INSTRUCTOR'S MANUAL

APPLIED PARTIAL DIFFERENTIAL EQUATIONS
with Fourier Series and Boundary Value Problems

Fourth Edition

Richard Haberman
Department of Mathematics
Southern Methodist University

PRENTICE HALL, Upper Saddle River, NJ 07458

Preface

In this manual there are solutions to most of the starred exercises of APPLIED PARTIAL DIFFERENTIAL EQUATIONS with Fourier Series and Boundary Value Problems, Fourth Edition, by Richard Haberman.

Over 1000 exercises of varying difficulty form an essential part of this text. It is hoped that these approximately 250 selected solutions will be useful for instructors and those contemplating adopting this text.

I would like to express my appreciation to Shari Webster and Nita Blanscet for the preparation of this manual using LaTeX.

Richard Haberman

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Chapter 1. Heat Equation

Section 1.2

1.2.9 (d) Circular cross section means that $P = 2\pi r, A = \pi r^2$, and thus $P/A = 2/r$, where r is the radius. Also $\gamma = 0$.

1.2.9 (e) $u(x,t) = u(t)$ implies that

$$c\rho\,\frac{du}{dt} = -\frac{2h}{r}u\,.$$

The solution of this first-order linear differential equation with constant coefficients, which satisfies the initial condition $u(0) = u_0$, is

$$u(t) = u_0 \exp\left[-\frac{2h}{c\rho r}t\right].$$

Section 1.3

1.3.2 $\partial u/\partial x$ is continuous if $K_0(x_0-) = K_0(x_0+)$, that is, if the conductivity is continuous.

Section 1.4

1.4.1 (a) Equilibrium satisfies (1.4.14), $d^2u/dx^2 = 0$, whose general solution is (1.4.17), $u = c_1 + c_2x$. The boundary condition $u(0) = 0$ implies $c_1 = 0$ and $u(L) = T$ implies $c_2 = T/L$ so that $u = Tx/L$.

1.4.1 (d) Equilibrium satisfies (1.4.14), $d^2u/dx^2 = 0$, whose general solution (1.4.17), $u = c_1 + c_2x$. From the boundary conditions, $u(0) = T$ yields $T = c_1$ and $du/dx(L) = \alpha$ yields $\alpha = c_2$. Thus $u = T + \alpha x$.

1.4.1 (f) In equilibrium, (1.2.9) becomes $d^2u/dx^2 = -Q/K_0 = -x^2$, whose general solution (by integrating twice) is $u = -x^4/12 + c_1 + c_2x$. The boundary condition $u(0) = T$ yields $c_1 = T$, while $du/dx(L) = 0$ yields $c_2 = L^3/3$. Thus $u = -x^4/12 + L^3x/3 + T$.

1.4.1 (h) Equilibrium satisfies $d^2u/dx^2 = 0$. One integration yields $du/dx = c_2$, the second integration yields the general solution $u = c_1 + c_2x$.

$$\begin{aligned} x = 0:&\quad c_2 - (c_1 - T) = 0 \\ x = L:&\quad c_2 = \alpha \text{ and thus } c_1 = T + \alpha. \end{aligned}$$

Therefore, $u = (T + \alpha) + \alpha x = T + \alpha(x + 1)$.

1.4.7 (a) For equilibrium:

$$\frac{d^2u}{dx^2} = -1 \text{ implies } u = -\frac{x^2}{2} + c_1x + c_2 \text{ and } \frac{du}{dx} = -x + c_1.$$

From the boundary conditions $\frac{du}{dx}(0) = 1$ and $\frac{du}{dx}(L) = \beta, c_1 = 1$ and $-L + c_1 = \beta$ which is consistent only if $\beta + L = 1$. If $\beta = 1 - L$, there is an equilibrium solution ($u = -\frac{x^2}{2} + x + c_2$). If $\beta \neq 1 - L$, there isn't an equilibrium solution. The difficulty is caused by the heat flow being specified at both ends and a source specified inside. An equilibrium will exist only if these three are in balance. This balance can be mathematically verified from conservation of energy:

$$\frac{d}{dt}\int_0^L c\rho u\,dx = -\frac{du}{dx}(0) + \frac{du}{dx}(L) + \int_0^L Q_0\,dx = -1 + \beta + L.$$

If $\beta + L = 1$, then the total thermal energy is constant and the initial energy = the final energy:

$$\int_0^L f(x)\,dx = \int_0^L \left(-\frac{x^2}{2} + x + c_2\right)\,dx, \quad \text{which determines} \quad c_2.$$

If $\beta + L \neq 1$, then the total thermal energy is always changing in time and an equilibrium is never reached.

Section 1.5

1.5.9 (a) In equilibrium, (1.5.14) using (1.5.19) becomes $\frac{d}{dr}\left(r\frac{du}{dr}\right) = 0$. Integrating once yields $rdu/dr = c_1$ and integrating a second time (after dividing by r) yields $u = c_1 \ln r + c_2$. An alternate general solution is $u = c_1 \ln(r/r_1) + c_3$. The boundary condition $u(r_1) = T_1$ yields $c_3 = T_1$, while $u(r_2) = T_2$ yields $c_1 = (T_2 - T_1)/\ln(r_2/r_1)$. Thus, $u = \frac{1}{\ln(r_2/r_1)}\left[(T_2 - T_1)\ln r/r_1 + T_1 \ln(r_2/r_1)\right]$.

1.5.11 For equilibrium, the radial flow at $r = a$, $2\pi a\beta$, must equal the radial flow at $r = b$, $2\pi b$. Thus $\beta = b/a$.

1.5.13 From exercise 1.5.12, in equilibrium $\frac{d}{dr}\left(r^2\frac{du}{dr}\right) = 0$. Integrating once yields $r^2du/dr = c_1$ and integrating a second time (after dividing by r^2) yields $u = -c_1/r + c_2$. The boundary conditions $u(4) = 80$ and $u(1) = 0$ yields $80 = -c_1/4 + c_2$ and $0 = -c_1 + c_2$. Thus $c_1 = c_2 = 320/3$ or $u = \frac{320}{3}\left(1 - \frac{1}{r}\right)$.

Chapter 2. Method of Separation of Variables

Section 2.3

2.3.1 (a) $u(r,t) = \phi(r)h(t)$ yields $\phi \frac{dh}{dt} = \frac{kh}{r}\frac{d}{dr}\left(r\frac{d\phi}{dr}\right)$. Dividing by $k\phi h$ yields $\frac{1}{kh}\frac{dh}{dt} = \frac{1}{r\phi}\frac{d}{dr}\left(r\frac{d\phi}{dr}\right) = -\lambda$ or $\frac{dh}{dt} = -\lambda kh$ and $\frac{1}{r}\frac{d}{dr}\left(r\frac{d\phi}{dr}\right) = -\lambda\phi$.

2.3.1 (c) $u(x,y) = \phi(x)h(y)$ yields $h\frac{d^2\phi}{dx^2} + \phi\frac{d^2h}{dy^2} = 0$. Dividing by ϕh yields $\frac{1}{\phi}\frac{d^2\phi}{dx^2} = -\frac{1}{h}\frac{d^2h}{dy^2} = -\lambda$ or $\frac{d^2\phi}{dx^2} = -\lambda\phi$ and $\frac{d^2h}{dy^2} = \lambda h$.

2.3.1 (e) $u(x,t) = \phi(x)h(t)$ yields $\phi(x)\frac{dh}{dt} = kh(t)\frac{d^4\phi}{dx^4}$. Dividing by $k\phi h$, yields $\frac{1}{kh}\frac{dh}{dt} = \frac{1}{\phi}\frac{d^4\phi}{dx^4} = \lambda$.

2.3.1 (f) $u(x,t) = \phi(x)h(t)$ yields $\phi(x)\frac{d^2h}{dt^2} = c^2h(t)\frac{d^2\phi}{dx^2}$. Dividing by $c^2\phi h$, yields $\frac{1}{c^2h}\frac{d^2h}{dt^2} = \frac{1}{\phi}\frac{d^2\phi}{dx^2} = -\lambda$.

2.3.2 (b) $\lambda = (n\pi/L)^2$ with $L = 1$ so that $\lambda = n^2\pi^2,\ n = 1, 2, \ldots$

2.3.2 (d)

(i) If $\lambda > 0, \phi = c_1\cos\sqrt{\lambda}x + c_2\sin\sqrt{\lambda}x$. $\phi(0) = 0$ implies $c_1 = 0$, while $\frac{d\phi}{dx}(L) = 0$ implies $c_2\sqrt{\lambda}\cos\sqrt{\lambda}L = 0$. Thus $\sqrt{\lambda}L = -\pi/2 + n\pi(n = 1, 2, \ldots)$.

(ii) If $\lambda = 0, \phi = c_1 + c_2x$. $\phi(0) = 0$ implies $c_1 = 0$ and $d\phi/dx(L) = 0$ implies $c_2 = 0$. Therefore $\lambda = 0$ is not an eigenvalue.

(iii) If $\lambda < 0$, let $\lambda = -s$ and $\phi = c_1\cosh\sqrt{s}x + c_2\sinh\sqrt{s}x$. $\phi(0) = 0$ implies $c_1 = 0$ and $d\phi/dx(L) = 0$ implies $c_2\sqrt{s}\cosh\sqrt{s}L = 0$. Thus $c_2 = 0$ and hence there are no eigenvalues with $\lambda < 0$.

2.3.2 (f) The simpliest method is to let $x' = x - a$. Then $d^2\phi/dx'^2 + \lambda\phi = 0$ with $\phi(0) = 0$ and $\phi(b-a) = 0$. Thus (from p. 46) $L = b - a$ and $\lambda = [n\pi/(b-a)]^2,\ n = 1, 2, \ldots$.

2.3.3 From (2.3.30), $u(x,t) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{L}e^{-k(n\pi/L)^2t}$. The initial condition yields $2\cos\frac{3\pi x}{L} = \sum_{n=1}^{\infty} B_n\sin\frac{n\pi x}{L}$. From (2.3.35), $B_n = \frac{2}{L}\int_0^L 2\cos\frac{3\pi x}{L}\sin\frac{n\pi x}{L}\,dx$.

2.3.4 (a) Total heat energy $= \int_0^L c\rho uA\,dx = c\rho A\sum_{n=1}^{\infty} B_n e^{-k\left(\frac{n\pi}{L}\right)^2 t}\frac{1-\cos n\pi}{\frac{n\pi}{L}}$, using (2.3.30) where B_n satisfies (2.3.35).

2.3.4 (b)

heat flux to right $= -K_0\partial u/\partial x$
total heat flow to right $= -K_0A\partial u/\partial x$
heat flow out at $x = 0 = K_0A\frac{\partial u}{\partial x}\Big|_{x=0}$
heat flow out $(x = L) = -K_0A\frac{\partial u}{\partial x}\Big|_{x=L}$

2.3.4 (c) From conservation of thermal energy, $\frac{d}{dt}\int_0^L u\,dx = k\frac{\partial u}{\partial x}\Big|_0^L = k\frac{\partial u}{\partial x}(L) - k\frac{\partial u}{\partial x}(0)$. Integrating from $t = 0$ yields

$$\underbrace{\int_0^L u(x,t)\,dx}_{\substack{\text{heat energy}\\ \text{at } t}} - \underbrace{\int_0^L u(x,0)\,dx}_{\substack{\text{initial heat}\\ \text{energy}}} = k\underbrace{\int_0^t \left[\frac{\partial u}{\partial x}(L)\right.}_{\substack{\text{integral of}\\ \text{flow in at}\\ x=L}} - \underbrace{\left.\frac{\partial u}{\partial x}(0)\right]dx}_{\substack{\text{integral of}\\ \text{flow out at}\\ x=L}}\ .$$

2.3.8 (a) The general solution of $k\frac{d^2u}{dx^2} = \alpha u\ (\alpha > 0)$ is $u(x) = a\cosh\sqrt{\frac{\alpha}{k}}x + b\sinh\sqrt{\frac{\alpha}{k}}x$. The boundary condition $u(0) = 0$ yields $a = 0$, while $u(L) = 0$ yields $b = 0$. Thus $u = 0$.

2.3.8 (b) Separation of variables, $u = \phi(x)h(t)$ or $\phi\frac{dh}{dt} + \alpha\phi h = kh\frac{d^2\phi}{dx^2}$, yields two ordinary differential equations (divide by $k\phi h$): $\frac{1}{kh}\frac{dh}{dt} + \frac{\alpha}{k} = \frac{1}{\phi}\frac{d^2\phi}{dx^2} = -\lambda$. Applying the boundary conditions, yields the eigenvalues $\lambda = (n\pi/L)^2$ and corresponding eigenfunctions $\phi = \sin\frac{n\pi x}{L}$. The time-dependent part are exponentials, $h = e^{-\lambda kt}e^{-\alpha t}$. Thus by superposition, $u(x,t) = e^{-\alpha t}\sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L}e^{-k(n\pi/L)^2 t}$, where the initial conditions $u(x,0) = f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L}$ yields $b_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}\,dx$. As $t \to \infty$, $u \to 0$, the only equilibrium solution.

2.3.9 (a) If $\alpha < 0$, the general equilibrium solution is $u(x) = a\cos\sqrt{\frac{-\alpha}{k}}x + b\sin\sqrt{\frac{-\alpha}{k}}x$. The boundary condition $u(0) = 0$ yields $a = 0$, while $u(L) = 0$ yields $b\sin\sqrt{\frac{-\alpha}{k}}L = 0$. Thus if $\sqrt{\frac{-\alpha}{k}}L \neq n\pi, u = 0$ is the only equilibrium solution. However, if $\sqrt{\frac{-\alpha}{k}}L = n\pi$, then $u = A\sin\frac{n\pi x}{L}$ is an equilibrium solution.

2.3.9 (b) Solution obtained in 2.3.8 is correct. If $-\frac{\alpha}{k} = \left(\frac{\pi}{L}\right)^2, u(x,t) \to b_1\sin\frac{\pi x}{L}$, the equilibrium solution. If $-\frac{\alpha}{k} < \left(\frac{\pi}{L}\right)^2$, then $u \to 0$ as $t \to \infty$. However, if $-\frac{\alpha}{k} > \left(\frac{\pi}{L}\right)^2, u \to \infty$ (if $b_1 \neq 0$). Note that $b_1 > 0$ if $f(x) \geq 0$. Other more unusual events can occur if $b_1 = 0$. [Essentially, the other possible equilibrium solutions are unstable.]

Section 2.4

2.4.1 The solution is given by (2.4.19), where the coefficients satisfy (2.4.21) and hence (2.4.23-24).

(a) $A_0 = \frac{1}{L}\int_{L/2}^{L} 1dx = \frac{1}{2}, A_n = \frac{2}{L}\int_{L/2}^{L}\cos\frac{n\pi x}{L}dx = \frac{2}{L}\cdot\frac{L}{n\pi}\sin\frac{n\pi x}{L}\Big|_{L/2}^{L} = -\frac{2}{n\pi}\sin\frac{n\pi}{2}$

(b) by inspection $A_0 = 6, A_3 = 4$, others $= 0$.

(c) $A_0 = \frac{-2}{L}\int_0^L \sin\frac{\pi x}{L}\,dx = \frac{2}{\pi}\cos\frac{\pi x}{L}\Big|_0^L = \frac{2}{\pi}(1 - \cos\pi) = 4/\pi, A_n = \frac{-4}{L}\int_0^L \sin\frac{\pi x}{L}\cos\frac{n\pi x}{L}\,dx$

(d) by inspection $A_8 = -3$, others $= 0$.

2.4.3 Let $x' = x - \pi$. Then the boundary value problem becomes $d^2\phi/dx'^2 = -\lambda\phi$ subject to $\phi(-\pi) = \phi(\pi)$ and $d\phi/dx'(-\pi) = d\phi/dx'(\pi)$. Thus, the eigenvalues are $\lambda = (n\pi/L)^2 = n^2\pi^2$, since $L = \pi, n = 0, 1, 2, ...$ with the corresponding eigenfunctions being both $\sin n\pi x'/L = \sin n(x-\pi) = (-1)^n \sin nx =>$ $\sin nx$ and $\cos n\pi x'/L = \cos n(x-\pi) = (-1)^n\cos nx => \cos nx$.

Section 2.5

2.5.1 (a) Separation of variables, $u(x,y) = h(x)\phi(y)$, implies that $\frac{1}{h}\frac{d^2h}{dx^2} = -\frac{1}{\phi}\frac{d^2\phi}{dy^2} = -\lambda$. Thus $d^2h/dx^2 = -\lambda h$ subject to $h'(0) = 0$ and $h'(L) = 0$. Thus as before, $\lambda = (n\pi/L)^2, n = 0, l, 2, \ldots$ with $h(x) = \cos n\pi x/L$. Furthermore, $\frac{d^2\phi}{dy^2} = \lambda\phi = \left(\frac{n\pi}{L}\right)^2\phi$ so that

$n = 0: \phi = c_1 + c_2 y$, where $\phi(0) = 0$ yields $c_1 = 0$

$n \neq 0: \phi = c_1\cosh\frac{n\pi y}{L} + c_2\sinh\frac{n\pi y}{L}$, where $\phi(0) = 0$ yields $c_1 = 0$.

The result of superposition is

$$u(x,y) = A_0 y + \sum_{n=1}^{\infty} A_n \cos\frac{n\pi x}{L}\sinh\frac{n\pi y}{L}.$$

The nonhomogeneous boundary condition yields

$$f(x) = A_0 H + \sum_{n=1}^{\infty} A_n \sinh\frac{n\pi H}{L}\cos\frac{n\pi x}{L},$$

so that

$$A_0 H = \frac{1}{L}\int_0^L f(x)\,dx \text{ and } A_n\sinh\frac{n\pi H}{L} = \frac{2}{L}\int_0^L f(x)\cos\frac{n\pi x}{L}\,dx.$$

2.5.1 (c) Separation of variables, $u = h(x)\phi(y)$, yields $\frac{1}{h}\frac{d^2h}{dx^2} = -\frac{1}{\phi}\frac{d^2\phi}{dy^2} = \lambda$. The boundary conditions $\phi(0) = 0$ and $\phi(H) = 0$ yield an eigenvalue problem in y, whose solution is $\lambda = (n\pi/H)^2$ with $\phi = \sin n\pi y/H, n = 1, 2, 3, \ldots$ The solution of the x-dependent equation is $h(x) = \cosh n\pi x/H$ using $dh/dx(0) = 0$. By superposition:

$$u(x,y) = \sum_{n=1}^{\infty} A_n \cosh\frac{n\pi x}{H}\sin\frac{n\pi y}{H}.$$

The nonhomogeneous boundary condition at $x = L$ yields $g(y) = \sum_{n=1}^{\infty} A_n \cosh\frac{n\pi L}{H}\sin\frac{n\pi y}{H}$, so that A_n is determined by $A_n \cosh\frac{n\pi L}{H} = \frac{2}{H}\int_0^H g(y)\sin\frac{n\pi y}{H}\,dy$.

2.5.1 (e) Separation of variables, $u = \phi(x)h(y)$, yields the eigenvalues $\lambda = (n\pi/L)^2$ and corresponding eigenfunctions $\phi = \sin n\pi x/L, n = 1, 2, 3, \ldots$ The y-dependent differential equation, $\frac{d^2h}{dy^2} = \left(\frac{n\pi}{L}\right)^2 h$, satisfies $h(0) - \frac{dh}{dy}(0) = 0$. The general solution $h = c_1\cosh\frac{n\pi y}{L} + c_2\sinh\frac{n\pi y}{L}$ obeys $h(0) = c_1$, while $\frac{dh}{dy} = \frac{n\pi}{L}\left(c_1\sinh\frac{n\pi y}{L} + c_2\cosh\frac{n\pi y}{L}\right)$ obeys $\frac{dh}{dy}(0) = c_2\frac{n\pi}{L}$. Thus, $c_1 = c_2\frac{n\pi}{L}$ and hence $h_n(y) = \cosh\frac{n\pi y}{L} + \frac{L}{n\pi}\sinh\frac{n\pi y}{L}$. Superposition yields

$$u(x,y) = \sum_{n=1}^{\infty} A_n h_n(y)\sin n\pi x/L,$$

where A_n is determined from the boundary condition, $f(x) = \sum_{n=1}^{\infty} A_n h_n(H)\sin n\pi x/L$, and hence

$$A_n h_n(H) = \frac{2}{L}\int_0^L f(x)\sin n\pi x/L\,dx\,.$$

2.5.2 (a) From physical reasoning (or exercise 1.5.8), the total heat flow across the boundary must equal zero in equilibrium (without sources, i.e. Laplace's equation). Thus $\int_0^L f(x)\,dx = 0$ for a solution.

2.5.3 In order for u to be bounded as $r \to \infty, c_1 = 0$ in (2.5.43) and $\bar{c}_2 = 0$ in (2.5.44). Thus,

$$u(r,\theta) = \sum_{n=0}^{\infty} A_n r^{-n}\cos n\theta + \sum_{n=1}^{\infty} B_n r^{-n}\sin n\theta.$$

(a) The boundary condition yields $A_0 = \ln 2, A_3 a^{-3} = 4$, other $A_n = 0, B_n = 0$.

(b) The boundary conditions yield (2.5.46) with a^{-n} replacing a^n. Thus, the coefficients are determined by (2.5.47) with a^n replaced by a^{-n}

2.5.4 By substituting (2.5.47) into (2.5.45) and interchanging the orders of summation and integration

$$u(r,\theta) = \frac{1}{\pi}\int_{-\pi}^{\pi} f(\bar{\theta})\left[\frac{1}{2} + \sum_{n=1}^{\infty}\left(\frac{r}{a}\right)^n(\cos n\theta\cos n\bar{\theta} + \sin n\theta\sin n\bar{\theta})\right]\,d\bar{\theta}.$$

Noting the trigonometric addition formula and $\cos z = R_e[e^{iz}]$, we obtain

$$u(r,\theta) = \frac{1}{\pi}\int_{-\pi}^{\pi} f(\bar{\theta})\left[-\frac{1}{2} + Re\sum_{n=0}^{\infty}\left(\frac{r}{a}\right)^n e^{in(\theta-\bar{\theta})}\right]\,d\bar{\theta}.$$

Summing the geometric series enables the bracketed term to be replaced by

$$-\frac{1}{2} + Re\frac{1}{1 - \frac{r}{a}e^{i(\theta-\bar{\theta})}} = -\frac{1}{2} + \frac{1 - \frac{r}{a}\cos(\theta-\bar{\theta})}{1 + \frac{r^2}{a^2} - \frac{2r}{a}\cos(\theta-\bar{\theta})} = \frac{\frac{1}{2} - \frac{1}{2}\frac{r^2}{a^2}}{1 + \frac{r^2}{a^2} - \frac{2r}{a}\cos(\theta-\bar{\theta})}.$$

2.5.5 (a) The eigenvalue problem is $d^2\phi/d\theta^2 = -\lambda\phi$ subject to $d\phi/d\theta(0) = 0$ and $\phi(\pi/2) = 0$. It can be shown that $\lambda > 0$ so that $\phi = \cos\sqrt{\lambda}\theta$ where $\phi(\pi/2) = 0$ implies that $\cos\sqrt{\lambda}\pi/2 = 0$ or $\sqrt{\lambda}\pi/2 = -\pi/2 + n\pi, n = 1, 2, 3, \ldots$ The eigenvalues are $\lambda = (2n-1)^2$. The radially dependent term satisfies (2.5.40), and hence the boundedness condition at $r = 0$ yields $G(r) = r^{2n-1}$. Superposition yields

$$u(r,\theta) = \sum_{n=1}^{\infty} A_n r^{2n-1}\cos(2n-1)\theta.$$

The nonhomogeneous boundary condition becomes

$$f(\theta) = \sum_{n=1}^{\infty} A_n \cos(2n-1)\theta \quad \text{or} \quad A_n = \frac{4}{\pi}\int_0^{\pi/2} f(\theta)\cos(2n-1)\theta\, d\theta.$$

2.5.5 (c) The boundary conditions of (2.5.37) must be replaced by $\phi(0) = 0$ and $\phi(\pi/2) = 0$. Thus, $L = \pi/2$, so that $\lambda = (n\pi/L)^2 = (2n)^2$ and $\phi = \sin\frac{n\pi\theta}{L} = \sin 2n\theta$. The radial part that remains bounded at $r = 0$ is $G = r^{\sqrt{\lambda}} = r^{2n}$. By superposition,

$$u(r,\theta) = \sum_{n=1}^{\infty} A_n r^{2n}\sin 2n\theta\ .$$

To apply the nonhomogeneous boundary condition, we differentiate with respect to r:

$$\frac{\partial u}{\partial r} = \sum_{n=1}^{\infty} A_n (2n) r^{2n-1}\sin 2n\theta\ .$$

The bc at $r = 1$, $f(\theta) = \sum_{n=1}^{\infty} 2nA_n \sin 2n\theta$, determines $A_n, 2nA_n = \frac{4}{\pi}\int_0^{\pi/2} f(\theta)\sin 2n\theta\, d\theta$.

2.5.6 (a) The boundary conditions of (2.5.37) must be replaced by $\phi(0) = 0$ and $\phi(\pi) = 0$. Thus $L = \pi$, so that the eigenvalues are $\lambda = (n\pi/L)^2 = n^2$ and corresponding eigenfunctions $\phi = \sin n\pi\theta/L = \sin n\theta, n = 1, 2, 3, \ldots$ The radial part which is bounded at $r = 0$ is $G = r^{\sqrt{\lambda}} = r^n$. Thus by superposition

$$u(r,\theta) = \sum_{n=1}^{\infty} A_n r^n \sin n\theta\ .$$

The bc at $r = a$, $g(\theta) = \sum_{n=1}^{\infty} A_n a^n \sin n\theta$, determines $A_n, A_n a^n = \frac{2}{\pi}\int_0^{\pi} g(\theta)\sin n\theta\, d\theta$.

2.5.7 (b) The boundary conditions of (2.5.37) must be replaced by $\phi'(0) = 0$ and $\phi'(\pi/3) = 0$. This will yield a cosine series with $L = \pi/3, \lambda = (n\pi/L)^2 = (3n)^2$ and $\phi = \cos n\pi\theta/L = \cos 3n\theta, n = 0, 1, 2, \ldots$. The radial part which is bounded at $r = 0$ is $G = r^{\sqrt{\lambda}} = r^{3n}$. Thus by superposition

$$u(r,\theta) = \sum_{n=0}^{\infty} A_n r^{3n}\cos 3n\theta\ .$$

The boundary condition at $r = a$, $g(\theta) = \sum_{n=0}^{\infty} A_n a^{3n}\cos 3n\theta$, determines A_n: $A_0 = \frac{3}{\pi}\int_0^{\pi/3} g(\theta)\, d\theta$ and $(n \neq 0) A_n a^{3n} = \frac{6}{\pi}\int_0^{\pi/3} g(\theta)\cos 3n\theta\, d\theta$.

2.5.8 (a) There is a full Fourier series in θ. It is easier (but equivalent) to choose radial solutions that satisfy the corresponding homogeneous boundary condition. Instead of r^n and r^{-n} (1 and $\ln r$ for $n = 0$), we choose $\phi_1(r)$ such that $\phi_1(a) = 0$ and $\phi_2(r)$ such that $\phi_2(b) = 0$:

$$\phi_1(r) = \begin{cases} \ln(r/a) & n = 0 \\ \left(\frac{r}{a}\right)^n - \left(\frac{a}{r}\right)^n & n \neq 0 \end{cases} \quad \phi_2(r) = \begin{cases} \ln(r/b) & n = 0 \\ \left(\frac{r}{b}\right)^n - \left(\frac{b}{r}\right)^n & n \neq 0 \end{cases}.$$

Then by superposition

$$u(r,\theta) = \sum_{n=0}^{\infty} \cos n\theta \left[A_n\phi_1(r) + B_n\phi_2(r)\right] + \sum_{n=1}^{\infty} \sin n\theta \left[C_n\phi_1(r) + D_n\phi_2(r)\right].$$

The boundary conditions at $r = a$ and $r = b$,

$$f(\theta) = \sum_{n=0}^{\infty} \cos n\theta \left[A_n\phi_1(a) + B_n\phi_2(a)\right] + \sum_{n=1}^{\infty} \sin n\theta \left[C_n\phi_1(a) + D_n\phi_2(a)\right]$$

$$g(\theta) = \sum_{n=0}^{\infty} \cos n\theta \left[A_n\phi_1(b) + B_n\phi_2(b)\right] + \sum_{n=1}^{\infty} \sin n\theta \left[C_n\phi_1(b) + D_n\phi_2(b)\right]$$

easily determine A_n, B_n, C_n, D_n since $\phi_1(a) = 0$ and $\phi_2(b) = 0 : D_n\phi_2(a) = \frac{1}{\pi}\int_{-\pi}^{\pi} f(\theta)\sin n\theta \, d\theta$, etc.

2.5.9 (a) The boundary conditions of (2.5.37) must be replaced by $\phi(0) = 0$ and $\phi(\pi/2) = 0$. This is a sine series with $L = \pi/2$ so that $\lambda = (n\pi/L)^2 = (2n)^2$ and the eigenfunctions are $\phi = \sin n\pi\theta/L = \sin 2n\theta, n = 1, 2, 3, \ldots$. The radial part which is zero at $r = a$ is $G = (r/a)^{2n} - (a/r)^{2n}$. Thus by superposition,

$$u(r,\theta) = \sum_{n=1}^{\infty} A_n \left[\left(\frac{r}{a}\right)^{2n} - \left(\frac{a}{r}\right)^{2n}\right] \sin 2n\theta.$$

The nonhomogeneous boundary condition, $f(\theta) = \sum_{n=1}^{\infty} A_n \left[\left(\frac{b}{a}\right)^{2n} - \left(\frac{a}{b}\right)^{2n}\right] \sin 2n\theta$, determines A_n : $A_n \left[\left(\frac{b}{a}\right)^{2n} - \left(\frac{a}{b}\right)^{2n}\right] = \frac{4}{\pi}\int_0^{\pi/2} f(\theta)\sin 2n\theta \, d\theta$.

2.5.9 (b) The two homogeneous boundary conditions are in r, and hence $\phi(r)$ must be an eigenvalue problem. By separation of variables, $u = \phi(r)G(\theta), d^2G/d\theta^2 = \lambda G$ and $r^2\frac{d^2\phi}{dr^2} + r\frac{d\phi}{dr} + \lambda\phi = 0$. The radial equation is equidimensional (see p.78) and solutions are in the form $\phi = r^p$. Thus $p^2 = -\lambda$ (with $\lambda > 0$) so that $p = \pm i\sqrt{\lambda}$. $r^{\pm i\sqrt{\lambda}} = e^{\pm i\sqrt{\lambda}\ln r}$. Thus real solutions are $\cos(\sqrt{\lambda}\ln r)$ and $\sin(\sqrt{\lambda}\ln r)$. It is more convenient to use independent solutions which simplify at $r = a, \cos[\sqrt{\lambda}\ln(r/a)]$ and $\sin[\sqrt{\lambda}\ln(r/a)]$. Thus the general solution is

$$\phi = c_1\cos[\sqrt{\lambda}\ln(r/a)] + c_2\sin[\sqrt{\lambda}\ln(r/a)].$$

The homogeneous condition $\phi(a) = 0$ yields $0 = c_1$, while $\phi(b) = 0$ implies $\sin[\sqrt{\lambda}\ln(r/a)] = 0$. Thus $\sqrt{\lambda}\ln(b/a) = n\pi$, $n = 1, 2, 3, \ldots$ and the corresponding eigenfunctions are $\phi = \sin\left[n\pi\frac{\ln(r/a)}{\ln(b/a)}\right]$. The solution of the θ -equation satisfying $G(0) = 0$ is $G = \sinh\sqrt{\lambda}\theta = \sinh\frac{n\pi\theta}{\ln(b/a)}$. Thus by superposition

$$u = \sum_{n=1}^{\infty} A_n \sinh\frac{n\pi\theta}{\ln(b/a)} \sin\left[n\pi\frac{\ln(r/a)}{\ln(b/a)}\right].$$

The nonhomogeneous boundary condition,

$$f(r) = \sum_{n=1}^{\infty} A_n \sinh\frac{n\pi^2}{2\ln(b/a)} \sin\left[n\pi\frac{\ln(r/a)}{\ln(b/a)}\right],$$

will determine A_n. One method (for another, see exercise 5.3.9) is to let $z = \ln(r/a)/\ln(b/a)$. Then $a < r < b$, lets $0 < z < 1$. This is a sine series in z (with $L = 1$) and hence

$$A_n \sinh\frac{n\pi^2}{2\ln(b/a)} = 2\int_0^1 f(r)\sin\left[n\pi\frac{\ln(r/a)}{\ln(b/a)}\right] dz.$$

But $dz = dr/r\ln(b/a)$. Thus

$$A_n \sinh\frac{n\pi^2}{2\ln(b/a)} = \frac{2}{\ln(b/a)}\int_0^1 f(r)\sin\left[n\pi\frac{\ln(r/a)}{\ln(b/a)}\right] dr/r.$$

Chapter 3. Fourier Series

Section 3.2

3.2.2 (a) $x \sim a_0 + \sum_{n=1}^{\infty} a_n \cos n\pi x/L + \sum_{n=1}^{\infty} b_n \sin n\pi x/L$. From (3.2.2), $a_0 = 0$ since $f(x)$ is odd, $(n \neq 0) a_n = 0$ since $f(x)$ is odd, and $b_n = \frac{1}{L}\int_{-L}^{L} x \sin n\pi x/L \, dx = \frac{2L}{n\pi}(-1)^{n+1}$.

3.2.2 (c) $\sin \pi x/L \sim a_0 + \sum_{n=1}^{\infty} a_n \cos n\pi x/L + \sum_{n=1}^{\infty} b_n \sin n\pi x/L$. By inspection, $b_1 = 1$, all others $= 0$.

3.2.2 (f) From (3.2.2),

$$\begin{array}{rrcl} & a_0 & = & \frac{1}{2L}\int_0^L dx = 1/2 \\ (n \neq 0) & a_n & = & \frac{1}{L}\int_0^L \cos n\pi x/L \, dx = 0 \\ & b_n & = & \frac{1}{L}\int_0^L \sin n\pi x/L \, dx = \frac{-1}{n\pi} \cos \frac{n\pi x}{L}\Big|_0^L = \frac{1-\cos n\pi}{n\pi} \end{array}$$

Thus $b_n = 2/n\pi$, n odd, but $b_n = 0$, n even.

3.3.2 (d) From (3.3.6), $B_n = \frac{2}{L}\int_0^{L/2} \sin n\pi x/L \, dx = \frac{-2}{n\pi} \cos n\pi x/L\Big|_0^{L/2} = \frac{2}{n\pi}\left(1 - \cos \frac{n\pi}{2}\right)$.

3.3.10 $f(-x) = \begin{cases} x^2 & -x < 0 \quad (\text{or } x > 0) \\ e^x & -x > 0 \quad (\text{or } x < 0). \end{cases}$ Thus.

$$\begin{aligned} f_e(x) &= \tfrac{1}{2}[f(x) + f(-x)] = \tfrac{1}{2}\begin{cases} x^2 + e^x & x < 0 \\ x^2 + e^{-x} & x > 0 \end{cases} \\ f_0(x) &= \tfrac{1}{2}[f(x) - f(-x)] = \tfrac{1}{2}\begin{cases} x^2 - e^x & x < 0 \\ e^{-x} - x^2 & x > 0 \end{cases} \end{aligned}$$

3.3.13 $b_n = \frac{2}{L}\int_0^L f(x) \sin n\pi x/L \, dx$, given that $f(x)$ is even around $L/2$. Note (perhaps by graphing) that $\sin n\pi x/L$ is odd around $L/2$ for n even. Thus $f(x) \sin n\pi x/L$ is odd around $L/2$ for n even, and hence $b_n = 0$ for n even.

Section 3.4

3.4.1 (a) $\int_a^b = \int_a^{c-} + \int_{c+}^b$. Thus

$$\int_a^b u \frac{dv}{dx} \, dx = uv\Big|_a^{c-} + uv\Big|_{c+}^b - \int_a^b v \frac{du}{dx} \, dx = uv\Big|_a^b + uv\Big|_{c+}^{c-} - \int_a^b v \frac{du}{dx} \, dx.$$

3.4.3 (a) We want to determine the sine coefficients of df/dx: $\frac{df}{dx} = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{L}$, where the cosine coefficients of f are given

$$f = \sum_{n=0}^{\infty} a_n \cos \frac{n\pi x}{L} \qquad (n \neq 0) \;\; a_n = \frac{2}{L}\int_0^L f \cos \frac{n\pi x}{L} \, dx.$$

Here by integration by parts

$$b_n = \frac{2}{L}\int_0^L \frac{df}{dx} \sin \frac{n\pi x}{L} \, dx = \frac{2}{L}\left[f \sin \frac{n\pi x}{L}\Big|_0^{x_0-} + f \sin \frac{n\pi x}{L}\Big|_{x_0+}^{L} - \frac{n\pi}{L}\int_0^L f \cos \frac{n\pi x}{L} \, dx \right].$$

Thus $b_n = \frac{2}{L} \sin \frac{n\pi x_0}{L}(\alpha - \beta) - \frac{n\pi}{L} a_n$.

3.4.9 $\frac{\partial u}{\partial x} = \sum_{n=1}^{\infty} \left(\frac{n\pi}{L}\right) b_n \cos\frac{n\pi x}{L}$ since $u(0) = 0$ and $u(L) = 0$. $\frac{\partial^2 u}{\partial x^2} \sim -\sum_{n=1}^{\infty} \left(\frac{n\pi}{L}\right)^2 b_n \sin\frac{n\pi x}{L}$. Thus from p. 119, $\sum_{n=1}^{\infty} \frac{db_n}{dt} \sin\frac{n\pi x}{L} \sim -k\sum_{n=1}^{\infty} \left(\frac{n\pi}{L}\right)^2 b_n \sin\frac{n\pi x}{L} + q$. Thus $\frac{db_n}{dt} + k\left(\frac{n\pi}{L}\right)^2 b_n = \frac{2}{L}\int_0^L q \sin\frac{n\pi x}{L}\, dx$.

3.4.12 The eigenfunctions of the related homogeneous problem are $\cos n\pi x/L, n = 0, 1, 2, \ldots$. Thus $u \sim \sum_{n=0}^{\infty} A_n(t) \cos n\pi x/L$, which can be differentiated (if u is continuous) since it is a cosine series: $\partial u/\partial x \sim \sum_{n=0}^{\infty} A_n(-n\pi/L) \sin\frac{n\pi x}{L}$. This can be differentiated again (if $\partial u/\partial x$ is continuous) only because $\partial u/\partial x(0) = 0$ and $\partial u/\partial x(L) = 0$: $\partial^2 u/\partial x^2 \sim -\sum_{n=0}^{\infty} A_n(n\pi/L)^2 \cos n\pi x/L$. Thus from p. 119

$$\sum_{n=0}^{\infty} \left[\frac{dA_n}{dt} + k\left(\frac{n\pi}{L}\right)^2 A_n\right] \cos\frac{n\pi x}{L} = e^{-t} + e^{-2t}\cos\frac{3\pi x}{L}.$$

The right hand side is a simple cosine series (with only two non-zero terms). Thus

$$\frac{dA_n}{dt} + k\left(\frac{n\pi}{L}\right)^2 A_n = \begin{cases} e^{-t} & n = 0 \\ e^{-2t} & n = 3 \\ 0 & \text{otherwise}. \end{cases}$$

The initial conditions are $A_0(0) = \frac{1}{L}\int_0^L f(x)\, dx$ and $(n \neq 0)$ $A_n(0) = \frac{2}{L}\int_0^L f(x)\cos\frac{n\pi x}{L}\, dx$. The solution of the differential equations are

$$\begin{aligned} n \neq 0, 3 \quad A_n(t) &= A_n(0)e^{-k(n\pi/L)^2 t} \\ A_0(t) &= A_0(0) + 1 - e^{-t}, \text{ obtained by integration} \\ A_3(t) &= A_3(0)e^{-k(n\pi/L)^2 t} + \frac{e^{-2t} - e^{-k(n\pi/L)^2 t}}{k\left(\frac{3\pi}{L}\right)^2 - 2}, \end{aligned}$$

obtained by using the method of undetermined coefficients (judicious guessing) for the particular solution. This works if e^{-2t} is not a homogeneous solution, i.e., $-2 \neq -k(3\pi/L)^2$.

Section 3.5

3.5.4 (a) Using 3.4.13 with $f(x) = \cosh x (f(0) = 1, f(L) = \cosh L)$,
$\sinh x \sim \frac{1}{L}(\cosh L - 1) + \sum_{n=1}^{\infty} \left[\frac{n\pi}{L} b_n + \frac{2}{L}((-1)^n \cosh L - 1)\right] \cos\frac{n\pi x}{L}$. Since this is a cosine series, it may be differentiated

$$\cosh x \sim -\sum_{n=1}^{\infty} \left[\left(\frac{n\pi}{L}\right)^2 b_n + \frac{2n\pi}{L^2}((-1)^n \cosh L - 1)\right] \sin\frac{n\pi x}{L}.$$

Thus $b_n = -\left(\frac{n\pi}{L}\right)^2 b_n - \frac{2n\pi}{L^2}[(-1)^n \cosh L - 1]$ or $b_n = \frac{2n\pi}{L^2} \frac{1 - (-1)^n \cosh L}{1 + (n\pi/L)^2}$.

3.5.4 (b) Integrating yields $\sinh x = A_0 + \sum_{n=1}^{\infty} -\frac{L}{n\pi} b_n \cos n\pi x/L$, where $A_0 = \frac{1}{L}\int_0^L \sinh x\, dx = \frac{1}{L}(\cosh L - 1)$. Integrating again yields $\cosh x - 1 = A_0 x + \sum_{n=1}^{\infty} -\left(\frac{L}{n\pi}\right)^2 b_n \sin n\pi x/L$. Thus

$$\sum_{n=1}^{\infty} b_n \left[1 + \left(\frac{L}{n\pi}\right)^2\right] \sin n\pi x/L = 1 + A_0 x.$$

Using (3.3.8) and (3.3.12) $b_n \left[1 + \left(\frac{L}{n\pi}\right)^2\right] = \frac{2}{n\pi}[1 - (-1)^n] + \frac{1}{L}(\cosh L - 1)\frac{2L}{n\pi}(-1)^{n+1}$ or

$$b_n = \frac{2}{n\pi} \frac{1 - (-1)^n \cosh L}{1 + \left(\frac{L}{n\pi}\right)^2}.$$

3.5.7 Evaluate (3.5.6) at $x = L/2$:

$$\frac{L^2}{8} = \frac{L^2}{4} - \frac{4L^2}{\pi^3}\left(1 - \frac{1}{3^3} + \frac{1}{5^3} - \ldots\right) \text{ or } 1 - \frac{1}{3^3} + \frac{1}{5^3} - \frac{1}{7^3} = \frac{L^2/8}{4L^2/\pi^3} = \pi^3/32.$$

Section 3.6

3.6.1 The complex Fourier coefficient is defined by (3.6.7):

$$c_n = \frac{1}{2L}\int_{-L}^{L} f(x)e^{in\pi x/L}dx = \frac{1}{2L\triangle}\int_{x_0}^{x_0+\triangle} e^{in\pi x/L}dx\,.$$

Thus

$$c_n = \frac{1}{2L\triangle}\frac{L}{in\pi}e^{in\pi x/L}\Big|_{x_0}^{x_0+\triangle} = \frac{1}{2in\pi\triangle}e^{in\pi x_0/L}(e^{in\pi\triangle/L}-1)\,.$$

Equivalently,

$$c_n = \frac{1}{2in\pi\triangle}e^{in\pi(x_0+\triangle/2)/L}(e^{in\pi\triangle/2L}-e^{-in\pi\triangle/2L}) \quad \text{or}$$

$$c_n = \frac{1}{n\pi\triangle}e^{in\pi(x_0+\triangle/2)/L}\sin(n\pi\triangle/2L)\,.$$

Chapter 4. Vibrating Strings and Membranes

Section 4.4

4.4.1 (a) Natural frequencies are $c\sqrt{\lambda}$, but $\lambda = (n\pi/L)^2$. Thus frequencies are $n\pi c/L, n = 1, 2, 3, ...$

4.4.1 (b) Natural frequencies are $c\sqrt{\lambda}$. The boundary condition $\phi(0) = 0$ implies $\phi = c_1 \sin\sqrt{\lambda}x$, while $d\phi/dx(H) = 0$ yields $\sqrt{\lambda}H = (m - \frac{1}{2})\pi$ with $m = 1, 2, 3$. Thus the frequencies are $(m - \frac{1}{2})\pi c/H$ and the eigenfunctions are $\sin(m - \frac{1}{2})\pi x/H$.

4.4.2 (c) By separation of variables, $u = \phi(x)h(t)$, $\frac{d^2h}{dt^2} = -\lambda h$ and $T_0\frac{d^2\phi}{dx^2} + (\alpha + \lambda\rho_0)\phi = 0$. With $\phi(0) = 0$ and $\phi(L) = 0, (\alpha + \lambda\rho_0)/T_0 = (n\pi/L)^2, n = 1, 2, 3, \ldots$ and $\phi = \sin n\pi x/L$. In general $h(t)$ involves a linear combination of $\sin\sqrt{\lambda}t$ and $\cos\sqrt{\lambda}t$, but the homogeneous initial condition $u(x, 0) = 0$ implies there are no cosines. Thus by superposition

$$u(x,t) = \sum_{n=1}^{\infty} A_n \sin\sqrt{\lambda_n}t \sin n\pi x/L,$$

where the frequencies of vibration are $\sqrt{\lambda_n} = \sqrt{\frac{(n\pi/L)^2T_0 - \alpha}{\rho_0}}$. The other initial condition, $f(x) = \sum_{n=1}^{\infty} A_n\sqrt{\lambda_n}\sin n\pi x/L$, determines A_n

$$A_n\sqrt{\lambda_n} = \frac{2}{L}\int_0^L f(x)\sin n\pi x/L\,dx.$$

4.4.3 (b) By separation of variables, $u = \phi(x)h(t)$, $\frac{\rho_0 h'' + \beta h'}{hT_0} = \frac{\phi''}{\phi} = -\lambda$. The boundary conditions $\phi(0) = 0$ and $\phi(L) = 0$ yield $\lambda = (n\pi/L)^2$ with $\phi = \sin n\pi x/L$, $n = 1, 2, 3, \ldots$. The time-dependent equation has constant coefficients,

$$\rho_0 h'' + \beta h' + \left(\frac{n\pi}{L}\right)^2 T_0 h = 0,$$

and hence can be solved by substitution $h = e^{rt}$. This yields the quadratic equation

$$\rho_0 r^2 + \beta r + \left(\frac{n\pi}{L}\right)^2 T_0 = 0,$$

whose roots are

$$r = \frac{-\beta \pm \sqrt{\beta^2 - 4\rho_0T_0(n\pi/L)^2}}{2\rho_0}.$$

Since $\beta^2 < 4\rho_0T_0(\pi/L)^2$, the discriminant is < 0 for all n:

$$r = -\frac{\beta}{2\rho_0} + iw_n, \text{ where } w_n = \sqrt{\frac{T_0}{\rho_0}\left(\frac{n\pi}{L}\right)^2 - \frac{\beta^2}{4\rho_0^2}}.$$

Real solutions are $h = e^{-\beta t/2\rho_0}(\sin w_n t, \cos w_n t)$. Thus by superposition

$$u = e^{-\beta t/2\rho_0}\sum_{n=1}^{\infty}(a_n\cos w_n t + b_n \sin w_n t)\sin\frac{n\pi x}{L}.$$

The initial condition $u(x, 0) = f(x)$ determines a_n, $a_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}dx$, while $\frac{\partial u}{\partial t}(x, 0) = g(x)$ is a little more complicated, $g(x) = \sum_{n=1}^{\infty} b_n w_n \sin\frac{n\pi x}{L} - \frac{\beta}{2\rho_0}\underbrace{\sum_{n=1}^{\infty} a_n \sin\frac{n\pi x}{L}}_{f(x)}$, and thus

$$b_n w_n = \frac{\beta a_n}{2\rho_0} + \frac{2}{L}\int_0^L g(x)\sin\frac{n\pi x}{L}\,dx.$$

Chapter 5. Sturm-Liouville Eigenvalue Problems

Section 5.3

5.3.1 By separation of variables, $u = \phi(x)h(t)$,

$$\frac{1}{h}\frac{d^2h}{dt^2} = \frac{1}{\rho_0\phi(x)}\left[T_0\frac{d^2\phi}{dx^2} + \alpha\phi\right] = -\lambda.$$

Thus,

$$\frac{d^2h}{dt^2} = -\lambda h \text{ and } T_0\frac{d^2\phi}{dx^2} + \alpha\phi + \lambda\rho_0\phi = 0.$$

For the time-dependent equation (if $\lambda > 0$), $h = c_1\cos\sqrt{\lambda}t + c_2\sin\sqrt{\lambda}t$. The spatial equation is in Sturm-Liouville form: $p = T_0$ constant, $q = \alpha(x), \sigma = \rho_0(x)$.

5.3.3 $Hd^2\phi/dx^2 = \frac{d}{dx}(Hd\phi/dx) - dH/dx\, d\phi/dx$. Thus $\frac{d}{dx}(Hd\phi/dx) + d\phi/dx(H\alpha - dH/dx) + (\lambda\beta+\gamma)H\phi = 0$. To be in standard Sturm-Liouville form,

$$\frac{dH}{dx} = \alpha H \text{ or } H = c_1\exp\left[\int^x \alpha(t)\, dt\right], \text{ let } c_1 = 1.$$

Then $p(x) = H,\ q(x) = \gamma H$, and $\sigma(x) = \beta H$.

5.3.4 (b) By separation of variables, $u = \phi(x)h(t)$, $\frac{1}{h}\frac{dh}{dt} = \frac{1}{\phi}\left(k\frac{d^2\phi}{dx^2} - v_0\frac{d\phi}{dx}\right) = -\lambda$. The boundary value problem is $k\frac{d^2\phi}{dx^2} - v_0\frac{d\phi}{dx} + \lambda\phi = 0$. $\phi = e^{rx}$ implies $kr^2 - v_0r + \lambda = 0$ or

$$r = \frac{v_0 \pm \sqrt{v_0^2 - 4\lambda k}}{2k} = \frac{v_0 \pm i\sqrt{4\lambda k - v_0^2}}{2k}.$$

To satisfy the boundary conditions $\phi(0) = 0$ and $\phi(L) = 0$:

$$\frac{\sqrt{4\lambda k - v_0^2}}{2k} = \frac{n\pi}{L} \text{ and } \phi(x) = e^{\frac{v_0}{2k}x}\sin\frac{n\pi x}{L}.$$

Thus, by superposition,

$$u = \sum_{n=1}^{\infty} A_n e^{\frac{v_0}{2k}x}\sin\frac{n\pi x}{L}e^{-\lambda_n t} = e^{\frac{v_0}{2k}x}\sum_{n=1}^{\infty} A_n\sin\frac{n\pi x}{L}e^{-\lambda_n t}.$$

The initial value problem can be solved

$$f(x) = e^{\frac{v_0}{2k}x}\sum_{n=1}^{\infty} A_n\sin\frac{n\pi x}{L}$$

so that

$$A_n = \frac{2}{L}\int_0^L f(x)e^{-\frac{v_0}{2k}x}\sin\frac{n\pi x}{L}\, dx.$$

Note that $\lambda = \frac{v_0^2}{4k} + k\left(\frac{n\pi}{L}\right)^2$.

5.3.9 (c) Since it is equidimensional, $\phi = x^r$, which implies $r(r-1)+r+\lambda = 0$ or $r^2 = -\lambda$. If $\lambda > 0, r = \pm i\sqrt{\lambda}$, where $x^{\pm i\sqrt{\lambda}} = e^{\pm i\sqrt{\lambda}\ln x}$. Thus the general solution is $\phi = c_1\cos\left(\sqrt{\lambda}\ln x\right) + c_2\sin\left(\sqrt{\lambda}\ln x\right)$. The boundary condition $\phi(1) = 0$ implies $c_1 = 0$, and hence $\phi(b) = 0$ yields $\sin\left(\sqrt{\lambda}\ln b\right) = 0$ or $\sqrt{\lambda}\ln b = n\pi,\ n = 1, 2, \ldots$. Thus $\lambda = (n\pi/\ln b)^2$. If $\lambda = 0$ the differential equation becomes $\frac{d}{dx}(xd\phi/dx) = 0$. Thus, the general solution is $\phi = c_1 + c_2\ln x$. The condition $\phi(1) = 0$ yields $c_1 = 0$, and then $\phi(b) = 0$ implies $c_2 = 0$. Thus $\lambda = 0$ is not an eigenvalue.

Section 5.4

5.4.2 The eigenfunctions satisfy (5.4.6) with the boundary conditions being $d\phi/dx(0) = 0$ and $d\phi/dx(L) = 0$. From the Rayleigh quotient (5.4.16) $\lambda \geq 0$. Also $\lambda = 0$ only if $d\phi/dx = 0$ for all x. The boundary conditions imply $\lambda = 0$ is an eigenvalue with $\phi =$ constant $= 1$ the corresponding eigenfunction. Thus by superposition using (5.4.5)

$$u = \sum_{n=1}^{\infty} a_n \phi_n(x) e^{-\lambda_n t} (\text{with } \lambda_1 = 0, \phi_1 = 1).$$

The initial conditions,

$$f(x) = \sum_{n=1}^{\infty} a_n \phi_n(x) \; ,$$

yield

$$a_n = \frac{\int_0^L f \phi_n c\rho \; dx}{\int_0^L \phi_n^2 c\rho \; dx},$$

since the eigenfunctions are orthogonal with weight $c\rho$. As $t \to \infty$,

$$u \to a_1 = \frac{\int_0^L f \; c\rho \; dx}{\int_0^L c\rho \; dx},$$

a weighted average of the initial temperature distribution. This can also be shown using conservation of total thermal energy. Since the ends are insulated, the final thermal energy ($t \to \infty$) equals the initial thermal energy ($t = 0$).

5.4.3 From (5.2.11) the eigenfunctions, denoted $\phi_n(r)$, are orthogonal with weight r. The time-dependent part satisfies (5.2.10), and hence $h(t) = e^{-\lambda k t}$. By superposition

$$u = \sum_{n=1}^{\infty} a_n \phi_n(r) e^{-\lambda k t}.$$

The initial condition,

$$f(r) = \sum_{n=1}^{\infty} a_n \phi_n(r) \; ,$$

determines a_n,

$$a_n = \frac{\int_0^a f(r) \phi_n(r) r \; dr}{\int_0^a \phi_n^2(r) r \; dr}.$$

5.4.6 By separation of variables, $u = \phi(x) h(t)$,

$$\begin{aligned} T_0 \frac{d^2\phi}{dx^2} + \lambda \rho_0 \phi &= 0 \\ \frac{d^2 h}{dt^2} + \lambda h &= 0 \, . \end{aligned}$$

The boundary conditions are of the Sturm-Liouville type, and therefore the eigenfunctions denoted $\phi_n(x)$ are orthogonal with weight $\rho_0(x)$. We call the eigenvalues λ_n. Then the time-dependent problem has solutions $\cos\sqrt{\lambda_n}t$ and $\sin\sqrt{\lambda_n}t$. Initially at rest means $\partial u/\partial t(x, 0) = 0$, so that only cosines are needed. Thus by superposition

$$u = \sum_{n=1}^{\infty} A_n \phi_n(x) \cos\sqrt{\lambda_n}t.$$

The initial position yields

$$f(x) = \sum_{n=1}^{\infty} A_n \phi_n(x) \; ,$$

determining A_n,

$$A_n = \frac{\int_0^L f(x)\phi_n(x)\rho_0(x)\,dx}{\int_0^L \phi_n^2 \rho_0\,dx}.$$

Section 5.5

5.5.1 (g) To be self-adjoint (if p is constant),

$$u(L)\frac{dV}{dx}(L) - v(L)\frac{du}{dx}(L) = u(0)\frac{dv}{dx}(0) - v((0)\frac{du}{dx}(0).$$

The derivatives at $x = L$ may be eliminated from the boundary conditions. After some algebra, we obtain

$$(\alpha\delta - \beta\gamma)\left[u(0)\frac{dv}{dx}(0) - v(0)\frac{du}{dx}(0)\right] = u(0)\frac{dv}{dx}(0) - v(0)\frac{du}{dx}(0).$$

Thus, it is self-adjoint only if $\alpha\delta - \beta\gamma = 1$.

5.5.9 Multiply the differential equation by ϕ and integrate:

$$\int_0^1 \phi\frac{d^4\phi}{dx^4}\,dx + \lambda\int_0^1 e^x\phi^2\,dx = 0.$$

Note that

$$\phi\frac{d^4\phi}{dx^4} = \frac{d}{dx}\left(\phi\frac{d^3\phi}{dx^3}\right) - \frac{d\phi}{dx}\frac{d^3\phi}{dx^3} = \frac{d}{dx}\left(\phi\frac{d^3\phi}{dx^3}\right) - \frac{d}{dx}\left(\frac{d\phi}{dx}\frac{d^2\phi}{dx^2}\right) + \left(\frac{d^2\phi}{dx^2}\right)^2.$$

Thus $\phi\frac{d^3\phi}{dx^3}\Big|_0^1 - \frac{d\phi}{dx}\frac{d^2\phi}{dx^2}\Big|_0^1 + \int_0^1\left(\frac{d^2\phi}{dx^2}\right)^2\,dx + \lambda\int_0^1 e^x\phi^2\,dx = 0.$

From the boundary conditions, the boundary contributions vanish. Thus

$$\lambda = \frac{-\int_0^1\left(\frac{d^2\phi}{dx^2}\right)^2\,dx}{\int_0^1 e^x\phi^2\,dx},$$

so that $\lambda \leq 0$.

Section 5.6

5.6.1 (c) From (5.6.6) using $p = 1, q = 0, \sigma = 1$ and using the boundary conditions,

$$\lambda_1 \leq \frac{u_T^2(1) + \int_0^1\left(\frac{du_T}{dx}\right)^2\,dx}{\int_0^1 u_T^2\,dx}.$$

Any trial function should satisfy the boundary conditions (and be continuous with no zeroes). Geometrically, a simple example is a parabola ($u_T(0) = 0, u_T(1) = 1, du_T/dx(1) = -1$, from the boundary conditions. To obtain this parabola, we substitute $u_T = ax + bx^2$ into the condition at $x = 1 : a + 2b + a + b = 0$, a simple choice being $a = 3, b = -2$ so that $u_T = 3x - 2x^2$ and $u_T/dx = 3 - 4x$. Thus,

$$\lambda_1 \leq \frac{1 + \int_0^1(3-4x)^2\,dx}{\int_0^1(3x-2x^2)^2\,dx} = \frac{1 - \frac{1}{12}(3-4x)^3\Big|_0^1}{\int_0^1(9x^2 - 12x^3 + 4x^4)\,dx} = \frac{1 - \frac{1}{12}(-1-27)}{3 - 3 + \frac{4}{5}} = \frac{40/12}{4/5} = 4\frac{1}{6}.$$

Section 5.7

5.7.1 $c^2 = T_0/\rho$. In this example $1 \leq c^2 \leq 1 + \alpha^2$. Thus from the bottom of page 197 with $L = 1$

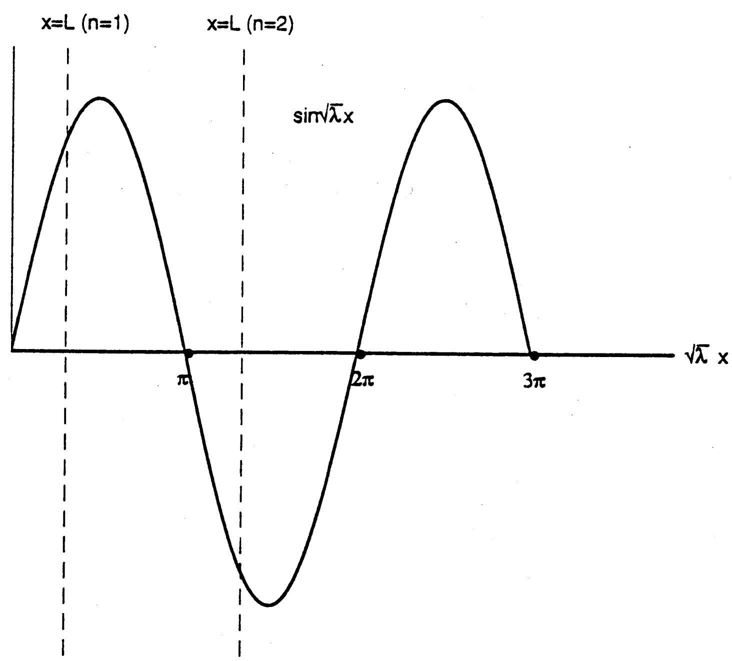
$$\pi^2 \leq \lambda_1 \leq \pi^2 \left(1 + \alpha^2\right) \text{ or } \pi \leq \sqrt{\lambda_1} \leq \pi\sqrt{1 + \alpha^2}.$$

The circular frequency (cycles per 2π units of time) is $\sqrt{\lambda}$, but the actual frequency is $\sqrt{\lambda}/2\pi$ (cycles per 1 unit of time):

$$\frac{1}{2} \leq \frac{\sqrt{\lambda}}{2\pi} \leq \frac{1}{2}\sqrt{1 + \alpha^2}.$$

Section 5.8

5.8.2 (c) By extension of Fig. 5.8.2 (a), $0 < \sqrt{\lambda_1}L < \frac{\pi}{2}, \pi < \sqrt{\lambda_2}L < \frac{3\pi}{2}, 2\pi < \sqrt{\lambda_3}L < \frac{5\pi}{2}$, etc. In general $(n-1)\pi < \sqrt{\lambda_n}L < \left(n - \frac{1}{2}\right)\pi$. The eigenfunction $\sin\sqrt{\lambda}x$ is graphed as a function of $\sqrt{\lambda}x$. The endpoint $x = L$ occurs at $\sqrt{\lambda}L$. For the first eigenfunction this occurs before $\pi/2$; the eigenfunction has no zeroes for $0 < x < L$. For the second eigenfunction, the end occurs before $3\pi/2$ and after π; the eigenfunction has one zero at $\sqrt{\lambda_2}x = \pi$. Etc.. In general, the nth eigenfunction has $n-1$ zeroes.



5.8.3 (b) If $\lambda > 0$, then $\phi = c_1 \cos\sqrt{\lambda}x + c_2 \sin\sqrt{\lambda}x$. $d\phi/dx(0) = 0$ implies $c_2 = 0$. The boundary condition at $x = L$ yields $-\sqrt{\lambda}\sin\sqrt{\lambda}L + h\cos\sqrt{\lambda}L = 0$, or $\tan\sqrt{\lambda}L = h/\sqrt{\lambda} = hL/\sqrt{\lambda}L$. Thus, the straight line in Fig. 5.8.1 must be replaced by the hyperbola ($z = c/x'$, where $c = hL$ and $x' = \sqrt{\lambda}L$). Finally, $0 < \sqrt{\lambda_1}L < \frac{\pi}{2}, \pi < \sqrt{\lambda_2}L < \frac{3\pi}{2}, \ldots$. In general, $(n-1)\pi < \sqrt{\lambda_n}L < \left(n - \frac{1}{2}\right)\pi,\ n = 1, 2, 3, \ldots$. Asymptotically for large n, $\sqrt{\lambda_n}L \sim (n-1)\pi$.

5.8.7 (a) $\int_0^\pi \left(u\frac{d^2v}{dx^2} - v\frac{d^2u}{dx^2}\right)\, dx = u\frac{dv}{dx} - v\frac{du}{dx}\Big|_0^\pi = 2\left[\frac{du}{dx}(0)\frac{dv}{dx}(\pi) - \frac{dv}{dx}(0)\frac{du}{dx}(\pi)\right].$

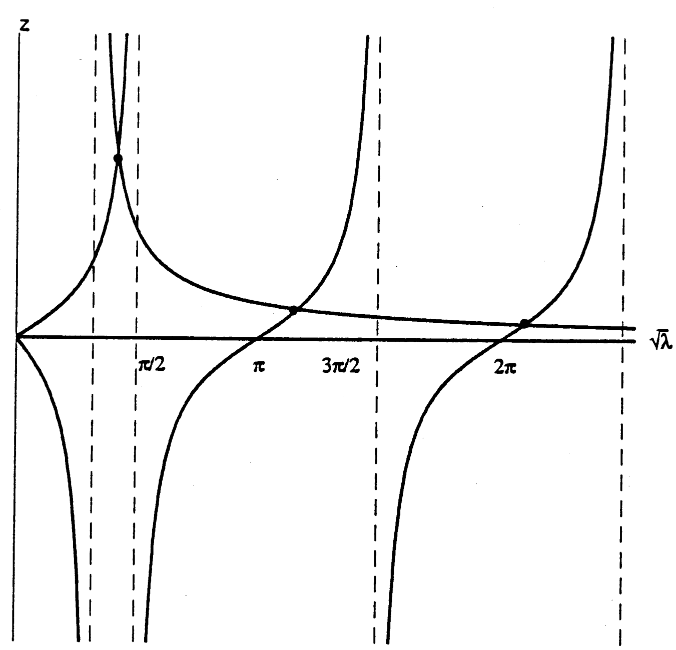
5.8.7 (b) If $\lambda > 0$, then $\phi(0) = 0$ implies $\phi = \sin\sqrt{\lambda}x$. The other condition yields $\sin\sqrt{\lambda}\pi = 2\sqrt{\lambda} = 2\sqrt{\lambda}\pi/\pi$. From a graphical solution, there is only one solution since the slope $2/\pi$ of the straight line is less than 1, the slope at the origin of the sinusoidal function. In fact, we note that $\lambda = 1/4$ (although this is not important).

5.8.7 (c) If $\lambda = -s < 0$, then $\phi(0) = 0$ implies $\phi = \sinh\sqrt{s}x$. The other condition yields $\sinh\sqrt{s}\pi = 2\sqrt{s} = 2\sqrt{s}\pi/\pi$. There are no solutions (by graphing the sinh function and the straight line) since the slope $2/\pi$ of the straight line is less than 1, the slope at the origin of the hyperbolic function.

5.8.7 (d) If $\lambda = 0$, then $\phi(0) = 0$ implies $\phi = x$. The other condition yields $\pi = 2$, and hence $\lambda = 0$ is not an eigenvalue.

5.8.7 (e) Since the boundary conditions for this eigenvalue problem are not of the form (5.3.2), this is not a regular Sturm-Liouville problem. Thus, there may be complex eigenvalues (with a non-zero imaginary part), not obtained in parts (b) - (d).

5.8.8 (c) If $\lambda > 0$, $\phi = c_1\cos\sqrt{\lambda}x + c_2\sin\sqrt{\lambda}x$, so that $d\phi dx = \sqrt{\lambda}(-c_1\sin\sqrt{\lambda}x + c_2\cos\sqrt{\lambda}x)$. The condition at $x = 0$, yields $c_1 = c_2\sqrt{\lambda}$, so that the eigenfunction is any multiple of $\phi = \sin\sqrt{\lambda}x + \sqrt{\lambda}\cos\sqrt{\lambda}x$. The boundary condition at $x = 1$ then yields $\tan\sqrt{\lambda} = 2\sqrt{\lambda}/(\lambda - 1)$. This is graphed using Fig. 5.8.1 with the straight line replaced by $z = \frac{2x'}{(x'^2-1)}$, where $x' = \sqrt{\lambda}$. Note the singularity at $\sqrt{\lambda} = 1 < \pi/2$. Thus $\sqrt{\lambda_n} \sim (n-1)\pi$.



5.8.10 (a) $h = 1, L = 1$ and thus, from (5.8.15), $\tan\sqrt{\lambda} = -\sqrt{\lambda}$ with $\pi/2 < \sqrt{\lambda_1} < \pi$ from (5.8.19). Using a PC, $\tan(2.03) = -2.022419$ and $\tan(2.02) = -2.074373$. Thus $\sqrt{\lambda_1} \sim 2.03$ or $\lambda_1 \approx 4.12$.

5.8.10 (b) $h = 1, L = 1$ and thus, from (5.8.15), $\tan\sqrt{\lambda_1} = -\sqrt{\lambda_1}$ with $\pi/2 < \sqrt{\lambda_1} < \pi$ from (5.8.19). Using Newton's method (letting $x = \sqrt{\lambda_1}$) with $f(x) = x + \tan x$, yields

$$x_{n+l} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{x_n + \tan x_n}{1 + \sec^2 x_n}.$$

In this manner, we obtain $\sqrt{\lambda_1} = \lim_{n\to\infty} x_n = 2.028758\ldots$ or $\lambda_1 = 4.11586\ldots$

5.8.13 $\int_0^L \sin^2\sqrt{\lambda}x\, dx = \frac{1}{2}\int_0^L (1-\cos 2\sqrt{\lambda}x)\, dx = \frac{L}{2} - \frac{\sin 2\sqrt{\lambda}L}{4\sqrt{\lambda}} = \frac{L}{2} - \frac{\sin\sqrt{\lambda}L\cos\sqrt{\lambda}L}{2\sqrt{\lambda}}$. But $\tan\sqrt{\lambda}L = -\sqrt{\lambda}/h$. Therefore, $\sin\sqrt{\lambda}L = \frac{-\frac{\sqrt{\lambda}}{h}}{\sqrt{1+(\lambda/h^2)}}$ and $\cos\sqrt{\lambda}L = \frac{1}{\sqrt{1+(\lambda/h^2)}}$. Thus $\int_0^L \sin^2\sqrt{\lambda}x\, dx = \frac{L}{2} + \frac{1}{2h[1+(\lambda/h^2)]}$.

Section 5.9

5.9.1 (b) To satisfy $\phi(0) = 0$, we use (5.9.9) and hence $d\phi/dx \approx (\sigma p)^{-1/4}(\lambda\sigma/p)^{1/2}\cos\left(\lambda^{1/2}\int_0^x (\sigma/p)^{1/2}\, dx_0\right)$. Thus

$$\lambda^{1/2}\int_0^L (\sigma/p)^{1/2}\, dx_0 \approx \left(n + \frac{1}{2}\right)\pi\,.$$

5.9.3 (a)

$$\begin{aligned}\phi' &= \left(A' + Ai\lambda^{1/2}\sigma^{1/2}\right)\exp[\ldots]\\ \phi'' &= \left[A'' + i\lambda^{1/2}\left(2A'\sigma^{1/2} + \tfrac{1}{2}\sigma^{-1/2}\sigma' A\right) - \lambda\sigma A\right]\exp[\ldots]\end{aligned}$$

By substituting into the differential equation, we obtain $A'' = i\lambda^{1/2}\left(2A'\sigma^{1/2} + \frac{1}{2}\sigma^{-1/2}\sigma' A\right) + qA = 0$.

5.9.3 (e) Let $A = \sum_{n=0}^{\infty} A_n\lambda^{-n/2}$. Thus $A'_{n+1}\sigma^{1/2} + \frac{1}{4}\sigma^{-1/2}\sigma' A_{n+1} = \frac{i}{2}\left(A''_n + qA_n\right)$ or $\frac{d}{dx}\left(\sigma^{1/4}A_{n+1}\right) = \frac{i}{2}\sigma^{-1/4}\left(A''_n + qA_n\right)$. By integrating, $A_{n+1} = \frac{i}{2}\sigma^{-1/4}\int_0^x \sigma^{-1/4}\left(A''_n + qA_n\right)\, dx_0$.

Section 5.10

5.10.2 (b) From (3.5.5) or (3.3.21-23), $x = \frac{L}{2} - \frac{4L}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\cos n\pi x/L$ (odd only). From (5.10.14) with $\sigma = 1$ for a cosine series

$$\int_0^L f^2\, dx = \sum_{n=0}^{\infty}\alpha_n^2\int_0^L \cos^2 n\pi x/L\, dx.$$

With $f = x, \alpha_0 = L/2$, (n odd) $\alpha_n = \frac{-4L}{n^2\pi^2}$, ($n$ even) $\alpha_n = 0$, we obtain (dividing by L^3) $\frac{1}{3} = \frac{1}{4} + \sum_{n=1}^{\infty}\frac{8}{\pi^4}\frac{1}{n^4}$(odd only) or $\frac{\pi^4}{96} = 1 + \frac{1}{3^4} + \frac{1}{5^4} + \frac{1}{7^4} + \ldots$

Chapter 6. Numerical Methods

Section 6.2

6.2.6 We use (6.2.13):

$$\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial}{\partial y}(\frac{\partial u}{\partial x}) = \frac{\partial}{\partial y}\frac{u(x+\Delta x, y) - u(x-\Delta x, y)}{2\Delta x}$$

$$= \frac{1}{2\Delta y\, 2\Delta x}[u(x+\Delta x, y+\Delta y) - u(x-\Delta x, y+\Delta y) - u(x+\Delta x, y-\Delta y) + u(x-\Delta x, y-\Delta y)].$$

Section 6.3

6.3.4 (a) At $t = 0$, $u_j^{(0)} = f(x_j) = f_j$. Thus, from (6.3.24)

$$f_j = \sum_{n=1}^{N-1} \beta_n \sin \frac{n\pi x_j}{L} = \sum_{n=1}^{N-1} \beta_n \sin \frac{n\pi j}{N},$$

since $x_j = j\Delta x$ and $\Delta x = L/N$. We multiply by $\sin\frac{m\pi j}{N}$ and sum from $j = 1$ to $j = N - 1$ (analogous to integrating over x):

$$\sum_{j=1}^{N-1} f_j \sin \frac{m\pi j}{N} = \sum_{n=1}^{N-1} \beta_n \sum_{j=1}^{N-1} \sin \frac{m\pi j}{N} \sin \frac{n\pi j}{N}.$$

The discrete eigenfunctions are orthogonal (see exercise 6.3.3), and hence the inner summation is nonzero only for $n = m$. Thus

$$\sum_{j=1}^{N-1} f_j \sin \frac{m\pi j}{N} = \beta_m \sum_{j=1}^{N-1} \sin^2 \frac{m\pi j}{N}.$$

6.3.4 (b) Using the double angle formula $\sum_{j=1}^{N-1} \sin^2\frac{m\pi j}{N} = \frac{1}{2}\sum_{j=1}^{N-1}(1-\sin\frac{2m\pi j}{N})$. However

$$\sum_{j=1}^{N-1} \sin \frac{2m\pi j}{N} = \mathrm{Im} \sum_{j=1}^{N-1} e^{i\frac{2m\pi j}{N}} = \mathrm{Im}\frac{e^{i\frac{2m\pi}{N}} - e^{i\frac{2m\pi N}{N}}}{1 - e^{i\frac{2m\pi}{N}}} = 0,$$

since $e^{i2m\pi} = 1$, where the sum of a finite geometric series has been used. Thus,

$$\sum_{j=1}^{N-1} \sin^2 \frac{m\pi j}{N} = \frac{1}{2}(N-1).$$

6.3.6 (d) From the top of page 238, the scheme is stable if and only if $s \leq \frac{1}{1-\cos\frac{(N-1)\pi}{N}}$. For $n = 10$, from a calculator or pc $s \leq \frac{1}{1-\cos\frac{9\pi}{10}} = 0.5125....$

6.3.9 (a) Using the centered difference, $f_1 = \frac{1}{(\Delta x)^2}(u_2 - 2u_1 + u_0)$ and $f_2 = \frac{1}{(\Delta x)^2}(u_3 - 2u_2 + u_1)$, where $\Delta x = \frac{L}{3}$. From the boundary conditions $u_0 = u_3 = 0$.

6.3.9 (b) From part (a), $\mathbf{A}\vec{u} = \vec{f}$, where $\mathbf{A} = \frac{1}{(\Delta x)^2}\begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix}$.

6.3.10 (a) As on page 240, we let $u_j^{(m)}$ be the probability that a particle is located at point j at time $m\Delta t$. Thus $u_j^{(m)} = a\,u_{j-1}^{(m-1)} + a\,u_{j+1}^{(m-1)} + bu_j^{(m-1)}$.

6.3.10 (b) Using the Taylor series of a function of two variables,

$$u_j^{(m-1)} = u_j^{(m)} - \Delta t\frac{\partial u}{\partial t} + O(\Delta t)^2$$

$$u_{j-1}^{(m-1)} = u_j^{(m)} - \Delta t\frac{\partial u}{\partial t} - \Delta x\frac{\partial u}{\partial x} + \frac{(\Delta x)^2}{2}\frac{\partial^2 u}{\partial x^2} + O[(\Delta t)^2, \Delta x\Delta t, (\Delta x)^3]$$

$$u_{j+1}^{(m-1)} = u_j^{(m)} - \Delta t\frac{\partial u}{\partial t} + \Delta x\frac{\partial u}{\partial x} + \frac{(\Delta x)^2}{2}\frac{\partial^2 u}{\partial x^2} + O[(\Delta t)^2, \Delta x\Delta t, (\Delta x)^3]$$

Thus,

$$u_j^{(m)} = (2a+b)\,u_j^{(m)} - (2a+b)\Delta t\frac{\partial u}{\partial t} + a(\Delta x)^2\frac{\partial^2 u}{\partial x^2} + O[(\Delta t)^2, \Delta x\Delta t, (\Delta x)^3].$$

From $2a+b=1$, we obtain $\frac{\partial u}{\partial t} = \frac{ka}{s}\frac{\partial^2 u}{\partial x^2}$ since $\lim_{\Delta x\to 0}$ and $_{\Delta t\to 0}\ \frac{(\Delta x)^2}{\Delta t} = \frac{k}{s}$.

6.3.14 (c) The eigenvalues are determined from (6.3.50):

$$\begin{vmatrix} 1-\lambda & 2 & -3 \\ 2 & 4-\lambda & -6 \\ 0 & 1/3 & 2-\lambda \end{vmatrix} = (1-\lambda)[(4-\lambda)(2-\lambda)+2] - 4(2-\lambda) - 2$$

$= -\lambda^3 + \lambda^2(1+6) + \lambda(-6-10+4) = -\lambda(\lambda^2 - 7\lambda + 12) = -\lambda(\lambda-4)(\lambda-3)$.

Thus, $\lambda = 0, 3, 4$. According to the Gershgorin circle theorem, every eigenvalue lies in the region composed of the interiors of 3 circles:

$$\left|\lambda - 1\right| \le 5, \left|\lambda - 4\right| \le 8, \quad \text{and} \quad \left|\lambda - 2\right| \le \frac{1}{3},$$

from (6.3.62). In this case all three eigenvalues lie in the first two circles, but none lie in the third circle.

Section 6.4

6.4.1 Here we do not assume $\Delta x = \Delta y$. From (6.2.16)

$$\frac{u_{j,l}^{(m+1)} - u_{j,l}^{(m)}}{\Delta t} = k[\frac{u_{j+1,l}^{(m)} - 2u_{j,l}^{(m)} + u_{j-1,l}^{(m)}}{(\Delta x)^2} + \frac{u_{j,l+1}^{(m)} - 2u_{j,l}^{(m)} + u_{j,l-1}^{(m)}}{(\Delta y)^2}].$$

We substitute (6.4.2) and obtain

$$\frac{Q-1}{\Delta t} = k[\frac{e^{i\alpha\Delta x} - 2 + e^{-i\alpha\Delta x}}{(\Delta x)^2} + \frac{e^{i\beta\Delta y} - 2 + e^{-i\beta\Delta y}}{(\Delta y)^2}].$$

Thus, $Q = 1 + 2k\Delta t[\frac{\cos(\alpha\Delta x)-1}{(\Delta x)^2} + \frac{\cos(\beta\Delta y)-1}{(\Delta y)^2}]$. To be stable $-1 < Q < 1$. Since $\cos(\alpha\Delta x) > -1$ and $\cos(\beta\Delta y) > -1$, we are guaranteed stability if $4k\Delta t[\frac{1}{(\Delta x)^2} + \frac{1}{(\Delta y)^2}] \le 2$, which yields the desired result.

Section 6.5

6.5.5 (a) The partial difference equation is $\frac{u_j^{(m+1)} - u_j^{(m)}}{\Delta t} + c\frac{u_{j+1}^{(m)} - u_{j-1}^{(m)}}{2\Delta x} = 0$ using (6.2.13).

6.5.5 (b) We substitute (6.5.6) into the results of part (a) and obtain $\frac{Q-1}{\Delta t} + c\frac{e^{i\alpha\Delta x} - e^{-i\alpha\Delta x}}{2\Delta x} = 0$. Thus $Q = 1 - c\frac{\Delta t}{\Delta x} i\sin(\alpha\Delta \text{x})$. Stability is determined by $\left|Q\right|$. Since $\left|Q\right| = \sqrt{1+\sigma^2} > 1$, where $\sigma = c\frac{\Delta t}{\Delta x}\sin(\alpha\Delta \text{x})$, this scheme is unstable.

6.5.6 Using (6.2.13), the partial difference equation is $\frac{u_j^{(m+1)} - u_j^{(m-1)}}{2\Delta t} + c\frac{u_{j+1}^{(m)} - u_{j-1}^{(m)}}{2\Delta x} = 0$. By substituting (6.5.6), we obtain $\frac{Q-\frac{1}{Q}}{2\Delta t} + c\frac{e^{i\alpha\Delta x} - e^{-i\alpha\Delta x}}{2\Delta x} = 0$. This is equivalent to $Q - \frac{1}{Q} + 2i\sigma = 0$, where $\sigma = c\frac{\Delta t}{\Delta x}\sin(\alpha\Delta \text{x})$. Thus $Q^2 + 2i\sigma Q - 1 = 0$ or $Q = -i\sigma \pm \sqrt{1-\sigma^2}$. If $\sigma^2 < 1$, then $\left|Q\right| = \sqrt{\sigma^2 + 1 - \sigma^2} = 1$. The numerical scheme is stable if $\sigma^2 < 1$. This (stability of the numerical scheme) is guaranteed if $c\frac{\Delta t}{\Delta x} < 1$ since $\left|\sin(\alpha\Delta \text{x})\right| \le 1$. However, if $\sigma^2 > 1$, then $Q = i(-\sigma \pm \sqrt{\sigma^2 - 1})$. In this case, the scheme is unstable since $\left|Q\right| > \left|\sigma\right|$ for one root and $\left|\sigma\right| > 1$ in this case.

Chapter 7. Higher Dimensional PDES

Section 7.3

7.3.1 (a) The result of product solutions $u(x,y,t) = \phi(x,y)h(t)$ is $dh/dt = -\lambda kh$ and $\nabla^2\phi = -\lambda\phi$, which is further separated $\phi(x,y) = f(x)g(y)$: $\frac{d^2f}{dx^2} = -\mu f$ with $f(0) = f(L) = 0$ yielding $\mu = \left(\frac{n\pi}{L}\right)^2, f = \sin\frac{n\pi x}{L}, n = 1,2,...$ and $\frac{d^2g}{dy^2} = -(\lambda-\mu)g$ with $g(0) = g(H) = 0$ yielding $\lambda - \mu = \left(\frac{m\pi}{H}\right)^2, g = \sin\frac{m\pi y}{H}, m = 1,2,\ldots$ Thus $\lambda_{nm} = \left(\frac{n\pi}{L}\right)^2 + \left(\frac{m\pi}{H}\right)^2, \phi = \sin\frac{n\pi x}{L}\sin\frac{m\pi y}{H}, n = 1,2,\ldots, m = 1,2,\ldots$ By superposition

$$u = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} a_{nm}\sin\frac{n\pi x}{L}\sin\frac{m\pi y}{H}e^{-\lambda_{nm}kt}.$$

The initial condition implies

$$f = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} a_{nm}\sin\frac{n\pi x}{L}\sin\frac{m\pi y}{H},$$

yielding (by two-dimensional orthogonality)

$$a_{nm} = \frac{4}{LH}\int_0^H\int_0^L f\sin\frac{n\pi x}{L}\sin\frac{m\pi y}{H}\,dx\,dy.$$

As $t \to \infty$, $u \to 0$ (the only equilibrium temperature if $u = 0$ along the entire boundary).

7.3.1 (c) The result of product solutions $u(x,y,t) = \phi(x,y)h(t)$ is $dh/dt = -\lambda kh$ and $\nabla^2\phi = -\lambda\phi$, which is further separated $\phi(x,y) = f(x)g(y)$:
$\frac{d^2f}{dx^2} = -\mu f$ with $f'(0) = f'(L) = 0$ yielding $\mu = \left(\frac{n\pi}{L}\right)^2, f = \cos\frac{n\pi x}{L}, n = 0,1,2,...$and $\frac{d^2g}{dy^2} = -(\lambda-\mu)g$ with $g(0) = g(H) = 0$ yielding $\lambda-\mu = \left(\frac{m\pi}{H}\right)^2, g = \sin\frac{m\pi y}{H}, m = 1,2,\ldots$ Thus $\lambda_{nm} = \left(\frac{n\pi}{L}\right)^2 + \left(\frac{m\pi}{H}\right)^2, \phi = \cos\frac{n\pi x}{L}\sin\frac{m\pi y}{H}, n = 0,1,2,\ldots, m = 1,2,\ldots$ By superposition

$$u = \sum_{n=0}^{\infty}\sum_{m=1}^{\infty} a_{nm}\cos\frac{n\pi x}{L}\sin\frac{m\pi y}{H}e^{-\lambda_{nm}kt}.$$

The initial condition implies

$$f = \sum_{n=0}^{\infty}\sum_{m=1}^{\infty} a_{nm}\cos\frac{n\pi x}{L}\sin\frac{m\pi y}{H},$$

yielding (by two-dimensional orthogonality)

$$a_{nm} = \frac{\int_0^H\int_0^L f\cos\frac{n\pi x}{L}\sin\frac{m\pi y}{H}\,dx\,dy}{\int_0^H\int_0^L \cos^2\frac{n\pi x}{L}\sin^2\frac{m\pi y}{H}\,dx\,dy = \begin{cases}\frac{1}{2}LH & n=0\\ \frac{1}{4}LH & n\neq 0\end{cases}}.$$

As $t \to \infty$, $u \to 0$ (the only equilibrium temperature if $u = 0$ along the entire boundary).

7.3.2 (b) The result of product solutions $u(x,y,z,t) = \phi(x,y,z)h(t)$ is $dh/dt = -\lambda kh$ and $\nabla^2\phi = -\lambda\phi$, which is further separated $\phi(x,y) = f(x)g(y)Q(z)$:

$\frac{d^2f}{dx^2} = -\mu f$ with $f'(0) = f'(L) = 0$ yielding $\mu = \left(\frac{n\pi}{L}\right)^2, f = \cos\frac{n\pi x}{L}, n = 0,1,2,\ldots$ $\frac{d^2g}{dy^2} = -\nu g$ with $g'(0) = g'(L) = 0$ yielding $\nu = \left(\frac{m\pi}{H}\right)^2, g = \cos\frac{m\pi y}{H}, m = 0,1,2,\ldots$ $\frac{d^2Q}{dz^2} = -(\lambda-\mu-\nu)Q$ with $Q'(0) = Q'(W) = 0$ yielding $\lambda-\mu-\nu = \left(\frac{\ell\pi}{W}\right)^2, Q = \cos\frac{\ell\pi z}{W}, \ell = 0,1,2,\ldots$ Thus $\lambda_{nm\ell} = \left(\frac{n\pi}{L}\right)^2 + \left(\frac{m\pi}{H}\right)^2 + \left(\frac{\ell\pi}{W}\right)^2, \phi = \cos\frac{n\pi x}{L}\cos\frac{m\pi y}{H}, \cos\frac{\ell\pi z}{W}, n = 0,1,2,\ldots, m = 0,1,2,\ldots, \ell = 0,1,2,\ldots$
By superposition

$$u = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\sum_{\ell=0}^{\infty} a_{nm\ell}\cos\frac{n\pi x}{L}\cos\frac{m\pi y}{H}\cos\frac{\ell\pi z}{W}e^{-\lambda_{nm\ell}kt}.$$

The initial condition implies

$$f = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \sum_{\ell=0}^{\infty} a_{nm\ell} \cos \frac{n\pi x}{L} \cos \frac{m\pi y}{H} \cos \frac{\ell \pi z}{W},$$

yielding (by three-dimensional orthogonality)

$$a_{nm\ell} = \frac{\int_0^W \int_0^H \int_0^L f \cos \frac{n\pi x}{L} \cos \frac{m\pi y}{H} \cos \frac{\ell\pi z}{W}\, dx\, dy\, dz}{\int_0^W \int_0^H \int_0^L f \cos^2 \frac{n\pi x}{L} \cos^2 \frac{m\pi y}{H} \cos^2 \frac{\ell\pi z}{W}\, dx\, dy\, dz}.$$

The normalization integrals in the denominator are easy, but one must distinguish between $n = 0$ and $n \neq 0$ (and similarly with m and ℓ). As $t \to \infty$ ($\lambda_{000} = 0$, all others > 0)

$$u(x, y, z, t) \to a_{000} = \frac{1}{LHW} \int_0^W \int_0^H \int_0^L f\, dx\, dy\, dz,$$

the average of the initial temperature distribution since all sides are insulated and thus all the initial energy must be in this mode.

7.3.4 (b) The eigenvalue problems both yield cosine series. Thus, by superposition

$$u(x, y, t) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} A_{nm} \cos \frac{n\pi x}{L} \cos \frac{m\pi y}{H} h_{nm}(t)\ .$$

However, the solution of (7.3.17) satisfying $h(0) = 0$ is

$$h_{nm}(t) = \begin{cases} t & n = 0, m = 0 \\ \sin \omega_{nm} t & \text{otherwise,} \end{cases}$$

where $\omega_{nm}^2 = c^2 \left[(n\pi/L)^2 + (m\pi/H)^2 \right]$. To satisfy the other initial condition,

$$f = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} A_{nm} \cos \frac{n\pi x}{L} \cos \frac{m\pi y}{H} h'_{nm}(0),$$

and thus by orthogonality (2-dimensional)

$$A_{nm} h'_{nm}(0) = \frac{\int_0^H \int_0^L f \cos \frac{n\pi x}{L} \cos \frac{m\pi y}{H}\, dx\, dy}{\int_0^H \int_0^L \cos^2 \frac{n\pi x}{L} \cos^2 \frac{m\pi y}{H}\, dx\, dy}.$$

We note that $h'_{nm}(0) = 1$ for $n = 0, m = 0$, but otherwise $h'_{nm}(0) = \omega_{nm}$.

7.3.6 (b) By separating the z-variable, $u(x, y, z) = \phi(x, y) h(z)$, $\frac{d^2 h}{dz^2} = \lambda h$ and $\nabla^2 \phi = -\lambda \phi$. From Section 7.3, $\phi = 0$ on the boundaries implies that $\lambda_{nm} = (n\pi/L)^2 + (m\pi/W)^2$ with the eigenfunctions being $\phi_{nm} = \sin n\pi x/L \sin m\pi y/W, n = 1, 2, 3, \ldots, m = 1, 2, 3, \ldots$. Since $dh/dz(0) = 0$, it follows that $h(z) = \cosh \sqrt{\lambda_{nm}} z$. Thus by superposition

$$u = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} A_{nm} \sin \frac{n\pi x}{L} \sin \frac{m\pi y}{W} \cosh \sqrt{\lambda_{nm}} z.$$

The nonhomogeneous boundary condition,

$$f = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} A_{nm} \cosh \sqrt{\lambda_{nm}} H \sin \frac{n\pi x}{L} \sin \frac{m\pi y}{W},$$

determines A_{nm}

$$A_{nm} \cosh \sqrt{\lambda_{nm}} H = \frac{\int_0^W \int_0^L f \sin \frac{n\pi x}{L} \sin \frac{m\pi y}{W}\, dx\, dy}{\int_0^W \int_0^L \sin^2 \frac{n\pi x}{L} \sin^2 \frac{m\pi y}{W}\, dx\, dy = \frac{LW}{4}}.$$

7.3.7 (c) A solution only exists if $\int_0^H \int_0^W f(y,z)\, dy\, dz = 0$, the condition for an equilibrium. By separation of variables

$$u(x,y,z) = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} A_{nm} \cos\frac{n\pi y}{W} \cos\frac{m\pi z}{H} \cosh\beta_{nm}x,$$

where $\beta_{nm}^2 = (n\pi/W)^2 + (m\pi/H)^2$. The nonhomogeneous boundary condition,

$$f = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} A_{nm}\beta_{nm} \sinh\beta_{nm}L \cos\frac{n\pi y}{W} \cos\frac{m\pi z}{H},$$

determines A_{nm}

$$A_{nm}\beta_{nm}\sinh\beta_{nm}L = \frac{\int_0^H \int_0^W f \cos\frac{n\pi y}{W} \cos\frac{m\pi z}{H}\, dy\, dz}{\int_0^H \int_0^W \cos^2\frac{n\pi y}{W} \cos^2\frac{m\pi z}{H}\, dy\, dz}.$$

Since $\beta_{00} = 0$, we note that if $\int_0^H \int_0^W f\, dy\, dz \neq 0$, there are no solutions. However, if $\int_0^H \int_0^W f\, dy\, dz = 0$, then A_{00} is arbitrary and the other coefficients uniquely determined.

7.3.7 (d) See the solution to 7.3.7(c). Instead the nonhomogeneous condition

$$g(y,z) = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} A_{nm} \cos\frac{n\pi y}{W} \cos\frac{m\pi z}{H} \cosh\beta_{nm}L,$$

always determines A_{nm} (without difficulty)

$$A_{nm}\cosh\beta_{nm}L = \frac{\int_0^H \int_0^W g \cos\frac{n\pi y}{W} \cos\frac{m\pi z}{H}\, dy\, dz}{\int_0^H \int_0^W \cos^2\frac{n\pi y}{W} \cos^2\frac{m\pi z}{H}\, dy\, dz}.$$

Section 7.4

7.4.1 (a) Equation (7.3.15) with $f'(0) = f'(L) = 0$ yields $\mu = (m\pi/L)^2$ with $f(x) = \cos m\pi x/L, m = 0, 1, 2, \ldots$. Equation (7.3.16) with $g(0) = g(H) = 0$ yields $\lambda - \mu = (n\pi/H)^2$ with $g(y) = \sin n\pi y/H, n = 1, 2, \ldots$. Thus $\lambda_{nm} = (n\pi/H)^2 + (m\pi/L)^2$, where $n = 1, 2, 3, \ldots$ and $m = 0, 1, 2, \ldots$.

Section 7.7

7.7.1 The product solutions are represented in (7.7.45). The initial condition $u(r,\theta,0) = 0$ implies that all terms with $\cos c\sqrt{\lambda_{nm}}t$ vanish. The other initial condition $\partial u/\partial t(r,\theta,0) = \alpha(r)\sin 3\theta$ implies that only $m = 3$ is needed (and in fact only the $\sin 3\theta$ terms). Thus, by superposition,

$$u(r,\theta,t) = \sum_{n=1}^{\infty} A_n J_3\left(\sqrt{\lambda_{3n}}r\right) \sin 3\theta \sin c\sqrt{\lambda_{3n}}t.$$

From the initial condition (cancelling the $\sin 3\theta$ term)

$$\alpha(r) = \sum_{n=1}^{\infty} A_n c\sqrt{\lambda_{3n}} J_3\left(\sqrt{\lambda_{3n}}r\right).$$

Using the orthogonality of $J_3\left(\sqrt{\lambda_{3n}}r\right)$ with weight r:

$$A_n c\sqrt{\lambda_{3n}} = \int_0^a \alpha(r) J_3\left(\sqrt{\lambda_{3n}}r\right) r\, dr \Big/ \int_0^a J_3^2\left(\sqrt{\lambda_{3n}}r\right) r\, dr .$$

7.7.2 (d) The boundary condition in (7.7.35) should be $f'(a) = 0$. From the Rayleigh quotient (7.6.5), $\lambda \geq 0$ with $\lambda = 0$ only if $\phi(r, \theta)$ is constant. Thus from (7.7.34) $\lambda = 0$ only for $m = 0$ in which case $\phi(r, \theta) = 1$. Otherwise $\lambda > 0$ and the transformation $z = \sqrt{\lambda} r$ may be used. Thus, first (7.7.37) is valid, then $f(r)$ satisfies (7.7.38). The boundary condition $f'(a) = 0$, then implies that the other λ_{mn} are determined by $J'_m\left(\sqrt{\lambda_{mn}}a\right) = 0$. The time-dependent part yields (7.7.5). Thus, the initial condition $u(r, \theta, 0) = 0$ implies that $h(0) = 0$ and hence $h(t) = \sin c\sqrt{\lambda_{mn}}t$. However, if $\lambda = 0, h(t) = t$ from (7.7.5). Thus, by superposition,

$$u(r, \theta, t) = \sum_{m=0}^{\infty}\sum_{n=1}^{\infty}\left(A_{mn}\cos m\theta + B_{mn}\sin m\theta\right)H_{mn}(r, t),$$

where

$$H_{mn}(r, t) = \begin{cases} t & m = 0, n = 1 \\ J_m\left(\sqrt{\lambda_{mn}}r\right)\sin c\sqrt{\lambda_{mn}}t & \text{otherwise.} \end{cases}$$

The other initial condition becomes

$$\beta(r, \theta) = \sum_{m=0}^{\infty}\sum_{n=1}^{\infty}\left(A_{mn}\cos m\theta + B_{mn}\sin m\theta\right)\phi_{mn}(r),$$

where

$$\phi_{mn}(r) = \begin{cases} 1 & m = 0, n = 1 \\ c\sqrt{\lambda_{mn}}J_m\left(\sqrt{\lambda_{mn}}r\right) & \text{otherwise.} \end{cases}$$

From orthogonality of these two-dimensional eigenfunctions

$$A_{mn} = \frac{\int_{-\pi}^{\pi}\int_0^a \beta(r, \theta)\phi_{mn}(r)\cos m\theta \; r dr \; d\theta}{\int_{-\pi}^{\pi}\int_0^a \phi_{mn}^2(r)\cos^2 m\theta \; r dr \; d\theta}$$

and B_{mn} is the same as A_{mn} with $\cos m\theta$ above replaced by $\sin m\theta$.

7.7.3 (a) The boundary conditions for (7.7.12) are $g(0) = g(\pi/2) = 0$. This is a standard eigenvalue problem with $L = \pi/2$. Thus, $\mu = (m\pi/L)^2 = 4m^2$ with eigenfunctions $g(\theta) = \sin 2m\theta, m = 1, 2, 3, \ldots$. Thus, the m^2 in (7.7.34) must be replaced by $4m^2$ [i.e., m replaced by $2m$ in (7.7.37) - (7.7.38)]. Thus, the frequencies of vibration are $c\sqrt{\lambda_{mn}}$ where $J_{2m}\left(\sqrt{\lambda_{mn}}a\right) = 0$.

7.7.5 The boundary conditions for (7.7.12) are $g(0) = g(\pi/2) = 0$. This is a standard eigenvalue problem with $L = \pi/2$. Thus, $\mu = (m\pi/L)^2 = 4m^2$ with eigenfunctions $g(\theta) = \sin 2m\theta, m = 1, 2, 3, \ldots$. Thus, the m^2 in (7.7.34) must be replaced by $4m^2$ [i.e., m replaced by $2m$ in (7.7.37)]. The boundary conditions for (7.7.37) are $f(a) = f(b) = 0$. Thus

$$\begin{aligned} c_1 J_{2m}\left(\sqrt{\lambda}a\right) + c_2 Y_{2m}\left(\sqrt{\lambda}a\right) &= 0 \\ c_1 J_{2m}\left(\sqrt{\lambda}b\right) + c_2 Y_{2m}\left(\sqrt{\lambda}b\right) &= 0 \,. \end{aligned}$$

Nontrivial solutions can be obtained from the determinant (or by elimination):

$$J_{2m}\left(\sqrt{\lambda}a\right)Y_{2m}\left(\sqrt{\lambda}b\right) - J_{2m}\left(\sqrt{\lambda}b\right)Y_{2m}\left(\sqrt{\lambda}a\right) = 0,$$

where the frequencies are $c\sqrt{\lambda}$.

7.7.8 For the heat equation, Section 7.7 is valid with (7.7.5) replaced by $dh/dt = -\lambda kh$. The boundary condition introduces more substantial changes. The Rayleigh quotient shows $\lambda \geq 0$ with $\lambda = 0$ only when $\phi(r, \theta)$ is constant, which means $m = 0$. The other eigenfunctions still satisfy (7.7.38) with the boundary condition $f'(a) = 0$ yielding $J'_m\left(\sqrt{\lambda}a\right) = 0$. Thus

$$\phi_{mn}(r, \theta) = \begin{cases} 1 & m = 0, n = 1 \\ J_m\left(\sqrt{\lambda}r\right)\cos m\theta & \text{otherwise.} \end{cases}$$

By superposition

$$u(r,\theta,t)=\sum_{n=1}^{\infty}\sum_{m=0}^{\infty}A_{mn}\phi_{mn}(r,\theta)e^{-\lambda_{mn}kt}+\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}B_{mn}J_m\left(\sqrt{\lambda_{mn}}r\right)\sin m\theta e^{-\lambda_{mn}kt}.$$

The initial conditions yield equations for A_{mn} and B_{mn}. Using the two-dimensional orthogonality of the eigenfunctions

$$A_{mn}=\frac{\int_{-\pi}^{\pi}\int_0^a f(r,\theta)\phi_{mn}(r,\theta)r\,dr\;d\theta}{\int_{-\pi}^{\pi}\int_0^a \phi_{mn}^2(r,\theta)r\,dr\;d\theta}$$

and a similar expression for B_{mn}. As $t\to\infty, e^{-\lambda_{mn}kt}\to 0$ except for $m=0, n=1$ since then $\lambda_{mn}=0$. Thus $u(r,\theta,t)\to A_{01}$, where $A_{01}=\left(\int_{-\pi}^{\pi}\int_0^a f(r,\theta)r\,dr\;d\theta\right)\Big/\left(\pi a^2\right)$, the average of the initial temperature distribution. Using physical reasoning, the equilibrium should be a constant. By conservation of thermal energy that constant should be the above value, since the boundaries are insulated.

7.7.9 (b) For the heat equation, Section 7.7 is valid with (7.7.5) replaced by $dh/dt=-\lambda kh$. The boundary conditions for (7.7.12) are $g'(0)=g'(\pi)=0$. This is a cosine series in θ with $L=\pi$, i.e. $\mu=(m\pi/L)^2=m^2$ (as before) but with $g=\cos m\theta$ only. Thus (7.7.18-20) is valid with the condition at $r=a$ being $f'(a)=0$. The Rayleigh quotient shows $\lambda\geq 0$ with $\lambda=0$ only when $m=0$ and $f(r)=1$. The other eigenfunctions satisfy (7.7.37) - (7.7.38), where now the boundary condition at $r=a$ yields $J_m'(\sqrt{\lambda}a)=0$. Thus the eigenfunctions are

$$\phi_{mn}(r)=\begin{cases}1 & m=0, n=1\\ J_m\left(\sqrt{\lambda_{mn}}r\right) & \text{otherwise.}\end{cases}$$

By superposition

$$u(r,\theta,t)=\sum_{n=1}^{\infty}\sum_{m=0}^{\infty}A_{mn}\phi_{mn}(r)\cos m\theta e^{-\lambda_{mn}kt}.$$

The initial conditions determine the coefficients A_{mn} (by orthogonality):

$$A_{mn}=\frac{\int_0^{\pi}\int_0^a f(r,\theta)\phi_{mn}(r)\cos m\theta r\,dr\,d\theta}{\int_0^{\pi}\int_0^a \phi_{mn}^2(r)\cos^2 m\theta r\,dr\,d\theta}.$$

As $t\to\infty, e^{-\lambda_{mn}kt}\to 0$ except for $m=0, n=1$ since then $\lambda_{mn}=0$. Thus $u(r,\theta,t)\to A_{01}$, where

$$A_{01}=\frac{\int_0^{\pi}\int_0^a f(r,\theta)r\,dr\,d\theta}{\frac{1}{2}\pi a^2},$$

the average of the initial temperature distribution. The temperature approaches an equilibrium. This can be obtained easily by conservation of thermal energy since the boundary is insulated.

7.7.10 Subsection 7.7.9 is valid with $h(t)$ instead solving $dh/dt=-\lambda kh$. Thus by superposition

$$u(r,t)=\sum_{n=1}^{\infty}a_nJ_0\left(\sqrt{\lambda_n}r\right)e^{-\lambda_nkt},$$

where (7.7.63) is satisfied. The initial condition yields (7.7.66) with $\alpha(r)$ replaced by $f(r)$. As $t\to\infty, u\to 0$ since $\lambda_n>0$. This can be shown to be the only equilibrium solution since $u(a,t)=0$, see exercise 1.5.10.

7.7.12 (a) $x^2y''-6y\approx 0$. Substituting $y=x^p$ yields $p(p-1)-6=0$ or $(p-3)(p+2)=0$ or $p=-2,3$. Thus $y=c_1(x^{-2}+\ldots)+c_2(x^3+\ldots)$.

7.7.12 (c) $x^2y''+xy'+4y\approx 0$. Substituting $y=x^p$ yields $p(p-1)+p+4=0$ or $p=\pm 2i$. Since $x^{\pm 2i}=e^{\pm 2i\ln x}$, we obtain real solutions $y=c_1\left[\cos(2\ln x)+\ldots\right]+c_2\left[\sin(2\ln x)+\ldots\right]$.

7.7.12 (e) $x^2y''-4xy'+6y\approx 0$. Substituting $y=x^p$ yields $p(p-1)-4p+6=0$ or $p^2-5p+6=0$ or $(p-3)(p-2)=0$ or $p=2,3$. Thus $y=c_1\left(x^2+\ldots\right)+c_2\left(x^3+\ldots\right)$.

Section 7.8

7.8.1 (b) Since $\lambda > 0$, (7.7.37) is valid. The boundary conditions yield

$$c_1 J_m\left(\sqrt{\lambda}\right) + c_2 Y_m\left(\sqrt{\lambda}\right) = 0$$
$$c_1 J_m\left(2\sqrt{\lambda}\right) + c_2 Y_m\left(2\sqrt{\lambda}\right) = 0\,.$$

By the determinant condition for a nontrivial solution (or by elimination), the eigenvalues are determined by

$$J_m\left(\sqrt{\lambda}\right) Y_m\left(2\sqrt{\lambda}\right) - J_m\left(2\sqrt{\lambda}\right) Y_m\left(\sqrt{\lambda}\right) = 0\,.$$

7.8.1 (d) Using the Rayleigh quotient minimization principle (5.6.5) with $p = r, q = -m^2/r$, and $\sigma = r$ (and $x = r$)

$$\lambda_1 = \min \frac{\int_1^2 [r\left(\frac{du}{dr}\right)^2 + m^2\frac{u^2}{r}]dr}{\int_1^2 u^2 r\ dr}.$$

The smallest occurs with $m = 0$. Upper and lower bounds can be obtained using the technique discussed in Section 5.7. Since $1 \le r \le 2$,

$$\frac{1}{2}\min\frac{\int_1^2\left(\frac{du}{dr}\right)^2\ dr}{\int_1^2 u^2\ dr} \le \lambda_1 \le \frac{2}{1}\min\frac{\int_1^2\left(\frac{du}{dr}\right)^2\ dr}{\int_1^2 u^2\ dr}.$$

But $\min\frac{\int_1^2\left(\frac{du}{dr}\right)^2\ dr}{\int_1^2 u^2\ dr} = \left(\frac{\pi}{L}\right)^2$, where $L = 2 - 1 = 1$. Thus

$$\frac{1}{2}\pi^2 \le \lambda_1 \le 2\pi^2.$$

7.8.2 (d) Section 7.7 is valid with $dh/dt = -\lambda k h$. The boundary conditions for (7.7.12) are $g(0) = g(\pi/2) = 0$. This is the standard sine series with $L = \pi/2$ so that $\lambda = (m\pi/L)^2 = 4m^2$ with $g(\theta) = \sin 2m\theta$. Thus (7.7.34) - (7.7.39) are valid with m replaced by $2m$. Thus by superposition

$$u(r,\theta,t) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} c_{mn} J_{2m}\left(\sqrt{\lambda_{mn}}r\right)\sin 2m\theta e^{-\lambda_{mn}kt},$$

where (7.7.39) is valid with m replaced by $2m$. The initial condition determines c_{mn}

$$c_{mn} = \frac{\int_0^{\pi/2}\int_0^a G(r,\theta) J_{2m}\left(\sqrt{\lambda_{mn}}r\right)\sin 2m\theta\ r dr\ d\theta}{\int_0^{\pi/2}\int_0^a J_{2m}^2\left(\sqrt{\lambda_{mn}}r\right)\sin^2 2m\theta\ r dr\ d\theta}.$$

7.8.8 If $m = \frac{1}{2}$, then $f = z^{-1/2}y$ where $y'' + y = 0$. Thus

$$J_{1/2}(z) = z^{-1/2}\left(c_1\cos z + c_2 \sin z\right) \quad \text{and} \quad Y_{1/2}(z) = z^{-1/2}\left(c_3\cos z + c_4\sin z\right), \quad \text{where}$$

we cannot determine c_i from differential equation. Equation (7.7.33) can be used as a definition. As $z \to 0$ we have obtained $J_{1/2} \sim c_1 z^{-1/2}$ and $Y_{1/2} \sim c_3 z^{-1/2}$ Thus from (7.7.33), $c_1 = 0$ so that $J_{1/2} \sim c_2 z^{1/2}$ resulting in $c_2 = 1/\left(2^{1/2}\frac{1}{2}!\right)$ and $c_3 = -2^{1/2}\left(-\frac{1}{2}\right)!/\pi$ but c_4 cannot be determined from this type of consideration. Gamma function results (see exercise 10.3.14) show that $(-1/2)! = \Gamma(1/2) = \sqrt{\pi}$ and $(1/2)! = \Gamma(3/2) = 1/2\Gamma(1/2) = 1/2\sqrt{\pi}$. Thus, $c_2 = \sqrt{2/\pi}$ and $c_3 = -\sqrt{2/\pi}$. Therefore $J_{1/2}(z) = \sqrt{2/\pi z}\sin z$ and $Y_{1/2}(z) = -\sqrt{2/\pi z}\cos z + c_4\sqrt{\pi/2}J_{1/2}(z)$. From (7.8.3), $c_4 = 0$.

Section 7.9

7.9.1 (b) This is similar to the problem in subsection 7.9.3. To satisfy $u(r,\theta,H) = 0$, instead we use $h(z) = \sinh\sqrt{\lambda}(z-H)$. The condition at $z = 0$ implies $m = 7$ only (and $\sin 7\theta$ only also). Thus by superposition

$$u(r,\theta,z) = \sum_{n=1}^{\infty} A_{mn} \sinh\sqrt{\lambda_{7n}}(H-z)\sin 7\theta J_7\left(\sqrt{\lambda_{7n}}r\right),$$

where (7.9.18) is satisfied with $m = 7$. It is not necessary to use notation λ_{7n}; instead λ_n may be used. The boundary condition at $z = 0$ (after cancelling $\sin 7\theta$) determines A_n

$$A_n \sinh\sqrt{\lambda_{7n}}H = \frac{\int_0^a \alpha(r) J_7\left(\sqrt{\lambda_{7n}}r\right)\, r\, dr}{\int_0^a J_7^2\left(\sqrt{\lambda_{7n}}r\right)\, r\, dr}.$$

7.9.2 (b) This has eigenvalue problems in θ and z and so is similar to subsection 7.9.4. The boundary conditions in θ are $g(0) = g(\pi) = 0$ so that $\mu = (m\pi/L)^2 = m^2$ since $L = \pi$ with $g(\theta) = \sin m\theta$ only. In z the boundary conditions are $h(0) = 0$ and $h'(H) = 0$. Thus $h(z) = \sin\sqrt{-\lambda}z$ with $\lambda = -(n-1/2)^2(\pi/H)^2, n = 1, 2, \ldots$. Thus (7.9.31) is valid with n replaced by $n - 1/2$. In this way (7.9.37) is valid ($n \to n - 1/2$ and $\sin m\theta$ only). By superposition

$$u(r,\theta,z) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} A_{mn} I_m\left[\left(n-\frac{1}{2}\right)\frac{\pi}{H}r\right]\sin\left(n-\frac{1}{2}\right)\frac{\pi z}{H}\sin m\theta.$$

The boundary condition at $r = a$ determines A_{mn}

$$A_{mn} I_m\left[\left(n-\frac{1}{2}\right)\frac{\pi}{H}a\right] = \frac{\int_0^\pi\int_0^H \beta(\theta,z)\sin m\theta \sin\left(n-\frac{1}{2}\right)\frac{\pi z}{H}\, dz\, d\theta}{\int_0^\pi\int_0^H \sin^2\left(n-\frac{1}{2}\right)\frac{\pi z}{H}\sin^2 m\theta\, dz\, d\theta}.$$

The normalization integral equals $\pi H/4$.

7.9.3 (b) By separation of variables, time only first, $\nabla_3^2\phi + \lambda\phi = 0$ and $dh/dt = -\lambda k h$. Here $\nabla_3^2\phi$ is the same operator as in Section 7.7 with an extra term ϕ_{zz}. If we next separate z, we know $Q_{zz} = -\nu Q$ with $\nu = (\ell\pi/H)^2$, because of the boundary conditions and $Q(z) = \cos\ell\pi z/H, \ell = 0, 1, 2, \ldots$. Thus $\nabla^2\phi + \lambda'\phi = 0$ where $\lambda' = \lambda - (\ell\pi/H)^2$ and here the Laplacian is the two-dimensional one previously analyzed. We follow Section 7.7. The boundary condition for (7.7.12) is $g'(0) = g'(\pi/2)$. This is a cosine series in θ, where $\mu = [m\pi/(\pi/2)]^2 = 4m^2$ and $g(\theta) = \cos 2m\theta, m = 0, 1, 2, \ldots$ [and thus m is replaced by $2m$ in (7.7.34) and (7.7.38)]. Note that $\lambda' = 0$ only for $m = 0$ with $f(r) = 1$. Otherwise (7.7.38) is valid. The boundary condition at $r = a$ yields $J'_{2m}\left(\sqrt{\lambda'_{mn}}a\right) = 0$. In this manner, the three-dimensional eigenfunctions are

$$\phi_{\ell mn}(r,\theta,z) = \begin{cases} 1 & m = 0, \ell = 0, n = 1 \\ \cos\frac{\ell\pi z}{H}\cos 2m\theta J_{2m}\left(\sqrt{\lambda'_{mn}}r\right) & \text{otherwise.} \end{cases}$$

By superposition

$$u(r,\theta,z,t) = \sum_{n=1}^{\infty}\sum_{m=0}^{\infty}\sum_{\ell=0}^{\infty} A_{\ell mn}\phi_{\ell mn}(r,\theta,z)e^{-\lambda_{\ell mn}kt},$$

where $\lambda_{\ell mn} = (\ell\pi/H)^2 + \lambda'_{mn}$. The initial condition determines $A_{\ell mn}$

$$A_{\ell mn} = \int\int\int f(r,\theta,z)\phi_{\ell mn}(r,\theta,z)\, rdr\, d\theta\, dz \Big/ \int\int\int \phi_{\ell mn}^2\, rdr\, d\theta\, dz\ .$$

As $t \to \infty, e^{-\lambda_{\ell mn}kt} \to \infty$ except for $\ell = 0, m = 0, n = 1$ since $\lambda_{001} = 0:$

$$u(r,\theta,z,t) \to A_{001} = \frac{1}{H\pi a^2/4}\int\int\int f(r,\theta,z)r\, dr\, d\theta\, dz,$$

the average of the initial temperature distribution. We expect this because with insulated boundaries the equilibrium temperature ought to be a constant (with the same thermal energy as the initial condition).

7.9.4 (a) By separation of variables in time $\nabla^2\phi + \lambda\phi = 0$ and $dh/dt = -\lambda kh$. Here $\nabla^2\phi$ is the same operator as in Section 7.7 with an extra term ϕ_{zz}. If we next separate z, we know $Q_{zz} = -\nu Q$ where from the boundary condition $\nu = (\ell\pi/H)^2$ with $\phi = \sin \ell\pi z/H, \ell = 1, 2, 3, \ldots \nabla^2\phi + \lambda'\phi = 0$ where $\lambda' = \lambda - \nu$ and here the Laplacian is the two-dimensional one previously analyzed. Since there is no θ-dependence, we may follow Section 7.7.9. The orthogonal eigenfunctions in the radial direction are $J_0\left(\sqrt{\lambda'_n}r\right)$ where $J_0\left(\sqrt{\lambda'_n}a\right) = 0$ determines λ'_n. By superposition

$$u(r,z,t) = \sum_{\ell=1}^{\infty}\sum_{n=1}^{\infty} A_{n\ell} J_0\left(\sqrt{\lambda'_n}r\right) \sin\frac{\ell\pi z}{H} e^{-\lambda kt},$$

where $\lambda = \lambda' + (\ell\pi/H)^2$. The initial condition determines $A_{n\ell}$

$$A_{n\ell} = \frac{\int\int f(r,z) J_0\left(\sqrt{\lambda'_n}r\right) \sin\frac{\ell\pi z}{H}\, r dr\, dz}{\int\int J_0^2\left(\sqrt{\lambda'_n}r\right) \sin^2\frac{\ell\pi z}{H}\, r dr\, dz}.$$

Chapter 8. Nonhomogeneous Problems

Section 8.2

8.2.1 (a) An equilibrium satisfies $d^2u_E/dx^2 = 0$ with $u_E(0) = A$ and $du_E/dx(L) = B$. Thus $u_E(x) = A+Bx$. We now introduce the displacement from equilibrium, defined by (8.2.9). $v(x,t)$ satisfies (8.2.10) subject to the boundary conditions, (8.2.11)and $\partial v/\partial x(L,t) = 0$, and the initial conditions (8.2.13). Since the boundary conditions and pde are homogeneous, it may be solved by separation of variables, $v(x,t) = \phi(x)h(t)$. Thus $dh/dt = -\lambda kh$ and $d^2\phi/dx^2 = -\lambda\phi$ subject to $\phi(0) = 0$ and $d\phi/dx(L) = 0$. The eigenfunctions are $\sin\sqrt{\lambda}x$, where $\sqrt{\lambda}L = (n-\frac{1}{2})\pi, n = 1,2,3,...$ Thus by superposition

$$v(x,t) = \sum_{n=1}^{\infty} a_n \sin(n-\frac{1}{2})\pi x/Le^{-\lambda_n kt}.$$

The initial conditions determine the coefficients a_n

$$a_n = \frac{2}{L}\int_0^L g(x)\sin(n-\frac{1}{2})\pi x/L\,dx,$$

where $g(x) = f(x) - u_E(x)$.

8.2.1 (d) An equilibrium satisfies $kd^2u_E/dx^2 = -Q = -k$ or $d^2u_E/dx^2 = -1$ subject to $u_E(0) = A$ and $u_E(L) = B$. Thus, $u_E(x) = -x^2/2 + A + \gamma x$, where $B = -L^2/2 + A + \gamma L$. The displacement from equilibrium, defined by (8.2.9), satisfies (8.2.10-13). Consequently the solution is given by (8.2.17).

8.2.2 (a) A reference temperature distribution $r(x,t)$ must satisfy the boundary conditions, $r_x(0,t) = A(t)$ and $r_x(L,t) = B(t)$. An example is the quadratic $r(x,t) = A(t)x+[B(t) - A(t)]x^2/2L$. By introducing the difference by (8.2.25), we have $u(x,t) = v(x,t) + r(x,t)$. Thus

$$\frac{\partial v}{\partial t} - k\frac{\partial^2 v}{\partial x^2} = Q - \frac{\partial r}{\partial t} + k\frac{\partial^2 r}{\partial x^2},$$

subject to the homogeneous boundary conditions $v_x(0,t) = v_x(L,t) = 0$ and the initial condition $v(x,0) = f(x) - r(x,0)$.

8.2.2 (c) A reference temperature distribution $r(x,t)$ must satisfy the boundary conditions, $r_x(0,t) = A(t)$ and $r(L,t) = B(t)$. The simplest example is the linear function $r(x,t) = A(t)x + B(t) - LA(t)$. By introducing the difference by (8.2.25), we have $u(x,t) = v(x,t) + r(x,t)$. Thus

$$\frac{\partial v}{\partial t} - k\frac{\partial^2 v}{\partial x^2} = Q - \frac{\partial r}{\partial t}$$

since $\partial^2 r/\partial x^2 = 0$, subject to the homogeneous boundary conditions, $v_x(0,t) = 0$ and $v(L,t) = 0$, and the initial condition $v(x,0) = f(x) - r(x,0)$.

8.2.6 (a) An equilibrium satisfies $d^2u_E/dx^2 = 0$ with $u_E(0) = A$ and $u_E(L) = B$. Thus $u_E(x) = A + (B - A)x/L$. By introducing the displacement, defined by (8.2.9), we have

$$\frac{\partial^2 v}{\partial t^2} = c^2\frac{\partial^2 v}{\partial x^2}$$

subject to $v(0,t) = v(L,t) = 0$ and the initial conditions $v(x,0) = f(x) - u_E(x)$ and $v_t(x,0) = g(x)$. From Section 4.4

$$v(x,t) = \sum_{n=1}^{\infty} (A_n \cos\, n\pi ct/L + B_n \sin n\pi ct/L)\sin n\pi x/L,$$

where from initial conditions
$A_n = \frac{2}{L}\int_0^L [f(x) - u_E(x)]\sin\frac{n\pi x}{L}dx$ and $B_n\frac{n\pi c}{L} = \frac{2}{L}\int_0^L g(x)\sin\frac{n\pi x}{L}dx$. For large $t, v(x,t)$ oscillates so that $u(x,t)\ \ u_E(x)$.

8.2.6 (d) An equilibrium satisfies $c^2d^2u_E/dx^2 = -\sin\pi x/L$ with $u_E(0) = u_E(L) = 0$. Thus $u_E(x) = (L/\pi c)^2\sin\,\pi x/L$. By introducing the displacement, defined by (8.2.9), we have the same result as exercise 8.2.6(a).

Section 8.3

8.3.1 (c) A reference temperature distribution $r(x,t)$ must satisfy $r(0,t) = A(t)$ and $r_x(L,t) = 0$. The simplest example is $r(x,t) = A(t)$. We introduce the difference $[v(x,t) = u(x,t) - r(x,t)]$, so that

$$\frac{\partial v}{\partial t} - k\frac{\partial^2 v}{\partial x^2} = Q(x,t) - \frac{dA}{dt},$$

since $\partial r/\partial t = dA/dt$ and $\partial^2 r/\partial x^2 = 0$. The boundary conditions are homogeneous $v(0,t) = 0$ and $v_x(L,t) = 0$. The initial condition is $v(x,0) = f(x) - A(0)$. The related homogeneous eigenfunctions are $\sin(n - \frac{1}{2})\pi x/L, n = 1,2,3...$ since $d^2\phi/dx^2 + \lambda\phi = 0$ subject to $\phi(0) = 0$ and $d\phi/dx(L) = 0$. According to the method of eigenfunction expansion

$$v(x,t) = \sum_{n=1}^{\infty} B_n(t)\sin(n - \frac{1}{2})\pi x/L.$$

This can be differentiated term-by-term since both $v(x,t)$ and the eigenfunctions satisfy the same set of homogeneous boundary conditions. Thus

$$\sum_{n=1}^{\infty}\left[\frac{dB_n}{dt} + k\frac{(n-\frac{1}{2})^2\pi^2}{L^2}B_n\right]\sin(n-\frac{1}{2})\frac{\pi x}{L} = Q(x,t) - \frac{dA}{dt}.$$

By orthogonality

$$\frac{dB_n}{dt} + k\lambda_n B_n = \bar{q}_n(t) = \frac{2}{L}\int_0^L \left[Q(x,t) - \frac{dA}{dt}\right]\sin(n-\frac{1}{2})\frac{\pi x}{L}dx.$$

The solution of this is given by (8.3.10).

8.3.1 (f) The related homogeneous eigenfunctions yield a Fourier cosine series. Thus, by the method of eigenfunction expansion

$$u(x,t) = \sum_{n=0}^{\infty} A_n(t)\cos\frac{n\pi x}{L}.$$

This can be differentiated term-by-term since both $u(x,t)$ and $\cos n\pi x/L$ satisfy the same set of homogeneous boundary conditions. Thus $A_n(t)$ satisfies (8.3.9) whose solution is given by (8.3.10).

8.3.3 The related homogeneous eigenfunctions satisfy $L(\phi) = \lambda\sigma\phi = 0$ (where $\sigma = c\rho$) and $\phi(0) = \phi(L) = 0$. According to the method of eigenfunction expansions

$$u(x,t) = \sum_{n=1}^{\infty} a_n(t)\phi_n(x).$$

This can be differentiated term-by-term since both $u(x,t)$ and $\phi_n(x)$ satisfy the same set of homogeneous boundary conditions. Thus

$$c\rho\sum_{n=1}^{\infty}\frac{da_n}{dt}\phi_n(x) = \sum_{n=1}^{\infty} a_n(t)L[\phi_n(x)] + f(x,t),$$

but $L(\phi_n) = -\lambda_n c\rho\phi_n$. Thus

$$c\rho\sum_{n=1}^{\infty}\phi_n(x)\left[\frac{da_n}{dt} + \lambda_n a_n\right] = f(x,t).$$

Since the eigenfunctions are orthogonal with weight $\sigma = c\rho$

$$\frac{da_n}{dt} + \lambda_n a_n = f_n(t) = \int_0^L f(x,t)\phi_n(x)dx \Big/ \int_0^L \phi_n^2 c\,\rho\,dx.$$

Now the integrating factor may be used to derive an expression similar to (8.3.10). The initial conditions yield

$$a_n(0) = \int_0^L g(x)\phi_n(x)c\rho dx \bigg/ \int_0^L \phi_n^2 c\rho dx.$$

8.3.4 (a) An equilibrium solution satisfies
$\frac{d}{dx}\left[K_0(x)\frac{du_E}{dx}\right] = 0$ subject to $u_E(0) = A$ and $u_E(L) = B$.
By integration $K_0(x)du_E/dx = c_1$. By integrating again $u_E(x) = c_2 + c_1\int_0^x \frac{d\bar{x}}{K_0(\bar{x})}$.
The boundary conditions yield $c_2 = A$ and $B - A = c_1\int_0^L d\bar{x}/K_0(\bar{x})$.

8.3.5 Since there is no θ-dependence, the corresponding homogeneous eigenfunctions are $J_0(\sqrt{\lambda_n}r)$ such that $J_0(\sqrt{\lambda_n}a) = 0$. We note that $\nabla^2\phi = -\lambda\phi$. According to the method of eigenfunction expansion

$$u(r,t) = \sum_{n=1}^{\infty} A_n(t)J_0(\sqrt{\lambda_n}r).$$

This can be differentiated term-by-term since both $u(x,t)$ and $J_0(\sqrt{\lambda_n}r)$ satisfy the same set of homogeneous boundary conditions. Thus

$$\sum_{n=1}^{\infty}\left[\frac{dA_n}{dt} + k\lambda_n A_n\right] J_0(\sqrt{\lambda_n}r) = f(r,t).$$

Since these Bessel functions are orthogonal with weight r

$$\frac{dA_n}{dt} + k\lambda_n A_n = f_n(t) = \frac{\int_0^a f(r,t)J_0(\sqrt{\lambda_n}r)rdr}{\int_0^a J_0^2(\sqrt{\lambda_n}r)rdr}.$$

This is in the form of (8.3.9) whose solution is given by (8.3.10) involving the initial conditons, $A_n(0) = 0$.

8.3.7 A reference temperature distribution $r(x,t)$ must satisfy $r(0,t) = 0$ and $r(L,t) = t$. The simplest example is $r(x,t) = x\,t/L$. The difference $[v(x,t) = u(x,t) - r(x,t)]$ satisfies

$$\frac{\partial v}{\partial t} - \frac{\partial^2 v}{\partial x^2} = -\frac{x}{L}\,.$$

subject to homogeneous boundary conditions $v(0,t) = v(L,t) = 0$ and the initial condition $v(x,0) = 0$. This problem has an equilibrium solution $v_E(x) = x^3/6L + c_1 x$ where $0 = L^2/6 + c_1 L$. This is equivalent to starting with the reference temperature distribution

$$r_1(x,t) = xt/L + x^3/6L - Lx/6\,.$$

In this case $[v_1(x,t) = u(x,t) - r_1(x,t)]$, the new difference satisfies $\frac{\partial v_1}{\partial t} - \frac{\partial^2 v_1}{\partial x^2} = 0$ with $v_1(0,t) = v_1(L,t) = 0$ and the initial condition $v_1(x,0) = Lx/6 - x^3/6L$. The solution is

$$v_1(x,t) = \sum_{n=1}^{\infty} a_n \sin\frac{n\pi x}{L}e^{-(n\pi/L)^2 t}\,,$$

where $a_n = \frac{2}{L}\int_0^L (Lx/6 - x^3/6L)\sin\frac{n\pi x}{L}dx$.

Section 8.4

8.4.1. (b) The related homogeneous eigenfunctions are cosines. Thus

$$u(x,t) = \sum_{n=0}^{\infty} A_n(t) \cos \ n\pi x/L.$$

From the pde

$$A_n'(t) = q_n + k\, I_n \int_0^L u_{xx} \cos n\pi x/L\, dx$$

where $I_0 = 1/L$, $(n \neq 0) I_n \ = \ 2/L$, and

$$q_n = I_n \int_0^L Q(x,t) \cos \ n\pi x/L \ dx \ \ .$$

Using Green's formula (8.4.11) with $v = \cos \ n\pi x/L$,

$$A_n' + k(n\pi/L)^2 A_n = q_n + kI_n[(-1)^n B - A].$$

Using the integrating factor

$$A_n(t) = A_n(0) e^{-\lambda_n kt} \ + e^{-\lambda_n kt} \int_0^t \ \{q_n \ + \ kI_n[(-1)^n B - A]\}\, e^{\lambda_n k\tau} d\tau$$

where $A_n(0) \ = \ I_n \ \int_0^L f(x) \cos \ n\pi x/L \ dx$.

Section 8.5

8.5.2 (b) The related eigenfunctions are sines. Thus

$$u(x,t) \ = \ \sum_{n=1}^{\infty} \ A_n(t) \sin \ n\pi x/L \ .$$

By term-by-term differentiation

$$A_n'' \ + \ c^2 \lambda_n A_n \ = \ q_n \cos \omega t,$$

where $\lambda_n \ = \ (n\pi/L)^2$ and $q_n \ = \ \frac{2}{L} \int_0^L g(x) \sin \ n\pi x/L \ dx$. If $\omega^2 \ =\neq \ c^2\lambda_n \ = \ (n\pi c/L)^2$, then (8.5.29) may be used. From the initial conditions, $c_2 \ = \ 0$ and

$$c_1 \ + \ [q_n/(c^2\lambda_n - \omega^2)] \ = \ \frac{2}{L} \int_0^L \ f(x) \ \sin \ n\pi x/L \ dx \ .$$

If $\omega^2 \ = \ (n\pi c/L)^2$, resonance, then only for that one value of n, (8.5.31) is valid. From the initial conditions, $c_2 \ = \ 0$ and

$$c_1 \ = \ \frac{2}{L} \int_0^L \ f(x) \sin \ n\pi x/L \ dx \ .$$

8.5.5 (c) The eigenfunctions for $\nabla^2 \phi \ = \ -\lambda\phi$ on a semi-circle are $J_m(\sqrt{\lambda_{mn}} r) \sin \ m\theta$, where $J_m(\sqrt{\lambda_{mn}} a) \ = \ 0$, by modifying Section 7.7 with the boundary conditions $g(0) \ = \ g(\pi) \ = \ 0$. Thus, the method of eigenfunction expansions yields

$$u(r,\theta,t) \ = \ \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} A_{mn}(t)\, J_m(\sqrt{\lambda_{mn}} r) \sin \ m\theta \, .$$

By substituting into the pde

$$A_{mn}'' \ + \ c^2 \lambda_{mn} A_{mn} \ = \ Q_{mn} \, ,$$

where

$$Q_{mn} = \int\int Q\, J_m(\sqrt{\lambda_{mn}}r) \sin\, m\theta\, r d\, r d\theta \Big/ \int\int J_m^2(\sqrt{\lambda_{mn}}r) \sin^2 m\theta\, r d r d\theta\ .$$

Using (8.5.19)

$$A_{mn} = c_{mn} \cos\, c\sqrt{\lambda_{mn}}t + \int_0^t Q_{mn}(\tau)\frac{\sin\, c\sqrt{\lambda_{mn}}(t-\tau)}{c\sqrt{\lambda_{mn}}}\, d\tau\ ,$$

where the sin $c\sqrt{\lambda_{mn}}\, t$ vanishes because of the initial condition and where

$$c_{mn} = \int\int f\, J_m(\sqrt{\lambda_{mn}}\, r) \sin\, m\theta\, r d\, r d\theta \Big/ \int\int J_m^2(\sqrt{\lambda_{mn}}r) \sin^2 m\theta\, r d r d\theta\ .$$

8.5.6 (a) Since ϕ are orthogonal with weight $1, dA = r d r d\theta, a'(0) = 0$ and $a(0) = \int\int H\phi r dr d\theta / \int\int \phi^2 r d r d\theta$. From the pde,

$$\frac{1}{c^2}a'' + \lambda a = \int\int g\,\phi r d r d\theta \Big/ \int\int \phi^2\, r d r d\theta\ .$$

Section 8.6

8.6.1 (b) Using two-dimensional eigenfunctions

$$u(x,y) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} A_{nm} \sin\, n\pi x/L\, \sin\, m\pi y/H.$$

Using (8.6.17), we need the normal derivative

$$\nabla\phi\cdot\hat{n} = \partial\phi/\partial x\ |_{x=L} = \frac{n\pi}{L}\cos\frac{n\pi x}{L}\sin\frac{m\pi y}{H}\ |_{x=L} = \frac{n\pi}{L}(-1)^n \sin\, m\pi y/H.$$

Thus

$$\oint u\nabla\phi\cdot\hat{n}\, d\, s = \frac{n\pi}{L}(-1)^n\int_0^H \sin\, m\pi y/H dy = \frac{nH}{mL}(-1)^n\,[1-(-1)^m],$$

so that

$$A_{nm} = \left\{-Q_{nm} - \frac{4n}{mL^2}(-1)^n[1-(-1)^m]\right\} / \lambda_{nm}$$

where $\lambda_{nm} = (\frac{n\pi}{L})^2 + (\frac{m\pi}{H})^2$, and where $Q_{nm} = \int\int Q\, \sin\, n\pi x/L \sin\, m\pi y/H$.

8.6.1 (d) Using two-dimensional eigenfunctions

$$u(x,y) = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} A_{nm} \cos\, n\pi x/L\, \cos\, m\pi y/H\ .$$

By term-by-term differentiation (valid since u and the eigenfunctions satisfy the same set of homogeneous boundary conditions):

$$-A_{nm}[(n\pi/L)^2 + (m\pi/H)^2] = \frac{\int\int Q\, \cos\, n\pi x/L\, \cos\, m\pi y/H\, dx\, dy}{\int\int \cos^2 n\pi x/L\, \cos^2\, m\pi y/H\, dx\, dy}\ .$$

Thus if $\int\int Q\, dx\, dy \neq 0$, there is no solution. However, if $\int\int Q\, dx\, dy = 0$, then A_{00} is arbitrary and the others are determined above.

8.6.3 (a) The two-dimensional eigenfunctions for $\nabla^2\phi = -\lambda\phi$ are the set $J_m(\sqrt{\lambda_{mn}}r)\left\{\begin{array}{c}\cos m\theta\\ \sin m\theta\end{array}\right\}$, where $J_m(\sqrt{\lambda_{mn}}a) = 0$. By the method of eigenfunction expansion

$$u(r,\theta) = \sum_{m=0}^{\infty}\sum_{n=1}^{\infty} A_{mn} \cos\, m\theta\, J_m(\sqrt{\lambda}r) + \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} B_{mn} \sin\, m\theta\, J_m(\sqrt{\lambda}r).$$

From (8.6.17)

$$\begin{pmatrix} A_{mn} \\ B_{mn} \end{pmatrix} = \frac{-\frac{1}{\lambda}\int\int Q \begin{pmatrix} \cos m\theta \\ \sin m\theta \end{pmatrix} J_m(\sqrt{\lambda}r) r\, dr\, d\theta}{\int\int \begin{pmatrix} \cos^2 m\theta \\ \sin^2 m\theta \end{pmatrix} J_m^2(\sqrt{\lambda}r) r\, dr\, d\theta} .$$

8.6.6 Using one-dimensional eigenfunctions

$$u(x,y) = \sum_{n=1}^{\infty} a_n(y) \sin n\, x .$$

Substituting into the pde yields (8.6.5)

$$\sum_{n=1}^{\infty} \left[\frac{d^2 a_n}{dy^2} - n^2 a_n \right] \sin nx = e^{2y} \sin x .$$

Thus for $n \neq 1, a_n(y) = \alpha_n \sinh ny + \beta_n \cosh n\, y$. However, for $n = 1$, $\frac{d^2 a_1}{dy^2} - a_1 = e^{2y}$ and thus $a_1(y) = \frac{1}{3} e^{2y} + \alpha_1 \sinh y + \beta_1 \cosh y$, using the method of undetermined coefficients for a particular solution. The other boundary conditions are $a_n(0) = 0$ and $a_n(L) = \frac{2}{\pi}\int_0^\pi f(x) \sin nx dx$. Thus $\frac{1}{3} + \beta_1 = 0$ and $(n \neq 1)\beta_n = 0$. Also $\frac{1}{3}e^{2L} + \alpha_1 \sinh L - \frac{1}{3}\cosh L = a_1(L)$ and $(n \neq 1)\alpha_n \sinh n\, L = a_n(L)$, determining all α_n.

Chapter 9. Time-Independent Green's Functions

Section 9.2

9.2.1 (d) Using the method of eigenfunction expansion with a cosine series

$$u(x,t) = \sum_{n=0}^{\infty} a_n(t) \cos n\pi x/L\,.$$

The pde becomes

$$\partial u/\partial t = \sum_{n=0}^{\infty} a_n'(t) \cos n\pi x/L = k\,\partial^2 u/\partial x^2 + Q(x,t)\,.$$

Thus $a_n'(t) = q_n(t) + \int_0^L k\,\frac{\partial^2 u}{\partial x^2} \cos n\pi x/L dx \Big/ \int_0^L \cos^2 n\pi x/L dx$,

where $q_n(t) = \int_0^L Q(x,t) \cos n\pi x/L\,dx \Big/ \int_0^L \cos^2 n\pi x/L\,dx$. Using Green's formula (8.4.12) with $v = \cos n\pi x/L$

$$\int_0^L \frac{\partial^2 u}{\partial x^2} \cos n\pi x/L\,dx = -(n\pi/L)^2 \int_0^L u \cos n\pi x/L\,dx + (-1)^n B(t) - A(t).$$

Thus $a_n'(t) + k(n\pi/L)^2 a_n = q_n(t) + k[(-1)^n B(t) - A(t)]/I_n$, where $I_0 = L$ and $(n \neq 0) I_n = L/2$. Using the usual integrating factor [see (9.2.16)],

$$a_n(t) = a_n(0)e^{-k(n\pi/L)^2 t} + \int_0^t \left[q_n(t_0) + k\frac{(-1)^n B(t_0) - A(t_0)}{I_n}\right] e^{-k(n\pi/L)^2(t-t_0)} dt_0,$$

where $a_n(0) = \int_0^L g(x) \cos \frac{n\pi x}{L} dx\,/I_n$. Therefore

$$\begin{aligned} u(x,t) = &\int_0^L g(x_0) \sum_{n=0}^{\infty} \frac{1}{I_n} \cos\frac{n\pi x}{L} \cos\frac{n\pi x_0}{L} e^{-k(n\pi/L)^2 t} dx_0 \\ &+ \int_0^t \sum_{n=0}^{\infty} k\frac{(-1)^n B(t_0) - A(t_0)}{I_n} \cos\frac{n\pi x}{L} e^{-k(n\pi/L)^2(t-t_0)} d\,t_0 \\ &+ \int_0^L \int_0^t Q(x_0,t_0) \sum_{n=0}^{\infty} \frac{1}{I_n} \cos\frac{n\pi x}{L} \cos\frac{n\pi x_0}{L} e^{-k(n\pi/L)^2(t-t_0)} d\,t_0 d\,x_0. \end{aligned}$$

We introduce the Green's function,

$$G(x,t;x_0,t_0) = \sum_{n=0}^{\infty} \frac{1}{I_n} \cos\frac{n\pi x}{L} \cos\frac{n\pi x_0}{L} e^{-k(n\pi/L)^2(t-t_0)},$$

so that

$$\begin{aligned} u(x,t) = &\int_0^L g(x_0)G(x,t;x_0,0)dx_0 + \int_0^L \int_0^t Q(x_0,t_0)G(x,t;x_0,t_0)dt_0 dx_0 \\ &+ \int_0^t kB(t_0)G(x,t;L,t_0)dt_0 - \int_0^t kA(t)G(x,t;0,t_0)dt_0. \end{aligned}$$

9.2.3 The method of eigenfunction expansion yields $u = \sum_{n=1}^{\infty} a_n(t) \sin n\pi x/L$. This can be term-by-term differentiated with respect to x since both u and $\sin n\pi x/L$ satisfy the same set of homogeneous boundary conditions. Therefore, $\frac{d^2a_n}{dt^2} = -c^2(\frac{n\pi}{L})^2 a_n + q_n(t)$, where $q_n(t) = \frac{2}{L}\int_0^L Q(x,t)\sin\frac{n\pi x}{L}dx$. Using (8.5.19)

$$a_n(t) = c_1\cos\frac{n\pi ct}{L} + c_2\sin\frac{n\pi ct}{L} + \frac{L}{n\pi c}\int_0^t q_n(t_0)\sin\frac{n\pi c}{L}(t-t_0)dt_0,$$

where $a_n(0) = c_1 = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}dx$ and $\frac{da_n}{dt}(0) = \frac{n\pi c}{L}c_2 = \frac{2}{L}\int_0^L g(x)\sin\frac{n\pi x}{L}dx$.
By substituting these back into the series, we obtain

$$u(x,t) = \int_0^L f(x_0)\sum_{n=1}^{\infty}\frac{2}{L}\sin\frac{n\pi x_0}{L}\sin\frac{n\pi x}{L}\cos\frac{n\pi ct}{L}dx_0$$

$$+\int_0^L g(x_0)\sum_{n=1}^{\infty}\frac{2}{L}\sin\frac{n\pi x_0}{L}\sin\frac{n\pi x}{L}\frac{\sin\frac{n\pi ct}{L}}{\frac{n\pi c}{L}}dx_0$$

$$+\int_0^L\int_0^t Q(x_0,t_0)\sum_{n=1}^{\infty}\frac{2}{L}\sin\frac{n\pi x_0}{L}\sin\frac{n\pi x}{L}\frac{\sin\frac{n\pi c}{L}(t-t_0)}{\frac{n\pi c}{L}}dt_0dx_0 .$$

We introduce a Green's function

$$G(x,t;x_0,t_0) = \sum_{n=1}^{\infty}\frac{2}{L}\sin\frac{n\pi x_0}{L}\sin\frac{n\pi x}{L}\frac{\sin\frac{n\pi c}{L}(t-t_0)}{\frac{n\pi c}{L}},$$

so that

$$u(x,t) = \int_0^L\int_0^t Q(x_0,t_0)G(x,t;x_0,t_0)dt_0dx_0$$

$$+\int_0^L g(x_0)G(x,t;x_0,0)\,dx_0 + \int_0^L f(x_0)\frac{\partial G}{\partial t}(x,t;x_0,0)\,dx_0 .$$

We must have some "faith" in these calculations since the series for $\partial G/\partial t$ does not converge. This result will be obtained later in a different way [see (11.2.24)].

Section 9.3

9.3.5 (a) $du/dx = \int_L^x f(\bar{x})d\bar{x}$ using the boundary condition at $x = L$. Integrating again gives $u(x) = \int_0^x\left(\int_L^{x_0} f(\bar{x})d\bar{x}\right)dx_0$ using $u(0) = 0$. Integration-by-parts,

$$\begin{array}{ll} dw = dx_0 & v = \int_L^{x_0} f(\bar{x})d\bar{x} \\ w = x_0 & dv = f(x_0)dx_0, \end{array}$$

yields

$$u(x) = x\int_L^x f(\bar{x})d\bar{x} - \int_0^x x_0 f(x_0)dx_0 ,$$

which is equivalent to answer given.

9.3.5 (b) Let u_1 be a homogeneous solution satisfying $u(0) = 0$, i.e., $u_1 = x$, while let u_2 be an independent one satisfying $du/dx(L) = 0$, i.e. $u_2 = 1$. Since $p = 1$ from (9.3.10-12) $\frac{dv_1}{dx} = f$ and $\frac{dv_2}{dx} = -xf$. Thus from (9.3.9)

$$u = x\int_0^x f(x_0)dx_0 - \int_0^x x_0 f(x_0)dx_0 + c_1x + c_2.$$

The boundary condition $u(0) = 0$ yields $c_2 = 0$, while $du/dx(L) = 0$ yields $c_1 = -\int_0^L f(x_0)dx_0$, which is equivalent to answer given.

9.3.5 (c) In order for $u(x) = \int_0^L f(x_0)G(x,x_0)dx_0$,

$$G(x,x_0) = \begin{cases} -x_0 & x_0 < x \\ -x & x_0 > x. \end{cases}$$

9.3.6 (a) If $x < x_0, G(x,x_0) = c_1x + c_2$, where $c_2 = 0$ from $G(0,x_0) = 0$. If $x > x_0, G(x,x_0) = c_3x + c_4$, where $c_3 = 0$ from $dG/dx(L,x_0) = 0$. From continuity at $x = x_0$: $c_1x_0 = c_4$. By integrating from x_0- to x_0+, $\left.\frac{dG}{dx}\right|_{x_0-}^{x_0+} = 1$ or $0 - c_1 = 1$. Thus $c_1 = -1$ and $c_4 = -x_0$, yielding

$$G(x,x_0) = \begin{cases} -x & x < x_0 \\ -x_0 & x > x_0 \end{cases} .$$

9.3.6 (b) See answer on p. 743.

9.3.9 (a) Let u_1 be a homogeneous solution satisfying $u(0) = 0$, i.e., $u_1 = \sin x$, while u_2 satisfies $u(L) = 0$, i.e. $u_2 = \sin(x-L)$. Note these are independent if $L \neq n\pi$. We also note that $u_1du_2/dx - u_2du_1/dx = \sin L \neq 0$. From (9.3.10-12) $dv_1/dx = -f\sin(x-L)/\sin L$ and $dv_2/dx = f\sin x/\sin L$.

Thus from (9.3.9)

$$u = \frac{-\sin x}{\sin L}\int_L^x f(x_0)\sin(x_0 - L)dx_0 + \frac{\sin(x-L)}{\sin L}\int_0^x f(x_0)\sin x_0 dx_0 + c_1\sin x + c_2\sin(x-L) .$$

The boundary condition $u(0) = 0$ yields $c_2 = 0$, while $u(L) = 0$ yields $c_1 = 0$.

9.3.9 (b) In order to put the result of part (a) into the form (9.3.15),

$$G(x,x_0) = \begin{cases} \sin x\sin(x_0 - L)/\sin L & x < x_0 \\ \sin x_0\sin(x-L)/\sin L & x > x_0 \end{cases} .$$

9.3.11 (a) For $x \neq x_0, d^2G/dx^2 + G = 0$. If $x < x_0, G(x,x_0) = b\sin x$ using $G(0,x_0) = 0$. If $x > x_0, G(x,x_0) =$ a $\sin(x-L)$ using $G(L,x_0) = 0$. To be continuous at $x = x_0$:

$$b\sin x_0 = a\sin(x_0 - L). \tag{1}$$

Integrating the defining differential equation from x_0- to x_0+ yields

$$\left.\frac{dG}{dx}\right|_{x_0-}^{x_0+} + \int_{x_0-}^{x_0+} G dx = 1.$$

This integral vanishes since G is continous, and thus

$$a\cos(x_0 - L) - b\sin x_0 = 1. \tag{2}$$

Equations (1) - (2) may be solved by elimination only if $L \neq n\pi$, yielding a $= \sin x_0/\sin L$ and b $= \sin(x_0 - L)/\sin L$, using a trigonometric addition formula. Thus

$$G(x,x_0) = \begin{cases} \sin x\sin(x_0 - L)/\sin L & x < x_0 \\ \sin x_0\sin(x-L)/\sin L & x > x_0. \end{cases}$$

Alternate method: The constants a and b may be redefined so that $G(x,x_0)$ is automatically continuous at $x = x_0$:

$$G(x,x_0) = \begin{cases} c\sin(x_0 - L)\sin x & x < x_0 \\ c\sin x_0\sin(x-L) & x > x_0, \end{cases}$$

where c is a constant independent of x_0. The jump condition on the derivative (9.3.45) determines c, yielding the same answer.

9.3.13 (b) For $x \neq x_0, d^2G/dx^2 + k^2G = 0$. A particularly convenient choice of homogeneous solutions is

$$G(x, x_0) = \begin{cases} c_1 e^{ik(x-x_0)} + c_2 e^{-ik(x-x_0)} & x < x_0 \\ c_3 e^{ik(x-x_0)} + c_4 e^{-ik(x-x_0)} & x > x_0 \,. \end{cases}$$

The corresponding solution of ϕ is obtained by multiplying u and thus $G(x, x_0)$ by $e^{-i\omega t}$. For example $e^{i(kx-\omega t)}$ is a right-going wave (assuming $\omega/k > 0$). For $x < x_0$, we want a left-going wave in order to be outward propagating and thus $c_1 = 0$. For $x > x_0$, it should be right-going and thus $c_4 = 0$:

$$G(x, x_0) = \begin{cases} c_2 e^{-ik(x-x_0)} & x < x_0 \\ c_3 e^{ik(x-x_0)} & x > x_0. \end{cases}$$

To be continuous at $x = x_0, c_2 = c_3$. The jump condition is obtained by integrating th defining differential equation from x_0- to x_0+:

$$\left.\frac{dG}{dx}\right|_{x_0-}^{x_0+} + k^2 \int_{x_0-}^{x_0+} G dx = 1 \,.$$

Since G is continuous

$$\left.\frac{dG}{dx}\right|_{x_0-}^{x_0+} = 1 \quad \text{or}$$

$c_3 ik + c_2 ik = 1$, yielding $c_2 = c_3 = \frac{1}{2ik}$. A particularly convenient form for the Green's function is

$$G(x, x_0) = \frac{1}{2ik} e^{ik|x-x_0|}.$$

9.3.14 (d) Using Green's formula with $v = G(x, x_0)$:

$$\int_0^L [uL(G) - GL(u)]dx = p(u\frac{dG}{dx} - G\frac{du}{dx})\Big|_0^L \,.$$

We note that $L(u) = f$ and $L(G) = \delta(x - x_0)$. Thus

$$u(x_0) = \int_0^L f(x)G(x, x_0)dx + p(u\frac{dG}{dx} - G\frac{du}{dx})\Big|_0^L \,.$$

The Green's function satisfies the related homogeneous boundary conditions, $G(0, x_0) = 0$ and $\frac{dG}{dx}(L, x_0) + h\,G(L, x_0) = 0$, while u satisfies the given conditions. Thus

$$u(x_0) = \int_0^L f(x)G(x, x_0)dx + p(L)G(L, x_0)(-hu(L) + hu(L) - \beta) - p(0)\alpha\frac{dG}{dx}(0, x_0) \,.$$

Switching x and x_0 and using symmetry yields the given answer.

9.3.15 (a) For $x \neq x_0, L(G) = 0$. This problem defines homogeneuous solutions needed to solve the boundary conditions for the Green's function. Thus

$$G(x, x_0) = \begin{cases} c_1 y_1(x) & x < x_0 \\ c_2 y_2(x) & x > x_0 \,. \end{cases}$$

Continuity at $x = x_0$ yields

$$c_1 y_1(x_0) = c_2 y_2(x_0) \,. \tag{3}$$

The jump condition for the derivative is obtained by integrating the defining differential equation, yielding

$$p\frac{dG}{dx}\Big|_{x_0-}^{x_0+} = 1$$

since $\int_{x_0-}^{x_0+} q\,G\,dx$ vanishes because G is continuous. In this manner

$$c_1\frac{dy_1}{dx}(x_0) - c_2\frac{dy_2}{dx}(x_0) = -1/p(x_0)\ . \qquad (4)$$

Solving (3) - (4) simultaneously yields

$$G(x, x_0) = \begin{cases} y_1(x)y_2(x_0)/c & x < x_0 \\ y_2(x)y_1(x_0)/c & x > x_0\ , \end{cases}$$

where c is the constant defined on page 387 and is related to the Wronskian

$$w = y_1\frac{dy_2}{dx} - y_2\frac{dy_1}{dx} = c/p(x)\ .$$

The constant c is determined from the calculations of y_1 and y_2 at either end: $c/p(0) = -y_2(0)$ or $c/p(L) = y_1(L)$.

9.3.21 If $x \neq x_0$, then $dG/dx = 0$. If $x < x_0, G(x, x_0) = 0$ in order to satisfy $G(0, x_0) = 0$. If $x > x_0$, the general solution is $G(x, x_0) = c$. The Green's function is not continuous, since its derivative is a Dirac delta function. The jump condition is obtained by integrating the defining differential equation

$$G\,|_{x_0-}^{x_0+} = 1\ .$$

Therefore $c - 0 = 1$. We conclude that

$$G(x, x_0) = \begin{cases} 0 & x < x_0 \\ 1 & x > x_0, \end{cases}$$

which is not symmetric.

9.3.25 (b) The simplest particular solution corresponds to the initial condition $u(0) = u^{'}(0) = u^{''}(0) = u^{'''}(0) = 0$. Taking the Laplace transform of the differential equation yields

$$s^4\bar{u}(s) = F(s) \quad \text{and thus} \quad \bar{u}(s) = F(s)/s^4\ .$$

Using the convolution theorem,

$$u(x) = \int_0^x f(\bar{x})g(x - \bar{x})d\bar{x}\ ,$$

where $g(x)$ is a function whose Laplace transform is $1/s^4$. From the table inside the front cover, $g(x) = x^3/3!$. Thus

$$u(x) = \int_0^x f(\bar{x})(x - \bar{x})^3/3!d\bar{x}.$$

Section 9.4

9.4.2 (a) Using Green's formula

$$\int_0^L [uL(\phi_h) - \phi_h L(u)]\,dx = p\left(u\frac{d\phi_h}{dx} - \phi_h\frac{du}{dx}\right)\Big|_0^L\ .$$

Since $L(u) = f, u(0) = \alpha, u(L) = \beta$ and $L(\phi_h) = 0, \phi_h(0) = 0, \phi_h(L) = 0$, it follows that

$$0 = \int_0^L f\phi_h dx + p(L)\beta\frac{d\phi_h(L)}{dx} - p(0)\alpha\frac{d\phi_h(0)}{dx}\ .$$

9.4.3 (b) Homogeneous solutions satisfy $d^2\phi/dx^2 + \phi = 0$ with $d\phi/dx(0) = d\phi/dx(\pi) = 0$. Thus $\phi_h = \cos x$ is a nontrivial homogeneous solution. According to the Fredholm alternative, solutions to the nonhomogeneous problem (subject to homogeneous boundary conditions) exist only if the right hand side is orthogonal to $\phi(x)$. Since

$$\int_0^\pi \cos x \sin x \, dx = 0,$$

there are an infinite number of solutions.

9.4.6 (a) The general solution of the differential equation is

$$u = 1 + c_1 \cos x + c_2 \sin x \; .$$

The boundary condition $u(0) = 0$ implies $0 = 1 + c_1$, while $u(\pi) = 0$ yields $0 = 1 - c_1$. These are inconsistent so that no solutions exist. To apply the Fredholm alternative, we note that $\phi_h = \sin x$ is a homogeneous solution since $\phi(0) = \phi(\pi) = 0$. Furthermore,

$$\int_0^\pi 1 \cdot \sin x \, dx = 2 \neq 0 \; .$$

Thus, there are no solutions since the right hand side $f(x) = 1$ is not orthogonal to $\phi_h = \sin x$.

9.4.6 (b) The general solution of the differential equation is

$$u = 1 + c_1 \cos x + c_2 \sin x \; .$$

The boundary condition $du/dx(0) = 0$ implies $0 = c_2$, while $du/dx(\pi) = 0$ yields $0 = -c_2$. This is possible with $c_2 = 0$ and c_1 arbitrary:

$$u = 1 + c_1 \cos x \; .$$

There are an infinite number of solutions. To apply the Fredholm alternative, we note that $\phi_h = \cos x$ is a homogeneous solution since $\phi'(0) = \phi'(\pi) = 0$. Since

$$\int_0^\pi 1 \cdot \cos x \, dx = 0 \; ,$$

the right-hand side $f(x) = 1$ is orthogonal to all homogeneous solutions $\phi_h = \cos x$. Thus there should be an infinite number of solutions.

9.4.6 (c) The general solution of the differential equation is

$$u = 1 + c_1 \cos x + c_2 \sin x \; .$$

The boundary condition $u(-\pi) = u(\pi)$ implies $1 - c_1 = 1 - c_1$, while $u'(-\pi) = u'(\pi)$ yields $-c_2 = -c_2$. Thus the above soltuion for u is valid with both c_1 and c_2 arbitrary. There are an infinite number of solutions. To apply the Fredholm alternative, we note that both $\phi_h = \sin x$ and $\phi_h = \cos x$ are homogeneous solutions satisfying the homogeneous boundary conditions. Here

$$\int_{-\pi}^{\pi} 1 \cdot \cos x \, dx = 0 \quad \text{and} \quad \int_{-\pi}^{\pi} 1 \cdot \sin x \, dx = 0 \; .$$

Thus, there are an infinite number of solutions.

9.4.8 (a) To obtain a particular solution, we substitute $u_p = A\, x \sin x$ into the differential equation in order to determine the constant A. Here $u'_p = A(x \cos x + \sin x)$ and $u_p'' = A(-x \sin x + 2\cos x)$.

Therefore $A(-x \sin x + 2\cos x + x \sin x) = \cos x$ or $A = \frac{1}{2}$. Thus the general solution is

$$u = x \sin x/2 + c_1 \cos x + c_2 \sin x \; .$$

The boundary condition $u(0) = 0$ yields $0 = c_1$, while $u(\pi) = 0$ also yields $0 = c_1$. Thus

$$u = \frac{1}{2}x \sin x + c_2 \sin x \ ,$$

with c_2 arbitrary, which is an infinite number of solutions. To apply the Fredholm alternative, we note that $\phi_h = \sin x$ is a homogeneous solution satisfying the boundary conditions. Since

$$\int_0^\pi \cos x \sin x \, dx = 0 \ ,$$

there are an infinite number of solutions.

9.4.10 Since $\phi_h = \sin x$ is a solution of both the corresponding homogeneous differential equation and boundary condition, a modified Green's function $G_m(x, x_0)$ must be introduced satisfying

$$\frac{d^2 G_m}{dx^2} + G_m = \delta(x - x_0) + c \sin x \ ,$$

subject to $G_m(0, x_0) = G_m(\pi, x_0) = 0$. The constant c is chosen so that the right-hand side is orthogonal to $\phi_h = \sin x$:

$$0 = \int_0^\pi \sin x[\delta(x - x_0) + c \sin x]dx \quad \text{or} \quad c = -\frac{2}{\pi} \sin x_0 \ .$$

If $x \neq x_0, G''_m + G_m = c \sin x$. The method of undetermined coefficients can be used to find a particular solution, $G_m = -\frac{c}{2}x \cos x$. Thus

$$G_m(x, x_0) = \frac{1}{\pi} \sin x_0 \, x \cos x + \begin{cases} c_1 \cos x + c_2 \sin x & x < x_0 \\ c_3 \cos x + c_4 \sin x & x > x_0 \ . \end{cases}$$

The boundary conditions yield

$$0 = c_1 \quad \text{and} \quad 0 = -\sin x_0 - c_3 \ ,$$

while continuity at $x = x_0$ yields

$$c_1 \cos x_0 + c_2 \sin x_0 = c_3 \cos x_0 + c_4 \sin x_0 \ ,$$

which simplifies to $c_2 = -\cos x_0 + c_4$. The jump condition $x = x_0$ is

$$\left. \frac{dG_m}{dx} \right|_{x_0-}^{x_0+} = 1$$

since $G_m(x, x_0)$ is continuous. The jump condition will be satisfied since c was picked so that $G_m(x, x_0)$ exists. Thus there are an infinite number of modified Green's functions

$$G_m(x, x_0) = \frac{1}{\pi} \sin x_0 \, x \cos x + \begin{cases} (c_4 - \cos x_0) \sin x & x < x_0 \\ c_4 \sin x - \sin x_0 \cos x & x > x_0 \ , \end{cases}$$

where c_4 is an arbitrary function of $x_0, c_4(x_0)$. We chose $G_m(x, x_0)$ to be symmetric, as is guaranteed by Exercise 9.4.9, so that (9.4.17) may be derived. For example, for $x < x_0 \, G_m(x, x_0) = G_m(x_0, x)$ yields

$$\frac{1}{\pi} \sin x_0 x \cos x + [c_4(x_0) - \cos x_0] \sin x$$
$$= \frac{1}{\pi} \sin x \, x_0 \cos x_0 + c_4(x) \sin x_0 - \sin x \cos x_0 \ .$$

As a function of $x, c_4(x)$ is some combination of $\sin x$ and $x \cos x$:

$$c_4(x) = a \sin x + b x \cos x \cos x \ :$$

where a and b are constants, independent of x and x_0. Equating the coefficients of $\sin x$ and $x\cos x$ yields

$\sin x:$ $\quad a\sin x_0 + b\,x_0\cos x_0 - \cos x_0 = \frac{1}{\pi}x_0\cos x_0 + a\sin x_0 - \cos x_0$

$x\cos x:$ $\quad \frac{1}{\pi}\sin x_0 = b\sin x_0\ .$

Thus $b = 1/\pi$ and a is arbitrary, yielding an infinite number of symmetric modified Green's functions:

$$G_m(x,x_0) = a\sin x\sin x_0 + \begin{cases} \frac{1}{\pi}(x\cos x\sin x_0 + x_0\cos x_0\sin x) \\ \qquad - \cos x_0\sin x & x < x_0 \\ \frac{1}{\pi}(x_0\cos x_0\sin x + x\cos x\sin x_0) \\ \qquad - \cos x\sin x_0 & x > x_0\ . \end{cases}$$

To obtain a formula for $u(x)$, we use Green's formula

$$\int_0^\pi [uL(G_m) - G_mL(u)]\,dx = u\frac{dG_m}{dx} - G_m\left.\frac{du}{dx}\right|_0^\pi\ .$$

Using the differential equations for $u(x)$ and $G_m(x,x_0)$, we obtain

$$u(x_0) = \int_0^\pi f(x)G_m(x,x_0)dx + \frac{2}{\pi}\sin x_0\int_0^\pi u(x)\sin x\,dx + \beta\frac{dG_m}{dx}(\pi) - \alpha\frac{dG_m}{dx}(0)\ .$$

We can use any modified Green's function. With $a = 0$

$$\frac{dG_m}{dx}(\pi) = -\frac{1}{\pi}(x_0\cos x_0 + \sin x_0)$$

$$\frac{dG_m}{dx}(0) = \frac{1}{\pi}(\sin x_0 + x_0\cos x_0) - \cos x_0\ .$$

Switching x and x_0, using the symmetry of $G_m(x,x_0)$, we obtain

$$u(x) = \int_0^\pi f(x_0)G_m(x,x_0)dx_0 - \frac{\beta}{\pi}(x\cos x + \sin x) - \alpha[\frac{1}{\pi}(\sin x + x\cos x) - \cos x] + k\sin x,$$

where the constant $k = \frac{2}{\pi}\int_0^\pi u(x_0)\sin x_0\,dx_0$. This constant is arbitrary, representing an arbitrary multiple of the homogeneous solution. This solution is consistent (based on calculating $\int_0^\pi u(x_0)\sin x_0\,dx_0$) is the original problem actually has a solution. From exercise 9.4.2 a solution only exists if

$$0 = \int_0^\pi f(x)\sin x\,dx - \alpha - \beta\ ,$$

since $\phi_h = \sin x$.

9.4.11 (a) By integrating the differential equation from 0 to L,

$$\left.\frac{dG_a}{dx}\right|_0^L = 1\ .$$

Applying the boundary conditions yields $c = 1$.

9.4.11 (b) The modified Green's function represents a steady-state heat flow problem with heat source $-\delta(x - x_0)$ insulated at $x = 0$ with heat flow to the right (out) at $x = L$ of $-c$. The total thermal energy generated inside per unit time is $\int_0^L -\delta(x - x_0)dx = -1$. For equilibrium, the heat generated inside must equal the heat flowing out. Thus

$$-1 = -c\ .$$

9.4.11 (c) From the differential equation

$$G_a(x,x_0) = \begin{cases} c_1 x + c_2 & x < x_0 \\ c_3 x + c_4 & x > x_0 \ . \end{cases}$$

$dG_a/dx(0) = 0$ implies $c_1 = 0$, while $dG_a/dx(L) = c = 1$ implies $c_3 = c = 1$.
To be continous at $x = x_0, c_1 x_0 + c_2 = c_3 x_0 + c_4$ or $c_2 = x_0 + c_4$. Thus

$$G_a(x,x_0) = \begin{cases} x_0 + c_4(x_0) & x < x_0 \\ x + c_4(x_0) & x > x_0 \ , \end{cases}$$

which automatically satisfies the jump condition $dG_a/dx\,|_{x_0+} - dG_a/dx\,|_{x_0-} = 1$. In general, this different modified Green's function is not symmetric.

9.4.11 (d) To be symmetric, for example $G_a(x,x_0) = G_a(x_0,x)$ for $x < x_0 : x_0 + c_4(x_0) = x_0 + c_4(x)$.
This is only valid if $c_4(x_0)$ is an arbitrary constant α, independent of x_0:

$$G_a(x,x_0) = \alpha + \begin{cases} x_0 & x < x_0 \\ x & x > x_0 \ . \end{cases}$$

There is an infinite number of symmetric modified Green's funciton of this type because if there is one, any multiple of the homogeneous solution $\phi_h(x_0)$ can be added.

9.4.11 (e) Using Green's formula with $v = G_a(x,x_0)$,

$$\int_0^L \left(u\frac{d^2G_a}{dx^2} - G_a(x,x_0)\frac{d^2u}{dx^2} \right) dx = u\frac{dG_a}{dx} - G_a(x,x_0)\frac{du}{dx}\bigg|_0^L \ .$$

Using the differential equations for u and $G_a(x,x_0)$ and the fundamental property of the Dirac delta function yields

$$u(x_0) = \int_0^L f(x)G_a(x,x_0)dx + u(L)\frac{dG_a}{dx}(x,x_0)\bigg|_{x=L} - u(0)\frac{dG_a}{dx}(x,x_0)\bigg|_{x=0} \ .$$

From part (d), $\frac{dG_a}{dx}(x,x_0)\big|_{x=L} = 1$ and $\frac{dG_a}{dx}(x,x_0)\big|_{x=0} = 0$, and thus

$$u(x_0) = \int_0^L f(x)G_a(x,x_0)dx + u(L) \ .$$

Switching x and x_0 using the symmetry of $G_a(x,x_0)$ yields

$$u(x) = \int_0^L f(x_0)G_a(x,x_0)dx_0 + u(L) \ .$$

To show this is consistent, we substitute $x = L$, in which case we obtain

$$\int_0^L f(x_0)dx_0 = 0$$

since $G_a(L,x_0)$ is independent of x_0. This is the condition for the solution to exist from the Fredholm alternative since $\phi_h(x) = 1$. Thus $u(L)$ is arbitrary above (and may be replaced by an arbitrary constant), representing an arbitrary multiple of the homogeneuous solution $\phi_h(x) = 1$.

Section 9.5

9.5.3 (c) On the semi-circle with $G = 0$ on the boundary, the eigenfunctions of $\nabla^2\phi = -\lambda\phi$ are the family $\phi_{mn} = \sin m\theta J_m(\sqrt{\lambda_{mn}}r)$ where λ_{mn} is determined from $J_m(\sqrt{\lambda_{mn}}a) = 0$. By the method of eigenfunction expansion,

$$G(r,\theta;r_0,\theta_0) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} A_{mn}\phi_{mn}(r,\theta)\ . \tag{5}$$

Since G and ϕ_{mn} satisfy the same set of homogeneous boundary conditions,

$$\nabla^2 G = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} A_{mn}\nabla^2\phi_{mn} = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} A_{mn}(-\lambda_{mn}\phi_{mn})\ .$$

Using the orthogonality of ϕ_{mn} and the differential equation for G:

$$-\lambda_{mn}A_{mn} = \frac{\int\int \phi_{mn}\nabla^2 G\,dx\,dy}{\int\int \phi_{mn}^2 dx\,dy} = \frac{\phi_{mn}(r_0,\theta_0)}{\int\int \phi_{mn}^2 dx\,dy}\ .$$

Substituting this into (5) yields

$$G(r,\theta;r_0,\theta_0) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \frac{\phi_{mn}(r,\theta)\phi_{mn}(r_0,\theta_0)}{-\lambda_{mn}\int\int \phi_{mn}^2 \bar{r}d\bar{r}d\bar{\theta}}\ .$$

9.5.4 Using Green's formula with $v = G(\vec{x},\vec{x}_0)$:

$$\iiint (u\nabla^2 G - G\nabla^2 u)dx\,dy\,dz = \oiint (u\nabla G - G\nabla u)\cdot\hat{n}\,dS\ .$$

Substituting $\nabla^2 G = \delta(\vec{x} - \vec{x}_0)$ with $G = 0$ on the boundary yields

$$u(\vec{x}_0) = \iiint G(\vec{x},\vec{x}_0)f(\vec{x})dx\,dy\,dz + \oiint h(\vec{x})\nabla G\cdot\hat{n}\,dS\ .$$

By switching $\vec{x}$ and $\vec{x}_0$ and using the symmetry of G, we obtain

$$u(\vec{x}) = \iiint G(\vec{x},\vec{x}_0)f(\vec{x}_0)dx_0dy_0dz_0 + \oiint h(\vec{x}_0)\nabla_{\vec{x}_0}G\cdot\hat{n}dS_0\ .$$

9.5.9 (b) The $\theta-$dependent eigenfunctions are $\sin m\theta, m = 1, 2, ...$ since $G = 0$ on $\theta = 0$ and $\theta = \pi$. Thus, the method of eigenfunction expansion yields

$$G = \sum_{m=1}^{\infty} A_m(r)\sin m\theta\ .$$

By taking the Laplacian and using the differential equation for the Green's function, we obtain

$$\nabla^2 G = \delta(\vec{x} - \vec{x}_0) = \sum_{m=1}^{\infty}\left[\frac{1}{r}\frac{d}{dr}\left(r\frac{dA_m}{dr}\right) - \frac{m^2}{r^2}A_m\right]\sin m\theta\ . \tag{6}$$

Since

$$\iint f(r,\theta)\delta(\vec{x} - \vec{x}_0)dx\,dy = \iint f(r,\theta)\delta(\vec{x} - \vec{x}_0)r\,dr\,d\theta = f(r_0,\theta_0),$$

it is helpful to note that

$$\delta(\vec{x}-\vec{x}_0)=\frac{1}{r_0}\delta(r-r_0)\delta(\theta-\theta_0)\,.$$

Thus from (6), using the orthogonality of sines,

$$\frac{1}{r}\frac{d}{dr}(r\frac{dA_m}{dr})-\frac{m^2}{r^2}A_m=\frac{2}{\pi}\int_0^\pi\delta(r-r_0)\delta(\theta-\theta_0)\frac{\sin\,m\theta}{r_0}d\theta=\frac{2\,\sin\,m\theta_0}{\pi r_0}\delta(r-r_0)\,.$$

This may be solved by the methods of Section 9.3 to obtain one-dimensional Green's functions. For $r\neq r_0$, the equation is equidimensional, whose solutions are r^p with $p=\pm m$. For $r<r_0$, the solution must be bounded, and thus it is proportional to r^m. For $r>r_0$, it satisfies the zero boundary condition at $r=a$. In this manner

$$A_m(r)=\begin{cases} c r^m\left[(\frac{r_0}{a})^m-(\frac{a}{r_0})^m\right] & r<r_0\\ c r_0^m\left[(\frac{r}{a})^m-(\frac{a}{r})^m\right] & r>r_0\,,\end{cases}$$

where we have made this function automatically continuous at $r=r_0$. The constant c is chosen to satisfy the jump-condition $r\frac{dA_m}{dr}\Big|_{r_0-}^{r_0+}=\frac{2}{\pi}\sin\,m\theta_0$. Thus

$$c\left[(\frac{r_0}{a})^{m-1}\frac{1}{a}+\frac{1}{a}(\frac{a}{r_0})^{m+1}-\frac{1}{r_0}(\frac{r_0}{a})^m+\frac{1}{r_0}(\frac{a}{r_0})^m\right]mr_0^m=\frac{2}{\pi r_0}\sin\,m\theta_0\,,$$

so that $c=a^{-m}\frac{\sin\,m\,\theta_0}{m\pi}$, yielding the answer.

9.5.10 (a)

$$\iiint\left[u(\nabla^2+k^2)v-v(\nabla^2+k^2)u\right]dV=\iiint(u\nabla^2v-v\nabla^2u)dV=\oiint(u\nabla v-v\nabla u)\cdot\hat{n}\,dS\,.$$

9.5.10 (b) Let $\rho=|\vec{x}-\vec{x}_0|$ in three dimensions as in (9.5.28b). Then $(\rho\neq 0)\nabla^2G+k^2G=0$. Spherically symmetric solutions will satisfy

$$\frac{1}{\rho^2}\frac{d}{d\rho}\left(\rho^2\frac{dG}{d\rho}\right)+k^2G=0\,.$$

Introduce the change of variable, $G=y/\rho$. This is admittedly motivated by the answer. Since $dG/d\rho=\frac{1}{\rho}dy/d\rho-y/\rho^2$, it follows that

$$\frac{d^2y}{d\rho^2}+k^2y=0\,.$$

Thus $y=c_1e^{ik\rho}+c_2e^{-ik\rho}$ or $G=(c_1e^{ik\rho}+c_2e^{-ik\rho})/\rho$.

The time-dependent problem is obtained by multiplying by $e^{-i\omega t}$. We note that $e^{i(k\rho-\omega t)}$ is outgoing if $\omega/k>0$, while $e^{-i(k\rho+\omega t)}$ is incoming. Thus, to be outgoing $c_2=0$

$$G=c_1e^{ik\rho}/\rho.$$

To determine c_1 we integrate the defining differential equation over a small sphere centered at $\vec{x}=\vec{x}_0$ with radius ρ:

$$\iiint(\nabla^2G+k^2G)dV=1\,.$$

It can be shown that $\lim\limits_{\rho\to 0}\iiint GdV=0$ and thus from the divergence theorem

$$\lim_{\rho\to 0}\oiint\nabla G\cdot\hat{n}\,dS=1\,.$$

On the surface of the sphere, $\nabla G\cdot\hat{n}=\partial G/\partial\rho$ is constant. Hence $\lim\limits_{\rho\to 0}\frac{\partial G}{\partial\rho}\oiint dS=1$. However, the surface area is $4\pi\rho^2$. Thus $\lim\limits_{\rho\to 0}4\pi\rho^2\frac{\partial G}{\partial\rho}=1$, which implies $c_1=-1/4\pi$.

9.5.10 (c) Let $r = |\vec{x} - \vec{x}_0|$ in two dimensions as in (9.5.28a). For $r \neq 0, \nabla^2 G + k^2 G = 0$. Circularly symmetric solutions will satisfy $\frac{1}{r}\frac{d}{dr}(r\frac{dG}{dr}) + k^2 G = 0$, which is equivalent to $r^2 d^2G/dr^2 + r dG/dr + k^2 r^2 G = 0$. This can be related to Bessel's differential equation, so that $G = c_1 J_0(kr) + c_2 Y_0(kr)$.

To investigate whether or not this solution is outgoing for large r, we use the asymptotic expansions for the Bessel functions (7.8.3). In this way, for large r

$$G \sim (2/\pi kr)^{1/2}[c_1 \cos(kr - \pi/4) + c_2 \sin(kr - \pi/4)] \ .$$

By using Euler's formulas, we obtain an equivalent expression

$$G \sim (2\pi kr)^{-1/2}[(c_1 - ic_2)e^{i(kr-\pi/4)} + (c_1 + ic_2)e^{-i(kr-\pi/4)}] \ .$$

The time-dependent problem is obtained by multiplying by $e^{-i\omega t}$. We note that $e^{i(kr-\omega t)}$ is outgoing if $\omega/k > 0$, while $e^{-i(kr+\omega t)}$ is incoming. Thus, to be outgoing

$$c_1 + ic_2 = 0 \ . \qquad (7)$$

To determine the remaining constant we must investigate the singularity at $r = 0$. If we integrate the defining differential equation over a circle of radius r centered at $\vec{x} = \vec{x}_0$, we obtain using the divergence theorem

$$\oint \nabla G \cdot \hat{n} ds + k^2 \iint G \, dA = 1 \ .$$

Since G is only a function of r, $\nabla G \cdot \hat{n} = dG/dr$, and $\oint ds = 2\pi r$, we have

$$2\pi r \frac{dG}{dr} + 2\pi k^2 \int_0^r G\bar{r} \, d\bar{r} = 1 \ .$$

Since rG is integrable, we take the limit as $r \to 0$ and obtain the simpler expression for the singularity condition

$$\lim_{r \to 0} 2\pi r \frac{dG}{dr} = 1 \ .$$

We use the asymptotic expansions for Bessel functions for small arguments, (7.7.33). In this manner

$$G = c_1[1 - (kr/2)^2 + ...] + c_2[2/\pi \log(kr) + ...]$$

and hence

$$\frac{dG}{dr} = 2c_2/\pi r + ...$$

Thus, $c_2 = 1/4$ to satisfy the singularity condition. By also using (7), we obtain

$$G = \frac{1}{4}[Y_0(kr) - iJ_0(kr)] \ .$$

9.5.13 (a) To satisfy the boundary condition that $\partial G/\partial y = 0$ at $y = 0$, we introduce a positive image source at $(x_0, -y_0, z_0)$. The Green's function for this half-space is the superposition of the two infinite space Green's functions

$$G(\vec{x}, \vec{x}_0) = -\frac{1}{4\pi}\left(\left[(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2\right]^{-1/2} + \left[(x - x))^2 + (y + y_0)^2 + (z - z_0)^2\right]^{-1/2}\right) \ .$$

It can be verified that $\partial G/\partial y = 0$ at $y = 0$.

9.5.13 (b) From the three-dimensional version of Green's formula

$$u(\vec{x}_0) = \iiint f(\vec{x})G(\vec{x}, \vec{x}_0)dV - \oiint G\nabla u \cdot \hat{n}\, dS ,$$

since $\partial G/\partial y = 0$ on $y = 0$. We note that $\hat{n} = -\hat{j}$. Thus, after switching $\vec{x}$ and $\vec{x}_0$ and using the symmetry of G

$$u(\vec{x}) = \iiint f(\vec{x}_0)G(\vec{x}, \vec{x}_0)dV_0 + \iint G(\vec{x}, \vec{x}_0)\bigg|_{y_0=0} \frac{\partial u}{\partial y_0}\bigg|_{y=0} dx_0 dz_0 ,$$

where $\partial u/\partial y_0|_{y_0=0} = h(x_0, z_0)$ and

$$G(\vec{x}, \vec{x}_0)|_{y_0=0} = -\frac{1}{2\pi}[(x-x_0)^2 + y^2 + (z-z_0)^2]^{-1/2} .$$

9.5.14 To satisfy the boundary conditions that $G = 0$ at $x = 0$, we use the method of images. We introduce three image sources, negative ones at $(-x_0, y_0)$ and $(x_0, -y_0)$ and a positive one at $(-x_0, -y_0)$. The Green's function in the first quadrant is the superposition of the four infinite-space two-dimensional Green's functions each satisfying (9.5.31a). By using properties of logarithms, we obtain

$$G(\vec{x}, \vec{x}_0) = \frac{1}{4\pi}\left\{ ln\frac{[(x-x_0)^2 + (y-y_0)^2][(x+x_0)^2 + (y+y_0)^2]}{[(x-x_0)^2 + (y+y_0)^2][(x+x_0)^2 + (y-y_0)^2]} \right\} .$$

It is easy to verify that $G = 0$ along $x = 0$ and along $y = 0$.

9.5.19 The Green's function inside a circle of radius a is given by (9.5.57), where ϕ is the angle between $\vec{x}$ and $\vec{x}_0$. We use polar coordinates with $\vec{x}$ being represented by (r, θ) and $\vec{x}_0$ by (r_0, θ_0). Thus $\phi = \theta - \theta_0$. For a semi-circle with $G = 0$ on the boundaries, image sources may be introduced. There are negative ones located at $(r_0, -\theta_0)$ and the corresponding point outside of the circle. In this case $\phi = \theta + \theta_0$. Thus

$$G(\vec{x}, \vec{x}_0) = \frac{1}{4\pi} ln\left[a^2 \frac{r^2 + r_0^2 - 2rr_0\cos(\theta - \theta_0)}{r^2 r_0^2 + a^4 - 2rr_0 a^2 \cos(\theta - \theta_0)}\right]$$
$$-\frac{1}{4\pi} ln\left[a^2 \frac{r^2 + r_0^2 - 2rr_0\cos(\theta + \theta_0)}{r^2 r_0^2 + a^4 - 2rr_0 a^2 \cos(\theta + \theta_0)}\right] .$$

At $y = 0$, $\theta = 0$ or $\theta = \pi$ in which case the above formula yields $G = 0$ as desired.

9.5.22 (c) In three dimensions the response to a Dirac delta function is $G = -1/4\pi\rho$, where $\rho = |\vec{x} - \vec{x}_0|$ is the distance from the source and where $\vec{x}_0 = (x_0, y_0, z_0)$. To satisfy $G = 0$ at $x = 0$ and at $x = L$, image sources must be introduced. To satisfy $G = 0$ at $x = 0$, a negative source must be introduced at $(-x_0, y_0, z_0)$. Note: to satisfy $G = 0$ at $x = L$, a negative image source at $(-x_0 + 2L, y_0, z_0)$ and a positive image source at $(x_0 + 2L, y_0, z_0)$ exists. This process must continue indefinitely. There will be positive sources at $x_0 + 2Ln$ and negative sources at $-x_0 + 2Ln$, where n is an arbitrary integer (positive or negative). Thus

$$G(\vec{x}, \vec{x}_0) = -\frac{1}{4\pi}\sum_{n=-\infty}^{\infty}\left(\frac{1}{|\vec{x} - \vec{\alpha}_n|} - \frac{1}{|\vec{x} - \vec{\beta}_n|}\right) ,$$

where $\vec{\alpha}_n = (x_0 + 2Ln, y_0, z_0)$ and $\vec{\beta}_n = (-x_0 + 2Ln, y_0, z_0)$.

Chapter 10. Fourier Transform

Section 10.2

10.2.1 $u = \int_0^\infty c(\omega)e^{-i\omega x}e^{-k\omega^2 t}d\omega + \int_{-\infty}^0 c(\omega)e^{-i\omega x}e^{-k\omega^2 t}d\omega$. In the latter integral, let $\omega' = -\omega$. Then $u = \int_0^\infty c(\omega)e^{-i\omega x}e^{-k\omega^2 t}d\omega + \int_0^\infty c(-\omega')e^{i\omega' x}e^{-k\omega'^2 t}d\omega'$. Since ω' is a "dummy" variable, it may be called ω. Now Euler's formula yields

$$u = \int_0^\infty [c(\omega) + c(-\omega)]\cos\omega x e^{-k\omega^2 t}d\omega + i\int_0^\infty [-c(\omega) + c(-\omega)]\sin\omega x e^{-k\omega^2 t}d\omega.$$

Comparing this with (10.2.9) yields (for $\omega > 0$)
$A(\omega) = c(\omega) + c(-\omega)$ and $B(\omega) = i[-c(\omega) + c(-\omega)]$.
By adding and subtracting these formulas (for $\omega > 0$),
$c(\omega) = \frac{1}{2}[A(\omega) + iB(\omega)]$ and $c(-\omega) = \frac{1}{2}[A(\omega) - iB(\omega)]$.
If A and B are real, then
$\bar{c}(\omega) = \frac{1}{2}[A(\omega) - iB(\omega)] = c(-\omega)$.

Section 10.3

10.3.6 The Fourier transform is obtained from (10.3.6) with $\gamma = 1$:

$$F(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(x)e^{i\omega x}dx = \frac{1}{2\pi}\int_{-a}^{a} e^{i\omega x}dx = \left.\frac{e^{i\omega x}}{2\pi i\omega}\right|_{-a}^{a} .$$

Thus, $F(\omega) = \frac{1}{2\pi i\omega}(e^{i\omega a} - e^{-i\omega a}) = \frac{1}{\pi\omega}\sin\omega a$.

10.3.7 The inverse Fourier transform is obtained from (10.3.7) with $\gamma = 1$:

$$f(x) = \int_{-\infty}^{\infty} F(\omega)e^{-i\omega x}d\omega = \int_{-\infty}^{\infty} e^{-|\omega|\alpha}e^{-i\omega x}d\omega$$

$$= \int_0^\infty e^{-\omega\alpha}e^{-i\omega x}d\omega + \int_{-\infty}^0 e^{\omega\alpha}e^{-i\omega x}d\omega .$$

Thus

$$f(x) = \left.\frac{e^{\omega(-\alpha-ix)}}{-\alpha-ix}\right|_0^\infty + \left.\frac{e^{\omega(\alpha-ix)}}{\alpha-ix}\right|_{-\infty}^0 = \frac{1}{\alpha+ix} + \frac{1}{\alpha-ix} = \frac{2\alpha}{\alpha^2+x^2} .$$

10.3.10 (a) By separation of variable ($u = h(t)\phi(r)$)

$$\frac{1}{kh}\frac{dh}{dt} = \frac{1}{r\phi}\frac{d}{dr}(r\frac{d\phi}{dr}) = -s^2 .$$

Thus $r^2\frac{d^2\phi}{dr^2} + r\frac{d\phi}{dr} + s^2r^2\phi = 0$, which is Bessel's differential equation of order 0. Thus

$$\phi = c_1 J_0(sr) + c_2 Y_0(sr) .$$

Since u is bounded at $r = 0, c_2 = 0$. By the superposition principle (see p. 447),

$$u(r,t) = \int_0^\infty B(s)J_0(sr)e^{-s^2kt}ds .$$

It is convenient (due to the weight function for Bessel functions) to let $B(s) = sA(s)$.

10.3.10 (b) One or two-dimensional Green's formula may be used. In one-dimension

$$\int_a^b [uL(v) - vL(u)]dr = p(u\frac{dv}{dr} - v\frac{du}{dr})\bigg|_a^b ,$$

where $L = \frac{d}{dr}(r\frac{d}{dr})$, so that $p = r$ and $a = 0$ and $b = L$. We note that

$$L[J_0(sr)] = -s^2 r J_0(sr) \quad \text{and} \quad L[J_0(s_1 r)] = -s_1^2 r J_0(s_1 r) .$$

Thus, letting $u = J_0(sr)$ and $v = J_0(s_1 r)$ in Green's formula yields

$$(s^2 - s_1^2) \int_0^L J_0(sr) J_0(s_1 r) r\, dr = L \left(u\frac{dv}{dr} - v\frac{du}{dr} \right)\bigg|_L .$$

For large L, the right-hand side may be approximated using (7.8.3):

$$u = J_0(sr) \sim \sqrt{\frac{2}{\pi sr}} \cos(sr - \pi/4) \quad \text{and thus} \quad \frac{du}{dr} \sim -s\sqrt{\frac{2}{\pi sr}} \sin(sr - \pi/4)$$

and a similar expression for v (with s replaced by s_1). Thus

$$(s^2 - s_1^2) \int_0^L J_0(sr) J_0(s_1 r) r dr \sim L\frac{2}{\pi L} \left[\sqrt{\frac{s}{s_1}} \sin(sL - \frac{\pi}{4}) \cos(s_1 L - \frac{\pi}{4}) - \sqrt{\frac{s_1}{s}} \sin(s_1 L - \frac{\pi}{4}) \cos(sL - \frac{\pi}{4}) \right] ,$$

so that $\int_0^L J_0(sr) J_0(s_1 r) r dr$

$$\sim \frac{2}{\pi} \frac{\sqrt{\frac{s}{s_1}} \sin(sL - \frac{\pi}{4}) \cos(s_1 L - \frac{\pi}{4}) - \sqrt{\frac{s_1}{s}} \sin(s_1 L - \frac{\pi}{4}) \cos(sL - \frac{\pi}{4})}{s^2 - s_1^2} .$$

10.3.10 (c) In order to use the results of part (b), we multiply equation on bottom of page 456 by $J_0(s_1 r) r$ and integrate from 0 to L:

$$\int_0^L f(r) J_0(s_1 r) r dr = \int_0^\infty A(s) \int_0^L J_0(sr) J_0(s_1 r) r dr\, s ds .$$

In the above expressions, we note that $A(s) = A(s_1) + A(s) - A(s_1)$. The contribution to the integral from $A(s) - A(s_1)$ vanishes as $L \to \infty$ by the Riemann-Lebesgue lemma (see first edition of this text) since $[A(s) - A(s_1)]/(s^2 - s_1^2)$ is continuous. This result can be shown as in exercise 10.3.9 (b) by integration-by-parts. Thus

$$\int_0^\infty f(r) J_0(s_1 r) r dr = \frac{2}{\pi} A(s_1) \lim_{L\to\infty} \int_0^\infty \frac{\sqrt{\frac{s}{s_1}} \sin(sL - \frac{\pi}{4}) \cos(s_1 L - \frac{\pi}{4}) - \sqrt{\frac{s_1}{s}} \sin(s_1 L - \frac{\pi}{4}) \cos(sL - \frac{\pi}{4})}{s^2 - s_1^2} s ds .$$

We also note that $\sqrt{\frac{s}{s_1}} = 1 + \sqrt{\frac{s}{s_1}} - 1$ and $\sqrt{\frac{s_1}{s}} = 1 + \sqrt{\frac{s_1}{s}} - 1$, where again terms $\sqrt{\frac{s}{s_1}} - 1$ and $\sqrt{\frac{s_1}{s}} - 1$, do not contribute due to the Riemann-Lebesgue lemma. Similarly, $s^2 - s_1^2 = (s - s_1)(s + s_1) = 2s_1(s - s_1) + (s - s_1)^2$, where the latter term will not contribute as $L \to \infty$, as well as $s = s_1 + s - s_1$. Thus,

$$\int_0^\infty f(r) J_0(s_1 r) r dr = \frac{2}{\pi} A(s_1) \lim_{L\to\infty} \int_0^\infty \frac{\sin(sL - \frac{\pi}{4}) \cos(s_1 L - \frac{\pi}{4}) - \sin(s_1 L - \frac{\pi}{4}) \cos(sL - \frac{\pi}{4})}{2s_1(s - s_1)} s_1 ds.$$

Using the trigonometric addition formulas, $\int_0^\infty f(r) J_0(s_1 r) r dr = \frac{1}{\pi} A(s_1) \lim_{L\to\infty} \int_0^\infty \frac{\sin L\,(s - s_1)}{s - s_1} ds$. The change of variables $w = L(s - s_1)$ shows that

$$\int_0^\infty f(r) J_0(s_1 r) r dr = \frac{A(s_1)}{\pi} \int_{-\infty}^\infty \frac{\sin w}{w} dw = A(s_1)$$

for $s_1 > 0$, using the sine integral (see exercise 10.3.9 (b) again). This gives a symmetric inversion formula

$$A(s) = \int_0^\infty f(r)J_0(sr)rdr \ .$$

10.3.16 Let $t = ky^n$, so that $dt = kny^{n-1}dy$. Thus

$$\int_0^\infty y^p e^{-ky^n} dy = \frac{1}{kn}\int_0^\infty y^p e^{-t}\frac{dt}{y^{n-1}} = \frac{1}{kn}\int_0^\infty (\frac{t}{k})^{\frac{1+p-n}{n}} e^{-t}dt \ .$$

Using the definition of the gamma function in exercise 10.3.14,

$$\int_0^\infty y^p e^{-ky^n} dy = \frac{1}{n}k^{-(1+p)/n}\Gamma(x) \ ,$$

where $x - 1 = \frac{1+p}{n} - 1$ or $x = (1+p)/n$.

Section 10.4

10.4.3 (a) Taking the Fourier transform yields

$$\frac{\partial \bar{u}}{\partial t} = -k\omega^2\bar{u} - i\omega c\bar{u} \ .$$

The solution of this initial value problem is

$$\bar{u}(\omega, t) = F(\omega)\underbrace{e^{-k\omega^2 t}e^{-i\omega ct}}_{G(\omega)} \ .$$

Inverting using the convolution theorem yields

$$u(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(\bar{x})g(x-\bar{x})d\bar{x} \ ,$$

where the shift theorem determines $g(x), g(x) = \sqrt{\frac{\pi}{kt}}e^{-(x+ct)^2/4kt}$. Thus

$$u(x,t) = \frac{1}{\sqrt{4\pi kt}}\int_{-\infty}^{\infty} f(\bar{x})e^{-(x-\bar{x}+ct)^2/4kt}d\bar{x} \ .$$

10.4.5 (a) Taking the Fourier transform yields

$$\frac{\partial \bar{u}}{\partial t} = -k\omega^2\bar{u} + \bar{Q} \ .$$

A particular solution is obtained using the usual integrating factor.

10.4.5 (b) The general solution is a particular solution plus an arbitrary multiple of the homogeneous solution

$$\bar{u} = ce^{-k\omega^2 t} + e^{-k\omega^2 t}\int_0^t \bar{Q}(\omega,\tau)e^{k\omega^2\tau}d\tau \ .$$

From the initial conditions, $c = \bar{u}(\omega, 0) = F(\omega)$, the Fourier transform of the initial conditions.

10.4.5 (c) Using the convolution theorem,

$$u(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(\bar{x})\sqrt{\frac{\pi}{kt}}e^{-(x-\bar{x})^2/4kt}d\bar{x} + \frac{1}{2\pi}\int_0^t\int_{-\infty}^{\infty} Q(\bar{x},\tau)\sqrt{\frac{\pi}{k(\tau-t)}}e^{-(x-\bar{x})^2/4k(\tau-t)}d\bar{x}d\tau \ .$$

10.4.6 Taking the Fourier transform yields

$$-\omega^2 Y + idY/d\omega = 0 \ ,$$

since $\mathcal{F}[xf(x)] = -idF/d\omega$. the general solution of this first-order differential equation is

$$Y(\omega) = ce^{-i\omega^3/3} \ .$$

By using the formula for the inverse Fourier transform, we obtain

$$y(x) = c\int_{-\infty}^{\infty} e^{-i\omega^3/3} e^{-i\omega x} d\omega = 2c\int_0^{\infty} \cos(\frac{\omega^3}{3} + \omega x) d\omega \ ,$$

where Euler's formula and symmetry has been used. The constant c can be determined from the initial conditon

$$3^{-2/3} \Big/ \Gamma(\frac{2}{3}) = y(0) = 2c\int_0^{\infty} \cos(\frac{\omega^3}{3}) d\omega = 2c\frac{\sqrt{3}}{2} 3^{-2/3}\Gamma(\frac{1}{3}) \ ,$$

from exercise 10.3.17. Thus,

$$c = \frac{1}{\sqrt{3}\,\Gamma(\frac{1}{3})\Gamma(\frac{2}{3})} = \frac{\sin \pi/3}{\pi\sqrt{3}} = \frac{1}{2\pi}$$

from exercise 10.3.15. In this manner

$$y(x) \equiv A_i(x) = \frac{1}{\pi}\int_0^{\infty} \cos(\frac{\omega^3}{3} + \omega x) d\omega \ .$$

10.4.7 (a) By taking the Fourier transform of the pde, we obtain

$$\frac{\partial \bar{u}}{\partial t} = k(-i\omega)^3 \bar{u} \ .$$

The solution of this ordinary differential equation is

$$\bar{u}(\omega, t) = c(\omega) e^{ik\omega^3 t} \ ,$$

where $c(\omega)$ is determined from the initial conditions, $\bar{u}(\omega, 0) = F(\omega) = c(\omega)$. Thus

$$u(x,t) = \int_{-\infty}^{\infty} F(\omega) e^{ik\omega^3 t} e^{-i\omega x} d\omega \ .$$

10.4.7 (b) If $g(x) = \int_{-\infty}^{\infty} e^{ik\omega^3 t} e^{-i\omega x} d\omega$, then by the convolution theorem

$$u(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(\bar{x}) g(x - \bar{x}) d\bar{x} \ .$$

10.4.7 (c) Using symmetry and the change of variables $k\omega^3 t = s^3/3$ [or $\omega = s/(3kt)^{1/3}$] yields

$$g(x) = \int_{-\infty}^{\infty} \cos(k\omega^3 t - \omega x) d\omega = \frac{2}{(3kt)^{1/3}}\int_0^{\infty} \cos\left(\frac{s^3}{3} - \frac{s}{(3kt)^{1/3}} x\right) ds \ .$$

By using exercise 10.4.6, we obtain $g(x) = \frac{2\pi}{(3kt)^{1/3}} A_i\left[\frac{-x}{(3kt)^{1/3}}\right]$, so that

$$u(x,t) = \frac{1}{(3kt)^{1/3}}\int_{-\infty}^{\infty} f(\bar{x}) A_i\left[\frac{\bar{x} - x}{(3kt)^{1/3}}\right] d\bar{x} \ .$$

Section 10.5

10.5.3 Only the Fourier cosine transform of $e^{-\alpha x^2}$ has a simple expression:

$$C[e^{-\alpha x^2}] = \frac{2}{\pi}\int_0^\infty e^{-\alpha x^2}\cos\omega x dx = \frac{1}{\pi}\int_{-\infty}^\infty e^{-\alpha x^2} e^{i\omega x} dx \ ,$$

using symmetry. Thus

$$C[e^{-\alpha x^2}] = 2\,\mathcal{F}[e^{-\alpha x^2}] = \frac{1}{\sqrt{\pi\alpha}} e^{-\omega^2/4\alpha} \ .$$

10.5.10 We use the inverse cosine transform of $e^{-\omega\alpha}$. From exercise 10.5.1 (see the table of Fourier cosine transforms), $\frac{\alpha}{x^2+\alpha^2} = \int_0^\infty e^{-\omega\alpha}\cos\omega x d\omega$. Differentiating this with respect to α yields

$$\frac{\partial}{\partial\alpha}\left[\frac{\alpha}{x^2+\alpha^2}\right] = \int_0^\infty (-\omega)e^{-\omega\alpha}\cos\omega x d\omega \ .$$

Thus

$$C^{-1}\left[\omega e^{-\omega\alpha}\right] = -\frac{\partial}{\partial\alpha}\left[\frac{\alpha}{x^2+\alpha^2}\right] = \frac{2\alpha^2-(x^2+\alpha^2)}{(x^2+\alpha^2)^2} \ .$$

10.5.11 (a) We take the Fourier sine transform and obtain

$$\frac{\partial\bar{u}}{\partial t} = k(\frac{2}{\pi}\omega - \omega^2\bar{u}) \ ,$$

where $\bar{u}$ is the sine transform and where (10.5.27) has been used. The general solution of this ordinary differential equation is

$$\bar{u}(\omega,t) = \frac{2}{\pi\omega} + A(\omega)e^{-k\omega^2 t} \ .$$

By using the initial condition, $\bar{u}(\omega,0) = F(\omega) = A(\omega) + \frac{2}{\pi\omega}$,

$$\bar{u}(\omega,t) = \frac{2}{\pi\omega} + \left[F(\omega) - \frac{2}{\pi\omega}\right] e^{-k\omega^2 t} \ .$$

To invert this, we note from the tables that $S[1] = 2/\pi\omega$. Furthermore, the remaining term is the form of a product, one the sine transform of a known function and the other the cosine transform of a known function, $C[e^{-\alpha x^2}] = e^{-\omega^2/4\alpha}/\sqrt{\pi\alpha}$, where we will let $4\alpha = 1/kt$ or $\alpha = 1/4kt$. Thus by the convolution theorem for sine transforms [see the table or exercise (10.5.6)]

$$u(x,t) = 1 + \frac{1}{\pi}\int_0^\infty [f(\bar{x})-1]\frac{1}{2}\left[\sqrt{\frac{\pi}{kt}}e^{-(x-\bar{x})^2/4kt} - \sqrt{\frac{\pi}{kt}}e^{-(x+\bar{x})^2/4kt}\right] dx.$$

10.5.11 (b) By letting $v(x,t) = u(x,t)$, we obtain $\frac{\partial v}{\partial t} = k\frac{\partial^2 v}{\partial x^2}$ subject to $v(0,t) = 0$ and $v(x,0) = f(x) - 1$. This problem has been solved in Section 10.5. From (10.5.39)

$$v(x,t) = \frac{1}{\sqrt{4\pi kt}}\int_0^\infty [f(\bar{x})-1][e^{-(x-\bar{x})^2/4kt} - e^{-(x+\bar{x})^2/4kt}]d\bar{x} \ .$$

10.5.11 (c) The answers are the same.

10.5.17 (a) By taking the sine transform of the pde, we obtain

$$\frac{\partial\bar{u}}{\partial t} = k\left[\frac{2}{\pi}\omega A e^{i\sigma_0 t} - \omega^2\bar{u}\right] \ ,$$

where (10.5.27) has been used and where $\bar{u}(\omega,t)$ is the sine transform of $u(x,t)$. Homogeneous solution of this equation is $e^{-k\omega^2 t}$, while a particular solution is obtained by the method of undetermined coefficients (judiciuous guessing). In this manner

$$\bar{u}(\omega,t) = c(\omega)e^{-k\omega^2 t} + \frac{\frac{2}{\pi}\omega A}{\omega^2 + \frac{i\sigma_0}{k}} e^{i\sigma_0 t} .$$

The initial condition implies $\bar{u}(\omega,0) = 0 = c(\omega) + \frac{\frac{2}{\pi}\omega A}{\omega^2+\frac{i\sigma_0}{k}}$.

10.5.17 (b) As $t \to \infty, e^{-k\omega^2 t} \to 0$. Thus for large t $\bar{u}(\omega,t) \sim \frac{\frac{2}{\pi}\omega A}{\omega^2+\frac{i\sigma_0}{k}} e^{i\sigma_0 t}$.

Section 10.6

10.6.1 (a) By taking the Fourier transform in x, we obtain

$$-\omega^2 \bar{u} + \frac{d^2\bar{u}}{dy^2} = 0.$$

The boundary conditions for this differential equation are $\bar{u}(\omega,0) = F_1(\omega)$ and $\bar{u}(\omega,H) = F_2(\omega)$. The solution of this equation, solving these boundary conditions, is

$$\bar{u}(\omega,y) = F_2(\omega)\frac{\sinh \omega y}{\sinh \omega H} + F_1(\omega)\frac{\sinh \omega(H-y)}{\sinh \omega H} .$$

The desired solution $u(x,y)$ is the inverse Fourier transform obtained by the defining integral, (10.3.7).

10.6.2 (b) The boundary conditions suggest the use of a cosine transform in y. By applying this transform, Laplace's equation becomes

$$\frac{d^2\bar{u}}{dx^2} - \omega^2 \bar{u} = 0 ,$$

where $\bar{u}(x,\omega)$ is the cosine transform in y of $u(x,y)$ and where (10.5.26) has been used, simplified by $\partial u/\partial y(x,0) = 0$. The solution of the differential equation is a linear combination of hyperbolic (or exponential) functions. Since $\partial u/\partial x(L,y) = 0, \partial\bar{u}/\partial x(L,\omega) = 0$. Thus

$$\bar{u}(x,\omega) = A(\omega)\cosh \omega(L-x) ,$$

where the other boundary condition $u(0,y) = g_1(y)$ becomes $\bar{u}(0,\omega) = G_1(\omega)$, the cosine transform of $g_1(y)$. Thus

$$G_1(\omega) = A(\omega)\cosh \omega L ,$$

determining $A(\omega)$. The solution $u(x,y)$ is the inverse cosine transform, determined by the defining integral, of $\bar{u}(x,\omega)$.

10.6.4 (a) By taking the sine transform, in x, we obtain

$$-\omega^2\bar{u} + d^2\bar{u}/dy^2 = 0 ,$$

where $\bar{u}(\omega,y)$ is the sine transform in x of $u(x,y)$. The general solution of this differential equation is

$$\bar{u}(\omega,y) = a(\omega)e^{-\omega y} + b(\omega)e^{\omega y} ,$$

where only $\omega > 0$ is needed. Since we assume $u \to 0$ as $y \to \infty$, it follows that $\bar{u}(\omega,y) \to 0$ as $y \to \infty$. Thus $b(\omega) = 0$, so that

$$\bar{u}(\omega,y) = a(\omega)e^{-\omega y} .$$

The nonhomogeneous boundary condition $\partial u/\partial y(x,0) = f(x)$, becomes $\partial\bar{u}/\partial y(\omega,0) = F(\omega)$ when the sine transform is introduced, where $F(\omega)$ is the sine transform of $f(x), F(\omega) = S[f(x)]$. Thus

$$-\omega a(\omega) = S[f(x)],$$

so that

$$\bar{u}(\omega, y) = -\frac{1}{\omega} S[f(x)] e^{-\omega y} \ .$$

The apparent singularity at $\omega = 0$ may cause some difficulty in calculating the inverse transform. The simplest way to determine $u(x, y)$ is to first eliminate the singularity. Note that

$$\frac{\partial}{\partial y} \bar{u}(\omega, y) = S[f(x)] e^{-\omega y} \ .$$

The convolution theorem for sine transforms may be used since

$$h(x) = C^{-1}[e^{-\omega y}] = \frac{y}{x^2 + y^2}$$

from the tables or exercise 10.5.1. Thus

$$\frac{\partial u}{\partial y}(x, y) = \frac{1}{\pi} \int_0^\infty f(\bar{x}) \left[\frac{y}{(x - \bar{x})^2 + y^2} - \frac{y}{(x + \bar{x})^2 + y^2} \right] d\bar{x} \ .$$

By integrating this with respect to y, we obtain

$$u(x, y) = \frac{1}{2\pi} \int_0^\infty f(\bar{x}) ln \frac{(x - \bar{x})^2 + y^2}{(x + \bar{x})^2 + y^2} d\bar{x} + Q(x) \ ,$$

where $Q(x)$ is an arbitrary function of x. Since $u \to 0$ as $y \to \infty$, we obtain $Q(x) = 0$.

10.6.11 (a) By taking the sine transform in x and the sine transform in y, we obtain

$$\frac{\partial \bar{u}}{\partial t} = -k\,(\omega_1^2 + \omega_2^2) \bar{u} \ ,$$

where $\bar{u}(\omega_1, \omega_2, t)$ is the appropriate "double" transform of $u(x, y, t)$. We introduce $\vec{\omega} \equiv (\omega_1, \omega_2)$ and $\omega \equiv |\vec{\omega}| = \sqrt{\omega_1^2 + \omega_2^2}$. Then

$$\bar{u}(\vec{\omega}) = c(\vec{\omega}) e^{-k\omega^2 t} \ ,$$

where $c(\vec{\omega}) = F(\vec{\omega})$, the double transform of the initial conditions. Thus

$$u(x, y, t) = \int_0^\infty \int_0^\infty F(\vec{\omega}) e^{-k\omega^2 t} \sin \omega_1 x \ \sin \omega_2 y \, d\omega_1 \, d\omega_2 \ ,$$

where $F(\vec{\omega}) = \frac{4}{\pi^2} \int_0^\infty \int_0^\infty f(x, y) \sin \omega_1 x \sin \omega_2 y \, dx \, dy$. We note that $F(\vec{\omega})$ is odd in ω_1 and odd in ω_2. Thus the integrand in the double integral for $u(x, y, t)$ is even in both ω_1 and ω_2. Furthermore we may extend $f(x, y)$ as an odd function of x and y. If we replace $\sin \omega_1 x$ by $e^{\pm i\omega_1 x} / \pm i$ in both double integrals, no error is made since the cosine term makes no contribution due to its symmetry. Similarly $\sin \omega_2 x$ may be replaced by $e^{\pm i\omega_2 x} / \pm i$. Thus

$$u(x, y, t) = \int_{-\infty}^\infty \int_{-\infty}^\infty -F(\vec{\omega}) e^{-k\omega^2 t} e^{-i\vec{\omega} \cdot \vec{r}} d\omega_1 d\omega_2$$

and

$$-F(\omega) = \int_{-\infty}^\infty \int_{-\infty}^\infty f(x, y) e^{i\vec{\omega} \cdot \vec{r}} dx \, dy \ .$$

These are related to the formulas for double Fourier transforms, where the convolution theorem may be used so that $u(x, y, t)$ satisfies (10.6.95). However, this answer depends on $f(x_0, y_0)$ for $-\infty < x_0 < \infty$ and $-\infty < y_0 < \infty$ while it is only naturally given in the first quadrant. We note that we have extended $f(x, y)$ as an odd function in both x and y. If we add up contributions from all four quadrants, we obtain the answer given.

10.6.12 (a) If we introduce first a sine transform in y, then

$$\frac{\partial \bar{u}}{\partial t} = k\left(\frac{\partial^2 \bar{u}}{\partial x^2} - \omega^2 \bar{u}\right)$$

with $\bar{u}(0,\omega,t) = \bar{u}(L,\omega,t) = 0$ and the initial conditions $\bar{u}(x,\omega,0) = F(x,\omega)$. Here $\bar{u}(x,\omega,t)$ is the sine transform in y of $u(x,y,t)$. The pde for $\bar{u}(x,\omega,t)$ is suited for the usual Fourier series in x

$$\bar{u}(x,\omega,t) = \sum_{n=1}^{\infty} A_n(\omega,t)\sin\frac{n\pi x}{L}\ , \qquad (8)$$

where the initial conditions yield

$$F(x,\omega) = \sum_{n=1}^{\infty} A_n(\omega,0)\sin\frac{n\pi x}{L} \quad \text{or} \quad A_n(\omega,0) = \frac{2}{L}\int_0^L F(x,\omega)\sin\frac{n\pi x}{L}dx.$$

By substituting (8) into the pde, we obtain the differential equation satisfied by $A_n(\omega,t)$:

$$dA_n/dt = -k[(n\pi/L)^2 + \omega^2]A_n\ .$$

Thus

$$A_n(\omega,t) = A_n(\omega,0)e^{-k[(n\pi/L)^2+\omega^2]t}\ .$$

By inversion of the sine transform

$$u(x,y,t) = \int_0^{\infty}\left(\sum_{n=1}^{\infty} A_n(\omega,t)\sin\frac{n\pi x}{L}\right)\sin\omega y\, d\omega\ ,$$

where $A_n(\omega,t)$ is given above with

$$A_n(\omega,0) = \frac{4}{L\pi}\int_0^L\int_0^{\infty} f(x,y)\sin\omega y\,\sin\frac{n\pi x}{L}dy\,dx\ ,$$

since $F(x,\omega)$ is the sine transform in y of $f(x,y)$.

10.6.15 (a) We introduce $\bar{u}(\vec{\omega},z)$ the double Fourier transform in x and y of $u(x,y,z)$:

$$-\omega^2\bar{u} + \frac{d^2\bar{u}}{dz^2} = 0\ ,$$

where $\omega = \sqrt{\omega_1^2+\omega_2^2} > 0$. The general solution of this equation is

$$\bar{u}(\vec{\omega},z) = A(\vec{\omega})e^{-\omega z} + B(\vec{\omega})e^{\omega z}\ .$$

Since $u \to 0$ as $z \to \infty$, if follows that $\bar{u} \to 0$ as $z \to \infty$, and hence $B(\vec{\omega}) = 0$. Furthermore, the boundary condition at $z = 0$ is that $\bar{u}$ equals $F(\vec{\omega})$ there, the double Fourier transform of $f(x,y)$:

$$\bar{u}(\vec{\omega},z) = F(\vec{\omega})e^{-\omega z}\ .$$

10.6.15 (b) By using the convolution theorem for double Fourier transforms

$$u(x,y,z) = \frac{1}{(2\pi)^2}\iint f(x,y)g(\vec{r}-\vec{r}_0)dx_0dy_0\ ,$$

where the double Fourier transform of $g(\vec{r})$ is known to equal $e^{-\omega z}$:

$$\mathcal{F}[g(\vec{r})] = e^{-\omega z}\ .$$

Determining $g(\vec{r})$ is not easy. From the general inversion formula

$$g(\vec{r}) = \iint e^{-\omega z} e^{-i\vec{\omega}\cdot\vec{r}} d\omega_1 d\omega_2 = \int_0^{2\pi} \int_0^{\infty} e^{-\omega z} e^{-i\omega r \cos(\phi-\theta)} \omega \, d\omega \, d\phi \ ,$$

where the latter integral is obtained using polar coordiantes. If polar coordinates were chosen oriented in the direction of $\vec{r}$ (which is θ) then $\vec{\omega} \cdot \vec{r} = \omega r \cos \phi$, which is equivalent to replacing θ by 0 in the double integral. The ω -integral (and what follows) is easier by integrating from ∞ to z:

$$\int_{\infty}^{z} g(\vec{r}) dz = -\int_0^{2\pi} \int_0^{\infty} e^{-\omega z} e^{-i\omega r \cos \phi} d\omega d\phi \ ,$$

since $\int_{\infty}^{z} e^{-\omega z} \omega dz = -e^{-\omega z} \big|_{\infty}^{z} = -e^{-\omega z}$. The ω integral is now easy

$$\int_{\infty}^{z} g(\vec{r}) dz = -\int_0^{2\pi} \frac{e^{-\omega(z+ir\cos\phi)}}{-(z+ir\cos\phi)} \bigg|_0^{\infty} d\phi = -\int_0^{2\pi} \frac{d\phi}{z+ir\cos\phi} \ .$$

Thus

$$\int_{\infty}^{z} g(\vec{r}) dz = -\int_0^{2\pi} \frac{z - ir\cos\phi}{z^2 + r^2\cos^2\phi} d\phi = -z\int_0^{2\pi} \frac{d\phi}{z^2+r^2\cos^2\phi} \ ,$$

since the $\cos\phi$ integral is zero by symmetry. The resulting integral can be put in the given form

$$\int_{\infty}^{z} g(\vec{r}) dz = -z\int_0^{2\pi} \frac{d\phi}{(z^2+r^2)\cos^2\phi + z^2\sin^2\phi} = \frac{-2\pi z}{z(z^2+r^2)^{1/2}} \ ,$$

for $z > 0$. Differentiating this yields $g(\vec{r}) = 2\pi z(z^2+r^2)^{-3/2}$, so that

$$u(x,y,z) = \frac{z}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{f(x_0,y_0)}{[(x-x_0)^2+(y-y_0)^2+z^2]^{3/2}} dx_0 dy_0 \ .$$

10.6.16 (b) We want the radial problem to be an eigenvalue problem. By separation of variables of Laplace's equation in polar coordinates, $u(r,\theta) = \phi(r)h(\theta)$,

$$\frac{d^2h}{d\theta^2} = \lambda h \quad \text{and} \quad r\frac{d}{dr}(r\frac{d\phi}{dr}) + \lambda\phi = 0 \ ;$$

see (2.5.37) and (2.5.40). The eigenvalue problem consists of the radial equation, to be solved subject to $\phi(0)$ bounded and $\phi(a) = 0$. this is in the form of a singular Sturm-Liouville problem, singular at $r = 0$. In some sense, though, the eigenfunctions will be orthogonal with weight $1/r$. The radial differential equation is equidimensional

$$r^2\frac{d^2\phi}{dr^2} + r\frac{d\phi}{dr} + \lambda\phi = 0 \ ,$$

so that solutions are in the form $\phi = r^p$ with $p(p-1)+p+\lambda = 0$ or $p = \pm i\sqrt{\lambda}$. Since $r^{\pm i\sqrt{\lambda}} = e^{\pm i\sqrt{\lambda}\ln r}$, real solutions are linear combinations of $\sin(\sqrt{\lambda}\ln r)$ and $\cos(\sqrt{\lambda}\ln r)$. However, the boundary condition at $r = a$ is most easily satisfied using the following set of independent solutions:

$$\phi = c_1\cos(\sqrt{\lambda}\ln\frac{r}{a}) + c_2\sin(\sqrt{\lambda}\ln\frac{r}{a}) \ .$$

Here $\phi(a) = 0$ implies that $c_1 = 0$, so that

$$\phi(r) = c_2\sin(\sqrt{\lambda}\ln\frac{r}{a}) \ .$$

The other condition, $\phi(0)$ is bounded, is satisfied for all $\lambda > 0$. We thus use the generalized superposition principle of integrating over λ. We introduce $\omega = \sqrt{\lambda}$ and integrate over ω instead:

$$u_2(r,\theta) = \int_0^{\infty} A(\omega)\sinh\omega\theta\,\sin(\omega\ln\frac{r}{a})d\omega \ ,$$

where we have noted that the θ-dependent part which satisfies $h(0) = 0$ is $\sinh \omega\,\theta$. The nonhomogeneous boundary condition becomes

$$g_2(r) = \int_0^\infty A(\omega) \sinh \omega \frac{\pi}{2} \sin(\omega \ln \frac{r}{a}) d\omega \, .$$

In order to determine $A(\omega)$, it is convenient to consider the variable $\rho = -\ln(r/a)$. As $0 < r < a$, it follows that $0 < \rho < \infty$, a semi-infinite variable similar to the type that transforms might be used for:

$$g_2(r) = \int_0^\infty [-A(\omega) \sinh \omega \frac{\pi}{2}] \sin(\omega\rho) d\omega \, .$$

Thus $-A(\omega) \sinh \omega\,\pi/2$ is the Fourier transform in ρ of $g_2(r)$, and hence by definition of its transform

$$-A(\omega) \sinh \omega \frac{\pi}{2} = \frac{2}{\pi} \int_0^\infty g_2(r) \sin \omega\,\rho\, d\,\rho \, .$$

We may return to the physically meaningful variable r, by noting that $d\rho = -dr/r$ in which case

$$-A(\omega) \sinh \omega \frac{\pi}{2} = -\frac{2}{\pi} \int_0^a g_2(r) \sin[\omega \ln r/a] \frac{dr}{r} \, .$$

Note the appearance of the "weight" $1/r$.

10.6.18 By taking the Fourier transform in x, we obtain

$$\frac{\partial^2 \bar{u}}{\partial t^2} = -c^2 \omega^2 \bar{u} \, ,$$

where $\bar{u}(\omega, t)$ is the Fourier transform in x of $u(x, t)$. The general solution of this differential equation is

$$\bar{u}(\omega, t) = A(\omega) \cos c\omega t + B(\omega) \sin c\,\omega\, t \, .$$

By transforming the initial conditions, we obtain

$$\bar{u}(\omega, 0) = 0 \quad \text{and} \quad \frac{\partial}{\partial t}\bar{u}(\omega, 0) = G(\omega) \, ,$$

where $G(\omega)$ is the Fourier transform in x of $g(x)$. Hence $0 = A(\omega)$ and $G(\omega) = c\,\omega\, B(\omega)$. Thus

$$\bar{u}(\omega, t) = G(\omega) \frac{\sin c\,\omega\, t}{c\omega} = \frac{\pi}{c} G(\omega) F(\omega) \, .$$

We note that in the tables (or by exercise 10.3.6) that if

$$f(x) = \begin{cases} 0 & |x| > ct \\ 1 & |x| < ct \, , \end{cases}$$

then $\mathcal{F}[f(x)] = F(\omega) = \sin c\omega t / \pi\omega$. This enables us to use the convolution theorem for Fourier transforms:

$$u(x, t) = \frac{\pi}{c} [\frac{1}{2\pi} \int_{-\infty}^\infty g(\bar{x}) f(x - \bar{x}) d\bar{x}] \, ,$$

where

$$f(x - \bar{x}) = \begin{cases} 0 & |x - \bar{x}| > ct \\ 1 & |x - \bar{x}| < ct \, . \end{cases}$$

Thus

$$u(x, t) = \frac{1}{2c} \int_{x-ct}^{x+ct} g(\bar{x}) d\bar{x} \, .$$

Chapter 11. Green's Functions for Wave and Heat Eqns

Section 11.2

11.2.6 (a) From (11.2.24),

$$u(x,t) = \int_0^t \int_{-\infty}^{\infty} G(x,t;x_0,t_0)Q(x_0,t_0)dx_0dt_0 \ ,$$

where from (11.2.31-32) $G = \begin{cases} 0 & |x-x_0| > c(t-t_0) \\ \frac{1}{2c} & |x-x_0| < c(t-t_0) \ . \end{cases}$

Contributions only occur if $x - c(t-t_0) < x_0 < x + c(t-t_0)$. Thus

$$u(x,t) = \frac{1}{2c}\int_0^t \int_{x-c(t-t_0)}^{x+c(t-t_0)} Q(x_0,t_0)dx_0dt_0 \ .$$

11.2.6 (b) If $x = x_1, t = t_1$, then $x_1 - c(t_1-t_0) < x_0 < x_1 + c(t_1 - t_0)$, as sketched on page 746 for $0 < t_0 < t$.

11.2.7 (a) From exercise 11.2.6,

$$u(x,t) = \frac{1}{2c}\int_0^t \int_{x-c(t-t_0)}^{x+c(t-t_0)} Q(x_0,t_0)dx_0dt_0 \ .$$

If $Q(x,t) = g(x)e^{-i\omega t}$, then

$$u(x,t) = \frac{1}{2c}\int_0^t \int_{x-c(t-t_0)}^{x+c(t-t_0)} g(x_0)e^{-i\omega t_0}dx_0dt_0 \ .$$

In order to do the t_0-integration first, the order of integration must be interchanged, requiring an analysis of the triangular region (see page 746). If x_0 is first fixed, there appears to be two cases, $x_0 < x$ in which case t_0 ranges from 0 to $t - \frac{x-x_0}{c}$ and $x_0 > x$ in which case t_0 ranges from 0 to $t - \frac{x_0-x}{c}$. Both correspond to t_0 ranging from 0 to $t - \frac{|x-x_0|}{c}$. Thus

$$u(x,t) = \frac{1}{2c}\int_{x-ct}^{x+ct} g(x_0) \int_0^{t-\frac{|x-x_0|}{c}} e^{-i\omega t_0}dt_0dx_0 \ .$$

By performing the t_0-integration, we obtain

$$u(x,t) = \frac{1}{2c}\int_{x-ct}^{x+ct} g(x_0)[1-e^{-i\omega(t-\frac{|x-x_0|}{c})}]\frac{dx_0}{i\omega}, \quad \text{or}$$

$$u(x,t) = \int_{-\infty}^{\infty} g(x_0)I(x,x_0)dx_0 \ ,$$

where the influence function $I(x,x_0)$ satisfies

$$I(x,x_0) = \begin{cases} 0 & |x-x_0| > ct \\ \frac{1-e^{-i\omega(t-\frac{|x-x_0|}{c})}}{2ic\omega} & |x-x_0| < ct \ . \end{cases}$$

If $x > x_0$, then $e^{-i\omega(t-\frac{x-x_0}{c})}$ occurs which is right-going (outward), while if $x < x_0$, then $e^{-i\omega(t-\frac{x_0-x}{c})}$ occurs which is left-going (also outward). This outward propagating influence function does not have an infinite spatial extent. Instead, it is cut-off at a distance corresponding to the propagation velocity $\pm c$.

11.2.8 (a) From (11.2.24)

$$u(\vec{x},t) = \int_0^t \iiint G(\vec{x},t;\vec{x}_0,t_0)Q(\vec{x}_0,t_0)dV_0dt_0 ,$$

where $Q(\vec{x}_0,t_0) = g(\vec{x}_0)e^{-i\omega t_0}$ and the Green's function satisfies (11.2.47) with $\rho = |\vec{x} - \vec{x}_0|$. Thus

$$u(\vec{x},t) = \int_0^t \iiint g(\vec{x}_0)\frac{1}{4\pi c\rho}\delta[\rho - c(t-t_0)]e^{-i\omega t_0}dV_0dt_0 .$$

The singularity occurs at $t_0 = t - \frac{|\vec{x}-\vec{x}_0|}{c}$. However, note that $\delta(ct) = \frac{1}{|c|}\delta(t)$ [see (9.3.34)]. In order for $t_0 > 0, t > \frac{|\vec{x}-\vec{x}_0|}{c}$. Thus

$$u(\vec{x},t) = \iiint g(\vec{x}_0)I(\vec{x},\vec{x}_0,t)dV_0 ,$$

where

$$I(\vec{x},\vec{x}_0,t) = \begin{cases} 0 & t < \frac{|\vec{x}-\vec{x}_0|}{c} \\ \frac{1}{4\pi c^2|\vec{x}-\vec{x}_0|}e^{-i\omega(t-\frac{|\vec{x}-\vec{x}_0|}{c})}, & t > \frac{|\vec{x}-\vec{x}_0|}{c} \end{cases} .$$

Notice that the region of influence is $|\vec{x} - \vec{x}_0| < ct$, the inside of a sphere expanding (outward) as t increases.

11.2.10 (a) By symmetry, a negative image source at $(-x_0, t_0)$ may be used so that

$$G(x,t;x_0,t_0) = G_\infty(x,t;x_0,t_0) - G_\infty(x,t;-x_0,t_0) ,$$

where G_∞ is given by (11.2.31-32).

11.2.10 (b) We use the one-dimensional version of (11.2.16). At $x = 0, \hat{n} = -\hat{i}$, and at $x = \infty, \hat{n} = i$. Thus

$$u(x,t) = -c^2\int_0^t \left(u\frac{\partial G}{\partial x_0} - G\frac{\partial u}{\partial x_0}\right)\Big|_0^\infty dt_0 .$$

Since $G = \partial G/\partial x_0 = 0$ at $x_0 = \infty$ and $G(x,0) = 0$ because $G(0,x) = 0$. Thus

$$u(x,t) = c^2\int_0^t h(t_0)\frac{\partial G}{\partial x_0}\Big|_{x_0=0} dt_0 , \tag{9}$$

since $u(0,t) = h(t)$. We evaluate $\partial G/\partial x_0$ from part (a):

$$\begin{aligned} G(x,t;x_0,t_0) &= \frac{1}{2c}\{-H[x-x_0-c(t-t_0)] + H[x-x_0+c(t-t_0)] \\ &\quad +H[x+x_0-c(t-t_0)] - H[x+x_0+c(t-t_0)]\} , \\ \frac{\partial G}{\partial x_0} &= \frac{1}{2c}\{\delta[x-x_0-c(t-t_0)] - \delta[x-x_0+c(t-t_0)] \\ &\quad +\delta[x+x_0-c(t-t_0)] - \delta[x+x_0+c(t-t_0)]\} , \\ \frac{\partial G}{\partial x_0}\Big|_{x_0=0} &= \frac{1}{c}\{\delta[x-c(t-t_0)] - \delta[x+c(t-t_0)]\} = \frac{1}{c}\delta[x-c(t-t_0)], \end{aligned}$$

since $\delta[x + c(t-t_0)] = 0$ because $x > 0$ and $t > t_0$. Thus from (9)

$$\begin{aligned} u(x,t) &= c^2\int_0^t h(t_0)\frac{1}{c}\delta[x-c(t-t_0)]\, dt_0 \\ &= \int_0^t h(t_0)\delta(t_0 + \frac{x}{c} - t)\, dt_0 , \end{aligned}$$

because of (9.3.34). Thus

$$u(x,t) = \begin{cases} 0 & \text{if } x > ct \\ h(t-\frac{x}{c}) & \text{if } 0 < x < ct . \end{cases}$$

11.2.12 (a) The solution is obtained from (11.2.24) using the three-dimensional infinite space Green's function for the wave equation (11.2.39). In this manner

$$G(x,y,t;x_1,y_1,t_1) = \frac{1}{4\pi c}\int_0^t \iiint \frac{1}{\rho}\delta[\rho - c(t-t_0)]\cdot\delta(x_0-x_1)\delta(y_0-y_1)\delta(t_0-t_1)dx_0dy_0dz_0dt_0 \ ,$$

where $\rho^2 = [(x-x_0)^2 + (y-y_0)^2 + (z-z_0)^2]$ and where we will need to show our answer does not depend on z. The x_0, y_0, t_0 integrals are easy due to the delta functions. If $t > t_1$, then

$$G(x,y,t;x_1,y_1,t_1) = \frac{1}{4\pi c}\int_{-\infty}^{\infty} \frac{1}{\rho_1}\delta[\rho_1 - c(t-t_1)]dz_0 \ ,$$

where $\rho_1^2 = r^2 + (z-z_0)^2$ and r is the two-dimensional distance from source point to response point

$$r = [(x-x_1)^2 + (y-y_1)^2]^{1/2} \ .$$

The integrand is symmetric around $z_0 = z, z_0 = z \pm (\rho_1^2 - r^2)^{1/2}$. Thus

$$G(x,y,t;x_1,y_1,t_1) = \frac{1}{2\pi c}\int_{z}^{\infty} \frac{1}{\rho_1}\delta[\rho_1 - c(t-t_1)]dz_0 \ .$$

It is now convenient to change variables (fix r), $\rho_1^2 = r^2 + (z-z_0)^2$ so that $2\rho_1 d\rho_1 = 2(z-z_0)(-dz_0)$. Thus $G(x,y,t;x_1,y_1,t_1) = \frac{1}{2\pi c}\int_r^\infty \frac{1}{z_0-z}\delta[\rho_1 - c(t-t_1)]d\rho_1$. There is an impulse at $\rho_1 = c(t-t_1)$. Thus

$$G(x,y,t;x_1,y_1,t_1) = \begin{cases} 0 & \text{if} \quad r > c(t-t_1) \\ \frac{1}{2\pi c}\frac{1}{\sqrt{c^2(t-t_1)^2 - r^2}} & \text{if} \quad r < c(t-t_1) \ . \end{cases}$$

Section 11.3

11.3.2 (a) $\int_a^b \underbrace{v[p\frac{d^2u}{dx^2} + r\frac{du}{dx} + qu]}_{L(u)}dx = u[rv - \frac{d}{dx}(pv)]\Big|_a^b + \frac{du}{dx}pv\Big|_a^b + \int_a^b u[qv + \frac{d^2}{dx^2}(pv) - \frac{d}{dx}(rv)]dx$,

since $vp\frac{d^2u}{dx^2} = \frac{d}{dx}(vp\frac{du}{dx}) - \frac{du}{dx}\frac{d}{dx}(vp)$, $\frac{du}{dx}\frac{d}{dx}(vp) = \frac{d}{dx}(u\frac{d}{dx}(vp)) - u\frac{d^2}{dx^2}(vp)$, and $vr\frac{du}{dx} = \frac{d}{dx}(vru) - u\frac{d}{dx}(rv)$. By letting

$$L^*(v) = \frac{d^2}{dx^2}(pv) - \frac{d}{dx}(rv) + qv = p\frac{d^2v}{dx^2} + (2\frac{dp}{dx} - r)\frac{dv}{dx} + (\frac{d^2p}{dx^2} - \frac{dr}{dx} + q)v \ ,$$

we obtain $\int_a^b [vL(u) - uL^*(v)]dx = -H\Big|_a^b$, where

$$H(x) = u[\frac{d}{dx}(vp) - vr] - pv\frac{du}{dx} = p(u\frac{dv}{dx} - v\frac{du}{dx}) + uv(\frac{dp}{dx} - r) \ .$$

If L = L*, then

$$p\frac{d^2}{dx^2} + (2\frac{dp}{dx} - r)\frac{d}{dx} + (\frac{d^2p}{dx^2} - \frac{dr}{dx} + q) = p\frac{d^2}{dx^2} + r\frac{d}{dx} + q \ .$$

This requires $2dp/dx - r = r$ and $d^2p/dx^2 - dr/dx = 0$. Consequently $r = dp/dx$, which results in

$$L = \frac{d}{dx}(p\frac{d}{dx}) + q \ .$$

11.3.6 (c) We use the one-dimensional version of (11.3.21). Since $\hat{n} = -\hat{i}$ at $x = 0$,

$$u(x,t) = k\int_0^t [u(0,t_0)\frac{\partial G}{\partial x_0}(x,t;0,t_0) - G(x,t;0,t_0)\frac{\partial u}{\partial x_0}(0,t_0)]dt_0.$$

Since the Green's fucntion satisfies the homogeneous boundary condition, $G(0,t;x_0,t_0)=0$, which because of symmetry yields $G(x_0,t_0;0,t)=0$. Thus,

$$u(x,t)=k\int_0^t u(0,t_0)\frac{\partial G}{\partial x_0}(x,t;0,t_0)dt_0 \ .$$

For the Green's function we use (11.3.34). Thus

$$\frac{\partial G}{\partial x_0}(x,t;x_0,t_0)=\frac{1}{\sqrt{4\pi k(t-t_0)}}\left\{\frac{(x-x_0)}{2k(t-t_0)}exp\left[-\frac{(x-x_0)^2}{4k(t-t_0)}\right]+\frac{(x+x_0)}{2k(t-t_0)}exp\left[-\frac{(x+x_0)^2}{4k(t-t_0)}\right]\right\}$$

and

$$\frac{\partial G}{\partial x_0}(x,t;0,t_0)=\frac{x/k(t-t_0)}{\sqrt{4\pi k(t-t_0)}}exp\left[-\frac{x^2}{4k(t-t_0)}\right] ,$$

so that in general since $u(0,t)=A(t)$,

$$u(x,t)=\frac{x}{\sqrt{4\pi k}}\int_0^t A(t_0)(t-t_0)^{-3/2}exp\left[\frac{-x^2}{4k(t-t_0)}\right]dt_0 \ .$$

In the given case where $A(t_0)=1$

$$u(x,t)=\frac{x}{\sqrt{4\pi k}}\int_0^t (t-t_0)^{-3/2}exp\left[\frac{-x^2}{4k(t-t_0)}\right]dt_0 \ .$$

This can be simplified by the change of variables $\eta=\frac{x}{\sqrt{4k(t-t_0)}}$ and thus $d\eta=\frac{x}{2\sqrt{4k}}(t-t_0)^{-3/2}dt_0$. In this manner, we obtain

$$u(x,t)=\frac{2}{\sqrt{\pi}}\int_{x/\sqrt{4kt}}^{\infty} e^{-\eta^2}d\eta \ .$$

11.3.7 The Green's function satisfies the related homogeneous boundary condition $\partial G/\partial x=0$ at $x=0$. This boundary condition can be satisfied using one-dimensional infinite space Green's functions (11.3.27), if we introduce a positive image source at $(-x_0,t_0)$. In this manner, the desired Green's function is

$$G(x,t;x_0,t_0)=\frac{1}{\sqrt{4\pi k(t-t_0)}}\left\{exp\left[-\frac{(x-x_0)^2}{4k(t-t_0)}\right]+exp\left[-\frac{(x+x_0)^2}{4k(t-t_0)}\right]\right\}.$$

Chapter 12. Method of Characteristics

Section 12.2

12.2.2 The general solution of the pde is

$$w(x,t) = P(x+3t)$$

using (12.2.12) since $c = -3$. In order to satisfy the initial condition, $w(x,0) = P(x) = \cos x$. Therefore

$$w(x,t) = \cos(x+3t)\ .$$

12.2.5 (b) Using the ideas of subsection 12.2.2, if $dx/dt = x$, then $dw/dt = 1$. The characteristics satisfy

$$x = x_0 e^t\ , \tag{10}$$

where the characteristics start ($t = 0$) at $x = x_0$. Along the characteristics

$$w(x,t) = t + w_0\ ,$$

where w_0 is the value of w at $t = 0$ where $x = x_0$, i.e., $w_0 = f(x_0)$. Thus

$$w(x,t) = t + f(x_0) = t + f(xe^{-t}),$$

using (10) to solve for x_0 as a function of x and t.

12.2.5 (d) Using the ideas of subsection 12.2.2, if $dx/dt = 3t$, then $dw/dt = w$. The characteristics satisfy

$$x = \frac{3}{2}t^2 + x_0\ , \tag{11}$$

where the characteristics start ($t = 0$) at $x = x_0$. Along the characteristics,

$$w(x,t) = w_0 e^t\ ,$$

where w_0 is the value of w at $t = 0$, where $x = x_0$, i.e., $w_0 = f(x_0)$. Thus

$$w(x,t) = f(x_0)e^t = f(x - \frac{3}{2}t^2)e^t\ ,$$

using (11) to solve for x_0 as a function of x and t.

12.2.6 Using the ideas of subsection 12.2.2, if $dx/dt = 2u$, then $du/dt = 0$. The latter equation states that u is constant along the characteristic. If the characteristic starts ($t = 0$) at x_0, then $u = f(x_0)$ there and everywhere on this characteristic. Thus

$$x = 2f(x_0)t + x_0\ ,$$

since $x = x_0$ at $t = 0$. These characteristics are straight lines with velocity $2f(x_0)$. However, they are not necessarily parallel to each other, as different lines have different slopes.

12.2.8 The characteristics satisfy $x = 2f(x_0)t + x_0$. If $x_0 > 0$, then $f(x_0) = 2$:

$$x = 4t + x_0\ .$$

Along these characteristics ($x_0 > 0$) $u = 2$. Thus

$$u = 2 \quad \text{for} \quad x > 4t\ .$$

If $x_0 < 0$, then $f(x_0) = 1$:

$$x = 2t + x_0\ .$$

Along these characteristics ($x_0 < 0$) $u = 1$. Thus

$$u = \begin{cases} 1 & \text{for} \quad x < 2t \\ 2 & \text{for} \quad x > 4t \, . \end{cases}$$

We have not determined what occurs for $2t < x < 4t$. In the limit as $L \to 0$ of exercise 12.2.7, all values of u between 1 and 2 occur at the discontinuity at $x_0 = 0$. Thus there are an infinite number of characteristics starting at $x_0 = 0$, each satisfying

$$x = 2f(x_0)t$$

$$u(x,t) = f(x_0) \, .$$

By eliminating $f(x_0)$ in this "fan-shaped" region

$$u(x,t) = x/2t \, .$$

Thus

$$u(x,t) = \begin{cases} 1 & x < 2t \\ x/2t & 2t < x < 4t \\ 2 & x > 4t \, . \end{cases}$$

This is similar to the result sketched in Figures 12.6.4 and 12.6.5.

Section 12.3

12.3.4 (a) $\frac{\partial u}{\partial t} = \frac{dF(x-ct)}{d(x-ct)}(-c)\Big|_{t=0} = -c\frac{dF}{dx}(x) \, .$

12.3.4 (b) $\frac{\partial u}{\partial x} = \frac{dF(x-ct)}{d(x-ct}\Big|_{x=0} = \frac{dF(-ct)}{d(-ct)} = -\frac{1}{c}\frac{d}{dt}F(-ct) \quad .$

Section 12.4

12.4.1 The general solution of the wave equation is given by (12.3.4)

$$u(x,t) = F(x-ct) + G(x+ct) \, . \tag{12}$$

Since for $x > 0$ $u(x,0) = 0$ and $\partial u/\partial t(x,0) = 0$, it follows from (12.3.10-11) that

$$F(x) = 0 \quad \text{and} \quad G(x) = 0 \quad \text{only for} \quad x > 0 \, .$$

The boundary condition, $u(0,t) = h(t)$ for $t > 0$, yields

$$h(t) = F(-ct) + G(ct) \quad \text{if} \quad t > 0 \, . \tag{13}$$

We now can evaluate (12). If $x > ct$, then the arguments of both F and G are positive and there $F = G = 0$. Thus

$$u(x,t) = 0 \quad \text{if} \quad x > ct \, .$$

However if $x < ct$, then the argument of F only is negative. From (13), letting $z = -ct$,

$$F(z) = h(-z/c) - G(-z) \quad \text{for} \quad z < 0 \, .$$

Thus

$$F(x-ct) = h(\frac{x-ct}{-c}) - G(ct-x),$$

resulting in

$$u(x,t) = h(t - \frac{x}{c}) - G(ct-x) + G(x+ct)$$

if $x < ct$. In this case both arguments of G are positive, and hence both G equal zero. Thus

$$u(x,t) = \begin{cases} 0 & x > ct \\ h(t - \frac{x}{c}) & 0 < x < ct \, . \end{cases}$$

It is very helpful in understanding this result to sketch characteristics (as for example in Figures 12.4.1 and 12.4.2).

12.4.2 The general solution of the wave equation is given by (12.3.4)

$$u(x,t) = F(x-ct) + G(x+ct) \ . \tag{14}$$

The initial conditons are only valid for $x < 0$. Thus (12.3.10-11) implies that

$$G(x) = \frac{1}{2}\cos x \quad \text{and} \quad F(x) = \frac{1}{2}\cos x \quad \text{for} \quad x < 0 \ . \tag{15}$$

The boundary condition, $u(0,t) = e^{-t}$ for $t > 0$ yields

$$e^{-t} = F(-ct) + G(ct) \quad \text{for} \quad t > 0 \ . \tag{16}$$

We now can evaluate (14). If $x < -ct$, then the arguments of both F and G are negative, and there $F = G = 1/2\cos x$. Thus

$$u(x,t) = \frac{1}{2}\cos(x-ct) + \frac{1}{2}\cos(x+ct) = \cos x \cos ct \ , x < -ct \ .$$

If $-ct < x < 0$, then the argument of G only is positive. From (16), letting $z = ct$

$$G(z) = e^{-z/c} - F(-z) \quad \text{for} \quad z > 0 \ .$$

Thus for $x + ct > 0$

$$G(x+ct) = e^{-(x+ct)/c} - F(-x-ct) \ ,$$

which yields

$$u(x,t) = e^{-(x+ct)/c} - F(-x-ct) + F(x-ct)$$

if $x + ct > 0$. In this case both arguments of F are negative, and hence (15) may be used:

$$u(x,t) = e^{-(t+x/c)} + \frac{1}{2}\cos(x-ct) - \frac{1}{2}\cos(-x-ct) \ .$$

In conclusion

$$u(x,t) = \begin{cases} \cos x \cos ct & x + ct < 0 \\ e^{-(t+x/c)} + \sin x \sin ct & x + ct > 0 \ . \end{cases}$$

Figure 12.4.1 reflected around the t-axis, i.e., x replaced by $-x$, shows the important characteristic, $x + ct = 0$.

12.4.6 The general solution of the wave equation is given by (12.3.4)

$$u(x,t) = F(x-ct) + G(x+ct) \ .$$

The zero initial conditions from (12.3.10-11) imply that

$$F(x) = 0 \quad \text{and} \quad G(x) = 0 \quad \text{only for} \quad x > 0 \ .$$

Since $x + ct > 0$, it follows that

$$u(x,t) = F(x-ct) \ .$$

To apply the nonhomogeneous boundary condition, we calculate:

$$\frac{\partial u}{\partial x}(x,t) = \frac{dF(x-ct)}{d(x-ct}$$

and thus

$$\frac{\partial u}{\partial x}(0,t) = \frac{dF(-ct)}{d(-ct)} = -\frac{1}{c}\frac{d}{dt}F(-ct) = h(t)$$

for $t > 0$. Consequently

$$F(-ct) = -c\int_0^t h(\bar{t})d\bar{t}\ ,$$

where we have used $F(0) = 0$ since we have assumed $F(x)$ is continuous and we already know $F(x) = 0$ for $x > 0$. We note that $(z = -ct)$ for $z < 0$

$$F(z) = -c\int_0^{-z/c} h(\bar{t})d\bar{t}$$

and hence

$$F(x-ct) = -c\int_0^{t-x/c} h(\bar{t})d\bar{t} \quad \text{if} \quad x - ct < 0\ .$$

In summary,

$$u(x,t) = \begin{cases} 0 & x > ct \\ -c\int_0^{t-\frac{x}{c}} h(\bar{t})d\bar{t} & x < ct\ . \end{cases}$$

12.4.7 The general solution of the wave equation is given by (12.3.4)

$$u(x,t) = F(x-ct) + G(x+ct)\ . \tag{17}$$

The initial conditions (valid for $x > 0$) are applied in (12.3.10-11). Since it is initially at rest,

$$F(x) = G(x) = \frac{1}{2}f(x) \quad \text{for} \quad x > 0\ .$$

The boundary condition requires $\partial u/\partial x$:

$$\frac{\partial u}{\partial x} = \frac{dF(x-ct)}{d(x-ct)} + \frac{dG(x+ct)}{d(x+ct)}\ .$$

Evaluating this at $x = 0$ yields

$$h(t) = -\frac{1}{c}\frac{d}{dt}F(-ct) + \frac{1}{c}\frac{d}{dt}G(ct), \quad \text{for} \quad t > 0\ .$$

By integrating this, we obtain for $t > 0$

$$c\int_0^t h(\bar{t})d\bar{t} = -F(-ct) + G(ct) \tag{18}$$

since $G(0) - F(0) = \frac{1}{2}f(0) - \frac{1}{2}f(0) = 0$ by continuity. In (17) G is known for positive arguments, but F is not known for negative arguments, needed if $x < ct$. However, from (18), letting $z = -ct$

$$F(z) = G(-z) - c\int_0^{-z/c} h(\bar{t})d\bar{t} \quad \text{for} \quad z < 0\ .$$

Consequently, for $x - ct < 0$ (i.e., $0 < x < ct$)

$$F(x-ct) = G(ct-x) - c\int_0^{t-x/c} h(\bar{t})d\bar{t}\ .$$

We thus can calculate u,

$$u(x,t) = \begin{cases} \frac{1}{2}[f(x+ct) + f(x-ct)] & x > ct \\ \frac{1}{2}[f(x+ct) + f(ct-x)] - c\int_0^{t-x/c} h(\bar{t})d\bar{t} & x < ct\ . \end{cases}$$

Section 12.5

12.5.1 (b) By separation of variables, $u(x,t)$ is given by (4.4.11) where the conditions for A_n and B_n are given in (4.4.12) and (4.4.13). If $g(x) = 0, B_n = 0$ and

$$u(x,t) = \sum_{n=1}^{\infty} A_n \sin \frac{n\pi x}{L} \cos \frac{n\pi ct}{L} ,$$

where $f(x) = \sum_{n=1}^{\infty} A_n \sin \frac{n\pi x}{L}$ for $0 < x < L$. However, the series does not equal $f(x)$ outside of $0 < x < L$. Instead, from the theory of Fourier sine series.

$$\sum_{n=1}^{\infty} A_n \sin \frac{n\pi x}{L} = \bar{f}(x) , \tag{19}$$

where $\bar{f}(x)$ is the odd periodic extension of $f(x)$. Since $\sin a \cos b = \frac{1}{2} \sin(a+b) + \frac{1}{2}\sin(a-b)$, it follows that

$$u(x,t) = \frac{1}{2} \sum_{n=1}^{\infty} A_n \sin \frac{n\pi}{L}(x+ct) + \frac{1}{2} \sum_{n=1}^{\infty} A_n \sin \frac{n\pi}{L}(x-ct) .$$

Thus from (19)

$$u(x,t) = \frac{1}{2}[\bar{f}(x+ct) + \bar{f}(x-ct)] ,$$

where $\bar{f}(x)$ is the odd periodic extension of $f(x)$. This is the result found on page 545.

Section 12.6

12.6.1 (a) By integration, $\rho(x,t) = c(x)$, where $c(x)$ is an arbitrary function of x. From the initial condition, $\rho(x,0) = f(x) = c(x)$. Thus the solution is

$$\rho(x,t) = f(x) .$$

12.6.1 (c) This is an ordinary differential equation, keeping x fixed, whose general solution is

$$\rho(x,t) = A(x)e^{-3xt} ,$$

where $A(x)$ is an arbitrary function of x. From the initial condition $\rho(x,0) = f(x) = A(x)$. Therefore

$$\rho(x,t) = f(x)e^{-3xt} .$$

The method of characteristics can also be used (if $dx/dt = 0$, then $d\rho/dt = -3x\rho$).

12.6.2 This is an ordinary differential equation, keeping x fixed, whose general solution is

$$\rho(x,t) = A(x)e^{t} ,$$

where $A(x)$ is an arbitrary function of x. The condition says that

$$1 + \sin x = A(x)e^{t} = A(x)e^{-x/2}$$

since this is given along $x = -2t$ (equivalent to $t = -x/2$). Thus

$$A(x) = e^{x/2}(1 + \sin x) ,$$

yielding the solution

$$\rho(x,t) = e^{x/2}(1 + \sin x)e^{t} = (1 + \sin x)e^{t+\frac{x}{2}} .$$

12.6.3 (a) The general solution of this pde is

$$\rho(x,t) = A(x - c_0 t) \ ,$$

where A is an arbitrary function. From the initial condition, $\rho(x,0) = A(x) = \sin x$. Therefore,

$$\rho(x,t) = \sin(x - c_0 t) \ .$$

By the method of characteristics, if $dx/dt = c_0$, then $d\rho/dt = 0$. The characteristics satisfy

$$x = c_0 t + x_0 \ ,$$

where $x(0) = x_0$. Also

$$\rho(x,t) = \rho(x_0, 0) = \sin x_0 \ ,$$

from the initial condition. Thus,

$$\rho(x,t) = \sin(x - c_0 t) \ .$$

12.6.3 (b) By the method of characteristics, if $dx/dt = c_0$, then $d\rho/dt = 0$. The characteristics that start at $x(0) = x_0$ for $x_0 > 0$ are

$$x = c_0 t + x_0 \ .$$

From the initial condition

$$\rho(x,t) = \rho(x_0, 0) = f(x_0) \quad \text{for} \quad x_0 > 0 \ .$$

Thus $\rho(x,t) = f(x - c_0 t)$ for $x_0 = x - c_0 t > 0$. However, there are characteristics that start at $x = 0$ at $t = t_0 < 0$. For these

$$x = c_0 (t - t_0) \ .$$

From the boundary condition

$$\rho(x,t) = \rho(0, t_0) = g(t_0) \quad \text{for} \quad t_0 = t - \frac{x}{c_0} > 0 \ .$$

$t_0 = t - x/c_0$ is called the retarded time. In summary,

$$\rho(x,t) = \begin{cases} f(x - c_0 t) & x > c_0 t \\ g(t - \frac{x}{c_0}) & c_0 t > x > 0 \ . \end{cases}$$

12.6.4 (a) The condition $u(0) = u_{max}$ yields $\alpha = u_{max}$, while $u(\rho_{max}) = 0$ yields $0 = \alpha + \beta \rho_{max}$ or $\beta = -u_{max}/\rho_{max}$. Thus $u(\rho) = u_{max}(1 - \rho/\rho_{max})$. The flow satisfies $q = \rho u$, and hence $q(\rho) = u_{max}\rho(1 - \rho/\rho_{max})$. This is a parabola which is concave down with $q(0) = q(\rho_{max}) = 0$.

12.6.4 (b) The flow is maximum where

$$q'(\rho) = u_{max}(1 - 2\rho/\rho_{max}) = 0 \ .$$

Thus $\rho = \rho_{max}/2, u = u_{max}/2$, and hence $q = \rho_{max} u_{max}/4$ is the maximum flow.

12.6.8 (a) By the method of characteristics, if $dx/dt = c$, then $d\rho/dt = e^{-3x}$. The characteristics satisfy

$$x = ct + x_0 \ , \tag{20}$$

where $x(0) = x_0$. Along the characteristics

$$d\rho/dt = e^{-3(ct+x_0)} \ .$$

Since x_0 is constant, by integration

$$\rho(x,t) = \frac{1}{-3c} e^{-3(ct+x_0)} + k \ , \tag{21}$$

where k is determined from the initial condition

$$\rho(x_0,0) = f(x_0) = -\frac{1}{3c}e^{-3x_0} + k \,. \tag{22}$$

Using (20) - (22),

$$\rho(x,t) = -\frac{1}{3c}e^{-3x} + \frac{1}{3c}e^{-3(x-ct)} + f(x-ct) \,.$$

12.6.8 (c) By the method of characteristics, if $dx/dt = t$, then $d\rho/dt = 5$. The characteristics satisfy

$$x = \frac{1}{2}t^2 + x_0 \,, \tag{23}$$

where $x(0) = x_0$. Along the characteristics,

$$\rho(x,t) = 5t + \rho(x_0,0) \,,$$

since at $t = 0, x = x_0$. From the initial condition and (23)

$$\rho(x,t) = 5t + f(x - \frac{1}{2}t^2) \,.$$

12.6.8 (e) By the method of characteristics, if $dx/dt = -t^2$, then $d\rho/dt = -\rho$. The characteristics satisfy

$$x = -\frac{1}{3}t^3 + x_0 \,, \tag{24}$$

where $x(0) = x_0$. Along the characteristics, ρ satisfies an ordinary differential equation whose general solution is

$$\rho(x,t) = \rho(x_0,0)e^{-t} \,,$$

since at $t = 0, x = x_0$. Furthermore, $\rho(x_0,0) = f(x_0)$. Therefore, from (24)

$$\rho(x,t) = f(x + \frac{1}{3}t^3)e^{-t} \,.$$

12.6.8 (g) By the method of characteristics, if $dx/dt = x$, then $d\,\rho/dt = t$. The characteristics satisfy a differential equation whose solution is

$$x = x_0e^t \,, \tag{25}$$

since $x(0) = x_0$. Along these characteristics,

$$\rho(x,t) = \frac{1}{2}t^2 + \rho(x_0,0) \,,$$

since at $t = 0, x = x_0$. Furthermore, $\rho(x_0,0) = f(x_0)$. Therefore, from (25)

$$\rho(x,t) = \frac{1}{2}t^2 + f(xe^{-t}) \,.$$

12.6.9 (a) By the method of characteristics, if $dx/dt = -\rho^2$, then $d\rho/dt = 3\rho$. Along the characteristics

$$\rho(x,t) = \rho(x_0,0)e^{3t} \,,$$

since we choose $x = x_0$ at $t = 0$. thus the characteristics satisfy

$$\frac{dx}{dt} = -\rho^2(x_0,0)e^{6t} \,.$$

Since x_0 is constant,

$$x = -\frac{1}{6}(e^{6t} - 1)\rho^2(x_0,0) + x_0 \,.$$

From the initial condition $\rho(x_0,0) = f(x_0)$, we have a parametric representation of the solution: $\rho(x,t) = f(x_0)e^{3t}$, where x_0 satisfies $x = -\frac{1}{6}(e^{6t} - 1)f^2(x_0) + x_0$.

12.6.9 (c) By the method of characteristics, if $dx/dt = t^2\rho$, then $d\rho/dt = -\rho$. Along the characteristics

$$\rho(x,t) = \rho(x_0, 0)e^{-t} ,$$

since we choose $x = x_0$ at $t = 0$. The characteristics satisfy

$$\frac{dx}{dt} = t^2 e^{-t}\rho(x_0, 0) .$$

Since x_0 is constant and $\rho(x_0, 0) = f(x_0)$,

$$x = f(x_0)\int_0^t \tau^2 e^{-\tau} d\tau + x_0 , \tag{26}$$

where the integral can be evaluated (by tables, symbolic integration routines, or repeated integration-by-parts). A parametric representation is

$$\rho(x,t) = f(x_0)e^{-t} ,$$

where x_0 satisfies (26).

12.6.14 (a) Since $\rho = f(x - Vt), \rho_t = -Vf', \rho_x = f'$, etc.. Therefore

$$-Vf' + u_{max}(1 - \frac{2f}{\rho_{max}})f' = \nu f'' .$$

12.6.14 (c) By integration

$$-Vf + u_{max}(f - \frac{f^2}{\rho_{max}}) = \nu f' + c.$$

If $f \to \rho_2$ as $x \to \infty$ such that $f' \to 0$, then $-V\rho_2 + u_{max}(\rho_2 - \frac{\rho_2^2}{\rho_{max}}) = c$. Similarly as $x \to -\infty$ $-V\rho_1 + u_{max}(\rho_1 - \frac{\rho_1^2}{\rho_{max}}) = c$. Subtracting these yields

$$V(\rho_1 - \rho_2) + u_{max}(\rho_2 - \frac{\rho_2^2}{\rho_{max}}) - u_{max}(\rho_1 - \frac{\rho_1^2}{\rho_{max}}) = 0 .$$

Thus

$$V = \frac{u_{max}(\rho_2 - \frac{\rho_2^2}{\rho_{max}}) - u_{max}(\rho_1 - \frac{\rho_1^2}{\rho_{max})}}{\rho_2 - \rho_1} ,$$

where long division yields the given answer. However, in this form $V = [q]/[\rho]$ since $q = \rho u$ and $u = u_{max}(1 - \rho/\rho_{max})$ if $\nu = 0$. This is the same as the shock velocity derived in (12.6.20).

12.6.17 (a) The pde is (12.6.12) where $c(\rho) = q'(\rho)$, where $q = \rho u = u_{max}(\rho - \rho^2/\rho_{max})$. Thus

$$c(\rho) = u_{max}(1 - 2\rho/\rho_{max}) .$$

By the method of characteristics, if $dx/dt = c(\rho)$, then $d\rho/dt = 0$. Since ρ is constant along the characteristics, the characteristics are straight lines. We use the notation $x(0) = x_0$. We need to calculate the density wave velocity for $\rho = \rho_{max}/5$ and $\rho = 3\rho_{max}/5$:

$$c(\rho_{max}/5) = 3u_{max}/5 \quad \text{and} \quad c(3\rho_{max}/5) = -u_{max}/5 .$$

Thus, if $x_0 > 0$, from the initial condition

$$\rho(x,t) = 3\rho_{max}/5 \quad \text{along} \quad x = -u_{max}t/5 + x_0 ,$$

while if $x_0 < 0$,

$$\rho(x,t) = \rho_{max}/5 \quad \text{along} \quad x = 3u_{max}t/5 + x_0 .$$

These two families of characteristics intersect as in Fig. 12.6.9 (a), though if $x > x_0$, the slopes of the lines should be negative here. Thus a shock exists separating $\rho = \rho_{max}/5$ from $\rho = 3\rho_{max}/5$, starting at $x_s = 0$ at $t = 0$. The shock velocity satisfies (12.6.20):

$$\frac{dx_s}{dt} = \frac{[q]}{[\rho]} = u_{max}\frac{\frac{3}{5}\rho_{max}\cdot\frac{2}{5}\rho_{max} - \frac{1}{5}\rho_{max}\cdot\frac{4}{5}\rho_{max}}{\frac{3}{5}\rho_{max} - \frac{1}{5}\rho_{max}} = \frac{u_{max}}{5},$$

since $q = \rho u$ and $u(\frac{3}{5}\rho_{max}) = \frac{2}{5}u_{max}$ and $u(\frac{1}{5}\rho_{max}) = \frac{4}{5}\rho_{max}$. Figure 12.6.9 (c) is valid with $x_s = \frac{u_{max}}{5}t$, and thus

$$\rho(x,t) = \begin{cases} \rho_{max}/5 & x < \frac{u_{max}}{5}t \\ 3\rho_{max}/5 & x > \frac{u_{max}}{5}t\,. \end{cases}$$

In addition Fig. 12.6.10 is valid with the changes reflected by the above formula.

12.6.18 (b) The pde is (12.6.12), where

$$\begin{aligned} u &= u_{max}(1 - \rho^2/\rho_{max}^2) \\ q = \rho u &= u_{max}(\rho - \rho^3/\rho_{max}^2) \\ c(\rho) = q'(\rho) &= u_{max}(1 - 3\rho^2/\rho_{max}^2). \end{aligned}$$

Since $q''(\rho) < 0, q(\rho)$ is concave down. Furthermore $q'(\rho)$ is a decreasing function of ρ. Thus, since initially

$$\rho(x,0) = \begin{cases} \rho_1 & x < 0 \\ \rho_2 & x > 0 \end{cases}$$

with $\rho_2 < \rho_1$, the faster density wave velocity is ahead of the slower density wave velocity. Thus density will spread out necessitating fan-like characteristics. By the method of characteristics, if $dx/dt = c(\rho)$, then $d\rho/dt = 0$. The characteristics are straight lines. We use the notation $x(0) = x_0$. Thus, if $x_0 > 0$ from the initial condition

$$\rho(x,t) = \rho_2 \quad \text{along} \quad x = c(\rho_2)t + x_0\,,$$

while if $x_0 < 0$

$$\rho(x,t) = \rho_1 \quad \text{along} \quad x = c(\rho_1)t + x_0\,.$$

Since these diverge, Figure 12.6.4 is valid. So far

$$\rho(x,t) = \begin{cases} \rho_2 & x > c(\rho_2)t \\ \rho_1 & x < c(\rho_1)t\,, \end{cases}$$

but we have not determined $\rho(x,t)$ for $c(\rho_1)t < x < c(\rho_2)t$, since $c(\rho_2) > c(\rho_1)$. There, by the method of characteristics, $x = c(\rho)t$, since at $t = 0, x = x_0 = 0$. We solve for $\rho(x,t)$ there as follows:

$$\frac{x}{t} = c(\rho) = u_{max}(1 - 3\rho^2/\rho_{max}^2)$$

and hence $3\rho^2/\rho_{max}^2 = 1 - \frac{x}{u_{max}t}$ or

$$\rho(x,t) = \frac{\rho_{max}}{\sqrt{3}}\sqrt{1 - \frac{x}{u_{max}t}}\,.$$

This is valid in the fan-shaped region, $c(\rho_1)t < x < c(\rho_2)t$.

Chapter 13. Laplace Transform

Section 13.2

13.2.4 Using the convolution theorem with $g(t-\bar{t})=1$ (and hence $g(t)=1$),

$$\mathcal{L}\left[\int_0^t f(\bar{t})d\bar{t}\right] = F(s)G(s) = F(s)/s \ ,$$

since $g(t)=1$ implies that $G(s)=1/s$. An alternate method to derive this result is to let

$$u(t) = \int_0^t f(\bar{t})d\bar{t} \ , \quad \text{so that} \quad du/dt = f(t) \ .$$

By taking the Laplace transform, we obtain $sU(s) = F(s)$ or $U(s) = F(s)/s$, since $u(0)=0$.

13.2.5 (b) We use

$$\mathcal{L}[-tf(t)] = dF/ds \ .$$

Here $f(t) = -\sin 4t$, so that $F(s) = -4/(s^2+16)$. Thus

$$\mathcal{L}[t\sin 4t] = \frac{d}{ds}\frac{-4}{s^2+16} = \frac{8s}{(s^2+16)^2} \ .$$

13.2.5 (d) We use

$$\mathcal{L}[e^{at}f(t)] = F(s-a)$$

with $a=3$ and $f(t)=\sin 4t$. Since from the tables

$$F(s) = \frac{4}{s^2+16}, \quad \text{we have} \quad \mathcal{L}[e^{3t}f(t)] = \frac{4}{(s-3)^2+16} = \frac{4}{s^2-6s+25} \ .$$

13.2.5 (f) It is possible to express $f(t)$ in terms of step functions:

$$f(t) = t^2[H(t-5) - H(t-8)] \ .$$

Now we use the formula

$$\mathcal{L}[f(t-b)H(t-b)] = e^{-bs}F(s) \ .$$

with b = 5. We want $f(t-5) = t^2$, and thus $f(t) = (t+5)^2 = t^2 + 10t + 25$. From the tables, $F(s) = \frac{2}{s^3} + \frac{10}{s^2} + \frac{25}{s}$. Thus

$$\mathcal{L}[t^2H(t-5)] = e^{-5s}(\frac{2}{s^3} + \frac{10}{s^2} + \frac{25}{s}) \ .$$

Also $f(t-8) = t^2$, and thus $f(t) = (t+8)^2 = t^2 + 16t + 64$. From the tables, $F(s) = \frac{2}{s^3} + \frac{16}{s^2} + \frac{64}{s}$. Thus

$$\mathcal{L}\{t^2[H(t-5) - H(t-8)]\} = e^{-5s}(\frac{2}{s^3} + \frac{10}{s^2} + \frac{25}{s}) - e^{-8s}(\frac{2}{s^3} + \frac{16}{s^2} + \frac{64}{s}) \ .$$

13.2.5 (h) We use the formula

$$\mathcal{L}[H(t-b)f(t-b)] = e^{-bs}F(s)$$

with $b=1$ and $f(t)=t^4$. Consequently from the tables

$$\mathcal{L}[(t-1)^4H(t-1)] = e^{-s}\frac{4!}{s^5} \ .$$

13.2.6 (e) We use the method of partial fractions

$$\frac{s}{s^2+8s+7} = \frac{s}{(s+7)(s+1)} = \frac{a}{s+7} + \frac{b}{s+1},$$

where from (13.2.4)

$$a = \lim_{s\to -7} \frac{s}{s+1} = \frac{7}{6} \quad \text{and} \quad b = \lim_{s\to -1} \frac{s}{s+7} = \frac{-1}{6}.$$

Thus

$$\mathcal{L}^{-1}\left[\frac{s}{s^2+8s+7}\right] = \frac{7}{6}e^{-7t} - \frac{1}{6}e^{-t}.$$

13.2.6 (j) First we let

$$F(s) = \frac{s+2}{s(s^2+9)} = \frac{a}{s} + \frac{b}{s+3i} + \frac{c}{s-3i},$$

using complex roots, where

$$a = \lim_{s\to 0} \frac{s+2}{s^2+9} = \frac{2}{9}, b = \lim_{s\to -3i} \frac{s+2}{s(s-3i)} = \frac{2-3i}{-18}, c = \lim_{s\to +3i} \frac{s+2}{s(s+3i)} = \frac{2+3i}{-18}.$$

Thus

$$f(t) = \frac{2}{9} + \frac{2+3i}{-18}e^{3it} + \frac{2-3i}{-18}e^{-3it} = \frac{2}{9} - \frac{2}{9}\cos 3t + \frac{3i}{18}(-2i\sin 3t),$$

using Euler's formulas. This can also be obtained by noting $F(s) = \frac{2/9}{s} + \frac{c_1 s + c_2}{s^2+9}$ and therefore

$$\frac{c_1 s + c_2}{s^2+9} = \frac{s+2}{s(s^2+9)} - \frac{2/9}{s} = \frac{s+2-\frac{2}{9}(s^2+9)}{s(s^2+9)} = \frac{s-\frac{2}{9}s^2}{s(s^2+9)} = \frac{1-\frac{2}{9}s}{s^2+9}.$$

In this manner using the tables

$$f(t) = \frac{2}{9} + \frac{1}{3}\sin 3t - \frac{2}{9}\cos 3t.$$

We also need to invert $-5e^{-4s}F(s)$. From the table the inversion of this is $-5H(t-4)f(t-4)$. Thus

$$\mathcal{L}^{-1}\left[\frac{s+2}{s(s^2+9)}(1-5e^{-4s})\right] = \frac{2}{9}+\frac{1}{3}\sin 3t - \frac{2}{9}\cos 3t - 5H(t-4)\left[\frac{2}{9} + \frac{1}{3}\sin 3(t-4) - \frac{2}{9}\cos 3\,(t-4)\right].$$

13.2.7 (b) By taking the Laplace transform of the differential equation, we obtain

$$sY(s) - 2 + Y(s) = 1/s,$$

since $y(0) = 2$. Thus

$$Y(s) = \frac{2+1/s}{s+1} = \frac{1+2s}{s(s+1)} = \frac{a}{s} + \frac{b}{s+1},$$

using partial fractions, where from (13.2.4)

$$a = \lim_{s\to 0} \frac{1+2s}{s+1} = 1 \quad \text{and} \quad b = \lim_{s\to -1} \frac{1+2s}{s} = 1.$$

Consequently

$$y(t) = 1 + e^{-t}.$$

13.2.7 (d) The right-hand side can be expressed in terms of a step function

$$f(t) = e^{-t}H(t-3)\ .$$

Its Laplace transform is

$$F(s) = G(s)e^{-3s}$$

where $g(t-3) = e^{-t}$ or $g(t) = e^{-(t+3)}$, so that $G(s) = e^{-3}/s+1$. Thus

$$F(s) = e^{-3(s+1)}/\,s+1\ ,$$

which can be obtained in other ways. By taking the Laplace transform of the differential equation, we obtain

$$s^2Y(s) - s3 - 7 + 5[sY(s) - 3] - 6Y(s) = e^{-(s+1)}/s + 1\ .$$

Consequently,

$$Y(s) = \frac{3s+22}{s^2+5s-6} + \frac{e^{-3(s+1)}}{(s+1)(s^2+5s-6)}\ .$$

We invert these separately:

$$Y_1(s) = \frac{3s+22}{s^2+5s-6} = \frac{3s+22}{(s+6)s-1)} = \frac{a}{s+6} + \frac{b}{s-1}\ ,$$

using partial fractions, where

$$a = \lim_{s\to-6} \frac{3s+22}{s-1} = -\frac{4}{7} \quad \text{and} \quad b = \lim_{s\to1} \frac{3s+22}{s+6} = \frac{25}{7}\ .$$

Thus $y_1(t) = \frac{25}{7}e^t - \frac{4}{7}e^{-6t}$. For the other term we first need

$$G_1(s) = \frac{1}{(s+1)(s^2+5s-6)} = \frac{1}{(s+1)(s+6)(s-1)} = \frac{c_1}{s+1} + \frac{c_2}{s+6} + \frac{c_3}{s-1}\ ,$$

where using partial fractions

$$c_1 = \lim_{s\to-1} \frac{1}{(s+6)(s-1)} = \frac{-1}{10}, c_2 = \lim_{s\to-6} \frac{1}{(s+1)(s-1)} = \frac{1}{35}, c_3 = \lim_{s\to1} \frac{1}{(s+1)(s+6)} = \frac{1}{14}\ .$$

Thus

$$g_1(t) = -\frac{1}{10}e^{-t} + \frac{1}{35}e^{-6t} + \frac{1}{14}e^t\ .$$

We use the formula

$$\mathcal{L}^{-1}[e^{-3s}G_1(s)] = H(t-3)g_1(t-3)\ .$$

Thus

$$y(t) = \frac{25}{7}e^t - \frac{4}{7}e^{-6t} + e^{-3}H(t-3)[-\frac{1}{10}e^{-(t-3)} + \frac{1}{35}e^{-6(t-3)} + \frac{1}{14}e^{t-3}]\ .$$

Note that for $t > 3$

$$y(t) = e^t\left[\frac{25}{7} + \frac{e^{-6}}{14}\right] + e^{-6t}[\frac{e^{15}}{35} - \frac{4}{7}] + \frac{1}{10}e^{-t}\ .$$

13.2.7 (f) By taking the Laplace transform of the differential equation, we obtain

$$s^2Y(s) + 4Y(s) = \frac{1}{s^2+1},$$

and thus

$$Y(s) = \frac{1}{(s^2+1)(s^2+4)}\ .$$

Partial fractions may be utilized. Alternatively

$$Y(s) = \frac{a}{s^2+1} + \frac{b}{s^2+4}$$

because $Y(s)$ is even is s. Thus by multiplying by $(s^2+1)(s^2+4)$

$$a(s^2+4) + b(s^2+1) = 1 \, .$$

Consequently $a + b = 0$ and $4a + b = 1$. Thus $a = -b = 1/3$. By inversion, we obtain

$$y(t) = \frac{1}{3} \sin t - \frac{1}{3}(\frac{1}{2} \sin 2t).$$

Section 13.3

13.3.2 By taking the Laplace transform, we obtain $s^2Y(s) + Y(s) = F(s)$ or $Y(s) = \frac{F(s)}{s^2+1}$. Using the convolution theorem

$$y(t) = \int_0^t f(\bar{t})g(t-\bar{t})d\bar{t} \, ,$$

where $G(s) = 1/s^2 + 1$. Thus $g(t) = \sin t$ so that

$$y(t) = \int_0^t f(\bar{t}) \sin(t-\bar{t})d\bar{t} \, .$$

If $f(t) = \delta(t - t_0)$, then $y(t) = G(t, t_0)$:

$$G(t,t_0) = \int_0^t \delta(\bar{t} - t_0) \sin(t-\bar{t})d\bar{t} = \sin(t-t_0),$$

for $t > t_0$. Thus

$$y(t) = \int_0^t f(\bar{t})G(t,\bar{t})d\bar{t} \, .$$

Section 13.4

13.4.3 By taking the Laplace transform in t, we obtain

$$s^2U(x,s) - s \sin x = c^2 \partial^2 U/\partial x^2 \, .$$

The general solution of this ordinary differential equation is

$$U(x,s) = \frac{s}{s^2+c^2} \sin x + A(s)e^{\frac{-sx}{c}} + B(s)e^{\frac{sx}{c}} \, ,$$

using the method of undetermined coefficients for a particular solution. Since $u(x,t)$ is bounded as $x \to \pm\infty$, its Laplace transform must also be bounded as $x \to \pm\infty$. Thus $A(s) = B(s) = 0$:

$$U(x,s) = \frac{s}{s^2+c^2} \sin x \, .$$

From the inverse Laplace transform (using the table), we obtain

$$u(x,t) = \sin x \cos ct \, .$$

13.4.4 By taking the Laplace transform in t, we obtain

$$sU = k\partial^2 U/\partial x^2 \quad \text{for} \quad x > 0$$

since $u(x,0)=0$. The general solution of this differential equation is

$$U(x,s)=A(s)e^{-\sqrt{s}x/\sqrt{k}}+B(s)e^{\sqrt{s}x/\sqrt{k}}\ .$$

The problems is defined for $0<x<\infty$. The boundary conditions are $u(0,t)=f(t)$ and $u(x,t)\to 0$ as $x\to\infty$. These are Laplace transformed in t, yielding

$$U(0,s)=F(s)\qquad\text{and}\quad U(x,s)\to 0\quad\text{as}\quad x\to\infty\ ,$$

where $F(s)$ is the Laplace transform of $f(t)$. The condition as $x\to\infty$ yields $B(s)=0$, while the condition at $x=0$ yields $A(s)=F(s)$:

$$U(x,s)=F(s)e^{-\sqrt{s}x/\sqrt{k}}\ .$$

To apply the convolution theorem, we must invert $G(s)=e^{-\sqrt{s}x/\sqrt{k}}$. From the table $g(t)=\frac{a}{2\sqrt{\pi}}t^{-3/2}e^{-a^2/4t}$, where $a=x/\sqrt{k}$. Thus

$$u(x,t)=\int_0^t f(\bar{t})\frac{x}{2\sqrt{k\pi}}(t-\bar{t})^{-3/2}e^{-x^2/4k(t-\bar{t})}d\bar{t}\ .$$

Section 13.5

13.5.3 By taking the Laplace transform in t, we obtain

$$s^2\bar{U}=c^2\partial^2\bar{U}/\partial x^2$$

since $u(x,0)=\frac{\partial}{\partial t}u(x,0)=0$. We also Laplace transform the boundary conditions

$$\bar{U}(0,s)=0\quad\text{and}\quad\bar{U}_x(L,s)=B(s)\ ,$$

where $B(s)$ is the Laplace transform of $b(t)$. The general solution of this ordinary differential equation is

$$\bar{U}(x,s)=c_1(s)\sinh\frac{s}{c}x+c_2(s)\cosh\frac{s}{c}x\ .$$

The boundary conditions yield $0=c_2(s)$ and $B(s)=c_1\frac{s}{c}\cosh\frac{s}{c}L$, and thus

$$\bar{U}(x,s)=\frac{c}{s}B(s)\frac{\sinh\frac{s}{c}x}{\cosh\frac{s}{c}L}\ .$$

From the convolution theorem

$$u(x,t)=\int_0^t b(\bar{t})f(t-\bar{t})d\bar{t}$$

where $F(s)=\frac{c}{s}\frac{\sinh\frac{s}{c}x}{\cosh\frac{s}{c}L}\equiv\frac{G(s)}{s}$. Thus from exercise 13.2.4,

$$f(t)=\int_0^t g(\bar{t})d\bar{t},$$

where

$$G(s)=c\frac{\sinh\frac{s}{c}x}{\cosh\frac{s}{c}L}=c\frac{e^{\frac{s}{c}x}-e^{-\frac{s}{c}x}}{e^{\frac{s}{c}L}+e^{-\frac{s}{c}L}}=c\left[e^{\frac{s}{c}(x-L)}-e^{-\frac{s}{c}(x+L)}\right]\sum_{n=0}^{\infty}(-1)^n e^{-\frac{s}{c}2Ln}$$
$$=c\sum_{n=0}^{\infty}(-1)^n\left[e^{-\frac{s}{c}(L-x+2nL)}-e^{-\frac{s}{c}(x+L+2Ln)}\right]\ .$$

Each term in this series is inverted easily:

$$g(t)=c\sum_{n=0}^{\infty}(-1)^n\left[\delta(t-\frac{L-x+2Ln}{c})-\delta(t-\frac{x+L+2Ln}{c})\right]\ .$$

By integration

$$f(t)=c\sum_{n=0}^{\infty}(-1)^n\left[H(t-\frac{L-x+2Ln}{c})-H(t-\frac{x+L+2Ln}{c})\right]\ .$$

Section 13.6

13.6.4 (a) For the Green's function, $q(x,t)$ is replaced by $\delta(x-x_0)\delta(t-t_0)$. The corresponding homogeneous boundary condition must be satisfied $G(0,t;x_0,t_0)=0$ and $G\to 0$ as $x\to\infty$, along with the causality condition $G(x,t;x_0,t_0)=0$ for $t<t_0$. by taking the Laplace transform in t, the pde becomes

$$s\bar{G}=k\partial^2\bar{G}/\partial x^2+\delta(x-x_0)e^{-st_0}$$

subject to the boundary conditions $\bar{G}(0,s;x_0,t_0)=0$ and $\bar{G}\to 0$ as $x\to\infty$. The differential equation for $\bar{G}$ may be solved using techniques for Green's functions. For $x\neq x_0, s\bar{G}=k\partial^2\bar{G}/\partial x^2$. We choose independent homogeneous solutions that each satisfy one of the boundary conditions ($\sinh\sqrt{\frac{s}{k}}x$ and $e^{-\sqrt{s/kx}}$). Thus

$$\bar{G}(x,s;x_0,t_0)=\begin{cases} A\,e^{-\sqrt{s/k}\,x_0}\sinh\sqrt{\frac{s}{k}}x & x<x_0 \\ A\,\sinh\sqrt{s/k}\,x_0\,e^{-\sqrt{s/k}\,x} & x>x_0\ , \end{cases}$$

where we have chosen coefficients such that $\bar{G}$ is automatically continuous at $x=x_0$. Here A is a constant, independent of x_0, obtained from the jump condition on the derivative:

$$0=k\left.\frac{d\bar{G}}{dx}\right|_{x_0-}^{x_0+}+e^{-st_0}\ ,$$

since $\bar{G}$ is continuous. In this way, we obtain

$$A=\frac{e^{-st_0}e^{-\sqrt{s/k}\,x_0}}{\sqrt{sk}}(\sinh\sqrt{s/k}\,x_0+\cosh\sqrt{s/k}\,x_0)=\frac{e^{-st_0}}{\sqrt{sk}}\ .$$

Section 13.7

13.7.1 (b) Here $F(s)=1/(s^2+9)$ has simple poles at $s=\pm 3i$. The inversion integral is

$$f(t)=\frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty}\frac{e^{st}}{s^2+9}ds\ ,$$

for $t>0$, where $\gamma>0$ to be to the right of all singularities. Closing the contour to the left for $t>0$ and neglecting the large semi-circle (as can be justified) yields (13.7.11)

$$f(t)=\sum_n \mathrm{res}(s_n)\ ,$$

where these are the residues of e^{st}/s^2+9. Since these are simple poles at $s=\pm 3i$

$$f(t)=\sum_n\frac{e^{s_n t}}{2s_n}=\frac{e^{3it}}{6i}+\frac{e^{-3it}}{-6i}=\frac{1}{3}\frac{e^{3it}-e^{-3it}}{2i}=\frac{1}{3}\sin 3t\ ,$$

using Euler's formula.

Section 13.8

13.8.1 One way to obtain this result is to use (11.2.24) in which case

$$u(x,t)=-\int_0^L f(x_0)\frac{\partial}{\partial t_0}G(x,t;x_0,0)dx_0\ .$$

This can also be obtained using Laplace transforms. Here we evaluate $\frac{\partial}{\partial t_0}G(x,t;x_0,0)$ using Laplace transforms. From (13.6.11-12)

$$\frac{\partial}{\partial t_0}\bar{G}(x,t;x_0,0)=\frac{-1}{c\sinh\frac{s}{c}L}\begin{cases}\sinh\frac{s}{c}(L-x_0)\sinh\frac{s}{c}x & x<x_0\\ \sinh\frac{s}{c}x_0\sinh\frac{s}{c}(L-x) & x>x_0\ .\end{cases}$$

To invert this, we note as in (13.8.5) that the singularities are only simple poles s_n, located at the zeros of the denominator $\sinh \frac{L}{c} s_n = 0$. Again $s = 0$ is not a pole. There are an infinite number of poles located at

$$\frac{L}{c} s_n = i\, n\, \pi \qquad n = \pm 1,\ \pm 2,\ \pm 3, \ldots .$$

The residue at each pole may be evaluated

$$\operatorname{res}(s_n) = \frac{R(s_n)}{Q'(s_n)} = \frac{P(s_n) e^{s_n t}}{L \cosh(\frac{L}{c} s_n)} ,$$

where if $x < x_0$

$$P(s_n) = -\sinh \frac{s_n}{c}(L - x_0) \sinh \frac{s_n}{c} x ,$$

and the result for $x > x_0$ can be obtained by symmetry. Since $\sinh ix = i \sin x$

$$P(s_n) = \sin \frac{n\pi}{L}(L - x_0) \sin \frac{n\pi x}{L} = -\cos n\pi \sin \frac{n\pi x_0}{L} \sin \frac{n\pi x}{L} ,$$

which is thus also the result for $x > x_0$. Consequently

$$\frac{\partial}{\partial t_0} G(x, t; x_0, 0) = \sum_{n=-\infty (n\neq 0)}^{\infty} \frac{-\cos n\pi \sin \frac{n\pi x_0}{L} \sin \frac{n\pi x}{L} e^{in\pi ct/L}}{L \cos(n\pi)} = -\frac{2}{L} \sum_{n=1}^{\infty} \sin \frac{n\pi x_0}{L} \sin \frac{n\pi x}{L} \cos \frac{n\pi ct}{L} .$$

Thus

$$u(x, t) = \int_0^L f(x_0) \frac{2}{L} \sum_{n=1}^{\infty} \sin \frac{n\pi x_0}{L} \sin \frac{n\pi x}{L} \cos \frac{n\pi ct}{L} dx_0 .$$

This is the result of separation of variables since $u(x, t) = \sum_{n=1}^{\infty} a_n \sin \frac{n\pi x}{L} \cos \frac{n\pi ct}{L}$, where $a_n = \frac{2}{L} \int_0^L f(x_0) \sin \frac{n\pi x_0}{L} dx_0$.

Chapter 14. Dispersive Waves

Section 14.2

14.2.1 (c) $u = e^{i(kx-\omega t)}$ implies $i(-i\omega) = (ik)^2$ or $\omega = k^2$.

14.2.5 Assume $\phi = A(y)e^{i(kx-\omega t)}$ and free surface $s = Be^{i(kx-\omega t)}$. From Laplace's equation $\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} = 0$, we have

$$-k^2 A + A''(y) = 0.$$

To solve the bc at the bottom $y = -h$, it is easier to use solutions which are functions of $y + h$ rather than y. The general solution is a linear combination of exponentials but hyperbolic functions are simpler:

$$A(y) = c_1 \cosh k(y+h) + c_2 \sinh k(y+h).$$

The bc at $y = -h$ becomes $A'(-h) = 0$ or $c_2 = 0$. With $c_1 = A$,

$$\phi = A \cosh k(y+h) e^{i(kx-\omega t)}.$$

The free surface conditions (at $y = 0$) become

$$\frac{\partial\phi}{\partial t} + gs = 0 \Rightarrow -i\omega A \cosh kh + gB = 0$$
$$\frac{\partial\phi}{\partial y} - g\frac{\partial s}{\partial t} = 0 \Rightarrow kA \sinh kh - i\omega B = 0$$

This linear combination (in A and B) has a nontrivial solution only if its determinant is zero

$$\begin{vmatrix} -i\omega \cosh kh & g \\ k \sinh kh & i\omega\, . \end{vmatrix}$$

Thus $\omega^2 \cosh kh - gk \sinh kh = 0$ or $\omega^2 = gk \tanh kh$. This can also be obtained by elimination.

14.2.7 Assume $\phi = A(y)e^{i(kx-\omega t)}$ and free surface $s = Be^{i(kx-\omega t)}$. From Laplace's equation $\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} = 0$, we have

$$-k^2 A + A''(y) = 0,$$

whose general solution is

$$A(y) = c_1 e^{ky} + c_2 e^{-ky}.$$

Since $\frac{\partial\phi}{\partial y} = A'(y)e^{i(kx-\omega t)}$, the bc becomes $A'(y) \to 0$ as $y \to -\infty$. Thus $c_2 = 0$ (assuming $k > 0$). With $c_1 = A$,

$$\phi = Ae^{ky}e^{i(kx-\omega t)}.$$

The free surface conditions (at $y = 0$) become

$$\frac{\partial\phi}{\partial t} + gs = 0 \Rightarrow -i\omega A + gB = 0$$

$$\frac{\partial \phi}{\partial y} - g\frac{\partial s}{\partial t} = 0 \to kA - i\omega B = 0$$

This linear combination (in A and B) has a nontrivial solution only if its determinant is zero

$$\begin{vmatrix} -i\omega & g \\ k & i\omega. \end{vmatrix}$$

Thus $\omega^2 - gk = 0$ or $\omega^2 = g\,k$. This can also be obtained by elimination.

Section 14.3

14.3.9 (a) Let $u = G(x)e^{-i\omega_f t}$ so that

$$-i\omega_f G = G''' + \delta(x).$$

The jump conditions are that G and G' are continuous at $x = 0$, but $[G''] = -1$. Homogeneous solutions ($G = e^{rx}$) satisfy $-i\omega_f = r^3$. One root is $r = i\omega_f^{1/3} \equiv ik$. The other two roots are $r = (\pm\sqrt{3} - i)k/2$. Since $k = \omega_f^{1/3}$ or $\omega_f = k^3$, in this problem the group velocity $\frac{d\omega_f}{dk} = 3k^2 > 0$. The radiation condition implies that the oscillatory term exists only for $x > 0$. Thus

$$G = \begin{cases} b_2 e^{-\frac{i}{2}kx} e^{\frac{\sqrt{3}}{2}|k|x} & x < 0 \\ a_1 e^{ikx} + b_1 e^{-\frac{i}{2}kx} e^{-\frac{\sqrt{3}}{2}|k|x} & x > 0, \end{cases}$$

where we have insisted that the non-oscillatory solutions decay. The conditions at $x = 0$ are (letting $x = sgnk$)

G continuous: $b_2 = a_1 + b_1$

G' continuous: $b_2(-\frac{i}{2} + \frac{\sqrt{3}}{2}s) = ia_1 + b_1(-\frac{i}{2} - \frac{\sqrt{3}}{2}s)$

$[G''] = -1$: $-a_1 + b_1(-\frac{1}{4} + \frac{3}{4}s^2) - b_2(-\frac{1}{4} + \frac{3}{4}s^2) = -1$

Solving this system yields $a_1 = 2/3$, $3b_1 = i\sqrt{3}s - 1$, and $3b_2 = i\sqrt{3}s + 1$.

Section 14.5

14.5.3 (b) $\omega' = k^2 - k$. The group velocity is a parabola with minimum when $\omega'' = 0$ or $k = \frac{1}{2}$. Here $\omega'(\frac{1}{2}) = -\frac{1}{4}$. There are two waves if $\frac{x}{t} > -\frac{1}{4}$ and there are no waves if $\frac{x}{t} < -\frac{1}{4}$.

Section 14.6

14.6.1 $k = \frac{\partial\theta}{\partial x} = k_0 + (x - \omega'(k_0)t)\frac{\partial k_0}{\partial x} = k_0$ and $-\omega = \frac{\partial \theta}{\partial t} = -\omega_0 + (x - \omega'(k_0)t)\frac{\partial k_0}{\partial t} = -\omega_0$ since $x = \omega'(k_0)t$.

Section 14.7

14.7.8 From exercise (14.7.7), the consistency equation is $u_t = -\frac{1}{2}P_{xxx} + 2P_x(u - \lambda) + Pu_x$. Here we assume $P = A + B\lambda + C\lambda^2$ with C assumed constant. Substituting this into the consistency equation yields

$$u_t = -\frac{1}{2}(A_{xxx} + B_{xxx}\lambda) + 2(A_x + B_x\lambda)(u - \lambda) + (A + B\lambda + C\lambda^2)u_x.$$

Since we do not wish $u(x,t)$ to depend on λ, the coefficients of each power of λ must be identical. The $O(\lambda^2)$ terms are $0 = -2B_x + Cu_x$ which can be immediately integrated to yield

$$2B = Cu + 2B_0(t).$$

The $O(\lambda)$ terms are $0 = -\frac{1}{2}B_{xxx} + 2uB_x - 2A_x + Bu_x$. In order to integrate to solve for A, we eliminate B to show that $2uB_x + Bu_x = Cuu_x + u_x(\frac{1}{2}Cu + B_0(t))$ may in fact be integrated. Thus,

$$2A = -\frac{1}{2}B_{xx} + \frac{3}{4}Cu^2 + B_0 u + 2A_0(t).$$

Then the $O(1)$ terms yield the following generally nonlinear partial differential equations that satisfy Lax's equations

$$u_t = -\frac{1}{2}A_{xxx} + 2uA_x + Au_x.$$

Section 14.8

14.8.6 (b) Substituting $u = e^{ikx+(\sigma-i\omega)t}$ yields $\sigma - i\omega = -1 + Rk^2 - k^4 - ik^3$. Collecting real and imaginary parts yields the answer.

14.8.7 (c) Substituting $u = e^{ikx+(\sigma-i\omega)t}$ yields $\sigma - i\omega = -i(ik) = k$. Thus, $\sigma = k$ and $\omega = 0$. This problem is ill-posed because the growth rate σ is unbounded as $k \to \infty$.

Section 14.9

14.9.3 In the method of multiple scales, (14.9.22) is substituted into the equation and we obtain one addition term to (14.9.23) due to the nonlinear damping:

$$\omega^2\frac{\partial^2 u}{\partial\theta^2} + \epsilon[2\omega(T)\frac{\partial}{\partial T}\frac{\partial u}{\partial\theta} + \omega'(T)\frac{\partial u}{\partial\theta}] + \epsilon^2\frac{\partial^2 u}{\partial T^2} + \omega^2 u = -\epsilon(\omega\frac{\partial u}{\partial\theta} + \epsilon\frac{\partial u}{\partial T})^3.$$

When the perturbation expansion (14.9.24) is introduced, to leading order we obtain

$$\omega^2(\frac{\partial^2 u_0}{\partial\theta^2} + u_0) = 0$$

whose general solution is $u_0(\theta, T) = A(T)e^{i\theta} + A^*(T)e^{-i\theta}$. The $O(\epsilon)$ equation is

$$\omega^2(\frac{\partial^2 u_1}{\partial\theta^2} + u_1) = -[2\omega(T)\frac{\partial}{\partial T}\frac{\partial u_0}{\partial\theta} + \omega'(T)\frac{\partial u_0}{\partial\theta}] - \omega^3[\frac{\partial u_0}{\partial\theta}]^3$$

$$= -[2\omega(T)A'(T) + \omega'(T)A(T)]ie^{i\theta} + i\omega^3[A^3e^{3i\theta} - 3A^2A^*e^{i\theta}] + (*)$$

after substituting the expression for u_0. The third harmonic terms $e^{\pm 3i\theta}$ are not secular. However the terms $e^{\pm i\theta}$ correspond to the forcing frequency equaling the natural frequency, are said to be secular, and must be eliminated:

$$-2\omega(T)A'(T) - \omega'(T)A(T) - 3\omega^3A^2A^* = 0,$$

which is a differential equation the complex amplitude $A(T)$ must satisfy. To obtain equations for the amplitude and phase, we substitute $A(T) = r(T)e^{i\psi(T)}$ and obtain

$$2\omega r'(T) + \omega'(T)r(T) + 3\omega^3 r^3 = 0,$$

and $\psi'(T) = 0$ which implies that the slow phase $\psi(T)$ is a constant.

14.9.5 We use the method of multiple scales with a phase θ satisfying $\frac{d\theta}{dt} = \omega(\epsilon t)$ where the slowly varying frequency is the usual frequency for a pendulum of length L

$$\omega(\epsilon t) = \sqrt{\frac{g}{L(\epsilon t)}}.$$

We substitute (14.9.22) into $\frac{d^2u}{dt^2} + \omega^2(\epsilon t)u = -\epsilon \frac{2}{L(T)} \frac{dL}{dT} \frac{du}{dt}$ and obtain

$$\omega^2 \frac{\partial^2 u}{\partial \theta^2} + \epsilon[2\omega(T) \frac{\partial}{\partial T} \frac{\partial u}{\partial \theta} + \omega'(T) \frac{\partial u}{\partial \theta}] + \epsilon^2 \frac{\partial^2 u}{\partial T^2} + \omega^2 u = -\epsilon \frac{2}{L(T)} \frac{dL}{dT} (\omega \frac{\partial u}{\partial \theta} + \epsilon \frac{\partial u}{\partial T}).$$

instead of (14.9.23). When the perturbation expansion (14.9.24) is introduced, to leading order we obtain

$$\omega^2 (\frac{\partial^2 u_0}{\partial \theta^2} + u_0) = 0$$

whose general solution is $u_0(\theta, T) = A(T)e^{i\theta} + A^*(T)e^{-i\theta}$. The $O(\epsilon)$ equation is

$$\omega^2 (\frac{\partial^2 u_1}{\partial \theta^2} + u_1) = -[2\omega(T) \frac{\partial}{\partial T} \frac{\partial u_0}{\partial \theta} + \omega'(T) \frac{\partial u_0}{\partial \theta}] - \omega \frac{2}{L(T)} \frac{dL}{dT} \frac{\partial u_0}{\partial \theta}$$

$$= -[2\omega(T)A'(T) + \omega'(T)A(T)]ie^{i\theta} - i\omega \frac{2}{L(T)} \frac{dL}{dT} Ae^{i\theta} + (*)$$

after substituting the expression for u_0. All the terms on the right hand side, corresponding to the forcing frequency equaling the natural frequency, are said to be secular and must be eliminated:

$$2\omega(T)A'(T) + \omega'(T)A(T) + \omega \frac{2}{L(T)} \frac{dL}{dT} A = 0,$$

which is a differential equation the amplitude $A(T)$ must satisfy. This equation is easily solved by first dividing by ωA:

$$2\frac{A'}{A} + \frac{\omega'}{\omega} + \frac{2L'}{L} = 0.$$

Integrating yields $2 \ln A + \ln \omega + 2 \ln L$ =constant, so that the amplitude satisfies

$$A(T) = c_1 \omega^{-1/2} L^{-1} = c_2 L^{1/4} L^{-1} = c_2 L^{-3/4},$$

using the expression for the frequency ω. Since $\frac{d\theta}{dt} = \omega(\epsilon t) = \sqrt{\frac{g}{L(\epsilon t)}}$,

it follows that $\theta = \int \sqrt{\frac{g}{L(\epsilon t)}} dt + \psi$. Thus

$$u_0 = cL^{-3/4} \cos(\int \sqrt{\frac{g}{L(\epsilon t)}} dt + \psi).$$

14.9.8 In the method of multiple scales, we assume the wave amplitude and frequency are slowly varying and satisfy $k = \frac{\partial \theta}{\partial x}$ and $\omega = -\frac{\partial \theta}{\partial t}$. Using (14.9.67) and (14.9.68), our equation $\frac{\partial u}{\partial t} = \frac{\partial^3 u}{\partial x^3}$ becomes

$$-\omega \frac{\partial u}{\partial \theta} + \epsilon \frac{\partial u}{\partial T} = (k \frac{\partial}{\partial \theta} + \epsilon \frac{\partial}{\partial X})^3 u = k^3 \frac{\partial^3 u}{\partial \theta^3} + \epsilon(3k^2 \frac{\partial^3 u}{\partial X \partial \theta^2} + 3kk_X \frac{\partial^2 u}{\partial \theta^2}) + O(\epsilon^2).$$

We claim that the perturbation method will not work unless the frequency satisfies the dispersion relation

$$\omega = k^3.$$

Conservation of waves $k_T + \omega_X = 0$ becomes $k_T + 3k^2 k_X = 0$. The perturbation expansion $u = u_0 + \epsilon u_1 + \ldots$, yields the $O(1)$ equation $-k^3(\frac{\partial u_0}{\partial \theta} + \frac{\partial^3 u_0}{\partial \theta^3}) = 0$, so that the slowly varying plane wave satisfies $u_0 = A(\mathrm{X},T)e^{i\theta} + A^*(\mathrm{X},T)e^{-i\theta}$. The $O(\epsilon)$ equation is

$$-k^3\left(\frac{\partial u_1}{\partial \theta} + \frac{\partial^3 u_1}{\partial \theta^3}\right) = -\frac{\partial u_0}{\partial T} + 3k^2 \frac{\partial^3 u_0}{\partial X \partial \theta^2} + 3kk_X \frac{\partial^2 u_0}{\partial \theta^2}$$

$$= -A_T e^{i\theta} - 3(k^2 A_X + kk_X A)e^{i\theta} + (*).$$

All terms on the right-hand side are secular (with forcing frequency equaling natural frequency) and must be eliminated, so that

$$A_T + 3k^2 A_X + 3kk_X A = 0.$$

Section 14.10

14.10.3 In this problem (as we will show), there can be a boundary layer at both boundaries $x = 0$ and $x = 1$. The leading-order solution of the outer expansion ($u = u_0 + \ldots$) is

$$u_0 = -\frac{x}{4}.$$

Inner (left): Near $x = 0$, we introduce the boundary layer scaling $x = \epsilon^{1/2} X_L$, so that the differential equation becomes

$$\frac{d^2 u}{dX_L^2} - 4u = \epsilon^{1/2} X_L.$$

The inner (left) expansion ($u = U_0(X_L) + \ldots$) yields to leading-order $\frac{d^2 U_0}{dX_L^2} - 4U_0 = 0$, whose general solution is $U_0 = c_1 e^{2X_L} + c_2 e^{-2X_L}$. In matching the left boundary layer solution to the outer solution, the limit $X_L \to +\infty$ is involved, so that to avoid exponential growth on the boundary layer scale, $c_1 = 0$ and thus $U_0 = c_2 e^{-2X_L}$. The boundary condition at $x = 0\,(X_L = 0)$ is satisfied if $c_2 = 1$ so that the leading-order solution in the left boundary layer is

$$U_0 = e^{-2X_L}.$$

Inner (right): Near $x = 1$, we introduce the boundary layer scaling $x - 1 = \epsilon^{1/2} X_R$, so that the differential equation becomes

$$\frac{d^2 u}{dX_R^2} - 4u = 1 + \epsilon^{1/2} X_R.$$

The inner (right) expansion ($u = V_0(X_R) + \ldots$) yields to leading-order $\frac{d^2 V_0}{dX_R^2} - 4V_0 = 1$, whose general solution is $V_0 = -\frac{1}{4} + c_3 e^{2X_R} + c_4 e^{-2X_R}$. In matching the right boundary layer solution to the outer solution, the limit $X_R \to -\infty$ is involved, so that to avoid exponential growth on the boundary layer scale, $c_4 = 0$ and thus $V_0 = -\frac{1}{4} + c_3 e^{2X_R}$. The boundary condition at $x = 1\,(X_R = 0)$ is satisfied if $2 = -\frac{1}{4} + c_3$ or $c_3 = \frac{9}{4}$ so that the leading-order solution in the right boundary layer is

$$V_0 = -\frac{1}{4} + \frac{9}{4} e^{2X_R}.$$

In this problem, the inner and outer solutions have been obtained without matching. However, one should always do at least a leading-order match. In this problem the matching of the leading-order inner (left) to the leading-order outer solution is $\lim_{X_L\to+\infty} U_0 = \lim_{x\to 0} u_0 = 0$ since $U_0 = e^{-2X_L}$ and $u_0 = -\frac{x}{4}$. In this problem the matching of the leading-order inner (right) to the leading-order outer solution is $\lim_{X_R\to-\infty} V_0 = \lim_{x\to 1} u_0 = -\frac{1}{4}$ since $V_0 = -\frac{1}{4} + \frac{9}{4}e^{2X_R}$ and $u_0 = -\frac{x}{4}$.

14.10.6 For the second-order differential equation $\epsilon\frac{d^2u}{dx^2} + (2x+1)\frac{du}{dx} + 2u = 0$, the differential equation is reduced to first-order when $\epsilon = 0$. Thus, we expect a boundary layer to be necessary. To determine the location of the boundary layer (either $x = 0$ or $x = 1$), we note that derivatives will be large in a boundary layer so that roughly the differential equation will be $\frac{d^2u}{dX^2} + (\text{positive})\frac{du}{dX} = 0$ (because $2x+1 > 0$) whose solution is $u = A + Be^{-(\text{positive})X}$. The limit $X \to -\infty$ must be not permitted so that a boundary layer can only occur at $x = 0$. The outer solution (away from the boundary layer) has an outer expansion $u = u_0(x) + \ldots$. The leading-order outer equation is $(2x+1)\frac{du_0}{dx} + 2u_0 = 0$. This first-order linear ordinary differential equation can be solved by many methods, but since it is an Euler equation perhaps the simplest method is to observe that some power $u_0 = c(2x+1)^r$ satisfies this equation, where by direct substitution $2r + 2 = 0$ or equivalently $r = -1$. The constant c can be determined from the boundary condition at $x = 1$ since we have shown that a boundary layer exists in this problem at $x = 0$. In this way $c = 6$ and the leading-order outer solution (which solves the boundary condition at $x = 1$) is

$$u_0 = 6(2x+1)^{-1}.$$

The boundary layer at $x = 0$ has the scaling $x = \epsilon X$ so that the differential equation in the boundary layer is $\frac{d^2u}{dX^2} + (2\epsilon X + 1)\frac{du}{dX} + 2\epsilon u = 0$. The inner expansion is $u = U_0(X) + \ldots$ and the leading-order inner solution satisfies $\frac{d^2U_0}{dX^2} + \frac{dU_0}{dX} = 0$, whose general solution is $U_0(X) = A + Be^{-X}$. The boundary condition at $x = 0$ $(X = 0)$ yields $1 = A + B$. Thus, we may express the leading-order inner solution as

$$U_0(X) = 1 - B + Be^{-X}.$$

The constant B can be determined by matching the leading-order inner and outer solutions. Here, $\lim_{X\to+\infty} U_0 = \lim_{x\to 0} u_0$, which yields $1 - B = 6$ or $B = -5$. Thus, the leading-order inner solution is $U_0(X) = 6 - 5e^{-X}$ where $x = \epsilon X$.