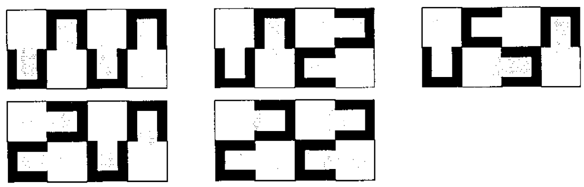

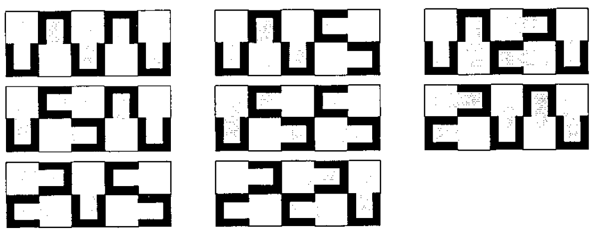

see also FIBONACCI NUMBER, GOMORY'S THEOREM, HEXOMINO, PENTOMINO, POLYOMINO, TETROMINO, TRIOMINO

References

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Gardner, M. "Polyominoes." Ch. 13 in *The Scientific American Book of Mathematical Puzzles & Diversions.* New York: Simon and Schuster, pp. 124–140, 1959.

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Lei, A. "Domino." http://www.cs.ust.hk/~philipl/omino/domino.html.

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Domino Problem

see WANG'S CONJECTURE

Donaldson Invariants

Distinguish between smooth MANIFOLDS in 4-D.

Donkin's Theorem

The product of three translations along the directed sides of a TRIANGLE through twice the lengths of these sides is the identity.

Donut

see TORUS

Doob's Theorem

A theorem proved by Doob (1942) which states that any random process which is both GAUSSIAN and MARKOV has the following forms for its correlation function, spectral density, and probability densities:

$$C_y(\tau) = {\sigma_y}^2 e^{-\tau/\tau_r}$$

$$G_y(f) = \frac{4{\tau_r}^{-1}{\sigma_y}^2}{(2\pi f)^2 + {\tau_r}^{-2}}$$

$$p_1(y) = \frac{1}{\sqrt{2\pi{\sigma_y}^2}} e^{-(y-\bar{y})^2/2{\sigma_y}^2}$$

$$p_2(y_1|y_2,\tau) = \frac{1}{\sqrt{2\pi(1-e^{-2\tau/\tau_r}){\sigma_y}^2}} \times \exp\left\{-\frac{[(y_2-\bar{y}) - e^{-\tau/\tau_r}(y_1-\bar{y})]^2}{2(1-e^{-2\tau/\tau_r}){\sigma_y}^2}\right\},$$

where $\bar{y}$ is the MEAN, σ_y the STANDARD DEVIATION, and τ_r the relaxation time.

References

Doob, J. L. "Topics in the Theory of Markov Chains." *Trans. Amer. Math. Soc.* **52**, 37–64, 1942.

Dot

The "dot" $\cdot$ has several meanings in mathematics, including MULTIPLICATION ($a \cdot b$ is pronounced "a times b"), computation of a DOT PRODUCT ($\mathbf{a}\cdot\mathbf{b}$ is pronounced "$\mathbf{a}$ dot $\mathbf{b}$"), or computation of a time DERIVATIVE ($\dot{a}$ is pronounced "a dot").

see also DERIVATIVE, DOT PRODUCT, TIMES

Dot Product

The dot product can be defined by

$$\mathbf{X} \cdot \mathbf{Y} = |\mathbf{X}|\,|\mathbf{Y}| \cos\theta, \tag{1}$$

where θ is the angle between the vectors. It follows immediately that $\mathbf{X}\cdot\mathbf{Y} = 0$ if $\mathbf{X}$ is PERPENDICULAR to $\mathbf{Y}$. The dot product is also called the INNER PRODUCT and written $\langle a, b\rangle$. By writing

$$A_x = A\cos\theta_A \qquad B_x = B\cos\theta_B \tag{2}$$

$$A_y = A\sin\theta_A \qquad B_y = B\sin\theta_B, \tag{3}$$

it follows that (1) yields

$$\begin{aligned}\mathbf{A}\cdot\mathbf{B} &= AB\cos(\theta_A - \theta_B)\\ &= AB(\cos\theta_A\cos\theta_B + \sin\theta_A\sin\theta_B)\\ &= A\cos\theta_A B\cos\theta_B + A\sin\theta_A B\sin\theta_B\\ &= A_xB_x + A_yB_y.\end{aligned} \tag{4}$$

So, in general,

$$\mathbf{X}\cdot\mathbf{Y} = x_1y_1 + \ldots + x_ny_n. \tag{5}$$

The dot product is COMMUTATIVE

$$\mathbf{X}\cdot\mathbf{Y} = \mathbf{Y}\cdot\mathbf{X}, \tag{6}$$

ASSOCIATIVE

$$(r\mathbf{X})\cdot\mathbf{Y} = r(\mathbf{X}\cdot\mathbf{Y}), \tag{7}$$

and DISTRIBUTIVE

$$\mathbf{X}\cdot(\mathbf{Y}+\mathbf{Z}) = \mathbf{X}\cdot\mathbf{Y} + \mathbf{X}\cdot\mathbf{Z}. \tag{8}$$

The DERIVATIVE of a dot product of VECTORS is

$$\frac{d}{dt}[\mathbf{r}_1(t)\cdot\mathbf{r}_2(t)] = \mathbf{r}_1(t)\cdot\frac{d\mathbf{r}_2}{dt} + \frac{d\mathbf{r}_1}{dt}\cdot\mathbf{r}_2(t). \tag{9}$$

The dot product is invariant under rotations

$$\begin{aligned}\mathbf{A}' \cdot \mathbf{B}' &= A_i' B_i' = a_{ij} A_j a_{ik} B_k = (a_{ij} a_{ik}) A_j B_k \\ &= \delta_{jk} A_j B_k = A_j B_j = \mathbf{A} \cdot \mathbf{B}, \qquad (10)\end{aligned}$$

where EINSTEIN SUMMATION has been used.

The dot product is also defined for TENSORS A and B by

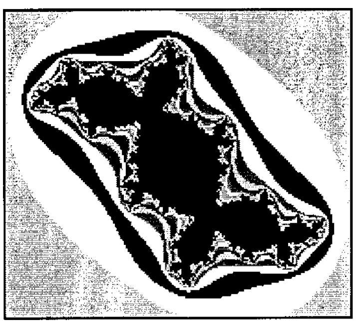
$$A \cdot B \equiv A^\alpha B_\alpha. \qquad (11)$$

see also CROSS PRODUCT, INNER PRODUCT, OUTER PRODUCT, WEDGE PRODUCT

References

Arfken, G. "Scalar or Dot Product." §1.3 in *Mathematical Methods for Physicists, 3rd ed.* Orlando, FL: Academic Press, pp. 13–18, 1985.

Douady's Rabbit Fractal



A JULIA SET with $c = -0.123 + 0.745i$, also known as the DRAGON FRACTAL.

see also SAN MARCO FRACTAL, SIEGEL DISK FRACTAL

References

Wagon, S. *Mathematica in Action.* New York: W. H. Freeman, p. 176, 1991.

Double Bubble

The planar double bubble (three circular arcs meeting in two points at equal 120° ANGLES) has the minimum PERIMETER for enclosing two equal areas (Foisy 1993, Morgan 1995).

see also APPLE, BUBBLE, DOUBLE BUBBLE CONJECTURE, SPHERE-SPHERE INTERSECTION

References

Campbell, P. J. (Ed.). Reviews. *Math. Mag.* **68**, 321, 1995.

Foisy, J.; Alfaro, M.; Brock, J.; Hodges, N.; and Zimba, J. "The Standard Double Soap Bubble in $\mathbb{R}^2$ Uniquely Minimizes Perimeter." *Pacific J. Math.* **159**, 47–59, 1993.

Morgan, F. "The Double Bubble Conjecture." *FOCUS* **15**, 6–7, 1995.

Peterson, I. "Toil and Trouble over Double Bubbles." *Sci. News* **148**, 101, Aug. 12, 1995.

Double Bubble Conjecture

Two partial SPHERES with a separating boundary (which is planar for equal volumes) separate two volumes of air with less AREA than any other boundary. The planar case was proved true for equal volumes by J. Hass and R. Schlafy in 1995 by reducing the problem to a set of 200,260 integrals which they carried out on an ordinary PC.

see also DOUBLE BUBBLE

References

Haas, J. and Schlafy, R. "Double Bubbles Minimize." Preprint, 1995.

Double Contraction Relation

A TENSOR t is said to satisfy the double contraction relation when

$$t_{ij}^{m*} t_{ij}^n = \delta_{mn}.$$

This equation is satisfied by

$$\begin{aligned}\hat{t}^0 &= \frac{2\hat{\mathbf{z}}\hat{\mathbf{z}} - \hat{\mathbf{x}}\hat{\mathbf{x}} - \hat{\mathbf{y}}\hat{\mathbf{y}}}{\sqrt{6}} \\ \hat{t}^{\pm 1} &= \mp\tfrac{1}{2}(\hat{\mathbf{x}}\hat{\mathbf{z}} + \hat{\mathbf{z}}\hat{\mathbf{x}}) - \tfrac{1}{2}i(\hat{\mathbf{y}}\hat{\mathbf{z}} - \hat{\mathbf{z}}\hat{\mathbf{y}}) \\ \hat{t}^{\pm 2} &= \mp\tfrac{1}{2}(\hat{\mathbf{x}}\hat{\mathbf{x}} + \hat{\mathbf{y}}\hat{\mathbf{y}}) - \tfrac{1}{2}i(\hat{\mathbf{x}}\hat{\mathbf{y}} - \hat{\mathbf{y}}\hat{\mathbf{x}}),\end{aligned}$$

where the hat denotes zero trace, symmetric unit TENSORS. These TENSORS are used to define the SPHERICAL HARMONIC TENSOR.

see also SPHERICAL HARMONIC TENSOR, TENSOR

References

Arfken, G. "Alternating Series." *Mathematical Methods for Physicists, 3rd ed.* Orlando, FL: Academic Press, p. 140, 1985.

Double Cusp

see DOUBLE POINT

Double Exponential Distribution

see FISHER-TIPPETT DISTRIBUTION, LAPLACE DISTRIBUTION

Double Exponential Integration

An excellent NUMERICAL INTEGRATION technique used by *Maple V R4*® (Waterloo Maple Inc.) for numerical computation of integrals.

see also INTEGRAL, INTEGRATION, NUMERICAL INTEGRATION

References

Davis, P. J. and Rabinowitz, P. *Methods of Numerical Integration, 2nd ed.* New York: Academic Press, p. 214, 1984.

Di Marco, G.; Favati, P.; Lotti, G.; and Romani, F. "Asymptotic Behaviour of Automatic Quadrature." *J. Complexity* **10**, 296–340, 1994.

Mori, M. *Developments in the Double Exponential Formula for Numerical Integration. Proceedings of the International Congress of Mathematicians, Kyoto 1990.* New York: Springer-Verlag, pp. 1585–1594, 1991.

Mori, M. and Ooura, T. "Double Exponential Formulas for Fourier Type Integrals with a Divergent Integrand." In *Contributions in Numerical Mathematics* (Ed. R. P. Agarwal). World Scientific Series in Applicable Analysis, Vol. 2, pp. 301–308, 1993.
Ooura, T. and Mori, M. "The Double Exponential Formula for Oscillatory Functions over the Half Infinite Interval." *J. Comput. Appl. Math.* **38**, 353–360, 1991.
Takahasi, H. and Mori, M. "Double Exponential Formulas for Numerical Integration." *Pub. RIMS Kyoto Univ.* **9**, 721–741, 1974.
Toda, H. and Ono, H. "Some Remarks for Efficient Usage of the Double Exponential Formulas." *Kokyuroku RIMS Kyoto Univ.* **339**, 74–109, 1978.

Double Factorial

The double factorial is a generalization of the usual FACTORIAL $n!$ defined by

$$n!! \equiv \begin{cases} n\cdot(n-2)\ldots 5\cdot 3\cdot 1 & n \text{ odd} \\ n\cdot(n-2)\ldots 6\cdot 4\cdot 2 & n \text{ even} \\ 1 & n=-1,0. \end{cases} \tag{1}$$

For $n = 0, 1, 2, \ldots$, the first few values are 1, 1, 2, 3, 8, 15, 48, 105, 384, ... (Sloane's A006882).

There are many identities relating double factorials to FACTORIALS. Since

$$\begin{aligned}&(2n+1)!!2^n n!\\ &= [(2n+1)(2n-1)\cdots 1][2n][2(n-1)][2(n-2)]\cdots 2(1)\\ &= [(2n+1)(2n-1)\cdots 1][2n(2n-2)(2n-4)\cdots 2]\\ &= (2n+1)(2n)(2n-1)(2n-2)(2n-3)(2n-4)\cdots 2(1)\\ &= (2n+1)!,\end{aligned} \tag{2}$$

it follows that $(2n+1)!! = \frac{(2n+1)!}{2^n n!}$. Since

$$\begin{aligned}(2n)!! &= (2n)(2n-2)(2n-4)\cdots 2\\ &= [2(n)][2(n-1)][2(n-2)]\cdots 2 = 2^n n!,\end{aligned} \tag{3}$$

it follows that $(2n)!! = 2^n n!$. Since

$$\begin{aligned}&(2n-1)!!2^n n!\\ &= [(2n-1)(2n-3)\cdots 1][2n][2(n-1)][2(n-2)]\cdots 2(1)\\ &= (2n-1)(2n-3)\cdots 1][2n(2n-2)(2n-4)\cdots 2]\\ &= 2n(2n-1)(2n-2)(2n-3)(2n-4)\cdots 2(1)\\ &= (2n)!,\end{aligned} \tag{4}$$

it follows that

$$(2n-1)!! = \frac{(2n)!}{2^n n!}. \tag{5}$$

Similarly, for $n = 0, 1, \ldots$,

$$(-2n-1)!! = \frac{(-1)^n}{(2n-1)!!} = \frac{(-1)n2^n n!}{(2n)!}. \tag{6}$$

For n ODD,

$$\begin{aligned}\frac{n!}{n!!} &= \frac{n(n-1)(n-2)\cdots(1)}{n(n-2)(n-4)\cdots(1)}\\ &= (n-1)(n-3)\cdots(1) = (n-1)!!.\end{aligned} \tag{7}$$

For n EVEN,

$$\begin{aligned}\frac{n!}{n!!} &= \frac{n(n-1)(n-2)\cdots(2)}{n(n-2)(n-4)\cdots(2)}\\ &= (n-1)(n-3)\cdots(2) = (n-1)!!.\end{aligned} \tag{8}$$

Therefore, for any n,

$$\frac{n!}{n!!} = (n-1)!! \tag{9}$$

$$n! = n!!(n-1)!!. \tag{10}$$

The FACTORIAL may be further generalized to the MULTIFACTORIAL

see also FACTORIAL, MULTIFACTORIAL

References

Sloane, N. J. A. Sequence A006882/M0876 in "An On-Line Version of the Encyclopedia of Integer Sequences."

Double Folium

see BIFOLIUM

Double-Free Set

A SET of POSITIVE integers is double-free if, for any integer x, the SET $\{x, 2x\} \not\subset S$ (or equivalently, if $x \in S$ IMPLIES $2x \notin S$). Define

$$r(n) = \max\{S : S \subset \{1, 2, \ldots, n\} \text{ is double-free}\}.$$

Then an asymptotic formula is

$$r(n) \sim \tfrac{2}{3}n + \mathcal{O}(\ln n)$$

(Wang 1989).

see also TRIPLE-FREE SET

References

Finch, S. "Favorite Mathematical Constants." `http://www.mathsoft.com/asolve/constant/triple/triple.html`.
Wang, E. T. H. "On Double-Free Sets of Integers." *Ars Combin.* **28**, 97–100, 1989.

Double Gamma Function

see DIGAMMA FUNCTION

Double Point

A point traced out twice as a closed curve is traversed. The maximum number of double points for a nondegenerate QUARTIC CURVE is three. An ORDINARY DOUBLE POINT is called a NODE.

Arnold (1994) gives pictures of spherical and PLANE CURVES with up to five double points, as well as other curves.

see also BIPLANAR DOUBLE POINT, CONIC DOUBLE POINT, CRUNODE, CUSP, ELLIPTIC CONE POINT, GAUSS'S DOUBLE POINT THEOREM, NODE (ALGEBRAIC CURVE), ORDINARY DOUBLE POINT, QUADRUPLE POINT RATIONAL DOUBLE POINT, SPINODE, TACNODE, TRIPLE POINT, UNIPLANAR DOUBLE POINT

References

Aicardi, F. Appendix to "Plane Curves, Their Invariants, Perestroikas, and Classifications." In *Singularities & Bifurcations* (V. I. Arnold). Providence, RI: Amer. Math. Soc., pp. 80–91, 1994.

Fischer, G. (Ed.). *Mathematical Models from the Collections of Universities and Museums.* Braunschweig, Germany: Vieweg, pp. 12–13, 1986.

Double Sixes

Two sextuples of SKEW LINES on the general CUBIC SURFACE such that each line of one is SKEW to one LINE in the other set. Discovered by Schläfli.

see also BOXCARS, CUBIC SURFACE, SOLOMON'S SEAL LINES

References

Fischer, G. (Ed.). *Mathematical Models from the Collections of Universities and Museums.* Braunschweig, Germany: Vieweg, p. 11, 1986.

Double Sum

A nested sum over two variables. Identities involving double sums include the following:

$$\sum_{p=0}^{\infty}\sum_{q=0}^{p} a_{q,p-q} = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} a_{n,m} = \sum_{r=0}^{\infty}\sum_{s=0}^{\lfloor r/2\rfloor} a_{s,r-2s}, \quad (1)$$

where

$$\lfloor r/2\rfloor = \begin{cases} \frac{1}{2}r & r \text{ even} \\ \frac{1}{2}(r-1) & r \text{ odd} \end{cases} \quad (2)$$

is the FLOOR FUNCTION, and

$$\sum_{i=1}^{n}\sum_{j=1}^{n} x_i x_j = n^2 \left\langle x^2 \right\rangle. \quad (3)$$

Consider the sum

$$S(a,b,c;s) = \sum_{(m,n)\neq(0,0)} (am^2+bmn+cn^2)^{-s} \quad (4)$$

over binary QUADRATIC FORMS. If S can be decomposed into a linear sum of products of DIRICHLET L-SERIES, it is said to be solvable. The related sums

$$S_1(a,b,c;s) = \sum_{(m,n)\neq(0,0)} (-1)^m (am^2+bmn+cn^2)^{-s} \quad (5)$$

$$S_2(a,b,c;s) = \sum_{(m,n)\neq(0,0)} (-1)^n (am^2+bmn+cn^2)^{-s} \quad (6)$$

$$S_{1,2}(a,b,c;s) = \sum_{(m,n)\neq(0,0)} (-1)^{m+n} (am^2+bmn+cn^2)^{-s} \quad (7)$$

can also be defined, which gives rise to such impressive FORMULAS as

$$S_1(1,0,58;1) = -\frac{\pi\ln(27+5\sqrt{29})}{\sqrt{58}}. \quad (8)$$

A complete table of the principal solutions of all solvable $S(a,b,c;s)$ is given in Glasser and Zucker (1980, pp. 126–131).

see also EULER SUM

References

Glasser, M. L. and Zucker, I. J. "Lattice Sums in Theoretical Chemistry." *Theoretical Chemistry: Advances and Perspectives, Vol. 5.* New York: Academic Press, 1980.

Zucker, I. J. and Robertson, M. M. "A Systematic Approach to the Evaluation of $\sum_{(m,n\neq 0,0)}(am^2+bmn+cn^2)^{-s}$." *J. Phys. A: Math. Gen.* **9**, 1215–1225, 1976.

Doublet Function

$$y = \delta'(x-a),$$

where $\delta(x)$ is the DELTA FUNCTION.

see also DELTA FUNCTION

References

von Seggern, D. *CRC Standard Curves and Surfaces.* Boca Raton, FL: CRC Press, p. 324, 1993.

Doubly Even Number

An even number N for which $N \equiv 0 \pmod 4$. The first few POSITIVE doubly even numbers are 4, 8, 12, 16, ... (Sloane's A008586).

see also EVEN FUNCTION, ODD NUMBER, SINGLY EVEN NUMBER

References

Sloane, N. J. A. Sequence A008586 in "An On-Line Version of the Encyclopedia of Integer Sequences."

Doubly Magic Square

see BIMAGIC SQUARE

Dougall-Ramanujan Identity

Discovered by Ramanujan around 1910. From Hardy (1959, pp. 102–103),

$$\sum_{n=0}^{\infty}(-1)^n(s+2n)\frac{s^{(n)}}{1^{(n)}}\frac{(x+y+z+u+2s+1)^{(n)}}{(x+y+z+u-s)_{(n)}} \times \prod_{x,y,z,u}\frac{x_{(n)}}{(x+s+1)^{(n)}}$$
$$= \frac{s}{\Gamma(s+1)\Gamma(x+y+z+u+s+1)} \times \prod_{x,y,z,u}\frac{\Gamma(x+s+1)\Gamma(y+z+u+s+1)}{\Gamma(z+u+s+1)}, \quad (1)$$

where

$$a^{(n)} \equiv a(a+1)\cdots(a+n-1) \quad (2)$$

$$a_{(n)} \equiv a(a-1)\cdots(a-n+1) \quad (3)$$

(here, the POCHHAMMER SYMBOL has been written $a^{(n)}$). This can be rewritten as

$$ {}_7F_6\left[\begin{matrix} s, 1+\frac{1}{2}s, -x-y, -z, -u, x-y+z+u+2s+1 \\ \frac{1}{2}s, x+s+1, y+s+1, z+s+1, u+s+1, \quad ;1 \\ -x-y-z-u-s \end{matrix}\right]$$
$$= \frac{1}{\Gamma(s+1)\Gamma(x+y+z+u+s+1)} \times \prod_{x,t,z,u}\frac{\Gamma(x+s+1)\Gamma(y+z+u+s+1)}{\Gamma(z+u+s+1)}. \quad (4)$$

In a more symmetric form, if $n = 2a_1+1 = a_2+a_3+a_4+a_5$, $a_6 = 1+a_1/2$, $a_7 = -n$, and $b_i = 1+a_1-a_{i+1}$ for $i = 1, 2, \ldots, 6$, then

$$ {}_7F_6\left[\begin{matrix} a_1, a_2, a_3, a_4, a_5, a_6, a_7 \\ b_1, b_2, b_3, b_4, b_5, b_6 \end{matrix}; 1\right]$$
$$\frac{(a_1+1)_n(a_1-a_2-a_3+1)_n}{(a_1-a_2+1)_n(a_1-a_3+1)_n} \times \frac{(a_1-a_2-a_4+1)_n(a_1-a_3-a_4+1)_n}{(a_1-a_4+1)_n(a_1-a_2-a_3-a_4+1)_n}, \quad (5)$$

where $(a)_n$ is the POCHHAMMER SYMBOL (Petkovšek *et al.* 1996).

The identity is a special case of JACKSON'S IDENTITY.

see also DIXON'S THEOREM, DOUGALL'S THEOREM, GENERALIZED HYPERGEOMETRIC FUNCTION, HYPERGEOMETRIC FUNCTION, JACKSON'S IDENTITY, SAALSCHÜTZ'S THEOREM

References

Dixon, A. C. "Summation of a Certain Series." *Proc. London Math. Soc.* **35**, 285–289, 1903.

Hardy, G. H. *Ramanujan: Twelve Lectures on Subjects Suggested by His Life and Work, 3rd ed.* New York: Chelsea, 1959.

Petkovšek, M.; Wilf, H. S.; and Zeilberger, D. *A=B.* Wellesley, MA: A. K. Peters, pp. 43, 126–127, and 183–184, 1996.

Dougall's Theorem

$$ {}_5F_4\left[\begin{matrix} \frac{1}{2}n+1, n, -x, -y, -z \\ \frac{1}{2}n, x+n+1, y+n+1, z+n+1 \end{matrix}\right] = \frac{\Gamma(x+n+1)\Gamma(y+n+1)\Gamma(z+n+1)\Gamma(x+y+z+n+1)}{\Gamma(n+1)\Gamma(x+y+n+1)\Gamma(y+z+n+1)\Gamma(x+z+n+1)},$$

where ${}_5F_4(a,b,c,d,e;f,g,h,i;z)$ is a GENERALIZED HYPERGEOMETRIC FUNCTION and $\Gamma(z)$ is the GAMMA FUNCTION.

see also DOUGALL-RAMANUJAN IDENTITY, GENERALIZED HYPERGEOMETRIC FUNCTION

Doughnut

see TORUS

Douglas-Neumann Theorem

If the lines joining corresponding points of two directly similar figures are divided proportionally, then the LOCUS of the points of the division will be a figure directly similar to the given figures.

References

Eves, H. "Solution to Problem E521." *Amer. Math. Monthly* **50**, 64, 1943.

Musselman, J. R. "Problem E521." *Amer. Math. Monthly* **49**, 335, 1942.

Dovetailing Problem

see CUBE DOVETAILING PROBLEM

Dowker Notation

A simple way to describe a knot projection. The advantage of this notation is that it enables a KNOT DIAGRAM to be drawn quickly.

For an oriented ALTERNATING KNOT with n crossings, begin at an arbitrary crossing and label it 1. Now follow the undergoing strand to the next crossing, and denote it 2. Continue around the knot following the same strand until each crossing has been numbered twice. Each crossing will have one even number and one odd number, with the numbers running from 1 to $2n$.

Now write out the ODD NUMBERS 1, 3, ..., $2n-1$ in a row, and underneath write the even crossing number corresponding to each number. The Dowker NOTATION is this bottom row of numbers. When the sequence of even numbers can be broken into two permutations of consecutive sequences (such as $\{4,6,2\}$ $\{10,12,8\}$), the knot is composite and is not uniquely determined by the Dowker notation. Otherwise, the knot is prime and the NOTATION uniquely defines a single knot (for amphichiral knots) or corresponds to a single knot or its MIRROR IMAGE (for chiral knots).

For general nonalternating knots, the procedure is modified slightly by making the sign of the even numbers

POSITIVE if the crossing is on the top strand, and NEGATIVE if it is on the bottom strand.

These data are available only for knots, but not for links, from Berkeley's gopher site.

References
Adams, C. C. *The Knot Book: An Elementary Introduction to the Mathematical Theory of Knots.* New York: W. H. Freeman, pp. 35–40, 1994.
Dowker, C. H. and Thistlethwaite, M. B. "Classification of Knot Projections." *Topol. Appl.* **16**, 19–31, 1983.

Down Arrow Notation

An inverse of the up ARROW NOTATION defined by

$$e \downarrow n = \ln n$$
$$e \downarrow\downarrow n = \ln^* n$$
$$e \downarrow\downarrow\downarrow n = \ln^{**} n,$$

where $\ln^* n$ is the number of times the NATURAL LOGARITHM must be iterated to obtain a value $\leq e$.

see also ARROW NOTATION

References
Vardi, I. *Computational Recreations in Mathematica.* Redwood City, CA: Addison-Wesley, pp. 12 and 231–232, 1991.

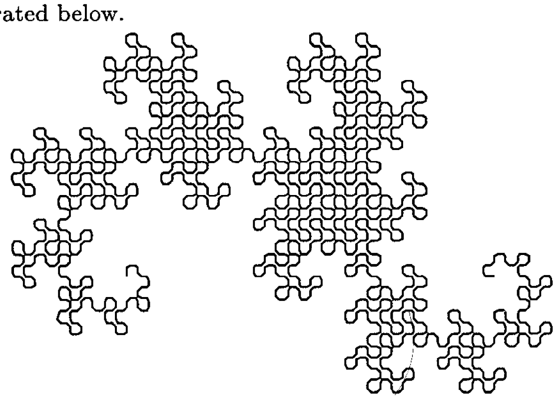
Dozen

12.

see also BAKER'S DOZEN, GROSS

Dragon Curve

Nonintersecting curves which can be iterated to yield more and more sinuosity. They can be constructed by taking a path around a set of dots, representing a left turn by 1 and a right turn by 0. The first-order curve is then denoted 1. For higher order curves, add a 1 to the end, then copy the string of digits preceding it to the end but switching its center digit. For example, the second-order curve is generated as follows: (1)1 → (1)1(0) → 110, and the third as: (110)1 → (110)1(100) → 1101100. Continuing gives 110110011100100... (Sloane's A014577). The OCTAL representation sequence is 1, 6, 154, 66344, ... (Sloane's A003460). The dragon curves of orders 1 to 9 are illustrated below.



This procedure is equivalent to drawing a RIGHT ANGLE and subsequently replacing each RIGHT ANGLE with another smaller RIGHT ANGLE (Gardner 1978). In fact, the dragon curve can be written as a LINDENMAYER SYSTEM with initial string `"FX"`, STRING REWRITING rules `"X" -> "X+YF+"`, `"Y" -> "-FX-Y"`, and angle 90°.

see also LINDENMAYER SYSTEM, PEANO CURVE

References
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Dixon, R. *Mathographics.* New York: Dover, pp. 180–181, 1991.
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Dubrovsky, V. "Nesting Puzzles, Part II: Chinese Rings Produce a Chinese Monster." *Quantum* **6**, 61–65 (Mar.) and 58–59 (Apr.), 1996.
Gardner, M. *Mathematical Magic Show: More Puzzles, Games, Diversions, Illusions and Other Mathematical Sleight-of-Mind from Scientific American.* New York: Vintage, pp. 207–209 and 215–220, 1978.
Lauwerier, H. *Fractals: Endlessly Repeated Geometric Figures.* Princeton, NJ: Princeton University Press, pp. 48–53, 1991.
Peitgen, H.-O. and Saupe, D. (Eds.). *The Science of Fractal Images.* New York: Springer-Verlag, p. 284, 1988.
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Vasilyev, N. and Gutenmacher, V. "Dragon Curves." *Quantum* **6**, 5–10, 1995.

Dragon Fractal

see DOUADY'S RABBIT FRACTAL

Draughts

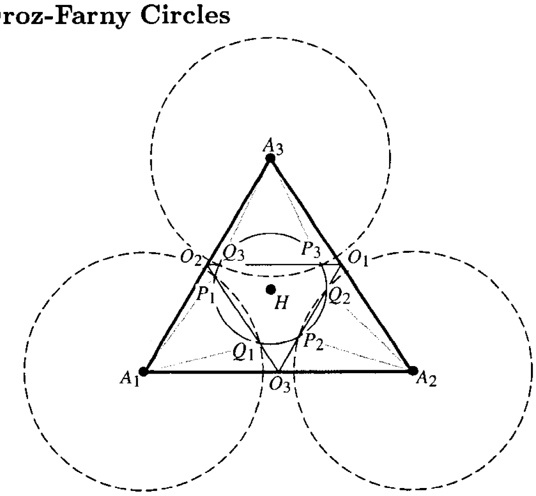
see CHECKERS

Drinfeld's Symmetric Space

A set of points which do not lie on any of a certain class of HYPERPLANES.

References
Teitelbaum, J. "The Geometry of p-adic Symmetric Spaces." *Not. Amer. Math. Soc.* **42**, 1120–1126, 1995.

Droz-Farny Circles

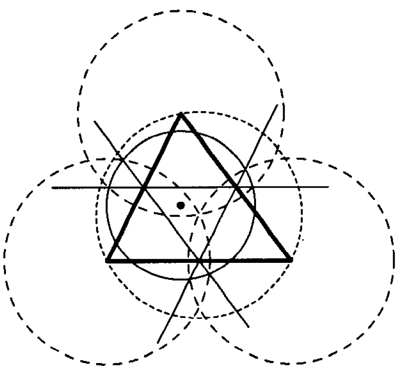


Draw a CIRCLE with center H which cuts the lines O_2O_3, O_3O_1, and O_1O_2 (where O_i are the MIDPOINTS) at P_1, Q_1; P_2, Q_2; and P_3, Q_3 respectively, then

$$\overline{A_1P_1} = \overline{A_2P_2} = \overline{A_3P_3} = \overline{A_1Q_1} = \overline{A_2Q_2} = \overline{A_3Q_3}.$$

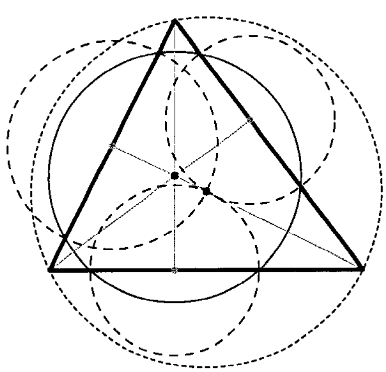
Conversely, if equal CIRCLES are drawn about the VERTICES of a TRIANGLE, they cut the lines joining the MIDPOINTS of the corresponding sides in six points. These points lie on a CIRCLE whose center is the ORTHOCENTER. If r is the RADIUS of the equal CIRCLES centered on the vertices A_1, A_2, and A_3, and R_0 is the RADIUS of the CIRCLE about H, then

$$R_1{}^2 = 4R^2 + r^2 - \tfrac{1}{2}(a_1{}^2 + a_2{}^2 + a_3{}^2).$$

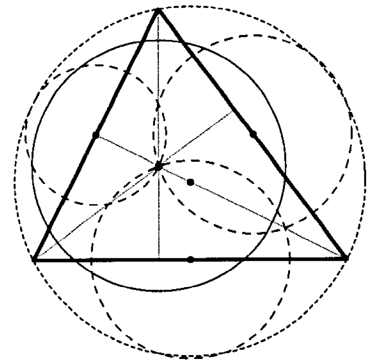


If the circles equal to the CIRCUMCIRCLE are drawn about the VERTICES of a triangle, they cut the lines joining midpoints of the adjacent sides in points of a CIRCLE R_2 with center H and RADIUS

$$R_2{}^2 = 5R^2 - \tfrac{1}{2}(a_1{}^2 + a_2{}^2 + a_3{}^2).$$



It is equivalent to the circle obtained by drawing circles with centers at the feet of the altitudes and passing through the CIRCUMCENTER. These circles cut the corresponding sides in six points on a circle R_2' whose center is H.



Furthermore, the circles about the midpoints of the sides and passing though H cut the sides in six points lying on another equivalent circle R_2'' whose center is O. In summary, the second Droz-Farny circle passes through 12 notable points, two on each of the sides and two on each of the lines joining midpoints of the sides.

References

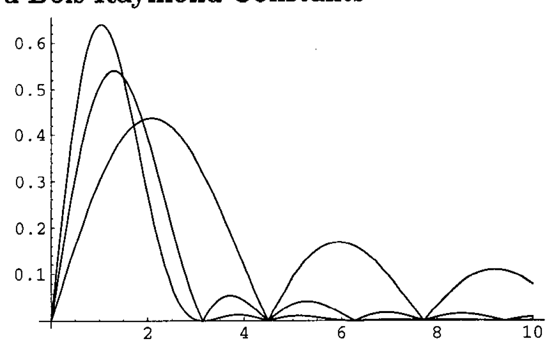
Goormaghtigh, R. "Droz-Farny's Theorem." *Scripta Math.* **16**, 268–271, 1950.

Johnson, R. A. *Modern Geometry: An Elementary Treatise on the Geometry of the Triangle and the Circle.* Boston, MA: Houghton Mifflin, pp. 256–258, 1929.

Drum

see ISOSPECTRAL MANIFOLDS

Du Bois Raymond Constants



The constants C_n defined by

$$C_n \equiv \int_0^\infty \left| \frac{d}{dt} \left(\frac{\sin t}{t} \right)^n \right| dt - 1$$

which are difficult to compute numerically. The first few are

$$C_1 \approx 455$$
$$C_2 \approx 0.1945$$
$$C_3 \approx 0.028254$$
$$C_4 \approx 0.00524054.$$

Rather surprisingly, the second Du Bois Raymond constant is given analytically by

$$C_2 = \tfrac{1}{2}(e^2 - 7) = 0.1945280494\ldots$$

(Le Lionnais 1983).

References
Le Lionnais, F. *Les nombres remarquables.* Paris: Hermann, p. 23, 1983.
Plouffe, S. "Dubois-Raymond 2nd Constant." http://lacim.uqam.ca/piDATA/dubois.txt.

Dual Basis

Given a CONTRAVARIANT BASIS $\{\vec{e}_1, \ldots, \vec{e}_n\}$, its dual COVARIANT basis is given by

$$\vec{e}^{\,\alpha} \cdot \vec{e}_\beta = g(\vec{e}^{\,\alpha}, \vec{e}_\beta) = \delta^\alpha_\beta,$$

where g is the METRIC and δ^α_β is the mixed KRONECKER DELTA. In EUCLIDEAN SPACE with an ORTHONORMAL BASIS,

$$\vec{e}^{\,j} = \vec{e}_j,$$

so the BASIS and its dual are the same.

Dual Bivector

A dual BIVECTOR is defined by

$$\tilde{X}_{ab} \equiv \tfrac{1}{2}\epsilon_{abcd}X^{cd},$$

and a self-dual BIVECTOR by

$$X^*_{ab} \equiv X_{ab} + i\tilde{X}_{ab}.$$

Dual Graph

The dual graph G^* of a POLYHEDRAL GRAPH G has VERTICES each of which corresponds to a face of G and each of whose faces corresponds to a VERTEX of G. Two nodes in G^* are connected by an EDGE if the corresponding faces in G have a boundary EDGE in common.

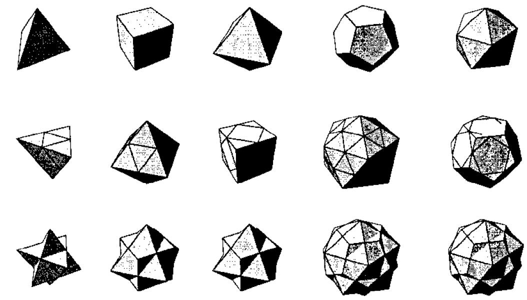
Dual Map

see PULLBACK MAP

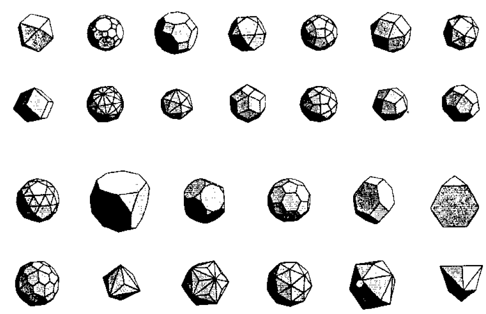
Dual Polyhedron

By the DUALITY PRINCIPLE, for every POLYHEDRON, there exists another POLYHEDRON in which faces and VERTICES occupy complementary locations. This POLYHEDRON is known as the dual, or RECIPROCAL. The dual polyhedron of a PLATONIC SOLID or ARCHIMEDEAN SOLID can be drawn by constructing EDGES tangent to the RECIPROCATING SPHERE (a.k.a. MIDSPHERE and INTERSPHERE) which are PERPENDICULAR to the original EDGES.

The dual of a general solid can be computed by connecting the midpoints of the sides surrounding each VERTEX, and constructing the corresponding tangent POLYGON. (The tangent polygon is the polygon which is tangent to the CIRCUMCIRCLE of the POLYGON produced by connecting the MIDPOINT on the sides surrounding the given VERTEX.) The process is illustrated below for the PLATONIC SOLIDS. The POLYHEDRON COMPOUNDS consisting of a POLYHEDRON and its dual are generally very attractive, and are also illustrated below for the PLATONIC SOLIDS.



The ARCHIMEDEAN SOLIDS and their duals are illustrated below.



The following table gives a list of the duals of the PLATONIC SOLIDS and KEPLER-POINSOT SOLIDS together with the names of the POLYHEDRON-dual COMPOUNDS.

Polyhedron	Dual
Császár polyhedron	Szilassi polyhedron
cube	octahedron
cuboctahedron	rhombic dodecahedron
dodecahedron	icosahedron
great dodecahedron	small stellated dodec.
great icosahedron	great stellated dodec.
great stellated dodec.	great icosahedron
icosahedron	dodecahedron
octahedron	cube
small stellated dodec.	great dodecahedron
Szilassi polyhedron	Császár polyhedron
tetrahedron	tetrahedron

polyhedron	compound
cube	cube-octahedron compound
dodecahedron	dodec.-icosahedron compound
great dodecahedron	great dodecahedron-small stellated dodec. compound
great icosahedron	great icosahedron-great stellated dodec. compound
great stellated dodec.	great icosahedron-great stellated dodec. compound
icosahedron	dodec.-icosahedron compound
octahedron	cube-octahedron compound
small stellated dodec.	great dodec.-small stellated dodec. compound
tetrahedron	stella octangula

see also DUALITY PRINCIPLE, POLYHEDRON COMPOUND, RECIPROCATING SPHERE

References
Weisstein, E. W. "Polyhedron Duals." http://www.astro.virginia.edu/~eww6n/math/notebooks/Duals.m.
Wenninger, M. *Dual Models.* Cambridge, England: Cambridge University Press, 1983.

Dual Scalar

Given a third RANK TENSOR,

$$V_{ijk} \equiv \det[\mathbf{A} \quad \mathbf{B} \quad \mathbf{C}],$$

where det is the DETERMINANT, the dual scalar is defined as

$$V \equiv \frac{1}{3!}\epsilon_{ijk}V_{ijk},$$

where ϵ_{ijk} is the LEVI-CIVITA TENSOR.

see also DUAL TENSOR, LEVI-CIVITA TENSOR

Dual Solid

see DUAL POLYHEDRON

Dual Tensor

Given an antisymmetric second RANK TENSOR C_{ij}, a dual pseudotensor C_i is defined by

$$C_i \equiv \tfrac{1}{2}\epsilon_{ijk}C_{jk}, \tag{1}$$

where

$$C_i \equiv \begin{bmatrix} C_{23} \\ C_{31} \\ C_{12} \end{bmatrix} \tag{2}$$

$$C_{jk} \equiv \begin{bmatrix} 0 & C_{12} & -C_{31} \\ -C_{12} & 0 & C_{23} \\ C_{31} & -C_{23} & 0 \end{bmatrix}. \tag{3}$$

see also DUAL SCALAR

References
Arfken, G. "Pseudotensors, Dual Tensors." §3.4 in *Mathematical Methods for Physicists, 3rd ed.* Orlando, FL: Academic Press, pp. 128–137, 1985.

Dual Voting

A term in SOCIAL CHOICE THEORY meaning each alternative receives equal weight for a single vote.

see also ANONYMOUS, MONOTONIC VOTING

Duality Principle

All the propositions in PROJECTIVE GEOMETRY occur in dual pairs which have the property that, starting from either proposition of a pair, the other can be immediately inferred by interchanging the parts played by the words "point" and "line." A similar duality exists for RECIPROCATION (Casey 1893).

see also BRIANCHON'S THEOREM, CONSERVATION OF NUMBER PRINCIPLE, DESARGUES' THEOREM, DUAL POLYHEDRON, PAPPUS'S HEXAGON THEOREM, PASCAL'S THEOREM, PERMANENCE OF MATHEMATICAL RELATIONS PRINCIPLE, PROJECTIVE GEOMETRY, RECIPROCATION

References
Casey, J. "Theory of Duality and Reciprocal Polars." Ch. 13 in *A Treatise on the Analytical Geometry of the Point, Line, Circle, and Conic Sections, Containing an Account of Its Most Recent Extensions, with Numerous Examples, 2nd ed., rev. enl.* Dublin: Hodges, Figgis, & Co., pp. 382–392, 1893.
Ogilvy, C. S. *Excursions in Geometry.* New York: Dover, pp. 107–110, 1990.

Duality Theorem

Dual pairs of LINEAR PROGRAMS are in "strong duality" if both are possible. The theorem was first conceived by John von Neumann. The first written proof was an Air Force report by George Dantzig, but credit is usually given to Tucker, Kuhn, and Gale.

see also LINEAR PROGRAMMING

Duffing Differential Equation

The most general forced form of the Duffing equation is

$$\ddot{x} + \delta\dot{x} + (\beta x^3 \pm {\omega_0}^2 x) = A\sin(\omega t + \phi). \tag{1}$$

If there is no forcing, the right side vanishes, leaving

$$\ddot{x} + \delta\dot{x} + (\beta x^3 \pm {\omega_0}^2 x) = 0. \tag{2}$$

If $\delta = 0$ and we take the plus sign,

$$\ddot{x} + {\omega_0}^2 x + \beta x^3 = 0. \tag{3}$$

This equation can display chaotic behavior. For $\beta > 0$, the equation represents a "hard spring," and for $\beta < 0$, it represents a "soft spring." If $\beta < 0$, the phase portrait curves are closed. Returning to (1), take $\beta = 1$, $\omega_0 = 1$, $A = 0$, and use the minus sign. Then the equation is

$$\ddot{x} + \delta\dot{x} + (x^3 - x) = 0 \tag{4}$$

(Ott 1993, p. 3). This can be written as a system of first-order ordinary differential equations by writing

$$\dot{x} = y, \tag{5}$$

$$\dot{y} = x - x^3 - \delta y. \tag{6}$$

The fixed points of these differential equations

$$\dot{x} = y = 0, \tag{7}$$

so $y = 0$, and

$$\dot{y} = x - x^3 - \delta y = x(1 - x^2) - 0 \tag{8}$$

giving $x = 0, \pm 1$. Differentiating,

$$\ddot{x} = \dot{y} = x - x^3 - \delta y \tag{9}$$

$$\ddot{y} = (1 - 3x^2)\dot{x} - \delta \dot{y} \tag{10}$$

$$\begin{bmatrix} \ddot{x} \\ \ddot{y} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 - 3x^2 & -\delta \end{bmatrix} \begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix}. \tag{11}$$

Examine the stability of the point (0,0):

$$\begin{vmatrix} 0 - \lambda & 1 \\ 1 & -\delta - \lambda \end{vmatrix} = \lambda(\lambda + \delta) - 1 = \lambda^2 + \lambda\delta - 1 = 0 \tag{12}$$

$$\lambda_{\pm}^{(0,0)} = \tfrac{1}{2}(-\delta \pm \sqrt{\delta^2 + 4}). \tag{13}$$

But $\delta^2 \geq 0$, so $\lambda_{\pm}^{(0,0)}$ is real. Since $\sqrt{\delta^2 + 4} > |\delta|$, there will always be one POSITIVE ROOT, so this fixed point is unstable. Now look at $(\pm 1, 0)$.

$$\begin{vmatrix} 0 - \lambda & 1 \\ -2 & -\delta - \lambda \end{vmatrix} = \lambda(\lambda + \delta) + 2 = \lambda^2 + \lambda\delta + 2 = 0 \tag{14}$$

$$\lambda_{\pm}^{(\pm 1,0)} = \tfrac{1}{2}(-\delta \pm \sqrt{\delta^2 - 8}). \tag{15}$$

For $\delta > 0$, $\Re[\lambda_{\pm}^{(\pm 1,0)}] < 0$, so the point is asymptotically stable. If $\delta = 0$, $\lambda_{\pm}^{(\pm 1,0)} = \pm i\sqrt{2}$, so the point is linearly stable. If $\delta \in (-2\sqrt{2}, 0)$, the radical gives an IMAGINARY PART and the REAL PART is > 0, so the point is unstable. If $\delta = -2\sqrt{2}$, $\lambda_{\pm}^{(\pm 1,0)} = \sqrt{2}$, which has a POSITIVE REAL ROOT, so the point is unstable. If $\delta < -2\sqrt{2}$, then $|\delta| < \sqrt{\delta^2 - 8}$, so both ROOTS are POSITIVE and the point is unstable. Summarizing,

$$\begin{cases} \text{asymptotically stable} & \delta > 0 \\ \text{linearly stable (superstable)} & \delta = 0 \\ \text{unstable} & \delta < 0. \end{cases} \tag{16}$$

Now specialize to the case $\delta = 0$, which can be integrated by quadratures. In this case, the equations become

$$\dot{x} = y \tag{17}$$

$$\dot{y} = x - x^3. \tag{18}$$

Differentiating (17) and plugging in (18) gives

$$\ddot{x} = \dot{y} = x - x^3. \tag{19}$$

Multiplying both sides by $\dot{x}$ gives

$$\ddot{x}\dot{x} - \dot{x}x + \dot{x}x^3 = 0 \tag{20}$$

$$\frac{d}{dt}(\tfrac{1}{2}\dot{x}^2 - \tfrac{1}{2}x^2 + \tfrac{1}{4}x^4) = 0, \tag{21}$$

so we have an invariant of motion h,

$$h \equiv \tfrac{1}{2}\dot{x}^2 - \tfrac{1}{2}x^2 + \tfrac{1}{4}x^4. \tag{22}$$

Solving for $\dot{x}^2$ gives

$$\dot{x}^2 = \left(\frac{dx}{dt}\right)^2 = 2h + x^2 - \tfrac{1}{2}x^4 \tag{23}$$

$$\frac{dx}{dt} = \sqrt{2h + x^2 + \tfrac{1}{2}x^2}, \tag{24}$$

so

$$t = \int dt = \int \frac{dx}{\sqrt{2h + x^2 + \frac{1}{2}x^2}}. \tag{25}$$

Note that the invariant of motion h satisfies

$$\dot{x} = \frac{\partial h}{\partial \dot{x}} = \frac{\partial h}{\partial y} \tag{26}$$

$$\frac{\partial h}{\partial x} = -x + x^3 = -\dot{y}, \tag{27}$$

so the equations of the Duffing oscillator are given by the HAMILTONIAN SYSTEM

$$\begin{cases} \dot{x} = \frac{\partial h}{\partial y} \\ \dot{y} = -\frac{\partial h}{\partial x}. \end{cases} \tag{28}$$

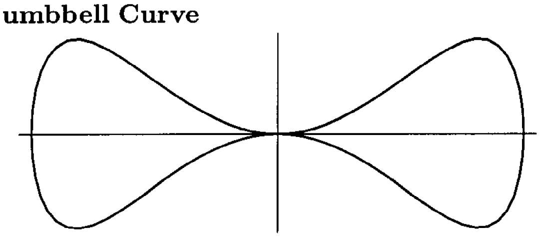
References

Ott, E. *Chaos in Dynamical Systems.* New York: Cambridge University Press, 1993.

Duhamel's Convolution Principle

Can be used to invert a LAPLACE TRANSFORM.

Dumbbell Curve



$$y^2 = a^2(x^4 - x^6).$$

see also BUTTERFLY CURVE, EIGHT CURVE, PIRIFORM

References

Cundy, H. and Rollett, A. *Mathematical Models, 3rd ed.* Stradbroke, England: Tarquin Pub., p. 72, 1989.

Duodecillion

In the American system, 10^{39}.

see also LARGE NUMBER

Dupin's Cyclide

see CYCLIDE

Dupin's Indicatrix

A pair of conics obtained by expanding an equation in MONGE'S FORM $z = F(x, y)$ in a MACLAURIN SERIES

$$\begin{aligned} z &= z(0,0) + z_1 x + z_2 y \\ &\quad + \tfrac{1}{2}(z_{11}x^2 + 2z_{12}xy + z_{22}y^2) + \ldots \\ &= \tfrac{1}{2}(b_{11}x^2 + 2b_{12}xy + b_{22}y^2). \end{aligned}$$

This gives the equation

$$b_{11}x^2 + 2b_{12}xy + b_{22}y^2 = \pm 1.$$

Amazingly, the radius of the indicatrix in any direction is equal to the SQUARE ROOT of the RADIUS OF CURVATURE in that direction (Coxeter 1969).

References

Coxeter, H. S. M. "Dupin's Indicatrix" §19.8 in *Introduction to Geometry, 2nd ed.* New York: Wiley, pp. 363–365, 1969.

Dupin's Theorem

In three mutually orthogonal systems of the surfaces, the LINES OF CURVATURE on any surface in one of the systems are its intersections with the surfaces of the other two systems.

Duplication of the Cube

see CUBE DUPLICATION

Duplication Formula

see LEGENDRE DUPLICATION FORMULA

Durand's Rule

The NEWTON-COTES FORMULA

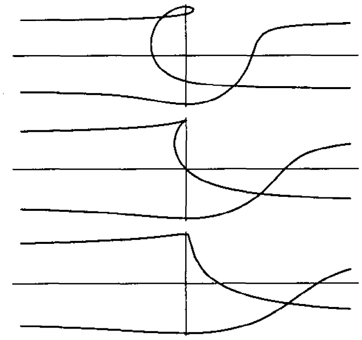
$$\int_{x_1}^{x_n} f(x)\,dx = h\left(\tfrac{2}{5}f_1 + \tfrac{11}{10}f_2 + f_3 + \ldots + f_{n-2} + \tfrac{11}{10}f_{n-1} + \tfrac{2}{5}f_n\right).$$

see also BODE'S RULE, HARDY'S RULE, NEWTON-COTES FORMULAS, SIMPSON'S 3/8 RULE, SIMPSON'S RULE, TRAPEZOIDAL RULE, WEDDLE'S RULE

References

Beyer, W. H. (Ed.) *CRC Standard Mathematical Tables, 28th ed.* Boca Raton, FL: CRC Press, p. 127, 1987.

Dürer's Conchoid



These curves appear in Dürer's work *Instruction in Measurement with Compasses and Straight Edge* (1525) and arose in investigations of perspective. Dürer constructed the curve by drawing lines QRP and $P'QR$ of length 16 units through $Q(q,0)$ and $R(r,0)$, where $q+r = 13$. The locus of P and P' is the curve, although Dürer found only one of the two branches of the curve.

The ENVELOPE of the lines QRP and $P'QR$ is a PARABOLA, and the curve is therefore a GLISSETTE of a point on a line segment sliding between a PARABOLA and one of its TANGENTS.

Dürer called the curve "Muschellini," which means CONCHOID. However, it is not a true CONCHOID and so is sometimes called DÜRER'S SHELL CURVE. The Cartesian equation is

$$\begin{aligned} 2y^2(x^2+y^2) - 2by^2(x+y) + (b^2 - 3a^2)y^2 - a^2x^2 \\ + 2a^2b(x+y) + a^2(a^2 - b^2) = 0. \end{aligned}$$

The above curves are for $(a, b) = (3, 1)$, $(3, 3)$, $(3, 5)$. There are a number of interesting special cases. If $b = 0$, the curve becomes two coincident straight lines $x = 0$. For $a = 0$, the curve becomes the line pair $x = b/2$, $x = -b/2$, together with the CIRCLE $x + y = b$. If $a = b/2$, the curve has a CUSP at $(-2a, a)$.

References

Lawrence, J. D. *A Catalog of Special Plane Curves.* New York: Dover, pp. 157–159, 1972.

Lockwood, E. H. *A Book of Curves.* Cambridge, England: Cambridge University Press, p. 163, 1967.

MacTutor History of Mathematics Archive. "Dürer's Shell Curves." http://www-groups.dcs.st-and.ac.uk/~history/Curves/Durers.html.

Dürer's Magic Square

16	3	2	13
5	10	11	8
9	6	7	12
4	15	14	1

Dürer's magic square is a MAGIC SQUARE with MAGIC CONSTANT 34 used in an engraving entitled *Melencolia I* by Albrecht Dürer (The British Museum). The engraving shows a disorganized jumble of scientific equipment lying unused while an intellectual sits absorbed in thought. Dürer's magic square is located in the upper left-hand corner of the engraving. The numbers 15 and 14 appear in the middle of the bottom row, indicating the date of the engraving, 1514.

References
Boyer, C. D. and Merzbach, U. C. *A History of Mathematics.* New York: Wiley, pp. 296–297, 1991.
Hunter, J. A. H. and Madachy, J. S. *Mathematical Diversions.* New York: Dover, p. 24, 1975.
Rivera, C. "Melancholia." http://www.sci.net.mx/~crivera/melancholia.htm.

Dürer's Shell Curve

see DÜRER'S CONCHOID

Durfee Polynomial

Let $F(n)$ be a family of PARTITIONS of n and let $F(n,d)$ denote the set of PARTITIONS in $F(n)$ with DURFEE SQUARE of size d. The Durfee polynomial of $F(n)$ is then defined as the polynomial

$$P_{F,n} = \sum |F(n,d)| y^d,$$

where $0 \leq d \leq \sqrt{n}$.

see also DURFEE SQUARE, PARTITION

References
Canfield, E. R.; Corteel, S.; and Savage, C. D. "Durfee Polynomials." *Electronic J. Combinatorics* **5**, No. 1, R32, 1–21, 1998. http://www.combinatorics.org/Volume_5/v5i1toc.html#R32.

Durfee Square

The length of the largest-sized SQUARE contained within the FERRERS DIAGRAM of a PARTITION.

see also DURFEE POLYNOMIAL, FERRERS DIAGRAM, PARTITION

Dvoretzky's Theorem

Each centered convex body of sufficiently high dimension has an "almost spherical" k-dimensional central section.

Dyad

Dyads extend VECTORS to provide an alternative description to second RANK TENSORS. A dyad $D(\mathbf{A},\mathbf{B})$ of a pair of VECTORS $\mathbf{A}$ and $\mathbf{B}$ is defined by $D(\mathbf{A},\mathbf{B}) \equiv \mathbf{AB}$. The DOT PRODUCT is defined by

$$\mathbf{A}\cdot\mathbf{BC} \equiv (\mathbf{A}\cdot\mathbf{B})\mathbf{C}$$

$$\mathbf{AB}\cdot\mathbf{C} \equiv \mathbf{A}(\mathbf{B}\cdot\mathbf{C}),$$

and the COLON PRODUCT by

$$\mathbf{AB}:\mathbf{CD} \equiv \mathbf{C}\cdot\mathbf{AB}\cdot\mathbf{D} = (\mathbf{A}\cdot\mathbf{C})(\mathbf{B}\cdot\mathbf{D}).$$

References
Morse, P. M. and Feshbach, H. "Dyadics and Other Vector Operators." §1.6 in *Methods of Theoretical Physics, Part I.* New York: McGraw-Hill, pp. 54–92, 1953.

Dyadic

A linear POLYNOMIAL of DYADS $\mathbf{AB}+\mathbf{CD}+\ldots$ consisting of nine components A_{ij} which transform as

$$(A_{ij})' = \sum_{m,n} \frac{h_m h_n}{h'_i h'_j} \frac{\partial x_m}{\partial x'_i} \frac{\partial x_n}{\partial x'_j} A_{mn} \tag{1}$$

$$= \sum_{m,n} \frac{h'_i h'_j}{h_m h_n} \frac{\partial x'_i}{\partial x_m} \frac{\partial x'_j}{\partial x_n} A_{mn} \tag{2}$$

$$= \sum_{m,n} \frac{h'_i h_n}{h_m h'_j} \frac{\partial x'_i}{\partial x_m} \frac{\partial x_n}{\partial x'_j} A_{mn}. \tag{3}$$

Dyadics are often represented by Gothic capital letters. The use of dyadics is nearly archaic since TENSORS perform the same function but are notationally simpler.

A unit dyadic is also called the IDEMFACTOR and is defined such that

$$\mathbf{I}\cdot\mathbf{A} \equiv \mathbf{A}. \tag{4}$$

In CARTESIAN COORDINATES,

$$\mathbf{I} = \hat{\mathbf{x}}\hat{\mathbf{x}} + \hat{\mathbf{y}}\hat{\mathbf{y}} + \hat{\mathbf{z}}\hat{\mathbf{z}}, \tag{5}$$

and in SPHERICAL COORDINATES

$$\mathbf{I} = \nabla\mathbf{r}. \tag{6}$$

see also DYAD, TETRADIC

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Dyck's Theorem

see VON DYCK'S THEOREM

Dye's Theorem

For any two ergodic measure-preserving transformations on nonatomic PROBABILITY SPACES, there is an ISOMORPHISM between the two PROBABILITY SPACES carrying orbits onto orbits.

Dymaxion

Buckminster Fuller's term for the CUBOCTAHEDRON.

see also CUBOCTAHEDRON, MECON

Dynamical System

A means of describing how one state develops into another state over the course of time. Technically, a dynamical system is a smooth action of the reals or the INTEGERS on another object (usually a MANIFOLD). When the reals are acting, the system is called a continuous dynamical system, and when the INTEGERS are acting, the system is called a discrete dynamical system. If f is any CONTINUOUS FUNCTION, then the evolution of a variable x can be given by the formula

$$x_{n+1} = f(x_n). \tag{1}$$

This equation can also be viewed as a difference equation

$$x_{n+1} - x_n = f(x_n) - x_n, \tag{2}$$

so defining

$$g(x) \equiv f(x) - x \tag{3}$$

gives

$$x_{n+1} - x_n = g(x_n) * 1, \tag{4}$$

which can be read "as n changes by 1 unit, x changes by $g(x)$." This is the discrete analog of the DIFFERENTIAL EQUATION

$$x'(n) = g(x(n)). \tag{5}$$

see also ANOSOV DIFFEOMORPHISM, ANOSOV FLOW, AXIOM A DIFFEOMORPHISM, AXIOM A FLOW, BIFURCATION THEORY, CHAOS, ERGODIC THEORY, GEODESIC FLOW

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Dynkin Diagram

A diagram used to describe CHEVALLEY GROUPS.

see also COXETER-DYNKIN DIAGRAM

References

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E

e

The base of the NATURAL LOGARITHM, named in honor of Euler. It appears in many mathematical contexts involving LIMITS and DERIVATIVES, and can be defined by

$$e \equiv \lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x, \tag{1}$$

or by the infinite sum

$$e = \sum_{k=0}^{\infty} \frac{1}{k!}. \tag{2}$$

The numerical value of e is

$$e = 2.718281828459045235360287471352662497757\ldots \tag{3}$$

(Sloane's A001113).

Euler proved that e is IRRATIONAL, and Liouville proved in 1844 that e does not satisfy any QUADRATIC EQUATION with integral COEFFICIENTS. Hermite proved e to be TRANSCENDENTAL in 1873. It is not known if $\pi + e$ or π/e is IRRATIONAL. However, it is known that $\pi + e$ and π/e do not satisfy any POLYNOMIAL equation of degree ≤ 8 with INTEGER COEFFICIENTS of average size 10^9 (Bailey 1988, Borwein *et al.* 1989).

The special case of the EULER FORMULA

$$e^{ix} = \cos x + i \sin x \tag{4}$$

with $x = \pi$ gives the beautiful identity

$$e^{i\pi} + 1 = 0, \tag{5}$$

an equation connecting the fundamental numbers i, PI, e, 1, and 0 (ZERO).

Some CONTINUED FRACTION representations of e include

$$e = 2 + \cfrac{1}{1 + \cfrac{1}{2 + \cfrac{2}{3 + \cfrac{3}{\ddots}}}} \tag{6}$$

$$= [2, 1, 2, 1, 1, 4, 1, 1, 6, \ldots] \tag{7}$$

(Sloane's A003417) and

$$\frac{e-1}{e+1} = [2, 6, 10, 14, \ldots] \tag{8}$$

$$e - 1 = [1, 1, 2, 1, 1, 4, 1, 1, 6, \ldots] \tag{9}$$

$$\tfrac{1}{2}(e-1) = [0, 1, 6, 10, 14, \ldots] \tag{10}$$

$$\sqrt{e} = [1, 1, 1, 1, 5, 1, 1, 1, 9, 1, \ldots]. \tag{11}$$

The first few convergents of the CONTINUED FRACTION are 3, 8/3, 11/4, 19/7, 87/32, 106/39, 193/71, ... (Sloane's A007676 and A007677).

Using the RECURRENCE RELATION

$$a_n = n(a_{n-1} + 1) \tag{12}$$

with $a_1 = a^{-1}$, compute

$$\prod_{n=1}^{\infty} (1 + a_n{}^{-1}). \tag{13}$$

The result is e^a. Gosper gives the unusual equation connecting π and e,

$$\sum_{n=1}^{\infty} \frac{1}{n^2} \cos\left(\frac{9}{n\pi + \sqrt{n^2\pi^2 - 9}}\right) = -\frac{\pi^2}{12e^3} = -0.040948222\ldots. \tag{14}$$

Rabinowitz and Wagon (1995) give an ALGORITHM for computing digits of e based on earlier DIGITS, but a much simpler SPIGOT ALGORITHM was found by Sales (1968). Around 1966, MIT hacker Eric Jensen wrote a very concise program (requiring less than a page of assembly language) that computed e by converting from factorial base to decimal.

Let $p(n)$ be the probability that a random ONE-TO-ONE function on the INTEGERS 1, ..., n has at least one FIXED POINT. Then

$$\lim_{n\to\infty} p(n) = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k!} = 1 - \frac{1}{e} = 0.6321205588\ldots. \tag{15}$$

STIRLING'S FORMULA gives

$$\lim_{n\to\infty} \frac{(n!)^{1/n}}{n} = \frac{1}{e}. \tag{16}$$

Castellanos (1988) gives several curious approximations to e,

$$e \approx 2 + \frac{54^2 + 41^2}{80^2} \tag{17}$$

$$\approx (\pi^4 + \pi^5)^{1/6} \tag{18}$$

$$\approx \frac{271801}{99990} \tag{19}$$

$$\approx \left(150 - \frac{87^3 + 12^5}{83^3}\right)^{1/5} \tag{20}$$

$$\approx 4 - \frac{300^4 - 100^4 - 1291^2 + 9^2}{91^5} \tag{21}$$

$$\approx \left(1097 - \frac{55^5 + 311^3 - 11^3}{68^5}\right)^{1/7}, \tag{22}$$

which are good to 6, 7, 9, 10, 12, and 15 digits respectively.

Examples of *e* MNEMONICS (Gardner 1959, 1991) include:

> "By omnibus I traveled to Brooklyn" (6 digits).
>
> "To disrupt a playroom is commonly a practice of children" (10 digits).
>
> "It enables a numskull to memorize a quantity of numerals" (10 digits).
>
> "I'm forming a mnemonic to remember a function in analysis" (10 digits).
>
> "He repeats: I shouldn't be tippling, I shouldn't be toppling here!" (11 digits).
>
> "In showing a painting to probably a critical or venomous lady, anger dominates. O take guard, or she raves and shouts" (21 digits). Here, the word "O" stands for the number 0.

A much more extensive mnemonic giving 40 digits is

> "We present a mnemonic to memorize a constant so exciting that Euler exclaimed: '!' when first it was found, yes, loudly '!'. My students perhaps will compute *e*, use power or Taylor series, an easy summation formula, obvious, clear, elegant!"

(Barel 1995). In the latter, 0s are represented with "!". A list of *e* mnemonics in several languages is maintained by A. P. Hatzipolakis.

Scanning the decimal expansion of *e* until all n-digit numbers have occurred, the last appearing is 6, 12, 548, 1769, 92994, 513311, ... (Sloane's A032511). These end at positions 21, 372, 8092, 102128, 1061613, 12108841, ...

see also CARLEMAN'S INEQUALITY, COMPOUND INTEREST, DE MOIVRE'S IDENTITY, EULER FORMULA, EXPONENTIAL FUNCTION, HERMITE-LINDEMANN THEOREM, NATURAL LOGARITHM

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e-Divisor

d is called an *e*-divisor (or EXPONENTIAL DIVISOR) of

$$n = {p_1}^{a_1} {p_2}^{a_2} \cdots {p_r}^{a_r}$$

if $d|n$ and

$$d = {p_1}^{b_1} {p_2}^{b_2} \cdots {p_r}^{b_r}$$

where $b_j | a_j$ with $1 \leq j \leq r$.

see also *e*-PERFECT NUMBER

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E_n-Function

The $\mathrm{E}_n(x)$ function is defined by the integral

$$\mathrm{E}_n(x) \equiv \int_1^\infty \frac{e^{-xt}\,dt}{t^n} \tag{1}$$

and is given by the *Mathematica*® (Wolfram Research, Champaign, IL) function `ExpIntegralE[n,x]`. Defining $t \equiv \eta^{-1}$ so that $dt = -\eta^{-2} d\eta$,

$$\mathrm{E}_n(x) = \int_0^1 e^{-x/\eta} \eta^{n-2}\,d\eta \tag{2}$$

$$\mathrm{E}_n(0) = \frac{1}{n-1}. \tag{3}$$

The function satisfies the RECURRENCE RELATIONS

$$\mathrm{E}_n'(x) = -\mathrm{E}_{n-1}(x) \tag{4}$$

$$n\,\mathrm{E}_{n+1}(x) = e^{-x} - x\,\mathrm{E}_n(x). \tag{5}$$

Equation (4) can be derived from

$$\mathrm{E}_n(x) = \int_1^\infty \frac{e^{-tx}}{t^n}\,dt \tag{6}$$

$$\begin{aligned}\mathrm{E}_n'(x) &= \frac{d}{dx}\int_1^\infty \frac{e^{-tx}}{t^n}\,dt = \int_1^\infty \frac{d}{dx}\left(\frac{e^{-tx}}{t^n}\right)dt \\ &= -\int_1^\infty t\frac{e^{-tx}}{t^n}\,dt \\ &= -\int_1^\infty \frac{e^{-tx}}{t^{n-1}}\,dt = -\,\mathrm{E}_{n-1}(x),\end{aligned} \tag{7}$$

and (5) using integrating by parts, letting

$$u = \frac{1}{t^n} \qquad dv = e^{-tx}\,dt \tag{8}$$

$$du = -\frac{n}{t^{n+1}}\,dt \qquad v = -\frac{e^{-tx}}{x}, \tag{9}$$

$$\begin{aligned}\mathrm{E}_n(x) &= \int u\,dv = uv - \int v\,du \\ &= -\frac{e^{-tx}}{xt^n} - \frac{n}{x}\int_1^\infty \frac{e^{-tx}\,dx}{t^{n+1}} \\ &= x\int_1^\infty \frac{e^{-tx}\,dt}{t^n} \\ &= -\left[\frac{1}{e^{tx}t^n}\right]_1^\infty - n\int_1^\infty \frac{e^{-tx}\,dx}{t^{n+1}} \\ &= x\,\mathrm{E}_n(x) = e^{-x} - n\,\mathrm{E}_{n+1}(x).\end{aligned} \tag{10}$$

Solving (10) for $n\,\mathrm{E}_n(x)$ gives (5). An asymptotic expansion gives

$$(n-1)!\,\mathrm{E}_n(x) = (-x)^{n-1}\,\mathrm{E}_1(x) + e^{-x}\sum_{s=0}^{n} -2(n-s-2)!(-x)^s, \tag{11}$$

so

$$\mathrm{E}_n(x) = \frac{e^{-x}}{x}\left[1 - \frac{n}{x} + \frac{n(n+1)}{x^2} + \ldots\right]. \tag{12}$$

The special case $n = 1$ gives

$$\mathrm{E}_1(x) \equiv -\,\mathrm{ei}\,(-x) = \int_1^\infty \frac{e^{-tx}\,dt}{t} = \int_x^\infty \frac{e^{-u}\,du}{u}, \tag{13}$$

where $\mathrm{ei}(x)$ is the EXPONENTIAL INTEGRAL, which is also equal to

$$\mathrm{E}_1(x) = -\gamma - \ln x - \sum_{n=1}^\infty \frac{(-1)^n x^n}{n!n}, \tag{14}$$

where γ is the EULER-MASCHERONI CONSTANT.

$$\mathrm{E}_1(0) = \infty \tag{15}$$

$$\mathrm{E}_1(ix) = -\,\mathrm{ci}(x) + i\,\mathrm{si}(x), \tag{16}$$

where $\mathrm{ci}(x)$ and $\mathrm{si}(x)$ are the COSINE INTEGRAL and SINE INTEGRAL.

see also COSINE INTEGRAL, E_t-FUNCTION, EXPONENTIAL INTEGRAL, GOMPERTZ CONSTANT, SINE INTEGRAL

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Spanier, J. and Oldham, K. B. "The Exponential Integral Ei(x) and Related Functions." Ch. 37 in *An Atlas of Functions.* Washington, DC: Hemisphere, pp. 351–360, 1987.

E_t-Function

A function which arises in FRACTIONAL CALCULUS.

$$E_t(\nu, a) = \frac{1}{\Gamma(\nu)}e^{at}\int_0^t x^{\nu-1}e^{-ax}\,dx = t^\nu e^{at}\gamma(\nu, at), \tag{1}$$

where γ is the incomplete GAMMA FUNCTION and Γ the complete GAMMA FUNCTION. The E_t function satisfies the RECURRENCE RELATION

$$E_t(\nu, a) = aE_t(\nu+1, a) + \frac{t^\nu}{\Gamma(\nu+1)}. \tag{2}$$

A special value is

$$E_t(0, a) = e^{at}. \tag{3}$$

see also E_n-FUNCTION

References

Abramowitz, M. and Stegun, C. A. (Eds.). "Exponential Integral and Related Functions." Ch. 5 in *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables, 9th printing.* New York: Dover, pp. 227–233, 1972.

e-Multiperfect Number

A number n is called a k e-perfect number if $\sigma_e(n) = kn$, where $\sigma_e(n)$ is the SUM of the e-DIVISORS of n.

see also e-DIVISOR, e-PERFECT NUMBER

References

Guy, R. K. "Exponential-Perfect Numbers." §B17 in *Unsolved Problems in Number Theory, 2nd ed.* New York: Springer-Verlag, pp. 73, 1994.

e-Perfect Number

A number n is called an e-perfect number if $\sigma_e(n) = 2n$, where $\sigma_e(n)$ is the SUM of the e-DIVISORS of n. If m is SQUAREFREE, then $\sigma_e(m) = m$. As a result, if n is e-perfect and m is SQUAREFREE with $m \perp b$, then mn is e-perfect. There are no ODD e-perfect numbers.

see also e-DIVISOR

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Subbarao, M. V. and Suryanarayan, D. "Exponential Perfect and Unitary Perfect Numbers." *Not. Amer. Math. Soc.* **18**, 798, 1971.

Ear

A PRINCIPAL VERTEX x_i of a SIMPLE POLYGON P is called an ear if the diagonal $[x_{i-1}, x_{i+1}]$ that bridges x_i lies entirely in P. Two ears x_i and x_j are said to overlap if

$$\operatorname{int}[x_{i-1}, x_i, x_{i+1}] \cap \operatorname{int}[x_{j-1}, x_j, x_{j+1}] = \varnothing.$$

The TWO-EARS THEOREM states that, except for TRIANGLES, every SIMPLE POLYGON has at least two nonoverlapping ears.

see also ANTHROPOMORPHIC POLYGON, MOUTH, TWO-EARS THEOREM

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Toussaint, G. "Anthropomorphic Polygons." *Amer. Math. Monthly* **122**, 31–35, 1991.

Early Election Results

Let Jones and Smith be the only two contestants in an election that will end in a deadlock when all votes for Jones (J) and Smith (S) are counted. What is the EXPECTATION VALUE of $X_k \equiv |S - J|$ after k votes are counted? The solution is

$$\begin{aligned}\langle X_k\rangle &= \frac{2N\binom{N-1}{\lfloor k/2\rfloor}\binom{N-1}{\lfloor k/2\rfloor-1}}{\binom{2N}{k}}\\ &= \begin{cases}\frac{k(2N-k)}{2N}\binom{N}{k/2}^2\binom{2N}{k}^{-1} & \text{for } k \text{ even}\\ \frac{k(2N-k+1)}{2N}\binom{N}{(k-1)/2}^2\binom{2N}{k-1}^{-1} & \text{for } k \text{ odd.}\end{cases}\end{aligned}$$

References

Handelsman, M. B. Solution to Problem 10248. "Early Returns in a Tied Election." *Amer. Math. Monthly* **102**, 554–556, 1995.

Eban Number

The sequence of numbers whose names (in English) do not contain the letter "e" (i.e., "e" is "banned"). The first few eban numbers are 2, 4, 6, 30, 32, 34, 36, 40, 42, 44, 46, 50, 52, 54, 56, 60, 62, 64, 66, 2000, 2002, 2004, ... (Sloane's A006933); i.e., two, four, six, thirty, etc.

References

Sloane, N. J. A. Sequence A006933/M1030 in "An On-Line Version of the Encyclopedia of Integer Sequences."

Eberhart's Conjecture

If q_n is the nth prime such that M_{q_n} is a MERSENNE PRIME, then

$$q_n \sim (3/2)^n.$$

It was modified by Wagstaff (1983) to yield

$$q_n \sim (2^{e^{-\gamma}})^n,$$

where γ is the EULER-MASCHERONI CONSTANT.

References

Ribenboim, P. *The Book of Prime Number Records, 2nd ed.* New York: Springer-Verlag, pp. 332–333, 1989.

Wagstaff, S. S. "Divisors of Mersenne Numbers." *Math. Comput.* **40**, 385–397, 1983.

Eccentric

Not CONCENTRIC.

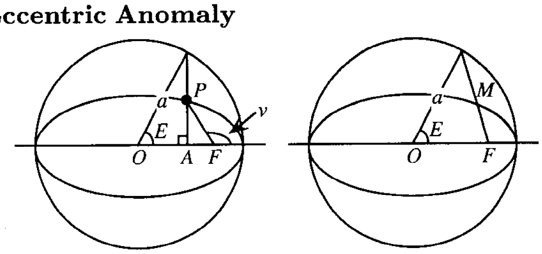
see also CONCENTRIC, CONCYCLIC

Eccentric Angle

The angle θ measured from the CENTER of an ELLIPSE to a point on the ELLIPSE.

see also ECCENTRICITY, ELLIPSE

Eccentric Anomaly



The ANGLE obtained by drawing the AUXILIARY CIRCLE of an ELLIPSE with center O and FOCUS F, and drawing a LINE PERPENDICULAR to the SEMIMAJOR AXIS and intersecting it at A. The ANGLE E is then defined as illustrated above. Then for an ELLIPSE with ECCENTRICITY e,

$$AF = OF - AO = ae - a\cos E. \tag{1}$$

But the distance AF is also given in terms of the distance from the FOCUS $r = FP$ and the SUPPLEMENT of the ANGLE from the SEMIMAJOR AXIS v by

$$AF = r\cos(\pi - v) = -r\cos v. \tag{2}$$

Equating these two expressions gives

$$r = \frac{a(\cos E - e)}{\cos v}, \tag{3}$$

which can be solved for $\cos v$ to obtain

$$\cos v = \frac{a(\cos E - e)}{r}. \tag{4}$$

To get E in terms of r, plug (4) into the equation of the ELLIPSE

$$r = \frac{a(1-e^2)}{1+e\cos v} \tag{5}$$

$$r(1+e\cos v) = a(1-e^2) \tag{6}$$

$$r\left(1+\frac{ae\cos E}{r}-\frac{e^2}{r}\right) = r + ae\cos E - e^2 = a(1-e^2) \tag{7}$$

$$r = a(1-e^2) - ea\cos E + e^2 a = a(1-e\cos E). \tag{8}$$

Differentiating gives

$$\dot{r} = ae\dot{E}\sin E. \tag{9}$$

The eccentric anomaly is a very useful concept in orbital mechanics, where it is related to the so-called mean anomaly M by KEPLER'S EQUATION

$$M = E - e\sin E. \tag{10}$$

M can also be interpreted as the AREA of the shaded region in the above figure (Finch).

see also ECCENTRICITY, ELLIPSE, KEPLER'S EQUATION

References

Danby, J. M. *Fundamentals of Celestial Mechanics, 2nd ed., rev. ed.* Richmond, VA: Willmann-Bell, 1988.
Finch, S. "Favorite Mathematical Constants." http://www.mathsoft.com/asolve/constant/lpc/lpc.html.

Eccentricity

A quantity defined for a CONIC SECTION which can be given in terms of SEMIMAJOR and SEMIMINOR AXES for an ELLIPSE. For an ELLIPSE with SEMIMAJOR AXIS a and SEMIMINOR AXIS b,

$$e \equiv \sqrt{1-\frac{b^2}{a^2}}.$$

The eccentricity can be interpreted as the fraction of the distance to the semimajor axis at which the FOCUS lies,

$$e = \frac{c}{a},$$

where c is the distance from the center of the CONIC SECTION to the FOCUS. The table below gives the type of CONIC SECTION corresponding to various ranges of eccentricity e.

e	Curve
$e=0$	circle
$0<e<1$	ellipse
$e=1$	parabola
$e>1$	hyperbola

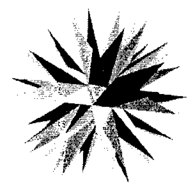
see also CIRCLE, CONIC SECTION, ECCENTRIC ANOMALY, ELLIPSE, FLATTENING, HYPERBOLA, OBLATENESS, PARABOLA, SEMIMAJOR AXIS, SEMIMINOR AXIS

Eccentricity (Graph)

The length of the longest shortest path from a VERTEX in a GRAPH.

see also DIAMETER (GRAPH)

Echidnahedron



ICOSAHEDRON STELLATION #4.

References

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Eckardt Point

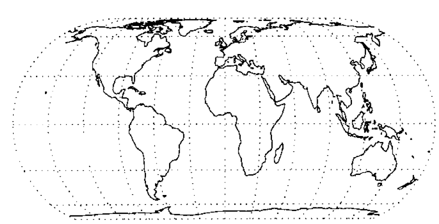
On the CLEBSCH DIAGONAL CUBIC, all 27 of the complex lines present on a general smooth CUBIC SURFACE are real. In addition, there are 10 points on the surface where three of the 27 lines meet. These points are called Eckardt points (Fischer 1986).

see also CLEBSCH DIAGONAL CUBIC, CUBIC SURFACE

References

Fischer, G. (Ed.). *Mathematical Models from the Collections of Universities and Museums.* Braunschweig, Germany: Vieweg, p. 11, 1986.

Eckert IV Projection



The equations are

$$x = \frac{2}{\sqrt{\pi(4+\pi)}}(\lambda-\lambda_0)(1+\cos\theta) \tag{1}$$

$$y = 2\sqrt{\frac{\pi}{4+\pi}}\sin\theta, \tag{2}$$

where θ is the solution to

$$\theta + \sin\theta\cos\theta + 2\sin\theta = (2 + \tfrac{1}{2}\pi)\sin\phi. \quad (3)$$

This can be solved iteratively using NEWTON'S METHOD with $\theta_0 = \phi/2$ to obtain

$$\Delta\theta = -\frac{\theta + \sin\theta\cos\theta + 2\sin\theta - (2 - \frac{1}{2}\pi)\sin\phi}{2\cos\theta(1+\cos\theta)}. \quad (4)$$

The inverse FORMULAS are

$$\phi = \sin^{-1}\left(\frac{\theta + \sin\theta\cos\theta + 2\sin\theta}{2 + \frac{1}{2}\pi}\right) \quad (5)$$

$$\lambda = \lambda_0 + \frac{\pi\sqrt{4+\pi}\,x}{1+\cos\theta}, \quad (6)$$

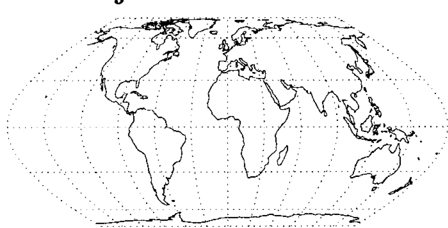
where

$$\theta = \sin^{-1}\left(\frac{y}{2}\sqrt{\frac{4+\pi}{\pi}}\right). \quad (7)$$

References

Snyder, J. P. *Map Projections—A Working Manual.* U. S. Geological Survey Professional Paper 1395. Washington, DC: U. S. Government Printing Office, pp. 253–258, 1987.

Eckert VI Projection



The equations are

$$x = \frac{(\lambda - \lambda_0)(1+\cos\theta)}{\sqrt{2+\pi}} \quad (1)$$

$$y = \frac{2\theta}{\sqrt{2+\pi}}, \quad (2)$$

where θ is the solution to

$$\theta + \sin\theta = (1 + \tfrac{1}{2}\pi)\sin\phi. \quad (3)$$

This can be solved iteratively using NEWTON'S METHOD with $\theta_0 = \phi$ to obtain

$$\Delta\theta = -\frac{\theta + \sin\theta - (1 + \frac{1}{2}\pi)\sin\phi}{1+\cos\theta}. \quad (4)$$

The inverse FORMULAS are

$$\phi = \sin^{-1}\left(\frac{\theta + \sin\theta}{1 + \frac{1}{2}\pi}\right) \quad (5)$$

$$\lambda = \lambda_0 + \frac{\sqrt{2+\pi}\,x}{1+\cos\theta}, \quad (6)$$

where

$$\theta = \tfrac{1}{2}\sqrt{2+\pi}\,y. \quad (7)$$

References

Snyder, J. P. *Map Projections—A Working Manual.* U. S. Geological Survey Professional Paper 1395. Washington, DC: U. S. Government Printing Office, pp. 253–258, 1987.

Economized Rational Approximation

A PADÉ APPROXIMATION perturbed with a CHEBYSHEV POLYNOMIAL OF THE FIRST KIND to reduce the leading COEFFICIENT in the ERROR.

Eddington Number

$$136 \cdot 2^{256} \approx 1.575 \times 10^{79}.$$

According to Eddington, the exact number of protons in the universe, where 136 was the RECIPROCAL of the fine structure constant as best as it could be measured in his time.

see also LARGE NUMBER

Edge-Coloring

An edge-coloring of a GRAPH G is a coloring of the edges of G such that adjacent edges (or the edges bounding different regions) receive different colors. BRELAZ'S HEURISTIC ALGORITHM can be used to find a good, but not necessarily minimal, edge-coloring.

see also BRELAZ'S HEURISTIC ALGORITHM, CHROMATIC NUMBER, k-COLORING

References

Saaty, T. L. and Kainen, P. C. *The Four-Color Problem: Assaults and Conquest.* New York: Dover, p. 13, 1986.

Edge Connectivity

The minimum number of EDGES whose deletion from a GRAPH disconnects it.

see also VERTEX CONNECTIVITY

Edge (Graph)

For an undirected GRAPH, an unordered pair of nodes which specify the line connecting them. For a DIRECTED GRAPH, the edge is an ordered pair of nodes.

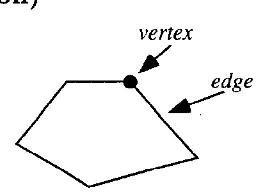
see also EDGE NUMBER, NULL GRAPH, TAIT COLORING, TAIT CYCLE, VERTEX (GRAPH)

Edge Number

The number of EDGES in a GRAPH, denoted $|E|$.

see also EDGE (GRAPH)

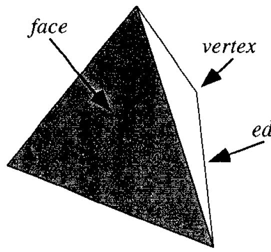
Edge (Polygon)



A LINE SEGMENT on the boundary of a FACE, also called a SIDE.

see also EDGE (POLYHEDRON), VERTEX (POLYGON)

Edge (Polyhedron)



A LINE SEGMENT where two FACES of a POLYHEDRON meet, also called a SIDE.

see also EDGE (POLYGON), VERTEX (POLYHEDRON)

Edge (Polytope)

A 1-D LINE SEGMENT where two 2-D FACES of an n-D POLYTOPE meet, also called a SIDE.

see also EDGE (POLYGON), EDGE (POLYHEDRON)

Edgeworth Series

Approximate a distribution in terms of a NORMAL DISTRIBUTION. Let

$$\phi(t) \equiv \frac{1}{\sqrt{2\pi}} e^{-t^2/2},$$

then

$$f(t) = \phi(t) + \tfrac{1}{3!}\gamma_1 \phi^{(3)}(t) + \left[\frac{\gamma_2}{4!}\phi^{(4)}(t) + \frac{10{\gamma_1}^2}{6!}\phi^{(6)}(t)\right] + \ldots.$$

see also CORNISH-FISHER ASYMPTOTIC EXPANSION, GRAM-CHARLIER SERIES

References

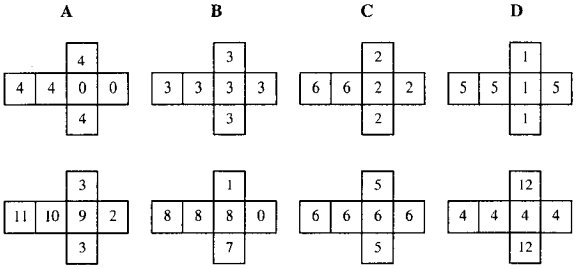
Abramowitz, M. and Stegun, C. A. (Eds.). *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables, 9th printing.* New York: Dover, p. 935, 1972.

Kenney, J. F. and Keeping, E. S. *Mathematics of Statistics, Pt. 2, 2nd ed.* Princeton, NJ: Van Nostrand, p. 108, 1951.

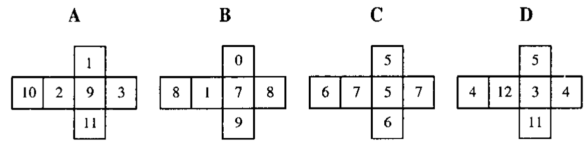
Edmonds' Map

A nonreflexible regular map of GENUS 7 with eight VERTICES, 28 EDGES, and eight HEPTAGONAL faces.

Efron's Dice



A set of four nontransitive DICE such that the probabilities of A winning against B, B against C, C against D, and D against A are all 2:1. A set in which ties may occur, in which case the DICE are rolled again, which gives ODDS of 11:6 is



see also DICE, SICHERMAN DICE

References

Gardner, M. "Mathematical Games: The Paradox of the Nontransitive Dice and the Elusive Principle of Indifference." *Sci. Amer.* **223**, 110–114, Dec. 1970.

Honsberger, R. "Some Surprises in Probability." Ch. 5 in *Mathematical Plums* (Ed. R. Honsberger). Washington, DC: Math. Assoc. Amer., pp. 94–97, 1979.

Egg

An OVAL with one end more pointed than the other.

see also ELLIPSE, MOSS'S EGG, OVAL, OVOID, THOM'S EGGS

Egyptian Fraction

see UNIT FRACTION

Ehrhart Polynomial

Let Δ denote an integral convex POLYTOPE of DIMENSION n in a lattice M, and let $l_\Delta(k)$ denote the number of LATTICE POINTS in Δ dilated by a factor of the integer k,

$$l_\Delta(k) = \#(k\Delta \cap M) \tag{1}$$

for $k \in \mathbb{Z}^+$. Then l_Δ is a polynomial function in k of degree n with rational coefficients

$$l_\Delta(k) = a_n k^n + a_{n-1}k^{n-1} + \ldots + a_0 \tag{2}$$

called the Ehrhart polynomial (Ehrhart 1967, Pommersheim 1993). Specific coefficients have important geometric interpretations.

1. a_n is the CONTENT of Δ.
2. a_{n-1} is half the sum of the CONTENTS of the $(n-1)$-D faces of Δ.
3. $a_0 = 1$.

Let $S_2(\Delta)$ denote the sum of the lattice lengths of the edges of Δ, then the case $n = 2$ corresponds to PICK'S THEOREM,

$$l_\Delta(k) = \mathrm{Vol}(\Delta)k^2 + \tfrac{1}{2}S_2(\Delta) + 1. \tag{3}$$

Let $S_3(\Delta)$ denote the sum of the lattice volumes of the 2-D faces of Δ, then the case $n = 3$ gives

$$l_\Delta(k) = \mathrm{Vol}(\Delta)k^3 + \tfrac{1}{2}S_3(\Delta)k^2 + a_1 k + 1, \tag{4}$$

where a rather complicated expression is given by Pommersheim (1993), since a_1 can unfortunately *not* be interpreted in terms of the edges of Δ. The Ehrhart polynomial of the tetrahedron with vertices at (0, 0, 0), (a, 0, 0), (0, b, 0), (0, 0, c) is

$$\begin{aligned} l_\Delta(k) = \tfrac{1}{6}abck^3 + \tfrac{1}{4}(ab+ac+bc+d)k^2 \\ + \left[\frac{1}{12}\left(\frac{ac}{b}+\frac{bc}{a}+\frac{ab}{c}+\frac{d^2}{abc}\right)\right. \\ +\tfrac{1}{4}(a+b+c+A+B+C) - As\left(\frac{bc}{d},\frac{aA}{d}\right) \\ \left. -Bs\left(\frac{ac}{d},\frac{bB}{d}\right) - Cs\left(\frac{ab}{d},\frac{cC}{d}\right)\right] k + 1, \quad (5) \end{aligned}$$

where $s(x,y)$ is a DEDEKIND SUM, $A = \gcd(b,c)$, $B = \gcd(a,c)$, $C = \gcd(a,b)$ (here, gcd is the GREATEST COMMON DENOMINATOR), and $d = ABC$ (Pommersheim 1993).

see also DEHN INVARIANT, PICK'S THEOREM

References
Ehrhart, E. "Sur une problème de géométrie diophantine linéaire." *J. Reine angew. Math.* **227**, 1–29, 1967.
MacDonald, I. G. "The Volume of a Lattice Polyhedron." *Proc. Camb. Phil. Soc.* **59**, 719–726, 1963.
McMullen, P. "Valuations and Euler-Type Relations on Certain Classes of Convex Polytopes." *Proc. London Math. Soc.* **35**, 113–135, 1977.
Pommersheim, J. "Toric Varieties, Lattices Points, and Dedekind Sums." *Math. Ann.* **295**, 1–24, 1993.
Reeve, J. E. "On the Volume of Lattice Polyhedra." *Proc. London Math. Soc.* **7**, 378–395, 1957.
Reeve, J. E. "A Further Note on the Volume of Lattice Polyhedra." *Proc. London Math. Soc.* **34**, 57–62, 1959.

Ei

see EXPONENTIAL INTEGRAL, E_n-FUNCTION

Eigenfunction

If $\tilde{L}$ is a linear OPERATOR on a FUNCTION SPACE, then f is an eigenfunction for $\tilde{L}$ and λ is the associated EIGENVALUE whenever $\tilde{L}f = \lambda f$.

see also EIGENVALUE, EIGENVECTOR

Eigenvalue

Let A be a linear transformation represented by a MATRIX A. If there is a VECTOR $\mathbf{X} \in \mathbb{R}^n \neq \mathbf{0}$ such that

$$\mathsf{A}\mathbf{X} = \lambda\mathbf{X} \quad (1)$$

for some SCALAR λ, then λ is the eigenvalue of A with corresponding (right) EIGENVECTOR $\mathbf{X}$. Letting A be a $k \times k$ MATRIX,

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1k} \\ a_{21} & a_{22} & \cdots & a_{2k} \\ \vdots & \vdots & \ddots & \vdots \\ a_{k1} & a_{k2} & \cdots & a_{kk} \end{bmatrix} \quad (2)$$

with eigenvalue λ, then the corresponding EIGENVECTORS satisfy

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1k} \\ a_{21} & a_{22} & \cdots & a_{2k} \\ \vdots & \vdots & \ddots & \vdots \\ a_{k1} & a_{k2} & \cdots & a_{kk} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_k \end{bmatrix} = \lambda \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_k \end{bmatrix}, \quad (3)$$

which is equivalent to the homogeneous system

$$\begin{bmatrix} a_{11}-\lambda & a_{12} & \cdots & a_{1k} \\ a_{21} & a_{22}-\lambda & \cdots & a_{2k} \\ \vdots & \vdots & \ddots & \vdots \\ a_{k1} & a_{k2} & \cdots & a_{kk}-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_k \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}. \quad (4)$$

Equation (4) can be written compactly as

$$(\mathsf{A} - \lambda\mathsf{I})\mathbf{X} = \mathbf{0}, \quad (5)$$

where I is the IDENTITY MATRIX.

As shown in CRAMER'S RULE, a system of linear equations has nontrivial solutions only if the DETERMINANT vanishes, so we obtain the CHARACTERISTIC EQUATION

$$|\mathsf{A} - \lambda\mathsf{I}| = 0. \quad (6)$$

If all k λs are different, then plugging these back in gives $k-1$ independent equations for the k components of each corresponding EIGENVECTOR. The EIGENVECTORS will then be orthogonal and the system is said to be nondegenerate. If the eigenvalues are n-fold DEGENERATE, then the system is said to be degenerate and the EIGENVECTORS are not linearly independent. In such cases, the additional constraint that the EIGENVECTORS be orthogonal,

$$\mathbf{X}_i \cdot \mathbf{X}_j = X_i X_j \delta_{ij}, \quad (7)$$

where δ_{ij} is the KRONECKER DELTA, can be applied to yield n additional constraints, thus allowing solution for the EIGENVECTORS.

Assume A has nondegenerate eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ and corresponding linearly independent EIGENVECTORS $\mathbf{X}_1, \mathbf{X}_2, \ldots, \mathbf{X}_k$ which can be denoted

$$\begin{bmatrix} x_{11} \\ x_{12} \\ \vdots \\ x_{1k} \end{bmatrix}, \begin{bmatrix} x_{21} \\ x_{22} \\ \vdots \\ x_{2k} \end{bmatrix}, \ldots \begin{bmatrix} x_{k1} \\ x_{k2} \\ \vdots \\ x_{kk} \end{bmatrix}. \quad (8)$$

Define the matrices composed of eigenvectors

$$\mathsf{P} \equiv [\mathbf{X}_1 \quad \mathbf{X}_2 \quad \cdots \quad \mathbf{X}_k] = \begin{bmatrix} x_{11} & x_{21} & \cdots & x_{k1} \\ x_{12} & x_{22} & \cdots & x_{k2} \\ \vdots & \vdots & \ddots & \vdots \\ x_{1k} & x_{2k} & \cdots & x_{kk} \end{bmatrix} \quad (9)$$

and eigenvalues

$$\mathsf{D} \equiv \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_k \end{bmatrix}, \tag{10}$$

where D is a DIAGONAL MATRIX. Then

$$\begin{aligned} \mathsf{AP} &= \mathsf{A}[\mathbf{X}_1 \quad \mathbf{X}_2 \quad \cdots \quad \mathbf{X}_k] \\ &= [\mathsf{A}\mathbf{X}_1 \quad \mathsf{A}\mathbf{X}_2 \quad \cdots \quad \mathsf{A}\mathbf{X}_k] \\ &= [\lambda_1\mathbf{X}_1 \quad \lambda_2\mathbf{X}_2 \quad \cdots \quad \lambda_k\mathbf{X}_k] \\ &= \begin{bmatrix} \lambda_1 x_{11} & \lambda_2 x_{21} & \cdots & \lambda_k x_{k1} \\ \lambda_1 x_{12} & \lambda_2 x_{22} & \cdots & \lambda_k x_{k2} \\ \vdots & \vdots & \ddots & \vdots \\ \lambda_1 x_{1k} & \lambda_2 x_{2k} & \cdots & \lambda_k x_{kk} \end{bmatrix} \\ &= \begin{bmatrix} x_{11} & x_{21} & \cdots & x_{k1} \\ x_{12} & x_{22} & \cdots & x_{k2} \\ \vdots & \vdots & \ddots & \vdots \\ x_{1k} & x_{2k} & \cdots & x_{kk} \end{bmatrix} \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_k \end{bmatrix} \\ &= \mathsf{PD}, \end{aligned} \tag{11}$$

so

$$\mathsf{A} = \mathsf{PDP}^{-1}. \tag{12}$$

Furthermore,

$$\begin{aligned} \mathsf{A}^2 &= (\mathsf{PDP}^{-1})(\mathsf{PDP}^{-1}) = \mathsf{PD}(\mathsf{P}^{-1}\mathsf{P})\mathsf{DP}^{-1} \\ &= \mathsf{PD}^2\mathsf{P}^{-1}. \end{aligned} \tag{13}$$

By induction, it follows that for $n > 0$,

$$\mathsf{A}^n = \mathsf{PD}^n\mathsf{P}^{-1}. \tag{14}$$

The inverse of A is

$$\mathsf{A}^{-1} = (\mathsf{PDP}^{-1})^{-1} = \mathsf{PD}^{-1}\mathsf{P}^{-1}, \tag{15}$$

where the inverse of the DIAGONAL MATRIX D is trivially given by

$$\mathsf{D}^{-1} = \frac{1}{k} \begin{bmatrix} {\lambda_1}^{-1} & 0 & \cdots & 0 \\ 0 & {\lambda_2}^{-1} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & {\lambda_k}^{-1} \end{bmatrix}. \tag{16}$$

Equation (14) therefore holds for both POSITIVE and NEGATIVE n.

A further remarkable result involving the matrices P and D follows from the definition

$$\begin{aligned} e^{\mathsf{A}} &\equiv \sum_{n=0}^{\infty} \frac{\mathsf{A}^n}{n!} = \sum_{n=0}^{\infty} \frac{\mathsf{PD}^n\mathsf{P}^{-1}}{n!} \\ &= \mathsf{P}\left(\frac{\sum_{n=0}^{\infty} \mathsf{D}^n}{n!}\right)\mathsf{P}^{-1} = \mathsf{P}e^{\mathsf{D}}\mathsf{P}^{-1}. \end{aligned} \tag{17}$$

Since D is a DIAGONAL MATRIX,

$$\begin{aligned} e^{\mathsf{D}} &= \sum_{n=0}^{\infty} \frac{\mathsf{D}^n}{n!} = \sum_{n=0}^{\infty} \frac{1}{n!} \begin{bmatrix} {\lambda_1}^n & 0 & \cdots & 0 \\ 0 & {\lambda_2}^n & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & {\lambda_k}^n \end{bmatrix} \\ &= \begin{bmatrix} \sum_{i=0}^{\infty} {\lambda_1}^i & 0 & \cdots & 0 \\ 0 & \sum_{i=0}^{\infty} {\lambda_2}^i & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \sum_{i=0}^{\infty} {\lambda_k}^i \end{bmatrix} \\ &= \begin{bmatrix} e^{\lambda_1} & 0 & \cdots & 0 \\ 0 & e^{\lambda_2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & e^{\lambda_k} \end{bmatrix}, \end{aligned} \tag{18}$$

e^{D} can be found using

$$\mathsf{D}^n = \begin{bmatrix} {\lambda_1}^n & 0 & \cdots & 0 \\ 0 & {\lambda_2}^n & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & {\lambda_k}^n \end{bmatrix}. \tag{19}$$

Assume we know the eigenvalue for

$$\mathsf{A}\mathbf{X} = \lambda\mathbf{X}. \tag{20}$$

Adding a constant times the IDENTITY MATRIX to A,

$$(\mathsf{A} + c\,\mathsf{I})\mathbf{X} = (\lambda + c)\mathbf{X} \equiv \lambda'\mathbf{X}, \tag{21}$$

so the new eigenvalues equal the old plus c. Multiplying A by a constant c

$$(c\,\mathsf{A})\mathbf{X} = c\,(\lambda\mathbf{X}) \equiv \lambda'\mathbf{X}, \tag{22}$$

so the new eigenvalues are the old multiplied by c.

Now consider a SIMILARITY TRANSFORMATION of A. Let $|\mathsf{A}|$ be the DETERMINANT of A, then

$$\begin{aligned} |\mathsf{Z}^{-1}\mathsf{AZ} - \lambda\mathsf{I}| &= |\mathsf{Z}^{-1}(\mathsf{A} - \lambda\mathsf{I})\mathsf{Z}| \\ &= |\mathsf{Z}|\,|\mathsf{A} - \lambda\mathsf{I}|\,|\mathsf{Z}^{-1}| = |\mathsf{A} - \lambda\mathsf{I}|, \end{aligned} \tag{23}$$

so the eigenvalues are the same as for A.

see also BRAUER'S THEOREM, CONDITION NUMBER, EIGENFUNCTION, EIGENVECTOR, FROBENIUS THEOREM, GERŠGORIN CIRCLE THEOREM, LYAPUNOV'S FIRST THEOREM, LYAPUNOV'S SECOND THEOREM, OSTROWSKI'S THEOREM, PERRON'S THEOREM, PERRON-FROBENIUS THEOREM, POINCARÉ SEPARATION THEOREM, RANDOM MATRIX, SCHUR'S INEQUALITIES, STURMIAN SEPARATION THEOREM, SYLVESTER'S INERTIA LAW, WIELANDT'S THEOREM

References

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Nash, J. C. "The Algebraic Eigenvalue Problem." Ch. 9 in *Compact Numerical Methods for Computers: Linear Algebra and Function Minimisation, 2nd ed.* Bristol, England: Adam Hilger, pp. 102–118, 1990.
Press, W. H.; Flannery, B. P.; Teukolsky, S. A.; and Vetterling, W. T. "Eigensystems." Ch. 11 in *Numerical Recipes in FORTRAN: The Art of Scientific Computing, 2nd ed.* Cambridge, England: Cambridge University Press, pp. 449–489, 1992.

Eigenvector

A right eigenvector satisfies

$$\mathsf{A}\mathbf{X} = \lambda\mathbf{X}, \tag{1}$$

where $\mathbf{X}$ is a column VECTOR. The right EIGENVALUES therefore satisfy

$$|\mathsf{A} - \lambda\mathsf{I}| = 0. \tag{2}$$

A left eigenvector satisfies

$$\mathbf{X}\mathsf{A} = \lambda\mathbf{X}, \tag{3}$$

where $\mathbf{X}$ is a row VECTOR, so

$$(\mathbf{X}\mathsf{A})^{\mathrm{T}} = \lambda_L \mathbf{X}^{\mathrm{T}} \tag{4}$$

$$\mathsf{A}^{\mathrm{T}}\mathbf{X}^{\mathrm{T}} = \lambda_L \mathbf{X}^{\mathrm{T}}, \tag{5}$$

where $\mathbf{X}^{\mathrm{T}}$ is the transpose of $\mathbf{X}$. The left EIGENVALUES satisfy

$$|\mathsf{A}^{\mathrm{T}} - \lambda_L \mathsf{I}| = |\mathsf{A}^{\mathrm{T}} - \lambda_L \mathsf{I}^{\mathrm{T}}| = |(\mathsf{A} - \lambda_L \mathsf{I})^{\mathrm{T}}| = |(\mathsf{A} - \lambda_L \mathsf{I})|, \tag{6}$$

(since $|\mathsf{A}| = |\mathsf{A}^{\mathrm{T}}|$) where $|\mathsf{A}|$ is the DETERMINANT of A. But this is the same equation satisfied by the right EIGENVALUES, so the left and right EIGENVALUES are the same. Let $\mathbf{X}_R$ be a MATRIX formed by the columns of the right eigenvectors and $\mathbf{X}_L$ be a MATRIX formed by the rows of the left eigenvectors. Let

$$\mathsf{D} \equiv \begin{bmatrix} \lambda_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \lambda_n \end{bmatrix}. \tag{7}$$

Then

$$\mathsf{A}\mathbf{X}_R = \mathbf{X}_R\mathsf{D} \qquad \mathbf{X}_L\mathsf{A} = \mathsf{D}\mathbf{X}_L \tag{8}$$

$$\mathbf{X}_L\mathsf{A}\mathbf{X}_R = \mathbf{X}_L\mathbf{X}_R\mathsf{D} \qquad \mathbf{X}_L\mathsf{A}\mathbf{X}_R = \mathsf{D}\mathbf{X}_L\mathbf{X}_R, \tag{9}$$

so

$$\mathbf{X}_L\mathbf{X}_R\mathsf{D} = \mathsf{D}\mathbf{X}_L\mathbf{X}_R. \tag{10}$$

But this equation is of the form $\mathsf{CD} = \mathsf{DC}$ where D is a DIAGONAL MATRIX, so it must be true that $\mathsf{C} \equiv \mathbf{X}_L\mathbf{X}_R$ is also diagonal. In particular, if A is a SYMMETRIC MATRIX, then the left and right eigenvectors are transposes of each other. If A is a SELF-ADJOINT MATRIX, then the left and right eigenvectors are conjugate HERMITIAN MATRICES.

Given a 3×3 MATRIX A with eigenvectors $\mathbf{x}_1$, $\mathbf{x}_2$, and $\mathbf{x}_3$ and corresponding EIGENVALUES λ_1, λ_2, and λ_3, then an arbitrary VECTOR $\mathbf{y}$ can be written

$$\mathbf{y} = b_1\mathbf{x}_1 + b_2\mathbf{x}_2 + b_3\mathbf{x}_3. \tag{11}$$

Applying the MATRIX A,

$$\begin{aligned} \mathsf{A}\mathbf{y} &= b_1\mathsf{A}\mathbf{x}_1 + b_2\mathsf{A}\mathbf{x}_2 + b_3\mathsf{A}\mathbf{x}_3 \\ &= \lambda_1\left(b_1\mathbf{x}_1 + \frac{\lambda_2}{\lambda_1}b_2\mathbf{x}_2 + \frac{\lambda_3}{\lambda_1}b_3\mathbf{x}_3\right), \end{aligned} \tag{12}$$

so

$$\mathsf{A}^n\mathbf{y} = \lambda_1{}^n\left[b_1\mathbf{x}_1 + \left(\frac{\lambda_2}{\lambda_1}\right)^n b_2\mathbf{x}_2 + \left(\frac{\lambda_3}{\lambda_1}\right)^n b_3\mathbf{x}_3\right]. \tag{13}$$

If $\lambda_1 > \lambda_2, \lambda_3$, it therefore follows that

$$\lim_{n\to\infty} \mathsf{A}^n\mathbf{y} = \lambda_1{}^n b_1\mathbf{x}_1, \tag{14}$$

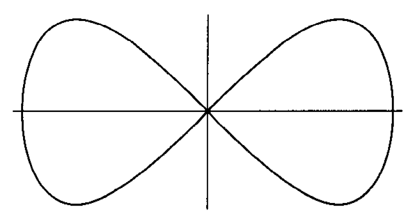
so repeated application of the matrix to an arbitrary vector results in a vector proportional to the EIGENVECTOR having the largest EIGENVALUE.

see also EIGENFUNCTION, EIGENVALUE

References

Arfken, G. "Eigenvectors, Eigenvalues." §4.7 in *Mathematical Methods for Physicists, 3rd ed.* Orlando, FL: Academic Press, pp. 229–237, 1985.
Press, W. H.; Flannery, B. P.; Teukolsky, S. A.; and Vetterling, W. T. "Eigensystems." Ch. 11 in *Numerical Recipes in FORTRAN: The Art of Scientific Computing, 2nd ed.* Cambridge, England: Cambridge University Press, pp. 449–489, 1992.

Eight Curve



A curve also known as the GERONO LEMNISCATE. It is given by CARTESIAN COORDINATES

$$x^4 = a^2(x^2 - y^2), \tag{1}$$

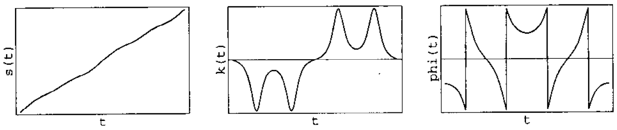
POLAR COORDINATES,

$$r^2 = a^2 \sec^4\theta \cos(2\theta), \tag{2}$$

and parametric equations

$$x = a\sin t \tag{3}$$

$$y = a\sin t\cos t. \tag{4}$$



The CURVATURE and TANGENTIAL ANGLE are

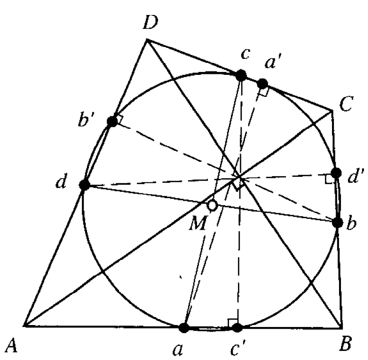
$$\kappa(t) = -\frac{3\sin t + \sin(3t)}{2[\cos^2 t + \cos^2(2t)]^{3/2}} \quad (5)$$

$$\phi(t) = -\tan^{-1}[\cos t \sec(2t)]. \quad (6)$$

see also BUTTERFLY CURVE, DUMBBELL CURVE, EIGHT SURFACE, PIRIFORM

References
Cundy, H. and Rollett, A. *Mathematical Models, 3rd ed.* Stradbroke, England: Tarquin Pub., p. 71, 1989.
Lawrence, J. D. *A Catalog of Special Plane Curves.* New York: Dover, pp. 124–126, 1972.
Lee, X. "Lemniscate of Gerono." `http://www.best.com/~xah/SpecialPlaneCurves_dir/LemniscateOfGerono_dir/lemniscateOfGerono.html`.
MacTutor History of Mathematics Archive. "Eight Curve." `http://www-groups.dcs.st-and.ac.uk/~history/Curves/Eight.html`.

Eight-Point Circle Theorem

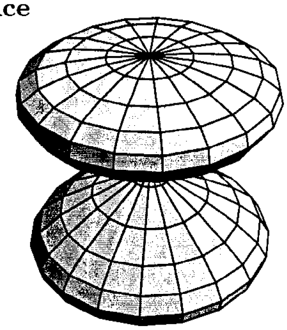


Let $ABCD$ be a QUADRILATERAL with PERPENDICULAR DIAGONALS. The MIDPOINTS of the sides (a, b, c, and d) determine a PARALLELOGRAM (the VARIGNON PARALLELOGRAM) with sides PARALLEL to the DIAGONALS. The eight-point circle passes through the four MIDPOINTS and the four feet of the PERPENDICULARS from the opposite sides a', b', c', and d'.

see also FEUERBACH'S THEOREM

References
Brand, L. "The Eight-Point Circle and the Nine-Point Circle." *Amer. Math. Monthly* **51**, 84–85, 1944.
Honsberger, R. *Mathematical Gems II.* Washington, DC: Math. Assoc. Amer., pp. 11–13, 1976.

Eight Surface



The SURFACE OF REVOLUTION given by the parametric equations

$$x(u,v) = \cos u \sin(2v) \quad (1)$$

$$y(u,v) = \sin u \sin(2v) \quad (2)$$

$$z(u,v) = \sin v \quad (3)$$

for $u \in [0, 2\pi)$ and $v \in [-\pi/2, \pi/2]$.

see also EIGHT CURVE

References
Gray, A. *Modern Differential Geometry of Curves and Surfaces.* Boca Raton, FL: CRC Press, pp. 209–210 and 224, 1993.

Eikonal Equation

$$\sum_{i=1}^{n} \left(\frac{\partial u}{\partial x_i}\right)^2 = 1.$$

Eilenberg-Mac Lane Space

For any ABELIAN GROUP G and any NATURAL NUMBER n, there is a unique SPACE (up to HOMOTOPY type) such that all HOMOTOPY GROUPS except for the nth are trivial (including the 0th HOMOTOPY GROUPS, meaning the SPACE is path-connected), and the nth HOMOTOPY GROUP is ISOMORPHIC to the GROUP G. In the case where $n = 1$, the GROUP G can be non-ABELIAN as well.

Eilenberg-Mac Lane spaces have many important applications. One of them is that every TOPOLOGICAL SPACE has the HOMOTOPY type of an iterated FIBRATION of Eilenberg-Mac Lane spaces (called a POSTNIKOV SYSTEM). In addition, there is a spectral sequence relating the COHOMOLOGY of Eilenberg-Mac Lane spaces to the HOMOTOPY GROUPS of SPHERES.

Eilenberg-Mac Lane-Steenrod-Milnor Axioms

see EILENBERG-STEENROD AXIOMS

Eilenberg-Steenrod Axioms

A family of FUNCTORS $H_n(\cdot)$ from the CATEGORY of pairs of TOPOLOGICAL SPACES and continuous maps, to the CATEGORY of ABELIAN GROUPS and group homomorphisms satisfies the Eilenberg-Steenrod axioms if the following conditions hold.

1. LONG EXACT SEQUENCE OF A PAIR AXIOM. For every pair (X, A), there is a natural long exact sequence

$$\ldots \to H_n(A) \to H_n(X) \to H_n(X, A) \to H_{n-1}(A) \to \ldots,$$

where the MAP $H_n(A) \to H_n(X)$ is induced by the INCLUSION MAP $A \to X$ and $H_n(X) \to H_n(X, A)$ is induced by the INCLUSION MAP $(X, \phi) \to (X, A)$. The MAP $H_n(X, A) \to H_{n-1}(A)$ is called the BOUNDARY MAP.

2. HOMOTOPY AXIOM. If $f : (X, A) \to (Y, B)$ is homotopic to $g : (X, A) \to (Y, B)$, then their INDUCED MAPS $f_* : H_n(X, A) \to H_n(Y, B)$ and $g_* : H_n(X, A) \to H_n(Y, B)$ are the same.

3. EXCISION AXIOM. If X is a SPACE with SUBSPACES A and U such that the CLOSURE of A is contained in the interior of U, then the INCLUSION MAP $(X\ U, A\ U) \to (X, A)$ induces an isomorphism $H_n(X\ U, A\ U) \to H_n(X, A)$.

4. DIMENSION AXIOM. Let X be a single point space. $H_n(X) = 0$ unless $n = 0$, in which case $H_0(X) = G$ where G are some GROUPS. The H_0 are called the COEFFICIENTS of the HOMOLOGY theory $H(\cdot)$.

These are the axioms for a generalized homology theory. For a cohomology theory, instead of requiring that $H(\cdot)$ be a FUNCTOR, it is required to be a co-functor (meaning the INDUCED MAP points in the opposite direction). With that modification, the axioms are essentially the same (except that all the induced maps point backwards).

see also ALEKSANDROV-ČECH COHOMOLOGY

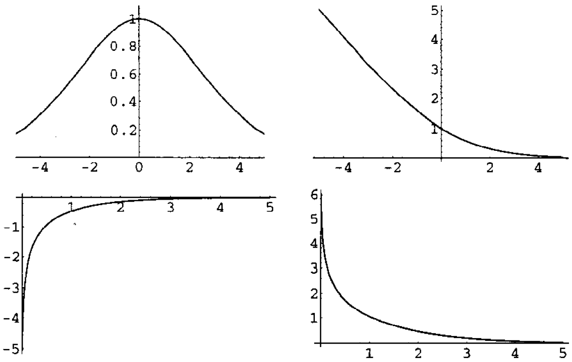
Ein Function

$$\mathrm{Ein}(z) \equiv \int_0^z \frac{(1 - e^{-t})\,dt}{t} = \mathrm{E}_1(z) + \ln z + \gamma,$$

where γ is the EULER-MASCHERONI CONSTANT and E_1 is the E_n-FUNCTION with $n = 1$.

see also E_n-FUNCTION

Einstein Functions



The functions $x^2 e^x/(e^x - 1)^2$, $x/(e^x - 1)$, $\ln(1 - e^{-x})$, and $x/(e^x - 1) - \ln(1 - e^{-x})$.

References

Abramowitz, M. and Stegun, C. A. (Eds.). "Debye Functions." §27.1 in *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables, 9th printing.* New York: Dover, pp. 999–1000, 1972.

Einstein Summation

The implicit convention that repeated indices are summed over so that, for example,

$$a_i a_i \equiv \sum_i a_i a_i.$$

Eisenstein Integer

The numbers $a + b\omega$, where

$$\omega \equiv \tfrac{1}{2}(-1 + i\sqrt{3})$$

is one of the ROOTS of $z^3 = 1$, the others being 1 and

$$\omega^2 = \tfrac{1}{2}(-1 - i\sqrt{3}).$$

Eisenstein integers are members of the QUADRATIC FIELD $\mathbb{Q}(\sqrt{-3})$, and the COMPLEX NUMBERS $\mathbb{Z}[\omega]$. Every Eisenstein integer has a unique factorization. Specifically, any NONZERO Eisenstein integer is uniquely the product of POWERS of -1, ω, and the "positive" EISENSTEIN PRIMES (Conway and Guy 1996). Every Eisenstein integer is within a distance $|n|/\sqrt{3}$ of some multiple of a given Eisenstein integer n.

Dörrie (1965) uses the alternative notation

$$J \equiv \tfrac{1}{2}(1 + i\sqrt{3}) \quad (1)$$

$$O \equiv \tfrac{1}{2}(1 - i\sqrt{3}). \quad (2)$$

for $-\omega^2$ and $-\omega$, and calls numbers of the form $aJ+bO$ G-NUMBERS. O and J satisfy

$$J+O=1 \quad (3)$$
$$JO=1 \quad (4)$$
$$J^2+O=0 \quad (5)$$
$$O^2+J=0 \quad (6)$$
$$J^3=-1 \quad (7)$$
$$O^3=-1. \quad (8)$$

The sum, difference, and products of G numbers are also G numbers. The norm of a G number is

$$N(aJ+bO)=a^2+b^2-ab. \quad (9)$$

The analog of FERMAT'S THEOREM for Eisenstein integers is that a PRIME NUMBER p can be written in the form

$$a^2-ab+b^2=(a+b\omega)(a+b\omega^2)$$

IFF $3 \nmid p+1$. These are precisely the PRIMES of the form $3m^2+n^2$ (Conway and Guy 1996).

see also EISENSTEIN PRIME, EISENSTEIN UNIT, GAUSSIAN INTEGER, INTEGER

References

Conway, J. H. and Guy, R. K. *The Book of Numbers.* New York: Springer-Verlag, pp. 220–223, 1996.

Cox, D. A. §4A in *Primes of the Form x^2+ny^2: Fermat, Class Field Theory and Complex Multiplication.* New York: Wiley, 1989.

Dörrie, H. "The Fermat-Gauss Impossibility Theorem." §21 in *100 Great Problems of Elementary Mathematics: Their History and Solutions.* New York: Dover, pp. 96–104, 1965.

Guy, R. K. "Gaussian Primes. Eisenstein-Jacobi Primes." §A16 in *Unsolved Problems in Number Theory, 2nd ed.* New York: Springer-Verlag, pp. 33–36, 1994.

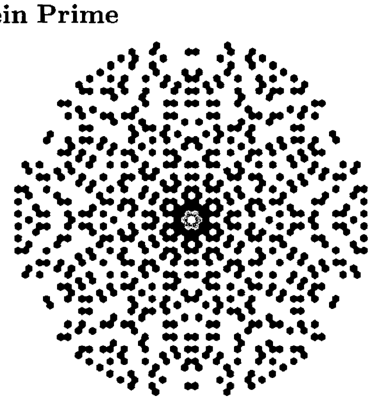
Riesel, H. Appendix 4 in *Prime Numbers and Computer Methods for Factorization, 2nd ed.* Boston, MA: Birkhäuser, 1994.

Wagon, S. "Eisenstein Primes." *Mathematica in Action.* New York: W. H. Freeman, pp. 278–279, 1991.

Eisenstein-Jacobi Integer

see EISENSTEIN INTEGER

Eisenstein Prime



Let ω be the CUBE ROOT of unity $(-1+i\sqrt{3})/2$. Then the Eisenstein primes are

1. Ordinary PRIMES CONGRUENT to 2 (mod 3),
2. $1-\omega$ is prime in $\mathbb{Z}[\omega]$,
3. Any ordinary PRIME CONGRUENT to 1 (mod 3) factors as $\alpha\alpha^*$, where each of α and α^* are primes in $\mathbb{Z}[\omega]$ and α and α^* are not "associates" of each other (where associates are equivalent modulo multiplication by an EISENSTEIN UNIT).

References

Cox, D. A. §4A in *Primes of the Form x^2+ny^2: Fermat, Class Field Theory and Complex Multiplication.* New York: Wiley, 1989.

Guy, R. K. "Gaussian Primes. Eisenstein-Jacobi Primes." §A16 in *Unsolved Problems in Number Theory, 2nd ed.* New York: Springer-Verlag, pp. 33–36, 1994.

Wagon, S. "Eisenstein Primes." *Mathematica in Action.* New York: W. H. Freeman, pp. 278–279, 1991.

Eisenstein Series

$$E_r(t)=\sum_{m,n}{}' \frac{1}{(mt+n)^{2r}},$$

where the sum Σ' excludes $m=n=0$, $\Im[t]>0$, and r is an INTEGER >2. The Eisenstein series satisfies the remarkable property

$$E_r\left(\frac{at+b}{ct+d}\right)=(ct+d)^{2r}E_r(t).$$

see also RAMANUJAN-EISENSTEIN SERIES

Eisenstein Unit

The Eisenstein units are the EISENSTEIN INTEGERS ±1, $\pm\omega$, $\pm\omega^2$, where

$$\omega=\tfrac{1}{2}(-1+i\sqrt{3})$$
$$\omega^2=\tfrac{1}{2}(-1-i\sqrt{3}).$$

see also EISENSTEIN INTEGER, EISENSTEIN PRIME

References

Conway, J. H. and Guy, R. K. *The Book of Numbers.* New York: Springer-Verlag, pp. 220–223, 1996.

Elastica

The elastica formed by bent rods and considered in physics can be generalized to curves in a RIEMANNIAN MANIFOLD which are a CRITICAL POINT for

$$F^\lambda(\gamma)=\int_\gamma(\kappa^2+\lambda),$$

where κ is the GEODESIC CURVATURE of γ, λ is a REAL NUMBER, and γ is closed or satisfies some specified boundary condition. The curvature of an elastica must satisfy

$$0 = 2\kappa''(s) + \kappa^3(s) + 2\kappa(s)G(s) - \lambda\kappa(s),$$

where κ is the signed curvature of γ, $G(s)$ is the GAUSSIAN CURVATURE of the oriented Riemannian surface M along γ, κ'' is the second derivative of κ with respect to s, and λ is a constant.

References

Barros, M. and Garay, O. J. "Free Elastic Parallels in a Surface of Revolution." *Amer. Math. Monthly* **103**, 149–156, 1996.
Bryant, R. and Griffiths, P. "Reduction for Constrained Variational Problems and $\int(k^2/s)\,ds$." *Amer. J. Math.* **108**, 525–570, 1986.
Langer, J. and Singer, D. A. "Knotted Elastic Curves in R^3." *J. London Math. Soc.* **30**, 512–520, 1984.
Langer, J. and Singer, D. A. "The Total Squared of Closed Curves." *J. Diff. Geom.* **20**, 1–22, 1984.

Elation

A perspective COLLINEATION in which the center and axis are incident.

see also HOMOLOGY (GEOMETRY)

Elder's Theorem

A generalization of STANLEY'S THEOREM. It states that the total number of occurrences of an INTEGER k among all unordered PARTITIONS of n is equal to the number of occasions that a part occurs k or more times in a PARTITION, where a PARTITION which contains r parts that each occur k or more times contributes r to the sum in question.

see also STANLEY'S THEOREM

References

Honsberger, R. *Mathematical Gems III.* Washington, DC: Math. Assoc. Amer, pp. 8–9, 1985.

Election

see EARLY ELECTION RESULTS, VOTING

Electric Motor Curve

see DEVIL'S CURVE

Element

If x is a member of a set A, then x is said to be an element of A, written $x \in A$. If x is not an element of A, this is written $x \notin A$. The term element also refers to a particular member of a GROUP, or entry in a MATRIX.

Elementary Function

A function built up of compositions of the EXPONENTIAL FUNCTION and the TRIGONOMETRIC FUNCTIONS and their inverses by ADDITION, MULTIPLICATION, DIVISION, root extractions (the ELEMENTARY OPERATIONS) under repeated compositions. Not all functions are elementary. For example, the NORMAL DISTRIBUTION FUNCTION

$$\Phi(x) \equiv \frac{1}{\sqrt{2\pi}} \int_0^x e^{-t^2/2}\,dt$$

is a notorious example of a nonelementary function. Nonelementary functions are called TRANSCENDENTAL FUNCTIONS.

see also ALGEBRAIC FUNCTION, ELEMENTARY OPERATION, ELEMENTARY SYMMETRIC FUNCTION, TRANSCENDENTAL FUNCTION

References

Shanks, D. *Solved and Unsolved Problems in Number Theory, 4th ed.* New York: Chelsea, p. 145, 1993.

Elementary Matrix

The elementary MATRICES are the PERMUTATION MATRIX p_{ij} and the SHEAR MATRIX e_{ij}^s.

Elementary Operation

One of the operations of ADDITION, SUBTRACTION, MULTIPLICATION, DIVISION, and root extraction.

see also ALGEBRAIC FUNCTION, ELEMENTARY FUNCTION

Elementary Symmetric Function

The elementary symmetric functions Π_n on n variables $\{x_1, \ldots, x_n\}$ are defined by

$$\Pi_1 = \sum_{1\leq i\leq n} x_i \tag{1}$$

$$\Pi_2 = \sum_{1\leq i<j\leq n} x_i x_j \tag{2}$$

$$\Pi_3 = \sum_{1\leq i<j<k\leq n} x_i x_j x_k \tag{3}$$

$$\Pi_4 = \sum_{1\leq i<j<k<l\leq n} x_i x_j x_k x_l \tag{4}$$

$$\vdots$$

$$\Pi_n = \prod_{i\leq i\leq n} x_i. \tag{5}$$

Alternatively, Π_j can be defined as the coefficient of x^{n-j} in the GENERATING FUNCTION

$$\prod_{1\leq i\leq n} (x + x_i). \tag{6}$$

The elementary symmetric functions satisfy the relationships

$$\sum_{i=1}^{n} {x_i}^2 = {\Pi_1}^2 - 2\Pi_2 \tag{7}$$

$$\sum_{i=1}^{n} {x_i}^3 = {\Pi_1}^3 - 3\Pi_1\Pi_2 + 3\Pi_3 \tag{8}$$

$$\sum_{i=1}^{n} {x_i}^4 = {\Pi_1}^4 - 4{\Pi_1}^2\Pi_2 + 2{\Pi_2}^2 + 4\Pi_1\Pi_3 - 4\Pi_4 \tag{9}$$

(Beeler *et al.* 1972, Item 6).

see also FUNDAMENTAL THEOREM OF SYMMETRIC FUNCTIONS, NEWTON'S RELATIONS, SYMMETRIC FUNCTION

References

Beeler, M.; Gosper, R. W.; and Schroeppel, R. *HAKMEM.* Cambridge, MA: MIT Artificial Intelligence Laboratory, Memo AIM-239, Feb. 1972.

Elements

The classic treatise in geometry written by Euclid and used as a textbook for more than 1,000 years in western Europe. *The Elements*, which went through more than 2,000 editions and consisted of 465 propositions, are divided into 13 "books" (an archaic word for "chapters").

Book	Contents
1	triangles
2	rectangles
3	Circles
4	polygons
5	proportion
6	similarity
7–10	number theory
11	solid geometry
12	pyramids
13	platonic solids

The elements started with 23 definitions, five POSTULATES, and five "common notions," and systematically built the rest of plane and solid geometry upon this foundation. The five EUCLID'S POSTULATES are

1. It is possible to draw a straight LINE from any POINT to another POINT.
2. It is possible to produce a finite straight LINE continuously in a straight LINE.
3. It is possible to describe a CIRCLE with any CENTER and RADIUS.
4. All RIGHT ANGLES are equal to one another.
5. If a straight LINE falling on two straight LINES makes the interior ANGLES on the same side less than two RIGHT ANGLES, the straight LINES (if extended indefinitely) meet on the side on which the ANGLES which are less than two RIGHT ANGLES lie.

(Dunham 1990). Euclid's fifth postulate is known as the PARALLEL POSTULATE. After more than two millennia of study, this POSTULATE was found to be independent of the others. In fact, equally valid NON-EUCLIDEAN GEOMETRIES were found to be possible by changing the assumption of this POSTULATE. Unfortunately, Euclid's postulates were not rigorously complete and left a large number of gaps. Hilbert needed a total of 20 postulates to construct a logically complete geometry.

see also PARALLEL POSTULATE

References

Casey, J. *A Sequel to the First Six Books of the Elements of Euclid, 6th ed.* Dublin: Hodges, Figgis, & Co., 1892.
Dixon, R. *Mathographics.* New York: Dover, pp. 26–27, 1991.
Dunham, W. *Journey Through Genius: The Great Theorems of Mathematics.* New York: Wiley, pp. 30–83, 1990.
Heath, T. L. *The Thirteen Books of the Elements, 2nd ed., Vol. 1: Books I and II.* New York: Dover, 1956.
Heath, T. L. *The Thirteen Books of the Elements, 2nd ed., Vol. 2: Books III–IX.* New York: Dover, 1956.
Heath, T. L. *The Thirteen Books of the Elements, 2nd ed., Vol. 3: Books X–XIII.* New York: Dover, 1956.
Joyce, D. E. "Euclid's Elements." `http://aleph0.clarku.edu/~djoyce/java/elements/elements.html`

Elevator Paradox

A fact noticed by physicist G. Gamow when he had an office on the second floor and physicist M. Stern had an office on the sixth floor of a seven-story building (Gamow and Stern 1958, Gardner 1986). Gamow noticed that about 5/6 of the time, the first elevator to stop on his floor was going down, whereas about the same fraction of time, the first elevator to stop on the sixth floor was going up. This actually makes perfect sense, since 5 of the 6 floors 1, 3, 4, 5, 6, 7 are above the second, and 5 of the 6 floors 1, 2, 3, 4, 5, 7 are below the sixth. However, the situation takes some unexpected turns if more than one elevator is involved, as discussed by Gardner (1986).

References

Gamow, G. and Stern, M. *Puzzle Math.* New York: Viking, 1958.
Gardner, M. "Elevators." Ch. 10 in *Knotted Doughnuts and Other Mathematical Entertainments.* New York: W. H. Freeman, pp. 123–132, 1986.

Elkies Point

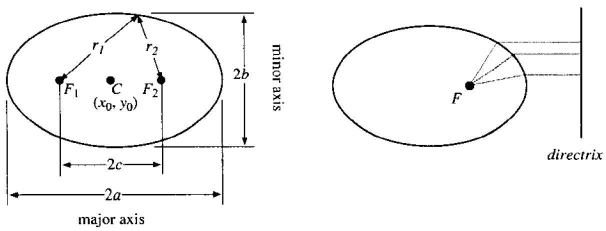
Given POSITIVE numbers s_a, s_b, and s_c, the Elkies point is the unique point Y in the interior of a TRIANGLE ΔABC such that the respective INRADII r_a, r_b, r_c of the TRIANGLES ΔBYC, ΔCYA, and ΔAYB satisfy $r_a : r_b : r_c = s_a : s_b : s_c$.

see also CONGRUENT INCIRCLES POINT, INRADIUS

References

Kimberling, C. "Central Points and Central Lines in the Plane of a Triangle." *Math. Mag.* **67**, 163–187, 1994.
Kimberling, C. and Elkies, N. "Problem 1238 and Solution." *Math. Mag.* **60**, 116–117, 1987.

Ellipse



A curve which is the LOCUS of all points in the PLANE the SUM of whose distances r_1 and r_2 from two fixed points F_1 and F_2 (the FOCI) separated by a distance of $2c$ is a given POSITIVE constant $2a$ (left figure). This results in the two-center BIPOLAR COORDINATE equation

$$r_1 + r_2 = 2a, \tag{1}$$

where a is the SEMIMAJOR AXIS and the ORIGIN of the coordinate system is at one of the FOCI. The ellipse can also be defined as the LOCUS of points whose distance from the FOCUS is proportional to the horizontal distance from a vertical line known as the DIRECTRIX (right figure).

The ellipse was first studied by Menaechmus, investigated by Euclid, and named by Apollonius. The FOCUS and DIRECTRIX of an ellipse were considered by Pappus. In 1602, Kepler believed that the orbit of Mars was OVAL; he later discovered that it was an ellipse with the Sun at one FOCUS. In fact, Kepler introduced the word "FOCUS" and published his discovery in 1609. In 1705 Halley showed that the comet which is now named after him moved in an elliptical orbit around the Sun (MacTutor Archive).

A ray passing through a FOCUS will pass through the other focus after a single bounce. Reflections not passing through a FOCUS will be tangent to a confocal HYPERBOLA or ELLIPSE, depending on whether the ray passes between the FOCI or not. Let an ellipse lie along the x-AXIS and find the equation of the figure (1) where r_1 and r_2 are at $(-c, 0)$ and $(c, 0)$. In CARTESIAN COORDINATES,

$$\sqrt{(x+c)^2 + y^2} + \sqrt{(x-c)^2 + y^2} = 2a. \tag{2}$$

Bring the second term to the right side and square both sides,

$$(x+c)^2 + y^2 = 4a^2 - 4a\sqrt{(x-c)^2 + y^2} + (x-c)^2 + y^2. \tag{3}$$

Now solve for the SQUARE ROOT term and simplify

$$\begin{aligned}\sqrt{(x-c)^2 + y^2} \\ = -\frac{1}{4a}(x^2 + 2xc + c^2 + y^2 - 4a^2 - x^2 + 2xc - c^2 - y^2) \\ = -\frac{1}{4a}(4xc - 4a^2) = a - \frac{c}{a}x. \end{aligned} \tag{4}$$

Square one final time to clear the remaining SQUARE ROOT,

$$x^2 - 2xc + c^2 + y^2 = a^2 - 2cx + \frac{c^2}{a^2}x^2. \tag{5}$$

Grouping the x terms then gives

$$x^2 \frac{a^2 - c^2}{a^2} + y^2 = a^2 - c^2, \tag{6}$$

which can be written in the simple form

$$\frac{x^2}{a^2} + \frac{y^2}{a^2 - c^2} = 1. \tag{7}$$

Defining a new constant

$$b^2 \equiv a^2 - c^2 \tag{8}$$

puts the equation in the particularly simple form

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1. \tag{9}$$

The parameter b is called the SEMIMINOR AXIS by analogy with the parameter a, which is called the SEMIMAJOR AXIS. The fact that b as defined above is actually the SEMIMINOR AXIS is easily shown by letting r_1 and r_2 be equal. Then two RIGHT TRIANGLES are produced, each with HYPOTENUSE a, base c, and height $b \equiv \sqrt{a^2 - c^2}$. Since the largest distance along the MINOR AXIS will be achieved at this point, b is indeed the SEMIMINOR AXIS.

If, instead of being centered at (0, 0), the CENTER of the ellipse is at (x_0, y_0), equation (9) becomes

$$\frac{(x - x_0)^2}{a^2} + \frac{(y - y_0)^2}{b^2} = 1. \tag{10}$$

As can be seen from the CARTESIAN EQUATION for the ellipse, the curve can also be given by a simple parametric form analogous to that of a CIRCLE, but with the x and y coordinates having different scalings,

$$x = a \cos t \tag{11}$$

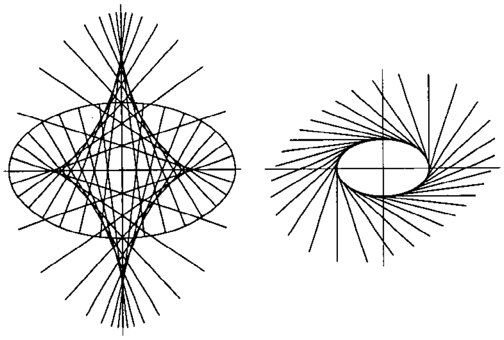
$$y = b \sin t. \tag{12}$$

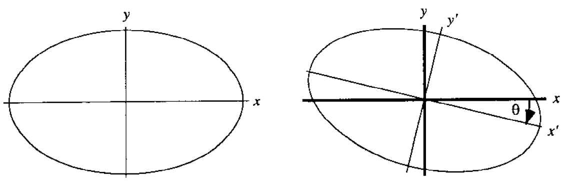
The unit TANGENT VECTOR of the ellipse so parameterized is

$$x_T(t) = -\frac{a \sin t}{\sqrt{b^2 \cos^2 t + a^2 \sin^2 t}} \tag{13}$$

$$y_T(t) = \frac{b \cos t}{\sqrt{b^2 \cos^2 t + a^2 \sin^2 t}}. \tag{14}$$

A sequence of NORMAL and TANGENT VECTORS are plotted below for the ellipse.

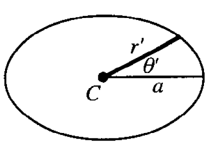




curve rotated by angle θ

For an ellipse centered at the ORIGIN but inclined at an arbitrary ANGLE θ to the x-AXIS, the parametric equations are

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} a\cos t \\ b\sin t \end{bmatrix}$$
$$= \begin{bmatrix} a\cos\theta\cos t + b\sin\theta\sin t \\ -a\sin\theta\cos t + b\cos\theta\sin t \end{bmatrix}. \qquad (15)$$



In POLAR COORDINATES, the ANGLE θ' measured from the *center* of the ellipse is called the ECCENTRIC ANGLE. Writing r' for the distance of a point from the ellipse center, the equation in POLAR COORDINATES is just given by the usual

$$x = r'\cos\theta' \qquad (16)$$
$$y = r'\sin\theta'. \qquad (17)$$

Here, the coordinates θ' and r' are written with primes to distinguish them from the more common polar coordinates for an ellipse which are centered on a *focus*. Plugging the polar equations into the Cartesian equation (9) and solving for r'^2 gives

$$r'^2 = \frac{b^2a^2}{b^2\cos^2\theta' + a^2\sin^2\theta'}. \qquad (18)$$

Define a new constant $0 \leq e < 1$ called the ECCENTRICITY (where $e = 0$ is the case of a CIRCLE) to replace b

$$e \equiv \sqrt{1 - \frac{b^2}{a^2}}, \qquad (19)$$

from which it also follows from (8) that

$$a^2e^2 = a^2 - b^2 \equiv c^2 \qquad (20)$$
$$c = ae \qquad (21)$$
$$b^2 = a^2(1-e^2). \qquad (22)$$

Therefore (18) can be written as

$$r'^2 = \frac{a^2(1-e^2)}{1 - e^2\cos^2\theta'} \qquad (23)$$

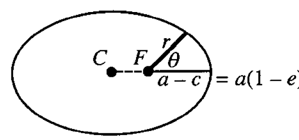
$$r' = a\sqrt{\frac{1-e^2}{1-e^2\cos^2\theta'}}. \qquad (24)$$

If $e \ll 1$, then

$$r' = a\{1 - \tfrac{1}{2}e^2\sin^2\theta' - \tfrac{1}{16}e^4[5 + 3\cos(2\theta')]\sin^2\theta' + \ldots\}, \qquad (25)$$

so

$$\frac{\Delta r'}{a} \equiv \frac{a - r'}{a} \approx \tfrac{1}{2}e^2\sin^2\theta'. \qquad (26)$$



If r and θ are measured from a FOCUS instead of from the center, as they commonly are in orbital mechanics, then the equations of the ellipse are

$$x = c + r\cos\theta \qquad (27)$$
$$y = r\sin\theta, \qquad (28)$$

and (9) becomes

$$\frac{(c + r\cos\theta)^2}{a^2} + \frac{r^2\sin^2\theta}{b^2} = 1. \qquad (29)$$

Clearing the DENOMINATORS gives

$$b^2(c^2 + 2cr\cos\theta + r^2\cos^2\theta) + a^2r^2\sin^2\theta = a^2b^2 \qquad (30)$$

$$b^2c^2 + 2rcb^2\cos\theta + b^2r^2\cos^2\theta + a^2r^2 - a^2r^2\cos^2\theta = a^2b^2. \qquad (31)$$

Plugging in (21) and (22) to re-express b and c in terms of a and e,

$$a^2(1-e^2)a^2e^2 + 2aea^2(1-e^2)r\cos\theta + a^2(1-e^2)r^2\cos^2\theta$$
$$+ a^2r^2 - a^2r^2\cos^2\theta = a^2[a^2(1-e^2)]. \qquad (32)$$

Simplifying,

$$-r^2 + [er\cos\theta - a(1-e^2)]^2 = 0 \qquad (33)$$

$$r = \pm[er\cos\theta - a(1-e^2)]. \qquad (34)$$

The sign can be determined by requiring that r must be POSITIVE. When $e = 0$, (34) becomes $r = \pm(-a)$, but since a is always POSITIVE, we must take the NEGATIVE sign, so (34) becomes

$$r = a(1-e^2) - er\cos\theta \tag{35}$$

$$r(1+e\cos\theta) = a(1-e^2) \tag{36}$$

$$r = \frac{a(1-e^2)}{1+e\cos\theta}. \tag{37}$$

The distance from a FOCUS to a point with horizontal coordinate x is found from

$$\cos\theta = \frac{c+x}{r}. \tag{38}$$

Plugging this into (37) yields

$$r + e(c+x) = a(1-e^2) \tag{39}$$

$$r = a(1-e^2) - e(c+x). \tag{40}$$

Summarizing relationships among the parameters characterizing an ellipsé,

$$b = a\sqrt{1-e^2} = \sqrt{a^2-c^2} \tag{41}$$

$$c = \sqrt{a^2-b^2} = ae \tag{42}$$

$$e = \sqrt{1-\frac{b^2}{a^2}} = \frac{c}{a}. \tag{43}$$

The ECCENTRICITY can therefore be interpreted as the position of the FOCUS as a fraction of the SEMIMAJOR AXIS.

In PEDAL COORDINATES with the PEDAL POINT at the FOCUS, the equation of the ellipse is

$$\frac{b^2}{p^2} = \frac{2a}{r} - 1. \tag{44}$$

To find the RADIUS OF CURVATURE, return to the parametric coordinates centered at the center of the ellipse and compute the first and second derivatives,

$$x' = -a\sin t \tag{45}$$

$$y' = b\cos t \tag{46}$$

$$x'' = -a\cos t \tag{47}$$

$$y'' = -b\sin t. \tag{48}$$

Therefore,

$$\begin{aligned} R &= \frac{(x'^2+y'^2)^{3/2}}{x'y''-x''y'} \\ &= \frac{(a^2\sin^2 t + b^2\cos^2 t)^{3/2}}{-a\sin t(-b\sin t) - (a\cos t)(b\cos t)} \\ &= \frac{(a^2\sin^2 t + b^2\cos^2 t)^{3/2}}{ab(\sin^2 t + \cos^2 t)} \\ &= \frac{(a^2\sin^2 t + b^2\cos^2 t)^{3/2}}{ab}. \end{aligned} \tag{49}$$

Similarly, the unit TANGENT VECTOR is given by

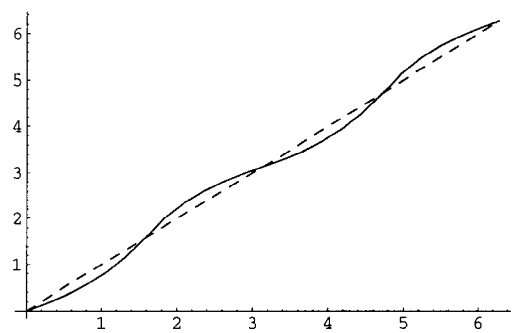
$$\hat{\mathbf{T}} = \begin{bmatrix} -a\sin t \\ b\cos t \end{bmatrix} \frac{1}{\sqrt{a^2\sin^2 t + b^2\cos^2 t}}. \tag{50}$$

The ARC LENGTH of the ellipse can be computed using

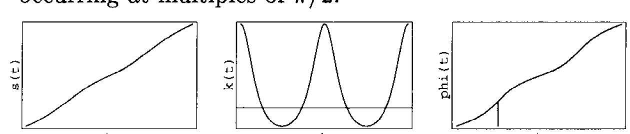
$$\begin{aligned} s &= \int \sqrt{x'^2+y'^2}\,dt = \int \sqrt{a^2\cos^2 t + b^2\sin^2 t}\,dt \\ &= a\int \sqrt{(1-\sin^2 t) + \frac{b^2}{a^2}\sin^2 t}\,dt \\ &= a\int \sqrt{1-\left(1-\frac{b^2}{a^2}\right)\sin^2 t}\,dt \\ &= a\int \sqrt{1-e^2\sin^2 t}\,dt = aE(t,e), \end{aligned} \tag{51}$$

where E is an incomplete ELLIPTIC INTEGRAL OF THE SECOND KIND. Again, note that t is a parameter which does not have a direct interpretation in terms of an ANGLE. However, the relationship between the polar angle from the ellipse center θ and the parameter t follows from

$$\theta = \tan^{-1}\left(\frac{y}{x}\right) = \tan^{-1}\left(\frac{b}{a}\tan t\right). \tag{52}$$



This function is illustrated above with θ shown as the solid curve and t as the dashed, with $b/a = 0.6$. Care must be taken to make sure that the correct branch of the INVERSE TANGENT function is used. As can be seen, θ weaves back and forth around t, with crossings occurring at multiples of $\pi/2$.



The CURVATURE and TANGENTIAL ANGLE of the ellipse are given by

$$\kappa = \frac{ab}{(b^2\cos^2 t + a^2\sin^2 t)^{3/2}} \tag{53}$$

$$\phi = -\tan^{-1}\left(\frac{b}{a}\cos t\right). \tag{54}$$

The entire PERIMETER p of the ellipse is given by setting $t = 2\pi$ (corresponding to $\theta = 2\pi$), which is equivalent to four times the length of one of the ellipse's QUADRANTS,

$$p = aE(2\pi, e) = 4aE(\tfrac{1}{2}\pi, e) = 4aE(e), \qquad (55)$$

where $E(e)$ is a complete ELLIPTIC INTEGRAL OF THE SECOND KIND with MODULUS k. The PERIMETER can be computed numerically by the rapidly converging GAUSS-KUMMER SERIES

$$\begin{aligned} p &= \pi(a+b)\sum_{n=0}^{\infty}\binom{\frac{1}{2}}{n}^2 h^{2n} \\ &= \pi(a+b)(1 + \tfrac{1}{4}h^2 + \tfrac{1}{64}h^4 + \tfrac{1}{256}h^6 + \ldots), \qquad (56) \end{aligned}$$

where

$$h \equiv \frac{a-b}{a+b} \qquad (57)$$

and $\binom{n}{k}$ is a BINOMIAL COEFFICIENT. Approximations to the PERIMETER include

$$\begin{aligned} p &\approx \pi\sqrt{2(a^2+b^2)} && (58) \\ &\approx \pi[3(a+b) - \sqrt{(a+3b)(3a+b)}] && (59) \\ &\approx \pi(a+b)\left(1 + \frac{3t}{10+\sqrt{4-3t}}\right), && (60) \end{aligned}$$

where the last two are due to Ramanujan (1913-14),

$$t \equiv \left(\frac{a-b}{a+b}\right)^2, \qquad (61)$$

and (60) is accurate to within $\sim 3 \cdot 2^{-17}t^5$.

The maximum and minimum distances from the FOCUS are called the APOAPSIS and PERIAPSIS, and are given by

$$\begin{aligned} r_+ &= r_{\text{apoapsis}} = a(1+e) && (62) \\ r_- &= \text{periapsis} = a(1-e). && (63) \end{aligned}$$

The AREA of an ellipse may be found by direct INTEGRATION

$$\begin{aligned} A &= \int_{-a}^{a}\int_{-b\sqrt{a^2-x^2}/a}^{b\sqrt{a^2-x^2}/a} dy\,dx = \int_{-a}^{a}\frac{2b}{a}\sqrt{a^2-x^2}\,dx \\ &= \frac{2b}{a}\left\{\frac{1}{2}\left[x\sqrt{a^2-x^2} + a^2\sin^{-1}\left(\frac{x}{|a|}\right)\right]\right\}_{x=-a}^{a} \\ &= ab[\sin^{-1}1 - \sin^{-1}(-1)] = ab\left[\frac{\pi}{2} - \left(\frac{\pi}{2}\right)\right] = \pi ab. \\ & \qquad (64) \end{aligned}$$

The AREA can also be computed more simply by making the change of coordinates $x' \equiv (b/a)x$ and $y' \equiv y$ from the elliptical region R to the new region R'. Then the equation becomes

$$\frac{1}{a^2}\left(\frac{a}{b}x'\right)^2 + \frac{y'^2}{b^2} = 1, \qquad (65)$$

or $x'^2 + y'^2 = b^2$, so R' is a CIRCLE of RADIUS b. Since

$$\frac{\partial x}{\partial x'} = \left(\frac{\partial x'}{\partial x}\right)^{-1} = \left(\frac{b}{a}\right)^{-1} = \frac{a}{b}, \qquad (66)$$

the JACOBIAN is

$$\left|\frac{\partial(x,y)}{\partial(x',y')}\right| = \begin{vmatrix} \frac{\partial x}{\partial x'} & \frac{\partial y'}{\partial x'} \\ \frac{\partial x}{\partial y'} & \frac{\partial y}{\partial y'} \end{vmatrix} = \begin{vmatrix} \frac{a}{b} & 0 \\ 0 & 1 \end{vmatrix} = \frac{a}{b}. \qquad (67)$$

The AREA is therefore

$$\begin{aligned} \iint_R dx\,dy &= \iint_{R'}\left|\frac{\partial(x,y)}{\partial(x',y')}\right| dx'\,dy' \\ &= \frac{a}{b}\iint_{R'} dx'dy' = \frac{a}{b}(\pi b^2) = \pi ab, \qquad (68) \end{aligned}$$

as before. The AREA of an arbitrary ellipse given by the QUADRATIC EQUATION

$$ax^2 + bxy + cy^2 = 1 \qquad (69)$$

is

$$A = \frac{2\pi}{\sqrt{4ac-b^2}}. \qquad (70)$$

The AREA of an ELLIPSE with semiaxes a and b with respect to a PEDAL POINT P is

$$A = \tfrac{1}{2}\pi(a^2 + b^2 + |OP|^2). \qquad (71)$$

The ellipse INSCRIBED in a given TRIANGLE and tangent at its MIDPOINTS is called the MIDPOINT ELLIPSE. The LOCUS of the centers of the ellipses INSCRIBED in a TRIANGLE is the interior of the MEDIAL TRIANGLE. Newton gave the solution to inscribing an ellipse in a convex QUADRILATERAL (Dörrie 1965, p. 217). The centers of the ellipses INSCRIBED in a QUADRILATERAL all lie on the straight line segment joining the MIDPOINTS of the DIAGONALS (Chakerian 1979, pp. 136–139).

The AREA of an ellipse with BARYCENTRIC COORDINATES (α, β, γ) INSCRIBED in a TRIANGLE of unit AREA is

$$\Delta = \pi\sqrt{(1-2\alpha)(1-2\beta)(1-2\gamma)}. \qquad (72)$$

(Chakerian 1979, pp. 142–145).

The LOCUS of the apex of a variable CONE containing an ellipse fixed in 3-space is a HYPERBOLA through the FOCI of the ellipse. In addition, the LOCUS of the apex of a CONE containing that HYPERBOLA is the original ellipse. Furthermore, the ECCENTRICITIES of the ellipse and HYPERBOLA are reciprocals. The LOCUS of centers of a PAPPUS CHAIN of CIRCLES is an ellipse. Surprisingly, the locus of the end of a garage door mounted on rollers along a vertical track but extending beyond the track is a quadrant of an ellipse (the envelopes of positions is an ASTROID).

see also CIRCLE, CONIC SECTION, ECCENTRIC ANOMALY, ECCENTRICITY, ELLIPTIC CONE, ELLIPTIC CURVE, ELLIPTIC CYLINDER, HYPERBOLA, MIDPOINT ELLIPSE, PARABOLA, PARABOLOID, QUADRATIC CURVE, REFLECTION PROPERTY, SALMON'S THEOREM, STEINER'S ELLIPSE

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Ellipse Caustic Curve

For an ELLIPSE given by

$$x = r\cos t \qquad (1)$$

$$y = \sin t \qquad (2)$$

with light source at $(x, 0)$, the CAUSTIC is

$$x = \frac{N_x}{D_x} \qquad (3)$$

$$y = \frac{N_y}{D_y}, \qquad (4)$$

where

$$\begin{aligned} N_x = {} & 2rx(3-5r^2) + (-6r^2+6r^4-3x^2+9r^2x^2)\cos t \\ & + 6rx(1-r^2)\cos(2t) \\ & + (-2r^2+2r^4-x^2-r^2x^2)\cos(3t) \qquad (5) \end{aligned}$$

$$\begin{aligned} D_x = {} & 2r(1+2r^2+4x^2) + 3x(1-5r^2)\cos t \\ & + (6r+6r^3)\cos(2t) + x(1-r^2)\cos(3t) \qquad (6) \end{aligned}$$

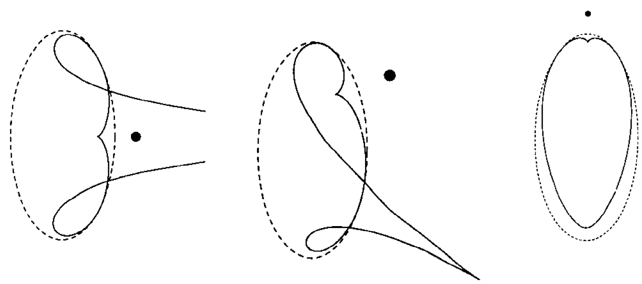
$$N_y = 8r(-1+r^2-x^2)\sin^3 t \qquad (7)$$

$$\begin{aligned} D_y = {} & 2r(-1-r^2-4x^2) + 3(-x+5r^2)\cos t \\ & + 6r(1-r^2)\cos(2t) + x(-1+r^2)\cos(3t). \qquad (8) \end{aligned}$$

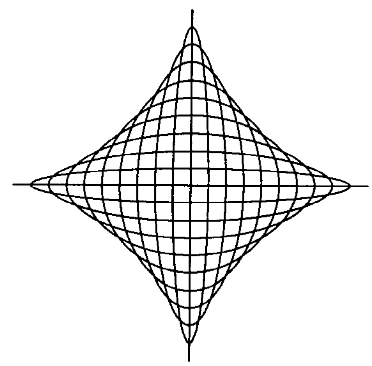
At $(\infty, 0)$,

$$x = \frac{\cos t[-1+5r^2-\cos(2t)(1+r^2)]}{4r} \qquad (9)$$

$$y = \sin^3 t. \qquad (10)$$



Ellipse Envelope



Consider the family of ELLIPSES

$$\frac{x^2}{c^2} + \frac{y^2}{(1-c)^2} - 1 = 0 \qquad (1)$$

for $c \in [0, 1]$. The PARTIAL DERIVATIVE with respect to c is

$$-\frac{2x^2}{c^3} + \frac{2y^2}{(1-c)^3} = 0 \qquad (2)$$

$$\frac{x^2}{c^3} - \frac{y^2}{(1-c)^3} = 0. \qquad (3)$$

Combining (1) and (3) gives the set of equations

$$\begin{bmatrix} \frac{1}{c^2} & \frac{1}{(1-c)^2} \\ \frac{1}{c^3} & -\frac{1}{(1-c)^3} \end{bmatrix} \begin{bmatrix} x^2 \\ y^2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad (4)$$

$$\begin{aligned} \begin{bmatrix} x^2 \\ y^2 \end{bmatrix} &= \frac{1}{\Delta} \begin{bmatrix} -\frac{1}{(1-c)^3} & -\frac{1}{(1-c)^2} \\ -\frac{1}{c^3} & \frac{1}{c^2} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\ &= \frac{1}{\Delta} \begin{bmatrix} -\frac{1}{(1-c)^3} \\ -\frac{1}{c^3} \end{bmatrix}, \qquad (5) \end{aligned}$$

where the DISCRIMINANT is

$$\Delta = -\frac{1}{c^2(1-c)^3} - \frac{1}{c^3(1-c)^2} = -\frac{1}{c^3(1-c)^3}, \qquad (6)$$

so (5) becomes

$$\begin{bmatrix} x^2 \\ y^2 \end{bmatrix} = \begin{bmatrix} c^3 \\ (1-c)^3 \end{bmatrix}. \quad (7)$$

Eliminating c then gives

$$x^{2/3} + y^{2/3} = 1, \quad (8)$$

which is the equation of the ASTROID. If the curve is instead represented parametrically, then

$$x = c \cos t \quad (9)$$

$$y = (1-c) \sin t. \quad (10)$$

Solving

$$\frac{\partial x}{\partial t}\frac{\partial y}{\partial c} - \frac{\partial x}{\partial c}\frac{\partial y}{\partial t} = (-c \sin t)(-\sin t) - (\cos t)[(1-c)\cos t] = c(\sin^2 t + \cos^2 t) - \cos^2 t = c - \cos^2 t = 0 \quad (11)$$

for c gives

$$c = \cos^2 t, \quad (12)$$

so substituting this back into (9) and (10) gives

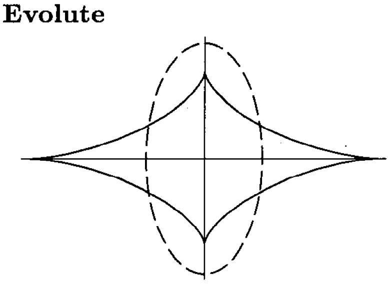
$$x = (\cos^2 t) \cos t = \cos^3 t \quad (13)$$

$$y = (1 - \cos^2 t) \sin t = \sin^3 t, \quad (14)$$

the parametric equations of the ASTROID.

see also ASTROID, ELLIPSE, ENVELOPE

Ellipse Evolute



The EVOLUTE of an ELLIPSE is given by the parametric equations

$$x = \frac{a^2 - b^2}{a} \cos^3 t \quad (1)$$

$$y = \frac{b^2 - a^2}{b} \sin^3 t, \quad (2)$$

which can be combined and written

$$(ax)^{2/3} + (by)^{2/3} = [(a^2 - b^2)\cos^3 t]^{2/3} + [(b^2 - a^2)\sin^3 t]^{2/3} = (a^2 - b^2)^{2/3}(\sin^2 t + \cos^2 t) = (a^2 - b^2)^{2/3} = c^{4/3}, \quad (3)$$

which is a stretched ASTROID called the LAMÉ CURVE. From a point inside the EVOLUTE, four NORMALS can be drawn to the ellipse, but from a point outside, only two NORMALS can be drawn.

see also ASTROID, ELLIPSE, EVOLUTE, LAMÉ CURVE

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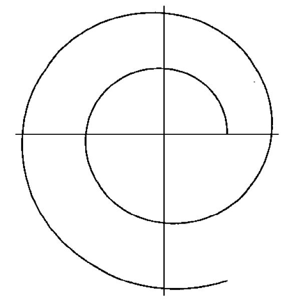
Ellipse Involute



From ELLIPSE, the TANGENT VECTOR is

$$\mathbf{T} = \begin{bmatrix} -a \sin t \\ b \cos t \end{bmatrix}, \quad (1)$$

and the ARC LENGTH is

$$s = a \int \sqrt{1 - e^2 \sin^2 t}\, dt = aE(t, e), \quad (2)$$

where $E(t,e)$ is an incomplete ELLIPTIC INTEGRAL OF THE SECOND KIND. Therefore,

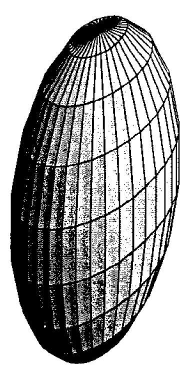
$$\mathbf{r}_i = \mathbf{r} - s\hat{\mathbf{T}} = \begin{bmatrix} a \cos t \\ b \sin t \end{bmatrix} - aeE(t,e) \begin{bmatrix} -a \sin t \\ b \cos t \end{bmatrix} \quad (3)$$

$$= \begin{bmatrix} a\{\cos t + aeE(t,e) \sin t\} \\ b\{\sin t - aeE(t,e) \cos t\}. \end{bmatrix} \quad (4)$$

Ellipse Pedal Curve

The pedal curve of an ellipse with a FOCUS as the PEDAL POINT is a CIRCLE.

Ellipsoid



A QUADRATIC SURFACE which is given in CARTESIAN COORDINATES by

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1, \quad (1)$$

where the semi-axes are of lengths a, b, and c. In SPHERICAL COORDINATES, this becomes

$$\frac{r^2 \cos^2\theta \sin^2\phi}{a^2} + \frac{r^2 \sin^2\theta \sin^2\phi}{b^2} + \frac{r^2 \cos^2\phi}{c^2} = 1. \quad (2)$$

The parametric equations are

$$x = a\cos\theta\sin\phi \quad (3)$$
$$y = b\sin\theta\sin\phi \quad (4)$$
$$z = c\cos\phi. \quad (5)$$

The SURFACE AREA (Bowman 1961, pp. 31–32) is

$$S = 2\pi c^2 + \frac{2\pi b}{\sqrt{a^2-c^2}}[(a^2-c^2)E(\theta) + c^2\theta], \quad (6)$$

where $E(\theta)$ is a COMPLETE ELLIPTIC INTEGRAL OF THE SECOND KIND,

$$e_1{}^2 \equiv \frac{a^2-c^2}{a^2} \quad (7)$$
$$e_2{}^2 \equiv \frac{b^2-c^2}{b^2} \quad (8)$$
$$k \equiv \frac{e_2}{a_1}, \quad (9)$$

and θ is given by inverting the expression

$$e_1 = \operatorname{sn}(\theta, k), \quad (10)$$

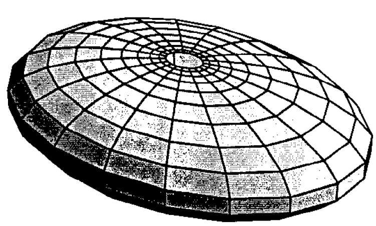
where $\operatorname{sn}(\theta, k)$ is a JACOBI ELLIPTIC FUNCTION. The VOLUME of an ellipsoid is

$$V = \tfrac{4}{3}\pi abc. \quad (11)$$

If two axes are the same, the figure is called a SPHEROID (depending on whether $c < a$ or $c > a$, an OBLATE SPHEROID or PROLATE SPHEROID, respectively), and if all three are the same, it is a SPHERE.

A different parameterization of the ellipsoid is the so-called stereographic ellipsoid, given by the parametric equations

$$x(u,v) = \frac{a(1-u^2-v^2)}{1+u^2+v^2} \quad (12)$$
$$y(u,v) = \frac{2bu}{1+u^2+v^2} \quad (13)$$
$$z(u,v) = \frac{2cv}{1+u^2+v^2}. \quad (14)$$



A third parameterization is the Mercator parameterization

$$x(u,v) = a\operatorname{sech} v\cos u \quad (15)$$
$$y(u,v) = b\operatorname{sech} v\sin u \quad (16)$$
$$z(u,v) = c\tanh v \quad (17)$$

(Gray 1993).

The SUPPORT FUNCTION of the ellipsoid is

$$h = \left(\frac{x^2}{a^4} + \frac{y^2}{b^4} + \frac{z^2}{c^4}\right)^{-1/2}, \quad (18)$$

and the GAUSSIAN CURVATURE is

$$K = \frac{h^4}{a^2b^2c^2} \quad (19)$$

(Gray 1993, p. 296).

see also CONVEX OPTIMIZATION THEORY, OBLATE SPHEROID, PROLATE SPHEROID, SPHERE, SPHEROID

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Ellipsoid Geodesic

An ELLIPSOID can be specified parametrically by

$$x = a\cos u\sin v \quad (1)$$
$$y = b\sin u\sin v \quad (2)$$
$$z = c\cos v. \quad (3)$$

The GEODESIC parameters are then

$$P = \sin^2 v(b^2\cos^2 u + a^2\sin^2 u) \quad (4)$$
$$Q = \tfrac{1}{4}(b^2-a^2)\sin(2u)\sin(2v) \quad (5)$$
$$R = \cos^2 v(a^2\cos^2 u + b^2\sin^2 u) + c^2\sin^2 v. \quad (6)$$

When the coordinates of a point are on the QUADRIC

$$\frac{x^2}{a} + \frac{y^2}{b} + \frac{z^2}{c} = 1 \quad (7)$$

and expressed in terms of the parameters p and q of the confocal quadrics passing through that point (in other words, having $a+p$, $b+p$, $c+p$, and $a+q$, $b+q$, $c+q$ for the squares of their semimajor axes), then the equation of a GEODESIC can be expressed in the form

$$\frac{q\,dq}{\sqrt{q(a+q)(b+q)(c+q)(\theta+q)}} \pm \frac{p\,dp}{\sqrt{p(a+p)(b+p)(c+p)(\theta+p)}} = 0, \quad (8)$$

with θ an arbitrary constant, and the ARC LENGTH element ds is given by

$$-2\frac{ds}{pq} = \frac{dq}{\sqrt{q(a+q)(b+q)(c+q)(\theta+q)}} \pm \frac{dp}{\sqrt{p(a+p)(b+p)(c+p)(\theta+p)}}, \quad (9)$$

where upper and lower signs are taken together.

see also OBLATE SPHEROID GEODESIC, SPHERE GEODESIC

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Eisenhart, L. P. *A Treatise on the Differential Geometry of Curves and Surfaces.* New York: Dover, pp. 236–241, 1960.

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Ellipsoidal Calculus

Ellipsoidal calculus is a method for solving problems in control and estimation theory having unknown but bounded errors in terms of sets of approximating ellipsoidal-value functions. Ellipsoidal calculus has been especially useful in the study of LINEAR PROGRAMMING.

References

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Ellipsoidal Coordinates

see CONFOCAL ELLIPSOIDAL COORDINATES

Ellipsoidal Harmonic

see ELLIPSOIDAL HARMONIC OF THE FIRST KIND, ELLIPSOIDAL HARMONIC OF THE SECOND KIND

Ellipsoidal Harmonic of the First Kind

The first solution to LAMÉ'S DIFFERENTIAL EQUATION, denoted $E_n^m(x)$ for $m = 1, \ldots, 2n+1$. They are also called LAMÉ FUNCTIONS. The product of two ellipsoidal harmonics of the first kind is a SPHERICAL HARMONIC. Whittaker and Watson (1990, pp. 536–537) write

$$\Theta_p = \frac{x^2}{a^2+\theta_p} + \frac{y^2}{b^2+\theta_p} + \frac{z^2}{c^2+\theta_p} - 1 \quad (1)$$

$$\Pi(\Theta) \equiv \Theta_1\Theta_2\cdots\Theta_m, \quad (2)$$

and give various types of ellipsoidal harmonics and their highest degree terms as

1. $\Pi(\Theta) : 2m$
2. $x\Pi(\Theta), y\Pi(\Theta), z\Pi(\Theta) : 2m+1$
3. $yz\Pi(\Theta), zx\Pi(\Theta), xy\Pi(\Theta) : 2m+2$
4. $xyz\Pi(\Theta) : 2m+3$.

A Lamé function of degree n may be expressed as

$$(\theta+a^2)^{\kappa_1}(\theta+b^2)^{\kappa_2}(\theta+c^2)^{\kappa_3}\prod_{p=1}^{m}(\theta-\theta_p), \quad (3)$$

where $\kappa_i = 0$ or $1/2$, θ_i are REAL and unequal to each other and to $-a^2$, $-b^2$, and $-c^2$, and

$$\tfrac{1}{2}n = m + \kappa_1 + \kappa_2 + \kappa_3. \quad (4)$$

Byerly (1959) uses the RECURRENCE RELATIONS to explicitly compute some ellipsoidal harmonics, which he denotes by $K(x)$, $L(x)$, $M(x)$, and $N(x)$,

$$\begin{aligned}
K_0(x) &= 1\\
L_0(x) &= 0\\
M_0(x) &= 0\\
N_0(x) &= 0\\
K_1(x) &= x\\
L_1(x) &= \sqrt{x^2-b^2}\\
M_1(x) &= \sqrt{x^2-c^2}\\
N_1(x) &= 0\\
K_2^{p_1}(x) &= x^2 - \tfrac{1}{3}[b^2+c^2-\sqrt{(b^2+c^2)^2-3b^2c^2}\,]\\
K_2^{p_2}(x) &= x^2 - \tfrac{1}{3}[b^2+c^2+\sqrt{(b^2+c^2)^2-3b^2c^2}\,]\\
L_2(x) &= x\sqrt{x^2-b^2}\\
M_2(x) &= x\sqrt{x^2-c^2}\\
N_2(x) &= \sqrt{(x^2-b^2)(x^2-c^2)}\\
K_3^{p_1}(x) &= x^3 - \tfrac{1}{5}x[2(b^2+c^2)\\
&\quad - \sqrt{4(b^2+c^2)^2-15b^2c^2}\,]\\
K_3^{p_2}(x) &= x^3 - \tfrac{1}{5}x[2(b^2+c^2)\\
&\quad + \sqrt{4(b^2+c^2)^2-15b^2c^2}\,]\\
L_3^{q_1}(x) &= \sqrt{x^2-b^2}[x^2 - \tfrac{1}{5}(b^2+2c^2\\
&\quad - \sqrt{(b^2+2c^2)^2-5b^2c^2}\,)]\\
L_3^{q_2}(x) &= \sqrt{x^2-b^2}[x^2 - \tfrac{1}{5}(b^2+2c^2\\
&\quad + \sqrt{(b^2+2c^2)^2-5b^2c^2}\,)]\\
M_3^{q_1}(x) &= \sqrt{x^2-c^2}[x^2 - \tfrac{1}{5}(2b^2+c^2\\
&\quad - \sqrt{(2b^2+c^2)^2-5b^2c^2}\,)]\\
M_3^{q_2}(x) &= \sqrt{x^2-c^2}[x^2 - \tfrac{1}{5}(2b^2+c^2\\
&\quad + \sqrt{(2b^2+c^2)^2-5b^2c^2}\,)]\\
M_3^{q_3}(x) &= x\sqrt{(x^2-b^2)(x^2-c^2)}
\end{aligned}$$

see also ELLIPSOIDAL HARMONIC OF THE SECOND KIND, STIELTJES' THEOREM

References

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Ellipsoidal Harmonic of the Second Kind

Given by

$$F_m^p(x) = (2m+1)E_m^p(x) \times \int_x^\infty \frac{dx}{(x^2-b^2)(x^2-c^2)[E_m^p(x)]^2}.$$

Elliptic Alpha Function

Elliptic alpha functions relate the complete ELLIPTIC INTEGRALS OF THE FIRST $K(k_r)$ and SECOND KINDS $E(k_r)$ at ELLIPTIC INTEGRAL SINGULAR VALUES k_r according to

$$\alpha(r) = \frac{E'(k_r)}{K(k_r)} - \frac{\pi}{4[K(k_r)]^2} \tag{1}$$

$$= \frac{\pi}{4[K(k_r)]^2} + \sqrt{r} - \frac{E(k_r)\sqrt{r}}{K(k_r)} \tag{2}$$

$$= \frac{\pi^{-1} - 4\sqrt{r}\, q \frac{d\vartheta_4(q)}{dq} \frac{1}{\vartheta_4(q)}}{\vartheta_3{}^4(q)}, \tag{3}$$

where $\vartheta_3(q)$ is a THETA FUNCTION and

$$k_r = \lambda^*(r) \tag{4}$$

$$q = e^{-\pi\sqrt{r}}, \tag{5}$$

and $\lambda^*(r)$ is the ELLIPTIC LAMBDA FUNCTION. The elliptic alpha function is related to the ELLIPTIC DELTA FUNCTION by

$$\alpha(r) = \tfrac{1}{2}[\sqrt{r} - \delta(r)]. \tag{6}$$

It satisfies

$$\alpha(4r) = (1+k_r)^2\alpha(r) - 2\sqrt{r}\, k_r, \tag{7}$$

and has the limit

$$\lim_{r\to\infty}\left[\alpha(r) - \frac{1}{\pi}\right] \approx 8\left(\sqrt{r} - \frac{1}{\pi}\right)e^{-\pi\sqrt{r}} \tag{8}$$

(Borwein *et al.* 1989). A few specific values (Borwein and Borwein 1987, p. 172) are

$$\alpha(1) = \tfrac{1}{2}$$
$$\alpha(2) = \sqrt{2} - 1$$
$$\alpha(3) = \tfrac{1}{2}(\sqrt{3} - 1)$$
$$\alpha(4) = 2(\sqrt{2} - 1)^2$$
$$\alpha(5) = \tfrac{1}{2}(\sqrt{5} - \sqrt{2\sqrt{5} - 2})$$
$$\alpha(6) = 5\sqrt{6} + 6\sqrt{3} - 8\sqrt{2} - 11$$
$$\alpha(7) = \tfrac{1}{2}(\sqrt{7} - 2)$$
$$\alpha(8) = 2(10 + 7\sqrt{2})(1 - \sqrt{\sqrt{8} - 2})^2$$
$$\alpha(9) = \tfrac{1}{2}[3 - 3^{3/4}\sqrt{2}(\sqrt{3} - 1)]$$
$$\alpha(10) = -103 + 72\sqrt{2} - 46\sqrt{5} + 33\sqrt{10}$$
$$\alpha(12) = 264 + 154\sqrt{3} - 188\sqrt{2} - 108\sqrt{6}$$
$$\alpha(13) = \tfrac{1}{2}(\sqrt{13} - \sqrt{74\sqrt{13} - 258})$$
$$\alpha(15) = \tfrac{1}{2}(\sqrt{15} - \sqrt{5} - 1)$$
$$\alpha(16) = \frac{4(\sqrt{8} - 1)}{(2^{1/4} + 1)^4}$$
$$\alpha(18) = -3057 + 2163\sqrt{2} + 1764\sqrt{3} - 1248\sqrt{6}$$
$$\alpha(22) = -12479 - 8824\sqrt{2} + 3762\sqrt{11} + 2661\sqrt{22}$$
$$\alpha(25) = \tfrac{5}{2}[1 - 25^{1/4}(7 - 3\sqrt{5})]$$
$$\alpha(27) = 3[\tfrac{1}{2}(\sqrt{3} + 1) - 2^{1/3}]$$
$$\begin{aligned}\alpha(30) = {} & \tfrac{1}{2}\{\sqrt{30} - (2 + \sqrt{5})^2(3 + \sqrt{10})^2 \\ & \times (-6 - 5\sqrt{2} - 3\sqrt{5} - 2\sqrt{10} + \sqrt{6}\sqrt{57 + 40\sqrt{2}}) \\ & \times [56 + 38\sqrt{2} + \sqrt{30}(2 + \sqrt{5})(3 + \sqrt{10})]\}\end{aligned}$$
$$\alpha(37) = \tfrac{1}{2}[\sqrt{37} - (171 - 25\sqrt{37})\sqrt{\sqrt{37} - 6}]$$
$$\begin{aligned}\alpha(49) = {} & \tfrac{7}{2} \\ & - \sqrt{7[\sqrt{2}\, 7^{3/4}(33011 + 12477\sqrt{7}) - 21(9567 + 3616\sqrt{7})]}\end{aligned}$$
$$\begin{aligned}\alpha(46) = {} & \tfrac{1}{2}[\sqrt{46} + (18 + 13\sqrt{2} + \sqrt{661 + 468\sqrt{2}})^2 \\ & \times (18 + 13\sqrt{2} - 3\sqrt{2}\sqrt{147 + 104\sqrt{2}} + \sqrt{661 + 468\sqrt{2}}) \\ & \times (200 + 14\sqrt{2} + 26\sqrt{23} + 18\sqrt{46} + \sqrt{46}\sqrt{661 + 468\sqrt{2}})]\end{aligned}$$
$$\begin{aligned}\alpha(58) & = [\tfrac{1}{2}(\sqrt{29} + 5)]^6(99\sqrt{29} - 444)(99\sqrt{2} - 70 - 13\sqrt{29}) \\ & = 3(-40768961 + 28828008\sqrt{2} - 7570606\sqrt{29} \\ & \quad + 5353227\sqrt{58})\end{aligned}$$
$$\alpha(64) = \frac{8[2(\sqrt{8} - 1) - (2^{1/4} - 1)^4]}{(\sqrt{\sqrt{2} + 1} + 2^{5/8})^4}.$$

J. Borwein has written an ALGORITHM which uses lattice basis reduction to provide algebraic values for $\alpha(n)$.

see also ELLIPTIC INTEGRAL OF THE FIRST KIND, ELLIPTIC INTEGRAL OF THE SECOND KIND, ELLIPTIC INTEGRAL SINGULAR VALUE, ELLIPTIC LAMBDA FUNCTION

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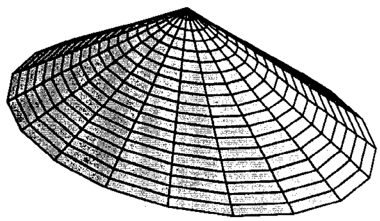
Elliptic Cone



A CONE with ELLIPTICAL CROSS-SECTION. The parametric equations for an elliptic cone of height h, SEMIMAJOR AXIS a, and SEMIMINOR AXIS b are

$$x = (h - z)a\cos\theta$$
$$y = (h - z)b\sin\theta$$
$$z = z,$$

where $\theta \in [0, 2\pi)$ and $z \in [0, h]$.

see also CONE, ELLIPTIC CYLINDER, ELLIPTIC PARABOLOID, HYPERBOLIC PARABOLOID

References

Fischer, G. (Ed.). Plate 68 in *Mathematische Modelle/Mathematical Models, Bildband/Photograph Volume.* Braunschweig, Germany: Vieweg, p. 63, 1986.

Elliptic Cone Point

see ISOLATED SINGULARITY

Elliptic Curve

Informally, an elliptic curve is a type of CUBIC CURVE whose solutions are confined to a region of space which is topologically equivalent to a TORUS. Formally, an elliptic curve over a FIELD K is a nonsingular CUBIC CURVE in two variables, $f(X, Y) = 0$, with a K-rational point (which may be a point at infinity). The FIELD K is usually taken to be the COMPLEX NUMBERS $\mathbb{C}$, REALS $\mathbb{R}$, RATIONALS $\mathbb{Q}$, algebraic extensions of $\mathbb{Q}$, p-ADIC NUMBERS $\mathbb{Q}_p$, or a FINITE FIELD.

By an appropriate change of variables, a general elliptic curve over a FIELD of CHARACTERISTIC $\neq 2, 3$

$$Ax^3 + Bx^2y + Cxy^2 + Dy^3 + Ex^2 + Fxy + Gy^2 + Hx + Iy + J = 0, \quad (1)$$

where A, B, ..., are elements of K, can be written in the form

$$y^2 = x^3 + ax + b, \quad (2)$$

where the right side of (2) has no repeated factors. If K has CHARACTERISTIC three, then the best that can be done is to transform the curve into

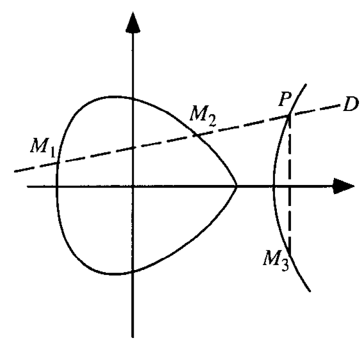
$$y^2 = x^3 + ax^2 + bx + c \quad (3)$$

(the x^2 term cannot be eliminated). If K has CHARACTERISTIC two, then the situation is even worse. A general form into which an elliptic curve over any K can be transformed is called the WEIERSTRAß FORM, and is given by

$$y^2 + ay = x^3 + bx^2 + cxy + dx + e, \quad (4)$$

where a, b, c, d, and e are elements of K. Luckily, $\mathbb{Q}$, $\mathbb{R}$, and $\mathbb{C}$ all have CHARACTERISTIC zero.

Whereas CONIC SECTIONS can be parameterized by the rational functions, elliptic curves cannot. The simplest parameterization functions are ELLIPTIC FUNCTIONS. ABELIAN VARIETIES can be viewed as generalizations of elliptic curves.



If the underlying FIELD of an elliptic curve is algebraically closed, then a straight line cuts an elliptic curve at three points (counting multiple roots at points of tangency). If two are known, it is possible to compute the third. If two of the intersection points are K-RATIONAL, then so is the third. Let (x_1, y_1) and (x_2, y_2) be two points on an elliptic curve E with DISCRIMINANT

$$\Delta_E = -16(4a^3 + 27b^2) \quad (5)$$

satisfying

$$\Delta_E \neq 0. \quad (6)$$

A related quantity known as the j-INVARIANT of E is defined as

$$j(E) \equiv \frac{2^8 3^3 a^3}{4a^3 + 27b^2}. \quad (7)$$

Now define

$$\lambda = \begin{cases} \frac{y_1 - y_2}{x_1 - x_2} & \text{for } x_1 \neq x_2 \\ \frac{3{x_1}^2 + a}{2y_1} & \text{for } x_1 = x_2. \end{cases} \quad (8)$$

Then the coordinates of the third point are

$$x_3 = \lambda^2 - x_1 - x_2 \quad (9)$$
$$y_3 = \lambda(x_3 - x_1) + y_1. \quad (10)$$

For elliptic curves over $\mathbb{Q}$, Mordell proved that there are a finite number of integral solutions. The MORDELL-WEIL THEOREM says that the GROUP of RATIONAL

POINTS of an elliptic curve over $\mathbb{Q}$ is finitely generated. Let the ROOTS of y^2 be r_1, r_2, and r_3. The discriminant is then

$$\Delta = k(r_1 - r_2)^2 (r_1 - r_3)^2 (r_2 - r_3)^2. \tag{11}$$

The amazing TANIYAMA-SHIMURA CONJECTURE states that all rational elliptic curves are also modular. This fact is far from obvious, and despite the fact that the conjecture was proposed in 1955, it was not proved until 1995. Even so, Wiles' proof surprised most mathematicians, who had believed the conjecture unassailable. As a side benefit, Wiles' proof of the TANIYAMA-SHIMURA CONJECTURE also laid to rest the famous and thorny problem which had baffled mathematicians for hundreds of years, FERMAT'S LAST THEOREM.

Curves with small CONDUCTORS are listed in Swinnerton-Dyer (1975) and Cremona (1997). Methods for computing integral points (points with integral coordinates) are given in Gebel *et al.* and Stroeker and Tzanakis (1994).

see also ELLIPTIC CURVE GROUP LAW, FERMAT'S LAST THEOREM, FREY CURVE, j-INVARIANT, MINIMAL DISCRIMINANT, MORDELL-WEIL THEOREM, OCHOA CURVE, RIBET'S THEOREM, SIEGEL'S THEOREM, SWINNERTON-DYER CONJECTURE, TANIYAMA-SHIMURA CONJECTURE, WEIERSTRAß FORM

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Elliptic Curve Factorization Method

A factorization method, abbreviated ECM, which computes a large multiple of a point on a random ELLIPTIC CURVE modulo the number to be factored N. It tends to be faster than the POLLARD ρ FACTORIZATION and POLLARD $p-1$ FACTORIZATION METHOD.

see also ATKIN-GOLDWASSER-KILIAN-MORAIN CERTIFICATE, ELLIPTIC CURVE PRIMALITY PROVING, ELLIPTIC PSEUDOPRIME

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Elliptic Curve Group Law

The GROUP of an ELLIPTIC CURVE which has been transformed to the form

$$y^2 = x^3 + ax + b$$

is the set of K-RATIONAL POINTS, including the single POINT AT INFINITY. The group law (addition) is defined as follows: Take 2 K-RATIONAL POINTS P and Q. Now 'draw' a straight line through them and compute the third point of intersection R (also a K-RATIONAL POINT). Then

$$P + Q + R = 0$$

gives the identity point at infinity. Now find the inverse of R, which can be done by setting $R = (a, b)$ giving $-R = (a, -b)$.

This remarkable result is only a special case of a more general procedure. Essentially, the reason is that this type of ELLIPTIC CURVE has a single point at infinity which is an inflection point (the line at infinity meets the curve at a single point at infinity, so it must be an intersection of multiplicity three).

Elliptic Curve Primality Proving

A class of algorithm, abbreviated ECPP, which provides certificates of primality using sophisticated results from the theory of ELLIPTIC CURVES. A detailed description and list of references are given by Atkin and Morain (1990, 1993).

Adleman and Huang (1987) designed an independent algorithm using elliptic curves of genus two.

see also ATKIN-GOLDWASSER-KILIAN-MORAIN CERTIFICATE, ELLIPTIC CURVE FACTORIZATION METHOD, ELLIPTIC PSEUDOPRIME

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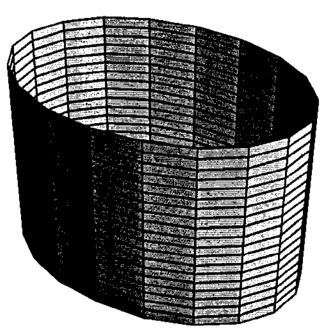
Elliptic Cylinder



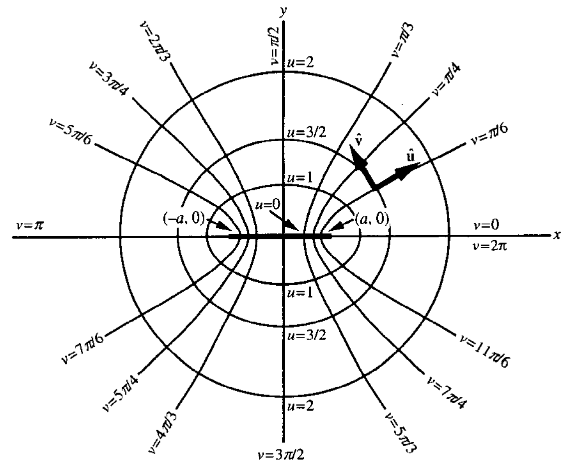
A CYLINDER with ELLIPTICAL CROSS-SECTION. The parametric equations for an elliptic cylinder of height h, SEMIMAJOR AXIS a, and SEMIMINOR AXIS b are

$$x = a\cos\theta$$
$$y = b\sin\theta$$
$$z = z,$$

where $\theta \in [0, 2\pi)$ and $z \in [0, h]$.

see also CONE, CYLINDER, ELLIPTIC CONE, ELLIPTIC PARABOLOID

Elliptic Cylindrical Coordinates



The v coordinates are the asymptotic angle of confocal PARABOLA segments symmetrical about the x axis. The u coordinates are confocal ELLIPSES centered on the origin.

$$x = a\cosh u \cos v \tag{1}$$
$$y = a\sinh u \sin v \tag{2}$$
$$z = z, \tag{3}$$

where $u \in [0, \infty)$, $v \in [0, 2\pi)$, and $z \in (-\infty, \infty)$. They are related to CARTESIAN COORDINATES by

$$\frac{x^2}{a^2\cosh^2 u} + \frac{y^2}{a^2\sinh^2 u} = 1 \tag{4}$$

$$\frac{x^2}{a^2\cos^2 v} - \frac{y^2}{a^2\sin^2 v} = 1. \tag{5}$$

The SCALE FACTORS are

$$h_1 = a\sqrt{\cosh^2 u \sin^2 v + \sinh^2 u \cos^2 v} \tag{6}$$
$$= a\sqrt{\frac{\cosh(2u) - \cos(2v)}{2}} \tag{7}$$
$$= a\sqrt{\sinh^2 u + \sin^2 v} \tag{8}$$
$$h_2 = a\sqrt{\sinh^2 u \sin^2 v + \sinh^2 u \cos^2 v} \tag{9}$$
$$= a\sqrt{\frac{\cosh(2u) - \cos(2v)}{2}} \tag{10}$$
$$= a\sqrt{\sinh^2 u + \sin^2 v} \tag{11}$$
$$h_3 = 1. \tag{12}$$

The LAPLACIAN is

$$\nabla^2 = \frac{1}{a^2(\sinh^2 u + \sin^2 v)}\left(\frac{\partial^2}{\partial u^2} + \frac{\partial^2}{\partial v^2}\right) + \frac{\partial^2}{\partial z^2}. \tag{13}$$

Let

$$q_1 = \cosh u \tag{14}$$

$$q_2 = \cos v \tag{15}$$

$$q_3 = z. \tag{16}$$

Then the new SCALE FACTORS are

$$h_{q_1} = a\sqrt{\frac{{q_1}^2 - {q_2}^2}{{q_1}^2 - 1}} \tag{17}$$

$$h_{q_2} = a\sqrt{\frac{{q_1}^2 - {q_2}^2}{1 - {q_1}^2}} \tag{18}$$

$$h_{q_3} = 1. \tag{19}$$

The HELMHOLTZ DIFFERENTIAL EQUATION is SEPARABLE.

see also CYLINDRICAL COORDINATES, HELMHOLTZ DIFFERENTIAL EQUATION—ELLIPTIC CYLINDRICAL COORDINATES

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Elliptic Delta Function

$$\delta(r) = \sqrt{r} - 2\alpha(r),$$

where α is the ELLIPTIC ALPHA FUNCTION.

see also ELLIPTIC ALPHA FUNCTION, ELLIPTIC INTEGRAL SINGULAR VALUE

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Elliptic Exponential Function

The inverse of the ELLIPTIC LOGARITHM

$$\operatorname{eln}(x) \equiv \int_x^\infty \frac{dt}{\sqrt{t^3 + at^2 + bt}}.$$

It is doubly periodic in the COMPLEX PLANE.

Elliptic Fixed Point (Differential Equations)

A FIXED POINT for which the STABILITY MATRIX is purely IMAGINARY, $\lambda_\pm = \pm i\omega$ (for $\omega > 0$).

see also DIFFERENTIAL EQUATION, FIXED POINT, HYPERBOLIC FIXED POINT (DIFFERENTIAL EQUATIONS), PARABOLIC FIXED POINT, STABLE IMPROPER NODE, STABLE NODE, STABLE SPIRAL POINT, STABLE STAR, UNSTABLE IMPROPER NODE, UNSTABLE NODE, UNSTABLE SPIRAL POINT, UNSTABLE STAR

References

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Elliptic Fixed Point (Map)

A FIXED POINT of a LINEAR TRANSFORMATION (MAP) for which the rescaled variables satisfy

$$(\delta - \alpha)^2 + 4\beta\gamma < 0.$$

see also HYPERBOLIC FIXED POINT (MAP), LINEAR TRANSFORMATION, PARABOLIC FIXED POINT

Elliptic Function

A doubly periodic function with periods $2\omega_1$ and $2\omega_2$ such that

$$f(z + 2\omega_1) = f(z + 2\omega_2) = f(z), \tag{1}$$

which is ANALYTIC and has no singularities except for POLES in the finite part of the COMPLEX PLANE. The ratio ω_1/ω_2 must not be purely real. If this ratio is real, the function reduces to a singly periodic function if it is rational and a constant if the ratio is irrational (Jacobi, 1835). ω_1 and ω_2 are labeled such that $\Im(\omega_2/\omega_1) > 0$. A "cell" of an elliptic function is defined as a parallelogram region in the COMPLEX PLANE in which the function is not multi-valued. Properties obeyed by elliptic functions include

1. The number of POLES in a cell is finite.
2. The number of ROOTS in a cell is finite.
3. The sum of RESIDUES in any cell is 0.
4. LIOUVILLE'S ELLIPTIC FUNCTION THEOREM: An elliptic function with no POLES in a cell is a constant.
5. The number of zeros of $f(z) - c$ (the "order") equals the number of POLES of $f(z)$.
6. The simplest elliptic function has order two, since a function of order one would have a simple irreducible POLE, which would need to have a NONZERO residue. By property (3), this is impossible.
7. Elliptic functions with a single POLE of order 2 with RESIDUE 0 are called WEIERSTRAß ELLIPTIC FUNCTIONS. Elliptic functions with two simple POLES having residues a_0 and $-a_0$ are called JACOBI ELLIPTIC FUNCTIONS.
8. Any elliptic function is expressible in terms of either WEIERSTRAß ELLIPTIC FUNCTION or JACOBI ELLIPTIC FUNCTIONS.
9. The sum of the AFFIXES of ROOTS equals the sum of the AFFIXES of the POLES.
10. An algebraic relationship exists between any two elliptic functions with the same periods.

The elliptic functions are inversions of the ELLIPTIC INTEGRALS. The two standard forms of these functions are known as JACOBI ELLIPTIC FUNCTIONS and WEIERSTRAß ELLIPTIC FUNCTIONS. JACOBI ELLIPTIC FUNCTIONS arise as solutions to differential equations of the form

$$\frac{d^2x}{dt^2} = A + Bx + Cx^2 + Dx^3, \tag{2}$$

and WEIERSTRAß ELLIPTIC FUNCTIONS arise as solutions to differential equations of the form

$$\frac{d^2x}{dt^2} = A + Bx + Cx^2. \tag{3}$$

see also ELLIPTIC CURVE, ELLIPTIC INTEGRAL, JACOBI ELLIPTIC FUNCTIONS, LIOUVILLE'S ELLIPTIC FUNCTION THEOREM, MODULAR FORM, MODULAR FUNCTION, NEVILLE THETA FUNCTION, THETA FUNCTION, WEIERSTRAß ELLIPTIC FUNCTIONS

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Elliptic Functional

see COERCIVE FUNCTIONAL

Elliptic Geometry

A constant curvature NON-EUCLIDEAN GEOMETRY which replaces the PARALLEL POSTULATE with the statement "through any point in the plane, there exist no lines PARALLEL to a given line." Elliptic geometry is sometimes also called RIEMANNIAN GEOMETRY. It can be visualized as the surface of a SPHERE on which "lines" are taken as GREAT CIRCLES. In elliptic geometry, the sum of angles of a TRIANGLE is $> 180°$.

see also EUCLIDEAN GEOMETRY, HYPERBOLIC GEOMETRY, NON-EUCLIDEAN GEOMETRY

Elliptic Group Modulo p

$E(a,b)/p$ denotes the elliptic GROUP modulo p whose elements are 1 and ∞ together with the pairs of INTEGERS (x,y) with $0 \le x, y < p$ satisfying

$$y^2 \equiv x^3 + ax + b \pmod{p} \tag{1}$$

with a and b INTEGERS such that

$$4a^3 + 27b^2 \not\equiv 0 \pmod{p}. \tag{2}$$

Given (x_1, y_1), define

$$(x_i, y_i) \equiv (x_1, y_1)^i \pmod{p}. \tag{3}$$

The ORDER h of $E(a,b)/p$ is given by

$$h = 1 + \sum_{x=1}^{p} \left[\left(\frac{x^3 + ax + b}{p} \right) + 1 \right], \tag{4}$$

where $(x^3 + ax + b/p)$ is the LEGENDRE SYMBOL, although this FORMULA quickly becomes impractical. However, it has been proven that

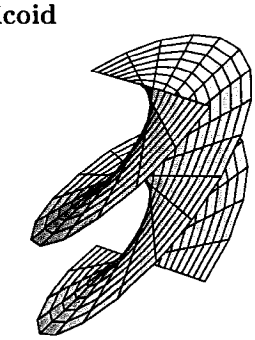
$$p + 1 - 2\sqrt{p} \le h(E(a,b)/p) \le p + 1 + 2\sqrt{p}. \tag{5}$$

Furthermore, for p a PRIME > 3 and and INTEGER n in the above interval, there exists a and b such that

$$h(E(a,b)/p) = n, \tag{6}$$

and the orders of elliptic GROUPS mod p are nearly uniformly distributed in the interval.

Elliptic Helicoid



A generalization of the HELICOID to the parametric equations

$$x(u,v) = av\cos u$$
$$y(u,v) = bv\sin u$$
$$z(u,v) = cu.$$

see also HELICOID

References

Gray, A. *Modern Differential Geometry of Curves and Surfaces.* Boca Raton, FL: CRC Press, p. 264, 1993.

Elliptic Hyperboloid

The elliptic hyperboloid is the generalization of the HYPERBOLOID to three distinct semimajor axes. The elliptic hyperboloid of one sheet is a RULED SURFACE and has Cartesian equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1, \tag{1}$$

and parametric equations

$$x(u,v) = a\sqrt{1+u^2}\cos v \tag{2}$$
$$y(u,v) = b\sqrt{1+u^2}\sin v \tag{3}$$
$$z(u,v) = cu \tag{4}$$

for $v \in [0, 2\pi)$, or

$$x(u,v) = a(\cos u \mp v\sin u) \tag{5}$$
$$y(u,v) = b(\sin u \pm v\cos u) \tag{6}$$
$$z(u,v) = \pm cv, \tag{7}$$

or

$$x(u,v) = a\cosh v\cos u \tag{8}$$
$$y(u,v) = b\cosh v\sin u \tag{9}$$
$$z(u,v) = c\sinh v. \tag{10}$$

The two-sheeted elliptic hyperboloid oriented along the z-AXIS has Cartesian equation

$$\frac{x^2}{a^2} + \frac{y^2}{a^2} - \frac{z^2}{c^2} = -1, \tag{11}$$

and parametric equations

$$x = a\sinh u\cos v \tag{12}$$
$$y = b\sinh u\sin v \tag{13}$$
$$z = c \pm \cosh u. \tag{14}$$

The two-sheeted elliptic hyperboloid oriented along the x-AXIS has Cartesian equation

$$\frac{x^2}{a^2} - \frac{y^2}{a^2} - \frac{z^2}{c^2} = 1 \tag{15}$$

and parametric equations

$$x = a\cosh u\cosh v \tag{16}$$
$$y = b\sinh u\cosh v \tag{17}$$
$$z = c\sinh v. \tag{18}$$

see also HYPERBOLOID, RULED SURFACE

References

Gray, A. *Modern Differential Geometry of Curves and Surfaces.* Boca Raton, FL: CRC Press, pp. 296–297, 1993.

Elliptic Integral

An elliptic integral is an INTEGRAL of the form

$$\int \frac{A(x) + B(x)\sqrt{S(x)}}{C(x) + D(x)\sqrt{S(x)}}\,dx, \tag{1}$$

or

$$\int \frac{A(x)\,dx}{B(x)\sqrt{S(x)}}, \tag{2}$$

where A, B, C, and D are POLYNOMIALS in x and S is a POLYNOMIAL of degree 3 or 4. Another form is

$$\int R(w,x)\,dx, \tag{3}$$

where R is a RATIONAL FUNCTION of x and y, w^2 is a function of x CUBIC or QUADRATIC in x, $R(w,x)$ contains at least one ODD POWER of w, and w^2 has no repeated factors.

Elliptic integrals can be viewed as generalizations of the TRIGONOMETRIC FUNCTIONS and provide solutions to a wider class of problems. For instance, while the ARC LENGTH of a CIRCLE is given as a simple function of the parameter, computing the ARC LENGTH of an ELLIPSE requires an elliptic integral. Similarly, the position of a pendulum is given by a TRIGONOMETRIC FUNCTION as a function of time for small angle oscillations, but the full solution for arbitrarily large displacements requires the use of elliptic integrals. Many other problems in electromagnetism and gravitation are solved by elliptic integrals.

A very useful class of functions known as ELLIPTIC FUNCTIONS is obtained by inverting elliptic integrals (by analogy with the inverse trigonometric functions). ELLIPTIC FUNCTIONS (among which the JACOBI ELLIPTIC FUNCTIONS and WEIERSTRASS ELLIPTIC FUNCTION are the two most common forms) provide a powerful tool for analyzing many deep problems in NUMBER THEORY, as well as other areas of mathematics.

All elliptic integrals can be written in terms of three "standard" types. To see this, write

$$R(w,x) \equiv \frac{P(w,x)}{Q(w,x)} = \frac{wP(w,x)Q(-w,x)}{wQ(w,x)Q(-w,x)}. \tag{4}$$

But since $w^2 = f(x)$,

$$Q(w,x)Q(-w,x) \equiv Q_1(w,x) = Q_1(w,x), \quad (5)$$

then

$$wP(w,x)Q(-w,x) = A + Bx + Cw + Dx^2 + Ewx + Fw^2 + Gw^2x + Hw^3x$$
$$= (A + Bx + Dx^2 + Fw^2 + Gw^2x) + w(c + Ex + Hw^2x + \ldots) = P_1(x) + wP_2(x), \quad (6)$$

so

$$R(w,x) = \frac{P_1(x) + wP_2(x)}{wQ_1(w)} = \frac{R_1(x)}{w} + R_2(x). \quad (7)$$

But any function $\int R_2(x)\,dx$ can be evaluated in terms of elementary functions, so the only portion that need be considered is

$$\frac{\int R_1(x)}{w}\,dx. \quad (8)$$

Now, any quartic can be expressed as S_1S_2 where

$$S_1 \equiv a_1x^2 + 2b_1x + c_1 \quad (9)$$
$$S_2 \equiv a_2x^2 + 2b_2x + c_2. \quad (10)$$

The COEFFICIENTS here are real, since pairs of COMPLEX ROOTS are COMPLEX CONJUGATES

$$[x - (R + Ii)][x - (R - Ii)] = x^2 + x(-R + Ii - R - Ii) + (R^2 - I^2 i) = x^2 - 2Rx + (R^2 + I^2). \quad (11)$$

If all four ROOTS are real, they must be arranged so as not to interleave (Whittaker and Watson 1990, p. 514). Now define a quantity λ such that $S_1 + \lambda S_2$

$$(a_1 - \lambda a_2)x^2 - (2b_1 - 2b_2\lambda)x + (c_1 - \lambda c_2) \quad (12)$$

is a SQUARE NUMBER and

$$2\sqrt{(a_1 - \lambda a_2)(c_1 - \lambda_2)} = 2(b_1 - b_2\lambda) \quad (13)$$

$$(a_1 - \lambda a_2)(c_1 - \lambda c_2) - (b_1 - \lambda b_2)^2 = 0. \quad (14)$$

Call the ROOTS of this equation λ_1 and λ_2, then

$$S_1 - \lambda_1 S_2 = \left[\sqrt{(a_1 - \lambda_1 a_2)x} + \sqrt{c_1 - \lambda c_2}\right]^2 = (a_1 - \lambda_1 a_2)\left(x + \sqrt{\frac{c_1 - \lambda_1 c_2}{a_1 - \lambda_1 a_2}}\right) \equiv (a_1 - \lambda_1 a_2)(x - \alpha)^2 \quad (15)$$

$$S_1 - \lambda_2 S_2 = \left[\sqrt{(a_1 - \lambda_1 a_2)x} + \sqrt{c_1 - \lambda c_2}\right]^2 = (a_1 - \lambda_1 a_2)\left(x + \sqrt{\frac{c_1 - \lambda_2 c_2}{a_1 - \lambda_2 a_2}}\right) \equiv (a_1 - \lambda_2 a_2)(x - \beta)^2. \quad (16)$$

Taking (15)-(16) and $\lambda_2(1) - \lambda_1(2)$ gives

$$S_2(\lambda_2 - \lambda_1) = (a_1 - \lambda_1 a_2)(x - \alpha)^2 - (a_1 - \lambda_2 a_2)(x - \beta)^2 \quad (17)$$

$$S_1(\lambda_2 - \lambda_1) = \lambda_2(a_1 - \lambda_1 a_2)(x - \alpha)^2 - \lambda_1(a_1 - \lambda_2 a_2)(x - \beta^2). \quad (18)$$

Solving gives

$$S_1 = \frac{a_1 - \lambda_1 a_2}{\lambda_2 - \lambda_1}(x - \alpha)^2 - \frac{a_1 - \lambda_2 a_2}{\lambda_2 - \lambda_1}(x - \beta)^2 \equiv A_1(x - \alpha)^2 + B_1(x - \beta)^2 \quad (19)$$

$$S_2 = \frac{\lambda_2(a_1 - \lambda_1 a_2)}{\lambda_2 - \lambda_1}(x - \alpha)^2 - \frac{\lambda_1(a_1 - \lambda_2 a_2)}{\lambda_2 - \lambda_1}(x - \beta)^2 \equiv A_2(x - \alpha)^2 + B_2(x - \beta)^2, \quad (20)$$

so we have

$$w^2 = S_1S_2 = [A_1(x - \alpha)^2 + B_1(x - \beta)^2][A^2(x - \alpha)^2 + B^2(x - \beta)^2]. \quad (21)$$

Now let

$$t \equiv \frac{x - \alpha}{x - \beta} \quad (22)$$

$$dy = [(x - \beta)^{-1} - (x - \alpha)(x - \beta)^{-2}]\,dx = \frac{(x - \beta) - (x - \alpha)}{(x - \beta)^2}\,dx = \frac{\alpha - \beta}{(x - \beta)^2}\,dx, \quad (23)$$

so

$$w^2 = (x - \beta)^4\left[A_1\left(\frac{x - \alpha}{x - \beta}\right)^2 + B_1\right] \times \left[A_2\left(\frac{x - \alpha}{x - \beta}\right) + B_2\right] = (x - \beta)^4(A_1t^2 + B_1)(A_2t^2 + B_2), \quad (24)$$

and

$$w = (x - \beta)^2\sqrt{(A_1t^2 + B_1)(A_2t^2 + B_2)} \quad (25)$$

$$\frac{dx}{w} = \left[\frac{(x - \beta)^2}{\alpha - \beta}\,dt\right]\frac{1}{(x - \beta)^2\sqrt{(A_1t^2 + B_1)(A_2t^2 + B_2)}} = \frac{dt}{(\alpha - \beta)\sqrt{(A_1t^2 + B_1)(A_2t^2 + B_2)}}. \quad (26)$$

Now let

$$R_3(t) \equiv \frac{R_1(x)}{\alpha - \beta}, \quad (27)$$

so

$$\int \frac{R_1(x)\,dx}{w} = \int \frac{R_3(t)\,dt}{\sqrt{(A_1t^2 + B_1)(A_2t^2 + B_2)}}. \quad (28)$$

Rewriting the EVEN and ODD parts

$$R_3(t) + R_3(-t) \equiv 2R_4(t^2) \tag{29}$$
$$R_3(t) - R_3(-t) \equiv 2tR_5(t^2), \tag{30}$$

gives

$$R_3(t) \equiv \tfrac{1}{2}(R_{\text{even}} - R_{\text{odd}}) = R_4(t^2) + tR_5(t^2), \tag{31}$$

so we have

$$\int \frac{R_1(x)\,dx}{w} = \int \frac{R_4(t^2)\,dt}{\sqrt{(A_1t^2+B_1)(A_2t^2+B_2)}} + \int \frac{R_5(t^2)t\,dt}{\sqrt{(A_1t^2+B_1)(A_2t^2+B_2)}}. \tag{32}$$

Letting

$$u \equiv t^2 \tag{33}$$
$$du = 2t\,dt \tag{34}$$

reduces the second integral to

$$\frac{1}{2}\int \frac{R_5(u)\,du}{\sqrt{(A_1u+B_1)(A_2u+B_2)}}, \tag{35}$$

which can be evaluated using elementary functions. The first integral can then be reduced by INTEGRATION BY PARTS to one of the three Legendre elliptic integrals (also called Legendre-Jacobi ELLIPTIC INTEGRALS), known as incomplete elliptic integrals of the first, second, and third kind, denoted $F(\phi,k)$, $E(\phi,k)$, and $\Pi(n;\phi,k)$, respectively (von Kármán and Biot 1940, Whittaker and Watson 1990, p. 515). If $\phi = \pi/2$, then the integrals are called complete elliptic integrals and are denoted $K(k)$, $E(k)$, $\Pi(n;k)$.

Incomplete elliptic integrals are denoted using a MODULUS k, PARAMETER $m \equiv k^2$, or MODULAR ANGLE $\alpha \equiv \sin^{-1} k$. An elliptic integral is written $I(\phi|m)$ when the PARAMETER is used, $I(\phi,k)$ when the MODULUS is used, and $I(\phi\backslash\alpha)$ when the MODULAR ANGLE is used. Complete elliptic integrals are defined when $\phi = \pi/2$ and can be expressed using the expansion

$$(1-k^2\sin^2\theta)^{-1/2} = \sum_{n=0}^{\infty} \frac{(2n-1)!!}{(2n)!!} k^{2n}\sin^{2n}\theta. \tag{36}$$

An elliptic integral in standard form

$$\int_a^x \frac{dx}{\sqrt{f(x)}}, \tag{37}$$

where

$$f(x) = a_4x^4 + a_3x^3 + a_2x^2 + a_1x + a_0, \tag{38}$$

can be computed analytically (Whittaker and Watson 1990, p. 453) in terms of the WEIERSTRAß ELLIPTIC FUNCTION with invariants

$$g_2 = a_0a_4 - 4a_1a_3 + 3{a_2}^2 \tag{39}$$
$$g_3 = a_0a_2a_4 - 2a_1a_2a_3 - a_4{a_1}^2 - {a_3}^2a_0. \tag{40}$$

If $a \equiv x_0$ is a root of $f(x) = 0$, then the solution is

$$x = x_0 + \tfrac{1}{4}f'(x_0)[\wp(z;g_2,g_3) - \tfrac{1}{24}f''(x_0)]^{-1}. \tag{41}$$

For an arbitrary lower bound,

$$x = a + \frac{\sqrt{f(a)}\wp'(z) + \frac{1}{2}f'(a)[\wp(z) - \frac{1}{24}f''(a)] + \frac{1}{24}f(a)f'''(a)}{2[\wp(z) - \frac{1}{24}f''(a)]^2 - \frac{1}{48}f(a)f^{(a)}(a)}, \tag{42}$$

where $\wp(z) \equiv \wp(z;g_2,g_3)$ is a WEIERSTRAß ELLIPTIC FUNCTION.

A generalized elliptic integral can be defined by the function

$$T(a,b) \equiv \frac{2}{\pi}\int_0^{\pi/2} \frac{d\theta}{\sqrt{a^2\cos^2\theta + b^2\sin^2\theta}} \tag{43}$$
$$= \frac{2}{\pi}\int_0^{\pi/2}\int \frac{d\theta}{\cos\theta\sqrt{a^2+b^2\tan^2\theta}} \tag{44}$$

(Borwein and Borwein 1987). Now let

$$t \equiv b\tan\theta \tag{45}$$
$$dt = b\sec^2\theta\,d\theta. \tag{46}$$

But

$$\sec\theta = \sqrt{1+\tan^2\theta}, \tag{47}$$

so

$$dt = \frac{b}{\cos\theta}\sec\theta\,d\theta = \frac{b}{\cos\theta}\sqrt{1+\tan^2\theta}\,d\theta = \frac{b}{\cos\theta}\sqrt{1+\left(\frac{t}{b}\right)^2}\,d\theta = \frac{d\theta}{\cos\theta}\sqrt{b^2+t^2}, \tag{48}$$

and

$$\frac{d\theta}{\cos\theta} = \frac{dt}{\sqrt{b^2+t^2}}, \tag{49}$$

and the equation becomes

$$T(a,b) = \frac{2}{\pi}\int_0^\infty \frac{dt}{\sqrt{(a^2+t^2)(b^2+t^2)}} = \frac{1}{\pi}\int_{-\infty}^\infty \frac{dt}{\sqrt{(a^2+t^2)(b^2+t^2)}}. \tag{50}$$

Now we make the further substitution $u \equiv \frac{1}{2}(t - ab/t)$. The differential becomes

$$du = \tfrac{1}{2}(1 + ab/t^2)\,dt, \tag{51}$$

but $2u = t - ab/t$, so

$$2u/t = 1 - ab/t^2 \tag{52}$$

$$ab/t^2 = 1 - 2u/t \tag{53}$$

and

$$1 + ab/t^2 = 2 - 2u/t = 2(1 - u/t). \tag{54}$$

However, the left side is always positive, so

$$1 + ab/t^2 = 2 - 2u/t = 2|1 - u/t| \tag{55}$$

and the differential is

$$dt = \frac{du}{\left|1 - \frac{u}{t}\right|}. \tag{56}$$

We need to take some care with the limits of integration. Write (50) as

$$\int_{-\infty}^{\infty} f(t)\,dt = \int_{-\infty}^{0^-} f(t)\,dt + \int_{0+}^{\infty} f(t)\,dt. \tag{57}$$

Now change the limits to those appropriate for the u integration

$$\int_{-\infty}^{\infty} g(u)\,du + \int_{-\infty}^{\infty} g(u)\,du = 2\int_{-\infty}^{\infty} g(u)\,du, \tag{58}$$

so we have picked up a factor of 2 which must be included. Using this fact and plugging (56) in (50) therefore gives

$$T(a,b) = \frac{2}{\pi}\int_{-\infty}^{\infty} \frac{du}{\left|1 - \frac{u}{t}\right| \sqrt{a^2b^2 + (a^2+b^2)t^2 + t^4}}. \tag{59}$$

Now note that

$$u^2 = \frac{t^4 - 2abt^2 + a^2b^2}{4t^2} \tag{60}$$

$$4u^2t^2 = t^4 - 2abt^2 + 2abt^2 \tag{61}$$

$$a^2b^2 + t^4 = 4u^2t^2 + 2abt^2. \tag{62}$$

Plug (62) into (59) to obtain

$$\begin{aligned} T(a,b) &= \frac{2}{\pi}\int_{-\infty}^{\infty} \frac{du}{\left|1 - \frac{u}{t}\right| \sqrt{4u^2t^2 + 2abt^2 + (a^2+b^2)t^2}} \\ &= \frac{2}{\pi}\int_{-\infty}^{\infty} \frac{du}{|t-u|\sqrt{4u^2 + (a+b)^2}}. \end{aligned} \tag{63}$$

But

$$2ut = t^2 - ab \tag{64}$$

$$t^2 - 2ut - ab = 0 \tag{65}$$

$$t = \tfrac{1}{2}(2u \pm \sqrt{4u^2 + 4ab}) = u \pm \sqrt{u^2 + ab}, \tag{66}$$

so

$$t - u = \pm\sqrt{u^2 + ab}, \tag{67}$$

and (63) becomes

$$\begin{aligned} T(a,b) &= \frac{2}{\pi}\int_{-\infty}^{\infty} \frac{du}{\sqrt{[4u^2 + (a+b)^2](u^2+ab)}} \\ &= \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{du}{\sqrt{\left[u^2 + \left(\frac{a+b}{2}\right)^2\right](u^2+ab)}}. \end{aligned} \tag{68}$$

We have therefore demonstrated that

$$T(a,b) = T(\tfrac{1}{2}(a+b), \sqrt{ab}). \tag{69}$$

We can thus iterate

$$a_{i+1} = \tfrac{1}{2}(a_i + b_i) \tag{70}$$

$$b_{i+1} = \sqrt{a_i b_i}, \tag{71}$$

as many times as we wish, without changing the value of the integral. But this iteration is the same as and therefore converges to the ARITHMETIC-GEOMETRIC MEAN, so the iteration terminates at $a_i = b_i = M(a_0, b_0)$, and we have

$$\begin{aligned} T(a_0,b_0) &= T(M(a_0,b_0), M(a_0,b_0)) \\ &= \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{dt}{M^2(a_0,b_0) + t^2} \\ &= \frac{1}{\pi M(a_0,b_0)}\left[\tan^{-1}\left(\frac{t}{M(a_0,b_0)}\right)\right]_{-\infty}^{\infty} \\ &= \frac{1}{\pi M(a_0,b_0)}\left[\frac{\pi}{2} - \left(\frac{\pi}{2}\right)\right] \\ &= \frac{1}{M(a_0,b_0)}. \end{aligned} \tag{72}$$

Complete elliptic integrals arise in finding the arc length of an ELLIPSE and the period of a pendulum. They also arise in a natural way from the theory of THETA FUNCTIONS. Complete elliptic integrals can be computed using a procedure involving the ARITHMETIC-GEOMETRIC MEAN. Note that

$$\begin{aligned} T(a,b) &\equiv \frac{2}{\pi}\int_0^{\pi/2} \frac{d\theta}{\sqrt{a^2\cos^2\theta + b^2\sin^2\theta}} \\ &= \frac{2}{\pi}\int_0^{\pi/2} \frac{d\theta}{a\sqrt{\cos^2\theta + \left(\frac{b}{a}\right)^2\sin^2\theta}} \\ &= \frac{2}{a\pi}\int_0^{\pi/2} \frac{d\theta}{\sqrt{1 - \left(1 - \frac{b^2}{a^2}\right)^2\sin^2\theta}}. \end{aligned} \tag{73}$$

So we have

$$T(a,b) = \frac{2}{a\pi} K\left(1 - \frac{b^2}{a^2}\right) = \frac{1}{M(a,b)}, \quad (74)$$

where $K(k)$ is the complete ELLIPTIC INTEGRAL OF THE FIRST KIND. We are free to let $a \equiv a_0 \equiv 1$ and $b \equiv b_0 \equiv k'$, so

$$\frac{2}{\pi} K(\sqrt{1-k'^2}) = \frac{2}{\pi} K(k) = \frac{1}{M(1,k')}, \quad (75)$$

since $k \equiv \sqrt{1-k'^2}$, so

$$K(k) = \frac{\pi}{2M(1,k')}. \quad (76)$$

But the ARITHMETIC-GEOMETRIC MEAN is defined by

$$a_i = \tfrac{1}{2}(a_{i-1} + b_{i-1}) \quad (77)$$

$$b_i = \sqrt{a_{i-1} b_{i-1}} \quad (78)$$

$$c_i = \begin{cases} \tfrac{1}{2}(a_{i-1} - b_{i-1}) & i > 0 \\ \sqrt{{a_0}^2 - {b_0}^2} & i = 0, \end{cases} \quad (79)$$

where

$$c_{n-1} = \tfrac{1}{2} a_n - b_n = \frac{{c_n}^2}{4a_{n+1}} \leq \frac{{c_n}^2}{4M(a_0,b_0)}, \quad (80)$$

so we have

$$K(k) = \frac{\pi}{2a_N}, \quad (81)$$

where a_N is the value to which a_n converges. Similarly, taking instead $a_0' = 1$ and $b_0' = k$ gives

$$K'(k) = \frac{\pi}{2a_N'}. \quad (82)$$

Borwein and Borwein (1987) also show that defining

$$U(a,b) \equiv \frac{\pi}{2} \int_0^{\pi/2} \sqrt{a^2 \cos^2 + b^2 \sin^2 \theta}\, d\theta = aE'\left(\frac{b}{a}\right) \quad (83)$$

leads to

$$2U(a_{n+1}, b_{n+1}) - U(a_n, b_n) = a_n b_n T(a_n, b_n), \quad (84)$$

so

$$\frac{K(k) - E(k)}{K(k)} = \tfrac{1}{2}({c_0}^2 + 2{c_1}^2 + 2^2 {c_2}^2 + \ldots + 2^n {c_n}^2) \quad (85)$$

for $a_0 \equiv 1$ and $b_0 \equiv k'$, and

$$\frac{K'(k) - E'(k)}{K'(k)} = \tfrac{1}{2}({c_0'}^2 + 2{c_1'}^2 + 2^2 {c_2'}^2 + \ldots + 2^n {c_n'}^2). \quad (86)$$

The elliptic integrals satisfy a large number of identities. The complementary functions and moduli are defined by

$$K'(k) \equiv K(\sqrt{1-k^2}) = K(k'). \quad (87)$$

Use the identity of generalized elliptic integrals

$$T(a,b) = T(\tfrac{1}{2}(a+b), \sqrt{ab}) \quad (88)$$

to write

$$\begin{aligned} \frac{1}{a} K\left(\sqrt{1 - \frac{b^2}{a^2}}\right) &= \frac{2}{a+b} K\left(\sqrt{1 - \frac{4ab}{(a+b)^2}}\right) \\ &= \frac{2}{a+b} K\left(\sqrt{\frac{a^2 + b^2 - 2ab}{(a+b)^2}}\right) \\ &= \frac{2}{a+b} K\left(\frac{a-b}{a+b}\right) \end{aligned} \quad (89)$$

$$K\left(\sqrt{1 - \frac{b^2}{a^2}}\right) = \frac{2}{1 + \frac{b}{a}} K\left(\frac{1 - \frac{b}{a}}{1 + \frac{b}{a}}\right). \quad (90)$$

Define

$$k' \equiv \frac{b}{a}, \quad (91)$$

and use

$$k \equiv \sqrt{1 - k'^2}, \quad (92)$$

so

$$K(k) = \frac{2}{1+k'} K\left(\frac{1-k'}{1+k'}\right). \quad (93)$$

Now letting $l \equiv (1-k')/(1+k')$ gives

$$l(1+k') = 1 - k' \Rightarrow k'(l+1) = 1 - l \quad (94)$$

$$k' = \frac{1-l}{1+l} \quad (95)$$

$$\begin{aligned} k = \sqrt{1-k'^2} &= \sqrt{1 - \left(\frac{1-l}{1+l}\right)^2} \\ &= \sqrt{\frac{(1+l)^2 - (1-l)^2}{(1+l)^2}} = \frac{2\sqrt{l}}{1+l}, \end{aligned} \quad (96)$$

and

$$\begin{aligned} \tfrac{1}{2}(1+k') = \frac{1}{2}\left(1 + \frac{1-l}{1+l}\right) &= \frac{1}{2}\left[\frac{(1+l) + (1-l)}{1+l}\right] \\ &= \frac{1}{1+l}. \end{aligned} \quad (97)$$

Writing k instead of l,

$$K(k) = \frac{1}{k+1} K\left(\frac{2\sqrt{k}}{1+k}\right). \quad (98)$$

Similarly, from Borwein and Borwein (1987),

$$E(k) = \frac{1+k}{2} E\left(\frac{2\sqrt{k}}{1+k}\right) + \frac{k'^2}{2} K(k) \tag{99}$$

$$E(k) = (1+k')E\left(\frac{1-k'}{1+k'}\right) - k'K(k). \tag{100}$$

Expressions in terms of the complementary function can be derived from interchanging the moduli and their complements in (93), (98), (99), and (100).

$$\begin{aligned} K'(k) = K(k') &= \frac{2}{1+k} K\left(\frac{1-k}{1+k}\right) \\ &= \frac{2}{1+k} K'\left(\sqrt{1-\left(\frac{1-k}{1+k}\right)^2}\right) \\ &= \frac{2}{1+k} K'\left(\frac{2\sqrt{k}}{1+k}\right) \end{aligned} \tag{101}$$

$$K'(k) = \frac{1}{1+k'} K\left(\frac{2\sqrt{k'}}{1+k'}\right) = \frac{1}{1+k'} K'\left(\frac{1-k'}{1+k'}\right), \tag{102}$$

and

$$E'(k) = (1+k)E'\left(\frac{2\sqrt{k}}{1+k}\right) - kK'(k) \tag{103}$$

$$E'(k) = \left(\frac{1+k'}{2}\right) E'\left(\frac{1-k'}{1+k'}\right) + \frac{k^2}{2} K'(k). \tag{104}$$

Taking the ratios

$$\frac{K'(k)}{K(k)} = 2\frac{K'\left(\frac{2\sqrt{k}}{1+k}\right)}{K\left(\frac{2\sqrt{k}}{1+k}\right)} = \frac{1}{2}\frac{K'\left(\frac{1-k'}{1+k'}\right)}{K\left(\frac{1-k'}{1+k'}\right)} \tag{105}$$

gives the MODULAR EQUATION of degree 2. It is also true that

$$K(x) = \frac{4}{(1+\sqrt{x'})^2} K\left(\left[\frac{1-\sqrt[4]{1-x^4}}{1+\sqrt[4]{1-x^4}}\right]^2\right). \tag{106}$$

see also ABELIAN INTEGRAL, AMPLITUDE, ARGUMENT (ELLIPTIC INTEGRAL), CHARACTERISTIC (ELLIPTIC INTEGRAL), DELTA AMPLITUDE, ELLIPTIC FUNCTION, ELLIPTIC INTEGRAL OF THE FIRST KIND, ELLIPTIC INTEGRAL OF THE SECOND KIND, ELLIPTIC INTEGRAL OF THE THIRD KIND, ELLIPTIC INTEGRAL SINGULAR VALUE, HEUMAN LAMBDA FUNCTION, JACOBI ZETA FUNCTION, MODULAR ANGLE, MODULUS (ELLIPTIC INTEGRAL), NOME, PARAMETER

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Elliptic Integral of the First Kind

Let the MODULUS k satisfy $0 < k^2 < 1$. (This may also be written in terms of the PARAMETER $m \equiv k^2$ or MODULAR ANGLE $\alpha \equiv \sin^{-1} k$.) The incomplete elliptic integral of the first kind is then defined as

$$F(\phi, k) = \int_0^\phi \frac{d\theta}{\sqrt{1-k^2\sin^2\theta}}. \tag{1}$$

Let

$$t \equiv \sin\theta \tag{2}$$

$$dt = \cos\theta\, d\theta = \sqrt{1-t^2}\, d\theta \tag{3}$$

$$\begin{aligned} F(\phi,k) &= \int_0^{\sin\phi} \frac{1}{\sqrt{1-k^2t^2}} \frac{dt}{\sqrt{1-t^2}} \\ &= \int_0^{\sin\phi} \frac{dt}{\sqrt{(1-k^2t^2)(1-t^2)}}. \end{aligned} \tag{4}$$

Let

$$v \equiv \tan\theta \tag{5}$$

$$dv \equiv \sec^2\theta\, d\theta = (1+v^2)\, d\theta, \tag{6}$$

so the integral can also be written as

$$\begin{aligned} F(\phi,k) &= \int_0^{\tan\phi} \frac{1}{\sqrt{1-k^2\frac{v^2}{1+u^2}}} \frac{du}{1+v^2} \\ &= \int_0^{\tan\phi} \frac{dv}{\sqrt{1+v^2}\sqrt{(1+v^2)-k^2v^2}} \qquad (7) \\ &= \int_0^{\tan\phi} \frac{dv}{\sqrt{(1+v^2)(1+k'v^2)}}, \qquad (8) \end{aligned}$$

where $k'^2 \equiv 1 - k^2$ is the complementary MODULUS.

The integral

$$I = \frac{1}{\sqrt{2}} \int_0^{\theta_0} \frac{d\theta}{\sqrt{\cos\theta - \cos\theta_0}}, \tag{9}$$

which arises in computing the period of a pendulum, is also an elliptic integral of the first kind. Use

$$\cos\theta = 1 - 2\sin^2(\tfrac{1}{2}\theta) \tag{10}$$

$$\sin(\tfrac{1}{2}\theta) = \sqrt{\frac{1-\cos\theta}{2}} \tag{11}$$

to write

$$\begin{aligned} \sqrt{\cos\theta - \cos\theta_0} &= \sqrt{1 - 2\sin^2(\tfrac{1}{2}\theta) - \cos\theta_0} \\ &= \sqrt{1-\cos\theta_0}\sqrt{1 - \frac{2}{1-\cos\theta_0}\sin^2(\tfrac{1}{2}\theta)} \\ &= \sqrt{2}\sin(\tfrac{1}{2}\theta_0)\sqrt{1 - \csc^2(\tfrac{1}{2}\theta_0)\sin^2(\tfrac{1}{2}\theta)}, \end{aligned} \tag{12}$$

so

$$I = \frac{1}{2}\int_0^{\theta_0} \frac{d\theta}{\sin(\frac{1}{2}\theta_0)\sqrt{1 - \csc^2(\frac{1}{2}\theta_0)\sin^2(\frac{1}{2}\theta)}}. \tag{13}$$

Now let

$$\sin(\tfrac{1}{2}\theta) = \sin(\tfrac{1}{2}\theta_0)\sin\phi, \tag{14}$$

so the angle θ is transformed to

$$\phi = \sin^{-1}\left(\frac{\sin\theta}{\theta_0}\right), \tag{15}$$

which ranges from 0 to $\pi/2$ as θ varies from 0 to θ_0. Taking the differential gives

$$\tfrac{1}{2}\cos(\tfrac{1}{2}\theta)\,d\theta = \sin(\tfrac{1}{2}\theta_0)\cos\phi\,d\phi, \tag{16}$$

or

$$\tfrac{1}{2}\sqrt{1 - \sin^2(\tfrac{1}{2}\theta_0)\sin^2\phi}\,d\theta = \sin(\tfrac{1}{2}\theta_0)\cos\phi\,d\phi. \tag{17}$$

Plugging this in gives

$$\begin{aligned} I &= \int_0^{\pi/2} \frac{1}{\sqrt{1 - \sin^2(\frac{1}{2}\theta_0)\sin^2\phi}} \frac{\sin(\frac{1}{2}\theta_0)\cos\phi\,d\phi}{\sin(\frac{1}{2}\theta_0)\sqrt{1-\sin^2\phi}} \\ &= \int_0^{\pi/2} \frac{d\phi}{\sqrt{1 - \sin^2(\frac{1}{2}\theta_0)\sin^2\phi}} = K(\sin(\tfrac{1}{2}\theta_0)), \end{aligned} \tag{18}$$

so

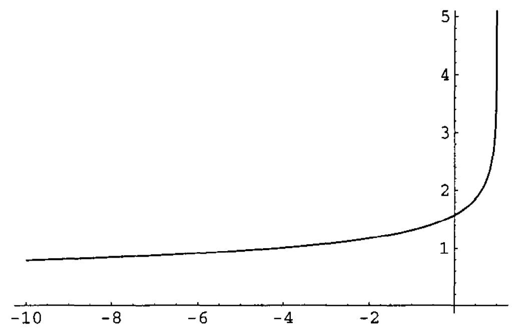
$$I = \frac{1}{\sqrt{2}}\int_0^{\theta_0} \frac{d\theta}{\sqrt{\cos\theta - \cos\theta_0}} = K(\sin(\tfrac{1}{2}\theta_0)). \tag{19}$$

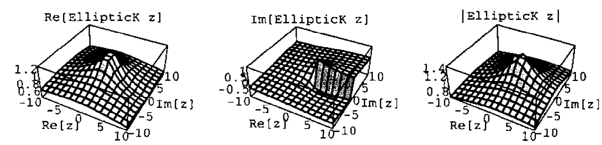
Making the slightly different substitution $\phi = \theta/2$, so $d\theta = 2\,d\phi$ leads to an equivalent, but more complicated expression involving an *incomplete* elliptic function of the first kind,

$$\begin{aligned} I &= 2\frac{1}{\sqrt{2}}\frac{1}{\sqrt{2}}\csc(\tfrac{1}{2}\theta_0)\int_0^{\theta_0} \frac{d\phi}{\sqrt{1 - \csc^2(\frac{1}{2}\theta_0)\sin^2\phi}} \\ &= \csc(\tfrac{1}{2}\theta_0)F(\tfrac{1}{2}\theta_0, \csc(\tfrac{1}{2}\theta_0)). \end{aligned} \tag{20}$$

Therefore, we have proven the identity

$$\csc x F(x, \csc x) = K(\sin x). \tag{21}$$





The complete elliptic integral of the first kind, illustrated above as a function of $m = k^2$, is defined by

$$\begin{aligned} K(k) &\equiv F(\tfrac{1}{2}\pi, k) && (22) \\ &= \sum_{n=0}^{\infty} \frac{(2n-1)!!}{(2n)!!}k^{2n}\int_0^{2\pi}\sin^{2n}\theta\,d\theta && (23) \\ &= \tfrac{1}{2}\pi{\vartheta_3}^2(q) && (24) \\ &= \sum_{n=0}^{\infty}\frac{(2n-1)!!}{(2n)!!}k^{2n}\frac{\pi}{2}\frac{(2n-1)!!}{(2n)!!} \\ &= \frac{\pi}{2}\sum_{n=0}^{\infty}\left[\frac{(2n-1)!!}{(2n)!!}\right]^2 k^{2n} && (25) \\ &= \tfrac{1}{2}\pi\,{}_2F_1(\tfrac{1}{2},\tfrac{1}{2},1;k^2) && (26) \\ &= \frac{\pi}{2\sqrt{1-k^2}}P_{-1/2}\left(\frac{1+k^2}{1-k^2}\right), && (27) \end{aligned}$$

where

$$q = e^{-\pi K'(k)/K(k)} \tag{28}$$

is the NOME (for $|q| < 1$), ${}_2F_1(a,b;c;x)$ is the HYPERGEOMETRIC FUNCTION, and $P_n(x)$ is a LEGENDRE POLYNOMIAL. $K(k)$ satisfies the LEGENDRE RELATION

$$E(k)K'(k) + E'(k)K(k) - K(k)K'(k) = \tfrac{1}{2}\pi, \tag{29}$$

where $E(k)$ and $K(k)$ are complete elliptic integrals of the first and SECOND KINDS, and $E'(k)$ and $K'(k)$ are the complementary integrals. The modulus k is often suppressed for conciseness, so that $E(k)$ and $K(k)$ are often simply written E and K, respectively.

The DERIVATIVE of $K(k)$ is

$$\frac{dK}{dk} \equiv \int_0^1 \frac{dt}{\sqrt{(1-t^2)(1-k'^2t^2)}} = \frac{E(k)}{k(1-k^2)} - \frac{K(k)}{k} \tag{30}$$

$$\frac{d}{dk}\left(kk'^2\frac{dK}{dk}\right) = kK, \tag{31}$$

so

$$E = k(1-k^2)\left(\frac{dK}{dk}+\frac{K}{k}\right) = (1-k^2)\left(k\frac{dK}{dk}+k\right) \tag{32}$$

(Whittaker and Watson 1990, pp. 499 and 521).

see also AMPLITUDE, CHARACTERISTIC (ELLIPTIC INTEGRAL), ELLIPTIC INTEGRAL SINGULAR VALUE, GAUSS'S TRANSFORMATION, LANDEN'S TRANSFORMATION, LEGENDRE RELATION, MODULAR ANGLE, MODULUS (ELLIPTIC INTEGRAL), PARAMETER

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Elliptic Integral of the Second Kind

Let the MODULUS k satisfy $0 < k^2 < 1$. (This may also be written in terms of the PARAMETER $m \equiv k^2$ or MODULAR ANGLE $\alpha \equiv \sin^{-1} k$.) The incomplete elliptic integral of the second kind is then defined as

$$E(\phi,k) \equiv \int_0^\phi \sqrt{1-k^2\sin^2\theta}\,d\theta. \tag{1}$$

A generalization replacing $\sin\theta$ with $\sinh\theta$ gives

$$-iE(i\phi,-k) = \int_0^\phi \sqrt{1-k^2\sinh^2\theta}\,d\theta. \tag{2}$$

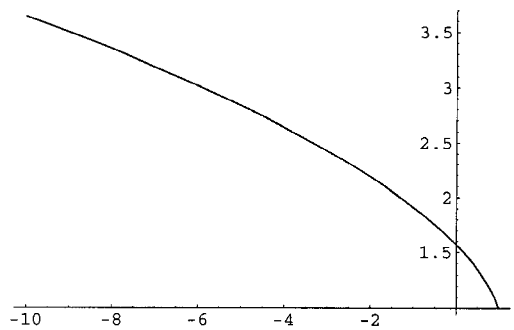
To place the elliptic integral of the second kind in a slightly different form, let

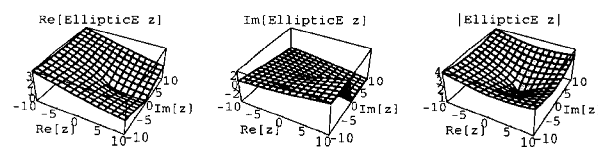
$$t \equiv \sin\theta \tag{3}$$

$$dt = \cos\theta\,d\theta = \sqrt{1-t^2}\,d\theta, \tag{4}$$

so the elliptic integral can also be written as

$$E(\phi,k) = \int_0^{\sin\phi} \sqrt{1-k^2t^2}\,\frac{dt}{\sqrt{1-t^2}} = \int_0^{\sin\phi} \sqrt{\frac{1-k^2t^2}{1-t^2}}\,dt. \tag{5}$$





The complete elliptic integral of the second kind, illustrated above as a function of the PARAMETER m, is defined by

$$E(k) \equiv E(\tfrac{1}{2}\pi,k) \tag{6}$$

$$= \frac{\pi}{2}\left\{1-\sum_{n=1}^\infty \left[\frac{(2n-1)!!}{(2n)!!)}\right]^2 \frac{k^{2n}}{2n-1}\right\} \tag{7}$$

$$= \tfrac{1}{2}\pi\,{}_2F_1(-\tfrac{1}{2},\tfrac{1}{2},1;k^2) \tag{8}$$

$$= \int_0^K \mathrm{dn}^2\,u\,du, \tag{9}$$

where ${}_2F_1(a,b;c;x)$ is the HYPERGEOMETRIC FUNCTION and $\mathrm{dn}\,u$ is a JACOBI ELLIPTIC FUNCTION. The complete elliptic integral of the second kind satisfies the LEGENDRE RELATION

$$E(k)K'(k)+E'(k)K(k)-K(k)K'(k) = \tfrac{1}{2}\pi, \tag{10}$$

where E and K are complete ELLIPTIC INTEGRALS OF THE FIRST and second kinds, and E' and K' are the complementary integrals. The DERIVATIVE is

$$\frac{dE}{dk} = \frac{E(k)-K(k)}{k} \tag{11}$$

(Whittaker and Watson 1990, p. 521). If k_r is a singular value (i.e.,

$$k_r = \lambda^*(r), \tag{12}$$

where λ^* is the ELLIPTIC LAMBDA FUNCTION), and $K(k_r)$ and the ELLIPTIC ALPHA FUNCTION $\alpha(r)$ are also known, then

$$E(k) = \frac{K(k)}{\sqrt{r}}\left[\frac{\pi}{3[K(k)]^2} - \alpha(r)\right] + K(k). \tag{13}$$

see also ELLIPTIC INTEGRAL OF THE FIRST KIND, ELLIPTIC INTEGRAL OF THE THIRD KIND, ELLIPTIC INTEGRAL SINGULAR VALUE

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Abramowitz, M. and Stegun, C. A. (Eds.). "Elliptic Integrals." Ch. 17 in *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables, 9th printing.* New York: Dover, pp. 587–607, 1972.

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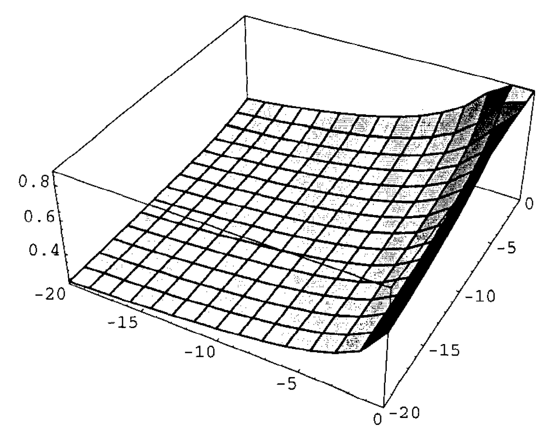
Elliptic Integral of the Third Kind

Let $0 < k^2 < 1$. The incomplete elliptic integral of the third kind is then defined as

$$\Pi(n;\phi,k) = \int_0^{\phi} \frac{d\theta}{(1-n\sin^2\theta)\sqrt{1-k^2\sin^2\theta}} \tag{1}$$

$$= \int_0^{\sin\phi} \frac{dt}{(1-nt^2)\sqrt{(1-t^2)(1-k^2t^2)}}, \tag{2}$$

where n is a constant known as the CHARACTERISTIC.



The complete elliptic integral of the second kind

$$\Pi(n|m) = \Pi(n;\tfrac{1}{2}\pi|m) \tag{3}$$

is illustrated above.

see also ELLIPTIC INTEGRAL OF THE FIRST KIND, ELLIPTIC INTEGRAL OF THE SECOND KIND, ELLIPTIC INTEGRAL SINGULAR VALUE

References

Abramowitz, M. and Stegun, C. A. (Eds.). "Elliptic Integrals" and "Elliptic Integrals of the Third Kind." Ch. 17 and §17.7 in *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables, 9th printing.* New York: Dover, pp. 587–607, 1972.

Elliptic Integral Singular Value

When the MODULUS k has a singular value, the complete elliptic integrals may be computed in analytic form in terms of GAMMA FUNCTIONS. Abel (quoted in Whittaker and Watson 1990, p. 525) proved that whenever

$$\frac{K'(k)}{K(k)} = \frac{a+b\sqrt{n}}{c+d\sqrt{n}}, \tag{1}$$

where a, b, c, d, and n are INTEGERS, $K(k)$ is a complete ELLIPTIC INTEGRAL OF THE FIRST KIND, and $K'(k) \equiv K(\sqrt{1-k^2})$ is the complementary complete ELLIPTIC INTEGRAL OF THE FIRST KIND, then the MODULUS k is the ROOT of an algebraic equation with INTEGER COEFFICIENTS.

A MODULUS k_r such that

$$\frac{K'(k_r)}{K(k_r)} = \sqrt{r}, \tag{2}$$

is called a singular value of the elliptic integral. The ELLIPTIC LAMBDA FUNCTION $\lambda^*(r)$ gives the value of k_r. Selberg and Chowla (1967) showed that $K(\lambda^*(r))$ and $E(\lambda^*(r))$ are expressible in terms of a finite number of GAMMA FUNCTIONS. The complete ELLIPTIC INTEGRALS OF THE SECOND KIND $E(k_r)$ and $E'(k_r)$ can be expressed in terms of $K(k_r)$ and $K'(k_r)$ with the aid of the ELLIPTIC ALPHA FUNCTION $\alpha(r)$.

The following table gives the values of $K(k_r)$ for small integral r in terms of GAMMA FUNCTIONS.

$$K(k_1) = \frac{\Gamma^2(\frac{1}{4})}{4\sqrt{\pi}}$$

$$K(k_2) = \frac{\sqrt{\sqrt{2}+1}\,\Gamma(\frac{1}{8})\Gamma(\frac{3}{8})}{2^{13/4}\sqrt{\pi}}$$

$$K(k_3) = \frac{3^{1/4}\Gamma^3(\frac{1}{3})}{2^{7/3}\pi}$$

$$K(k_4) = \frac{(\sqrt{2}+1)\Gamma^2(\frac{1}{4})}{2^{7/2}\sqrt{\pi}}$$

$$K(k_5) = (\sqrt{5}+2)^{1/4}\sqrt{\frac{\Gamma(\frac{1}{20})\Gamma(\frac{3}{20})\Gamma(\frac{7}{20})\Gamma(\frac{9}{20})}{160\pi}}$$

$$K(k_6) = \sqrt{(\sqrt{2}-1)(\sqrt{3}+\sqrt{2})(2+\sqrt{3})} \times \sqrt{\frac{\Gamma(\frac{1}{24})\Gamma(\frac{5}{24})\Gamma(\frac{7}{24})\Gamma(\frac{11}{24})}{384\pi}}$$

$$K(k_7) = \frac{\Gamma(\frac{1}{7})\Gamma(\frac{2}{7})\Gamma(\frac{4}{7})}{7^{1/4}\cdot 4\pi}$$

$$K(k_8) = \sqrt{\frac{2\sqrt{2}+\sqrt{1+5\sqrt{2}}}{4\sqrt{2}}}\,\frac{(\sqrt{2}+1)^{1/4}\Gamma(\frac{1}{8})\Gamma(\frac{3}{8})}{8\sqrt{\pi}}$$

$$K(k_9) = \frac{3^{1/4}\sqrt{2+\sqrt{3}}}{12\sqrt{\pi}\Gamma^2(\frac{1}{4})}$$

$$K(k_{10}) = \sqrt{(2+3\sqrt{2}+\sqrt{5})}$$
$$\times \sqrt{\frac{\Gamma(\frac{1}{40})\Gamma(\frac{7}{40})\Gamma(\frac{9}{40})\Gamma(\frac{11}{40})\Gamma(\frac{13}{40})\Gamma(\frac{19}{40})\Gamma(\frac{23}{40})\Gamma(\frac{37}{40})}{2560\pi^3}}$$

$$K(k_{11}) = [2 + (17+3\sqrt{33})^{1/3} - (3\sqrt{33}-17)^{1/3}]^2$$
$$\times \frac{\Gamma(\frac{1}{11})\Gamma(\frac{3}{11})\Gamma(\frac{4}{11})\Gamma(\frac{5}{11})\Gamma(\frac{9}{11})}{11^{1/4}144\pi^2}$$

$$K(k_{12}) = \frac{3^{1/4}(\sqrt{2}+1)(\sqrt{3}+\sqrt{2})\sqrt{2-\sqrt{3}}\Gamma^3(\frac{1}{3})}{2^{13/3}\pi}$$

$$K(k_{13}) = \frac{(18+5\sqrt{13})^{1/4}}{\sqrt{6656\pi^5}}$$
$$\times \sqrt{\Gamma(\tfrac{1}{52})\Gamma(\tfrac{7}{52})\Gamma(\tfrac{9}{52})\Gamma(\tfrac{11}{52})\Gamma(\tfrac{15}{52})\Gamma(\tfrac{17}{52})}$$
$$\times \sqrt{\Gamma(\tfrac{19}{52})\Gamma(\tfrac{25}{52})\Gamma(\tfrac{29}{52})\Gamma(\tfrac{31}{52})\Gamma(\tfrac{47}{52})\Gamma(\tfrac{49}{52})}$$

$$K(k_{15}) = \sqrt{\frac{(\sqrt{5}+1)\Gamma(\frac{1}{15})\Gamma(\frac{2}{15})\Gamma(\frac{4}{15})\Gamma(\frac{8}{15})}{240\pi}}$$

$$K(k_{16}) = \frac{(2^{1/4}+1)^2\Gamma^2(\frac{1}{4})}{2^{9/2}\sqrt{\pi}}$$

$$K(k_{17}) = C_1\left[\frac{\Gamma(\frac{1}{68})\Gamma(\frac{3}{68})\Gamma(\frac{7}{68})\Gamma(\frac{11}{68})\Gamma(\frac{13}{68})}{\Gamma(\frac{5}{68})\Gamma(\frac{15}{68})\Gamma(\frac{19}{68})\Gamma(\frac{29}{68})}\right]^{1/4}$$
$$\times [\Gamma(\tfrac{21}{68})\Gamma(\tfrac{25}{68})\Gamma(\tfrac{27}{68})\Gamma(\tfrac{31}{68})\Gamma(\tfrac{33}{68})]^{1/4}$$

$$K(k_{25}) = \frac{\sqrt{5}+2}{20}\frac{\Gamma^2(\frac{1}{4})}{\sqrt{\pi}},$$

where $\Gamma(z)$ is the GAMMA FUNCTION and C_1 is an algebraic number (Borwein and Borwein 1987, p. 298).

Borwein and Zucker (1992) give amazing expressions for singular values of complete elliptic integrals in terms of CENTRAL BETA FUNCTIONS

$$\beta(p) \equiv B(p,p). \tag{3}$$

Furthermore, they show that $K(k_n)$ is *always* expressible in terms of these functions for $n \equiv 1, 2 \pmod 4$. In such cases, the Γ functions appearing in the expression are of the form $\Gamma(t/4n)$ where $1 \leq t \leq (2n-1)$ and $(t, 4n) = 1$. The terms in the numerator depend on the sign of the KRONECKER SYMBOL $\{t/4n\}$. Values for the first few n are

$$K(k_1) = 2^{-2}\beta(\tfrac{1}{4})$$
$$K(k_2) = 2^{-13/4}\beta(\tfrac{1}{8})$$
$$K(k_3) = 2^{-4/3}3^{-1/4}\beta(\tfrac{1}{3}) = 2^{-5/3}3^{-3/4}\beta(\tfrac{1}{6})$$
$$K(k_5) = 2^{-33/20}5^{-5/8}(11+5\sqrt{5})^{1/4}\sin(\tfrac{1}{20}\pi)\beta(\tfrac{1}{2})$$
$$= 2^{-29/20}5^{-3/8}(1+\sqrt{5})^{1/4}\sin(\tfrac{3}{20}\pi)\beta(\tfrac{3}{20})$$
$$K(k_6) = 2^{-47/12}3^{-3/4}(\sqrt{2}-1)(\sqrt{3}+1)\beta(\tfrac{1}{24})$$
$$= 2^{-43/12}3^{-1/4}(\sqrt{3}-1)\beta(\tfrac{5}{24})$$
$$K(k_7) = 2\cdot 7^{-3/4}\sin(\tfrac{1}{7}\pi)\sin(\tfrac{2}{7}\pi)B(\tfrac{1}{7},\tfrac{2}{7})$$
$$= 2^{-2/7}7^{-1/4}\frac{\beta(\frac{1}{7})\beta(\frac{2}{7})}{\beta(\frac{1}{14})}$$

$$K(k_{10}) = 2^{-61/20}5^{-1/4}(\sqrt{5}-2)^{1/2}(\sqrt{10}+3)\frac{\beta(\frac{1}{8})\beta(\frac{7}{40})}{\beta(\frac{1}{3}40)}$$
$$= 2^{-15/4}5^{-3/4}(\sqrt{5}-2)^{1/2}\frac{\beta(\frac{1}{40})\beta(\frac{1}{9}40)}{\beta(\frac{3}{8})}$$
$$K(k_{11}) = R\cdot 2^{-7/11}\sin(\tfrac{1}{11}\pi)\sin(\tfrac{3}{11}\pi)B(\tfrac{1}{22},\tfrac{3}{22})$$
$$K(k_{13}) = 2^{-3}13^{-5/8}(5\sqrt{13}+18)^{1/4}$$
$$\times [\tan(\tfrac{1}{52}\pi)\tan(\tfrac{3}{52}\pi)\tan(\tfrac{9}{52}\pi)]^{1/2}\frac{\beta(\frac{1}{52})\beta(\frac{9}{52})}{\beta(\frac{23}{52})}$$
$$K(k_{14}) = \sqrt{\sqrt{4\sqrt{2}+2}+\sqrt{2}+\sqrt{2\sqrt{2}-1}}\cdot 2^{-13/4}7^{-3/8}$$
$$\times \left[\frac{\tan(\frac{5}{56}\pi)\tan(\frac{13}{56}\pi)}{\tan(\frac{11}{56}\pi)}\right]^{1/4}\sqrt{\frac{\beta(\frac{5}{56})\beta(\frac{13}{56})\beta(\frac{1}{8})}{\beta(\frac{11}{56})}}$$
$$K(k_{15}) = 2^{-1}3^{-3/4}5^{-7/12}B(\tfrac{1}{15},\tfrac{4}{15})$$
$$= \frac{2^{-2}3^{-3/4}5^{-3/4}(\sqrt{5}-1)\beta(\frac{1}{15})\beta(\frac{4}{15})}{\beta(\frac{1}{3})}$$
$$K(k_{17}) = C_2\left[\frac{\beta(\frac{1}{68})\beta(\frac{3}{68}(\beta(\frac{7}{68})\beta(\frac{9}{68})\beta(\frac{11}{68})\beta(\frac{13}{68})}{\beta(\frac{5}{68})\beta(\frac{15}{68})}\right]^{1/4},$$

where R is the REAL ROOT of

$$x^3 - 4x = 4 = 0 \tag{4}$$

and C_2 is an algebraic number (Borwein and Zucker 1992). Note that $K(k_{11})$ is the only value in the above list which cannot be expressed in terms of CENTRAL BETA FUNCTIONS.

Using the ELLIPTIC ALPHA FUNCTION, the ELLIPTIC INTEGRALS OF THE SECOND KIND can also be found from

$$E = \frac{\pi}{4\sqrt{r}K} + \left[1 - \frac{\alpha(r)}{\sqrt{r}}\right]K \tag{5}$$
$$E' = \frac{\pi}{4K} + \alpha(r)K, \tag{6}$$

and by definition,

$$K' = K\sqrt{n}. \tag{7}$$

see also CENTRAL BETA FUNCTION, ELLIPTIC ALPHA FUNCTION, ELLIPTIC DELTA FUNCTION, ELLIPTIC INTEGRAL OF THE FIRST KIND, ELLIPTIC INTEGRAL OF THE SECOND KIND, ELLIPTIC LAMBDA FUNCTION, GAMMA FUNCTION, MODULUS (ELLIPTIC INTEGRAL)

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Elliptic Integral Singular Value—k_1

The first SINGULAR VALUE k_1, corresponding to

$$K'(k_1) = K(k_1), \tag{1}$$

is given by

$$k_1 = \frac{1}{\sqrt{2}} \tag{2}$$

$$k_1' = \frac{1}{\sqrt{2}}. \tag{3}$$

As shown in LEMNISCATE FUNCTION,

$$\begin{aligned} K\left(\frac{1}{\sqrt{2}}\right) &\equiv \int_0^1 \frac{dt}{\sqrt{(1-t^2)\left(1-\frac{1}{2}t^2\right)}} \\ &= \sqrt{2}\int_0^1 \frac{dt}{\sqrt{1-t^4}}. \end{aligned} \tag{4}$$

Let

$$u \equiv t^4 \tag{5}$$

$$du = 4t^3\,dt = 4u^{3/4}\,dt \tag{6}$$

$$dt = \tfrac{1}{4}u^{-3/4}\,du, \tag{7}$$

then

$$\begin{aligned} K\left(\frac{1}{\sqrt{2}}\right) &= \frac{\sqrt{2}}{4}\int_0^1 u^{-3/4}(1-u)^{-1/2}\,du \\ &= \frac{\sqrt{2}}{4}B(\tfrac{1}{4},\tfrac{1}{2}) = \frac{\Gamma(\frac{1}{4})\Gamma(\frac{1}{2})}{\Gamma(\frac{3}{4})}\frac{\sqrt{2}}{4}, \end{aligned} \tag{8}$$

where $B(a,b)$ is the BETA FUNCTION and $\Gamma(z)$ is the GAMMA FUNCTION. Now use

$$\Gamma(\tfrac{1}{2}) = \sqrt{\pi} \tag{9}$$

and

$$\frac{1}{\Gamma(1-x)} = \frac{\sin(\pi x)}{\pi}\Gamma(x), \tag{10}$$

so

$$\frac{1}{\Gamma\left(\frac{3}{4}\right)} = \frac{1}{\Gamma\left(1-\frac{1}{4}\right)} = \frac{\sin\left(\frac{\pi}{4}\right)}{\pi}\Gamma(\tfrac{1}{4}) = \frac{1}{\pi\sqrt{2}}\Gamma(\tfrac{1}{4}). \tag{11}$$

Therefore,

$$K\left(\frac{1}{\sqrt{2}}\right) = \frac{\Gamma^2(\frac{1}{4})\sqrt{\pi}\sqrt{2}}{4\pi\sqrt{2}} = \frac{\Gamma^2(\frac{1}{4})}{4\sqrt{\pi}}. \tag{12}$$

Now consider

$$E\left(\frac{1}{\sqrt{2}}\right) \equiv \int_0^1 \sqrt{\frac{1-\frac{1}{2}t^2}{1-t^2}}\,dt. \tag{13}$$

Let

$$t^2 \equiv 1-u^2 \tag{14}$$

$$2t\,dt = -2u\,du \tag{15}$$

$$dt = -\frac{1}{t}u\,du = u(1-u^2)^{-1/2}\,du, \tag{16}$$

so

$$\begin{aligned} E\left(\frac{1}{\sqrt{2}}\right) &= \int_0^1 \sqrt{\frac{1-\frac{1}{2}(1-u^2)}{1-(1-u^2)}}\,u(1-u^2)^{-1/2}\,du \\ &= \int_0^1 \frac{\sqrt{\frac{1}{2}(1+u^2)}}{u}u(1-u^2)^{-1/2}\,du \\ &= \frac{1}{\sqrt{2}}\int_0^1 \sqrt{\frac{1+u^2}{1-u^2}}\,du. \end{aligned} \tag{17}$$

Now note that

$$\begin{aligned} \left(\frac{1}{\sqrt{1-u^4}} + \frac{u^2}{\sqrt{1-u^4}}\right)^2 &= \frac{(1+u^2)^2}{1-u^4} \\ &= \frac{(1+u^2)^2}{(1+u^2)(1-u^2)} = \frac{1+u^2}{1-u^2}, \end{aligned} \tag{18}$$

so

$$\begin{aligned} E\left(\frac{1}{\sqrt{2}}\right) &= \frac{1}{\sqrt{2}}\int_0^1 \sqrt{\frac{1+u^2}{1-u^2}}\,du \\ &= \frac{1}{\sqrt{2}}\int_0^1 \left(\frac{1}{\sqrt{1-u^4}} + \frac{u^2}{\sqrt{1-u^4}}\right)du \\ &= \frac{1}{2}K\left(\frac{1}{\sqrt{2}}\right) + \frac{1}{\sqrt{2}}\int_0^1 \frac{u^2\,du}{\sqrt{1-u^4}}. \end{aligned} \tag{19}$$

Now let

$$t \equiv u^4 \tag{20}$$

$$dt = 4u^3\,du, \tag{21}$$

so

$$\int_0^1 \frac{u^2\,du}{\sqrt{1-u^4}} = \frac{1}{4}\int_0^1 t^{1/2}t^{-3/4}(1-t)^{-1/2}\,dt$$
$$= \tfrac{1}{4}\int_0^1 t^{-1/4}(1-t)^{-1/2}\,dt$$
$$= \tfrac{1}{4}B(\tfrac{3}{4},\tfrac{1}{2}) = \frac{\Gamma(\frac{3}{4})\Gamma(\frac{1}{2})}{4\Gamma(\frac{5}{4})}. \qquad (22)$$

But

$$[\Gamma(\tfrac{5}{4})]^{-1} = [\tfrac{1}{4}\Gamma(\tfrac{1}{4})]^{-1} \qquad (23)$$
$$\Gamma(\tfrac{3}{4}) = \pi\sqrt{2}\,[\Gamma(\tfrac{1}{4})]^{-1} \qquad (24)$$
$$\Gamma(\tfrac{1}{2}) = \sqrt{\pi}, \qquad (25)$$

so

$$\int_0^1 \frac{u^2\,du}{\sqrt{1-u^4}} = \frac{1}{4}\frac{\pi\sqrt{2}\cdot 4\sqrt{\pi}}{\Gamma^2(\frac{1}{4})} = \frac{\sqrt{2}\pi^{3/2}}{\Gamma^2(\frac{1}{4})} \qquad (26)$$

$$E\left(\frac{1}{\sqrt{2}}\right) = \tfrac{1}{2}K + \frac{\pi^{3/2}}{\Gamma^2(\frac{1}{4})} = \frac{\Gamma^2(\frac{1}{4})}{8\sqrt{\pi}} + \frac{\pi^{3/2}}{\Gamma^2(\frac{1}{4})}$$
$$= \frac{1}{4}\sqrt{\frac{\pi}{2}}\left[\frac{\Gamma(\frac{1}{4})}{\Gamma(\frac{3}{4})} + \frac{\Gamma(\frac{3}{4})}{\Gamma(\frac{5}{4})}\right]. \qquad (27)$$

Summarizing (12) and (27) gives

$$K\left(\frac{1}{\sqrt{2}}\right) = \frac{\Gamma^2(\frac{1}{4})}{4\sqrt{\pi}}$$
$$K'\left(\frac{1}{\sqrt{2}}\right) = \frac{\Gamma^2(\frac{1}{4})}{4\sqrt{\pi}}$$
$$E\left(\frac{1}{\sqrt{2}}\right) = \frac{\Gamma^2(\frac{1}{4})}{8\sqrt{\pi}} + \frac{\pi^{3/2}}{\Gamma^2(\frac{1}{4})}$$
$$E'\left(\frac{1}{\sqrt{2}}\right) = \frac{\Gamma^2(\frac{1}{4})}{8\sqrt{\pi}} + \frac{\pi^{3/2}}{\Gamma^2(\frac{1}{4})}.$$

Elliptic Integral Singular Value—k_2

The second SINGULAR VALUE k_2, corresponding to

$$K'(k_2) = \sqrt{2}\,K(k_2), \qquad (1)$$

is given by

$$k_2 = \tan\left(\frac{\pi}{8}\right) = \sqrt{2}-1, \qquad (2)$$
$$k_2' = \sqrt{2}\,(\sqrt{2}-1). \qquad (3)$$

For this modulus,

$$E(\sqrt{2}-1) = \frac{1}{4}\sqrt{\frac{\pi}{4}}\left[\frac{\Gamma(\frac{1}{8})}{\Gamma(\frac{5}{8})} + \frac{\Gamma(\frac{5}{8})}{\Gamma(\frac{9}{8})}\right]. \qquad (4)$$

Elliptic Integral Singular Value—k_3

The third SINGULAR VALUE k_3, corresponding to

$$K'(k_3) = \sqrt{3}\,K(k_3), \qquad (1)$$

is given by

$$k_3 = \sin\left(\frac{\pi}{12}\right) = \tfrac{1}{4}(\sqrt{6}-\sqrt{2}). \qquad (2)$$

As shown by Legendre,

$$K(k_3) = \frac{\sqrt{\pi}}{2\cdot 3^{3/4}}\frac{\Gamma(\frac{1}{6})}{\Gamma(\frac{2}{3})} \qquad (3)$$

(Whittaker and Watson 1990, p. 525). In addition,

$$E(k_3) = \frac{\pi}{4\sqrt{3}}\frac{1}{K} + \frac{\sqrt{3}+1}{2\sqrt{3}}K$$
$$= \frac{1}{4}\left(\frac{\pi}{\sqrt{3}}\right)^{1/2}\left[\left(1+\frac{1}{\sqrt{3}}\right)\frac{\Gamma(\frac{1}{3})}{\Gamma(\frac{5}{6})} + \frac{2\Gamma(\frac{5}{6})}{\Gamma(\frac{1}{3})}\right], \qquad (4)$$

and

$$E'(k_3) = \frac{\pi\sqrt{3}}{4}\frac{1}{K'(k_3)} + \frac{\sqrt{3}-1}{2\sqrt{3}}K'(k_3). \qquad (5)$$

Summarizing,

$$K[\tfrac{1}{4}(\sqrt{6}-\sqrt{2})] = \frac{\sqrt{\pi}}{2\cdot 3^{3/4}}\frac{\Gamma(\frac{1}{6})}{\Gamma(\frac{2}{3})} \qquad (6)$$
$$K'[\tfrac{1}{4}(\sqrt{6}-\sqrt{2})] = \sqrt{3}K = \frac{\sqrt{\pi}}{2\cdot 3^{1/4}}\frac{\Gamma(\frac{1}{6})}{\Gamma(\frac{2}{3})} \qquad (7)$$
$$E[\tfrac{1}{4}(\sqrt{6}-\sqrt{2})]$$
$$= \frac{1}{4}\left(\frac{\pi}{\sqrt{3}}\right)^{1/2}\left[\left(1+\frac{1}{\sqrt{3}}\right)\frac{\Gamma(\frac{1}{3})}{\Gamma(\frac{5}{6})} + \frac{2\Gamma(\frac{5}{6})}{\Gamma(\frac{1}{3})}\right] \qquad (8)$$
$$E'[\tfrac{1}{4}(\sqrt{6}-\sqrt{2})]$$
$$= \frac{\sqrt{\pi}}{2}\left[3^{3/4}\frac{\Gamma(\frac{2}{3})}{\Gamma(\frac{1}{6})} + \frac{\sqrt{3}-1}{2\cdot 3^{3/4}}\frac{\Gamma(\frac{1}{6})}{\Gamma(\frac{2}{3})}\right]. \qquad (9)$$

(Whittaker and Watson 1990).

see also THETA FUNCTION

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Elliptic Lambda Function

The λ GROUP is the SUBGROUP of the GAMMA GROUP with a and d ODD; b and c EVEN. The function

$$\lambda(t) \equiv \lambda(q) \equiv k^2(q) = \left[\frac{\vartheta_2(q)}{\vartheta_3(q)}\right]^4, \tag{1}$$

where

$$q \equiv e^{i\pi t} \tag{2}$$

is a λ-MODULAR FUNCTION and ϑ_i are THETA FUNCTIONS.

$\lambda^*(r)$ gives the value of the MODULUS k_r for which the complementary and normal complete ELLIPTIC INTEGRALS OF THE FIRST KIND are related by

$$\frac{K'(k_r)}{K(k_r)} = \sqrt{r}. \tag{3}$$

It can be computed from

$$\lambda^*(r) \equiv k(q) = \frac{\vartheta_2{}^2(q)}{\vartheta_3{}^2(q)}, \tag{4}$$

where

$$q \equiv e^{-\pi\sqrt{r}}, \tag{5}$$

and ϑ_i is a THETA FUNCTION.

From the definition of the lambda function,

$$\lambda^*(r') = \lambda^*\left(\frac{1}{r}\right) = \lambda^{*\prime}(r). \tag{6}$$

For all rational r, $K(\lambda^*(r))$ and $E(\lambda^*(r))$ are expressible in terms of a finite number of GAMMA FUNCTIONS (Selberg and Chowla 1967). $\lambda^*(r)$ is related to the RAMANUJAN g- AND G-FUNCTIONS by

$$\lambda^*(n) = \tfrac{1}{2}\left(\sqrt{1+G_n^{-12}} - \sqrt{1-G_n^{-12}}\right) \tag{7}$$

$$\lambda^*(n) = g_n^6\left(\sqrt{g_n^{12}+g_n^{-12}} - g_n^6\right). \tag{8}$$

Special values are

$$\lambda^*(\tfrac{2}{29}) = (13\sqrt{58}-99)(\sqrt{2}+1)^6$$
$$\lambda^*(\tfrac{2}{5}) = (\sqrt{10}-3)(\sqrt{2}+1)^2$$
$$\lambda^*(\tfrac{2}{3}) = (2-\sqrt{3})(\sqrt{2}+\sqrt{3})$$
$$\lambda^*(\tfrac{3}{4}) = (\sqrt{3}-\sqrt{2})^2(\sqrt{2}+1)^2$$
$$\lambda^*(1) = \frac{1}{\sqrt{2}}$$
$$\lambda^*(2) = \sqrt{2}-1$$
$$\lambda^*(3) = \tfrac{1}{4}\sqrt{2}(\sqrt{3}-1)$$
$$\lambda^*(4) = 3-2\sqrt{2}$$
$$\lambda^*(5) = \tfrac{1}{2}\left(\sqrt{\sqrt{5}-1} - \sqrt{3-\sqrt{5}}\right)$$
$$\lambda^*(6) = (2-\sqrt{3})(\sqrt{3}-\sqrt{2})$$
$$\lambda^*(7) = \tfrac{1}{8}\sqrt{2}(3-\sqrt{7})$$
$$\lambda^*(8) = \left(\sqrt{2}+1-\sqrt{2\sqrt{2}+2}\right)^2$$
$$\lambda^*(9) = \tfrac{1}{2}(\sqrt{2}-3^{1/4})(\sqrt{3}-1)$$
$$\lambda^*(10) = (\sqrt{10}-3)(\sqrt{2}-1)^2$$
$$\lambda^*(11) = \tfrac{1}{12}\sqrt{6}\left(\sqrt{1+2x_{11}-4x_{11}{}^{-1}} - \sqrt{11+2x_{11}-4x_{11}{}^{-1}}\right)$$
$$\lambda^*(12) = (\sqrt{3}-\sqrt{2})^2(\sqrt{2}-1)^2 = 15-10\sqrt{2}+8\sqrt{3}-6\sqrt{6}$$
$$\lambda^*(13) = \tfrac{1}{2}\left(\sqrt{5\sqrt{13}-17} - \sqrt{19-5\sqrt{13}}\right)$$
$$\lambda^*(14) = -11-8\sqrt{2}-2(\sqrt{2}+2)\sqrt{5+4\sqrt{2}} + \sqrt{11+8\sqrt{2}}\left(2+2\sqrt{2}+\sqrt{2}\sqrt{5+4\sqrt{2}}\right)$$
$$\lambda^*(15) = \tfrac{1}{16}\sqrt{2}(3-\sqrt{5})(\sqrt{5}-\sqrt{3})(2-\sqrt{3})$$
$$\lambda^*(16) = \frac{(2^{1/4}-1)^2}{(2^{1/4}+1)^2}$$
$$\begin{aligned}\lambda^*(17) = \tfrac{1}{4}\sqrt{2}\Big(&\sqrt{42+10\sqrt{17}} \\ &-13\sqrt{-3+\sqrt{17}}\sqrt{5+\sqrt{17}} \\ &-3\sqrt{17}\sqrt{-3+\sqrt{17}}\sqrt{5+\sqrt{17}} \\ &-\sqrt{-38-10\sqrt{17}+13\sqrt{-3+\sqrt{17}}\sqrt{5+\sqrt{17}}} \\ &\overline{+3\sqrt{17}\sqrt{-3+\sqrt{17}}\sqrt{5+\sqrt{17}}}\Big)\end{aligned}$$
$$\lambda^*(18) = (\sqrt{2}-1)^3(2-\sqrt{3})^2$$
$$\lambda^*(22) = (3\sqrt{11}-7\sqrt{2})(10-3\sqrt{11})$$
$$\lambda^*(30) = (\sqrt{3}-\sqrt{2})^2(2-\sqrt{3})(\sqrt{6}-\sqrt{5})(4-\sqrt{15})$$
$$\lambda^*(34) = (\sqrt{2}-1)^2(3\sqrt{2}-\sqrt{17}) \times \left(\sqrt{297+72\sqrt{17}} - \sqrt{296+72\sqrt{17}}\right)$$
$$\lambda^*(42) = (\sqrt{2}-1)^2(2-\sqrt{3})^2(\sqrt{7}-\sqrt{6})(8-3\sqrt{7})$$
$$\lambda^*(58) = (13\sqrt{58}-99)(\sqrt{2}-1)^6$$
$$\lambda^*(210) = (\sqrt{2}-1)^2(2-\sqrt{3})(\sqrt{7}-\sqrt{6})^2(8-3\sqrt{7}) \times (\sqrt{10}-3)^2(4-\sqrt{15})^2(\sqrt{15}-\sqrt{14})(6-\sqrt{35}),$$

where

$$x_{11} \equiv (17+3\sqrt{33})^{1/3}.$$

In addition,

$$\lambda^*(1') = \frac{1}{\sqrt{2}}$$
$$\lambda^*(2') = \sqrt{2\sqrt{2}-2}$$
$$\lambda^*(3') = \tfrac{1}{4}\sqrt{2}(\sqrt{3}+1)$$
$$\lambda^*(4') = 2^{1/4}(2\sqrt{2}-2)$$
$$\lambda^*(5') = \tfrac{1}{2}\left(\sqrt{\sqrt{5}-1} + \sqrt{3-\sqrt{5}}\right)$$
$$\lambda^*(7') = \tfrac{1}{8}\sqrt{2}\,(3+\sqrt{7})$$
$$\lambda^*(9') = \tfrac{1}{2}(\sqrt{2}+3^{1/4})(\sqrt{3}-1)$$
$$\lambda^*(12') = 2\sqrt{-208+147\sqrt{2}-120\sqrt{3}+85\sqrt{6}}.$$

see also ELLIPTIC ALPHA FUNCTION, ELLIPTIC INTEGRAL OF THE FIRST KIND, MODULUS (ELLIPTIC INTEGRAL), RAMANUJAN g- AND G-FUNCTIONS, THETA FUNCTION

References
Borwein, J. M. and Borwein, P. B. *Pi & the AGM: A Study in Analytic Number Theory and Computational Complexity.* New York: Wiley, pp. 139 and 298, 1987.
Bowman, F. *Introduction to Elliptic Functions, with Applications.* New York: Dover, pp. 75, 95, and 98, 1961.
Selberg, A. and Chowla, S. "On Epstein's Zeta-Function." *J. Reine. Angew. Math.* **227**, 86–110, 1967.
Watson, G. N. "Some Singular Moduli (1)." *Quart. J. Math.* **3**, 81–98, 1932.

Elliptic Logarithm

A generalization of integrals of the form

$$\int_\infty^x \frac{dt}{\sqrt{t^2+at}},$$

which can be expressed in terms of logarithmic and inverse trigonometric functions to

$$\operatorname{eln}(x) \equiv \int_x^\infty \frac{dt}{\sqrt{t^3+at^2+bt}}.$$

The inverse of the elliptic logarithm is the ELLIPTIC EXPONENTIAL FUNCTION.

Elliptic Modular Function

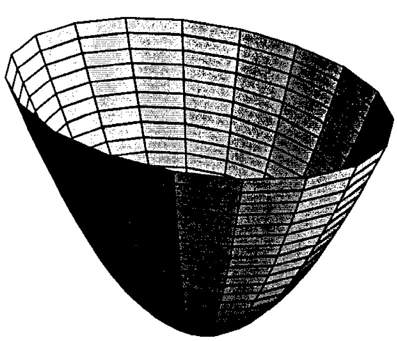
$$\varphi(z) = \left[\frac{{\vartheta_2}^4(0,z)}{{\vartheta_3}^4(0,z)}\right]^{1/8},$$

where ϑ is a THETA FUNCTION. A special case is

$$\varphi(-e^{-\pi\sqrt{3}}) = (4\sqrt{3}-7)^{1/8}.$$

see also MODULAR FUNCTION

Elliptic Paraboloid



A QUADRATIC SURFACE which has ELLIPTICAL CROSS-SECTION. The elliptic paraboloid of height h, SEMIMAJOR AXIS a, and SEMIMINOR AXIS b can be specified parametrically by

$$x = a\sqrt{u}\cos v$$
$$y = b\sqrt{u}\sin v$$
$$z = u.$$

for $v \in [0, 2\pi)$ and $u \in [0, h]$.

see also ELLIPTIC CONE, ELLIPTIC CYLINDER, PARABOLOID

References
Fischer, G. (Ed.). Plate 66 in *Mathematische Modelle/Mathematical Models, Bildband/Photograph Volume.* Braunschweig, Germany: Vieweg, p. 61, 1986.

Elliptic Partial Differential Equation

A second-order PARTIAL DIFFERENTIAL EQUATION, i.e., one of the form

$$Au_{xx} + 2Bu_{xy} + Cu_{yy} + Du_x + Eu_y + F = 0, \quad (1)$$

is called elliptic if the MATRIX

$$\mathsf{Z} \equiv \begin{bmatrix} A & B \\ B & C \end{bmatrix} \quad (2)$$

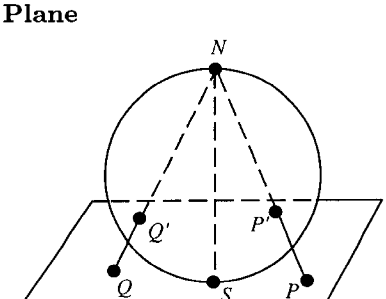
is POSITIVE DEFINITE. LAPLACE'S EQUATION and POISSON'S EQUATION are examples of elliptic partial differential equations. For an elliptic partial differential equation, BOUNDARY CONDITIONS are used to give the constraint $u(x,y) = g(x,y)$ on $\partial\Omega$, where

$$u_{xx} + u_{yy} = f(u_x, u_y, u, x, y) \quad (3)$$

holds in Ω.

see also HYPERBOLIC PARTIAL DIFFERENTIAL EQUATION, PARABOLIC PARTIAL DIFFERENTIAL EQUATION, PARTIAL DIFFERENTIAL EQUATION

Elliptic Plane



The REAL PROJECTIVE PLANE with elliptic METRIC where the distance between two points P and Q is defined as the RADIAN ANGLE between the projection of the points on the surface of a SPHERE (which is tangent to the plane at a point S) from the ANTIPODE N of the tangent point.

References
Coxeter, H. S. M. *Introduction to Geometry, 2nd ed.* New York: Wiley, p. 94, 1969.

Elliptic Point

A point $\mathbf{p}$ on a REGULAR SURFACE $M \in \mathbb{R}^3$ is said to be elliptic if the GAUSSIAN CURVATURE $K(\mathbf{p}) > 0$ or equivalently, the PRINCIPAL CURVATURES κ_1 and κ_2 have the same sign.

see also ANTICLASTIC, ELLIPTIC FIXED POINT (DIFFERENTIAL EQUATIONS), ELLIPTIC FIXED POINT (MAP), GAUSSIAN CURVATURE, HYPERBOLIC POINT, PARABOLIC POINT, PLANAR POINT, SYNCLASTIC

References

Gray, A. *Modern Differential Geometry of Curves and Surfaces.* Boca Raton, FL: CRC Press, p. 280, 1993.

Elliptical Projection

see MOLLWEIDE PROJECTION

Elliptic Pseudoprime

Let E be an ELLIPTIC CURVE defined over the FIELD of RATIONAL NUMBERS $\mathbb{Q}(\sqrt{-d})$ having equation

$$y^2 = x^3 + ax + b$$

with a and b INTEGERS. Let P be a point on E with integer coordinates and having infinite order in the additive group of rational points of E, and let n be a COMPOSITE NATURAL NUMBER such that $(-d/n) = -1$, where $(-d/n)$ is the JACOBI SYMBOL. Then if

$$(n+1)P \equiv 0 \pmod{n},$$

n is called an elliptic pseudoprime for (E, P).

see also ATKIN-GOLDWASSER-KILIAN-MORAIN CERTIFICATE, ELLIPTIC CURVE PRIMALITY PROVING, STRONG ELLIPTIC PSEUDOPRIME

References

Balasubramanian, R. and Murty, M. R. "Elliptic Pseudoprimes. II." Submitted.

Gordon, D. M. "The Number of Elliptic Pseudoprimes." *Math. Comput.* **52**, 231–245, 1989.

Gordon, D. M. "Pseudoprimes on Elliptic Curves." In *Théorie des nombres* (Ed. J. M. DeKoninck and C. Levesque). Berlin: de Gruyter, pp. 290–305, 1989.

Miyamoto, I. and Murty, M. R. "Elliptic Pseudoprimes." *Math. Comput.* **53**, 415–430, 1989.

Ribenboim, P. *The New Book of Prime Number Records, 3rd ed.* New York: Springer-Verlag, pp. 132–134, 1996.

Elliptic Rotation

Leaves the CIRCLE

$$x^2 + y^2 = 1$$

invariant.

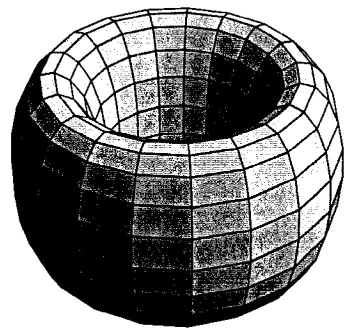
$$x' = x\cos\theta - y\sin\theta$$
$$y' = x\sin\theta + y\sin\theta.$$

see also EQUIAFFINITY

Elliptic Theta Function

see NEVILLE THETA FUNCTION, THETA FUNCTION

Elliptic Torus



A generalization of the ring TORUS produced by stretching or compressing in the z direction. It is given by the parametric equations

$$x(u,v) = (a + b\cos v)\cos u$$
$$y(u,v) = (a + b\cos v)\sin u$$
$$z(u,v) = c\sin v.$$

see also TORUS

References

Gray, A. "Tori." §11.4 in *Modern Differential Geometry of Curves and Surfaces.* Boca Raton, FL: CRC Press, pp. 218–220, 1993.

Elliptic Umbilic Catastrophe

A CATASTROPHE which can occur for three control factors and two behavior axes.

see also HYPERBOLIC UMBILIC CATASTROPHE

Ellipticity

Given a SPHEROID with equatorial radius a and polar radius c,

$$e \equiv \begin{cases} \sqrt{\frac{a^2-c^2}{a^2}} & a > c \quad \text{(oblate spheroid)} \\ \sqrt{\frac{c^2-a^2}{a^2}}. & a < c \quad \text{(prolate spheroid)} \end{cases}$$

see also FLATTENING, OBLATE SPHEROID, PROLATE SPHEROID, SPHEROID

Ellison–Mendès-France Constant

$$\sum_{n\leq x} \frac{1}{n} \ln\left(\frac{x}{n}\right) = \tfrac{1}{2}(\ln x)^2 + \gamma \ln x + D + \mathcal{O}(x^{-1}),$$

where γ is the EULER-MASCHERONI CONSTANT, and

$$D = 2.723\ldots$$

is the Ellision-Mendès-France constant.

References

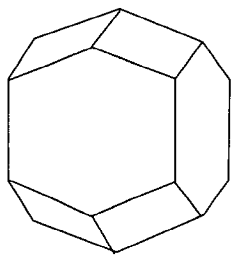
Ellison, W. J. and Mendès-France, M. *Les nombres premiers.* Paris: Hermann, 1975.

Le Lionnais, F. *Les nombres remarquables.* Paris: Hermann, p. 47, 1983.

Elongated Cupola
A n-gonal CUPOLA adjoined to a $2n$-gonal PRISM.

see also ELONGATED PENTAGONAL CUPOLA, ELONGATED SQUARE CUPOLA, ELONGATED TRIANGULAR CUPOLA

Elongated Dipyramid
see also ELONGATED PENTAGONAL DIPYRAMID, ELONGATED SQUARE DIPYRAMID, ELONGATED TRIANGULAR DIPYRAMID

Elongated Dodecahedron


A SPACE-FILLING POLYHEDRON and PARALLELOHEDRON.

References
Coxeter, H. S. M. *Regular Polytopes, 3rd ed.* New York: Dover, pp. 29–30 and 257, 1973.

Elongated Gyrobicupola
see ELONGATED PENTAGONAL GYROBICUPOLA, ELONGATED SQUARE GYROBICUPOLA, ELONGATED TRIANGULAR GYROBICUPOLA

Elongated Gyrocupolarotunda
see ELONGATED PENTAGONAL GYROCUPOLAROTUNDA

Elongated Orthobicupola
see ELONGATED PENTAGONAL ORTHOBICUPOLA, ELONGATED TRIANGULAR ORTHOBICUPOLA

Elongated Orthobirotunda
see ELONGATED PENTAGONAL ORTHOBIROTUNDA

Elongated Orthocupolarotunda
see ELONGATED PENTAGONAL ORTHOCUPOLAROTUNDA

Elongated Pentagonal Cupola
see JOHNSON SOLID

Elongated Pentagonal Dipyramid
see JOHNSON SOLID

Elongated Pentagonal Gyrobicupola
see JOHNSON SOLID

Elongated Pentagonal Gyrobirotunda
see JOHNSON SOLID

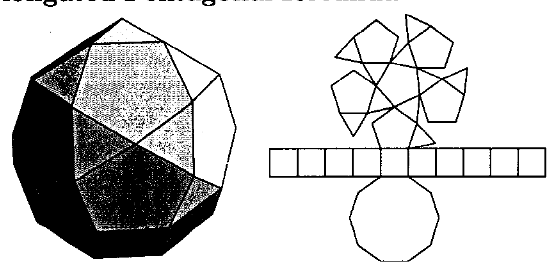
Elongated Pentagonal Gyrocupolarotunda
see JOHNSON SOLID

Elongated Pentagonal Orthobicupola
see JOHNSON SOLID

Elongated Pentagonal Orthobirotunda
see JOHNSON SOLID

Elongated Pentagonal Orthocupolarotunda
see JOHNSON SOLID

Elongated Pentagonal Pyramid
see JOHNSON SOLID

Elongated Pentagonal Rotunda


A PENTAGONAL ROTUNDA adjoined to a decagonal PRISM which is JOHNSON SOLID J_{21}.

Elongated Pyramid
An n-gonal PYRAMID adjoined to an n-gonal PRISM.

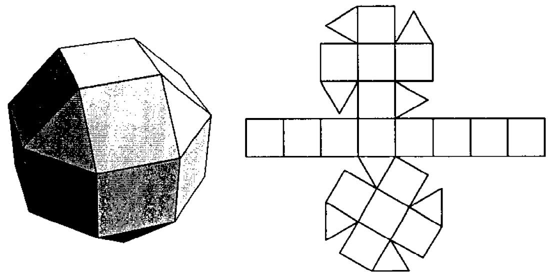
see also ELONGATED PENTAGONAL PYRAMID, ELONGATED SQUARE PYRAMID, ELONGATED TRIANGULAR PYRAMID, GYROELONGATED PYRAMID

Elongated Rotunda
see ELONGATED PENTAGONAL ROTUNDA

Elongated Square Cupola
see JOHNSON SOLID

Elongated Square Dipyramid
see JOHNSON SOLID

Elongated Square Gyrobicupola



A nonuniform POLYHEDRON obtained by rotating the bottom third of a SMALL RHOMBICUBOCTAHEDRON (Ball and Coxeter 1987, p. 137). It is also called MILLER'S SOLID, the MILLER-AŠKINUZE SOLID, or the PSEUDORHOMBICUBOCTAHEDRON, and is JOHNSON SOLID J_{37}.

see also SMALL RHOMBICUBOCTAHEDRON

References
Aškinuze, V. G. "O čisle polupravil'nyh mnogogrannikov." *Math. Prosvešč.* **1**, 107–118, 1957.
Ball, W. W. R. and Coxeter, H. S. M. *Mathematical Recreations and Essays, 13th ed.* New York: Dover, pp. 137–138, 1987.
Cromwell, P. R. *Polyhedra.* New York: Cambridge University Press, pp. 91–92, 1997.

Elongated Square Pyramid

see JOHNSON SOLID

Elongated Triangular Cupola

see JOHNSON SOLID

Elongated Triangular Dipyramid

see JOHNSON SOLID

Elongated Triangular Gyrobicupola

see JOHNSON SOLID

Elongated Triangular Orthobicupola

see JOHNSON SOLID

Elongated Triangular Pyramid

see JOHNSON SOLID

Elsasser Function

$$E(y,u) \equiv \int_{-1/2}^{1/2} \exp\left[-\frac{2\pi y u \sinh(2\pi y)}{\cosh(2\pi y) - \cos(2\pi x)}\right] dx.$$

Embeddable Knot

A KNOT K is an n-embeddable knot if it can be placed on a GENUS n standard embedded surface without crossings, but K cannot be placed on any standardly embedded surface of lower GENUS without crossings. Any KNOT is an n-embeddable knot for some n. The FIGURE-OF-EIGHT KNOT is a 2-EMBEDDABLE KNOT. A knot with BRIDGE NUMBER b is an n-embeddable knot where $n \leq b$.

see also TUNNEL NUMBER

Embedding

see EXTRINSIC CURVATURE, HYPERBOLOID EMBEDDING, INJECTION, SPHERE EMBEDDING

Empty Set

The SET containing no elements, denoted $\varnothing$. Strangely, the empty set is both OPEN and CLOSED for any SET X and TOPOLOGY. A GROUPOID, SEMIGROUP, QUASIGROUP, RINGOID, and SEMIRING can be empty. A MONOID, GROUP, and RINGS must have at least one element, while DIVISION RINGS and FIELDS must have at least two elements.

References
Conway, J. H. and Guy, R. K. *The Book of Numbers.* New York: Springer-Verlag, p. 266, 1996.

Enantiomer

Two objects which are MIRROR IMAGES of each other are called enantiomers. The term enantiomer is synonymous with ENANTIOMORPH.

see also AMPHICHIRAL KNOT, CHIRAL, DISYMMETRIC, HANDEDNESS, MIRROR IMAGE, REFLEXIBLE

References
Ball, W. W. R. and Coxeter, H. S. M. "Polyhedra." Ch. 5 in *Mathematical Recreations and Essays, 13th ed.* New York: Dover, pp. 130–161, 1987.

Enantiomorph

see ENANTIOMER

Encoding

An encoding is a way of representing a number or expression in terms of another (usually simpler) one. However, multiple expressions can also be encoded as a single expression, as in, for example,

$$(a,b) \equiv \tfrac{1}{2}[(a+b)^2 + 3a + b]$$

which encodes a and b uniquely as a single number.

a	b	(a,b)
0	0	0
0	1	1
1	0	2
0	2	3
1	2	4
2	0	5

see also CODE, CODING THEORY

Endogenous Variable

An economic variable which is independent of the relationships determining the equilibrium levels, but nonetheless affects the equilibrium.

see also EXOGENOUS VARIABLE

References

Iyanaga, S. and Kawada, Y. (Eds.). *Encyclopedic Dictionary of Mathematics.* Cambridge, MA: MIT Press, p. 458, 1980.

Endomorphism

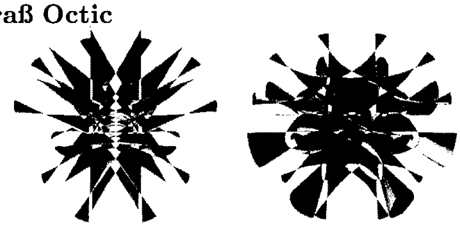
A SURJECTIVE MORPHISM from an object to itself. In ERGODIC THEORY, let X be a SET, F a SIGMA ALGEBRA on X and m a PROBABILITY MEASURE. A MAP $T : X \to X$ is called an endomorphism or MEASURE-PRESERVING TRANSFORMATION if

1. T is SURJECTIVE,
2. T is MEASURABLE,
3. $m(T^{-1}A) = m(A)$ for all $A \in F$.

An endomorphism is called ERGODIC if it is true that $T^{-1}A = A$ IMPLIES $m(A) = 0$ or 1, where $T^{-1}A = \{x \in X : T(x) \in A\}$.

see also MEASURABLE FUNCTION, MEASURE-PRESERVING TRANSFORMATION, MORPHISM, SIGMA ALGEBRA, SURJECTIVE

Endraß Octic



Endraß surfaces are a pair of OCTIC SURFACES which have 168 ORDINARY DOUBLE POINTS. This is the maximum number known to exist for an OCTIC SURFACE, although the rigorous upper bound is 174. The equations of the surfaces $X_8^\pm$ are

$$\begin{aligned}64(x^2-w^2)(y^2-w^2)[(x+y)^2-2w^2]\\ [(x-y)^2-2w^2]-\{-4(1\pm\sqrt{2})(x^2+y^2)^2\\ +[8(2\pm\sqrt{2})z^2+2(2\pm 7\sqrt{2})w^2](x^2+y^2)\\ -16z^4+8(1\mp 2\sqrt{2})z^2w^2-(1+12\sqrt{2})w^4\}^2=0,\end{aligned}$$

where w is a parameter taken as $w = 1$ in the above plots. All ORDINARY DOUBLE POINTS of X_8^+ are real, while 24 of those in X_8^- are complex. The surfaces were discovered in a 5-D family of octics with 112 nodes, and are invariant under the GROUP $D_8 \otimes Z_2$.

see also OCTIC SURFACE

References

Endraß, S. "Octics with 168 Nodes." `http://www.mathematik.uni-mainz.de/AlgebraischeGeometrie/docs/Eendrassoctic.shtml`.

Endraß, S. "Flächen mit vielen Doppelpunkten." *DMV-Mitteilungen* **4**, 17–20, 4/1995.

Endraß, S. "A Proctive Surface of Degree Eight with 168 Nodes." *J. Algebraic Geom.* **6**, 325–334, 1997.

Energy

The term energy has an important physical meaning in physics and is an extremely useful concept. A much more abstract mathematical generalization is defined as follows. Let Ω be a SPACE with MEASURE $\mu \geq 0$ and let $\Phi(P, Q)$ be a real function on the PRODUCT SPACE $\Omega \times \Omega$. When

$$\begin{aligned}(\mu, nu) &= \iint \Phi(P,Q)\,d\mu(Q)\,d\nu(P)\\ &= \int \Phi(P,\mu)\,d\nu(P)\end{aligned}$$

exists for measures $\mu, \nu \geq 0$, (μ, ν) is called the MUTUAL ENERGY and (μ, μ) is called the ENERGY.

see also DIRICHLET ENERGY, MUTUAL ENERGY

References

Iyanaga, S. and Kawada, Y. (Eds.). "General Potential." §335.B in *Encyclopedic Dictionary of Mathematics.* Cambridge, MA: MIT Press, p. 1038, 1980.

Engel's Theorem

A finite-dimensional LIE ALGEBRA all of whose elements are ad-NILPOTENT is itself a NILPOTENT LIE ALGEBRA.

Enneacontagon

A 90-sided POLYGON.

Enneacontahedron

A ZONOHEDRON constructed from the 10 diameters of the DODECAHEDRON which has 90 faces, 30 of which are RHOMBS of one type and the other 60 of which are RHOMBS of another. The enneacontahedron somewhat resembles a figure of Sharp.

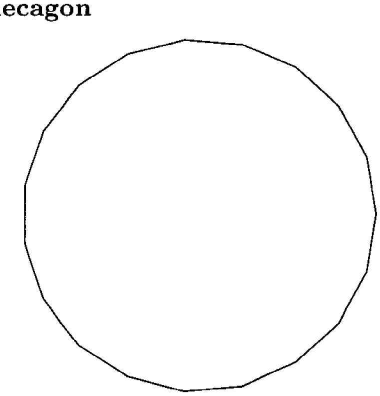
see also DODECAHEDRON, RHOMB, ZONOHEDRON

References

Ball, W. W. R. and Coxeter, H. S. M. *Mathematical Recreations and Essays, 13th ed.* New York: Dover, pp. 142–143, 1987.

Sharp, A. *Geometry Improv'd.* London, p. 87, 1717.

Enneadecagon



A 19-sided POLYGON, sometimes also called the ENNEAKAIDECAGON.

Enneagon

see NONAGON

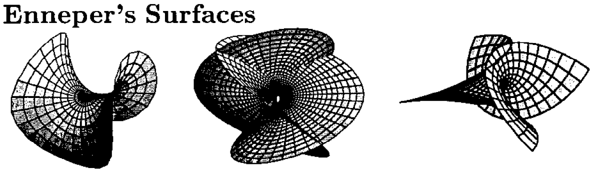
Enneagonal Number

see NONAGONAL NUMBER

Enneakaidecagon

see ENNEADECAGON

Enneper's Surfaces



The Enneper surfaces are a three-parameter family of surfaces with constant curvature. In general, they are described by elliptic functions. However, special cases which can be specified parametrically using ELEMENTARY FUNCTION include the KUEN SURFACE, REMBS' SURFACES, and SIEVERT'S SURFACE. The surfaces shown above can be generated using the ENNEPER-WEIERSTRAß PARAMETERIZATION with

$$f(\zeta) = 1 \qquad (1)$$
$$g(\zeta) = \zeta. \qquad (2)$$

Letting $z = re^{i\phi}$ and taking the REAL PART give

$$x = \Re[re^{i\phi} - \tfrac{1}{3}r^3e^{3i\phi}] \qquad (3)$$
$$y = \Re[ire^{i\phi} + \tfrac{1}{3}ir^3e^{3i\phi}] \qquad (4)$$
$$z = \Re[r^2e^{2i\phi}], \qquad (5)$$

where $r \in [0,1]$ and $\phi \in [-\pi, \pi)$. Letting $z = u + iv$ instead gives the figure on the right,

$$x = u - \tfrac{1}{3}u^3 + uv^2 \qquad (6)$$
$$y = -v - u^2v + \tfrac{1}{3}v^3 \qquad (7)$$
$$z = u^2 - v^2 \qquad (8)$$

(do Carmo 1986, Gray 1993, Nordstrand). This surface has a HOLE in its middle. Nordstrand gives the implicit form

$$\left(\frac{y^2 - x^2}{2z} + \tfrac{2}{9}z^2 + \tfrac{2}{3}\right)^3 - 6\left[\frac{(y^2 - x^2)}{4z} - \tfrac{1}{4}(x^2 + y^2 + \tfrac{8}{9}z^2) + \tfrac{2}{9}\right]^2 = 0. \qquad (9)$$

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Maeder, R. *The Mathematica Programmer.* San Diego, CA: Academic Press, pp. 150–151, 1994.
Nordstrand, T. "Enneper's Minimal Surface." `http://www.uib.no/people/nfytn/enntxt.htm`.
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Enneper-Weierstraß Parameterization

Gives a parameterization of a MINIMAL SURFACE.

$$\Re \int \begin{bmatrix} f(1-g^2) \\ if(1+g^2) \\ 2fg \end{bmatrix} d\zeta.$$

see also MINIMAL SURFACE

References
Dickson, S. "Minimal Surfaces." *Mathematica J.* **1**, 38–40, 1990.
do Carmo, M. P. *Mathematical Models from the Collections of Universities and Museums* (Ed. G. Fischer). Braunschweig, Germany: Vieweg, p. 41, 1986.
Weierstraß, K. "Über die Flächen deren mittlere Krümmung überall gleich null ist." *Monatsber. Berliner Akad.*, 612–625, 1866.

Enormous Theorem

see CLASSIFICATION THEOREM

Enriques Surfaces

An Enriques surface X is a smooth compact complex surface having irregularity $q(X) = 0$ and nontrivial canonical sheaf K_X such that $K_X^2 = O_X$ (Endraß). Such surfaces cannot be embedded in projective 3-space, but there nonetheless exist transformations onto singular surfaces in projective 3-space. There exists a family of such transformed surfaces of degree six which passes through each edge of a TETRAHEDRON twice. A subfamily with tetrahedral symmetry is given by the two-parameter (r, c) family of surfaces

$$f_r x_0x_1x_2x_3 + c({x_0}^2{x_1}^2{x_2}^2 + {x_0}^2{x_1}^2{x_3}^2 + {x_0}^2{x_2}^2{x_3}^2 + {x_1}^2{x_2}^2{x_3}^2 = 0$$

and the polynomial f_r is a sphere with radius r,

$$f_r = (3 - r)({x_0}^2 + {x_1}^2 + {x_2}^2 + {x_3}^2) - 2(1 + r)(x_0x_1 + x_0x_2 + x_0x_3 + x_1x_2 + x_1x_3 + x_2x_3)$$

(Endraß).

References
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Barth, W. and Peters, C. "Automorphisms of Enriques Surfaces." *Invent. Math.* **73**, 383–411, 1983.
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Enriques, F. *Le superficie algebriche.* Bologna, Italy: Zanichelli, 1949.
Enriques, F. "Sulla classificazione." *Atti Accad. Naz. Lincei* **5**, 1914.
Hunt, B. *The Geometry of Some Special Arithmetic Quotients.* New York: Springer-Verlag, p. 317, 1996.

Entire Function

If a function is ANALYTIC on $\mathbb{C}^*$, where $\mathbb{C}^*$ denotes the extended COMPLEX PLANE, then it is said to be entire.

see also ANALYTIC FUNCTION, HOLOMORPHIC FUNCTION, MEROMORPHIC

Entringer Number

The Entringer numbers $E(n,k)$ are the number of PERMUTATIONS of $\{1,2,\ldots,n+1\}$, starting with $k+1$, which, after initially falling, alternately fall then rise. The Entringer numbers are given by

$$E(0,0) = 1$$
$$E(n,0) = 0$$

together with the RECURRENCE RELATION

$$E(n,k) = E(n,k+1) + E(n-1,n-k).$$

The numbers $E(n) = E(n,n)$ are the SECANT and TANGENT NUMBERS given by the MACLAURIN SERIES

$$\sec x + \tan x = A_0 + A_1 x + A_2 \frac{x^2}{2!} + A_3 \frac{x^3}{3!} + A_4 \frac{x^4}{4!} + A_5 \frac{x^5}{5!} + \ldots.$$

see also ALTERNATING PERMUTATION, BOUSTROPHEDON TRANSFORM, EULER ZIGZAG NUMBER, PERMUTATION, SECANT NUMBER, SEIDEL-ENTRINGER-ARNOLD TRIANGLE, TANGENT NUMBER, ZAG NUMBER, ZIG NUMBER

References
Entringer, R. C. "A Combinatorial Interpretation of the Euler and Bernoulli Numbers." *Nieuw. Arch. Wisk.* **14**, 241–246, 1966.
Millar, J.; Sloane, N. J. A.; and Young, N. E. "A New Operation on Sequences: The Boustrophedon Transform." *J. Combin. Th. Ser. A* **76**, 44–54, 1996.
Poupard, C. "De nouvelles significations enumeratives des nombres d'Entringer." *Disc. Math.* **38**, 265–271, 1982.

Entropy

In physics, the word entropy has important physical implications as the amount of "disorder" of a system. In mathematics, a more abstract definition is used. The (Shannon) entropy of a variable X is defined as

$$H(X) \equiv -\sum_x p(x) \ln[p(x)],$$

where $p(x)$ is the probability that X is in the state x, and $p \ln p$ is defined as 0 if $p = 0$. The joint entropy of variables $X_1, \ldots, X_n$ is then defined by

$$H(X_1, \ldots, X_n) \equiv -\sum_{x_1} \cdots \sum_{x_n} p(x_1, \ldots, x_n) \ln[p(x_1, \ldots, x_n)].$$

see also KOLMOGOROV ENTROPY, KOLMOGOROV-SINAI ENTROPY, MAXIMUM ENTROPY METHOD, METRIC ENTROPY, ORNSTEIN'S THEOREM, REDUNDANCY, SHANNON ENTROPY, TOPOLOGICAL ENTROPY

References
Ott, E. "Entropies." §4.5 in *Chaos in Dynamical Systems.* New York: Cambridge University Press, pp. 138–144, 1993.

Entscheidungsproblem

see DECISION PROBLEM

Enumerative Geometry

Schubert's application of the CONSERVATION OF NUMBER PRINCIPLE.

see also CONSERVATION OF NUMBER PRINCIPLE, DUALITY PRINCIPLE, HILBERT'S PROBLEMS, PERMANENCE OF MATHEMATICAL RELATIONS PRINCIPLE

References
Bell, E. T. *The Development of Mathematics, 2nd ed.* New York: McGraw-Hill, p. 340, 1945.

Envelope

The envelope of a one-parameter family of curves given implicitly by

$$U(x,y,c) = 0, \tag{1}$$

or in parametric form by $(f(t,c), g(t,c))$, is a curve which touches every member of the family. For a curve represented by $(f(t,c), g(t,c))$, the envelope is found by solving

$$0 = \frac{\partial f}{\partial t}\frac{\partial g}{\partial c} - \frac{\partial f}{\partial c}\frac{\partial g}{\partial t}. \tag{2}$$

For a curve represented implicitly, the envelope is given by simultaneously solving

$$\frac{\partial U}{\partial c} = 0 \tag{3}$$

$$U(x,y,c) = 0. \tag{4}$$

see also ASTROID, CARDIOID, CATACAUSTIC, CAUSTIC, CAYLEYIAN CURVE, DÜRER'S CONCHOID, ELLIPSE ENVELOPE, ENVELOPE THEOREM, EVOLUTE, GLISSETTE, HEDGEHOG, KIEPERT'S PARABOLA, LINDELOF'S THEOREM, NEGATIVE PEDAL CURVE

References

Lawrence, J. D. *A Catalog of Special Plane Curves.* New York: Dover, pp. 33–34, 1972.

Lee, X. "Envelope." http://www.best.com/~xah/SpecialPlaneCurves_dir/Envelope_dir/envelope.html.

Yates, R. C. "Envelopes." *A Handbook on Curves and Their Properties.* Ann Arbor, MI: J. W. Edwards, pp. 75–80, 1952.

Envelope Theorem

Relates EVOLUTES to single paths in the CALCULUS OF VARIATIONS. Proved in the general case by Darboux and Zermelo (1894) and Kneser (1898). It states: "When a single parameter family of external paths from a fixed point O has an ENVELOPE, the integral from the fixed point to any point A on the ENVELOPE equals the integral from the fixed point to any second point B on the ENVELOPE plus the integral along the envelope to the first point on the ENVELOPE, $J_{OA} = J_{OB} + J_{BA}$."

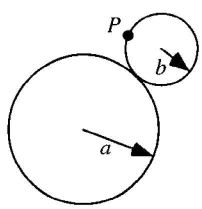
References

Kimball, W. S. *Calculus of Variations by Parallel Displacement.* London: Butterworth, p. 292, 1952.

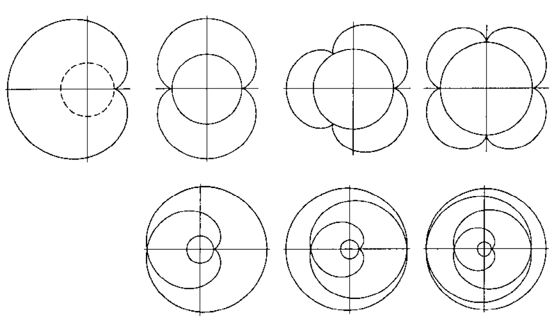
Envyfree

An agreement in which all parties feel as if they have received the best deal.

Epicycloid



The path traced out by a point P on the EDGE of a CIRCLE of RADIUS b rolling on the outside of a CIRCLE of RADIUS a.



It is given by the equations

$$x = (a+b)\cos\phi - b\cos\left(\frac{a+b}{b}\phi\right) \quad (1)$$

$$y = (a+b)\sin\phi - b\sin\left(\frac{a+b}{b}\phi\right) \quad (2)$$

$$x^2 = (a+b)^2\cos^2\phi - 2b(a+b)\cos\phi\cos\left(\frac{a+b}{b}\phi\right) + b^2\cos^2\left(\frac{a+b}{b}\phi\right) \quad (3)$$

$$y^2 = (a+b)^2\sin^2\phi - 2b(a+b)\sin\phi\sin\left(\frac{a+b}{b}\phi\right) + b^2\sin^2\left(\frac{a+b}{b}\phi\right) \quad (4)$$

$$\begin{aligned} r^2 = x^2 + y^2 = (a+b)^2 + b^2 \\ - 2b(a+b)\left\{\cos\left[\left(\frac{a}{b}+1\right)\phi\right]\cos\phi \right. \\ \left. + \sin\left[\left(\frac{a}{b}+1\right)\phi\right]\sin\phi\right\}. \end{aligned} \quad (5)$$

But

$$\cos\alpha\cos\beta + \sin\alpha\sin\beta = \cos(\alpha-\beta), \quad (6)$$

so

$$\begin{aligned} r^2 &= (a+b)^2 + b^2 - 2b(a+b)\cos\left[\left(\frac{a}{b}+1\right)\phi - \phi\right] \\ &= (a+b)^2 + b^2 - 2b(a+b)\cos\left(\frac{a}{b}\phi\right). \end{aligned} \quad (7)$$

Note that ϕ is the parameter here, *not* the polar angle. The polar angle from the center is

$$\tan\theta = \frac{y}{x} = \frac{(a+b)\sin\phi - b\sin\left(\frac{a+b}{b}\phi\right)}{(a+b)\cos\phi - b\cos\left(\frac{a+b}{b}\phi\right)}. \quad (8)$$

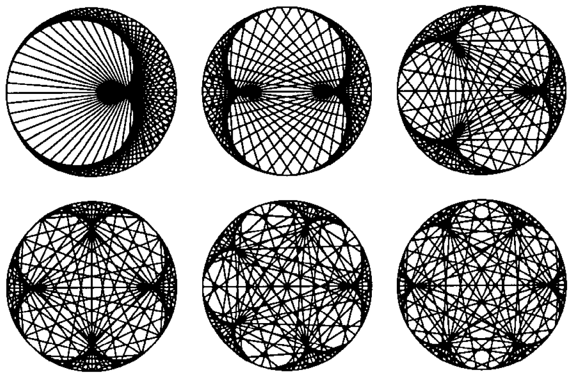
To get n CUSPS in the epicycloid, $b = a/n$, because then n rotations of b bring the point on the edge back to its starting position.

$$\begin{aligned} r^2 &= a^2\left[\left(1+\frac{1}{n}\right)^2 + \left(\frac{1}{n}\right)^2 - 2\left(\frac{1}{n}\right)\left(1+\frac{1}{n}\right)\cos(n\phi)\right] \\ &= a^2\left[1+\frac{2}{n}+\frac{1}{n^2}+\frac{1}{n^2} - \left(\frac{2}{n}\right)\left(\frac{n+1}{n}\right)\cos(n\phi)\right] \\ &= a^2\left[\frac{n^2+2n+2}{n^2} - \frac{2(n+1)}{n^2}\cos(n\phi)\right] \\ &= \frac{a^2}{n^2}\left[(n^2+2n+2) - 2(n+1)\cos(n\phi)\right], \end{aligned} \quad (9)$$

so

$$\begin{aligned} \tan\theta &= \frac{a\left(\frac{n+1}{n}\right)\sin\phi - \frac{a}{n}\sin[(n+1)\phi]}{a\left(\frac{n+1}{n}\right)\cos\phi - \frac{a}{n}\cos[(n+1)\phi]} \\ &= \frac{(n+1)\sin\phi - \sin[(n+1)\phi]}{(n+1)\cos\phi - \cos[(n+1)\phi]}. \end{aligned} \quad (10)$$

An epicycloid with one cusp is called a CARDIOID, one with two cusps is called a NEPHROID, and one with five cusps is called a RANUNCULOID.



n-epicycloids can also be constructed by beginning with the DIAMETER of a CIRCLE, offsetting one end by a series of steps while at the same time offsetting the other end by steps n times as large. After traveling around the CIRCLE once, an n-cusped epicycloid is produced, as illustrated above (Madachy 1979).

Epicycloids have TORSION

$$\tau = 0 \tag{11}$$

and satisfy

$$\frac{s^2}{a^2} + \frac{\rho^2}{b^2} = 1, \tag{12}$$

where ρ is the RADIUS OF CURVATURE ($1/\kappa$).

see also CARDIOID, CYCLIDE, CYCLOID, EPICYCLOID—1-CUSPED, HYPOCYCLOID, NEPHROID, RANUNCULOID

References

Bogomolny, A. "Cycloids." http://www.cut-the-knot.com/pythagoras/cycloids.html.

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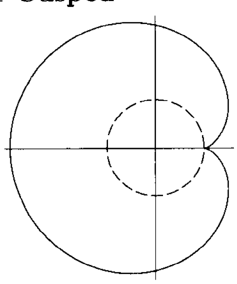
MacTutor History of Mathematics Archive. "Epicycloid." http://www-groups.dcs.st-and.ac.uk/~history/Curves/Epicycloid.html.

Madachy, J. S. *Madachy's Mathematical Recreations.* New York: Dover, pp. 219–225, 1979.

Wagon, S. *Mathematica in Action.* New York: W. H. Freeman, pp. 50–52, 1991.

Yates, R. C. "Epi- and Hypo-Cycloids." *A Handbook on Curves and Their Properties.* Ann Arbor, MI: J. W. Edwards, pp. 81–85, 1952.

Epicycloid—1-Cusped



A 1-cusped epicycloid has $b = a$, so $n = 1$. The radius measured from the center of the large circle for a 1-cusped epicycloid is given by EPICYCLOID equation (9) with $n = 1$ so

$$\begin{aligned} r^2 &= \frac{a^2}{n^2}\left[(n^2+2n+2) - 2(n+1)\cos(n\phi)\right] \\ &= a^2[(1^2+2\cdot 1+2) - 2(1+1)\cos(1\cdot\phi)] \\ &= a^2(5-4\cos\phi) \end{aligned} \tag{1}$$

$$r = a\sqrt{5-4\cos\phi}, \tag{2}$$

and

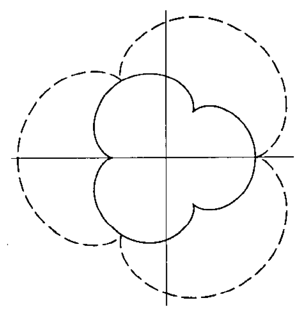
$$\tan\theta = \frac{2\sin\phi - \sin(2\phi)}{2\cos\phi - \cos(2\phi)}. \tag{3}$$

The 1-cusped epicycloid is just an offset CARDIOID.

Epicycloid—2-Cusped

see NEPHROID

Epicycloid Evolute



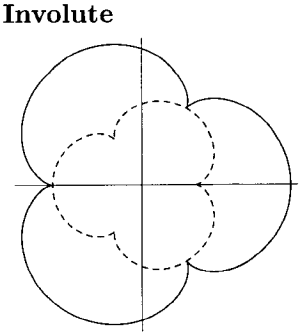
The EVOLUTE of the EPICYCLOID

$$x = (a+b)\cos t - b\cos\left[\left(\frac{a+b}{b}\right)t\right]$$
$$y = (a+b)\sin t - b\sin\left[\left(\frac{a+b}{b}\right)t\right]$$

is another EPICYCLOID given by

$$x = \frac{a}{a+2b}\left\{(a+b)\cos t + b\cos\left[\left(\frac{a+b}{b}\right)t\right]\right\}$$
$$y = \frac{a}{a+2b}\left\{(a+b)\sin t + b\cos\left[\left(\frac{a+b}{b}\right)t\right]\right\}.$$

Epicycloid Involute



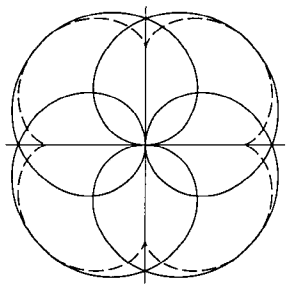
The INVOLUTE of the EPICYCLOID

$$x = (a+b)\cos t - b\cos\left[\left(\frac{a+b}{b}\right)t\right]$$
$$y = (a+b)\sin t - b\sin\left[\left(\frac{a+b}{b}\right)t\right]$$

is another EPICYCLOID given by

$$x = \frac{a+2b}{a}\left\{(a+b)\cos t + b\cos\left[\left(\frac{a+b}{b}\right)t\right]\right\}$$
$$y = \frac{a+2b}{a}\left\{(a+b)\sin t + b\cos\left[\left(\frac{a+b}{b}\right)t\right]\right\}.$$

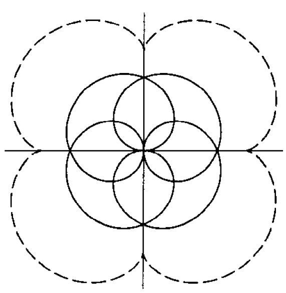
Epicycloid Pedal Curve



The PEDAL CURVE of an EPICYCLOID with PEDAL POINT at the center, shown for an epicycloid with four cusps, is not a ROSE as claimed by Lawrence (1972).

References
Lawrence, J. D. *A Catalog of Special Plane Curves.* New York: Dover, p. 204, 1972.

Epicycloid Radial Curve



The RADIAL CURVE of an EPICYCLOID is shown above for an epicycloid with four cusps. It is not a ROSE, as claimed by Lawrence (1972).

References
Lawrence, J. D. *A Catalog of Special Plane Curves.* New York: Dover, p. 202, 1972.

Epimenides Paradox

A PARADOX, also called the LIAR'S PARADOX, attributed to the philosopher Epimenides in the sixth century BC. "All Cretans are liers... One of their own poets has said so." A sharper version of the paradox is the EUBULIDES PARADOX, "This statement is false."

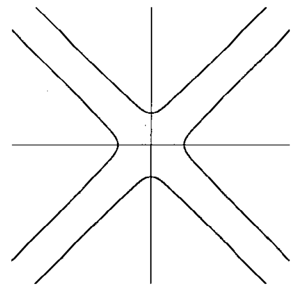
see also EUBULIDES PARADOX, SOCRATES' PARADOX

References
Hofstadter, D. R. *Gödel, Escher, Bach: An Eternal Golden Braid.* New York: Vintage Books, p. 17, 1989.

Epimorphism

A SURJECTIVE MORPHISM.

Epispiral



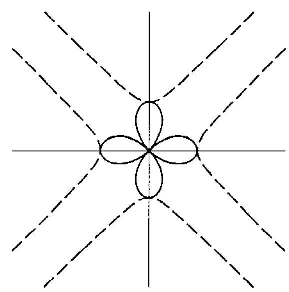
A plane curve with polar equation

$$r = \frac{a}{\cos(n\theta)}.$$

There are n sections if n is ODD and $2n$ if n is EVEN.

References
Lawrence, J. D. *A Catalog of Special Plane Curves.* New York: Dover, pp. 192–193, 1972.

Epispiral Inverse Curve



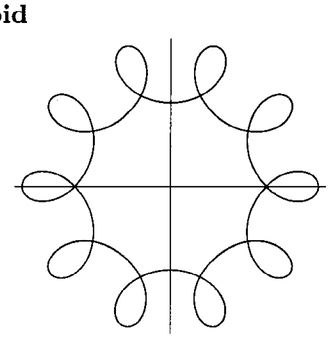
The INVERSE CURVE of the EPISPIRAL

$$r = a\sec(nt)$$

with INVERSION CENTER at the origin and inversion radius k is the ROSE

$$r = \frac{k\cos(nt)}{a}.$$

Epitrochoid



The ROULETTE traced by a point P attached to a CIRCLE of radius b rolling around the outside of a fixed

CIRCLE of radius a. These curves were studied by Dürer (1525), Desargues (1640), Huygens (1679), Leibniz, Newton (1686), L'Hospital (1690), Jakob Bernoulli (1690), la Hire (1694), Johann Bernoulli (1695), Daniel Bernoulli (1725), Euler (1745, 1781). An epitrochoid appears in Dürer's work *Instruction in Measurement with Compasses and Straight Edge* (1525). He called epitrochoids SPIDER LINES because the lines he used to construct the curves looked like a spider.

The parametric equations for an epitrochoid are

$$x = m\cos t - h\cos\left(\frac{m}{b}t\right)$$
$$y = m\sin t - h\sin\left(\frac{m}{b}t\right),$$

where $m \equiv a + b$ and h is the distance from P to the center of the rolling CIRCLE. Special cases include the LIMAÇON with $a = b$, the CIRCLE with $a = 0$, and the EPICYCLOID with $h = b$.

see also EPICYCLOID, HYPOTROCHOID, SPIROGRAPH

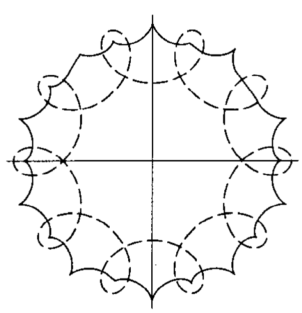
References

Lawrence, J. D. *A Catalog of Special Plane Curves.* New York: Dover, pp. 168–170, 1972.

Lee, X. "Epitrochoid." http://www.best.com/~xah/SpecialPlaneCurves_dir/Epitrochoid_dir/epitrochoid.html.

Lee, X. "Epitrochoid and Hypotrochoid Movie Gallery." http://www.best.com/~xah/SpecialPlaneCurves_dir/EpiHypoTMovieGallery_dir/epiHypoTMovieGallery.html.

Epitrochoid Evolute



Epsilon

In mathematics, a small POSITIVE INFINITESIMAL quantity whose LIMIT is usually taken to be 0. The late mathematician P. Erdős also used the term "epsilons" to refer to children.

Epsilon-Neighborhood

see NEIGHBORHOOD

Epstein Zeta Function

$$Z\begin{vmatrix}\mathbf{g}\\ \mathbf{h}\end{vmatrix}(q;s) = \sum_{\mathbf{l}} \frac{e^{-2\pi i\mathbf{h}\cdot\mathbf{l}}}{[q(\mathbf{l}+\mathbf{g})]^{s/2}},$$

where $\mathbf{g}$ and $\mathbf{h}$ are arbitrary VECTORS, the SUM runs over a d-dimensional LATTICE, and $\mathbf{l} = -\mathbf{g}$ is omitted if $\mathbf{g}$ is a lattice VECTOR.

see also ZETA FUNCTION

References

Glasser, M. L. and Zucker, I. J. "Lattice Sums in Theoretical Chemistry." *Theoretical Chemistry: Advances and Perspectives, Vol. 5.* New York: Academic Press, pp. 69–70, 1980.

Shanks, D. "Calculation and Applications of Epstein Zeta Functions." *Math. Comput.* **29**, 271–287, 1975.

Equal

Two quantities are said to be equal if they are, in some well-defined sense, equivalent. Equality of quantities a and b is written $a = b$.

A symbol with three horizontal line segments ($\equiv$) resembling the equals sign is used to denote both equality by definition (e.g., $A \equiv B$ means A is DEFINED to be equal to B) and CONGRUENCE (e.g., $13 \equiv 12 \pmod 1$) means 13 divided by 12 leaves a REMAINDER of 1—a fact known to all readers of analog clocks).

see also CONGRUENCE, DEFINED, DIFFERENT, EQUAL BY DEFINITION, EQUALITY, EQUIVALENT, ISOMORPHISM

Equal by Definition

see DEFINED

Equal Detour Point

The center of an outer SODDY CIRCLE. It has TRIANGLE CENTER FUNCTION

$$\alpha = 1 + \frac{2\Delta}{a(b+c-a)} = \sec(\tfrac{1}{2}A)\cos(\tfrac{1}{2}B)\cos(\tfrac{1}{2}C) + 1.$$

Given a point Y not between A and B, a detour of length

$$|AY| + |YB| - |AB|$$

is made walking from A to B via Y, the point is of equal detour if the three detours from one side to another via Y are equal. If ABC has no ANGLE $> 2\sin^{-1}(4/5)$, then the point given by the above TRILINEAR COORDINATES is the unique equal detour point. Otherwise, the ISOPERIMETRIC POINT is also equal detour.

References

Kimberling, C. "Central Points and Central Lines in the Plane of a Triangle." *Math. Mag.* **67**, 163–187, 1994.

Kimberling, C. "Isoperimetric Point and Equal Detour Point." http://www.evansville.edu/~ck6/tcenters/recent/isoper.html.

Veldkamp, G. R. "The Isoperimetric Point and the Point(s) of Equal Detour." *Amer. Math. Monthly* **92**, 546–558, 1985.

Equal Parallelians Point

The point of intersection of the three LINE SEGMENTS, each parallel to one side of a TRIANGLE and touching the other two, such that all three segments are of the same length. The TRILINEAR COORDINATES are

$$bc(ca+ab-bc):ca(ab+bc-ca):ab(bc+ca-ab).$$

References
Kimberling, C. "Equal Parallelians Point." http://www.evansville.edu/~ck6/tcenters/recent/eqparal.html.

Equality

A mathematical statement of the equivalence of two quantities. The equality "A is equal to B" is written $A = B$.

see also EQUAL, INEQUALITY

Equally Likely Outcomes Distribution

Let there be a set S with N elements, each of them having the same probability. Then

$$\begin{aligned} P(S) &= P\left(\bigcup_{i=1}^{N} E_i\right) = \sum_{i=1}^{N} P(E_i) \\ &= P(E_i)\sum_{i=1}^{N} 1 = NP(E_i). \end{aligned}$$

Using $P(S) \equiv 1$ gives

$$P(E_i) = \frac{1}{N}.$$

see also UNIFORM DISTRIBUTION

Equi-Brocard Center

The point Y for which the TRIANGLES BYC, CYA, and AYB have equal BROCARD ANGLES.

References
Kimberling, C. "Central Points and Central Lines in the Plane of a Triangle." *Math. Mag.* **67**, 163–187, 1994.

Equiaffinity

An AREA-preserving AFFINITY. Equiaffinities include the ELLIPTIC ROTATION, HYPERBOLIC ROTATION, HYPERBOLIC ROTATION (CROSSED), and PARABOLIC ROTATION.

Equiangular Spiral

see LOGARITHMIC SPIRAL

Equianharmonic Case

The case of the WEIERSTRAß ELLIPTIC FUNCTION with invariants $g_2 = 0$ and $g_3 = 1$.

see also LEMNISCATE CASE, PSEUDOLEMNISCATE CASE

References
Abramowitz, M. and Stegun, C. A. (Eds.). "Equianharmonic Case ($g_2 = 0$, $g_3 = 1$)." §18.13 in *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables, 9th printing.* New York: Dover, p. 652, 1972.

Equichordal Point

A point P for which all the CHORDS passing through P are of the same length. It satisfies

$$px + py = [\text{const}],$$

where p is the CHORD length. It is an open question whether a plane convex region can have two equichordal points.

see also EQUICHORDAL PROBLEM, EQUIPRODUCT POINT, EQUIRECIPROCAL POINT

Equichordal Problem

Is there a planar body bounded by a simple closed curve and star-shaped with respect to two interior points p and q whose point X-rays at p and q are both constant? Rychlik (1997) has answered the question in the negative.

see also EQUICHORDAL POINT

References
Rychlik, M. "The Equichordal Point Problem." *Elec. Res. Announcements Amer. Math. Soc.* **2**, 108–123, 1996.
Rychlik, M. "A Complete Solution to the Equichordal Problem of Fujiwara, Blaschke, Rothe, and Weitzenböck." *Invent. Math.* **129**, 141–212, 1997.

Equidecomposable

The ability of two plane or space regions to be DISSECTED into each other.

Equidistance Postulate

PARALLEL lines are everywhere equidistant. This POSTULATE is equivalent to the PARALLEL AXIOM.

References
Dunham, W. "Hippocrates' Quadrature of the Lune." Ch. 1 in *Journey Through Genius: The Great Theorems of Mathematics.* New York: Wiley, p. 54, 1990.

Equidistant Cylindrical Projection

see CYLINDRICAL EQUIDISTANT PROJECTION

Equidistributed Sequence

A sequence of REAL NUMBERS $\{x_n\}$ is equidistributed if the probability of finding x_n in any subinterval is proportional to the subinterval length.

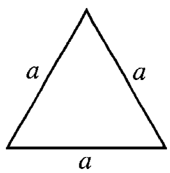
see also WEYL'S CRITERION

References
Kuipers, L. and Niederreiter, H. *Uniform Distribution of Sequences.* New York: Wiley, 1974.
Pólya, G. and Szegő, G. *Problems and Theorems in Analysis I.* New York: Springer-Verlag, p. 88, 1972.
Vardi, I. *Computational Recreations in Mathematica.* Reading, MA: Addison-Wesley, pp. 155–156, 1991.

Equilateral Hyperbola

see RECTANGULAR HYPERBOLA

Equilateral Triangle



An equilateral triangle is a TRIANGLE with all three sides of equal length s. An equilateral triangle also has three equal 60° ANGLES.

An equilateral triangle can be constructed by TRISECTING all three ANGLES of any TRIANGLE (MORLEY'S THEOREM). NAPOLEON'S THEOREM states that if three equilateral triangles are drawn on the LEGS of any TRIANGLE (either all drawn inwards or outwards) and the centers of these triangles are connected, the result is another equilateral triangle.

Given the distances of a point from the three corners of an equilateral triangle, a, b, and c, the length of a side s is given by

$$3(a^4+b^4+c^4+s^4)=(a^2+b^2+c^2+s^2)^2 \tag{1}$$

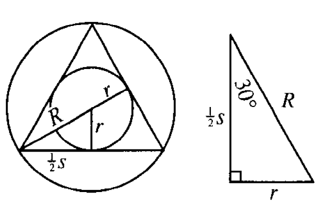
(Gardner 1977, pp. 56–57 and 63). There are infinitely many solutions for which a, b, and c are INTEGERS. In these cases, one of a, b, c, and s is DIVISIBLE by 3, one by 5, one by 7, and one by 8 (Guy 1994, p. 183).

The ALTITUDE h of an equilateral triangle is

$$h=\tfrac{1}{2}\sqrt{3}\,s, \tag{2}$$

where s is the side length, so the AREA is

$$A=\tfrac{1}{2}sh=\tfrac{1}{4}\sqrt{3}\,s^2. \tag{3}$$



The INRADIUS r, CIRCUMRADIUS R, and AREA A can be computed directly from the formulas for a general regular POLYGON with side length s and $n=3$ sides,

$$r=\tfrac{1}{2}s\cot\left(\frac{\pi}{3}\right)=\tfrac{1}{2}s\tan\left(\frac{\pi}{6}\right)=\tfrac{1}{6}\sqrt{3}\,s \tag{4}$$

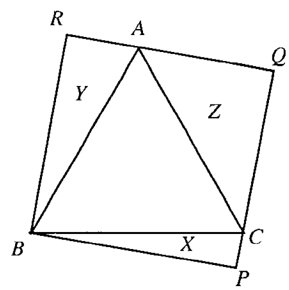
$$R=\tfrac{1}{2}s\csc\left(\frac{\pi}{3}\right)=\tfrac{1}{2}s\sec\left(\frac{\pi}{6}\right)=\tfrac{1}{3}\sqrt{3}\,s \tag{5}$$

$$A=\tfrac{1}{4}ns^2\cot\left(\frac{\pi}{3}\right)=\tfrac{1}{4}\sqrt{3}\,s^2. \tag{6}$$

The AREAS of the INCIRCLE and CIRCUMCIRCLE are

$$A_r=\pi r^2=\tfrac{1}{12}\pi s^2 \tag{7}$$

$$A_R=\pi R^2=\tfrac{1}{3}\pi s^2. \tag{8}$$

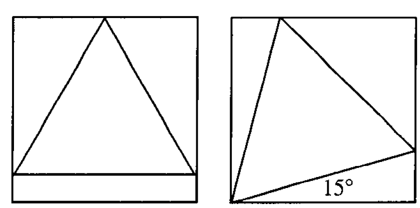


Let any RECTANGLE be circumscribed about an EQUILATERAL TRIANGLE. Then

$$X+Y=Z, \tag{9}$$

where X, Y, and Z are the AREAS of the triangles in the figure (Honsberger 1985).

Begin with an arbitrary TRIANGLE and find the EXCENTRAL TRIANGLE. Then find the EXCENTRAL TRIANGLE of that triangle, and so on. Then the resulting triangle approaches an equilateral triangle. The only RATIONAL TRIANGLE is the equilateral triangle (Conway and Guy 1996). A POLYHEDRON composed of only equilateral triangles is known as a DELTAHEDRON.



The largest equilateral triangle which can be inscribed in a UNIT SQUARE (left) has side length and area

$$s=1 \tag{10}$$

$$A=\tfrac{1}{4}\sqrt{3}. \tag{11}$$

The smallest equilateral triangle which can be inscribed (right) is oriented at an angle of 15° and has side length and area

$$s=\sec(15^\circ)=\sqrt{6}-\sqrt{2} \tag{12}$$

$$A=2\sqrt{3}-3 \tag{13}$$

(Madachy 1979).

see also ACUTE TRIANGLE, DELTAHEDRON, EQUILIC QUADRILATERAL, FERMAT POINT, GYROELONGATED SQUARE DIPYRAMID, ICOSAHEDRON, ISOGONIC CENTERS, ISOSCELES TRIANGLE, MORLEY'S THEOREM, OCTAHEDRON, PENTAGONAL DIPYRAMID, RIGHT TRIANGLE, SCALENE TRIANGLE, SNUB DISPHENOID, TETRAHEDRON, TRIANGLE, TRIANGULAR DIPYRAMID, TRIAUGMENTED TRIANGULAR PRISM, VIVIANI'S THEOREM

References

Beyer, W. H. (Ed.) *CRC Standard Mathematical Tables, 28th ed.* Boca Raton, FL: CRC Press, p. 121, 1987.

Conway, J. H. and Guy, R. K. "The Only Rational Triangle." In *The Book of Numbers.* New York: Springer-Verlag, pp. 201 and 228–239, 1996.

Dixon, R. *Mathographics.* New York: Dover, p. 33, 1991.

Gardner, M. *Mathematical Carnival: A New Round-Up of Tantalizers and Puzzles from Scientific American.* New York: Vintage Books, 1977.

Guy, R. K. "Rational Distances from the Corners of a Square." §D19 in *Unsolved Problems in Number Theory, 2nd ed.* New York: Springer-Verlag, pp. 181–185, 1994.

Honsberger, R. "Equilateral Triangles." Ch. 3 in *Mathematical Gems I.* Washington, DC: Math. Assoc. Amer., 1973.

Honsberger, R. *Mathematical Gems III.* Washington, DC: Math. Assoc. Amer., pp. 19–21, 1985.

Madachy, J. S. *Madachy's Mathematical Recreations.* New York: Dover, pp. 115 and 129–131, 1979.

Equilibrium Point

An equilibrium point in GAME THEORY is a set of strategies $\{\hat{x}_1, \ldots, \hat{x}_n\}$ such that the ith payoff function $K_i(\mathbf{x})$ is larger or equal for any other ith strategy, i.e.,

$$K_i(\hat{x}_1, \ldots, \hat{x}_n) \geq K_i(\hat{x}_1, \ldots, \hat{x}_{i-1}, x_i, \hat{x}_{i+1}, \ldots, \hat{x}_n).$$

see NASH EQUILIBRIUM

Equilic Quadrilateral

A QUADRILATERAL in which a pair of opposite sides have the same length and are inclined at 60° to each other (or equivalently, satisfy $\langle A\rangle + \langle B\rangle = 120°$). Some interesting theorems hold for such quadrilaterals. Let $ABCD$ be an equilic quadrilateral with $AD = BC$ and $\langle A\rangle + \langle B\rangle = 120°$. Then

1. The MIDPOINTS P, Q, and R of the diagonals and the side CD always determine an EQUILATERAL TRIANGLE.
2. If EQUILATERAL TRIANGLE PCD is drawn outwardly on CD, then ΔPAB is also an EQUILATERAL TRIANGLE.
3. If EQUILATERAL TRIANGLES are drawn on AC, DC, and DB away from AB, then the three new VERTICES P, Q, and R are COLLINEAR.

See Honsberger (1985) for additional theorems.

References

Garfunkel, J. "The Equilic Quadrilateral." *Pi Mu Epsilon J.*, 317–329, Fall 1981.

Honsberger, R. *Mathematical Gems III.* Washington, DC: Math. Assoc. Amer., pp. 32–35, 1985.

Equinumerous

Let A and B be two classes of POSITIVE integers. Let $A(n)$ be the number of integers in A which are less than or equal to n, and let $B(n)$ be the number of integers in B which are less than or equal to n. Then if

$$A(n) \sim B(n),$$

A and B are said to be equinumerous.

The four classes of PRIMES $8k+1$, $8k+3$, $8k+5$, $8k+7$ are equinumerous. Similarly, since $8k+1$ and $8k+5$ are both of the form $4k+1$, and $8k+3$ and $8k+7$ are both of the form $4k+3$, $4k+1$ and $4k+3$ are also equinumerous.

see also BERTRAND'S POSTULATE, CHOQUET THEORY, PRIME COUNTING FUNCTION

References

Shanks, D. *Solved and Unsolved Problems in Number Theory, 4th ed.* New York: Chelsea, pp. 21–22 and 31–32, 1993.

Equipollent

Two statements in LOGIC are said to be equipollent if they are deducible from each other. Two SETS with the same CARDINAL NUMBER are also said to be equipollent. The term EQUIPOTENT is sometimes used instead of equipollent.

Equipotent

see EQUIPOLLENT

Equipotential Curve

A curve in 2-D on which the value of a function $f(x, y)$ is a constant. Other synonymous terms are ISARITHM and ISOPLETH.

see also LEMNISCATE

Equiproduct Point

A point, such as interior points of a disk, such that

$$(px)(py) = [\text{const}],$$

where p is the CHORD length.

see also EQUICHORDAL POINT, EQUIRECIPROCAL POINT

Equireciprocal Point

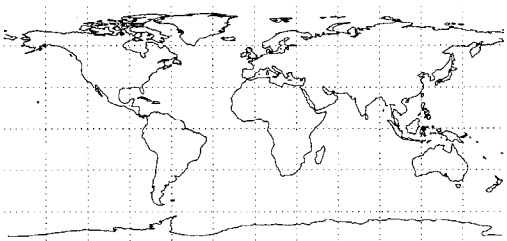
A point, such as the FOCI of an ELLIPSE, which satisfies

$$\frac{1}{px} + \frac{1}{py} = [\text{const}],$$

where p is the CHORD length.

see also EQUICHORDAL POINT, EQUIPRODUCT POINT

Equirectangular Projection



A MAP PROJECTION, also called a RECTANGULAR PROJECTION, in which the horizontal coordinate is the longitude and the vertical coordinate is the latitude.

Equiripple

A distribution of ERROR such that the ERROR remaining is always given approximately by the last term dropped.

Equitangential Curve

see TRACTRIX

Equivalence Class

An equivalence class is defined as a SUBSET of the form $\{x \in X : xRa\}$, where a is an element of X and the NOTATION "xRy" is used to mean that there is an EQUIVALENCE RELATION between x and y. It can be shown that any two equivalence classes are either equal or disjoint, hence the collection of equivalence classes forms a partition of X. For all $a, b \in X$, we have aRb IFF a and b belong to the same equivalence class.

A set of CLASS REPRESENTATIVES is a SUBSET of X which contains EXACTLY ONE element from each equivalence class.

For n a POSITIVE INTEGER, and a, b INTEGERS, consider the CONGRUENCE $a \equiv b \pmod{n}$, then the equivalence classes are the sets $\{\ldots, -2n, -n, 0, n, 2n, \ldots\}$, $\{\ldots, 1-2n, 1-n, 1, 1+n, 1+2n, \ldots\}$ etc. The standard CLASS REPRESENTATIVES are taken to be 0, 1, 2, ..., $n-1$.

see also CONGRUENCE, COSET

References

Shanks, D. *Solved and Unsolved Problems in Number Theory, 4th ed.* New York: Chelsea, pp. 56–57, 1993.

Equivalence Problem

see METRIC EQUIVALENCE PROBLEM

Equivalence Relation

An equivalence relation on a set X is a SUBSET of $X \times X$, i.e., a collection R of ordered pairs of elements of X, satisfying certain properties. Write "xRy" to mean (x, y) is an element of R, and we say "x is related to y," then the properties are

1. Reflexive: aRa for all $a \in X$,
2. Symmetric: aRb IMPLIES bRa for all $a, b \in X$
3. Transitive: aRb and bRc imply aRc for all $a, b, c \in X$,

where these three properties are completely independent. Other notations are often used to indicate a relation, e.g., $a \equiv b$ or $a \sim b$.

see also EQUIVALENCE CLASS, TEICHMÜLLER SPACE

References

Stewart, I. and Tall, D. *The Foundations of Mathematics.* Oxford, England: Oxford University Press, 1977.

Equivalent

If $A \Rightarrow B$ and $B \Rightarrow A$ (i.e, $A \Rightarrow B \wedge B \Rightarrow A$, where $\Rightarrow$ denotes IMPLIES), then A and B are said to be equivalent, a relationship which is written symbolically as $A \Leftrightarrow B$ or $A \rightleftharpoons B$. However, if A and B are "equivalent by definition" (i.e., A is DEFINED to be B), this is written $A \equiv B$, a notation which conflicts with that for a CONGRUENCE.

see also DEFINED, IFF, IMPLIES

Equivalent Matrix

An $m \times n$ MATRIX A is said to be equivalent to another $m \times n$ MATRIX B IFF

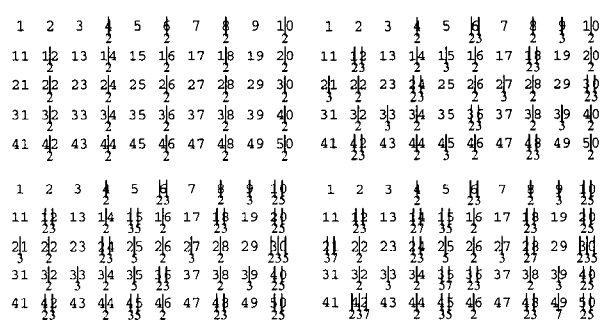
$$\mathsf{B} = \mathsf{PAQ}$$

for P and Q any $m \times n$ and $n \times n$ MATRICES, respectively.

References

Gradshteyn, I. S. and Ryzhik, I. M. *Tables of Integrals, Series, and Products, 5th ed.* San Diego, CA: Academic Press, p. 1103, 1979.

Eratosthenes Sieve



An ALGORITHM for making tables of PRIMES. Sequentially write down the INTEGERS from 2 to the highest number n you wish to include in the table. Cross out all numbers > 2 which are divisible by 2 (every second number). Find the smallest remaining number > 2. It is 3. So cross out all numbers > 3 which are divisible by 3 (every third number). Find the smallest remaining number > 3. It is 5. So cross out all numbers > 5 which are divisible by 5 (every fifth number).

Continue until you have crossed out all numbers divisible by $\lfloor \sqrt{n} \rfloor$, where $\lfloor x \rfloor$ is the FLOOR FUNCTION. The numbers remaining are PRIME. This procedure is illustrated in the above diagram which sieves up to 50, and therefore crosses out PRIMES up to $\lfloor\sqrt{50}\rfloor = 7$. If the procedure is then continued up to n, then the number of cross-outs gives the number of distinct PRIME factors of each number.

References
Conway, J. H. and Guy, R. K. *The Book of Numbers.* New York: Springer-Verlag, pp. 127–130, 1996.
Ribenboim, P. *The New Book of Prime Number Records.* New York: Springer-Verlag, pp. 20–21, 1996.

Erdős-Anning Theorem

If an infinite number of points in the PLANE are all separated by INTEGER distances, then all the points lie on a straight LINE.

Erdős-Kac Theorem

A deeper result than the HARDY-RAMANUJAN THEOREM. Let $N(x,a,b)$ be the number of INTEGERS in $[3,x]$ such that inequality

$$a \leq \frac{\omega(n) - \ln\ln n}{\sqrt{\ln\ln n}} \leq b$$

holds, where $\omega(n)$ is the number of different PRIME factors of n. Then

$$\lim_{x\to\infty} N(x,a,b) = \frac{(x+o(x))}{\sqrt{2\pi}} \int_a^b e^{-t^2/2}\,dt.$$

The theorem is discussed in Kac (1959).

References
Kac, M. *Statistical Independence in Probability, Analysis and Number Theory.* New York: Wiley, 1959.
Riesel, H. "The Erdős-Kac Theorem." *Prime Numbers and Computer Methods for Factorization, 2nd ed.* Boston, MA: Birkhäuser, pp. 158–159, 1994.

Erdős-Mordell Theorem

If O is any point inside a TRIANGLE ΔABC, and P, Q, and R are the feet of the perpendiculars from O upon the respective sides BC, CA, and AB, then

$$OA + OB + OC \geq 2(OP + OQ + OR).$$

Oppenheim (1961) and Mordell (1962) also showed that

$$OA \times OB \times OC \geq (OQ + OR)(OR + OP)(OP + OQ).$$

References
Bankoff, L. "An Elementary Proof of the Erdős-Mordell Theorem." *Amer. Math. Monthly* **65**, 521, 1958.
Brabant, H. "The Erdős-Mordell Inequality Again." *Nieuw Tijdschr. Wisk.* **46**, 87, 1958/1959.
Casey, J. *A Sequel to the First Six Books of the Elements of Euclid, 6th ed.* Dublin: Hodges, Figgis, & Co., p. 253, 1892.
Coxeter, H. S. M. *Introduction to Geometry, 2nd ed.* New York: Wiley, p. 9, 1969.
Erdős, P. "Problem 3740." *Amer. Math. Monthly* **42**, 396, 1935.
Fejes-Tóth, L. *Lagerungen in der Ebene auf der Kugel und im Raum.* Berlin: Springer, 1953.
Mordell, L. J. "On Geometric Problems of Erdős and Oppenheim." *Math. Gaz.* **46**, 213–215, 1962.
Mordell, L. J. and Barrow, D. F. "Solution to Problem 3740." *Amer. Math. Monthly* **44**, 252–254, 1937.
Oppenheim, A. "The Erdős Inequality and Other Inequalities for a Triangle." *Amer. Math. Monthly* **68**, 226–230 and 349, 1961.
Veldkamp, G. R. "The Erdős-Mordell Inequality." *Nieuw Tijdschr. Wisk.* **45**, 193–196, 1957/1958.

Erdős Number

An author's Erdős number is 1 if he has co-authored a paper with Erdős, 2 if he has co-authored a paper with someone who has co-authored a paper with Erdős, etc.

References
Grossman, J. and Ion, P. "The Erdős Number Project." `http://www.acs.oakland.edu/~grossman/erdoshp.html`.

Erdős Reciprocal Sum Constants

see A-SEQUENCE, $B2$-SEQUENCE, NONAVERAGING SEQUENCE

Erdős-Selfridge Function

The Erdős-Selfridge function $g(k)$ is defined as the least integer bigger than $k+1$ such that all prime factors of $\binom{g(k)}{k}$ exceed k (Ecklund *et al.* 1974). The best lower bound known is

$$g(k) \geq \exp\left(c\frac{\ln^3 k}{\ln\ln k}^{1/2}\right)$$

(Granville and Ramare 1996). Scheidler and Williams (1992) tabulated $g(k)$ up to $k=140$, and Lukes *et al.* (1997) tabulated $g(k)$ for $135 \leq k \leq 200$. The values for $n = 2, 3, \ldots$ are 4, 7, 7, 23, 62, 143, 44, 159, 46, 47, 174, 2239, ... (Sloane's A046105).

see also BINOMIAL COEFFICIENT, LEAST PRIME FACTOR

References
Ecklund, E. F. Jr.; Erdős, P.; and Selfridge, J. L. "A New Function Associated with the prime factors of $\binom{n}{k}$. *Math. Comput.* **28**, 647–649, 1974.
Erdős, P.; Lacampagne, C. B.; and Selfridge, J. L. "Estimates of the Least Prime Factor of a Binomial Coefficient." *Math. Comput.* **61**, 215–224, 1993.
Granville, A. and Ramare, O. "Explicit Bounds on Exponential Sums and the Scarcity of Squarefree Binomial Coefficients." *Mathematika* **43**, 73–107, 1996.
Lukes, R. F.; Scheidler, R.; and Williams, H. C. "Further Tabulation of the Erdős-Selfridge Function." *Math. Comput.* **66**, 1709–1717, 1997.
Scheidler, R. and Williams, H. C. "A Method of Tabulating the Number-Theoretic Function $g(k)$." *Math. Comput.* **59**, 251–257, 1992.
Sloane, N. J. A. Sequence A046105 in "An On-Line Version of the Encyclopedia of Integer Sequences."

Erdős Squarefree Conjecture

The CENTRAL BINOMIAL COEFFICIENT $\binom{2n}{n}$ is never SQUAREFREE for $n > 4$. This was proved true for all sufficiently large n by SÁRKÖZY'S THEOREM. Goetgheluck (1988) proved the CONJECTURE true for $4 < n \leq 2^{42205184}$ and Vardi (1991) for $4 < n < 2^{774840978}$. The conjecture was proved true in its entirely by Granville and Ramare (1996).

see also CENTRAL BINOMIAL COEFFICIENT

References

Erdős, P. and Graham, R. L. *Old and New Problems and Results in Combinatorial Number Theory.* Geneva, Switzerland: L'Enseignement Mathématique Université de Genève, Vol. 28, p. 71, 1980.

Goetgheluck, P. "Prime Divisors of Binomial Coefficients." *Math. Comput.* **51**, 325–329, 1988.

Granville, A. and Ramare, O. "Explicit Bounds on Exponential Sums and the Scarcity of Squarefree Binomial Coefficients." *Mathematika* **43**, 73–107, 1996.

Sander, J. W. "On Prime Divisors of Binomial Coefficients." *Bull. London Math. Soc.* **24**, 140–142, 1992.

Sander, J. W. "A Story of Binomial Coefficients and Primes." *Amer. Math. Monthly* **102**, 802–807, 1995.

Sárközy, A. "On Divisors of Binomial Coefficients. I." *J. Number Th.* **20**, 70–80, 1985.

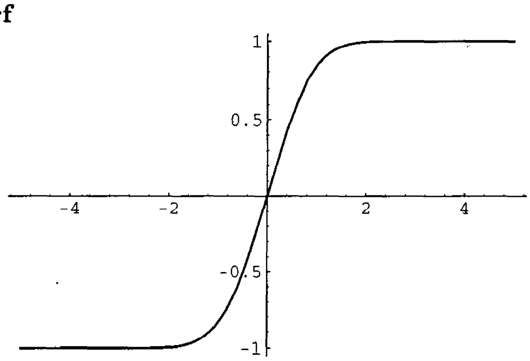
Vardi, I. "Applications to Binomial Coefficients." *Computational Recreations in Mathematica.* Reading, MA: Addison-Wesley, pp. 25–28, 1991.

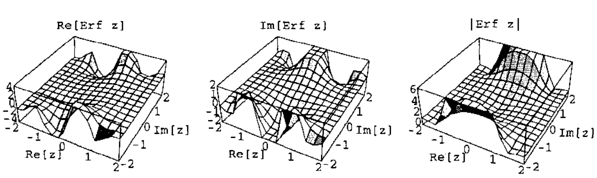
Erdős-Szekeres Theorem

Suppose $a, b \in \mathbb{N}$, $n = ab + 1$, and $x_1, \ldots, x_n$ is a sequence of n REAL NUMBERS. Then this sequence contains a MONOTONIC increasing (decreasing) subsequence of $a + 1$ terms or a MONOTONIC decreasing (increasing) subsequence of $b + 1$ terms. DILWORTH'S LEMMA is a generalization of this theorem.

see also COMBINATORICS

Erf





The "error function" encountered in integrating the GAUSSIAN DISTRIBUTION.

$$\operatorname{erf}(z) \equiv \frac{2}{\sqrt{\pi}} \int_0^z e^{-t^2}\,dt \tag{1}$$

$$= 1 - \operatorname{erfc}(z) \tag{2}$$

$$= \sqrt{\pi}\,\gamma(\tfrac{1}{2}, z^2), \tag{3}$$

where ERFC is the complementary error function and $\gamma(x, a)$ is the incomplete GAMMA FUNCTION. It can also be defined as a MACLAURIN SERIES

$$\operatorname{erf}(z) = \frac{2}{\sqrt{\pi}} \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n+1}}{n!(2n+1)}. \tag{4}$$

Erf has the values

$$\operatorname{erf}(0) = 0 \tag{5}$$

$$\operatorname{erf}(\infty) = 1. \tag{6}$$

It is an ODD FUNCTION

$$\operatorname{erf}(-z) = -\operatorname{erf}(z), \tag{7}$$

and satisfies

$$\operatorname{erf}(z) + \operatorname{erfc}(z) = 1. \tag{8}$$

Erf may be expressed in terms of a CONFLUENT HYPERGEOMETRIC FUNCTION OF THE FIRST KIND M as

$$\operatorname{erf}(z) = \frac{2z}{\sqrt{\pi}} M(\tfrac{1}{2}, \tfrac{3}{2}, -z^2) = \frac{2z}{\sqrt{\pi}} e^{-z^2} M(1, \tfrac{3}{2}, z^2). \tag{9}$$

Erf is bounded by

$$\frac{1}{x + \sqrt{x^2 + 2}} < e^{x^2} \int_x^\infty e^{-t^2}\,dt \leq \frac{1}{x + \sqrt{x^2 + \frac{4}{\pi}}}. \tag{10}$$

Its DERIVATIVE is

$$\frac{d^n}{dz^n} \operatorname{erf}(z) = (-1)^{n-1} \frac{2}{\sqrt{\pi}} H_n(z) e^{-z^2}, \tag{11}$$

where H_n is a HERMITE POLYNOMIAL. The first DERIVATIVE is

$$\frac{d}{dz} \operatorname{erf}(z) = \frac{2}{\sqrt{\pi}} e^{-z^2/2}, \tag{12}$$

and the integral is

$$\int \operatorname{erf}(z)\,dz = z \operatorname{erf}(z) + \frac{e^{-z^2}}{\sqrt{\pi}}. \tag{13}$$

For $x \ll 1$, erf may be computed from

$$\begin{aligned}
\operatorname{erf}(x) &= \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2}\,dt && (14)\\
&= \frac{2}{\sqrt{\pi}} \int_0^x \sum_{k=0}^{\infty} \frac{(-t^2)^k}{k!}\,dt\\
&= \frac{2}{\sqrt{\pi}} \int_0^x \sum_{k=0}^{\infty} \frac{(-1)^k t^{2k}}{k!}\,dt\\
&= \frac{2}{\sqrt{\pi}} \sum_{k=0}^{\infty} \frac{x^{2k+1}(-1)^k}{k!(2k+1)} && (15)\\
&= \frac{2}{\sqrt{\pi}}(x - \tfrac{1}{3}x^3 + \tfrac{1}{10}x^5 - \tfrac{1}{42}x^7 + \tfrac{1}{216}x^9\\
&\quad - \tfrac{1}{1320}x^{11} + \ldots) && (16)\\
&= \frac{2}{\sqrt{\pi}} e^{-x^2} x \left[1 + \frac{2x^2}{1\cdot 3} + \frac{(2x)^2}{1\cdot 3\cdot 5} + \ldots\right] && (17)
\end{aligned}$$

(Acton 1990). For $x \gg 1$,

$$\begin{aligned}
\operatorname{erf}(x) &= \frac{2}{\sqrt{\pi}} \left(\int_0^\infty e^{-t^2}\,dt - \int_x^\infty e^{-t^2}\,dt\right)\\
&= 1 - \frac{2}{\sqrt{\pi}} \int_x^\infty e^{-t^2}\,dt. && (18)
\end{aligned}$$

Using INTEGRATION BY PARTS gives

$$\begin{aligned}
\int_x^\infty e^{-t^2}\,dt &= -\frac{1}{2}\int_x^\infty \frac{1}{x}\,d(e^{-x^2})\\
&= -\frac{1}{2}\left[\frac{e^{-t^2}}{t}\right]_x^\infty - \frac{1}{2}\int_x^\infty \frac{e^{-t^2}\,dt}{t^2}\\
&= \frac{e^{-x^2}}{2x} + \frac{1}{4}\int_x^\infty \frac{1}{t^3}\,d(e^{-t^2})\\
&= \frac{e^{-x^2}}{2x} - \frac{e^{-x^2}}{4x^3} - \ldots, && (19)
\end{aligned}$$

so

$$\operatorname{erf}(x) = 1 - \frac{e^{-x^2}}{\sqrt{\pi}\,x}\left(1 - \frac{1}{2x^2} - \ldots\right) \qquad (20)$$

and continuing the procedure gives the ASYMPTOTIC SERIES

$$\begin{aligned}
\operatorname{erf}(x) = 1 - \frac{e^{-x^2}}{\sqrt{\pi}}&(x^{-1} - \tfrac{1}{2}x^{-3}\\
&+ \tfrac{3}{4}x^{-5} - \tfrac{15}{8}x^{-7} + \tfrac{105}{16}x^{-9} + \ldots). && (21)
\end{aligned}$$

A COMPLEX generalization of erf is defined as

$$\begin{aligned}
w(z) &\equiv e^{-z^2}\operatorname{erfc}(-iz) && (22)\\
&= e^{-z^2}\left(1 + \frac{2i}{\sqrt{\pi}} + \frac{2i}{\sqrt{\pi}}\int_0^z e^{t^2}\,dt\right) && (23)\\
&= \frac{i}{\pi}\int_{-\infty}^\infty \frac{e^{-t^2}\,dt}{z-t} = \frac{2iz}{\pi}\int_0^\infty \frac{e^{-t^2}\,dt}{z^2-t^2}. && (24)
\end{aligned}$$

see also DAWSON'S INTEGRAL, ERFC, ERFI, GAUSSIAN INTEGRAL, NORMAL DISTRIBUTION FUNCTION, PROBABILITY INTEGRAL

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Spanier, J. and Oldham, K. B. "The Error Function erf(x) and Its Complement erfc(x)." Ch. 40 in *An Atlas of Functions.* Washington, DC: Hemisphere, pp. 385–393, 1987.

Erfc

The "complementary error function"

$$\begin{aligned}
\operatorname{erfc}(x) &\equiv \frac{2}{\sqrt{\pi}}\int_x^\infty e^{-t^2}\,dt && (1)\\
&= 1 - \operatorname{erf}(x) && (2)\\
&= \sqrt{\pi}\,\gamma(\tfrac{1}{2}, z^2), && (3)
\end{aligned}$$

where γ is the incomplete GAMMA FUNCTION. It has the values

$$\operatorname{erfc}(0) = 1 \qquad (4)$$

$$\operatorname{erfc}(\infty) = 0 \qquad (5)$$

$$\operatorname{erfc}(-x) = 2 - \operatorname{erfc}(x) \qquad (6)$$

$$\int_0^\infty \operatorname{erfc}(x)\,dx = \frac{1}{\sqrt{\pi}} \qquad (7)$$

$$\int_0^\infty \operatorname{erfc}^2(x)\,dx = \frac{2-\sqrt{2}}{\sqrt{\pi}}. \qquad (8)$$

A generalization is obtained from the differential equation

$$\frac{d^2y}{dz^2} + 2z\frac{dy}{dz} - 2ny = 0. \qquad (9)$$

The general solution is then

$$y = A\operatorname{erfci}_n(z) + B\operatorname{erfci}_n(-z), \qquad (10)$$

where $\operatorname{erfci}_n(z)$ is the erfc integral. For integral $n \geq 1$,

$$\begin{aligned}
\operatorname{erfci}_n(z) &= \underbrace{\int\cdots\int}_{n} \operatorname{erfc}(z)\,dz && (11)\\
&= \frac{2}{\sqrt{\pi}}\int_0^\infty \frac{(t-z)^2}{n!}e^{-t^2}\,dt. && (12).
\end{aligned}$$

The definition can be extended to $n = -1$ and 0 using

$$\operatorname{erfci}_{-1}(z) = \frac{2}{\sqrt{\pi}} e^{-z^2} \quad (13)$$

$$\operatorname{erfci}_0(z) = \operatorname{erfc}(z). \quad (14)$$

see also ERF, ERFI

References

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Erfi

$$\operatorname{erfi}(z) \equiv -i \operatorname{erf}(iz).$$

see also ERF, ERFC

Ergodic Measure

An ENDOMORPHISM is called ergodic if it is true that $T^{-1}A = A$ IMPLIES $m(A) = 0$ or 1, where $T^{-1}A = \{x \in X : T(x) \in A\}$. Examples of ergodic endomorphisms include the MAP $X \to 2x \bmod 1$ on the unit interval with LEBESGUE MEASURE, certain AUTOMORPHISMS of the TORUS, and "Bernoulli shifts" (and more generally "Markov shifts").

Given a MAP T and a SIGMA ALGEBRA, there may be many ergodic measures. If there is only one ergodic measure, then T is called uniquely ergodic. An example of a uniquely ergodic transformation is the MAP $x \mapsto x + a \bmod 1$ on the unit interval when a is irrational. Here, the unique ergodic measure is LEBESGUE MEASURE.

Ergodic Theory

Ergodic theory can be described as the statistical and qualitative behavior of measurable group and semigroup actions on MEASURE SPACES. The GROUP is most commonly $\mathbb{N}$, $\mathbb{R}$, $\mathbb{R}^+$, and $\mathbb{Z}$.

Ergodic theory had its origins in the work of Boltzmann in statistical mechanics. Its mathematical origins are due to von Neumann, Birkhoff, and Koopman in the 1930s. It has since grown to be a huge subject and has applications not only to statistical mechanics, but also to number theory, differential geometry, functional analysis, etc. There are also many internal problems (e.g., ergodic theory being applied to ergodic theory) which are interesting.

see also AMBROSE-KAKUTANI THEOREM, BIRKHOFF'S ERGODIC THEOREM, DYE'S THEOREM, DYNAMICAL SYSTEM, HOPF'S THEOREM, ORNSTEIN'S THEOREM

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Ergodic Transformation

A transformation which has only trivial invariant SUBSETS is said to be invariant.

Erlanger Program

A program initiated by F. Klein in an 1872 lecture to describe geometric structures in terms of their group AUTOMORPHISMS.

References

Klein, F. "Vergleichende Betrachtungen über neuere geometrische Forschungen." 1872.

Yaglom, I. M. *Felix Klein and Sophus Lie: Evolution of the Idea of Symmetry in the Nineteenth Century.* Boston, MA: Birkhäuser, 1988.

Ermakoff's Test

The series $\sum f(n)$ for a monotonic nonincreasing $f(x)$ is convergent if

$$\overline{\lim_{x\to\infty}} \frac{e^x f(e^x)}{f(x)} < 1$$

and divergent if

$$\underline{\lim_{x\to\infty}} \frac{e^x f(e^x)}{f(x)} > 1.$$

References

Bromwich, T. J. I'a and MacRobert, T. M. *An Introduction to the Theory of Infinite Series, 3rd ed.* New York: Chelsea, p. 43, 1991.

Error

The difference between a quantity and its estimated or measured quantity.

see also ABSOLUTE ERROR, PERCENTAGE ERROR, RELATIVE ERROR

Error-Correcting Code

An error-correcting code is an algorithm for expressing a sequence of numbers such that any errors which are introduced can be detected and corrected (within certain limitations) based on the remaining numbers. The study of error-correcting codes and the associated mathematics is known as CODING THEORY.

Error detection is much simpler than error correction, and one or more "check" digits are commonly embedded in credit card numbers in order to detect mistakes. Early space probes like Mariner used a type of error-correcting code called a block code, and more recent space probes use convolution codes. Error-correcting codes are also used in CD players, high speed modems, and cellular phones. Modems use error detection when they compute CHECKSUMS, which are sums of the digits in a given transmission modulo some number. The ISBN used to identify books also incorporates a check DIGIT.

A powerful check for 13 DIGIT numbers consists of the following. Write the number as a string of DIGITS $a_1a_2a_3\cdots a_{13}$. Take $a_1+a_3+\ldots+a_{13}$ and double. Now add the number of DIGITS in ODD positions which are >4 to this number. Now add $a_2+a_4+\ldots+a_{12}$. The check number is then the number required to bring the last DIGIT to 0. This scheme detects all single DIGIT errors and all TRANSPOSITIONS of adject DIGITS except 0 and 9.

see also CHECKSUM, CODING THEORY, GALOIS FIELD, HADAMARD MATRIX, ISBN

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Error Curve

see GAUSSIAN FUNCTION

Error Function

see ERF, ERFC

Error Function Distribution

A NORMAL DISTRIBUTION with MEAN 0.

$$P(x)=\frac{h}{\sqrt{\pi}}e^{-h^2x^2}. \tag{1}$$

The CHARACTERISTIC FUNCTION is

$$\phi(t)=e^{-t^2/(4h^2)}. \tag{2}$$

The MEAN, VARIANCE, SKEWNESS, and KURTOSIS are

$$\mu=0 \tag{3}$$
$$\sigma^2=\frac{1}{2h^2} \tag{4}$$
$$\gamma_1=0 \tag{5}$$
$$\gamma_2=0. \tag{6}$$

The CUMULANTS are

$$\kappa_1=0 \tag{7}$$
$$\kappa_2=\frac{1}{2h^2} \tag{8}$$
$$\kappa_n=0 \tag{9}$$

for $n\geq 3$.

Error Propagation

Given a FORMULA $y=f(x)$ with an ABSOLUTE ERROR in x of dx, the ABSOLUTE ERROR is dy. The RELATIVE ERROR is dy/y. If $x=f(u,v)$, then

$$x_i-\bar{x}=(u_i-\bar{u})\frac{\partial x}{\partial u}+(v_i-\bar{v})\frac{\partial x}{\partial v}+\ldots, \tag{1}$$

so

$$\begin{aligned}{\sigma_x}^2&\equiv\frac{1}{N-1}\sum_i^N(x_i-\bar{x})^2\\&=\frac{1}{N-1}\sum_i^N\left[(u_i-\bar{u})^2\left(\frac{\partial x}{\partial u}\right)^2+(v_i-\bar{v})^2\left(\frac{\partial x}{\partial v}\right)^2\right.\\&\quad\left.+2(u_i-\bar{u})(v_i-\bar{v})\left(\frac{\partial x}{\partial u}\right)\left(\frac{\partial x}{\partial v}\right)+\ldots\right].\end{aligned} \tag{2}$$

The definitions of VARIANCE and COVARIANCE then give

$${\sigma_u}^2\equiv\frac{1}{N-1}\sum_{i=1}^N(u_i-\bar{u})^2 \tag{3}$$
$${\sigma_v}^2\equiv\frac{1}{N-1}\sum_{i=1}^N(v_i-\bar{v})^2 \tag{4}$$
$${\sigma_{uv}}^2\equiv\frac{1}{N-1}\sum_{i=1}^N(u_i-\bar{u})(v_i-\bar{v}), \tag{5}$$

so

$$\begin{aligned}{\sigma_x}^2&={\sigma_u}^2\left(\frac{\partial x}{\partial u}\right)^2+{\sigma_v}^2\left(\frac{\partial x}{\partial v}\right)^2\\&\quad+2\sigma_{uv}\left(\frac{\partial x}{\partial u}\right)\left(\frac{\partial x}{\partial v}\right)+\ldots.\end{aligned} \tag{6}$$

If u and v are uncorrelated, then $\sigma_{uv}=0$ so

$${\sigma_x}^2={\sigma_u}^2\left(\frac{\partial x}{\partial u}\right)^2+{\sigma_v}^2. \tag{7}$$

Now consider addition of quantities with errors. For $x = au \pm bv$, $\partial x/\partial u = a$ and $\partial x/\partial v = \pm b$, so

$$\sigma_x{}^2 = a^2\sigma_u{}^2 + b^2\sigma_v{}^2 \pm 2ab\sigma_{uv}{}^2. \tag{8}$$

For division of quantities with $x = \pm au/v$, $\partial x/\partial u = \pm a/v$ and $\partial x/\partial v = \mp au/v^2$, so

$$\sigma_x{}^2 = \frac{a^2}{v^2}\sigma_u{}^2 + \frac{a^2u^2}{\sigma_v{}^4} - 2\frac{a}{v}\frac{au}{v^2}\sigma_{uv}{}^2. \tag{9}$$

$$\begin{aligned}\left(\frac{\sigma_x}{x}\right)^2 &= \frac{a^2}{v^2}\frac{v^2}{a^2u^2}\sigma_u{}^2 + \frac{a^2u^2}{v^4}\frac{v^2}{a^2u^2} - 2\left(\frac{a}{v}\right)\left(\frac{au}{v^2}\right)\sigma_{uv}{}^2\\ &= \left(\frac{\sigma_u}{u}\right)^2 + \left(\frac{\sigma_v}{v}\right)^2 - 2\left(\frac{\sigma_{uv}}{u}\right)\left(\frac{\sigma_{uv}}{v}\right). \end{aligned} \tag{10}$$

For exponentiation of quantities with

$$x = a^{\pm bu} = (e^{\ln a})^{\pm bu} = e^{\pm b(\ln a)u}, \tag{11}$$

$$\frac{\partial x}{\partial u} = \pm b(\ln a)e^{\pm b\ln au} = \pm b(\ln a)x, \tag{12}$$

so

$$\sigma_x = \sigma_u b(\ln a)x \tag{13}$$

$$\frac{\sigma_x}{x} = b\ln a\sigma_u. \tag{14}$$

If $a = e$, then

$$\frac{\sigma_x}{x} = b\sigma_u. \tag{15}$$

For LOGARITHMS of quantities with $x = a\ln(\pm bu)$, $\partial x/\partial u = a(\pm b)/(\pm bu) = a/u$, so

$$\sigma_x{}^2 = \sigma_u{}^2\left(\frac{a^2}{u^2}\right) \tag{16}$$

$$\sigma_x = a\frac{\sigma_u}{u}. \tag{17}$$

For multiplication with $x = \pm auv$, $\partial x/\partial u = \pm av$ and $\partial x/\partial v = \pm au$, so

$$\sigma_x{}^2 = a^2v^2\sigma_u{}^2 + a^2u^2\sigma_v{}^2 + 2a^2uv\sigma_{uv}{}^2 \tag{18}$$

$$\begin{aligned}\left(\frac{\sigma_x}{x}\right)^2 &= \frac{a^2v^2}{a^2u^2v^2}\sigma_u{}^2 + \frac{a^2u^2}{a^2u^2v^2}\sigma_v{}^2 + \frac{2a^2uv}{a^2u^2v^2}\sigma_{uv}{}^2\\ &= \left(\frac{\sigma_u}{u}\right)^2 + \left(\frac{\sigma_v}{v}\right)^2 + 2\left(\frac{\sigma_{uv}}{u}\right)\left(\frac{\sigma_{uv}}{v}\right). \end{aligned} \tag{19}$$

For POWERS, with $x = au^{\pm b}$, $\partial x/\partial u = \pm abu^{\pm b-1} = \pm bx/u$, so

$$\sigma_x{}^2 = \sigma_u{}^2\frac{b^2x^2}{u^2} \tag{20}$$

$$\frac{\sigma_x}{x} = b\frac{\sigma_u}{u}. \tag{21}$$

see also ABSOLUTE ERROR, PERCENTAGE ERROR, RELATIVE ERROR

References

Abramowitz, M. and Stegun, C. A. (Eds.). *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables, 9th printing.* New York: Dover, p. 14, 1972.

Bevington, P. R. *Data Reduction and Error Analysis for the Physical Sciences.* New York: McGraw-Hill, pp. 58–64, 1969.

Escher's Map

$$f(z) \mapsto z^{(1+\cos\beta+i\sin\beta)/2}.$$

Escribed Circle

see EXCIRCLE

Essential Singularity

A SINGULARITY a for which $f(z)(z-a)^n$ is not DIFFERENTIABLE for any INTEGER $n > 0$.

see also PICARD'S THEOREM, WEIERSTRAß-CASORATI THEOREM

Estimate

An estimate is an educated guess for an unknown quantity or outcome based on known information. The making of estimates is an important part of statistics, since care is needed to provide as accurate an estimate as possible using as little input data as possible. Often, an estimate for the uncertainty ΔE of an estimate E can also be determined statistically. A rule that tells how to calculate an estimate based on the measurements contained in a sample is called an ESTIMATOR.

see also BIAS (ESTIMATOR), ERROR, ESTIMATOR

References

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Estimator

An estimator is a rule that tells how to calculate an ESTIMATE based on the measurements contained in a sample. For example, the "sample MEAN" AVERAGE $\bar{x}$ is an estimator for the population MEAN μ.

The mean square error of an estimator $\tilde{\theta}$ is defined by

$$\mathrm{MSE} \equiv \left\langle (\tilde{\theta} - \theta)^2 \right\rangle.$$

Let B be the BIAS, then

$$\begin{aligned}\mathrm{MSE} &= \left\langle [(\tilde{\theta} - \langle\tilde{\theta}\rangle) + B(\tilde{\theta})]^2 \right\rangle\\ &= \left\langle (\tilde{\theta} - \langle\tilde{\theta}\rangle)^2 \right\rangle + B^2(\tilde{\theta}) \equiv V(\tilde{\theta}) + B^2(\tilde{\theta}),\end{aligned}$$

where V is the estimator VARIANCE.

see also BIAS (ESTIMATOR), ERROR, ESTIMATE, k-STATISTIC

Eta Function

see DEDEKIND ETA FUNCTION, DIRICHLET ETA FUNCTION, THETA FUNCTION

Ethiopian Multiplication

see RUSSIAN MULTIPLICATION

Etruscan Venus Surface

A 3-D shadow of a 4-D KLEIN BOTTLE.

see also IDA SURFACE

References

Peterson, I. *Islands of Truth: A Mathematical Mystery Cruise.* New York: W. H. Freeman, pp. 42–44, 1990.

Eubulides Paradox

The PARADOX "This statement is false," stated in the fourth century BC. It is a sharper version of the EPIMENIDES PARADOX, "All Cretans are liers... One of their own poets has said so."

see also EPIMENIDES PARADOX, SOCRATES' PARADOX

References

Hofstadter, D. R. *Gödel, Escher, Bach: An Eternal Golden Braid.* New York: Vintage Books, p. 17, 1989.

Euclid's Axioms

see EUCLID'S POSTULATES

Euclid's Elements

see ELEMENTS

Euclid's Fifth Postulate

see EUCLID'S POSTULATES

Euclid Number

The nth Euclid number is defined by

$$E_n \equiv 1 + \prod_{i=1}^{n} p_i,$$

where p_i is the ith PRIME. The first few E_n are 3, 7, 31, 211, 2311, 30031, 510511, 9699691, 223092871, 6469693231, ... (Sloane's A006862). The largest factor of E_n are 3, 7, 31, 211, 2311, 509, 277, 27953, ... (Sloane's A002585). The n of the first few PRIME Euclid numbers E_n are 1, 2, 3, 4, 5, 11, 75, 171, 172, 384, 457, 616, 643, ... (Sloane's A014545) up to a search limit of 700. It is not known if there are an INFINITE number of PRIME Euclid numbers (Guy 1994, Ribenboim 1996).

see also SMARANDACHE SEQUENCES

References

Guy, R. K. *Unsolved Problems in Number Theory, 2nd ed.* New York: Springer-Verlag, 1994.

Ribenboim, P. *The New Book of Prime Number Records.* New York: Springer-Verlag, 1996.

Sloane, N. J. A. Sequences A014544, A006862/M2698, and A002585/M2697 in "An On-Line Version of the Encyclopedia of Integer Sequences."

Wagon, S. *Mathematica in Action.* New York: W. H. Freeman, pp. 35–37, 1991.

Euclid's Postulates

1. A straight LINE SEGMENT can be drawn joining any two points.
2. Any straight LINE SEGMENT can be extended indefinitely in a straight LINE.
3. Given any straight LINE SEGMENT, a CIRCLE can be drawn having the segment as RADIUS and one endpoint as center.
4. All RIGHT ANGLES are congruent.
5. If two lines are drawn which intersect a third in such a way that the sum of the inner angles on one side is less than two RIGHT ANGLES, then the two lines inevitably must intersect each other on that side if extended far enough. This postulate is equivalent to what is known as the PARALLEL POSTULATE.

Euclid's fifth postulate cannot be proven as a theorem, although this was attempted by many people. Euclid himself used only the first four postulates ("ABSOLUTE GEOMETRY") for the first 28 propositions of the *Elements*, but was forced to invoke the PARALLEL POSTULATE on the 29th. In 1823, Janos Bolyai and Nicolai Lobachevsky independently realized that entirely self-consistent "NON-EUCLIDEAN GEOMETRIES" could be created in which the parallel postulate *did not hold.* (Gauss had also discovered but suppressed the existence of non-Euclidean geometries.)

see also ABSOLUTE GEOMETRY, CIRCLE, *Elements*, LINE SEGMENT, NON-EUCLIDEAN GEOMETRY, PARALLEL POSTULATE, PASCH'S THEOREM, RIGHT ANGLE

References

Hofstadter, D. R. *Gödel, Escher, Bach: An Eternal Golden Braid.* New York: Vintage Books, pp. 88–92, 1989.

Euclid's Principle

see EUCLID'S THEOREMS

Euclid's Theorems

A theorem sometimes called "Euclid's First Theorem" or EUCLID'S PRINCIPLE states that if p is a PRIME and $p|ab$, then $p|a$ or $p|b$ (where $|$ means DIVIDES). A COROLLARY is that $p|a^n \Rightarrow p|a$ (Conway and Guy 1996). The FUNDAMENTAL THEOREM OF ARITHMETIC is another COROLLARY (Hardy and Wright 1979).

Euclid's Second Theorem states that the number of PRIMES is INFINITE. This theorem, also called the INFINITUDE OF PRIMES theorem, was proved by Euclid in Proposition IX.20 of the *Elements.* Ribenboim (1989) gives nine (and a half) proofs of this theorem. Euclid's elegant proof proceeds as follows. Given a finite sequence of consecutive PRIMES 2, 3, 5, ..., p, the number

$$N = 2 \cdot 3 \cdot 5 \cdots p + 1, \tag{1}$$

known as the ith EUCLID NUMBER when $p = p_i$ is the ith PRIME, is either a new PRIME or the product of PRIMES.

If N is a PRIME, then it must be greater than the previous PRIMES, since one plus the product of PRIMES must be greater than each PRIME composing the product. Now, if N is a product of PRIMES, then at least one of the PRIMES must be greater than p. This can be shown as follows. If N is COMPOSITE and not greater than p, then one of its factors (say F) must be one of the PRIMES in the sequence, 2, 3, 5, ..., p. It therefore DIVIDES the product $2 \cdot 3 \cdot 5 \cdots p$. However, since it is a factor of N, it also DIVIDES N. But a number which DIVIDES two numbers a and $b < a$ also DIVIDES their difference $a - b$, so F must also divide

$$N-(2\cdot3\cdot5\cdots p) = (2\cdot3\cdot5\cdots p+1)-(2\cdot3\cdot5\cdots p) = 1. \quad (2)$$

However, in order to divide 1, F must be 1, which is contrary to the assumption that it is a PRIME in the sequence 2, 3, 5, ... It therefore follows that if N is composite, it has at least one factor greater than p. Since N is either a PRIME greater than p or contains a factor greater than p, a PRIME larger than the largest in the finite sequence can always be found, so there are an infinite number of PRIMES. Hardy (1967) remarks that this proof is "as fresh and significant as when it was discovered" so that "two thousand years have not written a wrinkle" on it.

A similar argument shows that $p! \pm 1$ is PRIME, and

$$1 \cdot 3 \cdot 5 \cdot 7 \cdots p + 1 \quad (3)$$

must be either PRIME or be divisible by a PRIME $> p$. Kummer used a variation of this proof, which is also a proof by contradiction. It assumes that there exist only a finite number of PRIMES $N = p_1, p_2, \ldots, p_r$. Now consider $N - 1$. It must be a product of PRIMES, so it has a PRIME divisor p_i in common with N. Therefore, $p_i | N - (N-1) = 1$ which is nonsense, so we have proved the initial assumption is wrong by contradiction.

It is also true that there are runs of COMPOSITE NUMBERS which are arbitrarily long. This can be seen by defining

$$n \equiv j! = \prod_{i=1}^{j} i, \quad (4)$$

where $j!$ is a FACTORIAL. Then the $j - 1$ consecutive numbers $n + 2, n + 3, \ldots, n + j$ are COMPOSITE, since

$$n + 2 = (1 \cdot 2 \cdots j) + 2 = 2(1 \cdot 3 \cdot 4 \cdots n + 1) \quad (5)$$
$$n + 3 = (1 \cdot 2 \cdots j) + 3 = 3(1 \cdot 2 \cdot 4 \cdot 5 \cdots n + 1) \quad (6)$$
$$n + j = (1 \cdot 2 \cdots j) + j = j[1 \cdot 2 \cdots (j - 1) + 1]. \quad (7)$$

Guy (1981, 1988) points out that while $p_1 p_2 \cdots p_n + 1$ is not necessarily PRIME, letting q be the next PRIME after $p_1 p_2 \cdots p_n + 1$, the number $q - p_1 p_2 \cdots p_n + 1$ is almost always a PRIME, although it has not been proven that this must *always* be the case.

see also DIVIDE, EUCLID NUMBER, PRIME NUMBER

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Euclidean Algorithm

An ALGORITHM for finding the GREATEST COMMON DIVISOR of two numbers a and b, also called Euclid's algorithm. It is an example of a P-PROBLEM whose time complexity is bounded by a quadratic function of the length of the input values (Banach and Shallit). Let $a = bq + r$, then find a number u which DIVIDES both a and b (so that $a = su$ and $b = tu$), then u also DIVIDES r since

$$r = a - bq = su - qtu = (s - qt)u. \quad (1)$$

Similarly, find a number v which DIVIDES b and r (so that $b = s'v$ and $r = t'v$), then v DIVIDES a since

$$a = bq + r = s'vq + t'v = (s'q + t')v. \quad (2)$$

Therefore, every common DIVISOR of a and b is a common DIVISOR of b and r, so the procedure can be iterated as follows

$$a = bq_1 + r_1 \quad (3)$$
$$b = q_2 r_1 + r_2 \quad (4)$$
$$r_1 = q_3 r_2 + r_3 \quad (5)$$
$$r_{n-2} = q_n r_{n-1} + r_n \quad (6)$$
$$r_{n-1} = q_{n+1} r_n, \quad (7)$$

where r_n is $\mathrm{GCD}(a, b) \equiv (a, b)$. Lamé showed that the number of steps needed to arrive at the GREATEST COMMON DIVISOR for two numbers less than N is

$$\text{steps} \leq \frac{\log_{10} N}{\log_{10} \phi} + \frac{\log_{10} \sqrt{5}}{\log_{10} \phi} \quad (8)$$

where ϕ is the GOLDEN MEAN, or ≤ 5 times the number of digits in the smaller number. Numerically, Lamé's expression evaluates to

$$\text{steps} \leq 4.785 \log_{10} N + 1.6723. \tag{9}$$

As shown by LAMÉ'S THEOREM, the worst case occurs when the ALGORITHM is applied to two consecutive FIBONACCI NUMBERS. Heilbronn showed that the average number of steps is $12 \ln 2/\pi^2 \log_{10} n = 0.843 \log_{10} n$ for all pairs (n, b) with $b < n$. Kronecker showed that the shortest application of the ALGORITHM uses least absolute remainders. The QUOTIENTS obtained are distributed as shown in the following table (Wagon 1991).

Quotient	%
1	41.5
2	17.0
3	9.3

For details, see Uspensky and Heaslet (1939) or Knuth (1973). Let $T(m, n)$ be the number of divisions required to compute $\text{GCD}(m, n)$ using the Euclidean algorithm, and define $T(m, 0) = 0$ if $m \geq 0$. Then

$$T(m,n) = \begin{cases} 1 + T(n, m \bmod n) & \text{for } m \geq n \\ 1 + T(n,m) & \text{for } m < n. \end{cases} \tag{10}$$

Define the functions

$$T(n) = \frac{1}{n} \sum_{0 \leq m < n} T(m,n) \tag{11}$$

$$\tau(n) = \frac{1}{\phi(n)} \sum_{\substack{0 \leq m < n \\ \text{GCD}(m,n)=1}} T(m,n) \tag{12}$$

$$A(N) = \frac{1}{N^2} \sum_{\substack{1 \leq m \leq N \\ 1 \leq n \leq N}} T(m,n), \tag{13}$$

where ϕ is the TOTIENT FUNCTION, $T(n)$ is the average number of divisions when n is fixed and m chosen at random, $\tau(n)$ is the average number of divisions when n is fixed and m is a random number coprime to n, and $A(N)$ is the average number of divisions when m and n are both chosen at random in $[1, N]$. Norton (1990) showed that

$$T(n) = \frac{12 \ln 2}{\pi^2} \left[\ln n - \sum_{d|n} \frac{\Lambda(d)}{d} \right] + C + \frac{1}{n} \sum_{d|n} \phi(d) \mathcal{O}(d^{-1/6+\epsilon}), \tag{14}$$

where Λ is the VON MANGOLDT FUNCTION and C is PORTER'S CONSTANT. Porter (1975) showed that

$$\tau(n) = \frac{12 \ln 2}{\pi^2} \ln n + C + \mathcal{O}(n^{-1/6} + \epsilon), \tag{15}$$

and Norton (1990) proved that

$$A(N) = \frac{12 \ln 2}{\pi^2} \left[\ln N - \tfrac{1}{2} + \frac{6}{\pi^2} \zeta'(2) \right] + C - \tfrac{1}{2} + \mathcal{O}(N^{-1/6+\epsilon}). \tag{16}$$

There exist 22 QUADRATIC FIELDS in which there is a Euclidean algorithm (Inkeri 1947).

see also FERGUSON-FORCADE ALGORITHM

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Euclidean Construction

see GEOMETRIC CONSTRUCTION

Euclidean Geometry

A GEOMETRY in which EUCLID'S FIFTH POSTULATE holds, sometimes also called PARABOLIC GEOMETRY. 2-D Euclidean geometry is called PLANE GEOMETRY, and 3-D Euclidean geometry is called SOLID GEOMETRY. Hilbert proved the CONSISTENCY of Euclidean geometry.

see also ELLIPTIC GEOMETRY, GEOMETRIC CONSTRUCTION, GEOMETRY, HYPERBOLIC GEOMETRY, NON-EUCLIDEAN GEOMETRY, PLANE GEOMETRY

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Euclidean Group

The GROUP of ROTATIONS and TRANSLATIONS.

see also ROTATION, TRANSLATION

References

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Euclidean Metric

The FUNCTION $f : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ that assigns to any two VECTORS $(x_1, \ldots, x_n)$ and $(y_1, \ldots, y_n)$ the number

$$\sqrt{(x_1 - y_1)^2 + \ldots + (x_n - y_n)^2},$$

and so gives the "standard" distance between any two VECTORS in $\mathbb{R}^n$.

Euclidean Motion

A Euclidean motion of $\mathbb{R}^n$ is an AFFINE TRANSFORMATION whose linear part is an ORTHOGONAL TRANSFORMATION.

see also RIGID MOTION

References

Gray, A. *Modern Differential Geometry of Curves and Surfaces.* Boca Raton, FL: CRC Press, p. 105, 1993.

Euclidean Norm

see L_2-NORM

Euclidean Number

A Euclidean number is a number which can be obtained by repeatedly solving the QUADRATIC EQUATION. Euclidean numbers, together with the RATIONAL NUMBERS, can be constructed using classical GEOMETRIC CONSTRUCTIONS. However, the cases for which the values of the TRIGONOMETRIC FUNCTIONS SINE, COSINE, TANGENT, etc., can be written in closed form involving square roots of REAL NUMBERS are much more restricted.

see also ALGEBRAIC INTEGER, ALGEBRAIC NUMBER, CONSTRUCTIBLE NUMBER, RADICAL INTEGER

References

Conway, J. H. and Guy, R. K. "Three Greek Problems." In *The Book of Numbers.* New York: Springer-Verlag, pp. 192–194, 1996.

Euclidean Plane

The 2-D EUCLIDEAN SPACE denoted $\mathbb{R}^2$.

see also COMPLEX PLANE, EUCLIDEAN SPACE

Euclidean Space

Euclidean n-space is the SPACE of all n-tuples of REAL NUMBERS, $(x_1, x_2, \ldots, x_n)$ and is denoted $\mathbb{R}^n$. $\mathbb{R}^n$ is a VECTOR SPACE and has LEBESGUE COVERING DIMENSION n. Elements of $\mathbb{R}^n$ are called n-VECTORS. $\mathbb{R}^1 = \mathbb{R}$ is the set of REAL NUMBERS (i.e., the REAL LINE), and $\mathbb{R}^2$ is called the EUCLIDEAN PLANE. In Euclidean space, COVARIANT and CONTRAVARIANT quantities are equivalent so $\vec{e}^j = \vec{e}_j$.

see also EUCLIDEAN PLANE, REAL LINE, VECTOR

References

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Eudoxus's Kampyle

see KAMPYLE OF EUDOXUS

Euler's $6n + 1$ Theorem

Every PRIME of the form $6n + 1$ can be written in the form $x^2 + 3y^2$.

Euler's Addition Theorem

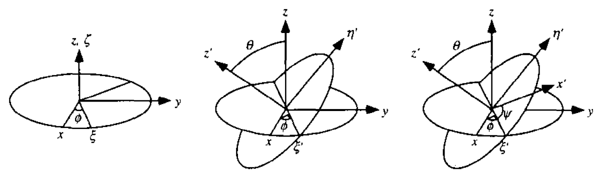
Let $g(x) \equiv (1 - x^2)(1 - k^2x^2)$. Then

$$\int_0^a \frac{dx}{\sqrt{g(x)}} + \int_0^b \frac{dx}{\sqrt{g(x)}} = \int_0^c \frac{dx}{\sqrt{g(x)}},$$

where

$$c \equiv \frac{b\sqrt{g(a)} + a\sqrt{g(b)}}{\sqrt{1 - k^2a^2b^2}}.$$

Euler Angles



According to EULER'S ROTATION THEOREM, any ROTATION may be described using three ANGLES. If the ROTATIONS are written in terms of ROTATION MATRICES B, C, and D, then a general ROTATION A can be written as

$$\mathsf{A} = \mathsf{BCD}. \tag{1}$$

The three angles giving the three rotation matrices are called Euler angles. There are several conventions for Euler angles, depending on the axes about which the rotations are carried out. Write the MATRIX A as

$$\mathsf{A} \equiv \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{11} & a_{12} & a_{13} \end{bmatrix}. \tag{2}$$

In the so-called "x-convention," illustrated above,

$$\mathsf{D} \equiv \begin{bmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{3}$$

$$\mathsf{C} \equiv \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{bmatrix} \tag{4}$$

$$\mathsf{B} \equiv \begin{bmatrix} \cos\psi & \sin\psi & 0 \\ -\sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{bmatrix}, \tag{5}$$

so

$$\begin{aligned}
a_{11} &= \cos\psi\cos\phi - \cos\theta\sin\phi\sin\psi \\
a_{12} &= \cos\psi\sin\phi + \cos\theta\cos\phi\sin\psi \\
a_{13} &= \sin\psi\sin\theta \\
a_{21} &= -\sin\psi\cos\phi - \cos\theta\sin\phi\cos\psi \\
a_{22} &= -\sin\psi\sin\phi + \cos\theta\cos\phi\cos\psi \\
a_{23} &= \cos\psi\sin\theta \\
a_{31} &= \sin\theta\sin\phi \\
a_{32} &= -\sin\theta\cos\phi \\
a_{33} &= \cos\theta
\end{aligned}$$

To obtain the components of the ANGULAR VELOCITY $\boldsymbol{\omega}$ in the body axes, note that for a MATRIX

$$\mathsf{A} \equiv [\,\mathbf{A}_1 \quad \mathbf{A}_2 \quad \mathbf{A}_3\,], \tag{6}$$

it is true that

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{bmatrix} = \begin{bmatrix} a_{11}\omega_x + a_{12}\omega_y + a_{13}\omega_z \\ a_{21}\omega_x + a_{22}\omega_y + a_{23}\omega_z \\ a_{31}\omega_x + a_{32}\omega_y + a_{33}\omega_z \end{bmatrix} \tag{7}$$

$$= \mathbf{A}_1\omega_x + \mathbf{A}_2\omega_y + \mathbf{A}_3\omega_z. \tag{8}$$

Now, ω_z corresponds to rotation about the ϕ axis, so look at the ω_z component of $\mathsf{A}\boldsymbol{\omega}$,

$$\boldsymbol{\omega}_\phi = \mathbf{A}_1\omega_z = \begin{bmatrix} \sin\psi\sin\theta \\ \cos\psi\sin\theta \\ \cos\theta \end{bmatrix}\dot{\phi}. \tag{9}$$

The line of nodes corresponds to a rotation by θ about the ξ-axis, so look at the ω_ξ component of $\mathsf{B}\boldsymbol{\omega}$,

$$\boldsymbol{\omega}_\theta = \mathbf{B}_1\omega_\xi = \mathbf{B}_1\dot{\theta} = \begin{bmatrix} \cos\psi \\ -\sin\psi \\ 0 \end{bmatrix}\dot{\theta}. \tag{10}$$

Similarly, to find rotation by ψ about the remaining axis, look at the ω_ψ component of $\mathsf{B}\boldsymbol{\omega}$,

$$\boldsymbol{\omega}_\psi = \mathbf{B}_3\omega_\psi = \mathbf{B}_3\dot{\psi} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\dot{\psi}. \tag{11}$$

Combining the pieces gives

$$\boldsymbol{\omega} = \begin{bmatrix} \sin\psi\sin\theta\dot{\phi} + \cos\psi\dot{\theta} \\ \cos\psi\sin\theta\dot{\phi} - \sin\psi \\ \cos\theta\dot{\phi} + \dot{\psi}. \end{bmatrix} \tag{12}$$

For more details, see Goldstein (1980, p. 176) and Landau and Lifschitz (1976, p. 111).

The x-convention Euler angles are given in terms of the CAYLEY-KLEIN PARAMETERS by

$$\begin{aligned}
\phi &= -2i\ln\left(\pm\frac{\alpha^{1/2}\gamma^{1/4}}{\beta^{1/4}(1+\beta\gamma)^{1/4}}\right), \\
&\quad - 2i\ln\left(\pm\frac{i\alpha^{1/2}\gamma^{1/4}}{\beta^{1/4}(1+\beta\gamma)^{1/4}}\right) && (13) \\
\psi &= -2i\ln\left(\pm\frac{\alpha^{1/2}\beta^{1/4}}{\gamma^{1/4}(1+\beta\gamma)^{1/4}}\right), \\
&\quad - 2i\ln\left(\pm\frac{i\alpha^{1/2}\beta^{1/4}}{\gamma^{1/4}(1+\beta\gamma)^{1/4}}\right) && (14) \\
\theta &= \pm 2\cos^{-1}(\pm\sqrt{1+\beta\gamma}). && (15)
\end{aligned}$$

In the "y-convention,"

$$\phi_x \equiv \phi_y + \tfrac{1}{2}\pi \tag{16}$$

$$\psi_x \equiv \psi_y - \tfrac{1}{2}\pi. \tag{17}$$

Therefore,

$$\sin\phi_x = \cos\phi_y \tag{18}$$

$$\cos\phi_x = -\sin\phi_y \tag{19}$$

$$\sin\psi_x = -\cos\psi_y \tag{20}$$

$$\cos\psi_x = \sin\psi_y \tag{21}$$

$$D \equiv \begin{bmatrix} -\sin\phi & \cos\phi & 0 \\ -\cos\phi & -\sin\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{22}$$

$$C \equiv \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{bmatrix} \tag{23}$$

$$B \equiv \begin{bmatrix} \sin\psi & -\cos\psi & 0 \\ \cos\psi & \sin\psi & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{24}$$

and A is given by

$$\begin{aligned}
a_{11} &= -\sin\psi\sin\phi + \cos\theta\cos\phi\cos\psi \\
a_{12} &= \sin\psi\cos\phi + \cos\theta\sin\phi\cos\psi \\
a_{13} &= -\cos\psi\sin\theta \\
a_{21} &= -\cos\psi\sin\phi - \cos\theta\cos\phi\sin\psi \\
a_{22} &= \cos\psi\cos\phi - \cos\theta\sin\phi\sin\psi \\
a_{23} &= \sin\psi\sin\theta \\
a_{31} &= \sin\theta\cos\phi \\
a_{32} &= \sin\theta\sin\phi \\
a_{33} &= \cos\theta.
\end{aligned}$$

In the "xyz" (pitch-roll-yaw) convention, θ is pitch, ψ is roll, and ϕ is yaw.

$$D \equiv \begin{bmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{25}$$

$$C \equiv \begin{bmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{bmatrix} \tag{26}$$

$$B \equiv \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\psi & \sin\psi \\ 0 & -\sin\psi & \cos\psi \end{bmatrix} \tag{27}$$

and A is given by

$$\begin{aligned}
a_{11} &= \cos\theta\cos\phi \\
a_{12} &= \cos\theta\sin\phi \\
a_{13} &= -\sin\theta \\
a_{21} &= \sin\psi\sin\theta\cos\phi - \cos\psi\sin\phi \\
a_{22} &= \sin\psi\sin\theta\sin\phi + \cos\psi\cos\phi \\
a_{23} &= \cos\theta\sin\psi \\
a_{31} &= \cos\psi\sin\theta\cos\phi + \sin\psi\sin\phi \\
a_{32} &= \cos\psi\sin\theta\sin\phi - \sin\psi\cos\phi \\
a_{33} &= \cos\theta\cos\psi.
\end{aligned}$$

A set of parameters sometimes used instead of angles are the EULER PARAMETERS e_0, e_1, e_2 and e_3, defined by

$$e_0 \equiv \cos\left(\frac{\phi}{2}\right) \tag{28}$$

$$\mathbf{e} \equiv \begin{bmatrix} e_1 \\ e_2 \\ e_3 \end{bmatrix} = \hat{\mathbf{n}}\sin\left(\frac{\phi}{2}\right). \tag{29}$$

Using EULER PARAMETERS (which are QUATERNIONS), an arbitrary ROTATION MATRIX can be described by

$$\begin{aligned}
a_{11} &= {e_0}^2 + {e_1}^2 - {e_2}^2 - {e_3}^2 \\
a_{12} &= 2(e_1e_2 + e_0e_3) \\
a_{13} &= 2(e_1e_3 - e_0e_2) \\
a_{21} &= 2(e_1e_2 - e_0e_3) \\
a_{22} &= {e_0}^2 - {e_1}^2 + {e_2}^2 - {e_3}^2 \\
a_{23} &= 2(e_2e_3 + e_0e_1) \\
a_{31} &= 2(e_1e_3 + e_0e_2) \\
a_{32} &= 2(e_2e_3 - e_0e_1) \\
a_{33} &= {e_0}^2 - {e_1}^2 - {e_2}^2 + {e_3}^2
\end{aligned}$$

(Goldstein 1960, p. 153).

If the coordinates of two pairs of n points $\mathbf{x}_i$ and $\mathbf{x}'_i$ are known, one rotated with respect to the other, then the Euler rotation matrix can be obtained in a straightforward manner using LEAST SQUARES FITTING. Write the points as arrays of vectors, so

$$[\mathbf{x}'_1 \quad \cdots \quad \mathbf{x}'_n] = A[\mathbf{x}_1 \quad \cdots \quad \mathbf{x}_n]. \tag{30}$$

Writing the arrays of vectors as matrices gives

$$X' = AX \tag{31}$$

$$X'X^{T} = AXX^{T}, \tag{32}$$

and solving for A gives

$$A = X'X^{T}(XX^{T})^{-1}. \tag{33}$$

However, we want the angles θ, ϕ, and ψ, not their combinations contained in the MATRIX A. Therefore, write the 3×3 MATRIX

$$A = \begin{bmatrix} f_1(\theta,\phi,\psi) & f_2(\theta,\phi,\psi) & f_3(\theta,\phi,\psi) \\ f_4(\theta,\phi,\psi) & f_5(\theta,\phi,\psi) & f_6(\theta,\phi,\psi) \\ f_7(\theta,\phi,\psi) & f_7(\theta,\phi,\psi) & f_9(\theta,\phi,\psi) \end{bmatrix} \tag{34}$$

as a 1×9 VECTOR

$$\mathbf{f} = \begin{bmatrix} f_1(\theta,\phi,\psi) \\ \vdots \\ f_9(\theta,\phi,\psi) \end{bmatrix}. \tag{35}$$

Now set up the matrices

$$\begin{bmatrix} \left.\frac{\partial f_1}{\partial\theta}\right|_{\theta_i,\phi_i,\psi_i} & \left.\frac{\partial f_1}{\partial\phi}\right|_{\theta_i,\phi_i,\psi_i} & \left.\frac{\partial f_1}{\partial\psi}\right|_{\theta_i,\phi_i,\psi_i} \\ \vdots & \vdots & \vdots \\ \left.\frac{\partial f_9}{\partial\theta}\right|_{\theta_i,\phi_i,\psi_i} & \left.\frac{\partial f_9}{\partial\phi}\right|_{\theta_i,\phi_i,\psi_i} & \left.\frac{\partial f_9}{\partial\psi}\right|_{\theta_i,\phi_i,\psi_i} \end{bmatrix} \begin{bmatrix} d\theta \\ d\phi \\ d\psi \end{bmatrix} = d\mathbf{f}. \tag{36}$$

Using NONLINEAR LEAST SQUARES FITTING then gives solutions which converge to (θ, ϕ, ψ).

see also CAYLEY-KLEIN PARAMETERS, EULER PARAMETERS, EULER'S ROTATION THEOREM, INFINITESIMAL ROTATION, QUATERNION, ROTATION, ROTATION MATRIX

References
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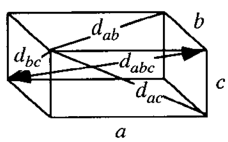
Goldstein, H. "The Euler Angles" and "Euler Angles in Alternate Conventions." §4-4 and Appendix B in *Classical Mechanics, 2nd ed.* Reading, MA: Addison-Wesley, pp. 143–148 and 606–610, 1980.
Landau, L. D. and Lifschitz, E. M. *Mechanics, 3rd ed.* Oxford, England: Pergamon Press, 1976.

Euler-Bernoulli Triangle

see SEIDEL-ENTRINGER-ARNOLD TRIANGLE

Euler Brick



A RECTANGULAR PARALLELEPIPED ("BRICK") with integral edges $a > b > c$ and face diagonals d_{ij} given by

$$d_{ab} = \sqrt{a^2 + b^2} \tag{1}$$

$$d_{ac} = \sqrt{a^2 + c^2} \tag{2}$$

$$d_{bc} = \sqrt{b^2 + c^2}. \tag{3}$$

The problem is also called the BRICK, DIAGONALS PROBLEM, PERFECT BOX, PERFECT CUBOID, or RATIONAL CUBOID problem.

Euler found the smallest solution, which has sides $a = 240$, $b = 117$, and $c = 44$ and face DIAGONALS $d_{ab} = 267$, $d_{ac} = 244$, and $d_{bc} = 125$. Kraitchik gave 257 cuboids with the ODD edge less than 1 million (Guy 1994, p. 174). F. Helenius has compiled a list of the 5003 smallest (measured by the longest edge) Euler bricks. The first few are (240, 117, 44), (275, 252, 240), (693, 480, 140), (720, 132, 85), (792, 231, 160), ... (Sloane's A031173, A031174, and A031175). Parametric solutions for Euler bricks are also known.

No solution is known in which the oblique SPACE DIAGONAL

$$d_{abc} = \sqrt{a^2 + b^2 + c^2} \tag{4}$$

is also an INTEGER. If such a brick exists, the smallest side must be at least 1,281,000,000 (R. Rathbun 1996). Such a solution is equivalent to solving the DIOPHANTINE EQUATIONS

$$A^2 + B^2 = C^2 \tag{5}$$

$$A^2 + D^2 = E^2 \tag{6}$$

$$B^2 + D^2 = F^2 \tag{7}$$

$$B^2 + E^2 = G^2. \tag{8}$$

A solution with integral SPACE DIAGONAL and two out of three face diagonals is $a = 672$, $b = 153$, and $c = 104$, giving $d_{ab} = 3\sqrt{52777}$, $d_{ac} = 680$, $d_{bc} = 185$, and $d_{abc} = 697$. A solution giving integral space and face diagonals with only a single nonintegral EDGE is $a = 18720$, $b = \sqrt{211773121}$, and $c = 7800$, giving $d_{ab} = 23711$, $d_{ac} = 20280$, $d_{bc} = 16511$, and $d_{abc} = 24961$.

see also CUBOID, CYCLIC QUADRILATERAL, DIAGONAL (POLYHEDRON), PARALLELEPIPED, PYTHAGOREAN QUADRUPLE

References
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Sloane, N. J. A. Sequences A031173, A031174, and A031175 in "An On-Line Version of the Encyclopedia of Integer Sequences."
Rathbun, R. L. Personal communication, 1996.
Spohn, W. G. "On the Integral Cuboid." *Amer. Math. Monthly* **79**, 57–59, 1972.
Spohn, W. G. "On the Derived Cuboid." *Canad. Math. Bull.* **17**, 575–577, 1974.
Wells, D. G. *The Penguin Dictionary of Curious and Interesting Numbers.* London: Penguin, p. 127, 1986.

Euler Chain

A CHAIN (GRAPH) whose EDGES consist of all graph EDGES.

Euler Characteristic

Let a closed surface have GENUS g. Then the POLYHEDRAL FORMULA becomes the POINCARÉ FORMULA

$$\chi \equiv V - E + F = 2 - 2g, \tag{1}$$

where χ is the Euler characteristic, sometimes also known as the EULER-POINCARÉ CHARACTERISTIC. In terms of the INTEGRAL CURVATURE of the surface K,

$$\iint K\,da = 2\pi\chi. \tag{2}$$

The Euler characteristic is sometimes also called the EULER NUMBER. It can also be expressed as

$$\chi = p_0 - p_1 + p_2, \tag{3}$$

where p_i is the ith BETTI NUMBER of the space.

see also CHROMATIC NUMBER, MAP COLORING

Euler's Circle

see NINE-POINT CIRCLE

Euler's Conjecture

$$g(k) = 2^k + \left\lfloor \left(\frac{3}{2}\right)^k \right\rfloor - 2,$$

where $g(k)$ is the quantity appearing in WARING'S PROBLEM, and $\lfloor x \rfloor$ is the FLOOR FUNCTION.

see also WARING'S PROBLEM

Euler Constant

see e, EULER-MASCHERONI CONSTANT, MACLAURIN-CAUCHY THEOREM

Euler's Criterion

Let $p = 2m + 1$ be an ODD PRIME and a a POSITIVE INTEGER with $p \nmid a$. Then

$$a^m \equiv 1 \pmod{p} \tag{1}$$

IFF there exists an INTEGER t such that

$$p \equiv t^2 \pmod{p}. \tag{2}$$

In other words,

$$a^{(p-1)/2} \equiv \frac{a}{p} \pmod{p}, \tag{3}$$

where (a/p) is the LEGENDRE SYMBOL.

see also QUADRATIC RESIDUE

References

Rosen, K. H. Ch. 9 in *Elementary Number Theory and Its Applications, 3rd ed.* Reading, MA: Addison-Wesley, 1993.

Shanks, D. *Solved and Unsolved Problems in Number Theory, 4th ed.* New York: Chelsea, pp. 33–37, 1993.

Wagon, S. *Mathematica in Action.* New York: W. H. Freeman, p. 293, 1991.

Euler Curvature Formula

$$\kappa = \kappa_1 \cos^2\theta + \kappa_2 \sin^2\theta,$$

where κ is the normal CURVATURE in a direction making an ANGLE θ with the first principle direction.

Euler Differential Equation

The general nonhomogeneous equation is

$$x^2 \frac{d^2y}{dx^2} + \alpha x \frac{dy}{dx} + \beta y = S(x). \tag{1}$$

The homogeneous equation is

$$x^2 y'' + \alpha x y' + \beta y = 0 \tag{2}$$

$$y'' + \frac{\alpha}{x} y' + \frac{\beta}{x^2} y = 0. \tag{3}$$

Now attempt to convert the equation from

$$y'' + p(x)y' + q(x)y = 0 \tag{4}$$

to one with constant COEFFICIENTS

$$\frac{d^2y}{dz^2} + A\frac{dy}{dz} + By = 0 \tag{5}$$

by using the standard transformation for linear SECOND-ORDER ORDINARY DIFFERENTIAL EQUATIONS. Comparing (3) and (5), the functions $p(x)$ and $q(x)$ are

$$p(x) \equiv \frac{\alpha}{x} = \alpha x^{-1} \tag{6}$$

$$q(x) \equiv \frac{\beta}{x^2} = \beta x^{-2}. \tag{7}$$

Let $B \equiv \beta$ and define

$$\begin{aligned} z &\equiv B^{-1/2} \int \sqrt{q(x)}\, dx = \beta^{-1/2} \int \sqrt{\beta x^{-2}}\, dx \\ &= \int x^{-1}\, dx = \ln x. \end{aligned} \tag{8}$$

Then A is given by

$$\begin{aligned} A &\equiv \frac{q'(x) + 2p(x)q(x)}{2[q(x)]^{3/2}} B^{1/2} \\ &= \frac{-2\beta x^{-3} + 2(\alpha x^{-1})(\beta x^{-2})}{2(\beta x^{-2})^{3/2}} \beta^{1/2} \\ &= \alpha - 1, \end{aligned} \tag{9}$$

which is a constant. Therefore, the equation becomes a second-order ODE with constant COEFFICIENTS

$$\frac{d^2y}{dz^2} + (\alpha - 1)\frac{dy}{dz} + \beta y = 0. \tag{10}$$

Define

$$\begin{aligned} r_1 &\equiv \tfrac{1}{2}\left(-A + \sqrt{A^2 - 4B}\right) \\ &= \tfrac{1}{2}\left[1 - \alpha + \sqrt{(\alpha-1)^2 - 4\beta}\right] \end{aligned} \tag{11}$$

$$\begin{aligned} r_2 &\equiv \tfrac{1}{2}\left(-A - \sqrt{A^2 - 4B}\right) \\ &= \tfrac{1}{2}\left[1 - \alpha - \sqrt{(\alpha-1)^2 - 4\beta}\right] \end{aligned} \tag{12}$$

and

$$a \equiv \tfrac{1}{2}(1 - \alpha) \tag{13}$$

$$b \equiv \tfrac{1}{2}\sqrt{4\beta - (\alpha-1)^2}. \tag{14}$$

The solutions are

$$y = \begin{cases} c_1 e^{r_1 z} + c_2 e^{r_2 z} & (\alpha-1)^2 > 4\beta \\ (c_1 + c_2 z)e^{az} & (\alpha-1)^2 = 4\beta \\ e^{az}[c_1\cos(bz) + c_2\sin(bz)] & (\alpha-1)^2 < 4\beta. \end{cases} \tag{15}$$

In terms of the original variable x,

$$y = \begin{cases} c_1|x|^{r_1} + c_2|x|^{r_2} & (\alpha-1)^2 > 4\beta \\ (c_1 + c_2\ln|x|)|x|^a & (\alpha-1)^2 = 4\beta \\ |x|^a[c_1\cos(b\ln|x|) + c_2\sin(b\ln|x|)] & (\alpha-1)^2 < 4\beta. \end{cases} \tag{16}$$

Euler's Displacement Theorem

The general displacement of a rigid body (or coordinate frame) with one point fixed is a ROTATION about some axis. Furthermore, a ROTATION may be described in any basis using three ANGLES.

see also EUCLIDEAN MOTION, EULER ANGLES, RIGID MOTION, ROTATION

Euler's Distribution Theorem

For signed distances,

$$\overline{AB} \cdot \overline{CD} + \overline{AC} \cdot \overline{DB} + \overline{AD} \cdot \overline{BC} = 0,$$

since

$$(b-a)(d-c) + (c-a)(b-d) + (d-a)(c-b) = 0.$$

References

Johnson, R. A. *Modern Geometry: An Elementary Treatise on the Geometry of the Triangle and the Circle.* Boston, MA: Houghton Mifflin, p. 3, 1929.

Euler Equation

see also EULER DIFFERENTIAL EQUATION, EULER FORMULA, EULER-LAGRANGE DIFFERENTIAL EQUATION

Euler's Factorization Method

Works by expressing N as a QUADRATIC FORM in two different ways. Then

$$N = a^2 + b^2 = c^2 + d^2, \tag{1}$$

so

$$a^2 - c^2 = d^2 - b^2 \tag{2}$$

$$(a-c)(a+c) = (d-b)(d+b). \tag{3}$$

Let k be the GREATEST COMMON DIVISOR of $a-c$ and $d-b$ so

$$a - c = kl \tag{4}$$

$$d - b = km \tag{5}$$

$$(l, m) = 1, \tag{6}$$

(where (l,m) denotes the GREATEST COMMON DIVISOR of l and m), and

$$l(a+c) = m(d+b). \tag{7}$$

But since $(l,m) = 1$, $m|a+c$ and

$$a + c = mn, \tag{8}$$

which gives

$$b + d = ln, \tag{9}$$

so we have

$$\begin{aligned}[(\tfrac{1}{2}k)^2 + (\tfrac{1}{2}n)^2](l^2+m^2) &= \tfrac{1}{4}(k^2+n^2)(l^2+m^2)\\ &= \tfrac{1}{4}[(kn)^2 + (kl)^2 + (nm)^2 + (nl)^2]\\ &= \tfrac{1}{4}[(d-b)^2 + (a-c)^2 + (a+c)^2 + (d+b)^2]\\ &= \tfrac{1}{4}(2a^2 + 2b^2 + 2c^2 + 2d^2)\\ &= \tfrac{1}{4}(2N + 2N) = N. \end{aligned} \tag{10}$$

see also PRIME FACTORIZATION ALGORITHMS

Euler's Finite Difference Transformation

A transformation for the acceleration of the convergence of slowly converging ALTERNATING SERIES,

$$\sum_{k=0}^{\infty} (-1)^k a_k = \sum_{n=0}^{\infty} \frac{\Delta^k a_0}{2^{n+1}}.$$

References

Iyanaga, S. and Kawada, Y. (Eds.). *Encyclopedic Dictionary of Mathematics.* Cambridge, MA: MIT Press, p. 1163, 1980.

Euler Formula

The Euler formula states

$$e^{ix} = \cos x + i \sin x, \tag{1}$$

where i is the IMAGINARY NUMBER. Note that the EULER POLYHEDRAL FORMULA is sometimes also called the Euler formula, as is the EULER CURVATURE FORMULA. The equivalent expression

$$ix = \ln(\cos x + i \sin x) \tag{2}$$

had previously been published by Cotes (1714). The special case of the formula with $x = \pi$ gives the beautiful identity

$$e^{i\pi} + 1 = 0, \tag{3}$$

an equation connecting the fundamental numbers i, PI, e, 1, and 0 (ZERO).

The Euler formula can be demonstrated using a series expansion

$$\begin{aligned} e^{ix} &= \sum_{n=0}^{\infty} \frac{(ix)^n}{n!}\\ &= \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} + i \sum_{n=1}^{\infty} \frac{(-1)^{n-1} x^{2n-1}}{(2n-1)!}\\ &= \cos x + i \sin x. \end{aligned} \tag{4}$$

It can also be proven using a COMPLEX integral. Let

$$z \equiv \cos\theta + i \sin\theta \tag{5}$$

$$dz = (-\sin\theta + i\cos\theta)\,d\theta = i(\cos\theta + i\sin\theta)\,d\theta = iz\,d\theta \tag{6}$$

$$\int \frac{dz}{z} = \int i\,d\theta \tag{7}$$

$$\ln z = i\theta, \tag{8}$$

so

$$z = e^{i\theta} \equiv \cos\theta + i\sin\theta. \tag{9}$$

see also DE MOIVRE'S IDENTITY, EULER POLYHEDRAL FORMULA

References
Castellanos, D. "The Ubiquitous Pi. Part I." *Math. Mag.* **61**, 67–98, 1988.
Conway, J. H. and Guy, R. K. "Euler's Wonderful Relation." *The Book of Numbers.* New York: Springer-Verlag, pp. 254-256, 1996.
Cotes, R. *Philosophical Transactions* **29**, 32, 1714.
Euler, L. *Miscellanea Berolinensia* **7**, 179, 1743.
Euler, L. *Introductio in Analysin Infinitorum, Vol. 1.* Lausanne, p. 104, 1748.

Euler Four-Square Identity

The amazing polynomial identity

$$\begin{aligned}(a_1{}^2+a_2{}^2+a_3{}^2+a_4{}^2)(b_1{}^2+b_2{}^2+b_3{}^2+b_4{}^2)\\ =(a_1b_1-a_2b_2-a_3b_3-a_4b_4)^2\\ +(a_1b_2+a_2b_1+a_3b_4-a_4b_3)^2\\ +(a_1b_3-a_2b_4+a_3b_1+a_4b_2)^2\\ +(a_1b_4+a_2b_3-a_3b_2+a_4b_1)^2,\end{aligned}$$

communicated by Euler in a letter to Goldbach on April 15, 1705. The identity also follows from the fact that the norm of the product of two QUATERNIONS is the product of the norms (Conway and Guy 1996).

see also FIBONACCI IDENTITY, LAGRANGE'S FOUR-SQUARE THEOREM

References
Conway, J. H. and Guy, R. K. *The Book of Numbers.* New York: Springer-Verlag, p. 232, 1996.
Petkovšek, M.; Wilf, H. S.; and Zeilberger, D. *A=B.* Wellesley, MA: A. K. Peters, p. 8, 1996.

Euler's Graeco-Roman Squares Conjecture

Euler conjectured that there do not exist GRAECO-ROMAN SQUARES (now known as EULER SQUARES) of order $n = 4k+2$ for $k = 1, 2, \ldots$. Such squares were found to exist in 1959, refuting the CONJECTURE.

see also EULER SQUARE, LATIN SQUARE

Euler Graph

A GRAPH containing an EULERIAN CIRCUIT. An undirected GRAPH is Eulerian IFF every VERTEX has EVEN DEGREE. A DIRECTED GRAPH is Eulerian IFF every VERTEX has equal INDEGREE and OUTDEGREE. A planar BIPARTITE GRAPH is DUAL to a planar Euler graph and vice versa. The number of Euler graphs with n nodes are 1, 1, 2, 3, 7, 16, 54, 243, ... (Sloane's A002854).

References
Sloane, N. J. A. Sequence A002854/M0846 in "An On-Line Version of the Encyclopedia of Integer Sequences."

Euler's Homogeneous Function Theorem

Let $f(x,y)$ be a HOMOGENEOUS FUNCTION of order n so that

$$f(tx,ty) = t^n(x,y). \tag{1}$$

Then define $x' \equiv xt$ and $y' \equiv yt$. Then

$$\begin{aligned}nt^{n-1}f(x,y) &= \frac{\partial f}{\partial x'}\frac{\partial x'}{\partial t}+\frac{\partial f}{\partial y'}\frac{\partial y'}{\partial t}\\ &= x\frac{\partial f}{\partial x'}+y\frac{\partial f}{\partial y'} = x\frac{\partial f}{\partial (xt)}+y\frac{\partial f}{\partial (yt)}. \quad (2)\end{aligned}$$

Let $t = 1$, then

$$x\frac{\partial f}{\partial x}+y\frac{\partial f}{\partial y} = nf(x,y). \tag{3}$$

This can be generalized to an arbitrary number of variables

$$x_i\frac{\partial f}{\partial x_i} = nf(\mathbf{x}), \tag{4}$$

where EINSTEIN SUMMATION has been used.

Euler's Hypergeometric Transformations

$${}_2F_1(a,b;c;z) = \int_0^1 \frac{t^{b-1}(1-t)^{c-b-1}}{(1-tz)^a}\,dt, \tag{1}$$

where ${}_2F_1(a,b;c;z)$ is a HYPERGEOMETRIC FUNCTION. The solution can be written using the Euler's transformations

$$t \to t \tag{2}$$
$$t \to 1-t \tag{3}$$
$$t \to (1-z-tz)^{-1} \tag{4}$$
$$t \to \frac{1-t}{1-tz} \tag{5}$$

in the equivalent forms

$$\begin{aligned}{}_2F_1(a,b;c;z) &= (1-z)^{-a}\,{}_2F_1(a,c-b;c;z/(z-1)) &(6)\\ &= (1-z)^{-b}\,{}_2F_1(c-a,b;c;z/(z-1)) &(7)\\ &= (1-z)^{c-a-b}\,{}_2F_1(c-a,c-b;c;z). &(8)\end{aligned}$$

see also HYPERGEOMETRIC FUNCTION

References
Morse, P. M. and Feshbach, H. *Methods of Theoretical Physics, Part I.* New York: McGraw-Hill, pp. 585–591, 1953.

Euler Identity

For $|z| < 1$,

$$\prod_{p=1}^\infty (1+z^p) = \prod_{q=1}^\infty (1-z^{2q-1})^{-1}.$$

see also JACOBI TRIPLE PRODUCT, q-SERIES

Euler's Idoneal Number

see IDONEAL NUMBER

Euler Integral

Euler integration was defined by Schanuel and subsequently explored by Rota, Chen, and Klain. The Euler integral of a FUNCTION $f : \mathbb{R} \to \mathbb{R}$ (assumed to be piecewise-constant with finitely many discontinuities) is the sum of

$$f(x) - \tfrac{1}{2}[f(x_+) + f(x_-)]$$

over the finitely many discontinuities of f. The n-D Euler integral can be defined for classes of functions $\mathbb{R}^n \to \mathbb{R}$. Euler integration is additive, so the Euler integral of $f+g$ equals the sum of the Euler integrals of f and g.

see also EULER MEASURE

Euler-Jacobi Pseudoprime

An Euler-Jacobi pseudoprime is a number n such that

$$2^{(n-1)/2} \equiv \frac{2}{n} \pmod{n}.$$

The first few are 561, 1105, 1729, 1905, 2047, 2465, ... (Sloane's A006971).

see also PSEUDOPRIME

References

Sloane, N. J. A. Sequence A006971/M5461 in "An On-Line Version of the Encyclopedia of Integer Sequences."

Euler L-Function

A special case of the ARTIN L-FUNCTION for the POLYNOMIAL x^2+1. It is given by

$$L(s) = \prod_{p \text{ odd prime}} \frac{1}{1-\chi^-(p)p^{-s}},$$

where

$$\chi^-(p) \equiv \begin{cases} 1 & \text{for } p \equiv 1 \pmod 4 \\ -1 & \text{for } p \equiv 3 \pmod 4 \end{cases} = \left(\frac{-1}{p}\right),$$

where $(-1/p)$ is a LEGENDRE SYMBOL.

References

Knapp, A. W. "Group Representations and Harmonic Analysis, Part II." *Not. Amer. Math. Soc.* **43**, 537–549, 1996.

Euler-Lagrange Differential Equation

A fundamental equation of CALCULUS OF VARIATIONS which states that if J is defined by an INTEGRAL of the form

$$J = \int f(x, y, \dot{y})\, dx, \tag{1}$$

where

$$\dot{y} \equiv \frac{dy}{dt}, \tag{2}$$

then J has a STATIONARY VALUE if the Euler-Lagrange differential equation

$$\frac{\partial f}{\partial y} - \frac{d}{dt}\left(\frac{\partial f}{\partial \dot{y}}\right) = 0 \tag{3}$$

is satisfied. If time DERIVATIVE NOTATION is replaced instead by space variable notation, the equation becomes

$$\frac{\partial f}{\partial y} - \frac{d}{dx}\frac{\partial f}{\partial y_x} = 0. \tag{4}$$

In many physical problems, f_x (the PARTIAL DERIVATIVE of f with respect to x) turns out to be 0, in which case a manipulation of the Euler-Lagrange differential equation reduces to the greatly simplified and partially integrated form known as the BELTRAMI IDENTITY,

$$f - y_x \frac{\partial f}{\partial y_x} = C. \tag{5}$$

For three independent variables (Arfken 1985, pp. 924–944), the equation generalizes to

$$\frac{\partial f}{\partial u} - \frac{\partial}{\partial x}\frac{\partial f}{\partial u_x} - \frac{\partial}{\partial y}\frac{\partial f}{\partial u_y} - \frac{\partial}{\partial z}\frac{\partial f}{\partial u_z} = 0. \tag{6}$$

Problems in the CALCULUS OF VARIATIONS often can be solved by solution of the appropriate Euler-Lagrange equation.

To derive the Euler-Lagrange differential equation, examine

$$\begin{aligned} \delta J \equiv \delta \int L(q, \dot{q}, t)\, dt &= \int \left(\frac{\partial L}{\partial q}\delta q + \frac{\partial L}{\partial \dot{q}}\delta \dot{q}\right) dt \\ &= \int \left[\frac{\partial L}{\partial q}\delta q + \frac{\partial L}{\partial \dot{q}}\frac{d(\delta q)}{dt}\right] dt, \end{aligned} \tag{7}$$

since $\delta \dot{q} = d(\delta q)/dt$. Now, integrate the second term by PARTS using

$$u = \frac{\partial L}{\partial \dot{q}} \qquad dv = d(\delta q) \tag{8}$$

$$du = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) dt \qquad v = \delta q, \tag{9}$$

so

$$\int \frac{\partial L}{\partial \dot{q}} \frac{d(\delta q)}{dt}\,dt = \int \frac{\partial L}{\partial \dot{q}}\,d(\delta q)$$
$$= \left[\frac{\partial L}{\partial \dot{q}}\delta q\right]_{t_1}^{t_2} - \int_{t_1}^{t_2} \left(\frac{d}{dt}\frac{\partial L}{\partial \dot{q}}\,dt\right)\delta q. \quad (10)$$

Combining (7) and (10) then gives

$$\delta J = \left[\frac{\partial L}{\partial \dot{q}}\delta q\right]_{t_1}^{t_2} + \int_{t_1}^{t_2} \left(\frac{\partial L}{\partial q} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}}\right)\delta q\,dt. \quad (11)$$

But we are varying the path only, not the endpoints, so $\delta q(t_1) = \delta q(t_2) = 0$ and (11) becomes

$$\delta J = \int_{t_1}^{t_2} \left(\frac{\partial L}{\partial q} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}}\right)\delta q\,dt. \quad (12)$$

We are finding the STATIONARY VALUES such that $\delta J = 0$. These must vanish for any small change δq, which gives from (12),

$$\frac{\partial L}{\partial q} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) = 0. \quad (13)$$

This is the Euler-Lagrange differential equation.

The variation in J can also be written in terms of the parameter κ as

$$\delta J = \int [f(x, y + \kappa v, \dot{y} + \kappa \dot{v}) - f(x, y, \dot{y})]\,dt$$
$$= \kappa I_1 + \tfrac{1}{2}\kappa^2 I_2 + \tfrac{1}{6}\kappa^3 I_3 + \tfrac{1}{24}\kappa^4 I_4 + \ldots, \quad (14)$$

where

$$v = \delta y \quad (15)$$
$$\dot{v} = \delta \dot{y} \quad (16)$$

and the first, second, etc., variations are

$$I_1 = \int (v f_y + \dot{v} f_{\dot{y}})\,dt \quad (17)$$
$$I_2 = \int (v^2 f_{yy} + 2v\dot{v} f_{y\dot{y}} + \dot{v}^2 f_{\dot{y}\dot{y}})\,dt \quad (18)$$
$$I_3 = \int (v^3 f_{yyy} + 3v^2\dot{v} f_{yy\dot{y}} + 3v\dot{v}^2 f_{y\dot{y}\dot{y}} + \dot{v}^3 f_{\dot{y}\dot{y}\dot{y}})\,dt \quad (19)$$
$$I_4 = \int (v^4 f_{yyyy} + 4v^3\dot{v} f_{yyy\dot{y}} + 6v^2\dot{v}^2 f_{yy\dot{y}\dot{y}} + 4v\dot{v}^3 f_{y\dot{y}\dot{y}\dot{y}} + \dot{v}^4 f_{\dot{y}\dot{y}\dot{y}\dot{y}})\,dt. \quad (20)$$

The second variation can be re-expressed using

$$\frac{d}{dt}(v^2\lambda) = v^2\dot{\lambda} + 2v\dot{v}\lambda, \quad (21)$$

so

$$I_2 + [v^2\lambda]_2^1 = \int_1^2 [v^2(f_{yy} + \dot{\lambda}) + 2v\dot{v}(f_{y\dot{y}} + \lambda) + \dot{v}^2 f_{\dot{y}\dot{y}}]\,dt. \quad (22)$$

But

$$[v^2\lambda]_2^1 = 0. \quad (23)$$

Now choose λ such that

$$f_{\dot{y}\dot{y}}(f_{yy} + \dot{\lambda}) = (f_{y\dot{y}} + \lambda)^2 \quad (24)$$

and z such that

$$f_{y\dot{y}} + \lambda = -\frac{f_{\dot{y}\dot{y}}}{z}\frac{dz}{dt} \quad (25)$$

so that z satisfies

$$f_{\dot{y}\dot{y}}\ddot{z} + \dot{f}_{\dot{y}\dot{y}}\dot{z} - (f_{yy} - \dot{f}_{y\dot{y}})z = 0. \quad (26)$$

It then follows that

$$I_2 = \int f_{\dot{y}\dot{y}}\left(\dot{v} + \frac{f_{y\dot{y}} + \lambda}{f_{\dot{y}\dot{y}}v}\right)^2 dt = \int f_{\dot{y}\dot{y}}\left(\dot{v} - \frac{v}{z}\frac{dz}{dt}\right)^2. \quad (27)$$

see also BELTRAMI IDENTITY, BRACHISTOCHRONE PROBLEM, CALCULUS OF VARIATIONS, EULER-LAGRANGE DERIVATIVE

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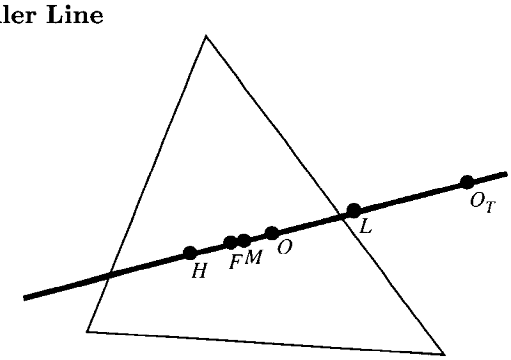
Euler-Lagrange Derivative

The derivative

$$\frac{\delta L}{\delta q} \equiv \frac{\partial L}{\partial q} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right)$$

appearing in the EULER-LAGRANGE DIFFERENTIAL EQUATION.

Euler Line



The line on which the ORTHOCENTER H, CENTROID M, CIRCUMCENTER O, DE LONGCHAMPS POINT L, NINE-POINT CENTER F, and the TANGENTIAL TRIANGLE CIRCUMCIRCLE O_T of a TRIANGLE lie. The INCENTER lies on the Euler line only if the TRIANGLE is an ISOSCELES TRIANGLE. The Euler line consists of all points with TRILINEAR COORDINATES $\alpha : \beta : \gamma$ which satisfy

$$\begin{vmatrix} \alpha & \beta & \gamma \\ \cos A & \cos B & \cos C \\ \cos B \cos C & \cos C \cos A & \cos A \cos B \end{vmatrix} = 0, \qquad (1)$$

which simplifies to

$$\alpha \cos A(\cos^2 B - \cos^2 C) + \beta \cos B(\cos^2 C - \cos^2 A) + \gamma \cos C(\cos^2 A - \cos^2 B) = 0. \qquad (2)$$

This can also be written

$$\alpha \sin(2A) \sin(B - C) + \beta \sin(2B) \sin(C - A) + \gamma \sin(2C) \sin(A - B) = 0. \qquad (3)$$

The Euler line may also be given parametrically by

$$P(\lambda) = O + \lambda H \qquad (4)$$

(Oldknow 1996).

λ	Center
-2	point at infinity
-1	de Longchamps point L
0	circumcenter O
1	centroid G
2	nine-point center F
∞	orthocenter H

The ORTHOCENTER is twice as far from the CENTROID as is the CIRCUMCENTER. The CIRCUMCENTER O, NINE-POINT CENTER F, CENTROID G, and ORTHOCENTER H form a HARMONIC RANGE.

The Euler line intersects the SODDY LINE in the DE LONGCHAMPS POINT, and the GERGONNE LINE in the EVANS POINT. The ISOTOMIC CONJUGATE of the Euler line is called JERABEK'S HYPERBOLA (Casey 1893, Vandeghen 1965).

see also CENTROID (TRIANGLE), CIRCUMCENTER, EVANS POINT, GERGONNE LINE, JERABEK'S HYPERBOLA, DE LONGCHAMPS POINT, NINE-POINT CENTER, ORTHOCENTER, SODDY LINE, TANGENTIAL TRIANGLE

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Euler-Lucas Pseudoprime

Let $U(P, Q)$ and $V(P, Q)$ be LUCAS SEQUENCES generated by P and Q, and define

$$D \equiv P^2 - 4Q.$$

Then

$$\begin{cases} U_{(n-(D/n))/2} \equiv 0 \pmod{n} & \text{when } (Q/n) = 1 \\ V_{(n-(D/n))/2} \equiv D \pmod{n} & \text{when } (Q/n) = -1, \end{cases}$$

where (Q/n) is the LEGENDRE SYMBOL. An ODD COMPOSITE NUMBER n such that $(n, QD) = 1$ (i.e., n and QD are RELATIVELY PRIME) is called an Euler-Lucas pseudoprime with parameters (P, Q).

see also PSEUDOPRIME, STRONG LUCAS PSEUDOPRIME

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Euler's Machin-Like Formula

The MACHIN-LIKE FORMULA

$$\tfrac{1}{4}\pi = \tan^{-1}(\tfrac{1}{2}) + \tan^{-1}(\tfrac{1}{3}).$$

The other 2-term MACHIN-LIKE FORMULAS are HERMANN'S FORMULA, HUTTON'S FORMULA, and MACHIN'S FORMULA.

see also INVERSE TANGENT

Euler-Maclaurin Integration Formulas

The first Euler-Maclaurin integration formula is

$$\int_0^1 f(x)\,dx = \tfrac{1}{2}[f(1) + f(0)] - \sum_{p=1}^{q} \frac{1}{(2p)!} B_{2p}[f^{(2p-1)}(1) - f^{(2p-1)}(0)] + \frac{1}{(2q)!} \int_0^1 f^{(2q)}(x) B_{2q}(x)\,dx, \qquad (1)$$

where B_n are BERNOULLI NUMBERS. SUMS may be converted to INTEGRALS by inverting the FORMULA to obtain

$$\sum_{m=1}^{n} f(m) = \int_1^n f(x)\,dx + \tfrac{1}{2}[f(1) + f(n)] + \frac{B_2}{2!}[f'(n) - f'(1)] + \ldots. \qquad (2)$$

For a more general case when $f(x)$ is tabulated at n values $f_1, f_2, \ldots, f_n$,

$$\int_{x_1}^{x_n} f(x)\,dx = h[\tfrac{1}{2}f_1 + f_2 + f_3 + \ldots + f_{n-1} + \tfrac{1}{2}f_n] - \sum_{k=1}^{\infty} \frac{B_{2k}h^{2k}}{(2k)!}[f_n{}^{(2k-1)} - f_1{}^{(2k-1)}]. \quad (3)$$

The Euler-Maclaurin formula is implemented in *Mathematica*® (Wolfram Research, Champaign, IL) as the function `NSum` with option `Method->Integrate`.

The second Euler-Maclaurin integration formula is used when $f(x)$ is tabulated at n values $f_{3/2}$, $f_{5/2}$, $\ldots$, $f_{n-1/2}$:

$$\int_{x_1}^{x_n} f(x)\,dx = h[f_{3/2} + f_{5/2} + f_{7/2} + \ldots + f_{n-3/2} + f_{n-1/2}] - \sum_{k=1}^{\infty} \frac{B_{2k}h^{2k}}{(2k)!}(1-2^{-2k+1})[f_n{}^{(2k-1)} - f_1{}^{(2k-1)}]. \quad (4)$$

see also SUM, WYNN'S EPSILON METHOD

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Euler-Mascheroni Constant

The Euler-Mascheroni constant is denoted γ (or sometimes C) and has the numerical value

$$\gamma \approx 0.577215664901532860606512090082402431042\ldots \quad (1)$$

(Sloane's A001620). The CONTINUED FRACTION of the Euler-Mascheroni constant is [0, 1, 1, 2, 1, 2, 1, 4, 3, 13, 5, 1, 1, 8, 1, 2, 4, 1, 1, 40, ...] (Sloane's A002852). The first few CONVERGENTS are 1, 1/2, 3/5, 4/7, 11/19, 15/26, 71/123, 228/395, 3035/5258, 15403/26685, ... (Sloane's A046114 and A046115). The positions at which the digits 1, 2, ... first occur in the CONTINUED FRACTION are 2, 4, 9, 8, 11, 69, 24, 14, 139, 52, 22, ... (Sloane's A033149). The sequence of largest terms in the CONTINUED FRACTION is 1, 2, 4, 13, 40, 49, 65, 399, 2076, ... (Sloane's A033091), which occur at positions 2, 4, 8, 10, 20, 31, 34, 40, 529, ... (Sloane's A033092).

It is not known if this constant is IRRATIONAL, let alone TRANSCENDENTAL. However, Conway and Guy (1996) are "prepared to bet that it is transcendental," although they do not expect a proof to be achieved within their lifetimes.

The Euler-Mascheroni constant arises in many integrals

$$\gamma \equiv -\int_0^\infty e^{-x}\ln x\,dx \quad (2)$$

$$= \int_0^\infty \left(\frac{1}{1-e^{-x}} - \frac{1}{x}\right) e^{-x}\,dx \quad (3)$$

$$= \int_0^\infty \frac{1}{x}\left(\frac{1}{1+x} - e^{-x}\right) dx. \quad (4)$$

and sums

$$\gamma \equiv 1 + \sum_{k=2}^{\infty}\left[\frac{1}{k} + \ln\left(\frac{k-1}{k}\right)\right] \quad (5)$$

$$= \lim_{m\to\infty}\left(\sum_{n=1}^{m}\frac{1}{n} - \ln m\right) \quad (6)$$

$$= \sum_{n=2}^{\infty}(-1)^n\frac{\zeta(n)}{n} \quad (7)$$

$$= \ln\left(\frac{4}{\pi}\right) - \sum_{n=1}^{\infty}\frac{(-1)^n\zeta(n+1)}{2^n(n+1)} \quad (8)$$

$$= \lim_{n\to\infty}\left[\sum_{k=1}^{n}k^{-1} - \ln n - \frac{1}{2n} + \sum_{k=1}^{n}\frac{B_{2k}}{(2k)n^{2k}}\right], \quad (9)$$

where $\zeta(z)$ is the RIEMANN ZETA FUNCTION and B_n are the BERNOULLI NUMBERS. It is also given by the EULER PRODUCT

$$e^\gamma = \lim_{n\to\infty}\frac{1}{\ln n}\prod_{i=1}^{n}\frac{1}{1-\frac{1}{p_i}}, \quad (10)$$

where the product is over PRIMES p. Another connection with the PRIMES was provided by Dirichlet's 1838 proof that the average number of DIVISORS of all numbers from 1 to n is asymptotic to

$$\frac{\sum_{i=1}^{n}\sigma_0(i)}{n} \sim \ln n + 2\gamma - 1 \quad (11)$$

(Conway and Guy 1996). de la Vallée Poussin (1898) proved that, if a large number n is divided by all PRIMES $\leq n$, then the average amount by which the QUOTIENT is less than the next whole number is γ.

INFINITE PRODUCTS involving γ also arise from the G-FUNCTION with POSITIVE INTEGER n. The cases $G(2)$ and $G(3)$ give

$$\prod_{n=1}^{\infty} e^{-1+1/(2n)}\left(1+\frac{1}{n}\right)^n = \frac{e^{1+\gamma/2}}{\sqrt{2\pi}} \tag{12}$$

$$\prod_{n=1}^{\infty} e^{-2+2/n}\left(1+\frac{2}{n}\right)^n = \frac{e^{3+2\gamma}}{2\pi}. \tag{13}$$

The Euler-Mascheroni constant is also given by the limits

$$\gamma = \lim_{s\to 1}\frac{\zeta(s)-1}{s-1} \tag{14}$$

$$= -\Gamma'(1) \tag{15}$$

$$= \lim_{x\to\infty}\left[x-\Gamma\left(\frac{1}{x}\right)\right] \tag{16}$$

(Le Lionnais 1983).

The difference between the nth convergent in (6) and γ is given by

$$\sum_{k=1}^{n}\frac{1}{k} - \ln n - \gamma = \int_n^\infty \frac{x-\lfloor x\rfloor}{x^2}\,dx, \tag{17}$$

where $\lfloor x\rfloor$ is the FLOOR FUNCTION, and satisfies the INEQUALITY

$$\frac{1}{2(n+1)} < \sum_{k=1}^{n}\frac{1}{k} - \ln n - \gamma < \frac{1}{2n} \tag{18}$$

(Young 1991). A series with accelerated convergence is

$$\gamma = \tfrac{3}{2} - \ln 2 - \sum_{m=2}^{\infty}(-1)^m\frac{m-1}{m}[\zeta(m)-1] \tag{19}$$

(Flajolet and Vardi 1996). Another series is

$$\gamma = \sum_{n=1}^{\infty}(-1)^n\frac{\lfloor \lg n\rfloor}{n} \tag{20}$$

(Vacca 1910, Gerst 1969), where LG is the LOGARITHM to base 2. The convergence of this series can be greatly improved using Euler's CONVERGENCE IMPROVEMENT transformation to

$$\gamma = \sum_{k=1}^{\infty}2^{-(k+1)}\sum_{j=0}^{k-1}\frac{1}{\binom{2^{k-j}+j}{j}}, \tag{21}$$

where $\binom{a}{b}$ is a BINOMIAL COEFFICIENT (Beeler *et al.* 1972, Item 120, with $k-j$ replacing the undefined i). Bailey (1988) gives

$$\gamma = \frac{2^n}{e^{2^n}}\sum_{m=0}^{\infty}\frac{2^{mn}}{(m+1)!}\sum_{t=0}^{m}\frac{1}{t+1} - n\ln 2 + \mathcal{O}\left(\frac{1}{2^n e^{2^n}}\right), \tag{22}$$

which is an improvement over Sweeney (1963).

The symbol γ is sometimes also used for

$$\gamma' \equiv e^\gamma \approx 1.781072 \tag{23}$$

(Gradshteyn and Ryzhik 1979, p. xxvii).

Odena (1982–1983) gave the strange approximation

$$(0.11111111)^{1/4} = 0.577350\ldots, \tag{24}$$

and Castellanos (1988) gave

$$(\tfrac{7}{83})^{2/9} = 0.57721521\ldots \tag{25}$$

$$\left(\frac{520^6+22}{524}\right)^{1/6} = 0.5772156634\ldots \tag{26}$$

$$\left(\frac{80^3+92}{61^4}\right)^{1/6} = 0.57721566457\ldots \tag{27}$$

$$\frac{990^3-55^3-79^2-4^2}{70^5} = 0.5772156649015295\ldots. \tag{28}$$

No quadratically converging algorithm for computing γ is known (Bailey 1988). 7,000,000 digits of γ have been computed as of Feb. 1998 (Plouffe).

see also EULER PRODUCT, MERTENS THEOREM, STIELTJES CONSTANTS

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Euler-Mascheroni Integrals

Define

$$I_n \equiv (-1)^n \int_0^\infty (\ln z)^n e^{-z}\, dz, \qquad (1)$$

then

$$I_0 = \int_0^\infty e^{-z}\, dz = [-e^{-z}]_0^\infty = (0+1) = 1 \qquad (2)$$

$$I_1 = -\int_0^\infty (\ln z)e^{-z}\, dz = \gamma \qquad (3)$$

$$I_2 = \gamma^2 + \tfrac{1}{6}\pi^2 \qquad (4)$$

$$I_3 = \gamma^3 + \tfrac{1}{2}\gamma\pi^2 + 2\zeta(3) \qquad (5)$$

$$I_4 = \gamma^4 + \gamma^2\pi^2 - \tfrac{3}{20}\pi^4 + 8\gamma\zeta(3), \qquad (6)$$

where γ is the EULER-MASCHERONI CONSTANT and $\zeta(3)$ is APÉRY'S CONSTANT.

Euler Measure

Define the Euler measure of a polyhedral set as the EULER INTEGRAL of its indicator function. It is easy to show by induction that the Euler measure of a closed bounded convex POLYHEDRON is always 1 (independent of dimension), while the Euler measure of a d-D relative-open bounded convex POLYHEDRON is $(-1)^d$.

Euler Number

The Euler numbers, also called the SECANT NUMBERS or ZIG NUMBERS, are defined for $|x| < \pi/2$ by

$$\operatorname{sech} x - 1 \equiv -\frac{E_1^* x^2}{2!} + \frac{E_2^* x^4}{4!} - \frac{E_3^* x^6}{6!} + \ldots \qquad (1)$$

$$\sec x - 1 \equiv \frac{E_1^* x^2}{2!} + \frac{E_2^* x^4}{4!} + \frac{E_3^* x^6}{6!} + \ldots, \qquad (2)$$

where sech is the HYPERBOLIC SECANT and sec is the SECANT. Euler numbers give the number of ODD ALTERNATING PERMUTATIONS and are related to GENOCCHI NUMBERS. The base e of the NATURAL LOGARITHM is sometimes known as Euler's number.

Some values of the Euler numbers are

$$\begin{aligned}
E_1^* &= 1\\
E_2^* &= 5\\
E_3^* &= 61\\
E_4^* &= 1{,}385\\
E_5^* &= 50{,}521\\
E_6^* &= 2{,}702{,}765\\
E_7^* &= 199{,}360{,}981\\
E_8^* &= 19{,}391{,}512{,}145\\
E_9^* &= 2{,}404{,}879{,}675{,}441\\
E_{10}^* &= 370{,}371{,}188{,}237{,}525\\
E_{11}^* &= 69{,}348{,}874{,}393{,}137{,}901\\
E_{12}^* &= 15{,}514{,}534{,}163{,}557{,}086{,}905
\end{aligned}$$

(Sloane's A000364). The first few PRIME Euler numbers E_n occur for $n = 2$, 3, 19, 227, 255, ... (Sloane's A014547) up to a search limit of $n = 1415$.

The slightly different convention defined by

$$E_{2n} = (-1)^n E_n^* \qquad (3)$$

$$E_{2n+1} = 0 \qquad (4)$$

is frequently used. These are, for example, the Euler numbers computed by the *Mathematica*® (Wolfram Research, Champaign, IL) function `EulerE[n]`. This definition has the particularly simple series definition

$$\operatorname{sech} x - 1 \equiv \sum_{k=0}^\infty \frac{E_k x^k}{k!} \qquad (5)$$

and is equivalent to

$$E_n = 2^n E_n(\tfrac{1}{2}), \qquad (6)$$

where $E_n(x)$ is an EULER POLYNOMIAL.

To confuse matters further, the EULER CHARACTERISTIC is sometimes also called the "Euler number."

see also BERNOULLI NUMBER, EULERIAN NUMBER, EULER POLYNOMIAL, EULER ZIGZAG NUMBER, GENOCCHI NUMBER

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Euler Parameters

The four parameters e_0, e_1, e_2, and e_3 describing a finite rotation about an arbitrary axis. The Euler parameters are defined by

$$e_0 \equiv \cos\left(\frac{\phi}{2}\right) \tag{1}$$

$$\mathbf{e} \equiv \begin{bmatrix} e_1 \\ e_2 \\ e_3 \end{bmatrix} = \hat{\mathbf{n}} \sin\left(\frac{\phi}{2}\right), \tag{2}$$

and are a QUATERNION in scalar-vector representation

$$(e_0, \mathbf{e}) = e_0 + e_1 x + e_2 j + e_3 k. \tag{3}$$

Because EULER'S ROTATION THEOREM states that an arbitrary rotation may be described by only three parameters, a relationship must exist between these four quantities

$$e_0{}^2 + \mathbf{e} \cdot \mathbf{e} = e_0{}^2 + e_1{}^2 + e_2{}^2 + e_3{}^2 = 1 \tag{4}$$

(Goldstein 1980, p. 153). The rotation angle is then related to the Euler parameters by

$$\cos\phi = 2e_0{}^2 - 1 = e_0{}^2 - \mathbf{e} \cdot \mathbf{e} = e_0{}^2 - e_1{}^2 - e_2{}^2 - e_3{}^2 \tag{5}$$

$$\hat{\mathbf{n}} \sin\phi = 2\mathbf{e}e_0. \tag{6}$$

The Euler parameters may be given in terms of the EULER ANGLES by

$$e_0 = \cos[\tfrac{1}{2}(\phi + \psi)] \cos(\tfrac{1}{2}\theta) \tag{7}$$

$$e_1 = \sin[\tfrac{1}{2}(\phi - \psi)] \sin(\tfrac{1}{2}\theta) \tag{8}$$

$$e_2 = \cos[\tfrac{1}{2}(\phi - \psi)] \sin(\tfrac{1}{2}\theta) \tag{9}$$

$$e_3 = \sin[\tfrac{1}{2}(\phi + \psi)] \cos(\tfrac{1}{2}\theta) \tag{10}$$

(Goldstein 1980, p. 155).

Using the Euler parameters, the ROTATION FORMULA becomes

$$\mathbf{r}' = \mathbf{r}(e_0{}^2 - e_1{}^2 - e_2{}^2 - e_3{}^2) + 2\mathbf{e}(\mathbf{e} \cdot \mathbf{r}) + (\mathbf{r} \times \hat{\mathbf{n}}) \sin\phi, \tag{11}$$

and the ROTATION MATRIX becomes

$$\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \mathsf{A} \begin{bmatrix} x \\ y \\ z \end{bmatrix}, \tag{12}$$

where the elements of the matrix are

$$a_{ij} = \delta_{ij}(e_0{}^2 - e_k e_k) + 2e_i e_j + 2\epsilon_{ijk} e_0 e_k. \tag{13}$$

Here, EINSTEIN SUMMATION has been used, δ_{ij} is the KRONECKER DELTA, and ϵ_{ijk} is the PERMUTATION SYMBOL. Written out explicitly, the matrix elements are

$$a_{11} = e_0{}^2 + e_1{}^2 - e_2{}^2 - e_3{}^2 \tag{14}$$

$$a_{12} = 2(e_1 e_2 + e_0 e_3) \tag{15}$$

$$a_{13} = 2(e_1 e_3 - e_0 e_2) \tag{16}$$

$$a_{21} = 2(e_1 e_2 - e_0 e_3) \tag{17}$$

$$a_{22} = e_0{}^2 - e_1{}^2 + e_2{}^2 - e_3{}^2 \tag{18}$$

$$a_{23} = 2(e_2 e_3 + e_0 e_1) \tag{19}$$

$$a_{31} = 2(e_1 e_3 + e_0 e_2) \tag{20}$$

$$a_{32} = 2(e_2 e_3 - e_0 e_1) \tag{21}$$

$$a_{33} = e_0{}^2 - e_1{}^2 - e_2{}^2 + e_3{}^2. \tag{22}$$

see also EULER ANGLES, QUATERNION, ROTATION MATRIX

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Euler's Pentagonal Number Theorem

$$\prod_{n=1}^{\infty} (1 - x^n) = \sum_{n=-\infty}^{\infty} (-1)^n x^{n(3n+1)/2}, \tag{1}$$

where $n(3n+1)/2$ are generalized PENTAGONAL NUMBERS. Related equalities are

$$\prod_{k=1}^{\infty} (1 - x^k t) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{n(n+1)/2} t^n}{\prod_{k=1}^{n} (1 - x^k)} \tag{2}$$

$$\prod_{k=1}^{\infty} (1 - x^k t)^{-1} = \sum_{n=0}^{\infty} \frac{t^n}{\prod_{k=1}^{n} (1 - x^k)}. \tag{3}$$

see also PARTITION FUNCTION P, PENTAGONAL NUMBER

Euler's Phi Function

see TOTIENT FUNCTION

Euler-Poincaré Characteristic

see EULER CHARACTERISTIC

Euler's Polygon Division Problem

The problem of finding in how many ways E_n a PLANE convex POLYGON of n sides can be divided into TRIANGLES by diagonals. Euler first proposed it to Christian Goldbach in 1751, and the solution is the CATALAN NUMBER $E_n = C_{n-2}$.

see also CATALAN NUMBER, CATALAN'S PROBLEM

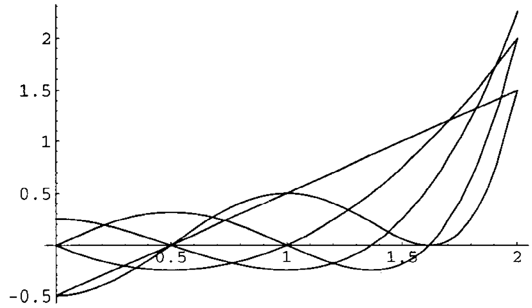
References

Guy, R. K. "Dissecting a Polygon Into Triangles." *Bull. Malayan Math. Soc.* **5**, 57–60, 1958.

Euler Polyhedral Formula

see POLYHEDRAL FORMULA

Euler Polynomial



A POLYNOMIAL $E_n(x)$ given by the sum

$$\frac{2e^{xt}}{e^t+1} \equiv \sum_{n=0}^{\infty} E_n(x)\frac{t^n}{n!}. \tag{1}$$

Euler polynomials are related to the BERNOULLI NUMBERS by

$$E_{n-1}(x) = \frac{2^n}{n}\left[B_n\left(\frac{x+1}{2}\right) - B_n\left(\frac{x}{2}\right)\right] \tag{2}$$

$$= \frac{2}{n}\left[B_n(x) - 2^n B_n\left(\frac{x}{2}\right)\right] \tag{3}$$

$$E_{n-2}(x) = 2\binom{n}{2}^{-1}\sum_{k=0}^{n-2}\binom{n}{k}[(2^{n-k}-1)B_{n-k}B_k(x)], \tag{4}$$

where $\binom{n}{k}$ is a BINOMIAL COEFFICIENT. Setting $x = 1/2$ and normalizing by 2^n gives the EULER NUMBER

$$E_n = 2^n E_n(\tfrac{1}{2}). \tag{5}$$

Call $E'_n = E_n(0)$, then the first few terms are $-1/2$, 0, $1/4$, $-1/2$, 0, $17/8$, 0, $31/2$, 0, ... The terms are the same but with the SIGNS reversed if $x = 1$. These values can be computed using the double sum

$$E_n(0) = 2^{-n}\sum_{j=1}^{n}\left[(-1)^{j+n+1}j^k\sum_{k=0}^{n-j}\binom{n+1}{k}\right]. \tag{6}$$

The BERNOULLI NUMBERS B_n for $n > 1$ can be expressed in terms of the E'_n by

$$B_n = -\frac{nE'_{n-1}}{2(2^n-1)}. \tag{7}$$

see also BERNOULLI POLYNOMIAL, EULER NUMBER, GENOCCHI NUMBER

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Euler Polynomial Identity

see EULER FOUR-SQUARE IDENTITY

Euler Power Conjecture

see EULER'S SUM OF POWERS CONJECTURE

Euler Product

For $\sigma > 1$,

$$\zeta(\sigma) \equiv \sum_{n=1}^{\infty}\frac{1}{n^\sigma} = \prod_p \frac{1}{1-\frac{1}{p^\sigma}},$$

where $\zeta(z)$ is the RIEMANN ZETA FUNCTION.

$$e^\gamma = \lim_{n\to\infty}\frac{1}{\ln n}\prod_{i=1}^{n}\frac{1}{1-\frac{1}{p_i}},$$

where the product is over PRIMES p, where γ is the EULER-MASCHERONI CONSTANT.

see also DEDEKIND FUNCTION

Euler Pseudoprime

Euler pseudoprimes to a base a are ODD COMPOSITE numbers such that $(a,n) = 1$ and the JACOBI SYMBOL satisfies

$$\left(\frac{a}{n}\right) \equiv a^{(n-1)/2} \pmod{n}.$$

No ODD COMPOSITE number is an Euler pseudoprime for all bases a RELATIVELY PRIME to it. This class includes some CARMICHAEL NUMBERS and all STRONG PSEUDOPRIMES to base a. An Euler pseudoprime is pseudoprime to at most 1/2 of all possible bases less than itself. The first few Euler pseudoprimes are 341, 561, 1105, 1729, 1905, 2047, ... (Sloane's A006970).

see also PSEUDOPRIME, STRONG PSEUDOPRIME

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Euler's Quadratic Residue Theorem

A number D that possesses no common divisor with a prime number p is either a QUADRATIC RESIDUE or nonresidue of p, depending whether $D^{(p-1)/2}$ is congruent mod p to ± 1.

Euler Quartic Conjecture

Euler conjectured that there are no POSITIVE INTEGER solutions to the quartic DIOPHANTINE EQUATION

$$A^4 + B^4 = C^4 + D^4.$$

This conjecture was disproved by N. D. Elkies in 1988, who found an infinite class of solutions.

see also DIOPHANTINE EQUATION—QUARTIC

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Euler's Rotation Theorem

An arbitrary ROTATION may be described by only three parameters.

see also EULER ANGLES, EULER PARAMETERS, ROTATION MATRIX

Euler's Rule

The numbers $2^n pq$ and $2^n r$ are AMICABLE NUMBERS if the three INTEGERS

$$\begin{aligned} p &\equiv 2^m(2^{n-m}+1)-1 \\ q &\equiv 2^m(2^{n-m}+1)-1 \\ r &\equiv 2^{n+m}(2^{n-m}+1)^2-1 \end{aligned}$$

are all PRIME numbers for some POSITIVE INTEGER m satisfying $1 \leq m \leq n-1$ (Dickson 1952, p. 42). However, there are exotic AMICABLE NUMBERS which do not satisfy Euler's rule, so it is a SUFFICIENT but not NECESSARY condition for amicability.

see also AMICABLE NUMBERS

References

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Euler's Series Transformation

Accelerates the rate of CONVERGENCE for an ALTERNATING SERIES

$$\begin{aligned} S &= \sum_{s=0}^{\infty} (-1)^s u_s \\ &= u_0 - u_1 + u_2 - \ldots - u_{n-1} + \sum_{s=0}^{\infty} \frac{(-1)^2}{2^{s+1}} [\Delta^s u_n] \quad (1) \end{aligned}$$

for n EVEN and Δ the FORWARD DIFFERENCE operator

$$\Delta^k u_n \equiv \sum_{m=0}^{k} (-1)^m \binom{k}{m} u_{n+k-m}, \quad (2)$$

where $\binom{k}{m}$ are BINOMIAL COEFFICIENTS. The POSITIVE terms in the series can be converted to an ALTERNATING SERIES using

$$\sum_{r=1}^{\infty} v_r = \sum_{r=1}^{\infty} (-1)^{r-1} w_r, \quad (3)$$

where

$$w_r \equiv v_r + 2v_{2r} + 4v_{4r} + 8v_{8r} + \ldots. \quad (4)$$

see also ALTERNATING SERIES

References

Abramowitz, M. and Stegun, C. A. (Eds.). *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables, 9th printing.* New York: Dover, p. 16, 1972.

Euler's Spiral

see CORNU SPIRAL

Euler Square

A square ARRAY made by combining n objects of two types such that the first and second elements form LATIN SQUARES. Euler squares are also known as GRAECO-LATIN SQUARES, GRAECO-ROMAN SQUARES, or LATIN-GRAECO SQUARES. For many years, Euler squares were known to exist for $n = 3$, 4, and for every ODD n except $n = 3k$. EULER'S GRAECO-ROMAN SQUARES CONJECTURE maintained that there do not exist Euler squares of order $n = 4k + 2$ for $k = 1, 2, \ldots$. However, such squares were found to exist in 1959, refuting the CONJECTURE.

see also LATIN RECTANGLE, LATIN SQUARE, ROOM SQUARE

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Euler Sum

In response to a letter from Goldbach, Euler considered DOUBLE SUMS of the form

$$\begin{aligned} s(m,n) &= \sum_{k=1}^{\infty} \left(1 + \frac{1}{2} + \ldots + \frac{1}{k}\right)^m (k+1)^{-n} \quad (1) \\ &= \sum_{k=1}^{\infty} [\gamma + \psi_0(k+1)]^m (k+1)^{-n} \quad (2) \end{aligned}$$

with $m \geq 1$ and $n \geq 2$ and where γ is the EULER-MASCHERONI CONSTANT and $\Psi(x) = \psi_0(x)$ is the DIGAMMA FUNCTION. Euler found explicit formulas in terms of the RIEMANN ZETA FUNCTION for $s(1,n)$ with $n \geq 2$, and E. Au-Yeung numerically discovered

$$\sum_{k=1}^{\infty}\left(1+\frac{1}{2}+\ldots+\frac{1}{k}\right)^2 k^{-2} = \tfrac{17}{4}\zeta(4), \quad (3)$$

where $\zeta(z)$ is the RIEMANN ZETA FUNCTION, which was subsequently rigorously proven true (Borwein and Borwein 1995). Sums involving k^{-n} can be re-expressed in terms of sums the form $(k+1)^{-n}$ via

$$\begin{aligned}\sum_{k=1}^{\infty}&\left(1+\frac{1}{2^m}+\ldots+\frac{1}{k^m}\right)k^{-n}\\ &=\sum_{k=0}^{\infty}\left[1+\frac{2}{2^m}+\ldots+\frac{1}{(k+1)^m}\right](k+1)^{-n}\\ &=\sum_{k=1}^{\infty}\left(1+\frac{1}{2^m}+\ldots+\frac{1}{k^m}\right)(k+1)^{-n}+\sum_{k=1}^{\infty}k^{-(m+n)}\\ &\equiv \sigma_h(m,n)+\zeta(m+n) \quad (4)\end{aligned}$$

and

$$\begin{aligned}\sum_{k=1}^{\infty}&\left(1+\frac{1}{2}+\ldots+\frac{1}{k}\right)^2 k^{-n}\\ &= s_h(2,n)+2s_h(1,n+1)+\zeta(n+2), \quad (5)\end{aligned}$$

where σ_h is defined below.

Bailey *et al.* (1994) subsequently considered sums of the forms

$$s_h(m,n)=\sum_{k=1}^{\infty}\left(1+\frac{1}{2}+\ldots+\frac{1}{k}\right)^m (k+1)^{-n} \quad (6)$$

$$s_a(m,n)=\sum_{k=1}^{\infty}\left[1-\frac{1}{2}+\ldots+\frac{(-1)^{k+1}}{k}\right]^m (k+1)^{-n} \quad (7)$$

$$a_h(m,n)=\sum_{k=1}^{\infty}\left(1+\frac{1}{2}+\ldots+\frac{1}{k}\right)^m (-1)^{k+1}(k+1)^{-n} \quad (8)$$

$$a_a(m,n)=\sum_{k=1}^{\infty}\left(1-\frac{1}{2}+\ldots+\frac{(-1)^{k+1}}{k}\right)^m (-1)^{k+1}(k+1)^{-n} \quad (9)$$

$$\sigma_h(m,n)=\sum_{k=1}^{\infty}\left(1+\frac{1}{2^m}+\ldots+\frac{1}{k^m}\right)(k+1)^{-n} \quad (10)$$

$$\sigma_a(m,n)=\sum_{k=1}^{\infty}\left(1-\frac{1}{2^m}+\ldots+\frac{(-1)^{k+1}}{k^m}\right)(k+1)^{-n} \quad (11)$$

$$\alpha_h(m,n)=\sum_{k=1}^{\infty}\left(1+\frac{1}{2^m}+\ldots+\frac{1}{k^m}\right)(-1)^{k+1}(k+1)^{-n} \quad (12)$$

$$\alpha_a(m,n)=\sum_{k=1}^{\infty}\left(1-\frac{1}{2^m}+\ldots+\frac{(-1)^{k+1}}{k^m}\right)(-1)^{k+1}(k+1)^{-n}, \quad (13)$$

where s_h and s_a have the special forms

$$s_h=\sum_{k=1}^{\infty}[\gamma+\psi_0(n+1)]^m k+1^{-n} \quad (14)$$

$$\begin{aligned}a_a=\sum_{k=1}^{\infty}&\{\ln 2+\tfrac{1}{2}(-1)^n\\ &\times[\psi_0(\tfrac{1}{2}n+\tfrac{1}{2})-\psi_0(\tfrac{1}{2}n+1)]\}^m(k+1)^{-m}. \quad (15)\end{aligned}$$

Analytic single or double sums over $\zeta(z)$ can be constructed for

$$s_h(1,n)=\tfrac{1}{2}n\zeta(n+1)-\tfrac{1}{2}\sum_{k=1}^{n-2}\zeta(n-k)\zeta(k+1) \quad (16)$$

$$\begin{aligned}s_h(2,n)&=\tfrac{1}{3}n(n+1)\zeta(n+2)+\zeta(2)\zeta(n)\\ &-\tfrac{1}{2}n\sum_{k=0}^{n-2}\zeta(n-k)\zeta(k+2)\\ &+\tfrac{1}{3}\sum_{k=2}^{n-2}\zeta(n-k)\sum_{j=1}^{k-1}\zeta(j+1)\zeta(k+1-j)+\sigma_h(2,n) \quad (17)\end{aligned}$$

$$\begin{aligned}s_h(2,2n-1)&=\tfrac{1}{6}(2n^2-7n-3)\zeta(2n+1)+\zeta(2)\zeta(2n-1)\\ &-\tfrac{1}{2}\sum_{k=1}^{n-2}(2k-1)\zeta(2n-1-2k)\zeta(2k+2)\\ &+\tfrac{1}{3}\sum_{k=1}^{n-2}\zeta(2k+1)\sum_{j=1}^{n-2-k}\zeta(2j+1)\zeta(2n-1-2k-2j) \quad (18)\end{aligned}$$

$$\sigma_h(1,n)=s_h(1,n) \quad (19)$$

$$\begin{aligned}\sigma_h(2,2n-1)&=-\tfrac{1}{2}(2n^2+n+1)\zeta(2n+1)+\zeta(2)\zeta(2n-1)\\ &+\sum_{k=1}^{n-1}2k\zeta(k+1)\zeta(2n-2k) \quad (20)\end{aligned}$$

$$\begin{aligned}\sigma_h(m\text{ even}, n\text{ odd})&=\tfrac{1}{2}\left[\binom{m+n}{m}-1\right]\zeta(m+n)+\zeta(m)\zeta(n)\\ &-\sum_{j=1}^{m+n}\left[\binom{2j-2}{m-1}+\binom{2j-2}{n-1}\right]\\ &\times\zeta(2j-1)\zeta(m+n-2j+1) \quad (21)\end{aligned}$$

$$\begin{aligned}\sigma_h(m\text{ odd}, n\text{ even})&=-\tfrac{1}{2}\left[\binom{m+n}{m}+1\right]\zeta(m+n)\\ &+\sum_{k=1}^{m+n}\left[\binom{2j-2}{m-1}+\binom{2j-2}{n-1}\right]\\ &\times\zeta(2j-1)\zeta(m+n-2j+1), \quad (22)\end{aligned}$$

where $\binom{n}{m}$ is a BINOMIAL COEFFICIENT. Explicit formulas inferred using the PSLQ ALGORITHM include

$$s_h(2,2)=\tfrac{3}{2}\zeta(4)+\tfrac{1}{2}[\zeta(2)]^2 \quad (23)$$

$$=\tfrac{11}{360}\pi^4 \quad (24)$$

$$s_h(2,4) = \tfrac{2}{3}\zeta(6) - \tfrac{1}{3}\zeta(2)\zeta(4) + \tfrac{1}{3}[\zeta(2)]^3 - [\zeta(3)]^2 \quad (25)$$
$$= \tfrac{37}{22680}\pi^6 - [\zeta(3)]^2 \quad (26)$$
$$s_h(3,2) = \tfrac{15}{2}\zeta(5) + \zeta(2)\zeta(3) \quad (27)$$
$$s_h(3,3) = -\tfrac{33}{16}\zeta(6) + 2[\zeta(3)]^2 \quad (28)$$
$$s_h(3,4) = \tfrac{119}{16}\zeta(7) - \tfrac{33}{4}\zeta(3)\zeta(4) + 2\zeta(2)\zeta(5) \quad (29)$$
$$s_h(3,6) = \tfrac{197}{24}\zeta(9) - \tfrac{33}{4}\zeta(4)\zeta(5) - \tfrac{37}{8}\zeta(3)\zeta(6) + [\zeta(3)]^3 + 3\zeta(2)\zeta(7) \quad (30)$$
$$s_h(4,2) = \tfrac{859}{24}\zeta(6) + 3[\zeta(3)]^2 \quad (31)$$
$$s_h(4,3) = -\tfrac{109}{8}\zeta(7) + \tfrac{37}{2}\zeta(3)\zeta(4) - 5\zeta(2)\zeta(5) \quad (32)$$
$$s_h(4,5) = -\tfrac{29}{2}\zeta(9) + \tfrac{37}{2}\zeta(4)\zeta(5) + \tfrac{33}{4}\zeta(3)\zeta(6) - \tfrac{8}{3}[\zeta(3)]^3 - 7\zeta(2)\zeta(7) \quad (33)$$
$$s_h(5,2) = \tfrac{1855}{16}\zeta(7) + 33\zeta(3)\zeta(4) + \tfrac{57}{2}\zeta(2)\zeta(5) \quad (34)$$
$$s_h(5,4) = \tfrac{890}{9}\zeta(9) + 66\zeta(4)\zeta(5) - \tfrac{4295}{24}\zeta(3)\zeta(6) - 5[\zeta(3)]^3 + \tfrac{265}{8}\zeta(2)\zeta(7) \quad (35)$$
$$s_h(6,3) = -\tfrac{3073}{12}\zeta(9) - 243\zeta(4)\zeta(5) + \tfrac{2097}{4}\zeta(3)\zeta(6) + \tfrac{67}{3}[\zeta(3)]^3 - \tfrac{651}{8}\zeta(2)\zeta(7) \quad (36)$$
$$s_h(7,2) = \tfrac{134701}{36}\zeta(9) + \tfrac{15697}{8}\zeta(4)\zeta(5) + \tfrac{29555}{24}\zeta(3)\zeta(6) + 56[\zeta(3)]^3 + \tfrac{3287}{4}\zeta(2)\zeta(7), \quad (37)$$

$$s_a(2,2) = 6\operatorname{Li}_4(\tfrac{1}{2}) + \tfrac{1}{4}(\ln 2)^4 - \tfrac{29}{8}\zeta(4) + \tfrac{3}{2}\zeta(2)(\ln 2)^2 \quad (38)$$
$$s_a(2,3) = 4\operatorname{Li}_5(\tfrac{1}{2}) - \tfrac{1}{30}(\ln 2)^5 - \tfrac{17}{32}\zeta(5) - \tfrac{11}{8}\zeta(4)\ln 2 + \tfrac{7}{4}\zeta(3)(\ln 2)^2 + \tfrac{1}{3}\zeta(2)(\ln 2)^3 - \tfrac{3}{4}\zeta(2)\zeta(3), \quad (39)$$
$$s_a(3,2) = -24\operatorname{Li}_5(\tfrac{1}{2}) + 6\ln 2\operatorname{Li}_4(\tfrac{1}{2}) + \tfrac{9}{20}(\ln 2)^5 + \tfrac{659}{32}\zeta(5) - \tfrac{285}{16}\zeta(4)\ln 2 + \tfrac{5}{2}\zeta(2)(\ln 2)^3 + \tfrac{1}{2}\zeta(2)\zeta(3), \quad (40)$$

$$a_h(2,2) = -2\operatorname{Li}_4(\tfrac{1}{2}) - \tfrac{1}{12}(\ln 2)^4 + \tfrac{99}{48}\zeta(4) - \tfrac{7}{4}\zeta(3)\ln 2 + \tfrac{1}{2}\zeta(2)(\ln 2)^2 \quad (41)$$
$$a_h(2,3) = -4\operatorname{Li}_5(\tfrac{1}{2}) - 4(\ln 2)\operatorname{Li}_4(\tfrac{1}{2}) - \tfrac{2}{15}(\ln 2)^5 + \tfrac{107}{32}\zeta(5) - \tfrac{7}{4}\zeta(3)(\ln 2)^2 + \tfrac{2}{3}\zeta(2)(\ln 2)^3 + \tfrac{3}{8}\zeta(2)\zeta(3) \quad (42)$$
$$a_h(3,2) = 6\operatorname{Li}_5(\tfrac{1}{2}) + 6(\ln 2)\operatorname{Li}_4(\tfrac{1}{2}) + \tfrac{1}{5}(\ln 2)^5 - \tfrac{33}{8}\zeta(5) + \tfrac{21}{8}\zeta(3)(\ln 2)^2 - \zeta(2)(\ln 2)^3 - \tfrac{15}{16}\zeta(2)\zeta(3), \quad (43)$$

and

$$a_a(2,2) = -4\operatorname{Li}_4(\tfrac{1}{2}) - \tfrac{1}{6}(\ln 2)^4 + \tfrac{37}{16}\zeta(4) + \tfrac{7}{4}\zeta(3)(\ln 2) - 2\zeta(2)(\ln 2)^2 \quad (44)$$
$$a_a(2,3) = 4(\ln 2)\operatorname{Li}_4(\tfrac{1}{2}) + \tfrac{1}{6}(\ln 2)^5 - \tfrac{79}{32}\zeta(5) + \tfrac{11}{8}\zeta(4)(\ln 2) - \zeta(2)(\ln 2)^3 + \tfrac{3}{8}\zeta(2)\zeta(3) \quad (45)$$
$$a_a(3,2) = 30\operatorname{Li}_5(\tfrac{1}{2}) - \tfrac{1}{4}(\ln 2)^5 - \tfrac{1813}{64}\zeta(5) + \tfrac{285}{16}\zeta(4)(\ln 2) + \tfrac{21}{8}\zeta(3)(\ln 2)^2 - \tfrac{7}{2}\zeta(2)(\ln 2)^3 + \tfrac{3}{4}\zeta(2)\zeta(3), \quad (46)$$

where Li_n is a POLYLOGARITHM, and $\zeta(z)$ is the RIEMANN ZETA FUNCTION (Bailey and Plouffe). Of these, only $s_h(3,2)$, $s_h(3,3)$ and the identities for $s_a(m,n)$, $a_h(m,n)$ and $a_a(m,n)$ have been rigorously established.

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Euler's Sum of Powers Conjecture

Euler conjectured that at least n nth POWERS are required for $n > 2$ to provide a sum that is itself an nth POWER. The conjecture was disproved by Lander and Parkin (1967) with the counterexample

$$27^5 + 84^5 + 110^5 + 133^5 = 144^5.$$

see also DIOPHANTINE EQUATION

References

Lander, L. J. and Parkin, T. R. "A Counterexample to Euler's Sum of Powers Conjecture." *Math. Comput.* **21**, 101–103, 1967.

Euler's Theorem

A generalization of FERMAT'S LITTLE THEOREM. Euler published a proof of the following more general theorem in 1736. Let $\phi(n)$ denote the TOTIENT FUNCTION. Then

$$a^{\phi(n)} \equiv 1 \pmod n$$

for all a RELATIVELY PRIME to n.

see also CHINESE HYPOTHESIS, EULER'S DISPLACEMENT THEOREM, EULER'S DISTRIBUTION THEOREM, FERMAT'S LITTLE THEOREM, TOTIENT FUNCTION

References

Shanks, D. *Solved and Unsolved Problems in Number Theory, 4th ed.* New York: Chelsea, p. 21 and 23–25, 1993.

Euler Totient Function

see TOTIENT FUNCTION

Euler's Totient Rule

The number of bases mod p in which $1/p$ has cycle length l is the same as the number of FRACTIONS $0/(p-1)$, $1/(p-1)$, ..., $(p-2)/(p-1)$ which have least DENOMINATOR l.

see also TOTIENT FUNCTION

References

Conway, J. H. and Guy, R. K. *The Book of Numbers.* New York: Springer-Verlag, pp. 167–168, 1996.

Euler's Transform

A technique for SERIES CONVERGENCE IMPROVEMENT which takes a convergent alternating series

$$\sum_{k=0}^{\infty}(-1)^k a_k = a_0 - a_1 + a_2 - \ldots \qquad (1)$$

into a series with more rapid convergence to the same value to

$$s = \sum_{k=0}^{\infty}\frac{(-1)^k\Delta^k a_0}{2^{k+1}}, \qquad (2)$$

where the FORWARD DIFFERENCE is defined by

$$\Delta^k a_0 = \sum_{m=0}^{k} \equiv (-1)^m \binom{k}{m} a_{k-m} \qquad (3)$$

(Abramowitz and Stegun 1972; Beeler *et al.* 1972, Item 120).

see also FORWARD DIFFERENCE

References

Abramowitz, M. and Stegun, C. A. (Eds.). *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables, 9th printing.* New York: Dover, p. 16, 1972.

Beeler, M.; Gosper, R. W.; and Schroeppel, R. *HAKMEM.* Cambridge, MA: MIT Artificial Intelligence Laboratory, Memo AIM-239, Feb. 1972.

Euler Transformation

see EULER'S FINITE DIFFERENCE TRANSFORMATION, EULER'S HYPERGEOMETRIC TRANSFORMATIONS, EULER'S TRANSFORM

Euler's Triangle

The triangle of numbers $A_{n,k}$ given by

$$A_{n,1} = A_{n,n} = 1$$

and the RECURRENCE RELATION

$$A_{n+1,k} = kA_{n,k} + (n+2-k)A_{n,k-1}$$

for $k \in [2, n]$, where $A_{n,k}$ are EULERIAN NUMBERS.

$$\begin{array}{c} 1 \\ 1 \quad 1 \\ 1 \quad 4 \quad 1 \\ 1 \quad 11 \quad 11 \quad 1 \\ 1 \quad 26 \quad 66 \quad 26 \quad 1 \\ 1 \quad 57 \quad 302 \quad 302 \quad 57 \quad 1 \\ 1\ 120\ 1191\ 2416\ 1191\ 120\ 1. \end{array}$$

The numbers 1, 1, 1, 1, 4, 1, 1, 11, 11, 1, ... are Sloane's A008292. Amazingly, the Z-TRANSFORMS of t^n

$$\frac{(z-1)^n}{T^n z} Z[t^n] = \frac{(1-z)^n}{T^n z}\lim_{x\to 0}\frac{\partial^n}{\partial x^n}\left(\frac{z}{z-e^{-xT}}\right)$$

are generators for Euler's triangle.

see also CLARK'S TRIANGLE, EULERIAN NUMBER, LEIBNIZ HARMONIC TRIANGLE, NUMBER TRIANGLE, PASCAL'S TRIANGLE, SEIDEL-ENTRINGER-ARNOLD TRIANGLE, Z-TRANSFORM

References

Sloane, N. J. A. Sequence A008292 in "An On-Line Version of the Encyclopedia of Integer Sequences."

Euler Triangle Formula

Let O and I be the CIRCUMCENTER and INCENTER of a TRIANGLE with CIRCUMRADIUS R and INRADIUS r. Let d be the distance between O and I. Then

$$d^2 = R^2 - 2rR.$$

Euler Walk

see EULERIAN TRAIL

Euler Zigzag Number

The number of ALTERNATING PERMUTATIONS for n elements is sometimes called an Euler zigzag number. Denote the number of ALTERNATING PERMUTATIONS on n elements for which the first element is k by $E(n,k)$. Then $E(1,1)$ and

$$E(n,k) = \begin{cases} 0 & \text{for } k \geq n \text{ or } k < 1 \\ E(n,k+1) & \text{otherwise.} \\ \quad +E(n-1,n-k) & \end{cases}$$

see also ALTERNATING PERMUTATION, ENTRINGER NUMBER, SECANT NUMBER, TANGENT NUMBER

References

Ruskey, F. "Information of Alternating Permutations." http://sue.csc.uvic.ca/~cos/inf/perm/Alternating.html.

Sloane, N. J. A. Sequence A000111/M1492 in "An On-Line Version of the Encyclopedia of Integer Sequences."

Eulerian Circuit

An EULERIAN TRAIL which starts and ends at the same VERTEX. In other words, it is a CYCLE which uses each EDGE exactly once. The term EULERIAN CYCLE is also used synonymously with Eulerian circuit. For technical reasons, Eulerian circuits are easier to study mathematically than are HAMILTONIAN CIRCUITS. As a generalization of the KÖNIGSBERG BRIDGE PROBLEM, Euler showed (without proof) that a CONNECTED GRAPH has an Eulerian circuit IFF it has no VERTICES of ODD DEGREE.

see also EULER GRAPH, HAMILTONIAN CIRCUIT

Eulerian Cycle

see EULERIAN CIRCUIT

Eulerian Integral of the First Kind

Legendre and Whittaker and Watson's (1990) term for the BETA FUNCTION.

References

Whittaker, E. T. and Watson, G. N. *A Course in Modern Analysis, 4th ed.* Cambridge, England: Cambridge University Press, 1990.

Eulerian Integral of the Second Kind

$$\Pi(z,n) \equiv \int_0^n \left(1-\frac{t}{n}\right)^n t^{z-1}\,dt = n^z \int_0^1 (1-\tau)^n \tau^{z-1}\,d\tau$$
$$= \frac{n!}{z(z+1)\cdots(z+n)} n^z.$$

Eulerian Number

The number of PERMUTATIONS of length n with $k \leq n$ RUNS, denoted $\left\langle {n \atop k} \right\rangle$, $A_{n,k}$, or $A(n,k)$. The Eulerian numbers are given explicitly by the sum

$$\left\langle {n \atop k} \right\rangle = \sum_{j=0}^{k} (-1)^j \binom{n+1}{j} (k-j)^n. \tag{1}$$

Making the definition

$$b_{n,1} = 1 \tag{2}$$
$$b_{1,n} = 1 \tag{3}$$

together with the RECURRENCE RELATION

$$b_{n,k} = nb_{n,k-1} + kb_{n-1,k} \tag{4}$$

for $n > k$ then gives

$$\left\langle {n \atop k} \right\rangle = b_{k,n-k+1}. \tag{5}$$

The arrangement of the numbers into a triangle gives EULER'S TRIANGLE, whose entries are 1, 1, 1, 1, 4, 1, 1, 11, 11, 1, ... (Sloane's A008292). Therefore, they represent a sort of generalization of the BINOMIAL COEFFICIENTS where the defining RECURRENCE RELATION weights the sum of neighbors by their row and column numbers, respectively.

The Eulerian numbers satisfy

$$\sum_{k=1}^{n} \left\langle {n \atop k} \right\rangle = n!. \tag{6}$$

Eulerian numbers also arise in the surprising context of integrating the SINC FUNCTION, and also in sums of the form

$$\sum_{k=1}^{\infty} k^n r^k = \mathrm{Li}_{-n}(r) = \frac{r}{(1-r)^{n+1}} \sum_{i=1}^{n} \left\langle {n \atop i} \right\rangle r^{n-i}, \tag{7}$$

where $\mathrm{Li}_m(z)$ is the POLYLOGARITHM function.

see also COMBINATION LOCK, EULER NUMBER, EULER'S TRIANGLE, EULER ZIGZAG NUMBER, POLYLOGARITHM, SINC FUNCTION, WORPITZKY'S IDENTITY, Z-TRANSFORM

References

Carlitz, L. "Eulerian Numbers and Polynomials." *Math. Mag.* **32**, 247–260, 1959.

Foata, D. and Schützenberger, M.-P. *Théorie Géométrique des Polynômes Eulériens.* Berlin: Springer-Verlag, 1970.

Kimber, A. C. "Eulerian Numbers." Supplement to *Encyclopedia of Statistical Sciences.* (Eds. S. Kotz, N. L. Johnson, and C. B. Read). New York: Wiley, pp. 59–60, 1989.

Salama, I. A. and Kupper, L. L. "A Geometric Interpretation for the Eulerian Numbers." *Amer. Math. Monthly* **93**, 51–52, 1986.

Sloane, N. J. A. Sequence A008292 in "An On-Line Version of the Encyclopedia of Integer Sequences."

Eulerian Trail

A WALK on the EDGES of a GRAPH which uses each EDGE exactly once. A CONNECTED GRAPH has an Eulerian trail IFF it has at most two VERTICES of ODD DEGREE.

see also EULERIAN CIRCUIT

Eutactic Star

An orthogonal projection of a CROSS onto a 3-D SUBSPACE. It is said to be normalized if the CROSS vectors are all of unit length.

see also HADWIGER'S PRINCIPAL THEOREM

Evans Point

The intersection of the GERGONNE LINE and the EULER LINE. It does not appear to have a simple parametric representation.

References

Oldknow, A. "The Euler-Gergonne-Soddy Triangle of a Triangle." *Amer. Math. Monthly* **103**, 319–329, 1996.

Eve

see APPLE, ROOT, SNAKE, SNAKE EYES, SNAKE OIL METHOD, SNAKE POLYIAMOND

Even Function

A function $f(x)$ such that $f(x) = f(-x)$. An even function times an ODD FUNCTION is odd.

Even Number

An INTEGER of the form $N = 2n$, where n is an INTEGER. The even numbers are therefore $\ldots, -4, -2, 0, 2, 4, 6, 8, 10, \ldots$ (Sloane's A005843). Since the even numbers are integrally divisible by two, $N \equiv 0 \pmod 2$ for even N. An even number N for which $N \equiv 2 \pmod 4$ is called a SINGLY EVEN NUMBER, and an even number N for which $N \equiv 0 \pmod 4$ is called a DOUBLY EVEN NUMBER. An integer which is not even is called an ODD NUMBER. The GENERATING FUNCTION of the even numbers is

$$\frac{2x}{(x-1)^2} = 2x + 4x^2 + 6x^3 + 8x^4 + \ldots.$$

see also DOUBLY EVEN NUMBER, EVEN FUNCTION, ODD NUMBER, SINGLY EVEN NUMBER

References

Sloane, N. J. A. Sequence A005843/M0985 in "An On-Line Version of the Encyclopedia of Integer Sequences."

Eventually Periodic

A PERIODIC SEQUENCE such as {1, 1, 1, 2, 1, 2, 1, 2, 1, 2, 1, 1, 2, 1, ...} which is periodic from some point onwards.

see also PERIODIC SEQUENCE

Everett's Formula

$$f_p = (1-p)f_0 + pf_1 + E_2\delta_0^2 + F_2\delta_1^2 + E_4\delta_0^4 + F_4\delta_1^4 + E_6\delta_0^6 + F_6\delta_1^6 + \ldots, \quad (1)$$

for $p \in [0,1]$, where δ is the CENTRAL DIFFERENCE and

$$E_{2n} \equiv G_{2n} - G_{2n+1} \equiv B_{2n} - B_{2n+1} \quad (2)$$

$$F_{2n} \equiv G_{2n+1} \equiv B_{2n} + B_{2n+1}, \quad (3)$$

where G_k are the COEFFICIENTS from GAUSS'S BACKWARD FORMULA and GAUSS'S FORWARD FORMULA and B_k are the COEFFICIENTS from BESSEL'S FINITE DIFFERENCE FORMULA. The E_ks and F_ks also satisfy

$$E_{2n}(p) = F_{2n}(q) \quad (4)$$

$$F_{2n}(p) = E_{2n}(q), \quad (5)$$

for

$$q \equiv 1 - p. \quad (6)$$

see also BESSEL'S FINITE DIFFERENCE FORMULA

References

Abramowitz, M. and Stegun, C. A. (Eds.). *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables, 9th printing.* New York: Dover, pp. 880–881, 1972.

Acton, F. S. *Numerical Methods That Work, 2nd printing.* Washington, DC: Math. Assoc. Amer., pp. 92–93, 1990.

Beyer, W. H. *CRC Standard Mathematical Tables, 28th ed.* Boca Raton, FL: CRC Press, p. 433, 1987.

Everett Interpolation

see EVERETT'S FORMULA

Eversion

A curve on the unit sphere S^2 is an eversion if it has no corners or cusps (but it may be self-intersecting). These properties are guaranteed by requiring that the curve's velocity never vanishes. A mapping $\boldsymbol{\sigma} : S^1 \to S^2$ forms an immersion of the CIRCLE into the SPHERE IFF, for all $\theta \in \mathbb{R}$,

$$\left|\frac{d}{d\theta}[\boldsymbol{\sigma}(e^{i\theta})]\right| > 0.$$

Smale (1958) showed it is possible to turn a SPHERE inside out (SPHERE EVERSION) using eversion.

see also SPHERE EVERSION

References

Smale, S. "A Classification of Immersions of the Two-Sphere." *Trans. Amer. Math. Soc.* **90**, 281–290, 1958.

Evolute

An evolute is the locus of centers of curvature (the envelope) of a plane curve's normals. The original curve is then said to be the INVOLUTE of its evolute. Given a plane curve represented parametrically by $(f(t), g(t))$, the equation of the evolute is given by

$$x = f - R\sin\tau \quad (1)$$

$$y = g + R\cos\tau, \quad (2)$$

where (x, y) are the coordinates of the running point, R is the RADIUS OF CURVATURE

$$R = \frac{(f'^2 + g'^2)^{3/2}}{f'g'' - f''g'}, \quad (3)$$

and τ is the angle between the unit TANGENT VECTOR

$$\hat{\mathbf{T}} = \frac{\mathbf{x}'}{|\mathbf{x}'|} = \frac{1}{\sqrt{f'^2 + g'^2}} \begin{bmatrix} f' \\ g' \end{bmatrix} \quad (4)$$

and the x-AXIS,

$$\cos\tau = \hat{\mathbf{T}} \cdot \hat{\mathbf{x}} \quad (5)$$

$$\sin\tau = \hat{\mathbf{T}} \cdot \hat{\mathbf{y}}. \quad (6)$$

Combining gives

$$x = f - \frac{(f'^2 + g'^2)g'}{f'g'' - f''g'} \tag{7}$$

$$y = g + \frac{(f'^2 + g'^2)g'}{f'g'' - f''g'}. \tag{8}$$

The definition of the evolute of a curve is independent of parameterization for any differentiable function (Gray 1993). If E is the evolute of a curve I, then I is said to be the INVOLUTE of E. The centers of the OSCULATING CIRCLES to a curve form the evolute to that curve (Gray 1993, p. 90).

The following table lists the evolutes of some common curves.

Curve	Evolute
astroid	astroid 2 times as large
cardioid	cardioid 1/3 as large
cayley's sextic	nephroid
circle	point (0, 0)
cycloid	equal cycloid
deltoid	deltoid 3 times as large
ellipse	Lamé curve
epicycloid	enlarged epicycloid
hypocycloid	similar hypocycloid
limaçon	circle catacaustic for a point source
logarithmic spiral	equal logarithmic spiral
nephroid	nephroid 1/2 as large
parabola	Neile's parabola
tractrix	catenary

see also INVOLUTE, OSCULATING CIRCLE

References

Cayley, A. "On Evolutes of Parallel Curves." *Quart. J. Pure Appl. Math.* **11**, 183–199, 1871.

Dixon, R. "String Drawings." Ch. 2 in *Mathographics.* New York: Dover, pp. 75–78, 1991.

Gray, A. "Evolutes." §5.1 in *Modern Differential Geometry of Curves and Surfaces.* Boca Raton, FL: CRC Press, pp. 76–80, 1993.

Jeffrey, H. M. "On the Evolutes of Cubic Curves." *Quart. J. Pure Appl. Math.* **11**, 78–81 and 145–155, 1871.

Lawrence, J. D. *A Catalog of Special Plane Curves.* New York: Dover, pp. 40 and 202, 1972.

Lee, X. "Evolute." `http://www.best.com/~xah/SpecialPlaneCurves_dir/Evolute_dir/evolute.html`.

Lockwood, E. H. "Evolutes and Involutes." Ch. 21 in *A Book of Curves.* Cambridge, England: Cambridge University Press, pp. 166–171, 1967.

Yates, R. C. "Evolutes." *A Handbook on Curves and Their Properties.* Ann Arbor, MI: J. W. Edwards, pp. 86–92, 1952.

Exact Covering System

A system of congruences $a_i \bmod n_i$ with $1 \leq i \leq k$ is called a COVERING SYSTEM if every INTEGER y satisfies $y \equiv a_i \pmod{n}$ for at least one value of i. A covering system in which each integer is covered by just one congruence is called an exact covering system.

see also COVERING SYSTEM

References

Guy, R. K. "Exact Covering Systems." §F14 in *Unsolved Problems in Number Theory, 2nd ed.* New York: Springer-Verlag, pp. 253–256, 1994.

Exact Differential

A differential of the form

$$df = P(x,y)\,dx + Q(x,y)\,dy \tag{1}$$

is exact (also called a TOTAL DIFFERENTIAL) if $\int df$ is path-independent. This will be true if

$$df = \frac{\partial f}{\partial x}\,dx + \frac{\partial f}{\partial y}\,dy, \tag{2}$$

so P and Q must be of the form

$$P(x,y) = \frac{\partial f}{\partial x} \qquad Q(x,y) = \frac{\partial f}{\partial y}. \tag{3}$$

But

$$\frac{\partial P}{\partial y} = \frac{\partial^2 f}{\partial y \partial x} \tag{4}$$

$$\frac{\partial Q}{\partial x} = \frac{\partial^2 f}{\partial x \partial y}, \tag{5}$$

so

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}. \tag{6}$$

see also PFAFFIAN FORM, INEXACT DIFFERENTIAL

Exact Period

see LEAST PERIOD

Exact Trilinear Coordinates

The TRILINEAR COORDINATES $\alpha : \beta : \gamma$ of a point P relative to a TRIANGLE are PROPORTIONAL to the directed distances $a' : b' : c'$ from P to the side lines (i.e, $a' : b' : c' = k\alpha : b' = k\beta : k\gamma$). Letting k be the constant of proportionality,

$$k \equiv \frac{2\Delta}{a\alpha + b\beta + c\gamma},$$

where Δ is the AREA of ΔABC and a, b, and c are the lengths of its sides. When the trilinears are chosen so that $k = 1$, the coordinates are known as exact trilinear coordinates.

see also TRILINEAR COORDINATES

Exactly One

"Exactly one" means "one and only one," sometimes also referred to as "JUST ONE." J. H. Conway has also humorously suggested "onee" (one and only one) by analogy with IFF (if and only if), "twoo" (two and only two), and "threee" (three and only three). This refinement is sometimes needed in formal mathematical discourse because, for example, if you have two apples, you also have one apple, but you do not have *exactly one* apple.

In 2-valued LOGIC, exactly one is equivalent to the exclusive or operator XOR,

$$P(E) \text{ XOR } P(F) = P(E) + P(F) - 2P(E \cap F).$$

see also IFF, PRECISELY UNLESS, XOR

Exactly When

see IFF

Excenter

The center J_i of an EXCIRCLE. There are three excenters for a given TRIANGLE, denoted J_1, J_2, J_3. The INCENTER I and excenters J_i of a TRIANGLE are an ORTHOCENTRIC SYSTEM.

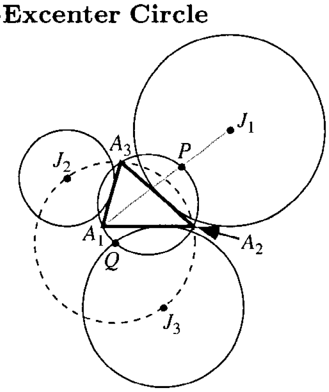
$$\overline{OI}^2 + \overline{OJ_1}^2 + \overline{OJ_2}^2 + \overline{OJ_3}^2 = 12R^2,$$

where O is the CIRCUMCENTER, J_i are the excenters, and R is the CIRCUMRADIUS (Johnson 1929, p. 190). Denote the MIDPOINTS of the original TRIANGLE M_1, M_2, and M_3. Then the lines J_1M_1, J_2M_2, and J_3M_3 intersect in a point known as the MITTENPUNKT.

see also CENTROID (ORTHOCENTRIC SYSTEM), EXCENTER-EXCENTER CIRCLE, EXCENTRAL TRIANGLE, EXCIRCLE, INCENTER, MITTENPUNKT

References
Dixon, R. *Mathographics.* New York: Dover, pp. 58–59, 1991.
Johnson, R. A. *Modern Geometry: An Elementary Treatise on the Geometry of the Triangle and the Circle.* Boston, MA: Houghton Mifflin, 1929.

Excenter-Excenter Circle



Given a TRIANGLE $\Delta A_1A_2A_3$, the points A_1, I, and J_1 lie on a line, where I is the INCENTER and J_1 is the EXCENTER corresponding to A_1. Furthermore, the circle with J_2J_3 as the diameter has Q as its center, where P is the intersection of A_1J_1 with the CIRCUMCIRCLE of $A_1A_2A_3$ and Q is the point opposite P on the CIRCUMCIRCLE. The circle with diameter J_2J_3 also passes through A_2 and A_3 and has radius

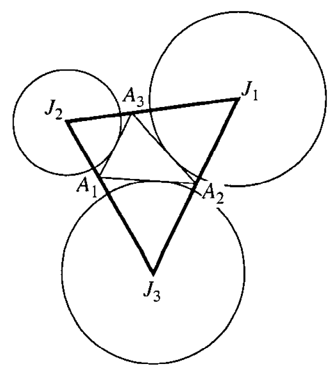
$$r = \tfrac{1}{2}a_1 \csc(\tfrac{1}{2}\alpha_1) = 2R\cos(\tfrac{1}{2}\alpha_1).$$

It arises because the points I, J_1, J_2, and J_3 form an ORTHOCENTRIC SYSTEM.

see also EXCENTER, INCENTER-EXCENTER CIRCLE, ORTHOCENTRIC SYSTEM

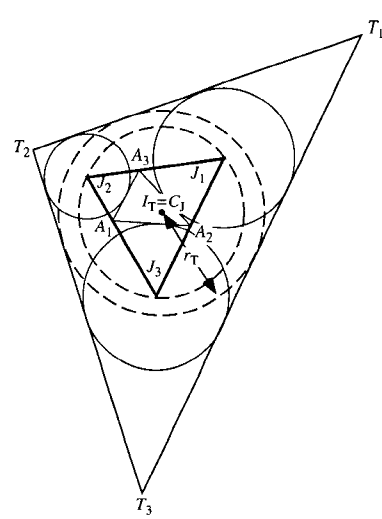
References
Johnson, R. A. *Modern Geometry: An Elementary Treatise on the Geometry of the Triangle and the Circle.* Boston, MA: Houghton Mifflin, pp. 185–186, 1929.

Excentral Triangle



The TRIANGLE $J = \Delta J_1J_2J_3$ with VERTICES corresponding to the EXCENTERS of a given TRIANGLE A, also called the TRITANGENT TRIANGLE.

Beginning with an arbitrary TRIANGLE A, find the excentral triangle J. Then find the excentral triangle $J'$of that TRIANGLE, and so on. Then the resulting TRIANGLE $J^{(\infty)}$ approaches an EQUILATERAL TRIANGLE.



Call T the TRIANGLE tangent externally to the EXCIRCLES of A. Then the INCENTER I_T of K coincides with the CIRCUMCENTER C_J of TRIANGLE $\Delta J_1 J_2 J_3$, where J_i are the EXCENTERS of A. The INRADIUS r_T of the INCIRCLE of T is

$$r_T = 2R + r = \tfrac{1}{2}(r + r_1 + r_2 + r_3),$$

where R is the CIRCUMRADIUS of A, r is the INRADIUS, and r_i are the EXRADII (Johnson 1929, p. 192).

see also EXCENTER, EXCENTER-EXCENTER CIRCLE, EXCIRCLE, MITTENPUNKT

References
Johnson, R. A. *Modern Geometry: An Elementary Treatise on the Geometry of the Triangle and the Circle.* Boston, MA: Houghton Mifflin, 1929.

Excess

see KURTOSIS

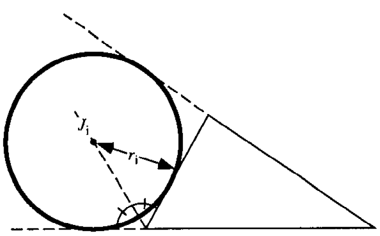
Excess Coefficient

see KURTOSIS

Excessive Number

see ABUNDANT NUMBER

Excircle



Given a TRIANGLE, extend two nonadjacent sides. The CIRCLE tangent to these two lines and to the other side of the TRIANGLE is called an ESCRIBED CIRCLE, or excircle. The CENTER J_i of the excircle is called the EXCENTER and lies on the external ANGLE BISECTOR of the opposite ANGLE. Every TRIANGLE has three excircles, and the TRILINEAR COORDINATES of the EXCENTERS are $-1 : 1 : 1$, $1 : -1 : 1$, and $1 : 1 : -1$. The RADIUS r_i of the excircle i is called its EXRADIUS.

Given a TRIANGLE with INRADIUS r, let h_i be the ALTITUDES of the excircles, and r_i their RADII (the EXRADII). Then

$$\frac{1}{h_1} + \frac{1}{h_2} + \frac{1}{h_3} = \frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3} = \frac{1}{r}$$

(Johnson 1929, p. 189).

see also EXCENTER, EXCENTER-EXCENTER CIRCLE, EXCENTRAL TRIANGLE, FEUERBACH'S THEOREM, NAGEL POINT, TRIANGLE TRANSFORMATION PRINCIPLE

References
Coxeter, H. S. M. and Greitzer, S. L. *Geometry Revisited.* Washington, DC: Math. Assoc. Amer., pp. 11-13, 1967.
Johnson, R. A. *Modern Geometry: An Elementary Treatise on the Geometry of the Triangle and the Circle.* Boston, MA: Houghton Mifflin, pp. 176–177 and 182–194, 1929.

Excision Axiom

One of the EILENBERG-STEENROD AXIOMS which states that, if X is a SPACE with SUBSPACES A and U such that the CLOSURE of A is contained in the interior of U, then the INCLUSION MAP $(X\ U, A\ U) \to (X, A)$ induces an isomorphism $H_n(X\ U, A\ U) \to H_n(X, A)$.

Excluded Middle Law

A law in (2-valued) LOGIC which states there is no third alternative to TRUTH or FALSEHOOD. In other words, every statement must be either A or not-A. This fact no longer holds in THREE-VALUED LOGIC or FUZZY LOGIC.

Excludent

A method which can be used to solve any QUADRATIC CONGRUENCE. This technique relies on the fact that solving

$$x^2 \equiv b \pmod{p}$$

is equivalent to finding a value y such that

$$b + py = x^2.$$

Pick a few small moduli m. If y mod m does not make $b+py$ a quadratic residue of m, then this value of y may be excluded. Furthermore, values of $y > p/4$ are never necessary.

Excludent Factorization Method

Also known as the difference of squares. It was first used by Fermat and improved by Gauss. Gauss looked for INTEGERS x and y satisfying

$$y^2 \equiv x^2 - N \pmod{E}$$

for various moduli E. This allowed the exclusion of many potential factors. This method works best when factors are of approximately the same size, so it is sometimes better to attempt mN for some suitably chosen value of m.

see also PRIME FACTORIZATION ALGORITHMS

Exclusive Or

see XOR

Exeter Point

Define A' to be the point (other than the VERTEX A) where the MEDIAN through A meets the CIRCUMCIRCLE of ABC, and define B' and C' similarly. Then the Exeter point is the PERSPECTIVE CENTER of the TRIANGLE $A'B'C'$ and the TANGENTIAL TRIANGLE. It has TRIANGLE CENTER FUNCTION

$$\alpha = a(b^4 + c^4 - a^4).$$

References

Kimberling, C. "Central Points and Central Lines in the Plane of a Triangle." *Math. Mag.* **67**, 163–187, 1994.

Kimberling, C. "Exeter Point." `http://www.evansville.edu/~ck6/tcenters/recent/exeter.html`.

Kimberling, C. and Lossers, O. P. "Problem 6557 and Solution." *Amer. Math. Monthly* **97**, 535–537, 1990.

Exhaustion Method

The method of exhaustion was a INTEGRAL-like limiting process used by Archimedes to compute the AREA and VOLUME of 2-D LAMINA and 3-D SOLIDS.

see also INTEGRAL, LIMIT

Existence

If at least one solution can be determined for a given problem, a solution to that problem is said to exist. Frequently, mathematicians seek to prove the existence of solutions and then investigate their UNIQUENESS.

see also EXISTS, UNIQUE

Existential Closure

A class of processes which attempt to round off a domain and simplify its theory by adjoining elements.

see also MODEL COMPLETION

References

Kenneth, M. "Domain Extension and the Philosophy of Mathematics." *J. Philos.* **86**, 553–562, 1989.

Exists

If there exists an A, this is written $\exists A$. Similarly, A does not exit is written $\nexists A$.

see also EXISTENCE, FOR ALL, QUANTIFIER

Exmedian

The line through the VERTEX of a TRIANGLE which is PARALLEL to the opposite side.

References

Johnson, R. A. *Modern Geometry: An Elementary Treatise on the Geometry of the Triangle and the Circle.* Boston, MA: Houghton Mifflin, p. 176, 1929.

Exmedian Point

The point of intersection of two EXMEDIANS.

References

Johnson, R. A. *Modern Geometry: An Elementary Treatise on the Geometry of the Triangle and the Circle.* Boston, MA: Houghton Mifflin, p. 176, 1929.

Exogenous Variable

An economic variable that is related to other economic variables and determines their equilibrium levels.

see also ENDOGENOUS VARIABLE

References

Iyanaga, S. and Kawada, Y. (Eds.). *Encyclopedic Dictionary of Mathematics.* Cambridge, MA: MIT Press, p. 458, 1980.

Exotic $\mathbb{R}^4$

Donaldson (1983) showed there exists an exotic smooth DIFFERENTIAL STRUCTURE on $\mathbb{R}^4$. Donaldson's result has been extended to there being precisely a CONTINUUM of nondiffeomorphic DIFFERENTIAL STRUCTURES on $\mathbb{R}^4$.

see also EXOTIC SPHERE

References

Donaldson, S. K. "Self-Dual Connections and the Topology of Smooth 4-Manifold." *Bull. Amer. Math. Soc.* **8**, 81–83, 1983.

Monastyrsky, M. *Modern Mathematics in the Light of the Fields Medals.* Wellesley, MA: A. K. Peters, 1997.

Exotic Sphere

Milnor (1963) found more than one smooth structure on the 7-D HYPERSPHERE. Generalizations have subsequently been found in other dimensions. Using SURGERY theory, it is possible to relate the number of DIFFEOMORPHISM classes of exotic spheres to higher homotopy groups of spheres (Kosinski 1992). Kervaire and Milnor (1963) computed a list of the number $N(d)$ of distinct (up to DIFFEOMORPHISM) DIFFERENTIAL STRUCTURES on spheres indexed by the DIMENSION d of the sphere. For $d = 1, 2, \ldots$, assuming the POINCARÉ CONJECTURE, they are 1, 1, 1, ≥ 1, 1, 1, 28, 2, 8, 6, 992, 1, 3, 2, 16256, 2, 16, 16, ... (Sloane's A001676). The status of $d = 4$ is still unresolved: at least one exotic structure exists, but it is not known if others do as well.

The only exotic Euclidean spaces are a CONTINUUM of EXOTIC $\mathbb{R}^4$ structures.

see also EXOTIC $\mathbb{R}^4$, HYPERSPHERE

References

Kervaire, M. A. and Milnor, J. W. "Groups of Homotopy Spheres: I." *Ann. Math.* **77**, 504–537, 1963.

Kosinski, A. A. §X.6 in *Differential Manifolds.* Boston, MA: Academic Press, 1992.

Milnor, J. "Topological Manifolds and Smooth Manifolds." *Proc. Internat. Congr. Mathematicians (Stockholm, 1962)* Djursholm: Inst. Mittag-Leffler, pp. 132–138, 1963.

Milnor, J. W. and Stasheff, J. D. *Characteristic Classes.* Princeton, NJ: Princeton University Press, 1973.

Monastyrsky, M. *Modern Mathematics in the Light of the Fields Medals.* Wellesley, MA: A. K. Peters, 1997.

Novikov, S. P. (Ed.). *Topology I.* New York: Springer-Verlag, 1996.

Sloane, N. J. A. Sequence A001676/M5197 in "An On-Line Version of the Encyclopedia of Integer Sequences."

Exp

see EXPONENTIAL FUNCTION

Expansion

An AFFINE TRANSFORMATION in which the scale is increased. It is the opposite of a DILATION (CONTRACTION).

see also DILATION

Expansive

Let ϕ be a MAP. Then ϕ is expansive if the DISTANCE $d(\phi^n x, \phi^n y) < \delta$ for all $n \in \mathbb{Z}$, then $x = y$. Equivalently, ϕ is expansive if the orbits of two points x and y are always very close.

Expectation Value

For one discrete variable,

$$\langle f(x) \rangle = \sum_x P(x). \tag{1}$$

For one continuous variable,

$$\langle f(x) \rangle = \int f(x)P(x)\,dx. \tag{2}$$

The expectation value satisfies

$$\langle ax + by \rangle = a\langle x \rangle + b\langle y \rangle \tag{3}$$

$$\langle a \rangle = a \tag{4}$$

$$\left\langle \sum x \right\rangle = \sum \langle x \rangle. \tag{5}$$

For multiple discrete variables

$$\langle f(x_1, \ldots, x_n) \rangle = \sum_{x_1, \ldots, x_n} P(x_1, \ldots, x_n). \tag{6}$$

For multiple continuous variables

$$\langle f(x_1, \ldots, x_n) \rangle = \int f(x_1, \ldots, x_n) P(x_1, \ldots, x_n)\,dx_1 \cdots dx_n. \tag{7}$$

The (multiple) expectation value satisfies

$$\begin{aligned} \langle (x - \mu_x)(y - \mu_y) \rangle &= \langle xy - \mu_x y - \mu_y x + \mu_x \mu_y \rangle \\ &= \langle xy \rangle - \mu_x \mu_y - \mu_y \mu_x + \mu_x \mu_y \\ &= \langle xy \rangle - \langle x \rangle \langle y \rangle, \end{aligned} \tag{8}$$

where μ_i is the MEAN for the variable i.

see also MEAN

Experimental Design

see DESIGN

Exploration Problem

see JEEP PROBLEM

Exponent

The POWER p in an expression a^p.

Exponent Laws

The laws governing the combination of EXPONENTS (POWERS) are

$$x^m \cdot x^n = x^{m+n} \tag{1}$$

$$\frac{x^m}{x^n} = x^{m-n} \tag{2}$$

$$(x^m)^n = x^{mn} \tag{3}$$

$$(xy)^m = x^m y^m \tag{4}$$

$$\left(\frac{x}{y}\right)^n = \frac{x^n}{y^n} \tag{5}$$

$$x^{-n} = \frac{1}{x^n} \tag{6}$$

$$\left(\frac{x}{y}\right)^{-n} = \left(\frac{y}{x}\right)^n, \tag{7}$$

where quantities in the DENOMINATOR are taken to be nonzero. Special cases include

$$x^1 = x \tag{8}$$

and

$$x^0 = 1 \tag{9}$$

for $x \neq 0$. The definition $0^0 = 1$ is sometimes used to simplify formulas, but it should be kept in mind that this equality is a definition and not a fundamental mathematical truth.

see also EXPONENT, POWER

Exponent Vector

Let p_i denote the ith PRIME, and write

$$m = \prod_i {p_i}^{v_i}.$$

Then the exponent vector is $\mathbf{v}(m) = (v_1, v_2, \ldots)$.

see also DIXON'S FACTORIZATION METHOD

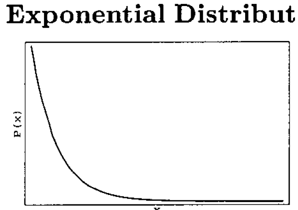
References

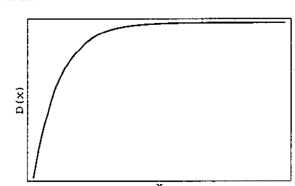
Pomerance, C. "A Tale of Two Sieves." *Not. Amer. Math. Soc.* **43**, 1473–1485, 1996.

Exponential Digital Invariant

see NARCISSISTIC NUMBER

Exponential Distribution





Given a POISSON DISTRIBUTION with rate of change λ, the distribution of waiting times between successive changes (with $k = 0$) is

$$D(x) \equiv P(X \leq x) = 1 - P(X > x) = 1 - \frac{(\lambda x)^0 e^{-\lambda x}}{0!} = 1 - e^{-\lambda x} \tag{1}$$

$$P(x) = D'(x) = \lambda e^{-\lambda x}, \tag{2}$$

which is normalized since

$$\int_0^\infty P(x)\,dx = \lambda \int_0^\infty e^{-\lambda x}\,dx = -[e^{-\lambda x}]_0^\infty = -(0-1) = 1. \tag{3}$$

This is the only MEMORYLESS RANDOM DISTRIBUTION. Define the MEAN waiting time between successive changes as $\theta \equiv \lambda^{-1}$. Then

$$P(x) = \begin{cases} \frac{1}{\theta} e^{-x/\theta} & x \geq 0 \\ 0 & x < 0. \end{cases} \tag{4}$$

The MOMENT-GENERATING FUNCTION is

$$M(t) = \int_0^\infty e^{tx} \left(\frac{1}{\theta}\right) e^{-x/\theta}\,dx = \frac{1}{\theta} \int_0^\infty e^{-(1-\theta t)x/\theta}\,dx = \left[\frac{e^{-(1-\theta t)x/\theta}}{1-\theta t}\right]_0^\infty = \frac{1}{1-\theta t} \tag{5}$$

$$M'(t) = \frac{\theta}{(1-\theta t)^2} \tag{6}$$

$$M''(t) = \frac{2\theta^2}{(1-\theta t)^3}, \tag{7}$$

so

$$R(t) \equiv \ln M(t) = -\ln(1-\theta t) \tag{8}$$

$$R'(t) = \frac{\theta}{1-\theta t} \tag{9}$$

$$R''(t) = \frac{\theta^2}{(1-\theta t)^2} \tag{10}$$

$$\mu = R'(0) = \theta \tag{11}$$

$$\sigma^2 = R''(0) = \theta^2. \tag{12}$$

The SKEWNESS and KURTOSIS are given by

$$\gamma_1 = 2 \tag{13}$$

$$\gamma_2 = 6. \tag{14}$$

The MEAN and VARIANCE can also be computed directly

$$\langle x \rangle \equiv \int_0^\infty P(x)\,dx = \frac{1}{s} \int_0^\infty x e^{-x/s}\,dx. \tag{15}$$

Use the integral

$$\int x e^{ax}\,dx = \frac{e^{ax}}{a^2}(ax-1) \tag{16}$$

to obtain

$$\langle x \rangle = \frac{1}{s} \left[\frac{e^{-x/s}}{\left(-\frac{1}{s}\right)^2} \left\{\left(-\frac{1}{s}\right)x - 1\right\}\right]_0^\infty = -s\left[e^{-x/s}\left(1+\frac{x}{s}\right)\right]_0^\infty = -s(0-1) = s. \tag{17}$$

Now, to find

$$\left\langle x^2 \right\rangle = \frac{1}{s} \int_0^\infty x^2 e^{-x/s}\,dx, \tag{18}$$

use the integral

$$\int x^2 e^{-x/s}\,dx = \frac{e^{ax}}{a^3}(2 - 2ax + a^2x^2) \tag{19}$$

$$\left\langle x^2 \right\rangle = \frac{1}{s} \left[\frac{e^{-x/s}}{\left(-\frac{1}{s}\right)^3} \left(2 + \frac{2}{s}x + \frac{1}{s^2}x^2\right)\right]_0^\infty = -s^2(0-2) = 2s^2, \tag{20}$$

giving

$$\sigma^2 \equiv \left\langle x^2 \right\rangle - \langle x \rangle^2 = 2s^2 - s^2 = s^2 \tag{21}$$

$$\sigma \equiv \sqrt{\operatorname{var}(x)} = s. \tag{22}$$

If a generalized exponential probability function is defined by

$$P_{(\alpha,\beta)}(x) = \frac{1}{\beta} e^{-(x-\alpha)/\beta}, \tag{23}$$

then the CHARACTERISTIC FUNCTION is

$$\phi(t) = \frac{e^{i\alpha t}}{1 - i\beta t}, \tag{24}$$

and the MEAN, VARIANCE, SKEWNESS, and KURTOSIS are

$$\mu = \alpha + \beta \tag{25}$$

$$\sigma^2 = \beta^2 \tag{26}$$

$$\gamma_1 = 2 \tag{27}$$

$$\gamma_2 = 6. \tag{28}$$

see also DOUBLE EXPONENTIAL DISTRIBUTION

References

Balakrishnan, N. and Basu, A. P. *The Exponential Distribution: Theory, Methods, and Applications.* New York: Gordon and Breach, 1996.

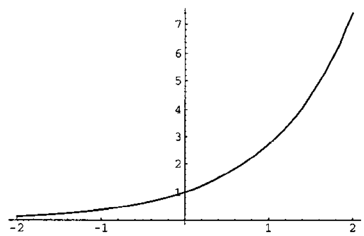
Beyer, W. H. *CRC Standard Mathematical Tables, 28th ed.* Boca Raton, FL: CRC Press, pp. 534–535, 1987.

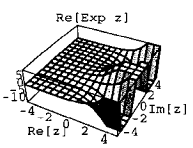
Spiegel, M. R. *Theory and Problems of Probability and Statistics.* New York: McGraw-Hill, p. 119, 1992.

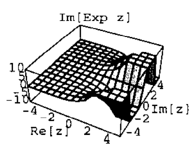
Exponential Divisor

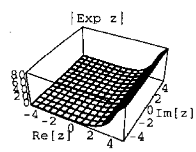
see e-DIVISOR

Exponential Function









The exponential function is defined by

$$\exp(x) \equiv e^x, \tag{1}$$

where e is the constant 2.718.... It satisfies the identity

$$\exp(x+y) = \exp(x)\exp(y). \tag{2}$$

If $z \equiv x + iy$,

$$e^z = e^{x+iy} = e^x e^{iy} = e^x(\cos y + i \sin y). \tag{3}$$

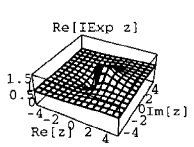
If

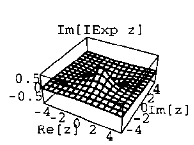
$$a + bi = e^{x+iy}, \tag{4}$$

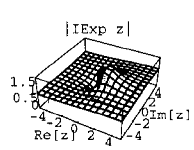
then

$$y = \tan^{-1}\left(\frac{b}{a}\right) \tag{5}$$

$$x = \ln\left\{b \csc\left[\tan^{-1}\left(\frac{b}{a}\right)\right]\right\} = \ln\left\{a \sec\left[\tan^{-1}\left(\frac{b}{a}\right)\right]\right\}. \tag{6}$$







The above plot shows the function $e^{1/z}$.

see also EULER FORMULA, EXPONENTIAL RAMP, FOURIER TRANSFORM—EXPONENTIAL FUNCTION, SIGMOID FUNCTION

References

Abramowitz, M. and Stegun, C. A. (Eds.). "Exponential Function." §4.2 in *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables, 9th printing.* New York: Dover, p. 69–71, 1972.

Fischer, G. (Ed.). Plates 127–128 in *Mathematische Modelle/Mathematical Models, Bildband/Photograph Volume.* Braunschweig, Germany: Vieweg, pp. 124–125, 1986.

Spanier, J. and Oldham, K. B. "The Exponential Function $\exp(bx+c)$" and "Exponentials of Powers $\exp(-ax^\nu)$." Chs. 26–27 in *An Atlas of Functions.* Washington, DC: Hemisphere, pp. 233–261, 1987.

Yates, R. C. "Exponential Curves." *A Handbook on Curves and Their Properties.* Ann Arbor, MI: J. W. Edwards, pp. 86–97, 1952.

Exponential Function (Truncated)

see EXPONENTIAL SUM FUNCTION

Exponential Inequality

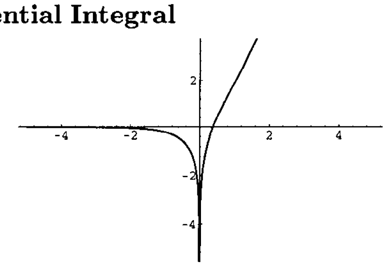
For $c < 1$,

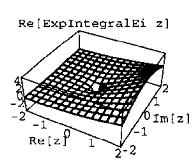
$$x^c < 1 + c(x-1).$$

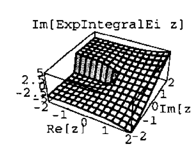
For $c > 1$,

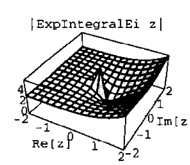
$$x^c > 1 + c(x-1).$$

Exponential Integral









Let $E_1(x)$ be the E_n-FUNCTION with $n = 1$,

$$E_1(x) \equiv \int_1^\infty \frac{e^{-tx}\,dt}{t} = \int_x^\infty \frac{e^{-u}\,du}{u}. \tag{1}$$

Then define the exponential integral $\mathrm{ei}(x)$ by

$$E_1(x) = -\mathrm{ei}(-x), \tag{2}$$

where the retention of the $-\mathrm{ei}(-x)$ NOTATION is a historical artifact. Then $\mathrm{ei}(x)$ is given by the integral

$$\mathrm{ei}(x) = -\int_{-x}^\infty \frac{e^{-t}\,dt}{t}. \tag{3}$$

This function is given by the *Mathematica*® (Wolfram Research, Champaign, IL) function `ExpIntegralEi[x]`. The exponential integral can also be written

$$\mathrm{ei}(ix) = \mathrm{ci}(x) + i\,\mathrm{si}(x), \tag{4}$$

where $\mathrm{ci}(x)$ and $\mathrm{si}(x)$ are COSINE and SINE INTEGRAL.

The real ROOT of the exponential integral occurs at 0.37250741078..., which is not known to be expressible in terms of other standard constants. The quantity $-e\,\mathrm{ei}(-1) = 0.596347362\ldots$ is known as the GOMPERTZ CONSTANT.

see also COSINE INTEGRAL, E_n-FUNCTION, GOMPERTZ CONSTANT, SINE INTEGRAL

References

Arfken, G. *Mathematical Methods for Physicists, 3rd ed.* Orlando, FL: Academic Press, pp. 566–568, 1985.

Morse, P. M. and Feshbach, H. *Methods of Theoretical Physics, Part I.* New York: McGraw-Hill, pp. 434–435, 1953.

Press, W. H.; Flannery, B. P.; Teukolsky, S. A.; and Vetterling, W. T. "Exponential Integrals." §6.3 in *Numerical Recipes in FORTRAN: The Art of Scientific Computing, 2nd ed.* Cambridge, England: Cambridge University Press, pp. 215–219, 1992.

Spanier, J. and Oldham, K. B. "The Exponential Integral Ei(x) and Related Functions." Ch. 37 in *An Atlas of Functions.* Washington, DC: Hemisphere, pp. 351–360, 1987.

Exponential Map

On a LIE GROUP, exp is a MAP from the LIE ALGEBRA to its LIE GROUP. If you think of the LIE ALGEBRA as the TANGENT SPACE to the identity of the LIE GROUP, $\exp(v)$ is defined to be $h(1)$, where h is the unique LIE GROUP HOMEOMORPHISM from the REAL NUMBERS to the LIE GROUP such that its velocity at time 0 is v.

On a RIEMANNIAN MANIFOLD, exp is a MAP from the TANGENT BUNDLE of the MANIFOLD to the MANIFOLD, and $\exp(v)$ is defined to be $h(1)$, where h is the unique GEODESIC traveling through the base-point of v such that its velocity at time 0 is v.

The three notions of exp (exp from COMPLEX ANALYSIS, exp from LIE GROUPS, and exp from Riemannian geometry) are all linked together, the strongest link being between the LIE GROUPS and Riemannian geometry definition. If G is a compact LIE GROUP, it admits a left and right invariant RIEMANNIAN METRIC. With respect to that metric, the two exp maps agree on their common domain. In other words, one-parameter subgroups are geodesics. In the case of the MANIFOLD $\mathbb{S}^1$, the CIRCLE, if we think of the tangent space to 1 as being the IMAGINARY axis (y-AXIS) in the COMPLEX PLANE, then

$$\begin{aligned}\exp_{\text{Riemannian geometry}}(v) &= \exp_{\text{Lie Groups}}(v)\\ &= \exp_{\text{complex analysis}}(v),\end{aligned}$$

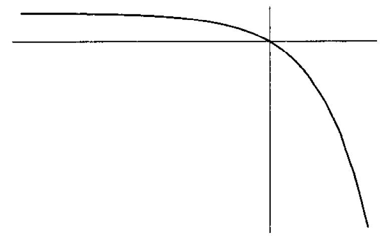
and so the three concepts of the exponential all agree in this case.

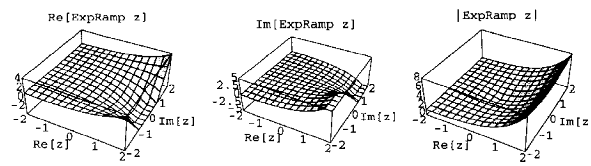
see also EXPONENTIAL FUNCTION

Exponential Matrix

see MATRIX EXPONENTIAL

Exponential Ramp





The curve

$$y = 1 - e^{ax}.$$

see also EXPONENTIAL FUNCTION, SIGMOID FUNCTION

References

von Seggern, D. *CRC Standard Curves and Surfaces.* Boca Raton, FL: CRC Press, p. 158, 1993.

Exponential Sum Formulas

$$\begin{aligned}\sum_{n=0}^{N-1} e^{iNx} &= \frac{1-e^{iNx}}{1-e^{ix}} = \frac{-e^{iNx/2}\left(e^{-iNx/2}-e^{iNx/2}\right)}{-e^{ix/2}\left(e^{-ix/2}-e^{ix/2}\right)}\\ &= \frac{\sin(\frac{1}{2}Nx)}{\sin(\frac{1}{2}x)}e^{ix(N-1)/2}, \qquad (1)\end{aligned}$$

where

$$\sum_{n=0}^{N-1} r^n = \frac{1-r^n}{1-r} \qquad (2)$$

has been used. Similarly,

$$\begin{aligned}\sum_{n=0}^{N-1} p^n e^{inx} &= \frac{1-p^Ne^{iNx}}{1-pe^{ix}} = \frac{(1-p^Ne^{iNx})(1-pe^{-ix})}{(1-pe^{ix})(1-pe^{-ix})}\\ &= \frac{1-p^Ne^{iNx}-pe^{-ix}+p^{N+1}e^{ix(N-1)}}{1-p(e^{ix}+e^{-ix})+p^2}\\ &= \frac{p^{N+1}e^{ix(N-1)}-p^Ne^{iNx}+1-pe^{-ix}}{1-2p\cos x+p^2}. \qquad (3)\end{aligned}$$

This gives

$$\sum_{n=0}^{\infty} p^n e^{inx} = \lim_{N\to\infty}\sum_{n=0}^{N-1} p^n e^{inx} = \frac{1-pe^{-ix}}{1-2p\cos x+p^2}. \qquad (4)$$

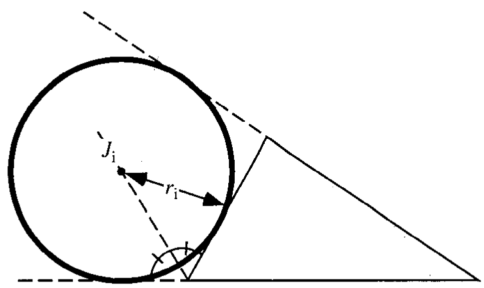
By looking at the REAL and IMAGINARY PARTS of these FORMULAS, sums involving sines and cosines can be obtained.

Exponential Sum Function

$$\mathrm{es}_n(x) \equiv \exp_n(x) \equiv \sum_{m=0}^{n} \frac{x^m}{m!}.$$

see also GAMMA FUNCTION

Exradius



The RADIUS of an EXCIRCLE. Let a TRIANGLE have exradius r_a (sometimes denoted ρ_a), opposite side of length a, AREA Δ, and SEMIPERIMETER s. Then

$$\begin{aligned} {r_a}^2 &= \left(\frac{\Delta}{s-a}\right)^2 &(1)\\ &= \frac{s(s-c)(s-b)}{s-a} &(2)\\ &= 4R\sin(\tfrac{1}{2}\alpha_1)\cos(\tfrac{1}{2}\alpha_2)\cos(\tfrac{1}{2}\alpha_3) &(3)\end{aligned}$$

(Johnson 1929, p. 189) where R is the CIRCUMRADIUS. Let r be the INRADIUS, then

$$4R = r_a + r_b + r_c - r \qquad (4)$$

$$\frac{1}{r_a} + \frac{1}{r_b} + \frac{1}{r_c} = \frac{1}{r} \qquad (5)$$

$$r r_a r_b r_c = \Delta^2. \qquad (6)$$

Some fascinating FORMULAS due to Feuerbach are

$$r_2 r_3 + r_3 r_1 + r_1 r_3 = s^2 \qquad (7)$$

$$r(r_2 r_3 + r_3 r_1 + r_1 r_2) = s\Delta = r_1 r_2 r_3 \qquad (8)$$

$$r(r_1 + r_2 + r_3) = a_2 a_3 + a_3 a_1 + a_1 a_2 - s^2 \qquad (9)$$

$$r r_1 + r r_2 + r r_3 + r_1 r_2 + r_2 r_3 + r_3 r_1 = a_2 a_3 + a_3 a_1 + a_1 a_2 \qquad (10)$$

$$r_2 r_3 + r_3 r_1 + r_1 r_2 - r r_1 - r r_2 - r r_3 - \tfrac{1}{2}({a_1}^2 + {a_2}^2 + {a_3}^2) \qquad (11)$$

(Johnson 1929, pp. 190–191).

see also CIRCLE, CIRCUMRADIUS, EXCIRCLE, INRADIUS, RADIUS

References

Johnson, R. A. *Modern Geometry: An Elementary Treatise on the Geometry of the Triangle and the Circle.* Boston, MA: Houghton Mifflin, 1929.

Mackay, J. S. "Formulas Connected with the Radii of the Incircle and Excircles of a Triangle." *Proc. Edinburgh Math. Soc.* **12**, 86–105.

Mackay, J. S. "Formulas Connected with the Radii of the Incircle and Excircles of a Triangle." *Proc. Edinburgh Math. Soc.* **13**, 103–104.

Exsecant

$$\operatorname{exsec} x \equiv \sec x - 1,$$

where $\sec x$ is the SECANT.

see also COVERSINE, HAVERSINE, SECANT, VERSINE

References

Abramowitz, M. and Stegun, C. A. (Eds.). *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables, 9th printing.* New York: Dover, p. 78, 1972.

Extended Cycloid

see PROLATE CYCLOID

Extended Goldbach Conjecture

see GOLDBACH CONJECTURE

Extended Greatest Common Divisor

see GREATEST COMMON DIVISOR

Extended Mean-Value Theorem

Let the functions f and g be DIFFERENTIABLE on the OPEN INTERVAL (a, b) and CONTINUOUS on the CLOSED INTERVAL $[a, b]$. If $g'(x) \neq 0$ for any $x \in (a, b)$, then there is at least one point $c \in (a, b)$ such that

$$\frac{f'(c)}{g'(c)} = \frac{f(b) - f(a)}{g(b) - g(a)}.$$

see also MEAN-VALUE THEOREM

Extended Riemann Hypothesis

The first quadratic nonresidue mod p of a number is always less than $2(\ln p)^2$.

see also RIEMANN HYPOTHESIS

References

Bach, E. *Analytic Methods in the Analysis and Design of Number-Theoretic Algorithms.* Cambridge, MA: MIT Press, 1985.

Wagon, S. *Mathematica in Action.* New York: W. H. Freeman, p. 295, 1991.

Extension

The definition of a SET by enumerating its members. An extensional definition can always be reduced to an INTENTIONAL one.

see also INTENSION

References

Russell, B. "Definition of Number." *Introduction to Mathematical Philosophy.* New York: Simon and Schuster, 1971.

Extension Problem

Given a SUBSPACE A of a SPACE X and a MAP from A to a SPACE Y, is it possible to extend that MAP to a MAP from X to Y?

see also LIFTING PROBLEM

Extensions Calculus

see EXTERIOR ALGEBRA

Extent

The RADIUS of the smallest CIRCLE centered at one of the points of an N-CLUSTER, which contains all the points in the N-CLUSTER.

see also N-CLUSTER

Exterior

That portion of a region lying "outside" a specified boundary.

see also INTERIOR

Exterior Algebra

The ALGEBRA of the EXTERIOR PRODUCT, also called an ALTERNATING ALGEBRA or GRASSMANN ALGEBRA. The study of exterior algebra is also called AUSDEHNUNGSLEHRE and EXTENSIONS CALCULUS. Exterior algebras are GRADED ALGEBRAS.

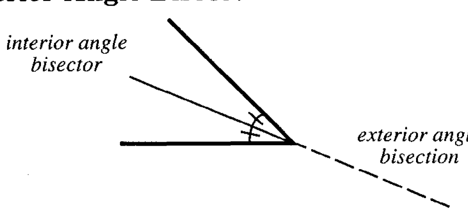
In particular, the exterior algebra of a VECTOR SPACE is the DIRECT SUM over k in the natural numbers of the VECTOR SPACES of alternating k-forms on that VECTOR SPACE. The product on this algebra is then the wedge product of forms. The exterior algebra for a VECTOR SPACE V is constructed by forming monomials u, $v \wedge w$, $x \wedge y \wedge z$, etc., where u, v, w, x, y, and z are vectors in V and $\wedge$ is asymmetric multiplication. The sums formed from linear combinations of the MONOMIALS are the elements of an exterior algebra.

References

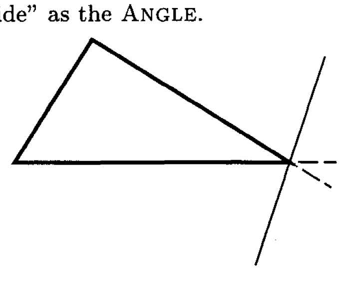
Forder, H. G. *The Calculus of Extension.* Cambridge, England: Cambridge University Press, 1941.

Lounesto, P. "Counterexamples to Theorems Published and Proved in Recent Literature on Clifford Algebras, Spinors, Spin Groups, and the Exterior Algebra." `http://www.hit.fi/~lounesto/counterexamples.htm`.

Exterior Angle Bisector



The exterior bisector of an ANGLE is the LINE or LINE SEGMENT which cuts it into two equal ANGLES on the opposite "side" as the ANGLE.



For a TRIANGLE, the exterior angle bisector bisects the SUPPLEMENTARY ANGLE at a given VERTEX. It also divides the opposite side externally in the ratio of adjacent sides.

see also ANGLE BISECTOR, ISODYNAMIC POINTS

Exterior Angle Theorem

In any TRIANGLE, if one of the sides is extended, the exterior angle is greater than both the interior and opposite angles.

References

Dunham, W. *Journey Through Genius: The Great Theorems of Mathematics.* New York: Wiley, p. 41, 1990.

Exterior Derivative

Consider a DIFFERENTIAL k-FORM

$$\omega^1 = b_1\,dx_1 + b_2\,dx_2. \tag{1}$$

Then its exterior derivative is

$$d\omega^1 = db_1 \wedge dx_1 + db_2 \wedge dx_2, \tag{2}$$

where $\wedge$ is the WEDGE PRODUCT. Similarly, consider

$$\omega^1 = b_1(x_1,x_2)\,dx_1 + b_2(x_1,x_2)\,dx_2. \tag{3}$$

Then

$$\begin{aligned} d\omega^1 &= db_1 \wedge dx_1 + db_2 \wedge dx_2 \\ &= \left(\frac{\partial b_1}{\partial x_1}dx_1 + \frac{\partial b_1}{\partial x_2}dx_2\right) \wedge dx_1 \\ &\quad + \left(\frac{\partial b_2}{\partial x_1}dx_1 + \frac{\partial b_2}{\partial x_2}dx_2\right) \wedge dx_2. \end{aligned} \tag{4}$$

Denote the exterior derivative by

$$Dt \equiv \frac{\partial}{\partial x} \wedge t. \tag{5}$$

Then for a 0-form t,

$$(Dt)_\mu \equiv \frac{\partial t}{\partial x^\mu}, \tag{6}$$

for a 1-form t,

$$(Dt)_{\mu\nu} \equiv \frac{1}{2}\left(\frac{\partial t_\nu}{\partial x^\mu} - \frac{\partial t_\mu}{\partial x^\nu}\right), \tag{7}$$

and for a 2-form t,

$$(Dt)_{ijk} \equiv \tfrac{1}{3}\epsilon_{ijk}\left(\frac{\partial t_{23}}{\partial x^1} + \frac{\partial t_{31}}{\partial x^2} + \frac{\partial t_{12}}{\partial x^3}\right), \tag{8}$$

where ϵ_{ijk} is the PERMUTATION TENSOR.

The second exterior derivative is

$$D^2 t = \frac{\partial}{\partial x} \wedge \left(\frac{\partial}{\partial x} \wedge t\right) = \left(\frac{\partial}{\partial x} \wedge \frac{\partial}{\partial x}\right) \wedge t = 0, \tag{9}$$

which is known as POINCARÉ'S LEMMA.

see also DIFFERENTIAL k-FORM, POINCARÉ'S LEMMA, WEDGE PRODUCT

Exterior Dimension

A type of DIMENSION which can be used to characterize FAT FRACTALS.

see also FAT FRACTAL

References

Grebogi, C.; McDonald, S. W.; Ott, E.; and Yorke, J. A. "Exterior Dimension of Fat Fractals." *Phys. Let. A* **110**, 1–4, 1985.

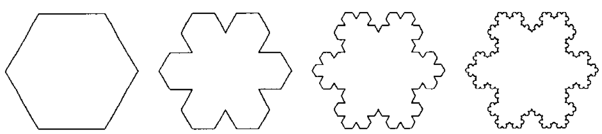
Grebogi, C.; McDonald, S. W.; Ott, E.; and Yorke, J. A. Erratum to "Exterior Dimension of Fat Fractals." *Phys. Let. A* **113**, 495, 1986.

Ott, E. *Chaos in Dynamical Systems.* New York: Cambridge University Press, p. 98, 1993.

Exterior Product

see WEDGE PRODUCT

Exterior Snowflake



A FRACTAL.

see also FLOWSNAKE FRACTAL, KOCH ANTISNOWFLAKE, KOCH SNOWFLAKE, PENTAFLAKE

References

Wagon, S. *Mathematica in Action.* New York: W. H. Freeman, pp. 193–195, 1991.

Weisstein, E. W. "Fractals." `http://www.astro.virginia.edu/~eww6n/math/notebooks/Fractal.m`.

Extra Strong Lucas Pseudoprime

Given the LUCAS SEQUENCE $U_n(b,-1)$ and $V_n(b,-1)$, define $\Delta = b^2 - 4$. Then an extra strong Lucas pseudoprime to the base b is a COMPOSITE NUMBER $n = 2^r s + (\Delta/n)$, where s is ODD and $(n, 2\Delta) = 1$ such that either $U_s \equiv 0 \pmod{n}$ and $V_s \equiv \pm 2 \pmod{n}$, or $V_{2^t s} \equiv 0 \pmod{n}$ for some t with $0 \leq t < r - 1$. An extra strong Lucas pseudoprime is a STRONG LUCAS PSEUDOPRIME with parameters $(b, -1)$. COMPOSITE n are extra strong pseudoprimes for at most 1/8 of possible bases (Grantham 1997).

see also LUCAS PSEUDOPRIME, STRONG LUCAS PSEUDOPRIME

References

Grantham, J. "Frobenius Pseudoprimes." `http://www.clark.net/pub/grantham/pseudo/pseudo.ps`

Grantham, J. "A Frobenius Probable Prime Test with High Confidence." 1997. `http://www.clark.net/pub/grantham/pseudo/pseudo2.ps`

Jones, J. P. and Mo, Z. "A New Primality Test Using Lucas Sequences." Preprint.

Extrapolation

see RICHARDSON EXTRAPOLATION

Extremal Coloring

see EXTREMAL GRAPH

Extremal Graph

A two-coloring of a COMPLETE GRAPH K_n of n nodes which contains exactly the number $N \equiv (R+B)_{\min}$ of MONOCHROMATIC FORCED TRIANGLES and no more (i.e., a minimum of $R+B$ where R and B are the numbers of red and blue TRIANGLES). Goodman (1959) showed that for an extremal graph,

$$N(n) = \begin{cases} \frac{1}{3}m(m-1)(m-2) & \text{for } n = 2m \\ \frac{1}{3}2m(m-1)(4m+1) & \text{for } n = 4m+1 \\ \frac{1}{3}2m(m+1)(4m-1) & \text{for } n = 4m+3. \end{cases}$$

This is sometimes known as GOODMAN'S FORMULA. Schwenk (1972) rewrote it in the form

$$N(n) = \binom{n}{3} - \left\lfloor \tfrac{1}{2}n \left\lfloor \tfrac{1}{4}(n-1)^2 \right\rfloor \right\rfloor,$$

sometimes known as SCHWENK'S FORMULA, where $\lfloor x \rfloor$ is the FLOOR FUNCTION. The first few values of $N(n)$ for $n = 1, 2, \ldots$ are 0, 0, 0, 0, 0, 2, 4, 8, 12, 20, 28, 40, 52, 70, 88, ... (Sloane's A014557).

see also BICHROMATIC GRAPH, BLUE-EMPTY GRAPH, GOODMAN'S FORMULA, MONOCHROMATIC FORCED TRIANGLE, SCHWENK'S FORMULA

References

Goodman, A. W. "On Sets of Acquaintances and Strangers at Any Party." *Amer. Math. Monthly* **66**, 778–783, 1959.

Schwenk, A. J. "Acquaintance Party Problem." *Amer. Math. Monthly* **79**, 1113–1117, 1972.

Sloane, N. J. A. Sequence A014557 in "An On-Line Version of the Encyclopedia of Integer Sequences."

Extremals

A field of extremals is a plane region which is SIMPLY CONNECTED by a one-parameter family of extremals. The concept was invented by Weierstraß.

Extreme and Mean Ratio

see GOLDEN MEAN

Extreme Value Distribution

N.B. A detailed on-line essay by S. Finch was the starting point for this entry.

Let M_n denote the "extreme" (i.e., largest) ORDER STATISTIC $X^{(n)}$ for a distribution of n elements X_i taken from a continuous UNIFORM DISTRIBUTION. Then the distribution of the M_n is

$$P(M_n < x) = \begin{cases} 0 & \text{if } x < 0 \\ x^n & \text{if } 0 \leq x \leq 1 \\ 1 & \text{if } x > 1, \end{cases} \tag{1}$$

and the MEAN and VARIANCE are

$$\mu = \frac{n}{n+1} \tag{2}$$

$$\sigma^2 = \frac{n}{(n+1)^2(n+2)}. \tag{3}$$

If X_i are taken from a STANDARD NORMAL DISTRIBUTION, then its cumulative distribution is

$$F(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-t^2/2}\,dt = \tfrac{1}{2} + \Phi(x), \tag{4}$$

where $\Phi(x)$ is the NORMAL DISTRIBUTION FUNCTION. The probability distribution of M_n is then

$$P(M_n < x) = [F(x)]^n = \frac{n}{\sqrt{2\pi}} \int_{-\infty}^{x} [F(t)]^{n-1} e^{-t^2/2}\,dt. \tag{5}$$

The MEAN $\mu(n)$ and VARIANCE $\sigma^2(n)$ are expressible in closed form for small n,

$$\mu(1) = 0 \tag{6}$$

$$\mu(2) = \frac{1}{\sqrt{\pi}} \tag{7}$$

$$\mu(3) = \frac{3}{2\sqrt{\pi}} \tag{8}$$

$$\mu(4) = \frac{3}{2\sqrt{\pi}} \left[1 + \frac{2}{\pi} \sin^{-1}(\tfrac{1}{3})\right] \tag{9}$$

$$\mu(5) = \frac{5}{4\sqrt{\pi}} \left[1 + \frac{6}{\pi} \sin^{-1}(\tfrac{1}{3})\right] \tag{10}$$

and

$$\sigma^2(1) = 1 \tag{11}$$

$$\sigma^2(2) = 1 - \frac{1}{\pi} \tag{12}$$

$$\sigma^2(3) = \frac{4\pi - 9 + 2\sqrt{3}}{4\pi} \tag{13}$$

$$\sigma^2(4) = 1 + \frac{\sqrt{3}}{\pi} - [\mu(4)]^2 \tag{14}$$

$$\sigma^2(5) = 1 + \frac{5\sqrt{3}}{4\pi} + \frac{5\sqrt{3}}{2\pi^2} \sin^{-1}(\tfrac{1}{4}) - [\mu(5)]^2. \tag{15}$$

No exact expression is known for $\mu(6)$ or $\sigma^2(6)$, but there is an equation connecting them

$$[\mu(6)]^2 + \sigma^2(6) = 1 + \frac{5\sqrt{3}}{4\pi} + \frac{15\sqrt{3}}{2\pi^2} \sin^{-1}(\tfrac{1}{4}). \tag{16}$$

An analog to the CENTRAL LIMIT THEOREM states that the asymptotic normalized distribution of M_n satisfies one of the three distributions

$$P(y) = \exp(-e^{-y}) \tag{17}$$

$$P(y) = \begin{cases} 0 & \text{if } y \le 0 \\ \exp(-y^{-a}) & \end{cases} \tag{18}$$

$$P(y) = \begin{cases} \exp[-(-y)^a] & \text{if } y \le 0 \\ 1 & \text{if } y > 0, \end{cases} \tag{19}$$

also known as GUMBEL, Fréchet, and WEIBULL DISTRIBUTIONS, respectively.

see also FISHER-TIPPETT DISTRIBUTION, ORDER STATISTIC

References

Balakrishnan, N. and Cohen, A. C. *Order Statistics and Inference.* New York: Academic Press, 1991.

David, H. A. *Order Statistics, 2nd ed.* New York: Wiley, 1981.

Finch, S. "Favorite Mathematical Constants." `http://www.mathsoft.com/asolve/constant/extval/extval.html`.

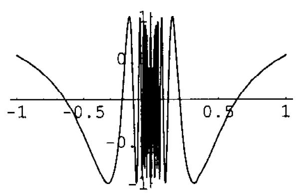
Gibbons, J. D. *Nonparametric Statistical Inference.* New York: McGraw-Hill, 1971.

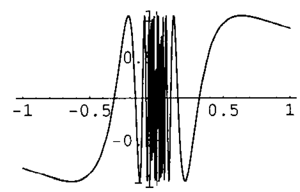
Extreme Value Theorem

If a function f is continuous on a closed interval $[a, b]$, then f has both a MAXIMUM and a MINIMUM on $[a, b]$. If f has an extreme value on an open interval (a, b), then the extreme value occurs at a CRITICAL POINT. This theorem is sometimes also called the WEIERSTRAß EXTREME VALUE THEOREM.

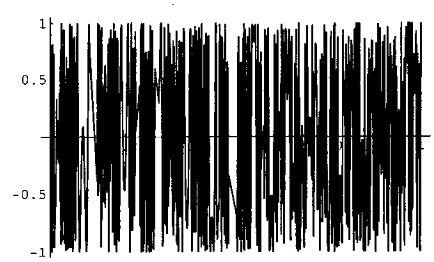
Extremum

A MAXIMUM or MINIMUM. An extremum may be LOCAL (a.k.a. a RELATIVE EXTREMUM; an extremum in a given region which is not the overall MAXIMUM or MINIMUM) or GLOBAL. Functions with many extrema can be very difficult to GRAPH. Notorious examples include the functions $\cos(1/x)$ and $\sin(1/x)$ near $x = 0$





and $\sin(e^{2x+9})$ near 0 and 1.



The latter has

$$\left\lfloor \frac{e^{11}}{\pi} - \frac{1}{2} \right\rfloor - \left\lceil \frac{e^{9}}{\pi} - \frac{1}{2} \right\rceil + 1 = 19058 - 2579 + 1 = 16480$$

extrema in the CLOSED INTERVAL [0,1] (Mulcahy 1996).

see also GLOBAL EXTREMUM, GLOBAL MAXIMUM, GLOBAL MINIMUM, KUHN-TUCKER THEOREM, LAGRANGE MULTIPLIER, LOCAL EXTREMUM, LOCAL MAXIMUM, LOCAL MINIMUM, MAXIMUM, MINIMUM

References

Abramowitz, M. and Stegun, C. A. (Eds.). *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables, 9th printing.* New York: Dover, p. 14, 1972.

Mulcahy, C. "Plotting and Scheming with Wavelets." *Math. Mag.* **69**, 323–343, 1996.

Tikhomirov, V. M. *Stories About Maxima and Minima.* Providence, RI: Amer. Math. Soc., 1991.

Extremum Test

Consider a function $f(x)$ in 1-D. If $f(x)$ has a relative extremum at x_0, then either $f'(x_0) = 0$ or f is not DIFFERENTIABLE at x_0. Either the first or second DERIVATIVE tests may be used to locate relative extrema of the first kind.

A NECESSARY condition for $f(x)$ to have a MINIMUM (MAXIMUM) at x_0 is

$$f'(x_0) = 0,$$

and

$$f''(x_0) \geq 0 \qquad (f''(x_0) \leq 0).$$

A SUFFICIENT condition is $f'(x_0) = 0$ and $f''(x_0) > 0$ ($f''(x_0) < 0$). Let $f'(x_0) = 0$, $f''(x_0) = 0$, ..., $f^{(n)}(x_0) = 0$, but $f^{(n+1)}(x_0) \neq 0$. Then $f(x)$ has a RELATIVE MAXIMUM at x_0 if n is ODD and $f^{(n+1)}(x_0) < 0$, and $f(x)$ has a RELATIVE MINIMUM at x_0 if n is ODD and $f^{(n+1)}(x_0) > 0$. There is a SADDLE POINT at x_0 if n is EVEN.

see also EXTREMUM, FIRST DERIVATIVE TEST, RELATIVE MAXIMUM, RELATIVE MINIMUM, SADDLE POINT (FUNCTION), SECOND DERIVATIVE TEST

Extrinsic Curvature

A curvature of a SUBMANIFOLD of a MANIFOLD which depends on its particular EMBEDDING. Examples of extrinsic curvature include the CURVATURE and TORSION of curves in 3-space, or the mean curvature of surfaces in 3-space.

see also CURVATURE, INTRINSIC CURVATURE, MEAN CURVATURE

F

F-Distribution

Arises in the testing of whether two observed samples have the same VARIANCE. Let ${\chi_m}^2$ and ${\chi_n}^2$ be independent variates distributed as CHI-SQUARED with m and n DEGREES OF FREEDOM. Define a statistic $F_{n,m}$ as the ratio of the dispersions of the two distributions

$$F_{n,m} \equiv \frac{{\chi_n}^2/n}{{\chi_m}^2/m}. \tag{1}$$

This statistic then has an F-distribution with probability function and cumulative distribution

$$F_{n,m}(x) = \frac{\Gamma(\frac{n+m}{2})n^{n/2}m^{m/2}}{\Gamma(\frac{n}{2})\Gamma(\frac{m}{2})}\frac{x^{n/2-1}}{(m+nx)^{(n+m)/2}} \tag{2}$$

$$= \frac{m^{m/2}n^{n/2}x^{n/2-1}}{(m+nx)^{(n+m)/2}B(\frac{1}{2}n,\frac{1}{2}m)} \tag{3}$$

$$= I(1;\tfrac{1}{2}m;\tfrac{1}{2}n) - I\left(\frac{m}{m+nx};\tfrac{1}{2}m;\tfrac{1}{2}n\right), \tag{4}$$

where $\Gamma(z)$ is the GAMMA FUNCTION, $B(a,b)$ is the BETA FUNCTION, and $I(a,b;x)$ is the REGULARIZED BETA FUNCTION. The MEAN, VARIANCE, SKEWNESS and KURTOSIS are

$$\mu = \frac{m}{m-2} \tag{5}$$

$$\sigma^2 = \frac{2m^2(m+n-2)}{n(m-2)^2(m-4)} \tag{6}$$

$$\gamma_1 = \frac{2(m+2n-2)}{m-6}\sqrt{\frac{2(m-4)}{n(m+n-2)}} \tag{7}$$

$$\gamma_2 = \frac{12(-16+20m-8m^2+m^3+44n)}{n(m-6)(m-8)(n+m-2)} + \frac{12(-32mn+5m^2n-22n^2+5mn^2)}{n(m-6)(m-8)(n+m-2)}. \tag{8}$$

The probability that F would be as large as it is if the first distribution has a smaller variance than the second is denoted $Q(F_{n,m})$.

The noncentral F-distribution is given by

$$P(x) = e^{-\lambda/2+(\lambda n_1 x)/[2(n_2+n_1 x)]} \times {n_1}^{n_1/2}{n_2}^{n_2/2}x^{n_1/2-1}(n_2+n_1x)^{-(n_1+n_2)/2} \times \frac{\Gamma(\frac{1}{2}n_1)\Gamma(1+\frac{1}{2}n_2)L_{n_2/2}^{n_1/2-1}\left(-\frac{\lambda n_1 x}{2(n_2+n_1x)}\right)}{B(\frac{1}{2}n_1,\frac{1}{2}n_2)\Gamma[\frac{1}{2}(n_1+n_2)]}, \tag{9}$$

where $\Gamma(z)$ is the GAMMA FUNCTION, $B(\alpha,\beta)$ is the BETA FUNCTION, and $L_m^n(z)$ is an associated LAGUERRE POLYNOMIAL.

see also BETA FUNCTION, GAMMA FUNCTION, REGULARIZED BETA FUNCTION, SNEDECOR'S F-DISTRIBUTION

References

Abramowitz, M. and Stegun, C. A. (Eds.). *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables, 9th printing.* New York: Dover, pp. 946–949, 1972.

Press, W. H.; Flannery, B. P.; Teukolsky, S. A.; and Vetterling, W. T. "Incomplete Beta Function, Student's Distribution, F-Distribution, Cumulative Binomial Distribution." §6.2 in *Numerical Recipes in FORTRAN: The Art of Scientific Computing, 2nd ed.* Cambridge, England: Cambridge University Press, pp. 219–223, 1992.

Spiegel, M. R. *Theory and Problems of Probability and Statistics.* New York: McGraw-Hill, pp. 117–118, 1992.

F-Polynomial

see KAUFFMAN POLYNOMIAL F

F-Ratio

The RATIO of two independent estimates of the VARIANCE of a NORMAL DISTRIBUTION.

see also F-DISTRIBUTION, NORMAL DISTRIBUTION, VARIANCE

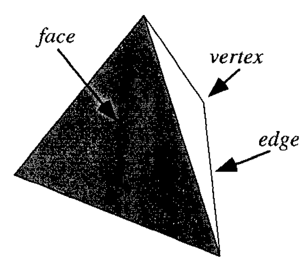
F-Ratio Distribution

see F-DISTRIBUTION

Fabry Imbedding

A representation of a PLANAR GRAPH as a planar straight line graph such that no two EDGES cross.

Face



The intersection of an n-D POLYTOPE with a tangent HYPERPLANE. 0-D faces are known as VERTICES (nodes), 1-D faces as EDGES, $(n-2)$-D faces as RIDGES, and $(n-1)$-D faces as FACETS.

see also EDGE (POLYHEDRON), FACET, POLYTOPE, RIDGE, VERTEX (POLYHEDRON)

Facet

An $(n-1)$-D FACE of an n-D POLYTOPE. A procedure for generating facets is known as FACETING.

Faceting

Using a set of corners of a SOLID that lie in a plane to form the VERTICES of a new POLYGON is called faceting. Such POLYGONS may outline new FACES that join to enclose a new SOLID, even if the sides of the POLYGONS do not fall along EDGES of the original SOLID.

References
Holden, A. *Shapes, Space, and Symmetry.* New York: Columbia University Press, p. 94, 1971.

Factor

A factor is a portion of a quantity, usually an INTEGER or POLYNOMIAL. The determination of factors is called FACTORIZATION (or sometimes "FACTORING"). It is usually desired to break factors down into the smallest possible pieces so that no factor is itself factorable. For INTEGERS, the determination of factors is called PRIME FACTORIZATION. For large quantities, the determination of all factors is usually very difficult except in exceptional circumstances.

see also DIVISOR, FACTORIZATION, GREATEST PRIME FACTOR, LEAST PRIME FACTOR, PRIME FACTORIZATION ALGORITHMS

Factor Base

The primes with LEGENDRE SYMBOL $(n/p) = 1$ (less than $N = \pi(d)$ for trial divisor d) which need be considered when using the QUADRATIC SIEVE FACTORIZATION METHOD.

see also DIXON'S FACTORIZATION METHOD

References
Morrison, M. A. and Brillhart, J. "A Method of Factoring and the Factorization of F_7." *Math. Comput.* **29**, 183–205, 1975.

Factor (Graph)

A 1-factor of a GRAPH with n VERTICES is a set of $n/2$ separate EDGES which collectively contain all n of the VERTICES of G among their endpoints.

Factor Group

see QUOTIENT GROUP

Factor Level

A grouping of statistics.

Factor Ring

see QUOTIENT RING

Factor Space

see QUOTIENT SPACE

Factorial

The factorial $n!$ is defined for a POSITIVE INTEGER n as

$$n! \equiv \begin{cases} n\cdot(n-1)\cdots 2\cdot 1 & n=1,2,\ldots \\ 1 & n=0. \end{cases} \tag{1}$$

The first few factorials for $n = 0, 1, 2, \ldots$ are 1, 1, 2, 6, 24, 120, ... (Sloane's A000142). An older NOTATION for the factorial is $\underline{\lfloor n}$ (Dudeney 1970, Gardner 1978, Conway and Guy 1996).

As n grows large, factorials begin acquiring tails of trailing ZEROS. To calculate the number of trailing ZEROS for $n!$, use

$$Z = \sum_{k=1}^{k_{\max}} \left\lfloor \frac{n}{5^k} \right\rfloor, \tag{2}$$

where

$$k_{\max} \equiv \left\lfloor \frac{\ln n}{\ln 5} \right\rfloor \tag{3}$$

and $\lfloor x \rfloor$ is the FLOOR FUNCTION (Gardner 1978, p. 63; Ogilvy and Anderson 1988, pp. 112–114). For $n = 1, 2, \ldots$, the number of trailing zeros are 0, 0, 0, 0, 1, 1, 1, 1, 1, 2, 2, 2, 2, 2, 3, 3, ... (Sloane's A027868). This is a special application of the general result that the POWER of a PRIME p dividing $n!$ is

$$\epsilon_p(n) = \sum_{r\geq 0} \left\lfloor \frac{n}{p^r} \right\rfloor \tag{4}$$

(Graham *et al.* 1994, Vardi 1991). Stated another way, the exact POWER of a PRIME p which divides $n!$ is

$$\frac{n - \text{sum of digits of the base-}p\text{ representation of } n}{p-1}. \tag{5}$$

By noting that

$$n! = \Gamma(n+1), \tag{6}$$

where $\Gamma(n)$ is the GAMMA FUNCTION for INTEGERS n, the definition can be generalized to COMPLEX values

$$z! \equiv \Gamma(z+1) \equiv \int_0^\infty e^{-t} t^z \, dt. \tag{7}$$

This defines $z!$ for all COMPLEX values of z, except when z is a NEGATIVE INTEGER, in which case $z! = \infty$. Using the identities for GAMMA FUNCTIONS, the values of $(\frac{1}{2}n)!$ (half integral values) can be written explicitly

$$(-\tfrac{1}{2})! = \sqrt{\pi} \tag{8}$$

$$(\tfrac{1}{2})! = \tfrac{1}{2}\sqrt{\pi} \tag{9}$$

$$(n-\tfrac{1}{2})! = \frac{\sqrt{\pi}}{2^n}(2n-1)!! \tag{10}$$

$$(n+\tfrac{1}{2})! = \frac{\sqrt{\pi}}{2^{n+1}}(2n+1)!!, \tag{11}$$

where $n!!$ is a DOUBLE FACTORIAL.

For INTEGERS s and n with $s < n$,

$$\frac{(s-n)!}{(2s-2n)!} = \frac{(-1)^{n-s}(2n-2s)!}{(n-s)!}. \tag{12}$$

The LOGARITHM of $z!$ is frequently encountered

$$\ln(z!) = \frac{1}{2}\ln\left[\frac{\pi z}{\sin(\pi z)}\right] - \gamma - \sum_{n=1}^{\infty}\frac{\zeta(2n+1)}{2n+1}z^{2n+1} \tag{13}$$

$$= \frac{1}{2}\ln\left[\frac{\pi z}{\sin(\pi z)}\right] - \frac{1}{2}\ln\left(\frac{1+z}{1-z}\right) + (1-\gamma)z - \sum_{n=1}^{\infty}[\zeta(2n+1)-1]\frac{z^{2n+1}}{2n+1} \tag{14}$$

$$= \ln\left[\lim_{n\to\infty}\frac{n!}{(z+1)(z+2)\cdots(z+n)}n^z\right] \tag{15}$$

$$= \lim_{n\to\infty}[\ln(n!) + z\ln n - \ln(z+1) - \ln(z+2) - \ldots - \ln(z+n)] \tag{16}$$

$$= \sum_{n=1}^{\infty}\frac{z^n}{n!}F_{n-1}(0) \tag{17}$$

$$= -\gamma z + \sum_{n=2}^{\infty}(-1)^n\frac{z^n}{n}\zeta(n) \tag{18}$$

$$= -\ln(1+z) + z(1-\gamma) + \sum_{n=2}^{\infty}(-1)^n[\zeta(n)-1]\frac{z^n}{n}, \tag{19}$$

where γ is the EULER-MASCHERONI CONSTANT, ζ is the RIEMANN ZETA FUNCTION, and F_n is the POLYGAMMA FUNCTION. The factorial can be expanded in a series

$$z! = \sqrt{2\pi}\, z^{z+1/2}e^{-z}(1 + \tfrac{1}{12}z^{-1} + \tfrac{1}{288}z^{-2} - \tfrac{139}{51840}z^{-3} + \ldots). \tag{20}$$

STIRLING'S SERIES gives the series expansion for $\ln(z!)$,

$$\ln(z!) = \tfrac{1}{2}\ln(2\pi) + \left(z+\tfrac{1}{2}\right)\ln z - z + \frac{B_2}{2z} + \ldots + \frac{B_{2n}}{2n(2n-1)z^{2n-1}} + \ldots$$

$$= \tfrac{1}{2}\ln(2\pi) + \left(z+\tfrac{1}{2}\right)\ln z - z + \tfrac{1}{12}z^{-1} - \tfrac{1}{360}z^{-3} + \tfrac{1}{1260}z^{-5} - \ldots, \tag{21}$$

where B_n is a BERNOULLI NUMBER.

Identities satisfied by sums of factorials include

$$\sum_{k=0}^{\infty}\frac{1}{k!} = e = 2.718281828\ldots \tag{22}$$

$$\sum_{k=0}^{\infty}\frac{(-1)^k}{k!} = e^{-1} = 0.3678794412\ldots \tag{23}$$

$$\sum_{k=0}^{\infty}\frac{1}{(k!)^2} = I_0(2) = 2.279585302\ldots \tag{24}$$

$$\sum_{k=0}^{\infty}\frac{(-1)^k}{(k!)^2} = J_0(2) = 0.2238907791\ldots \tag{25}$$

$$\sum_{k=0}^{\infty}\frac{1}{(2k)!} = \cosh 1 = 1.543080635\ldots \tag{26}$$

$$\sum_{k=0}^{\infty}\frac{(-1)^k}{(2k)!} = \cos 1 = 0.5403023059\ldots \tag{27}$$

$$\sum_{k=0}^{\infty}\frac{1}{(2k+1)!} = \sinh 1 = 1.175201194\ldots \tag{28}$$

$$\sum_{k=0}^{\infty}\frac{(-1)^k}{(2k+1)!} = \sin 1 = 0.8414709848\ldots \tag{29}$$

(Spanier and Oldham 1987), where I_0 is a MODIFIED BESSEL FUNCTION OF THE FIRST KIND, J_0 is a BESSEL FUNCTION OF THE FIRST KIND, cosh is the HYPERBOLIC COSINE, cos is the COSINE, sinh is the HYPERBOLIC SINE, and sin is the SINE.

Let h be the exponent of the greatest POWER of a PRIME p dividing $n!$. Then

$$h = \sum_{\substack{i=1\\ p^i\le n}}\left\lfloor\frac{n}{p^i}\right\rfloor. \tag{30}$$

Let g be the number of 1s in the BINARY representation of n. Then

$$g + h = n \tag{31}$$

(Honsberger 1976). In general, as discovered by Legendre in 1808, the POWER m of the PRIME p dividing $n!$ is given by

$$m = \sum_{k=0}^{\infty}\left\lfloor\frac{n}{p^k}\right\rfloor = \frac{n-(n_0+n_1+\ldots+n_N)}{p-1}, \tag{32}$$

where the INTEGERS $n_1, \ldots, n_N$ are the digits of n in base p (Ribenboim 1989).

The sum-of-factorials function is defined by

$$\Sigma(n) \equiv \sum_{k=1}^{n}k!$$

$$= \frac{-e + \mathrm{ei}(1) + \pi i + \mathrm{E}_{2n+1}(-1)\Gamma(n+2)}{e}, \tag{33}$$

$$= \frac{-e + \mathrm{ei}(1) + \Re[\mathrm{E}_{2n+1}(-1)]\Gamma(n+2)}{e}, \tag{34}$$

where $\mathrm{ei}(1) \approx 1.89512$ is the EXPONENTIAL INTEGRAL, E_n is the E_n-FUNCTION, and i is the IMAGINARY NUMBER. The first few values are 1, 3, 9, 33, 153, 873, 5913, 46233, 409113, ... (Sloane's A007489). $\Sigma(n)$ cannot be written as a hypergeometric term plus a constant (Petkovšek *et al.* 1996). However the sum

$$\Sigma'(n) \equiv \sum_{k=1}^{n} kk! = (n+1)! - 1 \tag{35}$$

has a simple form, with the first few values being 1, 5, 23, 119, 719, 5039, ... (Sloane's A033312).

The numbers $n! + 1$ are prime for $n = 1, 2, 3, 11, 27, 37, 41, 73, 77, 116, 154, \ldots$ (Sloane's A002981), and the numbers $n! - 1$ are prime for $n = 3, 4, 6, 7, 12, 14, 30, 32, 33, 38, 94, 166, \ldots$ (Sloane's A002982). In general, the power-product sequences (Mudge 1997) are given by $S_k^{\pm}(n) = (n!)^k \pm 1$. The first few terms of $S_2^+(n)$ are 2, 5, 37, 577, 14401, 518401, ... (Sloane's A020549), and $S_2^+(n)$ is PRIME for $n = 1, 2, 3, 4, 5, 9, 10, 11, 13, 24, 65, 76, \ldots$ (Sloane's A046029). The first few terms of $S_2^-(n)$ are 0, 3, 35, 575, 14399, 518399, ... (Sloane's A046030), but $S_2^-(n)$ is PRIME for only $n = 2$ since $S_2^-(n) = (n!)^2 - 1 = (n!+1)(n!-1)$ for $n > 2$. The first few terms of $S_3^-(n)$ are 0, 7, 215, 13823, 1727999, ..., and the first few terms of $S_3^+(n)$ are 2, 9, 217, 13825, 1728001, ... (Sloane's A19514).

There are only four INTEGERS equal to the sum of the factorials of their digits. Such numbers are called FACTORIONS. While no factorial is a SQUARE NUMBER, D. Hoey listed sums $< 10^{12}$ of distinct factorials which give SQUARE NUMBERS, and J. McCranie gave the one additional sum less than $21! = 5.1 \times 10^{19}$:

$$\begin{aligned}
0! + 1! + 2! &= 2^2 \\
1! + 2! + 3! &= 3^2 \\
1! + 4! &= 5^2 \\
1! + 5! &= 11^2 \\
4! + 5! &= 12^2 \\
1! + 2! + 3! + 6! &= 27^2 \\
1! + 5! + 6! &= 29^2 \\
1! + 7! &= 71^2 \\
4! + 5! + 7! &= 72^2 \\
1! + 2! + 3! + 7! + 8! &= 213^2 \\
1! + 4! + 5! + 6! + 7! + 8! &= 215^2 \\
1! + 2! + 3! + 6! + 9! &= 603^2 \\
1! + 4! + 8! + 9! &= 635^2 \\
1! + 2! + 3! + 6! + 7! + 8! + 10! &= 1917^2
\end{aligned}$$

$$\begin{aligned}
&1! + 2! + 3! + 7! + 8! + 9! + 10! + 11! + 12! \\
&\qquad + 13! + 14! + 15! = 1183893^2
\end{aligned}$$

(Sloane's A014597). The first few values for which the alternating SUM

$$\sum_{i=1}^{n} (-1)^{n-i} i! \tag{36}$$

is PRIME are 3, 4, 5, 6, 7, 8, 41, 59, 61, 105, 160, ... (Sloane's A014615, Guy 1994, p. 100). The only known factorials which are products of factorial in an ARITHMETIC SEQUENCE are

$$\begin{aligned}
0!1! &= 1! \\
1!2! &= 2! \\
0!1!2! &= 2! \\
6!7! &= 10! \\
1!3!5! &= 6! \\
1!3!5!7! &= 10!
\end{aligned}$$

(Madachy 1979).

There are no identities of the form

$$n! = a_1! a_2! \cdots a_r! \tag{37}$$

for $r \geq 2$ with $a_i \geq a_j \geq 2$ for $i < j$ for $n \leq 18160$ except

$$9! = 7!3!3!2! \tag{38}$$

$$10! = 7!6! = 7!5!3! \tag{39}$$

$$16! = 14!5!2! \tag{40}$$

(Guy 1994, p. 80).

There are three numbers less than 200,000 for which

$$(n-1)! + 1 \equiv 0 \pmod{n^2}, \tag{41}$$

namely 5, 13, and 563 (Le Lionnais 1983). BROWN NUMBERS are pairs (m, n) of INTEGERS satisfying the condition of BROCARD'S PROBLEM, i.e., such that

$$n! + 1 = m^2. \tag{42}$$

Only three such numbers are known: (5, 4), (11, 5), (71, 7). Erdős conjectured that these are the only three such pairs (Guy 1994, p. 193).

see also ALLADI-GRINSTEAD CONSTANT, BROCARD'S PROBLEM, BROWN NUMBERS, DOUBLE FACTORIAL, FACTORIAL PRIME, FACTORION, GAMMA FUNCTION, HYPERFACTORIAL, MULTIFACTORIAL, POCHHAMMER SYMBOL, PRIMORIAL, ROMAN FACTORIAL, STIRLING'S SERIES, SUBFACTORIAL, SUPERFACTORIAL

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Factorial Moment

$$\nu_{(r)} \equiv \sum_x x^{(r)} f(x),$$

where

$$x^{(r)} \equiv x(x-1)\cdots(x-r+1).$$

Factorial Number

see FACTORIAL

Factorial Prime

A PRIME of the form $n! \pm 1$. $n!+1$ is PRIME for 1, 2, 3, 11, 27, 37, 41, 73, 77, 116, 154, 320, 340, 399, 427, 872, 1477, ... (Sloane's A002981) up to a search limit 4850. $n!-1$ is PRIME for 3, 4, 6, 7, 12, 14, 30, 32, 33, 38, 94, 116, 324, 379, 469, 546, 974, 1963, 3507, 3610, ... (Sloane's A002982) up to a search limit of 4850.

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Factorial Sum

Sums with unity NUMERATOR and FACTORIALS in the DENOMINATOR which can be expressed analytically include

$$\sum_{i=1}^n \frac{1}{(n+i-k)!(n-i)!} = \frac{{}_2F_1(1,-n;1+n-k;-1)-1}{\Gamma(1+n)\Gamma(1+n-k)} \quad (1)$$

$$\sum_{i=1}^n \frac{1}{(n+i-1)!(n-i)!} = \frac{n\sqrt{\pi}}{2\Gamma(\frac{1}{2}+n)\Gamma(1+n)} \quad (2)$$

$$\sum_{i=1}^n \frac{1}{(n+i)!(n-i)!} = \frac{\sqrt{\pi}}{2\Gamma(\frac{1}{2}+n)\Gamma(1+n)} - \frac{1}{2\Gamma^2(1+n)} \quad (3)$$

$$\sum_{i=1}^n \frac{1}{(n+i+1)!(n-i)!} = \frac{\sqrt{\pi}}{2\Gamma(\frac{3}{2}+n)\Gamma(1+n)} - \frac{1}{2\Gamma(1+n)\Gamma(2+n)}, \quad (4)$$

where ${}_2F_1(a,b;c;z)$ is a HYPERGEOMETRIC FUNCTION and $\Gamma(z)$ is a GAMMA FUNCTION.

Sums with i in the NUMERATOR having analytic solutions include

$$\sum_{i=1}^n \frac{i}{(n+i-k)!(n-i)!} = \frac{n\,{}_2F_1(2,1-n;2-k+n;-1)}{(1-k+n)\Gamma(1+n)\Gamma(1-k+n)} \quad (5)$$

$$\sum_{i=1}^n \frac{i}{(n+i-1)!(n-i)!} = \frac{1}{2\Gamma(n)}\left[\frac{\sqrt{\pi}}{2\Gamma(\frac{1}{2}+n)} + \frac{n}{\Gamma(1+n)}\right] \quad (6)$$

$$\sum_{i=1}^n \frac{i}{(n+i)!(n-i)!} = \frac{n}{2\Gamma^2(1+n)} \quad (7)$$

$$\sum_{i=1}^{n} \frac{i}{(n+i+1)!(n-i)!} = \frac{1}{2\Gamma(1+n)} \left[\frac{1}{\Gamma(2+n)} - \frac{(n^2+3n+2)\sqrt{\pi}}{2\Gamma(\frac{3}{2}+n)} \right]. \quad (8)$$

A sum with i^2 in the NUMERATOR is

$$\sum_{i=1}^{n} \frac{i^2}{(n+i-k)!(n-i)!} = \frac{n}{(1-k+n)(2-k+n)\Gamma(1+n)\Gamma(1-k+n)} \times[(2-k+n)\,{}_2F_1(2,1-n;2-k+n;-1) + 2(n-1)\,{}_2F_1(3,2-n;3-k+n;-1)], \quad (9)$$

where ${}_2F_1(a,b;c;z)$ is the HYPERGEOMETRIC FUNCTION.

Sums of factorial POWERS include

$$\sum_{n=0}^{\infty} \frac{(n!)^2}{(2n)!} = \frac{4}{3} + \frac{2\pi}{9\sqrt{3}} \quad (10)$$

$$\sum_{n=0}^{\infty} \frac{(n!)^3}{(3n)!} = \int_0^1 [P(t) + Q(t)\cos^{-1} R(t)]\,dt, \quad (11)$$

where

$$P(t) = \frac{2(8+7t^2-7t^3)}{(4-t^2+t^3)^2} \quad (12)$$

$$Q(t) = \frac{4t(1-t)(5+t^2-t^3)}{(4-t^2+t^3)^2\sqrt{(1-t)(4-t^2+t^3)}} \quad (13)$$

$$R(t) = 1 - \tfrac{1}{2}(t^2-t^3) \quad (14)$$

(Beeler *et al.* 1972, Item 116).

References
Beeler, M.; Gosper, R. W.; and Schroeppel, R. *HAKMEM.* Cambridge, MA: MIT Artificial Intelligence Laboratory, Memo AIM-239, Feb. 1972.

Factoring

see FACTORIZATION

Factorion

A factorion is an INTEGER which is equal to the sum of FACTORIALS of its digits. There are exactly four such numbers:

$$1 = 1! \quad (1)$$

$$2 = 2! \quad (2)$$

$$145 = 1! + 4! + 5! \quad (3)$$

$$40,585 = 4! + 0! + 5! + 8! + 5! \quad (4)$$

(Gardner 1978, Madachy 1979, Pickover 1995). The factorion of an n-digit number cannot exceed $n \cdot 9!$ digits.

see also FACTORIAL

References
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Pickover, C. A. "The Loneliness of the Factorions." Ch. 22 in *Keys to Infinity.* New York: W. H. Freeman, pp. 169–171 and 319–320, 1995.

Factorization

The finding of FACTORS (DIVISORS) of a given INTEGER, POLYNOMIAL, etc. Factorization is also called FACTORING.

see also FACTOR, PRIME FACTORIZATION ALGORITHMS

Fagnano's Point

The point of coincidence of P and P' in FAGNANO'S PROBLEM.

Fagnano's Problem

In a given ACUTE-angled TRIANGLE ΔABC, INSCRIBE another TRIANGLE whose PERIMETER is as small as possible. The answer is the PEDAL TRIANGLE of ΔABC.

References
Coxeter, H. S. M. and Greitzer, S. L. *Geometry Revisited.* Washington, DC: Math. Assoc. Amer., pp. 88–89, 1967.

Fagnano's Theorem

If $P(x,y)$ and $P(x',y')$ are two points on an ELLIPSE

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \quad (1)$$

with ECCENTRIC ANGLES ϕ and ϕ' such that

$$\tan\phi\tan\phi' = \frac{b}{a} \quad (2)$$

and $A = P(a,0)$ and $B = P(0,b)$. Then

$$\operatorname{arc} BP + \operatorname{arc} BP' = \frac{e^2xx'}{a}. \quad (3)$$

This follows from the identity

$$E(u,k) + E(v,k) - E(k) = k^2 \operatorname{sn}(u,k)\operatorname{sn}(v,k), \quad (4)$$

where $E(u,k)$ is an incomplete ELLIPTIC INTEGRAL OF THE SECOND KIND, $E(k)$ is a complete ELLIPTIC INTEGRAL OF THE SECOND KIND, and $\operatorname{sn}(v,k)$ is a JACOBI ELLIPTIC FUNCTION. If P and P' coincide, the point where they coincide is called FAGNANO'S POINT.

Fair Game

A GAME which is not biased toward any player.

see also GAME, MARTINGALE

Fairy Chess

A variation of CHESS involving a change in the form of the board, the rules of play, or the pieces used. For example, the normal rules of chess can be used but with a cylindrical or MÖBIUS STRIP connection of the edges.

see also CHESS

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Fallacy

A fallacy is an incorrect result arrived at by apparently correct, though actually specious reasoning. The most common example of a mathematical fallacy is the "proof" that $1 = 2$ as follows. Let $a = b$, then

$$ab = a^2$$
$$ab - b^2 = a^2 - b^2$$
$$b(a-b) = (a+b)(a-b)$$
$$b = a + b$$
$$b = 2b$$
$$1 = 2.$$

The incorrect step is division by $a - b$ (equal to 0), which is invalid. Ball and Coxeter (1987) give other such examples in the areas of both arithmetic and geometry.

References

Ball, W. W. R. and Coxeter, H. S. M. *Mathematical Recreations and Essays, 13th ed.* New York: Dover, pp. 41–45 and 76–84, 1987.

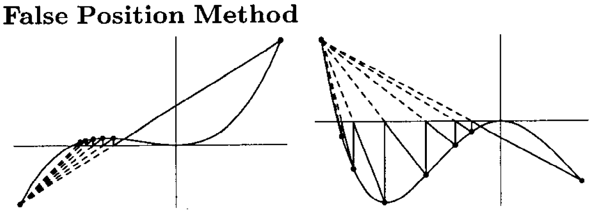
Pappas, T. "Geometric Fallacy & the Fibonacci Sequence." *The Joy of Mathematics.* San Carlos, CA: Wide World Publ./Tetra, p. 191, 1989.

False

A statement which is rigorously not TRUE. Regular two-valued LOGIC allows statements to be only TRUE or false, but FUZZY LOGIC treats "truth" as a continuum which can have a value between 0 and 1.

see also ALETHIC, FUZZY LOGIC, LOGIC, TRUE, TRUTH TABLE, UNDECIDABLE

False Position Method



An ALGORITHM for finding ROOTS which uses the point where the linear approximation crosses the axis as the next iteration and keeps the same initial point for each iteration. Using the two-point form of the line

$$y - y_1 = \frac{f(x_{n-1}) - f(x_1)}{x_{n-1} - x_1}(x_n - x_1)$$

with $y = 0$, using $y_1 = f(x_1)$, and solving for x_n therefore gives the iteration

$$x_n = x_1 - \frac{x_{n-1} - x_1}{f(x_{n-1}) - f(x_1)} f(x_1).$$

see also BRENT'S METHOD, RIDDERS' METHOD, SECANT METHOD

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Faltung (Form)

Let A and B be bilinear forms

$$A = A(x,y) = \sum\sum a_{ij}x_iy_i$$
$$B = B(x,y) = \sum\sum b_{ij}x_iy_i$$

and suppose that A and B are bounded in $[p,p']$ with bounds M and N. Then

$$F = F(A,B) = \sum\sum f_{ij}x_iy_j,$$

where the series

$$f_{ij} = \sum_k a_{ik}b_{kj}$$

is absolutely convergent, is called the faltung of A and B. F is bounded in $[p,p']$, and its bound does not exceed MN.

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Faltung (Function)

see CONVOLUTION

Fan

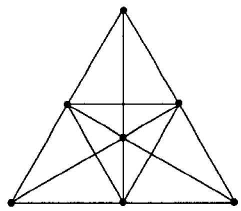
A SPREAD in which each node has a FINITE number of children.

see also SPREAD (TREE)

Fano's Axiom

The three diagonal points of a COMPLETE QUADRILATERAL are never COLLINEAR.

Fano Plane



The 2-D PROJECTIVE PLANE over $GF(2)$ ("of order two"), illustrated above. It is a BLOCK DESIGN with $\nu = 7$, $k = 3$, $\lambda = 1$, $r = 3$, and $b = 7$, and is also the STEINER TRIPLE SYSTEM $S(7)$.

The Fano plane also solves the TRANSYLVANIA LOTTERY, which picks three numbers from the INTEGERS 1–14. Using two Fano planes we can guarantee matching two by playing just 14 times as follows. Label the VERTICES of one Fano plane by the INTEGERS 1–7, the other plane by the INTEGERS 8–14. The 14 tickets to play are the 14 lines of the two planes. Then if (a, b, c) is the winning ticket, at least two of a, b, c are either in the interval [1, 7] or [8, 14]. These two numbers are on exactly one line of the corresponding plane, so one of our tickets matches them.

The Lehmers (1974) found an application of the Fano plane for factoring INTEGERS via QUADRATIC FORMS. Here, the triples of forms used form the lines of the PROJECTIVE GEOMETRY on seven points, whose planes are Fano configurations corresponding to pairs of residue classes mod 24 (Lehmer and Lehmer 1974, Guy 1975, Shanks 1985). The group of AUTOMORPHISMS (incidence-preserving BIJECTIONS) of the Fano plane is the SIMPLE GROUP of ORDER 168 (Klein 1870).

see also DESIGN, PROJECTIVE PLANE, STEINER TRIPLE SYSTEM, TRANSYLVANIA LOTTERY

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Far Out

A word used by Tukey to describe data points which are outside the outer FENCES.

References

Tukey, J. W. *Explanatory Data Analysis.* Reading, MA: Addison-Wesley, p. 44, 1977.

Far-Out Point

For a TRIANGLE with side lengths a, b, and c, the far-out point has TRIANGLE CENTER FUNCTION

$$\alpha = a(b^4 + c^4 - a^4 - b^2c^2).$$

As $a : b : c$ approaches $1 : 1 : 1$, this point moves out along the EULER LINE to infinity.

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Farey Sequence

The Farey sequence F_n for any POSITIVE INTEGER n is the set of irreducible RATIONAL NUMBERS a/b with $0 \leq a \leq b \leq n$ and $(a, b) = 1$ arranged in increasing order.

$$F_1 = \{\tfrac{0}{1}, \tfrac{1}{1}\} \tag{1}$$
$$F_2 = \{\tfrac{0}{1}, \tfrac{1}{2}, \tfrac{1}{1}\} \tag{2}$$
$$F_3 = \{\tfrac{0}{1}, \tfrac{1}{3}, \tfrac{1}{2}, \tfrac{2}{3}, \tfrac{1}{1}\} \tag{3}$$
$$F_4 = \{\tfrac{0}{1}, \tfrac{1}{4}, \tfrac{1}{3}, \tfrac{1}{2}, \tfrac{2}{3}, \tfrac{3}{4}, \tfrac{1}{1}\} \tag{4}$$
$$F_5 = \{\tfrac{0}{1}, \tfrac{1}{5}, \tfrac{1}{4}, \tfrac{1}{3}, \tfrac{2}{5}, \tfrac{1}{2}, \tfrac{3}{5}, \tfrac{2}{3}, \tfrac{3}{4}, \tfrac{4}{5}, \tfrac{1}{1}\}. \tag{5}$$

There is always an ODD number of terms, and the middle term is always 1/2. Let p/q, p'/q', and p''/q'' be three successive terms in a Farey series. Then

$$qp' - pq' = 1 \tag{6}$$

$$\frac{p'}{q'} = \frac{p + p''}{q + q''}. \tag{7}$$

These two statements are actually equivalent.

The number of terms $N(n)$ in the Farey sequence for the INTEGER n is

$$N(n) = 1 + \sum_{k=1}^{n} \phi(k) = 1 + \Phi(n), \tag{8}$$

where $\phi(k)$ is the TOTIENT FUNCTION and $\Phi(n)$ is the SUMMATORY FUNCTION of $\phi(k)$, giving 2, 3, 5, 7, 11, 13, 19, ... (Sloane's A005728). The asymptotic limit for the function $N(n)$ is

$$N(n) \sim \frac{3n^2}{\pi^2} = 0.3039635509n^2 \tag{9}$$

(Vardi 1991, p. 155). For a method of computing a successive sequence from an existing one of n terms, insert the MEDIANT fraction $(a + b)/(c + d)$ between terms a/c and b/d when $c + d \leq n$ (Hardy and Wright 1979, pp. 25–26; Conway and Guy 1996).

FORD CIRCLES provide a method of visualizing the Farey sequence. The Farey sequence F_n defines a subtree of the STERN-BROCOT TREE obtained by pruning unwanted branches (Graham *et al.* 1994).

see also FORD CIRCLE, MEDIANT, RANK (SEQUENCE), STERN-BROCOT TREE

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Farey Series

see FAREY SEQUENCE

Farkas's Lemma

The INEQUALITY $\langle f_0, x\rangle \leq 0$ follows from

$$\langle f_1, x\rangle \leq 0, \ldots, \langle f_n, x\rangle \leq 0$$

IFF there exist NONNEGATIVE numbers $\lambda_1, \ldots, \lambda_n$ with

$$\sum_{k=1}^{n} \lambda_k f_k = f_0.$$

This LEMMA is used in the proof of the KUHN-TUCKER THEOREM.

see also KUHN-TUCKER THEOREM, LAGRANGE MULTIPLIER

Faro Shuffle

see RIFFLE SHUFFLE

Fast Fibonacci Transform

For a general second-order recurrence equation

$$f_{n+1} = x f_n + y f_{n-1}, \tag{1}$$

define a multiplication rule on ordered pairs by

$$(A, B)(C, D) = (AD + BC + xAC, BD + yAC). \tag{2}$$

The inverse is then given by

$$(A, B)^{-1} = \frac{(-A, xA + B)}{B^2 + xAB - yA^2}, \tag{3}$$

and we have the identity

$$(f_1, y f_0)(1, 0)^n = (f_{n+1}, y f_n) \tag{4}$$

(Beeler *et al.* 1972, Item 12).

References
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Fast Fourier Transform

The fast Fourier transform (FFT) is a DISCRETE FOURIER TRANSFORM ALGORITHM which reduces the number of computations needed for N points from $2N^2$ to $2N \lg N$, where LG is the base-2 LOGARITHM. If the function to be transformed is not harmonically related to the sampling frequency, the response of an FFT looks like a SINC FUNCTION (although the integrated POWER is still correct). ALIASING (LEAKAGE) can be reduced by APODIZATION using a TAPERING FUNCTION. However, ALIASING reduction is at the expense of broadening the spectral response.

FFTs were first discussed by Cooley and Tukey (1965), although Gauss had actually described the critical factorization step as early as 1805 (Gergkand 1969, Strang 1993). A DISCRETE FOURIER TRANSFORM can be computed using an FFT by means of the DANIELSON-LANCZOS LEMMA if the number of points N is a POWER of two. If the number of points N is not a POWER of two, a transform can be performed on sets of points corresponding to the prime factors of N which is slightly degraded in speed. An efficient real Fourier transform algorithm or a fast HARTLEY TRANSFORM (Bracewell 1965) gives a further increase in speed by approximately a factor of two. Base-4 and base-8 fast Fourier transforms use optimized code, and can be 20–30% faster than base-2 fast Fourier transforms. PRIME factorization is slow when the factors are large, but discrete Fourier transforms can be made fast for $N = 2, 3, 4, 5, 7, 8, 11, 13$, and 16 using the WINOGRAD TRANSFORM ALGORITHM (Press *et al.* 1992, pp. 412–413, Arndt).

Fast Fourier transform algorithms generally fall into two classes: decimation in time, and decimation in frequency. The Cooley-Tukey FFT ALGORITHM first rearranges the input elements in bit-reversed order, then builds the output transform (decimation in time). The basic idea is to break up a transform of length N into two transforms of length $N/2$ using the identity

$$\sum_{n=0}^{N-1} a_n e^{-2\pi i n k/N} = \sum_{n=0}^{N/2-1} a_{2n} e^{-2\pi i (2n)k/N} + \sum_{n=0}^{N/2-1} a_{2n+1} e^{-2\pi i (2n+1)k/N}$$

$$= \sum_{n=0}^{N/2-1} a_n^{\text{even}} e^{-2\pi i n k/(N/2)} + e^{-2\pi i k/N} \sum_{n=0}^{N/2-1} a_n^{\text{odd}} e^{-2\pi i n k/(N/2)},$$

sometimes called the DANIELSON-LANCZOS LEMMA. The easiest way to visualize this procedure is perhaps via the FOURIER MATRIX.

The Sande-Tukey ALGORITHM (Stoer and Burlisch 1980) first transforms, then rearranges the output values (decimation in frequency).

see also DANIELSON-LANCZOS LEMMA, DISCRETE FOURIER TRANSFORM, FOURIER MATRIX, FOURIER TRANSFORM, HARTLEY TRANSFORM, NUMBER THEORETIC TRANSFORM, WINOGRAD TRANSFORM

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Fat Fractal

A CANTOR SET with LEBESGUE MEASURE greater than 0.

see also CANTOR SET, EXTERIOR DERIVATIVE, FRACTAL, LEBESGUE MEASURE

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Fatou Dust

see FATOU SET

Fatou's Lemma

If a SEQUENCE $\{f_n\}$ of NONNEGATIVE measurable functions is defined on a measurable set E, then

$$\int_E \liminf_{n\to\infty} f_n \, d\mu \leq \liminf_{n\to\infty} \int_E f_n \, d\mu.$$

References

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Fatou Set

A set consisting of the complementary set of complex numbers to a JULIA SET.

see also JULIA SET

References

Schroeder, M. *Fractals, Chaos, Power Laws.* New York: W. H. Freeman, p. 39, 1991.

Fatou's Theorems

Let $f(\theta)$ be LEBESGUE INTEGRABLE and let

$$f(r,\theta) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t) \frac{1-r^2}{1-2r\cos(t-\theta)+r^2} \, dt \tag{1}$$

be the corresponding POISSON INTEGRAL. Then ALMOST EVERYWHERE in $-\pi \leq \theta \leq \pi$,

$$\lim_{r\to 0^-} f(r,\theta) = f(\theta). \tag{2}$$

Let

$$F(z) = c_0 + c_1 z + c_2 z^2 + \ldots + c_n z^n + \ldots \tag{3}$$

be regular for $|z| < 1$, and let the integral

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} |F(re^{i\theta})|^2 \, d\theta \tag{4}$$

be bounded for $r < 1$. This condition is equivalent to the convergence of

$$|c_0|^2 + |c_1|^2 + \ldots + |c_n|^2 + \ldots. \tag{5}$$

Then almost everywhere in $-\pi \leq \theta \leq \pi$,

$$\lim_{r \to 0^-} F(re^{i\theta}) = F(e^{i\theta}). \tag{6}$$

Furthermore, $F(e^{i\theta})$ is measurable, $|F(e^{i\theta})|^2$ is LEBESGUE INTEGRABLE, and the FOURIER SERIES of $F(e^{i\theta})$ is given by writing $z = e^{i\theta}$.

References
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Faulhaber's Formula

In a 1631 edition of *Academiae Algebrae*, J. Faulhaber published the general formula for the SUM of pth POWERS of the first n POSITIVE INTEGERS,

$$\sum_{k=1}^{n} k^p = \frac{1}{p+1} \sum_{i=1}^{p+1} (-1)^{\delta_{ip}} \binom{p+1}{i} B_{p+1-i} n^i, \tag{1}$$

where δ_{ip} is the KRONECKER DELTA, $\binom{n}{i}$ is a BINOMIAL COEFFICIENT, and B_i is the ith BERNOULLI NUMBER. Computing the sums for $p = 1, \ldots, 10$ gives

$$\sum_{k=1}^{n} k = \tfrac{1}{2}(n^2 + n) \tag{2}$$

$$\sum_{k=1}^{n} k^2 = \tfrac{1}{6}(2n^3 + 3n^2 + n) \tag{3}$$

$$\sum_{k=1}^{n} k^3 = \tfrac{1}{4}(n^4 + 2n^3 + n^2) \tag{4}$$

$$\sum_{k=1}^{n} k^4 = \tfrac{1}{30}(6n^5 + 15n^4 + 10n^3 - n) \tag{5}$$

$$\sum_{k=1}^{n} k^5 = \tfrac{1}{12}(2n^6 + 6n^5 + 5n^4 - n^2) \tag{6}$$

$$\sum_{k=1}^{n} k^6 = \tfrac{1}{42}(6n^7 + 21n^6 + 21n^5 - 7n^3 + n) \tag{7}$$

$$\sum_{k=1}^{n} k^7 = \tfrac{1}{24}(3n^8 + 12n^7 + 14n^6 - 7n^4 + 2n^2) \tag{8}$$

$$\sum_{k=1}^{n} k^8 = \tfrac{1}{90}(10n^9 + 45n^8 + 60n^7 - 42n^5 + 20n^3 - 3n) \tag{9}$$

$$\sum_{k=1}^{n} k^9 = \tfrac{1}{20}(2n^{10} + 10n^9 + 15n^8 - 14n^6 + 10n^4 - 3n^2) \tag{10}$$

$$\sum_{k=1}^{n} k^{10} = \tfrac{1}{66}(6n^{11} + 33n^{10} + 55n^9 - 66n^7 + 66n^5 - 33n^3 + 5n). \tag{11}$$

see also POWER, SUM

References
Conway, J. H. and Guy, R. K. *The Book of Numbers.* New York: Springer-Verlag, p. 106, 1996.

Favard Constants

N.B. A detailed on-line essay by S. Finch was the starting point for this entry.

Let $T_n(x)$ be an arbitrary trigonometric POLYNOMIAL

$$T_n(x) = \tfrac{1}{2}a_0 + \left\{ \sum_{k=1}^{n} [a_k \cos(kx) + b_k \sin(kx)] \right\},$$

where the COEFFICIENTS are real. Let the rth derivative of $T_n(x)$ be bounded in $[-1, 1]$, then there exists a POLYNOMIAL $T_n(x)$ for which

$$|f(x) - T_n(x)| \leq \frac{K_r}{(n+1)^r},$$

for all x, where K_r is the rth Favard constant, which is the smallest constant possible.

$$K_r = \frac{4}{\pi} \sum_{k=0}^{\infty} \left[\frac{(-1)^k}{2k+1} \right]^{r+1}.$$

These can be expressed by

$$K_r = \begin{cases} \frac{4}{\pi}\lambda(r+1) & \text{for } r \text{ odd} \\ \frac{4}{\pi}\beta(r+1) & \text{for } r \text{ even,} \end{cases}$$

where λ is the DIRICHLET LAMBDA FUNCTION and β is the DIRICHLET BETA FUNCTION. Explicitly,

$$K_0 = 1$$
$$K_1 = \tfrac{1}{2}\pi$$
$$K_2 = \tfrac{1}{8}\pi^2$$
$$K_3 = \tfrac{1}{24}\pi^3.$$

References
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Feigenbaum Constant

A universal constant for functions approaching CHAOS via period doubling. It was discovered by Feigenbaum in 1975 and demonstrated rigorously by Lanford (1982) and Collet and Eckmann (1979, 1980). The Feigenbaum constant δ characterizes the geometric approach of the bifurcation parameter to its limiting value. Let μ_k be the point at which a period 2^k cycle becomes unstable.

Denote the converged value by μ_∞. Assuming geometric convergence, the difference between this value and μ_k is denoted

$$\lim_{k\to\infty} \mu_\infty - \mu_k = \frac{\Gamma}{\delta^k}, \tag{1}$$

where Γ is a constant and δ is a constant > 1. Solving for δ gives

$$\delta \equiv \lim_{n\to\infty} \frac{\mu_{n+1} - \mu_n}{\mu_{n+2} - \mu_{n+1}} \tag{2}$$

(Rasband 1990, p. 23). For the LOGISTIC EQUATION,

$$\delta = 4.669216091\ldots \tag{3}$$

$$\Gamma = 2.637\ldots \tag{4}$$

$$\mu_\infty = 3.5699456\ldots. \tag{5}$$

Amazingly, the Feigenbaum constant $\delta \approx 4.669$ is "universal" (i.e., the same) for all 1-D MAPS $f(x)$ if $f(x)$ has a single locally quadratic MAXIMUM. More specifically, the Feigenbaum constant is universal for 1-D MAPS if the SCHWARZIAN DERIVATIVE

$$D_{\text{Schwarzian}} \equiv \frac{f'''(x)}{f'(x)} - \frac{3}{2}\left[\frac{f''(x)}{f'(x)}\right]^2 \tag{6}$$

is NEGATIVE in the bounded interval (Tabor 1989, p. 220). Examples of maps which are universal include the HÉNON MAP, LOGISTIC MAP, LORENZ SYSTEM, Navier-Stokes truncations, and sine map $x_{n+1} = a\sin(\pi x_n)$. The value of the Feigenbaum constant can be computed explicitly using functional group renormalization theory. The universal constant also occurs in phase transitions in physics and, curiously, is very nearly equal to

$$\pi + \tan^{-1}(e^\pi) = 4.669201932\ldots. \tag{7}$$

The CIRCLE MAP is *not* universal, and has a Feigenbaum constant of $\delta \approx 2.833$. For an AREA-PRESERVING 2-D MAP with

$$x_{n+1} = f(x_n, y_n) \tag{8}$$

$$y_{n+1} = g(x_n, y_n), \tag{9}$$

the Feigenbaum constant is $\delta = 0.7210978\ldots$ (Tabor 1989, p. 225). For a function of the form

$$f(x) = 1 - a|x|^n \tag{10}$$

with a and n constant and n an INTEGER, the Feigenbaum constant for various n is given in the following table (Briggs 1991, Briggs *et al.* 1991), which updates the values in Tabor (1989, p. 225).

n	δ
2	5.9679
4	7.2846
6	8.3494
8	9.2962

An additional constant α, defined as the separation of adjacent elements of PERIOD DOUBLED ATTRACTORS from one double to the next, has a value

$$\lim_{n\to\infty} \frac{d_n}{d_{n+1}} \equiv -\alpha = -2.502907875\ldots \tag{11}$$

for "universal" maps (Rasband 1990, p. 37). This value may be approximated from functional group renormalization theory to the zeroth order by

$$1 - \alpha^{-1} = \frac{1-\alpha^{-2}}{[1-\alpha^{-2}(1-\alpha^{-1})]^2}, \tag{12}$$

which, when the QUINTIC EQUATION is numerically solved, gives $\alpha = -2.48634\ldots$, only 0.7% off from the actual value (Feigenbaum 1988).

see also ATTRACTOR, BIFURCATION, FEIGENBAUM FUNCTION, LINEAR STABILITY, LOGISTIC MAP, PERIOD DOUBLING

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Feigenbaum Function

Consider an arbitrary 1-D MAP

$$x_{n+1} = F(x_n) \tag{1}$$

at the onset of CHAOS. After a suitable rescaling, the Feigenbaum function

$$g(x) = \lim_{n\to\infty} \frac{1}{F^{(2^n)}(0)} F^{(2^n)}(xF^{(2^n)}(0)) \tag{2}$$

is obtained. This function satisfies

$$g(g(x)) = -\frac{1}{\alpha} g(\alpha x), \tag{3}$$

with $\alpha = 2.50290\ldots$, a quantity related to the FEIGENBAUM CONSTANT.

see also BIFURCATION, CHAOS, FEIGENBAUM CONSTANT

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Feit-Thompson Conjecture

Concerns PRIMES p and q for which $p^q - 1$ and $q^p - 1$ have a common factor. The only (p, q) pair with both values less than 400,000 is (17, 3313), with a common factor 112,643.

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Feit-Thompson Theorem

Every FINITE SIMPLE GROUP (which is not CYCLIC) has EVEN ORDER, and the ORDER of every FINITE SIMPLE noncommutative group is DOUBLY EVEN, i.e., divisible by 4 (Feit and Thompson 1963).

see also BURNSIDE PROBLEM, FINITE GROUP, ORDER (GROUP), SIMPLE GROUP

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Fejes Tóth's Integral

$$\frac{1}{2\pi(n+1)} \int_{-\pi}^{\pi} f(x) \left\{ \frac{\sin[\frac{1}{2}(n+1)x]}{\sin(\frac{1}{2}x)} \right\}^2 dx$$

gives the nth CESÀRO MEAN of the FOURIER SERIES of $f(x)$.

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Fejes Tóth's Problem

How can n points be distributed on a UNIT SPHERE such that they maximize the minimum distance between any pair of points? In 1943, Fejes Tóth proved that for N points, there always exist two points whose distance d is

$$d \leq \sqrt{4 - \csc^2\left[\frac{\pi N}{6(N-2)}\right]},$$

and that the limit is exact for $N = 3$, 4, 6, and 12.

For two points, the points should be at opposite ends of a DIAMETER. For four points, they should be placed at the VERTICES of an inscribed TETRAHEDRON. There is no best solution for five points since the distance cannot be reduced below that for six points. For six points, they should be placed at the VERTICES of an inscribed OCTAHEDRON. For seven points, the best solution is four equilateral spherical triangles with angles of 80°. For eight points, the best dispersal is *not* the VERTICES of the inscribed CUBE, but of a square ANTIPRISM with equal EDGES. The solution for nine points is eight equilateral spherical triangles with angles of $\cos^{-1}(1/4)$. For 12 points, the solution is an inscribed ICOSAHEDRON.

The general problem has not been solved.

see also THOMSON PROBLEM

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Whyte, L. L. "Unique Arrangement of Points on a Sphere." *Amer. Math. Monthly* **59**, 606–611, 1952.

Feller's Coin-Tossing Constants

see COIN TOSSING

Feller-Lévy Condition

Given a sequence of independent random variates X_1, X_2, ..., if ${\sigma_k}^2 = \mathrm{var}(X_k)$ and

$${\rho_n}^2 \equiv \max_{k\leq n} \left(\frac{{\sigma_k}^2}{{s_n}^2} \right),$$

then

$$\lim_{n\to\infty} {\rho_n}^2 = 0.$$

This means that if the LINDEBERG CONDITION holds for the sequence of variates $X_1, \ldots$, then the VARIANCE of an individual term in the sum S_n of X_k is asymptotically negligible. For such sequences, the LINDEBERG CONDITION is NECESSARY as well as SUFFICIENT for the LINDEBERG-FELLER CENTRAL LIMIT THEOREM to hold.

References

Zabell, S. L. "Alan Turing and the Central Limit Theorem." *Amer. Math. Monthly* **102**, 483–494, 1995.

Fence

Values one STEP outside the HINGES are called inner fences, and values two steps outside the HINGES are called outer fences. Tukey calls values outside the outer fences FAR OUT.

see also ADJACENT VALUE

References

Tukey, J. W. *Explanatory Data Analysis.* Reading, MA: Addison-Wesley, p. 44, 1977.

Fence Poset

A PARTIAL ORDER defined by $(i-1, j)$, $(i+1, j)$ for ODD i.

see also PARTIAL ORDER

References

Ruskey, F. "Information on Ideals of Partially Ordered Sets." http://sue.csc.uvic.ca/~cos/inf/pose/Ideals.html.

Ferguson-Forcade Algorithm

A practical algorithm for determining if there exist integers a_i for given real numbers x_i such that

$$a_1x_1 + a_2x_2 + \ldots + a_nx_n = 0,$$

or else establish bounds within which no such INTEGER RELATION can exist (Ferguson and Forcade 1979). A nonrecursive variant of the original algorithm was subsequently devised by Ferguson (1987). The Ferguson-Forcade algorithm has shown that there are no algebraic equations of degree ≤ 8 with integer coefficients having Euclidean norms below certain bounds for e/π, $e+\pi$, $\ln\pi$, γ, e^γ, γ/e, γ/π, and $\ln\gamma$, where e is the base for the NATURAL LOGARITHM, π is PI, and γ is the EULER-MASCHERONI CONSTANT (Bailey 1988).

Constant	Bound
e/π	6.1030×10^{14}
$e+\pi$	2.2753×10^{18}
$\ln\pi$	8.7697×10^{9}
γ	3.5739×10^{9}
e^γ	1.6176×10^{17}
γ/e	1.8440×10^{11}
γ/π	6.5403×10^{9}
$\ln\gamma$	2.6881×10^{10}

see also CONSTANT PROBLEM, EUCLIDEAN ALGORITHM, INTEGER RELATION, PSLQ ALGORITHM

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Fermat $4n+1$ Theorem

Every PRIME of the form $4n+1$ is a sum of two SQUARE NUMBERS in one unique way (up to the order of SUMMANDS). The theorem was stated by Fermat, but the first published proof was by Euler.

see also SIERPIŃSKI'S PRIME SEQUENCE THEOREM, SQUARE NUMBER

References

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Fermat's Algorithm

see FERMAT'S FACTORIZATION METHOD

Fermat Compositeness Test

Uses FERMAT'S LITTLE THEOREM

Fermat's Congruence

see FERMAT'S LITTLE THEOREM

Fermat Conic

A PLANE CURVE of the form $y = x^n$. For $n > 0$, the curve is a generalized PARABOLA; for $n < 0$ it is a generalized HYPERBOLA.

see also CONIC SECTION, HYPERBOLA, PARABOLA

Fermat's Conjecture

see FERMAT'S LAST THEOREM

Fermat Difference Equation

see PELL EQUATION

Fermat Diophantine Equation

see FERMAT DIFFERENCE EQUATION

Fermat Equation

The DIOPHANTINE EQUATION

$$x^n + y^n = z^n.$$

The assertion that this equation has no nontrivial solutions for $n > 2$ is called FERMAT'S LAST THEOREM.

see also FERMAT'S LAST THEOREM

Fermat-Euler Theorem

see FERMAT'S LITTLE THEOREM

Fermat's Factorization Method

Given a number n, look for INTEGERS x and y such that $n = x^2 - y^2$. Then

$$n = (x-y)(x+y) \tag{1}$$

and n is factored. Any ODD NUMBER can be represented in this form since then $n = ab$, a and b are ODD, and

$$a = x + y \tag{2}$$

$$b = x - y. \tag{3}$$

Adding and subtracting,

$$a + b = 2x \tag{4}$$

$$a - b = 2y, \tag{5}$$

so solving for x and y gives

$$x = \tfrac{1}{2}(a+b) \tag{6}$$

$$y = \tfrac{1}{2}(a-b). \tag{7}$$

Therefore,

$$x^2 - y^2 = \tfrac{1}{4}[(a+b)^2 - (a-b)^2] = ab. \tag{8}$$

As the first trial for x, try $x_1 \lceil\sqrt{n}\rceil$, where $\lceil x \rceil$ is the CEILING FUNCTION. Then check if

$$\Delta x_1 = {x_1}^2 - n \tag{9}$$

is a SQUARE NUMBER. There are only 22 combinations of the last two digits which a SQUARE NUMBER can assume, so most combinations can be eliminated. If Δx_1 is not a SQUARE NUMBER, then try

$$x_2 = x_1 + 1, \tag{10}$$

so

$$\begin{aligned}\Delta x_2 &= {x_2}^2 - n = (x_1+1)^2 - n = {x_1}^2 + 2x_1 + 1 - n \\ &= \Delta x_1 + 2x_1 + 1.\end{aligned} \tag{11}$$

Continue with

$$\begin{aligned}\Delta x_3 &= {x_3}^2 - n = (x_2+1)^2 - n = {x_2}^2 + 2x_2 + 1 - n \\ &= \Delta x_2 + 2x_2 + 1 = \Delta x_2 + 2x_1 + 3,\end{aligned} \tag{12}$$

so subsequent differences are obtained simply by adding two.

Maurice Kraitchik sped up the ALGORITHM by looking for x and y satisfying

$$x^2 \equiv y^2 \pmod{n}, \tag{13}$$

i.e., $n|(x^2 - y^2)$. This congruence has uninteresting solutions $x \equiv \pm y \pmod{n}$ and interesting solutions $x \not\equiv \pm y \pmod{n}$. It turns out that if n is ODD and DIVISIBLE by at least two different PRIMES, then at least half of the solutions to $x^2 \equiv y^2 \pmod{n}$ with xy COPRIME to n are interesting. For such solutions, $(n, x-y)$ is neither n nor 1 and is therefore a nontrivial factor of n (Pomerance 1996). This ALGORITHM can be used to prove primality, but is not practical. In 1931, Lehmer and Powers discovered how to search for such pairs using CONTINUED FRACTIONS. This method was improved by Morrison and Brillhart (1975) into the CONTINUED FRACTION FACTORIZATION ALGORITHM, which was the fastest ALGORITHM in use before the QUADRATIC SIEVE FACTORIZATION METHOD was developed.

see also PRIME FACTORIZATION ALGORITHMS, SMOOTH NUMBER

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Morrison, M. A. and Brillhart, J. "A Method of Factoring and the Factorization of F_7." *Math. Comput.* **29**, 183–205, 1975.

Pomerance, C. "A Tale of Two Sieves." *Not. Amer. Math. Soc.* **43**, 1473–1485, 1996.

Fermat's Last Theorem

A theorem first proposed by Fermat in the form of a note scribbled in the margin of his copy of the ancient Greek text *Arithmetica* by Diophantus. The scribbled note was discovered posthumously, and the original is now lost. However, a copy was preserved in a book published by Fermat's son. In the note, Fermat claimed to have discovered a proof that the DIOPHANTINE EQUATION $x^n + y^n = z^n$ has no INTEGER solutions for $n > 2$.

The full text of Fermat's statement, written in Latin, reads "Cubum autem in duos cubos, aut quadratoquadratum in duos quadratoquadratos & generaliter nullam in infinitum ultra quadratum potestatem in duos eiusdem nominis fas est diuidere cuius rei demonstrationem mirabilem sane detexi. Hanc marginis exiguitas non caperet." In translation, "It is impossible for a cube to be the sum of two cubes, a fourth power to be the sum of two fourth powers, or in general for any number that is a power greater than the second to be the sum of two like powers. I have discovered a truly marvelous demonstration of this proposition that this margin is too narrow to contain."

As a result of Fermat's marginal note, the proposition that the DIOPHANTINE EQUATION

$$x^n + y^n = z^n, \tag{1}$$

where x, y, z, and n are INTEGERS, has no NONZERO solutions for $n > 2$ has come to be known as Fermat's Last Theorem. It was called a "THEOREM" on the strength of Fermat's statement, despite the fact that no other mathematician was able to prove it for hundreds of years.

Note that the restriction $n > 2$ is obviously necessary since there are a number of elementary formulas for generating an infinite number of PYTHAGOREAN TRIPLES (x, y, z) satisfying the equation for $n = 2$,

$$x^2 + y^2 = z^2. \tag{2}$$

A first attempt to solve the equation can be made by attempting to factor the equation, giving

$$(z^{n/2} + y^{n/2})(z^{n/2} - y^{n/2}) = x^n. \tag{3}$$

Since the product is an exact POWER,

$$\begin{cases} z^{n/2} + y^{n/2} = 2^{n-1}p^n \\ z^{n/2} - y^{n/2} = 2q^n \end{cases} \text{ or } \begin{cases} z^{n/2} + y^{n/2} = 2p^n \\ z^{n/2} - y^{n/2} = 2^{n-1}q^n. \end{cases} \tag{4}$$

Solving for y and z gives

$$\begin{cases} z^{n/2} = 2^{n-2}p^n + q^n \\ y^{n/2} = 2^{n-2}p^n - q^n \end{cases} \text{ or } \begin{cases} z^{n/2} = p^n + 2^{n-2}q^n \\ y^{n/2} = p^n - 2^{n-2}q^n, \end{cases} \tag{5}$$

which give

$$\begin{cases} z = (2^{n-2}p^n + q^n)^{2/n} \\ y = (2^{n-2}p^n - q^n)^{2/n} \end{cases} \text{ or } \begin{cases} z = (p^n + 2^{n-2}q^n)^{2/n} \\ y = (p^n - 2^{n-2}q^n)^{2/n}. \end{cases} \tag{6}$$

However, since solutions to these equations in RATIONAL NUMBERS are no easier to find than solutions to the original equation, this approach unfortunately does not provide any additional insight.

It is sufficient to prove Fermat's Last Theorem by considering PRIME POWERS only, since the arguments can otherwise be written

$$(x^m)^p + (y^m)^p = (z^m)^p, \tag{7}$$

so redefining the arguments gives

$$x^p + y^p = z^p. \tag{8}$$

The so-called "first case" of the theorem is for exponents which are RELATIVELY PRIME to x, y, and z ($p \nmid x, y, z$) and was considered by Wieferich. Sophie Germain proved the first case of Fermat's Last Theorem for any ODD PRIME p when $2p + 1$ is also a PRIME. Legendre subsequently proved that if p is a PRIME such that $4p + 1$, $8p + 1$, $10p + 1$, $14p + 1$, or $16p + 1$ is also a PRIME, then the first case of Fermat's Last Theorem holds for p. This established Fermat's Last Theorem for $p < 100$. In 1849, Kummer proved it for all REGULAR PRIMES and COMPOSITE NUMBERS of which they are factors (Vandiver 1929, Ball and Coxeter 1987).

Kummer's attack led to the theory of IDEALS, and Vandiver developed VANDIVER'S CRITERIA for deciding if a given IRREGULAR PRIME satisfies the theorem. Genocchi (1852) proved that the first case is true for p if $(p, p-3)$ is not an IRREGULAR PAIR. In 1858, Kummer showed that the first case is true if either $(p, p-3)$ or $(p, p-5)$ is an IRREGULAR PAIR, which was subsequently extended to include $(p, p-7)$ and $(p, p-9)$ by Mirimanoff (1905). Wieferich (1909) proved that if the equation is solved in integers RELATIVELY PRIME to an ODD PRIME p, then

$$2^{p-1} \equiv 1 \pmod{p^2}. \tag{9}$$

(Ball and Coxeter 1987). Such numbers are called WIEFERICH PRIMES. Mirimanoff (1909) subsequently showed that

$$3^{p-1} \equiv 1 \pmod{p^2} \tag{10}$$

must also hold for solutions RELATIVELY PRIME to an ODD PRIME p, which excludes the first two WIEFERICH PRIMES 1093 and 3511. Vandiver (1914) showed

$$5^{p-1} \equiv 1 \pmod{p^2}, \tag{11}$$

and Frobenius extended this to

$$11^{p-1}, 17^{p-1} \equiv 1 \pmod{p^2}. \tag{12}$$

It has also been shown that if p were a PRIME of the form $6x - 1$, then

$$7^{p-1}, 13^{p-1}, 19^{p-1} \equiv 1 \pmod{p^2}, \tag{13}$$

which raised the smallest possible p in the "first case" to 253,747,889 by 1941 (Rosser 1941). Granville and Monagan (1988) showed if there exists a PRIME p satisfying Fermat's Last Theorem, then

$$q^{p-1} \equiv 1 \pmod{p^2} \tag{14}$$

for $q = 5, 7, 11, \ldots, 71$. This establishes that the first case is true for all PRIME exponents up to 714,591,416,091,398 (Vardi 1991).

The "second case" of Fermat's Last Theorem (for $p | x, y, z$) proved harder than the first case.

Euler proved the general case of the theorem for $n = 3$, Fermat $n = 4$, Dirichlet and Lagrange $n = 5$. In 1832, Dirichlet established the case $n = 14$. The $n = 7$ case was proved by Lamé (1839), using the identity

$$\begin{aligned} &(X + Y + Z)^7 - (X^7 + Y^7 + Z^7) \\ &\quad = 7(X + Y)(X + Z)(Y + Z) \\ &\qquad \times [(X^2 + Y^2 + Z^2 + XY + XZ + YZ)^2 \\ &\qquad\qquad + XYZ(X + Y + Z)]. \end{aligned} \tag{15}$$

Although some errors were present in this proof, these were subsequently fixed by Lebesgue (1840). Much additional progress was made over the next 150 years, but no completely general result had been obtained. Buoyed by false confidence after his proof that PI is TRANSCENDENTAL, the mathematician Lindemann proceeded to publish several proofs of Fermat's Last Theorem, all of them invalid (Bell 1937, pp. 464–465). A prize of 100,000 German marks (known as the Wolfskel Prize) was also offered for the first valid proof (Ball and Coxeter 1987, p. 72).

A recent false alarm for a general proof was raised by Y. Miyaoka (Cipra 1988) whose proof, however, turned out to be flawed. Other attempted proofs among both professional and amateur mathematicians are discussed by vos Savant (1993), although vos Savant erroneously claims that work on the problem by Wiles (discussed below) is invalid. By the time 1993 rolled around, the general case of Fermat's Last Theorem had been shown to be true for all exponents up to 4×10^6 (Cipra 1993). However, given that a proof of Fermat's Last Theorem requires truth for *all* exponents, proof for any finite number of exponents does not constitute any significant progress towards a proof of the general theorem (although the fact that no counterexamples were found for this many cases is highly suggestive).

In 1993, a bombshell was dropped. In that year, the general theorem was partially proven by Andrew Wiles (Cipra 1993, Stewart 1993) by proving the SEMISTABLE case of the TANIYAMA-SHIMURA CONJECTURE. Unfortunately, several holes were discovered in the proof shortly thereafter when Wiles' approach via the TANIYAMA-SHIMURA CONJECTURE became hung up on properties of the SELMER GROUP using a tool called an "Euler system." However, the difficulty was circumvented by Wiles and R. Taylor in late 1994 (Cipra 1994, 1995ab) and published in Taylor and Wiles (1995) and Wiles (1995). Wiles' proof succeeds by (1) replacing ELLIPTIC CURVES with Galois representations, (2) reducing the problem to a CLASS NUMBER FORMULA, (3) proving that FORMULA, and (4) tying up loose ends that arise because the formalisms fail in the simplest degenerate cases (Cipra 1995a).

The proof of Fermat's Last Theorem marks the end of a mathematical era. Since virtually all of the tools which were eventually brought to bear on the problem had yet to be invented in the time of Fermat, it is interesting to speculate about whether he actually was in possession of an elementary proof of the theorem. Judging by the temerity with which the problem resisted attack for so long, Fermat's alleged proof seems likely to have been illusionary.

see also ABC CONJECTURE, BOGOMOLOV-MIYAOKA-YAU INEQUALITY, MORDELL CONJECTURE, PYTHAGOREAN TRIPLE, RIBET'S THEOREM, SELMER GROUP, SOPHIE GERMAIN PRIME, SZPIRO'S CONJECTURE, TANIYAMA-SHIMURA CONJECTURE, VOJTA'S CONJECTURE, WARING FORMULA

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Fermat's Lesser Theorem

see FERMAT'S LITTLE THEOREM

Fermat's Little Theorem

If p is a PRIME number and a a NATURAL NUMBER, then

$$a^p \equiv a \pmod{p}. \tag{1}$$

Furthermore, if $p \nmid a$ (p does not divide a), then there exists some smallest exponent d such that

$$a^d - 1 \equiv 0 \pmod{p} \tag{2}$$

and d divides $p-1$. Hence,

$$a^{p-1} - 1 \equiv 0 \pmod{p}. \tag{3}$$

This is a generalization of the CHINESE HYPOTHESIS and a special case of EULER'S THEOREM. It is sometimes called FERMAT'S PRIMALITY TEST and is a NECESSARY but not SUFFICIENT test for primality. Although it was presumably proved (but suppressed) by Fermat, the first proof was published by Euler in 1749.

The theorem is easily proved using mathematical INDUCTION. Suppose $p|a^p - a$. Then examine

$$(a+1)^p - (a+1). \tag{4}$$

From the BINOMIAL THEOREM,

$$(a+1)^p = a^p + \binom{p}{1} a^{p-1} + \binom{p}{2} a^{p-2} + \ldots + \binom{p}{p-1} a + 1. \tag{5}$$

Rewriting,

$$(a+1)^p - a^p - 1 = \binom{p}{1} a^{p-1} + \binom{p}{2} a^{p-2} + \ldots + \binom{p}{p-1} a. \tag{6}$$

But p divides the right side, so it also divides the left side. Combining with the induction hypothesis gives that p divides the sum

$$[(a+1)^p - a^p - 1] + (a^p - a) = (a+1)^p - (a+1), \tag{7}$$

as assumed, so the hypothesis is true for any a. The theorem is sometimes called FERMAT'S SIMPLE THEOREM. WILSON'S THEOREM follows as a COROLLARY of Fermat's Little Theorem.

Fermat's little theorem shows that, if p is PRIME, there does not exists a base $a < p$ with $(a,p) = 1$ such that $a^{p-1} - 1$ possesses a nonzero residue modulo p. If such base a exists, p is therefore guaranteed to be composite. However, the lack of a nonzero residue in Fermat's little theorem does *not* guarantee that p is PRIME. The property of unambiguously certifying composite numbers while passing some PRIMES make Fermat's little theorem a COMPOSITENESS TEST which is sometimes called the FERMAT COMPOSITENESS TEST. COMPOSITE NUMBERS known as FERMAT PSEUDOPRIMES (or sometimes simply "PSEUDOPRIMES") have zero residue for some as and so are not identified as composite. Worse still, there exist numbers known as CARMICHAEL NUMBERS (the smallest of which is 561) which give zero residue for *any* choice of the base a RELATIVELY PRIME to p. However, FERMAT'S LITTLE THEOREM CONVERSE provides a criterion for certifying the primality of a number.

A number satisfying Fermat's little theorem for some nontrivial base and which is not known to be composite is called a PROBABLE PRIME. A table of the smallest PSEUDOPRIMES P for the first 100 bases a follows (Sloane's A007535).

a	P	a	P	a	P	a	P	a	P
2	341	22	69	42	205	62	63	82	91
3	91	23	33	43	77	63	341	83	105
4	15	24	25	44	45	64	65	84	85
5	124	25	28	45	76	65	133	85	129
6	35	26	27	46	133	66	91	86	87
7	25	27	65	47	65	67	85	87	91
8	9	28	87	48	49	68	69	88	91
9	28	29	35	49	66	69	85	89	99
10	33	30	49	50	51	70	169	90	91
11	15	31	49	51	65	71	105	91	115
12	65	32	33	52	85	72	85	92	93
13	21	33	85	53	65	73	111	93	301
14	15	34	35	54	55	74	75	94	95
15	341	35	51	55	63	75	91	95	141
16	51	36	91	56	57	76	77	96	133
17	45	37	45	57	65	77	95	97	105
18	25	38	39	58	95	78	341	98	99
19	45	39	95	59	87	79	91	99	145
20	21	40	91	60	341	80	81	100	259
21	55	41	105	61	91	81	85		

see also BINOMIAL THEOREM, CARMICHAEL NUMBER, CHINESE HYPOTHESIS, COMPOSITE NUMBER, COMPOSITENESS TEST, EULER'S THEOREM, FERMAT'S LITTLE THEOREM CONVERSE, FERMAT PSEUDOPRIME, MODULO MULTIPLICATION GROUP, PRATT CERTIFICATE, PRIMALITY TEST, PRIME NUMBER, PSEUDOPRIME, RELATIVELY PRIME, TOTIENT FUNCTION, WIEFERICH PRIME, WILSON'S THEOREM, WITNESS

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Fermat's Little Theorem Converse

The converse of FERMAT'S LITTLE THEOREM is also known as LEHMER'S THEOREM. It states that, if an INTEGER x is PRIME to m and $x^{m-1} \equiv 1 \pmod{m}$ and there is no INTEGER $e < m-1$ for which $x^e \equiv 1 \pmod{m}$, then m is PRIME. Here, x is called a WITNESS to the primality of m. This theorem is the basis for the PRATT PRIMALITY CERTIFICATE.

see also FERMAT'S LITTLE THEOREM, PRATT CERTIFICATE, PRIMALITY CERTIFICATE, WITNESS

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Fermat-Lucas Number

A number of the form $2^n + 1$ obtained by setting $x = 1$ in a FERMAT-LUCAS POLYNOMIAL. The first few are 3, 5, 9, 17, 33, ... (Sloane's A000051).

see also FERMAT NUMBER (LUCAS)

References
Sloane, N. J. A. Sequence A000051/M0717 in "An On-Line Version of the Encyclopedia of Integer Sequences."

Fermat Number

A BINOMIAL NUMBER of the form $F_n = 2^{2^n} + 1$. The first few for $n = 0, 1, 2, \ldots$ are 3, 5, 17, 257, 65537, 4294967297, ... (Sloane's A000215). The number of DIGITS for a Fermat number is

$$\begin{aligned} D(n) &= \lfloor [\log(2^{2^n}+1)] + 1 \rfloor \approx \lfloor \log(2^{2^n}) + 1 \rfloor \\ &= \lfloor 2^n \log 2 + 1 \rfloor. \end{aligned} \tag{1}$$

Being a Fermat number is the NECESSARY (but not SUFFICIENT) form a number

$$N_n \equiv 2^n + 1 \tag{2}$$

must have in order to be PRIME. This can be seen by noting that if $N_n = 2^n + 1$ is to be PRIME, then n cannot have any ODD factors b or else N_n would be a factorable number of the form

$$\begin{aligned} 2^n + 1 &= (2^a)^b + 1 \\ &= (2^a+1)[2^{a(b-1)} - 2^{a(b-2)} + 2^{a(b-3)} - \ldots + 1]. \end{aligned} \tag{3}$$

Therefore, for a PRIME N_n, n must be a POWER of 2.

Fermat conjectured in 1650 that every Fermat number is PRIME, but only COMPOSITE Fermat numbers F_n are known for $n \geq 5$. Eisenstein (1844) proposed as a problem the proof that there are an infinite number of Fermat primes (Ribenboim 1996, p. 88), but this has not yet been achieved. An anonymous writer proposed that numbers of the form 2^2+1, $2^{2^2}+1$, $2^{2^{2^2}}+1$ were PRIME. However, this conjecture was refuted when Selfridge (1953) showed that

$$F_{16} = 2^{2^{16}} + 1 = 2^{2^{2^{2^2}}} + 1 \tag{4}$$

is COMPOSITE (Ribenboim 1996, p. 88). Numbers of the form $a^{2^n} + b^{2^n}$ are called generalized Fermat numbers (Ribenboim 1996, pp. 359–360).

Fermat numbers satisfy the RECURRENCE RELATION

$$F_m = F_0 F_1 \cdots F_{m-1} + 2. \tag{5}$$

F_n can be shown to be PRIME iff it satisfies PÉPIN'S TEST

$$3^{(F_n-1)/2} \equiv -1 \pmod{F_n}. \tag{6}$$

PÉPIN'S THEOREM

$$3^{2^{2^n-1}} \equiv -1 \pmod{F_n} \tag{7}$$

is also NECESSARY and SUFFICIENT.

In 1770, Euler showed that any FACTOR of F_n must have the form

$$2^{n+1}K+1, \tag{8}$$

where K is a POSITIVE INTEGER. In 1878, Lucas increased the exponent of 2 by one, showing that FACTORS of Fermat numbers must be of the form

$$2^{n+2}L+1. \tag{9}$$

If

$$F = p_1 p_2 \cdots p_r \tag{10}$$

is the factored part of $F_n = FC$ (where C is the cofactor to be tested for primality), compute

$$A \equiv 3^{F_n-1} \pmod{F_n} \tag{11}$$
$$B \equiv 3^{F-1} \pmod{F_n} \tag{12}$$
$$R \equiv A - B \pmod{C}. \tag{13}$$

Then if $R \equiv 0$, the cofactor is a PROBABLE PRIME to the base 3^F; otherwise C is COMPOSITE.

In order for a POLYGON to be circumscribed about a CIRCLE (i.e., a CONSTRUCTIBLE POLYGON), it must have a number of sides N given by

$$N = 2^k F_0 \cdots F_n, \tag{14}$$

where the F_n are *distinct* Fermat primes. This is equivalent to the statement that the trigonometric functions $\sin(k\pi/N)$, $\cos(k\pi/N)$, etc., can be computed in terms of finite numbers of additions, multiplications, and square root extractions iff N is of the above form. The only known Fermat PRIMES are

$$F_0 = 3$$
$$F_1 = 5$$
$$F_2 = 17$$
$$F_3 = 257$$
$$F_4 = 65537$$

and it seems unlikely that any more exist.

Factoring Fermat numbers is extremely difficult as a result of their large size. In fact, only F_5 to F_{11} have been complete factored, as summarized in the following table. Written out explicitly, the complete factorizations are

$$F_5 = 641 \cdot 6700417$$
$$F_6 = 274177 \cdot 67280421310721$$
$$F_7 = 59649589127497217 \cdot 5704689200685129054721$$
$$F_8 = 1238926361552897 \cdot 93461639715357977769163 \cdots$$
$$\cdots 558199606896584051237541638188580280321$$
$$F_9 = 2424833 \cdot 7455602825647884208337395736200454918783366342657 \cdots$$
$$\cdots 54918783366342657 \cdot P99$$
$$F_{10} = 45592577 \cdot 6487031809 \cdot 46597757852200185 \cdots$$
$$\cdots 43264560743076778192897 \cdot P252$$
$$F_{11} = 319489 \cdot 974849 \cdot 167988556341760475137$$
$$\cdot 3560841906445833920513 \cdot P564.$$

Here, the final large PRIME is not explicitly given since it can be computed by dividing F_n by the other given factors.

F_n	Digits	Facts.	Digits	Reference
5	10	2	3, 7	Euler 1732
6	20	2	6, 14	Landry 1880
7	39	2	7, 22	Morrison and Brillhart 1975
8	78	2	16, 62	Brent and Pollard 1981
9	155	3	7, 49, 99	Manasse and Lenstra (In Cipra 1993)
10	309	4	8, 10, 40, 252	Brent 1995
11	617	5	6, 6, 21, 22, 564	Brent 1988

Tables of known factors of Fermat numbers are given by Keller (1983), Brillhart *et al.* (1988), Young and Buell (1988), Riesel (1994), and Pomerance (1996). Young and Buell (1988) discovered that F_{20} is COMPOSITE, and Crandall *et al.* (1995) that F_{22} is COMPOSITE. A current list of the known factors of Fermat numbers is maintained by Keller, and reproduced in the form of a *Mathematica*® notebook by Weisstein. In these tables, since all factors are of the form $k2^n+1$, the known factors are expressed in the concise form (k, n). The number of factors for Fermat numbers F_n for $n = 0, 1, 2, \ldots$ are 1, 1, 1, 1, 1, 2, 2, 2, 2, 3, 4, 5, ...

see also CULLEN NUMBER, PÉPIN'S TEST, PÉPIN'S THEOREM, POCKLINGTON'S THEOREM, POLYGON, PROTH'S THEOREM, SELFRIDGE-HURWITZ RESIDUE, WOODALL NUMBER

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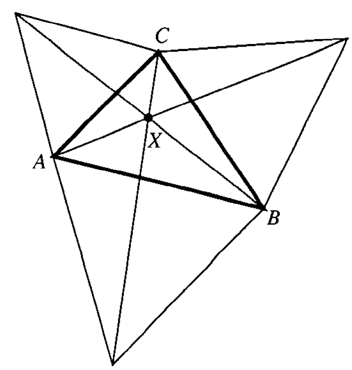
Fermat Number (Lucas)

A number of the form $2^n - 1$ obtained by setting $x = 1$ in a FERMAT POLYNOMIAL is called a MERSENNE NUMBER.

see also FERMAT-LUCAS NUMBER, MERSENNE NUMBER

Fermat Point



Also known as the first ISOGONIC CENTER and the TORRICELLI POINT. In a given ACUTE TRIANGLE ΔABC, the Fermat point is the point X which minimizes the sum of distances from A, B, and C,

$$|AX| + |BX| + |CX|. \tag{1}$$

This problem is called FERMAT'S PROBLEM or STEINER'S PROBLEM (Courant and Robbins 1941) and was proposed by Fermat to Torricelli. Torricelli's solution was published by his pupil Viviani in 1659 (Johnson 1929).

If all ANGLES of the TRIANGLE are less than 120° ($2\pi/3$), then the Fermat point is the interior point X from which each side subtends an ANGLE of 120°, i.e.,

$$\angle BXC = \angle CXA = \angle AXB = 120°. \tag{2}$$

The Fermat point can also be constructed by drawing EQUILATERAL TRIANGLES on the outside of the given TRIANGLE and connecting opposite VERTICES. The three diagonals in the figure then intersect in the Fermat point. The TRIANGLE CENTER FUNCTION of the Fermat point is

$$\alpha = \csc(A + \tfrac{1}{3}\pi) \quad (3)$$
$$= bc[c^2a^2 + (c^2 + a^2 - b^2)^2][a^2b^2 - (a^2 + b^2 - c^2)^2]$$
$$\times [4\Delta - \sqrt{3}(b^2 + c^2 - a^2)]. \quad (4)$$

The ANTIPEDAL TRIANGLE is EQUILATERAL and has AREA

$$\Delta' = 2\Delta \left[1 + \cot\omega \cot\left(\frac{\pi}{3}\right)\right], \quad (5)$$

where ω is the BROCARD ANGLE.

Given three POSITIVE REAL NUMBERS l, m, n, the "generalized" Fermat point is the point P of a given ACUTE TRIANGLE ΔABC such that

$$l \cdot PA + m \cdot PB + n \cdot PC \quad (6)$$

is a minimum (Greenberg and Robertello 1965, van de Lindt 1966, Tong and Chua 1995)

see also ISOGONIC CENTERS

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Fermat's Polygonal Number Theorem

In 1638, Fermat proposed that every POSITIVE INTEGER is a sum of *at most* three TRIANGULAR NUMBERS, four SQUARE NUMBERS, five PENTAGONAL NUMBERS, and n n-POLYGONAL NUMBERS. Fermat claimed to have a proof of this result, although Fermat's proof has never been found. Gauss proved the triangular case, and noted the event in his diary on July 10, 1796, with the notation

$$** E\Upsilon RHKA \qquad num = \Delta + \Delta + \Delta.$$

This case is equivalent to the statement that every number of the form $8m + 3$ is a sum of three ODD SQUARES (Duke 1997). More specifically, a number is a sum of three SQUARES IFF it is not of the form $4^b(8m + 7)$ for $b \geq 0$, as first proved by Legendre in 1798.

Euler was unable to prove the square case of Fermat's theorem, but he left partial results which were subsequently used by Lagrange. The square case was finally proved by Jacobi and independently by Lagrange in 1772. It is therefore sometimes known as LAGRANGE'S FOUR-SQUARE THEOREM. In 1813, Cauchy proved the proposition in its entirety.

see also FIFTEEN THEOREM, VINOGRADOV'S THEOREM, LAGRANGE'S FOUR-SQUARE THEOREM, WARING'S PROBLEM

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Fermat Polynomial

The POLYNOMIALS obtained by setting $p(x) = 3x$ and $q(x) = -2$ in the LUCAS POLYNOMIAL SEQUENCES. The first few Fermat polynomials are

$$\begin{aligned}\mathcal{F}_1(x) &= 1\\ \mathcal{F}_2(x) &= 3x\\ \mathcal{F}_3(x) &= 9x^2 - 2\\ \mathcal{F}_4(x) &= 27x^3 - 12x\\ \mathcal{F}_5(x) &= 81x^4 - 54x^2 + 4,\end{aligned}$$

and the first few Fermat-Lucas polynomials are

$$\begin{aligned}f_1(x) &= 3x\\ f_2(x) &= 9x^2 - 4\\ f_3(x) &= 27x^3 - 18x\\ f_4(x) &= 81x^4 - 72x^2 + 8\\ f_5(x) &= 243x^5 - 270x^3 + 60x.\end{aligned}$$

Fermat and Fermat-Lucas POLYNOMIALS satisfy

$$\mathcal{F}_n(1) = \mathcal{F}_n$$

$$f_n(1) = f_n$$

where $\mathcal{F}_n$ are FERMAT NUMBERS and f_n are FERMAT-LUCAS NUMBERS.

Fermat's Primality Test

see FERMAT'S LITTLE THEOREM

Fermat Prime

A FERMAT NUMBER $F_n = 2^{2^n} + 1$ which is PRIME.

see also CONSTRUCTIBLE POLYGON, FERMAT NUMBER

Fermat's Problem

In a given ACUTE TRIANGLE ΔABC, locate a point whose distances from A, B, and C have the smallest possible sum. The solution is the point from which each side subtends an angle of 120°, known as the FERMAT POINT.

see also ACUTE TRIANGLE, FERMAT POINT

Fermat Pseudoprime

A Fermat pseudoprime to a base a, written psp(a), is a COMPOSITE NUMBER n such that $a^{n-1} \equiv 1 \pmod{n}$ (i.e., it satisfies FERMAT'S LITTLE THEOREM, sometimes with the requirement that n must be ODD; Pomerance *et al.* 1980). psp(2)s are called POULET NUMBERS or, less commonly, SARRUS NUMBERS or FERMATIANS (Shanks 1993). The first few EVEN psp(2)s (including the PRIME 2 as a pseudoprime) are 2, 161038, 215326, ... (Sloane's A006935).

If base 3 is used in addition to base 2 to weed out potential COMPOSITE NUMBERS, only 4709 COMPOSITE NUMBERS remain $< 25 \times 10^9$. Adding base 5 leaves 2552, and base 7 leaves only 1770 COMPOSITE NUMBERS.

see also FERMAT'S LITTLE THEOREM, POULET NUMBER, PSEUDOPRIME

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Sloane, N. J. A. Sequence A006935/M2190 in "An On-Line Version of the Encyclopedia of Integer Sequences."

Fermat Quotient

The Fermat quotient for a number a and a PRIME base p is defined as

$$q_p(a) \equiv \frac{a^{p-1} - 1}{p}. \tag{1}$$

If $p \nmid ab$, then

$$q_p(ab) = q_p(a) + q_p(b) \tag{2}$$

$$q_p(p \pm 1) = \mp 1 \tag{3}$$

$$q_p(2) = \frac{1}{p}\left(1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \ldots - \frac{1}{p-1}\right), \tag{4}$$

all (mod p). The quantity $q_p(2) = (2^{p-1} - 1)/p$ is known to be SQUARE for only two PRIMES: the so-called WIEFERICH PRIMES 1093 and 3511 (Lehmer 1981, Crandall 1986).

see also WIEFERICH PRIME

References

Crandall, R. *Projects in Scientific Computation.* New York: Springer-Verlag, 1986.

Lehmer, D. H. "On Fermat's Quotient, Base Two." *Math. Comput.* **36**, 289–290, 1981.

Fermat's Right Triangle Theorem

The AREA of a RATIONAL RIGHT TRIANGLE cannot be a SQUARE NUMBER. This statement is equivalent to "a CONGRUUM cannot be a SQUARE NUMBER."

see also CONGRUUM, RATIONAL TRIANGLE, RIGHT TRIANGLE, SQUARE NUMBER

Fermat's Sigma Problem

Solve

$$\sigma(x^3) = y^2$$

and

$$\sigma(x^2) = y^3,$$

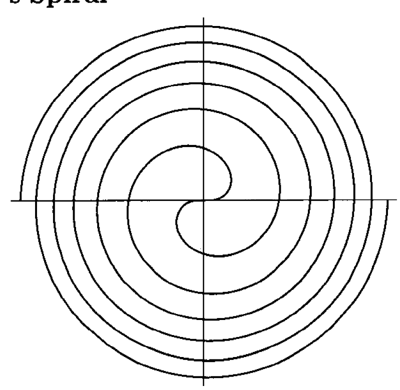
where σ is the DIVISOR FUNCTION.

see also WALLIS'S PROBLEM

Fermat's Simple Theorem

see FERMAT'S LITTLE THEOREM

Fermat's Spiral



An ARCHIMEDEAN SPIRAL with $m = 2$ having polar equation

$$r = a\theta^{1/2},$$

discussed by Fermat in 1636 (MacTutor Archive). It is also known as the PARABOLIC SPIRAL. For any given POSITIVE value of θ, there are two corresponding values of r of opposite signs. The resulting spiral is therefore symmetrical about the line $y = -x$. The CURVATURE is

$$\kappa(\theta) = \frac{\frac{3a^2}{4\theta} + a^2\theta}{\left(\frac{a^2}{4\theta} + a^2\theta\right)^{3/2}}.$$

References

Dixon, R. *Mathographics.* New York: Dover, p. 121, 1991.

Gray, A. *Modern Differential Geometry of Curves and Surfaces.* Boca Raton, FL: CRC Press, pp. 69–70, 1993.
Lee, X. "Equiangular Spiral." `http://www.best.com/~xah/SpecialPlaneCurves_dir/EquiangularSpiral_dir/equiangularSpiral.html`.
Lockwood, E. H. *A Book of Curves.* Cambridge, England: Cambridge University Press, p. 175, 1967.
MacTutor History of Mathematics Archive. "Fermat's Spiral." `http://www-groups.dcs.st-and.ac.uk/~history/Curves/Fermats.html`.
Wells, D. *The Penguin Dictionary of Curious and Interesting Geometry.* Middlesex, England: Penguin Books, 1991.

Fermat Spiral Inverse Curve

The INVERSE CURVE of FERMAT'S SPIRAL with the origin taken as the INVERSION CENTER is the LITUUS.

References
Lawrence, J. D. *A Catalog of Special Plane Curves.* New York: Dover, pp. 186–187, 1972.

Fermat Sum Theorem

The only whole number solution to the DIOPHANTINE EQUATION

$$y^3 = x^2 + 2$$

is $y = 3$, $x = \pm 5$. This theorem was offered as a problem by Fermat, who suppressed his own proof.

Fermat's Theorem

A PRIME p can be represented in an essentially unique manner in the form $x^2 + y^2$ for integral x and y IFF $p \equiv 1 \pmod 4$ or $p = 2$. It can be restated by letting

$$Q(x,y) \equiv x^2 + y^2,$$

then all RELATIVELY PRIME solutions (x, y) to the problem of representing $Q(x, y) = m$ for m any INTEGER are achieved by means of successive applications of the GENUS THEOREM and COMPOSITION THEOREM. There is an analog of this theorem for EISENSTEIN INTEGERS.

see also EISENSTEIN INTEGER, SQUARE NUMBER

References
Shanks, D. *Solved and Unsolved Problems in Number Theory, 4th ed.* New York: Chelsea, pp. 142–143, 1993.

Fermat's Two-Square Theorem

see FERMAT'S THEOREM

Fermatian

see POULET NUMBER

Fermi-Dirac Distribution

A distribution which arises in the study of half-integral spin particles in physics,

$$P(k) = \frac{k^s}{e^{k-\mu} + 1}.$$

Its integral is

$$\int_0^\infty \frac{k^s\,dk}{e^{k-\mu}+1} = e^\mu \Gamma(s+1)\Phi(-e^\mu, s+1, 1),$$

where Φ is the LERCH TRANSCENDENT.

Fern

see BARNSLEY'S FERN

Ferrari's Identity

$$(a^2 + 2ac - 2bc - b^2)^4 + (b^2 - 2ab - 2ac - c^2)^4$$
$$+(c^2 + 2ab + 2bc - a^2)^4 = 2(a^2 + b^2 + c^2 - ab + ac + bc)^4.$$

see also DIOPHANTINE EQUATION—QUARTIC

References
Berndt, B. C. *Ramanujan's Notebooks, Part IV.* New York: Springer-Verlag, pp. 96–97, 1994.

Ferrers Diagram

see YOUNG DIAGRAM

Ferrers' Function

An alternative name for an associated LEGENDRE POLYNOMIAL.

see also LEGENDRE POLYNOMIAL

References
Sansone, G. *Orthogonal Functions, rev. English ed.* New York: Dover, p. 246, 1991.

Ferrier's Prime

According to Hardy and Wright (1979), the largest PRIME found before the days of electronic computers is the 44-digit number

$$F \equiv \tfrac{1}{17}(2^{148} + 1)$$
$$= 20988936657440586486151264256610222593863921,$$

which was found using only a mechanical calculator.

References
Hardy, G. H. and Wright, E. M. *An Introduction to the Theory of Numbers, 5th ed.* Oxford, England: Clarendon Press, pp. 16–22, 1979.

Feuerbach Circle

see NINE-POINT CIRCLE

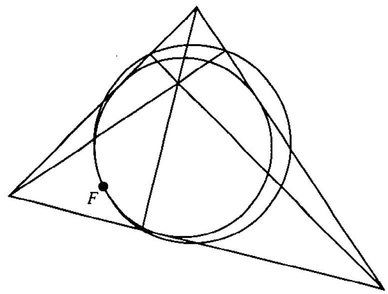
Feuerbach's Conic Theorem

The LOCUS of the centers of all CONICS through the VERTICES and ORTHOCENTER of a TRIANGLE (which are RECTANGULAR HYPERBOLAS when not degenerate), is a CIRCLE through the MIDPOINTS of the sides, the points half way from the ORTHOCENTER to the VERTICES, and the feet of the ALTITUDE.

see also ALTITUDE, CONIC SECTION, FEUERBACH'S THEOREM, KIEPERT'S HYPERBOLA, MIDPOINT, ORTHOCENTER, RECTANGULAR HYPERBOLA

References
Coolidge, J. L. *A Treatise on Algebraic Plane Curves.* New York: Dover, p. 198, 1959.

Feuerbach Point



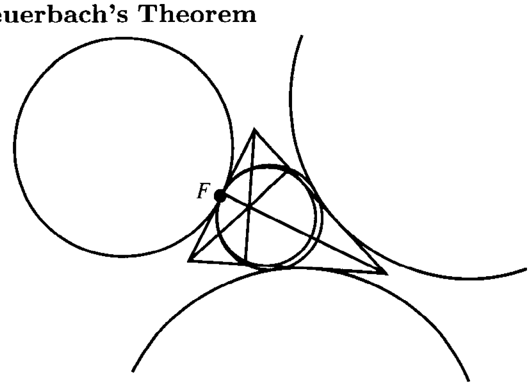
The point F at which the INCIRCLE and NINE-POINT CIRCLE are tangent. It has TRIANGLE CENTER FUNCTION

$$\alpha = 1 - \cos(B - C).$$

see also FEUERBACH'S THEOREM

References
Johnson, R. A. *Modern Geometry: An Elementary Treatise on the Geometry of the Triangle and the Circle.* Boston, MA: Houghton Mifflin, p. 200, 1929.
Kimberling, C. "Central Points and Central Lines in the Plane of a Triangle." *Math. Mag.* **67**, 163–187, 1994.
Salmon, G. *Conic Sections, 6th ed.* New York: Chelsea, p. 127, 1954.

Feuerbach's Theorem



1. The CIRCLE which passes through the feet of the PERPENDICULARS dropped from the VERTICES of any TRIANGLE on the sides opposite them passes also through the MIDPOINTS of these sides as well as through the MIDPOINT of the segments which join the VERTICES to the point of intersection of the PERPENDICULAR (a NINE-POINT CIRCLE).
2. The NINE-POINT CIRCLE of any TRIANGLE is TANGENT internally to the INCIRCLE and TANGENT externally to the three EXCIRCLES.

see also EXCIRCLE, FEUERBACH POINT, INCIRCLE, MIDPOINT, NINE-POINT CIRCLE, PERPENDICULAR, TANGENT

References
Coxeter, H. S. M. and Greitzer, S. L. *Geometry Revisited.* Washington, DC: Math. Assoc. Amer., pp. 117–119, 1967.
Dixon, R. *Mathographics.* New York: Dover, p. 59, 1991.

Feynman Point

The sequence of six 9s which begins at the 762th decimal place of PI,

$$\pi = 3.14159\ldots 134\underbrace{999999}837\ldots.$$

see also PI

FFT

see FAST FOURIER TRANSFORM

Fiber

A quantity F corresponding to a FIBER BUNDLE, where the FIBER BUNDLE is a MAP $f : E \to B$, with E the TOTAL SPACE of the FIBER BUNDLE and B the BASE SPACE of the FIBER BUNDLE.

see also FIBER BUNDLE, WHITNEY SUM

Fiber Bundle

A fiber bundle (also called simply a BUNDLE) with FIBER F is a MAP $f : E \to B$ where E is called the TOTAL SPACE of the fiber bundle and B the BASE SPACE of the fiber bundle. The main condition for the MAP to be a fiber bundle is that every point in the BASE SPACE $b \in B$ has a NEIGHBORHOOD U such that $f^{-1}(U)$ is HOMEOMORPHIC to $U \times F$ in a special way. Namely, if

$$h : f^{-1}(U) \to U \times F$$

is the HOMEOMORPHISM, then

$$\mathrm{proj}_U \circ h = f_{|f^{-1}(U)|},$$

where the MAP proj_U means projection onto the U component. The homeomorphisms h which "commute with projection" are called local TRIVIALIZATIONS for the fiber bundle f. In other words, E looks like the product $B \times F$ (at least locally), except that the fibers $f^{-1}(x)$ for $x \in B$ may be a bit "twisted."

Examples of fiber bundles include any product $B \times F \to B$ (which is a bundle over B with FIBER F), the MÖBIUS STRIP (which is a fiber bundle over the CIRCLE with FIBER given by the unit interval [0,1]; i.e, the BASE SPACE is the CIRCLE), and $\mathbb{S}^3$ (which is a bundle over $\mathbb{S}^2$ with fiber $\mathbb{S}^1$). A special class of fiber bundle is the VECTOR BUNDLE, in which the FIBER is a VECTOR SPACE.

see also BUNDLE, FIBER SPACE, FIBRATION

Fiber Space

A fiber space, depending on context, means either a FIBER BUNDLE or a FIBRATION.

see also FIBER BUNDLE, FIBRATION

Fibonacci Dual Theorem

Let F_n be the nth FIBONACCI NUMBER. Then the sequence $\{F_n\}_{n=2}^{\infty} = \{1, 2, 3, 5, 8, \ldots\}$ is COMPLETE, even if one is restricted to subsequences in which no two consecutive terms are both passed over (until the desired total is reached; Brown 1965, Honsberger 1985).

see also COMPLETE SEQUENCE, FIBONACCI NUMBER.

References

Brown, J. L. Jr. "A New Characterization of the Fibonacci Numbers." *Fib. Quart.* **3**, 1–8, 1965.

Honsberger, R. *Mathematical Gems III.* Washington, DC: Math. Assoc. Amer., p. 130, 1985.

Fibonacci Hyperbolic Cosine

Let

$$\psi \equiv 1 + \phi = \tfrac{1}{2}(3 + \sqrt{5}\,) \approx 2.618034 \tag{1}$$

where ϕ is the GOLDEN RATIO, and

$$\alpha = \ln \phi \approx 0.4812118. \tag{2}$$

Then define

$$\mathrm{cFh}(x) \equiv \frac{\psi^{x+1/2} + \psi^{-(x+1/2)}}{\sqrt{5}} \tag{3}$$

$$= \frac{\phi^{(2x+1)} + \phi^{-(2x+1)}}{\sqrt{5}} \tag{4}$$

$$= \frac{2}{\sqrt{5}} \cosh[(2x+1)\alpha]. \tag{5}$$

This function satisfies

$$\mathrm{cFh}(-x) = \mathrm{cFh}(x-1). \tag{6}$$

For $n \in \mathbb{Z}$, $\mathrm{cFh}(n) = F_{2n+1}$ where F_n is a FIBONACCI NUMBER.

References

Trzaska, Z. W. "On Fibonacci Hyperbolic Trigonometry and Modified Numerical Triangles." *Fib. Quart.* **34**, 129–138, 1996.

Fibonacci Hyperbolic Cotangent

$$\mathrm{ctFh}(x) \equiv \frac{\mathrm{cFh}(x)}{\mathrm{sFh}(x)},$$

where $\mathrm{cFh}(x)$ is the FIBONACCI HYPERBOLIC COSINE and $\mathrm{sFh}(x)$ is the FIBONACCI HYPERBOLIC SINE.

References

Trzaska, Z. W. "On Fibonacci Hyperbolic Trigonometry and Modified Numerical Triangles." *Fib. Quart.* **34**, 129–138, 1996.

Fibonacci Hyperbolic Sine

Let

$$\psi \equiv 1 + \phi = \tfrac{1}{2}(3 + \sqrt{5}\,) \approx 2.618034 \tag{1}$$

where ϕ is the GOLDEN RATIO, and

$$\alpha = \ln \phi \approx 0.4812118. \tag{2}$$

Then define

$$\mathrm{sFh}(x) \equiv \frac{\psi^{x} - \psi^{-x}}{\sqrt{5}} \tag{3}$$

$$= \frac{\phi^{2x} - \phi^{-2x}}{\sqrt{5}} \tag{4}$$

$$= \frac{2}{\sqrt{5}} \sinh[2x\alpha]. \tag{5}$$

For $n \in \mathbb{Z}$, $\mathrm{sFh}(n) = F_{2n}$ where F_n is a FIBONACCI NUMBER. The function satisfies

$$\mathrm{sFh}(-x) = -\,\mathrm{sFh}(x). \tag{6}$$

References

Trzaska, Z. W. "On Fibonacci Hyperbolic Trigonometry and Modified Numerical Triangles." *Fib. Quart.* **34**, 129–138, 1996.

Fibonacci Hyperbolic Tangent

$$\mathrm{tFh}(x) \equiv \frac{\mathrm{sFh}(x)}{\mathrm{cFh}(x)},$$

where $\mathrm{sFh}(x)$ is the FIBONACCI HYPERBOLIC SINE and $\mathrm{cFh}(x)$ is the FIBONACCI HYPERBOLIC COSINE.

References

Trzaska, Z. W. "On Fibonacci Hyperbolic Trigonometry and Modified Numerical Triangles." *Fib. Quart.* **34**, 129–138, 1996.

Fibonacci Identity

Since

$$|(a+ib)(c+id)| = |a+ib|\,|c+di| \tag{1}$$

$$|(ac-bd) + i(bc+ad)| = \sqrt{a^2+b^2}\sqrt{c^2+d^2}, \tag{2}$$

it follows that

$$(a^2+b^2)(c^2+d^2) = (ac-bd)^2 + (bc+ad)^2 \equiv e^2 + f^2. \tag{3}$$

This identity implies the 2-D CAUCHY-SCHWARZ SUM INEQUALITY.

see also CAUCHY-SCHWARZ SUM INEQUALITY, EULER FOUR-SQUARE IDENTITY

References

Petkovšek, M.; Wilf, H. S.; and Zeilberger, D. *A=B.* Wellesley, MA: A. K. Peters, p. 9, 1996.

Fibonacci Matrix

A SQUARE MATRIX related to the FIBONACCI NUMBERS. The simplest is the FIBONACCI Q-MATRIX.

Fibonacci n-Step Number

An n-step Fibonacci sequence is given by defining $F_k = 0$ for $k \leq 0$, $F_1 = F_2 = 1$, $F_3 = 2$, and

$$F_k = \sum_{i=1}^{k} F_{n-i} \tag{1}$$

for $k > 3$. The case $n = 1$ corresponds to the degenerate 1, 1, 2, 2, 2, 2 ..., $n = 2$ to the usual FIBONACCI NUMBERS 1, 1, 2, 3, 5, 8, ... (Sloane's A000045), $n = 3$ to the TRIBONACCI NUMBERS 1, 1, 2, 4, 7, 13, 24, 44, 81, ... (Sloane's A000073), $n = 4$ to the TETRANACCI NUMBERS 1, 1, 2, 4, 8, 15, 29, 56, 108, ... (Sloane's A000078), etc.

The limit $\lim_{k\to\infty} F_k/F_{k-1}$ is given by solving

$$x^n(2-x) = 1 \tag{2}$$

for x and taking the REAL ROOT $x > 1$. If $n = 2$, the equation reduces to

$$x^2(2-x) = 1 \tag{3}$$

$$x^3 - 2x^2 + 1 = (x-1)(x^2 - x - 1) = 0, \tag{4}$$

giving solutions

$$x = 1, \tfrac{1}{2}(1 \pm \sqrt{5}). \tag{5}$$

The ratio is therefore

$$x = \tfrac{1}{2}(1+\sqrt{5}) = \phi = 1.618\ldots, \tag{6}$$

which is the GOLDEN RATIO, as expected. Solutions for $n = 1$, 2, ... are given numerically by 1, 1.61803, 1.83929, 1.92756, 1.96595, ..., approaching 2 as $n \to \infty$.

see also FIBONACCI NUMBER, TRIBONACCI NUMBER

References

Sloane, N. J. A. Sequences A000045/M0692, A000073/M1074, and A000078/M1108 in "An On-Line Version of the Encyclopedia of Integer Sequences."

Fibonacci Number

The sequence of numbers defined by the U_n in the LUCAS SEQUENCE. They are companions to the LUCAS NUMBERS and satisfy the same RECURRENCE RELATION,

$$F_n \equiv F_{n-2} + F_{n-1} \tag{1}$$

for $n = 3$, 4, ..., with $F_1 = F_2 = 1$. The first few Fibonacci numbers are 1, 1, 2, 3, 5, 8, 13, 21, ... (Sloane's A000045). The Fibonacci numbers give the number of pairs of rabbits n months after a single pair begins breeding (and newly born bunnies are assumed to begin breeding when they are two months old).

The ratios of alternate Fibonacci numbers are given by the convergents to ϕ^{-2}, where ϕ is the GOLDEN RATIO, and are said to measure the fraction of a turn between successive leaves on the stalk of a plant (PHYLLOTAXIS): 1/2 for elm and linden, 1/3 for beech and hazel, 2/5 for oak and apple, 3/8 for poplar and rose, 5/13 for willow and almond, etc. (Coxeter 1969, Ball and Coxeter 1987). The Fibonacci numbers are sometimes called PINE CONE NUMBERS (Pappas 1989, p. 224)

Another RECURRENCE RELATION for the Fibonacci numbers is

$$F_{n+1} = \left\lfloor \frac{F_n(1+\sqrt{5})+1}{2} \right\rfloor = \left\lfloor \phi F_n + \tfrac{1}{2} \right\rfloor, \tag{2}$$

where $\lfloor x \rfloor$ is the FLOOR FUNCTION and ϕ is the GOLDEN RATIO. This expression follows from the more general RECURRENCE RELATION that

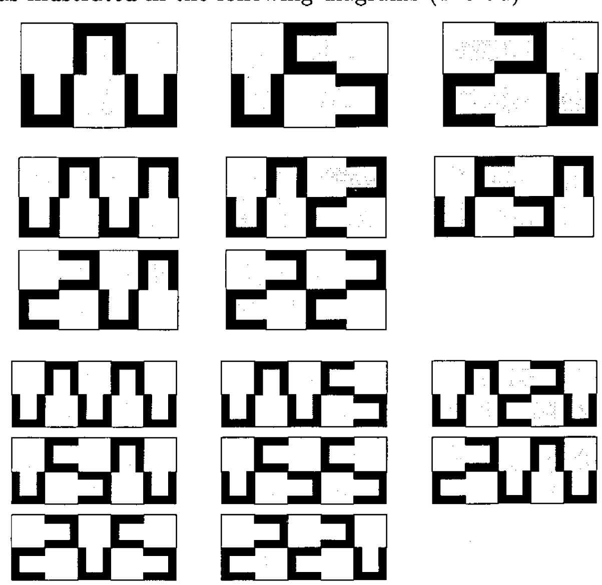
$$\begin{vmatrix} F_n & F_{n+1} & \cdots & F_{n+k} \\ F_{n+k+1} & F_{n+k+2} & \cdots & F_{n+2k} \\ \vdots & \vdots & \ddots & \vdots \\ F_{n+k(k-1)+1} & F_{n+k(k-1)+2} & \cdots & F_{n+k^2} \end{vmatrix} = 0. \tag{3}$$

The GENERATING FUNCTION for the Fibonacci numbers is

$$g(x) = \sum_{n=0}^{\infty} F_n x^n = \frac{1}{1-x-x^2}. \tag{4}$$

Yuri Matijasevič (1970) proved that the equation $n = F_{2m}$ is a DIOPHANTINE EQUATION. This led to the proof of the impossibility of the tenth of HILBERT'S PROBLEMS (does there exist a general method for solving DIOPHANTINE EQUATIONS?) by Julia Robinson and Martin Davis in 1970.

The Fibonacci number F_{n+1} gives the number of ways for 2×1 DOMINOES to cover a $2 \times n$ CHECKERBOARD, as illustrated in the following diagrams (Dickau).



The number of ways of picking a SET (including the EMPTY SET) from the numbers 1, 2, ..., n without picking two consecutive numbers is F_{n+2}. The number of ways of picking a set (including the EMPTY SET) from the numbers 1, 2, ..., n without picking two consecutive numbers (where 1 and n are now consecutive) is $L_n = F_{n+1} + F_{n-1}$, where L_n is a LUCAS NUMBER. The probability of not getting two heads in a row in n tosses of a COIN is $F_{n+2}/2^n$ (Honsberger 1985, pp. 120–122). Fibonacci numbers are also related to the number of ways in which n COIN TOSSES can be made such that there are not three consecutive heads or tails. The number of ideals of an n-element FENCE POSET is the Fibonacci number F_n.

Sum identities are

$$\sum_{k=1}^{n} F_k = F_{n+2} - 1. \qquad (5)$$

$$F_1 + F_3 + F_5 + \ldots + F_{2k+1} = F_{2k+2} \qquad (6)$$

$$1 + F_2 + F_4 + F_6 + \ldots + F_{2k} = F_{2k+1} \qquad (7)$$

$$\sum_{k=1}^{n} {F_k}^2 = F_n F_{n+1} \qquad (8)$$

$$F_{2n} = {F_{n+1}}^2 - {F_{n-1}}^2 \qquad (9)$$

$$F_{3n} = {F_{n+1}}^3 + {F_n}^3 + {F_{n-1}}^3. \qquad (10)$$

Additional RECURRENCE RELATIONS are CASSINI'S IDENTITY

$$F_{n-1}F_{n+1} - {F_n}^2 = (-1)^n \qquad (11)$$

and the relations

$$F_{2n+1} = 1 + F_2 + F_4 + \ldots + F_{2n} \qquad (12)$$

$${F_{n+1}}^2 = 4F_nF_{n-1} + {F_{n-2}}^2 \qquad (13)$$

(Brousseau 1972),

$$F_{n+m} = F_{n-1}F_m + F_nF_{m+1} \qquad (14)$$

$$F_{(k+1)n} = F_{n-1}F_{kn} + F_nF_{kn+1} \qquad (15)$$

(Honsberger 1985, p. 107),

$$F_n = F_lF_{n-l+1} + F_{l-1}F_{n-l}, \qquad (16)$$

so if $l = n - l + 1$, then $2l = n + 1$ and $l = (n+1)/2$

$$F_n = {F_{(n+1)/2}}^2 + {F_{(n-1)/2}}^2. \qquad (17)$$

Letting $k \equiv (n-1)/2$,

$$F_{2k+1} = {F_{k+1}}^2 + {F_k}^2 \qquad (18)$$

$${F_{n+2}}^2 - {F_{n+1}}^2 = F_nF_{n+3} \qquad (19)$$

$${F_n}^2 = {F_{n-1}}^2 + 3{F_{n-2}}^2 + 2F_{n-2}F_{n-3}. \qquad (20)$$

Sum FORMULAS for F_n include

$$F_n = \frac{1}{2^{n-1}}\left[\binom{n}{1} + 5\binom{n}{3} + 5^2\binom{n}{5} + \ldots\right] \qquad (21)$$

$$F_{n+1} = \binom{n}{0} + \binom{n-1}{1} + \binom{n-2}{2} + \ldots. \qquad (22)$$

Cesàro derived the FORMULAS

$$\sum_{k=0}^{n} \binom{n}{k} F_k = F_{2n} \qquad (23)$$

$$\sum_{k=0}^{n} \binom{n}{k} 2^k F_k = F_{3n} \qquad (24)$$

(Honsberger 1985, pp. 109–110). Additional identities can be found throughout the *Fibonacci Quarterly* journal. A list of 47 generalized identities are given by Halton (1965).

In terms of the LUCAS NUMBER L_n,

$$F_{2n} = F_nL_n \qquad (25)$$

$$F_{2n}({L_{2n}}^2 - 1) = F_{6n} \qquad (26)$$

$$F_{m+p} + (-1)^{p+1}F_{m-p} = F_pL_m \qquad (27)$$

$$\sum_{k=a+1}^{a+4n} F_k = F_{a+4n+2} - F_{a+2} = F_{2n}L_{a+2n+2} \qquad (28)$$

(Honsberger 1985, pp. 111–113). A remarkable identity is

$$\exp(L_1x + \tfrac{1}{2}L_2x^2 + \tfrac{1}{3}L_3x^3 + \ldots) = F_1 + F_2x + F_3x^2 + \ldots \qquad (29)$$

(Honsberger 1985, pp. 118-119). It is also true that

$$5{F_n}^2 = {L_n}^2 - 4(-1)^n \qquad (30)$$

and

$$\frac{{L_n}^2 - (-1)^a {L_{n+a}}^2}{{F_n}^2 - (-1)^a {F_{n+a}}^2} = 5 \qquad (31)$$

for a ODD, and

$$\frac{{L_n}^2 + {L_{n+a}}^2 - 8(-1)^n}{{F_n}^2 + {F_{n+a}}^2} = 5 \qquad (32)$$

for a EVEN (Freitag 1996).

The equation (1) is a LINEAR RECURRENCE SEQUENCE

$$x_n = Ax_{x-1} + Bx_{n-2} \qquad n \geq 3, \qquad (33)$$

so the closed form for F_n is given by

$$F_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}, \qquad (34)$$

where α and β are the roots of $x^2 = Ax + B$. Here, $A = B = 1$, so the equation becomes

$$x^2 - x - 1 = 0, \tag{35}$$

which has ROOTS

$$x = \tfrac{1}{2}(1 \pm \sqrt{5}). \tag{36}$$

The closed form is therefore given by

$$F_n = \frac{(1+\sqrt{5})^n - (1-\sqrt{5})^n}{2^n\sqrt{5}}. \tag{37}$$

This is known as BINET'S FORMULA. Another closed form is

$$F_n = \left[\frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n\right] = \left[\frac{\phi^n}{\sqrt{5}}\right], \tag{38}$$

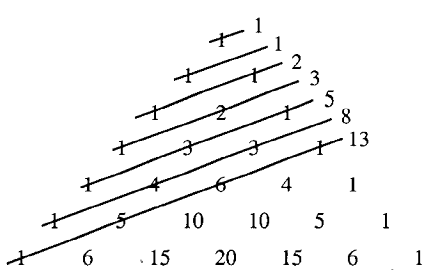
where $[x]$ is the NINT function.

From (1), the RATIO of consecutive terms is

$$\begin{aligned}\frac{F_n}{F_{n-1}} &= 1 + \frac{F_{n-2}}{F_{n-1}} = 1 + \frac{1}{\frac{F_{n-1}}{F_{n-2}}} \\ &= 1 + \frac{1}{1 + \frac{1}{\frac{F_{n-3}}{F_{n-2}}}} = \left[1, 1, \ldots, \frac{F_2}{F_1}\right] \\ &= [\underbrace{1, 1, \ldots, 1}_{n-1}],\end{aligned} \tag{39}$$

which is just the first few terms of the CONTINUED FRACTION for the GOLDEN RATIO ϕ. Therefore,

$$\lim_{n\to\infty} \frac{F_n}{F_{n-1}} = \phi. \tag{40}$$



The "SHALLOW DIAGONALS" of PASCAL'S TRIANGLE sum to Fibonacci numbers (Pappas 1989),

$$\sum_{k=1}^{n} \binom{k}{n-k} = \frac{(-1)^n {}_3F_2(1, 2, 1-n; \frac{1}{2}(3-n), 2-\frac{1}{2}n; -\frac{1}{4})}{\pi(2-3n+n^2)} = F_{n+1}, \tag{41}$$

where ${}_3F_2(a, b, c; d, e; z)$ is a GENERALIZED HYPERGEOMETRIC FUNCTION.

The sequence of final digits in Fibonacci numbers repeats in cycles of 60. The last two digits repeat in 300, the last three in 1500, the last four in 15,000, etc.

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{F_n F_{n+2}} = 2 - \sqrt{5} \tag{42}$$

(Clark 1995). A very curious addition of the Fibonacci numbers is the following addition tree,

```
0
 1
  1
   2
    3
     5
      8
      13
       21
        34
         55
          89
            ⋱
0112359550561...
```

which is equal to the fractional digits of 1/89,

$$\sum_{n=0}^{\infty} \frac{F_n}{10^{n+1}} = \frac{1}{89}. \tag{43}$$

For $n \geq 3$, $F_n | F_m$ IFF $n | m$. $L_n | L_m$ IFF n divides into m an EVEN number of times. $(F_m, F_n) = F_{(m,n)}$ (Michael 1964; Honsberger 1985, pp. 131-132). No ODD Fibonacci number is divisible by 17 (Honsberger 1985, pp. 132 and 242). No Fibonacci number > 8 is ever of the form $p - 1$ or $p + 1$ where p is a PRIME number (Honsberger 1985, p. 133).

Consider the sum

$$s_k = \sum_{n=2}^{k} \frac{1}{F_{n-1}F_{n+1}} = \sum_{n=2}^{k} \left(\frac{1}{F_{n-1}F_n} - \frac{1}{F_n F_{n+1}}\right). \tag{44}$$

This is a TELESCOPING SUM, so

$$s_k = 1 - \frac{1}{F_{k+1}F_{k+2}}, \tag{45}$$

thus

$$S \equiv \lim_{k\to\infty} s_k = 1 \tag{46}$$

(Honsberger 1985, pp. 134–135). Using BINET'S FORMULA, it also follows that

$$\frac{F_{n+r}}{F_n} = \frac{\alpha^{n+r} - \beta^{n+r}}{\alpha^n - \beta^n} = \frac{\alpha^{n+r}}{\alpha^n} \frac{1 - \left(\frac{\beta}{\alpha}\right)^{n+r}}{1 - \left(\frac{\beta}{\alpha}\right)^n}, \tag{47}$$

where

$$\alpha = \tfrac{1}{2}(1 + \sqrt{5}) \tag{48}$$

$$\beta = \tfrac{1}{2}(1 - \sqrt{5}) \tag{49}$$

so

$$\frac{F_{n+r}}{F_n} = \alpha^r. \tag{50}$$

$$S' = \sum_{n=1}^{\infty} \frac{F_n}{F_{n+1}F_{n+2}} = 1 \tag{51}$$

(Honsberger 1985, pp. 138 and 242–243). The MILLIN SERIES has sum

$$S'' \equiv \sum_{n=0}^{\infty} \frac{1}{F_{2^n}} = \tfrac{1}{2}(7-\sqrt{5}\,) \tag{52}$$

(Honsberger 1985, pp. 135–137).

The Fibonacci numbers are COMPLETE. In fact, dropping one number still leaves a COMPLETE SEQUENCE, although dropping two numbers does not (Honsberger 1985, pp. 123 and 126). Dropping two terms from the Fibonacci numbers produces a sequence which is not even WEAKLY COMPLETE (Honsberger 1985, p. 128). However, the sequence

$$F_n' \equiv F_n - (-1)^n \tag{53}$$

is WEAKLY COMPLETE, even with any finite subsequence deleted (Graham 1964). $\{F_n{}^2\}$ is not COMPLETE, but $\{F_n{}^2\} + \{F_n{}^2\}$ are. 2^{N-1} copies of $\{F_n{}^N\}$ are COMPLETE.

For a discussion of SQUARE Fibonacci numbers, see Cohn (1964), who proved that the only SQUARE NUMBER Fibonacci numbers are 1 and $F_{12} = 144$ (Cohn 1964, Guy 1994). Ming (1989) proved that the only TRIANGULAR Fibonacci numbers are 1, 3, 21, and 55. The Fibonacci and LUCAS NUMBERS have no common terms except 1 and 3. The only CUBIC Fibonacci numbers are 1 and 8.

$$(F_nF_{n+3}, 2F_{n+1}F_{n+2}, F_{2n+3} = F_{n+1}{}^2 + F_{n+2}{}^2) \tag{54}$$

is a PYTHAGOREAN TRIPLE.

$$F_{4n}{}^2 + 8F_{2n}(F_{2n} + F_{6n}) = (3F_{4n})^2 \tag{55}$$

is always a SQUARE NUMBER (Honsberger 1985, p. 243).

In 1975, James P. Jones showed that the Fibonacci numbers are the POSITIVE INTEGER values of the POLYNOMIAL

$$P(x,y) = -y^5 + 2y^4x + y^3x^2 - 2y^2x^3 - y(x^4-2) \tag{56}$$

for GAUSSIAN INTEGERS x and y (Le Lionnais 1983). If n and k are two POSITIVE INTEGERS, then between n^k and n^{k+1}, there can never occur more than n Fibonacci numbers (Honsberger 1985, pp. 104–105).

Every F_n that is PRIME has a PRIME n, but the converse is not necessarily true. The first few PRIME Fibonacci numbers are for n = 3, 4, 5, 7, 11, 13, 17, 23, 29, 43, 47, 83, 131, 137, 359, 431, 433, 449, 509, 569, 571, ... (Sloane's A001605; Dubner and Keller 1998). Gardner's statement that F_{531} is prime is incorrect, especially since 531 is not even PRIME (Gardner 1979, p. 161). It is not known if there are an INFINITE number of Fibonacci primes.

The Fibonacci numbers F_n, are SQUAREFUL for n = 6, 12, 18, 24, 25, 30, 36, 42, 48, 50, 54, 56, 60, 66, ..., 300, 306, 312, 324, 325, 330, 336, ... (Sloane's A037917) and SQUAREFREE for n = 1, 2, 3, 4, 5, 7, 8, 9, 10, 11, 13, ... (Sloane's A037918). The largest known SQUAREFUL Fibonacci number is F_{336}, and no SQUAREFUL Fibonacci numbers F_p are known with p PRIME.

see also CASSINI'S IDENTITY, FAST FIBONACCI TRANSFORM, FIBONACCI DUAL THEOREM, FIBONACCI n-STEP NUMBER, FIBONACCI Q-MATRIX, GENERALIZED FIBONACCI NUMBER, INVERSE TANGENT, LINEAR RECURRENCE SEQUENCE, LUCAS SEQUENCE, NEAR NOBLE NUMBER, PELL SEQUENCE, RABBIT CONSTANT, STOLARSKY ARRAY, TETRANACCI NUMBER, TRIBONACCI NUMBER, WYTHOFF ARRAY, ZECKENDORF REPRESENTATION, ZECKENDORF'S THEOREM

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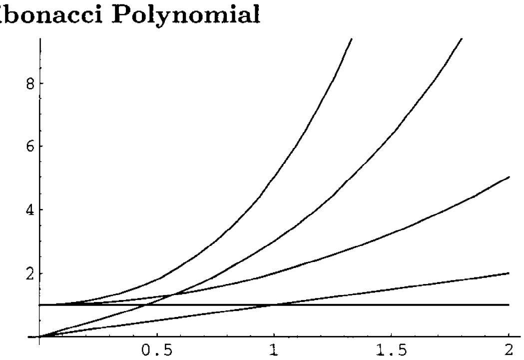
Fibonacci Polynomial



The W POLYNOMIALS obtained by setting $p(x) = x$ and $q(x) = 1$ in the LUCAS POLYNOMIAL SEQUENCE. (The corresponding w POLYNOMIALS are called LUCAS POLYNOMIALS.) The Fibonacci polynomials are defined by the RECURRENCE RELATION

$$F_{n+1}(x) = xF_n(x) + F_{n-1}(x), \tag{1}$$

with $F_1(x) = 1$ and $F_2(x) = x$. They are also given by the explicit sum formula

$$F_n(x) = \sum_{j=0}^{\lfloor (n-1)/2 \rfloor} \binom{n-j-1}{j} x^{n-2j-1}, \tag{2}$$

where $\lfloor x \rfloor$ is the FLOOR FUNCTION and $\binom{n}{m}$ is a BINOMIAL COEFFICIENT. The first few Fibonacci polynomials are

$$\begin{aligned} F_1(x) &= 1 \\ F_2(x) &= x \\ F_3(x) &= x^2 + 1 \\ F_4(x) &= x^3 + 2x \\ F_5(x) &= x^4 + 3x^2 + 1. \end{aligned}$$

The Fibonacci polynomials are normalized so that

$$F_n(1) = F_n, \tag{3}$$

where the F_ns are FIBONACCI NUMBERS.

The Fibonacci polynomials are related to the MORGAN-VOYCE POLYNOMIALS by

$$F_{2n+1}(x) = b_n(x^2) \tag{4}$$

$$F_{2n+n2}(x) = xB_n(x^2) \tag{5}$$

(Swamy 1968).

see also BRAHMAGUPTA POLYNOMIAL, FIBONACCI NUMBER, MORGAN-VOYCE POLYNOMIAL

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Fibonacci Pseudoprime

Consider a LUCAS SEQUENCE with $P > 0$ and $Q = \pm 1$. A Fibonacci pseudoprime is a COMPOSITE NUMBER n such that

$$V_n \equiv P \pmod{n}.$$

There exist no EVEN Fibonacci pseudoprimes with parameters $P = 1$ and $Q = -1$ (Di Porto 1993) or $P = Q = 1$ (André-Jeannin 1996). André-Jeannin (1996) also proved that if $(P, Q) \neq (1, -1)$ and $(P, Q) \neq (1, 1)$, then there exists at least one EVEN Fibonacci pseudoprime with parameters P and Q.

see also PSEUDOPRIME

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Fibonacci Q-Matrix

A FIBONACCI MATRIX of the form

$$\mathsf{M} = \begin{bmatrix} m & 1 \\ 1 & 0 \end{bmatrix}. \qquad (1)$$

If U and V are defined as BINET FORMS

$$U_n = mU_{n-1} + U_{n-2} \quad (U_0 = 0, U_1 = 1) \qquad (2)$$

$$V_n = mV_{n-1} + V_{n-2} \quad (V_0 = 2, V_1 = m), \qquad (3)$$

then

$$\mathsf{M} = \begin{bmatrix} U_{n+1} & U_n \\ U_n & U_{n-1} \end{bmatrix} \qquad (4)$$

$$\mathsf{M}^{-1} = \mathsf{M} - m\mathsf{I} = \begin{bmatrix} 0 & 1 \\ 1 & -m \end{bmatrix}. \qquad (5)$$

Defining

$$\mathsf{Q} \equiv \begin{bmatrix} F_2 & F_1 \\ F_1 & F_0 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}, \qquad (6)$$

then

$$\mathsf{Q}^n = \begin{bmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{bmatrix} \qquad (7)$$

(Honsberger 1985, pp. 106–107).

see also BINET FORMS, FIBONACCI NUMBER

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Fibonacci Sequence

see FIBONACCI NUMBER

Fibration

If $f : E \to B$ is a FIBER BUNDLE with B a PARACOMPACT TOPOLOGICAL SPACE, then f satisfies the HOMOTOPY LIFTING PROPERTY with respect to all TOPOLOGICAL SPACES. In other words, if $g : [0,1] \times X \to B$ is a HOMOTOPY from g_0 to g_1, and if g_0' is a LIFT of the MAP g_0 with respect to f, then g has a LIFT to a MAP g' with respect to f. Therefore, if you have a HOMOTOPY of a MAP into B, and if the beginning of it has a LIFT, then that LIFT can be extended to a LIFT of the HOMOTOPY itself.

A fibration is a MAP between TOPOLOGICAL SPACES $f : E \to B$ such that it satisfies the HOMOTOPY LIFTING PROPERTY.

see also FIBER BUNDLE, FIBER SPACE

Field

A field is any set of elements which satisfies the FIELD AXIOMS for both addition and multiplication and is a commutative DIVISION ALGEBRA. An archaic word for a field is RATIONAL DOMAIN. A field with a finite number of members is known as a FINITE FIELD or GALOIS FIELD.

Because the identity condition must be different for addition and multiplication, every field must have at least two elements. Examples include the COMPLEX NUMBERS ($\mathbb{C}$), RATIONAL NUMBERS ($\mathbb{Q}$), and REAL NUMBERS ($\mathbb{R}$), but *not* the INTEGERS ($\mathbb{Z}$), which form a RING. It has been proven by Hilbert and Weierstraß that all generalizations of the field concept to triplets of elements are equivalent to the field of COMPLEX NUMBERS.

see also ADJUNCTION, ALGEBRAIC NUMBER FIELD, COEFFICIENT FIELD, CYCLOTOMIC FIELD, FIELD AXIOMS, FIELD EXTENSION, FUNCTION FIELD, GALOIS FIELD, MAC LANE'S THEOREM, MODULE, NUMBER FIELD, QUADRATIC FIELD, RING, SKEW FIELD, VECTOR FIELD

Field Axioms

The field axioms are generally written in additive and multiplicative pairs.

Name	Addition	Multiplication
Commutivity	$a + b = b + a$	$ab = ba$
Associativity	$(a+b)+c = a+(b+c)$	$(ab)c = a(bc)$
Distributivity	$a(b+c) = ab + ac$	$(a+b)c = ac + bc$
Identity	$a + 0 = a = 0 + a$	$a \cdot 1 = a = 1 \cdot a$
Inverses	$a + (-a) = 0 = (-a) + a$	$aa^{-1} = 1 = a^{-1}a$ if $a \neq 0$

see also ALGEBRA, FIELD

Field Extension

A FIELD L is said to be a field extension of field K if K is a SUBFIELD of L. This is denoted L/K (note that this NOTATION conflicts with that of a QUOTIENT GROUP). The COMPLEX NUMBERS are a field extension of the REAL NUMBERS, and the REAL NUMBERS are a field extension of the RATIONAL NUMBERS.

see also FIELD

Fields Medal

The mathematical equivalent of the Nobel Prize (there is no Nobel Prize in mathematics) which is awarded by the International Mathematical Union every four years to one or more outstanding researchers, usually under 40 years of age. The first Fields Medal was awarded in 1936.

see also BURNSIDE PROBLEM, MATHEMATICS PRIZES, POINCARÉ CONJECTURE, ROTH'S THEOREM, TAU CONJECTURE

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Fifteen Theorem

A theorem due to Conway *et al.* (1997) which states that, if a POSITIVE definite QUADRATIC FORM with integral matrix entries represents all natural numbers up to 15, then it represents all natural numbers. This theorem contains LAGRANGE'S FOUR-SQUARE THEOREM, since every number up to 15 is the sum of at most four SQUARES.

see also INTEGER-MATRIX FORM, LAGRANGE'S FOUR-SQUARE THEOREM, QUADRATIC FORM

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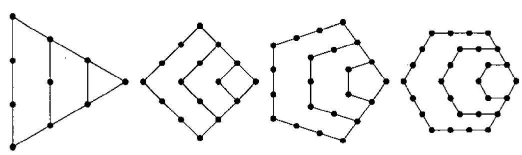
Figurate Number



A number which can be represented by a regular geometrical arrangement of equally spaced points. If the arrangement forms a REGULAR POLYGON, the number is called a POLYGONAL NUMBER. The polygonal numbers illustrated above are called triangular, square, pentagonal, and hexagon numbers, respectively. Figurate numbers can also form other shapes such as centered polygons, L-shapes, 3-dimensional solids, etc. The following table lists the most common types of figurate numbers.

Name	Formula
biquadratic	n^4
centered cube	$(2n-1)(n^2-n+1)$
centered pentagonal	$\frac{1}{2}(5n^2-5n+2)$
centered square	$n^2+(n-1)^2$
centered triangular	$\frac{1}{2}(3n^2-3n+2)$
cubic	n^3
decagonal	$4n^2-3n$
gnomic	$2n-1$
heptagonal	$\frac{1}{2}n(5n-3)$
heptagonal pyramidal	$\frac{1}{6}n(n+1)(5n-2)$
hex	$3n^2-3n+1$
hexagonal	$n(2n-1)$
hexagonal pyramidal	$\frac{1}{6}n(n+1)(4n-1)$
octagonal	$n(3n-2)$
octahedral	$\frac{1}{3}n(2n^2+1)$
pentagonal	$\frac{1}{2}n(3n-1)$
pentagonal pyramidal	$\frac{1}{2}n^2(n+1)$
pentatope	$\frac{1}{24}n(n+1)(n+2)(n+3)$
pronic number	$n(n+1)$
rhombic dodecahedral	$(2n-1)(2n^2-2n+1)$
square	n^2
stella octangula	$n(2n^2-1)$
tetrahedral	$\frac{1}{6}n(n+1)(n+2)$
triangular	$\frac{1}{2}n(n+1)$
truncated octahedral	$16n^3-33n^2+24n-6$
truncated tetrahedral	$\frac{1}{6}n(23n^2-27n+10)$

An n-D FIGURATE NUMBER can be defined by

$$f_{m,s}^{r}=\frac{(rs+m-s)(r+m-2)}{m!(r-1)!}.$$

see also BIQUADRATIC NUMBER, CENTERED CUBE NUMBER, CENTERED PENTAGONAL NUMBER, CENTERED POLYGONAL NUMBER, CENTERED SQUARE NUMBER, CENTERED TRIANGULAR NUMBER, CUBIC NUMBER, DECAGONAL NUMBER, FIGURATE NUMBER TRIANGLE, GNOMIC NUMBER, HEPTAGONAL NUMBER, HEPTAGONAL PYRAMIDAL NUMBER, HEX NUMBER, HEX PYRAMIDAL NUMBER, HEXAGONAL NUMBER, HEXAGONAL PYRAMIDAL NUMBER, NEXUS NUMBER, OCTAGONAL NUMBER, OCTAHEDRAL NUMBER, PENTAGONAL NUMBER, PENTAGONAL PYRAMIDAL NUMBER, PENTATOPE NUMBER, POLYGONAL NUMBER, PRONIC NUMBER, PYRAMIDAL NUMBER, RHOMBIC DODECAHEDRAL NUMBER, SQUARE NUMBER, STELLA OCTANGULA NUMBER, TETRAHEDRAL NUMBER, TRIANGULAR NUMBER, TRUNCATED OCTAHEDRAL NUMBER, TRUNCATED TETRAHEDRAL NUMBER

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Figurate Number Triangle

A PASCAL'S TRIANGLE written in a square grid and padded with zeroes, as written by Jakob Bernoulli (Smith 1984). The figurate number triangle therefore has entries

$$a_{ij} = \binom{i}{j},$$

where i is the row number, j the column number, and $\binom{i}{j}$ a BINOMIAL COEFFICIENT. Written out explicitly (beginning each row with $j = 0$),

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 1 & 2 & 1 & 0 & 0 & 0 & 0 & \cdots \\ 1 & 3 & 3 & 1 & 0 & 0 & 0 & \cdots \\ 1 & 4 & 6 & 4 & 1 & 0 & 0 & \cdots \\ 1 & 5 & 10 & 10 & 5 & 1 & 0 & \cdots \\ 1 & 6 & 15 & 20 & 15 & 6 & 1 & \cdots \\ 1 & 7 & 21 & 35 & 35 & 21 & 7 & \ddots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

Then we have the sum identities

$$\sum_{j=0}^{i} a_{ij} = 2^i$$

$$\sum_{j=1}^{i} a_{ij} = 2^i - 1$$

$$\sum_{i=0}^{n} a_{ij} = a_{(n+1),(j+1)} = \frac{n+1}{j+1} a_{nj}.$$

see also BINOMIAL COEFFICIENT, FIGURATE NUMBER, PASCAL'S TRIANGLE

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Smith, D. E. *A Source Book in Mathematics.* New York: Dover, p. 86, 1984.

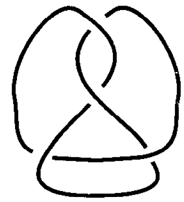
Figure Eight Knot

see FIGURE-OF-EIGHT KNOT

Figure Eight Surface

see EIGHT SURFACE

Figure-of-Eight Knot



The knot 04_{001}, which is the unique PRIME KNOT of four crossings, and which is a 2-EMBEDDABLE KNOT. It is AMPHICHIRAL. It is also known as the FLEMISH KNOT and SAVOY KNOT, and it has BRAID WORD $\sigma_1{\sigma_2}^{-1}\sigma_1{\sigma_2}^{-1}$.

References

Owen, P. *Knots.* Philadelphia, PA: Courage, p. 16, 1993.

Figures

A number x is said to have "n figures" if it takes n DIGITS to express it. The number of figures is therefore equal to one more than the POWER of 10 in the SCIENTIFIC NOTATION representation of the number. The word is most frequently used in reference to monetary amounts, e.g., a "six-figure salary" would fall in the range of \$100,000 to \$999,999.

see also DIGIT, SCIENTIFIC NOTATION, SIGNIFICANT FIGURES

Filon's Integration Formula

A formula for NUMERICAL INTEGRATION,

$$\begin{aligned} \int_{x_0}^{x_n} & f(x)\cos(tx)\,dx \\ &= h\{\alpha(th)[f_{2n}\sin(tx_{2n}) - f_0\sin(tx_0)] \\ &\quad +\beta(th)C_{2n} + \gamma(th)C_{2n-1} + \tfrac{2}{45}th^4 S'_{2n-1}\} - R_n, \end{aligned} \tag{1}$$

where

$$C_{2n} = \sum_{i=0}^{n} f_{2i}\cos(tx_{2i}) - \tfrac{1}{2}[f_{2n}\cos(tx_{2n}) + f_0\cos(tx_0)] \tag{2}$$

$$C_{2n-1} = \sum_{i=1}^{n} f_{2i-1}\cos(tx_{2i-1}) \tag{3}$$

$$S'_{2n-1} = \sum_{i=1}^{n} f^{(3)}_{2i-1}\sin(tx_{2i-1}) \tag{4}$$

$$\alpha(\theta) = \frac{1}{\theta} + \frac{\sin(2\theta)}{2\theta^2} - \frac{2\sin^2\theta}{\theta^3} \tag{5}$$

$$\beta(\theta) = 2\left[\frac{1+\cos^2\theta}{\theta^2} - \frac{\sin(2\theta)}{\theta^3}\right] \tag{6}$$

$$\gamma(\theta) = 4\left(\frac{\sin\theta}{\theta^3} - \frac{\cos\theta}{\theta^2}\right), \tag{7}$$

and the remainder term is

$$R_n = \tfrac{1}{90}nh^5 f^{(4)}(\xi) + \mathcal{O}(th^7). \tag{8}$$

References

Abramowitz, M. and Stegun, C. A. (Eds.). *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables, 9th printing.* New York: Dover, pp. 890–891, 1972.

Tukey, J. W. In *On Numerical Approximation: Proceedings of a Symposium Conducted by the Mathematics Research Center, United States Army, at the University of Wisconsin, Madison, April 21–23, 1958* (Ed. R. E. Langer). Madison, WI: University of Wisconsin Press, p. 400, 1959.

Filter

Formally, a filter is defined in terms of a SET X and a SET Φ of SUBSETS of X. Then Φ is called a filter if

1. $X \in \Phi$,
2. the EMPTY SET $\varnothing \notin \Phi$,
3. $A \subset B \subset X$ and $A \in \Phi$ IMPLIES $B \in \Phi$,
4. and $A, B \in \Phi$ IMPLIES $A \cup B \in \Phi$.

Informally, a filter is a function or procedure which removes unwanted parts of a signal. The concept of filtering and filter functions is particularly useful in engineering. One particularly elegant method of filtering FOURIER TRANSFORMS a signal into frequency space, performs the filtering operation there, then transforms back into the original space (Press *et al.* 1992).

see also SAVITZKY-GOLAY FILTER, WIENER FILTER

References

Press, W. H.; Flannery, B. P.; Teukolsky, S. A.; and Vetterling, W. T. "Digital Filtering in the Time Domain." §13.5 in *Numerical Recipes in FORTRAN: The Art of Scientific Computing, 2nd ed.* Cambridge, England: Cambridge University Press, pp. 551–556, 1992.

Fine's Equation

$$\prod_{n=1} \frac{(1-q^{2n})(1-q^{3n})(1-q^{8n})(1-q^{12n})}{(1-q^n)(1-q^{24n})} = 1 + \sum_{N=1} E_{1,5,7,11}(N;24)q^N,$$

where E is the sum of the DIVISORS of N CONGRUENT to 1, 5, 7, and 11 (mod 24) minus the sum of DIVISORS of N CONGRUENT to -1, -5, -7, and -11 (mod 24).

see also q-SERIES

Finite

A SET which contains a NONNEGATIVE integral number of elements is said to be finite. A SET which is not finite is said to be INFINITE. A finite or COUNTABLY INFINITE SET is said to be COUNTABLE. While the meaning of the term "finite" is fairly clear in common usage, precise definitions of FINITE and INFINITE are needed in technical mathematics and especially in SET THEORY.

see also COUNTABLE SET, COUNTABLY INFINITE SET, INFINITE, SET THEORY, UNCOUNTABLY INFINITE SET

Finite Difference

The finite difference is the discrete analog of the DERIVATIVE. The finite FORWARD DIFFERENCE of a function f_p is defined as

$$\Delta f_p \equiv f_{p+1} - f_p, \tag{1}$$

and the finite BACKWARD DIFFERENCE as

$$\nabla f_p \equiv f_p - f_{p-1}. \tag{2}$$

If the values are tabulated at spacings h, then the notation

$$f_p \equiv f(x_0 + ph) \equiv f(x) \tag{3}$$

is used. The kth FORWARD DIFFERENCE would then be written as $\Delta^k f_p$, and similarly, the kth BACKWARD DIFFERENCE as $\nabla^k f_p$.

However, when f_p is viewed as a discretization of the continuous function $f(x)$, then the finite difference is sometimes written

$$\Delta f(x) \equiv f(x+\tfrac{1}{2}) - f(x-\tfrac{1}{2}) = 2\,\mathrm{I_I}(x) * f(x), \tag{4}$$

where $*$ denotes CONVOLUTION and $\mathrm{I_I}(x)$ is the odd IMPULSE PAIR. The finite difference operator can therefore be written

$$\tilde{\Delta} = 2\,\mathrm{I_I}\,*. \tag{5}$$

An nth POWER has a constant nth finite difference. For example, take $n = 3$ and make a DIFFERENCE TABLE,

$$\begin{array}{llllll} x & x^3 & & & & \\ & & \Delta & & & \\ 1 & 1 & & \Delta^2 & & \\ & & 7 & & \Delta^3 & \\ 2 & 8 & & 12 & & \Delta^4 \\ & & 19 & & 6 & \\ 3 & 27 & & 18 & & 0 \\ & & 37 & & 6 & \\ 4 & 64 & & 24 & & \\ & & 61 & & & \\ 5 & 125 & & & & \end{array}. \tag{6}$$

The Δ^3 column is the constant 6.

Finite difference formulas can be very useful for extrapolating a finite amount of data in an attempt to find the general term. Specifically, if a function $f(n)$ is known at only a few discrete values $n = 0, 1, 2, \ldots$ and it is desired to determine the analytical form of f, the following procedure can be used if f is assumed to be a POLYNOMIAL function. Denote the nth value in the SEQUENCE of interest by a_n. Then define b_n as the FORWARD DIFFERENCE $\Delta_n \equiv a_{n+1} - a_n$, c_n as the second FORWARD DIFFERENCE $\Delta_n^2 \equiv b_{n+1} - b_n$, etc., constructing a table as follows

$$\begin{array}{ccccc} a_0 \equiv f(0) & a_1 \equiv f(1) & a_2 \equiv f(2) & \ldots & a_p \equiv f(p) \\ b_0 \equiv a_1 - a_0 & b_1 \equiv a_2 - a_1 & \ldots & b_{p-1} \equiv a_p - a_{p-1} & \\ & c_0 \equiv b_2 - b_1 & \ldots & \ldots & \\ & & \ddots & & \end{array} \tag{7}$$

Continue computing d_0, e_0, etc., until a 0 value is obtained. Then the POLYNOMIAL function giving the values a_n is given by

$$f(n) = \sum_{k=0}^{p} \alpha_k \binom{n}{k} \tag{8}$$

$$= a_0 + b_0 n + \frac{c_0 n(n-1)}{2} + \frac{d_0 n(n-1)(n-2)}{2 \cdot 3} + \ldots. \tag{9}$$

When the notation $\Delta_0 \equiv a_0$, $\Delta_0^2 \equiv b_0$, etc., is used, this beautiful equation is called NEWTON'S FORWARD DIFFERENCE FORMULA. To see a particular example, consider a SEQUENCE with first few values of 1, 19, 143, 607, 1789, 4211, and 8539. The difference table is then given by

$$\begin{array}{ccccccccccccc}
1 & & 19 & & 143 & & 607 & & 1789 & & 4211 & & 8539 \\
& 18 & & 124 & & 464 & & 1182 & & 2422 & & 4328 & \\
& & 106 & & 340 & & 718 & & 1240 & & 1906 & & \\
& & & 234 & & 378 & & 522 & & 666 & & & \\
& & & & 144 & & 144 & & 144 & & & & \\
& & & & & 0 & & 0 & & & & &
\end{array}$$

Reading off the first number in each row gives $a_0 = 1$, $b_0 = 18$, $c_0 = 106$, $d_0 = 234$, $e_0 = 144$. Plugging these in gives the equation

$$f(n) = 1 + 18n + 53n(n-1) + 39n(n-1)(n-2) + 6n(n-1)(n-2)(n-3), \tag{10}$$

which simplifies to $f(n) = 6n^4 + 3n^3 + 2n^2 + 7n + 1$, and indeed fits the original data exactly!

Beyer (1987) gives formulas for the derivatives

$$h^n \frac{d^n f(x_0 + ph)}{dx^n} \equiv h^n \frac{d^n f_p}{dx^n} \equiv \frac{d^n f_p}{dp^n} \tag{11}$$

(Beyer 1987, pp. 449–451) and integrals

$$\int_{x_0}^{x_n} f(x)\,dx = h \int_0^n f_p\,dp \tag{12}$$

(Beyer 1987, pp. 455-456) of finite differences.

Finite differences lead to DIFFERENCE EQUATIONS, finite analogs of DIFFERENTIAL EQUATIONS. In fact, UMBRAL CALCULUS displays many elegant analogs of well-known identities for continuous functions. Common finite difference schemes for PARTIAL DIFFERENTIAL EQUATIONS include the so-called Crank-Nicholson, Du Fort-Frankel, and Laasonen methods.

see also BACKWARD DIFFERENCE, BESSEL'S FINITE DIFFERENCE FORMULA, DIFFERENCE EQUATION, DIFFERENCE TABLE, EVERETT'S FORMULA, FORWARD DIFFERENCE, GAUSS'S BACKWARD FORMULA, GAUSS'S FORWARD FORMULA, INTERPOLATION, JACKSON'S DIFFERENCE FAN, NEWTON'S BACKWARD DIFFERENCE FORMULA, NEWTON-COTES FORMULAS, NEWTON'S DIVIDED DIFFERENCE INTERPOLATION FORMULA, NEWTON'S FORWARD DIFFERENCE FORMULA, QUOTIENT-DIFFERENCE TABLE, STEFFENSON'S FORMULA, STIRLING'S FINITE DIFFERENCE FORMULA, UMBRAL CALCULUS

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Milne-Thomson, L. M. *The Calculus of Finite Differences.* London: Macmillan, 1951.

Richardson, C. H. *An Introduction to the Calculus of Finite Differences.* New York: Van Nostrand, 1954.

Spiegel, M. *Calculus of Finite Differences and Differential Equations.* New York: McGraw-Hill, 1971.

Finite Field

A finite field is a FIELD with a finite ORDER (number of elements), also called a GALOIS FIELD. The order of a finite field is always a PRIME or a POWER of a PRIME (Birkhoff and Mac Lane 1965). For each PRIME POWER, there exists exactly one (up to an ISOMORPHISM) finite field GF(p^n), often written as $\mathbb{F}_{p^n}$ in current usage. GF(p) is called the PRIME FIELD of order p, and is the FIELD of RESIDUE CLASSES modulo p, where the p elements are denoted 0, 1, ..., $p-1$. $a = b$ in GF(p) means the same as $a \equiv b \pmod{p}$. Note, however, that $2 \times 2 \equiv 0 \pmod{4}$ in the RING of residues modulo 4, so 2 has no reciprocal, and the RING of residues modulo 4 is distinct from the finite field with four elements. Finite fields are therefore denoted GF(p^n), instead of GF$(p_1 \cdots p_n)$ for clarity.

The finite field GF(2) consists of elements 0 and 1 which satisfy the following addition and multiplications tables.

+	0	1
0	0	1
1	1	0

×	0	1
0	0	0
1	0	1

If a subset S of the elements of a finite field F satisfies the above AXIOMS with the same operators of F, then S

is called a SUBFIELD. Finite fields are used extensively in the study of ERROR-CORRECTING CODES.

When $n > 1$, $GF(p^n)$ can be represented as the FIELD of EQUIVALENCE CLASSES of POLYNOMIALS whose COEFFICIENTS belong to $GF(p)$. Any IRREDUCIBLE POLYNOMIAL of degree n yields the same FIELD up to an ISOMORPHISM. For example, for $GF(2^3)$, the modulus can be taken as $x^3+x^2+1=0$, x^3+x+1, or any other IRREDUCIBLE POLYNOMIAL of degree 3. Using the modulus x^3+x+1, the elements of $GF(2^3)$—written 0, x^0, x^1, ...—can be represented as POLYNOMIALS with degree less than 3. For instance,

$$\begin{aligned}
x^3 &\equiv -x-1 \equiv x+1 \\
x^4 &\equiv x(x^3) \equiv x(x+1) \equiv x^2+x \\
x^5 &\equiv x(x^2+x) \equiv x^3+x^2 \equiv x^2-x-1 \equiv x^2+x+1 \\
x^6 &\equiv x(x^2+x+1) \equiv x^3+x^2+x \equiv x^2-1 \equiv x^2+1 \\
x^7 &\equiv x(x^2+1) \equiv x^3+x \equiv -1 \equiv 1 \equiv x_0.
\end{aligned}$$

Now consider the following table which contains several different representations of the elements of a finite field. The columns are the power, polynomial representation, triples of polynomial representation COEFFICIENTS (the vector representation), and the binary INTEGER corresponding to the vector representation (the regular representation).

	Representation		
Power	Polynomial	Vector	Regular
0	0	(000)	0
x^0	1	(001)	1
x^1	x	(010)	2
x^2	x^2	(100)	4
x^3	$x+1$	(011)	3
x^4	x^2+x	(110)	6
x^5	x^2+x+1	(111)	7
x^6	x^2+1	(101)	5

The set of POLYNOMIALS in the second column is closed under ADDITION and MULTIPLICATION modulo x^3+x+1, and these operations on the set satisfy the AXIOMS of finite field. This particular finite field is said to be an extension field of degree 3 of GF(2), written $GF(2^3)$, and the field GF(2) is called the base field of $GF(2^3)$. If an IRREDUCIBLE POLYNOMIAL generates all elements in this way, it is called a PRIMITIVE IRREDUCIBLE POLYNOMIAL. For any PRIME or PRIME POWER q and any POSITIVE INTEGER n, there exists a PRIMITIVE IRREDUCIBLE POLYNOMIAL of degree n over $GF(q)$.

For any element c of $GF(q)$, $c^q = c$, and for any NONZERO element d of $GF(q)$, $d^{q-1}=1$. There is a smallest POSITIVE INTEGER n satisfying the sum condition $n \cdot 1 = 0$ in $GF(q)$, which is called the characteristic of the finite field $GF(q)$. The characteristic is a PRIME NUMBER for every finite field, and it is true that

$$(x+y)^p = x^p + y^p$$

over a finite field with characteristic p.

see also FIELD, HADAMARD MATRIX, RING, SUBFIELD

References

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Birkhoff, G. and Mac Lane, S. *A Survey of Modern Algebra, 3rd ed.* New York: Macmillan, p. 413, 1965.

Dickson, L. E. *History of the Theory of Numbers, Vol. 1: Divisibility and Primality.* New York: Chelsea, p. viii, 1952.

Finite Game

A GAME in which each player has a finite number of moves and a finite number of of choices at each move.

see also GAME, ZERO-SUM GAME

References

Dresher, M. *The Mathematics of Games of Strategy: Theory and Applications.* New York: Dover, p. 2, 1981.

Finite Group

A GROUP of finite ORDER. Examples of finite groups are the MODULO MULTIPLICATION GROUPS and the POINT GROUPS. The CLASSIFICATION THEOREM of finite SIMPLE GROUPS states that the finite SIMPLE GROUPS can be classified completely into one of five types.

There is no known FORMULA to give the number of possible finite groups as a function of the ORDER h. It is possible, however, to determine the number of ABELIAN GROUPS using the KRONECKER DECOMPOSITION THEOREM, and there is at least one ABELIAN GROUP for every finite order h.

The following table gives the numbers and names of the first few groups of ORDER h. In the table, N_{NA} denotes the number of non-Abelian groups, N_A denotes the number of ABELIAN GROUPS, and N the total number of groups. In addition, Z_n denotes an CYCLIC GROUP of ORDER n, A_n an ALTERNATING GROUP, D_n a DIHEDRAL GROUP, Q_8 the group of the QUATERNIONS, T the cubic group, and $\otimes$ a DIRECT PRODUCT.

h	Name	N_{NA}	N_A	N
1	$\langle e \rangle$	1	0	1
2	Z_2	1	0	1
3	Z_3	1	0	1
4	$Z_2 \otimes Z_2, Z_4$	2	0	2
5	Z_5	1	0	1
6	Z_6, D_3	1	1	2
7	Z_7	1	0	1
8	$Z_2 \otimes Z_2 \otimes Z_2, Z_2 \otimes Z_4, Z_8, Q_8, D_4$	3	2	5
9	$Z_3 \otimes Z_3, Z_9$	2	0	2
10	Z_{10}, D_5	1	1	2
11	Z_{11}	1	0	1
12	$Z_2 \otimes Z_6, Z_{12}, A_4, D_6, T$	2	3	5
13	Z_{13}	1	0	1
14	Z_{14}, D_7	1	1	2
15	Z_{15}	1	0	1

Miller (1930) gave the number of groups for orders 1–100, including an erroneous 297 as the number of groups of ORDER 64. Senior and Lunn (1934, 1935) subsequently completed the list up to 215, but omitted 128 and 192. The number of groups of ORDER 64 was corrected in Hall and Senior (1964). James *et al.* (1990) found 2328 groups in 115 ISOCLINISM families of ORDER 128, correcting previous work, and O'Brien (1991) found the number of groups of ORDER 256. The number of groups is known for orders up to 1000, with the possible exception of 512 and 768. Besche and Eick (1998) have determined the number of finite groups of orders less than 1000 which are not powers of 2 or 3. These numbers appear in the *Magma*® database. The numbers of nonisomorphic finite groups N of each ORDER h for the first few hundred orders are given in the following table (Sloane's A000001—the very first sequence).

The number of ABELIAN GROUPS of ORDER h is denoted N_A (Sloane's A000688). The smallest order for which there exist $n = 1, 2, \ldots$ nonisomorphic groups are 1, 4, 75, 28, 8, 42, ... (Sloane's A046057). The incrementally largest number of nonisomorphic finite groups are 1, 2, 5, 14, 15, 51, 52, 267, 2328, ... (Sloane's A046058), which occur for orders 1, 4, 8, 16, 24, 32, 48, 64, 128, ... (Sloane's A046059).

h	N	N_A	h	N	N_A	h	N	N_A	h	N	N_A
1	1	1	51	1	1	101	1	1	151	1	1
2	1	1	52	5	2	102	4	1	152	12	3
3	1	1	53	1	1	103	1	1	153	2	2
4	2	2	54	15	3	104	14	3	154	4	1
5	1	1	55	2	1	105	2	1	155	2	1
6	2	1	56	13	3	106	2	1	156	18	2
7	1	1	57	2	1	107	1	1	157	1	1
8	5	3	58	2	1	108	45	6	158	2	1
9	2	2	59	1	1	109	1	1	159	1	1
10	2	1	60	13	2	110	6	1	160	238	7
11	1	1	61	1	1	111	2	1	161	1	1
12	5	2	62	2	1	112	43	5	162	55	5
13	1	1	63	4	2	113	1	1	163	1	1
14	2	1	64	267	11	114	6	1	164	5	2
15	1	1	65	1	1	115	1	1	165	2	1
16	14	5	66	4	1	116	5	2	166	2	1
17	1	1	67	1	1	117	4	2	167	1	1
18	5	2	68	5	2	118	2	1	168	57	3
19	1	1	69	1	1	119	1	1	169	2	2
20	5	2	70	4	1	120	47	3	170	4	1
21	2	1	71	1	1	121	2	2	171	5	2
22	2	1	72	50	6	122	2	1	172	4	2
23	1	1	73	1	1	123	1	1	173	1	1
24	15	3	74	2	1	124	4	2	174	4	1
25	2	2	75	3	2	125	5	3	175	2	2
26	2	1	76	4	2	126	16	2	176	42	5
27	5	3	77	1	1	127	1	1	177	1	1
28	4	2	78	6	1	128	2328	15	178	2	1
29	1	1	79	1	1	129	2	1	179	1	1
30	4	1	80	52	5	130	4	1	180	37	4
31	1	1	81	15	5	131	1	1	181	1	1
32	51	7	82	2	1	132	10	2	182	4	1
33	1	1	83	1	1	133	1	1	183	2	1
34	2	1	84	15	2	134	2	1	184	12	3
35	1	1	85	1	1	135	5	3	185	1	1
36	14	4	86	2	1	136	15	3	186	6	1
37	1	1	87	1	1	137	1	1	187	1	1
38	2	1	88	12	3	138	4	1	188	4	2
39	2	1	89	1	1	139	1	1	189	13	3
40	14	3	90	10	2	140	11	2	190	4	1
41	1	1	91	1	1	141	1	1	191	1	1
42	6	1	92	4	2	142	2	1	192	1543	11
43	1	1	93	2	1	143	1	1	193	1	1
44	4	2	94	2	1	144	197	10	194	2	1
45	2	2	95	1	1	145	1	1	195	2	1
46	2	1	96	230	7	146	2	1	196	17	4
47	1	1	97	1	1	147	6	2	197	1	1
48	52	5	98	5	2	148	5	2	198	10	2
49	2	2	99	2	2	149	1	1	199	1	1
50	2	2	100	16	4	150	13	2	200	52	6

h	N	N_A	h	N	N_A	h	N	N_A	h	N	N_A
201	2	1	251	1	1	301	2	1	351	14	3
202	2	1	252	46	4	302	2	1	352	195	7
203	2	1	253	2	1	303	1	1	353	1	1
204	12	2	254	2	1	304	42	5	354	4	1
205	2	1	255	1	1	305	2	1	355	2	1
206	2	1	256	56092	22	306	10	2	356	5	2
207	2	2	257	1	1	307	1	1	357	2	1
208	51	5	258	6	1	308	9	2	358	2	1
209	1	1	259	1	1	309	2	1	359	1	1
210	12	1	260	15	2	310	6	1	360	162	6
211	1	1	261	2	2	311	1	1	361	2	2
212	5	2	262	2	1	312	61	3	362	2	1
213	1	1	263	1	1	313	1	1	363	3	2
214	2	1	264	39	3	314	2	1	364	11	2
215	1	1	265	1	1	315	4	2	365	1	1
216	177	9	266	4	1	316	4	2	366	6	1
217	1	1	267	1	1	317	1	1	367	1	1
218	2	1	268	4	2	318	4	1	368	42	5
219	2	1	269	1	1	319	1	1	369	2	2
220	15	2	270	30	3	320	1640	11	370	4	1
221	1	1	271	1	1	321	1	1	371	1	1
222	6	1	272	54	5	322	4	1	372	15	2
223	1	1	273	5	1	323	1	1	373	1	1
224	197	7	274	2	1	324	176	10	374	4	1
225	6	4	275	4	2	325	2	2	375	7	3
226	2	1	276	10	2	326	2	1	376	12	3
227	1	1	277	1	1	327	2	1	377	1	1
228	15	2	278	2	1	328	15	3	378	60	3
229	1	1	279	4	2	329	1	1	379	1	1
230	4	1	280	40	3	330	12	1	380	11	2
231	2	1	281	1	1	331	1	1	381	2	1
232	14	3	282	4	1	332	4	2	382	2	1
233	1	1	283	1	1	333	5	2	383	1	1
234	16	2	284	4	2	334	2	1	384	20169	15
235	1	1	285	2	1	335	1	1	385	2	1
236	4	2	286	4	1	336	228	5	386	2	1
237	2	1	287	1	1	337	1	1	387	4	2
238	4	1	288	1045	14	338	5	2	388	5	2
239	1	1	289	2	2	339	1	1	389	1	1
240	208	5	290	4	1	340	15	2	390	12	1
241	1	1	291	2	1	341	1	1	391	1	1
242	5	2	292	5	2	342	18	2	392	44	6
243	67	7	293	1	1	343	5	3	393	1	1
244	5	2	294	23	2	344	12	3	394	2	1
245	2	2	295	1	1	345	1	1	395	1	1
246	4	1	296	14	3	346	2	1	396	30	4
247	1	1	297	5	3	347	1	1	397	1	1
248	12	3	298	2	1	348	12	2	398	2	1
249	1	1	299	1	1	349	1	1	399	5	1
250	15	3	300	49	4	350	10	2	400	221	10

see also ABELIAN GROUP, ABEL'S THEOREM, ABHYANKAR'S CONJECTURE, ALTERNATING GROUP, BURNSIDE'S LEMMA, BURNSIDE PROBLEM, CHEVALLEY GROUPS, CLASSIFICATION THEOREM, COMPOSITION SERIES, DIHEDRAL GROUP, GROUP, JORDAN-HÖLDER THEOREM, KRONECKER DECOMPOSITION THEOREM, LIE GROUP, LIE-TYPE GROUP, LINEAR GROUP, MODULO MULTIPLICATION GROUP, ORDER (GROUP), ORTHOGONAL GROUP, p-GROUP, POINT GROUPS, SIMPLE GROUP, SPORADIC GROUP, SYMMETRIC GROUP, SYMPLECTIC GROUP, TWISTED CHEVALLEY GROUPS, UNITARY GROUP

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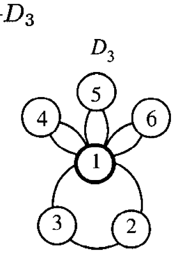
Finite Group—D_3



The DIHEDRAL GROUP D_3 is one of the two groups of ORDER 6. It the non-Abelian group of smallest ORDER. Examples of D_3 include the POINT GROUPS known as C_{3h}, C_{3v}, S_3, D_3, the symmetry group of the EQUILATERAL TRIANGLE, and the group of permutation of three objects. Its elements A_i satisfy $A_i{}^3 = 1$, and four of its elements satisfy $A_i{}^2 = 1$, where 1 is the IDENTITY ELEMENT. The CYCLE GRAPH is shown above, and the MULTIPLICATION TABLE is given below.

D_3	1	A	B	C	D	E
1	1	A	B	C	D	E
A	A	1	D	E	B	C
B	B	E	1	D	C	A
C	C	D	E	1	A	B
D	D	C	A	B	E	1
E	E	B	C	A	1	D

The CONJUGACY CLASSES are $\{1\}$, $\{A, B, C\}$

$$A^{-1}AA = A \tag{1}$$
$$B^{-1}AB = C \tag{2}$$
$$C^{-1}AC = B \tag{3}$$
$$D^{-1}AD = C \tag{4}$$
$$E^{-1}AE = B, \tag{5}$$

and $\{D, E\}$,

$$DA^{-1}D = E \tag{6}$$
$$B^{-1}DB = D. \tag{7}$$

A reducible 2-D representation using REAL MATRICES can be found by performing the spatial rotations corresponding to the symmetry elements of C_{3v}. Take the z-AXIS along the C_3 axis.

$$I = R_z(0) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \tag{8}$$

$$A = R_z(\tfrac{2}{3}\pi) = \begin{bmatrix} \cos(\frac{2}{3}\pi) & \sin(\frac{2}{3}\pi) \\ -\sin(\frac{2}{3}\pi) & \cos(\frac{2}{3}\pi) \end{bmatrix}$$
$$= \begin{bmatrix} -\frac{1}{2} & -\frac{1}{2}\sqrt{3} \\ \frac{1}{2}\sqrt{3} & -\frac{1}{2} \end{bmatrix} \tag{9}$$

$$B = R_z(\tfrac{4}{3}\pi) = \begin{bmatrix} -\frac{1}{2} & \frac{1}{2}\sqrt{3} \\ -\frac{1}{2}\sqrt{3} & -\frac{1}{2} \end{bmatrix} \tag{10}$$

$$C = R_C(\pi) = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \tag{11}$$

$$D = R_D(\pi) = CB = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2}\sqrt{3} \\ -\frac{1}{2}\sqrt{3} & -\frac{1}{2} \end{bmatrix} \tag{12}$$

$$E = R_E(\pi) = CA = \begin{bmatrix} \frac{1}{2} & \frac{1}{2}\sqrt{3} \\ \frac{1}{2}\sqrt{3} & -\frac{1}{2} \end{bmatrix}. \tag{13}$$

To find the irreducible representation, note that there are three CONJUGACY CLASSES. Rule 5 requires that there be three irreducible representations satisfying

$$h = l_1{}^2 + l_2{}^2 + l_3{}^2 = 6, \tag{14}$$

so it must be true that

$$l_1 = l_2 = 1, l_3 = 2. \tag{15}$$

By rule 6, we can let the first representation have all 1s.

D_3	1	A	B	C	D	E
Γ_1	1	1	1	1	1	1

To find representation orthogonal to the totally symmetric representation, we must have three $+1$ and three -1 CHARACTERS. We can also add the constraint that the components of the IDENTITY ELEMENT 1 be positive. The three CONJUGACY CLASSES have 1, 2, and 3 elements. Since we need a total of three $+1$s and we have required that a $+1$ occur for the CONJUGACY CLASS of ORDER 1, the remaining $+1$s must be used for the elements of the CONJUGACY CLASS of ORDER 2, i.e., A and B.

D_3	1	A	B	C	D	E
Γ_1	1	1	1	1	1	1
Γ_2	1	1	1	-1	-1	-1

Using the rule 1, we see that

$$1^2 + 1^2 + \chi_3{}^2(1) = 6, \tag{16}$$

so the final representation for 1 has CHARACTER 2. Orthogonality with the first two representations (rule 3) then yields the following constraints:

$$1 \cdot 1 \cdot 2 + 1 \cdot 2 \cdot \chi_2 + 1 \cdot 3 \cdot \chi_3 = 2 + 2\chi_2 + 3\chi_3 = 0 \tag{17}$$

$$1 \cdot 1 \cdot 2 + 1 \cdot 2 \cdot \chi_2 + (-1) \cdot 3 \cdot \chi_3 = 2 + 2\chi_2 - 3\chi_3 = 0. \tag{18}$$

Solving these simultaneous equations by adding and subtracting (18) from (17), we obtain $\chi_2 = -1$, $\chi_3 = 0$. The full CHARACTER TABLE is then

D_3	1	A	B	C	D	E
Γ_1	1	1	1	1	1	1
Γ_2	1	1	1	-1	-1	-1
Γ_3	2	-1	-1	0	0	0

Since there are only three CONJUGACY CLASSES, this table is conventionally written simply as

D_3	1	$A = B$	$C = D = E$
Γ_1	1	1	1
Γ_2	1	1	-1
Γ_3	2	-1	0

Writing the irreducible representations in matrix form then yields

$$1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{19}$$

$$A = \begin{bmatrix} -\frac{1}{2} & -\frac{1}{2}\sqrt{3} & 0 & 0 \\ \frac{1}{2}\sqrt{3} & -\frac{1}{2} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{20}$$

$$B = \begin{bmatrix} -\frac{1}{2} & \frac{1}{2}\sqrt{3} & 0 & 0 \\ -\frac{1}{2}\sqrt{3} & -\frac{1}{2} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{21}$$

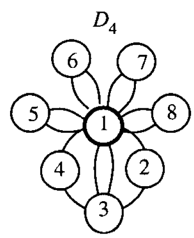
$$C = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \tag{22}$$

$$D = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2}\sqrt{3} & 0 & 0 \\ -\frac{1}{2}\sqrt{3} & -\frac{1}{2} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \tag{23}$$

$$E = \begin{bmatrix} \frac{1}{2} & \frac{1}{2}\sqrt{3} & 0 & 0 \\ \frac{1}{2}\sqrt{3} & -\frac{1}{2} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}. \tag{24}$$

see also Dihedral Group, Finite Group—D_4, Finite Group—Z_6

Finite Group—D_4



The Dihedral Group D_4 is one of the two non-Abelian groups of the five groups total of Order 8. It is sometimes called the octic group. Examples of D_4 include the symmetry group of the Square. The Cycle Graph is shown above.

see also Dihedral Group, Finite Group—D_3, Finite Group—Z_8, Finite Group—$Z_2 \otimes Z_2 \otimes Z_2$, Finite Group—$Z_2 \otimes Z_4$, Finite Group—$Z_8$,

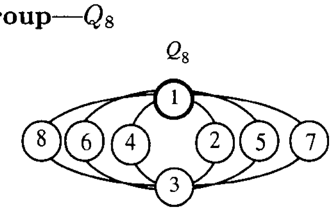
Finite Group—$\langle e \rangle$

The unique (and trivial) group of Order 1 is denoted $\langle e \rangle$. It is (trivially) Abelian and Cyclic. Examples include the Point Group C_1 and the integers modulo 1 under addition.

$\langle e \rangle$	1
1	1

The only class is $\{1\}$.

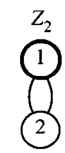
Finite Group—Q_8



One of the three Abelian groups of the five groups total of Order 8. The group Q_8 has the Multiplication Table of $\pm 1, i, j, k$, where 1, i, j, and k are the Quaternions. The Cycle Graph is shown above.

see also Finite Group—D_4, Finite Group—$Z_2 \otimes Z_2 \otimes Z_2$, Finite Group—$Z_2 \otimes Z_4$, Finite Group—$Z_8$, Quaternion

Finite Group—Z_2

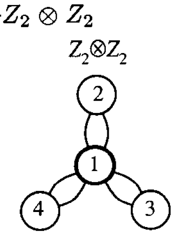


The unique group of Order 2. Z_2 is both Abelian and Cyclic. Examples include the Point Groups C_s, C_i, and C_2, the integers modulo 2 under addition, and the Modulo Multiplication Groups M_3, M_4, and M_6. The elements A_i satisfy $A_i{}^2 = 1$, where 1 is the Identity Element. The Cycle Graph is shown above, and the Multiplication Table is given below.

Z_2	1	A
1	1	A
A	A	1

The Conjugacy Classes are $\{1\}$ and $\{A\}$. The irreducible representation for the C_2 group is $\{1, -1\}$.

Finite Group—$Z_2 \otimes Z_2$



One of the two groups of Order 4. The name of this group derives from the fact that it is a Direct Product of two Z_2 Subgroups. Like the group Z_4, $Z_2 \otimes Z_2$ is an Abelian Group. Unlike Z_4, however, it is not Cyclic. In addition to satisfying $A_i{}^4 = 1$ for each element A_i, it also satisfies $A_i{}^2 = 1$, where 1 is the Identity Element. Examples of the $Z_2 \otimes Z_2$ group include the Viergruppe, Point Groups D_2, C_{2h}, and C_{2v}, and the Modulo Multiplication Groups M_8 and M_{12}. That M_8, the Residue Classes prime to 8 given by $\{1, 3, 5, 7\}$, are a group of type $Z_2 \otimes Z_2$ can be shown by verifying that

$$1^2 = 1 \quad 3^2 = 9 \equiv 1 \quad 5^2 = 25 \equiv 1 \quad 7^2 = 49 \equiv 1 \pmod{8} \tag{1}$$

and

$$3 \cdot 5 = 15 \equiv 7 \quad 3 \cdot 7 = 21 \equiv 5 \quad 5 \cdot 7 = 35 \equiv 3 \pmod{8}. \tag{2}$$

$Z_2 \otimes Z_2$ is therefore a MODULO MULTIPLICATION GROUP.

The CYCLE GRAPH is shown above, and the multiplication table for the $Z_2 \otimes Z_2$ group is given below.

$Z_2 \otimes Z_2$	1	A	B	C
1	1	A	B	C
A	A	1	C	B
B	B	C	1	A
C	C	B	A	1

The CONJUGACY CLASSES are $\{1\}$, $\{A\}$,

$$A^{-1}AA = A \tag{3}$$
$$B^{-1}AB = A \tag{4}$$
$$C^{-1}AC = A, \tag{5}$$

$\{B\}$,

$$A^{-1}BA = B \tag{6}$$
$$C^{-1}BC = B, \tag{7}$$

and $\{C\}$.

Now explicitly consider the elements of the C_{2v} POINT GROUP.

C_{2v}	E	C_2	σ_v	σ_v
E	E	C_2	σ_v	σ_v'
C_2	C_2	E	σ_v'	σ_v
σ_v	σ_v	σ_v'	E	C_2
σ_v'	σ_v'	σ_v	C_2	E

In terms of the VIERGRUPPE elements

V	I	V_1	V_2	V_3
I	V_1	V_2	V_3	V_4
V_1	V_1	I	V_3	V_2
V_2	V_2	V_3	I	V_1
V_3	V_3	V_2	V_1	I

A reducible representation using 2-D REAL MATRICES is

$$1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \tag{8}$$
$$A = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \tag{9}$$
$$B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \tag{10}$$
$$C = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}. \tag{11}$$

Another reducible representation using 3-D REAL MATRICES can be obtained from the symmetry elements of the D_2 group (1, $C_2(z)$, $C_2(y)$, and $C_2(x)$) or C_{2v} group (1, C_2, σ_v, and σ_v'). Place the C_2 axis along the z-axis, σ_v in the x-y plane, and σ_v' in the y-z plane.

$$1 = E = E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{12}$$
$$A = R_x(\pi) = \sigma_v = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{13}$$
$$C = R_z(\pi) = C_2 = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{14}$$
$$B = R_y(\pi) = \sigma_v' = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{15}$$

In order to find the irreducible representations, note that the traces are given by $\chi(1) = 3, \chi(C_2) = -1$, and $\chi(\sigma_v) = \chi(\sigma_v') = 1$. Therefore, there are at least three distinct CONJUGACY CLASSES. However, we see from the MULTIPLICATION TABLE that there are actually four CONJUGACY CLASSES, so group rule 5 requires that there must be four irreducible representations. By rule 1, we are looking for POSITIVE INTEGERS which satisfy

$$l_1{}^2 + l_2{}^2 + l_3{}^2 + l_4{}^2 = 4. \tag{16}$$

The only combination which will work is

$$l_1 = l_2 = l_3 = l_4 = 1, \tag{17}$$

so there are four one-dimensional representations. Rule 2 requires that the sum of the squares equal the ORDER $h = 4$, so each 1-D representation must have CHARACTER ± 1. Rule 6 requires that a totally symmetric representation always exists, so we are free to start off with the first representation having all 1s. We then use orthogonality (rule 3) to build up the other representations. The simplest solution is then given by

C_{2v}	1	C_2	σ_v	σ_v'
Γ_1	1	1	1	1
Γ_2	1	-1	-1	1
Γ_3	1	-1	1	-1
Γ_4	1	1	-1	-1

These can be put into a more familiar form by switching Γ_1 and Γ_3, giving the CHARACTER TABLE

C_{2v}	1	C_2	σ_v	σ_v'
Γ_3	1	-1	1	-1
Γ_2	1	-1	-1	1
Γ_1	1	1	1	1
Γ_4	1	1	-1	-1

The matrices corresponding to this representation are now

$$1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad (18)$$

$$C_2 = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad (19)$$

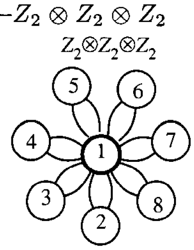
$$\sigma_v = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \quad (20)$$

$$\sigma_v' = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}, \quad (21)$$

which consist of the previous representation with an additional component. These matrices are now orthogonal, and the order equals the matrix dimension. As before, $\chi(\sigma_v) = \chi(\sigma_v')$.

see also FINITE GROUP—Z_4

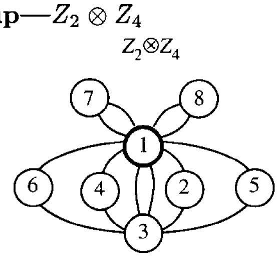
Finite Group—$Z_2 \otimes Z_2 \otimes Z_2$



One of the three Abelian groups of the five groups total of ORDER 8. Examples include the MODULO MULTIPLICATION GROUP M_{24}. The elements A_i of this group satisfy ${A_i}^2 = 1$, where 1 is the IDENTITY ELEMENT. The CYCLE GRAPH is shown above.

see also FINITE GROUP—D_4, FINITE GROUP—Q_8, FINITE GROUP—$Z_2 \otimes Z_4$, FINITE GROUP—Z_8

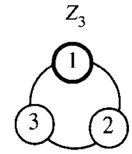
Finite Group—$Z_2 \otimes Z_4$



One of the three Abelian groups of the five groups total of ORDER 8. Examples include the MODULO MULTIPLICATION GROUPS M_{15}, M_{16}, M_{20}, and M_{30}. The elements A_i of this group satisfy ${A_i}^4 = 1$, where 1 is the IDENTITY ELEMENT, and four of the elements satisfy ${A_i}^2 = 1$. The CYCLE GRAPH is shown above.

see also FINITE GROUP—D_4, FINITE GROUP—Q_8, FINITE GROUP—$Z_2 \otimes Z_2 \otimes Z_2$, FINITE GROUP—$Z_8$

Finite Group—Z_3



The unique group of ORDER 3. It is both ABELIAN and CYCLIC. Examples include the POINT GROUPS C_3 and D_3 and the integer modulo 3. The elements A_i of the group satisfy ${A_i}^3 = 1$ where 1 is the IDENTITY ELEMENT. The CYCLE GRAPH is shown above, and the MULTIPLICATION TABLE is given below.

Z_3	1	A	B
1	1	A	B
A	A	B	1
B	B	1	A

The CONJUGACY CLASSES are $\{1\}$, $\{A\}$,

$$A^{-1}AA = A$$
$$B^{-1}AB = A,$$

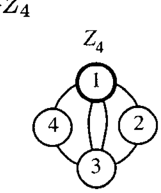
and $\{B\}$,

$$A^{-1}BA = B$$
$$B^{-1}BB = B.$$

The irreducible representation (CHARACTER TABLE) is therefore

Γ	1	A	B
Γ_1	1	1	1
Γ_2	1	1	-1
Γ_3	1	-1	1

Finite Group—Z_4



One of the two groups of ORDER 4. Like $Z_2 \otimes Z_2$, it is ABELIAN, but unlike $Z_2 \otimes Z_2$, it is a CYCLIC. Examples include the POINT GROUPS C_4 and S_4 and the MODULO MULTIPLICATION GROUPS M_5 and M_{10}. Elements A_i of the group satisfy ${A_i}^4 = 1$, where 1 is the IDENTITY ELEMENT, and two of the elements satisfy ${A_i}^2 = 1$.

The CYCLE GRAPH is shown above. The MULTIPLICATION TABLE for this group may be written in three equivalent ways—denoted here by $Z_4^{(1)}$, $Z_4^{(2)}$, and $Z_4^{(3)}$—by permuting the symbols used for the group elements.

$Z_4^{(1)}$	1	A	B	C
1	1	A	B	C
A	A	B	C	1
B	B	C	1	A
C	C	1	A	B

The MULTIPLICATION TABLE for $Z_4^{(2)}$ is obtained from $Z_4^{(1)}$ by interchanging A and B.

$Z_4^{(2)}$	1	A	B	C
1	1	A	B	C
A	A	1	C	B
B	B	C	A	1
C	C	B	1	A

The MULTIPLICATION TABLE for $Z_4^{(3)}$ is obtained from $Z_4^{(1)}$ by interchanging A and C.

$Z_4^{(3)}$	1	A	B	C
1	1	A	B	C
A	A	C	1	B
B	B	1	C	A
C	C	B	A	1

The CONJUGACY CLASSES of Z_4 are $\{1\}$, $\{A\}$,

$$A^{-1}AA = A \tag{1}$$
$$B^{-1}AB = A \tag{2}$$
$$C^{-1}AC = A, \tag{3}$$

$\{B\}$,

$$A^{-1}BA = B \tag{4}$$
$$B^{-1}BB = B \tag{5}$$
$$C^{-1}BC = B, \tag{6}$$

and $\{C\}$.

The group may be given a reducible representation using COMPLEX NUMBERS

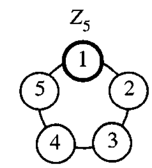
$$1 = 1 \tag{7}$$
$$A = i \tag{8}$$
$$B = -1 \tag{9}$$
$$C = -i, \tag{10}$$

or REAL MATRICES

$$1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \tag{11}$$
$$A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \tag{12}$$
$$B = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \tag{13}$$
$$C = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}. \tag{14}$$

see also FINITE GROUP—$Z_2 \otimes Z_2$

Finite Group—Z_5

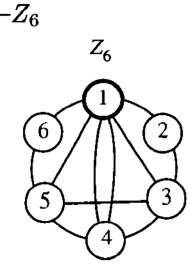


The unique GROUP of ORDER 5, which is ABELIAN. Examples include the POINT GROUP C_5 and the integers mod 5 under addition. The elements A_i satisfy $A_i{}^5 = 1$, where 1 is the IDENTITY ELEMENT. The CYCLE GRAPH is shown above, and the MULTIPLICATION TABLE is illustrated below.

Z_5	1	A	B	C	D
1	1	A	B	C	D
A	A	B	C	D	1
B	B	C	D	1	A
C	C	D	1	A	B
D	D	1	A	B	C

The CONJUGACY CLASSES are $\{1\}$, $\{A\}$, $\{B\}$, $\{C\}$, and $\{D\}$.

Finite Group—Z_6



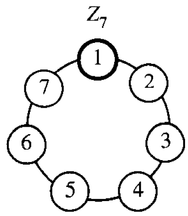
One of the two groups of ORDER 6 which, unlike D_3, is ABELIAN. It is also a CYCLIC. It is isomorphic to $Z_2 \otimes Z_3$. Examples include the POINT GROUPS C_6 and S_6, the integers modulo 6 under addition, and the MODULO MULTIPLICATION GROUPS M_7, M_9, and M_{14}. The elements A_i of the group satisfy $A_i{}^6 = 1$, where 1 is the IDENTITY ELEMENT, three elements satisfy $A_i{}^3 = 1$, and two elements satisfy $A_i{}^2 = 1$. The CYCLE GRAPH is shown above, and the MULTIPLICATION TABLE is given below.

Z_6	1	A	B	C	D	E
1	1	A	B	C	D	E
A	A	1	E	D	B	C
B	B	E	1	A	C	D
C	C	D	A	1	E	B
D	D	B	C	E	1	A
E	E	C	D	B	A	1

The CONJUGACY CLASSES are $\{1\}$, $\{A\}$, $\{B\}$, $\{C\}$, $\{D\}$, and $\{E\}$.

see also FINITE GROUP—D_3

Finite Group—Z_7

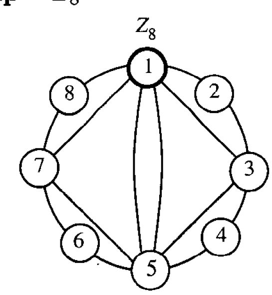


The unique GROUP of ORDER 7. It is ABELIAN and CYCLIC. Examples include the POINT GROUP C_7 and the integers modulo 7 under addition. The elements A_i of the group satisfy ${A_i}^7 = 1$, where 1 is the IDENTITY ELEMENT. The CYCLE GRAPH is shown above.

Z_7	1	A	B	C	D	E	F
1	1	A	B	C	D	E	F
A	A	B	C	D	E	F	1
B	B	C	D	E	F	1	A
C	C	D	E	F	1	A	B
D	D	E	F	1	A	B	C
E	E	F	1	A	B	C	D
F	F	1	A	B	C	D	E

The CONJUGACY CLASSES are $\{1\}$, $\{A\}$, $\{B\}$, $\{C\}$, $\{D\}$, $\{E\}$, and $\{F\}$.

Finite Group—Z_8



One of the three Abelian groups of the five groups total of ORDER 8. An example is the residue classes modulo 17 which QUADRATIC RESIDUES, i.e., $\{1, 2, 4, 8, 9, 13, 15, 16\}$ under multiplication modulo 17. The elements A_i satisfy ${A_i}^8 = 1$, four of them satisfy ${A_i}^4 = 1$, and two satisfy ${A_i}^2 = 1$. The CYCLE GRAPH is shown above.

see also FINITE GROUP—D_4, FINITE GROUP—Q_8, FINITE GROUP—$Z_2 \otimes Z_4$, FINITE GROUP—$Z_2 \otimes Z_2 \otimes Z_2$

Finite Mathematics

The branch of mathematics which does not involve infinite sets, limits, or continuity.

see also COMBINATORICS, DISCRETE MATHEMATICS

References

Hildebrand, F. H. and Johnson, C. G. *Finite Mathematics.* Boston, MA: Prindle, Weber, and Schmidt, 1970.

Kemeny, J. G.; Snell, J. L.; and Thompson, G. L. *Introduction to Finite Mathematics, 3rd ed.* Englewood Cliffs, NJ: Prentice-Hall, 1974.

Marcus, M. *A Survey of Finite Mathematics.* New York: Dover, 1993.

Finite Simple Group

see SIMPLE GROUP

Finite Simple Group Classification Theorem

see CLASSIFICATION THEOREM

Finite-to-One Factor

A MAP $\psi : M \to M$, where M is a MANIFOLD, is a finite-to-one factor of a MAP $\Psi : X \to X$ if there exists a continuous ONTO MAP $\pi : X \to M$ such that $\psi \circ \pi = \pi \circ \Psi$ and $\pi^{-1}(x) \subset X$ is finite for each $x \in M$.

Finsler Geometry

The geometry of FINSLER SPACE.

Finsler Manifold

see FINSLER SPACE

Finsler Metric

A continuous real function $L(x, y)$ defined on the TANGENT BUNDLE $T(M)$ of an n-D DIFFERENTIABLE MANIFOLD M is said to be a Finsler metric if

1. $L(x, y)$ is DIFFERENTIABLE at $x \neq y$,
2. $L(x, \lambda y) = |\lambda| L(x, y)$ for any element $(x, y) \in T(M)$ and any REAL NUMBER λ,
3. Denoting the METRIC

$$g_{ij}(x, y) = \frac{1}{2} \frac{\partial^2 [L(x, y)]^2}{\partial y^i \partial y^j},$$

then g_{ij} is a POSITIVE DEFINITE MATRIX.

A DIFFERENTIABLE MANIFOLD M with a Finsler metric is called a FINSLER SPACE.

see also DIFFERENTIABLE MANIFOLD, FINSLER SPACE, TANGENT BUNDLE

References

Iyanaga, S. and Kawada, Y. (Eds.). "Finsler Spaces." §161 in *Encyclopedic Dictionary of Mathematics.* Cambridge, MA: MIT Press, p. 540–542, 1980.

Finsler Space

A general space based on the LINE ELEMENT

$$ds = F(x^1, \ldots, x^n; dx^1, \ldots, dx^n),$$

with $F(x, y) > 0$ for $y \neq 0$ a function on the TANGENT BUNDLE $T(M)$, and homogeneous of degree 1 in y. Formally, a Finsler space is a DIFFERENTIABLE MANIFOLD possessing a FINSLER METRIC. Finsler geometry is RIEMANNIAN GEOMETRY *without* the restriction that the LINE ELEMENT be quadratic of the form

$$F^2 = g_{ij}(x)\, dx^i\, dx^j.$$

A compact boundaryless Finsler space is locally Minkowskian IFF it has 0 "flag curvature."

see also FINSLER METRIC, HODGE'S THEOREM, RIEMANNIAN GEOMETRY, TANGENT BUNDLE

References

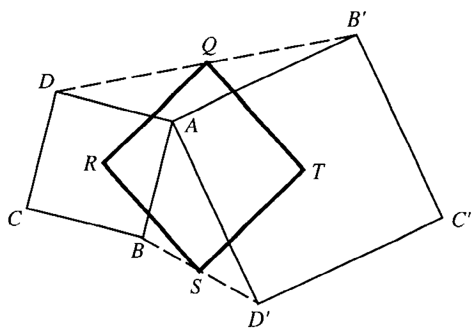
Akbar-Zadeh, H. "Sur les espaces de Finsler à courbures sectionnelles constantes." *Acad. Roy. Belg. Bull. Cl. Sci.* **74**, 281–322, 1988.

Bao, D.; Chern, S.-S.; and Shen, Z. (Eds.). *Finsler Geometry.* Providence, RI: Amer. Math. Soc., 1996.

Chern, S.-S. "Finsler Geometry is Just Riemannian Geometry without the Quadratic Restriction." *Not. Amer. Math. Soc.* **43**, 959–963, 1996.

Iyanaga, S. and Kawada, Y. (Eds.). "Finsler Spaces." §161 in *Encyclopedic Dictionary of Mathematics.* Cambridge, MA: MIT Press, p. 540–542, 1980.

Finsler-Hadwiger Theorem



Let the SQUARES $\square ABCD$ and $\square AB'C'D'$ share a common VERTEX A. The midpoints Q and S of the segments $B'D$ and BD' together with the centers of the original squares R and T then form another square $\square QRST$. This theorem is a special case of the FUNDAMENTAL THEOREM OF DIRECTLY SIMILAR FIGURES (Detemple and Harold 1996).

see also FUNDAMENTAL THEOREM OF DIRECTLY SIMILAR FIGURES, SQUARE

References

Detemple, D. and Harold, S. "A Round-Up of Square Problems." *Math. Mag.* **69**, 15–27, 1996.

Finsler, P. and Hadwiger, H. "Einige Relationen im Dreieck." *Comment. Helv.* **10**, 316–326, 1937.

Fisher, J. C.; Ruoff, D.; and Shileto, J. "Polygons and Polynomials." In *The Geometric Vein: The Coxeter Festschrift.* New York: Springer-Verlag, 321–333, 1981.

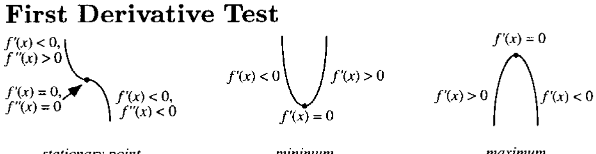
First-Countable Space

A TOPOLOGICAL SPACE in which every point has a countable BASE for its neighborhood system.

First Curvature

see CURVATURE

First Derivative Test



Suppose $f(x)$ is CONTINUOUS at a STATIONARY POINT x_0.

1. If $f'(x) > 0$ on an OPEN INTERVAL extending left from x_0 and $f'(x) < 0$ on an OPEN INTERVAL extending right from x_0, then f has a RELATIVE MAXIMUM (possibly a GLOBAL MAXIMUM) at x_0.
2. If $f'(x) < 0$ on an OPEN INTERVAL extending left from x_0 and $f'(x) > 0$ on an OPEN INTERVAL extending right from x_0, then f has a RELATIVE MINIMUM (possibly a GLOBAL MINIMUM) at x_0.
3. If $f'(x)$ has the same sign on an OPEN INTERVAL extending left from x_0 and on an OPEN INTERVAL extending right from x_0, then f does not have a RELATIVE EXTREMUM at x_0.

see also EXTREMUM, GLOBAL MAXIMUM, GLOBAL MINIMUM, INFLECTION POINT, MAXIMUM, MINIMUM, RELATIVE EXTREMUM, RELATIVE MAXIMUM, RELATIVE MINIMUM, SECOND DERIVATIVE TEST, STATIONARY POINT

References

Abramowitz, M. and Stegun, C. A. (Eds.). *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables, 9th printing.* New York: Dover, p. 14, 1972.

First Digit Law

see BENFORD'S LAW

First Digit Phenomenon

see BENFORD'S LAW

First Multiplier Theorem

Let D be a planar Abelian DIFFERENCE SET and t be any DIVISOR of n. Then t is a numerical multiplier of D, where a multiplier is defined as an automorphism α of G which takes D to a translation $g + D$ of itself for some $g \in G$. If α is of the form $\alpha : x \to tx$ for $t \in \mathbb{Z}$ relatively prime to the order of G, then α is called a numerical multiplier.

References

Gordon, D. M. "The Prime Power Conjecture is True for $n < 2,000,000$." *Electronic J. Combinatorics* **1**, R6, 1–7, 1994. `http://www.combinatorics.org/Volume_1/volume1.html#R6`.

Fischer's Baby Monster Group

see BABY MONSTER GROUP

Fischer Groups

The SPORADIC GROUPS Fi_{22}, Fi_{23}, and Fi'_{24}. These groups were discovered during the investigation of 3-TRANSPOSITION GROUPS.

see also SPORADIC GROUP

References

Wilson, R. A. "ATLAS of Finite Group Representation." `http://for.mat.bham.ac.uk/atlas/F22.html`, `F23.html`, and `F24.html`.

Fish Bladder

see LENS

Fisher-Behrens Problem

The determination of a test for the equality of MEANS for two NORMAL DISTRIBUTIONS with different VARIANCES given samples from each. There exists an exact test which, however, does not give a unique answer because it does not use all the data. There also exist approximate tests which do not use all the data.

see also NORMAL DISTRIBUTION

References

Fisher, R. A. "The Fiducial Argument in Statistical Inference." *Ann. Eugenics* **6**, 391–398, 1935.

Kenney, J. F. and Keeping, E. S. "The Behrens-Fisher Test." §9.8 in *Mathematics of Statistics, Pt. 2, 2nd ed.* Princeton, NJ: Van Nostrand, pp. 257–260 and 261–264, 1951.

Sukhatme, P. V. "On Fisher and Behrens' Test of Significance of the Difference in Means of Two Normal Samples." *Sankhya* **4**, 39, 1938.

Fisher's Block Design Inequality

A balanced incomplete BLOCK DESIGN (v, k, λ, r, b) exists only or $b \geq v$ (or, equivalently, $r \geq k$).

see also BRUCK-RYSER-CHOWLA THEOREM

References

Dinitz, J. H. and Stinson, D. R. "A Brief Introduction to Design Theory." Ch. 1 in *Contemporary Design Theory: A Collection of Surveys* (Ed. J. H. Dinitz and D. R. Stinson). New York: Wiley, pp. 1–12, 1992.

Fisher's Estimator Inequality

Given T an UNBIASED ESTIMATOR of θ so that $\langle T\rangle = \theta$. Then

$$\text{var}(T) \geq \frac{1}{N\int_{-\infty}^{\infty}\left[\frac{\partial(\ln f)}{\partial\theta}\right]^2 f\,dx},$$

where var is the VARIANCE.

Fisher's Exact Test

A STATISTICAL TEST used to determine if there are nonrandom associations between two CATEGORICAL VARIABLES. Let there exist two such variables X and Y, with m and n observed states, respectively. Now form an $n \times m$ MATRIX in which the entries a_{ij} represent the number of observations in which $x = i$ and $y = j$. Calculate the row and column sums R_i and C_j, respectively, and the total sum

$$N = \sum_i R_i = \sum_j C_j$$

of the MATRIX. Then calculate the conditional LIKELIHOOD (P-VALUE) of getting the actual matrix given the particular row and column sums, given by

$$P_{\text{crit}} = \frac{(R_1!R_2!\cdots R_m!)(C_1!C_2!\cdots C_n!)}{N!\prod_{i,j} a_{ij}!}$$

(which is a HYPERGEOMETRIC DISTRIBUTION). Now find all possible MATRICES of NONNEGATIVE INTEGERS consistent with the row and column sums R_i and C_j. For each one, calculate the associated P-VALUE using (0) (where the sum of these probabilities must be 1). Then the P-VALUE of the test is given by the sum of all P-VALUES which are $\leq P_{\text{crit}}$.

The test is most commonly applied to a 2×2 MATRICES, and is computationally unwieldy for large m or n.

As an example application of the test, let X be a journal, say either *Mathematics Magazine* or *Science*, and let Y be the number of articles on the topics of mathematics and biology appearing in a given issue of one of these journals. If *Mathematics Magazine* has five articles on math and one on biology, and *Science* has none on math and four on biology, then the relevant matrix would be

	Math. Mag.	*Science*	
math	5	0	$R_1 = 5$
biology	1	4	$R_2 = 5$
	$C_1 = 6$	$C_2 = 4$	$N = 10.$

Computing P_{crit} gives

$$P_{\text{crit}} = \frac{5!^2 6!4!}{10!(5!0!1!4!)} = 0.0238,$$

and the other possible matrices and their Ps are

$$\begin{bmatrix} 4 & 1 \\ 2 & 3 \end{bmatrix} \quad P = 0.2381$$

$$\begin{bmatrix} 3 & 2 \\ 3 & 2 \end{bmatrix} \quad P = 0.4762$$

$$\begin{bmatrix} 2 & 3 \\ 4 & 1 \end{bmatrix} \quad P = 0.2381$$

$$\begin{bmatrix} 1 & 4 \\ 5 & 0 \end{bmatrix} \quad P = 0.0238,$$

which indeed sum to 1, as required. The sum of P-values less than or equal to $P_{\text{crit}} = 0.0238$ is then 0.0476 which, because it is less than 0.05, is SIGNIFICANT. Therefore, in this case, there would be a statistically significant association between the journal and type of article appearing.

Fisher Index

The statistical INDEX

$$P_B \equiv \sqrt{P_L P_P},$$

where P_L is LASPEYRES' INDEX and P_P is PAASCHE'S INDEX.

see also INDEX

References

Kenney, J. F. and Keeping, E. S. *Mathematics of Statistics, Pt. 1, 3rd ed.* Princeton, NJ: Van Nostrand, p. 66, 1962.

Fisher Kurtosis

$$\gamma_2 \equiv b_2 \equiv \frac{\mu_4}{{\mu_2}^2} - 3 = \frac{\mu_4}{\sigma^4} - 3,$$

where μ_i is the ith MOMENT about the MEAN and $\sigma = \sqrt{\mu_2}$ is the STANDARD DEVIATION.

see also FISHER SKEWNESS, KURTOSIS, PEARSON KURTOSIS

Fisher Sign Test

A robust nonparametric test which is an alternative to the PAIRED t-TEST. This test makes the basic assumption that there is information only in the signs of the differences between paired observations, not in their sizes. Take the paired observations, calculate the differences, and count the number of +s n_+ and −s n_-, where

$$N \equiv n_+ + n_-$$

is the sample size. Calculate the BINOMIAL COEFFICIENT

$$B \equiv \binom{N}{n_+}.$$

Then $B/2^N$ gives the probability of getting exactly this many +s and −s if POSITIVE and NEGATIVE values are equally likely. Finally, to obtain the P-VALUE for the test, sum all the COEFFICIENTS that are $\leq B$ and divide by 2^N.

see also HYPOTHESIS TESTING

Fisher Skewness

$$\gamma_1 = \frac{\mu_3}{{\mu_2}^{3/2}} = \frac{\mu_3}{\sigma^3},$$

where μ_i is the i MOMENT about the MEAN, and $\sigma = \sqrt{\mu_2}$ is the STANDARD DEVIATION.

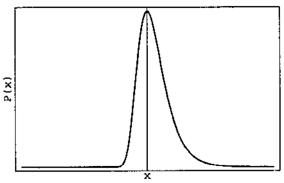
see also FISHER KURTOSIS, MOMENT, SKEWNESS, STANDARD DEVIATION

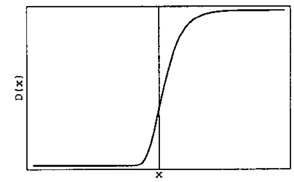
Fisher's Theorem

Let A be a sum of squares of n independent normal standardized variates x_i, and suppose $A = B + C$ where B is a quadratic form in the x_i, distributed as CHI-SQUARED with h DEGREES OF FREEDOM. Then C is distributed as χ^2 with $n - h$ DEGREES OF FREEDOM and is independent of B. The converse of this theorem is known as COCHRAN'S THEOREM.

see also CHI-SQUARED DISTRIBUTION, COCHRAN'S THEOREM

Fisher-Tippett Distribution





Also called the EXTREME VALUE DISTRIBUTION and LOG-WEIBULL DISTRIBUTION. It is the limiting distribution for the smallest or largest values in a large sample drawn from a variety of distributions.

$$P(x) = \frac{e^{(a-x)/b - e^{(a-x)/b}}}{b} \tag{1}$$

$$D(x) = e^{-e^{(a-x)/b}}. \tag{2}$$

These can be computed directly be defining

$$z \equiv \exp\left(\frac{a-x}{b}\right) \tag{3}$$

$$x = a - b\ln z \tag{4}$$

$$dz = -\frac{1}{b}\exp\left(\frac{a-x}{b}\right)\,dx. \tag{5}$$

Then the MOMENTS are

$$\begin{aligned}
\mu_n &\equiv \int_{-\infty}^{\infty} x^n P(x)\,dx \\
&= \frac{1}{b}\int_{-\infty}^{\infty} x^n \exp\left(\frac{a}{x-b}\right)\exp[-e^{(a-x)/b}]\,dx \\
&= -\int_{\infty}^{0} (a - b\ln z)^n e^{-z}\,dz \\
&= \int_0^{\infty} (a - b\ln z)^n e^{-z}\,dz \\
&= \sum_{k=0}^{n} \binom{n}{k}(-1)^k a^{n-k} b^k \int_0^{\infty} (\ln z)^k e^{-z}\,dz \\
&= \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k I(k),
\end{aligned} \tag{6}$$

where $I(k)$ are EULER-MASCHERONI INTEGRALS. Plugging in the EULER-MASCHERONI INTEGRALS $I(k)$ gives

$$\mu_0 = 1 \tag{7}$$

$$\mu_1 = a + b\gamma \tag{8}$$

$$\mu_2 = a^2 + 2ab\gamma + b^2(\gamma^2 + \tfrac{1}{6}\pi^2) \tag{9}$$

$$\begin{aligned}\mu_3 &= a^3 + 3a^2b\gamma + 3ab^2(\gamma^2 + \tfrac{1}{6}\pi^2) \\ &\quad + b^3[\gamma^3 + \tfrac{1}{2}\gamma\pi^2 + 2\zeta(3)]\end{aligned} \tag{10}$$

$$\begin{aligned}\mu_4 &= a^4 + 4a^3b\gamma + 6a^2b^2(\gamma^2 + \tfrac{1}{6}\pi^2) \\ &\quad + 4ab^3[\gamma^3 + \tfrac{1}{2}\gamma\pi^2 + 2\zeta(3)] \\ &\quad + b^4[\gamma^4 + \gamma^2\pi^2 + \tfrac{3}{20}\pi^4 + 8\gamma\zeta(3)],\end{aligned} \tag{11}$$

where γ is the EULER-MASCHERONI CONSTANT and $\zeta(3)$ is APÉRY'S CONSTANT. The MEAN, VARIANCE, SKEWNESS, and KURTOSIS are therefore

$$\mu = a + b\gamma \tag{12}$$

$$\sigma^2 = \mu_2 - \mu_1{}^2 = \tfrac{1}{6}\pi^2 b^2 \tag{13}$$

$$\begin{aligned}\gamma_1 &= \frac{\mu_3}{\sigma^3} \\ &= \frac{6\sqrt{6}}{b^3\pi^3}\{a^3 + 3a^2b\gamma + 3ab^2(\gamma^2 + \tfrac{1}{6}\pi^2) \\ &\quad + b^3[\gamma^3 + \tfrac{1}{2}\gamma\pi^2 + 2\zeta(3)]\}\end{aligned} \tag{14}$$

$$\begin{aligned}\gamma_2 &= \frac{\mu_4}{\sigma^4} - 3 \\ &= \frac{36}{b^4\pi^4}\{a^4 + 4a^3b\gamma + a^2b^2(6\gamma^2 + \pi^2) \\ &\quad + 4ab^3[\gamma^3 + \tfrac{1}{2}\gamma\pi^2 + 2\zeta(3)] \\ &\quad + b^4[\gamma^4 + \gamma^2\pi^2 + \tfrac{3}{20}\pi^4 + 8\gamma\zeta(3)]\}.\end{aligned} \tag{15}$$

The CHARACTERISTIC FUNCTION is

$$\phi(t) = \Gamma(1 - i\beta t)e^{i\alpha t}, \tag{16}$$

where $\Gamma(z)$ is the GAMMA FUNCTION. The special case of the Fisher-Tippett distribution with $a = 0$, $b = 1$ is called GUMBEL'S DISTRIBUTION.

see also EULER-MASCHERONI INTEGRALS, GUMBEL'S DISTRIBUTION

Fisher's z-Distribution

$$g(z) = \frac{2n_1{}^{n_1/2}n_2{}^{n_2/2}}{B\left(\frac{n_1}{2}, \frac{n_2}{2}\right)} \frac{e^{n_1 z}}{(n_1 e^{2z} + n_2)^{(n_1+n_1)/2}} \tag{1}$$

(Kenney and Keeping 1951). This general distribution includes the CHI-SQUARED DISTRIBUTION and STUDENT'S t-DISTRIBUTION as special cases. Let u^2 and v^2 be INDEPENDENT UNBIASED ESTIMATORS of the VARIANCE of a NORMALLY DISTRIBUTED variate. Define

$$z \equiv \ln\left(\frac{u}{v}\right) = \tfrac{1}{2}\ln\left(\frac{u^2}{v^2}\right). \tag{2}$$

Then let

$$F \equiv \frac{u^2}{v^2} = \frac{\frac{Ns_1{}^2}{n_1}}{\frac{Ns_2{}^2}{n_2}} \tag{3}$$

so that n_1F/n_2 is a ratio of CHI-SQUARED variates

$$\frac{n_1 F}{n_2} = \frac{\chi^2(n_1)}{\chi^2(n_2)}, \tag{4}$$

which makes it a ratio of GAMMA DISTRIBUTION variates, which is itself a BETA PRIME DISTRIBUTION variate,

$$\frac{\gamma\left(\frac{n_1}{2}\right)}{\gamma\left(\frac{n_2}{2}\right)} = \beta'\left(\tfrac{n_1}{2}, \tfrac{n_2}{2}\right), \tag{5}$$

giving

$$f(F) = \frac{\left(\frac{n_1 F}{n_2}\right)^{n_1/2-1}\left(1 + \frac{n_1 F}{n_2}\right)^{-(n_1+n_2)/2}\frac{n_1}{n_2}}{B\left(\frac{n_1}{2}, \frac{n_2}{2}\right)}. \tag{6}$$

The MEAN is

$$\langle F \rangle = \frac{n_2}{n_2 - 2}, \tag{7}$$

and the MODE is

$$\frac{n_2}{n_2 + 2}\frac{n_1 - 2}{n_1}. \tag{8}$$

see also BETA DISTRIBUTION, BETA PRIME DISTRIBUTION, CHI-SQUARED DISTRIBUTION, GAMMA DISTRIBUTION, NORMAL DISTRIBUTION, STUDENT'S t-DISTRIBUTION

References

Kenney, J. F. and Keeping, E. S. *Mathematics of Statistics, Pt. 2, 2nd ed.* Princeton, NJ: Van Nostrand, pp. 180–181, 1951.

Fisher's z'-Transformation

Let r be the CORRELATION COEFFICIENT. Then defining

$$z' \equiv \tanh^{-1} r \tag{1}$$

$$\zeta \equiv \tanh^{-1}\rho, \tag{2}$$

gives

$$\sigma_{z'} = (N - 3)^{-1/2} \tag{3}$$

$$\mathrm{var}(z') = \frac{1}{n} + \frac{4 - \rho^2}{2n^2} + \ldots \tag{4}$$

$$\gamma_1 = \frac{\rho\left|\rho^2 - \frac{9}{16}\right|}{n^{3/2}} \tag{5}$$

$$\gamma_2 = \frac{32 - 3\rho^4}{16N}, \tag{6}$$

where $n \equiv N - 1$.

see also CORRELATION COEFFICIENT

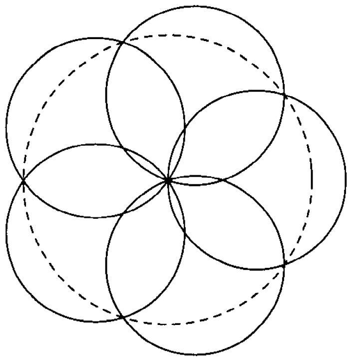
Fitting Subgroup

The unique smallest NORMAL NILPOTENT SUBGROUP of H, denoted $F(H)$. The generalized fitting subgroup is defined by $F^*(H) = F(H)E(H)$, where $E(H)$ is the commuting product of all components of H, and F is the fitting subgroup of H.

Five Cubes

see CUBE 5-COMPOUND

Five Disks Problem



Given five *equal* DISKS placed *symmetrically* about a given center, what is the smallest RADIUS r for which the RADIUS of the circular AREA covered by the five disks is 1? The answer is $r = \phi - 1 = 1/\phi = 0.6180340\ldots$, where ϕ is the GOLDEN RATIO, and the centers c_i of the disks $i = 1, \ldots, 5$ are located at

$$c_i = \begin{bmatrix} \frac{1}{\phi}\cos\left(\frac{2\pi i}{5}\right) \\ \frac{1}{\phi}\sin\left(\frac{2\pi i}{5}\right) \end{bmatrix}.$$

The GOLDEN RATIO enters here through its connection with the regular PENTAGON. If the requirement that the disks be symmetrically placed is dropped (the general DISK COVERING PROBLEM), then the RADIUS for $n = 5$ disks can be reduced slightly to $0.609383\ldots$ (Neville 1915).

see also ARC, DISK COVERING PROBLEM, FLOWER OF LIFE, SEED OF LIFE

References

Ball, W. W. R. and Coxeter, H. S. M. "The Five-Disc Problem." In *Mathematical Recreations and Essays, 13th ed.* New York: Dover, pp. 97–99, 1987.

Neville, E. H. "On the Solution of Numerical Functional Equations, Illustrated by an Account of a Popular Puzzle and of its Solution." *Proc. London Math. Soc.* **14**, 308–326, 1915.

Five Tetrahedra Compound

see TETRAHEDRON 5-COMPOUND

Fixed

When referring to a planar object, "fixed" means that the object is regarded as fixed in the plane so that it may not be picked up and flipped. As a result, MIRROR IMAGES are not necessarily equivalent for fixed objects.

see also FREE, MIRROR IMAGE

Fixed Element

see FIXED POINT (MAP)

Fixed Point

A point which does not change upon application of a MAP, system of DIFFERENTIAL EQUATIONS, etc.

see also FIXED POINT (DIFFERENTIAL EQUATIONS), FIXED POINT (MAP), FIXED POINT THEOREM

References

Shashkin, Yu. A. *Fixed Points.* Providence, RI: Amer. Math. Soc., 1991.

Fixed Point (Differential Equations)

Points of an AUTONOMOUS system of ordinary differential equations at which

$$\begin{cases} \frac{dx_1}{dt} = f_1(x_1, \ldots, x_n) = 0 \\ \vdots \\ \frac{dx_n}{dt} = f_n(x_1, \ldots, x_n) = 0. \end{cases}$$

If a variable is slightly displaced from a FIXED POINT, it may (1) move back to the fixed point ("asymptotically stable" or "superstable"), (2) move away ("unstable"), or (3) move in a neighborhood of the fixed point but not approach it ("stable" but not "asymptotically stable"). Fixed points are also called CRITICAL POINTS or EQUILIBRIUM POINTS. If a variable starts at a point that is not a CRITICAL POINT, it cannot reach a critical point in a finite amount of time. Also, a trajectory passing through at least one point that is not a CRITICAL POINT cannot cross itself unless it is a CLOSED CURVE, in which case it corresponds to a periodic solution.

A fixed point can be classified into one of several classes using LINEAR STABILITY analysis and the resulting STABILITY MATRIX.

see also ELLIPTIC FIXED POINT (DIFFERENTIAL EQUATIONS), HYPERBOLIC FIXED POINT (DIFFERENTIAL EQUATIONS), STABLE IMPROPER NODE, STABLE NODE, STABLE SPIRAL POINT, STABLE STAR, UNSTABLE IMPROPER NODE, UNSTABLE NODE, UNSTABLE SPIRAL POINT, UNSTABLE STAR

Fixed Point (Map)

A point x^* which is mapped to itself under a MAP G, so that $x^* = G(x^*)$. Such points are sometimes also called INVARIANT POINTS, or FIXED ELEMENTS (Woods 1961). Stable fixed points are called elliptical. Unstable fixed points, corresponding to an intersection of a stable and unstable invariant MANIFOLD, are called HYPERBOLIC (or SADDLE). Points may also be called asymptotically stable (a.k.a. superstable).

see also CRITICAL POINT, INVOLUNTARY

References

Shashkin, Yu. A. *Fixed Points.* Providence, RI: Amer. Math. Soc., 1991.

Woods, F. S. *Higher Geometry: An Introduction to Advanced Methods in Analytic Geometry.* New York: Dover, p. 14, 1961.

Fixed Point Theorem

If g is a continuous function $g(x) \in [a,b]$ FOR ALL $x \in [a,b]$, then g has a FIXED POINT in $[a,b]$. This can be proven by noting that

$$g(a) \geq a \qquad g(b) \leq b$$

$$g(a) - a \geq 0 \qquad g(b) - b \leq 0.$$

Since g is continuous, the INTERMEDIATE VALUE THEOREM guarantees that there exists a $c \in [a,b]$ such that

$$g(c) - c = 0,$$

so there must exist a c such that

$$g(c) = c,$$

so there must exist a FIXED POINT $\in [a,b]$.

see also BANACH FIXED POINT THEOREM, BROUWER FIXED POINT THEOREM, KAKUTANI'S FIXED POINT THEOREM, LEFSHETZ FIXED POINT FORMULA, LEFSHETZ TRACE FORMULA, POINCARÉ-BIRKHOFF FIXED POINT THEOREM, SCHAUDER FIXED POINT THEOREM

Fixed Point (Transformation)

see FIXED POINT (MAP)

Flag

A collection of FACES of an n-D POLYTOPE or simplicial COMPLEX, one of each DIMENSION $0, 1, \ldots, n-1$, which all have a common nonempty INTERSECTION. In normal 3-D, the flag consists of a half-plane, its bounding RAY, and the RAY's endpoint.

Flag Manifold

For any SEQUENCE of INTEGERS $0 < n_1 < \ldots < n_k$, there is a flag manifold of type $(n_1, \ldots, n_k)$ which is the collection of ordered pairs of vector SUBSPACES of $\mathbb{R}^{n_k}$ $(V_1, \ldots, V_k)$ with $\dim(V_i) = n_i$ and V_i a SUBSPACE of V_{i+1}. There are also COMPLEX flag manifolds with COMPLEX subspaces of $\mathbb{C}^{n_k}$ instead of REAL SUBSPACES of a REAL n_k-space. These flag manifolds admit the structure of MANIFOLDS in a natural way and are used in the theory of LIE GROUPS.

see also GRASSMANN MANIFOLD

References

Lu, J.-H. and Weinstein, A. "Poisson Lie Groups, Dressing Transformations, and the Bruhat Decomposition." *J. Diff. Geom.* **31**, 501–526, 1990.

Flat

A set in $\mathbb{R}^d$ formed by translating an affine subspace or by the intersection of a set of HYPERPLANES.

Flat Norm

The flat norm on a CURRENT is defined by

$$\mathcal{F}(S) = \int \{\text{Area } T + \text{vol } R : S - T = \partial R\},$$

where ∂R is the boundary of R.

see also COMPACTNESS THEOREM, CURRENT

References

Morgan, F. "What Is a Surface?" *Amer. Math. Monthly* **103**, 369–376, 1996.

Flat Space Theorem

If it is possible to transform a coordinate system to a form where the metric elements $g_{\mu\nu}$ are constants independent of x^μ, then the space is flat.

Flat Surface

A REGULAR SURFACE and special class of MINIMAL SURFACE for the GAUSSIAN CURVATURE vanishes everywhere. A TANGENT DEVELOPABLE, GENERALIZED CONE, and GENERALIZED CYLINDER are all flat surfaces.

see also MINIMAL SURFACE

References

Gray, A. *Modern Differential Geometry of Curves and Surfaces.* Boca Raton, FL: CRC Press, p. 280, 1993.

Flattening

The flattening of a SPHEROID (also called OBLATENESS) is denoted ε or f. It is defined as

$$\varepsilon \equiv \begin{cases} \frac{a-c}{a} = 1 - \frac{c}{a} & \text{oblate} \\ \frac{c-a}{a} = \frac{c}{a} - 1 & \text{prolate,} \end{cases}$$

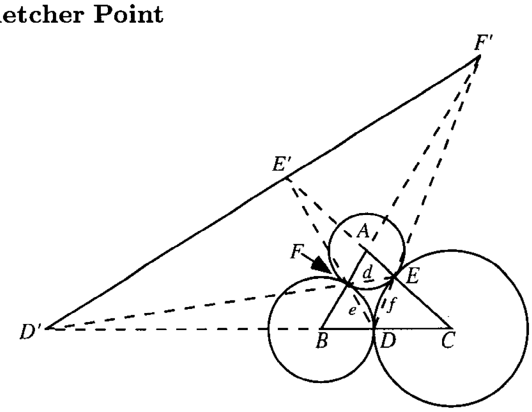
where c is the polar RADIUS and a is the equatorial RADIUS.

see also ECCENTRICITY, ELLIPSOID, OBLATE SPHEROID, PROLATE SPHEROID, SPHEROID

Flemish Knot

see FIGURE-OF-EIGHT KNOT

Fletcher Point



The intersection of the GERGONNE LINE and the SODDY LINE. It has TRILINEAR COORDINATES given by

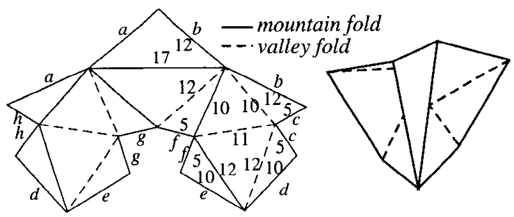
$$Fl = I - \frac{1}{3}\left(\frac{1}{d} + \frac{1}{e} + \frac{1}{f}\right) Ge,$$

where I is the INCENTER, Ge the GERGONNE POINT, and d, e, and f are the lengths of the sides of the CONTACT TRIANGLE ΔDEF.

see also CONTACT TRIANGLE, GERGONNE LINE, GERGONNE POINT, SODDY LINE

References
Oldknow, A. "The Euler-Gergonne-Soddy Triangle of a Triangle." *Amer. Math. Monthly* **103**, 319–329, 1996.

Flexible Polyhedron



The RIGIDITY THEOREM states that if the faces of a *convex* POLYHEDRON are made of metal plates and the EDGES are replaced by hinges, the POLYHEDRON would be RIGID. The theorem was stated by Cauchy (1813), although a mistake in this paper went unnoticed for more than 50 years. Concave polyhedra need not be RIGID, and such nonrigid polyhedra are called flexible polyhedra. Connelly (1978) found the first example of a reflexible polyhedron, consisting of 18 triangular faces. A flexible polyhedron with only 14 triangular faces and 9 vertices (shown above), believed to be the simplest possible composed of only triangles, was subsequently found by Steffen (Mackenzie 1998). There also exists a six-vertex eight-face flexible polyhedron (Wunderlich and Schwabe 1986, Cromwell 1997).

Connelly *et al.* (1997) proved that a flexible polyhedron must keep its VOLUME constant (Mackenzie 1998).

see also POLYHEDRON, QUADRICORN, RIGID, RIGIDITY THEOREM

References
Cauchy, A. L. "Sur les polygons et le polyhéders." *XVIe Cahier* **IX**, 87–89, 1813.
Connelly, R. "A Flexible Sphere." *Math. Intel.* **1**, 130–131, 1978.
Connelly, R.; Sabitov, I.; and Walz, A. "The Bellows Conjecture." *Contrib. Algebra Geom.* **38**, 1–10, 1997.
Cromwell, P. R. *Polyhedra.* New York: Cambridge University Press, 1997.
Mackenzie, D. "Polyhedra Can Bend But Not Breathe." *Science* **279**, 1637, 1998.
Wunderlich, W. and Schwabe, C. "Eine Familie von geschlossen gleichflachigen Polyhedern, die fast beweglich sind." *Elem. Math.* **41**, 88–98, 1986.

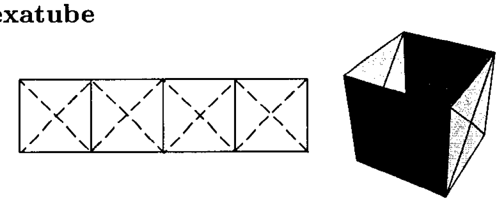
Flexagon

An object created by FOLDING a piece of paper along certain lines to form loops. The number of states possible in an n-FLEXAGON is a CATALAN NUMBER. By manipulating the folds, it is possible to hide and reveal different faces.

see also FLEXATUBE, FOLDING, HEXAFLEXAGON, TETRAFLEXAGON

References
Crampin, J. "On Note 2449." *Math. Gazette* **41**, 55–56, 1957.
Cundy, H. and Rollett, A. *Mathematical Models, 3rd ed.* Stradbroke, England: Tarquin Pub., pp. 205–207, 1989.
Madachy, J. S. *Madachy's Mathematical Recreations.* New York: Dover, pp. 62–84, 1979.
Gardner, M. "Hexaflexagons." Ch. 1 in *The Scientific American Book of Mathematical Puzzles & Diversions.* New York: Simon and Schuster, 1959.
Gardner, M. Ch. 2 in *The Second Scientific American Book of Mathematical Puzzles & Diversions: A New Selection.* New York: Simon and Schuster, pp. 24–31, 1961.
Maunsell, F. G. "The Flexagon and the Hexaflexagon." *Math. Gazette* **38**, 213–214, 1954.
Oakley, C. O. and Wisner, R. J. "Flexagons." *Amer. Math. Monthly* **64**, 143–154, 1957.
Wheeler, R. F. "The Flexagon Family." *Math. Gaz.* **42**, 1–6, 1958.

Flexatube



A FLEXAGON-like structure created by connecting the ends of a strip of four squares after folding along 45° diagonals. Using a number of folding movements, it is possible to flip the flexatube inside out so that the faces originally facing inward face outward. Gardner (1961) illustrated one possible solution, and Steinhaus (1983) gives a second.

see also FLEXAGON, HEXAFLEXAGON, TETRAFLEXAGON

References
Cundy, H. and Rollett, A. *Mathematical Models, 3rd ed.* Stradbroke, England: Tarquin Pub., p. 205, 1989.
Gardner, M. *The Second Scientific American Book of Mathematical Puzzles & Diversions: A New Selection.* New York: Simon and Schuster, pp. 29–31, 1961.
Steinhaus, H. *Mathematical Snapshots, 3rd American ed.* New York: Oxford University Press, pp. 177–181 and 190, 1983.

Flip Bifurcation

Let $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be a one-parameter family of C^3 maps satisfying

$$f(0,0) = 0$$

$$\left[\frac{\partial f}{\partial x}\right]_{\mu=0,x=0} = -1$$

$$\left[\frac{\partial^2 f}{\partial x^2}\right]_{\mu=0,x=0} < 0$$
$$\left[\frac{\partial^3 f}{\partial x^3}\right]_{\mu=0,x=0} < 0.$$

Then there are intervals $(\mu_1, 0)$, $(0, \mu_2)$, and $\epsilon > 0$ such that

1. If $\mu \in (0, \mu_2)$, then $f_\mu(x)$ has one unstable fixed point and one stable orbit of period two for $x \in (-\epsilon, \epsilon)$, and
2. If $\mu \in (\mu_1, 0)$, then $f_\mu(x)$ has a single stable fixed point for $x \in (-\epsilon, \epsilon)$.

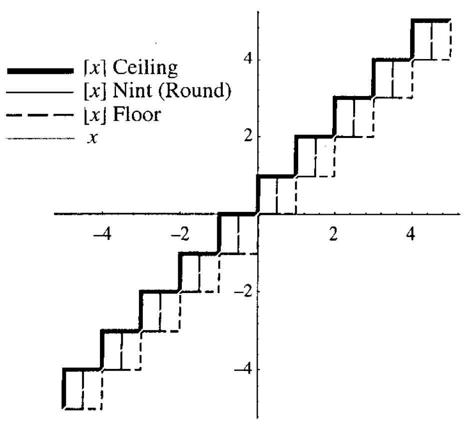
This type of BIFURCATION is known as a flip bifurcation. An example of an equation displaying a flip bifurcation is

$$f(x) = \mu - x - x^2.$$

see also BIFURCATION

References
Rasband, S. N. *Chaotic Dynamics of Nonlinear Systems.* New York: Wiley, pp. 27–30, 1990.

Floor Function



The function $\lfloor x \rfloor$ is the largest INTEGER $\leq x$, shown as the dashed curve in the above plot, and also called the GREATEST INTEGER FUNCTION. In many computer languages, the floor function is called the INTEGER PART function and is denoted `int(x)`. The name and symbol for the floor function were coined by K. E. Iverson (Graham *et al.* 1990).

Unfortunately, in many older and current works (e.g., Shanks 1993, Ribenboim 1996), the symbol $[x]$ is used instead of $\lfloor x \rfloor$. Because of the elegant symmetry of the floor function and CEILING FUNCTION symbols $\lfloor x \rfloor$ and $\lceil x \rceil$, and because $[x]$ is such a useful symbol when interpreted as an IVERSON BRACKET, the use of $[x]$ to denote the floor function should be deprecated. In this work, the symbol $[x]$ is used to denote the nearest integer NINT function since it naturally falls between the $\lfloor x \rfloor$ and $\lceil x \rceil$ symbols.

see also CEILING FUNCTION, FRACTIONAL PART, INT, IVERSON BRACKET, NINT

References
Graham, R. L.; Knuth, D. E.; and Patashnik, O. "Integer Functions." Ch. 3 in *Concrete Mathematics: A Foundation for Computer Science.* Reading, MA: Addison-Wesley, pp. 67–101, 1990.
Iverson, K. E. *A Programming Language.* New York: Wiley, p. 12, 1962.
Ribenboim, P. *The New Book of Prime Number Records.* New York: Springer-Verlag, pp. 180–182, 1996.
Shanks, D. *Solved and Unsolved Problems in Number Theory, 4th ed.* New York: Chelsea, p. 14, 1993.
Spanier, J. and Oldham, K. B. "The Integer-Value Int(x) and Fractional-Value frac(x) Functions." Ch. 9 in *An Atlas of Functions.* Washington, DC: Hemisphere, pp. 71–78, 1987.

Floquet Analysis

Given a system of periodic ORDINARY DIFFERENTIAL EQUATIONS of the form

$$\frac{d}{dt}\begin{bmatrix} x \\ y \\ v_x \\ v_y \end{bmatrix} = -\begin{bmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ \Phi_{xx} & \Phi_{yy} & 0 & 0 \\ \Phi_{xy} & \Phi_{yy} & 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ v_x \\ v_y \end{bmatrix}, \tag{1}$$

the solution can be written as a linear combination of functions of the form

$$\begin{bmatrix} x(t) \\ y(t) \\ v_x(t) \\ v_y(t) \end{bmatrix} = \begin{bmatrix} x_0 \\ y_0 \\ v_{x0} \\ v_{y0} \end{bmatrix} e^{\mu t} P_\mu(t), \tag{2}$$

where $P_\mu(t)$ is a function periodic with the same period T as the equations themselves. Given an ORDINARY DIFFERENTIAL EQUATION of the form

$$\ddot{x} + g(t)x = 0, \tag{3}$$

where $g(t)$ is periodic with period T, the ODE has a pair of independent solutions given by the REAL and IMAGINARY PARTS of

$$x(t) = w(t)e^{i\psi(t)} \tag{4}$$
$$\dot{x} = (\dot{w} + iw\dot{\psi})e^{i\psi} \tag{5}$$
$$\begin{aligned} \ddot{x} &= [\ddot{w} + i\dot{w}\dot{\psi} + i(\dot{w}\dot{\psi} + w\ddot{\psi} + iw\dot{\psi}^2)]e^{i\psi} \\ &= [(\ddot{w} - w\dot{\psi}^2) + i(2\dot{w}\dot{\psi} + w\ddot{\psi})]e^{i\psi}. \end{aligned} \tag{6}$$

Plugging these into (3) gives

$$\ddot{w} + 2i\dot{w}\dot{\psi} + w(g + i\ddot{\psi} - \dot{\psi}^2) = 0, \tag{7}$$

so the REAL and IMAGINARY PARTS are

$$\ddot{w} + w(g - \dot{\psi}^2) = 0 \tag{8}$$

$$2\dot{w}\dot{\psi} + w\ddot{\psi} = 0. \tag{9}$$

From (9),

$$\begin{aligned} \frac{2\dot{w}}{w} + \frac{\ddot{\psi}}{\dot{\psi}} &= 2\frac{d}{dt}(\ln w) + \frac{d}{dt}[\ln(\dot{\psi})] \\ &= \frac{d}{dt}\ln(\dot{\psi}w^2) = 0. \end{aligned} \tag{10}$$

Integrating gives

$$\dot{\psi} = \frac{C}{w^2}, \tag{11}$$

where C is a constant which must equal 1, so ψ is given by

$$\psi = \int_{t_0}^{t} \frac{dt}{w^2}. \tag{12}$$

The REAL solution is then

$$x(t) = w(t)\cos[\psi(t)], \tag{13}$$

so

$$\begin{aligned}\dot{x} &= \dot{w}\cos\psi - w\dot{\psi}\sin\psi = \dot{w}\frac{x}{w} - w\dot{\psi}\sin\psi \\ &= \dot{w}\frac{x}{w} - w\frac{1}{w^2}\sin\psi = \dot{w}\frac{x}{w} - \frac{1}{w}\sin\psi \end{aligned} \tag{14}$$

and

$$\begin{aligned}1 &= \cos^2\psi + \sin^2\psi = x^2 w^{-2} + \left[w\left(\dot{w}\frac{x}{w} - \dot{x}\right)\right]^2 \\ &= x^2 w^{-2} + (\dot{w}x - w\dot{x})^2 \equiv I(x, \dot{x}, t), \end{aligned} \tag{15}$$

which is an integral of motion. Therefore, although $w(t)$ is not explicitly known, an integral I always exists. Plugging (10) into (8) gives

$$\ddot{w} + g(t)w - \frac{1}{w^3} = 0, \tag{16}$$

which, however, is not any easier to solve than (3).

References
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Morse, P. M. and Feshbach, H. *Methods of Theoretical Physics, Part I.* New York: McGraw-Hill, pp. 556–557, 1953.

Floquet's Theorem

see FLOQUET ANALYSIS

Flow

An ACTION with $G = \mathbb{R}$. Flows are generated by VECTOR FIELDS and vice versa.

see also ACTION, AMBROSE-KAKUTANI THEOREM, ANOSOV FLOW, AXIOM A FLOW, CASCADE, GEODESIC FLOW, SEMIFLOW

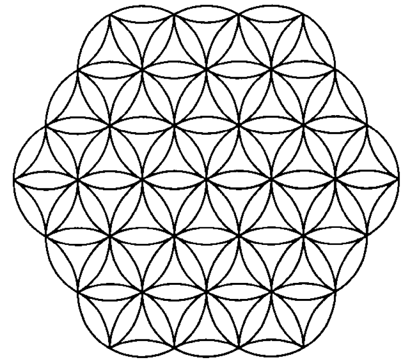
Flow Line

A flow line for a map on a VECTOR FIELD $\mathbf{F}$ is a path $\sigma(t)$ such that $\sigma'(t) = \mathbf{F}(\sigma(t))$.

Flower

see DAISY, FLOWER OF LIFE, ROSE

Flower of Life



One of the beautiful arrangements of CIRCLES found at the Temple of Osiris at Abydos, Egypt (Rawles 1997). The CIRCLES are placed with six-fold symmetry, forming a mesmerizing pattern of CIRCLES and LENSES.

see also FIVE DISKS PROBLEM, REULEAUX TRIANGLE, SEED OF LIFE, VENN DIAGRAM

References
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Wein, J. "The Flower of Life." `http://www2.cruzio.com/~flower`.
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Flowsnake

see PEANO-GOSPER CURVE

Flowsnake Fractal

see GOSPER ISLAND

Floyd's Algorithm

An algorithm for finding the shortest path between two VERTICES.

see also DIJKSTRA'S ALGORITHM

Fluent

Newton's term for a variable in his method of FLUXIONS (differential calculus).

Fluxion

The term for DERIVATIVE in Newton's CALCULUS.

Flype

A 180° rotation of a TANGLE.

see also FLYPING CONJECTURE, TANGLE

Flyping Conjecture

Also called the TAIT FLYPING CONJECTURE. Given two reduced alternating projections of the same knot, they are equivalent on the SPHERE IFF they are related by a series of FLYPES. It was proved by Menasco and Thistlethwaite (1991). It allows all possible REDUCED alternating projections of a given ALTERNATING KNOT to be drawn.

References
Adams, C. C. *The Knot Book: An Elementary Introduction to the Mathematical Theory of Knots.* New York: W. H. Freeman, pp. 164–165, 1994.
Menasco, W. and Thistlethwaite, M. "The Tait Flyping Conjecture." *Bull. Amer. Math. Soc.* **25**, 403–412, 1991.
Stewart, I. *The Problems of Mathematics, 2nd ed.* Oxford, England: Oxford University Press, pp. 284–285, 1987.

Focus

A point related to the construction and properties of CONIC SECTIONS.

see also ELLIPSE, ELLIPSOID, HYPERBOLA, HYPERBOLOID, PARABOLA, PARABOLOID, REFLECTION PROPERTY

References
Coxeter, H. S. M. and Greitzer, S. L. *Geometry Revisited.* Washington, DC: Math. Assoc. Amer., pp. 141–144, 1967.

Fold Bifurcation

Let $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be a one-parameter family of C^2 MAP satisfying

$$f(0,0) = 0$$
$$\left[\frac{\partial f}{\partial x}\right]_{\mu=0,x=0} = 1$$
$$\left[\frac{\partial^2 f}{\partial x^2}\right]_{\mu=0,x=0} > 0$$
$$\left[\frac{\partial f}{\partial \mu}\right]_{\mu=0,x=0} > 0,$$

then there exist intervals $(\mu_1, 0)$, $(0, \mu_2)$ and $\epsilon > 0$ such that

1. If $\mu \in (\mu_1, 0)$, then $f_\mu(x)$ has two fixed points in $(-\epsilon, \epsilon)$ with the positive one being unstable and the negative one stable, and
2. If $\mu \in (0, \mu_2)$, then $f_\mu(x)$ has no fixed points in $(-\epsilon, \epsilon)$.

This type of BIFURCATION is known as a fold bifurcation, sometimes also called a SADDLE-NODE BIFURCATION or TANGENT BIFURCATION. An example of an equation displaying a fold bifurcation is

$$\dot{x} = \mu - x^2$$

(Guckenheimer and Holmes 1997, p. 145).

see also BIFURCATION

References
Guckenheimer, J. and Holmes, P. *Nonlinear Oscillations, Dynamical Systems, and Bifurcations of Vector Fields, 3rd ed.* New York: Springer-Verlag, pp. 145–149, 1997.
Rasband, S. N. *Chaotic Dynamics of Nonlinear Systems.* New York: Wiley, pp. 27–28, 1990.

Fold Catastrophe

A CATASTROPHE which can occur for one control factor and one behavior axis.

Folding

The points accessible from c by a single fold which leaves $a_1, \ldots, a_n$ fixed are exactly those points interior to or on the boundary of the intersection of the CIRCLES through c with centers at a_i, for $i = 1, \ldots, n$. Given any three points in the plane a, b, and c, there is an EQUILATERAL TRIANGLE with VERTICES x, y, and z for which a, b, and c are the images of x, y, and z under a single fold. Given any four points in the plane a, b, c, and d, there is some SQUARE with VERTICES x, y, z, and w for which a, b, c, and d are the images of x, y, z, and w under a sequence of at most three folds. Also, any four collinear points are the images of the VERTICES of a suitable SQUARE under at most two folds. Every five (six) points are the images of the VERTICES of suitable regular PENTAGON (HEXAGON) under at most five (six) folds. The least number of folds required for $n \geq 4$ is not known, but some bounds are. In particular, every set of n points is the image of a suitable REGULAR n-gon under at most $F(n)$ folds, where

$$F(n) \leq \begin{cases} \frac{1}{2}(3n-2) & \text{for } n \text{ even} \\ \frac{1}{2}(3n-3) & \text{for } n \text{ odd.} \end{cases}$$

The first few values are 0, 2, 3, 5, 6, 8, 9, 11, 12, 14, 15, 17, 18, 20, 21, ... (Sloane's A007494).

see also FLEXAGON, MAP FOLDING, ORIGAMI

References
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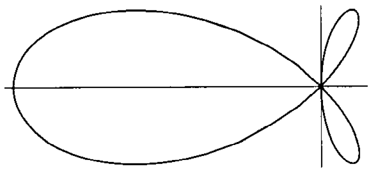
Foliation

Let M^n be an n-MANIFOLD and let $\mathsf{F} = \{F_\alpha\}$ denote a PARTITION of M into DISJOINT path-connected SUBSETS. Then F is called a foliation of M of codimension c (with $0 < c < n$) if there EXISTS a COVER of M by OPEN SETS U, each equipped with a HOMEOMORPHISM $h : U \to \mathbb{R}^n$ or $h : U \to \mathbb{R}^n_+$ which throws each nonempty component of $F_\alpha \cap U$ onto a parallel translation of the standard HYPERPLANE $\mathbb{R}^{n-c}$ in $\mathbb{R}^n$. Each F_α is then called a LEAF and is not necessarily closed or compact.

see also LEAF (FOLIATION), REEB FOLIATION

References
Rolfsen, D. *Knots and Links.* Wilmington, DE: Publish or Perish Press, p. 284, 1976.

Folium



The word "folium" means leaf-shaped. The polar equation is

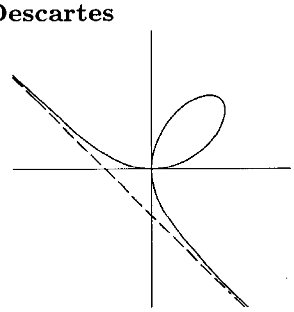
$$r = \cos\theta(4a\sin^2\theta - b).$$

If $b \geq 4a$, it is a single folium. If $b = 0$, it is a BIFOLIUM. If $0 < b < 4a$, it is a TRIFOLIUM. The simple folium is the PEDAL CURVE of the DELTOID where the PEDAL POINT is one of the CUSPS.

see also BIFOLIUM, FOLIUM OF DESCARTES, KEPLER'S FOLIUM, QUADRIFOLIUM, ROSE, TRIFOLIUM

References
Lawrence, J. D. *A Catalog of Special Plane Curves.* New York: Dover, pp. 152–153, 1972.
MacTutor History of Mathematics Archive. "Folium." http://www-groups.dcs.st-and.ac.uk/~history/Curves/Folium.html.

Folium of Descartes

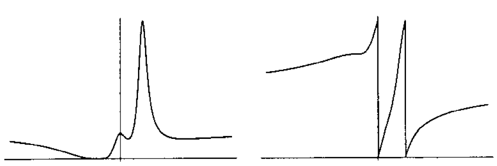


A plane curve proposed by Descartes to challenge Fermat's extremum-finding techniques. In parametric form,

$$x = \frac{3at}{1+t^3} \quad (1)$$

$$y = \frac{3at^2}{1+t^3}. \quad (2)$$

The curve has a discontinuity at $t = -1$. The left wing is generated as t runs from -1 to 0, the loop as t runs from 0 to ∞, and the right wing as t runs from $-\infty$ to -1.



The CURVATURE and TANGENTIAL ANGLE of the folium of Descartes, illustrated above, are

$$\kappa(t) = \frac{2(1+t^3)^4}{3(1+4t^2-4t^3-4t^5+4t^6+t^8)^{3/2}} \quad (3)$$

$$\phi(t) = \frac{1}{2}\left[\pi + \tan^{-1}\left(\frac{1-2t^3}{t^4-2t}\right) - \tan^{-1}\left(\frac{2t^3-1}{t^4-2t}\right)\right]. \quad (4)$$

Converting the parametric equations to POLAR COORDINATES gives

$$r^2 = \frac{(3at)^2(1+t^2)}{(1+t^3)^2} \quad (5)$$

$$\theta = \tan^{-1}\left(\frac{y}{x}\right) = \tan^{-1} t, \quad (6)$$

so

$$d\theta = \frac{dt}{1+t^2}. \quad (7)$$

The AREA enclosed by the curve is

$$A = \tfrac{1}{2}\int r^2\,d\theta = \tfrac{1}{2}\int_0^\infty \frac{(3at)^2(1+t^2)}{(1+t^3)^2}\frac{dt}{1+t^2} = \tfrac{3}{2}a^2\int_0^\infty \frac{3t^2\,dt}{(1+t^3)^2}. \quad (8)$$

Now let $u \equiv 1 + t^3$ so $du = 3t^2\,dt$

$$A = \tfrac{3}{2}a^2\int_1^\infty \frac{du}{u^2} = \tfrac{3}{2}a^2\left[-\frac{1}{u}\right]_1^\infty = \tfrac{3}{2}a^2(-0+1) = \tfrac{3}{2}a^2. \quad (9)$$

In CARTESIAN COORDINATES,

$$x^3 + y^3 = \frac{(3at)^3(1+t^3)}{(1+t^3)^3} = \frac{(3at)^3}{(1+t^3)^2} = 3axy \quad (10)$$

(MacTutor Archive). The equation of the ASYMPTOTE is

$$y = -a - x. \quad (11)$$

References
Gray, A. *Modern Differential Geometry of Curves and Surfaces.* Boca Raton, FL: CRC Press, pp. 59–62, 1993.
Lawrence, J. D. *A Catalog of Special Plane Curves.* New York: Dover, pp. 106–109, 1972.
MacTutor History of Mathematics Archive. "Folium of Descartes." http://www-groups.dcs.st-and.ac.uk/~history/Curves/Foliumd.html.
Stroeker, R. J. "Brocard Points, Circulant Matrices, and Descartes' Folium." *Math. Mag.* **61**, 172–187, 1988.
Yates, R. C. "Folium of Descartes." In *A Handbook on Curves and Their Properties.* Ann Arbor, MI: J. W. Edwards, pp. 98–99, 1952.

Follows

see SUCCEEDS

Fontené Theorems

1. If the sides of the PEDAL TRIANGLE of a point P meet the corresponding sides of a TRIANGLE $\Delta O_1O_2O_3$ at X_1, X_2, and X_3, respectively, then P_1X_1, P_2X_2, P_3X_3 meet at a point L common to the CIRCLES $O_1O_2O_3$ and $P_1P_2P_3$. In other words, L is one of the intersections of the NINE-POINT CIRCLE of $A_1A_2A_3$ and the PEDAL CIRCLE of P.
2. If a point moves on a fixed line through the CIRCUMCENTER, then its PEDAL CIRCLE passes through a fixed point on the NINE-POINT CIRCLE.

3. The PEDAL CIRCLE of a point is tangent to the NINE-POINT CIRCLE IFF the point and its ISOGONAL CONJUGATE lie on a LINE through the ORTHOCENTER. FEUERBACH'S THEOREM is a special case of this theorem.

see also CIRCUMCENTER, FEUERBACH'S THEOREM, ISOGONAL CONJUGATE, NINE-POINT CIRCLE, ORTHOCENTER, PEDAL CIRCLE

References
Johnson, R. A. *Modern Geometry: An Elementary Treatise on the Geometry of the Triangle and the Circle.* Boston, MA: Houghton Mifflin, pp. 245–247, 1929.

Foot

see PERPENDICULAR FOOT

For All

If a proposition P is true for all B, this is written $P\forall B$.

see also ALMOST ALL, EXISTS, QUANTIFIER

Forcing

A technique in SET THEORY invented by P. Cohen (1963, 1964, 1966) and used to prove that the AXIOM OF CHOICE and CONTINUUM HYPOTHESIS are independent of one another in ZERMELO-FRAENKEL SET THEORY.

see also AXIOM OF CHOICE, CONTINUUM HYPOTHESIS, SET THEORY, ZERMELO-FRAENKEL SET THEORY

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Cohen, P. J. "The Independence of the Continuum Hypothesis." *Proc. Nat. Acad. Sci. U. S. A.* **50**, 1143–1148, 1963.
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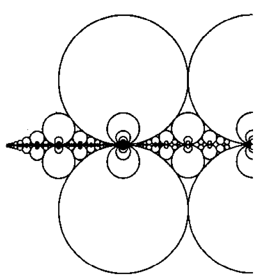
Ford Circle



Pick any two INTEGERS h and k, then the CIRCLE of RADIUS $1/(2k^2)$ centered at $(h/k, 1/(2k^2))$ is known as a Ford circle. No matter what and how many hs and ks are picked, none of the Ford circles intersect (and all are tangent to the x-AXIS). This can be seen by examining the squared distance between the centers of the circles with (h,k) and (h',k'),

$$d^2 = \left(\frac{h'}{k'} - \frac{h}{k}\right)^2 + \left(\frac{1}{2k'^2} - \frac{1}{2k^2}\right). \tag{1}$$

Let s be the sum of the radii

$$s = r_1 + r_2 = \frac{1}{2k^2} + \frac{1}{2k'^2}, \tag{2}$$

then

$$d^2 - s^2 = \frac{(h'k - hk')^2 - 1}{k^2k'^2}. \tag{3}$$

But $(h'k - k'h)^2 \geq 1$, so $d^2 - s^2 \geq 0$ and the distance between circle centers is $\geq$ the sum of the CIRCLE RADII, with equality (and therefore tangency) IFF $|h'k - k'h| = 1$. Ford circles are related to the FAREY SEQUENCE (Conway and Guy 1996).

see also ADJACENT FRACTION, FAREY SEQUENCE, STERN-BROCOT TREE

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Conway, J. H. and Guy, R. K. "Farey Fractions and Ford Circles." *The Book of Numbers.* New York: Springer-Verlag, pp. 152–154, 1996.
Ford, L. R. "Fractions." *Amer. Math. Monthly* **45**, 586–601, 1938.
Pickover, C. A. "Fractal Milkshakes and Infinite Archery." Ch. 14 in *Keys to Infinity.* New York: W. H. Freeman, pp. 117–125, 1995.
Rademacher, H. *Higher Mathematics from an Elementary Point of View.* Boston, MA: Birkhäuser, 1983.

Ford's Theorem

Let a, b, and k be INTEGERS with $k \geq 1$. For $j = 0, 1, 2$, let

$$S_j \equiv \sum_{\substack{i=0 \\ i\equiv j \pmod 3}} (-1)^j \binom{k}{i} a^{k-i} b^i.$$

Then

$$2(a^2 + ab + b^2)^{2k} = (S_0 - S_1)^4 + (S_1 - S_2)^4 + (S_2 - S_0)^4.$$

see also BHARGAVA'S THEOREM, DIOPHANTINE EQUATION—QUARTIC

References
Berndt, B. C. *Ramanujan's Notebooks, Part IV.* New York: Springer-Verlag, pp. 100–101, 1994.

Forest

A GRAPH without any CIRCUITS (CYCLES), which therefore consists only of TREES. A forest with k components and n nodes has $n - k$ EDGES.

Fork

see TREE

Form

see CANONICAL FORM, CUSP FORM, DIFFERENTIAL k-FORM, FORM (GEOMETRIC), FORM (POLYNOMIAL), MODULAR FORM, NORMAL FORM, PFAFFIAN FORM, QUADRATIC FORM

Form (Geometric)

A 1-D geometric object such as a PENCIL or RANGE.

Form (Polynomial)

A HOMOGENEOUS POLYNOMIAL in two or more variables.

see also DISCONNECTED FORM, k-FORM

Formal Logic

see SYMBOLIC LOGIC

Formosa Theorem

see CHINESE REMAINDER THEOREM

Formula

A mathematical equation or a formal logical expression. The correct Latin plural form of formula is "formulae," although the less pretentious-sounding "formulas" is used more commonly.

see also ARCHIMEDES' RECURRENCE FORMULA, BAYES' FORMULA, BENSON'S FORMULA, BESSEL'S FINITE DIFFERENCE FORMULA, BESSEL'S INTERPOLATION FORMULA, BESSEL'S STATISTICAL FORMULA, BINET'S FORMULA, BINOMIAL FORMULA, BRAHMAGUPTA'S FORMULA, BRENT-SALAMIN FORMULA, BRETSCHNEIDER'S FORMULA, BRIOSCHI FORMULA, CALDERÓN'S FORMULA, CARDANO'S FORMULA, CAUCHY'S FORMULA, CAUCHY'S COSINE INTEGRAL FORMULA, CAUCHY INTEGRAL FORMULA, CHASLES-CAYLEY-BRILL FORMULA, CHEBYSHEV APPROXIMATION FORMULA, CHRISTOFFEL-DARBOUX FORMULA, CHRISTOFFEL FORMULA, CLAUSEN FORMULA, CLENSHAW RECURRENCE FORMULA, DESCARTES-EULER POLYHEDRAL FORMULA, DESCARTES' FORMULA, DIRICHLET'S FORMULA, DIXON-FERRAR FORMULA, DOBIŃSKI'S FORMULA, DUPLICATION FORMULA, ENNEPER-WEIERSTRAß PARAMETERIZATION, EULER CURVATURE FORMULA, EULER FORMULA, EULER-MACLAURIN INTEGRATION FORMULAS, EULER POLYHEDRAL FORMULA, EULER TRIANGLE FORMULA, EVERETT'S FORMULA, EXPONENTIAL SUM FORMULAS, FAULHABER'S FORMULA, FRENET FORMULAS, GAUSS'S BACKWARD FORMULA, GAUSS-BONNET FORMULA, GAUSS'S FORMULA, GAUSS'S FORWARD FORMULA, GAUSS MULTIPLICATION FORMULA, GAUSS-SALAMIN FORMULA, GIRARD'S SPHERICAL EXCESS FORMULA, GOODMAN'S FORMULA, GREGORY'S FORMULA, GRENZ-FORMEL, GRINBERG FORMULA, HALLEY'S IRRATIONAL FORMULA, HALLEY'S RATIONAL FORMULA, HANSEN-BESSEL FORMULA, HERON'S FORMULA, HOOK LENGTH FORMULA, JACOBI ELLIPTIC FUNCTIONS, JENSEN'S FORMULA, JONAH FORMULA, KAC FORMULA, KNESER-SOMMERFELD FORMULA, KUMMER'S FORMULAS, LAISANT'S RECURRENCE FORMULA, LANDEN'S FORMULA, LEFSHETZ FIXED POINT FORMULA, LEFSHETZ TRACE FORMULA, LEGENDRE DUPLICATION FORMULA, LEGENDRE'S FORMULA, LEHMER'S FORMULA, LICHNEROWICZ FORMULA, LICHNEROWICZ-WEITZENBOCK FORMULA, LOBACHEVSKY'S FORMULA, LOGARITHMIC BINOMIAL FORMULA, LUDWIG'S INVERSION FORMULA, MACHIN'S FORMULA, MACHIN-LIKE FORMULAS, MEHLER'S BESSEL FUNCTION FORMULA, MEHLER'S HERMITE POLYNOMIAL FORMULA, MEISSEL'S FORMULA, MENSURATION FORMULA, MÖBIUS INVERSION FORMULA, MORLEY'S FORMULA, NEWTON'S BACKWARD DIFFERENCE FORMULA, NEWTON-COTES FORMULAS, NEWTON'S FORWARD DIFFERENCE FORMULA, NICHOLSON'S FORMULA, PASCAL'S FORMULA, PICK'S FORMULA, POINCARÉ FORMULA, POISSON'S BESSEL FUNCTION FORMULA, POISSON'S HARMONIC FUNCTION FORMULA, POISSON SUM FORMULA, POLYHEDRAL FORMULA, PROSTHAPHAERESIS FORMULAS, QUADRATIC FORMULA, QUADRATURE FORMULAS, RAYLEIGH'S FORMULAS, RIEMANN'S FORMULA, RODRIGUES FORMULA, ROTATION FORMULA, SCHLÄFLI'S FORMULA, SCHRÖTER'S FORMULA, SCHWENK'S FORMULA, SEGNER'S RECURRENCE FORMULA, SERRET-FRENET FORMULAS, SHERMAN-MORRISON FORMULA, SOMMERFELD'S FORMULA, SONINE-SCHAFHEITLIN FORMULA, STEFFENSON'S FORMULA, STIRLING'S FINITE DIFFERENCE FORMULA, STIRLING'S FORMULA, STRASSEN FORMULAS, THIELE'S INTERPOLATION FORMULA, WALLIS FORMULA, WATSON'S FORMULA, WATSON-NICHOLSON FORMULA, WEBER'S FORMULA, WEBER-SONINE FORMULA, WEYRICH'S FORMULA, WOODBURY FORMULA

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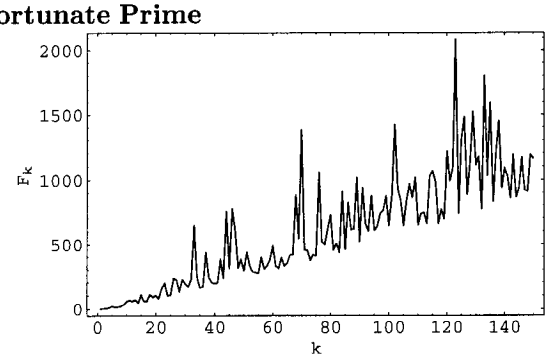
Fortunate Prime



Let

$$X_k \equiv 1 + p_k\#,$$

where p_k is the kth PRIME and $p\#$ is the PRIMORIAL, and let q_k be the NEXT PRIME (i.e., the smallest PRIME greater than X_k),

$$q_k = p_{1+\pi(X_k)} = p_{1+\pi(1+p_k\#)},$$

where $\pi(n)$ is the PRIME COUNTING FUNCTION. Then R. F. Fortune conjectured that $F_k \equiv q_k - X_k + 1$ is PRIME for all k. The first values of F_k are 3, 5, 7, 13, 23, 17, 19, 23, ... (Sloane's A005235), and all known values of F_k are indeed PRIME (Guy 1994). The indices of these primes are 2, 3, 4, 6, 9, 7, 8, 9, 12, 18, ... In numerical order with duplicates removed, the Fortunate primes are 3, 5, 7, 13, 17, 19, 23, 37, 47, 59, 61, 67, 71, 79, 89, ... (Sloane's A046066).

see also ANDRICA'S CONJECTURE, PRIMORIAL

References

Guy, R. K. *Unsolved Problems in Number Theory, 2nd ed.* New York: Springer-Verlag, p. 7, 1994.

Sloane, N. J. A. Sequences A046066 and A005235/M2418 in "An On-Line Version of the Encyclopedia of Integer Sequences."

Forward Difference

The forward difference is a FINITE DIFFERENCE defined by

$$\Delta f_p \equiv f_{p+1} - f_p. \tag{1}$$

Higher order differences are obtained by repeated operations of the forward difference operator, so

$$\begin{aligned}\Delta^2 f_p &= \Delta_p{}^2 = \Delta(\Delta_p) = \Delta(f_{p+1} - f_p)\\ &= \Delta_{p+1} - \Delta_p = f_{p+2} - 2f_{p+1} + f_p.\end{aligned} \tag{2}$$

In general,

$$\Delta_p^k \equiv \Delta^k f_p \equiv \sum_{m=0}^{k} (-1)^m \binom{k}{m} f_{p+k-m}, \tag{3}$$

where $\binom{k}{m}$ is a BINOMIAL COEFFICIENT.

NEWTON'S FORWARD DIFFERENCE FORMULA expresses f_p as the sum of the nth forward differences

$$f_p = f_0 + p\Delta_0 + \tfrac{1}{2!}p(p+1)\Delta_0^2 + \tfrac{1}{3!}p(p+1)(p+2)\Delta_0^3 + \ldots \tag{4}$$

where Δ_0^n is the first nth difference computed from the difference table.

see also BACKWARD DIFFERENCE, CENTRAL DIFFERENCE, DIFFERENCE EQUATION, DIVIDED DIFFERENCE, RECIPROCAL DIFFERENCE

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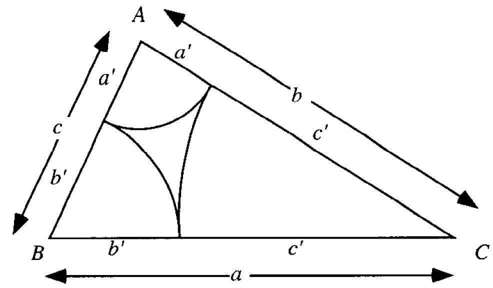
Fountain

An (n, k) fountain is an arrangement of n coins in rows such that exactly k coins are in the bottom row and each coin in the $(i+1)$st row touches exactly two in the ith row.

References

Berndt, B. C. *Ramanujan's Notebooks, Part III.* New York: Springer-Verlag, p. 79, 1985.

Four Coins Problem



Given three coins of possibly different sizes which are arranged so that each is tangent to the other two, find the coin which is tangent to the other three coins. The solution is the inner SODDY CIRCLE.

see also APOLLONIUS CIRCLES, APOLLONIUS' PROBLEM, ARBELOS, BEND (CURVATURE), CIRCUMCIRCLE, COIN, DESCARTES CIRCLE THEOREM, HART'S THEOREM, PAPPUS CHAIN, SODDY CIRCLES, SPHERE PACKING, STEINER CHAIN

References

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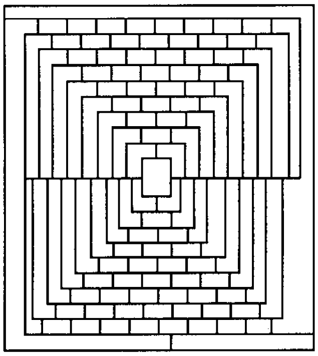
Four-Color Theorem

The four-color theorem states that any map in a PLANE can be colored using four-colors in such a way that regions sharing a common boundary (other than a single point) do not share the same color. This problem is sometimes also called GUTHRIE'S PROBLEM after F. Guthrie, who first conjectured the theorem in 1853. The CONJECTURE was then communicated to de Morgan and thence into the general community. In 1878, Cayley wrote the first paper on the conjecture.

Fallacious proofs were given independently by Kempe (1879) and Tait (1880). Kempe's proof was accepted for a decade until Heawood showed an error using a map with 18 faces (although a map with nine faces suffices to show the fallacy). The HEAWOOD CONJECTURE provided a very general result for map coloring, showing that in a GENUS 0 SPACE (i.e., either the SPHERE or PLANE), six colors suffice. This number can easily be reduced to five, but reducing the number of colors all the way to four proved very difficult.

Finally, Appel and Haken (1977) announced a computer-assisted proof that four colors were SUFFICIENT. However, because part of the proof consisted of an exhaustive analysis of many discrete cases by a computer, some mathematicians do not accept it. However, no flaws have yet been found, so the proof appears valid. A potentially independent proof has recently been constructed by N. Robertson, D. P. Sanders, P. D. Seymour, and R. Thomas.

Martin Gardner (1975) played an April Fool's joke by (incorrectly) claiming that the map of 110 regions illustrated below requires five colors and constitutes a counterexample to the four-color theorem.



see also CHROMATIC NUMBER, HEAWOOD CONJECTURE, MAP COLORING, SIX-COLOR THEOREM

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Four Travelers Problem

Let four LINES in a PLANE represent four roads in GENERAL POSITION, and let one traveler T_i be walking along each road at a constant (but not necessarily equal to any other traveler's) speed. Say that two travelers T_i and T_j have "met" if they were simultaneously at the intersection of their two roads. Then if T_1 has met all other three travelers (T_2, T_3, and T_4) and T_2, in addition to meeting T_1, has met T_3 and T_4, then T_3 and T_4 have also met!

References
Bogomolny, A. "Four Travellers Problem." `http://www.cut-the-knot.com/gproblems.html`.

Four-Vector

A four-element vector

$$a^\mu = \begin{bmatrix} a^0 \\ a^1 \\ a^2 \\ a^3 \end{bmatrix}, \tag{1}$$

which transforms under a LORENTZ TRANSFORMATION like the POSITION FOUR-VECTOR. This means it obeys

$$a'^\mu = \Lambda^\mu_\nu a^\nu \tag{2}$$

$$a_\mu \cdot b_\mu \equiv a_\mu b^\mu \tag{3}$$

$$a_\mu \cdot b^\mu = a'_\mu b'_\mu, \tag{4}$$

where Λ^μ_μ is the LORENTZ TENSOR. Multiplication of two four-vectors with the METRIC $g_{\mu\nu}$ gives products of the form

$$g_{\mu\nu} x^\mu x^\nu = (x^0)^2 - (x^1)^2 - (x^2)^2 - (x^3)^2. \tag{5}$$

In the case of the POSITION FOUR-VECTOR, $x^0 = ct$ (where c is the speed of light) and this product is an invariant known as the spacetime interval.

see also GRADIENT FOUR-VECTOR, LORENTZ TRANSFORMATION, POSITION FOUR-VECTOR, QUATERNION

References
Morse, P. M. and Feshbach, H. "The Lorentz Transformation, Four-Vectors, Spinors." §1.7 in *Methods of Theoretical Physics, Part I.* New York: McGraw-Hill, pp. 93–107, 1953.

Four-Vertex Theorem

A closed embedded smooth PLANE CURVE has at least four vertices, where a vertex is defined as an extremum of CURVATURE.

see also CURVATURE

References
Tabachnikov, S. "The Four-Vertex Theorem Revisited—Two Variations on the Old Theme." *Amer. Math. Monthly* **102**, 912–916, 1995.

Fourier-Bessel Series

see BESSEL FUNCTION FOURIER EXPANSION, SCHLÖMILCH'S SERIES

Fourier-Bessel Transform

see HANKEL TRANSFORM

Fourier Cosine Series

If $f(x)$ is an EVEN FUNCTION, then $b_n = 0$ and the FOURIER SERIES collapses to

$$f(x) = \tfrac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos(nx), \tag{1}$$

where

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{2}{\pi}\int_0^{\pi} f(x)\,dx \tag{2}$$

$$\begin{aligned} a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)\,dx \\ &= \frac{2}{\pi}\int_0^{\pi} f(x)\cos(nx)\,dx, \end{aligned} \tag{3}$$

where the last equality is true because

$$f(x)\cos(nx) = f(-x)\cos(-nx). \tag{4}$$

Letting the range go to L,

$$a_0 = \frac{1}{L}\int_0^L f(x)\,dx \tag{5}$$

$$a_n = \frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi x}{L}\right)\,dx. \tag{6}$$

see also EVEN FUNCTION, FOURIER COSINE TRANSFORM, FOURIER SERIES, FOURIER SINE SERIES

Fourier Cosine Transform

The Fourier cosine transform is the REAL PART of the full complex FOURIER TRANSFORM,

$$\mathcal{F}\cos[f(x)] = \Re[\mathcal{F}[f(x)]].$$

see also FOURIER SINE TRANSFORM, FOURIER TRANSFORM

References

Press, W. H.; Flannery, B. P.; Teukolsky, S. A.; and Vetterling, W. T. "FFT of Real Functions, Sine and Cosine Transforms." §12.3 in *Numerical Recipes in FORTRAN: The Art of Scientific Computing, 2nd ed.* Cambridge, England: Cambridge University Press, pp. 504–515, 1992.

Fourier Integral

see FOURIER TRANSFORM

Fourier Matrix

The $n \times n$ SQUARE MATRIX F_n with entries given by

$$F_{jk} = e^{2\pi ijk/n} \tag{1}$$

for $j, k = 1, 2, \ldots, n$, and normalized by $1/\sqrt{n}$ to make it a UNITARY. The Fourier matrix F_2 is given by

$$\mathsf{F}_2 = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & i^2 \end{bmatrix}, \tag{2}$$

and the F_4 matrix by

$$\begin{aligned} \mathsf{F}_4 &= \frac{1}{\sqrt{4}}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & i & i^2 & i^3 \\ 1 & i^2 & i^4 & i^6 \\ 1 & i^3 & i^6 & i^9 \end{bmatrix} \\ &= \frac{1}{2}\begin{bmatrix} 1 & & 1 & \\ & 1 & & i \\ 1 & & -1 & \\ & 1 & & -i \end{bmatrix}\begin{bmatrix} 1 & 1 & & \\ 1 & i^2 & & \\ & & 1 & 1 \\ & & 1 & i^2 \end{bmatrix}\begin{bmatrix} 1 & & & \\ & & 1 & \\ & 1 & & \\ & & & 1 \end{bmatrix}. \end{aligned} \tag{3}$$

In general,

$$\mathsf{F}_{2n} = \begin{bmatrix} \mathsf{I}_n & \mathsf{D}_n \\ \mathsf{I}_n & -\mathsf{D}_n \end{bmatrix}\begin{bmatrix} \mathsf{F}_n & \\ & \mathsf{F}_n \end{bmatrix}\begin{bmatrix} \text{even-odd} \\ \text{shuffle} \end{bmatrix}, \tag{4}$$

with

$$\begin{aligned} \begin{bmatrix} \mathsf{F}_n & \\ & \mathsf{F}_n \end{bmatrix} &= \begin{bmatrix} \mathsf{I}_{n/2} & \mathsf{D}_{n/2} & & \\ \mathsf{I}_{n/2} & -\mathsf{D}_{n/2} & & \\ & & \mathsf{I}_{n/2} & \mathsf{D}_{n/2} \\ & & \mathsf{I}_{n/2} & -\mathsf{D}_{n/2} \end{bmatrix} \\ &\times \begin{bmatrix} \mathsf{F}_{n/2} & & & \\ & \mathsf{F}_{n/2} & & \\ & & \mathsf{F}_{n/2} & \\ & & & \mathsf{F}_{n/2} \end{bmatrix}\begin{bmatrix} \text{even-odd} \\ 0,2 \pmod 4 \\ \text{even-odd} \\ 1,3 \pmod 4 \end{bmatrix}, \end{aligned} \tag{5}$$

where I_n is the $n \times n$ IDENTITY MATRIX. Note that the factorization (which is the basis of the FAST FOURIER TRANSFORM) has two copies of F_2 in the center factor MATRIX.

see also FAST FOURIER TRANSFORM, FOURIER TRANSFORM

References

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Fourier-Mellin Integral

The inverse of the LAPLACE TRANSFORM

$$F(t) = \mathcal{L}^{-1}[f(s)] = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} e^{st} f(s)\,ds$$

$$f(s) = \mathcal{L}[F(t)] = \int_0^{\infty} F(t)e^{-st}\,dt.$$

see also BROMWICH INTEGRAL, LAPLACE TRANSFORM

Fourier Series

Fourier series are expansions of PERIODIC FUNCTIONS $f(x)$ in terms of an infinite sum of SINES and COSINES

$$f(x) = \sum_{n=0}^{\infty} a_n \cos(nx) + \sum_{n=0}^{\infty} b_n \sin(nx). \quad (1)$$

Fourier series make use of the ORTHOGONALITY relationships of the SINE and COSINE functions, which can be used to calculate the coefficients a_n and b_n in the sum. The computation and study of Fourier series is known as HARMONIC ANALYSIS.

To compute a Fourier series, use the integral identities

$$\int_{-\pi}^{\pi} \sin(mx)\sin(nx)\,dx = \pi\delta_{mn} \qquad \text{for } n, m \neq 0 \quad (2)$$

$$\int_{-\pi}^{\pi} \cos(mx)\cos(nx)\,dx = \pi\delta_{mn} \qquad \text{for } n, m \neq 0 \quad (3)$$

$$\int_{-\pi}^{\pi} \sin(mx)\cos(nx)\,dx = 0 \quad (4)$$

$$\int_{-\pi}^{\pi} \sin(mx)\,dx = 0 \quad (5)$$

$$\int_{-\pi}^{\pi} \cos(mx)\,dx = 0, \quad (6)$$

where δ_{mn} is the KRONECKER DELTA. Now, expand your function $f(x)$ as an infinite series of the form

$$\begin{aligned} f(x) &= \sum_{n=0}^{\infty} a'_n \cos(nx) + \sum_{n=0}^{\infty} b_n \sin(nx) \\ &= \tfrac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos(nx) + \sum_{n=1}^{\infty} b_n \sin(nx), \quad (7) \end{aligned}$$

where we have relabeled the $a_0 = 2a'_0$ term for future convenience but left $a_n = a'_n$. Assume the function is periodic in the interval $[-\pi, \pi]$. Now use the orthogonality conditions to obtain

$$\begin{aligned} &\int_{-\pi}^{\pi} f(x)\,dx \\ &= \int_{-\pi}^{\pi} \left[\sum_{n=1}^{\infty} a_n \cos(nx) + \sum_{n=1}^{\infty} b_n \sin(nx) + \tfrac{1}{2}a_0\right] dx \\ &= \sum_{n=1}^{\infty} \int_{-\pi}^{\pi} [a_n \cos(nx) + b_n \sin(nx)]\,dx + \tfrac{1}{2}a_0 \int_{-\pi}^{\pi} dx \\ &= \sum_{n=1}^{\infty} (0+0) + \pi a_0 = \pi a_0 \quad (8) \end{aligned}$$

and

$$\begin{aligned} &\int_{-\pi}^{\pi} f(x)\sin(mx)\,dx \\ &= \int_{-\pi}^{\pi} \left[\sum_{n=1}^{\infty} a_n \cos(nx) + \sum_{n=1}^{\infty} b_n \sin(nx) + \tfrac{1}{2}a_0\right] \times \sin(mx)\,dx \\ &= \sum_{n=1}^{\infty} \int_{-\pi}^{\pi} [a_n \cos(nx)\sin(mx) + b_n \sin(nx)\sin(mx)]\,dx \\ &\quad + \tfrac{1}{2}a_0 \int_{-\pi}^{\pi} \sin(mx)\,dx \\ &= \sum_{n=1}^{\infty} (0 + b_n\pi\delta_{mn}) + 0 = \pi b_n, \quad (9) \end{aligned}$$

so

$$\begin{aligned} \int_{-\pi}^{\pi} f(x)\cos(mx)\,dx &= \int_{-\pi}^{\pi} \left[\sum_{n=1}^{\infty} a_n \cos(nx) + \sum_{n=1}^{\infty} b_n \sin(nx) + \tfrac{1}{2}a_0\right] \cos(mx)\,dx \\ &= \sum_{n=1}^{\infty} \int_{-\pi}^{\pi} [a_n \cos(nx)\cos(mx) \\ &\quad + b_n \sin(nx)\cos(mx)]\,dx + \tfrac{1}{2}a_0 \int_{-\pi}^{\pi} \cos(mx)\,dx \\ &= \sum_{n=1}^{\infty} (a_n\pi\delta_{mn} + 0) + 0 = \pi a_n. \quad (10) \end{aligned}$$

Plugging back into the original series then gives

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx \quad (11)$$

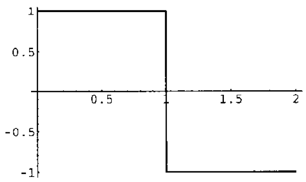
$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)\,dx \quad (12)$$

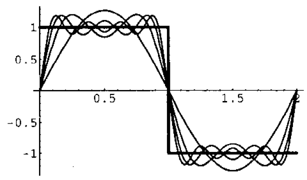
$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)\,dx \quad (13)$$

for $n = 1, 2, 3, \ldots$. The series expansion converges to the function $\bar{f}$ (equal to the original function at points of continuity or to the average of the two limits at points of discontinuity)

$$\bar{f} \equiv \begin{cases} \tfrac{1}{2}\left[\lim_{x\to x_0^-} f(x) + \lim_{x\to x_0^+} f(x)\right] \\ \quad \text{for } -\pi < x_0 < \pi \\ \tfrac{1}{2}\left[\lim_{x\to -\pi^+} f(x) + \lim_{x\to \pi_-} f(x)\right] \\ \quad \text{for } x_0 = -\pi, \pi \end{cases} \quad (14)$$

if the function satisfies the DIRICHLET CONDITIONS.





Near points of discontinuity, a "ringing" known as the GIBBS PHENOMENON, illustrated below, occurs. For a function $f(x)$ periodic on an interval $[-L, L]$, use a change of variables to transform the interval of integration to $[-1, 1]$. Let

$$x \equiv \frac{\pi x'}{L} \tag{15}$$

$$dx = \frac{\pi\, dx'}{L}. \tag{16}$$

Solving for x', $x' = Lx/\pi$. Plugging this in gives

$$f(x') = \tfrac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x'}{L}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x'}{L}\right) \tag{17}$$

$$\begin{cases} a_0 = \frac{1}{L}\int_{-L}^{L} f(x')\,dx' \\ a_n = \frac{1}{L}\int_{-L}^{L} f(x')\cos\left(\frac{n\pi x'}{L}\right)dx' \\ b_n = \frac{1}{L}\int_{-L}^{L} f(x')\sin\left(\frac{n\pi x'}{L}\right)dx' \end{cases}. \tag{18}$$

If a function is EVEN so that $f(x) = f(-x)$, then $f(x)\sin(nx)$ is ODD. (This follows since $\sin(nx)$ is ODD and an EVEN FUNCTION times an ODD FUNCTION is an ODD FUNCTION.) Therefore, $b_n = 0$ for all n. Similarly, if a function is ODD so that $f(x) = f(-x)$, then $f(x)\cos(nx)$ is ODD. (This follows since $\cos(nx)$ is EVEN and an EVEN FUNCTION times an ODD FUNCTION is an ODD FUNCTION.) Therefore, $a_n = 0$ for all n.

Because the SINES and COSINES form a COMPLETE ORTHOGONAL BASIS, the SUPERPOSITION PRINCIPLE holds, and the Fourier series of a linear combination of two functions is the same as the linear combination of the corresponding two series. The COEFFICIENTS for Fourier series expansions for a few common functions are given in Beyer (1987, pp. 411–412) and Byerly (1959, p. 51).

The notion of a Fourier series can also be extended to COMPLEX COEFFICIENTS. Consider a real-valued function $f(x)$. Write

$$f(x) = \sum_{n=-\infty}^{\infty} A_n e^{inx}. \tag{19}$$

Now examine

$$\begin{aligned}\int_{-\pi}^{\pi} f(x)e^{-imx}\,dx &= \int_{-\pi}^{\pi}\left(\sum_{n=-\infty}^{\infty} A_n e^{inx}\right)e^{-imx}\,dx \\ &= \sum_{n=-\infty}^{\infty} A_n \int_{-\pi}^{\pi} e^{i(n-m)x}\,dx \\ &= \sum_{n=-\infty}^{\infty} A_n \int_{-\pi}^{\pi}\{\cos[(n-m)x] + i\sin[(n-m)x]\}\,dx \\ &= \sum_{m=-\infty}^{\infty} A_n 2\pi\delta_{mn} = 2\pi A_m,\end{aligned} \tag{20}$$

so

$$A_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-inx}\,dx. \tag{21}$$

The COEFFICIENTS can be expressed in terms of those in the FOURIER SERIES

$$\begin{aligned} A_n &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)[\cos(nx) - i\sin(nx)]\,dx \\ &= \begin{cases} \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)[\cos(nx) + i\sin(nx)]\,dx & n < 0 \\ \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx & n = 0 \\ \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)[\cos(nx) - i\sin(nx)]\,dx & n > 0 \end{cases} \\ &= \begin{cases} \frac{1}{2}(a_n + ib_n) & n < 0 \\ \frac{1}{2}a_0 & n = 0 \\ \frac{1}{2}(a_n - ib_n) & n > 0. \end{cases} \end{aligned} \tag{22}$$

For a function periodic in $[-L, L]$, these become

$$f(x) = \sum_{n=-\infty}^{\infty} A_n e^{i(2\pi n x/L)} \tag{23}$$

$$A_n = \frac{1}{L}\int_{-L/2}^{L/2} f(x)e^{-i(2\pi n x/L)}\,dx. \tag{24}$$

These equations are the basis for the extremely important FOURIER TRANSFORM, which is obtained by transforming A_n from a discrete variable to a continuous one as the length $L \to \infty$.

see also DIRICHLET FOURIER SERIES CONDITIONS, FOURIER COSINE SERIES, FOURIER SINE SERIES, FOURIER TRANSFORM, GIBBS PHENOMENON, LEBESGUE CONSTANTS (FOURIER SERIES), LEGENDRE SERIES, RIESZ-FISCHER THEOREM, SCHLÖMILCH'S SERIES

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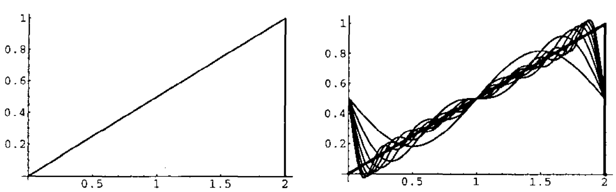
Fourier Series—Power Series

For $f(x) = x^k$ on the INTERVAL $[-L, L)$ and periodic with period $2L$, the FOURIER SERIES is given by

$$\begin{aligned} a_n &= \frac{1}{L}\int_{-L}^{L} x^k \cos\left(\frac{n\pi x}{L}\right) dx \\ &= \frac{2L^k}{1+k}\,{}_1F_2\left(\begin{matrix} & 1+\tfrac{1}{2}k \\ \tfrac{1}{2} & \tfrac{1}{2}(3+k)\end{matrix}; -\tfrac{1}{4}\pi^2 n^2\right) \\ b_n &= \frac{1}{L}\int_{-L}^{L} x^k \sin\left(\frac{n\pi x}{L}\right) dx \\ &= \frac{2n\pi L^k}{2+k}\,{}_1F_2\left(\begin{matrix} & 1+\tfrac{1}{2}k \\ \tfrac{3}{2} & 2+\tfrac{1}{2}k\end{matrix}; -\tfrac{1}{4}\pi^2 n^2\right), \end{aligned}$$

where ${}_1F_2(a; b, c; x)$ is a generalized HYPERGEOMETRIC FUNCTION.

Fourier Series—Right Triangle



Consider a string of length $2L$ plucked at the right end, then

$$\begin{aligned} a_0 &= \frac{1}{L}\int_0^{2L} \frac{x}{2L}\,dx = \frac{1}{2L^2}[\tfrac{1}{2}x^2]_0^L = \frac{1}{4L^2}(2L)^2 = 1 \\ a_n &= \frac{1}{L}\int_0^{2L} \frac{x}{2L}\cos\left(\frac{n\pi x}{L}\right) dx \\ &= \frac{[2n\pi\cos(n\pi) - \sin(n\pi)]\sin(n\pi)}{n^2\pi^2} = 0 \end{aligned}$$

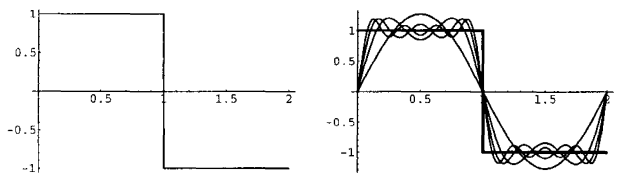
$$\begin{aligned} b_n &= \frac{1}{L}\int_0^{2L} \frac{x}{2L}\sin\left(\frac{n\pi x}{L}\right) dx \\ &= \frac{-2n\pi\cos(2n\pi) + \sin(2n\pi)}{2n^2\pi^2} = -\frac{1}{n\pi}. \end{aligned}$$

The Fourier series is therefore

$$f(x) = \tfrac{1}{2} - \frac{1}{\pi}\sum_{n=1}^{\infty} \frac{1}{n}\sin\left(\frac{n\pi x}{L}\right).$$

see also FOURIER SERIES

Fourier Series—Square Wave



Consider a square wave of length $2L$. Since the function is ODD, $a_0 = a_n = 0$, and

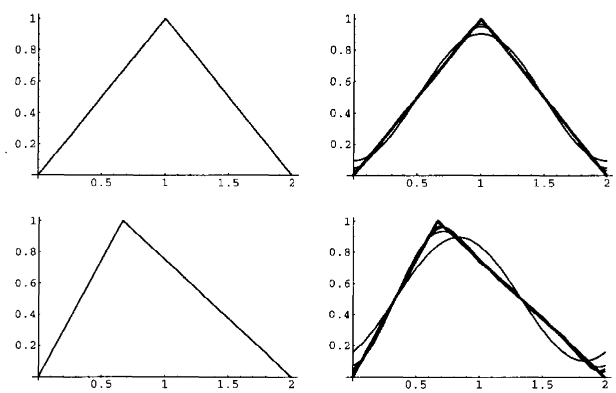
$$\begin{aligned} b_n &= \frac{2}{L}\int_0^{L} \sin\left(\frac{n\pi x}{L}\right) dx \\ &= \frac{4}{n\pi}\sin^2(\tfrac{1}{2}n\pi) = \frac{4}{n\pi}\begin{cases} 0 & n \text{ even} \\ 1 & n \text{ odd.}\end{cases} \end{aligned}$$

The Fourier series is therefore

$$f(x) = \frac{4}{\pi}\sum_{n=1,3,5,\ldots}^{\infty} \frac{1}{n}\sin\left(\frac{n\pi x}{L}\right).$$

see also FOURIER SERIES, SQUARE WAVE

Fourier Series—Triangle



Let a string of length $2L$ have a y-displacement of unity when it is pinned an x-distance which is $(1/m)$th of the way along the string. The displacement as a function of x is then

$$f_m(x) = \begin{cases} \frac{mx}{2L} & 0 \le x \le \frac{2L}{m} \\ \frac{m}{1-m}\left(\frac{x}{2L} - 1\right) & \frac{2L}{m} \le x \le 2L. \end{cases}$$

The COEFFICIENTS are therefore

$$\begin{aligned} a_0 &= \frac{1}{L}\left[\int_0^{2L/m} \frac{nx}{2L}\,dx + \int_{2L/m}^{2L} \frac{n}{1-n}\left(\frac{x}{2L}-1\right)dx\right] \\ &= 1 \\ a_n &= \frac{m\left[1-m-\cos(2\pi n)+m\cos\left(\frac{2n\pi}{m}\right)\right]}{2(m-1)n^2\pi^2} \\ &= \frac{m^2\left[\cos\left(\frac{2n\pi}{m}\right)-1\right]}{2(m-1)n^2\pi^2} \\ b_n &= \frac{m\left[m\sin\left(\frac{2\pi n}{m}\right)-\sin(2\pi n)\right]}{2(m-1)n^2\pi^2} \\ &= \frac{m^2\sin\left(\frac{2\pi n}{m}\right)}{2(m-1)n^2\pi^2}. \end{aligned}$$

The Fourier series is therefore

$$\begin{aligned} f_m(x) &= \tfrac{1}{2} + \frac{m^2}{2(m-1)\pi^2} \\ &\quad \times \sum_{n=1}^{\infty}\left\{\frac{1}{n^2}\left[\cos\left(\frac{2n\pi}{m}\right)-1\right]\cos\left(\frac{n\pi x}{L}\right)\right. \\ &\quad \left. + \frac{\sin\left(\frac{2\pi n}{m}\right)}{n^2}\sin\left(\frac{n\pi x}{L}\right)\right\}. \end{aligned}$$

If $m=2$, then a_n and b_n simplify to

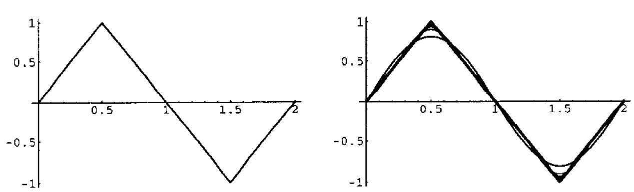
$$\begin{aligned} a_n &= -\frac{4}{n^2\pi^2}\sin^2(\tfrac{1}{2}n\pi) = -\frac{4}{n^2\pi^2}\begin{cases} 0 & n=0,\,2,\,\ldots \\ 1 & n=1,\,3,\,\ldots \end{cases} \\ b_n &= 0, \end{aligned}$$

giving

$$f_2(x) = \tfrac{1}{2} - \frac{4}{\pi^2}\sum_{n=1,3,5,\ldots}^{\infty}\frac{1}{n^2}\cos\left(\frac{n\pi x}{L}\right).$$

see also FOURIER SERIES

Fourier Series—Triangle Wave



Consider a triangle wave of length $2L$. Since the function is ODD, $a_0 = a_n = 0$, and

$$\begin{aligned} b_n &= \frac{2}{L}\left\{\int_0^{L/2}\frac{x}{L/2}\sin\left(\frac{n\pi x}{L}\right)dx\right. \\ &\quad \left. + \int_{L/2}^{L}\left[1-\frac{2}{L}(x-\tfrac{1}{2}L)\right]\sin\left(\frac{n\pi x}{L}\right)dx\right\}dx \\ &= \frac{32}{\pi^2 n^2}\cos(\tfrac{1}{4}n\pi)\sin^3(\tfrac{1}{4}n\pi) \\ &= \frac{32}{\pi^2 n^2}\begin{cases} 0 & n=0,\,4,\,\ldots \\ \frac{1}{4} & n=1,\,5,\,\ldots \\ 0 & n=2,\,6,\,\ldots \\ -\frac{1}{4} & n=3,\,7,\,\ldots \end{cases} \\ &= \frac{8}{\pi^2 n^2}\begin{cases} (-1)^{(n-1)/2} & \text{for } n \text{ odd} \\ 0 & \text{for } n \text{ even.} \end{cases} \end{aligned}$$

The Fourier series is therefore

$$f(x) = \frac{8}{\pi^2}\sum_{n=1,3,5,\ldots}^{\infty}\frac{(-1)^{(n-1)/2}}{n^2}\sin\left(\frac{n\pi x}{L}\right).$$

see also FOURIER SERIES

Fourier Sine Series

If $f(x)$ is an ODD FUNCTION, then $a_n = 0$ and the FOURIER SERIES collapses to

$$f(x) = \sum_{n=1}^{\infty} b_n \sin(nx), \tag{1}$$

where

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)\,dx = \frac{2}{\pi}\int_0^{\pi} f(x)\sin(nx)\,dx \tag{2}$$

for $n = 1, 2, 3, \ldots$. The last EQUALITY is true because

$$\begin{aligned} f(x)\sin(nx) &= [-f(-x)][-\sin(-nx)] \\ &= f(-x)\sin(-nx). \end{aligned} \tag{3}$$

Letting the range go to L,

$$b_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)dx. \tag{4}$$

see also FOURIER COSINE SERIES, FOURIER SERIES, FOURIER SINE TRANSFORM

Fourier Sine Transform

The Fourier sine transform is the IMAGINARY PART of the full complex FOURIER TRANSFORM,

$$\mathcal{F}\sin[f(x)] = \Im[\mathcal{F}[f(x)]].$$

see also FOURIER COSINE TRANSFORM, FOURIER TRANSFORM

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Fourier-Stieltjes Transform

Let $f(x)$ be a positive definite, measurable function on the INTERVAL $(-\infty, \infty)$. Then there exists a monotone increasing, real-valued bounded function $\alpha(t)$ such that

$$f(x) = \int_{-\infty}^{\infty} e^{itx}\,d\alpha(t)$$

for "ALMOST ALL" x. If $\alpha(t)$ is nondecreasing and bounded and $f(x)$ is defined as above, then $f(x)$ is called the Fourier-Stieltjes transform of $\alpha(t)$, and is both continuous and positive definite.

see also FOURIER TRANSFORM, LAPLACE TRANSFORM

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Fourier Transform

The Fourier transform is a generalization of the COMPLEX FOURIER SERIES in the limit as $L \to \infty$. Replace the discrete A_n with the continuous $F(k)\,dk$ while letting $n/L \to k$. Then change the sum to an INTEGRAL, and the equations become

$$f(x) = \int_{-\infty}^{\infty} F(k) e^{2\pi ikx}\,dk \tag{1}$$

$$F(k) = \int_{-\infty}^{\infty} f(x) e^{-2\pi ikx}\,dx. \tag{2}$$

Here,

$$F(k) = \mathcal{F}[f(x)] = \int_{-\infty}^{\infty} f(x) e^{-2\pi ikx}\,dx \tag{3}$$

is called the *forward* $(-i)$ Fourier transform, and

$$f(x) = \mathcal{F}^{-1}[F(k)] = \int_{-\infty}^{\infty} F(k) e^{2\pi ikx}\,dk \tag{4}$$

is called the *inverse* $(+i)$ Fourier transform. Some authors (especially physicists) prefer to write the transform in terms of angular frequency $\omega \equiv 2\pi\nu$ instead of the oscillation frequency ν. However, this destroys the symmetry, resulting in the transform pair

$$H(\nu) = \mathcal{F}[h(t)] = \int_{-\infty}^{\infty} h(t) e^{-i\omega t}\,dt \tag{5}$$

$$h(t) = \mathcal{F}^{-1}[H(\nu)] = \frac{1}{2\pi}\int_{-\infty}^{\infty} H(\nu) e^{i\omega t}\,d\omega. \tag{6}$$

In general, the Fourier transform pair may be defined using two arbitrary constants A and B as

$$F(\omega) = A\int_{-\infty}^{\infty} f(t) e^{Bi\omega t}\,dt \tag{7}$$

$$f(t) = \frac{B}{2\pi A}\int_{-\infty}^{\infty} F(\omega) e^{-Bi\omega t}\,d\omega. \tag{8}$$

The *Mathematica*® program (Wolfram Research, Champaign, IL) calls A the `$FourierOverallConstant` and B the `$FourierFrequencyConstant`, and defines $A = B = 1$ by default. Morse and Feshbach (1953) use $B = 1$ and $A = 1/\sqrt{2\pi}$. In this work, following Bracewell (1965, pp. 6–7), $A = 1$ and $B = -2\pi$ unless otherwise stated.

Since any function can be split up into EVEN and ODD portions $E(x)$ and $O(x)$,

$$f(x) = \tfrac{1}{2}[f(x)+f(-x)] + \tfrac{1}{2}[f(x)-f(-x)] = E(x)+O(x), \tag{9}$$

a Fourier transform can always be expressed in terms of the FOURIER COSINE TRANSFORM and FOURIER SINE TRANSFORM as

$$\mathcal{F}[f(x)] = \int_{-\infty}^{\infty} E(x)\cos(2\pi kx)\,dx - i\int_{-\infty}^{\infty} O(x)\sin(2\pi kx)\,dx. \tag{10}$$

A function $f(x)$ has a forward and inverse Fourier transform such that

$$f(x) = \begin{cases} \int_{-\infty}^{\infty} e^{2\pi ikx}\left[\int_{-\infty}^{\infty} f(x)e^{-2\pi ikx}\,dx\right]dk \\ \quad\text{for } f(x) \text{ continuous at } x \\ \frac{1}{2}[f(x_+)+f(x_-)] \\ \quad\text{for } f(x) \text{ discontinuous at } x, \end{cases} \tag{11}$$

provided that

1. $\int_{-\infty}^{\infty} |f(x)|\,dx$ exists.
2. Any discontinuities are finite.
3. The function has bounded variation. A SUFFICIENT weaker condition is fulfillment of the LIPSCHITZ CONDITION.

The smoother a function (i.e., the larger the number of continuous DERIVATIVES), the more compact its Fourier transform.

The Fourier transform is linear, since if $f(x)$ and $g(x)$ have FOURIER TRANSFORMS $F(k)$ and $G(k)$, then

$$\int [af(x)+bg(x)]e^{-2\pi ikx}\,dx = a\int_{-\infty}^{\infty} f(x)e^{-2\pi ikx}\,dx + b\int_{-\infty}^{\infty} g(x)e^{-2\pi ikx}\,dx = F(k)+G(k). \tag{12}$$

Therefore,

$$\mathcal{F}[af(x)+bg(x)] = a\mathcal{F}[f(x)]+b\mathcal{F}[g(x)] = aF(k)+bG(k). \tag{13}$$

The Fourier transform is also symmetric since $F(k) = \mathcal{F}[f(x)]$ implies $F(-k) = \mathcal{F}[f(x)]$.

Let $f*g$ denote the CONVOLUTION, then the transforms of convolutions of functions have particularly nice transforms,

$$\mathcal{F}[f*g] = \mathcal{F}[f]\mathcal{F}[g] \tag{14}$$

$$\mathcal{F}[fg] = \mathcal{F}[f]*\mathcal{F}[g] \tag{15}$$

$$\mathcal{F}[\mathcal{F}(f)+\mathcal{F}(g)] = f*g \tag{16}$$

$$\mathcal{F}[\mathcal{F}(f)*\mathcal{F}(g)] = fg. \tag{17}$$

The first of these is derived as follows:

$$\begin{aligned}\mathcal{F}[f*g] &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-2\pi ikx} f(x')g(x-x')\,dx'\,dx\\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}[e^{-2\pi ikx'}f(x')\,dx']\\ &\quad\times[e^{-2\pi ik(x-x')}g(x-x')\,dx]\\ &= \left[\int_{-\infty}^{\infty} e^{-2\pi ikx'}f(x')\,dx'\right]\\ &\quad\times\left[\int_{-\infty}^{\infty} e^{-2\pi ikx''}g(x'')\,dx''\right]\\ &= \mathcal{F}[f]\mathcal{F}[g],\end{aligned} \tag{18}$$

where $x'' \equiv x - x'$.

There is also a somewhat surprising and extremely important relationship between the AUTOCORRELATION and the Fourier transform known as the WIENER-KHINTCHINE THEOREM. Let $\mathcal{F}[f(x)] = F(k)$, and F^* denote the COMPLEX CONJUGATE of F, then the FOURIER TRANSFORM of the ABSOLUTE SQUARE of $F(k)$ is given by

$$\mathcal{F}[|F(k)|^2] = \int_{-\infty}^{\infty} f^*(\tau)f(\tau+x)\,d\tau. \tag{19}$$

The Fourier transform of a DERIVATIVE $f'(x)$ of a function $f(x)$ is simply related to the transform of the function $f(x)$ itself. Consider

$$\mathcal{F}[f'(x)] = \int_{-\infty}^{\infty} f'(x)e^{-2\pi ikx}\,dx. \tag{20}$$

Now use INTEGRATION BY PARTS

$$\int v\,du = [uv] - \int u\,dv \tag{21}$$

with

$$du = f'(x)\,dx \qquad v = e^{-2\pi ikx} \tag{22}$$

$$u = f(x) \qquad dv = -2\pi ike^{-2\pi ikx}\,dx, \tag{23}$$

then

$$\mathcal{F}[f'(x)] = [f(x)e^{-2\pi ikx}]_{-\infty}^{\infty} - \int_{-\infty}^{\infty} f(x)(-2\pi ike^{-2\pi ikx}\,dx). \tag{24}$$

The first term consists of an oscillating function times $f(x)$. But if the function is bounded so that

$$\lim_{x\to\pm\infty} f(x) = 0 \tag{25}$$

(as any physically significant signal must be), then the term vanishes, leaving

$$\mathcal{F}[f'(x)] = 2\pi ik\int_{-\infty}^{\infty} f(x)e^{-2\pi ikx}\,dx = 2\pi ik\mathcal{F}[f(x)]. \tag{26}$$

This process can be iterated for the nth DERIVATIVE to yield

$$\mathcal{F}[f^{(n)}(x)] = (2\pi ik)^n\mathcal{F}[f(x)]. \tag{27}$$

The important MODULATION THEOREM of Fourier transforms allows $\mathcal{F}[\cos(2\pi k_0x)f(x)]$ to be expressed in terms of $\mathcal{F}[f(x)] = F(k)$ as follows,

$$\begin{aligned}\mathcal{F}[\cos(2\pi k_0x)f(x)] &\equiv \int_{-\infty}^{\infty} f(x)\cos(2\pi k_0x)e^{-2\pi ikx}\,dx\\ &= \tfrac{1}{2}\int_{-\infty}^{\infty} f(x)e^{2\pi ik_0x}e^{-2\pi ikx}\,dx\\ &\quad+\tfrac{1}{2}\int_{-\infty}^{\infty} f(x)e^{-2\pi ik_0x}e^{-2\pi ikx}\,dx\\ &= \tfrac{1}{2}\int_{-\infty}^{\infty} f(x)e^{-2\pi i(k-k_0)x}\,dx\\ &\quad+\tfrac{1}{2}\int_{-\infty}^{\infty} f(x)e^{-2\pi i(k+k_0)x}\,dx\\ &= \tfrac{1}{2}[F(k-k_0)+F(k+k_0)].\end{aligned} \tag{28}$$

Since the DERIVATIVE of the FOURIER TRANSFORM is given by

$$F'(k) \equiv \frac{d}{dk}\mathcal{F}[f(x)] = \int_{-\infty}^{\infty}(-2\pi ix)f(x)e^{-2\pi ikx}\,dx, \tag{29}$$

it follows that

$$F'(0) = -2\pi i\int_{-\infty}^{\infty} xf(x)\,dx. \tag{30}$$

Iterating gives the general FORMULA

$$\mu_n \equiv \int_{-\infty}^{\infty} x^n f(x)\,dx = \frac{F^{(n)}(0)}{(-2\pi i)^n}. \tag{31}$$

The VARIANCE of a FOURIER TRANSFORM is

$$\sigma_f{}^2 = \left\langle (xf - \langle xf\rangle)^2\right\rangle, \tag{32}$$

and it is true that

$$\sigma_{f+g} = \sigma_f + \sigma_g. \tag{33}$$

If $f(x)$ has the FOURIER TRANSFORM $F(k)$, then the Fourier transform has the shift property

$$\begin{aligned}&\int_{-\infty}^{\infty} f(x-x_0)e^{-2\pi ikx}\,dx\\ &\quad= \int_{-\infty}^{\infty} f(x-x_0)e^{-2\pi i(x-x_0)k}e^{-2\pi i(kx_0)}\,d(x-x_0)\\ &\quad= e^{-2\pi ikx_0}F(k),\end{aligned} \tag{34}$$

so $f(x-x_0)$ has the FOURIER TRANSFORM

$$\mathcal{F}[f(x-x_0)] = e^{-2\pi ikx_0}F(k). \tag{35}$$

If $f(x)$ has a FOURIER TRANSFORM $F(k)$, then the Fourier transform obeys a similarity theorem.

$$\int_{-\infty}^{\infty} f(ax)e^{-2\pi ikx}\,dx = \frac{1}{|a|}\int_{-\infty}^{\infty} f(ax)e^{-2\pi i(ax)(k/a)}\,d(ax) = \frac{1}{|a|}F\left(\frac{k}{a}\right), \tag{36}$$

so $f(ax)$ has the FOURIER TRANSFORM $|a|^{-1}F\left(\frac{k}{a}\right)$.

The "equivalent width" of a Fourier transform is

$$w_e \equiv \frac{\int_{-\infty}^{\infty} f(x)\,dx}{f(0)} = \frac{F(0)}{\int_{-\infty}^{\infty} F(k)\,dk}. \tag{37}$$

The "autocorrelation width" is

$$w_a \equiv \frac{\int_{-\infty}^{\infty} f\star f^*\,dx}{[f\star f^*]_0} = \frac{\int_{-\infty}^{\infty} f\,dx\int_{-\infty}^{\infty} f^*\,dx}{\int_{-\infty}^{\infty} ff^*\,dx}, \tag{38}$$

where $f\star g$ denotes the CROSS-CORRELATION of f and g.

Any operation on $f(x)$ which leaves its AREA unchanged leaves $F(0)$ unchanged, since

$$\int_{-\infty}^{\infty} f(x)\,dx = \mathcal{F}[f(0)] = F(0). \tag{39}$$

In 2-D, the Fourier transform becomes

$$F(x,y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(k_x,k_y)e^{-2\pi i(k_xx+k_yy)}\,dk_x\,dk_y \tag{40}$$

$$f(k_x,k_y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F(x,y)e^{2\pi i(k_xx+k_yy)}\,dx\,dy. \tag{41}$$

Similarly, the n-D Fourier transform can be defined for $\mathbf{k}$, $\mathbf{x}\in\mathbb{R}^n$ by

$$F(\mathbf{x}) = \underbrace{\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}}_{n} f(\mathbf{k})e^{-2\pi i\mathbf{k}\cdot\mathbf{x}}\,d^n\mathbf{k} \tag{42}$$

$$f(\mathbf{k}) = \underbrace{\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}}_{n} F(\mathbf{x})e^{2\pi i\mathbf{k}\cdot\mathbf{x}}\,d^n\mathbf{x}. \tag{43}$$

see also AUTOCORRELATION, CONVOLUTION, DISCRETE FOURIER TRANSFORM, FAST FOURIER TRANSFORM, FOURIER SERIES, FOURIER-STIELTJES TRANSFORM, HANKEL TRANSFORM, HARTLEY TRANSFORM, INTEGRAL TRANSFORM, LAPLACE TRANSFORM, STRUCTURE FACTOR, WINOGRAD TRANSFORM

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Fourier Transform—1

The FOURIER TRANSFORM of the CONSTANT FUNCTION $f(x)=1$ is given by

$$\mathcal{F}[1] = \int_{-\infty}^{\infty} e^{-2\pi ikx}\,dx = \delta(k),$$

according to the definition of the DELTA FUNCTION.

see also DELTA FUNCTION

Fourier Transform—$1/x$

The FOURIER TRANSFORM of the function $1/x$ is given by

$$\begin{aligned}\mathcal{F}\left(-\frac{1}{\pi x}\right) &= -\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{e^{-2\pi ikx}}{x}\,dx \\ &= PV\int_{-\infty}^{\infty}\frac{\cos(2\pi kx)-i\sin(2\pi kx)}{x}\,dx \\ &= \begin{cases} -\frac{2i}{\pi}\int_0^\infty \frac{\sin(2\pi kx)}{x}\,dx & \text{for } k<0 \\ \frac{2i}{\pi}\int_0^\infty \frac{\sin(2\pi kx)}{x}\,dx & \text{for } k>0 \end{cases} \\ &= \begin{cases} -i & \text{for } k<0 \\ i & \text{for } k>0, \end{cases}\end{aligned} \tag{1}$$

which can also be written as the single equation

$$\mathcal{F}\left(-\frac{1}{\pi x}\right) = i[1-2H(-k)], \tag{2}$$

where $H(x)$ is the HEAVISIDE STEP FUNCTION. The integrals follow from the identity

$$\begin{aligned}\int_0^\infty \frac{\sin(2\pi kx)}{x}\,dx &= \int_0^\infty \frac{\sin(2\pi kx)}{2\pi kx}\,d(2\pi kx) \\ &= \int_0^\infty \operatorname{sinc} z\,dz = \tfrac{1}{2}\pi.\end{aligned} \tag{3}$$

Fourier Transform—Cosine

$$\begin{aligned}\mathcal{F}[\cos(2\pi k_0 x)] &= \int_{-\infty}^{\infty} e^{-2\pi ikx}\left(\frac{e^{2\pi ik_0x}+e^{-2\pi ik_0x}}{2}\right)dx \\ &= \tfrac{1}{2}\int_{-\infty}^{\infty}[e^{-2\pi i(k-k_0)x}+e^{-2\pi i(k+k_0)x}]\,dx \\ &= \tfrac{1}{2}[\delta(k-k_0)+\delta(k+k_0)],\end{aligned}$$

where $\delta(x)$ is the DELTA FUNCTION.

see also COSINE, FOURIER TRANSFORM—SINE

Fourier Transform—Delta Function

The FOURIER TRANSFORM of the DELTA FUNCTION is given by

$$\mathcal{F}[\delta(x-x_0)] = \int_{-\infty}^{\infty}\delta(x-x_0)e^{-2\pi ikx}\,dx = e^{-2\pi ikx_0}.$$

see also DELTA FUNCTION

Fourier Transform—Exponential Function

The FOURIER TRANSFORM of $e^{-k_0|x|}$ is given by

$$\begin{aligned}\mathcal{F}[e^{-k_0|x|}] &= \int_{-\infty}^{\infty} e^{-k_0|x|}e^{-2\pi ikx}\,dx \\ &= \int_{-\infty}^{0} e^{-2\pi ikx}e^{2\pi xk_0}\,dx + \int_0^\infty e^{-2\pi ikx}e^{-2\pi k_0x}\,dx \\ &= \int_{-\infty}^{0}[\cos(2\pi kx)-i\sin(2\pi kx)]e^{2\pi k_0x}\,dx \\ &\quad + \int_0^\infty [\cos(2\pi kx)-i\sin(2\pi kx)]e^{-2\pi k_0x}\,dx.\end{aligned} \tag{1}$$

Now let $u \equiv -x$ so $du = -dx$, then

$$\begin{aligned}\mathcal{F}[e^{-k_0|x|}] &= \int_0^\infty [\cos(2\pi ku)+i\sin(2\pi ku)]e^{-2\pi k_0u}\,du] \\ &\quad + \int_0^\infty [\cos(2\pi ku)-i\sin(2\pi ku)]e^{-2\pi k_0u}\,du] \\ &= 2\int_0^\infty \cos(2\pi ku)e^{-2\pi k_0u}\,du,\end{aligned} \tag{2}$$

which, from the DAMPED EXPONENTIAL COSINE INTEGRAL, gives

$$\mathcal{F}[e^{-2\pi k_0|x|}] = \frac{1}{\pi}\frac{k_0}{k^2+{k_0}^2}, \tag{3}$$

which is a LORENTZIAN FUNCTION.

see also DAMPED EXPONENTIAL COSINE INTEGRAL, EXPONENTIAL FUNCTION, LORENTZIAN FUNCTION

Fourier Transform—Gaussian

The FOURIER TRANSFORM of a GAUSSIAN FUNCTION $f(x) \equiv e^{-ax^2}$ is given by

$$\begin{aligned}F(k) &= \int_{-\infty}^{\infty} e^{-ax^2}e^{ikx}\,dx \\ &= \int_{-\infty}^{\infty} e^{-ax^2}[\cos(kx)+i\sin(kx)]\,dx \\ &= \int_{-\infty}^{\infty} e^{-ax^2}\cos(kx)\,dx + i\int_{-\infty}^{\infty} e^{-ax^2}\sin(kx)\,dx.\end{aligned}$$

The second integrand is EVEN, so integration over a symmetrical range gives 0. The value of the first integral is given by Abramowitz and Stegun (1972, p. 302, equation 7.4.6)

$$F(k) = \sqrt{\frac{\pi}{a}}\,e^{-k^2/4a},$$

so a GAUSSIAN transforms to a GAUSSIAN.

see also GAUSSIAN FUNCTION

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Fourier Transform—Heaviside Step Function

$$\mathcal{F}[H(x)] = \int_{-\infty}^{\infty} e^{-2\pi ikx} H(x)\,dx = \frac{1}{2}\left[\delta(k) - \frac{i}{\pi k}\right],$$

where $H(x)$ is the HEAVISIDE STEP FUNCTION and $\delta(k)$ is the DELTA FUNCTION.

see also HEAVISIDE STEP FUNCTION

Fourier Transform—Lorentzian Function

$$\mathcal{F}\left[\frac{1}{\pi}\frac{\frac{1}{2}\Gamma}{(x-x_0)^2 + (\frac{1}{2}\Gamma)^2}\right] = e^{-2\pi ikx_0 - \Gamma\pi|k|}.$$

see also LORENTZIAN FUNCTION

Fourier Transform—Ramp Function

Let $R(x)$ be the RAMP FUNCTION, then the FOURIER TRANSFORM of $R(x)$ is given by

$$\mathcal{F}[R(x)] = \int_{-\infty}^{\infty} e^{-2\pi ikx} R(x)\,dx = \pi i\delta'(2\pi k) - \frac{1}{4\pi^2 k^2},$$

where $\delta'(x)$ is the DERIVATIVE of the DELTA FUNCTION.

see also RAMP FUNCTION

Fourier Transform—Rectangle Function

Let $\Pi(x)$ be the RECTANGLE FUNCTION, then the FOURIER TRANSFORM is

$$\mathcal{F}[\Pi(x)] = \operatorname{sinc}(\pi k),$$

where $\operatorname{sinc}(x)$ is the SINC FUNCTION.

see also RECTANGLE FUNCTION, SINC FUNCTION

Fourier Transform—Sine

$$\begin{aligned}\mathcal{F}[\sin(2\pi k_0 x)] &= \int_{-\infty}^{\infty} e^{-2\pi ik_0 x}\left(\frac{e^{2\pi i\nu_0 t} - e^{-2\pi ik_0 x}}{2i}\right)dx \\ &= \tfrac{1}{2}i\int_{-\infty}^{\infty}[-e^{-2\pi i(k-k_0)x} + e^{-2\pi i(k+k_0)x}]\,dt \\ &= \tfrac{1}{2}i[\delta(k+k_0) - \delta(k-k_0)],\end{aligned}$$

where $\delta(x)$ is the DELTA FUNCTION.

see also FOURIER TRANSFORM—COSINE, SINE

Fox's H-Function

A very general function defined by

$$\begin{aligned}H(z) &= \mathbf{H}^{m,n}_{p,q}\left[z\left|\begin{matrix}(a_1,\alpha_1),\ldots,(a_p,\alpha_p)\\(b_1,\beta_1),\ldots,(b_p,\beta_p)\end{matrix}\right.\right] \\ &= \frac{1}{2\pi i}\int_C \frac{\prod_{j=1}^m \Gamma(b_j-\beta_i s)\prod_{j=1}^n \Gamma(1-a_j+\alpha_j s)}{\prod_{j=m+1}^q \Gamma(1-b_j+\beta_j s)\prod_{j=n+1}^{qp}\Gamma(a_j-\alpha_j s)} z^s\,ds,\end{aligned}$$

where $0 \le m \le q$, $0 \le n \le p$, $\alpha_j, \beta_j > 0$, and a_j, b_j are COMPLEX NUMBERS such that the pole of $\Gamma(b_j - \beta_j s)$ for $j = 1, 2, \ldots, m$ coincides with any POLE of $\Gamma(1 - a_j + \alpha_j s)$ for $j = 1, 2, \ldots, n$. In addition C, is a CONTOUR in the complex s-plane from $\omega - i\infty$ to $\omega + i\infty$ such that $(b_j + k)/\beta_j$ and $(a_j - 1 - k)/\alpha_j$ lie to the right and left of C, respectively.

see also MACROBERT'S E-FUNCTION, MEIJER'S G-FUNCTION

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Frac

see FRACTIONAL PART

Fractal

An object or quantity which displays SELF-SIMILARITY, in a somewhat technical sense, on all scales. The object need not exhibit *exactly* the same structure at all scales, but the same "type" of structures must appear on all scales. A plot of the quantity on a log-log graph versus scale then gives a straight line, whose slope is said to be the FRACTAL DIMENSION. The prototypical example for a fractal is the length of a coastline measured with different length RULERS. The shorter the RULER, the longer the length measured, a PARADOX known as the COASTLINE PARADOX.

see also BACKTRACKING, BARNSLEY'S FERN, BOX FRACTAL, BUTTERFLY FRACTAL, CACTUS FRACTAL, CANTOR SET, CANTOR SQUARE FRACTAL, CAROTID-KUNDALINI FRACTAL, CESÀRO FRACTAL, CHAOS GAME, CIRCLES-AND-SQUARES FRACTAL, COASTLINE PARADOX, DRAGON CURVE, FAT FRACTAL, FATOU SET, FLOWSNAKE FRACTAL, FRACTAL DIMENSION, H-FRACTAL, HÉNON MAP, ITERATED FUNCTION SYSTEM, JULIA FRACTAL, KAPLAN-YORKE MAP, KOCH ANTISNOWFLAKE, KOCH SNOWFLAKE, LÉVY FRACTAL, LÉVY TAPESTRY, LINDENMAYER SYSTEM, MANDELBROT SET, MANDELBROT TREE, MENGER SPONGE, MINKOWSKI SAUSAGE, MIRA FRACTAL, NEWTON'S METHOD, PENTAFLAKE, PYTHAGORAS TREE, RABINOVICH-FABRIKANT EQUATION, SAN MARCO FRACTAL, SIERPIŃSKI CARPET, SIERPIŃSKI CURVE, SIERPIŃSKI SIEVE, STAR FRACTAL, ZASLAVSKII MAP

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Fractal Dimension

The term "fractal dimension" is sometimes used to refer to what is more commonly called the CAPACITY DIMENSION (which is, roughly speaking, the exponent D in the expression $n(\epsilon) = \epsilon^{-D}$, where $n(\epsilon)$ is the minimum number of OPEN SETS of diameter ϵ needed to cover the set). However, it can more generally refer to any of the dimensions commonly used to characterize fractals (e.g., CAPACITY DIMENSION, CORRELATION DIMENSION, INFORMATION DIMENSION, LYAPUNOV DIMENSION, MINKOWSKI-BOULIGAND DIMENSION).

see also BOX COUNTING DIMENSION, CAPACITY DIMENSION, CORRELATION DIMENSION, FRACTAL DIMENSION, HAUSDORFF DIMENSION, INFORMATION DIMENSION, LYAPUNOV DIMENSION, MINKOWSKI-BOULIGAND DIMENSION, POINTWISE DIMENSION, q-DIMENSION

References

Rasband, S. N. "Fractal Dimension." Ch. 4 in *Chaotic Dynamics of Nonlinear Systems.* New York: Wiley, pp. 71–83, 1990.

Fractal Land

see CAROTID-KUNDALINI FRACTAL

Fractal Process

A 1-D MAP whose increments are distributed according to a NORMAL DISTRIBUTION. Let $y(t-\Delta t)$ and $y(t+\Delta t)$ be values, then their correlation is given by the BROWN FUNCTION

$$r = 2^{2H-1} - 1.$$

When $H = 1/2$, $r = 0$ and the fractal process corresponds to 1-D Brownian motion. If $H > 1/2$, then $r > 0$ and the process is called a PERSISTENT PROCESS. If $H < 1/2$, then $r < 0$ and the process is called an ANTIPERSISTENT PROCESS.

see also ANTIPERSISTENT PROCESS, PERSISTENT PROCESS

References

von Seggern, D. *CRC Standard Curves and Surfaces.* Boca Raton, FL: CRC Press, 1993.

Fractal Sequence

Given an INFINITIVE SEQUENCE $\{x_n\}$ with associated array $a(i,j)$, then $\{x_n\}$ is said to be a fractal sequence

1. If $i+1 = x_n$, then there exists $m < n$ such that $i = x_m$,
2. If $h < i$, then, for every j, there is exactly one k such that $a(i,j) < a(h,k) < a(i,j+1)$.

(As i and j range through N, the array $A = a(i,j)$, called the associative array of x, ranges through all of N.) An example of a fractal sequence is 1, 1, 1, 1, 2, 1, 2, 1, 3, 2, 1, 3, 2, 1, 3, ...

If $\{x_n\}$ is a fractal sequence, then the associated array is an INTERSPERSION. If x is a fractal sequence, then the UPPER-TRIMMED SUBSEQUENCE is given by $\lambda(x) = x$, and the LOWER-TRIMMED SUBSEQUENCE $V(x)$ is another fractal sequence. The SIGNATURE of an IRRATIONAL NUMBER is a fractal sequence.

see also INFINITIVE SEQUENCE

References

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Fractal Valley

see CAROTID-KUNDALINI FUNCTION

Fraction

A RATIONAL NUMBER expressed in the form a/b, where a is called the NUMERATOR and b is called the DENOMINATOR. A PROPER FRACTION is a fraction such that $a/b < 1$, and a LOWEST TERMS FRACTION is a fraction with common terms canceled out of the NUMERATOR and DENOMINATOR.

The Egyptians expressed their fractions as sums (and differences) of UNIT FRACTIONS. Conway and Guy (1999) give a table of Roman NOTATION for fractions, in which multiples of 1/12 (the UNCIA) were given separate names.

see also ADJACENT FRACTION, ANOMALOUS CANCELLATION, CONTINUED FRACTION, DENOMINATOR, EGYPTIAN FRACTION, FAREY SEQUENCE, GOLDEN RULE, HALF, LOWEST TERMS FRACTION, MEDIANT, NUMERATOR, PROPER FRACTION, PYTHAGOREAN FRACTION, QUARTER, RATIONAL NUMBER, UNIT FRACTION

References

Conway, J. H. and Guy, R. K. *The Book of Numbers.* New York: Springer-Verlag, pp. 22–23, 1996.

Courant, R. and Robbins, H. "Decimal Fractions. Infinite Decimals." §2.2.2 in *What is Mathematics?: An Elementary Approach to Ideas and Methods, 2nd ed.* Oxford, England: Oxford University Press, pp. 61–63, 1996.

Fractional Calculus

Denote the nth DERIVATIVE D^n and the n-fold INTEGRAL D^{-n}. Then

$$D^{-1}f(t) = \int_0^t f(\xi)\,d\xi. \tag{1}$$

Now, if

$$D^{-n}f(t) = \frac{1}{(n-1)!}\int_0^t (t-\xi)^{n-1} f(\xi)\,d\xi \tag{2}$$

is true for n, then

$$\begin{aligned} D^{-(n+1)}f(t) &= D^{-1}\left[\frac{1}{(n-1)!}\int_0^t (t-\xi)^{n-1}f(\xi)\,d\xi\right] \\ &= \int_0^t \left[\frac{1}{(n-1)!}\int_0^x (x-\xi)^{n-1}f(\xi)\,d\xi\right]dx. \end{aligned} \tag{3}$$

Interchanging the order of integration gives

$$D^{-(n+1)}f(t) = \frac{1}{n!}\int_0^t (t-\xi)^n f(\xi)\,d\xi. \tag{4}$$

But (2) is true for $n = 1$, so it is also true for all n by INDUCTION. The fractional integral of $f(t)$ can then be defined by

$$D^{-\nu}f(t) = \frac{1}{\Gamma(\nu)}\int_0^t (t-\xi)^{\nu-1} f(\xi)\,d\xi, \tag{5}$$

where $\Gamma(\nu)$ is the GAMMA FUNCTION.

The fractional integral can only be given in terms of elementary functions for a small number of functions. For example,

$$D^{-\nu}t^{-\lambda} = \frac{\Gamma(\lambda+1)}{\Gamma(\lambda+\nu+1)}t^{\lambda+\nu} \quad \text{for } \lambda > -1, \nu > 0 \tag{6}$$

$$D^{-\nu}e^{at} = \frac{1}{\Gamma(\nu)}e^{at}\int_0^t x^{\nu-1}e^{-ax}\,dx \equiv E_t(\nu,a), \tag{7}$$

where $E_t(\nu,a)$ is the E_t-FUNCTION. The fractional derivative of f (if it exists) can be defined by

$$D^\mu f(t) = D^m[D^{-(m-\mu)}f(t)]. \tag{8}$$

An example is

$$\begin{aligned} D^\mu t^\lambda &= \frac{\Gamma(\lambda+1)}{\Gamma(\lambda+m-\mu+1)} \\ &= \frac{\Gamma(\lambda+1)}{\Gamma(\lambda-\mu+1)}t^{\lambda-\mu} \quad \text{for } \lambda > -1, \mu > 0 \end{aligned} \tag{9}$$

$$D^\rho E_t(\nu,a) = E_t(\nu-\rho,a) \qquad \text{for } \nu > 0, \rho \neq 0. \tag{10}$$

It is always true that, for $\mu, \nu > 0$,

$$D^{-\mu} D^{-\nu} f(t) = D^{-(\mu+\nu)}, \tag{11}$$

but *not* always true that

$$D^{\mu} D^{\nu} = D^{\mu+\nu}. \tag{12}$$

see also DERIVATIVE, INTEGRAL

References
Love, E. R. "Fractional Derivatives of Imaginary Order." *J. London Math. Soc.* **3**, 241–259, 1971.
McBride, A. C. *Fractional Calculus.* New York: Halsted Press, 1986.
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Spanier, J. and Oldhan, K. B. *The Fractional Calculus.* New York: Academic Press, 1974.

Fractional Derivative

see FRACTIONAL CALCULUS

Fractional Differential Equation

The solution to the differential equation

$$[D^{2v} + aD^{v} + bD^{0}]y(t) = 0$$

is

$$y(t) = \begin{cases} e_\alpha(t) - e_\beta(t) \\ \quad \text{for } \alpha \neq \beta \\ te^{\alpha t}, \sum_{k=-(q-1)}^{q-1} \alpha^k (q - |k|) D^{1-(k+1)v}(te^{\alpha^q t}) \\ \quad \text{for } \alpha = \beta \neq 0 \\ \frac{t^{2\nu-1}}{\Gamma(2v)} \\ \quad \text{for } \alpha = \beta = 0, \end{cases}$$

where

$$q = \frac{1}{v}$$

$$e_\beta(t) = \sum_{k=0}^{q-1} \beta^{q-k-1} E_t(-kv, \beta^q),$$

$E_t(a, x)$ is the E_t-FUNCTION, and $\Gamma(n)$ is the GAMMA FUNCTION.

References
Miller, K. S. "Derivatives of Noninteger Order." *Math. Mag.* **68**, 183–192, 1995.

Fractional Fourier Transform

A z-TRANSFORM with

$$z \equiv e^{2\pi i \alpha / N}$$

for $\alpha \neq \pm 1$. This transform can be used to detect frequencies which are not INTEGER multiples of the lowest DISCRETE FOURIER TRANSFORM frequency.

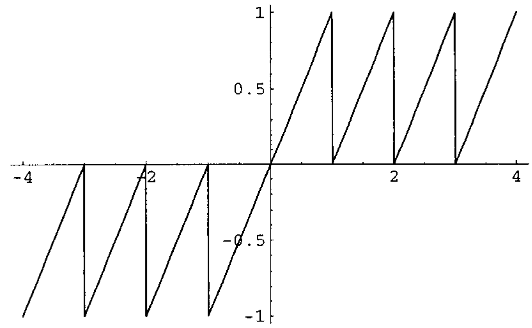
see also z-TRANSFORM

References
Graham, R. L.; Knuth, D. E.; and Patashnik, O. *Concrete Mathematics, 2nd ed.* Reading, MA: Addison-Wesley, 1994.

Fractional Integral

see FRACTIONAL CALCULUS

Fractional Part



The function giving the fractional (nonintegral) part of a number and defined as

$$\text{frac}(x) \equiv \begin{cases} x - \lfloor x \rfloor & x \geq 0 \\ x - \lfloor x \rfloor - 1 & x \leq 0, \end{cases}$$

where $\lfloor x \rfloor$ is the FLOOR FUNCTION.

see also CEILING FUNCTION, FLOOR FUNCTION, NINT, ROUND, TRUNCATE, WHOLE NUMBER

References
Spanier, J. and Oldham, K. B. "The Integer-Value Int(x) and Fractional-Value frac(x) Functions." Ch. 9 in *An Atlas of Functions.* Washington, DC: Hemisphere, pp. 71–78, 1987.

Fractran

Fractran is an algorithm applied to a given list f_1, f_2, ..., f_k of FRACTIONS. Given a starting INTEGER N, the Fractran algorithm proceeds by repeatedly multiplying the integer at a given stage by the first element f_i given an integer PRODUCT. The algorithm terminates when there is no such f_i.

The list

$$\frac{17}{91}, \frac{78}{85}, \frac{19}{51}, \frac{23}{38}, \frac{29}{33}, \frac{77}{29}, \frac{95}{23}, \frac{77}{19}, \frac{1}{17}, \frac{11}{13}, \frac{13}{11}, \frac{15}{2}, \frac{1}{7}, \frac{55}{1}$$

with starting integer $N = 2$ generates a sequence 2, 15, 825, 725, 1925, 2275, 425, 390, 330, 290, 770, ... Conway (1987) showed that the only other powers of 2 which occur are those with PRIME exponent: 2^2, 2^3, 2^5, 2^7, ...

References
Conway, J. H. "Unpredictable Iterations." In *Proc. Number Theory Conf., Boulder, CO,* pp. 49–52, 1972.
Conway, J. H. "Fractran: A Simple Universal Programming Language for Arithmetic." Ch. 2 in *Open Problems in Communication and Computation* (Ed. T. M. Cover and B. Gopinath). New York: Springer-Verlag, pp. 4–26, 1987.

Framework

Consider a finite collection of points $p = (p_1, \ldots, p_n)$, $p_i \in \mathbb{R}^d$ EUCLIDEAN SPACE (known as a CONFIGURATION) and a graph G whose VERTICES correspond to pairs of points that are constrained to stay the same distance apart. Then the graph G together with the configuration p, denoted $G(p)$, is called a framework.

see also BAR (EDGE), CONFIGURATION, RIGID

Franklin Magic Square

52	61	4	13	20	29	36	45
14	3	62	51	46	35	30	19
53	60	5	12	21	28	37	44
11	6	59	54	43	38	27	22
55	58	7	10	23	26	39	42
9	8	57	56	41	40	25	24
50	63	2	15	18	31	34	47
16	1	64	49	48	33	32	17

Benjamin Franklin constructed the above 8×8 PANMAGIC SQUARE having MAGIC CONSTANT 260. Any half-row or half-column in this square totals 130, and the four corners plus the middle total 260. In addition, bent diagonals (such as 52-3-5-54-10-57-63-16) also total 260 (Madachy 1979, p. 87).

see also MAGIC SQUARE, PANMAGIC SQUARE

References

Madachy, J. S. "Magic and Antimagic Squares." Ch. 4 in *Madachy's Mathematical Recreations.* New York: Dover, pp. 103–113, 1979.

Pappas, T. "The Magic Square of Benjamin Franklin." *The Joy of Mathematics.* San Carlos, CA: Wide World Publ./Tetra, p. 97, 1989.

Fransén-Robinson Constant

$$F \equiv \int_0^\infty \frac{dx}{\Gamma(x)} = 2.8077702420\ldots,$$

where $\Gamma(x)$ is the GAMMA FUNCTION. The above plots show the functions $\Gamma(x)$ and $1/\Gamma(x)$.

see also GAMMA FUNCTION

References

Finch, S. "Favorite Mathematical Constants." `http://www.mathsoft.com/asolve/constant/fran/fran.html`.

Fransén, A. "Accurate Determination of the Inverse Gamma Integral." *BIT* **19**, 137–138, 1979.

Fransén, A. "Addendum and Corrigendum to 'High-Precision Values of the Gamma Function and of Some Related Coefficients.'" *Math. Comput.* **37**, 233–235, 1981.

Fransén, A. and Wrigge, S. "High-Precision Values of the Gamma Function and of Some Related Coefficients." *Math. Comput.* **34**, 553–566, 1980.

Plouffe, S. "Fransen-Robinson Constant." `http://lacim.uqam.ca/piDATA/fransen.txt`.

Fréchet Bounds

Any bivariate distribution function with marginal distribution functions F and G satisfies

$$\max\{F(x)+G(y)-1,0\} \leq H(x,y) \leq \min\{F(x),G(y)\}.$$

Fréchet Derivative

A function f is Fréchet differentiable at a if

$$\lim_{x\to a} \frac{f(x)-f(a)}{x-a}$$

exists. This is equivalent to the statement that ϕ has a removable DISCONTINUITY at a, where

$$\phi(x) \equiv \frac{f(x)-f(a)}{x-a}.$$

Every function which is Fréchet differentiable is also Carathéodory differentiable.

see also CARATHÉODORY DERIVATIVE, DERIVATIVE

Fréchet Space

A complete metrizable SPACE, sometimes also with the restriction that the space be locally convex.

Fredholm Integral Equation of the First Kind

An INTEGRAL EQUATION of the form

$$f(x) = \int_{-\infty}^\infty K(x,t)\phi(t)\,dt$$

$$\phi(x) = \frac{1}{2\pi}\int_{-\infty}^\infty \frac{F(\omega)}{K(\omega)} e^{-i\omega x}\,d\omega.$$

see also FREDHOLM INTEGRAL EQUATION OF THE SECOND KIND, INTEGRAL EQUATION, VOLTERRA INTEGRAL EQUATION OF THE FIRST KIND, VOLTERRA INTEGRAL EQUATION OF THE SECOND KIND

References

Arfken, G. *Mathematical Methods for Physicists, 3rd ed.* Orlando, FL: Academic Press, p. 865, 1985.

Fredholm Integral Equation of the Second Kind

An INTEGRAL EQUATION of the form

$$\phi(x) = f(x) + \lambda \int_{-\infty}^\infty K(x,t)\phi(t)\,dt$$

$$\phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^\infty \frac{F(t)e^{-ixt}\,dt}{1-\sqrt{2\pi}\,\lambda K(t)}.$$

see also FREDHOLM INTEGRAL EQUATION OF THE FIRST KIND, INTEGRAL EQUATION, NEUMANN SERIES (INTEGRAL EQUATION), VOLTERRA INTEGRAL

EQUATION OF THE FIRST KIND, VOLTERRA INTEGRAL EQUATION OF THE SECOND KIND

References
Arfken, G. *Mathematical Methods for Physicists, 3rd ed.* Orlando, FL: Academic Press, p. 865, 1985.
Press, W. H.; Flannery, B. P.; Teukolsky, S. A.; and Vetterling, W. T. "Fredholm Equations of the Second Kind." §18.1 in *Numerical Recipes in FORTRAN: The Art of Scientific Computing, 2nd ed.* Cambridge, England: Cambridge University Press, pp. 782–785, 1992.

Free

When referring to a planar object, "free" means that the object is regarded as capable of being picked up out of the plane and flipped over. As a result, MIRROR IMAGES are equivalent for free objects.

A free abstract mathematical object is generated by n elements in a "free manner," i.e., such that the n elements satisfy no nontrivial relations among themselves. To make this more formal, an algebraic GADGET X is freely generated by a SUBSET G if, for any function $f : G \to Y$ where Y is any other algebraic GADGET, there exists a unique HOMOMORPHISM (which has different meanings depending on what kind of GADGETS you're dealing with) $g : X \to Y$ such that g restricted to G is f.

If the algebraic GADGETS are VECTOR SPACES, then G freely generates X IFF G is a BASIS for X. If the algebraic GADGETS are ABELIAN GROUPS, then G freely generates X IFF X is a DIRECT SUM of the INTEGERS, with G consisting of the standard BASIS.

see also FIXED, GADGET, MIRROR IMAGE, RANK

Free Group

The generators of a group G are defined to be the smallest subset of group elements such that all other elements of G can be obtained from them and their inverses. A GROUP is a free group if no relation exists between its generators (other than the relationship between an element and its inverse required as one of the defining properties of a group). For example, the additive group of whole numbers is free with a single generator, 1.

see also FREE SEMIGROUP

Free Semigroup

A SEMIGROUP with a noncommutative product in which no PRODUCT can ever be expressed more simply in terms of other ELEMENTS.

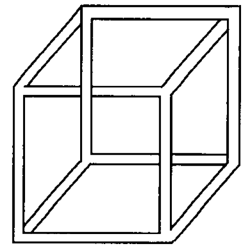
see also FREE GROUP, SEMIGROUP

Free Variable

An occurrence of a variable in a LOGIC FORMULA which is not inside the scope of a QUANTIFIER.

see also BOUND, SENTENCE

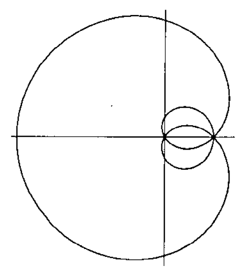
Freemish Crate



An IMPOSSIBLE FIGURE box which can be drawn but not built.

References
Fineman, M. *The Nature of Visual Illusion.* New York: Dover, p. 120–122, 1996.
Jablan, S. "Are Impossible Figures Possible?" `http://members.tripod.com/~modularity/kulpa.htm`.
Pappas, T. "The Impossible Tribar." *The Joy of Mathematics.* San Carlos, CA: Wide World Publ./Tetra, p. 13, 1989.

Freeth's Nephroid



A STROPHOID of a CIRCLE with the POLE O at the CENTER of the CIRCLE and the fixed point P on the CIRCUMFERENCE of the CIRCLE. In a paper published by the London Mathematical Society in 1879, T. J. Freeth described it and various other STROPHOIDS (MacTutor Archive). If the line through P PARALLEL to the y-AXIS cuts the NEPHROID at A, then ANGLE AOP is $3\pi/7$, so this curve can be used to construct a regular HEPTAGON. The POLAR equation is

$$r = a[1 + 2\sin(\tfrac{1}{2}\theta)].$$

see also STROPHOID

References
Lawrence, J. D. *A Catalog of Special Plane Curves.* New York: Dover, pp. 175 and 177–178, 1972.
MacTutor History of Mathematics Archive. "Freeth's Nephroid." `http://www-groups.dcs.st-and.ac.uk/~history/Curves/Freeths.html`.

Freiman's Constant

The end of the last gap in the LAGRANGE SPECTRUM, given by

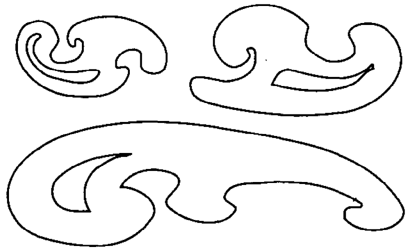
$$F \equiv \frac{2221564096 + 293748\sqrt{462}}{491993569} = 4.5278295661\ldots.$$

REAL NUMBERS greater than F are members of the MARKOV SPECTRUM.

see also LAGRANGE SPECTRUM, MARKOV SPECTRUM

References
Conway, J. H. and Guy, R. K. *The Book of Numbers.* New York: Springer-Verlag, pp. 188–189, 1996.

French Curve



French curves are plastic (or wooden) templates having an edge composed of several different curves. French curves are used in drafting (or were before computer-aided design) to draw smooth curves of almost any desired curvature in mechanical drawings. Several typical French curves are illustrated above.

see also CORNU SPIRAL

Frenet Formulas

Also known as the SERRET-FRENET FORMULAS

$$\begin{bmatrix} \dot{\mathbf{T}} \\ \dot{\mathbf{N}} \\ \dot{\mathbf{B}} \end{bmatrix} = \begin{bmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & \tau \\ 0 & -\tau & 0 \end{bmatrix} \begin{bmatrix} \mathbf{T} \\ \mathbf{N} \\ \mathbf{B} \end{bmatrix},$$

where $\mathbf{T}$ is the unit TANGENT VECTOR, $\mathbf{N}$ is the unit NORMAL VECTOR, $\mathbf{B}$ is the unit BINORMAL VECTOR, τ is the TORSION, κ is the CURVATURE, and $\dot{\mathbf{x}}$ denotes $d\mathbf{x}/ds$.

see also CENTRODE, FUNDAMENTAL THEOREM OF SPACE CURVES, NATURAL EQUATION

References
Frenet, F. "Sur les courbes à double courbure." Thèse. Toulouse, 1847. Abstract in *J. de Math.* **17**, 1852.
Gray, A. *Modern Differential Geometry of Curves and Surfaces.* Boca Raton, FL: CRC Press, p. 126, 1993.
Kreyszig, E. "Formulae of Frenet." §15 in *Differential Geometry.* New York: Dover, p. 40–43, 1991.
Serret, J. A. "Sur quelques formules relatives à la théorie des courbes à double courbure." *J. de Math.* **16**, 1851.

Frequency Curve

see GAUSSIAN FUNCTION

Fresnel's Elasticity Surface

A QUARTIC SURFACE given by

$$r = \sqrt{a^2x^2 + b^2y^2 + c^2z^2},$$

where

$$r^2 \equiv x'^2 + y'^2 + z'^2,$$

also known as FRESNEL'S WAVE SURFACE. It was introduced by Fresnel in his studies of crystal optics.

References
Fischer, G. (Ed.). *Mathematical Models from the Collections of Universities and Museums.* Braunschweig, Germany: Vieweg, p. 16, 1986.
Fischer, G. (Ed.). Plates 38–39 in *Mathematische Modelle/Mathematical Models, Bildband/Photograph Volume.* Braunschweig, Germany: Vieweg, pp. 38–39, 1986.
von Seggern, D. *CRC Standard Curves and Surfaces.* Boca Raton, FL: CRC Press, p. 304, 1993.

Fresnel Integrals

In physics, the Fresnel integrals are most often defined by

$$\begin{aligned} C(u) + iS(u) &\equiv \int_0^u e^{i\pi x^2/2}\,dx \\ &= \int_0^u \cos(\tfrac{1}{2}\pi x^2)\,dx + i\int_0^u \sin(\tfrac{1}{2}\pi x^2)\,dx, \end{aligned} \tag{1}$$

so

$$C(u) \equiv \int_0^u \cos(\tfrac{1}{2}\pi x^2)\,dx \tag{2}$$

$$S(u) \equiv \int_0^u \sin(\tfrac{1}{2}\pi x^2)\,dx. \tag{3}$$

They satisfy

$$C(\pm\infty) = -\tfrac{1}{2} \tag{4}$$

$$S(\pm\infty) = \tfrac{1}{2}. \tag{5}$$

Related functions are defined as

$$C_1(z) \equiv \sqrt{\frac{2}{\pi}} \int_0^x \cos t^2\,dt \tag{6}$$

$$S_1(z) \equiv \sqrt{\frac{2}{\pi}} \int_0^x \sin t^2\,dt \tag{7}$$

$$C_2(z) \equiv \frac{1}{\sqrt{2\pi}} \int \frac{\cos t}{\sqrt{t}}\,dt \tag{8}$$

$$S_2(z) \equiv \frac{1}{\sqrt{2\pi}} \int \frac{\sin t}{\sqrt{t}}\,dt. \tag{9}$$

An asymptotic expansion for $x \gg 1$ gives

$$C(u) \approx \frac{1}{2} + \frac{1}{\pi u}\sin(\tfrac{1}{2}\pi u^2) \tag{10}$$

$$S(u) \approx \frac{1}{2} - \frac{1}{\pi u}\cos(\tfrac{1}{2}\pi u^2). \tag{11}$$

Therefore, as $u \to \infty$, $C(u) = 1/2$ and $S(u) = 1/2$. The Fresnel integrals are sometimes alternatively defined as

$$x(t) = \int_0^t \cos(v^2)\,dv \tag{12}$$

$$y(t) = \int_0^t \sin(v^2)\,dv. \tag{13}$$

Letting $x \equiv v^2$ so $dx = 2v\,dv = 2\sqrt{x}\,dv$, and $dv = x^{-1/2}\,dx/2$

$$x(t) = \tfrac{1}{2}\int_0^{\sqrt{t}} x^{-1/2}\cos x\,dx \tag{14}$$

$$y(t) = \tfrac{1}{2}\int_0^{\sqrt{t}} x^{-1/2}\sin x\,dx. \tag{15}$$

In this form, they have a particularly simple expansion in terms of SPHERICAL BESSEL FUNCTIONS OF THE FIRST KIND. Using

$$j_0(x) = \frac{\sin x}{x} \tag{16}$$

$$n_1(x) = -j_{-1}(x) = -\frac{\cos x}{x}, \tag{17}$$

where $n_1(x)$ is a SPHERICAL BESSEL FUNCTION OF THE SECOND KIND

$$\begin{aligned} x(t^2) &= -\tfrac{1}{2}\int_0^t n_1(x)x^{1/2}\,dx \\ &= \tfrac{1}{2}\int_0^t j_{-1}(x)x^{1/2}\,dx = x^{1/2}\sum_{n=0}^{\infty} j_{2n}(x) \end{aligned} \tag{18}$$

$$\begin{aligned} y(t^2) &= \tfrac{1}{2}\int_0^t j_0(x)x^{1/2}\,dx \\ &= x^{1/2}\sum_{n=0}^{\infty} j_{2n+1}(x). \end{aligned} \tag{19}$$

see also CORNU SPIRAL

References

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Leonard, I. E. "More on Fresnel Integrals." *Amer. Math. Monthly* **95**, 431–433, 1988.

Press, W. H.; Flannery, B. P.; Teukolsky, S. A.; and Vetterling, W. T. "Fresnel Integrals, Cosine and Sine Integrals." §6.79 in *Numerical Recipes in FORTRAN: The Art of Scientific Computing, 2nd ed.* Cambridge, England: Cambridge University Press, pp. 248–252, 1992.

Spanier, J. and Oldham, K. B. "The Fresnel Integrals $S(x)$ and $C(x)$." Ch. 39 in *An Atlas of Functions.* Washington, DC: Hemisphere, pp. 373–383, 1987.

Fresnel's Wave Surface

see FRESNEL'S ELASTICITY SURFACE

Frey Curve

Let $a^p + b^p = c^p$ be a solution to FERMAT'S LAST THEOREM. Then the corresponding Frey curve is

$$y^2 = x(x - a^p)(x + b^p). \tag{1}$$

Frey showed that such curves cannot be MODULAR, so if the TANIYAMA-SHIMURA CONJECTURE were true, Frey curves couldn't exist and FERMAT'S LAST THEOREM would follow with b EVEN and $a \equiv -1 \pmod 4$. Frey curves are SEMISTABLE. Invariants include the DISCRIMINANT

$$(a^p - 0)^2(-b^p - 0)[a^p - (-b)^p]^2 = a^{2p}b^{2p}c^{2p}. \tag{2}$$

The MINIMAL DISCRIMINANT is

$$\Delta = 2^{-8}a^{2p}b^{2p}c^{2p}, \tag{3}$$

the CONDUCTOR is

$$N = \prod_{l|abc} l, \tag{4}$$

and the j-INVARIANT is

$$j = \frac{2^8(a^{2p} + b^{2p} + a^p b^p)^3}{a^{2p}b^{2p}c^{2p}} = \frac{2^8(c^{2p} - b^p c^p)^3}{(abc)^{2p}}. \tag{5}$$

see also ELLIPTIC CURVE, FERMAT'S LAST THEOREM, TANIYAMA-SHIMURA CONJECTURE

References

Cox, D. A. "Introduction to Fermat's Last Theorem." *Amer. Math. Monthly* **101**, 3–14, 1994.

Gouvêa, F. Q. "A Marvelous Proof." *Amer. Math. Monthly* **101**, 203–222, 1994.

Frey Elliptic Curve

see FREY CURVE

Friday the Thirteenth

The Gregorian calendar follows a pattern of leap years which repeats every 400 years. There are 4,800 months in 400 years, so the 13th of the month occurs 4,800 times in this interval. The number of times the 13th occurs on each weekday is given in the table below. As shown by Brown (1933), the thirteenth of the month is slightly more likely to be on a Friday than on any other day.

Day	Number of 13s	Fraction
Sunday	687	14.31%
Monday	685	14.27%
Tuesday	685	14.27%
Wednesday	687	14.31%
Thursday	684	14.25%
Friday	688	14.33%
Saturday	684	14.25%

see also 13, WEEKDAY

References

Ball, W. W. R. and Coxeter, H. S. M. *Mathematical Recreations and Essays, 13th ed.* New York: Dover, p. 27, 1987.

Brown, B. H. "Solution to Problem E36." *Amer. Math. Monthly* **40**, 607, 1933.

Press, W. H.; Flannery, B. P.; Teukolsky, S. A.; and Vetterling, W. T. *Numerical Recipes in FORTRAN: The Art of Scientific Computing, 2nd ed.* Cambridge, England: Cambridge University Press, pp. 14–15, 1992.

Friend

A friend of a number n is another number m such that (m, n) is a FRIENDLY PAIR.

see also FRIENDLY PAIR, SOLITARY NUMBER

References
Anderson, C. W. and Hickerson, D. Problem 6020. "Friendly Integers." *Amer. Math. Monthly* **84**, 65–66, 1977.

Friendly Giant Group

see MONSTER GROUP

Friendly Pair

Define

$$\Sigma(n) \equiv \frac{\sigma(n)}{n},$$

where $\sigma(n)$ is the DIVISOR FUNCTION. Then a PAIR of distinct numbers (k, m) is a friendly pair (and k is said to be a FRIEND of m) if

$$\Sigma(k) = \Sigma(m).$$

For example, 4320 and 4680 are a friendly pair, since $\sigma(4320) = 15120$, $\sigma(4680) = 16380$, and

$$\Sigma(4320) \equiv \tfrac{15120}{4320} = \tfrac{7}{2}$$
$$\Sigma(4680) \equiv \tfrac{16380}{4680} = \tfrac{7}{2}.$$

Numbers which do not have FRIENDS are called SOLITARY NUMBERS. SOLITARY NUMBERS satisfy $(\sigma(n), n) = 1$, where (a, b) is the GREATEST COMMON DIVISOR of a and b.

see also ALIQUOT SEQUENCE, FRIEND, SOLITARY NUMBER

References
Anderson, C. W. and Hickerson, D. Problem 6020. "Friendly Integers." *Amer. Math. Monthly* **84**, 65–66, 1977.

Frieze Pattern

$$\begin{array}{ccc} & b & \\ a & & d \\ & c & \end{array}$$

An arrangement of numbers at the intersection of two sets of perpendicular diagonals such that $a+d = b+c+1$ (for an additive frieze pattern) or $ad = bc + 1$ (for a multiplicative frieze pattern) in each diamond.

References
Conway, J. H. and Coxeter, H. S. M. "Triangulated Polygons and Frieze Patterns." *Math. Gaz.* **57**, 87–94, 1973.
Conway, J. H. and Guy, R. K. In *The Book of Numbers.* New York: Springer-Verlag, pp. 74–76 and 96–97, 1996.

Frobenius-König Theorem

The PERMANENT of an $n \times n$ MATRIX with all entries either 0 or 1 is 0 IFF the MATRIX contains an $r \times s$ submatrix of 0s with $r + s = n + 1$. This result follows from the KÖNIG-EGEVÁRY THEOREM.

see also KÖNIG-EGEVÁRY THEOREM, PERMANENT

Frobenius Map

A map $x \mapsto x^p$ where p is a PRIME.

Frobenius Method

If x_0 is an ordinary point of the ORDINARY DIFFERENTIAL EQUATION, expand y in a TAYLOR SERIES about x_0, letting

$$y = \sum_{n=0}^{\infty} a_n x^n. \tag{1}$$

Plug y back into the ODE and group the COEFFICIENTS by POWER. Now, obtain a RECURRENCE RELATION for the nth term, and write the TAYLOR SERIES in terms of the a_ns. Expansions for the first few derivatives are

$$y = \sum_{n=0}^{\infty} a_n x^n \tag{2}$$

$$y' = \sum_{n=1}^{\infty} n a_n x^{n-1} = \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n \tag{3}$$

$$y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n. \tag{4}$$

If x_0 is a regular singular point of the ORDINARY DIFFERENTIAL EQUATION,

$$P(x)y'' + Q(x)y' + R(x)y = 0, \tag{5}$$

solutions may be found by the Frobenius method or by expansion in a LAURENT SERIES. In the Frobenius method, assume a solution of the form

$$y = x^k \sum_{n=0}^{\infty} a_n x^n, \tag{6}$$

so that

$$y = x^k \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} a_n x^{n+k} \tag{7}$$

$$y' = \sum_{n=0}^{\infty} a_n (n+k) x^{k+n-1} \tag{8}$$

$$y'' = \sum_{n=0}^{\infty} a_n (n+k)(n+k-1) x^{k+n-2}. \tag{9}$$

Now, plug y back into the ODE and group the COEFFICIENTS by POWER to obtain a recursion FORMULA for the a_nth term, and then write the TAYLOR SERIES in terms of the a_ns. Equating the a_0 term to 0 will produce the so-called INDICIAL EQUATION, which will give the allowed values of k in the TAYLOR SERIES.

FUCHS'S THEOREM guarantees that at least one POWER series solution will be obtained when applying the Frobenius method if the expansion point is an ordinary, or regular, SINGULAR POINT. For a regular SINGULAR POINT, a LAURENT SERIES expansion can also be used. Expand y in a LAURENT SERIES, letting

$$y = c_{-n}x^{-n} + \ldots + c_{-1}x^{-1} + c_0 + c_1 x + \ldots + c_n x^n + \ldots. \quad (10)$$

Plug y back into the ODE and group the COEFFICIENTS by POWER. Now, obtain a recurrence FORMULA for the c_nth term, and write the TAYLOR EXPANSION in terms of the c_ns.

see also FUCHS'S THEOREM, ORDINARY DIFFERENTIAL EQUATION

References

Arfken, G. "Series Solutions—Frobenius' Method." §8.5 in *Mathematical Methods for Physicists, 3rd ed.* Orlando, FL: Academic Press, pp. 454–467, 1985.

Frobenius-Peron Equation

$$\rho_{n+1}(x) = \int \rho_n(y)\delta[x - M(y)]\,dy,$$

where $\delta(x)$ is a DELTA FUNCTION, $M(x)$ is a map, and ρ is the NATURAL DENSITY.

References

Ott, E. *Chaos in Dynamical Systems.* New York: Cambridge University Press, p. 51, 1993.

Frobenius Pseudoprime

Let $f(x)$ be a MONIC POLYNOMIAL of degree d with discriminant Δ. Then an ODD INTEGER n with $(n, f(0)\Delta) = 1$ is called a Frobenius pseudoprime with respect to $f(x)$ if it passes a certain algorithm given by Grantham (1996). A Frobenius pseudoprime with respect to a POLYNOMIAL $f(x) \in \mathbb{Z}[x]$ is then a composite Frobenius probably prime with respect to the POLYNOMIAL $x - a$.

While 323 is the first LUCAS PSEUDOPRIME with respect to the Fibonacci polynomial $x^2 - x - 1$, the first Frobenius pseudoprime is 5777. If $f(x) = x^3 - rx^2 + sx - 1$, then any Frobenius pseudoprime n with respect to $f(x)$ is also a PERRIN PSEUDOPRIME. Grantham (1997) gives a test based on Frobenius pseudoprimes which is passed by COMPOSITE NUMBERS with probability at most 1/7710.

see also PERRIN PSEUDOPRIME, PSEUDOPRIME, STRONG FROBENIUS PSEUDOPRIME

References

Grantham, J. "Frobenius Pseudoprimes." 1996. `http://www.clark.net/pub/grantham/pseudo/pseudo.ps`

Grantham, J. "A Frobenius Probable Prime Test with High Confidence." 1997. `http://www.clark.net/pub/grantham/pseudo/pseudo2.ps`

Grantham, J. "Pseudoprimes/Probable Primes." `http://www.clark.net/pub/grantham/pseudo.`

Frobenius Theorem

Let $\mathsf{A} = a_{ij}$ be a MATRIX with POSITIVE COEFFICIENTS so that $a_{ij} > 0$ for all $i, j = 1, 2, \ldots, n$, then A has a POSITIVE EIGENVALUE λ_0, and all its EIGENVALUES lie on the CLOSED DISK

$$|z| \leq \lambda_0.$$

see also CLOSED DISK, OSTROWSKI'S THEOREM

References

Gradshteyn, I. S. and Ryzhik, I. M. *Tables of Integrals, Series, and Products, 5th ed.* San Diego, CA: Academic Press, p. 1121, 1979.

Frobenius Triangle Identities

Let $C_{L,M}$ be a PADÉ APPROXIMANT. Then

$$C_{(L+1)/M}S_{(L-1)/M} - C_{L/(M+1)}S_{L/(M+1)} = C_{L/M}S_{L/M} \quad (1)$$

$$C_{L/(M+1)}S_{(L+1)/M} - C_{(L+1)/M}S_{L/(M+1)} = C_{(L+1)/(M+1)}xS_{L/M} \quad (2)$$

$$C_{(L+1)/M}S_{L/M} - C_{L/M}S_{(L+1)/M} = C_{(L+1)/(M+1)}xS_{L/(M-1)} \quad (3)$$

$$C_{L/(M+1)}S_{L/M} - C_{L/M}S_{L/(M+1)} = C_{(L+1)/(M+1)}xS_{(L-1)/M}, \quad (4)$$

where

$$S_{L/M} = G(x)P_L(x) + H(x)Q_M(x) \quad (5)$$

and C is the C-DETERMINANT.

see also C-DETERMINANT, PADÉ APPROXIMANT

References

Baker, G. A. Jr. *Essentials of Padé Approximants in Theoretical Physics.* New York: Academic Press, p. 31, 1975.

Frontier

see BOUNDARY

Frullani's Integral

If $f'(x)$ is continuous and the integral converges,

$$\int_0^\infty \frac{f(ax) - f(bx)}{x}\,dx = [f(0) - f(\infty)]\ln\left(\frac{b}{a}\right).$$

References

Spiegel, M. R. *Mathematical Handbook of Formulas and Tables.* New York: McGraw-Hill, 1968.

Frustum

The portion of a solid which lies between two PARALLEL PLANES cutting the solid. Degenerate cases are obtained for finite solids by cutting with a single PLANE only.

see also CONICAL FRUSTUM, PYRAMIDAL FRUSTUM, SPHERICAL SEGMENT

Fubini Principle

If the average number of envelopes per pigeonhole is a, then some pigeonhole will have at least a envelopes. Similarly, there must be a pigeonhole with at most a envelopes.

see also PIGEONHOLE PRINCIPLE

Fuchsian System

A system of linear differential equations

$$\frac{dy}{dz} = A(z)y,$$

with $A(z)$ an ANALYTIC $n \times n$ MATRIX, for which the MATRIX $A(z)$ is ANALYTIC in $\bar{\mathbb{C}}\backslash\{a_1, \ldots, a_N\}$ and has a POLE of order 1 at a_j for $j = 1, \ldots, N$. A system is Fuchsian IFF there exist $n \times n$ matrices $B_1, \ldots, B_N$ with entries in $\mathbb{Z}$ such that

$$A(z) = \sum_{j=1}^{N} \frac{B_j}{z - a_j}$$

$$\sum_{j=1}^{N} B_j = v.$$

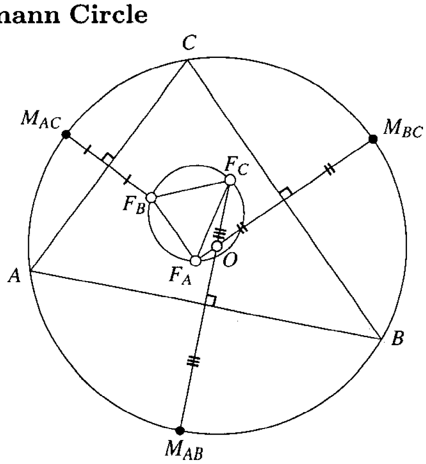
Fuchs's Theorem

At least one POWER SERIES solution will be obtained when applying the FROBENIUS METHOD if the expansion point is an ordinary, or regular, SINGULAR POINT. The number of ROOTS is given by the ROOTS of the INDICIAL EQUATION.

References

Arfken, G. *Mathematical Methods for Physicists, 3rd ed.* Orlando, FL: Academic Press, pp. 462–463, 1985.

Fuhrmann Circle



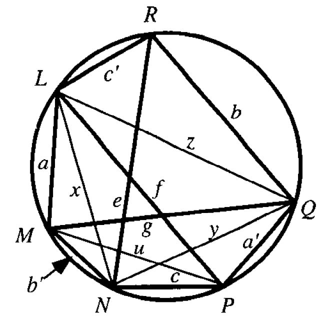
The CIRCUMCIRCLE of the FUHRMANN TRIANGLE.

see also FUHRMANN TRIANGLE, MID-ARC POINTS

References

Fuhrmann, W. *Synthetische Beweise Planimetrischer Sätze.* Berlin, p. 107, 1890.
Johnson, R. A. *Modern Geometry: An Elementary Treatise on the Geometry of the Triangle and the Circle.* Boston, MA: Houghton Mifflin, pp. 228–229, 1929.

Fuhrmann's Theorem



Let the opposite sides of a convex CYCLIC HEXAGON be a, a', b, b', c, and c', and let the DIAGONALS e, f, and g be so chosen that a, a', and e have no common VERTEX (and likewise for b, b', and f), then

$$efg = aa'e + bb'f + cc'g + abc + a'b'c'.$$

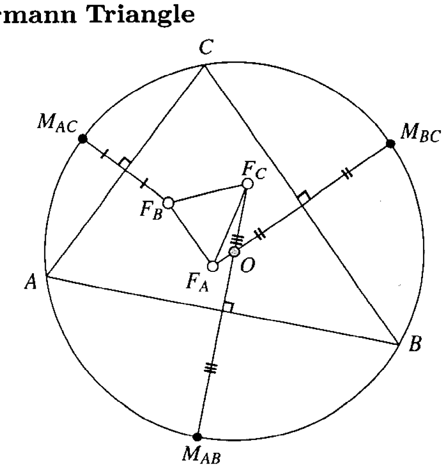
This is an extension of PTOLEMY'S THEOREM to the HEXAGON.

see also CYCLIC HEXAGON, HEXAGON, PTOLEMY'S THEOREM

References

Fuhrmann, W. *Synthetische Beweise Planimetrischer Sätze.* Berlin, p. 61, 1890.
Johnson, R. A. *Modern Geometry: An Elementary Treatise on the Geometry of the Triangle and the Circle.* Boston, MA: Houghton Mifflin, pp. 65–66, 1929.

Fuhrmann Triangle



The Fuhrmann triangle of a TRIANGLE ΔABC is the TRIANGLE $\Delta F_C F_B F_A$ formed by reflecting the MID-ARC POINTS M_{AB}, M_{AC}, M_{BC} about the lines AB, AC, and BC. The CIRCUMCIRCLE of the Fuhrmann triangle is called the FUHRMANN CIRCLE, and the lines $F_A M_{BC}$, $F_B M_{AC}$, and $F_C M_{AB}$ CONCUR at the CIRCUMCENTER O.

see also FUHRMANN CIRCLE, MID-ARC POINTS

References

Fuhrmann, W. *Synthetische Beweise Planimetrischer Sätze.* Berlin, p. 107, 1890.

Johnson, R. A. *Modern Geometry: An Elementary Treatise on the Geometry of the Triangle and the Circle.* Boston, MA: Houghton Mifflin, pp. 228–229, 1929.

Full Reptend Prime

A PRIME p for which $1/p$ has a maximal period DECIMAL EXPANSION of $p-1$ DIGITS. The first few numbers with maximal decimal expansions are 7, 17, 19, 23, 29, 47, 59, 61, 97, ... (Sloane's A001913).

References

Sloane, N. J. A. Sequence A001913/M4353 in "An On-Line Version of the Encyclopedia of Integer Sequences."

Full Width at Half Maximum

The full width at half maximum (FWHM) is a parameter commonly used to describe the width of a "bump" on a curve or function. It is given by the distance between points on the curve at which the function reaches half its maximum value. The following table gives the analytic and numerical full widths for several common curves.

Function	Formula	FWHM
Bartlett	$1-\frac{\lvert x\rvert}{a}$	a
Blackman		$0.810957a$
Connes	$\left(1-\frac{x^2}{a^2}\right)$	$\sqrt{4-2\sqrt{2}}\,a$
Cosine	$\cos\left(\frac{\pi x}{2a}\right)$	$\frac{4}{3}a$
Gaussian	$e^{-x^2/(2\sigma^2)}$	$2\sqrt{2\ln 2}\,\sigma$
Hamming		$1.05543a$
Hanning		a
Lorentzian	$\frac{\frac{1}{2}\Gamma}{x^2+(\frac{1}{2}\Gamma)^2}$	Γ
Welch	$1-\frac{x^2}{a^2}$	$\sqrt{2}\,a$

see also APODIZATION FUNCTION, MAXIMUM

Fuller Dome

see GEODESIC DOME

Function

A way of associating unique objects to every point in a given SET. A function from A to B is an object f such that for every $a \in A$, there is a unique object $f(a) \in B$. Examples of functions include $\sin x$, x, x^2, etc. The term MAP is synonymous with function.

Poincaré remarked with regard to the proliferation of pathological functions, "Formerly, when one invented a new function, it was to further some practical purpose; today one invents them in order to make incorrect the reasoning of our fathers, and nothing more will ever be accomplished by these inventions."

see also ABELIAN FUNCTION, ABSOLUTE VALUE, ACKERMANN FUNCTION, AIRY FUNCTIONS, ALGEBRAIC FUNCTION, ALGEBROIDAL FUNCTION, ALPHA FUNCTION, ANDREW'S SINE, ANGER FUNCTION, APODIZATION FUNCTION, APPARATUS FUNCTION, ARGUMENT (FUNCTION), ARTIN L-FUNCTION, AUTOMORPHIC FUNCTION, BACHELIER FUNCTION, BARNES G-FUNCTION, BARTLETT FUNCTION, BASSET FUNCTION, BATEMAN FUNCTION, BEI, BER, BERNOULLI FUNCTION, BESSEL FUNCTION OF THE FIRST KIND, BESSEL FUNCTION OF THE SECOND KIND, BESSEL FUNCTION OF THE THIRD KIND, BETA FUNCTION, BETA FUNCTION (EXPONENTIAL), BINOMIAL COEFFICIENT, BLACKMAN FUNCTION, BLANCMANGE FUNCTION, BOOLEAN FUNCTION, BOURGET FUNCTION, BOXCAR FUNCTION, BROWN FUNCTION, CAL, CANTOR FUNCTION, CARMICHAEL FUNCTION, CAROTID-KUNDALINI FUNCTION, CEILING FUNCTION, CENTER FUNCTION, CENTRAL BETA FUNCTION, CHARACTERISTIC FUNCTION, CHEBYSHEV FUNCTION, CIRCULAR FUNCTIONS, CLAUSEN FUNCTION, COMB FUNCTION, COMPLETE FUNCTIONS, COMPLEX CONJUGATE, COMPUTABLE FUNCTION, CONCAVE FUNCTION, CONFLUENT HYPERGEOMETRIC FUNCTION, CONFLUENT HYPERGEOMETRIC FUNCTION OF THE FIRST KIND, CONFLUENT HYPERGEOMETRIC FUNCTION OF THE SECOND KIND, CONFLUENT HYPERGEOMETRIC LIMIT FUNCTION, CONICAL FUNCTION, CONNES FUNCTION, CONSTANT FUNCTION, CONTIGUOUS FUNCTION, CONTINUOUS FUNCTION, CONVEX FUNCTION, COPULA, COSECANT, COSINE, COSINE APODIZATION FUNCTION, COTANGENT, COULOMB WAVE FUNCTION, COVERSINE, CUBE ROOT, CUBED, CUMULANT-GENERATING FUNCTION, CUMULATIVE DISTRIBUTION FUNCTION, CUNNINGHAM FUNCTION, CYLINDER FUNCTION, CYLINDRICAL FUNCTION, DEBYE FUNCTIONS, DECREASING FUNCTION, DEDEKIND ETA FUNCTION, DEDEKIND FUNCTION, DELTA FUNCTION, DIGAMMA FUNCTION, DILOGARITHM, DIRAC DELTA FUNCTION, DIRICHLET BETA FUNCTION, DIRICHLET ETA FUNCTION, DIRICHLET FUNCTION, DIRICHLET LAMBDA FUNCTION, DISTRIBUTION FUNCTION, DIVISOR FUNCTION, DOUBLE GAMMA FUNCTION, DOUBLET FUNCTION, E_n-FUNCTION, E_t-FUNCTION, EIGENFUNCTION, EIN FUNCTION, EINSTEIN FUNCTIONS, ELEMENTARY FUNCTION, ELLIPTIC ALPHA FUNCTION, ELLIPTIC DELTA FUNCTION, ELLIPTIC EXPONENTIAL FUNCTION, ELLIPTIC FUNCTION, ELLIPTIC FUNCTIONAL, ELLIPTIC LAMBDA FUNCTION, ELLIPTIC MODULAR FUNCTION, ELLIPTIC THETA FUNCTION, ELSASSER FUNCTION, ENTIRE FUNCTION, EPSTEIN ZETA FUNCTION, ERDŐS-SELFRIDGE FUNCTION, ERF, ERROR FUNCTION, EXPONENTIAL RAMP, EULER L-FUNCTION, EVEN FUNCTION, EXPONENTIAL FUNCTION, EXPONENTIAL FUNCTION (TRUNCATED), EXPONENTIAL SUM FUNCTION, EXSECANT, FLOOR FUNCTION, FOX'S H-FUNCTION,

Function Space, G-Function, Gamma Function, Gate Function, Gaussian Function, Gegenbauer Function, Generalized Function, Generalized Hyperbolic Functions, Generalized Hypergeometric Function, Generating Function, Gordon Function, Green's Function, Growth Function, Gudermannian Function, H-Function, Haar Function, Hamming Function, Hankel Function, Hankel Function of the First Kind, Hankel Function of the Second Kind, Hann Function, Hanning Function, Harmonic Function, Haversine, Heaviside Step Function, Hecke L-Function, Hemicylindrical Function, Hemispherical Function, Heuman Lambda Function, Hh Function, Hilbert Function, Holonomic Function, Homogeneous Function, Hurwitz Zeta Function, Hyperbolic Cosecant, Hyperbolic Cosine, Hyperbolic Cotangent, Hyperbolic Functions, Hyperbolic Secant, Hyperbolic Sine, Hyperbolic Tangent, Hyperelliptic Function, Hypergeometric Function, Identity Function, Implicit Function, Implicit Function Theorem, Incomplete Gamma Function, Increasing Function, Infinite Product, Instrument Function, Int, Inverse Cosecant, Inverse Cosine, Inverse Cotangent, Inverse Function, Inverse Hyperbolic Functions, Inverse Secant, Inverse Sine, Inverse Tangent, j-Function, Jacobi Elliptic Functions, Jacobi Function of the First Kind, Jacobi Function of the Second Kind, Jacobi Theta Function, Jacobi Zeta Function, Jinc Function, Joint Probability Density Function, Jonquière's Function, K-Function, Kei, Kelvin Functions, Ker, Koebe Function, L-Function, Lambda Function, Lambda Hypergeometric Function, Lambert's W-Function, Lamé Function, Legendre Function of the First Kind, Legendre Function of the Second Kind, Lemniscate Function, Lemniscate Function, Length Distribution Function, Lerch Transcendent, Lévy Function, Linearly Dependent Functions, Liouville Function, Lipschitz Function, Logarithm, Logarithmically Convex Function, Logit Transformation, Lommel Function, Lyapunov Function, MacRobert's E-Function, Mangoldt Function, Mathieu Function, Measurable Function, Meijer's G-Function, Meromorphic, Mertens Function, Mertz Apodization Function, Mittag-Leffler Function, Möbius Function, Möbius Periodic Function, Mock Theta Function, Modified Bessel Function of the First Kind, Modified Bessel Function of the Second Kind, Modified Spherical Bessel Function, Modified Struve Function, Modular Function, Modular Gamma Function, Modular Lambda Function, Moment-Generating Function, Monogenic Function, Monotonic Function, Mu Function, Multiplicative Function, Multivalued Function, Multivariate Function, Neumann Function, Nint, Nu Function, Null Function, Numeric Function, Oblate Spheroidal Wave Function, Odd Function, Omega Function, One-Way Function, Parabolic Cylinder Function, Partition Function P, Partition Function Q, Parzen Apodization Function, Pearson-Cunningham Function, Pearson's Function, Periodic Function, Planck's Radiation Function, Plurisubharmonic Function, Pochhammer Symbol, Poincaré-Fuchs-Klein Automorphic Function, Poisson-Charlier Function, Polygamma Function, Polygenic Function, Polylogarithm, Positive Definite Function, Potential Function, Power, Prime Counting Function, Prime Difference Function, Probability Density Function, Probability Distribution Function, Prolate Spheroidal Wave Function, Psi Function, Pulse Function, q-Beta Function, Q-Function, q-Gamma Function, Quasiperiodic Function, Rademacher Function, Ramanujan Function, Ramanujan g- and G- Functions, Ramanujan Theta Functions, Ramp Function, Rational Function, Real Function, Rectangle Function, Regular Function, Regularized Gamma Function, Restricted Divisor Function, Riemann Function, Riemann-Mangoldt Function, Riemann-Siegel Functions, Riemann Theta Function, Riemann Zeta Function, Ring Function, Sal, Sampling Function, Scalar Function, Schlomilch's Function, Secant, Sequency Function, Sgn, Shah Function, Siegel Modular Function, Sigma Function, Sigmoid Function, Sign, Sinc Function, Sine, Smarandache Function, Spence's Function, Spherical Bessel Function of the First Kind, Spherical Bessel Function of the Second Kind, Spherical Hankel Function of the First Kind, Spherical Hankel Function of the Second Kind, Spherical Harmonic, Spheroidal Wavefunction, Sprague-Grundy Function, Square Root, Squared, Step Function, Struve Function, Sturm Function, Summatory Function, Symmetric Function, TAK Function, Tangent, Tapering Function, Tau Function, Tetrachoric Function, Theta Function, Toroidal Function, Toronto Function, Total Function, Totient Function, Totient Valence Function, Transcendental Function, Transfer Function, Trapdoor Function, Triangle Center Function, Triangle Function, Tricomi Function, Trigonometric Functions, Uniform Apodization Function, Univalent Function, Vector Function, Versine, von Mangoldt Function, W-Function, Walsh Function, Weber Functions, Weierstraß Elliptic Function, Weierstraß Function, Weierstraß Sigma Function, Weierstraß Zeta Function, Weighting Function, Welch Apodization

FUNCTION, WHITTAKER FUNCTION, WIENER FUNCTION, WINDOW FUNCTION, XI FUNCTION, ZETA FUNCTION

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Function Field

see ALGEBRAIC FUNCTION FIELD

Function Space

$f(I)$ is the collection of all real-valued continuous functions defined on some interval I. $f^{(n)}(I)$ is the collection of all functions $\in f(I)$ with continuous nth DERIVATIVES. A function space is a TOPOLOGICAL VECTOR SPACE whose "points" are functions.

see also FUNCTIONAL, FUNCTIONAL ANALYSIS, OPERATOR

Functional

A mapping between FUNCTION SPACES if the range is on the REAL LINE or in the COMPLEX PLANE.

see also COERCIVE FUNCTIONAL, CURRENT, ELLIPTIC FUNCTIONAL, GENERALIZED FUNCTION, LAX-MILGRAM THEOREM, OPERATOR, RIESZ REPRESENTATION THEOREM

Functional Analysis

A branch of mathematics concerned with infinite dimensional spaces (mainly FUNCTION SPACES) and mappings between them. The SPACES may be of different, and possibly INFINITE, DIMENSIONS. These mappings are called OPERATORS or, if the range is on the REAL line or in the COMPLEX PLANE, FUNCTIONALS.

see also FUNCTIONAL, OPERATOR

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Kreyszig, E. *Introductory Functional Analysis with Applications.* New York: Wiley, 1989.
Yoshida, K. *Functional Analysis and Its Applications.* New York: Springer-Verlag, 1971.
Zeidler, E. *Nonlinear Functional Analysis and Its Applications.* New York: Springer-Verlag, 1989.
Zeidler, E. *Applied Functional Analysis: Applications to Mathematical Physics.* New York: Springer-Verlag, 1995.

Functional Calculus

An early name for CALCULUS OF VARIATIONS.

Functional Derivative

A generalization of the concept of the DERIVATIVE to GENERALIZED FUNCTIONS.

Functor

A function between CATEGORIES which maps objects to objects and MORPHISMS to MORPHISMS. Functors exist in both covariant and contravariant types.

see also CATEGORY, EILENBERG-STEENROD AXIOMS, MORPHISM, SCHUR FUNCTOR

Fundamental Class

The canonical generator of the nonvanishing HOMOLOGY GROUP on a TOPOLOGICAL MANIFOLD.

see also CHERN NUMBER, PONTRYAGIN NUMBER, STIEFEL-WHITNEY NUMBER

Fundamental Continuity Theorem

Given two POLYNOMIALS of the same order in one variable where the first p COEFFICIENTS (but not the first $p-1$) are 0 and the COEFFICIENTS of the second approach the corresponding COEFFICIENTS of the first as limits, then the second POLYNOMIAL will have exactly p roots that increase indefinitely. Furthermore, exactly k ROOTS of the second will approach each ROOT of multiplicity k of the first as a limit.

References
Coolidge, J. L. *A Treatise on Algebraic Plane Curves.* New York: Dover, p. 4, 1959.

Fundamental Discriminant

$-D$ is a fundamental discriminant if D is a POSITIVE INTEGER which is not DIVISIBLE by any square of an ODD PRIME and which satisfies $D \equiv 3 \pmod 4$ or $D \equiv 4, 8 \pmod{16}$.

see also DISCRIMINANT

References
Atkin, A. O. L. and Morain, F. "Elliptic Curves and Primality Proving." *Math. Comput.* **61**, 29–68, 1993.
Borwein, J. M. and Borwein, P. B. *Pi & the AGM: A Study in Analytic Number Theory and Computational Complexity.* New York: Wiley, p. 294, 1987.
Cohn, H. *Advanced Number Theory.* New York: Dover, 1980.
Dickson, L. E. *History of the Theory of Numbers, Vols. 1–3.* New York: Chelsea, 1952.

Fundamental Forms

There are three types of so-called fundamental forms. The most important are the first and second (since the third can be expressed in terms of these). The fundamental forms are extremely important and useful in determining the metric properties of a surface, such as LINE ELEMENT, AREA ELEMENT, NORMAL CURVATURE, GAUSSIAN CURVATURE, and MEAN CURVATURE. Let M be a REGULAR SURFACE with $\mathbf{v_p}, \mathbf{w_p}$ points on the TANGENT SPACE $M_\mathbf{p}$ of M. Then the first fundamental form is the INNER PRODUCT of tangent vectors,

$$\mathrm{I}(\mathbf{v_p}, \mathbf{w_p}) = \mathbf{v_p} \cdot \mathbf{w_p}. \tag{1}$$

For $M \in \mathbb{R}^3$, the second fundamental form is the symmetric bilinear form on the TANGENT SPACE $M_\mathbf{p}$,

$$\mathrm{II}(\mathbf{v_p}, \mathbf{w_p}) = S(\mathbf{v_p}) \cdot \mathbf{w_p}, \tag{2}$$

where S is the SHAPE OPERATOR. The third fundamental form is given by

$$\mathrm{III}(\mathbf{v_p}, \mathbf{w_p}) = S(\mathbf{v_p}) \cdot S(\mathbf{w_p}). \tag{3}$$

The first and second fundamental forms satisfy

$$\mathrm{I}(a\mathbf{x}_u + b\mathbf{x}_v, a\mathbf{x}_u + b\mathbf{x}_v) = Ea^2 + 2Fab + Gb^2 \tag{4}$$

$$\mathrm{II}(a\mathbf{x}_u + b\mathbf{x}_v, a\mathbf{x}_u + b\mathbf{x}_v) = ea^2 + 2fab + gb^2, \tag{5}$$

and so their ratio is simply the NORMAL CURVATURE

$$\kappa(\mathbf{v_p}) = \frac{\mathrm{II}(\mathbf{v_p})}{\mathrm{I}(\mathbf{v_p})} \tag{6}$$

for any nonzero TANGENT VECTOR. The third fundamental form is given in terms of the first and second forms by

$$\mathrm{III} - 2H\mathrm{II} + K\mathrm{I} = 0, \tag{7}$$

where H is the MEAN CURVATURE and K is the GAUSSIAN CURVATURE.

The first fundamental form (or LINE ELEMENT) is given explicitly by the RIEMANNIAN METRIC

$$ds^2 = E\,du^2 + 2F\,du\,dv + G\,dv^2. \tag{8}$$

It determines the ARC LENGTH of a curve on a surface. The coefficients are given by

$$E = \mathbf{x}_{uu} = \left|\frac{\partial \mathbf{x}}{\partial u}\right|^2 \tag{9}$$

$$F = \mathbf{x}_{uv} = \frac{\partial \mathbf{x}}{\partial u} \cdot \frac{\partial \mathbf{x}}{\partial v} \tag{10}$$

$$G = \mathbf{x}_{vv} = \left|\frac{\partial \mathbf{x}}{\partial v}\right|^2. \tag{11}$$

The coefficients are also denoted $g_{uu} = E$, $g_{uv} = F$, and $g_{vv} = G$. In CURVILINEAR COORDINATES (where $F = 0$), the quantities

$$h_u \equiv \sqrt{g_{uu}} = \sqrt{E} \tag{12}$$

$$h_v \equiv \sqrt{g_{vv}} = \sqrt{G} \tag{13}$$

are called SCALE FACTORS.

The second fundamental form is given explicitly by

$$e\,du^2 + 2f\,du\,dv + g\,dv^2 \tag{14}$$

where

$$e = \sum_i X_i \frac{\partial^2 x_i}{\partial u^2} \tag{15}$$

$$f = \sum_i X_i \frac{\partial^2 x_i}{\partial u \partial v} \tag{16}$$

$$g = \sum_i X_i \frac{\partial^2 x_i}{\partial v^2}, \tag{17}$$

and X_i are the DIRECTION COSINES of the surface normal. The second fundamental form can also be written

$$e = -\mathbf{N}_u \cdot \mathbf{x}_u = \mathbf{N} \cdot \mathbf{x}_{uu} \tag{18}$$

$$f = -\mathbf{N}_v \cdot \mathbf{x}_u = \mathbf{N} \cdot \mathbf{x}_{uv} = \mathbf{N}_{vu} \cdot \mathbf{x}_{vu}$$

$$= -\mathbf{N}_u \cdot \mathbf{x}_v \tag{19}$$

$$g = -\mathbf{N}_v \cdot \mathbf{x}_v = \mathbf{N} \cdot \mathbf{x}_{vv}, \tag{20}$$

where $\mathbf{N}$ is the NORMAL VECTOR, or

$$e = \frac{\det(\mathbf{x}_{uu}\mathbf{x}_u\mathbf{x}_v)}{\sqrt{EG - F^2}} \tag{21}$$

$$f = \frac{\det(\mathbf{x}_{uv}\mathbf{x}_u\mathbf{x}_v)}{\sqrt{EG - F^2}} \tag{22}$$

$$g = \frac{\det(\mathbf{x}_{vv}\mathbf{x}_u\mathbf{x}_v)}{\sqrt{EG - F^2}}. \tag{23}$$

see also ARC LENGTH, AREA ELEMENT, GAUSSIAN CURVATURE, GEODESIC, KÄHLER MANIFOLD, LINE OF CURVATURE, LINE ELEMENT, MEAN CURVATURE, NORMAL CURVATURE, RIEMANNIAN METRIC, SCALE FACTOR, WEINGARTEN EQUATIONS

References

Gray, A. "The Three Fundamental Forms." §14.6 in *Modern Differential Geometry of Curves and Surfaces.* Boca Raton, FL: CRC Press, pp. 251–255, 259–260, 275–276, and 282–291, 1993.

Fundamental Group

The fundamental group of a CONNECTED SET S is the QUOTIENT GROUP of the GROUP of all paths with initial and final points at a given point P and the SUBGROUP of all paths HOMOTOPIC to the degenerate path consisting of the point P.

The fundamental group of the CIRCLE is the INFINITE CYCLIC GROUP. Two fundamental groups having different points P are ISOMORPHIC. If the fundamental group consists only of the identity element, then the set S is simply connected.

see also MILNOR'S THEOREM

Fundamental Homology Class

see also FUNDAMENTAL CLASS

Fundamental Lemma of Calculus of Variations

If

$$\int_a^b M(x)h(x)\,dx = 0$$

$\forall\ h(x)$ with CONTINUOUS second PARTIAL DERIVATIVES, then

$$M(x) = 0$$

on the OPEN INTERVAL (a, b).

Fundamental System

A set of ALGEBRAIC INVARIANTS for a QUANTIC such that any invariant of the QUANTIC is expressible as a POLYNOMIAL in members of the set. In 1868, Gordan proved the existence of finite fundamental systems of algebraic invariants and covariants for any binary QUANTIC. In 1890, Hilbert (1890) proved the HILBERT BASIS THEOREM, which is a finiteness theorem for the related concept of SYZYGIES.

see also HILBERT BASIS THEOREM, SYZYGY

References

Hilbert, D. "Über die Theorie der algebraischen Formen." *Math. Ann.* **36**, 473–534, 1890.

Fundamental Theorem of Algebra

Every POLYNOMIAL equation having COMPLEX COEFFICIENTS and degree ≥ 1 has at least one COMPLEX ROOT. This theorem was first proven by Gauss. It is equivalent to the statement that a POLYNOMIAL $P(z)$ of degree n has n values of z (some of them possibly degenerate) for which $P(z) = 0$. An example of a POLYNOMIAL with a single ROOT of multiplicity > 1 is $z^2 - 2z + 1 = (z-1)(z-1)$, which has $z = 1$ as a ROOT of multiplicity 2.

see also DEGENERATE, POLYNOMIAL

References

Courant, R. and Robbins, H. "The Fundamental Theorem of Algebra." §2.5.4 in *What is Mathematics?: An Elementary Approach to Ideas and Methods, 2nd ed.* Oxford, England: Oxford University Press, pp. 101–103, 1996.

Fundamental Theorem of Arithmetic

Any POSITIVE INTEGER can be represented in exactly one way as a PRODUCT of PRIMES. The theorem is also called the UNIQUE FACTORIZATION THEOREM. The fundamental theorem of algebra is a COROLLARY of the first of EUCLID'S THEOREMS (Hardy and Wright 1979).

see also EUCLID'S THEOREMS, INTEGER, PRIME NUMBER

References

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Hardy, G. H. and Wright, E. M. "Statement of the Fundamental Theorem of Arithmetic," "Proof of the Fundamental Theorem of Arithmetic," and "Another Proof of teh Fundamental Theorem of Arithmetic." §1.3, 2.10 and 2.11 in *An Introduction to the Theory of Numbers, 5th ed.* Oxford, England: Clarendon Press, pp. 3 and 21, 1979.

Fundamental Theorems of Calculus

The first fundamental theorem of calculus states that, if f is CONTINUOUS on the CLOSED INTERVAL $[a, b]$ and F is the ANTIDERIVATIVE (INDEFINITE INTEGRAL) of f on $[a, b]$, then

$$\int_a^b f(x)\,dx = F(b) - F(a). \tag{1}$$

The second fundamental theorem of calculus lets f be CONTINUOUS on an OPEN INTERVAL I and lets a be any point in I. If F is defined by

$$F(x) = \int_a^x f(t)\,dt, \tag{2}$$

then

$$F'(x) = f(x) \tag{3}$$

at each point in I.

The complex fundamental theorem of calculus states that if $f(z)$ has a CONTINUOUS ANTIDERIVATIVE $F(z)$ in a region R containing a parameterized curve $\gamma : z = z(t)$ for $\alpha \leq t \leq \beta$, then

$$\int_\gamma f(z)\,dz = F(z(\beta)) - F(z(\alpha)). \tag{4}$$

see also CALCULUS, DEFINITE INTEGRAL, INDEFINITE INTEGRAL, INTEGRAL

Fundamental Theorem of Curves

The CURVATURE and TORSION functions along a SPACE CURVE determine it up to an orientation-preserving ISOMETRY.

Fundamental Theorem of Directly Similar Figures

Let F_0 and F_1 denote two directly similar figures in the plane, where $P_1 \in F_1$ corresponds to $P_0 \in F_0$ under the given similarity. Let $r \in (0,1)$, and define $F_r = \{(1-r)P_0 + rP_1 : P_0 \in F_0\}$. Then F_r is also directly similar to F_0.

see also FINSLER-HADWIGER THEOREM

References

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Eves, H. Solution to Problem E521. *Amer. Math. Monthly* **50**, 64, 1943.

Fundamental Theorem of Gaussian Quadrature

The ABSCISSAS of the N point GAUSSIAN QUADRATURE FORMULA are precisely the ROOTS of the ORTHOGONAL POLYNOMIAL for the same INTERVAL and WEIGHTING FUNCTION.

see also GAUSSIAN QUADRATURE

Fundamental Theorem of Genera

$$2^{\omega(d)-1}|h(-d)|,$$

where $\omega(d)$ is the genus of forms and $h(-d)$ is the CLASS NUMBER of an IMAGINARY QUADRATIC FIELD.

References

Arno, S.; Robinson, M. L.; and Wheeler, F. S. "Imaginary Quadratic Fields with Small Odd Class Number." `http://www.math.uiuc.edu/Algebraic-Number-Theory/0009/`.

Cohn, H. *Advanced Number Theory.* New York: Dover, p. 224, 1980.

Gauss, C. F. *Disquisitiones Arithmeticae.* New Haven, CT: Yale University Press, 1966.

Fundamental Theorem of Plane Curves

Two unit-speed plane curves which have the same CURVATURE differ only by a EUCLIDEAN MOTION.

see also FUNDAMENTAL THEOREM OF SPACE CURVES

References

Gray, A. *Modern Differential Geometry of Curves and Surfaces.* Boca Raton, FL: CRC Press, pp. 103 and 110–111, 1993.

Fundamental Theorem of Projective Geometry

A PROJECTIVITY is determined when three points of one RANGE and the corresponding three points of the other are given.

see also PROJECTIVE GEOMETRY

Fundamental Theorem of Space Curves

If two single-valued continuous functions $\kappa(s)$ (CURVATURE) and $\tau(s)$ (TORSION) are given for $s > 0$, then there exists EXACTLY ONE SPACE CURVE, determined except for orientation and position in space (i.e., up to a EUCLIDEAN MOTION), where s is the ARC LENGTH, κ is the CURVATURE, and τ is the TORSION.

see also ARC LENGTH, CURVATURE, EUCLIDEAN MOTION, FUNDAMENTAL THEOREM OF PLANE CURVES, TORSION (DIFFERENTIAL GEOMETRY)

References

Gray, A. "The Fundamental Theorem of Space Curves." §7.7 in *Modern Differential Geometry of Curves and Surfaces.* Boca Raton, FL: CRC Press, pp. 123 and 142–145, 1993.

Struik, D. J. *Lectures on Classical Differential Geometry.* New York: Dover, p. 29, 1988.

Fundamental Theorem of Symmetric Functions

Any symmetric polynomial (respectively, symmetric rational function) can be expressed as a POLYNOMIAL (respectively, RATIONAL FUNCTION) in the ELEMENTARY SYMMETRIC FUNCTIONS on those variables.

see also ELEMENTARY SYMMETRIC FUNCTION

References

Coolidge, J. L. *A Treatise on Algebraic Plane Curves.* New York: Dover, p. 2, 1959.

Herstein, I. N. *Noncommutative Rings.* Washington, DC: Math. Assoc. Amer., 1968.

Fundamental Unit

In a real QUADRATIC FIELD, there exists a special UNIT η known as the fundamental unit such that all units ρ are given by $\rho = \pm\eta^m$, for $m = 0, \pm1, \pm2, \ldots$. The following table gives the fundamental units for the first few real quadratic fields.

d	$\eta(d)$	d	$\eta(d)$
2	$1+\sqrt{2}$	51	$50+7\sqrt{51}$
3	$2+\sqrt{3}$	53	$\frac{1}{2}(7+\sqrt{53})$
5	$\frac{1}{2}(1+\sqrt{5})$	55	$89+12\sqrt{55}$
6	$5+2\sqrt{6}$	57	$151+20\sqrt{57}$
7	$8+3\sqrt{7}$	58	$99+13\sqrt{58}$
10	$3+\sqrt{10}$	59	$530+69\sqrt{59}$
11	$10+3\sqrt{11}$	61	$\frac{1}{2}(39+5\sqrt{61})$
13	$\frac{1}{2}(3+\sqrt{13})$	62	$63+8\sqrt{62}$
14	$15+4\sqrt{14}$	65	$8+\sqrt{65}$
15	$4+\sqrt{15}$	66	$65+8\sqrt{66}$
17	$4+\sqrt{17}$	67	$48842+5967\sqrt{67}$
19	$170+39\sqrt{19}$	69	$\frac{1}{2}(25+3\sqrt{69})$
21	$\frac{1}{2}(5+\sqrt{21})$	70	$251+30\sqrt{70}$
22	$197+42\sqrt{22}$	71	$3480+413\sqrt{71}$
23	$24+5\sqrt{23}$	73	$1068+125\sqrt{73}$
26	$5+\sqrt{26}$	74	$43+5\sqrt{74}$
29	$\frac{1}{2}(5+\sqrt{29})$	77	$\frac{1}{2}(9+\sqrt{77})$
30	$11+2\sqrt{30}$	78	$53+6\sqrt{78}$
31	$1520+273\sqrt{31}$	79	$80+9\sqrt{79}$
33	$5+4\sqrt{33}$	82	$9+\sqrt{82}$
34	$35+6\sqrt{34}$	83	$82+9\sqrt{83}$
35	$6+\sqrt{35}$	85	$\frac{1}{2}(9+\sqrt{85})$
37	$6+\sqrt{37}$	86	$10405+1122\sqrt{86}$
38	$37+6\sqrt{38}$	87	$28+3\sqrt{87}$
39	$25+4\sqrt{39}$	89	$501+54\sqrt{89}$
41	$32+5\sqrt{41}$	91	$1574+165\sqrt{91}$
42	$13+2\sqrt{42}$	93	$\frac{1}{2}(29+3\sqrt{93})$
43	$3482+531\sqrt{43}$	95	$39+4\sqrt{95}$
46	$24335+3588\sqrt{46}$	97	$5604+569\sqrt{97}$
47	$48+7\sqrt{47}$		

see also QUADRATIC FIELD, UNIT

References

Cohn, H. "Fundamental Units" and "Construction of Fundamental Units." §6.4 and 6.5 in *Advanced Number Theory.* New York: Dover, pp. 98–102, and 261–274, 1980.

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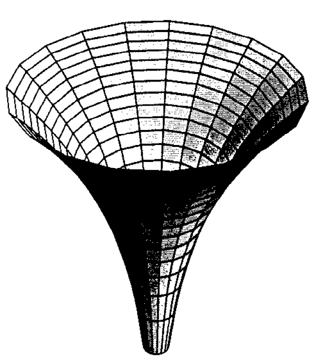
Funnel



The funnel surface is a REGULAR SURFACE defined by the Cartesian equation

$$z = \frac{1}{2}\ln(x^2+y^2) \tag{1}$$

and the parametric equations

$$x(r,\theta) = r\cos\theta \tag{2}$$

$$y(r,\theta) = r\sin\theta \tag{3}$$

$$z(r,\theta) = \ln r. \tag{4}$$

see also GABRIEL'S HORN, PSEUDOSPHERE, SINCLAIR'S SOAP FILM PROBLEM

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Fuss's Problem

see BICENTRIC POLYGON

Futile Game

A GAME which permits a draw ("tie") when played properly by both players.

Fuzzy Logic

An extension of two-valued LOGIC such that statements need not be TRUE or FALSE, but may have a degree of truth between 0 and 1. Such a system can be extremely useful in designing control logic for real-world systems such as elevators.

see also ALETHIC, FALSE, LOGIC, TRUE

References

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FWHM

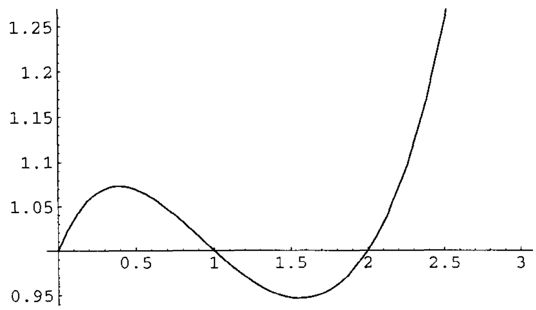
see FULL WIDTH AT HALF MAXIMUM

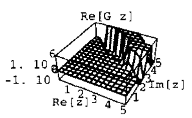
G

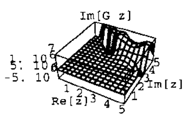
g-Function

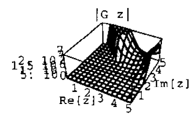
see RAMANUJAN g- AND G-FUNCTIONS

G-Function









Defined in Whittaker and Watson (1990, p. 264) and also called the BARNES G-FUNCTION.

$$G(z+1) \equiv (2\pi)^{z/2} e^{-[z(z+1)+\gamma z^2]/2} \times \prod_{n=1}^{\infty} \left[\left(1+\frac{z}{n}\right)^n e^{-z+z^2/(2n)} \right], \quad (1)$$

where γ is the EULER-MASCHERONI CONSTANT. This is an ANALYTIC CONTINUATION of the G function defined in the construction of the GLAISHER-KINKELIN CONSTANT

$$G(n+1) \equiv \frac{(n!)^n}{K(n+1)}, \quad (2)$$

which has the special values

$$G(n) = \begin{cases} 0 & \text{if } n = 0, -1, -2, \ldots \\ 1 & \text{if } n = 1 \\ 0!1!2!\cdots(n-2)! & \text{if } n = 2, 3, 4, \ldots \end{cases} \quad (3)$$

for INTEGER n. This function is what Sloane and Plouffe (1995) call the SUPERFACTORIAL, and the first few values for $n = 1, 2, \ldots$ are 1, 1, 1, 2, 12, 288, 34560, 24883200, 125411328000, 5056584744960000, ... (Sloane's A000178).

The G-function is the reciprocal of the DOUBLE GAMMA FUNCTION. It satisfies

$$G(z+1) = \Gamma(z)G(z) \quad (4)$$

$$\frac{(n!)^n}{G(n+1)} = 1^1 \cdot 2^2 \cdot 3^3 \cdots n^n \quad (5)$$

$$\frac{G'(z+1)}{G(z+1)} = \tfrac{1}{2}\ln(2\pi) - \tfrac{1}{2} - z + z\frac{\Gamma'(z)}{\Gamma(z)} \quad (6)$$

$$\ln\left[\frac{G(1-z)}{G(1+z)}\right] = \int_0^z \pi z \cot(\pi z)\, dz - z\ln(2\pi) \quad (7)$$

and has the special values

$$G(\tfrac{1}{2}) = A^{-3/2}\pi^{-1/4}e^{1/8}2^{1/24} \quad (8)$$

$$G(1) = 1, \quad (9)$$

where

$$A = \exp\left[-\frac{\zeta'(2)}{2\pi^2} + \frac{\ln(2\pi)}{12} + \frac{\gamma}{2}\right] = 1.28242713\ldots. \quad (10)$$

The G-function can arise in spectral functions in mathematical physics (Voros 1987).

An unrelated pair of functions are denoted g_n and G_n and are known as RAMANUJAN g- AND G-FUNCTIONS.

see also EULER-MASCHERONI CONSTANT, GLAISHER-KINKELIN CONSTANT, K-FUNCTION, MEIJER'S G-FUNCTION, RAMANUJAN g- AND G-FUNCTIONS, SUPERFACTORIAL

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G-Number

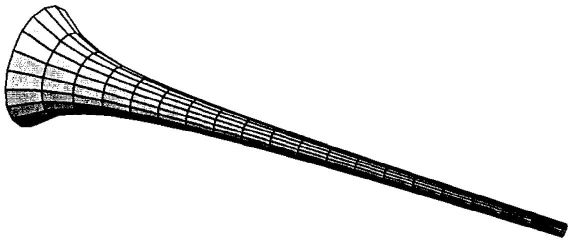
see EISENSTEIN INTEGER

G-Space

A G-space is a special type of HAUSDORFF SPACE. Consider a point x and a HOMEOMORPHISM of an open NEIGHBORHOOD V of x onto an OPEN SET of $\mathbb{R}^n$. Then a space is a G-space if, for any two such NEIGHBORHOODS V' and V'', the images of $V' \cup V''$ under the different HOMEOMORPHISMS are ISOMETRIC. If $n = 2$, the HOMEOMORPHISMS need only be conformal (but not necessarily orientation-preserving).

see also GREEN SPACE

Gabriel's Horn



The SURFACE OF REVOLUTION of the function $y = 1/x$ about the x-axis for $x \geq 1$. It has FINITE VOLUME

$$\begin{aligned} V &= \int_1^\infty \pi y^2\, dx = \pi \int_1^\infty \frac{dx}{x^2} \\ &= \pi \left[-\frac{1}{x}\right]_1^\infty = \pi[0-(-1)] = \pi, \end{aligned}$$

but INFINITE SURFACE AREA, since

$$\begin{aligned} S &= \int_1^\infty 2\pi y \sqrt{1+y'^2}\, dx \\ &> 2\pi \int_1^\infty y\, dx = 2\pi \int_1^\infty \frac{dx}{x} = 2\pi[\ln x]_1^\infty \\ &= 2\pi[\ln\infty - 0] = \infty. \end{aligned}$$

This leads to the paradoxical consequence that while Gabriel's horn can be filled up with π cubic units of paint, an INFINITE number of square units of paint are needed to cover its surface!

see also FUNNEL, PSEUDOSPHERE

Gabriel's Staircase

The SUM

$$\sum_{k=1}^\infty kr^k = \frac{r}{(1-r)^2},$$

valid for $0 < r < 1$.

Gadget

A term of endearment used by ALGEBRAIC TOPOLOGISTS when talking about their favorite power tools such as ABELIAN GROUPS, BUNDLES, HOMOLOGY GROUPS, HOMOTOPY GROUPS, k-THEORY, MORSE THEORY, OBSTRUCTIONS, stable homotopy theory, VECTOR SPACES, etc.

see also ABELIAN GROUP, ALGEBRAIC TOPOLOGY, BUNDLE, FREE, HOMOLOGY GROUP, HOMOTOPY GROUP, k-THEORY, OBSTRUCTION, MORSE THEORY, VECTOR SPACE

Gale-Ryser Theorem

Let p and q be PARTITIONS of a POSITIVE INTEGER, then there exists a (0,1)-matrix A such that $c(A) = p$, $r(A) = q$ IFF q is dominated by p^*.

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Ryser, H. J. "The Class $\mathcal{A}(\mathbf{R}, \mathbf{S})$." *Combinatorial Mathematics.* Buffalo, NY: Math. Assoc. Amer., pp. 61–65, 1963.

Galilean Transformation

A transformation from one reference frame to another moving with a constant VELOCITY v with respect to the first for classical motion. However, special relativity shows that the transformation must be modified to the LORENTZ TRANSFORMATION for relativistic motion. The forward Galilean transformation is

$$\begin{bmatrix} t' \\ x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -v & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} t \\ x \\ y \\ z \end{bmatrix},$$

and the inverse transformation is

$$\begin{bmatrix} t \\ x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ v & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} t' \\ x' \\ y' \\ z' \end{bmatrix}.$$

see also LORENTZ TRANSFORMATION

Gall's Stereographic Projection

A CYLINDRICAL PROJECTION which projects the equator onto a tangent cylinder which intersects the globe at $\pm$ 45°. The transformation equations are

$$\begin{aligned} x &= \lambda \\ y &= \tan(\tfrac{1}{2}\phi), \end{aligned}$$

where λ is the LONGITUDE and ϕ the LATITUDE.

see also STEREOGRAPHIC PROJECTION

References

Dana, P. H. "Map Projections." `http://www.utexas.edu/depts/grg/gcraft/notes/mapproj/mapproj.html`.

Gallows

Schroeder (1991) calls the CEILING FUNCTION symbols $\lceil$ and $\rceil$ the "gallows" because of their similarity in appearance to the structure used for hangings.

see also CEILING FUNCTION

References

Schroeder, M. *Fractals, Chaos, Power Laws: Minutes from an Infinite Paradise.* New York: W. H. Freeman, p. 57, 1991.

Gallucci's Theorem

If three SKEW LINES all meet three other SKEW LINES, any TRANSVERSAL to the first set of three meets any TRANSVERSAL to the second set of three.

see also SKEW LINES, TRANSVERSAL LINE

Galois Extension Field

The splitting FIELD for a separable POLYNOMIAL over a FINITE FIELD K, where L is a FIELD EXTENSION of K.

Galois Field

see FINITE FIELD

Galois Group

Let L be a FIELD EXTENSION of K, denoted L/K, and let G be the set of AUTOMORPHISMS of L/K, that is, the set of AUTOMORPHISMS σ of L such that $\sigma(x) = x$ for every $x \in K$, so that K is fixed. Then G is a GROUP of transformations of L, called the Galois group of L/K.

The Galois group of $(\mathbb{C}/\mathbb{R})$ consists of the IDENTITY ELEMENT and COMPLEX CONJUGATION. These functions both take a given REAL to the same real.

see also ABHYANKAR'S CONJECTURE, FINITE GROUP, GROUP

References

Jacobson, N. *Basic Algebra I, 2nd ed.* New York: W. H. Freeman, p. 234, 1985.

Galois Imaginary

A mathematical object invented to solve irreducible CONGRUENCES of the form

$$F(x) \equiv 0 \pmod{p},$$

where p is PRIME.

Galois's Theorem

An algebraic equation is algebraically solvable IFF its GROUP is SOLVABLE. In order that an irreducible equation of PRIME degree be solvable by radicals, it is NECESSARY and SUFFICIENT that all its ROOTS be rational functions of two ROOTS.

see also ABEL'S IMPOSSIBILITY THEOREM, SOLVABLE GROUP

Galois Theory

If there exists a ONE-TO-ONE correspondence between two SUBGROUPS and SUBFIELDS such that

$$\begin{aligned} G(E(G')) &= G' \\ E(G(E')) &= E', \end{aligned}$$

then E is said to have a Galois theory.

Galoisian

An algebraic extension E of F for which every IRREDUCIBLE POLYNOMIAL in F which has a single ROOT in E has *all* its ROOTS in E is said to be Galoisian. Galoisian extensions are also called algebraically normal.

Gambler's Ruin

Let two players each have a finite number of pennies (say, n_1 for player one and n_2 for player two). Now, flip one of the pennies (from either player), with each player having 50% probability of winning, and give the penny to the winner. If the process is repeated indefinitely, the probability that one or the other player will *eventually* lose all his pennies is unity. However, the chances that the individual players will be rendered penniless are

$$\begin{aligned} P_1 &= \frac{n_1}{n_1+n_2} \\ P_2 &= \frac{n_2}{n_1+n_2}. \end{aligned}$$

see also COIN TOSSING, MARTINGALE, SAINT PETERSBURG PARADOX

References

Cover, T. M. "Gambler's Ruin: A Random Walk on the Simplex." §5.4 in In *Open Problems in Communications and Computation.* (Ed. T. M. Cover and B. Gopinath). New York: Springer-Verlag, p. 155, 1987.

Hajek, B. "Gambler's Ruin: A Random Walk on the Simplex." §6.3 in In *Open Problems in Communications and Computation.* (Ed. T. M. Cover and B. Gopinath). New York: Springer-Verlag, pp. 204–207, 1987.

Kraitchik, M. "The Gambler's Ruin." §6.20 in *Mathematical Recreations.* New York: W. W. Norton, p. 140, 1942.

Game

A game is defined as a conflict involving gains and losses between two or more opponents who follow formal rules. The study of games belongs to a branch of mathematics known as GAME THEORY.

see also GAME THEORY

Game Expectation

Let the elements in a PAYOFF MATRIX be denoted a_{ij}, where the is are player A's STRATEGIES and the js are player B's STRATEGIES. Player A can get at least

$$\min_{j \leq n} a_{ij} \tag{1}$$

for STRATEGY i. Player B can force player A to get no more than $\max_{j \leq m} a_{ij}$ for a STRATEGY j. The best STRATEGY for player A is therefore

$$\min_{i \leq m} \min_{j \leq n} a_{ij}, \tag{2}$$

and the best STRATEGY for player B is

$$\min_{j \leq n} \max_{i \leq m} a_{ij}. \tag{3}$$

In general,

$$\min_{i\le m}\min_{j\le n} a_{ij} \le \min_{j\le n}\max_{i\le m} a_{ij}. \tag{4}$$

Equality holds only if a SADDLE POINT is present, in which case the quantity is called the VALUE of the game.

see also GAME, PAYOFF MATRIX, SADDLE POINT (GAME), STRATEGY, VALUE

Game of Life

see LIFE

Game Matrix

see PAYOFF MATRIX

Game Theory

A branch of MATHEMATICS and LOGIC which deals with the analysis of GAMES (i.e., situations in which parties are involved in situations where their interests conflict). In addition to the mathematical elegance and complete "solution" which is possible for simple games, the principles of game theory also find applications to complicated games such as cards, checkers, and chess, as well as real-world problems as diverse as economics, property division, politics, and warfare.

see also BOREL DETERMINACY THEOREM, CATEGORICAL GAME, CHECKERS, CHESS, DECISION THEORY, EQUILIBRIUM POINT, FINITE GAME, FUTILE GAME, GAME EXPECTATION, GO, HI-Q, IMPARTIAL GAME, MEX, MINIMAX THEOREM, MIXED STRATEGY, NASH EQUILIBRIUM, NASH'S THEOREM, NIM, NIM-VALUE, PARTISAN GAME, PAYOFF MATRIX, PEG SOLITAIRE, PERFECT INFORMATION, SADDLE POINT (GAME), SAFE, SPRAGUE-GRUNDY FUNCTION, STRATEGY, TACTIX, TIT-FOR-TAT, UNSAFE, VALUE, WYTHOFF'S GAME, ZERO-SUM GAME

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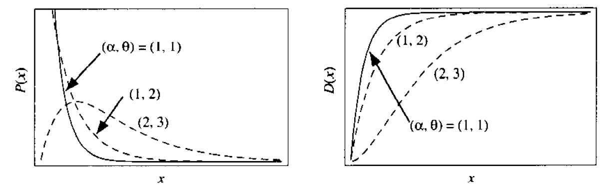
Gamma Distribution



A general type of statistical DISTRIBUTION which is related to the BETA DISTRIBUTION and arises naturally in processes for which the waiting times between POISSON DISTRIBUTED events are relevant. Gamma distributions have two free parameters, labeled α and θ, a few of which are illustrated above.

Given a POISSON DISTRIBUTION with a rate of change λ, the DISTRIBUTION FUNCTION $D(x)$ giving the waiting times until the hth change is

$$\begin{aligned} D(x) &= P(X\le x) = 1-P(X>x)\\ &= 1-\sum_{k=0}^{h-1}\frac{(\lambda x)^k e^{-\lambda x}}{k!}\\ &= 1-e^{-\lambda x}\sum_{k=0}^{h-1}\frac{(\lambda x)^k}{k!} \end{aligned} \tag{1}$$

for $x\ge 0$. The probability function $P(x)$ is then obtained by differentiating $D(x)$,

$$\begin{aligned} P(x) &= D'(x)\\ &= \lambda e^{-\lambda x}\sum_{k=0}^{h-1}\frac{(\lambda x)^k}{k!} - e^{-\lambda x}\sum_{k=0}^{h-1}\frac{k(\lambda x)^{k-1}\lambda}{k!}\\ &= \lambda e^{-\lambda x} + \lambda e^{-\lambda x}\sum_{k=1}^{h-1}\frac{(\lambda x)^k}{k!} - e^{-\lambda x}\sum_{k=1}^{h-1}\frac{k(\lambda x)^{k-1}\lambda}{k!}\\ &= \lambda e^{-\lambda x} - \lambda e^{-\lambda x}\sum_{k=1}^{h-1}\left[\frac{k(\lambda x)^{k-1}}{k!} - \frac{(\lambda x)^k}{k!}\right]\\ &= \lambda e^{-\lambda x}\left\{1-\sum_{k=1}^{h-1}\left[\frac{(\lambda x)^{k-1}}{(k-1)!} - \frac{(\lambda x)^k}{k!}\right]\right\}\\ &= \lambda e^{-\lambda x}\left\{1-\left[1-\frac{(\lambda x)^{h-1}}{(h-1)!}\right]\right\} = \frac{\lambda(\lambda x)^{h-1}}{(h-1)!}e^{-\lambda x}. \end{aligned} \tag{2}$$

Now let $\alpha \equiv h$ and define $\theta \equiv 1/\lambda$ to be the time between changes. Then the above equation can be written

$$P(x) = \begin{cases} \frac{x^{\alpha-1}e^{-x/\theta}}{\Gamma(\alpha)\theta^\alpha} & 0 \le x < \infty \\ 0 & x < 0. \end{cases} \tag{3}$$

The CHARACTERISTIC FUNCTION describing this distribution is

$$\phi(t) = (1-it)^{-p}, \tag{4}$$

and the MOMENT-GENERATING FUNCTION is

$$\begin{aligned} M(t) &= \int_0^\infty \frac{e^{tx}x^{\alpha-1}e^{-x/\theta}\,dx}{\Gamma(\alpha)\theta^\alpha} \\ &= \int_0^\infty \frac{x^{\alpha-1}e^{-(1-\theta t)x/\theta}\,dx}{\Gamma(\alpha)\theta^\alpha}. \end{aligned} \tag{5}$$

In order to find the MOMENTS of the distribution, let

$$y \equiv \frac{(1-\theta t)x}{\theta} \tag{6}$$

$$dy = \frac{1-\theta t}{\theta}\,dx, \tag{7}$$

so

$$\begin{aligned} M(t) &= \int_0^\infty \left(\frac{\theta y}{1-\theta t}\right)^{\alpha-1} \frac{e^{-y}}{\Gamma(\alpha)\theta^\alpha}\frac{\theta\,dy}{1-\theta t} \\ &= \frac{1}{(1-\theta t)^\alpha\Gamma(\alpha)}\int_0^\infty y^{\alpha-1}e^{-y}\,dy \\ &= \frac{1}{(1-\theta t)^\alpha}, \end{aligned} \tag{8}$$

and the logarithmic Moment-Generating function is

$$R(t) \equiv \ln M(t) = -\alpha\ln(1-\theta t) \tag{9}$$

$$R'(t) = \frac{\alpha\theta}{1-\theta t} \tag{10}$$

$$R''(t) = \frac{\alpha\theta^2}{(1-\theta t)^2}. \tag{11}$$

The MEAN, VARIANCE, SKEWNESS, and KURTOSIS are then

$$\mu = R'(0) = \alpha\theta \tag{12}$$

$$\sigma^2 = R''(0) = \alpha\theta^2 \tag{13}$$

$$\gamma_1 = \frac{2}{\sqrt{\alpha}} \tag{14}$$

$$\gamma_2 = \frac{6}{\alpha}. \tag{15}$$

The gamma distribution is closely related to other statistical distributions. If X_1, X_2, ..., X_n are independent random variates with a gamma distribution having parameters (α_1,θ), (α_2,θ), ..., (α_n,θ), then $\sum_{i=1}^n X_i$ is distributed as gamma with parameters

$$\alpha = \sum_{i=1}^n \alpha_i \tag{16}$$

$$\theta = \theta. \tag{17}$$

Also, if X_1 and X_2 are independent random variates with a gamma distribution having parameters (α_1,θ) and (α_2,θ), then $X_1/(X_1+X_2)$ is a BETA DISTRIBUTION variate with parameters (α_1,α_2). Both can be derived as follows.

$$P(x,y) = \frac{1}{\Gamma(\alpha_1)\Gamma(\alpha_2)}e^{x_1+x_2}{x_1}^{\alpha_1-1}{x_2}^{\alpha_2-1}. \tag{18}$$

Let

$$u = x_1 + x_2 \qquad x_1 = uv \tag{19}$$

$$v = \frac{x_1}{x_1+x_2} \qquad x_2 = u(1-v), \tag{20}$$

then the JACOBIAN is

$$J\left(\frac{x_1,x_2}{u,v}\right) = \begin{vmatrix} v & u \\ 1-v & -u \end{vmatrix} = -u, \tag{21}$$

so

$$g(u,v)\,du\,dv = f(x,y)\,dx\,dy = f(x,y)u\,du\,dv. \tag{22}$$

$$\begin{aligned} g(u,v) &= \frac{u}{\Gamma(\alpha_1)\Gamma(\alpha_2)}e^{-u}(uv)^{\alpha_1-1}u^{\alpha_2-1}(1-v)^{\alpha_2-1} \\ &= \frac{1}{\Gamma(\alpha_1)\Gamma(\alpha_2)}e^{-u}u^{\alpha_1+\alpha_2-1}v^{\alpha_1-1}(1-v)^{\alpha_2-1}. \end{aligned} \tag{23}$$

The sum $X_1 + X_2$ therefore has the distribution

$$f(u) = f(x_1+x_2) = \int_0^1 g(u,v)\,dv = \frac{e^{-u}u^{\alpha_1+\alpha_2-1}}{\Gamma(\alpha_1+\alpha_2)}, \tag{24}$$

which is a gamma distribution, and the ratio $X_1/(X_1+X_2)$ has the distribution

$$\begin{aligned} h(v) &= h\left(\frac{x_1}{x_1+x_2}\right) = \int_0^\infty g(u,v)\,du \\ &= \frac{v^{\alpha_1-1}(1-v)^{\alpha_2-1}}{B(\alpha_1,\alpha_2)}, \end{aligned} \tag{25}$$

where B is the BETA FUNCTION, which is a BETA DISTRIBUTION.

If X and Y are gamma variates with parameters α_1 and α_2, the X/Y is a variate with a BETA PRIME DISTRIBUTION with parameters α_1 and α_2. Let

$$u = x + y \qquad v = \frac{x}{y}, \tag{26}$$

then the JACOBIAN is

$$J\left(\frac{u,v}{x,y}\right) = \begin{vmatrix} 1 & 1 \\ \frac{1}{y} & -\frac{x}{y^2} \end{vmatrix} = -\frac{x+y}{y^2} = -\frac{(1+v)^2}{u}, \quad (27)$$

so

$$dx\,dy = \frac{u}{(1+v)^2}\,du\,dv \quad (28)$$

$$\begin{aligned} g(u,v) &= \frac{1}{\Gamma(\alpha_1)\Gamma(\alpha_2)} e^{-u} \left(\frac{uv}{1+v}\right)^{\alpha_1-1} \\ &\qquad \left(\frac{u}{1+v}\right)^{\alpha_2-1} \frac{u}{(1+v)^2} \\ &= \frac{1}{\Gamma(\alpha_1)\Gamma(\alpha_2)} e^{-u} u^{\alpha_1+\alpha_2-1} v^{\alpha_2-1} (1+v)^{-\alpha_1-\alpha_2}. \end{aligned} \quad (29)$$

The ratio X/Y therefore has the distribution

$$h(v) = \int_0^\infty g(u,v)\,du = \frac{v^{\alpha_1-1}(1+v)^{-\alpha_1-\alpha_2}}{B(\alpha_1,\alpha_2)}, \quad (30)$$

which is a BETA PRIME DISTRIBUTION with parameters (α_1, α_2).

The "standard form" of the gamma distribution is given by letting $y \equiv x/\theta$, so $dy = dx/\theta$ and

$$\begin{aligned} P(y)\,dy &= \frac{x^{\alpha-1}e^{-x/\theta}}{\Gamma(\alpha)\theta^\alpha}\,dx = \frac{(\theta y)^{\alpha-1}e^{-y}}{\Gamma(\alpha)\theta^\alpha}\,(\theta\,dy) \\ &= \frac{y^{\alpha-1}e^{-y}}{\Gamma(\alpha)}\,dy, \end{aligned} \quad (31)$$

so the MOMENTS about 0 are

$$\nu_r = \frac{1}{\Gamma(\alpha)} \int_0^\infty e^{-x} x^{\alpha-1+r}\,dx = \frac{\Gamma(\alpha+r)}{\Gamma(\alpha)} = (\alpha)_r, \quad (32)$$

where $(\alpha)_r$ is the POCHHAMMER SYMBOL. The MOMENTS about $\mu = \mu_1$ are then

$$\mu_1 = \alpha \quad (33)$$

$$\mu_2 = \alpha \quad (34)$$

$$\mu_3 = 2\alpha \quad (35)$$

$$\mu_4 = 3\alpha^2 + 6\alpha. \quad (36)$$

The MOMENT-GENERATING FUNCTION is

$$M(t) = \frac{1}{(1-t)^\alpha}, \quad (37)$$

and the CUMULANT-GENERATING FUNCTION is

$$K(t) = \alpha \ln(1-t) = \alpha\left(t + \tfrac{1}{2}t^2 + \tfrac{1}{3}t^3 + \ldots\right), \quad (38)$$

so the CUMULANTS are

$$\kappa_r = \alpha\Gamma(r). \quad (39)$$

If x is a NORMAL variate with MEAN μ and STANDARD DEVIATION σ, then

$$y \equiv \frac{(x-\mu)^2}{2\sigma^2} \quad (40)$$

is a standard gamma variate with parameter $\alpha = 1/2$.

see also BETA DISTRIBUTION, CHI-SQUARED DISTRIBUTION

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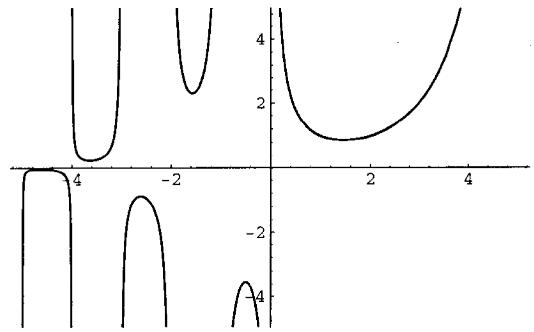
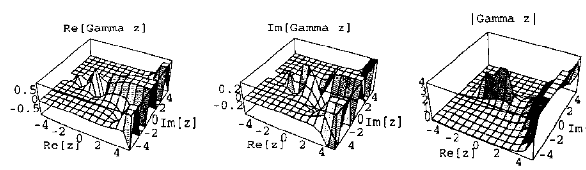
Gamma Function





The complete gamma function is defined to be an extension of the FACTORIAL to COMPLEX and REAL NUMBER arguments. It is ANALYTIC everywhere except at $z = 0$, -1, -2, ... It can be defined as a DEFINITE INTEGRAL for $\Re[z] > 0$ (Euler's integral form)

$$\Gamma(z) \equiv \int_0^\infty t^{z-1}e^{-t}\,dt \quad (1)$$

$$= 2\int_0^\infty e^{-t^2} t^{2z-1}\,dt, \quad (2)$$

or

$$\Gamma(z) \equiv \int_0^1 \left[\ln\left(\frac{1}{t}\right)\right]^{z-1} dt. \quad (3)$$

INTEGRATING (1) by parts for a REAL argument, it can be seen that

$$\begin{aligned} \Gamma(x) &= \int_0^\infty t^{x-1}e^{-t}\,dt \\ &= [-t^{x-1}e^{-t}]_0^\infty + \int_0^\infty (x-1)t^{x-2}e^{-t}\,dt \\ &= (x-1)\int_0^\infty t^{x-2}e^{-t}\,dt = (x-1)\Gamma(x-1). \end{aligned} \quad (4)$$

If x is an INTEGER $n = 1, 2, 3, \ldots$ then

$$\begin{aligned}\Gamma(n) &= (n-1)\Gamma(n-1) = (n-1)(n-2)\Gamma(n-2)\\ &= (n-1)(n-2)\cdots 1 = (n-1)!,\end{aligned} \tag{5}$$

so the gamma function reduces to the FACTORIAL for a POSITIVE INTEGER argument.

BINET'S FORMULA is

$$\ln\Gamma(a) = (a-\tfrac{1}{2})\ln a - a + \tfrac{1}{2}\ln(2\pi) + 2\int_0^\infty \frac{\tan\left(\frac{z}{a}\right)}{e^{2\pi z}-1}\,dz \tag{6}$$

for $\Re[a] > 0$ (Whittaker and Watson 1990, p. 251). The gamma function can also be defined by an INFINITE PRODUCT form (Weierstraß Form)

$$\Gamma(z) \equiv \left[ze^{\gamma z}\prod_{r=1}^\infty\left(1+\frac{z}{r}\right)e^{-z/r}\right]^{-1}, \tag{7}$$

where γ is the EULER-MASCHERONI CONSTANT. This can be written

$$\Gamma(z) = \frac{1}{z}\exp\left[\sum_{k=1}^\infty \frac{(-1)^k s_k}{k}x^k\right], \tag{8}$$

where

$$s_1 \equiv \gamma \tag{9}$$

$$s_k \equiv \zeta(k) \tag{10}$$

for $k \geq 2$, where ζ is the RIEMANN ZETA FUNCTION (Finch). Taking the logarithm of both sides of (7),

$$-\ln[\Gamma(z)] = \ln z + \gamma z + \sum_{n=1}^\infty\left[\ln\left(1+\frac{z}{n}\right) - \frac{z}{n}\right]. \tag{11}$$

Differentiating,

$$\begin{aligned}-\frac{\Gamma'(z)}{\Gamma(z)} &= \frac{1}{z} + \gamma + \sum_{n=1}^\infty\left(\frac{\frac{1}{n}}{1+\frac{z}{n}} - \frac{1}{n}\right)\\ &= \frac{1}{z} + \gamma + \sum_{n=1}^\infty\left(\frac{1}{n+z} - \frac{1}{n}\right)\end{aligned} \tag{12}$$

$$\Gamma'(z) = -\Gamma(z)\left[\frac{1}{z} + \gamma + \sum_{n=1}^\infty\left(\frac{1}{n+z} - \frac{1}{n}\right)\right] \tag{13}$$

$$\equiv \Gamma(z)\Psi(z) = \Gamma(z)\psi_0(z) \tag{14}$$

$$\begin{aligned}\Gamma'(1) &= -\Gamma(1) - \{1+\gamma+[(\tfrac{1}{2}-1)+(\tfrac{1}{3}-\tfrac{1}{2})\\ &\quad + \ldots + \left(\frac{1}{n+1} - \frac{1}{n}\right) + \ldots]\}\\ &= -(1+\gamma-1) = -\gamma\end{aligned} \tag{15}$$

$$\begin{aligned}\Gamma'(n) &= -\Gamma(n)\left\{\frac{1}{n} + \gamma + \left[\left(\frac{1}{1+n} - 1\right) + \left(\frac{1}{2+n} - \frac{1}{2}\right)\right.\right.\\ &\quad \left.\left. + \left(\frac{1}{3+n} - \frac{1}{3}\right) + \ldots\right]\right\}\\ &= -(n-1)!\left(\frac{1}{n} + \gamma - \sum_{k=1}^n \frac{1}{k}\right),\end{aligned} \tag{16}$$

where $\Psi(z)$ is the DIGAMMA FUNCTION and $\psi_0(z)$ is the POLYGAMMA FUNCTION. nth derivatives are given in terms of the POLYGAMMA FUNCTIONS ψ_n, ψ_{n-1}, ..., ψ_0.

The minimum value x_0 of $\Gamma(x)$ for REAL POSITIVE $x = x_0$ is achieved when

$$\Gamma'(x_0) = \Gamma(x_0)\psi_0(x_0) = 0 \tag{17}$$

$$\psi_0(x_0) = 0, \tag{18}$$

This can be solved numerically to give $x_0 = 1.46163\ldots$ (Sloane's A030169), which has CONTINUED FRACTION [1, 2, 6, 63, 135, 1, 1, 1, 1, 4, 1, 38, ...] (Sloane's A030170). At x_0, $\Gamma(x_0)$ achieves the value 0.8856031944... (Sloane's A030171), which has CONTINUED FRACTION [0, 1, 7, 1, 2, 1, 6, 1, 1, ...] (Sloane's A030172).

The Euler limit form is

$$\begin{aligned}\frac{1}{\Gamma(z)} &= z\left[\lim_{m\to\infty} e^{(1+1/2+\ldots+1/m-\ln m)z}\right]\\ &\quad\times\left[\lim_{m\to\infty}\prod_{n=1}^m\left\{\left(1+\frac{z}{n}\right)e^{-z/n}\right\}\right]\\ &= \frac{1}{z}\prod_{n=1}^\infty\left[\left(1+\frac{1}{n}\right)^z\left(1+\frac{z}{n}\right)^{-1}\right],\end{aligned} \tag{19}$$

so

$$\Gamma(z) \equiv \lim_{n\to\infty}\frac{1\cdot 2\cdot 3\cdots n}{z(z+1)(z+2)\cdots(z+n)}n^z. \tag{20}$$

The LANCZOS APPROXIMATION for $z > 0$ is

$$\begin{aligned}\Gamma(z+1) &= (z+\gamma+\tfrac{1}{2})^{z+1/2}e^{z+\gamma+1/2}\sqrt{2\pi}\\ &\quad\times\left[c_0 + \frac{c_1}{z+1} + \frac{c_2}{z+2} + \ldots + \frac{c_n}{z+n} + \epsilon\right].\end{aligned} \tag{21}$$

The complete gamma function $\Gamma(x)$ can be generalized to the incomplete gamma function $\Gamma(x,a)$ such that $\Gamma(x) = \Gamma(x,0)$. The gamma function satisfies the recurrence relations

$$\Gamma(1+z) = z\Gamma(z) \tag{22}$$

$$\Gamma(1-z) = -z\Gamma(-z). \tag{23}$$

Additional identities are

$$\Gamma(x)\Gamma(-x) = -\frac{\pi}{x\sin(\pi x)} \tag{24}$$

$$\Gamma(x)\Gamma(1-x) = \frac{\pi}{\sin(\pi x)} \tag{25}$$

$$\begin{aligned}\ln[\Gamma(x+iy+1)] &= \ln(x^2+y^2) + i\tan^{-1}\left(\frac{y}{x}\right)\\ &\quad + \ln[\Gamma(x+iy)]\end{aligned} \tag{26}$$

$$|(ix)!|^2 = \frac{\pi x}{\sinh(\pi x)} \tag{27}$$

$$|(n+ix)!| = \sqrt{\frac{\pi x}{\sinh(\pi x)}}\prod_{s=1}^n\sqrt{s^2+x^2}. \tag{28}$$

For integral arguments, the first few values are 1, 1, 2, 6, 24, 120, 720, 5040, 40320, 362880, ... (Sloane's A000142). For half integral arguments,

$$\Gamma(\tfrac{1}{2}) = \sqrt{\pi} \quad (29)$$

$$\Gamma(\tfrac{3}{2}) = \tfrac{1}{2}\sqrt{\pi} \quad (30)$$

$$\Gamma(\tfrac{5}{2}) = \tfrac{3}{4}\sqrt{\pi}. \quad (31)$$

In general, for m a POSITIVE INTEGER $m = 1, 2, \ldots$

$$\Gamma(\tfrac{1}{2}+m) = \frac{1\cdot 3\cdot 5\cdots(2m-1)}{2^m}\sqrt{\pi} = \frac{(2m-1)!!}{2^m}\sqrt{\pi} \quad (32)$$

$$\Gamma(\tfrac{1}{2}-m) = \frac{(-1)^m 2^m}{1\cdot 3\cdot 5\cdots(2m-1)}\sqrt{\pi} = \frac{(-1)^m 2^m}{(2m-1)!!}\sqrt{\pi}. \quad (33)$$

For $\Re[x] = -\tfrac{1}{2}$,

$$|(-\tfrac{1}{2}+iy)!|^2 = \frac{\pi}{\cosh(\pi y)}. \quad (34)$$

Gamma functions of argument $2z$ can be expressed using the LEGENDRE DUPLICATION FORMULA

$$\Gamma(2z) = (2\pi)^{-1/2}2^{2z-1/2}\Gamma(z)\Gamma(z+\tfrac{1}{2}). \quad (35)$$

Gamma functions of argument $3z$ can be expressed using a triplication FORMULA

$$\Gamma(3z) = (2\pi)^{-1}3^{3z-1/2}\Gamma(z)\Gamma(z+\tfrac{1}{3})\Gamma(z+\tfrac{2}{3}). \quad (36)$$

The general result is the GAUSS MULTIPLICATION FORMULA

$$\Gamma(z)\Gamma(z+\tfrac{1}{n})\cdots\Gamma(z+\tfrac{n+1}{n}) = (2\pi)^{(n-1)/2}n^{1/2-nz}\Gamma(nz). \quad (37)$$

The gamma function is also related to the RIEMANN ZETA FUNCTION ζ by

$$\Gamma\left(\frac{s}{2}\right)\pi^{-2/2}\zeta(s) = \Gamma\left(\frac{1-s}{2}\right)\pi^{-(1-s)/2}\zeta(1-s). \quad (38)$$

Borwein and Zucker (1992) give a variety of identities relating gamma functions to square roots and ELLIPTIC INTEGRAL SINGULAR VALUES k_n, i.e., MODULI k_n such that

$$\frac{K'(k_n)}{K(k_n)} = \sqrt{n}, \quad (39)$$

where $K(k)$ is a complete ELLIPTIC INTEGRAL OF THE FIRST KIND and $K'(k) = K(k') = K(\sqrt{1-k^2})$ is the complementary integral.

$$\Gamma(\tfrac{1}{3}) = 2^{7/9}3^{-1/12}\pi^{1/3}[K(k_3)]^{1/3} \quad (40)$$

$$\Gamma(\tfrac{1}{4}) = 2\pi^{1/4}[K(k_1)]^{1/2} \quad (41)$$

$$\Gamma(\tfrac{1}{6}) = 2^{-1/3}3^{1/2}\pi^{-1/2}[\Gamma(\tfrac{1}{3})]^2 \quad (42)$$

$$\Gamma(\tfrac{1}{8})\Gamma(\tfrac{3}{8}) = (\sqrt{2}-1)^{1/2}2^{13/4}\pi^{1/2}K(k_2) \quad (43)$$

$$\frac{\Gamma(\frac{1}{8})}{\Gamma(\frac{3}{8})} = 2(\sqrt{2}+1)^{1/2}\pi^{-1/4}[K(k_1)]^{1/2} \quad (44)$$

$$\Gamma(\tfrac{1}{12}) = 2^{-1/4}3^{3/8}(\sqrt{3}+1)^{1/2}\pi^{-1/2}\Gamma(\tfrac{1}{4})\Gamma(\tfrac{1}{3}) \quad (45)$$

$$\Gamma(\tfrac{5}{12}) = 2^{1/4}3^{-1/8}(\sqrt{3}-1)^{1/2}\pi^{1/2}\frac{\Gamma(\frac{1}{4})}{\Gamma(\frac{1}{3})} \quad (46)$$

$$\frac{\Gamma(\frac{1}{24})\Gamma(\frac{11}{24})}{\Gamma(\frac{5}{24})\Gamma(\frac{7}{24})} = \sqrt{3}\sqrt{2+\sqrt{3}} \quad (47)$$

$$\frac{\Gamma(\frac{1}{24})\Gamma(\frac{5}{24})}{\Gamma(\frac{7}{24})\Gamma(\frac{11}{24})} = 4\cdot 3^{1/4}(\sqrt{3}+\sqrt{2})\pi^{-1/2}K(k_1) \quad (48)$$

$$\frac{\Gamma(\frac{1}{24})\Gamma(\frac{7}{24})}{\Gamma(\frac{5}{24})\Gamma(\frac{11}{24})} = 2^{25/18}3^{1/3}(\sqrt{2}+1)\pi^{-1/3}[K(k_3)]^{2/3} \quad (49)$$

$$\Gamma(\tfrac{1}{24})\Gamma(\tfrac{5}{25})\Gamma(\tfrac{7}{24})\Gamma(\tfrac{11}{24}) = 384(\sqrt{2}+1)(\sqrt{3}-\sqrt{2})(2-\sqrt{3})\pi[K(k_6)]^2 \quad (50)$$

$$\Gamma(\tfrac{1}{10}) = 2^{-7/10}5^{1/4}(\sqrt{5}+1)^{1/2}\pi^{-1/2}\Gamma(\tfrac{1}{5})\Gamma(\tfrac{2}{5}) \quad (51)$$

$$\Gamma(\tfrac{3}{10}) = 2^{-3/5}(\sqrt{5}-1)\pi^{1/2}\frac{\Gamma(\frac{1}{5})}{\Gamma(\frac{2}{5})} \quad (52)$$

$$\frac{\Gamma(\frac{1}{15})\Gamma(\frac{4}{15})\Gamma(\frac{7}{15})}{\Gamma(\frac{2}{15})} = 2\cdot 3^{1/2}5^{1/6}\sin(\tfrac{2}{15}\pi)[\Gamma(\tfrac{1}{3})]^2 \quad (53)$$

$$\frac{\Gamma(\frac{1}{15})\Gamma(\frac{2}{15})\Gamma(\frac{7}{15})}{\Gamma(\frac{4}{15})} = 2^2\cdot 3^{2/5}\sin(\tfrac{1}{5}\pi)\sin(\tfrac{4}{15}\pi)[\Gamma(\tfrac{1}{5})]^2 \quad (54)$$

$$\frac{\Gamma(\frac{2}{15})\Gamma(\frac{4}{15})\Gamma(\frac{7}{15})}{\Gamma(\frac{1}{15})} = \frac{2^{-3/2}3^{-1/5}5^{1/4}(\sqrt{5}-1)^{1/2}[\Gamma(\frac{2}{5})]^2}{\sin(\frac{4}{15}\pi)} \quad (55)$$

$$\frac{\Gamma(\frac{1}{15})\Gamma(\frac{2}{15})\Gamma(\frac{4}{15})}{\Gamma(\frac{7}{15})} = 60(\sqrt{5}-1)\sin(\tfrac{7}{15}\pi)[K(k_{15})]^2 \quad (56)$$

$$\frac{\Gamma(\frac{1}{20})\Gamma(\frac{9}{20})}{\Gamma(\frac{3}{20})\Gamma(\frac{7}{20})} = 2^{-1}5^{1/4}(\sqrt{5}+1) \quad (57)$$

$$\frac{\Gamma(\frac{1}{20})\Gamma(\frac{3}{20})}{\Gamma(\frac{7}{20})\Gamma(\frac{9}{20})} = 2^{4/5}(10-2\sqrt{5})^{1/2}\pi^{-1}\sin(\tfrac{7}{20}\pi)\sin(\tfrac{9}{20}\pi)[\Gamma(\tfrac{1}{5})]^2 \quad (58)$$

$$\frac{\Gamma(\frac{1}{20})\Gamma(\frac{7}{20})}{\Gamma(\frac{3}{20})\Gamma(\frac{9}{20})}$$

$$= 2^{3/5}(10+2\sqrt{5})^{1/2}\pi^{-1}\sin(\tfrac{3}{20}\pi)\sin(\tfrac{9}{20}\pi)[\Gamma(\tfrac{2}{5})]^2 \tag{59}$$

$$\Gamma(\tfrac{1}{20})\Gamma(\tfrac{3}{20})\Gamma(\tfrac{7}{20})\Gamma(\tfrac{9}{20}) = 160(\sqrt{5}-2)^{1/2}\pi[K(k_5)]^2. \tag{60}$$

A few curious identities include

$$\prod_{n=1}^{8}\Gamma(\tfrac{1}{3}n) = \frac{640}{3^6}\left(\frac{\pi}{\sqrt{3}}\right)^3 \tag{61}$$

$$\frac{[\Gamma(\frac{1}{4})]^4}{16\pi^2} = \frac{3^2}{3^2-1}\frac{5^2-1}{5^2}\frac{7^2}{7^2-1}\cdots \tag{62}$$

$$\frac{\Gamma'(1)}{\Gamma(1)} - \frac{\Gamma'(\frac{1}{2})}{\Gamma(\frac{1}{2})} = 2\ln 2 \tag{63}$$

(Magnus and Oberhettinger 1949, p. 1). Ramanujan also gave a number of fascinating identities:

$$\frac{\Gamma^2(n+1)}{\Gamma(n+xi+1)\Gamma(n-xi+1)} = \prod_{k=1}^{\infty}\left[1+\frac{x^2}{(n+k)^2}\right] \tag{64}$$

$$\phi(m,n)\phi(n,m) = \frac{\Gamma^3(m+1)\Gamma^3(n+1)}{\Gamma(2m+n+1)\Gamma(2n+m+1)} \times\frac{\cosh[\pi(m+n)\sqrt{3}]-\cos[\pi(m-n)]}{2\pi^2(m^2+mn+n^2)}, \tag{65}$$

where

$$\phi(m,n) \equiv \prod_{k=1}^{\infty}\left[1+\left(\frac{m+n}{k+m}\right)^3\right], \tag{66}$$

$$\prod_{k=1}^{\infty}\left[1+\left(\frac{n}{k}\right)^3\right]\prod_{k=1}^{\infty}\left[1+3\left(\frac{n}{n+2k}\right)^2\right] = \frac{\Gamma(\frac{1}{2}n)}{\Gamma[\frac{1}{2}(n+1)]}\frac{\cosh(\pi n\sqrt{3})-\cos(\pi n)}{2^{n+2}\pi^{3/2}n} \tag{67}$$

(Berndt 1994).

The following ASYMPTOTIC SERIES is occasionally useful in probability theory (e.g., the 1-D RANDOM WALK):

$$\frac{\Gamma(J+\frac{1}{2})}{\Gamma(J)} = \sqrt{J}\left(1-\frac{1}{8J}+\frac{1}{128J^2} +\frac{5}{1024J^3}-\frac{21}{32768J^4}+\ldots\right) \tag{68}$$

(Graham *et al.* 1994). This series also gives a nice asymptotic generalization of STIRLING NUMBERS OF THE FIRST KIND to fractional values.

It has long been known that $\Gamma(\frac{1}{4})\pi^{-1/4}$ is TRANSCENDENTAL (Davis 1959), as is $\Gamma(\frac{1}{3})$ (Le Lionnais 1983), and Chudnovsky has apparently recently proved that $\Gamma(\frac{1}{4})$ is itself TRANSCENDENTAL.

The upper incomplete gamma function is given by

$$\Gamma(a,x) \equiv \int_x^{\infty} t^{a-1}e^{-t}\,dt = 1-\gamma(a,x), \tag{69}$$

where γ is the lower incomplete gamma function. For a an INTEGER n

$$\Gamma(n,x) = (n-1)!e^{-x}\sum_{s=0}^{n-1}\frac{x^s}{s!} = (n-1)!e^{-x}\,\mathrm{es}_{n-1}(x), \tag{70}$$

where es is the EXPONENTIAL SUM FUNCTION. The lower incomplete gamma function is given by

$$\begin{aligned}\gamma(a,x) &\equiv \Gamma(a)-\Gamma(a,x) = \int_0^x e^{-t}t^{a-1}\,dt \\ &= a^{-1}x^a e^{-x}{}_1F_1(1;1+a;x) \\ &= a^{-1}x^a{}_1F_1(a;1+a;-x),\end{aligned} \tag{71}$$

where ${}_1F_1(a;b;x)$ is the CONFLUENT HYPERGEOMETRIC FUNCTION OF THE FIRST KIND. For a an INTEGER n,

$$\begin{aligned}\gamma(n,x) &= (n-1)!\left(1-e^{-x}\sum_{s=0}^{n-1}\frac{x^s}{s!}\right) \\ &= (n-1)!\,[1-\mathrm{es}_{n-1}(x)].\end{aligned} \tag{72}$$

The function $\Gamma(a,z)$ is denoted `Gamma[a,z]` and the function $\gamma(a,z)$ is denoted `Gamma[a,0,z]` in *Mathematica*® (Wolfram Research, Champaign, IL).

see also DIGAMMA FUNCTION, DOUBLE GAMMA FUNCTION, FRANSÉN-ROBINSON CONSTANT G-FUNCTION, GAUSS MULTIPLICATION FORMULA, LAMBDA FUNCTION, LEGENDRE DUPLICATION FORMULA, MU FUNCTION, NU FUNCTION, PEARSON'S FUNCTION, POLYGAMMA FUNCTION, REGULARIZED GAMMA FUNCTION, STIRLING'S SERIES

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Gamma Group

The gamma group Γ is the set of all transformations w of the form

$$w(t) = \frac{at+b}{ct+d},$$

where a, b, c, and d are INTEGERS and $ad - bc = 1$.

see also KLEIN'S ABSOLUTE INVARIANT, LAMBDA GROUP, THETA FUNCTION

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Gamma-Modular

see MODULAR GAMMA FUNCTION

Gamma Statistic

$$\gamma_r \equiv \frac{\kappa_r}{\sigma^{r+2}},$$

where κ_r are CUMULANTS and σ is the STANDARD DEVIATION.

see also KURTOSIS, SKEWNESS

Garage Door

see ASTROID

Gårding's Inequality

Gives a lower bound for the inner product (Lu, u), where L is a linear elliptic REAL differential operator of order m, and u has compact support.

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Garman-Kohlhagen Formula

$$V_t = e^{-y\tau} S_t N(d_1) - e^{-r\tau} K N(d_2),$$

where N is the cumulative NORMAL DISTRIBUTION and

$$d_1, d_2 = \frac{\log\left(\frac{S_t}{K}\right) + (r - y \pm \frac{1}{2}\sigma^2)\tau}{\sigma\sqrt{\tau}}.$$

If $y = 0$, this is the standard form of the Black-Scholes formula.

see also BLACK-SCHOLES THEORY

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Gate Function

Bracewell's term for the RECTANGLE FUNCTION.

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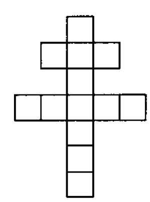
Gauche Conic

see SKEW CONIC

Gaullist Cross



A CROSS also called the CROSS OF LORRAINE or PATRIARCHAL CROSS.

see also CROSS, DISSECTION

Gauss's Backward Formula

$$f_p = f_0 + p\delta_{-1/2} + G_2^*\delta_0^2 + G_3\delta_{-1/2}^3 + G_4^*\delta_0^4 + G_5\delta_{-1/2}^5 + \ldots,$$

for $p \in [0,1]$, where δ is the CENTRAL DIFFERENCE and

$$G_{2n}^* = \binom{p+n}{2n}$$

$$G_{2n+1} = \binom{p+n}{2n+1},$$

where $\binom{n}{k}$ is a BINOMIAL COEFFICIENT.

see also CENTRAL DIFFERENCE, GAUSS'S FORWARD FORMULA

References

Beyer, W. H. *CRC Standard Mathematical Tables, 28th ed.* Boca Raton, FL: CRC Press, p. 433, 1987.

Gauss-Bodenmiller Theorem

The CIRCLES on the DIAGONALS of a COMPLETE QUADRILATERAL as DIAMETERS are COAXAL. Furthermore, the ORTHOCENTERS of the four TRIANGLES of a COMPLETE QUADRILATERAL are COLLINEAR on the RADICAL AXIS of the COAXAL CIRCLES.

see also COAXAL CIRCLES, COLLINEAR, COMPLETE QUADRILATERAL, DIAGONAL (POLYGON), ORTHOCENTER, RADICAL AXIS

References

Johnson, R. A. *Modern Geometry: An Elementary Treatise on the Geometry of the Triangle and the Circle.* Boston, MA: Houghton Mifflin, p. 172, 1929.

Gauss-Bolyai-Lobachevsky Space

A non-Euclidean space with constant NEGATIVE GAUSSIAN CURVATURE.

see also LOBACHEVSKY-BOLYAI-GAUSS GEOMETRY, NON-EUCLIDEAN GEOMETRY

Gauss-Bonnet Formula

The Gauss-Bonnet formula has several formulations. The simplest one expresses the total GAUSSIAN CURVATURE of an embedded triangle in terms of the total GEODESIC CURVATURE of the boundary and the JUMP ANGLES at the corners.

More specifically, if M is any 2-D RIEMANNIAN MANIFOLD (like a surface in 3-space) and if T is an embedded triangle, then the Gauss-Bonnet formula states that the integral over the whole triangle of the GAUSSIAN CURVATURE with respect to AREA is given by 2π minus the sum of the JUMP ANGLES minus the integral of the GEODESIC CURVATURE over the whole of the boundary of the triangle (with respect to ARC LENGTH),

$$\iint_T K\,dA = 2\pi - \sum \alpha_i - \int_{\partial T} \kappa_g\,ds, \tag{1}$$

where K is the GAUSSIAN CURVATURE, dA is the AREA measure, the α_is are the JUMP ANGLES of ∂T, and κ_g is the GEODESIC CURVATURE of ∂T, with ds the ARC LENGTH measure.

The next most common formulation of the Gauss-Bonnet formula is that for any compact, boundaryless 2-D RIEMANNIAN MANIFOLD, the integral of the GAUSSIAN CURVATURE over the entire MANIFOLD with respect to AREA is 2π times the EULER CHARACTERISTIC of the MANIFOLD,

$$\iint_M K\,dA = 2\pi\chi(M). \tag{2}$$

This is somewhat surprising because the total GAUSSIAN CURVATURE is differential-geometric in character, but the EULER CHARACTERISTIC is topological in character and does not depend on differential geometry at all. So if you distort the surface and change the curvature at any location, regardless of how you do it, the same total curvature is maintained.

Another way of looking at the Gauss-Bonnet theorem for surfaces in 3-space is that the GAUSS MAP of the surface has DEGREE given by half the EULER CHARACTERISTIC of the surface

$$\iint_M K\,dA = 2\pi\chi(M) - \sum \alpha_i - \int_{\partial M} \kappa_g\,ds, \tag{3}$$

which works only for ORIENTABLE SURFACES. This makes the Gauss-Bonnet theorem a simple consequence of the POINCARE-HOPF INDEX THEOREM, which is a nice way of looking at things if you're a topologist, but not so nice for a differential geometer. This proof can be found in Guillemin and Pollack (1974). Millman and Parker (1977) give a standard differential-geometric proof of the Gauss-Bonnet theorem, and Singer and Thorpe (1996) give a GAUSS'S THEOREMA EGREGIUM-inspired proof which is entirely intrinsic, without any reference to the ambient EUCLIDEAN SPACE.

A general Gauss-Bonnet formula that takes into account both formulas can also be given. For any compact 2-D RIEMANNIAN MANIFOLD with corners, the integral of the GAUSSIAN CURVATURE over the 2-MANIFOLD with respect to AREA is 2π times the EULER CHARACTERISTIC of the MANIFOLD minus the sum of the JUMP ANGLES and the total GEODESIC CURVATURE of the boundary.

References

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Guillemin, V. and Pollack, A. *Differential Topology.* Englewood Cliffs, NJ: Prentice-Hall, 1974.
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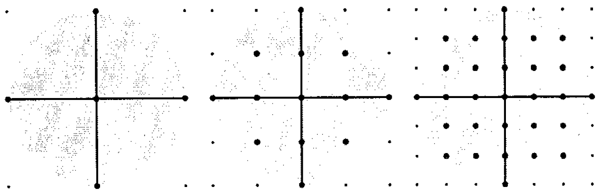
Singer, I. M. and Thorpe, J. A. *Lecture Notes on Elementary Topology and Geometry.* New York: Springer-Verlag, 1996.

Gauss-Bonnet Theorem

see GAUSS-BONNET FORMULA

Gauss's Circle Problem



Count the number of LATTICE POINTS $N(r)$ inside the boundary of a CIRCLE of RADIUS r with center at the origin. The exact solution is given by the SUM

$$N(r) = 1 + 4\lfloor r \rfloor + 4\sum_{i=1}^{\lfloor r \rfloor} \left\lfloor \sqrt{r^2 - i^2} \right\rfloor. \quad (1)$$

The first few values are 1, 5, 13, 29, 49, 81, 113, 149, ... (Sloane's A000328).

Gauss showed that

$$N(r) = \pi r^2 + E(r), \quad (2)$$

where

$$|E(r)| \leq 2\sqrt{2}\,\pi r. \quad (3)$$

Writing $|E(r)| \leq Cr^\theta$, the best bounds on θ are $1/2 < \theta \leq 46/73 \approx 0.630137$ (Huxley 1990). The problem has also been extended to CONICS and higher dimensions. The limit 1/2 was obtained by Hardy and Landau (1915), and the limit 46/73 improves previous values of $24/37 \approx 0.64864$ (Cheng 1963) and $34/53 \approx 0.64150$ (Vinogradov), and $7/11 \approx 0.63636$.

see also CIRCLE LATTICE POINTS

References
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Guy, R. K. "Gauß's Lattice Point Problem." §F1 in *Unsolved Problems in Number Theory, 2nd ed.* New York: Springer-Verlag, pp. 240–2417, 1994.
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Gauss's Class Number Conjecture

In his monumental treatise *Disquisitiones Arithmeticae,* Gauss conjectured that the CLASS NUMBER $h(-d)$ of an IMAGINARY quadratic field with DISCRIMINANT $-d$ tends to infinity with d. A proof was finally given by Heilbronn (1934), and Siegel (1936) showed that for any $\epsilon > 0$, there exists a constant $c_\epsilon > 0$ such that

$$h(-d) > c_\epsilon d^{1/2-\epsilon}$$

as $d \to \infty$. However, these results were not effective in actually determining the values for a given m of a complete list of fundamental discriminants $-d$ such that $h(-d) = m$, a problem known as GAUSS'S CLASS NUMBER PROBLEM.

Goldfeld (1976) showed that if there exists a "Weil curve" whose associated DIRICHLET L-SERIES has a zero of at least third order at $s = 1$, then for any $\epsilon > 0$, there exists an effectively computable constant c_ϵ such that

$$h(-d) > c_\epsilon (\ln d)^{1-\epsilon}.$$

Gross and Zaiger (1983) showed that certain curves must satisfy the condition of Goldfeld, and Goldfeld's proof was simplified by Oesterlé (1985).

see also CLASS NUMBER, GAUSS'S CLASS NUMBER PROBLEM, HEEGNER NUMBER

References
Arno, S.; Robinson, M. L.; and Wheeler, F. S. "Imaginary Quadratic Fields with Small Odd Class Number." http://www.math.uiuc.edu/Algebraic-Number-Theory/0009/.
Böcherer, S. "Das Gauß'sche Klassenzahlproblem." *Mitt.. Math. Ges. Hamburg* **11**, 565–589, 1988.
Gauss, C. F. *Disquisitiones Arithmeticae.* New Haven, CT: Yale University Press, 1966.
Goldfeld, D. M. "The Class Number of Quadratic Fields and the Conjectures of Birch and Swinnerton-Dyer." *Ann. Scuola Norm. Sup. Pisa* **3**, 623–663, 1976.
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Heilbronn, H. "On the Class Number in Imaginary Quadratic Fields." *Quart. J. Math. Oxford Ser.* **25**, 150–160, 1934.
Oesterlé, J. "Nombres de classes des corps quadratiques imaginaires." *Astérique* **121-122**, 309–323, 1985.
Siegel, C. L. "Über die Klassenzahl quadratischer Zahlkörper." *Acta. Arith.* **1**, 83–86, 1936.

Gauss's Class Number Problem

For a given m, determine a complete list of fundamental DISCRIMINANTS $-d$ such that the CLASS NUMBER is given by $h(-d) = m$. Heegner (1952) gave a solution for $m = 1$, but it was not completely accepted due to a number of apparent gaps. However, subsequent examination of Heegner's proof show it to be "essentially" correct (Conway and Guy 1996). Conway and Guy (1996) therefore call the nine values of $n(-d)$ having $h(-d) = 1$ where $-d$ is the DISCRIMINANT corresponding to a QUADRATIC FIELD $a + b\sqrt{-n}$ ($n = -1, -2, -3, -7, -11, -19, -43, -67$, and -163; Sloane's A003173) the HEEGNER NUMBERS. The HEEGNER NUMBERS have a number of fascinating properties.

Stark (1967) and Baker (1966) gave independent proofs of the fact that only nine such numbers exist; both proofs were accepted. Baker (1971) and Stark (1975) subsequently and independently solved the generalized class number problem completely for $m = 2$. Oesterlé (1985) solved the case $m = 3$, and Arno (1992) solved the case $m = 4$. Wagner (1996) solve the cases $n = 5$, 6, and 7. Arno *et al.* (1993) solved the problem for ODD m satisfying $5 \leq m \leq 23$.

see also CLASS NUMBER, GAUSS'S CLASS NUMBER CONJECTURE, HEEGNER NUMBER

References
Arno, S. "The Imaginary Quadratic Fields of Class Number 4." *Acta Arith.* **40**, 321–334, 1992.
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Stark, H. M. "A Complete Determination of the Complex Quadratic Fields of Class Number One." *Michigan Math. J.* **14**, 1–27, 1967.
Stark, H. M. "On Complex Quadratic Fields with Class Number Two." *Math. Comput.* **29**, 289–302, 1975.
Wagner, C. "Class Number 5, 6, and 7." *Math. Comput.* **65**, 785–800, 1996.

Gauss's Constant

The RECIPROCAL of the ARITHMETIC-GEOMETRIC MEAN of 1 and $\sqrt{2}$,

$$\begin{aligned}\frac{1}{M(1,\sqrt{2})} &= \frac{2}{\pi}\int_0^1 \frac{1}{\sqrt{1-x^4}}\,dx && (1)\\ &= \frac{2}{\pi}\int_0^{\pi/2} \frac{d\theta}{\sqrt{1+\sin^2\theta}} && (2)\\ &= \frac{\sqrt{2}}{\pi}K\left(\frac{1}{\sqrt{2}}\right)\\ &= \frac{1}{(2\pi)^{3/2}}[\Gamma(\tfrac{1}{4})]^2 && (3)\\ &= 0.83462684167\ldots, && (4)\end{aligned}$$

where $K(k)$ is the complete ELLIPTIC INTEGRAL OF THE FIRST KIND and $\Gamma(z)$ is the GAMMA FUNCTION.

see also ARITHMETIC-GEOMETRIC MEAN, GAUSS-KUZMIN-WIRSING CONSTANT

References
Finch, S. "Favorite Mathematical Constants." `http://www.mathsoft.com/asolve/constant/gauss/gauss.html`.

Gauss's Criterion

Let p be an ODD PRIME and b a POSITIVE INTEGER not divisible by p. Then for each POSITIVE ODD INTEGER $2k-1<p$, let r_i be

$$r_k \equiv (2k-1)b \pmod p$$

with $0 < r_k < p$, and let t be the number of EVEN r_is. Then

$$(b/p) = (-1)^t,$$

where (b/p) is the LEGENDRE SYMBOL.

References
Shanks, D. "Gauss's Criterion." §1.17 in *Solved and Unsolved Problems in Number Theory, 4th ed.* New York: Chelsea, pp. 38–40, 1993.

Gauss's Double Point Theorem

If a sequence of DOUBLE POINTS is passed as a CLOSED CURVE is traversed, each DOUBLE POINT appears once in an EVEN place and once in an ODD place.

References
Rademacher, H. and Toeplitz, O. *The Enjoyment of Mathematics: Selections from Mathematics for the Amateur.* Princeton, NJ: Princeton University Press, pp. 61–66, 1957.

Gauss Equations

If $\mathbf{x}$ is a regular patch on a REGULAR SURFACE in $\mathbb{R}^3$ with normal $\hat{\mathbf{N}}$, then

$$\begin{aligned}\mathbf{x}_{uu} &= \Gamma_{11}^1\mathbf{x}_u + \Gamma_{11}^2\mathbf{x}_v + e\hat{\mathbf{N}} && (1)\\ \mathbf{x}_{uv} &= \Gamma_{12}^1\mathbf{x}_u + \Gamma_{12}^2\mathbf{x}_v + f\hat{\mathbf{N}} && (2)\\ \mathbf{x}_{vv} &= \Gamma_{22}^1\mathbf{x}_u + \Gamma_{22}^2\mathbf{x}_v + g\hat{\mathbf{N}}, && (3)\end{aligned}$$

where e, f, and g are coefficients of the second FUNDAMENTAL FORM and Γ_{ij}^k are CHRISTOFFEL SYMBOLS OF THE SECOND KIND.

see also CHRISTOFFEL SYMBOL OF THE SECOND KIND, FUNDAMENTAL FORMS, MAINARDI-CODAZZI EQUATIONS

References
Gray, A. *Modern Differential Geometry of Curves and Surfaces.* Boca Raton, FL: CRC Press, pp. 398–400, 1993.

Gauss's Equation (Radius Derivatives)

Expresses the second derivatives of $\mathbf{r}$ in terms of the CHRISTOFFEL SYMBOL OF THE SECOND KIND.

$$\mathbf{r}_{ij} = \Gamma_{ij}^k\mathbf{r}_k + (\mathbf{r}_{ij}\cdot\mathbf{n})\mathbf{n}.$$

Gauss's Formula

$$4\frac{x^p - y^p}{x - y} = R^2(x,y) - (-1)^{(p-1)/2} p S^2(x,y),$$

where R and S are HOMOGENEOUS POLYNOMIALS in x and y with integral COEFFICIENTS.

see also AURIFEUILLEAN FACTORIZATION, GAUSS'S BACKWARD FORMULA, GAUSS'S FORWARD FORMULA

References

Shanks, D. *Solved and Unsolved Problems in Number Theory, 4th ed.* New York: Chelsea, p. 105, 1993.

Gauss's Formulas

Let a SPHERICAL TRIANGLE have sides a, b, and c with A, B, and C the corresponding opposite angles. Then

$$\frac{\sin[\frac{1}{2}(a-b)]}{\sin(\frac{1}{2}c)} = \frac{\sin[\frac{1}{2}(A-B)]}{\cos(\frac{1}{2}C)} \quad (1)$$

$$\frac{\sin[\frac{1}{2}(a+b)]}{\sin(\frac{1}{2}c)} = \frac{\cos[\frac{1}{2}(A-B)]}{\sin(\frac{1}{2}C)} \quad (2)$$

$$\frac{\cos[\frac{1}{2}(a-b)]}{\cos(\frac{1}{2}c)} = \frac{\sin[\frac{1}{2}(A+B)]}{\cos(\frac{1}{2}C)} \quad (3)$$

$$\frac{\cos[\frac{1}{2}(a+b)]}{\cos(\frac{1}{2}c)} = \frac{\cos[\frac{1}{2}(A+B)]}{\sin(\frac{1}{2}C)}. \quad (4)$$

see also SPHERICAL TRIGONOMETRY

Gauss's Forward Formula

$$f_p = f_0 + p\delta_{1/2} + G_2\delta_0^2 + G_3\delta_{1/2}^3 + G_4\delta_0^4 + G_5\delta_{1/2}^5 + \ldots,$$

for $p \in [0,1]$, where δ is the CENTRAL DIFFERENCE and

$$G_{2n} = \binom{p+n-1}{2n}$$

$$G_{2n+1} = \binom{p+n}{2n+1},$$

where $\binom{n}{k}$ is a BINOMIAL COEFFICIENT.

see also CENTRAL DIFFERENCE, GAUSS'S BACKWARD FORMULA

References

Beyer, W. H. *CRC Standard Mathematical Tables, 28th ed.* Boca Raton, FL: CRC Press, p. 433, 1987.

Gauss's Harmonic Function Theorem

If a function ϕ is HARMONIC in a SPHERE, then the value of ϕ at the center of the SPHERE is the ARITHMETIC MEAN of its value on the surface.

Gauss's Hypergeometric Theorem

$${}_2F_1(a,b;c;1) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}$$

for $\Re[c-a-b] > 0$, where ${}_2F_1(a,b;c;x)$ is a HYPERGEOMETRIC FUNCTION. If a is a NEGATIVE INTEGER $-n$, this becomes

$${}_2F_1(-n,b;c;1) = \frac{(c-b)_n}{(c)_n},$$

which is known as the VANDERMONDE THEOREM.

see also GENERALIZED HYPERGEOMETRIC FUNCTION, HYPERGEOMETRIC FUNCTION

References

Petkovšek, M.; Wilf, H. S.; and Zeilberger, D. *A=B.* Wellesley, MA: A. K. Peters, pp. 42 and 126, 1996.

Gauss's Inequality

If a distribution has a single MODE at μ_0, then

$$P(|x-\mu_0| \geq \lambda\tau) \leq \frac{4}{9\lambda^2},$$

where

$$\tau^2 \equiv \sigma^2 + (\mu - \mu_0)^2.$$

Gauss's Interpolation Formula

$$f(x) \approx t_n(x) = \sum_{k=0}^{2n} f_k \zeta_k(x),$$

where $t_n(x)$ is a trigonometric POLYNOMIAL of degree n such that $t_n(x_k) = f_k$ for $k = 0, \ldots, 2n$, and

$$\zeta_k(x) = \frac{\sin[\frac{1}{2}(x-x_0)]\cdots\sin[\frac{1}{2}(x-x_{k-1})]}{\sin[\frac{1}{2}(x_k-x_0)]\cdots\sin[\frac{1}{2}(x_k-x_{k-1})]} \frac{\sin[\frac{1}{2}(x-x_{k+1})]\cdots\sin[\frac{1}{2}(x-x_{2n})]}{\sin[\frac{1}{2}(x_k-x_{k+1})]\cdots\sin[\frac{1}{2}(x_k-x_{2n})]}.$$

References

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Beyer, W. H. (Ed.) *CRC Standard Mathematical Tables, 28th ed.* Boca Raton, FL: CRC Press, pp. 442–443, 1987.

Gauss-Jacobi Mechanical Quadrature

If $x_1 < x_2 < \ldots < x_n$ denote the zeros of $p_n(x)$, there exist REAL NUMBERS $\lambda_1, \lambda_2, \ldots, \lambda_n$ such that

$$\int_a^b \rho(x)\,d\alpha(x) = \lambda_1\rho(x_1) + \lambda_2\rho(x_2) + \ldots + \lambda_n\rho(x_n),$$

for an arbitrary POLYNOMIAL of order $2n-1$ and the $\lambda_n's$ are called CHRISTOFFEL NUMBERS. The distribution $d\alpha(x)$ and the INTEGER n uniquely determine these numbers λ_ν.

References

Szegő, G. *Orthogonal Polynomials, 4th ed.* Providence, RI: Amer. Math. Soc., p. 47, 1975.

Gauss-Jordan Elimination

A method for finding a MATRIX INVERSE. To apply Gauss-Jordan elimination, operate on a MATRIX

$$[\mathsf{A} \quad \mathsf{I}] \equiv \begin{bmatrix} a_{11} & \cdots & a_{1n} & 1 & 0 & \cdots & 0 \\ a_{21} & \cdots & a_{2n} & 0 & 1 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} & 0 & 0 & \cdots & 1 \end{bmatrix},$$

where I is the IDENTITY MATRIX, to obtain a MATRIX of the form

$$\begin{bmatrix} 1 & 0 & \cdots & 0 & b_{11} & \cdots & b_{1n} \\ 0 & 1 & \cdots & 0 & b_{21} & \cdots & b_{2n} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 & b_{n1} & \cdots & b_{nn} \end{bmatrix}.$$

The MATRIX

$$\mathsf{B} \equiv \begin{bmatrix} b_{11} & \cdots & b_{1n} \\ b_{21} & \cdots & b_{2n} \\ \vdots & \ddots & \vdots \\ b_{n1} & \cdots & b_{nn} \end{bmatrix}$$

is then the MATRIX INVERSE of A. The procedure is numerically unstable unless PIVOTING (exchanging rows and columns as appropriate) is used. Picking the largest available element as the pivot is usually a good choice.

see also GAUSSIAN ELIMINATION, LU DECOMPOSITION, MATRIX EQUATION

References

Press, W. H.; Flannery, B. P.; Teukolsky, S. A.; and Vetterling, W. T. "Gauss-Jordan Elimination" and "Gaussian Elimination with Backsubstitution." §2.1 and 2.2 in *Numerical Recipes in FORTRAN: The Art of Scientific Computing, 2nd ed.* Cambridge, England: Cambridge University Press, pp. 27–32 and 33–34, 1992.

Gauss-Kummer Series

$$\begin{aligned} {}_2F_1(-\tfrac{1}{2}, -\tfrac{1}{2}; 1; h^2) &= \sum_{n=0}^{\infty} \binom{\frac{1}{2}}{n}^2 h^{2n} \\ &= 1 + \tfrac{1}{4}h^2 + \tfrac{1}{64}h^4 + \tfrac{1}{256}h^6 + \ldots, \end{aligned}$$

where ${}_2F_1(a, b; c; x)$ is a HYPERGEOMETRIC FUNCTION. This can be derived using KUMMER'S QUADRATIC TRANSFORMATION.

Gauss-Kuzmin-Wirsing Constant

N.B. A detailed on-line essay by S. Finch was the starting point for this entry.

Let x_0 be a random number from $[0, 1]$ written as a simple CONTINUED FRACTION

$$x_0 = 0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \ldots}}}. \tag{1}$$

Define

$$\begin{aligned} x_n &= 0 + \cfrac{1}{a_{n+1} + \cfrac{1}{a_{n+2} + \cfrac{1}{a_{n+3} + \ldots}}} \\ &= \frac{1}{x_{n-1}} - \left\lfloor \frac{1}{x_{n-1}} \right\rfloor. \end{aligned} \tag{2}$$

Gauss (1800) showed that if $F(n, x)$ is the probability that $x_n < x$, then

$$\lim_{n\to\infty} F(n, x) = \frac{\ln(1+x)}{\ln 2}. \tag{3}$$

Kuzmin (1928) published the first proof, which was subsequently improved by Lévy (1929). Wirsing (1974) showed, among other results, that

$$\lim_{n\to\infty} \frac{F(n, x) - \frac{\ln(1+x)}{\ln 2}}{(-\lambda)^n} = \Psi(x), \tag{4}$$

where $\lambda = 0.3036630029\ldots$ and $\Psi(x)$ is an analytic function with $\Psi(0) = \Psi(1) = 0$. This constant is connected to the efficiency of the EUCLIDEAN ALGORITHM (Knuth 1981).

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Gauss-Laguerre Quadrature

see LAGUERRE-GAUSS QUADRATURE

Gauss's Lemma

Let the multiples $m, 2m, \ldots, [(p-1)/2]m$ of an INTEGER such that $p \nmid m$ be taken. If there are an EVEN number r of least POSITIVE RESIDUES mod p of these numbers $> p/2$, then m is a QUADRATIC RESIDUE of p. If r is ODD, m is a QUADRATIC NONRESIDUE. Gauss's lemma can therefore be stated as $(m|p) = (-1)^r$, where $(m|p)$ is the LEGENDRE SYMBOL. It was proved by Gauss as a step along the way to the QUADRATIC RECIPROCITY THEOREM.

see also QUADRATIC RECIPROCITY THEOREM

Gauss's Machin-Like Formula

The MACHIN-LIKE FORMULA

$$\tfrac{1}{4}\pi = 12\cot^{-1}18 + 8\cot^{-1}57 - 5\cot^{-1}239.$$

Gauss-Manin Connection

A connection defined on a smooth ALGEBRAIC VARIETY defined over the COMPLEX NUMBERS.

References
Iyanaga, S. and Kawada, Y. (Eds.). *Encyclopedic Dictionary of Mathematics.* Cambridge, MA: MIT Press, p. 81, 1980.

Gauss Map

The Gauss map is a function from an ORIENTABLE SURFACE in EUCLIDEAN SPACE to a SPHERE. It associates to every point on the surface its oriented NORMAL VECTOR. For surfaces in 3-space, the Gauss map of the surface has DEGREE given by half the EULER CHARACTERISTIC of the surface

$$\iint_M K\,dA = 2\pi\chi(M) - \sum \alpha_i - \int_{\partial M} \kappa_g\,ds,$$

which works only for ORIENTABLE SURFACES.

see also CURVATURE, NIRENBERG'S CONJECTURE, PATCH

References
Gray, A. "The Local Gauss Map" and "The Gauss Map via Mathematica." §10.3 and §15.3 in *Modern Differential Geometry of Curves and Surfaces.* Boca Raton, FL: CRC Press, pp. 193–194 and 310–316, 1993.

Gauss's Mean-Value Theorem

Let $f(z)$ be an ANALYTIC FUNCTION in $|z-a|<R$. Then

$$f(z) = \frac{1}{2\pi}\int_0^{2\pi} f(z+re^{i\theta})\,d\theta$$

for $0<r<R$.

Gauss Measure

The standard Gauss measure of a finite dimensional REAL HILBERT SPACE H with norm $||\cdot||_H$ has the BOREL MEASURE

$$\mu_H\,(dh) = (\sqrt{2\pi}\,)^{-\dim(H)}\exp(\tfrac{1}{2}||h||_H^2)\lambda_H\,(dh),$$

where λ_H is the LEBESGUE MEASURE on H.

Gauss Multiplication Formula

$$\begin{aligned}(2n\pi)^{(n-1)/2}n^{1/2-nz}\Gamma(nz) &= \Gamma(z)\Gamma(z+\tfrac{1}{n})\Gamma(z+\tfrac{2}{n})\cdots\Gamma(z+\tfrac{n-1}{n}) \\ &= \prod_{k=0}^{n-1}\Gamma\left(z+\frac{k}{n}\right),\end{aligned}$$

where $\Gamma(z)$ is the GAMMA FUNCTION.

see also GAMMA FUNCTION, LEGENDRE DUPLICATION FORMULA, POLYGAMMA FUNCTION

References
Abramowitz, M. and Stegun, C. A. (Eds.). *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables, 9th printing.* New York: Dover, p. 256, 1972.

Gauss Plane

see COMPLEX PLANE

Gauss's Polynomial Theorem

If a POLYNOMIAL

$$f(x) = x^N + C_1x^{N-1} + C_2x^{N-2} + \ldots + C_N$$

with integral COEFFICIENTS is divisible into a product of two POLYNOMIALS $f=\psi\phi$

$$\begin{aligned}\psi &= x^m + \alpha_1x^{m-1} + \ldots + \alpha_m \\ \phi &= x^n + \beta_1x^{n-1} + \ldots + \beta_n,\end{aligned}$$

then the COEFFICIENTS of this POLYNOMIAL are INTEGERS.

see also ABEL'S IRREDUCIBILITY THEOREM, ABEL'S LEMMA, KRONECKER'S POLYNOMIAL THEOREM, POLYNOMIAL, SCHOENEMANN'S THEOREM

References
Dörrie, H. *100 Great Problems of Elementary Mathematics: Their History and Solutions.* New York: Dover, p. 119, 1965.

Gauss's Reciprocity Theorem

see QUADRATIC RECIPROCITY THEOREM

Gauss-Salamin Formula

see BRENT-SALAMIN FORMULA

Gauss's Test

If $u_n>0$ and given $B(n)$ a bounded function of n as $n\to\infty$, express the ratio of successive terms as

$$\frac{u_n}{u_{n+1}} = 1 + \frac{h}{n} + \frac{B(n)}{n^2}.$$

The SERIES converges for $h>1$ and diverges for $h\le 1$.

see also CONVERGENCE TESTS

References
Arfken, G. *Mathematical Methods for Physicists, 3rd ed.* Orlando, FL: Academic Press, pp. 287–288, 1985.

Gauss's Theorem

see DIVERGENCE THEOREM

Gauss's Theorema Egregium

Gauss's theorema egregium states that the GAUSSIAN CURVATURE of a surface embedded in 3-space may be understood intrinsically to that surface. "Residents" of the surface may observe the GAUSSIAN CURVATURE of the surface without ever venturing into full 3-dimensional space; they can observe the curvature of the surface they live in without even knowing about the 3-dimensional space in which they are embedded.

In particular, GAUSSIAN CURVATURE can be measured by checking how closely the ARC LENGTH of small RADIUS CIRCLES correspond to what they should be in EUCLIDEAN SPACE, $2\pi r$. If the ARC LENGTH of CIRCLES tends to be smaller than what is expected in EUCLIDEAN SPACE, then the space is positively curved; if larger, negatively; if the same, 0 GAUSSIAN CURVATURE.

Gauss (effectively) expressed the theorema egregium by saying that the GAUSSIAN CURVATURE at a point is given by $-R(v,w)v,w$, where R is the RIEMANN TENSOR, and v and w are an orthonormal basis for the TANGENT SPACE.

see also CHRISTOFFEL SYMBOL OF THE SECOND KIND, GAUSS EQUATIONS, GAUSSIAN CURVATURE

References

Gray, A. "Gauss's Theorema Egregium." §20.2 in *Modern Differential Geometry of Curves and Surfaces*. Boca Raton, FL: CRC Press, pp. 395–397, 1993.

Reckziegel, H. In *Mathematical Models from the Collections of Universities and Museums* (Ed. G. Fischer). Braunschweig, Germany: Vieweg, pp. 31–32, 1986.

Gauss's Transformation

If

$$(1+x\sin^2\alpha)\sin\beta=(1+x)\sin\alpha,$$

then

$$(1+x)\int_0^\alpha \frac{d\phi}{\sqrt{1-x^2\sin^2\phi}}=\int_0^\beta \frac{d\phi}{\sqrt{1-\frac{4x}{(1+x)^2}\sin^2\phi}}.$$

see also ELLIPTIC INTEGRAL OF THE FIRST KIND, LANDEN'S TRANSFORMATION

Gaussian Approximation Algorithm

see ARITHMETIC-GEOMETRIC MEAN

Gaussian Bivariate Distribution

The Gaussian bivariate distribution is given by

$$P(x_1,x_2)=\frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}}\exp\left[-\frac{z}{2(1-\rho^2)}\right], \quad (1)$$

where

$$z\equiv\frac{(x_1-\mu_1)^2}{{\sigma_1}^2}-\frac{2\rho(x_1-\mu_1)(x_2-\mu_2)}{\sigma_1\sigma_2}+\frac{(x_2-\mu_2)^2}{{\sigma_2}^2}, \quad (2)$$

and

$$\rho\equiv\operatorname{cov}(x_1,x_2)=\frac{\langle x_1x_2\rangle-\langle x_1\rangle\langle x_2\rangle}{\sigma_1\sigma_2} \quad (3)$$

is the COVARIANCE. Let X_1 and X_2 be normally and independently distributed variates with MEAN 0 and VARIANCE 1. Then define

$$Y_1\equiv\mu_1+\sigma_{11}X_1+\sigma_{12}X_2 \quad (4)$$

$$Y_2\equiv\mu_2+\sigma_{21}X_1+\sigma_{22}X_2. \quad (5)$$

These new variates are normally distributed with MEAN μ_1 and μ_2, VARIANCE

$${\sigma_1}^2\equiv{\sigma_{11}}^2+{\sigma_{12}}^2 \quad (6)$$

$${\sigma_2}^2\equiv{\sigma_{21}}^2+{\sigma_{22}}^2, \quad (7)$$

and COVARIANCE

$$V_{12}\equiv\sigma_{11}\sigma_{21}+\sigma_{12}\sigma_{22}. \quad (8)$$

The COVARIANCE matrix is

$$V_{ij}=\begin{bmatrix}{\sigma_1}^2 & \rho\sigma_1\sigma_2\\ \rho\sigma_1\sigma_2 & {\sigma_2}^2\end{bmatrix}, \quad (9)$$

where

$$\rho\equiv\frac{V_{12}}{\sigma_1\sigma_2}=\frac{\sigma_{11}\sigma_{21}+\sigma_{12}\sigma_{22}}{\sigma_1\sigma_2}. \quad (10)$$

The joint probability density function for x_1 and x_2 is

$$f(x_1,x_2)\,dx_1\,dx_2=\frac{1}{2\pi}e^{-({x_1}^2+{x_2}^2)/2}\,dx_1\,dx_2. \quad (11)$$

However, from (4) and (5) we have

$$\begin{bmatrix}y_1-\mu_1\\ y_2-\mu_2\end{bmatrix}=\begin{bmatrix}\sigma_{11} & \sigma_{12}\\ \sigma_{21} & \sigma_{22}\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}. \quad (12)$$

Now, if

$$\begin{vmatrix}\sigma_{11} & \sigma_{12}\\ \sigma_{21} & \sigma_{22}\end{vmatrix}\neq 0, \quad (13)$$

then this can be inverted to give

$$\begin{bmatrix}x_1\\ x_2\end{bmatrix}=\begin{bmatrix}\sigma_{11} & \sigma_{12}\\ \sigma_{21} & \sigma_{22}\end{bmatrix}^{-1}\begin{bmatrix}y_1-\mu_1\\ y_2-\mu_2\end{bmatrix}$$
$$=\frac{1}{\sigma_{11}\sigma_{22}-\sigma_{12}\sigma_{21}}\begin{bmatrix}\sigma_{22} & -\sigma_{12}\\ -\sigma_{21} & \sigma_{11}\end{bmatrix}\begin{bmatrix}y_1-\mu_1\\ y_2-\mu_2\end{bmatrix}. \quad (14)$$

Therefore,

$${x_1}^2+{x_2}^2=\frac{[\sigma_{22}(y_1-\mu_1)-\sigma_{12}(y_2-\mu_2)]^2}{(\sigma_{11}\sigma_{22}-\sigma_{12}\sigma_{21})^2}+\frac{[-\sigma_{21}(y_1-\mu_1)+\sigma_{11}(y_2-\mu_2)]^2}{(\sigma_{11}\sigma_{22}-\sigma_{12}\sigma_{21})^2}. \quad (15)$$

Expanding the NUMERATOR gives

$$\sigma_{22}^2(y_1-\mu_1)^2 - 2\sigma_{12}\sigma_{22}(y_1-\mu_1)(y_2-\mu_2) + \sigma_{12}^2(y_2-\mu_2)^2 + \sigma_{21}^2(y_1-\mu_1)^2 - 2\sigma_{11}\sigma_{21}(y_1-\mu_1)(y_2-\mu_2) + \sigma_{11}^2(y_2-\mu_2)^2, \qquad (16)$$

so

$$\begin{aligned}(x_1^2+x_2^2)(\sigma_{11}\sigma_{22}-\sigma_{12}\sigma_{21})^2 &= (y_1-\mu_1)^2(\sigma_{21}^2+\sigma_{22}^2) - 2(y_1-\mu_1)(y_2-\mu_2)(\sigma_{11}\sigma_{21}+\sigma_{12}\sigma_{22}) + (y_2-\mu_2)^2(\sigma_{11}^2+\sigma_{12}^2)\\ &= \sigma_2^2(y_1-\mu_1)^2 - 2(y_1-\mu_1)(y_2-\mu_2)(\rho\sigma_1\sigma_2) + \sigma_1^2(y_2-\mu_2)^2\\ &= \sigma_1^2\sigma_2^2\left[\frac{(y_1-\mu_1)^2}{\sigma_1^2} - \frac{2\rho(y_1-\mu_1)(y_2-\mu_2)}{\sigma_1\sigma_2} + \frac{(y_2-\mu_2)^2}{\sigma_2^2}\right]. \qquad (17)\end{aligned}$$

But

$$\frac{1}{1-\rho^2} = \frac{1}{1-\frac{V_{12}^2}{\sigma_1^2\sigma_2^2}} = \frac{\sigma_1^2\sigma_2^2}{\sigma_1^2\sigma_2^2 - V_{12}^2} = \frac{\sigma_1^2\sigma_2^2}{(\sigma_{11}^2+\sigma_{12}^2)(\sigma_{21}^2+\sigma_{22}^2) - (\sigma_{11}\sigma_{21}+\sigma_{12}\sigma_{22})^2}. \qquad (18)$$

The DENOMINATOR is

$$\sigma_{11}^2\sigma_{21}^2 + \sigma_{11}^2\sigma_{22}^2 + \sigma_{12}^2\sigma_{21}^2 + \sigma_{12}^2\sigma_{22}^2 - \sigma_{11}^2\sigma_{21}^2 - 2\sigma_{11}\sigma_{12}\sigma_{21}\sigma_{22} - \sigma_{12}^2\sigma_{22}^2 = (\sigma_{11}\sigma_{22}-\sigma_{12}\sigma_{21})^2, \qquad (19)$$

so

$$\frac{1}{1-\rho^2} = \frac{\sigma_1^2\sigma_2^2}{(\sigma_{11}\sigma_{22}-\sigma_{12}\sigma_{21})^2} \qquad (20)$$

and

$$x_1^2+x_2^2 = \frac{1}{1-\rho^2} \times \left[\frac{(y_1-\mu_1)^2}{\sigma_1^2} - \frac{2\rho(y_1-\mu_1)(y_2-\mu_2)}{\sigma_1\sigma_2} + \frac{(y_2-\mu_2)^2}{\sigma_2^2}\right]. \qquad (21)$$

Solving for x_1 and x_2 and defining

$$\rho' \equiv \frac{\sigma_1\sigma_2\sqrt{1-\rho^2}}{\sigma_{11}\sigma_{22}-\sigma_{12}\sigma_{21}} \qquad (22)$$

gives

$$x_1 = \frac{\sigma_{22}(y_1-\mu_1) - \sigma_{12}(y_2-\mu_2)}{\rho'} \qquad (23)$$

$$x_2 = \frac{-\sigma_{21}(y_1-\mu_1) + \sigma_{11}(y_2-\mu_2)}{\rho'}. \qquad (24)$$

The JACOBIAN is

$$\begin{aligned}J\left(\frac{x_1,x_2}{y_1,y_2}\right) &= \begin{vmatrix}\frac{\partial x_1}{\partial y_1} & \frac{\partial x_1}{\partial y_2}\\ \frac{\partial x_2}{\partial y_1} & \frac{\partial x_2}{\partial y_2}\end{vmatrix} = \begin{vmatrix}\frac{\sigma_{22}}{\rho'} & -\frac{\sigma_{12}}{\rho'}\\ -\frac{\sigma_{21}}{\rho'} & \frac{\sigma_{11}}{\rho'}\end{vmatrix}\\ &= \frac{1}{\rho'^2}(\sigma_{11}\sigma_{22}-\sigma_{12}\sigma_{21})\\ &= \frac{1}{\rho'} = \frac{1}{\sigma_1\sigma_2\sqrt{1-\rho^2}}. \qquad (25)\end{aligned}$$

Therefore,

$$dx_1\,dx_2 = \frac{dy_1\,dy_2}{\sigma_1\sigma_2\sqrt{1-\rho^2}} \qquad (26)$$

and

$$\frac{1}{2\pi}e^{-(x_1^2+x_2^2)/2}\,dx_1\,dx_2 = \frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}}e^{-v/2}\,dy_1\,dy_2, \qquad (27)$$

where

$$v \equiv \frac{1}{1-\rho^2} \times \left[\frac{(y_1-\mu_1)^2}{\sigma_1^2} - \frac{2\rho(y_1-\mu_1)(y_2-\mu_2)}{\sigma_1\sigma_2} + \frac{(y_2-\mu_2)^2}{\sigma_2^2}\right]. \qquad (28)$$

Now, if

$$\begin{vmatrix}\sigma_{11} & \sigma_{12}\\ \sigma_{21} & \sigma_{22}\end{vmatrix} = 0, \qquad (29)$$

then

$$\sigma_{11}\sigma_{12} = \sigma_{12}\sigma_{21} \qquad (30)$$

$$y_1 = \mu_1 + \sigma_{11}x_1 + \sigma_{12}x_2 \qquad (31)$$

$$y_2 = \mu_2 + \frac{\sigma_{12}\sigma_{21}}{\sigma_{11}}x_2 = \mu_2 + \frac{\sigma_{11}\sigma_{21}x_1 + \sigma_{12}\sigma_{21}x_2}{\sigma_{11}} = \mu_2 + \frac{\sigma_{21}}{\sigma_{11}}(\sigma_{11}x_1 + \sigma_{12}x_2), \qquad (32)$$

so

$$y_1 = \mu_1 + x_3 \qquad (33)$$

$$y_2 = \mu_2 + \frac{\sigma_{21}}{\sigma_{11}}x_3, \qquad (34)$$

where

$$x_3 = y_1 - \mu_1 = \frac{\sigma_{11}}{\sigma_{21}}(y_2-\mu_2). \qquad (35)$$

The CHARACTERISTIC FUNCTION is given by

$$\begin{aligned}\phi(t_1,t_2) &\equiv \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{i(t_1x_1+t_2x_2)}P(x_1,x_2)\,dx_1\,dx_2\\ &= N\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{i(t_1x_1+t_2x_2)}\exp\left[-\frac{z}{2(1-\rho^2)}\right]dx_1\,dx_2, \qquad (36)\end{aligned}$$

where

$$z \equiv \left[\frac{(x_1-\mu_1)^2}{\sigma_1{}^2} - \frac{2\rho(x_1-\mu_1)(x_2-\mu_2)}{\sigma_1\sigma_2} + \frac{(x_2-\mu_2)^2}{\sigma_2{}^2}\right] \tag{37}$$

and

$$N \equiv \frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}}. \tag{38}$$

Now let

$$u \equiv x_1 - \mu_1 \tag{39}$$

$$w \equiv x_2 - \mu_2. \tag{40}$$

Then

$$\phi(t_1,t_2) = N' \int_{-\infty}^{\infty} \left(e^{it_2w} \exp\left[-\frac{1}{2(1-\rho^2)}\frac{w^2}{\sigma_2{}^2}\right]\right) \times \int_{-\infty}^{\infty} e^v e^{t_1u}\,du\,dw, \tag{41}$$

where

$$v \equiv -\frac{1}{2(1-\rho^2)}\frac{1}{\sigma_1{}^2}\left[u^2 - \frac{2\rho\sigma_1 w}{\sigma_2}u\right]$$

$$N' \equiv \frac{e^{i(t_1\mu_1+t_2\mu_2)}}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}}. \tag{42}$$

COMPLETE THE SQUARE in the inner integral

$$\int_{-\infty}^{\infty} \exp\left\{-\frac{1}{2(1-\rho^2)}\frac{1}{\sigma_1{}^2}\left[u^2 - \frac{2\rho\sigma_1 w}{\sigma_2}u\right]\right\} e^{t_1u}\,du$$
$$= \int_{-\infty}^{\infty} \exp\left\{-\frac{1}{2\sigma_1{}^2(1-\rho^2)}\left[u - \frac{\rho_1\sigma_1 w}{\sigma^2}\right]^2\right\}$$
$$\times \left\{\frac{1}{2\sigma_1{}^2(1-\rho^2)}\left(\frac{\rho_1\sigma_1 w}{\sigma_2}\right)^2\right\} e^{it_1u}\,du. \tag{43}$$

Rearranging to bring the exponential depending on w outside the inner integral, letting

$$v \equiv u - \rho\frac{\sigma_1 w}{\sigma_2}, \tag{44}$$

and writing

$$e^{it_1u} = \cos(t_1u) + i\sin(t_1u) \tag{45}$$

gives

$$\phi(t_1,t_2) = N' \int_{-\infty}^{\infty} e^{it_2w} \exp\left[-\frac{1}{2\sigma_2{}^2(1-\rho^2)}w^2\right]$$
$$\times \exp\left[\frac{\rho^2}{2\sigma_2{}^2(1-\rho^2)}w^2\right] \int_{-\infty}^{\infty} \exp\left[-\frac{1}{2\sigma_2{}^2(1-\rho^2)}v^2\right]$$
$$\times \left\{\cos\left[t_1\left(v + \frac{\rho\sigma_1 w}{\sigma_2}\right)\right]\right.$$
$$\left. + i\sin\left[t_1\left(v + \frac{\rho\sigma_1 w}{\sigma_2}\right)\right]\right\} dv\,dw. \tag{46}$$

Expanding the term in braces gives

$$\left[\cos(t_1v)\cos\left(\frac{\rho\sigma_1 wt_1}{\sigma_2}\right) - \sin(t_1v)\sin\left(\frac{\rho\sigma_1 w}{\sigma_2 t_1}\right)\right]$$
$$+ i\left[\sin(t_1v)\cos\left(\frac{\rho\sigma_1 w}{\sigma_2 t_1}\right) + \cos(t_1v)\sin\left(\frac{\rho\sigma_1 wt_1}{\sigma_2}\right)\right]$$
$$= \left[\cos\left(\frac{\rho\sigma_1 wt_1}{\sigma_2}\right) + i\sin\left(\frac{\rho\sigma_1 wt_1}{\sigma_2}\right)\right]$$
$$[\cos(t_1v) + i\sin(t_1v)]$$
$$= \exp\left(\frac{i\rho\sigma_1 w}{\sigma_2}t_1\right)[\cos(t_1v) + i\sin(t_1v)]. \tag{47}$$

But $e^{-ax^2}\sin(bx)$ is ODD, so the integral over the sine term vanishes, and we are left with

$$\phi(t_1,t_2) = N' \int_{-\infty}^{\infty} e^{it_2w} \exp\left[-\frac{w^2}{2\sigma_2{}^2}\right]$$
$$\times \exp\left[\frac{\rho^2 w^2}{2\sigma_2{}^2(1-\rho^2)}\right] \exp\left[\frac{i\rho\sigma_1 wt_1}{\sigma_2}\right] dw$$
$$\times \int_{-\infty}^{\infty} \exp\left[-\frac{v^2}{2\sigma_1{}^2(1-\rho^2)}\right] \cos(t_1v)\,dv$$
$$= N' \int_{-\infty}^{\infty} \exp\left[iw\left(t_2 + t_1\left(\rho\frac{\sigma_1}{\sigma_2}\right)\right)\right] \exp\left[-\frac{w^2}{2\sigma_2{}^2}\right] dw$$
$$\times \int_{-\infty}^{\infty} \exp\left[-\frac{v^2}{2\sigma_1{}^2(1-\rho^2)}\right] \cos(t_1v)\,dv. \tag{48}$$

Now evaluate the GAUSSIAN INTEGRAL

$$\int_{-\infty}^{\infty} e^{ikx}e^{-ax^2}\,dx = \int_{-\infty}^{\infty} e^{-ax^2}\cos(kx)\,dx$$
$$= \sqrt{\frac{\pi}{a}}e^{-k^2/4a} \tag{49}$$

to obtain the explicit form of the CHARACTERISTIC FUNCTION,

$$\phi(t_1,t_2) = \frac{e^{i(t_1\mu_1+t_2\mu_2)}}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}}$$
$$\times \left\{\sigma_2\sqrt{2\pi}\exp\left[-\frac{1}{4}\left(t_2 + \rho\frac{\sigma_1}{\sigma_2}t_1\right)^2 2\sigma_2{}^2\right]\right\}$$
$$\times \left\{\sigma_1\sqrt{2\pi(1-\rho^2)}\exp\left[-\tfrac{1}{4}t_1{}^2 2\sigma_1{}^2(1-\rho^2)\right]\right\}$$
$$= e^{i(t_1\mu_1+t_2\mu_2)}\exp\{-\tfrac{1}{2}[t_2{}^2\sigma_2{}^2 + 2\rho\sigma_1\sigma_2t_1t_2$$
$$+ \rho^2\sigma_1{}^2t_1{}^2 + (1-\rho^2)\sigma_1{}^2t_1{}^2]\}$$
$$= \exp[i(t_1\mu_1 + t_2\mu_2)$$
$$- \tfrac{1}{2}(\sigma_1{}^2t_1{}^2 + 2\rho\sigma_1\sigma_2t_1t_2 + \sigma_1{}^2t_1{}^2)]. \tag{50}$$

Let z_1 and z_2 be two independent Gaussian variables with MEANS $\mu_i = 0$ and $\sigma_i{}^2 = 1$ for $i = 1, 2$. Then the variables a_1 and a_2 defined below are Gaussian bivariates with unit VARIANCE and CROSS-CORRELATION COEFFICIENT ρ:

$$a_1 = \sqrt{\frac{1+\rho}{2}}\,z_1 + \sqrt{\frac{1-\rho}{2}}\,z_2 \tag{51}$$

$$a_2 = \sqrt{\frac{1+\rho}{2}}\, z_1 - \sqrt{\frac{1-\rho}{2}}\, z_2. \tag{52}$$

The conditional distribution is

$$P(x_2|x_1) = \frac{1}{\sigma_2\sqrt{2\pi(1-\rho^2)}} \exp\left[-\frac{(x^2-\mu'^2)^2}{2{\sigma'_2}^2}\right], \tag{53}$$

where

$$\mu'_2 \equiv \mu_2 + \rho\frac{\sigma_2}{\sigma_1}(x_1-\mu_1) \tag{54}$$

$$\sigma'_2 \equiv \sigma_2\sqrt{1-\rho_2}\,. \tag{55}$$

The marginal probability density is

$$\begin{aligned} P(x_2) &= \int_{-\infty}^{\infty} P(x_1,x_2)\,dx_1 \\ &= \frac{1}{\sigma_2\sqrt{2\pi}} \exp\left[-\frac{(x_2-\mu_2)^2}{2{\sigma_2}^2}\right]. \end{aligned} \tag{56}$$

see also BOX-MULLER TRANSFORMATION, GAUSSIAN DISTRIBUTION, MCMOHAN'S THEOREM, NORMAL DISTRIBUTION

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Abramowitz, M. and Stegun, C. A. (Eds.). *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables, 9th printing.* New York: Dover, pp. 936–937, 1972.

Spiegel, M. R. *Theory and Problems of Probability and Statistics.* New York: McGraw-Hill, p. 118, 1992.

Gaussian Brackets

Published by Gauss in *Disquisitiones Arithmeticae.* They are defined as follows.

$$[\,] = 1 \tag{1}$$

$$[a_1] = a_1 \tag{2}$$

$$[a_1, a_2] = [a_1]a_2 + [\,] \tag{3}$$

$$\begin{aligned} [a_1, a_2, \ldots, a_n] = {}& [a_1, a_2, \ldots, a_{n-1}]a_n \\ &+ [a_1, a_2, \ldots, a_{n-2}]. \end{aligned} \tag{4}$$

Gaussian brackets are useful for treating CONTINUED FRACTIONS because

$$\cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \ldots + \cfrac{1}{a_n}}}} = \frac{[a_2, a_n]}{[a_1, a_n]}. \tag{5}$$

The NOTATION $[x]$ conflicts with that of GAUSSIAN POLYNOMIALS and the NINT function.

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Herzberger, M. *Modern Geometrical Optics.* New York: Interscience Publishers, pp. 457–462, 1958.

Gaussian Coefficient

see q-BINOMIAL COEFFICIENT

Gaussian Coordinate System

A coordinate system which has a METRIC satisfying $g_{ii} = -1$ and $\partial g_{ij}/\partial x_j = 0$.

Gaussian Curvature

An intrinsic property of a space independent of the coordinate system used to describe it. The Gaussian curvature of a REGULAR SURFACE in $\mathbb{R}^3$ at a point $\mathbf{p}$ is formally defined as

$$K(\mathbf{p}) = \tfrac{1}{2}\det(S(\mathbf{p})), \tag{1}$$

where S is the SHAPE OPERATOR and det denotes the DETERMINANT.

If $\mathbf{x} : U \to \mathbb{R}^3$ is a REGULAR PATCH, then the Gaussian curvature is given by

$$K = \frac{eg - f^2}{EG - F^2}, \tag{2}$$

where E, F, and G are coefficients of the first FUNDAMENTAL FORM and e, f, and g are coefficients of the second FUNDAMENTAL FORM (Gray 1993, p. 282). The Gaussian curvature can be given entirely in terms of the first FUNDAMENTAL FORM

$$ds^2 = E\,du^2 + 2F\,du\,dv + G\,dv^2 \tag{3}$$

and the DISCRIMINANT

$$g \equiv EG - F^2 \tag{4}$$

by

$$K = \frac{1}{\sqrt{g}}\left[\frac{\partial}{\partial v}\left(\frac{\sqrt{g}}{E}\Gamma_{11}^2\right) - \frac{\partial}{\partial u}\left(\frac{\sqrt{g}}{E}\Gamma_{12}^2\right)\right], \tag{5}$$

where Γ_{ij}^k are the CONNECTION COEFFICIENTS. Equivalently,

$$\begin{aligned} K = {}& \frac{1}{g^2}\begin{vmatrix} E & F & \frac{\partial F}{\partial v} - \frac{1}{2}\frac{\partial G}{\partial u} \\ F & G & \frac{1}{2}\frac{\partial G}{\partial v} \\ \frac{1}{2}\frac{\partial E}{\partial u} & k_{23} & k_{33} \end{vmatrix} \\ & - \frac{1}{g^2}\begin{vmatrix} E & F & \frac{1}{2}\frac{\partial E}{\partial v} \\ F & G & \frac{1}{2}\frac{\partial G}{\partial u} \\ \frac{1}{2}\frac{\partial E}{\partial v} & \frac{1}{2}\frac{\partial G}{\partial v} & 0 \end{vmatrix}, \end{aligned} \tag{6}$$

where

$$k_{23} \equiv \frac{\partial F}{\partial u} - \frac{1}{2}\frac{\partial E}{\partial v} \tag{7}$$

$$k_{33} \equiv -\frac{1}{2}\frac{\partial^2 E}{\partial v^2} + \frac{\partial^2 F}{\partial u \partial v} - \frac{1}{2}\frac{\partial^2 G}{\partial u^2}. \tag{8}$$

Writing this out,

$$\begin{aligned} K = \frac{1}{2g}\left[2\frac{\partial^2 F}{\partial u \partial v} - \frac{\partial^2 E}{\partial v^2} - \frac{\partial^2 G}{\partial u^2}\right] \\ - \frac{G}{4g^2}\left[\frac{\partial E}{\partial u}\left(2\frac{\partial F}{\partial v} - \frac{\partial G}{\partial u}\right) - \left(\frac{\partial E}{\partial v}\right)^2\right] \\ + \frac{F}{4g_2}\left[\frac{\partial E}{\partial u}\frac{\partial G}{\partial v} - 2\frac{\partial E}{\partial v}\frac{\partial G}{\partial u}\right. \\ \left. + \left(2\frac{\partial F}{\partial u} - \frac{\partial E}{\partial v}\right)\left(2\frac{\partial F}{\partial v} - \frac{\partial G}{\partial u}\right)\right] \\ - \frac{E}{4g^2}\left[\frac{\partial G}{\partial v}\left(2\frac{\partial F}{\partial u} - \frac{\partial E}{\partial v}\right) - \left(\frac{\partial G}{\partial u}\right)^2\right]. \end{aligned} \quad (9)$$

The Gaussian curvature is also given by

$$K = \frac{\det(\mathbf{x}_{uu}\mathbf{x}_u\mathbf{x}_v)\det(\mathbf{x}_{vv}\mathbf{x}_u\mathbf{x}_v) - [\det(\mathbf{x}_{uv}\mathbf{x}_u\mathbf{x}_v)]^2}{[|\mathbf{x}_u|^2|\mathbf{x}_v|^2 - (\mathbf{x}_u \cdot \mathbf{x}_v)^2]^2} \quad (10)$$

(Gray 1993, p. 285), as well as

$$K = \frac{[\hat{\mathbf{N}}\,\hat{\mathbf{N}}_1\,\hat{\mathbf{N}}_2]}{\sqrt{g}} = \frac{\epsilon^{ij}[\hat{\mathbf{N}}\,\hat{\mathbf{T}}\,\hat{\mathbf{T}}_i]_j}{\sqrt{g}}, \quad (11)$$

where ϵ^{ij} is the LEVI-CIVITA SYMBOL, $\hat{\mathbf{N}}$ is the unit NORMAL VECTOR and $\hat{\mathbf{T}}$ is the unit TANGENT VECTOR. The Gaussian curvature is also given by

$$K = -\frac{R}{2} = \kappa_1\kappa_2 = \frac{1}{R_1 R_2}, \quad (12)$$

where R is the CURVATURE SCALAR, κ_1 and κ_2 the PRINCIPAL CURVATURES, and R_1 and R_2 the PRINCIPAL RADII OF CURVATURE. For a MONGE PATCH with $z = h(u,v)$,

$$K = \frac{h_{uu}h_{vv} - {h_{uv}}^2}{(1 + {h_u}^2 + {h_v}^2)^2}. \quad (13)$$

The Gaussian curvature K and MEAN CURVATURE H satisfy

$$H^2 \geq K, \quad (14)$$

with equality only at UMBILIC POINTS, since

$$H^2 - K^2 = \tfrac{1}{4}(\kappa_1 - \kappa_2)^2. \quad (15)$$

If $\mathbf{p}$ is a point on a REGULAR SURFACE $M \subset \mathbb{R}^3$ and $\mathbf{v_p}$ and $\mathbf{w_p}$ are tangent vectors to M at $\mathbf{p}$, then the Gaussian curvature of M at $\mathbf{p}$ is related to the SHAPE OPERATOR S by

$$S(\mathbf{v_p}) \times S(\mathbf{w_p}) = K(\mathbf{p})\mathbf{v_p} \times \mathbf{w_p}. \quad (16)$$

Let $\mathbf{Z}$ be a nonvanishing VECTOR FIELD on M which is everywhere PERPENDICULAR to M, and let V and W be VECTOR FIELDS tangent to M such that $V \times W = \mathbf{Z}$, then

$$K = \frac{\mathbf{Z} \cdot (D_V\mathbf{Z} \times D_W\mathbf{Z})}{2|\mathbf{Z}|^4} \quad (17)$$

(Gray 1993, pp. 291–292).

For a SPHERE, the Gaussian quadrature is $K = 1/a^2$. For EUCLIDEAN SPACE, the Gaussian quadrature is $K = 0$. For GAUSS-BOLYAI-LOBACHEVSKY SPACE, the Gaussian quadrature is $K = -1/a^2$. A FLAT SURFACE is a REGULAR SURFACE and special class of MINIMAL SURFACE on which Gaussian curvature vanishes everywhere.

A point $\mathbf{p}$ on a REGULAR SURFACE $M \in \mathbb{R}^3$ is classified based on the sign of $K(\mathbf{p})$ as given in the following table (Gray 1993, p. 280), where S is the SHAPE OPERATOR.

Sign	Point
$K(\mathbf{p}) > 0$	elliptic point
$K(\mathbf{p}) < 0$	hyperbolic point
$K(\mathbf{p}) = 0$ but $S(\mathbf{p}) \neq 0$	parabolic point
$K(\mathbf{p}) = 0$ and $S(\mathbf{p}) = 0$	planar point

A surface on which the Gaussian curvature K is everywhere POSITIVE is called SYNCLASTIC, while a surface on which K is everywhere NEGATIVE is called ANTICLASTIC. Surfaces with constant Gaussian curvature include the CONE, CYLINDER, KUEN SURFACE, PLANE, PSEUDOSPHERE, and SPHERE. Of these, the CONE and CYLINDER are the only FLAT SURFACES OF REVOLUTION.

see also ANTICLASTIC, BRIOSCHI FORMULA, DEVELOPABLE SURFACE, ELLIPTIC POINT, FLAT SURFACE, HYPERBOLIC POINT, INTEGRAL CURVATURE, MEAN CURVATURE, METRIC TENSOR, MINIMAL SURFACE, PARABOLIC POINT, PLANAR POINT, SYNCLASTIC, UMBILIC POINT

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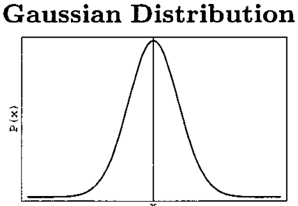
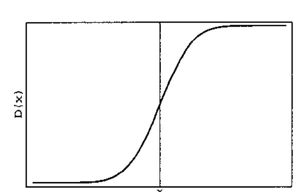
Gaussian Differential Equation

see HYPERGEOMETRIC DIFFERENTIAL EQUATION

Gaussian Distribution





The Gaussian probability distribution with MEAN μ and STANDARD DEVIATION σ is a GAUSSIAN FUNCTION of the form

$$P(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-(x-\mu)^2/2\sigma^2}, \quad (1)$$

where $P(x)\,dx$ gives the probability that a variate with a Gaussian distribution takes on a value in the range $[x, x+dx]$. This distribution is also called the NORMAL DISTRIBUTION or, because of its curved flaring shape, the BELL CURVE. The distribution $P(x)$ is properly normalized for $x \in (-\infty, \infty)$ since

$$\int_{-\infty}^{\infty} P(x)\,dx = 1. \quad (2)$$

The cumulative DISTRIBUTION FUNCTION, which gives the probability that a variate will assume a value $\leq x$, is then

$$D(x) \equiv \int_{-\infty}^{x} P(x)\,dx = \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{x} e^{-(x-\mu)^2/2\sigma^2}\,dx. \quad (3)$$

Gaussian distributions have many convenient properties, so random variates with unknown distributions are often assumed to be Gaussian, especially in physics and astronomy. Although this can be a dangerous assumption, it is often a good approximation due to a surprising result known as the CENTRAL LIMIT THEOREM. This theorem proves that the MEAN of any set of variates with any distribution having a finite MEAN and VARIANCE tends to the Gaussian distribution. Many common attributes such as test scores, height, etc., follow roughly Gaussian distributions, with few members at the high and low ends and many in the middle.

Making the transformation

$$z \equiv \frac{x-\mu}{\sigma} \quad (4)$$

so that $dz = dz/\sigma$ gives a variate with unit VARIANCE and 0 MEAN

$$P(x)\,dx = \frac{1}{\sqrt{2\pi}} e^{-z^2/2}\,dz, \quad (5)$$

known as a standard NORMAL DISTRIBUTION. So defined, z is known as a z-SCORE).

The NORMAL DISTRIBUTION FUNCTION gives the probability that a standard normal variate assumes a value in the interval $[0, z]$,

$$\Phi(x) \equiv \frac{1}{\sqrt{2\pi}} \int_0^x e^{-z^2/2}\,dz = \tfrac{1}{2}\operatorname{erf}\left(\frac{x}{\sqrt{2}}\right). \quad (6)$$

Here, ERF is a function sometimes called the error function. Neither $\Phi(z)$ nor ERF can be expressed in terms of finite additions, subtractions, multiplications, and root extractions, and so both must be either computed numerically or otherwise approximated. The value of a for which $P(x)$ falls within the interval $[-a, a]$ with a given probability P is called the P CONFIDENCE INTERVAL.

The Gaussian distribution is also a special case of the CHI-SQUARED DISTRIBUTION, since substituting

$$z \equiv \frac{(x-\mu)^2}{\sigma^2} \quad (7)$$

so that

$$dz = \tfrac{1}{2}\frac{2(x-\mu)}{\sigma}\,dx = \sqrt{\frac{z}{\sigma}}\,dx \quad (8)$$

(where an extra factor of 1/2 has been added to dz since z runs from 0 to ∞ instead of from $-\infty$ to ∞), gives

$$\begin{aligned} P(x)\,dx &= \frac{1}{\sqrt{2\pi}} e^{-(z/\sigma)/2} \left(\frac{z}{\sigma}\right)^{-1/2} d\left(\frac{z}{\sigma}\right)\,dz \\ &= \frac{1}{2^{1/2}\Gamma(\frac{1}{2})} e^{-(z/\sigma)/2} \left(\frac{z}{\sigma}\right)^{-1/2} d\left(\frac{z}{\sigma}\right)\,dz, \end{aligned} \quad (9)$$

which is a CHI-SQUARED DISTRIBUTION in z/σ with $r = 1$ (i.e., a GAMMA DISTRIBUTION with $\alpha = 1/2$ and $\theta = 2$).

Cramer showed in 1936 that if X and Y are INDEPENDENT variates and $X + Y$ has a Gaussian distribution, then both X and Y must be Gaussian (CRAMER'S THEOREM).

The ratio X/Y of independent Gaussian-distributed variates with zero MEAN is distributed with a CAUCHY DISTRIBUTION. This can be seen as follows. Let X and Y both have MEAN 0 and standard deviations of σ_x and σ_y, respectively, then the joint probability density function is the GAUSSIAN BIVARIATE DISTRIBUTION with $\rho = 0$,

$$f(x, y) = \frac{1}{2\pi\sigma_x\sigma_y} e^{-[x^2/(2\sigma_x{}^2)+y^2/(2\sigma_y{}^2)]}. \quad (10)$$

From RATIO DISTRIBUTION, the distribution of $U = Y/X$ is

$$\begin{aligned} P(u) &= \int_{-\infty}^{\infty} |x| f(x, ux)\,dx \\ &= \frac{1}{2\pi\sigma_x\sigma_y} \int_{-\infty}^{\infty} |x| e^{-[x^2/(2\sigma_x{}^2)+u^2x^2/(2\sigma_y{}^2)]} \\ &= \frac{1}{\pi\sigma_x\sigma_y} \int_0^{\infty} x \exp\left[-x^2\left(\frac{1}{2\sigma_x{}^2} + \frac{u^2}{2\sigma_y{}^2}\right)\right] dx. \end{aligned} \quad (11)$$

But

$$\int_0^{\infty} x e^{-ax^2}\,dx = \left[-\frac{1}{2a} e^{-ax^2}\right]_0^{\infty} = \frac{1}{2a}[0 - (-1)] = \frac{1}{2a}, \quad (12)$$

so

$$P(u) = \frac{1}{\pi\sigma_x\sigma_y}\frac{1}{2\left(\frac{1}{2\sigma_x{}^2}+\frac{u^2}{2\sigma_y{}^2}\right)} = \frac{1}{\pi}\frac{\sigma_x\sigma_y}{u^2\sigma_x{}^2+\sigma_y{}^2}$$
$$= \frac{1}{\pi}\frac{\frac{\sigma_y}{\sigma_x}}{u^2+\left(\frac{\sigma_y}{\sigma_x}\right)^2}, \tag{13}$$

which is a CAUCHY DISTRIBUTION with MEAN $\mu = 0$ and full width

$$\Gamma = \frac{2\sigma_y}{\sigma_x}. \tag{14}$$

The CHARACTERISTIC FUNCTION for the Gaussian distribution is

$$\phi(t) = e^{imt-\sigma^2t^2/2}, \tag{15}$$

and the MOMENT-GENERATING FUNCTION is

$$M(t) = \langle e^{tx}\rangle = \int_{-\infty}^{\infty}\frac{e^{tx}}{\sigma\sqrt{2\pi}}e^{-(x-\mu)^2/2\sigma^2}\,dx$$
$$= \frac{1}{\sigma\sqrt{2\pi}}\int_{-\infty}^{\infty}\exp\left\{-\frac{1}{2\sigma^2}\left[x^2-2(\mu+\sigma^2t)x+\mu^2\right]\right\}dx. \tag{16}$$

COMPLETING THE SQUARE in the exponent,

$$\frac{1}{2\sigma^2}[x^2-2(\mu+\sigma^2t)x+\mu^2]$$
$$= \frac{1}{2\sigma^2}\left\{[x-(\mu+\sigma^2t)]^2+[\mu^2-(\mu+\sigma^2t)^2]\right\}. \tag{17}$$

Let

$$y \equiv x-(\mu+\sigma^2t) \tag{18}$$
$$dy = dx \tag{19}$$
$$a \equiv \frac{1}{2\sigma^2}. \tag{20}$$

The integral then becomes

$$M(t) = \frac{1}{\sigma\sqrt{2\pi}}\int_{-\infty}^{\infty}\exp\left[-ay^2+\frac{2\mu\sigma^2t+\sigma^4t^2}{2\sigma^2}\right]dy$$
$$= \frac{1}{\sigma\sqrt{2\pi}}\int_{-\infty}^{\infty}\exp[-ay^2+\mu t+\tfrac{1}{2}\sigma^2t^2]\,dy$$
$$= \frac{1}{\sigma\sqrt{2\pi}}e^{\mu t+\sigma^2t^2/2}\int_{-\infty}^{\infty}e^{-ay^2}\,dy$$
$$= \frac{1}{\sigma\sqrt{2\pi}}\sqrt{\frac{\pi}{a}}\,e^{\mu t+\sigma^2t^2/2}$$
$$= \frac{\sqrt{2\sigma^2\pi}}{\sigma\sqrt{2\pi}}\,e^{\mu t+\sigma^2t^2/2} = e^{\mu t+\sigma^2t^2/2}, \tag{21}$$

so

$$M'(t) = (\mu+\sigma^2t)e^{\mu t+\sigma^2t^2/2} \tag{22}$$
$$M''(t) = \sigma^2e^{\mu t+\sigma^2t^2/2}+e^{\mu t+\sigma^2t^2/2}(\mu+t\sigma^2)^2, \tag{23}$$

and

$$\mu = M'(0) = \mu \tag{24}$$
$$\sigma^2 = M''(0)-[M'(0)]^2$$
$$= (\sigma^2+\mu^2)-\mu^2 = \sigma^2. \tag{25}$$

These can also be computed using

$$R(t) = \ln[M(t)] = \mu t+\tfrac{1}{2}\sigma^2t^2 \tag{26}$$
$$R'(t) = \mu+\sigma^2t \tag{27}$$
$$R''(t) = \sigma^2, \tag{28}$$

yielding, as before,

$$\mu = R'(0) = \mu \tag{29}$$
$$\sigma^2 = R''(0) = \sigma^2. \tag{30}$$

The moments can also be computed directly by computing the MOMENTS about the origin $\mu_n' \equiv \langle x^n\rangle$,

$$\mu_n' = \frac{1}{\sigma\sqrt{2\pi}}\int_{-\infty}^{\infty}x^ne^{-(x-\mu)^2/2\sigma^2}\,dx. \tag{31}$$

Now let

$$u \equiv \frac{x-\mu}{\sqrt{2}\sigma} \tag{32}$$
$$du = \frac{dx}{\sqrt{2}\sigma} \tag{33}$$
$$x = \sigma u\sqrt{2}+\mu, \tag{34}$$

giving

$$\mu_n' = \frac{\sqrt{2}\,\sigma}{\sigma\sqrt{2\pi}}\int_{-\infty}^{\infty}x^ne^{-u^2}\,du = \frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty}x^ne^{-u^2}\,du, \tag{35}$$

so

$$\mu_0' = 1 \tag{36}$$
$$\mu_1' = \frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty}xe^{-u^2}\,du$$
$$= \frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty}(\sqrt{2}\,\sigma u+\mu)e^{-u^2}\,du$$
$$= [\sqrt{2}\sigma H_1(1)+\mu H_0(1)] = (0+\mu) = \mu \tag{37}$$
$$\mu_2' = \frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty}x^2e^{-u^2}\,du$$
$$= \frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty}(2\sigma^2u^2+2\sqrt{2}\,\sigma\mu u+\mu^2)e^{-u^2}\,du$$
$$= [2\sigma^2H_2(1)+2\sqrt{2}\,\sigma\mu H_1(1)+\mu^2H_0(1)]$$
$$= (2\sigma^2\tfrac{1}{2}+0+\mu^2) = \mu^2+\sigma^2 \tag{38}$$
$$\mu_3' = \frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty}x^3e^{-u^2}\,du$$

$$= \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} (2\sqrt{2}\,\sigma^3 u^3 + 6\mu\sigma^2 u^2 + 3\sqrt{2}\,\mu^2\sigma u + \mu^3) e^{-u^2}\,du$$
$$= [2\sqrt{2}\,\sigma^3 H_3(1) + 6\mu\sigma^2 H_2(1) + 3\sqrt{2}\,\mu^2\sigma H_1(1) + \mu^3 H_0(1)]$$
$$= (0 + 6\mu^2\sigma^2 \tfrac{1}{2} + 0 + \mu^3) = \mu(\mu^2 + 3\sigma^2) \tag{39}$$
$$\mu_4' = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} x^3 e^{-u^2}\,du$$
$$= \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} (4\sigma^4 u^4 + 8\sqrt{2}\,\mu\sigma^3 u^3 + 12\mu^2\sigma^2 u^2 + 4\sqrt{2}\,\mu^3\sigma u + \mu^4) e^{-u^2}\,du$$
$$= [4\sigma^4 H_4(1) + 8\sqrt{2}\,\mu\sigma^3 H_3(1) + 12\mu^2\sigma^2 H_2(1) + 4\sqrt{2}\,\mu^3\sigma H_1(1) + \mu^4 H_0(1)]$$
$$= (4\sigma^4 \tfrac{3}{4} + 0 + 12\mu^2\sigma^2 \tfrac{1}{2} + 0 + \mu^4)$$
$$= \mu^4 + 6\mu^2\sigma^2 + 3\sigma^4, \tag{40}$$

where $H_n(a)$ are GAUSSIAN INTEGRALS.

Now find the MOMENTS about the MEAN,

$$\mu_1 \equiv 0 \tag{41}$$
$$\mu_2 \equiv \mu_2' - (\mu_1')^2 = (\mu^2 + \sigma^2) - \mu^2 = \sigma^2 \tag{42}$$
$$\mu_3 \equiv \mu_3' - 3\mu_2'\mu_1' + 2(\mu_1')^3$$
$$= \mu(\mu^2 + 3\sigma^2) - 3(\sigma^2 + \mu^2)\mu + 2\mu^3 = 0 \tag{43}$$
$$\mu_4 \equiv \mu_4' - 4\mu_3'\mu_1' + 6\mu_2'(\mu_1')^2 - 3(\mu_1')^4$$
$$= (\mu^4 + 6\mu^2\sigma^2 + 3\sigma^4) - 4(\mu^3 + 3\mu\sigma^2)\mu + 6(\mu^2 + \sigma^2)\mu^2 - 3\mu^4$$
$$= 3\sigma^4, \tag{44}$$

so the VARIANCE, STANDARD DEVIATION, SKEWNESS, and KURTOSIS are given by

$$\operatorname{var}(x) \equiv \mu_2 = \sigma^2 \tag{45}$$
$$\operatorname{stdv}(x) \equiv \sqrt{\operatorname{var}(x)} = \sigma \tag{46}$$
$$\gamma_1 = \frac{\mu_3}{\sigma^3} = 0 \tag{47}$$
$$\gamma_2 = \frac{\mu_4}{\sigma^4} - 3 = \frac{3\sigma^4}{\sigma^4} - 3 = 0. \tag{48}$$

The VARIANCE of the SAMPLE VARIANCE s^2 for a sample taken from a population with a Gaussian distribution is

$$\operatorname{var}(s^2) = \frac{(N-1)[(N-1)\mu_4' - (N-3)\mu_2'^2]}{N^3}$$
$$= \frac{(N-1)}{N^3}[(N-1)(\mu^4 + 6\mu^2\sigma^2 + 3\sigma^4) - (N-3)(\mu^2 + \sigma^2)^2]$$
$$= \frac{2(N-1)(\mu^4 + 2\mu^2 N\sigma^2 + N\sigma^4)}{N^3}. \tag{49}$$

If $\mu = 0$, this expression simplifies to

$$\operatorname{var}(s^2) = \frac{2(N-1)N\sigma^4}{N^3} = \frac{2\sigma^4(N-1)}{N^2}, \tag{50}$$

and the STANDARD ERROR is

$$[\text{standard error}] = \frac{\sqrt{2(N-1)}}{N}. \tag{51}$$

The CUMULANT-GENERATING FUNCTION for a Gaussian distribution is

$$K(h) = \ln(e^{\nu_1 h} e^{\sigma^2 h^2/2}) = \nu_1 h + \tfrac{1}{2}\sigma^2 h^2, \tag{52}$$

so

$$\kappa_1 = \nu_1 \tag{53}$$
$$\kappa_2 = \sigma^2 \tag{54}$$
$$\kappa_r = 0 \quad \text{for } r > 2. \tag{55}$$

For Gaussian variates, $\kappa_r = 0$ for $r > 2$, so the variance of k-STATISTIC k_3 is

$$\operatorname{var}(k_3) = \frac{\kappa_6}{N} + \frac{9\kappa_2\kappa_4}{N-1} + \frac{9\kappa_3^2}{N-1} + \frac{6\kappa_2^3}{N(N-1)(N-2)}$$
$$= \frac{6\kappa_2^3}{N(N-1)(N-2)}. \tag{56}$$

Also,

$$\operatorname{var}(k_4) = \frac{24k_2^4 N(N-1)^2}{(N-3)(N-2)(N+3)(N+5)} \tag{57}$$
$$\operatorname{var}(g_1) = \frac{6N(N-1)}{(N-2)(N+1)(N+3)} \tag{58}$$
$$\operatorname{var}(g_2) = \frac{24N(N-1)^2}{(N-3)(N-2)(N+3)(N+5)}, \tag{59}$$

where

$$g_1 \equiv \frac{k_3}{k_2^{3/2}} \tag{60}$$
$$g_2 \equiv \frac{k_4}{k_2^2}. \tag{61}$$

If $P(x)$ is a Gaussian distribution, then

$$D(x) = \frac{1}{2}\left[1 + \operatorname{erf}\left(\frac{x-\mu}{\sigma\sqrt{2}}\right)\right], \tag{62}$$

so variates x_i with a Gaussian distribution can be generated from variates y_i having a UNIFORM DISTRIBUTION in (0,1) via

$$x_i = \sigma\sqrt{2}\,\operatorname{erf}^{-1}(2y_i - 1) + \mu. \tag{63}$$

However, a simpler way to obtain numbers with a Gaussian distribution is to use the BOX-MULLER TRANSFORMATION.

The Gaussian distribution is an approximation to the BINOMIAL DISTRIBUTION in the limit of large numbers,

$$P(n_1) = \frac{1}{\sqrt{2\pi Npq}} \exp\left[-\frac{(n_1 - Np)^2}{2Npq}\right], \tag{64}$$

where n_1 is the number of steps in the POSITIVE direction, N is the number of trials ($N \equiv n_1 + n_2$), and p and q are the probabilities of a step in the POSITIVE direction and NEGATIVE direction ($q \equiv 1 - p$).

The differential equation having a Gaussian distribution as its solution is

$$\frac{dy}{dx} = \frac{y(\mu - x)}{\sigma^2}, \tag{65}$$

since

$$\frac{dy}{y} = \frac{\mu - x}{\sigma^2}\, dx \tag{66}$$

$$\ln\left(\frac{y}{y_0}\right) = -\frac{1}{2\sigma^2}(\mu - x)^2 \tag{67}$$

$$y = y_0 e^{-(x-\mu)^2/2\sigma^2} \tag{68}$$

This equation has been generalized to yield more complicated distributions which are named using the so-called PEARSON SYSTEM.

see also BINOMIAL DISTRIBUTION, CENTRAL LIMIT THEOREM, ERF, GAUSSIAN BIVARIATE DISTRIBUTION, LOGIT TRANSFORMATION, NORMAL DISTRIBUTION, NORMAL DISTRIBUTION FUNCTION, PEARSON SYSTEM, RATIO DISTRIBUTION, z-SCORE

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Gaussian Distribution—Linear Combination of Variates

If x is NORMALLY DISTRIBUTED with MEAN μ and VARIANCE σ^2, then a linear function of x,

$$y = ax + b, \tag{1}$$

is also NORMALLY DISTRIBUTED. The new distribution has MEAN $a\mu+b$ and VARIANCE $a^2\sigma^2$, as can be derived using the MOMENT-GENERATING FUNCTION

$$\begin{aligned} M(t) &= \left\langle e^{t(ax+b)}\right\rangle = e^{tb}\left\langle e^{atx}\right\rangle = e^{tb}e^{\mu at+\sigma^2(at)^2/2} \\ &= e^{tb+\mu at+\sigma^2a^2t^2/2} = e^{(b+a\mu)t+a^2\sigma^2t^2/2}, \end{aligned} \tag{2}$$

which is of the standard form with

$$\mu' = b + a\mu \tag{3}$$

$$\sigma'^2 = a^2\sigma^2. \tag{4}$$

For a weighted sum of independent variables

$$y \equiv \sum_{i=1}^n a_i x_i, \tag{5}$$

the expectation is given by

$$\begin{aligned} M(t) &= \left\langle e^{yt}\right\rangle = \left\langle \exp\left(t\sum_{i=1}^n a_i x_i\right)\right\rangle \\ &= \left\langle e^{a_1tx_1}e^{a_2tx_2}\cdots e^{a_ntx_n}\right\rangle \\ &= \prod_{i=1}^n \left\langle e^{a_itx_i}\right\rangle = \prod_{i=1}^n \exp(a_i\mu_it + \tfrac{1}{2}{a_i}^2{\sigma_i}^2t^2). \end{aligned} \tag{6}$$

Setting this equal to

$$\exp(\mu t + \tfrac{1}{2}\sigma^2t^2) \tag{7}$$

gives

$$\mu \equiv \sum_{i=1}^n a_i\mu_i \tag{8}$$

$$\sigma^2 \equiv \sum_{i=1}^n {a_i}^2{\sigma_i}^2. \tag{9}$$

Therefore, the MEAN and VARIANCE of the weighted sums of n RANDOM VARIABLES are their weighted sums.

If x_i are INDEPENDENT and NORMALLY DISTRIBUTED with MEAN 0 and VARIANCE σ^2, define

$$y_i \equiv \sum_j c_{ij}x_j, \tag{10}$$

where c obeys the ORTHOGONALITY CONDITION

$$c_{ik}c_{jk} = \delta_{ij}, \tag{11}$$

with δ the KRONECKER DELTA. Then y_i are also independent and normally distributed with MEAN 0 and VARIANCE σ^2.

Gaussian Elimination

A method for solving MATRIX EQUATIONS of the form

$$\mathsf{Ax} = \mathbf{b}. \tag{1}$$

Starting with the system of equations

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1k} \\ a_{21} & a_{22} & \cdots & a_{2k} \\ \vdots & \vdots & \ddots & \vdots \\ a_{k1} & a_{k2} & \cdots & a_{kk} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_k \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_k \end{bmatrix}, \qquad (2)$$

compose the augmented MATRIX equation

$$\left[\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1k} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2k} & b_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{k1} & a_{k2} & \cdots & a_{kk} & b_k \end{array}\right] \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_k \end{bmatrix}. \qquad (3)$$

Then, perform MATRIX operations to put the augmented MATRIX into the form

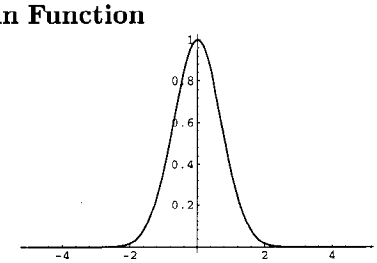
$$\left[\begin{array}{cccc|c} a'_{11} & a'_{12} & \cdots & a'_{1k} & b'_1 \\ 0 & a'_{22} & \cdots & a'_{2k} & b'_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & a'_{kk} & b'_k \end{array}\right] \begin{bmatrix} x'_1 \\ x'_2 \\ \vdots \\ x'_k \end{bmatrix}. \qquad (4)$$

Solve for a'_{kk}, then substitute back in to obtain solutions for $n = 1, 2, \ldots, k-1$,

$$x_i = \frac{1}{a'_{ii}} \left(b'_i - \sum_{j=i+1}^{k} a'_{ij} x'_j \right). \qquad (5)$$

see also GAUSS-JORDAN ELIMINATION, LU DECOMPOSITION, MATRIX EQUATION, SQUARE ROOT METHOD

Gaussian Function



In 1-D, the Gaussian function is the function from the GAUSSIAN DISTRIBUTION,

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-(x-\mu)^2/2\sigma^2}, \qquad (1)$$

sometimes also called the FREQUENCY CURVE. The FULL WIDTH AT HALF MAXIMUM (FWHM) for a Gaussian is found by finding the half-maximum points x_0. The constant scaling factor can be ignored, so we must solve

$$e^{-(x_0-\mu)^2/2\sigma^2} = \tfrac{1}{2} f(x_{\max}) \qquad (2)$$

But $f(x_{\max})$ occurs at $x_{\max} = \mu$, so

$$e^{-(x_0-\mu)^2/2\sigma^2} = \tfrac{1}{2} f(\mu) = \tfrac{1}{2}. \qquad (3)$$

Solving,

$$e^{-(x_0-\mu)^2/2\sigma^2} = 2^{-1} \qquad (4)$$

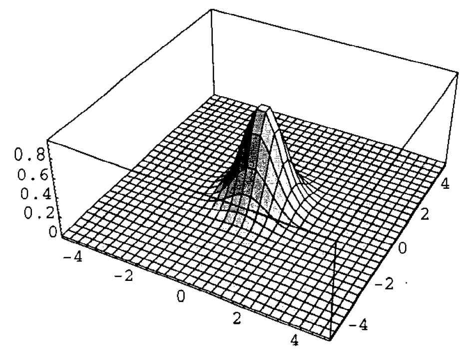
$$-\frac{(x_0-\mu)^2}{2\sigma^2} = -\ln 2 \qquad (5)$$

$$(x_0-\mu)^2 = 2\sigma^2 \ln 2 \qquad (6)$$

$$x_0 = \pm\sigma\sqrt{2\ln 2} + \mu. \qquad (7)$$

The FULL WIDTH AT HALF MAXIMUM is therefore given by

$$\text{FWHM} \equiv x_+ - x_- = 2\sqrt{2\ln 2}\,\sigma \approx 2.3548\sigma. \qquad (8)$$

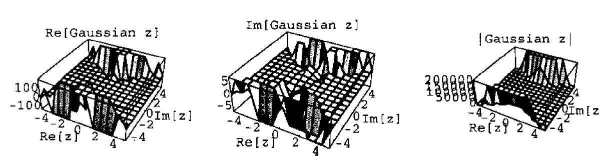


In 2-D, the circular Gaussian function is the distribution function for uncorrelated variables x and y having a GAUSSIAN BIVARIATE DISTRIBUTION and equal STANDARD DEVIATION $\sigma = \sigma_x = \sigma_y$,

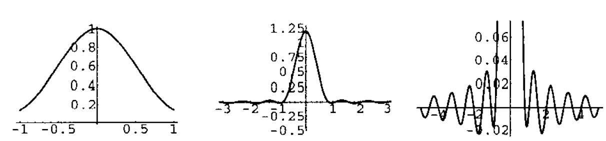
$$f(x, y) = \frac{1}{2\pi\sigma^2} e^{-[(x-\mu_x)^2 + (y-\mu_y)^2]/2\sigma^2}. \qquad (9)$$

The corresponding elliptical Gaussian function corresponding to $\sigma_x \neq \sigma_y$ is given by

$$f(x, y) = \frac{1}{2\pi\sigma_x\sigma_y} e^{-[(x-\mu_x)^2/2\sigma_x{}^2 + [(y-\mu_y)^2/2\sigma_y{}^2]}. \qquad (10)$$



The above plots show the real and imaginary parts of $(2\pi)^{-1/2} e^{-z^2}$ together with the complex absolute value $|(2\pi)^{-1/2} e^{-z^2}|$.



The Gaussian function can also be used as an APODIZATION FUNCTION, shown above with the corresponding INSTRUMENT FUNCTION.

The HYPERGEOMETRIC FUNCTION is also sometimes known as the Gaussian function.

see also ERF, ERFC, FOURIER TRANSFORM—GAUSSIAN, GAUSSIAN BIVARIATE DISTRIBUTION, GAUSSIAN DISTRIBUTION, NORMAL DISTRIBUTION

References

MacTutor History of Mathematics Archive. "Frequency Curve." http://www-groups.dcs.st-and.ac.uk/~history/Curves/Frequency.html.

Gaussian Hypergeometric Series

see HYPERGEOMETRIC FUNCTION

Gaussian Integer

A COMPLEX NUMBER $a+bi$ where a and b are INTEGERS. The Gaussian integers are members of the QUADRATIC FIELD $\mathbb{Q}(\sqrt{-1})$. The sum, difference, and product of two Gaussian integers are Gaussian integers, but $a+bi|c+di$ only if there is an $e+fi$ such that

$$(a+bi)(e+fi)=(ae-bf)+(af+be)i=c+di.$$

Gaussian INTEGERS can be uniquely factored in terms of other Gaussian INTEGERS up to POWERS of i and rearrangements.

The norm of a Gaussian integer is defined by

$$n(x+iy)=x^2+y^2.$$

GAUSSIAN PRIMES are Gaussian integers $a+ib$ for which $n(a+ib)=a^2+b^2$ is PRIME and a a PRIME INTEGER a such that $a\equiv 3 \pmod 4$.

1. If $2|n(x+iy)$, then $1+i$ and $1-i|x+iy$. These factors are equivalent since $-i(i-1)=i+1$. For example, $2=(1+i)(1-i)$ is not a Gaussian prime.
2. If $n(x+iy)\equiv 3 \pmod 4)\,|n(x+iy)$, then $n(a+ib)|x+iy$.
3. If $n(x+iy)\equiv 1 \pmod 4)\,|n(x+iy)$, then $a+ib$ or $b+ia|x+iy$. If both do, then $n(a+ib)|x+iy$.

The Gaussian primes with $|a|,|b|\leq 5$ are given by $-5-4i$, $-5-2i$, $-5+2i$, $-5+4i$, $-4-5i$, $-4-i$, $-4+i$, $-4+5i$, $-3-2i$, -3, $-3+2i$, $-2-5i$, $-2-3i$, $-2-i$, $-2+i$, $-2+3i$, $-2+5i$, $-1-4i$, $-1-2i$, $-1-i$, $-1+i$, $-1+2i$, $-1+4i$, $-3i$, $3i$, $1-4i$, $1-2i$, $1-i$, $1+i$, $1+2i$, $1+4i$, $2-5i$, $2-3i$, $2-i$, $2+i$, $2+3i$, $2+5i$, $3-2i$, 3, $3+2i$, $4-5i$, $4-i$, $4+i$, $4+5i$, $5-4i$, $5-2i$, $5+2i$, $5+4i$.

Every Gaussian integer is within $|n|/\sqrt{2}$ of a multiple of a Gaussian integer n.

see also COMPLEX NUMBER, EISENSTEIN INTEGER, GAUSSIAN PRIME, INTEGER, OCTONION

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Conway, J. H. and Guy, R. K. "Gauss's Whole Numbers." In *The Book of Numbers.* New York: Springer-Verlag, pp. 217–223, 1996.

Shanks, D. "Gaussian Integers and Two Applications." §50 in *Solved and Unsolved Problems in Number Theory, 4th ed.* New York: Chelsea, pp. 149–151, 1993.

Gaussian Integral

The Gaussian integral, also called the PROBABILITY INTEGRAL, is the integral of the 1-D Gaussian over $(-\infty,\infty)$. It can be computed using the trick of combining two 1-D Gaussians

$$\begin{aligned}\int_{-\infty}^{\infty} e^{-x^2}\,dx &= \sqrt{\left(\int_{-\infty}^{\infty} e^{-y^2}\,dy\right)\left(\int_{-\infty}^{\infty} e^{-x^2}\,dx\right)}\\ &= \sqrt{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-(x^2+y^2)}\,dy\,dx} \qquad (1)\end{aligned}$$

and switching to POLAR COORDINATES,

$$\begin{aligned}\int_{-\infty}^{\infty} e^{-x^2}\,dx &= \sqrt{\int_0^{2\pi}\int_0^{\infty} e^{-r^2} r\,dr\,d\theta}\\ &= \sqrt{2\pi\left[-\tfrac{1}{2}e^{-r^2}r\right]_0^{\infty}} = \sqrt{\pi}. \qquad (2)\end{aligned}$$

However, a simple proof can also be given which does not require transformation to POLAR COORDINATES (Nicholas and Yates 1950).

The integral from 0 to a finite upper limit a can be given by the CONTINUED FRACTION

$$\int_0^a e^{-x^2}\,dx = \frac{\sqrt{\pi}}{2}\,\frac{1}{a+}\,\frac{2}{2a+}\,\frac{3}{a+}\,\frac{4}{2a+\ldots}. \qquad (3)$$

The general class of integrals of the form

$$I_n(a)\equiv\int_0^{\infty} e^{-ax^2}x^n\,dx \qquad (4)$$

can be solved analytically by setting

$$x\equiv a^{-1/2}y \qquad (5)$$

$$dx = a^{-1/2}\,dy \qquad (6)$$

$$y^2=ax^2. \qquad (7)$$

Then

$$\begin{aligned}I_n(a) &= a^{-1/2}\int_0^{\infty} e^{-y^2}(a^{-1/2})^n\,dy\\ &= a^{-(1+n)/2}\int_0^{\infty} e^{-y^2}y^n\,dy. \qquad (8)\end{aligned}$$

For $n=0$, this is just the usual Gaussian integral, so

$$I_0(a)=\frac{\sqrt{\pi}}{2}a^{-1/2}=\frac{1}{2}\sqrt{\frac{\pi}{a}}. \qquad (9)$$

For $n=1$, the integrand is integrable by quadrature,

$$I_1(a)=a^{-1}\int_0^{\infty} e^{-y^2}y\,dy = a^{-1}[-\tfrac{1}{2}e^{-y^2}]_0^{\infty}=\tfrac{1}{2}a^{-1}. \qquad (10)$$

To compute $I_n(a)$ for $n > 1$, use the identity

$$\begin{aligned}-\frac{\partial}{\partial a} I_{n-2}(a) &= -\frac{\partial}{\partial a}\int_0^\infty e^{-ax^2}x^{n-2}\,dx \\ &= -\int_0^\infty -x^2e^{-ax^2}x^{n-2}\,dx \\ &= \int_0^\infty e^{-ax^2}x^n\,dx = I_n(a). \quad (11)\end{aligned}$$

For $n = 2s$ EVEN,

$$\begin{aligned}I_n(a) &= \left(-\frac{\partial}{\partial a}\right) I_{n-2}(a) = \left(-\frac{\partial}{\partial a}\right)^2 I_{n-4} \\ &= \ldots = \left(-\frac{\partial}{\partial a}\right)^{n/2} I_0(a) \\ &= \frac{\partial^{n/2}}{\partial a^{n/2}} I_0(a) = \frac{\sqrt{\pi}}{2}\frac{\partial^{n/2}}{\partial a^{n/2}} a^{-1/2}, \quad (12)\end{aligned}$$

so

$$\int_0^\infty x^{2s}e^{-ax^2}\,dx = \frac{(s-\frac{1}{2})!}{2a^{s+1/2}} = \frac{(2s-1)!!}{2^{s+1}a^s}\sqrt{\frac{\pi}{a}}. \quad (13)$$

If $n = 2s+1$ is ODD, then

$$\begin{aligned}I_n(a) &= \left(-\frac{\partial}{\partial a}\right) I_{n-2}(a) = \left(-\frac{\partial}{\partial a}\right)^2 I_{n-4}(a) \\ &= \ldots = \left(-\frac{\partial}{\partial a}\right)^{(n-1)/2} I_1(a) \\ &= \frac{\partial^{(n-1)/2}}{\partial a^{(n-1)/2}} I_1(a) = \frac{1}{2}\frac{\partial^{(n-1)/2}}{\partial a^{(n-1)/2}} a^{-1}, \quad (14)\end{aligned}$$

so

$$\int_0^\infty x^{2s+1}e^{-ax^2}\,dx = \frac{s!}{2a^{s+1}}. \quad (15)$$

The solution is therefore

$$\int_0^\infty e^{-ax^2}x^n\,dx = \begin{cases} \frac{(n-1)!!}{2^{n/2+1}a^{n/2}}\sqrt{\frac{\pi}{a}} & \text{for } n \text{ even} \\ \frac{[(n+1)/2]!}{2a^{(n+1)/2}} & \text{for } n \text{ odd.}\end{cases} \quad (16)$$

The first few values are therefore

$$I_0(a) = \frac{1}{2}\sqrt{\frac{\pi}{a}} \quad (17)$$

$$I_1(a) = \frac{1}{2a} \quad (18)$$

$$I_2(a) = \frac{1}{4a}\sqrt{\frac{\pi}{a}} \quad (19)$$

$$I_3(a) = \frac{1}{2a^2} \quad (20)$$

$$I_4(a) = \frac{3}{8a^2}\sqrt{\frac{\pi}{a}} \quad (21)$$

$$I_5(a) = \frac{1}{a^3} \quad (22)$$

$$I_6(a) = \frac{15}{16a^3}\sqrt{\frac{\pi}{a}}. \quad (23)$$

A related, often useful integral is

$$H_n(a) \equiv \frac{1}{\sqrt{\pi}}\int_{-\infty}^\infty e^{-ax^2}x^n\,dx, \quad (24)$$

which is simply given by

$$H_n = \begin{cases} \frac{2I_n(a)}{\sqrt{\pi}} & \text{for } n \text{ even} \\ 0 & \text{for } n \text{ odd.}\end{cases} \quad (25)$$

References

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Gaussian Integral (Linking Number)

see LINKING NUMBER

Gaussian Joint Variable Theorem

Also called the MULTIVARIATE THEOREM. Given an EVEN number of variates from a NORMAL DISTRIBUTION with MEANS all 0,

$$\langle x_1x_2\rangle = \langle x_1\rangle\langle x_2\rangle, \quad (1)$$

$$\begin{aligned}&\langle x_1x_2x_3x_4\rangle \\ &= \langle x_1x_2\rangle\langle x_3x_4\rangle + \langle x_1x_3\rangle\langle x_2x_4\rangle + \langle x_1x_4\rangle\langle x_2x_3\rangle, \quad (2)\end{aligned}$$

etc. Given an ODD number of variates,

$$\langle x_1\rangle = 0, \quad (3)$$

$$\langle x_1x_2x_3\rangle = 0, \quad (4)$$

etc.

Gaussian Mountain Range

see CAROTID-KUNDALINI FUNCTION

Gaussian Multivariate Distribution

see also GAUSSIAN BIVARIATE DISTRIBUTION, JOINT THEOREM, MULTIVARIATE THEOREM

Gaussian Polynomial

Defined by

$$[l] \equiv \frac{1-q^l}{1-q} \quad (1)$$

for integral l, and

$$\begin{bmatrix} n \\ k \end{bmatrix} \equiv \begin{cases} \prod_{l=1}^k \frac{[n-l+1]}{[l]} & \text{for } 0 \le k \le n \\ 0 & \text{otherwise.}\end{cases} \quad (2)$$

Unfortunately, the NOTATION conflicts with that of GAUSSIAN BRACKETS and the NEAREST INTEGER

FUNCTION. Gaussian POLYNOMIALS satisfy the identities

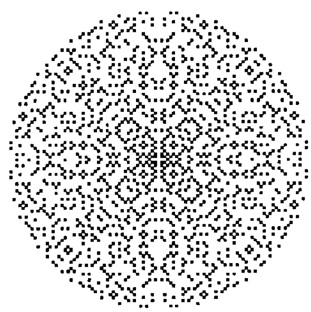
$$\frac{\begin{bmatrix} n+1 \\ k+1 \end{bmatrix}}{\begin{bmatrix} n \\ k+1 \end{bmatrix}} = \frac{1-q^{n+1}}{1-q^{n-k}} \tag{3}$$

$$\frac{\begin{bmatrix} n+1 \\ k+1 \end{bmatrix}}{\begin{bmatrix} n+1 \\ k \end{bmatrix}} = \frac{1-q^{n-k+1}}{1-q^{k+1}}. \tag{4}$$

For $q = 1$, the Gaussian polynomial turns into the BINOMIAL COEFFICIENT.

see also BINOMIAL COEFFICIENT, GAUSSIAN COEFFICIENT, q-SERIES

Gaussian Prime



Gaussian primes are GAUSSIAN INTEGERS $a + ib$ for which $n(a + ib) = a^2 + b^2$ is PRIME and a a PRIME INTEGER a such that $a \equiv 3 \pmod 4$. The above plot of the COMPLEX PLANE shows the Gaussian primes as filled squares.

see also EISENSTEIN INTEGER, GAUSSIAN INTEGER

References

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Wagon, S. "Gaussian Primes." §9.4 in *Mathematica in Action.* New York: W. H. Freeman, pp. 298–303, 1991.

Gaussian Quadrature

Seeks to obtain the best numerical estimate of an integral by picking optimal ABSCISSAS x_i at which to evaluate the function $f(x)$. The FUNDAMENTAL THEOREM OF GAUSSIAN QUADRATURE states that the optimal ABSCISSAS of the m-point GAUSSIAN QUADRATURE FORMULAS are precisely the roots of the orthogonal POLYNOMIAL for the same interval and WEIGHTING FUNCTION. Gaussian quadrature is optimal because it fits all POLYNOMIALS up to degree $2m$ exactly. Slightly less optimal fits are obtained from RADAU QUADRATURE and LAGUERRE QUADRATURE.

$W(x)$	Interval	x_i Are Roots Of
1	$(-1,1)$	$P_n(x)$
e^{-t}	$(0,\infty)$	$L_n(x)$
e^{-t^2}	$(-\infty,\infty)$	$H_n(x)$
$(1-t^2)^{-1/2}$	$(-1,1)$	$T_n(x)$
$(1-t^2)^{1/2}$	$(-1,1)$	$U_n(x)$
$x^{1/2}$	$(0,1)$	$x^{-1/2}P_{2n+1}(\sqrt{x})$
$x^{-1/2}$	$(0,1)$	$P_n(\sqrt{x})$

To determine the weights corresponding to the Gaussian ABSCISSAS, compute a LAGRANGE INTERPOLATING POLYNOMIAL for $f(x)$ by letting

$$\pi(x) \equiv \prod_{j=1}^{m}(x - x_j) \tag{1}$$

(where Chandrasekhar 1967 uses F instead of π), so

$$\pi'(x_j) = \left[\frac{d\pi}{dx}\right]_{x=x_j} = \prod_{\substack{i=1 \\ i\neq j}}^{m}(x_j - x_i). \tag{2}$$

Then fitting a LAGRANGE INTERPOLATING POLYNOMIAL through the m points gives

$$\phi(x) = \sum_{j=1}^{m} \frac{\pi(x)}{(x - x_j)\pi'(x_j)} f(x_j) \tag{3}$$

for arbitrary points x_i. We are therefore looking for a set of points x_j and weights w_j such that for a WEIGHTING FUNCTION $W(x)$,

$$\int_a^b \phi(x)W(x)\,dx = \int_a^b \sum_{j=1}^{m} \frac{\pi(x)W(x)}{(x-x_j)\pi'(x_j)}\,dx\, f(x_j)$$
$$\equiv \sum_{j=1}^{m} w_j f(x_j), \tag{4}$$

with WEIGHT

$$w_j = \frac{1}{\pi'(x_j)} \int_a^b \frac{\pi(x)W(x)}{x - x_j}\,dx. \tag{5}$$

The weights w_j are sometimes also called the CHRISTOFFEL NUMBER (Chandrasekhar 1967). For orthogonal POLYNOMIALS $\phi_j(x)$ with j=1, ..., n,

$$\phi_j(x) = A_j \pi(x) \tag{6}$$

(Hildebrand 1956, p. 322), where A_n is the COEFFICIENT of x^n in $\phi_n(x)$, then

$$w_j = \frac{1}{\phi_n'(x_j)} \int_a^b W(x) \frac{\phi(x)}{x - x_j}\,dx$$
$$= -\frac{A_{n+1}\gamma_n}{A_n \phi_n'(x_j)\phi_{n+1}(x)}, \tag{7}$$

where

$$\gamma_m \equiv \int [\phi_m(x)]^2 W(x)\,dx. \quad (8)$$

Using the relationship

$$\phi_{n+1}(x_i) = -\frac{A_{n+1}A_{n-1}}{A_n{}^2}\frac{\gamma_n}{\gamma_{n-1}}\phi_{n-1}(x_i) \quad (9)$$

(Hildebrand 1956, p. 323) gives

$$w_j = \frac{A_n}{A_{n-1}}\frac{\gamma_{n-1}}{\phi'_n(x_j)\phi_{n-1}(x_j)}. \quad (10)$$

(Note that Press *et al.* 1992 omit the factor A_n/A_{n-1}.) In Gaussian quadrature, the weights are all POSITIVE. The error is given by

$$E_n = \frac{f^{(2n)}(\xi)}{(2n)!}\int_a^b W(x)[\pi(x)]^2\,dx = \frac{\gamma_n}{A_n{}^2}\frac{f^{(2n)}(\xi)}{(2n)!}, \quad (11)$$

where $a < \xi < b$ (Hildebrand 1956, pp. 320–321).

Other curious identities are

$$\sum_{k=0}^{m}\frac{[\phi_k(x)]^2}{\gamma_k} = \frac{A_m}{A_{m+1}\gamma_m}[\phi'_{m+1}(x)\phi_m(x) - \phi'_m(x)\phi_{m+1}(x)] \quad (12)$$

and

$$\sum_{k=0}^{m}\frac{[\phi_k(x)]^2}{\gamma_k} = -\frac{A_m\phi'_m(x_i)\phi_{m+1}(x_i)}{A_{m+1}\gamma_m} = \frac{1}{w_i} \quad (13)$$

(Hildebrand 1956, p. 323).

In the NOTATION of Szegő (1975), let $x_{1n} < \ldots < x_{nn}$ be an ordered set of points in $[a,b]$, and let $\lambda_{1n}, \ldots, \lambda_{nn}$ be a set of REAL NUMBERS. If $f(x)$ is an arbitrary function on the CLOSED INTERVAL $[a,b]$, write the MECHANICAL QUADRATURE as

$$Q_n(f) = \sum_{\nu=1}^{n}\lambda_{\nu n} f(x_{\nu n}). \quad (14)$$

Here $x_{\nu n}$ are the ABSCISSAS and $\lambda_{\nu n}$ are the COTES NUMBERS.

see also CHEBYSHEV QUADRATURE, CHEBYSHEV-GAUSS QUADRATURE, CHEBYSHEV-RADAU QUADRATURE, FUNDAMENTAL THEOREM OF GAUSSIAN QUADRATURE, HERMITE-GAUSS QUADRATURE, JACOBI-GAUSS QUADRATURE, LAGUERRE-GAUSS QUADRATURE, LEGENDRE-GAUSS QUADRATURE, LOBATTO QUADRATURE, MEHLER QUADRATURE, RADAU QUADRATURE

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Whittaker, E. T. and Robinson, G. *The Calculus of Observations: A Treatise on Numerical Mathematics, 4th ed.* New York: Dover, pp. 152–163, 1967.

Gaussian Sum

$$S(p,q) \equiv \sum_{r=0}^{q-1} e^{-\pi i r^2 p/q}, \quad (1)$$

where p and q are RELATIVELY PRIME INTEGERS. If $(n,n') = 1$, then

$$S(m, nn') = S(mn', n)S(mn, n'). \quad (2)$$

Gauss showed

$$\sum_{r=0}^{q-1} e^{2\pi i r^2/q} = \frac{1-i^q}{1-i}\sqrt{q} \quad (3)$$

for ODD q. A more general result was obtained by Schaar. For p and q of opposite PARITY (i.e., one is EVEN and the other is ODD), SCHAAR'S IDENTITY states

$$\frac{1}{\sqrt{q}}\sum_{r=0}^{q-1} e^{-\pi i r^2 p/q} = \frac{e^{-\pi i/4}}{\sqrt{p}}\sum_{r=0}^{p-1} e^{\pi i r^2 q/p}. \quad (4)$$

Such sums are important in the theory of QUADRATIC RESIDUES.

see also KLOOSTERMAN'S SUM, SCHAAR'S IDENTITY, SINGULAR SERIES

References

Evans, R. and Berndt, B. "The Determination of Gauss Sums." *Bull. Amer. Math. Soc.* **5**, 107–129, 1981.
Katz, N. M. *Gauss Sums, Kloosterman Sums, and Monodromy Groups.* Princeton, NJ: Princeton University Press, 1987.
Riesel, H. *Prime Numbers and Computer Methods for Factorization, 2nd ed.* Boston, MA: Birkhäuser, pp. 132–134, 1994.

Gear Graph
A WHEEL GRAPH with a VERTEX added between each pair of adjacent VERTICES.

Gegenbauer Function
see ULTRASPHERICAL FUNCTION

Gegenbauer Polynomial
see ULTRASPHERICAL POLYNOMIAL

Gelfond-Schneider Constant
The number $2^{\sqrt{2}} = 2.66514414\ldots$ which is known to be TRANSCENDENTAL by GELFOND'S THEOREM.

References
Courant, R. and Robbins, H. *What is Mathematics?: An Elementary Approach to Ideas and Methods, 2nd ed.* Oxford, England: Oxford University Press, p. 107, 1996.

Gelfond-Schneider Theorem
see GELFOND'S THEOREM

Gelfond's Theorem
Also called the GELFOND-SCHNEIDER THEOREM. a^b is TRANSCENDENTAL if

1. a is ALGEBRAIC $\neq 0$, 1 and
2. b is ALGEBRAIC and IRRATIONAL.

This provides the solution to the seventh of HILBERT'S PROBLEMS.

see also ALGEBRAIC NUMBER, HILBERT'S PROBLEMS, IRRATIONAL NUMBER, TRANSCENDENTAL NUMBER

References
Baker, A. *Transcendental Number Theory.* London: Cambridge University Press, 1990.
Courant, R. and Robbins, H. *What is Mathematics?: An Elementary Approach to Ideas and Methods, 2nd ed.* Oxford, England: Oxford University Press, p. 107, 1996.

Genaille Rods
Numbered rods which can be used to perform multiplication.

see also NAPIER'S BONES

References
Gardner, M. "Napier's Bones." Ch. 7 in *Knotted Doughnuts and Other Mathematical Entertainments.* New York: W. H. Freeman, 1986.

Genera
see FUNDAMENTAL THEOREM OF GENERA

General Linear Group
The general linear group $GL_n(q)$ is the set of $n \times n$ MATRICES with entries in the FIELD $\mathbb{F}_q$ which have NONZERO DETERMINANT.

see also LANGLANDS RECIPROCITY, PROJECTIVE GENERAL LINEAR GROUP, PROJECTIVE SPECIAL LINEAR GROUP, SPECIAL LINEAR GROUP

References
Conway, J. H.; Curtis, R. T.; Norton, S. P.; Parker, R. A.; and Wilson, R. A. "The Groups $GL_n(q)$, $SL_n(q)$, $PGL_n(q)$, and $PSL_n(q) = L_n(q)$." §2.1 in *Atlas of Finite Groups: Maximal Subgroups and Ordinary Characters for Simple Groups.* Oxford, England: Clarendon Press, p. x, 1985.

General Orthogonal Group
The general orthogonal group $GO_n(q, F)$ is the SUBGROUP of all elements of the PROJECTIVE GENERAL LINEAR GROUP that fix the particular nonsingular QUADRATIC FORM F. The determinant of such an element is ± 1.

see also PROJECTIVE GENERAL LINEAR GROUP

References
Conway, J. H.; Curtis, R. T.; Norton, S. P.; Parker, R. A.; and Wilson, R. A. "The Groups $GO_n(q)$, $SO_n(q)$, $PGO_n(q)$, and $PSO_n(q)$, and $O_n(q)$." §2.4 in *Atlas of Finite Groups: Maximal Subgroups and Ordinary Characters for Simple Groups.* Oxford, England: Clarendon Press, pp. xi–xii, 1985.

General Position
An arrangement of points with no three COLLINEAR, or of lines with no three concurrent.

see also ORDINARY LINE, NEAR-PENCIL

References
Guy, R. K. "Unsolved Problems Come of Age." *Amer. Math. Monthly* **96**, 903–909, 1989.

General Prismatoid
A solid such that the AREA A_y of any section parallel to and a distance y from a fixed PLANE can be expressed as

$$A_y = ay^3 + by^2 + cy + d.$$

The volume of such a solid is the same as for a PRISMATOID,

$$V = \tfrac{1}{6}h(A_1 + 4M + A_2).$$

Examples include the CONE, CYLINDER, PRISMATOID, SPHERE, and SPHEROID.

see also PRISMATOID, PRISMOID

References
Beyer, W. H. *CRC Standard Mathematical Tables, 28th ed.* Boca Raton, FL: CRC Press, p. 132, 1987.

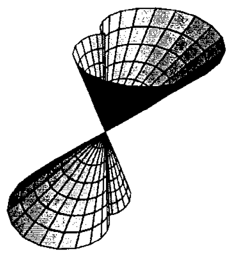
General Unitary Group

The general unitary group $GU_n(q)$ is the SUBGROUP of all elements of the GENERAL LINEAR GROUP $GL(q^2)$ that fix a given nonsingular Hermitian form. This is equivalent, in the canonical case, to the definition of GU_n as the group of UNITARY MATRICES.

References

Conway, J. H.; Curtis, R. T.; Norton, S. P.; Parker, R. A.; and Wilson, R. A. "The Groups $GU_n(q)$, $SU_n(q)$, $PGU_n(q)$, and $PSU_n(q) = U_n(q)$." §2.2 in *Atlas of Finite Groups: Maximal Subgroups and Ordinary Characters for Simple Groups.* Oxford, England: Clarendon Press, p. x, 1985.

Generalized Cone



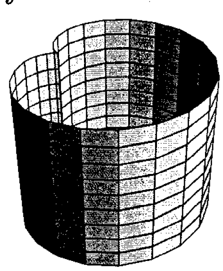
A RULED SURFACE is called a generalized cone if it can be parameterized by $\mathbf{x}(u,v) = \mathbf{p} + v\mathbf{y}(u)$, where $\mathbf{p}$ is a fixed point which can be regarded as the vertex of the cone. A generalized cone is a REGULAR SURFACE wherever $v\mathbf{y} \times \mathbf{y}' \neq \mathbf{0}$. The above surface is a generalized cylinder over a CARDIOID. A generalized cone is a FLAT SURFACE.

see also CONE

References

Gray, A. *Modern Differential Geometry of Curves and Surfaces.* Boca Raton, FL: CRC Press, pp. 341–342, 1993.

Generalized Cylinder



A RULED SURFACE is called a generalized cylinder if it can be parameterized by $\mathbf{x}(u,v) = v\mathbf{p} + \mathbf{y}(u)$, where $\mathbf{p}$ is a fixed point. A generalized cylinder is a REGULAR SURFACE wherever $\mathbf{y}' \times \mathbf{p} \neq \mathbf{0}$. The above surface is a generalized cylinder over a CARDIOID. A generalized cylinder is a FLAT SURFACE.

see also CYLINDER

References

Gray, A. *Modern Differential Geometry of Curves and Surfaces.* Boca Raton, FL: CRC Press, pp. 341–342, 1993.

Generalized Fibonacci Number

A generalization of the FIBONACCI NUMBERS defined by $1 = G_1 = G_2 = \ldots = G_{c-1}$ and the RECURRENCE RELATION

$$G_n = G_{n-1} + G_{n-c}. \tag{1}$$

These are the sums of elements on successive diagonals of a left-justified PASCAL'S TRIANGLE beginning in the left-most column and moving in steps of $c-1$ up and 1 right. The case $c = 2$ equals the usual FIBONACCI NUMBER. These numbers satisfy the identities

$$G_1 + G_2 + G_3 + \ldots + G_n = G_{n+3} - 1 \tag{2}$$

$$G_3 + G_6 + G_9 + \ldots + G_{3k} = G_{3k+1} - 1 \tag{3}$$

$$G_1 + G_4 + G_7 + \ldots + G_{3k+1} = G_{3k+2} \tag{4}$$

$$G_2 + G_5 + G_8 + \ldots + G_{3k+2} = G_{3k+3} \tag{5}$$

(Bicknell-Johnson and Spears 1996). For the special case $c = 3$,

$$G_{n+w} = G_{w-2}G_n + G_{w-3}G_{n+1} + G_{w-1}G_{n+2}. \tag{6}$$

Bicknell-Johnson and Spears (1996) give many further identities.

Horadam (1965) defined the generalized Fibonacci numbers $\{w_n\}$ as $w_n = w_n(a,b;p,q)$, where a, b, p, and q are INTEGERS, $w_0 = a$, $w_1 = b$, and $w_n = pw_{n-1} - qw_{n-2}$ for $n \geq 2$. They satisfy the identities

$$w_n w_{n+2r} - eq^n U_r = {w_{n+r}}^2 \tag{7}$$

$$4w_n {w_{n+1}}^2 w_{n+2} + (wq^n)^2 = (w_n w_{n+2} + {w_{n+1}}^2)^2 \tag{8}$$

$$w_n w_{n+1} w_{n+3} w_{n+4} = {w_{n+2}}^4 + eq^n(p^2+q){w_{n+2}}^2 + e^2 q^{2n+1} p^2 \tag{9}$$

$$4w_n w_{n+1} w_{n+2} w_{n+4} w_{n+5} w_{n+6} + e^2 q^{2n} (w_n U_4 U_5 - w_{n+1} U_2 U_6 - w_n U_1 U_8)^2 = (w_{n+1} w_{n+2} w_{n+6} + w_n w_{n+4} w_{n+5})^2, \tag{10}$$

where

$$e \equiv pab - qa^2 - b^2 \tag{11}$$

$$U_n \equiv w_n(0,1;p,q). \tag{12}$$

The final above result is due to Morgado (1987) and is called the MORGADO IDENTITY.

Another generalization of the Fibonacci numbers is denoted x_n. Given x_1 and x_2, define the generalized Fibonacci number by $x_n \equiv x_{n-2} + x_{n-1}$ for $n \geq 3$,

$$\sum_{i=1}^{n} x_n = x_{n+2} - x_2 \tag{13}$$

$$\sum_{i=1}^{10} x_n = 11x_7 \tag{14}$$

$$x_n^2 - x_{n-1}x_{n+2} = (-1)^n (x_2^2 - x_1^2 - x_1x_2), \tag{15}$$

where the plus and minus signs alternate.

see also FIBONACCI NUMBER

References

Bicknell, M. "A Primer for the Fibonacci Numbers, Part VIII: Sequences of Sums from Pascal's Triangle." *Fib. Quart.* **9**, 74–81, 1971.

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Horadam, A. F. "Generalization of a Result of Morgado." *Portugaliae Math.* **44**, 131–136, 1987.

Horadam, A. F. and Shannon, A. G. "Generalization of Identities of Catalan and Others." *Portugaliae Math.* **44**, 137–148, 1987.

Morgado, J. "Note on Some Results of A. F. Horadam and A. G. Shannon Concerning a Catalan's Identity on Fibonacci Numbers." *Portugaliae Math.* **44**, 243–252, 1987.

Generalized Function

The class of all regular sequences of PARTICULARLY WELL-BEHAVED FUNCTIONS equivalent to a given regular sequence (sometimes also called a DISTRIBUTION or FUNCTIONAL). A generalized function $p(x)$ has the properties

$$\int_{-\infty}^{\infty} p'(x)f(x)\,dx = -\int_{-\infty}^{\infty} p(x)f'(x)\,dx$$

$$\int_{-\infty}^{\infty} p^{(n)} f(x)\,dx = (-1)^n \int_{-\infty}^{\infty} p(x)f^{(n)}(x)\,dx.$$

The DELTA FUNCTION is a generalized function.

see also DELTA FUNCTION

Generalized Helicoid

The SURFACE generated by a twisted curve C when rotated about a fixed axis A and, at the same time, displaced PARALLEL to A so that the velocity of displacement is always proportional to the ANGULAR VELOCITY of ROTATION.

see also GENERALIZED HELIX, HELICOID, HELIX

References

do Carmo, M. P.; Fischer, G.; Pinkall, U.; and Reckziegel, H. "General Helicoids." §3.4.3 in *Mathematical Models from the Collections of Universities and Museums* (Ed. G. Fischer). Braunschweig, Germany: Vieweg, pp. 36–37, 1986.

Fischer, G. (Ed.). Plate 89 in *Mathematische Modelle/Mathematical Models, Bildband/Photograph Volume.* Braunschweig, Germany: Vieweg, p. 85, 1986.

Kreyszig, E. *Differential Geometry.* New York: Dover, p. 88, 1991.

Generalized Helix

The GEODESICS on a general cylinder generated by lines PARALLEL to a line l with which the TANGENT makes a constant ANGLE.

see also HELIX

Generalized Hyperbolic Functions

In 1757, V. Riccati first recorded the generalizations of the HYPERBOLIC FUNCTIONS defined by

$$F_{n,r}^{\alpha}(x) \equiv C \sum_{k=0}^{\infty} \frac{\alpha^k}{(nk+r)!} x^{nk+r}, \tag{1}$$

for $r = 0, \ldots, n-1$, where α is COMPLEX, and where the normalization is taken so that

$$F_{n,0}^{\alpha}(0) = 1. \tag{2}$$

This is called the α-hyperbolic function of order n of the kth kind. The functions $F_{n,r}^{\alpha}$ satisfy

$$F_{n,r}^{\alpha}(x) = (\sqrt[n]{\alpha})^{-r}(\sqrt[n]{\alpha}\,x) \tag{3}$$

and

$$f^{(k)}(x) = \alpha f(x), \tag{4}$$

where

$$f^{(k)}(0) = \begin{cases} 0 & k \neq r,\ 0 \leq k \leq n-1, \\ 1 & k = r. \end{cases} \tag{5}$$

In addition,

$$\frac{d}{dx} F_{n,r}^{\alpha}(x) = \begin{cases} F_{n,r-1}^{\alpha}(x) & \text{for } 0 < r \leq n-1 \\ \alpha F_{n,n-1}^{\alpha}(x) & \text{for } r = 0. \end{cases} \tag{6}$$

The functions give a generalized EULER FORMULA

$$e^{\sqrt[n]{\alpha}} = \sum_{r=0}^{n-1} (\sqrt[n]{\alpha})^r F_{n,r}^{\alpha}(x). \tag{7}$$

Since there are n nth roots of α, this gives a system of n linear equations. Solving for $F_{n,r}^{\alpha}$ gives

$$F_{n,r}^{\alpha}(x) = \frac{1}{n}(\sqrt[n]{\alpha})^{-r} \sum_{k=0}^{n-1} \omega_n{}^{-rk} \exp(\omega_n{}^k \sqrt[n]{\alpha}\,x), \tag{8}$$

where

$$\omega_n = \exp\left(\frac{2\pi i}{n}\right) \tag{9}$$

is a PRIMITIVE ROOT OF UNITY.

The LAPLACE TRANSFORM is

$$\int_0^{\infty} e^{-st} F_{n,r}^{\alpha}(at)\,dt = \frac{s^{n-r-1}a^r}{s^n + \alpha a_n}. \tag{10}$$

The generalized hyperbolic function is also related to the MITTAG-LEFFLER FUNCTION $E_\gamma(x)$ by

$$F^1_{n,0}(x) = E_n(x^n). \tag{11}$$

The values $n = 1$ and $n = 2$ give the exponential and circular/hyperbolic functions (depending on the sign of α), respectively.

$$F^\alpha_{1,0}(x) = e^{\alpha x} \tag{12}$$
$$F^\alpha_{2,0}(x) = \cosh(\sqrt{\alpha}\,x) \tag{13}$$
$$F^\alpha_{2,1}(x) = \frac{\sinh(\sqrt{\alpha}\,x)}{\sqrt{\alpha}}. \tag{14}$$

For $\alpha = 1$, the first few functions are

$$\begin{aligned}
F^1_{1,0}(x) &= e^x\\
F^1_{2,0}(x) &= \cosh x\\
F^1_{2,1}(x) &= \sinh x\\
F^1_{3,0}(x) &= \tfrac{1}{3}[e^x + 2e^{-x/2}\cos(\tfrac{1}{2}\sqrt{3}\,x)]\\
F^1_{3,1}(x) &= \tfrac{1}{3}[e^x + 2e^{-x/2}\cos(\tfrac{1}{2}\sqrt{3}\,x + \tfrac{1}{3}\pi)]\\
F^1_{3,2}(x) &= \tfrac{1}{3}[e^x + 2e^{-x/2}\cos(\tfrac{1}{2}\sqrt{3}\,x - \tfrac{1}{3}\pi)]\\
F^1_{4,0}(x) &= \tfrac{1}{2}(\cosh x + \cos x)\\
F^1_{4,1}(x) &= \tfrac{1}{2}(\sinh x + \sin x)\\
F^1_{4,2}(x) &= \tfrac{1}{2}(\cosh x - \cos x)\\
F^1_{4,3}(x) &= \tfrac{1}{2}(\sinh x - \sin x).
\end{aligned}$$

see also HYPERBOLIC FUNCTIONS, MITTAG-LEFFLER FUNCTION

References

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Muldoon, M. E. and Ungar, A. A. "Beyond Sin and Cos." *Math. Mag.* **69**, 3–14, 1996.

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Ungar, A. "Generalized Hyperbolic Functions." *Amer. Math. Monthly* **89**, 688–691, 1982.

Ungar, A. "Higher Order Alpha-Hyperbolic Functions." *Indian J. Pure. Appl. Math.* **15**, 301–304, 1984.

Generalized Hypergeometric Function

The generalized hypergeometric function is given by a HYPERGEOMETRIC SERIES, i.e., a series for which the ratio of successive terms can be written

$$\frac{a_{k+1}}{a_k} = \frac{P(k)}{Q(k)} = \frac{(k+a_1)(k+a_2)\cdots(k+a_p)}{(k+b_1)(k+b_2)\cdots(k+b_q)(k+1)}x. \tag{1}$$

(The factor of $k+1$ in the DENOMINATOR is present for historical reasons of notation.) The resulting generalized hypergeometric function is written

$$\sum_{k=0} a_k x^k = {}_pF_q\left[\begin{matrix} a_1, a_2, \cdots, a_p\\ b_1, b_2, \cdots, b_q\end{matrix}; x\right] \tag{2}$$

$$= \sum_{k=0}^\infty \frac{(a_1)_k (a_2)_k \cdots (a_p)_k}{(b_1)_k b (b_2)_k \cdots (b_q)_k}\frac{x^k}{k!}, \tag{3}$$

where $(a)_k$ is the POCHHAMMER SYMBOL or RISING FACTORIAL

$$(a)_k \equiv \frac{\Gamma(a+k)}{\Gamma(a)} = a(a+1)\cdots(a+k-1). \tag{4}$$

If the argument $x = 1$, then the function is abbreviated

$${}_pF_q\left[\begin{matrix} a_1, a_2 \ldots, a_p\\ b_1, b_2, \ldots, b_q\end{matrix}\right] \equiv {}_pF_q\left[\begin{matrix} a_1, a_2 \ldots, a_p\\ b_1, b_2, \ldots, b_q\end{matrix}; x\right]. \tag{5}$$

${}_2F_1(a,b;c;z)$ is "the" HYPERGEOMETRIC FUNCTION, and ${}_1F_1(a;b;z) \equiv M(z)$ is the CONFLUENT HYPERGEOMETRIC FUNCTION. A function of the form ${}_0F_1(;b;z)$ is called a CONFLUENT HYPERGEOMETRIC LIMIT FUNCTION.

The generalized hypergeometric function

$${}_{p+1}F_p\left[\begin{matrix} a_1, a_2, \ldots, a_{p+1}\\ b_1, b_2, \ldots, b_p\end{matrix}; z\right] \tag{6}$$

is a solution to the DIFFERENTIAL EQUATION

$$\begin{aligned}[\vartheta(\vartheta+b-1)\cdots(\vartheta+b_p-1)\\ -z(\vartheta+a_1)(\vartheta+a_2)\cdots(\vartheta+a_{p+1})]y = 0,\end{aligned} \tag{7}$$

where

$$\vartheta = z\frac{d}{dz}. \tag{8}$$

The other linearly independent solution is

$$z^{1-b_1}{}_{p+1}F_p\left[\begin{matrix} 1+a_1-b_1, 1-a_2-b_2,\\ \ldots, 1+a_{p+1}-b_1\\ 2-b_1, 1-b_2-b_1, \ldots,\\ 1-b_p-b_1\end{matrix}; z\right]. \tag{9}$$

A generalized hypergeometric equation is termed "well posed" if

$$1 + a_1 = b_1 + a_2 = \ldots = b_p + a_{p+1}. \tag{10}$$

Many sums can be written as generalized hypergeometric functions by inspection of the ratios of consecutive terms in the generating HYPERGEOMETRIC SERIES. For example, for

$$f(n) \equiv \sum_k (-1)^k \binom{2n}{k}^2, \tag{11}$$

the ratio of successive terms is

$$\frac{a_{k+1}}{a_k} = \frac{(-1)^{k+1}\binom{2n}{k+1}^2}{(-1)^k\binom{2n}{k}^2} = -\frac{(k-2n)^2}{(k+1)^2}, \tag{12}$$

yielding

$$f(n) = {}_2F_1\left[\begin{matrix}-2n, -2n\\ 1\end{matrix}; -1\right] = {}_2F_1(-2n, -2n; 1; -1) \tag{13}$$

(Petkovšek 1996, pp. 44–45).

Gosper (1978) discovered a slew unusual hypergeometric function identities, many of which were subsequently proven by Gessel and Stanton (1982). An important generalization of Gosper's technique, called ZEILBERGER'S ALGORITHM, in turn led to the powerful machinery of the WILF-ZEILBERGER PAIR (Zeilberger 1990).

Special hypergeometric identities include GAUSS'S HYPERGEOMETRIC THEOREM

$${}_2F_1(a, b; c; 1) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} \tag{14}$$

for $\Re[c-a-b] > 0$, KUMMER'S FORMULA

$${}_2F_1(a, b; c; -1) = \frac{\Gamma(\frac{1}{2}b+1)\Gamma(b-a+1)}{\Gamma(b+1)\Gamma(\frac{1}{2}b-a+1)}, \tag{15}$$

where $a - b + c = 1$ and b is a positive integer, SAALSCHÜTZ'S THEOREM

$${}_3F_2(a, b, c; d, e; 1) = \frac{(d-a)_{|c|}(d-b)_{|c|}}{d_{|c|}(d-a-b)_{|c|}} \tag{16}$$

for $d + e = a + b + c + 1$ with c a negative integer and $(a)_n$ the POCHHAMMER SYMBOL, DIXON'S THEOREM

$${}_3F_2(a, b, c; d, e; 1) = \frac{(\frac{1}{2}a)!(a-b)!(a-c)!(\frac{1}{2}a-b-c)!}{a!(\frac{1}{2}a-b)!(\frac{1}{2}a-c)!(a-b-c)!}, \tag{17}$$

where $1 + a/2 - b - c$ has a positive REAL PART, $d = a - b + 1$, and $e = a - c + 1$, the CLAUSEN FORMULA

$${}_4F_3\left[\begin{matrix}a & b & c & d\\ e & f & g & \end{matrix}; 1\right] = \frac{(2a)_{|d|}(a+b)_{|d|}(2b)_{|d|}}{(2a+2b)_{|d|}a_{|d|}b_{|d|}}, \tag{18}$$

for $a+b+c-d = 1/2$, $e = a+b+1/2$, $a+f = d+1 = b+g$, d a nonpositive integer, and the DOUGALL-RAMANUJAN IDENTITY

$$\begin{aligned}
&{}_7F_6\left[\begin{matrix}a_1, a_2, a_3, a_4, a_5, a_6, a_7\\ b_1, b_2, b_3, b_4, b_5, b_6\end{matrix}; 1\right]\\
&= \frac{(a_1+1)_n(a_1-a_2-a_3+1)_n}{(a_1-a_2+1)_n(a_1-a_3+1)_n}\\
&\quad\times\frac{(a_1-a_2-a_4+1)_n(a_1-a_3-a_4+1)_n}{(a_1-a_4+1)_n(a_1-a_2-a_3-a_4+1)_n},
\end{aligned} \tag{19}$$

where $n = 2a_1 + 1 = a_2 + a_3 + a_4 + a_5$, $a_6 = 1 + a_1/2$, $a_7 = -n$, and $b_i = 1 + a_1 - a_{i+1}$ for $i = 1, 2, \ldots, 6$. For all these identities, $(a)_n$ is the POCHHAMMER SYMBOL.

Gessel (1994) found a slew of new identities using WILF-ZEILBERGER PAIRS, including the following:

$${}_5F_4\left[\begin{matrix}-a-b, n+1, n+c+1, 2n-a-b+1, n+\frac{1}{2}(3-a-b)\\ n-a-b-c+1, n-a-b+1, 2n+2, n+\frac{1}{2}(1-a-b)\end{matrix}; 1\right] = 0 \tag{20}$$

$${}_3F_2\left[\begin{matrix}-3n, \frac{2}{3}-c, 3n+2\\ \frac{3}{2}, 1-3c\end{matrix}; \frac{3}{4}\right] = \frac{(c+\frac{2}{3})_n(\frac{1}{3})_n}{(1-c)_n(\frac{4}{3})_n} \tag{21}$$

$${}_3F_2\left[\begin{matrix}-3b, -\frac{3}{2}n, \frac{1}{2}(1-3n)\\ -3n, \frac{2}{3}-b-n\end{matrix}; \frac{4}{3}\right] = \frac{(\frac{1}{3}-b)_n}{(\frac{1}{3}+b)_n} \tag{22}$$

$${}_4F_3\left[\begin{matrix}\frac{3}{2}+\frac{1}{5}n, \frac{2}{3}, -n, 2n+2\\ n+\frac{11}{6}, \frac{4}{3}, \frac{1}{5}n+\frac{1}{2}\end{matrix}; \frac{2}{27}\right] = \frac{(\frac{5}{2})_n(\frac{11}{6})_n}{(\frac{3}{2})_n(\frac{7}{2})_n} \tag{23}$$

(Petkovšek *et al.* 1996, pp. 135–137).

see also CARLSON'S THEOREM, CLAUSEN FORMULA, CONFLUENT HYPERGEOMETRIC FUNCTION, CONFLUENT HYPERGEOMETRIC LIMIT FUNCTION, DIXON'S THEOREM, DOUGALL-RAMANUJAN IDENTITY, DOUGALL'S THEOREM, GOSPER'S ALGORITHM, HEINE HYPERGEOMETRIC SERIES, HYPERGEOMETRIC FUNCTION, HYPERGEOMETRIC IDENTITY, HYPERGEOMETRIC SERIES, JACKSON'S IDENTITY, KUMMER'S THEOREM, RAMANUJAN'S HYPERGEOMETRIC IDENTITY, SAALSCHÜTZ'S THEOREM, SAALSCHÜTZIAN, SISTER CELINE'S METHOD, THOMAE'S THEOREM, WATSON'S THEOREM, WHIPPLE'S TRANSFORMATION, WILF-ZEILBERGER PAIR, ZEILBERGER'S ALGORITHM

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Generalized Matrix Inverse

see MOORE-PENROSE GENERALIZED MATRIX INVERSE

Generalized Mean

A generalized version of the MEAN

$$m(t) \equiv \left(\frac{1}{n}\sum_{k=1}^{n} {a_k}^t\right)^{1/t} \tag{1}$$

with parameter t which gives the GEOMETRIC MEAN, ARITHMETIC MEAN, and HARMONIC MEAN as special cases:

$$\lim_{t\to 0} m(t) = G \tag{2}$$

$$m(1) = A \tag{3}$$

$$m(-1) = H. \tag{4}$$

see also MEAN

Generalized Remainder Method

An algorithm for computing a UNIT FRACTION.

Generating Function

A POWER SERIES

$$f(x) = \sum_{n=0}^{\infty} a_n x^n$$

whose COEFFICIENTS give the SEQUENCE $\{a_0, a_1, \ldots\}$. The *Mathematica*® (Wolfram Research, Champaign, IL) function `DiscreteMath'RSolve'PowerSum` gives the generating function of a given expression, and `ExponentialPowerSum` gives the exponential generating function.

Generating functions for the first few powers are

$$\begin{array}{lll} 1: & \frac{x}{1-x} & = x + x^2 + x^3 + \ldots \\ n: & \frac{x}{(1-x)^2} & = x + 2x^2 + 3x^3 + 4x^4 + \ldots \\ n^2: & \frac{x(x+1)}{(1-x)^3} & = x + 4x^2 + 9x^3 + 16x^4 + \ldots \\ n^3: & \frac{x(x^2+4x+1)}{(x-1)^4} & = x + 8x^2 + 27x^3 + \ldots \\ n^4: & \frac{x(x+1)(x^2+10x+1)}{(x-1)^5} & = x + 16x^2 + 81x^3 + \ldots . \end{array}$$

see also MOMENT-GENERATING FUNCTION, RECURRENCE RELATION

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Wilf, H. S. *Generatingfunctionology, 2nd ed.* New York: Academic Press, 1990.

Generation

In population studies, the direct offspring of a reference population (roughly) constitutes a single generation. For a CELLULAR AUTOMATON, the fundamental unit of time during which the rules of reproduction are applied once is called a generation.

Generator (Digitadition)

An INTEGER used to generate a DIGITADITION. A number can have more than one generator. If a number has no generator, it is called a SELF NUMBER.

Generator (Group)

An element of a CYCLIC GROUP, the POWERS of which generate the entire GROUP.

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Arfken, G. "Generators." §4.11 in *Mathematical Methods for Physicists, 3rd ed.* Orlando, FL: Academic Press, pp. 261–267, 1985.

Genetic Algorithm

An adaptive ALGORITHM involving search and optimization first used by John Holland. Holland created an electronic organism as a binary string ("chromosome"), and then used genetic and evolutionary principles of fitness-proportionate selection for reproduction (including random crossover and mutation) to search enormous solution spaces efficiently. So-called genetic programming languages apply the same principles, using an expression tree instead of a bit string as the "chromosome."

see also CELLULAR AUTOMATON

Genocchi Number

A number given by the GENERATING FUNCTION

$$\frac{2t}{e^t+1} = \sum_{n=1}^{\infty} G_n \frac{t^n}{n!}.$$

It satisfies $G_1 = 1$, $G_3 = G_5 = G_7 = \ldots$, and even coefficients are given by

$$\begin{aligned} G_{2n} &= 2(1-2^{2n})B_{2n} \\ &= 2nE_{2n-1}(0), \end{aligned}$$

where B_n is a BERNOULLI NUMBER and $E_n(x)$ is an EULER POLYNOMIAL. The first few Genocchi numbers for n EVEN are -1, 1, -3, 17, -155, 2073, ... (Sloane's A001469).

see also BERNOULLI NUMBER, EULER POLYNOMIAL

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Sloane, N. J. A. Sequence A001469/M3041 in "An On-Line Version of the Encyclopedia of Integer Sequences."

Gentle Diagonal

see PASCAL'S TRIANGLE

Gentle Giant Group

see MONSTER GROUP

Genus (Curve)

One of the PLÜCKER CHARACTERISTICS, defined by

$$p \equiv \tfrac{1}{2}(n-1)(n-2)-(\delta+\kappa) = \tfrac{1}{2}(m-1)(m-2)-(\tau+\iota),$$

where m is the class, n the order, δ the number of nodes, κ the number of CUSPS, ι the number of stationary tangents (INFLECTION POINTS), and τ the number of BITANGENTS.

see also RIEMANN CURVE THEOREM

References

Coolidge, J. L. *A Treatise on Algebraic Plane Curves.* New York: Dover, p. 100, 1959.

Genus (Knot)

The least genus of any SEIFERT SURFACE for a given KNOT. The UNKNOT is the only KNOT with genus 0.

Genus (Surface)

A topologically invariant property of a surface defined as the largest number of nonintersecting simple closed curves that can be drawn on the surface without separating it. Roughly speaking, it is the number of HOLES in a surface.

see also EULER CHARACTERISTIC

Genus Theorem

A DIOPHANTINE EQUATION

$$x^2+y^2=p$$

can be solved for p a PRIME IFF $p\equiv 1 \pmod 4$ or $p=2$. The representation is unique except for changes of sign or rearrangements of x and y.

see also COMPOSITION THEOREM, FERMAT'S THEOREM

Geocentric Latitude

An AUXILIARY LATITUDE given by

$$\phi_g = \tan^{-1}[(1-e^2)\tan\phi].$$

The series expansion is

$$\phi_g = \phi - e_2\sin(2\phi)+\tfrac{1}{2}{e_2}^2\sin(4\phi)+\tfrac{1}{3}{e_2}^3\sin(6\phi)+\ldots,$$

where

$$e_2 \equiv \frac{e^2}{2-e^2}.$$

see also LATITUDE

References

Adams, O. S. "Latitude Developments Connected with Geodesy and Cartography with Tables, Including a Table for Lambert Equal-Area Meridional Projections." Spec. Pub. No. 67. U. S. Coast and Geodetic Survey, 1921.

Snyder, J. P. *Map Projections—A Working Manual.* U. S. Geological Survey Professional Paper 1395. Washington, DC: U. S. Government Printing Office, pp. 17–18, 1987.

Geodesic

Given two points on a surface, the geodesic is defined as the shortest path on the surface connecting them. Geodesics have many interesting properties. The NORMAL VECTOR to any point of a GEODESIC arc lies along the normal to a surface at that point (Weinstock 1974, p. 65).

Furthermore, no matter how badly a SPHERE is distorted, there exist an infinite number of closed geodesics on it. This general result, demonstrated in the early 1990s, extended earlier work by Birkhoff, who proved in 1917 that there exists at least one closed geodesic on a distorted sphere, and Lyusternik and Schirelmann, who proved in 1923 that there exist at least three closed geodesics on such a sphere (Cipra 1993).

For a surface $g(x,y,z)=0$, the geodesic can be found by minimizing the ARC LENGTH

$$L \equiv \int ds = \int \sqrt{dx^2+dy^2+dz^2}. \tag{1}$$

But

$$dx = \frac{\partial x}{\partial u}du + \frac{\partial x}{\partial v}dv \tag{2}$$

$$dx^2 = \left(\frac{\partial x}{\partial u}\right)^2 du^2 + 2\frac{\partial x}{\partial u}\frac{\partial x}{\partial v}du\,dv + \left(\frac{\partial x}{\partial v}\right)^2 dv^2, \tag{3}$$

and similarly for dy^2 and dz^2. Plugging in,

$$\begin{aligned} L = \int \Bigg\{ &\left[\left(\frac{\partial x}{\partial u}\right)^2+\left(\frac{\partial y}{\partial u}\right)^2+\left(\frac{\partial z}{\partial u}\right)^2\right]du^2 \\ &+2\left[\frac{\partial x}{\partial u}\frac{\partial x}{\partial v}+\frac{\partial y}{\partial u}\frac{\partial y}{\partial v}+\frac{\partial z}{\partial u}\frac{\partial z}{\partial v}\right]du\,dv \\ &+\left[\left(\frac{\partial x}{\partial v}\right)^2+\left(\frac{\partial y}{\partial v}\right)^2+\left(\frac{\partial z}{\partial v}\right)^2\right]dv^2\Bigg\}^{1/2}. \end{aligned} \tag{4}$$

This can be rewritten as

$$L = \int \sqrt{P+2Qv'+Rv'^2}\,du \tag{5}$$

$$= \int \sqrt{Pu'^2+2Qu'+R}\,dv, \tag{6}$$

where

$$v' \equiv \frac{dv}{du} \tag{7}$$

$$u' \equiv \frac{du}{dv} \tag{8}$$

and

$$P \equiv \left(\frac{\partial x}{\partial u}\right)^2+\left(\frac{\partial y}{\partial u}\right)^2+\left(\frac{\partial z}{\partial u}\right)^2 \tag{9}$$

$$Q \equiv \frac{\partial x}{\partial u}\frac{\partial x}{\partial v}+\frac{\partial y}{\partial u}\frac{\partial y}{\partial v}+\frac{\partial z}{\partial u}\frac{\partial z}{\partial v} \tag{10}$$

$$R \equiv \left(\frac{\partial x}{\partial v}\right)^2+\left(\frac{\partial y}{\partial v}\right)^2+\left(\frac{\partial z}{\partial v}\right)^2. \tag{11}$$

Taking derivatives,

$$\frac{\partial L}{\partial v} = \tfrac{1}{2}(P + 2Qv' + Rv'^2)^{-1/2} \times \left(\frac{\partial P}{\partial v} + 2\frac{\partial Q}{\partial v}v' + \frac{\partial R}{\partial v}v'^2\right) \quad (12)$$

$$\frac{\partial L}{\partial v'} = \tfrac{1}{2}(P + 2Qv' + Rv'^2)^{-1/2}(2Q + 2Rv'), \quad (13)$$

so the EULER-LAGRANGE DIFFERENTIAL EQUATION then gives

$$\frac{\frac{\partial P}{\partial v} + 2v'\frac{\partial Q}{\partial v} + v'^2\frac{\partial R}{\partial v}}{2\sqrt{P + 2Qv' + Rv'^2}} - \frac{d}{du}\left(\frac{Q + Rv'}{\sqrt{P + 2Qv' + Rv'^2}}\right) = 0. \quad (14)$$

In the special case when P, Q, and R are explicit functions of u only,

$$\frac{Q + Rv'}{\sqrt{P + 2Qv' + Rv'^2}} = c_1 \quad (15)$$

$$\frac{Q^2 + 2QRv' + R^2v'^2}{P + 2Qv' + Rv'^2} = c_1{}^2 \quad (16)$$

$$v'^2R(R - c_1{}^2) + 2v'Q(R - c_1{}^2) + (Q^2 - Pc_1{}^2) = 0 \quad (17)$$

$$v' = \frac{1}{2R(R - c_1{}^2)}[2Q(c_1{}^2 - R) \pm\sqrt{4Q^2(R - c_1{}^2)^2 - 4R(R - c_1{}^2)(Q^2 - Pc_1{}^2)}]. \quad (18)$$

Now, if P and R are explicit functions of u only *and* $Q = 0$,

$$v' = \frac{\sqrt{4R(R - c_1{}^2)Pc_1{}^2}}{2R(R - c_1{}^2)} = c_1\sqrt{\frac{P}{R(R - c_1{}^2)}}, \quad (19)$$

so

$$v = c_1\int\sqrt{\frac{P}{R(R - c_1{}^2)}}\,du. \quad (20)$$

In the case $Q = 0$ where P and R are explicit functions of v only, then

$$\frac{\frac{\partial P}{\partial v} + v'^2\frac{\partial R}{\partial v}}{2\sqrt{P + Rv'^2}} - \frac{d}{du}\left(\frac{Rv'}{\sqrt{P + Rv'^2}}\right) = 0, \quad (21)$$

so

$$\frac{\partial P}{\partial v} + v'^2\frac{\partial R}{\partial v} - 2\sqrt{P + Rv'^2}R \times \left[\frac{v''}{\sqrt{P + Rv'^2}} + (-\tfrac{1}{2})\frac{v'(2Rv'v'')}{(P + Rv'^2)^{3/2}}\right] = 0 \quad (22)$$

$$\frac{\partial P}{\partial v} + v'^2\frac{\partial R}{\partial v} - 2Rv'' + \frac{2R^2v'^2v''}{P + Rv'^2} = 0 \quad (23)$$

$$\frac{Rv'^2}{\sqrt{P + Rv'^2}} - \sqrt{P + Rv'^2} = c_1 \quad (24)$$

$$Rv'^2 - (P + Rv'^2) = c_1\sqrt{P + Rv'^2} \quad (25)$$

$$\left(-\frac{P}{c_1}\right)^2 = P + Rv'^2 \quad (26)$$

$$\frac{P^2 - c_1{}^2P}{Rc_1{}^2} = v'^2, \quad (27)$$

and

$$u = c_1\int\sqrt{\frac{R}{P^2 - c_1{}^2P}}\,dv. \quad (28)$$

For a surface of revolution in which $y = g(x)$ is rotated about the x-axis so that the equation of the surface is

$$y^2 + z^2 = g^2(x), \quad (29)$$

the surface can be parameterized by

$$x = u \quad (30)$$

$$y = g(u)\cos v \quad (31)$$

$$z = g(u)\sin v. \quad (32)$$

The equation of the geodesics is then

$$v = c_1\int\frac{\sqrt{1 + [g'(u)]^2}\,du}{g(u)\sqrt{[g(u)]^2 - c_1{}^2}}. \quad (33)$$

see also ELLIPSOID GEODESIC, GEODESIC CURVATURE, GEODESIC DOME, GEODESIC EQUATION, GEODESIC TRIANGLE, GREAT CIRCLE, HARMONIC MAP, OBLATE SPHEROID GEODESIC, PARABOLOID GEODESIC

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Cipra, B. *What's Happening in the Mathematical Sciences, Vol. 1.* Providence, RI: Amer. Math. Soc., pp. 27, 1993.

Weinstock, R. *Calculus of Variations, with Applications to Physics and Engineering.* New York: Dover, pp. 26–28 and 45–46, 1974.

Geodesic Curvature

For a unit speed curve on a surface, the length of the surface-tangential component of acceleration is the geodesic curvature κ_g. Curves with $\kappa_g = 0$ are called GEODESICS. For a curve parameterized as $\boldsymbol{\alpha}(t) = \mathbf{x}(u(t), v(t))$, the geodesic curvature is given by

$$\kappa_g = \sqrt{EG - F^2}[-\Gamma_{11}^2u'^3 + \Gamma_{22}^1v'^3 - (2\Gamma_{12}^2 - \Gamma_{11}^1)u'^2v' + (2\Gamma_{12}^1 - \Gamma_{22}^2)u'v'^2 + u''v' - v''u'],$$

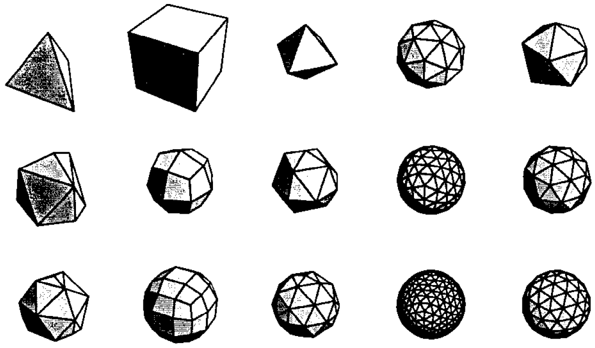
where E, F, and G are coefficients of the first FUNDAMENTAL FORM and Γ_{ij}^k are CHRISTOFFEL SYMBOLS OF THE SECOND KIND.

see also GEODESIC

References

Gray, A. "Geodesic Curvature." §20.5 in *Modern Differential Geometry of Curves and Surfaces.* Boca Raton, FL: CRC Press, pp. 402–407, 1993.

Geodesic Dome



A TRIANGULATION of a PLATONIC SOLID or other POLYHEDRON to produce a close approximation to a SPHERE. The nth order geodesation operation replaces each polygon of the polyhedron by the projection onto the CIRCUMSPHERE of the order n regular tessellation of that polygon. The above figure shows geodesations of orders 1 to 3 (from top to bottom) of the TETRAHEDRON, CUBE, OCTAHEDRON, DODECAHEDRON, and ICOSAHEDRON (from left to right).

R. Buckminster Fuller designed the first geodesic dome (i.e., geodesation of a HEMISPHERE). Fuller's dome was constructed from an ICOSAHEDRON by adding ISOSCELES TRIANGLES about each VERTEX and slightly repositioning the VERTICES. In such domes, neither the VERTICES nor the centers of faces necessarily lie at exactly the same distances from the center. However, these conditions are approximately satisfied.

In the geodesic domes discussed by Kniffen (1994), the sum of VERTEX angles is chosen to be a constant. Given a PLATONIC SOLID, let $e' \equiv 2e/v$ be the number of EDGES meeting at a VERTEX and n be the number of EDGES of the constituent POLYGON. Call the angle of the old VERTEX point A and the angle of the new VERTEX point F. Then

$$A = B \tag{1}$$

$$2e'A = nF \tag{2}$$

$$2A + F = 180^\circ. \tag{3}$$

Solving for A gives

$$2A + \frac{2e'}{n}A = 2A\left(1 + \frac{e'}{n}\right) = 180^\circ \tag{4}$$

$$A = 90^\circ \frac{n}{e' + n}, \tag{5}$$

and

$$F = \frac{2e'}{n}A = 180^\circ \frac{e'}{e' + n}. \tag{6}$$

The VERTEX sum is

$$\Sigma = nF = 180^\circ \frac{e'n}{e' + n}. \tag{7}$$

Solid	f	v	e'	n	A	F	Σ
tetrahedron			3	3	45°	90°	270°
cube	24	14	3	4	$51\frac{3}{7}^\circ$	$81\frac{3}{7}^\circ$	$308\frac{4}{7}^\circ$
octahedron			4	3	$38\frac{4}{7}^\circ$	$108\frac{4}{7}^\circ$	$308\frac{4}{7}^\circ$
dodecahedron	60	32	3	5	$56\frac{1}{4}^\circ$	$71\frac{1}{4}^\circ$	$337\frac{1}{2}^\circ$
icosahedron			5	3	$33\frac{3}{4}^\circ$	$118\frac{3}{4}^\circ$	$337\frac{1}{2}^\circ$

see also TRIANGULAR SYMMETRY GROUP

References

Kenner, H. *Geodesic Math and How to Use It.* Berkeley, CA: University of California Press, 1976.

Kniffen, D. "Geodesic Domes for Amateur Astronomers." *Sky and Telescope,* pp. 90–94, Oct. 1994.

Pappas, T. "Geodesic Dome of Leonardo da Vinci." *The Joy of Mathematics.* San Carlos, CA: Wide World Publ./Tetra, p. 81, 1989.

Geodesic Equation

$$d\tau^2 = -\eta_{\alpha\beta}\, d\xi^\alpha\, d\xi^\beta,$$

or

$$\frac{d^2\xi^\alpha}{d\tau^2} = 0.$$

see also GEODESIC

Geodesic Flow

A type of FLOW technically defined in terms of the TANGENT BUNDLE of a MANIFOLD.

see also DYNAMICAL SYSTEM

Geodesic Triangle

A TRIANGLE formed by the arcs of three GEODESICS on a smooth surface.

see also INTEGRAL CURVATURE

Geodetic Latitude

see LATITUDE

Geographic Latitude

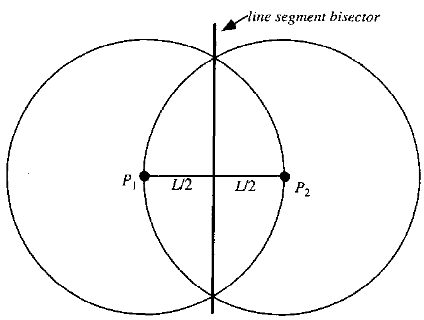
see LATITUDE

Geometric Construction

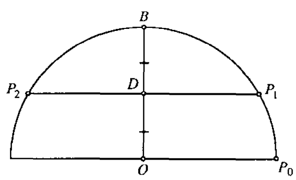
In antiquity, geometric constructions of figures and lengths were restricted to use of only a STRAIGHTEDGE and COMPASS. Although the term "RULER" is sometimes used instead of "STRAIGHTEDGE," no markings which could be used to make measurements were allowed according to the Greek prescription. Furthermore, the "COMPASS" could not even be used to mark off distances by setting it and then "walking" it along, so the COMPASS had to be considered to automatically collapse when not in the process of drawing a CIRCLE.

Because of the prominent place Greek geometric constructions held in Euclid's *Elements*, these constructions are sometimes also known as EUCLIDEAN CONSTRUCTIONS. Such constructions lay at the heart of the GEOMETRIC PROBLEMS OF ANTIQUITY of CIRCLE SQUARING, CUBE DUPLICATION, and TRISECTION of an ANGLE. The Greeks were unable to solve these problems, but it was not until hundreds of years later that the problems were proved to be actually impossible under the limitations imposed.

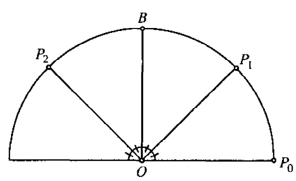
Simple algebraic operations such as $a+b$, $a-b$, ra (for r a RATIONAL NUMBER), a/b, ab, and $\sqrt{x}$ can be performed using geometric constructions (Courant and Robbins 1996). Other more complicated constructions, such as the solution of APOLLONIUS' PROBLEM and the construction of INVERSE POINTS can also accomplished.



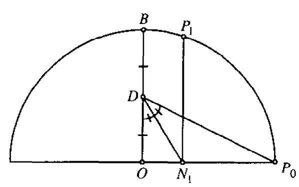
One of the simplest geometric constructions is the construction of a BISECTOR of a LINE SEGMENT, illustrated above.



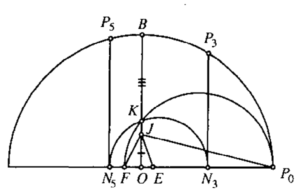
Equilateral Triangle



Square



Pentagon



17-gon

The Greeks were very adept at constructing POLYGONS, but it took the genius of Gauss to mathematically determine which constructions were possible and which were not. As a result, Gauss determined that a series of POLYGONS (the smallest of which has 17 sides; the HEPTADECAGON) had constructions unknown to the Greeks. Gauss showed that the CONSTRUCTIBLE POLYGONS (several of which are illustrated above) were closely related to numbers called the FERMAT PRIMES.

Wernick (1982) gave a list of 139 sets of three located points from which a TRIANGLE was to be constructed. Of Wernick's original list of 139 problems, 20 had not yet been solved as of 1996 (Meyers 1996).

It is possible to construct RATIONAL NUMBERS and EUCLIDEAN NUMBERS using a STRAIGHTEDGE and COMPASS construction. In general, the term for a number which can be constructed using a COMPASS and STRAIGHTEDGE is a CONSTRUCTIBLE NUMBER. Some IRRATIONAL NUMBERS, but *no* TRANSCENDENTAL NUMBERS, can be constructed.

It turns out that all constructions possible with a COMPASS and STRAIGHTEDGE can be done with a COMPASS alone, as long as a line is considered constructed when its two endpoints are located. The reverse is also true, since Jacob Steiner showed that all constructions possible with STRAIGHTEDGE and COMPASS can be done using only a straightedge, as long as a fixed CIRCLE and its center (or two intersecting CIRCLES without their centers, or three nonintersecting CIRCLES) have been drawn beforehand. Such a construction is known as a STEINER CONSTRUCTION.

GEOMETROGRAPHY is a quantitative measure of the simplicity of a geometric construction. It reduces geometric constructions to five types of operations, and seeks to reduce the total number of operations (called the "SIMPLICITY") needed to effect a geometric construction.

Dixon (1991, pp. 34–51) gives approximate constructions for some figures (the HEPTAGON and NONAGON) and lengths (PI) which cannot be rigorously constructed. Ramanujan (1913–14) and Olds (1963) give geometric constructions for $355/113 \approx \pi$. Gardner (1966, pp. 92–93) gives a geometric construction for

$$3+\tfrac{16}{113}=3.1415929\ldots\approx\pi.$$

Constructions for π are approximate (but inexact) forms of CIRCLE SQUARING.

see also CIRCLE SQUARING, COMPASS, CONSTRUCTIBLE NUMBER, CONSTRUCTIBLE POLYGON, CUBE DUPLICATION, ELEMENTS, FERMAT PRIME, GEOMETRIC PROBLEMS OF ANTIQUITY, GEOMETROGRAPHY, MASCHERONI CONSTRUCTION, NAPOLEON'S PROBLEM, NEUSIS CONSTRUCTION, PLANE GEOMETRY, POLYGON, PONCELET-STEINER THEOREM, RECTIFICATION, SIMPLICITY, STEINER CONSTRUCTION, STRAIGHTEDGE, TRISECTION

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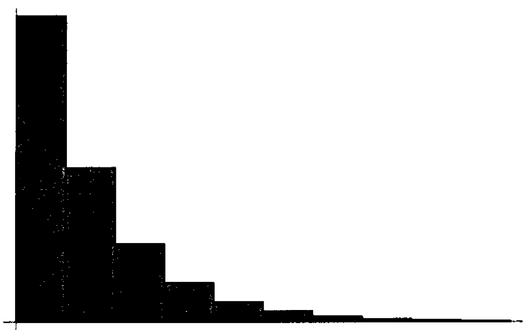
Geometric Distribution



A distribution such that

$$P(n) = q^{n-1}p = p(1-p)^{n-1}, \tag{1}$$

where $q \equiv 1-p$ and for $n = 1, 2, \ldots$. The distribution is normalized since

$$\sum_{n=1}^{\infty} P(n) = \sum_{n=1}^{\infty} q^{n-1}p = p\sum_{n=0}^{\infty} q^n = \frac{p}{1-q} = \frac{p}{p} = 1. \tag{2}$$

The MOMENT-GENERATING FUNCTION is

$$\phi(t) = p[1-(1-p)e^{it}]^{-1}, \tag{3}$$

or

$$\begin{aligned} M(t) = \langle e^{tn} \rangle &= \sum_{n=1}^{\infty} e^{tn}pq^{n-1} = p\sum_{n=0}^{\infty} e^{t(n+1)}q^n \\ &= pe^t \sum_{n=0}^{\infty} (e^t q)^n = \frac{pe^t}{1-e^t q} \end{aligned} \tag{4}$$

$$\begin{aligned} M'(t) &= p\left[\frac{(1-e^t q)e^t - e^t(-e^t q)}{(1-e^t q)^2}\right] \\ &= \frac{p(e^t - qe^{2t} + qe^{2t})}{(1-e^t q)^2} = \frac{pe^t}{(1-e^t q)^2} \end{aligned} \tag{5}$$

$$\begin{aligned} M''(t) &= p\frac{(1-e^t q)^2 e^t - e^t 2(1-e^t q)(-e^t q)}{(1-e^t q)^4} \\ &= p\frac{(1-2e^t q + e^{2t}q^2)e^t + 2qe^{2t}(1-e^t q)}{(1-e^t q)^4} \\ &= p\frac{e^t - 2e^{2t}q + e^{3t}q^2 + 2qe^{2t} - 2q^2e^{3t}}{(1-e^t q)^4} \\ &= p\frac{e^t - q^2e^{3t}}{(1-e^t q)^4} = \frac{pe^t(1-q^2e^{2t})}{(1-e^t q)^4} \\ &= \frac{pe^t(1+qe^t)}{(1-e^t q)^3} \end{aligned} \tag{6}$$

$$M'''(t) = \frac{pe^t[1+4e^t(1-p)+e^{2t}(1-p)^2]}{(1-e^t+e^t p)^4}. \tag{7}$$

Therefore,

$$M'(0) = \mu_1' = \mu = \frac{p}{(1-q)^2} = \frac{p}{p^2} = \frac{1}{p} \tag{8}$$

$$M''(0) = \mu_2' = \frac{p(1+q)}{(1+q)^3} = \frac{p(2-p)}{p^3} = \frac{2-p}{p^2} \tag{9}$$

$$M'''(0) = \mu_3' = \frac{(6-6p+p^2)}{p^3} \tag{10}$$

$$M^{(4)}(0) = \mu_4' = \frac{(p-2)(-p^2+12p-12)}{p^4}, \tag{11}$$

and

$$\begin{aligned} \mu_2 \equiv \mu_2' - (\mu_1')^2 &= \left(\frac{2}{p^2} - \frac{1}{p}\right) - \frac{1}{p^2} = \frac{1-p}{p^2} \\ &= \frac{q}{p^2} \end{aligned} \tag{12}$$

$$\begin{aligned} \mu_3 &\equiv \mu_3' - 3\mu_2'\mu_1' + 2(\mu_1')^3 \\ &= \frac{6-6p+p^2}{p^3} - 3\frac{2-p}{p^2}\frac{1}{p} + 2\left(\frac{1}{p}\right)^3 \\ &= \frac{6-6p+p^2-3(2-p)+2}{p^3} \\ &= \frac{(p-1)(p-2)}{p^3} \end{aligned} \tag{13}$$

$$\begin{aligned} \mu_4 &\equiv \mu_4' - 4\mu_3'\mu_1' + 6\mu_2'(\mu_1')^2 - 3(\mu_1')^4 \\ &= \frac{(p-2)(-p^2+12p-12)}{p^4} - 4\frac{6-6p+p^2}{p^3}\frac{1}{p} \\ &\quad + 6\frac{2-p}{p^2}\left(\frac{1}{p}\right)^2 - 3\left(\frac{1}{p}\right)^4 \\ &= \frac{(p-1)(-p^2+9p-9)}{p^4}, \end{aligned} \tag{14}$$

so the MEAN, VARIANCE, SKEWNESS, and KURTOSIS are given by

$$\mu \equiv \mu_1' = \frac{1}{p} \tag{15}$$

$$\sigma^2 = \mu_2 = \frac{q}{p^2} \tag{16}$$

$$\begin{aligned}\gamma_1 &= \frac{\mu_3}{\sigma^3} = \frac{(p-1)(p-2)}{p^3}\left(\frac{p^2}{1-p}\right)^{3/2} \\ &= \frac{(p-1)(p-2)}{(1-p)\sqrt{1-p}} = \frac{2-p}{\sqrt{1-p}} = \frac{2-p}{\sqrt{q}}\end{aligned} \tag{17}$$

$$\begin{aligned}\gamma_2 &= \frac{\mu_4}{\sigma^4} - 3 = \frac{(p-1)(-p^2+9p-9)}{p^4\frac{(1-p)^2}{p^4}} - 3 \\ &= \frac{-9+9p-p^2}{(p-1)} - 3 \\ &= \frac{p^2-6p+6}{1-p}.\end{aligned} \tag{18}$$

In fact, the moments of the distribution are given analytically in terms of the POLYLOGARITHM function,

$$\mu_k' \equiv \sum_{n=1}^{\infty} P(n)n^k = \sum_{n=1}^{\infty} p(1-p)^{n-1}n^k = \frac{p\,\mathrm{Li}_{-k}(1-p)}{1-p}. \tag{19}$$

For the case $p = 1/2$ (corresponding to the distribution of the number of COIN TOSSES needed to win in the SAINT PETERSBURG PARADOX) this formula immediately gives

$$\mu_1' = 2 \tag{20}$$

$$\mu_2' = 6 \tag{21}$$

$$\mu_3' = 26 \tag{22}$$

$$\mu_4' = 150, \tag{23}$$

so the MEAN, VARIANCE, SKEWNESS, and KURTOSIS in this case are

$$\mu = 2 \tag{24}$$

$$\sigma^2 = 2 \tag{25}$$

$$\gamma_1 = \tfrac{3}{2}\sqrt{2} \tag{26}$$

$$\gamma_2 = \tfrac{13}{2}. \tag{27}$$

The first CUMULANT of the geometric distribution is

$$\kappa_1 = \frac{1-p}{p}, \tag{28}$$

and subsequent CUMULANTS are given by the RECURRENCE RELATION

$$\kappa_{r+1} = (1-p)\frac{d\kappa_r}{dp}. \tag{29}$$

see also SAINT PETERSBURG PARADOX

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Geometric Mean

$$G \equiv \left(\prod_{i=1}^{n} a_i\right)^{1/n}.$$

Hoehn and Niven (1985) show that

$$G(a_1+c, a_2+c, \ldots, a_n+c) > c + G(a_1, a_2, \ldots, a_n)$$

for any POSITIVE constant c.

see also ARITHMETIC MEAN, ARITHMETIC-GEOMETRIC MEAN, CARLEMAN'S INEQUALITY, HARMONIC MEAN, MEAN, ROOT-MEAN-SQUARE

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Geometric Mean Index

The statistical INDEX

$$P_G \equiv \left[\prod\left(\frac{p_n}{p_0}\right)^{v_0}\right]^{1/\Sigma v_0},$$

where p_n is the price per unit in period n, q_n is the quantity produced in period n, and $v_n \equiv p_n q_n$ the value of the n units.

see also INDEX

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Geometric Probability Constants

see CUBE POINT PICKING, CUBE TRIANGLE PICKING

Geometric Problems of Antiquity

The Greek problems of antiquity were a set of geometric problems whose solution was sought using only COMPASS and STRAIGHTEDGE:

1. CIRCLE SQUARING.
2. CUBE DUPLICATION.
3. TRISECTION of an ANGLE.

Only in modern times, more than 2,000 years after they were formulated, were all three ancient problems proved insoluble using only COMPASS and STRAIGHTEDGE.

Another ancient geometric problem not proved impossible until 1997 is ALHAZEN'S BILLIARD PROBLEM. As Ogilvy (1990) points out, constructing the general REGULAR POLYHEDRON was really a "fourth" unsolved problem of antiquity.

see also ALHAZEN'S BILLIARD PROBLEM, CIRCLE SQUARING, COMPASS, CONSTRUCTIBLE NUMBER, CONSTRUCTIBLE POLYGON, CUBE DUPLICATION, GEOMETRIC CONSTRUCTION, REGULAR POLYHEDRON, STRAIGHTEDGE, TRISECTION

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Geometric Progression

see GEOMETRIC SEQUENCE

Geometric Sequence

A geometric sequence is a SEQUENCE $\{a_k\}$, $k = 1, 2, \ldots$, such that each term is given by a multiple r of the previous one. Another equivalent definition is that a sequence is geometric IFF it has a zero BIAS. If the multiplier is r, then the kth term is given by

$$a_k = ra_{k-1} = r^2 a_{k-2} = a_0 r^k.$$

Without loss of generality, take $a_0 = 1$, giving

$$a_k = r^k.$$

Geometric Series

A geometric series $\sum_k a_k$ is a series for which the ratio of each two consecutive terms a_{k+1}/a_k is a constant function of the summation index k, say r. Then the terms a_k are of the form $a_k = a_0 r^k$, so $a_{k+1}/a_k = r$. If $\{a_k\}$, with $k = 1, 2, \ldots$, is a GEOMETRIC SEQUENCE with multiplier $-1 < r < 1$ and $a_0 = 1$, then the geometric series

$$S_n = \sum_{k=0}^{n} a_k = \sum_{k=0}^{n} r^k \tag{1}$$

is given by

$$S_n \equiv \sum_{k=0}^{n} r^k = 1 + r + r^2 + \ldots + r^n, \tag{2}$$

so

$$rS_n = r + r^2 + r^3 + \ldots + r^{n+1}. \tag{3}$$

Subtracting

$$\begin{aligned}(1-r)S_n &= (1 + r + r^2 + \ldots + r^n) \\ &\quad - (r + r^2 + r^3 + \ldots + r^{n+1}) \\ &= 1 - r^{n+1},\end{aligned} \tag{4}$$

so

$$S_n \equiv \sum_{k=0}^{n} r^k = \frac{1 - r^{n+1}}{1 - r}. \tag{5}$$

As $n \to \infty$, then

$$S \equiv S_\infty = \sum_{k=0}^{\infty} r^k = \frac{1}{1-r}. \tag{6}$$

see also ARITHMETIC SERIES, GABRIEL'S STAIRCASE, HARMONIC SERIES, HYPERGEOMETRIC SERIES, WHEAT AND CHESSBOARD PROBLEM

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Geometrization Conjecture

see THURSTON'S GEOMETRIZATION CONJECTURE

Geometrography

A quantitative measure of the simplicity of a GEOMETRIC CONSTRUCTION which reduces geometric constructions to five steps. It was devised by È. Lemoine.

- S_1 Place a STRAIGHTEDGE's EDGE through a given POINT,
- S_2 Draw a straight LINE,
- C_1 Place a POINT of a COMPASS on a given POINT,
- C_2 Place a POINT of a COMPASS on an indeterminate POINT on a LINE,
- C_3 Draw a CIRCLE.

Geometrography seeks to reduce the number of operations (called the "SIMPLICITY") needed to effect a construction. If the number of the above operations are denoted m_1, m_2, n_1, n_2, and n_3, respectively, then the SIMPLICITY is $m_1 + m_2 + n_1 + n_2 + n_3$ and the symbol is $m_1 S_1 + m_2 S_2 + n_1 C_1 + n_2 C_2 + n_3 C_3$. It is apparently an unsolved problem to determine if a given GEOMETRIC CONSTRUCTION is of the smallest possible simplicity.

see also SIMPLICITY

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Geometry

Geometry is the study of figures in a SPACE of a given number of dimensions and of a given type. The most common types of geometry are PLANE GEOMETRY (dealing with objects like the LINE, CIRCLE, TRIANGLE, and POLYGON), SOLID GEOMETRY (dealing with objects like the LINE, SPHERE, and POLYHEDRON), and SPHERICAL GEOMETRY (dealing with objects like the SPHERICAL TRIANGLE and SPHERICAL POLYGON).

Historically, the study of geometry proceeds from a small number of accepted truths (AXIOMS or POSTULATES), then builds up true statements using a systematic and rigorous step-by-step PROOF. However, there is much more to geometry than this relatively dry textbook approach, as evidenced by some of the beautiful and unexpected results of PROJECTIVE GEOMETRY (not to mention Schubert's powerful but questionable ENUMERATIVE GEOMETRY).

Formally, a geometry is defined as a complete locally homogeneous RIEMANNIAN METRIC. In $\mathbb{R}^2$, the possible geometries are Euclidean planar, hyperbolic planar, and elliptic planar. In $\mathbb{R}^3$, the possible geometries include Euclidean, hyperbolic, and elliptic, but also include five other types.

see also ABSOLUTE GEOMETRY, AFFINE GEOMETRY, COORDINATE GEOMETRY, DIFFERENTIAL GEOMETRY, ENUMERATIVE GEOMETRY, FINSLER GEOMETRY, INVERSIVE GEOMETRY, MINKOWSKI GEOMETRY, NIL GEOMETRY, NON-EUCLIDEAN GEOMETRY, ORDERED GEOMETRY, PLANE GEOMETRY, PROJECTIVE GEOMETRY, SOL GEOMETRY, SOLID GEOMETRY, SPHERICAL GEOMETRY, THURSTON'S GEOMETRIZATION CONJECTURE

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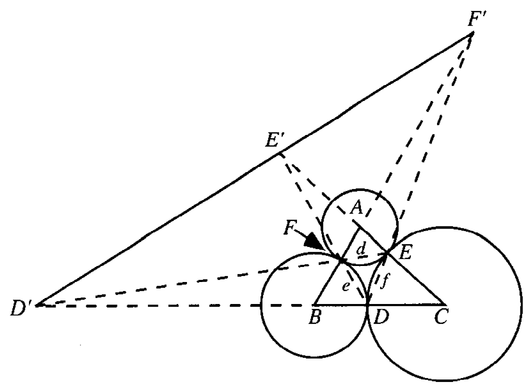
Gergonne Line



The perspective line for the CONTACT TRIANGLE ΔDEF and its TANGENTIAL TRIANGLE ΔABC. It is determined by the NOBBS POINTS D', E', and F'. In addition to the NOBBS POINTS, the FLETCHER POINT and EVANS POINT also lie on the Gergonne line where it intersects the SODDY LINE and EULER LINE, respectively. The D and D' coordinates are given by

$$D = B + \frac{f}{e}C$$
$$D' = B - \frac{f}{e}C,$$

so $BDCD'$ form a HARMONIC RANGE. The equation of the Gergonne line is

$$\frac{\alpha}{d} + \frac{\beta}{e} + \frac{\gamma}{f} = 0.$$

see also CONTACT TRIANGLE, EULER LINE, EVANS POINT, FLETCHER POINT, NOBBS POINTS, SODDY LINE, TANGENTIAL TRIANGLE

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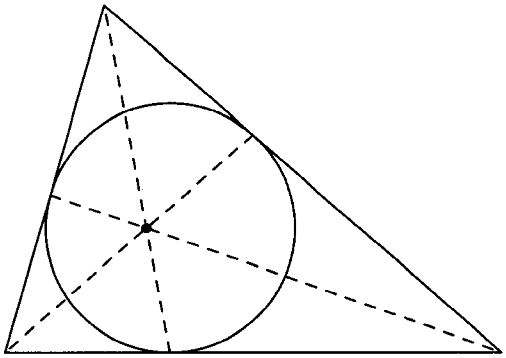
Gergonne Point



The common point of the CONCURRENT lines from the TANGENT points of a TRIANGLE'S INCIRCLE to the opposite VERTICES. It has TRIANGLE CENTER FUNCTION

$$\alpha = [a(b+c-a)]^{-1} = \tfrac{1}{2}\sec^2 A.$$

It is the ISOTOMIC CONJUGATE POINT of the NAGEL POINT. The CONTACT TRIANGLE and TANGENTIAL TRIANGLE are perspective from the Gergonne point.

see also GERGONNE LINE

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Germain Primes

see SOPHIE GERMAIN PRIME

Gerono Lemniscate

see EIGHT CURVE

Geršgorin Circle Theorem

Gives a region in the COMPLEX PLANE containing all the EIGENVALUES of a COMPLEX SQUARE MATRIX. Let

$$|x_k| = \max\{|x_i| : 1 \le i \le n\} > 0 \tag{1}$$

and define

$$R_i = \sum_{\substack{i=1 \\ j\ne i}}^{n} |a_{ij}|. \tag{2}$$

Then each EIGENVALUE of the MATRIX A of order n is in at least one of the disks

$$\{z : |z - a_{ii}| \le R_i\}. \tag{3}$$

The theorem can be made stronger as follows. Let r be an INTEGER with $1 \le r \le n$, then each EIGENVALUE of A is either in one of the disks Γ_1

$$\{z : |z - a_{jj}| \le S_j^{(r-1)}\}, \tag{4}$$

or in one of the regions

$$\left\{z : \sum_{i=1}^{r} |z - a_{ii}| \le \sum_{i=1}^{r} R_i\right\}, \tag{5}$$

where $S_j^{(r-1)}$ is the sum of magnitudes of the $r-1$ largest off-diagonal elements in column j.

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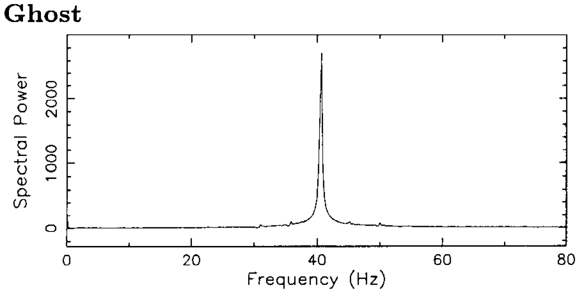
Ghost



If the sampling of an interferogram is modulated at a definite frequency instead of being uniformly sampled, spurious spectral features called "ghosts" are produced (Brault 1985). Periodic ruling or sampling errors introduce a modulation superposed on top of the expected fringe pattern due to uniform stage translation. Because modulation is a multiplicative process, spurious features are generated in spectral space at the sum and difference of the true fringe and ghost fringe frequencies, thus throwing power out of its spectral band.

Ghosts are copies of the actual spectrum, but appear at reduced strength. The above shows the power spectrum for a pure sinusoidal signal sampled by translating a Fourier transform spectrometer mirror at constant speed. The small blips on either side of the main peaks are ghosts.

In order for a ghost to appear, the process producing it must exist for most of the interferogram. However, if the ruling errors are not truly sinusoidal but vary across the length of the screw, a longer travel path can reduce their effect.

see also JITTER

References
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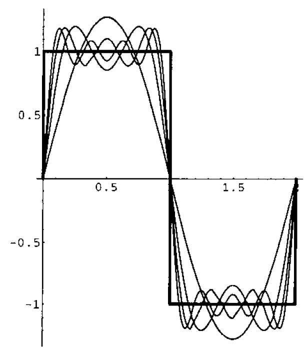
Gibbs Constant

see WILBRAHAM-GIBBS CONSTANT

Gibbs Effect

see GIBBS PHENOMENON

Gibbs Phenomenon



An overshoot of FOURIER SERIES and other EIGENFUNCTION series occurring at simple DISCONTINUITIES. It can be removed with the LANCZOS σ FACTOR.

see also FOURIER SERIES

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Gigantic Prime

A PRIME with 10,000 or more decimal digits. As of Nov. 15, 1995, 127 were known.

see also TITANIC PRIME

References
Caldwell, C. "The Ten Largest Known Primes." `http://www.utm.edu/research/primes/largest.html#largest`.

Gilbrat's Distribution

A CONTINUOUS DISTRIBUTION in which the LOGARITHM of a variable x has a NORMAL DISTRIBUTION,

$$P(x) = \frac{1}{\sqrt{2\pi}} e^{-(\ln x)^2/2}.$$

It is a special case of the LOG NORMAL DISTRIBUTION

$$P(x) = \frac{1}{S\sqrt{2\pi}} e^{-(\ln x - M)^2/2S^2}.$$

with $S = 1$ and $M = 0$.

see also LOG NORMAL DISTRIBUTION

Gilbreath's Conjecture

Let the DIFFERENCE of successive PRIMES be defined by $d_n \equiv p_{n+1} - p_n$, and d_n^k by

$$d_n^k \equiv \begin{cases} d_n & \text{for } k = 1 \\ |d_{n+1}^{k-1} - d_n^{k-1}| & \text{for } k > 1. \end{cases}$$

N. L. Gilbreath claimed that $d_1^k = 1$ for all k (Guy 1994). It has been verified for $k < 63419$ and all PRIMES up to $\pi(10^{13})$, where π is the PRIME COUNTING FUNCTION.

References
Guy, R. K. "Gilbreath's Conjecture." §A10 in *Unsolved Problems in Number Theory, 2nd ed.* New York: Springer-Verlag, pp. 25–26, 1994.

Gill's Method

A formula for numerical solution of differential equations,

$$\begin{aligned} y_{n+1} = y_n + \tfrac{1}{6}[k_1 + (2 - \sqrt{2})k_2 \\ + (2 + \sqrt{2})k_3 + k_4) + \mathcal{O}(h^5), \end{aligned}$$

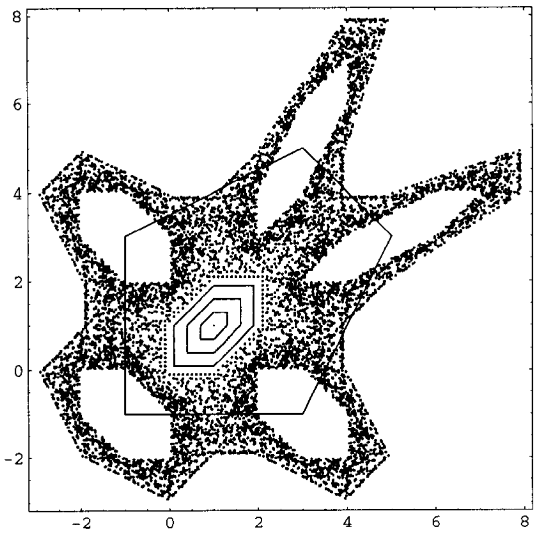
where

$$\begin{aligned} k_1 &= hf(x_n, y_n) \\ k_2 &= hf(x_n + \tfrac{1}{2}h, y_n + \tfrac{1}{2}k_1) \\ k_3 &= hf(x_n + \tfrac{1}{2}h, y_n + \tfrac{1}{2}(-1 + \sqrt{2})k_1 + (1 - \tfrac{1}{2}\sqrt{2})k_2) \\ k_4 &= hf(x_n + h, y_n - \tfrac{1}{2}\sqrt{2}\,k_2 + (1 + \tfrac{1}{2}\sqrt{2})k_3). \end{aligned}$$

see also ADAMS' METHOD, MILNE'S METHOD, PREDICTOR-CORRECTOR METHODS, RUNGE-KUTTA METHOD

References
Abramowitz, M. and Stegun, C. A. (Eds.). *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables, 9th printing.* New York: Dover, p. 896, 1972.

Gingerbreadman Map



A 2-D piecewise linear MAP defined by

$$x_{n+1} = 1 - y_n + |x_n|$$
$$y_{n+1} = x_n.$$

The map is chaotic in the filled region above and stable in the six hexagonal regions. Each point in the interior hexagon defined by the vertices (0, 0), (1, 0), (2, 1), (2, 2), (1, 2), and (0, 1) has an orbit with period six (except the point (1, 1), which has period 1). Orbits in the other five hexagonal regions circulate from one to the other. There is a unique orbit of period five, with all others having period 30. The points having orbits of period five are $(-1, 3)$, $(-1, -1)$, $(3, -1)$, $(5, 3)$, and $(3, 5)$, indicated in the above figure by the black line. However, there are infinitely many distinct periodic orbits which have an arbitrarily long period.

References

Devaney, R. L. "A Piecewise Linear Model for the Zones of Instability of an Area Preserving Map." *Physica D* **10**, 387–393, 1984.

Peitgen, H.-O. and Saupe, D. (Eds.). "A Chaotic Gingerbreadman." §3.2.3 in *The Science of Fractal Images*. New York: Springer-Verlag, pp. 149–150, 1988.

Girard's Spherical Excess Formula

Let a SPHERICAL TRIANGLE Δ have angles A, B, and C. Then the SPHERICAL EXCESS is given by

$$\Delta = A + B + C - \pi.$$

see also ANGULAR DEFECT, L'HUILIER'S THEOREM, SPHERICAL EXCESS

References

Coxeter, H. S. M. *Introduction to Geometry, 2nd ed.* New York: Wiley, pp. 94–95, 1969.

Girko's Circular Law

Let λ be EIGENVALUES of a set of RANDOM $n \times n$ MATRICES. Then $\lambda/\sqrt{n}$ is uniformly distributed on the DISK.

References

Girko, V. L. *Theory of Random Determinants.* Boston, MA: Kluwer, 1990.

Girth

The length of the shortest CYCLE in a GRAPH.

Girth	Example
3	tetrahedron
4	cube, $K_{3,3}$
5	Petersen graph

Giuga's Conjecture

If $n > 1$ and

$$n|1^{n-1} + 2^{n-1} + \ldots + (n-1)^{n-1} + 1,$$

is n necessarily a PRIME? In other words, defining

$$s_n \equiv \sum_{k=1}^{n-1} k^{n-1},$$

does there exist a COMPOSITE n such that $s_n \equiv -1 \pmod n$? It is known that $s_n \equiv -1 \pmod n$ IFF for each prime divisor p of n, $(p-1)|(n/p - 1)$ and $p|(n/p - 1)$ (Giuga 1950, Borwein *et al.* 1996); therefore, any counterexample must be SQUAREFREE. A composite INTEGER n satisfies $s_n \equiv -1 \pmod n$ IFF it is both a CARMICHAEL NUMBER and a GIUGA NUMBER. Giuga showed that there are no exceptions to the conjecture up to 10^{1000}. This was later improved to 10^{1700} (Bedocchi 1985) and 10^{13800} (Borwein *et al.* 1996).

see also ARGOH'S CONJECTURE

References

Bedocchi, E. "The $\mathbb{Z}(\sqrt{14})$ Ring and the Euclidean Algorithm." *Manuscripta Math.* **53**, 199–216, 1985.

Borwein, D.; Borwein, J. M.; Borwein, P. B.; and Girgensohn, R. "Giuga's Conjecture on Primality." *Amer. Math. Monthly* **103**, 40–50, 1996.

Giuga, G. "Su una presumibile propertietà caratteristica dei numeri primi." *Ist. Lombardo Sci. Lett. Rend. A* **83**, 511–528, 1950.

Ribenboim, P. *The Book of Prime Number Records, 2nd ed.* New York: Springer-Verlag, pp. 20–21, 1989.

Giuga Number

Any COMPOSITE NUMBER n with $p|(n/p - 1)$ for all PRIME DIVISORS p of n. n is a Giuga number IFF

$$\sum_{k=1}^{n-1} k^{\phi(n)} \equiv -1 \pmod n$$

where ϕ is the TOTIENT FUNCTION and IFF

$$\sum_{p|n} \frac{1}{p} - \prod_{p|n} \frac{1}{p} \in \mathbb{N}.$$

n is a Giuga number IFF

$$nB_{\phi(n)} \equiv -1 \pmod{n},$$

where B_k is a BERNOULLI NUMBER and ϕ is the TOTIENT FUNCTION. Every counterexample to Giuga's conjecture is a contradiction to ARGOH'S CONJECTURE and vice versa. The smallest known Giuga numbers are 30 (3 factors), 858, 1722 (4 factors), 66198 (5 factors), 2214408306, 24423128562 (6 factors), 432749205173838, 14737133470010574, 550843391309130318 (7 factors),

$$244197000982499715087866346,$$
$$554079914617070801288578559178$$

(8 factors), ... (Sloane's A007850).

It is not known if there are an infinite number of Giuga numbers. All the above numbers have sum minus product equal to 1, and any Giuga number of higher order must have at least 59 factors. The smallest ODD Giuga number must have at least nine PRIME factors.

see also ARGOH'S CONJECTURE, BERNOULLI NUMBER, TOTIENT FUNCTION

References

Borwein, D.; Borwein, J. M.; Borwein, P. B.; and Girgensohn, R. "Giuga's Conjecture on Primality." *Amer. Math. Monthly* **103**, 40–50, 1996.

Sloane, N. J. A. Sequence A007850 in "An On-Line Version of the Encyclopedia of Integer Sequences."

Giuga Sequence

A finite, increasing sequence of INTEGERS $\{n_1, \ldots, n_m\}$ such that

$$\sum_{i=1}^{m} \frac{1}{n_i} - \prod_{i=1}^{m} \frac{1}{n_i} \in \mathbb{N}.$$

A sequence is a Giuga sequence IFF it satisfies

$$n_i | (n_1 \cdots n_{i-1} \cdot n_{i+1} \cdot n_m - 1)$$

for $i = 1, \ldots, m$. There are no Giuga sequences of length 2, one of length 3 ({2, 3, 5}), two of length 4 ({2, 3, 7, 41} and {2, 3, 11, 13}), 3 of length 5 ({2, 3, 7, 43, 1805}, {2, 3, 7, 83, 85}, and {2, 3, 11, 17, 59}), 17 of length 6, 27 of length 7, and hundreds of length 8. There are infinitely many Giuga sequences. It is possible to generate longer Giuga sequences from shorter ones satisfying certain properties.

see also CARMICHAEL SEQUENCE

References

Borwein, D.; Borwein, J. M.; Borwein, P. B.; and Girgensohn, R. "Giuga's Conjecture on Primality." *Amer. Math. Monthly* **103**, 40–50, 1996.

Glaisher-Kinkelin Constant

N.B. A detailed on-line essay by S. Finch was the starting point for this entry.

Define

$$K(n+1) \equiv 0^0 1^1 2^2 3^3 \cdots n^n \tag{1}$$

$$G(n+1) \equiv \frac{(n!)^n}{K(n+1)} = \begin{cases} 1 & \text{if } n = 0 \\ 0!1!2! \cdots (n-1)! & \text{if } n > 0. \end{cases} \tag{2}$$

where G is the G-FUNCTION and K is the K-FUNCTION. Then

$$\lim_{n\to\infty} \frac{K(n+1)}{n^{n^2/2+n/2+1/2}e^{-n^2/4}} = A \tag{3}$$

$$\lim_{n\to\infty} \frac{G(n+1)}{n^{n^2/2-1/12}(2\pi)^{n/2}e^{-3n^2/4}} = \frac{e^{1/12}}{A}, \tag{4}$$

where

$$A = \exp\left[-\frac{\zeta'(2)}{2\pi^2} + \frac{\ln(2\pi)}{12} + \frac{\gamma}{2}\right] = 1.28242713\ldots, \tag{5}$$

where $\zeta(z)$ is the RIEMANN ZETA FUNCTION, π is PI, and γ is the EULER-MASCHERONI CONSTANT (Kinkelin 1860, Glaisher 1877, 1878, 1893, 1894). Glaisher (1877) also obtained

$$A = 2^{7/36}\pi^{-1/6} \exp\left\{\frac{1}{3} + \frac{2}{3}\int_0^{1/2} \ln[\Gamma(x+1)]\,dx\right\}. \tag{6}$$

Glaisher (1894) showed that

$$1^{1/1}2^{1/4}3^{1/9}4^{1/16}5^{1/25}\cdots = \left(\frac{A^{12}}{2\pi e^{\gamma}}\right)^{\pi^2/6} \tag{7}$$

$$1^{1/1}3^{1/9}5^{1/25}7^{1/49}9^{1/81}\cdots = \left(\frac{A^{12}}{2^{4/3}\pi e^{\gamma}}\right)^{\pi^2/8} \tag{8}$$

$$\frac{1^{1/1}5^{1/125}9^{1/729}\cdots}{3^{1/27}7^{1/343}11^{1/1331}\cdots} = \left(\frac{A}{2^{5/32}\pi^{1/32}e^{3/32+\gamma/48+s/4}}\right)^{\pi^3}, \tag{9}$$

where

$$s \equiv \frac{\zeta(3)}{3\cdot4\cdot5}\frac{1}{4^3} + \frac{\zeta(5)}{5\cdot6\cdot7}\frac{1}{4^5} + \frac{\zeta(7)}{7\cdot8\cdot9}\frac{1}{4^7} + \ldots. \tag{10}$$

see also G-FUNCTION, HYPERFACTORIAL, K-FUNCTION

References

Finch, S. "Favorite Mathematical Constants." `http://www.mathsoft.com/asolve/constant/glshkn/glshkn.html`.

Glaisher, J. W. L. "On a Numerical Continued Product." *Messenger Math.* **6**, 71–76, 1877.

Glaisher, J. W. L. "On the Product $1^1 2^2 3^3 \cdots n^n$." *Messenger Math.* **7**, 43–47, 1878.

Glaisher, J. W. L. "On Certain Numerical Products." *Messenger Math.* **23**, 145–175, 1893.

Glaisher, J. W. L. "On the Constant which Occurs in the Formula for $1^1 2^2 3^3 \cdots n^n$." *Messenger Math.* **24**, 1–16, 1894.

Kinkelin. "Über eine mit der Gammafunktion verwandte Transcendente und deren Anwendung auf die Integralrechnung." *J. Reine Angew. Math.* **57**, 122–158, 1860.

Glide

A product of a REFLECTION in a line and TRANSLATION along the same line.

see also REFLECTION, TRANSLATION

Glissette

The locus of a point P (or the envelope of a line) fixed in relation to a curve C which slides between fixed curves. For example, if C is a line segment and P a point on the line segment, then P describes an ELLIPSE when C slides so as to touch two ORTHOGONAL straight LINES. The glissette of the LINE SEGMENT C itself is, in this case, an ASTROID.

see also ROULETTE

References

Besant, W. H. *Notes on Roulettes and Glissettes, 2nd enl. ed.* Cambridge, England: Deighton, Bell & Co., 1890.

Lockwood, E. H. "Glissettes." Ch. 20 in *A Book of Curves.* Cambridge, England: Cambridge University Press, pp. 160–165, 1967.

Yates, R. C. "Glissettes." *A Handbook on Curves and Their Properties.* Ann Arbor, MI: J. W. Edwards, pp. 108–112, 1952.

Global $C(G;T)$ Theorem

If a SYLOW 2-SUBGROUP T of G lies in a unique maximal 2-local P of G, then P is a "strongly embedded" SUBGROUP of G, and G is known.

Global Extremum

A GLOBAL MINIMUM or GLOBAL MAXIMUM.

see also LOCAL EXTREMUM

Global Maximum

The largest overall value of a set, function, etc., over its entire range.

see also GLOBAL MINIMUM, LOCAL MAXIMUM, MAXIMUM

Global Minimum

The smallest overall value of a set, function, etc., over its entire range.

see also GLOBAL MAXIMUM, KUHN-TUCKER THEOREM, LOCAL MINIMUM, MINIMUM

Globe

A SPHERE which acts as a model of a spherical (or ellipsoidal) celestial body, especially the Earth, and on which the outlines of continents, oceans, etc. are drawn.

see also LATITUDE, LONGITUDE, SPHERE

Glove Problem

Let there be m doctors and $n \leq m$ patients, and let all mn possible combinations of examinations of patients by doctors take place. Then what is the minimum number of surgical gloves needed $G(m,n)$ so that no doctor must wear a glove contaminated by a patient and no patient is exposed to a glove worn by another doctor? In this problem, the gloves can be turned inside out and even placed on top of one another if necessary, but no "decontamination" of gloves is permitted. The optimal solution is

$$g(m,n) = \begin{cases} 2 & m = n = 2 \\ \frac{1}{2}(m+1) & n = 1,\ m = 2k+1 \\ \left\lceil \frac{1}{2}(m) + \frac{2}{3}n \right\rceil & \text{otherwise,} \end{cases}$$

where $\lceil x \rceil$ is the CEILING FUNCTION (Vardi 1991). The case $m = n = 2$ is straightforward since two gloves have a total of four surfaces, which is the number needed for $mn = 4$ examinations.

References

Gardner, M. *Aha! Aha! Insight.* New York: Scientific American, 1978.

Gardner, M. *Science Fiction Puzzle Tales.* New York: Crown, pp. 5, 67, and 104–150, 1981.

Hajnal, A. and Lovász, L. "An Algorithm to Prevent the Propagation of Certain Diseases at Minimum Cost." §10.1 in *Interfaces Between Computer Science and Operations Research* (Ed. J. K. Lenstra, A. H. G. Rinnooy Kan, and P. van Emde Boas). Amsterdam: Matematisch Centrum, 1978.

Orlitzky, A. and Shepp, L. "On Curbing Virus Propagation." Exercise 10.2 in Technical Memo. Bell Labs, 1989.

Vardi, I. "The Condom Problem." Ch. 10 in *Computational Recreations in Mathematica.* Redwood City, CA: Addison-Wesley, p. 203–222, 1991.

Glue Vector

A VECTOR specifying how layers are stacked in a LAMINATED LATTICE.

Gnomic Number

A FIGURATE NUMBER of the form $g_n = 2n - 1$ which are the areas of square gnomons, obtained by removing a SQUARE of side $n-1$ from a SQUARE of side n,

$$g_n = n^2 - (n-1)^2 = 2n - 1.$$

The gnomic numbers are therefore equivalent to the ODD NUMBERS, and the first few are 1, 3, 5, 7, 9, 11, ... (Sloane's A005408). The GENERATING FUNCTION for the gnomic numbers is

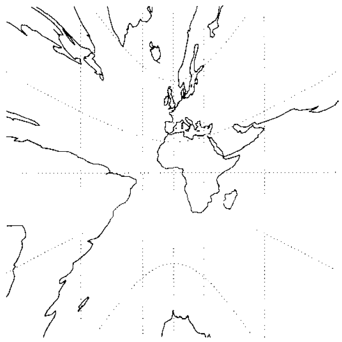
$$\frac{x(1+x)}{(x-1)^2} = x + 3x^2 + 5x^3 + 7x^4 + \ldots.$$

see also ODD NUMBER

References

Sloane, N. J. A. Sequence A005408/M2400 in "An On-Line Version of the Encyclopedia of Integer Sequences."

Gnomic Projection



A nonconformal projection from a SPHERE's center in which ORTHODROMES are straight LINES.

$$x = \frac{\cos\phi \sin(\lambda - \lambda_0)}{\cos c} \quad (1)$$

$$y = \frac{\cos\phi_1 \sin\phi - \sin\phi_1 \cos\phi \cos(\lambda - \lambda_0)}{\cos c}, \quad (2)$$

where

$$\cos c = \sin\phi_1 \sin\phi + \cos\phi_1 \cos\phi \cos(\lambda - \lambda_0). \quad (3)$$

The inverse FORMULAS are

$$\phi = \sin^{-1}(\cos c \sin\phi_1 + y \sin c \cos c \cos\phi_1) \quad (4)$$

$$\lambda = \lambda_0 + \tan^{-1}\left(\frac{x}{\cos\phi_1 - y \sin\phi_1}\right). \quad (5)$$

References
Coxeter, H. S. M. and Greitzer, S. L. *Geometry Revisited.* Washington, DC: Math. Assoc. Amer., pp. 150–153, 1967.
Snyder, J. P. *Map Projections—A Working Manual.* U. S. Geological Survey Professional Paper 1395. Washington, DC: U. S. Government Printing Office, pp. 164–168, 1987.

Gnomon

A shape which, when added to a figure, yields another figure SIMILAR to the original.

References
Shanks, D. *Solved and Unsolved Problems in Number Theory, 4th ed.* New York: Chelsea, p. 123, 1993.

Gnomon Magic Square

A 3×3 array of numbers in which the elements in each 2×2 corner have the same sum.

see also MAGIC SQUARE

Go

There are estimated to be about 4.63×10^{170} possible positions on a 19×19 board (Flammenkamp). The number of n-move Go games are 1, 362, 130683, 47046242, ... (Sloane's A007565).

References
Beeler, M.; Gosper, R. W.; and Schroeppel, R. *HAKMEM.* Cambridge, MA: MIT Artificial Intelligence Laboratory, Memo AIM-239, Item 96, Feb. 1972.
Flammenkamp, A. "A Short, Concise Ruleset of Go." `http://www.minet.uni-jena.de/~achim/gorules.html`.
Kraitchik, M. "Go." §12.4 in *Mathematical Recreations.* New York: W. W. Norton, pp. 279–280, 1942.
Sloane, N. J. A. Sequence A007565/M5447 in "An On-Line Version of the Encyclopedia of Integer Sequences."

Göbel's Sequence

Consider the RECURRENCE RELATION

$$x_n = \frac{1 + {x_0}^2 + {x_1}^2 + \ldots + {x_{n-1}}^2}{n}, \quad (1)$$

with $x_0 = 1$. The first few iterates of x_n are 1, 2, 3, 5, 10, 28, 154, ... (Sloane's A003504). The terms grow extremely rapidly, but are given by the asymptotic formula

$$x_n \approx (n^2 + 2n - 1 + 4n^{-1} - 21n^{-2} + 137n^{-3} - \ldots)C^{2^n}, \quad (2)$$

where

$$C = 1.04783144757641122955990946274313755459\ldots \quad (3)$$

(Zagier). It is more convenient to work with the transformed sequence

$$s_n = 2 + {x_1}^2 + {x_2}^2 + \ldots + {x_{n-1}}^2 = nx_n, \quad (4)$$

which gives the new recurrence

$$s_{n+1} = s_n + \frac{{s_n}^2}{n^2} \quad (5)$$

with initial condition $s_1 = 2$. Now, s_{n+1} will be nonintegral IFF $n \nmid s_n$. The smallest p for which $s_p \not\equiv 0 \pmod{p}$ therefore gives the smallest nonintegral s_{p+1}. In addition, since $p \nmid s_p$, $x_p = s_p/p$ is also the smallest nonintegral x_p.

For example, we have the sequences $\{s_n \pmod{k}\}_{n=1}^{k}$:

$$2, 6 \equiv 2, \tfrac{5}{4} \equiv 0, 0, 0 \pmod{5} \quad (6)$$

$$2, 6, 15 \equiv 1, \tfrac{5}{4} \equiv 0, 0, 0, 0 \pmod{7} \quad (7)$$

$$2, 6, 15 \equiv 4, \tfrac{52}{9} \equiv 7, \tfrac{161}{16} \equiv 8, \tfrac{264}{5} \equiv 0, 0, \ldots, 0 \pmod{11}. \quad (8)$$

Testing values of k shows that the first nonintegral x_n is x_{43}. Note that a direct verification of this fact is impossible since

$$x_{43} \approx 5.4093 \times 10^{178485291567} \quad (9)$$

(calculated using the asymptotic formula) is much too large to be computed and stored explicitly.

A sequence even more striking for remaining integral over many terms is the 3-Göbel sequence

$$x_n = \frac{1 + {x_0}^3 + {x_1}^3 + \ldots + {x_{n-1}}^3}{n}. \tag{10}$$

The first few terms of this sequence are 1, 2, 5, 45, 22815, ... (Sloane's A005166).

The Göbel sequences can be generalized to k powers by

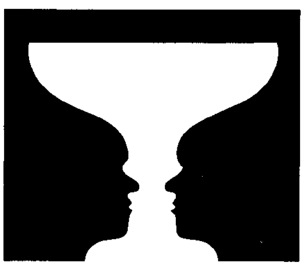
$$x_n = \frac{1 + {x_0}^k + {x_1}^k + \ldots + {x_{n-1}}^k}{n}. \tag{11}$$

see also SOMOS SEQUENCE

References

Guy, R. K. "The Strong Law of Small Numbers." *Amer. Math. Monthly* **95**, 697–712, 1988.
Guy, R. K. "A Recursion of Göbel." §E15 in *Unsolved Problems in Number Theory, 2nd ed.* New York: Springer-Verlag, pp. 214–215, 1994.
Sloane, N. J. A. Sequences A003504/M0728 and A005166/M1551 in "An On-Line Version of the Encyclopedia of Integer Sequences."
Zaiger, D. "Solution: Day 5, Problem 3." `http://www-groups.dcs.st-and.ac.uk/~john/Zagier/Solution5.3.html`.

Goblet Illusion



An ILLUSION in which the eye alternately sees two black faces, or a white goblet.

References

Fineman, M. *The Nature of Visual Illusion.* New York: Dover, pp. 111 and 115, 1996.
Rubin, E. *Synoplevede Figurer.* Copenhagen, Denmark: Gyldendalske, 1915.
What's Up with Kids Magazine. "Reversible Goblet." `http://wuwk.spurtek.com/COI_reversible_goblet.htm`.

Gödel's Completeness Theorem

If T is a set of AXIOMS in a first-order language, and a statement p holds for any structure M satisfying T, then p can be formally deduced from T in some appropriately defined fashion.

see also GÖDEL'S INCOMPLETENESS THEOREM

Gödel's Incompleteness Theorem

Informally, Gödel's incompleteness theorem states that all consistent axiomatic formulations of NUMBER THEORY include undecidable propositions (Hofstadter 1989). This is is sometimes called Gödel's first incompleteness theorem, and answers in the negative HILBERT'S PROBLEM asking whether mathematics is "complete" (in the sense that every statement in the language of NUMBER THEORY can be either proved or disproved). Formally, Gödel's theorem states, "To every ω-consistent recursive class κ of FORMULAS, there correspond recursive class-signs r such that neither (v Gen r) nor Neg(v Gen r) belongs to Flg(κ), where v is the FREE VARIABLE of r" (Gödel 1931).

A statement sometimes known as Gödel's second incompleteness theorem states that if NUMBER THEORY is consistent, then a proof of this fact does not exist using the methods of first-order PREDICATE CALCULUS. Stated more colloquially, any formal system that is interesting enough to formulate its own consistency can prove its own consistency IFF it is inconsistent.

Gerhard Gentzen showed that the consistency and completeness of arithmetic can be proved if "transfinite" induction is used. However, this approach does not allow proof of the consistency of all mathematics.

see also GÖDEL'S COMPLETENESS THEOREM, HILBERT'S PROBLEMS, KREISEL CONJECTURE, NATURAL INDEPENDENCE PHENOMENON, NUMBER THEORY, RICHARDSON'S THEOREM, UNDECIDABLE

References

Barrow, J. D. *Pi in the Sky: Counting, Thinking, and Being.* Oxford, England: Clarendon Press, p. 121, 1992.
Gödel, K. "Über Formal Unentscheidbare Sätze der *Principia Mathematica* und Verwandter Systeme, I." *Monatshefte für Math. u. Physik* **38**, 173–198, 1931.
Gödel, K. *On Formally Undecidable Propositions of Principia Mathematica and Related Systems.* New York: Dover, 1992.
Hofstadter, D. R. *Gödel, Escher, Bach: An Eternal Golden Braid.* New York: Vintage Books, p. 17, 1989.
Kolata, G. "Does Gödel's Theorem Matter to Mathematics?" *Science* **218**, 779–780, 1982.
Smullyan, R. M. *Gödel's Incompleteness Theorems.* New York: Oxford University Press, 1992.
Whitehead, A. N. and Russell, B. *Principia Mathematica.* New York: Cambridge University Press, 1927.

Gödel Number

A Gödel number is a unique number associated to a statement about arithmetic. It is formed as the PRODUCT of successive PRIMES raised to the POWER of the number corresponding to the individual symbols that comprise the sentence. For example, the statement $(\exists x)(x = sy)$ that reads "there EXISTS an x such that x is the immediate successor of y" is coded

$$(2^8)(3^4)(5^{13})(7^9)(11^8)(13^{13})(17^5)(19^7)(23^{16})(29^9),$$

where the numbers in the set (8, 4, 13, 9, 8, 13, 5, 7, 16, 9) correspond to the symbols that make up $(\exists x)(x = sy)$.

see also GÖDEL'S INCOMPLETENESS THEOREM

References

Hofstadter, D. R. *Gödel, Escher, Bach: An Eternal Golden Braid.* New York: Vintage Books, p. 18, 1989.

Goldbach Conjecture

Goldbach's original conjecture, written in a 1742 letter to Euler, states that every INTEGER > 5 is the SUM of three PRIMES. As re-expressed by Euler, an equivalent of this CONJECTURE (called the "strong" Goldbach conjecture) asserts that all POSITIVE EVEN INTEGERS ≥ 4 can be expressed as the SUM of two PRIMES. Schnirelmann (1931) proved that every EVEN number can be written as the sum of not more than 300,000 PRIMES (Dunham 1990), which seems a rather far cry from a proof for *four* PRIMES! The strong Goldbach conjecture has been shown to be true up to 4×10^{11} by Sinisalo (1993). Pogorzelski (1977) claimed to have proven the Goldbach conjecture, but his proof is not generally accepted (Shanks 1993).

The conjecture that all ODD numbers ≥ 9 are the SUM of three ODD PRIMES is called the "weak" Goldbach conjecture. Vinogradov proved that all ODD INTEGERS starting at some sufficiently large value are the SUM of three PRIMES (Guy 1994). The original "sufficiently large" $N \geq 3^{3^{15}} = e^{e^{16.573}}$ was subsequently reduced to $e^{e^{11.503}}$ by Chen and Wang (1989). Chen (1973, 1978) also showed that all sufficiently large EVEN NUMBERS are the sum of a PRIME and the PRODUCT of at most two PRIMES (Guy 1994, Courant and Robbins 1996).

It has been shown that if the weak Goldbach conjecture is false, then there are only a FINITE number of exceptions.

Other variants of the Goldbach conjecture include the statements that every EVEN number ≥ 6 is the SUM of two ODD PRIMES, and every INTEGER > 17 the sum of exactly three distinct PRIMES. Let $R(n)$ be the number of representations of an EVEN INTEGER n as the sum of two PRIMES. Then the "extended" Goldbach conjecture states that

$$R(n) \sim 2\Pi_2 \prod_{\substack{k=2 \\ p_k|n}} \frac{p_k - 1}{p_k - 2} \int_2^x \frac{dx}{(\ln x)^2},$$

where Π_2 is the TWIN PRIMES CONSTANT (Halberstam and Richert 1974).

If the Goldbach conjecture is true, then for every number m, there are PRIMES p and q such that

$$\phi(p) + \phi(q) = 2m,$$

where $\phi(x)$ is the TOTIENT FUNCTION (Guy 1994, p. 105).

Vinogradov (1937ab, 1954) proved that every sufficiently large ODD NUMBER is the sum of three PRIMES, and Estermann (1938) proves that almost all EVEN NUMBERS are the sums of two PRIMES.

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Golden Mean

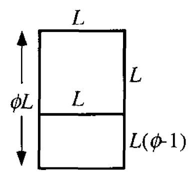
see GOLDEN RATIO

Golden Ratio

A number often encountered when taking the ratios of distances in simple geometric figures such as the DECAGON and DODECAGON. It is denoted ϕ, or sometimes τ (which is an abbreviation of the Greek "tome," meaning "to cut"). ϕ is also known as the DIVINE PROPORTION, GOLDEN MEAN, and GOLDEN SECTION and is a PISOT-VIJAYARAGHAVAN CONSTANT. It has surprising connections with CONTINUED FRACTIONS and the

EUCLIDEAN ALGORITHM for computing the GREATEST COMMON DIVISOR of two INTEGERS.

Given a RECTANGLE having sides in the ratio $1:\phi$, ϕ is defined such that partitioning the original RECTANGLE into a SQUARE and new RECTANGLE results in a new RECTANGLE having sides with a ratio $1:\phi$. Such a RECTANGLE is called a GOLDEN RECTANGLE, and successive points dividing a GOLDEN RECTANGLE into SQUARES lie on a LOGARITHMIC SPIRAL. This figure is known as a WHIRLING SQUARE.



This means that

$$\frac{1}{\phi-1}=\phi \tag{1}$$

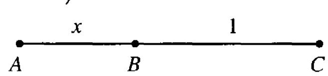
$$\phi^2-\phi-1=0. \tag{2}$$

So, by the QUADRATIC EQUATION,

$$\phi=\tfrac{1}{2}(1\pm\sqrt{1+4})=\tfrac{1}{2}(1+\sqrt{5}) \tag{3}$$

$$=1.6180339887498948482045868343656381177 20\ldots \tag{4}$$

(Sloane's A001622).



A geometric definition can be given in terms of the above figure. Let the ratio $x\equiv AB/BC$. The NUMERATOR and DENOMINATOR can then be taken as $\overline{AB}=x$ and $\overline{BC}=1$ without loss of generality. Now define the position of B by

$$\frac{BC}{AB}=\frac{AB}{AC}. \tag{5}$$

Plugging in gives

$$\frac{1}{x}=\frac{x}{1+x}, \tag{6}$$

or

$$x^2-x-1=0, \tag{7}$$

which can be solved using the QUADRATIC EQUATION to obtain

$$\phi\equiv x=\frac{1\pm\sqrt{1^2-(-4)}}{2}=\tfrac{1}{2}(1+\sqrt{5}). \tag{8}$$

ϕ is the "most" IRRATIONAL number because it has a CONTINUED FRACTION representation

$$\phi=[1,1,1,\ldots] \tag{9}$$

(Sloane's A000012). Another infinite representation in terms of a CONTINUED SQUARE ROOT is

$$\phi=\sqrt{1+\sqrt{1+\sqrt{1+\sqrt{1+\ldots}}}}. \tag{10}$$

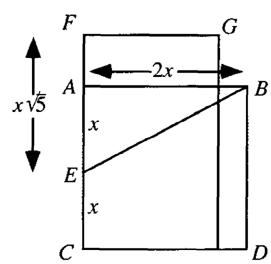
Ramanujan gave the curious CONTINUED FRACTION identities

$$\frac{1}{(\sqrt{\phi\sqrt{5}}\,)e^{2\pi/5}}=1+\cfrac{e^{-2\pi}}{1+\cfrac{e^{-4\pi}}{1+\cfrac{e^{-6\pi}}{1+\cfrac{e^{-8\pi}}{1+\cfrac{e^{-10\pi}}{1+\ldots}}}}} \tag{11}$$

$$\frac{1}{\left\{\frac{\sqrt{5}}{1+\left[5^{3/4}(\phi-1)^{5/2}-1\right]-\phi}\right\}e^{2\pi/\sqrt{5}}}=1+\cfrac{e^{-2\pi\sqrt{5}}}{1+\cfrac{e^{-4\pi\sqrt{5}}}{1+\cfrac{e^{-6\pi\sqrt{5}}}{1+\cfrac{e^{-8\pi\sqrt{5}}}{1+\cfrac{e^{-10\pi\sqrt{5}}}{1+\ldots}}}}} \tag{12}$$

(Ramanathan 1984).

The legs of a GOLDEN TRIANGLE are in a golden ratio to its base. In fact, this was the method used by Pythagoras to construct ϕ. Euclid used the following construction.



Draw the SQUARE $\square ABDC$, call E the MIDPOINT of AC, so that $AE=EC\equiv x$. Now draw the segment BE, which has length

$$x\sqrt{2^2+1^2}=x\sqrt{5}, \tag{13}$$

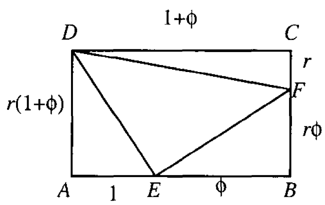
and construct EF with this length. Now construct $FG=EF$, then

$$\phi=\frac{FC}{CD}=\frac{EF+CE}{CD}=\frac{x(\sqrt{5}+1)}{2x}=\tfrac{1}{2}(\sqrt{5}+1). \tag{14}$$

The ratio of the CIRCUMRADIUS to the length of the side of a DECAGON is also ϕ,

$$\frac{R}{s} = \tfrac{1}{2}\csc\left(\frac{\pi}{10}\right) = \tfrac{1}{2}(1+\sqrt{5}) = \phi. \tag{15}$$

Similarly, the legs of a GOLDEN TRIANGLE (an ISOSCELES TRIANGLE with a VERTEX ANGLE of 36°) are in a GOLDEN RATIO to the base. Bisecting a GAULLIST CROSS also gives a golden ratio (Gardner 1961, p. 102).



In the figure above, three TRIANGLES can be INSCRIBED in the RECTANGLE $\square ABCD$ of arbitrary aspect ratio $1:r$ such that the three RIGHT TRIANGLES have equal areas by dividing AB and BC in the golden ratio. Then

$$K_{\Delta ADE} = \tfrac{1}{2}\cdot r(1+\phi)\cdot 1 = \tfrac{1}{2}r\phi^2 \tag{16}$$

$$K_{\Delta BEF} = \tfrac{1}{2}\cdot r\phi\cdot\phi = \tfrac{1}{2}r\phi^2 \tag{17}$$

$$K_{\Delta CDF} = \tfrac{1}{2}(1+\phi)\cdot r = \tfrac{1}{2}r\phi^2, \tag{18}$$

which are all equal.

The golden ratio also satisfies the RECURRENCE RELATION

$$\phi^{n+1} = \phi^{n-1} + \phi^n. \tag{19}$$

Taking $n = 0$ gives

$$\phi = \phi^{-1} + 1 \tag{20}$$

$$\phi^2 = 1 + \phi. \tag{21}$$

But this is the definition equation for ϕ (when the root with the plus sign is used). Squaring gives

$$\phi^2 = \tfrac{1}{4}(5+2\sqrt{5}+1) = \tfrac{1}{4}(6+2\sqrt{5}) = \tfrac{1}{2}(3+\sqrt{5})$$
$$= \tfrac{1}{2}(\sqrt{5}+1)+1 = \phi^0+\phi^1 \tag{22}$$

$$\phi^3 = (\phi^0+\phi^1)\phi^1 = \phi^0\phi^1 + (\phi^1)^2 = \phi^1+\phi^2, \tag{23}$$

and so on.

For the difference equations

$$\begin{cases} x_0 = 1 \\ x_n = 1 + \dfrac{1}{x_{n-1}} & \text{for } n = 1, 2, 3, \end{cases} \tag{24}$$

ϕ is also given by

$$\phi = \lim_{n\to\infty} x_n. \tag{25}$$

In addition,

$$\phi = \lim_{n\to\infty}\frac{F_n}{F_{n-1}}, \tag{26}$$

where F_n is the nth FIBONACCI NUMBER.

The SUBSTITUTION MAP

$$0 \to 01 \tag{27}$$

$$1 \to 0 \tag{28}$$

gives

$$0 \to 01 \to 010 \to 01001 \to \ldots, \tag{29}$$

giving rise to the sequence

$$0100101001001010010100100101\ldots \tag{30}$$

(Sloane's A003849). Here, the zeros occur at positions 1, 3, 4, 6, 8, 9, 11, 12, ... (Sloane's A000201), and the ones occur at positions 2, 5, 7, 10, 13, 15, 18, ... (Sloane's A001950). These are complementary BEATTY SEQUENCES generated by $\lfloor n\phi\rfloor$ and $\lfloor n\phi^2\rfloor$. The sequence also has many connections with the FIBONACCI NUMBERS.

Salem showed that the set of PISOT-VIJAYARAGHAVAN CONSTANTS is closed, with ϕ the smallest accumulation point of the set (Le Lionnais 1983).

see also BERAHA CONSTANTS, DECAGON, FIVE DISKS PROBLEM, GOLDEN RATIO CONJUGATE, GOLDEN TRIANGLE, ICOSIDODECAHEDRON, NOBLE NUMBER, PENTAGON, PENTAGRAM, PHI NUMBER SYSTEM, SECANT METHOD

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Golden Ratio Conjugate

The quantity

$$\phi_C \equiv \frac{1}{\phi} = \phi - 1 = \frac{\sqrt{5}-1}{2} \approx 0.6180339885, \tag{1}$$

where ϕ is the GOLDEN RATIO. The golden ratio conjugate is sometimes also called the SILVER RATIO. A quantity similar to the FEIGENBAUM CONSTANT can be found for the nth CONTINUED FRACTION representation

$$[a_0, a_1, a_2, \ldots]. \tag{2}$$

Taking the limit of

$$\delta_n \equiv \frac{\sigma_n - \sigma_{n-1}}{\sigma_n - \sigma_{n+1}} \tag{3}$$

gives

$$\delta \equiv \lim_{n\to\infty} = 1 + \phi = 2 + \phi_C. \tag{4}$$

see also GOLDEN RATIO, SILVER RATIO

Golden Rectangle

Given a RECTANGLE having sides in the ratio $1 : \phi$, the GOLDEN RATIO ϕ is defined such that partitioning the original RECTANGLE into a SQUARE and new RECTANGLE results in a new RECTANGLE having sides with a ratio $1 : \phi$. Such a RECTANGLE is called a golden rectangle, and successive points dividing a golden rectangle into SQUARES lie on a LOGARITHMIC SPIRAL.

see also GOLDEN RATIO, LOGARITHMIC SPIRAL, RECTANGLE

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Pappas, T. "The Golden Rectangle." *The Joy of Mathematics*. San Carlos, CA: Wide World Publ./Tetra, pp. 102–106, 1989.

Golden Rule

The mathematical golden rule states that, for any FRACTION, both NUMERATOR and DENOMINATOR may be multiplied by the same number without changing the fraction's value.

see also DENOMINATOR, FRACTION, NUMERATOR

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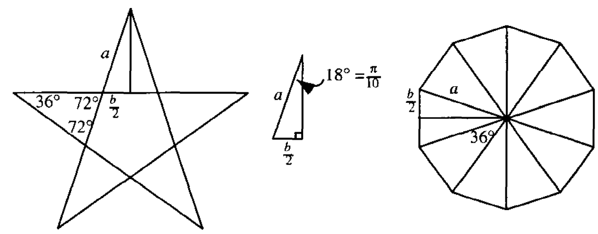
Golden Section

see GOLDEN RATIO

Golden Theorem

see QUADRATIC RECIPROCITY THEOREM

Golden Triangle



An ISOSCELES TRIANGLE with VERTEX angles 36°. Such TRIANGLES occur in the PENTAGRAM and DECAGON. The legs are in a GOLDEN RATIO to the base. For such a TRIANGLE,

$$\sin(18°) = \sin(\tfrac{1}{10}\pi) = \frac{\frac{1}{2}b}{l} \tag{1}$$

$$b = 2a\sin(\tfrac{1}{10}\pi) = 2a\frac{\sqrt{5}-1}{4} = \tfrac{1}{2}a(\sqrt{5}-1) \tag{2}$$

$$b + l = \tfrac{1}{2}a(\sqrt{5}+1) \tag{3}$$

$$\frac{b+a}{a} = \frac{\sqrt{5}+1}{2} = \phi. \tag{4}$$

see also DECAGON, GOLDEN RATIO, ISOSCELES TRIANGLE, PENTAGRAM

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Goldschmidt Solution

The discontinuous solution of the SURFACE OF REVOLUTION AREA minimization problem for surfaces connecting two CIRCLES. When the CIRCLES are sufficiently far apart, the usual CATENOID is no longer stable and the surface will break and form two surfaces with the CIRCLES as boundaries.

see also CALCULUS OF VARIATIONS, SURFACE OF REVOLUTION

Golomb Constant

see GOLOMB-DICKMAN CONSTANT

Golomb-Dickman Constant

N.B. A detailed on-line essay by S. Finch was the starting point for this entry.

Let Π be a PERMUTATION of n elements, and let α_i be the number of CYCLES of length i in this PERMUTATION. Picking Π at RANDOM gives

$$\left\langle \sum_{j=1}^{\infty} \alpha_j \right\rangle = \sum_{i=1}^{n} \frac{1}{i} = \ln n + \gamma + \mathcal{O}\left(\frac{1}{n}\right) \quad (1)$$

$$\operatorname{var}\left(\sum_{j=1}^{\infty} \alpha_j\right) = \sum_{i=1}^{n} \frac{i-1}{i^2} = \ln n + \gamma - \tfrac{1}{6}\pi^2 + \mathcal{O}\left(\frac{1}{n}\right) \quad (2)$$

$$\lim_{n\to\infty} P(\alpha_1 = 0) = \frac{1}{e} \quad (3)$$

(Shepp and Lloyd 1966, Wilf 1990). Goncharov (1942) showed that

$$\lim_{n\to\infty} P(\alpha_j = k) = \frac{1}{k!} e^{-1/j} j^{-k}, \quad (4)$$

which is a POISSON DISTRIBUTION, and

$$\lim_{n\to\infty} P\left[\left(\sum_{j=1}^{\infty} \alpha_j - \ln n\right)(\ln n)^{-1/2} \leq x\right] = \Phi(x), \quad (5)$$

which is a NORMAL DISTRIBUTION, γ is the EULER-MASCHERONI CONSTANT, and Φ is the NORMAL DISTRIBUTION FUNCTION. Let

$$M(\alpha) \equiv \max_{1\leq j<\infty} \alpha_j \quad (6)$$

$$m(\alpha) \equiv \min_{1\leq j<\infty} \alpha_j. \quad (7)$$

Golomb (1959) derived

$$\lambda \equiv \lim_{n\to\infty} \frac{\langle M(\alpha)\rangle}{n} = 0.6243299885\ldots, \quad (8)$$

which is known as the GOLOMB CONSTANT or Golomb-Dickman constant. Knuth (1981) asked for the constants b and c such that

$$\lim_{n\to\infty} n^b[\langle M(\alpha)\rangle - \lambda n - \tfrac{1}{2}\lambda] = c, \quad (9)$$

and Gourdon (1996) showed that

$$\langle M(\alpha)\rangle = \lambda(n+\tfrac{1}{2}) - \frac{e^\gamma}{24n} + \frac{\frac{1}{48}e^\gamma - \frac{1}{8}(-1)^n}{n^2} + \frac{\frac{17}{3840}e^\gamma + \frac{1}{8}(-1)^n + \frac{1}{6}j^{1+2n} + \frac{1}{6}j^{2+n}}{n^3}, \quad (10)$$

where

$$j \equiv e^{2\pi i/3}. \quad (11)$$

λ can be expressed in terms of the function $f(x)$ defined by $f(x) = 1$ for $1 \leq x \leq 2$ and

$$\frac{df}{dx} = -\frac{f(x-1)}{x-1} \quad (12)$$

for $x > 2$, by

$$\lambda = \int_1^\infty \frac{f(x)}{x^2}\,dx. \quad (13)$$

Shepp and Lloyd (1966) derived

$$\lambda = \int_0^\infty \exp\left(-x - \int_x^\infty \frac{e^{-y}}{y}\,dy\right) = \int_0^1 \exp\left(\int_0^x \frac{dy}{\ln y}\right) dx. \quad (14)$$

Mitchell (1968) computed λ to 53 decimal places.

Surprisingly enough, there is a connection between λ and PRIME FACTORIZATION (Knuth and Pardo 1976, Knuth 1981, pp. 367–368, 395, and 611). Dickman (1930) investigated the probability $P(x,n)$ that the largest PRIME FACTOR p of a random INTEGER between 1 and n satisfies $p < n^x$ for $x \in (0,1)$. He found that

$$F(x) \equiv \lim_{n\to\infty} P(x,n) = \begin{cases} 1 & \text{if } x \geq 1 \\ \int_0^x F\left(\frac{t}{1-t}\right)\frac{dt}{t} & \text{if } 0 \leq x < 1. \end{cases} \quad (15)$$

Dickman then found the average value of x such that $p = n^x$, obtaining

$$\mu \equiv \lim_{n\to\infty} \langle x\rangle = \lim_{n\to\infty}\left\langle \frac{\ln p}{\ln n}\right\rangle = \int_0^1 x\frac{dF}{dx}\,dx = \int_0^1 F\left(\frac{1}{1-t}\right) dt = 0.62432999, \quad (16)$$

which is λ.

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Golomb Ruler

A Golomb ruler is a set of NONNEGATIVE integers such that all pairwise POSITIVE differences are distinct. The optimum Golomb ruler with n marks is the Golomb ruler having the smallest possible maximum element ("length"). The set (0, 1, 3, 7) is an order four Golomb ruler since its differences are ($1 = 1 - 0$, $2 = 3 - 1$, $3 = 3-0$, $4 = 7-3$, $6 = 7-1$, $7 = 7-0$), all of which are distinct. However, the optimum 4-mark Golomb ruler is (0, 1, 4, 6), which measures the distances (1, 2, 3, 4, 5, 6) (and is therefore also a PERFECT RULER).

The lengths of the optimal n-mark Golomb rulers for $n = 2, 3, 4, \ldots$ are 1, 3, 6, 11, 17, 25, 34, ... (Sloane's A003022, Vanderschel and Garry). The lengths of the optimal n-mark Golomb rulers are not known for $n \geq 20$.

see also PERFECT DIFFERENCE SET, PERFECT RULER, RULER, TAYLOR'S CONDITION, WEIGHINGS

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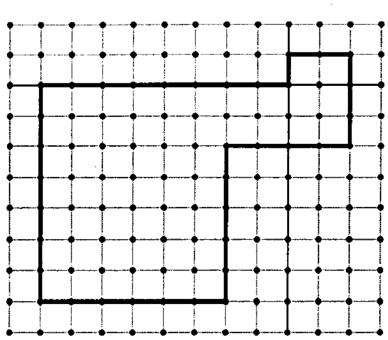
Golygon



A PLANE path on a set of equally spaced LATTICE POINTS, starting at the ORIGIN, where the first step is one unit to the north or south, the second step is two units to the east or west, the third is three units to the north or south, etc., and continuing until the ORIGIN is again reached. No crossing or backtracking is allowed. The simplest golygon is (0, 0), (0, 1), (2, 1), (2, −2), (−2, −2), (−2, −7), (−8, −7), (−8, 0), (0, 0).

A golygon can be formed if there exists an EVEN INTEGER n such that

$$\pm 1 \pm 3 \pm \ldots \pm (n-1) = 0 \qquad (1)$$

$$\pm 2 \pm 4 \pm \ldots \pm n = 0 \qquad (2)$$

(Vardi 1991). Gardner proved that all golygons are of the form $n = 8k$. The number of golygons of length n (EVEN), with each initial direction counted separately, is the PRODUCT of the COEFFICIENT of $x^{n^2/8}$ in

$$(1+x)(1+x^3)\cdots(1+x^{n-1}), \qquad (3)$$

with the COEFFICIENT of $x^{n(n/2+1)/8}$ in

$$(1+x)(1+x^2)\cdots(1+x^{n/2}). \qquad (4)$$

The number of golygons $N(n)$ of length $8n$ for the first few n are 4, 112, 8432, 909288, ... (Sloane's A006718) and is asymptotic to

$$N(n) \sim \frac{3 \cdot 2^{8n-4}}{\pi n^2 (4n+1)} \qquad (5)$$

(Sallows *et al.* 1991, Vardi 1991).

see also LATTICE PATH

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Gomory's Theorem

Regardless of where one white and one black square are deleted from an ordinary 8×8 CHESSBOARD, the reduced board can always be covered exactly with 31 DOMINOES (of dimension 2×1).

see also CHESSBOARD

Gompertz Constant

$$G \equiv \int_0^\infty \frac{e^{-u}}{1+u}\,du = -e\,\mathrm{ei}(-1) = 0.596347362\ldots,$$

where $\mathrm{ei}(x)$ is the EXPONENTIAL INTEGRAL. Stieltjes showed it has the CONTINUED FRACTION representation

$$G = \frac{1}{2-}\frac{1^2}{4-}\frac{2^2}{6-}\frac{3^2}{8-}\cdots.$$

see also EXPONENTIAL INTEGRAL

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Gompertz Curve

The function defined by

$$y = ab^{q^x}.$$

It is used in actuarial science for specifying a simplified mortality law. Using $s(x)$ as the probability that a newborn will achieve age x, the Gompertz law (1825) is

$$s(x) = \exp[-m(c^x - 1)],$$

for $c > 1$, $x \geq 0$.

see also LIFE EXPECTANCY, LOGISTIC GROWTH CURVE, MAKEHAM CURVE, POPULATION GROWTH

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Gonal Number

see POLYGONAL NUMBER

Good Path

see p-GOOD PATH

Good Prime

A PRIME p_n is called "good" if

$$p_n{}^2 > p_{n-i}p_{n+i}$$

for all $1 \leq i \leq n-1$ (there is a typo in Guy 1994 in which the is are replaced by 1s). There are infinitely many good primes, and the first few are 5, 11, 17, 29, 37, 41, 53, ... (Sloane's A028388).

see also ANDRICA'S CONJECTURE, PÓLYA CONJECTURE

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Goodman's Formula

A two-coloring of a COMPLETE GRAPH K_n of n nodes which contains exactly the number of MONOCHROMATIC FORCED TRIANGLES and no more (i.e., a minimum of $R+B$ where R and B are the number of red and blue TRIANGLES) is called an EXTREMAL GRAPH. Goodman (1959) showed that for an extremal graph,

$$R+B = \begin{cases} \frac{1}{3}m(m-1)(m-2) & \text{for } n = 2m \\ \frac{2}{3}m(m-1)(4m+1) & \text{for } n = 4m+1 \\ \frac{2}{3}m(m+1)(4m-1) & \text{for } n = 4m+3. \end{cases}$$

Schwenk (1972) rewrote the equation in the form

$$R+B = \binom{n}{3} - \left\lfloor \tfrac{1}{2}n \left\lfloor \tfrac{1}{4}(n-1)^2 \right\rfloor \right\rfloor,$$

where $\binom{n}{k}$ is a BINOMIAL COEFFICIENT and $\lfloor x \rfloor$ is the FLOOR FUNCTION.

see also BLUE-EMPTY GRAPH, EXTREMAL GRAPH, MONOCHROMATIC FORCED TRIANGLE

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Schwenk, A. J. "Acquaintance Party Problem." *Amer. Math. Monthly* **79**, 1113–1117, 1972.

Goodstein Sequence

Given a HEREDITARY REPRESENTATION of a number n in BASE, let $B[b](n)$ be the NONNEGATIVE INTEGER which results if we syntactically replace each b by $b+1$ (i.e., $B[b]$ is a base change operator that 'bumps the base' from b up to $b+1$). The HEREDITARY REPRESENTATION of 266 in base 2 is

$$\begin{aligned} 266 &= 2^8 + 2^3 + 2 \\ &= 2^{2^{2+1}} + 2^{2+1} + 2, \end{aligned}$$

so bumping the base from 2 to 3 yields

$$B[2](266) = 3^{3^{3+1}} + 3^{3+1} + 3.$$

Now repeatedly bump the base and subtract 1,

$$\begin{aligned} G_0(266) &= 266 = 2^{2^{2+1}} + 2^{2+1} + 2 \\ G_1(266) &= B[2](266) - 1 = 3^{3^{3+1}} + 3^{3+1} + 2 \\ G_2(266) &= B[3](G_1) - 1 = 4^{4^{4+1}} + 4^{4+1} + 1 \\ G_3(266) &= B[4](G_2) - 1 = 5^{5^{5+1}} + 5^{5+1} \\ G_4(266) &= B[5](G_3) - 1 = 6^{6^{6+1}} + 6^{6+1} - 1 \\ &= 6^{6^{6+1}} + 5 \cdot 6^6 + 5 \cdot 6^5 + \ldots + 5 \cdot 6 + 5 \\ G_5(266) &= B[6](G_4) - 1 \\ &= 7^{7^{7+1}} + 5 \cdot 7^7 + 5 \cdot 7^5 + \ldots + 5 \cdot 7 + 4, \end{aligned}$$

etc. Starting this procedure at an INTEGER n gives the Goodstein sequence $\{G_k(n)\}$. Amazingly, despite the apparent rapid increase in the terms of the sequence, GOODSTEIN'S THEOREM states that $G_k(n)$ is 0 for any n and any sufficiently large k.

see also GOODSTEIN'S THEOREM, HEREDITARY REPRESENTATION

Goodstein's Theorem

For all n, there exists a k such that the kth term of the GOODSTEIN SEQUENCE $G_k(n) = 0$. In other words, every GOODSTEIN SEQUENCE converges to 0.

The secret underlying Goodstein's theorem is that the HEREDITARY REPRESENTATION of n in base b mimics an ordinal notation for ordinals less than some number. For such ordinals, the base bumping operation leaves the ordinal fixed whereas the subtraction of one decreases the ordinal. But these ordinals are well-ordered, and this allows us to conclude that a Goodstein sequence eventually converges to zero.

Goodstein's theorem cannot be proved in PEANO ARITHMETIC (i.e., formal NUMBER THEORY).

see also NATURAL INDEPENDENCE PHENOMENON, PEANO ARITHMETIC

Googol

A LARGE NUMBER equal to 10^{100}, or

$$\begin{gathered}10000000000000000000000000\\0000000000000000000000000\\0000000000000000000000000\\0000000000000000000000000.\end{gathered}$$

see also GOOGOLPLEX, LARGE NUMBER

References
Kasner, E. and Newman, J. R. *Mathematics and the Imagination.* Redmond, WA: Tempus Books, pp. 20–27, 1989.
Pappas, T. "Googol & Googolplex." *The Joy of Mathematics.* San Carlos, CA: Wide World Publ./Tetra, p. 76, 1989.

Googolplex

A LARGE NUMBER equal to $10^{10^{100}}$.

see also GOOGOL, LARGE NUMBER

References
Kasner, E. and Newman, J. R. *Mathematics and the Imagination.* Redmond, WA: Tempus Books, pp. 23–27, 1989.
Pappas, T. "Googol & Googolplex." *The Joy of Mathematics.* San Carlos, CA: Wide World Publ./Tetra, p. 76, 1989.

Gordon Function

Another name for the CONFLUENT HYPERGEOMETRIC FUNCTION OF THE SECOND KIND, defined by

$$G(a|c|z) = e^{i\pi a}\frac{\Gamma(c)}{\Gamma(a)}\left\{\frac{\Gamma(1-c)}{\Gamma(1-a)}\left[e^{-\pi c}+\frac{\sin[\pi(a-c)]}{\sin(\pi a)}\right]\right.$$
$$\left.\times {}_1F_1(a;c;z) - 2\frac{\Gamma(c-1)}{\Gamma(c-a)}z^{1-c}{}_1F_1(a-c+1;2-c;z)\right\},$$

where $\Gamma(x)$ is the GAMMA FUNCTION and ${}_1F_1(a;b;z)$ is the CONFLUENT HYPERGEOMETRIC FUNCTION OF THE FIRST KIND.

see also CONFLUENT HYPERGEOMETRIC FUNCTION OF THE SECOND KIND

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Morse, P. M. and Feshbach, H. *Methods of Theoretical Physics, Part I.* New York: McGraw-Hill, pp. 671–672, 1953.

Gorenstein Ring

An algebraic RING which appears in treatments of duality in ALGEBRAIC GEOMETRY. Let A be a local ARTINIAN RING with $m \subset A$ its maximal IDEAL. Then A is a Gorenstein ring if the ANNIHILATOR of m has DIMENSION 1 as a VECTOR SPACE over $K = A/m$.

see also CAYLEY-BACHARACH THEOREM

References
Eisenbud, D.; Green, M.; and Harris, J. "Cayley-Bacharach Theorems and Conjectures." *Bull. Amer. Math. Soc.* **33**, 295–324, 1996.

Gosper's Algorithm

An ALGORITHM for finding closed form HYPERGEOMETRIC IDENTITIES The algorithm treats sums whose successive terms have ratios which are RATIONAL FUNCTIONS. Not only does it decide conclusively whether there exists a hypergeometric sequence z_n such that

$$t_n = z_{n+1} - z_n,$$

but actually produces z_n if it exists. If not, it produces $\sum_{k=0}^{n-1} t_n$. An outline of the algorithm follows (Petkovšek 1996):

1. For the ratio $r(n) = t_{n+1}/t_n$ which is a RATIONAL FUNCTION of n.
2. Write
$$r(n) = \frac{a(n)}{b(n)}\frac{c(n+1)}{c(n)},$$
where $a(n)$, $b(n)$, and $c(n)$ are polynomials satisfying
$$\mathrm{GCD}(a(n), b(n+h)) = 1$$
for all nonnegative integers h.
3. Find a nonzero polynomial solution $x(n)$ of
$$a(n)x(n+1) - b(n-1)x(n) = c(n),$$
if one exists.
4. Return $b(n-1)x(n)/c(n)t_n$ and stop.

Petkovšek *et al.* (1996) describe the algorithm as "one of the landmarks in the history of computerization of the problem of closed form summation." Gosper's algorithm is vital in the operation of ZEILBERGER'S ALGORITHM and the machinery of WILF-ZEILBERGER PAIRS.

see also HYPERGEOMETRIC IDENTITY, SISTER CELINE'S METHOD, WILF-ZEILBERGER PAIR, ZEILBERGER'S ALGORITHM

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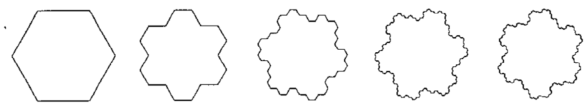
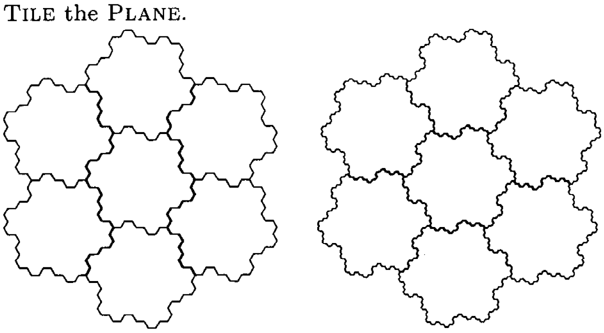
Gosper Island



A modification of the KOCH SNOWFLAKE which has FRACTAL DIMENSION

$$D = \frac{2\ln 3}{\ln 7} = 1.12915\ldots.$$

The term "Gosper island" was used by Mandelbrot (1977) because this curve bounds the space filled by the PEANO-GOSPER CURVE; Gosper and Gardner use the term FLOWSNAKE FRACTAL instead. Gosper islands can TILE the PLANE.



see also KOCH SNOWFLAKE, PEANO-GOSPER CURVE

References

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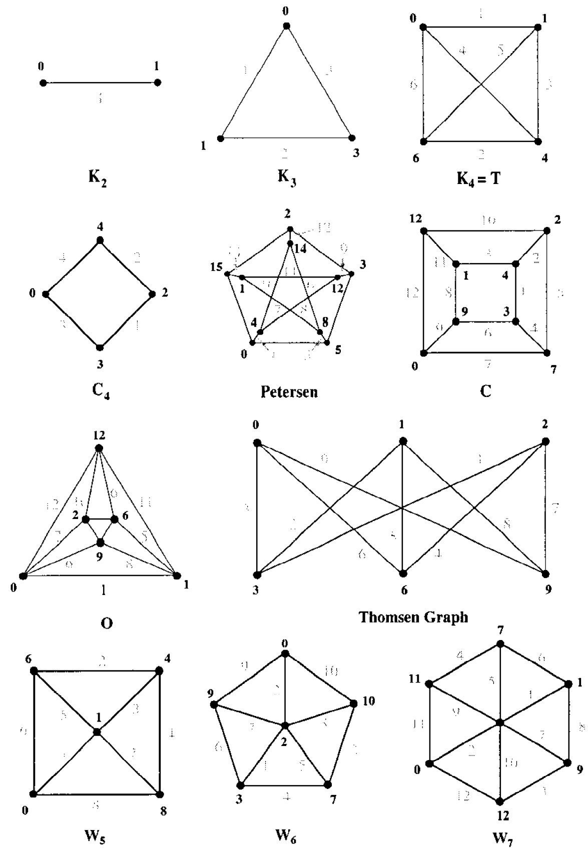
Gosper's Method

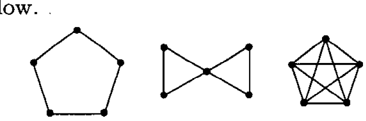
see GOSPER'S ALGORITHM

Graceful Graph

A LABELLED GRAPH which can be "gracefully numbered" is called a graceful graph. Label the nodes with distinct NONNEGATIVE INTEGERS. Then label the EDGES with the absolute differences between node values. If the EDGE numbers then run from 1 to e, the graph is gracefully numbered. In order for a graph to be graceful, it must be without loops or multiple EDGES.



Golomb showed that the number of EDGES connecting the EVEN-numbered and ODD-numbered sets of nodes is $\lfloor (e+1)/2 \rfloor$, where e is the number of EDGES. In addition, if the nodes of a graph are all of EVEN ORDER, then the graph is graceful only if $\lfloor (e+1)/2 \rfloor$ is EVEN. The only ungraceful simple graphs with ≤ 5 nodes are shown below.



There are exactly $e!$ graceful graphs with e EDGES (Sheppard 1976), where $e!/2$ of these correspond to different labelings of the same graph. Golomb (1974) showed that all complete bipartite graphs are graceful. CATERPILLAR GRAPHS; COMPLETE GRAPHS K_2, K_3, $K_4 = W_4 = T$ (and only these; Golomb 1974); CYCLIC GRAPHS C_n when $n \equiv 0$ or $3 \pmod 4$, when the number of consecutive chords $k = 2$, 3, or $n - 3$ (Koh and Punim 1982), or when they contain a P_k chord (Delorme *et al.* 1980, Koh and Yap 1985, Punnim and Pabhapote 1987); GEAR GRAPHS; PATH GRAPHS; the PETERSEN GRAPH; POLYHEDRAL GRAPHS $T = K_4 = W_4$, C, O, D, and I (Gardner 1983); STAR GRAPHS; the THOMSEN GRAPH (Gardner 1983); and WHEEL GRAPHS (Frucht 1988) are all graceful.

Some graceful graphs have only one numbering, but others have more than one. It is conjectured that all trees are graceful (Bondy and Murty 1976), but this has only been proved for trees with ≤ 16 VERTICES. It has also been conjectured that all unicyclic graphs are graceful.

An excellent on-line resource is Brundage (`http://www.math.washington.edu/~brundage/oldgraceful/`).

see also HARMONIOUS GRAPH, LABELLED GRAPH

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Graded Algebra

If A is a graded module and there EXISTS a degree-preserving linear map $\phi : A \otimes A \to A$, then (A, ϕ) is called a graded algebra.

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Gradian

A unit of angular measure in which the angle of an entire CIRCLE is 400 gradians. A RIGHT ANGLE is therefore 100 gradians.

see also DEGREE, RADIAN

Gradient

The gradient is a VECTOR operator denoted ∇ and sometimes also called DEL or NABLA. It most often is applied to a real function of three variables $f(u_1, u_2, u_3)$, and may be denoted

$$\nabla f \equiv \mathrm{grad(f)}. \qquad (1)$$

For general CURVILINEAR COORDINATES, the gradient is given by

$$\nabla\phi = \frac{1}{h_1}\frac{\partial\phi}{\partial u_1}\hat{\mathbf{u}}_1 + \frac{1}{h_2}\frac{\partial\phi}{\partial u_2}\hat{\mathbf{u}}_2 + \frac{1}{h_3}\frac{\partial\phi}{\partial u_3}\hat{\mathbf{u}}_3, \qquad (2)$$

which simplifies to

$$\nabla\phi(x,y,z) = \frac{\partial\phi}{\partial x}\hat{\mathbf{x}} + \frac{\partial\phi}{\partial y}\hat{\mathbf{y}} + \frac{\partial\phi}{\partial z}\hat{\mathbf{z}} \qquad (3)$$

in CARTESIAN COORDINATES.

The direction of ∇f is the orientation in which the DIRECTIONAL DERIVATIVE has the largest value and $|\nabla f|$ is the value of that DIRECTIONAL DERIVATIVE. Furthermore, if $\nabla f \neq 0$, then the gradient is PERPENDICULAR to the LEVEL CURVE through (x_0, y_0) if $z = f(x, y)$ and PERPENDICULAR to the level surface through (x_0, y_0, z_0) if $F(x, y, z) = 0$.

In TENSOR notation, let

$$ds^2 = g_\mu \, dx_\mu{}^2 \tag{4}$$

be the LINE ELEMENT in principal form. Then

$$\nabla_{\vec{e}_\alpha} \vec{e}_\beta = \nabla_\alpha \vec{e}_\beta = \frac{1}{\sqrt{g_\alpha}} \frac{\partial}{\partial x_\alpha} \vec{e}_\beta. \tag{5}$$

For a MATRIX A,

$$\nabla |\mathsf{Ax}| = \frac{(\mathsf{Ax})^{\mathrm{T}} \mathsf{A}}{|\mathsf{Ax}|}. \tag{6}$$

For expressions giving the gradient in particular coordinate systems, see CURVILINEAR COORDINATES.

see also CONVECTIVE DERIVATIVE, CURL, DIVERGENCE, LAPLACIAN, VECTOR DERIVATIVE

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Gradient Four-Vector

The 4-dimensional version of the GRADIENT, encountered frequently in general relativity and special relativity, is

$$\nabla_\mu = \begin{bmatrix} \frac{1}{c}\frac{\partial}{\partial t} \\ \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{bmatrix},$$

which can be written

$$(\nabla^\mu)^2 \equiv \Box^2,$$

where $\Box^2$ is the D'ALEMBERTIAN OPERATOR.

see also D'ALEMBERTIAN OPERATOR, GRADIENT, TENSOR, VECTOR

References

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Gradient Theorem

$$\int_b^a (\nabla f) \cdot d\mathbf{s} = f(b) - f(a),$$

where ∇ is the GRADIENT, and the integral is a LINE INTEGRAL. It is this relationship which makes the definition of a scalar potential function f so useful in gravitation and electromagnetism as a concise way to encode information about a VECTOR FIELD.

see also DIVERGENCE THEOREM, GREEN'S THEOREM, LINE INTEGRAL

Graeco-Latin Square

see EULER SQUARE

Graeco-Roman Square

see EULER SQUARE

Graeffe's Method

A ROOT-finding method which proceeds by multiplying a POLYNOMIAL $f(x)$ by $f(-x)$ and noting that

$$f(x) = (x - a_1)(x - a_2) \cdots (x - a_n) \tag{1}$$

$$f(-x) = (-1)^n (x + a_1)(x + a_2) \cdots (x + a_n) \tag{2}$$

so the result is

$$f(x)f(-x) = (-1)^n (x^2 - a_1{}^2)(x^2 - a_2{}^2) \cdots (x^2 - a_n{}^2). \tag{3}$$

Repeat ν times, then write this in the form

$$y^n + b_1 y^{n-1} + \ldots + b_n = 0 \tag{4}$$

where $y \equiv x^{2\nu}$. Since the coefficients are given by NEWTON'S RELATIONS

$$b_1 = -(y_1 + y_2 + \ldots + y_n) \tag{5}$$

$$b_2 = (y_1 y_2 + y_1 y_3 + \ldots + y_{n-1} y_n) \tag{6}$$

$$b_n = (-1)^n y_1 y_2 \cdots y_n, \tag{7}$$

and since the squaring procedure has separated the roots, the first term is larger than rest. Therefore,

$$b_1 \approx -y_1 \tag{8}$$

$$b_2 \approx y_1 y_2 \tag{9}$$

$$b_n \approx (-1)^n y_1 y_2 \cdots y_n, \tag{10}$$

giving

$$y_1 \approx -b_1 \tag{11}$$

$$y_2 \approx -\frac{b_2}{b_1} \tag{12}$$

$$y_n \approx -\frac{b_n}{b_{n-1}}. \tag{13}$$

Solving for the original roots gives

$$a_1 \approx \sqrt[2\nu]{-b_1} \tag{14}$$

$$a_2 \approx \sqrt[2\nu]{-\frac{b_2}{b_1}} \tag{15}$$

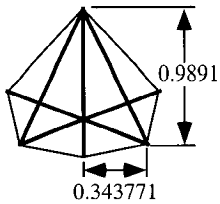
$$a_n \approx \sqrt[2\nu]{-\frac{b_n}{b_{n-1}}}. \tag{16}$$

This method works especially well if all roots are real.

References

von Kármán, T. and Biot, M. A. "Squaring the Roots (Graeffe's Method)." §5.8.c in *Mathematical Methods in Engineering: An Introduction to the Mathematical Treatment of Engineering Problems.* New York: McGraw-Hill, pp. 194–196, 1940.

Graham's Biggest Little Hexagon



The largest possible (not necessarily regular) HEXAGON for which no two of the corners are more than unit distance apart. In the above figure, the heavy lines are all of unit length. The AREA of the hexagon is $A = 0.674981\ldots$, where A is a ROOT of

$$\begin{aligned}4096A^{10} - 8192A^9 - 3008A^8 - 30,848A^7 + 21,056A^6\\ +146,496A^5 - 221,360A^4 + 1232A^3 + 144,464A^2\\ -78,488A + 11,993 = 0.\end{aligned}$$

see also CALABI'S TRIANGLE

References

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Graham's Number

The smallest dimension of a HYPERCUBE such that if the lines joining all pairs of corners are two-colored, a PLANAR COMPLETE GRAPH K_4 of one color will be forced. That an answer exists was proved by R. L. Graham and B. L. Rothschild. The actual answer is believed to be 6, but the best bound proved is

$$64\left\{\begin{array}{c}\underbrace{3\uparrow\uparrow\uparrow\uparrow 3}\\ \underbrace{3\uparrow 3}\\ \vdots\\ \underbrace{}\\ 3\uparrow 3,\end{array}\right.$$

where $\uparrow$ is stacked ARROW NOTATION. It is less than $3 \to 3 \to 3 \to 3$, where CHAINED ARROW NOTATION has been used.

see also ARROW NOTATION, CHAINED ARROW NOTATION, SKEWES NUMBER

References

Conway, J. H. and Guy, R. K. *The Book of Numbers.* New York: Springer-Verlag, pp. 61–62, 1996.

Gardner, M. "Mathematical Games." *Sci. Amer.* **237**, 18–28, Nov. 1977.

Gram-Charlier Series

Approximates a distribution in terms of a NORMAL DISTRIBUTION. Let

$$\phi(t) \equiv \frac{1}{\sqrt{2\pi}} e^{-t^2/2},$$

then

$$f(t) = \phi(t) + \tfrac{1}{6}\gamma_1\phi^{(3)}(t) + \tfrac{1}{24}\gamma_2\phi^{(4)}(t) + \ldots.$$

see also EDGEWORTH SERIES

References

Kenney, J. F. and Keeping, E. S. *Mathematics of Statistics, Pt. 2, 2nd ed.* Princeton, NJ: Van Nostrand, pp. 107–108, 1951.

Gram Determinant

The DETERMINANT

$$G(f_1, f_2, \ldots, f_n) = \begin{vmatrix} \int f_1^2\,dt & \int f_1 f_2\,dt & \cdots & \int f_1 f_n\,dt\\ \int f_2 f_1\,dt & \int f_2^2\,dt & \cdots & \int f_2 f_n\,dt\\ \vdots & \vdots & \ddots & \vdots\\ \int f_1 f_n\,dt & \int f_1 f_n\,dt & \cdots & \int f_n^2\,dt \end{vmatrix}.$$

see also GRAM-SCHMIDT ORTHONORMALIZATION, WRONSKIAN

References

Sansone, G. *Orthogonal Functions, rev. English ed.* New York: Dover, p. 2, 1991.

Gram's Inequality

Let $f_1(x), \ldots, f_n(x)$ be REAL INTEGRABLE FUNCTIONS over the CLOSED INTERVAL $[a, b]$, then the DETERMINANT of their integrals satisfies

$$\begin{vmatrix} \int_a^b f_1{}^2(x)\,dx & \int_a^b f_1(x)f_2(x)\,dx & \cdots & \int_a^b f_1(x)f_n(x)\,dx\\ \int_a^b f_2(x)f_1(x)\,dx & \int_a^b f_2{}^2(x)\,dx & \cdots & \int_a^b f_2(x)f_n(x)\,dx\\ \vdots & \vdots & \ddots & \vdots\\ \int_a^b f_n(x)f_1(x)\,dx & \int_a^b f_n(x)f_2(x)\,dx & \cdots & \int_a^b f_n(x)f_n(x)\,dx \end{vmatrix} \geq 0.$$

see also GRAM-SCHMIDT ORTHONORMALIZATION

References

Gradshteyn, I. S. and Ryzhik, I. M. *Tables of Integrals, Series, and Products, 5th ed.* San Diego, CA: Academic Press, p. 1100, 1979.

Gram Matrix

Given m points with n-D vector coordinates $\mathbf{v}_i$, let M be the $n \times m$ matrix whose jth column consists of the coordinates of the vector $\mathbf{v}_j$, with $j = 1, \ldots, m$. Then define the $m \times m$ Gram matrix of dot products $a_{ij} = \mathbf{v}_i \cdot \mathbf{v}_j$ as

$$\mathsf{A} = \mathsf{M}^{\mathrm{T}}\mathsf{M},$$

where A^{T} denotes the TRANSPOSE. The Gram matrix determines the vectors $\mathbf{v}_i$ up to ISOMETRY.

Gram-Schmidt Orthonormalization

A procedure which takes a nonorthogonal set of LINEARLY INDEPENDENT functions and constructs an ORTHOGONAL BASIS over an arbitrary interval with respect to an arbitrary WEIGHTING FUNCTION $w(x)$. Given an original set of linearly independent functions $\{u_n\}$, let $\{\psi_n\}$ denote the orthogonalized (but not normalized) functions and $\{\phi_n\}$ the orthonormalized functions.

$$\psi_0(x) \equiv u_1(x) \tag{1}$$

$$\phi_0(x) \equiv \frac{\psi_0(x)}{\sqrt{\int \psi_0{}^2(x)w(x)\,dx}}. \tag{2}$$

Take

$$\psi_1(x) = u_1(x) + a_{10}\phi_0(x), \tag{3}$$

where we require

$$\int \psi_1\phi_0 w\,dx = \int u_1\phi_0 w\,dx + a_{10}\int \phi_0{}^2 w\,dx = 0. \tag{4}$$

By definition,

$$\int \phi_0{}^2 w\,dx = 1, \tag{5}$$

so

$$a_{10} = -\int u_1\phi_0 w\,dx. \tag{6}$$

The first orthogonalized function is therefore

$$\psi_1 = u_1(x) - \left[\int u_1\phi_0 w\,dx\right]\phi_0, \tag{7}$$

and the corresponding normalized function is

$$\phi_1 = \frac{\psi_1(x)}{\sqrt{\int \psi_1{}^2 w\,dx}}. \tag{8}$$

By mathematical induction, it follows that

$$\phi_i(x) = \frac{\psi_i(x)}{\sqrt{\int \psi_i{}^2 w\,dx}}, \tag{9}$$

where

$$\psi_i(x) = u_i + a_{i0}\phi_0 + a_{i1}\phi_1 \ldots + a_{i,i-1}\phi_{i-1} \tag{10}$$

and

$$a_{ij} \equiv -\int u_i\phi_j w\,dx. \tag{11}$$

If the functions are normalized to N_j instead of 1, then

$$\int_a^b [\phi_j(x)]^2 w\,dx = N_j{}^2 \tag{12}$$

$$\phi_i(x) = N_i\frac{\psi_i(x)}{\sqrt{\int \psi_i{}^2 w\,dx}} \tag{13}$$

$$a_{ij} = -\frac{\int u_i\phi_j w\,dx}{N_j{}^2}. \tag{14}$$

ORTHOGONAL POLYNOMIALS are especially easy to generate using GRAM-SCHMIDT ORTHONORMALIZATION. Use the notation

$$\langle x_i|x_j\rangle \equiv \langle x_i|w|x_j\rangle \equiv \int_a^b x_i(x)x_j(x)w(x)\,dx, \tag{15}$$

where $w(x)$ is a WEIGHTING FUNCTION, and define the first few POLYNOMIALS,

$$p_0(x) \equiv 1 \tag{16}$$

$$p_1(x) = \left[x - \frac{\langle xp_0|p_0\rangle}{\langle p_0|p_0\rangle}\right]p_0. \tag{17}$$

As defined, p_0 and p_1 are ORTHOGONAL POLYNOMIALS, as can be seen from

$$\begin{aligned}\langle p_0|p_1\rangle &= \left\langle \left[x - \frac{\langle xp_0|p_0\rangle}{\langle p_0|p_0\rangle}\right]p_0\right\rangle \\ &= \langle xp_0\rangle - \frac{\langle xp_0|p_0\rangle}{\langle p_0|p_0\rangle}\langle p_0\rangle \\ &= \langle xp_0\rangle - \langle xp_0\rangle = 0.\end{aligned} \tag{18}$$

Now use the RECURRENCE RELATION

$$p_{i+1}(x) = \left[x - \frac{\langle xp_i|p_i\rangle}{\langle p_i|p_i\rangle}\right]p_i - \left[\frac{\langle p_i|p_i\rangle}{\langle p_{i-1}|p_{i-1}\rangle}\right]p_{i-1} \tag{19}$$

to construct all higher order POLYNOMIALS.

To verify that this procedure does indeed produce ORTHOGONAL POLYNOMIALS, examine

$$\begin{aligned}\langle p_{i+1}|p_i\rangle &= \left\langle \left[x - \frac{\langle xp_i|p_i\rangle}{\langle p_i|p_i\rangle}\right]p_i\middle|p_i\right\rangle \\ &\quad - \left\langle \frac{\langle p_i|p_i\rangle}{\langle p_{i-1}|p_{i-1}\rangle}p_{i-1}\middle|p_i\right\rangle \\ &= \langle xp_i|p_i\rangle - \frac{\langle xp_i|p_i\rangle}{\langle p_i|p_i\rangle}\langle p_i|p_i\rangle \\ &\quad - \frac{\langle p_i|p_i\rangle}{\langle p_{i-1}|p_{i-1}\rangle}\langle p_{i-1}|p_i\rangle \\ &= -\frac{\langle p_i|p_i\rangle}{\langle p_{i-1}|p_{i-1}\rangle}\langle p_{i-1}|p_i\rangle \\ &= -\frac{\langle p_i|p_i\rangle}{\langle p_{i-1}|p_{i-1}\rangle}\left[-\frac{\langle p_{i-1}|p_{j-1}\rangle}{\langle p_{j-2}|p_{j-2}\rangle}\langle p_{j-2}|p_{j-1}\rangle\right] \\ &= \ldots = (-1)^j\frac{\langle p_j|p_j\rangle}{\langle p_0|p_0\rangle}\langle p_0|p_1\rangle = 0,\end{aligned} \tag{20}$$

since $\langle p_0|p_1\rangle = 0$. Therefore, all the POLYNOMIALS $p_i(x)$ are orthogonal. Many common ORTHOGONAL POLYNOMIALS of mathematical physics can be generated in this manner. However, the process is numerically unstable (Golub and van Loan 1989).

see also GRAM DETERMINANT, GRAM'S INEQUALITY, ORTHOGONAL POLYNOMIALS

References
Arfken, G. "Gram-Schmidt Orthogonalization." §9.3 in *Mathematical Methods for Physicists, 3rd ed.* Orlando, FL: Academic Press, pp. 516–520, 1985.
Golub, G. H. and van Loan, C. F. *Matrix Computations, 3rd ed.* Baltimore, MD: Johns Hopkins, 1989.

Gram Series

$$R(x) = 1 + \sum_{k=1}^{\infty} \frac{(\ln x)^k}{kk!\zeta(k+1)},$$

where ζ is the RIEMANN ZETA FUNCTION. This approximation to the PRIME COUNTING FUNCTION is 10 times better than $\mathrm{Li}(x)$ for $x < 10^9$ but has been proven to be worse infinitely often by Littlewood (Ingham 1990). An equivalent formulation due to Ramanujan is

$$G(x) \equiv \frac{4}{\pi} \sum_{k=1}^{\infty} \frac{(-1)^{k-1} k}{B_{2k}(2k-1)} \left(\frac{\ln x}{2\pi}\right)^{2k-1} \sim \pi(x)$$

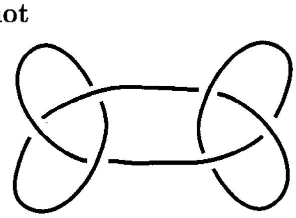
(Berndt 1994), where B_{2k} is a BERNOULLI NUMBER. The integral analog, also found by Ramanujan, is

$$J(x) \equiv \int_0^\infty \frac{(\ln x)^t\,dt}{t\Gamma(t+1)\zeta(t+1)} \sim \pi(x)$$

(Berndt 1994).

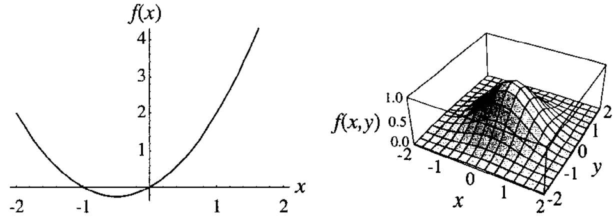
References
Berndt, B. C. *Ramanujan's Notebooks, Part IV.* New York: Springer-Verlag, pp. 124–129, 1994.
Gram, J. P. "Undersøgelser angaaende Maengden af Primtal under en given Graeense." *K. Videnskab. Selsk. Skr.* **2**, 183–308, 1884.
Ingham, A. E. Ch. 5 in *The Distribution of Prime Numbers.* New York: Cambridge, 1990.
Vardi, I. *Computational Recreations in Mathematica.* Reading, MA: Addison-Wesley, p. 74, 1991.

Granny Knot



A COMPOSITE KNOT of seven crossings consisting of a KNOT SUM of TREFOILS. The granny knot has the same ALEXANDER POLYNOMIAL $(x^2 - x + 1)^2$ as the SQUARE KNOT.

Graph (Function)



Technically, the graph of a function is its RANGE (a.k.a. image). Informally, given a FUNCTION $f(x_1, \ldots, x_n)$ defined on a DOMAIN U, the graph of f is defined as a CURVE or SURFACE showing the values taken by f over U (or some portion of U),

$$\operatorname{graph} f(x) \equiv \{(x, F(x)) \in \mathbb{R}^2 : x \in U\}$$

$$\operatorname{graph} f(x_1, \ldots, x_n) \equiv \{(x_1, \ldots, x_n, f(x_1, \ldots, x_n)) \in \mathbb{R}^{n+1} : (x_1, \ldots, x_n) \in U\}.$$

A graph is sometimes also called a PLOT.

Good routines for plotting graphs use adaptive algorithms which plot more points in regions where the function varies most rapidly (Wagon 1991, Math Works 1992, Heck 1993, Wickham-Jones 1994).

see also CURVE, EXTREMUM, GRAPH (GRAPH THEORY), HISTOGRAM, MAXIMUM, MINIMUM

References
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Heck, A. *Introduction to Maple, 2nd ed.* New York: Springer-Verlag, pp. 303–304, 1993.
Math Works. *Matlab Reference Guide.* Natick, MA: The Math Works, p. 216, 1992.
Tufte, E. R. *The Visual Display of Quantitative Information.* Cheshire, CN: Graphics Press, 1983.
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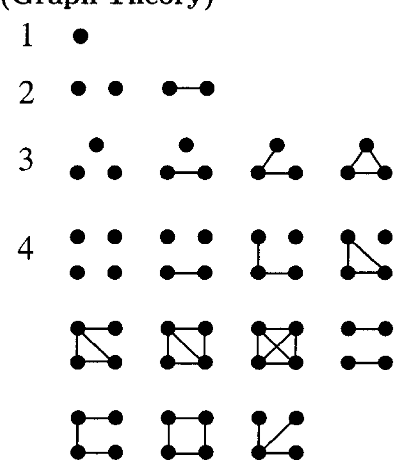
Wagon, S. *Mathematica in Action.* New York: W. H. Freeman, pp. 24–25, 1991.
Wickham-Jones, T. *Computer Graphics with Mathematica.* Santa Clara, CA: TELOS, pp. 579–584, 1994.
Yates, R. C. "Sketching." *A Handbook on Curves and Their Properties.* Ann Arbor, MI: J. W. Edwards, pp. 188–205, 1952.

Graph (Graph Theory)



A mathematical object composed of points known as VERTICES or NODES and lines connecting some (possibly empty) SUBSET of them, known as EDGES. The study of graphs is known as GRAPH THEORY. Graphs are 1-D COMPLEXES, and there are always an EVEN number of ODD NODES in a graph. The number of nonisomorphic graphs with v NODES is given by the PÓLYA ENUMERATION THEOREM. The first few values for $n = 1, 2, \ldots,$ are 1, 2, 4, 11, 34, 156, 1044, ... (Sloane's A000088; see above figure).

Graph sums, differences, powers, and products can be defined, as can graph eigenvalues.

Before applying PÓLYA ENUMERATION THEOREM, define the quantity

$$h_{\mathbf{j}} = \frac{p!}{\prod_{i=1}^{p} i^{j_i} j_i!}, \tag{1}$$

where $p!$ is the FACTORIAL of p, and the related polynomial

$$Z_p(S) = \sum_i h_{\mathbf{j}_i} \prod_{k=1}^{p} f_k{}^{(\mathbf{j}_i)_k}, \tag{2}$$

where the $\mathbf{j}_i = (j_1, \ldots, j_p)_i$ are all of the p-VECTORS satisfying

$$j_1 + 2j_2 + 3j_3 + \ldots + pj_p = p. \tag{3}$$

For example, for $p = 3$, the three possible values of $\mathbf{j}$ are

$$\mathbf{j}_1 = (3,0,0), \text{ since } (1\cdot 3) + (2\cdot 0) + (3\cdot 0) = 3,$$
$$\text{giving } h_{\mathbf{j}_1} = \frac{3!}{(1^3 3!)(2^0 0!)(3^0 0!)} = 1 \tag{4}$$
$$\mathbf{j}_2 = (1,1,0), \text{ since } (1\cdot 1) + (2\cdot 1) + (3\cdot 0) = 3,$$
$$\text{giving } h_{\mathbf{j}_2} = \frac{3!}{(1^1 1!)(2^1 1!)(3^0 0!)} = 3, \tag{5}$$
$$\mathbf{j}_3 = (0,0,1), \text{ since } (1\cdot 0) + (2\cdot 0) + (3\cdot 1) = 3$$
$$\text{giving } h_{\mathbf{j}_3} = \frac{3!}{(1^0 0!)(2^0 0!)(3^1 1!)} = 2. \tag{6}$$

Therefore,

$$Z_3(S) = f_1{}^3 + 3f_1 f_2 + 2f_3. \tag{7}$$

For small p, the first few values of $Z_p(S)$ are given by

$$Z_2(S) = f_1{}^2 + f_2 \tag{8}$$
$$Z_3(S) = f_1{}^3 + 3f_1 f_2 + 2f_3 \tag{9}$$
$$Z_4(S) = f_1{}^4 + 6f_1{}^2 f_2 + 3f_2{}^2 + 8f_1 f_3 + 6f_4 \tag{10}$$
$$\begin{aligned} Z_5(S) &= f_1{}^5 + 10f_1{}^3 f_2 + 15f_1 f_2{}^2 + 20f_1{}^2 f_3 \\ &\quad + 20f_2 f_3 + 30f_1 f_4 + 24f_5 \end{aligned} \tag{11}$$
$$\begin{aligned} Z_6(S) &= f_1{}^6 + 15f_1{}^4 f_2 + 45f_1{}^2 f_2{}^2 + 15f_2{}^3 \\ &\quad + 40f_1{}^3 f_3 + 120f_1 f_2 f_3 + 40f_3{}^2 \\ &\quad + 90f_1{}^2 f_4 + 90f_2 f_4 + 144f_1 f_5 + 120f_6 \end{aligned} \tag{12}$$
$$\begin{aligned} Z_7(S) &= f_1{}^7 + 21f_1{}^5 f_2 + 105f_1{}^3 f_2{}^2 + 105f_1 f_2{}^3 \\ &\quad + 70f_1{}^4 f_3 + 420f_1{}^2 f_2 f_3 + 210f_2{}^2 f_3 \\ &\quad + 280f_1 f_3{}^2 + 210f_1{}^3 f_4 + 630f_1 f_2 f_4 \\ &\quad + 420f_3 f_4 + 504f_1{}^2 f_5 + 504f_2 f_5 \\ &\quad + 840f_1 f_6 + 720f_7. \end{aligned} \tag{13}$$

Application of the PÓLYA ENUMERATION THEOREM then gives the formula

$$\begin{aligned} Z(R) &= \frac{1}{p!}\sum_{(j)} h_{\mathbf{j}} \prod_{n=0}^{\lfloor (p-1)/2 \rfloor} g_{2n+1}{}^{nj_{2n+1}+(2n+1)\binom{j_{2n+1}}{2}} \\ &\quad \times \prod_{n=1}^{\lfloor p/2 \rfloor} [(g_n g_{2n})^{n-1}]^{j_{2n}} g_{2n}{}^{2n\binom{j_{2n}}{2}} \\ &\quad \times \prod_{q=1}^{p} \prod_{r=q+1}^{p} g_{\mathrm{LCM}(q,r)}{}^{j_q j_r \mathrm{GCD}(q,r)}, \end{aligned} \tag{14}$$

where $\lfloor x \rfloor$ is the FLOOR FUNCTION, $\binom{n}{m}$ is a BINOMIAL COEFFICIENT, LCM is the LEAST COMMON MULTIPLE, GCD is the GREATEST COMMON DIVISOR, and the SUM (j) is over all $\mathbf{j}_i$ satisfying the sum identity described above. The first few generating functions $Z_p(R)$ are

$$Z_2(R) = 2g_1 \tag{15}$$
$$Z_3(R) = g_1{}^3 + 3g_1 g_2 + 2g_3 \tag{16}$$
$$Z_4(R) = g_1{}^6 + 9g_1{}^2 g_2{}^2 + 8g_3{}^2 + 6g_2 g_4 \tag{17}$$
$$\begin{aligned} Z_5(R) &= g_1{}^{10} + 10g_1{}^4 g_2{}^3 + 15g_1{}^2 g_2{}^4 + 20g_1 g_3{}^3 \\ &\quad + 30g_2 g_4{}^2 + 24g_5{}^2 + 20g_1 g_3 g_6 \end{aligned} \tag{18}$$
$$\begin{aligned} Z_6(R) &= g_1{}^{15} + 15g_1{}^7 g_2{}^4 + 60g_1{}^3 g_2{}^6 + 40g_1{}^3 g_3{}^4 \\ &\quad + 40g_3{}^5 + 180g_1 g_2 g_4{}^3 + 144g_5{}^3 \\ &\quad + 120g_1 g_2 g_3{}^2 g_6 + 120g_3 g_6{}^2 \end{aligned} \tag{19}$$
$$\begin{aligned} Z_7(R) &= g_1{}^{21} + 21g_1{}^{11} g_2{}^5 + 105g_1{}^5 g_2{}^8 \\ &\quad + 105g_1{}^3 g_2{}^9 + 70g_1{}^6 g_3{}^5 + 280g_3{}^7 \\ &\quad + 210g_1{}^3 g_2 g_4{}^4 + 630g_1 g_2{}^2 g_4{}^4 \\ &\quad + 504g_1 g_5{}^4 + 420g_1{}^2 g_2{}^2 g_3{}^3 g_6 \\ &\quad + 210g_1{}^2 g_2{}^2 g_3 g_6{}^2 + 840g_3 g_6{}^3 + 720g_7{}^3 \\ &\quad + 504g_1 g_5{}^2 g_{10} + 420g_2 g_3 g_4 g_{12}. \end{aligned} \tag{20}$$

Letting $g_i = 1 + x^i$ then gives a POLYNOMIAL $S_i(x)$, which is a GENERATING FUNCTION for (i.e., the terms of x^i give) the number of graphs with i EDGES. The total number of graphs having i edges is $S_i(1)$. The first few $S_i(x)$ are

$$S_2 = 1 + x \tag{21}$$
$$S_3 = 1 + x + x^2 + x^3 \tag{22}$$
$$S_4 = 1 + x + 2x^2 + 3x^3 + 2x^4 + x^5 + x^6 \tag{23}$$
$$\begin{aligned} S_5 &= 1 + x + 2x^2 + 4x^3 + 6x^4 + 6x^5 + 6x^6 \\ &\quad + 4x^7 + 2x^8 + x^9 + x^{10} \end{aligned} \tag{24}$$
$$\begin{aligned} S_6 &= 1 + x + 2x^2 + 5x^3 + 9x^4 + 15x^5 \\ &\quad + 21x^6 + 24x^7 + 24x^8 + 21x^9 \\ &\quad + 15x^{10} + 9x^{11} + 5x^{12} \\ &\quad + 2x^{13} + x^{14} + x^{15} \end{aligned} \tag{25}$$
$$\begin{aligned} S_7 &= 1 + x + 2x^2 + 5x^3 + 10x^4 + 21x^5 \\ &\quad + 21x^6 + 24x^7 + 41x^6 + 65x^7 + 97x^8 \end{aligned}$$

$$+ 131x^9 + 148x^{10} + 148x^{11} + 131x^{12} + 97x^{13} + 65x^{14} + 41x^{15} + 21x^{16} + 10x^{17} + 5x^{18} + 2x^{19} + x^{20} + x^{21}, \quad (26)$$

giving the number of graphs with n nodes as 1, 2, 4, 11, 34, 156, 1044, ... (Sloane's A000088). King and Palmer (cited in Read 1981) have calculated S_n up to $n = 24$, for which

$$S_{24} = 195,704,906,302,078,447,922,174,862,416,\cdots \cdots 726,256,004,122,075,267,063,365,754,368. \quad (27)$$

see also BIPARTITE GRAPH, CATERPILLAR GRAPH, CAYLEY GRAPH, CIRCULANT GRAPH, COCKTAIL PARTY GRAPH, COMPARABILITY GRAPH, COMPLEMENT GRAPH, COMPLETE GRAPH, CONE GRAPH, CONNECTED GRAPH, COXETER GRAPH, CUBICAL GRAPH, DE BRUIJN GRAPH, DIGRAPH, DIRECTED GRAPH, DODECAHEDRAL GRAPH, EULER GRAPH, EXTREMAL GRAPH, GEAR GRAPH, GRACEFUL GRAPH, GRAPH THEORY, HANOI GRAPH, HARARY GRAPH, HARMONIOUS GRAPH, HOFFMAN-SINGLETON GRAPH, ICOSAHEDRAL GRAPH, INTERVAL GRAPH, ISOMORPHIC GRAPHS, LABELLED GRAPH, LADDER GRAPH, LATTICE GRAPH, MATCHSTICK GRAPH, MINOR GRAPH, MOORE GRAPH, NULL GRAPH, OCTAHEDRAL GRAPH, PATH GRAPH, PETERSEN GRAPHS, PLANAR GRAPH, RANDOM GRAPH, REGULAR GRAPH, SEQUENTIAL GRAPH, SIMPLE GRAPH, STAR GRAPH, SUBGRAPH, SUPERGRAPH, SUPERREGULAR GRAPH, SYLVESTER GRAPH, TETRAHEDRAL GRAPH, THOMASSEN GRAPH, TOURNAMENT, TRIANGULAR GRAPH, TURAN GRAPH, TUTTE'S GRAPH, UNIVERSAL GRAPH, UTILITY GRAPH, WEB GRAPH, WHEEL GRAPH

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Sloane, N. J. A. and Plouffe, S. Extended entry in *The Encyclopedia of Integer Sequences.* San Diego: Academic Press, 1995.

Graph Theory

The mathematical study of the properties of the formal mathematical structures called GRAPHS.

see also ADJACENCY MATRIX, ADJACENCY RELATION, ARTICULATION VERTEX, BLUE-EMPTY COLORING, BRIDGE (GRAPH), CHROMATIC NUMBER, CHROMATIC POLYNOMIAL, CIRCUIT RANK, CROSSING NUMBER (GRAPH), CYCLE (GRAPH), CYCLOMATIC NUMBER, DEGREE, DIAMETER (GRAPH), DIJKSTRA'S ALGORITHM, ECCENTRICITY, EDGE-COLORING, EDGE CONNECTIVITY, EULERIAN CIRCUIT, EULERIAN TRAIL, FACTOR (GRAPH), FLOYD'S ALGORITHM, GIRTH, GRAPH TWO-COLORING, GROUP THEORY, HAMILTONIAN CIRCUIT, HASSE DIAGRAM, HUB, INDEGREE, INTEGRAL DRAWING, ISTHMUS, JOIN (GRAPH), LOCAL DEGREE, MONOCHROMATIC FORCED TRIANGLE, OUTDEGREE, PARTY PROBLEM, PÓLYA ENUMERATION THEOREM, PÓLYA POLYNOMIAL, RADIUS (GRAPH), RAMSEY NUMBER, RE-ENTRANT CIRCUIT, SEPARATING EDGE, TAIT COLORING, TAIT CYCLE, TRAVELING SALESMAN PROBLEM, TREE, TUTTE'S THEOREM, UNICURSAL CIRCUIT, VALENCY, VERTEX COLORING, WALK

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Graph Two-Coloring

Assignment of each EDGE of a GRAPH to one of two color classes ("red" or "green").

see also BLUE-EMPTY GRAPH, MONOCHROMATIC FORCED TRIANGLE

Graphical Partition

A graphical partition of order n is the DEGREE SEQUENCE of a GRAPH with $n/2$ EDGES and no isolated VERTICES. For $n = 2, 4, 6, \ldots$, the number of graphical partitions is 1, 2, 5, 9, 17, ... (Sloane's A000569).

References
Barnes, T. M. and Savage, C. D. "A Recurrence for Counting Graphical Partitions." *Electronic J. Combinatorics* **2**, R11, 1–10, 1995. `http://www.combinatorics.org/Volume_2/volume2.html#R11`.
Barnes, T. M. and Savage, C. D. "Efficient Generation of Graphical Partitions." Submitted.
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Grassmann Algebra

see EXTERIOR ALGEBRA

Grassmann Coordinates

An $(m+1)$-D SUBSPACE W of an $(n+1)$-D VECTOR SPACE V can be specified by an $(m+1)\times(n+1)$ MATRIX whose rows are the coordinates of a BASIS of W. The set of all $\binom{n+1}{m+1}$ $(m+1)\times(m+1)$ MINORS of this MATRIX are then called the Grassmann coordinates of w (where $\binom{a}{b}$ is a BINOMIAL COEFFICIENT).

see also CHOW COORDINATES

References
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Grassmann Manifold

A special case of a FLAG MANIFOLD. A Grassmann manifold is a certain collection of vector SUBSPACES of a VECTOR SPACE. In particular, $G_{n,k}$ is the Grassmann manifold of k-dimensional subspaces of the VECTOR SPACE $\mathbb{R}^n$. It has a natural MANIFOLD structure as an orbit-space of the STIEFEL MANIFOLD $V_{n,k}$ of orthonormal k-frames in $\mathbb{R}^n$. One of the main things about Grassmann manifolds is that they are classifying spaces for VECTOR BUNDLES.

Gray Code

An encoding of numbers so that adjacent numbers have a single DIGIT differing by 1. A BINARY Gray code with n DIGITS corresponds to a HAMILTONIAN PATH on an n-D HYPERCUBE (including direction reversals). The term Gray code is often used to refer to a "reflected" code, or more specifically still, the binary reflected Gray code.

To convert a BINARY number $d_1\, d_2 \cdots d_{n-1}\, d_n$ to its corresponding binary reflected Gray code, start at the right with the digit d_n (the nth, or last, DIGIT). If the d_{n-1} is 1, replace d_n by $1-d_n$; otherwise, leave it unchanged. Then proceed to d_{n-1}. Continue up to the first DIGIT d_1, which is kept the same since d_0 is assumed to be a 0. The resulting number $g_1\, g_2 \cdots g_{n-1}\, g_n$ is the reflected binary Gray code.

To convert a binary reflected Gray code $g_1\, g_2 \cdots g_{n-1}\, g_n$ to a BINARY number, start again with the nth digit, and compute

$$\Sigma_n \equiv \sum_{i=1}^{n-1} g_i \pmod 2.$$

If Σ_n is 1, replace g_n by $1-g_n$; otherwise, leave it the unchanged. Next compute

$$\Sigma_{n-1} \equiv \sum_{i=1}^{n-2} g_i \pmod 2,$$

and so on. The resulting number $d_1\, d_2 \cdots d_{n-1}\, d_n$ is the BINARY number corresponding to the initial binary reflected Gray code.

The code is called reflected because it can be generated in the following manner. Take the Gray code 0, 1. Write it forwards, then backwards: 0, 1, 1, 0. Then append 0s to the first half and 1s to the second half: 00, 01, 11, 10. Continuing, write 00, 01, 11, 10, 10, 11, 01, 00 to obtain: 000, 001, 011, 010, 110, 111, 101, 100, ... (Sloane's A014550). Each iteration therefore doubles the number of codes. The Gray codes corresponding to the first few nonnegative integers are given in the following table.

0	0	20	11110	40	111100
1	1	21	11111	41	111101
2	11	22	11101	42	111111
3	10	23	11100	43	111110
4	110	24	10100	44	111010
5	111	25	10101	45	111011
6	101	26	10111	46	111001
7	100	27	10110	47	111000
8	1100	28	10010	48	101000
9	1101	29	10011	49	101001
10	1111	30	10001	50	101011
11	1110	31	10000	51	101010
12	1010	32	110000	52	101110
13	1011	33	110001	53	101111
14	1001	34	110011	54	101101
15	1000	35	110010	55	101100
16	11000	36	110110	56	100100
17	11001	37	110111	57	100101
18	11011	38	110101	58	100111
19	11010	39	110100	59	100110

The binary reflected Gray code is closely related to the solution of the TOWERS OF HANOI as well as the BAGUENAUDIER.

see also BAGUENAUDIER, BINARY, HILBERT CURVE, MORSE-THUE SEQUENCE, RYSER FORMULA, TOWERS OF HANOI

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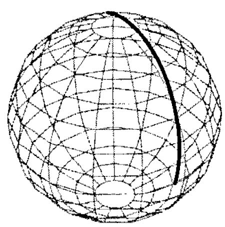
Great Circle



The shortest path between two points on a SPHERE, also known as an ORTHODROME. To find the great circle (GEODESIC) distance between two points located at LATITUDE δ and LONGITUDE λ of (δ_1, λ_1) and (δ_2, λ_2) on a SPHERE of RADIUS a, convert SPHERICAL COORDINATES to CARTESIAN COORDINATES using

$$\mathbf{r}_i = a \begin{bmatrix} \cos\lambda_i \cos\delta_i \\ \sin\lambda_i \cos\delta_i \\ \sin\delta_i \end{bmatrix}. \tag{1}$$

(Note that the LATITUDE δ is related to the COLATITUDE ϕ of SPHERICAL COORDINATES by $\delta = 90° - \phi$, so the conversion to CARTESIAN COORDINATES replaces $\sin\phi$ and $\cos\phi$ by $\cos\delta$ and $\sin\delta$, respectively.) Now find the ANGLE α between $\mathbf{r}_1$ and $\mathbf{r}_2$ using the DOT PRODUCT,

$$\begin{aligned}\cos\alpha &= \hat{\mathbf{r}}_1 \cdot \hat{\mathbf{r}}_2 \\ &= \cos\delta_1 \cos\delta_2 (\sin\lambda_1 \sin\lambda_2 + \cos\lambda_1 \cos\lambda_2) \\ &\quad + \sin\delta_1 \sin\delta_2 \\ &= \cos\delta_1 \cos\delta_2 \cos(\lambda_1 - \lambda_2) + \sin\delta_1 \sin\delta_2. \end{aligned} \tag{2}$$

The great circle distance is then

$$d = a\cos^{-1}[\cos\delta_1 \cos\delta_2 \cos(\lambda_1 - \lambda_2) + \sin\delta_1 \sin\delta_2]. \tag{3}$$

For the Earth, the *equatorial* RADIUS is $a \approx 6378$ km, or 3963 (statute) miles. Unfortunately, the FLATTENING of the Earth cannot be taken into account in this simple derivation, since the problem is considerable more complicated for a SPHEROID or ELLIPSOID (each of which has a RADIUS which is a function of LATITUDE).

The equation of the great circle can be explicitly computed using the GEODESIC formalism. Writing

$$u = \lambda \tag{4}$$

$$v = \delta = \tfrac{1}{2}\pi - \phi \tag{5}$$

gives the P, Q, and R parameters of the GEODESIC (which are just combinations of the PARTIAL DERIVATIVES) as

$$P \equiv \left(\frac{\partial x}{\partial u}\right)^2 + \left(\frac{\partial y}{\partial u}\right)^2 + \left(\frac{\partial z}{\partial u}\right)^2 = a^2 \sin^2 v \tag{6}$$

$$Q \equiv \frac{\partial x}{\partial u}\frac{\partial x}{\partial v} + \frac{\partial y}{\partial u}\frac{\partial y}{\partial v} + \frac{\partial z}{\partial u}\frac{\partial z}{\partial v} = 0 \tag{7}$$

$$R \equiv \left(\frac{\partial x}{\partial v}\right)^2 + \left(\frac{\partial y}{\partial v}\right)^2 + \left(\frac{\partial z}{\partial v}\right)^2 = a^2. \tag{8}$$

The GEODESIC differential equation then becomes

$$\cos v \sin^4 v + 2\cos v \sin^2 v v'^2 + \cos v v'^4 - \sin v v'' = 0. \tag{9}$$

However, because this is a special case of $Q = 0$ with P and R explicit functions of v only, the GEODESIC solution takes on the special form

$$\begin{aligned} v &= c_1 \int \sqrt{\frac{R}{P^2 - c_1{}^2 P}}\, dv = c_1 \int \frac{dv}{a^2 \sin^4 v - c_1{}^2 \sin^2 v} \\ &= \int \frac{dv}{\sin v \sqrt{\left(\frac{a}{c_1}\right)^2 \sin^2 v - 1}} \\ &= -\tan^{-1}\left[\frac{\cos v}{\sqrt{\left(\frac{a}{c_1}\right)^2 - 1}}\right] + c_2 \end{aligned} \tag{10}$$

(Gradshteyn and Ryzhik 1979, p. 174, eqn. 2.599.6), which can be rewritten as

$$v = -\sin^{-1}\left(\frac{\cot v}{\sqrt{\left(\frac{a}{c_1}\right)^2 - 1}}\right) + c_2. \tag{11}$$

It therefore follows that

$$(\sin c_2) a \sin v \cos u - (\cos c_2) a \sin v \sin u - \frac{a\cos v}{\sqrt{\left(\frac{a}{c_1}\right)^2 - 1}} = 0. \tag{12}$$

This equation can be written in terms of the CARTESIAN COORDINATES as

$$x \sin c_2 - y\cos c_2 - \frac{z}{\sqrt{\left(\frac{a}{c_1}\right)^2 - 1}} = 0, \tag{13}$$

which is simply a PLANE passing through the center of the SPHERE and the two points on the surface of the SPHERE.

see also GEODESIC, GREAT SPHERE, LOXODROME, MIKUSIŃSKI'S PROBLEM, ORTHODROME, POINT-POINT DISTANCE—2-D, SPHERE

References

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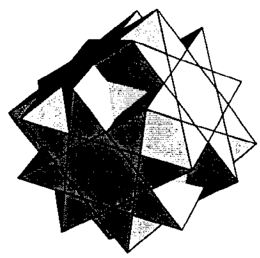
Weinstock, R. *Calculus of Variations, with Applications to Physics and Engineering.* New York: Dover, pp. 26–28 and 62–63, 1974.

Great Cubicuboctahedron



The UNIFORM POLYHEDRON U_{14} whose DUAL POLYHEDRON is the GREAT HEXACRONIC ICOSITETRAHEDRON. It has WYTHOFF SYMBOL $3\,4\,|\,\frac{4}{3}$. Its faces are $8\{3\}+6\{4\}+6\{\frac{8}{3}\}$. It is a FACETED version of the CUBE. The CIRCUMRADIUS of a great cubicuboctahedron with unit edge length is

$$R = \tfrac{1}{2}\sqrt{5-2\sqrt{2}}.$$

References

Wenninger, M. J. *Polyhedron Models.* Cambridge, England: Cambridge University Press, pp. 118–119, 1989.

Great Deltoidal Hexecontahedron

The DUAL of the GREAT RHOMBICOSIDODECAHEDRON (UNIFORM).

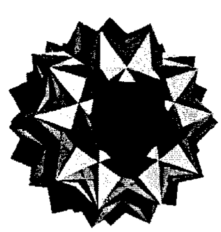
Great Deltoidal Icositetrahedron

The DUAL of the GREAT RHOMBICUBOCTAHEDRON (UNIFORM).

Great Dirhombicosidodecacron

The DUAL of the GREAT DIRHOMBICOSIDODECAHEDRON.

Great Dirhombicosidodecahedron



The UNIFORM POLYHEDRON U_{75} whose DUAL is the GREAT DIRHOMBICOSIDODECACRON. This POLYHEDRON is exceptional because it cannot be derived from SCHWARZ TRIANGLES and because it is the only UNIFORM POLYHEDRON with more than six POLYGONS surrounding each VERTEX (four SQUARES alternating with two TRIANGLES and two PENTAGRAMS). It has WYTHOFF SYMBOL $|\,\frac{3}{2}\,\frac{5}{3}\,3\,\frac{5}{2}$. Its faces are $40\{3\}+60\{4\}+24\{\frac{5}{2}\}$, and its CIRCUMRADIUS for unit edge length is

$$R = \tfrac{1}{2}\sqrt{2}.$$

References

Wenninger, M. J. *Polyhedron Models.* Cambridge, England: Cambridge University Press, pp. 200–203, 1989.

Great Disdyakis Dodecahedron

The DUAL of the GREAT TRUNCATED CUBOCTAHEDRON.

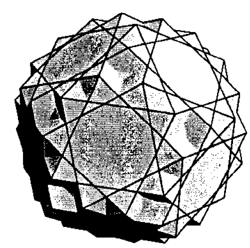
Great Disdyakis Triacontahedron

The DUAL of the GREAT TRUNCATED ICOSIDODECAHEDRON.

Great Ditrigonal Dodecacronic Hexecontahedron

The DUAL of the GREAT DITRIGONAL DODECICOSIDODECAHEDRON.

Great Ditrigonal Dodecicosidodecahedron



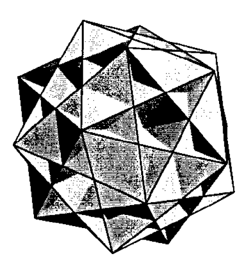
The UNIFORM POLYHEDRON U_{42} whose DUAL is the GREAT DITRIGONAL DODECACRONIC HEXECONTAHEDRON. It has WYTHOFF SYMBOL $3\,5\,|\,\frac{5}{3}$. Its faces are $20\{3\}+12\{5\}+12\{\frac{10}{3}\}$, and its CIRCUMRADIUS for unit edge length is

$$R = \tfrac{1}{4}\sqrt{34-6\sqrt{5}}.$$

References

Wenninger, M. J. *Polyhedron Models.* Cambridge, England: Cambridge University Press, p. 125, 1989.

Great Ditrigonal Icosidodecahedron



The UNIFORM POLYHEDRON U_{47} whose DUAL is the GREAT TRIAMBIC ICOSAHEDRON. It has WYTHOFF SYMBOL $\frac{3}{2}\,|\,3\,5$. Its faces are $20\{3\} + 12\{5\}$, and its CIRCUMRADIUS for unit edge length is

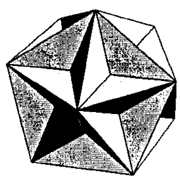
$$R = \tfrac{1}{2}\sqrt{3}.$$

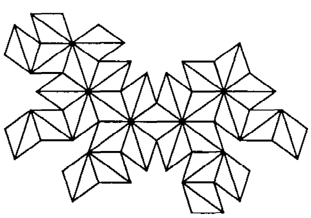
References
Wenninger, M. J. *Polyhedron Models.* Cambridge, England: Cambridge University Press, pp. 135–136, 1989.

Great Dodecacronic Hexecontahedron
The DUAL of the GREAT DODECICOSIDODECAHEDRON.

Great Dodecadodecahedron
see DODECADODECAHEDRON

Great Dodecahedron





The UNIFORM POLYHEDRON U_{35} which is the DUAL of the SMALL STELLATED DODECAHEDRON and one of the KEPLER-POINSOT SOLIDS. Its faces are $12\{5\}$. Its SCHLÄFLI SYMBOL is $\{5, \frac{5}{2}\}$, and its WYTHOFF SYMBOL is $\frac{5}{2}\,|\,2\,5$. Its faces are $12\{5\}$. Its CIRCUMRADIUS for unit edge length is

$$R = \tfrac{1}{2}5^{1/4}\phi^{1/2}a = \tfrac{1}{4}5^{1/4}\sqrt{2(1+\sqrt{5})},$$

where ϕ is the GOLDEN RATIO.

see also GREAT ICOSAHEDRON, GREAT STELLATED DODECAHEDRON, KEPLER-POINSOT SOLID, SMALL STELLATED DODECAHEDRON

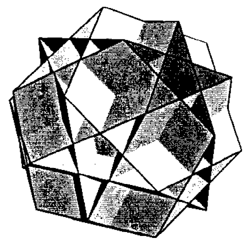
References
Fischer, G. (Ed.). Plate 105 in *Mathematische Modelle/Mathematical Models, Bildband/Photograph Volume.* Braunschweig, Germany: Vieweg, p. 104, 1986.

Great Dodecahedron-Small Stellated Dodecahedron Compound
A POLYHEDRON COMPOUND in which the GREAT DODECAHEDRON is interior to the SMALL STELLATED DODECAHEDRON.

see also POLYHEDRON COMPOUND

Great Dodecahemicosacron
The DUAL of the GREAT DODECAHEMICOSAHEDRON.

Great Dodecahemicosahedron



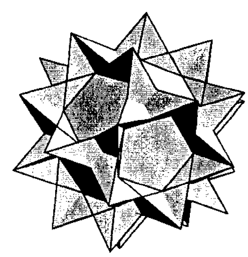
The UNIFORM POLYHEDRON U_{65} whose DUAL is the GREAT DODECAHEMICOSACRON. It has WYTHOFF SYMBOL $\frac{5}{3}\,\frac{5}{2}\,|\,\frac{5}{3}$. Its faces are $12\{\frac{5}{2}\} + 6\{\frac{10}{3}\}$. It is a FACETED DODECADODECAHEDRON. The CIRCUMRADIUS for unit edge length is

$$R = \tfrac{1}{2}\sqrt{3}.$$

References
Wenninger, M. J. *Polyhedron Models.* Cambridge, England: Cambridge University Press, pp. 106–107, 1989.

Great Dodecahemidodecacron
The DUAL of the GREAT DODECAHEMIDODECAHEDRON.

Great Dodecahemidodecahedron



The UNIFORM POLYHEDRON U_{70} whose DUAL is the GREAT DODECAHEMIDODECACRON. It has WYTHOFF SYMBOL $\frac{5}{3}\,\frac{5}{2}\,|\,\frac{5}{3}$. Its faces are $12\{\frac{5}{2}\} + 6\{\frac{10}{3}\}$. Its CIRCUMRADIUS for unit edge length is

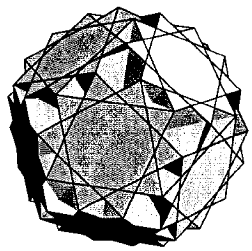
$$R = \phi^{-1},$$

where ϕ is the GOLDEN RATIO.

References
Wenninger, M. J. *Polyhedron Models.* Cambridge, England: Cambridge University Press, p. 165, 1989.

Great Dodecicosacron
The DUAL of the GREAT DODECICOSAHEDRON.

Great Dodecicosahedron



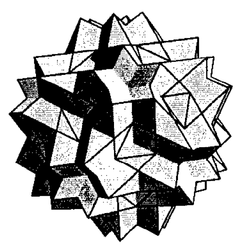
The UNIFORM POLYHEDRON U_{63} whose DUAL is the GREAT DODECICOSACRON. It has WYTHOFF SYMBOL $3\,\frac{5}{3}\,\Big|\,\begin{matrix}\frac{3}{2}\\ \frac{5}{2}\end{matrix}\Big|$. Its faces are $20\{6\} + 12\{\frac{10}{3}\}$. Its CIRCUMRADIUS for unit edge length is

$$R = \tfrac{1}{4}\sqrt{34 - 6\sqrt{5}}.$$

References

Wenninger, M. J. *Polyhedron Models.* Cambridge, England: Cambridge University Press, pp. 156–157, 1989.

Great Dodecicosidodecahedron



The UNIFORM POLYHEDRON U_{61} whose DUAL is the GREAT DODECACRONIC HEXECONTAHEDRON. Its WYTHOFF SYMBOL is $2\,\frac{5}{2}\mid 3$. Its faces are $20\{6\}+12\{\frac{5}{2}\}$, and its CIRCUMRADIUS for unit edge length is

$$R = \tfrac{1}{4}\sqrt{58 - 18\sqrt{5}}.$$

References

Wenninger, M. J. *Polyhedron Models.* Cambridge, England: Cambridge University Press, p. 148, 1989.

Great Hexacronic Icositetrahedron

The DUAL of the GREAT CUBICUBOCTAHEDRON.

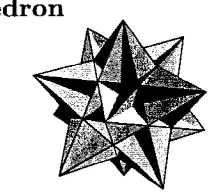
Great Hexagonal Hexecontahedron

The DUAL of the GREAT SNUB DODECICOSIDODECAHEDRON.

Great Icosacronic Hexecontahedron

The DUAL of the GREAT ICOSICOSIDODECAHEDRON.

Great Icosahedron



One of the KEPLER-POINSOT SOLIDS whose DUAL is the GREAT STELLATED DODECAHEDRON. Its faces are $20\{3\}$. It is also UNIFORM POLYHEDRON U_{53} and has WYTHOFF SYMBOL $3\,\frac{5}{2}\mid\frac{5}{3}$. Its faces are $20\{3\}+12\{\frac{5}{2}\}+12\{\frac{10}{3}\}$. Its CIRCUMRADIUS for unit edge length is

$$R = \tfrac{1}{2}\sqrt{11 - 4\sqrt{5}}.$$

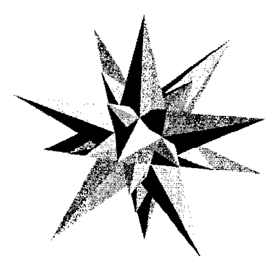
see also GREAT DODECAHEDRON, GREAT ICOSAHEDRON, GREAT STELLATED DODECAHEDRON, KEPLER-POINSOT SOLID, SMALL STELLATED DODECAHEDRON, TRUNCATED GREAT ICOSAHEDRON

References

Fischer, G. (Ed.). Plate 106 in *Mathematische Modelle/Mathematical Models, Bildband/Photograph Volume.* Braunschweig, Germany: Vieweg, p. 105, 1986.

Wenninger, M. J. *Polyhedron Models.* Cambridge, England: Cambridge University Press, p. 154, 1989.

Great Icosahedron-Great Stellated Dodecahedron Compound



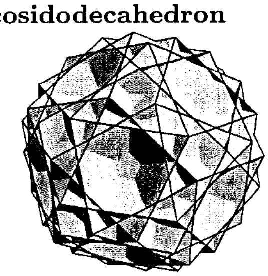
A POLYHEDRON COMPOUND most easily constructed by adding the VERTICES of a GREAT ICOSAHEDRON to a GREAT STELLATED DODECAHEDRON.

see also POLYHEDRON COMPOUND

References

Cundy, H. and Rollett, A. *Mathematical Models, 3rd ed.* Stradbroke, England: Tarquin Pub., pp. 132–133, 1989.

Great Icosicosidodecahedron



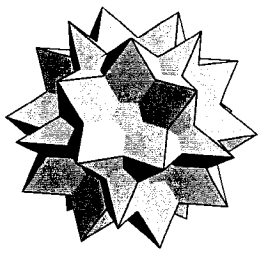
The UNIFORM POLYHEDRON U_{48} whose DUAL is the GREAT ICOSACRONIC HEXECONTAHEDRON. It has WYTHOFF SYMBOL $\frac{3}{2}\,5\,|\,3$. Its faces are $20\{3\}+20\{6\}+12\{5\}$. Its CIRCUMRADIUS for unit edge length is

$$R = \tfrac{1}{4}\sqrt{34-6\sqrt{5}}\,.$$

References

Wenninger, M. J. *Polyhedron Models.* Cambridge, England: Cambridge University Press, pp. 137–139, 1989.

Great Icosidodecahedron



A UNIFORM POLYHEDRON U_{54} whose DUAL is the GREAT RHOMBIC TRIACONTAHEDRON (also called the GREAT STELLATED TRIACONTAHEDRON). It is a STELLATED ARCHIMEDEAN SOLID. It has SCHLÄFLI SYMBOL $\left\{\begin{matrix}3\\ \frac{5}{2}\end{matrix}\right\}$ and WYTHOFF SYMBOL $2\,|\,3\,\frac{5}{2}$. Its faces are $20\{3\}+12\{\frac{5}{2}\}$. Its CIRCUMRADIUS for unit edge length is

$$R = \phi^{-1},$$

where ϕ is the GOLDEN RATIO.

References

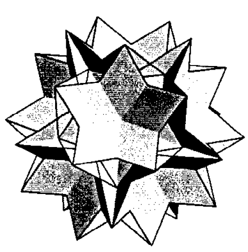
Cundy, H. and Rollett, A. *Mathematical Models, 3rd ed.* Stradbroke, England: Tarquin Pub., p. 124, 1989.

Wenninger, M. J. *Polyhedron Models.* Cambridge, England: Cambridge University Press, p. 147, 1989.

Great Icosihemidodecacron

The DUAL of the GREAT ICOSIHEMIDODECAHEDRON.

Great Icosihemidodecahedron



The UNIFORM POLYHEDRON U_{71} whose DUAL is the GREAT ICOSIHEMIDODECACRON. It has WYTHOFF SYMBOL $\frac{3}{2}\,3\,|\,\frac{5}{3}$. Its faces are $20\{3\}+6\{\frac{10}{3}\}$. For unit edge length, its CIRCUMRADIUS is

$$R = \phi^{-1},$$

where ϕ is the GOLDEN RATIO.

References

Wenninger, M. J. *Polyhedron Models.* Cambridge, England: Cambridge University Press, p. 164, 1989.

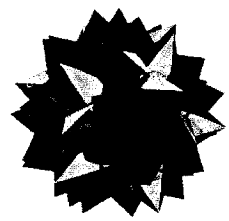
Great Inverted Pentagonal Hexecontahedron

The DUAL of the GREAT INVERTED SNUB ICOSIDODECAHEDRON.

Great Inverted Retrosnub Icosidodecahedron

see GREAT RETROSNUB ICOSIDODECAHEDRON

Great Inverted Snub Icosidodecahedron



The UNIFORM POLYHEDRON U_{69} whose DUAL is the GREAT INVERTED PENTAGONAL HEXECONTAHEDRON. It has WYTHOFF SYMBOL $|\,2\,3\,\frac{5}{2}$. Its faces are $80\{3\}+12\{\frac{5}{2}\}$. For unit edge length, it has CIRCUMRADIUS

$$R = \frac{1}{2}\sqrt{\frac{8\cdot 2^{2/3}-16x+2^{1/3}x^2}{8\cdot 2^{2/3}-10x+2^{1/3}x^2}} = 0.816080674799923,$$

where

$$x \equiv \left(49-27\sqrt{5}+3\sqrt{6}\sqrt{93-49\sqrt{5}}\right)^{1/3}.$$

References

Wenninger, M. J. *Polyhedron Models.* Cambridge, England: Cambridge University Press, p. 179, 1989.

Great Pentagonal Hexecontahedron

The DUAL of the GREAT SNUB ICOSIDODECAHEDRON.

Great Pentagrammic Hexecontahedron

The DUAL of the GREAT RETROSNUB ICOSIDODECAHEDRON.

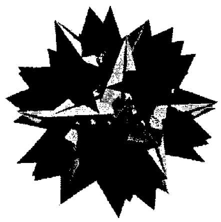
Great Pentakis Dodecahedron

The DUAL of the SMALL STELLATED TRUNCATED DODECAHEDRON.

Great Quasitruncated Icosidodecahedron

see GREAT TRUNCATED ICOSIDODECAHEDRON

Great Retrosnub Icosidodecahedron



The UNIFORM POLYHEDRON U_{74}, also called the GREAT INVERTED RETROSNUB ICOSIDODECAHEDRON, whose DUAL is the GREAT PENTAGRAMMIC HEXECONTAHEDRON. It has WYTHOFF SYMBOL $|\,2\,\frac{3}{2}\,\frac{5}{3}$. Its faces are $80\{3\}+12\{\frac{5}{2}\}$. For unit edge length, it has CIRCUMRADIUS

$$R=\tfrac{1}{2}\sqrt{\frac{2-x}{1-x}}\approx 0.5800015,$$

where x is the smaller NEGATIVE root of

$$x^3+2x^2-\phi^{-2}=0,$$

with ϕ the GOLDEN MEAN.

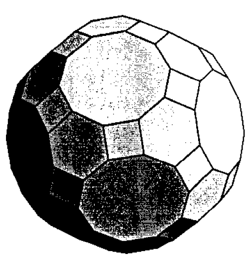
References
Wenninger, M. J. *Polyhedron Models.* Cambridge, England: Cambridge University Press, pp. 189–193, 1989.

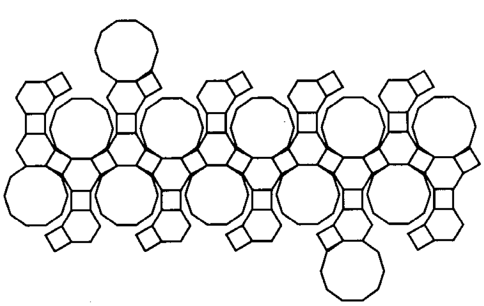
Great Rhombic Triacontahedron

A ZONOHEDRON which is the DUAL of the GREAT ICOSIDODECAHEDRON. It is also called the GREAT STELLATED TRIACONTAHEDRON.

References
Cundy, H. and Rollett, A. *Mathematical Models, 3rd ed.* Stradbroke, England: Tarquin Pub., p. 126, 1989.

Great Rhombicosidodecahedron (Archimedean)





An ARCHIMEDEAN SOLID also known as the RHOMBITRUNCATED ICOSIDODECAHEDRON. It is sometimes improperly called the TRUNCATED ICOSIDODECAHEDRON, a name which is inappropriate since TRUNCATION would yield RECTANGULAR instead of SQUARE. The great rhombicosidodecahedron is also UNIFORM POLYHEDRON U_{28}. Its DUAL is the DISDYAKIS TRIACONTAHEDRON, also called the HEXAKIS ICOSAHEDRON. It has SCHLÄFLI SYMBOL $t\begin{Bmatrix}3\\5\end{Bmatrix}$ and WYTHOFF SYMBOL $2\,3\,5\,|$. The INRADIUS, MIDRADIUS, and CIRCUMRADIUS for $a=1$ are

$$r=\tfrac{1}{241}(105+6\sqrt{5})\sqrt{31+12\sqrt{5}}\approx 3.73665$$

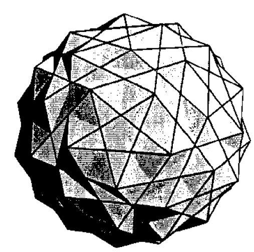
$$\rho=\tfrac{1}{2}\sqrt{30+12\sqrt{5}}\approx 3.76938$$

$$R=\tfrac{1}{2}\sqrt{31+12\sqrt{5}}\approx 3.80239.$$

see also SMALL RHOMBICOSIDODECAHEDRON

References
Ball, W. W. R. and Coxeter, H. S. M. *Mathematical Recreations and Essays, 13th ed.* New York: Dover, p. 137, 1987.

Great Rhombicosidodecahedron (Uniform)

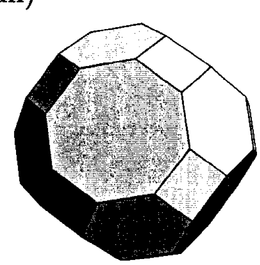


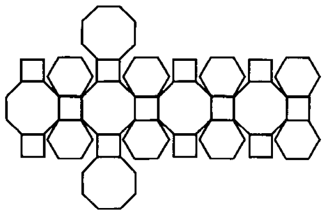
The UNIFORM POLYHEDRON U_{67}, also called the QUASIRHOMBICOSIDODECAHEDRON, whose DUAL is the GREAT DELTOIDAL HEXECONTAHEDRON. It has SCHLÄFLI SYMBOL $r'\begin{Bmatrix}3\\\frac{5}{2}\end{Bmatrix}$. It has WYTHOFF SYMBOL $3\,\frac{5}{3}\,|\,2$. Its faces are $20\{3\}+30\{4\}+12\{\frac{5}{2}\}$. For unit edge length, its CIRCUMRADIUS is

$$R=\tfrac{1}{2}\sqrt{11-4\sqrt{5}}.$$

References
Wenninger, M. J. *Polyhedron Models.* Cambridge, England: Cambridge University Press, pp. 162–163, 1989.

Great Rhombicuboctahedron (Archimedean)





An ARCHIMEDEAN SOLID sometimes (improperly) called the TRUNCATED CUBOCTAHEDRON and also called the RHOMBITRUNCATED CUBOCTAHEDRON. Its DUAL is the DISDYAKIS DODECAHEDRON, also called the HEXAKIS OCTAHEDRON. It has SCHLÄFLI SYMBOL $t\left\{{3 \atop 4}\right\}$. It is also UNIFORM POLYHEDRON U_{11} and has WYTHOFF SYMBOL $2\,3\,4\,|$. Its faces are $8\{6\}+12\{4\}+6\{8\}$. The SMALL CUBICUBOCTAHEDRON is a FACETED version. The INRADIUS, MIDRADIUS, and CIRCUMRADIUS for unit edge length are

$$r = \tfrac{3}{97}(14+\sqrt{2})\sqrt{13+6\sqrt{2}} \approx 2.20974$$
$$\rho = \tfrac{1}{2}\sqrt{12+6\sqrt{2}} \approx 2.26303$$
$$R = \tfrac{1}{2}\sqrt{13+6\sqrt{2}} \approx 2.31761.$$

Additional quantities are

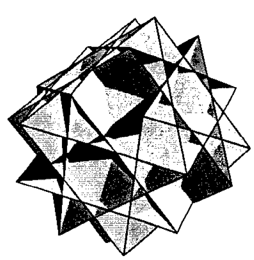
$$t = \tan(\tfrac{1}{8}\pi) = \sqrt{2}-1$$
$$l = 2t = 2(\sqrt{2}-1)$$
$$h = 1 + l\sin(\tfrac{1}{4}\pi) = 3-\sqrt{2}.$$

see also SMALL RHOMBICUBOCTAHEDRON, GREAT TRUNCATED CUBOCTAHEDRON

References

Ball, W. W. R. and Coxeter, H. S. M. *Mathematical Recreations and Essays, 13th ed.* New York: Dover, p. 138, 1987.

Great Rhombicuboctahedron (Uniform)



The UNIFORM POLYHEDRON U_{17}, also known as the QUASIRHOMBICUBOCTAHEDRON, whose DUAL is the GREAT DELTOIDAL ICOSITETRAHEDRON. It has SCHLÄFLI SYMBOL $r'\left\{{3 \atop 4}\right\}$ and WYTHOFF SYMBOL $\frac{3}{2}\,4\,|\,2$. Its faces are $8\{3\}+20\{4\}$. Its CIRCUMRADIUS for unit edge length is

$$R = \tfrac{1}{2}\sqrt{5-2\sqrt{2}}.$$

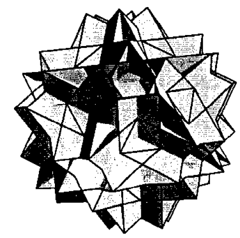
References

Wenninger, M. J. *Polyhedron Models.* Cambridge, England: Cambridge University Press, pp. 132–133, 1989.

Great Rhombidodecacron

The DUAL of the GREAT RHOMBIDODECAHEDRON.

Great Rhombidodecahedron



The UNIFORM POLYHEDRON U_{73} whose DUAL is the Great Rhombidodecacron. It WYTHOFF SYMBOL $2\,\frac{5}{3}\,|\,\frac{3}{2}\,\frac{5}{4}\,|$. Its faces are $30\{4\}+12\{\frac{10}{3}\}$. Its CIRCUMRADIUS for unit edge length is

$$R = \tfrac{1}{2}\sqrt{11-4\sqrt{5}}.$$

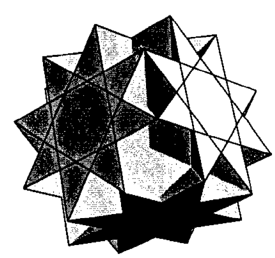
References

Wenninger, M. J. *Polyhedron Models.* Cambridge, England: Cambridge University Press, pp. 168–170, 1989.

Great Rhombihexacron

The DUAL of the GREAT RHOMBIHEXAHEDRON.

Great Rhombihexahedron



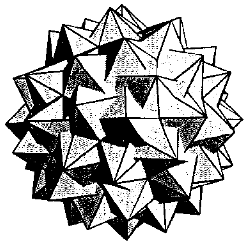
The UNIFORM POLYHEDRON U_{21} whose DUAL is the GREAT RHOMBIHEXACRON. It has WYTHOFF SYMBOL $2\,\frac{4}{3}\,\frac{3}{2}\,\frac{4}{2}\,|$. Its faces are $12\{4\}+6\{\frac{5}{3}\}$. Its CIRCUMRADIUS for unit edge length is

$$R = \tfrac{1}{2}\sqrt{5-2\sqrt{2}}.$$

References

Wenninger, M. J. *Polyhedron Models.* Cambridge, England: Cambridge University Press, pp. 159–160, 1989.

Great Snub Dodecicosidodecahedron

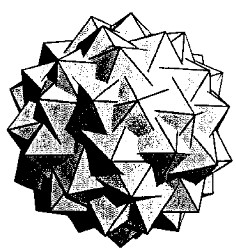


The UNIFORM POLYHEDRON U_{64} whose DUAL is the GREAT HEXAGONAL HEXECONTAHEDRON. It has WYTHOFF SYMBOL $|\,3\,\frac{5}{3}\,\frac{5}{2}$. Its faces are $80\{3\}+24\{\frac{5}{2}\}$. Its CIRCUMRADIUS for unit edge length is

$$R=\tfrac{1}{2}\sqrt{2}.$$

References
Wenninger, M. J. *Polyhedron Models.* Cambridge, England: Cambridge University Press, pp. 183–185, 1989.

Great Snub Icosidodecahedron



The UNIFORM POLYHEDRON U_{57} whose DUAL is the GREAT PENTAGONAL HEXECONTAHEDRON. It has WYTHOFF SYMBOL $|\,2\,3\,\frac{5}{3}$. Its faces are $80\{3\}+12\{\frac{5}{2}\}$. For unit edge length, it has CIRCUMRADIUS

$$R=\tfrac{1}{2}\sqrt{\frac{2-x}{1-x}}\approx 0.6450202,$$

where x is the most NEGATIVE ROOT of

$$x^3+2x^2-\phi^{-2}=0,$$

with ϕ the GOLDEN RATIO.

References
Wenninger, M. J. *Polyhedron Models.* Cambridge, England: Cambridge University Press, pp. 186–188, 1989.

Great Sphere

The great sphere on the surface of a HYPERSPHERE is the 3-D analog of the GREAT CIRCLE on the surface of a SPHERE. Let $2h$ be the number of reflecting SPHERES, and let great spheres divide a HYPERSPHERE into g 4-D TETRAHEDRA. Then for the POLYTOPE with SCHLÄFLI SYMBOL $\{p,q,r\}$,

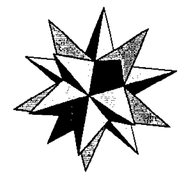
$$\frac{64h}{g}=12-p-2q-r+\frac{4}{p}+\frac{4}{r}.$$

see also GREAT CIRCLE

Great Stellapentakis Dodecahedron

The DUAL of the GREAT TRUNCATED ICOSAHEDRON.

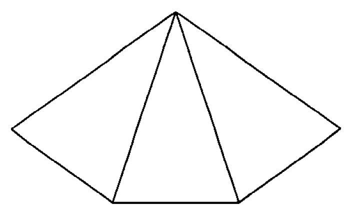
Great Stellated Dodecahedron



One of the KEPLER-POINSOT SOLIDS whose DUAL is the GREAT ICOSAHEDRON. Its SCHLÄFLI SYMBOL is $\{\frac{5}{2},3\}$. It is also UNIFORM POLYHEDRON U_{52} and has WYTHOFF SYMBOL $3\,|\,2\,\frac{5}{2}$. Its faces are $12\{\frac{5}{2}\}$. Its CIRCUMRADIUS for unit edge length is

$$R=\tfrac{1}{2}\sqrt{3}\phi^{-1}=\tfrac{1}{4}\sqrt{3}(\sqrt{5}-1).$$

The easiest way to construct it is to make 12 TRIANGULAR PYRAMIDS



with side length $\phi=(1+\sqrt{5})/2$ (the GOLDEN RATIO) times the base and attach them to the sides of an ICOSAHEDRON.

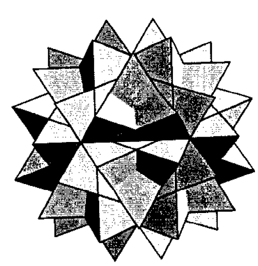
see also GREAT DODECAHEDRON, GREAT ICOSAHEDRON, GREAT STELLATED TRUNCATED DODECAHEDRON, KEPLER-POINSOT SOLID, SMALL STELLATED DODECAHEDRON

References
Fischer, G. (Ed.). Plate 104 in *Mathematische Modelle/Mathematical Models, Bildband/Photograph Volume.* Braunschweig, Germany: Vieweg, p. 103, 1986.

Great Stellated Triacontahedron

see GREAT RHOMBIC TRIACONTAHEDRON

Great Stellated Truncated Dodecahedron



The UNIFORM POLYHEDRON U_{66}, also called the QUASITRUNCATED GREAT STELLATED DODECAHEDRON, whose DUAL is the GREAT TRIAKIS ICOSAHEDRON. It has SCHLÄFLI SYMBOL t'$\{\frac{5}{2},3\}$ and WYTHOFF SYMBOL

$2\,3\,|\,\frac{5}{3}$. Its faces are $20\{3\}+12\{\frac{10}{3}\}$. Its CIRCUMRADIUS for unit edge length is

$$R = \tfrac{1}{4}\sqrt{74 - 30\sqrt{5}}\,.$$

References
Wenninger, M. J. *Polyhedron Models.* Cambridge, England: Cambridge University Press, p. 161, 1989.

Great Triakis Icosahedron

The DUAL of the GREAT STELLATED TRUNCATED DODECAHEDRON.

Great Triakis Octahedron

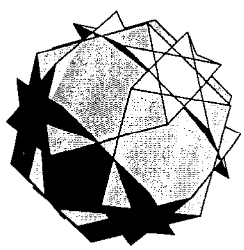
The DUAL of the STELLATED TRUNCATED HEXAHEDRON.

see also SMALL TRIAKIS OCTAHEDRON

Great Triambic Icosahedron

The DUAL of the GREAT DITRIGONAL ICOSIDODECAHEDRON.

Great Truncated Cuboctahedron

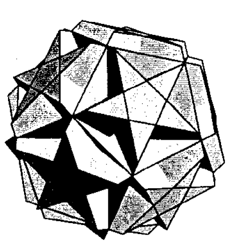


The UNIFORM POLYHEDRON U_{20}, also called the QUASITRUNCATED CUBOCTAHEDRON, whose DUAL is the GREAT DISDYAKIS DODECAHEDRON. It has SCHLÄFLI SYMBOL $t'\{\frac{3}{4}\}$ and WYTHOFF SYMBOL $2\,3\,\frac{4}{3}\,|$. Its CIRCUMRADIUS for unit edge length is

$$R = \tfrac{1}{2}\sqrt{13 - 6\sqrt{2}}\,.$$

References
Wenninger, M. J. *Polyhedron Models.* Cambridge, England: Cambridge University Press, pp. 145–146, 1989.

Great Truncated Icosahedron

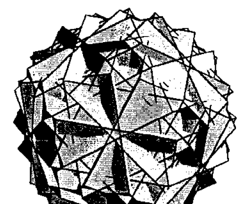


The UNIFORM POLYHEDRON U_{55}, also called the TRUNCATED GREAT ICOSAHEDRON, whose DUAL is the GREAT STELLAPENTAKIS DODECAHEDRON. It has SCHLÄFLI SYMBOL $t\{3,\frac{5}{2}\}$ and WYTHOFF SYMBOL $2\,\frac{5}{2}\,|\,3$. Its faces are $20\{6\}+12\{\frac{5}{2}\}$. Its CIRCUMRADIUS for unit edge length is

$$R = \tfrac{1}{4}\sqrt{58 - 18\sqrt{5}}\,.$$

References
Wenninger, M. J. *Polyhedron Models.* Cambridge, England: Cambridge University Press, p. 148, 1989.

Great Truncated Icosidodecahedron



The UNIFORM POLYHEDRON U_{68}, also called the GREAT QUASITRUNCATED ICOSIDODECAHEDRON, whose DUAL is the GREAT DISDYAKIS TRIACONTAHEDRON. It has SCHLÄFLI SYMBOL $t'\left\{\begin{matrix}3\\ \frac{5}{2}\end{matrix}\right\}$ and WYTHOFF SYMBOL $2\,3\,\frac{5}{3}\,|$. Its faces are $20\{6\}+30\{4\}+12\{\frac{10}{3}\}$. Its CIRCUMRADIUS for unit edge length is

$$R = \tfrac{1}{2}\sqrt{31 - 12\sqrt{5}}\,.$$

References
Wenninger, M. J. *Polyhedron Models.* Cambridge, England: Cambridge University Press, pp. 166–167, 1989.

Greater

A quantity a is said to be greater than b if a is larger than b, written $a > b$. If a is greater than or EQUAL to b, the relationship is written $a \geq b$. If a is MUCH GREATER than b, this is written $a \gg b$. Statements involving greater than and LESS than symbols are called INEQUALITIES.

see also EQUAL, GREATER THAN/LESS THAN SYMBOL, INEQUALITY, LESS, MUCH GREATER

Greater Than/Less Than Symbol

When applied to a system possessing a length R at which solutions in a variable r change character (such as the gravitational field of a sphere as r runs from the interior to the exterior), the symbols

$$r_> \equiv \max(r, R)$$

$$r_< \equiv \min(r, R)$$

are sometimes used.

see also EQUAL, GREATER, LESS

Greatest Common Denominator

see GREATEST COMMON DIVISOR

Greatest Common Divisor

The greatest common divisor of a and b GCD(a, b), sometimes written (a, b), is the largest DIVISOR common to a and b. Symbolically, let

$$a \equiv \prod_i p_i^{\alpha_i} \tag{1}$$

$$b \equiv \prod_i p_i^{\beta_i} \tag{2}$$

Then the greatest common divisor is given by

$$(a, b) = \prod_i p_i^{\min(\alpha_i, \beta_i)}, \tag{3}$$

where min denotes the MINIMUM. The GCD is DISTRIBUTIVE

$$(ma, mb) = m(a, b) \tag{4}$$

$$(ma, mb, mc) = m(a, b, c), \tag{5}$$

and ASSOCIATIVE

$$(a, b, c) = ((a, b), c) = (a, (b, c)) \tag{6}$$

$$(ab, cd) = (a, c)(b, d)\left(\frac{a}{(a, c)}, \frac{d}{(b, d)}\right)\left(\frac{c}{(a, c)}, \frac{b}{(b, d)}\right). \tag{7}$$

If $a = a_1(a, b)$ and $b = b_1(a, b)$, then

$$(a, b) = (a_1(a, b), b_1(a, b)) = (a, b)(a_1, b_1), \tag{8}$$

so $(a_1, b_1) = 1$ and a_1 and b_1 are said to be RELATIVELY PRIME. The GCD is also IDEMPOTENT

$$(a, a) = a, \tag{9}$$

COMMUTATIVE

$$(a, b) = (b, a), \tag{10}$$

and satisfies the ABSORPTION LAW

$$[a, (a, b)] = a. \tag{11}$$

The probability that two INTEGERS picked at random are RELATIVELY PRIME is $[\zeta(2)]^{-1} = 6/\pi^2$, where $\zeta(z)$ is the RIEMANN ZETA FUNCTION. Polezzi (1997) observed that $(m, n) = k$, where k is the number of LATTICE POINTS in the PLANE on the straight LINE connecting the VECTORS (0, 0) and (m, n) (excluding (m, n) itself). This observation is intimately connected with the probability of obtaining RELATIVELY PRIME integers, and also with the geometric interpretation of a REDUCED FRACTION y/x as a string through a LATTICE of points with ends at (1,0) and (x, y). The pegs it presses against (x_i, y_i) give alternate CONVERGENTS y_i/x_i of the CONTINUED FRACTION for y/x, while the other CONVERGENTS are obtained from the pegs it presses against with the initial end at (0, 1).

Knuth showed that

$$(2^p - 1, q^q - 1) = 2^{(p,q)} - 1 \tag{12}$$

for p, q PRIME.

The extended greatest common divisor of two INTEGERS m and n can be defined as the greatest common divisor of m and n which also satisfies the constraint $g = rm + sn$ for r and s given INTEGERS. It is used in solving LINEAR DIOPHANTINE EQUATIONS.

see also BEZOUT NUMBERS, EUCLIDEAN ALGORITHM, LEAST PRIME FACTOR

References

Polezzi, M. "A Geometrical Method for Finding an Explicit Formula for the Greatest Common Divisor." *Amer. Math. Monthly* **104**, 445–446, 1997.

Greatest Common Divisor Theorem

Given m and n, it is possible to choose c and d such that $cm + dn$ is a common factor of m and n.

Greatest Common Factor

see GREATEST COMMON DIVISOR

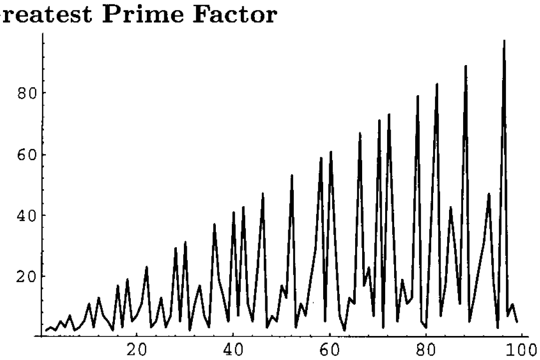
Greatest Integer Function

see FLOOR FUNCTION

Greatest Lower Bound

see INFIMUM, LEAST UPPER BOUND

Greatest Prime Factor



For an INTEGER $n \geq 2$, let gpf(x) denote the greatest prime factor of n, i.e., the number p_k in the factorization

$$n = p_1^{a_1} \cdots p_k^{a_k},$$

with $p_i < p_j$ for $i < j$. For $n = 2, 3, \ldots$, the first few are 2, 3, 2, 5, 3, 7, 2, 3, 5, 11, 3, 13, 7, 5, ... (Sloane's A006530). The greatest *multiple* prime factors for SQUAREFUL integers are 2, 2, 3, 2, 2, 3, 2, 2, 5, 3, 2, 2, 3, ... (Sloane's A046028).

see also DISTINCT PRIME FACTORS, FACTOR, LEAST COMMON MULTIPLE, LEAST PRIME FACTOR, MANGOLDT FUNCTION, PRIME FACTORS, TWIN PEAKS

References

Erdős, P. and Pomerance, C. "On the Largest Prime Factors of n and $n+1$." *Aequationes Math.* **17**, 211–321, 1978.

Guy, R. K. "The Largest Prime Factor of n." §B46 in *Unsolved Problems in Number Theory, 2nd ed.* New York: Springer-Verlag, pp. 101, 1994.

Heath-Brown, D. R. "The Largest Prime Factor of the Integers in an Interval." *Sci. China Ser. A* **39**, 449–476, 1996.

Mahler, K. "On the Greatest Prime Factor of $ax^m + by^n$." *Nieuw Arch. Wiskunde* **1**, 113–122, 1953.

Sloane, N. J. A. Sequence A006530/M0428 in "An On-Line Version of the Encyclopedia of Integer Sequences."

Grebe Point

see LEMOINE POINT

Greedy Algorithm

An algorithm used to recursively construct a SET of objects from the smallest possible constituent parts.

Given a SET of k INTEGERS (a_1, a_2, ..., a_k) with $a_1 < a_2 < \ldots < a_k$, a greedy algorithm can be used to find a VECTOR of coefficients (c_1, c_2, ..., c_k) such that

$$\sum_{i=1}^{k} c_i a_i = \mathbf{c} \cdot \mathbf{a} = n, \tag{1}$$

where $\mathbf{c} \cdot \mathbf{a}$ is the DOT PRODUCT, for some given INTEGER n. This can be accomplished by letting $c_i = 0$ for $i = 1$, ..., $k-1$ and setting

$$c_k = \left\lfloor \frac{n}{a_k} \right\rfloor. \tag{2}$$

Now define the difference between the representation and n as

$$\Delta \equiv n - \mathbf{c} \cdot \mathbf{a}. \tag{3}$$

If $\Delta = 0$ at any step, a representation has been found. Otherwise, decrement the NONZERO a_i term with least i, set all $a_j = 0$ for $j < i$, and build up the remaining terms from

$$c_j = \left\lfloor \frac{\Delta_j}{a_k} \right\rfloor \tag{4}$$

for $j = i-1$, ..., 1 until $\Delta = 0$ or all possibilities have been exhausted.

For example, MCNUGGET NUMBERS are numbers which are representable using only $(a_1, a_2, a_3) = (6, 9, 20)$. Taking $n = 62$ and applying the algorithm iteratively gives the sequence (0, 0, 3), (0, 2, 2), (2, 1, 2), (3, 0, 2), (1, 4, 1), at which point $\Delta = 0$. 62 is therefore a MCNUGGET NUMBER with

$$62 = (1 \cdot 6) + (4 \cdot 9) + (1 \cdot 20). \tag{5}$$

If *any* INTEGER n can be represented with $c_i = 0$ or 1 using a sequence (a_1, a_2, ...), then this sequence is called a COMPLETE SEQUENCE.

A greedy algorithm can also be used to break down arbitrary fractions into UNIT FRACTIONS in a finite number of steps. For a FRACTION a/b, find the least INTEGER x_1 such that $1/x_1 \leq a/b$, i.e.,

$$x_1 = \frac{\lceil b \rceil}{a}, \tag{6}$$

where $\lceil x \rceil$ is the CEILING FUNCTION. Then find the least INTEGER x_2 such that $1/x_2 \leq a/b - 1/x_1$. Iterate until there is no remainder. The ALGORITHM gives two or fewer terms for $1/n$ and $2/n$, three or fewer terms for $3/n$, and four or fewer for $4/n$.

Paul Erdős and E. G. Strays have conjectured that the DIOPHANTINE EQUATION

$$\frac{4}{n} = \frac{1}{a} + \frac{1}{b} + \frac{1}{c} \tag{7}$$

always can be solved, and W. Sierpiński conjectured that

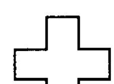
$$\frac{5}{n} = \frac{1}{a} + \frac{1}{b} + \frac{1}{c} \tag{8}$$

can be solved.

see also COMPLETE SEQUENCE, INTEGER RELATION, LEVINE-O'SULLIVAN GREEDY ALGORITHM, MCNUGGET NUMBER, REVERSE GREEDY ALGORITHM, SQUARE NUMBER, SYLVESTER'S SEQUENCE, UNIT FRACTION

References

Greek Cross



An irregular DODECAHEDRON CROSS in the shape of a PLUS SIGN.

see also CROSS, DISSECTION, DODECAHEDRON, LATIN CROSS, PLUS SIGN, SAINT ANDREW'S CROSS

Greek Problems

see GEOMETRIC PROBLEMS OF ANTIQUITY

Green's Function

Let

$$\tilde{L} = \tilde{D}^n + a_{n-1}(t)\tilde{D}^{n-1} + \ldots + a_1(t)\tilde{D} + a_0(t) \tag{1}$$

be a differential OPERATOR in 1-D, with $a_i(t)$ CONTINUOUS for $i = 0, 1, \ldots, n-1$ on the interval I, and assume we wish to find the solution $y(t)$ to the equation

$$\tilde{L}y(t) = h(t), \tag{2}$$

where $h(t)$ is a given CONTINUOUS on I. To solve equation (2), we look for a function $g : C^n(I) \mapsto C(I)$ such that $\tilde{L}(g(h)) = h$, where

$$y(t) = g(h(t)). \tag{3}$$

This is a CONVOLUTION equation of the form

$$y = g * h, \tag{4}$$

so the solution is

$$y(t) = \int_{t_0}^{t} g(t-x)h(x)\,dx, \tag{5}$$

where the function $g(t)$ is called the Green's function for $\tilde{L}$ on I.

Now, note that if we take $h(t) = \delta(t)$, then

$$y(t) = \int_{t_0}^{t} g(t-x)\delta(x)\,dx = g(t), \tag{6}$$

so the Green's function can be defined by

$$\tilde{L}g(t) = \delta(t). \tag{7}$$

However, the Green's function can be uniquely determined only if some initial or boundary conditions are given.

For an arbitrary linear differential operator $\tilde{L}$ in 3-D, the Green's function $G(\mathbf{r}, \mathbf{r}')$ is defined by analogy with the 1-D case by

$$\tilde{L}G(\mathbf{r}, \mathbf{r}') = \delta(\mathbf{r} - \mathbf{r}'). \tag{8}$$

The solution to $\tilde{L}\phi = f$ is then

$$\phi(\mathbf{r}) = \int G(\mathbf{r}, \mathbf{r}') f(\mathbf{r}')\,d^3\mathbf{r}'. \tag{9}$$

Explicit expressions for $G(\mathbf{r}, \mathbf{r}')$ can often be found in terms of a basis of given eigenfunctions $\phi_n(\mathbf{r}_1)$ by expanding the Green's function

$$G(\mathbf{r}_1, \mathbf{r}_2) = \sum_{n=0}^{\infty} a_n(\mathbf{r}_2)\phi_n(\mathbf{r}_1) \tag{10}$$

and DELTA FUNCTION,

$$\delta^3(\mathbf{r}_1 - \mathbf{r}_2) = \sum_{n=0}^{\infty} b_n \phi_n(\mathbf{r}_1). \tag{11}$$

Multiplying both sides by $\phi_m(\mathbf{r}_2)$ and integrating over $\mathbf{r}_1$ space,

$$\int \phi_m(\mathbf{r}_2)\delta^3(\mathbf{r}_1 - \mathbf{r}_2)\,d^3\mathbf{r}_1 = \sum_{n=0}^{\infty} b_n \int \phi_m(\mathbf{r}_2)\phi_n(\mathbf{r}_1)\,d^3\mathbf{r}_1 \tag{12}$$

$$\phi_m(\mathbf{r}_2) = \sum_{n=0}^{\infty} b_n \delta_{nm} = b_m, \tag{13}$$

so

$$\delta^3(\mathbf{r}_1 - \mathbf{r}_2) = \sum_{n=0}^{\infty} \phi_n(\mathbf{r}_1)\phi_n(\mathbf{r}_2). \tag{14}$$

By plugging in the differential operator, solving for the a_ns, and substituting into G, the original nonhomogeneous equation then can be solved.

References

Arfken, G. "Nonhomogeneous Equation—Green's Function," "Green's Functions—One Dimension," and "Green's Functions—Two and Three Dimensions." §8.7 and §16.5–16.6 in *Mathematical Methods for Physicists, 3rd ed.* Orlando, FL: Academic Press, pp. 480–491 and 897–924, 1985.

Green's Function—Helmholtz Differential Equation

The inhomogeneous HELMHOLTZ DIFFERENTIAL EQUATION is

$$\nabla^2\psi(\mathbf{r}) + k^2\psi(\mathbf{r}) = \rho(\mathbf{r}), \tag{1}$$

where the Helmholtz operator is defined as $\tilde{L} \equiv \nabla^2 + k^2$. The Green's function is then defined by

$$(\nabla^2 + k^2)G(\mathbf{r}_1, \mathbf{r}_2) = \delta^3(\mathbf{r}_1 - \mathbf{r}_2). \tag{2}$$

Define the basis functions ϕ_n as the solutions to the homogeneous HELMHOLTZ DIFFERENTIAL EQUATION

$$\nabla^2\phi_n(\mathbf{r}) + {k_n}^2\phi_n(\mathbf{r}) = 0. \tag{3}$$

The Green's function can then be expanded in terms of the ϕ_ns,

$$G(\mathbf{r}_1, \mathbf{r}_2) = \sum_{n=0}^{\infty} a_n(\mathbf{r}_2)\phi_n(\mathbf{r}_1), \tag{4}$$

and the DELTA FUNCTION as

$$\delta^3(\mathbf{r}_1 - \mathbf{r}_2) = \sum_{n=0}^{\infty} \phi_n(\mathbf{r}_1)\phi_n(\mathbf{r}_2). \tag{5}$$

Plugging (4) and (5) into (2) gives

$$\nabla^2\left[\sum_{n=0}^{\infty} a_n(\mathbf{r}_2)\phi_n(\mathbf{r}_1)\right] + k^2\sum_{n=0}^{\infty} a_n(\mathbf{r}_2)\phi_n(\mathbf{r}_1) = \sum_{n=0}^{\infty} \phi_n(\mathbf{r}_1)\phi_n(\mathbf{r}_2). \tag{6}$$

Using (3) gives

$$-\sum_{n=0}^{\infty} a_n(\mathbf{r}_2){k_n}^2\phi_n(\mathbf{r}) + k^2\sum_{n=0}^{\infty} a_n(\mathbf{r}_2)\phi_n(\mathbf{r}_1) = \sum_{n=0}^{\infty} \phi_n(\mathbf{r}_1)\phi_n(\mathbf{r}_2) \tag{7}$$

$$\sum_{n=0}^{\infty} a_n(\mathbf{r}_2)\phi_n(\mathbf{r}_1)(k^2 - k_n{}^2) = \sum_{n=0}^{\infty} \phi_n(\mathbf{r}_1)\phi_n(\mathbf{r}_2). \quad (8)$$

This equation must hold true for each n, so

$$a_n(\mathbf{r}_2)\phi_n(\mathbf{r}_1)(k^2 - k_n{}^2) = \phi_n(\mathbf{r}_1)\phi_n(\mathbf{r}_2) \quad (9)$$

$$a_n(\mathbf{r}_2) = \frac{\phi_n(\mathbf{r}_2)}{k^2 - k_n{}^2}, \quad (10)$$

and (4) can be written

$$G(\mathbf{r}_1, \mathbf{r}_2) = \sum_{n=0}^{\infty} \frac{\phi_n(\mathbf{r}_1)\phi_n(\mathbf{r}_2)}{k^2 - k_n{}^2}. \quad (11)$$

The general solution to (1) is therefore

$$\psi(\mathbf{r}_1) = \int G(\mathbf{r}_1, \mathbf{r}_2)\rho(\mathbf{r}_2)\, d^3\mathbf{r}_2$$
$$= \sum_{n=0}^{\infty} \int \frac{\phi_n(\mathbf{r}_1)\phi_n(\mathbf{r}_2)\rho(\mathbf{r}_2)}{k^2 - k_n{}^2}\, d^3\mathbf{r}_2. \quad (12)$$

References

Arfken, G. *Mathematical Methods for Physicists, 3rd ed.* Orlando, FL: Academic Press, pp. 529–530, 1985.

Green's Function—Poisson's Equation

POISSON'S EQUATION equation is

$$\nabla^2\phi = 4\pi\rho, \quad (1)$$

where ϕ is often called a potential function and ρ a density function, so the differential operator in this case is $\tilde{L} = \nabla^2$. As usual, we are looking for a Green's function $G(\mathbf{r}_1, \mathbf{r}_2)$ such that

$$\nabla^2 G(\mathbf{r}_1, \mathbf{r}_2) = \delta^3(\mathbf{r}_1 - \mathbf{r}_2). \quad (2)$$

But from LAPLACIAN,

$$\nabla^2\left(\frac{1}{|\mathbf{r} - \mathbf{r}'|}\right) = -4\pi\delta^3(\mathbf{r} - \mathbf{r}'), \quad (3)$$

so

$$G(\mathbf{r}, \mathbf{r}') = -\frac{1}{4\pi|\mathbf{r} - \mathbf{r}'|}, \quad (4)$$

and the solution is

$$\phi(\mathbf{r}) = \int G(\mathbf{r}, \mathbf{r}')[4\pi\rho(\mathbf{r}')]\, d^3\mathbf{r}' = -\int \frac{\rho(\mathbf{r}')\, d^3\mathbf{r}'}{|\mathbf{r} - \mathbf{r}'|}. \quad (5)$$

Expanding $G(\mathbf{r}_1, \mathbf{r}_2)$ in the SPHERICAL HARMONICS Y_l^m gives

$$G(\mathbf{r}_1, \mathbf{r}_2)$$
$$= \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \frac{1}{2l+1} \frac{r_<^l}{r_>^{l+1}} Y_l^m(\theta_1, \phi_1) Y_l^{m*}(\theta_2, \phi_2), \quad (6)$$

where $r_<$ and $r_>$ are GREATER THAN/LESS THAN SYMBOLS. This expression simplifies to

$$G(\mathbf{r}_1, \mathbf{r}_2) = \frac{1}{4\pi} \sum_{l=0}^{\infty} \frac{r_<^l}{r_>^{l+1}} P_l(\cos\gamma), \quad (7)$$

where P_l are LEGENDRE POLYNOMIALS, and $\cos\gamma \equiv \mathbf{r}_1 \cdot \mathbf{r}_2$. Equations (6) and (7) give the addition theorem for LEGENDRE POLYNOMIALS.

In CYLINDRICAL COORDINATES, the Green's function is much more complicated,

$$G(\mathbf{r}_1, \mathbf{r}_2) = \frac{1}{2\pi^2} \sum_{m=-\infty}^{\infty}$$
$$\int_0^\infty I_m(k\rho_<)K_m(k\rho_>)e^{im(\phi_1 - \phi_2)} \cos[k(z_1 - z_2)]\, dk, \quad (8)$$

where $I_m(x)$ and $K_m(x)$ are MODIFIED BESSEL FUNCTIONS OF THE FIRST and SECOND KINDS (Arfken 1985).

References

Arfken, G. *Mathematical Methods for Physicists, 3rd ed.* Orlando, FL: Academic Press, pp. 485–486, 905, and 912, 1985.

Green's Identities

Green's identities are a set of three vector derivative/integral identities which can be derived starting with the vector derivative identities

$$\nabla \cdot (\psi\nabla\phi) = \psi\nabla^2\phi + (\nabla\psi) \cdot (\nabla\phi) \quad (1)$$

and

$$\nabla \cdot (\phi\nabla\psi) = \phi\nabla^2\psi + (\nabla\phi) \cdot (\nabla\psi), \quad (2)$$

where $\nabla\cdot$ is the DIVERGENCE, ∇ is the GRADIENT, ∇^2 is the LAPLACIAN, and $\mathbf{a}\cdot\mathbf{b}$ is the DOT PRODUCT. From the DIVERGENCE THEOREM,

$$\int_V (\nabla \cdot \mathbf{F})\, dV = \int_S \mathbf{F} \cdot d\mathbf{a}. \quad (3)$$

Plugging (2) into (3),

$$\int_S \phi(\nabla\psi) \cdot d\mathbf{a} = \int_V [\phi\nabla^2\psi + (\nabla\phi) \cdot (\nabla\psi)]\, dV. \quad (4)$$

This is Green's first identity.

Subtracting (2) from (1),

$$\nabla \cdot (\phi\nabla\psi - \psi\nabla\phi) = \phi\nabla^2\psi - \psi\nabla^2\phi. \quad (5)$$

Therefore,

$$\int_V (\phi\nabla^2\psi - \psi\nabla^2\phi)\, dV = \int_S (\phi\nabla\psi - \psi\nabla\phi) \cdot d\mathbf{a}. \quad (6)$$

This is Green's second identity.

Let u have continuous first PARTIAL DERIVATIVES and be HARMONIC inside the region of integration. Then Green's third identity is

$$u(x,y) = \frac{1}{2\pi}\oint_C \left[\ln\left(\frac{1}{r}\right)\frac{\partial u}{\partial n} - u\frac{\partial}{\partial n}\ln\left(\frac{1}{r}\right)\right] ds \quad (7)$$

(Kaplan 1991, p. 361).

References

Kaplan, W. *Advanced Calculus, 4th ed.* Reading, MA: Addison-Wesley, 1991.

Greene's Method

A method for predicting the onset of widespread CHAOS. It is based on the hypothesis that the dissolution of an invariant torus can be associated with the sudden change from stability to instability of nearly closed orbits (Tabor 1989, p. 163).

see also OVERLAPPING RESONANCE METHOD

References

Tabor, M. *Chaos and Integrability in Nonlinear Dynamics: An Introduction.* New York: Wiley, 1989.

Green Space

A G-SPACE provides local notions of harmonic, hyperharmonic, and superharmonic functions. When there exists a nonconstant superharmonic function greater than 0, it is a called a Green space. Examples are $\mathbb{R}^n$ (for $n \geq 3$) and any bounded domain of $\mathbb{R}^n$.

Green's Theorem

Green's theorem is a vector identity which is equivalent to the CURL THEOREM in the PLANE. Over a region D in the plane with boundary ∂D,

$$\int_{\partial D} f(x,y)\,dx + g(x,y)\,dy = \iint_D \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right) dx\,dy$$

$$\int_{\partial D} \mathbf{F}\cdot d\mathbf{s} = \iint_D (\nabla\times\mathbf{F})\cdot\mathbf{k}\,dA.$$

If the region D is on the left when traveling around ∂D, then AREA of D can be computed using

$$A = \tfrac{1}{2}\int_{\partial D} x\,dy - y\,dx.$$

see also CURL THEOREM, DIVERGENCE THEOREM

References

Arfken, G. "Gauss's Theorem." §1.11 in *Mathematical Methods for Physicists, 3rd ed.* Orlando, FL: Academic Press, pp. 57–61, 1985.

Gregory's Formula

A series FORMULA for PI found by Gregory and Leibniz,

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} + \ldots.$$

It converges very slowly, but its convergence can be accelerated using certain transformations, in particular

$$\pi = \sum_{k=1}^{\infty}\frac{3^k-1}{4^k}\zeta(k+1),$$

where $\zeta(z)$ is the RIEMANN ZETA FUNCTION (Vardi 1991).

see also MACHIN'S FORMULA, MACHIN-LIKE FORMULAS, PI

References

Vardi, I. *Computational Recreations in Mathematica.* Reading, MA: Addison-Wesley, pp. 157–158, 1991.

Gregory Number

A number

$$t_x = \tan^{-1}(\tfrac{1}{x}) = \cot^{-1}x,$$

where x is an INTEGER or RATIONAL NUMBER, $\tan^{-1}x$ is the INVERSE TANGENT, and $\cot^{-1}x$ is the INVERSE COTANGENT. Gregory numbers arise in the determination of MACHIN-LIKE FORMULAS. Every Gregory number t_x can be expressed uniquely as a sum of t_ns where the ns are STØRMER NUMBERS.

References

Conway, J. H. and Guy, R. K. "Gregory's Numbers" In *The Book of Numbers.* New York: Springer-Verlag, pp. 241–242, 1996.

Grelling's Paradox

A semantic PARADOX, also called the HETEROLOGICAL PARADOX, which arises by defining "heterological" to mean "a word which does not describe itself." The word "heterological" is therefore heterological IFF it is not.

see also RUSSELL'S PARADOX

References

Hofstadter, D. R. *Gödel, Escher, Bach: An Eternal Golden Braid.* New York: Vintage Books, pp. 20–21, 1989.

Grenz-Formel

An equation derived by Kronecker:

$$\sum{}'(x^2+y^2+dz^2)^{-s} = 4\zeta(s)\eta(s) + \frac{2\pi}{s-1}\frac{\zeta(2s-2)}{d^{s-1}}$$

$$+\frac{2\pi^s}{\Gamma(s)}d^{(1-s)/2}\sum_{n=1}^{\infty}n^{(s-1)/2}\sum_{u^2|n}\frac{r\left(\frac{n}{u^2}\right)}{u^{2s-2}}\int_0^{\infty}e^{\pi\sqrt{nd}\,(y+y^{-1})}y^{s-2}\,dy,$$

where

$$r(n) = 4\sum_{d|n}\sin(\tfrac{1}{2}\pi d),$$

$\zeta(z)$, is the RIEMANN ZETA FUNCTION, $\eta(z)$ is the DIRICHLET ETA FUNCTION, $\Gamma(z)$ is the GAMMA FUNCTION, and the primed sum omits infinite terms (Selberg and Chowla 1967).

References
Borwein, J. M. and Borwein, P. B. *Pi & the AGM: A Study in Analytic Number Theory and Computational Complexity.* New York: Wiley, pp. 296–297, 1987.
Selberg, A. and Chowla, S. "On Epstein's Zeta-Function." *J. Reine. Angew. Math.* **227**, 86–110, 1967.

Griffiths Points

"The" Griffiths point is the fixed point in GRIFFITHS' THEOREM. Given four points on a CIRCLE and a line through the center of the CIRCLE, the four corresponding Griffiths points are COLLINEAR (Tabov 1995).

The points

$$Gr = I + 4Ge$$

$$Gr' = I - 4Ge,$$

are known as the first and second Griffiths points, where I is the INCENTER and Ge is the GERGONNE POINT.

see also GERGONNE POINT, GRIFFITHS' THEOREM, INCENTER, OLDKNOW POINTS, RIGBY POINTS

References
Oldknow, A. "The Euler-Gergonne-Soddy Triangle of a Triangle." *Amer. Math. Monthly* **103**, 319–329, 1996.
Tabov, J. "Four Collinear Griffiths Points." *Math. Mag.* **68**, 61–64, 1995.

Griffiths' Theorem

When a point P moves along a line through the CIRCUMCENTER of a given TRIANGLE Δ, the CIRCUMCIRCLE of the PEDAL TRIANGLE of P with respect to Δ passes through a fixed point (the GRIFFITHS POINT) on the NINE-POINT CIRCLE of Δ.

see also CIRCUMCENTER, GRIFFITHS POINTS, NINE-POINT CIRCLE, PEDAL TRIANGLE

Grimm's Conjecture

Grimm conjectures that if $n+1$, $n+2$, ..., $n+k$ are all COMPOSITE NUMBERS, then there are distinct PRIMES p_{i_j} such that $p_{i_j}|(n+j)$ for $1 \leq j \leq k$.

References
Guy, R. K. "Grimm's Conjecture." §B32 in *Unsolved Problems in Number Theory, 2nd ed.* New York: Springer-Verlag, p. 86, 1994.

Grinberg Formula

A formula satisfied by all HAMILTONIAN CIRCUITS with n nodes. Let f_j be the number of regions inside the circuit with j sides, and let g_j be the number of regions outside the circuit with j sides. If there are d interior diagonals, then there must be $d+1$ regions

$$[\# \text{ regions in interior}] = d+1 = f_2 + f_3 + \ldots + f_n. \quad (1)$$

Any region with j sides is bounded by j EDGES, so such regions contribute jf_j to the total. However, this counts each diagonal twice (and each EDGE only once). Therefore,

$$2f_2 + 3f_3 + \ldots + nf_n = 2d + n. \quad (2)$$

Take (2) $- 2 \times$ (1),

$$f_3 + 2f_4 + 3f_5 + \ldots + (n-2)f_n = n-2. \quad (3)$$

Similarly,

$$g_3 + 2g_4 + \ldots + (n-2)g_n = n-2, \quad (4)$$

so

$$(f_3-g_3)+2(f_4-g_4)+3(f_5-g_5)+\ldots+(n-2)(f_n-g_n) = 0. \quad (5)$$

Gröbner Basis

A Gröbner basis for a system of POLYNOMIAL equations is an equivalence system that possesses useful properties. It is very roughly analogous to computing an ORTHONORMAL BASIS from a set of BASIS VECTORS and can be described roughly as a combination of GAUSSIAN ELIMINATION (for linear systems) and the EUCLIDEAN ALGORITHM (for UNIVARIATE POLYNOMIALS over a FIELD).

Gröbner bases are useful in the construction of symbolic algebra algorithms. The algorithm for computing Gröbner bases is known as BUCHBERGER'S ALGORITHM.

see also BUCHBERGER'S ALGORITHM, COMMUTATIVE ALGEBRA

References
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Becker, T. and Weispfennig, V. *Gröbner Bases: A Computational Approach to Commutative Algebra.* New York: Springer-Verlag, 1993.
Cox, D.; Little, J.; and O'Shea, D. *Ideals, Varieties, and Algorithms: An Introduction to Algebraic Geometry and Commutative Algebra, 2nd ed.* New York: Springer-Verlag, 1996.
Eisenbud, D. *Commutative Algebra with a View toward Algebraic Geometry.* New York: Springer-Verlag, 1995.
Mishra, B. *Algorithmic Algebra.* New York: Springer-Verlag, 1993.

Groemer Packing

A honeycomb-like packing that forms HEXAGONS.

see also GROEMER THEOREM

References
Stewart, I. "A Bundling Fool Beats the Wrap." *Sci. Amer.* **268**, 142–144, 1993.

Groemer Theorem

Given n CIRCLES and a PERIMETER p, the total AREA of the CONVEX HULL is

$$A_{\text{Convex Hull}} = 2\sqrt{3}(n-1) + p(1 - \tfrac{1}{2}\sqrt{3}) + \pi(\sqrt{3}-1).$$

Furthermore, the actual AREA equals this value IFF the packing is a GROEMER PACKING. The theorem was proved in 1960 by Helmut Groemer.

see also CONVEX HULL

Gronwall's Theorem

Let $\sigma(n)$ be the DIVISOR FUNCTION. Then

$$\overline{\lim_{n\to\infty}} \frac{\sigma(n)}{n \ln\ln n} = e^\gamma,$$

where γ is the EULER-MASCHERONI CONSTANT. Ramanujan independently discovered a less precise version of this theorem (Berndt 1994). Robin (1984) showed that the validity of the inequality

$$\sigma(n) < e^\gamma n \ln\ln n$$

for $n \geq 5041$ is equivalent to the RIEMANN HYPOTHESIS.

References

Berndt, B. C. *Ramanujan's Notebooks: Part I.* New York: Springer-Verlag, p. 94, 1985.

Gronwall, T. H. "Some Asymptotic Expressions in the Theory of Numbers." *Trans. Amer. Math. Soc.* **37**, 113–122, 1913.

Nicholas, J.-L. "On Highly Composite Numbers." In *Ramanujan Revisited: Proceedings of the Centenary Conference* (Ed. G. E. Andrews, B. C. Berndt, and R. A. Rankin). Boston, MA: Academic Press, pp. 215–244, 1988.

Robin, G. "Grandes Valeurs de la foction somme des diviseurs et hypothèse de Riemann." *J. Math. Pures Appl.* **63**, 187–213, 1984.

Gross

A DOZEN DOZEN, or the SQUARE NUMBER 144.

see also 12, DOZEN

Grossencharacter

In the original formulation, a quantity associated with ideal class groups. According to Chevalley's formulation, a Grossencharacter is a MULTIPLICATIVE CHARACTER of the group of ADÉLES that is trivial on the diagonally embedded $k^\times$, where k is a number FIELD.

References

Knapp, A. W. "Group Representations and Harmonic Analysis, Part II." *Not. Amer. Math. Soc.* **43**, 537–549, 1996.

Grossman's Constant

Define the sequence $a_0 = 1$, $a_1 = x$, and

$$a_{n+2} = \frac{a_n}{1 + a_{n+1}}$$

for $n \geq 0$. Janssen and Tjaden (1987) showed that this sequence converges for exactly one value of x, $x = 0.73733830336929\ldots$, confirming Grossman's conjecture.

References

Finch, S. "Favorite Mathematical Constants." `http://www.mathsoft.com/asolve/constant/grssmn/grssmn.html`.

Janssen, A. J. E. M. and Tjaden, D. L. A. Solution to Problem 86-2. *Math. Intel.* **9**, 40–43, 1987.

Grothendieck's Majorant

The best known majorant of Grothendieck's constant. Let A be an $n \times n$ REAL SQUARE MATRIX such that

$$\left| \sum_{1\leq i,j\leq n} a_{ij} x_i y_j \right| \tag{1}$$

in which x_i and y_j have REAL ABSOLUTE VALUES < 1. Grothendieck has shown there exists a number K_G independent of A and n satisfying

$$\left| \sum_{1\leq i,j\leq n} a_{ij} \langle x_i, y_j \rangle \right| \tag{2}$$

in which the vectors x_i and y_j have a norm < 1 in HILBERT SPACE. The Grothendieck constant is the smallest REAL NUMBER for which this inequality has been proven. Krivine (1977) showed that

$$1.676\ldots \leq K_G \leq 1.782\ldots, \tag{3}$$

and has postulated that

$$K_G \equiv \frac{\pi}{2\ln(1+\sqrt{2})} = 1.7822139\ldots. \tag{4}$$

It is related to KHINTCHINE'S CONSTANT.

References

Krivine, J. L. "Sur las constante de Grothendieck." *C. R. A. S.* **284**, 8, 1977.

Le Lionnais, F. *Les nombres remarquables.* Paris: Hermann, p. 42, 1983.

Grothendieck's Theorem

Let E and F be paired spaces with S a family of absolutely convex bounded sets of F such that the sets of S generate F and, if $B_1, B_2 \in S$, then there exists a $B_3 \in S$ such that $B_3 \supset B_1$ and $B_3 \supset B_2$. Then E_S is complete IFF algebraic linear functional $f(y)$ of F that is weakly continuous on every $B \in S$ is expressed as $f(y) = \langle x, y \rangle$ for some $x \in E$. When E_S is not complete, the space of all linear functionals satisfying this condition gives the completion $\hat{E}_S$ of E_S.

see also MACKEY'S THEOREM

References

Iyanaga, S. and Kawada, Y. (Eds.). "Grothendieck's Theorem." §407L in *Encyclopedic Dictionary of Mathematics.* Cambridge, MA: MIT Press, p. 1274, 1980.

Ground Set

A PARTIALLY ORDERED SET is defined as an ordered pair $P = (X, \leq)$. Here, X is called the GROUND SET of P and $\leq$ is the PARTIAL ORDER of P.

see also PARTIAL ORDER, PARTIALLY ORDERED SET

Group

A group G is defined as a finite or infinite set of OPERANDS (called "elements") $A, B, C, \ldots$ that may be combined or "multiplied" via a BINARY OPERATOR to form well-defined products and which furthermore satisfy the following conditions:

1. Closure: If A and B are two elements in G, then the product AB is also in G.
2. Associativity: The defined multiplication is associative, i.e., for all $A, B, C \in G$, $(AB)C = A(BC)$.
3. Identity: There is an IDENTITY ELEMENT I (a.k.a. 1, E, or e) such that $IA = AI = A$ for every element $A \in G$.
4. Inverse: There must be an inverse or reciprocal of each element. Therefore, the set must contain an element $B = A^{-1}$ such that $AA^{-1} = A^{-1}A = I$ for each element of G.

A group is therefore a MONOID for which every element is invertible. A group must contain at least one element.

The study of groups is known as GROUP THEORY. If there are a finite number of elements, the group is called a FINITE GROUP and the number of elements is called the ORDER of the group.

Since each element A, B, C, ..., X, and Y is a member of the GROUP, GROUP property 1 requires that the product

$$D \equiv ABC \cdots XY \tag{1}$$

must also be a member. Now apply D to $Y^{-1}X^{-1} \cdots C^{-1}B^{-1}A^{-1}$,

$$D(Y^{-1}X^{-1} \cdots C^{-1}B^{-1}A^{-1}) = (ABC \cdots XY)(Y^{-1}X^{-1} \cdots C^{-1}B^{-1}A^{-1}). \tag{2}$$

But

$$\begin{aligned} ABC \cdots XYY^{-1}X^{-1} \cdots C^{-1}B^{-1}A^{-1} &= ABC \cdots XIX^{-1} \cdots C^{-1}B^{-1}A^{-1} \\ &= ABC \cdots C^{-1}B^{-1}A^{-1} = \ldots = AA^{-1} = I, \end{aligned} \tag{3}$$

so

$$I = D(Y^{-1}X^{-1} \cdots C^{-1}B^{-1}A^{-1}), \tag{4}$$

which means that

$$D^{-1} = Y^{-1}X^{-1} \cdots C^{-1}B^{-1}A^{-1} \tag{5}$$

and

$$(ABC \cdots XY)^{-1} = Y^{-1}X^{-1} \cdots C^{-1}B^{-1}A^{-1}. \tag{6}$$

An IRREDUCIBLE REPRESENTATION of a group is a representation for which there exists no UNITARY TRANSFORMATION which will transform the representation MATRIX into block diagonal form. The IRREDUCIBLE REPRESENTATION has some remarkable properties. Let the ORDER of a GROUP be h, and the dimension of the ith representation (the order of each constituent matrix) be l_i (a POSITIVE INTEGER). Let any operation be denoted R, and let the mth row and nth column of the matrix corresponding to a matrix R in the ith IRREDUCIBLE REPRESENTATION be $\Gamma_i(R)_{mn}$. The following properties can be derived from the GROUP ORTHOGONALITY THEOREM,

$$\sum_R \Gamma_i(R)_{mn}\Gamma_j(R)_{m'n'}{}^* = \frac{h}{\sqrt{l_i l_j}}\delta_{ij}\delta_{mm'}\delta_{nn'}. \tag{7}$$

1. The DIMENSIONALITY THEOREM:
$$h = \sum_i l_i{}^2 = l_1{}^2 + l_2{}^2 + l_3{}^2 + \ldots = \sum_i \chi_i{}^2(I), \tag{8}$$
where each l_i must be a POSITIVE INTEGER and χ is the CHARACTER (trace) of the representation.
2. The sum of the squares of the CHARACTERS in any IRREDUCIBLE REPRESENTATION i equals h,
$$h = \sum_R \chi_i{}^2(R). \tag{9}$$
3. ORTHOGONALITY of different representations
$$\sum_R \chi_i(R)\chi_j(R) = 0 \quad \text{for } i \neq j. \tag{10}$$
4. In a given representation, reducible or irreducible, the CHARACTERS of all MATRICES belonging to operations in the same class are identical (but differ from those in other representations).
5. The number of IRREDUCIBLE REPRESENTATIONS of a GROUP is equal to the number of CONJUGACY CLASSES in the GROUP. This number is the dimension of the Γ MATRIX (although some may have zero elements).
6. A one-dimensional representation with all 1s (totally symmetric) will always exist for any GROUP.
7. A 1-D representation for a GROUP with elements expressed as MATRICES can be found by taking the CHARACTERS of the MATRICES.
8. The number a_i of IRREDUCIBLE REPRESENTATIONS χ_i present in a reducible representation c is given by
$$a_i = \frac{1}{h}\sum_R \chi(R)\chi_i(R), \tag{11}$$

where h is the ORDER of the GROUP and the sum must be taken over all elements in each class. Written explicitly,

$$a_i = \frac{1}{h} \sum_R \chi(R) \chi_i{}'(R) n_R, \qquad (12)$$

where $\chi_i{}'$ is the CHARACTER of a single entry in the CHARACTER TABLE and n_R is the number of elements in the corresponding CONJUGACY CLASS.

see also ABELIAN GROUP, ADÉLE GROUP, AFFINE GROUP, ALTERNATING GROUP, ARTINIAN GROUP, ASCHBACHER'S COMPONENT THEOREM, B_p-THEOREM, BABY MONSTER GROUP, BETTI GROUP, BIMONSTER, BORDISM GROUP, BRAID GROUP, BRAUER GROUP, BURNSIDE PROBLEM, CENTER (GROUP), CENTRALIZER, CHARACTER (GROUP), CHARACTER (MULTIPLICATIVE), CHEVALLEY GROUPS, CLASSICAL GROUPS, COBORDISM GROUP, COHOMOTOPY GROUP, COMPONENT, CONJUGACY CLASS, COSET, CONWAY GROUPS, COXETER GROUP, CYCLIC GROUP, DIHEDRAL GROUP, DIMENSIONALITY THEOREM, DYNKIN DIAGRAM, ELLIPTIC GROUP MODULO p, ENGEL'S THEOREM, EUCLIDEAN GROUP, FEIT-THOMPSON THEOREM, FINITE GROUP, FISCHER GROUPS, FISCHER'S BABY MONSTER GROUP, FUNDAMENTAL GROUP, GENERAL LINEAR GROUP, GENERAL ORTHOGONAL GROUP, GENERAL UNITARY GROUP, GLOBAL $C(G;T)$ THEOREM, GROUPOID, GROUP ORTHOGONALITY THEOREM, HALL-JANKO GROUP, HAMILTONIAN GROUP, HARADA-NORTON GROUP, HEISENBERG GROUP, HELD GROUP, HERMANN-MAUGUIN SYMBOL, HIGMAN-SIMS GROUP, HOMEOMORPHIC GROUP, HYPERGROUP, ICOSAHEDRAL GROUP, IRREDUCIBLE REPRESENTATION, ISOMORPHIC GROUPS, JANKO GROUPS, JORDAN-HÖLDER THEOREM, KLEINIAN GROUP, KUMMER GROUP, $L_{p'}$-BALANCE THEOREM, LAGRANGE'S GROUP THEOREM, LOCAL GROUP THEORY, LINEAR GROUP, LYONS GROUP, MATHIEU GROUPS, MATRIX GROUP, MCLAUGHLIN GROUP, MÖBIUS GROUP, MODULAR GROUP, MODULO MULTIPLICATION GROUP, MONODROMY GROUP, MONOID, MONSTER GROUP, MULLIKEN SYMBOLS, NÉRON-SEVERI GROUP, NILPOTENT GROUP, NONCOMMUTATIVE GROUP, NORMAL SUBGROUP, NORMALIZER, O'NAN GROUP, OCTAHEDRAL GROUP, ORDER (GROUP), ORTHOGONAL GROUP, ORTHOGONAL ROTATION GROUP, OUTER AUTOMORPHISM GROUP, p-GROUP, p'-GROUP, p-LAYER, POINT GROUPS, POSITIVE DEFINITE FUNCTION, PRIME GROUP, PROJECTIVE GENERAL LINEAR GROUP, PROJECTIVE GENERAL ORTHOGONAL GROUP, PROJECTIVE GENERAL UNITARY GROUP, PROJECTIVE SPECIAL LINEAR GROUP, PROJECTIVE SPECIAL ORTHOGONAL GROUP, PROJECTIVE SPECIAL UNITARY GROUP, PROJECTIVE SYMPLECTIC GROUP, PSEUDOGROUP, QUASIGROUP, QUASISIMPLE GROUP, QUASITHIN THEOREM, QUASI-UNIPOTENT GROUP, REPRESENTATION, RESIDUE CLASS, RUBIK'S CUBE, RUDVALIS GROUP, SCHÖNFLIES SYMBOL, SCHUR MULTIPLIER, SEMISIMPLE, SIGNALIZER FUNCTOR THEOREM, SELMER GROUP, SEMIGROUP, SIMPLE GROUP, SOLVABLE GROUP, SPACE GROUPS, SPECIAL LINEAR GROUP, SPECIAL ORTHOGONAL GROUP, SPECIAL UNITARY GROUP, SPORADIC GROUP, STOCHASTIC GROUP, STRONGLY EMBEDDED THEOREM, SUBGROUP, SUBNORMAL, SUPPORT, SUZUKI GROUP, SYMMETRIC GROUP, SYMPLECTIC GROUP, TETRAHEDRAL GROUP, THOMPSON GROUP, TIGHTLY EMBEDDED, TITS GROUP, TRIANGULAR SYMMETRY GROUP, TWISTED CHEVALLEY GROUPS, UNIMODULAR GROUP, UNIPOTENT, UNITARY GROUP, VIERGRUPPE, VON DYCK'S THEOREM

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Wybourne, B. G. *Classical Groups for Physicists.* New York: Wiley, 1974.

Group Convolution

The convolution of two COMPLEX-valued functions on a GROUP G is defined as

$$(a * b)(g) = \sum_{k \in G} a(k) b(k^{-1} g)$$

where the SUPPORT (set which is not zero) of each function is finite.

References

Weinstein, A. "Groupoids: Unifying Internal and External Symmetry." *Not. Amer. Math. Soc.* **43**, 744–752, 1996.

Group Orthogonality Theorem

Let Γ be a representation for a GROUP of ORDER h, then

$$\sum_R \Gamma_i(R)_{mn} \Gamma_j(R)_{m'n'}{}^* = \frac{h}{\sqrt{l_i l_j}} \delta_{ij} \delta_{mm'} \delta_{nn'}.$$

The proof is nontrivial and may be found in Eyring *et al.* (1944).

References

Eyring, H.; Walker, J.; and Kimball, G. E. *Quantum Chemistry.* New York: Wiley, p. 371, 1944.

Group Ring

The set of sums $\sum_x a_x x$ ranging over a multiplicative GROUP and a_i are elements of a FIELD with all but a finite number of $a_i = 0$.

Group Theory

The study of GROUPS. Gauss developed but did not publish parts of the mathematics of group theory, but Galois is generally considered to have been the first to develop the theory. Group theory is a powerful formal method for analyzing abstract and physical systems in which SYMMETRY is present and has surprising importance in physics, especially quantum mechanics.

see also FINITE GROUP, GROUP, PLETHYSM, SYMMETRY

References

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Rose, J. S. *A Course on Group Theory.* New York: Dover, 1994.

Rotman, J. J. *An Introduction to the Theory of Groups, 4th ed.* New York: Springer-Verlag, 1995.

Groupoid

There are at least two definitions of "groupoid" currently in use.

The first type of groupoid is an algebraic structure on a SET with a BINARY OPERATOR. The only restriction on the operator is CLOSURE (i.e., applying the BINARY OPERATOR to two elements of a given set S returns a value which is itself a member of S). Associativity, commutativity, etc., are not required (Rosenfeld 1968, pp. 88–103). A groupoid can be empty. The numbers of nonisomorphic groupoids of this type having n elements are 1, 1, 10, 3330, 178981952, ... (Sloane's A001329), and the numbers of nonisomorphic and nonantiisimorphic groupoids are 1, 7, 1734, 89521056, ... (Sloane's A001424). An associative groupoid is called a SEMIGROUP.

The second type of groupoid is an algebraic structure first defined by Brandt (1926) and also known as a VIRTUAL GROUP. A groupoid with base B is a set G with mappings α and β from G onto B and a partially defined binary operation $(g,h) \mapsto gh$, satisfying the following four conditions:

1. gh is defined only when $\beta(G) = \alpha(h)$ for certain maps α and β from G onto $\mathbb{R}^2$ with $\alpha : (x,\gamma,y) \mapsto x$ and $\beta : (x,\gamma,y) \mapsto y$.
2. ASSOCIATIVITY: If either $(gh)k$ or $g(hk)$ is defined, then so is the other and $(gh)k = g(hk)$.
3. For each g in G, there are left and right IDENTITY ELEMENTS λ_g and ρ_g such that $\lambda_g g = g = g\rho_g$.
4. Each g in G has an inverse g^{-1} for which $gg^{-1} = \lambda_g$ and $g^{-1}g = \rho_g$

(Weinstein 1996). A groupoid is a small CATEGORY with every morphism invertible.

see also BINARY OPERATOR, INVERSE SEMIGROUP, LIE ALGEBROID, LIE GROUPOID, MONOID, QUASIGROUP, SEMIGROUP, TOPOLOGICAL GROUPOID

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Weinstein, A. "Groupoids: Unifying Internal and External Symmetry." *Not. Amer. Math. Soc.* **43**, 744–752, 1996.

Growth

A general term which refers to an increase (or decrease in the case of the oxymoron "NEGATIVE growth") in a given quantity.

see also GROWTH FUNCTION, GROWTH SPIRAL

Growth Function

see BLOCK GROWTH

Growth Spiral

see LOGARITHMIC SPIRAL

Grundy's Game

A special case of NIM played by the following rules. Given a heap of size n, two players alternately select a heap and divide it into two unequal heaps. A player loses when he cannot make a legal move because all heaps have size 1 or 2. Flammenkamp gives a table of the extremal SPRAGUE-GRUNDY VALUES for this game. The first few values of Grundy's game are 0, 0, 0, 1, 0, 2, 1, 0, 2, ... (Sloane's A002188).

References

Flammenkamp, A. "Sprague-Grundy Values of Grundy's Game." http://www.minet.uni-jena.de/~achim/grundy.html.

Sloane, N. J. A. Sequence A002188/M0044 in "An On-Line Version of the Encyclopedia of Integer Sequences."

Grundy-Sprague Number

see NIM-VALUE

Gudermannian Function

Denoted either $\gamma(x)$ or $\operatorname{gd}(x)$.

$$\operatorname{gd}(x) \equiv \tan^{-1}(\sinh x) = 2\tan^{-1}(e^x) - \tfrac{1}{2}\pi \qquad (1)$$

$$\operatorname{gd}^{-1}(x) = \ln[\tan(\tfrac{1}{4}\pi + \tfrac{1}{2}x)] = \ln(\sec x + \tan x). \qquad (2)$$

The derivatives are given by

$$\frac{d}{dx}\operatorname{gd}(x) = \operatorname{sech} x \qquad (3)$$

$$\frac{d}{dx}\operatorname{gd}^{-1}(x) = \sec x. \qquad (4)$$

Guldinus Theorem

see PAPPUS'S CENTROID THEOREM

Gumbel's Distribution

A special case of the FISHER-TIPPETT DISTRIBUTION with $a = 0$, $b = 1$. The MEAN, VARIANCE, SKEWNESS, and KURTOSIS are

$$\begin{aligned} \mu &= \gamma \\ \sigma^2 &= \tfrac{1}{6}\pi^2 \\ \gamma_1 &= \frac{12\sqrt{6}\,\zeta(3)}{\pi^3} \\ \gamma_2 &= \tfrac{12}{5}. \end{aligned}$$

where γ is the EULER-MASCHERONI CONSTANT, and $\zeta(3)$ is APÉRY'S CONSTANT.

see also FISHER-TIPPETT DISTRIBUTION

Guthrie's Problem

The problem of deciding if four-colors are sufficient to color any map on a plane or SPHERE.

see also FOUR-COLOR THEOREM

Gutschoven's Curve

see KAPPA CURVE

Guy's Conjecture

Guy's conjecture, which has not yet been proven or disproven, states that the CROSSING NUMBER for a COMPLETE GRAPH of order n is

$$\frac{1}{4}\left\lfloor\frac{n}{2}\right\rfloor\left\lfloor\frac{n-1}{2}\right\rfloor\left\lfloor\frac{n-2}{2}\right\rfloor\left\lfloor\frac{n-3}{2}\right\rfloor,$$

where $\lfloor x\rfloor$ is the FLOOR FUNCTION, which can be rewritten

$$\begin{cases} \frac{1}{64}n(n-2)^2(n-4) & \text{for } n \text{ even} \\ \frac{1}{64}(n-1)^2(n-3)^2 & \text{for } n \text{ odd.} \end{cases}$$

The first few values are 0, 0, 0, 0, 1, 3, 9, 18, 36, 60, ... (Sloane's A000241).

see also CROSSING NUMBER (GRAPH)

References

Sloane, N. J. A. Sequence A000241/M2772 in "An On-Line Version of the Encyclopedia of Integer Sequences."

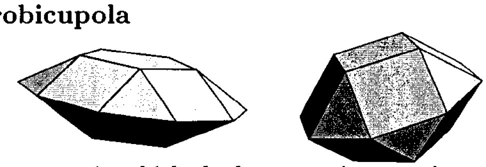
Gyrate Bidiminished Rhombicosidodecahedron

see JOHNSON SOLID

Gyrate Rhombicosidodecahedron

see JOHNSON SOLID

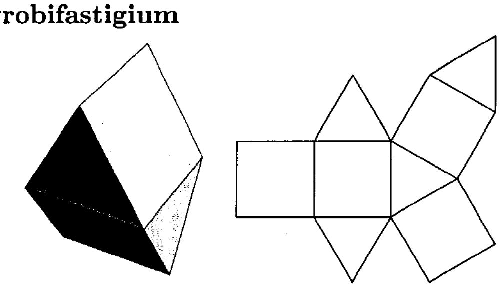
Gyrobicupola



A BICUPOLA in which the bases are in opposite orientations.

see also BICUPOLA, PENTAGONAL GYROBICUPOLA, SQUARE GYROBICUPOLA

Gyrobifastigium



JOHNSON SOLID J_{26}, consisting of two joined triangular PRISMS.

Gyrobirotunda

A BIROTUNDA in which the bases are in opposite orientations.

Gyrocupolarotunda

A CUPOLAROTUNDA in which the bases are in opposite orientations.

see also ORTHOCUPOLAROTUNDA

Gyroelongated Cupola

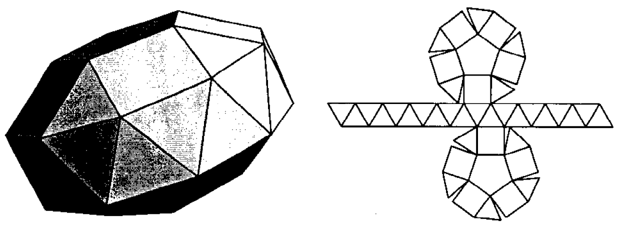
A n-gonal CUPOLA adjoined to a $2n$-gonal ANTIPRISM.

see also GYROELONGATED PENTAGONAL CUPOLA, GYROELONGATED SQUARE CUPOLA, GYROELONGATED TRIANGULAR CUPOLA

Gyroelongated Dipyramid

see GYROELONGATED PYRAMID, GYROELONGATED SQUARE DIPYRAMID

Gyroelongated Pentagonal Bicupola



JOHNSON SOLID J_{46}, which consists of a PENTAGONAL ROTUNDA adjoined to a decagonal ANTIPRISM.

Gyroelongated Pentagonal Birotunda

see JOHNSON SOLID

Gyroelongated Pentagonal Cupola

see JOHNSON SOLID

Gyroelongated Pentagonal Cupolarotunda

see JOHNSON SOLID

Gyroelongated Pentagonal Pyramid

see JOHNSON SOLID

Gyroelongated Pentagonal Rotunda

see JOHNSON SOLID

Gyroelongated Pyramid

An n-gonal pyramid adjoined to an n-gonal ANTIPRISM.

see also ELONGATED PYRAMID, GYROELONGATED DIPYRAMID, GYROELONGATED PENTAGONAL PYRAMID, GYROELONGATED SQUARE DIPYRAMID, GYROELONGATED SQUARE PYRAMID

Gyroelongated Rotunda

see GYROELONGATED PENTAGONAL ROTUNDA

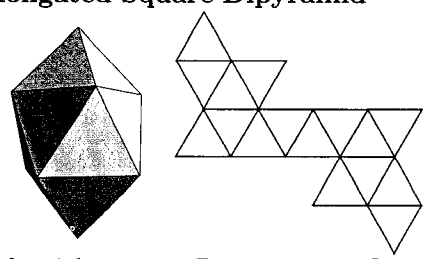
Gyroelongated Square Bicupola

see JOHNSON SOLID

Gyroelongated Square Cupola

see JOHNSON SOLID

Gyroelongated Square Dipyramid



One of the eight convex DELTAHEDRA. It consists of two oppositely faced SQUARE PYRAMIDS rotated 45° to each other and separated by a ribbon of eight side-to-side TRIANGLES. It is JOHNSON SOLID J_{17}.

Call the coordinates of the upper PYRAMID bases ($\pm$ 1, $\pm$ 1, h_1) and of the lower ($\pm\sqrt{2}$, 0, $-h_1$) and (0, $\pm\sqrt{2}$, $-h_1$). Call the PYRAMID apexes (0, 0, $\pm(h_1 + h_2)$). Consider the points (1, 1, 0) and (0, 0, $h_1 + h_2$). The height of the PYRAMID is then given by

$$\sqrt{1^2 + 1^2 + h_2{}^2} = \sqrt{2 + h_2{}^2} = 2 \tag{1}$$

$$h_2 = \sqrt{2}. \tag{2}$$

Now consider the points (1, 1, h_1) and ($\sqrt{2}$, 0, $-h_1$). The height of the base is given by

$$\begin{aligned}(1 - \sqrt{2})^2 + 1^2 + (2h_1)^2 &= 1 - 2\sqrt{2} + 2 + 1 + 4h_1{}^2 \\ &= 4 - 2\sqrt{2} + 4h_1{}^2 = 2^2 = 4 \end{aligned} \tag{3}$$

$$4h_1{}^2 = 2\sqrt{2} \tag{4}$$

$$h_1{}^2 = \frac{\sqrt{2}}{2} = \frac{1}{\sqrt{2}} = 2^{-1/2}, \tag{5}$$

so

$$h_1 = 2^{-1/4} \tag{6}$$

$$h_2 = 2^{1/2}. \tag{7}$$

Gyroelongated Square Pyramid

see JOHNSON SOLID

Gyroelongated Triangular Bicupola

see JOHNSON SOLID

Gyroelongated Triangular Cupola

see JOHNSON SOLID

H

h-Cobordism

An *h*-cobordism is a COBORDISM W between two MANIFOLDS M_1 and M_2 such that W is SIMPLY CONNECTED and the inclusion maps $M_1 \to W$ and $M_2 \to W$ are HOMOTOPY equivalences.

h-Cobordism Theorem

If W is a SIMPLY CONNECTED, COMPACT MANIFOLD with a boundary that has two components, M_1 and M_2, such that inclusion of each is a HOMOTOPY equivalence, then W is DIFFEOMORPHIC to the product $M_1 \times [0,1]$ for $\dim(M_1) \geq 5$. In other words, if M and M' are two simply connected MANIFOLDS of DIMENSION ≥ 5 and there exists an *h*-COBORDISM W between them, then W is a product $M \times I$ and M is DIFFEOMORPHIC to M'.

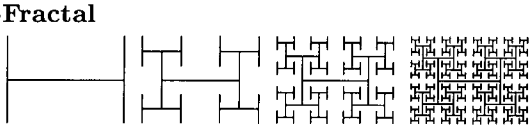
The proof of the *h*-cobordism theorem can be accomplished using SURGERY. A particular case of the *h*-cobordism theorem is the POINCARÉ CONJECTURE in dimension $n \geq 5$. Smale proved this theorem in 1961.

see also DIFFEOMORPHISM, POINCARÉ CONJECTURE, SURGERY

References

Smale, S. "Generalized Poincaré's Conjecture in Dimensions Greater than Four." *Ann. Math.* **74**, 391–406, 1961.

H-Fractal



The FRACTAL illustrated above.

References

Lauwerier, H. *Fractals: Endlessly Repeated Geometric Figures.* Princeton, NJ: Princeton University Press, pp. 1–2, 1991.

Weisstein, E. W. "Fractals." `http://www.astro.virginia.edu/~eww6n/math/notebooks/Fractal.m.`

H-Function

see FOX'S *H*-FUNCTION

H-Spread

The difference $H_2 - H_1$, where H_1 and H_2 are HINGES. It is the same as the INTERQUARTILE RANGE for $N = 5$, 9, 13, ... points.

see also HINGE, INTERQUARTILE RANGE, STEP

References

Tukey, J. W. *Explanatory Data Analysis.* Reading, MA: Addison-Wesley, p. 44, 1977.

H-Transform

A 2-D generalization of the HAAR TRANSFORM which is used for the compression of astronomical images. The algorithm consists of dividing the $2^N \times 2^N$ image into blocks of 2×2 pixels, calling the pixels in the block a_{00}, a_{10}, a_{01}, and a_{11}. For each block, compute the four coefficients

$$\begin{aligned} h_0 &\equiv \tfrac{1}{2}(a_{11} + a_{10} + a_{01} + a_{00}) \\ h_x &\equiv \tfrac{1}{2}(a_{11} + a_{10} - a_{01} - a_{00}) \\ h_y &\equiv \tfrac{1}{2}(a_{11} - a_{10} + a_{01} - a_{00}) \\ h_c &\equiv \tfrac{1}{2}(a_{11} - a_{10} - a_{01} + a_{00}). \end{aligned}$$

Construct a $2^{N-1} \times 2^{N-1}$ image from the h_0 values, and repeat until only one h_0 value remains. The H-transform can be performed in place and requires about $16N^2/3$ additions for an $N \times N$ image.

see also HAAR TRANSFORM

References

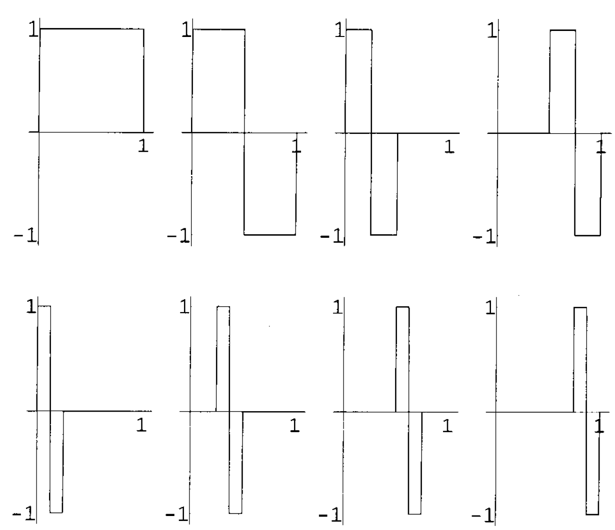
Capaccioli, M.; Held, E. V.; Lorenz, H.; Richter, G. M.; and Ziener, R. "Application of an Adaptive Filtering Technique to Surface Photometry of Galaxies. I. The Method Tested on NGC 3379." *Astron. Nachr.* **309**, 69–80, 1988.

Fritze, K.; Lange, M.; Möstle, G.; Oleak, H.; and Richter, G. M. "A Scanning Microphotometer with an On-Line Data Reduction for Large Field Schmidt Plates." *Astron. Nachr.* **298**, 189–196, 1977.

Richter, G. M. "The Evaluation of Astronomical Photographs with the Automatic Area Photometer." *Astron. Nachr.* **299**, 283–303, 1978.

White, R. L.; Postman, M.; and Lattanzi, M. G. "Compression of the Guide Star Digitised Schmidt Plates." In *Digitised Optical Sky Surveys: Proceedings of the Conference on "Digitised Optical Sky Surveys" held in Edinburgh, Scotland, 18–21 June 1991* (Ed. H. T. MacGillivray and E. B. Thompson). Dordrecht, Netherlands: Kluwer, pp. 167–175, 1992.

Haar Function



Define

$$\psi(x) \equiv \begin{cases} 1 & 0 \le x \le \frac{1}{2} \\ -1 & \frac{1}{2} \le x \le 1 \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

and

$$\psi_{jk}(x) \equiv \psi(2^j x - k), \tag{2}$$

where the FUNCTIONS plotted above are

$$\begin{aligned} \psi_{00} &= \psi(x) \\ \psi_{10} &= \psi(2x) \\ \psi_{11} &= \psi(2x-1) \\ \psi_{20} &= \psi(4x) \\ \psi_{21} &= \psi(4x-1) \\ \psi_{21} &= \psi(4x-2) \\ \psi_{21} &= \psi(4x-3). \end{aligned}$$

Then a FUNCTION $f(x)$ can be written as a series expansion by

$$f(x) = c_0 + \sum_{j=0}^{\infty} \sum_{k=0}^{2^j-1} c_{jk} \psi_{jk}(x). \tag{3}$$

The FUNCTIONS ψ_{jk} and ψ are all ORTHOGONAL in $[0,1]$, with

$$\int_0^1 \phi(x)\phi_{jk}(x)\,dx = 0 \tag{4}$$

$$\int_0^1 \phi_{jk}(x)\phi_{lm}(x)\,dx = 0. \tag{5}$$

These functions can be used to define WAVELETS. Let a FUNCTION be defined on n intervals, with n a POWER of 2. Then an arbitrary function can be considered as an n-VECTOR $\mathbf{f}$, and the COEFFICIENTS in the expansion $\mathbf{b}$ can be determined by solving the MATRIX equation

$$\mathbf{f} = \mathsf{W}_n \mathbf{b} \tag{6}$$

for $\mathbf{b}$, where W is the MATRIX of ψ basis functions. For example,

$$\begin{aligned} \mathsf{W}_4 &= \begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & -1 & 0 \\ 1 & -1 & 0 & 1 \\ 1 & -1 & 0 & -1 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 1 & & \\ 1 & -1 & & \\ & & 1 & 1 \\ & & 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ & & 1 & \\ & 1 & & \\ & & & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 & & \\ 1 & -1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix}. \end{aligned} \tag{7}$$

The WAVELET MATRIX can be computed in $\mathcal{O}(n)$ steps, compared to $\mathcal{O}(n \lg n)$ for the FOURIER MATRIX.

see also WAVELET, WAVELET TRANSFORM

References

Haar, A. "Zur Theorie der orthogonalen Funktionensysteme." *Math. Ann.* **69**, 331–371, 1910.

Strang, G. "Wavelet Transforms Versus Fourier Transforms." *Bull. Amer. Math. Soc.* **28**, 288–305, 1993.

Haar Integral

The INTEGRAL associated with the HAAR MEASURE.

see also HAAR MEASURE

Haar Measure

Any locally compact Hausdorff topological group has a unique (up to scalars) NONZERO left invariant measure which is finite on compact sets. If the group is Abelian or compact, then this measure is also right invariant and is known as the Haar measure.

Haar Transform

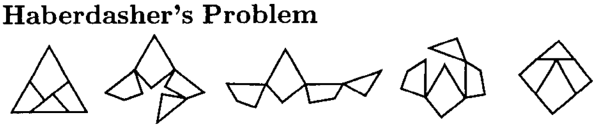
A 1-D transform which makes use of the HAAR FUNCTIONS.

see H-TRANSFORM, HAAR FUNCTION

References

Haar, A. "Zur Theorie der orthogonalen Funktionensysteme." *Math. Ann.* **69**, 331–371, 1910.

Haberdasher's Problem



With four cuts, DISSECT an EQUILATERAL TRIANGLE into a SQUARE. First proposed by Dudeney (1907) and discussed in Gardner (1961, p. 34) and Stewart (1987, p. 169). The solution can be hinged so that the three pieces collapse into either the TRIANGLE or the SQUARE.

see also DISSECTION

References

Gardner, M. *The Second Scientific American Book of Mathematical Puzzles & Diversions: A New Selection.* New York: Simon and Schuster, 1961.

Stewart, I. *The Problems of Mathematics, 2nd ed.* Oxford, England: Oxford University Press, 1987.

Hadamard Design

A SYMMETRIC BLOCK DESIGN $(4n+3, n+1, n)$ which is equivalent to a HADAMARD MATRIX of order $4n+4$. It is conjectured that Hadamard designs exist from all integers $n > 0$, but this has not yet been proven. This elusive proof (or disproof) remains one of the most important unsolved problems in COMBINATORICS.

References

Dinitz, J. H. and Stinson, D. R. "A Brief Introduction to Design Theory." Ch. 1 in *Contemporary Design Theory: A Collection of Surveys* (Ed. J. H. Dinitz and D. R. Stinson). New York: Wiley, pp. 1–12, 1992.

Hadamard's Inequality

Let $\mathsf{A} = a_{ii}$ be an arbitrary $n \times n$ nonsingular MATRIX with REAL elements and DETERMINANT $|\mathsf{A}|$, then

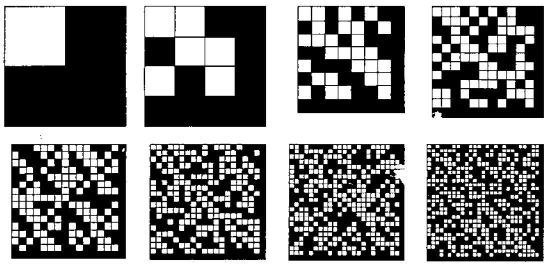
$$|\mathsf{A}|^2 \le \prod_{i=1}^{n} \left(\sum_{k=1}^{n} {a_{ik}}^2 \right).$$

see also HADAMARD'S THEOREM

References

Gradshteyn, I. S. and Ryzhik, I. M. *Tables of Integrals, Series, and Products, 5th ed.* San Diego, CA: Academic Press, p. 1110, 1979.

Hadamard Matrix



A class of SQUARE MATRIX invented by Sylvester (1867) under the name of ANALLAGMATIC PAVEMENT. A Hadamard matrix is a SQUARE MATRIX containing only 1s and -1s such that when any two columns or rows are placed side by side, HALF the adjacent cells are the same SIGN and half the other (excepting from the count an L-shaped "half-frame" bordering the matrix on two sides which is composed entirely of 1s). When viewed as pavements, cells with 1s are colored black and those with -1s are colored white. Therefore, the $n \times n$ Hadamard matrix H_n must have $n(n-1)/2$ white squares (-1s) and $n(n+1)/2$ black squares (1s).

This is equivalent to the definition

$$\mathsf{H}_n {\mathsf{H}_n}^{\mathrm{T}} = n\mathsf{I}_n, \tag{1}$$

where I_n is the $n \times n$ IDENTITY MATRIX. A Hadamard matrix of order $4n+4$ corresponds to a HADAMARD DESIGN ($4n+3$, $2n+1$, n).

PALEY'S THEOREM guarantees that there always exists a Hadamard matrix H_n when n is divisible by 4 and of the form $2^e(q^m+1)$, where p is an ODD PRIME. In such cases, the MATRICES can be constructed using a PALEY CONSTRUCTION. The PALEY CLASS k is undefined for the following values of $m < 1000$: 92, 116, 156, 172, 184, 188, 232, 236, 260, 268, 292, 324, 356, 372, 376, 404, 412, 428, 436, 452, 472, 476, 508, 520, 532, 536, 584, 596, 604, 612, 652, 668, 712, 716, 732, 756, 764, 772, 808, 836, 852, 856, 872, 876, 892, 904, 932, 940, 944, 952, 956, 964, 980, 988, 996.

Sawade (1985) constructed H_{268}. It is conjectured (and verified up to $n < 428$) that H_n exists for all n DIVISIBLE by 4 (van Lint and Wilson 1993). However, the proof of this CONJECTURE remains an important problem in CODING THEORY. The number of Hadamard matrices of order $4n$ are 1, 1, 1, 5, 3, 60, 487, ... (Sloane's A007299).

If H_n and H_m are known, then H_{nm} can be obtained by replacing all 1s in H_m by H_n and all -1s by $-\mathsf{H}_n$. For $n \leq 100$, Hadamard matrices with $n = 12$, 20, 28, 36, 44, 52, 60, 68, 76, 84, 92, and 100 cannot be built up from lower order Hadamard matrices.

$$\mathsf{H}_2 = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \tag{2}$$

$$\begin{aligned} \mathsf{H}_4 &= \begin{bmatrix} \mathsf{H}_2 & \mathsf{H}_2 \\ -\mathsf{H}_2 & \mathsf{H}_2 \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} & \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \\ -\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} & \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \end{bmatrix} \\ &= \begin{bmatrix} 1 & 1 & 1 & 1 \\ -1 & 1 & -1 & 1 \\ -1 & -1 & 1 & 1 \\ 1 & -1 & -1 & 1 \end{bmatrix}. \end{aligned} \tag{3}$$

H_8 can be similarly generated from H_4. Hadamard matrices can also be expressed in terms of the WALSH FUNCTIONS Cal and Sal

$$H_8 = \begin{bmatrix} \mathrm{Cal}(0,t) \\ \mathrm{Sal}(4,t) \\ \mathrm{Sal}(2,t) \\ \mathrm{Cal}(2,t) \\ \mathrm{Sal}(1,t) \\ \mathrm{Cal}(3,t) \\ \mathrm{Cal}(1,t) \\ \mathrm{Sal}(3,t) \end{bmatrix}. \tag{4}$$

Hadamard matrices can be used to make ERROR-CORRECTING CODES.

see also HADAMARD DESIGN, PALEY CONSTRUCTION, PALEY'S THEOREM, WALSH FUNCTION

References

Ball, W. W. R. and Coxeter, H. S. M. *Mathematical Recreations and Essays, 13th ed.* New York: Dover, pp. 107–109 and 274, 1987.

Beth, T.; Jungnickel, D.; and Lenz, H. *Design Theory.* New York: Cambridge University Press, 1986.

Colbourn, C. J. and Dinitz, J. H. (Eds.) "Hadamard Matrices and Designs." Ch. 24 in *CRC Handbook of Combinatorial Designs.* Boca Raton, FL: CRC Press, pp. 370–377, 1996.

Geramita, A. V. *Orthogonal Designs: Quadratic Forms and Hadamard Matrices.* New York: Marcel Dekker, 1979.

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Kimura, H. "Classification of Hadamard Matrices of Order 28." *Disc. Math.* **133**, 171–180, 1994.

Kimura, H. "Classification of Hadamard Matrices of Order 28 with Hall Sets." *Disc. Math.* **128**, 257–269, 1994.

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Ogilvie, G. A. "Solution to Problem 2511." *Math. Questions and Solutions* **10**, 74–76, 1868.

Paley, R. E. A. C. "On Orthogonal Matrices." *J. Math. Phys.* **12**, 311–320, 1933.

Ryser, H. J. *Combinatorial Mathematics.* Buffalo, NY: Math. Assoc. Amer., pp. 104–122, 1963.

Sawade, K. "A Hadamard Matrix of Order-268." *Graphs Combinatorics* **1**, 185–187, 1985.
Seberry, J. and Yamada, M. "Hadamard Matrices, Sequences, and Block Designs." Ch. 11 in *Contemporary Design Theory: A Collection of Surveys* (Eds. J. H. Dinitz and D. R. Stinson). New York: Wiley, pp. 431–560, 1992.
Sloane, N. J. A. Sequence A007299/M3736 in "An On-Line Version of the Encyclopedia of Integer Sequences."
Spence, E. "Classification of Hadamard Matrices of Order 24 and 28." *Disc. Math* **140**, 185–243, 1995.
Sylvester, J. J. "Thoughts on Orthogonal Matrices, Simultaneous Sign-Successions, and Tessellated Pavements in Two or More Colours, with Applications to Newton's Rule, Ornamental Tile-Work, and the Theory of Numbers." *Phil. Mag.* **34**, 461–475, 1867.
Sylvester, J. J. "Problem 2511." *Math. Questions and Solutions* **10**, 74, 1868.
van Lint, J. H. and Wilson, R. M. *A Course in Combinatorics.* New York: Cambridge University Press, 1993.

Hadamard's Theorem

Let $|\mathsf{A}|$ be an $n \times n$ DETERMINANT with COMPLEX (or REAL) elements a_{ij}, then $|\mathsf{A}| \neq 0$ if

$$|a_{ii}| > \sum_{\substack{j=1 \\ j\neq i}}^{n} |a_{ij}|.$$

see also HADAMARD'S INEQUALITY

References

Gradshteyn, I. S. and Ryzhik, I. M. *Tables of Integrals, Series, and Products, 5th ed.* San Diego, CA: Academic Press, p. 1110, 1979.

Hadamard Transform

A FAST FOURIER TRANSFORM-like ALGORITHM which produces a hologram of an image.

Hadamard-Vallée Poussin Constants

N.B. A detailed on-line essay by S. Finch was the starting point for this entry.

The sum of RECIPROCALS of PRIMES diverges, but

$$\lim_{n\to\infty}\left[\sum_{k=1}^{\pi(n)}\frac{1}{p_k}-\ln(\ln n)\right] = \gamma+\sum_{k=1}^{\infty}\left[\ln\left(1-\frac{1}{p_k}\right)+\frac{1}{p_k}\right] \equiv C_1 = 0.2614972128\ldots, \quad (1)$$

where $\pi(n)$ is the PRIME COUNTING FUNCTION and γ is the EULER-MASCHERONI CONSTANT (Le Lionnais 1983). Hardy and Wright (1985) show that, if $\omega(n)$ is the number of distinct PRIME factors of n, then

$$\lim_{n\to\infty}\left[\frac{1}{n}\sum_{k=1}^{n}\omega(k)-\ln(\ln n)\right] = C_1. \quad (2)$$

Furthermore, if $\Omega(n)$ is the total number of PRIME factors of n, then

$$\lim_{n\to\infty}\left[\frac{1}{n}\sum_{k=1}^{n}\Omega(k)-\ln(\ln n)\right] = C_1+\sum_{k=1}^{\infty}\frac{1}{p_k(p_k-1)} = 1.0346538819\ldots. \quad (3)$$

Similarly,

$$\lim_{n\to\infty}\left(\sum_{k=1}^{\pi(n)}\frac{\ln p_k}{p_k}-\ln n\right) = -\gamma-\sum_{j=2}^{\infty}\sum_{k=1}^{\infty}\frac{\ln p_k}{{p_k}^j} \equiv -C_2 = -1.3325822757\ldots. \quad (4)$$

References

Finch, S. "Favorite Mathematical Constants." `http://www.mathsoft.com/asolve/constant/hdmrd/hdmrd.html`.
Hardy, G. H. and Wright, E. M. *An Introduction to the Theory of Numbers, 5th ed.* Oxford, England: Clarendon Press, 1985.
Le Lionnais, F. *Les nombres remarquables.* Paris: Hermann, p. 24, 1983.
Rosser, J. B. and Schoenfeld, L. "Approximate Formulas for Some Functions of Prime Numbers." *Ill. J. Math.* **6**, 64–94, 1962.

Hadwiger's Principal Theorem

The VECTORS $\pm\mathbf{a}_1, \ldots, \pm\mathbf{a}_n$ in a 3-space form a normalized EUTACTIC STAR IFF $T\mathbf{x} = \mathbf{x}$ for all $\mathbf{x}$ in the 3-space.

Hadwiger Problem

What is the largest number of subcubes (not necessarily different) into which a CUBE cannot be divided by plane cuts? The answer is 47.

see also CUBE DISSECTION

Hafner-Sarnak-McCurley Constant

N.B. A detailed on-line essay by S. Finch was the starting point for this entry.

Given two randomly chosen INTEGER $n \times n$ matrices, what is the probability $D(n)$ that the corresponding determinants are coprime? Hafner *et al.* (1993) showed that

$$D(n) = \prod_{p_k}\left\{1-\left[1-\prod_{j=1}^{n}(1-{p_k}^{-j})\right]^2\right\}, \quad (1)$$

where the product is over PRIMES. The case $D(1)$ is just the probability that two random INTEGERS are coprime,

$$D(1) = \frac{6}{\pi^2} = 0.6079271019\ldots. \quad (2)$$

Vardi (1991) computed the limit

$$\sigma \equiv \lim_{n\to\infty} D(n) = 0.3532363719\ldots. \quad (3)$$

The speed of convergence is roughly $\sim 0.57^n$ (Flajolet and Vardi 1996).

References
Finch, S. "Favorite Mathematical Constants." http://www.mathsoft.com/asolve/constant/hafner/hafner.html.
Flajolet, P. and Vardi, I. "Zeta Function Expansions of Classical Constants." Unpublished manuscript. 1996. http://pauillac.inria.fr/algo/flajolet/Publications/landau.ps.
Hafner, J. L.; Sarnak, P.; and McCurley, K. "Relatively Prime Values of Polynomials." In *Contemporary Mathematics* Vol. 143 (Ed. M. Knopp and M. Seingorn). Providence, RI: Amer. Math. Soc., 1993.
Vardi, I. *Computational Recreations in Mathematica.* Redwood City, CA: Addison-Wesley, 1991.

Hahn-Banach Theorem

A linear FUNCTIONAL defined on a SUBSPACE of a VECTOR SPACE V and which is dominated by a sublinear function defined on V has a linear extension which is also dominated by the sublinear function.

References
Zeidler, E. *Applied Functional Analysis: Applications to Mathematical Physics.* New York: Springer-Verlag, 1995.

Hailstone Number

Sequences of INTEGERS generated in the COLLATZ PROBLEM. For example, for a starting number of 7, the sequence is 7, 22, 11, 34, 17, 52, 26, 13, 40, 20, 10, 5, 16, 8, 4, 2, 1, 4, 2, 1, ... Such sequences are called hailstone sequences because the values typically rise and fall, somewhat analogously to a hailstone inside a cloud.

While a hailstone eventually becomes so heavy that it falls to ground, every starting INTEGER ever tested has produced a hailstone sequence that eventually drops down to the number 1 and then "bounces" into the small loop 4, 2, 1, ...

see also COLLATZ PROBLEM

References
Schwartzman, S. *The Words of Mathematics: An Etymological Dictionary of Mathematical Terms Used in English.* Washington, DC: Math. Assoc. Amer., 1994.

Hairy Ball Theorem

There does not exist an everywhere NONZERO VECTOR FIELD on the 2-SPHERE $\mathbb{S}^2$. This implies that somewhere on the surface of the Earth, there is a point with zero horizontal wind velocity.

Half

The UNIT FRACTION 1/2.

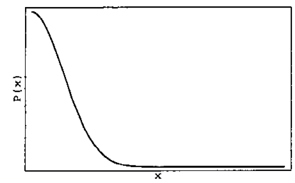
see also QUARTER, SQUARE ROOT, UNIT FRACTION

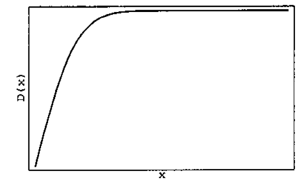
Half-Closed Interval

An INTERVAL in which one endpoint is included but not the other. A half-closed interval is denoted $[a, b)$ or $(a, b]$ and is also called a HALF-OPEN INTERVAL.

see also CLOSED INTERVAL, OPEN INTERVAL

Half-Normal Distribution





A NORMAL DISTRIBUTION with MEAN 0 and STANDARD DEVIATION $1/\theta$ limited to the domain $[0, \infty)$.

$$P(x) = \frac{2\theta}{\pi} e^{-x^2\theta^2/\pi} \quad (1)$$

$$D(x) = \operatorname{erf}\left(\frac{tx}{\sqrt{\pi}}\right). \quad (2)$$

The MOMENTS are

$$\mu_1 = \frac{1}{t} \quad (3)$$

$$\mu_2 = \frac{\pi}{2t^2} \quad (4)$$

$$\mu_3 = \frac{\pi}{t^3} \quad (5)$$

$$\mu_4 = \frac{3\pi^2}{4t^4}, \quad (6)$$

so the MEAN, VARIANCE, SKEWNESS, and KURTOSIS are

$$\mu = \frac{1}{\theta} \quad (7)$$

$$\sigma^2 \equiv \mu_2 - {\mu_1}^2 = \frac{\pi - 2}{2t^2} \quad (8)$$

$$\gamma_1 = 2\sqrt{\frac{2}{\pi}} \quad (9)$$

$$\gamma_2 = 0. \quad (10)$$

see also NORMAL DISTRIBUTION

Half-Open Interval

see HALF-CLOSED INTERVAL

Hall-Janko Group

The SPORADIC GROUP HJ, also denoted J_2.

see also JANKO GROUPS

Halley's Irrational Formula

A ROOT-finding ALGORITHM which makes use of a third-order TAYLOR SERIES

$$f(x) = f(x_n) + f'(x_n)(x - x_n) + \tfrac{1}{2}f''(x_n)(x - x_n)^2 + \ldots. \tag{1}$$

A ROOT of $f(x)$ satisfies $f(x) = 0$, so

$$0 \approx f(x_n) + f'(x_n)(x_{n+1} - x_n) + \tfrac{1}{2}f''(x_n)(x_{n+1} - x_n)^2. \tag{2}$$

Using the QUADRATIC EQUATION then gives

$$x_{n+1} = x_n + \frac{-f'(x_n) \pm \sqrt{[f'(x_n)]^2 - 2f(x_n)f''(x_n)}}{f''(x_n)}. \tag{3}$$

Picking the plus sign gives the iteration function

$$C_f(x) = x - \frac{1 - \sqrt{1 - \frac{2f(x)f''(x)}{[f'(x)]^2}}}{\frac{f''(x)}{f'(x)}}. \tag{4}$$

This equation can be used as a starting point for deriving HALLEY'S METHOD.

If the alternate form of the QUADRATIC EQUATION is used instead in solving (2), the iteration function becomes instead

$$C_f(x) = x - \frac{2f(x)}{f'(x) \pm \sqrt{[f'(x)]^2 - 2f(x)f''(x)}}. \tag{5}$$

This form can also be derived by setting $n = 2$ in LAGUERRE'S METHOD. Numerically, the SIGN in the DENOMINATOR is chosen to maximize its ABSOLUTE VALUE. Note that in the above equation, if $f''(x) = 0$, then NEWTON'S METHOD is recovered. This form of Halley's irrational formula has cubic convergence, and is usually found to be substantially more stable than NEWTON'S METHOD. However, it does run into difficulty when both $f(x)$ and $f'(x)$ or $f'(x)$ and $f''(x)$ are simultaneously near zero.

see also HALLEY'S METHOD, LAGUERRE'S METHOD, NEWTON'S METHOD

References

Qiu, H. "A Robust Examination of the Newton-Raphson Method with Strong Global Convergence Properties." Master's Thesis. University of Central Florida, 1993.

Scavo, T. R. and Thoo, J. B. "On the Geometry of Halley's Method." *Amer. Math. Monthly* **102**, 417–426, 1995.

Halley's Method

Also known as the TANGENT HYPERBOLAS METHOD or HALLEY'S RATIONAL FORMULA. As in HALLEY'S IRRATIONAL FORMULA, take the second-order TAYLOR POLYNOMIAL

$$f(x) = f(x_n) + f'(x_n)(x - x_n) + \tfrac{1}{2}f''(x_n)(x - x_n)^2 + \ldots. \tag{1}$$

A ROOT of $f(x)$ satisfies $f(x) = 0$, so

$$0 \approx f(x_n) + f'(x_n)(x_{n+1} - x_n) + \tfrac{1}{2}f''(x_n)(x_{n+1} - x_n)^2. \tag{2}$$

Now write

$$0 = f(x_n) + (x_{n+1} - x_n)[f'(x_n) + \tfrac{1}{2}f''(x_n)(x_{n+1} - x_n)], \tag{3}$$

giving

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n) + \frac{1}{2}f''(x_n)(x_{n+1} - x_n)}. \tag{4}$$

Using the result from NEWTON'S METHOD,

$$x_{n+1} - x_n = -\frac{f(x_n)}{f'(x_n)}, \tag{5}$$

gives

$$x_{n+1} = x_n - \frac{2f(x_n)f'(x_n)}{2[f'(x_n)]^2 - f(x_n)f''(x_n)}, \tag{6}$$

so the iteration function is

$$H_f(x) = x - \frac{2f(x)f'(x)}{2[f'(x)]^2 - f(x)f''(x)}. \tag{7}$$

This satisfies $H_f'(\alpha) = H_f''(\alpha) = 0$ where α is a ROOT, so it is third order for simple zeros. Curiously, the third derivative

$$H_f'''(\alpha) = -\left\{\frac{f'''(\alpha)}{f'(\alpha)} - \frac{3}{2}\left[\frac{f''(\alpha)}{f'(\alpha)}\right]^2\right\} \tag{8}$$

is the SCHWARZIAN DERIVATIVE. Halley's method may also be derived by applying NEWTON'S METHOD to $ff'^{-1/2}$. It may also be derived by using an OSCULATING CURVE of the form

$$y(x) = \frac{(x - x_n) + c}{a(x - x_n) + b}. \tag{9}$$

Taking derivatives,

$$f(x_n) = \frac{c}{b} \tag{10}$$

$$f'(x_n) = \frac{b - ac}{b^2} \tag{11}$$

$$f''(x_n) = \frac{2a(ac - b)}{b^3}, \tag{12}$$

which has solutions

$$a = -\frac{f''(x_n)}{2[f'(x_n)]^2 - f(x_n)f''(x_n)} \tag{13}$$

$$b = \frac{2f'(x_n)}{2[f'(x_n)]^2 - f(x_n)f''(x_n)} \tag{14}$$

$$c = \frac{2f(x_n)f'(x_n)}{2[f'(x_n)]^2 - f(x_n)f''(x_n)}, \tag{15}$$

so at a ROOT, $y(x_{n+1}) = 0$ and

$$x_{n+1} = x_n - c, \tag{16}$$

which is Halley's method.

see also HALLEY'S IRRATIONAL FORMULA, LAGUERRE'S METHOD, NEWTON'S METHOD

References
Scavo, T. R. and Thoo, J. B. "On the Geometry of Halley's Method." *Amer. Math. Monthly* **102**, 417–426, 1995.

Halley's Rational Formula

see HALLEY'S METHOD

Halphen Constant

see ONE-NINTH CONSTANT

Halphen's Transformation

A curve and its polar reciprocal with regard to the fixed CONIC have the same Halphen transformation.

References
Coolidge, J. L. *A Treatise on Algebraic Plane Curves.* New York: Dover, pp. 346–347, 1959.

Halting Problem

The determination of whether a TURING MACHINE will come to a halt given a particular input program. This problem is formally UNDECIDABLE, as first proved by Turing.

see also BUSY BEAVER, CHAITIN'S CONSTANT, TURING MACHINE, UNDECIDABLE

References
Chaitin, G. J. "Computing the Busy Beaver Function." §4.4 in *Open Problems in Communication and Computation* (Ed. T. M. Cover and B. Gopinath). New York: Springer-Verlag, pp. 108–112, 1987.
Davis, M. "What It a Computation." In *Mathematics Today: Twelve Informal Essays* (Ed. L. A. Steen). New York: Springer-Verlag, pp. 241–267, 1978.
Penrose, R. *The Emperor's New Mind: Concerning Computers, Minds, and the Laws of Physics.* Oxford, England: Oxford University Press, pp. 63–66, 1989.

Ham Sandwich Theorem

The volumes of any n n-D solids can always be simultaneously bisected by a $(n-1)$-D HYPERPLANE. Proving the theorem for $n = 2$ (where it is known as the PANCAKE THEOREM) is simple and can be found in Courant and Robbins (1978). The theorem was proved for $n > 3$ by Stone and Tukey (1942).

see also PANCAKE THEOREM

References
Chinn, W. G. and Steenrod, N. E. *First Concepts of Topology.* Washington, DC: Math. Assoc. Amer., 1966.
Courant, R. and Robbins, H. *What is Mathematics?: An Elementary Approach to Ideas and Methods.* Oxford, England: Oxford University Press, 1978.
Davis, P. J. and Hersh, R. *The Mathematical Experience.* Boston, MA: Houghton Mifflin, pp. 274–284, 1981.
Hunter, J. A. H. and Madachy, J. S. *Mathematical Diversions.* New York: Dover, pp. 67–69, 1975.
Stone, A. H. and Tukey, J. W. "Generalized 'Sandwich' Theorems." *Duke Math. J.* **9**, 356–359, 1942.

Hamilton's Equations

The equations defined by

$$\dot{q} = \frac{\partial H}{\partial p} \tag{1}$$

$$\dot{p} = -\frac{\partial H}{\partial q}, \tag{2}$$

where $\dot{x} \equiv dx/dt$ and H is the so-called Hamiltonian, are called Hamilton's equations. These equations frequently arise in problems of celestial mechanics. Another formulation related to Hamilton's equation is

$$p = \frac{\partial L}{\partial \dot{q}}, \tag{3}$$

where L is the so-called Lagrangian.

References
Morse, P. M. and Feshbach, H. "Hamilton's Principle and Classical Dynamics." §3.2 in *Methods of Theoretical Physics, Part I.* New York: McGraw-Hill, pp. 280–301, 1953.

Hamilton's Rules

The rules for the MULTIPLICATION of QUATERNIONS.

see also QUATERNION

Hamiltonian Circuit

A closed loop through a GRAPH that visits each node exactly once and ends adjacent to the initial point. The Hamiltonian circuit is named after Sir William Rowan Hamilton, who devised a puzzle in which such a path along the EDGES of an ICOSAHEDRON was sought (the ICOSIAN GAME).

All PLATONIC SOLIDS have a Hamiltonian circuit, as do planar 4-connected graphs. However, no foolproof method is known for determining whether a given general GRAPH has a Hamiltonian circuit. The number of Hamiltonian circuits on an n-HYPERCUBE is 2, 8, 96, 43008, ... (Sloane's A006069, Gardner 1986, pp. 23–24).

see also CHVÁTAL'S THEOREM, DIRAC'S THEOREM, EULER GRAPH, GRINBERG FORMULA, HAMILTONIAN GRAPH, HAMILTONIAN PATH, ICOSIAN GAME, KOZYREV-GRINBERG THEORY, ORE'S THEOREM, PÓSA'S THEOREM, SMITH'S NETWORK THEOREM

References
Chartrand, G. *Introductory Graph Theory.* New York: Dover, p. 68, 1985.
Gardner, M. "The Binary Gray Code." In *Knotted Doughnuts and Other Mathematical Entertainments.* New York: W. H. Freeman, pp. 23–24, 1986.
Sloane, N. J. A. Sequence A006069/M1903 in "An On-Line Version of the Encyclopedia of Integer Sequences."

Hamiltonian Cycle

see HAMILTONIAN CIRCUIT

Hamiltonian Graph

A GRAPH possessing a HAMILTONIAN CIRCUIT.

see also HAMILTONIAN CIRCUIT, HAMILTONIAN PATH

References

Chartrand, G. *Introductory Graph Theory.* New York: Dover, p. 68, 1985.

Chartrand, G.; Kapoor, S. F.; and Kronk, H. V. "The Many Facets of Hamiltonian Graphs." *Math. Student* **41**, 327–336, 1973.

Hamiltonian Group

A non-Abelian GROUP all of whose SUBGROUPS are self-conjugate.

References

Carmichael, R. D. "Hamiltonian Groups." §31 in *Introduction to the Theory of Groups of Finite Order.* New York: Dover, p. 113–116, 1956.

Hamiltonian Map

Consider a 1-D Hamiltonian MAP of the form

$$H(p,q) = \tfrac{1}{2}p^2 + V(q), \tag{1}$$

which satisfies HAMILTON'S EQUATIONS

$$\dot{q} = \frac{\partial H}{\partial p} \tag{2}$$

$$\dot{p} = -\frac{\partial H}{\partial q}. \tag{3}$$

Now, write

$$\dot{q}_i = \frac{(q_{i+1}-q_i)}{\Delta t}, \tag{4}$$

where

$$q_i = q(t) \tag{5}$$

$$q_{i+1} = q(t+\Delta t). \tag{6}$$

Then the equations of motion become

$$q_{i+1} = q_i + p_i\Delta t \tag{7}$$

$$p_{i+1} = p_i - \Delta t\left(\frac{\partial V}{\partial q_i}\right)_{q=q_i}. \tag{8}$$

Note that equations (7) and (8) are not AREA-PRESERVING, since

$$\frac{\partial(q_{i+1},p_{i+1})}{\partial(q_i,p_i)} = \begin{vmatrix} 1 & -\Delta t\frac{\partial^2 V}{\partial {q_i}^2} \\ \Delta t & 1 \end{vmatrix} = 1 + (\Delta t)^2\frac{\partial^2 V}{\partial {q_i}^2} \neq 1. \tag{9}$$

However, if we take instead of (7) and (8),

$$q_{i+1} = q_i + p_i\Delta t \tag{10}$$

$$p_{i+1} = p_i - \Delta t\left(\frac{\partial V}{\partial q_i}\right)_{q=q_{i+1}} \tag{11}$$

$$\begin{aligned}\frac{\partial(q_{i+1},p_{i+1})}{\partial(q_i,p_i)} &= \begin{vmatrix} 1 & -\Delta t\frac{\partial}{\partial q_i}\left(\frac{\partial V}{\partial q}\right)_{q=q_{i+1}} \\ \Delta t & 1 \end{vmatrix} \\ &= 1 + (\Delta t)^2\frac{\partial^2 V}{\partial {q_i}^2} = 1,\end{aligned} \tag{12}$$

which is AREA-PRESERVING.

Hamiltonian Path

A loop through a GRAPH that visits each node exactly once but does *not* end adjacent to the initial point. The number of Hamiltonian paths on an n-HYPERCUBE is 0, 0, 48, 48384, ... (Sloane's A006070, Gardner 1986, pp. 23–24).

see also HAMILTONIAN CIRCUIT, HAMILTONIAN GRAPH

References

Gardner, M. "The Binary Gray Code." In *Knotted Doughnuts and Other Mathematical Entertainments.* New York: W. H. Freeman, pp. 23–24, 1986.

Sloane, N. J. A. Sequence A006070/M5295 in "An On-Line Version of the Encyclopedia of Integer Sequences."

Hamiltonian System

A system of variables which can be written in the form of HAMILTON'S EQUATIONS.

Hammer-Aitoff Equal-Area Projection

A MAP PROJECTION whose inverse is defined using the intermediate variable

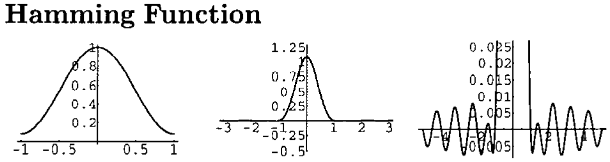
$$z \equiv \sqrt{1-(\tfrac{1}{4}x)^2-(\tfrac{1}{2}y)^2}.$$

Then the longitude and latitude are given by

$$\lambda = 2\tan^{-1}\left(\frac{zx}{2(2z^2-1)}\right)$$

$$\phi = \sin^{-1}(yz).$$

Hamming Function



An APODIZATION FUNCTION chosen to minimize the height of the highest sidelobe. The Hamming function is given by

$$A(x) = 0.54 + 0.46\cos\left(\frac{\pi x}{a}\right), \tag{1}$$

Its FULL WIDTH AT HALF MAXIMUM is $1.05543a$. The corresponding INSTRUMENT FUNCTION is

$$I(k) = \frac{a(1.08-0.64a^2k^2)\,\mathrm{sinc}(2\pi ak)}{1-4a^2k^2}. \tag{2}$$

This APODIZATION FUNCTION is close to the one produced by the requirement that the APPARATUS FUNCTION goes to 0 at $ka = 5/4$. From APODIZATION FUNCTION, a general symmetric apodization function $A(x)$ can be written as a FOURIER SERIES

$$A(x) = a_0 + 2\sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{b}\right), \tag{3}$$

where the COEFFICIENTS satisfy

$$a_0 + 2\sum_{n=1}^{\infty} a_n = 1. \tag{4}$$

The corresponding apparatus function is

$$I(t) = 2b\{a_0 \operatorname{sinc}(2\pi kb) + \sum_{n=1}^{\infty}[\operatorname{sinc}(2\pi kb + n\pi) + \operatorname{sinc}(2\pi kb - n\pi)]\}. \tag{5}$$

To obtain an APODIZATION FUNCTION with zero at $ka = 3/4$, use

$$a_0 + 2a_1 = 1, \tag{6}$$

so

$$a_0 \operatorname{sinc}(\tfrac{5}{2}\pi) + a_1[\operatorname{sinc}(\tfrac{7}{2}\pi) + \operatorname{sinc}(\tfrac{3}{2}\pi) = 0 \tag{7}$$

$$(1-2a_1)\frac{2}{5\pi} - a_1\left(\frac{2}{7\pi} + \frac{2}{3\pi}\right) = (1-2a_1)\tfrac{1}{5} - a_1(\tfrac{1}{7}+\tfrac{1}{3}) = 0 \tag{8}$$

$$a_1(\tfrac{1}{7} + \tfrac{1}{3} + \tfrac{2}{5}) = \tfrac{1}{5} \tag{9}$$

$$a_1 = \frac{\frac{1}{5}}{\frac{2}{5}+\frac{1}{7}+\frac{1}{3}} = \frac{7\cdot 3}{2\cdot 3\cdot 7 + 3\cdot 5 + 5\cdot 7} = \tfrac{21}{92} \approx 0.2283 \tag{10}$$

$$a_0 = 1 - 2a_1 = \frac{92 - 2\cdot 21}{92} = \frac{92-42}{92} = \tfrac{50}{92} = \tfrac{25}{46} \approx 0.5435. \tag{11}$$

The FWHM is 1.81522, the peak is 1.08, the peak NEGATIVE and POSITIVE sidelobes (in units of the peak) are -0.00689132 and 0.00734934, respectively.

see also APODIZATION FUNCTION, HANNING FUNCTION, INSTRUMENT FUNCTION

References

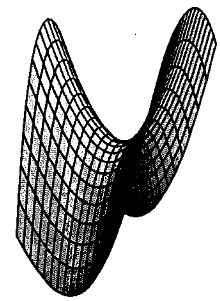
Blackman, R. B. and Tukey, J. W. "Particular Pairs of Windows." In *The Measurement of Power Spectra, From the Point of View of Communications Engineering.* New York: Dover, pp. 98–99, 1959.

Handedness

Objects which are identical except for a mirror reflection are said to display handedness and to be CHIRAL.

see also AMPHICHIRAL, CHIRAL, ENANTIOMER, MIRROR IMAGE

Handkerchief Surface



A surface given by the parametric equations

$$x(u,v) = u$$
$$y(u,v) = v$$
$$z(u,v) = \tfrac{1}{3}u^3 + uv^2 + 2(u^2 - v^2).$$

References

Gray, A. *Modern Differential Geometry of Curves and Surfaces.* Boca Raton, FL: CRC Press, p. 628, 1993.

Handle

Handles are to MANIFOLDS as CELLS are to CW-COMPLEXES. If M is a MANIFOLD together with a $(k-1)$-SPHERE $\mathbb{S}^{k-1}$ embedded in its boundary with a trivial TUBULAR NEIGHBORHOOD, we attach a k-handle to M by gluing the tubular NEIGHBORHOOD of the $(k-1)$-SPHERE $\mathbb{S}^{k-1}$ to the TUBULAR NEIGHBORHOOD of the standard $(k-1)$-SPHERE $\mathbb{S}^{k-1}$ in the $\dim(M)$-dimensional DISK.

In this way, attaching a k-handle is essentially just the process of attaching a fattened-up k-DISK to M along the $(k-1)$-SPHERE $\mathbb{S}^{k-1}$. The embedded DISK in this new MANIFOLD is called the k-handle in the UNION of M and the handle.

see also HANDLEBODY, SURGERY, TUBULAR NEIGHBORHOOD

Handlebody

A handlebody of type (n,k) is an n-D MANIFOLD that is attained from the standard n-DISK by attaching only k-D HANDLES.

see also HANDLE, HEEGAARD SPLITTING, SURGERY

References

Rolfsen, D. *Knots and Links.* Wilmington, DE: Publish or Perish Press, p. 46, 1976.

Hankel Function

A COMPLEX function which is a linear combination of BESSEL FUNCTIONS OF THE FIRST and SECOND KINDS.

see also HANKEL FUNCTION OF THE FIRST KIND, HANKEL FUNCTION OF THE SECOND KIND, SPHERICAL HANKEL FUNCTION OF THE FIRST KIND, SPHERICAL HANKEL FUNCTION OF THE SECOND KIND

References
Arfken, G. "Hankel Functions." §11.4 in *Mathematical Methods for Physicists, 3rd ed.* Orlando, FL: Academic Press, pp. 604–610, 1985.
Morse, P. M. and Feshbach, H. *Methods of Theoretical Physics, Part I.* New York: McGraw-Hill, pp. 623–624, 1953.

Hankel Function of the First Kind

$$H_n^{(1)}(z) \equiv J_n(z) + iY_n(z),$$

where $J_n(z)$ is a BESSEL FUNCTION OF THE FIRST KIND and $Y_n(z)$ is a BESSEL FUNCTION OF THE SECOND KIND. Hankel functions of the first kind can be represented as a CONTOUR INTEGRAL using

$$H_n^{(1)}(z) = \frac{1}{i\pi}\int_{0\ \text{[upper half plane]}}^{\infty} \frac{e^{(z/2)(t-1/t)}}{t^{n+1}}\,dt.$$

see also DEBYE'S ASYMPTOTIC REPRESENTATION, WATSON-NICHOLSON FORMULA, WEYRICH'S FORMULA

References
Arfken, G. "Hankel Functions." §11.4 in *Mathematical Methods for Physicists, 3rd ed.* Orlando, FL: Academic Press, pp. 604–610, 1985.
Morse, P. M. and Feshbach, H. *Methods of Theoretical Physics, Part I.* New York: McGraw-Hill, pp. 623–624, 1953.

Hankel Function of the Second Kind

$$H_n^{(2)}(z) \equiv J_n(z) - iY_n(z),$$

where $J_n(z)$ is a BESSEL FUNCTION OF THE FIRST KIND and $Y_n(z)$ is a BESSEL FUNCTION OF THE SECOND KIND. Hankel functions of the second kind can be represented as a CONTOUR INTEGRAL using

$$H_n^{(2)}(z) = \frac{1}{i\pi}\int_{-\infty\ \text{[lower half plane]}}^{0} \frac{e^{(z/2)(t-1/t)}}{t^{n+1}}\,dt.$$

see also WATSON-NICHOLSON FORMULA

References
Arfken, G. "Hankel Functions." §11.4 in *Mathematical Methods for Physicists, 3rd ed.* Orlando, FL: Academic Press, pp. 604–610, 1985.
Morse, P. M. and Feshbach, H. *Methods of Theoretical Physics, Part I.* New York: McGraw-Hill, pp. 623–624, 1953.

Hankel's Integral

$$J_m(x) = \frac{x^m}{2^{m-1}\sqrt{\pi}\,\Gamma\left(m+\frac{1}{2}\right)} \times \int_0^1 \cos(xt)(1-t^2)^{m-1/2}\,dt,$$

where $J_m(x)$ is a BESSEL FUNCTION OF THE FIRST KIND and $\Gamma(z)$ is the GAMMA FUNCTION. Hankel's integral can be derived from SONINE'S INTEGRAL.

see also POISSON INTEGRAL, SONINE'S INTEGRAL

Hankel Matrix

A MATRIX with identical values for each element in a given diagonal. Define H_n to be the Hankel matrix with leading column made up of the INTEGERS $1, \ldots, n$, then

$$\mathsf{H}_2 = \begin{bmatrix} 1 & 2 \\ 2 & 0 \end{bmatrix}$$
$$\mathsf{H}_3 = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 0 \\ 3 & 0 & 0 \end{bmatrix}.$$

Hankel Transform

Equivalent to a 2-D FOURIER TRANSFORM with a radially symmetric KERNEL, and also called the FOURIER-BESSEL TRANSFORM.

$$g(u,v) = \mathcal{F}[f(r)] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(r)e^{-2\pi i(ux+vy)}\,dx\,dy. \tag{1}$$

Let

$$x + iy = re^{i\theta} \tag{2}$$
$$u + iv = qe^{i\phi} \tag{3}$$

so that

$$x = r\cos\theta \tag{4}$$
$$y = r\sin\theta \tag{5}$$
$$r = \sqrt{x^2+y^2} \tag{6}$$

$$u = q\cos\phi \tag{7}$$
$$v = q\sin\phi \tag{8}$$
$$q = \sqrt{u^2+v^2}. \tag{9}$$

Then

$$\begin{aligned} g(q) &= \int_0^\infty \int_0^{2\pi} f(r)e^{-2\pi irq(\cos\phi\cos\theta+\sin\phi\sin\theta)} r\,dr\,d\theta \\ &= \int_0^\infty \int_0^{2\pi} f(r)e^{-2\pi irq\cos(\theta-\phi)} r\,dr\,d\theta \\ &= \int_0^\infty \int_{-\phi}^{2\pi-\phi} f(r)e^{-2\pi irq\cos\theta} r\,dr\,d\theta \\ &= \int_0^\infty \int_0^{2\pi} f(r)e^{-2\pi irq\cos\theta} r\,dr\,d\theta \\ &= \int_0^\infty f(r)\left[\int_0^{2\pi} e^{-2\pi irq\cos\theta}\,d\theta\right] r\,dr \\ &= 2\pi\int_0^\infty f(r)J_0(2\pi qr)r\,dr, \end{aligned} \tag{10}$$

where $J_0(z)$ is a zeroth order BESSEL FUNCTION OF THE FIRST KIND. Therefore, the Hankel transform pairs are

$$g(k) = \int_0^\infty f(x) J_0(kx) x \, dx \tag{11}$$

$$f(x) = \int_0^\infty g(k) J_0(kx) k \, dk. \tag{12}$$

see also BESSEL FUNCTION OF THE FIRST KIND, FOURIER TRANSFORM, LAPLACE TRANSFORM

References

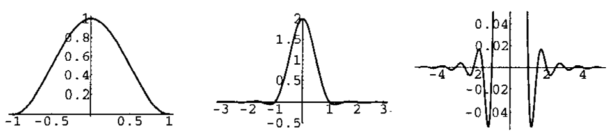
Arfken, G. *Mathematical Methods for Physicists, 3rd ed.* Orlando, FL: Academic Press, p. 795, 1985.

Bracewell, R. *The Fourier Transform and Its Applications.* New York: McGraw-Hill, pp. 244–250, 1965.

Hann Function

see HANNING FUNCTION

Hanning Function



An APODIZATION FUNCTION, also called the HANN FUNCTION, frequently used to reduce ALIASING in FOURIER TRANSFORMS. The illustrations above show the Hanning function, its INSTRUMENT FUNCTION, and a blowup of the INSTRUMENT FUNCTION sidelobes. The Hanning function is given by

$$f(x) = \cos^2\left(\frac{\pi x}{2a}\right) = \tfrac{1}{2} - \tfrac{1}{2}\cos\left(\frac{\pi x}{a}\right). \tag{1}$$

The INSTRUMENT FUNCTION for Hanning apodization can also be written

$$a[\operatorname{sinc}(2\pi ka) + \tfrac{1}{2}\operatorname{sinc}(2\pi ka - \pi) + \tfrac{1}{2}\operatorname{sinc}(2\pi ka + \pi)]. \tag{2}$$

Its FULL WIDTH AT HALF MAXIMUM is a. It has APPARATUS FUNCTION

$$\begin{aligned} A(x) &= \int_{-a}^{a} \left[\tfrac{1}{2} - \tfrac{1}{2}\cos\left(\frac{\pi x}{a}\right)\right] e^{-2\pi ikx} \, dx \\ &= \tfrac{1}{2}\int_{-a}^{a} e^{-2\pi ikx} \, dx - \tfrac{1}{2}\int_{-a}^{a} e^{-2\pi ikx} \, dx \\ &\equiv \tfrac{1}{2}(A_1 + A_2). \end{aligned} \tag{3}$$

The first integral is

$$I_1 = \int_{-a}^{a} e^{-2\pi ikx} \, dx = \frac{\sin(2\pi ka)}{\pi k} = 2a \operatorname{sinc}(2\pi ka). \tag{4}$$

The second integral can be rewritten

$$\begin{aligned} I_2 &= \int_{-a}^{0} \cos\left(\frac{\pi x}{a}\right) e^{-2\pi ikx} \, dx \\ &\quad + \int_0^a \cos\left(\frac{\pi x}{a}\right) e^{-2\pi ikx} \, dx \\ &= \int_0^a \cos\left(\frac{\pi x}{a}\right) (e^{2\pi ikx} + e^{-2\pi ikx}) \, dx \\ &= 2\int_0^a \cos\left(\frac{\pi x}{a}\right) \cos(2\pi kx) \, dx \\ &= 2\left\{\frac{\sin\left(\frac{\pi}{a} - 2\pi k\right)x}{2\left(\frac{\pi}{a} - 2\pi k\right)} + \frac{\sin\left(\frac{\pi}{a} + 2\pi k\right)x}{2\left(\frac{\pi}{a} + 2\pi k\right)}\right\}_0^a \\ &= a\left[\frac{\sin(\pi - 2\pi ka)}{\pi - 2\pi ka} + \frac{\sin(\pi + 2\pi ka)}{\pi + 2\pi ka}\right] \\ &= \frac{a}{\pi}\left[\frac{\sin(2\pi ka)}{1 - 2ka} - \frac{\sin(2\pi ka)}{1 + 2ka}\right] \\ &= a[\operatorname{sinc}(\pi - 2\pi ka) + \operatorname{sinc}(\pi + 2\pi ka)]. \end{aligned} \tag{5}$$

Combining (4) and (5) gives

$$\begin{aligned} A(x) = a[&\operatorname{sinc}(2\pi ka) + \tfrac{1}{2}\operatorname{sinc}(\pi - 2\pi ka) \\ &+ \tfrac{1}{2}\operatorname{sinc}(\pi + 2\pi ka)]. \end{aligned} \tag{6}$$

To find the extrema, define $x \equiv 2\pi ka$ and rewrite (6) as

$$A(x) = a[\sin x + \tfrac{1}{2}\operatorname{sinc}(x - \pi) + \tfrac{1}{2}\operatorname{sinc}(x + \pi)]. \tag{7}$$

Then solve

$$\frac{dA}{dx} = \frac{\pi^2(-x^3\cos x + 3x^2\sin x + \pi^2 x\cos x - \pi^2\sin x)}{x^2(\pi^2 - x^2)^2} = 0 \tag{8}$$

to find the extrema. The roots are $x = 7.42023$ and 10.7061, giving a peak NEGATIVE sidelobe of -0.026708 and a peak POSITIVE sidelobe (in units of a) of 0.00843441. The peak in units of a is 1, and the full-width at half maximum is given by setting (7) equal to 1/2 and solving for x, yielding

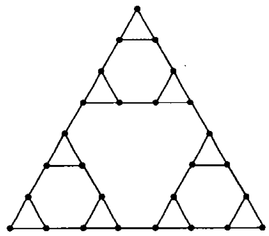
$$x_{1/2} = 2\pi k_{1/2} a = \pi. \tag{9}$$

Therefore, with $L \equiv 2a$, the FULL WIDTH AT HALF MAXIMUM is

$$\mathrm{FWHM} = 2k_{1/2} = \frac{1}{a} = \frac{2}{L}. \tag{10}$$

see also APODIZATION FUNCTION, HAMMING FUNCTION

Hanoi Graph



A GRAPH H_n arising in conjunction with the TOWERS OF HANOI problem. The above figure is the Hanoi graph H_3.

see also TOWERS OF HANOI

Hanoi Towers

see TOWERS OF HANOI

Hansen-Bessel Formula

$$\begin{aligned} J_n(z) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{iz\cos t}e^{in(t-\pi/2)}\,dt \\ &= \frac{i^{-n}}{\pi}\int_0^{\pi} e^{iz\cos t}\cos(nt)\,dt \\ &= \frac{1}{\pi}\int_0^{\pi}\cos(z\sin t - nt)\,dt \end{aligned}$$

for $n = 0, 1, 2, \ldots$, where $J_n(z)$ is a BESSEL FUNCTION OF THE FIRST KIND.

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Hansen Chain

An ADDITION CHAIN for which there is a SUBSET H of members such that each member of the chain uses the largest element of H which is less than the member.

see also ADDITION CHAIN, BRAUER CHAIN, HANSEN NUMBER

References

Guy, R. K. "Addition Chains. Brauer Chains. Hansen Chains." §C6 in *Unsolved Problems in Number Theory, 2nd ed.* New York: Springer-Verlag, pp. 111–113, 1994.

Hansen Number

A number n for which a shortest chain exists which is a HANSEN CHAIN is called a Hansen number.

References

Guy, R. K. *Unsolved Problems in Number Theory, 2nd ed.* New York: Springer-Verlag, pp. 111–112, 1994.

Hansen's Problem

A SURVEYING PROBLEM: from the position of two known but inaccessible points A and B, determine the position of two unknown accessible points P and P' by bearings from A, B, P' to P and A, B, P to P'.

see also SURVEYING PROBLEMS

References

Dörrie, H. "Annex to a Survey." §40 in *100 Great Problems of Elementary Mathematics: Their History and Solutions.* New York: Dover, pp. 193–197, 1965.

Happy Number

Let the sum of the SQUARES of the DIGITS of a POSITIVE INTEGER s_0 be represented by s_1. In a similar way, let the sum of the SQUARES of the DIGITS of s_1 be represented by s_2, and so on. If some $s_i = 1$ for $i \geq 1$, then the original INTEGER s_0 is said to be happy.

Once it is known whether a number is happy (or not), then any number in the sequence s_1, s_2, s_3, ... will also be happy (or not). A number which is not happy is called UNHAPPY. Unhappy numbers have EVENTUALLY PERIODIC sequences of s_i 4, 16, 37, 58, 89, 145, 42, 20, 4, ... which do not reach 1.

Any PERMUTATION of the DIGITS of an UNHAPPY or happy number must also be unhappy or happy. This follows from the fact that ADDITION is COMMUTATIVE. The first few happy numbers are 1, 7, 10, 13, 19, 23, 28, 31, 32, 44, 49, 68, 70, 79, 82, 86, 91, 94, 97, 100, ... (Sloane's A007770). These are also the numbers whose 2-RECURRING DIGITAL INVARIANT sequences have period 1.

see also KAPREKAR NUMBER, RECURRING DIGITAL INVARIANT , UNHAPPY NUMBER

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Harada-Norton Group

The SPORADIC GROUP *HN*.

References

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Harary Graph

The smallest k-connected GRAPH with n VERTICES.

Hard Hexagon Entropy Constant

N.B. A detailed on-line essay by S. Finch was the starting point for this entry.

A constant related to the HARD SQUARE ENTROPY CONSTANT. This constant is given by

$$\kappa_h \equiv \lim_{N\to\infty}[G(N)]^{1/N} = 1.395485972\ldots, \qquad (1)$$

where $G(N)$ is the number of configurations of nonattacking KINGS on an $n \times n$ chessboard with regular hexagonal cells, where $N \equiv n^2$. Amazingly, κ_h is algebraic and given by

$$\kappa_h \equiv \kappa_1\kappa_2\kappa_3\kappa_4, \qquad (2)$$

where

$$\kappa_1 \equiv 4^{-1}3^{5/4}11^{-5/12}c^{-2} \qquad (3)$$

$$\kappa_2 \equiv [1-\sqrt{1-c}+\sqrt{2+c+2\sqrt{1+c+c^2}}\,]^2 \qquad (4)$$

$$\kappa_3 \equiv [-1-\sqrt{1-c}+\sqrt{2+c+2\sqrt{1+c+c^2}}\,]^2 \qquad (5)$$

$$\kappa_4 \equiv [\sqrt{1-a}+\sqrt{2+a+2\sqrt{1+a+a^2}}\,]^{-1/2} \qquad (6)$$

$$a \equiv -\tfrac{124}{363}11^{1/3} \qquad (7)$$

$$b \equiv \tfrac{2501}{11979}33^{1/2} \qquad (8)$$

$$c \equiv \{\tfrac{1}{4}+\tfrac{3}{8}a[(b+1)^{1/3}-(b-1)^{1/3}]\}^{1/3}. \qquad (9)$$

(Baxter 1980, Joyce 1988).

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Hard Square Entropy Constant

N.B. A detailed on-line essay by S. Finch was the starting point for this entry.

Let $F(n^2)$ be the number of binary $n\times n$ MATRICES with no adjacent 1s (in either columns or rows). Define $N \equiv n^2$, then the hard square entropy constant is defined by

$$\kappa \equiv \lim_{N\to\infty}[F(N)]^{1/N} = 1.503048082\ldots.$$

The quantity $\ln\kappa$ arises in statistical physics (Baxter *et al.* 1980, Pearce and Seaton 1988), and is known as the entropy per site of hard squares. A related constant known as the HARD HEXAGON ENTROPY CONSTANT can also be defined.

References

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Hardy's Inequality

Let $\{a_n\}$ be a NONNEGATIVE SEQUENCE and $f(x)$ a NONNEGATIVE integrable FUNCTION. Define

$$A_n = \sum_{k=1}^{n} a_k \qquad (1)$$

$$B_n = \sum_{k=n}^{\infty} a_k \qquad (2)$$

and

$$F(x) = \int_0^x f(t)\,dt \qquad (3)$$

$$G(x) = \int_x^\infty f(t)\,dt, \qquad (4)$$

and take $p > 1$. For sums,

$$\sum_{n=1}^{\infty}\left(\frac{A_n}{n}\right)^p < \left(\frac{p}{p-1}\right)^p \sum_{n=1}^{\infty}(a_n)^p \qquad (5)$$

(unless all $a_n = 0$), and for integrals,

$$\int_0^\infty \left[\frac{F(x)}{x}\right]^p dx < \left(\frac{p}{p-1}\right)^p \int_0^\infty [f(x)]^p\,dx \qquad (6)$$

(unless f is identically 0).

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Hardy-Littlewood Conjectures

The first Hardy-Littlewood conjecture is called the k-TUPLE CONJECTURE. It states that the asymptotic number of PRIME CONSTELLATIONS can be computed explicitly.

The second Hardy-Littlewood conjecture states that

$$\pi(x+y)-\pi(x) \leq \pi(y)$$

for all x and y, where $\pi(x)$ is the PRIME COUNTING FUNCTION. Although it is not obvious, Richards (1974) proved that this conjecture is incompatible with the first Hardy-Littlewood conjecture.

see also PRIME CONSTELLATION, PRIME COUNTING FUNCTION

References

Richards, I. "On the Incompatibility of Two Conjectures Concerning Primes." *Bull. Amer. Math. Soc.* **80**, 419–438, 1974.

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Hardy-Littlewood Constants

see PRIME CONSTELLATION

Hardy-Littlewood Tauberian Theorem

Let $a_n \geq 0$ and suppose

$$\sum_{n=1}^{\infty} a_n e^{-an} \sim \frac{1}{a}$$

as $a \to 0^+$. Then

$$\sum_{n \leq x} a_n \sim x$$

as $x \to \infty$.

see also TAUBERIAN THEOREM

References

Berndt, B. C. *Ramanujan's Notebooks, Part IV.* New York: Springer-Verlag, pp. 118–119, 1994.

Hardy-Littlewood k-Tuple Conjecture

see PRIME PATTERNS CONJECTURE

Hardy-Ramanujan Number

The smallest nontrivial TAXICAB NUMBER, i.e., the smallest number representable in two ways as a sum of two CUBES. It is given by

$$1729 = 1^3 + 12^3 = 9^3 + 10^3.$$

The number derives its name from the following story G. H. Hardy told about Ramanujan. "Once, in the taxi from London, Hardy noticed its number, 1729. He must have thought about it a little because he entered the room where Ramanujan lay in bed and, with scarcely a hello, blurted out his disappointment with it. It was, he declared, 'rather a dull number,' adding that he hoped that wasn't a bad omen. 'No, Hardy,' said Ramanujan, 'it is a very interesting number. It is the smallest number expressible as the sum of two [POSITIVE] cubes in two different ways'" (Hofstadter 1989, Kanigel 1991, Snow 1993).

see also DIOPHANTINE EQUATION—CUBIC, TAXICAB NUMBER

References

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Hardy-Ramanujan Theorem

Let $\omega(n)$ be the number of DISTINCT PRIME FACTORS of n. If $\Psi(x)$ tends steadily to infinity with x, then

$$\ln\ln x - \Psi(x)\sqrt{\ln\ln x} < \omega(n) < \ln\ln x + \Psi(x)\sqrt{\ln\ln x}$$

for ALMOST ALL numbers $n < x$. "ALMOST ALL" means here the frequency of those INTEGERS n in the interval $1 \leq n \leq x$ for which

$$|\omega(n) - \ln\ln x| > \Psi(x)\sqrt{\ln\ln x}$$

approaches 0 as $x \to \infty$.

see also DISTINCT PRIME FACTORS, ERDŐS-KAC THEOREM

Hardy's Rule

Let the values of a function $f(x)$ be tabulated at intervals equally spaced by h about x_0, so that $f_{-3} = f(x_0 - 3h)$, $f_{-2} = f(x_0 - 2h)$, etc. Then Hardy's rule gives the approximation to the integral of $f(x)$ as

$$\int_{x_0-3h}^{x_0+3h} f(x)\,dx = \tfrac{1}{100}h(28f_{-3} + 162f_{-2} + 22f_0 + 162f_2 + 28f_3) + \tfrac{9}{1400}h^7[2f^{(4)}(\xi_2) - h^2 f^{(8)}(\xi_1)],$$

where the final term gives the error, with $\xi_1, \xi_2 \in [x_0 - 3h, x_0 + 3h]$.

see also BODE'S RULE, DURAND'S RULE, NEWTON-COTES FORMULAS, SIMPSON'S 3/8 RULE, SIMPSON'S RULE, TRAPEZOIDAL RULE, WEDDLE'S RULE

Harmonic Addition Theorem

To convert an equation of the form

$$f(\theta) = a\cos\theta + b\sin\theta \tag{1}$$

to the form

$$f(\theta) = c\cos(\theta + \delta), \tag{2}$$

expand (2) using the trigonometric addition formulas to obtain

$$f(\theta) = c\cos\theta\cos\delta - c\sin\theta\sin\delta. \tag{3}$$

Now equate the COEFFICIENTS of (1) and (3)

$$a = c\cos\delta \tag{4}$$

$$b = -c\sin\delta, \tag{5}$$

so

$$\tan\delta = -\frac{b}{a} \tag{6}$$

$$a^2 + b^2 = c^2, \tag{7}$$

and we have

$$\delta = \tan^{-1}\left(-\frac{b}{a}\right) \tag{8}$$

$$c = \sqrt{a^2 + b^2}. \tag{9}$$

Given two general sinusoidal functions with frequency ω:

$$\psi_1 = A_1 \sin(\omega t + \delta_1) \quad (10)$$
$$\psi_2 = A_2 \sin(\omega t + \delta_2), \quad (11)$$

their sum ψ can be expressed as a sinusoidal function with frequency ω

$$\begin{aligned}\psi \equiv \psi_1 + \psi_2 &= A_1[\sin(\omega t)\cos\delta_1 + \sin\delta_1 \cos(\omega t)] \\ &\quad + A_2[\sin(\omega t)\cos\delta_2 + \sin\delta_2\cos(\omega t)] \\ &= [A_1\cos\delta_1 + A_2\cos\delta_2]\sin(\omega t) \\ &\quad + [A_1 \sin\delta_1 + A_2\sin\delta_2]\cos(\omega t). \quad (12)\end{aligned}$$

Now, define

$$A\cos\delta \equiv A_1\cos\delta_1 + A_2\cos\delta_2 \quad (13)$$

$$A\sin\delta \equiv A_1\sin\delta_1 + A_2\sin\delta_2. \quad (14)$$

Then (12) becomes

$$A\cos\delta\sin(\omega t) + A\sin\delta\cos(\omega t) = A\sin(\omega t + \delta). \quad (15)$$

Square and add (13) and (14)

$$A_2 = {A_1}^2 + {A_2}^2 + 2A_1A_2\cos(\delta_2 - \delta_1). \quad (16)$$

Also, divide (14) by (13)

$$\tan\delta = \frac{A_1\sin\delta_1 + A_2\sin\delta_2}{A_1\cos\delta_1 + A_2\cos\delta_2}, \quad (17)$$

so

$$\psi = A\sin(\omega t + \delta), \quad (18)$$

where A and δ are defined by (16) and (17).

This procedure can be generalized to a sum of n harmonic waves, giving

$$\psi = \sum_{i=1}^{n} A_i \cos(\omega t + \delta_i) = A\cos(\omega t + \delta), \quad (19)$$

where

$$A^2 \equiv \sum_{i=1}^{n} {A_i}^2 + 2\sum_{j>i}^{n}\sum_{i=1}^{n} A_iA_j\cos(\delta_i - \delta_j) \quad (20)$$

and

$$\tan\delta = -\frac{\sum_{i=1}^{n} A_i\sin\delta_i}{\sum_{i=1}^{n} A_i\cos\delta_i}. \quad (21)$$

Harmonic Analysis

see also FOURIER SERIES

Harmonic Brick

A right-angled PARALLELEPIPED with dimensions $a \times ab \times abc$, where a, b, and c are INTEGERS.

see also BRICK, DE BRUIJN'S THEOREM, EULER BRICK

Harmonic Conjugate Function

The harmonic conjugate to a given function $u(x,y)$ is a function $v(x,y)$ such that

$$f(x,y) = u(x,y) + v(x,y)$$

is COMPLEX DIFFERENTIABLE (i.e., satisfies the CAUCHY-RIEMANN EQUATIONS). It is given by

$$v(z) = \int ux\,dy - uy\,dx.$$

Harmonic Conjugate Points

Given COLLINEAR points W, X, Y, and Z, Y and Z are harmonic conjugates with respect to W and X if

$$\frac{|WY|}{|YX|} = \frac{|WZ|}{|ZX|}.$$

The distances between such points are said to be in HARMONIC RATIO, and the LINE SEGMENT depicted above is called a HARMONIC SEGMENT.

Harmonic conjugate points are also defined for a TRIANGLE. If W and X have TRILINEAR COORDINATES $\alpha : \beta : \gamma$ and $\alpha' : \beta' : \gamma'$, then the TRILINEAR COORDINATES of the harmonic conjugates are

$$Y = \alpha + \alpha' : \beta + \beta' : \gamma + \gamma'$$
$$Z = \alpha - \alpha' : \beta - \beta' : \gamma - \gamma'$$

(Kimberling 1994).

see also HARMONIC RANGE, HARMONIC RATIO

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Harmonic Coordinates

Satisfy the condition

$$\Gamma^\lambda \equiv g^{\mu\nu}\Gamma_{\mu\nu}{}^\lambda = 0, \quad (1)$$

or equivalently,

$$\frac{\partial}{\partial x^\kappa}\left(\sqrt{g}\,g^{\lambda\kappa}\right) = 0. \quad (2)$$

It is always possible to choose such a system. Using the D'ALEMBERTIAN OPERATOR,

$$\Box^2\phi \equiv (g^{\lambda\kappa}\phi_{;\lambda})_{;\kappa} = g^{\lambda\kappa}\frac{\partial^2\phi}{\partial x^\lambda \partial x^\kappa} - \Gamma^\lambda \frac{\partial\phi}{\partial x^\lambda}. \quad (3)$$

But since $\Gamma^\lambda \equiv 0$ for harmonic coordinates,

$$\Box^2 x^\mu = 0. \quad (4)$$

Harmonic Decomposition

A POLYNOMIAL function in the elements of $\mathbf{x}$ can be uniquely decomposed into a sum of harmonic POLYNOMIALS times POWERS of $|\mathbf{x}|$.

Harmonic Divisor Number

A number n for which the HARMONIC MEAN of the DIVISORS of n, i.e., $nd(n)/\sigma(n)$, is an INTEGER, where $d(n)$ is the number of POSITIVE integral DIVISORS of n and $\sigma(n)$ is the DIVISOR FUNCTION. For example, the divisors of $n = 140$ are 1, 2, 4, 5, 7, 10, 14, 20, 28, 35, 70, and 140, giving

$$\begin{aligned} d(140) &= 12 \\ \sigma(140) &= 336 \\ \frac{140d(140)}{\sigma(140)} &= \frac{140 \cdot 12}{335} = 5, \end{aligned}$$

so 140 is a harmonic divisor number. Harmonic divisor numbers are also called ORE NUMBERS. Garcia (1954) gives the 45 harmonic divisor numbers less than 10^7. The first few are 1, 6, 140, 270, 672, 1638, ... (Sloane's A007340).

For distinct PRIMES p and q, harmonic divisor numbers are equivalent to EVEN PERFECT NUMBERS for numbers of the form $p^r q$. Mills (1972) proved that if there exists an ODD POSITIVE harmonic divisor number n, then n has a prime-POWER factor greater than 10^7.

Another type of number called "harmonic" is the HARMONIC NUMBER.

see also DIVISOR FUNCTION, HARMONIC NUMBER

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Harmonic Equation

see LAPLACE'S EQUATION

Harmonic Function

Any real-valued function $u(x,y)$ with continuous second PARTIAL DERIVATIVES which satisfies LAPLACE'S EQUATION

$$\nabla^2 u(x,y) = 0 \quad (1)$$

is called a harmonic function. Harmonic functions are called POTENTIAL FUNCTIONS in physics and engineering. Potential functions are extremely useful, for example, in electromagnetism, where they reduce the study of a 3-component VECTOR FIELD to a 1-component SCALAR FUNCTION. A scalar harmonic function is called a SCALAR POTENTIAL, and a vector harmonic function is called a VECTOR POTENTIAL.

To find a class of such functions in the PLANE, write the LAPLACE'S EQUATION in POLAR COORDINATES

$$u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0, \quad (2)$$

and consider only radial solutions

$$u_{rr} + \frac{1}{r}u_r = 0. \quad (3)$$

This is integrable by quadrature, so define $v \equiv du/dr$,

$$\frac{dv}{dr} + \frac{1}{r}v = 0 \quad (4)$$

$$\frac{dv}{v} = -\frac{dr}{r} \quad (5)$$

$$\ln\left(\frac{v}{A}\right) = -\ln r \quad (6)$$

$$\frac{v}{A} = \frac{1}{r} \quad (7)$$

$$v = \frac{du}{dr} = \frac{A}{r} \quad (8)$$

$$du = A\frac{dr}{r}, \quad (9)$$

so the solution is

$$u = A\ln r. \quad (10)$$

Ignoring the trivial additive and multiplicative constants, the general pure radial solution then becomes

$$u = \ln[(x-a)^2+(y-b)^2]^{1/2} = \tfrac{1}{2}\ln\left[(x-a)^2+(y-b)^2\right]. \quad (11)$$

Other solutions may be obtained by differentiation, such as

$$u = \frac{x-a}{(x-a)^2+(y-b)^2} \tag{12}$$

$$v = \frac{y-b}{(x-a)^2+(y-b)^2}, \tag{13}$$

$$u = e^x \sin y \tag{14}$$

$$v = e^x \cos y, \tag{15}$$

and

$$\tan^{-1}\left(\frac{y-b}{x-a}\right). \tag{16}$$

Harmonic functions containing azimuthal dependence include

$$u = r^n \cos(n\theta) \tag{17}$$

$$v = r^n \sin(n\theta). \tag{18}$$

The POISSON KERNEL

$$u(r,R,\theta,\phi) = \frac{R^2-r^2}{R^2-2rR\cos(\theta-\phi)+r^2} \tag{19}$$

is another harmonic function.

see also SCALAR POTENTIAL, VECTOR POTENTIAL

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Harmonic-Geometric Mean

Let

$$\alpha_{n+1} = \frac{2\alpha_n\beta_n}{\alpha_n+\beta_n}$$

$$\beta_{n+1} = \sqrt{\alpha_n\beta_n},$$

then

$$H(\alpha_0,\beta_0) \equiv \lim_{n\to\infty} a_n = \frac{1}{M({\alpha_0}^{-1},{\beta_0}^{-1})},$$

where M is the ARITHMETIC-GEOMETRIC MEAN.

see also ARITHMETIC MEAN, ARITHMETIC-GEOMETRIC MEAN, GEOMETRIC MEAN, HARMONIC MEAN

Harmonic Homology

A PERSPECTIVE COLLINEATION with center O and axis o not incident is called a HOMOLOGY. A HOMOLOGY is said to be harmonic if the points A and A' on a line through O are harmonic conjugates with respect to O and $o \cdot a$. Every PERSPECTIVE COLLINEATION of period two is a harmonic homology.

see also HOMOLOGY (GEOMETRY), PERSPECTIVE COLLINEATION

References

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Harmonic Logarithm

For all INTEGERS n and NONNEGATIVE INTEGERS t, the harmonic logarithms $\lambda_n^{(t)}(x)$ of order t and degree n are defined as the unique functions satisfying

1. $\lambda_0^{(t)}(x) = (\ln x)^t$,
2. $\lambda_n^{(t)}(x)$ has no constant term except $\lambda_0^{(0)}(x) = 1$,
3. $\frac{d}{dx}\lambda_n^{(t)}(x) = \lfloor n\rceil\, \lambda_{n-1}^{(t)}(x)$,

where the "ROMAN SYMBOL" $\lfloor n\rceil$ is defined by

$$\lfloor n\rceil \equiv \begin{cases} n & \text{for } n \neq 0 \\ 1 & \text{for } n = 0 \end{cases} \tag{1}$$

(Roman 1992). This gives the special cases

$$\lambda_n^{(0)}(x) = \begin{cases} x^n & \text{for } n \geq 0 \\ 0 & \text{for } n < 0 \end{cases} \tag{2}$$

$$\lambda_n^{(1)}(x) = \begin{cases} x^n(\ln x - H_n) & \text{for } n \geq 0 \\ x^n & \text{for } n < 0, \end{cases} \tag{3}$$

where H_n is a HARMONIC NUMBER

$$H_n \equiv \sum_{k=1}^n \frac{1}{k}. \tag{4}$$

The harmonic logarithm has the INTEGRAL

$$\int \lambda_n^{(1)}(x)\,dx = \frac{1}{\lfloor n+1\rceil}\lambda_{n+1}^{(1)}(x). \tag{5}$$

The harmonic logarithm can be written

$$\lambda_n^{(t)}(x) = \lfloor n\rceil!\tilde{D}^{-n}(\ln x)^t, \tag{6}$$

where $\tilde{D}$ is the DIFFERENTIAL OPERATOR, (so $\tilde{D}^{-n}$ is the nth INTEGRAL). Rearranging gives

$$\tilde{D}^k\lambda_n^{(t)}(x) = \left\lfloor \frac{\lfloor n\rceil!}{\lfloor n-k} \right\rceil!\lambda_{n-k}^{(t)}(x). \tag{7}$$

This formulation gives an analog of the BINOMIAL THEOREM called the LOGARITHMIC BINOMIAL FORMULA. Another expression for the harmonic logarithm is

$$\lambda_n^{(t)}(x) = x^n \sum_{j=0}^t (-1)^j (t)_j c_n^{(j)} (\ln x)^{t-j}, \tag{8}$$

where $(t)_j = t(t-1)\cdots(t-j+1)$ is a POCHHAMMER SYMBOL and $c_n^{(j)}$ is a two-index HARMONIC NUMBER (Roman 1992).

see also LOGARITHM, ROMAN FACTORIAL

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Harmonic Map

A harmonic map between RIEMANNIAN MANIFOLDS can be viewed as a generalization of a GEODESIC when the domain DIMENSION is one, or of a HARMONIC FUNCTION when the range is a EUCLIDEAN SPACE.

see also BOCHNER IDENTITY, EUCLIDEAN SPACE, GEODESIC, HARMONIC FUNCTION, RIEMANNIAN MANIFOLD

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Harmonic Mean

The harmonic mean $H(x_1, \ldots, x_n)$ of n points x_i (where $i = 1, \ldots, n$) is

$$\frac{1}{H} \equiv \frac{1}{n}\sum_{i=1}^{n}\frac{1}{x_i}. \tag{1}$$

The special case of $n = 2$ is therefore

$$\frac{1}{H} = \frac{1}{2}\left(\frac{1}{x_1}+\frac{1}{x_2},\right) \tag{2}$$

or

$$\frac{1}{H} = \frac{x_1+x_2}{2x_1x_2}. \tag{3}$$

The VOLUME-to-SURFACE AREA ratio for a cylindrical container with height h and radius r and the MEAN CURVATURE of a general surface are related to the harmonic mean.

Hoehn and Niven (1985) show that

$$H(a_1+c, a_2+c, \ldots, a_n+c) > c + H(a_1, a_2, \ldots, a_n) \tag{4}$$

for any POSITIVE constant c.

see also ARITHMETIC MEAN, ARITHMETIC-GEOMETRIC MEAN, GEOMETRIC MEAN, HARMONIC-GEOMETRIC MEAN, ROOT-MEAN-SQUARE

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Harmonic Mean Index

The statistical INDEX

$$P_H \equiv \frac{\sum p_0 q_0}{\sum \frac{p_0{}^2 q_0}{p_n}},$$

where p_n is the price per unit in period n, q_n is the quantity produced in period n, and $v_n \equiv p_n q_n$ the value of the n units.

see also INDEX

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Harmonic Number

A number of the form

$$H_n = \sum_{k=1}^{n}\frac{1}{k}. \tag{1}$$

This can be expressed analytically as

$$H_n = \gamma + \psi_0(n+1), \tag{2}$$

where γ is the EULER-MASCHERONI CONSTANT and $\Psi(x) = \psi_0(x)$ is the DIGAMMA FUNCTION. The number formed by taking alternate signs in the sum also has an analytic solution

$$H'_n = \sum_{k=1}^{n}\frac{(-1)^{k+1}}{k} \tag{3}$$

$$= \ln 2 + \tfrac{1}{2}(-1)^n[\psi_0(\tfrac{1}{2}n+\tfrac{1}{2}) - \psi_0(\tfrac{1}{2}n+1)]. \tag{4}$$

The first few harmonic numbers H_n are 1, 3/2, 11/6, 25/12, 137/60, ... (Sloane's A001008 and A002805). The HARMONIC NUMBER H_n is never an INTEGER (except for H_1), which can be proved by using the strong triangle inequality to show that the 2-ADIC VALUE of H_n is greater than 1 for $n > 1$. The harmonic numbers have ODD NUMERATORS and EVEN DENOMINATORS. The nth harmonic number is given asymptotically by

$$H_n \sim \ln n + \gamma + \frac{1}{2n}, \tag{5}$$

where γ is the EULER-MASCHERONI CONSTANT (Conway and Guy 1996). Gosper gave the interesting identity

$$\sum_{i=0}^{\infty}\frac{z^i H_i}{i!} = -e^z\sum_{k=1}^{\infty}\frac{(-z)^k}{kk!} = e^z[\ln z + \Gamma(0,z) + \gamma], \tag{6}$$

where $\Gamma(0,z)$ is the incomplete GAMMA FUNCTION and γ is the EULER-MASCHERONI CONSTANT. Borwein and Borwein (1995) show that

$$\sum_{n=1}^{\infty} \frac{{H_n}^2}{(n+1)^2} = \tfrac{11}{4}\zeta(4) = \tfrac{11}{360}\pi^4 \qquad (7)$$

$$\sum_{n=1}^{\infty} \frac{{H_n}^2}{n^2} = \tfrac{17}{4}\zeta(4) = \tfrac{17}{360}\pi^4 \qquad (8)$$

$$\sum_{n=1}^{\infty} \frac{H_n}{n^3} = \tfrac{5}{4}\zeta(4) = \tfrac{1}{72}\pi^4, \qquad (9)$$

where $\zeta(z)$ is the RIEMANN ZETA FUNCTION. The first of these had been previously derived by de Doelder (1991), and the last by Euler (1775). These identities are corollaries of the identity

$$\frac{1}{\pi}\int_0^{\pi} x^2\{\ln[2\cos(\tfrac{1}{2}x)]\}^2\,dx = \tfrac{11}{2}\zeta(4) = \tfrac{11}{180}\pi^4 \qquad (10)$$

(Borwein and Borwein 1995). Additional identities due to Euler are

$$\sum_{n=1}^{\infty} \frac{H_n}{n^2} = 2\zeta(3) \qquad (11)$$

$$2\sum_{n=1}^{\infty} \frac{H_n}{n^m} = (m+2)\zeta(m+1) - \sum_{n=1}^{m-2}\zeta(m-n)\zeta(n+1) \qquad (12)$$

for $m = 2, 3, \ldots$ (Borwein and Borwein 1995), where $\zeta(3)$ is APÉRY'S CONSTANT. These sums are related to so-called EULER SUMS.

Conway and Guy (1996) define the second harmonic number by

$$H_n^{(2)} \equiv \sum_{i=1}^{n} H_i = (n+1)(H_{n+1}-1) = (n+1)(H_{n+1}-H_1), \qquad (13)$$

the third harmonic number by

$$H_n^{(3)} \equiv \sum_{i=1}^{n} H_i^{(2)} = \binom{n+2}{2}(H_{n+2}-H_2), \qquad (14)$$

and the nth harmonic number by

$$H_n^{(k)} = \binom{n+k-1}{k-1}(H_{n+k-1}-H_{k-1}). \qquad (15)$$

A slightly different definition of a two-index harmonic number $c_n^{(j)}$ is given by Roman (1992) in connection with the HARMONIC LOGARITHM. Roman (1992) defines this by

$$c_n^{(0)} = \begin{cases} 1 & \text{for } n \geq 0 \\ 0 & \text{for } n < 0 \end{cases} \qquad (16)$$

$$c_0^{(j)} = \begin{cases} 1 & \text{for } j = 0 \\ 0 & \text{for } j \neq 0 \end{cases} \qquad (17)$$

plus the recurrence relation

$$cn_n^{(j)} = c_n^{(j-1)} + nc_{n-1}^{(j)}. \qquad (18)$$

For general $n > 0$ and $j > 0$, this is equivalent to

$$c_n^{(j)} = \sum_{i=1}^{n} \frac{1}{i} c_i^{(j-1)}, \qquad (19)$$

and for $n > 0$, it simplifies to

$$c_n^{(j)} = \sum_{i=1}^{n} \binom{n}{i}(-1)^{i-1}i^{-j}. \qquad (20)$$

For $n < 0$, the harmonic number can be written

$$c_n^{(j)} = (-1)^j \lfloor n\rceil!\, s(-n,j), \qquad (21)$$

where $\lfloor n\rceil!$ is the ROMAN FACTORIAL and s is a STIRLING NUMBER OF THE FIRST KIND.

A separate type of number sometimes also called a "harmonic number" is a HARMONIC DIVISOR NUMBER (or ORE NUMBER).

see also APÉRY'S CONSTANT, EULER SUM, HARMONIC LOGARITHM, HARMONIC SERIES, ORE NUMBER

References

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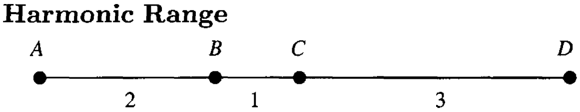
Harmonic Progression

see Harmonic Series

Harmonic Range



A set of four COLLINEAR points A, B, C, and D arranged such that

$$AB : BC = 2 : 1$$

$$AD : DC = 6 : 3.$$

Hardy (1967) uses the term HARMONIC SYSTEM OF POINTS to refer to a harmonic range.

see also EULER LINE, GERGONNE LINE, HARMONIC CONJUGATE POINTS, SODDY LINE

References

Hardy, G. H. *A Course of Pure Mathematics, 10th ed.* Cambridge, England: Cambridge University Press, pp. 99 and 106, 1967.

Harmonic Ratio

see HARMONIC CONJUGATE POINTS

Harmonic Segment

see HARMONIC CONJUGATE POINTS

Harmonic Series

The SUM

$$\sum_{k=1}^{\infty} \frac{1}{k} \tag{1}$$

is called the harmonic series. It can be shown to DIVERGE using the INTEGRAL TEST by comparison with the function $1/x$. The divergence, however, is very slow. In fact, the sum

$$\sum_{p} \frac{1}{p} \tag{2}$$

taken over all PRIMES also diverges. The generalization of the harmonic series

$$\zeta(n) \equiv \sum_{k=1}^{\infty} \frac{1}{k^n} \tag{3}$$

is known as the RIEMANN ZETA FUNCTION.

The sum of the first few terms of the harmonic series is given analytically by the nth HARMONIC NUMBER

$$H_n = \sum_{j=1}^{n} \frac{1}{j} = \gamma + \psi_0(n+1), \tag{4}$$

where γ is the EULER-MASCHERONI CONSTANT and $\Psi(x) = \psi_0(x)$ is the DIGAMMA FUNCTION. The number of terms needed to exceed 1, 2, 3, ... are 1, 4, 11, 31, 83, 227, 616, 1674, 4550, 12367, 33617, 91380, 248397, ... (Sloane's A004080). Using the analytic form shows that after 2.5×10^8 terms, the sum is still less than 20. Furthermore, to achieve a sum greater than 100, more than 1.509×10^{43} terms are needed!

Progressions of the form

$$\frac{1}{a_1}, \frac{1}{a_1+d}, \frac{1}{a_1+2d}, \ldots \tag{5}$$

are also sometimes called harmonic series (Beyer 1987).

The modified harmonic series, given by the sum

$$T = \sum_{k=1}^{\infty} \frac{1}{p_k}, \tag{6}$$

where p_k is the kth PRIME, diverges.

see also ARITHMETIC SERIES, BERNOULLI'S PARADOX, BOOK STACKING PROBLEM, EULER SUM, ZIPF'S LAW

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Harmonic System of Points

see HARMONIC RANGE

Harmonious Graph

A connected LABELLED GRAPH with n EDGES in which all VERTICES can be labeled with distinct INTEGERS (mod n) so that the sums of the PAIRS of numbers at the ends of each EDGE are also distinct (mod n). The LADDER GRAPH, FAN, WHEEL GRAPH, PETERSEN GRAPH, TETRAHEDRAL GRAPH, DODECAHEDRAL GRAPH, and ICOSAHEDRAL GRAPH are all harmonious (Graham and Sloane 1980).

see also GRACEFUL GRAPH, LABELLED GRAPH, POSTAGE STAMP PROBLEM, SEQUENTIAL GRAPH

References

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Guy, R. K. "The Corresponding Modular Covering Problem. Harmonious Labelling of Graphs." §C13 in *Unsolved Problems in Number Theory, 2nd ed.* New York: Springer-Verlag, pp. 127–128, 1994.

Harnack's Inequality

Let $D = D(z_0, R)$ be an OPEN DISK, and let u be a HARMONIC FUNCTION on D such that $u(z) \geq 0$ for all $z \in D$. Then for all $z \in D$, we have

$$0 \leq u(z) \leq \left(\frac{R}{R - |z - z_0|}\right)^2 u(z_0).$$

see also LIOUVILLE'S CONFORMALITY THEOREM

References

Flanigan, F. J. "Harnack's Inequality." §2.5.1 in *Complex Variables: Harmonic and Analytic Functions.* New York: Dover, pp. 88–90, 1983.

Harnack's Theorems

Harnack's first theorem states that a real irreducible curve of order n cannot have more than

$$\tfrac{1}{2}(n-1)(n-2) - \sum s_i(s_i - 1) + 1$$

circuits (Coolidge 1959, p. 57).

Harnack's second theorem states that there exists a curve of every order with the maximum number of circuits compatible with that order and with a certain number of double points, provided that number is not permissible for a curve of lower order (Coolidge 1959, p. 61).

References
Coolidge, J. L. *A Treatise on Algebraic Plane Curves.* New York: Dover, 1959.

Harshad Number

A POSITIVE INTEGER which is DIVISIBLE by the sum of its DIGITS, also called a NIVEN NUMBER (Kennedy *et al.* 1980). The first few are 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 12, 18, 20, 21, 24, ... (Sloane's A005349). Grundman (1994) proved that there is no sequence of more than 20 consecutive Harshad numbers, and found the smallest sequence of 20 consecutive Harshad numbers, each member of which has 44,363,342,786 digits.

Grundman (1994) defined an n-Harshad (or n-Niven) number to be a POSITIVE INTEGER which is DIVISIBLE by the sum of its digits in base $n \geq 2$. Cai (1996) showed that for $n = 2$ or 3, there exists an infinite family of sequences of consecutive n-Harshad numbers of length $2n$.

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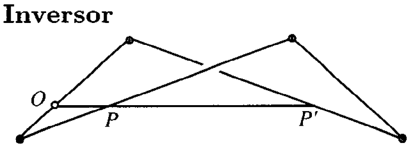
Hart's Inversor



A linkage which draws the inverse of a given curve. It can also convert circular to linear motion. The rods satisfy $AB = CD$ and $BC = DA$, and O, P, and P' remain COLLINEAR. Coxeter (1969, p. 428) shows that if $AO = \mu AB$, then

$$OP \times OP' = \mu(1-\mu)(AD^2 - AB^2).$$

see also PEAUCELLIER INVERSOR

References
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Rademacher, H. and Toeplitz, O. *The Enjoyment of Mathematics: Selections from Mathematics for the Amateur.* Princeton, NJ: Princeton University Press, pp. 124–129, 1957.

Hart's Theorem

Any one of the eight APOLLONIUS CIRCLES of three given CIRCLES is TANGENT to a CIRCLE C, as are the other three APOLLONIUS CIRCLES having (1) like contact with two of the given CIRCLES and (2) unlike contact with the third.

see also APOLLONIUS CIRCLES

References
Johnson, R. A. *Modern Geometry: An Elementary Treatise on the Geometry of the Triangle and the Circle.* Boston, MA: Houghton Mifflin, pp. 127–128, 1929.

Hartley Transform

An INTEGRAL TRANSFORM which shares some features with the FOURIER TRANSFORM, but which (in the discrete case), multiplies the KERNEL by

$$\cos\left(\frac{2\pi kn}{N}\right) - \sin\left(\frac{2\pi kn}{N}\right) \tag{1}$$

instead of

$$e^{-2\pi ikn/N} = \cos\left(\frac{2\pi kn}{N}\right) - i\sin\left(\frac{2\pi kn}{N}\right). \tag{2}$$

The Hartley transform produces REAL output for a REAL input, and is its own inverse. It therefore can have computational advantages over the DISCRETE FOURIER TRANSFORM, although analytic expressions are usually more complicated for the Hartley transform.

The discrete version of the Hartley transform can be written explicitly as

$$\mathcal{H}[a] \equiv \frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} a_n \left[\cos\left(\frac{2\pi kn}{N}\right) - \sin\left(\frac{2\pi kn}{N}\right)\right] \tag{3}$$

$$= \Re\mathcal{F}[a] - \Im\mathcal{F}[a], \tag{4}$$

where $\mathcal{F}$ denotes the FOURIER TRANSFORM. The Hartley transform obeys the CONVOLUTION property

$$\mathcal{H}[a*b]_k = \tfrac{1}{2}(A_kB_k - \bar{A}_k\bar{B}_k + A_k\bar{B}_k + \bar{A}_kB_k), \tag{5}$$

where

$$\bar{a}_0 \equiv a_0 \quad (6)$$

$$\bar{a}_{n/2} \equiv a_{n/2} \quad (7)$$

$$\bar{a}_k \equiv a_{n-k} \quad (8)$$

(Arndt). Like the FAST FOURIER TRANSFORM, there is a "fast" version of the Hartley transform. A decimation in time algorithm makes use of

$$\mathcal{H}_n^{\text{left}}[a] = \mathcal{H}_{n/2}[a^{\text{even}}] + \mathcal{X}\mathcal{H}_{n/2}[a^{\text{odd}}] \quad (9)$$

$$\mathcal{H}_n^{\text{right}}[a] = \mathcal{H}_{n/2}[a^{\text{even}}] - \mathcal{X}\mathcal{H}_{n/2}[a^{\text{odd}}], \quad (10)$$

where $\mathcal{X}$ denotes the sequence with elements

$$a_n \cos\left(\frac{\pi n}{N}\right) - \bar{a}_n \sin\left(\frac{\pi n}{N}\right). \quad (11)$$

A decimation in frequency algorithm makes use of

$$\mathcal{H}_n^{\text{even}}[a] = \mathcal{H}_{n/2}[a^{\text{left}} + a^{\text{right}}], \quad (12)$$

$$\mathcal{H}_n^{\text{odd}}[a] = \mathcal{H}_{n/2}[\mathcal{X}(a^{\text{left}} - a^{\text{right}})]. \quad (13)$$

The DISCRETE FOURIER TRANSFORM

$$A_k \equiv \mathcal{F}[a] = \sum_{n=0}^{N-1} e^{-2\pi ikn/N} a_n \quad (14)$$

can be written

$$\begin{bmatrix} A_k \\ A_{-k} \end{bmatrix} = \sum_{n=0}^{N-1} \underbrace{\begin{bmatrix} e^{-2\pi ikn/N} & 0 \\ 0 & e^{2\pi ikn/N} \end{bmatrix}}_{\mathsf{F}} \begin{bmatrix} a_n \\ a_n \end{bmatrix} \quad (15)$$

$$= \sum_{n=0}^{N-1} \underbrace{\frac{1}{2}\begin{bmatrix} 1-i & 1+i \\ 1+i & 1-i \end{bmatrix}}_{\mathsf{T}^{-1}} \underbrace{\begin{bmatrix} \cos\left(\frac{2\pi kn}{N}\right) & \sin\left(\frac{2\pi kn}{N}\right) \\ -\sin\left(\frac{2\pi kn}{N}\right) & \cos\left(\frac{2\pi kn}{N}\right) \end{bmatrix}}_{\mathsf{H}}$$

$$\times \underbrace{\frac{1}{2}\begin{bmatrix} 1+i & 1-i \\ 1-i & 1+i \end{bmatrix}}_{\mathsf{T}} \begin{bmatrix} a_n \\ a_n \end{bmatrix}, \quad (16)$$

so

$$\mathsf{F} = \mathsf{T}^{-1}\mathsf{H}\mathsf{T}. \quad (17)$$

see also DISCRETE FOURIER TRANSFORM, FAST FOURIER TRANSFORM, FOURIER TRANSFORM

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HashLife

A LIFE ALGORITHM that achieves remarkable speed by storing subpatterns in a hash table, and using them to skip forward, sometimes thousands of generations at a time. HashLife takes tremendous amounts of memory and can't show patterns at every step, but can quickly calculate the outcome of a pattern that takes millions of generations to complete.

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Hasse's Algorithm

see COLLATZ PROBLEM

Hasse's Conjecture

Define the ZETA FUNCTION of a VARIETY over a NUMBER FIELD by taking the product over all PRIME IDEALS of the ZETA FUNCTIONS of this VARIETY reduced modulo the PRIMES. Hasse conjectured that this product has a MEROMORPHIC continuation over the whole plane and a functional equation.

References
Lang, S. "Some History of the Shimura-Taniyama Conjecture." *Not. Amer. Math. Soc.* **42**, 1301–1307, 1995.

Hasse-Davenport Relation

Let F be a FINITE FIELD with q elements, and let F_s be a FIELD containing F such that $[F_s : F] = s$. Let χ be a nontrivial MULTIPLICATIVE CHARACTER of F and $\chi' = \chi \circ N_{F_s/F}$ a character of F_s. Then

$$(-g(\chi))^s = -g(\chi'),$$

where $g(x)$ is a GAUSSIAN SUM.

see also GAUSSIAN SUM, MULTIPLICATIVE CHARACTER

References
Ireland, K. and Rosen, M. "A Proof of the Hasse-Davenport Relation." §11.4 in *A Classical Introduction to Modern Number Theory, 2nd ed.* New York: Springer-Verlag, pp. 162–165, 1990.

Hasse Diagram

A graphical rendering of a PARTIALLY ORDERED SET displayed via the COVER relation of the PARTIALLY ORDERED SET with an implied upward orientation. A point is drawn for each element of the POSET, and line segments are drawn between these points according to the following two rules:

1. If $x < y$ in the poset, then the point corresponding to x appears lower in the drawing than the point corresponding to y.
2. The line segment between the points corresponding to any two elements x and y of the poset is included in the drawing IFF x covers y or y covers x.

Hasse diagrams are also called UPWARD DRAWINGS.

Hasse-Minkowski Theorem

Two nonsingular forms are equivalent over the rationals IFF they have the same DETERMINANT and the same p-SIGNATURES for all p.

Hasse Principle

A collection of equations satisfies the Hasse principle if, whenever one of the equations has solutions in $\mathbb{R}$ and all the $\mathbb{Q}_p$, then the equations have solutions in the RATIONALS $\mathbb{Q}$. Examples include the set of equations

$$ax^2 + bxy + cy^2 = 0$$

with a, b, and c INTEGERS, and the set of equations

$$x^2 + y^2 = a$$

for a rational. The trivial solution $x = y = 0$ is usually not taken into account when deciding if a collection of homogeneous equations satisfies the Hasse principle. The Hasse principle is sometimes called the LOCAL-GLOBAL PRINCIPLE.

see also LOCAL FIELD

Hasse's Resolution Modulus Theorem

The JACOBI SYMBOL $(a/y) = \chi(y)$ as a CHARACTER can be extended to the KRONECKER SYMBOL $(f(a)/y) = \chi^*(y)$ so that $\chi^*(y) = \chi(y)$ whenever $\chi(y) \neq 0$. When y is RELATIVELY PRIME to $f(a)$, then $\chi^*(y) \neq 0$, and for NONZERO values $\chi^*(y_1) = \chi^*(y_2)$ IFF $y_1 \equiv y_2 \bmod^+ f(a)$. In addition, $|f(a)|$ is the minimum value for which the latter congruence property holds in any extension symbol for $\chi(y)$.

see also CHARACTER (NUMBER THEORY), JACOBI SYMBOL, KRONECKER SYMBOL

References

Cohn, H. *Advanced Number Theory.* New York: Dover, pp. 35–36, 1980.

Hat

The hat is a caret-shaped symbol most commonly used to denote a UNIT VECTOR ($\hat{\mathbf{v}}$) or an ESTIMATOR ($\hat{x}$).

see also ESTIMATOR, UNIT VECTOR

Haupt-Exponent

The smallest exponent e for which $b^e \equiv 1 \pmod{p}$, where b and p are given numbers, is the haupt-exponent of $b \pmod{p}$. The number of bases having a haupt-exponent e is $\phi(e)$, where $\phi(e)$ is the TOTIENT FUNCTION. Cunningham (1922) published the haupt-exponents for primes to 25409 and bases 2, 3, 5, 6, 7, 10, 11, and 12.

see also COMPLETE RESIDUE SYSTEM, RESIDUE INDEX

References

Cunningham, A. *Haupt-Exponents, Residue Indices, Primitive Roots.* London: F. Hodgson, 1922.

Hausdorff Axioms

Describe subsets of elements x in a NEIGHBORHOOD SET E of x. The NEIGHBORHOOD is assumed to satisfy:

1. There corresponds to each point x at least one NEIGHBORHOOD $U(x)$, and each NEIGHBORHOOD $U(x)$ contains the point x.
2. If $U(x)$ and $V(x)$ are two NEIGHBORHOODS of the same point x, there must exist a NEIGHBORHOOD $W(x)$ that is a subset of both.
3. If the point y lies in $U(x)$, there must exist a NEIGHBORHOOD $U(y)$ that is a SUBSET of $U(x)$.
4. For two different points x and y, there are two corresponding NEIGHBORHOODS $U(x)$ and $U(y)$ with no points in common.

Hausdorff-Besicovitch Dimension

see CAPACITY DIMENSION

Hausdorff Dimension

Let A be a SUBSET of a METRIC SPACE X. Then the Hausdorff dimension $D(A)$ of A is the INFIMUM of $d \geq 0$ such that the d-dimensional HAUSDORFF MEASURE of A is 0. Note that this need not be an INTEGER.

In many cases, the Hausdorff dimension correctly describes the correction term for a resonator with FRACTAL PERIMETER in Lorentz's conjecture. However, in general, the proper dimension to use turns out to be the MINKOWSKI-BOULIGAND DIMENSION (Schroeder 1991).

see also CAPACITY DIMENSION, FRACTAL DIMENSION, MINKOWSKI-BOULIGAND DIMENSION

References

Federer, H. *Geometric Measure Theory.* New York: Springer-Verlag, 1969.

Hausdorff, F. "Dimension und äußeres Maß." *Math. Ann.* **79**, 157–179, 1919.

Ott, E. "Appendix: Hausdorff Dimension." *Chaos in Dynamical Systems.* New York: Cambridge University Press, pp. 100–103, 1993.

Schroeder, M. *Fractals, Chaos, Power Laws: Minutes from an Infinite Paradise.* New York: W. H. Freeman, pp. 41–45, 1991.

Hausdorff Measure

Let X be a METRIC SPACE, A be a SUBSET of X, and d a number ≥ 0. The d-dimensional Hausdorff measure of A, $H^d(A)$, is the INFIMUM of POSITIVE numbers y such that for every $r > 0$, A can be covered by a countable family of closed sets, each of diameter less than r, such that the sum of the dth POWERS of their diameters is less than y. Note that $H^d(A)$ may be infinite, and d need not be an INTEGER.

References

Federer, H. *Geometric Measure Theory.* New York: Springer-Verlag, 1969.

Ott, E. *Chaos in Dynamical Systems.* Cambridge, England: Cambridge University Press, p. 103, 1993.

Hausdorff Paradox

For $n \geq 3$, there exist no additive finite and invariant measures for the group of displacements in $\mathbb{R}^n$.

References

Le Lionnais, F. *Les nombres remarquables.* Paris: Hermann, p. 49, 1983.

Hausdorff Space

A TOPOLOGICAL SPACE in which any two points have disjoint NEIGHBORHOODS.

Haversine

$$\text{hav}(z) \equiv \tfrac{1}{2}\,\text{vers}(z) = \tfrac{1}{2}(1-\cos z),$$

where $\text{vers}(z)$ is the VERSINE and cos is the COSINE. Using a trigonometric identity, the haversine is equal to

$$\text{hav}(z) = \sin^2(\tfrac{1}{2}z).$$

see also COSINE, COVERSINE, EXSECANT, VERSINE

References

Abramowitz, M. and Stegun, C. A. (Eds.). *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables, 9th printing.* New York: Dover, p. 78, 1972.

Heads Minus Tails Distribution

A fair COIN is tossed $2n$ times. Let $D \equiv |H - T|$ be the absolute difference in the number of heads and tails obtained. Then the probability distribution is given by

$$P(D = 2k) = \begin{cases} (\frac{1}{2})^{2n}\binom{2n}{n} & k = 0 \\ 2(\frac{1}{2})^{2n}\binom{2n}{n+k} & k = 1,\, 2,\, \ldots, \end{cases}$$

where $P(D = 2k - 1) = 0$. The most probable value of D is $D = 2$, and the expectation value is

$$\langle D\rangle = \frac{n\binom{2n}{n}}{2^{2n-1}}.$$

see also BERNOULLI DISTRIBUTION, COIN, COIN TOSSING

References

Handelsman, M. B. Solution to Problem 436, "Distributing 'Heads' Minus 'Tails.'" *College Math. J.* **22**, 444–446, 1991.

Heap

A SET of N members forms a heap if it satisfies $a_{\lfloor j/2\rfloor} \geq a_j$ for $1 \leq \lfloor j/2\rfloor < j \leq N$, where $\lfloor x\rfloor$ is the FLOOR FUNCTION.

see also HEAPSORT

Heapsort

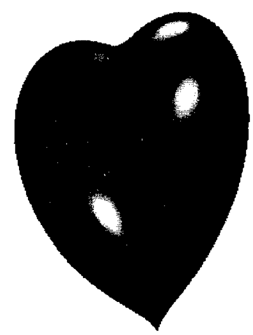
An $N \lg N$ SORTING ALGORITHM which is not quite as fast as QUICKSORT. It is a "sort-in-place" algorithm and requires no auxiliary storage, which makes it particularly concise and elegant to implement.

see also QUICKSORT, SORTING

References

Press, W. H.; Flannery, B. P.; Teukolsky, S. A.; and Vetterling, W. T. "Heapsort." §8.3 in *Numerical Recipes in FORTRAN: The Art of Scientific Computing, 2nd ed.* Cambridge, England: Cambridge University Press, pp. 327–329, 1992.

Heart Surface



A heart-shaped surface given by the SEXTIC EQUATION

$$(2x^2 + 2y^2 + z^2 - 1)^3 - \tfrac{1}{10}x^2z^3 - y^2z^3 = 0.$$

see also BONNE PROJECTION, PIRIFORM

References

Nordstrand, T. "Heart." `http://www.uib.no/people/nfytn/hearttxt.htm`.

Heat Conduction Equation

A diffusion equation of the form

$$\frac{\partial T}{\partial t} = \kappa\nabla^2 T. \tag{1}$$

Physically, the equation commonly arises in situations where κ is the thermal diffusivity and T the temperature.

The 1-D heat conduction equation is

$$\frac{\partial T}{\partial t} = \kappa\frac{\partial^2 T}{\partial x^2}. \tag{2}$$

This can be solved by SEPARATION OF VARIABLES using

$$T(x,t) = X(x)T(t). \tag{3}$$

Then

$$X\frac{dT}{dt} = \kappa T\frac{d^2X}{dx^2}. \tag{4}$$

Dividing both sides by κXT gives

$$\frac{1}{\kappa T}\frac{dT}{dt} = \frac{1}{X}\frac{d^2X}{dx^2} = -\frac{1}{\lambda^2}, \tag{5}$$

where each side must be equal to a constant. Anticipating the exponential solution in T, we have picked a negative separation constant so that the solution remains finite at all times and λ has units of length. The T solution is

$$T(t) = Ae^{-\kappa t/\lambda^2}, \tag{6}$$

and the X solution is

$$X(x) = C\cos\left(\frac{x}{\lambda}\right) + D\sin\left(\frac{x}{\lambda}\right). \tag{7}$$

The general solution is then

$$\begin{aligned} T(x,t) &= T(t)X(x) \\ &= Ae^{-\kappa t/\lambda^2}\left[C\cos\left(\frac{x}{\lambda}\right) + D\sin\left(\frac{x}{\lambda}\right)\right] \\ &= e^{-\kappa t/\lambda^2}\left[D\cos\left(\frac{x}{\lambda}\right) + E\sin\left(\frac{x}{\lambda}\right)\right]. \end{aligned} \tag{8}$$

If we are given the boundary conditions

$$T(0,t) = 0 \tag{9}$$

and

$$T(L,t) = 0, \tag{10}$$

then applying (9) to (8) gives

$$D\cos\left(\frac{x}{\lambda}\right) = 0 \Rightarrow D = 0, \tag{11}$$

and applying (10) to (8) gives

$$E\sin\left(\frac{L}{\lambda}\right) = 0 \Rightarrow \frac{L}{\lambda} = n\pi \Rightarrow \lambda = \frac{L}{n\pi}, \tag{12}$$

so (8) becomes

$$T_n(x,t) = E_n e^{-\kappa(n\pi/L)^2 t}\sin\left(\frac{n\pi x}{L}\right). \tag{13}$$

Since the general solution can have any n,

$$T(x,t) = \sum_{n=1}^{\infty} c_n \sin\left(\frac{n\pi x}{L}\right) e^{-\kappa(n\pi/L)^2 t}. \tag{14}$$

Now, if we are given an initial condition $T(x,0)$, we have

$$T(x,0) = \sum_{n=1}^{\infty} c_n \sin\left(\frac{n\pi x}{L}\right). \tag{15}$$

Multiplying both sides by $\sin(m\pi x/L)$ and integrating from 0 to L gives

$$\int_0^L \sin\left(\frac{m\pi x}{L}\right) T(x,0)\,dx = \int_0^L \sum_{n=1}^{\infty} c_n \sin\left(\frac{m\pi x}{L}\right)\sin\left(\frac{n\pi x}{L}\right)dx. \tag{16}$$

Using the ORTHOGONALITY of $\sin(nx)$ and $\sin(mx)$,

$$\begin{aligned} \sum_{n=1}^{\infty} c_n \int_0^L \sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{m\pi x}{L}\right)dx &= \sum_{n=1}^{\infty} \tfrac{1}{2}\pi\delta_{mn}c_n \\ &= \tfrac{1}{2}\pi c_m = \int_0^L \sin\left(\frac{m\pi x}{L}\right) T(x,0)\,dx, \end{aligned} \tag{17}$$

so

$$c_n = \frac{2}{\pi}\int_0^L \sin\left(\frac{m\pi x}{L}\right) T(x,0)\,dx. \tag{18}$$

If the boundary conditions are replaced by the requirement that the derivative of the temperature be zero at the edges, then (9) and (10) are replaced by

$$\left.\frac{\partial T}{\partial x}\right|_{(0,t)} = 0 \tag{19}$$

$$\left.\frac{\partial T}{\partial x}\right|_{(L,t)} = 0. \tag{20}$$

Following the same procedure as before, a similar answer is found, but with sine replaced by cosine:

$$T(x,t) = \sum_{n=1}^{\infty} c_n \cos\left(\frac{n\pi x}{L}\right) e^{-\kappa(n\pi/L)^2 t}, \tag{21}$$

where

$$c_n = \frac{2}{\pi}\int_0^L \cos\left(\frac{m\pi x}{L}\right) \left.\frac{\partial T(x,0)}{\partial x}\right|_{t=0} dx. \tag{22}$$

Heat Conduction Equation—Disk

To solve the HEAT CONDUCTION EQUATION on a 2-D disk of radius $R = 1$, try to separate the equation using

$$T(r,\theta,t) = R(r)\Theta(\theta)T(t). \tag{1}$$

Writing the θ and r terms of the LAPLACIAN in SPHERICAL COORDINATES gives

$$\nabla^2 = \frac{d^2R}{dr^2} + \frac{2}{r}\frac{dR}{dr} + \frac{1}{r^2}\frac{d^2\Theta}{d\theta^2}, \tag{2}$$

so the HEAT CONDUCTION EQUATION becomes

$$\frac{R\Theta}{\kappa}\frac{d^2T}{dt^2} = \frac{d^2R}{dr^2}\Theta T + \frac{2}{r}\frac{dR}{dr}\Theta T + \frac{1}{r^2}\frac{d^2\Theta}{d\theta^2}RT. \tag{3}$$

Multiplying through by $r^2/R\Theta T$ gives

$$\frac{r^2}{\kappa T}\frac{d^2T}{dt^2} = \frac{r^2}{R}\frac{d^2R}{dr^2} + \frac{2r}{R}\frac{dR}{dr} + \frac{d^2\Theta}{d\theta^2}\frac{1}{\Theta}. \tag{4}$$

The θ term can be separated.

$$\frac{d^2\Theta}{d\theta^2}\frac{1}{\Theta} = -n(n+1), \tag{5}$$

which has a solution

$$\Theta(\theta) = A\cos\left[\sqrt{n(n+1)}\,\theta\right] + B\sin\left[\sqrt{n(n+1)}\,\theta\right]. \tag{6}$$

The remaining portion becomes

$$\frac{r^2}{\kappa T}\frac{d^2T}{dt^2} = \frac{r^2}{R}\frac{d^2R}{dr^2} + \frac{2r}{R}\frac{dR}{dr} - n(n+1). \tag{7}$$

Dividing by r^2 gives

$$\frac{1}{\kappa T}\frac{d^2T}{dt^2} = \frac{1}{R}\frac{d^2R}{dr^2} + \frac{2}{rR}\frac{dR}{dr} - \frac{n(n+1)}{r^2} = -\frac{1}{\lambda^2}, \tag{8}$$

where a NEGATIVE separation constant has been chosen so that the t portion remains finite

$$T(t) = Ce^{-\kappa t/\lambda^2}. \tag{9}$$

The radial portion then becomes

$$\frac{1}{R}\frac{d^2R}{dr^2} + \frac{2}{rR}\frac{dR}{dr} - \frac{n(n+1)}{r^2} + \frac{1}{\lambda^2} = 0 \tag{10}$$

$$r^2\frac{d^2R}{dr^2} + 2r\frac{dR}{dr} + \left[\frac{r^2}{\lambda^2} - n(n+1)\right]R = 0, \tag{11}$$

which is the SPHERICAL BESSEL DIFFERENTIAL EQUATION. If the initial temperature is $T(r,0) = 0$ and the boundary condition is $T(1,t) = 1$, the solution is

$$T(r,t) = 1 - 2\sum_{n=1}^{\infty}\frac{J_0(\alpha_n r)}{\alpha_n J_1(\alpha_n)}e^{{\alpha_n}^2 t}, \tag{12}$$

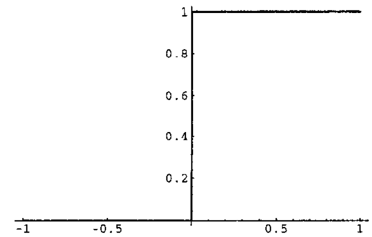
where α_n is the nth POSITIVE zero of the BESSEL FUNCTION OF THE FIRST KIND J_0.

Heaviside Calculus

A method of solving differential equations using FOURIER TRANSFORMS and LAPLACE TRANSFORMS.

see also FOURIER TRANSFORM, LAPLACE TRANSFORM

Heaviside Step Function



A discontinuous "step" function, also called the UNIT STEP, and defined by

$$H(x) = \begin{cases} 0 & x < 0 \\ \frac{1}{2} & x = 0 \\ 1 & x > 0. \end{cases} \tag{1}$$

It is related to the BOXCAR FUNCTION. The DERIVATIVE is given by

$$\frac{d}{dx}H(x) = \delta(x), \tag{2}$$

where $\delta(x)$ is the DELTA FUNCTION, and the step function is related to the RAMP FUNCTION $R(x)$ by

$$\frac{d}{dx}R(x) = -H(x). \tag{3}$$

Bracewell (1965) gives many identities, some of which include the following. Letting $*$ denote the CONVOLUTION,

$$H(x) * f(x) = \int_{-\infty}^{x} f(x')\,dx' \tag{4}$$

$$\begin{aligned} H(T) * H(T) &= \int_{-\infty}^{\infty} H(u)H(T-u)\,du \qquad (5) \\ &= H(0)\int_0^{\infty} H(T-u)\,du \\ &= H(0)H(T)\int_0^{T} du = TH(T). \qquad (6) \end{aligned}$$

Additional identities are

$$H(x)H(y) = \begin{cases} H(x) & x > y \\ H(y) & x < y \end{cases} \tag{7}$$

$$\begin{aligned} H(ax+b) &= H\left(x + \frac{b}{a}\right)H(a) + H\left(-x - \frac{b}{a}\right)H(-a) \\ &= \begin{cases} H\left(x + \frac{b}{a}\right) & a > 0 \\ H\left(-x - \frac{b}{a}\right) & a < 0. \end{cases} \qquad (8) \end{aligned}$$

The step function obeys the integral identities

$$\int_{-a}^{b} H(u-u_0)f(u)\,du = H(u_0)\int_{u_0}^{b} f(u)\,du \tag{9}$$

$$\int_{-a}^{b} H(u_1-u)f(u)\,du = H(u_1)\int_{-a}^{u_1} f(u)\,du \tag{10}$$

$$\int_{-a}^{b} H(u-u_0)H(u_1-u)f(u)\,du = H(u_0)H(u_1)\int_{u_0}^{u_1} f(u)\,du. \quad (11)$$

The Heaviside step function can be defined by the following limits,

$$\begin{aligned} H(x) &= \lim_{t\to 0}\left[\tfrac{1}{2}+\frac{1}{\pi}\tan^{-1}\left(-\frac{x}{t}\right)\right] && (12)\\ &= \tfrac{1}{2}\lim_{t\to 0}\operatorname{erfc}(-\tfrac{x}{t}) && (13)\\ &= \frac{1}{\sqrt{\pi}}\lim_{t\to 0}\int_{-s}^{\infty} t^{-1}e^{-u^2/t}\,du && (14)\\ &= \tfrac{1}{2}+\tfrac{1}{\pi}\lim_{t\to 0}\operatorname{si}(\tfrac{\pi x}{t}) && (15)\\ &= \lim_{t\to 0}\int_{-\infty}^{x} t^{-1}\operatorname{sinc}\left(\frac{u}{t}\right)du && (16)\\ &= \lim_{t\to 0}\begin{cases}\tfrac{1}{2}(1-e^{-x/t}) & x>0\\ -\tfrac{1}{2}(1-e^{-x/t}) & x<0\end{cases} && (17)\\ &= \lim_{t\to 0}\int_{-\infty}^{x} t^{-1}\Lambda\left(\frac{x-\frac{1}{2}t}{t}\right)dx, && (18)\end{aligned}$$

where Λ is the one-argument TRIANGLE FUNCTION and $\operatorname{si}(x)$ is the SINE INTEGRAL.

The FOURIER TRANSFORM of the Heaviside step function is given by

$$\mathcal{F}[H(x)] = \int_{-\infty}^{\infty} e^{-2\pi ikx}H(x)\,dx = \frac{1}{2}\left[\delta(k)-\frac{i}{\pi k}\right], \quad (19)$$

where $\delta(k)$ is the DELTA FUNCTION.

see also BOXCAR FUNCTION, DELTA FUNCTION, FOURIER TRANSFORM—HEAVISIDE STEP FUNCTION, RAMP FUNCTION, RAMP FUNCTION, RECTANGLE FUNCTION, SQUARE WAVE

References

Bracewell, R. *The Fourier Transform and Its Applications.* New York: McGraw-Hill, 1965.

Spanier, J. and Oldham, K. B. "The Unit-Step $u(x-a)$ and Related Functions." Ch. 8 in *An Atlas of Functions.* Washington, DC: Hemisphere, pp. 63–69, 1987.

Heawood Conjecture

The bound for the number of colors which are SUFFICIENT for MAP COLORING on a surface of GENUS g,

$$\chi(g) = \left\lfloor \tfrac{1}{2}(7+\sqrt{48g+1})\right\rfloor$$

is the best possible, where $\lfloor x\rfloor$ is the FLOOR FUNCTION. $\chi(g)$ is called the CHROMATIC NUMBER, and the first few values for $g = 0, 1, \ldots$ are 4, 7, 8, 9, 10, 11, 12, 12, 13, 13, 14, ... (Sloane's A000934).

The fact that $\chi(g)$ is also NECESSARY was proved by Ringel and Youngs (1968) with two exceptions: the SPHERE (PLANE), and the KLEIN BOTTLE (for which the Heawood FORMULA gives seven, but the correct bound is six). When the FOUR-COLOR THEOREM was proved in 1976, the KLEIN BOTTLE was left as the only exception. The four most difficult cases to prove were $g = 59$, 83, 158, and 257.

see also CHROMATIC NUMBER, FOUR-COLOR THEOREM, MAP COLORING, SIX-COLOR THEOREM, TORUS COLORING

References

Ringel, G. *Map Color Theorem.* New York: Springer-Verlag, 1974.

Ringel, G. and Youngs, J. W. T. "Solution of the Heawood Map-Coloring Problem." *Proc. Nat. Acad. Sci. USA* **60**, 438–445, 1968.

Sloane, N. J. A. Sequence A000934/M3292 in "An On-Line Version of the Encyclopedia of Integer Sequences."

Wagon, S. "Map Coloring on a Torus." §7.5 in *Mathematica in Action.* New York: W. H. Freeman, pp. 232–237, 1991.

Hebesphenomegacorona

see JOHNSON SOLID

Hecke Algebra

An associative RING, also called a HECKE RING, which has a technical definition in terms of commensurable SUBGROUPS.

Hecke L-Function

A generalization of the EULER L-FUNCTION associated with a GROSSENCHARACTER.

References

Knapp, A. W. "Group Representations and Harmonic Analysis, Part II." *Not. Amer. Math. Soc.* **43**, 537–549, 1996.

Hecke Operator

A family of operators on each SPACE of MODULAR FORMS. Hecke operators COMMUTE with each other.

Hecke Ring

see HECKE ALGEBRA

Hectogon

A 100-sided POLYGON.

Hedgehog

An envelope parameterized by its GAUSS MAP. The parametric equations for a hedgehog are

$$\begin{aligned} x &= p(\theta)\cos\theta + p'(\theta)\sin\theta\\ y &= p(\theta)\sin\theta + p'(\theta)\cos\theta. \end{aligned}$$

A plane convex hedgehog has at least four VERTICES where the CURVATURE has a stationary value. A plane convex hedgehog of constant width has at least six VERTICES (Martinez-Maure 1996).

References
Langevin, R.; Levitt, G.; and Rosenberg, H. "Hérissons et Multihérissons (Enveloppes paramétrées par leu application de Gauss." Warsaw: Singularities, 245–253, 1985. Banach Center Pub. 20, PWN Warsaw, 1988.
Martinez-Maure, Y. "A Note on the Tennis Ball Theorem." *Amer. Math. Monthly* **103**, 338–340, 1996.

Heegaard Diagram

A diagram expressing how the gluing operation that connects the HANDLEBODIES involved in a HEEGAARD SPLITTING proceeds, usually by showing how the meridians of the HANDLEBODY are mapped.

see also HANDLEBODY, HEEGAARD SPLITTING

References
Rolfsen, D. *Knots and Links.* Wilmington, DE: Publish or Perish Press, p. 239, 1976.

Heegaard Splitting

A Heegaard splitting of a connected orientable 3-MANIFOLD M is any way of expressing M as the UNION of two (3,1)-HANDLEBODIES along their boundaries. The boundary of such a (3,1)-HANDLEBODY is an orientable SURFACE of some GENUS, which determines the number of HANDLES in the (3,1)-HANDLEBODIES. Therefore, the HANDLEBODIES involved in a Heegaard splitting are the same, but they may be glued together in a strange way along their boundary. A diagram showing how the gluing is done is known as a HEEGAARD DIAGRAM.

References
Adams, C. C. *The Knot Book: An Elementary Introduction to the Mathematical Theory of Knots.* New York: W. H. Freeman, p. 255, 1994.

Heegner Number

The values of $-d$ for which QUADRATIC FIELDS $\mathbb{Q}(\sqrt{-d})$ are uniquely factorable into factors of the form $a + b\sqrt{-d}$. Here, a and b are half-integers, except for $d = 1$ and 2, in which case they are INTEGERS. The Heegner numbers therefore correspond to DISCRIMINANTS $-d$ which have CLASS NUMBER $h(-d)$ equal to 1, except for Heegner numbers -1 and -2, which correspond to $d = -4$ and -8, respectively.

The determination of these numbers is called GAUSS'S CLASS NUMBER PROBLEM, and it is now known that there are only nine Heegner numbers: -1, -2, -3, -7, -11, -19, -43, -67, and -163 (Sloane's A003173), corresponding to discriminants -4, -8, -3, -7, -11, -19, -43, -67, and -163, respectively.

Heilbronn and Linfoot (1934) showed that if a larger d existed, it must be $> 10^9$. Heegner (1952) published a proof that only nine such numbers exist, but his proof was not accepted as complete at the time. Subsequent examination of Heegner's proof show it to be "essentially" correct (Conway and Guy 1996).

The Heegner numbers have a number of fascinating connections with amazing results in PRIME NUMBER theory. In particular, the j-FUNCTION provides stunning connections between e, π, and the ALGEBRAIC INTEGERS. They also explain why Euler's PRIME-GENERATING POLYNOMIAL $n^2 - n + 41$ is so surprisingly good at producing PRIMES.

see also CLASS NUMBER, DISCRIMINANT (BINARY QUADRATIC FORM), GAUSS'S CLASS NUMBER PROBLEM, j-Function, PRIME-GENERATING POLYNOMIAL, QUADRATIC FIELD

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Heesch Number

The Heesch number of a closed plane figure is the maximum number of times that figure can be completely surrounded by copies of itself. The determination of the maximum possible (finite) Heesch number is known as HEESCH'S PROBLEM. The Heesch number of a TRIANGLE, QUADRILATERAL, regular HEXAGON, or any other shape that can TILE or TESSELLATE the plane, is infinity. Conversely, any shape with infinite Heesch number must tile the plane (Eppstein). The largest known (finite) Heesch number is 3, and corresponds to a tile invented by R. Ammann (Senechal 1995).

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Senechal, M. *Quasicrystals and Geometry.* New York: Cambridge University Press, 1995.

Heesch's Problem

How many times can a shape be completely surrounded by copies of itself without being able to TILE the entire plane, i.e., what is the maximum (finite) HEESCH NUMBER?

References
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Height

The vertical length of an object from top to bottom.

see also LENGTH (SIZE), WIDTH (SIZE)

Heilbronn Triangle Problem

N.B. A detailed on-line essay by S. Finch was the starting point for this entry.

Given any arrangement of n points within a UNIT SQUARE, let H_n be the smallest value for which there is at least one TRIANGLE formed from three of the points with AREA $\leq H_n$. The first few values are

$$\begin{aligned}
H_3 &= \tfrac{1}{2} \\
H_4 &= \tfrac{1}{2} \\
H_5 &= \tfrac{1}{9}\sqrt{3} \\
H_6 &= \tfrac{1}{8} \\
H_7 &\geq \tfrac{1}{12} \\
H_8 &\geq \tfrac{1}{4}(2-\sqrt{3}\,) \\
H_9 &\geq \tfrac{1}{21} \\
H_{10} &\geq \tfrac{1}{32}(3\sqrt{17}-11) \\
H_{11} &\geq \tfrac{1}{27} \\
H_{12} &\geq \tfrac{1}{33} \\
H_{13} &\geq 0.030 \\
H_{14} &\geq 0.022 \\
H_{15} &\geq 0.020 \\
H_{16} &\geq 0.0175.
\end{aligned}$$

Komlós *et al.* (1981, 1982) have shown that there are constants c such that

$$\frac{c \ln n}{n^2} \leq H_n \leq \frac{C}{n^{8/7}-\epsilon},$$

for any $\epsilon > 0$ and all sufficiently large n.

Using an EQUILATERAL TRIANGLE of unit AREA instead gives the constants

$$\begin{aligned}
h_3 &= 1 \\
h_4 &= \tfrac{1}{3} \\
h_5 &= 3-2\sqrt{2} \\
h_6 &= \tfrac{1}{8}.
\end{aligned}$$

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Heine-Borel Theorem

If a CLOSED SET of points on a line can be covered by a set of intervals so that every point of the set is an interior point of at least one of the intervals, then there exist a finite number of intervals with the covering property.

Heine Hypergeometric Series

$$
{}_r\phi_s\left[\begin{matrix}\alpha_1,\alpha_2,\ldots,\alpha_r\\ \beta_1,\ldots,\beta_s\end{matrix};z\right]
\equiv \sum_{n=0}^\infty \frac{(\alpha_1;q)_n(\alpha_2;q)_n\cdots(\alpha_r;q)_n}{(q;q)_n(\beta_1;q)_n\cdots(\beta_s;q)_n}z^n, \quad (1)
$$

where

$$(a;q)_n = (1-a)(1-aq)(1-aq^2)\cdots(1-aq^{n-1}), \quad (2)$$

$$(a;q)_0 = 1. \quad (3)$$

In particular,

$${}_2\psi_1(a,b;c;q,z) = \sum_{n=0}^\infty \frac{(a;q)_n(b;q)_n z^n}{(q;q)_n(c;q)_n} \quad (4)$$

(Andrews 1986, p. 10). Heine proved the transformation formula

$${}_2\phi_1(a,b;c;q,z) = \frac{(b;q)_\infty(az;q)_\infty}{(c;q)_\infty(z;q)_\infty}\,{}_2\phi_1(c/b,a;az;q,b), \quad (5)$$

and Rogers (1893) obtained the formulas

$$
\begin{aligned}
&{}_2\phi_1(a,b;c;q,z)\\
&\quad = \frac{(c/b;q)_\infty(bz;q)_\infty}{(z;q)_\infty(c;q)_\infty}\,{}_2\phi_1(b,abz/c;bz;q,c/b) \quad (6)\\
&{}_2\phi_1(a,b,c;q,z)\\
&\quad = (abz/c;q)_\infty(z;q)_\infty\,{}_2\phi_1(c/a,c/b;c;q,abz/c) \quad (7)
\end{aligned}
$$

(Andrews 1986, pp. 10–11).

see also q-SERIES

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Heisenberg Group

The Heisenberg group H^n in n COMPLEX variables is the GROUP of all (z,t) with $z \in \mathbb{C}^n$ and $t \in \mathbb{R}$ having multiplication

$$(w,t)(z,t') = (w+z, t+t' + \Im[w^{\mathrm{T}}z])$$

where w^{T} is the conjugate transpose. The Heisenberg group is ISOMORPHIC to the group of MATRICES

$$\begin{bmatrix} 1 & z^{\mathrm{T}} & \frac{1}{2}|z|^2 + it \\ 0 & 1 & z \\ 0 & 0 & 1 \end{bmatrix},$$

and satisfies

$$(z,t)^{-1} = (-z,-t).$$

Every finite-dimensional unitary representation is trivial on Z and therefore factors to a REPRESENTATION of the quotient $\mathbb{C}^n$.

see also NIL GEOMETRY

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Heisenberg Space

The boundary of COMPLEX HYPERBOLIC 2-SPACE.

see also HYPERBOLIC SPACE

Held Group

The SPORADIC GROUP *He*.

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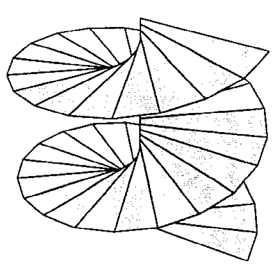
Helen of Geometers

see CYCLOID

Helicoid



The MINIMAL SURFACE having a HELIX as its boundary. It is the only RULED MINIMAL SURFACE other than the PLANE (Catalan 1842, do Carmo 1986). For many years, the helicoid remained the only known example of a complete embedded MINIMAL SURFACE of finite topology with infinite CURVATURE. However, in 1992 a second example, known as HOFFMAN'S MINIMAL SURFACE and consisting of a helicoid with a HOLE, was discovered (*Sci. News* 1992).

The equation of a helicoid in CYLINDRICAL COORDINATES is

$$z = c\theta. \tag{1}$$

In CARTESIAN COORDINATES, it is

$$\frac{y}{x} = \tan\left(\frac{z}{c}\right). \tag{2}$$

It can be given in parametric form by

$$x = u\cos v \tag{3}$$
$$y = u\sin v \tag{4}$$
$$z = cu, \tag{5}$$

which has an obvious generalization to the ELLIPTIC HELICOID. The differentials are

$$dx = \cos v\,du - u\sin v\,dv \tag{6}$$
$$dy = \sin v\,du + u\cos v\,dv \tag{7}$$
$$dz = 2cu\,dy, \tag{8}$$

so the LINE ELEMENT on the surface is

$$\begin{aligned} ds^2 &= dx^2 + dy^2 + dz^2 \\ &= \cos^2 v\,du^2 - 2u\sin v\cos v\,du\,dv + u^2\sin^2 v\,dv^2 \\ &\quad + \sin^2 v\,du^2 + 2u\sin v\cos v\,du\,dv + u^2\cos^2 v\,dv^2 \\ &\quad + 4c^2u^2\,du^2 \\ &= (1+4c^2u^2)\,du^2 + u^2\,dv^2, \end{aligned} \tag{9}$$

and the METRIC components are

$$g_{uu} = 1 + 4c^2u^2 \tag{10}$$
$$g_{uv} = 0 \tag{11}$$
$$g_{vv} = u^2. \tag{12}$$

From GAUSS'S THEOREMA EGREGIUM, the GAUSSIAN CURVATURE is then

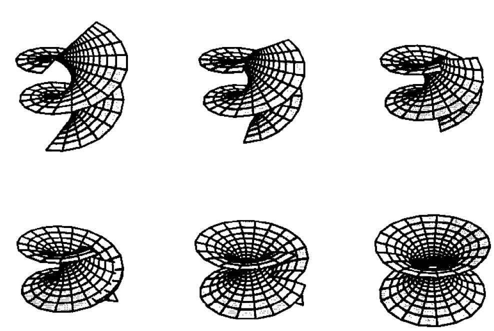
$$K = \frac{4c^2}{(1+4c^2u^2)^2}. \tag{13}$$

The MEAN CURVATURE is

$$H = 0, \tag{14}$$

and the equation for the LINES OF CURVATURE is

$$u = \pm c\sinh(v-k). \tag{15}$$



The helicoid can be continuously deformed into a CATENOID by the transformation

$$x(u,v) = \cos\alpha \sinh v \sin u + \sin\alpha \cosh v \cos u \quad (16)$$
$$y(u,v) = -\cos\alpha \sinh v \cos u + \sin\alpha \cosh v \sin u \quad (17)$$
$$z(u,v) = u\cos\alpha + v\sin\alpha, \quad (18)$$

where $\alpha = 0$ corresponds to a helicoid and $\alpha = \pi/2$ to a CATENOID.

If a twisted curve C (i.e., one with TORSION $\tau \neq 0$) rotates about a fixed axis A and, at the same time, is displaced parallel to A such that the speed of displacement is always proportional to the angular velocity of rotation, then C generates a GENERALIZED HELICOID.

see also CALCULUS OF VARIATIONS, CATENOID, ELLIPTIC HELICOID, GENERALIZED HELICOID, HELIX, HOFFMAN'S MINIMAL SURFACE, MINIMAL SURFACE

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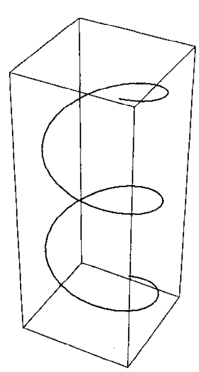
Helix



A helix is also called a CURVE OF CONSTANT SLOPE. It can be defined as a curve for which the TANGENT makes a constant ANGLE with a fixed line. The helix is a SPACE CURVE with parametric equations

$$x = r\cos t \quad (1)$$
$$y = r\sin t \quad (2)$$
$$z = ct, \quad (3)$$

where c is a constant. The CURVATURE of the helix is given by

$$\kappa = \frac{r}{r^2+c^2}, \quad (4)$$

and the LOCUS of the centers of CURVATURE of a helix is another helix. The ARC LENGTH is given by

$$s = \int \sqrt{x'^2 + y'^2 + z'^2}\,dt = \sqrt{r^2+c^2}\,t. \quad (5)$$

The TORSION of a helix is given by

$$\tau = \frac{1}{r^2(r^2+c^2)} \begin{vmatrix} -r\sin t & -r\cos t & r\sin t \\ r\cos t & -r\sin t & -r\cos t \\ c & 0 & 0 \end{vmatrix} = \frac{c}{r^2+c^2}, \quad (6)$$

so

$$\frac{\kappa}{\tau} = \frac{\frac{r}{r^2+c^2}}{\frac{c}{r^2+c^2}} = \frac{r}{c}, \quad (7)$$

which is a constant. In fact, LANCRET'S THEOREM states that a NECESSARY and SUFFICIENT condition for a curve to be a helix is that the ratio of CURVATURE to TORSION be constant. The OSCULATING PLANE of the helix is given by

$$\begin{vmatrix} z_1 - r\cos t & z_2 - r\sin t & z_3 - ct \\ -r\sin t & r\cos t & c \\ -r\cos t & -r\sin t & 0 \end{vmatrix} = 0 \quad (8)$$

$$z_1 c\sin t - z_2 c\cos t + (z_3 - ct)r = 0. \quad (9)$$

The MINIMAL SURFACE of a helix is a HELICOID.

see also GENERALIZED HELIX, HELICOID, SPHERICAL HELIX

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Helly Number

Given a Euclidean n-space,

$$H_n \equiv n+1.$$

see also EUCLIDEAN SPACE, HELLY'S THEOREM

Helly's Theorem

If F is a family of more than n bounded closed convex sets in Euclidean n-space $\mathbb{R}^n$, and if every H_n (where H_n is the HELLY NUMBER) members of F have at least one point in common, then all the members of F have at least one point in common.

see also CARATHÉODORY'S FUNDAMENTAL THEOREM, HELLY NUMBER

Helmholtz Differential Equation

A PARTIAL DIFFERENTIAL EQUATION which can be written in a SCALAR version

$$\nabla^2\psi + k^2\psi = 0, \tag{1}$$

or VECTOR form,

$$\nabla^2\mathbf{A} + k^2\mathbf{A} = 0, \tag{2}$$

where ∇^2 is the LAPLACIAN. When $k = 0$, the Helmholtz differential equation reduces to LAPLACE'S EQUATION. When $k^2 < 0$, the equation becomes the space part of the diffusion equation.

The Helmholtz differential equation can be solved by SEPARATION OF VARIABLES in only 11 coordinate systems, 10 of which (with the exception of CONFOCAL PARABOLOIDAL COORDINATES) are particular cases of the CONFOCAL ELLIPSOIDAL system: CARTESIAN, CONFOCAL ELLIPSOIDAL, CONFOCAL PARABOLOIDAL, CONICAL, CYLINDRICAL, ELLIPTIC CYLINDRICAL, OBLATE SPHEROIDAL, PARABOLOIDAL, PARABOLIC CYLINDRICAL, PROLATE SPHEROIDAL, and SPHERICAL COORDINATES (Eisenhart 1934). LAPLACE'S EQUATION (the Helmholtz differential equation with $k = 0$) is separable in the two additional BISPHERICAL COORDINATES and TOROIDAL COORDINATES.

If Helmholtz's equation is separable in a 3-D coordinate system, then Morse and Feshbach (1953, pp. 509–510) show that

$$\frac{h_1h_2h_3}{{h_n}^2} = f_n(u_n)g_n(u_i,u_j), \tag{3}$$

where $i \neq j \neq n$. The LAPLACIAN is therefore of the form

$$\nabla^2 = \frac{1}{h_1h_2h_3}\left\{g_1(u_2,u_3)\frac{\partial}{\partial u_1}\left[f_1(u_1)\frac{\partial}{\partial u_1}\right] + g_2(u_1,u_3)\frac{\partial}{\partial u_2}\left[f_2(u_2)\frac{\partial}{\partial u_2}\right] + g_3(u_1,u_2)\frac{\partial}{\partial u_3}\left[f_3(u_3)\frac{\partial}{\partial u_3}\right]\right\}, \tag{4}$$

which simplifies to

$$\nabla^2 = \frac{1}{{h_1}^2 f_1}\frac{\partial}{\partial u_1}\left[f_1(u_1)\frac{\partial}{\partial u_1}\right] + \frac{1}{{h_2}^2 f_2}\frac{\partial}{\partial u_2}\left[f_2(u_2)\frac{\partial}{\partial u_2}\right] + \frac{1}{{h_3}^2 f_3}\frac{\partial}{\partial u_3}\left[f_3(u_3)\frac{\partial}{\partial u_3}\right]. \tag{5}$$

Such a coordinate system obeys the ROBERTSON CONDITION, which means that the STÄCKEL DETERMINANT is of the form

$$S = \frac{h_1h_2h_3}{f_1(u_1)f_2(u_2)f_3(u_3)}. \tag{6}$$

Coordinate System	Variables	Solution Functions
Cartesian	$X(x)Y(y)Z(z)$	exponential, circular, hyperbolic
circular cylindrical	$R(r)\Theta(\theta)Z(z)$	Bessel, exponential, circular
conical		ellipsoidal harmonics, power
ellipsoidal	$\Lambda(\lambda)M(\mu)N(\nu)$	ellipsoidal harmonics
elliptic cylindrical	$U(u)V(v)Z(z)$	Mathieu, circular
oblate spheroidal	$\Lambda(\lambda)M(\mu)N(\nu)$	Legendre, circular
parabolic		Bessel, circular
parabolic cylindrical		Parabolic cylinder, Bessel, circular
paraboloidal	$U(u)V(v)\Theta(\theta)$	Baer functions, circular
prolate spheroidal	$\Lambda(\lambda)M(\mu)N(\nu)$	Legendre, circular
spherical	$R(r)\Theta(\theta)\Phi(\phi)$	Legendre, power, circular

see also LAPLACE'S EQUATION, POISSON'S EQUATION, SEPARATION OF VARIABLES, SPHERICAL BESSEL DIFFERENTIAL EQUATION

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Helmholtz Differential Equation—Bipolar Coordinates

In BIPOLAR COORDINATES, the HELMHOLTZ DIFFERENTIAL EQUATION is not separable, but LAPLACE'S EQUATION is.

see also LAPLACE'S EQUATION—BIPOLAR COORDINATES

Helmholtz Differential Equation—Cartesian Coordinates

In 2-D CARTESIAN COORDINATES, attempt SEPARATION OF VARIABLES by writing

$$F(x,y) = X(x)Y(y), \tag{1}$$

then the HELMHOLTZ DIFFERENTIAL EQUATION becomes

$$\frac{d^2X}{dx^2}Y + \frac{d^2Y}{dy^2}X + k^2XY = 0. \quad (2)$$

Dividing both sides by XY gives

$$\frac{1}{X}\frac{d^2X}{dx^2} + \frac{1}{Y}\frac{d^2Y}{dy^2} + k^2 = 0. \quad (3)$$

This leads to the two coupled ordinary differential equations with a separation constant m^2,

$$\frac{1}{X}\frac{d^2X}{dx^2} = m^2 \quad (4)$$

$$\frac{1}{Y}\frac{d^2Y}{dy^2} = -(m^2 + k^2), \quad (5)$$

where X and Y could be interchanged depending on the boundary conditions. These have solutions

$$X = A_m e^{mx} + B_m e^{-mx} \quad (6)$$

$$\begin{aligned} Y &= C_m e^{i\sqrt{m^2+k^2}\,y} + D_m e^{-i\sqrt{m^2+k^2}\,y} \\ &= E_m \sin(\sqrt{m^2+k^2}\,y) + F_m \cos(\sqrt{m^2+k^2}\,y). \end{aligned} \quad (7)$$

The general solution is then

$$F(x,y) = \sum_{m=1}^{\infty}(A_m e^{mx} + B_m e^{-mx}) \times [E_m \sin(\sqrt{m^2+k^2}\,y) + F_m \cos(\sqrt{m^2+k^2}\,y)]. \quad (8)$$

In 3-D CARTESIAN COORDINATES, attempt SEPARATION OF VARIABLES by writing

$$F(x,y,z) = X(x)Y(y)Z(z), \quad (9)$$

then the HELMHOLTZ DIFFERENTIAL EQUATION becomes

$$\frac{d^2X}{dx^2}YZ + \frac{d^2Y}{dy^2}XZ + \frac{d^2Z}{dz^2}XY + k^2XY = 0. \quad (10)$$

Dividing both sides by XYZ gives

$$\frac{1}{X}\frac{d^2X}{dx^2} + \frac{1}{Y}\frac{d^2Y}{dy^2} + \frac{1}{Z}\frac{d^2Z}{dz^2} + k^2 = 0. \quad (11)$$

This leads to the three coupled differential equations

$$\frac{1}{X}\frac{d^2X}{dx^2} = l^2 \quad (12)$$

$$\frac{1}{Y}\frac{d^2Y}{dy^2} = m^2 \quad (13)$$

$$\frac{1}{Z}\frac{d^2Z}{dz^2} = -(k^2 + l^2 + m^2), \quad (14)$$

where X, Y, and Z could be permuted depending on boundary conditions. The general solution is therefore

$$\begin{aligned} F(x,y,z) &= \sum_{l=1}^{\infty}\sum_{m=1}^{\infty}(A_l e^{lx} + B_l e^{-lx})(C_m e^{my} + D_m e^{-my}) \\ &\times (E_{lm} e^{-i\sqrt{k^2+l^2+m^2}\,z} + F_{lm} e^{i\sqrt{k^2+l^2+m^2}\,z}). \end{aligned} \quad (15)$$

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Helmholtz Differential Equation—Circular Cylindrical Coordinates

In CYLINDRICAL COORDINATES, the SCALE FACTORS are $h_r = 1$, $h_\theta = r$, $h_z = 1$ and the separation functions are $f_1(r) = r$, $f_2(\theta) = 1$, $f_3(z) = 1$, so the STÄCKEL DETERMINANT is 1. Attempt SEPARATION OF VARIABLES by writing

$$F(r,\theta,z) = R(r)\Theta(\theta)Z(z), \quad (1)$$

then the HELMHOLTZ DIFFERENTIAL EQUATION becomes

$$\frac{d^2R}{dr^2}\Theta Z + \frac{1}{r}\frac{dR}{dr}\Theta Z + \frac{1}{r^2}\frac{d^2\Theta}{d\theta^2}RZ + \frac{d^2Z}{dz^2}R\Theta + k^2R\Theta Z = 0. \quad (2)$$

Now divide by $R\Theta Z$,

$$\left(\frac{r^2}{R}\frac{d^2R}{dr^2} + \frac{r}{R}\frac{dR}{dr}\right) + \frac{d^2\Theta}{d\theta^2}\frac{1}{\Theta} + \frac{d^2Z}{dz^2}\frac{r^2}{Z} + k^2 = 0, \quad (3)$$

so the equation has been separated. Since the solution must be periodic in Θ from the definition of the circular cylindrical coordinate system, the solution to the second part of (3) must have a NEGATIVE separation constant

$$\frac{d^2\Theta}{d\theta^2}\frac{1}{\Theta} = -(k^2 + m^2), \quad (4)$$

which has a solution

$$\Theta(\theta) = C_m e^{-i\sqrt{k^2+m^2}\,\theta} + D_m e^{i\sqrt{k^2+m^2}\,\theta}. \quad (5)$$

Plugging (5) back into (3) gives

$$\frac{r^2}{R}\frac{d^2R}{dr^2} + \frac{r}{R}\frac{dR}{dr} - m^2 + \frac{d^2Z}{dz^2}\frac{r^2}{Z} = 0 \quad (6)$$

$$\frac{1}{R}\frac{d^2R}{dr^2} + \frac{1}{rR}\frac{dR}{dr} - \frac{m^2}{r^2} + \frac{d^2Z}{dz^2}\frac{1}{Z} = 0. \quad (7)$$

The solution to the second part of (7) must not be sinusoidal at $\pm\infty$ for a physical solution, so the differential equation has a POSITIVE separation constant

$$\frac{d^2Z}{dz^2}\frac{1}{Z} = n^2, \quad (8)$$

and the solution is

$$Z(z) = E_n e^{-nz} + F_n e^{nz}. \tag{9}$$

Plugging (9) back into (7) and multiplying through by R yields

$$\frac{d^2R}{dr^2} + \frac{1}{r}\frac{dR}{dr} + \left(n^2 - \frac{m^2}{r^2}\right) R = 0 \tag{10}$$

$$\frac{1}{n^2}\frac{d^2R}{dr^2} + \frac{1}{(nr)}\frac{1}{n}\frac{dR}{dr} + \left[1 - \frac{m^2}{(nr)^2}\right] R = 0 \tag{11}$$

$$\frac{d^2R}{d(nr)^2} + \frac{1}{(nr)}\frac{dR}{d(nr)} + \left[1 - \frac{m^2}{(nr)^2}\right] R = 0. \tag{12}$$

This is the BESSEL DIFFERENTIAL EQUATION, which has a solution

$$R(r) = A_{mn}J_m(nr) + B_{mn}Y_m(nr), \tag{13}$$

where $J_n(x)$ and $Y_n(x)$ are BESSEL FUNCTIONS OF THE FIRST and SECOND KINDS, respectively. The general solution is therefore

$$\begin{aligned} F(r,\theta,z) &= \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}[A_{mn}J_m(nr) + B_{mn}Y_m(nr)] \\ &\times (C_m e^{-i\sqrt{k^2+m^2}\,\theta} + D_m e^{i\sqrt{k^2+m^2}\,\theta})(E_n e^{-nz} + F_n e^{nz}). \end{aligned} \tag{14}$$

Actually, the HELMHOLTZ DIFFERENTIAL EQUATION is separable for general k of the form

$$k^2(r,\theta,z) = f(r) + \frac{g(\theta)}{r^2} + h(z) + k'^2. \tag{15}$$

see also CYLINDRICAL COORDINATES, HELMHOLTZ DIFFERENTIAL EQUATION

References

Morse, P. M. and Feshbach, H. *Methods of Theoretical Physics, Part I.* New York: McGraw-Hill, pp. 514 and 656–657, 1953.

Helmholtz Differential Equation—Confocal Ellipsoidal Coordinates

Using the NOTATION of Byerly (1959, pp. 252–253), LAPLACE'S EQUATION can be reduced to

$$\nabla^2 F = (\mu^2-\nu^2)\frac{\partial^2 F}{\partial\alpha^2} + (\lambda^2-\nu^2)\frac{\partial^2 F}{\partial\beta^2} + (\lambda^2-\mu^2)\frac{\partial^2 F}{\partial\gamma^2} = 0, \tag{1}$$

where

$$\begin{aligned} \alpha &= c\int_c^\lambda \frac{d\lambda}{\sqrt{(\lambda^2-b^2)(\lambda^2-c^2)}} \\ &= F\left[\frac{b}{c},\frac{\pi}{2}\right) - F\left(\frac{b}{c}, \sin^{-1}\left(\frac{c}{\lambda}\right)\right] \end{aligned} \tag{2}$$

$$\begin{aligned} \beta &= c\int_b^\mu \frac{d\mu}{\sqrt{(c^2-\mu^2)(\mu^2-b^2)}} \\ &= F\left[\sqrt{1-b^2-c^2}, \sin^{-1}\left(\sqrt{\frac{1-\frac{b^2}{\mu^2}}{1-\frac{b^2}{c^2}}}\right)\right] \end{aligned} \tag{3}$$

$$\begin{aligned} \gamma &= c\int_0^\nu \frac{d\nu}{\sqrt{(b^2-\nu^2)(c^2-\nu^2)}} \\ &= F\left(\frac{b}{c}, \sin^{-1}\left(\frac{\nu}{b}\right)\right). \end{aligned} \tag{4}$$

In terms of α, β, and γ,

$$\lambda = c\,\mathrm{dc}\left(\alpha, \frac{b}{c}\right) \tag{5}$$

$$\mu = b\,\mathrm{nd}\left(\beta, \sqrt{1-\frac{b^2}{c^2}}\right) \tag{6}$$

$$\nu = b\,\mathrm{sn}\left(\gamma, \frac{b}{c}\right). \tag{7}$$

Equation (1) is not separable using a function of the form

$$F = L(\alpha)M(\beta)N(\gamma), \tag{8}$$

but it is if we let

$$\frac{1}{L}\frac{d^2L}{d\alpha^2} = \sum a_k\lambda^k \tag{9}$$

$$\frac{1}{M}\frac{d^2M}{d\beta^2} = \sum b_k\mu^k \tag{10}$$

$$\frac{1}{N}\frac{d^2N}{d\gamma^2} = \sum c_k\nu^k. \tag{11}$$

These give

$$a_0 = -b_0 = c_0 \tag{12}$$

$$a_2 = -b_2 = c_2, \tag{13}$$

and all others terms vanish. Therefore (1) can be broken up into the equations

$$\frac{d^2L}{d\alpha^2} = (a_0 + a_2\lambda^2)L \tag{14}$$

$$\frac{d^2M}{d\beta^2} = -(a_0 + a_2\mu^2)M \tag{15}$$

$$\frac{d^2N}{d\gamma^2} = (a_0 + a_2\nu^2)N. \tag{16}$$

For future convenience, now write

$$a_0 = -(b^2 + c^2)p \tag{17}$$

$$a_2 = m(m+1), \tag{18}$$

then

$$\frac{d^2 L}{d\alpha^2} - [m(m+1)\lambda^2 - (b^2+c^2)p]L = 0 \tag{19}$$

$$\frac{d^2 M}{d\beta^2} + [m(m+1)\mu^2 - (b^2+c^2)p]M = 0 \tag{20}$$

$$\frac{d^2 N}{d\gamma^2} - [m(m+1)\nu^2 - (b^2+c^2)p]N = 0. \tag{21}$$

Now replace α, β, and γ to obtain

$$(\lambda^2 - b^2)(\lambda^2 - c^2)\frac{d^2 L}{d\lambda^2} + \lambda(\lambda^2 - b^2 + \lambda^2 - c^2)\frac{dL}{d\lambda} - [m(m+1)\lambda^2 - (b^2+c^2)p]L = 0 \tag{22}$$

$$(\mu^2 - b^2)(\mu^2 - c^2)\frac{d^2 M}{d\mu^2} + \mu(\mu^2 - b^2 + \mu^2 - c^2)\frac{dM}{d\mu} - [m(m+1)\mu^2 - (b^2+c^2)p]M = 0 \tag{23}$$

$$(\nu^2 - b^2)(\nu^2 - c^2)\frac{d^2 N}{d\nu^2} + \nu(\nu^2 - b^2 + \nu^2 - c^2)\frac{dN}{d\nu} - [m(m+1)\nu^2 - (b^2+c^2)p]N = 0. \tag{24}$$

Each of these is a LAMÉ'S DIFFERENTIAL EQUATION, whose solution is called an ELLIPSOIDAL HARMONIC. Writing

$$L(\lambda) = E_m^p(\lambda) \tag{25}$$

$$M(\lambda) = E_m^p(\mu) \tag{26}$$

$$N(\lambda) = E_m^p(\nu) \tag{27}$$

gives the solution to (1) as a product of ELLIPSOIDAL HARMONICS $E_m^p(x)$.

$$F = E_m^p(\lambda)E_m^p(\mu)E_m^p(\nu). \tag{28}$$

References

Arfken, G. "Confocal Ellipsoidal Coordinates (ξ_1, ξ_2, ξ_3)." §2.15 in *Mathematical Methods for Physicists, 2nd ed.* Orlando, FL: Academic Press, pp. 117–118, 1970.

Byerly, W. E. *An Elementary Treatise on Fourier's Series, and Spherical, Cylindrical, and Ellipsoidal Harmonics, with Applications to Problems in Mathematical Physics.* New York: Dover, pp. 251–258, 1959.

Morse, P. M. and Feshbach, H. *Methods of Theoretical Physics, Part I.* New York: McGraw-Hill, p. 663, 1953.

Helmholtz Differential Equation—Confocal Paraboloidal Coordinates

As shown by Morse and Feshbach (1953), the HELMHOLTZ DIFFERENTIAL EQUATION is separable in CONFOCAL PARABOLOIDAL COORDINATES.

see also CONFOCAL PARABOLOIDAL COORDINATES

References

Morse, P. M. and Feshbach, H. *Methods of Theoretical Physics, Part I.* New York: McGraw-Hill, p. 664, 1953.

Helmholtz Differential Equation—Conical Coordinates

In CONICAL COORDINATES, LAPLACE'S EQUATION can be written

$$\frac{\partial^2 V}{\partial \alpha^2} + \frac{\partial^2 V}{\partial \beta^2} + (\mu^2 - \nu^2)\frac{\partial}{\partial \lambda}\left(\lambda^2 \frac{\partial V}{\partial \lambda}\right) = 0, \tag{1}$$

where

$$\alpha = \int_a^\mu \frac{d\mu}{\sqrt{(\mu^2 - a^2)(b^2 - \mu^2)}} \tag{2}$$

$$\beta = \int_0^\nu \frac{d\nu}{\sqrt{(a^2 - \nu^2)(b^2 - \nu^2)}} \tag{3}$$

(Byerly 1959). Letting

$$V = U(u)R(r) \tag{4}$$

breaks (1) into the two equations,

$$\frac{d}{dr}\left(r^2 \frac{dR}{dr}\right) = m(m+1)R \tag{5}$$

$$\frac{\partial^2 U}{\partial \alpha^2} + \frac{\partial^2 U}{\partial \beta^2} + m(m+1)(\mu^2 - \nu^2)U = 0. \tag{6}$$

Solving these gives

$$R(r) = Ar^m + Br^{-m-1} \tag{7}$$

$$U(u) = E_m^p(\mu)E_m^p(\nu), \tag{8}$$

where E_m^p are ELLIPSOIDAL HARMONICS. The regular solution is therefore

$$V = Ar^m E_m^p(\mu)E_m^p(\nu). \tag{9}$$

However, because of the cylindrical symmetry, the solution $E_m^p(\mu)E_m^p(\nu)$ is an mth degree SPHERICAL HARMONIC.

References

Arfken, G. "Conical Coordinates (ξ_1, ξ_2, ξ_3)." §2.16 in *Mathematical Methods for Physicists, 2nd ed.* Orlando, FL: Academic Press, pp. 118–119, 1970.

Byerly, W. E. *An Elementary Treatise on Fourier's Series, and Spherical, Cylindrical, and Ellipsoidal Harmonics, with Applications to Problems in Mathematical Physics.* New York: Dover, p. 263, 1959.

Morse, P. M. and Feshbach, H. *Methods of Theoretical Physics, Part I.* New York: McGraw-Hill, pp. 514 and 659, 1953.

Helmholtz Differential Equation—Elliptic Cylindrical Coordinates

In ELLIPTIC CYLINDRICAL COORDINATES, the SCALE FACTORS are $h_u = h_v = a\sqrt{\sinh^2 u + \sin^2 v}$, $h_z = 1$, and the separation functions are $f_1(u) = f_2(v) = f_3(z) = 1$, giving a STÄCKEL DETERMINANT of $S = a^2(\sin^2 v + \sinh^2 u)$. The Helmholtz differential equation is

$$\frac{1}{a^2(\sinh^2 u + \sin^2 v)}\left(\frac{\partial^2 F}{\partial u^2} + \frac{\partial^2 F}{\partial v^2}\right) + \frac{\partial^2 F}{\partial z^2} + k^2 = 0. \tag{1}$$

Attempt SEPARATION OF VARIABLES by writing

$$F(u, v, z) = U(u)V(v)Z(z), \tag{2}$$

then the HELMHOLTZ DIFFERENTIAL EQUATION becomes

$$\frac{Z}{\sinh^2 u + \sin^2 v}\left(V\frac{d^2U}{du^2} + U\frac{d^2V}{dv^2}\right) + UV\frac{d^2Z}{dz^2} + k^2 UVZ = 0. \tag{3}$$

Now divide by UVZ to give

$$\frac{1}{\sinh^2 u + \sin^2 v}\left(\frac{1}{U}\frac{d^2U}{du^2} + \frac{1}{V}\frac{d^2V}{dv^2}\right) + \frac{1}{Z}\frac{d^2Z}{dz^2} + k^2 = 0. \tag{4}$$

Separating the Z part,

$$\frac{1}{Z}\frac{d^2Z}{dz^2} = -(k^2 + m^2) \tag{5}$$

$$\frac{1}{\sinh^2 u + \sin^2 v}\left(\frac{1}{U}\frac{d^2U}{du^2} + \frac{1}{V}\frac{d^2V}{dv^2}\right) = m^2, \tag{6}$$

so

$$\frac{d^2Z}{dz^2} = -(k^2 + m^2)Z, \tag{7}$$

which has the solution

$$Z(z) = A\cos(\sqrt{k^2 + m^2}\, z) + B\sin(\sqrt{k^2 + m^2}\, z). \tag{8}$$

Rewriting (6) gives

$$\left(\frac{1}{U}\frac{d^2U}{du^2} - m^2\sinh^2 u\right) + \left(\frac{1}{V}\frac{d^2V}{dv^2} - m^2\sin^2 v\right) = 0, \tag{9}$$

which can be separated into

$$\frac{1}{U}\frac{d^2U}{du^2} - m^2\sinh^2 u = c \tag{10}$$

$$c + \frac{1}{V}\frac{d^2V}{dv^2} - m^2\sin^2 v = 0, \tag{11}$$

so

$$\frac{d^2U}{du^2} - (c + m^2\sinh^2 u)U = 0 \tag{12}$$

$$\frac{d^2V}{dv^2} + (c - m^2\sin^2 v)V = 0. \tag{13}$$

Now use

$$\sinh^2 u = \tfrac{1}{2}[1 - \cosh(2u)] \tag{14}$$

$$\sin^2 v = \tfrac{1}{2}[1 - \cos(2v)] \tag{15}$$

to obtain

$$\frac{d^2U}{du^2} - \{c + \tfrac{1}{2}m^2[1 - \cosh(2u)]\}U = 0 \tag{16}$$

$$\frac{d^2V}{dv^2} + \{c + \tfrac{1}{2}m^2[1 - \cos(2v)]\}V = 0. \tag{17}$$

Regrouping gives

$$\frac{d^2U}{du^2} - [(c + \tfrac{1}{2}m^2) - \tfrac{1}{4}m^2 2\cosh(2u)]U = 0 \tag{18}$$

$$\frac{d^2V}{dv^2} + [(c + \tfrac{1}{2}m^2) - \tfrac{1}{4}m^2 2\cos(2v)]V = 0. \tag{19}$$

Let $b \equiv \frac{1}{2}m^2 + c$ and $q \equiv \frac{1}{4}m^2$, then these become

$$\frac{d^2U}{du^2} - [b - 2q\cosh(2u)]U = 0 \tag{20}$$

$$\frac{d^2V}{dv^2} + [b - 2q\cos(2v)]V = 0. \tag{21}$$

Here, (21) is the MATHIEU DIFFERENTIAL EQUATION and (20) is the modified MATHIEU DIFFERENTIAL EQUATION. These solutions are known as MATHIEU FUNCTIONS.

see also ELLIPTIC CYLINDRICAL COORDINATES, MATHIEU DIFFERENTIAL EQUATION, MATHIEU FUNCTION

References

Morse, P. M. and Feshbach, H. *Methods of Theoretical Physics, Part I.* New York: McGraw-Hill, pp. 514 and 657, 1953.

Helmholtz Differential Equation—Oblate Spheroidal Coordinates

As shown by Morse and Feshbach (1953) and Arfken (1970), the HELMHOLTZ DIFFERENTIAL EQUATION is separable in OBLATE SPHEROIDAL COORDINATES.

References

Arfken, G. "Oblate Spheroidal Coordinates (u, v, φ)." §2.11 in *Mathematical Methods for Physicists, 2nd ed.* Orlando, FL: Academic Press, pp. 107–109, 1970.

Morse, P. M. and Feshbach, H. *Methods of Theoretical Physics, Part I.* New York: McGraw-Hill, p. 662, 1953.

Helmholtz Differential Equation—Parabolic Coordinates

The SCALE FACTORS are $h_u = h_v = \sqrt{u^2 + v^2}$, $h_\theta = uv$ and the separation functions are $f_1(u) = u$, $f_2(v) = v$, $f_3(\theta) = 1$, given a STÄCKEL DETERMINANT of $S = u^2 + v^2$. The LAPLACIAN is

$$\frac{1}{u^2 + v^2}\left(\frac{1}{u}\frac{\partial F}{\partial u} + \frac{\partial^2 F}{\partial u^2} + \frac{1}{v}\frac{\partial F}{\partial v} + \frac{\partial^2 F}{\partial v^2}\right) + \frac{1}{u^2 v^2}\frac{\partial^2 F}{\partial \theta^2} + k^2 = 0. \tag{1}$$

Attempt SEPARATION OF VARIABLES by writing

$$F(u, v, z) \equiv U(u)V(v)\Theta(\theta), \tag{2}$$

then the HELMHOLTZ DIFFERENTIAL EQUATION becomes

$$\frac{1}{u^2+v^2}\left[V\Theta\left(\frac{1}{u}\frac{dU}{du}+\frac{d^2U}{du^2}\right) + U\Theta\left(\frac{1}{v}\frac{dV}{dv}+\frac{d^2V}{dv^2}\right)\right] + k^2UV\Theta = 0. \tag{3}$$

Now divide by $UV\Theta$,

$$\frac{u^2v^2}{u^2+v^2}\left[\frac{1}{U}\left(\frac{1}{u}\frac{dU}{du}+\frac{d^2U}{du^2}\right) + \frac{1}{V}\left(\frac{1}{v}\frac{dV}{dv}+\frac{d^2V}{dv^2}\right)\right] + \frac{1}{\Theta}\frac{d^2\Theta}{d\theta^2} + k^2 = 0. \tag{4}$$

Separating the Θ part,

$$\frac{1}{\Theta}\frac{d^2\Theta}{f\theta^2} = -(k^2+m^2) \tag{5}$$

$$\frac{u^2v^2}{u^2+v^2}\left[\frac{1}{U}\left(\frac{1}{u}\frac{dU}{du}+\frac{d^2U}{du^2}\right) + \frac{1}{V}\left(\frac{1}{v}\frac{dV}{dv}+\frac{d^2V}{dv^2}\right)\right] = k^2, \tag{6}$$

so

$$\frac{d^2\Theta}{d\theta^2} = -(k^2+m^2)\Theta, \tag{7}$$

which has solution

$$\Theta(\theta) = A\cos(\sqrt{k^2+m^2}\,\theta) + B\sin(\sqrt{k^2+m^2}\,\theta), \tag{8}$$

and

$$\left[\frac{1}{U}\left(\frac{1}{u}\frac{dU}{du}+\frac{d^2U}{du^2}\right)\right] + \left[\frac{1}{V}\left(\frac{1}{v}\frac{dV}{dv}+\frac{d^2V}{dv^2}\right)\right] - k^2\frac{u^2+v^2}{u^2v^2} = 0 \tag{9}$$

$$\left[\frac{1}{U}\left(\frac{1}{u}\frac{dU}{du}+\frac{d^2U}{du^2}\right) - \frac{k^2}{u^2}\right] + \left[\frac{1}{V}\left(\frac{1}{v}\frac{dV}{dv}+\frac{d^2V}{dv^2}\right) - \frac{k^2}{v^2}\right] = 0. \tag{10}$$

This can be separated

$$\frac{1}{U}\left(\frac{1}{u}\frac{dU}{du}+\frac{d^2U}{du^2}\right) - \frac{k^2}{u^2} = c \tag{11}$$

$$\frac{1}{V}\left(\frac{1}{v}\frac{dV}{dv}+\frac{d^2V}{dv^2}\right) - \frac{k^2}{v^2} = -c, \tag{12}$$

so

$$u^2\frac{d^2U}{du^2} + \frac{dU}{du} - (c+k^2)U = 0 \tag{13}$$

$$v^2\frac{d^2V}{dv^2} + \frac{dV}{dv} + (c-k^2)V = 0. \tag{14}$$

References

Arfken, G. "Parabolic Coordinates (ξ, η, ϕ)." §2.12 in *Mathematical Methods for Physicists, 2nd ed.* Orlando, FL: Academic Press, pp. 109–111, 1970.

Morse, P. M. and Feshbach, H. *Methods of Theoretical Physics, Part I.* New York: McGraw-Hill, pp. 514–515 and 660, 1953.

Helmholtz Differential Equation—Parabolic Cylindrical Coordinates

In PARABOLIC CYLINDRICAL COORDINATES, the SCALE FACTORS are $h_u = h_v = \sqrt{u^2+v^2}$, $h_z = 1$ and the separation functions are $f_1(u) = f_2(v) = f_3(z) = 1$, giving STÄCKEL DETERMINANT of $S = u^2+v^2$. The HELMHOLTZ DIFFERENTIAL EQUATION is

$$\frac{1}{u^2+v^2}\left(\frac{\partial^2F}{\partial u^2}+\frac{\partial^2F}{\partial v^2}\right) + \frac{\partial^2F}{\partial z^2} + k^2 = 0. \tag{1}$$

Attempt SEPARATION OF VARIABLES by writing

$$F(u, v, z) \equiv U(u)V(v)Z(z), \tag{2}$$

then the HELMHOLTZ DIFFERENTIAL EQUATION becomes

$$\frac{1}{u^2+v^2}\left(VZ\frac{d^2U}{du^2}+UZ\frac{d^2V}{dv^2}\right) + UV\frac{d^2Z}{dz^2} + k^2UVZ = 0. \tag{3}$$

Divide by UVZ,

$$\frac{1}{u^2+v^2}\left(\frac{1}{U}\frac{d^2U}{du^2}+\frac{1}{V}\frac{d^2V}{dv^2}\right) + \frac{1}{Z}\frac{d^2Z}{dz^2} + k^2 = 0. \tag{4}$$

Separating the Z part,

$$\frac{1}{Z}\frac{d^2Z}{dz^2} = -(k^2+m^2) \tag{5}$$

$$\frac{1}{u^2+v^2}\left(\frac{1}{U}\frac{d^2U}{du^2}+\frac{1}{V}\frac{d^2V}{dv^2}\right) - k^2 = 0 \tag{6}$$

$$\frac{1}{U}\frac{d^2U}{du^2}+\frac{1}{V}\frac{d^2V}{dv^2} - k^2(u^2+v^2) = 0, \tag{7}$$

so

$$\frac{d^2Z}{dz^2} = -(k^2+m^2)Z, \tag{8}$$

which has solution

$$Z(z) = A\cos(\sqrt{k^2+m^2}\,z) + B\sin(\sqrt{k^2+m^2}\,z), \tag{9}$$

and

$$\left(\frac{1}{U}\frac{d^2U}{du^2} - k^2u^2\right) + \left(\frac{1}{V}\frac{d^2V}{dv^2} - k^2v^2\right) = 0. \tag{10}$$

This can be separated

$$\frac{1}{U}\frac{d^2U}{du^2} - k^2u^2 = c \tag{11}$$

$$\frac{1}{V}\frac{d^2V}{dv^2} - k^2v^2 = -c, \tag{12}$$

so

$$\frac{d^2U}{du^2} - (c + k^2u^2)U = 0 \tag{13}$$

$$\frac{d^2V}{dv^2} + (c - k^2v^2)V = 0. \tag{14}$$

These are the WEBER DIFFERENTIAL EQUATIONS, and the solutions are known as PARABOLIC CYLINDER FUNCTIONS.

see also PARABOLIC CYLINDER FUNCTION, PARABOLIC CYLINDRICAL COORDINATES, WEBER DIFFERENTIAL EQUATIONS

References
Morse, P. M. and Feshbach, H. *Methods of Theoretical Physics, Part I.* New York: McGraw-Hill, pp. 515 and 658, 1953.

Helmholtz Differential Equation—Polar Coordinates

In 2-D POLAR COORDINATES, attempt SEPARATION OF VARIABLES by writing

$$F(r, \theta) = R(r)\Theta(\theta), \tag{1}$$

then the HELMHOLTZ DIFFERENTIAL EQUATION becomes

$$\frac{d^2R}{dr^2}\Theta + \frac{1}{r}\frac{dR}{dr}\Theta + \frac{1}{r^2}\frac{d^2\Theta}{d\theta^2}R + k^2R\Theta = 0. \tag{2}$$

Divide both sides by $R\Theta$

$$\left(\frac{r^2}{R}\frac{d^2R}{dr^2} + \frac{r}{R}\frac{dR}{dr}\right) + \left(\frac{1}{\Theta}\frac{d^2\Theta}{d\theta^2} + k^2\right) = 0. \tag{3}$$

The solution to the second part of (3) must be periodic, so the differential equation is

$$\frac{d^2\Theta}{d\theta^2}\frac{1}{\Theta} = -(k^2 + m^2), \tag{4}$$

which has solutions

$$\begin{aligned}\Theta(\theta) &= c_1 e^{i\sqrt{k^2+m^2}\,\theta} + c_2 e^{-i\sqrt{k^2+m^2}\,\theta} \\ &= c_3 \sin(\sqrt{k^2+m^2}\,\theta) + c_4 \cos(\sqrt{k^2+m^2}\,\theta).\end{aligned} \tag{5}$$

Plug (4) back into (3)

$$r^2R'' + rR' - m^2R = 0. \tag{6}$$

This is an EULER DIFFERENTIAL EQUATION with $\alpha \equiv 1$ and $\beta \equiv -m^2$. The roots are $r = \pm m$. So for $m = 0$, $r = 0$ and the solution is

$$R(r) = c_1 + c_2 \ln r. \tag{7}$$

But since $\ln r$ blows up at $r = 0$, the only possible physical solution is $R(r) = c_1$. When $m > 0$, $r = \pm m$, so

$$R(r) = c_1 r^m + c_2 r^{-m}. \tag{8}$$

But since r^{-m} blows up at $r = 0$, the only possible physical solution is $R_m(r) = c_1 r^m$. The solution for R is then

$$R_m(r) = c_m r^m \tag{9}$$

for $m = 0, 1, \ldots$ and the general solution is

$$\begin{aligned}F(r, \theta) = \sum_{m=0}^{\infty} [a_m r^m \sin(\sqrt{k^2+m^2}\,\theta) \\ + b_m r^m \cos(\sqrt{k^2+m^2}\,\theta)].\end{aligned} \tag{10}$$

References
Morse, P. M. and Feshbach, H. *Methods of Theoretical Physics, Part I.* New York: McGraw-Hill, pp. 502–504, 1953.

Helmholtz Differential Equation—Prolate Spheroidal Coordinates

As shown by Morse and Feshbach (1953) and Arfken (1970), the HELMHOLTZ DIFFERENTIAL EQUATION is separable in PROLATE SPHEROIDAL COORDINATES.

References
Arfken, G. "Prolate Spheroidal Coordinates (u, v, φ)." §2.10 in *Mathematical Methods for Physicists, 2nd ed.* Orlando, FL: Academic Press, pp. 103–107, 1970.
Morse, P. M. and Feshbach, H. *Methods of Theoretical Physics, Part I.* New York: McGraw-Hill, p. 661, 1953.

Helmholtz Differential Equation—Spherical Coordinates

In SPHERICAL COORDINATES, the SCALE FACTORS are $h_r = 1$, $h_\theta = r\sin\phi$, $h_\phi = r$, and the separation functions are $f_1(r) = r^2$, $f_2(\theta) = 1$, $f_3(\phi) = \sin\phi$, giving a STÄCKEL DETERMINANT of $S = 1$. The LAPLACIAN is

$$\begin{aligned}\nabla^2 \equiv \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \frac{1}{r^2\sin^2\phi}\frac{\partial^2}{\partial\theta^2} \\ + \frac{1}{r^2\sin\phi}\frac{\partial}{\partial\phi}\left(\sin\phi\frac{\partial}{\partial\phi}\right).\end{aligned} \tag{1}$$

To solve the HELMHOLTZ DIFFERENTIAL EQUATION in SPHERICAL COORDINATES, attempt SEPARATION OF VARIABLES by writing

$$F(r, \theta, \phi) = R(r)\Theta(\theta)\Phi(\phi). \tag{2}$$

Then the HELMHOLTZ DIFFERENTIAL EQUATION becomes

$$\frac{d^2R}{dr^2}\Phi\Theta + \frac{2}{r}\frac{dR}{dr}\Phi\Theta + \frac{1}{r^2\sin^2\phi}\frac{d^2\Theta}{d\theta^2}\Phi R + \frac{\cos\phi}{r^2\sin\phi}\frac{d\Phi}{d\phi}\Theta R + \frac{1}{r^2}\frac{d^2\Phi}{d\phi^2}\Theta R = 0. \quad (3)$$

Now divide by $R\Theta\Phi$,

$$\frac{r^2\sin^2\phi}{\Phi R\Theta}\Phi\Theta\frac{d^2R}{dr^2} + \frac{2}{r}\frac{r^2\sin^2\phi}{\Phi R\Theta}\Phi\Theta\frac{dR}{dr} + \frac{1}{r^2\sin^2\phi}\frac{r^2\sin^2\phi}{\Phi R\Theta}\Phi R\frac{d^2\Theta}{d\theta^2} + \frac{\cos\phi}{r^2\sin\phi}\frac{r^2\sin^2\phi}{\Phi\Theta R}\frac{d\Phi}{d\phi}\Theta R + \frac{1}{r^2}\frac{r^2\sin^2\phi}{\Phi R\Theta}\frac{d^2\Phi}{d\phi^2}\Theta R = 0 \quad (4)$$

$$\left(\frac{r^2\sin^2\phi}{R}\frac{d^2R}{dr^2} + \frac{2r\sin^2\phi}{R}\frac{dR}{dr}\right) + \left(\frac{1}{\Theta}\frac{d^2\Theta}{d\theta^2}\right) + \left(\frac{\cos\phi\sin\phi}{\Phi}\frac{d\Phi}{d\phi} + \frac{\sin^2\phi}{\Phi}\frac{d^2\Phi}{d\phi^2}\right) = 0. \quad (5)$$

The solution to the second part of (5) must be sinusoidal, so the differential equation is

$$\frac{d^2\Theta}{d\theta^2}\frac{1}{\Theta} = -m^2, \quad (6)$$

which has solutions which may be defined either as a COMPLEX function with $m = -\infty, \ldots, \infty$

$$\Theta(\theta) = A_m e^{im\theta}, \quad (7)$$

or as a sum of REAL sine and cosine functions with $m = -\infty, \ldots, \infty$

$$\Theta(\theta) = S_m \sin(m\theta) + C_m \cos(m\theta). \quad (8)$$

Plugging (6) back into (7),

$$\frac{r^2}{R}\frac{d^2R}{dr^2} + \frac{2r}{R}\frac{dR}{dr} - \frac{1}{\sin^2\phi}\left(m^2 + \frac{\cos\phi\sin\phi}{\Phi}\right)\frac{d\Phi}{d\phi} + \frac{\sin^2\phi}{\Phi}\frac{d^2\Phi}{d\phi^2} = 0. \quad (9)$$

The radial part must be equal to a constant

$$\frac{r^2}{R}\frac{d^2R}{dr^2} + \frac{2r}{R}\frac{dR}{dr} = l(l+1) \quad (10)$$

$$r^2\frac{d^2R}{dr^2} + 2r\frac{dR}{dr} = l(l+1)R. \quad (11)$$

But this is the EULER DIFFERENTIAL EQUATION, so we try a series solution of the form

$$R = \sum_{n=0}^{\infty} a_n r^{n+c}. \quad (12)$$

Then

$$r^2\sum_{n=0}^{\infty}(n+c)(n+c-1)a_n r^{n+c-2} + 2r\sum_{n=0}^{\infty}(n+c)a_n r^{n+c-1} - l(l+1)\sum_{n=0}^{\infty}a_n r^{n+c} = 0 \quad (13)$$

$$\sum_{n=0}^{\infty}(n+c)(n+c-1)a_n r^{n+c} + 2\sum_{n=0}^{\infty}(n+c)a_n r^{n+c} - l(l+1)\sum_{n=0}^{\infty}a_n r^{n+c} = 0 \quad (14)$$

$$\sum_{n=0}^{\infty}[(n+c)(n+c+1) - l(l+1)]a_n r^{n+c} = 0. \quad (15)$$

This must hold true for all POWERS of r. For the r^c term (with $n = 0$),

$$c(c+1) = l(l+1), \quad (16)$$

which is true only if $c = l, -l-1$ and all other terms vanish. So $a_n = 0$ for $n \neq l, -l-1$. Therefore, the solution of the R component is given by

$$R_l(r) = A_l r^l + B_l r^{-l-1}. \quad (17)$$

Plugging (17) back into (9),

$$l(l+1) - \frac{m^2}{\sin^2\phi} + \frac{\cos\phi}{\sin\phi}\frac{1}{\Phi}\frac{d\Phi}{d\phi} + \frac{1}{\Phi}\frac{d^2\Phi}{d\phi^2} = 0 \quad (18)$$

$$\Phi'' + \frac{\cos\phi}{\sin\phi}\Phi' + \left[l(l+1) - \frac{m^2}{\sin^2\phi}\right]\Phi = 0, \quad (19)$$

which is the associated LEGENDRE DIFFERENTIAL EQUATION for $x = \cos\phi$ and $m = 0, \ldots, l$. The general COMPLEX solution is therefore

$$\sum_{l=0}^{\infty}\sum_{m=-l}^{l}(A_l r^l + B_l r^{-l-1})P_l^m(\cos\phi)e^{-im\theta} \equiv \sum_{l=0}^{\infty}\sum_{m=-1}^{l}(A_l r^l + B_l r^{-l-1})Y_l^m(\theta,\phi), \quad (20)$$

where

$$Y_l^m(\theta,\phi) \equiv P_l^m(\cos\phi)e^{-im\theta} \quad (21)$$

are the (COMPLEX) SPHERICAL HARMONICS. The general REAL solution is

$$\sum_{l=0}^{\infty}\sum_{m=0}^{l}(A_l r^l + B_l r^{-l-1})P_l^m(\cos\phi) \times[S_m\sin(m\theta) + C_m\cos(m\theta)]. \quad (22)$$

Some of the normalization constants of P_l^m can be absorbed by S_m and C_m, so this equation may appear in the form

$$\sum_{l=0}^{\infty}\sum_{m=0}^{l}(A_l r^l + B_l r^{-l-1})P_l^m(\cos\phi)$$
$$\times[S_l^m \sin(m\theta) + C_l^m \cos(m\theta)]$$
$$\equiv \sum_{l=0}^{\infty}\sum_{m=0}^{l}(A_l r^l + B_l r^{-l-1})$$
$$\times[S_l^m Y_l^{m(o)}(\theta,\phi) + C_l^m Y_l^{m(e)}(\theta,\phi)], \quad (23)$$

where

$$Y_l^{m(o)}(\theta,\phi) \equiv P_l^m(\cos\theta)\sin(m\theta) \quad (24)$$

$$Y_l^{m(e)}(\theta,\phi) \equiv P_l^m(\cos\theta)\cos(m\theta) \quad (25)$$

are the EVEN and ODD (real) SPHERICAL HARMONICS. If azimuthal symmetry is present, then $\Theta(\theta)$ is constant and the solution of the Φ component is a LEGENDRE POLYNOMIAL $P_l(\cos\phi)$. The general solution is then

$$F(r,\phi) = \sum_{l=0}^{\infty}(A_l r^l + B_l r^{-l-1})P_l(\cos\phi). \quad (26)$$

Actually, the equation is separable under the more general condition that k^2 is of the form

$$k^2(r,\theta,\phi) = f(r) + \frac{g(\theta)}{r^2} + \frac{h(\phi)}{r^2\sin\theta} + k'^2. \quad (27)$$

References

Morse, P. M. and Feshbach, H. *Methods of Theoretical Physics, Part I.* New York: McGraw-Hill, p. 514 and 658, 1953.

Helmholtz Differential Equation—Spherical Surface

On the surface of a SPHERE, attempt SEPARATION OF VARIABLES in SPHERICAL COORDINATES by writing

$$F(\theta,\phi) = \Theta(\theta)\Phi(\phi), \quad (1)$$

then the HELMHOLTZ DIFFERENTIAL EQUATION becomes

$$\frac{1}{\sin^2\phi}\frac{d^2\Theta}{d\theta^2}\Phi + \frac{\cos\phi}{\sin\phi}\frac{d\Phi}{d\phi}\Theta + \frac{d^2\Phi}{d\phi^2}\Theta + k^2\Theta\Phi = 0. \quad (2)$$

Dividing both sides by $\Phi\Theta$,

$$\left(\frac{\cos\phi\sin\phi}{\Phi}\frac{d\Phi}{d\phi} + \frac{\sin^2\phi}{\Phi}\frac{d^2\Phi}{d\phi^2}\right) + \left(\frac{1}{\Theta}\frac{d^2\Theta}{d\theta^2} + k^2\right) = 0, \quad (3)$$

which can now be separated by writing

$$\frac{d^2\Theta}{d\theta^2}\frac{1}{\Theta} = -(k^2+m^2). \quad (4)$$

The solution to this equation must be periodic, so m must be an INTEGER. The solution may then be defined either as a COMPLEX function

$$\Theta(\theta) = A_m e^{i\sqrt{k^2+m^2}\,\theta} + B_m e^{-i\sqrt{k^2+m^2}\,\theta} \quad (5)$$

for $m = -\infty, \ldots, \infty$, or as a sum of REAL sine and cosine functions

$$\Theta(\theta) = S_m \sin(\sqrt{k^2+m^2}\,\theta) + C_m \cos(\sqrt{k^2+m^2}\,\theta) \quad (6)$$

for $m = 0, \ldots, \infty$. Plugging (4) into (3) gives

$$\frac{\cos\phi\sin\phi}{\Phi}\frac{d\Phi}{d\phi} + \frac{\sin^2\phi}{\Phi}\frac{d^2\Phi}{d\phi^2} + m^2 = 0 \quad (7)$$

$$\Phi'' + \frac{\cos\phi}{\sin\phi}\Phi' + \frac{m^2}{\sin^2\phi}\Phi = 0, \quad (8)$$

which is the LEGENDRE DIFFERENTIAL EQUATION for $x = \cos\phi$ with

$$m^2 \equiv l(l+1), \quad (9)$$

giving

$$l^2 + l - m^2 = 0 \quad (10)$$

$$l = \tfrac{1}{2}(-1 \pm \sqrt{1+4m^2}\,). \quad (11)$$

Solutions are therefore LEGENDRE POLYNOMIALS with a COMPLEX index. The general COMPLEX solution is then

$$F(\theta,\phi) = \sum_{m=-\infty}^{\infty} P_l(\cos\phi)(A_m e^{im\theta} + B_m e^{-im\theta}), \quad (12)$$

and the general REAL solution is

$$F(\theta,\phi) = \sum_{m=0}^{\infty} P_l(\cos\phi)[S_m \sin(m\theta) + C_m \cos(m\theta)]. \quad (13)$$

Note that these solutions depend on only a single variable m. However, on the surface of a sphere, it is usual to express solutions in terms of the SPHERICAL HARMONICS derived for the 3-D spherical case, which depend on the two variables l and m.

Helmholtz Differential Equation—Toroidal Coordinates

The HELMHOLTZ DIFFERENTIAL EQUATION is not separable.

see LAPLACE'S EQUATION—TOROIDAL COORDINATES

Helmholtz's Theorem

Any VECTOR FIELD $\mathbf{v}$ satisfying

$$[\nabla \cdot \mathbf{v}]_\infty = 0 \tag{1}$$

$$[\nabla \times \mathbf{v}]_\infty = 0 \tag{2}$$

may be written as the sum of an IRROTATIONAL part and a SOLENOIDAL part,

$$\mathbf{v} = -\nabla\phi + \nabla \times \mathbf{A}, \tag{3}$$

where for a VECTOR FIELD F,

$$\phi = -\int_V \frac{\nabla \cdot \mathbf{F}}{4\pi|\mathbf{r}' - \mathbf{r}|}\, d^3\mathbf{r}' \tag{4}$$

$$\mathbf{A} = \int_V \frac{\nabla \times \mathbf{F}}{4\pi|\mathbf{r}' - \mathbf{r}|} d^3\mathbf{r}'. \tag{5}$$

see also IRROTATIONAL FIELD, SOLENOIDAL FIELD, VECTOR FIELD

References

Arfken, G. "Helmholtz's Theorem." §1.15 in *Mathematical Methods for Physicists, 3rd ed.* Orlando, FL: Academic Press, pp. 78–84, 1985.

Gradshteyn, I. S. and Ryzhik, I. M. *Tables of Integrals, Series, and Products, 5th ed.* San Diego, CA: Academic Press, pp. 1084, 1980.

Helson-Szegő Measure

An absolutely continuous measure on ∂D whose density has the form $\exp(x + \bar{y})$, where x and y are real-valued functions in L^∞, $||y||_\infty < \pi/2$, exp is the EXPONENTIAL FUNCTION, and $||y||$ is the NORM.

Hemicylindrical Function

A function $S_n(z)$ which satisfies the RECURRENCE RELATION

$$S_{n-1}(z) - S_{n+1}(z) = 2S'_n(z)$$

together with

$$S_1(z) = -S'_0(z)$$

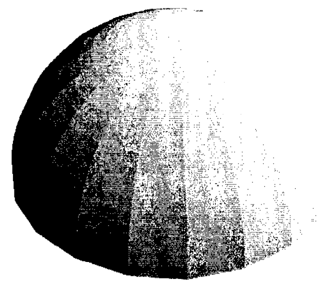
is called a hemicylindrical function.

References

Sonine, N. "Recherches sur les fonctions cylindriques et le développement des fonctions continues en séries." *Math. Ann.* **16**, 1–9 and 71–80, 1880.

Watson, G. N. "Hemi-Cylindrical Functions." §10.8 in *A Treatise on the Theory of Bessel Functions, 2nd ed.* Cambridge, England: Cambridge University Press, p. 353, 1966.

Hemisphere



Half of a SPHERE cut by a PLANE passing through its CENTER. A hemisphere of RADIUS r can be given by the usual SPHERICAL COORDINATES

$$x = r\cos\theta\sin\phi \tag{1}$$

$$y = r\sin\theta\sin\phi \tag{2}$$

$$z = r\cos\phi, \tag{3}$$

where $\theta \in [0, 2\pi)$ and $\phi \in [0, \pi/2]$. All CROSS-SECTIONS passing through the z-axis are SEMICIRCLES.

The VOLUME of the hemisphere is

$$V = \pi\int_0^r (r^2 - z^2)\, dz = \tfrac{2}{3}\pi r^3. \tag{4}$$

The weighted mean of z over the hemisphere is

$$\langle z\rangle = \pi\int_0^r z(r^2 - z^2)\, dz = \tfrac{1}{4}\pi r^2. \tag{5}$$

The CENTROID is then given by

$$\bar{z} = \frac{\langle z\rangle}{V} = \tfrac{3}{8}r \tag{6}$$

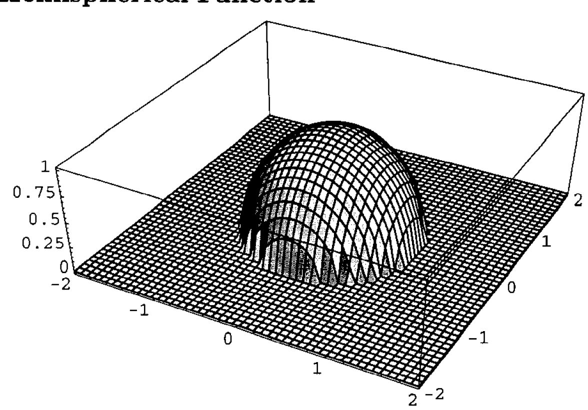
(Beyer 1987).

see also SEMICIRCLE, SPHERE

References

Beyer, W. H. (Ed.) *CRC Standard Mathematical Tables, 28th ed.* Boca Raton, FL: CRC Press, p. 133, 1987.

Hemispherical Function



The hemisphere function is defined as

$$H(x,y)=\begin{cases}\sqrt{a-x^2-y^2} & \text{for } \sqrt{x^2+y^2}\le a\\ 0 & \text{for } \sqrt{x^2+y^2}>a.\end{cases}$$

Watson (1966) defines a hemispherical function as a function S which satisfies the RECURRENCE RELATIONS

$$S_{n-1}(z)-S_{n+1}(z)=2S_n{}'(z)$$

with

$$S_1(z)=-S_0'(z).$$

see also CYLINDER FUNCTION, CYLINDRICAL FUNCTION

References

Watson, G. N. *A Treatise on the Theory of Bessel Functions, 2nd ed.* Cambridge, England: Cambridge University Press, p. 353, 1966.

Hempel's Paradox

A purple cow is a confirming instance of the hypothesis that all crows are black.

References

Carnap, R. *Logical Foundations of Probability.* Chicago, IL: University of Chicago Press, pp. 224 and 469, 1950.

Gardner, M. *The Scientific American Book of Mathematical Puzzles & Diversions.* New York: Simon and Schuster, pp. 52–54, 1959.

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Hempel, C. G. "A Purely Syntactical Definition of Confirmation." *J. Symb. Logic* **8**, 122–143, 1943.

Hempel, C. G. "Studies in Logic and Confirmation." *Mind* **54**, 1–26, 1945.

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Hosiasson-Lindenbaum, J. "On Confirmation." *J. Symb. Logic* **5**, 133–148, 1940.

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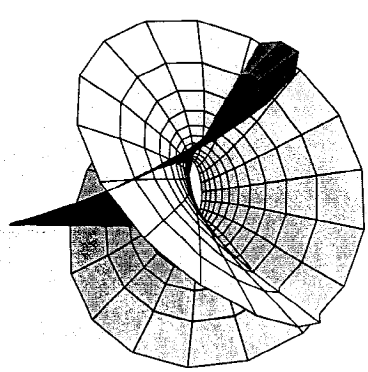
Hendecagon

see UNDECAGON

Henneberg's Minimal Surface



A double algebraic surface of 15th order and fifth class which can be given by parametric equations

$$x(u,v)=2\sinh u\cos v-\tfrac{2}{3}\sinh(3u)\cos(3v) \quad (1)$$

$$y(u,v)=2\sinh u\sin v-\tfrac{2}{3}\sinh(3u)\sin(3v) \quad (2)$$

$$z(u,v)=2\cosh(2u)\cos(2v). \quad (3)$$

It can also be obtained from the ENNEPER-WEIERSTRAß PARAMETERIZATION with

$$f=2-2z^{-4} \quad (4)$$

$$g=z. \quad (5)$$

see also MINIMAL SURFACE

References

Eisenhart, L. P. *A Treatise on the Differential Geometry of Curves and Surfaces.* New York: Dover, p. 267, 1960.

Gray, A. *Modern Differential Geometry of Curves and Surfaces.* Boca Raton, FL: CRC Press, pp. 446–448, 1993.

Nitsche, J. C. C. *Introduction to Minimal Surfaces.* Cambridge, England: Cambridge University Press, p. 144, 1989.

Hénon Attractor

see HÉNON MAP

Hénon-Heiles Equation

A nonlinear nonintegrable HAMILTONIAN SYSTEM with

$$\ddot{x}=-\frac{\partial V}{\partial x} \quad (1)$$

$$\ddot{y}=-\frac{\partial V}{\partial y}, \quad (2)$$

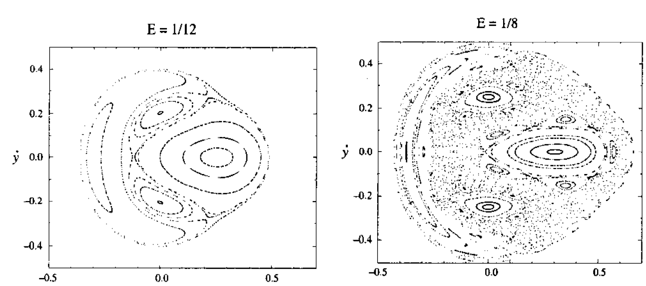
where

$$V(x,y)=\tfrac{1}{2}(x^2+y^2+2x^2y-\tfrac{2}{3}y^3) \quad (3)$$

$$V(r,\theta)=\tfrac{1}{2}r^2+\frac{1}{3}r^3\sin(3\theta). \quad (4)$$

The energy is

$$E=V(x,y)+\tfrac{1}{2}(\dot{x}^2+\dot{y}^2). \quad (5)$$



The above plots are SURFACES OF SECTION for $E=1/12$ and $E=1/8$. The Hamiltonian for a generalized Hénon-Heiles potential is

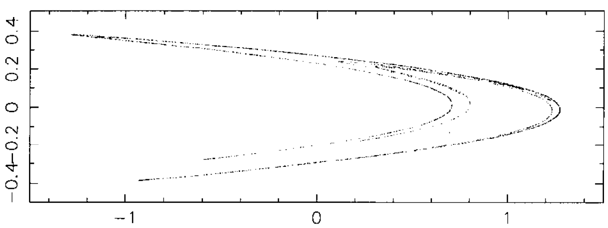
$$H=\tfrac{1}{2}(p_x{}^2+p_y{}^2+Ax^2+By^2)+Dx^2y-\tfrac{1}{3}Cy^3. \quad (6)$$

The equations of motion are integrable only for

1. $D/C = 0$,
2. $D/C = -1, A/B = 1$,
3. $D/C = -1/6$, and
4. $D/C = -1/16, A/B = 1/6$.

References
Gleick, J. *Chaos: Making a New Science.* New York: Penguin Books, pp. 144–153, 1988.
Hénon, M. and Heiles, C. "The Applicability of the Third Integral of Motion: Some Numerical Experiments." *Astron. J.* **69**, 73–79, 1964.

Hénon Map



A quadratic 2-D MAP given by the equations

$$x_{n+1} = 1 - \alpha {x_n}^2 + y_n \tag{1}$$
$$y_{n+1} = \beta x_n \tag{2}$$

or

$$x_{n+1} = x_n \cos\alpha - (y_n - {x_n}^2)\sin\alpha \tag{3}$$
$$y_{n+1} = x_n \sin\alpha + (y_n - {x_n}^2)\cos\alpha. \tag{4}$$

The above map is for $\alpha = 1.4$ and $\beta = 0.3$. The Hénon map has CORRELATION EXPONENT 1.25 ± 0.02 (Grassberger and Procaccia 1983) and CAPACITY DIMENSION 1.261 ± 0.003 (Russell *et al.* 1980). Hitzl and Zele (1985) give conditions for the existence of periods 1 to 6.

see also BOGDANOV MAP, LOZI MAP, QUADRATIC MAP

References
Dickau, R. M. "The Hénon Attractor." `http://forum.swarthmore.edu/advanced/robertd/henon.html`.
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Peitgen, H.-O. and Saupe, D. (Eds.). "A Chaotic Set in the Plane." §3.2.2 in *The Science of Fractal Images.* New York: Springer-Verlag, pp. 146–148, 1988.
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Hensel's Lemma

An important result in VALUATION THEORY which gives information on finding roots of POLYNOMIALS. Hensel's lemma is formally stated as follow. Let $(K, |\cdot|)$ be a complete non-Archimedean valuated field, and let R be the corresponding VALUATION RING. Let $f(x)$ be a POLYNOMIAL whose COEFFICIENTS are in R and suppose a_0 satisfies

$$|f(a_0)| < |f'(a_0)|^2, \tag{1}$$

where f' is the (formal) DERIVATIVE of f. Then there exists a unique element $a \in R$ such that $f(a) = 0$ and

$$|a - a_0| \leq \left|\frac{f(a_0)}{f'(a_0)}\right|. \tag{2}$$

Less formally, if $f(x)$ is a POLYNOMIAL with "INTEGER" COEFFICIENTS and $f(a_0)$ is "small" compared to $f'(a_0)$, then the equation $f(x) = 0$ has a solution "near" a_0. In addition, there are no other solutions near a_0, although there may be other solutions. The proof of the LEMMA is based around the Newton-Raphson method and relies on the non-Archimedean nature of the valuation.

Consider the following example in which Hensel's lemma is used to determine that the equation $x^2 = -1$ is solvable in the 5-adic numbers $\mathbb{Q}_5$ (and so we can embed the GAUSSIAN INTEGERS inside $\mathbb{Q}_5$ in a nice way). Let K be the 5-adic numbers $\mathbb{Q}_5$, let $f(x) = x^2 + 1$, and let $a_0 = 2$. Then we have $f(2) = 5$ and $f'(2) = 4$, so

$$|f(2)|_5 = \tfrac{1}{5} < |f'(2)|_5^2 = 1, \tag{3}$$

and the condition is satisfied. Hensel's lemma then tells us that there is a 5-adic number a such that $a^2 + 1 = 0$ and

$$|a - 2|_5 <= |\tfrac{5}{4}|_5 = \tfrac{1}{5}. \tag{4}$$

Similarly, there is a 5-adic number b such that $b^2 + 1 = 0$ and

$$|b - 3|_5 <= |\tfrac{10}{7}|_5 = \tfrac{1}{5}. \tag{5}$$

Therefore, we have found both the square roots of -1 in $\mathbb{Q}_5$. It is possible to find the roots of any POLYNOMIAL using this technique.

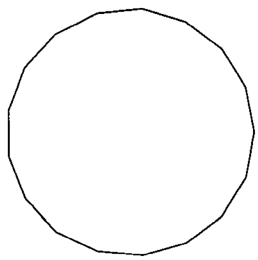
Henstock-Kurzweil Integral

see HK INTEGRAL

Heptacontagon

A 70-sided POLYGON.

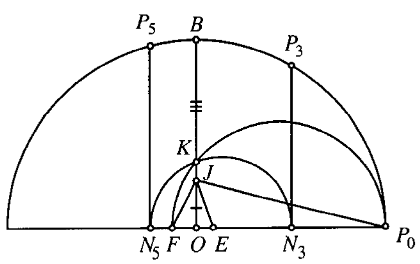
Heptadecagon



The REGULAR POLYGON of 17 sides is called the HEPTADECAGON, or sometimes the HEPTAKAIDECAGON. Gauss proved in 1796 (when he was 19 years old) that the heptadecagon is CONSTRUCTIBLE with a COMPASS and STRAIGHTEDGE. Gauss's proof appears in his monumental work *Disquisitiones Arithmeticae.* The proof relies on the property of irreducible POLYNOMIAL equations that ROOTS composed of a finite number of SQUARE ROOT extractions only exist when the order of the equation is a product of the form $2^a 3^b F_c \cdot F_d \cdots F_e$, where the F_n are distinct PRIMES of the form

$$F_n = 2^{2^n} + 1,$$

known as FERMAT PRIMES. Constructions for the regular TRIANGLE (3^1), SQUARE (2^2), PENTAGON ($2^{2^1}+1$), HEXAGON ($2^1 3^1$), etc., had been given by Euclid, but constructions based on the FERMAT PRIMES ≥ 17 were unknown to the ancients. The first explicit construction of a heptadecagon was given by Erchinger in about 1800.



17-gon

The following elegant construction for the heptadecagon (Yates 1949, Coxeter 1969, Stewart 1977, Wells 1992) was first given by Richmond (1893).

1. Given an arbitrary point O, draw a CIRCLE centered on O and a DIAMETER drawn through O.
2. Call the right end of the DIAMETER dividing the CIRCLE into a SEMICIRCLE P_0.
3. Construct the DIAMETER PERPENDICULAR to the original DIAMETER by finding the PERPENDICULAR BISECTOR OB.
4. Find J a QUARTER the way up OB.
5. Join JP_0 and find E so that $\angle OJE$ is a QUARTER of $\angle OJP_0$.
6. Find F so that $\angle EJF$ is 45°.
7. Construct the SEMICIRCLE with DIAMETER FP_0.
8. This SEMICIRCLE cuts OB at K.
9. Draw a SEMICIRCLE with center E and RADIUS EK.
10. This cuts the extension of OP_0 at N_3.
11. Construct a line PERPENDICULAR to OP_0 through N_3.
12. This line meets the original SEMICIRCLE at P_3.
13. You now have points P_0 and P_3 of a heptadecagon.
14. Use P_0 and P_3 to get the remaining 15 points of the heptadecagon around the original CIRCLE by constructing P_0, P_3, P_6, P_9, P_{12}, P_{15}, P_1, P_4, P_7, P_{10}, P_{13}, P_{16}, P_2, P_5, P_8, P_{11}, and P_{14}.
15. Connect the adjacent points P_i.

This construction, when suitably streamlined, has SIMPLICITY 53. The construction of Smith (1920) has a greater SIMPLICITY of 58. Another construction due to Tietze (1965) and reproduced in Hall (1970) has a SIMPLICITY of 50. However, neither Tietze (1965) nor Hall (1970) provides a proof that this construction is correct. Both Richmond's and Tietze's constructions require extensive calculations to prove their validity. De Temple (1991) gives an elegant construction involving the CARLYLE CIRCLES which has GEOMETROGRAPHY symbol $8S_1 + 4S_2 + 22C_1 + 11C_3$ and SIMPLICITY 45. The construction problem has now been automated to some extent (Bishop 1978).

see also 257-GON, 65537-GON, COMPASS, CONSTRUCTIBLE POLYGON, FERMAT NUMBER, FERMAT PRIME, REGULAR POLYGON, STRAIGHTEDGE, TRIGONOMETRY VALUES—$\pi/17$

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Stewart, I. "Gauss." *Sci. Amer.* **237**, 122–131, 1977.
Tietze, H. *Famous Problems of Mathematics.* New York: Graylock Press, 1965.
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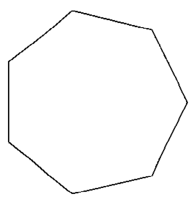
Heptagon



The unconstructible regular seven-sided POLYGON, illustrated above, has SCHLÄFLI SYMBOL $\{7\}$.

Although the regular heptagon is not a CONSTRUCTIBLE POLYGON, Dixon (1991) gives several close approximations. While the ANGLE subtended by a side is $360°/7 \approx 51.428571°$, Dixon gives constructions containing angles of $2\sin^{-1}(\sqrt{3}/4) \approx 51.317812°$, $\tan^{-1}(5/4) \approx 51.340191°$, and $30° + \sin^{-1}((\sqrt{3}-1)/2) \approx 51.470701°$.

Madachy (1979) illustrates how to construct a heptagon by folding and knotting a strip of paper.

see also EDMONDS' MAP, TRIGONOMETRY VALUES—$\pi/7$

References

Courant, R. and Robbins, H. "The Regular Heptagon." §3.3.4 in *What is Mathematics?: An Elementary Approach to Ideas and Methods, 2nd ed.* Oxford, England: Oxford University Press, pp. 138–139, 1996.
Dixon, R. *Mathographics.* New York: Dover, pp. 35–40, 1991.
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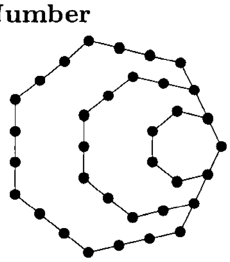
Heptagonal Number



A FIGURATE NUMBER of the form $n(5n-3)/2$. The first few are 1, 7, 18, 34, 55, 81, 112, ... (Sloane's A000566). The GENERATING FUNCTION for the heptagonal numbers is

$$\frac{x(4x+1)}{(1-x)^3} = x + 7x^2 + 18x^3 + 34x^4 + \ldots.$$

References

Sloane, N. J. A. Sequence A000566/M4358 in "An On-Line Version of the Encyclopedia of Integer Sequences."

Heptagonal Pyramidal Number

A PYRAMIDAL NUMBER of the form $n(n+1)(5n-2)/6$, The first few are 1, 8, 26, 60, 115, ... (Sloane's A002413). The GENERATING FUNCTION for the heptagonal pyramidal numbers is

$$\frac{x(4x+1)}{(x-1)^4} = x + 8x^2 + 26x^3 + 60x^4 + \ldots.$$

References

Sloane, N. J. A. Sequence A002413/M4498 in "An On-Line Version of the Encyclopedia of Integer Sequences."

Heptahedron

The regular heptahedron is a one-sided surface made from four TRIANGLES and three QUADRILATERALS. It is topologically equivalent to the ROMAN SURFACE (Wells 1991). While all of the faces are regular and vertices equivalent, the heptahedron is self-intersecting and is therefore not considered an ARCHIMEDEAN SOLID. There are three semiregular heptahedra: the pentagonal and pentagrammic PRISMS, and a FACETED OCTAHEDRON (Holden 1991).

References

Holden, A. *Shapes, Space, and Symmetry.* New York: Dover, p. 95, 1991.
Wells, D. *The Penguin Dictionary of Curious and Interesting Geometry.* New York: Viking Penguin, p. 98, 1992.

Heptakaidecagon

see HEPTADECAGON

Heptaparallelohedron

see CUBOCTAHEDRON

Heptomino

The heptominoes are the 7-POLYOMINOES. There are 108 different heptominoes.

see also HERSCHEL, PI HEPTOMINO, POLYOMINO

Herbrand's Theorem

Let an ideal class be in $\mathcal{A}$ if it contains an IDEAL whose lth power is PRINCIPAL. Let i be an ODD INTEGER $1 \leq i \leq l$ and define j by $i + j = 1$. Then $\mathcal{A}_1 = \langle e \rangle$. If $i \geq 3$ and $l \nmid B_j$, then $\mathcal{A}_i = \langle e \rangle$.

References

Ireland, K. and Rosen, M. "Herbrand's Theorem." §15.3 in *A Classical Introduction to Modern Number Theory, 2nd ed.* New York: Springer-Verlag, pp. 241–248, 1990.

Hereditary Representation

The representation of a number as a sum of powers of a BASE b, followed by expression of each of the exponents as a sum of powers of b, etc., until the process stops. For example, the hereditary representation of 266 in base 2 is

$$\begin{aligned}266 &= 2^8 + 2^3 + 2 \\ &= 2^{2^{2+1}} + 2^{2+1} + 2.\end{aligned}$$

see also GOODSTEIN SEQUENCE

Heredity

A property of a SPACE which is also true of each of its SUBSPACES. Being "COUNTABLE" is hereditary, but having a given GENUS is not.

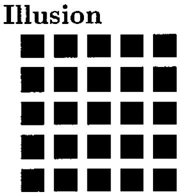
Hermann's Formula

The MACHIN-LIKE FORMULA

$$\tfrac{1}{4}\pi = 2\tan^{-1}(\tfrac{1}{2}) - \tan^{-1}(\tfrac{1}{7}).$$

The other 2-term MACHIN-LIKE FORMULAS are EULER'S MACHIN-LIKE FORMULA, HUTTON'S FORMULA, and MACHIN'S FORMULA.

Hermann Grid Illusion

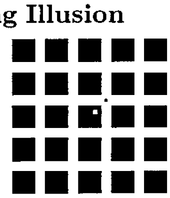


A regular 2-D arrangement of squares separated by vertical and horizontal "canals." Looking at the grid produces the illusion of gray spots in the white AREA between square VERTICES. The illusion was noted by Hermann (1870) while reading a book on sound by J. Tyndall.

References

Fineman, M. *The Nature of Visual Illusion.* New York: Dover, pp. 139–140, 1996.

Hermann-Hering Illusion



The illusion in view by staring at the small black dot for a half minute or so, then switching to the white dot. The black squares appear stationary when staring at the white dot, but a fainter grid of moving squares also appears to be present.

Hermann-Mauguin Symbol

A symbol used to represent the point and space groups (e.g., $2/m\bar{3}$). Some symbols have abbreviated form. The equivalence between Hermann-Mauguin symbols ("crystallographic symbol") and SCHÖNFLIES SYMBOLS for the POINT GROUPS is given by Cotton (1990).

see also POINT GROUPS

References

Cotton, F. A. *Chemical Applications of Group Theory, 3rd ed.* New York: Wiley, p. 379, 1990.

Hermit Point

see ISOLATED POINT

Hermite Constants

N.B. A detailed on-line essay by S. Finch was the starting point for this entry.

The Hermite constant is defined for DIMENSION n as the value

$$\gamma_n = \frac{\sup_f \min_{x_i} f(x_1, x_2, \ldots, x_n)}{[\mathrm{discriminant}(f)]^{1/n}}$$

(Le Lionnais 1983). In other words, they are given by

$$\gamma_n = 4\left(\frac{\delta_n}{V_n}\right)^{2/n},$$

where δ_n is the maximum *lattice* PACKING DENSITY for HYPERSPHERE PACKING and V_n is the CONTENT of the n-HYPERSPHERE. The first few values of $(\gamma_n)^n$ are 1, 4/3, 2, 4, 8, 64/3, 64, 256, ... Values for larger n are not known.

For sufficiently large n,

$$\frac{1}{2\pi e} \leq \frac{\gamma_n}{n} \leq \frac{1.744\ldots}{2\pi e}.$$

see also HYPERSPHERE PACKING, KISSING NUMBER, SPHERE PACKING

References

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Conway, J. H. and Sloane, N. J. A. *Sphere Packings, Lattices, and Groups, 2nd ed.* New York: Springer-Verlag, p. 20, 1993.

Le Lionnais, F. *Les nombres remarquables.* Paris: Hermann, p. 38, 1983.

Hermite Differential Equation

$$\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + \lambda y = 0. \qquad (1)$$

This differential equation has an irregular singularity at ∞. It can be solved using the series method

$$\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n - \sum_{n=1}^{\infty} 2na_nx^n + \sum_{n=0}^{\infty}\lambda a_nx^n = 0 \qquad (2)$$

$$(2a_2+\lambda a_4)+\sum_{n=1}^{\infty}[(n+2)(n+1)a_{n+2}-2na_n+\lambda a_n]x^n = 0. \tag{3}$$

Therefore,

$$a_2 = -\frac{\lambda a_0}{2} \tag{4}$$

and

$$a_{n+2} = \frac{2n-\lambda}{(n+2)(n+1)}a_n \tag{5}$$

for $n = 1, 2, \ldots$. Since (4) is just a special case of (5),

$$a_{n+2} = \frac{2n-\lambda}{(n+2)(n+1)}a_n \tag{6}$$

for $n = 0, 1, \ldots$. The linearly independent solutions are then

$$y_1 = a_0\left[1 - \frac{\lambda}{2!}x^2 - \frac{(4-\lambda)\lambda}{4!}x^4 - \frac{(8-\lambda)(4-\lambda)\lambda}{6!}x^6 - \ldots\right] \tag{7}$$

$$y_2 = a_1\left[x + \frac{(2-\lambda)}{3!}x^3 + \frac{(6-\lambda)(2-\lambda)}{5!}x^5 + \ldots\right]. \tag{8}$$

If $\lambda \equiv 4n = 0, 4, 8, \ldots$, then y_1 terminates with the POWER x^λ, and y_1 (normalized so that the COEFFICIENT of x^n is 2^n) is the regular solution to the equation, known as the HERMITE POLYNOMIAL. If $\lambda \equiv 4n+2 = 2, 6, 10, \ldots$, then y_2 terminates with the POWER x^λ, and y_2 (normalized so that the COEFFICIENT of x^n is 2^n) is the regular solution to the equation, known as the HERMITE POLYNOMIAL.

If $\lambda = 0$, then Hermite's differential equation becomes

$$y'' - 2xy' = 0, \tag{9}$$

which is of the form $P_2(x)y'' + P_1(x)y' = 0$ and so has solution

$$\begin{aligned} y &= c_1\int \frac{dx}{\exp\left(\int \frac{P_1}{P_2}\,dx\right)} + c_2 \\ &= c_1\int \frac{dx}{\exp\int -2x\,dx} + c_2 \\ &= c_1\int \frac{dx}{e^{-x^2}} + c_2 = c_1\int e^{x^2}\,dx + c_2. \end{aligned} \tag{10}$$

Hermite-Gauss Quadrature

Also called HERMITE QUADRATURE. A GAUSSIAN QUADRATURE over the interval $(-\infty, \infty)$ with WEIGHTING FUNCTION $W(x) = e^{-x^2}$. The ABSCISSAS for quadrature order n are given by the roots of the HERMITE POLYNOMIALS $H_n(x)$, which occur symmetrically about 0. The WEIGHTS are

$$w_i = -\frac{A_{n+1}\gamma_n}{A_n H_n'(x_i)H_{n+1}(x_i)} = \frac{A_n}{A_{n-1}}\frac{\gamma_{n-1}}{H_{n-1}(x_i)H_n'(x_i)}, \tag{1}$$

where A_n is the COEFFICIENT of x^n in $H_n(x)$. For HERMITE POLYNOMIALS,

$$A_n = 2^n, \tag{2}$$

so

$$\frac{A_{n+1}}{A_n} = 2. \tag{3}$$

Additionally,

$$\gamma_n = \sqrt{\pi}\,2^n n!, \tag{4}$$

so

$$\begin{aligned} w_i &= -\frac{2^{n+1}n!\sqrt{\pi}}{H_{n+1}(x_i)H_n'(x_i)} \\ &= \frac{2^n(n-1)!\sqrt{\pi}}{H_{n-1}(x_i)H_n'(x_i)}. \end{aligned} \tag{5}$$

Using the RECURRENCE RELATION

$$H_n'(x) = 2nH_{n-1}(x) = 2xH_n(x) - H_{n+1}(x) \tag{6}$$

yields

$$H_n'(x_i) = 2nH_{n-1}(x_i) = -H_{n+1}(x_i) \tag{7}$$

and gives

$$w_i = \frac{2^{n+1}n!\sqrt{\pi}}{[H_n'(x_i)]^2} = \frac{2^{n+1}n!\sqrt{\pi}}{[H_{n+1}(x_i)]^2}. \tag{8}$$

The error term is

$$E = \frac{n!\sqrt{\pi}}{2^n(2n)!}f^{(2n)}(\xi). \tag{9}$$

Beyer (1987) gives a table of ABSCISSAS and weights up to n=12.

n	x_i	w_i
2	±0.707107	0.886227
3	0	1.18164
	±1.22474	0.295409
4	±0.524648	0.804914
	±1.65068	0.0813128
5	0	0.945309
	±0.958572	0.393619
	±2.02018	0.0199532

The ABSCISSAS and weights can be computed analytically for small n.

n	x_i	w_i
2	$\pm\frac{1}{2}\sqrt{2}$	$\frac{1}{2}\sqrt{\pi}$
3	0	$\frac{2}{3}\sqrt{\pi}$
	$\pm\frac{1}{2}\sqrt{6}$	$\frac{1}{6}\sqrt{\pi}$
4	$\pm\sqrt{\frac{3-\sqrt{6}}{2}}$	$\frac{\sqrt{\pi}}{4(3-\sqrt{6})}$
	$\pm\sqrt{\frac{3+\sqrt{6}}{2}}$	$\frac{\sqrt{\pi}}{4(3+\sqrt{6})}$

References

Beyer, W. H. *CRC Standard Mathematical Tables, 28th ed.* Boca Raton, FL: CRC Press, p. 464, 1987.

Hildebrand, F. B. *Introduction to Numerical Analysis.* New York: McGraw-Hill, pp. 327–330, 1956.

Hermite Interpolation

see HERMITE'S INTERPOLATING FUNDAMENTAL POLYNOMIAL

Hermite's Interpolating Fundamental Polynomial

Let $l(x)$ be an nth degree POLYNOMIAL with zeros at $x_1, \ldots, x_m$. Then the fundamental POLYNOMIALS are

$$h_\nu^{(1)}(x) = \left[1 - \frac{l''(x_\nu)}{l'(x_\nu)}\right][l_\nu(x)]^2 \tag{1}$$

and

$$h_\nu^{(2)}(x) = (x - x_\nu)[l_\nu(x)]^2. \tag{2}$$

They have the properties

$$h_\nu^{(1)}(x_\mu) = \delta_{\nu\mu} \tag{3}$$
$$h^{(1)\prime}{}_\nu(x_\mu) = 0 \tag{4}$$
$$h^{(2)}(x_\mu) = 0 \tag{5}$$
$$h^{(2)\prime}(x_\mu) = \delta_{\nu\mu}. \tag{6}$$

Now let $f_1, \ldots, f_n$ and $f_1', \ldots, f_\nu'$ be values. Then the expansion

$$W_n(x) = \sum_{\nu=1}^n f_\nu h_\nu^{(1)}(x) + \sum_{\nu=1}^n f_\nu' h^{(2)}(x) \tag{7}$$

gives the unique HERMITE'S INTERPOLATING FUNDAMENTAL POLYNOMIAL for which

$$W_n(x_\nu) = f_\nu \tag{8}$$
$$W_n'(x_\nu) = f_\nu'. \tag{9}$$

If $f_\nu' = 0$, these are called STEP POLYNOMIALS. The fundamental POLYNOMIALS satisfy

$$h_1(x) + \ldots + h_n(x) = 1 \tag{10}$$

and

$$\sum_{\nu=1}^n x_\nu h_\nu^{(1)}(x) + \sum_{\nu=1}^n h_\nu^{(2)}(x) = x. \tag{11}$$

Also,

$$\int_a^b h_\nu^{(1)}(x)\,d\alpha(x) = \lambda_\nu \tag{12}$$
$$\int_a^b h_\nu^{(1)}(x)\,d\alpha(x) = 0 \tag{13}$$
$$\int_a^b x h_\nu'(x)\,d\alpha(x) = 0 \tag{14}$$
$$\int_a^b h_\nu^{(2)}(x)\,d\alpha(x) = 0 \tag{15}$$
$$\int_a^b h^{(2)\prime}{}_\nu\,d\alpha(x) = \lambda_\nu \tag{16}$$
$$\int_a^b x h^{(2)\prime}{}_\nu(x)\,dx = \lambda_\nu x_\nu, \tag{17}$$

for $\nu = 1, \ldots, n$.

References

Hildebrand, F. B. *Introduction to Numerical Analysis.* New York: McGraw-Hill, pp. 314–319, 1956.

Szegő, G. *Orthogonal Polynomials, 4th ed.* Providence, RI: Amer. Math. Soc., pp. 330–332, 1975.

Hermite-Lindemann Theorem

The expression

$$A_1 e^{\alpha_1} + A_2 e^{\alpha_2} + A_3 e^{\alpha_3} + \ldots,$$

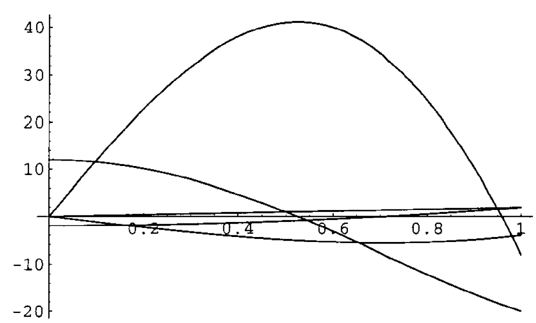
in which the COEFFICIENTS A_i differ from zero and in which the exponents α_i are ALGEBRAIC NUMBERS differing from each other, cannot equal zero.

see also ALGEBRAIC NUMBER, CONSTANT PROBLEM, INTEGER RELATION, LINDEMANN-WEIERSTRAß THEOREM

References

Dörrie, H. "The Hermite-Lindemann Transcendence Theorem." §26 in *100 Great Problems of Elementary Mathematics: Their History and Solutions.* New York: Dover, pp. 128–137, 1965.

Hermite Polynomial



A set of ORTHOGONAL POLYNOMIALS. The Hermite polynomials $H_n(x)$ are illustrated above for $x \in [0,1]$ and $n = 1, 2, \ldots, 5$.

The GENERATING FUNCTION for Hermite polynomials is

$$\exp(2xt - t^2) \equiv \sum_{n=0}^{\infty} \frac{H_n(x)t^n}{n!}. \tag{1}$$

Using a TAYLOR SERIES shows that,

$$\begin{aligned} H_n(x) &= \left[\left(\frac{\partial}{\partial t}\right)^n \exp(2xt - t^2)\right]_{t=0} \\ &= \left[e^{x^2}\left(\frac{\partial}{\partial t}\right)^n e^{-(x-t)^2}\right]_{t=0}. \end{aligned} \tag{2}$$

Since $\partial f(x-t)/\partial t = -\partial f(x-t)/\partial x$,

$$\begin{aligned} H_n(x) &= (-1)^n e^{x^2}\left[\left(\frac{\partial}{\partial x}\right)^n e^{-(x-t)^2}\right]_{t=0} \\ &= (-1)^n e^{x^2}\frac{d^n}{dx^n}e^{-x^2}. \end{aligned} \tag{3}$$

Now define operators

$$\tilde{O}_1 \equiv -e^{x^2}\frac{d}{dx}e^{-x^2} \tag{4}$$

$$\tilde{O}_2 \equiv e^{x^2/2}\left(x - \frac{d}{dx}\right)e^{-x^2/2}. \tag{5}$$

It follows that

$$\tilde{O}_1 f = -e^{x^2}\frac{d}{dx}[fe^{-x^2}] = 2xf - \frac{df}{dx} \tag{6}$$

$$\begin{aligned} \tilde{O}_2 f &= e^{x^2/2}\left(x - \frac{d}{dx}\right)[fe^{-x^2/2}] \\ &= xf + xf - \frac{df}{dx} = 2xf - \frac{df}{dx}, \end{aligned} \tag{7}$$

so

$$\tilde{O}_1 = \tilde{O}_2, \tag{8}$$

and

$$-e^{x^2}\frac{d}{dx}e^{-x^2} = e^{x^2/2}\left(x - \frac{d}{dx}\right)e^{-x^2/2}, \tag{9}$$

which means the following definitions are equivalent:

$$\exp(2xt - t^2) \equiv \sum_{n=0}^{\infty}\frac{H_n(x)t^n}{n!} \tag{10}$$

$$H_n(x) \equiv (-1)^n e^{x^2}\frac{d^n}{dx^n}e^{-x^2} \tag{11}$$

$$H_n(x) \equiv e^{x^2/2}\left(x - \frac{d}{dx}\right)ne^{-x^2/2}. \tag{12}$$

The Hermite POLYNOMIALS are related to the derivative of the ERROR FUNCTION by

$$H_n(z) = (-1)^2\frac{\sqrt{\pi}}{2}e^{z^2}\frac{d^{n+1}}{dz^{n+1}}\operatorname{erf}(z). \tag{13}$$

They have a contour integral representation

$$H_n(x) = \frac{n!}{2\pi i}\int e^{-t^2+2tx}t^{-n-1}\,dt. \tag{14}$$

They are orthogonal in the range $(-\infty, \infty)$ with respect to the WEIGHTING FUNCTION e^{-x^2}

$$\int_{-\infty}^{\infty} H_n(x)H_m(x)e^{-x^2}\,dx = \delta_{mn}2^n n!\sqrt{\pi}. \tag{15}$$

Define the associated functions

$$u_n(x) \equiv \sqrt{\frac{a}{\pi^{1/2}n!2^n}}\,H_n(ax)e^{-a^2x^2/2}. \tag{16}$$

These obey the orthogonality conditions

$$\int_{-\infty}^{\infty} u_n(x)\frac{du_m}{dx}\,dx = \begin{cases} a\sqrt{\frac{n+1}{2}} & m = n+1 \\ -a\sqrt{\frac{n}{2}} & m = n-1 \\ 0 & \text{otherwise} \end{cases} \tag{17}$$

$$\int_{-\infty}^{\infty} u_m(x)u_n(x)\,dx = \delta_{mn} \tag{18}$$

$$\int_{-\infty}^{\infty} u_m(x)xu_n(x)\,dx = \begin{cases} \frac{1}{a}\sqrt{\frac{n+!}{2}} & m = n+1 \\ \frac{1}{a}\sqrt{\frac{n}{2}} & m = n-1 \\ 0 & \text{otherwise} \end{cases} \tag{19}$$

$$\int_{-\infty}^{\infty} u_m(x)x^2u_n(x)\,dx = \begin{cases} \frac{2n+1}{2a^2} & m = n \\ \frac{\sqrt{(n+1)(n+2)}}{2a^2} & m = n+2 \\ 0 & m \neq n \neq n \pm 2 \end{cases} \tag{20}$$

$$\int_{-\infty}^{\infty} e^{-x^2}H_\alpha H_\beta H_\gamma\,dx = \sqrt{\pi}\frac{2^s\alpha!\beta!\gamma!}{(s-\alpha)!(s-\beta)!(s-\gamma)!}, \tag{21}$$

if $\alpha + \beta + \gamma = 2s$ is EVEN and $s \geq \alpha$, $s \geq \beta$, and $s \geq \gamma$. Otherwise, the last integral is 0 (Szegő 1975, p. 390).

They also satisfy the RECURRENCE RELATIONS

$$H_{n+1} = 2xH_n(x) - 2nH_{n-1}(x) \tag{22}$$

$$H_n'(x) = 2nH_{n-1}(x). \tag{23}$$

The DISCRIMINANT is

$$D_n = 2^{3n(n-1)/2}\prod_{\nu=1}^{n}\nu^\nu \tag{24}$$

(Szegő 1975, p. 143).

An interesting identity is

$$\sum_{\nu=0}^{n}\binom{n}{\nu}H_\nu(x)H_{n-\nu}(y) = 2^{n/2}H_n[2^{-1/2}(x+y)]. \tag{25}$$

The first few POLYNOMIALS are

$$\begin{aligned}
H_0(x) &= 1\\
H_1(x) &= 2x\\
H_2(x) &= 4x^2 - 2\\
H_3(x) &= 8x^3 - 12x\\
H_4(x) &= 16x^4 - 48x^2 + 12\\
H_5(x) &= 32x^5 - 160x^3 + 120x\\
H_6(x) &= 64x^6 - 480x^4 + 720x^2 - 120\\
H_7(x) &= 128x^7 - 1344x^5 + 3360x^3 - 1680x\\
H_8(x) &= 256x^8 - 3594x^6 + 13440x^4 - 13440x^2\\
&\quad + 160\\
H_9(x) &= 512x^9 - 9216x^7 + 48384x^5 - 80640x^3\\
&\quad + 30240x\\
H_{10}(x) &= 1024x^{10} - 23040x^8 + 161280x^6 - 403200x^4\\
&\quad + 302400x^2 - 30240.
\end{aligned}$$

A class of generalized Hermite POLYNOMIALS $\gamma_n^m(x)$ satisfying

$$e^{mxt-t^m} = \sum_{n=0}^{\infty} \gamma_n^m(x) t^n \tag{26}$$

was studied by Subramanyan (1990). A class of related POLYNOMIALS defined by

$$h_{n,m} = \gamma_n^m\left(\frac{2x}{m}\right) \tag{27}$$

and with GENERATING FUNCTION

$$e^{2xt-t^m} = \sum_{n=0}^{\infty} h_{n,m}(x) t^n \tag{28}$$

was studied by Djordjević (1996). They satisfy

$$H_n(x) = n! h_{n,2}(x). \tag{29}$$

A modified version of the HERMITE POLYNOMIAL is sometimes defined by

$$\mathrm{He}_n(x) \equiv H_n\left(\frac{x}{\sqrt{2}}\right). \tag{30}$$

see also MEHLER'S HERMITE POLYNOMIAL FORMULA, WEBER FUNCTIONS

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Hermite Quadrature

see HERMITE-GAUSS QUADRATURE

Hermite's Theorem

e is TRANSCENDENTAL.

Hermitian Form

A combination of variables x and y given by

$$axx^* + bxy^* + b^*x^*y + cyy^*,$$

where x^* and y^* are COMPLEX CONJUGATES.

Hermitian Matrix

If a MATRIX is SELF-ADJOINT, it is said to be a Hermitian matrix. Therefore, a Hermitian MATRIX is defined as one for which

$$\mathsf{A} = \mathsf{A}^\dagger, \tag{1}$$

where † denotes the ADJOINT MATRIX. Hermitian MATRICES have REAL EIGENVALUES with ORTHOGONAL EIGENVECTORS. For REAL MATRICES, Hermitian is the same as symmetrical. Any MATRIX C which is not Hermitian can be expressed as the sum of two Hermitian matrices

$$\mathsf{C} = \tfrac{1}{2}(\mathsf{C} + \mathsf{C}^\dagger) + \tfrac{1}{2}(\mathsf{C} - \mathsf{C}^\dagger). \tag{2}$$

Let U be a UNITARY MATRIX and A be a Hermitian matrix. Then the ADJOINT MATRIX of a SIMILARITY TRANSFORMATION is

$$\begin{aligned}(\mathsf{UAU}^{-1})^\dagger &= [(\mathsf{UA})(\mathsf{U}^{-1})]^\dagger = (\mathsf{U}^{-1})^\dagger(\mathsf{UA})^\dagger\\ &= (\mathsf{U}^\dagger)^\dagger(\mathsf{A}^\dagger\mathsf{U}^\dagger) = \mathsf{UAU}^\dagger = \mathsf{UAU}^{-1}.\end{aligned} \tag{3}$$

The specific matrix

$$\mathsf{H}(x,y,z) = \begin{bmatrix} z & x+iy \\ x-iy & -z \end{bmatrix} = x\mathsf{P}_1 + y\mathsf{P}_2 + z\mathsf{P}_3, \quad (4)$$

where P_i are PAULI SPIN MATRICES, is sometimes called "the" Hermitian matrix.

see also ADJOINT MATRIX, HERMITIAN OPERATOR, PAULI SPIN MATRICES

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Hermitian Operator

A Hermitian OPERATOR $\tilde{L}$ is one which satisfies

$$\int_a^b v^* \tilde{L} u \, dx = \int_a^b u \tilde{L} v^* \, dx. \quad (1)$$

As shown in STURM-LIOUVILLE THEORY, if $\tilde{L}$ is SELF-ADJOINT and satisfies the boundary conditions

$$[v^* p u']_{x=a} = [v^* p u']_{x=b}, \quad (2)$$

then it is automatically Hermitian. Hermitian operators have REAL EIGENVALUES, ORTHOGONAL EIGENFUNCTIONS, and the corresponding EIGENFUNCTIONS form a COMPLETE set when $\tilde{L}$ is second-order and linear. In order to prove that EIGENVALUES must be REAL and EIGENFUNCTIONS ORTHOGONAL, consider

$$\tilde{L} u_i + \lambda_i w u_i = 0. \quad (3)$$

Assume there is a second EIGENVALUE λ_j such that

$$\tilde{L} u_j + \lambda_j w u_j = 0 \quad (4)$$

$$\tilde{L} u_j^* + \lambda_j^* w u_j^* = 0. \quad (5)$$

Now multiply (3) by u_j^* and (5) by u_i

$$u_j^* \tilde{L} u_i + u_j^* \lambda_i w u_i = 0 \quad (6)$$

$$u_i \tilde{L} u_j^* + u_i \lambda_j^* w u_j^* = 0 \quad (7)$$

$$u_j^* \tilde{L} u_i - u_i \tilde{L} u_j^* = (\lambda_j^* - \lambda_i) w u_i u_j^*. \quad (8)$$

Now integrate

$$\int_a^b u_j^* \tilde{L} u_i - \int_a^b u_i \tilde{L} u_j^* = (\lambda_j^* - \lambda_i) \int_a^b w u_i u_j^*. \quad (9)$$

But because $\tilde{L}$ is Hermitian, the left side vanishes.

$$(\lambda_j^* - \lambda_i) \int_a^b w u_i u_j^* = 0. \quad (10)$$

If EIGENVALUES λ_i and λ_j are not degenerate, then $\int_a^b w u_i u_j^* = 0$, so the EIGENFUNCTIONS are ORTHOGONAL. If the EIGENVALUES are degenerate, the EIGENFUNCTIONS are not necessarily orthogonal. Now take $i = j$.

$$(\lambda_i^* - \lambda_i) \int_a^b w u_i u_i^* = 0. \quad (11)$$

The integral cannot vanish unless $u_i = 0$, so we have $\lambda_i^* = \lambda_i$ and the EIGENVALUES are real.

For a Hermitian operator $\tilde{O}$,

$$\langle \phi | \tilde{O} \psi \rangle = \langle \phi | \tilde{O} \psi \rangle^* = \langle \tilde{O} \phi | \psi \rangle. \quad (12)$$

In integral notation,

$$\int (\tilde{A}\phi)^* \psi \, dx = \int \phi^* \tilde{A} \psi \, dx. \quad (13)$$

Given Hermitian operators $\tilde{A}$ and $\tilde{B}$,

$$\langle \phi | \tilde{A}\tilde{B} \psi \rangle = \langle \tilde{A}\phi | \tilde{B}\psi \rangle = \langle \tilde{B}\tilde{A}\phi | \psi \rangle = \langle \phi | \tilde{B}\tilde{A}\psi \rangle^*. \quad (14)$$

Because, for a Hermitian operator $\tilde{A}$ with EIGENVALUE a,

$$\langle \psi | \tilde{A} \psi \rangle = \langle \tilde{A} \psi | \psi \rangle \quad (15)$$

$$a \langle \psi | \psi \rangle = a^* \langle \psi | \psi \rangle. \quad (16)$$

Therefore, either $\langle \psi | \psi \rangle = 0$ or $a = a^*$. But $\langle \psi | \psi \rangle = 0$ IFF $\psi = 0$, so

$$\langle \psi | \psi \rangle \neq 0, \quad (17)$$

for a nontrivial EIGENFUNCTION. This means that $a = a^*$, namely that Hermitian operators produce REAL expectation values. Every observable must therefore have a corresponding Hermitian operator. Furthermore,

$$\langle \psi_n | \tilde{A} \psi_m \rangle = \langle \tilde{A} \psi_n | \psi_m \rangle \quad (18)$$

$$a_m \langle \psi_n | \psi_m \rangle = a_n^* \langle \psi_n | \psi_m \rangle = a_n \langle \psi_n | \psi_m \rangle, \quad (19)$$

since $a_n = a_n^*$. Then

$$(a_m - a_n) \langle \psi_n | \psi_m \rangle = 0 \quad (20)$$

For $a_m \neq a_n$ (i.e., $\psi_n \neq \psi_m$),

$$\langle \psi_n | \psi_m \rangle = 0. \quad (21)$$

For $a_m = a_n$ (i.e., $\psi_n = \psi_m$),

$$\langle \psi_n | \psi_m \rangle = \langle \psi_n | \psi_n \rangle \equiv 1. \quad (22)$$

Therefore,

$$\langle \psi_n | \psi_m \rangle = \delta_{nm}, \quad (23)$$

so the basis of EIGENFUNCTIONS corresponding to a Hermitian operator are ORTHONORMAL. Given two Hermitian operators $\tilde{A}$ and $\tilde{B}$,

$$(\tilde{A}\tilde{B})^\dagger = \tilde{B}^\dagger \tilde{A}^\dagger = \tilde{B}\tilde{A} = \tilde{A}\tilde{B} + [\tilde{B}, \tilde{A}], \tag{24}$$

the operator $\tilde{A}\tilde{B}$ equals $(\tilde{A}\tilde{B})^\dagger$, and is therefore Hermitian, only if

$$[\tilde{B}, \tilde{A}] = 0. \tag{25}$$

Given an arbitrary operator $\tilde{A}$,

$$\begin{aligned} \langle \psi_1 | (\tilde{A} + \tilde{A}^\dagger)\psi_2 \rangle &= \langle (\tilde{A}^\dagger + \tilde{A})\psi_1 | \psi_2 \rangle \\ &= \langle (\tilde{A} + \tilde{A}^\dagger)\psi_1 | \psi_2 \rangle, \end{aligned} \tag{26}$$

so $\tilde{A} + \tilde{A}^\dagger$ is Hermitian.

$$\begin{aligned} \langle \psi_1 | i(\tilde{A} - \tilde{A}^\dagger)\psi_2 \rangle &= \langle -i(\tilde{A}^\dagger - \tilde{A})\psi_1 | \psi_2 \rangle \\ &= \langle i(\tilde{A} - \tilde{A}^\dagger)\psi_1 | \psi_2 \rangle, \end{aligned} \tag{27}$$

so $i(\tilde{A} - \tilde{A}^\dagger)$ is Hermitian. Similarly,

$$\langle \psi_1 | (\tilde{A}\tilde{A}^\dagger)\psi_2 \rangle = \langle \tilde{A}^\dagger \psi_1 | \tilde{A} \dagger \psi_2 \rangle = \langle (\tilde{A}\tilde{A}^\dagger)\psi_1 | \psi_2 \rangle, \tag{28}$$

so $\tilde{A}\tilde{A}^\dagger$ is Hermitian.

Define the Hermitian conjugate operator $\tilde{A}^\dagger$ by

$$\langle \tilde{A}\psi | \psi \rangle \equiv \langle \psi | \tilde{A}^\dagger \psi \rangle. \tag{29}$$

For a Hermitian operator, $\tilde{A} = \tilde{A}^\dagger$. Furthermore, given two Hermitian operators $\tilde{A}$ and $\tilde{B}$,

$$\begin{aligned} \langle \psi_2 | (\tilde{A}\tilde{B})^\dagger \psi_1 \rangle &= \langle (\tilde{A}\tilde{B})\psi_2 | \psi_1 \rangle = \langle \tilde{B}\psi_2 | \tilde{A}^\dagger \psi_1 \rangle \\ &= \langle \psi_2 | \tilde{B}^\dagger \tilde{A}^\dagger \psi_1 \rangle, \end{aligned} \tag{30}$$

so

$$(\tilde{A}\tilde{B})^\dagger = \tilde{B}^\dagger \tilde{A}^\dagger. \tag{31}$$

By further iterations, this can be generalized to

$$(\tilde{A}\tilde{B}\cdots\tilde{Z})^\dagger = \tilde{Z}^\dagger \cdots \tilde{B}^\dagger \tilde{A}^\dagger. \tag{32}$$

see also ADJOINT OPERATOR, HERMITIAN MATRIX, SELF-ADJOINT OPERATOR, STURM-LIOUVILLE THEORY

References

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Heron's Formula

Gives the AREA of a TRIANGLE in terms of the lengths of the sides a, b, and c and the SEMIPERIMETER

$$s \equiv \tfrac{1}{2}(a + b + c). \tag{1}$$

Heron's formula then states

$$\Delta = \sqrt{s(s-a)(s-b)(s-c)}. \tag{2}$$

Expressing the side lengths a, b, and c in terms of the radii a', b', and c' of the mutually tangent circles centered on the TRIANGLE vertices (which define the SODDY CIRCLES),

$$a = b' + c' \tag{3}$$

$$b = a' + c' \tag{4}$$

$$c = a' + b', \tag{5}$$

gives the particularly pretty form

$$\Delta = \sqrt{a'b'c'(a' + b' + c')} \tag{6}$$

The proof of this fact was discovered by Heron (ca. 100 BC–100 AD), although it was already known to Archimedes prior to 212 BC (Kline 1972). Heron's proof (Dunham 1990) is ingenious but extremely convoluted, bringing together a sequence of apparently unrelated geometric identities and relying on the properties of CYCLIC QUADRILATERALS and RIGHT TRIANGLES.

Heron's proof can be found in Proposition 1.8 of his work *Metrica.* This manuscript had been lost for centuries until a fragment was discovered in 1894 and a complete copy in 1896 (Dunham 1990, p. 118). More recently, writings of the Arab scholar Abu'l Raihan Muhammed al-Biruni have credited the formula to Heron's predecessor Archimedes (Dunham 1990, p. 127).

A much more accessible algebraic proof proceeds from the LAW OF COSINES,

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc}. \tag{7}$$

Then

$$\sin A = \frac{\sqrt{-a^4 - b^4 - c^4 + 2b^2c^2 + 2c^2a^2 + 2a^2b^2}}{2bc}, \tag{8}$$

giving

$$\begin{aligned} \Delta &= \tfrac{1}{2} bc \sin A & (9) \\ &= \tfrac{1}{4}\sqrt{-a^4 - b^4 - c^4 + 2b^2c^2 + 2c^2a^2 + 2a^2b^2} & (10) \\ &= \tfrac{1}{4}[(a+b+c)(-a+b+c)(a-b+c)(a+b-c)]^{1/2} & (11) \\ &= \sqrt{s(s-a)(s-b)(s-c)} & (12) \end{aligned}$$

(Coxeter 1969). Heron's formula contains the PYTHAGOREAN THEOREM.

see also BRAHMAGUPTA'S FORMULA, BRETSCHNEIDER'S FORMULA, HERONIAN TETRAHEDRON, HERONIAN TRIANGLE, SODDY CIRCLES, SSS THEOREM, TRIANGLE

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Heron Triangle

see HERONIAN TRIANGLE

Heronian Tetrahedron

A TETRAHEDRON with RATIONAL sides, FACE AREAS, and VOLUME. The smallest examples have pairs of opposite sides (148, 195, 203), (533, 875, 888), (1183, 1479, 1804), (2175, 2296, 2431), (1825, 2748, 2873), (2180, 2639, 3111), (1887, 5215, 5512), (6409, 6625, 8484), and (8619, 10136, 11275).

see also HERON'S FORMULA, HERONIAN TRIANGLE

References
Guy, R. K. "Simplexes with Rational Contents." §D22 in *Unsolved Problems in Number Theory, 2nd ed.* New York: Springer-Verlag, pp. 190–192, 1994.

Heronian Triangle

A TRIANGLE with RATIONAL side lengths and RATIONAL AREA. Brahmagupta gave a parametric solution for *integer* Heronian triangles (the three side lengths and area can be multiplied by their LEAST COMMON MULTIPLE to make them all INTEGERS): side lengths $c(a^2+b^2)$, $b(a^2+c^2)$, and $(b+c)(a^2-bc)$, giving SEMIPERIMETER

$$s = a^2(b+c)$$

and AREA

$$\Delta = abc(a+b)(a^2-bc).$$

The first few integer Hernonian triangles, sorted by increasing maximal side lengths, are (3, 4, 5), (6, 8, 10), (5, 12, 13), (9, 12, 15), (4, 13, 15), (13, 14, 15), (9, 10, 17), ... (Sloane's A046128, A046129, and A046130), having areas 6, 24, 30, 54, 24, 84, 36, ... (Sloane's A046131).

Schubert (1905) claimed that Heronian triangles with two rational MEDIANS do not exist (Dickson 1952). This was shown to be incorrect by Buchholz and Rathbun (1997), who discovered six such triangles.

see also HERON'S FORMULA, MEDIAN (TRIANGLE), PYTHAGOREAN TRIPLE, TRIANGLE

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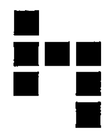
Herschel



A HEPTOMINO shaped like the astronomical symbol for Uranus (which was discovered by William Herschel).

Herschfeld's Convergence Theorem

For real, NONNEGATIVE terms x_n and REAL p with $0 < p < 1$, the expression

$$\lim_{k\to\infty} x_0 + (x_1 + (x_2 + (\ldots + (x_k)^p)^p)^p)^p$$

converges IFF $(x_n)^{p^n}$ is bounded.

see also CONTINUED SQUARE ROOT

References
Herschfeld, A. "On Infinite Radicals." *Amer. Math. Monthly* **42**, 419–429, 1935.
Jones, D. J. "Continued Powers and a Sufficient Condition for Their Convergence." *Math. Mag.* **68**, 387–392, 1995.

Hesse's Theorem

If two pairs of opposite VERTICES of a COMPLETE QUADRILATERAL are pairs of CONJUGATE POINTS, then the third pair of opposite VERTICES is likewise a pair of CONJUGATE POINTS.

Hessenberg Matrix

A matrix of the form

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1(n-1)} & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2(n-1)} & a_{2n} \\ 0 & a_{32} & a_{33} & \cdots & a_{3(n-1)} & a_{3n} \\ 0 & 0 & a_{43} & \cdots & a_{4(n-1)} & a_{4n} \\ 0 & 0 & 0 & \cdots & a_{5(n-1)} & a_{5n} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & a_{(n-1)(n-1)} & a_{(n-1)n} \\ 0 & 0 & 0 & 0 & a_{n(n-1)} & a_{nn} \end{bmatrix}.$$

References
Press, W. H.; Flannery, B. P.; Teukolsky, S. A.; and Vetterling, W. T. "Reduction of a General Matrix to Hessenberg Form." §11.5 in *Numerical Recipes in FORTRAN: The Art of Scientific Computing, 2nd ed.* Cambridge, England: Cambridge University Press, pp. 476–480, 1992.

Hessian Covariant

$$H \equiv |aa'a''|a_{x^{n-2}}a'_{x^{n-2}}a''_{x^{n-2}} = 0.$$

The nonsingular inflections of a curve are its nonsingular intersections with the Hessian.

References

Coolidge, J. L. *A Treatise on Algebraic Plane Curves.* New York: Dover, pp. 79, 95–98, and 151–161, 1959.

Hessian Determinant

The DETERMINANT

$$Hf(x,y) \equiv \begin{vmatrix} \frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial x \partial y} \\ \frac{\partial^2 f}{\partial y \partial x} & \frac{\partial^2 f}{\partial y^2} \end{vmatrix}$$

appearing in the SECOND DERIVATIVE TEST as $D \equiv Hf(x,y)$.

see also SECOND DERIVATIVE TEST

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Heteroclinic Point

If intersecting stable and unstable MANIFOLDS (SEPARATRICES) emanate from FIXED POINTS of different families, they are called heteroclinic points.

see also HOMOCLINIC POINT

Heterogeneous Numbers

Two numbers are heterogeneous if their PRIME factors are distinct.

see also DISTINCT PRIME FACTORS, HOMOGENEOUS NUMBERS

References

Le Lionnais, F. *Les nombres remarquables.* Paris: Hermann, p. 146, 1983.

Heterological Paradox

see GRELLING'S PARADOX

Heteroscedastic

A set of STATISTICAL DISTRIBUTIONS having different VARIANCES.

see also HOMOSCEDASTIC, VARIANCE

Heterosquare

9	8	7
2	1	6
3	4	5

1	2	3	4
5	6	7	8
9	10	11	12
13	14	16	15

A heterosquare is an $n \times n$ ARRAY of the integers from 1 to n^2 such that the rows, columns, and diagonals have different sums. (By contrast, in a MAGIC SQUARE, they have the *same* sum.) There are no heterosquares of order two, but heterosquares of every ODD order exist. They can be constructed by placing consecutive INTEGERS in a SPIRAL pattern (Fults 1974, Madachy 1979).

An ANTIMAGIC SQUARE is a special case of a heterosquare for which the sums of rows, columns, and main diagonals form a SEQUENCE of consecutive integers.

see also ANTIMAGIC SQUARE, MAGIC SQUARE, TALISMAN SQUARE

References

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Madachy, J. S. *Madachy's Mathematical Recreations.* New York: Dover, pp. 101–103, 1979.

Weisstein, E. W. "Magic Squares." `http://www.astro.virginia.edu/~eww6n/math/notebooks/MagicSquares.m`.

Heuman Lambda Function

$$\Lambda_0(\phi|m) \equiv \frac{F(\phi|1-m)}{K(1-m)} + \frac{2}{\pi}K(m)Z(\phi|1-m),$$

where ϕ is the AMPLITUDE, m is the PARAMETER, Z is the JACOBI ZETA FUNCTION, and $F(\phi|m')$ and $K(m)$ are incomplete and complete ELLIPTIC INTEGRALS OF THE FIRST KIND.

see also ELLIPTIC INTEGRAL OF THE FIRST KIND, JACOBI ZETA FUNCTION

References

Abramowitz, M. and Stegun, C. A. (Eds.). *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables, 9th printing.* New York: Dover, p. 595, 1972.

Heun's Differential Equation

$$\frac{d^2w}{dx^2} + \left(\frac{\gamma}{x} + \frac{\delta}{x-1} + \frac{\epsilon}{x-a}\right)\frac{dw}{dx} + \frac{\alpha\beta x - q}{x(x-1)(x-a)}w = 0,$$

where

$$\alpha + \beta - \gamma - \delta - \epsilon + 1 = 0.$$

References

Erdelyi, A.; Magnus, W.; Oberhettinger, F.; and Tricomi, F. G. *Higher Transcendental Functions, Vol. 3.* Krieger, pp. 57–62, 1981.

Whittaker, E. T. and Watson, G. N. *A Course in Modern Analysis, 4th ed.* Cambridge, England: Cambridge University Press, p. 576, 1990.

Heuristic

(1) Based on or involving trial and error. (2) Convincing without being rigorous.

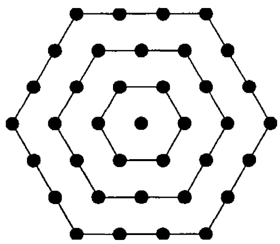
Hex Game

A two-player GAME. There is a winning strategy for the first player if there is an even number of cells on each side; otherwise, there is a winning strategy for the second player.

References

Gardner, M. Ch. 8 in *The Scientific American Book of Mathematical Puzzles & Diversions.* New York, NY: Simon and Schuster, 1959.

Hex Number



The CENTERED HEXAGONAL NUMBER given by

$$H_n = 1 + 6T_n = 2H_{n-1} - H_{n-2} + 6 = 3n^2 - 3n + 1,$$

where T_n is the nth TRIANGULAR NUMBER. The first few hex numbers are 1, 7, 19, 37, 61, 91, 127, 169, ... (Sloane's A003215). The GENERATING FUNCTION of the hex numbers is

$$\frac{x(x^2 + 4x + 1)}{(1-x)^3} = x + 7x^2 + 19x^3 + 37x^4 + \ldots.$$

The first TRIANGULAR hex numbers are 1 and 91, and the first few SQUARE ones are 1, 169, 32761, 6355441, ... (Sloane's A006051). SQUARE hex numbers are obtained by solving the DIOPHANTINE EQUATION

$$3x^2 + 1 = y^2.$$

The only hex number which is SQUARE and TRIANGULAR is 1. There are no CUBIC hex numbers.

see also MAGIC HEXAGON, CENTERED SQUARE NUMBER, STAR NUMBER, TALISMAN HEXAGON

References

Conway, J. H. and Guy, R. K. *The Book of Numbers.* New York: Springer-Verlag, p. 41, 1996.

Gardner, M. "Hexes and Stars." Ch. 2 in *Time Travel and Other Mathematical Bewilderments.* New York: W. H. Freeman, 1988.

Hindin, H. "Stars, Hexes, Triangular Numbers, and Pythagorean Triples." *J. Recr. Math.* **16**, 191–193, 1983–1984.

Sloane, N. J. A. Sequences A003215/M4362 and A006051/M5409 in "An On-Line Version of the Encyclopedia of Integer Sequences."

Hex (Polyhex)

see POLYHEX

Hex Pyramidal Number

A FIGURATE NUMBER which is equal to the CUBIC NUMBER n^3. The first few are 1, 8, 27, 64, ... (Sloane's A000578).

References

Conway, J. H. and Guy, R. K. *The Book of Numbers.* New York: Springer-Verlag, pp. 42–44, 1996.

Sloane, N. J. A. Sequence A000578/M4499 in "An On-Line Version of the Encyclopedia of Integer Sequences."

Hexa

see POLYHEX

Hexabolo

A 6-POLYABOLO.

Hexacontagon

A 60-sided POLYGON.

Hexacronic Icositetrahedron

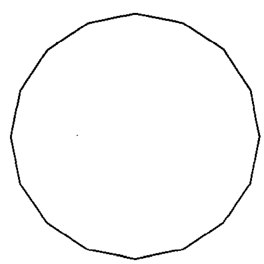
see GREAT HEXACRONIC ICOSITETRAHEDRON, SMALL HEXACRONIC ICOSITETRAHEDRON

Hexad

A SET of six.

see also MONAD, QUARTET, QUINTET, TETRAD, TRIAD

Hexadecagon



A 16-sided POLYGON, sometimes also called a HEXAKAIDECAGON.

see also POLYGON, REGULAR POLYGON, TRIGONOMETRY VALUES—$\pi/16$

Hexadecimal

The base 16 notational system for representing REAL NUMBERS. The digits used to represent numbers using hexadecimal NOTATION are 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, A, B, C, D, E, and F.

see also BASE (NUMBER), BINARY, DECIMAL, METADROME, OCTAL, QUATERNARY, TERNARY, VIGESIMAL

References

Weisstein, E. W. "Bases." `http://www.astro.virginia.edu/~eww6n/math/notebooks/Bases.m`.

Hexaflexagon

A FLEXAGON made by folding a strip into adjacent EQUILATERAL TRIANGLES. The number of states possible in a hexaflexagon is the CATALAN NUMBER $C_4 = 42$.

see also FLEXAGON, FLEXATUBE, TETRAFLEXAGON

References

Cundy, H. and Rollett, A. *Mathematical Models, 3rd ed.* Stradbroke, England: Tarquin Pub., pp. 205–207, 1989.

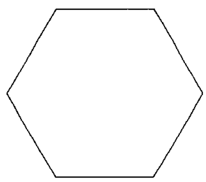
Gardner, M. Ch. 1 in *The Scientific American Book of Mathematical Puzzles & Diversions.* New York: Simon and Schuster, 1959.

Gardner, M. Ch. 2 in *The Second Scientific American Book of Mathematical Puzzles & Diversions: A New Selection.* New York: Simon and Schuster, 1961.

Maunsell, F. G. "The Flexagon and the Hexaflexagon." *Math. Gazette* **38**, 213–214, 1954.

Wheeler, R. F. "The Flexagon Family." *Math. Gaz.* **42**, 1–6, 1958.

Hexagon



A six-sided POLYGON. In proposition IV.15, Euclid showed how to inscribe a regular hexagon in a CIRCLE. The INRADIUS r, CIRCUMRADIUS R, and AREA A can be computed directly from the formulas for a general regular POLYGON with side length s and $n = 6$ sides,

$$r = \tfrac{1}{2}s\cot\left(\frac{\pi}{6}\right) = \tfrac{1}{2}\sqrt{3}\,s \tag{1}$$

$$R = \tfrac{1}{2}s\csc\left(\frac{\pi}{6}\right) = s \tag{2}$$

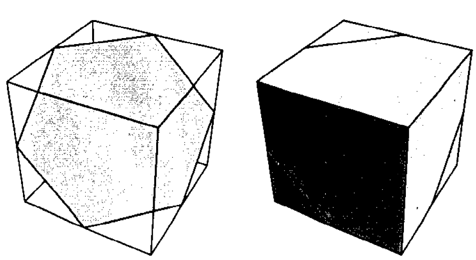
$$A = \tfrac{1}{4}ns^2\cot\left(\frac{\pi}{6}\right) = \tfrac{3}{2}\sqrt{3}\,s^2. \tag{3}$$

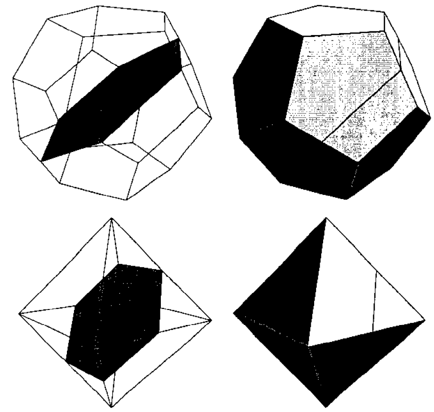
Therefore, for a regular hexagon,

$$\frac{R}{r} = \sec\left(\frac{\pi}{6}\right) = \frac{2}{\sqrt{3}}, \tag{4}$$

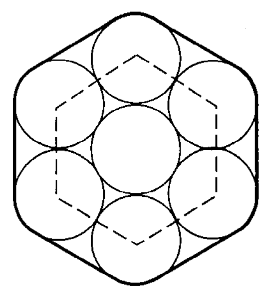
so

$$\frac{A_R}{A_r} = \left(\frac{R}{r}\right)^2 = \frac{4}{3}. \tag{5}$$





A PLANE PERPENDICULAR to a C_3 axis of a CUBE, DODECAHEDRON, or ICOSAHEDRON cuts the solid in a regular HEXAGONAL CROSS-SECTION (Holden 1991, pp. 22–23 and 27). For the CUBE, the PLANE passes through the MIDPOINTS of opposite sides (Steinhaus 1983, p. 170; Cundy and Rollett 1989, p. 157; Holden 1991, pp. 22–23). Since there are four such axes for the CUBE and OCTAHEDRON, there are four possibly hexagonal cross-sections. Since there are four such axes in each case, there are also four possibly hexagonal cross-sections.



Take seven CIRCLES and close-pack them together in a hexagonal arrangement. The PERIMETER obtained by wrapping a band around the CIRCLE then consists of six straight segments of length d (where d is the DIAMETER) and 6 arcs with total length 1/6 of a CIRCLE. The PERIMETER is therefore

$$p = (12 + 2\pi)r = 2(6 + \pi)r. \tag{6}$$

see also CUBE, CYCLIC HEXAGON, DISSECTION, DODECAHEDRON, GRAHAM'S BIGGEST LITTLE HEXAGON, HEXAGON POLYIAMOND, HEXAGRAM, MAGIC HEXAGON, OCTAHEDRON, PAPPUS'S HEXAGON THEOREM, PASCAL'S THEOREM, TALISMAN HEXAGON

References

Cundy, H. and Rollett, A. "Hexagonal Section of a Cube." §3.15.1 in *Mathematical Models, 3rd ed.* Stradbroke, England: Tarquin Pub., p. 157, 1989.

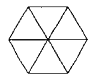
Dixon, R. *Mathographics.* New York: Dover, p. 16, 1991.

Holden, A. *Shapes, Space, and Symmetry.* New York: Dover, 1991.

Pappas, T. "Hexagons in Nature." *The Joy of Mathematics.* San Carlos, CA: Wide World Publ./Tetra, pp. 74–75, 1989.

Steinhaus, H. *Mathematical Snapshots, 3rd American ed.* New York: Oxford University Press, 1983.

Hexagon Polyiamond



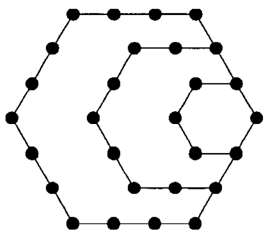
A 6-POLYIAMOND.

see also HEXAGON

References

Golomb, S. W. *Polyominoes: Puzzles, Patterns, Problems, and Packings, 2nd ed.* Princeton, NJ: Princeton University Press, p. 92, 1994.

Hexagonal Number



A FIGURATE NUMBER and 6-POLYGONAL NUMBER of the form $n(2n-1)$. The first few are 1, 6, 15, 28, 45, ... (Sloane's A000384). The GENERATING FUNCTION of the hexagonal numbers

$$\frac{x(3x+1)}{(1-x)^3} = x + 6x^2 + 15x^3 + 28x^4 + \ldots.$$

Every hexagonal number is a TRIANGULAR NUMBER since

$$r(2r-1) = \tfrac{1}{2}(2r-1)[(2r-1)+1].$$

In 1830, Legendre (1979) proved that every number larger than 1791 is a sum of four hexagonal numbers, and Duke and Schulze-Pillot (1990) improved this to three hexagonal numbers for every sufficiently large integer. The numbers 11 and 26 can only be represented as a sum using the maximum possible of six hexagonal numbers:

$$11 = 1+1+1+1+1+6$$
$$26 = 1+1+6+6+6+6.$$

see also FIGURATE NUMBER, HEX NUMBER, TRIANGULAR NUMBER

References

Duke, W. and Schulze-Pillot, R. "Representations of Integers by Positive Ternary Quadratic Forms and Equidistribution of Lattice Points on Ellipsoids." *Invent. Math.* **99**, 49–57, 1990.
Guy, R. K. "Sums of Squares." §C20 in *Unsolved Problems in Number Theory, 2nd ed.* New York: Springer-Verlag, pp. 136–138, 1994.
Legendre, A.-M. *Théorie des nombres, 4th ed., 2 vols.* Paris: A. Blanchard, 1979.
Sloane, N. J. A. Sequence A000384/M4108 in "An On-Line Version of the Encyclopedia of Integer Sequences."

Hexagonal Pyramidal Number

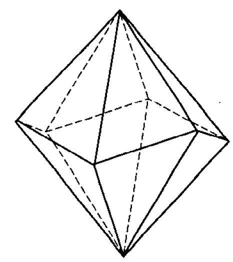
A PYRAMIDAL NUMBER of the form $n(n+1)(4n-1)/6$, The first few are 1, 7, 22, 50, 95, ... (Sloane's A002412). The GENERATING FUNCTION of the hexagonal pyramidal numbers is

$$\frac{x(3x+1)}{(x-1)^4} = x + 7x^2 + 22x^3 + 50x^4 + \ldots.$$

References

Sloane, N. J. A. Sequence A002412/M4374 in "An On-Line Version of the Encyclopedia of Integer Sequences."

Hexagonal Scalenohedron



An irregular DODECAHEDRON which is also a TRAPEZOHEDRON.

see also DODECAHEDRON, TRAPEZOHEDRON

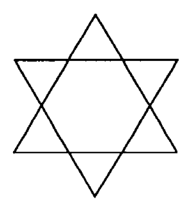
References

Cotton, F. A. *Chemical Applications of Group Theory, 3rd ed.* New York: Wiley, p. 63, 1990.

Hexagonal Tiling

see TILING

Hexagram



The STAR POLYGON $\{6,2\}$, also known as the STAR OF DAVID.

see also DISSECTION, PENTAGRAM, SOLOMON'S SEAL KNOT, STAR FIGURE, STAR OF LAKSHMI

Hexagrammum Mysticum Theorem

see PASCAL'S THEOREM

Hexahedron

A hexahedron is a six-sided POLYHEDRON. The regular hexahedron is the CUBE, although there are seven topologically different CONVEX hexahedra (Guy 1994).

see also CUBE

References

Guy, R. K. *Unsolved Problems in Number Theory, 2nd ed.* New York: Springer-Verlag, p. 189, 1994.

Hexahemioctahedron
The DUAL POLYHEDRON of the CUBOHEMIOCTAHEDRON.

Hexakaidecagon
see HEXADECAGON

Hexakis Icosahedron
see DISDYAKIS TRIACONTAHEDRON

Hexakis Octahedron
see DISDYAKIS DODECAHEDRON

Hexlet
Also called SODDY'S HEXLET. Consider three mutually tangent SPHERES A, B, and C. Then construct a chain of SPHERES tangent to each of A, B, and C threading and interlocking with the $A-B-C$ ring. Surprisingly, every chain closes into a "necklace" after six SPHERES regardless of where the first SPHERE is placed. This is a special case of KOLLROS' THEOREM. The centers of a Soddy hexlet always lie on an ELLIPSE (Ogilvy 1990, p. 63).

see also COXETER'S LOXODROMIC SEQUENCE OF TANGENT CIRCLES, KOLLROS' THEOREM, STEINER CHAIN

References
Coxeter, H. S. M. "Interlocking Rings of Spheres." *Scripta Math.* **18**, 113–121, 1952.
Gosset, T. "The Hexlet." *Nature* **139**, 251–252, 1937.
Honsberger, R. *Mathematical Gems II.* Washington, DC: Math. Assoc. Amer., pp. 49–50, 1976.
Morley, F. "The Hexlet." *Nature* **139**, 72–73, 1937.
Ogilvy, C. S. *Excursions in Geometry.* New York: Dover, pp. 60–72, 1990.
Soddy, F. "The Bowl of Integers and the Hexlet." *Nature* **139**, 77–79, 1937.
Soddy, F. "The Hexlet." *Nature* **139**, 154 and 252, 1937.

HexLife
An alternative LIFE game similar to Conway's, which is played on a hexagonal grid. No set of rules has yet emerged as uniquely interesting.

see also HIGHLIFE

References
Hensel, A. "A Brief Illustrated Glossary of Terms in Conway's Game of Life." `http://www.cs.jhu.edu/~callahan/glossary.html`.

Hexomino
One of the 35 6-POLYOMINOES.

References
Pappas, T. "Triangular, Square & Pentagonal Numbers." *The Joy of Mathematics.* San Carlos, CA: Wide World Publ./Tetra, p. 214, 1989.

Heyting Algebra
An ALGEBRA which is a special case of a LOGOS.

see also LOGOS, TOPOS

Hh Function
Let

$$Z(x) \equiv \frac{1}{\sqrt{2\pi}} e^{-x^2/2} \tag{1}$$

$$Q(x) \equiv \frac{1}{\sqrt{2\pi}} \int_x^\infty e^{-t^2/2}\, dt, \tag{2}$$

where Z and Q are closely related to the NORMAL DISTRIBUTION FUNCTION, then

$$\mathrm{Hh}_{-n}(x) = (-1)^{n-1}\sqrt{2\pi}\, z^{(n-1)}(x) \tag{3}$$

$$\mathrm{Hh}_n(x) = \frac{(-1)^n}{n!}\mathrm{Hh}_{-1}(x)\frac{d^n}{dx^n}\left[\frac{Q(x)}{Z(x)}\right]. \tag{4}$$

see also NORMAL DISTRIBUTION FUNCTION, TETRACHORIC FUNCTION

Hi-Q
A triangular version of PEG SOLITAIRE with 15 holes and 14 pegs. Numbering hole 1 at the apex of the triangle and thereafter from left to right on the next lower row, etc., the following table gives possible ending holes for a single peg removed (Beeler *et al.* 1972, Item 76). Because of symmetry, only the first five pegs need be considered. Also because of symmetry, removing peg 2 is equivalent to removing peg 3 and flipping the board horizontally.

remove	possible ending pegs
1	1, 7 = 10, 13
2	2, 6, 11, 14
4	3 = 12, 4, 9, 15
5	13

References
Beeler, M.; Gosper, R. W.; and Schroeppel, R. Item 75 in *HAKMEM.* Cambridge, MA: MIT Artificial Intelligence Laboratory, Memo AIM-239, Feb. 1972.

Higher Arithmetic
An archaic term for NUMBER THEORY.

Highest Weight Theorem
A theorem proved by É. Cartan (1913) which classifies the irreducible representations of COMPLEX semisimple LIE ALGEBRAS.

References
Knapp, A. W. "Group Representations and Harmonic Analysis, Part II." *Not. Amer. Math. Soc.* **43**, 537–549, 1996.

HighLife
An alternate set of LIFE rules similar to Conway's, but with the additional rule that six neighbors generate a birth. Most of the interest in this variant is due to the presence of a so-called replicator.

see also HEXLIFE, LIFE

References
Hensel, A. "A Brief Illustrated Glossary of Terms in Conway's Game of Life." `http://www.cs.jhu.edu/~callahan/glossary.html`.

Highly Abundant Number

see HIGHLY COMPOSITE NUMBER

Highly Composite Number

A COMPOSITE NUMBER (also called a SUPERABUNDANT NUMBER) is a number n which has more FACTORS than any other number less than n. In other words, $\sigma(n)/n$ exceeds $\sigma(k)/k$ for all $k < n$, where $\sigma(n)$ is the DIVISOR FUNCTION. They were called highly composite numbers by Ramanujan, who found the first 100 or so, and superabundant by Alaoglu and Erdős (1944).

There are an infinite numbers of highly composite numbers, and the first few are 2, 4, 6, 12, 24, 36, 48, 60, 120, 180, 240, 360, 720, 840, 1260, 1680, 2520, 5040, ... (Sloane's A002182). Ramanujan (1915) listed 102 up to 6746328388800 (but omitted 293, 318, 625, 600, and 29331862500). Robin (1983) gives the first 5000 highly composite numbers, and a comprehensive survey is given by Nicholas (1988).

If

$$N = 2^{a_2} 3^{a_3} \cdots p^{a_p} \tag{1}$$

is the prime decomposition of a highly composite number, then

1. The PRIMES 2, 3, ..., p form a string of consecutive PRIMES,
2. The exponents are nonincreasing, so $a_2 \geq a_3 \geq \ldots \geq a_p$, and
3. The final exponent a_p is always 1, except for the two cases $N = 4 = 2^2$ and $N = 36 = 2^2 \cdot 3^2$, where it is 2.

Let $Q(x)$ be the number of highly composite numbers $\leq x$. Ramanujan (1915) showed that

$$\lim_{x\to\infty} \frac{Q(x)}{\ln x} = \infty. \tag{2}$$

Erdős (1944) showed that there exists a constant $c_1 > 0$ such that

$$Q(x) \geq (\ln x)^{1+c_1} \tag{3}$$

Nicholas proved that there exists a constant $c_2 > 0$ such that

$$Q(x) \ll (\ln x)^{c_2}. \tag{4}$$

see also ABUNDANT NUMBER

References

Alaoglu, L. and Erdős, P. "On Highly Composite and Similar Numbers." *Trans. Amer. Math. Soc.* **56**, 448–469, 1944.

Andree, R. V. "Ramanujan's Highly Composite Numbers." *Abacus* **3**, 61–62, 1986.

Berndt, B. C. *Ramanujan's Notebooks, Part IV.* New York: Springer-Verlag, p. 53, 1994.

Dickson, L. E. *History of the Theory of Numbers, Vol. 1: Divisibility and Primality.* New York: Chelsea, p. 323, 1952.

Flammenkamp, A. `http://www.minet.uni-jena.de/~achim/highly.html`.

Honsberger, R. *Mathematical Gems I.* Washington, DC: Math. Assoc. Amer., p. 112, 1973.

Honsberger, R. "An Introduction to Ramanujan's Highly Composite Numbers." Ch. 14 in *Mathematical Gems III.* Washington, DC: Math. Assoc. Amer., pp. 193–207, 1985.

Kanigel, R. *The Man Who Knew Infinity: A Life of the Genius Ramanujan.* New York: Washington Square Press, p. 232, 1991.

Nicholas, J.-L. "On Highly Composite Numbers." In *Ramanujan Revisited: Proceedings of the Centenary Conference* (Ed. G. E. Andrews, B. C. Berndt, and R. A. Rankin). Boston, MA: Academic Press, pp. 215–244, 1988.

Ramanujan, S. "Highly Composite Numbers." *Proc. London Math. Soc.* **14**, 347–409, 1915.

Ramanujan, S. *Collected Papers.* New York: Chelsea, 1962.

Robin, G. "Méthodes d'optimalisation pour un problème de théories des nombres." *RAIRO Inform. Théor.* **17**, 239–247, 1983.

Sloane, N. J. A. Sequence A002182/M1025 in "An On-Line Version of the Encyclopedia of Integer Sequences."

Wells, D. *The Penguin Dictionary of Curious and Interesting Numbers.* New York: Penguin Books, p. 128, 1986.

Higman-Sims Group

The SPORADIC GROUP *HS*.

References

Wilson, R. A. "ATLAS of Finite Group Representation." `http://for.mat.bham.ac.uk/atlas/HS.html`.

Hilbert's Axioms

The 21 assumptions which underlie the GEOMETRY published in Hilbert's classic text *Grundlagen der Geometrie.* The eight INCIDENCE AXIOMS concern collinearity and intersection and include the first of EUCLID'S POSTULATES. The four ORDERING AXIOMS concern the arrangement of points, the five CONGRUENCE AXIOMS concern geometric equivalence, and the three CONTINUITY AXIOMS concern continuity. There is also a single parallel axiom equivalent to Euclid's PARALLEL POSTULATE.

see also CONGRUENCE AXIOMS, CONTINUITY AXIOMS, INCIDENCE AXIOMS, ORDERING AXIOMS, PARALLEL POSTULATE

References

Hilbert, D. *The Foundations of Geometry, 2nd ed.* Chicago, IL: Open Court, 1980.

Iyanaga, S. and Kawada, Y. (Eds.). "Hilbert's System of Axioms." §163B in *Encyclopedic Dictionary of Mathematics.* Cambridge, MA: MIT Press, pp. 544–545, 1980.

Hilbert Basis Theorem

If R is a NOETHERIAN RING, then $S = R[X]$ is also a NOETHERIAN RING.

see also ALGEBRAIC VARIETY, FUNDAMENTAL SYSTEM, SYZYGY

References

Hilbert, D. "Über die Theorie der algebraischen Formen." *Math. Ann.* **36**, 473–534, 1890.

Hilbert's Constants

N.B. A detailed on-line essay by S. Finch was the starting point for this entry.

Extend HILBERT'S INEQUALITY by letting $p, q > 1$ and

$$\frac{1}{p} + \frac{1}{q} \geq 1, \tag{1}$$

so that

$$0 < \lambda = 2 - \frac{1}{p} - \frac{1}{q} \leq 1. \tag{2}$$

Levin (1937) and Stečkin (1949) showed that

$$\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{a_m b_n}{(m+n)^\lambda} \leq \left\{ \pi \csc \left[\frac{\pi(q-1)}{\lambda q} \right] \right\}^\lambda \times \left(\int_0^\infty [f(x)]^p \, dx \right)^{1/p} \left(\int_0^\infty [g(x)]^q \, dx \right)^{1/q} \tag{3}$$

and

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{(x+y)^\lambda} \, dx \, dy < \pi \csc \left[\frac{\pi(q-1)}{p} \right]^\lambda \times \left(\int_0^\infty [f(x)]^p \, dx \right)^{1/p} \left(\int_0^\infty [g(x)]^q \, dx \right)^{1/q}. \tag{4}$$

Mitrinovic *et al.* (1991) indicate that this constant is the best possible.

see also HILBERT'S INEQUALITY

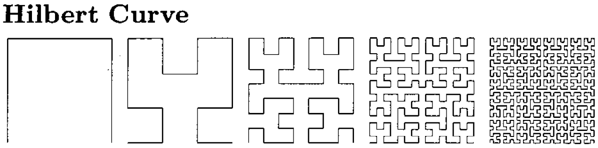
References

Finch, S. "Favorite Mathematical Constants." http://www.mathsoft.com/asolve/constant/hilbert/hilbert.html.

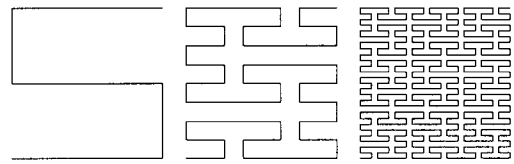
Mitrinovic, D. S.; Pecaric, J. E.; and Fink, A. M. *Inequalities Involving Functions and Their Integrals and Derivatives.* Dordrecht, Netherlands: Kluwer, 1991.

Stečkin, S. B. "On the Degree of Best Approximation to Continuous Functions." *Dokl. Akad. Nauk SSSR* **65**, 135–137, 1949.

Hilbert Curve



A LINDENMAYER SYSTEM invented by Hilbert (1891) whose limit is a PLANE-FILLING CURVE which fills a square. Traversing the VERTICES of an n-D HYPERCUBE in GRAY CODE order produces a generator for the n-D Hilbert curve (Goetz). The Hilbert curve can be simply encoded with initial string `"L"`, STRING REWRITING rules `"L" -> "+RF-LFL-FR+"`, `"R"->"-LF+RFR+FL-"`, and angle 90° (Peitgen and Saupe 1988, p. 278).



A related curve is the Hilbert II curve, shown above (Peitgen and Saupe 1988, p. 284). It is also a LINDENMAYER SYSTEM and the curve can be encoded with initial string `"X"`, STRING REWRITING rules `"X" -> "XFYFX+F+YFXFY-F-XFYFX"`, `"Y" -> "YFXFY-F-XFYFX+F+YFXFY"`, and angle 90°.

see also LINDENMAYER SYSTEM, PEANO CURVE, PLANE-FILLING CURVE, SIERPIŃSKI CURVE, SPACE-FILLING CURVE

References

Bogomolny, A. "Plane Filling Curves." http://www.cut-the-knot.com/do_you_know/hilbert.html.

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Peitgen, H.-O. and Saupe, D. (Eds.). *The Science of Fractal Images.* New York: Springer-Verlag, pp. 278 and 284, 1988.

Wagon, S. *Mathematica in Action.* New York: W. H. Freeman, pp. 198–206, 1991.

Hilbert Function

Let $\Gamma = \{p_1, \ldots, p_m\} \subset \mathbb{P}^2$ be a collection of m distinct points. Then the number of conditions imposed by Γ on forms of degree d is called the Hilbert function h_Γ of Γ. If curves X_1 and X_2 of degrees d and e meet in a collection Γ of $d \cdot e$ points, then for any k, the number $h_\Gamma(k)$ of conditions imposed by Γ on forms of degree k is independent of X_1 and X_2 and is given by

$$h_\Gamma(k) = \binom{k+2}{2} - \binom{k-d+2}{2} - \binom{k-e+2}{2} + \binom{k-d-e+2}{2},$$

where the BINOMIAL COEFFICIENT $\binom{a}{2}$ is taken as 0 if $a < 2$ (Cayley 1843).

References

Eisenbud, D.; Green, M.; and Harris, J. "Cayley-Bacharach Theorems and Conjectures." *Bull. Amer. Math. Soc.* **33**, 295–324, 1996.

Hilbert Hotel

Let a hotel have a DENUMERABLE set of rooms numbered 1, 2, 3, ... Then any finite number n of guests can be accommodated without evicting the current guests by moving the current guests from room i to room $i + n$. Furthermore, a DENUMERABLE number of guests can be similarly accommodated by moving the existing guests from i to $2i$, freeing up a DENUMERABLE number of rooms $2i-1$.

References
Lauwerier, H. "Hilbert Hotel." In *Fractals: Endlessly Repeated Geometric Figures.* Princeton, NJ: Princeton University Press, p. 22, 1991.
Pappas, T. "Hotel Infinity." *The Joy of Mathematics.* San Carlos, CA: Wide World Publ./Tetra, p. 37, 1989.

Hilbert's Inequality

Given a POSITIVE SEQUENCE $\{a_n\}$,

$$\sqrt{\sum_{j=-\infty}^{\infty}\left|\sum_{\substack{n=-\infty\\ n\neq j}}^{\infty}\frac{a_n}{j-n}\right|^2}\leq\pi\sqrt{\sum_{n=-\infty}^{\infty}|a_n|^2},$$

where the a_ns are REAL and "square summable."

Another INEQUALITY known as Hilbert's applies to NONNEGATIVE sequences $\{a_n\}$ and $\{b_n\}$,

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{a_m b_m}{m+n} < \pi\csc\left(\frac{\pi}{p}\right)\left(\sum_{m=1}^{\infty}{a_m}^p\right)^{1/p}\left(\sum_{n=1}^{\infty}{b_n}^q\right)^{1/q}$$

unless all a_n or all b_n are 0. If $f(x)$ and $g(x)$ are NONNEGATIVE integrable functions, then the integral form is

$$\int_0^\infty\int_0^\infty\frac{f(x)g(y)}{x+y}\,dx\,dy<\pi\csc\left(\frac{\pi}{p}\right)\times\left(\int_0^\infty[f(x)]^p\,dx\right)^{1/p}\left(\int_0^\infty[g(x)]^q\,dx\right)^{1/q}.$$

The constant $\pi\csc(\pi/P)$ is the best possible, in the sense that counterexamples can be constructed for any smaller value.

References
Hardy, G. H.; Littlewood, J. E.; and Pólya, G. *Inequalities, 2nd ed.* Cambridge, England: Cambridge University Press, pp. 308–309, 1988.

Hilbert Matrix

A MATRIX H with elements

$$H_{ij}\equiv(i+j-1)^{-1}$$

for $i,j=1,2,\ldots,n$. Although the MATRIX INVERSE is given analytically by

$$(H^{-1})_{ij}=\frac{(-1)^{i+j}}{i+j-1}\frac{(n+i-1)!(n+j-1)!}{[(i-1)!(j-1)!]^2(n-i)!(n-j)!},$$

Hilbert matrices are difficult to invert numerically. The DETERMINANTS for the first few values of H_n are given in the following table.

n	$\det(\mathsf{H}_n)$
1	1
2	8.33333×10^{-2}
3	4.62963×10^{-4}
4	1.65344×10^{-7}
5	3.74930×10^{-12}
6	5.36730×10^{-18}

Hilbert's Nullstellansatz

Let K be an algebraically closed field and let I be an IDEAL in $K(x)$, where $x=(x_1,x_2,\ldots,x_n)$ is a finite set of indeterminates. Let $p\in K(x)$ be such that for any $(c_1,\ldots,c_n)$ in K^n, if every element of I vanishes when evaluated if we set each $(x_i=c_i)$, then p also vanishes. Then p^j lies in I for some j. Colloquially, the theory of algebraically closed fields is a complete model.

Hilbert Number

see GELFOND-SCHNEIDER CONSTANT

Hilbert Polynomial

Let Γ be an ALGEBRAIC CURVE in a projective space of DIMENSION n, and let p be the PRIME IDEAL defining Γ, and let $\chi(p,m)$ be the number of linearly independent forms of degree m modulo p. For large m, $\chi(p,m)$ is a POLYNOMIAL known as the Hilbert polynomial.

References
Iyanaga, S. and Kawada, Y. (Eds.). *Encyclopedic Dictionary of Mathematics.* Cambridge, MA: MIT Press, p. 36, 1980.

Hilbert's Problems

A set of (originally) unsolved problems in mathematics proposed by Hilbert. Of the 23 total, ten were presented at the Second International Congress in Paris in 1900. These problems were designed to serve as examples for the kinds of problems whose solutions would lead to the furthering of disciplines in mathematics.

1a. Is there a transfinite number between that of a DENUMERABLE SET and the numbers of the CONTINUUM? This question was answered by Gödel and Cohen to the effect that the answer depends on the particular version of SET THEORY assumed.

1b. Can the CONTINUUM of numbers be considered a WELL-ORDERED SET? This question is related to Zermelo's AXIOM OF CHOICE. In 1963, the AXIOM OF CHOICE was demonstrated to be independent of all other AXIOMS in SET THEORY, so there appears to be no universally valid solution to this question either.

2. Can it be proven that the AXIOMS of logic are consistent? GÖDEL'S INCOMPLETENESS THEOREM indicated that the answer is "no," in the sense

that any formal system interesting enough to formulate its own consistency can prove its own consistency IFF it is inconsistent.

3. Give two TETRAHEDRA which cannot be decomposed into congruent TETRAHEDRA directly or by adjoining congruent TETRAHEDRA. Max Dehn showed this could not be done in 1902. W. F. Kagon obtained the same result independently in 1903.
4. Find GEOMETRIES whose AXIOMS are closest to those of EUCLIDEAN GEOMETRY if the ORDERING and INCIDENCE AXIOMS are retained, the CONGRUENCE AXIOMS weakened, and the equivalent of the PARALLEL POSTULATE omitted. This problem was solved by G. Hamel.
5. Can the assumption of differentiability for functions defining a continuous transformation GROUP be avoided? (This is a generalization of the CAUCHY FUNCTIONAL EQUATION.) Solved by John von Neumann in 1930 for bicompact groups. Also solved for the ABELIAN case, and for the solvable case in 1952 with complementary results by Montgomery and Zipin (subsequently combined by Yamabe in 1953). Andrew Glean showed in 1952 that the answer is also "yes" for all locally bicompact groups.
6. Can physics be axiomized?
7. Let $\alpha \neq 1 \neq 0$ be ALGEBRAIC and β IRRATIONAL. Is α^β then TRANSCENDENTAL? Proved true in 1934 by Aleksander Gelfond (GELFOND'S THEOREM; Courant and Robins 1996).
8. Prove the RIEMANN HYPOTHESIS. The CONJECTURE has still been neither proved nor disproved.
9. Construct generalizations of the RECIPROCITY THEOREM of NUMBER THEORY.
10. Does there exist a universal algorithm for solving DIOPHANTINE EQUATIONS? The impossibility of obtaining a general solution was proven by Julia Robinson and Martin Davis in 1970, following proof of the result that the equation $n = F_{2m}$ (where F_{2m} is a FIBONACCI NUMBER) is Diophantine by Yuri Matijasevich (Matijasevič 1970, Davis 1973, Davis and Hersh 1973, Matijasevič 1993).
11. Extend the results obtained for quadratic fields to arbitrary INTEGER algebraic fields.
12. Extend a theorem of Kronecker to arbitrary algebraic fields by explicitly constructing Hilbert class fields using special values. This calls for the construction of HOLOMORPHIC FUNCTIONS in several variables which have properties analogous to the exponential function and elliptic modular functions (Holtzapfel 1995).
13. Show the impossibility of solving the general seventh degree equation by functions of two variables.
14. Show the finiteness of systems of relatively integral functions.
15. Justify Schubert's ENUMERATIVE GEOMETRY (Bell 1945).
16. Develop a topology of REAL algebraic curves and surfaces. The SHIMURA-TANIYAMA CONJECTURE postulates just this connection. See Ilyashenko and Yakovenko (1995) and Gudkov and Utkin (1978).
17. Find a representation of definite form by SQUARES.
18. Build spaces with congruent POLYHEDRA.
19. Analyze the analytic character of solutions to variational problems.
20. Solve general BOUNDARY VALUE PROBLEMS.
21. Solve differential equations given a MONODROMY GROUP. More technically, prove that there always exists a FUCHSIAN SYSTEM with given singularities and a given MONODROMY GROUP. Several special cases had been solved, but a NEGATIVE solution was found in 1989 by B. Bolibruch (Anasov and Bolibruch 1994).
22. Uniformization.
23. Extend the methods of CALCULUS OF VARIATIONS.

References

Anasov, D. V. and Bolibruch, A. A. *The Riemann-Hilbert Problem.* Braunschweig, Germany: Vieweg, 1994.

Bell, E. T. *The Development of Mathematics, 2nd ed.* New York: McGraw-Hill, p. 340, 1945.

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Davis, M. "Hilbert's Tenth Problem is Unsolvable." *Amer. Math. Monthly* **80**, 233–269, 1973.

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Gudkov, D. and Utkin, G. A. *Nine Papers on Hilbert's 16th Problem.* Providence, RI: Amer. Math. Soc., 1978.

Holtzapfel, R.-P. *The Ball and Some Hilbert Problems.* Boston, MA: Birkhäuser, 1995.

Ilyashenko, Yu. and Yakovenko, S. (Eds.). *Concerning the Hilbert 16th Problem.* Providence, RI: Amer. Math. Soc., 1995.

Matijasevič, Yu. V. "Solution to of the Tenth Problem of Hilbert." *Mat. Lapok* **21**, 83–87, 1970.

Matijasevich, Yu. V. *Hilbert's Tenth Problem.* Cambridge, MA: MIT Press, 1993.

Hilbert-Schmidt Norm

The Hilbert-Schmidt norm of a MATRIX A is defined as

$$|\mathsf{A}|_2 \equiv \sqrt{\sum_{i,j} a_{ij}}.$$

Hilbert-Schmidt Theory

The study of linear integral equations of the Fredholm type with symmetric kernels

$$K(x,t) = K(t,x).$$

References

Arfken, G. "Hilbert-Schmidt Theory." §16.4 in *Mathematical Methods for Physicists, 3rd ed.* Orlando, FL: Academic Press, pp. 890–897, 1985.

Hilbert Space

A Hilbert space is VECTOR SPACE H with an INNER PRODUCT $\langle f, g\rangle$ such that the NORM defined by

$$|f| = \sqrt{\langle f, f\rangle}$$

turns H into a COMPLETE METRIC SPACE. If the INNER PRODUCT does not so define a NORM, it is instead known as an INNER PRODUCT SPACE.

Examples of FINITE-dimensional Hilbert spaces include

1. The REAL NUMBERS $\mathbb{R}^n$ with $\langle v, u\rangle$ the vector DOT PRODUCT of v and u.
2. The COMPLEX NUMBERS $\mathbb{C}^n$ with $\langle v, u\rangle$ the vector DOT PRODUCT of v and the COMPLEX CONJUGATE of u.

An example of an INFINITE-dimensional Hilbert space is L^2, the SET of all FUNCTIONS $f:\mathbb{R}\to\mathbb{R}$ such that the INTEGRAL of f^2 over the whole REAL LINE is FINITE. In this case, the INNER PRODUCT is

$$\langle f, g\rangle = \int f(x)g(x)\,dx.$$

A Hilbert space is always a BANACH SPACE, but the converse need not hold.

see also BANACH SPACE, L_2-NORM, L_2-SPACE, LIOUVILLE SPACE, PARALLELOGRAM LAW, VECTOR SPACE

References

Sansone, G. "Elementary Notions of Hilbert Space." §1.3 in *Orthogonal Functions, rev. English ed.* New York: Dover, pp. 5–10, 1991.

Stone, M. H. *Linear Transformations in Hilbert Space and Their Applications Analysis.* Providence, RI: Amer. Math. Soc., 1932.

Hilbert's Theorem

Every MODULAR SYSTEM has a MODULAR SYSTEM BASIS consisting of a finite number of POLYNOMIALS. Stated another way, for every order n there exists a nonsingular curve with the maximum number of circuits and the maximum number for any one nest.

References

Coolidge, J. L. *A Treatise on Algebraic Plane Curves.* New York: Dover, p. 61, 1959.

Hilbert Transform

$$g(y) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{f(x)\,dx}{x-y}$$
$$f(x) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{g(y)\,dy}{y-x}.$$

see also TITCHMARSH THEOREM

References

Bracewell, R. *The Fourier Transform and Its Applications.* New York: McGraw-Hill, pp. 267–272, 1965.

Hill Determinant

A DETERMINANT which arises in the solution of the second-order ORDINARY DIFFERENTIAL EQUATION

$$x^2\frac{d^2\psi}{dx^2} + x\frac{d\psi}{dx} + \left(\tfrac{1}{4}h^2x^2 + \tfrac{1}{2}h^2 - b + \frac{h^2}{4x^2}\right)\psi = 0. \quad (1)$$

Writing the solution as a POWER SERIES

$$\psi = \sum_{n=-\infty}^{\infty} a_n x^{s+2n} \quad (2)$$

gives a RECURRENCE RELATION

$$h^2a_{n+1} + [2h^2 - 4b + 16(n+\tfrac{1}{2}s)^2]a_n + h^2a_{n-1} = 0. \quad (3)$$

The value of s can be computed using the Hill determinant

$$\Delta(s) = \begin{vmatrix} \ddots & \vdots & \vdots & \vdots & \vdots & \iddots \\ \cdots & \frac{(\sigma+2)-\alpha^2}{4-\alpha^2} & \frac{\beta^2}{4-\alpha^2} & 0 & 0 & \cdots \\ \cdots & 0 & -\frac{\beta^2}{\alpha^2} & -\frac{\sigma^2-\alpha^2}{\alpha^2} & -\frac{\beta^2}{\alpha^2} & \cdots \\ \cdots & 0 & 0 & -\frac{\beta^2}{1-\alpha^2} & \frac{(\sigma-1)^2-\alpha^2}{1-\alpha^2} & \cdots \\ \iddots & \vdots & \vdots & \vdots & \vdots & \ddots \end{vmatrix}, \quad (4)$$

where

$$\sigma = \tfrac{1}{2}s \quad (5)$$
$$\alpha^2 = \tfrac{1}{4}b - \tfrac{1}{8}h^2 \quad (6)$$
$$\beta = \tfrac{1}{4}h, \quad (7)$$

and σ is the variable to solve for. The determinant can be given explicitly by the amazing formula

$$\Delta(s) = \Delta(0) - \frac{\sin^2(\pi s/2)}{\sin^2(\frac{1}{2}\pi\sqrt{b - \frac{1}{2}h^2})}, \tag{8}$$

where

$$\Delta(0) = \begin{vmatrix} \ddots & \vdots & \vdots & \vdots & \vdots & \iddots \\ \cdots & 1 & \frac{h^2}{144+2h^2-4b} & 0 & 0 & \cdots \\ \cdots & \frac{h^2}{64+2h^2-4b} & 1 & \frac{h^2}{64+2h^2-4b} & 0 & \cdots \\ \cdots & 0 & \frac{h^2}{16+2h^2-4b} & 1 & \frac{h^2}{16+2h^2-4b} & \cdots \\ \cdots & 0 & 0 & \frac{h^2}{2h^2-4b} & 1 & \cdots \\ \cdots & 0 & 0 & 0 & \frac{h^2}{16+2h^2-4b} & \cdots \\ \iddots & \vdots & \vdots & \vdots & \vdots & \ddots \end{vmatrix}, \tag{9}$$

leading to the implicit equation for s,

$$\sin^2(\tfrac{1}{2}\pi s) = \Delta(0)\sin^2\left(\tfrac{1}{2}\pi\sqrt{b - \tfrac{1}{2}h^2}\right). \tag{10}$$

see also HILL'S DIFFERENTIAL EQUATION

References

Morse, P. M. and Feshbach, H. *Methods of Theoretical Physics, Part I.* New York: McGraw-Hill, pp. 555–562, 1953.

Hill's Differential Equation

$$\frac{d^2x}{dt^2} = \phi(t)x,$$

where ϕ is periodic. It can be written as

$$\frac{d^2y}{dx^2} + \left[\theta_0 + 2\sum_{n=1}^{\infty}\theta_n\cos(2nz)\right] = 0,$$

where θ_n are known constants. A solution can be given by taking the "DETERMINANT" of an infinite MATRIX.

see also HILL DETERMINANT

Hillam's Theorem

If $f : [a,b] \to [a,b]$ (where $[a,b]$ denotes the CLOSED INTERVAL from a to b on the REAL LINE) satisfies a LIPSCHITZ CONDITION with constant K, i.e., if

$$|f(x) - f(y)| \leq K|x - y|$$

for all $x, y \in [a,b]$, then the iteration scheme

$$x_{n+1} = (1-\lambda)x_n + \lambda f(x_n),$$

where $\lambda = 1/(K+1)$, converges to a FIXED POINT of f.

References

Falkowski, B.-J. "On the Convergence of Hillam's Iteration Scheme." *Math. Mag.* **69**, 299–303, 1996.

Geist, R.; Reynolds, R.; and Suggs, D. "A Markovian Framework for Digital Halftoning." *ACM Trans. Graphics* **12**, 136–159, 1993.

Hillam, B. P. "A Generalization of Krasnoselski's Theorem on the Real Line." *Math. Mag.* **48**, 167–168, 1975.

Krasnoselski, M. A. "Two Remarks on the Method of Successive Approximations." *Uspehi Math. Nauk (N. S.)* **10**, 123–127, 1955.

Hindu Check

see CASTING OUT NINES

Hinge

1 M $4n+5$
150 895 1895
250 895 1099 1775
688 895 1166 1699
795 795 1333 1693
795 1499
H_1 H_2

The upper and lower hinges are descriptive statistics of a set of N data values, where N is of the form $N = 4n+5$ with $n = 0, 1, 2, \ldots$. The hinges are obtained by ordering the data in increasing order $a_1, \ldots, a_N$, and writing them out in the shape of a "w" as illustrated above. The values at the bottom legs are called the hinges H_1 and H_2 (and the central peak is the MEDIAN). In this ordering,

$$\begin{aligned} H_1 &= a_{n+2} = a_{(N+3)/4} \\ M &= a_{2n+3} = a_{(N+1)/2} \\ H_2 &= a_{3n+4} = a_{(3N+1)/4}. \end{aligned}$$

For N of the form $4n+5$, the hinges are identical to the QUARTILES. The difference $H_2 - H_1$ is called the H-SPREAD.

see also H-SPREAD, HABERDASHER'S PROBLEM, MEDIAN (STATISTICS), ORDER STATISTIC, QUARTILE, TRIMEAN

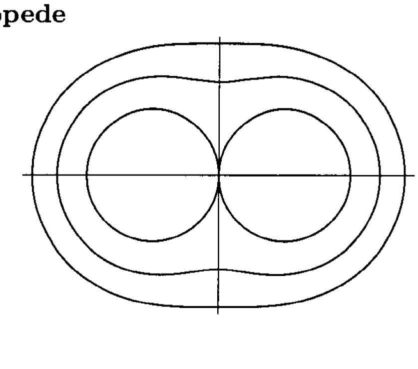
References

Tukey, J. W. *Explanatory Data Analysis.* Reading, MA: Addison-Wesley, pp. 32–34, 1977.

Hippias' Quadratrix

see QUADRATRIX OF HIPPIAS

Hippopede



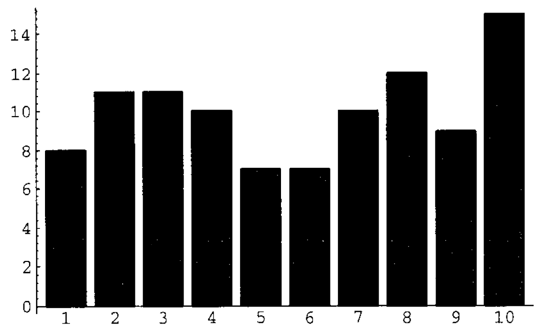
A curve also known as a HORSE FETTER and given by the polar equation

$$r^2 = 4b(a - b\sin^2\theta).$$

References

Lawrence, J. D. *A Catalog of Special Plane Curves.* New York: Dover, pp. 144–146, 1972.

Histogram



The grouping of data into bins (spaced apart by the so-called CLASS INTERVAL) plotting the number of members in each bin versus the bin number. The above histogram shows the number of variates in bins with CLASS INTERVAL 1 for a sample of 100 real variates with a UNIFORM DISTRIBUTION from 0 and 10. Therefore, bin 1 gives the number of variates in the range 0–1, bin 2 gives the number of variates in the range 1–2, etc.

see also OGIVE

Hitch

A KNOT that secures a rope to a post, ring, another rope, etc., but does not keep its shape by itself.

see also CLOVE HITCH, KNOT, LINK, LOOP (KNOT)

References

Owen, P. *Knots.* Philadelphia, PA: Courage, p. 17, 1993.

Hitting Set

Let S be a collection S of subsets of a finite set X. The smallest subset Y of X that meets every member of S is called the hitting set or VERTEX COVER. Finding the hitting set is an NP-COMPLETE PROBLEM.

Hjelmslev's Theorem

When all the points P on one line are related by an ISOMETRY to all points P' on another, the MIDPOINTS of the segments PP' are either distinct and collinear or coincident.

HJLS Algorithm

An algorithm for finding INTEGER RELATIONS whose running time is bounded by a polynomial in the number of real variables. It is much faster than other algorithms such as the FERGUSON-FORCADE ALGORITHM, LLL ALGORITHM, and PSOS ALGORITHM.

Unfortunately, it is numerically unstable and therefore requires extremely high precision. The cause of this instability is not known (Ferguson and Bailey 1992), but is believed to derive from its reliance on GRAM-SCHMIDT ORTHONORMALIZATION, which is know to be numerically unstable (Golub and van Loan 1989).

see also FERGUSON-FORCADE ALGORITHM, INTEGER RELATION, LLL ALGORITHM, PSLQ ALGORITHM, PSOS ALGORITHM

References

Ferguson, H. R. P. and Bailey, D. H. "A Polynomial Time, Numerically Stable Integer Relation Algorithm." RNR Techn. Rept. RNR-91-032, Jul. 14, 1992.

Golub, G. H. and van Loan, C. F. *Matrix Computations, 3rd ed.* Baltimore, MD: Johns Hopkins, 1996.

Hastad, J.; Just, B.; Lagarias, J. C.; and Schnorr, C. P. "Polynomial Time Algorithms for Finding Integer Relations Among Real Numbers." *SIAM J. Comput.* **18**, 859–881, 1988.

HK Integral

Named after Henstock and Kurzweil. Every LEBESGUE INTEGRABLE function is HK integrable with the same value.

References

Shenitzer, A. and Steprans, J. "The Evolution of Integration." *Amer. Math. Monthly* **101**, 66–72, 1994.

Hodge Star

On an oriented n-D RIEMANNIAN MANIFOLD, the Hodge star is a linear FUNCTION which converts alternating DIFFERENTIAL k-FORMS to alternating $(n-k)$-forms. If w is an alternating k-FORM, its Hodge star is given by

$$w(v_1, \ldots, v_k) = ({}^*w)(v_{k+1}, \ldots, v_n)$$

when $v_1, \ldots, v_n$ is an oriented orthonormal basis.

see also STOKES' THEOREM

Hodge's Theorem

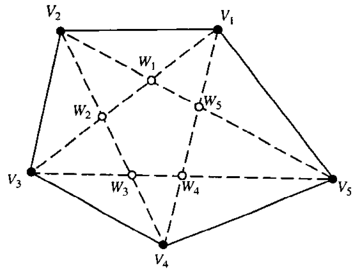
On a compact oriented FINSLER MANIFOLD without boundary, every COHOMOLOGY class has a UNIQUE harmonic representative. The DIMENSION of the SPACE of all harmonic forms of degree p is the pth BETTI NUMBER of the MANIFOLD.

see also BETTI NUMBER, COHOMOLOGY, DIMENSION, FINSLER MANIFOLD

References

Chern, S.-S. "Finsler Geometry is Just Riemannian Geometry without the Quadratic Restriction." *Not. Amer. Math. Soc.* **43**, 959–963, 1996.

Hoehn's Theorem



A geometric theorem related to the PENTAGRAM and also called the PRATT-KASAPI THEOREM.

$$\frac{|V_1W_1|}{|W_2V_3|}\frac{|V_2W_2|}{|W_3V_4|}\frac{|V_3W_3|}{|W_4V_5|}\frac{|V_4W_4|}{|W_5V_1|}\frac{|V_5W_5|}{|W_1V_2|}=1$$

$$\frac{|V_1W_2|}{|W_1V_3|}\frac{|V_2W_3|}{|W_2V_4|}\frac{|V_3W_4|}{|W_3V_5|}\frac{|V_4W_5|}{|W_4V_1|}\frac{|V_5W_1|}{|W_5V_2|}=1.$$

In general, it is also true that

$$\frac{|V_iW_i|}{|W_{i+1}V_{i+2}|}=\frac{|V_iV_{i+1}V_{i+4}|}{|V_iV_{i+1}V_{i+2}V_{i+4}|}\frac{|V_iV_{i+1}V_{i+2}V_{i+3}|}{|V_{i+2}V_{i+3}V_{i+1}|}.$$

This type of identity was generalized to other figures in the plane and their duals by Pinkernell (1996).

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Grünbaum, B. and Shepard, G. C. "Ceva, Menelaus, and the Area Principle." *Math. Mag.* **68**, 254–268, 1995.
Hoehn, L. "A Menelaus-Type Theorem for the Pentagram." *Math. Mag.* **68**, 254–268, 1995.
Pinkernell, G. M. "Identities on Point-Line Figures in the Euclidean Plane." *Math. Mag.* **69**, 377–383, 1996.

Hoffman's Minimal Surface

A minimal embedded surface discovered in 1992 consisting of a helicoid with a HOLE and HANDLE (*Science News* 1992). It has the same topology as a punctured sphere with a handle, and is only the second complete embedded minimal surface of finite topology and infinite total curvature discovered (the HELICOID being the first).

A three-ended minimal surface GENUS 1 is sometimes also called Hoffman's minimal surface (Peterson 1988).

see also HELICOID

References

Peterson, I. *Mathematical Tourist: Snapshots of Modern Mathematics.* New York: W. H. Freeman, pp. 57–59, 1988.
"Putting a Handle on a Minimal Helicoid." *Sci. News* **142**, 276, Oct. 24, 1992.

Hoffman-Singleton Graph

The only GRAPH of DIAMETER 2, GIRTH 5, and VALENCY 7. It contains many copies of the PETERSEN GRAPH.

References

Hoffman, A. J. and Singleton, R. R. "On Moore Graphs of Diameter Two and Three." *IBM J. Res. Develop.* **4**, 497–504, 1960.

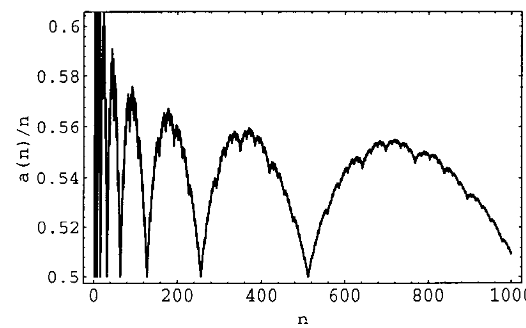
Hofstadter-Conway $10,000 Sequence

The INTEGER SEQUENCE defined by the RECURRENCE RELATION

$$a(n)=a(a(n-1))+a(n-a(n-1)),$$

with $a(1)=a(2)=1$. The first few values are 1, 1, 2, 2, 3, 4, 4, 4, 5, 6, ... (Sloane's A004001). Plotting $a(n)/n$ against n gives the BATRACHION plotted below. Conway (1988) showed that $\lim_{n\to\infty} a(n)/n = 1/2$ and offered a prize of $10,000 to the discoverer of a value of n for which $|a(i)/i - 1/2| < 1/20$ for $i > n$. The prize was subsequently claimed by Mallows, after adjustment to Conway's "intended" prize of $1,000 (Schroeder 1991), who found $n = 1489$.

$a(n)/n$ takes a value of 1/2 for n of the form 2^k with $k = 1, 2, \ldots$. Pickover (1996) gives a table of analogous values of n corresponding to different values of $|a(n)/n - 1/2| < e$.



see also BLANCMANGE FUNCTION, HOFSTADTER'S Q-SEQUENCE, MALLOW'S SEQUENCE

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Mallows, C. "Conway's Challenging Sequence." *Amer. Math. Monthly* **98**, 5–20, 1991.
Pickover, C. A. "The Drums of Ulupu." In *Mazes for the Mind: Computers and the Unexpected.* New York: St. Martin's Press, 1993.
Pickover, C. A. "The Crying of Fractal Batrachion 1,489." Ch. 25 in *Keys to Infinity.* New York: W. H. Freeman, pp. 183–191, 1995.
Schroeder, M. "John Horton Conway's 'Death Bet.'" *Fractals, Chaos, Power Laws.* New York: W. H. Freeman, pp. 57–59, 1991.
Sloane, N. J. A. Sequence A004001/M0276 in "An On-Line Version of the Encyclopedia of Integer Sequences."

Hofstadter Figure-Figure Sequence

Define $F(1) = 1$ and $S(1) = 2$ and write

$$F(n) = F(n-1) + S(n-1),$$

where the sequence $\{S(n)\}$ consists of those integers not already contained in $\{F(n)\}$. For example, $F(2) = F(1) + S(1) = 3$, so the next term of $S(n)$ is $S(2) = 4$, giving $F(3) = F(2) + S(2) = 7$. The next integer is 5, so $S(3) = 5$ and $F(4) = F(3) + S(3) = 12$. Continuing in this manner gives the "figure" sequence $F(n)$ as 1, 3, 7, 12, 18, 26, 35, 45, 56, ... (Sloane's A005228) and the "space" sequence as 2, 4, 5, 6, 8, 9, 10, 11, 13, 14, ... (Sloane's A030124).

References
Hofstadter, D. R. *Gödel, Escher, Bach: An Eternal Golden Braid.* New York: Vintage Books, p. 73, 1989.
Sloane, N. J. A. Sequences A030124 and A005288/M2629 in "An On-Line Version of the Encyclopedia of Integer Sequences."

Hofstadter G-Sequence

The sequence defined by $G(0) = 0$ and

$$G(n) = n - G(G(n-1)).$$

The first few terms are 1, 1, 2, 3, 3, 4, 4, 5, 6, 6, 7, 8, 8, 9, 9, ... (Sloane's A005206).

References
Hofstadter, D. R. *Gödel, Escher, Bach: An Eternal Golden Braid.* New York: Vintage Books, p. 137, 1989.
Sloane, N. J. A. Sequence A005206/M0436 in "An On-Line Version of the Encyclopedia of Integer Sequences."

Hofstadter H-Sequence

The sequence defined by $H(0) = 0$ and

$$H(n) = n - H(H(H(n-1))).$$

The first few terms are 1, 1, 2, 3, 4, 4, 5, 5, 6, 7, 7, 8, 9, 10, 10, 11, 12, 13, 13, 14, ... (Sloane's A005374).

References
Hofstadter, D. R. *Gödel, Escher, Bach: An Eternal Golden Braid.* New York: Vintage Books, p. 137, 1989.
Sloane, N. J. A. Sequence A005374/M0449 in "An On-Line Version of the Encyclopedia of Integer Sequences."

Hofstadter Male-Female Sequences

The pair of sequences defined by $F(0) = 1$, $M(0) = 0$, and

$$F(n) = n - M(F(n-1))$$
$$M(n) = n - F(M(n-1)).$$

The first few terms of the "male" sequence $M(n)$ are 0, 1, 2, 2, 3, 4, 4, 5, 6, 6, 7, 7, 8, 9, 9, ... (Sloane's A005379), and the first few terms of the "female" sequence $F(n)$ are 1, 2, 2, 3, 3, 4, 5, 5, 6, 6, 7, 8, 8, 9, 9, ... (Sloane's A005378).

References
Hofstadter, D. R. *Gödel, Escher, Bach: An Eternal Golden Braid.* New York: Vintage Books, p. 137, 1989.
Sloane, N. J. A. Sequences A005378/M0263 and A005379/M0278 in "An On-Line Version of the Encyclopedia of Integer Sequences."

Hofstadter Point

The r-HOFSTADTER TRIANGLE of a given TRIANGLE ΔABC is perspective to ΔABC, and the PERSPECTIVE CENTER is called the Hofstadter point. The TRIANGLE CENTER FUNCTION is

$$\alpha = \frac{\sin(rA)}{\sin(r - rA)}.$$

As $r \to 0$, the TRIANGLE CENTER FUNCTION approaches

$$\alpha = \frac{A}{a},$$

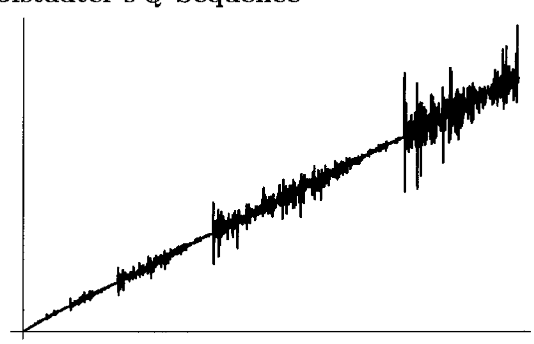
and as $r \to 1$, the TRIANGLE CENTER FUNCTION approaches

$$\alpha = \frac{a}{A}.$$

see also HOFSTADTER TRIANGLE

References
Kimberling, C. "Hofstadter Points." *Nieuw Arch. Wiskunder* **12**, 109–114, 1994.
Kimberling, C. "Major Centers of Triangles." *Amer. Math. Monthly* **104**, 431–438, 1997.
Kimberling, C. "Hofstadter Points." http://www.evansville.edu/~ck6/tcenters/recent/hofstad.html.

Hofstadter's Q-Sequence



The INTEGER SEQUENCE given by

$$Q(n) = Q(n - Q(n-1)) + Q(n - Q(n-2)),$$

with $Q(1) = Q(2) = 1$. The first few values are 1, 1, 2, 3, 3, 4, 5, 5, 6, 6, ... (Sloane's A005185; illustrated above). These numbers are sometimes called Q-NUMBERS.

see also HOFSTADTER-CONWAY \$10,000 SEQUENCE, MALLOW'S SEQUENCE

References
Conolly, B. W. "Meta-Fibonacci Sequences." In *Fibonacci and Lucas Numbers, and the Golden Section* (Ed. S. Vajda). New York: Halstead Press, pp. 127–138, 1989.

Guy, R. "Some Suspiciously Simple Sequences." *Amer. Math. Monthly* **93**, 186–191, 1986.
Hofstadter, D. R. *Gödel, Escher Bach: An Eternal Golden Braid.* New York: Vintage Books, pp. 137–138, 1980.
Pickover, C. A. "The Crying of Fractal Batrachion 1,489." Ch. 25 in *Keys to Infinity.* New York: W. H. Freeman, pp. 183–191, 1995.
Sloane, N. J. A. Sequence A005185/M0438 in "An On-Line Version of the Encyclopedia of Integer Sequences."

Hofstadter Sequences

Let $b_1 = 1$ and $b_2 = 2$ and for $n \geq 3$, let b_n be the least INTEGER $> b_{n-1}$ which can be expressed as the SUM of two or more consecutive terms. The resulting sequence is 1, 2, 3, 5, 6, 8, 10, 11, 14, 16, ... (Sloane's A005243). Let $c_1 = 2$ and $c_2 = 3$, form all possible expressions of the form $c_ic_j - 1$ for $1 \leq i \leq j \leq n$, and append them. The resulting sequence is 2, 3, 5, 9, 14, 16, 17, 18, ... (Sloane's A05244).

see also HOFSTADTER-CONWAY \$10,000 SEQUENCE, HOFSTADTER'S Q-SEQUENCE

References

Guy, R. K. "Three Sequences of Hofstadter." §E31 in *Unsolved Problems in Number Theory, 2nd ed.* New York: Springer-Verlag, pp. 231–232, 1994.
Sloane, N. J. A. Sequences A005243/M0623 and A00524/M0705 in "An On-Line Version of the Encyclopedia of Integer Sequences."

Hofstadter Triangle

For a NONZERO REAL NUMBER r and a TRIANGLE ΔABC, swing LINE SEGMENT BC about the vertex B towards vertex A through an ANGLE rB. Call the line along the rotated segment L. Construct a second line L' by rotating LINE SEGMENT BC about vertex C through an ANGLE rC. Now denote the point of intersection of L and L' by $A(r)$. Similarly, construct $B(r)$ and $C(r)$. The TRIANGLE having these points as vertices is called the Hofstadter r-triangle. Kimberling (1994) showed that the Hofstadter triangle is perspective to ΔABC, and calls PERSPECTIVE CENTER the HOFSTADTER POINT.

see also HOFSTADTER POINT

References

Kimberling, C. "Hofstadter Points." *Nieuw Arch. Wiskunde* **12**, 109–114, 1994.
Kimberling, C. "Hofstadter Points." http://www.evansville.edu/~ck6/tcenters/recent/hofstad.html.

Hölder Condition

A function $\phi(t)$ satisfies the Hölder condition on two points t_1 and t_2 on an arc L when

$$|\phi(t_2) - \phi(t_1)| \leq A|t_2 - t_1|^\mu,$$

with A and μ POSITIVE REAL constants.

Hölder Integral Inequality

If

$$\frac{1}{p} + \frac{1}{q} = 1$$

with p, $q > 1$, then

$$\int_a^b |f(x)g(x)|\,dx \leq \left[\int_a^b |f(x)|^p\,dx\right]^{1/p} \left[\int_a^b |g(x)|^q\,dx\right]^{1/q},$$

with equality when

$$|g(x)| = c|f(x)|^{p-1}.$$

If $p = q = 2$, this inequality becomes SCHWARZ'S INEQUALITY.

References

Abramowitz, M. and Stegun, C. A. (Eds.). *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables, 9th printing.* New York: Dover, p. 11, 1972.
Gradshteyn, I. S. and Ryzhik, I. M. *Tables of Integrals, Series, and Products, 5th ed.* San Diego, CA: Academic Press, p. 1099, 1993.
Hölder, O. "Über einen Mittelwertsatz." *Göttingen Nachr.*, 44, 1889.
Riesz, F. "Untersuchungen über Systeme integrierbarer Funktionen." *Math. Ann.* **69**, 456, 1910.
Riesz, F. "Su alcune disuguaglianze." *Boll. Un. Mat. It.* **7**, 77–79, 1928.
Sansone, G. *Orthogonal Functions, rev. English ed.* New York: Dover, pp. 32–33, 1991.

Hölder Sum Inequality

If

$$\frac{1}{p} + \frac{1}{q} = 1$$

with p, $q > 1$, then

$$\sum_{k=1}^n |a_kb_k| \leq \left(\sum_{k=1}^n |a_k|^p\right)^{1/p} \left(\sum_{k=1}^n |b_k|^q\right)^{1/q},$$

with equality when $|b_k| = c|a_k|^{p-1}$. If $p = q = 2$, this becomes the CAUCHY INEQUALITY.

References

Abramowitz, M. and Stegun, C. A. (Eds.). *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables, 9th printing.* New York: Dover, p. 11, 1972.
Gradshteyn, I. S. and Ryzhik, I. M. *Tables of Integrals, Series, and Products, 5th ed.* San Diego, CA: Academic Press, p. 1092, 1979.
Hardy, G. H.; Littlewood, J. E.; and Pólya, G. *Inequalities, 2nd ed.* Cambridge, England: Cambridge University Press, pp. 10–15, 1988.

Hole
A hole in a mathematical object is a TOPOLOGICAL structure which prevents the object from being continuously shrunk to a point. When dealing with TOPOLOGICAL SPACES, a DISCONNECTIVITY is interpreted as a hole in the space. Examples of holes are things like the hole in the "center" of a SPHERE or a CIRCLE and the hole produced in EUCLIDEAN SPACE cutting a KNOT out from it.

Singular HOMOLOGY GROUPS form a MEASURE of the hole structure of a SPACE, but they are one particular measure and they don't always pick up everything. HOMOTOPY GROUPS of a SPACE are another measure of holes in a SPACE, as well as BORDISM GROUPS, k-THEORY, COHOMOTOPY GROUPS, and so on.

There are many ways to measure holes in a space. Some holes are picked up by HOMOTOPY GROUPS that are not picked up by HOMOLOGY GROUPS, and some holes are picked up by HOMOLOGY GROUPS that are not picked up by HOMOTOPY GROUPS. (For example, in the TORUS, HOMOTOPY GROUPS "miss" the two-dimensional hole that is given by the TORUS itself, but the second HOMOLOGY GROUP picks that hole up.) In addition, HOMOLOGY GROUPS don't pick up the varying hole structures of the complement of KNOTS in 3-space, but the first HOMOTOPY GROUP (the fundamental group) does.

see also BRANCH CUT, BRANCH POINT, CORK PLUG, CROSS-CAP, GENUS (SURFACE), SINGULAR POINT (FUNCTION), SPHERICAL RING, TORUS

Holomorphic Function
A synonym for ANALYTIC FUNCTION.

see also ANALYTIC FUNCTION, HOMEOMORPHIC

Holonomic Constant
A limiting value of a HOLONOMIC FUNCTION near a SINGULAR POINT. Holonomic constants include APÉRY'S CONSTANT, CATALAN'S CONSTANT, PÓLYA'S RANDOM WALK CONSTANTS for $d > 2$, and PI.

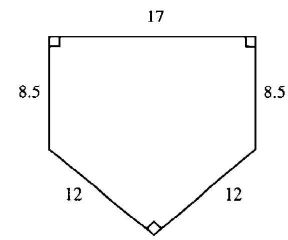
Holonomic Function
A solution of a linear homogeneous ORDINARY DIFFERENTIAL EQUATION with POLYNOMIAL COEFFICIENTS.

see also HOLONOMIC CONSTANT

References
Zeilberger, D. "A Holonomic Systems Approach to Special Function Identities." *J. Comput. Appl. Math.* **32**, 321–348, 1990.

Holonomy
A general concept in CATEGORY THEORY involving the globalization of topological or differential structures.

see also MONODROMY

Home Plate


Home plate in the game of BASEBALL is an irregular PENTAGON. However, the Little League rulebook's specification of the shape of home plate (Kreutzer and Kerley 1990), illustrated above, is not physically realizable, since it requires the existence of a (12, 12, 17) RIGHT TRIANGLE, whereas

$$12^2 + 12^2 = 288 \neq 289 = 17^2$$

(Bradley 1996).

see also BASEBALL COVER

References
Bradley, M. J. "Building Home Plate: Field of Dreams or Reality?" *Math. Mag.* **69**, 44–45, 1996.
Kreutzer, P. and Kerley, T. *Little League's Official How-to-Play Baseball Book.* New York: Doubleday, 1990.

Homeoid
A shell bounded by two similar ELLIPSOIDS having a constant ratio of axes. Given a CHORD passing through a homeoid, the distance between inner and outer intersections is equal on both sides. Since a spherical shell is a symmetric case of a homeoid, this theorem is also true for spherical shells (CONCENTRIC CIRCLES in the PLANE), for which it is easily proved by symmetry arguments.

see also CHORD, ELLIPSOID

Homeomorphic
There are two possible definitions:

1. Possessing similarity of form,
2. Continuous, ONE-TO-ONE, ONTO, and having a continuous inverse.

The most common meaning is possessing intrinsic topological equivalence. Two objects are homeomorphic if they can be deformed into each other by a continuous, invertible mapping. Homeomorphism ignores the space in which surfaces are embedded, so the deformation can be completed in a higher dimensional space than the surface was originally embedded. MIRROR IMAGES are homeomorphic, as are MÖBIUS BANDS with an EVEN number of half twists, and MÖBIUS BANDS with an ODD number of twists.

In CATEGORY THEORY terms, homeomorphisms are ISOMORPHISMS in the CATEGORY of TOPOLOGICAL SPACES and continuous maps.

see also HOMOMORPHIC, POLISH SPACE

Homeomorphic Group

If the ELEMENTS of two GROUPS are n to 1 and the correspondences satisfy the same GROUP multiplication table, the GROUPS are said to be homeomorphic.

see also ISOMORPHIC GROUPS

Homeomorphic Type

The following three pieces of information completely determine the homeomorphic type of the surface (Massey 1967):

1. Orientability,
2. Number of boundary components,
3. EULER CHARACTERISTIC.

see also ALGEBRAIC TOPOLOGY, EULER CHARACTERISTIC

References

Massey, W. S. *Algebraic Topology: An Introduction.* New York: Springer-Verlag, 1996.

Homeomorphism

see HOMEOMORPHIC, HOMEOMORPHIC GROUP, HOMEOMORPHIC TYPE, TOPOLOGICALLY CONJUGATE

HOMFLY Polynomial

A 2-variable oriented KNOT POLYNOMIAL $P_L(a,z)$ motivated by the JONES POLYNOMIAL (Freyd *et al.* 1985). Its name is an acronym for the last names of its co-discoverers: Hoste, Ocneanu, Millett, Freyd, Lickorish, and Yetter (Freyd *et al.* 1985). Independent work related to the HOMFLY polynomial was also carried out by Prztycki and Traczyk (1987). HOMFLY polynomial is defined by the SKEIN RELATIONSHIP

$$a^{-1}P_{L_+}(a,z) - aP_{L_-}(a,z) = zP_{L_0}(a,z) \qquad (1)$$

(Doll and Hoste 1991), where v is sometimes written instead of a (Kanenobu and Sumi 1993) or, with a slightly different relationship, as

$$\alpha P_{L_+}(\alpha,z) - \alpha^{-1}P_{L_-}(\alpha,z) = zP_{L_0}(\alpha,z) \qquad (2)$$

(Kauffman 1991). It is also defined as $P_L(\ell,m)$ in terms of SKEIN RELATIONSHIP

$$\ell P_{L_+} + \ell^{-1}P_{L_-} + mP_{L_0} = 0 \qquad (3)$$

(Lickorish and Millett 1988). It can be regarded as a nonhomogeneous POLYNOMIAL in two variables or a homogeneous POLYNOMIAL in three variables. In three variables the SKEIN RELATIONSHIP is written

$$xP_{L_+}(x,y,z) + yP_{L_-}(x,y,z) + zP_{L_0}(x,y,z) = 0. \qquad (4)$$

It is normalized so that $P_{\text{unknot}} = 1$. Also, for n unlinked unknotted components,

$$P_L(x,y,z) = \left(-\frac{x+y}{z}\right)^{n-1}. \qquad (5)$$

This POLYNOMIAL usually detects CHIRALITY but does not detect the distinct ENANTIOMERS of the KNOTS 09_{042}, 10_{048}, 10_{071}, 10_{091}, 10_{104}, and 10_{125} (Jones 1987). The HOMFLY polynomial of an oriented KNOT is the same if the orientation is reversed. It is a generalization of the JONES POLYNOMIAL $V(t)$, satisfying

$$V(t) = P(a=t, z=t^{1/2}-t^{-1/2}) \qquad (6)$$
$$V(t) = P(\ell = it^{-1}, m = i(t^{-1/2}-t^{1/2})). \qquad (7)$$

It is also a generalization of the ALEXANDER POLYNOMIAL $\nabla(z)$, satisfying

$$\Delta(z) = P(a=1, z=t^{1/2}-t^{-1/2}). \qquad (8)$$

The HOMFLY POLYNOMIAL of the MIRROR IMAGE K^* of a KNOT K is given by

$$P_{K^*}(\ell,m) = P_K(\ell^{-1},m), \qquad (9)$$

so P usually but not always detects CHIRALITY.

A split union of two links (i.e., bringing two links together without intertwining them) has HOMFLY polynomial

$$P(L_1 \cup L_2) = -(\ell+\ell^{-1})m^{-1}P(L_1)P(L_2). \qquad (10)$$

Also, the composition of two links

$$P(L_1 \# L_2) = P(L_1)P(L_2), \qquad (11)$$

so the POLYNOMIAL of a COMPOSITE KNOT factors into POLYNOMIALS of its constituent knots (Adams 1994).

MUTANTS have the same HOMFLY polynomials. In fact, there are infinitely many distinct KNOTS with the same HOMFLY POLYNOMIAL (Kanenobu 1986). Examples include (05_{001}, 10_{132}), (08_{008}, 10_{129}) (08_{016}, 10_{156}), and (10_{025}, 10_{056}) (Jones 1987). Incidentally, these also have the same JONES POLYNOMIAL.

M. B. Thistlethwaite has tabulated the HOMFLY polynomial for KNOTS up to 13 crossings.

see also ALEXANDER POLYNOMIAL, JONES POLYNOMIAL, KNOT POLYNOMIAL

References

Adams, C. C. *The Knot Book: An Elementary Introduction to the Mathematical Theory of Knots.* New York: W. H. Freeman, pp. 171–172, 1994.

Doll, H. and Hoste, J. "A Tabulation of Oriented Links." *Math. Comput.* **57**, 747–761, 1991.

Freyd, P.; Yetter, D.; Hoste, J.; Lickorish, W. B. R.; Millett, K.; and Oceanu, A. "A New Polynomial Invariant of Knots and Links." *Bull. Amer. Math. Soc.* **12**, 239–246, 1985.

Jones, V. "Hecke Algebra Representations of Braid Groups and Link Polynomials." *Ann. Math.* **126**, 335–388, 1987.
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Morton, H. R. and Short, H. B. "Calculating the 2-Variable Polynomial for Knots Presented as Closed Braids." *J. Algorithms* **11**, 117–131, 1990.
Przytycki, J. and Traczyk, P. "Conway Algebras and Skein Equivalence of Links." *Proc. Amer. Math. Soc.* **100**, 744–748, 1987.
Stoimenow, A. "Jones Polynomials." `http://www.informatik.hu-berlin.de/~stoimeno/ptab/j10.html`.
Weisstein, E. W. "Knots and Links." `http://www.astro.virginia.edu/~eww6n/math/notebooks/Knots.m`.

Homoclinic Point

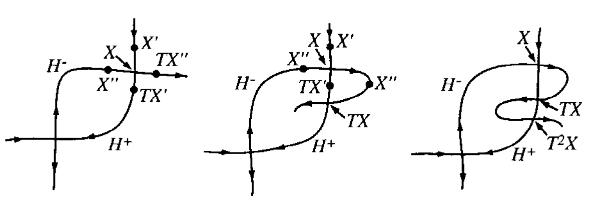
A point where a stable and an unstable separatrix (invariant manifold) from the same fixed point or same family intersect. Therefore, the limits

$$\lim_{k\to\infty} f^k(X)$$

and

$$\lim_{k\to-\infty} f^k(X)$$

exist and are equal.



Refer to the above figure. Let X be the point of intersection, with X_1 ahead of X on one MANIFOLD and X_2 ahead of X of the other. The mapping of each of these points TX_1 and TX_2 must be ahead of the mapping of X, TX. The only way this can happen is if the MANIFOLD loops back and crosses itself at a new homoclinic point. Another loop must be formed, with T^2X another homoclinic point. Since T^2X is closer to the hyperbolic point than TX, the distance between T^2X and TX is less than that between X and TX. Area preservation requires the AREA to remain the same, so each new curve (which is closer than the previous one) must extend further. In effect, the loops become longer and thinner. The network of curves leading to a dense AREA of homoclinic points is known as a homoclinic tangle or tendril. Homoclinic points appear where CHAOTIC regions touch in a hyperbolic FIXED POINT.

A small DISK centered near a homoclinic point includes infinitely many periodic points of different periods. Poincaré showed that if there is a single homoclinic point, there are an infinite number. More specifically, there are infinitely many homoclinic points in each small disk (Nusse and Yorke 1996).

see also HETEROCLINIC POINT

References

Nusse, H. E. and Yorke, J. A. "Basins of Attraction." *Science* **271**, 1376–1380, 1996.
Tabor, M. *Chaos and Integrability in Nonlinear Dynamics: An Introduction.* New York: Wiley, p. 145, 1989.

Homogeneous Coordinates

see TRILINEAR COORDINATES

Homogeneous Function

A function which satisfies

$$f(tx,ty) = t^n f(x,y)$$

for a fixed n. MEANS, the WEIERSTRAß ELLIPTIC FUNCTION, and TRIANGLE CENTER FUNCTIONS are homogeneous functions. A transformation of the variables of a TENSOR changes the TENSOR into another whose components are linear homogeneous functions of the components of the original TENSOR.

see also EULER'S HOMOGENEOUS FUNCTION THEOREM

Homogeneous Numbers

Two numbers are homogeneous if they have identical PRIME FACTORS. An example of a homogeneous pair is (6, 36), both of which share PRIME FACTORS 2 and 3:

$$6 = 2\cdot 3$$
$$36 = 2^2\cdot 3^2.$$

see also HETEROGENEOUS NUMBERS, PRIME FACTORS, PRIME NUMBER

References

Le Lionnais, F. *Les nombres remarquables.* Paris: Hermann, p. 146, 1983.

Homogeneous Polynomial

A multivariate polynomial (i.e., a POLYNOMIAL in more than one variable) with all terms having the same degree. For example, $x^3+xyz+y^2z+z^3$ is a homogeneous polynomial of degree three.

see also POLYNOMIAL

Homographic

see MÖBIUS TRANSFORMATION

Homography

A CIRCLE-preserving transformation composed of an EVEN number of inversions.

see also ANTIHOMOGRAPHY

Homological Algebra
An abstract ALGEBRA concerned with results valid for many different kinds of SPACES.

References
Hilton, P. and Stammbach, U. *A Course in Homological Algebra, 2nd ed.* New York: Springer-Verlag, 1997.
Weibel, C. A. *An Introduction to Homological Algebra.* New York: Cambridge University Press, 1994.

Homologous Points
The extremities of PARALLEL RADII of two CIRCLES are called homologous with respect to the SIMILITUDE CENTER collinear with them.

see also ANTIHOMOLOGOUS POINTS

References
Johnson, R. A. *Modern Geometry: An Elementary Treatise on the Geometry of the Triangle and the Circle.* Boston, MA: Houghton Mifflin, p. 19, 1929.

Homolographic Equal Area Projection
see MOLLWEIDE PROJECTION

Homology (Geometry)
A PERSPECTIVE COLLINEATION in which the center and axis are not incident.

see also ELATION, HARMONIC HOMOLOGY, PERSPECTIVE COLLINEATION

Homology Group
The term "homology group" usually means a singular homology group, which is an ABELIAN GROUP which partially counts the number of HOLES in a TOPOLOGICAL SPACE. In particular, singular homology groups form a MEASURE of the HOLE structure of a SPACE, but they are one particular measure and they don't always pick up everything.

In addition, there are "generalized homology groups" which are *not* singular homology groups.

Homology (Topology)
Historically, the term "homology" was first used in a topological sense by Poincaré. To him, it meant pretty much what is now called a COBORDISM, meaning that a homology was thought of as a relation between MANIFOLDS mapped into a MANIFOLD. Such MANIFOLDS form a homology when they form the boundary of a higher-dimensional MANIFOLD inside the MANIFOLD in question.

To simplify the definition of homology, Poincaré simplified the spaces he dealt with. He assumed that all the spaces he dealt with had a triangulation (i.e., they were "SIMPLICIAL COMPLEXES"). Then instead of talking about general "objects" in these spaces, he restricted himself to subcomplexes, i.e., objects in the space made up only on the simplices in the TRIANGULATION of the space. Eventually, Poincaré's version of homology was dispensed with and replaced by the more general SINGULAR HOMOLOGY. SINGULAR HOMOLOGY is the concept mathematicians mean when they say "homology."

In modern usage, however, the word homology is used to mean HOMOLOGY GROUP. For example, if someone says "X did Y by computing the homology of Z," they mean "X did Y by computing the HOMOLOGY GROUPS of Z." But sometimes homology is used more loosely in the context of a "homology in a SPACE," which corresponds to *singular* homology groups.

Singular homology groups of a SPACE measure the extent to which there are finite (compact) boundaryless GADGETS in that SPACE, such that these GADGETS are not the boundary of other finite (compact) GADGETS in that SPACE.

A generalized homology or cohomology theory must satisfy all of the EILENBERG-STEENROD AXIOMS with the exception of the DIMENSION AXIOM.

see also COHOMOLOGY, DIMENSION AXIOM, EILENBERG-STEENROD AXIOMS, GADGET, HOMOLOGICAL ALGEBRA, HOMOLOGY GROUP, SIMPLICIAL COMPLEX, SIMPLICIAL HOMOLOGY, SINGULAR HOMOLOGY

Homomorphic
Related to one another by a HOMOMORPHISM.

Homomorphism
A term used in CATEGORY THEORY to mean a general MORPHISM.

see also HOMEOMORPHISM, MORPHISM

Homoscedastic
A set of STATISTICAL DISTRIBUTIONS having the same VARIANCE.

see also HETEROSCEDASTIC

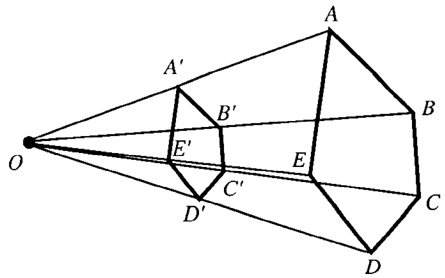
Homothecy
see DILATION

Homothetic
Two figures are homothetic if they are related by a DILATION (a dilation is also known as a HOMOTHECY). This means that they lie in the same plane and corresponding sides are PARALLEL; such figures have connectors of corresponding points which are CONCURRENT at a point known as the HOMOTHETIC CENTER. The HOMOTHETIC CENTER divides each connector in the same ratio k, known as the SIMILITUDE RATIO. For figures which are similar but do not have PARALLEL sides, a SIMILITUDE CENTER exists.

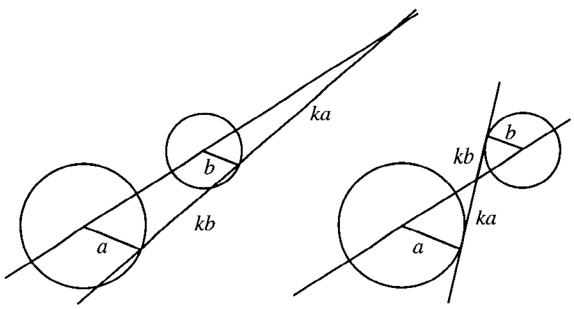
see also DILATION, HOMOTHETIC CENTER, PERSPECTIVE, SIMILITUDE RATIO

References
Johnson, R. A. *Modern Geometry: An Elementary Treatise on the Geometry of the Triangle and the Circle.* Boston, MA: Houghton Mifflin, 1929.

Homothetic Center



The meeting point of lines that connect corresponding points from HOMOTHETIC figures. In the above figure, O is the homothetic center of the HOMOTHETIC figures $ABCDE$ and $A'B'C'D'E'$. For figures which are similar but do not have PARALLEL sides, a SIMILITUDE CENTER exists (Johnson 1929, pp. 16–20).



Given two nonconcentric CIRCLES, draw RADII PARALLEL and in the same direction. Then the line joining the extremities of the RADII passes through a fixed point on the line of centers which divides that line externally in the ratio of RADII. This point is called the external homothetic center, or external center of similitude (Johnson 1929, pp. 19–20 and 41).

If RADII are drawn PARALLEL but instead in opposite directions, the extremities of the RADII pass through a fixed point on the line of centers which divides that line internally in the ratio of RADII (Johnson 1929, pp. 19–20 and 41). This point is called the internal homothetic center, or internal center of similitude (Johnson 1929, pp. 19–20 and 41).

The position of the homothetic centers for two circles of radii r_i, centers (x_i, y_i), and segment angle θ are given by solving the simultaneous equations

$$y - y_2 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_2)$$

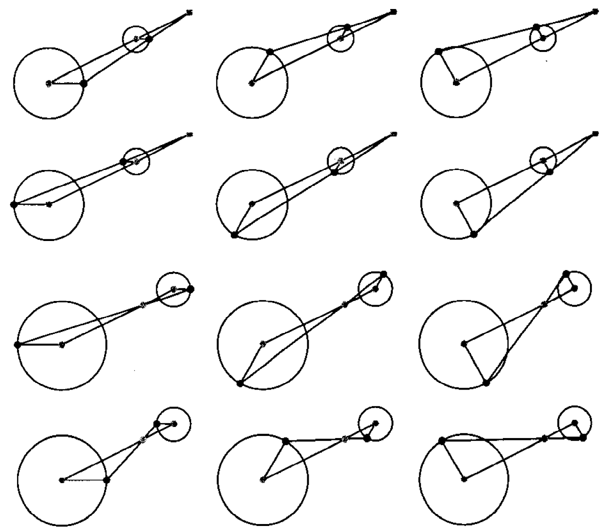
$$y - y_2^{\pm} = \frac{y_2^{\pm} - y_1^{\pm}}{x_2^{\pm} - x_1^{\pm}}(x - x_2^{\pm})$$

for (x, y), where

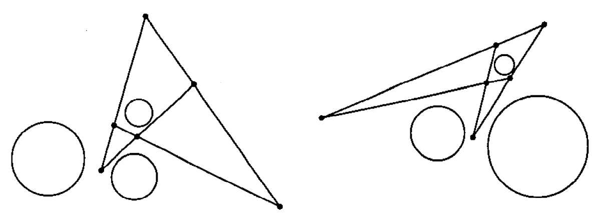
$$x_i^{\pm} \equiv x_i + (-1)^i r_i \cos\theta$$

$$y_i^{\pm} \equiv y_i + (-1)^i r_i \sin\theta,$$

and the plus signs give the external homothetic center, while the minus signs give the internal homothetic center.



As the above diagrams show, as the angles of the parallel segments are varied, the positions of the homothetic centers remain the same. This fact provides a (slotted) LINKAGE for converting circular motion with one radius to circular motion with another.



The six homothetic centers of three circles lie three by three on four lines (Johnson 1929, p. 120), which "enclose" the smallest circle.

The homothetic center of triangles is the PERSPECTIVE CENTER of HOMOTHETIC TRIANGLES. It is also called the SIMILITUDE CENTER (Johnson 1929, pp. 16–17).

see also APOLLONIUS' PROBLEM, PERSPECTIVE, SIMILITUDE CENTER

References

Johnson, R. A. *Modern Geometry: An Elementary Treatise on the Geometry of the Triangle and the Circle.* Boston, MA: Houghton Mifflin, 1929.

Weisstein, E. W. "Plane Geometry." `http://www.astro.virginia.edu/~eww6n/math/notebooks/PlaneGeometry.m`.

Homothetic Position

Two similar figures with PARALLEL homologous LINES and connectors of HOMOLOGOUS POINTS CONCURRENT at the HOMOTHETIC CENTER are said to be in homothetic position. If two SIMILAR figures are in the same plane but the corresponding sides are not PARALLEL, there exists a self-HOMOLOGOUS POINT which occupies the same homologous position with respect to the two figures.

Homothetic Triangles

Nonconcurrent TRIANGLES with PARALLEL sides are always HOMOTHETIC. Homothetic triangles are always PERSPECTIVE TRIANGLES. Their PERSPECTIVE CENTER is called their HOMOTHETIC CENTER.

Homotopy

A continuous transformation from one FUNCTION to another. A homotopy between two functions f and g from a SPACE X to a SPACE Y is a continuous MAP G from $X \in [0,1] \mapsto Y$ such that $G(x,0) = f(x)$ and $G(x,1) = g(x)$. Another way of saying this is that a homotopy is a path in the mapping SPACE $\mathrm{Map}(X,Y)$ from the first FUNCTION to the second.

see also h-COBORDISM

Homotopy Axiom

One of the EILENBERG-STEENROD AXIOMS which states that, if $f:(X,A)\to(Y,B)$ is homotopic to $g:(X,A)\to(Y,B)$, then their INDUCED MAPS $f_*:H_n(X,A)\to H_n(Y,B)$ and $g_*:H_n(X,A)\to H_n(Y,B)$ are the same.

Homotopy Group

A GROUP related to the HOMOTOPY classes of MAPS from SPHERES $\mathbb{S}^n$ into a SPACE X.

see also COHOMOTOPY GROUP

Homotopy Theory

The branch of ALGEBRAIC TOPOLOGY which deals with HOMOTOPY GROUPS.

References

Aubry, M. *Homotopy Theory and Models.* Boston, MA: Birkhäuser, 1995.

Honeycomb

A TESSELLATION in n-D, for $n \geq 3$. The only regular honeycomb in 3-D is $\{4,3,4\}$, which consists of eight cubes meeting at each VERTEX. The only quasiregular honeycomb (with regular cells and semiregular VERTEX FIGURES) has each VERTEX surrounded by eight TETRAHEDRA and six OCTAHEDRA and is denoted $\left\{\begin{matrix}3\\3,4\end{matrix}\right\}$.

There are many semiregular honeycombs, such as $\left\{\begin{matrix}3,3\\4\end{matrix}\right\}$, in which each VERTEX consists of two OCTAHEDRA $\{3,4\}$ and four CUBOCTAHEDRA $\left\{\begin{matrix}3\\4\end{matrix}\right\}$.

see also SPONGE, TESSELLATION

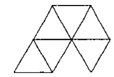
References

Bulatov, V. "Infinite Regular Polyhedra." `http://www.physics.orst.edu/~bulatov/polyhedra/infinite/`.

Hoof

see CYLINDRICAL WEDGE

Hook



A 6-POLYIAMOND.

References

Golomb, S. W. *Polyominoes: Puzzles, Patterns, Problems, and Packings, 2nd ed.* Princeton, NJ: Princeton University Press, p. 92, 1994.

Hook Length Formula

A FORMULA for the number of YOUNG TABLEAUX associated with a given YOUNG DIAGRAM. In each box, write the sum of one plus the number of boxes horizontally to the right and vertically below the box (the "hook length"). The number of tableaux is then $n!$ divided by the product of all "hook lengths". The `Combinatorica'NumberOfTableaux` function in *Mathematica*® implements the hook length formula.

see also YOUNG DIAGRAM, YOUNG TABLEAU

References

Jones, V. "Hecke Algebra Representations of Braid Groups and Link Polynomials." *Ann. Math.* **126**, 335–388, 1987.
Skiena, S. *Implementing Discrete Mathematics: Combinatorics and Graph Theory with Mathematica.* Reading, MA: Addison-Wesley, 1990.

Hopf Algebra

Let a graded module A have a multiplication ϕ and a co-multiplication ψ. Then if ϕ and ψ have the unity of k as unity and $\psi:(A,\phi)\to(A,\phi)\otimes(A,\phi)$ is an algebra homomorphism, then (A,ϕ,ψ) is called a Hopf algebra.

Hopf Bifurcation

The BIFURCATION of a FIXED POINT to a LIMIT CYCLE (Tabor 1989).

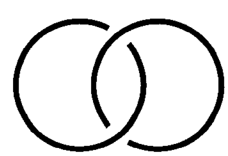
References

Guckenheimer, J. and Holmes, P. *Nonlinear Oscillations, Dynamical Systems, and Bifurcations of Vector Fields, 3rd ed.* New York: Springer-Verlag, pp. 150–154, 1997.
Marsden, J. and McCracken, M. *Hopf Bifurcation and Its Applications.* New York: Springer-Verlag, 1976.
Tabor, M. *Chaos and Integrability in Nonlinear Dynamics: An Introduction.* New York: Wiley, p. 197, 1989.

Hopf Circle

see HOPF MAP

Hopf Link



The LINK 2_1^2 which has JONES POLYNOMIAL

$$V(t) = -t - t^{-1}$$

and HOMFLY POLYNOMIAL

$$P(z,\alpha) = z^{-1}(\alpha^{-1} - \alpha^{-3}) + z\alpha^{-1}.$$

It has BRAID WORD ${\sigma_1}^2$.

Hopf Map

The first example discovered of a MAP from a higher-dimensional SPHERE to a lower-dimensional SPHERE which is not null-HOMOTOPIC. Its discovery was a shock to the mathematical community, since it was believed at the time that all such maps were null-HOMOTOPIC, by analogy with HOMOLOGY GROUPS. The Hopf map takes points (X_1, X_2, X_3, X_4) on a 3-sphere to points on a 2-sphere (x_1, x_2, x_3)

$$\begin{aligned} x_1 &= 2(X_1X_2 + X_3X_4) \\ x_2 &= 2(X_1X_4 - X_2X_3) \\ x_3 &= ({X_1}^2 + {X_3}^2) - ({X_2}^2 + {X_4}^2). \end{aligned}$$

Every point on the two SPHERES corresponds to a CIRCLE called the HOPF CIRCLE on the 3-SPHERE.

Hopf's Theorem

A NECESSARY and SUFFICIENT condition for a MEASURE which is quasi-invariant under a transformation to be equivalent to an invariant PROBABILITY MEASURE is that the transformation cannot (in a measure theoretic sense) compress the SPACE.

Horizontal

Oriented in position PERPENDICULAR to up-down, and therefore PARALLEL to a flat surface.

see also VERTICAL

Horizontal-Vertical Illusion

see VERTICAL-HORIZONTAL ILLUSION

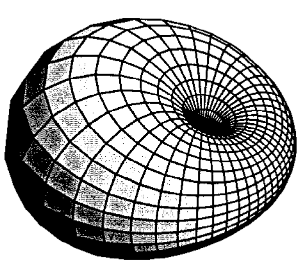
Horn Angle

The configuration formed by two curves starting at a point, called the VERTEX V, in a common direction. They are concrete illustrations of non-Archimedean geometries.

References

Kasner, E. "The Recent Theory of the Horn Angle." *Scripta Math* **11**, 263–267, 1945.

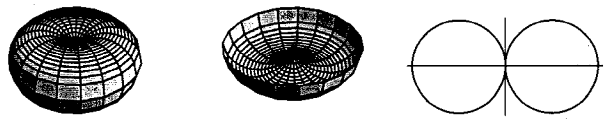
Horn Cyclide



The inversion of a HORN TORUS. If the inversion center lies on the torus, then the horn cyclide degenerates to a PARABOLIC HORN CYCLIDE.

see also CYCLIDE, HORN TORUS, PARABOLIC CYCLIDE, RING CYCLIDE, SPINDLE CYCLIDE, TORUS

Horn Torus



One of the three STANDARD TORI given by the parametric equations

$$x = (c + a\cos v)\cos u \qquad (1)$$

$$y = (c + a\cos v)\sin u \qquad (2)$$

$$z = a\sin v \qquad (3)$$

with $a = c$. The inversion of a horn torus is a HORN CYCLIDE (or PARABOLIC HORN CYCLIDE). The above left figure shows a horn torus, the middle a cutaway, and the right figure shows a CROSS-SECTION of the horn torus through the xz-plane.

see also CYCLIDE, HORN CYCLIDE, RING TORUS, SPINDLE TORUS, STANDARD TORI, TORUS

References

Gray, A. "Tori." §11.4 in *Modern Differential Geometry of Curves and Surfaces.* Boca Raton, FL: CRC Press, pp. 218–220, 1993.

Pinkall, U. "Cyclides of Dupin." §3.3 in *Mathematical Models from the Collections of Universities and Museums* (Ed. G. Fischer). Braunschweig, Germany: Vieweg, pp. 28–30, 1986.

Horned Sphere

see ALEXANDER'S HORNED SPHERE, ANTOINE'S HORNED SPHERE

Horner's Method

Let

$$P(x) = a_nx^n + \ldots + a_0 \qquad (1)$$

and $b_n \equiv a_n$. If we then define

$$b_k \equiv a_k + b_{k-1}x_0 \qquad (2)$$

for $k = n-1$, $n-2$, ..., 0, we obtain $b_0 = P(x_0)$. It therefore follows that

$$P(x) = (x - x_0)Q(x) + b_0, \qquad (3)$$

where

$$Q(x) \equiv b_nx^{n-1} + b_{n-1}x^{n-2} + \ldots + b_2x + b_1. \qquad (4)$$

In addition,

$$P'(x) = Q(x) + (x - x_0)Q'(x) \qquad (5)$$

$$P'(x_0) = Q(x_0). \qquad (6)$$

Horner's Rule

A rule for POLYNOMIAL computation which both reduces the number of necessary multiplications and results in less numerical instability due to potential subtraction of one large number from another. The rule simply factors out POWERS of x, giving

$$a_n x^n + a_{n-1} x^{n-1} + \ldots + a_0 = ((a_n x + a_{n-1})x + \ldots)x + a_0.$$

References

Vardi, I. *Computational Recreations in Mathematica.* Reading, MA: Addison-Wesley, p. 9, 1991.

Horocycle

The LOCUS of a point which is derived from a fixed point Q by continuous parallel displacement.

References

Coxeter, H. S. M. *Introduction to Geometry, 2nd ed.* New York: Wiley, p. 300, 1969.

Horse Fetter

see HIPPOPEDE

Horseshoe Map

see SMALE HORSESHOE MAP

Hough Transform

A technique used to detect boundaries in digital images.

Householder's Method

A ROOT-finding algorithm based on the iteration formula

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \left\{ 1 - \frac{[f(x_n)]^2 f''(x_n)}{2[f'(x_n)]^2} \right\}.$$

This method, like NEWTON'S METHOD, has poor convergence properties near any point where the DERIVATIVE $f'(x) = 0$.

see also NEWTON'S METHOD

References

Householder, A. S. *The Numerical Treatment of a Single Nonlinear Equation.* New York: McGraw-Hill, 1970.

Howell Design

Let S be a set of $n+1$ symbols, then a Howell design $H(s, 2n)$ on symbol set S is an $s \times s$ array H such that

1. Every cell of H is either empty or contains an unordered pair of symbols from S,
2. Every symbol of S occurs once in each row and column of H, and
3. Every unordered pair of symbols occurs in at most one cell of H.

References

Colbourn, C. J. and Dinitz, J. H. (Eds.) "Howell Designs." Ch. 26 in *CRC Handbook of Combinatorial Designs.* Boca Raton, FL: CRC Press, pp. 381–385, 1996.

Hub

The central point in a WHEEL GRAPH W_n. The hub has DEGREE $n-1$.

see also WHEEL GRAPH

Huffman Coding

A lossless data compression algorithm which uses a small number of bits to encode common characters. Huffman coding approximates the probability for each character as a POWER of 1/2 to avoid complications associated with using a nonintegral number of bits to encode characters using their actual probabilities.

References

Press, W. H.; Flannery, B. P.; Teukolsky, S. A.; and Vetterling, W. T. "Huffman Coding and Compression of Data." Ch. 20.4 in *Numerical Recipes in FORTRAN: The Art of Scientific Computing, 2nd ed.* Cambridge, England: Cambridge University Press, pp. 896–901, 1992.

Hull

see AFFINE HULL, CONVEX HULL

Humbert's Theorem

The NECESSARY and SUFFICIENT condition that an algebraic curve has an algebraic INVOLUTE is that the ARC LENGTH is a two-valued algebraic function of the coordinates of the extremities. Furthermore, this function is a ROOT of a QUADRATIC EQUATION whose COEFFICIENTS are rational functions of x and y.

References

Coolidge, J. L. *A Treatise on Algebraic Plane Curves.* New York: Dover, p. 195, 1959.

Hundkurve

see TRACTRIX

Hundred

$100 = 10^2$. Madachy (1979) gives a number of algebraic equations using the digits 1 to 9 which evaluate to 100, such as

$$\begin{aligned} (7-5)^2 + 96 + 8 - 4 - 3 - 1 &= 100 \\ 3^2 + 91 + 7 + 8 - 6 - 5 - 4 &= 100 \\ \sqrt{9} - 6 + 72 - (1)(3!) - 8 + 45 &= 100 \\ 123 - 45 - 67 + 89 &= 100, \end{aligned}$$

and so on.

see also 10, BILLION, HUNDRED, LARGE NUMBER, MILLION, THOUSAND

References

Madachy, J. S. *Madachy's Mathematical Recreations.* New York: Dover, pp. 156–159, 1979.

Hunt's Surface

An ALGEBRAIC SURFACE given by the implicit equation

$$4(x^2+y^2+z^2-13)^3+27(3x^2+y^2-4z^2-12)^2=0.$$

References
Hunt, B. "Algebraic Surfaces." `http://www.mathematik.uni-kl.de/~wwwagag/Galerie.html`.
Nordstrand, T. "Hunt's Surface." `http://www.uib.no/people/nfytn/hunttxt.htm`.

Huntington Equation

An equation proposed by Huntington (1933) as part of his definition of a BOOLEAN ALGEBRA,

$$n(n(x)+y)+n(n(x)+n(y))=x.$$

see also ROBBINS ALGEBRA, ROBBINS EQUATION

References
Huntington, E. V. "New Sets of Independent Postulates for the Algebra of Logic, with Special Reference to Whitehead and Russell's *Principia Mathematica*." *Trans. Amer. Math. Soc.* **35**, 274–304, 1933.
Huntington, E. V. "Boolean Algebra. A Correction." *Trans. Amer. Math. Soc.* **35**, 557–558, 1933.

Hurwitz Equation

The DIOPHANTINE EQUATION

$${x_1}^2+{x_2}^2+\ldots+{x_n}^2=ax_1x_2\cdots x_n$$

which has no INTEGER solutions for $a>n$.

see also LAGRANGE NUMBER (DIOPHANTINE EQUATION)

References
Guy, R. K. "Markoff Numbers." §D12 in *Unsolved Problems in Number Theory, 2nd ed.* New York: Springer-Verlag, pp. 166–168, 1994.

Hurwitz's Irrational Number Theorem

As Lagrange showed, any IRRATIONAL NUMBER α has an infinity of rational approximations p/q which satisfy

$$\left|\alpha-\frac{p}{q}\right|<\frac{1}{\sqrt{5}\,q^2}. \tag{1}$$

Similarly, if $\alpha\neq\frac{1}{2}(1+\sqrt{5}\,)$,

$$\left|\alpha-\frac{p}{q}\right|<\frac{1}{\sqrt{8}\,q^2}, \tag{2}$$

and if $\alpha\neq\frac{1}{2}(1+\sqrt{5}\,)\neq\sqrt{2}$,

$$\left|\alpha-\frac{p}{q}\right|<\frac{5}{\sqrt{221}}\frac{1}{q^2}. \tag{3}$$

In general, even tighter bounds of the form

$$\left|\alpha-\frac{p}{q}\right|<\frac{1}{L_nq^2} \tag{4}$$

can be obtained for the best rational approximation possible for an arbitrary irrational number α, where the L_n are called LAGRANGE NUMBERS and get steadily larger for each "bad" set of irrational numbers which is excluded.

see also HURWITZ'S IRRATIONAL NUMBER THEOREM, LIOUVILLE'S RATIONAL APPROXIMATION THEOREM, LIOUVILLE-ROTH CONSTANT, MARKOV NUMBER, ROTH'S THEOREM, SEGRE'S THEOREM, THUE-SIEGEL-ROTH THEOREM

References
Ball, W. W. R. and Coxeter, H. S. M. *Mathematical Recreations and Essays, 13th ed.* New York: Dover, p. 40, 1987.
Chandrasekharan, K. *An Introduction to Analytic Number Theory.* Berlin: Springer-Verlag, p. 23, 1968.
Conway, J. H. and Guy, R. K. *The Book of Numbers.* New York: Springer-Verlag, pp. 187–189, 1996.

Hurwitz Number

A number with a continued fraction whose terms are the values of one or more POLYNOMIALS evaluated on consecutive INTEGERS and then interleaved. This property is preserved by MÖBIUS TRANSFORMATIONS (Beeler *et al.* 1972, p. 44).

References
Beeler, M.; Gosper, R. W.; and Schroeppel, R. *HAKMEM.* Cambridge, MA: MIT Artificial Intelligence Laboratory, Memo AIM-239, Feb. 1972.

Hurwitz Polynomial

A POLYNOMIAL with REAL POSITIVE COEFFICIENTS and ROOTS which are either NEGATIVE or pairwise conjugate with NEGATIVE REAL PARTS.

Hurwitz-Radon Theorem

Determined the possible values of r and n for which there is an IDENTITY of the form

$$(x_1{}^2 + \ldots + x_r{}^2)(y_1{}^2 + \ldots + y_r{}^2) = z_1{}^2 + \ldots + z_n{}^2.$$

Hurwitz's Root Theorem

Let $\{f(x)\}$ be a SEQUENCE of ANALYTIC FUNCTIONS REGULAR in a region G, and let this sequence be UNIFORMLY CONVERGENT in every CLOSED SUBSET of G. If the ANALYTIC FUNCTION

$$\lim_{n\to\infty} f_n(x) = f(x)$$

does not vanish identically, then if $x = a$ is a zero of $f(x)$ of order k, a NEIGHBORHOOD $|x - a| < \delta$ of $x = a$ and a number N exist such that if $n > N$, $f_n(x)$ has exactly k zeros in $|x - a| < \delta$.

References

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Hurwitz Zeta Function

A generalization of the RIEMANN ZETA FUNCTION with a FORMULA

$$\zeta(s,a) \equiv \sum_{k=0}^{\infty} \frac{1}{(k+a)^s}, \tag{1}$$

where any term with $k + a = 0$ is excluded. The Hurwitz zeta function can also be given by the functional equation

$$\zeta\left(s, \frac{p}{q}\right) = 2\Gamma(1-s)(2\pi q)^{s-1} \sum_{n=1}^{q} \sin\left(\frac{\pi s}{2} + \frac{2\pi np}{q}\right) \zeta\left(1-s, \frac{n}{q}\right) \tag{2}$$

(Apostol 1976, Miller and Adamchik), or the integral

$$\zeta(s,a) = \tfrac{1}{2}a^{-3} + \frac{a^{1-s}}{s-1} + 2\int_0^\infty (a^2+y^2)^{-s/2} \left\{\sin\left[s\tan^{-1}\left(\frac{y}{a}\right)\right]\right\} \frac{dy}{e^{2\pi y}-1}. \tag{3}$$

If $\Re[z] < 0$, then

$$\zeta(z,a) = \frac{2\Gamma(1-z)}{(2\pi)^{1-z}} \left[\sin\left(\frac{\pi z}{2}\right) \sum_{n=1}^{\infty} \frac{\cos(2\pi an)}{n^{1-z}} + \cos\left(\frac{\pi z}{2}\right) \sum_{n=1}^{\infty} \frac{\sin(2\pi an)}{n^{1-z}}\right]. \tag{4}$$

The Hurwitz zeta function satisfies

$$\zeta(0,a) = \tfrac{1}{2} - a \tag{5}$$

$$\frac{d}{ds}\zeta(0,a) = \ln[\Gamma(a)] - \tfrac{1}{2}\ln(2\pi) \tag{6}$$

$$\frac{d}{ds}\zeta(0,0) = \tfrac{1}{2}\ln(2\pi), \tag{7}$$

where $\Gamma(z)$ is the GAMMA FUNCTION. The POLYGAMMA FUNCTION $\psi_m(z)$ can be expressed in terms of the Hurwitz zeta function by

$$\psi_m(z) = (-1)^{m+1} m!\,\zeta(1+m,z). \tag{8}$$

For POSITIVE integers k, p, and $q > p$,

$$\begin{aligned}\zeta'\left(-2k+1, \frac{p}{q}\right) &= \frac{[\psi(2k) - \ln(2\pi q)]B_{2k}(p/q)}{2k} - \frac{[\psi(2k) - \ln(2\pi)]B_{2k}}{q^{2k}2k} \\ &+ \frac{(-1)^{k+1}\pi}{(2\pi q)^{2k}} \sum_{n=1}^{q-1} \sin\left(\frac{2\pi pn}{q}\right) \psi_{(2k-1)}\left(\frac{n}{q}\right) \\ &+ \frac{(-1)^{k+1}2(2k-1)!}{(2\pi q)^{2k}} \sum_{n=1}^{q-1} \cos\left(\frac{2\pi pn}{q}\right) \zeta'\left(2k, \frac{n}{q}\right) \\ &+ \frac{\zeta'(-2k+1)}{q^{2k}},\end{aligned} \tag{9}$$

where B_n is a BERNOULLI NUMBER, $B_n(x)$ a BERNOULLI POLYNOMIAL, $\psi_n(z)$ is a POLYGAMMA FUNCTION, and $\zeta(z)$ is a RIEMANN ZETA FUNCTION (Miller and Adamchik). Miller and Adamchik also give the closed-form expressions

$$\zeta'(-2k+1, \tfrac{1}{2}) = -\frac{B_{2k}\ln 2}{4^k k} - \frac{(2^{2k-1}-1)\zeta'(-2k+1)}{2^{2k-1}} \tag{10}$$

$$\begin{aligned}\zeta'\left(-2k+1, \begin{matrix}\frac{1}{3}\\ \frac{2}{3}\end{matrix}\right) &= \mp\frac{(9^k-1)B_{2k}\pi}{\sqrt{3}(3^{2k-1}-1)8k} - \frac{B_{2k}\ln 3}{(3^{2k-1})4k} \\ &\mp \frac{(-1)^k\psi_{2k-1}(\frac{1}{3})}{2\sqrt{3}(6\pi)^{2k-1}} - \frac{(3^{2k-1}-1)\zeta'(-2k+1)}{2(3^{2k-1})}\end{aligned} \tag{11}$$

$$\begin{aligned}\zeta'\left(-2k+1, \begin{matrix}\frac{1}{4}\\ \frac{3}{4}\end{matrix}\right) &= \mp\frac{(4^k+1)B_{2k}\pi}{4^{k+1}k} + \frac{(4^{k-1}-1)B_{2k}\ln 2}{2^{3k-1}k} \\ &\mp \frac{(-1)^k\psi_{2k-1}(\frac{1}{4})}{4(8\pi)^{2k-1}} - \frac{(2^{2k-1}-1)\zeta'(-2k+1)}{2^{4k-1}}\end{aligned} \tag{12}$$

$$\begin{aligned}\zeta'\left(-2k+1, \begin{matrix}\frac{1}{6}\\ \frac{5}{6}\end{matrix}\right) &= \mp\frac{(9^k-1)(2^{2k-1}+1)B_{2k}\pi}{\sqrt{3}(6^{2k-1})8k} \\ &+ \frac{B_{2k}(3^{2k-1}-1)\ln 2}{(6^{2k-1})4k} + \frac{B_{2k}(2^{2k-1}-1)\ln 3}{(6^{2k-1})4k} \\ &\mp \frac{(-1)^k(2^{2k-1}+1)\psi_{2k-1}(\frac{1}{3})}{2\sqrt{3}\,(12\pi)^{2k-1}} \\ &+ \frac{(2^{2k-1}-1)(3^{2k-1}-1)\zeta'(-2k+1)}{2(6^{2k-1})}.\end{aligned} \tag{13}$$

see also KHINTCHINE'S CONSTANT, POLYGAMMA FUNCTION, PSI FUNCTION, RIEMANN ZETA FUNCTION, ZETA FUNCTION

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Hutton's Formula

The MACHIN-LIKE FORMULA

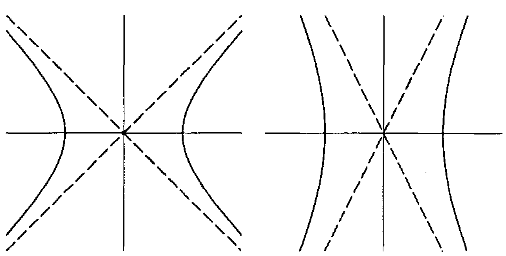
$$\tfrac{1}{4}\pi = 2\tan^{-1}(\tfrac{1}{3}) + \tan^{-1}(\tfrac{1}{7}).$$

The other two-term MACHIN-LIKE FORMULAS are EULER'S MACHIN-LIKE FORMULA, HERMANN'S FORMULA, and MACHIN'S FORMULA.

Hutton's Method

see LAMBERT'S METHOD

Hyperbola



In general, a hyperbola is defined as the LOCUS of all points in the PLANE the difference of whose distance from two fixed points (the FOCI F_1 and F_2) separated by a distance $2c$, where

$$c \equiv \sqrt{a^2 + b^2}, \tag{1}$$

is a given POSITIVE constant. By analogy with the definition of the ELLIPSE, the equation for a hyperbola with SEMIMAJOR AXIS a parallel to the x-AXIS and SEMIMINOR AXIS b parallel to the y-AXIS is given by

$$\frac{(x-x_0)^2}{a^2} - \frac{(y-y_0)^2}{b^2} = 1. \tag{2}$$

Unlike the ELLIPSE, no points of the hyperbola actually lie on the SEMIMINOR AXIS, but rather the ratio b/a determined the vertical scaling of the hyperbola. The ECCENTRICITY of the hyperbola is defined as

$$e \equiv \frac{c}{a} = \sqrt{1 + \frac{b^2}{a^2}}. \tag{3}$$

In the standard equation of the hyperbola, the center is located at (x_0, y_0), the FOCI are at $(x_0 \pm c, y_0)$, and the vertices are at $(x_0 \pm a, y_0)$. The so-called ASYMPTOTES (shown as the dashed lines in the above figures) can be found by substituting 0 for the 1 on the right side of the general equation (2),

$$y = \pm\frac{b}{a}(x - x_0) + y_0, \tag{4}$$

and therefore have SLOPES $\pm b/a$.

The special case $a = b$ (the left diagram above) is known as a RIGHT HYPERBOLA because the ASYMPTOTES are PERPENDICULAR.

In POLAR COORDINATES, the equation of a hyperbola centered at the ORIGIN (i.e., with $x_0 = y_0 = 0$) is

$$r^2 = \frac{a^2b^2}{b^2\cos^2\theta - a^2\sin^2\theta}. \tag{5}$$

In POLAR COORDINATES centered at a FOCUS,

$$r = \frac{a(e^2-1)}{1 - e\cos\theta}. \tag{6}$$

The two-center BIPOLAR COORDINATES equation with origin at a FOCUS is

$$r_1 - r_2 = \pm 2a. \tag{7}$$

The parametric equations for the hyperbola are

$$x = \pm a\cosh t \tag{8}$$
$$y = b\sinh t. \tag{9}$$

The CURVATURE and TANGENTIAL ANGLE are

$$\kappa(t) = -[\cosh(2t)]^{-3/2} \tag{10}$$
$$\phi(t) = -\tan^{-1}(\tanh t). \tag{11}$$

The special case of the RIGHT HYPERBOLA was first studied by Menaechmus. Euclid and Aristaeus wrote about the general hyperbola, but only studied one branch of it. The hyperbola was given its present name by Apollonius, who was the first to study both branches. The FOCUS and DIRECTRIX were considered by Pappus (MacTutor Archive). The hyperbola is the shape of an orbit of a body on an escape trajectory (i.e., a body with positive energy), such as some comets, about a fixed mass, such as the sun.

The LOCUS of the apex of a variable CONE containing an ELLIPSE fixed in 3-space is a hyperbola through the FOCI of the ELLIPSE. In addition, the LOCUS of the apex of a CONE containing that hyperbola is the original ELLIPSE. Furthermore, the ECCENTRICITIES of the ELLIPSE and hyperbola are reciprocals.

see also CONIC SECTION, ELLIPSE, HYPERBOLOID, JERABEK'S HYPERBOLA, KIEPERT'S HYPERBOLA, PARABOLA, QUADRATIC CURVE, RECTANGULAR HYPERBOLA, REFLECTION PROPERTY, RIGHT HYPERBOLA

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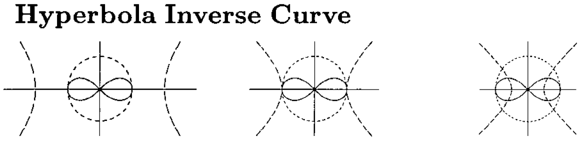
Hyperbola Evolute

The EVOLUTE of a RECTANGULAR HYPERBOLA is the LAMÉ CURVE

$$(ax)^{2/3} - (by)^{2/3} = (a+b)^{2/3}.$$

From a point between the two branches of the EVOLUTE, two NORMALS can be drawn to the HYPERBOLA. However, from a point beyond the EVOLUTE, four NORMALS can be drawn.

Hyperbola Inverse Curve

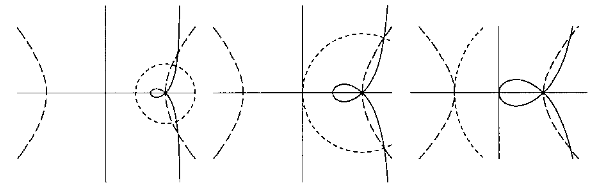


For a HYPERBOLA with $a = b$ with INVERSION CENTER at the center, the INVERSE CURVE

$$x = \frac{2k\cos t}{a[3-\cos(2t)]} \tag{1}$$

$$y = \frac{k\sin(2t)}{a[3-\cos(2t)]} \tag{2}$$

is a LEMNISCATE.

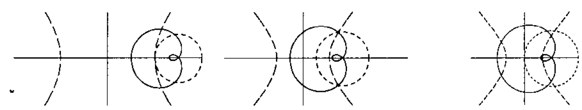


For an INVERSION CENTER at the VERTEX, the INVERSE CURVE

$$x = a + \frac{4k\cos t\sin^2(\frac{1}{2}t)}{a[5-4\cos t+\cos(2t)-2\sin(2t)]} \tag{3}$$

$$y = a + \frac{k(\tan t - 1)}{a[(\sec t - 1)^2 + (\tan t - 1)^2]} \tag{4}$$

is a RIGHT STROPHOID.

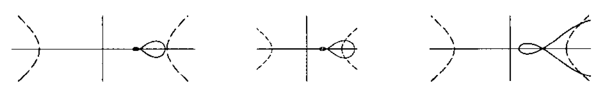


For an INVERSION CENTER at the FOCUS, the INVERSE CURVE

$$x = ae = \frac{k\cos t(1-e\cos t)}{a(\cos t - e)^2} \tag{5}$$

$$y = \frac{\sqrt{e^2-1}\,k\sin(2t)}{2a(\cos t - e)^2} \tag{6}$$

is a LIMAÇON, where e is the ECCENTRICITY.



For a HYPERBOLA with $a = \sqrt{3}\,b$ and INVERSION CENTER at the VERTEX, the INVERSE CURVE

$$x = b + \frac{2k\cos t(\sqrt{3}-\cos t)}{b[9-4\sqrt{3}\cos t+\cos(2t)-2\sin(2t)]} \tag{7}$$

$$y = b + \frac{k(\tan t - 1)}{b[(\sqrt{3}\sec t - 1)^2 + (\tan t - 1)^2]} \tag{8}$$

is a MACLAURIN TRISECTRIX.

References

Lawrence, J. D. *A Catalog of Special Plane Curves.* New York: Dover, p. 203, 1972.

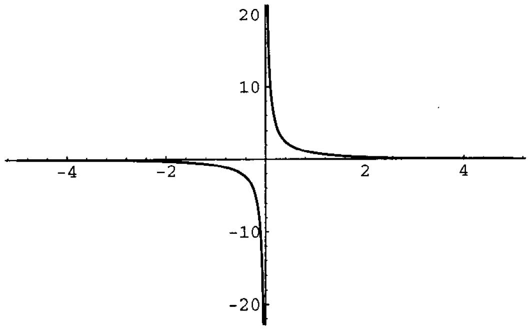
Hyperbola Pedal Curve

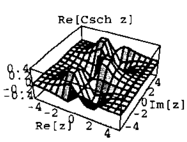
The PEDAL CURVE of a HYPERBOLA with the PEDAL POINT at the FOCUS is a CIRCLE. The PEDAL CURVE of a RECTANGULAR HYPERBOLA with PEDAL POINT at the center is a LEMNISCATE.

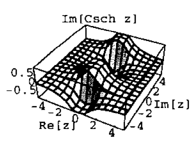
Hyperbolic Automorphism

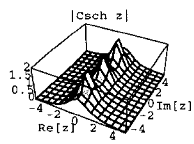
see ANOSOV AUTOMORPHISM

Hyperbolic Cosecant









The hyperbolic cosecant is defined as

$$\operatorname{csch} x \equiv \frac{1}{\sinh x} = \frac{2}{e^x - e^{-x}}.$$

see also BERNOULLI NUMBER, BIPOLAR COORDINATES, BIPOLAR CYLINDRICAL COORDINATES, COSECANT, HELMHOLTZ DIFFERENTIAL EQUATION—TOROIDAL COORDINATES, HYPERBOLIC SINE, POINSOT'S SPIRALS, SURFACE OF REVOLUTION, TOROIDAL FUNCTION

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Abramowitz, M. and Stegun, C. A. (Eds.). "Hyperbolic Functions." §4.5 in *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables, 9th printing.* New York: Dover, pp. 83–86, 1972.

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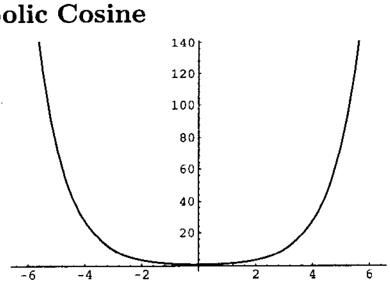
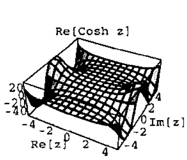
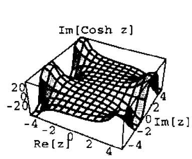
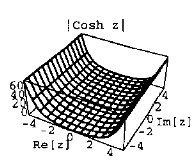
Hyperbolic Cosine









The hyperbolic cosine is defined as

$$\cosh x \equiv \tfrac{1}{2}(e^x + e^{-x}).$$

This function describes the shape of a hanging cable, known as the CATENARY.

see also BIPOLAR COORDINATES, BIPOLAR CYLINDRICAL COORDINATES, BISPHERICAL COORDINATES, CATENARY, CATENOID, CHI, CONICAL FUNCTION, CORRELATION COEFFICIENT—GAUSSIAN BIVARIATE DISTRIBUTION, COSINE, CUBIC EQUATION, DE MOIVRE'S IDENTITY, ELLIPTIC CYLINDRICAL COORDINATES, ELSASSER FUNCTION, FIBONACCI HYPERBOLIC COSINE, FIBONACCI HYPERBOLIC SINE, HYPERBOLIC GEOMETRY, HYPERBOLIC LEMNISCATE FUNCTION, HYPERBOLIC SINE, HYPERBOLIC SECANT, HYPERBOLIC TANGENT, INVERSIVE DISTANCE, LAPLACE'S EQUATION—BIPOLAR COORDINATES, LAPLACE'S EQUATION—BISPHERICAL COORDINATES, LAPLACE'S EQUATION—TOROIDAL COORDINATES, LEMNISCATE FUNCTION, LORENTZ GROUP, MATHIEU DIFFERENTIAL EQUATION, MEHLER'S BESSEL FUNCTION FORMULA, MERCATOR PROJECTION, MODIFIED BESSEL FUNCTION OF THE FIRST KIND, OBLATE SPHEROIDAL COORDINATES, PROLATE SPHEROIDAL COORDINATES, PSEUDOSPHERE, RAMANUJAN COS/COSH IDENTITY, SINE-GORDON EQUATION, SURFACE OF REVOLUTION, TOROIDAL COORDINATES

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Abramowitz, M. and Stegun, C. A. (Eds.). "Hyperbolic Functions." §4.5 in *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables, 9th printing.* New York: Dover, pp. 83–86, 1972.

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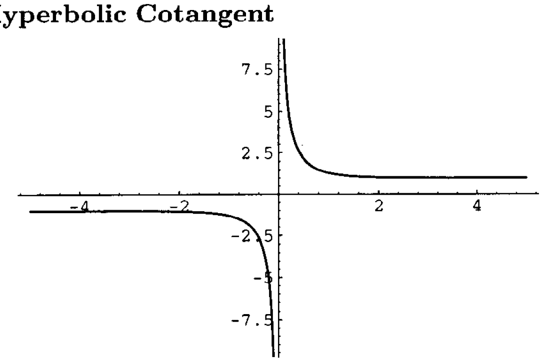
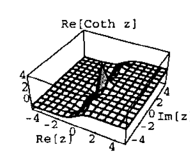
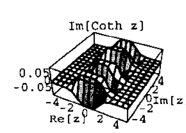
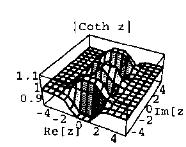
Hyperbolic Cotangent









The hyperbolic cotangent is defined as

$$\coth x \equiv \frac{e^x + e^{-x}}{e^x - e^{-x}} = \frac{e^{2x} + 1}{e^{2x} - 1}.$$

Its LAURENT SERIES is

$$\coth x = \frac{1}{x} + \tfrac{1}{3}x - \tfrac{1}{45}x^3 + \ldots.$$

see also BERNOULLI NUMBER, BIPOLAR COORDINATES, BIPOLAR CYLINDRICAL COORDINATES, COTANGENT, FIBONACCI HYPERBOLIC COTANGENT, HYPERBOLIC TANGENT, LAPLACE'S EQUATION—TOROIDAL COORDINATES, LEBESGUE CONSTANTS (FOURIER SERIES), PROLATE SPHEROIDAL COORDINATES, SURFACE OF REVOLUTION, TOROIDAL COORDINATES, TOROIDAL FUNCTION

References
Abramowitz, M. and Stegun, C. A. (Eds.). "Hyperbolic Functions." §4.5 in *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables, 9th printing.* New York: Dover, pp. 83–86, 1972.
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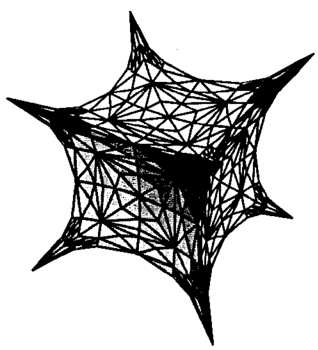
Hyperbolic Cube

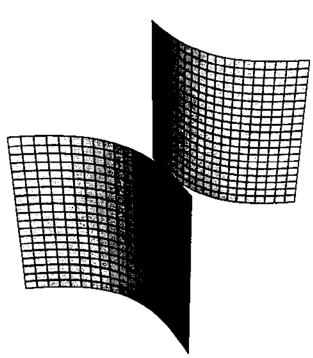


A hyperbolic version of the Euclidean CUBE.

see also HYPERBOLIC DODECAHEDRON, HYPERBOLIC OCTAHEDRON, HYPERBOLIC TETRAHEDRON

References
Rivin, I. "Hyperbolic Polyhedron Graphics." `http://www.mathsource.com/cgi-bin/MathSource/Applications/Graphics/3D/0201-788`.

Hyperbolic Cylinder

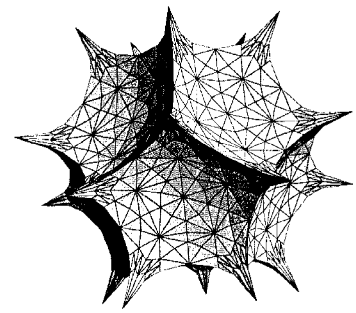


A QUADRATIC SURFACE given by the equation

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = -1.$$

see also ELLIPTIC PARABOLOID, PARABOLOID

Hyperbolic Dodecahedron



A hyperbolic version of the Euclidean DODECAHEDRON.

see also HYPERBOLIC CUBE, HYPERBOLIC OCTAHEDRON, HYPERBOLIC TETRAHEDRON

References
Rivin, I. "Hyperbolic Polyhedron Graphics." `http://www.mathsource.com/cgi-bin/MathSource/Applications/Graphics/3D/0201-788`.

Hyperbolic Fixed Point (Differential Equations)

A FIXED POINT for which the STABILITY MATRIX has EIGENVALUES $\lambda_1 < 0 < \lambda_2$, also called a SADDLE POINT.

see also ELLIPTIC FIXED POINT (DIFFERENTIAL EQUATIONS), FIXED POINT, STABLE IMPROPER NODE, STABLE SPIRAL POINT, STABLE STAR, UNSTABLE IMPROPER NODE, UNSTABLE NODE, UNSTABLE SPIRAL POINT, UNSTABLE STAR

References
Tabor, M. "Classification of Fixed Points." §1.4.b in *Chaos and Integrability in Nonlinear Dynamics: An Introduction.* New York: Wiley, pp. 22–25, 1989.

Hyperbolic Fixed Point (Map)

A FIXED POINT of a LINEAR TRANSFORMATION (MAP) for which the rescaled variables satisfy

$$(\delta - \alpha)^2 + 4\beta\gamma > 0.$$

see also ELLIPTIC FIXED POINT (MAP), LINEAR TRANSFORMATION, PARABOLIC FIXED POINT

Hyperbolic Functions

The hyperbolic functions sinh, cosh, tanh, csch, sech, coth (HYPERBOLIC SINE, HYPERBOLIC COSINE, etc.) share many properties with the corresponding CIRCULAR FUNCTIONS. The hyperbolic functions arise in many problems of mathematics and mathematical physics in which integrals involving $\sqrt{1+x^2}$ arise (whereas the CIRCULAR FUNCTIONS involve $\sqrt{1-x^2}$).

For instance, the HYPERBOLIC SINE arises in the gravitational potential of a cylinder and the calculation of the Roche limit. The HYPERBOLIC COSINE function is the shape of a hanging cable (the so-called CATENARY). The HYPERBOLIC TANGENT arises in the calculation of magnetic moment and rapidity of special relativity. All three appear in the Schwarzschild metric using external isotropic Kruskal coordinates in general relativity. The HYPERBOLIC SECANT arises in the profile of a laminar jet. The HYPERBOLIC COTANGENT arises in the Langevin function for magnetic polarization.

The hyperbolic functions are defined by

$$\sinh z \equiv \frac{e^z - e^{-z}}{2} = -\sinh(-z) \tag{1}$$

$$\cosh z \equiv \frac{e^z + e^{-z}}{2} = \cosh(-z) \tag{2}$$

$$\tanh z \equiv \frac{e^z - e^{-z}}{e^z + e^{-z}} = \frac{e^{2z}-1}{e^{2z}+1} \tag{3}$$

$$\operatorname{csch} z \equiv \frac{2}{e^z - e^{-z}} \tag{4}$$

$$\operatorname{sech} z \equiv \frac{2}{e^z + e^{-z}} \tag{5}$$

$$\coth z \equiv \frac{e^z + e^{-z}}{e^z - e^{-z}} = \frac{e^{2z}+1}{e^{2z}-1}. \tag{6}$$

For purely IMAGINARY arguments,

$$\sinh(iz) = i\sin z \tag{7}$$

$$\cosh(iz) = \cos z. \tag{8}$$

The hyperbolic functions satisfy many identities anomalous to the trigonometric identities (which can be inferred using OSBORNE'S RULE) such as

$$\cosh^2 x - \sinh^2 x = 1 \tag{9}$$

$$\cosh x + \sinh x = e^x \tag{10}$$

$$\cosh x - \sinh x = e^{-x}. \tag{11}$$

See also Beyer (1987, p. 168). Some half-angle FORMULAS are

$$\tanh\left(\frac{z}{2}\right) = \frac{\sinh x + i\sin y}{\cosh x + \cos y} \tag{12}$$

$$\coth\left(\frac{z}{2}\right) = \frac{\sinh x - i\sin y}{\cosh x - \cos y}. \tag{13}$$

Some double-angle FORMULAS are

$$\sinh(2x) = 2\sinh x\cosh x \tag{14}$$

$$\cosh(2x) = 2\cosh^2 x - 1 = 1 + 2\sinh^2 x. \tag{15}$$

Identities for COMPLEX arguments include

$$\sinh(x+iy) = \sinh x\cos y + i\cosh x\sin y \tag{16}$$

$$\cosh(x+iy) = \cosh x\cos y + i\sinh x\sin y. \tag{17}$$

The ABSOLUTE SQUARES for COMPLEX arguments are

$$|\sinh(z)|^2 = \sinh^2 x + \sin^2 y \tag{18}$$

$$|\cosh(z)|^2 = \sinh^2 x + \cos^2 y. \tag{19}$$

Integrals involving hyperbolic functions include

$$\begin{aligned}\int \frac{dx}{x\sqrt{a+bx}} &= \ln\left|\frac{\sqrt{a+bx}-\sqrt{a}}{\sqrt{a+bx}+\sqrt{a}}\right| \\ &= \ln\left|\frac{(\sqrt{a+bx}-\sqrt{a})^2}{(a+bx)-a}\right| \\ &= \ln\left|\frac{(a+bx)-2\sqrt{a(a+bx)}+a}{bx}\right|.\end{aligned} \tag{20}$$

If $b > 0$, then

$$\begin{aligned}\int \frac{dx}{x\sqrt{a+bx}} &= \ln\left|\frac{2a+bx-2\sqrt{a(a+bx)}}{bx}\right| \\ &= \ln\left|\left(\frac{2a}{bx}+1\right) - 2\sqrt{\frac{a}{bx}\left(\frac{a}{bx}+1\right)}\right|.\end{aligned} \tag{21}$$

Let $z \equiv 2a/bx + 1$, and $a/bx = (z-1)/2$ and

$$\begin{aligned}\int \frac{dx}{x\sqrt{a+bx}} &= \ln\left[z - 2\sqrt{\tfrac{1}{2}(z-1)\tfrac{1}{2}(z+1)}\right] \\ &= \ln\left[z - \sqrt{(z-1)(z+1)}\right] \\ &= \ln\left(z - \sqrt{z^2-1}\right) = \cosh^{-1}(z) \\ &= \cosh^{-1}\left(1 + \frac{2a}{bx}\right) \\ &= 2\tanh\left(-\sqrt{\frac{a}{a+bx}}\right).\end{aligned} \tag{22}$$

see also HYPERBOLIC COSECANT, HYPERBOLIC COSINE, HYPERBOLIC COTANGENT, GENERALIZED HYPERBOLIC FUNCTIONS, HYPERBOLIC INVERSE FUNCTIONS, HYPERBOLIC SECANT, HYPERBOLIC SINE, HYPERBOLIC TANGENT, HYPERBOLIC INVERSE FUNCTIONS, OSBORNE'S RULE

References

Abramowitz, M. and Stegun, C. A. (Eds.). "Hyperbolic Functions." §4.5 in *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables, 9th printing.* New York: Dover, pp. 83–86, 1972.

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Coxeter, H. S. M. and Greitzer, S. L. *Geometry Revisited.* Washington, DC: Math. Assoc. Amer., pp. 126–131, 1967.

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Hyperbolic Geometry

A NON-EUCLIDEAN GEOMETRY, also called LOBACHEVSKY-BOLYAI-GAUSS GEOMETRY, having constant SECTIONAL CURVATURE -1. This GEOMETRY satisfies all of EUCLID'S POSTULATES *except* the PARALLEL POSTULATE, which is modified to read: For any infinite straight LINE L and any POINT P not on it, there are *many other* infinitely extending straight LINES that pass through P and which do not intersect L.

In hyperbolic geometry, the sum of ANGLES of a TRIANGLE is less than 180°, and TRIANGLES with the same angles have the same areas. Furthermore, not all TRIANGLES have the same ANGLE sum (c.f. the AAA THEOREM for TRIANGLES in Euclidean 2-space). The best-known example of a hyperbolic space are SPHERES in Lorentzian 4-space. The POINCARÉ HYPERBOLIC DISK is a hyperbolic 2-space. Hyperbolic geometry is well understood in 2-D, but not in 3-D.

Geometric models of hyperbolic geometry include the KLEIN-BELTRAMI MODEL, which consists of an OPEN DISK in the Euclidean plane whose open chords correspond to hyperbolic lines. A 2-D model is the POINCARÉ HYPERBOLIC DISK. Felix Klein constructed an analytic hyperbolic geometry in 1870 in which a POINT is represented by a pair of REAL NUMBERS (x_1, x_2) with

$$x_1{}^2 + x_2{}^2 < 1$$

(i.e., points of an OPEN DISK in the COMPLEX PLANE) and the distance between two points is given by

$$d(x, X) = a\cosh^{-1}\left[\frac{1 - x_1X_1 - x_2X_2}{\sqrt{1 - x_1{}^2 - x_2{}^2}\sqrt{1 - X_1{}^2 - X_2{}^2}}\right].$$

The geometry generated by this formula satisfies all of EUCLID'S POSTULATES except the fifth. The METRIC of this geometry is given by the CAYLEY-KLEIN-HILBERT METRIC,

$$g_{11} = \frac{a^2(1 - x_2{}^2)}{(1 - x_1{}^2 - x_2{}^2)^2}$$

$$g_{12} = \frac{a^2 x_1 x_2}{(1 - x_1{}^2 - x_2{}^2)^2}$$

$$g_{22} = \frac{a^2(1 - x_1{}^2)}{(1 - x_1{}^2 - x_2{}^2)^2}.$$

Hilbert extended the definition to general bounded sets in a EUCLIDEAN SPACE.

see also ELLIPTIC GEOMETRY, EUCLIDEAN GEOMETRY, HYPERBOLIC METRIC, KLEIN-BELTRAMI MODEL, NON-EUCLIDEAN GEOMETRY, SCHWARZ-PICK LEMMA

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Hyperbolic Inverse Functions

$$\sinh^{-1}\left(\frac{a}{b}\right) = \ln\left(a + \sqrt{a^2 + b^2}\right) \tag{1}$$

$$\cosh^{-1} z = \ln\left(z \pm \sqrt{z^2 - 1}\right) \tag{2}$$

$$\tanh^{-1}\left(\frac{a}{b}\right) = \tfrac{1}{2}\ln\left(\frac{b + a}{b - a}\right) \tag{3}$$

$$\operatorname{csch}^{-1} z = \ln\left(1 \pm \sqrt{1 + z^2}\right) \tag{4}$$

$$\operatorname{sech}^{-1} z = \ln\left(\frac{1 \pm \sqrt{1 - z^2}}{z}\right) \tag{5}$$

$$\coth^{-1} z = \tfrac{1}{2}\ln\left(\frac{z + 1}{z - 1}\right). \tag{6}$$

References

Abramowitz, M. and Stegun, C. A. (Eds.). "Hyperbolic Functions." §4.6 in *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables, 9th printing.* New York: Dover, pp. 86–89, 1972.

Hyperbolic Knot

A hyperbolic knot is a KNOT that has a complement that can be given a metric of constant curvature -1. The only KNOTS which are not hyperbolic are TORUS KNOTS and SATELLITE KNOTS (including COMPOSITE KNOTS), as proved by Thurston in 1978. Therefore, all but six of the PRIME KNOTS with 10 or fewer crossings are hyperbolic. The exceptions with nine or fewer crossings are 03_{001} (the(3, 2)-TORUS KNOT), 05_{001}, 07_{001}, 08_{019} (the (4, 3)-TORUS KNOT), and 09_{001}.

Almost all hyperbolic knots can be distinguished by their hyperbolic volumes (exceptions being 05_{002} and a certain 12-crossing knot; see Adams 1994, p. 124). It has been conjectured that the smallest hyperbolic volume is 2.0298..., that of the FIGURE-OF-EIGHT KNOT.

MUTANT KNOTS have the same hyperbolic knot volume.

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Hyperbolic Lemniscate Function

By analogy with the LEMNISCATE FUNCTIONS, hyperbolic lemniscate functions can also be defined

$$\operatorname{arcsinhlemn} x \equiv \int_0^x (1+t^4)^{1/2}\,dt \tag{1}$$

$$\operatorname{arccoshlemn} x \equiv \int_x^1 (1+t^4)^{1/2}\,dt. \tag{2}$$

Let $0 \leq \theta \leq \pi/2$ and $0 \leq v \leq 1$, and write

$$\frac{\theta\mu}{2} = \int_0^v \frac{dt}{\sqrt{1+t^2}}, \tag{3}$$

where μ is the constant obtained by setting $\theta = \pi/2$ and $v = 1$. Then

$$\mu = \frac{2}{\pi} K\left(\frac{1}{\sqrt{2}}\right), \tag{4}$$

where $K(k)$ is a complete ELLIPTIC INTEGRAL OF THE FIRST KIND, and Ramanujan showed

$$2\tan^{-1} v = \theta + \sum_{n=1}^{\infty} \frac{\sin(2n\theta)}{n\cosh(n\pi)}, \tag{5}$$

$$\tfrac{1}{8}\pi - \tfrac{1}{2}\tan^{-1}(v^2) = \sum_{n=0}^{\infty} \frac{(-1)^n \cos[(2n+1)\theta]}{(2n+1)\cosh[\frac{1}{2}(2n+1)\pi]} \tag{6}$$

and

$$\ln\left(\frac{1+v}{1-v}\right) = \ln[\tan(\tfrac{1}{4}\pi + \tfrac{1}{2}\theta)] + 4\sum_{n=0}^{\infty} \frac{(-1)^n \sin[(2n+1)\theta]}{(2n+1)[e^{(2n+1)\pi} - 1]} \tag{7}$$

(Berndt 1994).

see also LEMNISCATE FUNCTION

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Berndt, B. C. *Ramanujan's Notebooks, Part IV.* New York: Springer-Verlag, pp. 255–258, 1994.

Hyperbolic Map

A linear MAP $\mathbb{R}^n$ is hyperbolic if none of its EIGENVALUES have modulus 1. This means that $\mathbb{R}^n$ can be written as a direct sum of two A-invariant SUBSPACES E^s and E^u (where s stands for stable and u for unstable). This means that there exist constants $C > 0$ and $0 < \lambda < 1$ such that

$$||A^n v|| \leq C\lambda^n ||v|| \quad \text{if } v \in E^s$$

$$||A^{-n} v|| \leq C\lambda^n ||v|| \quad \text{if } v \in E^u$$

for $n = 0, 1, \ldots$.

see also PESIN THEORY

Hyperbolic Metric

The METRIC for the POINCARÉ HYPERBOLIC DISK, a model for HYPERBOLIC GEOMETRY. The hyperbolic metric is invariant under conformal maps of the disk onto itself.

see also HYPERBOLIC GEOMETRY, POINCARÉ HYPERBOLIC DISK

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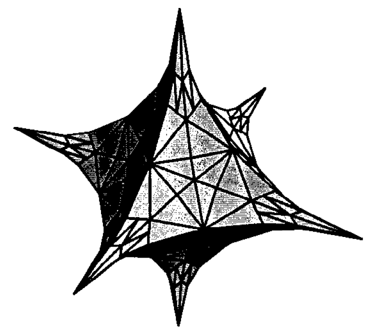
Hyperbolic Octahedron



A hyperbolic version of the Euclidean OCTAHEDRON, which is a special case of the ASTROIDAL ELLIPSOID with $a = b = c = 1$. It is given by the parametric equations

$$x = (\cos u \cos v)^3$$
$$y = (\sin u \cos v)^3$$
$$z = \sin^3 v$$

for $u \in [-\pi/2, \pi/2]$ and $v \in [-\pi, \pi]$.

see also ASTROIDAL ELLIPSOID, HYPERBOLIC CUBE, HYPERBOLIC DODECAHEDRON, HYPERBOLIC TETRAHEDRON

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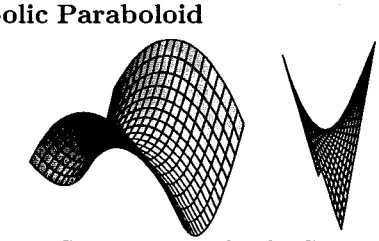
Rivin, I. "Hyperbolic Polyhedron Graphics." `http://www.mathsource.com/cgi-bin/MathSource/Applications/Graphics/3D/0201-788`.

Hyperbolic Paraboloid



A QUADRATIC SURFACE given by the Cartesian equation

$$z = \frac{y^2}{b^2} - \frac{x^2}{a^2} \tag{1}$$

(left figure). This form has parametric equations

$$x(u,v) = a(u+v) \tag{2}$$

$$y(u,v) = \pm bv \tag{3}$$

$$z(u,v) = u^2 + 2uv \tag{4}$$

(Gray 1993, p. 336). An alternative form is

$$z = xy \tag{5}$$

(right figure; Fischer 1986), which has parametric equations

$$x(u,v) = u \tag{6}$$

$$y(u,v) = v \tag{7}$$

$$z(u,v) = uv. \tag{8}$$

see also ELLIPTIC PARABOLOID, PARABOLOID, RULED SURFACE

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Hyperbolic Partial Differential Equation

A PARTIAL DIFFERENTIAL EQUATION of second-order, i.e., one of the form

$$Au_{xx} + 2Bu_{xy} + Cu_{yy} + Du_x + Eu_y + F = 0, \tag{1}$$

is called hyperbolic if the MATRIX

$$\mathsf{Z} \equiv \begin{bmatrix} A & B \\ B & C \end{bmatrix} \tag{2}$$

satisfies $\det(\mathsf{Z}) < 0$. The WAVE EQUATION is an example of a hyperbolic partial differential equation. Initial-boundary conditions are used to give

$$u(x,y,t) = g(x,y,t) \quad \text{for } x \in \partial\Omega, t > 0 \tag{3}$$

$$u(x,y,0) = v_0(x,y) \quad \text{in } \Omega \tag{4}$$

$$u_t(x,y,0) = v_1(x,y) \quad \text{in } \Omega, \tag{5}$$

where

$$u_{xy} = f(u_x, u_t, x, y) \tag{6}$$

holds in Ω.

see also ELLIPTIC PARTIAL DIFFERENTIAL EQUATION, PARABOLIC PARTIAL DIFFERENTIAL EQUATION, PARTIAL DIFFERENTIAL EQUATION

Hyperbolic Plane

In the hyperbolic plane $\mathbb{H}^2$, a pair of LINES can be PARALLEL (diverging from one another in one direction and intersecting at an IDEAL POINT at infinity in the other), can intersect, or can be HYPERPARALLEL (diverge from each other in both directions).

see also EUCLIDEAN PLANE, RIGID MOTION

Hyperbolic Point

A point **p** on a REGULAR SURFACE $M \in \mathbb{R}^3$ is said to be hyperbolic if the GAUSSIAN CURVATURE $K(\mathbf{p}) < 0$ or equivalently, the PRINCIPAL CURVATURES κ_1 and κ_2, have opposite signs.

see also ANTICLASTIC, ELLIPTIC POINT, GAUSSIAN CURVATURE, HYPERBOLIC FIXED POINT (DIFFERENTIAL EQUATIONS), HYPERBOLIC FIXED POINT (MAP), PARABOLIC POINT, PLANAR POINT, SYNCLASTIC

References

Gray, A. *Modern Differential Geometry of Curves and Surfaces.* Boca Raton, FL: CRC Press, p. 280, 1993.

Hyperbolic Polyhedron

A POLYHEDRON in a HYPERBOLIC GEOMETRY.

see HYPERBOLIC CUBE, HYPERBOLIC DODECAHEDRON, HYPERBOLIC OCTAHEDRON, HYPERBOLIC TETRAHEDRON

Hyperbolic Rotation

Also known as the LORENTZ TRANSFORMATION or PROCRUSTIAN STRETCH. Leaves each branch of the HYPERBOLA $x'y' = xy$ invariant and transforms CIRCLES into ELLIPSES with the same AREA.

$$x' = \mu^{-1}x$$

$$y' = \mu y.$$

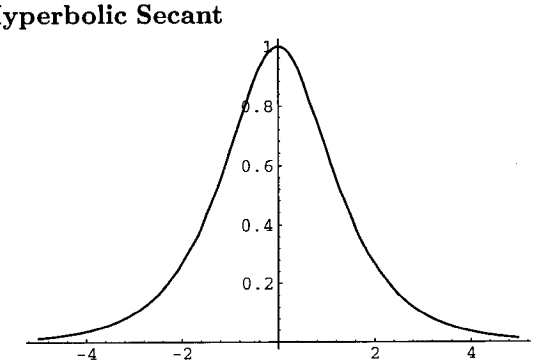
Hyperbolic Rotation (Crossed)

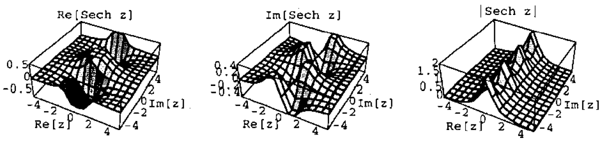
Exchanges branches of the HYPERBOLA $x'y' = xy$.

$$x' = \mu^{-1}x$$

$$y' = -\mu y.$$

Hyperbolic Secant





The hyperbolic secant is defined as

$$\operatorname{sech} x \equiv \frac{1}{\cosh x} = \frac{2}{e^x + e^{-x}}.$$

It has a MAXIMUM at $x = 0$ and inflection points at $x = \pm \operatorname{sech}^{-1}(1/\sqrt{2}) \approx 0.881374$.

see also BENSON'S FORMULA, CATENARY, CATENOID, EULER NUMBER, HYPERBOLIC COSINE, OBLATE SPHEROIDAL COORDINATES, PSEUDOSPHERE, SECANT, SURFACE OF REVOLUTION, TRACTRIX, TRACTROID

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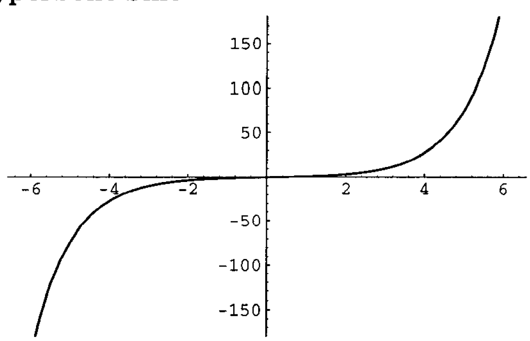
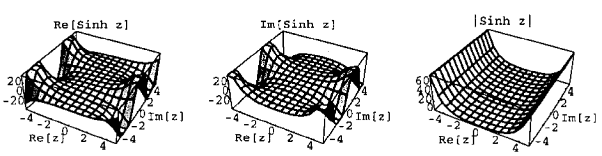
Hyperbolic Sine





The hyperbolic sine is defined as

$$\sinh x \equiv \tfrac{1}{2}(e^x - e^{-x}).$$

see also BETA FUNCTION (EXPONENTIAL), BIPOLAR COORDINATES, BIPOLAR CYLINDRICAL COORDINATES, BISPHERICAL COORDINATES, CATENARY, CATENOID, CONICAL FUNCTION, CUBIC EQUATION, DE MOIVRE'S IDENTITY, DIXON-FERRAR FORMULA, ELLIPTIC CYLINDRICAL COORDINATES, ELSASSER FUNCTION, FIBONACCI HYPERBOLIC COSINE, FIBONACCI HYPERBOLIC SINE, GUDERMANNIAN FUNCTION, HELICOID, HELMHOLTZ DIFFERENTIAL EQUATION—ELLIPTIC CYLINDRICAL COORDINATES, HYPERBOLIC COSECANT, LAPLACE'S EQUATION—BISPHERICAL COORDINATES, LAPLACE'S EQUATION—TOROIDAL COORDINATES, LEBESGUE CONSTANTS (FOURIER SERIES), LORENTZ GROUP, MERCATOR PROJECTION, MILLER CYLINDRICAL PROJECTION, MODIFIED BESSEL FUNCTION OF THE SECOND KIND, MODIFIED SPHERICAL BESSEL FUNCTION, MODIFIED STRUVE FUNCTION, NICHOLSON'S FORMULA, OBLATE SPHEROIDAL COORDINATES, PARABOLA INVOLUTE, PARTITION FUNCTION P, POINSOT'S SPIRALS, PROLATE SPHEROIDAL COORDINATES, RAMANUJAN'S TAU FUNCTION, SCHLÄFLI'S FORMULA, SHI, SINE, SINE-GORDON EQUATION, SURFACE OF REVOLUTION, TOROIDAL COORDINATES, TOROIDAL FUNCTION, TRACTRIX, WATSON'S FORMULA

References
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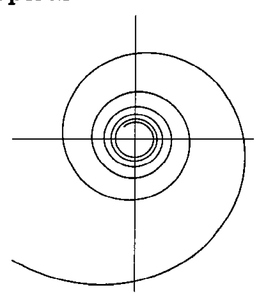
Spanier, J. and Oldham, K. B. "The Hyperbolic Sine sinh(x) and Cosine cosh(x) Functions." Ch. 28 in *An Atlas of Functions.* Washington, DC: Hemisphere, pp. 263–271, 1987.

Hyperbolic Space

see HYPERBOLIC GEOMETRY

Hyperbolic Spiral



An ARCHIMEDEAN SPIRAL with POLAR equation

$$r = \frac{a}{\theta}.$$

The hyperbolic spiral originated with Pierre Varignon in 1704 and was studied by Johann Bernoulli between 1710 and 1713, as well as by Cotes in 1722 (MacTutor Archive).

see also ARCHIMEDEAN SPIRAL, SPIRAL

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MacTutor History of Mathematics Archive. "Hyperbolic Spiral." `http://www-groups.dcs.st-and.ac.uk/~history/Curves/Hyperbolic.html`.

Hyperbolic Spiral Inverse Curve

Taking the pole as the INVERSION CENTER, the HYPERBOLIC SPIRAL inverts to ARCHIMEDES' SPIRAL

$$r = a\theta.$$

Hyperbolic Spiral Roulette

The ROULETTE of the pole of a HYPERBOLIC SPIRAL rolling on a straight line is a TRACTRIX.

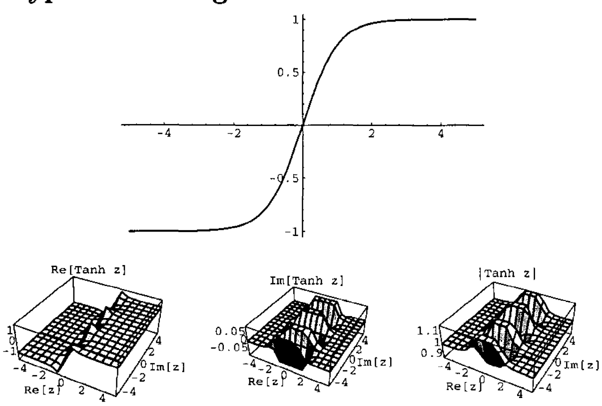
Hyperbolic Substitution

A substitution which can be used to transform integrals involving square roots into a more tractable form.

Form	Substitution
$\sqrt{x^2+a^2}$	$x = a\sinh u$
$\sqrt{x^2-a^2}$	$x = a\cosh u$

see also TRIGONOMETRIC SUBSTITUTION

Hyperbolic Tangent



By way of analogy with the usual TANGENT

$$\tan x \equiv \frac{\sin x}{\cos x},$$

the hyperbolic tangent is defined as

$$\tanh x \equiv \frac{\sinh x}{\cosh x} = \frac{e^x - e^{-x}}{e^x + e^{-x}} = \frac{e^{2x} - 1}{e^{2x} + 1},$$

where $\sinh x$ is the HYPERBOLIC SINE and $\cosh x$ is the HYPERBOLIC COSINE. The hyperbolic tangent can be written using a CONTINUED FRACTION as

$$\tanh x = \cfrac{x}{1 + \cfrac{x^2}{3 + \cfrac{x^2}{5 + \ldots}}}.$$

see also BERNOULLI NUMBER, CATENARY, CORRELATION COEFFICIENT—GAUSSIAN BIVARIATE DISTRIBUTION, FIBONACCI HYPERBOLIC TANGENT, FISHER'S z'-TRANSFORMATION, HYPERBOLIC COTANGENT, LORENTZ GROUP, MERCATOR PROJECTION, OBLATE SPHEROIDAL COORDINATES, PSEUDOSPHERE, SURFACE OF REVOLUTION, TANGENT, TRACTRIX, TRACTROID

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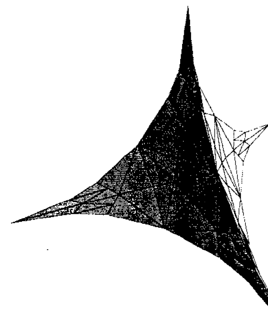
Spanier, J. and Oldham, K. B. "The Hyperbolic Tangent $\tanh(x)$ and Cotangent $\coth(x)$ Functions." Ch. 30 in *An Atlas of Functions.* Washington, DC: Hemisphere, pp. 279–284, 1987.

Hyperbolic Tetrahedron



A hyperbolic version of the Euclidean TETRAHEDRON.

see also HYPERBOLIC CUBE, HYPERBOLIC DODECAHEDRON, HYPERBOLIC OCTAHEDRON

References

Rivin, I. "Hyperbolic Polyhedron Graphics." `http://www.mathsource.com/cgi-bin/MathSource/Applications/Graphics/3D/0201-788`.

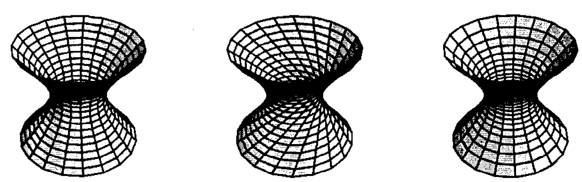
Hyperbolic Umbilic Catastrophe

A CATASTROPHE which can occur for three control factors and two behavior axes.

see also ELLIPTIC UMBILIC CATASTROPHE

Hyperboloid

A QUADRATIC SURFACE which may be one- or two-sheeted.



The one-sheeted circular hyperboloid is a doubly RULED SURFACE. When oriented along the z-AXIS, the one-sheeted circular hyperboloid has CARTESIAN COORDINATES equation

$$\frac{x^2}{a^2} + \frac{y^2}{a^2} - \frac{z^2}{c^2} = 1, \tag{1}$$

and parametric equation

$$x = a\sqrt{1+u^2}\cos v \quad (2)$$

$$y = a\sqrt{1+u^2}\sin v \quad (3)$$

$$z = cu \quad (4)$$

for $v \in [0, 2\pi)$ (left figure). Other parameterizations include

$$x(u,v) = a(\cos u \mp v \sin u) \quad (5)$$

$$y(u,v) = a(\sin u \pm v \cos u) \quad (6)$$

$$z(u,v) = \pm cv, \quad (7)$$

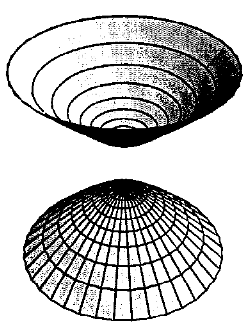
(middle figure), or

$$x(u,v) = a \cosh v \cos u \quad (8)$$

$$y(u,v) = a \cosh v \sin u \quad (9)$$

$$z(u,v) = c \sinh v \quad (10)$$

(right figure). An obvious generalization gives the one-sheeted ELLIPTIC HYPERBOLOID.



A two-sheeted circular hyperboloid oriented along the z-AXIS has CARTESIAN COORDINATES equation

$$\frac{x^2}{a^2} + \frac{y^2}{a^2} - \frac{z^2}{c^2} = -1. \quad (11)$$

The parametric equations are

$$x = a \sinh u \cos v \quad (12)$$

$$y = a \sinh u \sin v \quad (13)$$

$$z = \pm c \cosh u \quad (14)$$

for $v \in [0, 2\pi)$. Note that the plus and minus signs in z correspond to the upper and lower sheets. The two-sheeted circular hyperboloid oriented along the x-AXIS has Cartesian equation

$$\frac{x^2}{a^2} - \frac{y^2}{a^2} - \frac{z^2}{c^2} = 1 \quad (15)$$

and parametric equations

$$x = \pm a \cosh u \cosh v \quad (16)$$

$$y = a \sinh u \cosh v \quad (17)$$

$$z = c \sinh v \quad (18)$$

(Gray 1993, p. 313). Again, an obvious generalization gives the two-sheeted ELLIPTIC HYPERBOLOID.

The SUPPORT FUNCTION of the hyperboloid of one sheet

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1 \quad (19)$$

is

$$h = \left(\frac{x^2}{a^4} + \frac{y^2}{b^4} + \frac{z^2}{c^4}\right)^{-1/2}, \quad (20)$$

and the GAUSSIAN CURVATURE is

$$K = -\frac{h^4}{a^2b^2c^2}. \quad (21)$$

The SUPPORT FUNCTION of the hyperboloid of two sheets

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1 \quad (22)$$

is

$$h = \left(\frac{x^2}{a^4} - \frac{y^2}{b^4} + \frac{z^2}{c^4}\right)^{-1/2}, \quad (23)$$

and the GAUSSIAN CURVATURE is

$$K = \frac{h^4}{a^2b^2c^2} \quad (24)$$

(Gray 1993, pp. 296–297).

see also CATENOID, ELLIPSOID, ELLIPTIC HYPERBOLOID, HYPERBOLOID EMBEDDING, PARABOLOID, RULED SURFACE

References

Fischer, G. (Ed.). Plates 67 and 69 in *Mathematische Modelle/Mathematical Models, Bildband/Photograph Volume.* Braunschweig, Germany: Vieweg, pp. 62 and 64, 1986.

Gray, A. "The Hyperboloid of Revolution." §18.5 in *Modern Differential Geometry of Curves and Surfaces.* Boca Raton, FL: CRC Press, pp. 296–297, 311–314, and 369–370, 1993.

Hyperboloid Embedding

A 4-HYPERBOLOID has NEGATIVE CURVATURE, with

$$R^2 = x^2 + y^2 + z^2 - w^2 \quad (1)$$

$$2x\frac{dx}{dw} + 2y\frac{dy}{dw} + 2z\frac{dz}{dw} - 2w = 0. \quad (2)$$

Since

$$\mathbf{r} \equiv x\hat{\mathbf{x}} + y\hat{\mathbf{y}} + z\hat{\mathbf{z}}, \quad (3)$$

$$dw = \frac{x\,dx + y\,dy + z\,dz}{w} = \frac{\mathbf{r}\cdot d\mathbf{r}}{\sqrt{r^2 - R^2}}. \quad (4)$$

To stay on the surface of the HYPERBOLOID,

$$\begin{aligned} ds^2 &= dx^2 + dy^2 + dz^2 - dw^2 \\ &= dx^2 + dy^2 + dz^2 - \frac{r^2\,dr^2}{r^2 - R^2} \\ &= dr^2 + r^2\,d\Omega^2 + \frac{dr^2}{1 - \frac{R^2}{r^2}}. \end{aligned} \tag{5}$$

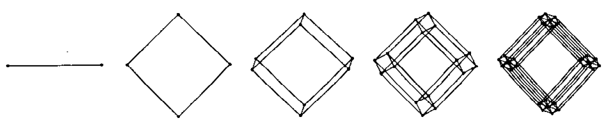
Hypercomplex Number

A number having properties departing from those of the REAL and COMPLEX NUMBERS. The most common examples are BIQUATERNIONS, EXTERIOR ALGEBRAS, GROUP algebras, MATRICES, OCTONIONS, and QUATERNIONS.

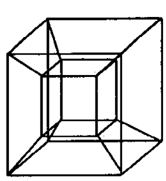
References

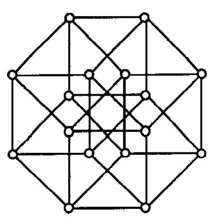
van der Waerden, B. L. *A History of Algebra from al-Khwarizmi to Emmy Noether.* New York: Springer-Verlag, pp. 177–217, 1985.

Hypercube



The generalization of a 3-CUBE to n-D, also called a MEASURE POLYTOPE. It is a regular POLYTOPE with mutually PERPENDICULAR sides, and is therefore an ORTHOTOPE. It is denoted γ_n and has SCHLÄFLI SYMBOL $\{4, \underbrace{3,3}_{n-2}\}$. The number of k-cubes contained in an n-cube can be found from the COEFFICIENTS of $(2k+1)^n$.





The 1-hypercube is a LINE SEGMENT, the 2-hypercube is the SQUARE, and the 3-hypercube is the CUBE. The hypercube in $\mathbb{R}^4$, called a TESSERACT, has the SCHLÄFLI SYMBOL $\{4,3,3\}$ and VERTICES $(\pm1, \pm1, \pm1, \pm1)$. The above figures show two visualizations of the TESSERACT. The figure on the left is a projection of the TESSERACT in 3-space (Gardner 1977), and the figure on the right is the GRAPH of the TESSERACT symmetrically projected into the PLANE (Coxeter 1973). A TESSERACT has 16 VERTICES, 32 EDGES, four SQUARES, and eight CUBES.

see also CROSS POLYTOPE, CUBE, HYPERSPHERE, ORTHOTOPE, PARALLELEPIPED, POLYTOPE, SIMPLEX, TESSERACT

References

Coxeter, H. S. M. *Regular Polytopes, 3rd ed.* New York: Dover, p. 123, 1973.

Gardner, M. "Hypercubes." Ch. 4 in *Mathematical Carnival: A New Round-Up of Tantalizers and Puzzles from Scientific American.* New York: Vintage Books, 1977.

Geometry Center. "The Tesseract (or Hypercube)." http://www.geom.umn.edu/docs/outreach/4-cube/.

Pappas, T. "How Many Dimensions are There?" *The Joy of Mathematics.* San Carlos, CA: Wide World Publ./Tetra, pp. 204–205, 1989.

Hyperdeterminant

A technically defined extension of the ordinary DETERMINANT to "higher dimensional" HYPERMATRICES. Cayley (1845) originally coined the term, but subsequently used it to refer to an ALGEBRAIC INVARIANT of a multilinear form. The hyperdeterminant of the $2\times2\times2$ HYPERMATRIX $A = a_{ijk}$ (for $i, j, k = 0, 1$) is given by

$$\begin{aligned} \det(A) = {} & ({a_{000}}^2{a_{111}}^2 + {a_{001}}^2{a_{110}}^2 + {a_{010}}^2{a_{101}}^2 + {a_{011}}^2{a_{100}}^2) \\ & - 2(a_{000}a_{001}a_{110}a_{111} + a_{000}a_{010}a_{101}a_{111} + a_{000}a_{011}a_{100}a_{111} \\ & + a_{001}a_{010}a_{101}a_{110} + a_{001}a_{011}a_{110}a_{100} + a_{010}a_{011}a_{101}a_{100}) \\ & + 4(a_{000}a_{011}a_{101}a_{110} + a_{001}a_{010}a_{100}a_{111}). \end{aligned}$$

The above hyperdeterminant vanishes IFF the following system of equations in six unknowns has a nontrivial solution,

$$\begin{aligned} a_{000}x_0y_0 + a_{010}x_0y_1 + a_{100}x_1y_0 + a_{110}x_1y_1 &= 0 \\ a_{001}x_0y_0 + a_{011}x_0y_1 + a_{101}x_1y_0 + a_{111}x_1y_1 &= 0 \\ a_{000}x_0z_0 + a_{001}x_0z_1 + a_{100}x_1z_0 + a_{101}x_1z_1 &= 0 \\ a_{010}x_0z_0 + a_{011}x_0z_1 + a_{110}x_1z_0 + a_{111}x_1z_1 &= 0 \\ a_{000}y_0z_0 + a_{001}y_0z_1 + a_{010}y_1z_0 + a_{011}y_1z_1 &= 0 \\ a_{100}y_0z_0 + a_{101}y_0z_1 + a_{110}y_1z_0 + a_{111}y_1z_1 &= 0. \end{aligned}$$

see also DETERMINANT, HYPERMATRIX

References

Cayley, A. "On the Theory of Linear Transformations." *Cambridge Math. J.* **4**, 193–209, 1845.

Gel'fand, I. M.; Kapranov, M. M.; and Zelevinsky, A. V. "Hyperdeterminants." *Adv. Math.* **96**, 226–263, 1992.

Schläfli, L. "Über die Resultante eine Systemes mehrerer algebraischer Gleichungen." *Denkschr. Kaiserl. Akad. Wiss., Math.-Naturwiss. Klasse* **4**, 1852.

Hyperellipse

$$y^{n/m} + c\left|\frac{x}{a}\right|^{n/m} - c = 0,$$

with $n/m > 2$. If $n/m < 2$, the curve is a HYPOELLIPSE.

see also ELLIPSE, HYPOELLIPSE, SUPERELLIPSE

References

von Seggern, D. *CRC Standard Curves and Surfaces.* Boca Raton, FL: CRC Press, p. 82, 1993.

Hyperelliptic Function

see ABELIAN FUNCTION

Hyperelliptic Integral

see ABELIAN INTEGRAL

Hyperfactorial

The function defined by

$$H(n) \equiv K(n+1) \equiv 1^1 2^2 3^3 \cdots n^n,$$

where K is the K-FUNCTION and the first few values for $n = 1, 2, \ldots$ are 1, 4, 108, 27648, 86400000, 4031078400000, 3319766398771200000, ... (Sloane's A002109), and these numbers are called hyperfactorials by Sloane and Plouffe (1995).

see also G-FUNCTION, GLAISHER-KINKELIN CONSTANT, K-FUNCTION

References

Sloane, N. J. A. Sequence A002109/M3706 in "An On-Line Version of the Encyclopedia of Integer Sequences."

Hypergeometric Differential Equation

$$x(x-1)\frac{d^2y}{dx^2} + [(1+\alpha+\beta)x - \gamma]\frac{dy}{dx} + \alpha\beta y = 0.$$

It has REGULAR SINGULAR POINTS at 0, 1, and ∞. Every ORDINARY DIFFERENTIAL EQUATION of second-order with at most three REGULAR SINGULAR POINTS can be transformed into the hypergeometric differential equation.

see also CONFLUENT HYPERGEOMETRIC DIFFERENTIAL EQUATION, CONFLUENT HYPERGEOMETRIC FUNCTION, HYPERGEOMETRIC FUNCTION

References

Morse, P. M. and Feshbach, H. *Methods of Theoretical Physics, Part I.* New York: McGraw-Hill, pp. 542–543, 1953.

Hypergeometric Distribution

Let there be n ways for a successful and m ways for an unsuccessful trial out of a total of $n+m$ possibilities. Take N samples and let x_i equal 1 if selection i is successful and 0 if it is not. Let x be the total number of successful selections,

$$x \equiv \sum_{i=1}^{N} x_i. \tag{1}$$

The probability of i successful selections is then

$$\begin{aligned} P(x=i) &= \\ &\frac{[\text{\# ways for } i \text{ successes}][\text{\# ways for } N-i \text{ unsuccesses}]}{[\text{total number of ways to select}]} \\ &= \frac{\binom{n}{i}\binom{m}{N-i}}{\binom{n+m}{N}} = \frac{\frac{n!}{i!(n-i)!}\frac{m!}{(m+i-N)!(N-i)!}}{\frac{(n+m)!}{N!(N-n-m)!}} \\ &= \frac{n!m!N!(N-m-n)!}{i!(n-i)!(m+i-N)!(N-i)!(n+m)!}. \end{aligned} \tag{2}$$

The ith selection has an equal likelihood of being in any trial, so the fraction of acceptable selections p is

$$p \equiv \frac{n}{n+m} \tag{3}$$

$$P(x_i = 1) = \frac{n}{n+m} \equiv p. \tag{4}$$

The expectation value of x is

$$\begin{aligned} \mu \equiv \langle x \rangle &= \left\langle \sum_{i=1}^{N} x_i \right\rangle = \sum_{i=1}^{N} \langle x_i \rangle \\ &= \sum_{i=1}^{N} \frac{n}{n+m} = \frac{nN}{n+m} = Np. \end{aligned} \tag{5}$$

The VARIANCE is

$$\operatorname{var}(x) \equiv \sum_{i=1}^{N} \operatorname{var}(x_i) + \sum_{i=1}^{N} \sum_{\substack{j=1 \\ j\neq i}}^{N} \operatorname{cov}(x_i, x_j). \tag{6}$$

Since x_i is a BERNOULLI variable,

$$\begin{aligned} \operatorname{var}(x_i) = p(1-p) &= \frac{n}{n+m}\left(1 - \frac{n}{n+m}\right) \\ &= \frac{n}{n+m}\left(1 - \frac{n}{n+m}\right) \\ &= \frac{n}{n+m}\left(\frac{n+m-n}{n+m}\right) = \frac{nm}{(n+m)^2}, \end{aligned} \tag{7}$$

so

$$\sum_{i=1}^{N} \operatorname{var}(x_i) = \frac{Nnm}{(n+m)^2}. \tag{8}$$

For $i < j$, the COVARIANCE is

$$\operatorname{cov}(x_i, x_j) = \langle x_i x_j \rangle - \langle x_i \rangle \langle x_j \rangle. \tag{9}$$

The probability that both i and j are successful for $i \neq j$ is

$$\begin{aligned} P(x_i = 1, x_j = 1) &= P(x_i = 1)P(x_j = 1 | x_i = 1) \\ &= \frac{n}{n+m}\frac{n-1}{n+m-1} \\ &= \frac{n(n-1)}{(n+m)(n+m-1)}. \end{aligned} \tag{10}$$

But since x_i and x_j are random BERNOULLI variables (each 0 or 1), their product is also a BERNOULLI variable. In order for $x_i x_j$ to be 1, both x_i and x_j must be 1,

$$\begin{aligned} \langle x_i x_j \rangle = P(x_i x_j = 1) &= P(x_i = 1, x_j = 1) \\ &= \frac{n}{n+m}\frac{n-1}{n+m-1} \\ &= \frac{n(n-1)}{(n+m)(n+m-1)}. \end{aligned} \tag{11}$$

Combining (11) with

$$\langle x_i\rangle\langle x_j\rangle=\frac{n}{n+m}\frac{n}{n+m}=\frac{n^2}{(n+m)^2}, \qquad (12)$$

gives

$$\begin{aligned}\operatorname{cov}(x_i,x_j)&=\frac{(n+m)(n^2-n)-n^2(n+m-1)}{(n+m)^2(n+m-1)}\\&=\frac{n^3+mn^2-n^2-mn-n^3-n^2m+n^2}{(n+m)^2(n+m-1)}\\&=-\frac{mn}{(n+m)^2(n+m-1)}.\end{aligned} \qquad (13)$$

There are a total of N^2 terms in a double summation over N. However, $i=j$ for N of these, so there are a total of $N^2-N=N(N-1)$ terms in the COVARIANCE summation

$$\sum_{i=1}^{N}\sum_{\substack{j=1\\ j\neq i}}^{N}\operatorname{cov}(x_i,x_j)=-\frac{N(N-1)mn}{(n+m)^2(n+m-1)}. \qquad (14)$$

Combining equations (6), (8), (11), and (14) gives the VARIANCE

$$\begin{aligned}\operatorname{var}(x)&=\frac{Nmn}{(n+m)^2}-\frac{N(N-1)mn}{(n+m)^2(n+m-1)}\\&=\frac{Nmn}{(m+n)^2}\left(1-\frac{N-1}{n+m-1}\right)\\&=\frac{Nmn}{(n+m)^2}\left(\frac{N+m-1-N+1}{n+m-1}\right)\\&=\frac{Nmn(n+m-N)}{(n+m)^2(n+m-1)},\end{aligned} \qquad (15)$$

so the final result is

$$\langle x\rangle=Np \qquad (16)$$

and, since

$$1-p=\frac{m}{n+m} \qquad (17)$$

and

$$np(1-p)=\frac{mn}{(n+m)^2}, \qquad (18)$$

we have

$$\begin{aligned}\sigma^2=\operatorname{var}(x)&=Np(1-p)\left(1-\frac{N-1}{n+m-1}\right)\\&=\frac{mnN(m+n-N)}{(m+n)^2(m+n-1)}.\end{aligned} \qquad (19)$$

The SKEWNESS is

$$\begin{aligned}\gamma_1&=\frac{q-p}{\sqrt{npq}}\sqrt{\frac{N-1}{N-m}}\left(\frac{N-2n}{N-2}\right)\\&=\frac{(m-n)(m+n-2N)}{m+n-2}\sqrt{\frac{m+n-1}{mnN(m+n-N)}},\end{aligned} \qquad (20)$$

and the KURTOSIS

$$\gamma_2=\frac{F(m,n,N)}{mnN(-3+m+n)(-2+m+n)(-m-n+N)}, \qquad (21)$$

where

$$\begin{aligned}F(m,n,N)=m^3&-m^5+3m^2n-6m^3n+m^4n+3mn^2\\&-12m^2n^2+8m^3n^2+n^3-6mn^3+8m^2n^3\\&+mn^4-n^5-6m^3N+6m^4N+18m^2nN\\&-6m^3nN+18mn^2N-24m^2n^2N-6n^3N\\&-6mn^3N+6n^4N+6m^2N^2-6m^3N^2\\&-24mnN^2+12m^2nN^2+6n^2N^2\\&+12mn^2N^2-6n^3N^2.\end{aligned} \qquad (22)$$

The GENERATING FUNCTION is

$$\phi(t)=\frac{\binom{m}{N}}{\binom{n+m}{N}}\,{}_2F_1(-N,-n;m-N+1;e^{it}), \qquad (23)$$

where ${}_2F_1(a,b;c;z)$ is the HYPERGEOMETRIC FUNCTION.

If the hypergeometric distribution is written

$$h_n(x,s)=\frac{\binom{np}{x}\binom{nq}{s-x}}{\binom{n}{s}}, \qquad (24)$$

then

$$\sum_{x=0}^{s}h_n(x,s)u^x=A\,{}_2F_1(-s,-np;nq-s+1;u). \qquad (25)$$

References

Beyer, W. H. *CRC Standard Mathematical Tables, 28th ed.* Boca Raton, FL: CRC Press, pp. 532–533, 1987.

Spiegel, M. R. *Theory and Problems of Probability and Statistics.* New York: McGraw-Hill, pp. 113–114, 1992.

Hypergeometric Function

A GENERALIZED HYPERGEOMETRIC FUNCTION ${}_pF_q(a_1,\ldots,a_p;b_1,\ldots,b_q;x)$ is a function which can be defined in the form of a HYPERGEOMETRIC SERIES, i.e., a series for which the ratio of successive terms can be written

$$\frac{a_{k+1}}{a_k}=\frac{P(k)}{Q(k)}=\frac{(k+a_1)(k+a_2)\cdots(k+a_p)}{(k+b_1)(k+b_2)\cdots(k+b_q)(k+1)}x. \qquad (1)$$

(The factor of $k+1$ in the DENOMINATOR is present for historical reasons of notation.) The function ${}_2F_1(a,b;c;x)$ corresponding to $p=2$, $q=1$ is the first hypergeometric function to be studied (and, in general, arises the most frequently in physical problems), and so is frequently known as "the" hypergeometric equation.

To confuse matters even more, the term "hypergeometric function" is less commonly used to mean CLOSED FORM.

The hypergeometric functions are solutions to the HYPERGEOMETRIC DIFFERENTIAL EQUATION, which has a REGULAR SINGULAR POINT at the ORIGIN. To derive the hypergeometric function based on the HYPERGEOMETRIC DIFFERENTIAL EQUATION, plug

$$y = \sum_{n=0}^{\infty} A_n z^n \tag{2}$$

$$y' = \sum_{n=0}^{\infty} n A_n z^{n-1} \tag{3}$$

$$y'' = \sum_{n=0}^{\infty} n(n-1) A_n z^{n-2} \tag{4}$$

into

$$z(1-z)y'' + [c-(a+b+1)a]y' - aby = 0 \tag{5}$$

to obtain

$$\sum_{n=0}^{\infty} n(n-1)A_n z^{n-1} - \sum_{n=0}^{\infty} n(n-1)A_n z^n + c\sum_{n=0}^{\infty} nA_n z^{n-1} + (a+b+1)\sum_{n=0}^{\infty} nA_n z^n - ab\sum_{n=0}^{\infty} A_n z^n = 0 \tag{6}$$

$$\sum_{n=2}^{\infty} n(n-1)A_n z^{n-1} - \sum_{n=0}^{\infty} n(n-1)A_n z^n + c\sum_{n=1}^{\infty} nA_n z^{n-1} - (a+b+1)\sum_{n=1}^{\infty} nA_n z^n - ab\sum_{n=0}^{\infty} A_n z^n = 0 \tag{7}$$

$$\sum_{n=0}^{\infty} (n+1)nA_{n+1} z^n - \sum_{n=0}^{\infty} n(n-1)A_n z^n + c\sum_{n=0}^{\infty} (n+1)A_{n+1} z^n - (a+b+1)\sum_{n=0}^{\infty} nA_n z^n - ab\sum_{n=0}^{\infty} A_n z^n = 0 \tag{8}$$

$$\sum_{n=0}^{\infty} [n(n+1)A_{n+1} - n(n-1)A_n + c(n+1)A_{n-1} - (a+b+1)nA_n - abA_n] z^n = 0 \tag{9}$$

$$\sum_{n=0}^{\infty} \{(n+1)(n+c)A_{n+1} - [n(n-1+a+b+1)+ab]A_n\} z^n = 0 \tag{10}$$

$$\sum_{n=0}^{\infty} \{(n+1)(n+c)A_{n+1} - [n^2+(a+b)n+ab]A_n\} z^n = 0, \tag{11}$$

so

$$A_{n+1} = \frac{(n+a)(n+b)}{(n+1)(n+c)} A_n \tag{12}$$

and

$$y = A_0 \left[1 + \frac{ab}{1!c} z + \frac{a(a+1)b(b+1)}{2!c(c+1)} z^2 + \ldots \right]. \tag{13}$$

This is the regular solution and is denoted

$$\begin{aligned} {}_2F_1(a,b;c;z) &= 1 + \frac{ab}{1!c} z + \frac{a(a+1)b(b+1)}{2!c(c+1)} z^2 + \ldots \\ &= \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{z^n}{n!}, \end{aligned} \tag{14}$$

where $(a)_n$ are POCHHAMMER SYMBOLS. The hypergeometric series is convergent for REAL $-1 < z < 1$, and for $z = \pm 1$ if $c > a+b$. The complete solution to the HYPERGEOMETRIC DIFFERENTIAL EQUATION is

$$y = A\,{}_2F_1(a,b;c;z) + Bz^{1-c}\,{}_2F_1(a+1-c, b+1-c; 2-c; z). \tag{15}$$

Derivatives are given by

$$\frac{d\,{}_2F_1(a,b;c;z)}{dz} = \frac{ab}{c}\,{}_2F_1(a+1,b+1;c+1;z) \tag{16}$$

$$\frac{d^2\,{}_2F_1(a,b;c;z)}{dz^2} = \frac{a(a+1)b(b+1)}{c(c+1)} \times {}_2F_1(a+2,b+2;c+2;z) \tag{17}$$

(Magnus and Oberhettinger 1949, p. 8). An integral giving the hypergeometric function is

$${}_2F_1(a,b;c;z) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 \frac{t^{b-1}(1-t)^{c-b-1}}{(1-tz)^a}\,dt \tag{18}$$

as shown by Euler in 1748.

A hypergeometric function can be written using EULER'S HYPERGEOMETRIC TRANSFORMATIONS

$$t \to t \tag{19}$$

$$t \to 1-t \tag{20}$$

$$t \to (1-z-tz)^{-1} \tag{21}$$

$$t \to \frac{1-t}{1-tz} \tag{22}$$

in any one of four equivalent forms

$$
\begin{aligned}
{}_2F_1(a,b;c;z) &= (1-z)^{-a}\,{}_2F_1(a,c-b;c;z/(z-1)) && (23)\\
&= (1-z)^{-b}\,{}_2F_1(c-a,b;c;z/(z-1)) && (24)\\
&= (1-z)^{c-a-b}\,{}_2F_1(c-a,c-b;c;z). && (25)
\end{aligned}
$$

It can also be written as a linear combination

$$
\begin{aligned}
{}_2F_1(a,b;c;z) &= \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}\,{}_2F_1(a,b;a+b+1-c;1-z)\\
&\quad+\frac{\Gamma(c)\Gamma(a+b-c)}{\Gamma(a)\Gamma(b)}(1-z)^{c-a-b}\\
&\quad\times{}_2F_1(c-a,c-b;1+c-a-b;1-z). \qquad (26)
\end{aligned}
$$

Kummer found all six solutions (not necessarily regular at the origin) to the HYPERGEOMETRIC DIFFERENTIAL EQUATION,

$$
\begin{aligned}
u_1(x) &= {}_2F_1(a,b;c;z)\\
u_2(x) &= {}_2F_1(a,b;a+b+1-c;1-z)\\
u_3(x) &= z^{-a}\,{}_2F_1(a,a+1-c;a+1-b;1/z)\\
u_4(x) &= z^{-b}\,{}_2F_1(b+1-c,b;b+1-a;1/z)\\
u_5(x) &= z^{1-c}\,{}_2F_1(b+1-c,a+1-c;2-c;z)\\
u_6(x) &= (1-z)^{c-a-b}\,{}_2F_1(c-a,c-b;c+1-a-b;1-z).
\end{aligned}
$$

Applying EULER'S HYPERGEOMETRIC TRANSFORMATIONS to the Kummer solutions then gives all 24 possible forms which are solutions to the HYPERGEOMETRIC DIFFERENTIAL EQUATION

$$
\begin{aligned}
u_1^{(1)}(x) &= {}_2F_1(a,b;c;z)\\
u_1^{(2)}(x) &= (1-z)^{-a}\,{}_2F_1(a,c-b;c;z/(z-1))\\
u_1^{(3)}(x) &= (1-z)^{-b}\,{}_2F_1(c-a,b;c;z/(z-1))\\
u_1^{(4)}(x) &= (1-z)^{c-a-b}\,{}_2F_1(c-a,c-b;c;z)\\
u_2^{(1)}(x) &= {}_2F_1(a,b;a+b+1-c;1-z)\\
u_2^{(2)}(x) &= z^{-a}\,{}_2F_1(a,a+1-c;a+b+1-c;1-1/z)\\
u_2^{(3)}(x) &= z^{-b}\,{}_2F_1(b+1-c,b;a+b+1-c;1-1/z)\\
u_2^{(4)}(x) &= z^{1-c}\,{}_2F_1(b+1-c,a+1-c;a+b+1-c;1-z)\\
u_3^{(1)}(x) &= z^{-a}\,{}_2F_1(a,a+1-c;a+1-b;1/z)\\
u_3^{(2)}(x) &= z^{-a}(1-1/1z)^{-a}\,{}_2F_1(a,c-b;a+1-b;1/(1-z))\\
u_3^{(3)}(x) &= z^{-a}(1-1/z)^{c-a-1}\\
&\quad\times{}_2F_1(1-b,a+1-c;a+1-b;1/(1-z))\\
u_3^{(4)}(x) &= z^{-a}(1-1/z)^{c-a-b}\,{}_2F_1(1-b,c-b;a+1-b;1/z)\\
u_4^{(1)}(x) &= z^{-a}\,{}_2F_1(b+1-c,b;b+1-a;1/z)\\
u_4^{(2)}(x) &= z^{-b}(1-1/z)^{c-b-1}\\
&\quad\times{}_2F_1(b_1-c,1-a;b+1-a;1/(1-z))\\
u_4^{(3)}(x) &= z^{-b}(1-1/z)^{-b}\,{}_2F_1(c-a,b;b+1-a;1/(1-z))\\
u_4^{(4)}(x) &= z^{-b}(1-1/z)^{c-a-b}\,{}_2F_1(c-a,1-a;b+1-a;1/z)\\
u_5^{(1)}(x) &= z^{1-c}\,{}_2F_1(b+1-c,a+1-c;2-c;z)\\
u_5^{(2)}(x) &= z^{1-c}(1-z)^{c-b-1}\,{}_2F_1(b+1-c,1-a;2-c;z/(z-1))\\
u_5^{(2)}(x) &= z^{1-c}(1-z)^{c-a-1}\,{}_2F_1(1-b,a+1-c;2-c;z/(z-1))\\
u_5^{(4)}(x) &= z^{1-c}(1-z)^{c-a-b}\,{}_2F_1(1-b,1-a;2-c;z)\\
u_6^{(1)}(x) &= (1-z)^{c-a-b}\,{}_2F_1(c-a,c-b;c+1-a-b;1-z)\\
u_6^{(2)}(x) &= z^{a-c}(1-z)^{c-a-b}\\
&\quad\times{}_2F_1(c-a,1-a;c+1-a-b;1-1/z)\\
u_6^{(2)}(x) &= z^{b-c}(1-z)^{c-a-b}\\
&\quad\times{}_2F_1(1-b,c-b;c+1-a-b;1-1/z)\\
u_6^{(4)}(x) &= z^{c-a-b}(1-z)^{c-a-b}\\
&\quad\times{}_2F_1(1-b,1-a;c+1-a-b;1-z).
\end{aligned}
$$

Goursat (1881) gives many hypergeometric transformation FORMULAS, including several cubic transformation FORMULAS.

Many functions of mathematical physics can be expressed as special cases of the hypergeometric functions. For example,

$$ {}_2F_1(-l,l+1,1;(1-z)/2) = P_l(z), \qquad (27) $$

where $P_l(z)$ is a LEGENDRE POLYNOMIAL.

$$ (1+z)^n = {}_2F_1(-n,b;b;-z) \qquad (28) $$

$$ \ln(1+z) = z\,{}_2F_1(1,1;2;-z) \qquad (29) $$

Complete ELLIPTIC INTEGRALS and the RIEMANN P-SERIES can also be expressed in terms of ${}_2F_1(a,b;c;z)$. Special values include

$$ {}_2F_1(a,b;a-b+1;-1) = 2^{-a}\sqrt{\pi}\frac{\Gamma(1+a+b)}{\Gamma(1+\frac{1}{2}a-b)\Gamma(\frac{1}{2}+\frac{1}{2}a)} \qquad (30) $$

$$ {}_2F_1(1,-a;a;-1) = \frac{\sqrt{\pi}}{2}\frac{\Gamma(a)}{\Gamma(a+\frac{1}{2})}+1 \qquad (31) $$

$$ {}_2F_1(a,b;c;\tfrac{1}{2}) = 2^a\,{}_2F_1(a,c-b;c;-1) \qquad (32) $$

$$ {}_2F_1(a,b;\tfrac{1}{2}(a+b+1);\tfrac{1}{2}) = \frac{\Gamma(\frac{1}{2})\Gamma[(\frac{1}{2}(1+a+b)]}{\Gamma[\frac{1}{2}(1+a)]\Gamma[\frac{1}{2}(1+b)]} \qquad (33) $$

$$ {}_2F_1(a,1-a;c;\tfrac{1}{2}) = \frac{\Gamma(\frac{1}{2}c)\Gamma[\frac{1}{2}(c+1)]}{\Gamma[\frac{1}{2}(a+c)]\Gamma[\frac{1}{2}(1+c-a)]} \qquad (34) $$

$$ {}_2F_1(a,b;c;1) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}. \qquad (35) $$

KUMMER'S FIRST FORMULA gives

$$ {}_2F_1(\tfrac{1}{2}+m-k,-n;2m+1;1) = \frac{\Gamma(2m+1)\Gamma(m+\frac{1}{2}+k+n)}{\Gamma(m+\frac{1}{2}+k)\Gamma(2m+1+n)}, \qquad (36) $$

where $m \neq -1/2, -1, -3/2, \ldots$. Many additional identities are given by Abramowitz and Stegun (1972, p. 557).

Hypergeometric functions can be generalized to GENERALIZED HYPERGEOMETRIC FUNCTIONS

$$ {}_nF_m(a_1,\ldots,a_n;b_1,\ldots,b_m;z). \quad (37)$$

A function of the form ${}_1F_1(a;b;z)$ is called a CONFLUENT HYPERGEOMETRIC FUNCTION, and a function of the form ${}_0F_1(;b;z)$ is called a CONFLUENT HYPERGEOMETRIC LIMIT FUNCTION.

see also APPELL HYPERGEOMETRIC FUNCTION, BARNES' LEMMA, BRADLEY'S THEOREM, CAYLEY'S HYPERGEOMETRIC FUNCTION THEOREM, CLAUSEN FORMULA, CLOSED FORM, CONFLUENT HYPERGEOMETRIC FUNCTION, CONFLUENT HYPERGEOMETRIC LIMIT FUNCTION, CONTIGUOUS FUNCTION, DARLING'S PRODUCTS, GENERALIZED HYPERGEOMETRIC FUNCTION, GOSPER'S ALGORITHM, HYPERGEOMETRIC IDENTITY, HYPERGEOMETRIC SERIES, JACOBI POLYNOMIAL, KUMMER'S FORMULAS, KUMMER'S QUADRATIC TRANSFORMATION, KUMMER'S RELATION, ORR'S THEOREM, RAMANUJAN'S HYPERGEOMETRIC IDENTITY, SAALSCHÜTZIAN, SISTER CELINE'S METHOD, ZEILBERGER'S ALGORITHM

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Hypergeometric Identity

A relation expressing a sum potentially involving BINOMIAL COEFFICIENTS, FACTORIALS, RATIONAL FUNCTIONS, and power functions in terms of a simple result. Thanks to results by Fasenmyer, Gosper, Zeilberger, Wilf, and Petkovšek, the problem of determining whether a given hypergeometric sum is expressible in simple closed form and, if so, finding the form, is now (subject to a mild restriction) completely solved. The algorithm which does so has been implemented in several computer algebra packages and is called ZEILBERGER'S ALGORITHM.

see also GENERALIZED HYPERGEOMETRIC FUNCTION, GOSPER'S ALGORITHM, HYPERGEOMETRIC SERIES, SISTER CELINE'S METHOD, WILF-ZEILBERGER PAIR, ZEILBERGER'S ALGORITHM

References
Petkovšek, M.; Wilf, H. S.; and Zeilberger, D. *A=B.* Wellesley, MA: A. K. Peters, p. 18, 1996.

Hypergeometric Polynomial

see JACOBI POLYNOMIAL

Hypergeometric Series

A hypergeometric series $\sum_k a_k$ is a series for which $a_0 = 1$ and the ratio of consecutive terms is a RATIONAL FUNCTION of the summation index k, i.e., one for which

$$\frac{a_{k+1}}{a_k} = \frac{P(k)}{Q(k)},$$

with $P(k)$ and $Q(k)$ POLYNOMIALS. The functions generated by hypergeometric series are called HYPERGEOMETRIC FUNCTIONS or, more generally, GENERALIZED HYPERGEOMETRIC FUNCTIONS. If the polynomials are completely factored, the ratio of successive terms can be written

$$\frac{a_{k+1}}{a_k} = \frac{P(k)}{Q(k)} = \frac{(k+a_1)(k+a_2)\cdot(k+a_p)}{(k+b_1)(k+b_2)\cdots(k+b_q)(k+1)}x,$$

where the factor of $k+1$ in the DENOMINATOR is present for historical reasons of notation, and the resulting GENERALIZED HYPERGEOMETRIC FUNCTION is written

$${}_pF_q\begin{bmatrix} a_1 & a_2 & \cdots & a_p \\ b_1 & b_2 & \cdots & b_q \end{bmatrix}; x\Bigg] = \sum_{k=0} a_k x^k.$$

If $p = 2$ and $q = 1$, the function becomes a traditional HYPERGEOMETRIC FUNCTION ${}_2F_1(a, b; c; x)$.

Many sums can be written as GENERALIZED HYPERGEOMETRIC FUNCTIONS by inspections of the ratios of consecutive terms in the generating hypergeometric series.

see also GENERALIZED HYPERGEOMETRIC FUNCTION, GEOMETRIC SERIES, HYPERGEOMETRIC FUNCTION, HYPERGEOMETRIC IDENTITY

References
Petkovšek, M.; Wilf, H. S.; and Zeilberger, D. "Hypergeometric Series," "How to Identify a Series as Hypergeometric," and "Software That Identifies Hypergeometric Series." §3.2–3.4 in *A=B*. Wellesley, MA: A. K. Peters, pp. 34–42, 1996.

Hypergroup

A MEASURE ALGEBRA which has many properties associated with the convolution MEASURE ALGEBRA of a GROUP, but no algebraic structure is assumed for the underlying SPACE.

References
Bloom, W. R.; and Heyer, H. *The Harmonic Analysis of Probability Measures on Hypergroups.* Berlin: de Gruyter, 1995.
Jewett, R. I. "Spaces with an Abstract Convolution of Measures." *Adv. Math.* **18**, 1–101, 1975.

Hypermatrix

A generalization of the MATRIX to an $n_1 \times n_2 \times \cdots$ array of numbers.

see also HYPERDETERMINANT

References
Gel'fand, I. M.; Kapranov, M. M.; and Zelevinsky, A. V. "Hyperdeterminants." *Adv. Math.* **96**, 226–263, 1992.

Hyperparallel

Two lines in HYPERBOLIC GEOMETRY which diverge from each other in both directions.

see also ANTIPARALLEL, IDEAL POINT, PARALLEL

Hyperperfect Number

A number n is called k-hyperperfect if

$$n = 1 + k \sum_i d_i,$$

where the summation is over the PROPER DIVISORS with $1 < d_i < n$, giving

$$k\sigma(n) = (k+1)n + k + 1,$$

where $\sigma(n)$ is the DIVISOR FUNCTION. The first few hyperperfect numbers are 21, 301, 325, 697, 1333, ... (Sloane's A007592). 2-hyperperfect numbers include 21, 2133, 19521, 176661, ... (Sloane's A007593), and the first 3-hyperperfect number is 325.

References
Guy, R. K. "Almost Perfect, Quasi-Perfect, Pseudoperfect, Harmonic, Weird, Multiperfect and Hyperperfect Numbers." §B2 in *Unsolved Problems in Number Theory, 2nd ed.* New York: Springer-Verlag, pp. 45–53, 1994.
Sloane, N. J. A. Sequences A007592/M5113 and A007593/M5121 in "An On-Line Version of the Encyclopedia of Integer Sequences."

Hyperplane

Let $a_1, a_2, \ldots, a_n$ be SCALARS not all equal to 0. Then the SET S consisting of all VECTORS

$$\mathbf{X} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

in $\mathbb{R}^n$ such that

$$a_1x_1 + a_2x_2 + \ldots + a_nx_n = 0$$

is a SUBSPACE of $\mathbb{R}^n$ called a hyperplane. More generally, a hyperplane is any co-dimension 1 vector SUBSPACE of a VECTOR SPACE. Equivalently, a hyperplane V in a VECTOR SPACE W is any SUBSPACE such that W/V is 1-dimensional. Equivalently, a hyperplane is the KERNEL of any NONZERO linear MAP from the VECTOR SPACE to the underlying FIELD.

Hyperreal Number

Hyperreal numbers are an extension of the REAL NUMBERS to include certain classes of infinite and infinitesimal numbers. A hyperreal number is said to be finite IFF $|x| < n$ for some INTEGER n. x is said to be infinitesimal IFF $|x| < 1/n$ for all INTEGERS n.

see also AX-KOCHEN ISOMORPHISM THEOREM, NONSTANDARD ANALYSIS

References
Apps, P. "The Hyperreal Line." `http://www.math.wisc.edu/~apps/line.html`.
Keisler, H. J. "The Hyperreal Line." In *Real Numbers, Generalizations of the Reals, and Theories of Continua* (Ed. P. Ehrlich). Norwell, MA: Kluwer, 1994.

Hyperspace

A SPACE having DIMENSION $n > 3$.

Hypersphere

The n-hypersphere (often simply called the n-sphere) is a generalization of the CIRCLE ($n = 2$) and SPHERE ($n = 3$) to dimensions $n \geq 4$. It is therefore defined as the set of n-tuples of points $(x_1, x_2, \ldots, x_n)$ such that

$$x_1{}^2 + x_2{}^2 + \ldots + x_n{}^2 = R^2, \tag{1}$$

where R is the RADIUS of the hypersphere. The CONTENT (i.e., n-D VOLUME) of an n-hypersphere of RADIUS R is given by

$$V_n = \int_0^R S_n r^{n-1}\,dr = \frac{S_n R^n}{n}, \tag{2}$$

where S_n is the hyper-SURFACE AREA of an n-sphere of unit radius. But, for a unit hypersphere, it must be true that

$$\begin{aligned} S_n \int_0^\infty e^{-r^2} r^{n-1}\,dr &= \underbrace{\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty}}_{n} e^{-({x_1}^2+\ldots+{x_n}^2)}\,dx_1 \cdots dx_m \\ &= \left(\int_{-\infty}^{\infty} e^{-x^2}\,dx\right)^n. \end{aligned} \tag{3}$$

But the GAMMA FUNCTION can be defined by

$$\Gamma(m) = 2\int_0^\infty e^{-r^2} r^{2m-1}\,dr, \tag{4}$$

so

$$\tfrac{1}{2} S_n \Gamma(\tfrac{1}{2}n) = [\Gamma(\tfrac{1}{2})]^n = (\pi^{1/2})^n \tag{5}$$

$$S_n = \frac{2\pi^{n/2}}{\Gamma(\frac{1}{2}n)}. \tag{6}$$

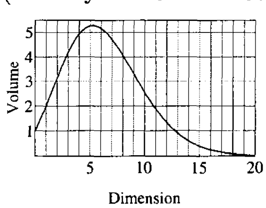
This gives the RECURRENCE RELATION

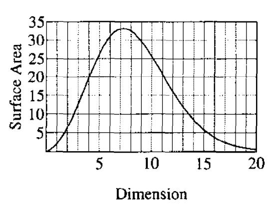
$$S_{n+2} = \frac{2\pi S_n}{n}. \tag{7}$$

Using $\Gamma(n+1) = n\Gamma(n)$ then gives

$$V_n = \frac{S_n R^n}{n} = \frac{\pi^{n/2} R^n}{(\frac{1}{2}n)\Gamma(\frac{1}{2}n)} = \frac{\pi^{n/2} R^n}{\Gamma(1+\frac{1}{2}n)} \tag{8}$$

(Conway and Sloane 1993).





Strangely enough, the hyper-SURFACE AREA and CONTENT reach MAXIMA and then decrease towards 0 as n increases. The point of MAXIMAL hyper-SURFACE AREA satisfies

$$\frac{dS_n}{dn} = \frac{\pi^{n/2}[\ln\pi - \psi_0(\frac{1}{2}n)]}{\Gamma(\frac{1}{2}n)} = 0, \tag{9}$$

where $\psi_0(x) \equiv \Psi(x)$ is the DIGAMMA FUNCTION. The point of MAXIMAL CONTENT satisfies

$$\frac{dV_n}{dn} = \frac{\pi^{n/2}[\ln\pi - \psi_0(1+\frac{1}{2}n)]}{2\Gamma(1+\frac{1}{2}n)} = 0. \tag{10}$$

Neither can be solved analytically for n, but the numerical solutions are $n = 7.25695\ldots$ for hyper-SURFACE AREA and $n = 5.25695\ldots$ for CONTENT. As a result, the 7-D and 5-D hyperspheres have MAXIMAL hyper-SURFACE AREA and CONTENT, respectively (Le Lionnais 1983).

n	V_n	$V_n/V_{n\text{-cube}}$	S_n
0	1	1	0
1	2	1	2
2	π	$\frac{1}{4}\pi$	2π
3	$\frac{4}{3}\pi$	$\frac{1}{6}\pi$	4π
4	$\frac{1}{2}\pi^2$	$\frac{1}{32}\pi^2$	$2\pi^2$
5	$\frac{8}{15}\pi^2$	$\frac{1}{60}\pi^2$	$\frac{8}{3}\pi^2$
6	$\frac{1}{6}\pi^3$	$\frac{1}{384}\pi^3$	π^3
7	$\frac{16}{105}\pi^3$	$\frac{1}{840}\pi^3$	$\frac{16}{15}\pi^3$
8	$\frac{1}{24}\pi^4$	$\frac{1}{6144}\pi^4$	$\frac{1}{3}\pi^4$
9	$\frac{32}{945}\pi^4$	$\frac{1}{15120}\pi^4$	$\frac{32}{105}\pi^4$
10	$\frac{1}{120}\pi^5$	$\frac{1}{122880}\pi^5$	$\frac{1}{12}\pi^5$

In 4-D, the generalization of SPHERICAL COORDINATES is defined by

$$x_1 = R\sin\psi\sin\phi\cos\theta \tag{11}$$

$$x_2 = R\sin\psi\sin\phi\sin\theta \tag{12}$$

$$x_3 = R\sin\psi\cos\phi \tag{13}$$

$$x_4 = R\cos\psi. \tag{14}$$

The equation for a 4-sphere is

$${x_1}^2 + {x_2}^2 + {x_3}^2 + {x_4}^2 = R^2, \tag{15}$$

and the LINE ELEMENT is

$$ds^2 = R^2[d\psi^2 + \sin^2\psi(d\phi^2 + \sin^2\phi\,d\theta^2)]. \tag{16}$$

By defining $r \equiv R\sin\psi$, the LINE ELEMENT can be rewritten

$$ds^2 = \frac{dr^2}{\left(1 - \frac{r^2}{R^2}\right)} + r^2(d\phi^2 + \sin^2\phi\,d\theta^2). \tag{17}$$

The hyper-SURFACE AREA is therefore given by

$$\begin{aligned} S_4 &= \int_0^\pi R\,d\psi \int_0^\pi R\sin\psi\,d\phi \int_0^{2\pi} R\sin\psi\sin\phi\,d\theta \\ &= 2\pi^2 R^3. \end{aligned} \tag{18}$$

see also CIRCLE, HYPERCUBE, HYPERSPHERE PACKING, MAZUR'S THEOREM, SPHERE, TESSERACT

References

Conway, J. H. and Sloane, N. J. A. *Sphere Packings, Lattices, and Groups, 2nd ed.* New York: Springer-Verlag, p. 9, 1993.

Le Lionnais, F. *Les nombres remarquables.* Paris: Hermann, p. 58, 1983.

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Hypersphere Packing

Draw unit n-spheres in an n-D space centered at all ± 1 coordinates. Then place an additional HYPERSPHERE at the origin tangent to the other HYPERSPHERES. Then the central HYPERSPHERE is contained with the HYPERSPHERE with VERTICES at the center of the other spheres for n between 2 and 8. However, for $n = 9$, the central HYPERSPHERE just touches the bounding HYPERSPHERE, and for $n > 9$, the HYPERSPHERE is partially outside the hypercube. This can be seen by finding the distance from the origin to the center of one of the HYPERSPHERES

$$\underbrace{\sqrt{(\pm 1)^2 + \ldots + (\pm 1)^2}}_{n} = \sqrt{n}.$$

The radius of the central sphere is therefore $\sqrt{n} - 1$. The distance from the origin to the center of the bounding hypercube is always 2 (two radii), so the center HYPERSPHERE is tangent when $\sqrt{n} - 1 = 2$, or $n = 9$, and outside for $n > 9$.

The analog of face-centered cubic packing is the densest lattice in 4- and 5-D. In 8-D, the densest lattice packing is made up of two copies of face-centered cubic. In 6- and 7-D, the densest lattice packings are cross-sections of the 8-D case. In 24-D, the densest packing appears to be the LEECH LATTICE. For high dimensions ($\sim$ 1000-D), the densest known packings are nonlattice. The densest lattice packings in n-D have been rigorously proved to have PACKING DENSITY 1, $\pi/(2\sqrt{3})$, $\pi/(3\sqrt{2})$, $\pi^2/16$, $\pi^2/(15\sqrt{2})$, $\pi^3/(48\sqrt{3})$, $\pi^3/105$, and $\pi^4/384$ (Finch).

The largest number of unit CIRCLES which can touch another is six. For SPHERES, the maximum number is 12. Newton considered this question long before a proof was published in 1874. The maximum number of hyperspheres that can touch another in n-D is the so-called KISSING NUMBER.

see also KISSING NUMBER, LEECH LATTICE, SPHERE PACKING

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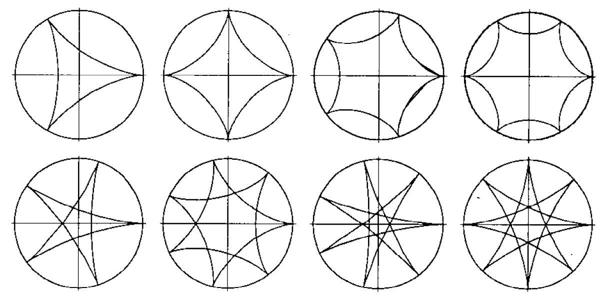
Hypervolume

see CONTENT

Hypocycloid



The curve produced by a small CIRCLE of RADIUS b rolling around the inside of a large CIRCLE of RADIUS $a > b$. A hypocycloid is a HYPOTROCHOID with $h = b$. To derive the equations of the hypocycloid, call the ANGLE by which a point on the small CIRCLE rotates about its center ϑ, and the ANGLE from the center of the large CIRCLE to that of the small CIRCLE ϕ. Then

$$(a-b)\phi = b\vartheta, \tag{1}$$

so

$$\vartheta = \frac{a-b}{b}\phi. \tag{2}$$

Call $\rho \equiv a - 2b$. If $x(0) = \rho$, then the first point is at minimum radius, and the Cartesian parametric equations of the hypocycloid are

$$\begin{aligned} x &= (a-b)\cos\phi - b\cos\vartheta \\ &= (a-b)\cos\phi - b\cos\left(\frac{a-b}{b}\phi\right) \qquad (3) \\ y &= (a-b)\sin\phi + b\sin\vartheta \\ &= (a-b)\sin\phi + b\sin\left(\frac{a-b}{b}\phi\right). \qquad (4) \end{aligned}$$

If $x(0) = a$ instead so the first point is at maximum radius (on the CIRCLE), then the equations of the hypocycloid are

$$x = (a-b)\cos\phi + b\cos\left(\frac{a-b}{b}\phi\right) \tag{5}$$

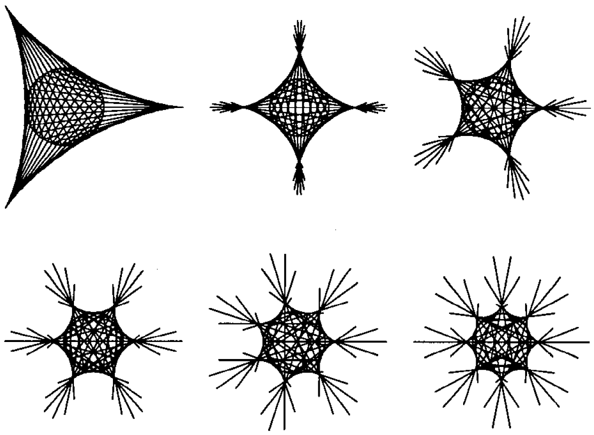
$$y = (a-b)\sin\phi - b\sin\left(\frac{a-b}{b}\phi\right). \tag{6}$$

An n-cusped non-self-intersecting hypocycloid has $a/b = n$. A 2-cusped hypocycloid is a LINE SEGMENT, as can be seen by setting $a = b$ in equations (3) and (4) and noting that the equations simplify to

$$x = a\sin\phi \tag{7}$$

$$y = 0. \tag{8}$$

A 3-cusped hypocycloid is called a DELTOID or TRICUSPOID, and a 4-cusped hypocycloid is called an ASTROID. If a/b is rational, the curve closes on itself and has b cusps. If a/b is IRRATIONAL, the curve never closes and fills the entire interior of the CIRCLE.



n-hypocycloids can also be constructed by beginning with the DIAMETER of a CIRCLE, offsetting one end by a series of steps while at the same time offsetting the other end by steps n times as large in the opposite direction and extending beyond the edge of the CIRCLE. After traveling around the CIRCLE once, an n-cusped hypocycloid is produced, as illustrated above (Madachy 1979).

Let r be the radial distance from a fixed point. For RADIUS OF TORSION ρ and ARC LENGTH s, a hypocycloid can given by the equation

$$s^2 + \rho^2 = 16r^2 \tag{9}$$

(Kreyszig 1991, pp. 63–64). A hypocycloid also satisfies

$$\sin^2\psi = \frac{\rho^2}{a^2-\rho^2}\frac{a^2-r^2}{r^2}, \tag{10}$$

where

$$r\frac{dr}{d\theta} = \tan\psi \tag{11}$$

and ψ is the ANGLE between the RADIUS VECTOR and the TANGENT to the curve.

The ARC LENGTH of the hypocycloid can be computed as follows

$$\begin{aligned} x' &= -(a-b)\sin\phi - (a-b)\sin\left(\frac{a-b}{b}\phi\right) \\ &= (a-b)\left[\sin\phi + \sin\left(\frac{a-b}{b}\phi\right)\right] \end{aligned} \tag{12}$$

$$\begin{aligned} y' &= (a-b)\cos\phi - (a-b)\cos\left(\frac{a-b}{a}\phi\right) \\ &= (a-b)\left[\cos\phi - \cos\left(\frac{a-b}{b}\phi\right)\right] \end{aligned} \tag{13}$$

$$\begin{aligned} x'^2 + y'^2 &= (a-b)^2\left[\sin^2\phi + 2\sin\phi\sin\left(\frac{a-b}{b}\phi\right)\right. \\ &\quad + \sin^2\left(\frac{a-b}{b}\phi\right) + \cos^2\phi - 2\cos\phi\cos\left(\frac{a-b}{b}\phi\right) \\ &\quad \left. + \cos^2\left(\frac{a-b}{b}\phi\right)\right] \\ &= (a-b)^2\left\{2 + 2\left[\sin\phi\sin\left(\frac{a-b}{a}\phi\right)\right.\right. \\ &\quad \left.\left. - \cos\phi\cos\left(\frac{a-b}{b}\phi\right)\right]\right\} \\ &= 2(a-b)^2\left[1 - \cos\left(\phi + \frac{a-b}{b}\phi\right)\right] \\ &= 4(a-b)^2\tfrac{1}{2}\left[1 - \cos\left(\frac{a}{b}\phi\right)\right] = 4(a-b)^2\sin^2\left(\frac{a\phi}{2b}\right), \end{aligned} \tag{14}$$

so

$$ds = \sqrt{x'^2 + y'^2}\,d\phi = 2(a-b)\sin\left(\frac{a\phi}{2b}\right)d\phi \tag{15}$$

for $\phi \leq (b/2a)\pi$. Integrating,

$$\begin{aligned} s(\phi) &= \int_0^\phi ds = 2(a-b)\left[-\frac{2b}{a}\cos\left(\frac{a\phi}{2b}\right)\right]_0^\phi \\ &= \frac{4b(a-b)}{a}\left[-\cos\left(\frac{a}{2b}\phi\right) + 1\right] \\ &= \frac{8b(a-b)}{a}\sin^2\left(\frac{a}{4b}\phi\right). \end{aligned} \tag{16}$$

The length of a single cusp is then

$$s\left(2\pi\frac{b}{a}\right) = \frac{8b(a-b)}{a}\sin^2\left(\frac{\pi}{2}\right) = \frac{8b(a-b)}{a}. \tag{17}$$

If $n \equiv a/b$ is rational, then the curve closes on itself without intersecting after n cusps. For $n \equiv a/b$ and with $x(0) = a$, the equations of the hypocycloid become

$$x = \frac{1}{n}[(n-1)\cos\phi - \cos[(n-1)\phi]a, \tag{18}$$

$$y = \frac{1}{n}[(n-1)\sin\phi + \sin[(n-1)\phi]a, \tag{19}$$

and

$$s_n = n\frac{8b(bn-b)}{nb} = 8b(n-1) = \frac{8a(n-1)}{n}. \tag{20}$$

Compute

$$\begin{aligned} xy' - yx' &= \left[(a-b)\cos\phi + b\cos\left(\frac{a-b}{a}\phi\right)\right](b-a) \\ &\quad \times \left[\sin\phi + \sin\left(\frac{a-b}{b}\phi\right)\right] \\ &\quad - \left[(a-b)\sin\phi - b\sin\left(\frac{a-b}{b}\phi\right)\right](a-b) \\ &\quad \times \left[\cos\phi - \cos\left(\frac{a-b}{b}\phi\right)\right] \\ &= 2(a^2 - 3ab + 2b^2)\sin^2\left(\frac{a\phi}{2b}\right). \end{aligned} \tag{21}$$

The AREA of one cusp is then

$$\begin{aligned} A &= \tfrac{1}{2}\int_0^{2\pi b/a}(xy' - yx')\,d\phi \\ &= (a^2 - 3ab + 2b^2)\left[\frac{at - b\sin\left(\frac{at}{b}\right)}{2a}\right]_a^{2\pi b/a} \\ &= (a^2 - 3ab + 2b^2)\left[\frac{a\left(2\pi\frac{b}{a}\right)}{2a}\right] \\ &= \frac{b(a^2 - 3ab + 2b^2)}{a}\pi. \end{aligned} \tag{22}$$

If $n = a/b$ is rational, then after n cusps,

$$\begin{aligned} A_n &= n\pi\frac{b(a^2 - 3ab + 2b^2)}{a} = n\pi\frac{\frac{a}{n}\left(a^2 - 3a\frac{a}{n} + 2\frac{a^2}{n^2}\right)}{a} \\ &= \frac{n^2 - 3n + 2}{n^2}\pi a^2 = \frac{(n-1)(n-2)}{n^2}\pi a^2. \end{aligned} \tag{23}$$

The equation of the hypocycloid can be put in a form which is useful in the solution of CALCULUS OF VARIATIONS problems with radial symmetry. Consider the case $x(0) = \rho$, then

$$\begin{aligned} r^2 &= x^2 + y^2 \\ &= \Big[(a-b)^2\cos^2\phi - 2(a-b)b\cos\phi\cos\left(\frac{a-b}{b}\phi\right) \\ &\quad + b^2\cos^2\left(\frac{a-b}{b}\phi\right) \\ &\quad + (a-b)^2\sin^2\phi + 2(a-b)b\sin\phi\sin\left(\frac{a-b}{b}\phi\right) \\ &\quad + b^2\sin^2\left(\frac{a-b}{b}\phi\right)\Big] \\ &= \Big\{(a-b)^2 + b^2 - 2(a-b)b \\ &\quad \times\left[\cos\phi\cos\left(\frac{a-b}{b}\phi\right) - \sin\phi\sin\left(\frac{a-b}{b}\phi\right)\right]\Big\} \\ &= (a-b)^2 + b^2 - 2(a-b)b\cos\left(\frac{a}{b}\phi\right). \end{aligned} \tag{24}$$

But $\rho = a - 2b$, so $b = (a-\rho)/2$, which gives

$$\begin{aligned} (a-b)^2 + b^2 &= [a - \tfrac{1}{2}(a-\rho)]^2 + [\tfrac{1}{2}(a-\rho)]^2 \\ &= [\tfrac{1}{2}(a+\rho)]^2 + [\tfrac{1}{2}(a-\rho)]^2 \\ &= \tfrac{1}{4}(a^2 + 2a\rho + \rho^2 + a^2 - 2a\rho + \rho^2) \\ &= \tfrac{1}{2}(a^2 + \rho^2) \end{aligned} \tag{25}$$

$$\begin{aligned} 2(a-b)b &= 2[a - \tfrac{1}{2}(a-\rho)]\tfrac{1}{2}(a-\rho) \\ &= \tfrac{1}{2}(a+\rho)(a-\rho) = \tfrac{1}{2}(a^2 - \rho^2). \end{aligned} \tag{26}$$

Now let

$$2\Omega t \equiv \frac{a}{b}\phi, \tag{27}$$

so

$$\phi = \frac{a-\rho}{a}\Omega t \tag{28}$$

$$\frac{\phi}{a-\rho} = \frac{\Omega t}{a}, \tag{29}$$

then

$$\begin{aligned} r^2 &= \tfrac{1}{2}(a^2 + \rho^2) - \tfrac{1}{2}(a^2 - \rho^2)\cos\left(\frac{a}{b}\phi\right) \\ &= \tfrac{1}{2}(a^2 + \rho^2) - \tfrac{1}{2}(a^2 - \rho^2)\cos(2\Omega t). \end{aligned} \tag{30}$$

The POLAR ANGLE is

$$\tan\theta \equiv \frac{y}{x} = \frac{(a-b)\sin\phi + b\sin\left(\frac{a-b}{a}\phi\right)}{(a-b)\cos\phi - b\cos\left(\frac{a-b}{a}\phi\right)}. \tag{31}$$

But

$$b = \tfrac{1}{2}(a-\rho) \tag{32}$$

$$a - b = \tfrac{1}{2}(a+\rho) \tag{33}$$

$$\frac{a-b}{b} = \frac{a+\rho}{a-\rho}, \tag{34}$$

so

$$\begin{aligned} \tan\theta &= \frac{\tfrac{1}{2}(a+\rho)\sin\phi + \tfrac{1}{2}(a-\rho)\sin\left(\frac{a+\rho}{z-\rho}\phi\right)}{\tfrac{1}{2}(a+\rho)\cos\phi - \tfrac{1}{2}(a-\rho)\cos\left(\frac{a+\rho}{a-\rho}\phi\right)} \\ &= \frac{(a+\rho)\sin\left(\frac{a-\rho}{a}\Omega t\right) + (a-\rho)\sin\left(\frac{a+\rho}{a}\Omega t\right)}{(a+\rho)\cos\left(\frac{a-\rho}{a}\Omega t\right) - (a-\rho)\cos\left(\frac{a+\rho}{a}\Omega t\right)} \\ &= \frac{a\left[\sin\left(\frac{a-\rho}{a}\Omega t\right) + \sin\left(\frac{a+\rho}{a}\Omega t\right)\right] + \rho\left[\sin\left(\frac{a-\rho}{a}\Omega t\right) - \sin\left(\frac{a+\rho}{a}\Omega t\right)\right]}{a\left[\cos\left(\frac{a-\rho}{a}\Omega t\right) - \cos\left(\frac{a+\rho}{a}\Omega t\right)\right] + \rho\left[\cos\left(\frac{a-\rho}{a}\Omega t\right) + \cos\left(\frac{a+\rho}{a}\Omega t\right)\right]} \\ &= \frac{2a\sin(\Omega t)\cos\left(\frac{\rho}{a}\Omega t\right) - 2\rho\cos(\Omega t)\sin\left(\frac{\rho}{a}\Omega t\right)}{2a\sin(\Omega t)\sin\left(\frac{\rho}{a}\Omega t\right) + 2\rho\cos(\Omega t)\cos\left(\frac{\rho}{a}\Omega t\right)} \\ &= \frac{a\tan(\Omega t) - \rho\tan\left(\frac{\rho}{a}\Omega t\right)}{a\tan(\Omega t)\tan\left(\frac{\rho}{a}\Omega t\right) + \rho}. \end{aligned} \tag{35}$$

Computing

$$\begin{aligned} \tan\left(\theta + \frac{\rho}{a}\Omega t\right) &= \frac{\left[a\tan(\Omega t) - \rho\tan\left(\frac{\rho}{a}\Omega t\right) + \tan\left(\frac{\rho}{a}\Omega t\right)\right] \times\left[a\tan(\Omega t)\tan\left(\frac{\rho}{a}\Omega t\right) + \rho\right]}{\left[a\tan(\Omega t)\tan\left(\frac{\rho}{a}\Omega t\right) + \rho\right] - \left[a\tan(\Omega t) - \rho\tan\left(\frac{\rho}{a}\Omega t\right)\right]\tan\left(\frac{\rho}{a}\Omega t\right)} \\ &= \frac{a\tan(\Omega t)\left[1 + \tan^2\left(\frac{\rho}{a}\Omega t\right)\right]}{\rho\left[1 + \tan^2\left(\frac{\rho}{a}\Omega t\right)\right]} \\ &= \frac{a}{\rho}\tan(\Omega t), \end{aligned} \tag{36}$$

then gives

$$\theta = \tan^{-1}\left[\frac{a}{\rho}\tan(\Omega t)\right] - \frac{\rho}{a}\Omega t. \tag{37}$$

Finally, plugging back in gives

$$\begin{aligned}\theta &= \tan^{-1}\left[\frac{a}{\rho}\tan\left(\frac{a}{a-\rho}\phi\right)\right] - \frac{\rho}{a}\frac{a}{a-\rho}\phi \\ &= \tan^{-1}\left[\frac{a}{\rho}\tan\left(\frac{a}{a-\rho}\phi\right)\right] - \frac{\rho}{a-\rho}\phi. \qquad (38)\end{aligned}$$

This form is useful in the solution of the SPHERE WITH TUNNEL problem, which is the generalization of the BRACHISTOCHRONE PROBLEM, to find the shape of a tunnel drilled through a SPHERE (with gravity varying according to Gauss's law for gravitation) such that the travel time between two points on the surface of the SPHERE under the force of gravity is minimized.

see also CYCLOID, EPICYCLOID

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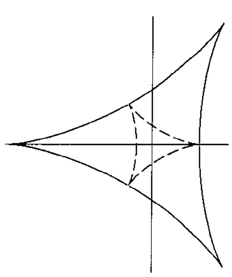
Hypocycloid—3-Cusped

see DELTOID

Hypocycloid—4-Cusped

see ASTROID

Hypocycloid Evolute



For $x(0) = a$,

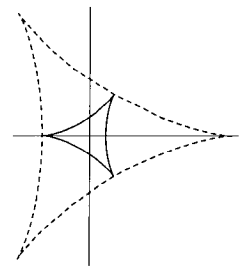
$$\begin{aligned}x &= \frac{a}{a-2b}\left[(a-b)\cos\phi - b\cos\left(\frac{a-b}{b}\phi\right)\right] \\ y &= \frac{a}{a-2b}\left[(a-b)\sin\phi + b\sin\left(\frac{a-b}{b}\phi\right)\right].\end{aligned}$$

If $a/b = n$, then

$$\begin{aligned}x &= \frac{1}{n-2}[(n-1)\cos\phi - \cos[(n-1)\phi]a \\ y &= \frac{1}{n-2}[(n-1)\sin\phi + \sin[(n-1)\phi]a.\end{aligned}$$

This is just the original HYPOCYCLOID scaled by the factor $(n-2)/n$ and rotated by $1/(2n)$ of a turn.

Hypocycloid Involute



The HYPOCYCLOID

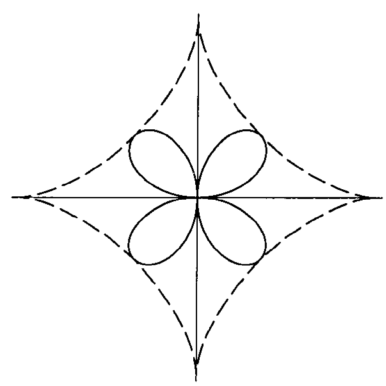
$$\begin{aligned}x &= \frac{a}{a-2b}\left[(a-b)\cos\phi - b\cos\left(\frac{a-b}{b}\phi\right)\right] \\ y &= \frac{a}{a-2b}\left[(a-b)\sin\phi + b\sin\left(\frac{a-b}{b}\phi\right)\right]\end{aligned}$$

has INVOLUTE

$$\begin{aligned}x &= \frac{a-2b}{a}\left[(a-b)\cos\phi + b\cos\left(\frac{a-b}{b}\phi\right)\right] \\ y &= \frac{a-2b}{a}\left[(a-b)\sin\phi - b\sin\left(\frac{a-b}{b}\phi\right)\right],\end{aligned}$$

which is another HYPOCYCLOID.

Hypocycloid Pedal Curve



The PEDAL CURVE for a PEDAL POINT at the center is a ROSE.

Hypoellipse

$$y^{n/m} + c\left|\frac{x}{a}\right|^{n/m} - c = 0,$$

with $n/m < 2$. If $n/m > 2$, the curve is a HYPERELLIPSE.

see also ELLIPSE, HYPERELLIPSE, SUPERELLIPSE

References
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Hypotenuse

The longest LEG of a RIGHT TRIANGLE (which is the side opposite the RIGHT ANGLE).

Hypothesis

A proposition that is consistent with known data, but has been neither verified nor shown to be false. It is synonymous with CONJECTURE.

see also BOURGET'S HYPOTHESIS, CHINESE HYPOTHESIS, CONTINUUM HYPOTHESIS, HYPOTHESIS TESTING, NESTED HYPOTHESIS, NULL HYPOTHESIS, POSTULATE, RAMANUJAN'S HYPOTHESIS, RIEMANN HYPOTHESIS, SCHINZEL'S HYPOTHESIS, SOUSLIN'S HYPOTHESIS

Hypothesis Testing

The use of statistics to determine the probability that a given hypothesis is true.

see also BONFERRONI CORRECTION, ESTIMATE, FISHER SIGN TEST, PAIRED t-TEST, STATISTICAL TEST, TYPE I ERROR, TYPE II ERROR, WILCOXON SIGNED RANK TEST

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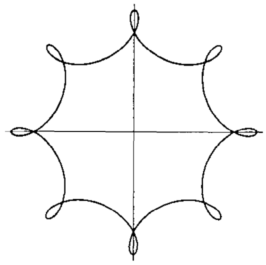
Hypotrochoid



The ROULETTE traced by a point P attached to a CIRCLE of radius b rolling around the inside of a fixed CIRCLE of radius a. The parametric equations for a hypotrochoid are

$$x = n\cos t + h\cos\left(\frac{n}{b}t\right) \quad (1)$$

$$y = n\sin t - h\sin\left(\frac{n}{b}t\right), \quad (2)$$

where $n \equiv a - b$ and h is the distance from P to the center of the rolling CIRCLE. Special cases include the HYPOCYCLOID with $h = b$, the ELLIPSE with $a = 2b$, and the ROSE with

$$a = \frac{2nh}{n+1} \quad (3)$$

$$b = \frac{(n-1)h}{n+1}. \quad (4)$$

see also EPITROCHOID, HYPOCYCLOID, SPIROGRAPH

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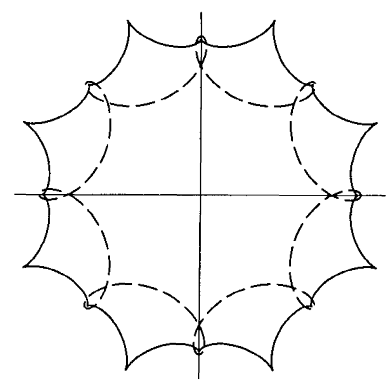
Hypotrochoid Evolute



The EVOLUTE of the HYPOTROCHOID is illustrated above.

Hyzer's Illusion

see FREEMISH CRATE

I

i

The IMAGINARY NUMBER i is defined as $i \equiv \sqrt{-1}$. However, for some reason engineers and physicists prefer the symbol j to i. Numbers of the form $z = x + iy$ where x and y are REAL NUMBERS are called COMPLEX NUMBERS, and when z is used to denote a COMPLEX NUMBER, it is sometimes (in older texts) called an "AFFIX."

The SQUARE ROOT of i is

$$\sqrt{i} = \pm\frac{i+1}{\sqrt{2}}, \tag{1}$$

since

$$\left[\frac{1}{\sqrt{2}}(i+1)\right]^2 = \tfrac{1}{2}(i^2 + 2i + 1) = i. \tag{2}$$

This can be immediately derived from the EULER FORMULA with $x = \pi/2$,

$$i = e^{i\pi/2} \tag{3}$$

$$\sqrt{i} = \sqrt{e^{i\pi/2}} = e^{i\pi/4} = \cos(\tfrac{1}{4}\pi) + i\sin(\tfrac{1}{4}\pi) = \frac{1+i}{\sqrt{2}}. \tag{4}$$

The PRINCIPAL VALUE of i^i is

$$i^i = (e^{i\pi/2})^i = e^{i^2\pi/2} = e^{-\pi/2} = 0.207879\ldots. \tag{5}$$

see also COMPLEX NUMBER, IMAGINARY IDENTITY, IMAGINARY NUMBER, REAL NUMBER, SURREAL NUMBER

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$\mathbb{I}$

see $\mathbb{Z}$

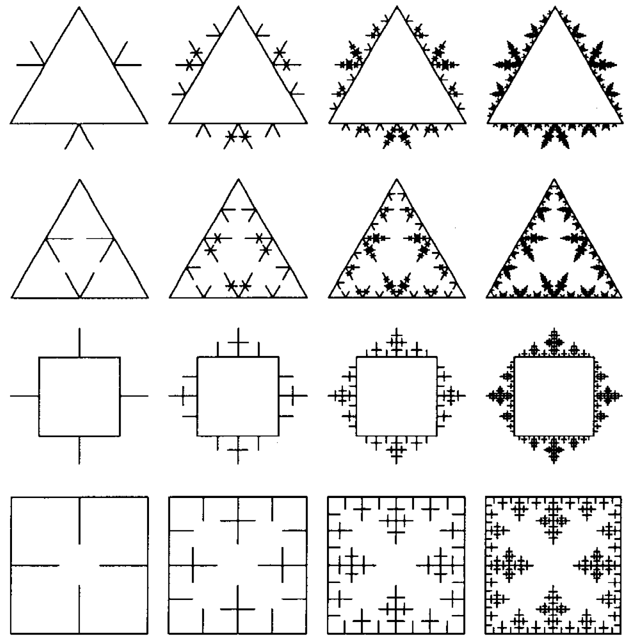
I-Signature

see SIGNATURE (RECURRENCE RELATION)

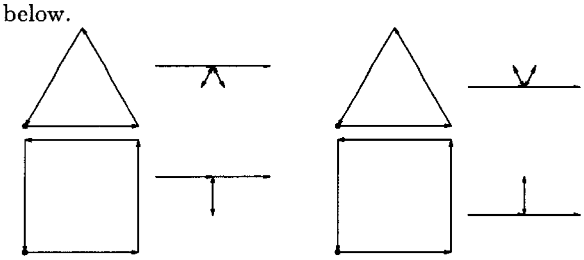
Iamond

see POLYIAMOND

Ice Fractal



A FRACTAL (square, triangle, etc.) based on a simple generating motif. The above plots show the ice triangle, antitriangle, square, and antisquare. The base curves and motifs for the fractals illustrated above are shown below.



see also FRACTAL

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Lauwerier, H. *Fractals: Endlessly Repeated Geometric Figures.* Princeton, NJ: Princeton University Press, p. 44, 1991.

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Icosagon

A 20-sided POLYGON. The SWASTIKA is an irregular icosagon.

see also SWASTIKA

Icosahedral Equation

Hunt (1996) gives the "dehomogenized" icosahedral equation as

$$[(z^{20}+1) - 228(z^{15} - z^5) + 494z^{10})^3 \\ +1728uz^5(z^{10} + 11z^5 - 1)^5 = 0.$$

Other forms include

$$\begin{aligned} I(u,v,Z) = u^5v^5(u^{10}+11u^5v^5-v^{10})^5 \\ -[u^{30}+v^{30}-10005(u^{20}v^{10}+u^{10}v^{20}) \\ +522(u^25v^5-u^5v^25)]^2Z=0 \end{aligned}$$

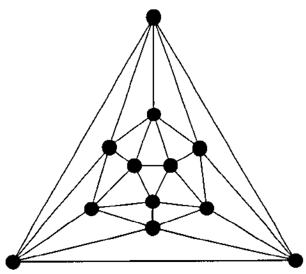
and

$$\begin{aligned} I(z,1,Z) = z^5(-1+11z^5+z^{10})^5 \\ -[1+z^{30}-10005(z^{10}+z^{20})+522(-z^5+z^{25})]^2Z=0. \end{aligned}$$

References

Hunt, B. *The Geometry of Some Special Arithmetic Quotients.* New York: Springer-Verlag, p. 146, 1996.

Icosahedral Graph



A POLYHEDRAL GRAPH.

see also CUBICAL GRAPH, DODECAHEDRAL GRAPH, OCTAHEDRAL GRAPH, TETRAHEDRAL GRAPH

Icosahedral Group

The GROUP I_h of symmetries of the ICOSAHEDRON and DODECAHEDRON. The icosahedral group consists of the symmetry operations E, $12C_5$, $12C_5^2$, $20C_3$, $15C_2$, i, $12S_{10}$, $12S_{10}^3$, $20S_6$, and 15σ (Cotton 1990).

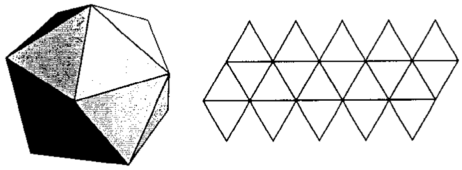
see also DODECAHEDRON, ICOSAHEDRON, OCTAHEDRAL GROUP, TETRAHEDRAL GROUP

References

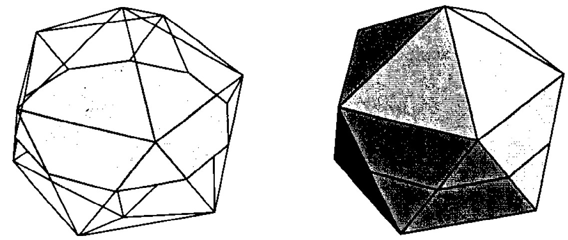
Cotton, F. A. *Chemical Applications of Group Theory, 3rd ed.* New York: Wiley, p. 48–50, 1990.

Lomont, J. S. "Icosahedral Group." §3.10.E in *Applications of Finite Groups.* New York: Dover, p. 82, 1987.

Icosahedron



A PLATONIC SOLID (P_5) with 12 VERTICES, 30 EDGES, and 20 equivalent EQUILATERAL TRIANGLE faces $20\{3\}$. It is described by the SCHLÄFLI SYMBOL $\{3,5\}$. It is also UNIFORM POLYHEDRON U_{22} and has WYTHOFF SYMBOL $5\,|\,2\,3$. The icosahedron has the ICOSAHEDRAL GROUP I_h of symmetries.



A plane PERPENDICULAR to a C_5 axis of an icosahedron cuts the solid in a regular DECAGONAL CROSS-SECTION (Holden 1991, pp. 24–25).

A construction for an icosahedron with side length $a = \sqrt{50-10\sqrt{5}}/5$ places the end vertices at $(0,0,\pm 1)$ and the central vertices around two staggered CIRCLES of RADII $\frac{2}{5}\sqrt{5}$ and heights $\pm\frac{1}{5}\sqrt{5}$, giving coordinates

$$\pm\left(\tfrac{2}{5}\sqrt{5}\cos(\tfrac{2}{5}i\pi), \tfrac{2}{5}\sqrt{5}\sin(\tfrac{2}{5}i\pi), \tfrac{1}{5}\sqrt{5}\right) \qquad (1)$$

for $i = 0, 1, \ldots, 4$, where all the plus signs or minus signs are taken together. Explicitly, these coordinates are

$$\mathbf{x}_0^\pm = \pm(\tfrac{2}{5}\sqrt{5}, 0, \tfrac{1}{5}\sqrt{5}) \qquad (2)$$

$$\mathbf{x}_1^\pm = \pm(\tfrac{1}{10}(5-\sqrt{5}), \tfrac{1}{10}\sqrt{50+10\sqrt{5}}, \tfrac{1}{5}\sqrt{5}) \qquad (3)$$

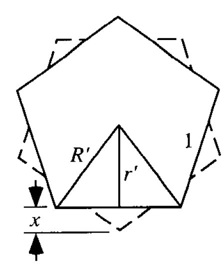
$$\mathbf{x}_2^\pm = \pm(-\tfrac{1}{10}(\sqrt{5}+5), \tfrac{1}{10}\sqrt{50-10\sqrt{5}}, \tfrac{1}{5}\sqrt{5}) \qquad (4)$$

$$\mathbf{x}_3^\pm = \pm(-\tfrac{1}{10}(\sqrt{5}-5), -\tfrac{1}{10}\sqrt{50-10\sqrt{5}}, \tfrac{1}{5}\sqrt{5}) \qquad (5)$$

$$\mathbf{x}_4^\pm = \pm(\tfrac{1}{10}(5-\sqrt{5}), -\tfrac{1}{10}\sqrt{50+10\sqrt{5}}, \tfrac{1}{5}\sqrt{5}). \qquad (6)$$

By a suitable rotation, the VERTICES of an icosahedron of side length 2 can also be placed at $(0,\pm\phi,\pm 1)$, $(\pm 1, 0, \pm\phi)$, and $(\pm\phi, \pm 1, 0)$, where ϕ is the GOLDEN RATIO. These points divide the EDGES of an OCTAHEDRON into segments with lengths in the ratio $\phi : 1$.

The DUAL POLYHEDRON of the icosahedron is the DODECAHEDRON. There are 59 distinct icosahedra when each TRIANGLE is colored differently (Coxeter 1969).



To derive the VOLUME of an icosahedron having edge length a, consider the orientation so that two VERTICES are oriented on top and bottom. The vertical distance between the top and bottom PENTAGONAL DIPYRAMIDS is then given by

$$z = \sqrt{\ell^2 - x^2}, \qquad (7)$$

where

$$\ell = \tfrac{1}{2}\sqrt{3}\,a \qquad (8)$$

is the height of an ISOSCELES TRIANGLE, and the SAGITTA $x \equiv R' - r'$ of the pentagon is

$$x = \tfrac{1}{2}a\tfrac{1}{10}\sqrt{25 - 10\sqrt{5}}\,a, \tag{9}$$

giving

$$x^2 = \tfrac{1}{20}\sqrt{5 - 2\sqrt{5}}\,a^2. \tag{10}$$

Plugging (8) and (10) into (7) gives

$$\begin{aligned} z &= a\sqrt{\tfrac{3}{4} - \tfrac{1}{20}(5 - 2\sqrt{5})} = a\sqrt{\frac{15 - (5 - 2\sqrt{5})}{20}} \\ &= a\sqrt{\frac{10 + 2\sqrt{5}}{20}} = \tfrac{1}{2}a\sqrt{\frac{10 + 2\sqrt{5}}{5}} \\ &= \tfrac{1}{10}\sqrt{50 + 10\sqrt{5}}\,a, \end{aligned} \tag{11}$$

which is identical to the radius of a PENTAGON of side a. The CIRCUMRADIUS is then

$$R = h + \tfrac{1}{2}z, \tag{12}$$

where

$$h = \tfrac{1}{10}\sqrt{50 - 10\sqrt{5}}\,a \tag{13}$$

is the height of a PENTAGONAL DIPYRAMID. Therefore,

$$\begin{aligned} R^2 &= (h + \tfrac{1}{2}z)^2 \\ &= (\tfrac{1}{10}\sqrt{50 - 10\sqrt{5}} + \tfrac{1}{20}\sqrt{50 + 10\sqrt{5}}\,)^2 a^2 \\ &= \left(\frac{5}{8} - \frac{3}{8\sqrt{5}} + \frac{\sqrt{20}}{10}\right)a^2 = \tfrac{1}{8}(5 + \sqrt{5}\,)a. \end{aligned} \tag{14}$$

Taking the square root gives the CIRCUMRADIUS

$$R = \sqrt{\tfrac{1}{8}(5 + \sqrt{5})}\,a = \tfrac{1}{4}\sqrt{10 + 2\sqrt{5}}\,a \approx 0.95105a. \tag{15}$$

The INRADIUS is

$$r = \tfrac{1}{12}(3\sqrt{3} + \sqrt{15}\,)a \approx 0.75576a. \tag{16}$$

The square of the INTERRADIUS is

$$\begin{aligned} \rho^2 &= (\tfrac{1}{2}z)^2 + {x_l}^2 \\ &= [(\tfrac{1}{4})(\tfrac{1}{100})(50 + 10\sqrt{5}\,) + \tfrac{1}{100}(25 + 10\sqrt{5}\,)]a^2 \\ &= \tfrac{1}{8}(3 + \sqrt{5})a^2, \end{aligned} \tag{17}$$

so

$$\rho = \sqrt{\tfrac{1}{8}(3 + \sqrt{5}\,)}\,a = \tfrac{1}{4}(1 + \sqrt{5}\,)a \approx 0.80901a. \tag{18}$$

The AREA of one face is the AREA of an EQUILATERAL TRIANGLE

$$A = \tfrac{1}{4}a^2\sqrt{3}. \tag{19}$$

The volume can be computed by taking 20 pyramids of height r

$$\begin{aligned} V &= 20[(\tfrac{1}{3}A)r] = 20\tfrac{1}{3}\tfrac{1}{4}\sqrt{3}\,a^2\,\tfrac{1}{12}(3\sqrt{3} + \sqrt{15}\,)a \\ &= \tfrac{5}{12}(3 + \sqrt{5}\,)a^3. \end{aligned} \tag{20}$$

Apollonius showed that

$$\frac{V_{\text{icosahedron}}}{V_{\text{dodecahedron}}} = \frac{A_{\text{icosahedron}}}{A_{\text{dodecahedron}}}, \tag{21}$$

where V is the volume and A the surface area.

see also AUGMENTED TRIDIMINISHED ICOSAHEDRON, DECAGON, DODECAHEDRON, GREAT ICOSAHEDRON, ICOSAHEDRON STELLATIONS, METABIDIMINISHED ICOSAHEDRON, TRIDIMINISHED ICOSAHEDRON, TRIGONOMETRY VALUES—$\pi/5$

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Icosahedron Stellations

Applying the STELLATION process to the ICOSAHEDRON gives

$$20 + 30 + 60 + 20 + 60 + 120 + 12 + 30 + 60 + 60$$

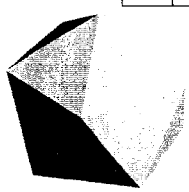
cells of ten different shapes and sizes in addition to the ICOSAHEDRON itself. After application of five restrictions due to J. C. P. Miller to define which forms should be considered distinct, 59 stellations are found to be possible. Miller's restrictions are

1. The faces must lie in the twenty bounding planes of the icosahedron.
2. The parts of the faces in the twenty planes must be congruent, but those parts lying in one place may be disconnected.
3. The parts lying in one plane must have threefold rotational symmetry with or without reflections.
4. All parts must be accessible, i.e., lie on the outside of the solid.
5. Compounds are excluded that can be divided into two sets, each of which has the full symmetry of the whole.

Of these, 32 have full icosahedral symmetry and 27 are ENANTIOMERIC forms. Four are POLYHEDRON COMPOUNDS, one is a KEPLER-POINSOT SOLID, and one is the DUAL POLYHEDRON of an ARCHIMEDEAN SOLID.

The only STELLATIONS of PLATONIC SOLIDS which are UNIFORM POLYHEDRA are the three DODECAHEDRON STELLATIONS the GREAT ICOSAHEDRON (stellation # 11).

n	name
1	icosahedron
2	triakisicosahedron
3	octahedron 5-compound
4	echidnahedron
11	great icosahedron
18	tetrahedron 10-compound
20	deltahedron-60
36	tetrahedron 5-compound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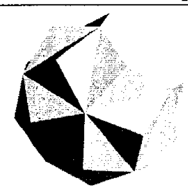
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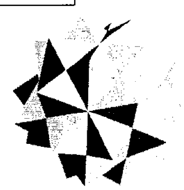
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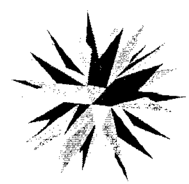
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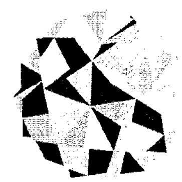
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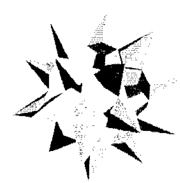
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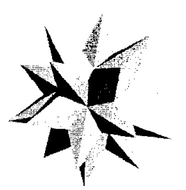
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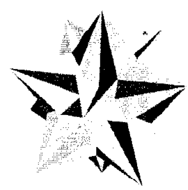
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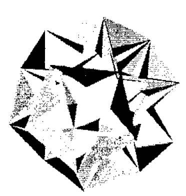
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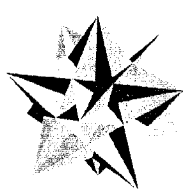
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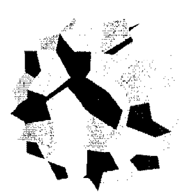
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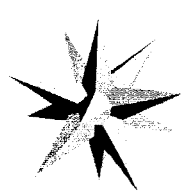
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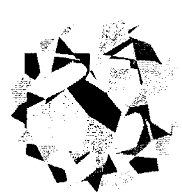
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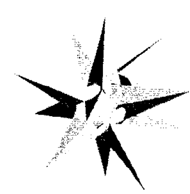
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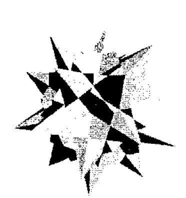
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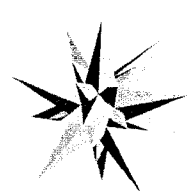
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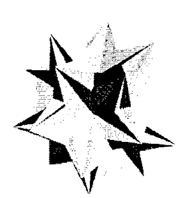
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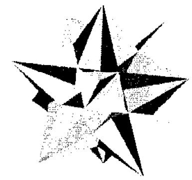
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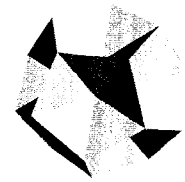
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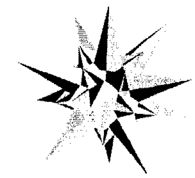
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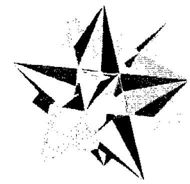
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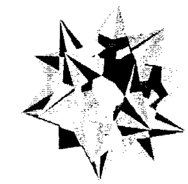
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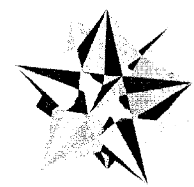
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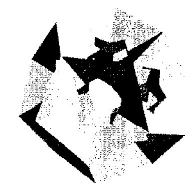
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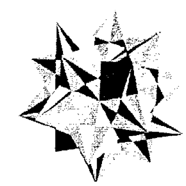
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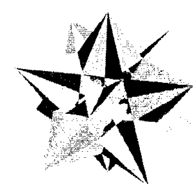
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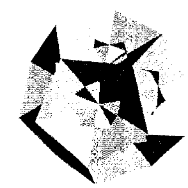
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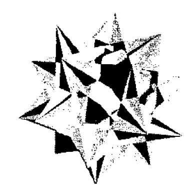
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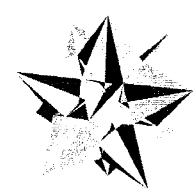
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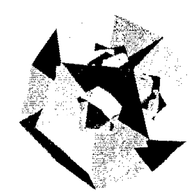
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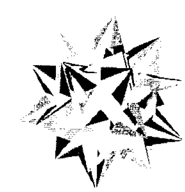
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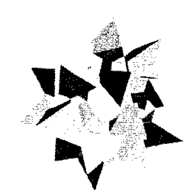
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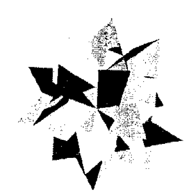
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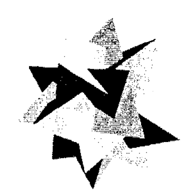
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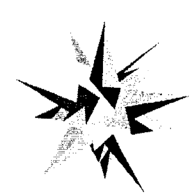
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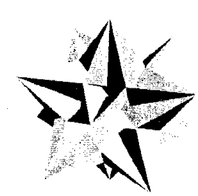
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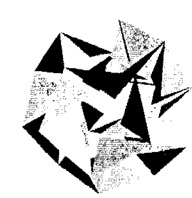
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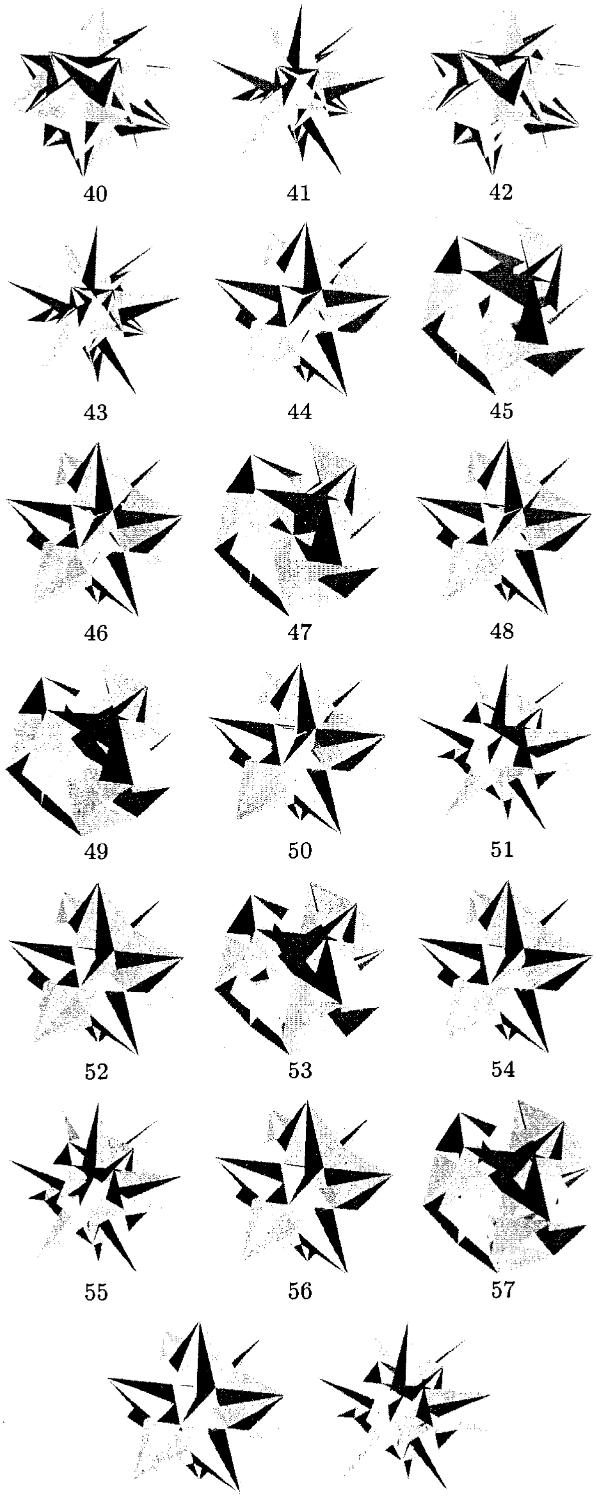

see also ARCHIMEDEAN SOLID STELLATION, DODECAHEDRON STELLATIONS, STELLATION

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Icosian Game

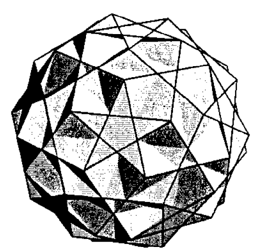
The problem of finding a HAMILTONIAN CIRCUIT along the edges of an ICOSAHEDRON, i.e., a path such that every vertex is visited a single time, no edge is visited twice, and the ending point is the same as the starting point.

see also HAMILTONIAN CIRCUIT, ICOSAHEDRON

References

Herschel, A. S. "Sir Wm. Hamilton's Icosian Game." *Quart. J. Pure Applied Math.* **5**, 305, 1862.

Icosidodecadodecahedron



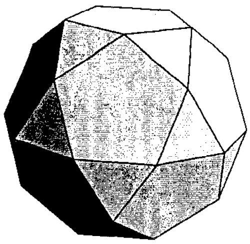
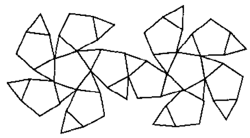
The UNIFORM POLYHEDRON U_{44} whose DUAL POLYHEDRON is the MEDIAL ICOSACRONIC HEXECONTAHEDRON. It has WYTHOFF SYMBOL $\frac{5}{3}\,5\,|\,3$. Its faces are $20\{6\}+12\{\frac{5}{2}\}+12\{5\}$. Its CIRCUMRADIUS for unit edge length is

$$R=\tfrac{1}{2}\sqrt{7}.$$

References

Wenninger, M. J. *Polyhedron Models.* Cambridge, England: Cambridge University Press, pp. 128–129, 1989.

Icosidodecahedron

An ARCHIMEDEAN SOLID whose DUAL POLYHEDRON is the RHOMBIC TRIACONTAHEDRON. It is one of the two convex QUASIREGULAR POLYHEDRA and has SCHLÄFLI SYMBOL $\left\{ {3 \atop 5} \right\}$. It is also UNIFORM POLYHEDRON U_{24} and has WYTHOFF SYMBOL $2\,|\,3\,5$. Its faces are $20\{3\}+12\{5\}$. The VERTICES of an icosidodecahedron of EDGE length $2\phi^{-1}$ are $(\pm 2, 0, 0)$, $(0, \pm 2, 0)$, $(0, 0, \pm 2)$, $(\pm 1, \pm\phi^{-1}, \pm 1)$, $(\pm 1, \pm\phi, \pm\phi^{-1})$, $(\pm\phi^{-1}, \pm 1, \pm\phi)$. The 30 VERTICES of an OCTAHEDRON 5-COMPOUND form an icosidodecahedron (Ball and Coxeter 1987). FACETED versions include the SMALL ICOSIHEMIDODECAHEDRON and SMALL DODECAHEMIDODECAHEDRON.

The faces of the icosidodecahedron consist of 20 triangles and 12 pentagons. Furthermore, its 60 edges are bisected perpendicularly by those of the reciprocal RHOMBIC TRIACONTAHEDRON (Ball and Coxeter 1987).

The INRADIUS, MIDRADIUS, and CIRCUMRADIUS for unit edge length are

$$r = \tfrac{1}{8}(5 + 3\sqrt{5}) \approx 1.46353$$
$$\rho = \tfrac{1}{2}\sqrt{5 + 2\sqrt{5}} \approx 1.53884$$
$$R = \tfrac{1}{2}(1 + \sqrt{5}) = \phi \approx 1.61803.$$

see also ARCHIMEDEAN SOLID, GREAT ICOSIDODECAHEDRON, QUASIREGULAR POLYHEDRON, SMALL ICOSIHEMIDODECAHEDRON, SMALL DODECAHEMIDODECAHEDRON

References
Ball, W. W. R. and Coxeter, H. S. M. *Mathematical Recreations and Essays, 13th ed.* New York: Dover, p. 137, 1987.
Wenninger, M. J. *Polyhedron Models.* Cambridge, England: Cambridge University Press, p. 73, 1989.

Icosidodecahedron Stellation

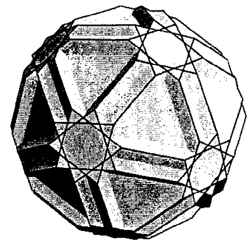
The first stellation is a DODECAHEDRON-ICOSAHEDRON COMPOUND.

References
Wenninger, M. J. *Polyhedron Models.* Cambridge, England: Cambridge University Press, pp. 73–96, 1989.

Icosidodecatruncated Icosidodecahedron

see ICOSITRUNCATED DODECADODECAHEDRON

Icositruncated Dodecadodecahedron



The UNIFORM POLYHEDRON U_{45} also called the ICOSIDODECATRUNCATED ICOSIDODECAHEDRON whose DUAL POLYHEDRON is the TRIDYAKIS ICOSAHEDRON. It has WYTHOFF SYMBOL $3\,\frac{5}{3}\,5\,|$. Its faces are $20\{6\}+12\{10\}+12\{\frac{10}{3}\}$. Its CIRCUMRADIUS for unit edge length is

$$R = 2.$$

References
Wenninger, M. J. *Polyhedron Models.* Cambridge, England: Cambridge University Press, pp. 130–131, 1989.

Ida Surface

A 3-D shadow of a 4-D KLEIN BOTTLE.

see also KLEIN BOTTLE

References
Peterson, I. *Islands of Truth: A Mathematical Mystery Cruise.* New York: W. H. Freeman, pp. 44–45, 1990.

Ideal

A subset I of elements in a RING R which forms an additive GROUP and has the property that, whenever x belongs to R and y belongs to I, then xy and yx belong to I. For example, the set of EVEN INTEGERS is an ideal in the RING of INTEGERS. Given an ideal I, it is possible to define a FACTOR RING R/I.

An ideal may be viewed as a lattice and specified as the finite list of algebraic integers that form a basis for the lattice. Any two bases for the same lattice are equivalent. Ideals have multiplication, and this is basically the Kronecker product of the two bases.

For any ideal I, there is an ideal I_i such that

$$II_i = z,$$

where z is a PRINCIPAL IDEAL, (i.e., an ideal of rank 1). Moreover there is a finite list of ideals I_i such that this equation may be satisfied for every I. The size of this list is known as the CLASS NUMBER. In effect, the above relation imposes an EQUIVALENCE RELATION on ideals, and the number of ideals modulo this relation is the class number. When the CLASS NUMBER is 1, the corresponding number RING has unique factorization and, in a sense, the class number is a measure of the failure of unique factorization in the original number ring.

Dedekind (1871) showed that every NONZERO ideal in the domain of INTEGERS of a FIELD is a unique product of PRIME IDEALS.

see also CLASS NUMBER, DIVISOR THEORY, IDEAL NUMBER, MAXIMAL IDEAL, PRIME IDEAL, PRINCIPAL IDEAL

References

Malgrange, B. *Ideals of Differentiable Functions.* London: Oxford University Press, 1966.

Ideal Number

A type of number involving the ROOTS OF UNITY which was developed by Kummer while trying to solve FERMAT'S LAST THEOREM. Although factorization over the INTEGERS is unique (the FUNDAMENTAL THEOREM OF ALGEBRA), factorization is *not* unique over the COMPLEX NUMBERS. Over the ideal numbers, however, factorization in terms of the COMPLEX NUMBERS becomes unique. Ideal numbers were so powerful that they were generalized by Dedekind into the more abstract IDEALS in general RINGS which are a key part of modern abstract ALGEBRA.

see also DIVISOR THEORY, FERMAT'S LAST THEOREM, IDEAL

Ideal (Partial Order)

An ideal I of a PARTIAL ORDER P is a subset of the elements of P which satisfy the property that if $y \in I$ and $x < y$, then $x \in I$. For k disjoint chains in which the ith chain contains n_i elements, there are $(1+n_1)(1+n_2)\cdots(1+n_k)$ ideals. The number of ideals of a n-element FENCE POSET is the FIBONACCI NUMBER F_n.

References

Ruskey, F. "Information on Ideals of Partially Ordered Sets." http://sue.csc.uvic.ca/~cos/inf/pose/Ideals.html.

Steiner, G. "An Algorithm to Generate the Ideals of a Partial Order." *Operat. Res. Let.* **5**, 317–320, 1986.

Ideal Point

A type of POINT AT INFINITY in which parallel lines in the HYPERBOLIC PLANE intersect at infinity in one direction, while diverging from one another in the other.

see also HYPERPARALLEL

Idele

The multiplicative subgroup of all elements in the product of the multiplicative groups $k_\nu^\times$ whose absolute value is 1 at all but finitely many ν, where k is a number FIELD and ν a PLACE.

see also ADÉLE

References

Knapp, A. W. "Group Representations and Harmonic Analysis, Part II." *Not. Amer. Math. Soc.* **43**, 537–549, 1996.

Idemfactor

see DYADIC

Idempotent

An OPERATOR $\tilde{A}$ such that $\tilde{A}^2 = \tilde{A}$ or an element of an ALGEBRA x such that $x^2 = x$.

see also AUTOMORPHIC NUMBER, BOOLEAN ALGEBRA, GROUP, SEMIGROUP

Identity

An identity is a mathematical relationship equating one quantity to another (which may initially appear to be different).

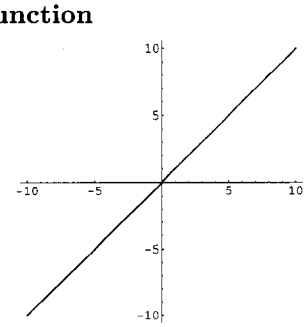
see also ABEL'S IDENTITY, ANDREWS-SCHUR IDENTITY, BAC-CAB IDENTITY, BEAUZAMY AND DÉGOT'S IDENTITY, BELTRAMI IDENTITY, BIANCHI IDENTITIES, BOCHNER IDENTITY, BRAHMAGUPTA IDENTITY, CASSINI'S IDENTITY, CAUCHY-LAGRANGE IDENTITY, CHRISTOFFEL-DARBOUX IDENTITY, CHU-VANDERMONDE IDENTITY, DE MOIVRE'S IDENTITY, DOUGALL-RAMANUJAN IDENTITY, EULER FOUR-SQUARE IDENTITY, EULER IDENTITY, EULER POLYNOMIAL IDENTITY, FERRARI'S IDENTITY, FIBONACCI IDENTITY, FROBENIUS TRIANGLE IDENTITIES, GREEN'S IDENTITIES, HYPERGEOMETRIC IDENTITY, IMAGINARY IDENTITY, JACKSON'S IDENTITY, JACOBI IDENTITIES, JACOBI'S DETERMINANT IDENTITY, LAGRANGE'S IDENTITY, LE CAM'S IDENTITY, LEIBNIZ IDENTITY, LIOUVILLE POLYNOMIAL IDENTITY, MATRIX POLYNOMIAL IDENTITY, MORGADO IDENTITY, NEWTON'S IDENTITIES, QUINTUPLE PRODUCT IDENTITY, RAMANUJAN 6-10-8 IDENTITY, RAMANUJAN COS/COSH IDENTITY, RAMANUJAN'S IDENTITY, RAMANUJAN'S SUM IDENTITY, REZNIK'S IDENTITY, ROGERS-RAMANUJAN IDENTITIES, SCHAAR'S IDENTITY, STREHL IDENTITY, SYLVESTER'S DETERMINANT IDENTITY, TRINOMIAL IDENTITY, VISIBLE POINT VECTOR IDENTITY, WATSON QUINTUPLE PRODUCT IDENTITY, WORPITZKY'S IDENTITY

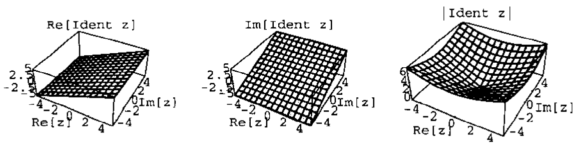
Identity Element

The identity element I (also denoted E, e, or 1) of a GROUP or related mathematical structure S is the unique elements such that $IA = AI = I$ for every element $A \in S$. The symbol "E" derives from the German word for unity, "Einheit."

see also BINARY OPERATOR, GROUP, INVOLUTION (GROUP), MONOID

Identity Function





The function $f(x) = x$ which assigns every REAL NUMBER x to the same REAL NUMBER x. It is identical to the IDENTITY MAP.

Identity Map

The MAP which assigns every REAL NUMBER to the same REAL NUMBER $\mathrm{id}_{\mathbb{R}}$. It is identical to the IDENTITY FUNCTION.

Identity Matrix

The identity matrix is defined as the MATRIX $\mathsf{1}$ (or I) such that

$$\mathsf{I}(\mathbf{X}) \equiv \mathbf{X}$$

for all VECTORS $\mathbf{X}$. The identity matrix is

$$I_{ij} = \delta_{ij}$$

for $i, j = 1, 2, \ldots, n$, where δ_{ij} is the KRONECKER DELTA. Written explicitly,

$$\mathsf{I} = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}.$$

Identity Operator

The OPERATOR $\tilde{I}$ which takes a REAL NUMBER to the same REAL NUMBER $\tilde{I}r = r$.

see also IDENTITY FUNCTION, IDENTITY MAP

Idoneal Number

A POSITIVE value of D for which the fact that a number is a MONOMORPH (i.e., the number is expressible in only one way as $x^2 + Dy^2$ or $x^2 - Dy^2$ where x^2 is RELATIVELY PRIME to Dy^2) guarantees it to be a PRIME, POWER of a PRIME, or twice one of these. The numbers are also called EULER'S IDONEAL NUMBERS, or SUITABLE NUMBERS.

The 65 idoneal numbers found by Gauss and Euler and conjectured to be the *only* such numbers (Shanks 1969) are 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 12, 13, 15, 16, 18, 21, 22, 24, 25, 28, 30, 33, 37, 40, 42, 45, 48, 57, 58, 60, 70, 72, 78, 85, 88, 93, 102, 105, 112, 120, 130, 133, 165, 168, 177, 190, 210, 232, 240, 253, 273, 280, 312, 330, 345, 357, 385, 408, 462, 520, 760, 840, 1320, 1365, and 1848 (Sloane's A000926).

References

Shanks, D. "On Gauss's Class Number Problems." *Math. Comput.* **23**, 151–163, 1969.

Sloane, N. J. A. Sequence A000926/M0476 in "An On-Line Version of the Encyclopedia of Integer Sequences."

Iff

If and only if (i.e., NECESSARY and SUFFICIENT). The terms "JUST IF" or "EXACTLY WHEN" are sometimes used instead. A iff B is written symbolically as $A \leftrightarrow B$. A iff B is also equivalent to $A \Rightarrow B$, together with $B \Rightarrow A$, where the symbol $\Rightarrow$ denotes "IMPLIES."

J. H. Conway believes that the word originated with P. Halmos and was transmitted through Kelley (1975). Halmos has stated, "To the best of my knowledge, I DID invent the silly thing, but I wouldn't swear to it in a court of law. So there—give me credit for it anyway" (Asimov 1997).

see also EQUIVALENT, EXACTLY ONE, IMPLIES, NECESSARY, SUFFICIENT

References

Asimov, D. "Iff." math-fun@cs.arizona.edu posting, Sept. 19, 1997.

Kelley, J. L. *General Topology.* New York: Springer-Verlag, 1975.

Ill-Conditioned

A system is ill-conditioned if the CONDITION NUMBER is too large (and singular if it is INFINITE).

see also CONDITION NUMBER

Illumination Problem

In the early 1950s, Ernst Straus asked

1. Is every POLYGONAL region illuminable from every point in the region?
2. Is every POLYGONAL region illuminable from at least one point in the region?

Here, illuminable means that there is a path from every point to every other by repeated reflections. Tokarsky (1995) showed that unilluminable rooms exist in the plane and 3-D, but question (2) remains open. The smallest known counterexample to (1) in the PLANE has 26 sides.

see also ART GALLERY THEOREM

References

Klee, V. "Is Every Polygonal Region Illuminable from Some Point?" *Math. Mag.* **52**, 180, 1969.

Tokarsky, G. W. "Polygonal Rooms Not Illuminable from Every Point." *Amer. Math. Monthly* **102**, 867–879, 1995.

Illusion

An object or drawing which appears to have properties which are physically impossible, deceptive, or counterintuitive.

see also BENHAM'S WHEEL, FREEMISH CRATE, GOBLET ILLUSION, HERMANN GRID ILLUSION, HERMANN-HERING ILLUSION, HYZER'S ILLUSION, IMPOSSIBLE FIGURE, IRRADIATION ILLUSION, KANIZSA TRIANGLE, MÜLLER-LYER ILLUSION, NECKER CUBE, ORBISON'S ILLUSION, PARALLELOGRAM ILLUSION, PENROSE

STAIRWAY, POGGENDORFF ILLUSION, PONZO'S ILLUSION, RABBIT-DUCK ILLUSION, TRIBAR, VERTICAL-HORIZONTAL ILLUSION, YOUNG GIRL-OLD WOMAN ILLUSION, ZOLLNER'S ILLUSION

References
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Ernst, B. *Optical Illusions.* New York: Taschen, 1996.
Fineman, M. *The Nature of Visual Illusion.* New York: Dover, 1996.
Gardner, M. "Optical Illusions." Ch. 1 in *Mathematical Circus: More Puzzles, Games, Paradoxes and Other Mathematical Entertainments from Scientific American.* New York: Knopf, 1979.
Gregory, R. L. *Eye and Brain, 5th ed.* Princeton, NJ: Princeton University Press, 1997.
"Illusions: Central Station." http://www.heureka.fi/i/Illusions_ctrl_station.html.en.
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Luckiesh, M. *Visual Illusions: Their Causes, Characteristics, and Applications.* New York: Dover, 1965.
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Tolansky, S. *Optical Illusions.* New York: Pergamon Press, 1964.

Image

see RANGE (IMAGE)

Imaginary Identity

see i

Imaginary Number

A COMPLEX NUMBER which has zero REAL PART, so that it can be written as a REAL NUMBER multiplied by the "imaginary unit" i (equal to $\sqrt{-1}$).

see also COMPLEX NUMBER, GALOIS IMAGINARY, GAUSSIAN INTEGER, i, REAL NUMBER

References
Conway, J. H. and Guy, R. K. *The Book of Numbers.* New York: Springer-Verlag, pp. 211–216, 1996.

Imaginary Part

The imaginary part $\Im$ of a COMPLEX NUMBER $z = x+iy$ is the REAL NUMBER multiplying i, so $\Im[x+iy] = y$. In terms of z itself,

$$\Im[z] = \frac{z - z^*}{2i},$$

where z^* is the COMPLEX CONJUGATE of z.

see also ABSOLUTE SQUARE, COMPLEX CONJUGATE, REAL PART

References
Abramowitz, M. and Stegun, C. A. (Eds.). *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables, 9th printing.* New York: Dover, p. 16, 1972.

Imaginary Point

A pair of values x and y one or both of which is COMPLEX.

References
Woods, F. S. *Higher Geometry: An Introduction to Advanced Methods in Analytic Geometry.* New York: Dover, p. 2, 1961.

Imaginary Quadratic Field

A QUADRATIC FIELD $\mathbb{Q}(\sqrt{D})$ with $D < 0$.

see also QUADRATIC FIELD

Immanant

For an $n \times n$ matrix, let S denote any permutation $e_1, e_2, \ldots, e_n$ of the set of numbers $1, 2, \ldots, n$, and let $\chi^{(\lambda)}(S)$ be the character of the symmetric group corresponding to the partition (λ). Then the immanant $|a_{mn}|^{(\lambda)}$ is defined as

$$|a_{mn}|^{(\lambda)} = \sum \chi^{(\lambda)}(S) P_S$$

where the summation is over the $n!$ permutations of the SYMMETRIC GROUP and

$$P_s = a_{1e_1} a_{2e_2} \cdots a_{ne_n}.$$

see also DETERMINANT, PERMANENT

References
Littlewood, D. E. and Richardson, A. R. "Group Characters and Algebra." *Philos. Trans. Roy. Soc. London A* **233**, 99–141, 1934.
Littlewood, D. E. and Richardson, A. R. "Immanants of Some Special Matrices." *Quart. J. Math. (Oxford)* **5**, 269–282, 1934.
Wybourne, B. G. "Immanants of Matrices." §2.19 in *Symmetry Principles and Atomic Spectroscopy.* New York: Wiley, pp. 12–13, 1970.

Immersed Minimal Surface

see ENNEPER'S SURFACES

Immersion

A special nonsingular MAP from one MANIFOLD to another such that at every point in the domain of the map, the DERIVATIVE is an injective linear map. This is equivalent to saying that every point in the DOMAIN has a NEIGHBORHOOD such that, up to DIFFEOMORPHISMS of the TANGENT SPACE, the map looks like the inclusion map from a lower-dimensional EUCLIDEAN SPACE to a higher-dimensional EUCLIDEAN SPACE.

see also BOY SURFACE, EVERSION, SMALE-HIRSCH THEOREM

References
Boy, W. "Über die Curvatura integra und die Topologie geschlossener Flächen." *Math. Ann* **57**, 151–184, 1903.
Pinkall, U. "Models of the Real Projective Plane." Ch. 6 in *Mathematical Models from the Collections of Universities and Museums* (Ed. G. Fischer). Braunschweig, Germany: Vieweg, pp. 63–67, 1986.

Impartial Game

A GAME in which the possible moves are the same for each player in any position. All positions in all impartial GAMES form an additive ABELIAN GROUP. For impartial games in which the last player wins (normal form games), the nim-value of the sum of two GAMES is the nim-sum of their nim-values. If the last player loses, the GAME is said to be in misère form and the analysis is much more difficult.

see also FAIR GAME, GAME, PARTISAN GAME

Implicit Function

A function which is not defined explicitly, but rather is defined in terms of an algebraic relationship (which can not, in general, be "solved" for the function in question). For example, the ECCENTRIC ANOMALY E of a body orbiting on an ELLIPSE with ECCENTRICITY e is defined implicitly in terms of the mean anomaly M by KEPLER'S EQUATION

$$M = E - e\sin E.$$

Implicit Function Theorem

Given

$$F_1(x,y,z,u,v,w) = 0$$

$$F_2(x,y,z,u,v,w) = 0$$

$$F_3(x,y,z,u,v,w) = 0,$$

if the JACOBIAN

$$JF(u,v,w) = \frac{\partial(F_1,F_2,F_3)}{\partial(u,v,w)} \neq 0,$$

then u, v, and w can be solved for in terms of x, y, and z and PARTIAL DERIVATIVES of u, v, w with respect to x, y, and z can be found by differentiating implicitly.

More generally, let A be an OPEN SET in $\mathbb{R}^{n+k}$ and let $f : A \to \mathbb{R}^n$ be a C^r FUNCTION. Write f in the form $f(x,y)$, where x and y are elements of $\mathbb{R}^k$ and $\mathbb{R}^n$. Suppose that (a, b) is a point in A such that $f(a,b) = 0$ and the DETERMINANT of the $n \times n$ MATRIX whose elements are the DERIVATIVES of the n component FUNCTIONS of f with respect to the n variables, written as y, evaluated at (a,b), is not equal to zero. The latter may be rewritten as

$$\operatorname{rank}(Df(a,b)) = n.$$

Then there exists a NEIGHBORHOOD B of a in $\mathbb{R}^k$ and a unique C^r FUNCTION $g : B \to \mathbb{R}^n$ such that $g(a) = b$ and $f(x,g(x)) = 0$ for all $x \in B$.

see also CHANGE OF VARIABLES THEOREM, JACOBIAN

References

Munkres, J. R. *Analysis on Manifolds.* Reading, MA: Addison-Wesley, 1991.

Implies

The symbol $\Rightarrow$ means "implies" in the mathematical sense. Let A be true. If this implies that B is also true, then the statement is written symbolically as $A \Rightarrow B$, or sometimes $A \subset B$. If $A \Rightarrow B$ and $B \Rightarrow A$ (i.e, $A \Rightarrow B \wedge B \Rightarrow A$), then A and B are said to be EQUIVALENT, a relationship which is written symbolically as $A \Leftrightarrow B$ or $A \rightleftharpoons B$.

see also EQUIVALENT

Impossible Figure

A class of ILLUSION in which an object which is physically unrealizable is apparently depicted.

see also FREEMISH CRATE, HOME PLATE, ILLUSION, NECKER CUBE, PENROSE STAIRWAY, TRIBAR

References

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Ernst, B. *Adventures with Impossible Figures.* Stradbroke, England: Tarquin, 1987.
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Fineman, M. *The Nature of Visual Illusion.* New York: Dover, pp. 119–122, 1996.
Jablan, S. "Impossible Figures." `http://members.tripod.com/~modularity/impos.htm` and "Are Impossible Figures Possible?" `http://members.tripod.com/~modularity/kulpa.htm`.
Kulpa, Z. "Are Impossible Figures Possible?" *Signal Processing* **5**, 201–220, 1983.
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Sugihara, K. "Classification of Impossible Objects." *Perception* **11**, 65–74, 1982.
Terouanne, E. "Impossible Figures and Interpretations of Polyhedral Figures." *J. Math. Psych.* **27**, 370–405, 1983.
Terouanne, E. "On a Class of 'Impossible' Figures: A New Language for a New Analysis." *J. Math. Psych.* **22**, 24–47, 1983.
Thro, E. B. "Distinguishing Two Classes of Impossible Objects." *Perception* **12**, 733–751, 1983.
Wilson, R. "Stamp Corner: Impossible Figures." *Math. Intell.* **13**, 80, 1991.

Impredicative

Definitions about a SET which depend on the entire SET.

Improper Integral

An INTEGRAL which has either or both limits INFINITE or which has an INTEGRAND which approaches INFINITY at one or more points in the range of integration.

see also DEFINITE INTEGRAL, INDEFINITE INTEGRAL, INTEGRAL, PROPER INTEGRAL

References
Press, W. H.; Flannery, B. P.; Teukolsky, S. A.; and Vetterling, W. T. "Improper Integrals." §4.4 in *Numerical Recipes in FORTRAN: The Art of Scientific Computing, 2nd ed.* Cambridge, England: Cambridge University Press, pp. 135–140, 1992.

Improper Node

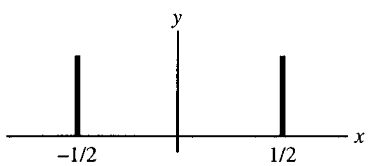
A FIXED POINT for which the STABILITY MATRIX has equal nonzero EIGENVECTORS.

see also STABLE IMPROPER NODE, UNSTABLE IMPROPER NODE

Improper Rotation

The SYMMETRY OPERATION corresponding to a a ROTATION followed by an INVERSION OPERATION, also called a ROTOINVERSION. This operation is denoted $\bar{n}$ for an improper rotation by $360°/n$, so the CRYSTALLOGRAPHY RESTRICTION gives only $\bar{1}$, $\bar{2}$, $\bar{3}$, $\bar{4}$, $\bar{6}$ for crystals. The MIRROR PLANE symmetry operation is $(x, y, z) \to (x, y, -z)$, etc., which is equivalent to $\bar{2}$.

Impulse Pair



The even impulse pair is the FOURIER TRANSFORM of $\cos(\pi k)$,

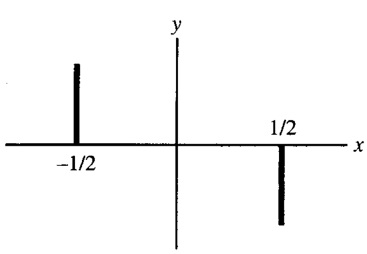
$$\text{II}(x) \equiv \tfrac{1}{2}\delta(x + \tfrac{1}{2}) + \tfrac{1}{2}\delta(x - \tfrac{1}{2}). \tag{1}$$

It satisfies

$$\text{II}(x) * f(x) = \tfrac{1}{2}f(x + \tfrac{1}{2}) + \tfrac{1}{2}f(x - \tfrac{1}{2}), \tag{2}$$

where $*$ denotes CONVOLUTION, and

$$\int_{-\infty}^{\infty} \text{II}(x)\, dx = 1. \tag{3}$$



The odd impulse pair is the FOURIER TRANSFORM of $i\sin(\pi s)$,

$$\text{I}_{\text{I}}(x) \equiv \tfrac{1}{2}\delta(x + \tfrac{1}{2}) - \tfrac{1}{2}\delta(x - \tfrac{1}{2}). \tag{4}$$

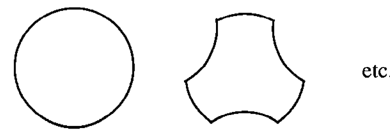
Impulse Symbol

Bracewell's term for the DELTA FUNCTION.

see also IMPULSE PAIR

References
Bracewell, R. *The Fourier Transform and Its Applications.* New York: McGraw-Hill, 1965.

In-and-Out Curve



A curve created by starting with a circle, dividing it into six arcs, and flipping three alternating arcs. The process is then repeated an infinite number of times.

Inaccessible Cardinal

An inaccessible cardinal is a CARDINAL NUMBER which cannot be expressed in terms of a smaller number of smaller cardinals.

Inaccessible Cardinals Axiom

see also LEBESGUE MEASURABILITY PROBLEM

Inadmissible

A word or string which is not ADMISSIBLE.

Incenter

The center I of a TRIANGLE'S INCIRCLE. It can be found as the intersection of ANGLE BISECTORS, and it is the interior point for which distances to the sidelines are equal. Its TRILINEAR COORDINATES are 1:1:1. The distance between the incenter and CIRCUMCENTER is $\sqrt{R(R-2r)}$.

The incenter lies on the EULER LINE only for an ISOSCELES TRIANGLE. It does, however, lie on the SODDY LINE. For an EQUILATERAL TRIANGLE, the CIRCUMCENTER O, CENTROID G, NINE-POINT CENTER F, ORTHOCENTER H, and DE LONGCHAMPS POINT Z all coincide with I.

The incenter and EXCENTERS of a TRIANGLE are an ORTHOCENTRIC SYSTEM. The POWER of the incenter with respect to the CIRCUMCIRCLE is

$$p = \frac{a_1 a_2 a_3}{a_1 + a_2 + a_3}$$

(Johnson 1929, p. 190). If the incenters of the TRIANGLES $\Delta A_1 H_2 H_3$, $\Delta A_2 H_3 A_1$, and $\Delta A_3 H_1 H_2$ are X_1, X_2, and X_3, then $X_2 X_3$ is equal and parallel to $I_2 I_3$, where H_i are the FEET of the ALTITUDES and I_i are the incenters of the TRIANGLES. Furthermore, X_1, X_2, X_3, are the reflections of I with respect to the sides of the TRIANGLE $\Delta I_1 I_2 I_3$ (Johnson 1929, p. 193).

If four points are on a CIRCLE (i.e., they are CONCYCLIC), the incenters of the four TRIANGLES form a RECTANGLE whose sides are parallel to the lines connecting the middle points of opposite arcs. Furthermore, the connectors pass through the center of the RECTANGLE (Fuhrmann 1890, p. 50; Johnson 1929, pp. 254–255). More generally, the 16 incenters and excenters of the TRIANGLES whose VERTICES are four points on a CIRCLE, are the intersections of two sets of four PARALLEL lines which are mutually PERPENDICULAR (Johnson 1929, p. 255).

see also CENTROID (ORTHOCENTRIC SYSTEM), CIRCUMCENTER, EXCENTER, GERGONNE POINT, INCIRCLE, INRADIUS, ORTHOCENTER

References

Carr, G. S. *Formulas and Theorems in Pure Mathematics, 2nd ed.* New York: Chelsea, p. 622, 1970.

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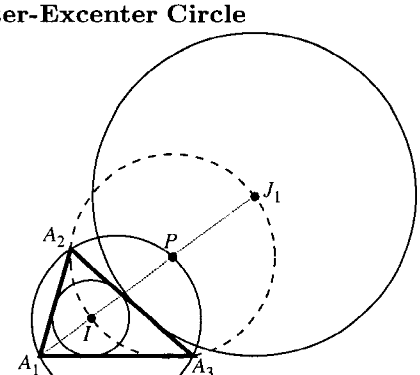
Fuhrmann, W. *Synthetische Beweise Planimetrischer Sätze.* Berlin, 1890.

Johnson, R. A. *Modern Geometry: An Elementary Treatise on the Geometry of the Triangle and the Circle.* Boston, MA: Houghton Mifflin, pp. 182–194, 1929.

Kimberling, C. "Central Points and Central Lines in the Plane of a Triangle." *Math. Mag.* **67**, 163–187, 1994.

Kimberling, C. "Incenter." `http://www.evansville.edu/~ck6/tcenters/class/incenter.html`.

Incenter-Excenter Circle



Given a triangle $\Delta A_1A_2A_3$, the points A_1, I, and J_1 lie on a line, where I is the INCENTER and J_1 is the EXCENTER corresponding to A_1. Furthermore, the CIRCLE with IJ_1 as the DIAMETER has P as its center, where P is the intersection of A_1J_1 with the CIRCUMCIRCLE of $\Delta A_1A_2A_3$, and passes through A_2 and A_3. This CIRCLE has RADIUS

$$r = \tfrac{1}{2}a_1 \sec(\tfrac{1}{2}\alpha_1) = 2R\sin(\tfrac{1}{2}\alpha_1).$$

It arises because $IJ_1J_2J_3$ forms an ORTHOCENTRIC SYSTEM.

see also CIRCUMCIRCLE, EXCENTER, EXCENTER-EXCENTER CIRCLE, INCENTER, ORTHOCENTRIC SYSTEM

References

Johnson, R. A. *Modern Geometry: An Elementary Treatise on the Geometry of the Triangle and the Circle.* Boston, MA: Houghton Mifflin, p. 185, 1929.

Incidence Axioms

The eight of HILBERT'S AXIOMS which concern collinearity and intersection; they include the first four of EUCLID'S POSTULATES.

see also ABSOLUTE GEOMETRY, CONGRUENCE AXIOMS, CONTINUITY AXIOMS, EUCLID'S POSTULATES, HILBERT'S AXIOMS, ORDERING AXIOMS, PARALLEL POSTULATE

References

Hilbert, D. *The Foundations of Geometry, 2nd ed.* Chicago, IL: Open Court, 1980.

Iyanaga, S. and Kawada, Y. (Eds.). "Hilbert's System of Axioms." §163B in *Encyclopedic Dictionary of Mathematics.* Cambridge, MA: MIT Press, pp. 544–545, 1980.

Incidence Matrix

For a k-D POLYTOPE Π_k, the incidence matrix is defined by

$$\eta_{ij}^k = \begin{cases} 1 & \text{if } \Pi_{k-1}^i \text{ belongs to } \Pi_k^j \\ 0 & \text{if } \Pi_{k-1}^i \text{ does not belong to } \Pi_k^j. \end{cases}$$

The ith row shows which Π_ks surround Π_{k-1}^i, and the jth column shows which Π_{k-1}s bound Π_k^j. Incidence matrices are also used to specify PROJECTIVE PLANES. The incidence matrices for a TETRAHEDRON $ABCD$ are

η^0	1	A	B	C
1	1	1	1	1

η^1	AD	BD	CD	BC	AC	AB
A	1	0	0	0	1	1
B	0	1	0	1	0	1
C	0	0	1	1	1	0
D	1	1	1	0	0	0

η^2	BCD	ACD	ABD	ABC
AD	0	1	1	0
BD	1	0	1	0
CD	1	1	0	0
BC	1	0	0	1
AC	0	1	0	1
AB	0	0	1	1

η^3	$ABCD$
BCD	1
ACD	1
ABD	1
ABC	1

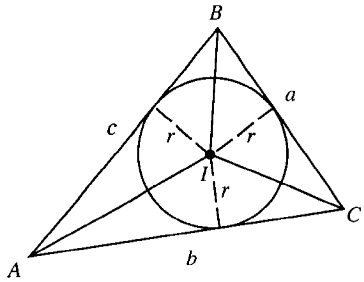
see also ADJACENCY MATRIX, k-CHAIN, k-CIRCUIT

Incident

Two objects which touch each other are said to be incident.

see also INCIDENCE MATRIX

Incircle



The INSCRIBED CIRCLE of a TRIANGLE ΔABC. The center I is called the INCENTER and the RADIUS r the INRADIUS. The points of intersection of the incircle with T are the VERTICES of the PEDAL TRIANGLE of T with the INCENTER as the PEDAL POINT (c.f. TANGENTIAL TRIANGLE). This TRIANGLE is called the CONTACT TRIANGLE.

The AREA K of the TRIANGLE ΔABC is given by

$$\begin{aligned} K &= \Delta AIC + \Delta CIB + \Delta AIB \\ &= \tfrac{1}{2}br + \tfrac{1}{2}ar + \tfrac{1}{2}cr = \tfrac{1}{2}(a+b+c)r = sr, \end{aligned}$$

where s is the SEMIPERIMETER.

Using the incircle of a TRIANGLE as the INVERSION CENTER, the sides of the TRIANGLE and its CIRCUMCIRCLE are carried into four equal CIRCLES (Honsberger 1976, p. 21). Pedoe (1995, p. xiv) gives a GEOMETRIC CONSTRUCTION for the incircle.

see also CIRCUMCIRCLE, CONGRUENT INCIRCLES POINT, CONTACT TRIANGLE, INRADIUS, TRIANGLE TRANSFORMATION PRINCIPLE

References
Coxeter, H. S. M. and Greitzer, S. L. *Geometry Revisited.* Washington, DC: Math. Assoc. Amer., pp. 11–13, 1967.
Honsberger, R. *Mathematical Gems II.* Washington, DC: Math. Assoc. Amer., 1976.
Johnson, R. A. *Modern Geometry: An Elementary Treatise on the Geometry of the Triangle and the Circle.* Boston, MA: Houghton Mifflin, pp. 182–194, 1929.
Pedoe, D. *Circles: A Mathematical View, rev. ed.* Washington, DC: Math. Assoc. Amer., 1995.

Inclusion-Exclusion Principle

If $A_1, \ldots, A_k$ are finite sets, then

$$\left|\bigcup_{i=1}^{k} A_i\right| = \sum_{i=1}^{k}(-1)^{i+1}\xi_i,$$

where ξ_i is the sum of the CARDINALITIES of the intersections of the sets taken i at a time.

Inclusion Map

Given a SUBSET B of a SET A, the INJECTION $f: B \to A$ defined by $f(b) = b$ for all $b \in B$ is called the inclusion map.

see also LONG EXACT SEQUENCE OF A PAIR AXIOM

Incommensurate

Two lengths are called incommensurate or incommensurable if their ratio cannot be expressed as a ratio of whole numbers. IRRATIONAL NUMBERS and TRANSCENDENTAL NUMBERS are incommensurate with the integers.

see also IRRATIONAL NUMBER, PYTHAGORAS'S CONSTANT, TRANSCENDENTAL NUMBER

Incomplete Gamma Function

see GAMMA FUNCTION

Incompleteness

A formal theory is said to be incomplete if it contains fewer theorems than would be possible while still retaining CONSISTENCY.

see also CONSISTENCY, GÖDEL'S INCOMPLETENESS THEOREM

References
Chaitin, G. J. "G. J. Chaitin's Home Page." http://www.cs.auckland.ac.nz/CDMTCS/chaitin.

Increasing Function

A function $f(x)$ increases on an INTERVAL I if $f(b) > f(a)$ for all $b > a$, where $a, b \in I$. Conversely, a function $f(x)$ decreases on an INTERVAL I if $f(b) < f(a)$ for all $b > a$ with $a, b \in I$.

If the DERIVATIVE $f'(x)$ of a CONTINUOUS FUNCTION $f(x)$ satisfies $f'(x) > 0$ on an OPEN INTERVAL (a, b), then $f(x)$ is increasing on (a, b). However, a function may increase on an interval without having a derivative defined at all points. For example, the function $x^{1/3}$ is increasing everywhere, including the origin $x = 0$, despite the fact that the DERIVATIVE is not defined at that point.

see also DECREASING FUNCTION, DERIVATIVE, NONDECREASING FUNCTION, NONINCREASING FUNCTION

Increasing Sequence

For a SEQUENCE $\{a_n\}$, if $a_{n+1} - a_n > 0$ for $n \geq x$, then a is increasing for $n \geq x$. Conversely, if $a_{n+1} - a_n < 0$ for $n \geq x$, then a is DECREASING for $n \geq x$. If $a_{n+1}/a_n > 1$ for all $n \geq x$, then a is increasing for $n \geq x$. Conversely, if $a_{n+1}/a_n < 1$ for all $n \geq x$, then a is decreasing for $n \geq x$.

Indefinite Integral

An INTEGRAL

$$\int f(x)\,dx$$

without upper and lower limits, also called an ANTIDERIVATIVE. The first FUNDAMENTAL THEOREM OF CALCULUS allows DEFINITE INTEGRALS to be computed in terms of indefinite integrals. If F is the indefinite integral for $f(x)$, then

$$\int_a^b f(x)\,dx = F(b) - F(a).$$

see also ANTIDERIVATIVE, CALCULUS, DEFINITE INTEGRAL, FUNDAMENTAL THEOREMS OF CALCULUS, INTEGRAL

Indefinite Quadratic Form

A QUADRATIC FORM $Q(\mathbf{x})$ is indefinite if it is less than 0 for some values and greater than 0 for others. The QUADRATIC FORM, written in the form $(\mathbf{x}, \mathsf{A}\mathbf{x})$, is indefinite if EIGENVALUES of the MATRIX A are of both signs.

see also POSITIVE DEFINITE QUADRATIC FORM, POSITIVE SEMIDEFINITE QUADRATIC FORM

References

Gradshteyn, I. S. and Ryzhik, I. M. *Tables of Integrals, Series, and Products, 5th ed.* San Diego, CA: Academic Press, p. 1106, 1979.

Indegree

The number of inward directed EDGES from a given VERTEX in a DIRECTED GRAPH.

see also LOCAL DEGREE, OUTDEGREE

Independence Axiom

A rational choice between two alternatives should depend only on how they differ.

Independence Complement Theorem

If sets E and F are INDEPENDENT, then so are E and F', where F' is the complement of F (i.e., the set of all possible outcomes not contained in F). Let $\cup$ denote "or" and $\cap$ denote "and." Then

$$\begin{aligned} P(E) &= P(EF \cup EF') &(1)\\ &= P(EF) + P(EF') - P(EF \cap EF'), &(2) \end{aligned}$$

where AB is an abbreviation for $A \cap B$. But E and F are independent, so

$$P(EF) = P(E)P(F). \tag{3}$$

Also, since F and F' are complements, they contain no common elements, which means that

$$P(EF \cap EF') = 0 \tag{4}$$

for any E. Plugging (4) and (3) into (2) then gives

$$P(E) = P(E)P(F) + P(EF'). \tag{5}$$

Rearranging,

$$P(EF') = P(E)[1 - P(F)] = P(E)P(F'), \tag{6}$$

Q.E.D.

see also INDEPENDENT STATISTICS

Independence Number

The number

$$\alpha(G) = \max(|U| : U \subset V \text{ independent})$$

for a GRAPH G. The independence number of the DE BRUIJN GRAPH of order n is given by 1, 2, 3, 7, 13, 28, ... (Sloane's A006946).

References

Sloane, N. J. A. Sequence A006946/M0834 in "An On-Line Version of the Encyclopedia of Integer Sequences."

Independent Equations

see LINEARLY INDEPENDENT

Independent Sequence

see STRONGLY INDEPENDENT, WEAKLY INDEPENDENT

Independent Statistics

Two variates A and B are statistically independent IFF the CONDITIONAL PROBABILITY $P(A|B)$ of A given B satisfies

$$P(A|B) = P(A), \tag{1}$$

in which case the probability of A and B is just

$$P(AB) = P(A \cap B) = P(A)P(B). \tag{2}$$

Similarly, n events $A_1, A_2, \ldots, A_n$ are independent IFF

$$P\left(\bigcap_{i=1}^n A_i\right) = \prod_{i=1}^n P(A_i). \tag{3}$$

Statistically independent variables are always UNCORRELATED, but the converse is not necessarily true.

see also BAYES' FORMULA, CONDITIONAL PROBABILITY, INDEPENDENCE COMPLEMENT THEOREM, UNCORRELATED

Independent Vertices

A set of VERTICES A of a GRAPH with EDGES V is independent if it contains no EDGES.

see also INDEPENDENCE NUMBER

Indeterminate Problems

see DIOPHANTINE EQUATION—LINEAR

Index

A statistic which assigns a single number to several individual statistics in order to quantify trends. The best-known index in the United States is the consumer price index, which gives a sort of "average" value for inflation based on the price changes for a group of selected products.

Let p_n be the price per unit in period n, q_n be the quantity produced in period n, and $v_n \equiv p_n q_n$ be the value of the n units. Let q_a be the estimated relative importance of a product. There are several types of indices defined, among them those listed in the following table.

Index	Abbr.	Formula
Bowley index	P_B	$\frac{1}{2}(P_L + P_P)$
Fisher index	P_F	$\sqrt{P_L P_P}$
Geometric mean index	P_G	$\left[\prod \left(\frac{p_n}{p_0}\right)^{v_0}\right]^{1/\sum v_0}$
Harmonic mean index	P_H	$\frac{\sum p_0 q_0}{\sum \frac{p_0{}^2 q_0}{p_n}}$
Laspeyres's index	P_L	$\frac{\sum p_n q_0}{\sum p_0 q_0}$
Marshall-Edgeworth index	P_{ME}	$\frac{\sum p_n (q_0 + q_n)}{\sum (v_0 + v_n)}$
Mitchell index	P_M	$\frac{\sum p_n q_a}{\sum p_0 q_a}$
Paasche's index	P_P	$\frac{\sum p_n q_n}{\sum p_0 q_n}$
Walsh index	P_W	$\frac{\sum \sqrt{q_0 q_n}\, p_n}{\sum \sqrt{q_0 q_n}\, p_0}$

see also Bowley Index, Fisher Index, Geometric Mean Index, Harmonic Mean Index, Laspeyres' Index, Marshall-Edgeworth Index, Mitchell Index, Paasche's Index, Residue Index, Walsh Index

References

Fisher, I. *The Making of Index Numbers: A Study of Their Varieties, Tests and Reliability, 3rd ed.* New York: Augustus M. Kelly, 1967.

Kenney, J. F. and Keeping, E. S. "Index Numbers." Ch. 5 in *Mathematics of Statistics, Pt. 1, 3rd ed.* Princeton, NJ: Van Nostrand, pp. 64–74, 1962.

Mudgett, B. D. *Index Numbers.* New York: Wiley, 1951.

Index Set

A Stochastic Process is a family of Random Variables $\{x(t, \bullet), t \in \mathcal{J}\}$ from some Probability Space $(S, \mathbb{S}, P)$ into a State Space $(S', \mathbb{S}')$, where $\mathcal{J}$ is the index set of the process.

References

Doob, J. L. "The Development of Rigor in Mathematical Probability (1900–1950)." *Amer. Math. Monthly* **103**, 586–595, 1996.

Index Theory

A branch of Topology dealing with topological invariants of Manifolds.

References

Roe, J. *Index Theory, Coarse Geometry, and Topology of Manifolds.* Providence, RI: Amer. Math. Soc., 1996.

Upmeier, H. *Toeplitz Operators and Index Theory in Several Complex Variables.* Boston, MA: Birkhäuser, 1996.

Indicatrix

A spherical image of a curve. The most common indicatrix is Dupin's Indicatrix.

see also Dupin's Indicatrix

Indicial Equation

The Recurrence Relation obtained during application of the Frobenius Method of solving a second-order ordinary differential equation. The indicial equation (also called the Characteristic Equation) is obtained by noting that, by definition, the lowest order term x^k (that corresponding to $n = 0$) must have a Coefficient of zero. For an example of the construction of an indicial equation, see Bessel Differential Equation.

1. If the two Roots are equal, only one solution can be obtained.
2. If the two Roots differ by a noninteger, two solutions can be obtained.
3. If the two Roots differ by an Integer, the larger will yield a solution. The smaller may or may not.

References

Morse, P. M. and Feshbach, H. *Methods of Theoretical Physics, Part I.* New York: McGraw-Hill, pp. 532–534, 1953.

Indifference Principle

see Insufficient Reason Principle

Induced Map

If $f : (X, A) \to (Y, B)$ is homotopic to $g : (X, A) \to (Y, B)$, then $f_* : H_n(X, A) \to H_n(Y, B)$ and $g_* : H_n(X, A) \to H_n(Y, B)$ are said to be the induced maps.

see also Eilenberg-Steenrod Axioms

Induced Norm

see Natural Norm

Induction

The use of the Induction Principle in a Proof. Induction used in mathematics is often called Mathematical Induction.

References

Buck, R. C. "Mathematical Induction and Recursive Definitions." *Amer. Math. Monthly* **70**, 128–135, 1963.

Induction Axiom

The fifth of Peano's Axioms, which states: If a Set S of numbers contains zero and also the successor of every number in S, then every number is in S.

see also Peano's Axioms

Induction Principle

The truth of an INFINITE sequence of propositions P_i for $i = 1, \ldots, \infty$ is established if (1) P_1 is true, and (2) P_k IMPLIES P_{k+1} for all k.

References

Courant, R. and Robbins, H. "The Principle of Mathematical Induction" and "Further Remarks on Mathematical Induction." §1.2.1 and 1.7 in *What is Mathematics?: An Elementary Approach to Ideas and Methods, 2nd ed.* Oxford, England: Oxford University Press, pp. 9–11 and 18–20, 1996.

Inequality

A mathematical statement that one quantity is greater than or less than another. "a is less than b" is denoted $a < b$, and "a is greater than b" is denoted $a > b$. "a is less than or equal to b" is denoted $a \leq b$, and "a is greater than or equal to b" is denoted $a \geq b$. The symbols $a \ll b$ and $a \gg b$ are used to denote "a is much less than b" and "a is much greater than b," respectively.

Solutions to the inequality $|x - a| < b$ consist of the set $\{x : -b < x - a < b\}$, or equivalently $\{x : a - b < x < a + b\}$. Solutions to the inequality $|x - a| > b$ consist of the set $\{x : x - a > b\} \cup \{x : x - a < -b\}$. If a and b are both POSITIVE or both NEGATIVE and $a < b$, then $1/a > 1/b$.

see also ABC CONJECTURE, ARITHMETIC-LOGARITHMIC-GEOMETRIC MEAN INEQUALITY, BERNOULLI INEQUALITY, BERNSTEIN'S INEQUALITY, BERRY-OSSEEN INEQUALITY, BIENAYMÉ-CHEBYSHEV INEQUALITY, BISHOP'S INEQUALITY, BOGOMOLOV-MIYAOKA-YAU INEQUALITY, BOMBIERI'S INEQUALITY, BONFERRONI'S INEQUALITY, BOOLE'S INEQUALITY, CARLEMAN'S INEQUALITY, CAUCHY INEQUALITY, CHEBYSHEV INEQUALITY, CHI INEQUALITY, COPSON'S INEQUALITY, ERDŐS-MORDELL THEOREM, EXPONENTIAL INEQUALITY, FISHER'S BLOCK DESIGN INEQUALITY, FISHER'S ESTIMATOR INEQUALITY, GÅRDING'S INEQUALITY, GAUSS'S INEQUALITY, GRAM'S INEQUALITY, HADAMARD'S INEQUALITY, HARDY'S INEQUALITY, HARNACK'S INEQUALITY, HÖLDER INTEGRAL INEQUALITY, HÖLDER'S SUM INEQUALITY, ISOPERIMETRIC INEQUALITY, JARNICK'S INEQUALITY, JENSEN'S INEQUALITY, JORDAN'S INEQUALITY, KANTROVICH INEQUALITY, MARKOV'S INEQUALITY, MINKOWSKI INTEGRAL INEQUALITY, MINKOWSKI SUM INEQUALITY, MORSE INEQUALITIES, NAPIER'S INEQUALITY, NOSARZEWSKA'S INEQUALITY, OSTROWSKI'S INEQUALITY, PTOLEMY INEQUALITY, ROBBIN'S INEQUALITY, SCHRÖDER-BERNSTEIN THEOREM, SCHUR'S INEQUALITIES, SCHWARZ'S INEQUALITY, SQUARE ROOT INEQUALITY, STEFFENSEN'S INEQUALITY, STOLARSKY'S INEQUALITY, STRONG SUBADDITIVITY INEQUALITY, TRIANGLE INEQUALITY, TURÁN'S INEQUALITIES, WEIERSTRAß PRODUCT INEQUALITY, WIRTINGER'S INEQUALITY, YOUNG INEQUALITY

References

Abramowitz, M. and Stegun, C. A. (Eds.). *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables, 9th printing.* New York: Dover, p. 16, 1972.

Beckenbach, E. F. and Bellman, Richard E. *An Introduction to Inequalities.* New York: Random House, 1961.

Beckenbach, E. F. and Bellman, Richard E. *Inequalities, 2nd rev. print.* Berlin: Springer-Verlag, 1965.

Hardy, G. H.; Littlewood, J. E.; and Pólya, G. *Inequalities, 2nd ed.* Cambridge, England: Cambridge University Press, 1952.

Kazarinoff, N. D. *Geometric Inequalities.* New York: Random House, 1961.

Mitrinovic, D. S. *Analytic Inequalities.* New York: Springer-Verlag, 1970.

Mitrinovic, D. S.; Pecaric, J. E.; and Fink, A. M. *Classical & New Inequalities in Analysis.* Dordrecht, Netherlands: Kluwer, 1993.

Mitrinovic, D. S.; Pecaric, J. E.; Fink, A. M. *Inequalities Involving Functions & Their Integrals & Derivatives.* Dordrecht, Netherlands: Kluwer, 1991.

Mitrinovic, D. S.; Pecaric, J. E.; and Volenec, V. *Recent Advances in Geometric Inequalities.* Dordrecht, Netherlands: Kluwer, 1989.

Inexact Differential

An infinitesimal which is not the differential of an actual function and which cannot be expressed as

$$dz = \left(\frac{\partial z}{\partial x}\right)_y dx + \left(\frac{\partial z}{\partial y}\right)_z dy,$$

the way an EXACT DIFFERENTIAL can. Inexact differentials are denoted with a bar through the d. The most common example of an inexact differential is the change in heat $đQ$ encountered in thermodynamics.

see also EXACT DIFFERENTIAL, PFAFFIAN FORM

References

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Inf

see INFIMUM, INFIMUM LIMIT

Infimum

The greatest lower bound of a set. It is denoted

$$\inf_S .$$

see also INFIMUM LIMIT, SUPREMUM

Infimum Limit

The limit infimum of a set is the greatest lower bound of the CLOSURE of a set. It is denoted

$$\lim_S \inf .$$

see also INFIMUM, SUPREMUM

Infinary Divisor

p^x is an infinary divisor of p^y (with $y > 0$) if $p^x|_{y-1}p^y$. This generalizes the concept of the k-ARY DIVISOR.

see also INFINARY PERFECT NUMBER, k-ARY DIVISOR

References

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Cohen, G. and Hagis, P. "Arithmetic Functions Associated with the Infinary Divisors of an Integer." *Internat. J. Math. Math. Sci.* **16**, 373–383, 1993.

Guy, R. K. *Unsolved Problems in Number Theory, 2nd ed.* New York: Springer-Verlag, p. 54, 1994.

Infinary Multiperfect Number

Let $\sigma_\infty(n)$ be the SUM of the INFINARY DIVISORS of a number n. An infinary k-multiperfect number is a number n such that $\sigma_\infty(n) = kn$. Cohen (1990) found 13 infinary 3-multiperfects, seven 4-multiperfects, and two 5-multiperfects.

see also INFINARY PERFECT NUMBER

References

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Guy, R. K. *Unsolved Problems in Number Theory, 2nd ed.* New York: Springer-Verlag, p. 54, 1994.

Infinary Perfect Number

Let $\sigma_\infty(n)$ be the SUM of the INFINARY DIVISORS of a number n. An infinary perfect number is a number n such that $\sigma_\infty(n) = 2n$. Cohen (1990) found 14 such numbers. The first few are 6, 60, 90, 36720, ... (Sloane's A007257).

see also INFINARY MULTIPERFECT NUMBER

References

Cohen, G. L. "On an Integer's Infinary Divisors." *Math. Comput.* **54**, 395–411, 1990.

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Infinite

Greater than any assignable quantity of the sort in question. In mathematics, the concept of the infinite is made more precise through the notion of an INFINITE SET.

see also COUNTABLE SET, COUNTABLY INFINITE SET, FINITE, INFINITE SET, INFINITESIMAL, INFINITY

Infinite Product

N.B. A detailed on-line essay by S. Finch was the starting point for this entry.

A PRODUCT involving an INFINITE number of terms. Such products can converge. In fact, for POSITIVE a_n, the PRODUCT $\prod_{n=1}^\infty a_n$ converges to a NONZERO number IFF $\sum_{n=1}^\infty \ln a_n$ converges.

Infinite products can be used to define the COSINE

$$\cos x = \prod_{n=1}^\infty \left[1 - \frac{4x^2}{\pi^2(2n-1)^2}\right], \tag{1}$$

GAMMA FUNCTION

$$\Gamma(z) = \left[ze^{\gamma z} \prod_{r=1}^\infty \left(1+\frac{z}{r}\right)e^{-z/r}\right]^{-1}, \tag{2}$$

SINE, and SINC FUNCTION. They also appear in the POLYGON CIRCUMSCRIBING CONSTANT

$$K = \prod_{n=3}^\infty \frac{1}{\cos\left(\frac{\pi}{n}\right)}. \tag{3}$$

An interesting infinite product formula due to Euler which relates π and the nth PRIME p_n is

$$\pi = \frac{2}{\prod_{i=n}^\infty \left[1 + \frac{\sin(\frac{1}{2}\pi p_n)}{p_n}\right]} \tag{4}$$

$$= \frac{2}{\prod_{i=n}^\infty \left[1 + \frac{(-1)^{(p_n-1)/2}}{p_n}\right]} \tag{5}$$

(Blatner 1997).

The product

$$\prod_{n=1}^\infty \left(1+\frac{1}{n^p}\right) \tag{6}$$

has closed form expressions for small POSITIVE integral $p \geq 2$,

$$\prod_{n=1}^\infty \left(1+\frac{1}{n^2}\right) = \frac{\sinh\pi}{\pi} \tag{7}$$

$$\prod_{n=1}^\infty \left(1+\frac{1}{n^3}\right) = \frac{1}{\pi}\cosh(\tfrac{1}{2}\pi\sqrt{3}\,) \tag{8}$$

$$\prod_{n=1}^\infty \left(1+\frac{1}{n^4}\right) = \frac{\cosh(\pi\sqrt{2}\,) - \cos(\pi\sqrt{2}\,)}{2\pi^2} \tag{9}$$

$$\prod_{n=1}^\infty \left(1+\frac{1}{n^5}\right) = \left|\Gamma[\exp(\tfrac{2}{5}\pi i)]\Gamma[\exp(\tfrac{6}{5}\pi i)]\right|^{-2}. \tag{10}$$

The d-ANALOG expression

$$[\infty!]_d = \prod_{n=3}^\infty \left(1 - \frac{2^d}{n^d}\right) \tag{11}$$

also has closed form expressions,

$$\prod_{n=3}^{\infty}\left(1-\frac{4}{n^2}\right)=\frac{1}{6} \tag{12}$$

$$\prod_{n=3}^{\infty}\left(1-\frac{8}{n^3}\right)=\frac{\sinh(\pi\sqrt{3})}{42\pi\sqrt{3}} \tag{13}$$

$$\prod_{n=3}^{\infty}\left(1-\frac{16}{n^4}\right)=\frac{\sinh(2\pi)}{120\pi} \tag{14}$$

$$\prod_{n=3}^{\infty}\left(1-\frac{32}{n^5}\right)=\left|\Gamma[\exp(\tfrac{1}{5}\pi i)]\Gamma[2\exp(\tfrac{7}{5}\pi i)]\right|^{-2}. \tag{15}$$

see also COSINE, DIRICHLET ETA FUNCTION, EULER IDENTITY, GAMMA FUNCTION, ITERATED EXPONENTIAL CONSTANTS, POLYGON CIRCUMSCRIBING CONSTANT, POLYGON INSCRIBING CONSTANT, Q-FUNCTION, SINE

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Infinite Series

A SERIES with an INFINITE number of terms.

see also SERIES

Infinite Set

A SET of S elements is said to be infinite if the elements of a PROPER SUBSET S' can be put into ONE-TO-ONE correspondence with the elements of S. An infinite set whose elements can be put into a ONE-TO-ONE correspondence with the set of INTEGERS is said to be COUNTABLY INFINITE; otherwise, it is called UNCOUNTABLY INFINITE.

see also ALEPH-0, ALEPH-1, CARDINAL NUMBER, COUNTABLY INFINITE SET, CONTINUUM, FINITE, INFINITE, INFINITY, ORDINAL NUMBER, TRANSFINITE NUMBER, UNCOUNTABLY INFINITE SET

References

Courant, R. and Robbins, H. *What is Mathematics?: An Elementary Approach to Ideas and Methods, 2nd ed.* Oxford, England: Oxford University Press, p. 77, 1996.

Infinitesimal

A quantity which yields 0 after the application of some LIMITING process. The understanding of infinitesimals was a major roadblock to the acceptance of CALCULUS and its placement on a firm mathematical foundation.

see also INFINITE, INFINITY, NONSTANDARD ANALYSIS

Infinitesimal Analysis

An archaic term for CALCULUS.

Infinitesimal Matrix Change

Let B, A, and e be square matrices with e small, and define

$$\mathsf{B}\equiv\mathsf{A}(\mathsf{I}+\mathsf{e}), \tag{1}$$

where I is the IDENTITY MATRIX. Then the inverse of B is approximately

$$\mathsf{B}^{-1}=(\mathsf{I}-\mathsf{e})\mathsf{A}^{-1}. \tag{2}$$

This can be seen by multiplying

$$\begin{aligned}\mathsf{BB}^{-1}&=(\mathsf{A}+\mathsf{Ae})(\mathsf{A}^{-1}-\mathsf{eA}^{-1})\\&=\mathsf{AA}^{-1}-\mathsf{AeA}^{-1}+\mathsf{AeA}^{-1}-\mathsf{Ae}^2\mathsf{A}^{-1}\\&=\mathsf{I}-\mathsf{Ae}^2\mathsf{A}^{-1}\approx 1.\end{aligned} \tag{3}$$

Note that if we instead let $\mathsf{B}'\equiv\mathsf{A}+\mathsf{e}$, and look for an inverse of the form $\mathsf{B}'^{-1}=\mathsf{A}^{-1}+\mathsf{C}$, we obtain

$$\begin{aligned}\mathsf{B}'\mathsf{B}'^{-1}&=(\mathsf{A}+\mathsf{e})(\mathsf{A}^{-1}+\mathsf{C})=\mathsf{AA}^{-1}+\mathsf{AC}+\mathsf{eA}^{-1}+\mathsf{eC}\\&=\mathsf{I}+\mathsf{AC}+\mathsf{e}(\mathsf{C}+\mathsf{A}^{-1})\equiv\mathsf{I}.\end{aligned} \tag{4}$$

In order to eliminate the e term, we require $\mathsf{C}=-\mathsf{A}^{-1}$. However, then $\mathsf{AC}=-\mathsf{I}$, so $\mathsf{BB}^{-1}=0$ so there can be no inverse of this form.

The exact inverse of B can be found as follows.

$$\mathsf{B}=\mathsf{A}(\mathsf{I}+\mathsf{e})=\mathsf{A}(\mathsf{I}+\mathsf{A}^{-1}\mathsf{e}), \tag{5}$$

so

$$\mathsf{B}^{-1}=[\mathsf{A}(\mathsf{I}+\mathsf{A}^{-1}\mathsf{e})]^{-1}. \tag{6}$$

Using a general MATRIX INVERSE identity then gives

$$\mathsf{B}^{-1}=(\mathsf{I}+\mathsf{A}^{-1}\mathsf{e})^{-1}\mathsf{A}^{-1}. \tag{7}$$

Infinitesimal Rotation

An infinitesimal transformation of a VECTOR $\mathbf{r}$ is given by

$$\mathbf{r}'=(\mathsf{I}+\mathsf{e})\mathbf{r}, \tag{1}$$

where the MATRIX e is infinitesimal and I is the IDENTITY MATRIX. (Note that the infinitesimal transformation may not correspond to an inversion, since inversion is a discontinuous process.) The COMMUTATIVITY of infinitesimal transformations e_1 and e_2 is established by the equivalence of

$$(\mathsf{I}+\mathsf{e}_1)(\mathsf{I}+\mathsf{e}_2) = \mathsf{I}^2+\mathsf{e}_1\mathsf{I}+\mathsf{I}\mathsf{e}_2+\mathsf{e}_1\mathsf{e}_2 \approx \mathsf{I}+\mathsf{e}_1+\mathsf{e}_2 \quad (2)$$

$$(\mathsf{I}+\mathsf{e}_2)(\mathsf{I}+\mathsf{e}_1) = \mathsf{I}^2+\mathsf{e}_2\mathsf{I}+\mathsf{I}\mathsf{e}_1+\mathsf{e}_2\mathsf{e}_1 \approx \mathsf{I}+\mathsf{e}_2+\mathsf{e}_1. \quad (3)$$

Now let

$$\mathsf{A} \equiv \mathsf{I}+\mathsf{e}. \quad (4)$$

The inverse A^{-1} is then $\mathsf{I}-\mathsf{e}$, since

$$\mathsf{A}\mathsf{A}^{-1} = (\mathsf{I}+\mathsf{e})(\mathsf{I}-\mathsf{e}) = \mathsf{I}^2-\mathsf{e}^2 \approx \mathsf{I}. \quad (5)$$

Since we are defining our infinitesimal transformation to be a rotation, ORTHOGONALITY of ROTATION MATRICES requires that

$$\mathsf{A}^{\mathrm{T}} = \mathsf{A}^{-1}, \quad (6)$$

but

$$\mathsf{A}^{-1} = \mathsf{I}-\mathsf{e} \quad (7)$$

$$(\mathsf{I}+\mathsf{e})^{\mathrm{T}} = \mathsf{I}^{\mathrm{T}}+\mathsf{e}^{\mathrm{T}} = \mathsf{I}+\mathsf{e}^{\mathrm{T}}, \quad (8)$$

so $\mathsf{e} = -\mathsf{e}^{\mathrm{T}}$ and the infinitesimal rotation is ANTISYMMETRIC. It must therefore have a MATRIX of the form

$$\mathsf{e} = \begin{bmatrix} 0 & d\Omega_3 & -d\Omega_2 \\ -d\Omega_3 & 0 & d\Omega_1 \\ d\Omega_2 & -d\Omega_1 & 0 \end{bmatrix}. \quad (9)$$

The differential change in a vector $\mathbf{r}$ upon application of the ROTATION MATRIX is then

$$d\mathbf{r} \equiv \mathbf{r}' - \mathbf{r} = (\mathsf{I}+\mathsf{e})\mathbf{r} - \mathbf{r} = \mathsf{e}\mathbf{r}. \quad (10)$$

Writing in MATRIX form,

$$\begin{aligned} d\mathbf{r} &= \begin{bmatrix} x \\ y \\ z \end{bmatrix} \begin{bmatrix} 0 & d\Omega_3 & -d\Omega_2 \\ -d\Omega_3 & 0 & d\Omega_1 \\ d\Omega_2 & -d\Omega_1 & 0 \end{bmatrix} \\ &= \begin{bmatrix} y\,d\Omega_3 - z\,d\Omega_2 \\ z\,d\Omega_1 - x\,d\Omega_3 \\ x\,d\Omega_2 - y\,d\Omega_1 \end{bmatrix} \quad (11) \\ &= (y\,d\Omega_3 - z\,d\Omega_2)\hat{\mathbf{x}} + (z\,d\Omega_1 - x\,d\Omega_3)\hat{\mathbf{y}} \\ &\quad + (x\,d\Omega_2 - y\,d\Omega_1)\hat{\mathbf{z}} \\ &= \mathbf{r} \times d\mathbf{\Omega}. \quad (12) \end{aligned}$$

Therefore,

$$\left(\frac{d\mathbf{r}}{dt}\right)_{\text{rotation, body}} = \mathbf{r} \times \frac{d\mathbf{\Omega}}{dt} = \mathbf{r} \times \boldsymbol{\omega}, \quad (13)$$

where

$$\boldsymbol{\omega} \equiv \frac{d\mathbf{\Omega}}{dt} = \hat{\mathbf{n}}\frac{d\phi}{dt}. \quad (14)$$

The total rotation observed in the stationary frame will be a sum of the rotational velocity and the velocity in the rotating frame. However, note that an observer in the stationary frame will see a velocity opposite in direction to that of the observer in the frame of the rotating body, so

$$\left(\frac{d\mathbf{r}}{dt}\right)_{\text{space}} = \left(\frac{d\mathbf{r}}{dt}\right)_{\text{body}} + \boldsymbol{\omega} \times \mathbf{r}. \quad (15)$$

This can be written as an operator equation, known as the ROTATION OPERATOR, defined as

$$\left(\frac{d}{dt}\right)_{\text{space}} = \left(\frac{d}{dt}\right)_{\text{body}} + \boldsymbol{\omega} \times . \quad (16)$$

see also ACCELERATION, EULER ANGLES, ROTATION, ROTATION MATRIX, ROTATION OPERATOR

Infinitive Sequence

A sequence $\{x_n\}$ is called an infinitive sequence if, for every i, $x_n = i$ for infinitely many n. Write $a(i,j)$ for the jth index n for which $x_n = i$. Then as i and j range through N, the array $A = a(i,j)$, called the associative array of x, ranges through all of N.

see also FRACTAL SEQUENCE

References

Kimberling, C. "Fractal Sequences and Interspersions." *Ars Combin.* **45**, 157–168, 1997.

Infinitude of Primes

see EUCLID'S THEOREMS

Infinity

An unbounded number greater than every REAL NUMBER, most often denoted as ∞. The symbol ∞ had been used as an alternative to M (1,000) in ROMAN NUMERALS until 1655, when John Wallis suggested it be used instead for infinity.

Infinity is a very tricky concept to work with, as evidenced by some of the counterintuitive results which follow from Georg Cantor's treatment of INFINITE SETS. Informally, $1/\infty = 0$, a statement which can be made rigorous using the LIMIT concept,

$$\lim_{x\to\infty} \frac{1}{x} = 0.$$

Similarly,

$$\lim_{x\to 0^+} \frac{1}{x} = \infty,$$

where the notation 0^+ indicates that the LIMIT is taken from the POSITIVE side of the REAL LINE.

see also ALEPH, ALEPH-0, ALEPH-1, CARDINAL NUMBER, CONTINUUM, CONTINUUM HYPOTHESIS, HILBERT HOTEL, INFINITE, INFINITE SET, INFINITESIMAL, LINE AT INFINITY, L'HOSPITAL'S RULE, POINT AT INFINITY, TRANSFINITE NUMBER, UNCOUNTABLY INFINITE SET, ZERO

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Inflection Point

A point on a curve at which the SIGN of the CURVATURE (i.e., the concavity) changes. The FIRST DERIVATIVE TEST can sometimes distinguish inflection points from EXTREMA for DIFFERENTIABLE functions $f(x)$.

see also CURVATURE, DIFFERENTIABLE, EXTREMUM, FIRST DERIVATIVE TEST, STATIONARY POINT

Information Dimension

Define the "information function" to be

$$I \equiv -\sum_{i=1}^{N} P_i(\epsilon)\ln[P_i(\epsilon)], \tag{1}$$

where $P_i(\epsilon)$ is the NATURAL MEASURE, or probability that element i is populated, normalized such that

$$\sum_{i=1}^{N} P_i(\epsilon) = 1. \tag{2}$$

The information dimension is then defined by

$$\begin{aligned} d_{\text{inf}} &\equiv -\lim_{\epsilon\to 0^+}\frac{I}{\ln(\epsilon)} \\ &= \lim_{\epsilon\to 0^+}\sum_{i=1}^{N}\frac{P_i(\epsilon)\ln[P_i(\epsilon)]}{\ln(\epsilon)}. \end{aligned} \tag{3}$$

If every element is equally likely to be visited, then $P_i(\epsilon)$ is independent of i, and

$$\sum_{i=1}^{N} P_i(\epsilon) = NP_i(\epsilon) = 1, \tag{4}$$

so

$$P_i(\epsilon) = \frac{1}{N}, \tag{5}$$

and

$$\begin{aligned} d_{\text{inf}} &= \lim_{\epsilon\to 0^+}\frac{\sum_{i=1}^{N}\frac{1}{N}\ln\left(\frac{1}{N}\right)}{\ln\epsilon} \\ &= \lim_{\epsilon\to 0^+}\frac{\ln(N^{-1})}{\ln\epsilon} = -\lim_{\epsilon\to 0^+}\frac{\ln N}{\ln(\epsilon)} = d_{\text{cap}}, \end{aligned} \tag{6}$$

where d_{cap} is the CAPACITY DIMENSION.

see also CORRELATION EXPONENT

References
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Nayfeh, A. H. and Balachandran, B. *Applied Nonlinear Dynamics: Analytical, Computational, and Experimental Methods.* New York: Wiley, pp. 545–547, 1995.

Information Entropy

see ENTROPY

Information Theory

The branch of mathematics dealing with the efficient and accurate storage, transmission, and representation of information.

see also CODING THEORY, ENTROPY

References
Goldman, S. *Information Theory.* New York: Dover, 1953.
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Initial Value Problem

An initial value problem is a problem that has its conditions specified at some time $t = t_0$. Usually, the problem is an ORDINARY DIFFERENTIAL EQUATION or a PARTIAL DIFFERENTIAL EQUATION. For example,

$$\begin{cases} \frac{\partial^2 u}{\partial t^2} - \nabla^2 u = f & \text{in } \Omega \\ u = u_0 & t = t_0 \\ u = u_1 & \text{on } \partial\Omega, \end{cases}$$

where $\partial\Omega$ denotes the boundary of Ω, is an initial value problem.

see also BOUNDARY CONDITIONS, BOUNDARY VALUE PROBLEM, PARTIAL DIFFERENTIAL EQUATION

References
Eriksson, K.; Estep, D.; Hansbo, P.; and Johnson, C. *Computational Differential Equations.* Lund, Sweden: Studentlitteratur, 1996.

Injection

see ONE-TO-ONE

Injective

A MAP is injective when it is ONE-TO-ONE, i.e., f is injective when $x \neq y$ IMPLIES $f(x) \neq f(y)$.

see also ONE-TO-ONE, SURJECTIVE

Injective Patch

An injective patch is a PATCH such that $\mathbf{x}(u_1, v_1) = \mathbf{x}(u_2, v_2)$ implies that $u_1 = u_2$ and $v_1 = v_2$. An example of a PATCH which is injective but not REGULAR is the function defined by (u^3, v^3, uv) for $u, v \in (-1, 1)$. However, if $\mathbf{x} : U \to \mathbb{R}^n$ is an injective regular patch, then $\mathbf{x}$ maps U diffeomorphically onto $\mathbf{x}(U)$.

see also PATCH, REGULAR PATCH

References

Gray, A. *Modern Differential Geometry of Curves and Surfaces.* Boca Raton, FL: CRC Press, p. 187, 1993.

Inner Automorphism Group

A particular type of AUTOMORPHISM GROUP which exists only for GROUPS. For a GROUP G, the inner automorphism group is defined by

$$\mathrm{Inn}(G) = \{\sigma_a : a \in G\} \subset \mathrm{Aut}(G)$$

where σ_a is an AUTOMORPHISM of G defined by

$$\sigma_a(x) = axa^{-1}.$$

see also AUTOMORPHISM, AUTOMORPHISM GROUP

Inner Product

see DOT PRODUCT

Inner Product Space

An inner product space is a VECTOR SPACE which has an INNER PRODUCT. If the INNER PRODUCT defines a NORM, then the inner product space is called a HILBERT SPACE.

see also HILBERT SPACE, INNER PRODUCT, NORM

Inradius

The radius of a TRIANGLE'S INCIRCLE or of a POLYHEDRON'S INSPHERE, denoted r. For a TRIANGLE,

$$r = \frac{1}{2}\sqrt{\frac{(b+c-a)(c+a-b)(a+b-c)}{a+b+c}} = \frac{\Delta}{s} \tag{1}$$

$$= 4R\sin(\tfrac{1}{2}\alpha_1)\sin(\tfrac{1}{2}\alpha_2)\sin(\tfrac{1}{2}\alpha_3), \tag{2}$$

where Δ is the AREA of the TRIANGLE, a, b, and c are the side lengths, s is the SEMIPERIMETER, and R is the CIRCUMRADIUS (Johnson 1929, p. 189).

Equation (1) can be derived easily using TRILINEAR COORDINATES. Since the INCENTER is equally spaced from all three sides, its trilinear coordinates are 1:1:1, and its exact trilinear coordinates are $r : r : r$. The ratio k of the exact trilinears to the homogeneous coordinates is given by

$$k = \frac{2\Delta}{a+b+c} = \frac{\Delta}{s}. \tag{3}$$

But since $k = r$ in this case,

$$r = k = \frac{\Delta}{s}, \tag{4}$$

Q. E. D.

Other equations involving the inradius include

$$Rr = \frac{abc}{4s} \tag{5}$$

$$\Delta^2 = rr_1r_2r_3 \tag{6}$$

$$\cos A + \cos B + \cos C = 1 + \frac{r}{R} \tag{7}$$

$$r = 2R\cos A\cos B\cos C \tag{8}$$

$$a^2 + b^2 + c^2 = 4rR + 8R^2, \tag{9}$$

where r_i are the EXRADII (Johnson 1929, pp. 189–191).

As shown in RIGHT TRIANGLE, the inradius of a RIGHT TRIANGLE of integral side lengths x, y, and z is also integral, and is given by

$$r = \frac{xy}{x+y+z}, \tag{10}$$

where z is the HYPOTENUSE. Let d be the distance between inradius r and CIRCUMRADIUS R, $d = \overline{rR}$. Then

$$R^2 - d^2 = 2Rr \tag{11}$$

$$\frac{1}{R-d} + \frac{1}{R+d} = \frac{1}{r} \tag{12}$$

(Mackay 1886–87). These and many other identities are given in Johnson (1929, pp. 186–190).

Expressing the MIDRADIUS ρ and CIRCUMRADIUS R in terms of the midradius gives

$$r = \frac{\rho^2}{\sqrt{\rho^2 + \frac{1}{4}a^2}} \tag{13}$$

$$r = \frac{R^2 - \frac{1}{4}a^2}{R} \tag{14}$$

for an ARCHIMEDEAN SOLID.

see also CARNOT'S THEOREM, CIRCUMRADIUS, MIDRADIUS

References

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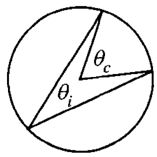
Mackay, J. S. "Formulas Connected with the Radii of the Incircle and Excircles of a Triangle." *Proc. Edinburgh Math. Soc.* **13**, 103–104.

Inscribed

A geometric figure which touches only the sides (or interior) of another figure.

see also CIRCUMSCRIBED, INCENTER, INCIRCLE, INRADIUS

Inscribed Angle



The ANGLE with VERTEX on a CIRCLE'S CIRCUMFERENCE formed by two points on a CIRCLE'S CIRCUMFERENCE. For ANGLES with the same endpoints,

$$\theta_c = 2\theta_i,$$

where θ_c is the CENTRAL ANGLE.

see also CENTRAL ANGLE

References

Pedoe, D. *Circles: A Mathematical View, rev. ed.* Washington, DC: Math. Assoc. Amer., pp. xxi–xxii, 1995.

Inside-Outside Theorem

Let $P(z)$ and $Q(z)$ be POLYNOMIALS with COMPLEX arguments and $\deg(Q) \geq \deg(P+2)$. Then

$$\int_\gamma \frac{P(z)}{Q(z)}\,dz = \begin{cases} 2\pi i \sum_{\text{inside }\gamma} \operatorname{Res} f(z) & \text{inside } \gamma \\ -2\pi i \sum_{\text{outside }\gamma} \operatorname{Res} f(z) & \text{outside } \gamma, \end{cases}$$

where Res are the RESIDUES.

Insphere

A SPHERE INSCRIBED in a given solid.

see also CIRCUMSPHERE, MIDSPHERE

Instrument Function

The finite FOURIER COSINE TRANSFORM of an APODIZATION FUNCTION, also known as an APPARATUS FUNCTION. The instrument function $I(k)$ corresponding to a given APODIZATION FUNCTION $A(x)$ is then given by

$$I(k) = \int_{-a}^{a} \cos(2\pi kx) A(x)\,dx.$$

see also APODIZATION FUNCTION, FOURIER COSINE TRANSFORM

Insufficient Reason Principle

A principle also called the INDIFFERENCE PRINCIPLE which was first enunciated by Johann Bernoulli. The insufficient reason principle states that, if we are ignorant of the ways an event can occur and therefore have no reason to believe that one way will occur preferentially to another, it will occur equally likely in any way.

Int

see INTEGER PART

Integer

One of the numbers $\ldots, -2, -1, 0, 1, 2, \ldots$. The SET of INTEGERS forms a RING which is denoted $\mathbb{Z}$. A given INTEGER n may be NEGATIVE ($n \in \mathbb{Z}^-$), NONNEGATIVE ($n \in \mathbb{Z}^*$), ZERO ($n = 0$), or POSITIVE ($n \in \mathbb{Z}^+ = \mathbb{N}$). The RING $\mathbb{Z}$ has CARDINALITY of $\aleph_0$. The GENERATING FUNCTION for the POSITIVE INTEGERS is

$$f(x) = \frac{1}{(1-x)^2} = x + 2x^2 + 3x^3 + 4x^4 + \ldots.$$

There are several symbols used to perform operations having to do with conversion between REAL NUMBERS and integers. The symbol $\lfloor x \rfloor$ ("FLOOR x") means "the largest integer not greater than x," i.e., `int(x)` in computer parlance. The symbol $[x]$ means "the nearest integer to x" (NINT), i.e., `nint(x)` in computer parlance. The symbol $\lceil x \rceil$ ("CEILING x") means "the smallest integer not smaller x," or `-int(-x)`, where `int(x)` is the INTEGER PART of x.

see also ALGEBRAIC INTEGER, ALMOST INTEGER, COMPLEX NUMBER, COUNTING NUMBER, CYCLOTOMIC INTEGER, EISENSTEIN INTEGER, GAUSSIAN INTEGER, $\mathbb{N}$, NATURAL NUMBER, NEGATIVE, POSITIVE, RADICAL INTEGER, REAL NUMBER, WHOLE NUMBER, $\mathbb{Z}$, $\mathbb{Z}^-$, $\mathbb{Z}^+$, $\mathbb{Z}^*$, ZERO

Integer Division

DIVISION in which the fractional part (remainder) is discarded is called integer division and is sometimes denoted $\backslash$. Integer division can be defined as $a\backslash b \equiv \lfloor a/b \rfloor$, where "/" denotes normal division and $\lfloor x \rfloor$ is the FLOOR FUNCTION. For example,

$$\begin{aligned} 10/3 &= 3 + 1/3 \\ 10\backslash 3 &= 3. \end{aligned}$$

Integer Factorization

see PRIME FACTORIZATION

Integer-Matrix Form

Let $Q(x) \equiv Q(\mathbf{x}) = Q(x_1, x_2, \ldots, x_n)$ be an integer-valued n-ary QUADRATIC FORM, i.e., a POLYNOMIAL with integer COEFFICIENTS which satisfies $Q(x) > 0$ for REAL $x \neq 0$. Then $Q(x)$ can be represented by

$$Q(x) = \mathbf{x}^{\mathrm{T}} \mathsf{A} \mathbf{x},$$

where

$$\mathsf{A} = \frac{1}{2} \frac{\partial^2 Q(x)}{\partial x_i \partial x_j}$$

is a POSITIVE symmetric matrix (Duke 1997). If A has POSITIVE entries, then $Q(x)$ is called an integer matrix form. Conway *et al.* (1997) have proven that, if a POSITIVE integer matrix quadratic form represents each of 1, 2, 3, 5, 6, 7, 10, 14, and 15, then it represents all POSITIVE INTEGERS.

see also FIFTEEN THEOREM

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Integer Module

see ABELIAN GROUP

Integer Part

The function $\text{int}(x)$ gives the INTEGER PART of x. In many computer languages, the function is denoted `int(x)`, but in mathematics, it is usually called the FLOOR FUNCTION and denoted $\lfloor x \rfloor$.

see also CEILING FUNCTION, FLOOR FUNCTION, NINT

Integer Relation

A set of REAL NUMBERS $x_1, \ldots, x_n$ is said to possess an integer relation if there exist integers a_i such that

$$a_1x_1 + a_2x_2 + \ldots + a_nx_n = 0,$$

with not all $a_i = 0$. An interesting example of such a relation is the 17-VECTOR $(1, x, x^2, \ldots, x^{16})$ with $x = 3^{1/4} - 2^{1/4}$, which has an integer relation (1, 0, 0, 0, −3860, 0, 0, 0, −666, 0, 0, 0, −20, 0, 0, 0, 1), i.e.,

$$1 - 3860x^4 - 666x^8 - 20x^{12} + x^{16} = 0.$$

This is a special case of finding the polynomial of degree $n = rs$ satisfied by $x = 3^{1/r} - 2^{1/s}$.

Algorithms for finding integer relations include the FERGUSON-FORCADE ALGORITHM, HJLS ALGORITHM, LLL ALGORITHM, PSLQ ALGORITHM, PSOS ALGORITHM, and the algorithm of Lagarias and Odlyzko (1985). Perhaps the simplest (and unfortunately most inefficient) such algorithm is the GREEDY ALGORITHM. Plouffe's "Inverse Symbolic Calculator" site includes a huge 54 million database of REAL NUMBERS which are algebraically related to fundamental mathematical constants.

see also CONSTANT PROBLEM, FERGUSON-FORCADE ALGORITHM, GREEDY ALGORITHM, HERMITE-LINDEMANN THEOREM, HJLS ALGORITHM, LATTICE REDUCTION, LLL ALGORITHM, PSLQ ALGORITHM, PSOS ALGORITHM, REAL NUMBER, LINDEMANN-WEIERSTRAß THEOREM

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Integer Sequence

A SEQUENCE whose terms are INTEGERS. The most complete printed references for such sequences are Sloane (1973) and its update, Sloane and Plouffe (1995). Sloane also maintains the sequences from both works together with many additional sequences in an on-line listing. In this listing, sequences are identified by a unique 6-DIGIT A-number. Sequences appearing in Sloane and Plouffe (1995) are ordered lexicographically and identified with a 4-DIGIT M-number, and those appearing in Sloane (1973) are identified with a 4-DIGIT N-number.

Sloane's huge (and enjoyable) database is accessible by either e-mail or web browser. To look up sequences by e-mail, send a message to either `sequences@research.att.com` or `superseeker@research.att.com` containing lines of the form `lookup 5 14 42 132 ...`. To use the browser version, point to `http://www.research.att.com/~njas/sequences/eisonline.html`.

see also ARONSON'S SEQUENCE, COMBINATORICS, CONSECUTIVE NUMBER SEQUENCES, CONWAY SEQUENCE, EBAN NUMBER, HOFSTADTER-CONWAY \$10,000 SEQUENCE, HOFSTADTER'S Q-SEQUENCE, LEVINE-O'SULLIVAN SEQUENCE, LOOK AND SAY SEQUENCE, MALLOW'S SEQUENCE, MIAN-CHOWLA SEQUENCE, MORSE-THUE SEQUENCE, NEWMAN-CONWAY SEQUENCE, NUMBER, PADOVAN SEQUENCE, PERRIN SEQUENCE, RATS SEQUENCE, SEQUENCE, SMARANDACHE SEQUENCES

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Integrable

A function for which the INTEGRAL can be computed is said to be integrable.

see also DIFFERENTIABLE, INTEGRAL, INTEGRATION

Integral

An integral is a mathematical object which can be interpreted as an AREA or a generalization of AREA. Integrals, together with DERIVATIVES, are the fundamental objects of CALCULUS. Other words for integral include ANTIDERIVATIVE and PRIMITIVE. The RIEMANN INTEGRAL is the simplest integral definition and the only one usually encountered in elementary CALCULUS. The RIEMANN INTEGRAL of the function $f(x)$ over x from a to b is written

$$\int_a^b f(x)\,dx. \tag{1}$$

Every definition of an integral is based on a particular MEASURE. For instance, the RIEMANN INTEGRAL is based on JORDAN MEASURE, and the LEBESGUE INTEGRAL is based on LEBESGUE MEASURE. The process of computing an integral is called INTEGRATION (a more archaic term for INTEGRATION is QUADRATURE), and the approximate computation of an integral is termed NUMERICAL INTEGRATION.

There are two classes of (Riemann) integrals: DEFINITE INTEGRALS

$$\int_a^b f(x)\,dx, \tag{2}$$

which have upper and lower limits, and INDEFINITE INTEGRALS, which are written without limits. The first FUNDAMENTAL THEOREM OF CALCULUS allows DEFINITE INTEGRALS to be computed in terms of INDEFINITE INTEGRALS, since if F is the INDEFINITE INTEGRAL for $f(x)$, then

$$\int_a^b f(x)\,dx = F(b) - F(a). \tag{3}$$

Wolfram Research (`http://www.integrals.com`) maintains a web site which will integrate many common (and not so common) functions. However, it cannot solve some simple integrals such as

$$\int \left[\frac{d}{dx}(x\sqrt{\sin x})\right] dx = \int \left(\frac{x\cos x}{2\sqrt{\sin x}} + \sqrt{\sin x}\right) dx \tag{4}$$

$$\int \left[\frac{d}{dx}L_2(x\ln x)\right] dx = -\int \left[\frac{(\ln x+1)\ln(1-x\ln x)}{x\ln x}\right] dx, \tag{5}$$

where L_2 is the DILOGARITHM. Furthermore, it gives an incorrect answer of $\pi^{1-2\sqrt{3}}/(\sqrt{3}\cdot 4^{\sqrt{3}})$ to

$$I(\sqrt{3}\,) = \int_0^{\pi/2} \frac{dx}{1+(\tan x)^{\sqrt{3}}} = \tfrac{1}{2}\pi. \tag{6}$$

This integral and, in fact, the generalized integral for arbitrary a

$$I(a) = \int_0^{\pi/2} \frac{dx}{1+(\tan x)^a}, \tag{7}$$

have a "trick" solution which takes advantage of the trigonometric identity

$$\tan(\tfrac{1}{2}\pi - x) = \cot x. \tag{8}$$

Letting $z \equiv (\tan x)^a$,

$$\begin{aligned} I(a) &= \int_0^{\pi/4} \frac{dx}{1+z} + \int_{\pi/4}^{\pi/2} \frac{dx}{1+1} \\ &= \int_0^{\pi/4} \frac{dx}{1+z} + \int_0^{\pi/4} \frac{dx}{1+\frac{1}{z}} \\ &= \int_0^{\pi/4} \left(\frac{1}{1+z} + \frac{1}{1+\frac{1}{z}}\right) = \int_0^{\pi/4} dx \\ &= \tfrac{1}{4}\pi. \end{aligned} \tag{9}$$

Here is a list of common INDEFINITE INTEGRALS:

$$\int x^r\,dx = \frac{x^{r+1}}{r+1} + C \tag{10}$$

$$\int \frac{dx}{x} = \ln|x| + C \tag{11}$$

$$\int a^x\,dx = \frac{a^x}{\ln a} + C \tag{12}$$

$$\int \sin x\,dx = -\cos x + C \tag{13}$$

$$\int \cos x\,dx = \sin x + C \tag{14}$$

$$\int \tan x\,dx = \ln|\sec x| + C \tag{15}$$

$$\int \csc x\,dx = \ln|\csc x - \cot x| + C \tag{16}$$

$$= \ln\left[\tan(\tfrac{1}{2}x)\right] + C \tag{17}$$

$$= \frac{1}{2}\ln\left(\frac{1-\cos x}{1+\cos x}\right) + C \tag{18}$$

$$\int \sec x\,dx = \ln|\sec x + \tan x| + C$$

$$= \mathrm{gd}^{-1}(x) + C \tag{19}$$

$$\int \cot x\,dx = \ln|\sin x| + C \tag{20}$$

$$\int \sec^2 x\,dx = \tan x + C \tag{21}$$

$$\int \csc^2 x\,dx = -\cot x + C \tag{22}$$

$$\int \sec x\,\tan x\,dx = \sec x + C \tag{23}$$

$$\int \cos^{-1} x\,dx = x\cos^{-1} x - \sqrt{1-x^2} + C \tag{24}$$

$$\int \sin^{-1} x\,dx = x\sin^{-1} x + \sqrt{1-x^2} + C \tag{25}$$

$$\int \tan^{-1} x\,dx = x\tan^{-1} x - \tfrac{1}{2}\ln(1+x^2) + C \tag{26}$$

$$\int \frac{dx}{\sqrt{a^2-x^2}} = \sin^{-1}\left(\frac{x}{a}\right) + C \tag{27}$$

$$\int \frac{dx}{\sqrt{a^2-x^2}} = \cos^{-1}\left(\frac{x}{a}\right) + C \tag{28}$$

$$\int \frac{dx}{a^2+x^2} = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + C \tag{29}$$

$$\int \frac{dx}{a^2+x^2} = -\frac{1}{a}\cot^{-1}\left(\frac{x}{a}\right) + C \tag{30}$$

$$\int \frac{dx}{x\sqrt{x^2-a^2}} = \frac{1}{a}\sec^{-1}\left(\frac{x}{a}\right) + C \tag{31}$$

$$\int \frac{dx}{x\sqrt{x^2-a^2}} = -\frac{1}{a}\csc^{-1}\left(\frac{x}{a}\right) + C \tag{32}$$

$$\int \sin^2(ax)\,dx = \frac{x}{2} - \frac{1}{4a}\sin(2ax) + C \tag{33}$$

$$\int \mathrm{sn}\,u\,du = k^{-1}\ln(\mathrm{dn}\,u - k\,\mathrm{cn}\,u) + C \tag{34}$$

$$\int \mathrm{sn}^2\,u\,du = \frac{u - E(u)}{k^2} + C \tag{35}$$

$$\int \mathrm{cn}\,u\,du = k^{-1}\sin^{-1}(k\,\mathrm{sn}\,u) + C \tag{36}$$

$$\int \mathrm{dn}\,u\,du = \sin^{-1}(\mathrm{sn}\,u) + C, \tag{37}$$

where $\sin x$ is the SINE; $\cos x$ is the COSINE; $\tan x$ is the TANGENT; $\csc x$ is the COSECANT; $\sec x$ is the SECANT; $\cot x$ is the COTANGENT; $\cos^{-1} x$ is the INVERSE COSINE; $\sin^{-1} x$ is the INVERSE SINE; $\tan^{-1}$ is the INVERSE TANGENT; $\mathrm{sn}\,u$, $\mathrm{cn}\,u$, and $\mathrm{dn}\,u$ are JACOBI ELLIPTIC FUNCTIONS; $E(u)$ is a complete ELLIPTIC INTEGRAL OF THE SECOND KIND; and $\mathrm{gd}(x)$ is the GUDERMANNIAN FUNCTION.

To derive (15), let $u \equiv \cos x$, so $du = -\sin x\,dx$ and

$$\begin{aligned}\int \tan x &= \int \frac{\sin x}{\cos x}\,dx = -\int \frac{du}{u}\\ &= -\ln|u| + C = -\ln|\cos x| + C\\ &= \ln|\cos x|^{-1} + C = \ln|\sec x| + C.\end{aligned} \tag{38}$$

To derive (18), let $u \equiv \csc x - \cot x$, so $du = (-\csc x\cot x + \csc^2 x)\,dx$ and

$$\begin{aligned}\int \csc x\,dx &= \int \csc x\frac{\csc x - \cot x}{\csc x - \cot x}\,dx\\ &= \int \frac{\csc^2 x + \cot x\csc x}{\csc x + \cot x}\,dx\\ &= \int \frac{du}{u} = \ln|u| + C\\ &= \ln|\csc x - \cot x| + C.\end{aligned} \tag{39}$$

To derive (19), let

$$u \equiv \sec x + \tan x, \tag{40}$$

so

$$du = (\sec x\tan x + \sec^2 x)\,dx \tag{41}$$

and

$$\begin{aligned}\int \sec x\,dx &= \int \sec x\frac{\sec x + \tan x}{\sec x + \tan x}\,dx\\ &= \int \frac{\sec^2 x + \sec x + \tan x}{\sec x + \tan x}\,dx\\ &= \int \frac{du}{u} = \ln|u| + C\\ &= \ln|\sec x + \tan x| + C.\end{aligned} \tag{42}$$

To derive (20), let $u \equiv \sin x$, so $du = \cos x\, dx$ and

$$\int \cot x\, dx = \int \frac{\cos x}{\sin x}\, dx = \int \frac{du}{u} = \ln|u| + C = \ln|\sin x| + C. \quad (43)$$

Differentiating integrals leads to some useful and powerful identities, for instance

$$\frac{d}{dx}\int_a^x f(x)\, dx = f(x), \quad (44)$$

which is the first FUNDAMENTAL THEOREM OF CALCULUS.

$$\frac{d}{dx}\int_x^b f(x)\, dx = -f(x) \quad (45)$$

$$\frac{d}{dx}\int_a^b f(x,t)\, dt = \int_a^b \frac{\partial}{\partial x} f(x,t)\, dt \quad (46)$$

$$\frac{d}{dx}\int_a^x f(x,t)\, dt = f(x,t) + \int_a^x \frac{\partial}{\partial x} f(x,t)\, dt. \quad (47)$$

If $f(x,t)$ is singular or INFINITE, then

$$\frac{d}{dx}\int_a^x f(x,t)\, dx = \frac{1}{x-a}\int_a^x \left[(x-a)\frac{\partial f}{\partial x} + (t-a)\frac{\partial f}{\partial t} + f\right] dt. \quad (48)$$

The LEIBNIZ IDENTITY is

$$\frac{d}{dx}\int_{u(x)}^{v(x)} f(x,t)\, dt = v'(x) f(x, v(x)) - u' f(x, u(x)) + \int_{u(x)}^{v(x)} \frac{\partial}{\partial x} f(x,t)\, dt. \quad (49)$$

Other integral identities include

$$\int_a^x \int_a^x f(t)\, dt\, dx = \int_a^x (x-t) f(t)\, dt \quad (50)$$

$$\int_0^x dt_n \int_0^{t_n} dt_{n-1} \cdots \int_0^{t_3} dt_2 \int_0^{t_2} f(t_1)\, dt_1 = \frac{1}{(n-1)!}\int_0^x (x-t)^{n-1} f(t)\, dt \quad (51)$$

$$\frac{\partial}{\partial x_k}(x_j J_k) = \delta_{jk} J_k + x_j \frac{\partial}{\partial x_k} J_k = \mathbf{J} + \mathbf{r}\nabla\cdot\mathbf{J} \quad (52)$$

$$\int_V \mathbf{J}\, d^3\mathbf{r} = \int_V \frac{\partial}{\partial x_k}(x_i J_k) - \int_V \mathbf{r}\nabla\cdot\mathbf{J}\, d^3\mathbf{r} = -\int_V \mathbf{r}\nabla\cdot\mathbf{J}\, d^3\mathbf{r}. \quad (53)$$

Integrals of the form

$$\int_a^b f(x)\, dx \quad (54)$$

with one INFINITE LIMIT and the other NONZERO may be expressed as finite integrals over transformed functions. If $f(x)$ decreases at least as fast as $1/x^2$, then let

$$t \equiv \frac{1}{x} \quad (55)$$

$$dt = -\frac{dx}{x^2} \quad (56)$$

$$dx = -x^2\, dt = -\frac{dt}{t^2}, \quad (57)$$

and

$$\int_a^b f(x)\, dx = -\int_{1/a}^{1/b} \frac{1}{t^2} f\left(\frac{1}{t}\right) dt = \int_{1/b}^{1/a} \frac{1}{t^2} f\left(\frac{1}{t}\right) dt. \quad (58)$$

If $f(x)$ diverges as $(x-a)^\gamma$ for $\gamma \in [0, 1]$, let

$$x \equiv t^{1/(1-\gamma)} + a \quad (59)$$

$$dx = \frac{1}{1-\gamma} t^{(1/1-\gamma)-1}\, dt = \frac{1}{1-\gamma} t^{[1-(1-\gamma)]/(1-\gamma)}\, dt = \frac{1}{\gamma-1} t^{\gamma/(1-\gamma)}\, dt \quad (60)$$

$$t = (x-a)^{1-\gamma}, \quad (61)$$

and

$$\int_a^b f(x)\, dx = \frac{1}{1-\gamma} = \int_0^{(b-a)^{1-\gamma}} t^{\gamma/(1-\gamma)} f(t^{1/(1-\gamma)} + a)\, dt. \quad (62)$$

If $f(x)$ diverges as $(x+b)^\gamma$ for $\gamma \in [0, 1]$, let

$$x \equiv b - t^{1/(1-\gamma)} \quad (63)$$

$$dx = -\frac{1}{\gamma-1} t^{\gamma/(1-\gamma)}\, dt \quad (64)$$

$$t = (b-x)^{1-\gamma}, \quad (65)$$

and

$$\int_a^b f(x)\, dx = \frac{1}{1-\gamma} = \int_0^{(b-a)^{1-\gamma}} t^{\gamma/(1-\gamma)} f(b - t^{1/(1-\gamma)})\, dt. \quad (66)$$

If the integral diverges exponentially, then let

$$t \equiv e^{-x} \quad (67)$$

$$dt = -e^{-x}\, dx \quad (68)$$

$$x = -\ln t, \quad (69)$$

and

$$\int_a^\infty f(x)\,dx = \int_0^{e^{-a}} f(-\ln t)\frac{dt}{t}. \qquad (70)$$

Integrals with rational exponents can often be solved by making the substitution $u = x^{1/n}$, where n is the LEAST COMMON MULTIPLE of the DENOMINATOR of the exponents.

Integration rules include

$$\int_a^a f(x)\,dx = 0 \qquad (71)$$

$$\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx. \qquad (72)$$

For $c \in (a,b)$,

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx. \qquad (73)$$

If g' is continuous on $[a,b]$ and f is continuous and has an antiderivative on an INTERVAL containing the values of $g(x)$ for $a \leq x \leq b$, then

$$\int_a^b f(g(x))g'(x)\,dx = \int_{g(a)}^{g(b)} f(u)\,du. \qquad (74)$$

Liouville showed that the integrals

$$\int e^{-x^2}\,dx \quad \int \frac{e^x}{x}\,dx \quad \int \frac{\sin x}{x}\,dx \quad \int \frac{dx}{\ln x} \qquad (75)$$

cannot be expressed as terms of a finite number of elementary functions. Other irreducibles include

$$\int x^x\,dx \quad \int x^{-x}\,dx \quad \int \sqrt{\sin x}\,dx. \qquad (76)$$

Chebyshev proved that if U, V, and W are RATIONAL NUMBERS, then

$$\int x^U(A+Bx^V)^W\,dx \qquad (77)$$

is integrable in terms of elementary functions IFF $(U+1)/V$, W, or $W+(U+1)/V$ is an INTEGER (Ritt 1948, Shanks 1993).

There are a wide range of methods available for NUMERICAL INTEGRATION. A good source for such techniques is Press *et al.* (1992). The most straightforward numerical integration technique uses the NEWTON-COTES FORMULAS (also called QUADRATURE FORMULAS), which approximate a function tabulated at a sequence of regularly spaced INTERVALS by various degree POLYNOMIALS. If the endpoints are tabulated, then the 2- and 3-point formulas are called the TRAPEZOIDAL RULE and SIMPSON'S RULE, respectively. The 5-point formula is called BODE'S RULE. A generalization of the TRAPEZOIDAL RULE is ROMBERG INTEGRATION, which can yield accurate results for many fewer function evaluations.

If the analytic form of a function is known (instead of its values merely being tabulated at a fixed number of points), the best numerical method of integration is called GAUSSIAN QUADRATURE. By picking the optimal ABSCISSAS at which to compute the function, Gaussian quadrature produces the most accurate approximations possible. However, given the speed of modern computers, the additional complication of the GAUSSIAN QUADRATURE formalism often makes it less desirable than the brute-force method of simply repeatedly calculating twice as many points on a regular grid until convergence is obtained. An excellent reference for GAUSSIAN QUADRATURE is Hildebrand (1956).

see also A-INTEGRABLE, ABELIAN INTEGRAL, CALCULUS, CHEBYSHEV-GAUSS QUADRATURE, CHEBYSHEV QUADRATURE, DARBOUX INTEGRAL, DEFINITE INTEGRAL, DENJOY INTEGRAL, DERIVATIVE, DOUBLE EXPONENTIAL INTEGRATION, EULER INTEGRAL, FUNDAMENTAL THEOREM OF GAUSSIAN QUADRATURE, GAUSS-JACOBI MECHANICAL QUADRATURE, GAUSSIAN QUADRATURE, HAAR INTEGRAL, HERMITE-GAUSS QUADRATURE, HERMITE QUADRATURE, HK INTEGRAL, INDEFINITE INTEGRAL, INTEGRATION, JACOBI-GAUSS QUADRATURE, JACOBI QUADRATURE, LAGUERRE-GAUSS QUADRATURE, LAGUERRE QUADRATURE, LEBESGUE INTEGRAL, LEBESGUE-STIELTJES INTEGRAL, LEGENDRE-GAUSS QUADRATURE, LEGENDRE QUADRATURE, LOBATTO QUADRATURE, MECHANICAL QUADRATURE, MEHLER QUADRATURE, NEWTON-COTES FORMULAS, NUMERICAL INTEGRATION, PERON INTEGRAL, QUADRATURE, RADAU QUADRATURE, RECURSIVE MONOTONE STABLE QUADRATURE, RIEMANN-STIELTJES INTEGRAL, ROMBERG INTEGRATION, RIEMANN INTEGRAL, STIELTJES INTEGRAL

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Shanks, D. *Solved and Unsolved Problems in Number Theory, 4th ed.* New York: Chelsea, p. 145, 1993.
Wolfram Research. "The Integrator." `http://www.integrals.com`

Integral Brick

see EULER BRICK

Integral Calculus

That portion of "the" CALCULUS dealing with INTEGRALS.

see also CALCULUS, DIFFERENTIAL CALCULUS, INTEGRAL

Integral Cuboid

see EULER BRICK

Integral Current

A RECTIFIABLE CURRENT whose boundary is also a RECTIFIABLE CURRENT.

Integral Curvature

Given a GEODESIC TRIANGLE (a triangle formed by the arcs of three GEODESICS on a smooth surface),

$$\int_{ABC} K\,da = A + B + C - \pi.$$

Given the EULER CHARACTERISTIC χ,

$$\iint K\,da = 2\pi\chi,$$

so the integral curvature of a closed surface is not altered by a topological transformation.

see also GAUSS-BONNET FORMULA, GEODESIC TRIANGLE

Integral Domain

A RING that is COMMUTATIVE under multiplication, has a unit element, and has no divisors of 0. The INTEGERS form an integral domain.

see also FIELD, RING

Integral Drawing

A GRAPH drawn such that the EDGES have only INTEGER lengths. It is conjectured that every PLANAR GRAPH has an integral drawing.

References

Harborth, H. and Möller, M. "Minimum Integral Drawings of the Platonic Graphs." *Math. Mag.* **67**, 355–358, 1994.

Integral Equation

If the limits are fixed, an integral equation is called a Fredholm integral equation. If one limit is variable, it is called a Volterra integral equation. If the unknown function is only under the integral sign, the equation is said to be of the "first kind." If the function is both inside and outside, the equation is called of the "second kind." A Fredholm equation of the first kind is of the form

$$f(x) = \int_a^b K(x,t)\phi(t)\,dt. \tag{1}$$

A Fredholm equation of the second kind is of the form

$$\phi(x) = f(x) + \lambda \int_a^b K(x,t)\phi(t)\,dt. \tag{2}$$

A Volterra equation of the first kind is of the form

$$f(x) = \int_a^x K(x,t)\phi(t)\,dt. \tag{3}$$

A Volterra equation of the second kind is of the form

$$\phi(x) = f(x) + \int_a^x K(x,t)\phi(t)\,dt, \tag{4}$$

where the functions $K(x,t)$ are known as KERNELS. Integral equations may be solved directly if they are SEPARABLE. Otherwise, a NEUMANN SERIES must be used.

A KERNEL is separable if

$$K(x,t) = \lambda \sum_{j=1}^n M_j(x)N_j(t). \tag{5}$$

This condition is satisfied by all POLYNOMIALS and many TRANSCENDENTAL FUNCTIONS. A FREDHOLM INTEGRAL EQUATION OF THE SECOND KIND with separable KERNEL may be solved as follows:

$$\begin{aligned}\phi(x) &= f(x) + \int_a^b K(x,t)\phi(t)\,dt \\ &= f(x) + \lambda \sum_{j=1}^n M_j(x) \int_a^b N_j(t)\phi(t)\,dt \\ &= f(x) + \lambda \sum_{j=1}^n c_j M_j(x),\end{aligned} \tag{6}$$

where

$$c_j \equiv \int_a^b N_j(t)\phi(t)\,dt. \tag{7}$$

Now multiply both sides of (7) by $N_i(x)$ and integrate over dx.

$$\int_a^b \phi(x)N_i(x)\,dx = \int_a^b f(x)N_i(x)\,dx + \lambda \sum_{j=1}^n c_j \int_a^b M_j(x)N_i(x)\,dx. \tag{8}$$

By (7), the first term is just c_i. Now define

$$b_i \equiv \int_a^b N_i(x) f(x)\, dx \tag{9}$$

$$a_{ij} = \int_a^b N_i(x) M_j(x)\, dx, \tag{10}$$

so (8) becomes

$$c_i = b_i + \lambda \sum_{j=1}^n a_{ij} c_j. \tag{11}$$

Writing this in matrix form,

$$\mathbf{C} = \mathbf{B} + \lambda \mathsf{A}\mathbf{C}, \tag{12}$$

so

$$(\mathsf{I} - \lambda \mathsf{A})\mathbf{C} = \mathbf{B} \tag{13}$$

$$\mathbf{C} = (\mathsf{I} - \lambda \mathsf{A})^{-1}\mathbf{B}. \tag{14}$$

see also FREDHOLM INTEGRAL EQUATION OF THE FIRST KIND, FREDHOLM INTEGRAL EQUATION OF THE SECOND KIND, VOLTERRA INTEGRAL EQUATION OF THE FIRST KIND, VOLTERRA INTEGRAL EQUATION OF THE SECOND KIND

References

Corduneanu, C. *Integral Equations and Applications.* Cambridge, England: Cambridge University Press, 1991.

Davis, H. T. *Introduction to Nonlinear Differential and Integral Equations.* New York: Dover, 1962.

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Lovitt, W. V. *Linear Integral Equations.* New York: Dover, 1950.

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Porter, D. and Stirling, D. S. G. *Integral Equations: A Practical Treatment, from Spectral Theory to Applications.* Cambridge, England: Cambridge University Press, 1990.

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Tricomi, F. G. *Integral Equations.* New York: Dover, 1957.

Integral of Motion

A function of the coordinates which is constant along a trajectory in PHASE SPACE. The number of DEGREES OF FREEDOM of a DYNAMICAL SYSTEM such as the DUFFING DIFFERENTIAL EQUATION can be decreased by one if an integral of motion can be found. In general, it is very difficult to discover integrals of motion.

Integral Sign

The symbol $\int$ used to denote an INTEGRAL $\int f(x)\, dx$. The symbol was chosen to be a stylized script "S" to stand for "summation."

see also INTEGRAL

Integral Test

Let $\sum u_k$ be a series with POSITIVE terms and let $f(x)$ be the function that results when k is replaced by x in the FORMULA for u_k. If f is decreasing and continuous for $x \geq 1$ and

$$\lim_{x \to \infty} f(x) = 0,$$

then

$$\sum_{k=1}^\infty u_k$$

and

$$\int_t^\infty f(x)\, dx$$

both converge or diverge, where $1 \leq t < \infty$. The test is also called the CAUCHY INTEGRAL TEST or MACLAURIN INTEGRAL TEST.

see also CONVERGENCE TESTS

References

Arfken, G. *Mathematical Methods for Physicists, 3rd ed.* Orlando, FL: Academic Press, pp. 283–284, 1985.

Integral Transform

A general integral transform is defined by

$$g(\alpha) = \int_a^b f(t) K(\alpha, t)\, dt,$$

where $K(\alpha, t)$ is called the KERNEL of the transform.

see also FOURIER TRANSFORM, FOURIER-STIELTJES TRANSFORM, H-TRANSFORM, HADAMARD TRANSFORM, HANKEL TRANSFORM, HARTLEY TRANSFORM, HOUGH TRANSFORM, OPERATIONAL MATHEMATICS, RADON TRANSFORM, WAVELET TRANSFORM, *Z*-TRANSFORM

References

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Carslaw, H. S. and Jaeger, J. C. *Operational Methods in Applied Mathematics.*

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Zayed, A. I. *Handbook of Function and Generalized Function Transformations.* Boca Raton, FL: CRC Press, 1996.

Integrand

The quantity being INTEGRATED, also called the KERNEL. For example, in $\int f(x)\, dx$, $f(x)$ is the integrand.

see also INTEGRAL, INTEGRATION

Integrating Factor

A FUNCTION by which an ORDINARY DIFFERENTIAL EQUATION is multiplied in order to make it integrable.

see also ORDINARY DIFFERENTIAL EQUATION

References

Morse, P. M. and Feshbach, H. *Methods of Theoretical Physics, Part I.* New York: McGraw-Hill, pp. 526–529, 1953.

Integration

The process of computing or obtaining an INTEGRAL. A more archaic term for integration is QUADRATURE.

see also CONTOUR INTEGRATION, INTEGRAL, INTEGRATION BY PARTS, MEASURE THEORY, NUMERICAL INTEGRATION

Integration Lattice

A discrete subset of $\mathbb{R}^s$ which is closed under addition and subtraction and which contains $\mathbb{Z}^s$ as a SUBSET.

see also LATTICE

References

Sloan, I. H. and Joe, S. *Lattice Methods for Multiple Integration.* New York: Oxford University Press, 1994.

Integration by Parts

A first-order (single) integration by parts uses

$$d(uv) = u\,dv + v\,du \tag{1}$$

$$\int d(uv) = uv = \int u\,dv + \int v\,du, \tag{2}$$

so

$$\int u\,dv = uv - \int v\,du \tag{3}$$

and

$$\int_a^b u\,dv = [uv]_a^b - \int_{f(a)}^{f(b)} v\,du. \tag{4}$$

Now apply this procedure n times to $\int f^{(n)}(x)g(x)\,dx$.

$$u = g(x) \qquad dv = f^{(n)}(x)\,dx \tag{5}$$

$$du = g'(x)\,dx \qquad v = f^{(n-1)}(x). \tag{6}$$

Therefore,

$$\int f^{(n)}g(x)\,dx = g(x)f^{(n-1)}(x) - \int f^{(n-1)}(x)g'(x)\,dx. \tag{7}$$

But

$$\int f^{(n-1)}(x)g'(x)\,dx = g'(x)f^{(n-2)}(x) - \int f^{(n-2)}(x)g''(x)\,dx \tag{8}$$

$$\int f^{(n-2)}(x)g''(x)\,dx = g''(x)f^{(n-3)}(x) - \int f^{(n-3)}(x)g^{(3)}(x)\,dx, \tag{9}$$

so

$$\int f^{(n)}(x)g(x)\,dx = g(x)f^{(n-1)}(x) - g'(x)f^{(n-2)}(x) + g''(x)f^{(n-3)}(x) - \ldots + (-1)^n \int f(x)g^{(n)}(x)\,dx. \tag{10}$$

Now consider this in the slightly different form $\int f(x)g(x)\,dx$. Integrate by parts a first time

$$u = f(x) \qquad dv = g(x)\,dx \tag{11}$$

$$du = f'(x)\,dx \qquad v = \int g(x)\,dx, \tag{12}$$

so

$$\int f(x)g(x)\,dx = f(x)\int g(x)\,dx - \int \left[\int g(x)\,dx\right] f'(x)\,dx. \tag{13}$$

Now integrate by parts a second time,

$$u = f'(x) \qquad dv = \int g(x)(dx)^2 \tag{14}$$

$$du = f''(x)\,dx \qquad v = \iint g(x)(dx)^2, \tag{15}$$

so

$$\int f(x)g(x)\,dx = f(x)\int g(x)\,dx - f'(x)\iint g(x)(dx)^2 + \int \left[\iint g(x)(dx)^2\right] f''(x)\,dx. \tag{16}$$

Repeating a third time,

$$\int f(x)g(x)\,dx = f(x)\int g(x)\,dx - f'(x)\iint g(x)(dx)^2 + f''(x)\iiint g(x)(dx)^3 - \int \left[\iiint g(x)(dx)^3\right] f'''(x)\,dx. \tag{17}$$

Therefore, after n applications,

$$\begin{aligned}\int f(x)g(x)\,dx &= f(x)\int g(x)\,dx - f'(x)\iint g(x)(dx)^2 \\ &\quad + f''(x)\iiint g(x)(dx)^3 - \ldots \\ &\quad + (-1)^{n+1} f^{(n)}(x) \underbrace{\int\cdots\int}_{n+1} g(x)(dx)^{n+1} \\ &\quad + (-1)^n \int \left[\underbrace{\int\cdots\int}_{n+1} g(x)(dx)^{n+1}\right] f^{(n+1)}(x)\,dx. \qquad (18)\end{aligned}$$

If $f^{(n+1)}(x) = 0$ (e.g., for an nth degree POLYNOMIAL), the last term is 0, so the sum terminates after n terms and

$$\begin{aligned}\int f(x)g(x)\,dx &= f(x)\int g(x)\,dx \\ &\quad - f'(x)\iint g(x)(dx)^2 + f''(x)\iiint g(x)(dx)^3 - \ldots \\ &\quad + (-1)^{n+1} f^{(n)}(x) \underbrace{\int\cdots\int}_{n+1} g(x)(dx)^{n+1}. \qquad (19)\end{aligned}$$

References

Abramowitz, M. and Stegun, C. A. (Eds.). *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables, 9th printing.* New York: Dover, p. 12, 1972.

Intension

A definition of a SET by mentioning a defining property.

see also EXTENSION

References

Russell, B. "Definition of Number." *Introduction to Mathematical Philosophy.* New York: Simon and Schuster, 1971.

Interchange Graph

see LINE GRAPH

Interest

Interest is a fee (or payment) made for the borrowing (or lending) of money. The two most common types of interest are SIMPLE INTEREST, for which interest is paid only on the initial PRINCIPAL, and COMPOUND INTEREST, for which interest earned can be re-invested to generate further interest.

see also COMPOUND INTEREST, CONVERSION PERIOD, RULE OF 72, SIMPLE INTEREST

References

Kellison, S. G. *Theory of Interest, 2nd ed.* Burr Ridge, IL: Richard D. Irwin, 1991.

Interior

That portion of a region lying "inside" a specified boundary. For example, the interior of the SPHERE is a BALL.

see also EXTERIOR

Interior Angle Bisector

see ANGLE BISECTOR

Intermediate Value Theorem

If f is continuous on a CLOSED INTERVAL $[a, b]$ and c is any number between $f(a)$ and $f(b)$ inclusive, there is at least one number x in the CLOSED INTERVAL such that $f(x) = c$.

see also WEIERSTRAß INTERMEDIATE VALUE THEOREM

Internal Bisectors Problem

see STEINER-LEHMUS THEOREM

Internal Knot

One of the knots $t_{p+1}, \ldots, t_{m-p-1}$ of a B-SPLINE with control points $\mathbf{P}_0, \ldots, \mathbf{P}_n$ and KNOT VECTOR

$$\mathbf{T} = \{t_0, t_1, \ldots, t_m\},$$

where

$$p \equiv m - n - 1.$$

see also B-SPLINE, KNOT VECTOR

Interpolation

The computation of points or values between ones that are known or tabulated using the surrounding points or values.

see also AITKEN INTERPOLATION, BESSEL'S INTERPOLATION FORMULA, EVERETT INTERPOLATION, EXTRAPOLATION, FINITE DIFFERENCE, GAUSS'S INTERPOLATION FORMULA, HERMITE INTERPOLATION, LAGRANGE INTERPOLATING POLYNOMIAL, NEWTON-COTES FORMULAS, NEWTON'S DIVIDED DIFFERENCE INTERPOLATION FORMULA, OSCULATING INTERPOLATION, THIELE'S INTERPOLATION FORMULA

References

Abramowitz, M. and Stegun, C. A. (Eds.). "Interpolation." §25.2 in *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables, 9th printing.* New York: Dover, pp. 878–882, 1972.

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Interquartile Range

Divide a set of data into two groups (high and low) of equal size at the MEDIAN if there is an EVEN number of data points, or two groups consisting of points on either side of the MEDIAN itself plus the MEDIAN if there is an ODD number of data points. Find the MEDIANS of the low and high groups, denoting these first and third quartiles by Q_1 and Q_3. The interquartile range is then defined by

$$\mathrm{IQR} \equiv Q_3 - Q_1.$$

see also H-SPREAD, HINGE, MEDIAN (STATISTICS)

Interradius

see MIDRADIUS

Intersection

The intersection of two sets A and B is the set of elements common to A and B. This is written $A \cap B$, and is pronounced "A intersection B" or "A cap B." The intersection of sets A_1 through A_n is written $\bigcap_{i=1}^{n} A_i$. The intersection of lines AB and CD is written $\overline{AB} \cap \overline{CD}$.

see also AND, UNION

Interspersion

An ARRAY $A = a_{ij}$, $i, j \geq 1$ of POSITIVE INTEGERS is called an interspersion if

1. The rows of A comprise a PARTITION of the POSITIVE INTEGERS,
2. Every row of A is an increasing sequence,
3. Every column of A is a (possibly FINITE) increasing sequence,
4. If (u_j) and (v_j) are distinct rows of A and if p and q are any indices for which $u_p < v_q < u_{p+1}$, then $u_{p+1} < v_{q+1} < u_{p+2}$.

If an array $A = a_{ij}$ is an interspersion, then it is a DISPERSION. If an array $A = a(i,j)$ is an interspersion, then the sequence $\{x_n\}$ given by $\{x_n = i : n = (i,j)\}$ for some j is a FRACTAL SEQUENCE. Examples of interspersion are the STOLARSKY ARRAY and WYTHOFF ARRAY.

see also DISPERSION (SEQUENCE), FRACTAL SEQUENCE, STOLARSKY ARRAY

References

Kimberling, C. "Interspersions and Dispersions." *Proc. Amer. Math. Soc.* **117**, 313–321, 1993.

Kimberling, C. "Fractal Sequences and Interspersions." *Ars Combin.* **45**, 157–168, 1997.

Intersphere

see MIDSPHERE

Interval

A collection of points on a LINE SEGMENT. If the endpoints a and b are FINITE and are included, the interval is called CLOSED and is denoted $[a, b]$. If one of the endpoints is $\pm\infty$, then the interval still contains all of its LIMIT POINTS, so $[a, \infty)$ and $(-\infty, b]$ are also closed intervals. If the endpoints are not included, the interval is called OPEN and denoted (a, b). If one endpoint is included but not the other, the interval is denoted $[a, b)$ or $(a, b]$ and is called a HALF-CLOSED (or HALF-OPEN) interval.

see also CLOSED INTERVAL, HALF-CLOSED INTERVAL, LIMIT POINT, OPEN INTERVAL, PENCIL

Interval Graph

A GRAPH $G = (V, E)$ is an interval graph if it captures the INTERSECTION RELATION for some set of INTERVALS on the REAL LINE. Formally, P is an interval graph provided that one can assign to each $v \in V$ an interval I_v such that $I_u \cap I_v$ is nonempty precisely when $uv \in E$.

see also COMPARABILITY GRAPH

References

Booth, K. S. and Lueker, G. S. "Testing for the Consecutive Ones Property, Interval Graphs, and Graph Planarity using PQ-Tree Algorithms." *J. Comput. System Sci.* **13**, 335–379, 1976.

Fishburn, P. C. *Interval Orders and Interval Graphs: A Study of Partially Ordered Sets.* New York: Wiley, 1985.

Gilmore, P. C. and Hoffman, A. J. "A Characterization of Comparability Graphs and of Interval Graphs." *Canad. J. Math.* **16**, 539–548, 1964.

Lekkerkerker, C. G. and Boland, J. C. "Representation of a Finite Graph by a Set of Intervals on the Real Line." *Fund. Math.* **51**, 45–64, 1962.

Interval Order

A POSET $P = (X, \leq)$ is an interval order if it is ISOMORPHIC to some set of INTERVALS on the REAL LINE ordered by left-to-right precedence. Formally, P is an interval order provided that one can assign to each $x \in X$ an INTERVAL $[x_L, x_R]$ such that $x_R < y_L$ in the REAL NUMBERS IFF $x < y$ in P.

see also PARTIALLY ORDERED SET

References

Fishburn, P. C. *Interval Orders and Interval Graphs: A Study of Partially Ordered Sets.* New York: Wiley, 1985.

Wiener, N. "A Contribution to the Theory of Relative Position." *Proc. Cambridge Philos. Soc.* **17**, 441–449, 1914.

Intrinsic Curvature

A CURVATURE such as GAUSSIAN CURVATURE which is detectable to the "inhabitants" of a surface and not just outside observers. An EXTRINSIC CURVATURE, on the other hand, is not detectable to someone who can't study the 3-dimensional space surrounding the surface on which he resides.

see also CURVATURE, EXTRINSIC CURVATURE, GAUSSIAN CURVATURE

Intrinsic Equation

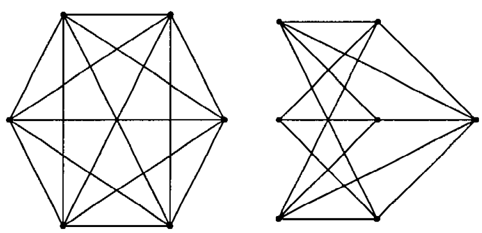
An equation which specifies a CURVE in terms of intrinsic properties such as ARC LENGTH, RADIUS OF CURVATURE, and TANGENTIAL ANGLE instead of with reference to artificial coordinate axes. Intrinsic equations are also called NATURAL EQUATIONS.

see also CESÀRO EQUATION, NATURAL EQUATION, WHEWELL EQUATION

References

Yates, R. C. "Intrinsic Equations." *A Handbook on Curves and Their Properties.* Ann Arbor, MI: J. W. Edwards, pp. 123–126, 1952.

Intrinsically Linked



A GRAPH is intrinsically linked if any embedding of it in 3-D contains a nontrivial LINK. A GRAPH is intrinsically linked IFF it contains one of the seven PETERSEN GRAPHS (Robertson *et al.* 1993).

The COMPLETE GRAPH K_6 (left) is intrinsically linked because it contains at least two linked TRIANGLES. The COMPLETE k-PARTITE GRAPH $K_{3,3,1}$ (right) is also intrinsically linked.

see also COMPLETE GRAPH, COMPLETE k-PARTITE GRAPH, PETERSEN GRAPHS

References

Adams, C. C. *The Knot Book: An Elementary Introduction to the Mathematical Theory of Knots.* New York: W. H. Freeman, pp. 217–221, 1994.

Robertson, N.; Seymour, P. D.; and Thomas, R. "Linkless Embeddings of Graphs in 3-Space." *Bull. Amer. Math. Soc.* **28**, 84–89, 1993.

Invariant

A quantity which remains unchanged under certain classes of transformations. Invariants are extremely useful for classifying mathematical objects because they usually reflect intrinsic properties of the object of study.

see ADIABATIC INVARIANT, ALEXANDER INVARIANT, ALGEBRAIC INVARIANT, ARF INVARIANT, INTEGRAL OF MOTION

References

Hunt, B. "Invariants." Appendix B.1 in *The Geometry of Some Special Arithmetic Quotients.* New York: Springer-Verlag, pp. 282–290, 1996.

Invariant Density

see NATURAL INVARIANT

Invariant (Elliptic Function)

The invariants of a WEIERSTRAß ELLIPTIC FUNCTION are defined by

$$g_2 \equiv 60\Sigma'\Omega_{mn}{}^{-4}$$
$$g_3 \equiv 140\Sigma'\Omega_{mn}{}^{-6}.$$

Here,

$$\Omega_{mn} \equiv 2m\omega_1 - 2n\omega_2,$$

where ω_1 and ω_2 are the periods of the ELLIPTIC FUNCTION.

Invariant Manifold

When stable and unstable invariant MANIFOLDS intersect, they do so in a HYPERBOLIC FIXED POINT (SADDLE POINT). The invariant MANIFOLDS are then called SEPARATRICES. A HYPERBOLIC FIXED POINT is characterized by two ingoing stable MANIFOLDS and two outgoing unstable MANIFOLDS. In integrable systems, incoming W^s and outgoing W^u MANIFOLDS all join up smoothly.

A stable invariant MANIFOLD W^s of a FIXED POINT Y^* is the set of all points Y_0 such that the trajectory passing through Y_0 tends to Y^* as $j \to \infty$.

An unstable invariant MANIFOLD W^u of a FIXED POINT Y^* is the set of all points Y_0 such that the trajectory passing through Y_0 tends to Y^* as $j \to -\infty$.

see also HOMOCLINIC POINT

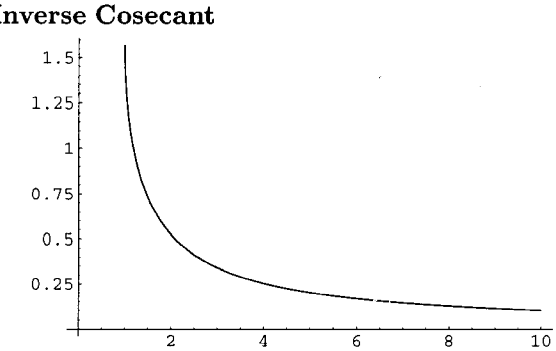
Invariant Point

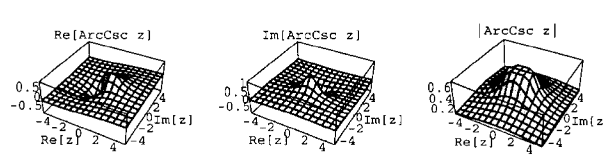
see FIXED POINT (TRANSFORMATION)

Invariant Subgroup

see NORMAL SUBGROUP

Inverse Cosecant





The function $\csc^{-1} x$, also denoted $\operatorname{arccsc}(x)$, where $\csc x$ is the COSECANT and the SUPERSCRIPT -1 denotes an

INVERSE FUNCTION, *not* the multiplicative inverse. The inverse cosecant satisfies

$$\csc^{-1} x = \sec^{-1}\left(\frac{x}{\sqrt{x^2-1}}\right) \tag{1}$$

for POSITIVE or NEGATIVE x, and

$$\csc^{-1} x = \pi + \csc^{-1}(-x) \tag{2}$$

for $x \geq 0$. The inverse cosecant is given in terms of other inverse trigonometric functions by

$$\csc^{-1} = \cos^{-1}\left(\frac{\sqrt{x^2-1}}{x}\right) \tag{3}$$

$$= \cot^{-1}(\sqrt{x^2-1}) \tag{4}$$

$$= \tfrac{1}{2}\pi - \sec^{-1} x = -\tfrac{1}{2}\pi - \sec^{-1}(-x) \tag{5}$$

$$= \sin^{-1}\left(\frac{1}{x}\right) \tag{6}$$

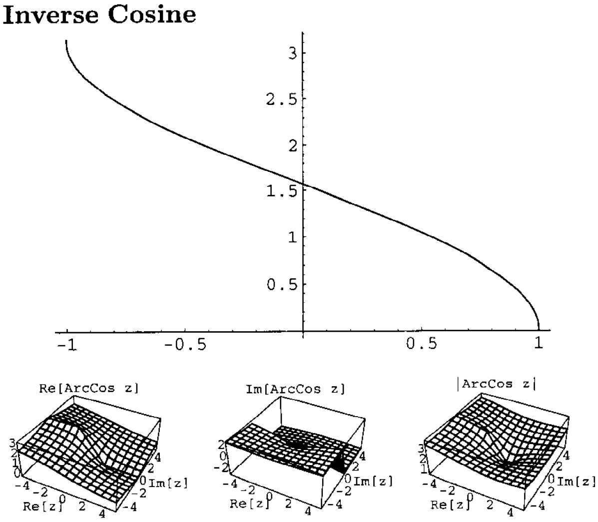
for $x \geq 0$.

see also COSECANT INVERSE SINE, SINE

References

Beyer, W. H. *CRC Standard Mathematical Tables, 28th ed.* Boca Raton, FL: CRC Press, pp. 142–143, 1987.

Inverse Cosine



The function $\cos^{-1} x$, also denoted $\arccos(x)$, where $\cos x$ is the COSINE and the superscript -1 denotes an INVERSE FUNCTION, *not* the multiplicative inverse. The MACLAURIN SERIES for the inverse cosine range $-1 < x < 1$ is

$$\cos^{-1} x = \tfrac{1}{2}\pi - x - \tfrac{1}{6}x^3 - \tfrac{3}{40}x^5 - \tfrac{5x}{112}x^7 - \tfrac{35}{1152}x^9 - \ldots. \tag{1}$$

The inverse cosine satisfies

$$\cos^{-1} x = \pi - \cos^{-1}(-x) \tag{2}$$

for POSITIVE and NEGATIVE x, and

$$\cos^{-1} = \tfrac{1}{2}\pi - \cos^{-1}(\sqrt{1-x^2}) \tag{3}$$

for $x \geq 0$. The inverse cosine is given in terms of other inverse trigonometric functions by

$$\cos^{-1} x = \cot^{-1}\left(\frac{x}{\sqrt{1-x^2}}\right) \tag{4}$$

$$= \tfrac{1}{2}\pi + \sin^{-1}(-x) = \tfrac{1}{2}\pi - \sin^{-1} x \tag{5}$$

$$= \tfrac{1}{2}\pi - \tan^{-1}\left(\frac{x}{\sqrt{1-x^2}}\right) \tag{6}$$

for POSITIVE or NEGATIVE x, and

$$\cos^{-1} x = \csc^{-1}\left(\frac{1}{\sqrt{1-x^2}}\right) \tag{7}$$

$$= \sec^{-1}\left(\frac{1}{x}\right) \tag{8}$$

$$= \sin^{-1}(\sqrt{1-x^2}) \tag{9}$$

$$= \tan^{-1}\left(\frac{\sqrt{1-x^2}}{x}\right) \tag{10}$$

for $x \geq 0$.

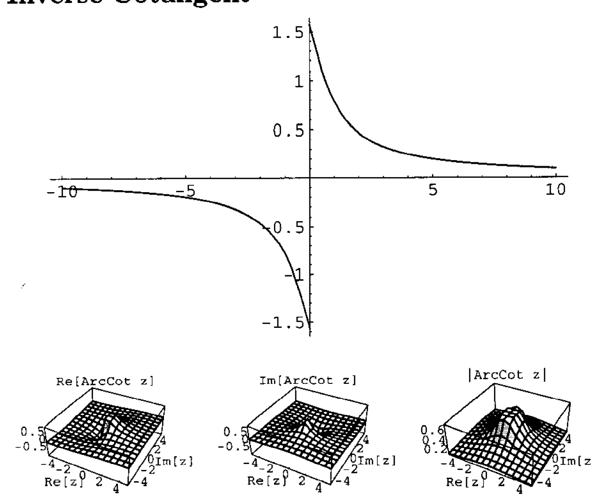
see also COSINE, INVERSE SECANT

References

Abramowitz, M. and Stegun, C. A. (Eds.). "Inverse Circular Functions." §4.4 in *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables, 9th printing.* New York: Dover, pp. 79–83, 1972.

Beyer, W. H. *CRC Standard Mathematical Tables, 28th ed.* Boca Raton, FL: CRC Press, pp. 142–143, 1987.

Inverse Cotangent



The function $\cot^{-1} x$, also denoted $\text{arccot}(x)$, where $\cot x$ is the COTANGENT and the superscript -1 denotes an INVERSE FUNCTION and *not* the multiplicative inverse. The MACLAURIN SERIES is given by

$$\cot^{-1} x = \tfrac{1}{2}\pi - x + \tfrac{1}{3}x^3 - \tfrac{1}{5}x^5 + \tfrac{1}{7}x^7 - \tfrac{1}{9}x^9 + \ldots, \tag{1}$$

and POWER SERIES by

$$\cot^{-1} x = x^{-1} - \tfrac{1}{3}x^{-3} + \tfrac{1}{5}x^{-5} - \tfrac{1}{7}x^{-7} + \tfrac{1}{9}x^{-9} + \ldots. \quad (2)$$

Euler derived the INFINITE series

$$\cot^{-1} x = x\left[\frac{1}{x^2+1} + \frac{2}{3(x^2+1)^2} + \frac{2\cdot 4}{3\cdot 5(x^2+1)^3} + \ldots\right] \quad (3)$$

(Wetherfield 1996).

The inverse cotangent satisfies

$$\cot^{-1} x = \pi - \cot^{-1}(-x) \quad (4)$$

for POSITIVE and NEGATIVE x, and

$$\cot^{-1} = \tfrac{1}{2}\pi - \cot^{-1}\left(\frac{1}{x}\right) \quad (5)$$

for $x \geq 0$. The inverse cotangent is given in terms of other inverse trigonometric functions by

$$\cot^{-1} x = \cos^{-1}\left(\frac{x}{\sqrt{x^2+1}}\right) \quad (6)$$

$$= \tfrac{1}{2}\pi - \sin^{-1}\left(\frac{x}{\sqrt{x^2+1}}\right) \quad (7)$$

$$= \tfrac{1}{2}\pi + \tan^{-1}(-x) = \tfrac{1}{2}\pi - \tan^{-1} x \quad (8)$$

for POSITIVE or NEGATIVE x, and

$$\cot^{-1} x = \csc^{-1}(\sqrt{x^2+1}) \quad (9)$$

$$= \sec^{-1}\left(\frac{\sqrt{x^2+1}}{x}\right) \quad (10)$$

$$= \sin^{-1}\left(\frac{1}{\sqrt{x^2+1}}\right) \quad (11)$$

$$= \tan^{-1}\left(\frac{1}{x}\right) \quad (12)$$

for $x \geq 0$.

A number

$$t_x = \cot^{-1} x, \quad (13)$$

where x is an INTEGER or RATIONAL NUMBER, is sometimes called a GREGORY NUMBER. Lehmer (1938a) showed that $\cot^{-1}(a/b)$ can be expressed as a finite sum of inverse cotangents of INTEGER arguments

$$\cot^{-1}\left(\frac{a}{b}\right) = \sum_{i=1}^{k}(-1)^{i-1}\cot^{-1} n_i, \quad (14)$$

where

$$n_i = \left\lfloor \frac{a_i}{b_i} \right\rfloor, \quad (15)$$

with $\lfloor x \rfloor$ the FLOOR FUNCTION, and

$$a_{i+1} = a_i n + i + b_i \quad (16)$$

$$b_{i+1} = a_i - n_i b_i, \quad (17)$$

with $a_0 = a$ and $b_0 = b$, and where the recurrence is continued until $b_{k+1} = 0$. If an INVERSE TANGENT sum is written as

$$\tan^{-1} n = \sum_{k=1} f_k \tan^{-1} n_k + f \tan^{-1}, \quad (18)$$

then equation (14) becomes

$$\cot^{-1} n = \sum_{k=1} f_k \cot^{-1} n_k + c\cot^{-1} 1, \quad (19)$$

where

$$c = 2 - f - 2\sum_{k=1} f_r. \quad (20)$$

Inverse cotangent sums can be used to generate MACHIN-LIKE FORMULAS.

An interesting inverse cotangent identity attributed to Charles Dodgson (Lewis Carroll) by Lehmer (1938b; Bromwich 1965, Castellanos 1988ab) is

$$\cot^{-1}(p+r) + \tan^{-1}(p+q) = \tan^{-1} p, \quad (21)$$

where

$$1 + p^2 = qr. \quad (22)$$

Other inverse cotangent identities include

$$2\cot^{-1}(2x) - \cot^{-1} x = \cot^{-1}(4x^3 + 3x) \quad (23)$$

$$3\cot^{-1}(3x) - \cot^{-1} x = \cot^{-1}\left(\frac{27x^4 + 18x^2 - 1}{8x}\right), \quad (24)$$

as well as many others (Bennett 1926, Lehmer 1938b).

see also COTANGENT, INVERSE TANGENT, MACHIN'S FORMULA, MACHIN-LIKE FORMULAS, TANGENT

References

Abramowitz, M. and Stegun, C. A. (Eds.). "Inverse Circular Functions." §4.4 in *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables, 9th printing.* New York: Dover, pp. 79–83, 1972.

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Lehmer, D. H. "On Arccotangent Relations for π." *Amer. Math. Monthly* **45**, 657–664, 1938b.

Weisstein, E. W. "Arccotangent Series." `http://www.astro.virginia.edu/~eww6n/math/notebooks/CotSeries.m`.

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Inverse Curve

Given a CIRCLE C with CENTER O and RADIUS k, then two points P and Q are inverse with respect to C if $OP \cdot OQ = k^2$. If P describes a curve C_1, then Q describes a curve C_2 called the inverse of C_1 with respect to the circle C (with INVERSION CENTER O). If the POLAR equation of C is $r(\theta)$, then the inverse curve has polar equation

$$r = \frac{k^2}{r(\theta)}.$$

If $O = (x_0, y_0)$ and $P = (f(t), g(t))$, then the inverse has equations

$$x = x_0 + \frac{k^2(f - x_0])}{(f - x_0)^2 + (g - y_0)^2}$$
$$y = y_0 + \frac{k^2(g - y_0)}{(f - x_0)^2 + (g - y_0)^2}.$$

Curve	Inversion Center	Inverse Curve
Archimedean spiral	origin	Archimedean spiral
cardioid	cusp	parabola
circle	any pt.	another circle
cissoid of Diocles	cusp	parabola
cochleoid	origin	quadratrix of Hippias
epispiral	origin	Rose
Fermat's spiral	origin	lituus
hyperbola	center	lemniscate
hyperbola	vertex	right strophoid
hyperbola with $a = \sqrt{3}$	vertex	Maclaurin trisectrix
lemniscate	center	hyperbola
lituus	origin	Fermat spiral
logarithmic spiral	origin	logarithmic spiral
Maclaurin trisectrix	focus	Tschirnhausen's cubic
parabola	focus	cardioid
parabola	vertex	cissoid of Diocles
quadratrix of Hippias		cochleoid
right strophoid	origin	the same right strophoid
sinusoidal spiral	origin	sinusoidal spiral inverse curve
Tschirnhausen cubic		sinusoidal spiral

see also INVERSION, INVERSION CENTER, INVERSION CIRCLE

References

Lee, X. "Inversion." http://www.best.com/~xah/SpecialPlaneCurves_dir/Inversion_dir/inversion.html.

Lee, X. "Inversion Gallery." http://www.best.com/~xah/SpecialPlaneCurves_dir/InversionGallery_dir/inversionGallery.html.

Yates, R. C. "Inversion." *A Handbook on Curves and Their Properties.* Ann Arbor, MI: J. W. Edwards, pp. 127–134, 1952.

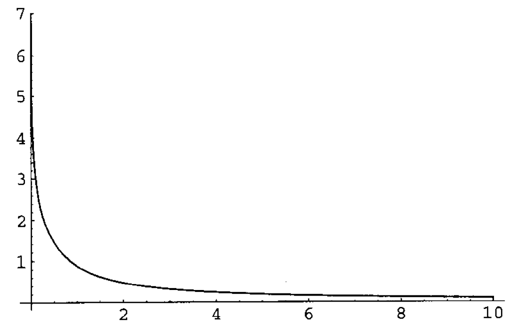
Inverse Filter

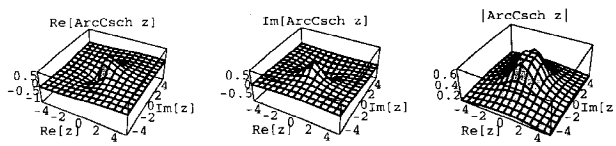
A linear DECONVOLUTION ALGORITHM.

Inverse Function

Given a FUNCTION $f(x)$, its inverse $f^{-1}(x)$ is defined by $f(f^{-1}(x)) \equiv x$. Therefore, $f(x)$ and $f^{-1}(x)$ are reflections about the line $y = x$.

Inverse Hyperbolic Cosecant

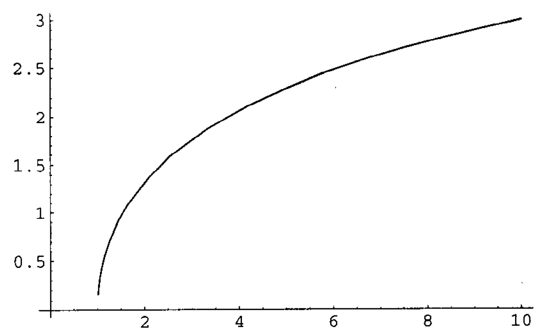


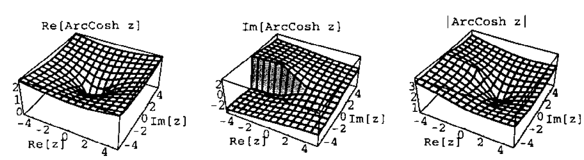


The INVERSE FUNCTION of the HYPERBOLIC COSECANT, denoted $\operatorname{csch}^{-1} x$.

see also HYPERBOLIC COSECANT

Inverse Hyperbolic Cosine

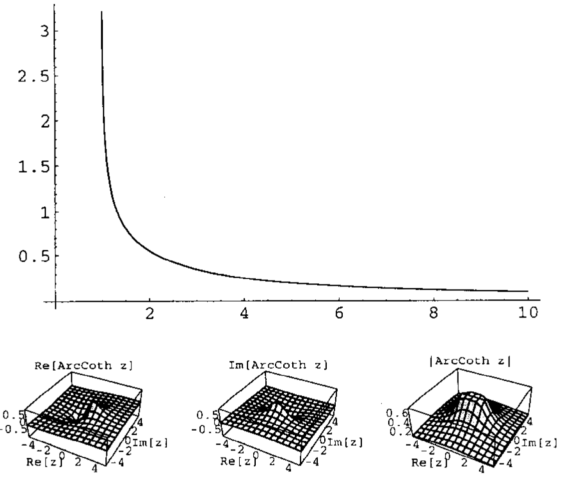




The INVERSE FUNCTION of the HYPERBOLIC COSINE, denoted $\cosh^{-1} x$.

see also HYPERBOLIC COSINE

Inverse Hyperbolic Cotangent



The INVERSE FUNCTION of the HYPERBOLIC COTANGENT, denoted $\coth^{-1} x$.

see also HYPERBOLIC COTANGENT

Inverse Hyperbolic Functions

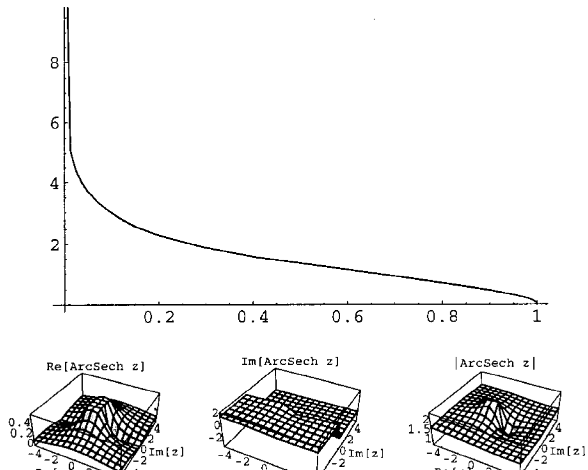
The INVERSE of the HYPERBOLIC FUNCTIONS, denoted $\cosh^{-1} x$, $\coth^{-1} x$, $\operatorname{csch}^{-1} x$, $\operatorname{sech}^{-1} x$, $\sinh^{-1} x$, and $\tanh^{-1} x$.

see also HYPERBOLIC FUNCTIONS

References

Spanier, J. and Oldham, K. B. "The Inverse Hyperbolic Functions." Ch. 31 in *An Atlas of Functions.* Washington, DC: Hemisphere, pp. 285–293, 1987.

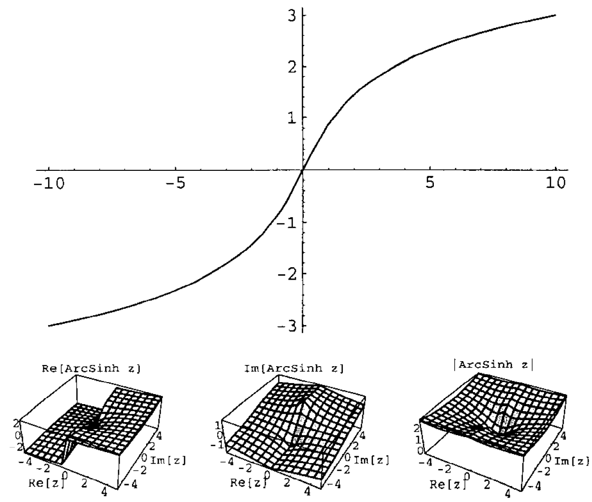
Inverse Hyperbolic Secant



The INVERSE FUNCTION of the HYPERBOLIC SECANT, denoted $\operatorname{sech}^{-1} x$.

see also HYPERBOLIC SECANT

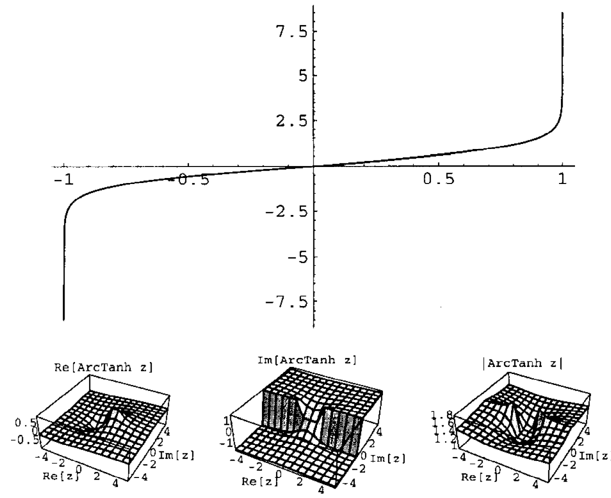
Inverse Hyperbolic Sine



The INVERSE FUNCTION of the HYPERBOLIC SINE, denoted $\sinh^{-1} x$.

see also HYPERBOLIC SINE

Inverse Hyperbolic Tangent



The INVERSE FUNCTION of the HYPERBOLIC TANGENT, denoted $\tanh^{-1} x$.

see also HYPERBOLIC TANGENT

Inverse Matrix

see also MATRIX INVERSE

Inverse Points

Points which are transformed into each other through INVERSION about a given INVERSION CIRCLE. The point P' which is the inverse point of a given point P with respect to an INVERSION CIRCLE C may be constructed geometrically using a COMPASS only (Courant and Robbins 1996).

see also GEOMETRIC CONSTRUCTION, INVERSION, POLAR, POLE (GEOMETRY)

References
Courant, R. and Robbins, H. "Geometrical Construction of Inverse Points." §3.4.3 in *What is Mathematics?: An Elementary Approach to Ideas and Methods, 2nd ed.* Oxford, England: Oxford University Press, pp. 144–145, 1996.

Inverse Quadratic Interpolation

The use of three prior points in a ROOT-finding ALGORITHM to estimate the zero crossing.

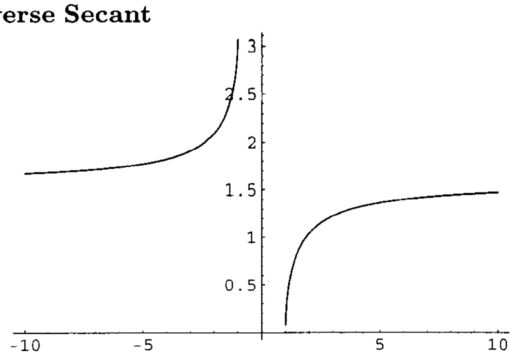
Inverse Scattering Method

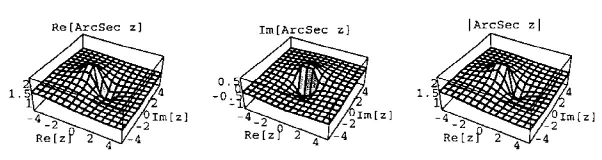
A method which can be used to solve the initial value problem for certain classes of nonlinear PARTIAL DIFFERENTIAL EQUATIONS. The method reduces the initial value problem to a linear INTEGRAL EQUATION in which time appears only implicitly. However, the solutions $u(x,t)$ and various of their derivatives must approach zero as $x \to \pm\infty$ (Infeld and Rowlands 1990).

see also ABLOWITZ-RAMANI-SEGUR CONJECTURE, BÄCKLUND TRANSFORMATION

References
Infeld, E. and Rowlands, G. "Inverse Scattering Method." §7.4 in *Nonlinear Waves, Solitons, and Chaos.* Cambridge, England: Cambridge University Press, pp. 192–196, 1990.
Miura, R. M. (Ed.) *Bäcklund Transformations, the Inverse Scattering Method, Solitons, and Their Applications.* New York: Springer-Verlag, 1974.

Inverse Secant





The function $\sec^{-1} x$, where $\sec x$ is the SECANT and the superscript -1 denotes the INVERSE FUNCTION, *not* the multiplicative inverse. The inverse secant satisfies

$$\sec^{-1} x = \csc^{-1}\left(\frac{x}{\sqrt{x^2-1}}\right) \tag{1}$$

for POSITIVE or NEGATIVE x, and

$$\sec^{-1} x = \pi + \sec^{-1}(-x) \tag{2}$$

for $x \geq 0$. The inverse secant is given in terms of other inverse trigonometric functions by

$$\begin{aligned} \sec^{-1} x &= \cos^{-1}\left(\frac{1}{x}\right) && (3)\\ &= \cot^{-1}\left(\frac{1}{\sqrt{x^2-1}}\right) && (4)\\ &= \tfrac{1}{2}\pi - \csc^{-1} x = -\tfrac{1}{2}\pi - \csc^{-1}(-x) && (5)\\ &= \sin^{-1}\left(\frac{\sqrt{x^2-1}}{x}\right) && (6)\\ &= \tan^{-1}(\sqrt{x^2-1}\,) && (7) \end{aligned}$$

for $x \geq 0$.

see also INVERSE COSECANT, SECANT

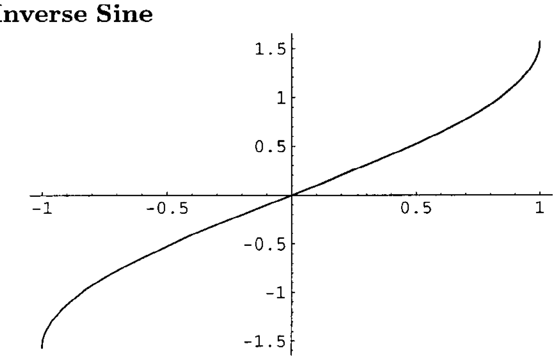
References
Beyer, W. H. *CRC Standard Mathematical Tables, 28th ed.* Boca Raton, FL: CRC Press, pp. 141–143, 1987.

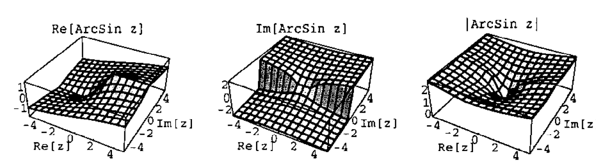
Inverse Semigroup

The abstract counterpart of a PSEUDOGROUP formed by certain subsets of a GROUPOID which admit a MULTIPLICATION.

References
Weinstein, A. "Groupoids: Unifying Internal and External Symmetry." *Not. Amer. Math. Soc.* **43**, 744–752, 1996.

Inverse Sine





The function $\sin^{-1} x$, where $\sin x$ is the SINE and the superscript -1 denotes the INVERSE FUNCTION, *not* the multiplicative inverse. The inverse sine satisfies

$$\sin^{-1} x = -\sin^{-1}(-x) \tag{1}$$

for POSITIVE and NEGATIVE x, and

$$\sin^{-1} = \tfrac{1}{2}\pi - \sin^{-1}(\sqrt{1-x^2}\,) \tag{2}$$

for $x \geq 0$. The inverse sine is given in terms of other inverse trigonometric functions by

$$\sin^{-1} x = \cos^{-1}(-x) - \tfrac{1}{2}\pi = \tfrac{1}{2}\pi - \cos^{-1} x \quad (3)$$

$$= \tfrac{1}{2}\pi - \cot^{-1}\left(\frac{x}{\sqrt{1-x^2}}\right) \quad (4)$$

$$= \tan^{-1}\left(\frac{x}{\sqrt{1-x^2}}\right) \quad (5)$$

for POSITIVE or NEGATIVE x, and

$$\sin^{-1} x = \cos^{-1}(\sqrt{1-x^2}\,) \quad (6)$$

$$= \cot^{-1}\left(\frac{\sqrt{1-x^2}}{x}\right) \quad (7)$$

$$= \csc^{-1}\left(\frac{1}{x}\right) \quad (8)$$

$$= \sec^{-1}\left(\frac{1}{\sqrt{1-x^2}}\right) \quad (9)$$

for $x \geq 0$.

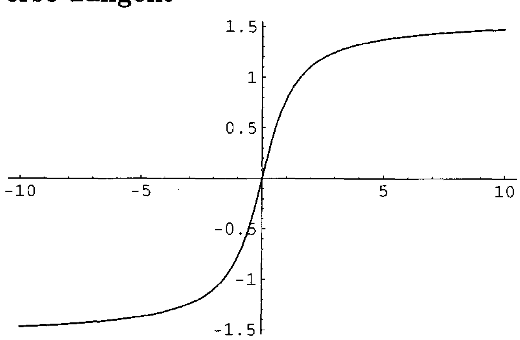
see also INVERSE COSINE, SINE

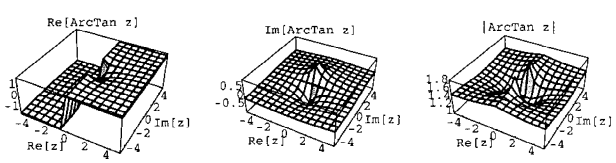
References

Abramowitz, M. and Stegun, C. A. (Eds.). "Inverse Circular Functions." §4.4 in *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables, 9th printing.* New York: Dover, pp. 79–83, 1972.

Beyer, W. H. *CRC Standard Mathematical Tables, 28th ed.* Boca Raton, FL: CRC Press, pp. 142–143, 1987.

Inverse Tangent





The inverse tangent is also called the arctangent and is denoted either $\tan^{-1} x$ or $\arctan x$. It has the MACLAURIN SERIES

$$\tan^{-1} x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1} = x - \tfrac{1}{3}x^3 + \tfrac{1}{5}x^5 - \tfrac{1}{7}x^7 + \ldots. \quad (1)$$

A more rapidly converging form due to Euler is given by

$$\tan^{-1} x = \sum_{n=0}^{\infty} \frac{2^{2n}(n!)^2}{(2n+1)!} \frac{x^{2n+1}}{(+x^2)^{n+1}} \quad (2)$$

(Castellanos 1988). The inverse tangent satisfies

$$\tan^{-1} x = -\tan^{-1}(-x) \quad (3)$$

for POSITIVE and NEGATIVE x, and

$$\tan^{-1} = \tfrac{1}{2}\pi - \tan^{-1}\left(\frac{1}{x}\right) \quad (4)$$

for $x \geq 0$. The inverse tangent is given in terms of other inverse trigonometric functions by

$$\tan^{-1} x = \tfrac{1}{2}\pi - \cos^{-1}\left(\frac{x}{\sqrt{x^2+1}}\right) \quad (5)$$

$$= \cot^{-1}(-x) - \tfrac{1}{2}\pi = \tfrac{1}{2}\pi - \cot^{-1} x \quad (6)$$

$$= \sin^{-1}\left(\frac{x}{\sqrt{x^2+1}}\right) \quad (7)$$

for POSITIVE or NEGATIVE x, and

$$\tan^{-1} x = \cos^{-1}\left(\frac{1}{\sqrt{x^2+1}}\right) \quad (8)$$

$$= \cot^{-1}\left(\frac{1}{x}\right) \quad (9)$$

$$= \csc^{-1}\left(\frac{\sqrt{x^2+1}}{x}\right) \quad (10)$$

$$= \sec^{-1}(\sqrt{x^2+1}\,) \quad (11)$$

for $x \geq 0$.

In terms of the HYPERGEOMETRIC FUNCTION,

$$\tan^{-1} x = x\,{}_2F_1(1, \tfrac{1}{2}; \tfrac{3}{2}; -x^2) \quad (12)$$

$$= \frac{x}{1+x^2}\,{}_2F_1\left(1, 1; \tfrac{3}{2}; \frac{x^2}{1+x^2}\right) \quad (13)$$

(Castellanos 1988). Castellanos (1986, 1988) also gives some curious formulas in terms of the FIBONACCI NUMBERS,

$$\tan^{-1} x = \sum_{n=0}^{\infty} \frac{(-1)^n F_{2n+1} t^{2n+1}}{5^n(2n+1)} \quad (14)$$

$$= 5\sum_{n=0}^{\infty} \frac{(-1)^n {F_{2n+1}}^2}{(2n+1)(u+\sqrt{u^2+1}\,)^{2n+1}} \quad (15)$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n 5^{n+2} {F_{2n+1}}^3}{(2n+1)(v+\sqrt{v^2+5}\,)^{2n+1}}, \quad (16)$$

where

$$t \equiv \frac{2x}{1+\sqrt{\frac{4x^2}{5}}} \tag{17}$$

$$u \equiv \frac{5}{4x}\left(1+\sqrt{1+\tfrac{24}{25}x^2}\right), \tag{18}$$

and v is the largest POSITIVE ROOT of

$$8xv^4 - 100v^3 - 450xv^2 + 875v + 625x = 0. \tag{19}$$

The inverse tangent satisfies the addition FORMULA

$$\tan^{-1} x + \tan^{-1} y = \tan^{-1}\left(\frac{x+y}{1-xy}\right) \tag{20}$$

as well as the more complicated FORMULAS

$$\tan^{-1}\left(\frac{1}{a-b}\right) = \tan^{-1}\left(\frac{1}{a}\right) + \tan^{-1}\left(\frac{b}{a^2-ab+1}\right) \tag{21}$$

$$\tan^{-1}\left(\frac{1}{a}\right) = 2\tan^{-1}\left(\frac{1}{2a}\right) - \tan^{-1}\left(\frac{1}{4a^3+3a}\right) \tag{22}$$

$$\tan^{-1}\left(\frac{1}{p}\right) = \tan^{-1}\left(\frac{1}{p+q}\right) + \tan^{-1}\left(\frac{q}{p^2+pq+1}\right), \tag{23}$$

the latter of which was known to Euler. The inverse tangent FORMULAS are connected with many interesting approximations to PI

$$\tan^{-1}(1+x) = \tfrac{1}{4}\pi + \tfrac{1}{2}x - \tfrac{1}{4}x^2 + \tfrac{1}{12}x^3 + \tfrac{1}{40}x^5 + \tfrac{1}{48}x^6 + \tfrac{1}{112}x^7 + \ldots. \tag{24}$$

Euler gave

$$\tan^{-1} x = \frac{y}{x}\left(\frac{2}{3}y + \frac{2\cdot 4}{3\cdot 5}y^2 + \frac{2\cdot 4\cdot 6}{3\cdot 5\cdot 7}y^3 + \ldots\right), \tag{25}$$

where

$$y \equiv \frac{x^2}{1+x^2}. \tag{26}$$

The inverse tangent has CONTINUED FRACTION representations

$$\tan^{-1} x = \cfrac{x}{1+\cfrac{x^2}{3+\cfrac{4x^2}{5+\cfrac{9x^2}{7+\cfrac{16x^2}{9+\ldots}}}}} \tag{27}$$

$$= \cfrac{x}{1+\cfrac{x^2}{3-x^2+\cfrac{9x^2}{5-3x^2+\cfrac{25x^2}{7-5x^2+\ldots}}}}. \tag{28}$$

To find $\tan^{-1} x$ numerically, the following ARITHMETIC-GEOMETRIC MEAN-like ALGORITHM can be used. Let

$$a_0 = (1+x^2)^{-1/2} \tag{29}$$

$$b_0 = 1. \tag{30}$$

Then compute

$$a_{i+1} = \tfrac{1}{2}(a_i + b_i) \tag{31}$$

$$b_{i+1} = \sqrt{a_{i+1}b_i}, \tag{32}$$

and the inverse tangent is given by

$$\tan^{-1} x = \lim_{n\to\infty} \frac{x}{\sqrt{1+x^2}\, a_n} \tag{33}$$

(Acton 1990).

An inverse tangent $\tan^{-1} n$ with integral n is called reducible if it is expressible as a finite sum of the form

$$\tan^{-1} n = \sum_{k=1} f_k \tan^{-1} n_k, \tag{34}$$

where f_k are POSITIVE or NEGATIVE INTEGERS and n_i are iINTEGERS $< n$. $\tan^{-1} m$ is reducible IFF all the PRIME factors of $1+m^2$ occur among the PRIME factors of $1+n^2$ for $n = 1, \ldots, m-1$. A second NECESSARY and SUFFICIENT condition is that the largest PRIME factor of $1+m^2$ is less than $2m$. Equivalent to the second condition is the statement that every GREGORY NUMBER $t_x = \cot^{-1} x$ can be uniquely expressed as a sum in terms of t_ms for which m is a STØRMER NUMBER (Conway and Guy 1996). To find this decomposition, write

$$\arg(1+in) = \arg \prod_{k=1} (1+n_k i)^{f_k}, \tag{35}$$

so the ratio

$$r = \frac{\prod_{k=1}(1+n_k i)^{f_k}}{1+in} \tag{36}$$

is a RATIONAL NUMBER. Equation (36) can also be written

$$r^2(1+n^2) = \prod_{k=1} (1+n_k{}^2)^{f_k}. \tag{37}$$

Writing (34) in the form

$$\tan^{-1} n = \sum_{k=1} f_k \tan^{-1} n_k + f\tan^{-1} 1 \tag{38}$$

allows a direct conversion to a corresponding INVERSE COTANGENT FORMULA

$$\cot^{-1} n = \sum_{k=1} f_k \cot^{-1} n_k + c\cot^{-1} 1, \tag{39}$$

where

$$c = 2 - f - 2\sum_{k=1} f_r. \tag{40}$$

Todd (1949) gives a table of decompositions of $\tan^{-1} n$ for $n \leq 342$. Conway and Guy (1996) give a similar table in terms of STØRMER NUMBERS.

Arndt and Gosper give the remarkable inverse tangent identity

$$\sin\left(\sum_{k=1}^{2n+1} \tan^{-1} a_k\right) = \frac{(-1)^n}{2n+1} \frac{\sum_{k=1}^{2n+1} \prod_{j=1}^{2n+1} \left[a_j - \tan\left(\frac{\pi(j-k)}{2n+1}\right)\right]}{\sqrt{\prod_{j=1}^{2n+1} ({a_j}^2+1)}}. \quad (41)$$

see also INVERSE COTANGENT, TANGENT

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Inverse Trigonometric Functions

INVERSE FUNCTIONS of the TRIGONOMETRIC FUNCTIONS written $\cos^{-1} x$, $\cot^{-1} x$, $\csc^{-1} x$, $\sec^{-1} x$, $\sin^{-1} x$, and $\tan^{-1} x$.

see also INVERSE COSECANT, INVERSE COSINE, INVERSE COTANGENT, INVERSE SECANT, INVERSE SINE, INVERSE TANGENT

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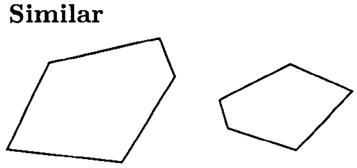
Spanier, J. and Oldham, K. B. "Inverse Trigonometric Functions." Ch. 35 in *An Atlas of Functions.* Washington, DC: Hemisphere, pp. 331–341, 1987.

Inversely Similar

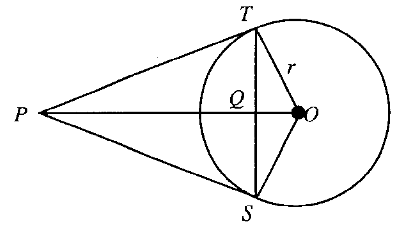


inversely similar

Two figures are said to be SIMILAR when all corresponding ANGLES are equal, and are inversely similar when all corresponding ANGLES are equal and described in the opposite rotational sense.

see also DIRECTLY SIMILAR, SIMILAR

Inversion



Inversion is the process of transforming points to their INVERSE POINTS. This sort of inversion was first systematically investigated by Jakob Steiner. Two points are said to be inverses with respect to an INVERSION CIRCLE with INVERSION CENTER $O = (x_0, y_0)$ and INVERSION RADIUS k if PT and PS are line segments symmetric about OP and tangent to the CIRCLE, and Q is the intersection of OP and ST. The curve to which a given curve is transformed under inversion is called its INVERSE CURVE.

Note that a point on the CIRCUMFERENCE of the INVERSION CIRCLE is its own inverse point. The inverse points obey

$$\frac{OP}{k} = \frac{k}{OQ}, \quad (1)$$

or

$$k^2 = OP \times OQ, \quad (2)$$

where k^2 is called the POWER. The equation for the inverse of the point (x, y) relative to the INVERSION CIRCLE with INVERSION CENTER (x_0, y_0) and inversion radius k is therefore

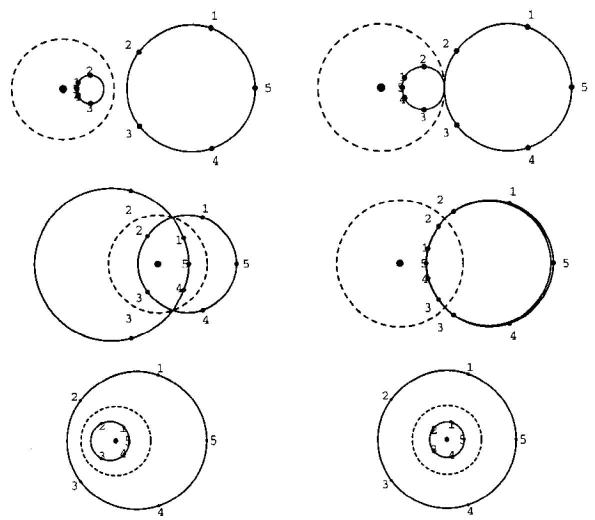
$$x' = x_0 + \frac{k^2(x - x_0)}{(x - x_0)^2 + (y - y_0)^2} \quad (3)$$

$$y' = y_0 + \frac{k^2(y - y_0)}{(x - x_0)^2 + (y - y_0)^2}. \quad (4)$$

In vector form,

$$\mathbf{x}' = \mathbf{x}_0 + \frac{k^2(\mathbf{x} - \mathbf{x}_0)}{|\mathbf{x} - \mathbf{x}_0|^2}. \quad (5)$$

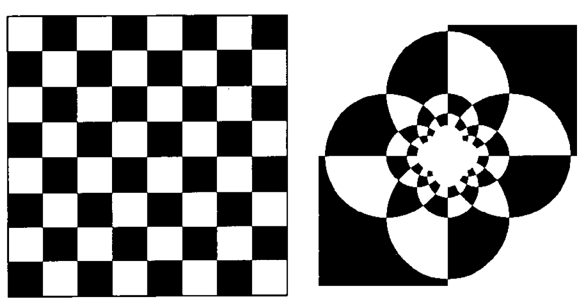
Any ANGLE inverts to an opposite ANGLE.



Treating LINES as CIRCLES of INFINITE RADIUS, all CIRCLES invert to CIRCLES. Furthermore, any two nonintersecting circles can be inverted into concentric circles by taking the INVERSION CENTER at one of the two limiting points (Coxeter 1969), and ORTHOGONAL CIRCLES invert to ORTHOGONAL CIRCLES (Coxeter 1969).

The inverse of a CIRCLE of RADIUS a with CENTER (x, y) with respect to an inversion circle with INVERSION CENTER $(0,0)$ and INVERSION RADIUS k is another CIRCLE with CENTER $(x', y') = (sx, sy)$ and RADIUS $r' = |s|a$, where

$$s \equiv \frac{k^2}{x^2 + y^2 - a^2}. \qquad (6)$$



The above plot shows a checkerboard centered at (0, 0) and its inverse about a small circle also centered at (0, 0) (Dixon 1991).

see also ARBELOS, HEXLET, INVERSE CURVE, INVERSION CIRCLE, INVERSION OPERATION, INVERSION RADIUS, INVERSIVE DISTANCE, INVERSIVE GEOMETRY, MIDCIRCLE, PAPPUS CHAIN, PEAUCELLIER INVERSOR, POLAR, POLE (GEOMETRY), POWER (CIRCLE), RADICAL LINE, STEINER CHAIN, STEINER'S PORISM

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Inversion Center

The point that INVERSION OF A CURVE is performed with respect to.

see also INVERSE POINTS, INVERSION CIRCLE, INVERSION RADIUS, INVERSIVE DISTANCE, POLAR, POLE (GEOMETRY), POWER (CIRCLE)

Inversion Circle

The CIRCLE with respect to which a INVERSE CURVE is computed or relative to which INVERSE POINTS are computed.

see also INVERSE POINTS, INVERSION CENTER, INVERSION RADIUS, INVERSIVE DISTANCE, MIDCIRCLE, POLAR, POLE (GEOMETRY), POWER (CIRCLE)

Inversion Operation

The SYMMETRY OPERATION $(x, y, z) \to (-x, -y, -z)$. When used in conjunction with a ROTATION, it becomes an IMPROPER ROTATION.

Inversion Radius

The RADIUS used in performing an INVERSION with respect to an INVERSION CIRCLE.

see also INVERSE POINTS, INVERSION CENTER, INVERSION CIRCLE, INVERSIVE DISTANCE, POLAR, POLE (GEOMETRY), POWER (CIRCLE)

Inversive Distance

The inversive distance is the NATURAL LOGARITHM of the ratio of two concentric circles into which the given circles can be inverted. Let c be the distance between the centers of two nonintersecting CIRCLES of RADII a and $b < a$. Then the inversive distance is

$$\delta = \cosh^{-1}\left|\frac{a^2 + b^2 - c^2}{2ab}\right|$$

(Coxeter and Greitzer 1967).

The inversive distance between the SODDY CIRCLES is given by

$$\delta = 2\cosh^{-1} 2,$$

and the CIRCUMCIRCLE and INCIRCLE of a TRIANGLE with CIRCUMRADIUS R and INRADIUS r are at inversive distance

$$\delta = 2\sinh^{-1}\left(\frac{1}{2}\sqrt{\frac{r}{R}}\right)$$

(Coxeter and Greitzer 1967, pp. 130–131).

References
Coxeter, H. S. M. and Greitzer, S. L. *Geometry Revisited.* Washington, DC: Math. Assoc. Amer., pp. 123–124 and 127–131, 1967.

Inversive Geometry

The GEOMETRY resulting from the application of the INVERSION operation. It can be especially powerful for solving apparently difficult problems such as STEINER'S PORISM and APOLLONIUS' PROBLEM.

see also HEXLET, INVERSE CURVE, INVERSION, PEAUCELLIER INVERSOR, POLAR, POLE (GEOMETRY), POWER (CIRCLE), RADICAL LINE

References
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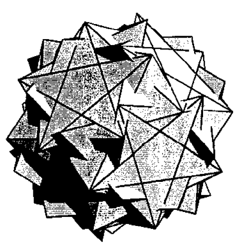
Inverted Funnel

see also FUNNEL, SINCLAIR'S SOAP FILM PROBLEM

Inverted Snub Dodecadodecahedron



The UNIFORM POLYHEDRON U_{60} whose DUAL POLYHEDRON is the MEDIAL INVERTED PENTAGONAL HEXECONTAHEDRON. It has WYTHOFF SYMBOL $|\,2\,\frac{5}{3}\,5$. Its faces are $12\{\frac{5}{3}\} + 60\{3\} + 12\{5\}$. It has CIRCUMRADIUS for unit edge length of

$$R \approx 0.8516302.$$

References
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Invertible Knot

A knot which can be deformed into itself but with the orientation reversed. The simplest noninvertible knot is 08_{017}. No general technique is known for determining if a KNOT is invertible. Burde and Zieschang (1985) give a tabulation from which it is possible to extract the invertible knots up to 10 crossings.

see also AMPHICHIRAL KNOT

References
Burde, G. and Zieschang, H. *Knots.* Berlin: de Gruyter, 1985.

Involuntary

A LINEAR TRANSFORMATION of period two. Since a LINEAR TRANSFORMATION has the form,

$$\lambda' = \frac{\alpha\lambda+\beta}{\gamma\lambda+\delta}, \tag{1}$$

applying the transformation a second time gives

$$\lambda'' = \frac{\alpha\lambda'+\beta}{\gamma\lambda'+\delta} = \frac{(\alpha^2+\beta\gamma)\lambda+\beta(\alpha+\delta)}{(\alpha+\delta)\gamma\lambda+\beta\gamma+\delta^2}. \tag{2}$$

For an involuntary, $\lambda'' = \lambda$, so

$$\gamma(\alpha+\delta)\lambda^2 + (\delta^2-\alpha^2)\lambda - (\alpha+\delta)\beta = 0. \tag{3}$$

Since each COEFFICIENT must vanish separately,

$$\alpha\gamma+\gamma\delta = 0 \tag{4}$$
$$\delta^2-\alpha^2 = 0 \tag{5}$$
$$\alpha\beta+\beta\delta = 0. \tag{6}$$

The first equation gives $\delta = \pm\alpha$. Taking $\delta = \alpha$ would require $\gamma = \beta = 0$, giving $\lambda = \lambda'$, the identity transformation. Taking $\delta = -\alpha$ gives $\delta = -\alpha$, so

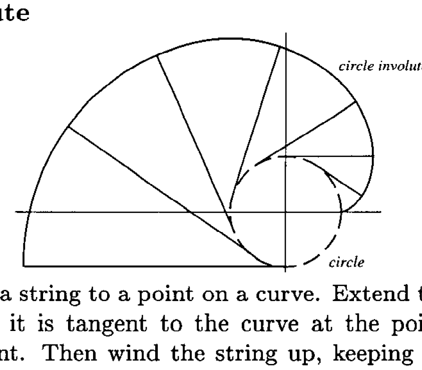
$$\lambda' = \frac{\alpha\lambda+\beta}{\gamma\lambda-\alpha}, \tag{7}$$

the general form of an INVOLUTION.

see also CROSS-RATIO, INVOLUTION (LINE)

References
Woods, F. S. *Higher Geometry: An Introduction to Advanced Methods in Analytic Geometry.* New York: Dover, pp. 14–15, 1961.

Involute



Attach a string to a point on a curve. Extend the string so that it is tangent to the curve at the point of attachment. Then wind the string up, keeping it always taut. The LOCUS of points traced out by the end of the string is the involute of the original curve, and the original curve is called the EVOLUTE of its involute. Although a curve has a unique EVOLUTE, it has infinitely many involutes corresponding to different choices of initial point. An involute can also be thought of as any curve ORTHOGONAL to all the TANGENTS to a given curve.

The equation of the involute is

$$\mathbf{r}_i = \mathbf{r} - s\hat{\mathbf{T}}, \tag{1}$$

where $\hat{\mathbf{T}}$ is the TANGENT VECTOR

$$\hat{\mathbf{T}} = \frac{\frac{d\mathbf{r}}{dt}}{\left|\frac{d\mathbf{r}}{dt}\right|} \tag{2}$$

and s is the ARC LENGTH

$$s = \int ds = \int \frac{ds}{dt}\,dt = \int \frac{\sqrt{ds^2}}{dt}\,dt = \int \sqrt{f'^2 + g'^2}\,dt. \tag{3}$$

This can be written for a parametrically represented function $(f(t), g(t))$ as

$$x(t) = f - \frac{sf'}{\sqrt{f'^2 + g'^2}} \tag{4}$$

$$y(t) = g - \frac{sg'}{\sqrt{f'^2 + g'^2}}. \tag{5}$$

Curve	Involute
astroid	astroid 1/2 as large
cardioid	cardioid 3 times as large
catenary	tractrix
circle catacaustic for a point source	limaçon
circle	circle involute (a spiral)
cycloid	equal cycloid
deltoid	deltoid 1/3 as large
ellipse	ellipse involute
epicycloid	reduced epicycloid
hypocycloid	similar hypocycloid
logarithmic spiral	equal logarithmic spiral
Neile's parabola	parabola
nephroid	Cayley's sextic
nephroid	nephroid 2 times as large

see also EVOLUTE, HUMBERT'S THEOREM

References

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Involution (Group)

An element of order 2 in a GROUP (i.e., an element A of a GROUP such that $A^2 = I$, where I is the IDENTITY ELEMENT).

see also GROUP, IDENTITY ELEMENT

Involution (Line)

Pairs of points of a line, the product of whose distances from a FIXED POINT is a given constant. This is more concisely defined as a PROJECTIVITY of period two.

see also INVOLUNTARY

Involution (Operator)

An OPERATOR of period 2, i.e., an OPERATOR $*$ which satisfies $((a)^*)^* = a$.

Involution (Set)

An involution of a SET S is a PERMUTATION of S which does not contain any cycles of length > 2. The PERMUTATION MATRICES of an involution are SYMMETRIC. The number of involutions $I(n)$ of a SET containing the first n integers is given by the RECURRENCE RELATION

$$I(n) = I(n-1) + (n-1)I(n-2).$$

For $n = 1, 2, \ldots$, the first few values of $I(n)$ are 1, 2, 4, 10, 26, 76, ... (Sloane's A000085). The number of involutions on n symbols cannot be expressed as a fixed number of hypergeometric terms (Petkovšek *et al.* 1996, p. 160).

see also PERMUTATION

References

Petkovšek, M.; Wilf, H. S.; and Zeilberger, D. *A=B.* Wellesley, MA: A. K. Peters, 1996.

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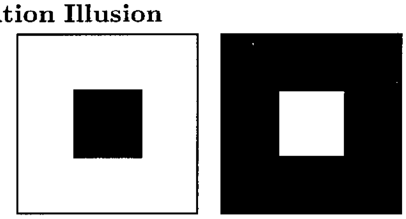
Involution (Transformation)

A TRANSFORMATION of period 2.

Irradiation Illusion



The ILLUSION shown above which was discovered by Helmholtz in the 19th century. Despite the fact that the two above figures are identical in size, the white hole looks bigger than the black one in this ILLUSION.

References

Pappas, T. "Irradiation Optical Illusion." *The Joy of Mathematics.* San Carlos, CA: Wide World Publ./Tetra, p. 199, 1989.

Irrational Number

A number which cannot be expressed as a FRACTION p/q for any INTEGERS p and q. Every TRANSCENDENTAL NUMBER is irrational. Numbers of the form $n^{1/m}$ are irrational unless n is the mth POWER of an INTEGER.

Numbers of the form $\log_n m$, where log is the LOGARITHM, are irrational if m and n are INTEGERS, one of which has a PRIME factor which the other lacks. e^r is irrational for rational $r \neq 0$. The irrationality of e was proven by Lambert in 1761; for the general case, see Hardy and Wright (1979, p. 46). π^n is irrational for POSITIVE integral n. The irrationality of π was proven by Lambert in 1760; for the general case, see Hardy and Wright (1979, p. 47). APÉRY'S CONSTANT $\zeta(3)$ (where $\zeta(z)$ is the RIEMANN ZETA FUNCTION) was proved irrational by Apéry (Apéry 1979, van der Poorten 1979).

From GELFOND'S THEOREM, a number of the form a^b is TRANSCENDENTAL (and therefore irrational) if a is ALGEBRAIC $\neq 0$, 1 and b is irrational and ALGEBRAIC. This establishes the irrationality of e^π (since $(-1)^{-i} = (e^{i\pi})^{-i} = e^\pi$), $2^{\sqrt{2}}$, and $e\pi$. Nesterenko (1996) proved that $\pi + e^\pi$ is irrational. In fact, he proved that π, e^π and $\Gamma(1/4)$ are algebraically independent, but it was not previously known that $\pi + e^\pi$ was irrational.

Given a POLYNOMIAL equation

$$x^m + c_{m-1}x^{m-1} + \ldots + c_0, \tag{1}$$

where c_i are INTEGERS, the roots x_i are either integral or irrational. If $\cos(2\theta)$ is irrational, then so are $\cos\theta$, $\sin\theta$, and $\tan\theta$.

Irrationality has not yet been established for 2^e, π^e, $\pi^{\sqrt{2}}$, or γ (where γ is the EULER-MASCHERONI CONSTANT).

QUADRATIC SURDS are irrational numbers which have periodic CONTINUED FRACTIONS.

HURWITZ'S IRRATIONAL NUMBER THEOREM gives bounds of the form

$$\left|\alpha - \frac{p}{q}\right| < \frac{1}{L_n q^2} \tag{2}$$

for the best rational approximation possible for an arbitrary irrational number α, where the L_n are called LAGRANGE NUMBERS and get steadily larger for each "bad" set of irrational numbers which is excluded.

The SERIES

$$\sum_{n=1}^\infty \frac{\sigma_k(n)}{n!}, \tag{3}$$

where $\sigma_k(n)$ is the DIVISOR FUNCTION, is irrational for $k = 1$ and 2. The series

$$\sum_{n=1}^\infty \frac{1}{2^n - 1} = \sum_{n=1}^\infty \frac{d(n)}{2^n}, \tag{4}$$

where $d(n)$ is the number of divisors of n, is also irrational, as are

$$\sum_{n=1}^\infty \frac{1}{q^n + r} \qquad \text{for} \qquad \sum_{n=1}^\infty \frac{(-1)^n}{q^n + r} \tag{5}$$

for q an INTEGER other than p, ± 1, and r a RATIONAL NUMBER other than 0 or $-q^n$ (Guy 1994).

see also ALGEBRAIC INTEGER, ALGEBRAIC NUMBER, ALMOST INTEGER, DIRICHLET FUNCTION, FERGUSON-FORCADE ALGORITHM, GELFOND'S THEOREM, HURWITZ'S IRRATIONAL NUMBER THEOREM, NEAR NOBLE NUMBER, NOBLE NUMBER, PYTHAGORAS'S THEOREM, QUADRATIC IRRATIONAL NUMBER, RATIONAL NUMBER, SEGRE'S THEOREM, TRANSCENDENTAL NUMBER

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van der Poorten, A. "A Proof that Euler Missed... Apéry's Proof of the Irrationality of $\zeta(3)$." *Math. Intel.* **1**, 196–203, 1979.

Irrationality Measure

see LIOUVILLE-ROTH CONSTANT

Irrationality Sequence

A sequence of POSITIVE INTEGERS $\{a_n\}$ such that $\sum 1/(a_n b_n)$ is IRRATIONAL for all integer sequences $\{b_n\}$. Erdős showed that $\{2^{2^n}\}$ is an irrationality sequence.

References

Guy, R. K. "Irrationality Sequence." §E24 in *Unsolved Problems in Number Theory, 2nd ed.* New York: Springer-Verlag, p. 225, 1994.

Irreducible Matrix

A SQUARE MATRIX which is not REDUCIBLE is said to be irreducible.

Irreducible Polynomial

A POLYNOMIAL or polynomial equation is said to be irreducible if it cannot be factored into polynomials of lower degree over the same FIELD.

The number of binary irreducible polynomials of degree n is equal to the number of n-bead fixed NECKLACES of two colors: 1, 2, 3, 4, 6, 8, 14, 20, 36, ... (Sloane's A000031), the first few of which are given in the following table.

n	Polynomials
1	x
2	x, $x+1$
3	x, x^2+x+1, $x+1$
4	x, x^3+x+1, x^3+x^2+1, $x+1$

see also FIELD, GALOIS FIELD, NECKLACE, POLYNOMIAL, PRIMITIVE IRREDUCIBLE POLYNOMIAL

References

Sloane, N. J. A. Sequences A000031/M0564 in "An On-Line Version of the Encyclopedia of Integer Sequences."

Sloane, N. J. A. and Plouffe, S. Extended entry in *The Encyclopedia of Integer Sequences.* San Diego: Academic Press, 1995.

Irreducible Representation

An irreducible representation of a GROUP is a representation for which there exists no UNITARY TRANSFORMATION which will transform the representation MATRIX into block diagonal form. The irreducible representation has a number of remarkable properties.

see also GROUP, ITÔ'S THEOREM, UNITARY TRANSFORMATION

Irreducible Semiperfect Number

see PRIMITIVE PSEUDOPERFECT NUMBER

Irreducible Tensor

Given a general second RANK TENSOR A_{ij} and a METRIC g_{ij}, define

$$\theta \equiv A_{ij}g^{ij} = A^i_i \tag{1}$$

$$\omega^i \equiv \epsilon^{ijk}A_{jk} \tag{2}$$

$$\sigma_{ij} \equiv \tfrac{1}{2}(A_{ij}+A_{ji}) - \tfrac{1}{3}g_{ij}A^k_k, \tag{3}$$

where δ_{ij} is the KRONECKER DELTA and ϵ^{ijk} is the LEVI-CIVITA SYMBOL. Then

$$\begin{aligned}\sigma_{ij} &+ \tfrac{1}{3}\theta g_{ij} + \tfrac{1}{2}\epsilon_{ijk}\omega^k \\ &= [\tfrac{1}{2}(A_{ij}+A_{ji}) - \tfrac{1}{3}g_{ij}A^k_k] + \tfrac{1}{3}A^k_k g_{ij} + \tfrac{1}{2}\epsilon_{ijk}[\epsilon^{\lambda\mu k}A_{\lambda\mu}] \\ &= \tfrac{1}{2}(A_{ij}+A_{ji}) + \tfrac{1}{2}(\delta^\lambda_i\delta^\mu_j - \delta^\mu_i\delta^\lambda_j)A_{\lambda\mu} \\ &= \tfrac{1}{2}(A_{ij}+A_{ji}) + \tfrac{1}{2}(A_{ij}-A_{ji}) = A_{ij}, \end{aligned}\tag{4}$$

where θ, ω^i, and σ_{ij} are TENSORS of RANK 0, 1, and 2.

see also TENSOR

Irredundant Ramsey Number

Let G_1, G_2, ..., G_t be a t-EDGE coloring of the COMPLETE GRAPH K_n, where for each $i = 1, 2, \ldots$, t, G_i is the spanning SUBGRAPH of K_n consisting of all EDGES colored with the ith color. The irredundant Ramsey number $s(q_1,\ldots,q_t)$ is the smallest INTEGER n such that for any t-EDGE coloring of K_n, the COMPLEMENT GRAPH $\overline{G_i}$ has an irredundant set of size q_i for at least one $i = 1, \ldots, t$. Irredundant Ramsey numbers were introduced by Brewster *et al.* (1989) and satisfy

$$s(q_1,\ldots,q_t) \leq R(q_1,\ldots,q_t).$$

For a summary, see Mynhardt (1992).

s	Bounds	Reference
$s(3,3)$	6	Brewster *et al.* 1989
$s(3,4)$	8	Brewster *et al.* 1989
$s(3,5)$	12	Brewster *et al.* 1989
$s(3,6)$	15	Brewster *et al.* 1990
$s(3,7)$	18	Chen and Rousseau 1995, Cockayne *et al.* 1991
$s(4,4)$	13	Cockayne *et al.* 1992
$s(3,3,3)$	13	Cockayne and Mynhardt 1994

References

Brewster, R. C.; Cockayne, E. J.; and Mynhardt, C. M. "Irredundant Ramsey Numbers for Graphs." *J. Graph Theory* **13**, 283–290, 1989.

Brewster, R. C.; Cockayne, E. J.; and Mynhardt, C. M. "The Irredundant Ramsey Number $s(3,6)$." *Quaest. Math.* **13**, 141–157, 1990.

Chen, G. and Rousseau, C. C. "The Irredundant Ramsey Number $s(3,7)$." *J. Graph. Th.* **19**, 263–270, 1995.

Cockayne, E. J.; Exoo, G.; Hattingh, J. H.; and Mynhardt, C. M. "The Irredundant Ramsey Number $s(4,4)$." *Util. Math.* **41**, 119–128, 1992.

Cockayne, E. J.; Hattingh, J. H.; and Mynhardt, C. M. "The Irredundant Ramsey Number $s(3,7)$." *Util. Math.* **39**, 145–160, 1991.

Cockayne, E. J. and Mynhardt, C. M. "The Irredundant Ramsey Number $s(3,3,3) = 13$." *J. Graph. Th.* **18**, 595–604, 1994.

Hattingh, J. H. "On Irredundant Ramsey Numbers for Graphs." *J. Graph Th.* **14**, 437–441, 1990.

Mynhardt, C. M. "Irredundant Ramsey Numbers for Graphs: A Survey." *Congres. Numer.* **86**, 65–79, 1992.

Irreflexive

A RELATION R on a SET S is irreflexive provided that no element is related to itself; in other words, xRx for no x in S.

see also RELATION

Irregular Pair

If p divides the NUMERATOR of the BERNOULLI NUMBER B_{2k} for $0 < 2k < p-1$, then $(p, 2k)$ is called an irregular pair. For $p < 30000$, the irregular pairs of various forms are $p = 16843$ for $(p, p-3)$, $p = 37$ for $(p, p-5)$, none for $(p, p-7)$, and $p = 67,877$ for $(p, p-9)$.

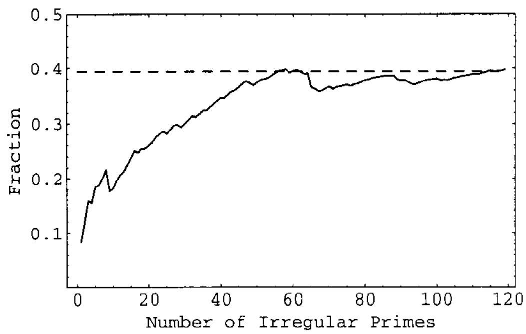
see also BERNOULLI NUMBER, IRREGULAR PRIME

References

Johnson, W. "Irregular Primes and Cyclotomic Invariants." *Math. Comput.* **29**, 113–120, 1975.

Irregular Prime

PRIMES for which Kummer's theorem on the unsolvability of FERMAT'S LAST THEOREM does not apply. An irregular prime p divides the NUMERATOR of one of the BERNOULLI NUMBERS B_{10}, B_{12}, ..., B_{2p-2}, as shown by Kummer in 1850. The FERMAT EQUATION has no solutions for REGULAR PRIMES.



An INFINITE number of irregular PRIMES exist, as proven in 1915 by Jensen. The first few irregular primes are 37, 59, 67, 101, 103, 131, 149, 157, ... (Sloane's A000928). Of the 283,145 PRIMES less than 4×10^6, 111,597 (or 39.41%) are regular. The conjectured FRACTION is $1 - e^{-1/2} \approx 39.35\%$ (Ribenboim 1996, p. 415).

see also BERNOULLI NUMBER, FERMAT'S LAST THEOREM, IRREGULAR PAIR, REGULAR PRIME

References

Buhler, J.; Crandall, R.; Ernvall, R.; and Metsänkylä, T. "Irregular Primes and Cyclotomic Invariants to Four Million." *Math. Comput.* **60**, 151–153, 1993.

Hardy, G. H. and Wright, E. M. *An Introduction to the Theory of Numbers, 5th ed.* Oxford, England: Clarendon Press, p. 202, 1979.

Johnson, W. "Irregular Primes and Cyclotomic Invariants." *Math. Comput.* **29**, 113–120, 1975.

Ribenboim, P. *The New Book of Prime Number Records.* New York: Springer-Verlag, pp. 325–329 and 414–425, 1996.

Sloane, N. J. A. Sequence A000928/M5260 in "An On-Line Version of the Encyclopedia of Integer Sequences."

Stewart, C. L. "A Note on the Fermat Equation." *Mathematika* **24**, 130–132, 1977.

Irregular Singularity

Consider a second-order ORDINARY DIFFERENTIAL EQUATION

$$y'' + P(x)y' + Q(x)y = 0.$$

If $P(x)$ and $Q(x)$ remain FINITE at $x = x_0$, then x_0 is called an ORDINARY POINT. If either $P(x)$ or $Q(x)$ diverges as $x \to x_0$, then x_0 is called a singular point. If $P(x)$ diverges more quickly than $1/(x - x_0)$, so $(x - x_0)P(x)$ approaches INFINITY as $x \to x_0$, or $Q(x)$ diverges more quickly than $1/(x - x_0)^2 Q$ so that $(x - x_0)^2 Q(x)$ goes to INFINITY as $x \to x_0$, then x_0 is called an IRREGULAR SINGULARITY (or ESSENTIAL SINGULARITY).

see also ORDINARY POINT, REGULAR SINGULAR POINT, SINGULAR POINT (DIFFERENTIAL EQUATION)

References

Arfken, G. "Singular Points." §8.4 in *Mathematical Methods for Physicists, 3rd ed.* Orlando, FL: Academic Press, pp. 451–453 and 461–463, 1985.

Irrotational Field

A VECTOR FIELD $\mathbf{v}$ for which the CURL vanishes,

$$\nabla \times \mathbf{v} = \mathbf{0}.$$

see also BELTRAMI FIELD, CONSERVATIVE FIELD, SOLENOIDAL FIELD, VECTOR FIELD

Isarithm

see EQUIPOTENTIAL CURVE

ISBN

Publisher	Digits
Addison-Wesley	0201
Amer. Math. Soc.	0821
Cambridge University Press	0521
CRC Press	0849
Dover	0486
McGraw-Hill	0070
Oxford University Press	0198
Springer-Verlag	0387
Wiley	0471

The International Standard Book Number (ISBN) is a 10-digit CODE which is used to identify a book uniquely. The first four digits specify the publisher, the next five digits the book, and the last digit d_{10} is a check digit which may be in the range 0–9 or X (where X equals 10). The check digit is computed from the equation

$$10d_1 + 9d_2 + 8d_3 + \ldots + 2d_9 + d_{10} \equiv 0 \pmod{11}.$$

For example, the number for this book is 0-8493-9640-9, and

$$\begin{aligned}&10 \cdot 0 + 9 \cdot 8 + 8 \cdot 4 + 7 \cdot 9 + 6 \cdot 3 + 5 \cdot 9 \\ &+4 \cdot 6 + 3 \cdot 4 + 2 \cdot 0 + 1 \cdot 9 = 275 = 25 \cdot 11 \equiv 0 \pmod{11},\end{aligned}$$

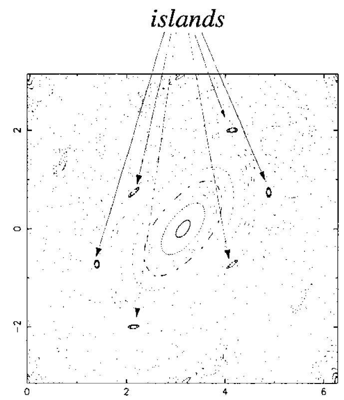
as required.

see also CODE

References

Press, W. H.; Flannery, B. P.; Teukolsky, S. A.; and Vetterling, W. T. *Numerical Recipes in FORTRAN: The Art of Scientific Computing, 2nd ed.* Cambridge, England: Cambridge University Press, p. 894, 1992.

Island



If an integrable QUASIPERIODIC system is slightly perturbed so that it becomes nonintegrable, only a finite number of n-CYCLES remain as a result of MODE LOCKING. One will be elliptical and one will be hyperbolic.

Surrounding the ELLIPTIC FIXED POINT is a region of stable ORBITS which circle it, as illustrated above in the STANDARD MAP with $K = 1.5$. As the map is iteratively applied, the island is mapped to a similar structure surrounding the next point of the elliptic cycle. The map thus has a chain of islands, with the FIXED POINT alternating between ELLIPTIC (at the center of the islands) and HYPERBOLIC (between islands). Because the unperturbed system goes through an INFINITY of rational values, the perturbed system must have an INFINITE number of island chains.

see also MODE LOCKING, ORBIT (MAP), QUASIPERIODIC FUNCTION

Isobaric Polynomial

A POLYNOMIAL in which the sum of SUBSCRIPTS is the same in each term.

see also HOMOGENEOUS POLYNOMIAL

Isochronous Curve

see SEMICUBICAL PARABOLA, TAUTOCHRONE PROBLEM

Isoclinal

see ISOCLINE

Isocline

A graphical method of solving an ORDINARY DIFFERENTIAL EQUATION of the form

$$\frac{dy}{dx} = f(x, y)$$

by plotting a series of curves $f(x, y) = [\text{const}]$, then drawing a curve PERPENDICULAR to each curve such that it satisfies the initial condition. This curve is the solution to the ORDINARY DIFFERENTIAL EQUATION.

References

Kármán, T. von and Biot, M. A. *Mathematical Methods in Engineering: An Introduction to the Mathematical Treatment of Engineering Problems.* New York: McGraw-Hill, pp. 3 and 7, 1940.

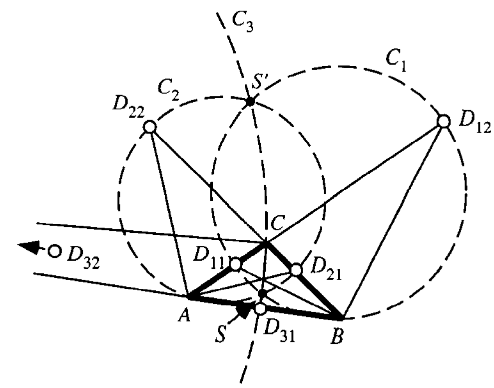
Isoclinic Groups

Two GROUPS G and H are said to be isoclinic if there are isomorphisms $G/Z(G) \to H/Z(H)$ and $G' \to H'$, where $Z(G)$ is the CENTER of the group, which identify the two commutator maps.

References

Conway, J. H.; Curtis, R. T.; Norton, S. P.; Parker, R. A.; and Wilson, R. A. "Isoclinism." §6.7 in *Atlas of Finite Groups: Maximal Subgroups and Ordinary Characters for Simple Groups.* Oxford, England: Clarendon Press, pp. xxiii–xxiv, 1985.

Isodynamic Points



The first and second isodynamic points of a TRIANGLE ΔABC can be constructed by drawing the triangle's ANGLE BISECTORS and EXTERIOR ANGLE BISECTORS. Each pair of bisectors intersects a side of the triangle (or its extension) in two points D_{i1} and D_{i2}, for $i = 1$, 2, 3. The three CIRCLES having $D_{11}D_{12}$, $D_{21}D_{22}$, and $D_{31}D_{32}$ as DIAMETERS are the APOLLONIUS CIRCLES C_1, C_2, and C_3. The points S and S' in which the three APOLLONIUS CIRCLES intersect are the first and second isodynamic points, respectively.

S and S' have TRIANGLE CENTER FUNCTIONS

$$\alpha = \sin(A \pm \tfrac{1}{3}\pi),$$

respectively. The ANTIPEDAL TRIANGLES of both points are EQUILATERAL and have AREAS

$$\Delta' = 2\Delta[\cot\omega \cot(\tfrac{1}{3}\pi)],$$

where ω is the BROCARD ANGLE.

The isodynamic points are ISOGONAL CONJUGATES of the ISOGONIC CENTERS. They lie on the BROCARD AXIS. The distances from either isodynamic point to the VERTICES are inversely proportional to the sides. The PEDAL TRIANGLE of either isodynamic point is an EQUILATERAL TRIANGLE. An INVERSION with either isodynamic point as the INVERSION CENTER transforms the triangle into an EQUILATERAL TRIANGLE.

The CIRCLE which passes through both the isodynamic points and the CENTROID of a TRIANGLE is known as the PARRY CIRCLE.

see also APOLLONIUS CIRCLES, BROCARD AXIS, CENTROID (TRIANGLE), ISOGONIC CENTERS, PARRY CIRCLE

References

Gallatly, W. *The Modern Geometry of the Triangle, 2nd ed.* London: Hodgson, p. 106, 1913.

Johnson, R. A. *Modern Geometry: An Elementary Treatise on the Geometry of the Triangle and the Circle.* Boston, MA: Houghton Mifflin, pp. 295–297, 1929.

Kimberling, C. "Central Points and Central Lines in the Plane of a Triangle." *Math. Mag.* **67**, 163–187, 1994.

Isoenergetic Nondegeneracy

The condition for isoenergetic nondegeneracy for a Hamiltonian

$$H = H_0(\mathbf{I}) + \epsilon H_1(\mathbf{I}, \boldsymbol{\theta})$$

is

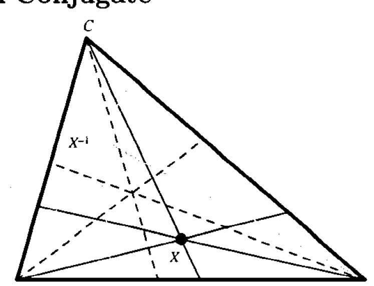
$$\begin{vmatrix} \frac{\partial^2 H_0}{\partial I_i \partial I_j} & \frac{\partial H_0}{\partial I_i} \\ \frac{\partial H_0}{\partial I_j} & 0 \end{vmatrix} \neq 0,$$

which guarantees the EXISTENCE on every energy level surface of a set of invariant tori whose complement has a small MEASURE.

References

Tabor, M. *Chaos and Integrability in Nonlinear Dynamics: An Introduction.* New York: Wiley, pp. 113–114, 1989.

Isogonal Conjugate



The isogonal conjugate X^{-1} of a point X in the plane of the TRIANGLE ΔABC is constructed by reflecting the lines AX, BX, and CX about the ANGLE BISECTORS at A, B, and C. The three reflected lines CONCUR at the isogonal conjugate. The TRILINEAR COORDINATES of the isogonal conjugate of the point with coordinates

$$\alpha : \beta : \gamma$$

are

$$\alpha^{-1} : \beta^{-1} : \gamma^{-1}.$$

Isogonal conjugation maps the interior of a TRIANGLE onto itself. This mapping transforms lines onto CONIC SECTIONS that CIRCUMSCRIBE the TRIANGLE. The type of CONIC SECTION is determined by whether the line d meets the CIRCUMCIRCLE C',

1. If d does not intersect C', the isogonal transform is an ELLIPSE;
2. If d is tangent to C', the transform is a PARABOLA;
3. If d cuts C', the transform is a HYPERBOLA, which is a RECTANGULAR HYPERBOLA if the line passes through the CIRCUMCENTER

(Casey 1893, Vandeghen 1965).

The isogonal conjugate of a point on the CIRCUMCIRCLE is a POINT AT INFINITY (and conversely). The sides of the PEDAL TRIANGLE of a point are PERPENDICULAR to the connectors of the corresponding VERTICES with the isogonal conjugate. The isogonal conjugate of a set of points is the LOCUS of their isogonal conjugate points.

The product of ISOTOMIC and isogonal conjugation is a COLLINEATION which transforms the sides of a TRIANGLE to themselves (Vandeghen 1965).

see also ANTIPEDAL TRIANGLE, COLLINEATION, ISOGONAL LINE, ISOTOMIC CONJUGATE POINT, LINE AT INFINITY, SYMMEDIAN LINE

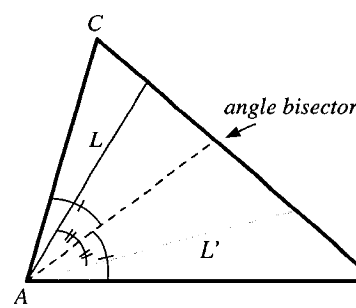
References

Casey, J. *A Treatise on the Analytical Geometry of the Point, Line, Circle, and Conic Sections, Containing an Account of Its Most Recent Extensions with Numerous Examples, 2nd rev. enl. ed.* Dublin: Hodges, Figgis, & Co., 1893.

Johnson, R. A. *Modern Geometry: An Elementary Treatise on the Geometry of the Triangle and the Circle.* Boston, MA: Houghton Mifflin, pp. 153–158, 1929.

Vandeghen, A. "Some Remarks on the Isogonal and Cevian Transforms. Alignments of Remarkable Points of a Triangle." *Amer. Math. Monthly* **72**, 1091–1094, 1965.

Isogonal Line



The line L' through a TRIANGLE VERTEX obtained by reflecting an initial line L (also through a VERTEX) about the ANGLE BISECTOR. If three lines from the VERTICES of a TRIANGLE ΔABC are CONCURRENT at $X = L_1L_2L_3$, then their isogonal lines are also CONCURRENT, and the point of concurrence $X' = L_1'L_2'L_3'$ is called the ISOGONAL CONJUGATE point.

see also ISOGONAL CONJUGATE

References

Johnson, R. A. *Modern Geometry: An Elementary Treatise on the Geometry of the Triangle and the Circle.* Boston, MA: Houghton Mifflin, pp. 153–157, 1929.

Isogonic Centers

The first isogonic center F_1 of a TRIANGLE is the FERMAT POINT. The second isogonic center F_2 is constructed analogously with the first isogonic center except that for F_2, the EQUILATERAL TRIANGLES are constructed on the same side of the opposite VERTEX. The second isogonic center has TRIANGLE CENTER FUNCTION

$$\alpha = \csc(A - \tfrac{1}{3}\pi).$$

Its ANTIPEDAL TRIANGLE is EQUILATERAL and has AREA

$$2\Delta = [-1 + \cot\omega\cot(\tfrac{1}{3}\pi)],$$

where ω is the BROCARD ANGLE.

The first and second isogonic centers are ISOGONAL CONJUGATES of the ISODYNAMIC POINTS.

see also BROCARD ANGLE, EQUILATERAL TRIANGLE, FERMAT POINT, ISODYNAMIC POINTS, ISOGONAL CONJUGATE

References

Gallatly, W. *The Modern Geometry of the Triangle, 2nd ed.* London: Hodgson, p. 107, 1913.

Kimberling, C. "Central Points and Central Lines in the Plane of a Triangle." *Math. Mag.* **67**, 163–187, 1994.

Isograph

The substitution of $re^{i\theta}$ for z in a POLYNOMIAL $p(z)$. $p(z)$ is then plotted as a function of θ for a given r in the COMPLEX PLANE. By varying r so that the curve passes through the ORIGIN, it is possible to determine a value for one ROOT of the POLYNOMIAL.

Isohedral Tiling

Let $S(T)$ be the group of symmetries which map a MONOHEDRAL TILING T onto itself. The TRANSITIVITY CLASS of a given tile T is then the collection of all tiles to which T can be mapped by one of the symmetries of $S(T)$. If T has k TRANSITIVITY CLASSES, then T is said to be k-isohedral. Berglund (1993) gives examples of k-isohedral tilings for $k = 1$, 2, and 4.

see also ANISOHEDRAL TILING

References

Berglund, J. "Is There a k-Anisohedral Tile for $k \geq 5$?" *Amer. Math. Monthly* **100**, 585-588, 1993.

Grünbaum, B. and Shephard, G. C. "The 81 Types of Isohedral Tilings of the Plane." *Math. Proc. Cambridge Philos. Soc.* **82**, 177-196, 1977.

Isohedron

A convex POLYHEDRON with symmetries acting transitively on its faces. Every isohedron has an EVEN number of faces (Grünbaum 1960).

References

Grünbaum, B. "On Polyhedra in E^3 Having All Faces Congruent." *Bull. Research Council Israel* **8F**, 215–218, 1960.

Grünbaum, B. and Shepard, G. C. "Spherical Tilings with Transitivity Properties." In *The Geometric Vein: The Coxeter Festschrift* (Ed. C. Davis, B. Grünbaum, and F. Shenk). New York: Springer-Verlag, 1982.

Isolated Point

A point on a curve, also known as an ACNODE or HERMIT POINT, which has no other points in its NEIGHBORHOOD.

Isolated Singularity

An isolated singularity is a SINGULARITY for which there exists a (small) REAL NUMBER ϵ such that there are no other SINGULARITIES within a NEIGHBORHOOD of radius ϵ centered about the SINGULARITY.

The types of isolated singularities possible for CUBIC SURFACES have been classified (Schläfli 1864, Cayley 1869, Bruce and Wall 1979) and are summarized in the following table from Fischer (1986).

Double Pt. Name	Symbol	Normal Form	Coxeter Diagram
conic	C_2	$x^2+y^2+z^2$	A_1
biplanar	B_3	$x^2+y^2+z^3$	A_2
biplanar	B_4	$x^2+y^2+z^4$	A_3
biplanar	B_5	$x^2+y^2+z^5$	A_4
biplanar	B_6	$x^2+y^2+z^6$	A_5
uniplanar	U_6	$x^2+z(y^2+z^2)$	D_4
uniplanar	U_7	$x^2+z(y^2+z^3)$	D_5
uniplanar	U_8	$x^2+y^3+z^4$	E_6
elliptic cone pt.	—	xy^2-4z^3 $-g_2x^2y+g_3x^3$	$\tilde{E}_6$

see also CUBIC SURFACE, RATIONAL DOUBLE POINT, SINGULARITY

References

Bruce, J. and Wall, C. T. C. "On the Classification of Cubic Surfaces." *J. London Math. Soc.* **19**, 245–256, 1979.

Cayley, A. "A Memoir on Cubic Surfaces." *Phil. Trans. Roy. Soc.* **159**, 231–326, 1869.

Fischer, G. (Ed.). *Mathematical Models from the Collections of Universities and Museums.* Braunschweig, Germany: Vieweg, pp. 12–13, 1986.

Morse, P. M. and Feshbach, H. *Methods of Theoretical Physics, Part I.* New York: McGraw-Hill, pp. 380–381, 1953.

Schläfli, L. "On the Distribution of Surfaces of Third Order into Species." *Phil. Trans. Roy. Soc.* **153**, 193–247, 1864.

Isolating Integral

An integral of motion which restricts the PHASE SPACE available to a DYNAMICAL SYSTEM.

Isometry

A BIJECTIVE MAP between two METRIC SPACES that preserves distances, i.e.,

$$d(f(x), f(y)) = d(x, y),$$

where f is the MAP and $d(a, b)$ is the DISTANCE function.

An isometry of the PLANE is a linear transformation which preserves length. Isometries include ROTATION, TRANSLATION, REFLECTION, GLIDES, and the IDENTITY MAP. If an isometry has more than one FIXED

POINT, it must be either the identity transformation or a reflection. Every isometry of period two (two applications of the transformation preserving lengths in the original configuration) is either a reflection or a half turn rotation. Every isometry in the plane is the product of at most three reflections (at most two if there is a FIXED POINT). Every finite group of isometries has at least one FIXED POINT.

see also DISTANCE, EUCLIDEAN MOTION, HJELMSLEV'S THEOREM, LENGTH (CURVE), REFLECTION, ROTATION, TRANSLATION

References

Gray, A. "Isometries of Surfaces." §13.2 in *Modern Differential Geometry of Curves and Surfaces.* Boca Raton, FL: CRC Press, pp. 255–258, 1993.

Isometric Latitude

An AUXILIARY LATITUDE which is directly proportional to the spacing of parallels of LATITUDE from the equator on an ellipsoidal MERCATOR PROJECTION. It is defined by

$$\psi = \ln \left| \tan(\tfrac{1}{4}\pi + \tfrac{1}{2}\phi) \left(\frac{1 - e\sin\phi}{1 + e\sin\phi} \right)^{e/2} \right|, \qquad (1)$$

where the symbol τ is sometimes used instead of ψ. The isometric latitude is related to the CONFORMAL LATITUDE by

$$\psi = \ln\tan(\tfrac{1}{4}\pi + \tfrac{1}{2}\chi). \qquad (2)$$

The inverse is found by iterating

$$\phi = 2\tan^{-1}\left[\exp(\psi) \left(\frac{1 + e\sin\phi}{1 - e\sin\phi} \right)^{e/2} \right] - \tfrac{1}{2}\pi, \qquad (3)$$

with the first trial as

$$\phi_0 = 2\tan^{-1}(e^{\psi}) - \tfrac{1}{2}\pi. \qquad (4)$$

see also LATITUDE

References

Adams, O. S. "Latitude Developments Connected with Geodesy and Cartography with Tables, Including a Table for Lambert Equal-Area Meridional Projections." Spec. Pub. No. 67. U. S. Coast and Geodetic Survey, 1921.

Snyder, J. P. *Map Projections—A Working Manual.* U. S. Geological Survey Professional Paper 1395. Washington, DC: U. S. Government Printing Office, p. 15, 1987.

Isomorphic Graphs

Two GRAPHS which contain the same number of VERTICES connected in the same way are said to be isomorphic. Formally, two graphs G and H with VERTICES $V_n = \{1, 2, \ldots, n\}$ are said to be isomorphic if there is a PERMUTATION p of V_n such that $\{u, v\}$ is in the set of EDGES $E(G)$ IFF $\{p(u), p(v)\}$ is in the set of EDGES $E(H)$.

References

Chartrand, G. "Isomorphic Graphs." §2.2 in *Introductory Graph Theory.* New York: Dover, pp. 32–40, 1985.

Isomorphic Groups

Two GROUPS are isomorphic if the correspondence between them is ONE-TO-ONE and the "multiplication" table is preserved. For example, the POINT GROUPS C_2 and D_1 are isomorphic GROUPS, written $C_2 \cong D_1$ or $C_2 \rightleftharpoons D_1$ (Shanks 1993). Note that the symbol $\cong$ is also used to denote geometric CONGRUENCE.

References

Shanks, D. *Solved and Unsolved Problems in Number Theory, 4th ed.* New York: Chelsea, 1993.

Isomorphic Posets

Two POSETS are said to be isomorphic if their "structures" are entirely analogous. Formally, POSETS $P = (X, \leq)$ and $Q = (X', \leq')$ are isomorphic if there is a BIJECTION f from X to X' such that $x \leq x'$ precisely when $f(x) \leq' f(x')$.

Isomorphism

Isomorphism is a very general concept which appears in several areas of mathematics. Formally, an isomorphism is BIJECTIVE MORPHISM. Informally, an isomorphism is a map which preserves sets and relations among elements.

A space isomorphism is a VECTOR SPACE in which addition and scalar multiplication are preserved. An isomorphism of a TOPOLOGICAL SPACE is called a HOMEOMORPHISM.

Two groups G_1 and G_2 with binary operators $+$ and $\times$ are isomorphic if there exists a map $f : G_1 \mapsto G_2$ which satisfies

$$f(x + y) = f(x) \times f(y).$$

An isomorphism preserves the identities and inverses of a GROUP. A GROUP which is isomorphic to itself is called an AUTOMORPHISM.

see also AUTOMORPHISM, AX-KOCHEN ISOMORPHISM THEOREM, HOMEOMORPHISM, MORPHISM

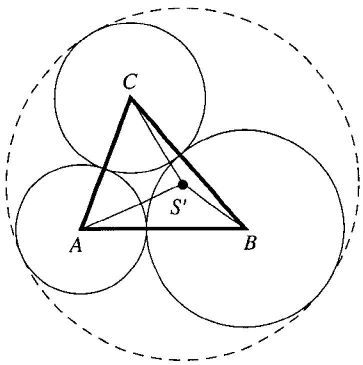
Isoperimetric Inequality

Let a PLANE figure have AREA A and PERIMETER p. Let the CIRCLE of PERIMETER p have RADIUS r. Then

$$\frac{4\pi A}{p^2} \leq 1,$$

where the quantity on the left is known as the ISOPERIMETRIC QUOTIENT.

Isoperimetric Point



The point S' which makes the PERIMETERS of the TRIANGLES $\Delta BS'C$, $\Delta CS'A$, and $\Delta AS'B$ equal. The isoperimetric point exists IFF the largest ANGLE of the triangle satisfies

$$\max(A, B, C) < 2\sin^{-1}(\tfrac{4}{5}) \approx 1.85459 \text{ rad} \approx 106.26°,$$

or equivalently

$$a + b + c > 4R + r,$$

where a, b, and c are the side lengths of ΔABC, r is the INRADIUS, and R is the CIRCUMRADIUS. The isoperimetric point is also the center of the outer SODDY CIRCLE of ΔABC and has TRIANGLE CENTER FUNCTION

$$\alpha = 1 - \frac{2\Delta}{a(b+c-a)} = \sec(\tfrac{1}{2}A)\cos(\tfrac{1}{2}B)\cos(\tfrac{1}{2}C) - 1.$$

see also EQUAL DETOUR POINT, PERIMETER, SODDY CIRCLES

References
Kimberling, C. "Central Points and Central Lines in the Plane of a Triangle." *Math. Mag.* **67**, 163–187, 1994.
Kimberling, C. "Isoperimetric Point and Equal Detour Point." `http://www.evansville.edu/~ck6/tcenters/recent/isoper.html`.
Kimberling, C. and Wagner, R. W. "Problem E 3020 and Solution." *Amer. Math. Monthly* **93**, 650–652, 1986.
Veldkamp, G. R. "The Isoperimetric Point and the Point(s) of Equal Detour." *Amer. Math. Monthly* **92**, 546–558, 1985.

Isoperimetric Problem

Find a closed plane curve of a given length which encloses the greatest AREA. The solution is a CIRCLE. If the class of curves to be considered is limited to smooth curves, the isoperimetric problem can be stated symbolically as follows: find an arc with parametric equations $x = x(t)$, $y = y(t)$ for $t \in [t_1, t_2]$ such that $x(t_1) = x(t_2)$, $y(t_1) = y(t_2)$ (where no further intersections occur) constrained by

$$l = \int_{t_1}^{t_2} \sqrt{x'^2 + y'^2}\, dt$$

such that

$$A = \tfrac{1}{2}\int_{t_1}^{t_2} (xy' - x'y)\, dt$$

is a MAXIMUM.

see also DIDO'S PROBLEM, ISOVOLUME PROBLEM

References
Bogomolny, A. "Isoperimetric Theorem and Inequality." `http://www.cut-the-knot.com/do_you_know/isoperimetric.html`.
Isenberg, C. Appendix V in *The Science of Soap Films and Soap Bubbles.* New York: Dover, 1992.

Isoperimetric Quotient

A quantity defined in the ISOPERIMETRIC INEQUALITY

$$Q \equiv \frac{4\pi A}{p^2}.$$

see also ISOPERIMETRIC INEQUALITY

Isoperimetric Theorem

Of all convex n-gons of a given PERIMETER, the one which maximizes AREA is the regular n-gon.

see also ISOPERIMETRIC INEQUALITY, ISOPERIMETRIC PROBLEM

Isopleth

see EQUIPOTENTIAL CURVE

Isoptic Curve

For a given curve C, consider the locus of the point P from where the TANGENTS from P to C meet at a fixed given ANGLE. This is called an isoptic curve of the given curve.

Curve	Isoptic
cycloid	curtate or prolate cycloid
epicycloid	epitrochoid
hypocycloid	hypotrochoid
parabola	hyperbola
sinusoidal spiral	sinusoidal spiral

see also ORTHOPTIC CURVE

References
Lawrence, J. D. *A Catalog of Special Plane Curves.* New York: Dover, pp. 58–59 and 206, 1972.
Yates, R. C. "Isoptic Curves." *A Handbook on Curves and Their Properties.* Ann Arbor, MI: J. W. Edwards, pp. 138–140, 1952.

Isosceles Tetrahedron

A nonregular TETRAHEDRON in which each pair of opposite EDGES are equal such that all triangular faces are congruent. A TETRAHEDRON is isosceles IFF the sum of the face angles at each VERTEX is 180°, and IFF its INSPHERE and CIRCUMSPHERE are concentric.

The only way for all the faces of a TETRAHEDRON to have the same PERIMETER or to have the same AREA is for them to be fully congruent, in which case the tetrahedron is isosceles.

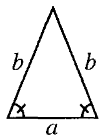
see also CIRCUMSPHERE, INSPHERE, ISOSCELES TRIANGLE, TETRAHEDRON

References

Brown, B. H. "Theorem of Bang. Isosceles Tetrahedra." *Amer. Math. Monthly* **33**, 224–226, 1926.

Honsberger, R. "A Theorem of Bang and the Isosceles Tetrahedron." Ch. 9 in *Mathematical Gems II*. Washington, DC: Math. Assoc. Amer., pp. 90–97, 1976.

Isosceles Triangle

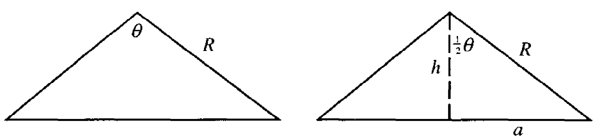


A TRIANGLE with two equal sides (and two equal ANGLES). The name derives from the Greek *iso* (same) and *skelos* (LEG). The height of the above isosceles triangle can be found from the PYTHAGOREAN THEOREM as

$$h = \sqrt{b^2 - \tfrac{1}{4}a^2}. \tag{1}$$

The AREA is therefore given by

$$A = \tfrac{1}{2}ah = \tfrac{1}{2}a\sqrt{b^2 - \tfrac{1}{4}a^2}. \tag{2}$$



There is a surprisingly simple relationship between the AREA and VERTEX ANGLE θ. As shown in the above diagram, simple TRIGONOMETRY gives

$$h = R\cos(\tfrac{1}{2}\theta) \tag{3}$$
$$a = R\sin(\tfrac{1}{2}\theta), \tag{4}$$

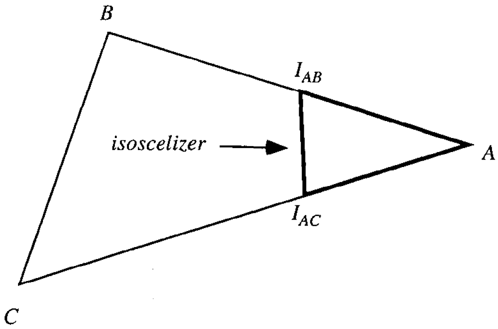
so the AREA is

$$A = \tfrac{1}{2}(2a)h = ah = R^2\cos(\tfrac{1}{2}\theta)\sin(\tfrac{1}{2}\theta) = \tfrac{1}{2}R^2\sin\theta. \tag{5}$$

No set of $n > 6$ points in the PLANE can determine only ISOSCELES TRIANGLES.

see also ACUTE TRIANGLE, EQUILATERAL TRIANGLE, INTERNAL BISECTORS PROBLEM, ISOSCELES TETRAHEDRON, ISOSCELIZER, OBTUSE TRIANGLE, POINT PICKING, PONS ASINORUM, RIGHT TRIANGLE, SCALENE TRIANGLE, STEINER-LEHMUS THEOREM

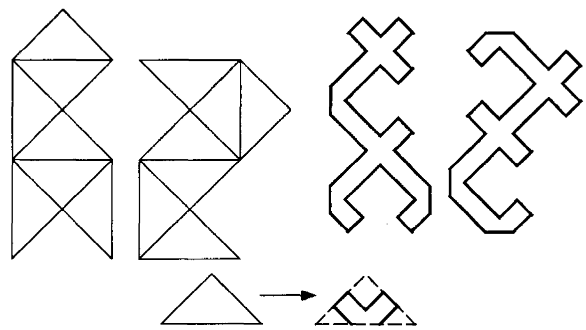
Isoscelizer



An isoscelizer of an ANGLE A in a TRIANGLE ΔABC is a LINE SEGMENT $I_{AB}I_{AC}$ where I_{AB} lies on AB and I_{AC} on AC such that $\Delta AI_{AB}I_{AC}$ is an ISOSCELES TRIANGLE.

see also CONGRUENT ISOSCELIZERS POINT, ISOSCELES TRIANGLE, YFF CENTER OF CONGRUENCE

Isospectral Manifolds



DRUMS that sound the same, i.e., have the same eigenfrequency spectrum. Two drums with differing AREA, PERIMETER, or GENUS can always be distinguished. However, Kac (1966) asked if it was possible to construct differently shaped drums which have the same eigenfrequency spectrum. This question was answered in the affirmative by Gordon *et al.* (1992). Two such isospectral manifolds are shown in the right figure above (Cipra 1992).

References

Chapman, S. J. "Drums That Sound the Same." *Amer. Math. Monthly* **102**, 124–138, 1995.

Cipra, B. "You Can't Hear the Shape of a Drum." *Science* **255**, 1642–1643, 1992.

Gordon, C.; Webb, D.; and Wolpert, S. "Isospectral Plane Domains and Surfaces via Riemannian Orbifolds." *Invent. Math.* **110**, 1–22, 1992.

Gordon, C.; Webb, D.; and Wolpert, S. "You Cannot Hear the Shape of a Drum." *Bull. Amer. Math. Soc.* **27**, 134–138, 1992.

Kac, M. "Can One Hear the Shape of a Drum?" *Amer. Math. Monthly* **73**, 1–23, 1966.

Isothermal Parameterization

A parameterization is isothermal if, for $\zeta \equiv u + iv$ and

$$\phi_k(\zeta) = \frac{\partial x_k}{\partial u} - i\frac{\partial x_k}{\partial v},$$

the identity

$$\phi_1{}^2(\zeta) + \phi_2{}^2(\zeta) + \phi_3{}^2(\zeta) = 0$$

holds.

see also MINIMAL SURFACE, TEMPERATURE

Isotomic Conjugate Point

The point of concurrence Q of the ISOTOMIC LINES relative to a point P. The isotomic conjugate $\alpha' : \beta' : \gamma'$ of a point with TRILINEAR COORDINATES $\alpha : \beta : \gamma$ is

$$(a^2\alpha)^{-1} : (b^2\beta)^{-1} : (c^2\gamma)^{-1}. \quad (1)$$

The isotomic conjugate of a LINE d having trilinear equation

$$l\alpha + m\beta + n\gamma \quad (2)$$

is a CONIC SECTION circumscribed on the TRIANGLE ΔABC (Casey 1893, Vandeghen 1965). The isotomic conjugate of the LINE AT INFINITY having trilinear equation

$$a\alpha + b\beta + c\gamma = 0 \quad (3)$$

is STEINER'S ELLIPSE

$$\frac{\beta'\gamma'}{a} + \frac{\gamma'\alpha'}{b} + \frac{\alpha'\beta'}{c} = 0 \quad (4)$$

(Vandeghen 1965). The type of CONIC SECTION to which d is transformed is determined by whether the line d meets STEINER'S ELLIPSE E.

1. If d does not intersect E, the isotomic transform is an ELLIPSE.
2. If d is tangent to E, the transform is a PARABOLA.
3. If d cuts E, the transform is a HYPERBOLA, which is a RECTANGULAR HYPERBOLA if the line passes through the isotomic conjugate of the ORTHOCENTER

(Casey 1893, Vandeghen 1965).

There are four points which are isotomically self-conjugate: the CENTROID M and each of the points of intersection of lines through the VERTICES PARALLEL to the opposite sides. The isotomic conjugate of the EULER LINE is called JERABEK'S HYPERBOLA (Casey 1893, Vandeghen 1965).

Vandeghen (1965) calls the transformation taking points to their isotomic conjugate points the CEVIAN TRANSFORM. The product of isotomic and ISOGONAL is a COLLINEATION which transforms the sides of a TRIANGLE to themselves (Vandeghen 1965).

see also CEVIAN TRANSFORM, GERGONNE POINT, ISOGONAL CONJUGATE, JERABEK'S HYPERBOLA, NAGEL POINT, STEINER'S ELLIPSE

References

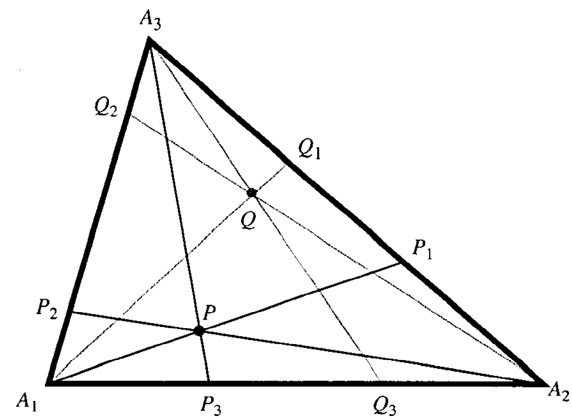
Casey, J. *A Treatise on the Analytical Geometry of the Point, Line, Circle, and Conic Sections, Containing an Account of Its Most Recent Extensions with Numerous Examples, 2nd rev. enl. ed.* Dublin: Hodges, Figgis, & Co., 1893.

Eddy, R. H. and Fritsch, R. "The Conics of Ludwig Kiepert: A Comprehensive Lesson in the Geometry of the Triangle." *Math. Mag.* **67**, 188–205, 1994.

Johnson, R. A. *Modern Geometry: An Elementary Treatise on the Geometry of the Triangle and the Circle.* Boston, MA: Houghton Mifflin, pp. 157–159, 1929.

Vandeghen, A. "Some Remarks on the Isogonal and Cevian Transforms. Alignments of Remarkable Points of a Triangle." *Amer. Math. Monthly* **72**, 1091–1094, 1965.

Isotomic Lines



Given a point P in the interior of a TRIANGLE $\Delta A_1A_2A_3$, draw the CEVIANS through P from each VERTEX which meet the opposite sides at P_1, P_2, and P_3. Now, mark off point Q_1 along side A_2A_3 such that $A_3P_1 = A_2Q_1$, etc., i.e., so that Q_i and P_i are equidistance from the MIDPOINT of A_jA_k. The lines A_1Q_1, A_2Q_2, and A_3Q_3 then coincide in a point Q known as the ISOTOMIC CONJUGATE POINT.

see also CEVIAN, ISOTOMIC CONJUGATE POINT, MIDPOINT

Isotone Map

A MAP which is monotone increasing and therefore order-preserving.

Isotope

To rearrange without cutting or pasting.

Isotopy

A HOMOTOPY from one embedding of a MANIFOLD M in N to another such that at every time, it is an embedding. The notion of isotopy is category independent, so notions of topological, piecewise-linear, smooth, isotopy (and so on) exist. When no explicit mention is made, "isotopy" usually means "smooth isotopy."

see also AMBIENT ISOTOPY, REGULAR ISOTOPY

Isotropic Tensor

A TENSOR which has the same components in all rotated coordinate systems.

rank	isotropic tensors
0	all
1	none
2	Kronecker delta
3	1
4	3

Isovolume Problem

Find the surface enclosing the maximum volume per unit surface AREA $I \equiv V/S$. The solution is a SPHERE, which has

$$I_{\text{sphere}} = \frac{\frac{4}{3}\pi r^3}{4\pi r^2} = \frac{1}{3}r.$$

see also DIDO'S PROBLEM, ISOPERIMETRIC PROBLEM

References

Bogomolny, A. "Isoperimetric Theorem and Inequality." http://www.cut-the-knot.com/do_you_know/isoperimetric.html.

Isenberg, C. Appendix VI in *The Science of Soap Films and Soap Bubbles.* New York: Dover, 1992.

Isthmus

see BRIDGE (GRAPH)

Iterated Exponential Constants

N.B. A detailed on-line essay by S. Finch was the starting point for this entry.

Euler (Le Lionnais 1983) and Eisenstein (1844) showed that the function $h(x) = x^{x^{x^{\cdot^{\cdot^{\cdot}}}}}$, where x^{x^x} is an abbreviation for $x^{(x^x)}$, converges only for $e^{-e} \leq x \leq e^{1/e}$, that is, $0.0659\ldots \leq x \leq 1.44466\ldots$. The value it converges to is the inverse of $x^{1/x}$, which has a closed form expression in terms of LAMBERT'S W-FUNCTION,

$$h(z) = \frac{W(-\ln z)}{-\ln z} \tag{1}$$

(Corless *et al.*). Knoebel (1981) gives

$$h(z) = 1 + \ln x + \frac{3^2(\ln z)^2}{3!} + \frac{4^3(\ln z)^3}{4!} + \ldots \tag{2}$$

(Vardi 1991). A CONTINUED FRACTION due to Khovanskii (1963) is

$$x^{1/x} = 1 + \cfrac{2(x-1)}{x^2+1-\cfrac{(x^2-1)(x-1)^2}{3x(x+1)-\cfrac{(4x^2-1)(x-1)^2}{5x(x+1)-\cfrac{(9x^2-1)(x-1)^2}{7x(x+1)-\ldots}}}}. \tag{3}$$

The function $g(x) = x^{(1/x)^{(1/x)^{\cdot^{\cdot^{\cdot}}}}}$ converges only for $e^{-1/e} \leq x \leq e^e$, that is, $0.692\ldots \leq x \leq 15.154\ldots$. The value it converges to is the inverse of x^x.

Some interesting related integrals are

$$\int_0^1 x^x\,dx = \sum_{n=1}^\infty \frac{(-1)^{n+1}}{n^n} = 0.7834305107\ldots \tag{4}$$

$$\int_0^1 x^{-x}\,dx = \sum_{n=1}^\infty \frac{1}{n^n} = 1.2912859971\ldots \tag{5}$$

(Spiegel 1968, Abramowitz and Stegun 1972).

see also LAMBERT'S W-FUNCTION

References

Abramowitz, M. and Stegun, C. A. (Eds.). *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables, 9th printing.* New York: Dover, 1972.

Baker, I. N. and Rippon, P. J. "A Note on Complex Iteration." *Amer. Math. Monthly* **92**, 501–504, 1985.

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de Villiers, J. M. and Robinson, P. N. "The Interval of Convergence and Limiting Functions of a Hyperpower Sequence." *Amer. Math. Monthly* **93**, 13–23, 1986.

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Knoebel, R. A. "Exponentials Reiterated." *Amer. Math. Monthly* **88**, 235–252, 1981.

Le Lionnais, F. *Les nombres remarquables.* Paris: Hermann, pp. 22 and 39, 1983.

Mauerer, H. "Über die Funktion $x^{x^{\cdot^{\cdot^{\cdot}}}}$ für ganzzahliges Argument (Abundanzen)." *Mitt. Math. Gesell. Hamburg* **4**, 33–50, 1901.

Spiegel, M. R. *Mathematical Handbook of Formulas and Tables.* New York: McGraw-Hill, 1968.

Vardi, I. *Computational Recreations in Mathematica.* Reading, MA: Addison-Wesley, p. 12, 1991.

Iterated Function System

A finite set of contraction maps w_i for $i = 1, 2, \ldots, N$, each with a contractivity factor $s < 1$, which map a compact METRIC SPACE onto itself. It is the basis for FRACTAL image compression techniques.

see also BARNSLEY'S FERN, SELF-SIMILARITY

References
Barnsley, M. F. "Fractal Image Compression." *Not. Amer. Math. Soc.* **43**, 657–662, 1996.
Barnsley, M. *Fractals Everywhere, 2nd ed.* Boston, MA: Academic Press, 1993.
Barnsley, M. F. and Demko, S. G. "Iterated Function Systems and the Global Construction of Fractals." *Proc. Roy. Soc. London, Ser. A* **399**, 243–275, 1985.
Barnsley, M. F. and Hurd, L. P. *Fractal Image Compression.* Wellesley, MA: A. K. Peters, 1993.
Diaconis, P. M. and Shashahani, M. "Products of Random Matrices and Computer Image Generation." *Contemp. Math.* **50**, 173–182, 1986.
Fisher, Y. *Fractal Image Compression.* New York: Springer-Verlag, 1995.
Hutchinson, J. "Fractals and Self-Similarity." *Indiana Univ. J. Math.* **30**, 713–747, 1981.
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Iterated Radical

see NESTED RADICAL

Iteration Sequence

A SEQUENCE $\{a_j\}$ of POSITIVE INTEGERS is called an iteration sequence if there EXISTS a strictly increasing sequence $\{s_k\}$ of POSITIVE INTEGERS such that $a_1 = s_1 \geq 2$ and $a_j = s_{a_{j-1}}$ for $j = 2, 3, \ldots$. A NECESSARY and SUFFICIENT condition for $\{a_j\}$ to be an iteration sequence is

$$a_j \geq 2a_{j-1} - a_{j-2}$$

for all $j \geq 3$.

References
Kimberling, C. "Interspersions and Dispersions." *Proc. Amer. Math. Soc.* **117**, 313–321, 1993.

Itô's Lemma

$$V_t - V_0 = \int_0^t f_x(S_u, T-u)\,dS_u - \int_0^t f_\tau(S_u, T-u)\,du + \tfrac{1}{2}\sigma^2 \int_0^t {S_u}^2 f_{xx}(S_u, T-u)\,du,$$

where $V_t = f(S_t, \tau)$ for $0 \leq \tau \equiv T - t \leq T$, and $f \in C^{2,1}((0,\infty) \times [0,T])$.

References
Price, J. F. "Optional Mathematics is Not Optional." *Not. Amer. Math. Soc.* **43**, 964–971, 1996.

Itô's Theorem

The dimension d of any IRREDUCIBLE REPRESENTATION of a GROUP G must be a DIVISOR of the index of each maximal normal Abelian SUBGROUP of G.

see also ABELIAN GROUP, IRREDUCIBLE REPRESENTATION, SUBGROUP

References
Lomont, J. S. *Applications of Finite Groups.* New York: Dover, p. 55, 1993.

Iverson Bracket

Let S be a mathematical statement, then the Iverson bracket is defined by

$$[S] \equiv \begin{cases} 0 & \text{if } S \text{ is true} \\ 1 & \text{if } S \text{ is false.} \end{cases}$$

This notation conflicts with the brackets sometimes used to denote the FLOOR FUNCTION. For this reason, and because of the elegant symmetry of the FLOOR FUNCTION and CEILING FUNCTION symbols $\lfloor x \rfloor$ and $\lceil x \rceil$, the use of $[x]$ to denote the FLOOR FUNCTION should be deprecated.

see also CEILING FUNCTION, FLOOR FUNCTION

References
Graham, R. L.; Knuth, D. E.; and Patashnik, O. *Concrete Mathematics: A Foundation for Computer Science.* Reading, MA: Addison-Wesley, p. 24, 1990.
Iverson, K. E. *A Programming Language.* New York: Wiley, p. 11, 1962.

Iwasawa's Theorem

Every finite-dimensional LIE ALGEBRA of characteristic $p \neq 0$ has a faithful finite-dimensional representation.

References
Jacobson, N. *Lie Algebras.* New York: Dover, pp. 204–205, 1979.

J

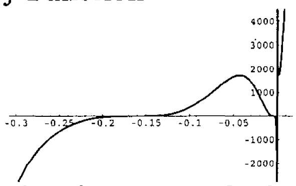
j

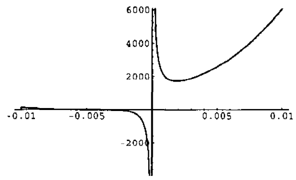
The symbol used by engineers and some physicists to denote i, the IMAGINARY NUMBER $\sqrt{-1}$.

j-Conductor

see FREY CURVE

j-Function





The j-function is defined as

$$j(q) \equiv 1728 J(\sqrt{q}), \tag{1}$$

where

$$J(q) \equiv \frac{4}{27} \frac{[1-\lambda(q)+\lambda^2(q)]^3}{\lambda^2(q)[1-\lambda(q)]^2} \tag{2}$$

is KLEIN'S ABSOLUTE INVARIANT, $\lambda(q)$ the ELLIPTIC LAMBDA FUNCTION

$$\lambda(q) \equiv k^2(q) = \left[\frac{\vartheta_2(q)}{\vartheta_3(q)}\right]^4, \tag{3}$$

and ϑ_i a THETA FUNCTION. This function can also be specified in terms of the WEBER FUNCTIONS f, f_1, f_2, γ_2, and γ_3 as

$$\begin{aligned} j(z) &= \frac{[f^{24}(z)-16]^3}{f^{24}(z)} && (4)\\ &= \frac{[{f_1}^{24}(z)+16]^3}{{f_1}^{24}(z)} && (5)\\ &= \frac{[{f_2}^{24}(z)+16]^3}{{f_2}^{24}(z)} && (6)\\ &= {\gamma_2}^3(z) && (7)\\ &= {\gamma_3}^2(z)+1728 && (8) \end{aligned}$$

(Weber 1902, p. 179; Atkin and Morain 1993).

The j-function is MEROMORPHIC function on the upper half of the COMPLEX PLANE which is invariant with respect to the SPECIAL LINEAR GROUP $SL(2,Z)$. It has a FOURIER SERIES

$$j(q) = \sum_{n=-\infty}^{\infty} a_n q^n, \tag{9}$$

for the NOME

$$q \equiv e^{2\pi i t} \tag{10}$$

with $\Im[t] > 0$. The coefficients in the expansion of the j-function satisfy:

1. $a_n = 0$ for $n < -1$ and $a_{-1} = 1$,
2. all a_ns are INTEGERS with fairly limited growth with respect to n, and
3. $j(q)$ is an ALGEBRAIC NUMBER, sometimes a RATIONAL NUMBER, and sometimes even an INTEGER at certain very special values of q (or t).

The latter result is the end result of the massive and beautiful theory of COMPLEX multiplication and the first step of Kronecker's so-called "JUGENDTRAUM."

Then all of the COEFFICIENTS in LAURENT SERIES

$$\begin{aligned} j(q) &= \frac{1}{q} + 744 + 196884q + 21494760q^2 \\ &+864299970q^3+20245856256q^4+333202640600q^5+\ldots \end{aligned} \tag{11}$$

(Sloane's A000521) are POSITIVE INTEGERS (Rankin 1977). Let d be a POSITIVE SQUAREFREE INTEGER, and define

$$t \equiv \begin{cases} i\sqrt{d} & \text{for } d \equiv 1 \text{ or } 2 \pmod 4 \\ \frac{1}{2}(1+i\sqrt{d}) & \text{for } d \equiv 3 \pmod 4. \end{cases} \tag{12}$$

Then the NOME is

$$\begin{aligned} q \equiv e^{i\pi\tau} &= \begin{cases} e^{2\pi i(i\sqrt{d})} \\ e^{2\pi i(1+i\sqrt{d})/2} \end{cases} \\ &= \begin{cases} e^{-2\pi\sqrt{d}} & \text{for } d \equiv 1 \text{ or } 2 \pmod 4 \\ -e^{-\pi\sqrt{d}} & \text{for } d \equiv 3 \pmod 4. \end{cases} \end{aligned} \tag{13}$$

It then turns out that $j(q)$ is an ALGEBRAIC INTEGER of degree $h(-d)$, where $h(-d)$ is the CLASS NUMBER of the DISCRIMINANT $-d$ of the QUADRATIC FIELD $\mathbb{Q}(\sqrt{n})$ (Silverman 1986). The first term in the LAURENT SERIES is then $q^{-1} = e^{-2\pi\sqrt{n}}$ or $-e^{-\pi\sqrt{n}}$, and all the later terms are POWERS of q^{-1}, which are small numbers. The larger n, the faster the series converges. If $h(-d) = 1$, then $j(q)$ is a ALGEBRAIC INTEGER of degree 1, i.e., just a plain INTEGER. Furthermore, the INTEGER is a perfect CUBE.

The numbers whose LAURENT SERIES give INTEGERS are those with CLASS NUMBER 1. But these are precisely the HEEGNER NUMBERS -1, -2, -3, -7, -11, -19, -43, -67, -163. The greater (in ABSOLUTE VALUE) the HEEGNER NUMBER d, the closer to an INTEGER is the expression $e^{\pi\sqrt{-n}}$, since the initial term in $j(q)$ is the largest and subsequent terms are the smallest. The best approximations with $h(-d) = 1$ are therefore

$$\begin{aligned} e^{\pi\sqrt{43}} &\approx 960^3 + 744 - 2.2 \times 10^{-4} && (14)\\ e^{\pi\sqrt{67}} &\approx 5280^3 + 744 - 1.3 \times 10^{-6} && (15)\\ e^{\pi\sqrt{163}} &\approx 640320^3 + 744 - 7.5 \times 10^{-13}. && (16) \end{aligned}$$

The exact values of $j(q)$ corresponding to the HEEGNER NUMBERS are

$$j(-e^{-\pi}) = 12^3 \quad (17)$$

$$j(e^{-2\pi\sqrt{2}}) = 20^3 \quad (18)$$

$$j(-e^{-\pi\sqrt{3}}) = 0^3 \quad (19)$$

$$j(-e^{-\pi\sqrt{7}}) = -15^3 \quad (20)$$

$$j(-e^{-\pi\sqrt{11}}) = -32^3 \quad (21)$$

$$j(-e^{-\pi\sqrt{19}}) = -96^3 \quad (22)$$

$$j(-e^{-\pi\sqrt{43}}) = -960^3 \quad (23)$$

$$j(-e^{-\pi\sqrt{67}}) = -5280^3 \quad (24)$$

$$j(-e^{-\pi\sqrt{163}}) = -640320^3. \quad (25)$$

(The number 5280 is particularly interesting since it is also the number of feet in a mile.) The ALMOST INTEGER generated by the last of these, $e^{\pi\sqrt{163}}$ (corresponding to the field $\mathbb{Q}(\sqrt{-163})$ and the IMAGINARY quadratic field of maximal discriminant), is known as the RAMANUJAN CONSTANT.

$e^{\pi\sqrt{22}}$, $e^{\pi\sqrt{37}}$, and $e^{\pi\sqrt{58}}$ are also ALMOST INTEGERS. These correspond to binary quadratic forms with discriminants -88, -148, and -232, all of which have CLASS NUMBER two and were noted by Ramanujan (Berndt 1994).

It turns out that the j-function also is important in the CLASSIFICATION THEOREM for finite simple groups, and that the factors of the orders of the SPORADIC GROUPS, including the celebrated MONSTER GROUP, are also related.

see also ALMOST INTEGER, KLEIN'S ABSOLUTE INVARIANT, WEBER FUNCTIONS

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j-Invariant

An invariant of an ELLIPTIC CURVE closely related to the DISCRIMINANT and defined by

$$j(E) \equiv \frac{2^8 3^3 a^3}{4a^3 + 27b^2}.$$

The determination of j as an ALGEBRAIC INTEGER in the QUADRATIC FIELD $\mathbb{Q}(j)$ is discussed by Greenhill (1891), Weber (1902), Berwick (1928), Watson (1938), Gross and Zaiger (1985), and Dorman (1988). The norm of j in $\mathbb{Q}(j)$ is the CUBE of an INTEGER in $\mathbb{Z}$.

see also DISCRIMINANT (ELLIPTIC CURVE), ELLIPTIC CURVE, FREY CURVE

References

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Jackson's Difference Fan

If, after constructing a DIFFERENCE TABLE, no clear pattern emerges, turn the paper through an ANGLE of 60° and compute a new table. If necessary, repeat the process. Each ROTATION reduces POWERS by 1, so the sequence $\{k^n\}$ multiplied by any POLYNOMIAL in n is reduced to 0s by a k-fold difference fan.

References

Conway, J. H. and Guy, R. K. "Jackson's Difference Fans." In *The Book of Numbers.* New York: Springer-Verlag, pp. 84–85, 1996.

Jackson's Identity

A q-SERIES identity involving

$$\frac{(aq)_q^m (aqde)_q^m (adec)_q^m (aqcd)_q^m}{(aqc)_q^m (aqd)_q^m (aqe)_q^m (aqcde)_q^m},$$

where

$$(a)_q^n = (1-a)(1-aq)\cdots(1-aq^{n-1}).$$

see also q-SERIES

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Jackson, F. H. "Summation of q-Hypergeometric Series." *Messenger Math.* **47**, 101–112, 1917.

Jackson's Theorem

Jackson's theorem is a statement about the error $E_n(f)$ of the best uniform approximation to a REAL FUNCTION $f(x)$ on $[-1,1]$ by REAL POLYNOMIALS of degree at most n. Let $f(x)$ be of bounded variation in $[-1,1]$ and let M' and V' denote the least upper bound of $|f(x)|$ and the total variation of $f(x)$ in $[-1,1]$, respectively. Given the function

$$F(x) = F(-1) + \int_{-1}^{x} f(x)\,dx, \tag{1}$$

then the coefficients

$$a_n = \tfrac{1}{2}(2n+1)\int_{-1}^{1} F(x)P_n(x)\,dx \tag{2}$$

of its LEGENDRE SERIES, where $P_n(x)$ is a LEGENDRE POLYNOMIAL, satisfy the inequalities

$$|a_n| < \begin{cases} \frac{6}{\sqrt{\pi}}(M'+V')n^{-3/2} & \text{for } n \geq 1 \\ \frac{4}{\sqrt{\pi}}(M'+V')n^{-3/2} & \text{for } n \geq 2. \end{cases} \tag{3}$$

Moreover, the LEGENDRE SERIES of $F(x)$ converges uniformly and absolutely to $F(x)$ in $[-1,1]$.

Bernstein strengthened Jackson's theorem to

$$2nE_{2n}(\alpha) \leq \frac{4n}{\pi(2n+1)} < \frac{2}{\pi} = 0.6366. \tag{4}$$

A specific application of Jackson's theorem shows that if

$$\alpha(x) = |x|, \tag{5}$$

then

$$E_n(\alpha) \leq \frac{6}{n}. \tag{6}$$

see also LEGENDRE SERIES, PICONE'S THEOREM

References

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Jaco-Shalen-Johannson Torus Decomposition

Irreducible orientable COMPACT 3-MANIFOLDS have a canonical (up to ISOTOPY) minimal collection of disjointly EMBEDDED incompressible TORI such that each component of the 3-MANIFOLD removed by the TORI is either "atoroidal" or "Seifert-fibered."

Jacobi Algorithm

A method which can be used to solve a TRIDIAGONAL MATRIX equation with largest absolute values in each row and column dominated by the diagonal element. Each diagonal element is solved for, and an approximate value plugged in. The process is then iterated until it converges. This algorithm is a stripped-down version of the JACOBI METHOD of matrix diagonalization.

see also JACOBI METHOD, TRIDIAGONAL MATRIX

References

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Jacobi-Anger Expansion

$$e^{iz\cos\theta} = \sum_{n=-\infty}^{\infty} i^n J_n(z) e^{in\theta},$$

where $J_n(z)$ is a BESSEL FUNCTION OF THE FIRST KIND. The identity can also be written

$$e^{iz\cos\theta} = J_0(z) + 2\sum_{n=1}^{\infty} i^n J_n(z)\cos(n\theta).$$

This expansion represents an expansion of plane waves into a series of cylindrical waves.

see also BESSEL FUNCTION OF THE FIRST KIND

Jacobi's Curvature Theorem

The principal normal indicatrix of a closed space curve with nonvanishing curvature bisects the AREA of the unit sphere if it is embedded.

Jacobi's Determinant Identity

Let

$$\mathsf{A} = \begin{bmatrix} \mathsf{B} & \mathsf{D} \\ \mathsf{E} & \mathsf{C} \end{bmatrix} \tag{1}$$

$$\mathsf{A}^{-1} = \begin{bmatrix} \mathsf{W} & \mathsf{X} \\ \mathsf{Y} & \mathsf{Z} \end{bmatrix}, \tag{2}$$

where B and W are $k \times k$ MATRICES. Then

$$(\det \mathsf{Z})(\det \mathsf{A}) = \det \mathsf{B}. \tag{3}$$

The proof follows from equating determinants on the two sides of the block matrices

$$\begin{bmatrix} \mathsf{B} & \mathsf{D} \\ \mathsf{E} & \mathsf{C} \end{bmatrix}\begin{bmatrix} \mathsf{I} & \mathsf{X} \\ \mathsf{O} & \mathsf{Z} \end{bmatrix} = \begin{bmatrix} \mathsf{B} & \mathsf{O} \\ \mathsf{E} & \mathsf{I} \end{bmatrix}, \tag{4}$$

where I is the IDENTITY MATRIX and O is the zero matrix.

References

Gantmacher, F. R. *The Theory of Matrices, Vol. 1.* New York: Chelsea, p. 21, 1960.

Horn, R. A. and Johnson, C. R. *Matrix Analysis.* Cambridge, England: Cambridge University Press, p. 21, 1985.

Jacobi Differential Equation

$$(1-x^2)y''+[\beta-\alpha-(\alpha+\beta+2)x]y'+n(n+\alpha+\beta+1)y=0 \quad (1)$$

or

$$\frac{d}{dx}[(1-x)^{\alpha+1}(1+x)^{\beta+1}y'] +n(n+\alpha+\beta+1)(1-x)^\alpha(1+x)^\beta y=0. \quad (2)$$

The solutions are JACOBI POLYNOMIALS. They can be transformed to

$$\frac{d^2u}{dx^2}+\left[\frac{1}{4}\frac{1-\alpha^2}{(1-x)^2}+\frac{1}{4}\frac{1-\beta^2}{(1+x)^2} +\frac{n(n+\alpha+\beta+1)+\frac{1}{2}(\alpha+1)(\beta+1)}{1-x^2}\right]u=0, \quad (3)$$

where

$$u=u(x)=(1-x)^{(\alpha+1)/2}(1+x)^{(\beta+1)/2}P_n^{(\alpha,\beta)}(x), \quad (4)$$

and

$$\frac{d^2u}{d\theta^2}+\left[\frac{\frac{1}{4}-\alpha^2}{4\sin^2(\frac{1}{2}\theta)}+\frac{\frac{1}{4}-\beta^2}{4\cos^2(\frac{1}{2}\theta)} +\left(n+\frac{\alpha+\beta+1}{2}\right)^2\right]u=0, \quad (5)$$

where

$$u=u(\theta)=\sin^{\alpha+1/2}(\tfrac{1}{2}\theta)\cos^{\beta+1/2}(\tfrac{1}{2}\theta)P_n^{(\alpha,\beta)}(\cos\theta). \quad (6)$$

Jacobi Differential Equation (Calculus of Variations)

$$\frac{d}{dx}\Omega_{\eta'}-\Omega_\eta=\frac{d}{dx}(f_{y'y}\eta+f_{y'y}\eta')-(f_{yy}\eta+f_{yy'}\eta')=0,$$

where

$$\Omega(x,\eta,\eta')\equiv\tfrac{1}{2}(f_{yy}\eta^2+2f_{yy'}\eta\eta'+f_{y'y}\eta'^2).$$

This equations arises in the CALCULUS OF VARIATIONS.

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Jacobi Elliptic Functions

The Jacobi elliptic functions are standard forms of ELLIPTIC FUNCTIONS. The three basic functions are denoted $\mathrm{cn}(u,k)$, $\mathrm{dn}(u,k)$, and $\mathrm{sn}(u,k)$, where k is known as the MODULUS. In terms of THETA FUNCTIONS,

$$\mathrm{sn}(u,k)=\frac{\vartheta_3}{\vartheta_4}\frac{\vartheta_1(u\vartheta_3{}^{-2})}{\vartheta_4(u\vartheta_3{}^{-2})} \quad (1)$$

$$\mathrm{cn}(u,k)=\frac{\vartheta_4}{\vartheta_2}\frac{\vartheta_2(u\vartheta_3{}^{-2})}{\vartheta_4(u\vartheta_3{}^{-2})} \quad (2)$$

$$\mathrm{dn}(u,k)=\frac{\vartheta_4}{\vartheta_3}\frac{\vartheta_3(u\vartheta_3{}^{-2})}{\vartheta_4(u\vartheta_3{}^{-2})} \quad (3)$$

(Whittaker and Watson 1990, p. 492), where $\vartheta_i\equiv\vartheta_i(0)$ (Whittaker and Watson 1990, p. 464). Ratios of Jacobi elliptic functions are denoted by combining the first letter of the NUMERATOR elliptic function with the first of the DENOMINATOR elliptic function. The multiplicative inverses of the elliptic functions are denoted by reversing the order of the two letters. These combinations give a total of 12 functions: cd, cn, cs, dc, dn, ds, nc, nd, ns, sc, sd, and sn. The AMPLITUDE ϕ is defined in terms of $\mathrm{sn}\,u$ by

$$y=\sin\phi=\mathrm{sn}(u,k). \quad (4)$$

The k argument is often suppressed for brevity so, for example, $\mathrm{sn}(u,k)$ can be written $\mathrm{sn}\,u$.

The Jacobi elliptic functions are periodic in $K(k)$ and $K'(k)$ as

$$\mathrm{sn}(u+2mK+2niK',k)=(-1)^m\,\mathrm{sn}(u,k) \quad (5)$$

$$\mathrm{cn}(u+2mK+2niK',k)=(-1)^{m+n}\,\mathrm{cn}(u,k) \quad (6)$$

$$\mathrm{dn}(u+2mK+2niK',k)=(-1)^n\,\mathrm{dn}(u,k), \quad (7)$$

where $K(k)$ is the complete ELLIPTIC INTEGRAL OF THE FIRST KIND, $K'(k)\equiv K(k')$, and $k'\equiv\sqrt{1-k^2}$ (Whittaker and Watson 1990, p. 503).

The $\mathrm{cn}\,x$, $\mathrm{dn}\,x$, and $\mathrm{sn}\,x$ functions may also be defined as solutions to the differential equations

$$\frac{d^2y}{dx^2}=-(1+k^2)y+2k^2y^3 \quad (8)$$

$$\frac{d^2y}{dx^2}=-(1-2k^2)y-2k^2y^3 \quad (9)$$

$$\frac{d^2y}{dx^2}=(2-k^2)y-2y^3. \quad (10)$$

The standard Jacobi elliptic functions satisfy the identities

$$\mathrm{sn}^2\,u+\mathrm{cn}^2\,u=1 \quad (11)$$

$$k^2\,\mathrm{sn}^2\,u+\mathrm{dn}^2\,u=1 \quad (12)$$

$$k^2\,\mathrm{cn}^2\,u+k'^2=\mathrm{dn}^2\,u \quad (13)$$

$$\mathrm{cn}^2\,u+k'^2\,\mathrm{sn}^2\,u=\mathrm{dn}^2\,u. \quad (14)$$

Jacobi Elliptic Functions

Special values are

$$\operatorname{cn}(0) = 1 \tag{15}$$
$$\operatorname{dn}(0) = 1 \tag{16}$$
$$\operatorname{cn}(K) = 0 \tag{17}$$
$$\operatorname{dn}(K) = k' \equiv \sqrt{1-k^2}, \tag{18}$$
$$\operatorname{sn}(K) = 1, \tag{19}$$

where $K = K(k)$ is a complete ELLIPTIC INTEGRAL OF THE FIRST KIND and k' is the complementary MODULUS (Whittaker and Watson 1990, pp. 498–499).

In terms of integrals,

$$u = \int_0^{\operatorname{sn} u} (1-t^2)^{1-/2}(1-k^2t^2)^{-1/2}\,dt \tag{20}$$
$$= \int_{\operatorname{ns} u}^{\infty} (t^2-1)^{-1/2}(t^2-l^2)^{-1/2}\,dt \tag{21}$$
$$= \int_{\operatorname{cn} u}^{1} (1-t^2)^{-1/2}(k'^2+k^2t^2)^{-1/2}\,dt \tag{22}$$
$$= \int_{1}^{\operatorname{nc} u} (t^2-1)^{-1/2}(k'^2t^2+k^2)^{-1/2}\,dt \tag{23}$$
$$= \int_{\operatorname{dn} u}^{1} (1-t^2)^{-1/2}(t^2-k'^2)^{-1/2}\,dt \tag{24}$$
$$= \int_{1}^{\operatorname{nd} u} (t^2-1)^{-1/2}(1-k'^2t^2)^{-1/2}\,dt \tag{25}$$
$$= \int_{0}^{\mathrm{U}} (1+t^2)^{-1/2}(1+k'^2t^2)^{-1/2}\,dt \tag{26}$$
$$= \int_{\operatorname{cs} u}^{\infty} (t^2+1)^{-1/2}(t^2+k'^2)^{-1/2}\,dt \tag{27}$$
$$= \int_{0}^{\operatorname{sd} u} (1-k'^2t^2)^{-1/2}(1+k^2t^2)^{-1/2}\,dt \tag{28}$$
$$= \int_{\operatorname{ds} u}^{\infty} (t^2-k'^2)^{-1/2}(t^2+k^2)^{-1/2}\,dt \tag{29}$$
$$= \int_{1}^{\operatorname{cd} u} (1-t^2)^{-1/2}(1-k^2t^2)^{-1/2}\,dt \tag{30}$$
$$= \int_{\operatorname{dc} u}^{1} (t^2-1)^{-1/2}(t^2-k^2)^{-1/2}\,dt \tag{31}$$

(Whittaker and Watson 1990, p. 494).

Jacobi elliptic functions addition formulas include

$$\operatorname{sn}(u+v) = \frac{\operatorname{sn} u \operatorname{cn} v \operatorname{dn} v + \operatorname{sn} v \operatorname{cn} u \operatorname{dn} u}{1 - k^2 \operatorname{sn}^2 u \operatorname{sn}^2 v} \tag{32}$$
$$\operatorname{cn}(u+v) = \frac{\operatorname{cn} u \operatorname{cn} v - \operatorname{sn} u \operatorname{sn} v \operatorname{dn} u \operatorname{dn} v}{1 - k^2 \operatorname{sn}^2 u \operatorname{sn}^2 v} \tag{33}$$
$$\operatorname{dn}(u+v) = \frac{\operatorname{dn} u \operatorname{dn} v - k^2 \operatorname{sn} u \operatorname{sn} v \operatorname{cn} u \operatorname{cn} v}{1 - k^2 \operatorname{sn}^2 u \operatorname{sn}^2 v}. \tag{34}$$

Extended to integral periods,

$$\operatorname{sn}(u+K) = \frac{\operatorname{cn} u}{\operatorname{dn} u} \tag{35}$$
$$\operatorname{cn}(u+K) = -\frac{k' \operatorname{sn} u}{\operatorname{dn} u} \tag{36}$$
$$\operatorname{dn}(u+K) = \frac{k'}{\operatorname{dn} u} \tag{37}$$

$$\operatorname{sn}(u+2K) = -\operatorname{sn} u \tag{38}$$
$$\operatorname{cn}(u+2K) = -\operatorname{cn} u \tag{39}$$
$$\operatorname{dn}(u+2K) = \operatorname{dn} u \tag{40}$$

For COMPLEX arguments,

$$\operatorname{sn}(u+iv) = \frac{\operatorname{sn}(u,k)\operatorname{dn}(v,k')}{1-\operatorname{dn}^2(u,k)\operatorname{sn}^2(v,k')} + \frac{i\operatorname{cn}(u,k)\operatorname{dn}(u,k)\operatorname{sn}(v,k')\operatorname{cn}(v,k')}{1-\operatorname{dn}^2(u,k)\operatorname{sn}^2(v,k')} \tag{41}$$
$$\operatorname{cn}(u+iv) = \frac{\operatorname{cn}(u,k)\operatorname{cn}(v,k')}{1-\operatorname{dn}^2(u,k)\operatorname{sn}^2(v,k')} - \frac{i\operatorname{sn}(u,k)\operatorname{dn}(u,k)\operatorname{sn}(v,k')\operatorname{dn}(v,k')}{1-\operatorname{dn}^2(u,k)\operatorname{sn}^2(v,k')} \tag{42}$$
$$\operatorname{dn}(u+iv) = \frac{\operatorname{dn}(u,k)\operatorname{cn}(v,k')\operatorname{dn}(v,k')}{1-\operatorname{dn}^2(u,k)\operatorname{sn}^2(v,k')} - \frac{ik^2\operatorname{sn}(u,k)\operatorname{cn}(u,k)\operatorname{sn}(v,k')}{1-\operatorname{dn}^2(u,k)\operatorname{sn}^2(v,k')}. \tag{43}$$

DERIVATIVES of the Jacobi elliptic functions include

$$\frac{d\operatorname{sn} u}{du} = \operatorname{cn} u \operatorname{dn} u \tag{44}$$
$$\frac{d\operatorname{cn} u}{du} = \operatorname{sn} u \operatorname{dn} u \tag{45}$$
$$\frac{d\operatorname{dn} u}{du} = -k^2 \operatorname{sn} u \operatorname{cn} u. \tag{46}$$

Double-period formulas involving the Jacobi elliptic functions include

$$\operatorname{sn}(2u) = \frac{2\operatorname{sn} u \operatorname{cn} u \operatorname{dn} u}{1-k^2\operatorname{sn}^4 u} \tag{47}$$
$$\operatorname{cn}(2u) = \frac{1-2\operatorname{sn}^2 u + k^2 \operatorname{sn}^4 u}{1-k^2\operatorname{sn}^4 u} \tag{48}$$
$$\operatorname{dn}(2u) = \frac{1-2k^2\operatorname{sn}^2 u + k^2 \operatorname{sn}^4 u}{1-k^2\operatorname{sn}^4 u}. \tag{49}$$

Half-period formulas involving the Jacobi elliptic functions include

$$\operatorname{sn}(\tfrac{1}{2}K) = \frac{1}{\sqrt{1+k'}} \tag{50}$$
$$\operatorname{cn}(\tfrac{1}{2}K) = \sqrt{\frac{k'}{1+k'}} \tag{51}$$
$$\operatorname{dn}(\tfrac{1}{2}K) = \sqrt{k'}. \tag{52}$$

Squared formulas include

$$\operatorname{sn}^2 u = \frac{1-\operatorname{cn}(2u)}{1+\operatorname{dn}(2u)} \tag{53}$$

$$\operatorname{cn}^2 u = \frac{\operatorname{dn}(2u)+\operatorname{cn}(2u)}{1+\operatorname{dn}(2u)} \tag{54}$$

$$\operatorname{dn}^2 u = \frac{\operatorname{dn}(2u)+\operatorname{cn}(2u)}{1+\operatorname{cn}(2u)}. \tag{55}$$

see also AMPLITUDE, ELLIPTIC FUNCTION, JACOBI'S IMAGINARY TRANSFORMATION, THETA FUNCTION, WEIERSTRASS ELLIPTIC FUNCTION

References

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Jacobi Function of the First Kind

see JACOBI POLYNOMIAL

Jacobi Function of the Second Kind

$$Q_n^{(\alpha,\beta)}(x) = 2^{-n-1}(x-1)^{-\alpha}(x+1)^{-\beta} \times \int_{-1}^{1}(1-t)^{n+\alpha}(1+t)^{n+\beta}(x-t)^{-n-1}\,dt.$$

In the exceptional case $n=0$, $\alpha+\beta+1=0$, a nonconstant solution is given by

$$Q^{(\alpha)}(x) = \ln(x+1) + \pi^{-1}\sin(\pi\alpha)(x-1)^{-\alpha}(x+1)^{-\beta} \times \int_{-1}^{1}\frac{(1-t)^\alpha(1+t)^\beta}{x-t}\ln(1+t)\,dt.$$

References

Szegő, G. "Jacobi Polynomials." Ch. 4 in *Orthogonal Polynomials, 4th ed.* Providence, RI: Amer. Math. Soc., pp. 73–79, 1975.

Jacobi-Gauss Quadrature

Also called JACOBI QUADRATURE or MEHLER QUADRATURE. A GAUSSIAN QUADRATURE over the interval $[-1,1]$ with WEIGHTING FUNCTION $W(x)=(1-x)^\alpha(1+x)^\beta$. The ABSCISSAS for quadrature order n are given by the roots of the JACOBI POLYNOMIALS $P_n^{(\alpha,\beta)}(x)$. The weights are

$$\begin{aligned} w_i &= -\frac{A_{n+1}\gamma_n}{A_n P_n^{(\alpha,\beta)\prime}(x_i)P_{n+1}^{(\alpha,\beta)}(x_i)} \\ &= \frac{A_n}{A_{n-1}}\frac{\gamma_{n-1}}{P_{n-1}^{(\alpha,\beta)}(x_i)P_n^{(\alpha,\beta)\prime}(x_i)}, \end{aligned} \tag{1}$$

where A_n is the COEFFICIENT of x^n in $P_n^{(\alpha,\beta)}(x)$. For JACOBI POLYNOMIALS,

$$A_n = \frac{\Gamma(2n+\alpha+\beta+1)}{2^n n!\Gamma(n+\alpha+\beta+1)}, \tag{2}$$

where $\Gamma(z)$ is a GAMMA FUNCTION. Additionally,

$$\gamma_n = \frac{1}{2^{2n}(n!)^2}\frac{2^{2n+\alpha+\beta+1}n!}{2n+\alpha+\beta+1} \times \frac{\Gamma(n+\alpha+1)\Gamma(n+\beta+1)}{\Gamma(n+\alpha+\beta+1)}, \tag{3}$$

so

$$\begin{aligned} w_i &= \frac{2n+\alpha+\beta+2}{n+\alpha+\beta+1}\frac{\Gamma(n+\alpha+1)\Gamma(n+\beta+1)}{\Gamma(n+\alpha+\beta+1)} \times \frac{2^{2n+\alpha+\beta+1}n!}{V_n'(x_i)V_{n+1}(x_i)} \qquad (4) \\ &= \frac{\Gamma(n+\alpha+1)\Gamma(n+\beta+1)}{\Gamma(n+\alpha+\beta+1)}\frac{2^{2n+\alpha+\beta+1}n!}{(1-x_i{}^2)[V_n'(x_i)]^2}, \qquad (5) \end{aligned}$$

where

$$V_m \equiv P_n^{(\alpha,\beta)}(x)\frac{2^n n!}{(-1)^n}. \tag{6}$$

The error term is

$$E_n = \frac{\Gamma(n+\alpha+1)\Gamma(n+\beta+1)\Gamma(n+\alpha+\beta+1)}{(2n+\alpha+\beta+1)[\Gamma(2n+\alpha+\beta+1)]^2} \times \frac{2^{2n+\alpha+\beta+1}n!}{(2n)!}f^{(2n)}(\xi) \tag{7}$$

(Hildebrand 1959).

References

Hildebrand, F. B. *Introduction to Numerical Analysis.* New York: McGraw-Hill, pp. 331–334, 1956.

Jacobi Identities

"The" Jacobi identity is a relationship

$$[A,[B,C]]+[B,[C,A]]+[C,[A,B]]=0, \qquad (1),$$

between three elements A, B, and C, where $[A,B]$ is the COMMUTATOR. The elements of a LIE GROUP satisfy this identity.

Relationships between the Q-FUNCTIONS Q_i are also known as Jacobi identities:

$$Q_1Q_2Q_3=1, \qquad (2)$$

equivalent to the JACOBI TRIPLE PRODUCT (Borwein and Borwein 1987, p. 65) and

$$Q_2{}^8=16qQ_1{}^8+Q_3{}^8, \qquad (3)$$

where

$$q\equiv e^{-\pi K'(k)/K(k)}, \qquad (4)$$

$K=K(k)$ is the complete ELLIPTIC INTEGRAL OF THE FIRST KIND, and $K'(k)=K(k')=K(\sqrt{1-k^2}\,)$. Using WEBER FUNCTIONS

$$f_1=q^{-1/24}Q_3 \qquad (5)$$

$$f_2=2^{1/2}q^{1/12}Q_1 \qquad (6)$$

$$f=q^{-1/24}Q_2, \qquad (7)$$

(5) and (6) become

$$f_1f_2f=\sqrt{2} \qquad (8)$$

$$f^8=f_1{}^8+f_2{}^8 \qquad (9)$$

(Borwein and Borwein 1987, p. 69).

see also COMMUTATOR, JACOBI TRIPLE PRODUCT, Q-FUNCTION, WEBER FUNCTIONS

References

Borwein, J. M. and Borwein, P. B. *Pi & the AGM: A Study in Analytic Number Theory and Computational Complexity.* New York: Wiley, 1987.

Jacobi's Imaginary Transformation

For JACOBI ELLIPTIC FUNCTIONS $\operatorname{sn}u$, $\operatorname{cn}u$, and $\operatorname{dn}u$,

$$\operatorname{sn}(iu,k)=i\frac{\operatorname{sn}(u,k')}{\operatorname{cn}(u,k')}$$

$$\operatorname{cn}(iu,k)=\frac{1}{\operatorname{cn}(u,k')}$$

$$\operatorname{dn}(iu,k)=\frac{\operatorname{dn}(u,k')}{\operatorname{cn}(u,k')}$$

(Abramowitz and Stegun 1972, Whittaker and Watson 1990).

see also JACOBI ELLIPTIC FUNCTIONS

References

Abramowitz, M. and Stegun, C. A. (Eds.). *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables, 9th printing.* New York: Dover, pp. 592 and 595, 1972.

Whittaker, E. T. and Watson, G. N. *A Course in Modern Analysis, 4th ed.* Cambridge, England: Cambridge University Press, p. 505, 1990.

Jacobi Matrix

see JACOBI ROTATION MATRIX, JACOBIAN

Jacobi Method

A method of diagonalizing MATRICES using JACOBI ROTATION MATRICES. It consists of a sequence of ORTHOGONAL SIMILARITY TRANSFORMATIONS, each of which eliminates one off-diagonal element.

see also JACOBI ALGORITHM, JACOBI ROTATION MATRIX

References

Press, W. H.; Flannery, B. P.; Teukolsky, S. A.; and Vetterling, W. T. "Jacobi Transformation of a Symmetric Matrix." §11.1 in *Numerical Recipes in FORTRAN: The Art of Scientific Computing, 2nd ed.* Cambridge, England: Cambridge University Press, pp. 456–462, 1992.

Jacobi Polynomial

Also known as the HYPERGEOMETRIC POLYNOMIALS, they occur in the study of ROTATION GROUPS and in the solution to the equations of motion of the symmetric top. They are solutions to the JACOBI DIFFERENTIAL EQUATION. Plugging

$$y=\sum_{\nu=0}^{\infty}a_\nu(x-1)^\nu \qquad (1)$$

into the differential equation gives the RECURRENCE RELATION

$$[\gamma-\nu(\nu+\alpha+\beta+1)]a_\nu-2(\nu+1)(\nu+\alpha+1)a_{\nu+1}=0 \quad (2)$$

for $\nu=0,\,1,\,\ldots,$ where

$$\gamma\equiv n(n+\alpha+\beta+1). \qquad (3)$$

Solving the RECURRENCE RELATION gives

$$P_n^{(\alpha,\beta)}(x)=\frac{(-1)^n}{2^n n!}(1-x)^{-\alpha}(1+x)^{-\beta} \times\frac{d^n}{dx^n}[(1-x)^{\alpha+n}(1+x)^{\beta+n}] \quad (4)$$

for $\alpha,\beta>-1$. They form a complete orthogonal system in the interval $[-1,1]$ with respect to the weighting function

$$w_n(x)=(1-x)^\alpha(1+x)^\beta, \qquad (5)$$

and are normalized according to

$$P_n^{(\alpha,\beta)}(1)=\binom{n+\alpha}{n}, \qquad (6)$$

where $\binom{n}{k}$ is a BINOMIAL COEFFICIENT. Jacobi polynomials can also be written

$$P_n^{(\alpha,\beta)}=\frac{\Gamma(2n+\alpha+\beta+1)}{n!\Gamma(n+\alpha+\beta+1)} \times G_n(\alpha+\beta+1,\beta+1,\tfrac{1}{2}(x+1)), \qquad (7)$$

where $\Gamma(z)$ is the GAMMA FUNCTION and

$$G_n(p,q,x) \equiv \frac{n!\Gamma(n+p)}{\Gamma(2n+p)} P_n^{(p-q,q-1)}(2x-1). \tag{8}$$

Jacobi polynomials are ORTHOGONAL satisfying

$$\int_{-1}^{1} P_m^{(\alpha,\beta)} P_n^{(\alpha,\beta)} (1-x)^\alpha (1+x)^\beta \, dx = \frac{2^{\alpha+\beta+1}}{2n+\alpha+\beta+1} \frac{\Gamma(n+\alpha+1)\Gamma(n+\beta+1)}{n!\Gamma(n+\alpha+\beta+1)} \delta_{mn}. \tag{9}$$

The COEFFICIENT of the term x^n in $P_n^{(\alpha,beta)}(x)$ is given by

$$A_n = \frac{\Gamma(2n+\alpha+\beta+1)}{2^n n!\Gamma(n+\alpha+\beta+1)}. \tag{10}$$

They satisfy the RECURRENCE RELATION

$$\begin{aligned} &2(n+1)(n+\alpha+\beta+1)(2n+\alpha+\beta)P_{n+1}^{(\alpha,\beta)}(x) \\ &= [(2n+\alpha+\beta+1)(\alpha^2-\beta^2)+(2n+\alpha+\beta)_3 x]P_n^{(\alpha,\beta)}(x) \\ &\quad -2(n+\alpha)(n+\beta)(2n+\alpha+\beta+2)P_{n-1}^{(\alpha,\beta)}(x), \end{aligned} \tag{11}$$

where $(m)_n$ is the RISING FACTORIAL

$$(m)_n \equiv m(m+1)\cdots(m+n-1) = \frac{(m+n-1)!}{(m-1)!}. \tag{12}$$

The DERIVATIVE is given by

$$\frac{d}{dx}[P_n^{(\alpha,\beta)}(x)] = \tfrac{1}{2}(n+\alpha+\beta+1)P_{n-1}^{(\alpha+1,\beta+1)}(x). \tag{13}$$

The ORTHOGONAL POLYNOMIALS with WEIGHTING FUNCTION $(b-x)^\alpha(x-a)^\beta$ on the CLOSED INTERVAL $[a,b]$ can be expressed in the form

$$[\text{const.}]P_n^{(\alpha,\beta)}\left(2\frac{x-a}{b-a}-1\right) \tag{14}$$

(Szegő 1975, p. 58).

Special cases with $\alpha = \beta$ are

$$\begin{aligned} P_{2\nu}^{(\alpha,\alpha)}(x) &= \frac{\Gamma(2\nu+\alpha+1)\Gamma(\nu+1)}{\Gamma(\nu+\alpha+1)\Gamma(2\nu+1)} P_\nu^{(\alpha,-1/2)}(2x^2-1) \\ &= (-1)^\nu \frac{\Gamma(2\nu+\alpha+1)\Gamma(\nu+1)}{\Gamma(\nu+\alpha+1)\Gamma(2\nu+1)} P_\nu^{(-1/2,\alpha)}(1-2x^2) \end{aligned} \tag{15}$$

$$\begin{aligned} P_{2\nu+1}^{(\alpha,\alpha)}(x) &= \frac{\Gamma(2\nu+\alpha+2)\Gamma(\nu+1)}{\Gamma(\nu+\alpha+1)\Gamma(2\nu+2)} x P_\nu^{(\alpha,1/2)}(2x^2-1) \\ &= (-1)^\nu \frac{\Gamma(2\nu+\alpha+2)\Gamma(\nu+1)}{\Gamma(\nu+\alpha+1)\Gamma(2\nu+2)} x P_\nu^{(1/2,\alpha)}(1-2x^2). \end{aligned} \tag{16}$$

Further identities are

$$P_n^{(\alpha+1,\beta)}(x) = \frac{2}{2n+\alpha+\beta+2} \times \frac{(n+\alpha+1)P_n^{(\alpha,\beta)} - (n+1)P_{n+1}^{(\alpha,\beta)}(x)}{1-x} \tag{17}$$

$$P_n^{(\alpha,\beta+1)}(x) = \frac{2}{2n+\alpha+\beta+2} \frac{(n+\beta+1)P_n^{(\alpha,\beta)}(x) + (n+1)P_{n+1}^{(\alpha,\beta)}(x)}{1+x} \tag{18}$$

$$\begin{aligned} &\sum_{\nu=0}^{n} \frac{2\nu+\alpha+\beta+1}{2^{\alpha+\beta+1}} \frac{\Gamma(\nu+1)\Gamma(\nu+\alpha+\beta+1)}{\Gamma(\nu+\alpha+1)\Gamma(\nu+\beta+1)} \times P_\nu^{(\alpha,\beta)}(x) Q_\nu^{(\alpha,\beta)}(y) \\ &= \frac{1}{2}\frac{(y-1)^{-\alpha}(y+1)^{-\beta}}{y-x} + \frac{2^{-\alpha-\beta}}{2n+\alpha+\beta+2} \times \frac{\Gamma(n+2)\Gamma(n+\alpha+\beta+2)}{\Gamma(n+\alpha+1)\Gamma(n+\beta+1)} \\ &\quad \frac{P_{n+1}^{(\alpha,\beta)}(x)Q_n^{(\alpha,\beta)}(y) - P_n^{(\alpha,\beta)}(x)Q_{n+1}^{\alpha,\beta}(y)}{x-y} \end{aligned} \tag{19}$$

(Szegő 1975, p. 79).

The KERNEL POLYNOMIAL is

$$K_n^{(\alpha,\beta)}(x,y) = \frac{2^{-\alpha-\beta}}{2n+\alpha+\beta+2} \times \frac{\Gamma(n+2)\Gamma(n+\alpha+\beta+2)}{\Gamma(n+\alpha+1)\Gamma(n+\beta+1)} \times \frac{P_{n+1}^{(\alpha,\beta)}(x)P_n^{(\alpha,\beta)}(y) - P_n^{(\alpha,\beta)}(x)P_{n+1}^{(\alpha,\beta)}(y)}{x-y} \tag{20}$$

(Szegő 1975, p. 71).

The DISCRIMINANT is

$$D_n^{(\alpha,\beta)} = 2^{-n(n-1)} \prod_{\nu=1}^{n} \nu^{\nu-2n+2}(\nu+\alpha)^{\nu-1}(\nu+\beta)^{\nu-1} \times (n+\nu+\alpha+\beta)^{n-\nu} \tag{21}$$

(Szegő 1975, p. 143).

For $\alpha = \beta = 0$, $P_n^{(0,0)}(x)$ reduces to a LEGENDRE POLYNOMIAL. The GEGENBAUER POLYNOMIAL

$$G_n(p,q,x) = \frac{n!\Gamma(n+p)}{\Gamma(2n+p)} P_n^{(p-q,q-1)}(2x-1) \tag{22}$$

and CHEBYSHEV POLYNOMIAL OF THE FIRST KIND can also be viewed as special cases of the Jacobi POLYNOMIALS. In terms of the HYPERGEOMETRIC FUNCTION,

$$P_n^{(\alpha,\beta)}(x) = \binom{n+\alpha}{n} \times {}_2F_1(-n, n+\alpha+\beta; \alpha+1; \tfrac{1}{2}(1-x)) \tag{23}$$

$$P_n^{(\alpha,\beta)}(x) = \binom{n+\alpha}{n}\left(\frac{x+1}{2}\right)^2 \times {}_2F_1\left(-n, -n-\beta; \alpha+1; \frac{x-1}{x+1}\right). \tag{24}$$

Let N_1 be the number of zeros in $x \in (-1, 1)$, N_2 the number of zeros in $x \in (-\infty, -1)$, and N_3 the number of zeros in $x \in (1, \infty)$. Define Klein's symbol

$$E(u) = \begin{cases} 0 & \text{if } u \leq 0 \\ \lfloor u \rfloor & \text{if } u \text{ positive and nonintegral} \\ u-1 & \text{if } u = 1, 2, \ldots, \end{cases} \tag{25}$$

where $\lfloor x \rfloor$ is the FLOOR FUNCTION, and

$$X(\alpha, \beta) = E[\tfrac{1}{2}(|2n+\alpha+\beta+1| - |\alpha| - |\beta| + 1)] \tag{26}$$

$$Y(\alpha, \beta) = E[\tfrac{1}{2}(-|2n+\alpha+\beta+1| + |\alpha| - |\beta| + 1)] \tag{27}$$

$$Z(\alpha, \beta) = E[\tfrac{1}{2}(-|2n+\alpha+\beta+1| - |\alpha| + |\beta| + 1)]. \tag{28}$$

If the cases $\alpha = -1, -2, \ldots, -n$, $\beta = -1, -2, \ldots, -n$, and $n+\alpha+\beta = -1, -2, \ldots, -n$ are excluded, then the number of zeros of $P_n^{(\alpha,\beta)}$ in the respective intervals are

$$N_1(\alpha, \beta) = \begin{cases} 2\left\lfloor \frac{1}{2}(X+1) \right\rfloor & \text{for } (-1)^n \binom{n+\alpha}{n}\binom{n+\beta}{n} > 0 \\ 2\left\lfloor \frac{1}{2}X \right\rfloor + 1 & \text{for } (-1)^n \binom{n+\alpha}{n}\binom{n+\beta}{n} < 0 \end{cases} \tag{29}$$

$$N_2(\alpha, \beta) = \begin{cases} 2\left\lfloor \frac{1}{2}(Y+1) \right\rfloor & \text{for } \binom{2n+\alpha+\beta}{n}\binom{n+\beta}{n} > 0 \\ 2\left\lfloor \frac{1}{2}Y \right\rfloor + 1 & \text{for } \binom{2n+\alpha+\beta}{n}\binom{n+\beta}{n} < 0 \end{cases} \tag{30}$$

$$N_3(\alpha, \beta) = \begin{cases} 2\left\lfloor \frac{1}{2}(Z+1) \right\rfloor & \text{for } \binom{2n+\alpha+\beta}{n}\binom{n+\alpha}{n} > 0 \\ 2\left\lfloor \frac{1}{2}Z \right\rfloor + 1 & \text{for } \binom{2n+\alpha+\beta}{n}\binom{n+\alpha}{n} < 0 \end{cases} \tag{31}$$

(Szegő 1975, pp. 144–146).

The first few POLYNOMIALS are

$$P_0^{(\alpha,\beta)}(x) = 1$$

$$P_1^{(\alpha,\beta)}(x) = \tfrac{1}{2}[2(\alpha+1) + (\alpha+\beta+2)(x-1)]$$

$$P_2^{(\alpha,\beta)}(x) = \tfrac{1}{8}[4(\alpha+1)_2 + 4(\alpha+\beta+3)(\alpha+2)(x-1) + (\alpha+\beta+2)_2(x-1)^2],$$

where $(m)_n$ is a RISING FACTORIAL. See Abramowitz and Stegun (1972, pp. 782–793) and Szegő (1975, Ch. 4) for additional identities.

see also CHEBYSHEV POLYNOMIAL OF THE FIRST KIND, GEGENBAUER POLYNOMIAL, JACOBI FUNCTION OF THE SECOND KIND, RISING FACTORIAL, ZERNIKE POLYNOMIAL

References

Abramowitz, M. and Stegun, C. A. (Eds.). "Orthogonal Polynomials." Ch. 22 in *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables, 9th printing.* New York: Dover, pp. 771–802, 1972.

Iyanaga, S. and Kawada, Y. (Eds.). "Jacobi Polynomials." Appendix A, Table 20.V in *Encyclopedic Dictionary of Mathematics.* Cambridge, MA: MIT Press, p. 1480, 1980.

Szegő, G. "Jacobi Polynomials." Ch. 4 in *Orthogonal Polynomials, 4th ed.* Providence, RI: Amer. Math. Soc., 1975.

Jacobi Quadrature

see JACOBI-GAUSS QUADRATURE

Jacobi Rotation Matrix

A MATRIX used in the JACOBI TRANSFORMATION method of diagonalizing MATRICES. It contains $\cos\phi$ in p rows and columns and $\sin\phi$ in q rows and columns,

$$P_{pq} \equiv \begin{bmatrix} 1 & & & & & & & & 0 \\ & \ddots & & & \vdots & & & \iddots & \\ & & \cos\phi & \cdots & 0 & \cdots & \sin\phi & & \\ & \cdots & 0 & \cdots & 1 & \cdots & 0 & \cdots & \\ & & \sin\phi & \cdots & 0 & \cdots & \cos\phi & & \\ & \iddots & & & \vdots & & & \ddots & \\ 0 & & & & & & & & 1 \end{bmatrix}.$$

see also JACOBI TRANSFORMATION

References

Press, W. H.; Flannery, B. P.; Teukolsky, S. A.; and Vetterling, W. T. "Jacobi Transformation of a Symmetric Matrix." §11.1 in *Numerical Recipes in FORTRAN: The Art of Scientific Computing, 2nd ed.* Cambridge, England: Cambridge University Press, pp. 456–462, 1992.

Jacobi Symbol

The product of LEGENDRE SYMBOLS (n/p_i) for each of the PRIME factors p_i such that $m = \prod_i p_i$, denoted (n/m). When m is a PRIME, the Jacobi symbol reduces to the LEGENDRE SYMBOL. The Jacobi symbol satisfies the same rules as the LEGENDRE SYMBOL

$$(n/m)(n/m') = (n/(mm')) \tag{1}$$

$$(n/m)(n'/m) = ((nn')/m) \tag{2}$$

$$(n^2/m) = (n/m^2) = 1 \quad \text{if } (m,n) = 1 \tag{3}$$

$$(n/m) = (n'/m) \quad \text{if } n \equiv n' \pmod{m} \tag{4}$$

$$(-1/m) = (-1)^{(m-1)/2} = \begin{cases} 1 & \text{for } m \equiv 1 \pmod 4 \\ -1 & \text{for } m \equiv -1 \pmod 4 \end{cases} \tag{5}$$

$$(2/m) = (-1)^{(m^2-1)/8} = \begin{cases} 1 & \text{for } m \equiv \pm 1 \pmod 8 \\ -1 & \text{for } m \equiv \pm 3 \pmod 8 \end{cases} \tag{6}$$

$$(n/m) = \begin{cases} (m/n) & \text{for } m \text{ or } n \equiv 1 \pmod 4 \\ -(m/n) & \text{for } m, n \equiv 3 \pmod 4. \end{cases} \tag{7}$$

Written another way, for m and n RELATIVELY PRIME ODD INTEGERS with $n \geq 3$,

$$(m/n) = (-1)^{(m-1)(n-1)/4}(n/m). \tag{8}$$

The Jacobi symbol is not defined if $m \leq 0$ or m is EVEN.

Bach and Shallit (1996) show how to compute the Jacobi symbol in terms of the SIMPLE CONTINUED FRACTION of a RATIONAL NUMBER a/b.

see also KRONECKER SYMBOL

References

Bach, E. and Shallit, J. *Algorithmic Number Theory, Vol. 1: Efficient Algorithms.* Cambridge, MA: MIT Press, pp. 343–344, 1996.

Guy, R. K. "Quadratic Residues. Schur's Conjecture." §F5 in *Unsolved Problems in Number Theory, 2nd ed.* New York: Springer-Verlag, pp. 244–245, 1994.

Riesel, H. "Jacobi's Symbol." *Prime Numbers and Computer Methods for Factorization, 2nd ed.* Boston, MA: Birkhäuser, pp. 281–284, 1994.

Jacobi Tensor

$$J^{\mu}_{\nu\alpha\beta} = J^{\mu}_{\nu\beta\alpha} \equiv \tfrac{1}{2}(R^{\mu}_{\alpha\nu\beta} + R^{\mu}_{\beta\nu\alpha}),$$

where R is the RIEMANN TENSOR.

see also RIEMANN TENSOR

Jacobi's Theorem

Let M_r be an r-rowed MINOR of the nth order DETERMINANT $|\mathsf{A}|$ associated with an $n \times n$ MATRIX $\mathsf{A} = a_{ij}$ in which the rows $i_1, i_2, \ldots, i_r$ are represented with columns $k_1, k_2, \ldots, k_r$. Define the complementary minor to M_r as the $(n-k)$-rowed MINOR obtained from $|\mathsf{A}|$ by deleting all the rows and columns associated with M_r and the signed complementary minor $M^{(r)}$ to M_r to be

$$\begin{aligned} M^{(r)} = {} & (-1)^{i_1+i_2+\ldots+i_r+k_1+k_2+\ldots+k_r} \\ & \times [\text{complementary minor to} M_r]. \end{aligned}$$

Let the MATRIX of cofactors be given by

$$\Delta = \begin{vmatrix} A_{11} & A_{12} & \cdots & A_{1n} \\ A_{21} & A_{22} & \cdots & A_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ A_{n1} & A_{n2} & \cdots & A_{nn} \end{vmatrix},$$

with M_r and M_r' the corresponding r-rowed minors of $|\mathsf{A}|$ and Δ, then it is true that

$$M_r' = |\mathsf{A}|^{r-1} M^{(r)}.$$

References

Gradshteyn, I. S. and Ryzhik, I. M. *Tables of Integrals, Series, and Products, 5th ed.* San Diego, CA: Academic Press, pp. 1109–1100, 1979.

Jacobi Theta Function

see THETA FUNCTION

Jacobi Transformation

see JACOBI METHOD

Jacobi Triple Product

The Jacobi triple product is the beautiful identity

$$\prod_{n=1}^{\infty}(1-x^{2n})(1+x^{2n-1}z^2)\left(1+\frac{x^{2n-1}}{z^2}\right) = \sum_{m=-\infty}^{\infty} x^{m^2} z^{2m}. \quad (1)$$

In terms of the Q-FUNCTIONS, (1) is written

$$Q_1Q_2Q_3 = 1, \quad (2)$$

which is one of the two JACOBI IDENTITIES. For the special case of $z=1$, (1) becomes

$$\begin{aligned} \varphi(x) \equiv G(1) &= \prod_{n=1}^{\infty}(1+x^{2n-1})^2(1-x^{2n}) \\ &= \sum_{m=-\infty}^{\infty} x^{m^2} = 1 + 2\sum_{m=0}^{\infty} x^{m^2}, \end{aligned} \quad (3)$$

where $\varphi(x)$ is the one-variable RAMANUJAN THETA FUNCTION.

To prove the identity, define the function

$$\begin{aligned} F(z) &\equiv \prod_{n=1}^{\infty}(1+x^{2n-1}z^2)\left(1+\frac{x^{2n-1}}{z^2}\right) \\ &= (1+xz^2)\left(1+\frac{x}{z^2}\right)(1+x^3z^2)\left(1+\frac{x^3}{z^2}\right) \\ &\quad \times (1+x^5z^2)\left(1+\frac{x^5}{z^2}\right)\cdots. \end{aligned} \quad (4)$$

Then

$$\begin{aligned} F(xz) = {} & (1+x^3z^2)\left(1+\frac{1}{xz^2}\right)(1+x^5z^2)\left(1+\frac{x}{z^2}\right) \\ & (1+x^7z^2)\left(1+\frac{x^3}{z^2}\right)\cdots. \end{aligned} \quad (5)$$

Taking (5) ÷ (4),

$$\begin{aligned} \frac{F(xz)}{F(z)} &= \left(1+\frac{1}{xz^2}\right)\left(\frac{1}{1+xz^2}\right) \\ &= \frac{xz^2+1}{xz^2}\frac{1}{1+xz^2} = \frac{1}{xz^2}, \end{aligned} \quad (6)$$

which yields the fundamental relation

$$xz^2F(xz) = F(z). \quad (7)$$

Now define

$$G(z) \equiv F(z)\prod_{n=1}^{\infty}(1-x^{2n}) \quad (8)$$

$$G(xz) = F(xz) \prod_{n=1}^{\infty} (1 - x^{2n}). \tag{9}$$

Using (7), (9) becomes

$$G(xz) = \frac{F(z)}{xz^2} \prod_{n=1}^{\infty} (1 - x^{2n}) = \frac{G(z)}{xz^2}, \tag{10}$$

so

$$G(z) = xz^2 G(xz). \tag{11}$$

Expand G in a LAURENT SERIES. Since G is an EVEN FUNCTION, the LAURENT SERIES contains only even terms.

$$G(z) = \sum_{m=-\infty}^{\infty} a_m z^{2m}. \tag{12}$$

Equation (11) then requires that

$$\sum_{m=-\infty}^{\infty} a_m z^{2m} = xz^2 \sum_{m=-\infty}^{\infty} a_m (xz)^{2m}$$
$$= \sum_{m=-\infty}^{\infty} a_m x^{2m+1} z^{2m+2}. \tag{13}$$

This can be re-indexed with $m' \equiv m - 1$ on the left side of (13)

$$\sum_{m=-\infty}^{\infty} a_m z^{2m} = \sum_{m=-\infty}^{\infty} a_m x^{2m-1} z^{2m}, \tag{14}$$

which provides a RECURRENCE RELATION

$$a_m = a_{m-1} x^{2m-1}, \tag{15}$$

so

$$a_1 = a_0 x \tag{16}$$
$$a_2 = a_1 x^3 = a_0 x^{3+1} = a_0 x^4 = a_0 x^{2^2} \tag{17}$$
$$a_3 = a_2 x^5 = a_0 x^{5+4} = a_0 x^9 = a_0 x^{3^2}. \tag{18}$$

The exponent grows greater by $(2m-1)$ for each increase in m of 1. It is given by

$$\sum_{n=1}^{m} (2m - 1) = 2\frac{m(m+1)}{2} - m = m^2. \tag{19}$$

Therefore,

$$a_m = a_0 x^{m^2}. \tag{20}$$

This means that

$$G(z) = a_0 \sum_{m=-\infty}^{\infty} x^{m^2} z^{2m}. \tag{21}$$

The COEFFICIENT a_0 must be determined by going back to (4) and (8) and letting $z = 1$. Then

$$F(1) = \prod_{n=1}^{\infty} (1 + x^{2n-1})(1 + x^{2n-1})$$
$$= \prod_{n=1}^{\infty} (1 + x^{2n-1})^2 \tag{22}$$

$$G(1) = F(1) \prod_{n=1}^{\infty} (1 - x^{2n})$$
$$= \prod_{n=1}^{\infty} (1 + x^{2n-1})^2 \prod_{n=1}^{\infty} (1 - x^{2n})$$
$$= \prod_{n=1}^{\infty} (1 + x^{2n-1})^2 (1 - x^{2n}), \tag{23}$$

since multiplication is ASSOCIATIVE. It is clear from this expression that the a_0 term must be 1, because all other terms will contain higher POWERS of x. Therefore,

$$a_0 = 1, \tag{24}$$

so we have the Jacobi triple product,

$$G(z) = \prod_{n=1}^{\infty} (1 - x^{2n})(1 + x^{2n-1} z^2) \left(1 + \frac{x^{2n-1}}{z^2}\right)$$
$$= \sum_{m=-\infty}^{\infty} x^{m^2} z^{2m}. \tag{25}$$

see also EULER IDENTITY, JACOBI IDENTITIES, Q-FUNCTION, QUINTUPLE PRODUCT IDENTITY, RAMANUJAN PSI SUM, RAMANUJAN THETA FUNCTIONS, SCHRÖTER'S FORMULA, THETA FUNCTION

References

Andrews, G. E. *q-Series: Their Development and Application in Analysis, Number Theory, Combinatorics, Physics, and Computer Algebra.* Providence, RI: Amer. Math. Soc., pp. 63–64, 1986.

Borwein, J. M. and Borwein, P. B. "Jacobi's Triple Product and Some Number Theoretic Applications." Ch. 3 in *Pi & the AGM: A Study in Analytic Number Theory and Computational Complexity.* New York: Wiley, pp. 62–101, 1987.

Jacobi, C. G. J. *Fundamentia Nova Theoriae Functionum Ellipticarum.* Regiomonti, Sumtibus fratrum Borntraeger, p. 90, 1829.

Whittaker, E. T. and Watson, G. N. *A Course in Modern Analysis, 4th ed.* Cambridge, England: Cambridge University Press, p. 470, 1990.

Jacobi Zeta Function

Denoted $\mathrm{zn}(u,k)$ or $Z(u)$.

$$Z(\phi|m) \equiv E(\phi|m) - \frac{E(m)F(\phi|m)}{K(m)},$$

where ϕ is the AMPLITUDE, m is the PARAMETER, and F and K are ELLIPTIC INTEGRALS OF THE FIRST KIND, and E is an ELLIPTIC INTEGRAL OF THE SECOND KIND. See Gradshteyn and Ryzhik (1980, p. xxxi) for expressions in terms of THETA FUNCTIONS.

see also ZETA FUNCTION

References

Abramowitz, M. and Stegun, C. A. (Eds.). *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables, 9th printing.* New York: Dover, p. 595, 1972.

Gradshteyn, I. S. and Ryzhik, I. M. *Tables of Integrals, Series, and Products, 5th ed.* San Diego, CA: Academic Press, 1979.

Jacobian

Given a set $\mathbf{y} = \mathbf{f}(\mathbf{x})$ of n equations in n variables x_1, ..., x_n, written explicitly as

$$\mathbf{y} \equiv \begin{bmatrix} f_1 \\ f_2 \\ \vdots \\ f_n \end{bmatrix}, \tag{1}$$

or more explicitly as

$$\begin{cases} y_1 = f_1(x_1, \ldots, x_n) \\ \vdots \\ y_n = f_n(x_1, \ldots, x_n), \end{cases} \tag{2}$$

the Jacobian matrix, sometimes simply called "the Jacobian" (Simon and Blume 1994) is defined by

$$\mathsf{J}(x_1, x_2, x_3) = \begin{bmatrix} \frac{\partial y_1}{\partial x_1} & \cdots & \frac{\partial y_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial y_n}{\partial x_1} & \cdots & \frac{\partial y_n}{\partial x_n} \end{bmatrix}. \tag{3}$$

The DETERMINANT of J is the JACOBIAN DETERMINANT (confusingly, often called "the Jacobian" as well) and is denoted

$$J = \left| \frac{\partial(y_1, \ldots, y_n)}{\partial(x_1, \ldots, x_n)} \right|. \tag{4}$$

Taking the differential

$$d\mathbf{y} = \mathbf{y}_{\mathbf{x}}\, d\mathbf{x} \tag{5}$$

shows that J is the DETERMINANT of the MATRIX $\mathbf{y}_{\mathbf{x}}$, and therefore gives the ratios of n-D volumes (CONTENTS) in y and x,

$$dy_1 \cdots dy_n = \left| \frac{\partial(y_1, \ldots, y_n)}{\partial(x_1, \ldots, x_n)} \right| dx_1 \cdots dx_n. \tag{6}$$

The concept of the Jacobian can also be applied to n functions in more than n variables. For example, considering $f(u, v, w)$ and $g(u, v, w)$, the Jacobians

$$\frac{\partial(f, g)}{\partial(u, v)} = \begin{vmatrix} f_u & f_v \\ g_u & g_v \end{vmatrix} \tag{7}$$

$$\frac{\partial(f, g)}{\partial(u, w)} = \begin{vmatrix} f_u & f_w \\ g_u & g_w \end{vmatrix} \tag{8}$$

can be defined (Kaplan 1984, p. 99).

For the case of $n = 3$ variables, the Jacobian takes the special form

$$Jf(x_1, x_2, x_3) \equiv \left| \frac{\partial \mathbf{y}}{\partial x_1} \cdot \frac{\partial \mathbf{y}}{\partial x_2} \times \frac{\partial \mathbf{y}}{\partial x_3} \right|, \tag{9}$$

where $\mathbf{a} \cdot \mathbf{b}$ is the DOT PRODUCT and $\mathbf{b} \times \mathbf{c}$ is the CROSS PRODUCT, which can be expanded to give

$$\left| \frac{\partial(y_1, y_2, y_3)}{\partial(x_1, x_2, x_3)} \right| = \begin{vmatrix} \frac{\partial y_1}{\partial x_1} & \frac{\partial y_1}{\partial x_2} & \frac{\partial y_1}{\partial x_3} \\ \frac{\partial y_2}{\partial x_1} & \frac{\partial y_2}{\partial x_2} & \frac{\partial y_2}{\partial x_3} \\ \frac{\partial y_3}{\partial x_1} & \frac{\partial y_3}{\partial x_2} & \frac{\partial y_3}{\partial x_3} \end{vmatrix}. \tag{10}$$

see also CHANGE OF VARIABLES THEOREM, CURVILINEAR COORDINATES, IMPLICIT FUNCTION THEOREM

References

Kaplan, W. *Advanced Calculus, 3rd ed.* Reading, MA: Addison-Wesley, pp. 98–99, 123, and 238–245, 1984.

Simon, C. P. and Blume, L. E. *Mathematics for Economists.* New York: W. W. Norton, 1994.

Jacobian Conjecture

If $\det[F'(x)] = 1$ for a POLYNOMIAL mapping F (where det is the DETERMINANT), then F is BIJECTIVE with POLYNOMIAL inverse.

Jacobian Curve

The Jacobian of a linear net of curves of order n is a curve of order $3(n - 1)$. It passes through all points common to all curves of the net. It is the LOCUS of points where the curves of the net touch one another and of singular points of the curve.

see also CAYLEYIAN CURVE, HESSIAN COVARIANT, STEINERIAN CURVE

References

Coolidge, J. L. *A Treatise on Algebraic Plane Curves.* New York: Dover, p. 149, 1959.

Jacobian Determinant

see JACOBIAN

Jacobian Group

The Jacobian group of a 1-D linear series is given by intersections of the base curve with the JACOBIAN CURVE of itself and two curves cutting the series.

References

Coolidge, J. L. *A Treatise on Algebraic Plane Curves.* New York: Dover, p. 283, 1959.

Jacobsthal-Lucas Number

see JACOBSTHAL NUMBER

Jacobsthal-Lucas Polynomial

see JACOBSTHAL POLYNOMIAL

Jacobsthal Number

The Jacobsthal numbers are the numbers obtained by the U_ns in the LUCAS SEQUENCE with $P = 1$ and $Q = -2$, corresponding to $a = 2$ and $b = -1$. They and the Jacobsthal-Lucas numbers (the V_ns) satisfy the RECURRENCE RELATION

$$J_n = J_{n-1} + 2J_{n-2}. \tag{1}$$

The Jacobsthal numbers satisfy $J_0 = 0$ and $J_1 = 1$ and are 0, 1, 1, 3, 5, 11, 21, 43, 85, 171, 341, ... (Sloane's A001045). The Jacobsthal-Lucas numbers satisfy $j_0 = 2$ and $j_1 = 1$ and are 2, 1, 5, 7, 17, 31, 65, 127, 257, 511, 1025, ... (Sloane's A014551). The properties of these numbers are summarized in Horadam (1996). They are given by the closed form expressions

$$J_n = \sum_{r=0}^{\lfloor (n-1)/2 \rfloor} \binom{n-1-r}{r} 2^r \tag{2}$$

$$j_n = \sum_{r=0}^{\lfloor n/2 \rfloor} \frac{n}{n-r} \binom{n-r}{r} 2^r, \tag{3}$$

where $\lfloor x \rfloor$ is the FLOOR FUNCTION and $\binom{n}{k}$ is a BINOMIAL COEFFICIENT. The Binet forms are

$$J_n = \tfrac{1}{3}(a^n - b^n) = \tfrac{1}{3}[2^n - (-1)^n] \tag{4}$$

$$j_n = a^n + b^n = 2^n + (-1)^n. \tag{5}$$

The GENERATING FUNCTIONS are

$$\sum_{i=1}^{\infty} J_i x^{i-1} = (1 - x - 2x^2)^{-1} \tag{6}$$

$$\sum_{i=1}^{\infty} j_i x^{i-1} = (1 + 4x)(1 - x - 2x^2)^{-1}. \tag{7}$$

The Simson FORMULAS are

$$J_{n+1}J_{n-1} - {J_n}^2 = (-1)^n 2^{n-1} \tag{8}$$

$$j_{n+1}j_{n-1} - {j_n}^2 = 9(-1)^{n-1}2^{n-1} = -9(J_{n+1}J_{n-1} - {J_n}^2). \tag{9}$$

Summation FORMULAS include

$$\sum_{i=2}^{n} J_i = \tfrac{1}{2}(J_{n+2} - 3) \tag{10}$$

$$\sum_{i=1}^{n} j_i = \tfrac{1}{2}(j_{n+2} - 5). \tag{11}$$

Interrelationships are

$$j_n J_n = J_{2n} \tag{12}$$

$$j_n = J_{n+1} + 2J_{n-1} \tag{13}$$

$$9J_n = j_{n+1} + 2j_{n-1} \tag{14}$$

$$j_{n+1} + j_n = 3(J_{n+1} + J_n) = 3 \cdot 2^n \tag{15}$$

$$j_{n+1} - j_n = 3(J_{n+1} - J_n) + 4(-1)^{n+1} = 2^n + 2(-1)^{n+1} \tag{16}$$

$$j_{n+1} - 2j_n = 3(2J_n - J_{n+1}) = 3(-1)^{n+1} \tag{17}$$

$$2j_{n+1} + j_{n-1} = 3(2J_{n+1} + J_{n-1}) + 6(-1)^{n+1} \tag{18}$$

$$j_{n+r} + j_{n-r} = 3(J_{n+r} + J_{n-r}) + 4(-1)^{n-r} \tag{19}$$

$$= 2^{n-r}(2^{2r} + 1) + 2(-1)^{n-r} \tag{20}$$

$$j_{n+r} - j_{n-r} = 3(J_{n+r} - J_{n-r}) = 2^{n-r}(2^{2r} - 1) \tag{21}$$

$$j_n = 3J_n + 2(-1)^n \tag{22}$$

$$3J_n + j_n = 2^{n+1} \tag{23}$$

$$J_n + j_n = 2J_{n+1} \tag{24}$$

$$j_{n+2}j_{n-2} - {j_n}^2 = -9(J_{n+2}J_{n-2} - J_n)^2 = 9(-1)^n 2^{n-2} \tag{25}$$

$$J_m j_n + J_n j_m = 2J_{m+n} \tag{26}$$

$$j_m j_n + 9J_m J_n = 2j_{m+n} \tag{27}$$

$${j_n}^2 + 9{J_n}^2 = 2j_{2n} \tag{28}$$

$$J_m j_n - J_n j_m = (-1)^n 2^{n+1} J_{m-n} \tag{29}$$

$$j_m j_n - 9J_m J_n = (-1)^n 2^{n+1} j_{m-n} \tag{30}$$

$${j_n}^2 - 9{J_n}^2 = (-1)^n 2^{n+2} \tag{31}$$

(Horadam 1996).

References

Horadam, A. F. "Jacobsthal and Pell Curves." *Fib. Quart.* **26**, 79–83, 1988.

Horadam, A. F. "Jacobsthal Representation Numbers." *Fib. Quart.* **34**, 40–54, 1996.

Sloane, N. J. A. Sequences A014551 and A001045/M2482 in "An On-Line Version of the Encyclopedia of Integer Sequences."

Jacobsthal Polynomial

The Jacobsthal polynomials are the POLYNOMIALS obtained by setting $p(x) = 1$ and $q(x) = 2x$ in the LUCAS POLYNOMIAL SEQUENCE. The first few Jacobsthal polynomials are

$$J_1(x) = 1$$
$$J_2(x) = 1$$
$$J_3(x) = 1 + 2x$$
$$J_4(x) = 1 + 4x$$
$$J_5(x) = 4x^2 + 6x + 1,$$

and the first few Jacobsthal-Lucas polynomials are

$$\begin{aligned} j_1(x) &= 1 \\ j_2(x) &= 4x+1 \\ j_3(x) &= 6x+1 \\ j_4(x) &= 8x^2+8x+1 \\ j_5(x) &= 20x^2+10x+1. \end{aligned}$$

Jacobsthal and Jacobsthal-Lucas polynomials satisfy

$$J_n(1) = J_n$$

$$j_n(1) = j_n$$

where J_n is a JACOBSTHAL NUMBER and j_n is a JACOBSTHAL-LUCAS NUMBER.

Janko Groups

The SPORADIC GROUPS J_1, J_2, J_3 and J_4. The Janko group J_2 is also known as the HALL-JANKO GROUP.

see also SPORADIC GROUP

References

Wilson, R. A. "ATLAS of Finite Group Representation." `http://for.mat.bham.ac.uk/atlas/J1.html`, `J2.html`, `J3.html`, and `J4.html`.

Japanese Triangulation Theorem

Let a convex CYCLIC POLYGON be TRIANGULATED in any manner, and draw the INCIRCLE to each TRIANGLE so constructed. Then the sum of the INRADII is a constant independent of the TRIANGULATION chosen. This theorem can be proved using CARNOT'S THEOREM. It is also true that if the sum of INRADII does not depend on the TRIANGULATION of a POLYGON, then the POLYGON is CYCLIC.

see also CARNOT'S THEOREM, CYCLIC POLYGON, INCIRCLE, INRADIUS, TRIANGULATION

References

Honsberger, R. *Mathematical Gems III.* Washington, DC: Math. Assoc. Amer., pp. 24–26, 1985.

Lambert, T. "The Delaunay Triangulation Maximizes the Mean Inradius." *Proc. Sixth Canadian Conf. Comput. Geometry.* Saskatoon, Saskatchewan, Canada, pp. 201–206, August 1994.

Jarnick's Inequality

Given a CONVEX plane region with AREA A and PERIMETER p, then

$$|N - A| < p,$$

where N is the number of enclosed LATTICE POINTS.

see also LATTICE POINT, NOSARZEWSKA'S INEQUALITY

Jeep Problem

Maximize the distance a jeep can penetrate into the desert using a given quantity of fuel. The jeep is allowed to go forward, unload some fuel, and then return to its base using the fuel remaining in its tank. At its base, it may refuel and set out again. When it reaches fuel it has previously stored, it may then use it to partially fill its tank. This problem is also called the EXPLORATION PROBLEM (Ball and Coxeter 1987).

Given $n+f$ (with $0 \leq f < 1$) drums of fuel at the edge of the desert and a jeep capable of holding one drum (and storing fuel in containers along the way), the maximum one-way distance which can be traveled (assuming the jeep travels one unit of distance per drum of fuel expended) is

$$\begin{aligned} d &= \frac{f}{2n+1} + \sum_{i=1}^{n} \frac{1}{2i-1} \\ &= \frac{f}{2n+1} + \tfrac{1}{2}[\gamma + 2\ln 2 + \psi_0(\tfrac{1}{2}+n)], \end{aligned}$$

where γ is the EULER-MASCHERONI CONSTANT and $\psi_n(z)$ the POLYGAMMA FUNCTION.

For example, the farthest a jeep with $n=1$ drum can travel is obviously 1 unit. However, with $n=2$ drums of gas, the maximum distance is achieved by filling up the jeep's tank with the first drum, traveling 1/3 of a unit, storing 1/3 of a drum of fuel there, and then returning to base with the remaining 1/3 of a tank. At the base, the tank is filled with the second drum. The jeep then travels 1/3 of a unit (expending 1/3 of a drum of fuel), refills the tank using the 1/3 of a drum of fuel stored there, and continues an additional 1 unit of distance on a full tank, giving a total distance of 4/3. The solutions for $n = 1, 2, \ldots$ drums are 1, 4/3, 23/15, 176/105, 563/315, ..., which can also be written as $a(n)/b(n)$, where

$$\begin{aligned} a(n) &= \left(\frac{1}{1} + \frac{1}{3} + \ldots + \frac{1}{2n-1}\right) \operatorname{LCM}(1,3,5,\ldots,2n-1) \\ b(n) &= \operatorname{LCM}(1,3,5,\ldots,2n-1) \end{aligned}$$

(Sloane's A025550 and A025547).

see also HARMONIC NUMBER

References

Alway, G. C. "Crossing the Desert." *Math. Gaz.* **41**, 209, 1957.

Ball, W. W. R. and Coxeter, H. S. M. *Mathematical Recreations and Essays, 13th ed.* New York: Dover, p. 32, 1987.

Bellman, R. Exercises 54–55 *Dynamic Programming.* Princeton, NJ: Princeton University Press, p. 103, 1955.

Fine, N. J. "The Jeep Problem." *Amer. Math. Monthly* **54**, 24–31, 1947.

Gale, D. "The Jeep Once More or Jeeper by the Dozen." *Amer. Math. Monthly* **77**, 493–501, 1970.

Gardner, M. *The Second Scientific American Book of Mathematical Puzzles & Diversions: A New Selection.* New York: Simon and Schuster, pp. 152 and 157–159, 1961.

Haurath, A.; Jackson, B.; Mitchem, J.; and Schmeichel, E. "Gale's Round-Trip Jeep Problem." *Amer. Math. Monthly* **102**, 299–309, 1995.
Helmer, O. "A Problem in Logistics: The Jeep Problem." Project Rand Report No. Ra 15015, Dec. 1947.
Phipps, C. G. "The Jeep Problem, A More General Solution." *Amer. Math. Monthly* **54**, 458–462, 1947.

Jenkins-Traub Method

A complicated POLYNOMIAL ROOT-finding algorithm which is used in the *IMSL®* (IMSL, Houston, TX) library and which Press *et al.* (1992) describe as "practically a standard in black-box POLYNOMIAL ROOT-finders."

References
IMSL, Inc. *IMSL Math/Library User's Manual.* Houston, TX: IMSL, Inc.
Press, W. H.; Flannery, B. P.; Teukolsky, S. A.; and Vetterling, W. T. *Numerical Recipes in FORTRAN: The Art of Scientific Computing, 2nd ed.* Cambridge, England: Cambridge University Press, p. 369, 1992.
Ralston, A. and Rabinowitz, P. §8.9–8.13 in *A First Course in Numerical Analysis, 2nd ed.* New York: McGraw-Hill, 1978.

Jensen's Formula

$$\int_0^{2\pi} \ln|z+e^{i\theta}|\,d\theta = 2\pi \ln^+|z|,$$

where

$$\ln^+ \equiv \max(0, \ln x)$$

and $\ln x$ is the NATURAL LOGARITHM.

Jensen's Inequality

For a REAL CONTINUOUS CONCAVE FUNCTION

$$\frac{\sum f(x_i)}{n} \leq f\left(\frac{\sum x_i}{n}\right)$$

if f is concave down,

$$\frac{\sum f(x_i)}{n} \geq f\left(\frac{\sum x_i}{n}\right)$$

if f is concave up, and

$$\frac{\sum f(x_i)}{n} = f\left(\frac{\sum x_i}{n}\right)$$

IFF $x_1 = x_2 = \ldots = x_n$. A special case is

$$\sqrt[n]{x_1 x_2 \cdots x_n} \leq \frac{x_1 + x_2 + \ldots + x_n}{n},$$

with equality IFF $x_1 = x_2 = \ldots = x_n$.

see also CONCAVE FUNCTION, MAHLER'S MEASURE

References
Gradshteyn, I. S. and Ryzhik, I. M. *Tables of Integrals, Series, and Products, 5th ed.* San Diego, CA: Academic Press, p. 1101, 1979.

Jensen Polynomial

Let $f(x)$ be a real ENTIRE FUNCTION of the form

$$f(x) = \sum_{k=0}^{\infty} \gamma_k \frac{x^k}{k!},$$

where the γ_ks are POSITIVE and satisfy TURÁN'S INEQUALITIES

$$\gamma_k{}^2 - \gamma_{k-1}\gamma_{k+1} \geq 0$$

for $k = 1, 2, \ldots$. The Jensen polynomial $g(t)$ associated with $f(x)$ is then given by

$$g_n(t) = \sum_{k=0}^{n} \binom{n}{k} \gamma_k t^k,$$

where $\binom{a}{b}$ is a BINOMIAL COEFFICIENT.

References
Csordas, G.; Varga, R. S.; and Vincze, I. "Jensen Polynomials with Applications to the Riemann ζ-Function." *J. Math. Anal. Appl.* **153**, 112–135, 1990.

Jerabek's Hyperbola

The ISOGONAL CONJUGATE of the EULER LINE. It passes through the the vertices of a TRIANGLE, the ORTHOCENTER, CIRCUMCENTER, the LEMOINE POINT, and the ISOGONAL CONJUGATE points of the NINE-POINT CENTER and DE LONGCHAMPS POINT.

see also CIRCUMCENTER, DE LONGCHAMPS POINT, EULER LINE, ISOGONAL CONJUGATE, LEMOINE POINT, NINE-POINT CENTER, ORTHOCENTER

References
Casey, J. *A Treatise on the Analytical Geometry of the Point, Line, Circle, and Conic Sections, Containing an Account of Its Most Recent Extensions with Numerous Examples, 2nd rev. enl. ed.* Dublin: Hodges, Figgis, & Co., 1893.
Pinkernell, G. M. "Cubic Curves in the Triangle Plane." *J. Geom.* **55**, 141–161, 1996.
Vandeghen, A. "Some Remarks on the Isogonal and Cevian Transforms. Alignments of Remarkable Points of a Triangle." *Amer. Math. Monthly* **72**, 1091–1094, 1965.

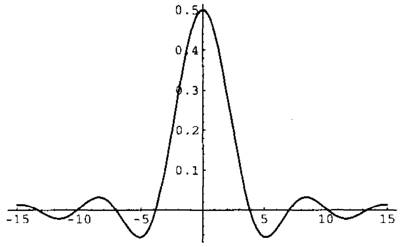
Jerk

The jerk $\mathbf{j}$ is defined as the time DERIVATIVE of the VECTOR ACCELERATION $\mathbf{a}$,

$$\mathbf{j} \equiv \frac{d\mathbf{a}}{dt}.$$

see also ACCELERATION, VELOCITY

Jinc Function



The jinc function is defined as

$$\operatorname{jinc}(x) \equiv \frac{J_1(x)}{x},$$

where $J_1(x)$ is a BESSEL FUNCTION OF THE FIRST KIND, and satisfies $\lim_{x\to 0} \operatorname{jinc}(x) = 1/2$. The DERIVATIVE of the jinc function is given by

$$\operatorname{jinc}'(x) = -\frac{J_2(x)}{x}.$$

The function is sometimes normalized by multiplying by a factor of 2 so that $\operatorname{jinc}(0) = 1$ (Siegman 1986, p. 729).

see also BESSEL FUNCTION OF THE FIRST KIND, SINC FUNCTION

References
Siegman, A. E. *Lasers.* Sausalito, CA: University Science Books, 1986.

Jitter

A SAMPLING phenomenon produced when a waveform is not sampled uniformly at an interval t each time, but rather at a series of slightly shifted intervals $t + \Delta t_i$ such that the average $\langle \Delta t_i \rangle = 0$.

see also GHOST, SAMPLING

Joachimsthal's Equation

Using CLEBSCH-ARONHOLD NOTATION,

$$\begin{aligned}\xi_1^n a_y^n + \xi_1^{n-1}\xi_2 a_y^{n-1} a_x + \tfrac{1}{2}n(n-1)\xi_1^{n-2}\xi_2^2 a_y^{n-2} a_x^2 + \ldots \\ + n\xi_1\xi_2^{n-1} a_y a_x^{n-1} + \xi_2^n a_x^n = 0.\end{aligned}$$

References
Coolidge, J. L. *A Treatise on Algebraic Plane Curves.* New York: Dover, p. 89, 1959.

Johnson Circle

The CIRCUMCIRCLE in JOHNSON'S THEOREM.

see also JOHNSON'S THEOREM

Johnson's Equation

The PARTIAL DIFFERENTIAL EQUATION

$$\frac{\partial}{\partial x}\left(u_t + uu_x + \tfrac{1}{2}u_{xxx} + \frac{u}{2t}\right) + \frac{3\alpha^2}{2t^2}u_{yy} = 0$$

which arises in the study of water waves.

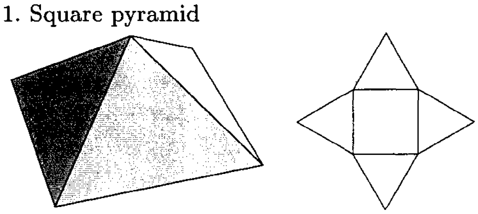
References
Infeld, E. and Rowlands, G. *Nonlinear Waves, Solitons, and Chaos.* Cambridge, England: Cambridge University Press, p. 223, 1990.

Johnson Solid

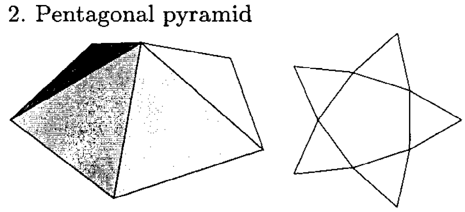
The Johnson solids are the CONVEX POLYHEDRA having regular faces (with the exception of the completely regular PLATONIC SOLIDS, the "SEMIREGULAR" ARCHIMEDEAN SOLIDS, and the two infinite families of PRISMS and ANTIPRISMS). There are 28 simple (i.e., cannot be dissected into two other regular-faced polyhedra by a plane) regular-faced polyhedra in addition to the PRISMS and ANTIPRISMS (Zalgaller 1969), and Johnson (1966) proposed and Zalgaller (1969) proved that there exist exactly 92 Johnson solids in all.

A database of solids and VERTEX NETS of these solids is maintained on the Bell Laboratories Netlib server, but a few errors exist in several entries. A concatenated and corrected version of the files is given by Weisstein, together with *Mathematica*® (Wolfram Research, Champaign, IL) code to display the solids and nets. The following table summarizes the names of the Johnson solids and gives their images and nets.

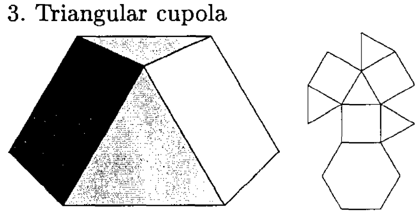
1. Square pyramid



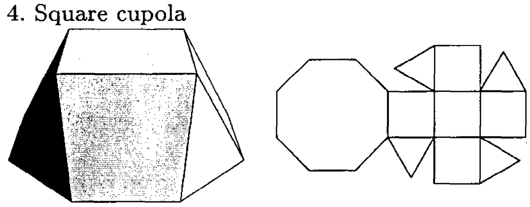
2. Pentagonal pyramid



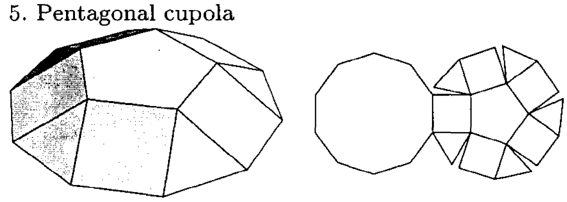
3. Triangular cupola



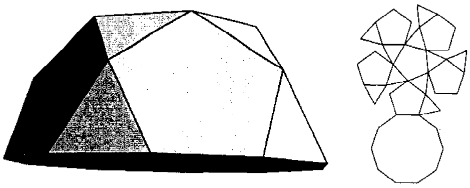
4. Square cupola



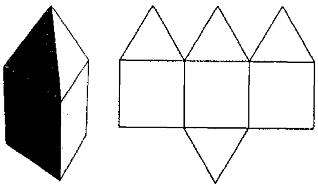
5. Pentagonal cupola



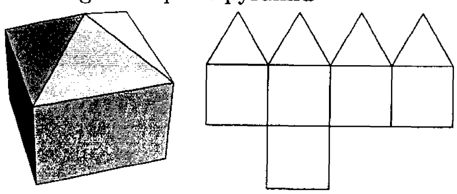
6. Pentagonal rotunda



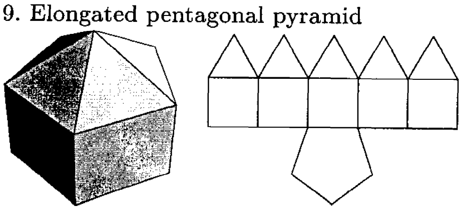
7. Elongated triangular pyramid



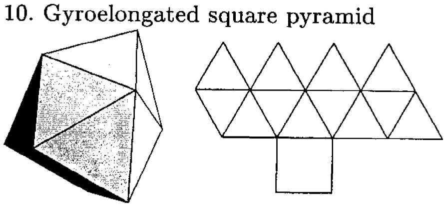
8. Elongated square pyramid



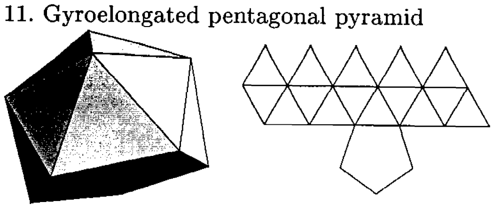
9. Elongated pentagonal pyramid



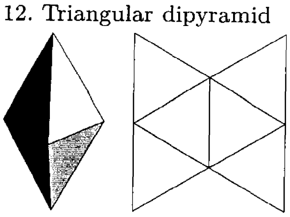
10. Gyroelongated square pyramid



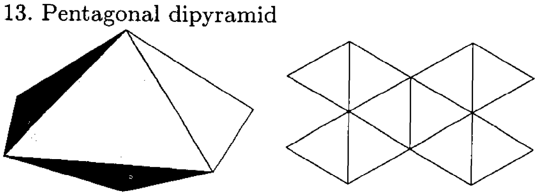
11. Gyroelongated pentagonal pyramid



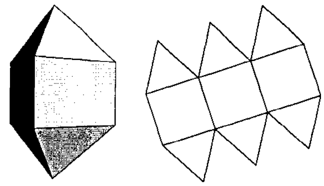
12. Triangular dipyramid



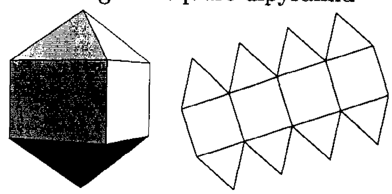
13. Pentagonal dipyramid



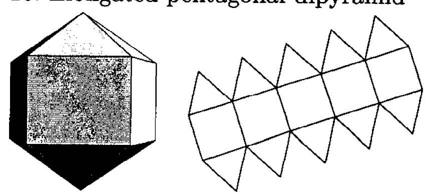
14. Elongated triangular dipyramid



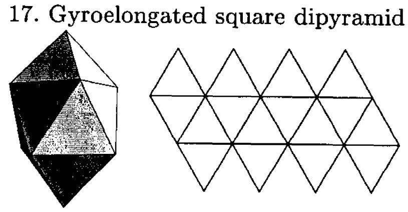
15. Elongated square dipyramid



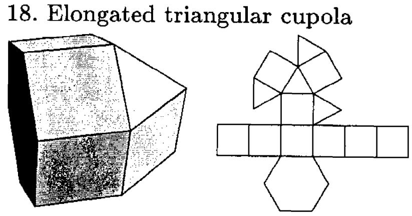
16. Elongated pentagonal dipyramid



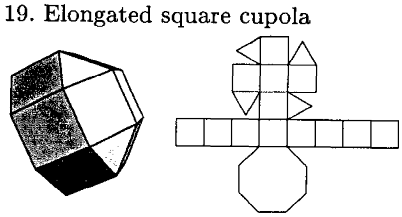
17. Gyroelongated square dipyramid



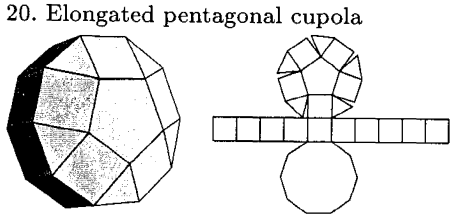
18. Elongated triangular cupola



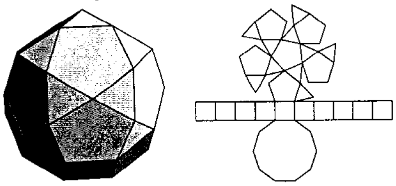
19. Elongated square cupola



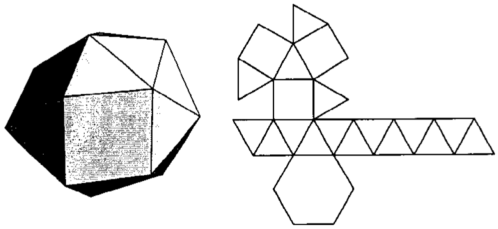
20. Elongated pentagonal cupola



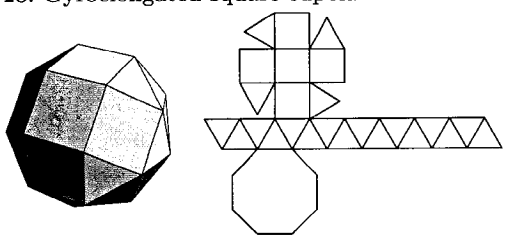
21. Elongated pentagonal rotunda



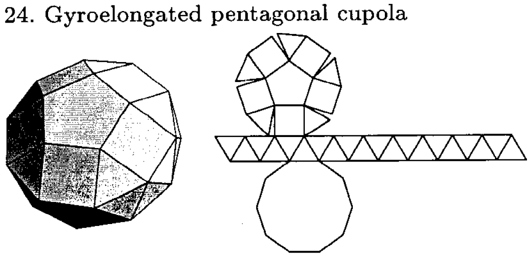
22. Gyroelongated triangular cupola



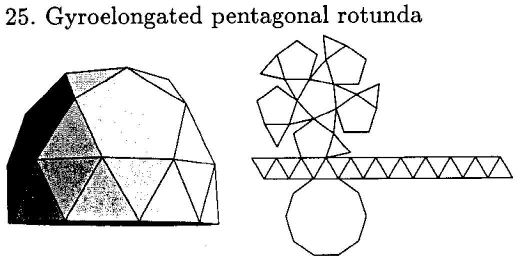
23. Gyroelongated square cupola



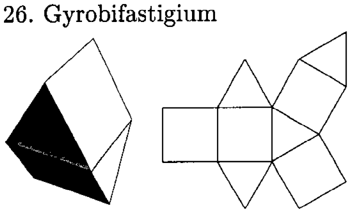
24. Gyroelongated pentagonal cupola



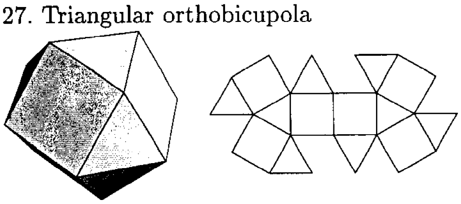
25. Gyroelongated pentagonal rotunda



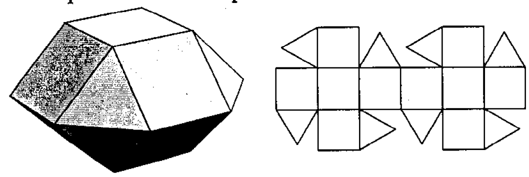
26. Gyrobifastigium



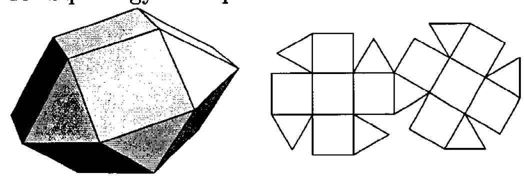
27. Triangular orthobicupola



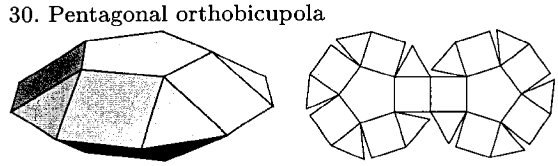
28. Square orthobicupola



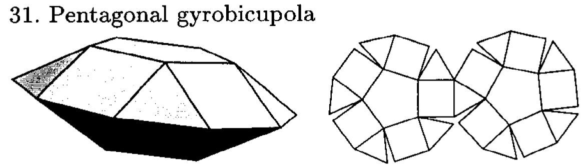
29. Square gyrobicupola



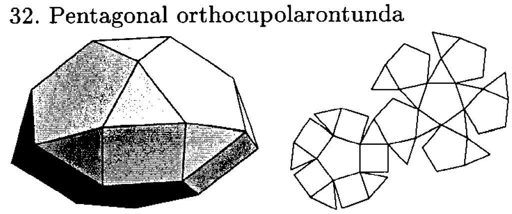
30. Pentagonal orthobicupola



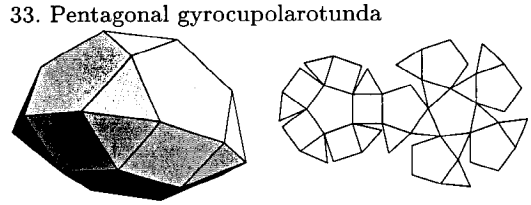
31. Pentagonal gyrobicupola



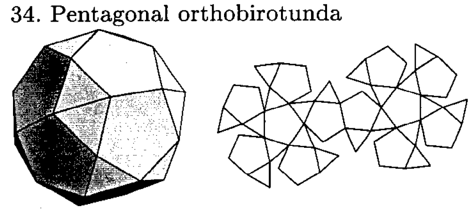
32. Pentagonal orthocupolarontunda



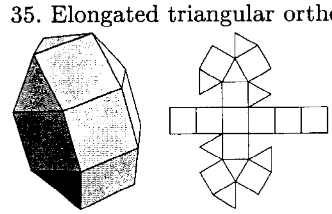
33. Pentagonal gyrocupolarotunda



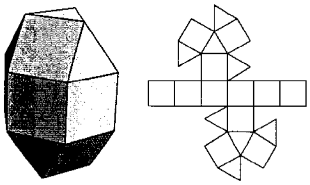
34. Pentagonal orthobirotunda



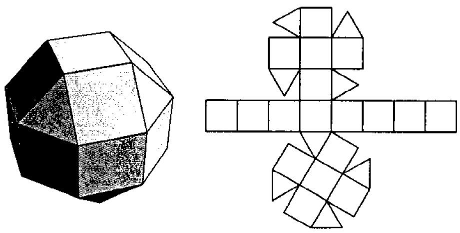
35. Elongated triangular orthobicupola



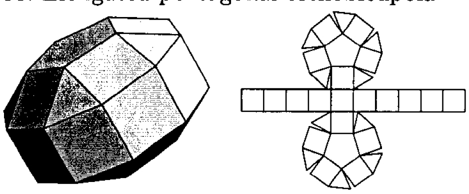
36. Elongated triangular gyrobicupola



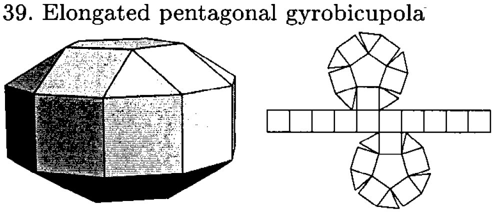
37. Elongated square gyrobicupola



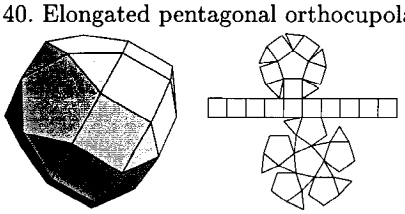
38. Elongated pentagonal orthobicupola



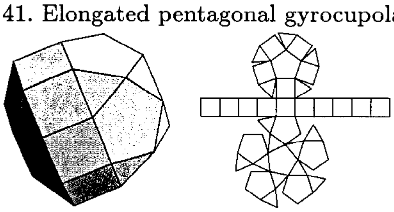
39. Elongated pentagonal gyrobicupola



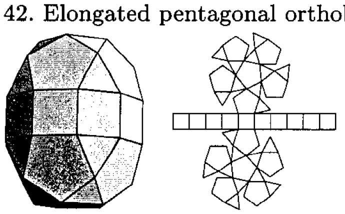
40. Elongated pentagonal orthocupolarotunda



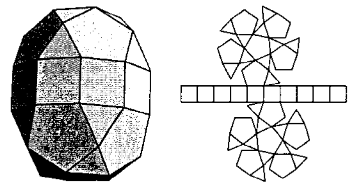
41. Elongated pentagonal gyrocupolarotunda



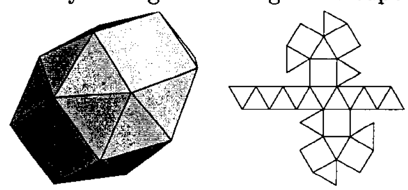
42. Elongated pentagonal orthobirotunda



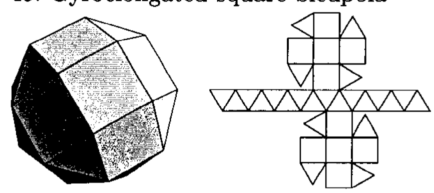
43. Elongated pentagonal gyrobirotunda



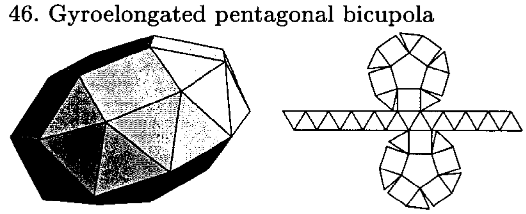
44. Gyroelongated triangular bicupola



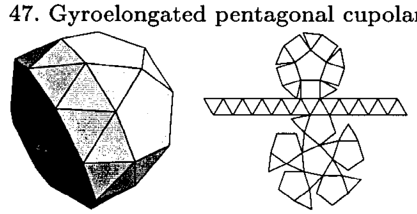
45. Gyroelongated square bicupola



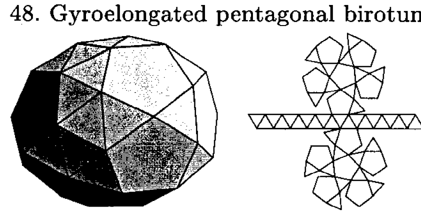
46. Gyroelongated pentagonal bicupola



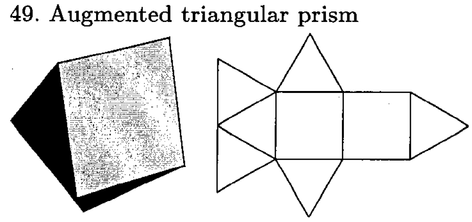
47. Gyroelongated pentagonal cupolarotunda



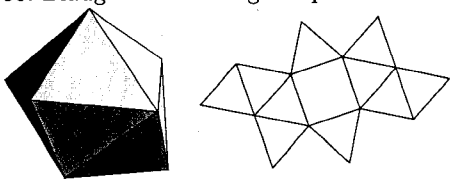
48. Gyroelongated pentagonal birotunda



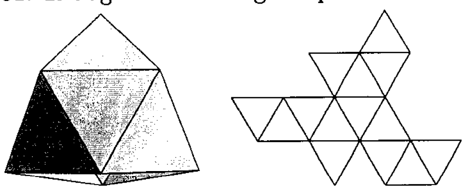
49. Augmented triangular prism



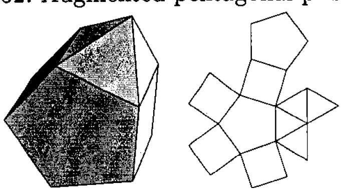
50. Biaugmented triangular prism



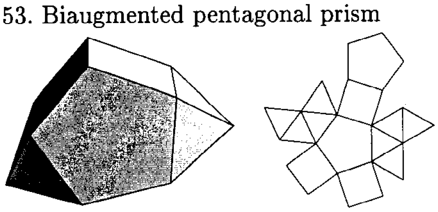
51. Triaugmented triangular prism



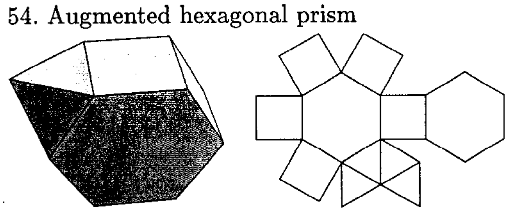
52. Augmented pentagonal prism



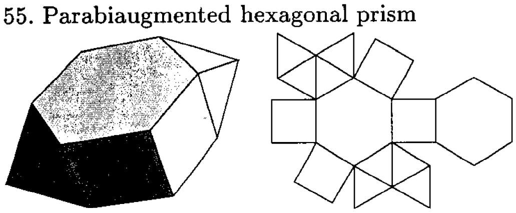
53. Biaugmented pentagonal prism



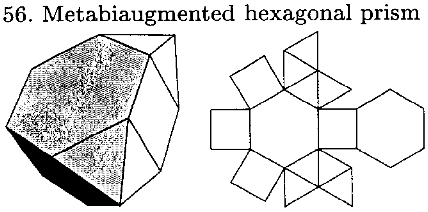
54. Augmented hexagonal prism



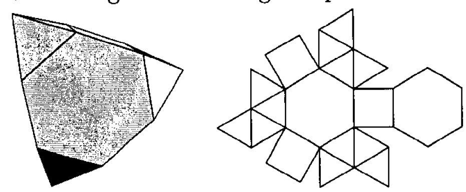
55. Parabiaugmented hexagonal prism



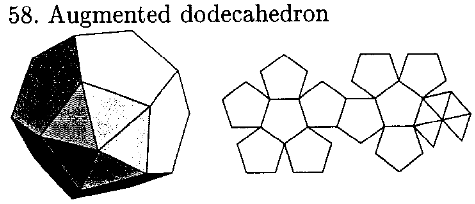
56. Metabiaugmented hexagonal prism



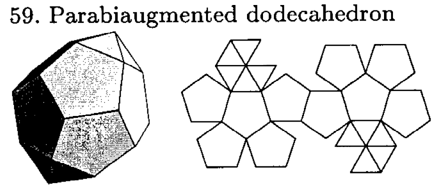
57. Triaugmented hexagonal prism



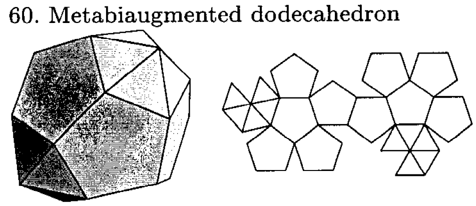
58. Augmented dodecahedron



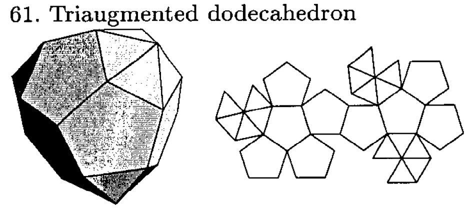
59. Parabiaugmented dodecahedron



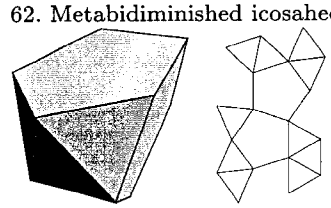
60. Metabiaugmented dodecahedron



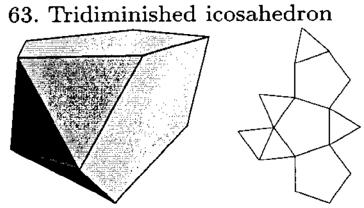
61. Triaugmented dodecahedron



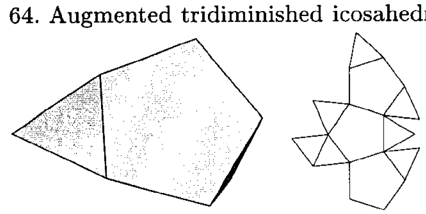
62. Metabidiminished icosahedron



63. Tridiminished icosahedron

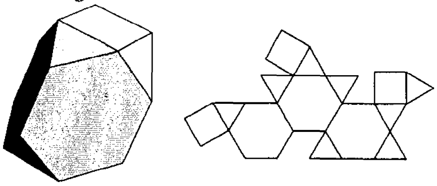


64. Augmented tridiminished icosahedron

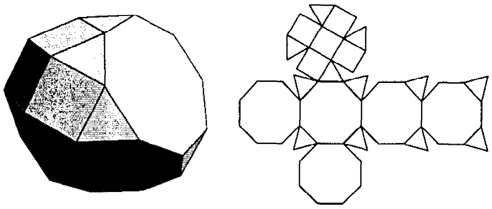


Johnson Solid

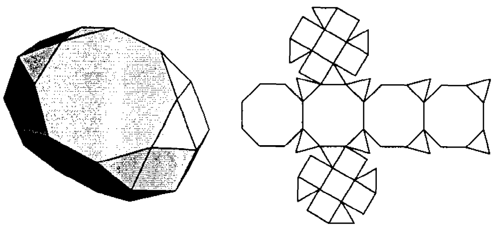
65. Augmented truncated tetrahedron



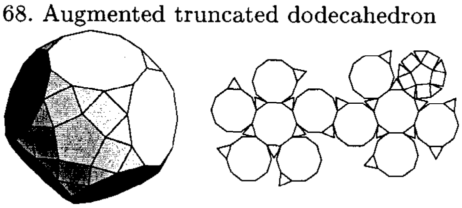
66. Augmented truncated cube



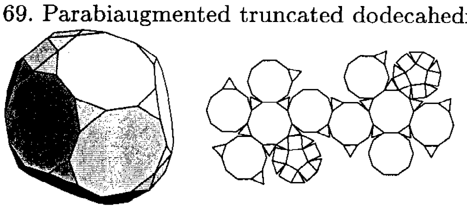
67. Biaugmented truncated cube



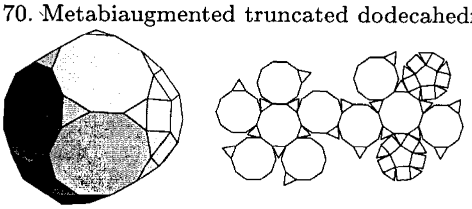
68. Augmented truncated dodecahedron



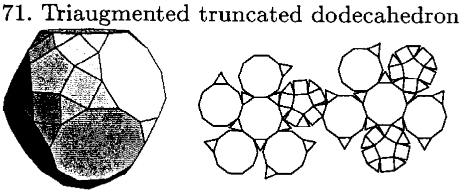
69. Parabiaugmented truncated dodecahedron



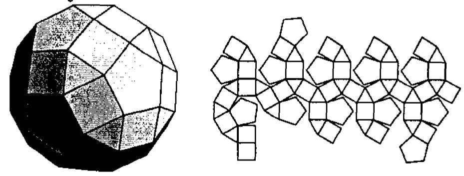
70. Metabiaugmented truncated dodecahedron



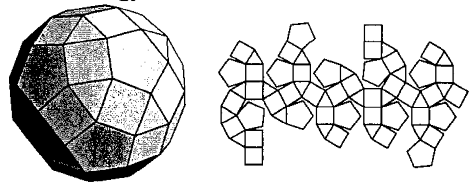
71. Triaugmented truncated dodecahedron



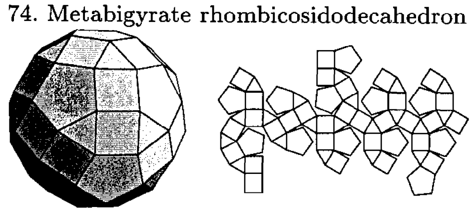
72. Gyrate rhombicosidodecahedron



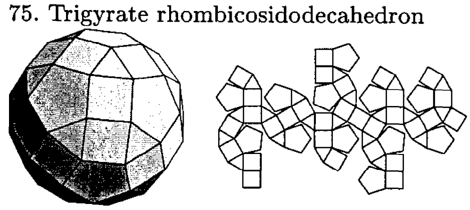
73. Parabigyrate rhombicosidodecahedron



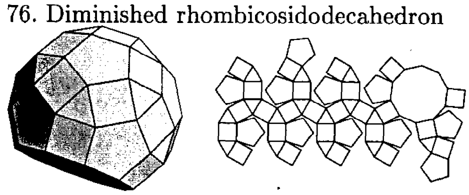
74. Metabigyrate rhombicosidodecahedron



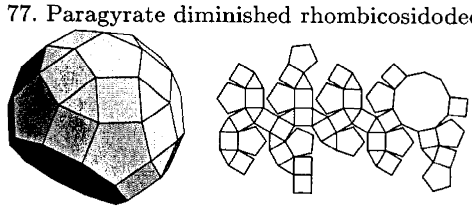
75. Trigyrate rhombicosidodecahedron



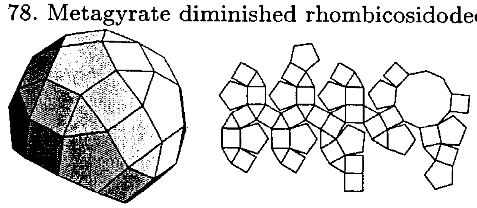
76. Diminished rhombicosidodecahedron



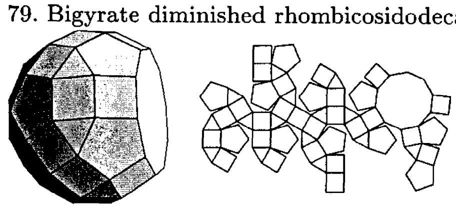
77. Paragyrate diminished rhombicosidodecahedron



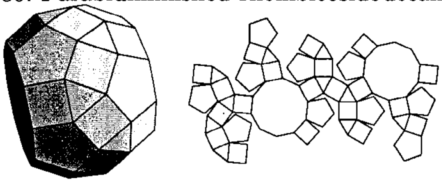
78. Metagyrate diminished rhombicosidodecahedron



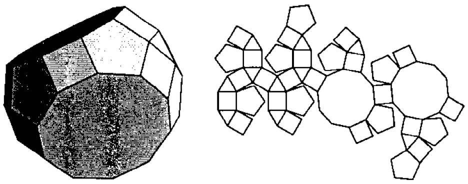
79. Bigyrate diminished rhombicosidodecahedron



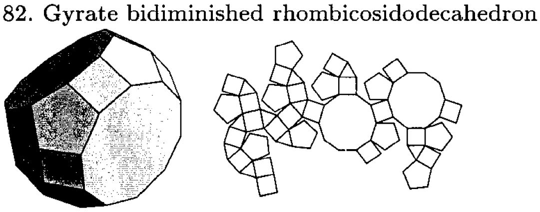
80. Parabidiminished rhombicosidodecahedron



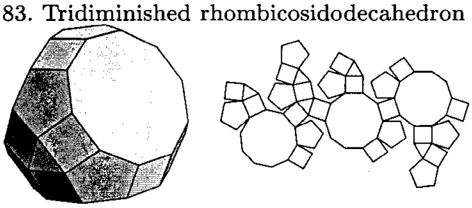
81. Metabidiminished rhombicosidodecahedron



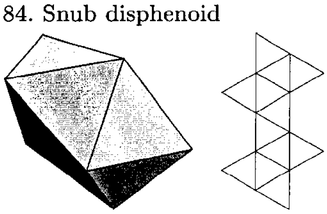
82. Gyrate bidiminished rhombicosidodecahedron



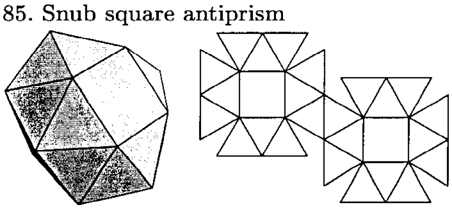
83. Tridiminished rhombicosidodecahedron



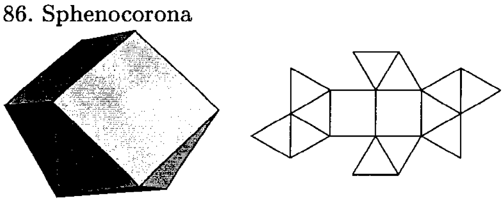
84. Snub disphenoid



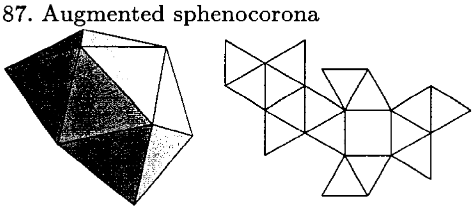
85. Snub square antiprism



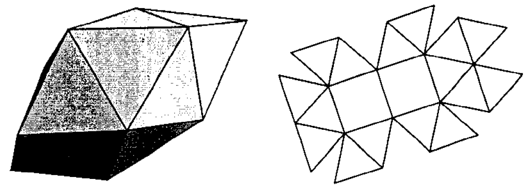
86. Sphenocorona



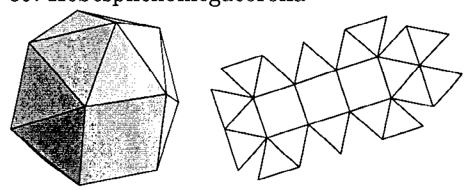
87. Augmented sphenocorona



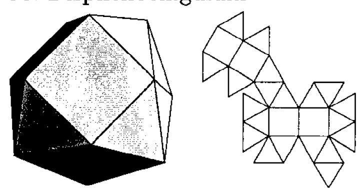
88. Sphenomegacorona



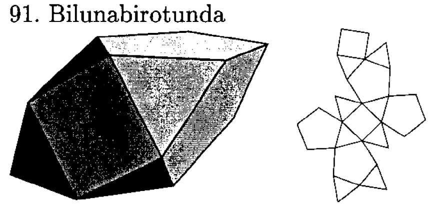
89. Hebesphenomegacorona



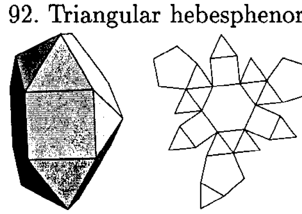
90. Disphenocingulum



91. Bilunabirotunda



92. Triangular hebesphenorotunda



The number of constituent n-gons ($\{n\}$) for each Johnson solid are given in the following table.

J_n	{3}	{4}	{5}	{6}	{8}	{10}
1	4	1				
2	5		1			
3	4	3		1		
4	4	5			1	
5	5	5	1			1
6	10		6			1
7	4	3				
8	4	5				
9	5	5	1			
10	12	1				
11	15		1			
12	6					
13	10					
14	6	3				
15	8	4				
16	10	5				
17	16					
18	4	9		1		
19	4	13			1	
20	5	15	1			1
21	10	10	6			1
22	16	3		1		
23	20	5			1	
24	25	5	1			1
25	30		6			1
26	4	4				
27	8	6				
28	8	10				
29	8	10				
30	10	10	2			
31	10	10	2			
32	15	5	7			
33	15	5	7			
34	20		12			
35	8	12				
36	8	12				
37	8	18				
38	10	20	2			
39	10	20	2			
40	15	15	7			
41	15	15	7			
42	20	10	12			
43	20	10	12			
44	20	6				
45	24	10				
46	30	10	2			
47	35	5	7			
48	40		12			
49	6	2				
50	10	1				
51	14					
52	4	4	2			
53	8	3	2			
54	4	5		2		
55	8	4		2		
56	8	4		2		
57	12	3		2		
58	5		11			
59	10		10			
60	10		10			
61	15		9			
62	10		2			
63	5		3			
64	7		3			
65	8	3		3		
66	12	5			5	
67	16	10			4	
68	25	5	1			11
69	30	10	2			10
70	30	10	2			10
71	35	15	3			9
72	20	30	12			
73	20	30	12			
74	20	30	12			
75	20	30	12			
76	15	25	11			1
77	15	25	11			1
78	15	25	11			1
79	15	25	11			1
80	10	20	10			2
81	10	20	10			2
82	10	20	10			2
83	5	15	9			3
84	12					
85	24	2				
86	12	2				
87	16	1				
88	16	2				
89	18	3				
90	20	4				
91	8	2	4			
92	13	3	3	1		

see also ANTIPRISM, ARCHIMEDEAN SOLID, CONVEX POLYHEDRON, KEPLER-POINSOT SOLID, POLYHEDRON, PLATONIC SOLID, PRISM, UNIFORM POLYHEDRON

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Weisstein, E. W. "Johnson Solid Netlib Database." `http://www.astro.virginia.edu/~eww6n/math/notebooks/JohnsonSolids.dat`.

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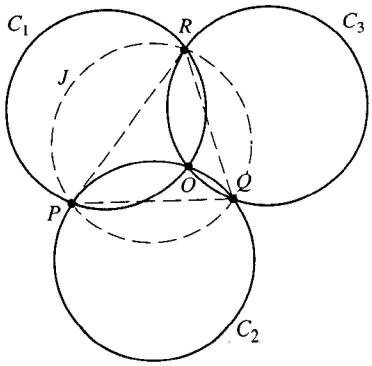
Johnson's Theorem



Let three equal CIRCLES with centers C_1, C_2, and C_3 intersect in a single point O and intersect pairwise in the points P, Q, and R. Then the CIRCUMCIRCLE J of ΔPQR (the so-called JOHNSON CIRCLE) is congruent to the original three.

see also CIRCUMCIRCLE, JOHNSON CIRCLE

References

Emch, A. "Remarks on the Foregoing Circle Theorem." *Amer. Math. Monthly* **23**, 162–164, 1916.

Honsberger, R. *Mathematical Gems II.* Washington, DC: Math. Assoc. Amer., pp. 18–21, 1976.

Johnson, R. "A Circle Theorem." *Amer. Math. Monthly* **23**, 161–162, 1916.

Join (Graph)

Let x and y be distinct nodes of G which are not joined by an EDGE. Then the graph G/xy which is formed by adding the EDGE (x, y) to G is called a join of G.

Join (Spaces)

Let X and Y be TOPOLOGICAL SPACES. Then their join is the factor space

$$X * Y = (X \times Y \times I)/\sim,$$

where $\sim$ is the EQUIVALENCE RELATION

$$(x,y,t) \sim (x',y',t') \Leftrightarrow \begin{cases} t = t' = 0 \text{ and } x = x' \\ \text{or} \\ t = t' = 1 \text{ and } y = y. \end{cases}$$

see also CONE (SPACE), SUSPENSION

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Rolfsen, D. *Knots and Links.* Wilmington, DE: Publish or Perish Press, p. 6, 1976.

Joint Distribution Function

A joint distribution function is a DISTRIBUTION FUNCTION in two variables defined by

$$D(x,y) \equiv P(X \leq x, Y \leq y) \tag{1}$$

$$D_x(x) \equiv D(x,\infty) \tag{2}$$

$$D_y(y) \equiv D(\infty, y) \tag{3}$$

so that the joint probability function

$$P[(x,y) \in C)] = \iint_{(x,y) \in C} P(x,y)\,dx\,dy \tag{4}$$

$$P(x \in A, y \in B) = \int_B \int_A P(x,y)\,dx\,dy \tag{5}$$

$$P(x,y) = P\{x \in (-\infty, x], y \in (-\infty, y]\} = \int_{-\infty}^{b} \int_{-\infty}^{a} P(x,y)\,dx\,dy \tag{6}$$

$$P(a \leq x \leq a+da, b \leq y \leq b+db) = \int_b^{b+db} \int_a^{a+da} P(x,y)\,dx\,dy \approx P(a,b)\,da\,db. \tag{7}$$

A multiple distribution function is of the form

$$D(a_1, \ldots, a_n) \equiv P(x_1 \leq a_1, \ldots, x_n \leq a_n). \tag{8}$$

see also DISTRIBUTION FUNCTION

Joint Probability Density Function

see JOINT DISTRIBUTION FUNCTION

Joint Theorem

see GAUSSIAN JOINT VARIABLE THEOREM

Jonah Formula

A formula for the generalized CATALAN NUMBER $_pd_{qi}$. The general formula is

$$\binom{n-q}{k-1} = \sum_{i=1}^{k} {}_pd_{qi} \binom{n-pi}{k-i},$$

where $\binom{n}{k}$ is a BINOMIAL COEFFICIENT, although Jonah's original formula corresponded to $p=2$, $q=0$ (Hilton and Pederson 1991).

References

Hilton, P. and Pederson, J. "Catalan Numbers, Their Generalization, and Their Uses." *Math. Intel.* **13**, 64–75, 1991.

Jones Polynomial

The second KNOT POLYNOMIAL discovered. Unlike the first-discovered ALEXANDER POLYNOMIAL, the Jones polynomial can sometimes distinguish handedness (as can its more powerful generalization, the HOMFLY POLYNOMIAL). Jones polynomials are LAURENT POLYNOMIALS in t assigned to an $\mathbb{R}^3$ KNOT. The Jones polynomials are denoted $V_L(t)$ for LINKS, $V_K(t)$ for KNOTS, and normalized so that

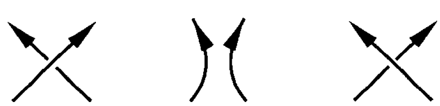
$$V_{\text{unknot}}(t) = 1. \tag{1}$$

For example, the Jones polynomial of the TREFOIL KNOT is given by

$$V_{\text{trefoil}}(t) = t + t^3 - t^4. \tag{2}$$

If a LINK has an ODD number of components, then V_L is a LAURENT POLYNOMIAL over the INTEGERS; if the number of components is EVEN, $V_L(t)$ is $t^{1/2}$ times a LAURENT POLYNOMIAL. The Jones polynomial of a KNOT SUM $L_1 \# L_2$ satisfies

$$V_{L_1 \# L_2} = (V_{L_1})(V_{L_2}). \tag{3}$$



L_+ L_0 L_-

The SKEIN RELATIONSHIP for under- and overcrossings is

$$t^{-1}V_{L_+} - tV_{L_-} = (t^{1/2} - t^{-1/2})V_{L_0}. \tag{4}$$

Combined with the link sum relationship, this allows Jones polynomials to be built up from simple knots and links to more complicated ones.

Some interesting identities from Jones (1985) follow. For any LINK L,

$$V_L(-1) = \Delta_L(-1), \tag{5}$$

where Δ_L is the ALEXANDER POLYNOMIAL, and

$$V_L(1) = (-2)^{p-1}, \tag{6}$$

where p is the number of components of L. For any KNOT K,

$$V_K(e^{2\pi i/3}) = 1 \tag{7}$$

and

$$\frac{d}{dt}V_K(1) = 0. \tag{8}$$

Let K^* denote the MIRROR IMAGE of a KNOT K. Then

$$V_{K^*}(t) = V_K(t^{-1}). \tag{9}$$

For example, the right-hand and left-hand TREFOIL KNOTS have polynomials

$$V_{\text{trefoil}}(t) = t + t^3 - t^4 \tag{10}$$
$$V_{\text{trefoil}^*}(t) = t^{-1} + t^{-3} - t^{-4}. \tag{11}$$

Jones defined a simplified trace invariant for knots by

$$W_K(t) = \frac{1 - V_K(t)}{(1-t^3)(1-t)}. \tag{12}$$

The ARF INVARIANT of W_K is given by

$$\text{Arf}(K) = W_K(i) \tag{13}$$

(Jones 1985), where i is $\sqrt{-1}$. A table of the W polynomials is given by Jones (1985) for knots of up to eight crossings, and by Jones (1987) for knots of up to 10 crossings. (Note that in these papers, an additional polynomial which Jones calls V is also tabulated, but it is not the conventionally defined Jones polynomial.)

Jones polynomials were subsequently generalized to the two-variable HOMFLY POLYNOMIALS, the relationship being

$$V(t) = P(a = t, x = t^{1/2} - t^{-1/2}) \tag{14}$$

$$V(t) = P(\ell = it, m = i(t^{-1/2} - t^{1/2})). \tag{15}$$

They are related to the KAUFFMAN POLYNOMIAL F by

$$V(t) = F(-t^{-3/4}, t^{-1/4} + t^{1/4}). \tag{16}$$

Jones (1987) gives a table of BRAID WORDS and W polynomials for knots up to 10 crossings. Jones polynomials for KNOTS up to nine crossings are given in Adams (1994) and for oriented links up to nine crossings by Doll and Hoste (1991). All PRIME KNOTS with 10 or fewer crossings have distinct Jones polynomials. It is not known if there is a nontrivial knot with Jones polynomial 1. The Jones polynomial of an (m,n)-TORUS KNOT is

$$\frac{t^{(m-1)(n-1)/2}(1 - t^{m+1} - t^{n+1} + t^{m+n})}{1 - t^2}. \tag{17}$$

Let k be one component of an oriented LINK L. Now form a new oriented LINK L^* by reversing the orientation of k. Then

$$V_{L^*} = t^{-3\lambda} V(L),$$

where V is the Jones polynomial and λ is the LINKING NUMBER of k and $L - k$. No such result is known for HOMFLY POLYNOMIALS (Lickorish and Millett 1988).

Birman and Lin (1993) showed that substituting the POWER SERIES for e^x as the variable in the Jones polynomial yields a POWER SERIES whose COEFFICIENTS are VASSILIEV POLYNOMIALS.

Let L be an oriented connected LINK projection of n crossings, then

$$n \geq \text{span } V(L), \tag{18}$$

with equality if L is ALTERNATING and has no REMOVABLE CROSSING (Lickorish and Millett 1988).

There exist distinct KNOTS with the same Jones polynomial. Examples include (05_{001}, 10_{132}), (08_{008}, 10_{129}), (08_{016}, 10_{156}), (10_{025}, 10_{056}), (10_{022}, 10_{035}), (10_{041}, 10_{094}), (10_{043}, 10_{091}), (10_{059}, 10_{106}), (10_{060}, 10_{083}), (10_{071}, 10_{104}), (10_{073}, 10_{086}), (10_{081}, 10_{109}), and (10_{137}, 10_{155}) (Jones 1987). Incidentally, the first four of these also have the same HOMFLY POLYNOMIAL.

Witten (1989) gave a heuristic definition in terms of a topological quantum field theory, and Sawin (1996) showed that the "quantum group" $U_q(sl_2)$ gives rise to the Jones polynomial.

see also ALEXANDER POLYNOMIAL, HOMFLY POLYNOMIAL, KAUFFMAN POLYNOMIAL F, KNOT, LINK, VASSILIEV POLYNOMIAL

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Jonquière's Function

see POLYGAMMA FUNCTION

Jordan Algebra

A nonassociative algebra with the product of elements A and B defined by the ANTICOMMUTATOR $\{A, B\} = AB + BA$.

see also ANTICOMMUTATOR

Jordan Curve

A Jordan curve is a plane curve which is topologically equivalent to (a HOMEOMORPHIC image of) the UNIT CIRCLE.

It is not known if every Jordan curve contains all four VERTICES of some SQUARE, but it has been proven true for "sufficiently smooth" curves and closed convex curves (Schnirelmann). For every TRIANGLE T and Jordan curve J, J has an INSCRIBED TRIANGLE similar to T.

see also JORDAN CURVE THEOREM, UNIT CIRCLE

Jordan Curve Theorem

If J is a simple closed curve in $\mathbb{R}^2$, then $\mathbb{R}^2 - J$ has two components (an "inside" and "outside"), with J the BOUNDARY of each.

see also JORDAN CURVE, SCHÖNFLIES THEOREM

References

Rolfsen, D. *Knots and Links.* Wilmington, DE: Publish or Perish Press, p. 9, 1976.

Jordan Decomposition Theorem

Let $V \neq (0)$ be a finite dimensional VECTOR SPACE over the COMPLEX NUMBERS, and let A be a linear operator on V. Then V can be expressed as a DIRECT SUM of cyclic subspaces.

References

Gohberg, I. and Goldberg, S. "A Simple Proof of the Jordan Decomposition Theorem for Matrices." *Amer. Math. Monthly* **103**, 157–159, 1996.

Jordan-Hölder Theorem

The composition quotient groups belonging to two COMPOSITION SERIES of a FINITE GROUP G are, apart from their sequence, ISOMORPHIC in pairs. In other words, if

$$I \subset H_s \subset \ldots \subset H_2 \subset H_1 \subset G$$

is one COMPOSITION SERIES and

$$I \subset K_t \subset \ldots \subset K_2 \subset K_1 \subset G$$

is another, then $t = s$, and corresponding to any composition quotient group K_j/K_{j+1}, there is a composition quotient group H_i/H_{i+1} such that

$$\frac{K_j}{K_{j+1}} = \frac{H_i}{H_{i+1}}.$$

This theorem was proven in 1869–1889.

see also COMPOSITION SERIES, FINITE GROUP, ISOMORPHIC GROUPS

References
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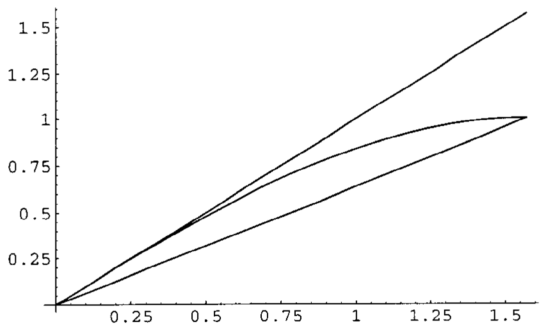
Jordan's Inequality



For $0 \leq x \leq \pi/2$,

$$\frac{2}{\pi}x \leq \sin x \leq x.$$

References
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Jordan's Lemma

Jordan's lemma shows the value of the INTEGRAL

$$I \equiv \int_{-\infty}^{\infty} f(x)e^{iax}\,dx \tag{1}$$

along the REAL AXIS is 0 for "nice" functions which satisfy $\lim_{R\to\infty} |f(Re^{i\theta})| = 0$. This is established using a CONTOUR INTEGRAL I_R which satisfies

$$\lim_{R\to\infty} |I_R| \leq \frac{\pi}{a} \lim_{R\to\infty} \epsilon = 0. \tag{2}$$

To derive the lemma, write

$$x \equiv Re^{i\theta} = R(\cos\theta + i\sin\theta) \tag{3}$$
$$dx = iRe^{i\theta}\,d\theta \tag{4}$$

and define the CONTOUR INTEGRAL

$$I_R = \int_0^{\pi} f(Re^{i\theta})e^{iaR\cos\theta - aR\sin\theta} iRe^{i\theta}\,d\theta \tag{5}$$

Then

$$\begin{aligned} |I_R| &= R\int_0^{\pi} |f(Re^{i\theta})|\,|e^{iaR\cos\theta}|\,|e^{-aR\sin\theta}|\,|i|\,|e^{i\theta}|\,d\theta \\ &= R\int_0^{\pi} |f(Re^{i\theta})|e^{-aR\sin\theta}\,d\theta \\ &= 2R\int_0^{\pi/2} |f(Re^{i\theta})|e^{-aR\sin\theta}\,d\theta. \end{aligned} \tag{6}$$

Now, if $\lim_{R\to\infty} |f(Re^{i\theta})| = 0$, choose an ϵ such that $|f(Re^{i\theta})| \leq \epsilon$, so

$$|I_R| \leq 2R\epsilon \int_0^{\pi/2} e^{-aR\sin\theta}\,d\theta. \tag{7}$$

But, for $\theta \in [0, \pi/2]$,

$$\frac{2}{\pi}\theta \leq \sin\theta, \tag{8}$$

so

$$\begin{aligned} |I_R| &\leq 2R\epsilon \int_0^{\pi/2} e^{-2aR\theta/\pi}\,d\theta \\ &= 2\epsilon R \frac{1 - e^{-aR}}{\frac{2aR}{\pi}} = \frac{\pi\epsilon}{a}(1 - e^{-aR}). \end{aligned} \tag{9}$$

As long as $\lim_{R\to\infty} |f(z)| = 0$, Jordan's lemma

$$\lim_{R\to\infty} |I_R| \leq \frac{\pi}{a} \lim_{R\to\infty} \epsilon = 0 \tag{10}$$

then follows.

see also CONTOUR INTEGRATION

References
Arfken, G. *Mathematical Methods for Physicists, 3rd ed.* Orlando, FL: Academic Press, pp. 406–408, 1985.

Jordan Measure

Let the set M correspond to a bounded, NONNEGATIVE function f on an interval $0 \leq f(x) \leq c$ for $x \in [a, b]$. The Jordan measure, when it exists, is the common value of the outer and inner Jordan measures of M.

The outer Jordan measure is the greatest lower bound of the areas of the covering of M, consisting of finite unions of RECTANGLES. The inner Jordan measure of M is the difference between the AREA $c(a-b)$ of the RECTANGLE S with base $[a, b]$ and height c, and the outer measure of the complement of M in S.

References
Shenitzer, A. and Steprans, J. "The Evolution of Integration." *Amer. Math. Monthly* **101**, 66–72, 1994.

Jordan Polygon

see SIMPLE POLYGON

Josephus Problem

Given a group of n men arranged in a CIRCLE under the edict that every mth man will be executed going around the CIRCLE until only one remains, find the position $L(n, m)$ in which you should stand in order to be the last survivor (Ball and Coxeter 1987). The original problem consisted of a CIRCLE of 41 men with every third man killed ($n = 41$, $m = 3$). In order for the lives of the last two men to be spared, they must be placed at positions 31 (last) and 16 (second-to-last).

The following array gives the original position of the last survivor out of a group of $n = 1, 2, \ldots$, if every mth man is killed:

1									
2	1								
3	3	2							
4	1	1	2						
5	3	4	1	2					
6	5	1	5	1	4				
7	7	4	2	6	3	5			
8	1	7	6	3	1	4	4		
9	3	1	1	8	7	2	3	8	
10	5	4	5	3	3	9	1	7	8

(Sloane's A032434). The survivor for $m = 2$ can be given analytically by

$$L(n, 2) = 1 + 2n - 2^{1+\lfloor \lg n \rfloor},$$

where $\lfloor n \rfloor$ is the FLOOR FUNCTION and LG is the LOGARITHM to base 2. The first few solutions are therefore 1, 1, 3, 1, 3, 5, 7, 1, 3, 5, 7, 9, 11, 13, 15, 1, ... (Sloane's A006257).

Mott-Smith (1954) discusses a card game called "Out and Under" in which cards at the top of a deck are alternately discarded and placed at the bottom. This is a Josephus problem with parameter $m = 2$, and Mott-Smith hints at the above closed-form solution.

The original position of the second-to-last survivor is given in the following table for $n = 2, 3, \ldots$:

1	1								
2	1	1							
3	1	1	2						
4	3	2	1	2					
5	1	1	5	1	4				
6	3	1	2	1	3	4			
7	1	4	6	3	1	3	4		
8	3	1	1	2	7	1	3	7	
9	5	4	5	3	3	8	1	6	4

(Sloane's A032435).

Another version of the problem considers a CIRCLE of two groups (say, "A" and "B") of 15 men each, with every ninth man cast overboard. To save all the members of the "A" group, the men must be placed at positions 1, 2, 3, 4, 10, 11, 13, 14, 15, 17, 20, 21, 25, 28, 29, giving the ordering

$$AAAABBBBBAABAAABABBAABBBABBAAB$$

which can be remembered with the aid of the MNEMONIC "From numbers' aid and art, never will fame depart." Consider the vowels only, assign $a = 1$, $e = 2$, $i = 3$, $o = 4$, $u = 5$, and alternately add a number of letters corresponding to a vowel value, so 4A (o), 5B (u), 2A (e), etc. (Ball and Coxeter 1987).

If every tenth man is instead thrown overboard, the men from the "A" group must be placed in positions 1, 2, 4, 5, 6, 12, 13, 16, 17, 18, 19, 21, 25, 28, 29, giving the sequence

$$AABAAABBBBBAABBAAAABABBBABBAAB$$

which can be constructed using the MNEMONIC "Rex paphi cum gente bona dat signa serena" (Ball and Coxeter 1987).

see also KIRKMAN'S SCHOOLGIRL PROBLEM, NECKLACE

References

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Ball, W. W. R. and Coxeter, H. S. M. *Mathematical Recreations and Essays, 13th ed.* New York: Dover, pp. 32–36, 1987.

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Jug

see THREE JUG PROBLEM

Jugendtraum

Kronecker proved that all the Galois extensions of the RATIONALS $\mathbb{Q}$ with ABELIAN Galois groups are SUBFIELDS of cyclotomic fields $Q(\mu_n)$, where μ_n is the group of nth ROOTS OF UNITY. He then sought to find a similar function whose division values would generate the Abelian extensions of an arbitrary NUMBER FIELD. He discovered that the j-FUNCTION works for IMAGINARY quadratic number fields K, but the completion of this problem, known as Kronecker's Jugendtraum ("dream of youth"), for other fields remains one of the great unsolved problems in NUMBER THEORY.

see also j-FUNCTION

References

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Juggling

The throwing and catching of multiple objects such that at least one is always in the air. Some aspects of juggling turn out to be quite mathematical. The best examples are the two-handed asynchronous juggling sequences known as "SITESWAPS."

see also SITESWAP

References

Buhler, J.; Eisenbud, D.; Graham, R.; and Wright, C. "Juggling Drops and Descents." *Amer. Math. Monthly* **101**, 507–519, 1994.

Donahue, B. "Jugglers Now Juggle Numbers to Compute New Tricks for Ancient Art." *New York Times*, pp. B5 and B10, Apr. 16, 1996.

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Julia Fractal

see JULIA SET

Julia Set

Let $R(z)$ be a rational function

$$R(z) \equiv \frac{P(z)}{Q(z)}, \tag{1}$$

where $z \in \mathbb{C}^*$, $\mathbb{C}^*$ is the RIEMANN SPHERE $\mathbb{C}\cup\{\infty\}$, and P and Q are POLYNOMIALS without common divisors. The "filled-in" Julia set J_R is the set of points z which do not approach infinity after $R(z)$ is repeatedly applied. The true Julia set is the boundary of the filled-in set (the set of "exceptional points"). There are two types of Julia sets: connected sets and CANTOR SETS.

For a Julia set J_c with $c \ll 1$, the CAPACITY DIMENSION is

$$d_{\text{capacity}} = 1 + \frac{|c|^2}{4\ln 2} + \mathcal{O}(|c|^3). \tag{2}$$

For small c, J_c is also a JORDAN CURVE, although its points are not COMPUTABLE.

Quadratic Julia sets are generated by the quadratic mapping

$$z_{n+1} = {z_n}^2 + c \tag{3}$$

for fixed c. The special case $c = -0.123 + 0.745i$ is called DOUADY'S RABBIT FRACTAL, $c = -0.75$ is called the SAN MARCO FRACTAL, and $c = -0.391 - 0.587i$ is the SIEGEL DISK FRACTAL. For every c, this transformation generates a FRACTAL. It is a CONFORMAL TRANSFORMATION, so angles are preserved. Let J be the JULIA SET, then $x' \mapsto x$ leaves J invariant. If a point P is on J, then all its iterations are on J. The transformation has a two-valued inverse. If $b = 0$ and y is started at 0, then the map is equivalent to the LOGISTIC MAP. The set of all points for which J is connected is known as the MANDELBROT SET.

see also DENDRITE FRACTAL, DOUADY'S RABBIT FRACTAL, FATOU SET, MANDELBROT SET, NEWTON'S METHOD, SAN MARCO FRACTAL, SIEGEL DISK FRACTAL

References

Dickau, R. M. "Julia Sets." `http://forum.swarthmore.edu/advanced/robertd/julias.html`.

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Jump

A point of DISCONTINUITY.

see also DISCONTINUITY, JUMP ANGLE, JUMPING CHAMPION

Jump Angle

A GEODESIC TRIANGLE with oriented boundary yields a curve which is piecewise DIFFERENTIABLE. Furthermore, the TANGENT VECTOR varies continuously at all but the three corner points, where it changes suddenly. The angular difference of the tangent vectors at these corner points are called the jump angles.

see also ANGULAR DEFECT, GAUSS-BONNET FORMULA

Jumping Champion

An integer n is called a JUMPING CHAMPION if n is the most frequently occurring difference between consecutive primes $n \leq N$ for some N (Odlyzko *et al.*). This term was coined by J. H. Conway in 1993. There are occasionally several jumping champions in a range. Odlyzko *et al.* give a table of jumping champions for $n \leq 1000$, consisting mainly of 2, 4, and 6. 6 is the jumping champion up to about $n \approx 1.74 \times 10^{35}$, at which point 30 dominates. At $n \approx 10^{425}$, 210 becomes champion, and subsequent PRIMORIALS are conjectured to take over at larger and larger n. Erdős and Straus (1980) proved that the jumping champions tend to infinity under the assumption of a quantitative form of the k-tuples conjecture.

see also PRIME DIFFERENCE FUNCTION, PRIME GAPS, PRIME NUMBER, PRIMORIAL

References

Erdős, P.; and Straus, E. G. "Remarks on the Differences Between Consecutive Primes." *Elem. Math.* **35**, 115–118, 1980.

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Nelson, H. "Problem 654." *J. Recr. Math.* **11**, 231, 1978–1979.
Odlyzko, A.; Rubinstein, M.; and Wolf, M. "Jumping Champions." `http://www.research.att.com/~amo/doc/recent.html`.

Jung's Theorem

Every finite set of points with SPAN d has an enclosing CIRCLE with RADIUS no greater than $\sqrt{3}\,d/3$.

References

Le Lionnais, F. *Les nombres remarquables.* Paris: Hermann, p. 28, 1983.
Rademacher, H. and Toeplitz, O. *The Enjoyment of Mathematics: Selections from Mathematics for the Amateur.* Princeton, NJ: Princeton University Press, pp. 103–110, 1957.

Just If

see IFF

Just One

see EXACTLY ONE

K

k-ary Divisor

Let a DIVISOR d of n be called a 1-ary divisor if $d \perp n/d$. Then d is called a k-ary divisor of n, written $d|_k n$, if the GREATEST COMMON $(k-1)$-ary divisor of d and (n/d) is 1.

In this notation, $d|n$ is written $d|_0 n$, and $d||n$ is written $d|_1 n$. p^x is an INFINARY DIVISOR of p^y (with $y > 0$) if $p^x|_{y-1}p^y$.

see also DIVISOR, GREATEST COMMON DIVISOR, INFINARY DIVISOR

References

Guy, R. K. *Unsolved Problems in Number Theory, 2nd ed.* New York: Springer-Verlag, p. 54, 1994.

k-Chain

Any sum of a selection of Π_ks, where Π_k denotes a k-D POLYTOPE.

see also k-CIRCUIT

k-Circuit

A k-CHAIN whose bounding $(k-1)$-CHAIN vanishes.

k-Coloring

A k-coloring of a GRAPH G is an assignment of one of k possible colors to each vertex of G such that no two adjacent vertices receive the same color.

see also COLORING, EDGE-COLORING

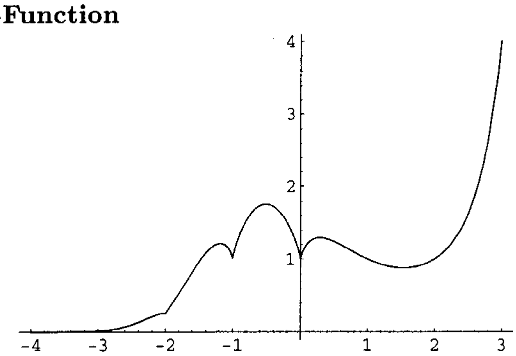
References

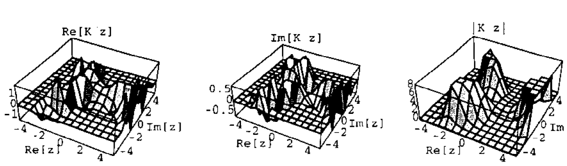
Saaty, T. L. and Kainen, P. C. *The Four-Color Problem: Assaults and Conquest.* New York: Dover, p. 13, 1986.

k-Form

see DIFFERENTIAL k-FORM

K-Function





An extension of the K-function

$$K(n+1) \equiv 0^0 1^1 2^2 3^3 \cdots n^n \tag{1}$$

defined by

$$K(z) = \frac{[\Gamma(z)]^z}{G(z)}. \tag{2}$$

Here, $G(z)$ is the G-FUNCTION defined by

$$G(n+1) \equiv \frac{(n!)^n}{K(n+1)} = \begin{cases} 1 & \text{if } n = 0 \\ 0!1!2!\cdots(n-1)! & \text{if } n > 0. \end{cases} \tag{3}$$

The K-function is given by the integral

$$K(z) = (2\pi)^{-(z-1)/2} \exp\left[\binom{z}{2} + \int_0^{z-1} \ln(t!)\,dt\right] \tag{4}$$

and the closed-form expression

$$K(z) = \exp[\zeta'(-1,z) - \zeta'(-1)], \tag{5}$$

where $\zeta(z)$ is the RIEMANN ZETA FUNCTION, $\zeta'(z)$ its DERIVATIVE, $\zeta(a,z)$ is the HURWITZ ZETA FUNCTION, and

$$\zeta'(a,z) \equiv \left[\frac{d\zeta(s,z)}{ds}\right]_{s=a}. \tag{6}$$

$K(z)$ also has a STIRLING-like series

$$K(z+1) = (2^{1/3}\pi_1 z)^{1/12} z^{\binom{z+1}{2}} \times \exp\left(\tfrac{1}{4}z^2 + \tfrac{1}{12} - \frac{B_4}{2\cdot 3\cdot 4z^2} - \frac{B_6}{4\cdot 5\cdot 6z^4} - \ldots\right), \tag{7}$$

where

$$\pi_1 \equiv [K(\tfrac{1}{2})]^8 \tag{8}$$

$$= e^{-(\ln 2)/3 - 12\zeta'(-1)} \tag{9}$$

$$= 2^{2/3}\pi e^{\gamma - 1 - \zeta'(2)/\zeta(2)}, \tag{10}$$

and γ is the EULER-MASCHERONI CONSTANT (Gosper).

The first few values of $K(n)$ for $n = 2, 3, \ldots$ are 1, 4, 108, 27648, 86400000, 4031078400000, ... (Sloane's A002109). These numbers are called HYPERFACTORIALS by Sloane and Plouffe (1995).

see also G-FUNCTION, GLAISHER-KINKELIN CONSTANT, HYPERFACTORIAL, STIRLING'S SERIES

References

Sloane, N. J. A. Sequence A002109/M3706 in "An On-Line Version of the Encyclopedia of Integer Sequences."

K-Graph

The GRAPH obtained by dividing a set of VERTICES $\{1, \ldots, n\}$ into $k-1$ pairwise disjoint subsets with VERTICES of degree $n_1, \ldots, n_{k-1}$, satisfying

$$n = n_1 + \ldots + n_{k-1},$$

and with two VERTICES joined IFF they lie in distinct VERTEX sets. Such GRAPHS are denoted $K_{n_1,\ldots,n_k}$.

see also BIPARTITE GRAPH, COMPLETE GRAPH, COMPLETE k-PARTITE GRAPH, k-PARTITE GRAPH

k-Matrix

A k-matrix is a kind of CUBE ROOT of the IDENTITY MATRIX defined by

$$\mathsf{k} = \begin{bmatrix} 0 & 0 & -i \\ i & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}.$$

It satisfies

$$\mathsf{k}^3 = \mathsf{I},$$

where I is the IDENTITY MATRIX.

see also CUBE ROOT, QUATERNION

k-Partite Graph

A k-partite graph is a GRAPH whose VERTICES can be partitioned into k disjoint sets so that no two vertices within the same set are adjacent.

see also COMPLETE k-PARTITE GRAPH, K-GRAPH

References

Saaty, T. L. and Kainen, P. C. *The Four-Color Problem: Assaults and Conquest.* New York: Dover, p. 12, 1986.

k-Statistic

An UNBIASED ESTIMATOR of the CUMULANTS κ_i of a DISTRIBUTION. The expectation values of the k-statistics are therefore given by the corresponding CUMULANTS

$$\langle k_1 \rangle = \kappa_1 \quad (1)$$

$$\langle k_2 \rangle = \kappa_2 \quad (2)$$

$$\langle k_3 \rangle = \kappa_3 \quad (3)$$

$$\langle k_4 \rangle = \kappa_4 \quad (4)$$

(Kenney and Keeping 1951, p. 189). For a sample of size, N, the first few k-statistics are given by

$$k_1 = m_1 \quad (5)$$

$$k_2 = \frac{N}{N-1} m_2 \quad (6)$$

$$k_3 = \frac{N^2}{(N-1)(N-2)} m_3 \quad (7)$$

$$k_4 = \frac{N^2[(N+1)m_4 - 3(N-1){m_2}^2]}{(N-1)(N-2)(N-3)}, \quad (8)$$

where m_1 is the sample MEAN, m_2 is the sample VARIANCE, and m_i is the sample ith MOMENT about the MEAN (Kenney and Keeping 1951, pp. 109–110, 163–165, and 189; Kenney and Keeping 1962). These statistics are obtained from inverting the relationships

$$\langle m_1 \rangle = \mu \quad (9)$$

$$\langle m_2 \rangle = \frac{N-1}{N} \mu_2 \quad (10)$$

$$\left\langle {m_2}^2 \right\rangle = \frac{(N-1)[(N-1)\mu_4 + (N^2 - 2N + 3){\mu_2}^2]}{N^3} \quad (11)$$

$$\langle m_3 \rangle = \frac{(N-1)(N-2)}{N^2} \mu_3 \quad (12)$$

$$\langle m_4 \rangle = \frac{(N-1)[(N^2 - 3N + 3)\mu_4 + 3(2N-3){\mu_2}^2]}{N^3}. \quad (13)$$

The first moment (sample MEAN) is

$$m_1 \equiv \langle x \rangle = \frac{1}{N} \sum_{i=1}^{N} x_i, \quad (14)$$

and the expectation value is

$$\langle m_1 \rangle = \left\langle \frac{1}{N} \sum_{i=1}^{N} x_i \right\rangle = \mu. \quad (15)$$

The second MOMENT (sample STANDARD DEVIATION) is

$$\begin{aligned} m_2 &\equiv \left\langle (x-\mu)^2 \right\rangle = \left\langle x^2 \right\rangle - 2\mu \langle x \rangle + \mu^2 = \left\langle x^2 \right\rangle - \mu^2 \\ &= \frac{1}{N} \sum_{i=1}^{N} {x_i}^2 - \left(\frac{1}{N} \sum_{i=1}^{N} x_i \right)^2 \\ &= \frac{1}{N} \sum_{i=1}^{N} {x_i}^2 - \frac{1}{N^2} \left(\sum_{i=1}^{N} {x_i}^2 + \sum_{\substack{i,j=1 \\ i \neq j}}^{N} x_i x_j \right) \\ &= \frac{N-1}{N^2} \sum_{i=1}^{N} {x_i}^2 - \frac{1}{N^2} \sum_{\substack{i,j=1 \\ i \neq j}}^{N} x_i x_j, \end{aligned} \quad (16)$$

and the expectation value is

$$\begin{aligned} \langle m_2 \rangle &= \frac{N-1}{N} \left\langle \frac{1}{N} \sum_{i=1}^{N} {x_i}^2 \right\rangle - \frac{1}{N^2} \left\langle \sum_{\substack{i,j=1 \\ i \neq j}}^{N} x_i x_j \right\rangle \\ &= \frac{N-1}{N} \mu_2' - \frac{N(N-1)}{N^2} \mu^2, \end{aligned} \quad (17)$$

since there are $N(N-1)$ terms $x_i x_j$, using

$$\langle x_i x_j \rangle = \langle x_i \rangle \langle x_j \rangle = \langle x_i \rangle^2, \quad (18)$$

and where μ_2' is the MOMENT about 0. Using the identity

$$\mu_2' = \mu_2 + \mu^2 \tag{19}$$

to convert to the MOMENT μ_2 about the MEAN and simplifying then gives

$$\langle m_2 \rangle = \frac{N-1}{N}\mu_2 \tag{20}$$

The factor $(N-1)/N$ is known as BESSEL'S CORRECTION.

The third MOMENT is

$$\begin{aligned} m_3 &\equiv \left\langle (x-\mu)^3 \right\rangle = \left\langle x^3 - 3\mu x^2 + 3\mu^2 x - \mu^3 \right\rangle \\ &= \left\langle x^3 \right\rangle - 3\mu \left\langle x^2 \right\rangle + 3\mu^2 \left\langle x \right\rangle - \mu^3 \\ &= \left\langle x^3 \right\rangle - 3\mu \left\langle x^2 \right\rangle + 3\mu^3 - \mu^3 \\ &= \left\langle x^3 \right\rangle - 3\mu \left\langle x^2 \right\rangle + 2\mu^3 \\ &= \frac{1}{N}\sum x_i^3 - 3\left(\frac{1}{N}\sum x_i\right)\left(\frac{1}{N}\sum x_j^2\right) \\ &\quad + 2\left(\frac{1}{N}\sum x_i\right)^3 \\ &= \frac{1}{N}\sum x_i^3 - \frac{3}{N^2}\left(\sum x_i\right)\left(\sum x_j^2\right) \\ &\quad + \frac{2}{N^3}\left(\sum x_i\right)^3. \end{aligned} \tag{21}$$

Now use the identities

$$\left(\sum x_i^2\right)\left(\sum x_j\right) = \sum x_i^3 + \sum x_i^2 x_j \tag{22}$$

$$\left(\sum x_i\right)^3 = \sum x_i^3 + 3\sum x_i^2 x_j + 6\sum x_i x_j x_k, \tag{23}$$

where it is understood that sums over products of variables exclude equal indices. Plugging in

$$\begin{aligned} m_3 &= \left(\frac{1}{N} - \frac{3}{N^2} + \frac{2}{N^3}\right)\sum x_i^3 \\ &+ \left(-\frac{3}{N^2} + 3\cdot\frac{2}{N^3}\right)\sum x_i^2 x_j + 6\cdot\frac{2}{N^3}\sum x_i x_j x_k. \end{aligned} \tag{24}$$

The expectation value is then given by

$$\begin{aligned} \langle m_3 \rangle &= \left(\frac{1}{N} - \frac{3}{N^2} + \frac{2}{N^3}\right) N\mu_3' \\ &+ \left(-\frac{3}{N^2} + \frac{6}{N^3}\right) N(N-1)\mu_2'\mu + \frac{12}{N^3}\tfrac{1}{6}N(N-1)(N-2)\mu^3 \end{aligned} \tag{25}$$

where μ_2' is the MOMENT about 0. Plugging in the identities

$$\mu_2' = \mu_2 + \mu^2 \tag{26}$$

$$\mu_3' = \mu_3 + 3\mu_2\mu + \mu^3 \tag{27}$$

and simplifying then gives

$$\langle m_3 \rangle = \frac{(N-1)(N-2)}{N^2}\mu_3 \tag{28}$$

(Kenney and Keeping 1951, p. 189).

The fourth MOMENT is

$$\begin{aligned} m_4 &= \left\langle (x-\mu)^4 \right\rangle = \left\langle x^4 - 4x^3\mu + 6x^2\mu^2 - 4x\mu^3 + \mu^4 \right\rangle \\ &= \left\langle x^4 \right\rangle - 4\mu \left\langle x^3 \right\rangle + 6\mu^2 \left\langle x^2 \right\rangle - 3\mu^4 \\ &= \frac{1}{N}\sum x_i^4 - \frac{4}{N^2}\left(\sum x_i\right)\left(\sum x_j^3\right) \\ &\quad + \frac{6}{N^3}\left(\sum x_i\right)^2\left(\sum x_j^2\right) - \frac{3}{N^4}\left(\sum x_i\right)^4 \end{aligned} \tag{29}$$

Now use the identities

$$\left(\sum x_i\right)\left(\sum x_j^3\right) = \sum x_i^4 + \sum x_i^3 x_j \tag{30}$$

$$\begin{aligned} \left(\sum x_i\right)^2\left(\sum x_j^2\right) &= \sum x_i^4 + 2\sum x_i^3 x_j \\ &\quad + 2\sum x_i^2 x_j^2 + 2\sum x_i^2 x_j x_k \end{aligned} \tag{31}$$

$$\begin{aligned} \left(\sum x_i\right)^4 &= \sum x_i^4 + 4\sum x_i^3 x_j + 6\sum x_i^2 x_j^2 \\ &\quad + 12\sum_{x_i}^{2} x_j x_k + 24\sum x_i x_j x_k x_l. \end{aligned} \tag{32}$$

Plugging in,

$$\begin{aligned} m_4 &= \left(\frac{1}{N} - \frac{4}{N^2} + \frac{6}{N^3} - \frac{3}{N^4}\right)\sum x_i^4 \\ &+ \left(-\frac{4}{N^2} + 2\cdot\frac{6}{N^3} - 4\cdot\frac{3}{N^4}\right)\sum x_i^3 x_j \\ &+ \left(2\cdot\frac{6}{N^3} - 6\cdot\frac{3}{N^4}\right)\sum x_i^2 x_j^2 \\ &+ \left(2\cdot\frac{6}{N^3} - 12\cdot\frac{3}{N^4}\right)\sum x_i^2 x_j x_k \\ &- 24\cdot\frac{3}{N^4}\sum x_i x_j x_k x_l. \end{aligned} \tag{33}$$

The expectation value is then given by

$$\begin{aligned} \langle m_4 \rangle &= \left(\frac{1}{N} - \frac{4}{N^2} + \frac{6}{N^3} - \frac{3}{N^4}\right) N\mu_4' \\ &+ \left(-\frac{4}{N^2} + \frac{12}{N^3} - \frac{12}{N^4}\right) N(N-1)\mu_3'\mu \\ &+ \left(\frac{12}{N^3} - \frac{18}{N^4}\right)\tfrac{1}{2}N(N-1)\mu_2'^2 \\ &+ \left(\frac{18}{N^3} - \frac{36}{N^4}\right)\tfrac{1}{2}N(N-1)(N-2)\mu_2'\mu^2 \\ &- \frac{72}{N^4}\tfrac{1}{24}N(N-1)(N-2)(N-3)\mu^4, \end{aligned} \tag{34}$$

where μ_i' are MOMENTS about 0. Using the identities

$$\mu_2' = \mu_2 + \mu^2 \tag{35}$$

$$\mu_3' = \mu_3 + 3\mu_2\mu + \mu^3 \tag{36}$$

$$\mu_4' = \mu_4 + 4\mu_3\mu + 6\mu_2\mu^2 + \mu^4 \tag{37}$$

and simplifying gives

$$\langle m_4\rangle = \frac{(N-1)[(N^2-3N+3)\mu_4+3(2N-3)\mu_2{}^2]}{N^3} \tag{38}$$

(Kenney and Keeping 1951, p. 189).

The square of the second moment is

$$\begin{aligned} m_2{}^2 &= (\langle x^2\rangle - \mu^2)^2 = \langle x^2\rangle^2 - 2\mu^2\langle x^2\rangle + \mu^4 \\ &= \left(\frac{1}{N}\sum x_i{}^2\right)^2 - 2\left(\frac{1}{N}\sum x_i\right)^2\left(\frac{1}{N}\sum x_i{}^2\right) \\ &\quad + \left(\frac{1}{N}\sum x_i\right)^4 \\ &= \frac{1}{N^2}\left(\sum x_i{}^2\right)^2 - \frac{2}{N^3}\left(\sum x_i\right)^2\left(\sum x_j{}^2\right) \\ &\quad + \frac{1}{N^4}\left(\sum x_i\right)^4. \end{aligned} \tag{39}$$

Now use the identities

$$\left(\sum x_i{}^2\right)^2 = \sum x_i{}^4 + 2\sum x_i{}^2 x_j{}^2 \tag{40}$$

$$\left(\sum x_i\right)^2\left(\sum x_j{}^2\right) = \sum x_i{}^4 + 2\sum x_i{}^2x_j{}^2 + 2\sum x_i{}^3x_j + 2\sum x_i{}^2x_jx_k \tag{41}$$

$$\left(\sum x_i\right)^4 = \sum x_i{}^4 + 6\sum x_i{}^2x_j{}^2 + 4\sum x_i{}^3x_j + 12\sum_{x_i}^{2} x_jx_k + 24\sum x_ix_jx_kx_l. \tag{42}$$

Plugging in,

$$\begin{aligned} m_2{}^2 &= \left(\frac{1}{N^2} - \frac{2}{N^3} + \frac{1}{N^4}\right)\sum x_i{}^4 \\ &\quad + \left(2\cdot\frac{1}{N^2} - 2\cdot\frac{2}{N^3} + 6\cdot\frac{1}{N^4}\right)\sum x_i{}^2x_j{}^2 \\ &\quad + \left(-2\cdot\frac{2}{N^3} + 4\cdot\frac{1}{N^4}\right)\sum x_i{}^3x_j \\ &\quad + \left(-2\cdot\frac{2}{N^3} + 12\cdot\frac{1}{N^4}\right)\sum x_i{}^2x_jx_k \\ &\quad + \frac{24}{N^4}\sum x_ix_jx_kx_l \end{aligned} \tag{43}$$

The expectation value is then given by

$$\begin{aligned} \langle m_2{}^2\rangle &= \left(\frac{1}{N^2} - \frac{2}{N^3} + \frac{1}{N^4}\right)N\mu_4' \\ &\quad + \left(\frac{2}{N^2} - \frac{4}{N^3} + \frac{6}{N^4}\right)\tfrac{1}{2}N(N-1)\mu_2'^2 \\ &\quad + \left(-\frac{4}{N^3} + \frac{4}{N^4}\right)N(N-1)\mu_3'\mu \\ &\quad + \left(-\frac{4}{N^3} + \frac{12}{N^4}\right)\tfrac{1}{2}N(N-1)(N-2)\mu_2'\mu^2 \\ &\quad + \frac{24}{N^4}\tfrac{1}{24}N(N-1)(N-2)(N-3)\mu^4 \end{aligned} \tag{44}$$

where μ_i' are MOMENTS about 0. Using the identities

$$\mu_2' = \mu_2 + \mu^2 \tag{45}$$

$$\mu_3' = \mu_3 + 3\mu_2\mu + \mu^3 \tag{46}$$

$$\mu_4' = \mu_4 + 4\mu_3\mu + 6\mu_2\mu^2 + \mu^4 \tag{47}$$

and simplifying gives

$$\langle m_2{}^2\rangle = \frac{(N-1)[(N-1)\mu_4 + (N^2-2N+3)\mu_2{}^2]}{N^3} \tag{48}$$

(Kenney and Keeping 1951, p. 189).

The VARIANCE of k_2 is given by

$$\operatorname{var}(k_2) = \frac{\kappa_4}{N} + \frac{2}{(N-1)\kappa_2{}^2}, \tag{49}$$

so an unbiased estimator of $\operatorname{var}(k_2)$ is given by

$$\hat{\operatorname{var}}(k_2) = \frac{2k_2{}^2N + (N-1)k_4}{N(N+1)} \tag{50}$$

(Kenney and Keeping 1951, p. 189). The VARIANCE of k_3 can be expressed in terms of CUMULANTS by

$$\operatorname{var}(k_3) = \frac{\kappa_6}{N} + \frac{9\kappa_2\kappa_4}{N-1} + \frac{9\kappa_3{}^2}{N-1} + \frac{6\kappa_2{}^3}{N(N-1)(N-2)}. \tag{51}$$

An UNBIASED ESTIMATOR for $\operatorname{var}(k_3)$ is

$$\hat{\operatorname{var}}(k_3) = \frac{6k_2{}^2N(N-1)}{(N-2)(N+1)(N+3)} \tag{52}$$

(Kenney and Keeping 1951, p. 190).

Now consider a finite population. Let a sample of N be taken from a population of M. Then UNBIASED ESTIMATORS M_2 for the population MEAN μ, M_2 for the population VARIANCE μ_2, G_1 for the population SKEWNESS γ_1, and G_2 for the population KURTOSIS γ_2 are

$$M_1 = \mu \tag{53}$$

$$M_2 = \frac{M-N}{N(M-1)}\mu_2 \tag{54}$$

$$G_1 = \frac{M-2N}{M-2}\sqrt{\frac{M-1}{N(M-N)}}\,\gamma_1 \tag{55}$$

$$G_2 = \frac{(M-1)(M^2-6MN+M+6N^2)\gamma_2}{N(M-2)(M-3)(M-N)} - \frac{6M(MN+M-N^2-1)}{N(M-2)(M-3)(M-N)} \tag{56}$$

(Church 1926, p. 357; Carver 1930; Irwin and Kendall 1944; Kenney and Keeping 1951, p. 143), where γ_1 is the sample SKEWNESS and γ_2 is the sample KURTOSIS.

see also GAUSSIAN DISTRIBUTION, KURTOSIS, MEAN, MOMENT, SKEWNESS, VARIANCE

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Kenney, J. F. and Keeping, E. S. "The *k*-Statistics." §7.9 in *Mathematics of Statistics, Pt. 1, 3rd ed.* Princeton, NJ: Van Nostrand, pp. 99–100, 1962.

k-Subset

A *k*-subset is a SUBSET containing exactly k elements.

see also SUBSET

k-Theory

A branch of mathematics which brings together ideas from algebraic geometry, LINEAR ALGEBRA, and NUMBER THEORY. In general, there are two main types of k-theory: topological and algebraic.

Topological k-theory is the "true" k-theory in the sense that it came first. Topological k-theory has to do with VECTOR BUNDLES over TOPOLOGICAL SPACES. Elements of a k-theory are STABLE EQUIVALENCE classes of VECTOR BUNDLES over a TOPOLOGICAL SPACE. You can put a RING structure on the collection of STABLY EQUIVALENT bundles by defining ADDITION through the WHITNEY SUM, and MULTIPLICATION through the TENSOR PRODUCT of VECTOR BUNDLES. This defines "the reduced real topological k-theory of a space."

"The reduced k-theory of a space" refers to the same construction, but instead of REAL VECTOR BUNDLES, COMPLEX VECTOR BUNDLES are used. Topological k-theory is significant because it forms a generalized COHOMOLOGY theory, and it leads to a solution to the vector fields on spheres problem, as well as to an understanding of the J-homeomorphism of HOMOTOPY THEORY.

Algebraic k-theory is somewhat more involved. Swan (1962) noticed that there is a correspondence between the CATEGORY of suitably nice TOPOLOGICAL SPACES (something like regular HAUSDORFF SPACES) and C^*-ALGEBRAS. The idea is to associate to every SPACE the C^*-ALGEBRA of CONTINUOUS MAPS from that SPACE to the REALS.

A VECTOR BUNDLE over a SPACE has sections, and these sections can be multiplied by CONTINUOUS FUNCTIONS to the REALS. Under Swan's correspondence, VECTOR BUNDLES correspond to modules over the C^*-ALGEBRA of CONTINUOUS FUNCTIONS, the MODULES being the modules of sections of the VECTOR BUNDLE. This study of MODULES over C^*-ALGEBRA is the starting point of algebraic k-theory.

The QUILLEN-LICHTENBAUM CONJECTURE connects algebraic k-theory to Étale cohomology.

see also C^*-ALGEBRA

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k-Tuple Conjecture

The first of the HARDY-LITTLEWOOD CONJECTURES. The k-tuple conjecture states that the asymptotic number of PRIME CONSTELLATIONS can be computed explicitly. In particular, unless there is a trivial divisibility condition that stops p, $p+a_1$, ..., $p+a_k$ from consisting of PRIMES infinitely often, then such PRIME CONSTELLATIONS will occur with an asymptotic density which is computable in terms of a_1, ..., a_k. Let $0 < m_1 < m_2 < \ldots < m_k$, then the k-tuple conjecture predicts that the number of PRIMES $p \leq x$ such that $p+2m_1$, $p+2m_2$, ..., $p+2m_k$ are all PRIME is

$$P(x; m_1, m_2, \ldots, m_k) \sim C(m_1, m_2, \ldots, m_k) \int_2^x \frac{dt}{\ln^{k+1} t}, \tag{1}$$

where

$$C(m_1, m_2, \ldots, m_k) = 2^k \prod_q \frac{1 - \frac{w(q; m_1, m_2, \ldots, m_k)}{q}}{\left(1 - \frac{1}{q}\right)^{k+1}}, \tag{2}$$

the product is over ODD PRIMES q, and

$$w(q; m_1, m_2, \ldots, m_k) \tag{3}$$

denotes the number of distinct residues of 0, m_1, ..., m_k (mod q) (Halberstam and Richert 1974, Odlyzko). If $k = 1$, then this becomes

$$C(m) = 2 \prod_q \frac{q(q-2)}{(q-1)^2} \prod_{q|m} \frac{q-1}{q-2}. \tag{4}$$

This conjecture is generally believed to be true, but has not been proven (Odlyzko *et al.*). The following special case of the conjecture is sometimes known as the PRIME PATTERNS CONJECTURE. Let S be a FINITE set of INTEGERS. Then it is conjectured that there exist infinitely many k for which $\{k + s : s \in S\}$ are all PRIME IFF S does not include all the RESIDUES of any PRIME. The TWIN PRIME CONJECTURE is a special case of the prime patterns conjecture with $S = \{0, 2\}$. This conjecture also implies that there are arbitrarily long ARITHMETIC PROGRESSIONS of PRIMES.

see also ARITHMETIC PROGRESSION, DIRICHLET'S THEOREM, HARDY-LITTLEWOOD CONJECTURES, k-TUPLE CONJECTURE, PRIME ARITHMETIC PROGRESSION, PRIME CONSTELLATION, PRIME QUADRUPLET,

PRIME PATTERNS CONJECTURE, TWIN PRIME CONJECTURE, TWIN PRIMES

References

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Kabon Triangles

The largest number $N(n)$ of nonoverlapping TRIANGLES which can be produced by n straight LINE SEGMENTS. The first few terms are 1, 2, 5, 7, 11, 15, 21, ... (Sloane's A006066).

References

Sloane, N. J. A. Sequence A006066/M1334 in "An On-Line Version of the Encyclopedia of Integer Sequences."

Kac Formula

The expected number of REAL zeros E_n of a RANDOM POLYNOMIAL of degree n is

$$E_n = \frac{1}{\pi}\int_{-\infty}^{\infty}\sqrt{\frac{1}{(t^2-1)^2}-\frac{(n+1)^2t^{2n}}{(t^{2n+2}-1)^2}}\,dt \quad (1)$$

$$= \frac{4}{\pi}\int_0^1\sqrt{\frac{1}{(1-t^2)^2}-\frac{(n+1)^2t^{2n}}{(1-t^{2n+2})^2}}\,dt. \quad (2)$$

As $n \to \infty$,

$$E_n = \frac{2}{\pi}\ln n + C_1 + \frac{2}{\pi n} + \mathcal{O}(n^{-2}), \quad (3)$$

where

$$C_1 = \frac{2}{\pi}\left[\ln 2 + \int_0^\infty \left(\sqrt{\frac{1}{x^2}-\frac{4e^{-2x}}{(1-e^{-2x})^2}}-\frac{1}{x+1}\right)dx\right] = 0.6257358072\ldots. \quad (4)$$

The initial term was derived by Kac (1943).

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Kac Matrix

The $(n+1)\times(n+1)$ TRIDIAGONAL MATRIX (also called the CLEMENT MATRIX) defined by

$$S_n = \begin{bmatrix} 0 & n & 0 & 0 & \cdots & 0 \\ 1 & 0 & n-1 & 0 & \cdots & 0 \\ 0 & 2 & 0 & n-2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & n-1 & 0 & 1 \\ 0 & 0 & 0 & 0 & n & 0 \end{bmatrix}.$$

The EIGENVALUES are $2k-n$ for $k = 0, 1, \ldots, n$.

Kähler Manifold

A manifold for which the EXTERIOR DERIVATIVE of the FUNDAMENTAL FORM Ω associated with the given Hermitian metric vanishes, so $d\Omega = 0$.

References

Amorós, J. *Fundamental Groups of Compact Kähler Manifolds.* Providence, RI: Amer. Math. Soc., 1996.

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Kakeya Needle Problem

What is the plane figure of least AREA in which a line segment of width 1 can be freely rotated (where translation of the segment is also allowed)? Besicovitch (1928) proved that there is *no* MINIMUM AREA. This can be seen by rotating a line segment inside a DELTOID, star-shaped 5-oid, star-shaped 7-oid, etc. When the figure is restricted to be convex, Cunningham and Schoenberg (1965) found there is still *no* minimum AREA. However, the smallest *simple convex* domain in which one can put a segment of length 1 which will coincide with itself when rotated by 180° is

$$\tfrac{1}{24}(5-2\sqrt{2})\pi = 0.284258\ldots$$

(Le Lionnais 1983).

see also CURVE OF CONSTANT WIDTH, LEBESGUE MINIMAL PROBLEM, REULEAUX POLYGON, REULEAUX TRIANGLE

References

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Kakutani's Fixed Point Theorem

Every correspondence that maps a compact convex subset of a locally convex space into itself with a closed graph and convex nonempty images has a fixed point.

see also FIXED POINT THEOREM

Kakutani's Problem

see COLLATZ PROBLEM

Kalman Filter

An ALGORITHM in CONTROL THEORY introduced by R. Kalman in 1960 and refined by Kalman and R. Bucy. It is an ALGORITHM which makes optimal use of imprecise data on a linear (or nearly linear) system with Gaussian errors to continuously update the best estimate of the system's current state.

see also WIENER FILTER

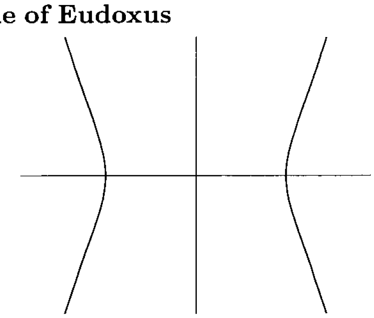
References

Chui, C. K. and Chen, G. *Kalman Filtering: With Real-Time Applications, 2nd ed.* Berlin: Springer-Verlag, 1991.
Grewal, M. S. *Kalman FIltering: Theory & Practice.* Englewood Cliffs, NJ: Prentice-Hall, 1993.

KAM Theorem

see KOLMOGOROV-ARNOLD-MOSER THEOREM

Kampyle of Eudoxus



A curve studied by Eudoxus in relation to the classical problem of CUBE DUPLICATION. It is given by the polar equation

$$r\cos^2\theta = a,$$

and the parametric equations

$$x = a\sec t$$
$$y = a\tan t\sec t$$

with $t \in [-\pi/2, \pi/2]$.

References

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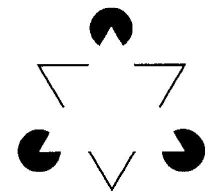
Kanizsa Triangle



An optical ILLUSION, illustrated above, in which the eye perceives a white upright EQUILATERAL TRIANGLE where none is actually drawn.

see also ILLUSION

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Kantrovich Inequality

Suppose $x_1 < x_2 < \ldots < x_n$ are given POSITIVE numbers. Let $\lambda_1, \ldots, \lambda_n \geq 0$ and $\sum \lambda_j = 1$. Then

$$\left(\sum \lambda_j x_j\right)\left(\sum \lambda_j {x_j}^{-1}\right) \leq A^2 G^{-2},$$

where

$$A = \tfrac{1}{2}(x_1 + x_n)$$
$$G = \sqrt{x_1 x_n}$$

are the ARITHMETIC and GEOMETRIC MEAN, respectively, of the first and last numbers.

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Kaplan-Yorke Conjecture

There are several versions of the Kaplan-Yorke conjecture, with many of the higher dimensional ones remaining unsettled. The original Kaplan-Yorke conjecture (Kaplan and Yorke 1979) proposed that, for a two-dimensional mapping, the CAPACITY DIMENSION D equals the KAPLAN-YORKE DIMENSION D_{KY},

$$D = D_{KY} = d_{\text{Lya}} = 1 + \frac{\sigma_1}{\sigma_2},$$

where σ_1 and σ_2 are the LYAPUNOV CHARACTERISTIC EXPONENTS. This was subsequently proven to be true in 1982. A later conjecture held that the KAPLAN-YORKE DIMENSION is generically equal to a probabilistic dimension which appears to be identical to the INFORMATION DIMENSION (Frederickson *et al.* 1983). This conjecture is partially verified by Ledrappier (1981). For invertible 2-D maps, $\nu = \sigma = D$, where ν is the CORRELATION EXPONENT, σ is the INFORMATION DIMENSION, and D is the CAPACITY DIMENSION (Young 1984).

see also CAPACITY DIMENSION, KAPLAN-YORKE DIMENSION, LYAPUNOV CHARACTERISTIC EXPONENT, LYAPUNOV DIMENSION

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Young, L. S. "Dimension, Entropy, and Lyapunov Exponents in Differentiable Dynamical Systems." *Phys. A* **124**, 639–645, 1984

Kaplan-Yorke Dimension

$$D_{\mathrm{KY}} \equiv j + \frac{\sigma_1 + \ldots + \sigma_j}{|\sigma_{j+1}|},$$

where $\sigma_1 \leq \sigma_n$ are LYAPUNOV CHARACTERISTIC EXPONENTS and j is the largest INTEGER for which

$$\lambda_1 + \ldots + \lambda_j \geq 0.$$

If $\nu = \sigma = D$, where ν is the CORRELATION EXPONENT, σ the INFORMATION DIMENSION, and D the HAUSDORFF DIMENSION, then

$$D \leq D_{\mathrm{KY}}$$

(Grassberger and Procaccia 1983).

References
Grassberger, P. and Procaccia, I. "Measuring the Strangeness of Strange Attractors." *Physica D* **9**, 189–208, 1983.

Kaplan-Yorke Map

$$x_{n+1} = 2x_n$$
$$y_{n+1} = \alpha y_n + \cos(4\pi x_n),$$

where x_n, y_n are computed mod 1. (Kaplan and Yorke 1979). The Kaplan-Yorke map with $\alpha = 0.2$ has CORRELATION EXPONENT 1.42 ± 0.02 (Grassberger Procaccia 1983) and CAPACITY DIMENSION 1.43 (Russell *et al.* 1980).

References
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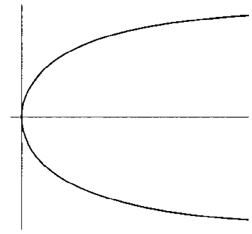
Kaplan, J. L. and Yorke, J. A. In *Functional Differential Equations and Approximations of Fixed Points* (Ed. H.-O. Peitgen and H.-O. Walther). Berlin: Springer-Verlag, p. 204, 1979.
Russell, D. A.; Hanson, J. D.; and Ott, E. "Dimension of Strange Attractors." *Phys. Rev. Let.* **45**, 1175–1178, 1980.

Kappa Curve



A curve also known as GUTSCHOVEN'S CURVE which was first studied by G. van Gutschoven around 1662 (MacTutor Archive). It was also studied by Newton and, some years later, by Johann Bernoulli. It is given by the Cartesian equation

$$(x^2 + y^2)y^2 = a^2x^2, \tag{1}$$

by the polar equation

$$r = a\cot\theta, \tag{2}$$

and the parametric equations

$$x = a\cos t\cot t \tag{3}$$
$$y = a\cos t. \tag{4}$$

References
Lawrence, J. D. *A Catalog of Special Plane Curves.* New York: Dover, pp. 136 and 139–141, 1972.
MacTutor History of Mathematics Archive. "Kappa Curve." `http://www-groups.dcs.st-and.ac.uk/~history/Curves/Kappa.html`.

Kaprekar Number

Consider an n-digit number k. Square it and add the right n digits to the left n or $n-1$ digits. If the resultant sum is k, then k is called a Kaprekar number. The first few are 1, 9, 45, 55, 99, 297, 703, ... (Sloane's A006886).

$$9^2 = 81 \qquad 8 + 1 = 9$$

$$297^2 = 88{,}209 \qquad 88 + 209 = 297.$$

see also DIGITAL ROOT, DIGITADITION, HAPPY NUMBER, KAPREKAR ROUTINE, NARCISSISTIC NUMBER, RECURRING DIGITAL INVARIANT

References
Sloane, N. J. A. Sequence A006886/M4625 in "An On-Line Version of the Encyclopedia of Integer Sequences."

Kaprekar Routine

A routine discovered in 1949 by D. R. Kaprekar for 4-digit numbers, but which can be generalized to k-digit numbers. To apply the Kaprekar routine to a number n, arrange the digits in descending (n') and ascending (n'') order. Now compute $K(n) \equiv n' - n''$ and iterate. The algorithm reaches 0 (a degenerate case), a constant, or a cycle, depending on the number of digits in k and the value of n.

For a 3-digit number n in base 10, the Kaprekar routine reaches the number 495 in at most six iterations. In base r, there is a unique number $((r-2)/2, r-1, r/2)_r$ to which n converges in at most $(r+2)/2$ iterations IFF r is EVEN. For any 4-digit number n in base-10, the routine terminates on the number 6174 after seven or fewer steps (where it enters the 1-cycle $K(6174) = 6174$).

2. 0, 0, 9, 21, {(45), (49)}, ...,
3. 0, 0, (32, 52), 184, (320, 580, 484), ...,
4. 0, 30, {201, (126, 138)}, (570, 765), {(2550), (3369), (3873)}, ...,
5. 8, (48, 72), 392, (1992, 2616, 2856, 2232), (7488, 10712, 9992, 13736, 11432), ...,
6. 0, 105, (430, 890, 920, 675, 860, 705), {5600, (4305, 5180)}, {(27195), (33860), (42925), (16840, 42745, 35510)}, ...,
7. 0, (144, 192), (1068, 1752, 1836), (9936, 15072, 13680, 13008, 10608), (55500, 89112, 91800, 72012, 91212, 77388), ...,
8. 21, 252, {(1589, 3178, 2723), (1022, 3122, 3290, 2044, 2212)}, {(17892, 20475), (21483, 25578, 26586, 21987)}, ...,
9. (16, 48), (320, 400), {(2256, 5312, 3856), (3712, 5168, 5456)}, {41520, (34960, 40080, 55360, 49520, 42240)}, ...,
10. 0, 495, 6174, {(53955, 59994), (61974, 82962, 75933, 63954), (62964, 71973, 83952, 74943)}, ...,

see also 196-ALGORITHM, KAPREKAR NUMBER, RATS SEQUENCE

References
Eldridge, K. E. and Sagong, S. "The Determination of Kaprekar Convergence and Loop Convergence of All 3-Digit Numbers." *Amer. Math. Monthly* **95**, 105–112, 1988.
Kaprekar, D. R. "An Interesting Property of the Number 6174." *Scripta Math.* **15**, 244-245, 1955.
Trigg, C. W. "All Three-Digit Integers Lead to..." *The Math. Teacher*, **67**, 41–45, 1974.
Young, A. L. "A Variation on the 2-digit Kaprekar Routine." *Fibonacci Quart.* **31**, 138–145, 1993.

Kaps-Rentrop Methods

A generalization of the RUNGE-KUTTA METHOD for solution of ORDINARY DIFFERENTIAL EQUATIONS, also called ROSENBROCK METHODS.

see also RUNGE-KUTTA METHOD

References
Press, W. H.; Flannery, B. P.; Teukolsky, S. A.; and Vetterling, W. T. *Numerical Recipes in FORTRAN: The Art of Scientific Computing, 2nd ed.* Cambridge, England: Cambridge University Press, pp. 730–735, 1992.

Kapteyn Series

A series of the form

$$\sum_{n=0}^{\infty} \alpha_n J_{\nu+n}[(\nu+n)z],$$

where $J_n(z)$ is a BESSEL FUNCTION OF THE FIRST KIND. Examples include Kapteyn's original series

$$\frac{1}{1-z} = 1 + 2\sum_{n=1}^{\infty} J_n(nz)$$

and

$$\frac{z^2}{2(1-z^2)} = \sum_{n=1}^{\infty} J_{2n}(2nz).$$

see also BESSEL FUNCTION OF THE FIRST KIND, NEUMANN SERIES (BESSEL FUNCTION)

References
Iyanaga, S. and Kawada, Y. (Eds.). *Encyclopedic Dictionary of Mathematics.* Cambridge, MA: MIT Press, p. 1473, 1980.

Karatsuba Multiplication

It is possible to perform MULTIPLICATION of LARGE NUMBERS in (many) fewer operations than the usual brute-force technique of "long multiplication." As discovered by Karatsuba and Ofman (1962), MULTIPLICATION of two n-DIGIT numbers can be done with a BIT COMPLEXITY of less than n^2 using identities of the form

$$\begin{aligned}(a+b\cdot 10^n)(c+d\cdot 10^n)\\ = ac + [(a+b)(c+d) - ac - bd]10^n + bd\cdot 10^{2n}. \quad (1)\end{aligned}$$

Proceeding recursively then gives BIT COMPLEXITY $\mathcal{O}(n^{\lg 3})$, where $\lg 3 = 1.58\ldots < 2$ (Borwein *et al.* 1989). The best known bound is $\mathcal{O}(n \lg n \lg\lg n)$ steps for $n \gg 1$ (Schönhage and Strassen 1971, Knuth 1981). However, this ALGORITHM is difficult to implement, but a procedure based on the FAST FOURIER TRANSFORM is straightforward to implement and gives BIT COMPLEXITY $\mathcal{O}((\lg n)^{2+\epsilon} n)$ (Brigham 1974, Borodin and Munro 1975, Knuth 1981, Borwein *et al.* 1989).

As a concrete example, consider MULTIPLICATION of two numbers each just two "digits" long in base w,

$$N_1 = a_0 + a_1 w \quad (2)$$
$$N_2 = b_0 + b_1 w, \quad (3)$$

then their PRODUCT is

$$\begin{aligned}P &\equiv N_1 N_2\\ &= a_0 b_0 + (a_0 b_1 + a_1 b_0) w + a_1 b_1 w^2\\ &= p_0 + p_1 w + p_2 w^2. \quad (4)\end{aligned}$$

Instead of evaluating products of individual digits, now write

$$q_0 = a_0 b_0 \quad (5)$$

$$q_1 = (a_0 + a_1)(b_0 + b_1) \quad (6)$$

$$q_2 = a_1 b_1. \quad (7)$$

The key term is q_1, which can be expanded, regrouped, and written in terms of the p_j as

$$q_1 = p_1 + p_0 + p_2. \quad (8)$$

However, since $p_0 = q_0$, and $p_2 = q_2$, it immediately follows that

$$p_0 = q_0 \quad (9)$$

$$p_1 = q_1 - q_0 - q_2 \quad (10)$$

$$p_2 = q_2, \quad (11)$$

so the three "digits" of p have been evaluated using three multiplications rather than four. The technique can be generalized to multidigit numbers, with the trade-off being that more additions and subtractions are required.

Now consider four-"digit" numbers

$$N_1 = a_0 + a_1 w + a_2 w^2 + a_3 w^3, \quad (12)$$

which can be written as a two-"digit" number represented in the base w^2,

$$N_1 = (a_0 + a_1 w) + (a_2 + a_3 w) * w^2. \quad (13)$$

The "digits" in the new base are now

$$a_0' = a_0 + a_1 w \quad (14)$$

$$a_1' = a_2 + a_3 w, \quad (15)$$

and the Karatsuba algorithm can be applied to N_1 and N_2 in this form. Therefore, the Karatsuba algorithm is not restricted to multiplying two-digit numbers, but more generally expresses the multiplication of two numbers in terms of multiplications of numbers of half the size. The asymptotic speed the algorithm obtains by recursive application to the smaller required subproducts is $\mathcal{O}(n^{\lg 3})$ (Knuth 1981).

When this technique is recursively applied to multidigit numbers, a point is reached in the recursion when the overhead of additions and subtractions makes it more efficient to use the usual $\mathcal{O}(n^2)$ MULTIPLICATION algorithm to evaluate the partial products. The most efficient overall method therefore relies on a combination of Karatsuba and conventional multiplication.

see also COMPLEX MULTIPLICATION, MULTIPLICATION, STRASSEN FORMULAS

References

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Zuras, D. "More on Squaring and Multiplying Large Integers." *IEEE Trans. Comput.* **43**, 899–908, 1994.

Katona's Problem

Find the minimum number $f(n)$ of subsets in a SEPARATING FAMILY for a SET of n elements, where a SEPARATING FAMILY is a SET of SUBSETS in which each pair of adjacent elements is found separated, each in one of two disjoint subsets. For example, the 26 letters of the alphabet can be separated by a family of nine:

$$\begin{array}{ccc} (abcdefghi) & (jklmnopqr) & (stuvwxyz) \\ (abcjklstu) & (defmnovwx) & (ghipqryz) \\ (adgjmpsvy) & (behknqtwz) & (cfilorux) \end{array}.$$

The problem was posed by Katona (1973) and solved by C. Mao-Cheng in 1982,

$$f(n) = \min\left\{2p + 3\left\lceil \log_3\left(\frac{n}{2^p}\right)\right\rceil : p = 0, 1, 2\right\},$$

where $\lceil x \rceil$ is the CEILING FUNCTION. $f(n)$ is nondecreasing, and the values for $n = 1, 2, \ldots$ are 0, 2, 3, 4, 5, 5, 6, 6, 6, 7, ... (Sloane's A07600). The values at which $f(n)$ increases are 1, 2, 3, 4, 5, 7, 10, 13, 19, 28, 37, ... (Sloane's A007601), so $f(26) = 9$, as illustrated in the preceding example.

see also SEPARATING FAMILY

References

Honsberger, R. "Cai Mao-Cheng's Solution to Katona's Problem on Families of Separating Subsets." Ch. 18 in *Mathematical Gems III.* Washington, DC: Math. Assoc. Amer., pp. 224–239, 1985.

Katona, G. O. H. "Combinatorial Search Problem." In *A Survey of Combinatorial Theory* (Ed. J. N. Srivasta *et al.*). Amsterdam, Netherlands: North-Holland, pp. 285–308, 1973.
Sloane, N. J. A. Sequences A007600/M0456 and A007601/M0525 in "An On-Line Version of the Encyclopedia of Integer Sequences."

Kauffman Polynomial F

A semi-oriented 2-variable KNOT POLYNOMIAL defined by

$$F_L(a,z) = a^{-w(L)} \langle |L| \rangle, \tag{1}$$

where L is an oriented LINK DIAGRAM, $w(L)$ is the WRITHE of L, $|L|$ is the unoriented diagram corresponding to L, and $\langle L \rangle$ is the BRACKET POLYNOMIAL. It was developed by Kauffman by extending the BLM/HO POLYNOMIAL Q to two variables, and satisfies

$$F(1,x) = Q(x). \tag{2}$$

The Kauffman POLYNOMIAL is a generalization of the JONES POLYNOMIAL $V(t)$ since it satisfies

$$V(t) = F(-t^{-3/4}, t^{-1/4} + t^{1/4}), \tag{3}$$

but its relationship to the HOMFLY POLYNOMIAL is not well understood. In general, it has more terms than the HOMFLY POLYNOMIAL, and is therefore more powerful for discriminating KNOTS. It is a semi-oriented POLYNOMIAL because changing the orientation only changes F by a POWER of a. In particular, suppose L^* is obtained from L by reversing the orientation of component k, then

$$F_{L^*} = a^{4\lambda} F_L, \tag{4}$$

where λ is the LINKING NUMBER of k with $L-k$ (Lickorish and Millett 1988). F is unchanged by MUTATION.

$$F_{L_1+F_{L_2}} = F(L_1)F(L_2) \tag{5}$$

$$F_{L_1 \cup L_2} = [(a^{-1}+a)x^{-1} - 1] F_{L_1} F_{L_2}. \tag{6}$$

M. B. Thistlethwaite has tabulated the Kauffman 2-variable POLYNOMIAL for KNOTS up to 13 crossings.

References

Lickorish, W. B. R. and Millett, B. R. "The New Polynomial Invariants of Knots and Links." *Math. Mag.* **61**, 1–23, 1988.
Stoimenow, A. "Kauffman Polynomials." http://www.informatik.hu-berlin.de/~stoimeno/ptab/k10.html.
Weisstein, E. W. "Knots and Links." http://www.astro.virginia.edu/~eww6n/math/notebooks/Knots.m.

Kauffman Polynomial X

A 1-variable KNOT POLYNOMIAL denoted X or $\mathcal{L}$.

$$\mathcal{L}_L(A) \equiv (-A^3)^{-w(L)} \langle L \rangle, \tag{1}$$

where $\langle L \rangle$ is the BRACKET POLYNOMIAL and $w(L)$ is the WRITHE of L. This POLYNOMIAL is invariant under AMBIENT ISOTOPY, and relates MIRROR IMAGES by

$$\mathcal{L}_{L^*} = \mathcal{L}_L(A^{-1}). \tag{2}$$

It is identical to the JONES POLYNOMIAL with the change of variable

$$\mathcal{L}(t^{-1/4}) = V(t). \tag{3}$$

The X POLYNOMIAL of the MIRROR IMAGE K^* is the same as for K but with A replaced by A^{-1}.

References

Kauffman, L. H. *Knots and Physics.* Singapore: World Scientific, p. 33, 1991.

Kei

The IMAGINARY PART of

$$e^{-\nu\pi i/2} K_\nu(xe^{\pi i/4}) = \mathrm{ker}_\nu(x) + i\,\mathrm{kei}_\nu(x).$$

see also BEI, BER, KER, KELVIN FUNCTIONS

References

Abramowitz, M. and Stegun, C. A. (Eds.). "Kelvin Functions." §9.9 in *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables, 9th printing.* New York: Dover, pp. 379–381, 1972.

Keith Number

A Keith number is an n-digit INTEGER N such that if a Fibonacci-like sequence (in which each term in the sequence is the sum of the n previous terms) is formed with the first n terms taken as the decimal digits of the number N, then N itself occurs as a term in the sequence. For example, 197 is a Keith number since it generates the sequence 1, 9, 7, 17, 33, 57, 107, 197, ... (Keith). Keith numbers are also called REPFIGIT NUMBERS.

There is no known general technique for finding Keith numbers except by exhaustive search. Keith numbers are much rarer than the PRIMES, with only 52 Keith numbers with < 15 digits: 14, 19, 28, 47, 61, 75, 197, 742, 1104, 1537, 2208, 2580, 3684, 4788, 7385, 7647, 7909, ... (Sloane's A007629). In addition, three 15-digit Keith numbers are known (Keith 1994). It is not known if there are an INFINITE number of Keith numbers.

References

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Heleen, B. "Finding Repfigits—A New Approach." *J. Recr. Math.* **26**, 184–187, 1994.

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Shirriff, K. "Computing Replicating Fibonacci Digits." *J. Recr. Math.* **26**, 191–193, 1994.
Sloane, N. J. A. Sequence A007629/M4922 in "An On-Line Version of the Encyclopedia of Integer Sequences."
"Table: Repfigit Numbers (Base 10*) Less than 10^{15}." *J. Recr. Math.* **26**, 195, 1994.

Keller's Conjecture

Keller conjectured that tiling an n-D space with n-D HYPERCUBES of equal size yields an arrangement in which at least two hypercubes have an entire $(n-1)$-D "side" in common. The CONJECTURE has been proven true for $n = 1$ to 6, but disproven for $n \geq 10$.

References
Cipra, B. "If You Can't See It, Don't Believe It." *Science* **259**, 26–27, 1993.
Cipra, B. *What's Happening in the Mathematical Sciences, Vol. 1.* Providence, RI: Amer. Math. Soc., pp. 24, 1993.

Kelvin's Conjecture

What space-filling arrangement of similar polyhedral cells of equal volume has minimal surface AREA? Kelvin proposed the 14-sided TRUNCATED OCTAHEDRON. Wearie and Phelan (1994) discovered another 14-sided POLYHEDRON that has 3% less SURFACE AREA.

References
Gray, J. "Parsimonious Polyhedra." *Nature* **367**, 598–599, 1994.
Wearie, D. and Phelan, R. "A Counter-Example to Kelvin's Conjecture on Minimal Surfaces." *Philos. Mag. Let.* **69**, 107–110, 1994.

Kelvin Functions

Kelvin defined the Kelvin functions BEI and BER according to

$$J_\nu(xe^{3\pi i/4}) = \mathrm{ber}_\nu(x) + i\,\mathrm{bei}_\nu(x), \qquad (1)$$

where $J_\nu(s)$ is a BESSEL FUNCTION OF THE FIRST KIND, and the functions KEI and KER by

$$e^{-\nu\pi i/2} K_\nu(xe^{\pi i/4}) = \mathrm{ker}_\nu(x) + i\,\mathrm{kei}_\nu(x), \qquad (2)$$

where $K_\nu(x)$ is a MODIFIED BESSEL FUNCTION OF THE SECOND KIND. For the special case $\nu = 0$,

$$J_0(i\sqrt{i}\,x) = J_0(\tfrac{1}{2}\sqrt{2}(i-1)x) \equiv \mathrm{ber}(x) + i\,\mathrm{bei}(x). \qquad (3)$$

see also BEI, BER, KEI, KER

References
Abramowitz, M. and Stegun, C. A. (Eds.). "Kelvin Functions." §9.9 in *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables, 9th printing.* New York: Dover, pp. 379–381, 1972.
Spanier, J. and Oldham, K. B. "The Kelvin Functions." Ch. 55 in *An Atlas of Functions.* Washington, DC: Hemisphere, pp. 543–554, 1987.

Kelvin Transformation

The transformation

$$v(x_1', \ldots, x_n') = \left(\frac{a}{r'}\right)^{n-2} u\left(\frac{a^2 x_1'}{r'^2}, \ldots, \frac{a^2 x_n'}{r'^2}\right),$$

where

$$r'^2 = {x_1'}^2 + \ldots + {x_n'}^2.$$

If $u(x_1, \ldots, x_n)$ is a HARMONIC FUNCTION on a DOMAIN D of $\mathbb{R}^n$ (with $n \geq 3$), then $v(x_1', \ldots, x_n')$ is HARMONIC on D'.

References
Iyanaga, S. and Kawada, Y. (Eds.). *Encyclopedic Dictionary of Mathematics.* Cambridge, MA: MIT Press, p. 623, 1980.

Kempe Linkage

A double rhomboid LINKAGE which gives rectilinear motion from circular without an inversion.

References
Rademacher, H. and Toeplitz, O. *The Enjoyment of Mathematics: Selections from Mathematics for the Amateur.* Princeton, NJ: Princeton University Press, pp. 126–127, 1957.

Kepler Conjecture

In 1611, Kepler proposed that close packing (cubic or hexagonal) is the densest possible SPHERE PACKING (has the greatest η), and this assertion is known as the Kepler conjecture. Finding the densest (not necessarily periodic) packing of spheres is known as the KEPLER PROBLEM.

A putative proof of the Kepler conjecture was put forward by W.-Y. Hsiang (Hsiang 1992, Cipra 1993), but was subsequently determined to be flawed (Conway *et al.* 1994, Hales 1994). According to J. H. Conway, nobody who has read Hsiang's proof has any doubts about its validity: it is nonsense.

see also DODECAHEDRAL CONJECTURE, KEPLER PROBLEM

References
Cipra, B. "Gaps in a Sphere Packing Proof?" *Science* **259**, 895, 1993.
Conway, J. H.; Hales, T. C.; Muder, D. J.; and Sloane, N. J. A. "On the Kepler Conjecture." *Math. Intel.* **16**, 5, Spring 1994.
Eppstein, D. "Sphere Packing and Kissing Numbers." `http://www.ics.uci.edu/~eppstein/junkyard/spherepack.html`.
Hales, T. C. "The Sphere Packing Problem." *J. Comput. Appl. Math.* **44**, 41–76, 1992.
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Hsiang, W.-Y. "On Soap Bubbles and Isoperimetric Regions in Noncompact Symmetrical Spaces. 1." *Tôhoku Math. J.* **44**, 151–175, 1992.
Hsiang, W.-Y. "A Rejoinder to Hales's Article." *Math. Intel.* **17**, 35–42, Winter 1995.

Kepler's Equation

Let M be the mean anomaly and E the ECCENTRIC ANOMALY of a body orbiting on an ELLIPSE with ECCENTRICITY e, then

$$M = E - e \sin E. \tag{1}$$

For M not a multiple of π, Kepler's equation has a unique solution, but is a TRANSCENDENTAL EQUATION and so cannot be inverted and solved directly for E given an arbitrary M. However, many algorithms have been derived for solving the equation as a result of its importance in celestial mechanics.

Writing a E as a POWER SERIES in e gives

$$E = M + \sum_{n=1}^{\infty} a_n e^n, \tag{2}$$

where the coefficients are given by the LAGRANGE INVERSION THEOREM as

$$a_n = \frac{1}{2^{n-1} n!} \sum_{k=0}^{\lfloor n/2 \rfloor} (-1)^k \binom{n}{k} (n-2k)^{n-1} \sin[(n-2k)M] \tag{3}$$

(Wintner 1941, Moulton 1970, Henrici 1974, Finch). Surprisingly, this series diverges for

$$e > 0.6627434193\ldots, \tag{4}$$

a value known as the LAPLACE LIMIT. In fact, E converges as a GEOMETRIC SERIES with ratio

$$r = \frac{e}{1+\sqrt{1+e^2}} \exp(\sqrt{1+e^2}\,) \tag{5}$$

(Finch).

There is also a series solution in BESSEL FUNCTIONS OF THE FIRST KIND,

$$E = M + \sum_{n=1}^{\infty} \frac{2}{n} J_n(ne) \sin(nM). \tag{6}$$

This series converges for all $e < 1$ like a GEOMETRIC SERIES with ratio

$$r = \frac{e}{1+\sqrt{1-e^2}} \exp(\sqrt{1-e^2}\,). \tag{7}$$

The equation can also be solved by letting ψ be the ANGLE between the planet's motion and the direction PERPENDICULAR to the RADIUS VECTOR. Then

$$\tan\psi = \frac{e \sin E}{\sqrt{1-e^2}}. \tag{8}$$

Alternatively, we can define e in terms of an intermediate variable ϕ

$$e \equiv \sin\phi, \tag{9}$$

then

$$\sin[\tfrac{1}{2}(v-E)] = \sqrt{\frac{r}{p}} \sin(\tfrac{1}{2}\phi) \sin v \tag{10}$$

$$\sin[\tfrac{1}{2}(v+E)] = \sqrt{\frac{r}{p}} \cos(\tfrac{1}{2}\phi) \sin v. \tag{11}$$

Iterative methods such as the simple

$$E_{i+1} = M + e \sin E_i \tag{12}$$

with $E_0 = 0$ work well, as does NEWTON'S METHOD,

$$E_{i+1} = E_i + \frac{M + e \sin E_i - E_i}{1 - e \cos E_i}. \tag{13}$$

In solving Kepler's equation, Stieltjes required the solution to

$$e^x (x-1) = e^{-x} (x+1), \tag{14}$$

which is 1.1996678640257734... (Goursat 1959, Le Lionnais 1983).

see also ECCENTRIC ANOMALY

References

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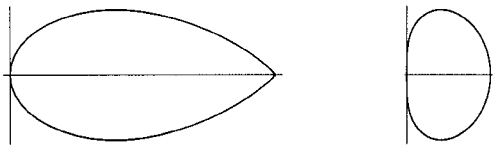
Kepler's Folium



The curve with implicit equation

$$[(x-b)^2+y^2][x(x-b)+y^2]-4a(x-b)y^2.$$

References

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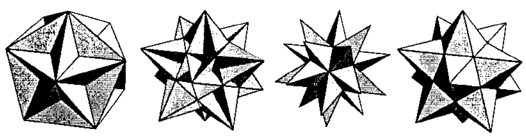
Kepler-Poinsot Solid



The Kepler-Poinsot solids are the four regular CONCAVE POLYHEDRA with intersecting facial planes. They are composed of regular CONCAVE POLYGONS and were unknown to the ancients. Kepler discovered two of them about 1619. These two were subsequently rediscovered by Poinsot, who also discovered the other two, in 1809. As shown by Cauchy, they are stellated forms of the DODECAHEDRON and ICOSAHEDRON.

The Kepler-Poinsot solids, illustrated above, are known as the GREAT DODECAHEDRON, GREAT ICOSAHEDRON, GREAT STELLATED DODECAHEDRON, and SMALL STELLATED DODECAHEDRON. Cauchy (1813) proved that these four exhaust all possibilities for regular star polyhedra (Ball and Coxeter 1987).

A table listing these solids, their DUALS, and COMPOUNDS is given below.

Polyhedron	Dual
great dodecahedron	small stellated dodec.
great Icosahedron	great stellated dodec.
great stellated dodec.	great icosahedron
small stellated dodec.	great dodecahedron

Polyhedron	Compound
great dodecahedron	great dodecahedron-small stellated dodec.
great icosahedron	great icosahedron-great stellated dodec.
great stellated dodec.	great icosahedron-great stellated dodec.
small stellated dodec.	great dodecahedron-small stellated dodec.

The polyhedra $\{\frac{5}{2},5\}$ and $\{5,\frac{5}{2}\}$ fail to satisfy the POLYHEDRAL FORMULA

$$V-E+F=2,$$

where V is the number of faces, E the number of edges, and F the number of faces, despite the fact that formula holds for all ordinary polyhedra (Ball and Coxeter 1987). This unexpected result led none less than Schläfli (1860) to conclude that they could not exist.

In 4-D, there are 10 Kepler-Poinsot solids, and in n-D with $n \geq 5$, there are none. In 4-D, nine of the solids have the same VERTICES as $\{3,3,5\}$, and the tenth has the same as $\{5,3,3\}$. Their SCHLÄFLI SYMBOLS are $\{\frac{5}{2},5,3\}$, $\{3,5,\frac{5}{2}\}$, $\{5,\frac{5}{2},5\}$, $\{\frac{5}{2},3,5\}$, $\{5,3,\frac{5}{2}\}$, $\{\frac{5}{2},5,\frac{5}{2}\}$, $\{5,\frac{5}{2},3\}$, $\{3,\frac{5}{2},5\}$, $\{\frac{5}{2},3,3\}$, and $\{3,3,\frac{5}{2}\}$.

Coxeter *et al.* (1954) have investigated star "Archimedean" polyhedra.

see also ARCHIMEDEAN SOLID, DELTAHEDRON, JOHNSON SOLID, PLATONIC SOLID, POLYHEDRON COMPOUND, UNIFORM POLYHEDRON

References

Ball, W. W. R. and Coxeter, H. S. M. *Mathematical Recreations and Essays, 13th ed.* New York: Dover, pp. 144–146, 1987.

Cauchy, A. L. "Recherches sur les polyèdres." *J. de l'École Polytechnique* **9**, 68–86, 1813.

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Kepler Problem

Finding the densest not necessarily periodic SPHERE PACKING.

see also KEPLER CONJECTURE, SPHERE PACKING

Kepler Solid

see KEPLER-POINSOT SOLID

Ker

The REAL PART of

$$e^{-\nu\pi i/2}K_\nu(xe^{\pi i/4})=\mathrm{ker}_\nu(x)+i\,\mathrm{kei}_\nu(x),$$

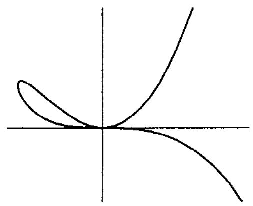
where $K_\nu(x)$ is a MODIFIED BESSEL FUNCTION OF THE SECOND KIND.

see also BEI, BER, KEI, KELVIN FUNCTIONS

References

Abramowitz, M. and Stegun, C. A. (Eds.). "Kelvin Functions." §9.9 in *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables, 9th printing.* New York: Dover, pp. 379–381, 1972.

Keratoid Cusp



The PLANE CURVE given by the Cartesian equation

$$y^2 = x^2 y + x^5.$$

References
Cundy, H. and Rollett, A. *Mathematical Models, 3rd ed.* Stradbroke, England: Tarquin Pub., p. 72, 1989.

Kernel (Integral)

The function $K(\alpha, t)$ in an INTEGRAL or INTEGRAL TRANSFORM

$$g(\alpha) = \int_a^b f(t) K(\alpha, t)\, dt.$$

see also BERGMAN KERNEL, POISSON KERNEL

Kernel (Linear Algebra)

see NULLSPACE

Kernel Polynomial

The function

$$K_n(x_0, x) = \overline{K_n(x, x_0)} = K_n(\bar{x}, \bar{x}_0)$$

which is useful in the study of many POLYNOMIALS.

References
Szegő, G. *Orthogonal Polynomials, 4th ed.* Providence, RI: Amer. Math. Soc., 1975.

Kervaire's Characterization Theorem

Let G be a GROUP, then there exists a piecewise linear KNOT K^{n-2} in $\mathbb{S}^n$ for $n \geq 5$ with $G = \pi_1(\mathbb{S}^n - K)$ IFF G satisfies

1. G is finitely presentable,
2. The Abelianization of G is infinite cyclic,
3. The normal closure of some single element is all of G,
4. $H_2(G) = 0$; the second homology of the group is trivial.

References
Rolfsen, D. *Knots and Links.* Wilmington, DE: Publish or Perish Press, pp. 350–351, 1976.

Ket

A CONTRAVARIANT VECTOR, denoted $|\psi\rangle$. The ket is DUAL to the COVARIANT BRA 1-VECTOR $\langle\psi|$. Taken together, the BRA and ket form an ANGLE BRACKET (bra+ket = bracket) $\langle\psi|\psi\rangle$. The ket is commonly encountered in quantum mechanics.

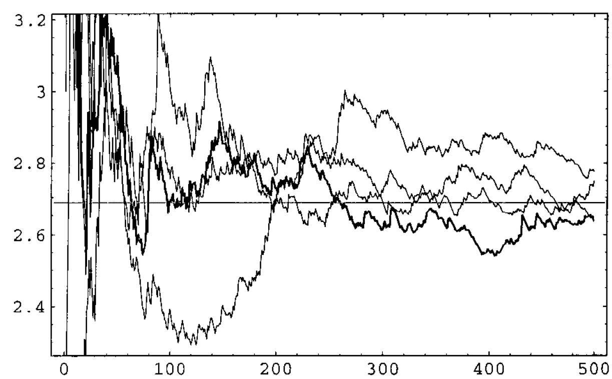
see also ANGLE BRACKET, BRA, BRACKET PRODUCT, CONTRAVARIANT VECTOR, COVARIANT VECTOR, DIFFERENTIAL k-FORM, ONE-FORM

Khinchin Constant

see KHINTCHINE'S CONSTANT

Khintchine's Constant

N.B. A detailed on-line essay by S. Finch was the starting point for this entry.



Let

$$x = [q_0, q_1, \ldots] = q_0 + \cfrac{1}{q_1 + \cfrac{1}{q_2 + \cfrac{1}{q_3 + \ldots}}} \tag{1}$$

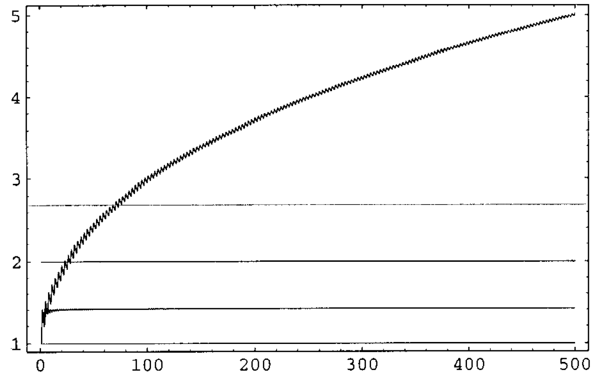
be the SIMPLE CONTINUED FRACTION of a REAL NUMBER x, where the numbers q_i are called PARTIAL QUOTIENTS. Khintchine (1934) considered the limit of the GEOMETRIC MEAN

$$G_n(x) = (q_1 q_2 \cdots q_n)^{1/n} \tag{2}$$

as $n \to \infty$. Amazingly enough, this limit is a constant *independent* of x—except if x belongs to a set of MEASURE 0–given by

$$K = 2.685452001\ldots \tag{3}$$

(Sloane's A002210), as proved in Kac (1959). The values $G_n(x)$ are plotted above for $n = 1$ to 500 and $x = \pi$, $1/\pi$, $\sin 1$, the EULER-MASCHERONI CONSTANT γ, and the COPELAND-ERDŐS CONSTANT. REAL NUMBERS x for which $\lim_{n\to\infty} G_n(x) \neq K$ include $x = e$, $\sqrt{2}$, $\sqrt{3}$, and the GOLDEN RATIO ϕ, all of which have periodic PARTIAL QUOTIENTS, plotted below.



The CONTINUED FRACTION for K is [2, 1, 2, 5, 1, 1, 2, 1, 1, ...] (Sloane's A002211). It is not known if K is IRRATIONAL, let alone TRANSCENDENTAL. Bailey *et al.* (1995) have computed K to 7350 DIGITS.

Explicit expressions for K include

$$K = \prod_{n=1}^{\infty} \left[1 + \frac{1}{n(n+2)}\right]^{\ln n/\ln 2} \tag{4}$$

$$\ln 2 \ln K = \tfrac{1}{12}\pi^2 + \tfrac{1}{2}(\ln 2)^2 + \int_0^{\pi} \frac{\ln(\theta|\cot\theta|)\, d\theta}{\theta} \tag{5}$$

$$\ln K = \frac{1}{\ln 2} \sum_{m=1}^{\infty} \frac{h_{m-1}}{m}[\zeta(2m) - 1], \tag{6}$$

where $\zeta(z)$ is the RIEMANN ZETA FUNCTION and

$$h_m = \sum_{j=1}^{m} \frac{(-1)^{j-1}}{j} \tag{7}$$

(Shanks and Wrench 1959). Gosper gave

$$\ln K = \frac{1}{\ln 2} \sum_{j=2}^{\infty} \frac{(-1)^j (2 - 2^j)\zeta'(j)}{j}, \tag{8}$$

where $\zeta'(z)$ is the DERIVATIVE of the RIEMANN ZETA FUNCTION. An extremely rapidly converging sum also due to Gosper is

$$\begin{aligned}\ln K = \frac{1}{\ln 2} \sum_{k=0}^{\infty} \Bigg\{ &- \ln(k+1)[\ln(k+3) \\ &-2\ln(k+2) + \ln(k+1)] \\ &- \frac{(-1)^k(2-2^{k+2})}{k+2} \left[\frac{\ln(k+1)}{(k+1)^{k+2}} - \zeta'(k+2, k+2)\right] \\ &+ \ln(k+1) \left[\sum_{s=1}^{k+2} \frac{(-1)^s(2-2^s)}{(k+1)^s s}\right]\Bigg\},\end{aligned} \tag{9}$$

where $\zeta(s, a)$ is the HURWITZ ZETA FUNCTION.

Khintchine's constant is also given by the integral

$$\ln 2 \ln(\tfrac{1}{2}K) = \int_0^1 \frac{1}{x(1+x)} \ln\left[\frac{\pi x(1-x^2)}{\sin(\pi x)}\right] dx. \tag{10}$$

If P_n/Q_n is the nth CONVERGENT of the CONTINUED FRACTION of x, then

$$\lim_{n\to\infty} (Q_n)^{1/n} = \lim_{n\to\infty} \left(\frac{P_n}{x}\right)^{1/n} = e^{\pi^2/(12\ln 2)} \approx 3.27582 \tag{11}$$

for almost all REAL x (Lévy 1936, Finch). This number is sometimes called the LÉVY CONSTANT, and the argument of the exponential is sometimes called the KHINTCHINE-LÉVY CONSTANT.

Define the following quantity in terms of the kth partial quotient q_k,

$$M(s, n, x) = \left(\frac{1}{n}\sum_{k=1}^{n} {q_k}^s\right)^{1/s}. \tag{12}$$

Then

$$\lim_{n\to\infty} M(1, n, x) = \infty \tag{13}$$

for almost all real x (Khintchine, Knuth 1981, Finch), and

$$M(1, n, x) \sim \mathcal{O}(\ln n). \tag{14}$$

Furthermore, for $s < 1$, the limiting value

$$\lim_{n\to\infty} M(s, n, x) = K(s) \tag{15}$$

exists and is a constant $K(s)$ with probability 1 (Rockett and Szüsz 1992, Khintchine 1997).

see also CONTINUED FRACTION, CONVERGENT, KHINTCHINE-LÉVY CONSTANT, LÉVY CONSTANT, PARTIAL QUOTIENT, SIMPLE CONTINUED FRACTION

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Vardi, I. "Khinchin's Constant." §8.4 in *Computational Recreations in Mathematica.* Reading, MA: Addison-Wesley, pp. 163–171, 1991.
Wrench, J. W. "Further Evaluation of Khintchine's Constant." *Math. Comput.* **14**, 370–371, 1960.

Khintchine-Lévy Constant

A constant related to KHINTCHINE'S CONSTANT defined by

$$KL \equiv \frac{\pi^2}{12\ln 2} = 1.1865691104\ldots.$$

see also KHINTCHINE'S CONSTANT, LÉVY CONSTANT

References

Plouffe, S. "Khintchine-Levy Constant." `http://lacim.uqam.ca/piDATA/klevy.txt`.

Khovanski's Theorem

If $f_1, \ldots, f_m : \mathbb{R}^n \to \mathbb{R}$ are exponential polynomials, then $\{x \in \mathbb{R}^n : f_1(x) = \cdots f_n(x) = 0\}$ has finitely many connected components.

References

Marker, D. "Model Theory and Exponentiation." *Not. Amer. Math. Soc.* **43**, 753–759, 1996.

Kiepert's Conics

see KIEPERT'S HYPERBOLA, KIEPERT'S PARABOLA

Kiepert's Hyperbola

A curve which is related to the solution of LEMOINE'S PROBLEM and its generalization to ISOSCELES TRIANGLES constructed on the sides of a given TRIANGLE. The VERTICES of the constructed TRIANGLES are

$$A' = -\sin\phi : \sin(C+\phi) : \sin(B+\phi) \quad (1)$$

$$B' = \sin(\mathbf{C}+\phi) : -\sin\phi : \sin(A+\phi) \quad (2)$$

$$C' = \sin(B+\phi) : \sin(A+\phi) : -\sin\phi, \quad (3)$$

where ϕ is the base ANGLE of the ISOSCELES TRIANGLE. Kiepert showed that the lines connecting the VERTICES of the given TRIANGLE and the corresponding peaks of the ISOSCELES TRIANGLES CONCUR. The TRILINEAR COORDINATES of the point of concurrence are

$$\sin(B+\phi)\sin(C+\phi) : \sin(C+\phi)\sin(A+\phi) : \sin(A+\phi)\sin(B+\phi). \quad (4)$$

The locus of this point as the base ANGLE varies is given by the curve

$$\frac{\sin(B-C)}{\alpha} + \frac{\sin(C-A)}{\beta} + \frac{\sin(A-B)}{\gamma} = \frac{bc(c^2-c^2)}{\alpha} + \frac{ca(c^2-a^2)}{\beta} + \frac{ab(a^2-b^2)}{\gamma} = 0. \quad (5)$$

Writing the TRILINEAR COORDINATES as

$$\alpha_i = d_i s_i, \quad (6)$$

where d_i is the distance to the side opposite α_i of length s_i and using the POINT-LINE DISTANCE FORMULA with (x_0, y_0) written as (x, y),

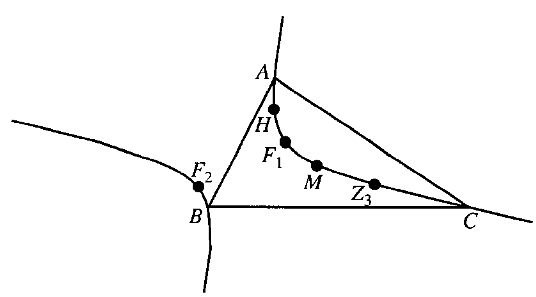
$$d_i = \frac{|(y_{i+2}-y_{i+1})(x-x_{i+1})}{s_i} - \frac{(x_{i+2}-x_{i+1})(y-y_{i+1})|}{s_i}, \quad (7)$$

where $y_4 \equiv y_1$ and $y_5 \equiv y_2$ gives the FORMULA

$$\sum_{i=1}^3 s_{i+1}s_{i+2}(s_{i+1}^2 - s_{i+2}^2) \times \frac{s_i}{(y_{i+2}-y_{i+1})(x-x_{i+1}) - (x_{i+2}-x_{i+1})(y-y_{i+1})} = 0 \quad (8)$$

$$\sum_{i=1}^3 \frac{(s_{i+1}^2 - s_{i+2}^2)}{(y_{i+2}-y_{i+1})(x-x_{i+1}) - (x_{i+2}-x_{i+1})(y-y_{i+1})} = 0. \quad (9)$$

Bringing this equation over a common DENOMINATOR then gives a quadratic in x and y, which is a CONIC SECTION (in fact, a HYPERBOLA). The curve can also be written as $\csc(A+t) : \csc(B+t) : \csc(C+t)$, as t varies over $[-\pi/4, \pi/4]$.



Kiepert's hyperbola passes through the triangle's CENTROID M ($\phi = 0$), ORTHOCENTER H ($\phi = \pi/2$), VERTICES A ($\phi = -\alpha$ if $\alpha \leq \pi/2$ and $\phi = \pi - \alpha$ if $\alpha > \pi/2$), B ($\phi = -\beta$), C ($\phi = -\gamma$), FERMAT POINT F_1 ($\phi = \pi/3$), second ISOGONIC CENTER F_2 ($\phi = -\pi/3$), ISOGONAL CONJUGATE of the BROCARD MIDPOINT ($\phi = \omega$), and BROCARD'S THIRD POINT Z_3 ($\phi = \omega$), where ω is the BROCARD ANGLE (Eddy and Fritsch 1994, p. 193).

The ASYMPTOTES of Kiepert's hyperbola are the SIMSON LINES of the intersections of the BROCARD AXIS with the CIRCUMCIRCLE. Kiepert's hyperbola is a RECTANGULAR HYPERBOLA. In fact, all nondegenerate conics through the VERTICES and ORTHOCENTER of a TRIANGLE are RECTANGULAR HYPERBOLAS the centers of which lie halfway between the ISOGONIC CENTERS and on the NINE-POINT CIRCLE. The LOCUS of centers of these HYPERBOLAS is the NINE-POINT CIRCLE.

The ISOGONAL CONJUGATE curve of Kiepert's hyperbola is the BROCARD AXIS. The center of the INCIRCLE of the TRIANGLE constructed from the MIDPOINTS of the sides of a given TRIANGLE lies on Kiepert's hyperbola of the original TRIANGLE.

see also BROCARD ANGLE, BROCARD AXIS, BROCARD POINTS, CENTROID (TRIANGLE), CIRCUMCIRCLE, ISOGONAL CONJUGATE, ISOGONIC CENTERS, ISOSCELES TRIANGLE, LEMOINE'S PROBLEM, NINE-POINT CIRCLE, ORTHOCENTER, SIMSON LINE

References

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Mineuer, A. "Sur les asymptotes de l'hyperbole de Kiepert." *Mathesis* **49**, 30–33, 1935.

Rigby, J. F. "A Concentrated Dose of Old-Fashioned Geometry." *Math. Gaz.* **57**, 296–298, 1953.

Vandeghen, A. "Some Remarks on the Isogonal and Cevian Transforms. Alignments of Remarkable Points of a Triangle." *Amer. Math. Monthly* **72**, 1091–1094, 1965.

Kiepert's Parabola

Let three similar ISOSCELES TRIANGLES $\Delta A'BC$, $\Delta AB'C$, and $\Delta ABC'$ be constructed on the sides of a TRIANGLE ΔABC. Then the ENVELOPE of the axis of the TRIANGLES ΔABC and $\Delta A'B'C'$ is Kiepert's parabola, given by

$$\frac{\sin A(\sin^2 B - \sin^2 C)}{u} + \frac{\sin B(\sin^2 C - \sin^2 A)}{v} + \frac{\sin C(\sin^2 A - \sin^2 B)}{w} = 0 \quad (1)$$

$$\frac{a(b^2 - c^2)}{u} + \frac{b(c^2 - a^2)}{v} + \frac{c(a^2 - b^2)}{w} = 0, \quad (2)$$

where $[u, v, w]$ are the TRILINEAR COORDINATES for a line tangent to the parabola. It is tangent to the sides of the TRIANGLE, the line at infinity, and the LEMOINE LINE. The FOCUS has TRIANGLE CENTER FUNCTION

$$\alpha = \csc(B - C). \quad (3)$$

The EULER LINE of a triangle is the DIRECTRIX of Kiepert's parabola. In fact, the DIRECTRICES of all parabolas inscribed in a TRIANGLE pass through the ORTHOCENTER. The BRIANCHON POINT for Kiepert's parabola is the STEINER POINT.

see also BRIANCHON POINT, ENVELOPE, EULER LINE, ISOSCELES TRIANGLE, LEMOINE LINE, STEINER POINTS

Kieroid

Let the center B of a CIRCLE of RADIUS a move along a line BA. Let O be a fixed point located a distance c away from AB. Draw a SECANT LINE through O and D, the MIDPOINT of the chord cut from the line DE (which is parallel to AB) and a distance b away. Then the LOCUS of the points of intersection of OD and the CIRCLE P_1 and P_2 is called a kieroid.

Special Case	Curve
$b = 0$	conchoid of Nicomedes
$b = a$	cissoid plus asymptote
$b = a = -c$	strophoid plus asymptote

References

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Killing's Equation

The equation defining KILLING VECTORS.

$$\mathcal{L}_X g_{ab} = X_{a;b} + X_{b;a} = 2X_{(a;b)} = 0,$$

where $\mathcal{L}$ is the LIE DERIVATIVE.

see also KILLING VECTORS

Killing Vectors

If any set of points is displaced by $X^i dx_i$ where all distance relationships are unchanged (i.e., there is an ISOMETRY), then the VECTOR field is called a Killing vector.

$$g_{ab} = \frac{\partial x'^c}{\partial x^a}\frac{\partial x'^d}{\partial x^b} g_{cd}(x'), \quad (1)$$

so let

$$x'^a = x^a + \epsilon x^a$$
$$\frac{\partial x'^a}{\partial x^b} = \delta^a_b + \epsilon x^a{}_{,b} \quad (2)$$

$$\begin{aligned} g_{ab}(x) &= (\delta^c_a + \epsilon x^c{}_{,a})\left(\delta^d_b + \epsilon x^d{}_{,b}\right) g_{cd}(x^e + \epsilon X^e) \\ &= (\delta^c_a + \epsilon x^c{}_{,a})\left(\delta^d_b + \epsilon x^d{}_{,b}\right)[g_{cd}(x) + \epsilon X^e g_{cd}(x)_{,e} + \ldots] \\ &= g_{ab}(x) + \epsilon[g_{ad}X^d{}_{,b} + g_{bd}X^d{}_{,a} + X^e g_{ab,e}] + \mathcal{O}(\epsilon^2) \\ &= \mathcal{L}_X g_{ab}, \end{aligned} \quad (3)$$

where $\mathcal{L}$ is the LIE DERIVATIVE. An ordinary derivative can be replaced with a covariant derivative in a LIE DERIVATIVE, so we can take as the definition

$$g_{ab;c} = 0 \quad (4)$$

$$g_{ab}g^{bc} = \delta^c_a, \quad (5)$$

which gives KILLING'S EQUATION

$$\mathcal{L}_X g_{ab} = X_{a;b} + X_{b;a} = 2X_{(a;b)} = 0. \quad (6)$$

A Killing vector X^b satisfies

$$g^{bc}X_{c;ab} - R_{ab}X^b = 0 \tag{7}$$

$$X_{a;bc} = R_{abcd}X^d \tag{8}$$

$$X^{a;b}{}_{;b} + R^a_c X^c = 0, \tag{9}$$

where R_{ab} is the RICCI TENSOR and R_{abcd} is the RIEMANN TENSOR.

A 2-sphere with METRIC

$$ds^2 = d\theta^2 + \sin^2\theta\, d\phi^2 \tag{10}$$

has three Killing vectors, given by the angular momentum operators

$$\tilde{L}_x = -\cos\phi\frac{\partial}{\partial\theta} + \cot\theta\sin\phi\frac{\partial}{\partial\phi} \tag{11}$$

$$\tilde{L}_y = \sin\phi\frac{\partial}{\partial\theta} + \cot\theta\cos\phi\frac{\partial}{\partial\phi} \tag{12}$$

$$\tilde{L}_z = \frac{\partial}{\partial\phi}. \tag{13}$$

The Killing vectors in Euclidean 3-space are

$$x^1 = \frac{\partial}{\partial x} \tag{14}$$

$$x^2 = \frac{\partial}{\partial y} \tag{15}$$

$$x^3 = \frac{\partial}{\partial z} \tag{16}$$

$$x^4 = y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y} \tag{17}$$

$$x^5 = z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z} \tag{18}$$

$$x^6 = x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}. \tag{19}$$

In MINKOWSKI SPACE, there are 10 Killing vectors

$$X_i^\mu = a_i{}^\mu \quad \text{for } i = 1,2,3,4 \tag{20}$$

$$X_k^0 = 0 \tag{21}$$

$$X_k^l = \epsilon^{lkm}x_m \quad \text{for } k = 1,2,3 \tag{22}$$

$$X_\mu^k = \delta_\mu{}^{[0_x k]} \quad \text{for } k = 1,2,3. \tag{23}$$

The first group is TRANSLATION, the second ROTATION, and the final corresponds to a "boost."

Kimberling Sequence

A sequence generated by beginning with the POSITIVE integers, then iteratively applying the following algorithm:

1. In iteration i, discard the ith element,
2. Alternately write the $i+k$ and $i-k$th elements until $k = i$,
3. Write the remaining elements in order.

The first few iterations are therefore

$\boxed{1}$	2	3	4	5	6	7	8	9	10	11
2	$\boxed{3}$	4	5	6	7	8	9	10	11	12
4	2	$\boxed{5}$	6	7	8	9	10	11	12	13.
6	2	7	$\boxed{4}$	8	9	10	11	12	13	14
8	7	9	2	$\boxed{10}$	6	11	12	13	14	15

The diagonal elements form the sequence 1, 3, 5, 4, 10, 7, 15, ... (Sloane's A007063).

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Kimberling Shuffle

see also KIMBERLING SEQUENCE

Kings Problem

Kg		Kg		Kg		Kg	
Kg		Kg		Kg		Kg	
Kg		Kg		Kg		Kg	
Kg		Kg		Kg		Kg	

The problem of determining how many nonattacking kings can be placed on an $n \times n$ CHESSBOARD. For $n = 8$, the solution is 16, as illustrated above (Madachy 1979). In general, the solutions are

$$K(n) = \begin{cases} \frac{1}{4}n^2 & n \text{ even} \\ \frac{1}{4}(n+1)^2 & n \text{ odd} \end{cases} \tag{1}$$

(Madachy 1979), giving the sequence of doubled squares 1, 1, 4, 4, 9, 9, 16, 16, ... (Sloane's A008794). This sequence has GENERATING FUNCTION

$$\frac{1+x^2}{(1-x^2)^2(1-x)} = 1 + x + 4x^2 + 4x^3 + 9x^4 + 9x^5 + \ldots. \tag{2}$$

Kg			Kg			Kg	
Kg			Kg			Kg	
Kg			Kg			Kg	

The minimum number of kings needed to attack or occupy all squares on an 8×8 CHESSBOARD is nine, illustrated above (Madachy 1979).

see also BISHOPS PROBLEM, CHESS, HARD HEXAGON ENTROPY CONSTANT, KNIGHTS PROBLEM, QUEENS PROBLEM, ROOKS PROBLEM

References
Madachy, J. S. *Madachy's Mathematical Recreations.* New York: Dover, p. 39, 1979.

King Walk

see DELANNOY NUMBER

Kinney's Set

A set of plane MEASURE 0 that contains a CIRCLE of every RADIUS.

References
Falconer, K. J. *The Geometry of Fractal Sets.* New York: Cambridge University Press, 1985.
Fejzić, H. "On Thin Sets of Circles." *Amer. Math. Monthly* **103**, 582–585, 1996.
Kinney, J. R. "A Thin Set of Circles." *Amer. Math. Monthly* **75**, 1077–1081, 1968.

Kinoshita-Terasaka Knot

The KNOT with BRAID WORD

$$\sigma_1{}^3\sigma_3{}^2\sigma_2\sigma_3{}^{-1}\sigma_1{}^{-2}\sigma_2\sigma_1{}^{-1}\sigma_3{}^{-1}\sigma_2{}^{-1}.$$

Its JONES POLYNOMIAL is

$$t^{-4}(-1+2t-2t^2+2t^3+t^6-2t^7+2t^8-2t^9+t^{10}),$$

the *same* as for CONWAY'S KNOT. It has the same ALEXANDER POLYNOMIAL as the UNKNOT.

References
Kinoshita, S. and Terasaka, H. "On Unions of Knots." *Osaka Math. J.* **9**, 131–153, 1959.

Kinoshita-Terasaka Mutants

References
Adams, C. C. *The Knot Book: An Elementary Introduction to the Mathematical Theory of Knots.* New York: W. H. Freeman, pp. 49–50, 1994.

Kirby Calculus

The manipulation of DEHN SURGERY descriptions by a certain set of operations.

References
Adams, C. C. *The Knot Book: An Elementary Introduction to the Mathematical Theory of Knots.* New York: W. H. Freeman, p. 263, 1994.

Kirby's List

A list of problems in low-dimensional TOPOLOGY maintained by R. C. Kirby. The list currently runs about 380 pages.

References
Kirby, R. "Problems in Low-Dimensional Topology." `http://math.berkeley.edu/~kirby/`.

Kirkman's Schoolgirl Problem

In a boarding school there are fifteen schoolgirls who always take their daily walks in rows of threes. How can it be arranged so that each schoolgirl walks in the same row with every other schoolgirl exactly once a week? Solution of this problem is equivalent to constructing a KIRKMAN TRIPLE SYSTEM of order $n = 2$. The following table gives one of the 7 distinct (up to permutations of letters) solutions to the problem.

Sun	Mon	Tue	Wed	Thu	Fri	Sat
ABC	ADE	AFG	AHI	AJK	ALM	ANO
DHL	BIK	BHJ	BEG	CDF	BEF	BDG
EJN	CMO	CLN	BMN	CLO	CIJ	CHK
FIO	FHN	DIM	DJO	EHM	DKN	EIL
GKM	GJL	EKO	FKL	GIN	GHO	FJM

(The table of Dörrie 1965 contains a misprint in which the $a_1 = B$ and $a_2 = C$ entries for Wednesday and Thursday are written simply as a.)

see also JOSEPHUS PROBLEM, KIRKMAN TRIPLE SYSTEM, STEINER TRIPLE SYSTEM

References
Abel, R. J. R. and Furino, S. C. "Kirkman Triple Systems." §I.6.3 in *The CRC Handbook of Combinatorial Designs* (Ed. C. J. Colbourn and J. H. Dinitz). Boca Raton, FL: CRC Press, pp. 88–89, 1996.
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Peirce, B. "Cyclic Solutions of the School-Girl Puzzle." *Astron. J.* **6**, 169–174, 1859–1861.
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Kirkman Triple System

A Kirkman triple system of order $v = 6n+3$ is a STEINER TRIPLE SYSTEM with parallelism (Ball and Coxeter 1987), i.e., one with the following additional stipulation: the set of $b = (2n+1)(3n+1)$ triples is partitioned into $3n+1$ components such that each component is a $(2n+1)$-subset of triples and each of the v elements appears exactly once in each component. The