2

Classes of groups and their properties

2.1 Classes of groups and closure operators

A *group theoretical class* or *class of groups* $\mathfrak{X}$ is a collection of groups with the property that if $G \in \mathfrak{X}$, then every group isomorphic to G belongs to $\mathfrak{X}$. The groups which belong to a class $\mathfrak{X}$ are referred to as $\mathfrak{X}$-*groups*.

Following K. Doerk and T. O. Hawkes [DH92], we denote the empty class of groups by $\emptyset$ whereas the Fraktur (Gothic) font is used when a single capital letter denotes a class of groups. If $\mathcal{S}$ is a set of groups, we use $(\mathcal{S})$ to denote the smallest class of groups containing $\mathcal{S}$, and when $\mathcal{S} = \{G_1, \dots, G_n\}$, a finite set, $(G_1, \dots, G_n)$ rather than $(\{G_1, \dots, G_n\})$.

Since certain natural classes of groups recur frequently, it is convenient to have a short fixed alphabet of classes:

- $\emptyset$ denotes the empty class of groups;
- $\mathfrak{A}$ denotes the class of all abelian groups;
- $\mathfrak{N}$ denotes the class of all nilpotent groups;
- $\mathfrak{U}$ denotes the class of all supersoluble groups;
- $\mathfrak{S}$ denotes the class of all soluble groups;
- $\mathfrak{J}$ denotes the class of all simple groups;
- $\mathbb{P}$ denotes either the class $\mathfrak{A} \cap \mathfrak{J}$ of all cyclic groups of prime order or the set of all primes;
- $\mathfrak{P}$ denote the class of all primitive groups;
- $\mathfrak{P}_i$ denotes the class of all primitive groups of type i, $1 \leq i \leq 3$;
- $\mathfrak{E}$ denotes the class of all finite groups.

The group classes are, of course, partially ordered by inclusion and the notation

$$\mathfrak{X} \subseteq \mathfrak{Y}$$

will be used to denote the fact that $\mathfrak{X}$ is a subclass of the class $\mathfrak{Y}$.

Sometimes it is preferable to deal with group theoretical properties or properties of groups: A *group theoretical property* $\mathcal{P}$ is a property pertaining to groups such that if a group G has $\mathcal{P}$, then every isomorphic image of G has $\mathcal{P}$. The groups which have a given group theoretical property form a class of groups and to belong to a given group theoretical class is a group theoretical property. Consequently, there is a one-to-one correspondence between the group classes and the group theoretical properties; for this reason we will often not distinguish between a group theoretical property and the class of groups that possess it.

Note that we do not require that a class of groups contains groups of order 1.

Definition 2.1.1. *Let G be a group and let $\mathfrak{X}$ be a class of groups.*

1. *We define*
$$\pi(G) = \{p : p \in \mathbb{P} \text{ and } p \mid |G|\}, \quad \text{and}$$
$$\pi(\mathfrak{X}) = \bigcup\{\pi(G) : G \in \mathfrak{X}\}.$$
2. *We also define*
$$\mathrm{K}\,\mathfrak{X} = \{S \in \mathfrak{J} : S \text{ is a composition factor of an } \mathfrak{X}\text{-group}\}$$
and
$$\operatorname{char}\mathfrak{X} = \{p : p \in \mathbb{P} \text{ and } C_p \in \mathfrak{X}\};$$
we say that $\operatorname{char}(\mathfrak{X})$ *is the* characteristic *of* $\mathfrak{X}$.

Obviously $\operatorname{char}\mathfrak{X}$ is contained in $\pi(\mathfrak{X})$, but the equality does not hold in general. If $\mathfrak{X} = \big(G : G = \mathrm{O}^{p'}(G)\big)$ is the class of all p'-perfect groups for some prime p, then $\operatorname{char}\mathfrak{X} = \{p\} \neq \pi(\mathfrak{X}) = \mathbb{P}$. Note that $\operatorname{char}\mathfrak{X}$, regarded as a subclass of $\mathfrak{J}$, is contained in $\mathrm{K}\,\mathfrak{X}$. The class of all p'-perfect groups shows that the inclusion is proper.

Definition 2.1.2. *If $\mathfrak{X}$ and $\mathfrak{Y}$ are two classes of groups, the* product class *$\mathfrak{X}\mathfrak{Y}$ is defined as follows: a group G belongs to $\mathfrak{X}\mathfrak{Y}$ if and only if there is a normal subgroup N of G such that $N \in \mathfrak{X}$ and $G/N \in \mathfrak{Y}$. Groups in the class $\mathfrak{X}\mathfrak{Y}$ are called* $\mathfrak{X}$-by-$\mathfrak{Y}$-groups*.*

If $\mathfrak{X} = \emptyset$ or $\mathfrak{Y} = \emptyset$, we have the obvious interpretation $\mathfrak{X}\mathfrak{Y} = \emptyset$.

It should be observed that this binary algebraic operation on the class of all classes of groups is neither associative nor commutative. For instance, let G be the alternating group of degree 4. Then $G \in (\mathfrak{C}\mathfrak{C})\mathfrak{C}$, where $\mathfrak{C}$ is the class of all cyclic groups. However G has no non-trivial normal cyclic subgroups, so $G \notin \mathfrak{C}(\mathfrak{C}\mathfrak{C})$.

On the other hand, the inclusion $\mathfrak{X}(\mathfrak{Y}\mathfrak{Z}) \subseteq (\mathfrak{X}\mathfrak{Y})\mathfrak{Z}$ is universally valid and, indeed, follows at once from our definition.

For the powers of a class $\mathfrak{X}$, we set $\mathfrak{X}^0 = (1)$, and for $n \in \mathbb{N}$ make the inductive definition $\mathfrak{X}^n = (\mathfrak{X}^{n-1})\mathfrak{X}$. A group in $\mathfrak{X}^2$ is sometimes denoted *meta*-$\mathfrak{X}$.

The past decades have seen the introduction of a very large number of classes of groups and it would be quite impossible to use a systematic alphabet for them. However, one soon observes that many of these classes are obtainable from simpler classes by certain uniform procedures. From this observation stems the importance for our purposes of the concept of closure operation. The first systematic use of closure operations in group theory occurs in papers of P. Hall [Hal59, Hal63] although the ideas are implicit in earlier papers of R. Baer and also in B. I. Plotkin [Plo58].

By an *operation* we mean a function C assigning to each class of groups $\mathfrak{X}$ a class of groups $\mathrm{C}\,\mathfrak{X}$ subject to the following conditions:

1. $\mathrm{C}\,\emptyset = \emptyset$, and
2. $\mathfrak{X} \subseteq \mathrm{C}\,\mathfrak{X} \subseteq \mathrm{C}\,\mathfrak{Y}$ whenever $\mathfrak{X} \subseteq \mathfrak{Y}$.

Should it happen that $\mathfrak{X} = \mathrm{C}\,\mathfrak{X}$, the class $\mathfrak{X}$ is said to be C-*closed*. By 1 and 2, the classes $\emptyset$ and $\mathfrak{E}$ are C-closed when C is any operation.

A partial ordering of operations is defined as follows: $\mathrm{C}_1 \leq \mathrm{C}_2$ means that $\mathrm{C}_1\,\mathfrak{X} \subseteq \mathrm{C}_2\,\mathfrak{X}$ for every class of groups $\mathfrak{X}$. Products of operations are formed according to the rule

$$(\mathrm{C}_1\,\mathrm{C}_2)\mathfrak{X} = \mathrm{C}_1(\mathrm{C}_2\,\mathfrak{X}).$$

An operation C is called a *closure operation* if it is idempotent, that is, if

3. $\mathrm{C} = \mathrm{C}^2$.

If C is a closure operation, then by Condition 2 and Condition 3, the class $\mathrm{C}\,\mathfrak{X}$ is the uniquely determined, smallest C-closed class that contains $\mathfrak{X}$. Thus if A and B are closure operations, $\mathrm{A} \leq \mathrm{B}$ if and only if B-closure invariably implies A-closure.

A closure operation can be determined by specifying the classes of groups that are closed. Let $\mathcal{S}$ be a class of classes of groups and suppose that every intersection of members of $\mathcal{S}$ belongs to $\mathcal{S}$: for example, $\mathcal{S}$ might consist of the closed classes of a closure operation. $\mathcal{S}$ determines a closure operation C defined as follows: for any class of groups $\mathfrak{X}$, let $\mathrm{C}\,\mathfrak{X}$ be the intersection of all those members of $\mathcal{S}$ that contain $\mathfrak{X}$. The C-closed classes are precisely the members of $\mathcal{S}$.

Now we list some of the most commonly used closure operations.

For a class $\mathfrak{X}$ of groups, we define:

$$\begin{aligned}
\mathrm{S}\,\mathfrak{X} &= (G : G \leq H \text{ for some } H \in \mathfrak{X});\\
\mathrm{Q}\,\mathfrak{X} &= (G : \text{there exist } H \in \mathfrak{X} \text{ and an epimorphism from } H \text{ onto } G);\\
\mathrm{S}_n\,\mathfrak{X} &= (G : G \text{ is subnormal in } H \text{ for some } H \in \mathfrak{X});\\
\mathrm{R}_0\,\mathfrak{X} &= \bigl(G : \text{there exist } N_i \trianglelefteq G \ (i = 1, \dots, r)\\
&\qquad \text{with } G/N_i \in \mathfrak{X} \text{ and } \bigcap_{i=1}^{r} N_i = 1\bigr).
\end{aligned}$$

Note that a group $G \in \mathrm{R}_0\,\mathfrak{X}$ if and only if G is isomorphic with a subdirect product of a direct product of a finite set of $\mathfrak{X}$-groups ([DH92, II, 1.18]).

$$\begin{aligned}
\mathrm{N}_0\,\mathfrak{X} &= \bigl(G : \text{there exist } K_i \text{ subnormal in } G\ (i = 1, \dots, r)\\
&\qquad\qquad \text{with } K_i \in \mathfrak{X} \text{ and } G = \langle K_1, \dots, K_r\rangle\bigr);\\
\mathrm{D}_0\,\mathfrak{X} &= (G : G = H_1 \times \cdots \times H_r \text{ with each } H_i \in \mathfrak{X});\\
\mathrm{E}_{\Phi}\,\mathfrak{X} &= (G : \text{there exists } N \trianglelefteq G \text{ with } N \le \Phi(G) \text{ and } G/N \in \mathfrak{X}).
\end{aligned}$$

The operations S_n and Q, and N_0 and R_0 are dual in the well-known duality between normal subgroup and factor group: this will become more apparent in the context of Fitting classes and formations in next sections.

Lemma 2.1.3 ([DH92, II, 1.6]). *The operations defined in the above list are all closure operations.*

We shall say that a class $\mathfrak{X}$ is *subgroup-closed* if $\mathfrak{X} = \mathrm{S}\,\mathfrak{X}$, that is, if every subgroup of an $\mathfrak{X}$-group is again an $\mathfrak{X}$-group; if $\mathfrak{X} = \mathrm{Q}\,\mathfrak{X}$, we shall say that $\mathfrak{X}$ is an *homomorph*, that is, every epimorphic image of an $\mathfrak{X}$ is an $\mathfrak{X}$-group. If $\mathfrak{X} = \mathrm{S}_n\,\mathfrak{X}$, we might say that $\mathfrak{X}$ is *subnormal subgroup-closed* and if $\mathfrak{X} = \mathrm{R}_0\,\mathfrak{X}$, we could say that $\mathfrak{X}$ is *residually closed*. An E_{Φ}-closed class is called *saturated*.

The product of two closure operations need not be a closure operation since it may easily fail to be idempotent. This leads us to make the following definition. Let $\{\mathrm{A}_\lambda : \lambda \in \Lambda\}$ be a set of operations (not necessarily closure operations). We define $\mathrm{C} = \langle \mathrm{A}_\lambda : \lambda \in \Lambda\rangle$, the *closure operation generated by the* A_λ, as that closure operation whose closed classes are the classes of groups that are A_λ-closed for every $\lambda \in \Lambda$. That is, $\mathrm{C}\,\mathfrak{X} = \bigcap\{\mathfrak{Y} : \mathfrak{X} \subseteq \mathfrak{Y} = \mathrm{A}_\lambda\,\mathfrak{Y} \text{ for all } \lambda \in \Lambda\}$ for any class $\mathfrak{X}$ of groups.

It is easily verified that C is the uniquely determined least closure operation such that $\mathrm{A}_\lambda \le \mathrm{C}$ for every $\lambda \in \Lambda$.

Of particular interest are $\langle \mathrm{A}\rangle$, the closure operation generated by the operation A, and also $\langle \mathrm{A}, \mathrm{B}\rangle$. In the latter case $\mathrm{A}\,\mathrm{B}$ and $\mathrm{B}\,\mathrm{A}$ may differ from $\langle \mathrm{A}, \mathrm{B}\rangle$, even although A and B are closure operations.

Now follows a simple but useful criterion for the product of two closure operations to be a closure operation.

Proposition 2.1.4 ([DH92, II, 1.16]). *If* A *and* B *are closure operations, any two of the following statements are equivalent:*

1. $\mathrm{A}\,\mathrm{B}$ *is a closure operation;*
2. $\mathrm{B}\,\mathrm{A} \le \mathrm{A}\,\mathrm{B}$*;*
3. $\mathrm{A}\,\mathrm{B} = \langle \mathrm{A}, \mathrm{B}\rangle$*.*

Next we give a list of some situations in which the criterion may be applied.

Lemma 2.1.5 ([DH92, II, 1.17 and 1.18]).

1. $\mathrm{Q}\,\mathrm{E}_{\Phi} \le \mathrm{E}_{\Phi}\,\mathrm{Q}$. *Thus* $\mathrm{E}_{\Phi}\,\mathrm{Q}$ *is a closure operation.*
2. $\mathrm{D}_0\,\mathrm{S} \le \mathrm{S}\,\mathrm{D}_0$. *Hence* $\mathrm{S}\,\mathrm{D}_0$ *is a closure operation.*

3. $\mathrm{D}_0 \mathrm{E}_\Phi \leq \mathrm{E}_\Phi \mathrm{D}_0$. *Hence* $\mathrm{E}_\Phi \mathrm{D}_0$ *is a closure operation.*
4. $\mathrm{R}_0 \mathrm{Q} \leq \mathrm{Q} \mathrm{R}_0$, *whence* $\mathrm{Q} \mathrm{R}_0$ *is a closure operation. Moreover,* $\mathrm{R}_0 \leq \mathrm{S} \mathrm{D}_0$, *whence every* $\mathrm{S} \mathrm{D}_0$*-closed class is* R_0*-closed.*

We shall adhere to the conventions about the empty class exposed in [DH92, II, p. 271].

2.2 Formations: Basic properties and results

Some of the most important classes of groups are formations. They are considered in some detail in the present section. We gather together facts of a general nature about formations and we give some important examples. Some classical results are also included.

Definition 2.2.1. *A* formation *is a class of groups which is both* Q*-closed and* R_0*-closed, that is, a class of groups* $\mathfrak{F}$ *is a formation if* $\mathfrak{F}$ *has the following two properties:*

1. *If* $G \in \mathfrak{F}$ *and* $N \trianglelefteq G$, *then* $G/N \in \mathfrak{F}$;
2. *If* N_1, $N_2 \trianglelefteq G$ *with* $N_1 \cap N_2 = 1$ *and* $G/N_i \in \mathfrak{F}$ *for* $i = 1, 2$, *then* $G \in \mathfrak{F}$.

By Lemma 2.1.5, $\mathrm{Q} \mathrm{R}_0 = \langle \mathrm{Q}, \mathrm{R}_0 \rangle$. Hence a class $\mathfrak{F}$ is a formation if and only if $\mathfrak{F} = \mathrm{Q} \mathrm{R}_0 \mathfrak{F}$. If $\mathfrak{X}$ is a class of groups, we shall sometimes write $\operatorname{form} \mathfrak{X}$ instead of $\mathrm{Q} \mathrm{R}_0 \mathfrak{X}$ for the *formation generated by* $\mathfrak{X}$.

Note that a class of groups which is simultaneously closed under S, Q, and D_0 is a formation by Lemma 2.1.5. Therefore the class $\mathfrak{N}_c$ of nilpotent groups of class at most c, the class $\mathfrak{S}^{(d)}$ of soluble groups of derived length at most d, the class $\mathfrak{E}(n)$ of groups of exponent at most n, the class $\mathfrak{U}$ of supersoluble groups, and the class $\mathfrak{A}$ of abelian groups are the most classical examples of formations. They are $\langle \mathrm{S}, \mathrm{Q}, \mathrm{D}_0 \rangle$-closed classes of groups.

The following elementary fact is useful in establishing the structure of minimal counterexamples in proofs involving Q- and R_0-closed classes.

Proposition 2.2.2 ([DH92, II, 2.5]). *Let* $\mathfrak{X}$ *and* $\mathfrak{Y}$ *be classes of groups.*

1. *Let* $\mathfrak{X} = \mathrm{Q} \mathfrak{X}$, $\mathfrak{Y} = \mathrm{R}_0 \mathfrak{Y}$, *and let* G *be a group of minimal order in* $\mathfrak{X} \setminus \mathfrak{Y}$. *Then* G *is monolithic (i.e.* G *has a unique minimal normal subgroup). If, in addition,* $\mathfrak{Y}$ *is saturated, then* G *is primitive.*
2. *Let* G *be a group of minimal order in* $\mathrm{R}_0 \mathfrak{X} \setminus \mathfrak{X}$. *Then* G *has a normal subgroups* N_1 *and* N_2 *such that* $G/N_i \in \mathfrak{X}$ *for* $i = 1, 2$ *and* $N_1 \cap N_2 = 1$. *If* $\mathfrak{X} = \mathrm{Q} \mathfrak{X}$, *then* N_1 *and* N_2 *can be chosen to be minimal normal subgroups of* G.

The next lemma provides some more examples of formations.

Lemma 2.2.3. 1. *If* S *is a non-abelian simple group, then* $\mathrm{D}_0\big((S) \cup (1)\big) = \mathrm{D}_0(S, 1)$ *is a* $\langle \mathrm{S}_n, \mathrm{N}_0 \rangle$*-closed formation. Hence* $\operatorname{form}(S) = \mathrm{D}_0(S, 1)$.

2. If $\mathfrak{F}$ and $\mathfrak{G}$ are formations and $\mathfrak{F} \cap \mathfrak{G} = (1)$, then $\mathrm{D}_0(\mathfrak{F} \cup \mathfrak{G}) = \mathrm{R}_0(\mathfrak{F} \cup \mathfrak{G})$.
3. Let $\emptyset \neq \mathfrak{F}$ be a formation and let S be a non-abelian simple group. Then $\mathrm{Q}\,\mathrm{R}_0(\mathfrak{F}, S) = \mathrm{D}_0(\mathfrak{F}, S) = \mathrm{D}_0\big(\mathfrak{F} \cup (S)\big)$.

Proof. 1. Write $\mathfrak{D} = \mathrm{D}_0(S, 1)$. Applying [DH92, A, 4.13], every normal subgroup of a $\mathfrak{D}$-group is a direct product of a subset of direct components isomorphic with S. Hence $\mathfrak{D}$ is S_n-closed. In addition, every normal subgroup N of a group $G \in \mathfrak{D}$ satisfies $G = N \times \mathrm{C}_G(N)$. Hence $G/N \in \mathfrak{D}$ and $\mathfrak{D}$ is Q-closed.

Assume that $\mathrm{R}_0\,\mathfrak{D} \neq \mathfrak{D}$ and derive a contradiction. Let G be a group of minimal order in $\mathrm{R}_0\,\mathfrak{D} \setminus \mathfrak{D}$. Then, by Proposition 2.2.2, G has minimal normal subgroups N_1 and N_2 such that $G/N_i \in \mathfrak{D}$, $i = 1, 2$, and $N_1 \cap N_2 = 1$. Consider the normal subgroup N_2N_1/N_1 of G/N_1. Since $G/N_1 \in \mathfrak{D}$, it follows that $G/N_1 = N_2N_1/N_1 \times R/N_1$ and N_2N_1/N_1 and R/N_1 are direct products of copies of S. In particular, $G = (N_1N_2)R$ and $R \cap N_1N_2 = N_1$. It implies that $R \cap N_2 = 1$ and $G = RN_2$. But $G/N_2 \in \mathfrak{D}$ and so $R \in \mathfrak{D}$. Hence $G \in \mathfrak{D}$, contrary to our initial supposition. Consequently $\mathfrak{D}$ is R_0-closed and hence $\mathfrak{D}$ is a formation. It is clear then that $\mathfrak{D} = \mathrm{form}(S)$.

Finally we show that $\mathfrak{D}$ is N_0-closed. Let N_1 and N_2 be normal subgroups of a group $G = N_1N_2$ such that $N_i \in \mathfrak{D}$, $i = 1, 2$. Then $M = N_1 \cap N_2 \in \mathfrak{D}$ and $G/M \in \mathrm{D}_0\,\mathfrak{D} = \mathfrak{D}$. Moreover if $C_i = \mathrm{C}_{M_i}(M)$, it is clear that $C_1 \cap C_2 \leq \mathrm{C}_M(M) = 1$ and $|C_i| = |N_i : M|$, $i = 1, 2$. Hence $C_1C_2 = \mathrm{C}_G(M)$ is isomorphic to G/M. Consequently $G = M \times \mathrm{C}_G(M) \in \mathfrak{D}$. We can conclude that $\mathfrak{D}$ is N_0-closed.

2. Clearly $\mathrm{D}_0(\mathfrak{F} \cup \mathfrak{G}) \subseteq \mathrm{R}_0(\mathfrak{F} \cup \mathfrak{G})$. Let $G \in \mathrm{R}_0(\mathfrak{F} \cup \mathfrak{G})$. Then G has normal subgroups N_i, $i = 1, \dots, n$, such that $G/N_i \in \mathfrak{F}$ and G has normal subgroups M_i, $i = 1, \dots, m$, such that $G/M_i \in \mathfrak{G}$. Moreover $\big(\bigcap_{i=1}^n N_i\big) \cap \big(\bigcap_{j=1}^m M_j\big) = 1$. Put $N = \bigcap_{i=1}^n N_i$ and $M = \bigcap_{j=1}^m M_j$. Then $G/N \in \mathrm{R}_0\,\mathfrak{F} = \mathfrak{F}$ and $G/M \in \mathrm{R}_0\,\mathfrak{G} = \mathfrak{G}$. Hence $G/MN \in \mathrm{Q}\,\mathfrak{F} \cap \mathrm{Q}\,\mathfrak{G} = \mathfrak{F} \cap \mathfrak{G} = (1)$. It follows that $G = MN \cong M \times N$ and $G \in \mathrm{D}_0(\mathfrak{F} \cup \mathfrak{G})$. Hence $\mathrm{D}_0(\mathfrak{F} \cup \mathfrak{G}) = \mathrm{R}_0(\mathfrak{F} \cup \mathfrak{G})$.

3. Denote $\mathfrak{D} = \mathrm{D}_0(\mathfrak{F}, S) = \mathrm{D}_0\big(\mathfrak{F} \cup (S)\big)$. Clearly we may assume $S \notin \mathfrak{F}$. In this case, $\mathrm{D}_0(S, 1) \cap \mathfrak{F} = (1)$ and $\mathfrak{D} = \mathrm{D}_0\big(\mathfrak{F}, \mathrm{D}_0(S, 1)\big) = \mathrm{R}_0\big(\mathfrak{F}, \mathrm{D}_0(S, 1)\big)$ by Statement 2. In particular, $\mathfrak{D}$ is R_0-closed.

Let $G \in \mathfrak{D}$ and N a normal subgroup of G. Since $G \in \mathfrak{D}$, we have that $G = M_1 \times M_2$, $M_1 \in \mathfrak{F}$ and $M_2 \in \mathrm{D}_0(S, 1)$. If N is contained in either M_1 or M_2, then $G/N \in \mathfrak{D}$ and if $M_1 \cap N = M_2 \cap N = 1$, then $N \leq \mathrm{Z}(G) = \mathrm{Z}(M_1) \times \mathrm{Z}(M_2)$. Since groups in $\mathrm{D}_0(S, 1)$ have trivial centre, we have that $N \leq M_1$, with contradicts $N \cap M_1 = 1$. Hence either $N \leq M_1$ or $N \leq M_2$. In both cases, $G/N \in \mathfrak{D}$. This implies that $\mathfrak{D}$ is Q-closed and so $\mathfrak{D}$ is indeed a formation. □

An important result in the theory of formations is the theorem of D. W. Barnes and O. H. Kegel that shows that a if a group with a prescribed action appears as a Frattini chief factor of a group in a given formation, then it will also appear as a complemented chief factor of a group in the same formation. The proof of this result depends on the following lemma.

Lemma 2.2.4 ([BBPR96a]). *Let the group $G = NB$ be the product of two subgroups N and B. Assume that N is normal in G. Since B acts by conjugation on N, we can construct the semidirect product, $X = [N]B$, with respect to this action. Then the natural map $\alpha\colon X \longrightarrow G$ given by $(nb)^\alpha = nb$, for every $n \in N$ and $b \in B$, is an epimorphism,* $\operatorname{Ker}(\alpha) \cap N = 1$ *and* $\operatorname{Ker}(\alpha) \le \mathrm{C}_X(N)$.

Corollary 2.2.5 ([BK66]). *Let $\mathfrak{F}$ be a formation. Let M and N be normal subgroups of a group $G \in \mathfrak{F}$. Assume that $M \le \mathrm{C}_G(N)$ and form the semidirect product $H = [N](G/M)$ with respect to the action of G/M on N by conjugation. Then $H \in \mathfrak{F}$.*

Proof. Consider G acting on N by conjugation and construct $X = [N]G$, the corresponding semidirect product. By Lemma 2.2.4, there exists an epimorphism $\alpha\colon X \longrightarrow G = NG$ such that $\operatorname{Ker}(\alpha) \cap N = 1$. Since $X/\operatorname{Ker}(\alpha) \cong G \in \mathfrak{F}$ and $X/N \cong G \in \mathfrak{F}$, it follows that $X \in \mathrm{R}_0\,\mathfrak{F} = \mathfrak{F}$. Now M is a normal subgroup of X contained in G and $X/M \cong [N](G/M)$. Hence $X/M \in \mathrm{Q}\,\mathfrak{F} = \mathfrak{F}$. $\square$

Let G be a group in a formation $\mathfrak{F}$ and let N be an abelian normal subgroup of G. Suppose that U is a subgroup of G such that $G = UN$. Then, by Lemma 2.2.4, G is an epimorphic image of $X = [N]U$, where U acts on N by conjugation. If $Z = N \cap U$, we have that $Z \le \mathrm{C}_G(N)$ and it is a normal subgroup of X. Moreover, $X/Z \cong [N](U/Z) \cong [N](G/N) \in \mathfrak{F}$ by Corollary 2.2.5. Since X has a normal subgroup, X_1 say, such that $X/X_1 \cong G \in \mathfrak{F}$ and $X_1 \cap U = 1$, it follows that $X \in \mathfrak{F}$. In particular, $U \in \mathfrak{F}$.

This result is a particular case of the following theorem of R. M. Bryant, R. A. Bryce, and B. Hartley.

Theorem 2.2.6 ([BBH70]). *Let U be a subgroup of a group G such that $G = UN$ for some nilpotent normal subgroup N of G. If G belongs to a formation $\mathfrak{F}$, then U is an $\mathfrak{F}$-group.*

The proof of this result also involves an application of Lemma 2.2.4. We need to prove a preliminary lemma.

Assume that G is a group and N a normal subgroup of G. Let N^* be a copy of the subgroup N and consider G acting by conjugation on N^*. Denote $X = [N^*]G$ the semidirect product of N^* with G with respect to this action.

If G is a group and n is a positive integer, denote $\mathrm{K}_1(G) = G$ and $\mathrm{K}_n(G) = [G, \mathrm{K}_{n-1}(G)]$ ([Hup67, III, 1.9]).

Lemma 2.2.7. *With the above notation*

$$\mathrm{K}_n([N, N^*]N) \le \mathrm{K}_{n+1}(N^*)\,\mathrm{K}_n(N) \qquad \textit{for all } n \in \mathbb{N}.$$

Proof. We use induction on n. We write a star $(^*)$ to denote the image by the G-isomorphism between N and N^*. Let x, $y \in N$. Then $[x, y^*] = x^{-1}(y^*)^{-1}xy^* = x^{-1}(y^{-1})^*xy^* = \bigl((y^{-1})^*\bigr)^x y^* = \bigl((y^{-1})^x y\bigr)^* = [x, y]^* = [x^*, y^*]$. This argument shows that if A and B are subgroups of N, then

$[A, B^*] = [A^*, B^*]$. In particular, $[N, N^*] = (N^*)'$ and so $\mathrm{K}_1([N, N^*]N) = [N, N^*]N = (N^*)'N = \mathrm{K}_2(N^*)\,\mathrm{K}_1(N)$. Now assume that the lemma holds for a given value of $n \geq 1$. Then

$$\begin{aligned}
\mathrm{K}_{n+1}([N, N^*]N) &= \bigl[\mathrm{K}_n([N, N^*]N), [N, N^*]N\bigr] && \text{by definition}\\
&\leq \bigl[\mathrm{K}_{n+1}(N^*)\,\mathrm{K}_n(N), [N, N^*]N\bigr] && \text{by inductive hypothesis}\\
&= \bigl[\mathrm{K}_{n+1}(N^*), [N, N^*]N\bigr]\\
&\quad\cdot \bigl[\mathrm{K}_n(N), [N, N^*]N\bigr] && \text{by [DH92, A, 7.4 (f)]}\\
&= \bigl[\mathrm{K}_{n+1}(N^*), [N, N^*]\bigr][\mathrm{K}_{n+1}(N^*), N]\\
&\quad\cdot \bigl[\mathrm{K}_n(N), [N, N^*]\bigr][\mathrm{K}_n(N), N] && \text{by [DH92, A, 7.4 (f)]}\\
&\leq \mathrm{K}_{n+2}(N^*)\,\mathrm{K}_{n+1}(N) && \text{because } \bigl[\mathrm{K}_n(N), [N, N^*]\bigr]\\
& && \quad = [\mathrm{K}_n(N^*), \mathrm{K}_2(N^*)]
\end{aligned}$$

because of the preceeding argument and applying [Hup67, III, 2.11].This completes the induction step and with it the proof of the lemma. ☐

Proof (of Theorem 2.2.6). Assume that the result is not true and let G be a counterexample of minimal order. Then there exists a nilpotent normal subgroup N of G and a proper subgroup U of G such that $G = NU$, $G \in \mathfrak{F}$, and $U \notin \mathfrak{F}$. Among the pairs (N, U) of subgroups of G satisfying the above condition, we choose a pair such that $|G : U| + \mathrm{cl}(N)$ is minimal (here $\mathrm{cl}(N)$ denotes the nilpotency class of N). Let V be a maximal subgroup of G containing U. Then $V = U(V \cap N)$ and $G = VN$. If $U \neq V$, then $|G : V| + \mathrm{cl}(N) < |G : U| + \mathrm{cl}(N)$ and so $V \in \mathfrak{F}$ by the choice of the pair (N, U). Therefore $U \in \mathfrak{F}$ by minimality of G, contrary to the choice of G. Therefore $U = V$ is a maximal subgroup of G. If $Z = \mathrm{Z}(N)$ were not contained in U, then $G = U\,\mathrm{Z}(N)$ and U would be in $\mathfrak{F}$ by the above argument. This would contradict the choice of G. Consequently $\mathrm{Z}(N)$ is contained in U. Denote $X = [N^*]U$ the semidirect product of a copy of N with U as usual. By Lemma 2.2.4, there exists an epimorphism $\alpha \colon X \longrightarrow UN = G$ and $\mathrm{Ker}(\alpha) \cap N^* = \mathrm{Ker}(\alpha) \cap U = 1$. It is clear that Z is a normal subgroup of G and $X/Z \cong [N^*](U/Z)$. Now we consider the group $T = [N^*](G/Z)$. Note that $T \in \mathfrak{F}$ by Corollary 2.2.5 and $[N^*](U/Z)$ is a supplement of $\langle (N/Z)^T \rangle$ in T. Moreover $\langle (N/Z)^T \rangle = [N/Z, T](N/Z) = [N/Z, N^*][N/Z, G/Z](N/Z) = [N/Z, N^*](N/Z)$. If $c = \mathrm{cl}(N)$, we have that $\mathrm{K}_c\bigl(\langle (N/Z)^T \rangle\bigr) = \mathrm{K}_c([N, N^*]N)Z/Z$ is contained in $\mathrm{K}_{c+1}(N^*)\,\mathrm{K}_c(N)Z/Z$ by Lemma 2.2.7. Since $\mathrm{K}_{c+1}(N^*) = 1$ and $\mathrm{K}_c(N) \leq Z$, it follows that $\mathrm{K}_c\bigl(\langle (N/Z)^T \rangle\bigr) = 1$ and $\langle (N/Z)^T \rangle$ is a normal nilpotent subgroup of T whose nilpotency class is less than c. Consequently, since $T \in \mathfrak{F}$, we have that $[N^*](U/Z) \in \mathfrak{F}$ by the minimal choice of G. Hence $X \in \mathrm{R}_0\,\mathfrak{F} = \mathfrak{F}$. This contradicts the choice of G and shows that U is, like G, and $\mathfrak{F}$-group. ☐

Let $\mathfrak{F}$ be a non-empty formation. Each group G has a smallest normal subgroup whose quotient belongs to $\mathfrak{F}$; this is called the $\mathfrak{F}$-*residual* of G and it is denoted by $G^{\mathfrak{F}}$. Clearly $G^{\mathfrak{F}}$ is a characteristic subgroup of G and $G^{\mathfrak{F}} = \bigcap\{N \trianglelefteq G : G/N \in \mathfrak{F}\}$. Consequently $G^{\mathfrak{F}} = 1$ if and only if $G \in \mathfrak{F}$.

The following proposition will be useful for later applications.

Proposition 2.2.8. *Let $\mathfrak{F}$ be a non-empty formation and let G be a group. If N is normal subgroup of G, we have:*

1. $(G/N)^{\mathfrak{F}} = G^{\mathfrak{F}}N/N$.
2. *If U is a subgroup of $G = UN$, then $U^{\mathfrak{F}}N = G^{\mathfrak{F}}N$.*
3. *If N is nilpotent and $G = UN$, then $U^{\mathfrak{F}}$ is contained in $G^{\mathfrak{F}}$.*

Proof. 1. Denote $R/N = (G/N)^{\mathfrak{F}}$. It is clear that $G/R \in \mathfrak{F}$. Hence $G^{\mathfrak{F}}N$ is contained in R. Moreover $G/G^{\mathfrak{F}}N \in \mathfrak{F}$. It implies that $(G/N)/(G^{\mathfrak{F}}N/N) \in \mathfrak{F}$ and so $R/N \leq G^{\mathfrak{F}}N/N$. Therefore $R = G^{\mathfrak{F}}N$.

2. Let θ denote the canonical isomorphism from $G/N = UN/N$ to $U/(U \cap N)$. Then $\big((G/N)^{\mathfrak{F}}\big)^{\theta} = \big(U/(U \cap N)\big)^{\mathfrak{F}}$, which is equal to $U^{\mathfrak{F}}(U \cap N)/(U \cap N)$ by Statement 1. Hence $U^{\mathfrak{F}}N/N = (G/N)^{\mathfrak{F}} = G^{\mathfrak{F}}N/N$ and $U^{\mathfrak{F}}N = G^{\mathfrak{F}}N$.

3. We have $G/G^{\mathfrak{F}} = (UG^{\mathfrak{F}}/G^{\mathfrak{F}})(NG^{\mathfrak{F}}/G^{\mathfrak{F}}) \in \mathfrak{F}$. Applying Theorem 2.2.6, it follows that $UG^{\mathfrak{F}}/G^{\mathfrak{F}} \in \mathfrak{F}$. Therefore $U^{\mathfrak{F}}$ is contained in $U \cap G^{\mathfrak{F}}$. □

Remark 2.2.9. We shall use henceforth the property of the $\mathfrak{F}$-residual stated in Statement 1 without further comment.

In general, the product class of two formations is not a formation in general ([DH92, IV, 1.6]). Fortunately we know a way of modifying the definition of a product to ensure that the corresponding product of two formations is again a formation. It was due to W. Gaschütz ([Gas69]).

Definition 2.2.10. *Let $\mathfrak{F}$ and $\mathfrak{G}$ be formations. We define $\mathfrak{F} \circ \mathfrak{G} := (G : G^{\mathfrak{G}} \in \mathfrak{F})$, and call $\mathfrak{F} \circ \mathfrak{G}$ the* formation product *of $\mathfrak{F}$ with $\mathfrak{G}$.*

This product enjoys the following properties ([DH92, IV, pages 337–338]).

Proposition 2.2.11. *Let $\mathfrak{F}$, $\mathfrak{G}$, and $\mathfrak{H}$ be formations. Then:*

1. *$\mathfrak{F} \circ \mathfrak{G} \subseteq \mathfrak{F}\mathfrak{G}$, and $\mathfrak{G} \subseteq \mathfrak{F} \circ \mathfrak{G}$ if $\mathfrak{F}$ is non-empty,*
2. *if $\mathfrak{F}$ is S_n-closed, then $\mathfrak{F} \circ \mathfrak{G} = \mathfrak{F}\mathfrak{G}$,*
3. *$\mathfrak{F} \circ \mathfrak{G}$ is a formation,*
4. *$G^{\mathfrak{F} \circ \mathfrak{G}} = (G^{\mathfrak{G}})^{\mathfrak{F}}$ for all $G \in \mathfrak{E}$, and*
5. *$(\mathfrak{F} \circ \mathfrak{G}) \circ \mathfrak{H} = \mathfrak{F} \circ (\mathfrak{G} \circ \mathfrak{H})$.*

Example 2.2.12. Let $\mathfrak{F}$ and $\mathfrak{G}$ be formations such that $\pi(\mathfrak{F}) \cap \pi(\mathfrak{G}) = \emptyset$. Denote $\pi_1 = \pi(\mathfrak{F})$ and $\pi_2 = \pi(\mathfrak{G})$. Then $\mathfrak{F} \times \mathfrak{G} = \big(G : G = \mathrm{O}_{\pi_1}(G) \times \mathrm{O}_{\pi_2}(G), \mathrm{O}_{\pi_1}(G) \in \mathfrak{F}, \mathrm{O}_{\pi_2}(G) \in \mathfrak{G}\big)$ is a formation. Moreover, if $\mathfrak{F}$ and $\mathfrak{G}$ are saturated, then $\mathfrak{F} \times \mathfrak{G}$ is saturated and, if $\mathfrak{F}$ and $\mathfrak{G}$ are subgroup-closed, then $\mathfrak{F} \times \mathfrak{G}$ is also subgroup-closed.

Proof. Note that $\mathfrak{F} \times \mathfrak{G} = (\mathfrak{F} \circ \mathfrak{G}) \cap (\mathfrak{G} \circ \mathfrak{F})$. Hence $\mathfrak{F} \times \mathfrak{G}$ is a formation by Proposition 2.2.11 (3).

Assume that $\mathfrak{F}$ and $\mathfrak{G}$ are saturated, then $\mathfrak{F} \circ \mathfrak{G}$ and $\mathfrak{G} \circ \mathfrak{F}$ are saturated by [DH92, IV, 3.13]. Hence $\mathfrak{F} \times \mathfrak{G}$ is saturated. □

Remark 2.2.13. Example 2.2.12 could be generalised along the following lines: Let $\mathcal{I}$ be a non-empty set. For each $i \in \mathcal{I}$, let $\mathfrak{F}_i$ be a subgroup-closed saturated formation. Assume that $\pi(\mathfrak{F}_i) \cap \pi(\mathfrak{F}_j) = \emptyset$ for all $i, j \in \mathcal{I}$, $i \neq j$. Denote $\pi_i = \pi(\mathfrak{F}_i)$, $i \in \mathcal{I}$. Then

$$\mathsf{X}_{i\in\mathcal{I}}\,\mathfrak{F}_i := \bigl(G = \mathrm{O}_{\pi_{i_1}}(G) \times \cdots \times \\ \mathrm{O}_{\pi_{i_n}}(G) : \mathrm{O}_{\pi_{i_j}}(G) \in \mathfrak{F}_{i_j}, 1 \leq j \leq n, \{i_1, \ldots, i_n\} \subseteq \mathcal{I}\bigr)$$

is a subgroup-closed saturated formation.

One of the most important results in the theory of classes of groups is the one stating the equivalence between saturated and local formations. W. Gaschütz introduced the local method to generate saturated formations in the soluble universe. Later, his student U. Lubeseder [Lub63] proved that every saturated formation in the soluble universe can be described in that way. Lubeseder's proof requires elementary ideas from the theory of modular representations, which are dispensed with in the account of the theorem in Huppert's book [Hup67]. In 1978 P. Schmid [Sch78] showed that solubility is not necessary for Lubeseder's result, although his proof reinstates the facts about blocks used by Lubeseder and also makes essential use of a theorem of W. Gaschütz, about the existence of certain non-split extensions. In an unpublished manuscript, R. Baer has investigated a different definition of local formation. It is more flexible than the one studied by P. Schmid in that the simple components, rather than the primes dividing its order, are used to label chief factors and its automorphism group. Hence the actions allowed on the insoluble chief factors can be independent of those on the abelian chief factors. R. Baer's approach leads to a family of formations called Baer-local formations. Local formations are a special case of Baer-local formations. Moreover, in the universe of soluble groups the two definitions coincide. The price to be paid for the greater generality of Baer's approach is that the Baer-local formations are no longer saturated. However, there is a suitable substitute for saturation. We say that a formation is *solubly saturated* if it is closed under taking extensions by the Frattini subgroup of the soluble radical. Of course solubly saturation is weaker than saturation. But it evidently coincides with saturation for classes of finite soluble groups, and it plays a precisely analogous role in Baer's generalisation: the Baer-local formations are precisely the solubly saturated ones.

Another approach to the Gaschütz-Lubeseder theorem in the finite universe is due to L. A. Shemetkov (see [She78, She97, She00]). He uses functions assigning a certain formation to each group (he recently calls them *satellites*) and introduces the notion of composition formation. It turns out that the composition formations are exactly the Baer-local formations ([She97]).

Any function $f\colon \mathbb{P} \longrightarrow \{\text{formations}\}$ is called a *formation function*. Given a formation function f, we define the class $\mathrm{LF}(f)$ as the class of all groups satisfying the following condition:

$G \in \mathrm{LF}(f)$ if, for all chief factors H/K of G and for all primes p dividing $|H/K|$, we have that $\mathrm{Aut}_G(H/K) = G/\mathrm{C}_G(H/K) \in f(p)$. (2.1)

The class $\mathrm{LF}(f)$ is a formation ([DH92, IV, 3.3]).

Definition 2.2.14. *A class of groups $\mathfrak{F}$ is called a* local formation *if there exists a formation function f such that $\mathfrak{F} = \mathrm{LF}(f)$.*

Theorem 2.2.15 (Gaschütz-Lubeseder-Schmid, [DH92, IV, 4.6]). *A formation $\mathfrak{F}$ is saturated if and only if $\mathfrak{F}$ is local.*

A map $f\colon \mathfrak{J} \longrightarrow \{\text{classes of groups}\}$ is called a *Baer function* provided that $f(J)$ is a formation for all simple groups J.

If f is a Baer function, then the class of all groups G satisfying that $\mathrm{Aut}_G(H/K)$ belongs to $f(J)$ if H/K is a chief factor of G whose composition factor is isomorphic to J is a formation. We call this formation the *Baer-local formation defined by* f, and we denote it by $\mathrm{BLF}(f)$. A class $\mathfrak{B}$ is called a *Baer-local formation* if $\mathfrak{B} = \mathrm{BLF}(f)$ for some Baer function f.

Theorem 2.2.16 ([DH92, IV, 4.12]). *The solubly saturated formations are precisely the Baer-local formations.*

Example 2.2.17. Let $\mathfrak{Q}$ be the solubly saturated formation locally defined by the Baer function f given by

$$f(S) = \begin{cases} (1) & \text{when } S \cong C_p, \text{ and} \\ \mathrm{D}_0(1, S) & \text{when } S \in \mathfrak{J} \setminus \mathbb{P}. \end{cases}$$

The formation in Example 2.2.17 is characterised as the class $\mathfrak{Q}$ of all groups G such that $G = \mathrm{C}^*_G(H/K)$ for every chief factor H/K of G, i.e. the class of all groups which only induce inner automorphisms on each chief factor (see [Ben70]). Groups in $\mathfrak{Q}$ are called *quasinilpotent*. It is clear that a nilpotent group is just a soluble quasinilpotent group. $\mathfrak{Q}$ is also S_n-closed and N_0-closed, that is, $\mathfrak{Q}$ is a Fitting class (see Section 2.3). Each group G has a largest normal $\mathfrak{Q}$-subgroup. This subgroup is called the *generalised Fitting subgroup* of G, and it is denoted by $\mathrm{F}^*(G)$. Applying [HB82b, X, 13.9, 13.10], $\mathrm{F}^*(G)$ is the intersection of the innerisers of the chief factors of G.

The main properties of the generalised Fitting subgroup are analysed in many books, for example in Section 13 of Chapter X of the book of B. Huppert and N. Blackburn [HB82b] or, more recently, in Section 6.5 of the book of H. Kurzweil and B. Stellmacher [KS04]. Let us summarise here the most relevant.

Definitions 2.2.18. *1. A group G is said to be* quasisimple *if G is perfect, i.e. $G' = G$, and $G/\mathrm{Z}(G)$ is a non-abelian simple group.*

2. *A subgroup H of a group G is said to be a* component *of G if H is a quasisimple subnormal subgroup of G.*
3. *The* cosocle *of a group G, $\mathrm{Cosoc}(G)$, is the intersection of all maximal normal subgroups of G.*
4. *A group G is said to be* comonolithic *if G has a unique maximal normal subgroup.*
5. *If G is a comonolithic group and $M = \mathrm{Cosoc}(G)$ is the unique maximal normal subgroup of G, then the quotient G/M is said to be the* head *of G.*

It is clear that if G is a quasisimple group, then G is comonolithic and $\mathrm{Cosoc}(G) = \mathrm{Z}(G)$. Also it is easy to see that if K is a normal subgroup of a quasisimple group G, then G/K is also a quasisimple group.

The next result, due to H. Wielandt, will be extremely useful.

Theorem 2.2.19 ([Wie39]). *If H and K are subnormal subgroups of a group G, H is perfect and comonolithic and H is not contained in K, then K normalises H.*

Proposition 2.2.20 (see [KS04, 6.5.3]). *If H and K are components of a group G, then either $H = K$ or $[H,K] = 1$.*

Definition 2.2.21. *The* layer *of a group G is the subgroup $\mathrm{E}(G)$ generated by all components of G, i.e. the product of all components of G.*

Proposition 2.2.22. *Let G be a group.*

1. *We have that $\mathrm{F}^*(G) = \mathrm{F}(G)\,\mathrm{E}(G)$ and $[\mathrm{F}(G), \mathrm{E}(G)] = 1$; in fact*
$$\mathrm{C}_{\mathrm{F}^*(G)}\big(\mathrm{E}(G)\big) = \mathrm{F}(G)$$
(see [HB82b, X, 13.15]).
2. $\mathrm{E}(G)$ *is the central product of all components of G, but not the product of any proper subset of them* (see [HB82b, X, 13.18] or [KS04, 6.5.6]).
3. $\mathrm{F}^*(G)/\mathrm{F}(G) = \mathrm{Soc}\Big(\mathrm{C}_G\big(\mathrm{F}(G)\big)\,\mathrm{F}(G)/\mathrm{F}(G)\Big)$ (see [HB82b, X, 13.13]).
4. $\mathrm{C}_G\big(\mathrm{F}^*(G)\big) \leq \mathrm{F}^*(G)$ (see [HB82b, X, 13.12] or [KS04, 6.5.8]).

2.3 Schunck classes and projectors

The starting point of the theory of classes of groups is the attempt to develop a generalised Sylow theory, which leads to an investigation into the problem of the existence of certain conjugacy classes of subgroups in finite groups.

Perhaps the most well-known existence and conjugacy theorem is Sylow's theorem which says, in its simplest form, that if p is a prime and G is a group, then the maximal p-subgroups of G are conjugate in G.

The beginnings of this particular area of finite group theory came with P. Hall's generalisation of Sylow's theorem for soluble groups.

Theorem 2.3.1 ([Hal28]). *Let G be a soluble group and π any set of primes. Then the maximal π-subgroups of G are conjugate in G.*

In a soluble group G, the π-subgroups of G with π'-index in G are exactly the maximal π-subgroups of G and they are referred as the *Hall π-subgroups of G*. Of course, this is the terminology we shall use here and we also use $\mathrm{Hall}_\pi(G)$ to denote the set of all Hall π-subgroups of G.

By considering the order and index of Hall π-subgroups, it is easy to see that they satisfy the following three conditions.

Let N be a normal subgroup of a soluble group G. Then:

1. $\mathrm{Hall}_\pi(G/N) = \{SN/N : S \in \mathrm{Hall}_\pi(G)\}$.
2. $\mathrm{Hall}_\pi(N) = \{S \cap N : S \in \mathrm{Hall}_\pi(G)\}$.
3. If $T/N \in \mathrm{Hall}_\pi(G/N)$ and $S \in \mathrm{Hall}_\pi(T)$, then $S \in \mathrm{Hall}_\pi(G)$.

In particular, Hall π-subgroups behave well as we pass from G to a factor group G/N or to a normal subgroup N. It is these three properties that have led to wide generalisations, the first and third properties leading to the theory of saturated formations and Schunck classes and the associated projectors and the second property to the theory of Fitting classes and injectors.

Both generalisations lead to conjugacy classes of subgroups in soluble groups which share another important property of Hall subgroups:

If $S \in \mathrm{Hall}_\pi(G)$ and $S \leq H \leq G$, then $S \in \mathrm{Hall}_\pi(H)$.

The results of P. L. M. Sylow and P. Hall seemed to be suggestive of certain arithmetic properties of groups. In 1937, P. Hall [Hal37] discovered the so-called Hall systems of a soluble group G by choosing a set of Hall p'-subgroups of G, one for each prime p, and taking their intersections. He proved that if $\varSigma$ and $\varSigma^*$ are two Hall systems of G, there exists an element $g \in G$ such that $\varSigma^* = \varSigma^g$. That is, G acts transitively by conjugation on the set of its Hall systems. Therefore the number of Hall systems of a soluble group is the index in G of the stabiliser of a Hall system with respect to the action of G. This stabiliser is what P. Hall called the *system normaliser*. P. Hall observed that all system normalisers are nilpotent, they are preserved under epimorphisms, and form a conjugacy class of subgroups. It is important to remark that system normalisers, defined in terms of the genuine Sylow structure of a soluble group, cover the central chief factors of the group and avoid the eccentric ones. Hence they are the natural connection between the two characterisations of soluble groups, the arithmetic and the normal (or commutator) structure, and afford a “measure of the nilpotence” of the group.

Despite of system normalisers, there was a little evidence to suggest the huge proliferation of results in the area. However, in 1961, R. W. Carter [Car61] introduced another conjugacy class of subgroups in each soluble group.

A *Carter subgroup* of a group G is a nilpotent subgroup C of G such that $\mathrm{N}_G(C) = C$. He proves:

Theorem 2.3.2 (R. W. Carter). *A soluble group G has a Carter subgroup and any two Carter subgroups of G are conjugate in G.*

It is clear that a Carter subgroup of a group G is a maximal nilpotent subgroup of G. However, if G is a non-nilpotent soluble group, then G has a maximal nilpotent subgroup which is not a Carter subgroup. Consequently, regarding maximality, the Carter subgroups are not to the class $\mathfrak{N}$ of all nilpotent groups as the Hall subgroups are to the class $\mathfrak{S}_\pi$ of all soluble π-groups.

However, there is a close relation between the abovementioned conjugacy classes: in a group G of nilpotent length 2, the Carter subgroups of G are exactly the system normalisers of G. Carter's theorem would then follow from this observation using induction on the nilpotent length.

W. Gaschütz viewed the Carter subgroups as analogues of the Sylow and Hall subgroups of a soluble groups and in 1963 published a seminal paper [Gas63] where a broad extension of the Hall and Carter subgroups was presented. The theory of formations was born. The new "covering subgroups" had many of the properties of the Sylow and Hall subgroups, but the theory was not arithmetic one, based on the orders of subgroups. Instead, the important idea was concerned with group classes having the same properties. He introduces the concepts of formation and $\mathfrak{F}$-covering subgroup, for a class $\mathfrak{F}$ of groups. He then proved that if $\mathfrak{F}$ is a formation of soluble groups, then every soluble group has an $\mathfrak{F}$-covering subgroup if and only if $\mathfrak{F}$ is saturated and, in this case, the $\mathfrak{F}$-covering subgroups form a unique conjugacy class of subgroups. These $\mathfrak{F}$-covering subgroups coincided with the Sylow p-subgroups, the Hall π-subgroups, and the Carter subgroups in the respective classes$\mathfrak{S}_p$, $\mathfrak{S}_\pi$, and $\mathfrak{N}$. Subsequently, H. Schunck in his Kiel Dissertation [Sch66], written under the direction of W. Gaschütz and H. Schubert, discovered precisely which classes $\mathfrak{Z}$, of soluble groups, always gave rise to $\mathfrak{Z}$-covering subgroups; he showed that these classes can be characterised in terms of their primitive groups and that they form a considerably larger family of classes than the saturated formations [Sch67]. They are known as Schunck classes and are the main concern of this section.

Two years later, W. Gaschütz [Gas69] defined the notion of $\mathfrak{Z}$-projector of some class of soluble groups $\mathfrak{Z}$ and showed that for Schunck classes $\mathfrak{Z}$ the notions of $\mathfrak{Z}$-projector and $\mathfrak{Z}$-covering subgroup coincided. Since then the term "projector" has been widely adopted in this context in preference to "covering subgroup."

The first serious attempt to broaden the study of Schunck classes and their projectors and take it outside the soluble universe was made by P. Förster [För84b], [För85b], and [För85c]. However, it should be remarked that the study of projective classes outside the soluble universe had been observed and treated previously by R. P. Erickson [Eri82] and P. Schmid [Sch74].

In the first part of the section we gather some of the basic facts about Schunck classes and projectors. The book of K. Doerk and T. O. Hawkes

[DH92] presents, in its Chapter III, an excellent treatment of this theme. Hence we refer to it for the proof of some of the results we include here. In the second part, we study the relationship between Schunck classes and formations and some Schunck classes which are close to saturated formations.

Definitions 2.3.3. *Let $\mathfrak{H}$ be a class of groups.*

1. *A subgroup X of a group G is said to be* $\mathfrak{H}$-maximal subgroup *of G if $X \in \mathfrak{H}$ and if $X \leq K \in \mathfrak{H}$, then $X = K$.*
   Denote by $\mathrm{Max}_{\mathfrak{H}}(G)$ *the set of all $\mathfrak{H}$-maximal subgroups of G.*
2. *A subgroup U of a group G is called an* $\mathfrak{H}$-projector *of G if UN/N is $\mathfrak{H}$-maximal in G/N for all $N \trianglelefteq G$.*
   We shall use $\mathrm{Proj}_{\mathfrak{H}}(G)$ *to denote the (possibly empty) set of $\mathfrak{H}$-projectors of a group G.*
3. *An* $\mathfrak{H}$-covering subgroup *of a group G is a subgroup E of G satisfying the following two conditions:*
   a) $E \in \mathrm{Max}_{\mathfrak{H}}(G)$, and
   b) if $T \leq G$, $E \leq T$, $N \trianglelefteq T$, and $T/N \in \mathfrak{H}$, then $T = NE$.
   The set of all $\mathfrak{H}$-covering subgroups of a group G will be denoted by $\mathrm{Cov}_{\mathfrak{H}}(G)$.

Consider the case where $\mathfrak{H} = \mathfrak{E}_\pi$, the class of all π-groups. Then, for each soluble group G,

$$\mathrm{Max}_{\mathfrak{H}}(G) = \mathrm{Proj}_{\mathfrak{H}}(G) = \mathrm{Cov}_{\mathfrak{H}}(G) = \mathrm{Hall}_\pi(G) \neq \emptyset.$$

However, the set $\mathrm{Hall}_\pi(G)$ can be empty for a non-soluble group G. In fact, P. Förster [För85b] showed that if π a non-empty set of primes such that, for each group G, $\mathrm{Hall}_\pi(G) \neq \emptyset$ then, either $\pi = \{p\}$, p a prime, or $\pi = \mathbb{P}$.

Definitions 2.3.4. *1. A class $\mathfrak{H}$ is called* projective *if* $\mathrm{Proj}_{\mathfrak{H}}(G) \neq \emptyset$ *for each group G.*

2. *A class $\mathfrak{H}$ will be called a* Gaschütz class *if* $\mathrm{Cov}_{\mathfrak{H}}(G) \neq \emptyset$ *for each group G.*
3. *A class $\mathfrak{H}$ is said to be a* Schunck class *if $\mathfrak{H}$ is a homomorph that comprises precisely those groups whose primitive epimorphic images are in $\mathfrak{H}$.*

Remark 2.3.5. If $\mathfrak{H}$ is a Schunck class, then $\mathfrak{H}$ is a saturated homomorph, that is, $\mathrm{E}_\Phi\,\mathfrak{H} = \mathfrak{H} = \mathrm{Q}\,\mathfrak{H}$.

It is clear that a saturated formation is a Schunck class. However, the family of all Schunck classes is considerably larger than the one of all saturated formations. Moreover, the fundamental role of the local definition of saturated formations, and therefore the arithmetic properties, are substituted in the case of Schunck classes by the primitive quotients of the group, and therefore by the role of maximal subgroups. In 1974, K. Doerk [Doe71, Doe74] introduced the concept of the *boundary* of a Schunck class, which plays a fundamental role in the study of Schunck classes.

Definitions 2.3.6. *1. For a class $\mathfrak{H}$ of groups, define*

$$\mathrm{b}(\mathfrak{H}) := (G \in \mathfrak{E} \setminus \mathfrak{H} : G/N \in \mathfrak{H} \textit{ for all } 1 \neq N \trianglelefteq G).$$

Obviously, $\mathrm{b}(\emptyset) = \mathrm{b}(\mathfrak{E}) = \emptyset$.
$\mathrm{b}(\mathfrak{H})$ *is said to be the* boundary *of* $\mathfrak{H}$.
We say that a class of groups $\mathfrak{B}$ *is a* boundary *if* $\mathfrak{B} = \mathrm{b}(\mathfrak{H})$ *for some class of groups* $\mathfrak{H}$.

2. If $\mathfrak{Y}$ *is a class of groups, define*

$$\mathrm{h}(\mathfrak{Y}) := \bigl(G \in \mathfrak{E} : \mathrm{Q}(G) \cap \mathfrak{Y} = \emptyset\bigr),$$

that is, the class of $\mathfrak{Y}$*-perfect groups.*
Clearly $\mathrm{h}(\emptyset) = \mathfrak{E}$ *and* $\mathrm{h}(\mathfrak{E}) = \emptyset$ *if* $1 \in \mathfrak{Y}$. *Moreover* $\mathfrak{Y} \cap \mathrm{h}(\mathfrak{Y}) = \emptyset$ *and* $\mathrm{h}(\mathfrak{Y})$ *is a homomorph.*

Theorem 2.3.7. *1. Let* $\mathfrak{H}$ *be a homomorph. Then* $\mathrm{h}\bigl(\mathrm{b}(\mathfrak{H})\bigr) = \mathfrak{H}$.
2. Let $\mathfrak{B}$ *be a boundary. Then* $\mathrm{b}\bigl(\mathrm{h}(\mathfrak{B})\bigr) = \mathfrak{B}$.

Proof. 1. Clearly $\mathfrak{H} \subseteq \mathrm{h}\bigl(\mathrm{b}(\mathfrak{H})\bigr)$. Suppose that $\mathrm{h}\bigl(\mathrm{b}(\mathfrak{H})\bigr)$ is not contained in $\mathfrak{H}$ and let G be a group in $\mathrm{h}\bigl(\mathrm{b}(\mathfrak{H})\bigr) \setminus \mathfrak{H}$ of minimal order. Since $\mathrm{h}\bigl(\mathrm{b}(\mathfrak{H})\bigr)$ is a homomorph, it follows that $G \in \mathrm{b}(\mathfrak{H})$. This is a contradiction. Therefore $\mathfrak{H} = \mathrm{h}\bigl(\mathrm{b}(\mathfrak{H})\bigr)$.

2. If $\mathfrak{B} = \mathrm{b}(\mathfrak{X})$ for some class of groups $\mathfrak{X}$, it follows that every proper epimorphic image of a group in $\mathfrak{B}$ does not belong to $\mathfrak{B}$. Hence $\mathfrak{B} \subseteq \mathrm{b}\bigl(\mathrm{h}(\mathfrak{B})\bigr)$.

Assume that $G \in \mathrm{b}\bigl(\mathrm{h}(\mathfrak{B})\bigr)$. Then $G \notin \mathrm{h}(\mathfrak{B})$ and so there exists a normal subgroup N of G such that $G/N \in \mathfrak{B}$. Suppose that $N \neq 1$. In this case $G/N \in \mathrm{h}(\mathfrak{B})$ by definition of boundary. This contradicts our choice of G. Consequently $N = 1$ and $G \in \mathfrak{B}$. This means that $\mathfrak{B} = \mathrm{b}\bigl(\mathrm{h}(\mathfrak{B})\bigr)$. □

Theorem 2.3.8. *Let* $\emptyset \neq \mathfrak{H}$ *be a class of groups.* $\mathfrak{H}$ *is a Schunck class if and only if* $\mathfrak{H}$ *is a homomorph and* $\mathrm{b}(\mathfrak{H}) \subseteq \mathfrak{P}$.

Proof. If $\mathfrak{H}$ is a Schunck class, then $\mathfrak{H}$ is a homomorph. Suppose that $G \in \mathrm{b}(\mathfrak{H})$ but G is not primitive. Then every epimorphic image of G belongs to $\mathfrak{H}$. Hence $G \in \mathfrak{H}$, contrary to the choice of G. Consequently G is primitive and $\mathrm{b}(\mathfrak{H}) \subseteq \mathfrak{P}$.

Conversely suppose that $\mathfrak{H}$ is a homomorph and $\mathrm{b}(\mathfrak{H}) \subseteq \mathfrak{P}$. Let G be a group whose epimorphic primitive images lie in $\mathfrak{H}$. Suppose that G does not belong to $\mathfrak{H}$. Then $G \in \mathrm{b}(\mathfrak{H})$ by [DH92, III, 2.2 (c)]. In this case G is primitive. This implies $G \in \mathfrak{H}$, which contradicts the fact that $G \in \mathrm{b}(\mathfrak{H})$. Therefore $G \in \mathfrak{H}$. □

Corollary 2.3.9. *For each class* $\mathfrak{X}$*, the class*

$$\mathrm{P}\,\mathrm{Q}\,\mathfrak{X} = \bigl(G : \mathrm{Q}(G) \cap \mathfrak{P} \subseteq \mathrm{Q}\,\mathfrak{X}\bigr)$$

is the smallest Schunck class containing $\mathfrak{X}$. *Therefore* $\mathfrak{X}$ *is a Schunck class if and only if* $\mathfrak{X} = \mathrm{P}\,\mathrm{Q}\,\mathfrak{X}$.

Proof. Clearly $\mathfrak{X} \subseteq \mathrm{P}\,\mathrm{Q}\,\mathfrak{X}$ and $\mathrm{P}\,\mathrm{Q}\,\mathfrak{X}$ is a homomorph. Moreover if $G \in \mathrm{b}(\mathrm{P}\,\mathrm{Q}\,\mathfrak{X})$, then $G \notin \mathrm{P}\,\mathrm{Q}\,\mathfrak{X}$. Hence $\mathrm{Q}(G) \cap \mathfrak{P}$ is not contained in $\mathrm{Q}\,\mathfrak{X}$. Since $G/N \in \mathrm{P}\,\mathrm{Q}\,\mathfrak{X}$ for all $1 \neq N \trianglelefteq G$, it follows that $\mathrm{Q}(G/N) \cap \mathfrak{P} \subseteq \mathrm{Q}\,\mathfrak{X}$. Therefore G should be primitive. Applying Theorem 2.3.8, $\mathrm{P}\,\mathrm{Q}\,\mathfrak{X}$ is a Schunck class. Now if $\mathfrak{H}$ is a Schunck class and $\mathfrak{X} \subseteq \mathfrak{H}$, then $\mathrm{Q}\,\mathfrak{X} \subseteq \mathrm{Q}\,\mathfrak{H} = \mathfrak{H}$. Hence $\mathrm{P}\,\mathrm{Q}\,\mathfrak{X} \subseteq \mathrm{P}\,\mathrm{Q}\,\mathfrak{H} = \mathfrak{H}$. $\square$

Remark 2.3.10. The above corollary shows, in particular, that $\mathrm{P}\,\mathrm{Q}$ is a closure operation.

For another closure operation for Schunck classes related to crowns, the reader is referred to [Haw73] and [Laf84a].

Combining Theorem 2.3.7 and Theorem 2.3.8, we have:

Corollary 2.3.11. *If $\mathfrak{Z}$ is a boundary composed of primitive groups, then* $\mathrm{h}(\mathfrak{Z})$ *is a Schunck class.*

In general, Schunck classes are not R_0-closed, as the following example shows:

Example 2.3.12. Let E be a non-abelian simple group. Then $\mathfrak{Z} = (E \times E)$ is a boundary composed of a primitive group. Hence $\mathrm{h}(\mathfrak{Z})$ is a Schunck class by Corollary 2.3.11. Clearly $E \in \mathrm{h}(\mathfrak{Z})$ and $E \times E \in \mathrm{R}_0\,\mathrm{h}(\mathfrak{Z}) \setminus \mathrm{h}(\mathfrak{Z})$.

This example also shows that $\mathrm{h}(\mathfrak{Z})$ is not D_0-closed.

Suppose that $\mathfrak{H}$ is a projective class. If G is a group in $\mathfrak{H}$, then $\mathrm{Proj}_{\mathfrak{H}}(G) = \{G\}$. Hence, for each normal subgroup N of G, we have that $\mathrm{Proj}_{\mathfrak{H}}(G) = \{G/N\}$ by definition of $\mathfrak{H}$-projector. Therefore $G/N \in \mathfrak{H}$. Moreover, if G is a group such that every primitive epimorphic images of G is in $\mathfrak{H}$, then G must be an $\mathfrak{H}$-group because otherwise an $\mathfrak{H}$-projector E of G would be contained in a maximal subgroup M of G. Since $G/\operatorname{Core}_G(M)$ is primitive, it would follow that $G/\operatorname{Core}_G(M) \in \mathfrak{H}$, and so $G = E\operatorname{Core}_G(M) = M$. This contradiction yields that $G \in \mathfrak{H}$ and $\mathfrak{H}$ is a Schunck class. It is proved in [DH92, III, 3.10] that the converse is also true.

Theorem 2.3.13 ([DH92, III, 3.10]). *A class $\mathfrak{H} \neq \emptyset$ is projective if and only if it is a Schunck class.*

Förster's proof of the above theorem depends on the following property of the projectors and covering subgroups. This property, usually called $\mathfrak{H}$-inductivity, allows him to translate the question of the universal existence of $\mathfrak{H}$-projectors and $\mathfrak{H}$-covering subgroups to the groups in the boundary of $\mathfrak{H}$ (see [DH92, III, 3.8]).

Proposition 2.3.14 ([DH92, III, 3.7]). *Let $\mathfrak{H}$ be a homomorph. Let* f *denote a function which assigns to each group G a possibly empty set* $\mathrm{f}(G)$ *of subgroups of G. If* f *is either of the functions* $\mathrm{Proj}_{\mathfrak{H}}(\cdot)$ *or* $\mathrm{Cov}_{\mathfrak{H}}(\cdot)$*, then it satisfies the following two conditions:*

1. *$G \in \mathrm{f}(G)$ if and only if $G \in \mathfrak{H}$;*
2. *whenever $N \trianglelefteq G$, $N \leq V \leq G$, $U \in \mathrm{f}(V)$, and $V/N \in \mathrm{f}(G/N)$, then $U \in \mathrm{f}(G)$.*

W. Gaschütz [Gas69] actually proved that in the soluble universe the Schunck classes are exactly the Gaschütz classes. However, in the general finite universe, they are no longer coincidental. For instance, the alternating group of degree 5 has no $\mathfrak{N}$-covering subgroups. However, we have:

Theorem 2.3.15 ([DH92, III, 3.11]). *A Schunck class whose boundary contains no primitive groups of type 2 is a Gaschütz class.*

The conjugacy question can be also resolved partially by examining the groups in the boundary. This approach works well for covering subgroups (see [DH92, III, 3.13]), but in the case of projectors, we must work with Schunck classes of monolithic boundary (see [DH92, III, 3.19]). In this context, the following result turns out to be crucial. It will also be used in other chapters.

Proposition 2.3.16 ([För84b]). *Let $\mathfrak{H}$ be a Schunck class. Then $\mathrm{b}(\mathfrak{H}) \cap \mathfrak{P}_3 = \emptyset$ if and only if $\mathfrak{H}$ satisfies the following property:*

Let H be an $\mathfrak{H}$-maximal subgroup of G such that $G = H\,\mathrm{F}^*(G)$. Then H is an $\mathfrak{H}$-projector of G. (2.2)

Proof. Assume that $\mathrm{b}(\mathfrak{H}) \cap \mathfrak{P}_3 = \emptyset$. Let G be a group with an $\mathfrak{H}$-maximal subgroup H such that $G = H\,\mathrm{F}^*(G)$. We prove that H is an $\mathfrak{H}$-projector of G by induction on $|G|$. First we claim:

For all $N \trianglelefteq G$, the hypotheses are inherited from H, G to H, HN. (2.3)

Since $G = H\,\mathrm{F}^*(G)$, we have that $HN = H\big(HN \cap \mathrm{F}^*(G)\big) = H\,\mathrm{F}^*(HN)$ as $HN \cap \mathrm{F}^*(G)$ is a normal quasinilpotent subgroup of HN.

For all minimal normal subgroups M of G such that $G/M \notin \mathfrak{H}$, the hypotheses are inherited from H, G to HM/M, G/M. (2.4)

It follows that $G/M = (HM/M)\big(\mathrm{F}^*(G)M/M\big)$ and $\mathrm{F}^*(G)M/M$ is a normal quasinilpotent subgroup of G/M. Hence $G/M = (HM/M)\,\mathrm{F}^*(G/M)$.

Assume that K/M is an $\mathfrak{H}$-maximal subgroup of G/M such that $HM/M \leq K/M$. Since $G/M \notin \mathfrak{H}$, we have that K is a proper subgroup of G. Moreover, if $K \in \mathfrak{H}$, we have $H = K$ by the $\mathfrak{H}$-maximality of K. Therefore we may assume that $K \notin \mathfrak{H}$. Since $\mathrm{F}^*(G)$ is contained in the inneriser of M, it follows that $G = H\,\mathrm{F}^*(G) = HM\,\mathrm{C}_G(M)$ and so $K = HM\,\mathrm{C}_K(M)$.

Assume that M is abelian. Then $K = H\,\mathrm{C}_K(M)$ and M is a minimal normal subgroup of K. Since $K \notin \mathfrak{H}$, we have that there exists a normal subgroup C of K such that $K/C \in \mathrm{b}(\mathfrak{H}) \subseteq \mathfrak{P}_1 \cup \mathfrak{P}_2$ by [DH92, III, 2.2c]. It is clear that M is not contained in C. Hence $K/C \in \mathfrak{P}_1$ and $\mathrm{Soc}(K/C) = MC/C$. Consequently $MC = \mathrm{C}_K(MC/C) = \mathrm{C}_K(M)$. Moreover $HC \neq K$ because $K/C \notin \mathfrak{H}$. This implies that HC is a maximal subgroup of K, $K = (HC)(MC)$ and $HC \cap MC = C$. On the other hand, $HC \cap M = 1$ and

$HC \cong HCM/M = K/M \in \mathfrak{H}$. The $\mathfrak{H}$-maximality of H in G implies that C is contained in H and so $K = HM$.

Suppose that M is not abelian. Put $C = \mathrm{C}_G(M)$. Then G/C is a primitive group of type 2 and $\mathrm{Soc}(G/C) = MC/C$ by Proposition 1.1.14. Suppose, by way of contradiction, that $G/C \in \mathfrak{H}$. Then $K/\mathrm{C}_K(M) = HM\,\mathrm{C}_K(M)/\mathrm{C}_K(M) \cong G/C \in \mathfrak{H}$. Let N be a normal subgroup of K such that $K/N \in \mathrm{b}(\mathfrak{H})$ ([DH92, III, 2.2 (c)]). Since $N \cap M = 1$, we have that $\mathrm{Soc}(K/N) = NM/N$. Hence $N = \mathrm{C}_G(NM/M) = C$. This contradicts $K/C \in \mathfrak{H}$. Therefore $G/C \notin \mathfrak{H}$ and HC is a proper subgroup of G. By induction, H is an $\mathfrak{H}$-projector of HC. Hence $H(HC \cap M)/(HC \cap M)$ is an $\mathfrak{H}$-projector of $HC/(HC \cap M) \cong G/M$. Therefore HM/M is an $\mathfrak{H}$-projector of G/M. In particular $HM/M = K/M$. This completes the proof of (2.4).

Assume that $G \in \mathfrak{H}$. Then $H = G$ is an $\mathfrak{H}$-projector of G. Hence we may assume that $G \notin \mathfrak{H}$. If $G/M \in \mathfrak{H}$ for all minimal normal subgroups of G, it follows that $G \in \mathrm{b}(\mathfrak{H})$. Hence G is a monolithic primitive group, $\mathrm{Soc}(G)$ is a minimal normal subgroup of G and $\mathrm{C}_G\big(\mathrm{Soc}(G)\big) \leq \mathrm{Soc}(G)$ by Proposition 1.1.12 and Proposition 1.1.14. Then $G = H\,\mathrm{Soc}(G)$, and from $H \in \mathrm{Max}_{\mathfrak{H}}(G)$ and the fact that $\mathrm{Soc}(G)$ is the unique minimal normal subgroup of G, we derive the claim of the proposition: $H \in \mathrm{Proj}_{\mathfrak{H}}(G)$.

Therefore we may suppose that $G/M \notin \mathfrak{H}$ for some minimal normal subgroup M of G. Then, in view of (2.3) and (2.4), the inductive hypothesis can be applied to yield that $H \in \mathrm{Proj}_{\mathfrak{H}}(HM)$ and $HM/M \in \mathrm{Proj}_{\mathfrak{H}}(G/M)$. By $\mathfrak{H}$-inductivity, $H \in \mathrm{Proj}_{\mathfrak{H}}(G)$.

Conversely assume that $\mathfrak{H}$ satisfies Property 2.2. Suppose that $\mathrm{b}(\mathfrak{H}) \cap \mathfrak{P}_3 \neq \emptyset$ and derive a contradiction. Consider $G \in \mathrm{b}(\mathfrak{H}) \cap \mathfrak{P}_3$. Then, by Theorem 1, $S = \mathrm{Soc}(G) = A \times B$, where A and B are the two unique minimal normal subgroups of G and both are complemented by a core-free maximal subgroup U of G. Consider the subgroup $T = U \cap S$. Then T is isomorphic to A and B. Since U is primitive by Corollary 1.1.13, it follows that T is not contained in $\Phi(U)$. Let Y be a proper subgroup of U such that $U = TY$. Write $X = YB$. Then $XA = YS = YTS = US = G$. Hence $X/(X \cap A) \cong G/A \in \mathfrak{H}$. Let L be a minimal supplement of $X \cap A$ in X. Clearly $X \cap A \cap L$ is contained in $\Phi(L)$ and so $L \in \mathrm{E}_{\Phi}\,\mathfrak{H} = \mathfrak{H}$. Let H be an $\mathfrak{H}$-maximal subgroup of G containing L. Since $G = XA = LA$, it follows that $G = HA$. Applying (2.2), H is an $\mathfrak{H}$-projector of G. Since $G/B \in \mathfrak{H}$, we have that $G = HB$. Therefore H is a core-free maximal subgroup of G such that $H \cap B = H \cap A = 1$. In particular, $L = H$. This implies that $X = G$ and so $Y = U$, contrary to the choice of Y. Consequently $\mathrm{b}(\mathfrak{H}) \cap \mathfrak{P}_3 = \emptyset$. □

The same arguments to those used in the proof of Proposition 2.3.16 lead to the following result.

Proposition 2.3.17. *Let $\mathfrak{H}$ be a Schunck class. If H is an $\mathfrak{H}$-maximal subgroup of a group G such that $G = H\,\mathrm{F}(G)$, then H is an $\mathfrak{H}$-projector of G.*

We now direct our attention towards certain formations that may be naturally associated with a Schunck class. In fact, our next objective is to prove that a Schunck class $\mathfrak{H}$ contains a unique largest formation. This result was proved independently by U. Kattwinkel [Kat77] and K.-U. Schaller [Sch77] in the soluble universe and by J. Lafuente [Laf84a] in the general case. We begin with a definition.

Definition 2.3.18. *Let $\mathfrak{H}$ be a class of groups.*

A chief factor H/K of a group G is said to be $\mathfrak{H}$-central *in G if $[H/K]*G$ is in $\mathfrak{H}$. Otherwise, the chief factor H/K is said to be* $\mathfrak{H}$-eccentric *in G.*

Note that if $\mathfrak{H}$ is a saturated formation and H is the canonical local definition of $\mathfrak{H}$ (see [DH92, IV, 3.9]), then a chief factor H/K of a group G is $\mathfrak{H}$-central in G if and only if H/K is H-central in G in the sense of [DH92, IV, 3.1].

Let $\mathfrak{H}$ be a class of groups. Denote by $f(\mathfrak{H})$ the class of all groups G in which every chief factor is $\mathfrak{H}$-central. The class $f_1(\mathfrak{H})$ is defined to be the class of all groups such that $[H/K]\bigl(G/\operatorname{C}_G(H/K)\bigr) \in \mathfrak{H}$ for every chief factor H/K of G.

It follows that $f_1(\mathfrak{H})$ is contained in $f(\mathfrak{H})$ but the equality does not hold in general.

Example 2.3.19. Let S be a non-abelian simple group. Consider the class $\mathfrak{H}$ of all groups with no quotient isomorphic to the direct product $S \times S$ of two copies of S, i.e. the Schunck class of all $(S \times S)$-perfect groups, is a Schunck class whose boundary is $\mathrm{b}(\mathfrak{H}) = (S \times S)$. The group $S \times S \in f(\mathfrak{H}) \setminus f_1(\mathfrak{H})$. Note that $f_1(\mathfrak{H})$ is contained in $\mathfrak{H}$.

Theorem 2.3.20. *Let $\mathfrak{H}$ be a class of groups. Then:*

1. *$f(\mathfrak{H})$ and $f_1(\mathfrak{H})$ are formations.*
2. *If $\mathfrak{F}$ is a formation contained in $\mathfrak{H}$, then $\mathfrak{F}$ is contained in $f(\mathfrak{H})$.*
3. *Let $\mathfrak{H}$ be a Schunck class. Then $\mathrm{b}(\mathfrak{H}) \cap \mathfrak{P}_3 = \emptyset$ if and only if $f(\mathfrak{H})$ is the largest formation contained in $\mathfrak{H}$.*
4. *Let $\mathfrak{H}$ be a Schunck class. Then $f_1(\mathfrak{H})$ is the largest formation contained in $\mathfrak{H}$.*

Proof. 1. Clearly $f(\mathfrak{H})$ and $f_1(\mathfrak{H})$ are formations.

2. Suppose, arguing for contradiction, that $\mathfrak{F}$ is a formation contained in $\mathfrak{H}$ such that $\mathfrak{F}$ is not contained in $f(\mathfrak{H})$. Let G be a group in $\mathfrak{F} \setminus f(\mathfrak{H})$ of minimal order. Then G has a unique minimal normal subgroup N and $G/N \in f(\mathfrak{H})$ by [DH92, II, 2.5]. Assume that N is non-abelian. Then $\operatorname{C}_G(N) = 1$ and G is isomorphic to $[N]*G$. Hence $[N]*G \in \mathfrak{H}$. Now if N is an abelian, we have that $[N]\bigl(G/\operatorname{C}_G(N)\bigr) \in \mathfrak{F} \subseteq \mathfrak{H}$ by Corollary 2.2.5. Therefore $G \in f(\mathfrak{H})$, contrary to our supposition. Hence $\mathfrak{F}$ is contained in $f(\mathfrak{H})$ and Statement 2 holds.

3. Let $\mathfrak{H}$ be a Schunck class such that $\mathrm{b}(\mathfrak{H}) \cap \mathfrak{P}_3 = \emptyset$. Assume that $f(\mathfrak{H})$ is not contained in $\mathfrak{H}$. Then a group of minimal order in the non-empty class $f(\mathfrak{H}) \setminus \mathfrak{H}$ is in the boundary of $\mathfrak{H}$. Hence G is a monolithic primitive group. If G is a primitive group of type 1, then G is isomorphic to

$[\mathrm{Soc}(G)]\Big(G/\,\mathrm{C}_G\big(\mathrm{Soc}(G)\big)\Big) \in \mathfrak{H}$ by Proposition 1.1.12 and if G is a primitive group of type 2, then $G = [\mathrm{Soc}(G)] * G \in \mathfrak{H}$. In both cases, we have that $G \in \mathfrak{H}$. This contradiction yields $f(\mathfrak{H}) \subseteq \mathfrak{H}$.

Conversely assume that $f(\mathfrak{H})$ is the largest formation contained in $\mathfrak{H}$. If G is in the boundary of $\mathfrak{H}$ and G is a primitive group of type 3, then G/A and G/B are $\mathfrak{H}$-groups, where A and B are the minimal normal subgroups of G. This implies that G/A and G/B belong to $f(\mathfrak{H})$ because all their factors are $\mathfrak{H}$-central. Hence $G \in \mathfrak{H}$, contrary to assumption. Consequently, $\mathrm{b}(\mathfrak{H}) \cap \mathfrak{P}_3 = \emptyset$.

4. Consider a group $G \in f_1(\mathfrak{H})$ and let N be a normal subgroup of G such that G/N is primitive. If G/N is a primitive group of type 1 or 3, then G/N belongs to $\mathfrak{H}$ by Proposition 1.1.12 (3). If $X = G/N$ is a primitive group of type 2 and $Z = \mathrm{Soc}(G/N)$, then $[Z]X$ is a primitive group of type 3 by Proposition 1.1.12 (3). Hence $[Z]X \in \mathfrak{H}$ and so $X \in \mathfrak{H}$ since $\mathfrak{H}$ is Q-closed. Consequently every primitive epimorphic image of G belongs to $\mathfrak{H}$ and then G is an $\mathfrak{H}$-group.

Let $\mathfrak{F}$ be a formation contained in $\mathfrak{H}$. Then $\mathfrak{F}$ is contained in $f(\mathfrak{H})$ by Statement 2. Let G be an $\mathfrak{F}$-group. Then every abelian chief factor of G is $\mathfrak{H}$-central in G. Suppose that H/K is a non-abelian chief factor of G. Denote $X = [H/K]\big(G/\,\mathrm{C}_G(H/K)\big)$. Then X is a primitive group of type 3 by Proposition 1.1.12 (3) with two minimal normal subgroups X_1 and X_2 such that $X_1 \cap X_2 = 1$ and $X/X_i \in \mathfrak{F}$, $1 \leq i \leq 2$. Hence $X \in \mathrm{R}_0\mathfrak{F} = \mathfrak{F}$. Therefore $G \in f_1(\mathfrak{H})$ and $\mathfrak{F}$ is contained in $f_1(\mathfrak{H})$. □

Example 3.1.37 shows that a class of groups $\mathfrak{H}$ does not contain a unique largest formation in general.

Example 2.3.21. Every Schunck class whose boundary consists of primitive groups of type 2 is a saturated formation.

Proof. By Theorem 2.3.20 (3), $f(\mathfrak{H})$ is contained in $\mathfrak{H}$. Now if G is a group in $\mathfrak{H}$ and H/K is an abelian chief factor of G, then $[H/K] * G$ is an $\mathfrak{H}$-group because it is not in the boundary of $\mathfrak{H}$. Since every non-abelian chief factor of G is $\mathfrak{H}$-central in G, it follows that $G \in f(\mathfrak{H})$ and $\mathfrak{H}$ is a saturated formation.

We bring the section to a close by studying a concrete family of Schunck classes with an eye to a subsequent application in Chapter 4.

Consider a formation $\mathfrak{F}$. Then, by Lemma 2.1.5 (1), $\mathfrak{H} = \mathrm{E}_\Phi\mathfrak{F}$ is a Schunck class and it is the smallest Schunck class containing $\mathfrak{F}$. Note that a primitive group is in $\mathfrak{H}$ if and only if it is in $\mathfrak{F}$. Hence $\mathfrak{H}$ has monolithic boundary and so $f(\mathfrak{H})$ is the largest formation contained in $\mathfrak{H}$ by Theorem 2.3.20 (3). It follows that $\mathfrak{H} = \mathrm{E}_\Phi f(\mathfrak{H})$, but $\mathfrak{F}$ is not equal to $f(\mathfrak{H})$ in general: if $\mathfrak{F} = \mathfrak{A}$ is the formation of all abelian groups, then $f(\mathrm{E}_\Phi\,\mathfrak{F})$ is the class of all nilpotent groups.

Schunck classes $\mathfrak{H}$ of the form $\mathrm{E}_\Phi\mathfrak{F}$ for some formation $\mathfrak{F}$ can be characterised by the property that each group not in $\mathfrak{H}$ always has a special critical subgroup.

Definition 2.3.22. *Let $\mathfrak{H}$ be a class of groups.*

1. *A maximal subgroup U of a group G is said to be* $\mathfrak{H}$-normal *in G if the primitive group $G/\operatorname{Core}_G(U)$ is in $\mathfrak{H}$. Otherwise, U is said to be* $\mathfrak{H}$-abnormal *in G.*
2. *A maximal subgroup U of a group G is said to be* $\mathfrak{H}$-critical *in G if U is an $\mathfrak{H}$-abnormal critical subgroup of G.*

Note that an $\mathfrak{H}$-critical subgroup is a monolithic maximal subgroup supplementing an $\mathfrak{H}$-eccentric chief factor.

Lemma 2.3.23. *Let $\mathfrak{H}$ be a Schunck class and let G be a group, N a normal subgroup of G, and $M \leq G$. If U is $\mathfrak{H}$-critical in M and $M \cap N \leq U$, then UN/N is $\mathfrak{H}$-critical in MN/N.*

Proof. By Proposition 1.4.10, UN/N is critical in MN/N. Since $N \cap M \leq U$, we have that $\operatorname{Core}_M(U)N/N = \operatorname{Core}_{MN/N}(UN/N)$ and so $M/\operatorname{Core}_M(U) \cong (MN/N)\big/\operatorname{Core}_{MN/N}(UN/N)$ is not in $\mathfrak{H}$. In other words, UN/N is an $\mathfrak{H}$-critical subgroup of MN/N. □

Let S be a non-abelian simple group and the Schunck class $\mathfrak{H}$ of all groups with no quotient isomorphic to the direct product $S \times S$ of two copies of S. All monolithic maximal subgroups of the group $G = S \times S$ are $\mathfrak{H}$-normal in G. Hence $G \notin \mathfrak{H}$ and G has no $\mathfrak{H}$-critical subgroups.

The following theorem characterises the Schunck classes of the form $\mathfrak{H} = \mathrm{E}_\Phi\mathfrak{F}$, for some formation $\mathfrak{F}$, among the Schunck classes for which every group which is not in $\mathfrak{H}$ possesses $\mathfrak{H}$-critical subgroups.

Theorem 2.3.24. *For a Schunck class $\mathfrak{H}$, the following statements are pairwise equivalent:*

1. *every group which is not in $\mathfrak{H}$ possesses $\mathfrak{H}$-critical subgroups;*
2. *$\mathfrak{H} = \mathrm{E}_\Phi \mathrm{Q}\, \mathrm{R}_0 \mathcal{P}(\mathfrak{H})$, where $\mathcal{P}(\mathfrak{H})$ is the class of all primitive groups in $\mathfrak{H}$;*
3. *$\mathfrak{H} = \mathrm{E}_\Phi\mathfrak{F}$, for some formation $\mathfrak{F}$;*
4. *a group G belongs to $\mathfrak{H}$ if and only if every minimal normal subgroup of $G/\Phi(G)$ is $\mathfrak{H}$-central in G.*

Proof. 1 implies 2. Since, for every group G, $\Phi(G)$ is the intersection of all normal subgroups N of G such that G/N is primitive and $\mathfrak{H}$ is a homomorph, it follows that $\mathfrak{H} \subseteq \mathrm{E}_\Phi \mathrm{Q}\, \mathrm{R}_0 \mathcal{P}(\mathfrak{H})$.

Let $G \in \mathrm{R}_0\mathcal{P}(\mathfrak{H})$. Then there exist normal subgroups $N_1, \dots, N_t$ of G such that $\bigcap_{i=1}^{t} N_i = 1$ and G/N_i is a primitive group in $\mathfrak{H}$, $1 \leq i \leq n$. Consequently, $\Phi(G) = 1$. Suppose that $G \notin \mathfrak{H}$ and let U be an $\mathfrak{H}$-critical subgroup of G. Then U is monolithic and $G = UN$ for some minimal normal subgroup of G. Moreover $N \cap N_i = 1$ for some i. Therefore NN_i/N_i is a chief factor of G which is G-isomorphic to N. If G/N_i is a primitive group of type 1, then $G/N_i \cong [N]\big(G/\operatorname{C}_G(N)\big) \cong G/\operatorname{Core}_G(U) \in \mathfrak{H}$ by Proposition 1.1.12. If G/N_i

is a primitive group of type 2, then $N_i = \mathrm{C}_G(N) = \mathrm{Core}_G(U)$ by Proposition 1.1.14 and $G/N_i = G/\mathrm{Core}_G(U) \in \mathfrak{H}$. Suppose that G/N_i is a primitive group of type 3. Then, by Proposition 1.1.13, NN_i/N_i and $C = \mathrm{C}_G(N)/N_i$ are the minimal normal subgroups of G/N_i and G/C and G/NN_i are primitive groups of type 2. Moreover $N_i = C \cap NN_i$. Assume that $G = UN_i$. Then $N \mathrm{Core}_G(U)$ is contained in $N_i \mathrm{Core}_G(U)$ because $N \mathrm{Core}_G(U)/\mathrm{Core}_G(U) = \mathrm{Soc}\big(N \mathrm{Core}_G(U)/\mathrm{Core}_G(U)\big)$. Hence N is abelian. This contradiction shows that N_i is contained in U. Hence $G/\mathrm{Core}_G(U) \in \mathrm{Q}(G/N_i) \subseteq \mathfrak{H}$. In any case, we have that $G/\mathrm{Core}_G(U)$ is an $\mathfrak{H}$-group, contrary to the choice of U. Therefore $G \in \mathfrak{H}$ and the equality $\mathfrak{H} = \mathrm{E}_{\Phi}\mathrm{Q}\,\mathrm{R}_0\mathcal{P}(\mathfrak{H})$ holds.

Since for every class $\mathfrak{X}$ of groups, the class $\mathrm{Q}\,\mathrm{R}_0\mathfrak{X}$ is a formation, it is clear that 2 implies 3.

3 implies 4. Let G be a group in $\mathfrak{H}$. If $N/\Phi(G)$ is a minimal normal subgroup of $G/\Phi(G)$, then $N/\Phi(G)$ is a supplemented chief factor of G and the primitive group associated with $N/\Phi(G)$ is isomorphic to a quotient group of G. Hence $[N/\Phi(G)] * G \in \mathfrak{H}$ and the chief factor $N/\Phi(G)$ is $\mathfrak{H}$-central in G.

Conversely, assume that every minimal normal subgroup of $G/\Phi(G)$ is $\mathfrak{H}$-central in G. Without loss of generality we may suppose that $\Phi(G) = 1$. Let N be a normal subgroup of G such that G/N is a monolithic primitive group. Then $\mathrm{Soc}(G/N) = AN/N$ for some minimal normal subgroup A of G. Since A is $\mathfrak{H}$-central in G, it follows that $G/N \in \mathfrak{H}$ and so $G/N \in \mathfrak{F}$. Therefore $G/\Phi(G) = G \in \mathrm{Q}\,\mathrm{R}_0\mathfrak{F} = \mathfrak{F}$.

4 implies 1. Let G be a group which is not in $\mathfrak{H}$. Assume first that $\Phi(G) = 1$. Suppose that all critical subgroups of G are $\mathfrak{H}$-normal in G. This means that each minimal normal subgroup is $\mathfrak{H}$-central in G. By hypothesis, G is in $\mathfrak{H}$. This is a contradiction. Therefore G has an $\mathfrak{H}$-critical subgroup.

For the case $\Phi(G) \neq 1$, consider the group $G^* = G/\Phi(G)$ which is not in $\mathfrak{H}$ either. Since $\Phi(G^*) = 1$, the G^* possesses an $\mathfrak{H}$-critical subgroup $U^* = U/\Phi(G)$. Clearly U is $\mathfrak{H}$-critical in G. □

The "soluble" version of Theorem 2.3.24 was proved by P. Förster in [För78].

2.4 Fitting classes, Fitting sets, and injectors

The theory of Fitting classes began when B. Fischer in his Habilitationschrift [Fis66] wanted to see how far it is possible to dualise the theory of saturated formations and projectors by interchanging the roles of normal subgroups and quotients groups. From this point of view the closure operations S_n and N_0 are the natural duals of Q and R_0, and so a Fitting class, i.e. a $\langle \mathrm{S}_n, \mathrm{N}_0\rangle$-closed class, should be regarded as the dual of a formation. However, in the soluble universe, it turns out that Fitting classes parallel Schunck classes more closely in the dual theory because they are precisely the classes for which a theory of injectors, dual of projectors, is possible. At the time of Fischer's initial investigation the projectors were still known by covering subgroups and by close analogy the dual concept chosen by Fischer was the so-called Fischer subgroup: if $\mathfrak{F}$ is a class of groups, a Fischer $\mathfrak{F}$-subgroup belongs to $\mathfrak{F}$ and contains each $\mathfrak{F}$-subgroup that it normalises. For an arbitrary Fitting class $\mathfrak{F}$, Fischer was able to prove that the existence of Fischer $\mathfrak{F}$-subgroups in every soluble group. However, he was not able to prove that the Fischer subgroups of a soluble group are all conjugate. Some years later, R. S. Dark [Dar72] gave an example of a Fitting class $\mathfrak{F}$ and a soluble group which has two conjugacy classes of Fischer $\mathfrak{F}$-subgroups.

As it turned out, the definition of projector, rather than covering subgroup, is the right thing to dualise in order to guarantee conjugacy. In 1967 the concept of injector appears in the celebrated paper "Injektoren endlicher auflösbarer Gruppen" by B. Fischer, W. Gaschütz, and B. Hartley [FGH67]. They prove that a class of soluble groups $\mathfrak{F}$ is a Fitting class if and only if every soluble group has an $\mathfrak{F}$-injector. Moreover, the $\mathfrak{F}$-injectors then form a single conjugacy class.

When $\mathfrak{F}$ is the Fitting class of all soluble π-groups, π a set of primes, the $\mathfrak{F}$-injectors of a soluble group, like its $\mathfrak{F}$-projectors, turn out to be the Hall π-subgroups. This is the only situation in which the injectors and projectors coincide, and so the two theories are quite independent generalisations of the classical Sylow and Hall subgroups.

In fact, as we see in Section 2.2, in the general, non-necessarily soluble, universe, projective classes and Schunck classes remain equivalent concepts. However, in Chapter 7, we shall show that there exist non-injective Fitting classes.

Definition 2.4.1. *A* Fitting class *is a class of groups which is both* S_n*-closed and* N_0*-closed, that is, a class of groups* $\mathfrak{F}$ *is a Fitting class if* $\mathfrak{F}$ *has the following two properties:*

1. *if* $G \in \mathfrak{F}$ *and* N *is a subnormal subgroup of* G*, then* $N \in \mathfrak{F}$*, and*
2. *if* N_1 *and* N_2 *are subnormal subgroups of a group* G *and* $G = \langle N_1, N_2 \rangle$*, then* $G \in \mathfrak{F}$*.*

Hence a class $\mathfrak{F}$ is a Fitting class if and only if $\mathfrak{F} = \langle \mathrm{S}_n, \mathrm{N}_0 \rangle \mathfrak{F}$.

As usual for classes defined by closure operations, the intersection of a family of Fitting classes is again a Fitting class, and the union of a family of Fitting classes which is a directed set with respect to the partial order of inclusion is also a Fitting class.

In particular, if $\mathfrak{Z}$ is a class of groups, the intersection $\operatorname{Fit} \mathfrak{Z}$ of all Fitting classes containing $\mathfrak{Z}$ is the smallest Fitting class containing $\mathfrak{Z}$; $\operatorname{Fit} \mathfrak{Z} = \langle \mathrm{S}_n, \mathrm{N}_0 \rangle \mathfrak{Z}$ is the *Fitting class generated by* $\mathfrak{Z}$. Note that if S is a non-abelian simple group, then $\operatorname{Fit}(S) = \operatorname{form}(S) = \mathrm{D}_0(1, S)$ by Example 2.2.3 (1).

Historically, the first example of a Fitting class is the class $\mathfrak{N}$ of all nilpotent groups. This was proved by H. Fitting in 1938. The formations $\mathfrak{N}_c$ and $\mathfrak{S}^{(d)}$ are also Fitting classes and, in general, since $\mathrm{D}_0 \leq \mathrm{N}_0$ and $\mathrm{R}_0 \leq \mathrm{SD}_0$, a subgroup closed Fitting class is R_0-closed. However a formation does not need to be a Fitting class. The formations $\mathfrak{A}$ of all abelian groups and $\mathfrak{U}$ of all supersoluble groups are not N_0-closed. Nevertheless, the following result can be used in some contexts as a substitute of the R_0-closure. It is known as the "quasi-R_0-lemma."

Lemma 2.4.2 ([DH92, IX, 1.13]). *Let N_1 and N_2 be normal subgroups of a group G such that $N_1 \cap N_2 = 1$ and G/N_1N_2 is nilpotent. Suppose that $\mathfrak{F}$ is a Fitting class such that $G/N_1 \in \mathfrak{F}$. Then $G \in \mathfrak{F}$ if and only if $G/N_2 \in \mathfrak{F}$.*

Definition 2.4.3. *If $\mathfrak{F}$ is a Fitting class and G is a group, then the subgroup*

$$G_{\mathfrak{F}} = \langle S : S \text{ is a subnormal } \mathfrak{F}\text{-subgroup of } G\rangle$$

is a normal $\mathfrak{F}$-subgroup of G, and it is called the $\mathfrak{F}$-radical *of G.*

Remark 2.4.4. If N is a normal subgroup of G and $\mathfrak{F}$ is a Fitting class, then $N_{\mathfrak{F}} = N \cap G_{\mathfrak{F}}$.

As might be expected, the class product of Fitting classes need not be a Fitting class in general (see Step 7 in [DH92, IX, 2.14 (b)]). A special product can be defined, which is dual to the formation product of Definition 2.2.10, which preserves the Fitting class property.

Definitions and notation 2.4.5. *Let $\mathfrak{X}$ and $\mathfrak{Y}$ be Fitting classes.*

1. *$\mathfrak{X} \diamond \mathfrak{Y}$ is the class of all groups G such that $G/G_{\mathfrak{X}} \in \mathfrak{Y}$. (This product, called* Fitting product*, was introduced by Gaschütz, see [DH92, IX, 1.10])*
2. *$\mathfrak{X} \cdot \mathfrak{Y}$ is the class of all groups G such that $G = G_{\mathfrak{X}}G_{\mathfrak{Y}}$.*

Proposition 2.4.6 (see [DH92, IX, 1.12]). *Let $\mathfrak{F}$, $\mathfrak{G}$, and $\mathfrak{H}$ be Fitting classes. Then:*

1. *$\mathfrak{F} \diamond \mathfrak{G} \subseteq \mathfrak{F}\mathfrak{G}$, and $\mathfrak{F} \subseteq \mathfrak{F} \diamond \mathfrak{G}$ if $\mathfrak{G}$ is non-empty,*
2. *if the class $\mathfrak{G}$ is a homomorph, then $\mathfrak{F} \diamond \mathfrak{G} = \mathfrak{F}\mathfrak{G}$,*
3. *$\mathfrak{F} \diamond \mathfrak{G}$ is a Fitting class,*
4. *for all $G \in \mathfrak{E}$, the $\mathfrak{G}$-radical of $G/G_{\mathfrak{F}}$ is $G_{\mathfrak{F}\diamond\mathfrak{G}}/G_{\mathfrak{F}}$, and*
5. *$(\mathfrak{F} \diamond \mathfrak{G}) \diamond \mathfrak{H} = \mathfrak{F} \diamond (\mathfrak{G} \diamond \mathfrak{H})$.*

On the other hand, if $\mathfrak{X}$ and $\mathfrak{Y}$ are Fitting classes, then the class $\mathfrak{X} \cdot \mathfrak{Y}$ is not necessarily a Fitting class (see [DH92, page 575]).

If $\mathfrak{X}$ and $\mathfrak{Y}$ are Fitting classes such that $\mathfrak{X} \subseteq \mathfrak{Y}$ and $\mathfrak{F}$ is a Fitting class, we write that $\mathfrak{F} \in \mathrm{Sec}(\mathfrak{X}, \mathfrak{Y})$ if $\mathfrak{X} \subseteq \mathfrak{F} \subseteq \mathfrak{Y}$; in this case we say that $\mathfrak{F}$ is in the *section* of $\mathfrak{X}$ and $\mathfrak{Y}$. The most known section of Fitting classes is the Lockett section.

In [Loc71], Lockett exploited the aberrant behaviour of radicals in direct products and show how to associate with each Fitting class $\mathfrak{X}$ another containing it, called $\mathfrak{X}^*$, such that $(G \times H)_{\mathfrak{X}^*} = G_{\mathfrak{X}^*} \times H_{\mathfrak{X}^*}$. Lockett's universe was the soluble one, but the definition of $\mathfrak{X}^*$, its Fitting character and its behaviour with respect to direct products still hold in the general finite universe (see [DH92, X, Section 1]). Thus $\mathfrak{X}^*$ is the class of all groups G such that $(G \times G)_{\mathfrak{X}}$ is subdirect in $G \times G$. We now define $\mathfrak{X}_*$ as the intersection of all Fitting classes $\mathfrak{F}$ such that $\mathfrak{F}^* = \mathfrak{X}^*$. Obviously $\mathfrak{X}_*$ is a Fitting class and it has the remarkable property that $(\mathfrak{X}_*)^* = \mathfrak{X}^*$ by [DH92, X, 1.13].

Definition 2.4.7. *Let $\mathfrak{X}$ be a Fitting class.*

1. *$\mathfrak{X}$ is a* Lockett class *if $\mathfrak{X} = \mathfrak{X}^*$.*
2. *The* Lockett section of $\mathfrak{X}$ *is* $\operatorname{Locksec}(\mathfrak{X}) = \operatorname{Sec}(\mathfrak{X}_*, \mathfrak{X}^*)$*.*

Observe that if $\mathfrak{X}$ is a Fitting class, each group $G \notin \mathfrak{X}$ such that every proper subnormal subgroup of G is in $\mathfrak{X}$ has to be comonolithic, by the N_0-closure of G, and $G_{\mathfrak{X}} = \operatorname{Cosoc}(G)$. Hence the following definition makes sense.

As we have seen in Section 2.3 boundaries play an important role in the study of Schunck classes. In fact, they provide a method to construct Schunck classes by exploiting the one-to-one correspondence between homomorphs and boundaries given by the maps b and h (Theorem 2.3.7). It is clear how the analogous maps b and h for Fitting classes must be defined.

Definitions and notation 2.4.8. *Let $\mathfrak{X}$ be a Fitting class.*

1. *The* boundary *of $\mathfrak{X}$, $\mathrm{b}(\mathfrak{X})$, is the class of all groups $X \notin \mathfrak{X}$ such that every proper subnormal subgroup of X is an $\mathfrak{X}$-group.*
2. *$\bar{\mathrm{b}}(\mathfrak{X}) = \big(G \in \mathrm{b}(\mathfrak{X}) : G = G'\big)$.*
3. *For a prime p, we denote $\mathrm{b}_p\,\mathfrak{X} = (G \in \mathrm{b}(\mathfrak{X}) : G/\operatorname{Cosoc}(G) \in \mathfrak{S}_p)$.*
4. *$\mathfrak{X}^{\mathrm{b}}$ denotes the Fitting class generated by the cosocles of all groups $G \in \mathrm{b}(\mathfrak{X})$:*
$$\mathfrak{X}^{\mathrm{b}} = \operatorname{Fit}\big(\operatorname{Cosoc}(G) : G \in \mathrm{b}(\mathfrak{X})\big).$$

Definition 2.4.9. *If $\mathfrak{Y}$ is a class of groups, denote*
$$\mathrm{h}(\mathfrak{Y}) = \big(H : \mathrm{s}_n(H) \cap \mathfrak{F} = \emptyset\big).$$

Remark 2.4.10. It reasonable to think that to use the same notation for distinct concepts of boundary introduces considerable ambiguity. However, we shall rely on the context to make the meaning clear. The same applies to the map h.

Definition 2.4.11. *A* preboundary *is a class $\mathfrak{m}$ of groups satisfying the following properties:*

1. *$\mathfrak{m}$ is* subnormally independent*, that is, if M is a proper subnormal subgroup of a group $X \in \mathfrak{m}$, then $M \notin \mathfrak{m}$;*
2. *if $X \in \mathfrak{m}$, then X is comonolithic.*

The maps b and h bear the same relation to the closure operation s_n as the maps b and h of Section 2.3 bear to the closure operation Q. The following theorem is the Fitting class version of Theorem 2.3.7.

Theorem 2.4.12 ([DH92, XI, 4.4]).

1. *If $\mathfrak{F}$ is a Fitting class, then* $\mathrm{h}\big(\mathrm{b}(\mathfrak{F})\big) = \mathfrak{F}$.
2. *If $\mathfrak{B}$ is a boundary of a Fitting class, then* $\mathrm{b}\big(\mathrm{h}(\mathfrak{B})\big) = \mathfrak{B}$.
3. *If $\mathfrak{F}$ is a Fitting class such that $\mathfrak{F} = \mathfrak{F}\mathfrak{S}$, then $\mathrm{b}(\mathfrak{F})$ is a preboundary of perfect groups and if $\mathfrak{B}$ is a preboundary of perfect groups, then $\mathrm{h}(\mathfrak{B})$ is a Fitting class such that $\mathrm{h}(\mathfrak{B}) = \mathrm{h}(\mathfrak{B})\mathfrak{S}$*

Therefore if $\mathfrak{T}$ is Fitting class, then $\mathfrak{T}\mathfrak{S} = \mathfrak{T}$ if and only if $\mathrm{b}(\mathfrak{F})$ is a preboundary of perfect groups.

Lemma 2.4.13. *Let E be a comonolithic perfect group. Then*

$$\mathrm{N}(E) = [E, \mathrm{Cosoc}(E)]$$

is the smallest normal subgroup of E contained in $\mathrm{Cosoc}(E)$ *such that*

$$\mathrm{Cosoc}(E)/\,\mathrm{N}(E) = \mathrm{Z}\big(E/\,\mathrm{N}(E)\big).$$

Proof. Put $M = \mathrm{Cosoc}(E)$. Observe first that $\mathrm{N}(E)$ is a normal subgroup of E such that $N = \mathrm{N}(E) \leq M$ and $M/N \leq \mathrm{Z}(E/N)$. Since E is perfect, we have that $M/N = \mathrm{Z}(E/N)$ by the maximality of M. Let N_1 be a normal subgroup of G such that $N_1 \leq M$ and $M/N_1 = \mathrm{Z}(E/N_1)$. Then $[E, M] = \mathrm{N}(E)$ is contained in N_1. ☐

Hence, if E is a comonolithic perfect group, then $E/\,\mathrm{N}(E)$ is quasisimple.

Definition 2.4.14. *Let $\mathfrak{F}$ be a Fitting class. A comonolithic perfect subnormal subgroup E of a group G is said to be an* $\mathfrak{F}$-component *of G if $E \notin \mathfrak{F}$ and* $\mathrm{N}(E) = [E, \mathrm{Cosoc}(E)] \in \mathfrak{F}$.

The subgroup generated by of all $\mathfrak{F}$-components of G is denoted by $\mathrm{E}_{\mathfrak{F}}(G)$.

Note that for the trivial Fitting class $\mathfrak{F} = (1)$, we have that the (1)-components of any group G are exactly the usual components and $\mathrm{E}_{(1)}(G) = \mathrm{E}(G)$ (see Definition 2.2.18 (2) and Definition 2.2.21).

Definitions and notation 2.4.15. *Let G be a group and $\mathfrak{m}$ a preboundary. We denote*

1. $\mathrm{b}_{\mathfrak{m}}(G)$ *for the set of all subnormal subgroups X of G such that $X \in \mathfrak{m}$.*
2. $\mathrm{E}_{\mathfrak{m}}(G)$ *for the subgroup generated by all subnormal subgroups X of G such that $X \in \mathrm{b}_{\mathfrak{m}}(G)$.*

If $\mathfrak{F}$ is a Fitting class such that $\mathfrak{F}\mathfrak{S} = \mathfrak{F}$ and X is an $\mathfrak{F}$-component of a group G, then X is a comonolithic perfect subnormal subgroup such that $\mathrm{N}(X) \leq X_{\mathfrak{F}} \leq \mathrm{Cosoc}(X)$. However, $X_{\mathfrak{F}} = \mathrm{Cosoc}(X)$, since $\mathrm{Cosoc}(X)/\,\mathrm{N}(X)$ is abelian. In other words, $X \in \mathrm{b}(\mathfrak{F})$. Therefore if $\mathfrak{m} = \mathrm{b}(\mathfrak{F})$, then $\mathrm{E}_{\mathfrak{m}}(G) = E_{\mathfrak{F}}(G)$, for every group G, and $\mathrm{b}_{\mathfrak{m}}(G)$ is the set of all $\mathfrak{F}$-components of G.

W. Anderson introduced the concept of Fitting sets in a successful attempt to localise the theory of Fitting classes to individual groups. He could adapt the general method of B. Fischer, W. Gaschütz, and B. Hartley to prove the existence of injectors, for Fitting sets, in each soluble group (see [DH92, VIII, 2.9]). In the proofs of both theorems, a lemma due to B. Hartley involving Carter subgroups turns out to be crucial (see [DH92, VIII, 2.8]). I. Hawthorn published in [Haw98] a completely original proof which only depends on some easy results on strongly closed p-subgroups. We present here this proof of the fundamental result of B. Fischer, W. Gaschütz, B. Hartley, and W. Anderson and we even go a bit further.

Definition 2.4.16. *Let G be a group. A* Fitting set *$\mathcal{F}$ of G is a non-empty set of subgroups of G such that*

1. *if $H \in \mathcal{F}$ and $g \in G$, then $H^g \in \mathcal{F}$,*
2. *if $H \in \mathcal{F}$ and S is a subnormal subgroup of H, then $S \in \mathcal{F}$, and*
3. *if N_1 and N_2 are normal $\mathcal{F}$-subgroups of the product N_1N_2, then $N_1N_2 \in \mathcal{F}$.*

If $\mathfrak{F}$ is a Fitting class and G is a group, then the set $\mathrm{Tr}_{\mathfrak{F}}(G) = \{H \leq G : H \in \mathfrak{F}\}$ (which is called the *trace* of $\mathfrak{F}$ in G) of all $\mathfrak{F}$-subgroups of G is a Fitting set of G. But not every Fitting set arise in this manner (see [DH92, VIII, 2.2]).

Definition 2.4.17. *If $\mathcal{F}$ is a Fitting set of G, then the subgroup*

$$G_{\mathcal{F}} = \langle S : S \text{ is a subnormal } \mathcal{F}\text{-subgroup of } G\rangle$$

is a normal $\mathcal{F}$-subgroup of G and it is called the $\mathcal{F}$-radical *of G (see [DH92, VIII, 2.4]).*

Remark 2.4.18. Let $\mathcal{F}$ be a Fitting set of a group G. If $H \leq G$, then the set

$$\mathcal{F}_H = \{S \leq H : S \in \mathcal{F}\}$$

is a Fitting set of H. When there is no danger of confusion we shall usually denote $\mathcal{F}_H$ simply by $\mathcal{F}$.

Definitions 2.4.19. *Let $\mathcal{F}$ be a non-empty set of subgroups of a group G.*

1. *The subgroups in $\mathcal{F}$ are called* $\mathcal{F}$-subgroups *of G. An $\mathcal{F}$-subgroup is said to be* $\mathcal{F}$-maximal *in G if for any $\mathcal{F}$-subgroup T such that $S \leq T$, we have that $S = T$.*
2. *An $\mathcal{F}$-subgroup S is said to be an $\mathcal{F}$-injector of G if $S \cap N$ is $\mathcal{F}$-maximal in N for any subnormal subgroup N of G.*

The, possibly empty, set of $\mathcal{F}$-injectors of a group G will be denoted by $\mathrm{Inj}_{\mathcal{F}}(G)$.

If $\mathfrak{F}$ is a Fitting class, we talk about $\mathfrak{F}$-maximal subgroups and of $\mathfrak{F}$-injectors. The, possibly empty, set of $\mathfrak{F}$-injectors of a group G will be denoted by $\mathrm{Inj}_{\mathfrak{F}}(G)$.

Definitions 2.4.20. *A set of subgroups $\mathcal{F}$ of a group G is said to be an* injective set *if G possesses $\mathcal{F}$-injectors.*

A class $\mathfrak{F}$ of groups is said to be an injective class *in a universe $\mathfrak{X}$ if every group $G \in \mathfrak{X}$ possesses $\mathfrak{F}$-injectors.*

Definition 2.4.21. *Let G be a group and p a prime. Consider a p-subgroup P_0 of G and suppose that $P_0 \leq P$, for $P \in \mathrm{Syl}_p(G)$. We say that P_0 is* strongly closed in P with respect to G, *if $P_0^g \cap P \leq P_0$, for all $g \in G$.*

Remark 2.4.22. Let G be a group and p a prime. Let P_0 be a p-subgroup of G such that $P_0 \leq P \in \mathrm{Syl}_p(G)$. Suppose that P_0 is strongly closed in P with respect to G. Then:

1. P_0 is a normal subgroup of P.
2. $P_0 \cap \mathrm{O}_p(G)$ is a normal subgroup of G.

Lemma 2.4.23. *Let G be a group and p a prime. Let P_0 be a p-subgroup of G such that $P_0 \leq P \in \mathrm{Syl}_p(G)$. Suppose that P_0 is strongly closed in P with respect to G.*

1. *If $P_0 \leq P^x$, for some $x \in G$, then P_0 is strongly closed in P^x with respect to G.*
2. *If N is a normal subgroup of G, then P_0N/N is strongly closed in PN/N with respect to G/N.*

Proof. 1. Observe that $P_0^{x^{-1}} = P_0^{x^{-1}} \cap P \leq P_0$. Hence $x \in N_G(P_0)$. If $g \in G$, we have
$$P_0^g \cap P^x = (P_0^{gx^{-1}} \cap P)^x \leq P_0^x = P_0.$$
This means that P_0 is strongly closed in P^x with respect to G.

2. Observe that, for each $g \in G$, there exists an element $x \in N$ such that
$$P_0^g \cap PN = P_0^g \cap P^x = (P_0^{gx^{-1}} \cap P)^x \leq P_0^x \leq P_0N.$$
The assertion easily follows. □

Lemma 2.4.24 (M. E. Harris, [Har72]). *Let G be a soluble group and p a prime. Let P_0 be a p-subgroup of G such that $P_0 \leq P \in \mathrm{Syl}_p(G)$. If P_0 is strongly closed in P with respect to G, then P_0 is a normally embedded subgroup of G* (see [DH92, Section I, 7]).

Proof. We use induction on the order of G. If M is a non-trivial normal subgroup of G, for any $H \leq G$ we write $\bar{H}$ to denote the subgroup HM/M of the quotient group $\bar{G} = G/M$.

By Lemma 2.4.23 (2), we have that $\bar{P}_0$ is strongly closed in $\bar{P}$ with respect to $\bar{G}$. By induction, the subgroup $\bar{P}_0$ is normally embedded in $\bar{G}$, that is, there exists a normal subgroup $\bar{N}$ of $\bar{G}$, such that $\bar{P} \cap \bar{N} = \bar{P}_0$. This means that there exists a normal subgroup N of G such that $P_0M = (P \cap N)M$. Then $P \cap P_0M = P \cap (P \cap N)M = (P \cap N)(P \cap M)$.

If either $\mathrm{Core}_G(P_0) \neq 1$ or $\mathrm{O}_{p'}(G) \neq 1$, then put either $M = \mathrm{Core}_G(P_0)$ or $M = \mathrm{O}_{p'}(G)$. In this case, $P_0 = P \cap N$ and the assertion follows. Hence we may assume that $\mathrm{Core}_G(P_0) = \mathrm{O}_{p'}(G) = 1$. Then, by [KS04, 6.4.4], we have that $\mathrm{C}_G\big(\mathrm{O}_p(G)\big) \leq \mathrm{O}_p(G)$ inasmuch as G is soluble. If $M = P_0 \cap \mathrm{O}_p(G) \neq 1$, then M is a non-trivial normal subgroup of G by Remark 2.4.22 (2). This contradicts $\mathrm{Core}_G(P_0) = 1$. Hence we can assume that P_0 and $\mathrm{O}_p(G)$ have trivial intersection. Since P_0 is normal in P by Remark 2.4.22 (1), it follows that $P_0 \leq \mathrm{C}_G\big(\mathrm{O}_p(G)\big) \leq \mathrm{O}_p(G)$. Hence $P_0 = 1$ and the lemma follows. □

Applying a result of P. Lockett (see [DH92, I, 7.8]) we have the following lemma.

Lemma 2.4.25. *Let G be a soluble group and p and q two primes. Let P_0 be a p-subgroup of G such that $P_0 \leq P \in \mathrm{Syl}_p(G)$ and Q_0 a q-subgroup of G such that $Q_0 \leq Q \in \mathrm{Syl}_q(G)$. If P_0 is strongly closed in P with respect to G and Q_0 is strongly closed in Q with respect to G, then there exists an element $g \in G$ such that $P_0^gQ_0 = Q_0P_0^g$.*

Theorem 2.4.26 (B. Fischer, W. Gaschütz, B. Hartley, and W. Anderson). *If G is a soluble group and $\mathcal{F}$ is a Fitting set of G, then G has a unique conjugacy class of $\mathcal{F}$-injectors.*

Proof (I. Hawthorn). We apply induction on the order of G and assume the result is true for all groups of smaller order.

Since G is soluble, there exists a prime p such that $\mathrm{O}^p(G)$ is a proper subgroup of G. By induction, $\mathrm{O}^p(G)$ possesses a unique conjugacy class of $\mathcal{F}$-injectors. Let S be one of them. Note that if $g \in G$, the subgroup S^g is also an $\mathfrak{F}$-injector of $\mathrm{O}^p(G)$ and then there exists an element $h \in \mathrm{O}^p(G)$ such that $S^g = S^h$. Consequently the Frattini argument holds and $G = \mathrm{N}_G(S)\,\mathrm{O}^p(G)$. In fact, if P is a Sylow p-subgroup of $\mathrm{N}_G(S)$, then $G = P\,\mathrm{O}^p(G)$.

Let R be the subgroup generated by the $\mathcal{F}$-subgroups of PS containing S. Since any such subgroup is subnormal in PS, we have that $R \in \mathcal{F}$.

Let T be an $\mathcal{F}$-subgroup of G such that S is contained in T. Observe that $T \cap \mathrm{O}^p(G)$ is an $\mathcal{F}$-subgroup. The $\mathcal{F}$-maximality of S in $\mathrm{O}^p(G)$ implies that $S = T \cap \mathrm{O}^p(G)$. Hence T is contained in $\mathrm{N}_G(S)$. Therefore any Sylow p-subgroup of T is conjugate in $\mathrm{N}_G(S)$ to a subgroup of P. Since $T/S \cong T\,\mathrm{O}^p(G)/\mathrm{O}^p(G)$ is a p-group, it follows that T is conjugate in $\mathrm{N}_G(S)$ to a group of the form P_0S, for some subgroup P_0 of P. Hence, all extensions of S which are elements of $\mathcal{F}$ are conjugate in $\mathrm{N}_G(S)$ to subgroups of R. In particular if G has in $\mathcal{F}$-injector, then it is conjugate to R.

It remains to show that R is an $\mathcal{F}$-injector of G. Since R is $\mathcal{F}$-maximal in G, it is enough to prove that R contains an $\mathcal{F}$-injector of M for every maximal normal subgroup M of G.

Suppose that $|G : M| = q$, q a prime, and let T be an $\mathcal{F}$-injector of M. The subgroups $T \cap M \cap \mathrm{O}^p(G) = T \cap \mathrm{O}^p(G)$ and $S \cap M \cap \mathrm{O}^p(G) = M \cap S$ are

$\mathcal{F}$-injectors of the normal subgroup $M \cap \mathrm{O}^p(G)$. Therefore they are conjugate in $M \cap \mathrm{O}^p(G)$. Choose T in such a way that $T \cap \mathrm{O}^p(G) = M \cap S = U$. Let $P_1 \in \mathrm{Syl}_p(T)$ and $Q_1 \in \mathrm{Syl}_q(S)$ so that $T = P_1U$ and $S = Q_1U$. Since S and T are subgroups of $\mathrm{N}_G(U)$, there exist a Sylow p-subgroup P of $\mathrm{N}_G(U)$ such that $P_1 \le P$ and a Sylow q-subgroup Q of $\mathrm{N}_G(U)$ such that $Q_1 \le Q$. If $g \in \mathrm{N}_G(U)$, then $(P_1^g \cap P)U \le T^g \in \mathcal{F}$. Since $(P_1^g \cap P)U$ and T are subnormal subgroups of PU, we have that $\langle P_1^g \cap P, P_1\rangle U$ is an $\mathcal{F}$-subgroup of PU. Moreover $T \le \langle P_1^g \cap P, P_1\rangle U \le \langle T^g, T\rangle \le M$. The $\mathcal{F}$-maximality of T in M yields $P_1^g \cap P \le P_1$. This is to say that P_1 is strongly closed in P with respect to $\mathrm{N}_G(U)$. Analogously it can be shown that Q_1 is strongly closed in Q with respect to $\mathrm{N}_G(U)$. By Lemma 2.4.25, there exists an element $g \in \mathrm{N}_G(U)$ such that the product $P_1^gQ_1$ is a subgroup of $\mathrm{N}_G(U)$.

Consider the subgroup $K = P_1^gQ_1U = (P_1U)^g(Q_1U) = T^gS$. Observe that $K \cap \mathrm{O}^p(G) = T^gS \cap \mathrm{O}^p(G) = \bigl(T^g \cap \mathrm{O}^p(G)\bigr)S = \bigl(T \cap \mathrm{O}^p(G)\bigr)^gS = U^gS = US = S$ and similarly $K \cap M = T^g$. Hence S and T^g are normal $\mathfrak{F}$-subgroups of K and therefore K is an $\mathcal{F}$-group. Since S is contained in K, we have that R contains a conjugate of K. This concludes the proof. $\square$

Theorem 2.4.27. *Let $\mathcal{F}$ be a Fitting set of a group G such that $G/G_{\mathcal{F}}$ is soluble. Then G has a unique conjugacy class of $\mathcal{F}$-injectors.*

Proof. Denote $N = G_{\mathcal{F}}$. The set $\mathcal{F}^* = \{H/N : H \in \mathcal{F}, N \le H\}$ is a Fitting set of the soluble group G/N. Moreover, using the arguments of [DH92, VIII, 2.17 (a)], we have that

$$\mathcal{F}_0 = \{S \le G : SN/N \in \mathcal{F}^* \text{ and } S \text{ is subnormal in } SN\}$$

is a Fitting set of G . Observe that $\mathcal{F}_0 \subseteq \mathcal{F}$ and for any subnormal subgroup S of G, we have that $S_{\mathcal{F}_0} = S_{\mathcal{F}}$.

We apply now the arguments of [DH92, VIII, 2.17 (b)], which hold in the non-soluble case, to conclude that if V/N is an $\mathcal{F}^*$-injector of G/N, then V is an $\mathcal{F}_0$-injector of G. We claim that, indeed, V is an $\mathcal{F}$-injector of G. To see that, we prove that for any subnormal subgroup S of G, the subgroup $V \cap S$ is $\mathcal{F}$-maximal in S. Suppose that there exists $W \in \mathcal{F}$ such that $V \cap S \le W \le S$. Then $(V \cap S)N/N = (V/N) \cap (SN/N) \le WN/N \le SN/N$. Since $S_{\mathcal{F}} = S_{\mathcal{F}_0} \le V \cap S \in \mathrm{Inj}_{\mathcal{F}_0}(S)$, then $S_{\mathcal{F}} \le W$. Recall that $N \cap S = S_{\mathcal{F}}$, by [DH92, VIII, 2.4 (d)]. Therefore $WN \cap S = W(N \cap S) = WS_{\mathcal{F}} = W$. Hence W is subnormal in WN and then $WN \in \mathcal{F}$. Consequently, $WN/N \in \mathcal{F}^*$. Since $(V/N) \cap (SN/N)$ is $\mathcal{F}^*$-maximal in SN/N, we have that $(V \cap S)N = WN$. This implies that

$$V \cap S = (V \cap S)(N \cap S) = (V \cap S)N \cap S = WN \cap S = W,$$

and then $V \cap S$ is $\mathcal{F}$-maximal in S. Thus, we deduce that $V \in \mathrm{Inj}_{\mathcal{F}}(G)$ as claimed.

On the other hand, applying [DH92, VIII, 2.15], if $V \in \mathrm{Inj}_{\mathcal{F}}(G)$, then V/N is an $\mathcal{F}^*$-injector of the soluble group G/N. By Theorem 2.4.26, the $\mathcal{F}^*$-injectors of G/N are conjugate in G/N. Consequently the $\mathcal{F}$-injectors of G form a conjugacy class of subgroups of G. $\square$

Corollary 2.4.28 ([BCMV84]). *If $\mathfrak{F}$ is a Fitting class, every group in $\mathfrak{F}\mathfrak{S}$ has a unique conjugacy class of $\mathfrak{F}$-injectors.*

One line taken in the study of Fitting classes in the soluble universe has been their classification according to the embedding properties of their injectors, and in this direction the pursuit of those with normal injectors has been especially fruitful. In this context the following definition makes sense.

Definition 2.4.29. *Let $\mathfrak{X}$ be a class of groups which is closed under taking subnormal subgroups, and let $1 \neq \mathfrak{F}$ be a Fitting class contained in $\mathfrak{X}$.*

1. *We say that $\mathfrak{F}$ is* normal *in $\mathfrak{X}$ or $\mathfrak{X}$-*normal *if $G_{\mathfrak{F}}$ is $\mathfrak{F}$-maximal in G for all $G \in \mathfrak{X}$.*
2. *$\mathfrak{F}$ is said to be* dominant *in $\mathfrak{X}$ or $\mathfrak{X}$-*dominant *if for all $H \in \mathfrak{X}$ any two $\mathfrak{F}$-maximal subgroups of H containing $H_{\mathfrak{F}}$ are conjugate in H.*

If $\mathfrak{X} = \mathfrak{E}$, we simply say that $\mathfrak{F}$ is a normal *(respectively* dominant*) Fitting class.*

It is clear that if $\mathfrak{F}$ is $\mathfrak{X}$-normal, then every group G has a unique $\mathfrak{F}$-injector, namely the $\mathfrak{F}$-radical. Moreover, applying [DH92, IX, 4.2], if $\mathfrak{F}$ is $\mathfrak{X}$-dominant, then every $\mathfrak{X}$-group has a unique conjugacy class of $\mathfrak{F}$-injectors, namely the $\mathfrak{F}$-maximal subgroups of H containing $H_{\mathfrak{F}}$.

The first investigation in normal Fitting classes was carried out by D. Blessenohl and W. Gaschütz in [BG70]. They quickly settle the question of which Schunck classes of soluble groups have normal projectors — these turn out to be the classes of all π-perfect groups (the projector in G being $\mathrm{O}^{\pi}(G)$) — and then go to lay the foundations for the much more complex dual theory (see [DH92, X, Section 3]).

2.5 Fitting formations

We have seen that many of the examples of Fitting classes are formations too. Naturally such classes are called *Fitting formations*. The class $\mathfrak{N}$, of all nilpotent groups, the classes $\mathfrak{E}_{\pi}$, of all π-groups, the class $\mathfrak{E}_{\pi}\mathfrak{E}_{\pi'}$, of all groups with a normal Hall π-subgroup, for any set π of prime numbers, are examples of Fitting formations.

We will be interested in the following example:

Example 2.5.1. Let $\mathcal{I}$ be a non-empty set. For each $i \in \mathcal{I}$, let $\mathfrak{F}_i$ be a subgroup-closed Fitting formation. Assume that $\pi(\mathfrak{F}_i) \cap \pi(\mathfrak{F}_j) = \emptyset$ for all $i, j \in \mathcal{I}$, $i \neq j$. Then $\mathsf{X}_{i \in \mathcal{I}} \mathfrak{F}_i$ is a saturated Fitting formation (see Remark 2.2.13).

The most remarkable milestone in the theory of Fitting formations was settled by R. A. Bryce and J. Cossey in 1982.

Theorem 2.5.2 (R. A. Bryce and J. Cossey, [BC82]). *Every subgroup-closed Fitting class of finite soluble groups is a saturated formation.*

The way towards the proof of this impressive result started ten years before. In [BC72] the same authors proved the following.

Theorem 2.5.3 (R. A. Bryce and J. Cossey, [BC72]). *Every subgroup-closed Fitting formation of finite soluble groups is saturated.*

An outline of the proof of these two results appears in Chapter XI of [DH92].

Unfortunately the above theorem is not true in the general universe of all finite groups as it is pointed out in [DH92, IX, 1.6]. In [BBE98], the authors found necessary and sufficient conditions for a subgroup-closed Fitting formation to be saturated.

Theorem 2.5.4 ([BBE98]). *For a subgroup-closed Fitting formation $\mathfrak{F}$ the following are equivalent:*

1. *If $G \in \mathfrak{F}$ is a primitive group of type 2 and E_p is the maximal Frattini extension of G with p-elementary abelian kernel, then $E_p \in \mathfrak{F}$, for every prime p dividing $|\mathrm{Soc}(G)|$,*
2. *$\mathfrak{F}$ is saturated.*

Up to now, no classification of the Fitting formations is known. However many of the known Fitting formations are gathered in two types: solubly saturated Fitting formations and Fitting formations defined by Fitting families of modules.

The search for a soluble non-saturated Fitting formation led to T. O. Hawkes to the introduction of what he called (see [Haw70]) the class of p-soluble groups, p a prime, whose absolute arithmetic p-rank is a p'-number. After that, and extending Hawkes's methods, many examples of soluble non-saturated Fitting formations have been introduced by different authors. The method presented by J. Cossey and C. Kanes in [CK87] and modified by Cossey in [Cos89] includes all previous constructions. Motivated by the local (or Baer) functions, the criterion to decide whether a particular p-soluble group belongs to one of these Cossey-Kanes classes is defined by imposing some conditions of a certain class of modules associated with the p-chief factors. That the classes so defined are Fitting formations is a consequence of the closure properties of the family of modules.

Definition 2.5.5. *Let K be a field. We associate with each group G of a suitable universe $\mathfrak{V}$ a class $\mathfrak{M}(G)$ of irreducible KG-modules. The class $\mathfrak{M} = \bigcup_G \mathfrak{M}(G)$ is said to be a* Fitting family of modules over K *if it satisfies the following properties:*

1. *If $V \in \mathfrak{M}(G)$ and N is a normal subgroup of G such that $N \leq \mathrm{C}_G(V)$, then V, regarded in the natural way as a $K(G/N)$-module, is in $\mathfrak{M}(G/N)$.*

2. *If $V \in \mathfrak{M}(H)$ and H is an epimorphic image of G, then V, regarded in the natural way as a KG-module, is in $\mathfrak{M}(G)$.*
3. *If $V \in \mathfrak{M}(G)$, N is a subnormal subgroup of G and U is an irreducible constituent of V_N, then $U \in \mathfrak{M}(N)$.*
4. *If N_1 and N_2 are normal subgroups of G such that $G = N_1N_2$ and V is an irreducible KG-module such that all composition factors of V_{N_i} are in $\mathfrak{M}(N_i)$, for $i = 1, 2$, then $V \in \mathfrak{M}(G)$.*

Clearly if $\mathfrak{M}(G)$ is non-empty, then the trivial KG-module K_G is in $\mathfrak{M}(G)$.

With this definition we can construct Fitting formations with the following procedure.

Theorem 2.5.6. *Fix a prime r. Let K be an extension field of $k = \mathrm{GF}(r)$. For any r-soluble group G, we denote $\mathfrak{T}_K(G)$ the class of all irreducible KG-modules V such that V is a composition factor of the module $W^K = W \otimes K$, where W is an r-chief factor of G. If, for every r-soluble group G, a class of irreducible KG-modules $\mathfrak{M}(G)$ is defined, and $\mathfrak{M} = \bigcup_G \mathfrak{M}(G)$, the class*

$$\mathfrak{T}(1, \mathfrak{M}) = \big(G : G \text{ is } r\text{-soluble and } \mathfrak{T}_K(G) \subseteq \mathfrak{M}(G)\big)$$

is a Fitting formation provided $\mathfrak{M}$ is a Fitting family in the r-soluble universe.

A proof of this theorem is presented in [DH92, IX, 2.18].

Thus, given a Fitting family $\mathfrak{M}$ in the r-soluble universe, r a prime, we have a way to distinguish between the abelian chief factors of a soluble group G: an r-chief factor M of G can be such that all composition factors of M^K are in $\mathfrak{M}(G)$ or not.

The family of modules proposed by J. Cossey and C. Kanes [CK87] is motivated by the class of characters, called π-factorable characters, introduced by I. M. Isaacs in [Isa84]. D. Gajendragadkar introduced in [Gaj79] the idea of π-special characters and established their basic properties. This idea was considerably developed and refined by Isaacs. The definition of π-special modules is derived from the definition of π-special characters and the properties are similar to those of Isaacs and Gajendragadkar.

We therefore specify that *for the rest of this section all groups considered are soluble.*

Definition 2.5.7. *Let K be an algebraically closed field of characteristic $r > 0$, π a set of primes, and G a group.*

1. *An irreducible KG-module V is called* π-special *if the dimension of V is a π-number and whenever S is a subnormal subgroup of G and U is a composition factor of V_S, then* $\det(x \,\mathrm{on}\, U) = 1$ *for all π'-elements x of S.*
2. *Suppose that $\mathcal{P} = \{\pi_i : i \in \mathcal{I}\}$ is a partition of $\mathbb{P}$, the set of all primes. An irreducible KG-module V is called* $\mathcal{P}$-factorable *if $V = U_{j_1} \otimes \cdots \otimes U_{j_n}$ for some π_{j_i}-special modules U_{j_i}, $\pi_{j_i} \in \mathcal{P}$, $i = 1, \ldots, n$, and $j_i \neq j_k$ when $i \neq k$.*

3. *An irreducible KG-module V is called* π-factorable *if V is $\mathcal{P}$-factorable for $\mathcal{P} = \{\pi, \pi'\}$.*

It turns out that if U and W are respectively π-special and π'-special irreducible KG-modules, then $U \otimes W$ is irreducible. Moreover, if U' and W' are respectively π-special and π'-special irreducible KG-modules, and $U \otimes W \cong U' \otimes W'$, then $U \cong U'$ and $W \cong W'$ (see [CK87, 2.4] for more details and notation). It is also true the following:

Lemma 2.5.8 ([CK87, 2.2]). *Let G be a group, K a field, π a set of primes and V be π-special KG-module. If S is a subnormal subgroup of G, then every irreducible constituent of V_S is π-special.*

The next lemma equips us with the basic arguments to prove closure properties of "Fitting type." Its proof is rather technical and can be seen in [CK87].

Lemma 2.5.9. *Let G be a group, K an algebraically closed field and $\mathcal{P} = \{\pi_i : i \in \mathcal{I}\}$ a partition of the set $\mathbb{P}$ of all primes.*

1. *If V is a $\mathcal{P}$-factorable KG-module and N is a normal subgroup of G, then any irreducible KN-submodule of V_N is a $\mathcal{P}$-factorable KN-module.*
2. *Suppose that M and N are normal subgroups of G such that $G = MN$. Let V be an irreducible KG-module such that all irreducible KM-submodules of V_M and all irreducible KN-submodules of V_N are $\mathcal{P}$-factorable. Then V is a $\mathcal{P}$-factorable KG-module.*

And now we prove the main result.

Theorem 2.5.10. *Let K be an algebraically closed field of characteristic a prime p. Let $\mathcal{P} = \{\pi_i : i \in \mathcal{I}\}$ be a partition of the set $\mathbb{P}$ of all primes. For each $i \in \mathcal{I}$, let $\mathfrak{X}_i$ be a Fitting formation. Denote $\mathcal{X} = \{\mathfrak{X}_i : i \in \mathcal{I}\}$. For every soluble group G, denote by $\mathfrak{M}(G)$ the class of all irreducible $\mathcal{P}$-factorable KG-modules V such that $V = V_1 \otimes \cdots \otimes V_{n(V)}$. Suppose further that*

1. *V_j is a $\pi_{i(j)}$-special KG-module, and*
2. *$G/\mathrm{C}_G(V_j) \in \mathfrak{X}_{i(j)}$, for $j = 1, \ldots, n(V)$.*

Then $\mathfrak{M} = \mathfrak{M}(K, \mathcal{P}, \mathcal{X}) = \bigcup_G \mathfrak{M}(G)$ is a Fitting family in the universe $\mathfrak{S}$.

Proof. 1. Let G be a group, let N be a normal subgroup of G, and let V be a KG-module in $\mathfrak{M}(G)$ such that $N \leq \mathrm{C}_G(V)$. Suppose that $V = V_1 \otimes \cdots \otimes V_n$ is a $\mathcal{P}$-factorisation of V where V_j is a $\pi_{i(j)}$-special KG-module and $G/\mathrm{C}_G(V_j) \in \mathfrak{X}_{i(j)}$, for each $j = 1, \ldots, n$. Then $\mathrm{C}_G(V) = \bigcap_{j=1}^n \mathrm{C}_G(V_j)$, and so $N \leq \mathrm{C}_G(V_j)$ for each j. Consider V_j as a $K(G/N)$-module. It is clear that V_j is also a $\pi_{i(j)}$-special $K(G/N)$-module. Therefore the $K(G/N)$-module V is $\mathcal{P}$-factorable. Since $(G/N)/\mathrm{C}_{G/N}(V_j) \cong G/\mathrm{C}_G(V_j)$, we have that $V \in \mathfrak{M}(G/N)$.

2. Let G and H be two groups such that $\varphi\colon G \longrightarrow H$ is an epimorphism, and consider an KH-module $V \in \mathfrak{M}(H)$. Suppose that $V = V_1 \otimes \cdots \otimes V_n$ is a $\mathcal{P}$-factorisation of V where V_j is a $\pi_{i(j)}$-special KH-module and $H/\operatorname{C}_H(V_j) \in \mathfrak{X}_{i(j)}$, for each $j = 1, \dots, n$. Each V_j is considered as a KG-module via φ. Let S be a subnormal subgroup of G. Since φ is an epimorphism, the image S^φ is a subnormal subgroup of H. Moreover, U is a composition factor of $(V_j)_S$ if and only if U is a composition factor of $(V_j)_{S^\varphi}$. For any $\pi(j)'$-element x of S, we have that x^φ is a $\pi(j)'$-element of S^φ and $\det(x \operatorname{on} U) = \det(x^\varphi \operatorname{on} U) = 1$. Therefore the KG-module V_j is $\pi_{i(j)}$-special and V is a $\mathcal{P}$-factorable KG-module. Finally, observe that $\operatorname{Ker}(\varphi) \leq \operatorname{C}_G(V) = \bigcap_{j=1}^n \operatorname{C}_G(V_j)$. Then $G/\operatorname{C}_G(V_j) \cong \bigl(G/\operatorname{Ker}(\varphi)\bigr)/\bigl(\operatorname{C}_G(V_j)/\operatorname{Ker}(\varphi)\bigr) \cong H/\operatorname{C}_H(V_j) \in \mathfrak{X}_{i(j)}$. Therefore $V \in \mathfrak{M}(G)$.

3. Let G be a group, N be a normal subgroup of G and V a KG-module in $\mathfrak{M}(G)$. Let U be an irreducible KN-submodule of V_N. Then U is a $\mathcal{P}$-factorable KN-module by Lemma 2.5.9 (1). In fact, if $V = V_1 \otimes \cdots \otimes V_n$ is a $\mathcal{P}$-factorisation of V where V_j is a $\pi_{i(j)}$-special KG-module then there exists a KN-submodule U_j of V_j such that $U = U_1 \otimes \cdots \otimes U_n$ is a $\mathcal{P}$-factorisation of U where each U_j is a $\pi_{i(j)}$-special KN-module, $1 \leq j \leq n$. Since each $\mathfrak{X}_{i(j)}$ is a Fitting class and $G/\operatorname{C}_G(V_j) \in \mathfrak{X}_{i(j)}$, then the normal subgroup $N\operatorname{C}_G(V_j)/\operatorname{C}_G(V_j)$ is in $\mathfrak{X}_{i(j)}$. This is to say that $N/\operatorname{C}_N(V_j) \in \mathfrak{X}_{i(j)}$. Since U_j is a KN-submodule of V_j, we have that $\operatorname{C}_N(V_j)$ is a normal subgroup of $\operatorname{C}_N(U_j)$ and then $N/\operatorname{C}_N(U_j)$ is an epimorphic image of $N/\operatorname{C}_N(V_j)$. Since each $\mathfrak{X}_{i(j)}$ is also a formation, we have that $N/\operatorname{C}_N(U_j) \in \mathfrak{X}_{i(j)}$. Therefore we deduce that $U \in \mathfrak{M}(N)$.

4. Let G be a group and suppose that M and N are normal subgroups of G such that $G = MN$. Let V be an irreducible KG-module such that all irreducible KM-submodules of V_M are in $\mathfrak{M}(M)$ and all irreducible KN-submodules of V_N are in $\mathfrak{M}(N)$. By Lemma 2.5.9 (2), V is $\mathcal{P}$-factorable KG-module. Suppose that $V = V_1 \otimes \cdots \otimes V_n$ is a $\mathcal{P}$-factorisation of V where V_j is a $\pi_{i(j)}$-special KG-module. By Clifford's theorem [DH92, B, 7.3], $(V_j)_M$ and $(V_j)_N$ are completely reducible. Suppose that $(V_j)_M = Z_{j,1} \oplus \cdots \oplus Z_{j,r(j)}$ is a decomposition of $(V_j)_M$ in irreducible KM-submodules. By Lemma 2.5.8 every $Z_{j,t}$ is a $\pi_{i(j)}$-special KM-module. Therefore

$$V_M = \bigoplus_{k(1)=1}^{r(1)} \cdots \bigoplus_{k(n)=1}^{r(n)} \bigl[Z_{1,k(1)} \otimes \cdots \otimes Z_{n,k(n)}\bigr]$$

Any $Z_{1,k(1)} \otimes \cdots \otimes Z_{n,k(n)}$ is a $\mathcal{P}$-factorisation of an irreducible constituent of V_M. Then $Z_{1,k(1)} \otimes \cdots \otimes Z_{n,k(n)} \in \mathfrak{M}(M)$. Therefore $M/\operatorname{C}_M(Z_{j,l}) \in \mathfrak{X}_{i(j)}$ for any par (j,l). It is clear that $\operatorname{C}_M(V_j) = \bigcap_{i=1}^{r(j)} \operatorname{C}_M(Z_{j,i})$. Since $\mathfrak{X}_{i(j)}$ is a formation, the group $M/\operatorname{C}_M(V_j)$ is in $\mathfrak{X}_{i(j)}$. We can argue analogously with V_N and deduce that $N/\operatorname{C}_N(V_j) \in \mathfrak{X}_{i(j)}$. Moreover since

$$M\operatorname{C}_N(V_j)/\operatorname{C}_M(V_j)\operatorname{C}_N(V_j) \cong M/\operatorname{C}_M(V_j) \in \mathfrak{X}_{i(j)}$$

and

$$N\,\mathrm{C}_M(V_j)/\,\mathrm{C}_M(V_j)\,\mathrm{C}_N(V_j) \cong N/\,\mathrm{C}_N(V_j) \in \mathfrak{X}_{i(j)},$$

and $\mathfrak{X}_{i(j)}$ is a Fitting class, we have that

$$G/\,\mathrm{C}_M(V_j)\,\mathrm{C}_N(V_j) = \\ [M\,\mathrm{C}_N(V_j)/\,\mathrm{C}_M(V_j)\,\mathrm{C}_N(V_j)][N\,\mathrm{C}_M(V_j)/\,\mathrm{C}_M(V_j)\,\mathrm{C}_N(V_j)]$$

is in $\mathfrak{X}_{i(j)}$. Finally, notice that $\mathrm{C}_M(V_j)\,\mathrm{C}_N(V_j)$ is a normal subgroup of $\mathrm{C}_G(V_j)$ and then $G/\,\mathrm{C}_G(V_j)$ is isomorphic to a quotient group of $G/\,\mathrm{C}_M(V_j)\,\mathrm{C}_N(V_j)$. Since $\mathfrak{X}_{i(j)}$ is a formation, we have that $G/\,\mathrm{C}_G(V_j) \in \mathfrak{X}_{i(j)}$. This implies that $V \in \mathfrak{M}(G)$. □

Examples and remarks 2.5.11. The Cossey-Kanes construction covers many of the known constructions of Fitting formations. For instance:

1. Let p be a prime and K an algebraically closed field of characteristic p. If $\mathcal{P} = \{\pi_1 = \{p\}, \pi_2 = p'\}$ and $\mathcal{X} = \{\mathfrak{X}_1 = (1), \mathfrak{X}_2 = \mathfrak{S}\}$, then $\mathfrak{M}^p = \mathfrak{M}(K, \mathcal{P}, \mathcal{X})$ is a Fitting family of modules in the universe $\mathfrak{S}$. The Fitting formation $\mathfrak{T} = \mathfrak{T}(\mathfrak{M}^p)$ is the one introduced by Hawkes in [Haw70].
2. The Fitting formations studied by K. L. Haberl and H. Heineken ([HH84] or [DH92, IX, 2.26]) are constructed using a not necessarily algebraically closed field K. Nevertheless they can also be included in the Cossey-Kanes construction thanks to a modification made by Cossey in [Cos89]. According to [HH84, 4.1], every Haberl-Heineken Fitting formation can be seen as a Fitting formation constructed by the Cossey-Kanes method with $\mathfrak{X}_1 = \mathfrak{S}, \mathfrak{X}_2 = (1)$.
3. The non-saturated Fitting formations introduced by T. R. Berger and J. Cossey in [BC78] are defined in terms of the Cossey-Kanes procedure. The Fitting formations of Berger-Cossey are the first examples of non-saturated Fitting formations composed of soluble groups whose p-length is less or equal to 1 for all primes p.
4. A result due to L. G. Kovács, which appears in [CK87, 4.2], characterises the saturation of the Fitting formations $\mathfrak{T}\big(\mathfrak{M}(K, \mathcal{P}, \mathcal{X})\big)$. This means that some of the Fitting formations constructed by the Cossey-Kanes procedure can be saturated.
5. Let $\mathfrak{M} = \bigcup_G \mathfrak{M}(G)$ be a Fitting family. In Theorem 2.5.6, assume that, instead of the class $\Gamma_K(G)$, we work with the class $\Delta_K(G)$ of all irreducible KG-modules V such that V is a composition factor of the module $W^K = W \otimes K$, where W is a complemented r-chief factor of G (r is a prime, K is a field with $\operatorname{char} K = r$ and G is r-soluble). Then the class

$$\mathfrak{C}(\mathfrak{M}) = \big(G : G \text{ is } r\text{-soluble and } \Delta_K(G) \subseteq \mathfrak{M}(G)\big)$$

is a Fitting class and a Schunck class (see [CO87]). Moreover, in this paper a criterion to decide which of these classes is a formation is presented.