i des sections méromorphes dont les diviseurs ont une intersection vide, avec les relations suivantes. Si $0 \leq i \leq N$, et que les $\mathrm{div}(\ell_j)$ $(j \neq i)$ ne se coupent qu'en un nombre fini de points, définissant un 0-cycle $\sum a_\alpha P_\alpha = \prod_{j\neq i} \mathrm{div}(\ell_j)$, on veut

(8.1.1) $\langle \ell_0, \ldots, f\ell_i, \ldots, \ell_N\rangle = \prod f(P_\alpha)^{a_\alpha} . \langle \ell_0, \ldots, \ell_N\rangle$.

Le fait que $\langle \ell_0, \ldots, \ell_N\rangle$ n'est défini que si l'intersection des $\mathrm{div}(\ell_i)$ est vide, et que (8.1.1) n'est imposé que si l'intersection des $\mathrm{div}(\ell_j)$ $(j \neq i)$ a un sens en tant que 0-cycle, crée une difficulté inessentielle. Si on la néglige, il est clair que deux quelconques $\langle \ell_1, \ldots, \ell_{N+1}\rangle$ sont proportionnels: joindre $\langle \ell_1, \ldots, \ell_{N+1}\rangle$ à $\langle m_1, \ldots, m_{N+1}\rangle$ par une chaîne de $\langle n_0^a, \ldots, n_N^a\rangle$ $(0 \leq a \leq N+1)$ avec $n_i = \ell_i$ ou m_i, en ne changeant à chaque étape qu'un seul argument. Si deux chaînes coïncident sauf pour une valeur de a, qu'elles imposent le même rapport $\langle m\rangle/\langle \ell\rangle := \langle m_0, \ldots, m_N\rangle/\langle \ell_0, \ldots, \ell_N\rangle$ résulte du théorème de Weil $f(\mathrm{div}\, g) = g(\mathrm{div}\, f)$ sur une intersection de $(N-1)$ diviseurs. Ne changeant qu'un n^a à la fois, on peut passer d'une chaîne à une autre en un nombre fini d'étapes. Toutes les chaînes imposent donc le même rapport $\langle m\rangle/\langle \ell\rangle$. Pour que la construction ne s'effondre pas, il faut encore vérifier que, pour $\langle m\rangle/\langle \ell\rangle$ ainsi calculé, on a bien

(1) $$(\langle n\rangle/\langle m\rangle)\,(\langle m\rangle/\langle \ell\rangle) = \langle n\rangle/\langle \ell\rangle .$$

Si pour un i $(0 \leq i \leq N)$ on a $\ell_i = m_i = n_i =: s$, avec $\mathrm{div}(s) = A - B$, l'assertion se réduit à (1) pour $(\ell_0, \ldots, \hat{\ell}_i, \ldots, \ell_N)$, $(m_0, \ldots, \hat{m}_i, \ldots, m_N)$ et $(n_0, \ldots, \hat{n}_i, \ldots, n_N)$ sur A et B. Choisissant une section méromorphe s de L_0, on se ramène à ce cas en comparant $\ell := (\ell_0, \ldots, \ell_N)$ à $\ell' := (s, \ell_1, \ldots, \ell_N)$ et de même pour m et n : il faut une compatibilité entre rapports dans le carré

$$\begin{array}{ccc} \langle \ell\rangle & \text{———} & \langle \ell'\rangle \\ | & & | \\ \langle m\rangle & \text{———} & \langle m'\rangle \end{array}$$

Joignant ℓ à m par une chaîne, on peut supposer que $\ell_j = m_j$ sauf pour une valeur i de j. L'assertion se ramène alors à une question analogue sur l'intersection des $\mathrm{div}(\ell_j)$ $(j \neq 0 , i)$, i.e. au cas $N = 0$ ou 1 déjà traité.

8.2 Bouchons les trous de cet argument. Pour éviter de parler de sections méromorphes, il y a intérêt à introduire un faisceau inversible très ample X tel que les $L_i \otimes X$ soient aussi très amples. Au lieu d'une section méromorphe ℓ_i de L_i, on prendra des sections ℓ_i' de $L_i \otimes X$ et ℓ_i'' de X, supposées localement non diviseur de zéro.

Formellement, $\ell_i = \ell_i' / \ell_i''$ et $\mathrm{div}(\ell_i) = \mathrm{div}(\ell_i') - \mathrm{div}(\ell_i'')$. On a gagné que $\mathrm{div}(\ell_i')$ et $\mathrm{div}(\ell_i'')$ sont des diviseurs de Cartier.

Une chaîne de ℓ à m peut contenir des n interdits, i.e. avec une intersection des $\mathrm{div}(n_i') \cup \mathrm{div}(n_i'')$ non vides. Pour obvier à ce problème, on pose

$$\langle\ell\rangle/\langle m\rangle := (\langle\ell\rangle/\langle r\rangle)/(\langle m\rangle/\langle r\rangle)$$

pour un r choisi assez général et le membre de droite défini comme en 8.1. Avec cette règle, (1) est clair. L'indépendance de r résulte de 8.1 (1) pour ℓ, r, r_1 et m, r, r_1, avec r et r_1 assez généraux.

L'hypothèse "assez général" a un sens et suffit à faire marcher les arguments si le corps de base est infini. Le cas d'un corps de base fini demande de plus une descente galoisienne.

Un autre problème est le sens à attribuer à une intersection de $(N - 1)$ $\mathrm{div}(\ell_i)$. On veut que ce soit une courbe à laquelle on puisse appliquer le théorème de Weil. Le plus simple ici est de ne considérer que des systèmes $\ell = (\ell_0 , \ldots, \ell_N)$, $\ell_i = (\ell_i' , \ell_i'')$ tels que chaque ℓ_i' ou ℓ_i'' soit non diviseur de zéro sur toute intersection de $\mathrm{div}\,(\ell_j') + \mathrm{div}\,(\ell_j'')$ $(j \neq i)$.

Ainsi modifié, l'argument 5.1 fonctionne encore sur une base S, pour autant qu'on le localise pour la topologie étale sur S. Dans 5.1.1, le produit des $f(P_\alpha)^{a_\alpha}$ est à remplacer par la norme de f, de l'intersection des $(\mathrm{div}(\ell_j') - \mathrm{div}(\ell_j''))$ $(j \neq i)$ à S. L'intersection considérée est un revêtement fini et plat virtuel de S.

8.3 Supposons que $S = \mathrm{Spec}(\mathbb{C})$, que X est lisse et que les L_i sont munis de métriques. On définit une métrique sur $\langle L_0 , \ldots, L_N\rangle$ par la formule suivante. Soient $\ell_i = \ell_i'/ \ell_i''$ comme en 5.2, avec $D_i' := \mathrm{div}\,(\ell_i')$ et $D_i'' := \mathrm{div}(\ell_i'')$ lisses et les D'_i et D''_i $(0 \leq i \leq N)$ en position générale. Pour $J \subset [0,N]$, soit $D(J)$ l'intersection des $D_j := D'_j - D''_j$. On aura à intégrer sur $D(_J)$. Par définition, une intégration sur $D(J)$ est la somme sur les $J_1 \subset J$ des $(-1)^{\# J_1}$ $\int_{D(J,J_1)}$, où $D(J , J_1)$ est l'intersection des $D'_j(j \in J - J_1)$ et des D''_j $(j \in J_1)$. On pose

(8.3.1) $\log\|\langle \ell_0 ,\ldots, \ell_N\rangle\| = \sum \int_{D(J)} \alpha_J$,

où la somme porte sur les $J \subset I$, $J \neq I$ et où, pour un $j(J)$ quelconque dans $[0,N] - J$, α_J est le produit des $\frac{1}{2\pi i}$ d'd" $\log\|\ell_j\|$ $(j \notin J , j \neq j(J))$ et de $\log\|\ell_{j(J)}\|$. La forme α_J dépend du choix de $j(J)$ mais non son intégrale. Le d'd" est pris au sens des distributions. On a encore (cf (4.3.2)).

(8.3.2) $\log\|\langle \ell_0 ,\ldots, \ell_N\rangle\| = \sum_{0 \leq i \leq N} \int_{D(]i,N])} \bigwedge_{j<i} \underline{\underline{c}}^1(L_j) \cdot \log\|\ell_i\|$.

Cette construction se généralise au cas où X est projectif et lisse sur S de type fini sur $\mathbb{C}$.

8.4 Les constructions 8.1 à 8.3 sont symétriques en les L_i : pour toute permutation σ de $[0,N]$, on dispose de

$$a(\sigma) : \langle L_0 ,\ldots, L_N\rangle \xrightarrow{\sim} \langle L_{\sigma(0)} ,\ldots, L_{\sigma(N)}\rangle .$$

Si $L_i = L_j$ (avec $i \neq j$) et que σ est la transposition qui échange i et j , on a $a(\sigma) = (-1)^n$ avec n un entier localement constant sur S . Cet entier est

$$n = \int_{X/S} c^1(L_i) \cdot \bigwedge_{k\neq i,j} c^1(L_k) \quad ,$$

ainsi qu'on le vérifie par réduction au cas $N = 1$.

<u>Proposition</u> 8.5 Pour X lisse sur S , supposé lisse de type fini sur $\mathbb{C}$, et les L_i munis de métriques, on a

$$\underline{\underline{c}}^1(\langle L_0 ,\ldots, L_N\rangle) = \int_{X/S} \bigwedge \underline{\underline{c}}^1(L_i) \quad .$$

Comme en 6.6, on se ramène au cas où chaque L_i est soit un $\mathcal{O}(D)$, avec D lisse, muni de la métrique singulière pour laquelle $\|1\| = 1$, soit est isomorphe à $\mathcal{O}$. Le cas où un L_i est un $\mathcal{O}(D)$ se ramène à un énoncé analogue sur D : on a une isométrie

$$\langle \mathcal{O}(D) , L_1 ,\ldots, L_N\rangle = \langle L_1 ,\ldots, L_N\rangle \text{ sur } D .$$

Le cas où tous les L_i sont isomorphes à $\mathcal{O}$ se traite comme en 6.6.

9. IC^2 et isomorphisme de Riemann-Roch

9.1 Soit $f: X \to S$ propre et plat purement de dimension relative 1 . Pour P le polynôme C^2 , 2.1 nous demande de construire un foncteur (pour les isomorphismes) $I_{X/S}C^2$ des fibrés vectoriels sur X vers les fibrés en droites sur S . La classe de Chern c^2 vérifie

(9.1.1) $\quad c^2(E) = 0$ pour E de rang 1 , et

(9.1.2) $\quad c^2(E) = c^2(E') + c^2(E'') + c^1(E')\, c^1(E'')$

pour E une extension de E'' par E' . Ceci conduit à vouloir l'existence d'isomorphismes canoniques

(9.1.3) $\quad I_{X/S}C^2(E) = \mathcal{O}$ pour E de rang 1 ;

(9.1.4) $\quad I_{X/S}C^2(E) \simeq I_{X/S}C^2(E') \otimes I_{X/S}C^2(E'') \otimes \langle \det(E') , \det(E'')\rangle$.

9.2 Considérons la catégorie de Picard commutative suivante:

<u>objets</u>: triples (n , L , M) où $n \in \Gamma(X,\mathbf{Z})$ (un entier localement constant sur X) , où L est un faisceau inversible sur X et où M est un faisceau inversible sur S ;

<u>flèches</u>: isomorphismes ;

<u>addition des objets</u>: $(n' , L' , M') + (n'' , L'' , M'')$ est le triple $(n' + n'' , L' \otimes L'' , M' \otimes M'' \otimes \langle L' , L''\rangle)$.

Définissons l'isomorphisme d'associativité $(A_1 + A_2) + A_3 = A_1 + (A_2 + A_3)$. Posons $A_i = (n_i , L_i , M_i)$ $(i = 1 , 2 , 3)$, $(A_1 + A_2) + A_3 = (n' , L' , M')$ et $A_1 + (A_2 + A_3) = (n'' , L'' , M'')$. On a

$$n' = n_1 + n_2 + n_3 = n''$$
$$L' = (L_1 \otimes L_2) \otimes L_3 \simeq L_1 \otimes (L_2 \otimes L_3) = L''$$
$$M' = (M_1 \otimes M_2 \otimes \langle L_1 , L_2\rangle) \otimes M_3 \otimes \langle L_1 \otimes L_2 , L_3\rangle$$
$$M'' = M_1 \otimes (M_2 \otimes M_3 \otimes \langle L_2 , L_3\rangle) \otimes \langle L_1 , L_2 \otimes L_3\rangle$$

et la bimultiplicativité de $\langle\ \rangle$ jointe aux isomorphismes usuels d'associativité et de commutativité pour $\otimes$ identifient tant M' que M'' à $\bigotimes_i M_i \otimes \bigotimes_{i<j} \langle L_i , L_j\rangle$.

Définissons l'isomorphisme de commutativité $\sigma : A_1 + A_2 = A_2 + A_1$. Posons $A_i = (n_i , L_i , M_i)$ $(i = 1 , 2)$, $A_1 + A_2 = (n' , L' , M')$, $A_2 + A_1 = (n'' , L'' , M'')$. On a

$$n' = n_1 + n_2 = n''$$

$$L' = L_1 \otimes L_2 \quad , \; L'' = L_2 \otimes L_1 \quad \text{et}$$

$L' \to L''$ est $\ell_1 \ell_2 \longmapsto (-1)^{n_1 n_2} \ell_2 \ell_1$

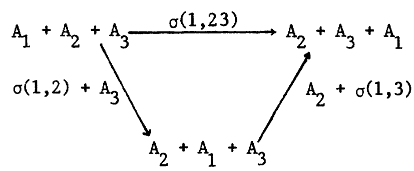
$$M' = M_1 \otimes M_2 \otimes \langle L_1 , L_2 \rangle$$

$$M'' = M_2 \otimes M_1 \otimes \langle L_2 , L_1 \rangle \quad \text{et}$$

$M' \to M''$ est $m_1 \otimes m_2 \otimes \langle \ell_1 , \ell_2 \rangle \longmapsto (-1)^N \, m_2 \otimes m_1 \otimes \langle \ell_2 , \ell_1 \rangle$

avec $N = \int_{X/S} n_1 n_2 (c^1(L_1) + c^1(L_2))$. Ce signe est justifié par le

<u>Lemme</u> 9.2.1 Les données d'associativité et de commutativité ci-dessus sont compatibles



Posons $A_i = (n_i , L_i , M_i)$, $c(i) = c^1(L_i)$ et soit $N(\, , \,)$ l'exposant de (-1) dans la définition de l'action de $\sigma(\, , \,)$ sur la troisième composante. Calculons les images d'une section locale $s = \langle \ell_1 , \ell_2 \rangle \langle \ell_1 , \ell_3 \rangle \langle \ell_2 , \ell_3 \rangle m_1 m_2 m_3$ de la troisième composante de $A_1 + A_2 + A_3$. Les m_i jouent un rôle de figurants et on les omettra de la notation. Pour calculer $\sigma(1,23)(s)$, on récrit $s = \langle \ell_1 , \ell_2 \ell_3 \rangle \langle \ell_2 \ell_3 \rangle$:

$$\begin{aligned} \sigma(1,23)\,(s) &= (-1)^{N(1,23)} \langle \ell_2 \ell_3 , \ell_1 \rangle \langle \ell_2 , \ell_3 \rangle \\ &= (-1)^{N(1,23)} \langle \ell_2 , \ell_3 \rangle \langle \ell_2 , \ell_1 \rangle \langle \ell_3 , \ell_1 \rangle . \end{aligned}$$

Pour calculer $(\sigma(1,2) + A_3)(s)$, on récrit $s = \langle \ell_1 , \ell_2 \rangle \langle \ell_1 \ell_2 , \ell_3 \rangle$. Par $\sigma(1,2)$, $\langle \ell_1 , \ell_2 \rangle$ donne $(-1)^{N(1,2)} \langle \ell_2 , \ell_1 \rangle$ et $\ell_1 \ell_2$ donne $(-1)^{n_1 n_2} \ell_2 \ell_1$, d'où

$$\begin{aligned} (\sigma(1,2) + A_3)\,(s) &= (-1)^{N(1,2)} \langle \ell_2 , \ell_1 \rangle \langle (-1)^{n_1 n_2} \ell_2 \ell_1 , \ell_3 \rangle \\ &= (-1)^{N(1,2) + \int n_1 n_2 c(3)} \langle \ell_2 , \ell_1 \rangle \langle \ell_2 , \ell_3 \rangle \langle \ell_1 , \ell_3 \rangle . \end{aligned}$$

De même,

$$(A_2 + \sigma(1,3))\ (\langle \ell_2 , \ell_1\rangle \langle \ell_2 , \ell_3\rangle \langle \ell_1 , \ell_3\rangle)$$
$$= (-1)^{N(1,3) + \int n_1 n_3 c(2)} .$$

Il reste à vérifier une congruence modulo 2

$$N(1,23) \equiv N(1,2) + \int n_1 n_2 c(3) + N(1,3) + \int n_1 n_3 c(2) .$$

Avant intégration de X à S, on a en effet

$$n_1(n_2 + n_3)\ (c(1) + c(2) + c(3)) = n_1 n_2 (c(1) + c(2)) + n_1 n_2 c(3)$$
$$+ n_1 n_3 (c(1) + c(3)) + n_1 n_3 c(2) .$$

Pour la deuxième composante, on retrouve la compatibilité entre associativité et commutativité pour les faisceaux inversibles gradués. Ceci termine la preuve de 9.2.1.

La vérification, plus facile, des autres axiomes de catégories de Picard commutatives est laissée au lecteur.

9.3 Une version précisée de (9.1.4) est qu'on veut que le foncteur $E \to (\mathrm{rg}(E)$, $\det(E)$, $I_{X/S}C^2(E))$ se factorise par un morphisme de catégories de Picard commutative des fibrés virtuels sur X dans la catégorie 9.2 - la compatibilité à $+$ prolongeant celle de

$$(\text{fibrés virtuels}) \longrightarrow (\mathrm{rg}(E) , \det(E)) .$$

On veut de plus une compatibilité à tout changement de base S'/S (i.e. un morphisme de préchamps en catégories de Picard commutatives de $f_*\underline{K}(X)$ vers le champ sur S de catégories 9.2). La proposition suivante est un résultat d'existence et d'unicité pour un tel foncteur IC_2, muni de données additionnelles et satisfaisant à des axiomes additionnels. On y suppose f lisse. Localement sur S pour la topologie étale, X est alors quasi-projectif et un faisceau cohérent Q sur X de Tor-dimension finie sur S définit un fibré vectoriel virtuel $[Q]$ sur X. On posera $IC_2(Q) := IC_2([Q])$.

Proposition 9.4 Soit f comme en 9.1, supposé lisse. A isomorphisme unique près, il existe un et une seul foncteur IC_2, muni, de façon compatible à tout changement de base, de données du type suivant.

(i) Le foncteur $T: E \longmapsto (\mathrm{rg}(E) , \det E, IC_2(E))$ est muni, pour chaque suite exacte courte $E' \to E \to E''$ d'un isomorphisme $T(E) \sim T(E') + T(E'')$. Cette donnée prolonge la donnée analogue pour $E \longmapsto (\mathrm{rg}\ E, \det E)$, et est compatible à l'associativité et à la commutativité: T se factorise par un morphisme de $\underline{\underline{K}}(X)$ dans la catégorie de Picard commutative 9.2.

(ii) Pour L un fibré en droites, $IC_2(L)$ est trivialisé.

(iii) Soient s une section de X/S , Q' un fibré en droites sur S , $Q = s_*Q'$. Soit L un fibré en droites sur X prolongeant Q (i.e. tel que $Q' = s^*L$) . La suite exacte courte

$$0 \to L(-s(S)) \to L \to Q \to 0$$

et (i) fournissent un isomorphisme

$$IC_2(L(-s(S)))\ IC_2(Q)\ \langle L(-s(S)), \det(Q)\rangle = IC_2(L) ,$$

d'où par (ii) et parce que $\det(Q) = \mathcal{O}(s(S))$ un isomorphisme

$$IC_2(Q)\ s^*(L(-s(S))) = 0 , \quad \text{i.e.}$$

$$IC_2(Q) \simeq (s^*Q)^{-1}\ s^*\mathcal{O}(s(S)) .$$

On exige que cet isomorphisme soit indépendant de L .

Remarques: (a) La condition (iii) n'est pas conséquence de (i) (ii). En effet, si on modifie les données de trivialisation (ii) par un facteur $\lambda(\deg L)$, elle ne reste vérifiée que si $n \longmapsto \lambda(n)$ est de la forme $n \longmapsto \lambda^n$.

(b) La formulation de (iii) utilise que, par (i) , T est prolongé à $\underline{\underline{K}}(X)$. Soient L_1 et L_2 deux fibrés en droites munis d'un isomorphisme $s^*L_i = Q$. Comparons les présentations $Q = L_i/L_i(-s)$ à $0 \to E \to L_1 \oplus L_2 \to Q \to 0$.

D'après 4.8 appliqué au diagramme des 9

$$\begin{array}{ccccc} L_1(-s) & \to & L_1 & \to & Q \\ \downarrow & & \downarrow & & \| \\ E & \to & L_1 \oplus L_2 & \to & Q \\ \downarrow & & \downarrow & & \downarrow \\ L_2 & = & L_2 & \to & 0 \end{array} ,$$

le diagramme

$$\begin{array}{ccc} T(L_1 \oplus L_2) & = & T(L_1) + T(L_2) \\ \| & & \| \\ T(E) + T(Q) & = & T(L_1(-s)) + T(L_2) + T(Q) \end{array}$$

est commutatif. L'isomorphisme

$$T(L_1) - T(L_1(-s)) = T(Q) = T(L_1 \oplus L_2) = T(E)$$

est donc déduit des isomorphismes $T(L_1 \oplus L_2) = T(L_1) + T(L_2)$ et $T(E) = T(L_1(-s)) + T(L_2)$: les soustraire membre à membre. En termes où seuls apparaissent des fibré vectoriels, (iii) affirme la commutativité du diagramme

$$\begin{array}{ccc} T(L_1) - T(L_1(-s)) & = T(L_1) + T(L_2) - T(E) = & T(L_2) + T(L_2(-s)) \\ & \searrow \qquad \swarrow & \\ & (s^*(Q))^{-1}\, s^*\mathcal{O}(s(S)) & . \end{array}$$

(c) Nous écrirons la preuve sous l'hypothèse que S est le spectre d'un corps algébriquement clos et que X est connexe. Le cas général se traite de même: remplacer "point" par "section" et parsemer de "localement pour la topologie étale". Nous donnerons ensuite une seconde preuve de l'existence, sans cette hypothèse simplificatrice.

9.5 Preuve de l'unicité Tout fibré vectoriel E admet une filtration $0 \subset F_1 \subset \dots \subset F_d = E$, avec F_i un fibré vectoriel de rang i localement facteur direct dans E. On peut construire une telle filtration comme suit. Si E est de rang ≤ 1 , il n'y a rien à faire. Sinon, on choisit un fibré en droite L . Si L est assez négatif pour que $E \otimes L^{-1}$ soit engendré par ses sections, une section générale s ne s'annulle nulle part et définit $f: L \to E$ localement facteur direct. On prend $F_1 = f(L)$ et on recommence avec E/F_1 . Si une filtration F est donnée et qu'on construit G par le procédé précédent, général, et avec L chaque fois choisi assez négatif, F et G seront en position générale au sens suivant. En dehors d'un ensemble fini S de points, F et G sont opposées: il existe une décomposition $E = \bigoplus_1^d L_i$ avec $F_a = \bigoplus_{i \leq a} L_i$, $G_a = \bigoplus_{i > d-a} L_i$. Au voisinage de chaque $s \in S$, il existe une seule valeur a de i $(1 \leq a < d)$ pour laquelle $F_i \oplus G_{d-i+1}$ ne soit pas tout E , et $\mathrm{coker}(F_a \oplus G_{d-a+1} \to E)$ est ponctuel de rang 1 . Au voisinage de s , (E , F , G) est alors isomorphe au modèle suivant: le faisceau modèle M est le sous-faisceau de $\mathcal{O}(s)^d$ de sections les $(f_1 \cdots f_d)$ avec f_i dans $\mathcal{O}$ pour $i \neq a , a+1$ et f_a , f_{a+1} de mêmes parties polaires. Les filtrations F , G sont induites par les filtrations opposées évidentes (cf. ci-dessus) de $\mathcal{O}(s)^d$.

Notations 9.5.1 Pour F et G deux filtrations en position générale de E , nous utiliserons les notations suivantes: $L_i := F_i \cap G_{d-i+1}$, $E' = \oplus L_i$, S l'ensemble des points où F et G ne sont pas opposées, a(s) l'entier i tel que $F_i \oplus G_{d-i+1} \xrightarrow{\neq} E$ en s , Q_s les faisceaux ponctuels de rang un, avec Q_s localisé en s , donnant lieu à la suite exacte

$$0 \to E' \to E \to \bigoplus_{s \in S} Q_s \to 0 \ .$$

Pour L un fibré en droite, soit $T(L) = (1 , L , 0)$ dans la catégorie 9.2. Pour Q ponctuel de rang un, en s , soit $T(Q) = (0 , \mathcal{O}(s) , (s^*Q)^{-1} \otimes \tau_s)$, où τ_s est l'espace tangent en s . Pour une suite exacte $0 \to L(-s) \to L \to Q \to 0$, on définit

$$T(L(-s)) + T(Q) \xrightarrow{\sim} T(L) \tag{9.5.2}$$

par la flèche évidente $L(-s) \otimes \mathcal{O}(s) \simeq L$ et par

$$(s^*Q)^{-1} \otimes \tau_s \otimes \langle L(-s) , \det Q\rangle \xrightarrow{\sim} \mathcal{O}$$

défini comme suit: $\det Q = \mathcal{O}(s)$, $\langle L(-s) , \det Q\rangle = s^*L(-s) = s^*L \otimes s^*\mathcal{O}(-s) = s^*Q \otimes m_s/m_s^2$, et on prend l'isomorphisme évident

$$(s^*Q)^{-1} \otimes \tau_s \otimes s^*Q \otimes m_s/m_s^2 \to \mathcal{O} \ .$$

Soient T_1 et T_2 deux foncteurs $(\mathrm{rg}\ E, \det E , IC_2(E))$ comme en 9.4. Pour E de rang 1, la donnée (ii) définit un isomorphisme $T_i(E) = T(E)$. Pour Q ponctuel de rang un, et L un fibré en droites muni d'un isomorphisme $Q_s = L_s$, soit $T_i(Q) \xrightarrow{\sim} T(Q)$ l'isomorphisme rendant commutatif

$$\begin{array}{ccc} T_i(L(-s)) + T_i(Q) & \xrightarrow{\sim} & T_i(L) \\ \big| & & \| \\ T(L(-s)) + T(Q) & \xrightarrow{\sim} & T(L) \end{array} \ .$$

D'après 9.4 (iii), il ne dépend pas du choix de L .

Pour F une filtration de E de quotients successifs des fibrés en droites, soit $\phi(F)$ l'isomorphisme composé

$$T_1(E) = \sum T_1(\mathrm{Gr}_i^F E) = \sum T(\mathrm{Gr}_i^F E) = \sum T_2(\mathrm{Gr}_i^F E) = T_2(E) \ ,$$

somme, rel 9.4 (ii), des $T_1(\mathrm{Gr}_i^F E) = T(\mathrm{Gr}_i^F E) = T_2(\mathrm{Gr}_i^F E)$.

Soit $E_1 \subset E$ avec un quotient ponctuel de rang un, en un point s :

$$0 \to E_1 \to E \to Q \to 0$$

La filtration F de E induit une filtration, encore notée F , de E_1 . Sauf pour une valeur, a , de i , on a $\mathrm{Gr}_i^F(E_1) \xrightarrow{\sim} \mathrm{Gr}_i^F(E)$. Pour $i = a$, on a

$$0 \longrightarrow \mathrm{Gr}_i^F E_1 \longrightarrow \mathrm{Gr}_i^F E \longrightarrow Q \longrightarrow 0 \ .$$

Appliquant 4.7 aux filtrations F et $E \supset E' \supset \{0\}$ de E, on trouve un diagramme commutatif dans $\underline{K}(X)$:

$$\begin{array}{ccc} [E] & \text{———} & \sum[\mathrm{Gr}_i^F E] \\ | & & | \\ [E_1]+[Q] & \text{—} & (\sum[\mathrm{Gr}_i^F E_1])+[Q] \end{array}$$

(flèche verticale droite déduite de $[\mathrm{Gr}_a^F E_1]+[Q]=[\mathrm{Gr}_a^F E])$. Passant par le coin supérieur droit, on trouve que $\phi(F)$ est compatible aux

$$T_i([E]) = \sum T_i([\mathrm{Gr}_i^F\ E_1]) + T_i([Q]) = \sum T(\mathrm{Gr}_i^F\ E_1) + T(Q) \ .$$

Passant par le coin inférieur gauche, on en déduit une compatibilité entre $\phi(F)$ pour E et E_1 :

$$\begin{array}{ccc} T_1(E) & \xrightarrow{\ \phi(F)\ } & T_2(E) \\ | & & | \\ T_1(E_1)+T(Q) & \xrightarrow{\ \phi(F')\ } & T_2(E_1)+T(Q) \end{array}$$

(flèches verticales: additivité de T_i pour $E_1 \to E \to Q$ et $T_i(Q) = T(Q))$.

Montrons que $\phi(F)$ est indépendant de F . Deux filtrations étant toujours en position générale par rapport à une même troisième, il suffit de vérifier que $\phi(F) = \phi(G)$ si F et G sont en position générale. On passe de E à E' (notations 9.5.1) par une suite de modifications du type considéré plus haut. Il suffit donc de prouver que $\phi(F) = \phi(G)$ pour E' , i.e. pour une somme $\bigoplus\limits_i L_i$ munie des deux filtrations opposées évidentes. Celà résulte de ce que T_1 et T_2 sont supposés compatibles à la commutativité:

$$\begin{array}{ccccc} & \sum T_1(L_i) & \text{——} & \sum T_2(L_i) & \\ {}^{F}\nearrow & & & & \searrow^{F} \\ T_1(E_1) & & & & T_2(E_1) \\ {}_{G}\searrow & & & & \nearrow_{G} \\ & \sum T_1(L_i) & \text{——} & \sum T_2(L_i) & \end{array} \quad .$$

Les $\phi(F)$ définissent un isomorphisme de foncteurs $T_1 \xrightarrow{\sim} T_2$, compatible aux trivialisations 9.4 (ii). Pour vérifier qu'il est compatible à l'additivité par suites exactes $E_1 \to E \to E_2$, prendre F avec E_1 = l'un des F_a .

9.6 Preuve de l'existence Nous allons prouver l'assertion 9.4 d'existence par une méthode suggérée par la preuve 9.5 de l'unicité. Pour simplifier la rédaction, on continue à supposer X connexe sur k algébriquement clos.

Soient E un fibré de rang d sur X et F une filtration de E de quotients successifs des fibrés en droites. On pose

$$T(E\ ,\ F) := \sum T(\mathrm{Gr}_i^F\ E)$$

Soit $E_1 \subset E$ avec $Q := E/E_1$ ponctuel de rang un. Pour chaque valeur de i sauf une, on a $\mathrm{Gr}_i^F E_1 \xrightarrow{\sim} \mathrm{Gr}_i^F E$. Pour la valeur exceptionnelle, i = a , on a une suite exacte courte $\mathrm{Gr}_i^F E_1 \to \mathrm{Gr}_i^F E \to Q$ et (9.5.2) fournit un isomorphisme

$$T(E\ ,\ F) = T(E_1\ ,\ F) + T(Q)\ . \tag{9.6.1}$$

Soient L un fibré en droites s et t , deux points distincts, $Q_s = L/L(-s)$ et $Q_t = L/L(-t)$. On vérifie que le diagramme d'isomorphismes (9.5.2) (= (9.6.1))

$$\begin{array}{ccc} T(L) & \text{———} & T(L(-s)) + T(Q_s) \\ | & & | \\ T(L(-t)) + T(Q_t) & \text{———} & T(L(-s-t)) + T(Q_s) + T(Q_t) \end{array} \tag{9.6.2}$$

est commutatif.

Soient Q_1 et Q_2 ponctuels de rang 1, de supports des points distincts, E' le noyau d'un épimorphisme de E dans $Q_1 + Q_2$ et $E_i := \mathrm{Ker}(E \to Q_i)$. Soit a(i) l'entier tel que $\mathrm{Gr}_{a(i)}^F E_i \to \mathrm{Gr}_{a(i)}^F E$ ne soit pas un isomorphisme. Le diagramme d'isomorphismes 9.6.1

$$\begin{array}{ccc} T(E\ ,\ F) & \text{———} & T(E_1\ ,\ F) + T(Q_1) \\ | & & | \\ T(E_2\ ,\ F) + T(Q_2) & \text{———} & T(E'\ ,\ F) + T(Q_1) + T(Q_2) \end{array} \tag{9.6.3}$$

est commutatif. Si $a(1) \neq a(2)$, c'est trivial (plutôt: exprime que les contraintes d'associativité et de commutativité de la catégorie 9.2 sont compatibles). Si $a(1) = a(2)$, cela résulte de (9.6.2).

Soient F et G comme en 9.5.1, dont nous reprenons les notations. Itérant la construction (9.6.1), on obtient un isomorphisme

$$T(E\ ,\ F) = T(E'\ ,\ F) + \sum T(Q_s) = \sum T(L_i) + \sum T(Q_s)$$

et de même pour G . L'itération suppose le choix d'un ordre total sur S mais il résulte de (9.6.3) que l'isomorphisme obtenu n'en dépend pas. Le membre de droite étant le même pour F et G , on obtient un isomorphisme

(9.6.4) $\qquad T(E\ ,\ F) \xrightarrow{\sim} T(E\ ,\ G)$.

Disons que deux filtrations F et G sont <u>adjacentes</u> si il existe a tel que $F_i = G_i$ pour $i \neq a$ et que F et G induisent sur F_{a+1}/F_{a-1} des filtrations en position générales. On a $T(E\ ,\ F) = \sum_{i\neq a,a+1} T(Gr_i^F(E)) + T(F_{a+1}/F_{a-1}\ ,\ F)$, de même pour G , et l'isomorphisme (9.6.4) pour $F_{a+1}/F_{a-1} = G_{a+1}/G_{a-1}$ fournit

(9.6.5) $\qquad T(E\ ,\ F) \xrightarrow{\sim} T(E\ ,\ G)$.

Soient Q ponctuel de rang un, localisé en s , et E_1 le noyau d'un épimorphisme de E dans Q . Dans divers cas où cela a un sens, les isomorphismes (9.6.1) et (9.6.4) (resp et (9.6.5)) donnent lieu à un diagramme commutatif

(9.6.6)
$$\begin{array}{ccc} T(E\ ,\ F) & \text{———} & T(E_1\ ,\ F) + T(Q) \\ | & & | \\ T(E\ ,\ G) & \text{———} & T(E_1\ ,\ G) + T(Q) \end{array} .$$

<u>Cas</u> 1: F et G en position général, et, avec les notations 9.5.1

<u>sous-cas</u> (a): $s \notin S$ et , pour un a , $L_i \to Q$ est non trivial si et seulement si $i = a$

<u>sous-cas</u> (b): $s \notin S$ et , pour un a , $L_i \to Q$ est non trivial si et seulement si $i = a$ ou $a + 1$

<u>sous-cas</u> (c): $s \notin S$ et en s , $E_1 = E'$.

<u>Cas</u> 2: F et G adjacentes, avec $F_a \neq G_a$ et, posant $F_1 = F_{a+1}(E_1)/F_{a-1}(E_1)$, $F = F_{a+1}(E)/F_{a-1}(E)$, on a soit

<u>sous-cas</u> (a) (b) (c): $F_1 \to F$ n'est pas un isomorphisme et F , muni des filtrations F , G et de F_1 est justifiable des cas 1 (a) (b) (c)

<u>sous-cas</u> (d): $F_1 \xrightarrow{\sim} F$, i.e. $F_{a+1}(E_1) \xrightarrow{\sim} F_{a+1}(E)$ ou $F_{a-1}(E_1)$ ne s'envoie pas isomorphiquement sur $F_{a-1}(E)$.

Le cas 1 (a) se déduit de (9.6.3) et le cas 1 (c) est la définition de 9.6.4. Prouvons 1 (b). Appliquant 9.6.3, on se ramène à supposer que $E = \oplus L_i$. On se ramène aussi au cas où $d = 2$. Soient $L = F_1$, $M = G_1$. On a $L_s \sim Q_s \sim M_s$. On a $T(E\ ,\ F) = T(L) + T(M)$, $T(E\ ,\ G) = T(M) + T(L)$, (9.6.4) = symétrie. On a $T(E_1\ ,\ F) = T(L(-s)) + T(M)$, $T(E_1\ ,\ G) = T(M(-s)) + T(L)$ et (9.6.4) est $T(L(-s)) + T(M) = T(L(-s)) + T(M(-s)) + T(Q) = T(M(-s)) + T(L(-s)) + T(Q) = T(M(-s)) + T(L)$. Il s'agit de vérifier la commutativité du diagramme

$$\begin{array}{ccc}
T(\mathcal{L}) + T(\mathcal{M}) - (T(\mathcal{L}(-s)) + T(\mathcal{M})) + T(Q) & - & (T(\mathcal{L}(-s)) + T(\mathcal{M}(-s))) + T(Q) + T(Q) \\
| & & | \\
T(\mathcal{M}) + T(\mathcal{L}) - (T(\mathcal{L}) + T(\mathcal{M}(-s))) + T(Q) & - & (T(\mathcal{M}(-s)) + T(\mathcal{L}(-s)) + T(Q) + T(Q)
\end{array}$$

(à droite, symétrie pour $T(\mathcal{L}(-s)) + T(\mathcal{M}(-s)))$. Elle résulte de ce que l'automorphisme de symétrie de $T(Q) + T(Q)$ est l'identité.

Les cas 2(a) (b) (c) se ramènent à 1 (a) (b) (c) et 2 (d) est trivial.

Soient F , G en position générale et reprenons les notations (9.5.1). Pour chaque permutation σ de $\{1 ,\dots, d\}$, soit F_σ la filtration de $\mathcal{E}$, de quotients successifs des fibrés en droites, qui induit sur $\mathcal{E}'$ la filtration par les $\bigoplus_{i \leq a} \mathcal{L}_{\sigma(i)}$. Pour τ la transposition de deux entiers consécutifs, F_σ et $F_{\sigma\tau}$ sont adjacentes. Pour $\sigma = (1 ,\dots, d)$ (resp $(d ,\dots, z)$) , $F_\sigma = F$ (resp G) . Joignons F à G par une chaîne $F = F(0)$, $F(1) ,\dots, F(N) = G$ de F_σ , avec $F(i + 1)$ adjacent à $F(i)$.

<u>Lemme</u> 9.6.7 L'isomorphisme (9.6.4) est le composé des isomorphismes (9.6.5) :

$$T(\mathcal{E} , F(0)) \to T(\mathcal{E} , F(1)) \to \dots \to T(\mathcal{E} , F(N)) .$$

Pour $s \in S$, si $\mathcal{E}_1 = \mathrm{Ker}(\mathcal{E} \to Q_s)$, on vérifie que (9.6.6) s'applique à $\mathcal{E}_1 \subset \mathcal{E}$ et aux paires de filtrations (F , G) (cas 1 (b)) ou (F(i) , F(i + 1)) (cas 2) . Ceci ramène l'énoncé pour $\mathcal{E}$ au même énoncé pour $\mathcal{E}_1$. Itérant, on se ramène au cas trivial où $\mathcal{E} = \bigoplus \mathcal{L}_i$.

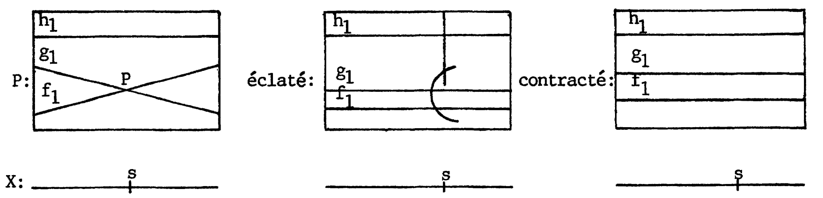
<u>Lemme</u> 9.6.8 Pour F , G , H trois filtrations en position générale (en un sens qui sera précisé), le triangle d'isomorphismes (9.6.4)

$$\begin{array}{ccc}
T(\mathcal{E} , G) & \text{———} & T(\mathcal{E} , G) \\
& \diagdown \quad \diagup & \\
& T(\mathcal{E} , H) &
\end{array}$$

est commutatif.

Le cas où $\mathcal{E}$ est de rang ≤ 1 est trivial. Supposons $\mathcal{E}$ de rang 2 et prenons comme hypothèse de position générale que F , G , H sont deux à deux en position générale avec des lieux exceptionnels disjoints. Soit P le fibré en droites projectives $\mathbb{P}(\mathcal{E}^\vee)$. Les droites F_1 , G_1 , H_1 définissent trois sections f_1 , g_1 , h_1 de P/X . L'hypothèse de position générale est qu'elles ne se coupent que deux à deux, et transversalement. Si f_1 et g_1 se coupent en p au-dessus de $s \in X$, la modification, en s , qui remplace $\mathcal{E}$ par $\mathcal{E}_1 = F_1 + G_1$, remplace

P par un contracté de l'éclaté de P en p : contracter le transformé par de la fibre en s . Après cette modification, la trace $H_1(-s)$ de H_1 sur $\mathcal{E}'$ reste, en s , un supplément de F_1 et G_1 . Dessin sur P :



Soit $Q := \mathcal{E}/\mathcal{E}_1$. Par définition (ou 9.6.6, cas 1 (b)) ,le diagramme

$$\begin{array}{ccc} T(\mathcal{E}\ ,\ F) & \xrightarrow{\sim} & T(\mathcal{E}\ ,\ G) \\ | & & | \\ T(\mathcal{E}'\ ,\ F) + T(Q) & \xrightarrow{\sim} & T(\mathcal{E}'\ ,\ G) + T(Q) \end{array}$$

est commutatif. Par 9.6.6 (cas 1 (a)) ,

$$\begin{array}{ccc} T(\mathcal{E}\ ,F) & \text{———} & T(\mathcal{E}\ ,\ H) \\ | & & | \\ T(\mathcal{E}'\ ,\ F) + T(Q) & \text{——} & T(\mathcal{E}'\ ,\ H) + T(Q) \end{array}$$

l'est également et de même pour G et H . La commutativité 9.6.8 pour $\mathcal{E}$ équivaut donc à la même commutativité pour $\mathcal{E}'$. Itérant cette construction, on se ramène à supposer les sections f_1 , g_1 , h_1 de P disjointes. Cela signifie que $\mathcal{E} = \mathcal{L} + \mathcal{L}$, avec $F_1 = \mathcal{L} + 0$, $G_1 = 0 + \mathcal{L}$, H_1 = image diagonale de $\mathcal{L}$. Traitons ce cas. Pour $\mathcal{M}$ un sous-fibré en droites de $\mathcal{E}$ localement facteur direct, la troisième composante de $T(\mathcal{E}$, filtration par $\mathcal{M}$) est $\langle \mathcal{M}\ ,\ \mathcal{E}/\mathcal{M} \rangle$, d'élément typique $\langle a\ ,\ b \rangle$ avec a (resp b) une section rationnelle de $\mathcal{M}$ (resp $\mathcal{E}/\mathcal{M}$) . Pour $\tilde{b}$ une section rationnelle de $\mathcal{E}$ d'image b dans $\mathcal{E}/\mathcal{M}$, on écrira encore $\langle a\ ,\ \tilde{b} \rangle$ pour $\langle a\ ,\ b \rangle$. Avec ces notations, les isomorphismes du triangle 9.6.8 induisent sur la troisième composante des $T(\mathcal{E}\ ,\ *)$, la seule qui pose problème, les isomorphismes suivants. Pour a et b des sections rationnelles de $\mathcal{L}$ à diviseurs disjoints, on a

$$\begin{array}{ccc} \langle (a,0)\ ,\ (0,b) \rangle & \longmapsto & \langle (0,b)\ ,\ (a,0) \rangle \\ \| & & \| \\ \langle (a,0)\ ,\ (b,b) \rangle & & \langle (0,b)\ ,\ (a,a) \rangle \\ \downarrow & & \downarrow \\ \langle b,b)\ ,\ (a,0) \rangle = \langle (b,b)\ ,\ (0,-a) \rangle & & \langle (a,a)\ ,\ (0,b) \rangle \end{array}$$

car le signe $(-1)^N$, $N = 1.1.(\deg L + \deg L)$ est $+1$. La commutativité (9.6.8) affirme donc que dans $\langle L , L\rangle$ on a

$$\langle a,b\rangle = \langle b,-a\rangle \ .$$

On a en effet $\langle b,-a\rangle = (-1)^{\deg L}\langle b,a\rangle$ et $\langle a,b\rangle = (-1)^{\deg L}\langle b,a\rangle$.

Supposons maintenant E de rang ≥ 3 et prenons comme hypothèse de position générale que F et G sont en position générale et que, pour une chaîne F(i) comme en 9.6.7, H est en position générale avec chacun des F(i) , et opposé aux F(i) aux points exceptionnels pour (F , G) . La factorisation 9.6.7 ramène 9.6.8 au lemme suivant

<u>Lemme</u> 9.6.9 Pour F et G adjacentes et H en position générale par rapport à F et à G , et opposé à F et G aux points exceptionnels pour (F , G) , le triangle d'isomorphismes 9.6.4 et 9.6.5

$$\begin{array}{ccc} T(E , F) & \text{———} & T(E , G) \\ & \diagdown \quad \diagup & \\ & T(E , H) & \end{array}$$

est commutatif.

Soit a l'entier tel que $F_a \neq G_a$. Pour $i \neq a , a+1$, soit $L_i = F_i \cap H_{d-1+1}$. Soit $F = F_{a+1} = H_{d-a+1}$, de rang 2, et $L^F = F_a \cap H_{d-a+1}$, $L^G = G_a \cap H_{d-a+1}$, $L^H = F_{a+1} \cap H_{d-a}$ les traces de F_a , G_a et H_{d-a} sur F . En dehors d'un nombre fini de points, on a $E = \oplus L_i \oplus F$, $F = L^F \oplus L^G = L^F \oplus L^H = L^G \oplus L^H$, et F , G , H sont donnés de façon évidentes en terme de ces décompositions.

On laisse au lecteur le soin de vérifier qu'en les points exceptionnels s , on a soit

(a) $E = \oplus L_i \oplus F$ et L^F , L^G , L^H sont en position générale dans F (cas où en s $F_a = G_a$, F_a n'est pas un supplémentaire de H_{d-a+1} , ou G_a n'est pas un supplémentaire de H_{d-a+1})

(b) F_i n'est pas un supplémentaire de H_{d-i+1} pour

(b1) $i = a-1$, (b2) $i = a+1$, (b3) $i \neq a , a-1 , a+1$.

Si s est du type (b) et que $E_1 = E$ en dehors de s et vaut $\oplus L_i \oplus F$ en s , (9.6.6) s'applique à $E_1 \subset E$ et à (F , G) , (F , H) , (G , H) . L'assertion (9.6.9) pour E équivaut donc à la même pour E_1 . Pour E_1 , s n'est plus exceptionnel, ou est du type (a) (si s était du type (b1)). Le cas où tous les points exceptionnels sont du type (a) se ramène au rang deux, déjà traité.

Fin de la preuve d'existence. Soient F et G deux filtrations de E . On définit

$$T(E , F) \xrightarrow{\sim} T(E , G) \tag{9.6.10}$$

comme le composé des isomorphismes (9.6.4)

$$T(E , F) \simeq T(E , H) \simeq T(E , G)$$

pour H en position générale par rapport à F et G .

D'après (9.6.8) appliqué à (F , H , H") et à (G , H , H') , deux filtrations H , H' donnent lieu au même isomorphisme (9.6.10) si (F , H , H') , (G , H , H') sont en position générale, donc toujours. D'après (9.6.8) (resp 9.6.9), l'isomorphisme (9.6.10) coïncide avec (9.6.4) (resp (9.6.5)) si ce dernier est défini. Les isomorphismes (9.6.10) forment un système transitif d'isomorphisme et on note $T(E)$ la valeur commune des $T(E , F)$. Pour une suite exacte courte $E' \to E \to E''$, une filtration F avec $E' = F_a$ pour un a définit un isomorphisme

$$T(E) = T(E , F) = T(E' , F) + T(E'' , F) = T(E') + T(E'') .$$

Cet isomorphisme est indépendant du choix de F (comparer deux F par une chaîne de filtrations adjacentes). C'est une donnée 9.4 (i) sur T . Pour E de rang un, F est unique et $T(E) = T(E , F) = T(E)$ déjà défini, d'où une donnée 9.4 (ii) vérifiant 9.4 (iii). La troisième composante de T est le foncteur IC_2 voulu.

9.7 Autre preuve de l'existence.

Si v est un fibré virtuel de rang zéro et de déterminant trivial sur X : $ch^0(v) = ch^1(v) = 0$, on a $ch^2(v) = -c^2(v)$ et la formule de Riemann-Roch s'écrit

$$c_1\, f_* v = \int_{X/S} - c^2(v) .$$

Soit $R(v) = (v - \mathcal{O}^{rg E}) - (\det v - \mathcal{O})$. On a

$$v = \mathcal{O}^{rg\, v} + (\det v - \mathcal{O}) + R(v) \tag{9.7.1}$$

et, appliquant c^* à l'égalité correspondante dans K_0 , on obtient $ch^0 R(v) = c^1 R(v) = 0$, $c^2 R(v) = c^2(v)$.

Ces formules suggèrent de définir

(9.7.2) $$I_{X/S}C^2(E) := \det f_*(-R(E)) := \det f_*((\det E - \mathcal{O}) - (E - \mathcal{O}^{\mathrm{rg}E})) .$$

et plus généralement, pour v dans $\underline{K}(X)$, $I_{X/S}C^2(v) = \det f_*(-R(v))$.

Etudions l'additivité en v des foncteurs $v \longmapsto \mathcal{O}^{\mathrm{rg}\, v}$ et $v \longmapsto (\det v - \mathcal{O})$, et par là, par 9.7.1, de $v \longmapsto R(v)$.

Pour n et m deux entiers, définissons

(9.7.3) $$\mathcal{O}^n \oplus \mathcal{O}^m \xrightarrow{\sim} \mathcal{O}^{n+m}$$

par $(x_1, \ldots, x_n), (y_1, \ldots, y_m) \longmapsto (x_1, \ldots, x_n, y_1, \ldots, y_m)$. Soit $(\mathbf{Z})$ la catégorie de Picard commutative d'ensemble d'objets $\Gamma(X, \mathbf{Z})$, de flèches les isomorphismes identiques, et d'addition $+$. Alors, $v \longmapsto \mathrm{rg}(v)$ et $n \longmapsto [\mathcal{O}^n]$ (muni de 9.7.3) sont des morphismes de catégories de Picard. Le premier est compatible à la commutativité, le second non: il s'en faut de $\varepsilon(\mathcal{O})^{nm}$. Le foncteur $v \longmapsto [\mathcal{O}^{\mathrm{rg}(v)}]$ a donc pour défaut de compatibilité à la commutativité

$$\begin{array}{ccc} \mathcal{O}^{\mathrm{rg}(v)} + \mathcal{O}^{\mathrm{rg}\, w} & \text{———} & \mathcal{O}^{\mathrm{rg}\, w} + \mathcal{O}^{\mathrm{rg}\, v} \\ {\scriptstyle (9.7.2)} \searrow & & \swarrow {\scriptstyle (9.7.2)} \\ & \mathcal{O}^{\mathrm{rg}(v+w)} & \end{array}$$

l'élément $\varepsilon(\mathcal{O})^{\mathrm{rg}(v)\,\mathrm{rg}(w)}$ de K_1 , représenté par l'automorphisme $(-1)^{\mathrm{rg}(v)\,\mathrm{rg}(w)}$ de $\mathcal{O}$.

Pour L et M deux fibrés en droites, plusieurs définitions d'un isomorphisme d'objets virtuels

(9.7.4) $$(L \otimes M - \mathcal{O}) = (L - \mathcal{O}) + (M - \mathcal{O}) + (L - \mathcal{O}) \otimes (M - \mathcal{O})$$

sont possibles ($(-\mathcal{O}) \otimes (-\mathcal{O}) \sim \mathcal{O}$ à droite, ordre des $\mathcal{O}$ à gauche) . Les isomorphismes obtenus diffèrent par $\varepsilon(\mathcal{O})$. On choisit la définition pour laquelle, si $L = \mathcal{O}(D)$ et $M = \mathcal{O}(E)$ (d'où $\mathcal{O}(D) - \mathcal{O} = [\mathcal{O}_D(D)]$, ...) et que D et E sont disjoints, l'isomorphisme obtenu est déduit des l'isomorphismes $\mathcal{O}_{D+E}(D + E) = \mathcal{O}_D(D) \oplus \mathcal{O}_E(E)$ et de la trivialisation

$$(\mathcal{O}(D) - \mathcal{O})\ (\mathcal{O}(E) - \mathcal{O}) = \mathcal{O}_D(D) \otimes \mathcal{O}_E(E) = 0$$

(car les supports sont disjoints). L'isomorphisme 9.7.4 a les compatibilités suivantes à l'associativité et à la commutativité (on omet les signes $\otimes$):

$$(9.7.5)\qquad \begin{array}{ccc} & LMN - O & \\ / & & \backslash \\ (L - O) + (MN - O) + (L - O)(MN - O) & & (LM - O) + (N - O) + (LM - O)(N - O) \\ \backslash & & / \\ \multicolumn{3}{c}{(L - O) + (M - O) + (N - O) + (L - O)(M - O) + (L - O)(N - O) + (M - O)(N - O)} \\ \multicolumn{3}{c}{+ (L - O)(M - O)(N - O)} \end{array}$$

$$(9.7.6)\qquad \begin{array}{ccc} (LM - O) & \text{———} & (L - O) + (M - O) + (L - O)(M - O) \\ | & & | \\ (LM - O) & \text{———} & (M - O) + (L - O) + (M - O)(L - O) \end{array}$$

Pour le foncteur $v \longmapsto (\det v - O)$, l'isomorphisme $\det(v + w) = \det(v) \otimes \det(w)$ et (9.7.3) fournissent

$$(9.7.7)\quad (\det(v + w) - O) = (\det v - O) + (\det w - O) + (\det v - O)\ (\det w - O)$$

et pour cet isomorphisme une compatibilité à la (9.7.5) à l'associativité. Par contre, det n'étant pas compatible à la commutativité, (9.7.7) ne l'est pas non plus: il s'en faut de l'élément de K_1 représenté par l'automorphisme $(-1)^{rg(v)rg(w)}$ de $\det(v + w)$.

L'isomorphisme (9.7.1) et les compatibilités à l'additivité (9.7.3) (9.7.7) fournissent pour $R(v)$ un isomorphisme

$$(9.7.8)\qquad R(v + w) = R(v) + R(w) - (\det v - O)\ (\det w - O) ,$$

avec une compatibilité à l'associativité (cf 9.7.5) mais non à la commutativité: défaut $\varepsilon(\det v \det w - O)^{rg(v)rg(w)}$.

Le fibré virtuel $R(v)$ est fibre à fibre de rang et de degré nul, donc d'Euler-Poincaré nul et le fibré en droite gradué $\det f_*R(v)$ est de degré 0 . Par applications des foncteurs det, f_* et "-" , (9.7.8) et (7.3) fournissent un isomorphisme

$$(9.7.9)\qquad I_{X/S}C^2(v + w) = I_{X/S}C^2(v) \otimes I_{X/S}C^2(w) \otimes \langle \det v , \det w\rangle$$

avec défaut de compatibilité à la commutativité $(-1)^N$,

$$N = \int_{X/S} rg(v)\ rg(w)\ (c_1(v) + c_1(w)) .$$

On en déduit une compatibilité à l'additivité pour

$$v \longmapsto (\mathrm{rg}(v)\ ,\ \det(v)\ ,\ I_{X/S}C^2(v))$$

qui fait de ce foncteur un morphisme de catégories de Picard commutatives de $\underline{K}(X)$ dans la catégorie 9.2. De là, une donnée d'additivité pour $E \longmapsto (\mathrm{rg}\ E\ ,\ \det E\ ,\ I_{X/S}C^2(E))$. Pour E de rang 1 , $R(E)$ est trivialisé, d'où une donnée 9.4(ii). On vérifie qu'elle vérifie 9.4(iii). Ceci achève la deuxième preuve de l'existence.

9.8 Appliquons $\det f_*$ à (9.7.1):

(9.8.1) $\det f_* v = \det Rf_*\mathcal{O}^{\mathrm{rg}\ v} \otimes \det f_*(\det v - \mathcal{O}) \otimes I_{X/S}C^2(v)^{-1}$.

D'après (7.5), le second facteur est une racine carrée de $\langle \det v, \det v \otimes \omega_{X/S}^{-1}\rangle$. D'après Mumford [17] § 5 , $\det Rf_*\mathcal{O}$ est une racine douzième de $\langle \omega\ ,\ \omega\rangle$. On obtient ainsi des isomorphismes de Riemann-Roch:

<u>Théorème</u> 9.9 Soit X propre et lisse sur S , à fibres géométriques connexes. Pour v dans $\underline{K}(X)$, on a des isomorphismes:

(i) Si v est de rang 0 , et de $\det(v)$ trivialisé:

$$\det f_* v = I_{X/S}C^2(v)$$

(ii) Si v est de rang 0 ,

$$(\det f_* v)^{\otimes 2} = \langle \det v\ ,\ \det v \otimes \omega_{X/S}^{-1}\rangle \otimes I_{X/S}C^2(v)^{-2}$$

(iii) En général,

$$(\det f_* v)^{\otimes 12} = \langle \omega\ ,\ \omega\rangle^{\otimes \mathrm{rg}(v)} \otimes \langle \det v\ ,\ \det v \otimes \omega_{X/S}^{-1}\rangle^{\otimes 6} \otimes I_{X/S}C^2(v)^{-12}\ .$$

10. Métriser IC^2 .

10.1 Soient X une variété analytique complexe, Ξ une suite exacte courte $E' \to E \to E''$ de fibrés vectoriels sur X , h une métrique sur E et h' , h'' les métriques induite et quotient sur E' et E'' . Au §5, nous avons rappelé la définition d'un "terme correctif" $\delta^i(h) := \delta^i(h\,;\,h'\,,\,h'')$ dans $M(X)$ vérifiant

$$ch^i(E\,,\,h) = ch^i(E'\,,\,h') + ch^i(E''\,,\,h'') + \frac{1}{2\pi i}\, d'd''\,\delta^i(h)\ .$$

Les formules $\delta^1 = 0$, $c^2 = -ch^2 + \frac{1}{2}(ch^1)^2$ donnent que

$$c^2(E,h) = c^2(E',h') + c^1(E'\,,\,h')\,c^1(E'',h'') + c^2(E'',h'') - \frac{1}{2\pi i}\, d'd''\delta^2(h)$$

et d'après (5.11.3), avec les notations de 5.11,

$$\delta^2(h) = \int_1^0 \frac{-1}{2\pi i}\,(-\|\lambda\|\alpha^*\alpha)\ d\,\log\|\lambda\|$$

$$= \frac{-1}{2\pi i}\ \mathrm{Tr}(\alpha^*\wedge\alpha)$$

(où $\alpha\in\Omega^{0,1}(\underline{Hom}(E''\,,\,E'))$)

Théorème 10.2 Soit X une courbe projective non singulière. Il est possible, d'une et d'une seule façon, d'attacher à chaque fibré vectoriel métrique $(E\,,h)$ sur X une métrique sur l'espace vectoriel de rang un $I_X C^2(E)$ de sorte que

(i) Pour E de rang 1, on obtient la métrique 1 sur l'espace vectoriel $I_X C^2(E) = \mathbb{C}$.

(ii) Pour toute suite exacte courte $E' \to E \to E''$, définissant un isomorphisme

$$IC_2(E) \simeq IC_2(E')\ IC_2(E'')\ \langle \det E'\,,\,\det E''\rangle\ ,$$

si x à gauche correspond à y à droite, les longueurs carrées sont liées par $\|x\| = \|y\|\ \exp(a)$ avec

$$a = \frac{1}{2\pi i}\int_X \mathrm{Tr}(\alpha^*\wedge\alpha)$$

(notations de 10.1).

10.3 Unicité: Soient $(E\,,\,h)$ et F une filtration de E de quotients successifs des fibrés en droites: $0 \subset F_1 \subset \ \dots \subset F_d = E$. La définition de IC_2 fournit un isomorphisme

$$IC_2(E) = \bigotimes_{i<j} \langle Gr_i^F(E) \ , \ Gr_j^F(E) \rangle$$

et un usage itéré de 10.2 (i) (ii) détermine le rapport entre la métrique 10.2 à gauche et le produit des métriques à droite, lorsque chaque Gr_i^F est muni de la métrique h_i déduite de celle de E . Ce rapport exp(a) est le suivant. Le fibré virtuel $[E]$ est canoniquement isomorphe à la somme des $[Gr_i^F(E)]$, et la métrique virtuelle $[h]$ est la somme des $[h_i]$ plus un terme correctif δ . On a

$$a = -\int_X \delta^2 \ .$$

Avec paramètres, si $f: X \to S$ est une famille de courbes projectives non singulières paramétrée par S , E un fibré vectoriel sur X muni d'une métrique h et que F est une filtration comme plus haut de E , la construction ci-dessus fournit une métrique g^F sur $I_{X/S}C^2(E)$. D'après 10.1, le terme correctif introduit dans 10.2 (ii) assure que

Lemme 10.4 La forme de Chern $c^1(I_{X/S}C^2(E) \ , \ g^F)$ est l'intégrale $\int_{X/S}$ de $c^2(E \ , \ h)$.

10.5 Existence. Il s'agit de montrer que la métrique g^F ne dépend pas de F . Soient F et G deux filtrations. Pour toute application $g(z)$ de $\mathbb{C}$ dans l'espace des métriques, 10.4 donne

$$d'd'' \log(g^F(z)/g^G(z)) = 0 \ .$$

Il en résulte (cf 2.4.4) que $u(F \ , \ G) := \log(g^F/g^G)$ est indépendant de la métrique g . Pour trois filtrations F , G , H , on a $u(F \ , \ H) = u(F \ , \ G) + u(G \ , \ H)$. Pour voir que $u = 0$, il suffit donc de le vérifier pour F et G en position générale. Ce cas se réduit à celui où F et G sont adjacentes (cf 9.6.7), puis au cas où E est de rang deux.

Pour E de rang deux, la donnée d'une filtration F équivaut à celle d'un fibré en droites $L \subset E$ localement facteur direct. Que les fibrés L et M correspondent à des filtrations en position générale F , G signifie qu'il existe un diviseur réduit E et un isomorphisme φ entre L et M sur E tel que

$$L \oplus M \subset E \subset L(\mathrm{E}) \oplus M(\mathrm{E})$$

et que les sections locales de E soient les $(\ell \ , \ m)$, avec ℓ (resp m) section locale de $L(\mathrm{E})$ (resp $M(\mathrm{E})$) et ℓ et m de même partie polaire (rel. φ) .

Les suites exactes $L \to E \to M(E)$ et $M \to E \to L(E)$ fournissent des isomorphismes

(10.5.1) $\quad \alpha_F : \langle L , M(E)\rangle \to IC_2(E)$

(10.5.2) $\quad \alpha_G : \langle M , L(E)\rangle \to IC_2(E)$.

On laisse au lecteur le soin de vérifier le

Lemme 10.6 Soient ℓ et m des sections de L et M . Supposons que ℓ et m ne s'annullent pas sur E , donc définissent $\langle \ell , m\rangle \in \langle L , M(E)\rangle$ et $\langle m , \ell\rangle \in \langle M , L(E)\rangle$. Si $\ell|E$ et $m|E$ se correspondent par φ , alors, au signe près, $\langle \ell , m\rangle$ et $\langle m , \ell\rangle$ définissent le même élément de $IC^2(E)$.

La métrique g^F sur $IC^2(E)$ correspond par (10.5.1) à la métrique de $\langle L , M\rangle$ modifiée par un terme correctif, et $\log \|\alpha_F(\langle \ell , m\rangle)\|_{g_F}$ est la somme de (a) l'intégrale qui apparaît dans la définition de $\|\langle \ell , m\rangle\|$, (b) les termes ponctuels qui y apparaissent, (c) l'intégrale qui définit le terme correctif.

Choisissons un recouvrement $X = U \cup V$ avec $U \subset\subset X\text{-div}(\ell) -\text{div}(m)$ et $V \subset\subset X - E$. Il existe une métrique g sur E telle que, sur U , g soit invariante par l'automorphisme de E qui échange ℓ et m , que, sur V , $L \perp M$, que localement sur V , soit $\|\ell\| = 1$, soit $\|m\| = 1$, et que $\|\ell\| = \|m\| = 1$ sur E . Pour une telle métrique, les termes ponctuels (b) sont nuls, les intégrands dans (a) (c) sont nuls et $\log \|\alpha_F\langle \ell , m\rangle\|_{g_F} = \log\|\alpha_G\langle m , \ell\rangle\|_{g_G}$ par symétrie. Ceci termine la preuve 10.5 et celle de 10.2.

11. Le théorème.

11.1 Soient X une courbe projective et lisse sur $\mathbb{C}$ et E un fibré vectoriel sur X. Si ω_X et E sont munis de métriques h et g, les deux membres de l'isomorphisme de Riemann-Roch 9.9

(11.1.1) $$(\det H\rangle(X, E))^{\otimes 12} = \langle\omega, \omega\rangle^{\otimes \mathrm{rg} E} \otimes \langle\det E, \det E \otimes \omega_{X/S}^{-1}\rangle^{\otimes 6} \otimes I_{X/S}C^2(v)^{\otimes(-12)}$$

sont métrisés. A gauche, la métrique "torsion analytique", normalisée comme en 1.2. A droite, le produit tensoriel de métriques (6.3) et (10.2). Pour x, à gauche, correspondant à y, à droite, soit

$$\phi(X, E, h, g) = \log(\|x\|/\|y\|) .$$

Rassemblons les informations dont on dispose sur ϕ. Si X et E dépendent de paramètres, i.e. si on part de $f\colon X \to S$ et de E sur X, (11.1.1) correspond à un isomorphisme de fibrés en droites. Si $\omega_{X/S}$ et E sont métrisés, la première forme de Chern de $\det Rf_*E$, pour la métrique "torsion analytique" a été calculée par [3] [6]. Elle est donnée par la formule de Riemann Roch, au niveau des formes, et coïncide donc (6.6.1) (10.4) avec la première forme de Chern du membre de droite. On conclut:

(11.1.2) En présence de paramètres, la fonction ϕ sur S vérifie

$$d'd''\phi = 0 .$$

Pour $E = E' \oplus E''$ et une métrique g sur E somme directe orthogonale de métrique g' sur E' et g'' sur E'', l'isomorphisme (11.1.1) est compatible à l'additivité des deux membres, et les métriques sont compatibles à l'additivité. On a donc

(11.1.3) $$\phi(X, E, h, g) = \phi(X, E', h, g') + \phi(X, E'', h, g'') .$$

Enfin, la compatibilité à la dualité de Serre pour les fibrés en droites donne

(11.1.4) $$\phi(X, L, h, g) = \phi(X, \omega L^\vee, h, h \otimes \check{g}) .$$

11.2 Comme expliqué en 2.4.4, (11.1.2) implique que $\phi(X, E, h, g)$ ne dépend pas de g et h. Posons $\phi(X, E) := \phi(X, E, h, g)$. Si E extension de E'' par E', la somme directe $E' \oplus E''$ est une limite de fibrés isomorphes à E. Par (11.1.3), on a donc

$$\phi(X, E) = \phi(X, E') + \phi(X, E'')$$

de sorte que $\phi(X, E)$ ne dépend que de la classe de E dans $K_0(X)$. On a

$$K^\circ(X) = \mathrm{Pic}(X) \times \mathbf{Z}$$

(les seuls invariants stables d'un fibré sont sa dimension et son déterminant).
A X fixé, la fonction $\phi(X, E)$ se factorise donc par $\mathrm{Pic}(X) \times \mathbf{Z}$. Etant additive, elle est de la forme

$$\phi(X, E) = a(X)\, \mathrm{rg}(E) + b(X)\, \deg(E) .$$

et (11.1.2) affirme que a et b sont des fonctions pluriharmoniques sur l'espace de module des courbes.

La compatibilité à la dualité de Serre (11.1.4) équivaut à $b(X) = 0$.

11.3 En genre $g \neq 1, 2$, une fonction pluriharmonique est constante: en tout genre, l'espace de modules M des courbes de genre g vérifie $H^1(M, \mathbf{R}) = 0$, de sorte qu'une fonction pluriharmonique réelle ϕ est la partie réelle d'une fonction holomorphe f. Pour $g \neq 1, 2$ M admet une compactification (normale) $\overline{M}$ avec $\overline{M} - M$ de codimension ≥ 2, f se prolonge à $\overline{M}$ et f est donc constante. On a prouvé:

<u>Théorème</u> 11.4 A un facteur $\exp(a(X)\, \mathrm{rg}(E))$ près, (11.1.1) est une isométrie. On a $d'd''a = 0$ et, en genre $g \neq 1, 2$, $a(X)$ ne dépend que du genre de X.

<u>Remarque</u> 11.5 En genre 1, on vérifie que $\phi = 0$. Ceci laisse penser que, pour une constante c, on

$$a(X) = c\chi(X) .$$

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p- ADIC PERIODS AND p- ADIC ETALE COHOMOLOGY

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Introduction

Classically, the relation between the Betti and de Rham cohomology of a projective and smooth variety defined over a number field is expressed in terms of the periods and via Hodge theory. We shall present evidence indicating that, in a precise sense, there is an analogous relation between étale and de Rham cohomology.

Let E be a number field , $\bar{E}$ an algebraic closure of E, and X be a proper smooth variety defined over E. Fix a prime number p and $\mathfrak{p}$ (resp. $\bar{\mathfrak{p}}$) a place of E (resp. $\bar{E}$) lying over p (resp. $\mathfrak{p}$). Let $V = H^m_{et}(X_{\bar{E}}, \mathbf{Q}_p)$ viewed as a representation of the decomposition group associated to $\bar{\mathfrak{p}}$ and $D = H^m_{DR}(X) \otimes E_{\mathfrak{p}}$ endowed with its Hodge filtration. In [31], Tate proved that, if m = 1 and X has good reduction at $\mathfrak{p}$, then setting $C = \hat{\bar{E}}_{\mathfrak{p}}$, $C \otimes V$ admits a Hodge-Tate decomposition; and conjectured that this holds for all m without the good reduction hypothesis. Faltings has recently announced a proof of this conjecture [9].

Changing notation, we are led to consider K a characteristic zero, non-archimedian local field of residue characteristic p and X a proper, smooth variety defined over K. We continue to denote by V (resp. D) the associated p-adic étale (resp. de Rham) cohomology of X. Using a basic result of Tate [31] and Dieudonné theory, Grothendieck proved in [21] that, if X has good reduction, then, for m = 1, V viewed as a representation of Gal ($\bar{K}$ /K) determines D endowed with its Hodge filtration and its "Frobenius structure" (coming from crystalline cohomology) and conversely. This mutual determination was not direct or explicit (but rather used the intermediary of the p-divisible group associated to the Albanese variety of X) and Grothendieck raised the problem of finding an explicit recipe for passing between V and D as well as the problem of obtaining similar results for m ≥ 2. This is his problem of the *mysterious functor.* The case m = 1 was resolved by one of us in [11,14] and a conjectural recipe in the case of m ≥ 2 was also given in [14].

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We summarize here the progress that has been made towards establishing the "crystalline conjecture" of [14] and, in addition, indicate some applications. Intuitively the idea is that, just as p-divisible groups serve as the bridge connecting V and D when m = 1, an appropriate "generalized p-divisible group" should serve to connect them for m ≥ 2 as well. In the following this intuition is not made explicit but rather we use the concepts and techniques which have developed from it.

The paper is divided into three sections. In the first, we state our main results and give both arithmetic and geometric applications. In the second, we introduce the syntomic topology and explain how to calculate crystalline cohomology using it, as well as give an essentially complete proof of the main part of Theorem A (of part I). In the third, we define certain sheaves S_n^r for the syntomic (or syntomic-étale topology) which intuitively can be thought of as an "intelligent version" of $\mathrm{Sym}^k \mu_{p^n}$ and give an outline of how our main result, (theorem B of part I) is proved.[2]

I. Crystalline Representations and the Construction of p-Adic Etale Cohomology

Throughout this paper, p is a fixed prime number, k a perfect field of characteristic p, W its ring of Witt vectors, K the fraction of field W, σ the absolute Frobenius of K ($\sigma(x) = x^p$) as well as (abusively) the induced automorphism of W (resp. K). Let $\overline{K}$ be an algebraic closure of K, G = Gal ($\overline{K}$/K), I the inertia subgroup, $\mathbf{Q_p}(1)$ the one dimensional $\mathbf{Q_p}$-vector space $\mathbf{Q_p} \otimes_{\mathbf{Z_p}} \mathrm{Hom}\,(\mathbf{Q_p}/\mathbf{Z_p}, \overline{K}^*)$, χ the cylotomic character giving the action of G on $\mathbf{Q_p}(1)$.

1. Crystalline Representations

1.1. A *p-adic representation* is a topological $\mathbf{Q_p}$-vector space equipped with a continuous linear action of G. The *dimension* of the representation is the dimension of the underlying space.

A *filtered module* is a K-vector space D equipped with

i) an exhaustive, separated decreasing filtration indexed by $\mathbf{Z}$ (i.e. a family of K-subspaces $\mathrm{Fil}^r D$ satisfying $\mathrm{Fil}^r D \supset \mathrm{Fil}^{r+1} D$, $\cup \mathrm{Fil}^r D = D$, $\cap \mathrm{Fil}^r D = 0$),

ii) a Frobenius, i.e. an injective σ-semilinear endomorphism

$$\phi : D \to D \quad .$$

[2] We assume that $p \neq 2$ in the third section. In terms of the "main case" of theorem B, namely when $p > \dim(X)$, this is not a restricition, because for H^1's the result is known.

The *dimension* of the filtered module is the dimension of the underlying K-space. With the obvious definition of morphism, the filtered modules form a category which is additive (but not abelian), $\mathbf{Q_p}$-linear, has direct sums, tensor products, kernels, cokernels and in which the notion of short exact sequence can be defined. If D is a finite dimensional filtered module, then the dual vector space is endowed with a natural structure of filtered module, denoted $D^{\vee}$.

1.2. EXAMPLE : Let X be a proper, smooth variety defined over K and let $\overline{X} = X \otimes \overline{K}$; for each $m \in \mathbf{N}$, $H^m_{ét}(\overline{X}, \mathbf{Q_p})$ is a finite dimensional p-adic representation. We denote the direct sum of these spaces by $H_{et}(X)$ and (for brevity) the m^{th} summand by $H^m_{et}(X)$.

We say that X has *good reduction* provided there exists a proper smooth W-scheme $\mathcal{X}$ such that $\mathcal{X} \otimes K = X$. If $\mathcal{X}_k$ is the special fiber of $\mathcal{X}$, then $K \underset{W}{\otimes} H^m_{cris}(\mathcal{X}_k/W)$ is endowed with a (bijective) Frobenius and is canonically isomorphic to $H^m_{DR}(X/K)$, and thus is endowed, by transport of structure, with a filtration, the Hodge filtration. Thus, we obtain a filtered module, $H^m_{cris}(X)$, which by [19], is independent of the choice of $\mathcal{X}$. As above, we denote the direct sum of these by $H_{cris}(X)$.

1.3. For any commutative ring A, let $A_n = A/p^nA$ Let $\mathcal{O}_{\overline{K}}$ be the ring of integers of $\overline{K}$ and $\tilde{\mathcal{O}}_{\overline{K}}$ be $\mathcal{O}_{\overline{K},1}$. The ring, $W_n(\tilde{\mathcal{O}}_{\overline{K}})$, of Witt vectors of length n with coefficients in $\tilde{\mathcal{O}}_{\overline{K}}$ is endowed with an action of G and a Frobenius. We view $W_n(\tilde{\mathcal{O}}_{\overline{K}})$ as a W_n-algebra via the composite

$$W_n \xrightarrow{\sigma^{-n}} W_n \longrightarrow W_n(\tilde{\mathcal{O}}_{\overline{K}}) \quad .$$

We define a surjective homomorphism of W_n-algebras

$$\theta_n\colon W_n(\tilde{\mathcal{O}}_{\overline{K}}) \longrightarrow \mathcal{O}_{\overline{K},n}$$

by $\theta_n(a_o, \dots, a_{n-1}) = \hat{a}_o^{p^n} + \dots + p^{n-1}\hat{a}_{n-1}^{p}$

(where $\hat{a}_j \in \mathcal{O}_{\overline{K},n}$ is any lifting of a_j). Note θ_n commutes with the action of G.

Denote by $W_n^{DP}(\tilde{\mathcal{O}}_{\overline{K}})$ the *divided power envelope* of the ideal Ker (θ_n) (compatible with the natural divided powers on (p)) and by $J_n(\mathcal{O}_{\overline{K}})$ the corresponding divided power ideal. Thus we have an exact sequence

$$0 \longrightarrow J_n(\mathcal{O}_{\overline{K}}) \longrightarrow W_n^{DP}(\tilde{\mathcal{O}}_{\overline{K}}) \longrightarrow \mathcal{O}_{\overline{K},n} \longrightarrow 0 \quad .$$

By functoriality G acts on this sequence and since $\phi(\text{Ker}(\theta_n)) \subset \text{Ker}(\theta_n) + p\cdot W_n(\tilde{\mathcal{O}}_{\overline{K}})$, it follows that ϕ extends to $W_n^{DP}(\tilde{\mathcal{O}}_{\overline{K}})$.

1.4. For any $n \geq 1$ we have a commutative diagram of W-algebras

$$\begin{array}{ccc} W_{n+1}(\mathcal{O}_{\bar K}) & \xrightarrow{\ \theta_{n+1}\ } & \mathcal{O}_{\bar K, n+1} \\ {\scriptstyle \upsilon_n}\downarrow & & \downarrow \\ W_n(\mathcal{O}_{\bar K}) & \xrightarrow[\ \theta_n\ ]{} & \mathcal{O}_{\bar K, n} \end{array}$$

where the right vertical arrow is reduction modulo p^n and $\upsilon_n(a_o,\ldots, a_n) = (a_o^p,\ldots,a_{n-1}^p)$. Note υ_n extends to the divided power envelopes.

We define $B^+_{cris} = K \underset{W}{\otimes} \varprojlim W_n^{DP}(\mathcal{O}_{\bar K})$; this is a topological K-algebra endowed with a continuous action of G and the structure of a filtered module. The filtration is defined by

$$\mathrm{Fil}^r B^+_{cris} = \begin{cases} B^+_{cris} & \text{if } r \le 0 \\ \text{the image of } K \underset{W}{\otimes} \varprojlim J_n^{[r]}(\mathcal{O}_{\bar K}) & \text{if } r > 0, \end{cases}$$

where $J_n^{[r]}(\mathcal{O}_{\bar K})$ denotes the r^{th} divided power of $J_n(\mathcal{O}_{\bar K})$.

1.5. For any $n \ge 1$, if $\varepsilon \in \boldsymbol{\mu}_{p^n}(\bar K)$ and $\tilde\varepsilon$ is its image in $\tilde{\mathcal{O}}_{\bar K}$, the element $[\tilde\varepsilon] = (\tilde\varepsilon,0,\ldots,0) \in W_n(\mathcal{O}_{\bar K})$ belongs to $1 + J_n(\mathcal{O}_{\bar K})$ and hence $\log([\tilde\varepsilon])$ is an element of $W_n^{DP}(\tilde{\mathcal{O}}_{\bar K})$. This construction defines a homomorphism $\boldsymbol{\mu}_{p^n}(\bar K) \longrightarrow W_n^{DP}(\mathcal{O}_{\bar K})$ and passing to the limit, we obtain an embedding $\mathbf{Q_p}(1) \longrightarrow B^+_{cris}$. We view $\mathbf{Q_p}(1)$ as included in B^+_{cris} via this map (which is G compatible) and, if t denotes any non-zero element of $\mathbf{Q_p}(1)$, we define B_{cris} to be $B^+_{cris}[t^{-1}]$. G acts on B_{cris}, $\phi(t) = p^{-1}\cdot t^{-1}$. Finally, we extend the definition of the filtration to B_{cris} by setting

$$\mathrm{Fil}^r B_{cris} = \bigcup_{s\ge 0} t^{-s}\mathrm{Fil}^{r+s} B^+_{cris} \quad .$$

1.6. Let D be a finite dimensional filtered module. We say D is *weakly admissible* if there is a lattice M in D such that $\Sigma p^{-r}\phi(M \cap \mathrm{Fil}^r D) = M$. The category of weakly admissible modules is stable under sub and quotient objects, extensions, tensor products and taking duals; it is abelian and in fact a $\mathbf{Q_p}$-linear, neutral tannakian category (cf. [12,24]).

If D is a weakly admissible filtered module, we set

$$V(D) = \{ x \in \mathrm{Fil}^0(B_{cris} \underset{K}{\otimes} D) \mid \phi(x) = x \} \quad .$$

This is a p-adic representation of dimension at most that of D. We say D is *admissible* provided these dimensions are equal. Conjecturally, all weakly admissible modules are admissible.

If V is a p-adic representation, we set

$$D(V) = (B_{cris} \otimes_{\mathbf{Q}_p} V)^G \quad .$$

This is a filtered module of dimension at most that of V. We say V is a *crystalline representation* if these dimensions are equal.

1.7. PROPOSITION ([12,13,16]): *The category of admissible filtered modules (viewed as a full subcategory of the category of weakly admissible modules) is stable under sub and quotient objects, tensor products, taking duals, extensions. Any weakly admissible module whose filtration has length less than* p *(i.e. such that for some* j, $Fil^j D = D$, $Fil^{j+p} D = 0$) *is admissible. The functor V induces a ⊗-equivalence between the category of admissible filtered modules and the category of crystalline representations. The functor* D *is a quasi-inverse functor.*

2. Crystalline Representations and Etale Cohomology

2.1. Let X be a proper, smooth K-variety having good reduction; we say X is *admissible* if $H_{cris}(X)$ is admissible. We conjecture that any such X is admissible. One has the following partial result.

2.2. THEOREM A: i) *The product of two admissible varieties is admissible; the standard cellular varieties* ($\mathbf{P}^N, \mathbf{Gr}_{k,N}, \ldots$) *are admissible.*

ii) *If* X *is a variety having good reduction and if one of the following conditions holds*

a) $p > \dim(X)$,

b) X *is a curve or an abelian variety,*

c) *there is a proper smooth model* $\mathcal{X}/W$ *with torsion-free Hodge cohomology and with ordinary* [4] *special fiber,*

then X *is admissible.*

Statement i) is a consequence of the Künneth formula, the stability of admissible modules under tensor product and the fact that certain "elementary" filtered modules (e.g. $S^i = K$ with $\phi = p^i \cdot \sigma$, $Fil^i = K$, $Fil^{i+1} = 0$) are admissible. In cases b) and c) statement ii) is established in [14,16]. We shall indicate the proof in case a) in II.2.7,2.8 (also see [22] when X is projective).

2.3. The main interest of Theorem A derives from:

THEOREM B: *There are defined on the category of admissible varieties canonical and functorial isomorphisms (compatible with multiplicative structure* *and cycle classes) of p-adic representations (resp. filtered modules)*

$$V(H_{cris}(X)) = H_{et}(X) \quad ,$$
$$D(H_{et}(X)) = H_{cris}(X) \quad ,$$

and consequently $B_{cris} \underset{K}{\otimes} H_{cris}(X) = B_{cris} \underset{\mathbf{Q}_p}{\otimes} H_{et}(X)$.

REMARK: If X has good reduction, then in fact the conclusion of the theorem is valid for H^m provided $m < p$.

We shall give the idea of the proof of theorem B in III.6. Here are several applications of theorems A and B.

3. "Arithmetical" Applications

3.1. Hodge-Tate Decomposition. Let $C = \hat{\overline{K}}$ and for any $r \in \mathbf{Z}$ set $C(r) = C \otimes \mathbf{Q}_p(1)^{\otimes r}$. For any admissible X, we have canonical, functorial isomorphisms compatible with ring structure and cycle classes

$$C \otimes H^m_{et}(X) = \bigoplus_{j=0}^{m} C(-j) \underset{K}{\otimes} H^{m-j}(X,\Omega^j_X) \quad .$$

This isomorphism is obtained by writing $gr^o(B_{cris} \otimes H^m_{et}(X)) = gr^o(B_{cris} \otimes H^m_{cris}(X))$ and using the fact [12], that $gr^r B_{cris} = C(r)$.

REMARK: This result was conjectured by Tate for all proper, smooth varieties over K and had been proved in many particular cases [1,4,5,7,15,22,31]. Recently Faltings has announced the result in the general case [9].

3.2. Action of Tame Inertia. Let V be a G-stable lattice in $H^m_{et}(X)$ and $\tilde{V}$ be the semi-simplification of V/pV; with respect to the action of I. Then, if X is admissible, the action of I (through its tame quotient) is given by characters of the form $\chi_h^{-(i_0 + pi_1 + \ldots + p^{h-1} i_{h-1})}$ where $h \geq 1$ and χ_h is a *fundamental character* of level h, i.e. a character of I with values in $\mathbf{F}_{p^h}^*$ which factors through F_h^* (where $I^{tame} = \varprojlim F_r^*$, F_r being the subfield of p^r elements of $\bar{k}$) and extends to an isomorphism $F_h \xrightarrow{\sim} \mathbf{F}_{p^h}$. Then the integers i_j satisfy $0 \leq i_j \leq m$.

This follows from a general property of the crystalline representations associated to filtered modules satisfying $Fil^o D = D$, $Fil^{m+1} D = 0$, cf [13]. This result had been conjectured by Serre [30] and proven by Raynaud for m = 1, [27], and Kato in a more precise form provided dim (X) < p - 1 and the special fiber $\mathcal{X}_k$ is Hodge-Witt [22].

3.3. Bounds for the Discriminant. With the same notation as in 3.2, let H_n be the kernel of the representation of G on V/p^nV, $L_n = \bar{K}^{H_n}$ and $\mathfrak{D}_{L_n/K}$ the corresponding different. Then, with the valuation v of L_n normalized so that v(p) = 1, we conjecture that $v(\mathfrak{D}_{L_n/K}) < n + m/(p-1)$. For m = 1 this is proven in [17]. The general case should follow from theorem B because this inequality should hold for any crystalline representation. This is true at least when n = 1 and m < p-1, cf [18].

As a consequence, by using methods analogous to those which prove there is no non-trivial abelian scheme over **Z** [17], one deduces the following result:

If X is a proper, smooth variety over **Q** which has good reduction everywhere, and if $i,j \in \mathbf{N}$ and satisfy $i \neq j$, $i + j \leq 3$, then $H^j(X,\Omega^i_X) = 0$. In particular, if the dimension of X is at most three, all its cohomology is algebraic [18].

3.4. Image of the Galois Group.

PROPOSITION: Let X *be an admissible variety and let* G_o *be the Zariski closure of the image of* I *in* $GL_{H^m_{et}(X)}$.

i) *The image of* I *is open in* $G_o(\mathbf{Q}_p)$ (*for its topology of* p-*adic Lie group*);

ii) G_o *is a connected group.*

The first property is a consequence of the fact that $H^m_{et}(X)$ is a Hodge-Tate representation [29]. The second property holds for any crystalline representation. It implies that if V is any I-stable subquotient of $H^m_{et}(X)$, then the action of I on the determinant of V is via χ^{-r} for some $r \geq 0$. For m = 1 this was proven by Raynaud [27]. Note that the fact that $H^m_{et}(X)$ is Hodge-Tate implies this last result only up to multiplication by a character of finite order.

4. Geometric Applications

4.1. Etale Cohomology of the Special Fiber. Let $\mathcal{X}$ be a proper smooth W-scheme such that $\mathcal{X}_K = X$ is admissible. The specialization map induces an isomorphism:

$$H^*_{et}(X_{\bar{k}}, \mathbf{Q_p}) \xrightarrow{\sim} H_{et}(X)^I .$$

This follows because the source is naturally identified with the fixed points of ϕ in $H_{cris}(X) \otimes \mathrm{Frac}\,(W(\bar{k}))$ and the isomorphism of theorem B transforms this into the target. When $m = 1$, this holds even if K is not absolutely unramified and with $\mathbf{Q_p}$ replaced by $\mathbf{Z_p}$. Can it be proven for $m \geq 2$ and K an arbitrary local field by "purely" étale cohomological methods? What is the local analogue of this result?

4.2. The Crystalline Discriminant. Assume $p \neq 2$ and that k is algebraically closed. Let Y be a proper, smooth k-variety of even dimension d. Set $H_{cris}(Y) = K \underset{W}{\otimes} H_{cris}(Y/W)$.

In [25], Ogus defines an invariant, the crystalline discriminant, to be the Legendre symbol of the reduction mod p of $p^{-\mathrm{ord}<\alpha,\alpha>} \cdot <\alpha,\alpha>$ where α is a generator of the determinant of $H^d_{cris}(Y)(\text{-}d/2)$ which is fixed by ϕ. Further, he gives a conjectural formula for this in terms of the ℓ-adic Betti numbers of Y.

Suppose now that Y admits a lifting to W whose generic fiber X is admissible. by theorem B, α is then identified with an element of the determinant of $H^d_{et}(X)(d/2)$. It now follows immediately from the Hodge index theorem that Ogus' crystalline discriminant conjecture is true for Y.

4.3. Absolute Hodge and Absolute Tate Cycles. Let X be a proper smooth variety defined over an algebraically closed field E of characteristic zero. Recall [25], an element ξ of $\mathrm{Fil}^r\ H^{2r}_{DR}(X)$ is said to be *absolutely Tate* provided there is a smooth **Z**-algebra $R \subset E$, a proper, smooth model of X, X_R, defined over R, having locally-free de Rham cohomology, an element $\xi_R \in H^{2r}_{DR}(X_R)$ giving back ξ and such that, for any perfect field of finite characteristic k, and any homomorphism $R \xrightarrow{\alpha} W$, the image of ξ_R, $\xi_W \in H^{2r}_{DR}(X_W)$ satisfies $\phi(\xi_W) = p^r\, \xi_W$. Given such a ξ, fix R as above; then, for almost all p, $p > \dim(X)$, and we may choose α to be an injection. Extending α to the algebraic closure of Frac(R) in E and applying theorem B, we see that ξ_W defines an element of $H^{2r}_{et}(X_{\bar{K}}, \mathbf{Q_p}(r))^G$, which by the proper base change theorem we may view as an element ξ_p of $H^{2r}_{et}(X, \mathbf{Q_p}(r))$. Conjecturally, [25], there is an absolute Hodge cycle $y = (y_{DR}, (y_\ell)_{\ell\ \mathrm{prime}})$ such that $\xi = y_{DR}$. It is reasonable to conjecture that y_p is equal to ξ_p for all p for which ξ_p is defined (and in particular that ξ_p is independent of *all* the choices made).

4.4. Transcendental Results. Let $\mathcal{X}$ be a proper, smooth W-scheme such that $\dim(\mathcal{X}_k) < p$. The proof of theorem A (part ii a) cf. II.2.6, in fact shows that $H_{DR}(\mathcal{X})$ is an object of the category MF_{tf}, [32], and that the Hodge to de Rham spectral sequence for $\mathcal{X}$

degenerates at E_1. If $\mathcal{X}$ is projective, as a consequence of the hard Lefschetz theorem for crystalline cohomology, [23], one deduces its validity for the Hodge cohomology of $\mathcal{X}_K$ (morphisms in MF_{tf} are strictly compatible with the filtrations). It follows that $h^{i,j} = h^{j,i}$ for $\mathcal{X}_K$. Now if X is any proper, smooth variety over a characteristic zero field E, we can choose $R, \mathcal{X}_R$, $\alpha: R \longrightarrow W$ (where p > dim X) as in 4.3 and immediately deduce the degeneration of the Hodge to de Rham spectral sequence (for X) at E_1, the hard Lefschetz theorem for the Hodge (or de Rham) cohomology of X (assuming X is projective) and, as Gabber suggested, using Hironaka, the Hodge symmetry even if X is "only" proper. The proof of part ii a of theorem A in fact gives the following result: if Y is a smooth variety of dimension < p defined over k and Y is liftable to W_2, then there is a canonical semi-linear quasi-isomorphism between $\oplus\, \Omega^j_Y[-j]$ and $\Omega^{\bullet}_Y$. Recently, Deligne and Illusie, [8], have found an incredibly elementary explicit proof of this fact and Raynaud has deduced from this a proof of the Kodaira-Nakano vanishing theorem for Y. By using the same method as above, this gives an *algebraic* proof of the result, valid for any X/E as above.

II. *p*-Adic Hodge Structures

1. The Syntomic Site, Crystalline Cohomology and the Cartier Isomorphism

1.1. Recall, [35], that a morphism f: $X \longrightarrow S$ of schemes is *locally a complete intersection* provided, locally on X there is a *regular* closed immersion into a smooth S-scheme through which f factors. A morphism is said to be *syntomic* provided it is flat and locally a complete intersection. This terminology is due to Mazur.

Let Y be a scheme. The big (resp. small) syntomic site Y_{SYN} (resp. Y_{syn}) of Y consists of the category of Y-schemes (resp. the full subcategory of the Y-schemes Z such that $Z \longrightarrow Y$ is syntomic) endowed with the topology (cf. [33, exposé IV]) generated by the surjective syntomic Y-morphisms of affine schemes. The big site is functorial in Y; for the small site there is the "usual" difficulty, cf. [34] or [2]; nevertheless cohomological calculations can be made using either.

1.2. For any k-scheme Z we write $\mathcal{O}_n^{cris}(Z) = H^0((Z/W_n)_{cris}, \mathcal{O}_{Z/W_n})$; this is a commutative W_n-algebra endowed with a Frobenius endomorphism, ϕ.

1.3. PROPOSITION: *The functor* $Z \longmapsto \mathcal{O}_n^{cris}(Z)$ *is a sheaf on the big syntomic site of* Spec(k). *For any* k*-scheme* Y, $H^*(Y_{SYN}, \mathcal{O}_n^{cris})$ *is canonically isomorphic* (*compatibly with Frobenius*) *to* $H^*((Y/W_n)_{cris}, \mathcal{O}_{Y/W_n})$.

The first statement follows easily from the fact that given a divided power thickening

$U \hookrightarrow T$ and a syntomic morphism $U' \longrightarrow U$ we can, locally on U', find a lifting to a syntomic morphism $T' \longrightarrow T$. The second statement follows easily from this plus the explicit description of crystalline cohomology in terms of the de Rham complex of a divided power envelope.

1.4. REMARK: If A is a k-algebra, we denote by $W_n^{DP}(A)$ the divided power envelope (compatible with the standard divided powers on $V\cdot W_n(A)$) of the ideal formed of all $(a_0,\dots,a_{n-1}) \in W_n(A)$ such that $a_0^p = 0$; this notation is consistent with that of I.1.3 when $A = \mathcal{O}_{\overline{K}}$. Passing to the associated sheaf for the Zariski topology, we find a natural homomorphism: $W_n^{DP}(A) \longrightarrow \mathcal{O}_n^{cris}(A)$, defined exactly as in I.1.3. One verifies that if the Frobenius of A is surjective then $W_n^{DP}(A) \xrightarrow{\sim} \mathcal{O}_n^{cris}(A)$ and this implies that the associated sheaf for the syntomic topology, $\widetilde{W}_n^{DP}$, is isomorphic to $\mathcal{O}_n^{cris}$.

1.5. We now restrict attention to the small syntomic site of k. If n and n' are integers such that $n \geq n' \geq 1$, and $n - n' = c$, there is an epimorphism, denoted generically by υ, $\mathcal{O}_n^{cris} \longrightarrow \mathcal{O}_{n'}^{cris}$, which is induced by the map $W_n(A) \longrightarrow W_{n'}(A)$ given by $(a_0,\dots,a_{n-1}) \longmapsto (a_0^{p^c},\dots,a_{n'-1}^{p^c})$. Using some standard facts about divided power envelopes of local complete intersections, it is shown that $\mathcal{O}_n^{cris}$ is flat as a W_n-algebra. Thus, given integers $n', n'' \geq 1$, we find a short exact sequence

$$0 \longrightarrow \mathcal{O}_{n''}^{cris} \xrightarrow{\ \pi\ } \mathcal{O}_{n'+n''}^{cris} \xrightarrow{\ \upsilon\ } \mathcal{O}_{n'}^{cris} \longrightarrow 0$$

where π is defined by $\pi x = p^{n'}\cdot\hat{x}$ if $\hat{x}$ is a lifting of x in $\mathcal{O}_{n'+n''}^{cris}$.

1.6. Let $\mathcal{O}_1$ be the "structure sheaf" on Spec(k), i.e. $\mathcal{O}_1(Y) = \Gamma(Y,\mathcal{O}_Y)$ for any k-scheme Y. There is an epimorphism of syntomic sheaves $\mathcal{O}_1^{cris} \longrightarrow \mathcal{O}_1$ and the kernel J_1 is a divided power ideal. For any $r \in \mathbf{N}$, let $J_1^{[r]}$ be the r^{th} divided power of J_1. The $J_1^{[r]}$ form a decreasing filtration of $\mathcal{O}_1^{cris}$,

$$\mathcal{O}_1^{cris} = J_1^{[0]} \supset J_1 = J_1^{[1]} \supset J_1^{[2]} \supset \dots,$$

and their intersection is zero. By an elementary calculation of the divided power envelopes of certain complete intersections, it is easy to check that for $r \leq p-1$ the natural map $\mathrm{Sym}^r(J_1/J_1^{[2]}) \longrightarrow J_1^{[r]}/J_1^{[r+1]}$ is an isomorphism, cf. 1.7.

For any integer $r \geq 1$, let $\hat{I}_1^{<r>}$ be the kernel of the composite mapping

$$\mathcal{O}_{r+1}^{cris} \xrightarrow{\ \phi\ } \mathcal{O}_{r+1}^{cris} \xrightarrow{\ \upsilon\ } \mathcal{O}_r^{cris}.$$

Set $\hat{I}_1^{<o>} = \mathcal{O}_1^{cris}$. There are two homomorphisms of $\hat{I}_1^{<r>}$ in $\mathcal{O}_1^{cris}$:

(i) υ_r is the composite $\hat{I}_1^{<r>} \longrightarrow \mathcal{O}_{r+1}^{cris} \longrightarrow \mathcal{O}_1^{cris}$

(ii) a σ-semi-linear homomorphism, $\hat{f}_r$, defined by $\hat{f}_r(x) = y$ if $\phi(x) = \pi(y)$.

Let $F^r\mathcal{O}_1^{cris} = \mathrm{Im}\upsilon_r$, $F_r\mathcal{O}_1^{cris} = \mathrm{Im}\ \hat{f}_r$ and set $F_{-1}\mathcal{O}_1^{cris} = 0$.

One verifies that for all $r \in \mathbf{N}$, $F^r\mathcal{O}_1^{cris} \supset F^{r+1}\mathcal{O}_1^{cris}$ and $F_{r-1}\mathcal{O}_1^{cris} \subset F_r\mathcal{O}_1^{cris}$ and that $\hat{f}_r$ induces a σ-semi-linear isomorphism:

$$f_r\colon F^r\mathcal{O}_1^{cris}/F^{r+1}\mathcal{O}_1^{cris} \xrightarrow{\sim} F_r\mathcal{O}_1^{cris}/F_{r-1}\mathcal{O}_1^{cris} .$$

1.7. PROPOSITION:

i) *For all* r, $F^r\mathcal{O}_1^{cris} = J_1^{[r]}$.

ii) $\cup F_r\mathcal{O}_1^{cris} = \mathcal{O}_1^{cris}$.

Working locally for the syntomic topology, we are lead to consider a k-algebra A of the following type. Let $(P_1,\dots,P_d)$ be a regular sequence in a polynomial ring $k[X_1,\dots,X_m]$. Let A be the quotient of the perfection of this polynomial ring by the ideal generated by $P_1,\dots,P_d$. If π_j denotes the image in A of the p^{th} root of P_j in $k[X_1,\dots,X_m]$, then one checks that

$$\mathcal{O}_1^{cris}(A) = W_1^{DP}(A) = \bigoplus_{m_1,\dots,m_d \in \mathbf{N}} A\gamma_{pm_1}(\pi_1)\dots\gamma_{pm_d}(\pi_d) .$$

and that $J_1^{[r]}(A)$ is the sub-A-module of $\mathcal{O}_1^{cris}(A)$ generated by the $\gamma_{m_1}(\pi_1)\dots\gamma_{m_d}(\pi_d)$ with $\Sigma m_j \geq r$, and that, if $\tilde{A} = A/(\pi_1,\dots,\pi_d)$ then

$$J_1^{[r]}/J_1^{[r+1]} = \bigoplus_{m_1+\dots+m_d = r} \tilde{A}\gamma_{m_1}(\pi_1)\dots\gamma_{m_d}(\pi_d) .$$

Let π'_j be a p^{rth} root of π_j in A and let $\hat{\pi}_j$ be the image of $(\pi'_j,\dots,0) \in W_{r+1}(A)$ in $W_{r+1}^{DP}(A) = \mathcal{O}_{r+1}^{cris}(A)$. As $\phi(\hat{\pi}_j) = p!\gamma_p(\hat{\pi}_j)$, we find that if $x \in A$ and $\hat{x} \in \mathcal{O}_{r+1}^{cris}(A)$ is a lifting and $\Sigma m_j = r$, then $\hat{u} = \hat{x}\gamma_{m_1}(\hat{\pi}_1)\dots\gamma_{m_d}(\hat{\pi}_d)$ is a lifting of $u = x\gamma_{m_1}(\pi_1)\dots\gamma_{m_d}(\pi_d)$ in $\mathcal{O}_{r+1}^{cris}(A)$ which satisfies:

$$\phi(\hat{u}) = (p!)^r\phi(\hat{x})\eta_{m_1}\dots\eta_{m_d}\gamma_{pm_1}(\hat{\pi}_1)\dots\gamma_{pm_d}(\hat{\pi}_d)$$

(where $\eta_m = (pm)!/(p!)^m m!$ is a p-adic unit).

This immediately gives statement ii). Using this calculation one shows inductively that $F_r\mathcal{O}_1^{cris}(A) = \bigoplus_{\Sigma m_j \leq r} A\gamma_{pm_1}(\pi_1)\dots\gamma_{pm_d}(\pi_d)$ and this easily gives the proposition.

2. Crystalline and de Rham Cohomology

2.1. Let s be an integer ≥ 1. If i: $\mathrm{Spec}(k)_{SYN} \longrightarrow \mathrm{Spec}(W_s)_{SYN}$ then one verifies that i_* is exact. We shall (by abuse of notation) continue to write $\mathcal{O}_n^{cris}$ for $i_*(\mathcal{O}_n^{cris})$ restricted to $\mathrm{Spec}(W_s)_{syn}$. Denote by $\mathcal{O}_s$ the "structural sheaf" on $\mathrm{Spec}(W_s)_{syn}$ and for any $n \leq s$ set $\mathcal{O}_n = \mathcal{O}_s/p^n.\mathcal{O}_s$. There is a homomorphism of W_n-algebras

$$\mathcal{O}_n^{cris} \longrightarrow \mathcal{O}_n$$

which, in terms of Witt vectors, is induced by the homomorphism $W_n(\mathcal{O}_1) \longrightarrow \mathcal{O}_n$ given by $(a_0,\dots,a_{n-1}) \longmapsto \hat{a}_0^{p^n}+\dots+p^{n-1}\hat{a}_{n-1}^p$, where $\hat{a}_j$ is a lifting of a_j to $\mathcal{O}_n$; this homomorphism was considered in the special case of $\mathcal{O}_{\overline{K},n}$ in I.1.3. This is an epimorphism of sheaves and its kernel J_n is a divided power ideal whose r^{th} divided power we denote by $J_n^{[r]}$.

For all $n \leq s$ we thus have a decreasing filtration

$$\mathcal{O}_n^{cris} = J_n^{[0]} \supset J_n = J_n^{[1]} \supset \dots$$

and $\cap J_n^{[r]} = (0)$. For n = 1, the just defined $J_1^{[r]}$ is the direct image by i of the $J_1^{[r]}$ of the preceding paragraph.

One checks that the $J_n^{[r]}$ are flat as sheaves of W_n-modules and thus if $n',n'' \in \mathbf{N}$ and $n' + n'' \leq s$ we have a short exact sequence

$$0 \longrightarrow J_{n''}^{[r]} \longrightarrow J_{n'+n''}^{[r]} \longrightarrow J_{n'}^{[r]} \longrightarrow 0 \quad .$$

2.2. Let Y_s be a smooth W_s-scheme of dimension d and (for $n \leq s$) set $Y_n = Y_s \otimes W_n$ and let $\mathcal{O}_{Y_n}^{cris}$ (resp. $J_{Y_n}^{[r]}$) be the restriction to $(Y_s)_{syn}$ of $\mathcal{O}_n^{cris}$ (resp. $J_n^{[r]}$). Denote by α the evident morphism of sites

$$\alpha: Y_{s,syn} \longrightarrow Y_{s,Zar} \quad .$$

PROPOSITION: *There are canonical isomorphisms* (for $1 \leq n \leq s$)

i) $R\alpha_*\mathcal{O}_{Y_n}^{cris} \xrightarrow{\sim} \Omega^{\bullet}_{Y_n}$.

ii) for $r \in \mathbf{N}$, $R\alpha_* J_{Y_n}^{[r]} \xrightarrow{\sim} \sigma_{\geq r}\Omega^{\bullet}_{Y_n}$ (where $\sigma_{\geq r}\Omega^{\bullet}_{Y_n} = 0 \to \dots \to 0 \to \Omega^r_{Y_n} \to \dots$) and $R\alpha_*(J_{Y_n}^{[r]}/J_{Y_n}^{[r+1]}) \xrightarrow{\sim} \Omega^r_{Y_n}[-r]$.

This is an easy consequence of the fact that crystalline cohomology can be calculated using the syntomic site and (of course) Berthelot's fundamental comparison theorem between crystalline and de Rham cohomology, [2].

2.3. Since $\phi(J_1) = 0$ it follows that $\phi(J_n^{[r]}) \subset p^r \cdot \mathcal{O}_n^{cris}$ provided $r \leq p-1$. Let r and n be two integers which satisfy $r \leq p-1$ and $n+r \leq s$. If $x \in J_n^{[r]}$ and $\hat{x} \in J_{n+r}^{[r]}$ is a lifting then $\phi\hat{x} = p^r \cdot \hat{y}$ with $\hat{y} \in \mathcal{O}_{n+r}^{cris}$ and the image of $\hat{y}$ in $\mathcal{O}_n^{cris}$ is well-defined (independent of the choice of $\hat{x}$ and $\hat{y}$). We thus obtain a σ-semi-linear map

$$\phi_r : J_n^{[r]} \longrightarrow \mathcal{O}_n^{cris} .$$

REMARK: Observe that for $n = 1$, $\phi_r J_1^{[r+1]} = 0$ provided $r<p-1$ and since $\mathrm{Sym}^r(J_1/J_1^{[2]}) \xrightarrow{\sim} J_1^{[r]}/J_1^{[r+1]}$ we see that ϕ_r is determined by ϕ_1 and thus may be defined provided $s \geq 2$. When $r = p-1$, this no longer applies, but nevertheless, we may define a map $J_1^{[p-1]}/J_1^{[p]} \longrightarrow \mathcal{O}_1^{cris}$ directly using only ϕ_1.

2.4. Fix two integers t and n such that $t \leq p-1$ and $n+t \leq s$. Let Λ_n^t be the cokernel of the injective map

$$\bigoplus_{r=1}^{t} J_n^{[r]} \longrightarrow \bigoplus_{r=0}^{t} J_n^{[r]}$$

given by $(x_1,\ldots,x_t) \longmapsto (x_1, x_2-p\,x_1,\ldots,x_t-p\,x_{t-1}, -p\,x_t)$. Since $\phi_{r-1}(x) = p\,\phi_r(x)$ if $x \in J_n^{[r]}$, the maps ϕ_r induce a semi-linear map

$$\bar{\phi} : \Lambda_n^t \longrightarrow \mathcal{O}_n^{cris} .$$

LEMMA: $\bar{\phi}(\Lambda_1^t) = F_t\mathcal{O}_1^{cris}$ *and the kernel of* $\bar{\phi}$ *is isomorphic to* $J_1^{[t+1]}$.

This is proved by induction on t, the case $t = 0$ being obvious. We have

$$\Lambda_1^t = \left(\bigoplus_{r=0}^{t-1} J_1^{[r]}/J_1^{[r+1]}\right) \oplus J_1^{[t]}$$

and it is clear that the following diagram commutes:

$$\begin{array}{ccc} J_1^{[t]} & \xrightarrow{\ \phi_t\ } & F_t\mathcal{O}_1^{cris} \\ \downarrow & & \downarrow \\ J_1^{[t]}/J_1^{[t+1]} & \xrightarrow{\ f_t\ } & F_t\mathcal{O}_1^{cris}/F_{t-1}\mathcal{O}_1^{cris} \end{array} .$$

If $t + 1 < p$, as $\phi_t(J_1^{[t+1]}) = 0$, we immediately find a short exact sequence

$$0 \longrightarrow J_1^{[t+1]} \longrightarrow \Lambda_1^t \longrightarrow F_t \mathcal{O}_1^{cris} \longrightarrow 0$$

where $J_1^{[t+1]} \longrightarrow \Lambda_1^t$ is the map $x \longmapsto ((0,\dots,0),x))$. If $t + 1 = p$, we still have $\bar{\phi}(\Lambda_1^{p-1}) = F_{p-1}\mathcal{O}_1^{cris}$ and if $x \in J_1^{[p]}$, one checks easily that there is a $y \in \mathcal{O}_1^{cris}$ such that $\phi_{p-1}(x) = \phi_0(y)$ and that the image of y in $\mathcal{O}_1^{cris}/J_1$ is independent of the choice of y. Denoting this image by $\beta(x)$ we define a morphism $J_1^{[p]} \longrightarrow \Lambda_1^{p-1}$ by $x \longmapsto ((-\beta(x),\dots,0),x)$ and obtain as above the short exact sequence for $t = p-1$ also.

REMARK: If $t = p-1$, we can also use the remark of 2.3 to define a map $J_1^{[p-1]}/J_1^{[p]} \longrightarrow F_{p-1}$ which lifts f_{p-1} and is defined in terms of ϕ_1. This gives an isomorphism

$$\bigoplus_{r=0}^{p-1} J_1^{[r]}/J_1^{[r+1]} \xrightarrow{\sim} F_{p-1}\mathcal{O}_1^{cris} .$$

2.5. From now on by abuse of notation we will write $H^m(F)$ for $H^m(Y_{S,syn},F)$ if F is a syntomic sheaf on Y_S. The exact sequence, deduced from 1.7,

$$0 \longrightarrow F_{j-1}\mathcal{O}_1^{cris} \longrightarrow F_j\mathcal{O}_1^{cris} \longrightarrow J_1^{[j]}/J_1^{[j+1]} \longrightarrow 0$$

yields a long exact cohomology sequence and thus isomorphisms $H^m(F_{j-1}\mathcal{O}_1^{cris}) \xrightarrow{\sim} H^m(F_j\mathcal{O}_1^{cris})$ provided $H^{m-1}(J_1^{[j]}/J_1^{[j+1]}) = H^m(J_1^{[j]}/J_1^{[j+1]}) = 0$. By 2.2 this is equivalent to the vanishing of $H^{m-1-j}(\Omega^j_{Y_1})$ and $H^{m-j}(\Omega^j_{Y_1})$ and hence holds provided $j > \inf(m,d)$. This condition insures that $H^m(F_{j-1}\mathcal{O}_1^{cris}) = H^m(\mathcal{O}_1^{cris})$. Similarly, if we consider the short exact sequence of 2.4, we obtain sufficient conditions for the map induced on cohomology by $\bar{\phi}$ to be injective (resp. bijective). We summarize:

LEMMA: *The mapping* $H^m(\Lambda_1^t) \longrightarrow H^m(\mathcal{O}_1^{cris})$ *induced by* $\bar{\phi}$ *is injective if* $m \le t$ *and bijective if* $d \le t$. *If* $t \ge \inf(m,d)$, $H^m(F_t\mathcal{O}_1^{cris}) \xrightarrow{\sim} H^m(\mathcal{O}_1^{cris})$.

As a consequence of this we see that if Y_2 is a smooth scheme over W_2 of dimension $d < p$, then the natural map $\bigoplus_{r=0}^{d} J_1^{[r]}/J_1^{[r+1]} \longrightarrow \mathcal{O}_1^{cris}$ induced by $\bar{\phi}$ if $d < p-1$ (or as in the remark of 2.4 if $d = p-1$) is a quasi-isomorphism. This implies:

COROLLARY: *If* X *is a smooth* k*-scheme of dimension* $< p$, *which is liftable to*

W_2, *then each choice of lifting defines in the derived category a quasi-isormorphism* :

$$\oplus\, \Omega_X^j[-j] \xrightarrow{\sim} \Omega_X^{\bullet} \ .$$

If X *is proper, then its Hodge to de Rham spectral sequence degenerates at* E_1.

REMARK: A completely elementary and explicit proof of this last result has been given by Deligne and Illusie [8], who in addition obtain interesting results about the liftability of smooth varieties of dimension $< p$.

2.6. THEOREM: *Let* Y_s *be a proper, smooth* W_s*-scheme whose special fiber has dimension* d. *For all* $m \geq 0$ *and all pairs* $(t,n) \in \mathbf{N}^2$ *which satisfy*

a) $\inf(m,d) \leq t \leq p-1$

b) $1 \leq n \leq s-t$

we have

i) *The mapping* $\bar{\phi}$ *induces a semi-linear isomorphism*

$$H^m(Y_{s,syn}, \Lambda_n^t) \xrightarrow{\sim} H^m(Y_{s,syn}, \mathcal{O}_n^{cris}) \ .$$

ii) *The exact sequence*

$$0 \longrightarrow \bigoplus_{r=1}^{t} J_n^{[r]} \longrightarrow \bigoplus_{r=0}^{t} J_n^{[r]} \longrightarrow \Lambda_n^t \longrightarrow 0$$

induces an exact sequence

$$0 \longrightarrow \bigoplus_{r=1}^{t} H^m(J_n^{[r]}) \longrightarrow \bigoplus_{r=0}^{t} H^m(J_n^{[r]}) \longrightarrow H^m(\Lambda_n^t) \longrightarrow 0 \ .$$

We prove this by induction on m (it being clearly true for m = -1). First, we show that the map induced by $\bar{\phi}$, $H^m(\Lambda_n^t) \longrightarrow H^m(\mathcal{O}_n^{cris})$, is injective. To do this we use an induction on n, the case n = 1 having been established in the lemma of 2.5. Suppose n ≥ 2; the flatness of the $J_n^{[r]}$'s as W_n-modules implies that of Λ_n^t and thus we have a commutative diagram with exact rows:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \Lambda_{n-1}^t & \longrightarrow & \Lambda_n^t & \longrightarrow & \Lambda_1^t & \longrightarrow & 0 \\ & & \downarrow \bar{\phi} & & \downarrow \bar{\phi} & & \downarrow \bar{\phi} & & \\ 0 & \longrightarrow & \mathcal{O}_{n-1}^{cris} & \longrightarrow & \mathcal{O}_n^{cris} & \longrightarrow & \mathcal{O}_1^{cris} & \longrightarrow & 0 \ ; \end{array}$$

which induces another such diagram:

$$\begin{array}{ccccccc} H^{m-1}(\Lambda_1^t) & \longrightarrow & H^m(\Lambda_{n-1}^t) & \longrightarrow & H^m(\Lambda_n^t) & \longrightarrow & H^m(\Lambda_1^t) \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ H^{m-1}(\mathcal{O}_1^{cris}) & \longrightarrow & H^m(\mathcal{O}_{n-1}^{cris}) & \longrightarrow & H^m(\mathcal{O}_n^{cris}) & \longrightarrow & H^m(\mathcal{O}_1^{cris}) \ . \end{array}$$

As the left hand vertical arrow is an isomorphism and the second and fourth vertical arrows are injective, it follows that the third vertical arrow is injective too.

Thus $\lg_{W_n} H^m(\Lambda_n^t) \le \lg_{W_n} H^m(\mathcal{O}_n^{cris})$. The induction hypothesis implies the exactness of the sequence

$$0 \longrightarrow \bigoplus_{r=1}^{t} H^m(J_n^{[r]}) \longrightarrow \bigoplus_{r=0}^{t} H^m(J_n^{[r]}) \longrightarrow H^m(\Lambda_n^t)$$

Thus $\lg_{W_n} H^m(\mathcal{O}_n^{cris}) = \lg_{W_n} H^m(J_n^{[o]}) \le \lg_{W_n} H^m(\Lambda_n^t)$ giving an equality between these lengths and the two assertions of the theorem follow for m.

2.7. COROLLARY: *Let* $s \in \mathbf{N} \cup \{\infty\}$ *and let* Y_S *be a proper smooth (formal) scheme on* W_S *(with the convention that* $W_\infty = W$*) whose special fiber has dimension* d.

i) *If* n *(finite or not)* $\le s\text{-}\inf(m,d)$ *and if* $\inf(m,d) \le \inf(p-1,s-1)$, *then, for all* r, *the mapping*

$$H^m(Y_{n,\sigma\ge r+1}\Omega^{\cdot}_{Y_n}) \longrightarrow H^m(Y_{n,\sigma\ge r}\Omega^{\cdot}_{Y_n})$$

is injective,

ii) *under the same hypothesis,* $H^m_{DR}(Y_n)$ *endowed with the filtration defined by* $H^m(Y_{n,\sigma\ge r}\Omega^{\cdot}_{Y_n})$ *and the mappings* ϕ_r: $H^m(Y_{n,\sigma\ge r}\Omega^{\cdot}_{Y_n}) \longrightarrow H^m_{DR}(Y_n)$ *is an object of the category* MF_{tf} *of* [32].

iii) *If* $d \le p-1$ *and if* n *(finite or not)* $\le s-d$, *the Hodge to de Rham spectral sequence*

$$E_1^{r,m} = H^m(Y_n, \Omega^r_{Y_n}) \Rightarrow H^*_{DR}(Y_n)$$

degenerates at E_1.

Let $t = \inf(m,d)$. When s is finite statement i) follows from the injectivity of

$$\bigoplus_{r=1}^{t} H^m(J_n^{[r]}) \longrightarrow \bigoplus_{r=0}^{t} H^m(J_n^{[r]})$$

and iii) follows from i). Assertion ii) is equivalent to the exactness of the sequence

$$0 \longrightarrow \bigoplus_{r=1}^{t} H^m(J_n^{[r]}) \longrightarrow \bigoplus_{r=0}^{t} H^m(J_n^{[r]}) \xrightarrow{\bar{\phi}} H^m(\mathcal{O}_n^{cris}) \longrightarrow 0 \quad .$$

The case where $n = s = \infty$ follows from this by passing to the limit.

2.8. REMARK: i) When Y_S is projective this result was obtained by Kato [22].

ii) Assertion ii) of 2.7 enables us to define a canonical splitting of the Hodge filttration of $H^m_{DR}(Y_n)$, the Wintenberger splitting, [32]. Further, one may associate to $H^m_{DR}(Y_n)$ a representation of G, cf. [13]. When $n = s = \infty$ we obtain the weak admissibility and hence, by [13] the admissibility of $K \underset{W}{\otimes} H^m_{DR}(Y_\infty)$.

III. *p*-Adic Periods and the Syntomic-Etale Topology

1. The Sheaves $\tilde{S}^r_n$ and Crystalline Cohomology

1.1. We shall work in this paragraph on the site $Spf(W)_{nil,SYN}$ whose objects consist of all W-schemes on which *p* is locally nilpotent, this category being endowed with the syntomic topology. For any X in this category, we denote by X_n its reduction modulo p^n and we recall that we have sheaves $\mathcal{O}_n$, $\mathcal{O}^{cris}_n$, J_n, $J^{[r]}_n$ defined as follows:

$$\mathcal{O}_n(X) = \Gamma(X_n, \mathcal{O}_{X_n})$$
$$\mathcal{O}^{cris}_n(X) = \mathcal{O}^{cris}_n(X_1) = H^0((X_1/W_n)_{cris}, \mathcal{O}_{X_1/W_n})$$
$$J_n = \mathrm{Ker}\,(\mathcal{O}^{cris}_n \longrightarrow \mathcal{O}_n)$$
$$J^{[r]}_n = r^{th} \text{ divided power of } J_n\ .$$

Recall that $\mathcal{O}^{cris}_n$ is endowed with a σ-semi-linear endomorphism ϕ, the Frobenius, and we define $\tilde{S}^r_n$ as follows

$$\tilde{S}^r_n = \mathrm{Ker}\,(J^{[r]}_n \xrightarrow{\phi - p^r} \mathcal{O}^{cris}_n)\ .$$

Thus we have a short exact sequence (denoting by $\mathcal{O}^{cris,r}_n$ the image of $\phi - p^r$)

$$0 \longrightarrow \tilde{S}^r_n \longrightarrow J^{[r]}_n \xrightarrow{\phi - p^r} \mathcal{O}^{cris,r}_n \longrightarrow 0\ .$$

LEMMA: $\mathcal{O}^{cris}_n / \mathcal{O}^{cris,r}_n$ *is killed by* p^r.

This is proved by explicit (laborious) computations using the fact that $\mathcal{O}^{cris}_n = \tilde{W}^{DP}_n$ (cf. II.1.4) and that for an appropriate finite extension L of K, there is an explicitly constructed element $u \in W^{DP}_n(\mathcal{O}_{L,1})$ such that $u \in J^{[p-1]}$ and $\phi u = p^{p-1}u$. (Viewing u in $W^{DP}_n(\tilde{\mathcal{O}}_{\bar{K}})$, it is the reduction modulo p^n of $t^{p-1}/p \in H^0_{cris}(\tilde{\mathcal{O}}_{\bar{K}}/W)$, where t is a generator for the Tate module $\mathbf{Z_p}(1)$ inside this ring.)

1.2. Let X be a proper, smooth W-scheme and $\bar{X} = X \otimes \mathcal{O}_{\bar{K}}$. Using the fact that syntomic morphisms can, locally, always be lifted modulo a nilpotent ideal, it follows that $H^*(\bar{X}_m, \tilde{S}^r_n)$, $H^*(\bar{X}_m, J^{[r]}_n)$, $H^*(\bar{X}_m, \mathcal{O}^{cris}_n)$, $H^*(\bar{X}_m, \tilde{\mathcal{O}}^{cris,r}_n)$ are all independent of the choice of $m \geq n$ and we will denote them by $H^*(\bar{X}, \tilde{S}^r_n)$ (resp...). Further, given any projective system of sheaves of $\mathbf{Z_p}$-modules $(\mathcal{F}_n)_{n\geq 1}$ we will write

$$H^*(\bar{X}, \mathcal{F}_{\mathbf{Z}_p}) = \varprojlim H^*(\bar{X}, \mathcal{F}_n)$$

$$H^*(\bar{X}, \mathcal{F}_{\mathbf{Q}_p}) = \mathbf{Q}_p \otimes_{\mathbf{Z}_p} H^*(\bar{X}, \mathcal{F}_{\mathbf{Z}_p}) \ .$$

1.3. PROPOSITION: i) $H^*(\bar{X}, \mathcal{O}_n^{cris}) = W_n^{DP}(\tilde{\mathcal{O}}_{\bar{K}}) \otimes H^*_{DR}(X_n)$

ii) $H^*(\bar{X}, \mathcal{O}^{cris}_{\mathbf{Q}_p}) = B^+_{cris} \otimes_K H^*_{DR}(X_K)$.

Statement ii) follows immediately from i); statement i) is the Künneth formula for $H^*_{cris}(X_n \otimes \mathcal{O}_{\bar{K},n}/W_n)$ and is proved by an elementary computation using the de Rham complex of a divided power envelope as well as the fact that $\mathrm{Spec}(\tilde{\mathcal{O}}_{\bar{K}})$ has trivial higher cohomology, [16].

1.4. LEMMA: i) $H^*(\bar{X}, \mathcal{O}^{cris,r}_{\mathbf{Q}_p}) = H^*(\bar{X}, \mathcal{O}^{cris}_{\mathbf{Q}_p})$

ii) *there is an integer* j (*independent of* n) *such that* p^j *kills the cokernel of* $H^*(\bar{X}, \mathcal{O}^{cris,r}_{\mathbf{Z}_p}) \longrightarrow H^*(\bar{X}, \mathcal{O}_n^{cris,r})$.

The first statement follows immediately from the lemma of 1.1 while the second statement follows from the fact that the cokernel of $H^*_{DR}(X) \longrightarrow H^*_{DR}(X_n)$ is killed by a power of *p* independent of n, together with statement i) of the proposition of 1.3.

1.5. Denote by π the morphism $\bar{X}_n \longrightarrow X_n$. One verifies directly that $R^j\pi_* J_n^{[r]} = 0$ for j > 0, and that one has a short exact sequence on $X_{n,syn}$

$$0 \longrightarrow \bigoplus_{\substack{i+j=r+1 \\ i,j\geq 1}} J_n^{[i]} \otimes_{W_n} J_n^{[j]}(\mathcal{O}_{\bar{K},n}) \longrightarrow \bigoplus_{i+j=r} J_n^{[i]} \otimes_{W_n} J_n^{[j]}(\mathcal{O}_{\bar{K},n}) \longrightarrow \pi_*(J_n^{[r]}) \longrightarrow 0 \ .$$

PROPOSITION: i) *If the dimension of* X_1 *is strictly less than* p, *then for any* m

$$\mathrm{Fil}^r(W_n^{DP}(\tilde{\mathcal{O}}_{\bar{K}}) \otimes H^m_{DR}(X_n)) \xrightarrow{\sim} H^m(\bar{X}_n, J_n^{[r]}) \ .$$

ii) $\mathrm{Fil}^r(B^+_{cris} \otimes H^m_{DR}(X)) \xrightarrow{\sim} H^m(\bar{X}, J^{[r]}_{\mathbf{Q}_p})$.

The first statement is an immediate consequence of the degeneration of the Hodge to de Rham spectral sequence (for X_n) at E_1 because then the long exact cohomology sequence associated to the above short exact sequence will decompose into short exact sequences. The second statement is a consequence of the fact that

$$\bigoplus_{\substack{i+j=r+1 \\ i,j\geq 1}} H^{m+1}(X, J^{[i]}_{\mathbf{Z}_p}) \otimes \mathrm{Fil}^j \varprojlim W_n^{DP}(\tilde{\mathcal{O}}_{\bar{K}}) \longrightarrow \bigoplus_{i+j=r} H^m(\bar{X}, J^{[i]}_{\mathbf{Z}_p}) \otimes \mathrm{Fil}^j \varprojlim W_n^{DP}(\tilde{\mathcal{O}}_{\bar{K}})$$

has its kernel killed by some power of *p*.

1.6. THEOREM: *Assume* X_K *is admissible and let* $r \geq m$. *The sequence*

$$0 \longrightarrow H^m(\bar{X}, \tilde{S}^r_{\mathbf{Q}_p}) \longrightarrow H^m(\bar{X}, J^{[r]}_{\mathbf{Q}_p}) \longrightarrow H^m(\bar{X}, \mathcal{O}^{cris,r}_{\mathbf{Q}_p}) \longrightarrow 0$$

is exact.

COROLLARY: *With the above hypothesis we have (for* $r \geq m$*) an exact sequence*

$$0 \longrightarrow H^m(\bar{X}, \tilde{S}^r_{\mathbf{Q}_p}) \longrightarrow \mathrm{Fil}^r(B^+_{cris} \underset{K}{\otimes} H^m_{DR}(X_K)) \xrightarrow{\phi - p^r} H^m_{DR}(X_K) \longrightarrow 0 \quad .$$

The corollary follows immediately from the theorem and 1.3 and 1.5. For the proof of the theorem we use the following:

LEMMA: *If* X_K *is admissible and* $r \geq m$ *then there is an* $s \geq 0$ *such that the cokernel of* $H^m(\bar{X}, J^{[r]}_{\mathbf{Z}_p}) \xrightarrow{\phi - p^r} H^m(\bar{X}, \mathcal{O}^{cris}_{\mathbf{Z}_p})$ *is killed by* p^s.

The lemma implies the theorem for $m = 0$ and thus, if $m \geq 1$, it suffices to verify the exactness of the sequence obtained by omitting the right hand zero. Using the lemma together with 1.4.ii) it is easy to check that there is an integer s' such that for any n the kernel of $H^m(\bar{X}, \tilde{S}^r_n) \longrightarrow H^m(\bar{X}, J^{[r]}_n)$ is killed by $p^{s'}$. Assuming this, an elementary and standard argument gives the theorem.

2. An Indication of the Proof of the Lemma of 1.6

2.1. LEMMA: *For* $i \geq 0$ $\mathrm{coker}(\mathrm{Fil}^i(\varprojlim W^{DP}_n(\tilde{\mathcal{O}}_{\bar{K}})) \xrightarrow{\phi - p^i} \varprojlim W^{DP}_n(\tilde{\mathcal{O}}_{\bar{K}}))$ *is killed by* p^i.

This follows from the lemma of 1.1 but may also be proved directly.

2.2. Recall we have defined in I.1.6. "*admissible filtered module* ". If D is such a module, then we have $B_{cris} \underset{\mathbf{Q}_p}{\otimes} V(D) = B_{cris} \underset{K}{\otimes} D$ (in a manner compatible with filtrations, Frobenius and the action of G).

2.3. PROPOSITION: *If* D *is admissible*, *then the map*

$$\mathrm{Fil}^0(B_{cris} \otimes D) \xrightarrow{\phi - 1} B_{cris} \otimes D$$

is surjective.

This follows immediately from 2.1, and implies that for any $i \in \mathbf{Z}$, $\mathrm{Fil}^i(B_{cris} \otimes D) \xrightarrow{\phi - p^i} B_{cris} \otimes D$ is surjective.

2.4. PROPOSITION: *If* D *is admissible and satisifies* $\mathrm{Fil}^0 D = D$, $\mathrm{Fil}^{i+1} D = 0$, *then*

$\mathrm{Fil}^i(B^+_{cris} \otimes D) \xrightarrow{\phi - p^i} B^+_{cris} \otimes D$ *is surjective.*

This is proved by a computation, using the basic property of B^+_{cris} that $t \cdot B^+_{cris} = \{x \in B^+_{cris} | \phi^r x \in \mathrm{Fil}^1 B^+_{cris}, \forall r\}$. In fact the proof yields the fact that $V(D) \subset t^{-i} B^+_{cris} \otimes D$.

2.5. Let D satisfy the hypothesis of 2.4 and choose (cf. [12,14]) a strongly divisible lattice M in D.

PROPOSITION: *There is an integer* $s \geq 0$ *such that the cokernel of* $\mathrm{Fil}^i(\varprojlim W_n^{DP}(\tilde{\mathcal{O}}_{\bar{K}}) \otimes M) \xrightarrow{\phi - p^i} \varprojlim W_n^{DP}(\tilde{\mathcal{O}}_{\bar{K}}) \otimes M$ *is killed by* p^s.

This is proved by a long involved computational argument, which we do not attempt to elucidate here.

2.6. Let $M \subset H^m_{DR}(X)/\mathrm{torsion}$ be a strongly divisible lattice, chosen sufficiently small so that the left hand vertical arrow in the following commutative diagram is defined:

$$\begin{array}{ccc} \mathrm{Fil}^r(\varprojlim W_n^{DP}(\tilde{\mathcal{O}}_{\bar{K}}) \otimes M) & \xrightarrow{\phi - p^r} & \varprojlim W_n^{DP}(\tilde{\mathcal{O}}_{\bar{K}}) \otimes M \\ \downarrow & & \downarrow \\ \text{Image of } H^m(\bar{X}, J^{[r]}_{\mathbb{Z}_p}) \text{ in } H^m(\bar{X}, \mathcal{O}^{cris}_{\mathbb{Z}_p})/\mathrm{torsion} & \xrightarrow{\phi - p^r} & H^m(\bar{X}, \mathcal{O}^{cris}_{\mathbb{Z}_p})/\mathrm{torsion} \end{array}$$

As the upper horizontal and right hand vertical arrows both have cokernels killed by fixed powers of p, the lemma of 1.6 follows.

2.7. REMARK: The proof actually shows that the results of 1.6 are valid provided $r \geq \inf(m, \text{length of the Hodge filtration of } H^m_{DR}(X_K))$.

3. The Sheaves S^r_n

3.1. The natural epimorphism $J^{[r]}_{n+r} \longrightarrow J^{[r]}_n$ induces a map $\tilde{S}^r_{n+r} \longrightarrow \tilde{S}^r_n$. We denote its image by S^r_n. It follows easily from the lemma of 1.1 that for $m \geq n+r$ this image coincides with $\mathrm{im}(\tilde{S}^r_m \longrightarrow \tilde{S}^r_n)$. The following is obvious:

LEMMA: $H^*(\bar{X}, \tilde{S}^r_{\mathbb{Z}_p}) \xrightarrow{\sim} H^*(\bar{X}, S^r_{\mathbb{Z}_p})$.

Notice that we have natural maps $S^r_n \times S^{r'}_n \longrightarrow S^{r+r'}_n$ which endow $\bigoplus_{r \geq 0} S^r_n$ with the structure of an associative, commutative $\mathbf{Z}/p^n\mathbf{Z}$-algebra with unit. We now define a map $\mu_{p^n} \longrightarrow S_n^1$ as follows: For any W-algebra A, we write A_{n+1} as a quotient of the polynomial ring over W_{n+1} and let $\mathfrak{D}$ be the corresponding divided power envelope so that we have an epimorphism $\mathfrak{D} \longrightarrow A_{n+1}$. Let $\zeta' \in \mathfrak{D}$ be a lifting of an element of $\mu_{p^{n+1}}(A_{n+1})$. Then $\zeta'^{p^{n+1}} \in 1 + J_{n+1}(A)$ and its logarithm $\log(\zeta'^{p^{n+1}})$ is well-defined. We define in this way a homomorphism $\mu_{p^{n+1}}(A) \longrightarrow \mu_{p^{n+1}}(A_{n+1}) \longrightarrow J_{n+1}(A)$ and it is clear that the image is contained in $\widetilde{S}_{n+1}^1(A)$, as well as that the image of $\mu_p(A)$ is contained in $\mathrm{Ker}\,(\widetilde{S}_{n+1}^1(A) \longrightarrow S_n^1(A))$. Passing to the associated sheaves, we thus obtain a map $\mu_{p^n} \longrightarrow S_n^1$.

3.2. PROPOSITION: *If* A *is a* p*-adically separated and complete flat* W*-algebra, such that* A_1 *is a syntomic k-algebra, then* $\mu_{p^n}(A) \xrightarrow{\sim} S_n^1(A)$.

This may be proved by explicit calculations using the Artin-Hasse logarithm.

4. The Syntomic Etale Site

4.1. We will say that a morphism $\mathcal{U} \longrightarrow \mathcal{V}$ of p-adic formal schemes over SpfW is syntomic provided $\mathcal{U}_n \longrightarrow \mathcal{V}_n$ is syntomic for all $n \geq 1$. Recall that if L is a complete non-archimedian valued field with ring of integers $\mathcal{O}_L$ then we may associate to any formal scheme of finite type over $\mathcal{O}_L$, a rigid analytic space over L, its generic fiber, cf. [28]. If $\mathfrak{Z}$ is any (finite type) formal $\mathcal{O}_L$-scheme we define the *syntomic-étale* site of $\mathfrak{Z}$, denoted $\mathfrak{Z}_{syn,et}$ as follows: Objects are morphisms $\mathcal{U} \longrightarrow \mathfrak{Z}$ which are syntomic, quasi-finite and have étale generic fiber (in the sense of rigid geometry). We shall be primarily concerned with the following situation. Given X our proper smooth W-scheme and $\overline{X} = X \otimes \mathcal{O}_{\overline{K}}$, let $\mathfrak{Z} = \hat{\overline{X}} = \varinjlim \overline{X}_n$ (so here L = C).

PROPOSITION: *If for any* $n \geq 1$, i *denotes the inclusion* $\overline{X}_n \longrightarrow \mathfrak{Z}$, *then* i_* *is exact for the syntomic-étale topology.*

It suffices to show that, if $\mathcal{A}$ is a p-adically complete W-algebra and $\mathcal{A}_n \longrightarrow B$ is syntomic and quasi-finite, then, locally, we may lift B to $\mathcal{B}$ such that $\mathcal{A} \longrightarrow \mathcal{B}$ is syntomic, quasi-finite with étale generic fiber. If $B = \mathcal{A}_n[X_1,\dots,X_d]/(f_1,\dots,f_d)$. then we may replace B by $B' = \mathcal{A}_n[Y_1,\dots,Y_d]/(f_1(X),\dots,f_d(X))$ where $X_i \longmapsto Y_i^{p^{n+1}}$ defines the map

$B \longrightarrow B'$ and we lift B' to $\mathcal{B}' = \mathcal{A}\{Y_1,\dots,Y_d\}/(\hat{f}_i(Y^{p^{n+1}}) + p^n.Y_i)$, where $\hat{f}_i$ lifts f_i.

REMARK: By abuse of notation, we will write S_n^r instead of $i_*S_n^r$ where $i:\bar{X}_{n+r} \longrightarrow \mathcal{Y}$.

4.2 We introduce also the syntomic -étale site of $\bar{X}$, where objects are morphisms $U \longrightarrow \bar{X}$ which are syntomic, quasi-finite and with $U \otimes \bar{K}$ étale. We have then morphisms (where $\mathcal{Y} = \hat{\bar{X}}$)

$$\mathcal{Y} \xrightarrow{\ i\ } \bar{X} \xleftarrow{\ j\ } X_{\bar{K}}$$

and these induce morphisms of topoi: $j: X_{\bar{K},et} \longrightarrow \bar{X}_{syn\text{-}et}$, $i: \mathcal{Y}_{syn\text{-}et} \longrightarrow \bar{X}_{syn\text{-}et}$. The definition of j is obvious since the category of schemes étale over $X_{\bar{K}}$ is a full sub-category of the category underlying the syntomic-étale site of $\bar{X}$. The definition of i_* is determined by $i_*\mathcal{F}(\mathrm{Spec}(B)) = \mathcal{F}(\mathrm{Spf}\ \hat{B})$ where Spec(B) is an affine object of the syntomic-étale site of $\bar{X}$ and $\hat{B} = \varprojlim B_n$. The definition of i^* is more subtle and requires some preliminary results.

4.3. PROPOSITION: *Let* Spec(A) *be Zariski open in* $\bar{X}$, $\mathcal{A} = \hat{A} = \varprojlim A_n$ *and* $\mathcal{A} \longrightarrow \mathcal{B}$ *be a syntomic, quasi-finite morphism* (*where* $\mathcal{B}$ *is a* p*-adically separated, complete algebra*) *with étale generic fiber. Then there is a syntomic, quasi-finite morphism* $A \longrightarrow B$ *with étale generic fiber such that* $\hat{B} = \mathcal{B}$.

If we write $\mathcal{B} = \mathcal{A}\{X_1,\dots,X_d\}/(P_1,\dots,P_d)$ where the P_i's are restricted power series and if we take polynomials $Q_1,\dots,Q_d$ in $A[X_1,\dots,X_d]$ which are congruent to the P's modulo p^r for some $r >> 0$, then $B = A[X_1,\dots,X_d]/(Q_1,\dots,Q_d)$ satisfies $\hat{B} = \mathcal{B}$, [17]. The facts that B is finitely presented and $\mathcal{B}$ is p-torsion-free imply that B is p-torsion-free. If j is the jacobian, then for some $g \in \mathcal{B}$, $j \cdot g = p^n$. If we write $g = g' + p^{n+1}g''$ with $g' \in B$, $g'' \in \mathcal{B}$, then $jg' = p^n(1 - pg'')$ and $1 - pg'' \in B$ so replacing B by $B[1/1 - pg'']$ we force the generic fiber to be étale. It now follows that $A \longrightarrow B$ is flat and hence this morphism satisfies all the conditions of the proposition.

We refer to a B as in the above proposition as a *decompletion* of $\mathcal{B}$.

4.4. If B is as constructed in 4.3, then we denote its henselization with respect to p by B^h. It is now easy to show, using the results of [26], that this ring depends only on $\mathcal{B}$ and neither on the choice of $\hat{A} \longrightarrow \mathcal{B}$ nor on the choice of B. We define the inverse image functor i^*, applied to a $\mathcal{L}$ on $\bar{X}_{syn\text{-}ét}$, by sheafifying the presheaf $\mathcal{B} \longmapsto \mathcal{L}(B^h)$ where this last term is defined as the (filtered) inductive limit, $\varinjlim \mathcal{Z}(B')$, the limit taken over all commutative diagrams:

$$\begin{array}{ccc} & & \mathcal{B} \\ & \nearrow & \uparrow \\ B & \longrightarrow & B' \end{array}$$

where $B \longrightarrow B'$ is étale.

Now notice that given a B' such as in the diagram, then the pair B' and B_K constitute a covering of B. This implies that the following diagram is cartesian for any sheaf $\mathcal{Z}$ on $\bar{X}_{\text{syn-ét}}$

$$\begin{array}{ccc} \mathcal{Z} & \longrightarrow & j_*j^*\mathcal{Z} \\ \downarrow & & \downarrow \\ i_*i^*\mathcal{Z} & \xrightarrow[i_*(\alpha)]{} & i_*i^*j_*j^*\mathcal{Z} \end{array}$$

and hence, just as in the case of the étale topology, we deduce:

PROPOSITION: *The functor* $\mathcal{Z} \longmapsto (i^*\mathcal{Z}, j^*\mathcal{Z}, \alpha)$ *establishes an equivalence of categories between sheaves on* $\bar{X}_{\text{syn,ét}}$ *and the category of triples* $(\mathcal{F},\mathcal{H},\alpha)$ *consisting of a sheaf* $\mathcal{F}$ *on* $\mathcal{Y}_{\text{syn-ét}}$, *a sheaf* $\mathcal{H}$ *on* $X_{\bar{K},\text{et}}$ *and a morphism* $\alpha: \mathcal{F} \longrightarrow i^*j_*\mathcal{H}$. *The functor* i_* *is exact as is the functor* $j_!$ *defined by* $j_!(\mathcal{H}) = (0,\mathcal{H},0)$. *For any sheaf* $\mathcal{Z}$ *we have an exact sequence*

$$0 \longrightarrow j_!j^*\mathcal{Z} \longrightarrow \mathcal{Z} \longrightarrow i_*i^*\mathcal{Z} \longrightarrow 0 \quad .$$

REMARK: It is useful to note that given $X' \longrightarrow \bar{X}$ a syntomic, quasi-finite morphism with étale generic fiber, there is a canonical decomposition $X' = X'_1 \coprod X'_2$ where X'_1 is p-adically separated and on X'_2 p is invertible.

5. Construction of the Sheaves $\mathcal{S}_n^r$

5.1. We construct sheaves, $\mathcal{S}_n^r$ on $\bar{X}_{\text{syn-ét}}$ such that $i^*\mathcal{S}_n^r = S_n^r$, $j^*\mathcal{S}_n^r = \mathbf{Z}/p^n\mathbf{Z}(r)$. By 4.4 this is equivalent to defining a map $\alpha: S_n^r \longrightarrow i^*j_*\,\mathbf{Z}/p^n\mathbf{Z}(r)$. Given $\hat{A} \longrightarrow \mathcal{B}$, B and B^h as in 4.3 and 4.4, it suffices to define a map $S_n^r(\mathcal{B}) \longrightarrow \mathbf{Z}/p^n\mathbf{Z}(r)(B^h[1/p])$. If we descend A to an A_0 defined over $\mathcal{O}_L$, L a finite extension of K in $\bar{K}$ and we descend B also, then we may write $B = \varinjlim B_i$ where

$B_i = B_0 \otimes_{A_0} \mathcal{O}_{L_i}$ where L_i runs through the finite extensions of L contained in $\bar{K}$.

Then $S_n^r(\mathcal{B}) = S_n^r(\mathcal{B}_{n+r}) = S_n^r(B_{n+r}) = \varinjlim S_n^r(B_{i,n+r})$. In addition, we have $B^h = \varinjlim B_i^h$ and thus $\mathbf{Z/p^nZ}(r)(B^h[1/p]) = \varinjlim \mathbf{Z/p^nZ}(r)(B_i^h[1/p])$. Thus, we may "work at finite level" and it suffices to define a map $S_n^r(B_{i,n+r}) \longrightarrow \mathbf{Z/p^nZ}(r)(B_i^h[1/p])$.
We change notation, and now write A for a smooth, p- adically separated $\mathcal{O}_L$-algebra, B for a p- adically separated syntomic, quasi-finite A-algebra having étale generic fiber. Further, we may assume that if π is a uniformizing parameter for $\mathcal{O}_L$ that $A/\pi A$ is an integral domain. Thus, A is endowed with a valuation, $\upsilon(a) = \max\{j | a \in \pi^j A\}$, inducing its p- adic topology. Let $\mathcal{B} = \hat{B}, \mathcal{A} = \hat{A}$. Because $\pi_0(\mathrm{Spec}(B^h[1/p])) = \pi_0(\mathrm{Spec}(\mathcal{B}[1/p]))$,[10], we may work with $\mathcal{B}$. $\mathcal{B}[1/p]$ is an étale $\mathcal{A}[1/p]$-algebra, and thus is a finite product of integral domains, each of which is regular and hence normal. We write $\mathcal{B}[1/p] = B_1 \times \ldots \times B_t$. Each B_j is a Tate algebra, and we denote by $\overset{\circ}{B}_j$ the subring of elements whose spectral norm is at most equal to 1, cf. [6]. Each $\overset{\circ}{B}_j$ is integrally closed in B_j and hence is a p- adically separated normal domain which contains the image of $\mathcal{B}$ in B_j. We have a natural map

$$S_n^r(\mathcal{B}_{n+r}) \longrightarrow \prod_{j=1}^{t} S_n^r(\overset{\circ}{B}_j/p^{n+r}).$$

Assume now without loss of generality, that L contains $\boldsymbol{\mu}_{p^n}(\bar{K})$ so that

$$\mathbf{Z/p^nZ}(r)(\mathcal{B})[1/p] = \prod_{j=1}^{t} (\boldsymbol{\mu}_{p^n}(\bar{K})^{\otimes r})_j \quad .$$

Thus, to define α, we must define a map $S_n^r(\overset{\circ}{B}_j) \longrightarrow \boldsymbol{\mu}_{p^n}(\bar{K})^{\otimes r}$.

5.2. Since B_j is a $W[\zeta_n]$-algebra, where ζ_n is a primitive $p^{n^{th}}$ root of 1, we have a natural map

$$\boldsymbol{\mu}_{p^n}(\bar{K})^{\otimes r} \longrightarrow \mathrm{Sym}^r(\boldsymbol{\mu}_{p^n}(\overset{\circ}{B}_j)) \longrightarrow S_n^r(\overset{\circ}{B}_j/p^{n+r})$$

where the last map is induced from $\boldsymbol{\mu}_{p^n} \longrightarrow S_n^1$, cf. 3.1. We refer to this homomorphism as the *natural homomorphism* .

THEOREM: *Let* C *be a* $W[\zeta_n]$*-algebra which is a normal domain and such that* p *is not invertible in* C. *Set* $r = (p-1)a + b$ *with* $a, b \geq 0$, $b < p-1$ *and let*

$c = a + \upsilon_p(a!)$. *Then there is a functorial isomorphism of* $\mu_{p^n}(\bar{K})^{\otimes r}$ *onto* $S_n^r(C/p^{n+r})$ *such that the natural homomorphism is* p^c *times this isomorphism.*

We indicate the definition of this isomorphism. Let $(\zeta_i)_{i\geq 1}$ be a generator for the Tate module $\mathbf{Z_p}(1)(\mathcal{O}_{\bar{K}})$ and let $t \in H^o_{cris}(\mathrm{Spec}(\tilde{\mathcal{O}}_{\bar{K}})/W)$ be the corresponding element. Consider the element $u = t^{p-1}/p \in \mathrm{Fil}^{p-1} H^o_{cris}(\mathrm{Spec}(\tilde{\mathcal{O}}_{\bar{K}})/W)$ and let t_n (resp. u_n) be the image in $W_n^{DP}(\tilde{\mathcal{O}}_{\bar{K}})$. In fact, they belong to $W_n^{DP}(W[\zeta_n]/p^n)$. Then $t_n^b\gamma_a(u_n) \in S_n^r(W[\zeta_n]/p^{n+r})$ and if $a! = p^{\upsilon(a!)}a'$, then $t_n^r = a' p^c t_n^b \gamma_a(u_n)$. The homomorphism $\mu_{p^n}(\bar{K})^{\otimes r} \longrightarrow S_n^r(C/p^{n+r})$ is now defined by sending $\zeta_n^{\otimes r}$ to $a't_n^b\gamma_a(u_n)$ viewed as an element in $S_n^r(C/p^{n+r})$.

5.3. From the theorem, the definition of α is immediate. There is an alternative procedure for defining α. Namely, in the notation of 5.1, we let E be the fraction field of A, a discretely valued field, and $\mathbf{C}$ be the completion of the algebraic closure of $\hat{E}$. For each $j = 1,\ldots,t$ we choose an embedding of $\mathrm{Frac}(B_j)$ into $\mathbf{C}$ which induces an embedding of $\overset{o}{B}_j$ into $\mathcal{O}_{\mathbf{C}}$. Thus, we obtain a map from $S_n^r(\overset{o}{B}_j/p^{n+r})$ to $S_n^r(\mathcal{O}_{\mathbf{C},n+r})$. But the field $\mathbf{C}$ contains $C = \hat{\bar{K}}$ and is completely analogous in the sense that, if $\mathbf{B}_{cris}$ is the ring associated to the perfect closure of $k(X_1)$, then $\mathbf{C}$ plays the role of C. Thus, the results of [14] can be applied in this context, and they imply that $S_n^r(\mathcal{O}_{\mathbf{C},n+r}) = \mu_{p^n}(\bar{K})^{\otimes r}$.

6. Construction of *p*- Adic Etale Cohomology

6.1. We shall indicate how the above results allow us to prove theorem B of I.2.3. Recall X is a proper, smooth W-scheme such that X_K is admissible. For any admissible filtered module D and any integer i write $V_i(D)$ for $\{x \in B_{cris}\otimes D \mid \phi(x) = p^i x, x \in \mathrm{Fil}^i\}$. Multiplication by t induces an isomorphism $V_i(D) \xrightarrow{\sim} V_{i+1}(D)$ which can be viewed intrinsically as a canonical isomoprhism $V_i(D)(-i) \xrightarrow{\sim} V_{i+1}(D)(-i-1)$. Assume $D = \mathrm{Fil}^o D$ and $\mathrm{Fil}^{i+1}(D) = 0$. Then, the proof of 2.4 gives the fact that $V_i(D) = \{x \in \mathrm{Fil}^i B^+_{cris} \otimes D) \mid \phi x = p^i x\}$. Applying this with $D = H^m_{cris}(X_K)$ and taking $r \geq \inf(m$, length of the Hodge filtration) we obtain from 1.6 and 3.2 an isomorphism (where we write $V_r^m(X)$ for $V_r(H^m_{cris}(X_K))$)

$$H^m(\bar{X}, S^r_{\mathbf{Q}_p}) \; = \; V^m_r(X) \;\; .$$

Note that, since X is admissible, $\bigoplus_{m=0}^{2d} V^m_m(X)(-m)$ is an anti-commutative graded algebra which satisfies Poincaré duality, cf [12,14]. We wish to prove that for r and m as above $V^m_r(X) \xrightarrow{\sim} H^m_{et}(X_{\bar{K}}, \mathbf{Q}_p(r))$.

6.2. PROPOSITION: *The natural map* $H^*(\bar{X}_{syn\text{-}et}, \mathcal{S}^r_n) \longrightarrow H^*(\mathcal{Y}_{syn\text{-}et}, S^r_n)$ *is an isomorphism.*

By 4.4 this is equivalent to the assertion that $H^*(j_! \mathbf{Z}/p^n\mathbf{Z}) = 0$. But using the exact sequence

$$0 \longrightarrow j_! \, \mathbf{Z}/p^n\mathbf{Z} \longrightarrow \mathbf{Z}/p^n\mathbf{Z} \longrightarrow i_* \, \mathbf{Z}/p^n\mathbf{Z} \longrightarrow 0$$

and recalling that i_* is exact, we see that this is equivalent to the fact that $H^*(\bar{X}_{syn\text{-}et}, \mathbf{Z}/p^n\mathbf{Z}) \xrightarrow{\sim} H^*(\mathcal{Y}_{syn\text{-}et}, \mathbf{Z}/p^n\mathbf{Z})$. Analysis of the proof of Grothendieck's comparison theorem, [20], shows that we may replace syn-et by étale in this last assertion. Now the proper base change theorem for étale cohomology yields the desired conclusion.

6.3. Using 6.1 and 4.1 we see that the map α constructed in 5 permits us to define a map

$$\beta\colon V^m_r(X) \longrightarrow H^m_{ét}(X_{\bar{K}}, \mathbf{Q}_p(r)) \;\; .$$

PROPOSITION: *The morphism* β *is an isomorphism.*

Since source and target have the same dimension, it suffices to show β is injective. The morphisms β are compatible with the multiplicative structure. Thus, using Poincaré duality on the source, it suffices to show that on $V^{2d}_d(X)$, β is injective (here $\dim(X_K = d)$. After replacing W by a finite unramified extension, we may assume that X has a W-valued point, x. Its crystalline cohomology class satisfies $\phi(cl(x)) = p^d\,(cl(x))$ and $cl(x) \in Fil^d H^{2d}_{cris}(X_K)$. Thus it suffices to show β takes this crystalline cycle class to the corresponding étale cycle class. Let us blow up the point x in X to obtain $\tilde{X}$ with exceptional divisor equal to $\mathbf{P}^{d-1}_W$. Observe that $\tilde{X}_K$ is admissible and that the étale cohomology of $X_{\bar{K}}$ injects into that of $\tilde{X}_{\bar{K}}$. Hence it suffices to prove that $\beta_{\tilde{X}}$ takes the crystalline class of a point to the étale cycle class. Since the exceptional divisor has self-intersection equal to - H, where H is a hyperplane in $\mathbf{P}^{d-1}_W$, it follows that the class of a point in either theory is given by $-\,c_1(\mathcal{O}_{\tilde{X}}(1))^d$. Thus, it suffices to show β transforms the crystalline Chern classes of a line bundle into its étale Chern class. To verify this, we recall that the crystalline Chern class (relative to W_n) is defined using the "exponential sequence":

$$\begin{array}{ccccccccc} 0 & \longrightarrow & 1 + J_n & \longrightarrow & \mathcal{O}_n^{cris*} & \longrightarrow & \mathcal{O}_n^* & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \log} & & & & & & \\ & & J_n & , & & & & & \end{array}$$

while the étale Chern class ("modulo p^n") is defined using the Kummer sequence:

$$0 \longrightarrow \mu_{p^n} \longrightarrow G_m \xrightarrow{p^n} G_m \longrightarrow 0 \ .$$

But, working over W_n, the Kummer sequence maps to the exponential sequence, since there is the natural map $\mathcal{O}_n^* \longrightarrow \mathcal{O}_n^{cris*}$ given by $\zeta \longmapsto \hat{\zeta}^{p^n}$ where $\hat{\zeta}$ (locally) lifts ζ. Hence, the desired compatibility follows from the fact that $S_n^1 = \mu_{p^n}$.

6.4. REMARK: Combining our techniques with those of Kato, [22], we can prove, even if K is no longer assumed absolutely unramified, that, if X is a proper and smooth $\mathcal{O}_K$-scheme, then, for $r < p - 1$ and $i \le r$, we have $H^i(\bar{X}, S_n^r) \xrightarrow{\sim} H^i_{et}(X_{\bar{K}}, \mathbf{Z}/p^n\mathbf{Z}(r))$. There remains though, the problem of relating the source to crystalline cohomology. When $e = 1$ this has been done and thus, using [13], we obtain the fact that, for $m < p - 1$, the invariant factors for $H^m_{et}(X_{\bar{K}}, \mathbf{Z}/p^n\mathbf{Z})$ coincide with those of $H^m_{DR}(X_n)$, and in particular that for $m \le p - 1$ the invariant factors of the torsion in $H^m_{et}(X_{\bar{K}}, \mathbf{Z}_p)$ depend only on the special fiber.

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AN INTRODUCTION TO HIGHER DIMENSIONAL ARAKELOV THEORY

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ABSTRACT. For arithmetic varieties, compactified at infinity in the style of Arakelov we define Chow groups and intersection products. These definitions extend constructions of Arakelov and Beilinson. We also construct a theory of characteristic classes for vector bundles on arithmetic varieties, equipped with Hermitian metrics. This is joint work with C. Soulé.

INTRODUCTION. These notes are an introduction to the joint work of Christophe Soulé and myself on higher dimensional Arakelov theory. They are an expanded version of the talks that I gave at the Arcata conference, and are intended both as a record of those talks and as an intermediate step between our two Comptes Rendues notes ([G-S 1]) and the final published version of our work. They are not, however, intended to replace this final version, and for this reason proofs have either merely been sketched, or else have been omitted.

I have omitted any special discussion of intersection theory on arithmetic surfaces, and recommend the original articles by Arakelov [Ar] and Faltings [Fa], together with the articles by Szpiro and Silverman in this volume. Also I have not discussed Arakelov K-theory (see the second of our Comptes-Rendues notes, [G-S 1]), nor Riemann Roch-theorems.

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1. CLASSICAL INTERSECTION THEORY.

Before discussing intersection theory on arithmetic varieties, I want to review the basic ideas of 'classical' intersection theory on varieties over fields. For detailed definitions and further references, I recommend the book [Fu].

If X is a smooth variety over a field, which for simplicity we suppose equidimensional, let us write $Z^p(X)$ for the group of codimension p cycles on X and $CH^p(X)$ for the Chow group of codimension p cycles modulo rational equivalence. The graded group $CH^*(X) = \oplus_p CH^p(X)$ can be made into a ring, the <u>Chow ring</u>. There are several methods of constructing this ring structure, the most old fashioned (i.e., ≈ 30 years old!) of which I shall now sketch. We start by considering two cycles [Y] and [Z], where Y and Z are integral subschemes of X, of codimension p and q respectively, which intersect properly, i.e., so that every component S of $Y \cap Z$ has codimension $p + q$. Then

$$[Y] \cdot [Z] = \sum_S \mu_S(Y,Z)[S]$$

where $\mu_S(Y,Z)$ is the intersection multiplicity of Serre ([Se] VC). If $\eta = \Sigma n_i[Y_i]$ and $\xi = \Sigma m_j[Z_j]$ are cycles such that Y_i and Z_j intersect properly for all i and j, then we define $\eta \cdot \xi$ by requiring that the product is bilinear. In order to define a product

$$CH^p(X) \otimes CH^q(X) \longrightarrow CH^{p+q}(X)$$

it is sufficient to know two things:

a) Every pair $\alpha \in CH^p(X)$, $\beta \in CH^q(X)$ of rational equivalence classes may be represented by a pair $\xi \in Z^p(X)$, $\eta \in Z^q(X)$ of algebraic cycles which meet properly.

b) The class of $\xi \cdot \eta \in CH^{p+q}(X)$ depends only on the pair (α, β), not on the choice of representative cycles ξ and η. For this it is sufficient to know that if ξ' is rationally equivalent to ξ and also intersects η properly, then $\xi' \cdot \eta$ and $\xi \cdot \eta$ are rationally equivalent.

It follows from the moving lemma, as proved in [Rl] that every smooth, quasi-projective, variety over a field satisfies a) and b).

Let me also mention that there are now other methods for doing intersection theory available; deformation to the normal cone and Bloch's formula (involving higher algebraic K-theory). These methods are frequently more powerful than the moving lemma.

In order to get numerical data from the ring structure on $CH^*(X)$, one uses the degree map $CH^n(X) \longrightarrow \mathbf{Z}$ (for $n = \dim X$), which is defined only if X is projective (or more generally, complete) over a field k. Composing the degree map with intersection product, one has the <u>intersection pairing</u> (for $p+q = \dim X$):

$$\langle\ ,\ \rangle : CH^p(X) \otimes CH^q(X) \longrightarrow \mathbf{Z}.$$

The degree map is defined at the level of 0-cycles by:

$$\deg : Z^d(X) \longrightarrow \mathbf{Z}$$

$$\Sigma\, n_i[P_i] \longrightarrow \Sigma\, n_i[k(P_i) : k].$$

If X is complete, then the degree of a 0-cycle on X depends only on its rational equivalence class. However, if X is not complete this fails to be true; e.g. the cycle $[\{0\}]$ on the affine line $\mathbf{A}^1_k = \mathrm{Spec}(k[x])$ has degree 1 but is the divisor of the rational function x, and is therefore rationally equivalent to zero.

If all we are interested in is the intersection pairing, then the following proposition gives us a direct route to its construction, at the cost of some geometric content.

PROPOSITION. Let Y, Z be two subschemes of codimensions p and q respectively of a complete nonsingular variety X over k. Then

$$\langle [Y],[Z] \rangle = \sum_{i,j=0}^{d} (-1)^{i+j} \dim_k \left(H^i(X, \underline{\mathrm{Tor}}_j^{\mathcal{O}_X}(\mathcal{O}_Y, \mathcal{O}_Z))\right).$$

Note that we do not suppose that Y and Z intersect properly.

PROOF. Let $K_0(X)$ be the Grothendieck group of locally free coherent sheaves of $\mathcal{O}_X$ modules on X; since X is nonsingular this is isomorphic to the Grothendieck group of all coherent sheaves on X. Let $F^pK_0(X)$ be the subgroup generated by classes of coherent sheaves having support of codimension at least p in X. So $[\mathcal{O}_Y] \in F^pK_0(X)$ and $[\mathcal{O}_Z] \in F^qK_0(X)$. Now $K_0(X)$ is a ring, with

$$[\mathcal{M}] \cdot [\mathcal{N}] = \sum_{i=0}^{d} (-1)^i [\underline{\mathrm{Tor}}_i^{\mathcal{O}_X}(\mathcal{M},\mathcal{N})]$$

and admits a trace or Euler characteristic map

$$\chi : K_0(X) \longrightarrow \mathbf{Z}$$
$$\chi : [\mathcal{M}] \longrightarrow \sum_{i=0}^{d} (-1)^i \dim_k(H^i(X,\mathcal{M})).$$

Note that $H^i(X,\mathcal{M})$ is finite dimensional since X is complete. Following [SGA 6] and [Gl], we know that the filtration $F^{\cdot}K_0(X)$ is compatible with the product structure, i.e.,

$$F^pK_0(X) \cdot F^qK_0(X) \subset F^{p+q}K_0(X)$$

hence there is an induced product on the associated graded

$$\mathrm{Gr}^pK_0(X) \otimes \mathrm{Gr}^qK_0(X) \longrightarrow \mathrm{Gr}^{p+q}K_0(X).$$

Also, since $F^{d+1}K_0(X) = 0$, $\mathrm{Gr}^dK_0(X) = F^dK_0(X)$. There are maps for $p \geq 0$

$$\varphi : CH^p \longrightarrow \mathrm{Gr}^p(X)$$
$$[Z] \longrightarrow [\mathcal{O}_Z]$$

which are compatible with both the product structures on domain and codomain (op. cit), and with the degree map:

$$\varphi(\eta \cdot \xi) = \varphi(\eta) \cdot \varphi(\xi)$$
$$\chi(\varphi(\xi)) = \deg(\xi) \ , \ \xi \in CH^d(X).$$

Hence if $\eta = [Y] \in CH^p(X)$, $\xi = [Z] \in CH^q(X)$ for $p+q = \dim(X)$.

$$\begin{aligned} \langle \eta, \xi \rangle &= \deg(\eta \cdot \xi) \\ &= \chi(\varphi(\eta \cdot \xi)) \\ &= \chi(\varphi(\eta) \cdot \varphi(\xi)) \\ &= \sum_{j=0}^{d} (-1)^j \chi(\underline{\mathrm{Tor}}_j^{\mathcal{O}_X}(\mathcal{O}_Y,\mathcal{O}_Z)) \\ &= \sum_{i,j=0}^{d} (-1)^{i+j} \dim_k H^i(X, \underline{\mathrm{Tor}}_j^{\mathcal{O}_X}(\mathcal{O}_Y,\mathcal{O}_Z)). \end{aligned}$$

QED

REMARK: If $k = \mathbf{F}_q$ is a finite field, and V is a finite dimensional k vector space,

$$\dim_k V = \log_q(\#(V)),$$

here $\log_q a = x$ if $q^x = a$. Hence we can rewrite the formula of the proposition as

$$\langle[Y],[Z]\rangle = \sum_{i,j=0}^{d} (-1)^{i+j} \log_q(\#(H^i(X, \underline{\mathrm{Tor}}_j^{\mathcal{O}_X}(\mathcal{O}_Y, \mathcal{O}_Z)))).$$

Note also that the degree map $Z^d(X) \longrightarrow \mathbf{Z}$ can be defined in a similar fashion:

$$\deg(\Sigma\, n_i[P_i]) = \Sigma\, n_i \log_q(\#k(P_i)).$$

2. ARITHMETIC VARIETIES.

Suppose now that X, rather than being a variety over a field is an arithmetic variety, by which I mean a scheme, flat and of finite type over spec $\mathbf{Z}$; $\pi : X \longrightarrow \mathrm{Spec}(\mathbf{Z})$. If X has dimension d, we can view X as a family of $(d-1)$ dimensional varieties X_F, one for each homomorphism from $\mathbf{Z}$ to a field F. In particular for each nonzero prime p in $\mathbf{Z}$ we have the closed fibre $X_p = X_{\mathbf{F}_p}$ over (p), as well as the generic fibre $X_{\mathbf{Q}}$.

If K is a number field, and $\mathcal{O}_K$ the ring of integers in K, we can also consider arithmetic varieties which are schemes flat and of finite type over $\mathrm{Spec}(\mathcal{O}_K)$.

If we want to do intersection theory on a nonsingular (= regular) arithmetic variety, we run into two problems. The first is that, as yet, no one has succeeded in making $CH^*(X)$ into a ring unless X is a surface or is smooth over $\mathrm{Spec}(\mathcal{O}_K)$ for $\mathcal{O}_K$ the ring of integers in a number field. However one can make $CH^*(X)_{\mathbf{Q}}(= CH^*(X) \otimes_{\mathbf{Z}} \mathbf{Q})$ into a ring ([G-S 2]). The second, and more serious, obstacle is to construct an appropriate degree map. Since an arithmetic variety does not have a ground field we could start with the remark at the end of section 1. and define by anology, a homomorphism, for $d = \dim(X)$:

$$\deg : Z^d(X) \longrightarrow \mathbf{R}$$
$$\Sigma\, n_i[P_i] \longrightarrow \Sigma\, n_i \log(\#k(P_i)) .$$

However this homomorphism is not compatible with rational equivalence. To remedy this we must introduce cycles (at ∞), an idea which we will discuss in the next 4 sections.

3. RINGS OF ALGEBRAIC INTEGERS.

The lack of a degree map is first apparent for $X = \mathrm{Spec}(\mathbf{Z})$ itself. A codimension one cycle on $\mathrm{Spec}(\mathbf{Z})$ is a finite formal sum

$$\mathfrak{z} = \Sigma\, n_p[p]$$

and

$$\deg(\mathfrak{z}) = \Sigma\, n_p \log(p) = \log(\Pi p^{n_p}).$$

Since $\mathbf{Z}$ is a P.I.D. any such cycle is the divisor of a 'function' in $\mathbf{Q}$, namely $f = \Pi p^{n_p}$. Following A. Weil, this defect is remedied by adding a point "at ∞" to $\mathrm{Spec}(\mathbf{Z})$. More generally, if K is a number field let $X = \mathrm{Spec}(\mathcal{O}_K)$, let $\bar{X} = X \cup \{v_1, \ldots, v_{r_1+r_2}\}$ where each v_i is a complex conjugacy class of embeddings $K \subset \mathbf{C}$. Let $Z^1(\bar{X}) = Z^1(X) \oplus \mathbf{R}^{r_1+r_2}$ where we take $\mathbf{R}^{r_1+r_2}$ to have basis $v_1, \ldots, v_{r_1+r_2}$ and if $f \in K^*$ define $\overline{\mathrm{div}}(f) \in Z^1(\bar{X})$ to be

$$\mathrm{div}(f) \oplus \sum_{i=1}^{r_1+r_2} \alpha_i[v_i]$$

where

$$\alpha_i = -\log|v_i(f)|.$$

Let $CH^1(\bar{X})$ be the quotient of $Z^1(\bar{X})$ by the subgroup of divisors of the form $\overline{\mathrm{div}}(f)$. Now, define the degree map

$$\overline{\deg} : Z^1(\bar{X}) \longrightarrow \mathbf{R}$$

by $\overline{\deg}(\mathfrak{z} \oplus \Sigma\, \alpha_i[v_i]) = \deg(\mathfrak{z}) + \Sigma\, \alpha_i \deg[v_i]$ for $\mathfrak{z} \in Z^1(X)$, where $\deg([v_i]) = 1$ or 2 depending on whether v_i is real or complex.

LEMMA. $\overline{\deg}$ factors through $CH^1(\bar{X})$.

PROOF. This is just a fancy way of stating the product formula c.f. [We].

A more familiar description of $CH^1(\bar{X})$, at least for number theorists, is given by

PROPOSITION. Let $\mathcal{O}_K$ be the ring of integers in a number field K. If $X = \mathrm{Spec}(\mathcal{O}_K)$, then

i) $CH^1(\bar{X}) \approx K^* \backslash GL_1(\mathbf{A}_k) / U$ where $\mathbf{A}_K$ is the ring of adeles of K, so $GL_1(\mathbf{A}_K)$ is the group of ideles, U is the maximal compact subgroup of the ideles and K^* is embedded into the ideles via the diagonal map.

ii) There is an exact sequence:

$$\mathcal{O}_K^* \xrightarrow{\rho} \mathbf{R}^{r_1+r_2} \longrightarrow CH^1(\bar{X}) \longrightarrow Cl(\mathcal{O}_K) \longrightarrow 0$$

where ρ is the classical regulator map and $Cl(\mathcal{O}_K)$ is the ideal class group of K.

iii) The kernel $CH^1(\bar{X})_0$ of the degree map $CH^1(\bar{X}) \longrightarrow \mathbf{R}$ is a compact Lie group.

PROOF. i) First observe that if K is a local field, and G is the maximal compact subgroup of K^*, then G/K^* is isomorphic to $\mathbf{Z}$ if K is non-archimedian and to $\mathbf{R}$ if K is archimedian. Hence for K global, $GL_1(\mathbf{A}_K)/U$ is isomorphic to the group of cycles on $\overline{\mathrm{Spec}(\mathcal{O}_K)}$, and under this isomorphism the image of K^* is the group of cycles rationally equivalent to zero.

ii) Noting that $CH^1(\mathrm{Spec}(\mathcal{O}_K))$ is just fancy notation for the ideal class group $Cl(\mathcal{O}_K)$, and that

$$\mathcal{O}_K^* = \mathrm{Ker}(\mathrm{div} : K^* \longrightarrow Z^1(\mathrm{Spec}(\mathcal{O}_K))),$$

the result is an elementary computation.

iii) From ii) one sees that there is an exact sequence

$$\mathcal{O}_K^* \longrightarrow \mathbf{R}^{r_1+r_2} \longrightarrow CH^1(\bar{X}) \longrightarrow CH^1(X) \longrightarrow 0.$$

Now observe that $CH^1(X) = Cl(\mathcal{O}_K)$ is finite and that the image of the regulator is a discrete cocompact subgroup of $\mathbf{R}^{r_1+r_2}$ (see [L] for example).

The group $CH^1(\bar{X})$ is naturally isomorphic to $\mathrm{Pic}(\bar{X})$, the group of rank 1 projective $\mathcal{O}_K$ modules L, equipped with Hermitian metrics on $L_v = L \otimes_{\mathcal{O}_K} \mathbf{C}$ for each $v:K \longrightarrow \mathbf{C}$ which are preserved under the σ-linear isomorphism $L_v \approx L_{\sigma v}$ where $\sigma : \mathbf{C} \longrightarrow \mathbf{C}$ is complex conjugation.

This leads naturally to defining a <u>vector bundle</u> on $\bar{X}$ as a projective $\mathcal{O}_K$ module P together with a hermitian metric on

$P \otimes_{\mathbb{Z}} \mathbb{C}$, invariant under complex conjugation (note that this means that the corresponding fundamental form is anti-invariant). The set of isomorphism classes of rank n vector bundles on $\overline{\mathrm{Spec}}(\mathcal{O}_K)$ is the double coset space

$$GL_n(K) \setminus GL_n(\mathbb{A}_K) / U$$

where U is the maximal compact subgroup of $GL_n(\mathbb{A}_K)$.

4. GREEN'S CURRENTS.

Suppose that X is an equidimensional compact complex manifold. The space $A^{p,q}(X)$ of C^{∞} complex valued differential forms on X of type (p,q) has a natural topology ([De R], §9). The topological dual of this space is the space $\mathcal{D}_{p,q}$ of complex valued currents of cotype (p,q). If $d = \dim X$, there is a natural injective map

$$\begin{aligned} A^{d-p,d-q}(X) &\longrightarrow \mathcal{D}_{p,q}(X) \\ \alpha &\longrightarrow \left\{\beta \longrightarrow \int_X \alpha \wedge \beta\right\} \end{aligned}$$

with dense image. We therefore write $\mathcal{D}_{p,q}(X) = \mathcal{D}^{d-p,d-q}(X)$, thinking of elements of this space as differential forms of type $(d-p,d-q)$ with coefficients which are distributions in the sense of Schwartz ([Sch]). The operators ∂, $\bar{\partial}$ on forms then extend to currents, and one can compute the cohomology of X with currents instead of forms.

If $Y \subset X$ is a closed codimension p analytic subspace, then one can associate a current $\delta_Y \in \mathcal{D}^{p,p}(X)$ to Y by

$$\delta_Y : \alpha \longrightarrow \int_Y \alpha|_Y \qquad \alpha \in A^{d-p,d-p}(X).$$

The integral makes sense since Y, even though it may be singular, has a triangulation. Note that δ_Y is a real current in the sense it takes real values on forms with real coefficients. If β is a form, then $\beta \wedge \delta_Y$ is defined:

$$\beta \wedge \delta_Y : \alpha \longrightarrow \int_Y (\beta \wedge \alpha)|_Y.$$

In particular, $\beta \wedge \delta_X = \beta$ (viewed as a current).

If X is provided with a Kahler metric h, recall that the subspace $\mathcal{H}^{p,q}(X) \subset A^{p,q}(X)$ consisting of harmonic (p,q) forms is canonically isomorphic to $H^{p,q}(X,C)$ and that there is a projection operator $H : \mathcal{A}^{p,q}(X) \longrightarrow \mathcal{H}^{p,q}(X)$. If $Y \subset X$ is a closed analytic subset, then $H(\delta_Y)$ is the harmonic representative of the fundamental class of Y in $H^{p,p}(X,\mathbf{R})$. (The real coefficients are possible because δ_Y and $H(\delta_Y)$ are both real).

Suppose that $\gamma \in \mathcal{A}^n(X)$, ($\mathcal{A}^n(X) = \bigoplus_{p+q=n} \mathcal{A}^{p,q}(X)$) is a current which is closed (i.e., $\partial\gamma = 0$ and $\bar{\partial}\gamma = 0$) and cohomologous to zero (i.e., $\exists\beta$ such that $\gamma = (\partial + \bar{\partial})\beta$). Then (see [G-H], p.149) there exists a solution, in $\mathcal{A}^{n-2}(X)$, of the equation

$$\partial\bar{\partial}u = \gamma.$$

The difference of two such solutions u', u'' may be written:

$$u' - u'' = \partial\alpha + \bar{\partial}\beta + \omega$$

where $\alpha, \beta \in \mathcal{A}^{n-3}(X)$ and ω is a harmonic form. Hence if we require that u be antiharmonic, i.e., $Hu = 0$, then u is unique up to addition of terms the form $\partial\alpha + \bar{\partial}\beta$.

DEFINITION. If $Y \subset X$ is a closed analytic subset of codimension p in the Kahler manifold (X,h), an <u>admissible Green's current</u> for Y is an element $g_Y \in \mathcal{A}^{p-1,p-1}(X)$ which is an antiharmonic (i.e., $H(g_Y) = 0$) solution of

$$\frac{1}{\pi i}\,\partial\bar{\partial}u = \delta_Y - H(\delta_Y).$$

Note that any two admissible Green's currents for Y differ by a current of the form $\partial\alpha + \bar{\partial}\beta$. Also since $\frac{1}{\pi i}\,\partial\bar{\partial}$ is a real operator, and δ_Y and $H(\delta_Y)$ are both real, g_Y can be chosen to be real (see [G-H], p.149). If we do not require g_Y to be antiharmonic, we shall refer to it as simply a Green's current for Y. Note that any Green's current differs from an admissible one by a harmonic form.

LEMMA. If Y, X are as above, there exist admissible Green's currents g_Y of Y which are C^{∞} outside of Y.

PROOF. Use the construction of the Green's current given in [G-H], p.149, together with the fact that the Green's operator associated to a Kahler metric has kernel which is C^{∞} off the diagonal, see [De R], (Th. 23).

5. THE CHOW GROUPS OF AN ARAKELOV VARIETY.

Now let $\pi : X \longrightarrow \mathbf{Z}$ be a regular arithmetic variety which we suppose has complete generic fibre $X_{\mathbf{Q}}$. If X has dimension d, then $X(\mathbf{C})$ is a complex manifold of dimension $(d-1)$ equipped with an anti-holomorphic (and hence orientation reversing) involution F_{∞} induced by complex conjugation.

DEFINITION. i) An (Arakelov) compactification of X consists of a choice of Kahler metric h on $X(\mathbf{C})$ which is invariant under F_{∞}, or equivalently, such that the associated real Kahler form ω, is anti-invariant: $F_{\infty}^{*}\omega = -\omega$. We write $\bar{X} = (X, \omega)$.

ii) A codimension p cycle on $\bar{X} = (X, \omega)$ consists of a pair (ξ,α) where ξ is a codimension p cycle on X and α is a harmonic form which is real, of type $(p-1,p-1)$ and such that $F_{\infty}^{*}\alpha = (-1)^{p-1}\alpha$. We shall write $\mathcal{H}^{p-1,p-1}(X_{\mathbf{R}})$ for the space of such forms.

Suppose that $W \subset X$ is an integral (i.e., reduced and irreducible) subscheme of codimension $(p-1)$ and let $f \in k(W)^{*}$ be a non-zero rational function on W. Then the divisor $\mathrm{div}(f)$ of f is a codimension p cycle on X, see [Fu], ch.1. The set of complex points $W(\mathbf{C})$ of W is either empty or a codimension $(p-1)$ analytic subset of $X(\mathbf{C})$, in the latter case $\log|f|$ is an L^{1} function on the nonsingular locus of $W(\mathbf{C})$, hence defines a current $i_{*}\log|f|$ on $X(\mathbf{C})$. Let $H(i_{*}\log|f|)$ be the $(p-1,p-1)$ component of the harmonic projection of this current. (Of course this depends on the choice of Kahler metric, see [de R]). Observe that $H(i_{*}\log|f|) \in \mathcal{H}^{p-1,p-1}(X_{\mathbf{R}})$; the key point is that if we write

$$[X] \cap : \mathcal{A}^{n}(X(C)) \longrightarrow \mathcal{D}_{2d-n}(X(C))$$

for the natural map, then (since $F_{\infty}^{2} = 1$)

$$F_{\infty *}([X] \cap \alpha) = (-1)^{d}[X] \cap F_{\infty}^{*}\alpha.$$

Now we can extend the notion of divisor to the Arakelov setting.

DEFINITION. i) For $f \in k(W)^*$, $W \subset X$ an integral codimension $(p-1)$ subscheme, set

$$\widehat{\mathrm{div}}(f) = (\mathrm{div}(f), -H(i_*\log|f|))$$

in $Z^p(\bar{X})$. Here $H(i_*\log|f|)$ is understood to be zero if $W(\mathbf{C}) = \emptyset$.

ii) Define $CH^p(\bar{X})$ to be the quotient of $Z^p(\bar{X})$ by the subgroup generated by all cycles of the form $\widehat{\mathrm{div}}(f)$.

Before stating the next theorem, I want to describe the 'K_1-type' Chow groups $CH^{p,p-1}(X)$. If X is a noetherian separated scheme, then $CH^p(X)$ can be described as the cokernel of the map

$$d : \bigoplus_{w\in X^{(p-1)}} k(w)^* \longrightarrow \bigoplus_{x\in X^{(p)}} \mathbf{Z}$$

where $X^{(i)}$ is the set of points $x \in X$ such that $\{\bar{x}\}$ has codimension i in X (i.e., the Krull dimension of $\mathcal{O}_{X,x}$ is i). This homomorphism forms part of a complex ([Q], §7) with $Z^p(X) = \bigoplus_{x\in X^{(p)}} \mathbf{Z}$ in degree p:

$$\cdots \longrightarrow \bigoplus_{x\in X^{(p-2)}} K_2k(x) \xrightarrow{d^{p-1}} \bigoplus_{w\in X^{(p-1)}} k(w)^* \xrightarrow{d^p} Z^p(X) \longrightarrow 0$$

where $K_2k(x)$ is the group defined by Milnor ([Mi]) and the map d^{p-1} is essentially a sum of Tame symbols ([Q], p.98). We write

$$CH^{p,p-1}(X) = \mathrm{Ker}(d^p)/\mathrm{Im}(d^{p-1}).$$

Up to torsion, $CH^{p-1,p}(X)$ is a subquotient of $K_1(X)$, just as $CH^p(X)$ is a subquotient of $K_0(X)$ ([So]). We can now describe the relationship between $CH^p(\bar{X})$ and $CH^p(X)$.

THEOREM. Let X be a regular arithmetic scheme with generic fibre projective over $\mathrm{Spec}(\mathbf{Q})$. For $p \geq 0$, there is an exact sequence

$$CH^{p-1,p}(X) \xrightarrow{\rho} H^{p-1,p-1}(X_{\mathbf{R}}) \xrightarrow{i} CH^p(\bar{X}) \xrightarrow{\epsilon} CH^p(X) \longrightarrow 0.$$

PROOF. (Sketch) ϵ is the forgetful map, $\epsilon(\xi,\omega) = \xi$, while $i(\omega) = (0,\omega)$. It is easy to see that $\mathrm{Ker}\ \epsilon = \mathrm{Im}\ i$, while the kernel of i is the image of the map

$$\tilde{\rho} : \mathrm{Ker}(d^p) \longrightarrow \mathcal{H}^{p-1,p-1}(X_{\mathbf{R}})$$

$$\sum_\alpha \{f_\alpha\} \longrightarrow \sum_\alpha - \mathcal{H}(j_{\alpha *} \log|f_\alpha|)$$

where $f_\alpha \in k(w_\alpha)$, w_α being the generic point of W_α, and $j_\alpha : W_\alpha \longrightarrow X$. We need to show that $\tilde{\rho}$ vanishes on the image of $d^{p-1}(\alpha)$ for $\alpha \in K_2(k(Z))$ where $Z \subset X$ is a codimension $(p-2)$ integral subscheme. It suffices to show therefore that for $f, g \in k(Z)^*$,

$$\tilde{\rho}(d^{p-1}(\{f,g\})) = 0.$$

One may reduce to the case where f and g have no divisors in common, in which case one may show that $\tilde{\rho}(d^{p-1}(\{f,g\}))$ is the harmonic projection of the current

$$j_*(\log|f|\delta_{\mathrm{div}(g)} - \log|g|\delta_{\mathrm{div}(f)})$$

(here $j : Z(\mathbf{C}) \longrightarrow X(\mathbf{C})$ is the natural map) which is zero by Stoke's theorem.

6. THE INTERSECTION PAIRING.

I now want to sketch the construction, for a nonsingular projective Arakelov variety $\bar{X} = (X, \omega)$ of dimension d, of the pairing

$$< , > : CH^p(\bar{X}) \otimes CH^{d-p}(\bar{X}) \longrightarrow \mathbf{R}.$$

First, observe that any integral subscheme $Z \subset X$ is either 'horizontal' i.e., flat over Spec $\mathbf{Z}$, or vertical, i.e., contained in a closed fibre. Hence we may decompose $Z^p(X)$ as a direct sum $Z^p(X)_h \oplus Z^p(X)_v$, so that every cycle ξ is written uniquely as the sum $\xi_h + \xi_v$ of a horizontal cycle and a vertical cycle.

Since the generic fibre $X_{\mathbf{Q}}$ of X is a smooth projective variety over a field, the moving lemma holds for cycles on $X_{\mathbf{Q}}$, hence if ξ, η are cycles on X, ξ is rationally equivalent to a cycle ξ', such that ξ'_h and η_h meet properly on $X_{\mathbf{Q}}$. In particular if ξ and η are cycles of complementary codimension on X, ξ is rationally equivalent to a cycle ξ' such that the intersection of ξ' and η is supported on a finite number of closed fibres. Now suppose that (ξ,α) and (η,β) are cycles in $Z^p(\bar{X})$; $Z^q(\bar{X})$ respectively, with $p+q = d$, such that ξ_h and η_h do not intersect on the generic fibre. Then we define the intersection $\langle(\xi,\alpha), (\eta,\beta)\rangle$ as the sum of four terms:

$$\langle(\xi,\alpha),(\eta,\beta)\rangle = \langle\xi,\eta\rangle_f + \langle\xi,\eta\rangle_\infty + \langle\xi,\beta\rangle + \langle\alpha,\eta\rangle.$$

Note that we set $\langle\alpha,\beta\rangle = 0$.

We now discuss each term separately:

First, $\langle\xi,\eta\rangle_f$. Since ξ_h and η_h have empty intersection $X_{\mathbf{Q}}$, ξ and η can be written as sums $\Sigma\, n_i[Z_i]$ and $\Sigma\, m_j[Y_j]$ where Z_i and Y_j are integral subschemes of X, with $Z_i \cap Y_j$ contained in a finite number of closed fibres. We then write

$$\langle\xi,\eta\rangle = \Sigma\, n_i m_j \langle Z_i, Y_j\rangle.$$

Now suppose $\xi = [Z]$, $\eta = [Y]$ with $Z \cap Y$ as above. Then because X is projective over $\mathrm{Spec}(\mathbf{Z})$ and $Z \cap Y$ is contained in a finite set of closed fibres,

$$H^i(X, \underline{\mathrm{Tor}}_j^{\mathcal{O}_X}(\mathcal{O}_Z, \mathcal{O}_Y))$$

is a finite abelian group, and is zero for i or $j > d$. Now set

$$\langle Z,Y\rangle_f = \sum_{i,j} (-1)^{i+j} \log(\# H^i(X, \underline{\mathrm{Tor}}_j^{\mathcal{O}_X}(\mathcal{O}_Z, \mathcal{O}_Y))).$$

Next, observe that ξ restricts to $X(\mathbf{C})$ to give a codimension p chain, thus an element of $H^{2p}(X(\mathbf{C}),\mathbf{Z})$, while β defines a class in $H^{2d-2p-2}(X(\mathbf{C}),\mathbf{R})$. Since $X(\mathbf{C})$ is 2d-2 dimensional as a manifold, we can intersect these two classes to get a number, which we call $\langle\xi,\beta\rangle$. We define $\langle\alpha,\eta\rangle$ analogously. Finally, we turn to the $\langle\xi,\eta\rangle_\infty$ term. This only depends on the restrictions of ξ and η to $X(\mathbf{C})$, hence on ξ_h and η_h. Writing $\xi_h = \Sigma\, n_i[Z_i]$, $\eta_h = \Sigma\, m_j[Y_j]$, then

$$\langle\xi,\eta\rangle_\infty = \Sigma\, n_i m_j \langle Z_i, Y_j\rangle_\infty$$

and we define

$$\langle Z_i, Y_j\rangle_\infty = -\int_{Z_i(\mathbf{C})} g_{Y_j} \; .$$

Notice that since $Z_i(C) \cap Y_j(C) = \emptyset$, g_{Y_j} is C^∞ on $Z_j(C)$. (see §4).

THEOREM. The intersection number described above induces a pairing, for $p \geq 0$,

$$CH^p(\bar{X}) \otimes CH^{d-p}(\bar{X}) \longrightarrow \mathbf{R}$$

if $\bar{X} = (X, \omega)$ is an equidimensional nonsingular projective Arakelov variety of dimension d.

The key point is to show that the pairing respects rational equivalence. This involves, in part, a special case of Serre's conjecture on intersection multiplicities, see [G-S 1], [G-S 2], [Se], [R2].

It is natural to ask two questions. The first is, to what extent does this pairing depend on the choice of metric? Suppose that (X, ω_1) and (X, ω_2) are two different Arakelov compactifications of the same arithmetic variety (which we suppose satisfies the assumptions of the theorem above). Then there is a canonical isomorphism:

$$\theta : CH^*(X, \omega_1) \xrightarrow{\sim} CH^*(X, \omega_2)$$

$$\theta : (Z, \alpha) \longrightarrow (Z, H_2(\alpha) + \tfrac{1}{2} H_2(H_1(g_Z^2) - g_Z^1)).$$

Here g_Z^i are the Green's currents of Z with respect to ω_i for $i = 1,2$, while H_i are the corresponding harmonic projections. Note that while θ preserves the exact sequence of §5, it mixes the 'finite' and 'at infinity' parts of $CH^*(\bar{X})$.

PROPOSITION. If X satisfies the conditions of the theorem, and ω_1, ω_2 are two Kahler metrics on $X(C)$, then the isomorphism

$$\theta : CH^*(X, \omega_1) \longrightarrow CH^*(X, \omega_2)$$

preserves the intersection pairing of the theorem.

PROOF. A manipulation using Stoke's theorem.

If we let $CH_c^*(X)$ be the kernel of the map

$$CH^*(\bar{X}) \longrightarrow H^*(X(C))$$

$$(Z, h) \longrightarrow H(Z),$$

then we get a pairing, for $p + q = d$,

$$CH^p_c(X) \otimes CH^q_c(X) \longrightarrow \mathbf{R}$$

which factors through a pairing

$$A^p(X) \otimes A^q(X) \longrightarrow \mathbf{R}$$

where $A^p(X) = CH^p_c(X)/H^{p-1,p-1}(X_{\mathbf{R}})$ is the subgroup of $CH^p(X)$ consisting of cycles homologically equivalent to zero, and this pairing does not depend on the choice of Kahler form ω. This pairing is the Beilinson height pairing, which generalizes to all the Chow groups the Neron height pairing (the case $p=1$, $q=d-1$). See [Be].

The second question regarding $CH^*(\bar{X})$ is whether it has a ring structure. The obvious product fails to be associative because a wedge product of harmonic forms is not in general harmonic. However if X is a hermitian symmetric space (i.e., a product of complex Grassmannians) equipped with an invariant metric, the harmonic forms are the invariant forms and form a subring of $A^*(X)$. In that case one can show that $CH^*(\bar{X})$ is an associative ring.

7. VECTOR BUNDLES AND CHARACTERISTIC CLASSES.

DEFINITION. If $\bar{X} = (X,\omega)$ is an Arakelov variety, $\mathrm{Pic}(\bar{X})$ is the group of (isometric) isomorphism classes of hermitian line bundles on X having harmonic curvature. (Such an $(L, \|\ \|)$ is said to be admissible). The following generalizes the result of Arakelov [Ar] for surfaces.

PROPOSITION. Let $\bar{X} = (X,\omega)$ be a projective nonsingular Arakelov variety. Then there is an isomorphism

$$C_1 : \mathrm{Pic}(\bar{X}) \longrightarrow CH^1(\bar{X}).$$

PROOF. If $(L, \|\ \|)$ is an admissible hermitian line bundle, and $s \in \Gamma(X,L)$ then we define $\overline{\mathrm{div}}(s) = (\mathrm{div}(s), -H(\log\|s\|)) \in Z^1(\bar{X})$. If $s' \in \Gamma(X,L)$, is another section, $s' = fs$ for $f \in k(X)^*$, and

$$\overline{\mathrm{div}}(s') = \overline{\mathrm{div}}(f) + \overline{\mathrm{div}}(s).$$

Hence $(L, \|\ \|) \longrightarrow \overline{\mathrm{div}}(s)$ gives a well defined map $\mathrm{Pic}(\bar{X}) \longrightarrow CH^1(\bar{X})$. It is easy to see that C_1 is an isomorphism.

This result is the starting point for a theory of Chern classes. One approach might be to have a definition of admissible hermitian vector bundle $(E, \| \ \|)$ which would have Chern classes $C_i(E, \| \ \|) \in CH^i(\bar{X})$. See Manin's artcile [Ma] for some discussion of what admissibility might mean. However, the problems with $CH^*(\bar{X})$ discussed at the end of §6, make it difficult to see how such a theory might work.

A different approach is suggested by the following result of Deligne. First define $\hat{Pic}(X)$ to be the group of isomorphism classes of hermitian line bundles, with no requirement of admissibility; note $Pic(X,\omega) \subset \hat{Pic}(X)$ for any ω.

THEOREM (Deligne, [De]). Let X be a regular proper arithmetic surface. There is a pairing

$$\hat{Pic}(X) \otimes \hat{Pic}(X) \longrightarrow \mathbf{R}$$

which for any choice of $\bar{X} = (X,\omega)$ restricts to Arakelov's pairing on $Pic(\bar{X})$.

DEFINITION. i) If X is a regular projective arithmetic variety $\hat{Z}^p(X)$ is the subspace of $Z^p(X) \oplus \mathcal{A}^{p-1,p-1}(X_{\mathbf{R}})/(\mathrm{Im}\,\partial + \mathrm{Im}\,\bar{\partial})$ consisting of pairs (ξ, g) such that

$$\frac{1}{\pi i}\partial\bar{\partial}g = \delta_{\xi} - \omega$$

for ω some C^{∞} form (necessarily real and closed of type (p,p)).

ii) If $W \subset X$ is an integral subscheme and $f \in k(W)^*$, then set

$$\hat{div}(f) = (div(f),\ i_*\log|f|)$$

where $i: W(C) \longrightarrow X(C)$ is the inclusion.

iii) Define $\hat{CH}^p(X)$ to be $\hat{Z}^p(X)$ modulo the subgroup generated by elements $\hat{div}(f)$.

PROPOSITION. If X is a regular projective arithmetic variety, there is an exact sequence

$$CH^{p,p-1}(X) \xrightarrow{\rho} A^{p-1,p-1}(X_{\mathbf{R}})/\mathrm{Im}(\partial,\bar{\partial}) \longrightarrow \hat{CH}^p(X) \longrightarrow \hat{CH}^p(X) \longrightarrow 0$$

where ρ is the regulator of Beilinson ([Be]); note ρ has image in

$$H^{p-1,p-1}(X_{\mathbb{R}}) \subset A^{p-1,p-1}(X_{\mathbb{R}})/\mathrm{Im}(\partial,\bar{\partial}).$$

PROOF. Same argument as in §5.

While these groups are larger than the $CH^*(\bar{X})$, they have the great advantage:

THEOREM. Let X be a regular projective arithmetic variety. Then

i) $\hat{CH}^*(X)_{\mathbb{Q}}$ is an associative ring.

ii) If $f:Y \longrightarrow X$ is a morphism between such varieties, there is a pullback map:

$$f^* : \hat{CH}^*(X)_{\mathbb{Q}} \longrightarrow \hat{CH}^*(Y)_{\mathbb{Q}}$$

and $(f\cdot g)^* = g^*f^*$ when defined.

iii) If X is smooth over the ring of integers in a number field, then f^* and the ring structure are defined without tensoring with $\mathbb{Q}$.

iv) If $f:X \longrightarrow Y$ is proper, there is a natural map $f_*:\hat{CH}^*(X) \longrightarrow \hat{CH}^*(Y)$ so that f_* and f^* satisfy the projection formula.

PROOF. The basic idea for i) is to define

$$(\xi,g)(\eta,h) = (\xi\cdot h,\ \delta_\xi h + g\omega)$$

where $\frac{1}{\pi i}\partial\bar{\partial}h = \delta_h - \omega$.

These groups $\hat{CH}^*$ are a plausible target for Chern classes. For if $(E, \|\ \|)$ is a hermitian vector bundle, and $s_i \in \Gamma(X,E)$ for $i = 1,\ldots,n$ are n sufficiently general sections then $c_k(E)$ has two types of representatives: C^∞ forms ω_k obtained from the curvature of the canonical unitary connection of E, and cycles σ_k defined by the degeneracy of the sections $\{s_i\}$ of E. Now δ_{σ_k} and ω_k are homologous currents, so we can ask if there is an canonical solution (modulo $\mathrm{Im}\,\partial + \mathrm{Im}\,\bar{\partial}$) of

$$\frac{1}{\pi i}\partial\bar{\partial}u = \delta_{\sigma_k} - \omega_k$$

and hence a class $(\sigma_k,u) \in \hat{CH}^k(X)$. This is by analogy with the existence of a canonical solution of

$$\frac{1}{\pi i}\partial\bar{\partial}u = \omega_k(E,\ \|\ \|) - \omega_k(E,\ \|\ \|')$$

for two different hermitian metrics on $\| \ \|$, see ([B-C]).

If $(L, \| \ \|)$ is a hermitian line bundle, $s \in \Gamma(X,L)$ and $\omega = \mathrm{curv}(L, \| \ \|)$, we know that

$$\frac{1}{\pi i} \log\|s\| = \delta_{\mathrm{div}(s)} - \omega$$

so it is reasonable to set

$$\hat{c}_1(L, \| \ \|) = (\mathrm{div}(s), \log\|s\|) \in \hat{CH}^1(X).$$

It is also reasonable to require that if the Whitney sum formula hold for orthogonal direct sums. The surprise is:

THEOREM. On the category of regular quasi-projective arithmetic varieties having projective generic fibre there exists a unique theory of Chern classes for hermitian vector bundles:

$$\hat{c}_k(E, \| \ \|) \in \hat{CH}^k(X)$$
$$\hat{c}\,.\,(E, \| \ \|) = \sum_{k \geq 0} \hat{c}_k(E, \| \ \|)$$

for $(E, \| \ \|)$ a hermitian bundle on X (i.e., E is a vector bundle on X, equipped with a hermitian metric over $X(\mathbb{C})$ invariant under F_∞) with the following properties.

i) Functoriality

If $f: Y \longrightarrow X$, then

$$\hat{c}.(f^*(E, \| \ \|)) = f^*\hat{c}.(E, \| \ \|).$$

ii) Normalization. If $(L, \| \ \|)$ is a hermitian line bundle, $\hat{c}_1(L, \| \ \|)$ is the class previously defined.

iii) Whitney sum formula. If $\oplus^\perp$ denotes orthogonal direcy sum:

$$\hat{c}.((E, \| \ \|) \oplus^\perp (F, \| \ \|)) = \hat{c}.(E, \| \ \|)\,\hat{c}.(F, \| \ \|).$$

iv) Twisting. If $(E, \| \ \|)$ is a hermitian vector bundle, and $(L, \| \ \|)$ a hermitian line bundle, then

$$\hat{c}.(E \otimes L, \| \ \| \otimes \| \ \|) = \hat{c}.(E, \| \ \|) * \hat{c}.(L, \| \ \|)$$

where $*$ is the product defined in [SGA 6], Exp. 0.1.16.

Notice that if a vector bundle E is equipped with two metrics $\| \ \|$ and $\| \ \|'$ then

$$\hat{c}_k(E, \|\ \|) - \hat{c}_k(E, \|\ \|') = (0, \beta)$$

where $\beta \in A^{p-1,p-1}/\mathrm{Im}(\partial, \bar{\partial})$, i.e., we obtain a solution for the equation

$$\frac{1}{\pi i} \partial\bar{\partial}\beta = \omega_k(E, \|\ \|) - \omega_k(E, \|\ \|')$$

describing the fact that two Chern forms for the same bundle are cohomologous. A canonical solution for this equation was originally described by Bott and Chern ([B-C]), and the two solutions coincide.

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DIOPHANTINE PROBLEMS IN COMPLEX HYPERBOLIC ANALYSIS

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ABSTRACT. We discuss two results concerning holomorphic maps into hyperbolic complex manifolds, or hyperbolically imbedded manifolds. One is an extension of a theorem of Noguchi, dealing with the uniform convergence of certain sequences of mappings; the other is a theorem of Cartan, the Second Main Theorem of Nevanlinna theory in the context of maps into projective space and the complement of certain hyperplanes. We improve Cartan's theorem by giving a better form to the ramification term.

0. INTRODUCTION. After Manin's proof of the Mordell conjecture in the function field case [Ma], Grauert not only gave another proof, but also gave an idea for still another over the complex numbers, which developed from Grauert-Reckziegel [G-R] to Riebesehl's recent paper [Ri].

On the other hand, in [L 2], partly motivated by similar considerations (like [Ko], Theorem 3.2 of Chapter V), I conjectured that a hyperbolic variety is mordellic, i.e. has only a finite number of rational points in every finitely generated field over the rationals. "Hyperbolic" here means Kobayashi hyperbolic [Ko]. As usual, the conjecture has a function field analogue. Noguchi has written recently a series of papers dealing with the function field case, under various hypotheses, and especially in the last one he pushes Grauert's idea much further, in the context of Kobayashi hyperbolicity.

I first review some definitions concerning hyperbolic spaces, and then prove an extension of Noguchi's theorem. I discuss afterward more precisely (and technically) in what ways it is an extension, and I also indicate how it is related to the Mordell conjecture. This is of interest to arithmeticians, but the extension also takes place in the analytic context of a theorem of Kwack, which we shall also recall.

In the second part of the paper I state an extension of a theorem of

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Cartan [Ca] having to do with diophantine questions in the context of Nevanlinna theory. Vojta [Vo] had the great insight to draw an analogy between the Second Main Theorem of this theory and the theory of heights, Roth's theorem, Schmidt's theorem, and the Mordell conjecture as well as other diophantine questions of algebraic number theory. I shall indicate briefly how the extension of Cartan's theorem fits in Vojta's program.

1. HYPERBOLICITY AND NOGUCHI'S THEOREM. Let X be a complex space and let $\mathbb{D}$ be the unit disc with its hyperbolic metric. The Kobayashi semi distance d_X is defined as follows. Given $x, y \in X$ we consider finite sequences of holomorphic maps

$$f_i \colon \mathbb{D} \longrightarrow X, \quad (i = 1,\dots,m),$$

points p_i, q_i in $\mathbb{D}$ such that $f_i(q_i) = f_{i+1}(p_i)$, and $f_1(p_1) = x$, $f_m(q_m) = y$. In other words, we join x to y by a chain of holomorphic discs. We let $d_{\mathbb{D}}$ be the hyperbolic distance on $\mathbb{D}$. We define the Kobayashi semi distance to be

$$d_X(x,y) = \inf \sum_{i=1}^{m} d_{\mathbb{D}}(p_i,q_i)$$

where the inf is taken for all choices of such chains of discs joining x to y. If for instance $X = \mathbb{C}$ then $d_X(x,y) = 0$ for all x, y. It is immediate that holomorphic maps are Kobayashi semi distance decreasing. Furthermore, if $X = \mathbb{D}$ then the Kobayashi semi distance is the usual hyperbolic (Poincaré-Lobatchevsky) distance.

We say that X is Kobayashi hyperbolic if d_X is a distance, that is $d_X(x,y) > 0$ for $x \neq y$.

Let X be a complex subspace of Y. We say that X is hyperbolically imbedded in Y if X is hyperbolic, and given two sequences $\{x_n\}$, $\{y_n\}$ in X converging to points x, y in $\bar{X}$ respectively, if $d_X(x_n,y_n) \longrightarrow 0$ then $x = y$.

Let Z be a complex manifold. By a length function H on Z (or on its tangent bundle TZ) we mean a function

$$H \colon TZ \longrightarrow \mathbb{R}_{\geqq 0}$$

such that $H(v) > 0$ for all $v \in TZ$, $v \neq 0$, and

$$H(cv) = |c|\, H(v) \quad \text{for } c \in \mathbb{C}.$$

Finally we require that H be continuous. The norm associated with a hermitian metric is a length function.

If H is a length function, then we can define the length of curves in the usual way, and we can then define a semi distance d_H, whereby $d_H(x,y)$ is the inf of the lengths of curves joining x to y.

If Y is a complex space rather than a complex manifold, then one can define the similar notion for Y, either by accepting to imbed Y globally in a complex manifold (which is OK in all the applications I know of), or by localizing the construction of a length function, using only such local imbeddings. I don't want to go into such technical details at this point.

It is easy to show that if X is hyperbolically imbedded in Y, then there exists a length function H on Y such that $d_H \leqq d_X$ on X. If X is relatively compact in Y, then for every length function H, we have $d_H \ll d_X$, the sign $\ll$ meaning that there exists a constant $C > 0$ such that $d_H \leqq Cd_X$. In other words, the Kobayashi distance on X is bounded from below by a hermitian distance on an ambient global complex manifold.

Let d be a distance function on a set, which is thus a metric space. Then one associates with d the Hausdorff measure μ_d^m for every positive integer m. Suppose now that M is a complex hermitian manifold of dimension m, with its positive (1,1)-form ω, and volume form $\Omega = \omega^m/m!$. Then we also have the measure μ_Ω associated with Ω, and it is a fact that

$$\mu_\Omega = \mu_d^{2m},$$

where d is the Riemannian distance associated with the hermitian metric. This is basically standard for euclidean space, and the proof for manifolds is essentially the same.

We shall also need a foundational fact due to Lelong [Le 1] and [Le 2]. For an elegant exposition, see also Griffiths [Gr], p. 13.

<u>LELONG'S THEOREM</u>. *Let μ_{2m} be the euclidean measure induced on an m-dimensional complex subspace X of $\mathbb{C}^N$. Let $\bar{X}$ be the closure, and assume that $\bar{X}-X$ lies outside the ball $B(R_0)$, say centered at the origin. Assume that $0 \in X$. Then for $r < R_0$ we have*

$$\mu_{2m}(X_r) \geqq v_{2m} r^{2m},$$

where v_{2m} is the Lebesgue volume of the unit ball in $\mathbb{R}^{2m}$, and $X_r = X \cap B(r)$ is the part of X in the ball of radius r.

Next we recall Kwack's theorem as extended by Kobayashi and Kiernan.

THE K^3-THEOREM, KIERNAN-KOBAYASHI-KWACK. *Let A be a divisor with normal crossings on a complex manifold M. Let $X \subset Y$ be a relatively compact hyperbolically imbedded complex subspace. Then every holomorphic map*

$$f: M-A \longrightarrow X$$

extends to a holomorphic map of M into Y.

This theorem was proved by Kwack [Kw] when $X = Y$ is compact. In this case, A can be arbitrary, no assumption on the nature of the singularities is needed since one can extend the map to M-Sing(A), then to M-Sing(Sing(A)), and so forth. Kobayashi introduced the concept of "hyperbolically imbedded", and generalized Kwack's result to the case when X is hyperbolically imbedded in Y, and A is non-singular. Kiernan treated the case when A has simple normal crossings (normal crossings for short), cf. [Ko 3] p. 93 as compared to [Ki 3]. The method of proof is the same as Kwack's, and part of the method goes back to Grauert-Reckziegel.

THEOREM 1. *Let:*

$X \subset Y$ *be a relatively compact, hyperbolically imbedded complex subspace;*

M *be a complex manifold of dimension* m;

A *be a divisor with normal crossings in* M.

Let

$$f_n: M-A \longrightarrow X$$

be a sequence of holomorphic maps, which converges uniformly on compact subsets of M-A to a holomorphic map $f: M-A \longrightarrow X$. Let $\bar{f}_n$, $\bar{f}$ be their holomorphic extensions from M into Y. Then the sequence $\{\bar{f}_n\}$ converges uniformly to $\bar{f}$ on every compact subset of M itself.

Before we give the proof, we point to the example when $M = \underline{D}$ is the unit disc, A is the origin, so $M-A = \underline{D}^*$ is the punctured unit disc. Then $m = 1$, the Hausdorff measure associated with the Kobayashi distance on $\underline{D}^*$ is 2-dimensional, and is the hyperbolic measure. Observe that the Hausdorff measure of punctured discs around the origin tends to 0 as the radius tends to 0. This is easily proved by representing the disc as the quotient of the upper half plane, and using the hyperbolic measure on the upper half plane.

We shall now prove the theorem. The question of convergence arises in the neighborhood of a point $a \in A$. Without loss of generality, we may assume that complex coordinates $z_1,\dots,z_m$ are chosen such that

$$M = \mathbf{D}^m, \qquad a = 0, \qquad \text{and } A \text{ is defined by } z_1\cdots z_p = 0.$$

Assume first that A is defined by $z_1 = 0$, so

$$A = \mathbf{D}^* \times \mathbf{D}^{m-1}.$$

Neighborhoods of 0 are given by

$$U_{k,r} = \mathbf{D}_{1/k} \times \mathbf{D}_r^{m-1} \qquad \text{with } 0 < r < 1.$$

We let $S_{1/k}$ denote the circle of radius $1/k$. Let $f(0) = y_0$. We let W denote a small open neighborhood of y_0 in Y. We can identify $W \subset B(y_0,1)$ with a complex subspace of a ball of radius 1 in some $\mathbf{C}^N$. We write coordinates of $z \in \mathbf{D}^m$ as

$$z = (z_1,z') \qquad \text{with } z_1 \in \mathbf{D} \text{ and } z' \in \mathbf{D}^{m-1}.$$

If the sequence $\{\bar{f}_n\}$ does not converge uniformly on some neighborhood of 0, then we can pick W as above such that, given k, r there are infinitely many n for which

$$\bar{f}_n(U_{k,r}) \not\subset W.$$

For each $w \in B(y_0,1)$ and $t > 0$ we let $B(w,t)$ be the open ball of radius t and center w, and we let $S(w,t)$ be the sphere of center w and radius t. There exists k_0 and r_0 such that

$$\bar{f}(\bar{\mathbf{D}}_{1/k_0} \times \mathbf{D}_{r_0}^{m-1}) \subset B(y_0,1/8)$$

simply by the continuity of $\bar{f}$. Since $\{\bar{f}_n\}$ converges uniformly to $\bar{f}$ on $S_{1/k} \times \mathbf{D}_{r_0}^{m-1}$, there exists a subsequence $\{f_{n_k}\}$ and a sequence

$\{z_k'\}$ of points $z_k' \in \underline{\underline{D}}_{r_0}^{m-1}$ such that

$$\lim z_k' = 0$$

and

(1) $f_{n_k}(S_{1/k}, z_k') \subset B(y_0, 1/4)$

(2) $\bar{f}_{n_k}(\underline{\underline{D}}_{1/k}, z_k') \not\subset B(y_0, 1)$.

Hence for each $k \geqq k_0$ there is a point $z_{1k} \in \underline{\underline{D}}_{1/k}$ such that

(3) the point $x_k = \bar{f}_{n_k}(z_{1k}, z_k')$ lies on $S(y_0, 1/2)$.

Let

$$E_k = \bar{f}_{n_k}(\underline{\underline{D}}_{1/k}, z_k') \cap B(x_k, 1/8).$$

Then E_k is a one-dimensional complex subspace of $B(x_k, 1/8)$. We note that $f_{n_k}(S_{1/k}, z_k')$ lies outside $B(x_k, 1/8)$. By Lelong's theorem, it follows that the euclidean measure of E_k satisfies

$$c_0 \leqq \mu^2_{\text{euc}}(E_k) \quad \text{for all } k = 1, 2, \ldots .$$

On the compact set $\bar{B}(y_0, 1) \cap Y$ all length functions are equivalent, i.e. each is less than a constant times the other, so the corresponding distance functions are equivalent, and so are the corresponding Hausdorff measures. If H is a length function on Y, with distance d_H, we let:

μ^2_H = 2-dimensional Hausdorff measure defined by d_H, restricted to X;

μ^2_X = 2-dimensional Hausdorff measure defined by d_X on X.

Then we get

$$\mu^2_{\text{euc}}(E_k) \leqq c_1 \mu^2_H(E_k).$$

Let $E_k^* = E_k - f_{n_k}(0, z_k')$. Then $\mu^2_H(E_k) = \mu^2_H(E_k^*)$. Note that $E_k^* \subset X$. Since $d_H \ll d_X$ on X because X is hyperbolically imbedded in Y, it follows that

$$\mu_H^2(E_k) \leqq C_2 \mu_X^2(E_k^*),$$

where μ_X^2 is the Hausdorff measure associated with the Kobayashi distance d_X. On the other hand, since

$$f_{n_k} : M\text{-}A = \underline{\underline{D}}^* \times \underline{\underline{D}}^{m-1} \longrightarrow X$$

is distance decreasing from d_{M-A} to d_X, it follows that

$$\mu_X^2(E_k^*) \leqq C_3 \mu_{M-A}^2(\underline{\underline{D}}_{1/k}^*, z_k').$$

But $\underline{\underline{D}}^*$ is imbedded on $(\underline{\underline{D}}^*, z_k')$ in M-A, and the imbedding is Kobayashi distance decreasing. Therefore we obtain the final inequality

$$\mu_{M-A}^2(\underline{\underline{D}}_{1/k}^*, z_k') \leqq C_4 \mu_{\underline{\underline{D}}^*}^2(\underline{\underline{D}}_{1/k}^*) \longrightarrow 0 \quad \text{as} \quad k \longrightarrow \infty.$$

This contradicts the first inequality in this successive chain, namely that $C_0 \leqq \mu_{euc}^2(E_k)$, and concludes the proof of the theorem in case $A = \underline{\underline{D}}^* \times \underline{\underline{D}}^{m-1}$.

The case when $A = \underline{\underline{D}}^{*p} \times \underline{\underline{D}}^{m-p}$ is then done by induction, since the sequence $\{f_n\}$ converges uniformly to f on compact subsets of $\underline{\underline{D}}^{*p-1} \times \underline{\underline{D}}^{m-p+1}$. This concludes the proof of the theorem.

The theorem was formulated by Noguchi for sections in a proper family. An analysis of the proof showed that the hypothesis of having sections was irrelevant, and that Noguchi's arguments applied without change to arbitrary maps as formulated above. For the formulation in terms of sections, see below.

Let $\pi : X \longrightarrow Y$ be a proper holomorphic map of complex spaces. We view π as representing a family of complex spaces $\pi^{-1}(y)$ for $y \in Y$. One wants conditions under which the set of sections $\mathrm{Sec}(\pi)$ is locally compact for the compact open topology, that is given any sequence of sections, there exists a subsequence which converges uniformly on every compact subset of Y.

Suppose that $\pi^{-1}(y)$ is hyperbolic for all $y \in Y$. Then for every point $y \in Y$ there is an open neighborhood V which is hyperbolic (in fact some closed subspace of a polydisc chart) and such that $\pi^{-1}(V)$ is hyperbolic. This follows immediately from Brody's criterion for hyperbolicity [Br] which he proves is an open condition. Since a holomorphic map, in this case a section, is Kobayashi distance decreasing, it follows that a family of sections

$$f: V \longrightarrow \pi^{-1}(V)$$

is equicontinuous, and satisfies the hypotheses of Ascoli's theorem, so is locally compact. This is the Kobayashi hyperbolicity version of Grauert's original idea, pursued by Riebesehl in the context where "hyperbolic" means "negative Gauss curvature" in some form, so one could call it "Gauss hyperbolic". Riebesehl falls short of analyzing what happens at "bad" points $y \in Y$, when the fiber $\pi^{-1}(y)$ is not Gauss hyperbolic. Noguchi made substantial progress by proving the following theorem.

NOGUCHI'S THEOREM [No]. *Let $\bar{R}$ be a Riemann surface (complex manifold of dimension 1), and let*

$$\bar{\pi} : \bar{X} \longrightarrow \bar{R}$$

be a proper holomorphic map. Let S be a discrete subset of $\bar{R}$, and let $R = \bar{R} - S$. Let $X = \pi^{-1}(R)$ and $\pi : X \longrightarrow R$ the restriction of $\bar{\pi}$ to X. Assume:

(a) *For all $y \in R$ the fiber $\pi^{-1}(y)$ is hyperbolic.*

(b) *For each point $y \in \bar{R} - R$ there is an open neighborhood V such that $\pi^{-1}(V - \{y\})$ is hyperbolically imbedded in $\pi^{-1}(V)$.*

Then the set of sections $\mathrm{Sec}(\bar{\pi})$ is locally compact.

This follows in the manner described above. Actually, Noguchi formulates a version with a higher dimensional base, and a divisor with normal crossings in the base where the fibers may be bad. He then defines the notion of (X,π,R) being hyperbolically imbedded in $(\bar{X},\bar{\pi},\bar{R})$, and gets the theorem in that case. For simplicity of formulation, in order to rely on the standard definition of hyperbolically imbedded, I limited myself to the case of a one-dimensional base, which is the essential case for applications to the Mordell type problems.

Noguchi also proved in the case of surfaces fibered by curves whose generic member has genus at least 2 that there always exists a model satisfying the hypothesis of the above theorem. The existence of such a model remains open in the case of higher dimensional hyperbolic fibers.

Suppose that for the proper map

$$\pi : X \longrightarrow Y$$

given $y \in Y$, the fiber $\pi^{-1}(y)$ is hyperbolic, or there exists a neighborhood V of y which is hyperbolic, and such that $\pi^{-1}(V - \{y\})$ is hyperbolically imbedded in $\pi^{-1}(V)$. Then the set of sections is locally compact by Noguchi's theorem.

<u>Suppose in addition that</u> X, Y <u>are projective. Then the set of sections is compact, and the projective degrees of the sections are bounded</u>.

This is the desired conclusion corresponding to Mordell's conjecture, that the heights of rational points are bounded in the case of number fields, cf. [L 1]. We are thus led naturally into the second part of this paper.

<u>2. ON CARTAN'S THEOREM</u>. Let F be a field with a family of absolute values $\{v\}$. If $a \in F^*$ then we write $\|a\|_v$ for a suitably normalized multiplicative absolute value, and

$$v(a) = -\log \|a\|_v .$$

We assume that the set of absolute values satisfies the product formula, that is for $a \neq 0$,

$$\prod_v \|a\|_v = 1,$$

or in additive terms,

$$\sum_v v(a) = 0.$$

Let $P = (a_0, \ldots, a_n) \in \underline{\underline{P}}^n(F)$ be a point in projective space, with projective coordinates $a_i \in F$. We define its <u>height</u>

$$h(P) = \sum_v \log \max_i \|a_i\|_v .$$

<u>Examples</u>. (Cf. [La 4], Chapter III, §1 and §3.) Let F be a number field, and let $\{v\}$ be the set of absolute values extending the ordinary absolute value on $\underline{\underline{Q}}$, or a p-adic absolute value. Normalizing $\|a\|_v$ suitably the product formula holds (Artin-Whaples).

Let F be the function field of a projective variety X, which we assume non-singular for simplicity. Say X is defined over an algebraically closed field, or even $\underline{\underline{C}}$. To each irreducible divisor W on X we have the associated order function

$$v_W(f) = \mathrm{ord}_W(f)\deg(W),$$

where ord_W is the order at W, and $\deg(W)$ is the projective degree. Again the product formula holds.

As a third example, we start from Vojta's remark that Jensen's formula is the analogue of the product formula. Thus let r be a fixed positive number, and let F be the field of meromorphic functions on the disc $\underline{\underline{D}}_r$. For each θ we have the absolute value

$$\|f\|_{\theta,r} = |f(re^{i\theta})|, \qquad \text{so} \qquad v_{\theta,r}(f) = -\log |f(re^{i\theta})|.$$

We allow ∞ to be a value. Given a point $a \in \underline{\underline{D}}_r$ we have the absolute value such that in logarithmic form

$$v_{a,r}(f) = (\mathrm{ord}_a f)\log \left|\frac{r}{a}\right|.$$

The factor $\log |r/a|$ is a normalizing factor, like $\log p$ in number theory when p is a prime number. Let c_f be the leading coefficient of f in the power series expansion at 0. Then we also let

$$v_0(f) = -\log |c_f|.$$

Strictly speaking, v_0 is not quite a valuation, but let that pass. Then

Jensen's formula is the sum formula (fixing r). If $f \neq 0$ then

$$\int_0^{2\pi} v_{\theta,r}(f)\,\frac{d\theta}{2\pi} + \sum_{a \in \underline{D}_r} v_{a,r}(f) - v_0(f) = 0.$$

The theory of the height (as function of r) associated with this sum formula is, by definition, Nevanlinna theory.

In the number field case, it is easily proved that a set of points of bounded height is finite. In the second example, called the function field case, bounding the height, i.e. the projective degree, implies that the points lie in a finite number of Chow families.

The Mordell conjecture (Faltings' theorem) was (is) that if X is a curve of genus at least 2 defined over a number field F, and say in projective space, then its set of rational points X(F) in F has bounded height - and therefore is finite.

My conjecture is that if X is a projective variety defined over F, and hyperbolic in some imbedding of F in the complex numbers, then again the points of X(F) have bounded height, and so X(F) is finite.

In Nevanlinna theory, given a holomorphic map $f: \underline{C} \longrightarrow X$ into some projective variety, bounding the height of the map implies that the map is constant.

In each case, finding conditions under which the height is bounded is regarded as an end in itself. Such conditions amount conjecturally to conditions of "hyperbolicity", in one form or another.

I shall now discuss the Nevanlinna case at greater length. The height of a map

$$f: \underline{C} \longrightarrow \underline{P}^n$$

is usually denote by T_f, so T_f is a function of r. Suppose that f is represented by coordinate functions $(f_0,\ldots,f_n)$ where f_i are entire functions without common zeros. Then the height T_f is given by

$$T_f(r) = \int_0^{2\pi} \log \max_i \|f_i\|_{\theta,r} \frac{d\theta}{2\pi} - \log \max_i |f_i(0)|.$$

We call this the Cartan height, since Cartan defined it in 1929 [Ca]. It is similar to the Weil height in number theory (also defined by Siegel at about the same time). More generally, if f is represented by meromorphic coordinates $(g_0, g_1, \ldots, g_n)$ then the Cartan height is equal to

$$T_f(r) = \int_0^{2\pi} \log \max_i \|g_i\|_{\theta,r} \frac{d\theta}{2\pi} - \log \max_{i \in M} |c_{g_i}| + \sum_{a \in \underline{D}_r} \log \max_i \|g_i\|_{a,r}$$

where the indices $i \in M$ in the term with the leading coefficients c_{g_i} are taken in an appropriate subset of all indices. This height T_f is independent of homogeneous coordinates, because if we change $(g_0, \ldots, g_n)$ by multiplying with a meromorphic function h, then all the max in all the terms of the formula change by $\log \|h\|_v$, where v ranges over v_θ, v_a, and $-\log |c_h|$. The sum of these is 0 by Jensen's formula, i.e. by the sum formula. Cartan also observed that if f is a meromorphic function, and $(1,f)$ is the corresponding map into $\underline{P}^1$, then

$$T_{(1,f)} = T_f + O(1),$$

where here T_f is the "characteristic function" defined by Nevanlinna for meromorphic functions. The height satisfies similar properties to those in number theory, better codified in this case than by the analysts. First:

Let f *and* $g\colon \underline{C} \longrightarrow \underline{P}^n$ *be two holomorphic maps and* $g = A \circ f$ *where* A *is a projective linear transformation. Then*

$$T_f = T_g + O(1).$$

We just defined the height of a map into projective space. More generally we can define the height of a map into an arbitrary projective variety. We just state what goes on without proofs.

Let $f: \mathbb{C} \longrightarrow X$ be a holomorphic map into a projective variety. To each Cartier divisor D on X one can associate a function $T_{f,D}$ of real numbers $\geqq 1$, depending only on the linear equivalence class of D, and uniquely determined up to a bounded function by the following properties. [The notation $O(1)$ always refers to $r \longrightarrow \infty$.]

H 1. The map $D \longmapsto T_{f,D}$ is a homomorphism mod $O(1)$.

H 2. If E is very ample, and $\psi: X \longrightarrow \mathbb{P}^n$ is one of the associated imbeddings into projective space, then

$$T_{f,E} = T_{\psi\circ f} + O(1),$$

where $T_{\psi\circ f}$ is the Cartan-Nevanlinna height.

In addition, the height satisfies further properties as follows.

H 3. For any Cartier divisor D and ample E, we have

$$T_{f,D} = O(T_{f,E}).$$

H 4. If D is effective and f meets D properly, then $T_{f,D} \geqq -O(1)$.

H 5. The association $(f,D) \longmapsto T_{f,D}$ is functorial in (X,D). In other words, if $\psi: X \longrightarrow Y$ is a morphism of varieties, and $D = \psi^{-1}D'$ where D' is a divisor on Y, then

$$T_{f,D} = T_{\psi\circ f,D'}.$$

Other properties of the height can be copied from those of number theory, cf. for instance [La 4], Chapter 4, §1, §2, §3, including the analogue of Weil's theorem, namely Proposition 2.1 of that chapter.

Next we come to the definition of other functions in Nevanlinna theory. Let X be a projective variety. Let D be an effective Cartier divisor on X, represented locally for the Zariski topology by regular functions. Say on a Zariski open set U, D is represented by the function φ. This means $D|U = (\varphi)|U$. A Weil function is a function

$$\lambda_D: X - \mathrm{supp}(D) \longrightarrow \mathbb{R}$$

which is continuous, and is such that if D is represented by φ on U, then there exists a continuous function $\alpha : U \longrightarrow \mathbb{R}$ such that for all $x \notin \operatorname{supp}(D)$ we have

$$\lambda_D(x) = -\log |\varphi(x)| + \alpha(x).$$

The difference of two Weil functions is the restriction to X-supp(D) of a continuous function on X, and so is bounded. Thus two Weil functions differ by $O(1)$. If L is a line bundle over X and s a holomorphic section whose divisor is D, and ρ is a metric on L, then for instance

$$\lambda_D(x) = -\log |s(x)|_\rho$$

is a Weil function associated with D.

Suppose that $f(\mathbb{C})$ is not contained in D. This is equivalent with the fact that f meets D discretely, i.e. in any disc $\mathbb{D}_r$ there are only a finite number of points $a \in \mathbb{D}_r$ such that $f(a) \in D$. Given a Weil function λ_D associated with the effective divisor D, we define the <u>proximity function</u>

$$m_f(r,D) = \int_0^{2\pi} \lambda_D(f(re^{i\theta})) \frac{d\theta}{2\pi}.$$

Then λ_D and $m_f(r,D)$ are additive in D mod $O(1)$.

Let D again be an effective divisor. We let

$$D_f = f^{-1}(D)$$

as a divisor on $\mathbb{C}$, which exists by our basic assumption that f meets D discretely. If D is represented by φ on U, and $f(a) \in U$, then we let

$$\nu_f(a,D) = \nu(a,D_f) = \operatorname{ord}_a(\varphi \circ f) \quad (\geqq 0 \text{ since } D \text{ is effective})$$

Note that $\varphi \circ f$ is a holomorphic map of $\mathbb{C}$ into $\mathbb{P}^1$. Then we define

$$N_f(r,D) = \sum_{\substack{a \in \mathbb{D}_r \\ a \neq 0}} \nu_f(a,D) \log \left|\frac{r}{a}\right| + \nu_f(0,D) \log r.$$

Thus $N_f(r,D)$ can be viewed as the "r-degree" of the divisor D_f, with the weighting factors $\log |r/a|$ similar to $\log p$ in number theory when p is a prime number.

We call $N_f(r,D)$ the <u>integrated counting function</u>. It is additive in D and $\geqq 0$ for $r \geqq 1$. It measures the zeros of f in D.

Finally we define the <u>height</u>, or <u>Cartan-Nevanlinna height</u>, to be

$$T_f(r,D) = m_f(r,D) + N_f(r,D).$$

This is well defined mod O(1).

We extend Weil functions, the proximity function and the integrated counting function to all Cartier divisors by additivity.

<u>FIRST MAIN THEOREM</u>. *If* $D = (\varphi)$ *is linearly equivalent to* 0, *then* $T_f(r,D) = O(1)$, i.e. $T_f(r,D)$ *is a bounded function of* r.

This is proved first for $\underline{\underline{P}}^1$ (Nevanlinna) and in general by functoriality.

The SECOND MAIN THEOREM is today only a conjecture.

CONJECTURE. *Let* D *be an effective divisor with normal crossings on a projective non-singular variety* X. *There exists a divisor* D' *having the following property. Let* $f: \underline{\underline{C}} \longrightarrow X$ *be holomorphic such that* $f(\underline{\underline{C}}) \not\subset D'$. *Let* K *be the canonical class. Let* E *be ample. Then*

$$m_f(r,D) + T_f(r,K) \leqq O_{exc}(\log r + \log T_f(r,E)),$$

where O_{exc} *means the usual* O *with the exception of* r *lying in a set of finite Lebesgue measure.*

Only very special cases of this conjecture are known today, essentially for maps into projective space or Grassmanians, in a linear situation. Furthermore, a ramification term is missing in the above inequality, but should be present in the same way that it is present in Nevanlinna's theory in the one-dimensional case. We now state one of the basic known results.

THEOREM 2. *Let* $f = (f_0,\ldots,f_n): \underline{\underline{C}} \longrightarrow \underline{\underline{P}}^n$ *be a holomorphic map, with* $f_0,\ldots,f_n$ *entire without common zeros. Assume that the image of* f *is not contained in any hyperplane. Let* $H_1,\ldots,H_q$ *be hyperplanes in general position with* $q \geqq n+2$ *(meaning any* n+1 *are linearly independent). Let* $W(f) = W(f_0,\ldots,f_n) = W$ *be the Wronskian. Let* H *be any hyperplane. Then*

$$\sum_{k=1}^{q} m_f(r,H_k) - (n+1)T_f(r,H) + N_W(r,0) \leqq O_{exc}(\log r + \log T_f(r)),$$

where O_{exc} has the usual O meaning, for $r \longrightarrow \infty$ but r lying outside some set of finite Lebesgue measure.

This theorem is due to Cartan [Ca] except for the term $N_W(r,0)$, which constitutes an improvement. Cartan has in its place a term also reflecting the ramification, but depending on the hyperplanes $H_1,\ldots,H_q$. This is the "wrong" structure for this term, which should reflect only the ramification of the mapping f, independently of the divisor D, which in this case is $\sum H_k$. For $n = 1$, the theorem is due to Nevanlinna who started it all.

As an example of a diophantine application, we show how a theorem of Borel is an immediate consequence of Cartan's theorem.

COROLLARY. *Let $g_1,\ldots,g_n$ be entire functions without zeros (so units in the ring of entire functions). Suppose that*

$$g_1+\ldots+g_n = 1.$$

Then $g_1,\ldots,g_n$ are linearly dependent.

Proof. Let $g\colon \mathbf{C} \longrightarrow \mathbf{P}^{n-1}$ be the map $(g_1,\ldots,g_n)$. Let $x_1,\ldots,x_n$ be the homogeneous variables of $\mathbf{P}^{n-1}$. Let H_k be the hyperplane $x_k = 0$ for $k = 1,\ldots,n$ and $x_1+\ldots+x_n = 0$ for $k = n+1$. Then g does not meet these hyperplanes. Hence $m_g(r,H_k) = T_g(r,H_k) + O(1)$ for $k = 1,\ldots,n+1$. The canonical class is the class of nH for any hyperplane H. If $g_1,\ldots,g_n$ are linearly independent, then by Cartan's theorem

$$(n+1)T_g \leqq nT_g + O_{exc}.$$

Hence $T_g(r) = O_{exc}(\log r)$, and it is then easy to show that all coordinates $g_1,\ldots,g_n$ are polynomials, so are constant since g_i is a unit for each i. This concludes the proof of the corollary.

Cartan's theorem itself is proved according to the Wronskian technique apparently first used by Nevanlinna, also in connection with Borel's theorem.

Vojta [Vo] has translated the Second Main Theorem into a similar conjecture for a height inequality in algebraic number theory. He shows how theorems like those of Roth, Schmidt, Faltings, and conjectures like those of Hall, Lang, Lang-Waldschmidt, and what is neither, like Fermat's Last Theorem, would follow (or more or less follow) from his translation. The "more or less" is due to the presence of an exceptional set, similar to the exceptional set of r of finite Lebesgue measure in Nevanlinna theory. For details, the reader can read Vojta's extensive exposition.

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Cubic Exponential Sums and Galois Representations

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0. Introduction The ς-function of an algebraic variety over a number field is always conjectured to be expressible in terms of automorphic forms. The examples for which such a description is known, however, are very few. Generally these are varieties with an explicit uniformization, or varieties known to be related to them by an algebraic correspondence. We shall discuss here one variety of a different type. For $n \geq 1$ set

$$W_n = \left\{(x_1, \ldots, x_n) \in \mathbb{P}^{n-1} \mid \sum x_i = \sum x_i^3 = 0\right\}.$$

Let $\eta(z) = q^{1/24} \prod_{n=1}^{\infty}(1 - q^n)$ where $q = \exp(2\pi i z)$ be the Dedekind eta-function, and define

$$\phi(z) = (2 - T(3))\left(\eta(z)\eta(2z)\eta(5z)^3\eta(10z)^3\right),$$

where $T(m)$ is the m'th Hecke operator. We will show

0.1 PROPOSITION. *ϕ is the unique newform of weight 4 and level 10 with trivial character.*

0.2 THEOREM. *Let $\varsigma(s)$ be Riemann's ς function. Then*

$$\varsigma(W_{10}, s) = \frac{\varsigma(s)\varsigma(s-1)\varsigma(s-2)\varsigma(s-3)^{-84}\varsigma(s-4)^{42}\varsigma(s-5)\varsigma(s-6)\varsigma(s-7)}{L(\phi, s-2)}$$

The proof of the theorem is based on the cohomological interpretation of the ς-function. The varieties W_n arise in connection with a conjecture of Birch [B] about the average distribution of cubic exponential sums. The cohomology of W_n was calculated, and Birch's conjecture proved, in [L]. We recall this in Section 1 and obtain the numerator of $\varsigma(W_{10}, s)$. To calculate the denominator we use a criterion of Serre (theorem 4.3), based on Faltings' method, for the equivalence of 2-dimensional, 2-adic Galois representations with even trace. The case when the trace is not even was previously treated by Serre in a letter to Tate and his 1984 course at the College de France. In a letter to C. Schoen Serre explained how to use that criterion to determine the ς-function of a certain threefold. The present results of Serre are described in section 4. They are used to prove

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theorem 0.2 up to Euler factors at 2, 3, and 5. The theorem then follows from Carayol's results [Car]. We also remark that while the Tate conjecture would imply the existence of an algebraic correspondence between W_{10} and a variety with an explicit uniformization (of the Kuga-Sato type), I have not been able to construct any such.

Next, let $\gamma(p)$, for any prime p, be the eigenvalue of $T(p)$ acting on ϕ. Set $\alpha(z) = \eta(8z)^4\eta(40z)$. We will prove

0.3 THEOREM. *a. α is form of weight $2\frac{1}{2}$, level 320 and character χ_5.*
b. α is an eigenfunction for all the $T(p^2)$.
c. The Shimura correspondence maps α to $\phi \otimes \chi_{-1}$.

Together with the previous results this implies a conjecture of Atkin [A] (conjecture 1.5 below).

The proof of 0.3a uses the Deligne-Serre theory. Next, machine calculations show that

$$T(p^2)\alpha - \left(\tfrac{-1}{p}\right)\gamma(p)\alpha$$

vanishes to a high order at ∞ for $p = 3$ or $7 \le p \le 31$ and hence must be zero. Theorem 4.3 and Shimura's correspondence of forms of half integral weight to forms of integral weight then give 0.3b and 0.3c. Serre's results are described in section 4. I would like to thank Coates, Katz and Ribet for helpful conversations, and Serre for allowing his work to be described here and for numerous suggestions to improve the manuscript.

1. Cubic exponential sums and $\mathbf{W_n}$ We recall results of [L]. For $a, b \in \mathbb{Z}$ and a prime p put

$$B(a,b,p) = \sum_{x (\operatorname{mod} p)} \exp\left(\frac{2\pi i}{p}(ax^3 + bx)\right)$$

If $p \nmid a$ $\quad |B(a,b,p)| < 2\sqrt{p}$ (Weil) and it is natural to ask for the distribution in p of

$$x_p = \cos\theta_p = \frac{B(a,b,p)}{2\sqrt{p}},$$

where $\theta_p \in [0,\pi]$. For $a \neq 0$

$$B(a,0,p) = 2\,\mathrm{Re}\left(g_3(p)\left(\tfrac{a}{p}\right)_3\right),$$

where $g_3(p)$ is the cubic Gauss sum. Hence $B(a,o,p) = 0$ if $p \equiv 2 (\operatorname{mod} 3)$ and one has the deep

THEOREM ([HB-P]). *The θ_p's are uniformly equidistributed over $p \equiv 1 (\operatorname{mod} 3)$.*

When $ab \neq 0$ one expects equidistribution with respect to the Sato-Tate measure

$$d\mu^{S-T} = \frac{2}{\pi}\sin^2\theta d\theta = \frac{2}{\pi}\sqrt{1-x^2}\,dx, \quad \text{where } x = \cos\theta.$$

Equivalently, let μ_n^{S-T} be the n'th moment of $d\mu^{S-T}$

$$\mu_n^{S-T} = \int_{-1}^{1} x^n d\mu^{S-T} = \begin{cases} 0 & n \text{ odd} \\ \frac{1}{2^{2R}(R+1)}\binom{2R}{R} & n = 2R \end{cases}$$

and denote the number of primes $\leq X$ by $\pi(X)$. One can then make the

CONJECTURE. *Suppose $ab \neq 0$. Then for any $n \geq 0$*

$$\frac{1}{\pi(X)} \sum_{p \leq X} \left(\frac{B(a,b,p)}{2\sqrt{p}}\right)^n \xrightarrow[X\to\infty]{} \mu_n^{S-T}.$$

As this seems out of reach presently, Birch proposed the

1.1 CONJECTURE. *For $n \geq 0$ and even*

$$\frac{1}{p(p-1)} \sum_{\substack{a,b (\bmod p) \\ a \not\equiv 0 (\bmod p)}} \left(\frac{B(a,b,p)}{2\sqrt{p}}\right)^n \xrightarrow[p\to\infty]{} \mu_n^{S-T}.$$

Of course one can make this conjecture for n odd as well. More precisely set

$$V_n(p) = \sum_{\substack{a,b (\bmod p) \\ a \not\equiv 0 (\bmod p)}} B(a,b,p)^n$$

$$f_n(p) = \begin{cases} 0 & n \text{ odd} \\ \frac{1}{(R+1)}\binom{2R}{R}(p-1)(p-\frac{R+1}{2})p^R & n = 2R, \end{cases}$$

and

$$E_n(p) = V_n(p) - f_n(p).$$

Birch [B] proved that

1.2 $E_n(p) = 0$ for $n = 2,4,6$ and 8, and conjectured that

1.3 $E_{12}(p) = 11pE_{10}(p)$ (for $p > 5$) and that

1.4 $E_{10}(p) = -(p-1)p^4\gamma_{10}(p)$,
where $\gamma_{10}(p)$ is an integer satisfying $|\gamma_{10}(p)| < 2p^{9/2}$. Atkin subsequently made the

1.5 CONJECTURE. *For $p > 5$ $T(p^2)\alpha = \left(\frac{-1}{p}\right)\gamma_{10}(p)\alpha$, with α as before.*

The connection between the varieties W_n and conjecture 1.1 is elementary. In fact we have

1.6 PROPOSITION. $V_n(p) = -p^n + p^2 + p^2(p-1)\,{}^{\#}W_n(\mathbb{F}_p)$.

The proof is routine, see e.g. [L], lemma 4.1.

When n is odd W_n is nonsingular in characteristic $p > n$. The Lefschetz hyperplane theorem then gives that

$$E_n(p) = p^2(p-1)\ \mathrm{Tr}\ Frob_p | H^{n-3}_{prim}(W_n \times \mathrm{spec}\, \overline{\mathbb{F}}_p, Q_l),$$

and Deligne's bounds give conjecture 1.1. On the other hand when $n = 2R$ the singular locus $T = T_{2R}$ of W_{2R} is not empty. Suppose $p > R > 2$. Then T consists of $\frac{1}{2}\binom{2R}{R}$ ordinary double points, and the blow up $\widetilde{W}_{2R} = Bl_T W_{2R}$ is nonsingular. Using Schoen's Hodge theoretic results [Sch] the cohomology of $\hat{W}_{2R}$ was determined in [L], giving in this case

$$(1.7) \qquad E_{2R}(p) = -p^2(p-1)\operatorname{Tr} Frob_p | H^{2R-3}(\widetilde{W}_{2R} \times \operatorname{spec} \overline{\mathbb{F}}_p, \mathbb{Q}_l),$$

and again conjecture 1.1 for $n = 2R$ follows from Deligne's bounds. It turns out that many Hodge groups of $\widetilde{W}_{2R}$ and W_n for n odd vanish, which gives

$$p^\alpha | E_n(p), \quad \text{where } \alpha = \alpha_n = \left[\frac{n+4}{3}\right],$$

by Mazur's results on Hodge and Newton polygons. Moreover, for $R \geq 3$

$$\dim H^{2R}(\widetilde{W}_{2R-3}) = \frac{2^{2R-1}-2}{3} - 2\binom{2R-1}{R-2}.$$

This is in accord with 1.2 for $n = 6$ or 8, and proves 1.4. To get 1.3 put $z_i = x_i + x_{R+i}$ and $w_i = x_i - x_{R+i}$ for $1 \leq i \leq R$. The equations defining W_{2R} then become

$$\sum z_i = 0 \quad \text{and} \quad 3\sum z_i w_i^2 = -\sum z_i^3.$$

Hence, for every $z_1, \ldots, z_R$ satisfying $\sum z_i = 0$ we have a quadric in the w_i's. It is easy to conclude that $^{\#}W_{2R}(\mathbb{F}_p)$ can be expressed in terms of $^{\#}Y_i(\mathbb{F}_p)$, $i = 1, \ldots, R$, where for any R

$$(1.8) \qquad Y_R = \left\{ (z_1, \ldots, z_R, \lambda) \,\Big|\, \sum z_i = 0 \quad \text{and} \quad \lambda^2 = 3^R \prod z_i (\sum z_i^3) \right\}.$$

When R is even Y_i is rational, as can be seen by setting $z_i = z_1 z_i'$ for $i \geq 2$. Working this out for $R = 6$ gives 1.3.

We remark in passing that it is possible to analyze the action of the symmetric group, acting on W_n by permuting coordinates, on cohomology. This gives that for $5 \leq R \leq 10$ $H^{2R-3}(\widetilde{W}_{2R} \times \operatorname{spec} \overline{\mathbb{Q}}, \mathbb{Q}_l)$ is a sum of two dimensional Galois representations. The general problem of understanding the Galois representation on $H^{2R-3}(\widetilde{W}_{2R} \times \operatorname{spec} \overline{\mathbb{Q}}, \mathbb{Q}_l)$ will be discussed elsewhere. The other cohomology groups of $\widetilde{W}_{2R}$ are spanned by algebraic cycles rational over $\mathbb{Z}[1/R!]$. Their dimensions have been calculated in [L]. In terms of L-functions, this gives

1.9 THEOREM. *Set* $t = {}^{\#}T = \frac{1}{2}\binom{2R}{R}$, $h = \frac{1}{(R+1)}\binom{2R}{R}$. *Then*

$$\text{(a)} \qquad \zeta(\widetilde{W}_{2R}, s) = \frac{\prod_{i=0}^{2R-3} \zeta(s-i)^{d_i}}{L(H^{2R-3}(\widetilde{W}_{2R} \times \operatorname{spec} \overline{\mathbb{Q}}, \mathbb{Q}_l), s)}$$

$$\text{(b)} \qquad \zeta(W_{2R}, s) = \frac{\prod_{i=0}^{2R-3} \zeta(s-i)^{e_i}}{L(H^{2R-3}(\widetilde{W}_{2R} \times \operatorname{spec} \overline{\mathbb{Q}}, \mathbb{Q}_l), s)}$$

where $d_0 = d_{2R-3} = 1$, $d_{R-2} = d_{R-3} = 1 + t + h$ *and* $d_i = 1 + t$ *otherwise, whereas*
$e_{R-2} = h - t$, $e_{R-1} = h$ *and* $e_i = 1$ *otherwise.*

PROOF. Up to Euler factors for primes $\leq R$ (a) is proven in [L]. Observe that $\varsigma(s-i)$ is the L-function of a compatible system of Galois representations associated to the i'th power of the cyclotomic character. But two compatible systems of Galois representations whose L-functions agree up to finitely many primes must have isomorphic semi-simplifications, and each compatible system determines its L-function. We thus see that the left and the right sides of (a) above agree. (b) follows from (a); see the proof of lemma 4.6 in [L].

1.10 COROLLARY. *To prove theorem 0.2 we need only to show that*

$$L(H^7(\widetilde{W}_{10} \times \operatorname{spec}\overline{Q}, Q_l), s) = L(\phi, s-2)$$

This will be done in Section 3.

2. The Shimura Correspondence Let us now prove proposition 0.1. The vector space $S_4(\Gamma_0(5))$ is 1-dimensional, spanned by $r(z) = (\eta(z)\eta(5z))^4$, and $S_4(\Gamma_0(10))$ is 3-dimensional [Sh1]. $\Gamma_0(10)$ has 4 cusps, represented by ∞, $1/2$, $1/5$ and 0. For $d/10$ the order of $\eta(dz)$ at the cusps is given by the table

	∞	$1/2$	$1/5$	0
$\eta(z)$	1/24	5/24	2/24	10/24
$\eta(2z)$	2/24	10/24	1/24	5/24
$\eta(5z)$	5/24	1/24	10/24	2/24
$\eta(10z)$	10/24	2/24	5/24	1/24

Recall now the following theorem of Newman:

THEOREM ([N]). *Suppose* $d_i|N$ *and* $r_i \in \mathbb{Z}$ *for* $i = 1, \ldots, n$. *Assume*

a. $\sum r_i = 0$.
b. $\prod d_i^{r_i} \in (Q^\times)^2$.
c. $h(z) = \prod \eta(d_i z)^{r_i}$ *has integral order at each cusp of* $\Gamma_0(N)$.

Then h(z) descends to the quotient and defines a meromorphic function on $X_0(N)$.

Abusing language, it follows that $h(z) = \eta(z)^{-3}\eta(2z)\eta(5z)^{-1}\eta(10z)^3$ is a function on $X_0(10)$. Hence $r(z)$, $r(2z)$ and $r(z)h(z)$ are modular forms of weight 4 for $\Gamma_0(10)$. In fact, they clearly have no poles in the upper half plane, and their orders at the cusps are given by the table

	∞	$1/2$	$1/5$	0
$r(z)$	1	1	2	2
$r(2z)$	2	2	1	1
$r(z)h(z)$	2	1	2	1

which shows that the three functions constitute a base for $S_4(\Gamma_0(10))$. Expanding gives

$$r(z)h(z) = q^2 - q^3 - 2q^4 + q^5 - q^7 + 4q^8 + 6q^9 + \dots$$

and it is easy to verify that

$$\phi = (2 - T(3))(rh) = q + 2q^2 - 8q^3 + \cdots \tag{2.1}$$

is indeed the normalized newform in $S_4(\Gamma_0(10))$. Proposition 0.1 follows.

The next step is the following

2.2 LEMMA. *Let* $g(z) = \eta(4z)\eta(20z)$. *Then* $g \in S_1^{new}(\Gamma_0(80), \chi_{-20})$.

PROOF. It is well known that

$$\eta(24z) = \sum_{n>0} \chi_{12}(n)q^{n^2} = \sum_{n\equiv 1 (\mathrm{mod}\ 12)} q^{n^2} - \sum_{n\equiv 7 (\mathrm{mod}\ 12)} q^{n^2}. \text{ Hence}$$

$$g(z) = \sum_{n\equiv 1(\mathrm{mod}\ 6)} \left[\sum_{m\equiv n(\mathrm{mod}\ 12)} q^{(m^2+5n^2)/6} - \sum_{m\equiv n+6(\mathrm{mod}\ 12)} q^{(m^2+5n^2)/6} \right]$$

$$= \sum_{n\equiv 1(\mathrm{mod}\ 6)} \sum_j (-1)^j q^{(n+j)^2+5j^2} \quad , \text{ where } m = n + 6j.$$

Let $K = \mathbf{Q}(\sqrt{-5})$, $\mathcal{O} = \mathcal{O}_K$ the ring of integers of K, $\sigma : K \longrightarrow K$ the non trivial automorphism, $N : K \longrightarrow \mathbf{Q}$ the norm, and $\mathcal{O}_2$ the completion of $\mathcal{O}$ at the prime above 2. The class number of $\mathcal{O}$ is 2. Let $\chi_0 : \mathbb{A}_K^\times / K^\times \longrightarrow \{\pm 1\}$ be the non-trivial unramified character. Set $\pi = 1 + \sqrt{-5}$. Then $\mathcal{O} = \mathbb{Z}[\pi]$ and $\mathcal{O}_2 = \mathbb{Z}_2[\pi]$. Let $\omega : \mathcal{O}_2^\times \longrightarrow \{\pm 1\}$ be the character of conductor (2). Note that $N(a+b\pi) = (a+b)^2 + 5b^2$, so that for $a, b \in \mathbb{Z}_2$ $a + b\pi \in \mathcal{O}_2^\times$ if and only if $a+b$ is odd, and then $\omega(a + b\pi) = (-1)^b$. It follows that

$$g(z) = \sum_{u \in 1+\pi\mathcal{O}} \omega(u) q^{N(u)}.$$

There are two extensions, χ and χ', of ω to a character (of finite order) of $\mathbb{A}_K^\times / K^\times$ unramified outside 2. They have order 4 and satisfy $\chi' = \bar{\chi} = \chi^\sigma$ and $\chi^2 = (\chi')^2 = \chi_0$. Denote the class of an ideal $I \subset \mathcal{O}$ by $[I]$, and let $\wp_3$ stand for an ideal of norm 3 in $\mathcal{O}$. Then

$$\begin{aligned}\sum_{u\in 1+\pi\mathcal{O}} \omega(u)N(u)^{-s} &= \sum_{(I,\pi)=1,|I|=1} \chi(I)N(I)^{-s}\\ &= \sum_{(I,\pi)=1} \frac{\chi(I)+\chi^{\sigma}(I)}{2}N(I)^{-s}\\ &= \frac{1}{2}\left[\sum_{(I,\pi)=1}\chi(I)N(I)^{-s} + \sum_{(J,\pi^{\sigma})=1}\chi(J)N(J)^{-s}\right]\\ &\qquad\qquad\qquad\qquad\qquad\text{(with } J=I^{\sigma}\text{)}\\ &= \frac{1}{2}L(\chi,s)\left[1-\chi(\wp_3)3^{-s}+1-\chi(\wp_3^{\sigma})3^{-s}\right]\\ &= L(\chi,s), \quad \text{since } \chi(\wp_3) \text{ has order 4,}\end{aligned}$$

and this proves the Lemma, by [SD], (7.2.1).

2.3 REMARK. The Galois representation attached to g is dihedral of type D_2. The extension of Q it defines is $L = Q(i, \sqrt{-5}, \sqrt{1+2\pi})$. One way to see this is as follows.

1. $K(i)/K$ is unramified, so that $L \supset K(i)$ and L is the Galois closure over Q of a quadratic extension $F/Q(i)$, cut out by a quadradic character $\psi : \mathbb{A}^{\times}_{Q(i)}/Q(i)^{\times} \longrightarrow \{\pm 1\}$. Again by [SD], (7.2.1), we may suppose that the conductor of ψ is $2(1+2i)$.
2. The extension $F/Q(i)$ is ramified at most at $(1+i)$ and $(1+2i)$. Hence $F = Q(i)(x)$, where $x^2 = 1+2i$, $i(1+2i)$, $(1+i)(1+2i)$ or $i(1+i)(1+2i)$.
3. $\psi((3)) = -1$ and $\psi((1-2i)) = 1$. Hence only $x^2 = 1+2i$ is possible.

We can now prove 0.3a. By Jacobi's triple product identity

$$\eta(8z)^3 = \sum \chi_{-4}(n) n q^{n^2},$$

so that $\eta(8z)^3 \in S_{3/2}(\Gamma_0(64), \chi_1)$ by [Sh 2]. From the dimension formulas of [C-O] we see that 64 is the exact level of $\eta(8z)^3$ (compare [Tun]). Hence $\alpha(z) = \eta(8z)^3 g(z)$ has level 320 and character $\chi_{-20}\,\chi_1\,\chi_{-4} = \chi_5$, as claimed.

To continue we now need the standard

2.4 PROPOSITION. *Let N be an integer divisible by 4 and k an odd integer. Denote the genus of $X_0(N)$ by g and the number of cusps by c. Suppose $f \in S_{k/2}(\Gamma_0(N), \chi)$. Then $f = 0$, or*

$$\mathrm{ord}_{\infty} f \le (2g-2+c)k/4 - c + 1$$

PROOF. Let $l = 4 \operatorname{ord} \chi$. Then $f^l \in S_{kl/2}(\Gamma_0(N))$. Since $4/N$ there are no elliptic elements in $\Gamma_0(N)$, so that the $kl/4$ (symmetric) differential $\omega = f^l (dz)^{kl/4}$ has no poles in the upper half plane. At a cusp $c \ne \infty$ we have $\mathrm{ord}_c \omega \ge l - kl/4$, and $\mathrm{ord}_{\infty}\omega = l\,\mathrm{ord}_{\infty} f - kl/4$. Hence

$$(2g-2)kl/4 = \deg(\omega) = \sum_{x\in X_0(N)} \mathrm{ord}_x w \ge (c-1)(l-kl/4) + l\,\mathrm{ord}_{\infty} f - kl/4,$$

proving the proposition.

Recall that since $4|N$

$$2g-2+c = N/6\prod_{p/N}(1+1/p) \quad \text{and} \quad c=\prod_{p/N}\lambda(N,p)$$

where

$$\lambda(N,p)=\begin{cases}(1+1/p)p^{v_p(N)/2} & \text{if } v_p(N) \text{ is even}\\ 2p^{(v_p(N)-1)/2} & \text{if } v_p(N) \text{ is odd}\end{cases}$$

(see [Sh1], [C-O]). In particular for $N = 320$ there are $c = 24$ cusps and $2g-2+c = 96$. For $f \in S_{5/2}(\Gamma_0(320),\chi)$ we therefore get that $\text{ord}_\infty f \leq 97$ or $f = 0$. By machine it is easy to verify that

$$(2.5) \qquad \text{ord}_\infty\big(T(p^2)\alpha - \chi_{-4}(p)\gamma(p)\alpha\big) \geq 98 \quad \text{for } p=3 \text{ or } 7 \leq p \leq 31.$$

Indeed, let us expand $\alpha(z) = \sum \alpha_n q^n$. It is easily seen that $\alpha_n = 0$ unless $n \equiv 3 (\text{mod } 8)$. Hence to verify 2.5 above, it is enough to calculate $[(98+5)/8]31^2 = 11532$ such α_n, and Atkin [A] has already gone much further (we nonetheless confirmed this!). Hence for $p = 3$ or $7 \leq p \leq 31$

$$T(p^2)\alpha = \chi_{-4}(p)\gamma(p)\alpha.$$

The space $S_{5/2}(\Gamma_0(320),\chi_5)$ is 60-dimensional by [C-O], and it has a basis $\beta_1,\ldots,\beta_{60}$ diagonal for all the $T(p^2)$. Writing $\alpha = \sum a_i\beta_i$ we see that the only $a_i \neq 0$ are the coefficients of those of the β_i which satisfy

$$T(p^2)\beta_i = \chi_{-4}(p)\gamma(p)\beta_i \quad \text{for } p=3 \text{ or } 7\leq p\leq 31,$$

so that by [Sh2],

$$T(p)\text{Sh}(\beta_i) = \chi_{-4}(p)\gamma(p)\text{Sh}(\beta_i)$$

for these p, where $\text{Sh}(\beta_i)$ is the image of β_i under the Shimura correspondence. Denote by $\rho_2(\phi)$, $\rho_2(\beta_i)$ the 2-dimensional 2-adic Galois representations associated by Deligne [D] to ϕ and to $\text{Sh}(\beta_i)$. Since the level of $\text{Sh}(\beta_i)$ divides 320 we see that these representations are unramified outside 2 and 5 and furthermore satisfy

a. $\det\rho_2(\beta_i)(Frob_p) = \det\rho_2(\phi)(Frob_p)$ for any prime $p \neq 2,5$
(in fact this determinant is the third power of the cyclotomic character).

b. $\text{Tr}\rho_2(\beta_i)(Frob_p) = \text{Tr}\rho_2(\phi)(Frob_p)$ for a prime $p=3$ or $7 \leq p \leq 31$.

c. For $p=3$ the above trace is even (see 2.1).

For 2-dimensional, 2-adic Galois representations unramified outside 2 and 5 it turns out that if their trace is even for $Frob_3$ it is always even (proposition 4.10). In our case we see that the traces of $\rho_2(\phi)\otimes\chi_{-1}$ and $\rho_2(\beta_i)$ are even. We now invoke the crucial theorem 4.3: if the traces and determinants of such representations agree for sufficiently many Frobenius elements, they are equal for all $p \neq 2,5$. Proposition 4.11a guarantees that the set $\{Frob_p\}$, for $p = 3$ or $7 \leq p \leq 31$ suffices for this purpose. This shows that $\rho_2(\phi)\otimes\chi_{-1}$ and

$\rho_2(\beta_i)$ when $a_i \neq 0$, all have isomorphic semi-simplifications. Going back to the corresponding β_i we see they all have the same eigenvalues for all the $T(p^2)$ for $p \neq 2, 5$. We conclude that α is an eigenform for these $T(p^2)$ with the same eigenvalues, which shows that $\mathrm{Sh}(\alpha) = \phi \otimes \chi_{-1}$, proving the rest of theorem 0.3.

2.6 REMARK. It is possible to prove theorem 0.3 without the use of Galois representations, but this seems to require either knowing a base for $S_4(\Gamma_0(160))$, (where $\mathrm{Sh}(\beta_i)$ must lie by results of Niwa), or at least knowing how many Fourier coefficients of a form in this space determine it. One could then proceed as above, but with a much heavier calculation. The method we chose, therefore, seems the most convenient.

3. The L-function To be able to apply theorem 4.3 we first prove the following

3.1 PROPOSITION. *$\#W_{10}(\mathbb{F}_p)$ is even for $p > 5$.*

PROOF. The symmetric group S_{10} acts on W_{10} by permuting coordinates. The orbit of a point $P \in \mathbb{P}^9$ consists of an odd number of points if and only if P is fixed by a 2-Sylow subgroup H of S_{10}. We may assume H is the standard one, generated by (12), (13)(24), (15)(26)(37)(48) and (9,10). But then there are precisely two points on $W_{10}(\mathbb{F}_p)$ fixed by H when $p > 5$, namely (0,0,0,0,0,0,0,0,1,-1) and (1,1,1,1,-1,-1,-1,-1,0,0). The proposition follows.

3.2 COROLLARY. *For $p > 5$ $\mathrm{Tr}\, Frob_p \mid H^7(\widetilde{W}_{10} \times \mathrm{spec}\, \overline{\mathbb{Q}}, \mathbb{Q}_l)$ is even.*

PROOF. By theorem 1.8b

$$\begin{aligned}\#W_{10}(\mathbb{F}_p) &= \sum_{i=0}^{7} p^i - 84p^3 + 42p^4 - \mathrm{Tr}\, Frob_p \mid H^7(\widetilde{W}_{10} \times \mathrm{spec}\, \overline{\mathbb{Q}}, \mathbb{Q}_l) \\ &\equiv \mathrm{Tr}\, Frob_p \mid H^7(\widetilde{W}_{10} \times \mathrm{spec}\, \overline{\mathbb{Q}}, \mathbb{Q}_l) (\mathrm{mod}\ 2)\end{aligned}$$

(this is a congruence between integers!) q.e.d.

PROOF OF THEOREM 0.2. Let $\rho = \{\rho_l\}$ be the compatible system of two dimensional Galois representations attached to $H^7(\widetilde{W}_{10} \times \mathrm{spec}\, \overline{\mathbb{Q}}, \mathbb{Q}_2)$. Then

1 ρ is unramified outside 2,3, and 5, since $\widetilde{W}_{10}$ has good reduction modulo $p > 5$.

2 $\mathrm{Tr}\, \rho \equiv 0 (\mathrm{mod}\ 2)$ by 3.2.

3 By Poincaré duality $\det \rho = \chi^7$, where χ is the cyclotomic character.

On the other hand let $\rho' = \chi^2 \rho(\phi)$, with $\rho(\phi) = \{\rho_l(\phi)\}$ the compatible system of two dimensional Galois representations, attached to ϕ by Deligne ([D]). Here

1' ρ' is unramified outside 2 and 5.

2' $\mathrm{Tr}\, \rho' \equiv 0 (\mathrm{mod}\ 2)$. Indeed by 4.10 we need only to verify that

$$\mathrm{Tr}\, \rho'(Frob_3) = \chi^2(Frob_3) \mathrm{Tr}\, \rho(\phi)(Frob_3) = 3^2(-8) \quad \text{(see 2.1)}$$

is even (compare section 2).

3' Again $\det \rho' = \chi^4 \det \rho(\phi) = \chi^7$.

By 4.11b it will follow that $L(\rho_2, s) = L(\rho'_2, s)$ up to Euler factors at 2, 3 and 5 provided $\operatorname{Tr}\rho(Frob_p) = \operatorname{Tr}\rho'(Frob_p)$ for $7 \le p \le 73$. This was easily verified on a machine, by calculating $V_{10}(p)$ and using 1.7 for the left hand side, and by expanding ϕ for the right hand side. Atkin's calculation [A] is an independent confirmation, since we now know theorem 0.3.

As in the proof of theorem 1.9 we obtain that ρ and ρ' have isomorphic semi-simplifications so that $L(\rho, s) = L(\rho', s) = L(\rho(\phi), s-2)$. But by [Car] $L(\rho(\phi), s) = L(\phi, s)$. Theorem 0.2 follows.

3.3 REMARK. It follows from theorem 0.2 that ρ is unramified at 3. One might try to prove this directly, since such a proof would eliminate the need for proposition 3.1 and reduce the number of primes needed to be checked. However in characteristic 3 the variety W_{10} acquires new singularities, such as $(\varepsilon_i)_{i=1}^{10} = (-1, -1, 1, 1, 1, 1, 1, 1, 1, 1)$. Using Y_5 (see 1.8) one can obtain a scheme Z over $\mathbb{Z}_3$ whose special fiber Z_0 has only ordinary double points and which satisfies $\dim H^3(Z_0, \mathbb{Q}_2) = 2 = \dim H^3(Z \times \overline{\mathbb{Q}}_3, \mathbb{Q}_2)$. However the local invariant cycle theorem, which would imply $H^3(Z_0, \mathbb{Q}_2) \simeq H^3(Z \times \operatorname{spec}\overline{\mathbb{Q}}_3, \mathbb{Q}_2)^{I_3}$, where I_3 is the inertia group, is not known in the mixed characteristic case. Therefore we must content ourselves with our global proof that ρ is unramified at 3.

4. A theorem of Serre on Galois representations

We describe here, with some additions and generalizations, results of Serre communicated in letters [LFS1-3] and conversations. They are based on "Faltings' method" [F], which gives, in principle, a way to verify the equivalence of two l-adic representations. Serre's result is an instance of this method, and gives a simple and easily checkable criterion in a special case.

The following terminology will be useful.

4.1 DEFINITION. *A subset T of a (finite dimensional) vector space V is non-quadratic (respectively non-cubic) if every homogeneous polynomial of degree $d = 2$ (respectively $d = 3$) on V which vanishes on T vanishes on V.*

Observe that if T is non-cubic it is a fortiori non-quadratic, and that the image of a non-quadratic (respectively non-cubic) subset $T \subset V$ under a surjective linear map is non-quadratic (respectively non-cubic). In case V is a vector space over $\mathbb{Z}/2\mathbb{Z}$ we moreover have

4.2. A function $f : V \longrightarrow \mathbb{Z}/2\mathbb{Z}$ is represented by a homogeneous polynomial of degree d if and only if $\sum_{I \subset \{0,\dots,d\}} f(\sum_{i \in I} v_i) = 0$ for any subset $\{v_i\}_{i=0}^{d} \subset V$, and $f(0) = 0$.

4.3 THEOREM. *Let K be a global field, S a finite set of primes of K, and E a finite extension of $\mathbb{Q}_2$. Denote the maximal ideal in the ring of integers of E by $\wp$ and the compositum of all quadratic extensions of K unramified outside S by K_S. Suppose $\rho_1, \rho_2 : \operatorname{Gal}(\overline{K}/K) \longrightarrow \operatorname{GL}_2(E)$ are continuous representations, unramified outside S, and furthermore satisfying*

1. $\mathrm{Tr}\,\rho_1 \equiv \mathrm{Tr}\,\rho_2 \equiv 0 \pmod{\wp}$ *and* $\det \rho_1 \equiv \det \rho_2 \pmod{\wp}$.
2. *There exists a set* T *of primes of* K, *disjoint from* S, *for which*
   i. *The image of the set* $\{Frob_t\}_{t\in T}$ *in (the* $\mathbb{Z}/2\mathbb{Z}$*-vector space)* $\mathrm{Gal}(K_S/K)$ *is non-cubic.*
   ii. $\mathrm{Tr}\,\rho_1(Frob_t) = \mathrm{Tr}\,\rho_2(Frob_t)$ *and* $\det \rho_1(Frob_t) = \det \rho_2(Frob_t)$ *for all* $t \in T$.

Then ρ_1 *and* ρ_2 *have isomorphic semi-simplifications.*

Recall first that the Frattini subgroup G^* of a finite p-group or a pro-p group is defined by the natural exact sequence

4.4. $$1 \longrightarrow G^* \longrightarrow G \longrightarrow G^{ab}/(G^{ab}) \longrightarrow 1$$

where G^{ab}, in the pro-p case, stands for the separated abelianization.

4.5 LEMMA. *Let* G *be a finite 2-group and* $T \subset G/G^*$ *a non-quadratic subset. Assume each* $t \in T$ *has a lift to an element of order 2 in* G. *Then* $G^* = 1$.

PROOF. Assume $G^* \neq 1$. Then G^* contains a subgroup N of index 2 which is normal in G. The assumptions of the lemma still hold if we replace G by G/N, except that now the Frattini subgroup has order 2.

We may thus suppose that $G^* \simeq \mathbb{Z}/2\mathbb{Z}$. The extension 4.4 (with $p = 2$) cannot be trivial, since the Frattini subgroup of $G/G^* \times \mathbb{Z}/2\mathbb{Z}$ is 1. Let $q \in H^2(G/G^*, \mathbb{Z}/2\mathbb{Z})$ be its class. The assumption of the lemma means that q, restricted to each subgroup $\langle t \rangle$, for $t \in T$, is trivial. It is well known that $H^*(G/G^*, \mathbb{Z}/2\mathbb{Z})$ is a polynomial algebra over the $\mathbb{Z}/2\mathbb{Z}$-vector space G/G^*. In particular q may be considered as a quadratic homogeneous polynomial. As such it is non-trivial and yet vanishes on the non-quadratic subset T, a contradiction.

4.6 REMARKS. a. The analog of lemma 4.5 for a prime $p \neq 2$ is false: the group of unipotent, $n \times n$ upper triangular matrices over $\mathbb{F}_p$ is non-abelian for $n \geq 3$ yet has no element of order $> p$.

b. It is certainly possible to write the above proof without using any cohomology.

We shall use Faltings' method to prove the following

4.7 PROPOSITION. *Let* G *be a pro-2 group which is topologically finitely generated, and let* E *be a finite extension of* $\mathbb{Q}_2$ *with ring of integers* $\mathcal{O}$ *and maximal ideal* $\wp$. *Suppose* $\rho_1, \rho_2 : G \longrightarrow \mathrm{GL}_2(E)$ *are continuous representations and* $\Sigma \subset G$ *a subset satisfying*

a. *The image of* Σ *in (the* $\mathbb{Z}/2\mathbb{Z}$*-vector space)* G/G^* *is non-cubic.*

b. $\mathrm{Tr}\,\rho_1(\sigma) = \mathrm{Tr}\,\rho_2(\sigma)$ *and* $\det \rho_1(\sigma) = \det \rho_2(\sigma)$ *for all* $\sigma \in \Sigma$.

Then ρ_1 *and* ρ_2 *have isomorphic semi-simplifications.*

PROOF. Since G is compact it preserves a full lattice in E^2 when acting via either ρ_i, for $i = 1, 2$. Since $\mathcal{O}$ is a discrete valuation ring such a lattice is free over $\mathcal{O}$. Hence we may assume that $\rho_i(G) \subset \mathrm{GL}_2(\mathcal{O})$.

Set $k = \mathcal{O}/\wp$ and pick any $g \in G$. As $|k^\times|$ is odd and G is a pro-2 group det $\rho_i(g) \equiv 1(\mathrm{mod}\,\wp)$, and for the same reason Tr $\rho_i(g) \equiv 0(\mathrm{mod}\,\wp)$.

Set $M_2 = \mathrm{Mat}_{2\times 2}(\mathcal{O})$. For $g \in G$ let $\rho : G \longrightarrow M_2 \times M_2$ be the map $\rho(g) = (\rho_1(g), \rho_2(g))$. and let M be the (linear) $\mathcal{O}$-span of $\rho(G)$. Clearly M is a sub-algebra with unity of $M_2 \times M_2$, free of rank ≤ 8 over $\mathcal{O}$, and spanned by $\{\rho(g)|g \in G\}$ as an $\mathcal{O}$-module.

Let $R = M/\wp M$ and denote the image of an element $g \in G$ in R by $\bar{g}$. Set $\Gamma = \{\bar{g}|g \in G\}$. Then $\Gamma \subset R^\times$ and R is a k-algebra, spanned as a k-vector space by Γ, and satisfying $\dim_k R \leq 8$. We will show the following

ASSERTION. *R is spanned over k by* $\{\bar{\sigma}|\sigma \in \Sigma \cup \{1\}\}$.

PROOF. Let $\sigma \in \Sigma$. We first claim that $\bar{\sigma}^2 = \bar{1}$ in Γ. To see this set $d = \det \rho_1(\sigma) = \det \rho_2(\sigma), t = \mathrm{Tr}\, \rho_1(\sigma) = \mathrm{Tr}\, \rho_2(\sigma)$. By the Cayley-Hamilton theorem (for two by two matrices!) we have $\rho(\sigma)^2 = t\rho(\sigma) - d(I, I)$ in $M_2 \times M_2$, where I is the identity matrix. Since (I, I), $\rho(\sigma)$ and $\rho(\sigma)^2$ belong to M this equality holds in M. Reducing modulo $\wp M$ we obtain $\bar{\sigma}^2 = 1$ in R, since $t \in \wp$ and $d \equiv 1(\mathrm{mod}\,\wp)$ as explained above. Hence $\bar{\sigma} = 1$ in Γ as well, as claimed.

We are now in a position to apply lemma 4.5. Since $G/G^* \longrightarrow \Gamma/\Gamma^*$ is surjective, the image of Σ in Γ/Γ^* is certainly non-quadratic. By lemma 4.5 $\Gamma^* = 1$ and hence $\Gamma \simeq (\mathbb{Z}/2\mathbb{Z})^r$ for some r.

In particular we see that Γ is commutative; since Γ spans R, R is a commutative ring. Let $r \in R$ be any element and write $r = \sum_{\gamma \in \Gamma} k_\gamma \gamma$, with $k_\gamma \in k$. Then $r^2 = (\sum k_\gamma^2)\bar{1}$. Hence $r^2 = 0$ or else r is invertible.

It follows that R is a local Artin algebra over k with maximal ideal

$$P = \{r \in R | r^2 = 0\},$$

and $R/P \simeq k$. We claim that $P^4 = 0$: if not, pick $x_1, \ldots, x_4 \in P$ such that $\prod x_i \neq 0$. Let $T_1, \ldots, T_4$ be variables, and set

$$R_0 = k[T_1, \ldots, T_4]/(T_1^2, \ \ldots, T_4^2).$$

The map of k-algebras $R_0 \longrightarrow R$ defined by $T_i \longrightarrow x_i$ would then be well-defined and injective, but $\dim_k R_0 = 16 > 8 \geq \dim_k R$, a contradiction. Observe in passing that $\bar{1} + \gamma$ belongs to P for any $\gamma \in \Gamma$, since $(\bar{1} + \gamma)^2 = 0$.

Let $\alpha : R \longrightarrow \mathbb{Z}/2\mathbb{Z}$ be any additive map satisfying $\alpha(\bar{1}) = 0$. We claim that the restriction of α to Γ is a homogeneous cubic polynomial with respect to the (multiplicative!) $\mathbb{Z}/2\mathbb{Z}$ vector space structure of Γ. Indeed by 4.2 we need only verify that for any subset $\{\gamma_i\}_{i=0}^3 \subset \Gamma$

$$\sum_{I \subset \{0,\ldots,3\}} \alpha\Big(\prod_{i \in I} \gamma_i\Big) = \alpha\Big(\prod_{i=0}^{3}(\bar{1} + \gamma_i)\Big) = 0,$$

but $\prod_{i=0}^3(\bar{1} + \gamma_i) \in P^4 = 0$, proving the assertion.

Now let $\alpha : R \longrightarrow k$ be a k-linear map such that $\alpha(\bar{\sigma}) = 0$ for any $\sigma \in \Sigma \cup \{1\}$, and let $\beta : k \longrightarrow \mathbb{Z}/2\mathbb{Z}$ be any additive map. Since the image of Σ in G/G^*

is non-cubic, $\{\bar{\sigma}|\sigma \in \Sigma\}$ is non-cubic in Γ. By the above $\beta(\alpha(\gamma)) = 0$ for any $\gamma \in \Gamma$, so that $\alpha(\gamma) = 0$ as well. Since the k-span of Γ is all of R, $\alpha = 0$. This proves that $\{\bar{\sigma}|\sigma \in \Sigma \cup \{1\}\}$ spans R as a k- vector space.

From Nakayama's lemma we now see that the $\mathcal{O}$-span of $\{\rho(\sigma)|\sigma \in \Sigma \cup \{1\}\}$ is all of M. In particular let $\alpha : M \longrightarrow \mathcal{O}$ be the map $\alpha(a,b) = \mathrm{Tr}\ a - \mathrm{Tr}\ b$. It is clear that α is $\mathcal{O}$-linear and that $\alpha(I,I) = 0$. By assumption, $\alpha(\rho(\sigma)) = 0$ for any $\sigma \in \Sigma$. It follows that $\alpha = 0$, and in particular $\mathrm{Tr}\ \rho_1(g) = \mathrm{Tr}\ \rho_2(g)$ for any $g \in G$. This concludes the proof of the proposition.

4.8 REMARK. Serre proved proposition 4.7 for $K = \mathbb{Q}$, $E = \mathbb{Q}_2$ and assuming that Σ *surjects* onto G/G^*. He suggested to merely assume that the image of Σ in G/G^* is non-quadratic. Surprisingly, this turns out not to be sufficient. Here is a counterexample: set

$$a = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad b = \begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix} \quad c = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

and define $\sigma_1 = (a,a)$, $\sigma_2 = (b,b)$ and $\sigma_3 = (c,-c)$. Let G be the closure of the subgroup generated by σ_1, σ_2, and σ_3 in $\mathrm{GL}_2(\mathbb{Z}_2)^2$ and let ρ_i denote the projection on the i'th coordinate, $i = 1,2$. Put

$$\Sigma = \{\sigma_1, \sigma_2, \sigma_1\sigma_2, \sigma_3, \sigma_1\sigma_3, \sigma_2\sigma_3\}.$$

Then $\det \rho_1(g) = \det \rho_2(g)$ for any $g \in G$ and $\mathrm{Tr}\,\rho_1(\sigma) = \mathrm{Tr}\,\rho_2(\sigma)$ for any $\sigma \in \Sigma$. One checks that, in the notation of the proof of 4.7

$$M = \{(x,y) \in P \times P \mid x - y \in 2P\} \quad \text{where}$$
$$P = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{Mat}_{2\times 2}(\mathbb{Z}_2) \;\middle|\; 2|c,\ 2|a-d \right\},$$

that $R = M/2M \simeq \mathbb{Z}/2\mathbb{Z}[T_1,T_2,T_3]/(T_1^2,T_2^2,T_3^2)$ under the map sending T_i to $\bar{\sigma}_i + \bar{1}$, that $G/G^* \simeq \Gamma \simeq (\mathbb{Z}/2\mathbb{Z})^3$ with basis $1+T_i$, $i = 1,2,3$, and that the image of Σ in Γ is therefore non-quadratic. However

$$\sigma_1\sigma_2\sigma_3 = \left(\begin{pmatrix} 1 & 1 \\ 2 & 1 \end{pmatrix}, - \begin{pmatrix} 1 & 1 \\ 2 & 1 \end{pmatrix} \right),$$

so that $\mathrm{Tr}\ \rho_1(\sigma_1\sigma_2\sigma_3) \neq \mathrm{Tr}\ \rho_2(\sigma_1\sigma_2\sigma_3)$. Hence ρ_1 and ρ_2 do not have isomorphic semi-simplifications.

PROOF OF THEOREM 4.3. Let k be the residue field of E and

$$\bar{\chi} : \mathrm{Gal}(\overline{K}/K) \longrightarrow k^\times$$

a character satisfying $\bar{\chi}^2 = \det \rho_i (\mathrm{mod}\ \wp)$ for $i = 1,2$. Such a $\bar{\chi}$ exists and is unique since $|k^\times|$ is odd. Denote by $\chi : \mathrm{Gal}(\overline{K}/K) \longrightarrow E^\times$ the Teichmüller lift of $\bar{\chi}$. Then

$$\det(\rho_i(g)\chi^{-1}(g)) \equiv 1 (\mathrm{mod}\ \wp)$$

and

$$\mathrm{Tr}(\rho_i(g)\chi^{-1}(g)) \equiv 0 (\mathrm{mod}\ \wp)$$

for any $g \in \mathrm{Gal}(\overline{K}/K)$, $i = 1, 2$. It follows that the image G of $\mathrm{Gal}(\overline{K}/K)$ in $\mathrm{Mat}_{2\times 2}(E)^2$ under the map $g \longrightarrow \chi^{-1}(g)(\rho_1(g), \rho_2(g))$ is a pro-2 group. Applying proposition 4.7 to G with the two projections to $\mathrm{GL}_2(E)$ in place of ρ_1 and ρ_2 gives the theorem.

4.9 REMARK. In case the field K in theorem 4.3 is a totally real number field the conditions $\mathrm{Tr}\,\rho(\sigma) = 0$ and $\det \rho(\sigma) = -1$ are in fact equivalent for any representation $\rho : \mathrm{Gal}(\overline{K}/K) \longrightarrow \mathrm{GL}(2, E)$ and any complex conjugation, i.e. an element $\sigma \in \mathrm{Gal}(\overline{K}/K)$ of order 2.

From now till the rest of the section let $K = \boldsymbol{Q}$, $E = \boldsymbol{Q}_2$. We will use the notation of theorem 4.3. We are interested in the cases when the set of primes of ramification is $S = \{2, 5\}$ or $S = \{2, 3, 5\}$. First we have

4.10 PROPOSITION. *Suppose* $\rho : \mathrm{Gal}(\overline{\boldsymbol{Q}}/\boldsymbol{Q}) \longrightarrow \mathrm{GL}_2(\boldsymbol{Q}_2)$ *is unramified outside* $S = \{2, 5\}$ *and* $\mathrm{Tr}\,\rho(Frob_3) \equiv 0 (\mathrm{mod}\ 2)$. *Then* $\mathrm{Tr}\,\rho \equiv 0 (\mathrm{mod}\ 2)$ *identically*.

PROOF. Let $\bar{\rho} : \mathrm{Gal}(\overline{\boldsymbol{Q}}/\boldsymbol{Q}) \longrightarrow \mathrm{GL}_2(\boldsymbol{Z}/2\boldsymbol{Z})$ be the reduction modulo 2 of ρ, and denote by L the extension of $\boldsymbol{Q}$ cut out by $Ker\ \bar{\rho}$. If $\mathrm{Tr}\,\rho \not\equiv 0 (\mathrm{mod}\ 2)$, L is a cyclic cubic extension or an S_3 extension of $\boldsymbol{Q}$, unramified outside 2 and 5. The case $L/\boldsymbol{Q}$ cyclic is manifestly impossible. In fact by [Bu], appendix 4, p. 109, L must be the splitting field of $f(x) = x^3 - x^2 + 2x + 2 = 0$. Since f modulo 3 is irreducible $Frob_3$ has order 3 in $S_3 = \mathrm{Gal}(L/\boldsymbol{Q}) \simeq \mathrm{GL}_2(\boldsymbol{Z}/2\boldsymbol{Z})$. Hence $\mathrm{Tr}\,\rho(Frob_3) = 1$ in $\boldsymbol{Z}/2\boldsymbol{Z}$, a contradiction.

Let S be given as in theorem 4.3. We shall say that a set of primes T, disjoint from S, is *sufficient* for S if T satisfies condition 2a of the theorem.

4.11 PROPOSITION. *a.* $T = \{3, 7, 11, 13, 17, 29, 31\}$ *is sufficient for* $S = \{2, 5\}$. *Moreover 31 can be replaced by* ∞, *where* $Frob_\infty$ *means complex conjugation.*
b. $T = \{7, 11, 13, 17, 19, 23, 29, 31, 41, 43, 53, 61, 71, 73\}$ *is sufficient for* $S = \{2, 3, 5\}$. *Here we can replace 71 by* ∞.

PROOF. For $s \in S \cup \{-1\}$ consider the homomorphism $f_s : \mathrm{Gal}(\boldsymbol{Q}_S/\boldsymbol{Q}) \longrightarrow \boldsymbol{Z}/2\boldsymbol{Z}$ defined by $f_s(g) = (\chi_s(g) + 1)/2$, where χ_s is the quadratic Galois character cutting out $\boldsymbol{Q}(\sqrt{s})$. The map $F = (f_s)_{s \in S} : \mathrm{Gal}(\boldsymbol{Q}_S/\boldsymbol{Q}) \longrightarrow (\boldsymbol{Z}/2\boldsymbol{Z})^r$, where $r = |S| + 1$, is an isomorphism. It is convenient to think of χ_s as a Dirichlet character, so that $\chi_s(Frob_t) = \left(\frac{s}{t}\right)$ for a finite prime $t \in T$, $\chi_s(Frob_\infty) = \left(\frac{-1}{s}\right)$ for $s \in S$, and $\chi_{-1}(Frob_\infty) = -1$. Condition 2a for the given T's follows then by explicitly writing down $\{F(t)\}_{t \in T}$ in each case. In fact in case a this last set is all of $(\boldsymbol{Z}/2\boldsymbol{Z})^r - 0$. In case b it is non-cubic.

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Lettre à J.-F. Mestre

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Cher Mestre,

Je vais essayer de dresser une liste des conjectures (ou "questions") que l'on peut faire dans la direction "formes modulaires - représentations galoisiennes".

Si f est une forme modulaire $(\bmod p)$ sur $\Gamma_0(N)$, de poids k, fonction propre des opérateurs de Hecke $T_{p'}$, pour p' ne divisant pas N, et à coefficients dans $\mathbf{F}_p$, je noterai ρ_f la représentation de $\mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$ à valeurs dans $\mathrm{GL}_2(\mathbf{F}_p)$ correspondant à f. Et je dirai qu'une telle représentation est "modulaire"; et je dirai aussi, si j'en ai besoin, qu'elle est "de niveau N et de poids k". La question la plus ambitieuse que l'on pourrait se poser serait de donner un critère portant sur une représentation

$$\rho : \mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q}) \longrightarrow \mathrm{GL}_2(\mathbf{F}_p)$$

qui permette d'affirmer que cette représentation est bien modulaire de niveau N et de poids k. J'y reviendrai à la fin de cette lettre. Pour l'instant, je vais me concentrer sur les problèmes que pose le poids 2. C'est ce dont on a besoin, si l'on veut prouver que "Weil + $\varepsilon \Rightarrow$ Fermat".

1. Les conjectures sur les formes de poids 2

1.1. Je commence par une conjecture d'aspect assez innocent, où l'on part d'une forme modulaire, et où l'on "diminue" son niveau:

C_1 - Soit ρ une représentation modulaire [1] de poids 2 de niveau N_1N_2, avec $(N_1, pN_2) = 1$. Si ρ est non ramifiée en tous les diviseurs premiers de N_1, alors ρ est modulaire de poids 2 de niveau N_2 (i.e. on "peut enlever N_1").

(Je précise que tous les énoncés C_1, $C_2, \ldots$ sont des *conjectures*.)

Noter que j'ai supposé que p ne divise pas N_1. L'énoncé C_1 n'est donc utile que pour le premier cas de Fermat (plus précisément, Weil + $C_1 \Rightarrow 1^{er}$ cas de Fermat.)

Attention à ceci: C_1 ne signifie pas que la forme modulaire f de poids 2 sur $\Gamma_0(N)$ telle que $\rho = \rho_f$ "provient" de $\Gamma_0(N_2)$, i.e. est une "vieille forme". Cela signifie simplement qu'il existe une forme modulaire f' sur $\Gamma_0(N_2)$, $\bmod p$, fonction propre des opérateurs de Hecke, qui donne lieu à la même représentation ρ. En général, les facteurs

[1] ici, et dans toute la suite, je suppose ρ absolument irréductible, pour éviter des ennuis avec les séries d'Eiseinstein, cf. Remarque 1.1.

locaux de f' relatifs aux diviseurs premiers de N_1 seront différents de ceux de f (ils sont imposés par les valeurs propres des Frobenius de ces nombres premiers).

Il y a de nombreux exemples numériques (nous en avons vérifié ensemble un certain nombre) où l'on peut tester C_1. J'ai donc une certaine confiance en C_1.

Remarque 1.1. - La conjecture C_1 serait fausse si j'acceptais les représentations réductibles. Exemple: prenons $p = 5$, $N = N_1 = 11$, de sorte que la représentation $1 \oplus \chi$ (χ = caractère cyclotomique $\mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q}) \to \mathbf{F}_p^*$) est modulaire de poids 2. Cette représentation est non ramifiée en 11. On devrait donc pouvoir "enlever" 11 de son conducteur, si C_1 s'appliquait. Mais ce n'est pas possible vu que $\Gamma_0(1)$ n'a pas de série d'Eiseinstein en poids 2 (cet ennui disparaîtrait si l'on convenait que la série d'Eiseinstein E_{p+1} "est de poids $2 \bmod p$". Peut-être faudrait-il faire cette convention, i.e. modifier la notion de poids? Je n'y ai pas sérieusement réfléchi.)

1.2. On suppose maintenant que $N = pN'$, avec $(p, N') = 1$, et l'on désire enlever p de N, ce que ne permettait pas C_1. Pour cela, on va regarder ce qu'est l'action de l'inertie en p dans la représentation $\rho = \rho_f$ (où f est sur $\Gamma_0(N)$ et de poids 2, comme d'habitude). On peut prouver que la représentation ρ, restreinte au groupe d'inertie I_p de p, est de l'un des types suivants:

a) $I_p \longrightarrow \mathbf{F}_{p^2}^* \longrightarrow \mathrm{GL}_2(\mathbf{F}_p)$, où $I_p \to \mathbf{F}_{p^2}^*$ est le "caractère fondamental de niveau 2", au sens de mon article dans *Invent.* 1972; je dirai que ce cas est celui du "niveau 2". On démontre qu'il se produit si et seulement si la valeur propre de f par U_p est 0.

b) Une extension $\left(\begin{smallmatrix} \chi & * \\ 0 & 1 \end{smallmatrix}\right)$ du caractère unité (en quotient) par le caractère cyclotomique χ (en sous-objet). Une telle extension est classée par un élément de K^*/K^{*p}, où K est l'extension non ramifiée maximale de $\mathbf{Q}_p$; la valuation de K définit un homomorphisme $K^*/K^{*p} \longrightarrow \mathbf{Z}/p\mathbf{Z}$, d'où un *élément canonique*[2] $e \in \mathbf{Z}/p\mathbf{Z}$ attaché à l'extension. (Dans le cas que vous connaissez bien, où f provient d'une courbe elliptique de conducteur N, cet élément e n'est autre que l'*exposant de* p dans le discriminant du modèle minimal de la courbe, réduit $\bmod p$ - à moins que ce ne soit son opposé, je n'ai pas le temps de faire le calcul ...) Lorsque cet invariant e est 0, je dirai que ρ est "peu ramifiée en p". Cette notion peut d'ailleurs se présenter d'autres façons: elle est équivalente, par exemple, à ce que la représentation ρ se prolonge en un schéma en groupes de type (p,p) *fini* et plat en p.

(De façon générale, je crois qu'il y aurait intérêt à introduire le terme "ρ est finie en p" pour dire que ρ se prolonge en un schéma en groupes de type (p,p) fini et plat en p. C'est le cas par exemple dans le cas a) ci-dessus.)

Si f est de niveau N', il est clair que ρ est finie en p, donc est soit du type a), soit du type b) peu ramifié. La conjecture C_2 dit que la réciproque est vraie:

C_2 **- Soit** ρ **une représentation modulaire (absolument irréductible) de poids 2 de niveau** $N = pN'$ **avec** $(p, N') = 1$. **Si** ρ **est finie en** p **au sens ci-dessus, alors** ρ **est modulaire de poids 2 de niveau** N'.

[2]pas tout à fait canonique! Il dépend d'une certaine identification. Mais sa nullité, ou non nullité, est vraiment canonique.

Cette conjecture paraît bien plus accessible que C_1. Je ne serais pas surpris que le cas où ρ est de type a) (niveau 2) soit déjà dans la littérature, sous une forme un peu déguisée. Quant au cas b) peu ramifié, Mazur devrait pouvoir le démontrer ...

Quoiqu'il en soit, il est clair que:

Weil $+C_1 + C_2 \Rightarrow$ Fermat.

1.3. On pourrait supposer que $N = p^2N'$, avec $(p, N') = 1$, et vouloir "enlever p^2" du conducteur de la représentation. Je n'ai pas suffisamment d'exemples de ce cas pour être sûr de la conjecture à faire. L'énoncé le plus naturel serait essentiellement le même que C_2, i.e.:

C_3 - **Si ρ est modulaire de poids 2 de niveau N, et si ρ est finie en p, alors ρ est modulaire de poids 2 de niveau N'.**

On n'en a pas besoin pour Fermat.

2. Conjectures générales sur les représentations

J'ai fait ces conjectures en 1974, mais je n'en ai publié qu'une forme édulcorée, en me bornant au niveau 1 (Astérisque 24-25, 1975, p. 109-117). Voici la forme la plus optimiste:

2.1. Définitions et conjectures

Je pars d'une représentation

$$\rho \ : \ \mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q}) \longrightarrow \mathrm{GL}(V),$$

où V est un espace vectoriel de dimension 2 sur $\mathbf{F}_q$. Le déterminant

$$\det \rho \ : \ \mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q}) \longrightarrow \mathbf{F}_q^*$$

est un caractère de degré 1. Je supposerai que ce caractère est *impair*, i.e. que $\det(\rho(c)) = -1$, où c est la multiplication complexe, vue comme élement de $\mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$. (Formulation équivalente: les valeurs propres de $\rho(c)$ sont 1 et -1.)

Sous sa forme la plus brutale, la conjecture dit :

C_4 -**La représentation ρ est modulaire.**

On a envie de préciser ça, en donnant le niveau, le caractère (car il s'agira de formes "de Nebentypus") et si possible le poids.

Commençons par définir le niveau (i.e. le *conducteur d'Artin*) de ρ; ici, l'hypothèse $\dim V = 2$ ne joue aucun rôle. On copie Artin, mais en se bornant aux nombres premiers $\neq p$. Autrement dit, si $l \neq p$, et si G_i $(i = 0, 1, \ldots)$ désignent les groupes de ramification en l (dans le groupe $G = \mathrm{Im}(\rho)$, disons), on pose

$$n(l) = \sum_{i=0}^{\infty} \frac{g_i}{g_0} \operatorname{codim} V^{G_i},$$

avec $g_i = |G_i|$.

C'est l'exposant du conducteur. Le conducteur lui-même est bien sûr:

$$N = N(\rho) = \prod_{l \neq p} l^{n(l)}$$

Par définition même, il est premier à p.

Passons à la définition du caractère ε. On constate facilement que le conducteur de $\det(\rho)$ est de la forme N' ou pN', avec N' divisant N. On en déduit sans mal que $\det(\rho)$ peut se décomposer de façon unique sous la forme

$$\det(\rho) = \varepsilon\chi^{k-1},$$

où χ est le caractère cyclotomique $\mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q}) \longrightarrow \mathbf{F}_p^* \subset \mathbf{F}_q^*$, où ε est un caractère $(\mathbf{Z}/N\mathbf{Z})^* \longrightarrow \mathbf{F}_q^*$ (vu comme caractère de $\mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$, bien sûr), et où k est un élément de $\mathbf{Z}/(p-1)\mathbf{Z}$ tel que $\varepsilon(-1) = (-1)^k$ (i.e. k et ε ont la même parité).

[Noter le cas un peu effrayant $p = 2$, où la condition $\det(\rho(c)) = -1$ est vide du fait que $-1 = 1$!, et où $\chi = 1$. J'y reviendrai.]

Je peux maintenant préciser C_4 ainsi:

C_5 - La représentation ρ (supposée de déterminant impair) est modulaire de niveau N, caractère ε, et poids $k \in \mathbf{Z}/(p-1)\mathbf{Z}$.

(Autrement dit, elle provient d'une forme modulaire sur $\Gamma_0(N)$ à coefficients dans $\mathbf{F}_q$, de type (k, ε), pour un certain poids k appartenant à la classe voulue dans $\mathbf{Z}/(p-1)\mathbf{Z}$.)

C'est vraiment une conjecture très optimiste (Deligne, en 1974, était très sceptique). Il est difficile de la tester, vu que nous ne savons guère construire des représentations autrement qu'avec des formes modulaires. J'ai tout de même vérifié un certain nombre de cas (par exemple $q = 2$, où $\mathrm{GL}(V) = S_3$, ce qui conduit à des corps cubiques - qui peuvent être totalement réels, ça marche quand même).

Voici quelques exemples qu'il serait intéressant d'étudier:

Exemple 1 - On part d'une courbe de genre 2 sur $\mathbf{Q}$, qui donne lieu à une représentation $(\bmod p)$ de degré 4, quel que soit p. On choisit p de telle sorte que la représentation soit extension de deux représentations de degré 2 (ça doit se produire de temps en temps, je suppose - et ça ne doit pas être trop difficile à tester). Les représentations ainsi obtenues sont-elles modulaires?

Exemple 2 - On prend $q = 4$, de sorte que $\mathrm{SL}(V) = A_5$. Se donner ρ revient donc à choisir un corps quintique à groupe de Galois A_5 (si l'on prend $\varepsilon = 1$). Si ce corps est imaginaire (et si la conjecture d'Artin est vraie), il y a une forme modulaire correspondante, et C_5 est vraie; inutile donc de tester ce cas. Mais si le corps quintique est totalement réel, on ne peut pas appliquer cet argument. C'est ce cas qu'il faudrait examiner sur des exemples, car cela permettrait peut-être de contre-exempler C_5 pour $p = 2$. Vous devriez en parler à Buhler.

(Je précise ce qu'il faudrait faire: partir d'un corps quintique totalement réel K, à groupe de Galois A_5, et non ramifié en 2 (pour simplifier). Calculer le conducteur N de la représentation correspondante. Chercher s'il existe une forme modulaire de poids 2 (cf. ci-dessous) sur $\Gamma_0(N)$, à coefficients dans $\mathbf{F}_4$, qui donne cette représentation. Si N n'est pas trop grand, cela devrait pouvoir se faire, vu qu'il s'agit de calculs en caractéristique 2.)

2.2. Le poids

Vous remarquerez que la conjecture C_5 ne précise pas exactement le poids k, mais seulement sa classe $\text{mod}(p-1)$. Il devrait être possible de faire mieux, en fonction du *type de ramification* en p de la représentation ρ. La question est liée, d'une part aux caractères fondamentaux de mon article d'Inventiones, d'autre part aux "cycles de Tate" décrits dans N. Jochnowitz, TAMS 270 (1982), p.253-267. Je n'ai pas envie d'entrer là-dedans pour le moment (il faudra bien que je le fasse un jour - ou que quelqu'un d'autre s'en charge...). Je vais me borner au cas du poids 2:

C_6 - **Dans les conditions de C_5, supposons $k \equiv 2 \text{ mod}(p-1)$, et supposons ρ absolument irréductible. Alors, pour que l'on puisse choisir k égal à 2, il faut et il suffit que la représentation ρ soit finie en p, au sens de 1.2.**

Cette conjecture *entraîne* les conjectures C_1 et C_2 de la partie 1. Elle est bien plus forte: à elle seule, elle *suffit à entraîner Fermat*, sans que l'on ait besoin de Weil! C'est dire qu'elle est probablement trop optimiste: il faut essayer de la démolir par des contre-exemples...

(Autre conséquence de la conjecture: inexistence de schémas en groupes finis et plats sur $\mathbf{Z}$, de type (p,p), et irréductibles.)

Une dernière remarque, avant d'aller mettre cette lettre à la boîte: je n'ai pas précisé ce que j'entends par "forme modulaire $\text{mod}\, p$" sur $\Gamma_0(N)$; est-ce quelque chose qui est défini seulement en caractéristique p, ou est-ce la réduction $(\text{mod}\, p)$ d'une forme en caractéristique 0? Ce n'est pas pareil; vous savez bien qu'il y a une forme modulaire de niveau 1 et de poids 1 $(\text{mod}\, 2)$, à savoir "a_1", et aussi une forme de niveau 1 et de poids 2 $(\text{mod}\, 3)$. Mais je crois que la différence entre les deux est de nature trop triviale pour que cela ait de l'importance, tant qu'on se borne aux représentations absolument irréductibles (mais je ne l'ai pas vérifié: c'est juste un sentiment!). Notez d'ailleurs que ce problème ne se pose guère que pour C_6, où l'on précise le poids.

Bien à vous, et bon séjour à Arcata,

J.-P. Serre

P.S - Vous pourriez être surpris par la formulation de C_5, où l'on se place en niveau premier à p. Il faut se rappeler que toute forme modulaire sur $\Gamma_0(p^\alpha N)$ est p-adiquement

sur $\Gamma_0(N)$: cela se voit par voie géométrique (cf. par exemple Katz, probablement dans LN 350 et 601), ou par voie élémentaire (j'ai donné l'argument, dans le cas particulier $N = 1$, dans LN 350 pour $\alpha = 1$, et pour $\alpha \geq 2$ dans l'Ens.Math. 22 (1976), p.227-260). Mais, bien sûr, cette réduction de niveau entraîne en général une augmentation du poids. Le cas le plus simple est celui des formes de poids 2 sur $\Gamma_0(pN)$ qui deviennent de poids $p + 1 \mod p$ sur $\Gamma_0(N)$.

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A SURVEY OF THE THEORY OF HEIGHT FUNCTIONS

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ABSTRACT. In this paper we give a brief survey, without proofs, of some of the main facts and Diophantine applications of the theory of height functions. The following topics are covered.
1. Weil heights
2. Metrized line bundles
3. The degree of a metrized line bundle
4. The modular height of an abelian variety
5. Canonical heights on abelian varieties
6. The difference of the Weil and canonical heights
7. Lower bounds for the canonical height
8. Specialization theorems

1. Weil Heights. The *Weil height* of a point in projective space is defined by the following formula.

$$h : \mathbb{P}^n(\bar{\mathbb{Q}}) \longrightarrow \mathbb{R}$$

$$h(x) = h([x_0,\dots,x_n]) = {}^1/_{[K:\mathbb{Q}]} \sum_{v\in M_K} [K_v:\mathbb{Q}_v] \log \max_{0\le i\le n} \{ |x_i| \}$$

Here $K/\mathbb{Q}$ is any number field such that $x \in \mathbb{P}^n(K)$, and M_K is a complete set of inequivalent absolute values on K normalized in the usual manner (cf. [5]).

Let $K/\mathbb{Q}$ be a number field, let V/K be an (irreducible, projective) variety, and let $L \in \mathrm{Pic}(V)$ be a very ample line bundle on V. Choose an embedding $\varphi_L : V \to \mathbb{P}^n$ corresponding to L. (I.e. $\varphi_L^* \mathcal{O}_{\mathbb{P}}(1) \cong L$.) Then we define the *Weil height on V corresponding to L* by composition,

$$h_L : V(\bar{K}) \longrightarrow \mathbb{R}$$

$$h_L(P) = h \circ \varphi_L(P) .$$

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Then, writing an arbitrary line bundle $L \in \mathrm{Pic}(V)$ as a difference of very ample line bundles, $L \cong L_1 \otimes L_2^{-1}$, we define

$$h_L = h_{L_1} - h_{L_2} .$$

The principal facts about Weil height functions are summarized in the following theorem.

Theorem 1.1. *Let* V/K *be a variety defined over a number field.*

(a) *For every line bundle* $L \in \mathrm{Pic}(V)$, *the Weil height* h_L *is well-defined up to* $O(1)$ *(i.e. up to addition of a function bounded on* $V(\bar{K})$.*)*

(b) *(Additivity) If* $L = L_1 \otimes L_2$, *then*

$$h_L = h_{L_1} + h_{L_2} + O(1) .$$

(c) *(Finiteness) Let* $L \in \mathrm{Pic}(V)$ *be an ample line bundle. Then for any constants* C, d, *the set*

$$\{ P \in V(\bar{K}) : h_L(P) \le C \text{ and } [K(P):K] \le d \}$$

contains only finitely many points.

(d) *(Positivity) Let* $L \in \mathrm{Pic}(V)$, *and let* $B \subset V$ *be the set of basepoints of* L. *Then*

$$h_L(P) \ge O(1)$$

for all $P \in V(\bar{K})$ *with* $P \notin B$.

(e) *(Functoriality) Let* $f : W \to V$ *be a morphism of varieties, and let* $L \in \mathrm{Pic}(V)$. *Then*

$$h_{f^*L} = h_L \circ f + O(1) .$$

Proof. (a),(b),(e) [4, ch. 4, thm. 5.1]
(c) [4, ch. 3, thm. 2.6]
(d) [4, ch. 4, thm. 5.2]

2. Metrized Line Bundles. Let V/K be a variety, and let $L \in \mathrm{Pic}(V)$. We say that L is a *metrized line bundle* if it comes equipped with a v-adic metric $\| \ \|_v$ on $L \otimes_K K_v$ for each infinite place $v \in M_K^{\infty}$.

Example 2.1. Let A/K be an abelian variety of dimension g, and consider the sheaf of holomorphic g-forms on A/K, which we denote by

$$\omega_{A/K} = \Lambda^g \Omega^1{}_{A/K} .$$

We make $\omega_{A/K}$ into a metrized line bundle as follows. For each $v \in M_K^\infty$, and each $\alpha \in H^0(A, \omega_{A/K} \otimes_K K_v)$, define $\|\alpha\|_v$ by

$$\|\alpha\|_v = \left[({}^i/_2)^g \int_{A(\bar{K}_v)} \alpha \wedge \bar{\alpha} \right]^{\frac{1}{2}} .$$

3. The Degree of a Metrized Line Bundle. Let R be the ring of integers of K, and let M be a line bundle on Spec(R). Thus M is a projective R-module of rank 1. As above, we say that M is a *metrized line bundle* if it comes equipped with a v-adic metric on $M \otimes_R K_v$ for each $v \in M_K^\infty$. If M is a metrized line bundle, then we define the *degree of M* by the formula

$$\deg M = \log \# {}^M/_{Rx} - \sum_{v \in M_K^\infty} [K_v:\mathbb{Q}_v] \log \|x\|_v .$$

Here $x \in M$ is any non-zero element of M. One easily verifies, using the product formula, that deg M is well-defined independent of the choice of x.

Now let V/K be a variety, $L \in \mathrm{Pic}(V)$ a metrized line bundle, and $P \in V(K)$. Choose a model $\mathcal{V}$/Spec(R) for V/K, and extend L to a line bundle $\mathcal{L} \in \mathrm{Pic}(\mathcal{V})$ and P to a section $P \in \mathcal{V}(\mathrm{Spec}\, R)$. Then $P^*\mathcal{L} \in \mathrm{Pic}(\mathrm{Spec}\, R)$, so we can calculate its degree.

Proposition 3.1. *With notation as above,*

$$[K:\mathbb{Q}]\ h_L(P) = \deg P^*\mathcal{L} + O(1) .$$

(Here the O(1) *naturally depends on the choice of* $\mathcal{V}$ *and* $\mathcal{L}$ *, but is independent of* P*.)*

Proof. [14, prop. 7.2]

4. The Modular Height of an Abelian Variety. Let A/K be an abelian variety, let $\mathcal{A}$/Spec(R) be a Néron model for A/K, and let $s \in \mathcal{A}(R)$ be the zero section. Recall (example 2.1) that we have defined a metric on the line bundle

$$\omega_{\mathcal{A}/R} \otimes_R K_v \cong \omega_{A/K} \otimes_K K_v$$

for each $v \in M_K^\infty$. We can thus compute the degree of the metrized line bundle $s^*\omega_{\mathcal{A}/R} \in \mathrm{Pic}(\mathrm{Spec}\, R)$. Define the *modular height of A/K* by the formula

$$h(A/K) = {}^1/_{[K:\mathbb{Q}]} \deg s^*\omega_{\mathcal{A}/R} .$$

Note that h(A/K) is *not* independent of the field K, since the Néron model $\mathcal{A}$ may change drastically under field extension. However, it is not hard to check that if $\mathcal{A}$/R has everywhere semi-stable reduction, then h(A/K) will not change under field extension.

Example 4.1. Let E/$\mathbb{Q}$ be an elliptic curve, and choose an isomorphism $E(\mathbb{C}) \cong \mathbb{C}/_{\mathbb{Z}+\tau\mathbb{Z}}$ for some τ in the upper half-plane. Further, let $\mathcal{D}_{E/\mathbb{Q}}$ be the minimal discriminant of E/$\mathbb{Q}$ (cf. [13, VIII §8]), let $q = e^{2\pi i\tau}$, and let $\Delta(\tau) = (2\pi)^{-12} q \prod (1-q^n)^{24}$ be the usual modular form. Then

$$12\, h(E/\mathbb{Q}) = \log |\mathcal{D}_{E/\mathbb{Q}}| - \log (|\Delta(\tau)|(\mathrm{Im}\, \tau)^6) .$$

(There is a similar formula for arbitrary number fields. See, e.g., [14, prop. 1.1].)

The two most important facts about the modular height are given by the following two theorems, which were used by Faltings in his proof of the isogeny and Shafarevich conjectures.

Theorem 4.2. *Let $\mathcal{A}/\mathbb{Q}$ be the moduli space of principally polarized abelian varieties of dimension* g, *and let* $h_{\mathcal{A}}$ *be a Weil height function on* $\mathcal{A}$ *corresponding to a fixed embedding* $\mathcal{A} \subset \mathbb{P}^n$. *For* $A/\bar{\mathbb{Q}}$ *an abelian variety and* η *a principal polarization on* A, *let* $j(A,\eta)$ *be the corresponding point of* $\mathcal{A}$.

There exist constants $c_i > 0$ *such that for all abelian varieties* A/K *having everywhere semi-stable reduction and all principal polarizations* η *on* A/K,

$$c_1 h_{\mathcal{A}}(j(A,\eta)) - c_2 \le h(A/K) \le c_3 h_{\mathcal{A}}(j(A,\eta)) + c_4 .$$

Proof. [4],[1]

Corollary 4.3. *For every constant* C, *there are only finitely many* K-*isomorphism classes of abelian varieties* A/K *satisfying* $h(A/K) \le C$.
Proof. [4],[1]

Theorem 4.4. *Let* A/K *be an abelian variety. There are constants* c,d, *depending on* A/K, *such that if* $\varphi : A \to B$ *is an isogeny of abelian varieties over* K *with* $\deg(\varphi) \ge d$, *then*

$$h(B/K) \le h(A/K) + c .$$

Proof. [4],[1]

5. Canonical Heights on Abelian Varieties. Let A/K be an abelian variety, and let $L \in \operatorname{Pic}(A)$. If L is symmetric (respectively anti-symmetric) we define the *canonical height on A relative to L* as follows.

$$\hat{h}_L : A(\bar{K}) \longrightarrow \mathbb{R}$$

$$\hat{h}_L(P) = \lim_{n \to \infty} \begin{cases} n^{-2}\, h_L(nP) & \text{if } [-1]^*L \cong L \\ n^{-1}\, h_L(nP) & \text{if } [-1]^*L \cong L^{-1} . \end{cases}$$

For a general line bundle, we define the canonical height by additivity,

$$\hat{h}_L = \tfrac{1}{2}\left(\hat{h}_{L \otimes [-1]^*L} + \hat{h}_{L \otimes ([-1]^*L)^{-1}} \right)$$

Theorem 5.1. (Néron,Tate) *Let* A/K *be an abelian variety, and let* $L \in \operatorname{Pic}(A)$ *be a symmetric, ample line bundle.*

(a) $\hat{h}_L$ *is a positive definite quadratic form on* $A(\bar{K}) \otimes \mathbb{R}$.

(b) $\hat{h}_L = h_L + O(1)$.

Proof. (a) [5, ch. 5, thm. 7.2]

(b) [5, ch. 5, thm. 3.1]

Both parts of theorem 5.1 give rise to interesting questions. Part (a) suggests the problem of determining the minimum value of $\hat{h}_L$ on the non-torsion points of $A(K)$, while part (b) leads one to inquire how the $O(1)$ bound for the difference $\hat{h}_L - h_L$ depends on the abelian variety A. These questions will be discussed in more detail in the next two sections.

6. The Difference of the Weil and Canonical Heights. The $O(1)$ bound in theorem 5.1b depends on the abelian variety A. In the case of elliptic curves, Dem'janenko [2] and Zimmer [17] have given estimates for this bound in terms of the coefficients of a given Weierstrass equation. Zimmer obtains the following precise estimate.

Theorem 6.1. *Let* E/K *be an elliptic curve given by a Weierstrass equation*

$$E : y^2 = x^3 + ax + b .$$

Let L *be the line bundle corresponding to this embedding. (I.e.* $L = L(3(O))$. *)* *Then for all* $P \in E(\bar{K})$,

$$| \hat{h}_L(P) - h([x(P), y(P), 1]) | \le \tfrac{1}{2} h([a^3, b^2, 1]) + 6 \log 2 .$$

Proof. [17]

There is a similar result for abelian varieties, due to Manin and Zarhin [7]. They estimate the difference between $\hat{h}_L$ and h_L for a particular embedding described by Mumford [8].

More generally, one can consider an algebraic family of abelian varieties, and ask how the difference $\hat{h}_L - h_L$ varies from fiber to fiber. Thus let A/K and T/K be projective varieties, and let $\pi : A \to T$ be a morphism such that almost all fibers are abelian varieties. (I.e. There is a non-empty Zariski open subset $T^0 \subset T$ such that $A^0 = \pi^{-1}(T^0)$ is an abelian scheme over T^0.) Let $L \in \mathrm{Pic}(A)$, and for each $t \in T(\bar{K})$, let $L(t)$ be the restriction of L to the fiber $A(t) = \pi^{-1}(t)$. Finally, fix a Weil height h_ξ on T corresponding to an ample $\xi \in \mathrm{Pic}(T)$.

Theorem 6.2. (Silverman-Tate) *With notation as above, there are constants* c_1, c_2 *such that for all* $t \in T^0(\bar{K})$ *and all* $P \in A(t)(\bar{K})$,

$$| \hat{h}_{L(t)}(P) - h_L(P) | \le c_1 h_\xi(t) + c_2 .$$

Proof. [11]

7. Lower Bounds for the Canonical Height. Let A/K be an abelian variety, and let $L \in \mathrm{Pic}(A)$ be a symmetric, ample line bundle. Then for all $P \in A(\bar{K})$, theorem 5.1a says that $\hat{h}(P) \geq 0$, and further $\hat{h}(P) = 0$ if and only if $P \in A_{tors}$. One might ask for a lower bound for non-zero values of $\hat{h}$.

Conjecture 7.1 (Lang-Silverman) *Let* A/K *be an abelian variety, and let* $L \in \mathrm{Pic}(A)$ *be a symmetric, very ample line bundle. There is a constant* $c > 0$, *depending only on* $\dim(A)$ *and* K, *such that for all non-torsion points* $P \in A(K)$,

$$\hat{h}_L(P) \geq c\, h(A/K) .$$

In words, this conjecture says that if A is arithmetically complicated (as measured by $h(A/K)$), then non-torsion points are also arithmetically complicated (as measured by $\hat{h}(P)$.) The following special case is known for elliptic curves. (See also [12] for another special case dealing with twists of a fixed abelian variety.)

Theorem 7.2. *Let* E/K *be an elliptic curve, and let* $L = L((O)) \in \mathrm{Pic}(E)$ *be the line bundle corresponding to the divisor* (O). *There is a constant* $c > 0$, *depending only on* $[K:\mathbb{Q}]$ *and the number of primes of* K *for which the j-invariant of* E *is not integral, such that for all non-torsion points* $P \in E(K)$,

$$\hat{h}_L(P) \geq c\, h(E/K) .$$

In particular, conjecture 7.1 is true if one restricts attention to elliptic curves with integral j-invariant.

Proof. [10]

Lower bounds such as those given in theorem 7.2 have applications to the problem of giving quantitative versions of classical finiteness theorems in Diophantine geometry. For example, theorem 7.2 plays a crucial role in the proof of the following version of Siegel's theorem for elliptic curves.

Theorem 7.3. *Let* $E/\mathbb{Q}$ *be an elliptic curve given by a minimal Weierstrass equation*

$$y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6 .$$

Let S *be a finite set of primes of* $\mathbb{Z}$, *and let* $\mathbb{Z}_S$ *be the ring of S-integers of* $\mathbb{Q}$.

Further, let

$$r(E) = \text{rank } E(\mathbb{Q});$$
$$E(\mathbb{Z}_S) = \{ P \in E(\mathbb{Q}) : x(P) \in \mathbb{Z}_S \};$$
$$\delta(E) = 1 + \#\{ p \in \mathbb{Z} : \text{ord}_p(j_E) < 0 \} .$$

There exists an absolute, *effectively computable constant* C, *such that*

$$\# E(\mathbb{Z}_S) \le C^{r(E)\delta(E) + \#S + 1} .$$

Proof. [15]

8. Specialization Theorems. In section 6 we discussed how the difference of the Weil height and the canonical height varies for points lying on an algebraic family of abelian varieties. If the points themselves also form a (one-dimensional) algebraic family, then one can say quite a bit more. Thus let C/K be a (smooth, projective) curve, and let $\pi : A \to C$ be a family of abelian varieties. (I.e. For some non-empty Zariski open subset $C^o \subset C$, $A^o = \pi^{-1}(C^o)$ is an abelian scheme over C^o.) Let $L \in \text{Pic}(A/C)$. Note that the function field $K(C)$ comes equipped with a standard set of absolute values satisfying the product formula, and so for any section $P \in A(C)$, we can compute the canonical height $\hat{h}_L(P)$. Further, for any $t \in C^o(\bar{K})$, we have a point $P(t) \in A(t)$, and so can compute its canonical height relative to $L(t) = L|_{A(t)}$. The next theorem gives a relation between these two canonical heights. (It generalizes results of Dem'janenko [3] and Manin [6], which deal with the case that the family $\pi : A \to C$ splits as a product, $A \cong A' \times C$.)

Theorem 8.1. *With notation as above, let* $\xi \in \text{Pic}(C)$ *be a line bundle of degree 1. Then*

$$\lim_{\substack{t \in C^o(\bar{K}) \\ h_\xi(t) \to \infty}} \frac{\hat{h}_{L(t)}(P(t))}{h_\xi(t)} = \hat{h}_L(P) .$$

Proof. [11]

Continuing with the above notation, notice that for each $t \in C(\bar{K})$, we have a *specialization homomorphism,*

$$\sigma_t : A(C) \longrightarrow A(t)(\bar{K}) \qquad P \longrightarrow P(t) .$$

Néron [9] proved that this map is frequently injective. Theorem 8.1 can be used to prove the following strengthening of Néron's result.

Theorem 8.2. *With notation as above, assume that the family* $\pi : A \to C$ *has no constant part. (I.e. the* $K(C)/K$ *trace of* A *is zero.) Then*

$$\{ \ t \in C^0(\bar{K}) \ : \ \sigma_t \textit{ is not injective } \}$$

is a set of bounded height. In particular, for any number d, σ_t *is injective for all but finitely many points of the set*

$$\bigcup_{[L:K] \le d} C(L) .$$

Proof. [11]

In the special case that $\pi : A \to C$ is a family of elliptic curves, Tate has given a stronger version of theorem 8.1; and his proof will work in the general case provided one can construct a sufficiently nice compactification of the Néron model of an abelian variety. (See [5, ch. 12, S3,4,5].)

Theorem 8.3. (Tate) *With notation as above, assume that* $\pi : A \to C$ *is a family of elliptic curves. (I.e. the fibers* $A(t)$ *have dimension 1.) Let* $P \in A(C)$. *Then there is a* $\lambda \in \mathrm{Pic}(C) \otimes \mathbb{Q}$, *depending on* P *and* L, *such that for all* $t \in C(\bar{K})$,

$$\hat{h}_{L(t)}(P(t)) = h_\lambda(t) + O(1) .$$

Corollary 8.4. (Tate) *With notation as above, let* $\xi \in \mathrm{Pic}(C)$ *be a line bundle of degree 1. Then for all* $t \in C(\bar{K})$,

$$\hat{h}_{L(t)}(P(t)) = \hat{h}_L(P)\, h_\xi(t) + O\left(\sqrt{h_\xi(t)}\right) + O(1) .$$

Proof. [16]

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PRÉSENTATION DE LA THÉORIE D'ARAKÉLOV

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Nous reprenons ci-dessous un manuscrit non publié datant d'avril 1983. Essentiellement un mois avant que G. Faltings prouve la conjecture de Mordell, j'avais estimé nécessaire d'écrire une dizaine de pages résumant un séminaire à Columbia que nous avions organisé avec S. Lang et P. Hriljac. Il y a deux raisons pour finalement publier aux comptes-rendus de l'école d'été d'Arcata sur la géométrie arithmétique ces quelques pages :

1) On voit dans ces notes et surtout dans les conjectures du §V qu'un des buts de la théorie d'Arakelov est d'obtenir un "Mordell effectif". On sait que Faltings a démontré la conjecture en question, bien que de façon non effective. Il dit cependant lui-même que l'idée d'Arakelov lui a permis de "penser" le problème.

2) Ces notes étaient destinées aux participants du séminaire mais ont un peu circulé. Après le coup d'éclat de Faltings, on m'a souvent demandé une telle introduction; je pense qu'elles peuvent servir à cet effet.

Nous avons en outre "actualisé" dans les huit notes présentées à la fin de ce travail les conjectures du §V.

RÉSUMÉ DE LA THÉORIE D'ARAKÉLOV ET DE SES DÉVELOPPEMENTS(*)

Dans ce résumé, j'ai essayé de mettre en lumière les propriétés "géométriques" de la théorie des intersections d'Arakelov sur une surface arithmétique. A cet effet, j'ai pensé qu'il était utile de développer clairement (?) le §1 qui ne semble pas connu sous cette forme. Les propriétés les plus "géométriques" qu'on obtient sont :

- le théorème de l'index (P. Hriljac, G. Faltings)
- le lemme 4 $(-(E.E)\ 4g(g-1)) \geqq (\omega_{X/\mathcal{O}} \cdot \omega_{X/\mathcal{O}})$
- le théorème d'existence de Faltings $(L.L) > 0$ $\deg L > 0$; alors $L^{\otimes m}$ a une section d'Arakelov pour $m >> 0$.
- $(\omega_{X/\mathcal{O}} \cdot \omega_{X/\mathcal{O}}) \geqq 0$.

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Par contre le lecteur doit se méfier des propriétés non "géométriques" (Rmq page 6, Z page 7 et le terme supplémentaire dans la formule (9)'). Le plan est le suivant :

I. DEGRÉ, RIEMANN-ROCH, DUALITÉ POUR LES FIBRÉS INVERSIBLES HERMITIENS SUR UN ANNEAU D'ENTIERS ALGÉBRIQUES.

K un corps de nombre de $d^o n$, $\mathcal{O}$ son anneau d'entiers, D la valeur absolue du discriminant

$r_1 = \#$ places réelles

$r_2 = \#$ places complexes

$\emptyset = r_1$ places réelles et r_2 places complexes t.q $\sigma \in \emptyset$ et $\bar{\sigma}$ complexe alors $\bar{\sigma} \notin \emptyset$.

Un faisceau inversible compactifié sur $\mathcal{O}$ est un module projectif L de rang 1 sur $\mathcal{O}$ muni pour tout σ d'un produit scalaire hermitien sur $L_\sigma = L \otimes_K K_\sigma$ ($K_\sigma = \mathbb{R}$ ou $\mathbb{C}$ selon ...).

$\mathrm{Pic}_c(\mathcal{O})$ est l'ensemble des classes d'isométries des faisceaux inversibles compactifiés. On a la suite exacte suivante

$$0 \longrightarrow \mathbb{R}^{n+2} / \log \mathcal{O} \longrightarrow \mathrm{Pic}_c(\mathcal{O}) \longrightarrow \mathbb{C}\ell(\mathcal{O}) \longrightarrow 0$$

Un diviseur compactifié est un élément de la forme $D = \Sigma\, n_v\, [v] + \sum_{\sigma \in \emptyset} \lambda_\sigma\, [\sigma]$ où Y est un idéal premier de $\mathcal{O}$, $n_v \in \mathbb{Z}$ (zéro p.p.t.v) et $\lambda_\sigma \in \mathbb{R}$

$$f \in K \qquad \mathrm{Div}(f) = (f) = \Sigma\, v_v(f)\, [v] - \Sigma \log \|f\|_\sigma\, [\sigma]$$

où $\|f\|_\sigma$ est la valeur absolue si σ réelle, la valeur absolue au carré si σ complexe.

DÉFINITION 3.- *Le degré du diviseur compactifié* D *ci-dessus est égal à :* $\Sigma\, n_v \log N(v) + \Sigma\, \lambda_\sigma$.

En particulier la formule du produit donne $f \in K \quad \deg(f) = 0$.

On notera $P(\mathcal{O})$ le sous-groupe de $\mathrm{Div}_c(\mathcal{O})$ engendré par les (f) .

PROPOSITION 1.- *On a un isomorphisme canonique*

$$\mathrm{Div}_c(\mathcal{O})/P(\mathcal{O}) \xrightarrow{\sim} \mathrm{Pic}_c(\mathcal{O}) \quad .$$

On va expliciter l'application inverse :

Soit L un faisceau inversible compactifié, $s \in L$ donne $\mathcal{O} \longrightarrow L$ par $1 \longrightarrow s$, on identifie ainsi par l'application duale L^{-1} à un idéal a_s de $\mathcal{O}$.

On a :

$$\deg L = \log \frac{N(a_s)}{\Pi \|s\|_\sigma} \tag{2}$$

DÉFINITION 4.- *Soit* $L \in \mathrm{Pic}_c(\mathcal{O})$ $H^\circ(L) = \{s \in L \mid |s|_\sigma \leq 1 \ \forall \sigma\}$.

Exemple : $H^\circ(\mathcal{O}) = 0 + \mu(\mathcal{O})$ (racines de l'unité).

LEMME 1.- *Si* $\deg L < 0$ *alors* $H^\circ(L) = 0$

LEMME 2.- *Si* $\deg L = 0$ *et* $H^\circ(L) \neq 0$ *alors* $L \simeq \mathcal{O}$ *avec sa métrique canonique* $(\|1\|_\sigma = 1)$.

Ces deux lemmes sont conséquences faciles de (2) .

DÉFINITION 5.- $\chi(L) = -\log \mathrm{vol}(\mathbb{R}^n/L)$ *le volume étant mesuré grâce aux métriques de* L. *On voit facilement la formule de Riemann Roch* $\boxed{\chi(L) = \deg L + \chi(\mathcal{O})}$ (3).

$$\chi'(L) = \log \frac{2^{r_1}\pi^{r_2}}{2^n \mathrm{vol}(L)} \qquad (\mathrm{vol}(L) =: \mathrm{vol}(\mathbb{R}^n/L)) \ .$$

On a par exemple $\chi'(\mathcal{O}) = -r_2 \log \frac{2}{\pi} + \frac{1}{2}\log D$) l'introduction de χ' se justifie par le lemme suivant :

LEMME 3.- (Minkowski). *Si* $\deg L \geq -\chi'(\mathcal{O})$ *alors* $H^\circ(L) \neq 0$.

Démonstration. On a tout fait pour appliquer le lemme de Minkowski sur les points d'un réseau dans la boule $\|s\|_\sigma \leq 1$ $\forall\sigma$! L'interprétation géométrique de χ' est agréable : Si L_t est obtenu en multipliant les métriques par $1/t$ on a $\chi'(L) = \lim_{n\to\infty} \# H^\circ(L_t)/t^n . 2^n$.

Il est maintenant plus facile et plus économique que dans la littérature de montrer le théorème suivant en jouant le lemme 3 contre le lemme 1 ou 2.

THÉORÈME 1.- (Minkowski-Hermite).

a) *si* $n > 1$ *alors* $D > 1$.

b) $\log \mathcal{O}^{\times}$ *est de rang* $r_1 + r_2 - 1$

c) $C\ell(\mathcal{O})$ *est fini*

d) *l'ensemble des corps des nombres* K' *de degré* m *fixé sur* K *et dont* le support *du discriminant relatif* $D_{K'/k}$ *est fixé, est fini.*

Pour illustrer je montre a) (c'est là qu'on est plus économique).

Lemme 3 contre lemme 1 donne $d - \chi'(\mathcal{O}) \geq 0$

si $r_2 \neq 0$ $D^{1/2} \geq (\frac{\pi}{2})^{r_2}$ C.Q.F.D

si $r_2 = 0$ alors $r_1 \geq 2$; il existe alors des nombres réels > 0 $\alpha_1 \cdots \alpha_{r_1}$ tel qu'il n'existe aucune unité u de $\mathcal{O}$ t-φ $\alpha_i = |u|_i$ et $\Sigma \log \alpha_i = i$ (il est facile de voir que $\log \mathcal{O}^{\times}$ est discret dans $\mathbb{R}^{r_1 - 1}$).

Prenons alors L isomorphe à $\mathcal{O}$ comme module mais tel que $\|1\|_i = \alpha_i$ si $\chi'(\mathcal{O}) = 0$; alors L a une section (lemme 3), et est de degré 0 ($\Sigma \log \alpha_i = 0$) ce qui contredit le lemme 2 (L n'est pas isométrique à $\mathcal{O}$).

DÉFINITION 6.- *Le faisceau dualisant* $\omega_{\mathcal{O}}$ *, est égal à la différente inverse i.e* $\omega_{\mathcal{O}} = \mathrm{Hom}_{\mathbb{Z}}(\mathcal{O}, \mathbb{Z})$ *. On met sur* $(\omega_\sigma)_\sigma$ *la métrique duale de celle de* $\mathcal{O}$ *. On a donc* $\log \mathrm{vol}(\omega_\sigma) = -\log \mathrm{vol}(\mathcal{O})$ *i.e :*

$$-\chi(\omega_{\mathcal{O}}) = \chi(\mathcal{O})$$

ou encore par R.R.

$$(4) \qquad \deg \omega_{\mathcal{O}} = -2\, \chi(\mathcal{O}) \quad .$$

Remarque. On aurait pu faire la même théorie pour $\mathcal{O}$ un ordre d'un corps de nombres.

II. LA THÉORIE DES INTERSECTIONS D'ARAKELOV SUR LES SURFACES ARITHMÉTIQUES (Métriques permises).

Soit $X \xrightarrow{f} \mathcal{O}$ une surface arithmétique i.e X est un schéma régulier. f est un morphisme projectif dont la fibre générique est une courbe de genre y sur K . Au vue de I il est naturel de considérer les fibrés de rang 1 sur X munis pour tout $\sigma \in \emptyset$ d'un produit scalaire hermitien $<,>_\sigma$ sur les fibres de L_σ — $X_\sigma = X \otimes K_\sigma$. Il faut cependant imposer une condition supplémentaire dont l'idée est due à Parshin.

On choisit des mesures $d\mu_\sigma$ sur X_σ t.φ .

(5) $\int_{X_\sigma} d\mu = 1$ $\forall \sigma$ et on impose la condition suivante sur la courbure de la métrique sur L .

(6) $\frac{1}{i\pi}\partial\bar{\partial}\log\|s\|_\sigma = \deg L \; d\mu_\sigma \quad \forall\sigma \quad (\deg L = \deg L_K / X_K)$

C'est un théorème à la Hodge que pour tout choix de mesures $d\mu_\sigma$ satisfaisant (5) on peut résoudre l'équation (6). En fait si $\deg L = 0$ la théorie de Hodge dit directement qu'on peut choisir des cocycles de valeur absolue 1. Comme on peut en choisissant une métrique sur un fibré L_o de degré >0 résoudre facilement (6) en prenant $d\mu_o = \frac{1}{2i\pi}\partial\partial \log \|s_o\| / \deg L_o$, il reste à résoudre une équation aux dérivées partielles $\Delta f = \rho$ avec $\rho = 0$ et Δ un "opérateur auto adjoint". C'est alors classique.

On définit donc le groupe des classes d'isométries de fibrés hermitiens satisfaisant (6) : $\mathrm{Pic}_c(X,d\mu_\sigma)$. On a une suite exacte :

$$0 \longrightarrow \mathbb{R}^{n+r_2}/\log \mathcal{O}^\times \longrightarrow \mathrm{Pic}_c(X,d\mu_\sigma) \longrightarrow \mathrm{Pic}(X) \longrightarrow 0 \quad .$$

DÉFINITION 6.- *Un diviseur compactifié de* X *est une somme formelle* $\Sigma\, n_Y[Y] + \Sigma\, \lambda_\sigma[X_\sigma]$ *où* $n_Y \in \mathbb{Z}$, $\lambda_\sigma \in \mathbb{R}$, Y *est un diviseur réduit irréductible de schéma* X $(n_Y = 0$ p.p.t. $Y)$.

Si $f \in K(X)$ est une fonction rationnelle on pose

$$(f) = \Sigma\, v_Y(f)\; [Y] - \Sigma \int_{X_\sigma} \log\|f\|_\sigma d\mu_\sigma \quad [X_\sigma] \; .$$

DÉFINITION 7.- *Soit* D *un diviseur irréductible de* X, *soit* $L \in \mathrm{Pic}_c(X,d\mu_\sigma)$

$\deg L/_D = (L \,.\, \mathcal{O}_X(D))$ (7)

Si D est horizontal le degré est celui du paragraphe I, si D vertical (voir Rmq a).

PROPOSITION 2 (Arakelov [A]).- *L'accouplement* $(L \,.\, \mathcal{O}_X(D))$ *est symétrique.*

COROLLAIRE.- $(L.(f)) = 0$ *pour toute fonction rationnelle* f *sur* X .

Pour goûter le sel de ces choses, il faut expliciter un peu.

Si L est dans $\mathrm{Pic}(X,d\mu)$ et $s \in f_*L$ on lui associe le diviseur compactifié D_s

$$D_s = s_f - \Sigma\left(\int \log \|s\|_\sigma \, d\mu_\sigma\right)[X_\sigma]$$

où s_f est le diviseur schématique associé à s ("à distance finie").

Si D est un diviseur sans composante à l'infini et si on écrit $D = \Sigma\, n_i\, P_i$ sur une clôture algébrique de K on a :

$$(8) \qquad (L.\mathcal{O}_X(D)) = \log N\,(s_f \cap D) - \sum_\sigma \sum_i n_i \log\|s(D_i\|_\sigma - \deg D\,.\sum_\sigma \int \log\|s\|_\sigma\, d\mu\sigma$$

Remarques. a) si D est contenu dans une fibre de f disons $D \longrightarrow f^{-1}(Y)$ alors on a

$$(7)' \qquad (L.\mathcal{O}_X(D)) = \deg L_{|D}\,.\log N\,(Y)$$

b) Si E_1 et E_2 sont deux sections de $f : X \longrightarrow \mathcal{O}$ si la section de ${}_X(E_i)$ satisfaisant à

$$\log\ |s_i|_\sigma\ d\mu_\sigma = 0 \qquad \forall\sigma$$

alors $(E_1.E_2) = \log N\,(E_1 \cap E_2) - \sum_\sigma \log\ |s_1(P_2)|_\sigma$.

c) Les deux propriétés de s vues en b) permettent d'identifier

$$g_\sigma(P_1,P_2) = \log\ |s_1(P_2)|_\sigma = -\log\ |s_2(P_1)|_\sigma$$

à une fonction de Green. On trouvera dans Gross [G] un très agréable traitement des fonctions de Green .

d) Si $P(X,d\mu)$ indique le groupe des diviseurs principaux, on a en vertu du corollaire de la proposition 2 un isomorphisme canonique

$$\mathrm{Pic}(X,d\mu) \simeq \mathrm{Div}_c(X)/_{P(X,d\mu)} \quad .$$

Il est facile maintenant de vérifier la proposition suivante :

PROPOSITION 3.- *Soit* $\mathrm{Spec}\ \mathcal{O}' \longrightarrow \mathrm{Spec}\ \mathcal{O}$ *un morphisme fini de degré* n , $X' \xrightarrow{f'} \mathcal{O}'$ *la résolution des singularités de* $X \times_{\mathcal{O}} \mathcal{O}'$ (Abhyancar, Lipman) $(d\mu'_\sigma)_{\sigma\in\emptyset'}$ *l'extension évidente des* $(d\mu_\sigma)_{\sigma\in\emptyset}$ *alors on a si* $\pi : X' \longrightarrow X$

$$(\pi^*L\,.\,\pi^*M) = n(L.\mu)$$

pour tout couple d'éléments de $\mathrm{Pic}(X,d\mu)$.

Remarques ennuyantes

i) Si D_1 et D_2 sont deux diviseurs compactifiés de X sans composantes communes (même à l'∞) les composantes locales à l'∞ des $(D_1.D_2)$ n'ont aucune raison d'être positive. En fait si on impose $\int \log |s(P)|_\sigma \, d\mu_\sigma = 0$ et $\deg L > 0$ $s \in f_* L$ il y a des points algébriques sur K tels que

$$\log |s(P)|_\sigma > 0 \; !!$$

ii) Si on a un morphisme $Y \xrightarrow{\pi} X$ sur $\mathcal{O}$ les formules attendues pour $(\pi^* L . \pi^* M)$ ne sont valides que si on prend $d\mu_Y = \frac{1}{\deg \pi} \pi^* d\mu_X$.

On verra plus bas qu'on a un choix canonique de métriques sur chaque X et qui ne satisfera pas cette dernière formule.

III. FORMULE D'ADJONCTION ET THÉORÈME DE RIEMANN-ROCH

Soit $\omega_{X/\mathcal{O}}$ le faisceau dualisant (schématique) relatif de $f : X \longrightarrow \mathcal{O}$. Ses "fibres à l'infini" sont $\Omega^1_{X_\sigma/\mathbb{C}}$. On veut faire un choix "raisonnable" pour les métriques sur les $\Omega^1_{X_\sigma}$ (et donc pour les $d\mu_\sigma$) . Considérons le produit $X_K \times X_K$ pour chaque choix de $d\mu_\sigma$ et tout couple de points rationnels P,Q $P \neq Q$ de $X(K)$. La fonction $G_\sigma(P,Q) = \exp g_\sigma(P,Q) = |(s_P(Q)|_\sigma = |s_Q(P)|_\sigma$ définit une métrique hermitienne sur $\mathcal{O}_{X_\sigma \times X_\sigma}(\Delta_\sigma)$ (Δ = diagonale), par restriction au fibré normal. On a donc une métrique sur $\Omega^1_{X_\sigma}$. Le problème est de savoir pour quels $d\mu_\sigma$ elle est permise (i.e satisfait (6)).

<u>THÉORÈME 2</u> (Arakelov [A]).- *Les métriques* $d\mu_\sigma = \frac{1}{g} \sum_{i=1}^{g} \omega_i \wedge \bar{\omega}_i$ *, où* (ω_i) *est une base orthonormale de* $H^\circ(X_\sigma, \Omega^1_{X_\sigma})$ *pour* $(\omega,\eta) = \int_{X_\sigma} \omega \wedge \bar{\eta}$ *, sont les seules pour lesquelles la <u>formule d'adjonction</u> suivante est vérifiée : soit* E *une <u>section</u> de* $p : X \longrightarrow \mathcal{O}$ *alors*

$$(\omega_{X/\mathcal{O}} . \mathcal{O}_X(E)) = -(E.E) \tag{9}$$

On voit facilement que $d\mu_\sigma$ est induite sur X_σ par la métrique plate sur la jacobienne $J(X_\sigma)$ pour tout plongement $X_\sigma \longrightarrow J(X_\sigma)$.

<u>Attention</u> . Dans l'énoncé du théorème 2 E est une section i.e de degré un sur $\mathcal{O}$. Si par exemple on a un diviseur $D = E_1 + E_2$ E_i sections s_i la section de $\mathcal{O}_X(E_i)$ donnant E_i (i.e $\int \log |s_i|_\sigma \, d\mu_\sigma = 0$) on a

$$(\omega_{X/\mathcal{O}} . \mathcal{O}_X(D)) + (D.D) = \log v\,(d_{D/\mathcal{O}}) - \sum_\sigma \log |s_1(P_2)|_\sigma + \log |s_2(P_1)|_\sigma$$

où $d_{D/\mathcal{O}}$ est le discriminant relatif de D sur $\mathcal{O}$ $(d_{D/\mathcal{O}} = \mathrm{Hom}_\mathcal{O}(\mathcal{O}_D, \mathcal{O}))$.

<u>Remarquons</u> cependant que si <u>D est contenu dans une fibre</u> de $f : X \longrightarrow \mathcal{O}$ la formule d'adjonction géométrique reste vraie à la multiplication par $\log N(Y)$ près.

A partir de maintenant, sauf mention du contraire, les X seront toujours munies des métriques $d\mu_\sigma$ du th.2 que nous appellerons canoniques.

Soit D un diviseur horizontal et effectif de S. Ecrivons, sur une clôture algébrique $\overline{K}$ de K $D_{\overline{K}} = \Sigma\, n_i P_i \quad P_i \in X(\overline{K})$. Soit $\omega_{D/\mathcal{O}} = \mathrm{Hom}_{\mathcal{O}}(\mathcal{O}_D, \mathcal{O})$ la différente relative inverse. Il est utile d'énoncer la formule suivante qui est un corollaire immédiat des définitions et de la formule (9).

Formule d'adjonction.- $(\omega_{X/\mathcal{O}} \cdot \mathcal{O}_X(D)) + (D.D) = \log|N(\omega_{D/\mathcal{O}})| - \sum_\sigma \sum_{i\neq j} g\,(P_i, P_j) n_i n_j$ (9)'.

Soit D un diviseur effectif de X. Posons (G. Faltings [G])
$\lambda(D) = \overset{\max}{\Lambda}\ f_* \mathcal{O}_X(D) \otimes \overset{\max}{\Lambda}\ (R^1 f_* \mathcal{O}_X(D))^\vee$. Pour chaque σ, $\lambda(D_\sigma)$ est muni d'une métrique hermitienne grâce aux deux formules suivantes

a) $\lambda(\mathcal{O}_X)_\sigma = \mathbb{C} \otimes \overset{g}{\Lambda}\, H^\circ(X, \Omega^1_{X_\sigma}) \qquad H^\circ(X_\sigma, \Omega^1_\sigma)$ muni de $\int_{X_\sigma} \omega \wedge \overline{\eta}$

b) si $P \in X(K_\sigma) \quad \lambda(D_\sigma) = \lambda(D_\sigma - P) \otimes \mathcal{O}_{X_\sigma}(D_\sigma)/P$.

DÉFINITION 8 (G. Faltings [F]) $\chi(D) = -\log \dfrac{\mathrm{vol}(\lambda(D))}{\#\,\mathrm{torsion}\ R^1 f_* \mathcal{O}_X(D)}$

Nous définissons alors $d^\circ\lambda(D) = \chi(D) - (d^\circ D + 1 - g)\,\chi(\mathcal{O})$ (10).
Il est facile de voir que si E est une section de $f : X \longrightarrow \mathcal{O}$ on a

$$d^\circ(\lambda(D+E)) = d^\circ\lambda(D) + (D \cdot E) + (E.E) \quad .$$

THÉORÈME 3 (Riemann-Roch).-

$$\chi(D) = \frac{1}{2}\,(D.D - K) + \chi(\mathcal{O}_X)$$

où K *est un diviseur d'Arakelov tel que* $\mathcal{O}_X(K) = \omega_X = \omega_{X/\mathcal{O}} \otimes f^*\omega_{\mathcal{O}}$.

La démonstration de ce théorème est formelle à partir de ce que l'on sait. Elle se fait de la façon suivante : (c'est encore un bénéfice du §1 en utilisant (10)).

(O) la formule est vraie pour $\chi(\mathcal{O}_X)$.

(I) la formule se comporte bien par changement de base $\mathrm{Spec}\ \mathcal{O}' \longrightarrow \mathrm{Spec}\ \mathcal{O}$, il suffit pour le vérifier d'appliquer la proposition 3 et de noter que (π^*D . diviseur contenu fibre except. de π) = 0.

(II) la formule est vraie pour D si et seulement si elle est vraie pour $D+E$ où E est une section (appliquer le th.2).

(III) la formule est vraie pour D si et seulement si elle est vraie pour $D+F$ où F est contenu dans une fibre.

On voit donc que si $\mathcal{O}_X(D) = \mathcal{O}_X(D')$ dans $\mathrm{Pic}(X,d\mu)$ $\chi(D) = \chi(D')$ et donc que si on a une seule place à l'infini $\lambda(D)$ et $\lambda(D')$ donnent le même élément dans $\mathrm{Pic}_c(\mathcal{O})$. On ne sera donc pas étonné du résultat suivant :

PROPOSITION 4 (G. Faltings [F]).- *Si* $\mathcal{O}_X(D) \cong \mathcal{O}_X(D')$ *dans* $\mathrm{Pic}(X,d\mu)$ *alors* $\lambda(D_\sigma)$ *et* $\lambda(D'_\sigma)$ *sont isomètres pour tout* σ.

La démonstration se fait avec l'idée de la thèse de G. Kempf [K]. Il suffit de démontrer la proposition 4 pour une surface de Riemann X et un faisceau inversible L muni de métriques permises pour les $d\mu$ canoniques et $d^\circ L = g-1$. Soit J_{g-1} la "jacobienne" des faisceaux inversibles de degré $g-1$ sur X, $\mathcal{L}$ le fibré de Poincaré sur $J_{g-1} \times X$, q la projection $J_{g-1} \times X \longrightarrow J_{q-1}$, alors il existe des fibrés vectoriels G et F (de rang g en fait) et $\varphi : G \longrightarrow F$ sur J_{g-1} tel que pour point Q de J_{g-1} on ait la suite exacte : (A. Grothendieck)

$$0 \longrightarrow H^\circ(X,\mathcal{L}(Q)) \longrightarrow G(Q) \longrightarrow F(Q) \longrightarrow H^1(X,\mathcal{L}(Q)) \longrightarrow 0$$

de plus on sait (D. Mumford) que l'application φ induit $0 \longrightarrow \overset{g}{\Lambda} G \otimes \overset{g}{\Lambda} F^\vee \longrightarrow \mathcal{O}_{J_{g-1}} \longrightarrow \mathcal{O} \longrightarrow 0$ où θ est le diviseur théta, i.e les faisceaux inversibles spéciaux de degré $g-1$. Donc $\lambda(\mathcal{L}(Q)) = \mathcal{O}_{J_{g-1}}(-\theta)_{/Q}$.
Le faisceau $\mathcal{O}_{J_{g-1}}(-\theta)$ est muni canoniquement d'une métrique dont la courbure est exactement proportionnelle au $d\mu$ canonique. La métrique de Faltings sur $\lambda(\mathcal{L}(Q))$ est donc bien celle-là.

THÉORÈME 4 (G. Faltings [F]).- *Soit* X *une surface de Riemann munie de sa métrique canonique* $d\mu$, L *un faisceau inversible sur* X *muni d'une métrique permise pour* $d\mu$. *Supposons que* $\deg(L) > 0$ *alors pour tout* $\varepsilon > 0$, *il existe* m_o *tel que pour tout* $m \geq m_o$ *on ait*

$$\mathrm{vol}(\{s \in H^\circ(X,L^{\otimes m}) \mid \int \|s\|\, d\mu \leq 1\}) \geq e^{-\varepsilon m^2}.$$

Le volume dans le théorème 4 est bien entendu pris par rapport à l'élément de volume canonique sur $H^\circ(X,L^{\otimes n})$ dont l'existence est assurée par la proposition 4.

Ce théorème est le plus fin de ceux que j'ai indiqué jusqu'ici. Un géomètre algébriste ne sera pas étonné du corollaire suivant :

COROLLAIRE.- *Soit* $f : X \longrightarrow \mathcal{O}$ *une surface arithmétique munie de ses métriques permises* $d\mu_\sigma$. *Soit* L *un élément de* $\mathrm{Pic}(X,d\mu_\sigma)$ *tel que* $(L.L) > 0$ *et*

$\deg(L) > 0$, *alors pour* m *suffisamment grand, il existe* $s \in f_* L^{\otimes m}$ *tel que*

$$\int \log \|s\|_\sigma \, d\mu_\sigma \leq 0 \quad \textit{pour tout} \quad \sigma$$

i.e s *correspond à un diviseur d'Arakelov effectif.*

Démonstration. Pour avoir $\int \log \|s\|_\sigma d\mu_\sigma \leq 0$ il suffit de vérifier $\int \|s\|_\sigma d\mu_\sigma \leq 1$. Soit $B_n = \underset{\sigma \in \emptyset}{X} B_{m,\sigma}$ où $B_{m,\sigma} = \{s \in H^\circ(X_\sigma, L_\sigma^{\otimes m}) \mid \int \|s\|_\sigma d\mu_\sigma$) par le lemme de Minkowski (lemme 3 au §I), il suffit d'assurer que $\mathrm{vol}(B_m) \geq 2^{(m \deg L - g+1)n} \mathrm{vol}(f_* L^{\otimes m})$ comme $R^1 f_* L^{\otimes m} = 0$ pour $m >> 0$. Le théorème de Riemann-Roch permet de conclure en prenant dans le théorème 4 $\varepsilon < \frac{n}{2}(L.L)$.

IV. LE THÉORÈME DE L'INDEX

Le théorème suivant permet de comparer les intersections d'Arakelov aux hauteurs de Néron-Tate.

THÉORÈME 5 (P. Hriljac et G. Faltings).- *Soit* $f : X \longrightarrow \mathcal{O}$ *une surface arithmétique munie de ses métriques canoniques* $d\mu_\sigma$. *Soit* (,) *la forme d'intersection d'Arakelov sur* $\mathrm{Pic}(X, d\mu)$ *et* q *la forme bilinéaire de Néron-Tate sur la jacobienne de* X_K . *Pour tout diviseur de degré zéro sur* X *notons* $\emptyset(D)$ *une combinaison linéaire de composantes de fibres de* $f : X \longrightarrow \mathcal{O}$ *telle que* $(D + \emptyset(D).F) = 0$ *pour tout* F *composante d'une fibre. Alors on a pour tout couple* (D,E) *de diviseurs de degré zéro sur* X

$$\boxed{(11) \quad (D + \emptyset(D).E) = -q(D,E)\,[K:\mathbb{Q}]}$$

Ce théorème, fondamental pour ce qui suit, se montre en comparant les fonctions de Néron locales aux composantes locales des intersections d'Arakelov. C'est la partie substantielle de la thèse de P. Hriljac (*). Comme on sait que la forme bilinéaire de Néron-Tate est définie positive sur $\mathrm{Jac}(X_K)_{/\mathrm{torsion}}$, on déduit aisément, du théorème 5, le théorème suivant qui est l'exact analogue du théorème de l'index de Hodge pour les surfaces algébriques. (1)

THÉORÈME 6 (P. Hriljac, G. Faltings).- *La signature de la forme d'intersection d'Arakelov possède un et un seul signe + .*

(*) La démonstration est reproduite dans la prochaine édition du livre de S. Lang [L] , cf. aussi [G] .

Remarquons que la démonstration même du théorème 6 ne fait appel qu'aux diviseurs de degré zéro sur X et qu'on voit ainsi que le théorème de l'index est vrai pour la forme d'intersection d'Arakelov sur $\mathrm{Pic}(X, d\mu_\sigma)$ dès que les $d\mu_\sigma$ satisfaisant à (5).

Avec le théorème 6 et le corollaire du théorème 4, on peut commencer à "mimer" les démonstrations géométriques qu'on trouve par exemple dans [S]. Par exemple le lemme suivant qui se trouve dans [S] (exposé III §3, proposition 2) :

LEMME 4.- *Soit* $f : X \longrightarrow \mathcal{O}$ *une surface arithmétique munie de ses métriques canoniques. Soit* D *un diviseur effectif d'Arakelov sur* X. *Alors on a*

$$(12) \quad (\omega_{X/\mathcal{O}} \cdot \omega_{X/\mathcal{O}}) \deg D \leq (\omega_{X/\mathcal{O}} \cdot D)\, 4g\,(g-1)$$

Démonstration. On peut facilement vérifier qu'il suffit de montrer la formule pour D sans composantes verticales, $g \geq 2$, et même D une section E de $f : X \longrightarrow \mathcal{O}$. On écrit alors le théorème de l'index pour $\omega_{X/\mathcal{O}}$, E, F où F est une fibre à l'infini. Compte tenu du théorème d'adjonction on obtient

$$\det \begin{pmatrix} \omega_{X/\mathcal{O}} \cdot \omega_{X/\mathcal{O}} & -E^2 & 2g-2 \\ -E^2 & E^2 & 1 \\ 2g-2 & 1 & 0 \end{pmatrix} \geq 0$$

Ce qui donne (12). Notons la formule pour E section

$$\boxed{(13) \quad (\omega_{X/\mathcal{O}} \cdot \omega_{X/\mathcal{O}}) \leq -(E.E)\, 4g\,(g-1)\,.}$$

COROLLAIRE (G. Faltings [F]).- *Soit* $f : X \longrightarrow \mathcal{O}$ *une surface arithmétique de genre* $g \geq 1$ *alors, si on munit* X *de ses métriques canoniques, on a*

$$(\omega_{X/\mathcal{O}} \cdot \omega_{X/\mathcal{O}}) \geq 0 \quad .$$

Bien entendu si $g = 1$ $\omega_{X/\mathcal{O}}$ provient de $\mathcal{O}$ et a donc self-intersection zéro. En genre supérieur $(\omega_{X/C} \cdot \omega_{X/C}) = 0$ caractérise, dans la situation géométrique, les fibrations isotriviales (cf. $[A]_2$ et [S]).

Nous n'avons pas dans la situation arithmétique de notion de fibration isotriviale. Ce qui lui ressemble le plus est la multiplication complexe. La démonstration, due à G. Faltings, de ce corollaire est élégante, nous l'indiquons ci-dessous.

Soit r_o tel que $(\omega_{X/\mathcal{O}}(r_oF))^2 = 0$ où F est une fibre à l'infini. Pour tout $r > r_o$ et $n \gg 0$ $(\omega_{X/\mathcal{O}}(rF))^{\otimes n}$ correspond à un diviseur d'Arakelov effectif par le corollaire du th.4 . On a donc par le lemme 4, en faisant tendre r vers r_o ,

$$(\omega_{X/\mathcal{O}} \cdot \omega_{X/\mathcal{O}})(2g-2) \quad ((\omega_{X/\mathcal{O}} \cdot \omega_{X/\mathcal{O}} + r_o(2g-2))2g(2g-2)$$

Comme $(\omega_{X/\mathcal{O}} \cdot \omega_{X/\mathcal{O}}) + 2r_o(2g-2) = 0$ on obtient $(\omega_{X/\mathcal{O}} \cdot \omega_{X/\mathcal{O}}) \quad g(\omega_{X/\mathcal{O}} \cdot \omega_{X/\mathcal{O}})$!

V. QUESTIONS ET ANALOGIES

Les questions que je pose ci-dessous sont motivées par la solution du problème de Mordell que j'ai donnée dans [S]. Elles ont toutes pour objet de majorer, minorer, ou interpréter les deux nombres d et $(\omega_{X/\mathcal{O}} \cdot \omega_{X/\mathcal{O}})$ (cf. ci-dessous)

$$d = \deg \overset{g}{\Lambda} f_* \omega_{X/\mathcal{O}} = -\log \text{vol}\ (f_*\omega_{X/\mathcal{O}}) - g\ \chi(\mathcal{O})$$

le volume sur $(f_*\omega_{X/\mathcal{O}})_\sigma = H^\circ(X_\sigma, \Omega^1_{X_\sigma})$ étant calculé pour la forme volume associée au produit scalaire hermitien $\int_X \omega \wedge \bar{\eta} = \langle \omega, \eta \rangle$.

Si $\omega_1 \dots \omega_g$ est une base de $f_* \omega_{X/\mathcal{O}}$ sur $\mathcal{O}$ (après changement de base on a

$$d = \log \frac{\text{vol}(\mathcal{O})^g}{(2^n \det(\sum_\sigma \int_{X_\sigma} \omega_i \wedge \bar{\omega}_\sigma))^{1/2}}$$

1) Questions sur $d = d^\circ f_* \omega_{X/\mathcal{O}}$

.1a) Interpréter $\exp(-d)$ en fonction de la distribution pour qu'un idéal premier v de $\mathcal{O}$ soit tel que la réduction $X(v)$ ne soit pas ordinaire (cf. [S] III.5).

.1b) Montrer que $d \geq 0$ (2).

.1c) Montrer que $d \leq g(\frac{1}{2} \sum_{v \in S} \log Nv - \chi(\mathcal{O}) + \frac{1}{2} s_\infty)$ où S = l'ensemble des places de $\mathcal{O}$ où X a mauvaise réduction et s_∞ une contribution des places $\sigma \in \emptyset$ (cf. [F] et plus bas) (3).

2) Questions sur $(\omega_{X/\mathcal{O}} \cdot \omega_{X/\mathcal{O}})$

2a) Interpréter la condition $(\omega_{X/\mathcal{O}} \cdot \omega_{X/\mathcal{O}}) = 0$ (par exemple $(\omega_{X/\mathcal{O}} \cdot \omega_{X/\mathcal{O}}) = 0$

implique-t-il quand X est semi-stable que X est lisse et qu'il existe un point rationnel E -après changement de base- tel que $(E.E) = 0$?) (4).

2b) Donner une borne supérieure pour $(\omega_{X/\mathcal{O}}.\omega_{X/\mathcal{O}})$ de la forme

$$(\omega_{X/\mathcal{O}}.\omega_{X/\mathcal{O}}) \leq 8g(g-1)\left(\frac{1}{2}\sum_{v\in S}\log N(v) - \chi(\mathcal{O}) + v_\infty\right) \quad (X \text{ semi-stable})(5) .$$

3) Question sur les points rationnels

3a) Donner une borne effective pour $|h_\omega(P) - (\omega_{X/\mathcal{O}}.E_p)|$ en termes de g, S et $\mathcal{O}$ et d où $h_\omega(P)$ est la hauteur naïve du point P pour une base $\omega_1 \dots \omega_g$ de $H^\circ(X_K, \Omega^1_{X_K})$ et E_p la section de $f : X \longrightarrow \mathcal{O}$ correspondante.

3b) Montrer qu'il existe -après changement de base- une section de $f : X \longrightarrow \mathcal{O}$ telle que $(\omega_{X/\mathcal{O}}.E) = -(E.E) \leq (d^\circ\omega + \sum_{v\in S}\log N(v) + s_\infty)[K:\mathbb{Q}]$ (cf. [S] dém. du th. 3 pour l'utilisation de la classe de Kodaira-Spencer à cet effet dans le cas géométrique). (6)

Relations entre ces questions

- G. Faltings dans F a introduit un nombre s tel que $(\omega_{X/\mathcal{O}}.\omega_{X/\mathcal{O}}) = 12d - \delta - s_\infty$ (Riemann-Roch 2) où $\delta = \sum_{v\in S} n_v \log N(v)$ pour $X \longrightarrow \mathcal{O}$ semi-stable et N_v = nombre de points singuliers dans la fibre de v. Ceci explique les relations entre 1c) et 2b). Par le lemme 4 on voit que 3b) implique 2b).
- Par la construction de Parshin-Kodaira (cf. [P] ou [S] III 8 Prop. 2) 3b) ou 2b) implique la conjecture de Mordell par théorème de l'index (cf. [S] III 8 Cor. 2) (7) .
- On aimerait aussi montrer qu'il n'y a pas de courbe lisse de genre ≥ 2 sur $\mathbb{Z}$ (8).

NOTES(*)

(1) Cette comparaison a aussi été démontrée par Gross dans [G].

(2) P. Deligne a donné dans son exposé n°616 (nov. 83) au Séminaire Bourbaki un contrexemple à cet énoncé pour la courbe elliptique à multiplication complexe par $\sqrt{-3}$.

(*) Le 15 octobre 1985.

(3) C'est cette quantité (appelée hauteur modulaire) qui a été montrée ne prendre qu'un nombre fini de valeurs quand K et S sont fixés par G. Faltings dans sa preuve de la conjecture de Mordell. Une borne effective de la nature indiquée en 1 c) n'a toujours pas été prouvée (cf. les notes (5) et (6)).

(4) et (5) Deux ans après, j'estime que les formules 1 c), 2 b) et 3 b) sont un peu naïves. Il serait plus conforme à la réalité de rajouter des "ε" . Par exemple, pour la conjecture "des petits points" 3b), je dirais aujourd'hui : pour tout $\varepsilon > 0$ il existe un point $P \in X_K(K)$ et un réel $C(K,\varepsilon)$ tel que la section E_p correspondante satisfasse :

$$\exp\left(\frac{-E_p^2}{K(P) : K}\right) \leq C(K, \ ) \prod_{v \in S} N(v)^{1+\varepsilon} \quad .$$

En particulier, la technique de la classe de Kodaira-Spencer S , appliquée en genre 1 donne sur les corps de fonctions pour une courbe elliptique semi-stable sur une courbe C complète $\deg \Delta = 12 \deg \omega \leq ((2g(C)-2)+s)6$ où Δ est le discriminant et s le nombre de points de C dont la fibre est singulière . L'analogue sans ε est déjà faux sur $\mathbb{Q}$ où on peut trouver (tables d'Anvers) une courbe elliptique semi-stable telle que $\frac{\log \Delta}{\log N} = 7{,}8\ldots$ N étant le conducteur (analogue de S) .

Conjecture : soit E une courbe elliptique semi-stable sur un corps de nombres K de conducteur N et de discriminant Δ , alors pour tout $\varepsilon > 0$ il existe un réel $C(K,\varepsilon)$ tel que

$$\Delta \leq C(K,\varepsilon) N^{6+\varepsilon}$$

Une conjecture plus faible est : sous les mêmes hypothèses, il existe une constante $c(K)$ ne dépendant que de K t.q $\Delta \leq N^{c(K)}$.
Cette conjecture implique celle de Fermat ! (appliquer l'astuce de Frey).

(6) On sait que G. Faltings a démontré la conjecture de Mordell un mois après que j'aie écrit ce manuscrit. Sa démonstration (merveilleuse) ne donne cependant pas de borne effective pour la hauteur des points rationnels.

(7) J-M Fontaine a montré en 1984 qu'il n'y a pas de variété abélienne sur $\mathbb{Z}$.

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